On the approximation of D.I.Y. water rocket dynamics including air drag

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Abstract. If you want to get accurate predictions for the motion of water and air propelled D.I.Y rockets, neglecting air resistance is not an option. But the theoretical analysis including air drag leads to a system of differential equations which can only be solved numerically. We propose an approximation which simply works by the estimate of a definite integral and which is even feasible for undergraduate physics courses. The results only slightly deviate from the reference data (received by the Runge-Kutta method). The motion is divided into several flight phases that are discussed separately and the resulting equations are solved by analytic and numeric methods. The different results from the flight phases are collected and are compared to data that has been achieved by well explained and documented experiments. Furthermore, we theoretically estimate the rocket’s drag coefficient. The result is confirmed by a wind tunnel experiment.

1 Introduction

The classical rocket equation [15, eq. 2.12] is attributed to Konstantin Eduardovitch Tsiolkovsky [23]. If the propellant is exhausted with constant speed the rocket equation can easily be integrated. In case of water and air propelled rockets the exhaust speed decreases together with the internal pressure and mass of the rocket.
Physics of water rockets have been the subject of several investigations. In an early paper [17, p. 152], Nelson and Wilson claim that rocket thrust and mass as a function of time, as well as the drag coefficient must be determined experimentally. Later, Finney assumes in [9] that “the air in the rocket behaves as an ideal gas and [...] expands isothermally”. He uses Bernoulli’s equation to determine an equation for the propellant’s mass flow rate, see [9, eq. 3]. Furthermore, Finney deduces an equation for the pressure as a function of time and calculated the rocket’s burn time, see [9, eq. 7, 8]. Therewith, he proposes the estimate

\[ h = \frac{1}{8} gt^2 - \frac{D}{64 M} g^2 t^4 \]  

for the rocket’s height, where \( D = \frac{1}{2} \rho_{\text{air}} c_D A_R \) contains the parameters of the drag force \( F_D \) and \( M \) is the mass of the empty rocket, see [9, eq. 14]. As mentioned in the appendix of his paper “an adiabatic approximation would seem more natural.” Gommes slightly improves the thrust prediction for water rockets: “the gas expansion has to be modeled as an adiabatic process.”, see [10]. He also includes the fact, that “Air expansion [...] is accompanied by vapor condensation”. A more meticulous investigation of the thermodynamics of the water rocket’s thrust phase was published by Romanelli, Bove and Madina in [19]. Indeed, the value of the polytropic exponent \( n \) in \( pV^n = \text{constant} \) affects the rocket’s maximum altitude. There are several other effects that raise the inaccuracy of the predictions more than that. Prusa used \( n = 1.4 \) for dry air, see [18, p. 724]. In our paper we will take into account the enhanced value for \( n \). Since \( n \) is a constant, this doesn’t cause additional difficulties. Now let us get back to the rocket’s movement: Prusa derived an equation for the rocket’s acceleration, cf. [18, eq. 2.2], and proposed a numerical algorithm to solve it, see [18, p. 723]. Prusa neglected additional thrust from compressed air which is left at the end of the water propulsion phase. An even more accurate consideration of the water rocket physics was given by Barrio-Perott et. al. in 2010, see [3]. They worked out a set of differential equations for the water thrust, cf. [3, eq. 18-21], as well as equations for the air thrust [3, eq. 28-31]. Both articles, [18] and [3] use numerical methods to achieve the solution.

In our paper the rocket’s ascent is divided into water thrust phase, air thrust phase and upward coasting phase. We also set up the equations of motion for the rocket in Section 2. The equation that governs the gas expansion inside the rocket tank is solved analytically in Subsection 3.2, but it is not possible to analytically solve the complete system of differential equations that describe the rocket’s launch. First and foremost, we study the water thrust phase in Section 4. On the one hand, we apply the Runge-Kutta method to the corresponding initial value problem to have some proper reference data. On the other hand, we deduce a simple method to approximate the results with high accuracy: Our approximation simply works by
the calculation of a definite integral. One advantage is that this calculation can be done with a simple graphing calculator without a computer algebra system and actually is feasible for undergraduate physics courses. Nonetheless, one receives very accurate results which only slightly deviate from the reference data (received by the Runge-Kutta method). Section 5 is concerned with the air thrust phase. D.I.Y. rockets are chaotic systems and there is no way to receive analytic results. Therefore, we make reasonable assumptions to simplify the underlying equations and introduce an efficiency factor for the air thrust. In Section 6 the upward coasting phase is discussed. A collection of the previous results is given in Section 7. Subsequently, our paper contains a detailed theoretical discussion of the drag coefficient and we verify the results by experimental data from a wind tunnel experiment, see Section 8. Finally, we compare our formula of the rocket’s maximum altitude with the experimental data from our D.I.Y. rocket launch experiments.

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2 The rocket’s acceleration and exhaust velocity

2.1 The acceleration

Consider a rocket moving with velocity \( v \) in vertical direction. Let \( m \) be the mass of the rocket including its propellant at a given time \( t \). During the interval of time \( dt \), the rocket ejects the mass element \( dm \) with the exhaust speed \( v_e \). According to the law of conservation of momentum, the velocity \( v \) of the rocket thereby increases by

\[
dv = -v_e \frac{dm}{m} dt,
\]

see e.g. [15, eq. 2.11]. Consequently, the thrust acceleration is given by

\[
a_I = -\frac{dm}{dt} \cdot \frac{v_e}{m}.
\]  

(2)

Since the total mass is shrinking, i.e. \( \frac{dm}{dt} < 0 \), it is \( a_I > 0 \). A rocket within the terrestrial atmosphere additionally experiences deceleration from gravity and air drag. Both of them decrease with the rocket’s height. However, a simple water rocket does only reach a maximum altitude of a few meters\(^1\). Consequently, the altitude dependence of gravity and air drag can be neglected. For the magnitude

\(^1\)It should be mentioned that a group of scientists at the University of Cape Town built a water and air propelled rocket that reached 830 m in 2015. This is the current record, cf. http://www.wra2.org/ and https://www.news.uct.ac.za/article/-2015-10-07-uct-team-smashes-eight-year-water-rocket-world-altitude-record. Certainly, a simple home-made rocket built from a 1 or 2 liter PET bottle is not even in a position to achieve this altitude.
\[ g = GM_{\text{Earth}}r^{-2} \]

of barycentric gravitational acceleration we use \( g = 9.81 \text{ms}^{-2} \) during numerical calculations. Air resistance is modeled by the drag force \( F_D = \frac{1}{2} \rho_{\text{air}} c_D A_R v^2 \) where \( \rho_{\text{air}} \) is the density of air, \( c_D \) the drag coefficient, \( A_R \) the reference area (later we use the rocket’s cross sectional area \( A_{cs} \) perpendicular to the direction of movement), and \( v \) the rocket’s velocity. Let us adopt the abbreviation

\[ D := \frac{1}{2} \rho_{\text{air}} c_D A_R \]  

(3)

from [9]. The drag force \( F_D = Dv^2 \) leads to a deceleration

\[ a_D = \frac{F_D}{m} = \frac{D}{m} v^2. \]  

(4)

An estimate of the drag coefficient of our model rocket is given in Section 8. Deceleration from gravity and drag point on the one hand and thrust acceleration on the other hand point in opposite directions. Therefore, the rocket’s total acceleration is given by

\[ a = a_I - g - a_D = -\frac{v_e dm}{m dt} - g - \frac{D}{m} v^2. \]  

(5)

For a water rocket, the working mass (water) represents the major part of the rocket’s total mass \( m \). Anyway, the total mass strongly decreases with time. In order to integrate (5) it is necessary to know the dependence of mass \( m \) and time \( t \).

2.2 Mass and exhaust velocity

Let \( \tilde{m}(t) \) be the time dependent working mass, namely the mass water in the rocket tank. Further let \( M \) denote the constant mass of the empty rocket. The time dependent mass of the compressed air doesn’t contribute significantly to the total mass of the rocket. Despite this fact, we will take it into account. Section (5) is concerned with the additional air propulsion after the water thrust phase. During the water thrust phase we treat the mass of compressed air as a constant, say \( m_{\text{air}} \). This approach is based on the idea that all water is expelled before the air escapes. Of course, this is somehow physically unrealistic. Indeed, an mixture of water and air is expelled, which yields a non-calculable chaos, in particular towards the end of the water thrust phase. Within our model the rocket’s mass is given by \( m(t) = M + m_{\text{air}} + \tilde{m}(t) \) during the water thrust phase. Let \( \rho \) denote the density of the working mass, in case of water this is \( \rho \approx 1000 \text{kgm}^{-3} \), and \( V(t) \) its volume. \( V_b \) is the volume of the rocket’s tank and \( V(t) = V_b - \tilde{V}(t) \) the part which is filled with air (or another gas, maybe water vapor etc.). Therefore, the mass \( m \) and the differential \( dm \) read

\[ m = M + m_{\text{air}} + \rho (V_b - V(t)) \]  

\[ \Rightarrow \quad dm = -\rho dV. \]  

(6)
Thereby, the dependence of mass $m$ and time $t$ is determined by the gas expansion during the water thrust phase. Let $p_a$ denote the atmospheric pressure and $p = p(V)$ the pressure of propellant and gas inside the rocket. Initial values at rocket launch for gas pressure and volume are denoted by $p_0$ and $V_0$, respectively. Bernoulli’s equation $p = p_a + v_e^2 \rho / 2$, see [22], relates the exhaust speed $v_e$ to the pressure difference $p - p_a$:

$$v_e = \sqrt{\frac{2(p(V) - p_a)}{\rho}}. \tag{7}$$

Indeed, the barometric pressure $p_a$ slightly depends on the altitude, see [12]. At altitudes that can be reached by a water rocket, $p_a$ decreases by about $12 \text{ Pa/m} = 12^{-5} \text{ bar/m}$. Since the pressure of the propellant will usually be about a few bar we can neglect the altitude dependence of the barometric pressure. Another point is that the fluid’s velocity before leaving the nozzle is assumed to be zero in (7). Strictly speaking, we have to take into account the rejuvenation of the bottle. Consider an incompressible fluid (like water) which laminar flows from a point inside the bottle with cross-section $A_R$ through the nozzle with cross-section $A$ at velocity $w_e$. Let $\bar{w}$ denote the velocity of the fluid inside the bottle. The continuity equation leads to $\bar{w} = w_e A / A_R = w_e r^2 / R^2$ where $R$ and $r$ are the radii of bottle and nozzle, respectively. Usually, the radius of the nozzle will be small against the rocket’s radius, that is $r \ll R$. From Bernoulli’s equation $p + \left( w_e r^2 / R^2 \right)^2 \rho / 2 = p_a + w_e^2 \rho / 2$ we receive the slightly more accurate exhaust velocity

$$w_e = \sqrt{\frac{2(p(V) - p_a)}{\rho}} \cdot \frac{1}{\sqrt{1 - \left( \frac{r}{R} \right)^4}} \right) = v_e \left[ 1 + \frac{r^4}{2R^4} + \mathcal{O} \left( \frac{r^8}{R^8} \right) \right].$$

As shown above\(^1\), equation \((7)\) represents the third order Taylor approximation w. r. t. $r / R$ for the exhaust velocity $w_e$. But what is the error while using \((7)\) instead of the latter equation? The radius of a commercially available PET bottle can be estimated to about 4 to 5 cm. Our nozzle has a diameter of about 9 mm. That leads to $R \approx 10r$ and we get $w_e \approx 1.00005 v_e$. The error for the exhaust velocity is about 0.005%. Even for a nozzle with $r = R / 3$, which seems to be unrealistic large, the error will be less than 1%. Obviously, approximation \((7)\) is good enough for our concern. The working mass pressure and therewith the pressure difference depend on the gas volume $V$. The propellant is expelled through a nozzle with a cross sectional area $A$ at speed $v_e$. Let $d\bar{A}$ be the vector surface element normal to $A$. In our case, the velocity vector $\bar{v}_e$ is perpendicular to $A$ as well. Therefore, the volumetric flow

\(^1\)The corresponding Taylor series is $\frac{1}{\sqrt{1 - x^4}} = 1 + \frac{1}{2} x^4 + \mathcal{O} (x^8)$ with $x = r / R$. 

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rate through the plane surface \( A \) reduces to
\[
\frac{dV}{dt} = \int_A \langle \vec{v}_e, dA \rangle = A v_e \tag{8}
\]
and from (6) we deduce the mass flow rate
\[
\frac{dm}{dt} = \frac{d}{dt} \left[ M + m_{\text{air}} + \rho (V_b - V(t)) \right] = -\rho \frac{dV}{dt} \tag{8} = -\rho A v_e. \tag{9}
\]
From (7) one gets \( \rho v_e^2 = 2(p(V) - p_a) \). With the above considerations the rocket’s acceleration (5) takes the form
\[
a = \frac{\rho A v_e^2}{m} - g - \frac{D}{m} v_e^2 = \frac{2A (p(V) - p_a) - D v_e^2}{M + m_{\text{air}} + \rho (V_b - V)} - g \tag{10}
\]
where \( V \) is the gas volume as a function of time \( t \). In order to solve the latter equation it is also necessary to know the relation \( p(V) \) of pressure and volume.

3 Gas Expansion gas during the thrust phase

“Many real processes undergone by gases or vapours are approximately polytropic with a polytropic index typically between 1.0 and 1.7 [...]”, cf. [8]. It is argued in [10] that the gas expansion in a model rocket can also be described by a polytropic process. Pressure \( p \) and Volume \( V \) are related by \( pV^n = \text{constant} \). If \( p_0 \) and \( V_0 \) denote the corresponding initial values this is
\[
p(V) = p_0 \left( \frac{V_0}{V} \right)^n. \tag{11}
\]
Volumetric flow rate (8) and exhaust speed (7) lead to
\[
\frac{dV}{dt} = A v_e = A \sqrt{\frac{2(p(V) - p_a)}{\rho}} \tag{12}
\]
which together with (11) yields
\[
\frac{dV}{dt} = A \sqrt{\frac{2(p_0 V_0^n V - p_a)}{\rho}}. \tag{13}
\]
In case of \( V = V_b \) we receive the rocket’s burn time by numerical integration of
\[
t_b = \frac{1}{A} \sqrt{\frac{\rho}{2}} \int_{V_0}^{V_b} \frac{dV}{\sqrt{p_0 V_0^n V - p_a}}. \tag{14}
\]
In fact, there is still left some compressed air after the whole water is expelled. This provides an additional air thrust. Strictly speaking, (14) gives the duration of the water-thrust phase. However, this time will be very short. We will discuss the air boost in further detail later in Section 4.

3.1 Polytropic index

Specific investigation of the gas expansion requires the value of the polytropic exponent. Frequently, the polytropic index is chosen as \( n = 1.4 \), see [14, 18] for example. This corresponds to the assumption that the gas in the rocket is dry air. As mentioned in [10] “the air expansion in the rocket is accompanied by water vapor condensation, which provides an extra thrust”. The water vapor pressure at which water vapor is in thermodynamic equilibrium with the water depends on the temperature. For moist air the relation of saturation vapor pressure and temperature can be well approximated by an equation given by Arden Buck in [5]. If \( T \) denotes the air temperature in °C, the saturated vapor pressure \( p_v \) is given by

\[
p_v = K \cdot \exp \left[ \left( 18.678 - \frac{T}{234.5} \right) \left( \frac{T}{257.14 + T} \right) \right],
\]

\( K = 611.21 \frac{N}{m^2} = 6.1121 \cdot 10^{-3} \text{bar}. \) (15)

In [19, 10] an approximation of the polytropic index that depends on the water vapor pressure is given by

\[
n = 1.15 + (1.4 - 1.15) \exp \left( -36 \frac{p_v}{p_0} \right).
\]

(16)

For example, \( p_0 = 3 \text{ bar} \) and \( T = 15^\circ\text{C} \) yield \( n \approx 1.35 \). Figure 1 shows the polytropic index in dependence of temperature and pressure, i.e. the function \( n(T, p_0) \) given by (16) together with (15). For the expansion of moist air in water rockets the polytropic index is about \( 1.1 \leq n \leq 1.4 \).

3.2 Analytic solution of the gas expansion equation

First, the question arises if there is an analytical solution for (13). If that is not the case, the corresponding equation can be solved numerically. The following approach is based on the Gaussian hypergeometric function. A little manipulation of the ordinary differential (13) for the volume yields

\[
V_n^\frac{n}{2} \frac{dV}{dt} = A \sqrt{\frac{2p_0 V_0^n}{\rho}} \sqrt{1 - \frac{p_a}{p_0 V_0^n}} V^n.
\]
Figure 1: Polytropic index \( n(T, p_0) \) where temperature varies from 0\(^\circ\)C to 40\(^\circ\)C. The pressure range is \( 0 \leq p_0 \leq 1 \) bar and \( 0 \leq p_0 \leq 25 \) bar, respectively.

Together with \( \frac{d}{dt} \left[ V^{n+2} \right] = \frac{n+2}{2} V^{n} \frac{dV}{dt} \) the latter equation leads to

\[
\frac{d}{dt} \left[ \left( \frac{p_a}{p_0 V_0^n} \right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}} \right] = \frac{(n+2)A}{2} \left( \frac{p_a}{p_0 V_0^n} \right)^{\frac{n+2}{2n}} \left( 2p_0 V_0^n \right)^{\frac{n+2}{2n}} \sqrt{\frac{2}{\rho}} \left( 1 - \left[ \left( \frac{p_a}{p_0 V_0^n} \right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}} \right] \right)^{-\frac{2n}{n+2}}.
\]

By using

\[
u := \left( \frac{p_a}{p_0 V_0^n} \right)^{\frac{n+2}{2n}} V^{\frac{n+2}{2}}, \quad k := \frac{(n+2)A}{2} \left( \frac{p_a}{p_0 V_0^n} \right)^{\frac{n+2}{2n}} \left( 2p_0 V_0^n \right)^{\frac{n+2}{2n}} \sqrt{\frac{2}{\rho}}, \quad \alpha := \frac{2n}{n+2}
\]

we finally get

\[
\frac{du}{\sqrt{1 - u^\alpha}} = k \, dt.
\]

This equation admits the solution

\[
u \cdot H\left( \frac{1}{2}, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; u^\alpha \right) = kt + \xi_0
\]

where \( \xi_0 = u_0 \cdot H\left( \frac{1}{2}, \frac{1}{\alpha}; \frac{1}{\alpha} + 1; u_0^\alpha \right) \) and \( u_0 = u(0) \). Here \( H(a, b; c; x) = \sum_{i=0}^{\infty} \frac{\Gamma(a+i)\Gamma(b)\Gamma(c)}{\Gamma(a)\Gamma(b+i)\Gamma(c+i)} x^i \) is a special generalized hypergeometric function aka Gauss hypergeometric function. More precisely, this function is a special solution of Euler’s hypergeometric differential equation, see [4, eq. 1.498],

\[
x(x-1)w'' + ((a+b+1)x-c)w' + abw = 0
\]
which can also be written as \((x \frac{d}{dx} + a)(x \frac{d}{dx} + b)w = (x \frac{d}{dx} + c)w'\). The function \(H(a, b; c; x)\) obeys

\[
\frac{d}{dx}(x^{c-1}H(a, b; c; x)) = (c - 1)x^{c-2}H(a, b; c - 1; x),
\]

\[
H(a, b; c; x) = (1 - x)^{-a},
\]

see [21, eq. 1.4.1.6] and [21, eq. 1.5.1]. Therefore, by writing \(g(x) = x^{1/2}H(\frac{1}{2}; \frac{1}{a}; \frac{1}{a}; 1; x)\) and \(\tilde{g}(u) = g(x(u))\) we get

\[
\frac{dg}{du} = \frac{dg}{dx}\bigg|_{x=u^\alpha} \cdot \frac{dx}{du} \quad \frac{d\tilde{g}}{du} = \frac{1}{\alpha} x^{1/\alpha - 1} H\left(\frac{1}{2}; \frac{1}{a}; \frac{1}{a}; x\right)\bigg|_{x=u^\alpha} \cdot \alpha u^{a-1}
\]

as stated. Although a solution of (13) is represented by (18), its implicit form is inexpedient for our concerns. Of course, the burn time \(t_b\) can be calculated from (18). But we also need a closed-form expression for the gas volume as a function of time. Hence, the expansion equation (13) will be included into the system of equations of motion for our rocket. However, let us take a look at the separate numerical solution of (13) now. Within our calculations with the Runge Kutta method the temperature of water and vapor is chosen to be 15°C. The volume of the bottle and the initial water volume are 1 dm³ and 0.35 dm³, respectively. The corresponding Figure 2 was created with wxMaxima. The source code is available at https://github.com/tguent/code.

**Figure 2:** Expansion of water vapor from 0.65 dm³ to 1 dm³ at 15°C. The left side shows the expansion with an initial pressure of 3 bar. This leads to a polytropic exponent of \(n \approx 1.35\). In the other case, the initial pressure is 10 bar, which leads to \(n \approx 1.39\). The polytropic exponent was determined by (16).
4 Movement during water thrust phase

The flight of the rocket can be divided into three parts: Thrust phase, coasting phase with upwards motion, and free-fall phase. We model the thrust phase by dividing it again into two parts: The main thrust is provided by the expelled water. During the water thrust phase the acceleration of the water rocket is modeled by (5). But usually there is still some compressed air left at the end of the water thrust phase. This gives rise to an additional nonzero air thrust. We refer to this as the air thrust phase in the following. After all propellant (water and air) is exhausted, the rocket can be regarded as an object that is thrown vertically upwards. The corresponding initial velocity is determined by the velocity at the end of the thrust phase. After reaching the maximum altitude, the rocket enters the free-fall phase. During the upward coasting and free-fall phase air drag has a crucial influence on the rocket’s movement. As we will see in the following, air drag is negligible during the thrust phase of a water rocket.

**Remark 1 (Example data).** For our numerical calculations we use the following data: The empty rocket has a mass of $1/8 \, kg$ and a volume of $1 \, dm^3$, its nozzle has a diameter of $9 \, mm$. The rocket radius is $4 \, cm$. Initial values for propellant volume and pressure are $V_0 = 0.35 \, dm^3$ and $p_0 = 3 \, bar$ at a temperature of $15^\circ C$. The mass of the compressed air is approximately $1.8 \, g$. The air drag coefficient was set to $c_D = 1$.

4.1 Water thrust phase including air drag

As mentioned in Section 3, pressure $p$ and volume $V$ are related by polytropic expansion (11) where $V$ as a function of time $t$ is determined by (13). The rocket’s velocity can be expressed as the rate of change of its position $h$ by $v = \frac{dh}{dt}$. Analogously, the acceleration (10) can be expressed as the rate of change of its velocity $a = \frac{dv}{dt}$. Finally, this leads to the following first order system of differential equations for gas volume $V$, altitude $h$ and velocity $v$:

\[
\begin{align*}
\frac{dV}{dt} &= A \sqrt{\frac{2 (p_0 V_0^n V^{-n} - p_a)}{\rho}}, \\
\frac{dh}{dt} &= v, \\
\frac{dv}{dt} &= \frac{2 A (p_0 V_0^n V^{-n} - p_a) - D v^2}{M + m_{air} + \rho V_b} - g.
\end{align*}
\]

(21)

The latter can be solved numerically. We use the fourth order Runge Kutta method, see [1], which is implemented in the computer algebra system **wxMaxima**, see [24]. The step-size was set to 0.01. The corresponding numerical results for the reference data given in Remark 1 are presented in Figure 3. In order to create comparability
we use the following reference values $v_b \approx 15.32 \frac{m}{s}$ at an altitude of $h_b \approx 2.71 m$ at the end of the water thrust phase. For the corresponding wxMaxima source code see again https://github.com/tguent/code.

Figure 3: Velocity and altitude of the rocket during the water thrust phase for the reference data, see Remark 1. At the end of the water thrust phase the rocket has a velocity of $v_b \approx 15.32 \frac{m}{s}$ at an altitude of $h_b \approx 2.71 m$.

4.2 Simple estimate - Water thrust phase approximation with zero drag coefficient

Solving the equations of motion (21) require advanced numerical calculations. In the following we present a more simple estimation of altitude and velocity at the end of the water thrust phase. The first equation of (21) determines the gas expansion. Figure 2 shows the gas expansion for $0 \leq t \leq t_b$, where $t_b$ is determined by (14). This indicates that the gas volume as a function of time is not too far from being linear, at least as a rough estimate. If we use

$$V(t) = \frac{V_b - V_0}{t_b} t + V_0$$

for our calculations the corresponding equation\(^\dagger\) of motion predicts a velocity of about $15.17 \frac{m}{s}$ at an altitude of $2.70 m$ at the end of the thrust phase (again for the data given in Remark 1). This is only a minor deviation downwards from the more accurate reference values of $v_b \approx 15.32 \frac{m}{s}$ and $h_b \approx 2.71 m$. The relative deviation of velocity and altitude at the end of the thrust phase is about 0.9% and 0.2%.

\(^\dagger\)With (22) the equations of motion (21) reduce to $\frac{dh}{dt} = v$, $\frac{dv}{dt} = \frac{2A}{M + m_{air} + \rho V_b} - \frac{p_0}{\left( \frac{V_b}{V_0} \right)^n} - p_a - Dv^2 - g$. 

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respectively. Thus, the linear approach (22) can be used as a good approximation. Now consider the last equation of (21) which incorporates thrust and air resistance. Although air drag has crucial influence on the maximum altitude of the rocket it has few influence during the thrust phase. During the subsequent free flight phase when the propellant is exhausted it is very important to incorporate air drag again. Next we calculate the data at the end of the water thrust phase in case of $c_D = 0$. Together with (22), equation (21) leads to

$$v_n(t) = \int_0^t \frac{2A}{M + m_{air} + \rho V_b - \rho \left(\frac{V_b - V_0}{t_b} x + V_0\right)} dx - gt \quad (23)$$

at a given time $t \leq t_b$. Integration over the burn time $t_b$, see (14), gives the velocity $v_{nb}$ at the end of the water thrust phase. The altitude at the end of a water rocket’s thrust phase is small. We use the mean acceleration $\bar{a} = v_{nb}/t_b$ and

$$h_{nb} := \frac{1}{2} \bar{a} t_b^2 = \frac{v_{nb} t_b}{2} \quad (24)$$

as a rough estimate of the altitude at the end of the water thrust phase. With the above used data from Remark 1, this leads to $v_{nb} \approx 15.60 \frac{m}{s}$ for $h_{nb} \approx 2.84 \text{ m}$. The results exceed the reference values $v_b \approx 15.32 \frac{m}{s}$ and $h_b \approx 2.71 \text{ m}$ by 1.8% and 4.9%, respectively. Such a simple analysis of the motion of a water rocket can actually be carried out in undergraduate physics courses. All necessary calculations for the method figured out in Section 4.2 can be done with a graphing calculator.

5 Movement during air thrust phase

5.1 Prelimarily remarks

As mentioned before, the mass of air inside the rocket is small, i.e. $m_{air} \ll M$. Even if we neglect the mass of the compressed air during the water thrust phase the error will be very small. Using (11) the mass of air inside the rocket can be calculated from

$$m_{air} = \rho_{air} \sqrt{\frac{p_0}{p_a}} V_0 \quad (25)$$

where $\rho_{air} \approx 1.23 \frac{g}{dm^3}$ is the density of air at atmospheric pressure $p_a \approx 1 \text{ bar}$. For example: An initial air volume of $V_0 = 0.65 \text{ dm}^3$ at $p_0 = 3 \text{ bar}$ contributes to the total mass of the rocket with about $2 \text{ g}$. In our case, the rocket has a curb weight of about $1/8 \text{ kg}$ plus $350 \text{ g}$ propellant (water). The additional mass of $2\text{ g}$ leads to a
deviation of about some centimeters of calculated altitude. I.e., the additional mass of the compressed air doesn’t have a noticeable influence on the water thrust phase where the rocket is propelled by the water. But what effect has the expelled air after the water thrust phase? On the one hand, there is only a tiny working mass left to induce a change in momentum. On the other hand, the air escapes very rapidly. Indeed, it already was a matter of discussion if the air boost can be neglected or not. Prusa notes: “Due to its low density, the thrust provided by air alone is negligible, and in a launch without water, the rocket is barely able to lift off of the air pump seal.”, cf. [18, p. 719]. On the other hand, Barrio-Perotti et. al. state: “This air is expelled through the nozzle causing an additional increase in the rocket momentum that sometimes cannot be neglected: for example, a rocket with air pressurized at 2 bars can reach a distance of about 10 m [...]”, cf. [3, p. 1138]. In [3], the air propulsion is also described by a system of differential equations, see [3, eq. 29, 30, 31] together with the algebraic equation [3, eq. 28] which have to be solved numerically. In order to decide whether the air boost has to be taken into account or not we filled our model rocket with compressed air (no water) at about 2 bar pressure. Actually, the rocket lifted off and reached a significant altitude of at least two meters. Thus we decided not to neglect the air propulsion. But since the effect is small compared to the water thrust, we choose a straightforward approximation of the air thrust phase in the following.

5.2 Rocket equation with simplified model assumptions

The gas volume equals the bottle’s volume $V_b$ now. According to (11), initial pressure $p_0$ and initial air density have reduced to

\[ p_b = p_0 \left( \frac{V_0}{V_b} \right)^n \quad \text{and} \quad \rho_b = \frac{\rho_{\text{air}} V_0}{V_b} \sqrt{\frac{p_0}{p_a}} \]  

(26)

The change in momentum can be calculated from the rocket equation (5). In order to avoid complicated numerical methods we neglect the air drag during the air thrust phase. Barrio-Perotti et. al. evaluate water and air thrust in [3]. In comparison with the duration of the water thrust phase, the air thrust phase is short. Beyond that, the rocket acceleration is strongly decreasing during the air propulsion, see [3, Fig. 12]: If the air propulsion lasts about 0.025 seconds, the major contribution may be over within a hundredth of a second. Compared with this, the water thrust in [3, Fig. 12] lasts six times longer (and leads to higher acceleration anyway). Based on the short duration of the air propulsion, we assume that the exhaust velocity $v_e$ can be treated as a constant. With these simplifications the rocket equation $\frac{dv}{dt} = -\frac{v_e dm}{m dt} - g$
can be integrated.

\[
\int_{v_b}^{v_{\text{max}}} dv = -v_e \int_{M+m_{\text{air}}}^{M} \frac{dm}{m} - \int_{\text{duration of air propulsion}} g \, dt.
\]  
(27)

Now let us rate the two terms on the right side of (27). Because the change of mass can be huge the integral over the mass may have a significant contribution to the result, despite of the fact that the duration of the air propulsion is very short. Now consider the last integral. The product of gravitational acceleration and duration of air propulsion will be also small. So we neglect this part. The remaining terms result in \(v_{\text{max}} \approx v_e \ln \left( \frac{M+m_{\text{air}}}{M} \right) + v_b\) where \(v_b\) is the rocket velocity at the end of the water thrust phase. Indeed, the above considerations are further based on the assumption that no air escapes before all water is expelled. This seems to be unrealistic. Particularly towards the end of the water thrust phase the propellant will be a mixture of water and air. Therefore, the mass of the remaining air at the end of water thrust will be less than the initial amount of air inside the rocket given by (25). In order to take into account the loss of air during the water thrust phase we include an efficiency factor \(\eta\) and receive

\[
v_{\text{max}} = v_e \ln \left( \frac{M+\eta \cdot m_{\text{air}}}{M} \right) + v_b.
\]  
(28)

Amongst other things, the efficiency factor \(\eta\) presumably depends on the initial proportion of water and air in the rocket, that is \(V_0/V_b\). If the initial amount of air equals the bottle volume, there is no water propulsion and the air provides the whole thrust. Obviously the air thrust efficiency should be \(\eta = 1\) in this case. If the whole bottle is filled with water, \(\eta = 0\) should indicate that there will be no air thrust. Furthermore, the efficiency strongly depends on the construction of the D.I.Y rocket. Since we consider a highly chaotic system, there should be a parameter to readjust our calculation to the experimental data of a special model rocket. Based on the preceding considerations we choose

\[
\eta = \left( \frac{V_0}{V_b} \right)^\mu.
\]  
(29)

As a first estimate one can use \(\mu = 1\). In order to fine-tune this estimate for a special D.I.Y. rocket, the exponent \(\mu \geq 0\) can be determined from the experimental data: For our rocket a value of \(\mu \approx 3\) works quite well.

Now we still need the exhaust velocity \(v_e\). We cannot simply calculate \(v_e\) from (7), since it is based on Bernoulli’s equation for incompressible flow. The density of a compressible flow is not constant. But the derivation of Bernoulli’s equation from
\[ \frac{dp}{\rho} + v \, dv + g \, dz = 0 \] can be modified for compressible flow. With \( p \propto \rho^n \), see [7, eq. 3.20], we receive the relationship \( \frac{n}{n-1} \cdot \rho^n + \frac{1}{2}v^2 = \text{constant} \) which applies to compressible adiabatic flows, see [7, eq. 3.21]. We use again the above notation \( \rho_{\text{air}} \) for the air density at atmospheric pressure \( p_a \). Density and pressure at the beginning of the air thrust phase are denoted by \( \rho_b \) and \( p_b \), respectively. Let us assume that the speed inside the bottle can be neglected. Therewith, we receive for the exhaust velocity:

\[
v_e = \sqrt{\frac{2n}{n-1} \left( \frac{p_b}{\rho_b} - \frac{p_a}{\rho_{\text{air}}} \right)} \quad \text{(26)}
\]

Combining (28) and (30) results in

\[
v_{\text{max}} \approx \sqrt{\frac{2n \, p_a}{(n-1) \, \rho_{\text{air}}} \left[ \left( \frac{p_0}{p_a} \right)^{1-\frac{1}{n}} \left( \frac{V_0}{V_b} \right)^{n-1} - 1 \right] \ln \left( \frac{M + \eta \cdot m_{\text{air}}}{M} \right) + v_b} \quad \text{(31)}
\]

Since the duration of the air propulsion is very short, we neglect the additional height which the rocket attains during this phase. But we take into account the enhanced velocity from the air thrust. This significantly affects the upward coasting.

6 Movement in the coasting phase

By the time the working mass is expelled the rocket has already reached its top speed. From this point we can regard the rocket as an object that is thrown vertically upwards. Air resistance has a significant impact on the movement after the thrust phase. Luckily, the corresponding equations of motion can be solved analytically.

The kinematics of an upward movement including air resistance are well known. As the rocket’s propellant is exhausted the remaining mass \( M \) of the rocket is constant. With the notation

\[
\psi = \frac{D}{M} = \frac{\rho_{\text{air}} \, c_D \, A_R}{2M},
\]

the air drag deceleration (4) takes the form \( a_D = \psi v^2 \) during the free flight phase, see (3). For \( v \leq v_{\text{max}} \) the equation of motion \( \frac{dv}{dt} = -(g + \psi v^2) \) leads to the integral equation

\[
\int_{v_{\text{max}}}^{v} \frac{dx}{g + \psi x^2} = - \int_{0}^{t} dt
\]

where \( t = 0 \) corresponds to the end of the thrust phase. The solution is given by

\[
v(t) = \sqrt{\frac{g}{\psi}} \tan \left( \arctan \left( \sqrt{\frac{\psi}{g} v_{\text{max}}} \right) - \sqrt{\psi g} t \right). \quad \text{(33)}
\]
With $v = 0$ the duration of the upward coasting phase is given by

$$t_c = \frac{1}{\sqrt{\psi g}} \arctan \left( \frac{\sqrt{\psi}}{g} v_{\text{max}} \right). \tag{34}$$

The upward coasting begins after the thrust phase and ends when the rocket has reached the maximum altitude. The covered distance during the free flight phase is given by

$$h_c = \int_0^{t_c} \sqrt{\frac{g}{\psi}} \tan \left( \arctan \left( \frac{\sqrt{\psi}}{g} v_{\text{max}} \right) - \sqrt{\psi g} t \right) dt$$

$$= \frac{1}{\psi} \left\{ \ln \left| \cos \left( \arctan \left( \frac{\sqrt{\psi}}{g} v_{\text{max}} \right) - \sqrt{\psi g} t_c \right) \right| - \ln \left| \cos \left( \arctan \left( \frac{\sqrt{\psi}}{g} v_{\text{max}} \right) \right) \right| \right\}$$

$$= -\frac{1}{\psi} \ln \left| \cos \left( \arctan \left( \frac{\sqrt{\psi}}{g} v_{\text{max}} \right) \right) \right|. \tag{35}$$

The first term in the second row cancels out by using (34) for $t_c$. Using $\cos (\arctan x) = \frac{1}{\sqrt{1+x^2}}$, i.e.\(^1\)

$$\ln |\cos (\arctan (x))| = \ln \left( \frac{1}{\sqrt{1+x^2}} \right) = -\frac{1}{2} \ln (1+x^2),$$

one finally gets

$$h_c = \frac{1}{2\psi} \ln \left( 1 + \frac{\psi}{g} v_{\text{max}}^2 \right) \tag{35}$$

where $\psi = \frac{\rho_{\text{air}} c_D A_R}{2M}$, see (32).

### 7 Collection of the results and maximum altitude

Let us summarize the previous results and therewith derive a method to estimate the maximum altitude of the rocket. First we have to calculate the burn time of the water thrust phase by

$$t_b = \frac{1}{A} \sqrt{\frac{\rho}{2}} \int_{V_0}^{V_b} \frac{dV}{\sqrt{p_0 V_0^n V^{-n} - p_a}},$$

see (14). Here $A$ is the nozzle cross sectional area, $\rho$ the water density, $n$ the polytropic exponent, $p_a$ the atmospheric pressure and $p_0$ the initial pressure in the

\(^1\)The absolute value function can be abandoned since $x > 0$. 

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rocket tank. The tank has the volume \( V_b \). At launch, \( V_0 < V_b \) is filled with air and \( V_b - V_0 \) is the initial water volume. The water thrust phase can be modeled by the system of differential equations (21). Its numerical solution gives velocity \( v_b \) and altitude \( h_b \) at the end of the water thrust phase. Alternatively, for a rough estimate we receive these values from

\[
v_b \approx v_{nb} = \int_0^{t_b} \frac{2A \left[ \frac{p_0 V_0^n}{(V_b - V_0 x + V_0)} - p_a \right]}{M + m_{air} + \rho V_b - \rho \left( \frac{V_b - V_0}{t_b} x + V_0 \right)} dx - g t_b ,
\]

\[
h_b \approx h_{nb} = \frac{v_b t_b}{2},
\]

see (23) and (24). \( M \) is the rocket’s curb mass, \( m_{air} \) the initial mass of air in the tank and \( g \) the gravitational acceleration. After the water thrust phase, the rocket experiences an additional acceleration due to the air propulsion. We only consider the enhanced velocity (31) within our model

\[
v_{max} \approx \sqrt{\frac{2 n p_a}{(n-1) \rho_{air}} \left[ \left( \frac{p_0}{p_a} \right)^{\frac{1}{n}} \left( \frac{V_0}{V_b} \right)^{n-1} - 1 \right]} \ln \left( \frac{M + \eta \cdot m_{air}}{M} \right) + v_b
\]

where \( \eta \) is the efficiency factor of air propulsion, see (29). The rocket enters the upward coasting phase with the velocity \( v_{max} \) at the altitude \( h_b \). During the upward coasting the rocket reaches an additional height of

\[
h_c = \frac{M}{\rho_{air} c_D A_R} \ln \left( 1 + \frac{\rho_{air} c_D A_R}{2 M g v_{max}^2} \right),
\]

see (35). Finally, the maximum altitude of the rocket can be calculated from \( h_{max} = h_b + h_c \).

8 Drag analysis - An estimate of the drag coefficient

The drag coefficient can be divided into its components which arise from pressure drag and friction drag, see [25, eq. 7.63]. Friction drag is caused by the viscosity of the surrounding air in our case. Pressure drag is the “difference between the high pressure in the front stagnation region and the low pressure in the rear separated region [...],” cf. [25, p. 448]. Basically, the drag coefficient varies with the Reynolds number \( Re = v L / \nu \) where \( v \) is the rocket’s velocity, \( L \) is its characteristic length,
and \( \nu \) is the kinematic viscosity\(^1\) of the surrounding atmosphere, see [25, eq. 7.61]. A simple home-made rocket built from a PET bottle achieves a maximum speed of about 20 m/s. From \( Re = vL/\nu \) we see that the Reynolds number will not exceed \( 5 \cdot 10^5 \) in this case. Of course, a more professional constructed water rocket might receive a maximum Reynolds number which is about ten times higher. The relation\(^2\) of drag coefficient and Reynolds number is usually obtained from laboratory experiments, see for example [25, fig. 7.16]. Since such precise aerodynamic considerations are beyond the scope of this paper we try to get an estimate of a constant drag coefficient of our rocket in the following. “The drag analysis of rockets [...] is usually simplified by considering the rocket to be made up of several simple basic components” [11]. We will confine our considerations to the drag analysis of nose cone, body tube, base, fins, and a (small) constant value for the launch lugs. The latter segmentation of a model rocket is shown in [11, fig. 17]. Due to interference drag the total drag of the rocket amounts to more than the sum of the components: “[...] additional amount of drag is caused by the joining of the fins to the rocket body. [...] Interference drag can be as much as 10% above the sum of the fin and body tube drag”, cf. [11, p. 9].

In this paper, the drag coefficients of nose cone, body tube, base, fins, interference and launch lugs are denoted by \( c_{nose} \), \( c_{tube} \), \( c_{base} \), \( c_{fin} \), \( c_{int} \), and \( c_{lau} \) respectively. The total drag coefficient \( c_D = c_{nose} + c_{tube} + c_{base} + c_{fin} + c_{int} + c_{lau} \) represents the case that the rocket is moving at zero angle to the wind direction. Any nonzero angle leads to an additional induced drag.

8.1 Nose cone and body tube of the rocket

The nose cone is exposed to pressure drag and skin friction drag. As mentioned in [11, p. 10] a flat nose cone (solely) would result in a drag coefficient of \( c_N = 0.8 \) due to pressure. Gregorek compares the latter case to the order of magnitude of drag coefficients of several shapes. [11, fig. 23] indicates that rounding the nose reduces the corresponding drag coefficient by about 90% or even more. Nose cone and body tube of the rocket are additionally exposed to friction drag. Luckily, there exists an equation, see [11, eq. 8], which includes the drag of the nose cone as well as the body tube. Let \( A_{cs} \) be the cross-sectional area of the body tube and \( A_{ws} \) the wetted

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\(^1\)The kinematic viscosity of air at 15°C is \( \approx 1.5 \cdot 10^{-5} \text{ m}^2/\text{s} \). The NASA provides a “Similarity Parameter Calculator” which calculates the Reynolds number: [https://www.grc.nasa.gov/www/k-12/airplane/viscosity.html](https://www.grc.nasa.gov/www/k-12/airplane/viscosity.html)

\(^2\)It is worth mentioning that the drag force \( F_D = \frac{1}{2} \rho_{air} c_D A v^2 \) increases with increasing speed of the rocket. Although if the drag coefficient \( c_D \) usually decreases.
surface area of the rocket. Therewith, \[11, \text{eq. 8}\] takes the form

\[
C_1 := c_{\text{nose}} + c_{\text{tube}} = 1.02 c_f \left(1 + \frac{3}{2} \left(\frac{L}{d}\right)^2\right) \frac{A_{\text{ws}}}{A_{\text{cs}}}
\]

(37)

where $c_f$ is the skin friction coefficient and $L/d$ the length to diameter ratio of the rocket.

### 8.2 Base and fins of the rocket

Flow separation causes low pressure at the rear of the rocket which results in base drag. An equation that estimates the base drag is given by

\[
c_{\text{base}} = 0.029 \sqrt{c_{\text{nose}} + c_{\text{tube}}},
\]

(38)

see \[11, \text{eq. 9}\]. Fins cause additional friction drag, pressure drag and induced drag. But they substantially enhance the flight stability of our rocket. Generally, the fin drag depends on various parameters like the thickness to chord ratio, planform area and cross-sections. We present a rough estimate in this paper. In order to get some upper limit we use fairly pessimistic assumptions for the zero lift fin drag coefficient. Following \[11, \text{p. 17}\], the fin thickness to chord ratio will rarely be greater than 0.1 for a typical model rocket. Average values for the zero lift fin drag coefficient are given in \[11, \text{fig. 40}\] for rectangular, rounded, and streamlined cross-sections. Certainly, we cannot choose a streamlined\(^1\) cross-section for a pessimistic estimate. Let $\tau$ denote the thickness to chord ratio and $c^*_{\text{fin}}$ the zero lift fin drag coefficient in the following. The plotted data in \[11, \text{fig. 40}\] suggests that $c^*_{\text{fin}}$ depends linearly on $\tau$ in for $0.03 < \tau < 0.136$. Within this interval we deduced the following relations:

\[
c^*_{\text{fin}} \approx 0.875 \tau \quad \text{(rectangular cross-section),}
\]

(39)

\[
c^*_{\text{fin}} \approx 0.5 \tau \quad \text{(rounded cross-section)}
\]

(40)

The left side of figure 4 shows that the graphic illustration of (39) and (40) reproduces the corresponding lines in \[11, \text{fig. 40}\]. Let us assume in the following that the fins have rectangular cross-section. The zero lift fin drag coefficient $c^*_{\text{fin}}$ is based on the fin area $A_{\text{fin}}$ whereas the other drag coefficients are based on the body tube cross-sectional area $A_{\text{cs}}$. Therefore, we have to adjust the coefficient $c^*_{\text{fin}}$ by

\[
c_{\text{fin}} = c^*_{\text{fin}} \frac{A_{\text{fin}}}{A_{\text{cs}}} = 0.875 \frac{A_{\text{fin}}}{A_{\text{cs}}} \tau,
\]

(41)

\(^1\)According to \[11, \text{fig. 41}\], the zero lift fin drag coefficient for streamlined cross-sections at ratio 0.1 doesn’t exceed 0.019 (laminar) to 0.024 (turbulent) for $30 \frac{\text{m}}{\text{s}}$ ($100 \frac{\text{ft}}{\text{sec}}$).
As above mentioned, rocket body and fins jointly cause additional intereferenz drag. Due to [11, eq. 21], the interference drag can be estimated by
\[
c_{\text{int}} = c_{\text{fin}}^* C_R \frac{d}{2A_{cs}} \cdot N \quad \text{(39)}
\]
where \(C_R\) is the root chord of the fin, \(d\) the diameter of the body tube, \(A_{cs}\) again its cross-sectional area, and \(N\) is the number of fins. The more accurate drag calculation given in [11] also includes drag from launch lugs. The examples given in [11, p. 44, 49] lead to a small launch lugs drag coefficient of 0.02 to 0.03. For our rough estimate we incorporate this kind of drag by adding \(c_{\text{lau}} = 0.03\).

### 8.3 Total drag coefficient

Collecting the above discussion, we receive a base value for the drag coefficient of the model rocket by summation of \(c_{\text{nose}}, c_{\text{tube}}, c_{\text{base}}, c_{\text{fin}}, c_{\text{int}}\) and \(c_{\text{lau}}\), see (37), (38), (39), (41), and (42). However, we have not considered the surface texture of the rocket so far, “roughness can cause drag to increase by about 25 percent”, c.f. [11, p. 43]. Since our rocket’s nose consists of half a tennis ball, fins and launch lugs are fixed with hot glue and tape, we presuppose that the surface might be rather rough. This leads to a significant uncertainty which we incorporate as an error in the following. Finally, we receive a more or less adequate formula for the drag coefficient of our D.I.Y. rocket by
\[
c_D = \left[ C_l + \frac{0.029}{\sqrt{C_l}} + \frac{0.875\tau}{A_{cs}} \left( A_{\text{fin}} + \frac{C_R d}{2} \right) \right] N + 0.03 \cdot (1.125 \pm 0.125) \quad \text{(43)}
\]

---

**Figure 4:** Left side: Zero lift fin drag coefficients for fins with rectangular and rounded cross-sections. Right side: Skin friction coefficient as a function of Reynolds number given by Prandtl’s law, see [25, eq. 7.43].
with $C_l$ from (37). Beside the ingredients that enter into $C_l$, $\tau$ is the thickness to chord ratio of the fins, $C_R$ the root chord of the fin, and $N$ the number of the fins. The skin friction coefficient $\bar{c}_f$ depends on the kind of air flow and varies greatly with the Reynolds number, see right side of Figure 4. As $c_f$ depends on the rocket’s speed it would be the best to include the skin friction into the system of differential equations (21). However, our estimate of the total drag coefficient seems to be too vague to justify this approach. Therefore, we decided to use a constant value for the skin friction coefficient in the following calculations.

8.4 An estimate of the drag coefficient of our D.I.Y. rocket

We get an estimate of the drag coefficient by (43). The maximum speed of our rocket is about $17 \frac{m}{s}$. The characteristic length of our rocket is 0.35 m. Hence, a kinematic air viscosity of $1.5 \cdot 10^{-5} \frac{m^2}{s}$ (at 15°C) leads to a maximum Reynolds number of about $4 \cdot 10^5$. Based on Prandtl’s law [25, eq. 7.43], a mean value for the skin friction coefficient on this scale is given by

$$\bar{c}_f = \frac{1}{4 \cdot 10^5} \int_0^{4 \cdot 10^5} \frac{0.027}{Re_x^{1/7}} dRe_x \approx 0.005.$$\n
Indeed, this value is in accordance with the skin friction coefficient at a Reynolds number of $4 \cdot 10^5$ given in [11, fig. A-1]. Our rocket has a diameter of 8 cm which leads to a cross-sectional area of $A_{cs} \approx 0.005 m^2$. The rocket’s surface area (taken as a cylinder with patched-up half of a tennis ball) amounts to $A_{ws} \approx 0.1 m^2$. The rocket has $N = 3$ fins. Each has a root chord of 7.5 cm and a fin area of $A_{fin} \approx 0.0035 m^2$. The fin-thickness-to-chord ratio is about 0.08. The drag coefficient (43) is

$$c_D = 0.57 \pm 0.06,$$

for a wxMaxima source code see again https://github.com/tguent/code.

8.5 Wind tunnel experiment at TU Dortmund University

Prof. Dr. Andreas Brümmer of the chair of Fluidics at TU Dortmund University kindly provided us the opportunity to check our estimate of the drag coefficient in a wind tunnel. The drag force was measured with a dynamometer with an error of $0.05 N$. We always had to do an additional measurement for the calibration. Thus, in the worst case scenario our error adds up to $\Delta F_D = 0.1 N$. The wind speed was given to the first decimal. Hence, we use an error of $\Delta v = 0.05 \frac{m}{s}$ in the following considerations. The drag coefficient $c_D$ is related to drag force $F_D$ and wind speed $v$.

\[\text{Drag coefficients of a non spinning tennis balls have been measured to } 0.65 \pm 0.05, \text{ see [13].}\]
by \( c_D = \frac{2 F_D}{\rho v^2 A R v^2} \). Therefore, the error of the drag coefficient can be calculated by

\[
\Delta c_D = c_D \left( \frac{\Delta F_D}{F_D} + 2 \frac{\Delta v}{v} \right).
\]

| Wind speed \( v/\text{m/s} \) | Drag Force \( F_D/N \) | Drag coefficient \( c_D \) |
|-----------------------------|----------------|----------------|
| 9.6 ± 0.05                  | 0.20 ± 0.1     | 0.7 ± 0.4      |
| 11.3 ± 0.05                 | 0.20 ± 0.1     | 0.5 ± 0.3      |
| 12.8 ± 0.05                 | 0.25 ± 0.1     | 0.5 ± 0.2      |
| 13.8 ± 0.05                 | 0.30 ± 0.1     | 0.5 ± 0.2      |
| 14.6 ± 0.05                 | 0.40 ± 0.1     | 0.6 ± 0.2      |
| 15.9 ± 0.05                 | 0.40 ± 0.1     | 0.5 ± 0.1      |
| 16.8 ± 0.05                 | 0.45 ± 0.1     | 0.5 ± 0.1      |
| 18.3 ± 0.05                 | 0.60 ± 0.1     | 0.6 ± 0.2      |
| 18.9 ± 0.05                 | 0.6 ± 0.10     | 0.6 ± 0.1      |
| 19.8 ± 0.05                 | 0.65 ± 0.1     | 0.6 ± 0.1      |
| 20.0 ± 0.05                 | 0.70 ± 0.1     | 0.6 ± 0.1      |

Table 1: The results from the wind tunnel experiment

From the wind tunnel experiment we receive a drag coefficient of

\[ c_D = 0.6 \pm 0.2, \]

see Table 1. In fact, this is a remarkable low value for our D.I.Y. rocket. However, from Figure 5 we see that the stream line flow pattern suggests very good aerodynamics properties.

Figure 5: Stream line flow pattern made visible by smoke in the wind tunnel at the Department of Fluidics at TU Dortmund University.
9 The rocket launch experiment

The rocket launch was repeated with different initial values for pressure, amount of water, and temperature (of water). The model rocket which was used for the experiment has weight 143 g. It was constructed from a bottle with a volume of 1000 ml. At launch the rocket was filled with an amount of water $V_l$ and air $V_0$ at pressure $p_0$ and temperature $T$. The difference to atmospheric pressure $p_a \approx 1\, bar$ is $p_0 - p_a$. The pressure $p_0$ was measured with an error of $\Delta p = 0.05\, bar$. The error is based on the manometer's scale. The rocket has a radius of 40 mm. We drew a scale for the water level which is legible up to about $\pm 2\, mm$. Within the relevant scope the rocket is cylindric. Accordingly, the initial amount of water $V_l$ – and therewith the air volume $V_0$ – is determined up to an error of $\Delta V = 10\, ml$. Due to the precision of temperature measurements the error of temperature doesn’t affect the results. Determining the rocket’s altitude $h_{mes}$ makes up the major part in measuring inaccuracy within our experiment. We placed a measuring rod beside the launching device. The whole flight phase was filmed and we provide links to the corresponding videos in Table 4. By evaluating the videos we receive the rocket’s maximum altitude by comparing to the length of the measuring rod. The extrapolation leads to an error of $\Delta h_{med} = 0.5\, m$. The measured altitudes can be confirmed by the given video links.

It is the aim of this section to compare the experimental data with the theoretical results, see Table 3. We use the mean values of the experimental data for the theoretical results. These are calculated as described in Section 4.1. Thereby, $h_{calc}$ is calculated by using the system of differential equations (21) for the thrust phase. We get $h_{est}$ by calculating the part of the altitude that is obtained from the water thrust phase from (36). This can be done with a simple graphing calculator. The rocket’s drag coefficient was set to $c_D = 0.6$. For the air thrust efficiency factor we use $\mu = 3$ in (29).

A wxMaxima source code for the calculations from Table 3 is available at https://github.com/tguent/code.

| $T/°C$ | $V_l/\text{ml}$ | $V_0/\text{ml}$ | $p_0/\text{bar}$ | $h_{mes}/\text{m}$ |
|--------|----------------|----------------|-----------------|-------------------|
| 1      | 17             | 350 ± 10       | 650 ± 10        | 3.5 ± 0.05        | 17.5 ± 0.5 m     |
| 2      | 17             | 325 ± 10       | 675 ± 10        | 3.5 ± 0.05        | 17.1 ± 0.5 m     |
| 3      | 17             | 255 ± 10       | 745 ± 10        | 3.0 ± 0.05        | 14.1 ± 0.5 m     |
| 4      | 17             | 255 ± 10       | 745 ± 10        | 3.5 ± 0.05        | 17.4 ± 0.5 m     |
| 5      | 20             | 0              | 1000            | 2.5 ± 0.05        | > 2 m            |
| 6      | 35             | 250 ± 10       | 750 ± 10        | 2.5 ± 0.05        | 7.6 ± 0.5 m      |

Table 2: The results from the rocket experiment
Table 3: Theoretical predictions

| $T/°C$ | $V_i/ml$ | $V_0/ml$ | $p_0/bar$ | $h_{calc}/m$ | $h_{est}/m$ |
|--------|----------|----------|-----------|--------------|-------------|
| 1      | 17       | 350      | 650       | 3.5          | 17.2 $m$    | 17.5 $m$    |
| 2      | 17       | 325      | 675       | 3.5          | 17.4 $m$    | 17.6 $m$    |
| 3      | 17       | 255      | 745       | 3.0          | 13.3 $m$    | 13.4 $m$    |
| 4      | 17       | 255      | 745       | 3.5          | 17.2 $m$    | 17.3 $m$    |
| 5      | 20       | 0        | 1000      | 2.5          | 2.4 $m$     | 2.4 $m$     |
| 6      | 35       | 250      | 750       | 2.5          | 9.5 $m$     | 9.5 $m$     |

Table 4: Video links

| video link |
|------------|
| 1 https://youtu.be/YXGb75eOqbI |
| 2 https://youtu.be/9XY_6iSHXPE |
| 3 https://youtu.be/3l5JJHO0Tp8 |
| 4 https://youtu.be/RUkJwDCq8uU |
| 5 https://youtu.be/MifjHZH3Z7Y |
| 6 https://youtu.be/8MRufGWCUjM |

10 Conclusion

The method introduced in this paper yields accurate theoretical predictions. The predictions in launches 1, 2 and 4 from Table 4 lie within the experimental error, see Table 2. Launch 5 was hard to evaluate because the rocket crashed into the ceiling. Indeed, the main goal of launch 5 was to show that there is significant thrust if the rocket is filled with pressured air only. Launch 3 yielded an altitude of 13.6 $m$ at the lower bound which differs from the predicted value of 13.4 $m$. The question comes up whether the error in altitude has to be enlarged. On the one hand, the camera position has to be far enough from the experiment that the elevation angle remains small. On the other hand, huge distance leads to more blurred pictures. It stands to reason that one has to find a more suitable method to evaluate the maximum altitude. In case of launch 6, theoretical and experimental results don’t coincide. Indeed, we launched the rocket several times without getting exploitable data because the rocket didn’t lift off correctly. In some cases the rocket stuck too long to the launching device due to friction. In other cases the rocket lurched through the air. Furthermore, our model D.I.Y. rocket began to leak after a huge amount of experiments.

Putting it all together, our method is suitable to predict the altitude in case that the rocket lifts off perfectly. But even in this case the water rocket physics represents a
highly chaotic system. On this basis, the simple estimation proposed in this paper yields amazingly good results.

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