Abstract. Let $X$ denote a simply connected compact Riemannian symmetric space, $U$ the universal covering of the identity component of the group of automorphisms of $X$, and $LU$ the loop group of $U$. In this paper we prove the existence (and conjecture the uniqueness) of an $LU$-invariant probability measure on a distributional completion of the loop space of $X$.

§0. Introduction.

Let $K$ denote a simply connected compact Lie group, and let $G$ denote the complexification of $K$. In [Pi1] we proved the following

Theorem (0.1). There exists a $L_{pol} K$-biinvariant probability measure $\mu$ on the formal completion, $L_{formal} G$, of the complex loop group $LG$.

We also conjectured that there is a unique such biinvariant measure.

One purpose of this paper is to present a more transparent proof of this theorem, especially in the simplest case, $K = SU(2)$ (see §3). A second purpose is to generalize the theorem to a context in which $K$ is replaced by a simply connected compact symmetric space $X$. Before stating this generalization, we will clarify the meaning of various terms.

First $L_{pol} K$ and $L_{an} K$ denote the groups consisting of maps from $S^1$ into $K$ which are polynomial (i.e. have finite Fourier series) and real analytic, respectively, with pointwise multiplication. The complexified loop group, $L_{an} G = H^0(S^1, G)$, is a complex Lie group. A neighborhood of the identity consists of those loops which have a unique (triangular or Birkhoff or Riemann-Hilbert) factorization

$$g = g_- \cdot g_0 \cdot g_+,$$  \hspace{1cm} (0.2)
where \( g_- \in H^0(D^*, \infty; G, 1) \), \( g_0 \in G \), \( g_+ \in H^0(D, 0; G, 1) \), and \( D \) and \( D^* \) denote the closed disks centered at 0 and \( \infty \), respectively (thus (0.2) is an equality of holomorphic functions which holds on a thin collar of \( S^1 \), the collar depending upon \( g \)). A model for this neighborhood is

\[
H^1(D^*, g) \times G \times H^1(D, g),
\]

(0.3)

where the linear coordinates are determined by \( \theta_+ = g_+^{-1}(\partial g_+) \), \( \theta_- = (\partial g_-)g_-^{-1} \). The (left or right) translates of this neighborhood by elements of \( L_{pol} K \) cover \( H^0(S^1, G) \).

The hyperfunction completion, \( L_{hyp} G \), is modelled on the space

\[
H^1(\Delta^*, g) \times G \times H^1(\Delta, g),
\]

(0.4)

where \( \Delta \) and \( \Delta^* \) denote the open disks centered at 0 and \( \infty \), respectively, and the transition functions are obtained by continuously extending the transition functions for the analytic loop space of the preceding paragraph. The global definition is

\[
L_{hyp} G = \lim_{r \downarrow 1} H^0(\{1 < |z| < r\}, G) \times H^0(S^1, G) \lim_{r \uparrow 1} H^0(\{r < |z| < 1\})
\]

(0.5)

From the global definition it is clear that the group \( H^0(S^1, G) \) acts naturally from both the left and right of \( L_{hyp} G \).

The formal completion is defined in a similar way, where \( H^1(\Delta, g) \) is replaced by the corresponding space of formal power series

\[
H^1_{formal}(\Delta, g) = \{\theta_+ = (\theta_1 + \theta_2 z + ..)dz, \quad \theta_i \in g\} \simeq \prod g.
\]

(0.6)

The global definition of the formal completion is

\[
L G = L_{formal} G = G(\mathbb{C}((z^{-1}))) \times_{G(\mathbb{C}(z))} G(\mathbb{C}((z))),
\]

(0.7)

where \( \mathbb{C}((z)) \) is the field of formal Laurent series \( \sum a_n z^n \), \( a_n = 0 \) for \( n << 0 \), and \( G(\mathbb{C}(z)) = L_{pol} G \).

To summarize, there is an equivariant way to form various “distributional” completions of the complex loop group, respecting its homotopy and complex structure,

\[
L_{pol} G \to L_{an} G \to L_{C^0 \cap W^{1/2}} G \to L_{hyp} G \to L G
\]

(0.8)
where $LG$ and $L_{hyp}G$ are “dual” to $L_{pol}G$ and $L_{an}G$, respectively, and $L_{C^0\cap W^{1/2}}G$ is essentially “self-dual”.

An aesthetic advantage of the hyperfunction completion is that $D$, the group of analytic orientation-preserving diffeomorphisms of $S^1$, acts equivariantly. If $\mu$ is supported on $L_{hyp}G$, then the conjectured uniqueness of $\mu$ implies the conjecture that $\mu$ is $D$-invariant.

Now suppose that $X$ is a simply connected compact symmetric space with a fixed basepoint. From this we obtain groups as in the following diagram,

$$
\begin{array}{ccc}
G & \rightarrow & U, \\
\uparrow & & \uparrow \\
G_{\mathbb{R}} & \rightarrow & U, \\
\downarrow & & \downarrow \\
K & \rightarrow & G
\end{array}
$$

(0.10)

where $U$ is a simply connected compact group acting isometrically and transitively on $X$, $X \simeq U/K$, $G$ is the complexification of $U$, and $G_{\mathbb{R}}$ acts by automorphisms of the dual (nonunitary type) symmetric space $G_{\mathbb{R}}/K$. There are equivariant totally geodesic embeddings

$$
\begin{array}{ccc}
U/K & \rightarrow & U \\
\downarrow & & \downarrow \\
G/G_{\mathbb{R}} & \rightarrow & G
\end{array}
$$

(0.11)

of symmetric spaces. The main point is that the antiholomorphic involution which fixes $L(G/G_{\mathbb{R}}) \rightarrow LG$ extends naturally to our distributional completions of $LG$. We define the hyperfunction and formal completions of $L(G/G_{\mathbb{R}})$ as the identity components of the fixed point sets of these extensions. We obtain equivariant homotopy equivalences

$$
L_{pol}(U/K) \rightarrow L_{pol}(G/G_{\mathbb{R}}) \rightarrow L_{an}(G/G_{\mathbb{R}}) \rightarrow L_{hyp}(G/G_{\mathbb{R}}) \rightarrow L(G/G_{\mathbb{R}}).
$$

(0.12)

The main result is the following

**Theorem (0.13).** There exists a $L_{pol}U$-invariant probability measure on the formal completion, $L(G/G_{\mathbb{R}})$.

We conjecture that there is a unique such invariant measure.

A general simply connected $X$ can be written uniquely as a product of irreducible $X$ (irreducible means that $u$ does not have any $\Theta$-invariant ideals, where $\Theta$ is the involution of $U$ which fixes $K$). There are two types of irreducible $X$. 
In the type I case $u$ is simple. In the type II case $u = \mathfrak{k} \oplus \mathfrak{k}$, where $\mathfrak{k}$ is simple, $\Theta(x, y) = (y, x)$, and $X = K$. Thus the essential novelty of this paper concerns the type I case.

The plan of the paper is the following. In §1 and §2 we introduce the basic notation used throughout the paper. In §1 we review some basic facts about triangular decompositions and symmetric spaces (readers interested in basic cases such as $S^2$ and $SU(2)$ can probably skip this section entirely). A more detailed treatment of the statements in this section will appear in [Pi4]. In §2 we consider loop spaces and their completions, and prove the minimal structural results that we will need.

In §3 we will give a relatively elementary proof of Theorem (0.1) in the case $K = SU(2)$. This proof reveals that we can a priori compute many distributions for the measure $\mu$ in Riemann-Hilbert coordinates, (0.4). We briefly discuss the issue of uniqueness, but this remains unresolved.

In §4 we will give a corresponding elementary proof of Theorem (0.13) in the case $S^2$.

In §5 and §6 we prove Theorem (0.13) in general. The proofs in §3-§6 depend upon a certain compactness result for Wiener measures parameterized by temperature (or radius of the circle); this depends heavily on ideas of the Malliavins ([MM]).

In §7 we discuss a conjectural formula for the “diagonal distribution”, and some of its potential implications.

§1. Symmetric Spaces and Triangular Decompositions.

Throughout the remainder of this paper, $U$ will denote a simply connected compact Lie group, $\Theta$ will denote an involution of $U$, with fixed point set $K$, and $X$ will denote the quotient, $U/K$. The space $X$ has the structure of a simply connected symmetric space of compact type. Conversely, given such a space $X$, together with the choice of basepoint, there is a symmetric pair $(U, K)$, satisfying the conditions above, such that $X \simeq U/K$. To pin down $U$ in terms of $X$, we could choose $U$ to be the universal covering of the identity component of the group of automorphisms of $X$; but for technical reasons, we will not assume this at the outset.

There is a unique extension of $\Theta$ to a holomorphic automorphism of $G$, the complexification of $U$. Corresponding to the diagram of groups in (0.10), there is a
Lie algebra diagram

\[ g = u \oplus iu \]

\[ g_{\mathbb{R}} = \mathfrak{k} \oplus p \]

\[ U = \mathfrak{k} \oplus i\mathfrak{p} \]

\[ \mathfrak{k} \]

where \( \Theta \), acting on the Lie algebra level, is +1 on \( \mathfrak{k} \) and −1 on \( \mathfrak{p} \). We let \((\cdot)^{-*}\) denote the Cartan involution for the pair \((G, U)\) (so that \((\cdot)^*\) is an antiholomorphic antiinvolution). The Cartan involution for the pair \((G, G_{\mathbb{R}})\) is given by \( \tau(g) = g^{-*\Theta} \). Since \(*\), \(\Theta\), \(\tau\), and \((\cdot)^{-1}\) commute, our practice of writing \(g^{\Theta}\) for \(\Theta(g)\), etc, should not cause any confusion.

There are natural maps

\[ K \rightarrow U \rightarrow U/K \]

\[ G_{\mathbb{R}} \rightarrow G \rightarrow G/G_{\mathbb{R}} \]

The vertical arrows (given by inclusion) are homotopy equivalences; more precisely, there are diffeomorphisms (polar or Cartan decompositions)

\[ K \times p \rightarrow G_{\mathbb{R}}, \quad U \times i\mathfrak{u} \rightarrow G, \quad U \times K \rightarrow G/G_{\mathbb{R}}, \]

in each case given by the formula \((g, X) \rightarrow gexp(X) \mod G_{\mathbb{R}}\) in the last case. In turn there are totally geodesic embeddings (or morphisms) of symmetric spaces

\[ U/K \xrightarrow{\phi} U : gK \rightarrow gg^{-*\Theta} \]

\[ G/G_{\mathbb{R}} \xrightarrow{\phi} G : gg_{\mathbb{R}} \rightarrow gg^{*\Theta} \]

The map \(\phi\) is equivariant, where \(g \in G\) acts on \(g_0 \in G\) by \(g_0 \rightarrow g_0 g^{*\Theta}\). The \(\phi\)-images are defined by simple algebraic equations, modulo connectedness issues,

\[ \phi(U/K) = \{g \in G : g^{-1} = g^* = g^{\Theta}\}_0 \rightarrow U = \{g^{-1} = g^*\} \]

\[ \phi(G/G_{\mathbb{R}}) = \{g^* = g^{\Theta}\}_0 \rightarrow G \]

where the 0-subscript denotes the connected component containing the identity (Note: the maps (1.4) exist for arbitrary automorphisms \(\Theta\); the simple characterization of the images is peculiar to automorphisms of order 2).
Fix a maximal abelian subalgebra \( t_0 \subset \mathfrak{k} \). We then obtain \( \Theta \)-stable Cartan subalgebras
\[
\mathfrak{h}_0 = \mathcal{Z}_{\mathfrak{g}_R}(t_0) = t_0 + \mathfrak{a}_0, \quad t = t_0 + \mathfrak{i}\mathfrak{a}_0
\]
and \( \mathfrak{h} = \mathfrak{h}_0^C \), for \( \mathfrak{g}_R, \mathfrak{u}, \) and \( \mathfrak{g} \), respectively, where \( \mathfrak{a}_0 \subset \mathfrak{p} \). We let \( T_0 \) and \( T \) denote the maximal tori in \( K \) and \( U \) corresponding to \( t_0 \) and \( t \), respectively.

Let \( \Delta \) denote the roots for \( \mathfrak{h} \) acting on \( \mathfrak{g} \); \( \Delta \subset \mathfrak{h}^* \), where \( \mathfrak{h}^*_R = \mathfrak{a}_0 \oplus \mathfrak{t}_0 \). We choose a Weyl chamber \( C^+ \) which is \( \Theta \)-stable (to prove that \( C^+ \) exists, we must show that \( \mathfrak{i}t_0 \), the +1 eigenspace of \( \Theta \) acting on \( \mathfrak{h}_R \), intersects the regular part of \( \mathfrak{h}_R \); since \( t_0 \) is maximal abelian, we can find regular elements in \( t_0 \)). Since \( \tau = -(-\cdot)^\Theta \) and \((-\cdot)^\Theta\) is the identity on \( \mathfrak{h}_R \), \( \tau(C^+) = -C^+ \).

Given our choice of \( C^+ \), we obtain a \( \Theta \)-stable triangular decomposition \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \), so that \( \tau(n^\pm) = n^{\mp} \). Let \( N^\pm = \exp(n^\pm) \), \( H = \exp(\mathfrak{h}) \), and \( B^\pm = HN^\pm \).

At the group level, we have the Birkhoff or triangular or LDU decomposition for \( G \),
\[
G = \bigsqcup W \hat{\Sigma}_w, \quad \hat{\Sigma}_w = N^- wHN^+,
\]
where \( \hat{\Sigma}_w \) is diffeomorphic to \((N^- \cap wN^-w^{-1}) \times H \times N^+ \). The intersection of this decomposition with the spaces in (1.5) is considered in detail in [Pi4]. Here we will only prove what we need for \( \phi(G/G_R) \).

(1.8) Proposition. Fix \( w \in W \).

(a) The intersection \( \{g^* = g^{\Theta}\} \cap \hat{\Sigma}_w \) is nonempty if and only if there exists \( w \in w \subset N_U(T) \) such that \( w^{*\Theta} = w \); \( w \) is unique modulo the action
\[
T \times \{w \in N_U(T) : w^{*\Theta} = w\} \to \{w^{*\Theta} = w\} : \lambda, w \to \lambda w^{*\Theta}.
\]

(b) The orbits of \( B^- \) in \( \{g^* = g^{\Theta}\} \cap \hat{\Sigma}_w \) are open and indexed by
\[
\pi_0(\{w \in w : w^{*\Theta} = w\}) \simeq \{w \in w : w^{*\Theta} = w\}/T.
\]

(c) The \( B^- \)-orbit through \( w \) belongs to \( \phi(G/G_R) \) if and only if \( Ad(w) \circ \Theta \) is equivalent to \( \Theta \) through conjugation by \( Ad(K) \).

(d) For the top stratum, the map
\[
N^- \times (T_0^{(2)} \times T_0^{(2)} \cap \exp(i\mathfrak{a}_0))^\Theta \times \exp(i\mathfrak{t}_0) \to \{g^* = g^{\Theta}\} \cap \hat{\Sigma}_1
\]
\[
l, [w, m], a_\emptyset \to g = lwm a_\emptyset l^{*\Theta}
\]
is a diffeomorphism.

In this paper we will only use (d). To prove this suppose that \( g \in \tilde{\Sigma}_1 \). There is a unique decomposition \( g = lhu \), where \( l \in N^- \), \( h \in H \), and \( u \in N^+ \). If \( g = g^* \Theta \), then \( lhu = u^* \Theta h^* \Theta l^* \Theta \), and because \((\cdot)^* \Theta \) interchanges \( N^\pm \), uniqueness of the decomposition implies \( u = l^* \Theta \) and \( h = h^* \Theta \). This leads to (d) in a routine way.

**Proposition (1.10).** For the action of \( U \) on \( G \) given by \( u, g \to ugu^{-\Theta} \), for each \( U \)-orbit \( O \), the intersection of \( O \) with \( \tilde{\Sigma}_1 \) is dense in \( O \).

**Proof of (1.10).** Let \( v \) denote a highest weight vector for the highest weight representation corresponding to the sum of the dominant integral functionals. Then

\[
\tilde{\Sigma}_1 = \{ g \in G : \langle g \cdot v, v \rangle \neq 0 \}, \tag{1.11}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the essentially unique \( U \)-invariant Hermitian inner product.

Fix \( g_0 \in G \). We must show that there exist \( g \in U \), arbitrarily close to 1, such that \( \langle gg_0 g^{-\Theta} v, v \rangle \neq 0 \). Suppose that this is not the case. Then

\[
\langle gg_0 g^{-\Theta} v, v \rangle = 0, \tag{1.12}
\]

for all \( g \in U \) sufficiently close to 1. Since the left hand side of this equation is a holomorphic function of \( g \in G \), and since \( U \) is a real form of \( G \), (1.12) holds for all \( g \in G \). If we take \( g = b \in B^+ \), then \( b^{-\Theta} \in B^+ \), \( b^{-\Theta} \cdot v \) is a multiple of \( v \), and \( b^* \in B^- \). Hence (1.12) implies

\[
\langle g_0 v, b^* v \rangle = 0, \quad \forall b^* \in B^- . \tag{1.13}
\]

But \( \{ b^* v : b^* \in B^- \} \) spans the space of the representation. This is a contradiction. \( \square \)

**Corollary (1.14).** Let \( U \) act on \( G \) as in (1.10). Suppose that \( \nu \) is a \( U \)-quasi-invariant measure on \( G \). Then the \( \nu \)-measure of the complement of \( \tilde{\Sigma}_1 \) is zero.

**Proof of (1.14).** The measure \( \nu \) will have a disintegration along the orbits of \( U \), and for a.e. such orbit, the derivative \( \nu_O \) will belong to the unique (Lebesgue) invariant measure class. Proposition (1.10) implies that the \( \nu_O \)-measure of \( O \setminus \tilde{\Sigma}_1 \) is zero. This implies (1.14). \( \square \)

§2. Loop Spaces and Completions.
In the introduction we recalled the definitions of $\mathbf{L}G$ and $L_{hyp}G$ (see Part III, chapter 2, of [Pi1]). To consider the finer properties of loop groups and their completions, it is useful to adopt the Kac-Moody point of view, where many structural results carry over from the theory of simple Lie algebras. This point of view will be needed only occasionally (in the proof of Lemma (6.5) and Appendix B). Otherwise it will suffice to adopt a more naive approach, developed here.

These completions have generalized Birkhoff factorizations

$$\mathbf{L}G = \bigsqcup \Sigma^\lambda_{\text{formal}}, \quad \Sigma^\lambda_{\text{formal}} = G(\mathbb{C}[[z^{-1}]]) \cdot \lambda \cdot G(\mathbb{C}[[z]]) \quad (2.1)$$

where $\mathbb{C}[[\zeta]]$ denotes formal power series in $\zeta$, and

$$L_{hyp}G = \bigsqcup \Sigma^\lambda_{\text{hyp}}, \quad \Sigma^\lambda_{\text{hyp}} = H^0(\Delta^*, G) \cdot \lambda \cdot H^0(\Delta, G), \quad (2.2)$$

which restrict to the standard Birkhoff factorization of $\mathbf{L}G$ (see chapter 8 of [PS]). It is the existence of these coherent decompositions, corresponding to different smoothness conditions, which imply that these various completions are all homotopy equivalent (see (8.6.6) in [PS]).

In both the formal and hyperfunction cases, the top stratum (the piece with $\lambda = 1$ above) is open and dense, and for each point $g$ in the top stratum, there is a unique factorization as in (0.2), where in the hyperfunction case $g_{\pm}$ are $G$-valued holomorphic functions in the the open disks $\Delta$ and $\Delta^*$, respectively, and in the formal case $g_{\pm}$ are simply formal power series satisfying the appropriate algebraic equations determined by $G$. We will refer to $g_-, g_0, g_+ (\theta_-, g_0, \theta_+, \text{respectively})$ as the Riemann-Hilbert coordinates (linear Riemann-Hilbert coordinates, respectively) of $g$.

Both $(\cdot)^*$ and $\Theta$ act on $\mathbf{L}G$, by acting on a loop pointwise.

**Lemma (2.3).** Both $(\cdot)^*$ and $\Theta$ extend continuously to involutions of $L_{hyp}G$ and $\mathbf{L}G$, and in general both permute the Birkhoff strata.

**Proof of (2.3).** We will consider the hyperfunction case for definiteness. Suppose that $g \in L_{hyp}G$. This means that $g = [g_l, g_r]$ is an equivalence class represented by a pair $(g_l, g_r)$, where $g_l$ ($g_r$) is a holomorphic map $\{1 - \epsilon < |z| < 1\} \to G$ ($\{1 < |z| < 1 + \epsilon\} \to G$, respectively), for some $\epsilon > 0$, and this pair is equivalent to any other pair of the form $(g_l h, h^{-1} g_r)$, for some $h \in H^0(S^1, G)$. The extension of $(\cdot)^*$ is given by

$$g^* = [g_r (\bar{z}^{-1})^*, g_l (\bar{z}^{-1})^*]. \quad (2.4)$$
This is well-defined and restricts to the pointwise action of $(\cdot)^*$ on $H^0(S^1, G)$, simply because $z^{-1} = \bar{z}$ on $S^1$.

From the formula (2.4), it is clear that $(\cdot)^*$ maps the $\lambda$ stratum to the $\lambda^* = \lambda^{-1}$ stratum.

For $\Theta$, the extension is given by $g^\Theta = [g^\Theta_l, g^\Theta_r]$ (i.e. we are simply applying $\Theta$ pointwise). Again it is clear this is well-defined, extends the pointwise application of $\Theta$ to $H^0(S^1, G)$, and the $\lambda$ stratum is mapped into the $\lambda^\Theta$ stratum. □

**Definition (2.5).** $L(G/G_R)$ and $L_{hyp}(G/G_R)$ are the identity components of the fixed point sets of $(\cdot)^*\Theta$ acting on $L^G$ and $L_{hyp}G$, respectively. The intersections of $\tilde{\Sigma}_1^{\text{formal}}$ and $\tilde{\Sigma}_1^{\text{hyp}}$ with $L(G/G_R)$ and $L_{hyp}(G/G_R)$, respectively, will be referred to as the top strata.

**Proposition (2.6).** (a) The intersection $\tilde{\Sigma}_\lambda^{\text{formal}} \cap L(G/G_R)$ is nonempty if and only if $\lambda^{*\Theta} = \lambda$ in $\text{Hom}(S^1, T)$.

(b) The top stratum is open and dense and diffeomorphic to $G/G_R \times H^1_{\text{formal}}(\Delta, g)$.

The analogous statements hold in the hyperfunction case.

We only need Part (b). This follows directly from the uniqueness of the Birkhoff factorization when $\lambda = 1$.

Finally we recall some facts about automorphisms.

Suppose that the group $U$ is the universal covering of the identity component of $\text{Aut}(X)$. Let $\pi$ denote the projection, so that we have an exact sequence

$$0 \to \text{ker}(\pi) \to U \xrightarrow{\pi} \text{Aut}(X) \xrightarrow{\pi_0} \pi_0(\text{Aut}(X)) \to 0. \quad (2.7)$$

Let $A_0$ denote the identity component of $\text{Aut}(X)$. Define

$$\mathcal{L}A_0 = \{g : \mathbb{R} \to U : \exists \Delta \in \text{ker}(\pi) \quad \text{with} \quad g(t + 1) = g(t)\Delta, \forall t\}, \quad (2.8)$$

where some degree of smoothness ($> 1/2$) is implicitly fixed. There is an exact sequence

$$0 \to \text{ker}(\pi) \to \mathcal{L}A_0 \to \mathcal{L}\text{Aut}(X) \to \pi_0(\text{Aut}(X)) \to 0, \quad (2.9)$$

where $g \in \mathcal{L}A_0$ maps to the loop $\pi(g(e^{2\pi it}))$. We will indicate the degree of smoothness by attaching a subscript, e.g. $\mathcal{L}_{\text{pol}}A_0$ denotes the group of elements $g$ such that $\pi(g(e^{2\pi it}))$ is polynomial, i.e. has a finite Fourier series relative to some matrix representation of $\text{Aut}(X)$. 

It is straightforward to check that $L_{pol}A_0$ acts naturally on $L(G/G_R)$ (these actions will be written out explicitly in subsequent sections).

These generalized loops will play a role in this paper for the following reason. When one fixes a triangular decomposition for $g$, there is no residual $Ad(G)$ symmetry. However outer automorphisms occasionally exist that respect this decomposition. For the loop algebra, $Lg$, elements of the center $C(K)$ give rise to outer automorphisms (see (3.4.4) of [PS]) that interact in a relatively simple way with Riemann-Hilbert factorization. These outer automorphisms are represented by multivalued loops.

§3. The Group Case, $X = K = SU(2)$.

In this section we will give a relatively elementary proof of Theorem (0.1), in the case $K = SU(2)$. We will then discuss a possible method for computing the measure explicitly.

**Existence.**

Suppose that

$$g = g_- \cdot g_0 \cdot g_+ ,$$

is a point in the top stratum of $L_{SL}(2, \mathbb{C})$. We write

$$g_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \quad a_0 d_0 - b_0 c_0 = 1, \quad (3.2)$$

$$g_+ = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = 1 + \begin{pmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{pmatrix} z + .. \quad (3.3)$$

and

$$g_- = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = 1 + \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} z^{-1} + .. \quad (3.4)$$

The action of the constants $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ on $L_{SL}(2, \mathbb{C})$ is completely transparent in these coordinates: for $g_l, g_r \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$,

$$g_l \cdot g \cdot g_r^{-1} = [g_l g - g_l^{-1}] \cdot [g_l g_0 g_r^{-1}] \cdot [g_r g + g_r^{-1}] . \quad (3.5)$$

Let $\sigma$ denote the outer automorphism of $LSL(2, \mathbb{C})$ given by

$$\sigma\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix} . \quad (3.6)$$
i.e. $\sigma$ is conjugation by the multivalued loop $\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}$. We will write out the action of $\sigma$ on $L_{SL}(2, \mathbb{C})$ in the Lemma below.

Let $w_0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and

$$i_0(SL(2, \mathbb{C})) = w_0 SL(2, \mathbb{C})^{\sigma} w_0^{-1} = \{ \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} : ad - bc = 1 \}. \quad (3.7)$$

A basic fact is that $L_{pol} SL(2, \mathbb{C})$ is generated by $SL(2, \mathbb{C}) i_0(SL(2, \mathbb{C}))$ (see (5.2.5) of [PS]; $w_0$ is perhaps a distraction in this context; it appears here to align our notation with that in §5, and its significance is somewhat illuminated by Appendix B).

**Lemma.** Suppose that $h = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \in i_0(SL(2, \mathbb{C}))$, and $g \in LG$ is in the top stratum, as in (3.1)-(3.4).

- **a)** If $\alpha = a + bB_1 \neq 0$, then $h \cdot g$ is also in the top stratum and has factorization

$$[\begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} b \begin{pmatrix} \alpha & 0 \\ \gamma(z) & \alpha^{-1} \end{pmatrix}] \cdot [\begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma_0 & \alpha \end{pmatrix} \cdot g_0] \cdot [\begin{pmatrix} 1 & 0 \\ b \alpha & 1 \end{pmatrix} g_0 g_0 \cdot g]$$

where $\gamma_0 = \frac{-2abA_1 + b^2(B_2 - A_1B_1)}{\alpha}$ and $\gamma(z) = \gamma_0 - bz$.

- **b)** If $\alpha' = d - cC_1 \neq 0$, then $g \cdot h^{-1}$ is in the top stratum and has factorization

$$[g \cdot g_0 \begin{pmatrix} 1 & -\alpha' \cdot \alpha z^{-1} \\ 0 & 1 \end{pmatrix} g_0^{-1}] \cdot [\begin{pmatrix} \alpha & 0 \\ \beta & \alpha' \end{pmatrix} \cdot g_0] \cdot [\begin{pmatrix} 1 & \beta \cdot (z^{-1}) \alpha' \cdot \alpha \end{pmatrix} \cdot g_0 g_0]$$

where $\beta = \frac{2cda_1 + c^2(z^2 - a_1c)}{\alpha'}$ and $\beta(z^{-1}) = \beta_0 + cz^{-1}$.

- **c)** If $a_0 \neq 0$, then $g^\sigma$ is also in the top stratum, and

$$g^\sigma = (g^-)^\sigma \begin{pmatrix} \frac{1}{a_0} & z^{-1} \\ 1 - \frac{B_1}{a_0} & z^{-1} \end{pmatrix},$$

$$g^\sigma_0 = \begin{pmatrix} a_0 + \frac{B_1c_1}{a_0} & \frac{B_1}{a_0} \\ \frac{c_1}{a_0} & \frac{1}{a_0} \end{pmatrix}, \quad (g^\sigma)_0^+ = \begin{pmatrix} \frac{1}{-c_1} & \frac{b_0}{a_0} c_1 z \\ 1 - \frac{c_1b_0}{a_0} \end{pmatrix} (g^\sigma)_0^+$$

**Proof of (3.8).** These are straightforward matrix calculations (they also follow from the more abstract calculations in §5 below).//

Part a) implies

$$B((h \cdot g)_-) = (dB + cDz^{-1})/\alpha \quad (3.9)$$
\[ D((h \cdot g)_-) = (bBz + aD)/\alpha \quad (3.10) \]

or in terms of components
\[ B_n((h \cdot g)_-) = (dB_n + cD_{n-1})/\alpha \quad (3.11) \]
\[ D_{n-1}((h \cdot g)_-) = (bB_n + aD_{n-1})/\alpha \quad (3.12) \]

This leads to the following

**Corollary (3.13).** The space of variables \( \{B_1, D_{n-1}, B_n\} \) is invariant under the action of \( h \). Define \( B'_n = B_n/D_{n-1} \). Then
\[ B'_n(h \cdot g) = \frac{c + dB'_n}{a + bB'_n}, \]
i.e. \( B'_n \) is equivariant with respect to the action of \( i_0(SL(2, \mathbb{C})) \) on \( \mathbf{LG} \) and the linear fractional action of \( SL(2, \mathbb{C}) \) on \( \hat{\mathbb{C}} \).

**Proof of Theorem (0.1) in the case \( K = SU(2) \).** Let \( \nu_\beta \) denote Wiener probability measure for \( L_{C^0}K \) with inverse temperature \( \beta \) (see Appendix A). Via the inclusion \( L_{C^0}K \to \mathbf{LG} \), we can view the \( \nu_\beta \) as probability measures on the formal completion, and these measures are quasi-invariant with respect to the left and right action of \( L_{pol}K \), and the action of \( \sigma \). This family of measures is asymptotically invariant with respect to these actions, meaning for example that
\[ \int |1 - \frac{d\nu_\beta(g^\sigma)}{d\nu_\beta(g)}|d\nu_\beta(g) \to 0 \quad (3.14) \]
as \( \beta \to 0 \), with similar results for \( g^\sigma \) replaced by \( l \cdot g \) or \( g \cdot r \), for \( l, r \in L_{pol}K \) (see Appendix A).

**Lemma (3.15).** The top stratum of \( L_{pol}G \) has full measure with respect to \( \nu_\beta \).

This is true for any \( L_{pol}K \)-biinvariant measure; see (2.1) of Part I of [Pi1]. We will reproduce this argument below when we consider symmetric spaces (see Lemma (6.5)).

**Lemma (3.16).** The measures \( \nu_\beta \) have weak limits, as \( \beta \to 0 \), with respect to \( BC = BC(\check{\Sigma}_1^{formal}) \), bounded continuous functions in the linear Riemann-Hilbert coordinate space
\[ H^1_{formal}(\Delta^*, g) \times G \times H^1_{formal}(\Delta, g). \]
Proof of (3.16). We will write
\[ \theta - = (\theta_1 + \theta_2w + ..)dw, \]  
where \( w = -1/z \), and the coordinates \( \theta_i \in \mathfrak{g} \). The formula \( \theta - = (\partial g_-)g_-^{-1} \) is equivalent to relations of the form
\[ g_1 = \theta_1 \]  
(3.18)
\[ 2g_2 = \theta_2 + \theta_1g_1 = \theta_2 + \theta_1^2 = \theta_2 + \text{det}(\theta_1) \]  
(3.19)
\[ 3g_3 = \theta_3 + \theta_2g_1 + \theta_1g_2 = \theta_3 + \theta_2\theta_1 + \frac{1}{2}\theta_1(\theta_2 + \text{det}\theta_1), \]  
(3.20)
i.e. \( ng_n = \theta_n \) plus a polynomial in lower order terms.

Slightly abusing standard terminology, we will say that a coordinate, or more generally a function of such coordinates, say \( \lambda \), is tight if the mass of the measures \( \lambda_*\nu_\beta \) does not escape to infinity, i.e. given \( \epsilon > 0 \), there is a compact set \( K_\epsilon \) in the target space of \( \lambda \) such that \( \lambda_*\nu_\beta(K_\epsilon) > 1 - \epsilon \) for all \( \beta \). Sums and products of such tight variables are also tight. By a theorem of Prohorov (see chapter 2, §6, of [Bi]), to show that \( \nu_\beta \) has weak limits with respect to \( BC(\tilde{\Sigma}_{\text{formal}}) \), it is necessary and sufficient to show that \( g_0 \), each of the individual coordinates \( \theta_n \), and the corresponding coordinates for \( \theta_+ \), are tight.

Since the functions \( B'_n \) are equivariant with respect to the action of \( i_0(SU(2)) \), it follows from asymptotic invariance that
\[
\lim_{\beta \to 0} (B'_n)_*\nu_\beta = \frac{1}{Z}(1 + |B'_n|^2)^{-2}dm(B'_n), \tag{3.21}
\]
the unique invariant probability for unitary linear fractional transformations. In particular each of the \( B'_n \) is tight.

To show that each \( \theta_n \) is tight, we argue as follows (this is the same argument as in [Pi1], but it is more explicit). We know that \( B_1 \) is tight (since we know the distribution of the limit, by (3.21)). The invariance with respect to \( SU(2) \) implies that \( g_1 = \theta_1 \) is tight (From an abstract perspective, \( B_1 \) is a linear function on the \( \{\theta_1\} \); \( SU(2) \) acts irreducibly by the adjoint action on \( \{\theta_1\} \), hence the \( SU(2) \) orbit of \( B_1 \), which consists of linear functions all having the same distributional properties, spans the dual of \( \{\theta_1\} \); this implies that \( \theta_1 \) is tight). This implies that \( D_1 \) is tight. Since \( B'_2 \) is tight, this implies that \( B_2 = D_1B'_2 \) is tight. Since \( \theta_2 = 2g_2 \) plus a polynomial function of \( g_1 \), it follows that \( \beta_2 \) is tight (in fact a special feature

of the $SU(2)$ case is that $\beta_2 = 2B_2$, i.e. there are no lower order terms). The invariance with respect to $SU(2)$ implies that $\theta_2$ is tight. This implies that $D_2$ is tight; together with the tightness of $B'_3$ this implies $B_3$ is tight, etc.

The tightness of the coefficients of $\theta_+$ is proved in a similar way.

The new aspect of our argument is a simple way to see that $g_0$ is tight. Because the $(g_0)_{\nu_\beta}$ are $SU(2)$-biinvariant, it suffices to show that $a_0$ is tight.

The basic observation is that the formula for $(g_\sigma)_{0}$ implies that $(b_0/a_0) = (B_1/a_0) = B_1/a_0 + B_1c_1$ is tight. But we already know that $B_1$ and $c_1$ are tight, and the limiting distribution of $B_1$ is in the Lebesgue class. Hence it follows that $a_0$ is tight. This proves Lemma (3.16). □

We now complete the proof of (0.1) in the following way (this part of the argument will not be repeated in subsequent sections). We identify $BC$ with a space of Borel functions on $LG$, by extending each such function to be zero on the complement of $\tilde{\Sigma}^{formal}_1$. The Lemma implies that the family $\{\nu_\beta\}$ has weak limits with respect to $BC$. If $\nu$ is a limit point, i.e. for some sequence $\beta_j \to 0$, $\nu_{\beta_j}(f) \to \nu(f)$, for all $f \in BC$, then for $g \in L_{pol}K$, asymptotic invariance implies that $g \nu_{\beta_j}(f) \to \nu(f)$ as well. In other words $\nu_{\beta_j}$ has a weak limit with respect to $g^*BC$, for each $g \in L_{pol}K$. Let $\mathcal{F}$ denote the $L_{pol}K$-invariant space generated by $BC$, which we can identify with a space of Borel functions on $LG$. We then have a surjective map $\bigoplus g^*BC \to \mathcal{F}$ and an injective map

$$\mathcal{F}^* \to \prod_{L_{pol}K} \{g^*BC(\tilde{\Sigma}^{formal}_1)^*\}^*.$$ (3.23)

The image of the family $\{\nu_\beta\}$ is precompact in each factor of (3.23), hence the image of the family under the map (3.23) is precompact. Suppose that $\nu$ is a limit point. To check that $\nu \in \mathcal{F}^*$, suppose that $\sum g_i^*f_i = 0$ in $\mathcal{F}$, where $f_i \in BC$ and $g_i \in L_{pol}K$, $i = 1,..,n$. Then

$$\nu(\sum g_i^*f_i) = \sum \lim_{\beta_j \to 0} \nu_{\beta_j}(g_i^*f_i) = \lim_{\beta_j \to 0} \nu_{\beta_j}(0) = 0.$$ (3.24)

Thus $\nu$ is a linear functional on $\mathcal{F}$, and we can interpret $\nu$ as a probability measure, supported on $\tilde{\Sigma}^{formal}_1$ (or any of its $L_{pol}K$-translates).

Asymptotic invariance implies that any such limit point will be $L_{pol}K$-invariant. □
Remarks (3.25). (a) There is another approach to the tightness of the variables \( \theta_n \), for \( n > 1 \), which uses the action of \( \sigma \), rather than the auxiliary variables \( B_n' \). As before it is first necessary to observe that \( B_1 \) and \( c_1 \) are tight. We use the \( \sigma \) action to get tightness of \( a_0 \), hence of \( g_0 \). We then move from degree to degree using the action of \( \sigma \). The basic fact is the following:

\[
B(g^\sigma) = A((g_-)^\sigma)(-B_1) + B((g_-)^\sigma)(1 - \frac{B_1c_0}{a_0}z^{-1}) \tag{3.26}
\]

\[
= -A(g_-)B_1 + zB(g_-)(1 - \frac{B_1c_0}{a_0}z^{-1}). \tag{3.27}
\]

This implies

\[
B_n(g^\sigma) = -A_nB_1 + B_{n+1} - \frac{B_nB_1c_0}{a_0}. \tag{3.28}
\]

Using asymptotic invariance with respect to \( \sigma \), this formula shows that \( B_{n+1} \) will be tight if we know that the coefficients of \( g_- \) are tight up to order \( n \).

This argument works without change in the \( S^2 \) case, as we will observe in §4.

(b) There is another approach without change in the \( S^2 \) case, as we will observe in §4.

(b) There is another approach to the tightness of \( a_0 \), also using the existence of \( \sigma \). The formula for \( g_0^\sigma \) shows that \( a_0 \) and \( 1/a_0 \) have the same limiting distributional properties (a symmetry which we explore in Appendix B). Thus if \( a_0 \) is not tight, then probabilistic mass escapes to \( a_0 = 0 \). This leads to a contradiction with Corollary (1.14), modulo details that will be handled in Lemma (5.15).

**On Calculating the Invariant Measure.**

Suppose that \( \mu \) is a \( L_{pol}K \)-biinvariant probability measure on \( LG \). We would like to calculate the \( \theta_- \) distribution of \( \mu \).

The proof of Lemma (3.16) suggests the following approach. We know the \( B_1 \) distribution. We use \( SU(2) \)-invariance to determine the \( \theta_1 \) distribution. This determines the joint distribution of \( D_1 \) and \( B_1 \). By (3.13) we know the \( B'_2 \) distribution. The invariant action (3.13) of \( i_0(SU(2)) \) then determines the joint distribution of \( D_1, B_1 \) and \( B_2 \). In this \( SU(2) \) case we have the special fact that \( 2B_2 = \beta_2 \) (see (3.19)). The knowledge of the \( 2B_2 = \beta_2 \) distribution, together with \( SU(2) \)-invariance, then determines the \( \theta_2 \) distribution. We should then have enough information to compute the joint distribution of \( \theta_1 \) and \( \theta_2 \) (see (3.29) below). At least in principle, this determines the joint distribution of \( D_2 \) and \( B_1 \). Using the action (3.13) again, we determine the joint distribution of \( D_2, B_1 \), and \( B_3 \). This determines \( \beta_3 \), and we continue in this way.
This strategy depends upon several conjectural uniqueness statements, and in practice the calculations rapidly become prohibitive. Here I will simply summarize a partial result, and state a conjecture.

**Proposition (3.29).** The probability measure

\[ Z^{-1}(1 + |\theta_1|^2 + \frac{1}{2}|\theta_2|^2)^{-7}dm(\theta_1, \theta_2) \]  

(3.30)

has the property that it restricts to SU(2)-invariant distributions for \( \theta_1 \) and \( \theta_2 \), a \( i_0(SU(2)) \)-invariant distribution on \( \{B_1, D_1, B_2\} \), and has \( B_1 \) and \( B_2' \) distributions given by (3.21).

As stated, this is a straightforward calculation. This uses a special fact about \( sl(2, \mathbb{C}) \), namely that \( 2B_2 = \beta_2 \) (see (3.19)). This proposition suggests, but does not guarantee, that (3.30) is the \( \{\theta_1, \theta_2\} \)-distribution of \( \mu \), because there is not an accompanying uniqueness statement (the \( \{\theta_i, i \leq 3\} \)-distribution definitely has a more complicated form).

Our conjecture that \( \mu \) is unique, hence \( \mathcal{D} \)-invariant, implies the conjecture that the distributions of the matrix coefficients of \( \theta_- \) are \( PSU(1,1) \)-invariant. The current state of knowledge regarding \( PSU(1,1) \)-invariant probability measures on \( H^1(\Delta) \) (or more generally, \( H^m(\Delta) \)) is at the stage of constructing interesting examples ([Pi3]). If this symmetry has some power (and I emphasize that I have little feeling for whether this is true), then (3.29) suggests the following

**Possible Conjecture (3.31).** The \( \beta \)-distribution of \( \mu \) is the probability measure with \( \beta_1, .., \beta_N \) distribution

\[ Z^{-1}(1 + |\beta_1|^2 + .. + \frac{1}{N}|\beta_N|^2)^{-1-3N}dm(\beta_1, .., \beta_N) \]  

for each \( N \).

This measure is a quotient of a Gaussian. This suggests that there might be some kind of “free field” (or Verma module type) construction of the \( \theta_- \) distribution of \( \mu \).

**Remark (3.32).** The measure \( \mu \) has a deformation \( \mu_l \) parameterized by “level” \( l \) (extensively discussed, but not completely proven to exist, in Part III of [Pi1]). This parameter fits into the formulae above in a very natural way. For example in the case of (3.31), one simply replaces the exponent \( 1+3N \) by \( 1+3N+l \); the corresponding measures are finite (and coherent) provided \( l > -1 \).
§4. The $S^2$ Case

In this case (0.10) specializes to

$$
G = SL(2, \mathbb{C}) \quad \Rightarrow \quad G_{\mathbb{R}} = SU(1, 1) \quad \Leftarrow \quad U = SU(2, \mathbb{C}),
$$

(4.1)

where

$$
(\begin{pmatrix} a & b \\ c & d \end{pmatrix})^\Theta = (\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}).
$$

(4.2)

and $(\cdot)^*$ is the usual Hermitian conjugate.

The top stratum of $L(SL(2, \mathbb{C})/SU(1, 1))$ consists of $g \in LSL(2, \mathbb{C})$ as in (3.1)–(3.4) with

$$
g_0 = \begin{pmatrix} a_0 & b_0 \\ -b_0 & d_0 \end{pmatrix}, \quad a_0, d_0 \in \mathbb{R}
$$

(4.3)

and $g_+ = g^{*\Theta}$, so that $a_i = \bar{A}_i$, $b_i = -\bar{C}_i$, $c_i = -\bar{B}_i$, $d_i = \bar{D}_i$.

The action of $\sigma$, from §3, commutes with the pointwise action of $\Theta$ and $(\cdot)^*$. Hence $\sigma$ will act on the formal completion of $L(G/G_{\mathbb{R}})$.

The group $L_{pol}SO(3)$ is generated by $SU(2)$ (the constants), and the map

$$
\tilde{\sigma} : t \rightarrow \begin{pmatrix} e^{i\pi t} & 0 \\ 0 & e^{-i\pi t} \end{pmatrix};
$$

note that the image of $\tilde{\sigma}$ in $L_{pol}Aut(S^2)_0 = L_{pol}SO(3)$ is $\sigma$. This follows from (5.2.5) of [PS].

In this special case we will now write out the action of $L_{pol}U$ on $L(G/G_{\mathbb{R}})$ in Riemann-Hilbert coordinates. These calculations follow directly from the group case of the preceding section.

**Lemma (4.4).** Suppose that $h = \begin{pmatrix} d & cz^{-1} \\ b & a \end{pmatrix} \in i_0(SL(2, \mathbb{C}))$, and $g \in L(G/G_{\mathbb{R}})$ is in the top stratum, as in (3.1)–(3.4) and (4.2). If $|a + bB_1|^2 \neq |b|^2a_0^2$ and $a + bB_1 \neq 0$, then $g' = h \cdot g \cdot h^{*\Theta}$ is also in the top stratum and has factorization

$$
g' = g'_- \cdot g'_0 \cdot g'^{*\Theta},
$$

where

$$
g'_- = \begin{pmatrix} d & cz^{-1} \\ b & a \end{pmatrix} g_0 \left( \begin{pmatrix} 1 & -\frac{b}{a+bB_1} z^{-1} \\ 0 & 1 \end{pmatrix} \right) g_0^{-1} \begin{pmatrix} \alpha & 0 \\ \gamma(z) & \alpha^{-1} \end{pmatrix},
$$

$$
g'_0 = \begin{pmatrix} \alpha^{-1} & 0 \\ -\gamma_0 & \alpha \end{pmatrix} g_0 \begin{pmatrix} \alpha' & \beta_0 \\ \alpha' & \alpha' \end{pmatrix},
$$
\[
\alpha = a + b(B_1 - \frac{\bar{b}}{a + bB_1}a_0^2) = \frac{|a + bB_1|^2 - |b|^2a_0^2}{a + bB_1},
\]
\[
\gamma_0 = -\frac{2abA_1 + b^2(B'_0 - A'_1A_1)}{\alpha}, \quad \gamma(z) = \gamma_0 - bz, \quad \alpha' = \bar{a} + \bar{b}B_1, \quad \text{and} \quad \beta_0 = \frac{2cd\alpha_1 + c^2(c_2 - a_1\alpha_1)}{\alpha}.
\]

The action of \(\sigma\) is the same as in the group case.

We will not use the precise formulas for \(\beta_0\) and \(\gamma_0\).

Let \(b = \bar{b}/(\bar{a} + \bar{b}B_1)\), \(g'_- = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}\). Lemma (4.4) implies the formulas:

\[
B(g'_-) = [(dB + cz^{-1}D) - (dA + cz^{-1}C)ba_0^2z^{-1} + (dB + cz^{-1}D)b\bar{b}_0a_0z^{-1}] / \alpha \tag{4.5}
\]

\[
D(g'_-) = [(bzB + aD) - (bzA + aC)\{ba_0^2z^{-1}\} + (bzB + aD)b\bar{b}_0a_0z^{-1}] / \alpha \tag{4.6}
\]

or

\[
B_n(g'_-) = [(dB_n + cD_{n-1}) - (dA_{n-1} + cC_{n-2})ba_0^2 + (dB_{n-1} + cD_{n-2})b\bar{b}_0a_0] / \alpha \tag{4.7}
\]

\[
D_{n-1}(g'_-) = [(BB_n + aD_{n-1}) - (bA_{n-1} + aC_{n-2})ba_0^2 + (BB_{n-1} + aD_{n-2})b\bar{b}_0a_0] / \alpha \tag{4.8}
\]

**Corollary (4.9).** The space of variables \(\{a_0, B_1\}\) is invariant under the action of \(h \in \iota_0(SL_2(\mathbb{C}))\); the action is given by

\[
a(g'_0) = \frac{a_0}{\alpha\alpha'} = \frac{a_0}{|a + bB_1|^2 - |b|^2a_0^2}, \tag{4.10}
\]

\[
B_1(g'_-) = [dB_1 + c - \frac{d\bar{b}a_0^2}{\bar{a} + bB_1}] / \alpha. \tag{4.11}
\]

The space of variables \(\{a_0, B_1, D_1, B_2\}\) is invariant; the action is given by (4.10), (4.11), and

\[
B_2(g'_-) = [dB_2 + cD_1 - (d(-D_1) + c)ba_0^2 + (dB_1 + c)b\bar{b}_0a_0] / \alpha \tag{4.12}
\]

\[
D_1(g'_-) = [BB_2 + aD_1 - b(-D_1)ba_0^2 + (BB_1 + a)b\bar{b}_0a_0] / \alpha \tag{4.13}
\]

The abstract explanation for the existence of this equivariant projection, in the case \(n = 1\), is given by the diagram (6.10) in §6.

**Proof of Theorem (0.13) for \(X = S^2\).**

Let \(\tilde{\nu}_\beta\) denote the projection of the Wiener measure \(\nu_\beta\) on \(L_{C^0}SU(2)\) to \(L_{C^0}S^2\). We view \(\tilde{\nu}_\beta\) as a measure on \(L(SU(2, \mathbb{C})/SU(1, 1))\).
**Lemma (4.14).** \( \tilde{\nu}_\beta \) has full measure on the top stratum.

This is a special case of Lemma (6.5) (I do not know of an argument that takes advantage of the small rank).

As in the group case, it suffices to show that the coordinates \( g_0, \theta_1, \ldots \) are tight, relative to the family of measures \( \tilde{\nu}_\beta \). We will first show that \( g_0 \) is tight.

Recall the Cartan isomorphism

\[
U \times K \rightarrow G \rightarrow \phi(G/G_R) : \left[ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \right] \rightarrow \left( \begin{array}{cc} a_0^2 e^{2x} - |b|^2 e^{-2x} & ab(e^{2x} + e^{-2x}) \\ -\bar{a}b(e^{2x} + e^{-2x}) & |a|^2 e^{-2x} - |b|^2 e^{2x} \end{array} \right).
\]

(4.15)

The corresponding \( U \)-equivariant projection

\[
\phi(G/G_R) \rightarrow U/K
\]

(4.16)

is given by

\[
\left( \begin{array}{cc} a_0 & b_0 \\ -\bar{b}_0 & d_0 \end{array} \right) \rightarrow \zeta = \frac{-\bar{b}_0}{a_0 + d_0 + \sqrt{1 + (\frac{a_0 - d_0}{2})^2}}
\]

(4.17)

\[
= \frac{-\bar{b}_0 a_0}{a_0^2 + 1 - |b_0|^2 + \frac{1}{2}(4a_0^2 + (a_0^2 + |b_0|^2 - 1)^2)^{1/2}}
\]

(4.18)

where \( \zeta = -\bar{b}/a \) is the usual affine coordinate for \( \mathbb{C}P^1 \). To see this, note that (4.15) implies

\[
a_0 + d_0 = 2(|a|^2 - |b|^2)ch(2x) = 2(|a|^2 - 1)ch(2x)
\]

(4.19)

\[
a_0 - d_0 = 2sh(2x), \quad b_0 = 2abch(2x)
\]

(4.20)

We can solve for \( sh(2x) \) and \( ch(2x) \) using the second equation (note that \( a_0 - d_0 \) will be our basic \( U \)-invariant quantity; we will use this below). We can then solve the first equation for \( |a|^2 \) and the last equation for \( ab \). We then divide to obtain the formula above.

Since \( \tilde{\nu}_\beta \) is \( SU(2) \)-invariant, the \( \zeta \)-distribution of \( \nu_\beta \) is the usual invariant measure \( Z^{-1}(1 + |\zeta|^2)^{-2}dm(\zeta) \).

To show that \( g_0 \) is tight, it suffices to show \( a_0 \), as a map into \( \mathbb{R} \), is tight (\( a_0 \) and \( d_0 \) have the same distribution and \( a_0 d_0 = 1 - |b_0|^2 \)). To accomplish this we will use the action of \( \sigma \). An important technical point is that the map

\[
LSU(2) \rightarrow \phi(LS^2) : g \rightarrow gg^{-\Theta}
\]

(4.21)
is $\sigma$ equivariant, since $\sigma$ is conjugation by a multivalued loop which is fixed by $\Theta$ for each $t$. This justifies our use of asymptotic invariance results for $\sigma$ and $\nu_\beta$.

Using the formula for $(g^\sigma)_{0\ell}$ in part (c) of (3.8) (which relates $1/a_0$ and $d_0$), and then conjugating by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, we see that $a_0$ has the same limiting distribution properties as $1/a_0$. It thus suffices to show that $1/a_0$, as a map into $\mathbb{R}$, is tight.

**Lemma (4.22).** We identify $\text{SL}(2, \mathbb{C})/\text{SU}(1, 1)$ with its $\phi$-image. Suppose that we are given a family $\{\lambda_\beta : \beta > 0\}$ of probability measures on $\text{SL}(2, \mathbb{C})/\text{SU}(1, 1)$ which is quasi-invariant and asymptotically invariant with respect to $\text{SU}(1, 1)$. Then $1/a_0$, as a map into $\mathbb{R}$, is tight.

Note that in our present context, the $\lambda_\beta = (g_0)_* \nu_\beta$ are $SU(2)$-invariant.

**Proof of (4.22).** Note that by Corollary (1.14), for each $\beta$, $a_0$ is nonzero a.e. $[\lambda_\beta]$. By way of contradiction, suppose that $1/a_0$ is not tight. Then there exists an $\epsilon > 0$ such that for all $k$ there is $\beta_k \to 0$ such that $\nu_{\beta_k}\{|a_0| < k^{-1}\} < \epsilon$. We would like to assert that this means that positive probability accumulates on $\{(0 \ b_o \ b_0) : |b_o|^2 = 1\}$, in the limit $\beta_k \to 0$, and this positive mass will be $SU(2)$-invariant, contradicting Corollary (1.14). However $|a_0| < M$ is not compact, so mass could possibly escape along the noncompact directions of the lower strata. We will use $\zeta$ to fill this gap.

We estimate $\zeta$ using (4.18):

$$|\zeta| \leq |a_0| \left| \frac{|b_0|}{|b_0|^2 - 1} \right| = |a_0| \left| \frac{1}{|b_0| - |b_0|^{-1}} \right|. \quad (4.23)$$

Since $\zeta$ is an affine coordinate for the 2-sphere, and represents a $SU(2)$-equivariant map, the $\zeta$-distribution of $\nu_\beta$, in the limit $\beta \to 0$, is the invariant measure $Z^{-1}(1 + |\zeta|^2)^{-2} dm(\zeta)$. Thus we can find $M = M_\epsilon$ such that $\nu_\beta\{\frac{1}{M} \leq |\zeta| \} \geq 1 - \epsilon/2$, for all sufficiently small $\beta$. So with $\nu_{\beta_k}$ probability $> \epsilon/2$,

$$||b_0| - \frac{1}{|b_0|}| < \frac{M}{k} \implies ||b_0| - 1| < \frac{M}{k}. \quad (4.24)$$

In the same way, using the (4.17), we obtain

$$\frac{1}{M} \leq |\zeta| \leq \frac{|b_0|}{|a_0 + d_0|} \implies |a_0 + d_0| \leq M|b_0|. \quad (4.25)$$

Thus we do obtain a nontrivial $SU(2)$-invariant measure on the lower strata, in the limit $k \to \infty$, contradicting (1.14). $\square$
We now know that $g_0$ is tight. To show that the $\theta_j$ are tight, we can proceed as in the $SU(2)$-case, for which we offered two arguments. The first is to note that because $a_0$ is tight, taking $h = \begin{pmatrix} 0 & z^{-1} \\ -z & 0 \end{pmatrix}$ in (4.10), we see that $B_1$ is tight. Using $SU(2)$-invariance we obtain that $\theta_1$ is tight. This implies $a_0$, $B_1$, and $D_1$ are tight. Using (4.13) with $h$ above, we see that $B_2$ is tight. We now continue, using (4.8) in general.

The alternative argument is to observe that the $\sigma$-action implies that $B_1$ has the same distributional properties as $b_0/a_0$, hence that $B_1$ is tight. The argument then proceeds exactly as in our second argument in the $SU(2)$ case, part (a) of Remark (3.25), to show that $\theta_\cdot$ is tight. □

On Calculating the Invariant Measure.

Suppose that $\nu$ is a $L_{pol}SU(2)$-invariant probability measure on $L(SL(2, \mathbb{C})/SU(1, 1))$. The distributions which we can a priori compute are $\zeta$ in (4.17), and translates of this (which are given by more complicated formulas). This contrasts sharply with the group case, where we a priori know the distribution of relatively simple variables such as the $B_n'$.

In §7 we will discuss a conjectural formula for the $g_0$ distribution. In terms of the Cartan and matrix coordinates in (4.15) (with $z = b_0/a_0$), this conjecture reads

$$ (g_0)_* \nu = \frac{1}{Z} \text{sech}^3(2x) \text{ch}^2(2x) dk \times dx $$

(4.26)

$$ = \frac{1}{Z} (1 + \left(\frac{|a_0 - d_0|}{2}\right)^{-3/2} a_0 d_0 dm(z) $$

(4.27)

$$ = \frac{1}{Z} (1 + \left(\frac{(1 + |z|^2) a_0^2 - 1}{2a_0}\right)^{-3/2} d(a_0^2) dm(z) $$

(4.28)

[The first expression is from Lemma (7.5); the second is obtained using (4.20) to convert $\text{sech}^3(2x)$ into matrix coordinates, and (C.19) of Appendix C to get an expression for the $G$-invariant measure in terms of $a_0$ and $z$].

Now the distribution of $z = b_0/a_0$ is the same as $B_1$ (using the formula for $\sigma$). Therefore the diagonal distribution conjecture implies the conjecture that the $B_1$ distribution is given by

$$ \frac{1}{Z} \frac{1}{(1 + |B_1|^2)^{3/2}} F(\frac{1}{1 + |B_1|^2}) dm(B_1), \quad F(\rho) = \int_0^\infty \frac{\rho}{(\rho + \frac{(x-1)^2}{x})^{3/2}} dx $$

(4.29)

The algebraic part of (4.29) is reminiscent of (3.31) with the level $l = -1/2$ inserted (see Remark (3.32) for the meaning of level).
The function $F(\rho)$ tends to the finite limit $\int_{-\infty}^{\infty} (1 + x^2)^{3/2} \, dx$ as $\rho \downarrow 0$; it is a decreasing function, and asymptotically behaves like $\rho^{-1}$ as $\rho \to \infty$ [To see this observe that the integral outside the interval $|x - 1| < \delta$ vanishes in the limit $\rho \to 0$; for $x$ close to 1 one can get rid of the solitary $x$ in the denominator of the integrand of (4.29)].

At this point I am stuck. I do not see how to use $SU(2)$-invariance to generate an expression for the $\theta_1$ distribution, and in thinking about $PSU(1,1)$-invariant measures, I have never come across something like $F$.

§5. General Group Case

The point of this section is to indicate how to generalize the argument given in §3, especially to identify the analogue of the variables $B'_n$, and the generalization of the distributional symmetry of $a_0$ and $1/a_0$.

We assume that $\mathfrak{g}$ is simple. In this case there is a highest root $\theta$. Let $h_\theta$ denote the coroot, and choose $e_{\pm \theta}$ in the $\pm \theta$ root spaces such that $e_{-\theta}, h_\theta, e_\theta$ satisfy the canonical $sl(2, \mathbb{C})$ relations. We define $i_0 : sl(2, \mathbb{C}) \to L_{pol} \mathfrak{g}$ by

\[
i_0\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = e_\theta z^{-1}, \quad i_0\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = -h_\theta, \quad i_0\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e_{-\theta} z. \quad (5.1)
\]

We denote the corresponding group homomorphism by the same symbol, $i_0 : SL(2, \mathbb{C}) \to L_{pol} G$. When $G = SL(2, \mathbb{C})$, this agrees with our notation in §3 and §4. The group $L_{pol} G$ is generated by $G$ and the image of $i_0$, and a similar statement applies with $K$ in place of $G$ ((5.2.5) of [PS]).

Lemma (5.2). Suppose that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}) \) and $g = g_- \cdot g_0 \cdot g_+$ is a point in the top stratum of $L G$. Let $h = i_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$, write

\[
g_- = \exp(x_1 z^{-1} + x_2 z^{-2} + \ldots), \quad (5.3)
\]

and $x_n = x'_n + Z_n e_\theta$ for the partial root space decomposition of $x_n$.

(a) If $a + b Z_1 \neq 0$, then the point

\[
g' = h \cdot \{g_- \cdot g_0 \cdot g_+\}
\]

is also in the top stratum, and has triangular factorization $g' = g'_- \cdot g'_0 \cdot g'_+$, where

\[
g'_- = h g_- \exp\left(-\frac{b}{a + b Z_1} e_{-\theta} z\right) (a + b Z_1)^{h_0} l_0^{-1},
\]
\[ g' = l_0(a + bZ_1)^{-h_\theta}g_0, \quad g'_+ = g_0^{-1} \exp\left( \frac{b}{a + bZ_1}e^{-\theta z} \right)g_0g_+, \]

and

\[ l_0 = \exp(-ab[e_{-\theta}, a^{-ad(h_\theta)}(x'_1)]) - b^2 Z_2 e_{-\theta}) \in N^- . \]

(b) If

\[ g_+ = \exp(y_1 z^{-1} + y_2 z^{-2} + ..), \]

\[ y_n = y'_n + W_n e_{-\theta}, \text{ and } a + cW_1 \neq 0, \text{ then the point} \]

\[ g' = \{g_- \cdot g_0 \cdot g_+\} \cdot h \]

is also in the top stratum, and has triangular factorization \( g' = g'_- \cdot g'_0 \cdot g'_+ \), where

\[ g'_- = g_- g_0 \exp\left( \frac{c}{a + cW_1}e^z \right)g_0^{-1} , \]

\[ g'_0 = g_0(a + cW_1)^{-h_\theta}u_0 , \]

\[ g'_+ = u_0^{-1}(a + cW_1)^{h_\theta} \exp\left( \frac{-c}{a + cW_1}e^z \right)g_+ h , \]

and

\[ u_0 = \exp(-ac[e_{\theta}, a^{-ad(h_\theta)}(y'_1)]) - c^2 W_2 e_{\theta}) \in N^+ . \]

Proof of (5.2). This is an easy calculation, once one understands the basic idea. In terms of Lie algbras,

\[ \{x \in H^0(D^*, \mathfrak{g}) : x(\infty) \in b^- \} \oplus \mathbb{C} e_{-\theta z} \]

is a parabolic subalgebra, with a semidirect decomposition

\[ i_0(sl(2, \mathbb{C})) \propto \{x \in H^0(D^*, \mathfrak{g}) : x(\infty) \in b^-, h^\theta(x(\infty)) = 0, Z_1 = 0\} . \]

(this is explained in more abstract terms in Part I of [Pi1]). By factoring \( g_- \) along this decomposition, we can reduce the calculation to one inside \( i_0(Sl(2, \mathbb{C})) \).

We therefore organize \( g' \) as

\[ g' = (\tilde{g}_- g_- \cdot (\infty)^{-1}) \cdot (\tilde{g}_- (\infty) g_0) \cdot (g_0^{-1} \exp\left( \frac{b}{a + bZ_1}e_{-\theta z} \right)g_0g_+) , \]

where

\[ \tilde{g}_- = hg_\theta \exp\left(- \frac{b}{a + bZ_1}e_{-\theta z} \right) . \]
We have organized the factors (in braces) to show that \( \tilde{g}_- \in G(\mathbb{C}((z^{-1}))) \).

We calculate

\[
\tilde{g}_-(\infty) = a^{-h_\theta} e^{a^{-1} b[e_{-\theta} x_1]} + a^{-1} b^2 [e_{-\theta} x_2] (1 + a^{-1} b Z_1)^{-h_\theta} = e^{ab[e_{-\theta} a^{-d(h_\theta)} (x_1^\prime)]} + b^2 Z_2 e^{-\theta} (a + b Z_1)^{-h_\theta}.
\] (5.9)

When we plug this into (5.6), we obtain part (a).

Part (b) is proven in the same way. \( \square \)

We now focus on the left action of \( h \) on \( g_- \), and seek the analogue of Lemma (3.13).

Suppose that \( \pi = \pi_\lambda \) is an irreducible highest weight representation of \( G \), corresponding to the dominant integral weight \( \lambda \). We will write \( \pi_* \) for the Lie algebra representation. We can then write

\[
\pi(g_-) = 1 + g_1 z^{-1} + g_2 z^{-2} + ..
\]

\[
= 1 + \pi_*(x_1) z^{-1} + (\pi_*(x_2) + \frac{1}{2} \pi_*(x_1)^2) z^{-2} + ..
\] (5.10)

where the \( g_n \in \mathcal{L}(V(\pi)) \). We will write \( g_0 = 1 \).

To generalize the discussion in §3 as directly as possible, we suppose that we can choose \( \lambda \) so that \( \lambda(h_\theta) = 1 \) (for the classical algebras the \( \lambda \) corresponding to the defining representation has this property). Let \( v_-\lambda \) denote a lowest weight vector. The assumption \( \lambda(h_\theta) = 1 \) implies that \( v_-\lambda \) and \( \pi_*(e_\theta) v_-\lambda \) span a subrepresentation for \( i_0(sl(2, \mathbb{C})) \) (\( \simeq sl(2, \mathbb{C}) \)) isomorphic to the defining representation.

Let \( D \) and \( B \) (\( D_n \) and \( B_n \), resp.) denote the functions of \( g_- \) (\( g_n \), resp.) which pick out the \( v_-\lambda \) and \( \pi_*(e_\theta) v_-\lambda \) components of \( \pi(g_-) v_-\lambda \) (\( \pi(g_n) v_-\lambda \), resp.), where we split off the other components using the essentially unique \( K \)-invariant Hermitian form. Note that \( D \) and \( B \) are formal power series in \( z^{-1} \), and \( D_0 = 1 \), since \( g_0 = 1 \) by convention.

Because \( v_-\lambda \) is lowest weight, and \( l_0 \) and \( e_-\theta \) are lower triangular, using (a) of Lemma (5.2), we obtain

\[
\pi(g'_*)(v_-\lambda) = \pi(h) \pi(g_-) (a + b Z_1)^{\pi_*(h_\theta)} (v_-\lambda)
\]

\[
= (a + b Z_1)^{-1} \pi(h) (\sum_{n=0}^{\infty} g_n v_-\lambda z^{-n}).
\] (5.12)

This implies the following generalization of (3.13).
Lemma (5.13). For the left action of $h$, as in (a) of Lemma (5.2), we have
\[
\begin{pmatrix}
B(g'_-) \\
D(g'_-)
\end{pmatrix} = \left(a + bZ_1\right)^{-1} \begin{pmatrix}
d & cz^{-1} \\
bz & a
\end{pmatrix} \begin{pmatrix}
B(g_-) \\
D(g_-)
\end{pmatrix},
\]
hence for $n \geq 1$,
\[
\begin{pmatrix}
B_n(g'_-) \\
D_{n-1}(g'_-)
\end{pmatrix} = \left(a + bZ_1\right)^{-1} \begin{pmatrix}
dB_n(g_-) + cD_{n-1}(g_-) \\
bB_n(g_-) + aD_{n-1}(g_-)
\end{pmatrix},
\]
and
\[
B'_n(g'_-) = \frac{c + dB'_n(g_-)}{a + bB'_n(g_-)}.
\]
where $B'_1 = B_1$, $B'_n = B_n/D_{n-1}$, $n > 1$.

Proof of Theorem (0.1). We first suppose that $g \neq E_8$. Assuming this, we can find a representation $\pi$, as above, such that $\lambda(h_\theta) = 1$; for the classical algebras we can choose the defining representation, and for the exceptional algebras, other than $E_8$, we can choose the smallest nontrivial representation.

We can now proceed as in the $SU(2)$ case. We again introduce the Wiener measures parameterized by inverse temperature $\beta$. By asymptotic invariance and the $i_0(SU(2))$-equivariance of $B'_n$ in Lemma (5.13), each of the $B'_n$ has the canonical invariant distribution, $Z^{-1}(1+|B'_n|^2)^{-2} dm(B'_n)$, in the limit $\beta \to 0$, and hence each is tight. The tightness of $B_1$, and the irreducible action of $K$ on $g \simeq \{\theta_1\}$, implies that $\theta_1 = g_1$ is tight. Thus $D_1$ is tight. Lemma (5.13) now implies that $B_2$ is tight. This implies that the analogous coefficient of $\theta_2$ is tight. The action of $K$ now implies that $\theta_2$ is tight, and so on.

To avoid excluding $E_8$, we consider the adjoint representation in place of $\pi$ above. We then consider the three-dimensional $i_0(sl(2,\mathbb{C})) \subset g$ in place of $span\{v_{-\lambda}, \pi_*(e_\theta)v_{-\lambda}\}$, and an affine coordinate analogous to $B'_n$ for the corresponding 2-sphere of (projective images of) lowest root vectors for $sl(2,\mathbb{C})$ inside $\mathbb{P}(i_0(sl(2,\mathbb{C})))$.

One approach to the tightness of $g_0$ is given in §4.3 of Part III of [Pi1]. Here we will consider an argument akin to that in §3. This will apply provided $g \neq G_2, F_4, E_8$. At the end we will give another argument that works in general.

To prove that $g_0$ is tight, it suffices to show that the projection of $g_0$ into $G$ modulo a finite subgroup is tight. Thus it suffices to show that $\pi(g_0)$ is tight, for some irreducible (not necessarily faithful) representation $(\pi, V(\pi))$ of $G$. To show this, we claim it suffices to prove that a single matrix coefficient $\langle \pi(g_0)v, w \rangle$ is tight,
for some \(v, w \in V(\pi)\). To see this, note that \(K \times K\) leaves each \(\pi(g_0) \ast \nu_\beta\) invariant, and acts irreducibly on \(\mathcal{L}(V(\pi))\). Thus if one matrix coefficient is tight, then all matrix coefficients are tight, because the span of the orbit of this single coefficient will be the entire linear dual of \(\mathcal{L}(V(\pi))\). This implies that \(\pi(g_0)\) will be tight, viewed as a map into \(\mathcal{L}(V(\pi))\) [Thus in the limit \(\beta \to 0\), we will definitely obtain \(K \times K\)-invariant probability measures on this space of matrices, although we do not a priori know they are supported on \(\pi(G)\)]. Since \(\pi(G)\) is an algebraic group, it is cut out by polynomial equations in the matrix coefficients, and hence these polynomials represent tight variables. Since these equations are satisfied for each \(\beta\), they will be satisfied in the limit. Thus \(\pi(g_0)\) will be tight, viewed as a map into \(\pi(G)\), and hence \(g_0\) will be tight.

With probability one, for each \(\beta\), we can uniquely factor \(g_0\), \(g_0 = l_0m_0a_0u_0\), where \(l_0 \in N^-\), \(u_0 \in N^+, m_0 \in T\), and \(a_0 \in \exp(h_\mathbb{R})\) (by Corollary (1.14)). The factors \(l_0\) and \(u_0\) are affine coordinates for the flag spaces \(K/T \simeq G/B^+\) and \(T\setminus K \simeq B^-\setminus G\), respectively. Since each Wiener measure is \(K \times K\)-invariant, \(l_0\) and \(u_0\) have the unique \(K\)-invariant distributions for all \(\beta\), hence are tight. Similarly \(m_0\) has the uniform distribution for all \(\beta\). The key issue concerns the noncompact directions, \(a_0\).

In the \(SU(2)\) and \(S^2\) cases we proved the tightness of \(a_0\) by observing that \(1/a_0\) has the same limiting distributional properties as \(a_0\). A generalization of this is the following

**Lemma (5.14).** Write \(a_0 = \prod r_j a_j^{h_j}\), where the \(h_j\) are the coroots of the simple positive roots \(\alpha_j\), and \(r\) denotes the rank of \(\mathfrak{g}\). Suppose that \(\mathfrak{g} \neq G_2, F_4, E_8\). Then for some pair \(i, j\), \(a_i\) and \(1/a_j\) will have the same limiting distribution properties with respect to the \(\nu_\beta\).

These symmetries arise from outer automorphisms (parameterized by \(C(K)\), which is vacuous in the excluded cases). This is discussed in Appendix B.

Now suppose that \(g_0\) is not tight. Then for each \(j\), all the matrix coefficients for \(\pi_{\Lambda_j}\) will not be tight, and in particular each \(a_j\) will not be tight. The Lemma implies that for some \(j\), \(1/a_j\) is not tight, i.e. there exists \(\epsilon > 0\) such that for all \(k\), there is \(\beta_k \rightarrow 0\) such that \(\nu_{\beta_k}\{a_j < \frac{1}{k}\} > \epsilon\).

We claim that this implies that positive probability accumulates on the lower strata of \(G\), as \(\beta \to 0\). The argument is essentially the same as in \(\S 3\). We can reduce to that situation using the following procedure (which will be used again in the proof of Lemma (6.5) below). Introduce the parabolic subgroup \(P_{(j)}\) of \(G\) which corresponds to the root \(\alpha_j\). Let \(P_{(j)}^+\) denote the nilradical, and let \(P_{(j)}^-\) denote
the opposite nilpotent subalgebra. We then consider the $i_{\alpha_j}(SL(2, \mathbb{C}))$-equivariant projection $G \to P_{(j)}^--G/P_{(j)}^+$. This projection is injective on $i_{\alpha_j}(SL(2, \mathbb{C}))$ itself. When we project the $\nu_\beta$ to this space, we obtain $SU(2) \times SU(2)$-invariant measures on $SL(2, \mathbb{C})$ (identified with the image of $i_{\alpha_j}$).

This now leads to a contradiction with the following Lemma, which will complete the proof of Theorem (0.1), at least in the cases $g \neq G_2, F_4, E_8$.

Lemma (5.15). Suppose that $\{\lambda_\beta : \beta > 0\}$ is a family of probability measures on $SL(2, \mathbb{C})$ quasi-invariant and asymptotically invariant with respect to $SU(2) \times SU(2)$. Then $1/a$, as a map into $\mathbb{R}$, is tight.

Proof of (5.15). By way of contradiction, suppose that $1/a$ is not tight. Then there exists $\epsilon > 0$ such that for all $k$, there is $\beta_k \to 0$ such that $\lambda_{\beta_k}\{a_j < \frac{1}{k}\} > \epsilon$.

When we factor $(a \ b \ c \ d) = (1 \ 0 \ z \ 1) (a \ 0 \ a^{-1} \ 1) (1 \ w \ 0 \ 1)$, (5.16)

for each $\beta$, $z$ and $w$ will have the standard $\mathcal{Z}^{-1}(1+|\zeta|^2)^{-2}dm(\zeta)$ distributions, in the limit $\beta \to 0$. Thus there is an $M = M_\epsilon$ such that $|z|, |w| < M$ with $\lambda_{\beta_k}$-probability $> 1 - \epsilon/3$ for all sufficiently small $\beta$. Since $b = aw$, $c = az$, these will accumulate positive probability as $\beta \to 0$.

To control $d$ we need to write down an analogue of $\zeta$ from §4. This is straightforward, but somewhat messy. We write

$$(a \ b \ c \ d) = k_1p, \quad k_1 \in S(2, \mathbb{C}), \quad p > 0,$$  

(5.17)

$$p^2 = \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ ab + cd & |b|^2 + |d|^2 \end{pmatrix}. \quad (5.18)$$

There is a $k_2 \in SU(2, \mathbb{C})$ which will diagonalize $p$, or equivalently $p^2$. We seek an affine coordinate $\zeta$ for $k_2U(1) \in SU(2)/U(1)$. This will represent an equivariant map to the 2-sphere for the diagonal copy of $SU(2)$.

The eigenvalues of $p^2$ are $\lambda = (A + D \pm ((A + D)^2 - 4))^{1/2}/2$ Then $p^2 - \lambda$ equals

$$\begin{pmatrix} \frac{A-D}{2} & \pm \sqrt{(\frac{A+D}{2})^2 - 1} \\ B & \frac{D-A}{2} & \pm \sqrt{(\frac{A+D}{2})^2 - 1} \end{pmatrix}. \quad (5.19)$$

We can take

$$\zeta = \frac{B}{D - A + \sqrt{(A+D)^2 - 4}} = \frac{a\bar{b} + z(1 + bc)}{D - A + \sqrt{(A+D)^2 - 4}}, \quad (5.20)$$
where we used \(ad - bc = 1\) to rewrite \(\bar{B}\) in terms of \(z\) (from (5.16)) and other variables not including \(d\). We then have

\[
D - A + \sqrt{(A + D)^2 - 4} = (\bar{a}b + z(1 + bc))/\zeta
\]  
(5.21)

Since \(D \geq 0\), we obtain

\[
\sqrt{(A + D)^2 - 4} \leq (\bar{a}b + z(1 + bc))/\zeta,
\]  
(5.22)

implying

\[
(A + D)^2 \leq 4 + \{(\bar{a}b + z(1 + bc))/\zeta\}^2
\]  
(5.23)

In this expression, only \(D = |b|^2 + |d|^2\) involves \(d\).

Since \(\zeta\) represents an equivariant map to the 2-sphere for the diagonal \(SU(2)\), \(\zeta\) has the usual invariant distribution in the limit \(\beta \to 0\). Hence we can find \(M' = M'_\epsilon\) such that such that \(|1/\zeta| < M'\) with \(\lambda_\beta\)-probability \(> 1 - \epsilon/3\) for all sufficiently small \(\beta\). We now see that with positive probability \(b, c,\) and \(d\) will remain finite with positive probability as \(k \to \infty\). This means that we do obtain a \(SU(2)\)-invariant measure on the lower strata, in the limit \(k \to \infty\), and this contradicts (1.14). \(\square\)

If \(g = G_2, F_4\) or \(E_8\), then this argument does not work, because we cannot apply (5.14). In general we consider \(a_0^{-\theta}\). This is the “\(|a|^{2\nu}\)” variable for \(i_0(SL(2, \mathbb{C}))\); this is due to the minus sign appearing in front of \(h_\theta\) in (5.1) (see (6.11) below). If none of the \(a_j\) are tight, then \(a_0^{-\theta}\) tends to zero with some positive probability uniform in \(\beta\). The Lemma (5.15) then applies and we obtain a contradiction. \(\square\)

§6. General Symmetric Space Case.

We continue to assume that \(g\) is simple (so that we will be considering the type I symmetric space case). The homomorphism \(i_0\) in (5.1) will continue to play a critical role. Because \(\Theta\) preserves the triangular decomposition of \(g\), \(\Theta(\theta) = \theta\), and \(\Theta(g_\theta) = g_\theta\), hence \(\Theta(e_\theta) = -e_\theta\), where \(\epsilon = \pm 1\) (it appears that both possibilities are unavoidable; for example for \(Sp(n)/U(n)\), it appears that we must have \(\epsilon = +1\), and for the real Grassmannian of oriented \(p\)-planes in \(\mathbb{R}^{p+q}\), it appears we must have \(\epsilon = -1\)). Thus the extension of \(\Theta\) to loop space will map the image of \(i_0\) into itself, and when \(\epsilon = 1\), this action will be isomorphic to (4.2) via \(i_0\), otherwise it is trivial.
Lemma (6.1). Suppose that \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \), and let \( h = i_0(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \). Suppose also that \( g = g_- \cdot g_0 \cdot g_+ \) is a point in the top stratum of \( \mathbf{L}(G/G_R) \), so that
\[
g_0^{*-\Theta} = g_0, \quad \text{and} \quad g_+ = g_-^{*-\Theta}.
\]

We write \( g_- \) as in (5.3), and \( x_n = x'_n + Z_n e_\theta \). Let
\[
Z'_1 = Z_1 - \frac{eb}{\bar{a} + ebZ_1} \langle \text{Ad}(g_0)e_\theta, e_\theta \rangle / \langle e_\theta, e_\theta \rangle.
\]

If both \( a + ebZ_1 \) and \( a + bZ'_1 \) are not zero, then the point
\[
g' = h \cdot \{ g_- \cdot g_0 \cdot g_+ \} \cdot h^{*-\Theta}
\]
is also in the top stratum, and the triangular factorization \( g' = g_- \cdot g'_0 \cdot g'_+ \) is given by
\[
g'_- = hg_-g_0 \exp(-eb/\bar{a} + ebZ_1)e_\theta z^{-1}g_0^{-1} \exp(-b/\bar{a} + bZ_1)e_\theta z)(a + bZ'_1)^{h_\theta} l_0^{-1},
\]
\[
g'_0 = l'_0(a + bZ'_1)^{-h_\theta} g_0(\bar{a} + ebZ_1)^{-h_\theta} u_0,
\]
\[
u_0 = \exp(-ac[e_\theta, a^{-ad(h_\theta)(y'_1)}] - c^2 W_2 e_\theta) \in N^+,
\]
and \( l'_0 \) is lower triangular.

Remark (6.2). Note that for generic \( g_0 \), with triangular decomposition \( g_0 = l_0 \text{diag}(g_0) u_0 \), we can write
\[
\langle \text{Ad}(g_0)e_\theta, e_\theta \rangle / \langle e_\theta, e_\theta \rangle = \text{diag}(g_0)^\theta,
\]
and
\[
a + bZ'_1 = \frac{|a + bZ_1|^2 + eb \text{diag}(g_0)^\theta}{\bar{a} + ebZ_1}.
\]
These formulas specialize to those in Lemma (4.4) (with \( \epsilon = 1 \)).

Proof of (6.1). This formula is obtained from Lemma (5.2) by first applying \( h^{*-\Theta} \) on the right, then \( h \) on the left (one obtains formulas with considerably different appearance if one does this in the opposite order, and then it is not so easy to compare with (4.4)). \( \square \)

Proof of Theorem (0.13). Let \( \nu_\beta \) denote Wiener measure for \( L_{C^0}U \) with inverse temperature \( \beta \), and let \( \tilde{\nu}_\beta \) denote the projection to \( L_{C^0}(U/K) \); we view \( \tilde{\nu}_\beta \) as a probability measure on \( \mathbf{L}(G/G_R) \). Because the map \( LU \to L(U/K) \): \( g \to gg^{-\Theta} \) is equivariant for the left actions of \( L_{pol}U \), the measures \( \tilde{\nu}_\beta \) form an asymptotically invariant family for this action.
Lemma (6.5). The top stratum \( L(G/G_R) \cap \bar{\Sigma}^\text{formal}_1 \) has full measure with respect to \( \tilde{\nu}_\beta \).

Proof of (6.5). The argument is the same as in Proposition (2.1.1) of Part I of [Pi1], for which it is convenient to use Kac-Moody technology. In the proof, for notational clarity, we will suppose \( \Theta(e_\theta) = -e_\theta \); we will note the minor changes necessary in the opposite case at the end of the proof.

There is a transitive action
\[
G(\mathbb{C}((z^{-1}))) \times L(G/G_R) \to L(G/G_R) : g, h \mapsto g \cdot h \cdot g^{*\Theta}.
\] (6.6)

This action is well-defined, because if \( g \in G(\mathbb{C}((z^{-1}))) \), then \( g^{*\Theta} \in G(\mathbb{C}((z))) \), and these groups act from the left and right of \( LG \), respectively. We let \( B^\pm \) denote the upper and lower Borel subgroups of \( G(\mathbb{C}((z^{-1}))) \), respectively.

The decomposition for \( L \) strictly analogous to (1.7) is given by
\[
L = \bigsqcup_{W \alpha \text{Hom}(S^1, T)} \bar{\Sigma}^L_w, \text{ where } \bar{\Sigma}^L_w = B^- w B^+,
\] (6.7)

and the superscript \( L \) is there to remind us of loop space. This induces a decomposition
\[
L(G/G_R) = \bigsqcup S_w, \text{ where } S_w = \bar{\Sigma}^L_w \cap L(G/G_R).
\] (6.8)

We are mainly interested in \( S_1 \), the “top stratum”, which consists of \( g \) in the formal completion that can be written uniquely as \( g = l \cdot h \cdot l^{*\Theta} \), where \( l \in \mathcal{N}^- = G(\mathbb{C}[[z^{-1}]])_1 \) (the formal completion of the profinite nilpotent algebra spanned by the negative roots of \( L_{pol}g \)), and \( h = h^{*\Theta} \in H \). To say that \( g \) belongs to \( S_1 \) is thus equivalent to the two conditions \( g \in \bar{\Sigma}^\text{formal}_1 \) and \( g_0 \in \bar{\Sigma}_1 \).

Let \( P^-_{(0)} \) denote the parabolic subgroup, with Levi decomposition,
\[
P^-_{(0)} = i_0(SL(2, \mathbb{C}))B^- = i_0(SL(2, \mathbb{C}))\alpha R^-_{(0)}.
\] (6.9)

\( (R^-_{(0)}) \) is the subgroup of \( \mathcal{N}^- \) that corresponds to the span of the negative root spaces, \( \mathbb{C}e_\theta z^{-1} \) excluded.

We consider the diagram
\[
\begin{array}{ccc}
L(G/G_R) & \xrightarrow{q} & \mathcal{R}^-_{(0)} \backslash L(G/G_R) \\
\uparrow & & \uparrow \\
S_1 \cup w_{(0)}S_1 & \xrightarrow{i_0(\phi(SL(2, \mathbb{C})/SU(1, 1)))} & i_0(\bar{\Sigma}_1) \\
\uparrow & & \uparrow \\
S_1 & \rightarrow & i_0(\bar{\Sigma}_1)
\end{array}
\] (6.10)
where \( w_{(0)} = i_0\left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \). The map \( q \) is \( i_0(SL(2, \mathbb{C})) \) equivariant, where the actions in question are induced by (6.6), because this group normalizes \( \mathcal{R}_{(0)}^- \). The up arrows are all injective (the only question here might involve the upper right up arrow; if \( g, h \in i_0(SL(2, \mathbb{C})) \), then \( gl^\tau = lh \), an equality in \( \mathcal{P}_{(0)}^- \); when we quotient out by \( \mathcal{R}_{(0)}^- \), we obtain \( g = h \)).

The sets on the left of (6.10) are the full \( q \)-inverse images of the sets on the right. For if \( g \) is in the formal completion and can be written uniquely as \( l \cdot h \cdot l^*\Theta \), we can factor \( l = rl_0 \) along the Levi decomposition, and then \( g' = l_0hl_0^*\Theta \in i_0(SL(2, \mathbb{C})) \) and \( g'g'^\tau = 1 \). Conversely if we have a factorization for \( g' \), we clearly obtain an element in \( S_1 \).

Now suppose that \( \nu \) is a \( L_{pol}K \)-quasi-invariant probability measure on the formal completion. Consider \( q_*\nu \) restricted to \( i_0(\phi(SL(2, \mathbb{C})/SU(1, 1)) \). This measure (which may have total mass less than one) is \( i_0(SU(2)) \)-quasi-invariant. By Corollary (1.14), it is supported on the top stratum. Therefore the top stratum and its \( w_{(0)} \)-translate are equivalent, up to a set of \( q_*\nu \)-measure zero. Together with the fact that the sets on the left of (6.10) are the full \( q \)-inverse images of the sets on the right, this implies that the \( w_{(0)} \)-translate of \( S_1 \) and \( S_1 \) are equivalent up to a set of \( \nu \)-measure zero.

Now recall that \( S_1 \) consists of \( g \in \tilde{S}^{formal}_1 \cap L(G/G_R) \) such that \( g_0 \) is in the top stratum of \( G/G_R \). By Corollary (1.14), applied to the \( U \)-quasi-invariant measure \( g_0_*\nu \), we conclude that \( S_1 \) and the top stratum are the same, up to a set of \( \nu \)-measure zero. But the union of these translates is the whole space. Therefore the complement of the top stratum must have \( \nu \)-measure zero.

In the event that \( \Theta(e_\theta) = e_\theta \), we replace \( SL(2, \mathbb{C})/SU(1, 1) \) by \( SL(2, \mathbb{C})/SU(2) \), and the same argument applies. \( \square \)

We now turn to the proof that \( g_0 \) is tight. With \( \nu_\beta \) probability one, by (1.14), we can write \( g_0 = l_0\text{wma}_0l_0^*\Theta \), as in (d) of (1.8). As in the group case it suffices to show that in some representation, there is some matrix coefficient which is tight. If we write \( a_0 = \prod a_j^{h_j} \), it suffices to show one of the \( a_j \) is tight (as a function into \( \mathbb{R} \)).

As something of a digression, consider first a case in which \( \Theta \) is an inner automorphism (this is equivalent to \( a_0 = 0 \)). In this case the action of \( \Theta \) on \( C(K) \)
and $\mathfrak{h}$ is trivial. In the notation of Appendix B, the corresponding automorphism, $\sigma_\Delta$, will map $L(G/G_\mathbb{R})$ into itself. Lemma (5.14) applies. Thus if all of the $a_j$ are not tight, then for some $i$, $1/a_i$ is not tight. This leads to a contradiction, using either (4.22) or (5.15), depending upon whether $\Theta(e_j) = \pm e_j$.

Now suppose that $\Theta$ is an outer automorphism. In some cases (e.g. $SU(2n)/SO(2n)$), there still exist nontrivial $\Delta \in C(K)$ fixed by $\Theta$. This is illustrated in Appendix B.

In general we argue as follows. If all of the $a_j$ are not tight, this implies that $a_j^\theta$ is not tight. Thus $a_j^\theta$ tends to zero with some positive probability uniform in $\beta$. But this is the “$|a|$” variable for $i_0(SL(2, \mathbb{C}))$: given an element in $SL(2, \mathbb{C})$ as in (5.16), when we apply $i_0$, as defined by (5.1),

$$i_0\left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right) = a^{-h_\theta} \quad \implies \quad |a|^2 = a_0^{-\theta}.$$  

We now consider the diagram as in (6.10). When we push our measures forward, we obtain an asymptotically invariant family of quasi-invariant measures for $SU(2)$ acting on $SL(2, \mathbb{C})/SU(1, 1)$ or $SL(2, \mathbb{C})/SU(2)$ (using the isomorphism $i_0$). We then apply Lemmas (4.22) and (5.15), respectively. This leads to a contradiction.

Thus we know that $g_0$ is tight. To complete the proof we need to show that the coefficients of $\theta_-$, or equivalently the coefficients of $x$ (where $g_- = exp(x)$), are tight. The argument is the same as the first argument we gave in the $S^2$ case.

We consider the action of $h$ in Lemma (6.1). The formula for $g_0'$ in Lemma (6.1) implies

$$\langle Ad(g_0')e_\theta, e_\theta\rangle = (a + bZ_1')^{-2}(a + \bar{b}Z_1)^{-2}\langle Ad(g_0)e_\theta, e_\theta\rangle$$  

$$= \frac{1}{(|a + bZ_1|^2 + e|b|^2\langle Ad(g_0)e_\theta, e_\theta\rangle)^2}\langle Ad(g_0)e_\theta, e_\theta\rangle,$$

where the second equality uses (6.4). We have already proven that $g_0$ is tight, so that $\langle Ad(g_0)e_\theta, e_\theta\rangle$ is tight. Asymptotic invariance implies that $g_0'$ (for a given $h$) is also tight. If we take $h$ with $a = 0$ and $b = 1$, we then see that $Z_1$ is tight.

Now $Z_1$ is the $e_\theta$ component of $x_1$. The constants $K$ act irreducibly by the adjoint representation on $\{x_1\}$. Since the constants leave the measures $\nu_\beta$ invariant, it follows that $x_1$ is tight.

We then apply the formula for $g_-'$ to compute the adjoint matrix coefficient

$$(Ad(g')_e)e_\theta, e_\theta = \langle hg_0 exp(\beta e_\theta z^{-1})g_0^{-1}(a + bZ_1')^{-2}e_\theta, e_\theta\rangle$$  

(6.14)
We again take $h$ with $a = 0$, $b = 1$. We move $h$ to the right side of the inner product. In the adjoint representation for $i_0$ we calculate that $h^{-1}e_{-\theta} = -z^{-2}e_{\theta}$. Also $ad(e_{-\theta})^3 = 0$. We then see that (6.14) equals

$$\left(Z'_1\right)^{-2}(g_{-\theta}e^{\beta e_{\theta}g_{0}z^{-1}}e_{-\theta}, e_{-\theta}, -z^{-2}e_{\theta}) = (Z'_1)^{-2}((\sum_{n=0}^{\infty} g_n z^{-n})(z^2 + \beta ad(e_{-\theta}) z + \frac{1}{2} \beta^2 ad(e_{-\theta})^2) e_{-\theta}, e_{\theta}).$$

(6.16)

where we are doing this calculation for $z \in S^1$, so that $z^{-1} = \bar{z}$ (these calculations closely mirror, but do not quite reduce to, the calculations (4.5)–(4.13), because we are considering the adjoint representation, rather than the analogue of the defining representation). We now expand the left hand side of (6.11) and (6.16) in powers of $z^{-1}$. By isolating the terms of order $-n$, we obtain

$$\langle g'_n e_{-\theta}, e_{-\theta}\rangle = (\langle g_{n+1} e_{-\theta}, e_{\theta}\rangle + \ldots)/ (Z'_1)^2,$$

(6.17)

where the trailing terms involve $g_m$, $m \leq n$.

We know that $Z'_1$ and $g_1 = x_1$ are tight. This implies that one matrix coefficient for $g_2$ is tight, and hence that a matrix coefficient for $x_2$ is tight. Using the action of the constants $K$, we conclude $x_2$ is tight. We then continue in this same way. □

§7. Diagonal Distribution Conjecture.

For $g \in \phi(G/G_R) \cap \tilde{\Sigma}_1$, we can write $g = lwma_{\Theta}$, as in (d) of (1.8). Let $\tilde{\Sigma}_w$ denote the set of $g$ in this intersection for fixed $w \in T_0^{(2)}/T_0^{(2)} \cap \exp(i a_0)$. The following is proven in [Pi4].

**Proposition (7.1).** Suppose that $U/K$ is Hermitian symmetric or a group. Then for $\lambda \in (i t_0)^*$,

$$\int_{\phi(U/K) \cap \tilde{\Sigma}_w} a_\phi(g)^{-i \lambda} = \frac{1}{M} \prod_{w} \frac{\langle \delta, \alpha \rangle}{\langle \delta - i \lambda, \alpha \rangle}$$

where $\delta$ is half the sum of the positive complex roots, and the product is over positive roots which are imaginary and noncompact with respect $Ad(w)\Theta$. Therefore

$$\int_{\phi(U/K)} a_\phi(g)^{-i \lambda} = \frac{1}{\#\{[w]\}} \sum_{[w]} \prod_{w} \frac{\langle \rho, \alpha \rangle}{\langle \delta - i \lambda, \alpha \rangle}$$
where the sum is over equivalence classes in $T_0^{(2)}/T_0^{(2)} \cap \exp(ia_0)$ such that $w \in \phi(U/K)$.

Example (7.2). We write $g \in \phi(S^2)$ as $g = \begin{pmatrix} a_0 & b_0 \\ -\bar{b}_0 & a_0 \end{pmatrix}$, where $a_0 \in \mathbb{R}$, $a_0^2 + |b_0|^2 = 1$. We have $a_\phi = \begin{pmatrix} |a_0| & 0 \\ 0 & |a_0|^{-1} \end{pmatrix}$, and $w$ is $\pm 1$, corresponding to the upper and lower hemispheres. Write $\lambda = \lambda \alpha_1$, where $\alpha_1 = \lambda_1 - \lambda_2$ is unique noncompact root.

We calculate

$$\int_{\phi(S^2) \cap \Sigma_\pm} a_\phi^{-i\lambda} = \int_0^1 t^{-i2\lambda} dt = \frac{1}{1 - i\lambda} = \frac{\langle \frac{1}{2} \alpha_1, \alpha_1 \rangle}{\langle \frac{1}{2} \alpha_1 - i\lambda, \alpha_1 \rangle}$$

(7.3)

(and note $\rho = \alpha_1$). The fact that the integrals corresponding to $w = \pm 1$ are the same is peculiar to this example.

Now suppose that we consider $L(U/K)$. Let $\nu$ denote an $L_{pol}U$-invariant probability measure on the formal completion. We think of this loop space as a symmetric space, with $k$ replaced by $Lk$, $p$ by $Lp$, and so on. We then write down the analogue of the formula in (7.1) (for the group case, see §4.3 of Part III of [Pi1]). Using the standard product formula for the sine function, we obtain the following

**Conjecture (7.4).** Given $g$ in the top stratum of $L(G/G_{\mathbb{R}})$, we write $g_0 = l_0wma_0l_0^*\Theta$ as in (d) of (1.8). Then

$$\int_{L(G/G_{\mathbb{R}})} a_\phi^{-i\lambda} d\nu = \frac{1}{\#\{w\}} \sum_{w} \prod_{w} \frac{\sin(\pi \langle \delta, \alpha \rangle)}{\sin(\pi \langle \delta - i\lambda, \alpha \rangle)}$$

where here the inner product is normalized so that a long root has length $\sqrt{2}$.

In the $S^2$ case it is a routine exercise to explicitly compute $(g_0)_*\nu$ itself.

**Lemma (7.5).** In the $S^2$ case, in terms of the coordinates (4.15), with $k = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, the Conjecture (7.4) is equivalent to

$$(g_0)_*\nu = Z^{-1} \text{sech}(2x) dk \times dx$$

**Sketch of the calculation.** By (4.19) – (4.20), $a_0 = (2|a|^2 - 1) \text{ch}(2x) + \text{sh}(2x)$, and by (C.6) of Appendix C, the $SL(2, \mathbb{C})$-invariant measure on $SL(2, \mathbb{C})/SU(1, 1)$ is given by $d\eta = \text{cosh}^2(2x) dk \times dx$. If $\delta(2x)d\eta$ is the $g_0$ distribution, then we must compute

$$\int |a_0|^{-i\lambda} = \int |(2|a|^2 - 1) \text{ch}(2x) + \text{sh}(2x)|^{-i\lambda} \delta(2x) \text{ch}^2(2x) dk dx$$
where \( u = 2|a|^2 - 1 \), and we used the fact that the \( a \)-distribution of Haar measure for \( SU(2) \) is Lebesgue measure on the unit disk. We do the \( u \)-integral, and we eventually obtain

\[
\int_{x=-\infty}^{+\infty} \int_{u=-1}^{1} |uch(2x) + sh(2x)|^{-i\lambda} ch^q(2x) dx du
\]  

(7.6)

According to Conjecture (7.5), this equals \( 1/sin(\frac{\pi}{2}(1 - i\lambda)) \). When we apply the inverse Fourier transform, we find \( \delta(2x) = sech^3(2x) \).

In general, to obtain a formula for the \( g_0 \) distribution, it is necessary to use DeLorme’s inversion formula for the “diagonal transform” of harmonic analysis for \( U \)-invariant functions on \( G/G_R \) (see [D] and references). This will be considered elsewhere (in the group case one can instead use Harish-Chandra’s spherical transform, which is more elementary).

In [Pi2], I described a possible application of the invariant measure for \( LK \) to the construction of the state space for the two dimensional sigma model with target \( K \). That discussion remains essentially unchanged when \( K \) is replaced by \( U/K \). The conjectural formula above for the \( g_0 \) (or zero-mode) distribution leads to the conjecture that (in the large radius limit) the radial part of the zero-mode Hamiltonian for the \( S^2 \) sigma model is equivalent to

\[
-(\frac{d}{dx})^2 + \frac{1}{4} - \frac{3}{4} sech^2(x), \quad x \in \mathbb{R}
\]  

(3.10)

(the zero-mode Hamiltonian is itself a \( U \)-invariant Laplace type operator on \( G/G_R \)).

One theme of this paper is that, in considering natural measures associated to the loop space \( L(U/K) \), Riemann-Hilbert factorization is important, and when \( U/K \) is positively curved, the zero-mode becomes the noncompact space \( G/G_R \). An amusing example is

\[
Sp(2, \mathbb{C}) \quad \uparrow \quad Sp(2, \mathbb{R}) \quad \begin{array}{c} \nearrow \end{array} \quad K = U(2) \quad \begin{array}{c} \searrow \end{array} \quad Sp(2)
\]  

(1.19)

where the compact six dimensional space \( Sp(2)/U(2) \) (isomorphic to the Grassmannian of oriented 2-planes in \( \mathbb{R}^5 \), via an exceptional low rank isomorphism) is
embedded in the 10 dimensional space $Sp(2, \mathbb{C})/Sp(2, \mathbb{R})$, with 4 noncompact dimensions.

In connection with sigma models, it is natural ask about loops into other ideal spaces, such as $T$, a flat torus, $\pi \backslash G_{\mathbb{R}}/K$, a compact locally symmetric space with negative curvature, or a Ricci flat space. In the nonpositively curved cases, in the framework of this paper, compactness fails. This implies that the existence, or lack thereof, of compactness for Wiener measures (for ideal spaces) corresponds to asymptotic freedom, or the lack thereof, in renormalization group analysis of the corresponding sigma models. Whether there is a tangible connection is unclear, as is the question of how to extend this to more general spaces (or to the supersymmetric setting).

Appendix A. Asymptotic Invariance of Wiener Measure

In this paper we will adopt the following

**Definition** (A.1). Given a group $A$ of automorphisms of a measure space $(M, \mathcal{M})$, and a family $\{\lambda_\beta : \beta > 0\}$ of quasi-invariant probability measures on $M$, this family is asymptotically invariant if

$$\int |1 - \frac{d\sigma \ast \lambda_\beta}{d\lambda_\beta}| d\lambda_\beta \to 0 \quad \text{as} \quad \beta \downarrow 0,$$

for each $\sigma \in A$.

If the measures $\lambda_\beta$ have a weak limit (relative to bounded continuous functions, for example, in a topological context), then the limit will represent an invariant object. Given an $A$-equivariant map $\phi : M \to N$, the family $\{\phi \ast \lambda_\beta\}$ will be asymptotically invariant.

In this appendix we will discuss a slight extension of the asymptotic invariance results in Part III of [Pi1].

Let $X = K$, a simply connected compact Lie group, which we view as a Riemannian symmetric space (using our normalized $Ad(K)$-invariant inner product). Let $\nu_\beta$ denote the Wiener probability measure on $L^{C^0}X$ with inverse temperature $\beta$.

**Proposition (A.2).** Suppose that $\sigma \in \mathcal{L}_{W^1, Aut}(X)_0$. Then

$$\int |1 - \frac{d\nu_\beta(g^\sigma)}{d\nu_\beta(g)}| d\nu_\beta(g) \leq 2^{3/2} p_T^2(1) \beta^{1/2} \mathcal{E}(\sigma)^{1/2},$$
where $E$ denotes the kinetic energy of $\sigma$, $\frac{1}{2} \int_0^1 \sigma^{-1} \sigma' \sigma^{-1} \sigma' - dt$, $T = 1/\beta$, and $p$ is the heat kernel.

**Proof of (A.2).** This result is already known if $\sigma$ is either a left or right multiplication. The essential point is to treat left and right multiplication simultaneously. We first consider unconditioned paths.

Let $\nu = \nu^{1,*}_\beta$ denote the Wiener probability measure on $Path_{C^0}^{1,*} K$. Given $l, r \in Path_{W_1}^{1,*} K$, we compute

$$
\int |1 - \frac{d\nu(lgr)}{d\nu(g)}|d\nu(g) \leq \int |1 - \frac{d\nu(gr)}{d\nu(g)}|d\nu(g) + \int |1 - \frac{d\nu(lgr)}{d\nu(gr)}\frac{d\nu(gr)}{d\nu(g)}|d\nu(g) \tag{A.3}
$$

$$
= \int |1 - \frac{d\nu(gr)}{d\nu(g)}|d\nu(g) + \int |1 - \frac{d\nu(lgr)}{d\nu(gr)}\frac{d\nu(gr)}{d\nu(g)}|d\nu(g) \tag{A.4}
$$

$$
= \int |1 - \frac{d\nu(gr)}{d\nu(g)}|d\nu(g) + \int |1 - \frac{d\nu(lgr)}{d\nu(gr)}|d\nu(g) \tag{A.5}
$$

$$
\leq 2\beta^{1/2}(E(r)^{1/2} + E(l)^{1/2}) \leq 2^{3/2}\beta^{1/2}\mathcal{E}(l, r)^{1/2} \tag{A.6}
$$

where the inequality in the fourth line uses (4.11.11) of [P1].

Suppose that $\Delta \in C(K)$, and $l$ and $r$ satisfy $l(t+1) = l(t)\Delta$, $r(t+1) = r(t)\Delta$, for all $t$. We let $\sigma(g) = lgr^{-1}$, for $g \in Path_{C^0}^{1,*} K$. Given $k \in K$, let $\nu^{1,k}_\beta$ denote the Wiener probability measure on $Path_{C^0}^{1,k} K$. Note that $\sigma$ acts on this space. We have the disintegration formula

$$
\nu^{1,*}_\beta = \int_K \nu^{1,k}_\beta p_T(k)dk. \tag{A.7}
$$

The inequality (A.3) – (A.6) implies

$$
\int_K \{ \int |1 - \frac{d\nu^{1,k}_\beta(g^\sigma)}{d\nu^{1,k}_\beta(g)}|d\nu^{1,k}_\beta(k)p_T(k)dk \leq 2^{3/2}(\beta E(\sigma))^{1/2}, \tag{A.8}
$$

where a priori we only know that $\sigma_* \nu^{1,k}_\beta$ has the same measure class as $\nu^{1,k}_\beta$ for a.e. $k$. We need to translate this statement about an average over $K$ to a statement about each $k$, especially $k = 1$.

The basic idea is that we can express $\nu^{1,1}_\beta$ as an integral,

$$
\nu^{1,1}_\beta = \int_K (\nu^{1,k}_2 \ast \nu^{1,k-1}_2) \frac{p^2_{T/2}(k)}{p_T(k)}dk. \tag{A.9}
$$
The argument following (4.1.11) of Part III of [Pi1] applies verbatim with \( \sigma(g) \) in place of the left multiplication \( g_L g \). □

Remark (A.10). It is important to extend the asymptotic invariance result above in a number of directions. Unfortunately the method in [Pi] depends on the fact that the target is a group (the methods in [MM] are more robust). This explains why Wiener measure for the target \( U/K \) has not made any appearance in this paper.

Appendix B. Symmetries of the Diagonal Distribution.

The Group Case.

In this appendix it is convenient to use the Kac-Moody extension \( \mathbb{C} \rightarrow \hat{L}_{pol} g \rightarrow L_{pol} g \). Our notation will be consistent with [KW], which compiles numerical information we will need. We very briefly recall that \( g \), which we assume is simple, is generated by \( r = \text{rank}(g) \) copies of \( sl(2, \mathbb{C}) \), with standard bases \( h_j, e_j, f_j, 1 \leq j \leq r \), satisfying the Chevalley-Serre relations; \( \hat{L}_{pol} g \) is generated by \( g \) (or the \( r \) copies of \( sl(2, \mathbb{C}) \)) and one additional copy of \( sl(2, \mathbb{C}) \), with standard basis \( h_0 = c - h_\theta, e_0 = e_{-\theta} \otimes z, f_0 = e_\theta \otimes z^{-1} \). The Dynkin diagram of \( \hat{L}_{pol} g \), which is the extended diagram for \( g \), encodes the (generalized Chevalley-Serre) relations among the different copies of \( sl(2, \mathbb{C}) \).

The automorphisms of the Dynkin diagram of \( \hat{L}_{pol} g \) (see Table II of [KW]) correspond exactly to \( Out(K) \alpha C(K) \). Given an element of the center, we obtain an outer automorphism of \( \hat{L}_{pol} g \), and also of \( L_{pol} g \) (and its completions). There is a canonical way to realize this outer automorphism by permuting the \( r + 1 \) copies of \( sl(2, \mathbb{C}) \) which generate \( \hat{L}_{pol} g \). This automorphism preserves the triangular decomposition of \( \hat{L}_{pol} g \), it commutes with the unitary involution, and it also lifts to an automorphism of \( \hat{L}_{pol} G \) (and its completions). At the level of loops, this automorphism is realized by a multivalued loop in \( L_{pol} Aut(K) \) (such as conjugation by \( w_0 \left( \begin{array}{cc} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{array} \right) \), which we used in §2 – 3); see (3.4.2) and (3.4.4) of [PS].

Asymptotic invariance of Wiener measure implies that the limiting distribution properties of \( g \in L_G \), and in particular \( a_0 \), are invariant under this symmetry. The formulas we obtain below for this symmetry imply Lemma (5.14). This requires a case by case analysis.

With \( \nu_\beta \) probability one, \( g = g_- \cdot g_0 \cdot g_+ \) is in the top stratum of \( L_G \), and in turn \( g_0 = l_0 m_0 a_0 u_0 \), where \( l_0 \in N^-, m_0 \in T, a_0 \in exp(h_\mathbb{R}) \), and \( u_0 \in N^+ \). We can write \( a_0 = \prod_1^r d_j^h \), where the \( h_j \) are the coroots of the simple positive roots \( \alpha_j \), and
\( a_j = |\sigma_j(\hat{g})/\sigma_0(\hat{g})^{a_j}| = |\langle \pi_{\Lambda_j}(g_0)v_{\Lambda_j}, v_{\Lambda_j} \rangle|, \) \hspace{1cm} (B.1)

where the \( \sigma_j \) are the fundamental matrix coefficients, viewed as functions on \( \hat{L}G, \) \( \hat{g} \in \hat{L}G \) projects to \( g, \) the integers \( \hat{a}_j \) are determined by \( h_\theta = \sum \hat{a}_jh_j \) (see Table I of [KW]), the \( \Lambda_j \) are the fundamental dominant weights for \( g = n^- \oplus h \oplus n^+ \) corresponding to the \( \alpha_j, \) and the \( \pi_{\Lambda_j} \) denote the corresponding irreducible representations (for the first equality in (B.1), see §4.3 of Part III of [Pi1]).

The main point is that in all cases with \( C(K) \neq 0, \) there is a corresponding automorphism which interchanges \( \sigma_0 \) and some other \( \sigma_j, \ j > 0 \) (see Table II of [KW]; note that automorphisms which fix the 0 node correspond to outer automorphisms of \( K \)). This is what leads to the inversion formulae in Lemma (5.14).

Suppose that \( K = SU(n). \) In this case \( C(SU(n)) = \mathbb{Z}_n\Delta, \) where \( \Delta = \exp(2\pi i/n). \) The generator \( \Delta \) corresponds to the Dynkin diagram symmetry \( \alpha_j \to \alpha_{j+1}, \) where \( j \) is read mod \( n-1. \) All \( \hat{a}_j = 1. \) We want to write down the multivalued loop which corresponds to the automorphism of \( \hat{L}g \) which sends \( h_j, e_j, f_j \) to \( h_{j+1}, e_{j+1}, f_{j+1}, \) where \( j \) is read mod \( n-1. \) In the defining representation of \( sl(n, \mathbb{C}), \) where \( h \) is the diagonal subalgebra, this multivalued loop is given by

\[
\sigma_\Delta(t) = \lambda w \prod_{j=1}^{n-1} e^{2\pi i(\frac{j}{n})th_j} \\
= \lambda \begin{pmatrix}
0 & 0 & \ldots & 0 & \exp(-2\pi i\frac{n-1}{n}t) \\
\exp(2\pi it/n) & 0 & \ldots & \exp(-2\pi i\frac{n-1}{n}t) \\
\ldots & \exp(2\pi it/n) & 0 & 0 & \ldots & \exp(2\pi it/n) & 0
\end{pmatrix}
\]

where \( w \) is the matrix representing the cyclic permutation \((12..n), \) and the factor \( \lambda \) guarantees \( det = 1, \) and is otherwise irrelevant. It is straightforward to check that conjugation by \( \sigma_\Delta \) implements the automorphism described above.

The automorphism of the Kac-Moody extension cyclically permutes \( \sigma_0, \sigma_1, \ldots, \sigma_{n-1}. \) Together with (B.1), this implies that \( \sigma_\Delta \) maps \( (a_j) \) to \((a_2/a_1, a_3/a_1, \ldots, a_{n-1}/a_1, a_1^{-1}). \) Thus in this case \( a_1^{-1} \) has the same limiting distribution properties as \( a_{n-1}. \)

In the case of \( C_l, C(Sp(l)) = \mathbb{Z}_2\Delta, \) where \( \Delta = -1 \) (as a matrix in the defining representation). This transposes \( \sigma_0 \) and \( \sigma_1. \) Hence \( (a_j) \) maps to \((a_1/a_l, \ldots, a_{l-1}/a_l, 1/a_l). \) We have

\[
\sigma_\Delta(t) = w \prod_{j=1}^{l} \exp(i\pi tjh_j)
\]
where (if we realize $C_l$ as in [V], page 300) $w$ is a particular representative for the Weyl group element that maps $(\lambda_1, \ldots, \lambda_l)$ to $(-\lambda_l, \ldots, -\lambda_1)$ (so that $h_i \leftrightarrow h_{l-i}$, $i < l$, and $h_l \leftrightarrow -h_\theta$). The formula is

$$w = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

where $A = \text{diag}(i, -i, i, \ldots)$, $B$ has 1’s on the antidiagonal, and $A$ and $B$ are $l \times l$ matrices.

In the case of $B_l$, $C(\text{Spin}(2l+1)) = \mathbb{Z}_2 \Delta$, the kernel of the projection to $SO(2l+1)$. The corresponding automorphism transposes $\sigma_1$ and $\sigma_0$; hence $(a_j)$ maps to $(1/a_1, a_2/a_1, \ldots, a_l/a_1)$. We must have

$$\sigma_\Delta(t) = w \exp(i\pi th_l),$$

where $w$ is a representative for the Weyl group element that changes the sign of $\lambda_1$ (this implies $h_1 \leftrightarrow -h_\theta$).

The other cases are similar.

The Type I Case.

Suppose that there exists $\Delta \in C(U)$ which is fixed by $\Theta$. In terms of the classification of type I symmetric spaces, this applies in all cases except the family $SU(n)/SO(n)$, for odd $n$ (because $SO(n)$ does not have a center for odd $n$), $E_6/Sp(4)$ (the central element of $Sp(4)$, of order 2, is not contained in $C(E_6) \simeq \mathbb{Z}_3$), $E_6/F_4$, and all cases in which $U = G_2, F_4$ or $E_8$ (because these do not have central elements).

Given $\Delta$ fixed by $\Theta$, the corresponding automorphism of $\hat{L}g$ will commute with the unique lift of $\Theta$ which fixes $c$, the central element. We therefore obtain a loop $\sigma$ which will act on $L(U/K)$ and our distributional completions. We write $g_0 = l_0 w ma_0 h_0^s \Theta$.

In cases in which $\Theta$ is an inner automorphism, $h_0 = t_0 = t$, and the formulae for the action the $a_j$ above apply.

Suppose $U/K = SU(2n)/SO(2n)$, where $\Theta$ is realized via the canonical (outer) automorphism which interchanges $\alpha_1 \leftrightarrow \alpha_{2n-1}$, $\alpha_2 \leftrightarrow \alpha_{2n-2}$, ... We have $\Delta = -1$. Then $a_0$ is of the form

$$a_0 = a_1^{h_1} \cdots a_n^{h_n} a_{n-1}^{h_{n-1}} \cdots a_1^{h_{2n-1}}$$

and the automorphism corresponding to $\Delta = -1$ (the $n^{th}$ power of the automorphism corresponding to $\sigma$ in (B.2)) maps this to the sequence of $a_j$'s

$$(a_2/a_n, \ldots, a_{n-1}/a_n, a_1/a_n, 1/a_n, a_1, a_n, a_{n-1}/a_n, \ldots, a_2/a_n)$$
Thus $1/a_n$ is equivalent to $a_{n-1}$ in distribution in the limit $\beta \to 0$.

The case $U/K = SU(2n)/Sp(n)$ is similar.

Appendix C. Integral Formulas for $G/G_\mathbb{R}$.

Consider the Cartan decomposition

$$\psi : U \times_K i\mathfrak{t} \to G/G_\mathbb{R} : [g, x] \to ge^xG_\mathbb{R} \quad (C.1)$$

(here $K$ acts on $U$ on the right and on $i\mathfrak{t}$ by the adjoint action, and given $g \in U$, $x \in i\mathfrak{t}$, $[g, x]$ denotes the image point in $U \times_K i\mathfrak{t}$). We want to express the $G$-invariant measure on $G/G_\mathbb{R}$ (which is unique up to a scalar) in terms of these coordinates. To do this we recall that there is a natural $U$-invariant connection on the vector bundle

$$U \times_K i\mathfrak{t} \to U/K. \quad (C.2)$$

The horizontal subspace at the point $[g, x] \in U \times_K i\mathfrak{t}$ is defined to be the image of the map

$$i\mathfrak{p} \to Hor|_{[g, x]} : \zeta \to \frac{d}{dt}|_{t=0}[ge^{t\zeta}, x]; \quad (C.3)$$

this map depends upon the choice of $(g, x)$, but the image is independent of this choice. Given this horizontal distribution, we obtain a Riemannian structure on our vector bundle, using the inner products on $i\mathfrak{p}$ and $i\mathfrak{t}$ (the vertical direction) induced by the Killing form. We let $dV$ denote the corresponding volume element.

Let $dV_{G/G_\mathbb{R}}$ denote the $G$-invariant volume for $G/G_\mathbb{R}$ (this exists because the adjoint action of $G_\mathbb{R}$ on $i\mathfrak{g}_\mathbb{R}$ admits an essentially unique invariant volume form).

Proposition (C.4). We have

$$\psi^*(dV_{G/G_\mathbb{R}}) = c \prod_{\alpha > 0}^p \cosh^2(\alpha(x_0)) \prod_{\alpha' > 0}^\mathfrak{t} \left| \frac{\sinh\alpha'(x_0)}{\alpha'(x_0)} \right|^2 dV([g, x])$$

where $x \in i\mathfrak{t}$ is $K$-conjugate to $x_0 \in i\mathfrak{t}_0$, and the products are over the positive roots for $i\mathfrak{t}_0$ acting on $\mathfrak{p}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$, respectively.

Examples (C.5). (a) If $X = S^2$, then

$$dV_{SL(2)/SU(1, 1)} = \cosh^2(2|x|)dV; \quad (C.6)$$
(b) In the group case \( X = K \), there is a more direct formulation of this result. In this case there is a commutative diagram
\[
(K \times K) \times_{\Delta(K)} \mathfrak{t} \times \mathfrak{k} \rightarrow \mathfrak{K} \times \mathfrak{K} / \{(g, g^{-*}) : g \in \mathfrak{K}\}
\]
where the first vertical arrow is given by
\[
(K \times K) \times_{\Delta(K)} \mathfrak{t} \times \mathfrak{k} \rightarrow K \times \mathfrak{k} : [(g, h), x] \rightarrow (gh^{-1}, 2hxh^{-1})
\]
the second vertical arrow is given by \([g, h] \rightarrow gh^*\), and the horizontal arrows are of the form \((g, x) \rightarrow ge^x\). Note the essential appearance of the “2” in (C.8). In terms of the coordinates \( g = ke^{x'} \), for \( g \in \mathfrak{K} \),
\[
dg = c \prod_{\alpha > 0} |\sinh(\alpha(x'_0))|^{2}dk \times dx'
\]
This is equivalent to (C.4), because up to conjugation \( x' = 2x \) and \( \sinh(2x) = 2\cosh(x)\sinh(x) \), and the \( p \) and \( k \) roots are the same.

**Proof of (C.4).** Fix \( g \in U \) and \( x \in i\mathfrak{k} \). Given these choices, using the map \( \psi \) in (C.1), we can identify
\[
T(U \times K \mathfrak{t})|_{[g, x]} = \text{Hor} \oplus \text{Vert} = i\mathfrak{p} \oplus \mathfrak{t} = i\mathfrak{g}_{\mathbb{R}}
\]
We also have an identification
\[
i\mathfrak{g}_{\mathbb{R}} \rightarrow T(G/G_{\mathbb{R}})|_{G_{\mathbb{R}}}.
\]
Thus given our choice of representative \( ge^x \) for the coset \( ge^xG_{\mathbb{R}} \), we obtain an identification
\[
T(G/G_{\mathbb{R}})|_{ge^xG_{\mathbb{R}}} = i\mathfrak{p} \oplus i\mathfrak{k} = i\mathfrak{g}_{\mathbb{R}},
\]
where \( X \in i\mathfrak{g}_{\mathbb{R}} \) corresponds to
\[
\frac{d}{dt}|_{t=0}ge^{x}e^{tX} \cdot G_{\mathbb{R}}.
\]
The \( G \)-invariant volume form at \( ge^xG_{\mathbb{R}} \) corresponds to the \( G_{\mathbb{R}} \)-invariant volume form on \( i\mathfrak{g}_{\mathbb{R}} \) via this identification.
With these identifications understood, we claim that
\[
d\psi|_{[g,x]} : i\mathfrak{p} \oplus i\mathfrak{k} \rightarrow i\mathfrak{p} \oplus i\mathfrak{k} : \zeta, y \rightarrow \text{proj}_{i\mathfrak{g}_\mathbb{R}}(e^{-ad_{\mathfrak{g}}}(\zeta) + \frac{1 - e^{-ad_{\mathfrak{g}}}}{ad_{\mathfrak{g}}}(y)). \quad (C.14)
\]
To verify this we calculate
\[
\frac{d}{dt}\bigg|_{t=0} e^{-x}g^{-1}ge^{t\zeta}e^x+ty = \frac{d}{dt}\bigg|_{t=0}(e^{A\text{d}(e^{-x})(\zeta)}e^{-xe^x+ty}) \quad (C.15)
\]
\[
= e^{-ad_{\mathfrak{g}}}(\zeta) + \frac{1 - e^{-ad_{\mathfrak{g}}}}{ad_{\mathfrak{g}}}(y). \quad (C.16)
\]
When we project, we obtain the claim.

Now we observe that since \( x \in i\mathfrak{k}, \) \( ad(x) \) maps \( i\mathfrak{g}_\mathbb{R} \) to \( \mathfrak{g}_\mathbb{R} \), and \( ad(x)^2 \) maps \( i\mathfrak{g}_\mathbb{R} \) into itself. Thus with respect to the decomposition \( i\mathfrak{p} \oplus i\mathfrak{k}, \)
\[
d\psi|_{[g,x]} = \begin{pmatrix}
\cosh(ad_{\mathfrak{g}}) & 0 \\
0 & \sinh(ad_{\mathfrak{g}}) / ad_{\mathfrak{g}}
\end{pmatrix}
\]
We then have
\[
det_{\mathbb{R}}(\frac{\sinh(ad_{\mathfrak{g}})}{ad_{\mathfrak{g}}}) : i\mathfrak{k} \rightarrow i\mathfrak{k} = det_{\mathbb{C}}(\frac{\sinh(ad_{\mathfrak{g}})}{ad_{\mathfrak{g}}} : \mathfrak{k}^C \rightarrow \mathfrak{k}^C) = \prod_{\alpha > 0} (\frac{\sinh(\alpha(x_0))}{\alpha(x_0)})^2. \quad (C.18)
\]
and similarly for \( \cosh(ad_{\mathfrak{g}}) \). This proves \((C.4)\). □

We now consider the triangular decomposition.

**Proposition (C.19).** We write an element of the top stratum of \( \phi(G/G_{\mathbb{R}}) \) as in \((d)\) of \((1.8), g = l\mathfrak{w}m\phi l^*\Theta\). In these coordinates
\[
dV = a_\phi dm(a_0)dm(l)dm(m)dm(w),
\]
where \( \rho \) denotes the sum of the positive complex roots.

**Proof of (C.19).** \( B^- \) has open orbits in the top stratum. The \( G \)-invariant measure in these orbits will be determined by \( B^- \)-invariance. The measure \( dV \) is clearly \( B^- \)-invariant. □

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