HYPERFINITE CONSTRUCTION OF $G$-EXPECTATION

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Abstract. The hyperfinite $G$-expectation is a nonstandard discrete analogue of $G$-expectation (in the sense of Robinsonian nonstandard analysis). A lifting of a continuous-time $G$-expectation operator is defined as a hyperfinite $G$-expectation which is infinitely close, in the sense of nonstandard topology, to the continuous-time $G$-expectation. We develop the basic theory for hyperfinite $G$-expectations and prove an existence theorem for liftings of (continuous-time) $G$-expectation. For the proof of the lifting theorem, we use a new discretization theorem for the $G$-expectation (also established in this paper, based on the work of Dolinsky, Nutz and Soner [Stoch. Proc. Appl. 122, (2012), 664–675]).

Keywords: $G$-expectation; Volatility uncertainty; Weak limit theorem; Lifting theorem; Nonstandard analysis; Hyperfinite discretization.

1. Introduction

Dolinsky et al. [8] showed a Donsker-type result for $G$-Brownian motion by introducing a notion of volatility uncertainty in discrete time and defined a discrete version of Peng’s $G$-expectation. In the continuous-time limit, the resulting sublinear expectation converges weakly to $G$-expectation. In their discretization, Dolinsky et al. [8] allow for martingale laws whose support is the whole set of reals in a $d$-dimensional setting. In other words, they only discretize the time line, but not the state space of the canonical process. Now for certain applications, for example, a hyperfinite construction of $G$-expectation in the sense of Robinsonian nonstandard analysis, a discretization of the state space would be necessary. Thus, we develop a modification of the construction by Dolinsky et al. [8] which even ensures that the sublinear expectation operator for the discrete-time canonical process corresponding to this discretization of the state space (whence the martingale laws are supported by a finite lattice only) converges to the $G$-expectation. Further, we prove a lifting theorem, in the sense of Robinsonian nonstandard analysis, for the $G$-expectation. Herein, we use the discretization result for the $G$-expectation.

Nonstandard analysis makes consistent use of infinitesimals in mathematical analysis based on techniques from mathematical logic. This approach is very promising because it also allows, for instance, to study continuous-time stochastic processes as formally finite objects. Many authors have applied nonstandard analysis to problems in measure theory, probability theory and mathematical economics (see for example, Anderson and Raimondo [3] and the references therein or the contribution in Berg [4]), especially after Loeb [20] converted nonstandard...
measures (i.e. the images of standard measures under the nonstandard embedding *) into real-valued, countably additive measures, by means of the standard part operator and Caratheodory’s extension theorem. One of the main ideas behind these applications is the extension of the notion of a finite set known as hyperfinite set or more causally, a formally finite set. Very roughly speaking, hyperfinite sets are sets that can be formally enumerated with both standard and nonstandard natural numbers up to a (standard or nonstandard, i.e. unlimited) natural number.

Anderson [2], Keisler [16], Lindstrøm [19], Hoover and Perkins [14], a few to mention, used Loeb’s [20] approach to develop basic nonstandard stochastic analysis and in particular, the nonstandard Itô calculus. Loeb [20] also presents the construction of a Poisson processes using nonstandard analysis. Anderson [2] showed that Brownian motion can be constructed from a hyperfinite number of coin tosses, and provides a detailed proof using a special case of Donsker’s theorem. Anderson [2] also gave a nonstandard construction of stochastic integration with respect to his construction of Brownian motion. Keisler [16] uses Anderson’s [2] result to obtain some results on stochastic differential equations. Lindstrøm [19] gave the hyperfinite construction (lifting) of $L^2$ standard martingales. Using nonstandard stochastic analysis, Perkins [24] proved a global characterization of (standard) Brownian local time. In this paper, we do not work on the Loeb space because the $G$-expectation and its corresponding $G$-Brownian motion are not based on a classical probability measure, but on a set of martingale laws.

The aim of this paper is to give two approximation results on $G$-expectation. First, to refine the discretization of $G$-expectation by Dolinsky et al. [8], in order to obtain a discretization of the sublinear expectation where the martingale laws are defined on a finite lattice rather than the whole set of reals. Second, to give an alternative, combinatorially inspired construction of the $G$-expectation based on the discretization result. We hope that this result may eventually become useful for applications in financial economics (especially existence of equilibrium on continuous-time financial markets with volatility uncertainty) and provides additional intuition for Peng’s $G$-stochastic calculus. We begin the nonstandard treatment of the $G$-expectation by defining a notion of $S$-continuity, a standard part operator, and proving a corresponding lifting (and pushing down) theorem. Thereby, we show that our hyperfinite construction is the appropriate nonstandard analogue of the $G$-expectation.

The rest of this paper is divided into two parts: in the first part, Section 2, we define Peng’s $G$-expectation and introduce a discrete-time analogue of a $G$-expectation in the spirit of Dolinsky et al. [8]. Unlike in Dolinsky et al. [8], we require the discretization of the martingale laws to be defined on a finite lattice rather than the whole set of reals. In the continuous-time limit, the resulting sublinear expectation converges weakly to the continuous-time $G$-expectation. In the second part, Section 3, we develop the basic theory for hyperfinite $G$-expectations and prove an existence theorem for liftings of (continuous-time) $G$-expectation. We extend the discrete time analogue of the $G$-expectation in Section 2 to a hyperfinite time analogue. Then, we use the characterization of convergence in nonstandard analysis to prove that the hyperfinite discrete-time analogue of the $G$-expectation is infinitely close in the sense of nonstandard topology to the continuous-time $G$-expectation.
2. Weak approximation of $G$-expectation with discrete state space

Peng [23] introduced a sublinear expectation on a well-defined space $L^1_G$, the completion of $\text{Lip}_{b,cyl}(\Omega)$ (bounded and Lipschitz cylinder function) under the norm $\|\cdot\|_{L^1_G}$, under which the increments of the canonical process $(B_t)_{t \geq 0}$ are zero-mean, independent and stationary and can be proved to be $(G)$-normally distributed. This type of process is called $G$-Brownian motion and the corresponding sublinear expectation is called $G$-expectation.

The $G$-expectation $\xi \mapsto \mathcal{E}^G(\xi)$ is a sublinear operator defined on a class of random variables on $\Omega$. The symbol $G$ refers to a given function

$$G(\gamma) := \frac{1}{2} \sup_{c \in \mathcal{D}} c\gamma : \mathbb{R} \to \mathbb{R}$$

where $\mathcal{D} = [r_D, R_D]$ is a nonempty, compact and convex set, and $0 \leq r_D \leq R_D < \infty$ are fixed numbers. The construction of the $G$-expectation is as follows. Let $\xi = f(B_T)$, where $B_T$ is the $G$-Brownian motion and $f$ a sufficiently regular function. Then $\mathcal{E}^G(\xi)$ is defined to be the initial value $u(0,0)$ of the solution of the nonlinear backward heat equation,

$$-\partial_t u - G(\partial_{xx}^2 u) = 0,$$

with terminal condition $u(\cdot, T) = f$, Pardoux and Peng [22]. The mapping $\mathcal{E}^G$ can be extended to random variables of the form $\xi = f(B_{t_1}, \ldots, B_{t_n})$ by a step-wise evaluation of the PDE and then to the completion $L^1_G$ of the space of all such random variables (cf. Dolinsky et al. [8]). Denis et al. [7] showed that $L^1_G$ is the completion of $\mathcal{C}_b(\Omega)$ and $\text{Lip}_{b,cyl}(\Omega)$ under the norm $\|\cdot\|_{L^1_G}$, and that $L^1_G$ is the space of the so-called quasi-continuous function and contains all bounded continuous functions on the canonical space $\Omega$, but not all bounded measurable functions are included. Ruan [27] introduced the invariance principle of $G$-Brownian motion using the theory of sublinear expectation. There also exists an equivalent alternative representation of the $G$-expectation known as the dual view on $G$-expectation via volatility uncertainty, see Denis et al. [7]:

$$\mathcal{E}^G(\xi) = \sup_{P \in \mathcal{P}^G} \mathbb{E}^P[\xi], \quad \xi = f(B_T),$$

where $\mathcal{P}^G$ is defined as the set of probability measures on $\Omega$ such that, for any $P \in \mathcal{P}^G$, $B$ is a martingale with the volatility $d\langle B \rangle_t / dt \in D P \otimes dt$ a.e.

2.1. Continuous-time construction of sublinear expectation. Let $\Omega = \{\omega \in \mathcal{C}([0,T]; \mathbb{R}) : \omega_0 = 0\}$ be the canonical space endowed with the uniform norm $\|\omega\|_\infty = \sup_{0 \leq t \leq T} |\omega_t|$, where $|\cdot|$ denotes the absolute value on $\mathbb{R}$. Let $B$ be the canonical process $B_t(\omega) = \omega_t$, and $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ the filtration generated by $B$. A probability measure $P$ on $\Omega$ is a martingale law provided $B$ is a $P$-martingale and $B_0 = 0$ $P$ a.s. Then, $\mathcal{P}_D$ is the set of martingale laws on $\Omega$ and the volatility takes values in $D, P \otimes dt$ a.e.;

$$\mathcal{P}_D = \{P \text{ martingale law on } \Omega: d\langle B \rangle_t / dt \in D, P \otimes dt \text{ a.e.}\}.$$

2.2. Discrete-time construction of sublinear expectation. We denote

$$\mathcal{L}_n = \left\{ \frac{j}{n\sqrt{n}}, \quad -n^2\sqrt{R_D} \leq j \leq n^2\sqrt{R_D}, \quad \text{for } j \in \mathbb{Z} \right\},$$
and $\mathcal{L}_{n+1} = \mathcal{L}_n \times \cdots \times \mathcal{L}(n+1)$ times, for $n \in \mathbb{N}$. Let $X^n = (X^n_k)_{k=0}^n$ be the canonical process $X^n_k(x) = x_k$ defined on $\mathcal{L}_{n+1}$ and $(\mathcal{F}_k^n)_{k=0}^n = \sigma(X^n_l, l = 0, \ldots, k)$ be the filtration generated by $X^n$. We note that $R_{D_n} = \sup_{x \in D} |x|$. 

\[ D'_n = D \cap \left(\frac{1}{n} \mathbb{N}\right)^2 \]

is a nonempty bounded set of volatilities. A probability measure $P$ on $\mathcal{L}_{n+1}$ is a martingale law provided $X^n$ is a $P$-martingale and $X^n_0 = 0$ $P$ a.s. The increment $\Delta X^n_k = X^n_k - X^n_{k-1}$. Let $\mathcal{P}_D^n$ be the set of martingale laws of $X^n$ on $\mathbb{R}^{n+1}$, i.e.,

\[ \mathcal{P}_D^n = \left\{ P \text{ martingale law on } \mathbb{R}^{n+1}; r_D \leq |\Delta X^n_k|^2 \leq R_D, P \text{ a.s.} \right\}, \]

such that for all $n$, $\mathcal{L}_{n+1} \subseteq \mathbb{R}^{n+1}$.

In order to establish a relation between the continuous-time and discrete-time settings, we obtained a continuous-time process $\hat{x}_t \in \Omega$ from any discrete path $x \in \mathcal{L}_{n+1}$ by linear interpolation. i.e.,

\[ \hat{x}_t := \left(\lfloor nt/T \rfloor + 1 - nt/T\right)x_{\lfloor nt/T \rfloor} + \left(nt/T - \lfloor nt/T \rfloor\right)x_{\lfloor nt/T \rfloor + 1} \]

where $\hat{\cdot} : \mathcal{L}_{n+1} \to \Omega$ is the linear interpolation operator, $x = (x_0, \ldots, x_n) \mapsto \hat{x} = \{(\hat{x})_{0 \leq t \leq T}\}$, and $|y|$ denotes the greatest integer less than or equal to $y$. If $X^n$ is the canonical process on $\mathcal{L}_{n+1}$ and $\xi$ is a random variable on $\Omega$, then $\xi(\hat{X}^n)$ defines a random variable on $\mathcal{L}_{n+1}$.

2.3. Strong formulation of volatility uncertainty. We consider martingale laws generated by stochastic integrals with respect to a fixed Brownian motion as in Dolinsky et al. [8], Nutz [21] and a fixed random walk as in Dolinsky et al. [8].

Continuous-time construction; let $\mathcal{Q}_D$ be the set of martingale laws:

\[ \mathcal{Q}_D = \left\{ P_0 \circ (M)^{-1}; M = \int f(t, B) dB_t, \text{ and } f \in C([0, T] \times \Omega; \sqrt{D}) \text{ is adapted} \right\}. \]

$B$ is the canonical process under the Wiener measure $P_0$.

Discrete-time construction; we fix $n \in \mathbb{N}$, $\Omega_n = \{\omega = (\omega_1, \ldots, \omega_n) : \omega_i \in \{\pm 1\}, \ i = 1, \ldots, n\}$ equipped with the power set and let

\[ P_n = \frac{\delta_{-1} + \delta_{+1}}{2} \otimes \cdots \otimes \frac{\delta_{-1} + \delta_{+1}}{2} \]

be the product probability associated with the uniform distribution where $\delta_x(A)$ is a Dirac measure for any $A \subseteq \mathbb{R}$ and a given $x \in A$. Let $\xi_1, \ldots, \xi_n$ be an i.i.d sequence of $\{\pm 1\}$-valued random variables. The components of $\xi_k$ are orthonormal in $L^2(P_n)$ and the associated scaled random walk is

\[ X = \frac{1}{\sqrt{n}} \sum_{i=1}^k \xi_i. \]

We denote by $\mathcal{Q}_{D_n}^n$ the set of martingale laws of the form:

\[ (3) \quad \mathcal{Q}_{D_n}^n = \left\{ P_n \circ (M_{X, n})^{-1}; f : \{0, \ldots, n\} \times \mathcal{L}_{n+1} \to \sqrt{D_n} \text{ is } \mathcal{F}_n \text{-adapted.} \right\} \]

where $M_{X, n} = \left(\sum_{l=1}^k f(l-1, X)\Delta X_l\right)_{k=0}^n$. 
2.4. Results and proofs. Theorem 1 states that a sublinear expectation with discrete-time volatility uncertainty on our finite lattice converges to the $G$-expectation.

Lemma 2.1. $Q^n_D = \{ P_n \circ (M^{f,x})^{-1} : f : \{0, \ldots, n\} \times \mathbb{R}^{n+1} \to \sqrt{D} \text{ is adapted} \}$. Then $Q^n_D \subseteq P^n_D$.

Proposition 2.2. Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$. Then,

(i) \[
\lim_{n \to \infty} \sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] = \sup_{P \in P^n_D} E^P[\xi].
\]

(ii) \[
\sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] = \max_{Q \in Q^n_{D/n}} E^Q[\hat{X}^n].
\]

To prove (4), we prove two separate inequalities together with a density argument. The left-hand side of (5) can be written as

\[
\sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] = \sup_{f \in A} E^{P_n(o(M^{f,x})^{-1})}[\xi(\hat{X}^n)],
\]

where $A = \{ f : \{0, \ldots, n\} \times L^{n+1} \to \sqrt{D}/n \text{ is } \mathcal{F}^n\text{-adapted} \}$. We prove that $A$ is a compact subset of a finite-dimensional vector space, and that $f \mapsto E^{P_n(o(M^{f,x})^{-1})}[\xi(\hat{X}^n)]$ is continuous. Before then, we introduce a smaller space $L^1_G$ that is defined as the completion of $C_b(\Omega; \mathbb{R})$ under the norm (cf. Dolinsky et al. \[8\])

\[
\| \xi \|_{L^1_G} = \sup_{Q \in Q} E^Q[\xi], \quad Q := P \cup \{ P \circ (\hat{X}^n)^{-1} : P \in P^n_D, n \in \mathbb{N} \}.
\]

This is because Proposition 2.2 will not hold if $\xi$ just belong to $L^1_G$, which is the completion of $C_b(\Omega; \mathbb{R})$ under the norm

\[
\| \xi \|_{L^1_G} := \sup_{P \in P^n_D} E^P[|\xi|].
\]

Proof of Proposition 2.2. First inequality (for $\preceq$ in (4)):

(7) \[
\lim_{n \to \infty} \sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] \leq \sup_{P \in P^n_D} E^P[\xi].
\]

For all $n$, $\sqrt{D'/n} \subseteq \sqrt{D}/n$ and $Q^n_{D/n} \subseteq Q^n_D$. It is shown in Dolinsky et al. \[8\] that

\[
\lim_{n \to \infty} \sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] \leq \sup_{P \in P^n_D} E^P[\xi].
\]

Since $Q^n_D \subseteq P^n_D$ (see Dolinsky et al. \[8, Remark 3.6\]) and $Q^n_D \subseteq P^n_D$ (see Lemma 2.1), (7) follows.

Second inequality (for $\succeq$ in (4)): It remains to show that

\[
\lim_{n \to \infty} \sup_{Q \in Q^n_{D/n}} E^Q[\xi(\hat{X}^n)] \geq \sup_{P \in P^n_D} E^P[\xi].
\]

For arbitrary $P \in Q^*_D$, we construct a sequence $(P^n)_{n}$ such that for all $n$,

(8) \[
P^n \in Q^n_{D/n},
\]
and
\[ \mathbb{E}^P[\xi] \leq \liminf_{n \to \infty} \mathbb{E}^{P^n}[\xi(\hat{X}^n)]. \]

For fixed \( n \), we want to construct martingales \( M^n \) whose laws are in \( \mathcal{Q}_{\mathbb{D}_n/n}^n \) and the laws of their interpolations tend to \( P \). Thus, we introduce a scaled random walk with the piecewise constant càdlàg property,

\[ W^n_t := \frac{1}{\sqrt{n}} \sum_{l=1}^{|nt/T|} \xi_l = \frac{1}{\sqrt{n}} Z^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T, \]

and we denote the continuous version of (10) obtained by linear interpolation by

\[ \tilde{W}^n_t := \frac{1}{\sqrt{n}} \tilde{Z}^n_{\lfloor nt/T \rfloor}, \quad 0 \leq t \leq T. \]

By the central limit theorem; \( (W^n, \tilde{W}^n) \to (W, \tilde{W}) \) as \( n \to \infty \) on \( D([0, T]; \mathbb{R}^2) \) \((\Rightarrow \) implies convergence in distribution\). i.e., the law \( (P_n) \) converges to the law \( P_0 \) on the Skorohod space \( D([0, T]; \mathbb{R}^2) \) Billingsley [5, Theorem 27.1]. Let \( g \in C([0, T] \times \Omega, \sqrt{\mathbb{D}}) \) such that

\[ P = P_0 \circ \left( \int_{\mathbb{R}^d} g(t, W) dW_t \right)^{-1}. \]

Since \( g \) is continuous and \( \tilde{W}^n_t \) is the interpolated version of (10),

\[ \left( W^n, \left( g \left( \lfloor nt/T \rfloor T/n, \tilde{W}^n_t \right) \right)_{t \in [0, T]} \right) \Rightarrow \left( W, g(t, W_t) \right)_{t \in [0, T]} \] as \( n \to \infty \) on \( D([0, T]; \mathbb{R}^2) \).

We introduce martingales with discrete-time integrals,

\[ M^n_k := \sum_{l=1}^k g \left( (t-1)T/n, W^n \right) \tilde{W}^n_{tT/n} - \tilde{W}^n_{(t-1)T/n}. \]

In order to construct \( M^n \) which is “close” to \( M \) and also is such that \( P_n \circ (M^n)^{-1} \in \mathcal{Q}_{\mathbb{D}_{n/n}}^n \).

We choose \( \tilde{h}_n : \{0, \cdots, n\} \times \Omega \to \sqrt{\mathbb{D}_{n/n}} \) such that

\[ d_{\mathcal{J}_1} \left( \left( \tilde{h}_n(\lfloor nt/T \rfloor T/n, \tilde{W}^n_t) \right)_{t \in [0, T]} ; \left( g(\lfloor nt/T \rfloor T/n, \tilde{W}^n_t) \right)_{t \in [0, T]} \right) \]

is minimal (this is possible because there are only finitely many choices for \( \left( \tilde{h}_n(\lfloor nt/T \rfloor T/n, \tilde{W}^n_t) \right)_{t \in [0, T]} \)) and \( d_{\mathcal{J}_1} \) is the Kolmogorov metric for the Skorohod \( \mathcal{J}_1 \) topology. From Billingsley [6, Theorem 4.3 and Definition 4.1], it follows that

\[ \left( W^n, \left( \tilde{h}_n(\lfloor nt/T \rfloor T/n, \tilde{W}^n_t) \right)_{t \in [0, T]} \right) \Rightarrow \left( W, g(t, W_t) \right)_{t \in [0, T]} \] on \( D([0, T]; \mathbb{R}^2) \).

We then define \( g_n : \{0, \cdots, n\} \times \mathbb{R}_{n+1} \to \sqrt{\mathbb{D}_{n/n}} \) by \( g_n : (t, \tilde{X}) \mapsto \tilde{h}_n(t, \tilde{X}) \). Let \( M^n \) be defined by

\[ M^n_k = \sum_{l=1}^k g_n \left( (l-1, 1/\sqrt{n} Z^n_\cdot) \right) \frac{1}{\sqrt{n}} \Delta Z^n_l, \quad \forall k \in \{0, \cdots, n\}. \]
By stability of stochastic integral (see Duffie and Protter [9, Theorem 4.3 and
Definition 4.1]),
\[
\left( M^n_{[nt/T]} \right)_{t \in [0,T]} \Rightarrow M \quad \text{as } n \to \infty \text{ on } D([0,T]; \mathbb{R})
\]
because
\[
M^n_{[nt/T]} = \sum_{l=1}^{[nt/T]} \tilde{h}_n \left( (l-1)T/n, \left( \tilde{W}_{kT/n} \right)_{k=0}^n \right) \Delta \tilde{W}_{lT/n}.
\]
In addition, as \( n \) goes to \( \infty \), the increments of \( M^n \) uniformly tend to 0. Thus, \( \tilde{M}^n \Rightarrow M \) on \( \Omega \). Since \( \xi \) is bounded and continuous,
\[
\lim_{n \to \infty} E^{P_n \circ (M^n)^{-1}}[\xi(\hat{X}^n)] = E^{P_0 \circ M^{-1}}[\xi].
\]
Therefore, (8) is satisfied for \( P^n = P_n \circ (M^n)^{-1} \in Q_{D_n}^\infty \). Taking the \( \lim \inf \) as \( n \) tends to \( \infty \) and the supremum over \( P \in D \), (13) becomes
\[
\sup_{P \in D} E^P[\xi] \leq \lim \inf_{n \to \infty} \sup_{Q \in Q_{D_n}^\infty} E^Q[\xi(\hat{X}^n)].
\]
Combining (7) and (14),
\[
\sup_{P \in D} E^P[\xi] \geq \lim \sup_{n \to \infty} \sup_{Q \in Q_{D_n}^\infty} E^Q[\xi(\hat{X}^n)] \geq \lim \inf_{n \to \infty} \sup_{Q \in Q_{D_n}^\infty} E^Q[\xi(\hat{X}^n)] \geq \sup_{P \in D} E^P[\xi].
\]
Therefore,
\[
\sup_{P \in D} E^P[\xi] = \lim_{n \to \infty} \sup_{Q \in Q_{D_n}^\infty} E^Q[\xi(\hat{X}^n)].
\]

Density argument: (4) is established for all \( \xi \in C_b(\Omega, \mathbb{R}) \). Since \( Q_D \subseteq P_D \) (see Dolinsky et al. [8, Remark 3.6]) and \( Q_{D_n}^\infty \subseteq P_D \) (see Lemma 2.1), \( Q_{D_n}^\infty \subseteq Q \) and \( Q_D \subseteq Q \). Thus, (4) holds for all \( \xi \in \mathbb{L}^1 \), and hence, holds for all \( \xi \) that satisfy condition of Proposition 2.2.

First part of 5: \( \mathcal{A} \) is closed and obviously bounded with respect to the norm \( \| \cdot \|_\infty \) as \( D_n^\infty \) is bounded. By Heine-Borel theorem, \( \mathcal{A} \) is a compact subset of a \( N(n,n) \)-dimensional vector space\(^1\) equipped with the norm \( \| \cdot \|_\infty \).

Second part of 5: Here, we show that \( F : f \mapsto E^{P_n \circ (M^{f,X})^{-1}}[\xi(\hat{X}^n)] \) is continuous. From Proposition 2.2 we know that \( \xi \) is continuous, \( \hat{X}^n \) is the interpolated canonical process, i.e., \( \hat{X} : \mathcal{L}_n^{n+1} \to \Omega \), thus \( \hat{X}^n \) is continuous and \( P_n \) takes it values from the set of real numbers. For \( F : f \mapsto E^{P_n \circ (M^{f,X})^{-1}}[\xi(\hat{X}^n)] \) to be continuous, \( \psi : f \mapsto M^{f,X} \) has to be continuous. Since \( \mathcal{A} \) is a compact subset of a \( N(n,n) \)-dimensional vector space for fixed \( n \in \mathbb{N} \) and \( M^{f,X} : \Omega_n \to \mathcal{L}_n^{n+1} \), for all \( f, g \in \mathcal{A} \),
\[
|M^{f,X} - M^{g,X}| = \|f - g\|_\infty \leq \|f - g\|_\infty.
\]
Thus, \( \psi \) is continuous with respect to the norm \( \| \cdot \|_{\infty} \). Hence \( F \) is continuous with respect to any norm on \( \mathbb{R}^{N(n,n)} \).

\(^1\)The cardinality of \( \mathcal{L}_n, \#\mathcal{L}_n = 2n + 1 \), \#\( \mathcal{L}_n^{n+1} = (2n + 1)^{n+1} \), and \#(\( \{0, \ldots, n\} \times \mathcal{L}_n^{n+1} \)) = \( (n+1)(2n+1)^{n+1} = N(n,n) \).
Theorem 1. Let $\xi : \Omega \to \mathbb{R}$ be a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_\infty)^b$ for some constants $a, b > 0$. Then,

$$\sup_{P \in \mathcal{G}_D} \mathbb{E}^P[\xi] = \lim_{n \to \infty} \max_{Q \in \mathcal{G}_{D_{1/n}}} \mathbb{E}^Q[\xi(\widetilde{X}^n)].$$

Proof. The proof follows directly from Proposition 2.2. 

3. Nonstandard construction of $G$-expectation

3.1. Hyperfinite-time setting. Here we present the nonstandard version of the discrete-time setting of the sublinear expectation and the strong formulation of volatility uncertainty on the hyperfinite timeline.

Definition 3.1. $^*\Omega$ is the $^*$-image of $\Omega$ endowed with the $^*$-extension of the maximum norm $^*\| \cdot \|_\infty$.

$^*D = ^*[^{\mathfrak{fr}}D, R_D]$ is the $^*$-image of $D$, and as such it is internal.

It is important to note that $st : ^*\Omega \to \Omega$ is the standard part map, and $st(\omega)$ will be referred to as the standard part of $\omega$, for every $\omega \in ^*\Omega$. $^z\omega$ denotes the standard part of a hyperreal $\omega$.

Definition 3.2. For every $\omega \in \Omega$, if there exists $\tilde{\omega} \in ^*\Omega$ such that $\|\tilde{\omega} - ^z\omega\|_\infty \simeq 0$, then $\tilde{\omega}$ is a nearstandard point in $^*\Omega$. This will be denoted as $ns(\tilde{\omega}) \in ^*\Omega$.

For all hypernatural $N$, let

$$L_N = \left\{-\frac{K}{N\sqrt{N}}, \ldots, -N^2\sqrt{R_D} \leq K \leq N^2\sqrt{R_D}, K \in ^*\mathbb{Z}\right\},$$

and the hyperfinite timeline

$$T = \left\{0, \frac{T}{N}, \ldots, -\frac{T}{N} + T, T\right\}.$$

We consider $L_N$ as the canonical space of paths on the hyperfinite timeline, and $X_N = (X_N^N)_{k=0}^N$ as the canonical process denoted by $X_k^N(\tilde{\omega}) = \tilde{\omega}_k$ for $\tilde{\omega} \in L_N$. $F_X$ is the internal filtration generated by $X_N$. The linear interpolation operator can be written as

$$\tilde{\omega}^{-} : \gamma^{-1} \to ^*\Omega, \text{ for } L_N \subseteq ^*\Omega,$$

where

$$\tilde{\omega}(t) := ([Nt/T] + 1 - Nt/T)\omega_{\lfloor Nt/T \rfloor} + (Nt/T - \lfloor Nt/T \rfloor)\omega_{\lfloor Nt/T \rfloor + 1},$$

for $\omega \in L_N$ and for all $t \in ^*[0,T]$. $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to $\gamma$ and $\gamma : T \to \{0, \ldots, N\}$ for $\gamma : t \to Nt/T$.

For the hyperfinite strong formulation of the volatility uncertainly, fix $N \in ^*\mathbb{N} \setminus \mathbb{N}$. Consider $\left\{\pm\frac{1}{N}\right\}^\mathbb{T}$, and let $P_N$ be the uniform counting measure on $\left\{\pm\frac{1}{N}\right\}^\mathbb{T}$. $P_N$ can also be seen as a measure on $L_N$, concentrated on $\left\{\pm\frac{1}{N}\right\}^\mathbb{T}$.

Let $\Omega_N = \{\omega = (\omega_1, \ldots, \omega_N) : \omega_i = \{\pm 1\}, i = 1, \ldots, N\}$, and let $\Xi_1, \ldots, \Xi_N$ be a $^*$-independent sequence of $\{\pm 1\}$-valued random variables on $\Omega_N$ and the components
of $\Xi_k$ are orthonormal in $L^2(P_N)$. We denote the hyperfinite random walk by
\[ \mathcal{X}_t = \frac{1}{\sqrt{N}} \sum_{l=1}^{N_{t/T}} \Xi_l \quad \text{for all } t \in \mathbb{T}. \]

The hyperfinite-time stochastic integral of some $F : \mathbb{T} \times \mathcal{L}_{N}^\ast \rightarrow \ast \mathbb{R}$ with respect to the hyperfinite random walk is given by
\[ \sum_{s=0}^{t} F(s, \mathcal{X}) \Delta \mathcal{X}_s : \Omega_N \rightarrow \ast \mathbb{R}, \quad \omega \in \Omega_N \mapsto \sum_{s=0}^{t} F(s, \mathcal{X}(\omega)) \Delta \mathcal{X}_s(\omega). \]

Thus, the hyperfinite set of martingale laws can be defined by
\[ \hat{Q}_{D_N}^N = \{ P_N \circ (M^{F,X})^{-1} : F : \mathbb{T} \times \mathcal{L}_{N}^\ast \rightarrow \sqrt{D_N} \} \]
where
\[ D'_N = \mathbb{D} \cap \left( \frac{1}{N} \mathbb{N} \right)^2 \]
and
\[ M^{F,X} = \left( \sum_{s=0}^{t} F(s, \mathcal{X}) \Delta \mathcal{X}_s \right)_{t \in \mathbb{T}}. \]

**Remark 3.1.** Up to scaling, $\hat{Q}_{D_N}^N = Q_{D_n}^N$.

3.2. Results and proofs.

**Definition 3.3 ((Uniform lifting of $\xi$)).** Let $\Xi : \mathcal{L}_{N}^\ast \rightarrow \ast \mathbb{R}$ be an internal function, and let $\xi : \Omega \rightarrow \mathbb{R}$ be a continuous function. $\Xi$ is said to be a *uniform lifting* of $\xi$ if and only if
\[ \forall \bar{\omega} \in \mathcal{L}_{N}^\ast \left( \bar{\omega} \in ns(\ast \Omega) \Rightarrow \ast \Xi(\bar{\omega}) = \xi(\ast \xi(\bar{\omega})) \right), \]
where $st(\bar{\omega})$ is defined with respect to the topology of uniform convergence on $\Omega$.

In order to construct the hyperfinite version of the $G$-expectation, we need to show that the *-image of $\xi$, $\xi$, with respect to $\bar{\omega} \in ns(\ast \Omega)$, is the canonical lifting of $\xi$ with respect to $st(\bar{\omega}) \in \Omega$. i.e., for every $\bar{\omega} \in ns(\ast \Omega)$, $\xi(\ast \xi(\bar{\omega})) = \xi(st(\bar{\omega}))$. To do this, we need to show that $\ast \xi$ is $S$-continuous in every nearstandard point $\bar{\omega}$.

It is easy to prove that there are two equivalent characteristics of $S$-continuity on $\ast \Omega$.

**Remark 3.2.** The following are equivalent for an internal function $\Phi : \ast \Omega \rightarrow \ast \mathbb{R}$;
\begin{enumerate}
    \item $\forall \omega' \in \ast \Omega \left( \ast \|\omega - \omega'\|_{\infty} \simeq 0 \Rightarrow \ast |\Phi(\omega) - \Phi(\omega')| \simeq 0 \right)$.  
    \item $\forall \varepsilon \gg 0, \exists \delta > 0 : \forall \omega' \in \ast \Omega \left( \ast \|\omega - \omega'\|_{\infty} < \delta \Rightarrow \ast |\Phi(\omega) - \Phi(\omega')| < \varepsilon \right)$.  
\end{enumerate}

(The case of Remark 3.2 where $\Omega = \mathbb{R}$ is well known and proved in Stroyan and Luxemburg [28, Theorem 5.1.1])

**Definition 3.4.** Let $\Phi : \ast \Omega \rightarrow \ast \mathbb{R}$ be an internal function. We say $\Phi$ is $S$-continuous in $\omega \in \ast \Omega$, if and only if it satisfies one of the two equivalent conditions of Remark 3.2.

**Proposition 3.3.** If $\xi : \Omega \rightarrow \mathbb{R}$ is a continuous function satisfying $|\xi(\omega)| \leq a(1 + \|\omega\|_{\infty})^b$, for $a, b > 0$, then, $\Xi = \ast \xi \circ \ast$ is a uniform lifting of $\xi$. 
Proof: Fix $\omega \in \Omega$. By definition, $\xi$ is continuous on $\Omega$, i.e., for all $\omega \in \Omega$, and for every $\varepsilon > 0$, there is a $\delta > 0$, such that for every $\omega' \in \Omega$, if
\begin{equation}
\| \omega - \omega' \|_\infty < \delta, \text{ then } |\xi(\omega) - \xi(\omega')| < \varepsilon.
\end{equation}
(19)
By the Transfer Principle: For all $\omega \in \Omega$, and for every $\varepsilon > 0$, there is a $\delta > 0$, such that for every $\omega' \in *\Omega$, (19) becomes,
\begin{equation}
*|\| \omega - \omega' \|_\infty < \delta, \text{ and } *|\xi(\omega) - \xi(\omega')| < \varepsilon.
\end{equation}
(20)
So, $*\xi$ is $S$-continuous in $*\omega$ for all $\omega \in \Omega$. Applying the equivalent characterization of $S$-continuity, Remark 3.2, (20) can be written as
\begin{equation}
*|\| \omega - \omega' \|_\infty \simeq 0, \text{ and } *|\xi(\omega) - \xi(\omega')| \simeq 0.
\end{equation}
We assume $\tilde{\omega}$ to be a nearstandard point. By Definition 3.2, this simply implies,
\begin{equation}
\forall \tilde{\omega} \in ns(*\Omega), \exists \omega \in \Omega : *|\| \tilde{\omega} - \omega \|_\infty \simeq 0.
\end{equation}
(21)
Thus, by $S$-continuity of $*\xi$ in $*\omega$,
\begin{equation}
*|\| \tilde{\omega} - \omega' \|_\infty \simeq 0.
\end{equation}
Using the triangle inequality, if $\omega' \in *\Omega$ with $*|\| \tilde{\omega} - \omega' \|_\infty \simeq 0$,
\begin{equation}
*|\| \omega - \omega' \|_\infty \simeq 0.
\end{equation}
and therefore again by the $S$-continuity of $*\xi$ in $*\omega$,
\begin{equation}
*|\xi(\omega) - \xi(\omega')| \simeq 0.
\end{equation}
And so,
\begin{equation}
*|\xi(\omega) - \xi(\omega')| \simeq 0.
\end{equation}
Thus, for all $\tilde{\omega} \in ns(*\Omega)$ and $\omega' \in *\Omega$, if $*|\| \tilde{\omega} - \omega' \|_\infty \simeq 0$, then,
\begin{equation}
*|\xi(\tilde{\omega}) - \xi(\omega')| \simeq 0.
\end{equation}
Hence, $*\xi$ is $S$-continuous in $\tilde{\omega}$. Equation (21) also implies
\begin{equation}
\tilde{\omega} \in m(\omega) \left( m(\omega) = \bigcap \{ *\Omega : \Omega \text{ is an open neighbourhood of } \omega \} \right)
\end{equation}
such that $\omega$ is unique, and in this case $st(\tilde{\omega}) = \omega$.
Therefore,
\begin{equation}
*\xi(\tilde{\omega}) = \xi(st(\tilde{\omega})).
\end{equation}
\Box

Definition 3.5. Let $\mathcal{E} : *\mathbb{R}^E_{\mathcal{N}} \to *\mathbb{R}$. We say that $\mathcal{E}$ lifts $\mathcal{E}^G$ if and only if for every $\xi : \Omega \to \mathbb{R}$ that satisfies $|\xi(\omega)| \leq a(1 + \| \omega \|_\infty)^b$ for some $a, b > 0$,
\begin{equation}
\mathcal{E}(*\xi \circ \cdot) \simeq \mathcal{E}^G(\xi).
\end{equation}

Theorem 2.
\begin{equation}
\max_{Q \in Q^G_{\mathcal{N}}} \mathbb{E}^Q[\cdot] \text{ lifts } \mathcal{E}^G(\xi).
\end{equation}
(22)
Proof. From Theorem 1,
\begin{equation}
\max_{Q \in \mathcal{Q}_{D_n}^N} \mathbb{E}_Q[\xi(\hat{X}^n)] \to \mathcal{E}_G(\xi), \quad \text{as } n \to \infty.
\end{equation}

For all $N \in \mathbb{N} \setminus N$, we know that (23) holds if and only if
\begin{equation}
\max_{Q \in \mathcal{Q}_{D_N}^N} \mathbb{E}_Q[\xi(\hat{X}^N)] \simeq \mathcal{E}_G(\xi),
\end{equation}
(see Albeverio et al. [1], Proposition 1.3.1). Now, we want to express (24) in terms of $\bar{\mathcal{Q}}_{D_N}^N$. i.e., to show that
\begin{equation}
\max_{Q \in \bar{\mathcal{Q}}_{D_N}^N} \mathbb{E}_{\bar{Q}}[\xi \circ \tilde{\cdot}] \simeq \mathcal{E}_G(\xi).
\end{equation}
To do this, use
\begin{equation}
\mathbb{E}_{\bar{Q}}[\xi \circ \tilde{\cdot}] = \mathbb{E}_{\bar{Q}}[\xi \circ \iota^{-1} \circ \iota]
\end{equation}
and
\begin{align*}
\mathbb{E}_{\bar{Q}}[\xi \circ \iota^{-1} \circ \iota] &= \mathbb{E}_{\bar{Q}}[\xi \circ \iota]
\end{align*}
\begin{align*}
&= \int_{\mathbb{R}^{N+1}} \xi \circ \iota dQ, \quad \text{(transforming measure)}
\end{align*}
\begin{align*}
&= \int_{\mathbb{T}} \xi \circ d(Q \circ j),
\end{align*}
\begin{align*}
&= \mathbb{E}_{\bar{Q}}[\xi \circ \tilde{\cdot}]
\end{align*}
for $j : \mathbb{R}^T \to \mathbb{R}^{N+1}$, $(xt)_{t \in T} \mapsto \left(\frac{z_{Nt}}{\sqrt{\nu}}\right)_{t \in \mathbb{R}^{N+1}}$.
Thus,
\begin{equation}
\bar{\mathcal{Q}}_{D_N}^N = \{Q \circ j : Q \in \mathcal{Q}_{D_N}^N\}.
\end{equation}
This implies,
\begin{equation}
\max_{Q \in \bar{\mathcal{Q}}_{D_N}^N} \mathbb{E}_{\bar{Q}}[\xi \circ \tilde{\cdot}] = \max_{Q \in \mathcal{Q}_{D_N}^N} \mathbb{E}_Q[\xi \circ \tilde{\cdot}].
\end{equation}
\[\square\]

APPENDIX

Proof of Lemma 2.1. From the above equation, we can say that $\Delta M_k = f(k, X)\xi_k$. And by the orthonormality property of $\xi_k$, we have
\begin{equation}
\mathbb{P}_n[f(k, X)^2 \xi_k^2 \xi^2_k] = \mathbb{P}_n[f(k, X)^2 | \mathcal{F}_k^n] \leq \mathbb{P}_n[\{(\sqrt{R_D})^2 | \mathcal{F}_k^n\} = R_D \quad P_n \text{ a.s.,}
\end{equation}
as $|\xi_k| = 1$, $f(\cdots)^2 \in \mathcal{D}$ implies
\begin{equation}
|\Delta M_k|^2 = |f(k, X)|^2 \in [r_D, R_D] \quad P_n \text{ a.s.}
\end{equation}
\[\square\]
Density argument verification. Let

\[ f : \xi \mapsto \sup_{P \in \mathcal{Q}} \mathbb{E}^P[\xi] \]

and

\[ g : \xi \mapsto \lim_{n \to \infty} \sup_{Q \in \mathcal{Q}} \mathbb{E}^{Q}[\xi(\hat{X}^n)]. \]

From (15), we know that for all \( \xi \in \mathcal{C}_b(\Omega, \mathbb{R}) \), \( f(\xi) = g(\xi) \). Since \( L^1_* \) is the completion of \( \mathcal{C}_b(\Omega, \mathbb{R}) \) under the norm \( \| \cdot \|_* \), \( \mathcal{C}_b(\Omega, \mathbb{R}) \) is dense in \( L^1_* \); and we want to prove for all \( \xi \in L^1_* \), \( f(\xi) = g(\xi) \). To prove this, it is sufficient to show that \( f \) and \( g \) are continuous with respect to the norm \( \| \cdot \|_* \).

For continuity of \( f \): For all \( P \in \mathcal{Q}_D \) and \( \xi, \xi' \in L^1_* \),

\[ \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi'] \leq \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[|\xi - \xi'|]. \]

Since, \( \mathcal{Q}_D \subseteq \mathcal{Q} \),

\[ (25) \quad \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi'] \leq \|\xi - \xi'\|_. \]

Interchanging \( \xi \) and \( \xi' \),

\[ (26) \quad \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi'] - \sup_{P \in \mathcal{Q}_D} \mathbb{E}^P[\xi] \leq \|\xi' - \xi\|_. \]

Adding (25) and (26), we have \( |f(\xi) - f(\xi')| \leq \|\xi - \xi'\|_*. \)

For continuity of \( g \): We follow the same argument as above.

Proof of Remark 3.2. Let \( \Phi \) be an internal function such that condition (1) holds. To show that (1) \( \Rightarrow \) (2), fix \( \varepsilon \gg 0 \). We shall show there exists a \( \delta \) for this \( \varepsilon \) as in condition (2). Since \( \Phi \) is internal, the set

\[ I = \left\{ \delta \in \mathbb{R}_{>0} : \forall \omega' \in *\Omega \ (\|\omega - \omega'\|_* < \delta \Rightarrow *|\Phi(\omega) - \Phi(\omega')| < \varepsilon) \right\}, \]

is internal by the Internal Definition Principle and also contains every positive infinitesimal. By Overspill (cf. Albeverio et al. [1, Proposition 1.27]) \( I \) must then contain some positive \( \delta \in \mathbb{R} \).

Conversely, suppose condition (1) does not hold, that is, there exists some \( \omega' \in *\Omega \) such that

\[ *\|\omega - \omega'\|_* \approx 0 \text{ and } *|\Phi(\omega) - \Phi(\omega')| \text{ is not infinitesimal.} \]

If \( \varepsilon = \min(1, *|\Phi(\omega) - \Phi(\omega')|)/2) \), we know that for each standard \( \delta > 0 \), there is a point \( \omega' \) within \( \delta \) of \( \omega \) at which \( \Phi(\omega') \) is farther than \( \varepsilon \) from \( \Phi(\omega) \). This shows that condition (2) cannot hold either. \( \square \)
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