The Lawson number of a semitopological semilattice

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Abstract
For a Hausdorff topologized semilattice $X$ its Lawson number $\bar{\Lambda}(X)$ is the smallest cardinal $\kappa$ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of $x$ in $X$ such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a subsemilattice of $X$ that does not contain $y$. It follows that $\bar{\Lambda}(X) \leq \bar{\psi}(X)$, where $\bar{\psi}(X)$ is the smallest cardinal $\kappa$ such that for any point $x \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of $x$ in $X$ such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$. We prove that a compact Hausdorff semitopological semilattice $X$ is Lawson (i.e., has a base of the topology consisting of subsemilattices) if and only if $\bar{\Lambda}(X) = 1$. Each Hausdorff topological semilattice $X$ has Lawson number $\bar{\Lambda}(X) \leq \omega$. On the other hand, for any infinite cardinal $\lambda$ we construct a Hausdorff zero-dimensional semitopological semilattice $X$ such that $|X| = \lambda$ and $\bar{\Lambda}(X) = \bar{\psi}(X) = \text{cf}(\lambda)$. A topologized semilattice $X$ is called (i) $\omega$-Lawson if $\bar{\Lambda}(X) \leq \omega$; (ii) complete if each non-empty chain $C \subseteq X$ has inf $C \in \overline{C}$ and sup $C \in \overline{C}$. We prove that for any complete subsemilattice $X$ of an $\omega$-Lawson semitopological semilattice $Y$, the partial order $\leq_X = \{(x, y) \in X \times X : xy = x\}$ of $X$ is closed in $Y \times Y$ and hence $X$ is closed in $Y$. This implies that for any continuous homomorphism $h : X \to Y$ from a complete topologized semilattice $X$ to an $\omega$-Lawson semitopological semilattice $Y$ the image $h(X)$ is closed in $Y$.
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1 Introduction

In this paper we introduce a new cardinal invariant $\bar{\Lambda}(X)$ of a Hausdorff topologized semilattice $X$, called the Lawson number of $X$. This was motivated by studying the closedness properties of complete topologized semilattices, see [1–6]. It turns out that complete semitopological semilattices share many common properties with compact topological semilattices, in particular their continuous homomorphic images in Hausdorff topological semilattices are closed.

A semilattice is any commutative semigroup of idempotents (an element $x$ of a semigroup is called an idempotent if $xx = x$).

Each semilattice $X$ carries the natural partial order $\leq_X$ defined by $x \leq_X y$ iff $xy = x = yx$. Many properties of semilattices are defined in the language of the natural partial order. In particular, for any point $x \in X$ we can consider its upper and lower sets

$$\uparrow x := \{y \in X : xy = x\} \text{ and } \downarrow x := \{y \in X : xy = y\}$$

in the partially ordered set $(X, \leq_X)$.

A subset $C$ of a semilattice $X$ is called a chain if $xy \in \{x, y\}$ for any $x, y \in C$. A semilattice $X$ is called chain-finite if each chain in $X$ is finite. A semilattice is called linear if it is a chain in itself.

A semilattice endowed with a topology is called a topologized semilattice. A topologized semilattice $X$ is called

- chain-compact if each closed chain in $X$ is compact;
- complete if each non-empty chain $C \subseteq X$ has $\inf C \in \bar{C}$ and $\sup C \in \bar{C}$.

Here $\bar{C}$ stands for the closure of $C$ in $X$. Chain-compact and complete topologized semilattices appeared to be very helpful in studying the closedness properties of topologized semilattices, see [1–6,11]. By Theorem 3.1 [1], a Hausdorff semitopological
semilattice is chain-compact if and only if it is complete (see also Theorem 4.3 [5] for generalization of this characterization to topologized posets). In [1] the first two authors proved the following closedness property of complete topologized semilattices.

**Theorem 2** (Banakh, Bardyla) *For any continuous homomorphism* $h : X \to Y$ *from a complete topologized semilattice* $X$ *to a Hausdorff topological semilattice* $Y$, *the image* $h[X]$ *is closed in* $Y$.

Theorems 1 and 2 motivate the following (still) open problem.

**Problem 1** Assume that $h : X \to Y$ is a continuous homomorphism from a complete topologized semilattice $X$ to a Hausdorff semitopological semilattice $Y$. Is $h[X]$ closed in $Y$?

In [4] the first two authors gave the following partial answer to Problem 1.

**Theorem 3** (Banakh, Bardyla) *For any continuous homomorphism* $h : X \to Y$ *from a complete topologized semilattice* $X$ *to a sequential Hausdorff semitopological semilattice* $Y$, *the image* $h[X]$ *is closed in* $Y$.

Another partial result to Problem 1 was given in [6].

**Theorem 4** (Banakh, Bardyla, Ravsky) *For any continuous homomorphism* $h : X \to Y$ *from a complete topologized semilattice* $X$ *to a functionally Hausdorff semitopological semilattice* $Y$, *the image* $h[X]$ *is closed in* $Y$.

Let us recall that a topological space $X$ is *functionally Hausdorff* if for any distinct points $x, y \in X$ there exists a continuous map $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

In fact, in [6] Theorem 4 was derived from the following closedness property of the partial order of a complete subsemilattice of a functionally Hausdorff semitopological semilattice.

**Theorem 5** (Banakh, Bardyla, Ravsky) *For any complete subsemilattice* $X$ *of a functionally Hausdorff semitopological semilattice* $Y$, *the partial order* $\leq_X$ *of* $X$ *is a closed subset of* $Y \times Y$.

In this paper we shall show that the answer to Problem 1 is affirmative under the additional condition that the semitopological semilattice $Y$ is $\omega$-Lawson. We shall also prove a counterpart of Theorem 5 for complete subsemilattices of $\omega$-Lawson semitopological semilattices.

We define a topologized semilattice $X$ to be $\omega$-Lawson if for any distinct points $x, y \in X$ there exists a countable family $\mathcal{U}$ of closed neighborhoods of $x$ such that $\bigcap \mathcal{U}$ is a subsemilattice of $X$ that does not contain $y$. A topologized semilattice $X$ is $\omega$-Lawson if and only if it is Hausdorff and has at most countable Lawson number $\bar{\Lambda}(X)$.

The *Lawson number* $\bar{\Lambda}(X)$ of a Hausdorff topologized semilattice $X$ is defined as the smallest cardinal $\kappa$ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of $x$ such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U}$ is a subsemilattice of $X$ that does not contain $y$. 

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The Lawson number will be studied in more details in Sect. 2. In that section we shall prove that every Hausdorff topological semilattice $X$ has $\Lambda(X) \leq \omega$. On the other hand, for any infinite cardinal $\lambda$ we shall construct a Hausdorff zero-dimensional semitopological semilattice $X$ of cardinality $|X| = \lambda$ and Lawson number $\Lambda(X) = \text{cf}(\lambda)$. In Sect. 4 we prove the main result of this paper:

**Theorem 6** For any complete subsemilattice $X$ of an $\omega$-Lawson semitopological semilattice $Y$, the natural partial order $\leq_X$ of $X$ is a closed subset of $Y \times Y$. Consequently, $X$ is closed in $Y$.

**Remark 1** By [7] (and [8]), there exists a metrizable (Lawson) semitopological semilattice $X$ whose partial order $\leq_X$ is not closed in $X \times X$.

Since the completeness is preserved by continuous homomorphisms into Hausdorff semitopological semilattices (see Lemma 1), Theorem 6 implies the following corollary giving a partial answer to Problem 1.

**Corollary 1** For any continuous homomorphism $h : X \to Y$ from a complete topologized semilattice to an $\omega$-Lawson semitopological semilattice $Y$ the image $h[X]$ is closed in $Y$.

**Problem 2** Let $X$ be a complete subsemilattice of an $\omega_1$-Lawson semitopological semilattice $Y$. Is $X$ closed in $Y$? Is the natural partial order $\leq_X$ of $X$ closed in $X \times X$?

## 2 The Lawson number of a Hausdorff topologized semilattice

In this section we study the Lawson number of a topologized semilattice in more details. We start with recalling the principal definition.

**Definition 1** The *Lawson number* $\Lambda(X)$ of a Hausdorff topologized semilattice $X$ is the smallest cardinal $\kappa$ such that for any distinct points $x, y \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of $x$ such that $|\mathcal{U}| \leq \kappa$ and the intersection $\bigcap \mathcal{U}$ is a closed subsemilattice of $X$ that does not contain $y$.

For any Hausdorff topologized semilattice $X$ the Lawson number $\Lambda(X)$ is well-defined and does not exceed the closed pseudocharacter $\check{\psi}(X)$ of $X$, defined as the smallest cardinal $\kappa$ such that for any point $x \in X$ there exists a family $\mathcal{U}$ of closed neighborhoods of $x$ such that $|\mathcal{U}| \leq \kappa$ and $\bigcap \mathcal{U} = \{x\}$. Therefore, $\Lambda(X) \leq \check{\psi}(X)$ for any Hausdorff topologized semilattice $X$.

Observe that a topologized semilattice $X$ is $\omega$-Lawson if and only if $X$ is Hausdorff and has $\Lambda(X) \leq \omega$.

**Definition 2** A topologized semilattice $X$ is defined to be $\kappa$-*Lawson* for a cardinal $\kappa$ if $X$ is Hausdorff and $\Lambda(X) \leq \kappa$.

The Lawson number admits the following simple characterization.

**Proposition 1** The Lawson number of a Hausdorff topologized semilattice $X$ is equal to the smallest cardinal $\kappa$ such that for any distinct points $x, y \in X$ there exist a closed subsemilattice $L$ in $X$ and a family $\mathcal{V}$ of closed neighborhoods of $x$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$.
We should prove that $\tilde{\Lambda}(X) = \Lambda(X)$ where $\Lambda(X)$ is the smallest cardinal $\kappa$ such that for any distinct points $x, y \in X$ there exist a closed subsemilattice $L$ in $X$ and a family $\mathcal{V}$ of closed neighborhoods of $x$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$.

To see that $\Lambda(X) \leq \tilde{\Lambda}(X)$, for any distinct points $x, y \in X$ use the definition of $\tilde{\Lambda}(X)$ and find a family $\mathcal{U}$ of closed neighborhoods of $x$ such that $|\mathcal{U}| \leq \tilde{\Lambda}(X)$ and $L := \bigcap \mathcal{U}$ is a closed subsemilattice of $X$ that does not contain $y$. Then $\bigcap \mathcal{U} = L \subseteq X \setminus \{y\}$, witnessing that $\Lambda(X) \leq \tilde{\Lambda}(X)$.

To see that $\tilde{\Lambda}(X) \leq \Lambda(X)$, for any distinct points $x, y \in X$ use the definition of $\Lambda(X)$ and find a closed subsemilattice $L$ of $X$ and a family $\mathcal{V}$ of closed neighborhoods of $x$ such that $|\mathcal{V}| \leq \Lambda(X)$ and $\bigcap \mathcal{V} \subseteq L \subseteq X \setminus \{y\}$. Then $\mathcal{U} := \{ U \cup L : V \in \mathcal{V} \}$ is a family of closed neighborhoods of $x$ such that $|\mathcal{U}| \leq |\mathcal{V}| \leq \Lambda(X)$ and $\bigcap \mathcal{U} = (\bigcap \mathcal{V}) \cup L = L$ is a closed subsemilattice of $X$ that does not contain $y$ and witnesses that $\tilde{\Lambda}(X) \leq \Lambda(X)$. □

The notion of a 1-Lawson semilattice extends the well-known notion of a Lawson semilattice (or else a topologized semilattice with small subsemilattices), introduced and studied by Lawson [13] (see also [10, Chapter 2]). Following [10, p. 12], we define a topologized semilattice $X$ to be

- Lawson if it has a base of the topology consisting of open subsemilattices;
- a $V$-semilattice if for any points $x \in X$ and $y \in X \setminus \uparrow x$ there exists a point $v \in X \setminus \downarrow y$ such that $\uparrow v$ is a neighborhood of $x$ in $X$;
- $\mathbb{I}$-separated if for any distinct points $x, y \in X$ there exists a continuous homomorphism $f : X \to \mathbb{I}$ such that $f(x) \neq f(y)$.

Here by $\mathbb{I}$ we denote the unit interval $[0, 1]$ endowed with the semilattice operation min.

**Proposition 2.** For a Hausdorff semitopological semilattice $X$, consider the following conditions:

1. $X$ is 1-Lawson;
2. $X$ is Lawson;
3. $X$ is a $V$-semilattice;
4. $X$ is $\mathbb{I}$-separated.

Any of the conditions (2), (3), (4) implies (1). If the space $X$ is compact, then the conditions (1)–(4) are equivalent.

**Proof.** To prove that $X$ is 1-Lawson, fix any distinct points $x, y \in X$. We need to find a closed subsemilattice $L$ in $X$ that contains $x$ in its interior but does not contain $y$.

(2) $\Rightarrow$ (1) Assume that $X$ is Lawson. Since $X$ is Hausdorff, there exists a closed neighborhood $N_x \subseteq X$ of $x$ such that $y \notin N_x$. Since $X$ is Lawson, there exists an open subsemilattice $V \subseteq X$ such that $x \in V \subseteq N_x$. Since $X$ is a semitopological semilattice, the closure $\overline{V}$ of the semilattice $V$ is a closed subsemilattice that contains $x$ in its interior but does not contain $y$. Therefore, the semilattice $X$ is 1-Lawson.

(3) $\Rightarrow$ (1) Assume that $X$ is a $V$-semilattice. If $x \notin \downarrow y$, then $y \notin \uparrow x$ and there exists an element $v \in X \setminus \downarrow y$ such that the upper set $\uparrow v$ contains $x$ in its interior. Since $X$ is a Hausdorff semitopological semilattice the upper set $\uparrow v = \{ z \in X : zv = v \}$
is closed. Then the closed subsemilattice $\uparrow v$ is a neighborhood of $x$ that does not contain $y$. If $x \not\in \downarrow y$, then $x \not\in \uparrow y$. Since $X$ is a $V$-semilattice, there exists an element $u \in X \setminus \downarrow x$ such that the upper set $\uparrow u$ contains $y$ in its interior. Observe that the complement $U := X \setminus \uparrow u$ is an open subsemilattice of $X$, containing $x$. Then $\overline{U}$ is a closed subsemilattice of $X$, which is a neighborhood of $x$ that does not contain $y$.

(4) $\Rightarrow$ (1) Assuming that $X$ is $\mathbb{I}$-separated, we can find a continuous homomorphism $f : X \to \mathbb{I}$ such that $f(x) \neq f(y)$. Choose any closed neighborhood $N \subset \mathbb{I}$ of $f(x)$ such that $f(y) \not\in N$. Then $f^{-1}(N)$ is a closed subsemilattice in $X$ that contains $x$ in its interior but does not contain $y$.

Now assume that $X$ is compact. In this case the conditions (1)–(4) are equivalent by Theorem 7.1 in [2]. In fact, the equivalence of the conditions (2) and (4) is a classical result of Lawson [13,14]. $\square$

**Example 1** The topological semilattice $\mathbb{Z}^\omega$ with the Tychonoff product topology and coordinatewise operation of minimum is Lawson and $\mathbb{I}$-separated but not a $V$-semilattice.

By [10, Example 2.21], there exists a metrizable compact topological semilattice, which is not Lawson and hence is not $1$-Lawson. However, such a semilattice necessarily is $\omega$-Lawson as shown by the following simple proposition.

**Proposition 3** Each Hausdorff topological semilattice $X$ is $\omega$-Lawson.

**Proof** Given two distinct points $x, y \in X$, choose a decreasing sequence $(U_n)_{n \in \omega}$ of open neighborhoods of $x$ such that $y \not\in \overline{U_0}$ and $U_n \cdot U_n \subseteq U_{n-1}$ and hence $\overline{U_n} \cdot \overline{U_n} \subseteq \overline{U_{n-1}}$ for all $n \in \mathbb{N}$. The choice of $U_0$ is possible by the Hausdorff property of $X$, and the choice of the neighborhoods $U_n$ is possible by the continuity of the semilattice operation at $(x, x)$. It follows that the intersection $\bigcap_{n \in \omega} \overline{U_n}$ is a closed subsemilattice of $X$ containing $x$ but not $y$. $\square$

**Corollary 2** Each compact Hausdorff semitopological semilattice is $\omega$-Lawson.

**Proof** By [14], each compact Hausdorff semitopological semilattice is a topological semilattice and by Proposition 3, is $\omega$-Lawson. $\square$

Let us also notice the following trivial (but useful) fact.

**Proposition 4** Each (Hausdorff) linear topologized semilattice $X$ is Lawson (and 1-Lawson).

Let us recall that a topological space $X$ is Urysohn if any distinct points in $X$ have disjoint closed neighborhoods. It is clear that each Urysohn space is Hausdorff.

We define a topological space $X$ to be $\kappa$-Urysohn for a cardinal $\kappa$ if for any distinct points $x, y \in X$ there are families $\mathcal{U}_x$ and $\mathcal{U}_y$ of open sets on $X$ such that $\max(|\mathcal{U}_x|, |\mathcal{U}_y|) \leq \kappa$, $x \in \bigcap \mathcal{U}_x$, $y \in \bigcap \mathcal{U}_y$ and the sets $\bigcap_{U \in \mathcal{U}_x} \overline{U}$ and $\bigcap_{V \in \mathcal{U}_y} \overline{V}$ are disjoint.

It is easy to see that a topological space is Urysohn if and only if it is 1-Urysohn.

**Example 2** For every infinite cardinal $\kappa$, there exists a Hausdorff Lawson topological semilattice, which is not $\kappa$-Urysohn.
Proof. Take any ordinal \( \lambda \) of cofinality \( \text{cf}(\lambda) > \kappa \) (for example, put \( \lambda := \kappa^+ \)). Consider the set \( L = \{ x_\alpha \}_{\alpha \leq \lambda} \cup \{ z \} \cup \{ y_\alpha \}_{\alpha \leq \lambda} \) of pairwise distinct points endowed with the linear order in which \( x_\alpha < x_\beta < z < y_\beta < y_\alpha \) for any ordinals \( \alpha < \beta \leq \lambda \). Let \( \tilde{L} := L \setminus \{ x_\lambda, y_\lambda \} \). On the set

\[
X = (\tilde{L} \times [0, \lambda)) \cup ([x_\lambda, y_\lambda] \times \{ \lambda \})
\]

consider the semilattice operation

\[
(x, \alpha) \cdot (y, \beta) := \begin{cases} 
(\min\{x, y\}, \min\{\alpha, \beta\}) & \text{if } \alpha, \beta < \lambda; \\
(\min\{x, z\}, \alpha) & \text{if } \alpha < \lambda = \beta; \\
(\min\{z, y\}, \beta) & \text{if } \beta < \lambda = \alpha; \\
(\min\{x, y\}, \lambda) & \text{if } \alpha = \lambda = \beta.
\end{cases}
\]

Endow \( X \) with the topology \( \tau \) consisting of all sets \( U \subseteq X \) satisfying the following three conditions:

- if \( (z, \alpha) \in U \) for some \( \alpha \in [0, \lambda) \), then \( \{ (x_\gamma, \alpha), (y_\gamma, \alpha) : \beta < \gamma < \lambda \} \subseteq U \) for some \( \beta \in [0, \lambda) \);
- if \( (x_\lambda, \lambda) \in U \), then \( \{ (x_\beta, \gamma) : \beta, \gamma \in [\alpha, \lambda) \} \subseteq U \) for some \( \alpha \in [0, \lambda) \);
- if \( (y_\lambda, \lambda) \in U \), then \( \{ (y_\beta, \gamma) : \beta, \gamma \in [\alpha, \lambda) \} \subseteq U \) for some \( \alpha \in [0, \lambda) \).

Taking into account that \( \text{cf}(\lambda) > \kappa \), we can show that \( (X, \tau) \) is a required Hausdorff Lawson topological semilattice which is not \( \kappa \)-Urysohn.

Now we construct Hausdorff zero-dimensional semitopological semilattices having an arbitrarily large Lawson number. We recall that a topological space is zero-dimensional if it has a base of the topology consisting of open-and-closed sets.

**Example 3** For any infinite cardinal \( \lambda \) there exists a Hausdorff zero-dimensional semitopological semilattice \( X \) such that \( |X| = \lambda \) and \( \bar{\Lambda}(X) = \bar{\psi}(X) = \text{cf}(\lambda) \).

**Proof** Consider the set

\[
X := \{ A \subseteq \lambda : A = \lambda \text{ or } A \text{ is finite} \}
\]

endowed with the semilattice operation of union. This semilattice has cardinality \( |X| = \lambda \). Here we identify the cardinal \( \lambda \) with the set \( [0, \lambda) \) of all ordinals smaller than \( \lambda \).

Now the trick is to introduce an appropriate topology on the semilattice \( X \). For this we define several kinds of sets in \( \lambda \).

A finite subset \( A \subseteq \lambda \) is defined to be sparse if \( |A \cap [\alpha, \alpha + \omega)| \leq 1 \) for any ordinal \( \alpha \in \lambda \).

For a set \( A \in X \) and an ordinal \( \alpha \in \lambda \) consider the set

\[
S[A; \alpha] := \{ B \in X : B \cap [0, \alpha) = A \cap [0, \alpha) \text{ and } B \cap [\alpha, \lambda) \text{ is sparse} \},
\]

and observe that \( \lambda \notin S[A; \alpha] \).
Let $\alpha \in \lambda$ be an ordinal, $n$ be a finite ordinal and $\varepsilon$ be a positive real number. A subset $A \subset \lambda$ is called $(\alpha, n, \varepsilon)$-fat if there exists a limit ordinal $\beta \in [\alpha, \lambda)$ and a finite ordinal $m > n$ such that

(i) the set $A \cap [\beta + \omega, \lambda)$ is sparse;

(ii) $[\beta, \beta + \omega) \cap A = [\beta, \beta + m]$;

(iii) the set $[0, \beta) \cap A$ is finite and has cardinality $< \varepsilon \cdot m$.

The conditions (i),(ii) ensure that the ordinal $\beta$ is unique.

Consider the subset

$$F[\alpha, n, \varepsilon] := \{\lambda\} \cup \{A \in X : A \text{ is } (\alpha, n, \varepsilon)\text{-fat}\}$$

of $X$.

Now we define a topology $\tau$ on the semilattice $X$. This topology consists of the sets $U \subseteq X$ satisfying the following two conditions:

(a) for any finite subset $A \in U$ of $\lambda$ there exists an ordinal $\alpha \in \lambda$ such that $A \in S[A, \alpha] \subseteq U$;

(b) if $\lambda \in U$, then there exist ordinals $\alpha \in \lambda, k \in \omega$, and a positive real number $\varepsilon$ such that $F[\alpha, k, \varepsilon] \subseteq U$.

It is easy to see that $\tau$ is a well-defined topology on $X$. Now we show that this topology is Hausdorff and zero-dimensional. Subsets $U \in \tau$ of $X$ will be called $\tau$-open.

**Claim 1** For any element $A \in X$ and ordinal $\alpha \in \lambda$ the set $S[A; \alpha]$ is open in $(X, \tau)$.

**Proof** To see that $S[A; \alpha]$ is open, take any element $B \in S[A; \alpha]$ and observe that $B \cap [0, \alpha) = A \cap [0, \alpha)$ and $B \cap [\alpha, \lambda)$ is sparse. Then $B \in S[B; \alpha] \subseteq S[A; \alpha] \subseteq X \setminus \{\lambda\}$ and the set $S[A; \alpha]$ is open by the definition of the topology $\tau$.

**Claim 2** For any ordinals $\alpha \in \lambda, n \in \omega$ and a positive real number $\varepsilon$, the set $F[\alpha, n, \varepsilon]$ is open and closed in $(X, \tau)$.

**Proof** Given any element $A \in F[\alpha, n, \varepsilon] \setminus \{\lambda\}$, find a unique limit ordinal $\beta \geq \alpha$ witnessing that $A$ is $(\alpha, n, \varepsilon)$-fat. Then $S[A; \beta + \omega] \subseteq F[\alpha, n, \varepsilon]$, witnessing that the set $F[\alpha, n, \varepsilon]$ is open in $(X, \tau)$.

To see that this set is closed in $(X, \tau)$, choose any set $A \in X \setminus F[\alpha, n, \varepsilon]$. It follows that $A$ is a finite subset of $\lambda$, which is not $(\alpha, n, \varepsilon)$-fat. Let $\beta$ be the smallest limit ordinal such that the intersection $A \cap [\beta + \omega, \lambda)$ is sparse. Let $m \in \omega$ be the smallest finite ordinal such that $A \cap [\beta, \beta + \omega) \subseteq [\beta, \beta + m]$. Since $A$ is not $(\alpha, n, \varepsilon)$-fat, one of the following conditions holds:

1. $\beta < \alpha$;
2. $m \leq n$;
3. $\beta \geq \alpha$ and $m > n$ but $[\beta, \beta + m] \nsubseteq A$;
4. $\beta \geq \alpha$, $m > n$, $[\beta, \beta + m] \subseteq A$ but $|A \cap [0, \beta)| \geq \varepsilon \cdot m$.

In all these cases $S[A; \beta + \omega]$ is a $\tau$-open neighborhood of $A$ such that $S[A; \beta + \omega] \cap F[\alpha, n, \varepsilon] = \emptyset$. 

$\blacksquare$
Claim 3 For any finite set $A \in X$ of $\lambda$ and any ordinal $\alpha \in \lambda$, the set $S[A; \alpha]$ is closed in $(X, \tau)$.

Proof Take any element $B \in X \setminus S[A; \alpha]$. If $B = \lambda$, then $F[\alpha, 1, 1]$ is a neighborhood of $B$, disjoint with $S[A; \alpha]$.

If $B \neq \lambda$, then $B \notin S[A; \alpha]$ implies that either $B \cap [\alpha, \lambda]$ is not sparse or $B \cap [\alpha, \lambda]$ is sparse but $B \cap [0, \alpha) \neq A \cap [0, \alpha)$. In the latter case $S[B; \alpha]$ is a $\tau$-open neighborhood of $B$, disjoint with $S[A; \alpha]$. So, we assume that $B \cap [\alpha, \lambda]$ is not sparse. In this case we can choose any ordinal $\beta \in [\lambda]$ with $B \subset [0, \beta)$ and observe that $S[B, \beta]$ is a $\tau$-open neighborhood of $B$ such that $S[B, \beta] \cap S[A; \alpha] = \emptyset$.

Claim 4 The topology $\tau$ is Hausdorff.

Proof Take any distinct elements $A, B \in X$. If $A$ and $B$ are finite subsets of $X$, then we can find an ordinal $\alpha \in \lambda$ such that $A \cup B \subseteq [0, \alpha)$ and observe that $S[A; \alpha]$ and $S[B; \alpha]$ are disjoint $\tau$-open neighborhoods of the elements $A$ and $B$, respectively.

If $B = \lambda$, then $A$ is a finite set, contained in $[0, \alpha)$ for some ordinal $\alpha \in \lambda$. In this case $S[A; \alpha]$ and $F[\alpha, 1, 1]$ are disjoint $\tau$-open neighborhoods of $A$ and $B$, respectively.

The case $A = \lambda$ can be considered by analogy.

Claims 1–4 show that the topology $\tau$ is Hausdorff and zero-dimensional.

Claim 5 The topologized semilattice $(X, \tau)$ is semitopological.

Proof Given any element $a \in X$, we should prove that the shift $s_a : X \to X$, $s_a : x \mapsto ax$, is continuous. If $a = \lambda$, then $s_a(X) = \{\lambda\}$ is a singleton, so the continuity of $s_a$ is trivial. So, we assume that $a$ is a finite subset of $\lambda$. To check the continuity of the shift $s_a$ at a point $x \in X$, fix any neighborhood $O_{ax} \in \tau$ of the point $ax = x \cup a$.

If $x \neq \lambda$, then $ax \neq \lambda$ and by the definition of the topology $\tau$, there exists an ordinal $\alpha \in \lambda$ such that $ax \in S[ax; \alpha] \subseteq O_{ax}$. Replacing $\alpha$ by a larger ordinal, we can assume that $ax \subseteq [0, \alpha)$. Then $O_x := S[x; \alpha]$ is a $\tau$-open neighborhood of $x$ such that $s_a(O_x) \subseteq S[ax; \alpha] \subseteq O_{ax}$.

If $x = \lambda$, then $ax = \lambda$ and by the definition of the topology $\tau$, there exist $\alpha \in \lambda$, $k \in \omega$ and $\varepsilon > 0$ such that $F[\alpha, k, \varepsilon] \subseteq O_{ax}$. Replacing $\alpha$ by a larger ordinal, we can assume that $a \subseteq [0, \alpha)$. Replacing $k$ by a larger number, we can assume that $|a| \leq \frac{1}{2}k\varepsilon$. In this case $O_x := F[\alpha, k, \frac{1}{2}\varepsilon]$ is a $\tau$-open neighborhood of $x = \lambda$ such that $s_a(O_x) \subseteq F[\alpha, k, \varepsilon] \subseteq O_{ax}$.

Claim 6 Let $\mathcal{U} \subseteq \tau$ be a family of open sets and $L$ be a $\tau$-closed subsemilattice in $X$ such that $|\mathcal{U}| < \text{cf}(\lambda)$ and $\emptyset \neq \bigcap \mathcal{U} \subseteq L$. Then $\lambda \in L$.

Proof Fix any element $x \in \bigcap \mathcal{U}$. If $x = \lambda$, then $\lambda = x \in \bigcap \mathcal{U} \subseteq L$ and we are done. So, we assume that $x$ is a finite subset of $\lambda$. Since the set $L$ is $\tau$-closed, the inclusion $\lambda \in L$ will follow as soon as we show that each neighborhood $O_\lambda \in \tau$ of $\lambda$ meets the set $L$. By Claim 2, there exist ordinals $\alpha \in \lambda$, $k \in \omega$ and a positive real number $\varepsilon$ such that $F[\alpha, k, \varepsilon] \subseteq O_\lambda$. 

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By Claim 1 and $|U| \leq \kappa < \text{cf}(\lambda)$, there exists a limit ordinal $\beta \in [\alpha, \lambda)$ such that $x \in [0, \beta)$ and $x \in S[x; \beta] \subseteq \bigcap U \subseteq L$. Choose a finite ordinal $n > k$ such that $|x| < \varepsilon n$. Observe that for every ordinal $\gamma \in [\beta, \beta + n]$ the set $x \cup \{\gamma\}$ belongs to the semilattice $L \supseteq S[x; \beta]$. Since $L$ is a subsemilattice, the $(\beta, k, \varepsilon)$-fat set $x \cup [\beta, \beta + n]$ belongs to $L \cap F[\beta, k, \varepsilon] \subseteq L \cap O_\lambda$. □

Claim 7 The semitopological semilattice $(X, \tau)$ has $\bar{\Lambda}(X, \tau) = \bar{\psi}(X, \tau) = \text{cf}(\lambda)$.

Proof Claim 6 implies that $\text{cf}(\lambda) \leq \bar{\Lambda}(X, \tau)$. Since $\bar{\Lambda}(X, \tau) \leq \bar{\psi}(X, \tau)$, it remains to prove that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$.

Choose a cofinal subset $C \subseteq \lambda$ of cardinality $|C| = \text{cf}(\lambda)$. To see that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$, take any $A \in X$. If $A$ is a finite subset of $\lambda$, then $A \subseteq [0, \alpha)$ for some ordinal $\alpha \in \lambda$. Then $\{A\} = \bigcap_{\alpha \leq \gamma \in C} S[A; \gamma]$. If $A = \lambda$, then $\{A\} = \lambda \subseteq \bigcap_{\gamma \in C} F[\gamma; 1, 1]$. In both cases the singleton $\{A\}$ is the intersection of $\text{cf}(\lambda)$ many closed neighborhoods of $A$, witnessing that $\bar{\psi}(X, \tau) \leq \text{cf}(\lambda)$. □

3 Complete topologized semilattices

In this section we recall some known properties and characterizations of complete topologized semilattices.

By a poset we understand a set endowed with a partial order. A topologized poset is a poset endowed with a topology. So, each topologized semilattice is a topologized poset.

A subset $D$ of a poset $(X, \leq)$ is called

- a chain if any elements $x, y \in D$ are comparable in the sense that $x \leq y$ or $y \leq x$;
- up-directed if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$;
- down-directed if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$.

It is clear that each chain in a poset is both up-directed and down-directed.

A topologized poset $X$ is defined to be

- up-complete if any nonempty up-directed subset $U \subseteq X$ has the least upper bound $\sup U \in \overline{U}$ in $X$;
- down-complete if any nonempty down-directed subset $D \subseteq X$ has the greatest lower bound $\inf D \in \overline{D}$ in $X$.

The proof of the following classical characterization can be found in [9,12,15] or [5, 2.2].

Proposition 5 For a topologized poset $X$ the following conditions are equivalent:

1. $X$ is up-complete;
2. Each non-empty chain $C \subseteq X$ has the least upper bound $\sup C \in \overline{C}$ in $X$.

Proposition 5 implies the following useful characterization of completeness in topologized semilattices.

Corollary 3 A topologized semilattice $X$ is complete if and only if it is up-complete and down-complete.
This corollary implies that each closed subsemilattice of a complete topologized semilattice has the smallest element.

A topologized semilattice \( Y \) is called \( \uparrow \)-closed if for every \( y \in Y \) the upper set \( \uparrow y = \{ x \in Y : xy = y \} \) is closed in \( Y \). It is easy to see that each \( T_1 \) semitopological semilattice is \( \uparrow \)-closed.

The following lemma (that can be derived from Corollary 3) is proved in [5], Lemma 5.3.

**Lemma 1** Let \( h : X \to Y \) be a continuous surjective homomorphism between topologized semilattices. If \( X \) is complete and \( Y \) is \( \uparrow \)-closed, then the topologized semilattice \( Y \) is complete.

### 4 Proof of Theorem 6 and Corollary 1

The proof of Theorem 6 is based on the following lemma.

**Lemma 2** Let \( X \) be a complete subsemilattice of a semitopological semilattice \( Y \). Let a pair \((x, y) \in Y \times Y\) belong to the closure of the natural partial order \( \leq_X \) of \( X \) in \( Y \times Y \), and let \( \{U_n\}_{n \in \omega} \), \( \{V_n\}_{n \in \omega} \) be sequences of closed neighborhoods of the points \( x \) and \( y \) in \( Y \), respectively. Then there exist points \( x' \in X \cap \bigcap_{n \in \omega} U_n \) and \( y' \in X \cap \bigcap_{n \in \omega} V_n \) such that \( x' \leq y' \).

**Proof** Replacing each set \( U_n \) by \( \bigcap_{i \leq n} U_i \), we can assume that \( U_{n+1} \subseteq U_n \) for all \( n \in \omega \). By the same reason, we can assume that the sequence \( \{V_n\}_{n \in \omega} \) is decreasing. For every \( n \in \omega \) denote by \( U_n^\circ \) and \( V_n^\circ \) the interiors of the sets \( U_n \) and \( V_n \) in \( Y \).

By induction we shall construct sequences \( (x_n)_{n \in \omega} \) and \( (y_n)_{n \in \omega} \) of points of \( X \) such that for every \( n \in \omega \) the following conditions are satisfied:

\[
\begin{align*}
(1_n) & \quad x_n \leq y_n; \\
(2_n) & \quad (x_i \cdots x_n, x_i \cdots x_n x) \subseteq U_i^\circ \text{ for all } i \leq n; \\
(3_n) & \quad (y_i \cdots y_n, y_i \cdots y_n y) \subseteq V_i^\circ \text{ for all } i \leq n.
\end{align*}
\]

To choose the initial points \( x_0, y_0 \), use the separate continuity of the semilattice operation and find neighborhoods \( U_0^\circ \subseteq U_0^\circ \) and \( V_0^\circ \subseteq V_0^\circ \) of \( x \) and \( y \) in \( Y \) such that \( U_0^\circ x \subseteq U_0^\circ \) and \( V_0^\circ y \subseteq V_0^\circ \). By our assumption, there are points \( x_0 \in X \cap U_0^\circ \) and \( y_0 \in X \cap V_0^\circ \) such that \( x_0 \leq y_0 \). The choice of the neighborhoods \( U_0' \) and \( V_0' \) ensures that the conditions (2\(_0\)) and (3\(_0\)) are satisfied.

Now assume that for some \( n \in \mathbb{N} \) points \( x_0, \ldots, x_{n-1} \) and \( y_0, \ldots, y_{n-1} \) of \( X \) are chosen so that the conditions (1\(_{n-1}\))–(3\(_{n-1}\)) are satisfied. The condition (2\(_{n-1}\)) implies that for every \( i \leq n \) we have the inclusion \( x_i \cdots x_{n-1} x = x_i \cdots x_{n-1} x \in U_i^\circ \) (if \( i = n \), then we understand that \( x_i \cdots x_{n-1} x \in x \)). Using the continuity of the shift \( s_x : Y \to Y, s_x : z \mapsto xz \), we can find a neighborhood \( U_n^\circ \subseteq Y \) of \( x \) such that \( x_i \cdots x_{n-1} \cdot (U_n^\circ \cup U_n^\circ x) \subseteq U_i^\circ \) for every \( i \leq n \). By analogy, we can find a neighborhood \( V_n^\circ \subseteq Y \) of \( y \) such that \( y_i \cdots y_{n-1} \cdot (V_n^\circ \cup V_n^\circ y) \subseteq V_i^\circ \) for every \( i \leq n \). By our assumption, there are points \( x_n \in X \cap U_n' \) and \( y_n \in X \cap V_n' \) such that \( x_n \leq y_n \). The choice of the neighborhoods \( U_n' \) and \( V_n' \) ensures that the conditions (2\(_n\)) and (3\(_n\)) are satisfied. This completes the inductive step.
The Lawson number of a semitopological semilattice

Now for every $i \in \omega$ consider the chain $C_i = \{x_i \cdots x_n : n \geq i\} \subseteq U_i^\circ$ in $X$. By the completeness of $X$, this chain has inf $C_i \in X \cap \overline{C_i} \subseteq X \cap \overline{U_i^\circ} \subseteq X \cap U_i$. Observing that inf $C_i \leq x_i x_{i+1} \cdots x_n \leq x_{i+1} \cdots x_n$ for all $i > n$, we see that inf $C_i$ is a lower bound of the chain $C_{i+1}$ and hence inf $C_i \leq$ inf $C_{i+1}$. By the completeness of $X$, for every $i \in \omega$ the chain $D_i := \{\text{inf } C_i : j \geq i\} \subseteq U_i$ has sup $D_i \in X \cap \overline{D_i} \subseteq X \cap U_i$. Since the sequence $(\text{inf } C_i)_{i \in \omega}$ is increasing, we get sup $D_0 = \sup D_i \in X \cap U_i$ for all $i \in \omega$. Consequently, sup $D_0 \in X \cap \bigcap_{i \in \omega} U_i$.

By analogy, for every $k \in \omega$ consider the chain $E_i = \{y_i \cdots y_n : n \geq i\} \subseteq V_i^\circ$ in $X$. By the completeness of $X$, this chain has inf $E_i \in X \cap \overline{E_i} \subseteq X \cap \overline{V_i^\circ} \subseteq X \cap V_i$. By the completeness of $X$, for every $i \in \omega$ the chain $F_i := \{\text{inf } E_i : j \geq i\} \subseteq V_i$ has sup $F_i \in X \cap \overline{F_i} \subseteq X \cap \overline{V_i} = X \cap V_i$. Since the sequence $(\text{inf } E_i)_{i \in \omega}$ is increasing, we get sup $F_0 = \sup F_i \in X \cap V_i$ for all $i \in \omega$. Consequently, sup $F_0 \in X \cap \bigcap_{i \in \omega} V_i$.

To finish the proof of Lemma 2, it suffices to show that sup $D_0 \leq$ sup $F_0$. The inductive conditions $(1_\omega)$, $n \in \omega$, imply that inf $C_i \leq$ inf $E_i$ for all $i \in \omega$ and sup $D_0 = \sup (\text{inf } C_i : i \in \omega) \leq \sup (\text{inf } E_i : i \in \omega) = \sup F_0$. □

The following two lemmas imply Theorem 6.

Lemma 3 Let $Y$ be an $\omega$-Lawson semitopological semilattice. For any complete subsemilattice $X \subseteq Y$ the natural partial order $\leq_X$ of $X$ is closed in $Y \times Y$.

Proof By Corollary 3, the complete semitopological semilattice $X$ is both up-complete and down-complete. To show that the partial order $\leq_X := \{(x, y) \in X \times X : x \leq y\}$ is closed in $Y \times Y$, take any pair $(y_1, y_2)$ in the closure of the set $\leq_X$ in $Y \times Y$. For every $i \in \{1, 2\}$, let $U_i$ be the set of all countable families $U$ of closed neighborhoods of $y_i$ in $Y$ such that $\bigcap U$ is a subsemilattice of $Y$. By Lemma 2, for any $U_1 \in U_1$ and $U_2 \in U_2$ there are points $x_1 \in X \cap \bigcap U_1$ and $x_2 \in X \cap \bigcap U_2$ such that $x_1 \leq x_2$. In particular, the closed subsemilattice $X \cap \bigcap U_1$ is not empty and has the smallest element inf $(X \cap \bigcap U_1) \in X$ (by the down-completeness of $X$). Denote this smallest element by $x(U_1)$. It follows that $x(U_1) := \text{inf } (X \cap \bigcap U_1) \leq x_1 \leq x_2$. Consequently, the closed subsemilattice $(\langle x(U_1) \rangle) \cap (X \cap U_2) \ni x_2$ is not empty and has the smallest element (by down-completeness of $X$), which will be denoted by $y(U_1, U_2)$. Observe that $x(U_1) \in X \cap \bigcap U_1$, $y(U_1, U_2) \in X \cap \bigcap U_2$ and $x(U_1) \leq y(U_1, U_2)$. For any families $U_1 \in U_1$ and $U_2, U'_2 \in U_2$ with $U_2 \subseteq U'_2$ we have $y(U_1, U_2) \leq y(U_1, U'_2)$. Therefore, the set $y(U_1, U_2) : U_2 \in U_2 \subseteq X$ is up-directed and by the up-completeness of $X$, it has the smallest upper bound in $X$, which will be denoted by $y(U_1)$. It follows that $x(U_1) \leq y(U_1)$. We claim that $y(U_1) = y_2$. In the opposite case we can use the $\omega$-Lawson property of $Y$ and choose a countable family $U_2' \in U_2$ such that $y(U_1) \notin \bigcup U_2'$. Taking into account that the set $\{y(U_1, U_2) : U_2 \in U_2\}$ is cofinal in $\{y(U_1, U_2) : U_2 \in U_2\}$, we conclude that

$$y(U_1) = \sup \{y(U_1, U_2) : U_2 \in U_2\} = \sup \{y(U_1, U_2 \cap U_2') : U_2 \in U_2\} \in \bigcup U_2',$$

which contradicts the choice of the family $U_2'$. This contradiction shows that $y_2 = y(U_1) \in X$. Now we see that $x(U_1) \leq y(U_1) = y_2$ for every $U_1 \in U_1$. By the up-completeness of the semitopological semilattice $X$, the up-directed subset $\{x(U_1) :
\( \mathcal{U}_1 \in \mathcal{U}_1 \) has the smallest upper bound \( x \in X \). It follows from \( x(\mathcal{U}_1) \leq y_2 \) for all \( \mathcal{U}_1 \in \mathcal{U}_1 \) that \( x \leq y_2 \).

It remains to check that \( x = y_1 \). In the opposite case, using the \( \omega \)-Lawson property of \( X \), we can find a countable family \( \mathcal{U}_1' \in \mathcal{U}_1 \) such that \( x \not\in \bigcap \mathcal{U}_1' \). Taking into account that the family \( \{x(\mathcal{U}_1 \cap \mathcal{U}_1') : \mathcal{U}_1 \in \mathcal{U}_1\} \) is cofinal in \( \{x(\mathcal{U}_1) : \mathcal{U}_1 \in \mathcal{U}_1\} \), we conclude that

\[ x = \sup \{x(\mathcal{U}_1) : \mathcal{U}_1 \in \mathcal{U}_1\} = \sup \{x(\mathcal{U}_1 \cap \mathcal{U}_1') : \mathcal{U}_1 \in \mathcal{U}_1\} \in \bigcap \mathcal{U}_1', \]

which contradicts the choice of \( \mathcal{U}_1' \). This contradiction shows that \( y_1 = x \in X \). Therefore we obtain that \( (y_1, y_2) \in X \times X \) and \( y_1 = x \leq y_2 \), which means that \( (y_1, y_2) \in \leq_X \).

**Lemma 4** Each complete subsemilattice \( X \) of an \( \omega \)-Lawson semitopological semilattice \( Y \) is closed in \( Y \).

**Proof** By Lemma 3, the partial order \( \leq_X : = \{(x, y) \in X \times X : xy = x\} \) is a closed subset of \( Y \times Y \). By Corollary 3, the complete semilattice \( X \) has the smallest element \( \min X \in X \). Consider the continuous map \( f : Y \to Y \times Y, f : y \mapsto (\min X, y) \), and observe that \( X = f^{-1}(\leq_X) \) is a closed subset of \( X \), being the preimage of the closed set \( \leq_X \) under the continuous map \( f \).

Finally, we prove Corollary 1.

**Lemma 5** For every continuous homomorphism \( h : X \to Y \) from a complete topologized semilattice \( X \) to an \( \omega \)-Lawson semitopological semilattice \( Y \), the image \( h[X] \) is closed in \( Y \).

**Proof** Observe that the \( \omega \)-Lawson property of \( Y \) implies that the semitopological semilattice \( Y \) is Hausdorff and hence \( \uparrow \)-closed. By Lemma 1, the semitopological semilattice \( h[X] \) of \( Y \) is complete and by Lemma 4, \( h[X] \) is closed in \( Y \).

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