Local Differential Privacy for Physical Sensor Data and Sparse Recovery

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Abstract

In this work we explore the utility of locally differentially private thermal sensor data. We exploit the ill-posedness of the inverse heat source localisation problem to design a locally differentially private recovery algorithm for the 1-dimensional, discrete heat source location problem and analyse its performance in terms of the Earth Mover Distance error. Our work indicates that it is possible to produce locally private sensor measurements that both keep the exact locations of the heat sources private and permit recovery of the “general geographic vicinity” of the sources. We also discuss the relationship between the property of an inverse problem being ill-conditioned and the amount of noise needed to maintain privacy.

1 Introduction

In recent years, wireless technology has allowed the power of lightweight (thermal, light, motion, etc.) sensors to be explored. This data offers important benefits to society. For example, thermal sensor data now plays an important role in controlling HVAC systems and minimising energy consumption in smart buildings [LFA02, BEC13]. Simultaneously, we have begun to understand the extent to which our privacy is compromised by allowing this increased level of data collection. In particular, allowing sensors into the home has resulted in considerable privacy concerns. The field of privacy-preserving data analytics has developed to help alleviate these privacy concerns [DR14]. A particular notion of privacy called differential privacy has emerged as a gold standard for privacy.

In this work we exploit the ill-posedness of the inverse heat source location problem to design an algorithm to release locally differentially private thermal sensor data. In traditional inverse problems, ill-posedness is an “insurmountable problem”. We exploit this traditional problem and use it to our advantage to achieve privacy. Our aim is to release useful thermal data while keeping the exact locations of heat sources private. For example, one might consider these heat sources to be people, whose locations we aim to keep private. We consider an idealised, discrete setting of the heat equation. Our work indicates that it is possible to produce locally private sensor measurements that both keep the exact locations of the heat sources private and permit recovery of the “general vicinity” of the sources. That is, the locally private data can be used to recover an estimate, \( \hat{f} \), that is close to the true source locations, \( f_0 \), in the Earth Mover Distance (EMD).

The heat kernel is an example of a severely ill-conditioned inverse problem [Web81]. That is, it is well-known that adding noise to thermal sensor data results in poor recovery error in classical norms like the \( \ell_1 \) and \( \ell_2 \) norms. This has resulted in a lack of interest in developing theoretical bounds and, as a result, the mathematical analysis and numerical algorithms for inverse heat source problems are still very limited. Intuitively, this should mean that we do not have to perturb the sensor data very much to achieve privacy. It turns out that while partially true, the definition of differential privacy is too strong for this to be true for general ill-conditioned inverse problems. We discuss the connections between the property of being ill-conditioned and how much noise we need to add to make measurements private.

The “local” part refers to the fact that the measurements are made private before they are sent to and collated by the data analyst. This is desirable for sensor measurements since the data analyst (e.g. landlord, building manager, utility company) is often the person the consumer would like to be protected against. In addition to removing the need to trust the data analyst, local differential privacy is attractive from an implementation perspective. It is often the case with
Table 1: Asymptotic upper bounds for private recovery assuming $\sqrt{T} (\frac{\sqrt{m}}{n} + ke^{-A^2/4T}) \leq c < 1$.

| VARIABLE | EMD$(\frac{f_0}{\|f_0\|_1}, \frac{f}{\|f\|_1})$ |
|----------|------------------------------------------|
| n        | $O(1 + \frac{1}{n^2})$ |
| m        | $O(1)$ |
| T        | $\min\{1, O(1 + \frac{1}{n} + T^2.5e^{-A^2/4T})\}$ |

wireless sensors that the data must be communicated via some untrusted channel \[WLSC07, FTC15\]. Usually this step would involve encrypting the data, incurring significant computational and communication overhead. However, if the data is made private prior to being sent, then an argument can then be made that it no longer needs to be encrypted. In addition, the dataset that is produced from this process can act like a synthetic data set for robust statistics.

1.1 Our Contribution

In this work we formulate a notion of local privacy for physical measurements and demonstrate that it can be used to produce data that allows a data analyst to recover approximately where a heat source is but prevents them from determining precisely where the heat source is. The problem of interest is stated precisely in Problem \[2\]. We design a locally differentially private recovery algorithm for the 1-dimensional, discrete heat source location problem using the Gaussian mechanism and $\ell_1$ constrained minimisation. Under some assumptions stated in Corollary \[10\] Table \[1\] shows the asymptotics of our theoretical upper bound on EMD error of the private recovery algorithm, where $f_0$ is the true source vector, $\hat{f}$ is our recovered estimate, $n$ is the size of the discrete universe, $m$ is the number of sensors, $t$ is the time lapse before the measurement is taken, $\mu$ is the diffusion constant and $A$ is a measure of separation between the sources. The full bound can be found in Corollary \[10\].

Our proof of Corollary \[10\] travels via Theorem \[9\] which is an upper bound on the EMD error for recovery of heat sources from noisy sensor data. Our proof of Theorem \[9\] is a generalization of the work of \[LOT14\] to more than one heat source. We then provide a lower bound for the EMD error of recovery in Theorem \[11\]. Our lower bound asymptotically matches our upper bound in its dependence on the standard deviation of the noise. It also matches experimental results in its dependence on the number of sensors.

Finally, we explore the relationship between the condition number of a matrix, $M$, and the amount of noise we need to add to the measurements to maintain privacy. That is, if we have a vector $x \in \mathbb{R}^n$ and measurement vector $y = Mx$, how much noise do we need to add to $y$ to keep $x$ private? We find that the amount of noise is greater than $1/(\sqrt{n}\kappa_2(M))$, where $\kappa_2(M)$ is the condition number. That is, if a problem is well-conditioned then we necessarily need to add a significant amount of noise to maintain privacy. The converse however is not generally true. It is possible to have a ill-conditioned matrix such that we still need to add a considerable amount of noise to maintain privacy.

2 Background and Problem Formulation

2.1 The sparse source recovery problem

We consider the 1-dimensional heat equation on an unbounded domain. Let $u(x, t) : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ be the temperature at location $x$ at time $t$, $f(x)$ be the initial temperature (with bounded support) and $\mu$ be the diffusion constant. Consider the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \mu \nabla^2 u$$

where $u$ obeys the boundary conditions

$$u(x, 0) = f(x) \quad \forall x \in \mathbb{R} \quad \text{and} \quad \lim_{|x| \rightarrow \infty} u(x, t) = 0 \quad \forall t \in [0, \infty).$$

Let $g(x, t) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{|x|^2}{4\mu t}}$ then from standard potential theory \[GL96\] we know we can write the solution as

$$u(x, t) = (g * f)(x, t) = \int_{\mathbb{R}} g(x - y, t) f(y) dy.$$
We will let \( T = \mu t \) for the remainder of the paper. Now, we consider the discretization of this problem. Let \( n > 0 \) and suppose the support of \( f \) is contained in the discrete set \( \{ \frac{1}{m}, \cdots, 1 \} \). The function \( f \) can be represented by a vector \( f_0 \in \mathbb{R}^n \) where \( f(x) = \sum_{i=1}^{n} (f_0)_i \delta(\frac{i}{n} - x) \) so

\[
u(x, t) = \sum_{i=1}^{n} (f_0)_i g \left( \frac{i}{n} - t \right).
\]

Let \( m > 0 \) and suppose we take \( m \) measurements at locations \( \frac{1}{m}, \cdots, 1 \) so \( y_i = u(\frac{i}{m}, t) \) is the measurement of the sensor at location \( \frac{i}{m} \). Since Equation (1) is linear we have

\[
y = Af_0 \quad \text{where} \quad A_{ij} = g \left( \frac{i}{n} - \frac{j}{m}, t \right).
\]

An obvious question then is: given the diffusion constant, time \( t \) and the (possibly noisy) measurements \( y = Af_0 \), how well can we reconstruct \( f_0 \)? If the measurements are not noisy then the answer is we can recover \( f_0 \) exactly if we have enough measurements [CR10]. However this problem is severely ill-posed [Web81], that is, small errors in the measurement data can lead to very large errors in the recovered source vector.

The ill-posedness of the heat source inversion problem is essentially derived from the fact that as heat dissipates, the measurement vectors for different source vectors become increasingly close. Thus, we only need to add a small amount of noise to make it difficult to determine between them. If the sources were originally far apart then this effect is less pronounced. Intuitively, if we know there are not very many heat sources, then we should be able to recover the general vicinity of the sources, even after we add a small amount of noise. For this reason, the heat source localisation problem has not received a lot of attention in the recent literature. With this in mind, we impose the following two changes to the problem to make it more tractable:

- Assume the source vector is sparse, that is \( \|f_0\|_0 = k \) where \( k \) is small.
- Measure the error in terms of the Earth-Mover distance (EMD).

The EMD, which we will define rigorously later, gives a measure of how “geographically similar” two distributions are. In the context of the heat source location problem, \( \text{EMD}(f_0, f) \) being small means that even though we may not be able to pinpoint exactly where the heat sources are, we can say approximately where they are. Achieving small error in traditional norms, such as \( \ell_1 \) or \( \ell_2 \), requires \( f \) to have most of its weight in the support \( f_0 \). That is, one must recover the locations exactly, which our privacy constraint prohibits. The EMD is thus a more amenable metric since it only requires us to determine the approximate, rather than exact, locations of sources.

**Problem 1.** Suppose that \( \|f_0\|_0 = k \) and \( \sigma > 0 \). Given measurements \( \tilde{y} = Af_0 + N(0, \sigma^2 I_m) \) can we produce an estimate \( \hat{f} \) such that \( \text{EMD}(f_0, \hat{f}) \) is small?

### 2.2 Differential Privacy

Differential privacy has emerged over the past decade as the leading definition of privacy for privacy-preserving data analysis. We give a very brief introduction to differential privacy in this section, a more in-depth introduction can be found in [DR14]. A database is a vector \( D \in D^n \) for some data universe \( D \). We call two databases \( D, D' \) adjacent or “neighbouring” if \( \|D - D'\|_0 = 1 \).

**Definition 1** ((\( \epsilon, \delta \))-Differential Privacy). [DMNS06] A randomised algorithm \( A \) is \((\epsilon, \delta)\)-differentially private if for all adjacent databases \( D, D' \) and events \( E \),

\[
P(A(D) \in E) \leq e^\epsilon P(A(D') \in E) + \delta.
\]

To understand this definition suppose the database \( D \) contains some sensitive information about Charlie and the data analyst, Lucy, produces some statistic \( A(D) \) about the database \( D \) via a differentially private algorithm. Then Lucy can give Charlie the following guarantee: an adversary given access to the output \( A(D) \) can not determine whether it was sampled from the probability distribution generated by the algorithm when the database is \( D \) or the
distribution generated by $D'$, where $D$ has Charlie’s true data and $D'$ has Charlie’s data replaced with an arbitrary element of $D$.

The definition of differential privacy is particularly attractive for machine learning because of two main properties. The first is that the guarantee holds regardless of what side-information an adversary has. The second is that Lucy can do adaptive data analysis and retain her privacy guarantee to Charlie. These properties combine to give differentially private algorithms the ability to be used as building blocks in designing algorithms.

A locally differentially private algorithm is a private algorithm in which Charlie creates a differentially private version of his data before sending it to Lucy. While each individual data point is not disclosive, in the aggregate, locally differentially private data analysis can still produce useful statistical insights \cite{RYY16, KOV16}.

### 2.3 Locally differentially private heat source location

Let us return to the sparse source recovery problem. Assume $\mu$ and $t$ are fixed and known. We would like to design a locally differentially private algorithm to solve Problem 1. Firstly, we need to clarify exactly what “data” we would like to keep private. We are going to consider the coordinates of $f_0$ to be our data, that is the locations of the heat sources are what we would like to keep private. We can not hope to keep the existence of a heat source private and also recover an estimation to $f_0$ that is close in the EMD. Therefore, we are going narrow our definition of “neighbouring” databases.

**Definition 2.** Two source vectors $f_0$ and $f'_0$ are neighbours if there exists $i \in [n]$ such that $(f_0)_j = (f'_0)_j$ for all $j \neq i, i+1$ and $(f_0)_i = (f'_0)_{i+1}$ and $(f_0)_{i+1} = (f'_0)_i$.

That is, two source vectors are neighbours if the differ in location of a single source by one unit. For example, the following two source vectors are neighbours (where $n = 5$).

\[
\begin{array}{cccccc}
0 & 1/5 & 2/5 & 3/5 & 4/5 & 1 \\
0 & 1/5 & 2/5 & 3/5 & 4/5 & 1 \\
\end{array}
\]

We have access to the source location vectors through the sensor measurements so the “local” part of our problem is that we will require each sensor to compute a differentially private version of its measurement before it sends its data to a central node. We then wish to use this locally differentially private data to recover an estimate to the source vector that is close in the EMD. The structure of the problem is outlined in the following diagram:

\[f_0 \xrightarrow{A} \{ y_1 \xrightarrow{A} \hat{y}_1 \} \xrightarrow{R} \hat{f}\]

**Problem 2.** Design algorithms $A$ and $R$ such that:

1. (Privacy) For all neighbouring source vectors $f_0$ and $f'_0$, sensor locations $s$ and Borel measurable sets $E$ we have

   \[
P(A(u_{f_0}(s, t)) \in E) \leq e^\epsilon P(A(u_{f'_0}(s, t)) \in E) + \delta.
   \]

2. (Utility) $EMD(f_0, \hat{f})$ is small.

### 2.4 Related work

An in-depth survey on differential privacy and its links to machine learning and signal processing can be found in \cite{SC13}. The Gaussian mechanism was folklore originally observed in \cite{DMT07} and a proof can now be found in \cite{DR14}. The body of literature on general and local differential privacy is vast and so we only mention here work that is directly related. There is growing body of literature of differentially private sensor data, for example \cite{LSWR12, LZJ15, WZY16, JB14, EE16}. Much of this work is concerned with differentially private release of aggregate data.
statistics derived from sensor data and the difficulty in maintaining privacy over a period of time (called the continual monitoring problem).

Connections between privacy and signal recovery have been explored previously in the literature. In [DMT07], Dwork et al. considered the recovery problem with noisy measurements where the matrix $M$ has i.i.d. standard Gaussian entries. Let $x \in \mathbb{R}^n$, $y = Mx \in \mathbb{R}^m$ where $m = \Omega(n)$, $\rho < 0.239$.... Suppose $y'$ is a perturbed version of $y$ such that a $\rho$ fraction of the measurements are perturbed arbitrary and the remaining measurements are correct to within an error of $\alpha$. Then [DMT07] concludes that w.h.p. the constrained $\ell_1$-minimisation, $\min \|y' - y\|_1$ s.t. $Mx = y$, recovers an estimate, $\hat{x}$, s.t. $\|x - \hat{x}\|_1 \leq O(\alpha)$. This is a negative result for privacy. In particular, when $\alpha = 0$ it says that providing reasonably accurate answers to a 0.761 fraction of randomly generated weighted subset sum queries is blatantly non-private. Newer results of Bun et al. [BUV14] can be interpreted in a similar light where $M$ is a binary matrix. Compressed sensing has also been used in the privacy literature as a way to reduce the amount of noise needed to maintain privacy [LZWY11, RBVC16].

There are also several connections between sparse signal recovery and inverse problems [FHE13, BLTT10, Hab08, LTT11]. The heat source identification problem is severely ill-conditioned and hence it is known that noisy recovery is impossible in the common norms like $\ell_1$ and $\ell_2$. This has resulted in a lack of interest in developing theoretical bounds [LOT14], thus the mathematical analysis and numerical algorithms for inverse heat source problems are still very limited. We exploit the ill-conditioned nature of the heat kernel to help us achieve privacy. We do not know of another paper that makes the connection between ill-posedness and privacy.

To the best of the authors knowledge, the two papers that are most closely related to this paper are Li et al. [LOT14] and Beddiaf et al. [BAPJ15]. Both of these papers attempt to circumvent the condition number lower bounds by changing the error metric to capture “the recovered solution is geographically close to the true solution”, as in this paper. Our algorithm is the same as Li et al., who also consider the Earth Mover Distance (EMD). Our upper bound is a generalisation of theirs to source vectors with more than one source. When we consider only a single source, our paper however contains no theoretical performance bounds.

3 Private Algorithm $A$

In this section we discuss the design of the $(\epsilon, \delta)$-differentially private algorithm $A$ from Problem 2. Since we are considering the application of lightweight sensors, the algorithm is simply going to be that each sensor will locally add Gaussian noise to their measurement before sending to the central node. There has been some work on the optimal type of noise to add to achieve privacy [GV16]. The reason Gaussian noise is attractive in this setting is that sensors already have inherent noise that is often modeled by Gaussian noise. Our upper bound on the EMD will only depend on the $\ell_1$-norm of the noise added to the measurement vector (not the type of noise) so any result reducing the standard deviation of noise added would immediately result in an improvement of our private EMD error bound, Corollary 10.

The question then is, in the case of Gaussian noise, how much noise should we add to maintain privacy? The following lemma says, essentially, that the standard deviation of the noise added to a statistic should be proportional to how much the statistic can vary between neighbouring data sets. Let $f : D^n \rightarrow \mathbb{R}^n$ be a function and let $\Delta_2 f = \max_{D, D'} \|f(D) - f(D')\|_2$ (called the $\ell_2$ sensitivity of $f$).

Lemma 3 (The Gaussian Mechanism). [DR14] Let $\epsilon > 0$, $\delta > 0$ and $\sigma = \frac{2\ln(1.25/\delta)\Delta_2 f}{\epsilon}$ then

$$A(D) \sim f(D) + N(0, \sigma I_n)$$

is an $(\epsilon, \delta)$-differentially private algorithm.

Let $A_i$ be the $i$th column of $A$.

Proposition 4. With the definition of neighbours presented in Definition 2 and restricting to $f_0 \in [0, 1]^n$ we have

$$A(u_{f_0}(s, t)) \sim u_{f_0}(s, t) + \frac{2\log(1.25/\delta)\Delta_2(A)}{\epsilon} N(0, 1)$$
is a $(\epsilon, \delta)$-differentially private algorithm where
\[
\Delta_2(A) = \max_{i \in [n]} \|A_i - A_i+1\|_2 = O\left(\frac{\sqrt{m}}{nT^{1.5}}\right)
\]

Proof. Suppose $f_0$ and $f'_0$ are neighbours where $a = (f_0)_i = (f'_0)_i$ and $b = (f_0)_{i+1} = (f'_0)_{i+1}$. Then $\|A f_0 - A f'_0\|_2 = |a - b|\|A_i - A_{i+1}\|_2$ and so $\Delta_2(A) = \max_{i \in [n]} \|A_i - A_{i+1}\|_2$. Then the fact that the algorithm is $(\epsilon, \delta)$-differentially private follows from Lemma 3. The proof of the upper bound on $\Delta_2(A)$ in Proposition 4 is a straightforward calculation and can be found in Appendix B. $\square$

This result is a little misleading in the parameter $n$ since our notion of neighbours depends on $n$. That is, two neighbours are at EMD at most $\frac{1}{n}$. Computational experiments found in Appendix B indicate that this analysis is asymptotically tight in $m$, $n$ and $T$.

By design, this data has lower utility than the original, unperturbed measurements, although we will see that it still allows us to recover an estimate to $f_0$ that is close in the EMD. In Appendix A we discuss how we can use pseudo-randomness to build a “backdoor” into this algorithm. That is, we can design a secret key that will give only trusted parties access to the unperturbed data, without us having to encrypt the sensor measurements.

### 3.1 Preserving Privacy for Ill-Conditioned Inverse Problems

We digress for a moment to discuss the intuition for general inverse problems. As mentioned earlier, the heat source location problem is ill-conditioned, that is, it behaves poorly under addition of noise to the sensor measurements. Intuitively, this should mean we only need to add a small amount of noise to mask the original data. In this discussion we will see that this statement is partially true. There is however, a fundamental difference between the notion of a problem being ill-conditioned (as defined by the condition number) and being easily kept private.

Let $M$ be an $m \times n$ matrix and consider the general inverse problem of recovering $x_0$ from the measurement vector $M x_0$. Let $(s_1, \ldots, s_{\min(m,n)})$ be the spectrum of $M$, enumerated such that $s_1 \geq s_{i+1}$. The condition number, $\kappa_2(M)$, is a measure of how ill-conditioned this inverse problem is. It is defined as
\[
\kappa_2(M) := \max_{e,b \in \mathbb{R}^m \setminus \{0\}} \frac{\|M^+ b\|_2 \|e\|_2}{\|M^+ e\|_2 \|b\|_2} = \frac{s_1(M)}{s_{\min(m,n)}(M)}
\]
where $M^+$ is the pseudo inverse of $M$. The larger the condition number the more ill-conditioned the problem is [BKW80]. The following matrix illustrates the difference between how ill-conditioned a matrix is and how much noise we need to add to maintain privacy. Suppose $M = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$ where $\rho < 1$ is small. While this problem is ill-conditioned, $\kappa_2(M) = 1/\rho$ is large, we still need to add considerable noise to the first coordinate of $M x_0$ to maintain privacy. A necessary condition for $\delta_2(M)$ to be small is that the matrix $M$ is almost rank 1, that is, the spectrum should be almost 1-sparse. In contrast the condition that $\kappa_2(M)$ is large is only a condition on the maximum and minimum singular values. The following lemma says that if the amount of noise we need to add, $\Delta_2(M)$, is small then the problem is necessarily ill-conditioned. Let $(s_1, \ldots, s_{\min(m,n)})$ be the spectrum of $M$, enumerated such that $s_i \geq s_{i+1}$.

**Lemma 5.** Let $M$ be a matrix such that $\|M\|_2 = 1$ and suppose the domain is $[0,1]^n$, then
\[
\Delta_2(M) \geq \frac{1}{\kappa_2(M)\sqrt{n}}.
\]

The following lemma gives a characterization of $\Delta_2(M)$ in terms of the spectrum of $M$. It essentially says that the matrix $M$ must be almost rank 1, in the sense that the spectrum should be dominated by the largest singular value.

**Lemma 6.** Furthermore, if $\Delta_2(M) \leq \nu$ then $\left|\sum_{i \neq 1} s_i\right| \leq (n + 1)^{3/2} \nu$ and $\|M_1 - M_{i+1}\|_2 \leq \nu$. Conversely, if $\left|\sum_{i \neq 1} s_i\right| \leq \nu$ and $\|M_1 - M_{i+1}\|_2 \leq \nu$ then $\Delta_2(M) \leq 4\nu$.

The proofs of both of the above lemmas follow from direct computation and can be found in the Appendix B. Our definitions of neighbours was not particularly important for Lemma 5 or Lemma 6. Indeed, the same statement would hold for any more general notion of neighbours such as $\|x_0 - x_0'\|_2 \leq 1$ or $\|x_0 - x_0'\|_0 \leq 1$. 

6
4 Noisy Recovery

In this section we present and analyse an algorithm, $R$, to solve Problem 1. We will then use this algorithm to explore how well we can recover the heat source locations from the locally private data. As has been studied extensively in the compressed sensing literature, one can solve $\ell_0$ problems (the problem of finding a sparse vector) by reducing to the $\ell_1$ convex relaxation if the matrix $A$ satisfies the restricted isometry property (RIP) [Don06]. Although the heat kernel matrix $A$ does not satisfy the RIP, reducing to the $\ell_1$ convex relaxation still promotes sparsity while allowing the problem to be computationally tractable. This use of constrained $\ell_1$-minimisation to recover heat sources was introduced in Li et al. [LOT14], who studied the case of a 1-sparse source vector. Li et al. also discussed the use of Bregman iteration to solve the $\ell_1$ minimisation problem as a sequence of unconstrained problems. We will not touch on specific algorithms for finding the minimiser in this work.

![Algorithm 1](image)

Figure 1: The red line represents $f_0$ and the blue (dashed) line represents $\hat{f} = R(\tilde{y})$ (both normalised to have unit $\ell_1$ norm) where $n = 100$, $m = 50$, $T = 0.5$ and $\sigma = 0.5$.

Suppose $\tilde{y} \sim A f_0 + N(0, \sigma^2 I_m)$. The recovery algorithm $R$ is outlined in Algorithm 1. The constraint $\|Af - \tilde{y}\|_2 \leq \sigma \sqrt{m}$ is derived from the following lemma.

**Lemma 7.** [HKTZ12] Let $\nu \sim N(0, \sigma^2 I_m)$ then for all $t > 0$,

$$P(\|\nu\|_2^2 > \sigma^2 (m + 2\sqrt{mt} + 2t)) \leq e^{-t}.$$  

So for large $m$ and small $\rho$, we have $\|\nu\|_2 \leq (1 + \rho)\sigma \sqrt{m}$ with high probability.

Lemma 7 implies that for large enough $m$, w.h.p. we have $f_0$ is a feasible point for the constrained optimisation in Algorithm 1. Figure 1 demonstrates the typical behaviour of the algorithm $R$. As can be seen in the figure, this algorithm returns an estimate $\hat{f}$ that is indeed close to $f_0$ in the EMD but not necessarily close in more traditional norms like the $\ell_1$ and $\ell_2$ norms. This phenomenon was noticed by Li et al., who proved that if $f_0$ consists of a single source then EMD$(f_0, f)$ is small where $f = R(\tilde{y})$ [LOT14].

4.1 Earth Mover Distance error for constrained $\ell_1$-minimisation

As mentioned earlier, the error norm we are concerned with is the Earth Mover Distance (EMD). The EMD can be defined between any two probability distributions on a finite discrete metric space $(\Omega, d(\cdot, \cdot))$. It can be understood as follows: imagine each probability distribution as representing the depth of dirt on the space then the EMD between the two distributions is the amount of “effort” it takes to transform one distribution into the other. The amount of effort depends on both how much dirt is being moved and how far it is being moved.

**Definition 8.** [RTG00] Let $P = \{(x_1, p_1), \ldots, (x_n, p_n)\}$ and $Q = \{(x_1, q_1), \ldots, (x_n, q_n)\}$ be two probability distributions on the discrete space $\{x_1, \ldots, x_n\}$. Now, let

$$f^* = \arg \min_{f \in [0,1]^{n \times n}} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \|x_i - x_j\|_2$$

st. $f_{ij} \geq 0 \ \forall i,j \in [m]$, $\sum_{j=1}^{n} f_{ij} \leq p_i \ \forall i \in [n]$, $\sum_{i=1}^{n} f_{ij} \leq q_i \ \forall i \in [n]$, and $\sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} = 1$.

then $\text{EMD}(P, Q) = \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_i - x_j\|_2$.

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Table 2: Asymptotic upper bounds for noisy recovery assuming $\sqrt{T}(\sigma + ke^{-A^2/4T}) \leq c < 1$.

| VARIABLE | EMD($f_0/\|f_0\|_1, \hat{f}/\|\hat{f}\|_1$) |
|----------|-----------------------------------------------|
| $n$      | $O(1)$                                        |
| $m$      | $O(1)$                                        |
| $T$      | $\min\{1, O(T^{1.5} + \sqrt{T})\}$            |
| $\sigma$ | $\min\{1, O(\sigma)\}$                      |

We can interpret $f_{ij}$ as the amount of dirt that is moved from $x_i$ in $Q$ to $x_j$ in $P$. We can now state our recovery bound.

**Theorem 9.** Suppose that $f_0$ is a source vector, $\hat{y} = R(\tilde{y})$ and assume the following:

1. $m\sqrt{T}/2 > 1$
2. $\sqrt{2T} < 1$
3. $|x_i - x_j| > \sqrt{2T} + 2A$ for some $A > 0$

then w.h.p.

$$\text{EMD}(f_0/\|f_0\|_1, \hat{f}/\|\hat{f}\|_1) \leq \min\{1, O\left(\frac{1}{1 - \min\{1, C\}} \left(\frac{1}{k}\sqrt{\frac{T^{1.5}C}{\sqrt{T}+1}} + k\min\{1, C\} + \frac{T^2C}{(T+1)k}\right)\right)\}$$

where $C = \min\{k, \sqrt{T}\left[\sigma + ke^{-A^2/4T}\right]\}$.

First, let us parse the assumptions of Theorem 9. A picture of the set-up for $k = 2$ is provided below:

```
  X                       X
 / \   \                 / \   \   \
 x1  2A  x2
  \   \                 \   \   \
 \sqrt{2T}            \sqrt{2T}
```

Assumption 1 states that $m$ needs to be large enough that for each possible source, there are sensors near it. Assumption 2 says that you need to take the measurements reasonably quickly. The exact bound of 1 comes from the fact that we are looking at the interval $[0, 1]$ so we can interpret this assumption as saying that you should take the measurements before the heat mass leaves the interval. Assumption 3 says that the sources need to be sufficiently far apart. We will discuss later how the algorithm fails if the sources are too close. Note that, if we assume $\sqrt{T}(\sigma + ke^{-A^2/4T}) \leq c < 1$ then the factor $\frac{1}{1-c}$ is upper bounded by $\frac{1}{1-c}$. This requirement intuitively says that if we increase the noise or the number of sources, then we need to take the measurements sooner to compensate.

Table 2 describes the asymptotics of Theorem 9 if all variables except the variable denoted in the left column are held constant and we assume $\sqrt{T}(\sigma + ke^{-A^2/4T}) \leq c < 1$. In some cases, the upper bound is expressed as $\min\{1, O(f(x))\}$ where $f$ is an increasing function of the parameter, $x$. While obviously $O(1)$, we retain the minimisation in the upper bound because experiments suggest that the constant in front of the function $f$ is small enough that this function provides the dominate behaviour for small $x$. The result is a generalisation of a result of Li et al. [LOT14] to source vectors with more than one source. Our proof is a generalisation of Li et al.’s proof. The details can be found in Appendix C.

Finally, we can state our EMD error bound for recovery of heat source locations from locally differentially private thermal sensor data.
Corollary 10. Suppose Assumptions (1) - (3) from Theorem 9 hold. Then there exists a locally differentially private algorithm $A$ and recovery algorithm $R$ such that if $\hat{f} = R(\tilde{A}(y_1), \cdots, \tilde{A}(y_m))$ then w.h.p.

$$EMD \left( \frac{f_0}{\|f_0\|_1}, \frac{\hat{f}}{\|\hat{f}\|_1} \right) \leq \min \left\{ 1, O \left[ \frac{1}{1 - \min\{1, C\}} \left( \frac{1}{\sqrt{T}} \sqrt{1.5C} + k \min\{1, C\} + \frac{T^2C}{(T+1)k} \right) \right] \right\}$$

where $C = \min\{k, \frac{\sqrt{m}\log(1/\delta)}{n\epsilon} + \sqrt{T}ke^{-2/4T}\}$.

The asymptotics of this bound were contained in Table 1 in Section 1.1. It is interesting to note that, unlike in the constant $\sigma$ case, the error increases as $T \to 0$. This is because as $T \to 0$ the inverse problem becomes less ill-conditioned so we need to add more noise. As the heat dissipates we don’t need to add as much noise to the measurements to keep the source locations private, but the unperturbed measurements become less useful so the EMD begins to increase again.

We will discuss our empirical observations on the tightness of the bounds in Theorem 9 and Corollary 10 in Section 5 but we point out that the dependence on $m$ is not what we expect. That is, we expect the error to decay with $m$ but the upper bound in Theorem 9 has no dependence on $m$. Figure 3b suggests that the EMD error decays like $O(1 + 1/\sqrt{m})$.

Variations of constrained $\ell_1$-minimisation have been studied, under the name basis pursuit denoising, extensively in the compressed sensing literature [Tro06, BRT09, BHEE10, Wai09b, Wai09a]. In [BHEE10], Ben-Haim et al. observed that for unconstrained basis pursuit denoising (where one solves the regularised version, $\min \|Af - \tilde{y}\|_2 + \gamma \|f\|_1$ for some parameter $\gamma$), there is a gap between the $\ell_2$-error achievable when we only know $\|Af_0 - \tilde{y}\| \leq \sigma \sqrt{m}$ compared to when we know the noise is white gaussian noise. Preliminary analysis suggests that this gap corresponds to the $1/\sqrt{m}$ error in our analysis. Unfortunately, the results from compressed sensing are not directly applicable to our setting since they analyse the $\ell_2$-error, which we know is very large for the heat kernel. It is possible that the techniques used may be applicable and we would like to explore this direction in the future.

In the private case, our analysis proves that the EMD is $O(1)$ in the parameter $m$, and our experiments suggest that this is tight (Figure 6b). Experiments with small $k$ indicate that the EMD error (for reasonable parameter settings) undergoes a sharp transition when there are “enough” sensors. A natural question to ask then is, how many sensors are enough? Intuition from compressed sensing would lead us to believe that as $k$ increases, the number of required sensors increases. Figure 5b (in Appendix E) indicates that this is not necessarily the case. Empirical results suggest that, after a certain point, as we increase $k$, it becomes easier to find the approximate locations of the sources. Figure 5a shows that for both the normalised $\ell_1$-norm and the EMD after an initial increase for small $k$, the error decreases as $k$ increases. One possible reason for this is that the space becomes saturated and any “guess” is not too far from a true source location.

### 4.2 Lower bound on Estimation Error

The following theorem gives a lower bound on the estimation error of the noisy recovery problem.

Theorem 11. We have

$$\inf_{\hat{f}} \sup_{f_0} \mathbb{E}[EMD(f_0, \hat{f})] = \Omega \left( \min\left\{ \frac{1}{2}, \frac{T^{1.5}\sigma}{\sqrt{m}} \right\} \right).$$

where $\inf_{\hat{f}}$ is the infimum over all estimators $\hat{f} : \mathbb{R}^m \to [0,1]^n$, $\sup_{f_0}$ is the supremum over all source vectors in $[0,1]^n$ and $\tilde{y}$ is sampled from $y + N(0, \sigma^2 I_m)$.

Note that this lower bound matches our upper bound asymptotically in $\sigma$ and is slightly loose in $T$. It varies by a factor of $\sqrt{m}$ from our theoretical upper bound but matches our experimental results in $m$. The proof of Theorem 11 is an application of Fano’s lemma. We first upper bound the KL-divergence between the distributions on measurements generated by two source vectors $f_0$ and $f'_0$ in terms of $EMD(f_0, f'_0)$. We then find a small class $T$ of source vectors such that for every pair in $T$ we can upper bound the KL-divergence and lower bound the EMD. The details can be found in Appendix D.
empirical and theoretical results imply that the EMD error increases both as
nature in
bound of \( O \) suggest that Theorem 9 is asymptotically tight in
was computed exactly for each trial. The results suggest that Theorem 10 is asymptotically tight in
rate behaves something like
results suggest tightness is
\( \Delta \)
intervals computed after performing 10 trials with fixed parameters. The sensitivity between two vectors is computed by first scaling both vectors to have unit

Thus our measurements, \( y_i \) are evaluations of a GMM. It has been noted in the GMM literature that it is difficult to determine between the sum of two overlapping normal distributions and the normal distribution centred between them \([Das99]\). If two sources are close enough together then the algorithm will often output an estimate that has the mass centred between the true source locations. In Figure 5 we can see the difference between the behaviour of the algorithm when the sources are close together (Figure 2a) compared to when they are further apart (Figure 2b).

A consequence of Theorem 11 is that that if two peaks are too close together, roughly at a distance of \( O \left( \min\left( \frac{1}{2}, \frac{T_1 \sqrt{\sigma}}{\sqrt{m}} \right) \right) \), then it is impossible for an estimator to differentiate between the true source vector and the source vector that has a single peak located in the middle.

We have not included the parameter \( k \) in the asymptotic tables because our theoretical bound does not adequately explain the experimental results we will present in Section 5. It appears from empirical results that the EMD error decays as we increase \( k \). The authors have not yet found an analytical explanation for this phenomenon however, we postulate that is it caused by the algorithm failing to differentiate between close peaks. If the source locations are chosen randomly then as we increase \( k \), the sources will become close enough that this effect kicks in. So, suppose that \( \hat{f} \) always has the mass centred between the true source locations. Then as \( k \) increases, the EMD error between \( f_0 \) and \( \hat{f} \) will decrease.

5 Experimental Results

In this section we present experimental results. The source vectors are \( n \) dimensional vectors where the possible source locations are \( \{ \frac{1}{m}, \cdots, 1 \} \). The sensors are placed at locations \( \{ \frac{1}{m}, \cdots, 1 \} \). The diffusion constant, \( \mu \), is equal to 1/2 for all experiments. In all graphs, all but one of the variables \( n \), \( m \), \( t \), \( \sigma \), \( \epsilon \) or \( \delta \) are kept constant. The EMD between two vectors is computed by first scaling both vectors to have unit \( \ell_1 \) norm, then computing the EMD between the normalised vectors. In experiments where \( k = 1 \), the source is always at 0.5. If \( k > 1 \) then \( k \) source locations are chosen uniformly at random. All sources have unit intensity.

All code was run in MATLAB R2016a. In all cases where confidence bars are present, the bars are 95% confidence intervals computed after performing 10 trials with fixed parameters. The sensitivity \( \Delta_2(A) = \max_i \| A_i - A_{i+1} \|_2 \) was computed exactly for each trial.

Figure 3 shows the results of simulations of Algorithm 1 on noisy thermal measurements. These simulations suggest that Theorem 9 is asymptotically tight in \( n \) and \( \sigma \). It is more difficult to determine whether the empirical results suggest tightness is \( T \), although it certainly does not contradict it. The result for \( m \) is inconclusive, our upper bound of \( O(1) \) and lower bound of \( O(1/\sqrt{m}) \) differ by a factor of \( \sqrt{m} \). The simulations suggest that the true error rate behaves something like \( 1 + \frac{1}{\sqrt{m}} \) since the error decays but not all the way to 0.

Figure 6 (in Appendix F) shows the results of simulations of our locally differentially private recovery algorithm. The results suggest that Theorem 10 is asymptotically tight in \( n \) and \( m \). It is difficult to determine the asymptotic nature in \( T \) from the empirical result however the shape is consistent with the upper bound given. In particular both empirical and theoretical results imply that the EMD error increases both as \( T \rightarrow 0 \) and \( T \rightarrow \infty \).
Figure 3: Empirical results for the EMD error of experiments running Algorithm $R$ on measurement data with added gaussian noise with variance $\sigma^2$. Unless specified otherwise, $n = 100$, $m = 50$, $T = 0.5$, $\sigma = 0.1$ and $k = 1$. In (3c), $\sigma = 0.2$. In (3d), $T = 0.05$ and $\sigma = 0.2$.

6 Future Work

As discussed earlier, the heat source location problem is related to the problem of estimating GMMs. A GMM is a probability distribution that is a weighted sum of gaussian distributions. The problem here is to reconstruct the probability distribution from a dataset sampled i.i.d. from the distribution. This problem is well-studied and there exist algorithms that w.h.p. recover accurate estimates of the means [Das99, KMV12]. It would be interesting in the future to explore how these algorithms (for example, Expectation-Maximisation) compare to $\ell_1$-minimisation for noisy recovery.

In this paper we discussed the relationship between ill-conditioned inverse problems and the amount of noise we need to achieve privacy. We found that while the condition number and the sensitivity are related, it is possible to have an ill-conditioned problem that still requires a lot of noise to achieve privacy. We would be interesting in exploring this relationship further. For example, what happens if we relax our definition of privacy to only requiring a fraction of the source locations to be kept private? Also, how does the distribution of the singular values affect the amount of noise needed for privacy?

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A Backdoor access via Pseudo-randomness

It has been explored previously in the privacy literature that replacing a random noise generator with cryptographically secure pseudorandom noise generator in an efficient differentially private algorithm creates an algorithm that satisfies a weaker version of privacy, computational differential privacy [MPRV09]. While differential privacy is secure against any adversary, computational differential privacy is secure against a computationally bounded adversary. In the following definition, \( \kappa \) is a security parameter that controls various quantities in our construction.

**Definition 12** (Simulation-based Computational Differential Privacy (SIM-CDP) [MPRV09]). A family, \( \{ \mathcal{M}_\kappa \}_{\kappa \in \mathbb{N}} \) of probabilistic algorithms \( \mathcal{M}_\kappa : \mathcal{D}^n \rightarrow \mathcal{R}_\kappa \) is \( \epsilon \)-SIM-CDP if there exists a family of \( \epsilon_\kappa \)-differentially private algorithms \( \{ \mathcal{A}_\kappa \}_{\kappa \in \mathbb{N}} \) such that for every probabilistic polynomial-time adversary \( \mathcal{P} \), every polynomial \( p(\cdot) \), every sufficiently large \( \kappa \in \mathbb{N} \), every dataset \( D \in \mathcal{D}^n \) with \( n \leq p(\kappa) \), and every advice string \( z_\kappa \) of size at \( p(\kappa) \), it holds that

\[
|\mathbb{P}[\mathcal{P}_\kappa(\mathcal{M}_\kappa(D) = 1)] - \mathbb{P}[\mathcal{P}_\kappa(\mathcal{A}_\kappa(D) = 1)]| \leq \text{negl}(\kappa).
\]

That is, \( \mathcal{M}_\kappa \) and \( \mathcal{A}_\kappa \) are computationally indistinguishable.

The transition from pseudo-randomness, of course, has the obvious advantage that pseudo-random noise is easier to generate than truly random noise. In our case, it also has the additional benefit that, given access to the seed value, pseudo-random noise can be removed, allowing us to build a “backdoor” into the algorithm. Suppose we have a trusted data analyst that wants access to the most accurate measurement data but does not have the capacity to protect sensitive data from being intercepted in transmission. Then suppose we allow this party is allowed to store the seed value of each sensor then use our locally private algorithm \( \mathcal{A} \) with a pseudo-random number generator. Then, the consumers are protected against an eavesdropping computationally bounded adversary, and the trusted party has access to the noiseless measurement data. This solution may be preferable to simply encrypting the data during transmission since there may be untrusted parties who we wish to give access to the private version of the data.

**Corollary 13** (Informal). There exists an algorithm, utilising only pseudo-randomness, that satisfies Problem 2 where local differential privacy is replaced with local simulation-based computational differential privacy. In addition, any trusted party with access to the seed of the random number generator can use the output of the private algorithm to generate the original data.

B Proof of upper bound in Proposition 4

**Proposition 4** With the definition of neighbours presented in Definition 2 and restricting to \( f_0 \in [0,1]^n \) we have

\[
\mathcal{A}(u_{f_0}(s,t)) \sim u_{f_0}(s,t) + \frac{2 \log(1.25/\delta) \Delta_2(A)}{\epsilon} \mathcal{N}(0,1)
\]

is a \((\epsilon, \delta)\)-differentially private algorithm where

\[
\Delta_2(A) = \max_{i \in [n]} \|A_i - A_{i+1}\|_2 = O\left(\frac{\sqrt{m}}{nT^{1/5}}\right)
\]

**Proof.** For all \( i \in [n] \) we have

\[
\|A_i - A_{i+1}\|_2^2 = \frac{1}{4\pi T} \sum_{j=1}^{m} \left( e^{-\frac{(x - \hat{x})^2}{4T}} - e^{-\frac{(x + 1 - \hat{x})^2}{4T}} \right)^2
\]

\[
= \frac{1}{4\pi T} \sum_{j=1}^{m} e^{-\frac{(x - \hat{x})^2}{2T}} \left( 1 - e^{-\frac{(x - \hat{x})^2 - (x + 1 - \hat{x})^2}{2T}} \right)^2
\]

\[
\leq \frac{1}{4\pi T} \max_{i \in [n]} \max_{j \in [m]} \left( 1 - e^{-\frac{(x - \hat{x})^2 - (x + 1 - \hat{x})^2}{2T}} \right)^2 \sum_{j=1}^{m} e^{-\frac{(x - \hat{x})^2}{2T}}
\]

\[\text{This data may still be corrupted by sensor noise that was not intentionally injected}\]
Figure 4: Empirical results of computation of $\Delta_2(A)$. Unless specified otherwise, $m = 500$ and $t = 0.1$. In (4a), $n = 500$ and in (4c), $n = 1000$.

Now, $\sum_{j=1}^{m} e^{-\frac{1}{4} \left( \frac{i-j}{\sqrt{m}} \right)^2} \leq m$ and

$$\max_{i \in [n]} \max_{j \in [m]} \left( 1 - e^{-\frac{1}{4} \left( \frac{i-j}{\sqrt{m}} \right)^2} \right)^2 \leq \max\{ (1 - e^{-\frac{1}{n^2}})^2, (1 - e^{-\frac{1}{T}})^2 \} = O \left( \frac{1}{n^2 T^2} \right).$$

Therefore,

$$\|A_i - A_{i+1}\|_2 = O \left( \frac{\sqrt{m}}{n T^{1/2}} \right).$$

Figure 4 shows calculations of $\Delta_2(A)$ with varying parameters. The vertical axes are scaled to emphasis the asymptotics. These calculations suggest that the analysis in Proposition 4 is asymptotically tight in $m$, $n$ and $T$.

**Lemma 5.** Let $M$ be a matrix such that $\|M\|_2 = 1$ and suppose the domain is $[0, 1]^n$, then

$$\Delta_2(M) \geq \frac{1}{\kappa_2(M) \sqrt{n}}.$$

**Proof.** Firstly,

$$\frac{1}{\kappa_2(M)} = \min_{\text{rank} E < \min\{m,n\}} \|M - E\|_2 \leq \|M - [M_1 M_2 \cdots M_{n-1}]\|_2 \leq \sqrt{n} \max_i \|M_i - M_{i+1}\|_2 = \sqrt{n} \Delta_2(M).$$

**Lemma 6.** Furthermore, if $\Delta_2(M) \leq \nu$ then $|\sum_{i \neq 1} s_i| \leq (n + 1)^{3/2} \nu$ and $\|M_i\|_2 - \|M_{i+1}\|_2 \leq \nu$. Conversely, if $|\sum_{i \neq 1} s_i| \leq \nu$ and $\|M_i\|_2 - \|M_{i+1}\|_2 \leq \nu$ then $\Delta_2(M) \leq 4\nu$.

**Proof.** Now, assume $\Delta_2(M) \leq \nu n$ then $\|M_i\|_2 - \|M_{i+1}\|_2 \leq \nu$. Suppose wlog that $\max_i \|M_i\|_2 = \|M_1\|_2$ and let $M' = [M_1 \cdots M_1]$ be the matrix whose columns are all duplicates of the first column of $M$. Recall that the trace norm of a matrix is the sum of its singular values and for any matrix, $\|M\|_{tr} \leq \sqrt{\min\{m,n\}} \|M\|_F$ and $\|M\|_2 \leq \|M\|_F$. Since $M'$ is rank 1, $\|M'\|_{tr} = \|M'\|_2$, thus,

$$\min_{\text{rank} E < \min\{m,n\}} \|M - E\|_2 \leq \|M - [M_1 M_2 \cdots M_{n-1}]\|_2 \leq \sqrt{n} \max_i \|M_i - M_{i+1}\|_2 = \sqrt{n} \Delta_2(M).$$

Conversely, suppose $|\sum_{i \neq 1} s_i| \leq \nu$ and $\|M_i\|_2 - \|M_{i+1}\|_2 \leq \nu$. Using the SVD we know, $M = \sum s_i U_i \otimes V_i$ where $U_i$ and $V_i$ are columns of unitary matrices $U$ and $V$. Thus, $\|M_i - M_{i+1}\|_2 = \| \sum s_j (V_j)_i U_j - \sum s_j (V_j)_{i+1} U_j \|_2 \leq s_1 |(V_1)_i - (V_1)_{i+1}| + \nu$. Also, $\nu \geq \|M_i\|_2 - \|M_{i+1}\|_2 \geq s_1 (V_i)_i - \nu s_1 (V_i)_{i+1} - \nu$ so $|(V_i)_i - (V_i)_{i+1}| \leq 3\nu/s_1$. 

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C Proof of Theorem\textsuperscript{9}

The following lemma is from [LOT14]. Since $T = \mu t$ is fixed we will let $g(x) = g(x,t)$.

Lemma 14. [LOT14] Suppose $s_1 < x < s_2$ and $|s_1 - s_2| \leq \sqrt{2T}$ and consider the function $W(z) = -g'(s_2 - x)g(z - s_1) - g'(x - s_1)g(s_2 - z)$. Then $W(z)$ has a single maximum at $x$ and

$$W(x) - W(z) = \begin{cases} > W(x) - W(s_2 - \sqrt{2T}) & \text{for } z \leq s_2 - \sqrt{2T} \\ \geq C_1||z - x||_2^2 & \text{for } z \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}] \\ > W(x) - W(s_1 + \sqrt{2T}) & \text{for } z \geq s_1 + \sqrt{2T} \end{cases}$$

where $C_1 = \inf_{z\in[s_2-\sqrt{2T},s_1+\sqrt{2T}]} \left[-W''(z)/2\right] > 0$.

The following two lemmas are necessary for our proof of Theorem\textsuperscript{9}. For all $i \in [k]$ and $j \in [p]$, let $W_{ij}(z) = -g'(s_{ij+p} - x_i)g(z - s_{ij}) - g'(x_i - s_{ij})g(s_{ij+p} - z)$. Let $p = m\sqrt{T}/2$. We will often replace the distance between $s_{ij}$ and $s_{ij+p}$ with $\sqrt{T}/2$ since it is asymptotically equal to the true distance $p/m = [m\sqrt{T}]/m$ in $m$.

Lemma 15. Using the assumptions of Theorem\textsuperscript{9} we have

$$\sum_{j=1}^{p} \inf_{z\in[s_{ij+p} - \sqrt{2T}, s_{ij} + \sqrt{2T}]} \left[-W''_{ij}(z)/2\right] \geq \Omega \left(\frac{m\sqrt{T}/8 + 1}{T^{2.5}}\right)$$

Proof. Note first that $s_{ij+p} - x_1 = \sqrt{T}/2 - (x_1 - s_{ij})$ and $s_{ij+p} - z = \sqrt{T}/2 - (z - s_{ij})$ for any $z \in [s_{ij+p} - \sqrt{2T}, s_{ij} + \sqrt{2T}]$. Let $z \in [s_{ij+p} - \sqrt{2T}, s_{ij} + \sqrt{2T}]$ then

$$-W''_{ij}(z) = g'(s_{ij+p} - x_i)g''(z - s_{ij}) + g'(x_i - s_{ij})g''(s_{ij+p} - z)$$

$$= \frac{1}{16\pi T^3} \left[ (s_{ij+p} - x_i) \left( 1 - \frac{(z - s_{ij})^2}{4T} \right) e^{-\frac{(s_{ij+p} - x_i)^2 - (z - s_{ij})^2}{4T}} \\ + (x_i - s_{ij}) \left( 1 - \frac{(s_{ij+p} - z)^2}{4T} \right) e^{-\frac{(s_{ij+p} - s_{ij})^2 - (s_{ij+p} - z)^2}{4T}} \right]$$

$$\geq \frac{1}{2T\sqrt{2\pi T}} \frac{1}{2T\sqrt{2\pi T}} e^{-\frac{\pi}{16\pi T}} \min\{(s_{ij+p} - x_i), (x_i - s_{ij})\} \left( 1 - \frac{(z - s_{ij})^2}{4T} \right) \left( 1 - \frac{(s_{ij+p} - z)^2}{4T} \right)$$

$$\geq \frac{e^{-\frac{\pi}{16\pi T}}}{16\pi T^3} \min\{(s_{ij+p} - x_i), (x_i - s_{ij})\} \left( 1 - \frac{(z - s_{ij})^2}{4T} \right) \left( 1 - \frac{(s_{ij+p} - z)^2}{4T} \right)$$

Therefore,

$$\sum_{j=1}^{p} \inf_{z\in[s_{ij+p} - \sqrt{2T}, s_{ij} + \sqrt{2T}]} \left[-W''_{ij}(z)/2\right] \geq \frac{3e^{-\frac{\pi}{16\pi T}}}{64\pi T^3} \sum_{j=1}^{p} \min\{(s_{ij+p} - x_i), (x_i - s_{ij})\}$$

$$= \frac{3e^{-\frac{\pi}{16\pi T}}}{64\pi T^3} 2 \sum_{i=1}^{p/2} \frac{i}{m}$$

$$= \frac{3e^{-\frac{\pi}{16\pi T}}}{64\pi T^3} 2 \sqrt{T/8} (m\sqrt{T/8} + 1)$$

\qed
Lemma 16. Using the assumptions of Theorem 9 we have
\[
\min_{i \in [k]} \min_{l \in \mathcal{S}_i} \sum_{j=1}^p (W_{ij}(x_i) - W_{ij}(l/n)) = \Omega \left( \frac{m \sqrt{T/2} (m \sqrt{T/2} + 1)^2}{m^2} \right).
\]

Proof. From Lemma 14 we know for all \( l \in \mathcal{S}_i \) we have
\[
W_{ij}(x_i) - W_{ij}(l/n) \geq \min \{W_{ij}(x_i) - W_{ij}(s_{i,j+p} - \sqrt{2T}), W_{ij}(x_i) - W_{ij}(s_i + \sqrt{2T})\}.
\]

Let’s start with
\[
W_{ij}(x_i) - W_{ij}(s_{i,j+p} - \sqrt{2T})
\]
\[
= -g'(s_{i,j+p} - x_i) \left( g(x_i - s_{i,j+p}) - g(s_{i,j+p} - \sqrt{2T} - s_i) \right) - g'(x_i - s_i) \left( g(s_{i,j+p} - x_i) - g(\sqrt{2T}) \right).
\]

Now, \( g(z) \) is concave down for \( z \in [-\sqrt{2T}, \sqrt{2T}] \) and \( \lambda \)-strongly concave on the interval \([-\sqrt{T/2}, \sqrt{T/2}]\) with \( \lambda = \frac{-7}{16 \sqrt{4\pi T^{1.5}}} \) so
\[
g(x) - g(y) \begin{cases} \geq -g'(x)(y - x) & \text{for } x, z \in [-\sqrt{2T}, \sqrt{2T}] \\ \geq -g'(x)(y - x) + \frac{7}{32 \sqrt{4\pi T^{1.5}}} e^{\frac{1}{T}} (y - x)^2 & \text{for } x, z \in [-\sqrt{T/2}, \sqrt{T/2}] \end{cases}
\]

Thus, since \( |s_{i,j+p} - s_i| = p/m \sim \sqrt{T/2} \) and \( |x_i - s_i| \leq \sqrt{T/2} \) and \( |s_{i,j+p} - x_i| \leq \sqrt{T/2} \) we have
\[
W_{ij}(x_i) - W_{ij}(s_{i,j+p} - \sqrt{2T})
\]
\[
\geq -g'(s_{i,j+p} - x_i) \left( -g'(x_i - s_{i,j+p})(s_{i,j+p} - x_i - \sqrt{2T}) + \frac{7}{32 \sqrt{4\pi T^{1.5}}} e^{\frac{1}{T}} (s_{i,j+p} - x_i - \sqrt{2T})^2 \right)
\]
\[
- g'(x_i - s_i) \left( -g'(s_{i,j+p} - x_i)(\sqrt{2T} - s_{i,j+p} - x_i) \right)
\]
\[
= g'(s_{i,j+p} - x_i) g'(x_i - s_i) \frac{7}{32 \sqrt{4\pi T^{1.5}}} e^{\frac{1}{T}} \left( s_{i,j+p} - x_i - \sqrt{2T} \right)^2
\]
\[
\geq (s_{i,j+p} - x_i)(x_i - s_i) e^{-\frac{1}{4T}} \frac{7}{512 \sqrt{4\pi T^{3.5}}} e^{\frac{1}{T}}
\]

where the last inequality follow since \( 0 \leq x_i - s_i = \sqrt{T/2} - (s_{i,j+p} - x_i) \leq \sqrt{T/2} \). Now,
\[
\sum_{j=1}^p (s_{i,j+p} - x_i)(x_i - s_i) \geq \sum_{j=1}^p \frac{j}{m} (\sqrt{T/2} - \frac{j}{m})
\]
\[
= \frac{p(p+1)(3m \sqrt{T/2} - 2p - 1)}{6m^2}
\]
\[
\sim \frac{m \sqrt{T/2} (m \sqrt{T/2} + 1)^2}{6m^2}
\]

We can now turn to our proof of the upper bound on the EMD error of Algorithm 1.

Theorem 9. Suppose that \( f_0 \) is a source vector, \( \hat{y} = R(\hat{y}) \) and assume the following:
1. \( m \sqrt{T/2} > 1 \)

2. \( \sqrt{2T} < 1 \)

3. \( |x_i - x_j| > \sqrt{2T} + 2A \text{ for some } A > 0 \)

then w.h.p.

\[
E(MD) \left( \frac{f_0}{\|f_0\|_1}, \frac{\hat{f}}{\|\hat{f}\|_1} \right) \leq \min \left\{ 1, O \left( 1 - \min\{1, C\} \cdot \left( 1 + k \min\{1, C\} + \frac{T^2 C}{(T + 1)k} \right) \right) \right\}
\]

where \( C = \min \left\{ k, \sqrt{T} \left( \alpha + ke^{-A^2/4T} \right) \right\}. \)

**Proof.** Note that from Lemma 7, w.h.p. \( f_0 \) is a feasible point so \( \|\hat{f}\|_1 \leq k. \) Let \( S_i = (x_i - \sqrt{T/2}, x_i + \sqrt{T/2}) \cap [0, 1] \) for \( i \in [k] \). Then we have that the \( S_i \)'s are disjoint and each interval \( S_i \) contains \( \sqrt{2T}m \) sensors. Let \( p = \lfloor \sqrt{T}/2m \rfloor \) and let \( s_{i_1} < \cdots < s_{i_p} \) be the locations of the sensors in \( S_i \) to the left of \( x_i \) and \( s_{i_1} > \cdots > s_{i_p} \) be the locations to the right. By Condition 3 we know that for any pair \( l \in T_i \) and \( s_{c_l} \) where \( l \neq c \) we have \( |l/n - s_{c_l}| \geq A. \)

For all \( i \in [k] \) and \( j \in [p] \), let \( W_{ij}(z) = -g'(s_{i+j} - x_i)g(z - s_i) - g'(x_i - s_i)g(s_{i+j} - z). \) Let \( T_i \) be the set of all \( l \in [n] \) such that \( l/n \in (x_i - \frac{x_i - x_{i-1}}{2}) \cap [0, 1] \) then

\[
g(x_i - s_i) - \sum_{l \in T_i} \hat{f}_l g(l/n - s_i) \leq y_{ij} - \hat{y}_{ij} + \frac{\|\hat{f}\|_1}{\sqrt{4\pi T}} e^{\frac{-A^2}{T}}.
\]

Therefore,

\[
\sum_{i=1}^{k} \sum_{j=1}^{p} W_{ij}(x_i) - \sum_{l \in T_i} \hat{f}_l W_{ij}(l/n)
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{p} \left[ -g'(s_{i+j} - x_i) g(x_i - s_i) - \sum_{l \in T_i} \hat{f}_l g(l/n - s_i) \right] - g'(x_i - s_i) \left[ g(s_{i+j} - x_i) - \sum_{l \in T_i} \hat{f}_l g(s_i - l/n) \right]
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{p} \left[ ||y - \hat{y}||_1 + \frac{2p\|\hat{x}\|_1}{\sqrt{4\pi T}} e^{\frac{-A^2}{T}} \right]
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{p} \left[ \sqrt{m} ||y - \hat{y}||_2 + \frac{2p\|\hat{x}\|_1}{\sqrt{4\pi T}} e^{\frac{-A^2}{T}} \right]
\]

\[
\leq \sum_{i=1}^{k} \sum_{j=1}^{p} \left[ 2\sigma m + \frac{mk}{\sqrt{2\pi}} e^{\frac{-A^2}{T}} \right]
\]

where \( C_2 = \max_{i \in [k]} \max_{j \in [p]} [-g'(|s_i - x_i|)] \) and the last inequality holds with high probability from Lemma 7.

Also,

\[
W_{ij}(x_i) \leq -g'(s_{i+j} - x_i) g(x_i - s_i) - g'(x_i - s_i) g(s_{i+j} - x_i)
\]

\[
\leq \frac{1}{16\pi T} |s_{i+j} - x_i| + \frac{1}{16\pi T^2} |x_i - s_i| 
\]

\[
\leq \frac{1}{8\pi T^{1.5}}
\]

Therefore, since \( \sum_{l \in T_i} \hat{f}_l \leq 1 \) we have

\[
\sum_{i=1}^{k} \sum_{j=1}^{p} W_{ij}(x_i) - \sum_{l \in T_i} \hat{f}_l W_{ij}(l/n) \leq \min \left\{ km \frac{m}{8\pi T} , C_2 \left[ 2\sigma m + \frac{mk}{\sqrt{2\pi}} e^{\frac{-A^2}{T}} \right] \right\} = B.
\]
Conversely, by Lemma 14, we have
\[
\sum_{i=1}^{k} \sum_{j=1}^{p} \left| W_{ij}(x_i) - \sum_{l \in T_i} \hat{f}_l W_{ij}(l/n) \right| 
\geq \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_{ij}(x_i) + \sum_{i=1}^{k} \sum_{l \in T_i} \hat{f}_l W_{ij}(x_i) - W_{ij}(l/n)) \right) 
\geq \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_{ij}(x_i) + \sum_{i=1}^{k} \sum_{l \in T_i} \hat{f}_l C_3 l(x_i - l/n)^2 + \sum_{i=1}^{k} \sum_{l \in T_i} C_3 \hat{f}_l 
\geq \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_{ij}(x_i) + C_5 \sum_{i=1}^{k} \sum_{l \in T_i} \hat{f}_l (x_i - l/n)^2 + \sum_{i=1}^{k} \sum_{l \in T_i} C_3 \hat{f}_l 
\]
where \( C_3 = \min_{i \in [k]} \min_{l \notin S_i} \sum_{j=1}^{p} \left(W_{ij}(x_i) - W_{ij}(l/n)\right) \geq 0. \)

Now, by the uniformity of the sensor locations, \( W_j = W_{ij}(x_i) = W_{ij}(x_i) \) so \( \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_{ij}(x_i) = \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) \geq \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_j \geq \sum_{j=1}^{p} \sum_{i=1}^{k} \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) W_j (k - \|f\|_1) \geq 0. \) Similarly the other two terms are both positive. Therefore, \( \sum_{i=1}^{k} \sum_{l \notin T_i} \hat{f}_l \leq B/C_3 \) or equivalently,
\[
\sum_{i=1}^{k} \sum_{l \notin T_i} \hat{f}_l \geq \|f\|_1 - \min \{k, B/C_3\}.
\]

This implies that most of the weight of the estimate \( \hat{f} \) is contained in the intervals \( S_1, \ldots, S_k \). Also,
\[
B \geq \sum_{j=1}^{p} \left| W_{ij}(x_i) - \sum_{l \in T_i} \hat{f}_l W_{ij}(l/n) \right| 
\geq \sum_{j=1}^{p} W_{ij}(x_i) - \sum_{l \in T_i} \hat{f}_l W_{ij}(x_i) 
\geq \left( \sum_{j=1}^{p} W_{ij}(x_i) \right) \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) = C_4 \left( 1 - \sum_{l \in T_i} \hat{f}_l \right) 
\]
Therefore, \( \sum_{l \in T_i} \hat{f}_l \geq 1 - \min \{1, B/C_4\} \) and
\[
\sum_{l \notin T_i} \hat{f}_l \leq \sum_{l \notin T_i} \hat{f}_l = \|f\|_1 - \sum_{a \notin T_i} \hat{f}_l \leq \|f\|_1 - (k - 1) (1 - \min \{1, B/C_4\}) \leq 1 + (k - 1) \min \{1, B/C_4\}. 
\]

This implies that the weight of estimate \( \hat{f} \) contained in the interval \( S_i \) is not too much larger than the true weight of 1. Also,
\[
\frac{\sum_{l \notin S_i} \hat{f}_l}{\|f\|_1} \leq \frac{1 + (k - 1) \min \{1, B/C_4\}}{k(1 - \min \{1, B/C_4\})} = \frac{1}{k} + \frac{\min \{1, B/C_4\}}{1 - \min \{1, B/C_4\}} 
\]
In order to upper bound the EMD(\( \frac{\hat{f}}{\|f\|_1} \)) we need a flow, we are going to assign weight \( \min \{\frac{\hat{f}_l}{\|f\|_1}, \frac{1}{k}\} \) to travel to \( x_i \) from within \( S_i \). The remaining unassigned weight is at most \( k \min \{1, B/C_4\} + \frac{\min \{1, B/C_4\}}{k(1 - \min \{1, B/C_4\})} \) and this weight can travel at most 1 unit in any flow. Therefore,
\[
\text{EMD} \left( \frac{f_0}{k}, \frac{\hat{f}}{\|f\|_1} \right) \leq \sum_{i=1}^{k} \sum_{l \notin S_i} \hat{f}_l \left| x_i - l/n \right| + k \min \{1, B/C_4\} + \frac{\min \{1, B/C_4\}}{1 - \min \{1, B/C_4\}} 
\leq \frac{1}{k(1 - \min \{1, B/C_4\})} \sqrt{B/C_5} \left( k \min \{1, B/C_4\} + \frac{\min \{1, B/C_4\}}{1 - \min \{1, B/C_4\}} \right) 
\]
Now, we need bounds on \( C_1, C_2, C_3 \) and \( C_4 \). Firstly, recall \( C_1 = \inf_{x \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}]} [-W''(z)/2] > 0 \). The sensors \( s_i \) and \( s_{i+p} \) are at a distance of \( p/m \) and recall that we only chose the sensors such that \( |x_i - s_i| \geq 1/m \).

Thus, any \( z \in [s_2 - \sqrt{2T}, s_1 + \sqrt{2T}] \) we have either \( |z - s_i| \leq \sqrt{T/8} \) or \( |z - s_{i+p}| \leq \sqrt{T/8} \) so

\[
-W''(z) = g'(s_{i+p} - x_i)g''(z - s_i) + g'(x_i - s_i)g''(s_{i+p} - z) \geq \frac{1}{16\pi T^3} \left( \frac{1}{m} \left( 1 - \frac{T}{27} \right) e^{-\frac{\sqrt{T}}{8}} \right)
\]

Therefore, \( C_1 \geq \frac{17\pi \sqrt{T}}{512} \frac{1}{n} \frac{1}{T^2} \). Next,

\[
C_2 = \max_{i \in [k]} \max_{j \in [2p]} -g'(|x_i - s_j|) \leq \frac{1}{2\sqrt{4\pi T}} e^{-\frac{1}{2\sqrt{4\pi T}}}
\]

By Lemma 16 we have, \( C_3 = \min_{i \in [k]} \min_{l \in nS} \| W_{ij}(x_i) - W_{ij}(l/n) \| = \Omega \left( \frac{m\sqrt{T/2}(m\sqrt{T/2}+1)^2}{n^2} \right) \).

Finally,

\[
C_4 = \sum_{j=1}^{p} W_{ij}(x_i)
\]

\[
= \frac{1}{8\pi T^2} \sum_{j=1}^{p} (s_{i+p} - x_i) e^{-\frac{(s_{i+p} - x_i)^2 - (x_i - s_i)^2}{4}} + (x_i - s_i) e^{-\frac{(s_{i+p} - x_i)^2 - (x_i - s_i)^2}{4}}
\]

\[
\geq \frac{1}{8\pi T^2} e^{-\frac{1}{2\sqrt{4\pi T}}} \sum_{j=1}^{p} (s_{i+p} - s_i)
\]

\[
\geq \Omega \left( \frac{me}{16\pi T} \right)
\]

Lemma 15 gives \( C_5 = \Omega \left( \frac{m\sqrt{T/8}+1}{T^{2/5}} \right) \). Putting all our bounds into (2) we gain the final result.

\[\square\]

**D Proof of Theorem 11**

The following generalisation of the upper bound in Proposition 4 will aid in our proof of Theorem 11.

**Lemma 17.** Suppose \( \| f_0 \|_1 = \| f'_0 \|_1 = 1 \) then

\[
\| Af_0 - A f'_0 \|_2 = O \left( \frac{\sqrt{m}}{T^{1.5}} EMD(f_0, f'_0) \right)
\]

**Proof.** Firstly, consider the single peak vectors \( e_i \) and \( e_j \). Then noting that \( Ae_i = A_i \), we have from Proposition 4 that

\[
\| Ae_i - Ae_j \|_2 \leq \sum_{l=0}^{j-i-1} \| Ae_{i+l} - Ae_{i+1+l} \|_2 \leq O \left( \frac{|i-j| \sqrt{m}}{nT^{1.5}} \right)
\]

Now, let \( f_{ij} \) be the optimal flow from \( f_0 \) to \( f'_0 \) as described in Definition 8 so \( f_0 = \sum_{i,j} f_{ij} e_i \) and \( f'_0 = \sum_{i,j} f_{ij} e_j \). Then

\[
\| Af_0 - A f'_0 \|_2 \leq \sum_{i,j} f_{ij} \| Ae_i - Ae_j \|_2
\]

\[
\leq O \left( \sum_{i,j} f_{ij} \frac{|i-j|}{n} \frac{\sqrt{m}}{T^{1.5}} \right)
\]

\[
= O \left( \frac{\sqrt{m}}{T^{1.5}} EMD(f_0, f'_0) \right)
\]

\[\square\]
Suppose \( p, q \) are probability distributions on the same space. Then the Kullback-Leibler (KL) divergence of \( p \) and \( q \) is defined by \( D(p||q) = \int (\log \frac{dp}{dq}) dp \). For a collection \( T \) of probability distributions, the KL diameter is defined by

\[
d_{KL}(T) = \sup_{p,q \in T} D(p||q).
\]

If \((\Omega, d)\) is a metric space, \( \epsilon > 0 \) and \( T \subset \Omega \), then we define the \( \epsilon \)-packing number of \( T \) to be the largest number of disjoint balls of radius \( \epsilon \) that can fit in \( T \), denoted by \( M(\epsilon, T, d) \). The following version of Fano’s lemma is found in [Yu97].

**Lemma 18** (Fano’s Inequality.). Let \((\Omega, d)\) be a metric space and \( \{P_\theta | \theta \in \Omega\} \) be a collection of probability measures. For any totally bounded \( T \subset \Omega \) and \( \epsilon > 0 \),

\[
\inf_{\theta} \sup_{\theta \in \Omega} \mathbb{P}_\theta \left( d^2(\hat{\theta}(X), \theta) \geq \frac{\epsilon^2}{4} \right) \geq 1 - \frac{d_{KL}(T) + 1}{\log M(\epsilon, T, d)}
\]

where the infimum is over all estimators.

**Theorem 11** We have

\[
\inf_{\hat{f}} \sup_{f_0} \mathbb{E}[\text{EMD}(f_0, \hat{f})] = \Omega \left( \min \left\{ \frac{1}{2}, \frac{T^{1.5} \sigma}{\sqrt{m}} \right\} \right).
\]

where \( \inf_{\hat{f}} \) is the infimum over all estimators \( \hat{f} : \mathbb{R}^m \to [0, 1]^n \), \( \sup_{f_0} \) is the supremum over all source vectors in \([0, 1]^n\) and \( y \) is sampled from \( y + N(0, \sigma^2 I_m) \).

**Proof of Theorem 11** For any source vector \( f_0 \), let \( P_{f_0} \) be the probability distribution induced on \( \mathbb{R}^m \) by the process \( Af_0 + N(0, \sigma^2 I_m) \). Then the inverse problem becomes estimating which distribution \( P_{f_0} \) the perturbed measurement vector is sampled from. Let \( f_0 \) and \( f'_0 \) be two source vectors. Then

\[
D(P_{f_0}||P_{f'_0}) = \sum_{i=1}^{m} \frac{((Af_0)_i - (Af'_0)_i)^2}{2\sigma^2}
\]

\[
= \frac{1}{2\sigma^2} \|Af_0 - Af'_0\|_2^2
\]

\[
\leq C \frac{m}{T^{3/2}} \sigma^2 \text{EMD}(f_0, f'_0)^2
\]

for some constant \( C \), where we use the fact that the KL-divergence is additive over independent random variables, along with Lemma 17. Now, let \( a = \min \left\{ \frac{1}{2}, \frac{T^{1.5} \sigma}{\sqrt{2Cm}} \right\} \). Let \( T \) be the set consisting of the following source vectors:

\[
e_{1/2}, (1/2)e_{1/2-a/2} + (1/2)e_{1/2+a/2}, (1/4)e_{1/2-a} + (1/2)e_{1/2} + (1/4)e_{1/2+a}, (1/2)e_{1/2} + (1/2)e_{1/2+a}, \text{which are all at an EMD } a \text{ from each other. Then } d_{KL}(T) + 1 \leq 3/2 \text{ and } \log M(a, T, \text{EMD}) = 2. \text{ Thus, by Lemma 18}
\]

\[
\inf_{\hat{f}} \sup_{f_0} \mathbb{E}[\text{EMD}(f_0, \hat{f})] \geq \frac{3}{4} a = \Omega \left( \min \left\{ \frac{1}{2}, \frac{T^{1.5} \sigma}{\sqrt{m}} \right\} \right).
\]
E Further Experimental Results

Figure 5: Each graph shows the result of the recovery algorithm $R$. Unless stated otherwise, $n = 100$, $m = 10$, $T = 0.05$, $\sigma = 0.1$, $\epsilon = 10$, $\delta = 0.01$ and $k = 4$.

Figure 6: Empirical results for the EMD error of experiments running Algorithm 1 on the locally differentially private thermal measurements. Unless specified otherwise, $n = 100$, $m = 50$, $t = 0.1$, $\delta = 0.1$ and $\epsilon = 1$. 