Deformations of dispersionless KdV hierarchies

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Abstract
The obstructions to the existence of a hierarchy of hydrodynamic conservation laws are studied for a multicomponent dispersionless KdV system. It is shown that if an underlying algebra is Jordan, then the lowest obstruction vanishes and that all higher obstructions automatically vanish. Deformations of these multicomponent dispersionless KdV-type equations are also studied. No new obstructions appear, and hence the existence of a fully deformed hierarchy depends on the existence of a single purely hydrodynamic conservation law.

1 Introduction
Consider the system of hydrodynamic type
\[ u^i_t = a^i_{jk} u^j u^k_x, \quad i, j, k = 1, \ldots, N, \]  
(1)
where the \( a^i_{jk} \) are constant. These may be regarded as the structure constants, with respect to some basis \( e_i \), of an algebra \( \mathcal{F} \),
\[ e_i \circ e_j = a^k_{ij} e_k. \]
Introducing an \( \mathcal{F} \)-valued field \( \mathcal{U} = w^i e_i \) equation (1) takes the simple form
\[ \mathcal{U}_t = \mathcal{U} \circ \mathcal{U}_x. \]
In this paper the following conditions on the algebra will be assumed:
- \( \mathcal{F} \) is a commutative algebra;
- \( \mathcal{F} \) possess a unity element \( e_1 \), so \( e_1 \circ e_i = e_i \);
• $\mathcal{F}$ is further equipped with a non-degenerate inner product $< , >$ such that

$$< a \circ b, c > = < a, b \circ c >$$

This is known as the Frobenius condition.

In terms of local coordinates in which the metric is written as $\eta_{ij}$, this last condition may be written as

$$\eta_{in} a^n_{jk} = \eta_{kn} a^n_{ji}.$$

which is equivalent to the condition that the tensor $a_{ijk}$ (where indices are raised and lowered using $\eta$ and $\eta^{-1}$) is totally symmetric. Note that no further conditions, such as the algebra being associative or Jordan, are imposed. In what follows it will be useful to introduce the so-called associator $\Delta_{ijk}^s$ defined by

$$(e_i \circ e_j) \circ e_k - e_i \circ (e_j \circ e_k) = \Delta_{ijk}^s e_s,$$

or, in components, by

$$\Delta_{ijk}^s = a_i^r a_{rk}^s - a_j^r a_{ir}^s.$$

This is a measure of the deviation of the commutative algebra $\mathcal{F}$ from being associative. Certain algebraic properties of the associator will be required later, and these are given in the appendix.

With these conditions equation (1) may be written in Hamiltonian form

$$u^i_t = \eta^{ij} \frac{d}{dx} \frac{\delta}{\delta u^j} \int h^{(3)} dx,$$

where

$$h^{(3)} = \frac{1}{3!} a_{ijk} u^i u^j u^k,$$

or as $u^i_t = \{ H^{(3)}, u^i \}$ where generically,

$$H = \int h dx$$

and the Hamiltonian structure is defined by

$$\{ H, G \} = \int \frac{\delta H}{\delta \dot{u}^i} \left( \eta^{ij} \frac{d}{dx} \right) \frac{\delta G}{\delta u^j} dx.$$

The fact that this defines a Hamiltonian structure follows from the fundamental theorem of Dubrovin and Novikov [DN].

## 2 Generalized Symmetries and Jordan algebras

Suppose one wishes to construct a hierarchy of hydrodynamic Hamiltonians which commute with this lowest (non-trivial) Hamiltonian, that is, to find functionals $H^{(n)} = \int h^{(n)} dx$ such that

$$\{ H^{(n)}, H^{(3)} \} = 0.$$
This implies that
\[
\int \frac{\partial h^{(n)}}{\partial u^i} a_{jk}^i u^h u^k_x \, dx = 0 ,
\]
so, under the assumption of rapidly decreasing, or periodic, boundary conditions, the integrand must be a total \(x\)-derivative. Thus
\[
\mathcal{E} \left\{ \frac{\partial h^{(n)}}{\partial u^i} a_{jk}^i u^h u^k_x \right\} = 0
\]
where \(\mathcal{E}\) is the Euler operator. This implies the following overdetermined set of equations for the Hamiltonian density \(h^{(n)}\):
\[
a_{jk}^i u^j \frac{\partial^2 h^{(n)}}{\partial u^i \partial u^p} = a_{jp}^i u^j \frac{\partial^2 h^{(n)}}{\partial u^i \partial u^k} . \tag{2}
\]
It will be assumed that the densities are homogeneous functions of degree \(n\), so by Euler’s theorem
\[
E \left( h^{(n)} \right) = nh^{(n)} .
\]
where \(E\) is the Euler vector field
\[
E = u^i \frac{\partial}{\partial u^i}
\]
As shown in [St], this, together with the earlier assumptions on the algebra \(\mathcal{F}\) imply
the following recursion equations connecting the densities:
\[
h^{(n+1)} = a_{jk}^i u^j u^k \frac{\partial h^{(n)}}{\partial u^i} . \tag{3}
\]
and
\[
\frac{\partial h^{(n)}}{\partial u^i} = h^{(n-1)} . \tag{4}
\]
The derivation of (3) (obtained by substituting \(k = 1\) into (2) and using homogeneity) only uses a subset of the equations in the overdetermined systems (3). One must therefore check that the \(h^{(n)}\) obtained by the recursive application of (3) does satisfy the full overdetermined system. In general there is an obstruction:

**Proposition 1** Suppose that \(h^{(n)}\) is a conserved density. The function \(h^{(n+1)}\) obtained by using (3) is a conserved density if and only if
\[
\Delta_{ir} s^u u^r \frac{\partial h^{(n)}}{\partial u^s} = 0 . \tag{5}
\]
Thus for an associative algebra all obstructions vanish.

Equation (3) may be viewed as extra generalized homogeneity conditions on the densities. Alternatively they may be viewed as forcing the flows to live in some reduced ring. The following results is easily obtained by direct calculation. Let
\[
\Xi_{ij} = \Delta_{ir} s^u u^r \frac{\partial}{\partial u^s} .
\]
Then:

\[ \Xi_{ii} = 0, \]
\[ \Xi_{ij} + \Xi_{ji} = 0, \]
\[ [\Xi_{ij}, E] = 0, \]
\[ [\Xi_{ij}, \Xi_{rs}] = (\Delta_{iq} q \Delta_{rps} - \Delta_{rips} q \Delta_{iq} n)u^p \partial_n. \]

For an arbitrary algebra \( F \) equation (5) will not have a solution. However, a necessary condition for the existence of a solution to (5), and hence for an unobstructed hierarchy is given by Frobenius' Theorem. Thus a necessary condition for the existence of a solution to (5) is that the commutator of the vector fields \( \Xi_{ij} \) must be a linear combination of the same vector fields:

\[ [\Xi_{ij}, \Xi_{rs}] = c_{ij,rs}^{pq} \Xi_{pq}. \]

**Example** Let \( F \) be the Jordan algebra \( D_N \) defined by

\[ e_1 \circ e_i = +e_i, \]
\[ e_i \circ e_i = -e_1, \quad i = 2, \ldots, N, \]
\[ e_i \circ e_j = 0, \quad \text{otherwise}. \]

(Such a multiplication comes from an underlying Clifford multiplication). Then

\[ \Xi_{ij} = \begin{cases} 0 & \text{if } i \text{ or } j = 1, \\ u^i \partial_j - u^j \partial_i & \text{otherwise} \end{cases} \]

then, for \( i, j, r, s \neq 1 \),

\[ [\Xi_{ij}, \Xi_{rs}] = \delta_{jr} \Xi_{is} + \delta_{is} \Xi_{jr} - \delta_{sj} \Xi_{ir} - \delta_{ri} \Xi_{js}, \]

and so the necessary condition is satisfied. Note that the vector fields define a Lie algebra isomorphic is \( u(1) \oplus so(N - 1) \). In the case the condition is also sufficient [St], for example when \( N = 3 \) they imply that

\[ h^{(n)} = h^{(n)}[u^1, (u^2)^2 + (u^3)^2]. \]

**Example** The condition (5) is not sufficient to ensure a solution to equation (2). Starting with the cubic prepotential

\[ F = \frac{1}{6} u^3 + \frac{1}{2} u(v^2 + w^2) + \frac{1}{6} \alpha_1 v^3 + \frac{1}{2} \alpha_2 v^2 w + \frac{1}{2} \alpha_3 vw^2 + \frac{1}{6} \alpha_4 w^3 \]

one may define the algebra \( F \) by the equations

\[ a_{ijk} = \frac{\partial^3 F}{\partial u^i \partial u^j \partial u^k}. \]
with \( \eta_{ij} = a_{1ij} \). This, together with its inverse, is then used to lower and raise indices, so \( a^i_{jk} = \eta^{ir} a_{rjk} \). All algebras \( \mathcal{F} \) arise in this way [St]. The only non-zero vector fields are

\[
\Xi_{23} = -\Xi_{32} = \Delta(v\partial_w - w\partial_v),
\]

where \( \Delta = \alpha_2^2 - \alpha_1 \alpha_3 + \alpha_2 \alpha_4 - 1 \), and hence the conditions in Proposition 2 are trivially satisfied. Since \( h^{(2)} \) satisfies \( \Xi_{23}(h^{(2)}) = 0 \) then \( h^{(3)} \) is a conserved density (as was known already). However, \( \Xi_{23}(h^{(3)}) = 0 \) if and only if:

(a) \( \Delta = 0 \);

or

(b) \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \).

Algebraically these mean the algebra \( \mathcal{F} \) is either

(a) associative (and hence trivially Jordan)

or

(b) the Jordan algebra \( D_3 \).

Clearly if \( \mathcal{F} \) is associative then the \( h^{(n)} \) constructed recursively via (3) are all conserved densities. If \( \mathcal{F} \) is the Jordan algebra \( D_3 \) then

\[
\Xi_{23} h^{(n+1)} = 2u\Xi_{23} h^{(n)}
\]

and hence all the obstructions vanish.

The results obtained in the above example hold more generally. Consider the obstruction to the existence of \( h^{(2)} \):

\[
\Xi_{ij} h^{(1)} = \Delta_{irj}^s u^r \partial_s (\eta_{1p} u^p),
\]

\[
= \Delta_{irj}^s u^r ,
\]

\[
= 0 .
\]

Hence \( h^{(2)} \) exists (which was already known). Note that the Casimirs \( h^{(1)}_\bullet = \eta_{\bullet p} u^p \), where \( \bullet = 1, \ldots, N \) are all conserved densities. In the case when \( \mathcal{F} \) is associative, when all obstructions vanish, these given rise, via (3), to \( N \)-independent families of conserved densities \( h^{(n)}_\bullet \). For an arbitrary algebra it is only when \( \bullet = 1 \) that an \( h^{(2)} \) exists automatically. The obstruction to the existence of \( h^{(3)} \) is similar:

\[
\Xi_{ij} h^{(2)} = \Delta_{irj}^s u^r \partial_s (\frac{1}{2} \eta_{pq} u^p u^q),
\]

\[
= \Delta_{ipjq} u^p u^q ,
\]

\[
= 0
\]

by symmetry/antisymmetry. Hence \( h^{(3)} \) exists (which again was already known). However there is an obstruction to the existence of \( h^{(4)} \):

\[
\Xi_{ij} h^{(3)} = \frac{1}{2} c_{pq} \Delta_{ris}^j u^r u^p u^q .
\]
Hence
\[ \{ \Xi_{ij} h^{(3)} \} \iff \left\{ c_{(pq)}^{\Delta_{rs}} j = 0 \right\} . \]

However this is just the condition for the algebra \( F \) to be Jordan. Hence:

**Proposition 2**
\[ \{ h^{(4)} \text{ exists } \} \iff \{ F \text{ is Jordan} \} . \]

Note, it is not immediately clear that, even in \( h^{(4)} \) exists, \( h^{(n)} \) exists for arbitrary \( n \); higher-order conditions on the algebra \( F \) could arise. However this is not the case.

**Proposition 3**
\[ \{ F \text{ is Jordan} \} \iff \{ h^{(n)} \text{ exists for } \forall n \} . \]

**Proof** It will be useful to define
\[ \partial h^{(n)} = \eta^{rs}_t \partial_t h^{(n)} e_s , \]
so, from equation 3
\[ (n - 1)\partial h^{(n)} = U \circ \partial h^{(n-1)} \]
and hence by induction
\[ \partial h^{(n+1)} = \frac{1}{n!} U^n \]
(7)
where \( U^n = U \circ U^{n-1} \) (at this stage there is no assumption of power associativity on \( F \)). With this the obstruction \( (\exists) \) may be written as an algebraic condition on the algebra \( F \). Let \( V = v^i e_i \), \( W = w^i e_i \), and consider
\[ O^{(n+1)} = v^i w^j \Delta_{rs} u^r \frac{\partial h^{(n)}}{\partial u^s} \]
(i.e. \( O^{(n+1)} \) is the obstruction to the existence of \( h^{(n+1)} \)). Using this definition and (7)
\[ O^{(n+1)} = \frac{1}{(n-1)!} \langle (V \circ U) \circ W - V \circ (U \circ W), U^{n-1} \rangle , \]
\[ = \frac{1}{(n-1)!} \langle W, (V \circ U) \circ U^{n-1} - (V \circ U^{n-1}) \circ U \rangle . \]
on using the Frobenius condition. Since \( W \) is arbitrary, it follows from the non-degeneracy of the inner product that
\[ \{ O^{(n+1)} = 0 \} \iff \{ (V \circ U) \circ U^{n-1} = (V \circ U^{n-1}) \circ U \} . \]
If \( n = 3 \) one just recovers proposition 3. However, if \( F \) is a Jordan algebra then
\[ (V \circ U) \circ U^{n-1} = (V \circ U^{n-1}) \circ U \quad \forall n \geq 2 \]
automatically. The proof of this fact is not immediately obvious (except in the case of special Jordan algebras) [Sc]. Hence the result.

Thus one has, in general, an infinite number of obstructions $O^{(n)}$ to the existence of $h^{(n)}$, but if $O^{(4)} = 0$ then all higher obstructions vanish automatically. So, if and only if $F$ is a Jordan algebra, starting from $h^{(3)}$ (or even from $h^{(2)}$ which generates the trivial flow) one may construct recursively all the Hamiltonians starting from the Casimir $h^{(1)}$

$$h^{(1)} \implies \cdots \implies h^{(n)} \implies h^{(n+1)} \implies \cdots .$$

Having established conditions for the integrability of equation (1) integrable deformations of this hydrodynamic system will now be considered. It will turn out the no new algebraic conditions will be required.

## 3 Deformations of hydrodynamic flows

### 3.1 First Order Deformations

Consider the dispersive KdV system

$$u^i_t = a^i_{jk} u^j u^k_x + \varepsilon u^i_{xxx}, \quad (8)$$

where $\varepsilon$ is a formal parameter. This too may be written in Hamiltonian form

$$u^i_t = \eta^{ij} \frac{d}{dx} \frac{\delta}{\delta u^j} \int \left\{ h^{(3)} - \varepsilon \left( \frac{1}{2} \eta_{ij} u^i_x u^j_x \right) \right\} dx.$$

Thus the dispersionless Hamiltonian density undergoes a first order deformation

$$h^{(3)} \mapsto h^{(3)}(\varepsilon) = h^{(3)} - \varepsilon \left( \frac{1}{2} \eta_{ij} u^i_x u^j_x \right).$$

The higher Hamiltonian densities, if they are to commute with the above deformed Hamiltonian will also undergo such a deformation, though this will contain higher-order terms, not just first-order terms,

$$h^{(n)} \mapsto h^{(n)}(\varepsilon) = h^{(n)} + \sum_{m=1}^{\infty} \varepsilon^m \delta^m h^{(n)}.$$

The main result of this section is the following recursion formula for $\delta h^{(n)}$:

$$n \chi^{(n)}_{ij} = a_{irs} u^r \chi^{(n-1)}_{sj} - 3 \frac{\partial^2 h^{(n-1)}}{2 \partial u^i \partial u^j}, \quad (9)$$

where $\delta h^{(n)} = \chi^{(n)}_{ij} u^i_x u^j_x$. The motivation for these calculations, and comments on the form of higher order deformations, will be postponed until later.
To second order

\[ \{ H^{(n)} + \varepsilon \delta H^{(n)}, H^{(3)} + \varepsilon \delta H^{(3)} \} = O(\varepsilon^2). \]

The zeroth order terms have already been constructed and the first order deformation \( \delta H^{(n)} \) must satisfy the equation

\[ \{ H^{(n)}, \delta H^{(3)} \} + \{ \delta H^{(n)}, H^{(3)} \} = 0. \]

Unpacking the various definition yields, on integrating by parts once to eliminate terms involving \( u_{xxx}^i \), again using rapidly decreasing, or periodic, boundary conditions:

\[
\int \left\{ A_{ij}^{(n)} u_{xx}^i u_x^j + B_{ijk}^{(n)} u_x^i u_x^j u_x^k \right\} dx = 0 \tag{10}
\]

where

\[ A_{ij}^{(n)} = -\partial_i \partial_j h^{(n)} - 2a_{sj}^r u^s \chi_{ir}^{(n)}, \quad B_{ijk}^{(n)} = \frac{1}{3!} \sum_{(ijk) \in S_3} a_{sk}^r u^s \left\{ \frac{\partial \chi_{ij}^{(n)}}{\partial u^r} - 2 \frac{\partial \chi_{ri}^{(n)}}{\partial u^j} \right\}. \tag{11} \]

It follows from the homogeneity of the densities \( h^{(n)} \) that the \( \chi_{ij}^{(n)} \) are homogeneous of degree \( n - 3 \), so

\[ u^r \frac{\partial \chi_{ij}^{(n)}}{\partial u^r} = (n - 3) \chi_{ij}^{(n)}, \tag{12} \]

and from the existence of the unity element in the algebra \( F \) it follows that

\[ \frac{\partial \chi_{ij}^{(n)}}{\partial u^1} = \chi_{ij}^{(n-1)}. \tag{13} \]

For the integrand in (10) to be a total \( x \)-derivative requires the following conditions\footnote{Equation (14) has been used in the derivation of (15).}, obtained using the Euler operator \( \mathcal{E} \) :

\[
A_{ij}^{(n)} = A_{ji}^{(n)}, \quad 6B_{ijk}^{(n)} = A_{ij,k}^{(n)} + A_{jk,i}^{(n)} + A_{ki,j}^{(n)} \tag{14-15} \]

The procedure to solve this overdetermined system is to use equations (14-15) with special values of \( i, j, k \) to derive (9), and then find conditions, if any, on the constants \( a_{jk}^i \) so that it solves the full set of equations.

Equation (15) simplifies to

\[
\sum_{\text{cyclic }i,j,k} a_{sk}^r u^s \left\{ \partial_r \chi_{ij}^{(n)} - \partial_j \chi_{ri}^{(n)} \right\} + a_{ij}^r \chi_{kr}^{(n)} = -\frac{3}{2} \partial_i \partial_j \partial_k h^{(n)}. \]

If \( i = j = k = 1 \) one obtains

\[
u_r \left\{ \partial_r \chi_{11}^{(n)} - \partial_1 \chi_{1r}^{(n)} \right\} + \chi_{11}^{(n)} = -\frac{1}{2} h_{(n-3)}. \]
If \( i = j = 1, k \neq 1 \) one obtains, on using the relation \( A_{ik} = A_{k1} \),
\[
2(n - 1)\chi_{1k}^{(n)} - \partial_k (u^r \chi_{1r}^{(n)}) = -\frac{3}{2} \partial_k h^{(n-2)} + 2a_{sk}^r u^s \chi_{1r}^{(n-1)} - a_{sk}^r \partial_r \chi_{11}^{(n)}.
\]
Assuming that
\[
\chi_{11}^{(n)} = -\frac{1}{2} h^{n-3}
\]
one finds that
\[
u^r \chi_{1r}^{(n)} = -\frac{1}{2} (n - 2) h^{n-2}
\]
and hence the recursion relation
\[
(n - 1) \chi_{1k}^{(n)} = -\partial_k h^{(n-2)} + a_{sk}^r u^s \chi_{1r}^{(n-1)}.
\]

Using the initial condition \( \chi_{ij}^{(2)} = -1/2 \eta_{ij} \) one may then prove by induction that
\[
\chi_{1k}^{(n)} = -\frac{1}{2} \partial_k h^{(n-2)}.
\]

Finally, if \( i = 1, j, k \neq 1 \), one obtains the required relation \( \Box \) on using the derivative of the relation \( A_{ik} = A_{k1} \). The first few solutions, starting with are \( \chi_{ij}^{(2)} = 0 \), are
\[
\chi_{ij}^{(3)} = -\frac{1}{2} \eta_{ij}, \quad \chi_{ij}^{(4)} = -\frac{1}{2} c_{ijr} u^r, \quad \chi_{ij}^{(5)} = -\frac{1}{20} \{ a_{ij}^q a^q_{rs} + 4a_{ij}^q a^q_{js} \} u^r u^s.
\]
These results, and more generally the recursion formula \( \Box \) have been derived using a subset of the full governing equations, and so one must check whether or not they satisfy the full set of equations. This will be done below.

### 3.2 Higher Order Deformations

In principle one may continue this procedure to find, if they exist, second order and higher deformations \( \delta^i H^{(n)} \) to the unperturbed Hamiltonians. These would have to satisfy the equation
\[
\{ \delta^i H^{(n)}, H^{(3)} \} + \{ \delta^{i-1} H^{(n)}, \delta H^{(3)} \} = 0.
\]
The number of terms in such deformations grows very rapidly. For \( i = 2 \),
\[
\delta^{(2)} H^{(n)} = \Xi^{(n,0)}_{ij} u^i_{xxx} u^j_x + \Xi^{(n,1)}_{ij} u^i_{xx} u^j_{xx} + \Xi^{(n,2)}_{ijk} u^i_x u^j_x u^k_x,
\]
and in general the number of such terms grows as the number of partitions of \( n \). For \( n = 4 \) and 5 equation \( \Box \) may be solved perturbatively with no constraint on the algebra \( \mathcal{F} \), the results being
\[
h^{(4)}(\varepsilon) = h^{(4)} + \left( -\frac{1}{2} a_{ijk} u^i_x u^j_x u^k_x \right) \varepsilon + \left( \frac{3}{10} \eta_{ij} u^i_x u^j_x \right) \varepsilon^2
\]
and
\[
h^{(5)}(\varepsilon) = h^{(5)} + \left( -\frac{1}{20} \{ a_{ij}^q a^q_{rs} + 4a_{ir}^q a^q_{js} \} u^r u^s u^i_x u^j_x \right) \varepsilon + \left( \frac{3}{10} a_{ijk} u^i_x u^j_x u^k_x \right) \varepsilon^2 + \left( -\frac{9}{70} \eta_{ij} u^i_x u^j_x \right) \varepsilon^3.
\]
3.3 Vanishing Obstructions and the Recursion Operator

Proceeding in this way one would expect new obstructions appearing at each order. However, if $\mathcal{F}$ is Jordan then there are no new obstructions at any order. This follows, not directly, as in section 2, but from the use of a recursion operator. The algebra $\mathcal{F}$ may be used to define an operator

$$
\mathcal{R}^i_j = \delta^i_j \left( \frac{d}{dx} \right)^2 + \left\{ \frac{2}{3} a_{jk} u^k \left( \frac{d}{dx} \right)^{-1} + \frac{1}{3} a_{jk} u^k \left( \frac{d}{dx} \right)^{-1} \right\} + \frac{1}{9} \Delta_{kl} u^l \left( \frac{d}{dx} \right)^{-1} \left\{ u^k \left( \frac{d}{dx} \right)^{-1} \right\},
$$

and it has been shown in [GK,Sv] that this defines a recursion operator if and only if $\mathcal{F}$ is Jordan. Hence one may obtain a bi-Hamiltonian hierarchy

$$
u^n_t = \mathcal{R}^n_x u_x.$$

Note however, that the condition that $\mathcal{F}$ is Jordan comes from the existence of a hydrodynamic conservation at the lowest possible degree at zeroth order; that is, it is the underlying dispersionless equation (1) which governs the properties and existence of the full dispersive hierarchy. Thus

$$
\left\{ \begin{array}{l}
\text{The existence of the purely hydrodynamic density } h^{(4)} \\
\text{The existence of a fully deformed dispersive hierarchy at all order}
\end{array} \right. \iff
\left\{ \begin{array}{l}
\text{The existence of the purely hydrodynamic density } h^{(4)} \\
\text{The existence of a fully deformed dispersive hierarchy at all order}
\end{array} \right.,
$$

there being no new obstruction.

4 Comments

The motivation of this paper came from the work of Eguchi et al. [EYY] who studied deformations of hydrodynamic systems associated to topological field theories, via Frobenius manifolds. They found, in specific examples, that first order deformations always exist, but obstructions will in general occur at second order. This has now been proved in general for semi-simple Frobenius manifolds [DZ]. Little work has been done on the existence of higher-order deformations for systems other than the KdV equation [DM]. Artificial examples may be constructed by scaling known dispersive hierarchies, but this does not resolve the basic problem of when a given hydrodynamic system may be deformed, or conversely, is the dispersionless limit of an integrable dispersive hierarchy.

The multicomponent KdV-type equations in this paper are not associated to Frobenius manifolds, unless the algebra $\mathcal{F}$ is associative [St], though they may be formulated in terms of a Jordan manifold [St]. However the same generic features remain. The equations for first order deformations, although overdetermined, still admit a solution. Pivotal in this derivation is the existence of a unity element in the algebra $\mathcal{F}$; for Frobenius manifolds this is automatic.

The form of these results bear a close resemblance to ideas used in deformation quantization; first order deformations define a $\partial$-operator, and the obstructions at higher
order are measured by a cohomology group. One might therefore expect that the results of this paper, and those of [EYY,DM,DZ], could be given some cohomological interpretation with the Jordan condition being equivalent to the vanishing of some cohomology group. Alternatively, the overdetermined systems (3) could be written as an exterior differential system and its integrability expressed in terms of vanishing curvature and torsions.

**Appendix**

The following results follow immediately from the definition of the associator:

\[
\Delta_{ijk}^s + \Delta_{kji}^s = 0,
\]
\[
\Delta_{ijk}^s + \Delta_{jki}^s + \Delta_{kij}^s = 0.
\]

The tensor \(\Delta_{ijks}\) is defined by

\[
\Delta_{ijks} = \langle \Delta_{ijk}^r e_r, e_s \rangle,
\]
\[
= \Delta_{ijk}^r \eta_{rs}.
\]

Using the Frobenius conditions

\[
\Delta_{ijks} = \langle (e_i \circ e_j) \circ e_k, e_s \rangle - \langle e_i \circ (e_j \circ e_k), e_s \rangle,
\]
\[
= \langle e_i \circ e_j, e_k \circ e_s \rangle - \langle e_j \circ e_k, e_i \circ e_s \rangle.
\]

From this and the commutativity of the algebra the following to results are immediate:

\[
\Delta_{ijks} = \Delta_{jisk},
\]
\[
\Delta_{ijks} = -\Delta_{iskj}.
\]

These results show that the associator is an algebraic curvature tensor under the identification

\[
R_{jik}^s \longleftrightarrow \Delta_{ijk}^s.
\]

More specifically, the pair \(\{\eta_{ij}, c_{ij}^k\}\) define a metric and a torsion free (though not metric) connection. The Bianchi identify will automatically be satisfied, but this may also be checked by direct calculation.

One further results will be required:

\[
\Delta_{ijk1} = c_{ij}^r c_{rk1} - c_{jk}^r c_{ri1},
\]
\[
= c_{ij}^r \eta_{rk} - c_{jk}^r \eta_{ri},
\]
\[
= c_{ijk} - c_{jki} = 0.
\]
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