Statistical Curse of the Second Half Rank, Eulerian numbers and Stirling numbers

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Abstract

I describe the occurrence of Eulerian numbers and Stirling numbers of the second kind in the combinatorics of the ’’Statistical Curse of the Second Half Rank’’ problem [1].

Keywords: stochastic processes, random permutations, Eulerian numbers, Stirling numbers

1 Introduction

The ’’Statistical Curse of the Second Half Rank’’ problem [1] stems from real life considerations leading to rather complex combinatorics. One is primarily concerned with rank expectations in sailing boats regattas, bearing in mind that the issue discussed here is quite general and can apply to rank expectations of students taking exams, or other types of similar endeavors as well.

Consider as a typical example the Spi Ouest-France regatta which takes place each year during 4 days at Easter in La Trinité-sur-Mer, Brittany (France). It involves a ”large” number \( n_b \) of identical boats, say \( n_b = 90 \), running a ”large” number \( n_r \) of races, say \( n_r = 10 \) (that is to say 2, 3 races per day, weather permitting, see Fig. 1). In each race each boat gets a rank \( 1 \leq \text{rank} \leq 90 \) with the condition that there are no ex aequo.
Figure 1: Just before the start of a race at the *Spi Ouest-France*: a big number of identical boats is going to cross a virtual starting line in a few seconds.

Once the last race is over, to determine the final rank of a boat and thus the winner of the regatta one proceeds as follows:

1) one adds each boat’s rank in each race → its score \( n_t \): here \( n_b = 90, n_r = 10 \) so that \( 10 \leq n_t \leq 900 \)

\( n_t = 10 \) is the lowest possible score → the boat was always ranked 1\(^{st}\)

\( n_t = 900 \) is the highest possible score → the boat was always ranked 90\(^{th}\)

\( n_t = 10 \times (1 + 90)/2 \simeq 450 \) the middle score → the boat was on average ranked 45\(^{th}\)

2) one orders the boats according to their score → their final rank

the boat with the lowest score → 1\(^{st}\) (the winner)

the boat with next to the lowest score → 2\(^{nd}\) etc...

What is the ”Statistical Curse of the Second Half Rank”? In the *Spi Ouest-France* 2009 results sheet (see Fig. 2 for boats with final rank between 60\(^{th}\) and 84\(^{th}\)), consider for example the boat with final rank 70\(^{th}\): its ranks in the 10 races are 51, 67, 76, 66, 55, 39, 67, 59, 66, 54 so that its score is \( n_t = 600 \). The crew of this boat might naively expect that since its mean rank is \( 600/10 = 60 \) and so it has been on average ranked 60\(^{th}\), its final rank should be around 60\(^{th}\). No way, the boat ends up being 70\(^{th}\).

This ”curse” phenomenon is quite general and one would like to understand its origin and be able to evaluate it. A qualitative explanation is simple: in a given race given the rank of the boat considered above, assume that the ranks of the other boats are random variables with a uniform distribution. The random rank assumption is good if, bearing in mind that all boats are identical, the crews can also be considered as more or less equally worthy, which for sure is partially the case. Since there are no ex aequo it means that the ranks of the other boats, in the first race, are a random permutation of \((1, 2, 3, \ldots, 50, 52, \ldots, 90)\), in the second race, a random permutation of \((1, 2, 3, \ldots, 50, 52, \ldots, 90)\) in parenthesis: this is because each boat’s worst rank is removed from the final counting. It is as if they were only 9 races with, for the boat considered here, ranks 51, 67, 66, 65, 59, 67, 59, 66, 54, ”improved” score \( n_t = 524 \) and mean rank \( 524/9 \simeq 58 \). So, on average, the boat is ranked 58\(^{th}\) even though it ends up being 70\(^{th}\).
| Race | Boat Name          | Skipper | Overall Score | 2ND | 3RD | 4TH | 5TH | 6TH | 7TH | 8TH | 9TH | 10TH |
|------|--------------------|---------|---------------|-----|-----|-----|-----|-----|-----|-----|-----|------|
| 60   | BMW SAILING CUP N°1|         |               |     |     |     |     |     |     |     |     |      |
| 61   | BMW SAILING CUP N°2|         |               |     |     |     |     |     |     |     |     |      |
| 62   | BMW SAILING CUP N°3|         |               |     |     |     |     |     |     |     |     |      |
| 63   | BMW SAILING CUP N°4|         |               |     |     |     |     |     |     |     |     |      |
| 64   | BMW SAILING CUP N°5|         |               |     |     |     |     |     |     |     |     |      |
| 65   | BMW SAILING CUP N°6|         |               |     |     |     |     |     |     |     |     |      |
| 66   | BMW SAILING CUP N°7|         |               |     |     |     |     |     |     |     |     |      |
| 67   | BMW SAILING CUP N°8|         |               |     |     |     |     |     |     |     |     |      |
| 68   | BMW SAILING CUP N°9|         |               |     |     |     |     |     |     |     |     |      |
| 69   | BMW SAILING CUP N°10|       |               |     |     |     |     |     |     |     |     |      |

Figure 2: Starting from the left the first column gives the final rank of the boat, the second column its score, the third column its "improved" score, the fourth column its name, the fifth column the name of its skipper, and the next 10 columns its ranks in each of the 10 races.
Each race is obviously independent from the others, so that the scores are the sums of 10 independent random variables. But 10 is already a large number in probability calculus so that the Central Limit Theorem applies. It follows that the scores are random variables with a gaussian probability density centered around the middle score \( \approx 450 \). A gaussian distribution implies a lot of boats with scores packed around the middle score. Since the score 600 of the boat considered here is larger than the middle score 450, this packing implies that its final rank is pushed upward from its mean rank: this is the statistical "curse". On the contrary if the boat’s score had been lower than the middle score, its final rank would have been pushed downward from its mean rank: its crew would have enjoyed a statistical "blessing".

Let us rewrite things more precisely by asking, given the score \( n_t \) of the boat considered among the \( n_b \) boats, what is the probability distribution \( P_{n_t}(m) \) for its final rank to be \( m \in [1, n_b] \)?

A complication arises as soon as \( n_r \geq 3 \): \( P_{n_t}(m) \) does not only depend on the score \( n_t \) but also on the ranks of the boat in each race. For example for \( n_r = 3 \), take \( n_b = 3 \) and the score \( n_t = 6 \), it is easy to check by complete enumeration that \( P_{6=2+2+2}(m) \neq P_{6=1+2+3}(m) \). The distributions are of course similar but slightly differ. To avoid this complication let us from now on consider \( n_b \) boats with in each race random ranks given by a random permutation of \((1, 2, 3, \ldots, n_b)\) ⊕ an additional "virtual" boat only specified by its score \( n_t \) and ask the question again: given the score \( n_t \) of this virtual boat what is the probability distribution \( P_{n_t}(m) \) for its final rank to be \( m \in [1, n_b + 1] \)? This problem is almost the same \(^2\) yet a little bit simpler since, by construction, it does not have the complication discussed above.

In a given race \( k \) call \( n_{i,k} \) the rank of the boat \( i \) with \( 1 \leq i \leq n_b \) and \( 1 \leq k \leq n_r \). There are no ex aequo in a given race: the \( n_{i,k} \)'s are a random permutation of \((1, 2, 3, \ldots, n_b)\) so that they are correlated random variables with

\[
\text{sum rule} \quad \sum_{i=1}^{n_b} n_{i,k} = 1 + 2 + 3 + \ldots + n_b = \frac{n_b(1 + n_b)}{2} \\
\text{mean} \quad \langle n_{i,k} \rangle = \frac{1 + n_b}{2} \\
\text{fluctuations} \quad \langle n_{i,k}n_{j,k} \rangle - \langle n_{i,k} \rangle \langle n_{j,k} \rangle = \frac{1 + n_b}{12} (n_b \delta_{i,j} - 1)
\]

Now, the score \( n_t \) of boat \( i \) is defined as \( n_t \equiv \sum_{k=1}^{n_r} n_{i,k} \), the middle score being \( n_r (1 + n_b) / 2 \). In the large \( n_r \) limit the Central Limit Theorem applies here for correlated random variables to yield the scores joint density probability distribution

\[
f(n_1, \ldots, n_{n_b}) = \]

\(^2\)In the 2-race case it is in fact the same problem.
\[
\sqrt{2\pi \lambda n_b} \left( \frac{1}{2\pi \lambda} \right)^{n_b} \delta \left( \sum_{i=1}^{n_b} (n_i - n_r \frac{1 + n_b}{2}) \right) \exp \left[ -\frac{1}{2\lambda} \sum_{i=1}^{n_b} (n_i - n_r \frac{1 + n_b}{2})^2 \right]
\]

with \( \lambda = n_r n_b (1 + n_b) / 12 \).

Now consider the virtual boat with score \( n_t \): \( P_{n_t}(m) \) is the probability for \( m - 1 \) boats among the \( n_b \)'s to have a score \( n_i < n_t \) and for the other \( n_b - m + 1 \) boats to have a score \( n_i \geq n_t \)

\[
P_{n_t}(m) = \frac{n_b}{m - 1} \int_{-\infty}^{n_t} dn_1 \ldots dn_{m-1} \int_{n_t}^{\infty} dn_m \ldots dn_b f(n_1, \ldots, n_b)
\]

Let us take the large number of boats limit: a saddle point approximation finally [1] gives \( \langle m \rangle \) as the cumulative probability distribution of a normal variable

\[
\langle m \rangle = \frac{n_b}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\bar{n}_t} \exp \left[ -\frac{n^2}{2\lambda} \right] dn
\]

where

\[
\bar{n}_t = n_t - n_r \frac{(1 + n_b)}{2}
\]

and \( n_r \leq n_t \leq n_r n_b \rightarrow -n_r \frac{n_t}{2} \leq \bar{n}_t \leq n_r \frac{n_t}{2} \). In Fig. 3 a plot of \( \langle m \rangle / n_b \) is displayed in the case \( n_r = 30, n_b = 200 \). The curse and blessing effects are clearly visible on the sharp increase around the middle score - a naive expectation would claim a linear increase. The variance can be obtained along similar lines with a manifest damping due

\[
\langle r \rangle
\]

Figure 3: \( \langle r \rangle = \frac{\langle m \rangle}{n_b} \), \( n_r = 30, n_b = 200 \), middle score 3000, dots = numerics
to the correlations effects.

So far we have dealt with the ”curse” which is a large number of races and boats effect. Let us now turn to the combinatorics of a small number of races \( n_r = 2, 3, \ldots \) for a given
number of boats \( n_b = 1, 2, \ldots \). The simplest situation is the 2-race case \( n_r = 2 \) which happens to be solvable -it can be viewed as a solvable "2-body" problem- with an exact solution for \( P_{n_t}(m) \). To obtain in this simple situation the probability distribution one proceeds as follows (see Fig. 4):

1) one represents the possible ranks configurations of any boat among the \( n_b \)'s in the two races by points on a \( n_b \times n_b \) square lattice (in the 3-race case one would have a cubic lattice, etc): since there are no ex aequo there is exactly 1 point per line and per column, so there are \( n_b! \) such configurations (in the \( n_r \) races case one would have \((n_b!)^{n_r-1}\) such configurations)

2) one enumerates all the configurations with \( m-1 \) points below the diagonal \( n_t \): this is the number of configurations with final rank \( m \) for the virtual boat\(^3\)

Figure 4: The 2-race case: a \( m = 3 \) configuration for \( n_b = 6 \) and \( n_t = 6 \). The dashed line is the diagonal \( n_t = 6 \).

The problem has been narrowed down to a combinatorial enumeration which is doable \[\text{[1]}\]: one finds for \( 2 \leq n_t \leq 1 + n_b \)

\[
P_{n_t}(m) = (1 + n_b) \sum_{k=0}^{m-1} (-1)^k (1 + n_b - n_t + m - k)^{n_t-1} \frac{(n_b - n_t + m - k)!}{k!(1 + n_b - k)!(m - k - 1)!}
\]

and for \( n_t = n_b + 1 + i \in [n_b + 2, 2n_b + 1] \) with \( i = 1, 2, \ldots , n_b \), by symmetry, \( P_{n_b+1+i}(m) = P_{n_b+2-i}(n_b + 2 - m) \).

So far no particular numbers, be they Eulerian or Stirling, have occured. Let us concentrate on the middle score \( n_t = 2(1 + n_b)/2 = 1 + n_b \) to get

\[^3\text{Contrary to the no ex aequo rule in a given race, boats of course have equal final ranks if they have the same score.}\]
\[ P_{n=1+n_b}(m) = (1 + n_b) \sum_{k=0}^{m-1} (-1)^k \frac{(m - k)^{n_b}}{k!(1 + n_b - k)!} \]

Let us tabulate \( P_{n=1+n_b}(m) \), with \( m \in [1, n_b + 1] \), for \( n_b = 1, 2, \ldots 7 \):

\[
\begin{align*}
\{1, 0\} \\
\frac{1}{2!}\{1, 1, 0\} \\
\frac{1}{3!}\{1, 4, 1, 0\} \\
\frac{1}{4!}\{1, 11, 11, 1, 0\} \\
\frac{1}{5!}\{1, 26, 66, 26, 1, 0\} \\
\frac{1}{6!}\{1, 57, 302, 302, 57, 1, 0\} \\
\frac{1}{7!}\{1, 120, 1191, 2416, 120, 1, 0\}
\end{align*}
\]

The numbers between brackets happen to be known as the Eulerian\((n_b, k)\) numbers with \( k = m - 1 \in [0, n_b - 1] \) (here one has dropped the trivial 0’s obtained for \( m = n_b + 1 \) i.e. \( k = n_b \)). An Eulerian number (see Fig. 5) is the number of permutations of the numbers 1 to \( n \) in which exactly \( m \) elements are greater than the previous element (permutations with \( m \) "ascents") as illustrated in Fig. 6. Their generating function is

\[
g(x, y) = \frac{e^x - e^{xy}}{e^y - e^x}
\]

Figure 5: The Eulerian numbers.
Figure 6: Ascents for \( n = 1, 2, 3 \). For \( n = 4 \), the permutation \((1, 4, 2, 3)\) has \( m = 2 \) ascents.

with a series expansion \( x \simeq 0 \)

\[
g(x, y) \simeq x + \frac{1}{2!} x^2 (y + 1) + \frac{1}{3!} x^3 \left( y^2 + 4y + 1 \right) + \frac{1}{4!} x^4 \left( y^3 + 11y^2 + 11y + 1 \right) + \ldots
\]

\[
+ \frac{1}{5!} x^5 \left( y^4 + 26y^3 + 66y^2 + 26y + 1 \right) + \frac{1}{6!} x^6 \left( y^5 + 57y^4 + 302y^3 + 302y^2 + 57y + 1 \right) + O \left( x^7 \right)
\]

It is not difficult to realize that all the information in \( P_{n_t}(m) \) is contained in \( P_{n_t=1+n_b}(m) \) that is to say in the generating function \( g(x, y) \). Rephrased more precisely, \( P_{n_t=n_b+1}(m) \) for \( n_b = 1, 2, \ldots \) is generated by \( yg(x, y) \) with the \( x \) exponent being the number of boats \( n_b \) and, for a given \( n_b \), the \( y \) exponent being the rank \( m \in [1, n_b+1] \). Similarly \( P_{n_t=n_b}(m) \) for \( n_b = 2, 3, \ldots \) is generated by

\[
(y - 1) \int_0^x g(x, y) \, dy + g(x, y) - x = \frac{e^x - e^{xy}}{e^y - e^x} + \frac{(y - 1) \left( \log(1 - y) - \log(e^{xy} - e^x) \right)}{y} - x
\]

\[
= x^2 y + \frac{1}{3} x^3 y(y + 2) + \frac{1}{12} x^4 y \left( y^2 + 7y + 4 \right) + \frac{1}{60} x^5 y \left( y^3 + 18y^2 + 33y + 8 \right)
\]

\[
+ \frac{1}{360} x^6 y \left( y^4 + 41y^3 + 171y^2 + 131y + 16 \right) + O \left( x^7 \right)
\]

This procedure can be repeated for \( n_t = n_b - 1, n_b - 2, \ldots \) with expressions involving double, triple, \ldots integrals of \( g(x, y) \).

Why Eulerian numbers should play a role here can be explained from scratch by a simple combinatorial argument\footnote{I thank S. Wagner for drawing my attention to this explanation.} let us use the notations that the boat which came up \( i \)th in
the first race had rank \(a(i)\) in the second race: \((a(1), a(2), \ldots, a(i), \ldots, a(n_b))\) is a random permutation of \((1, 2, \ldots, i, \ldots, n_b)\). In the case of interest where the score of the virtual boat is the middle score \(n_b + 1\), the boat that came \(i\)th in the first race beats the virtual boat if \(a(i) \leq n_b - i\), that is to say if it is better than \(i\)th, counting from the bottom, in the second race. Therefore, the counting problem of \(P_{n_i=n_b+1}(m)\) is equivalent to counting so-called excedances: an excedance in a permutation \((a(1), a(2), \ldots, a(i), \ldots, a(n_b))\) is an element such that \(a(i) > i\). The number of permutations with precisely \(m\) excedances is known to be an Eulerian number (thus excedances are what is called an Eulerian statistic, see for example [2] p. 23). As an illustration look at the case \(n_b = 3\): the six permutations of \((1, 2, 3)\) are \((1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\). The numbers of excedances (here defined as \(a(i) \leq n_b - i\)) are respectively 1, 1, 2, 1, 1, 0 which indeed yields the Eulerian \((3, k)\) numbers

\[1, 4, 1\]

for \(k = 0, 1, 2\) (1 permutation with 0 excedance, 4 permutations with 1 excedance, 1 permutation with 2 excedances) appearing in \(P_{n_i=n_b+1}(m)\), \(m \in [1, n_b + 1]\), \(n_b = 3\).

There is still another way to look at the problem in terms of Stirling numbers of the second kind, here defined as

\[n_m(i) = \frac{1}{(−i + n_t − 1)!} \sum_{j=1}^{n_t−1} j^{n_t−2}(-1)^{−i+j+n_t} \binom{−i+n_t−1}{j−1}\]

Stirling numbers count in how many ways the numbers \((1, 2, \ldots, n_t−1)\) can be partitioned in \(i\) groups: for example for \(n_t = 5\)

\[\rightarrow 1 \text{ way to split the numbers } (1, 2, 3, 4) \text{ into } 4 \text{ groups}
\]

\[(1), (2), (3), (4)\]

\[\rightarrow 6 \text{ ways to split the numbers } (1, 2, 3, 4) \text{ into } 3 \text{ groups}
\]

\[(1), (2), (3, 4); (1), (3), (2, 4); (1), (4), (2, 3); (2), (3), (1, 4); (2), (4), (1, 3); (3), (4), (1, 2)\]

\[\rightarrow 7 \text{ ways to split the numbers } (1, 2, 3, 4) \text{ into } 2 \text{ groups}
\]

\[(1), (2, 3, 4); (2), (1, 3, 4); (3), (1, 2, 4); (4), (1, 2, 3); (1, 2), (3, 4); (1, 3), (2, 4); (1, 4), (2, 3)\]

\[\rightarrow 1 \text{ way to split the numbers } (1, 2, 3, 4) \text{ into } 1 \text{ group}
\]

\[(1, 2, 3, 4)\]

so that one obtains

\[1, 6, 7, 1\]

The probability distribution \(P_{n_i}(m)\) can indeed be rewritten [3] as

\[P_{n_i}(m) = \frac{1}{n_b!} \sum_{i=m}^{n_t−1} (-1)^{i+m} n_m(i)(1 + n_b - i)! \left(\frac{i - 1}{m - 1}\right)\]

Why Stirling numbers should play a role here arises [3] from graph counting considerations on the \(n_b \times n_b\) lattice when one now includes all the points below the diagonal. As an example let us still consider the case \(n_t = 5\): below the diagonal \(n_t = 5\) they are 6 points \(a, b, c, d, e, f\) labelled by their lattice coordinates \(a = (1, 1), b = (1, 2), c = (1, 3), d =\)
One draws a graph according to the no ex aequo exclusion rule: starting say from the point \( a \) one links it to another point if it obeys the no ex aequo exclusion rule with respect to \( a \), that is to say if its coordinates are not \((1,1)\). In our example there is only one such point \( e \). Then one can link the point \( e \) to the points \( c \) and \( f \), which can also be linked together. Finally one can link \( c \) to \( d \) and \( f \) to \( b \) with also a link between \( d \) and \( b \). The numbers \( n_{nt}=5(i+1) \) count in the graph just obtained the number of subgraphs with either \( i=1 \) point (this is the number of points 6), \( i=2 \) points linked (there are 7 such cases), \( i=3 \) points fully linked (there is 1 such case), \( i=4 \) points fully linked (there is no such case), \ldots, a counting which finally gives the Stirling-like numbers

\[ 1, 6, 7, 1 \]

where the 1 on the left is by convention the number of subgraphs with \( i=0 \) point.

It still remains to be shown why this subgraph counting is indeed equivalent to the Stirling counting. To do this one has simply to notice that the former is encapsulated in a recurrence relation obtained by partitioning the \( n_{nt+1}(i+1) \) lattice counting as:

- either there is 0 point on the diagonal \( n_{t} \rightarrow n_{t}(i+1) \left( \begin{array}{c} n_{t} - 1 \\ 0 \end{array} \right) \)
- either there is 1 point on the diagonal \( n_{t} \rightarrow n_{t-1}(i) \left( \begin{array}{c} n_{t} - 1 \\ 1 \end{array} \right) \)
- either there are 2 points on the diagonal \( n_{t} \rightarrow n_{t-2}(i-1) \left( \begin{array}{c} n_{t} - 1 \\ 2 \end{array} \right) \)

etc

It follows that the numbers \( n_{nt+1}(i+1) \) have to obey the recurrence relation

\[
 n_{nt+1}(i+1) = \sum_{k'=0}^{i} n_{nt-k'}(i+1-k') \left( \begin{array}{c} n_{t} - 1 \\ k' \end{array} \right)
\]

valid for \( i \in [0, nt-2] \), bearing in mind that when \( i = nt - 1 \) trivially \( n_{nt+1}(nt) = 1 \). But this recurrence relation can be mapped on a more standard recurrence relation for the Stirling numbers of the second kind: if one sets \( k = nt - i \in [2, nt] \) and defines now Stirling\((nt,k) \equiv n_{nt+1}(i+1) \) one obtains for the Stirling\((nt,k) \)'s

\[
 \text{Stirling}(nt, k) = \sum_{k'=0}^{nt-k} \text{Stirling}(nt - k' - 1, k' - 1) \left( \begin{array}{c} n_{t} - 1 \\ k' \end{array} \right)
\]

which via \( k'' = nt - k' \in [k, nt] \) rewrites as

\[
 \text{Stirling}(nt, k) = \sum_{k''=k}^{nt} \text{Stirling}(k'' - 1, k' - 1) \left( \begin{array}{c} n_{t} - 1 \\ k'' \end{array} \right)
\]

that is to say finally

\[
 \text{Stirling}(nt+1, k + 1) = \sum_{k''=k}^{nt} \text{Stirling}(k'' - 1, k - 1) \left( \begin{array}{c} n_{t} - 1 \\ k'' \end{array} \right)
\]

This therefore establishes that the Stirling\((nt,k) \)'s, i.e. the \( n_{n}(i) \)'s, are indeed Stirling numbers of the second kind. This is of course not a surprise that both Eulerian and
Stirling numbers do play a role since there is a correspondence between them

\[
\text{Eulerian}(n_b, k) = \sum_{j=1}^{k+1} (-1)^{k-j+1} \binom{n-j}{n-k-1} j! \text{Stirling}(n_b, j)
\]

with \( k \in [0, n_b - 1] \).

Why does one rewrite \( P_n(m) \) in terms of Stirling numbers in the 2-race case? Because this rewriting can be generalized \([3]\) to the \( n_r \)-race case. The generalisation is formal since one does not know what the \( n_r \)-dependant "generalized Stirling" numbers which control the probability distribution are. Again this is like moving from a solvable "2-body" problem to a so far non solvable "\( n_r \)-body" problem.

Still it remains quite fascinating that well-known numbers in combinatorics, such as Eulerian and Stirling numbers, should be at the heart of the understanding of rank expectations in regattas, at least in the 2-race case.

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\(^5\)For Eulerian, Stirling, and other well-known numbers in combinatorics see for example \([2]\).

\(^6\)The lattice becomes larger with the number of races: as said above when \( n_r = 3 \) one has a cubic lattice. The corresponding graph structure becomes more and more involved, and so the counting of the associated fully connected subgraphs.