ASYMPTOTIC OPTIMALITY OF FINITE APPROXIMATIONS TO MARKOV DECISION PROCESSES WITH BOREL SPACES

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Abstract. Calculating optimal policies is known to be computationally difficult for Markov decision processes with Borel state and action spaces and for partially observed Markov decision processes even with finite state and action spaces. This paper studies finite-state approximations of discrete time Markov decision processes with Borel state and action spaces, for both discounted and average costs criteria. The stationary policies thus obtained are shown to approximate the optimal stationary policy with arbitrary precision under mild technical conditions. For compact-state MDPs, we obtain explicit rates of convergence bounds quantifying how the approximation improves as the size of the approximating finite state space increases. Using information theoretic arguments, the order optimality of the obtained rates of convergence is established for a large class of problems. We also show that, as a pre-processing setup, action space can taken to be finite with sufficiently large number points for the finite-state approximation problem; thereby, well known algorithms, such as value or policy iteration, Q-learning, etc., can be used to calculate near optimal policies.

Key words. Markov decision processes, stochastic control, finite state approximation, quantization.

AMS subject classifications. 93E20, 90C40, 90C39.

1. Introduction. In this paper, our aim is to study the finite-state approximation problem for computing near optimal policies for discrete time Markov decision processes (MDPs) with Borel state and action spaces, under discounted and average costs criteria.

In the theory of stochastic optimal control, although existence and structural properties of optimal policies have been studied extensively in the literature, computing such policies is generally a challenging problem for systems with uncountable state spaces. This situation also arises in the fully observed reduction of a partially observed Markov decision process even when the original system has finite state and action spaces.

In this paper, we show that one way to compute approximately optimal solutions for such MDPs is to construct a reduced model with a new transition probability and a one-stage cost function by quantizing the state/action spaces, i.e., by discretizing them on a finite grid. We exhibit that when the one-stage cost function and the transition probability of the original model satisfy certain continuity properties, the optimal policy for the approximating finite model applied to the original model has cost that converges to the optimal cost, as the discretization becomes finer. Moreover, under additional continuity conditions on the transition probability and the one stage cost function we also obtain bounds for a rate of approximation in terms of the number of points used to discretize the state space, thereby providing a tradeoff between the computation cost and the performance loss in the system. In particular, we study the following two problems.

(Q1) Under what conditions on the components of the MDP do the true costs correspond to the optimal policies obtained from finite models converge to the
optimal value function as the number of grid points goes to infinity? For this problem, we are only concerned with the convergence of the approximation; that is, we do not establish bounds for a rate of approximation.

(Q2) Can we obtain explicit bounds on the performance loss due to the discretization in terms of the number of grid points if we strengthen the conditions sufficient in (Q1)?

Combined with our recent work [25], where we investigated the asymptotic optimality of the quantization of action sets, the results in this paper lead to a constructive algorithm for obtaining approximately optimal solutions. First the action space is quantized with small error, and then the state space is quantized with small error, which results in a finite model that well approximates the original MDP. When the state space is compact, we can also obtain rates of convergence for both approximations, and using information theoretic tools we can establish that the obtained rates of convergence are order-optimal for a given class of MDPs. Since there exist various computational algorithms for finite-state Markov decision problems, the analysis in this paper can be considered to be constructive.

Answering these problems can also prove useful in networked control applications where perfect transmission of actions to an actuator is not realistic when there is an information transmission constraint (due to the presence of a communication channel) between a plant, a controller, and an actuator (see, e.g., [31]). Here, accurate discrete approximations imply that if there is an information transmission constraint in the system, then one can obtain a near optimal policy that can be reliably transmitted to the actuator by losslessly encoding its finite actions. Even though the problem of optimal quantization for information transmission from a plant/sensor to a controller for both stabilization and optimization has been studied extensively (see, e.g., [31]), these type of results appear to be new in the networked control literature when the problem of transmitting signals from a controller to an actuator is considered. In particular, such schemes require a simple encoding/decoding rule since an actuator does not have the computational capabilities of a controller to apply the adaptive encoding/decoding rules needed in optimal coding schemes. The quantization based approximation framework considered in this paper is particularly suitable for such applications. Furthermore, tools from information theory may allow for obtaining lower bounds on the approximation performance; using such an argument we show that the construction in this paper is order-optimal.

Various methods have been developed to compute near optimal policies in the literature. A partial list of these techniques is as follows: approximate dynamic programming, approximate value or policy iteration, simulation-based techniques, neuro-dynamic programming (or reinforcement learning), state aggregation, etc. For rather complete surveys of these techniques, we refer the reader to [5, 22, 20, 29, 3, 9, 10] and references therein. With the exception of [10, 20], these works in general study either a finite horizon cost or a discounted infinite horizon cost. As well, the majority of these results are for MDPs with discrete (i.e., finite or countable) state and action spaces, or a bounded one-stage cost function (see, e.g., [23, 5, 22, 20, 3]). Those that consider general state and action spaces (see, e.g., [8, 9, 10, 3]) assume in general Lipschitz type continuity conditions on the components of the control model, in order to provide a rate of convergence analysis for the approximation error.

Our paper differs from these results in the following ways: (i) we consider a general setup, where the state and action spaces are Borel, and the one-stage cost function is possibly unbounded, (ii) since we do not aim to provide rate of convergence result
in the first problem (Q1), the continuity assumptions we impose on the components of the control model are weaker than the conditions imposed in prior works that considered general state and action spaces, (iii) we also consider the challenging average cost criterion under reasonable assumptions. The price we pay for imposing weaker assumptions in (Q1) is that we cannot obtain explicit performance bounds in terms of the number of grid points used in the approximations. However, such bounds can be obtained under further assumptions on the transition probability and the one-stage cost functions; this is considered in problem (Q2) for compact-state MDPs.

In a related work [10], the authors develop a sequence of approximations using empirical distributions of a probability measure \( \psi \) with respect to which the transition probability of the MDP is absolutely continuous. They also consider Borel state and action spaces with a possibly unbounded one-stage cost function. By imposing Lipschitz type continuity conditions on the components of the control model, [10] obtains a concentration inequality type upper bound on the accuracy of the approximation based on the Wasserstein distance of order 1 between the probability measure \( \psi \) and its empirical estimate. These conditions are stronger than what we impose for the problem (Q1). We note that [10] adopts a simulation based approximation leading to probabilistic guarantees on the approximation, whereas we adopt a quantization based approach leading to deterministic approximation guarantees. For a review of further simulation based methods, see e.g., [6, 19].

Our approach to solve problem (Q1) can be summarized as follows: (i) first, we obtain approximation results for the compact-state case, (ii) we find conditions under which a compact representation leads to near optimality for non-compact state MDPs, (iii) we prove the convergence of the finite-state models to non-compact models. As a by-product of this analysis, we obtain compact-state-space approximations for an MDP with non-compact Borel state space. In particular, our findings directly lead to finite models if the state space is countable; similar problems in the countable context have been studied in the literature for the discounted cost [21, Section 6.10.2].

We note that the proposed method for solving the approximation problem for compact-state MDPs with the discounted cost is partly inspired by [23]. Specifically, we generalize the operator proposed for an approximate value iteration algorithm in [23] to uncountable state spaces. Then, unlike in [23], we use this operator as a transition step between the original optimality operator and the optimality operator of the approximate model. In [20], a similar construction was given for finite state-action MDPs. Our method to obtain finite-state MDPs from the compact-state model can be seen as a generalization of this construction.

The rest of the paper is organized as follows. In Section 2 we review the definition of discrete time Markov decision processes. In Section 3 we tackle the approximation problem (Q1) for MDPs with compact state space. In Section 4 an analogous approximation result is obtained for MDPs with non-compact state space. Discretization of the action space is considered in Section 5 for a general state space. In Section 6 we derive quantitative bounds on the approximation error in terms of the number of points used to discretize the state space for the compact-state case. In Section 7 the order optimality of the obtained bounds on the approximation errors is established. In Section 8 we present an example to numerically illustrate our results. Section 9 concludes the paper.

1.1. Notation and Conventions. For a metric space \( E \), the Borel \( \sigma \)-algebra (the smallest \( \sigma \)-algebra that contains the open sets of \( E \)) is denoted by \( B(E) \). We let \( B(E) \) and \( C_b(E) \) denote the set of all bounded Borel measurable and continuous real
functions on E, respectively. For any \( u \in C_b(E) \) or \( u \in B(E) \), let \( \|u\| := \sup_{e \in E} |u(e)| \) which turns \( C_b(E) \) and \( B(E) \) into Banach spaces. Given any Borel measurable function \( w : E \to [1, \infty) \) and any real valued Borel measurable function \( u \) on \( E \), we define the \( w \)-norm of \( u \) as

\[
\|u\|_w := \sup_{e \in E} \frac{|u(e)|}{w(e)},
\]

and let \( B_w(E) \) denote the Banach space of all real valued measurable functions \( u \) on \( E \) with finite \( w \)-norm [16]. Let \( \mathcal{P}(E) \) denote the set of all probability measures on \( E \). A sequence \( \{\mu_n\} \) of probability measures on \( E \) is said to converge weakly (resp., setwise) [17] to a probability measure \( \mu \) if \( \int_E g(e)\mu_n(de) \to \int_E g(e)\mu(de) \) for all \( g \in C_b(E) \) (resp., for all \( g \in B(E) \)). For any \( \mu, \nu \in \mathcal{P}(E) \), the total variation distance between \( \mu \) and \( \nu \), denoted as \( \|\mu - \nu\|_{TV} \), is equivalently defined as

\[
\|\mu - \nu\|_{TV} := 2 \sup_{D \in B(E)} |\mu(D) - \nu(D)| = \sup_{\|g\|_1 \leq 1} \left| \int_E g(e)\mu(de) - \int_E g(e)\nu(de) \right|.
\]

Unless otherwise specified, the term ‘measurable’ will refer to Borel measurability in the rest of the paper.

2. Markov Decision Processes. A discrete-time Markov decision process (MDP) can be described by a five-tuple

\[
(X, A, \{A(x) : x \in X\}, p, c),
\]

where Borel spaces (i.e., Borel subsets of complete and separable metric spaces) \( X \) and \( A \) denote the state and action spaces, respectively. The collection \( \{A(x) : x \in X\} \) is a family of nonempty subsets \( A(x) \) of \( A \), which give the admissible actions for the state \( x \in X \). The stochastic kernel \( p(\cdot | x, a) \) denotes the transition probability of the next state given that previous state-action pair is \( (x, a) \) [15]. Hence, it satisfies: (i) \( p(\cdot | x, a) \) is an element of \( \mathcal{P}(X) \) for all \( (x, a) \), and (ii) \( p(D | \cdot, \cdot) \) is a measurable function from \( X \times A \) to \( [0, 1] \) for each \( D \in B(X) \). The one-stage cost function \( c \) is a measurable function from \( X \times A \) to \( [0, \infty) \). In this paper, it is assumed that \( A(x) = A \) for all \( x \in X \).

Define the history spaces \( H_0 = X \) and \( H_t = (X \times A)^t \times X \), \( t = 1, 2, \ldots \) endowed with their product Borel \( \sigma \)-algebras generated by \( B(X) \) and \( B(A) \). A policy is a sequence \( \pi = \{\pi_t\} \) of stochastic kernels on \( A \) given \( H_t \). The set of all policies is denoted by \( \Pi \). Let \( \Phi \) denote the set of stochastic kernels \( \varphi \) on \( A \) given \( X \), and let \( F \) denote the set of all measurable functions \( f \) from \( X \) to \( A \). A randomized Markov policy is a sequence \( \pi = \{\pi_t\} \) of stochastic kernels on \( A \) given \( X \). A deterministic Markov policy is a sequence of stochastic kernels \( \pi = \{\pi_t\} \) on \( A \) given \( X \) such that \( \pi_t(\cdot | x) = \delta_{f_t(x)}(\cdot) \) for some \( f_t \in F \), where \( \delta_z \) denotes the point mass at \( z \). The set of randomized and deterministic Markov policies are denoted by \( \Pi \) and \( \Pi \), respectively. A randomized stationary policy is a constant sequence \( \pi = \{\pi_t\} \) of stochastic kernels on \( A \) given \( X \) such that \( \pi_t(\cdot | x) = \varphi(\cdot | x) \) for all \( t \) and some \( \varphi \in \Phi \). A deterministic stationary policy is a constant sequence of stochastic kernels \( \pi = \{\pi_t\} \) on \( A \) given \( X \) such that \( \pi_t(\cdot | x) = \delta_{f(x)}(\cdot) \) for all \( f \). The set of randomized and deterministic stationary policies are identified with the sets \( \Phi \) and \( \Phi \), respectively.

According to the Ionescu Tulcea theorem [15], an initial distribution \( \mu \) on \( X \) and a policy \( \pi \) define a unique probability measure \( P_T^\pi \) on \( H_\infty = (X \times A)^\infty \). The expectation with respect to \( P_T^\pi \) is denoted by \( E_T^\pi \). If \( \mu = \delta_x \), we write \( P_x^\pi \) and \( E_x^\pi \) instead of \( P_T^\pi \), respectively.
and $\mathbb{E}_\pi^x$. The cost functions to be minimized in this paper are the $\beta$-discounted cost and the average cost, respectively given by

$$J(\pi, x) = \mathbb{E}_\pi^x \left[ \sum_{t=0}^{\infty} \beta^t c(x_t, a_t) \right],$$

$$V(\pi, x) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_\pi^x \left[ \sum_{t=0}^{T-1} c(x_t, a_t) \right].$$

With this notation, the discounted and average value functions of the control problem are defined as

$$J^*(x) := \inf_{\pi \in \Pi} J(\pi, x),$$

$$V^*(x) := \inf_{\pi \in \Pi} V(\pi, x).$$

A policy $\pi^*$ is said to be optimal if $J(\pi^*, x) = J^*(x)$ (or $V(\pi^*, x) = V^*(x)$ for the average cost) for all $x \in X$. Under fairly mild conditions, the set $\mathcal{F}$ of deterministic stationary policies contains an optimal policy for discounted cost (see, e.g., [15, 11]) and average cost optimal control problems (under somewhat stronger continuity/recurrence conditions, see, e.g., [11]).

3. Finite State Approximations of MDPs with Compact State Space.

In this section we consider (Q1) for the MDPs with compact state space. To distinguish compact-state MDPs from non-compact ones, the state space of the compact-state MDPs will be denoted by $Z$ instead of $X$. We impose the assumptions below on the components of the Markov decision process; additional new assumptions will be made for the average cost problem in Section 3.2.

**Assumption 1.**

(a) The one-stage cost function $c$ is in $C_b(Z \times A)$.

(b) The stochastic kernel $p(\cdot | z, a)$ is weakly continuous in $(z, a)$ and setwise continuous in $a$, i.e., for all $z$ and $a$, $p(\cdot | z_k, a_k) \to p(\cdot | z, a)$ weakly when $(z_k, a_k) \to (z, a)$ and $p(\cdot | z, a_k) \to p(\cdot | z, a)$ setwise when $a_k \to a$.

(c) $Z$ and $A$ are compact.

Before proceeding with the main results, we first describe the procedure used to obtain finite-state models.

Let $d_Z$ denote the metric on $Z$. Since the state space $Z$ is assumed to be compact and thus totally bounded, one can find a sequence $\{z_{n,i}\}_{i=1}^{k_n}$ of finite grids in $Z$ such that for all $n$,

$$\min_{i \in \{1, \ldots, k_n\}} d_Z(z, z_{n,i}) < 1/n$$

for all $z \in Z$. The finite grid $\{z_{n,i}\}_{i=1}^{k_n}$ is called an $1/n$-net in $Z$. Let $Z_n := \{z_{n,1}, \ldots, z_{n,k_n}\}$ and define function $Q_n$ mapping $Z$ to $Z_n$ by

$$Q_n(z) := \arg \min_{z_{n,i} \in Z_n} d_Z(z, z_{n,i}),$$

where ties are broken so that $Q_n$ is measurable. In the literature, $Q_n$ is often called a nearest neighborhood quantizer with respect to distortion measure $d_Z$ [13]. For each $n$, $Q_n$ induces a partition $\{S_{n,i}\}_{i=1}^{k_n}$ of the state space $Z$ given by

$$S_{n,i} = \{z \in Z : Q_n(z) = z_{n,i}\},$$
with diameter \( \text{diam}(S_{n,i}) := \sup_{z,y \in S_{n,i}} d_Z(z,y) < 2/n \). Let \( \nu \) be a probability measure on \( Z \) satisfying

\[
\nu(S_{n,i}) > 0 \text{ for all } i,n. \tag{3.1}
\]

The existence of such a probability measure follows from the separability of \( Z \) and the fact that \( \text{int} S_{n,i} \neq \emptyset \) for all \( i,n \), where \( \text{int} B \) denotes the interior of the set \( B \). For example, if \( \{z_n\}_{n \geq 1} \) is a countable dense subset of \( Z \) one can define \( \nu \) as

\[
\nu = \sum_{n=1}^{\infty} 2^{-n} \delta_{z_n}.
\]

Then, \( \nu(B) > 0 \) for any nonempty open set \( B \) and therefore (3.1) holds.

We let \( \nu_{n,i} \) be the restriction of \( \nu \) to \( S_{n,i} \) defined by

\[
\nu_{n,i}(\cdot) := \frac{\nu(\cdot)}{\nu(S_{n,i})}.
\]

The measures \( \nu_{n,i} \) will be used to define a sequence of finite-state MDPs, denoted as MDP\(_n \) \((n \geq 1)\), to approximate the original model. To this end, for each \( n \) define the one-stage cost function \( c_n : Z_n \times A \to [0, \infty) \) and the transition probability \( p_n \) on \( Z_n \) given \( Z_n \times A \) by

\[
c_n(z_n,i,a) := \int_{S_{n,i}} c(z,a) \nu_{n,i}(dz), \tag{3.2}
\]

\[
p_n(\cdot | z_n,i,a) := \int_{S_{n,i}} Q_n \ast p(\cdot | z,a) \nu_{n,i}(dz),
\]

where \( Q_n \ast p(\cdot | z,a) \in \mathcal{P}(Z_n) \) is the pushforward of the measure \( p(\cdot | z,a) \) with respect to \( Q_n \); that is,

\[
Q_n \ast p(z_n,j|z,a) = p(\{z \in Z : Q_n(z) = z_n,j\}|z,a),
\]

for all \( z_n,j \in Z_n \). For each \( n \), we define MDP\(_n \) as a Markov decision process with the following components: \( Z_n \) is the state space, \( A \) is the action space, \( p_n \) is the transition probability and \( c_n \) is the one-stage cost function. History spaces, policies and cost functions are defined in a similar way as in the original model.

**Remark.** To avoid measurability problems associated with the operators that will be defined in the sequel, it is necessary to impose the setwise continuity of the transition probability with respect to the action variable. However, for the purposes of this section, without loss of generality, we can only assume the weak continuity of the transition probability in Assumption 1-(b). The reason is that any MDP with compact action space can be approximated with arbitrary precision by an MDP having a finite action space under weak continuity assumption [24], and in the case of finite action space, weak continuity and Assumption 1-(b) are equivalent. However, for the sake of completeness we use Assumption 1-(b) as it appears above.

### 3.1. Discounted Cost

Here we consider \((Q1)\) for the discounted cost criterion with a discount factor \( \beta \in (0, 1) \).

Define the operator \( T \) on \( B(Z) \) by

\[
Tu(z) := \min_{a \in A} \left[ c(z,a) + \beta \int_Z u(y)p(dy | z,a) \right]. \tag{3.2}
\]
In the literature $T$ is called the Bellman optimality operator. It can be proved that under Assumption 1-(a)(b), $T$ is a contraction operator with modulus $\beta$ mapping $C_b(Z)$ into itself (see [14, Theorem 2.8]); that is, $Tu \in C_b(Z)$ for all $u \in C_b(Z)$ and

$$
\|Tu -Tv\| \leq \beta \|u -v\| \text{ for all } u, v \in C_b(Z).
$$

The following theorem is a widely known result in the theory of Markov decision processes (see again [14, Theorem 2.8]) which also holds without a compactness assumption on the state space and with a weak continuity assumption instead of Assumption 1-(b).

**Theorem 3.1.** Suppose Assumption 1 holds. Then the value function $J^*$ is the unique fixed point in $C_b(Z)$ of the contraction operator $T$, i.e.,

$$
J^* = TJ^*.
$$

Furthermore, a deterministic stationary policy $f^*$ is optimal if and only if it satisfies the optimality equation, i.e.,

$$
J^*(z) = c(z, f^*(z)) + \beta \int_Z J^*(y)p(dy|z, f^*(z)). \tag{3.3}
$$

Finally, there exists a deterministic stationary policy $f^*$ which is optimal, so it satisfies (3.3).

Define, for all $n \geq 1$, the operator $T_n$, which is the Bellman optimality operator for MDP$_n$, by

$$
T_n u(z_{n,i}) := \min_{a \in A} \left[ c_n(z_{n,i}, a) + \beta \sum_{j=1}^{k_n} u(z_{n,j})p_n(z_{n,j}|z_{n,i}, a) \right],
$$

or equivalently,

$$
T_n u(z_{n,i}) = \min_{a \in A} \int_{S_{n,i}} \left[ c(z, a) + \beta \int_Z \hat{u}(y)p(dy|z, a) \right] \nu_{n,i}(dz),
$$

where $u : Z_n \rightarrow \mathbb{R}$ and $\hat{u}$ is the piecewise constant extension of $u$ to $Z$ given by $\hat{u}(z) = u \circ Q_n(z)$. For each $n$, under Assumption 1, [14, Lemma 2.5 and Theorem 2.2] implies the following: (i) $T_n$ is a contraction operator with modulus $\beta$ mapping $B(Z_n)$ (= $C_b(Z_n)$) into itself, (ii) the fixed point of $T_n$ is the value function $J_n^*$ of MDP$_n$, and (iii) there exists an optimal stationary policy $f_n^*$ for MDP$_n$, which therefore satisfies the optimality equation. Hence, we have

$$
J_n^* = T_n J_n^* = T_n J_n(f_n^*, \cdot) = J_n(f_n^*, \cdot),
$$

where $J_n$ denotes the discounted cost for MDP$_n$. Let us extend the optimal policy $f_n^*$ for MDP$_n$ to $X$ by letting $\hat{f}_n(z) = f_n^* \circ Q_n(z) \in \mathbb{F}$.

The following theorem is the main result of this section. It states that the cost function of the policy $\hat{f}_n$ converges to the value function $J^*$ as $n \rightarrow \infty$.

**Theorem 3.2.** The discounted cost of the policy $\hat{f}_n$, obtained by extending the optimal policy $f_n^*$ of MDP$_n$ to $Z$, converges to the optimal value function $J^*$ of the original MDP

$$
\lim_{n \rightarrow \infty} \|J(\hat{f}_n, \cdot) - J^*\| = 0.
$$
Hence, to find a near optimal policy for the original MDP, it is sufficient to compute the optimal policy of $\text{MDP}_n$ for sufficiently large $n$, and then extend this policy to the original state space.

To prove Theorem 3.2 we need a series of technical results. We first define an operator $\hat{T}_n$ on $B(Z)$ by extending $T_n$ to $B(Z)$:

$$\hat{T}_n u(z) := \min_{a \in A} \int_{Z_{n,i_n(z)}} \left[ c(x, a) + \beta \int_z u(y)p(dy|x, a) \right] \mu_{n,i_n(z)}(dx), \quad (3.4)$$

where $i_n : Z \to \{1, \ldots, k_n\}$ maps $z$ to the index of the partition $\{S_{n,i}\}$ it belongs to.

**Remark.** In the rest of this paper, when we take the integral of any function with respect to $\mu_{n,i_n(z)}$, it is tacitly assumed that the integral is taken over all set $S_{n,i_n(z)}$. Hence, we can drop $S_{n,i_n(z)}$ in the integral for the ease of notation.

Since the expression inside the minimization in (3.4) is a continuous function of $a$, by Assumption 1-(b), $\hat{T}_n$ maps $B(Z)$ into itself by [15, Proposition D.5]. Furthermore, it is a contraction operator with modulus $\beta$ which can be shown using [14, Proposition A.2]. Hence, it has a unique fixed point $\hat{J}_n^*$ that belongs to $B(Z)$, and this fixed point must be constant over the sets $S_{n,i}$ because of the averaging operation on each $S_{n,i}$. Furthermore, since $\hat{T}_n(u \circ Q_n) = (T_n u) \circ Q_n$ for all $u \in B(Z_n)$, we have

$$\hat{T}_n(\hat{J}_n^* \circ Q_n) = (T_n \hat{J}_n^*) \circ Q_n = \hat{J}_n^* \circ Q_n.$$

Hence, the fixed point of $\hat{T}_n$ is the piecewise constant extension of the fixed point of $T_n$, i.e.,

$$\hat{J}_n^* = J_n^* \circ Q_n.$$

**Remark.** This is the point where we need the setwise continuity of the transition probability $p$ with respect to $a$, because if we only assume that the stochastic kernel $p$ is weakly continuous, then it will be no longer true that $\hat{T}_n$ maps $B(Z)$ into itself (see [15, Proposition D.5]).

We now define another operator $F_n$ on $B(Z)$ by simply interchanging the order of the minimum and the integral in (3.4), i.e.,

$$F_n u(z) := \int \min_{a \in A} \left[ c(x, a) + \beta \int_z u(y)p(dy|x, a) \right] \mu_{n,i_n(z)}(dx) = \Gamma_n T u(z),$$

where

$$\Gamma_n u(z) := \int u(x) \mu_{n,i_n(z)}(dx).$$

We note that $F_n$ is the extension (to infinite state spaces) of the operator defined in [23, p. 236] for the proposed approximate value iteration algorithm. However, unlike in [23], $F_n$ will serve here as an intermediate point between $T$ and $\hat{T}_n$ (or $T_n$) to solve (Q1) for the discounted cost. To this end, we first note that $F_n$ is a contraction operator on $B(Z)$ with modulus $\beta$. Indeed it is clear that $F_n$ maps $B(Z)$ into itself. Furthermore, for any $u, v \in B(Z)$, we clearly have $\|\Gamma_n u - \Gamma_n v\| \leq \|u - v\|$. Hence, since $T$ is a contraction operator on $B(Z)$ with modulus $\beta$, $F_n$ is also a contraction operator on $B(Z)$ with modulus $\beta$. 


The following theorem states that the fixed point, say \( u_n^* \), of \( F_n \) converges to the fixed point \( J^* \) (i.e., the value function) of \( T \) as \( n \) goes to infinity.

**Theorem 3.3.** If \( u_n^* \) is the unique fixed point of \( F_n \), then \( \lim_{n \to \infty} \|u_n^* - J^*\| = 0 \).

The proof of Theorem 3.3 requires two lemmas.

**Lemma 3.4.** For any \( u \in B(Z) \), we have
\[
\|u - \Gamma_n u\| \leq 2 \inf_{r \in Z^{k_n}} \|u - \Phi_r\|,
\]
where \( \Phi_r(z) = \sum_{i=1}^{k_n} r_i 1_{S_{n,i}}(z) \), \( r = (r_1, \ldots, r_{k_n}) \), and \( 1_B \) denotes the indicator function of the set \( B \).

**Proof.** Fix any \( r \in Z^{k_n} \). Then, using the identity \( \Gamma_n \Phi_r = \Phi_r \), we obtain
\[
\|u - \Gamma_n u\| \leq \|u - \Phi_r\| + \|\Phi_r - \Gamma_n u\| = \|u - \Phi_r\| + \|\Gamma_n \Phi_r - \Gamma_n u\| \leq \|u - \Phi_r\| + \|\Phi_r - u\|.
\]
Since \( r \) is arbitrary, this completes the proof. \( \square \)

Notice that because of the operator \( \Gamma_n \), the fixed point \( u_n^* \) of \( F_n \) must be constant over the sets \( S_{n,i} \). We use this property to prove the next lemma.

**Lemma 3.5.** We have
\[
\|u_n^* - J^*\| \leq \frac{2}{1 - \beta} \inf_{r \in Z^{k_n}} \|J^* - \Phi_r\|.
\]

**Proof.** Note that \( \Gamma_n u_n^* = u_n^* \) since \( u_n^* \) is constant over the sets \( S_{n,i} \). Then, we have
\[
\|u_n^* - J^*\| \leq \|u_n^* - \Gamma_n J^*\| + \|\Gamma_n J^* - J^*\|
\]
\[
= \|F_n u_n^* - \Gamma_n T J^*\| + \|\Gamma_n J^* - J^*\| = \|\Gamma_n T u_n^* - \Gamma_n T J^*\| + \|\Gamma_n J^* - J^*\| \leq \|T u_n^* - T J^*\| + \|\Gamma_n J^* - J^*\| (\text{since } \|\Gamma_n u - \Gamma_n v\| \leq \|u - v\|)
\]
\[
\leq \beta \|u_n^* - J^*\| + \|\Gamma_n J^* - J^*\|.
\]
Hence, we obtain \( \|u_n^* - J^*\| \leq \frac{1}{1 - \beta} \|\Gamma_n J^* - J^*\| \). The result now follows from Lemma 3.4. \( \square \)

**Proof.** (Proof of Theorem 3.3) Recall that since \( Z \) is compact, the function \( J^* \) is uniformly continuous and \( \text{diam}(S_{n,i}) < 2/n \) for all \( i = 1, \ldots, k_n \). Hence, \( \lim_{n \to \infty} \inf_{r \in Z^{k_n}} \|J^* - \Phi_r\| = 0 \) which completes the proof in view of Lemma 3.5. \( \square \)

The next step is to show that the fixed point \( \hat{J}_n \) of \( \hat{T}_n \) converges to the fixed point \( J^* \) of \( T \). To this end, we first prove the following result.

**Lemma 3.6.** For any \( u \in C_b(Z) \), \( \|\hat{T}_n u - F_n u\| \to 0 \) as \( n \to \infty \).

**Proof.** Note that since \( \int_Z u(x)p(dx|y,a) \) is continuous as a function of \( (y,a) \) by Assumption 1-(b), it is sufficient to prove that for any \( l \in C_b(Z \times A) \)
\[
\left\| \min_a \int l(y,a)\nu_{n,i_{n}(x)}(dy) - \int l(y,a)\nu_{n,i_{n}(z)}(dy) \right\| = \sup_{\nu \in Z} \left\| \min_a \int l(y,a)\nu_{n,i_{n}(x)}(dy) - \int l(y,a)\nu_{n,i_{n}(z)}(dy) \right\| \to 0
\]
as \( n \to \infty \). Fix any \( \varepsilon > 0 \). Define \( \{ z_i \}_{i=1}^{\infty} := \bigcup_n Z_n \) and let \( \{ a_i \}_{i=1}^{\infty} \) be a sequence in \( A \) such that \( \min_{a \in A} l(z_i, a) = l(z_i, a_i) \); such \( a_i \) exists for each \( z_i \) because \( l(z_i, \cdot) \) is continuous and \( A \) is compact. Define \( g(y) := \min_{a \in A} l(y, a) \), which can be proved to be continuous, and therefore uniformly continuous since \( Z \) is compact. Thus by the uniform continuity of \( l \), there exists \( \delta > 0 \) such that \( d_{Z \times A}((y, a), (y', a')) < \delta \) implies \( |g(y) - g(y')| < \varepsilon/2 \) and \( |l(y, a) - l(y', a')| < \varepsilon/2 \). Choose \( n_0 \) such that \( 2/n_0 < \delta \). Then for all \( n \geq n_0 \), \( \max_{a \in \{ 1, \ldots, k_n \}} \text{diam}(S_{n,i}) < 2/n_0 < \delta \). Hence, for all \( y \in S_{n,i} \) we have

\[
|l(y, a_i) - \min_{a \in A} l(y, a)| \leq |l(y, a_i) - l(z_i, a_i)| + |\min_{a \in A} l(z_i, a) - \min_{a \in A} l(y, a)| = |l(y, a_i) - l(z_i, a_i)| + |g(z_i) - g(y)| < \varepsilon.
\]

This implies

\[
\left\| \min_{a} \int l(y, a)\nu_{n,i,n}(z)(dy) - \min_{a} l(y, a)\nu_{n,i,n}(z)(dy) \right\|
\leq \left\| \int l(y, a_i)\nu_{n,i,n}(z)(dy) - \min_{a} l(y, a)\nu_{n,i,n}(z)(dy) \right\|
\leq \sup_{z \in Z} \sup_{y \in S_{n,i,n}} |l(y, a_i) - \min_{a} l(y, a)|\nu_{n,i,n}(z)(dy) < \varepsilon.
\]

This completes the proof. \( \square \)

**Theorem 3.7.** The fixed point \( \hat{J}^*_n \) of \( \hat{T}_n \) converges to the fixed point \( J^* \) of \( T \).

**Proof.** We have

\[
\| \hat{J}^*_n - J^* \| \leq \| \hat{T}_n \hat{J}^*_n - \hat{T}_n J^* \| + \| \hat{T}_n J^* - F_n J^* \| + \| F_n J^* - F_n u^*_n \|
\]

\[
\leq \beta \| \hat{J}^*_n - J^* \| + \| \hat{T}_n J^* - F_n J^* \| + \beta \| J^* - u^*_n \| + \| u^*_n - J^* \|.
\]

Hence

\[
\| \hat{J}^*_n - J^* \| \leq \frac{\| \hat{T}_n J^* - F_n J^* \| + (1 + \beta)\| J^* - u^*_n \|}{1 - \beta}.
\]

The theorem now follows from Theorem 3.3 and Lemma 3.6. \( \square \)

Recall the optimal stationary policy \( f^*_n \) for MDP \( n \) and its extension \( \hat{f}_n(z) = f^*_n \circ Q_n(z) \) to \( Z \). Since \( \hat{J}^*_n = J^*_n \circ Q_n \), it is straightforward to prove that \( \hat{f}_n \) is the optimal selector of \( \hat{T}_n \hat{J}^*_n \); that is,

\[
\hat{T}_n \hat{J}^*_n = \hat{J}^*_n = \hat{T}_n \hat{f}_n \hat{J}^*_n,
\]

where \( \hat{T}_n \hat{f}_n \) is defined as

\[
\hat{T}_n \hat{f}_n u(z) := \int \left[ c(x, \hat{f}_n(x)) + \beta \int_Z u(y)p(dy|x, \hat{f}_n(x)) \right] \nu_{n,i,n}(z)(dx).
\]

Define analogously

\[
T_n \hat{f}_n u(z) := c(z, \hat{f}_n(z)) + \beta \int_Z u(y)p(dy|z, \hat{f}_n(z)).
\]

It can be proved that both \( \hat{T}_n \hat{f}_n \) and \( T_n \hat{f}_n \) are contraction operators on \( B(Z) \) with modulus \( \beta \), and it is known that the fixed point of \( T_n \hat{f}_n \) is the true cost function of the stationary policy \( \hat{f}_n \) (i.e., \( J(\hat{f}_n, z) \)).

**Lemma 3.8.** \( \| \hat{T}_n u - T_n u \| \to 0 \) as \( n \to \infty \), for any \( u \in C_b(Z) \).
Proof. The statement follows from the uniform continuity of the function \( c(z,a) + \beta \int_Z u(y)p(dy|z,a) \) and the fact that \( \hat{f}_n \) is constant over the sets \( S_{n,i} \).

Now, we prove the main result of this section.

Proof. (Proof of Theorem 3.2) We have

\[
\|J(\hat{f}_n, \cdot) - J^*\| \leq \|T_{f_n} J(\hat{f}_n, \cdot) - T_{f_n} J^*\| + \|T_{f_n} J^* - \hat{T}_{f_n} J^*\|
\]

\[
\quad + \|\hat{T}_{f_n} J^* - \hat{T}_{f_n} J_n^*\| + \|J_n^* - J^*\|
\]

\[
\leq \beta \|J(\hat{f}_n, \cdot) - J^*\| + \|T_{f_n} J^* - \hat{T}_{f_n} J^*\|
\]

\[
\quad + \beta \|J^* - J_n^*\| + \|J_n^* - J^*\|
\]

Hence, we obtain

\[
\|J(\hat{f}_n, \cdot) - J^*\| \leq \frac{\|T_{f_n} J^* - \hat{T}_{f_n} J^*\| + (1 + \beta) \|J_n^* - J^*\|}{1 - \beta}
\]

The result follows from Lemma 3.8 and Theorem 3.7.

3.2. Average Cost. In this section we impose some new conditions on the components of the original MDP in addition to Assumption 1 to solve (Q1) for the average cost. A version of first two conditions were imposed in [27] to show the existence of the solution to the Average Cost Optimality Equation (ACOE) and the optimal stationary policy by using the fixed point approach.

Assumption 2. Suppose Assumption 1 holds with item (b) replaced by condition (f) below. In addition, there exist a non-trivial finite measure \( \zeta \) on \( Z \), a nonnegative measurable function \( \theta \) on \( Z \times A \), and a constant \( \lambda \in (0,1) \) such that for all \( (z,a) \in Z \times A \)

\[
d) \ p(B|z,a) \geq \zeta(B)\theta(z,a) \quad \text{for all } B \in B(Z),
\]

\[
e) \ \frac{\lambda}{\zeta(z)} \leq \theta(z,a),
\]

\[
f) \ The stochastic kernel \ p(\cdot|z,a) \ is continuous in \ (z,a) \ with respect to the total variation distance.
\]

Observe that any deterministic stationary policy \( f \) defines a stochastic kernel \( p(\cdot|z,f(z)) \) on \( Z \) given \( Z \) which is the transition probability of the Markov chain \( \{z_i\}_{i=1}^\infty \) (state process) induced by \( f \). For any \( t \geq 1 \), let us write \( p^t(\cdot|z,f(z)) \) to denote the \( t \)-step transition probability of this Markov chain given the initial point \( z \); that is, \( p^t(\cdot|z,f(z)) \) is recursively defined as

\[
p^{t+1}(\cdot|z,f(z)) = \int_Z p(\cdot|x,f(x))p^t(dx|z,f(z)).
\]

The following theorem is a consequence of [12, Lemma 3.4 and Theorem 2.6] and [27, Theorems 3.3], which also holds with Assumption 2-(f) replaced by Assumption 1-(b).

Theorem 3.9. Under Assumptions 2 the following holds.

(i) For each \( f \in F \), the stochastic kernel \( p(\cdot|z,f(z)) \) is positive Harris recurrent with unique invariant probability measure \( \mu_f \). Therefore, we have

\[
V(f,z) = \int_Z c(z,f(z))\mu_f(dz) =: \rho_f.
\]

(ii) There exist positive real numbers \( R \) and \( \kappa < 1 \) such that for every \( z \in Z \)

\[
\sup_{f \in F} \|p^t(\cdot|z,f(z)) - \mu_f\|_\tau \leq R\kappa^t,
\]

where \( R \) and \( \kappa \) continuously depend on \( \zeta(Z) \) and \( \lambda \).
(iii) There exist \( f^* \in \mathbb{F} \) and \( h^* \in \mathcal{B}(\mathbb{Z}) \) such that the triplet \( (h^*, f^*, \rho_{f^*}) \) satisfies the average cost optimality inequality (ACOI), i.e.,

\[
\rho_{f^*} + h^*(z) \geq \min_{a \in A} \left[ c(z, a) + \int_{\mathbb{Z}} h^*(y)p(dy|z, a) \right] = c(z, f^*(z)) + \int_{\mathbb{Z}} h^*(y)p(dy|z, f^*(z)),
\]

and therefore,

\[
\inf_{\pi \in \Pi} V(\pi, z) =: V^*(z) = \rho_{f^*}.
\]

For each \( n \), define the one-stage cost function \( b_n : \mathbb{Z} \times A \to [0, \infty) \) and the stochastic kernel \( q_n \) on \( \mathbb{Z} \) given \( \mathbb{Z} \times A \) as

\[
b_n(z, a) := \int c(x, a)\nu_{n,i_n(z)}(dx),
\]

\[
q_n(\cdot | z, a) := \int p(\cdot | x, a)\nu_{n,i_n(z)}(dx).
\]

Observe that \( c_n \) (i.e., the one stage cost function of \( \tilde{\text{MDP}}_n \)) is the restriction of \( b_n \) to \( \mathbb{Z}_n \), and \( p_n \) (i.e., the stochastic kernel of \( \tilde{\text{MDP}}_n \)) is the pushforward of the measure \( q_n \) with respect to \( Q_n \); that is, \( c_n(z_{n,i}, a) = b_n(z_{n,i}, a) \) for all \( i = 1, \ldots, k_n \) and \( p_n(\cdot | z_{n,i}, a) = Q_n \ast q_n(\cdot | z_{n,i}, a) \).

For each \( n \), let \( \tilde{\text{MDP}}_n \) be defined as a Markov decision process with the following components: \( \mathbb{Z} \) is the state space, \( A \) is the action space, \( q_n \) is the transition probability, and \( c \) is the one-stage cost function. Similarly, let \( \text{MDP}_n \) be defined as a Markov decision process with the following components: \( \mathbb{Z} \) is the state space, \( A \) is the action space, \( q_n \) is the transition probability, and \( b_n \) is the one-stage cost function. History spaces, policies and cost functions are defined in a similar way as before. The models \( \tilde{\text{MDP}}_n \) and \( \text{MDP}_n \) are used as transitions between the original MDP and \( \text{MDP}_n \) in a similar way as the operators \( F_n \) and \( T_n \) were used as transitions between \( T \) and \( T_n \) for the discounted cost. We note that a similar technique was used in the proof of [20, Theorem 2], which studied the approximation problem for finite state-action MDPs. In [20] the one-stage cost function is first perturbed and then the transition probability is perturbed. We first perturb the transition probability and then the cost function. However, our proof method is otherwise quite different from that of [20, Theorem 2] since [20] assumes finite state and action spaces.

We note that a careful analysis of \( \tilde{\text{MDP}}_n \) reveals that its Bellman optimality operator is essentially the operator \( \tilde{T}_n \). Hence, the value function of \( \tilde{\text{MDP}}_n \) is the piecewise constant extension of the value function of \( \text{MDP}_n \) for the discounted cost. A similar conclusion will be made for the average cost in Lemma 3.10.

First, notice that if we define

\[
\theta_n(z, a) := \int \theta(y, a)\nu_{n,i_n(z)}(dy),
\]

\[
\zeta_n := Q_n \ast \zeta \text{ (i.e., pushforward of } \zeta \text{ with respect to } Q_n),
\]

then it is straightforward to prove that for all \( n \), both \( \tilde{\text{MDP}}_n \) and \( \text{MDP}_n \) satisfy Assumption 2-(d),(e) when \( \theta \) is replaced by \( \theta_n \), and Assumption 2-(d),(e) is true for \( \text{MDP}_n \) when \( \theta \) and \( \zeta \) are replaced by the restriction of \( \theta_n \) to \( \mathbb{Z}_n \) and \( \zeta_n \), respectively.
Hence, Theorem 3.9 holds (with the same $R$ and $\kappa$) for $\tilde{\text{MDP}}_n$, $\tilde{\text{MDP}}_n$, and $\text{MDP}_n$ for all $n$. Therefore, we denote by $\tilde{f}_n^*$, $\tilde{\rho}_n^*$ and $f_n^*$ the optimal stationary policies of $\text{MDP}_n$, $\tilde{\text{MDP}}_n$, and $\text{MDP}_n$ with the corresponding average costs $\tilde{\rho}_n^*$, $\tilde{\rho}_n^*$ and $\rho_n^*$, respectively.

Furthermore, we also write $\tilde{\rho}_n^*$, $\tilde{\rho}_n^*$, and $\rho_n^*$ to denote the average cost of any stationary policy $f$ for $\tilde{\text{MDP}}_n$, $\text{MDP}_n$, and $\text{MDP}_n$, respectively. The corresponding invariant probability measures are also denoted in a similar manner, with $\mu$ replacing $\rho$.

The following lemma essentially says that $\text{MDP}_n$ and $\tilde{\text{MDP}}_n$ are not very different.

**Lemma 3.10.** The stationary policy given by the piecewise constant extension of the optimal policy $f_n^*$ of $\text{MDP}_n$ to $\mathbb{Z}$ (i.e., $f_n^* \circ Q_n$) is optimal for $\tilde{\text{MDP}}_n$ with the same cost function $\tilde{\rho}_n^*$. Hence, $\tilde{f}_n^* = f_n^* \circ Q_n$ and $\tilde{\rho}_n^* = \rho_n^*$.

**Proof.** Note that by Theorem 3.9 there exists $h_n^* \in B(\mathbb{Z})$ such that the triplet $(h_n^*, f_n^*, \rho_n^*)$ satisfies the ACOI for $\text{MDP}_n$. But it is straightforward to show that the triplet $(h_n^*, f_n^*, \rho_n^*)$ satisfies the ACOI for $\tilde{\text{MDP}}_n$. By [12, Theorem 2.6 and Lemma 5.2], this implies that $f_n^* \circ Q_n$ is an optimal stationary policy for $\tilde{\text{MDP}}_n$ with cost function $\tilde{\rho}_n^*$. Hence $\tilde{f}_n^* = f_n^* \circ Q_n$ and $\tilde{\rho}_n^* = \rho_n^*$. \(\Box\)

The following theorem is the main result of this section. It states that if one applies the piecewise constant extension of the optimal stationary policy of $\text{MDP}_n$ to the original MDP, the resulting cost function will converge to the value function of the original MDP.

**Theorem 3.11.** The average cost of the optimal policy $\tilde{f}_n^*$ for $\tilde{\text{MDP}}_n$, obtained by extending the optimal policy $f_n^*$ of $\text{MDP}_n$ to $\mathbb{Z}$, converges to the optimal value function $J^* = \rho_f$, of the original MDP, i.e.,

$$\lim_{n \to \infty} |\tilde{\rho}_n^* - \rho_f| = 0.$$

Hence, to find a near optimal policy for the original MDP, it is sufficient to compute the optimal policy of $\text{MDP}_n$ for sufficiently large $n$, and then extend this policy to the original state space.

To show the statement of Theorem 3.11 we will prove a series of auxiliary results.

**Lemma 3.12.** For all $t \geq 1$ we have

$$\lim_{n \to \infty} \sup_{(y,f) \in \mathbb{Z} \times \mathcal{F}} \|p^t(\cdot | y, f(y)) - q^t(\cdot | y, f(y))\|_{TV} = 0.$$

**Proof.** We will prove the lemma by induction. Note that if one views the stochastic kernel $p(\cdot | z, a)$ as a mapping from $\mathbb{Z} \times A$ to $\mathcal{P}(\mathbb{Z})$, then Assumption 2·(f) implies that this mapping is continuous, and therefore uniformly continuous, when $\mathcal{P}(\mathbb{Z})$ is equipped with the metric induced by the total variation distance.

For $t = 1$ the claim holds by the following argument:

$$\sup_{(y,f) \in \mathbb{Z} \times \mathcal{F}} \|p(\cdot | y, f(y)) - q(\cdot | y, f(y))\|_{TV}$$

$$= 2 \sup_{(y,f) \in \mathbb{Z} \times \mathcal{F}} \sup_{D \in \mathcal{B}(\mathbb{Z})} |p(D| y, f(y)) - q(D| y, f(y))|$$

$$\leq 2 \sup_{(y,f) \in \mathbb{Z} \times \mathcal{F}} \sup_{D \in \mathcal{B}(\mathbb{Z})} \left\{ \int |p(D| y, f(y)) - p(D| z, f(y))| \nu_{\alpha, \hat{\iota}(y)}(dz) \right\}$$

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where $R$ Lemma 3.12. Assume the claim is true for value function of the original MDP.

As the mapping $p : Z \times A \rightarrow \mathcal{P}(Z)$ is uniformly continuous with respect to the total variation distance and $\max_{n,i} \text{diam}(\mathcal{S}_{n,i}) \rightarrow 0$ as $n \rightarrow \infty$, the result follows. Assume the claim is true for $t \geq 1$. Then we have

\[
\sup_{(y,f) \in \mathcal{Z} \times \mathcal{F}} \left\| p^{t+1}(\cdot | y, f(y)) - q^{t+1}_n(\cdot | y, f(y)) \right\|_{TV}
\]

\[
= 2 \sup_{(y,f) \in \mathcal{Z} \times \mathcal{F}} \sup_{D \in \mathcal{B}(\mathcal{Z})} \int \left| p(D|z, f(z))p^t(dz|y, f(y)) - \int p(D|z, f(z))q^t_n(dz|y, f(y)) \right| dz
\]

\[
+ 2 \sup_{D \in \mathcal{B}(\mathcal{Z})} \int \left| q_n(D|z, f(z))q^t_n(dz|y, f(y)) - \int q_n(D|z, f(z))q^t_n(dz|y, f(y)) \right| dz
\]

\[
\leq 2 \sup_{(y,f) \in \mathcal{Z} \times \mathcal{F}} \left\| p^t(\cdot | y, f(y)) - q^t_n(\cdot | y, f(y)) \right\|_{TV} + \sup_{(z,f) \in \mathcal{Z} \times \mathcal{F}} \left\| p(\cdot | z, f(z)) - q_n(\cdot | z, f(z)) \right\|_{TV}
\]

where the last inequality follows from the property of the total variation distance: for any $h \in \mathcal{B}(\mathcal{Z})$ and $\mu, \nu \in \mathcal{P}(\mathcal{Z})$ we have $\left| \int h(z) \mu(dz) - \int h(z) \nu(dz) \right| \leq \|h\| \|\mu - \nu\|_{TV}$. By the first step of the proof and the induction hypothesis, the last term converges to zero as $n \rightarrow \infty$. This completes the proof. \qed

Using Lemma 3.12 we prove the following result.

**Lemma 3.13.** We have $\sup_{f \in \mathcal{F}} |\hat{\rho}^n_f - \rho_f| \rightarrow 0$ as $n \rightarrow \infty$, where $\hat{\rho}^n_f$ is the cost function of the policy $f$ for $\widehat{MDP}_n$ and $\rho_f$ is the cost function of the policy $f$ for the original MDP.

**Proof.** For any $t \geq 1$ and $y \in \mathcal{Z}$ we have

\[
\sup_{f \in \mathcal{F}} |\hat{\rho}^n_f - \rho_f| = \sup_{f \in \mathcal{F}} \left| \int c(z, f(z))\hat{\mu}^n_f(dz) - \int c(z, f(z))\mu_f(dz) \right|
\]

\[
\leq \sup_{f \in \mathcal{F}} \left| \int c(z, f(z))\hat{\mu}^n_f(dz) - \int c(z, f(z))q^t_n(dz|y, f(y)) \right|
\]

\[
+ \sup_{f \in \mathcal{F}} \left| \int c(z, f(z))q^t_n(dz|y, f(y)) - \int c(z, f(z))p^t(dz|y, f(y)) \right|
\]

\[
+ \sup_{f \in \mathcal{F}} \left| \int c(z, f(z))p^t(dz|y, f(y)) - \int c(z, f(z))\mu_f(dz) \right|
\]

\[
\leq 2Rc^t\|c\| + \|c\| \sup_{(y,f) \in \mathcal{Z} \times \mathcal{F}} \left\| q^t_n(\cdot | y, f(y)) - p^t(\cdot | y, f(y)) \right\|_{TV} \quad \text{(by Theorem 3.9-(ii))}
\]

where $R$ and $\kappa$ are the constants in Theorem 3.9. Then, the result follows from Lemma 3.12. \qed

The following theorem states that the value function of $\widehat{MDP}_n$ converges to the value function of the original MDP.

**Lemma 3.14.** We have $|\hat{\rho}^n_{f^*} - \rho_{f^*}| \rightarrow 0$ as $n \rightarrow \infty$.

**Proof.** Notice that

\[
|\hat{\rho}^n_{f^*} - \rho_{f^*}| = \max(|\hat{\rho}^n_{f^*} - \rho_{f^*}, \rho_{f^*} - \hat{\rho}^n_{f^*})
\]

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This completes the proof.

**Lemma 3.15.** We have \( \sup_{f \in \mathcal{F}} |\tilde{\rho}^n_f - \hat{\rho}_n^f| \to 0 \) as \( n \to \infty \).

**Proof.** It is straightforward to show that \( b_n \to c \) uniformly. Since the probabilistic structure of \( \tilde{\text{MDP}}_n \) and \( \hat{\text{MDP}}_n \) are the same (i.e., \( \tilde{\rho}_n^f = \hat{\rho}_n^f \) for all \( f \)), we have

\[
\sup_{f \in \mathcal{F}} |\tilde{\rho}^n_f - \hat{\rho}_n^f| = \sup_{f \in \mathcal{F}} \left| \int_{\mathcal{Z}} b_n(z, f(z))\hat{\mu}_n^f(dz) - \int_{\mathcal{Z}} c(z, f(z))\hat{\mu}_n^f(dz) \right|
\]

\[
\leq \sup_{f \in \mathcal{F}} \int_{\mathcal{Z}} |b_n(z, f(z)) - c(z, f(z))|\hat{\mu}_n^f(dz)
\]

\[
\leq \|b_n - c\|.
\]

This completes the proof.

The next lemma states that the difference between the value functions of \( \tilde{\text{MDP}}_n \) and \( \hat{\text{MDP}}_n \) converges to zero.

**Lemma 3.16.** We have \( |\tilde{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| \to 0 \) as \( n \to \infty \).

**Proof.** See the proof of Lemma 3.14.

The following result states that if we apply the optimal policy of \( \tilde{\text{MDP}}_n \) to \( \hat{\text{MDP}}_n \), then the resulting cost converges to the value function of \( \hat{\text{MDP}}_n \).

**Lemma 3.17.** We have \( |\tilde{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| \to 0 \) as \( n \to \infty \).

**Proof.** Since \( |\tilde{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| \leq |\tilde{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| + |\tilde{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| + |\hat{\rho}^n_{f_n} - \hat{\rho}_n^{f_n}| \), the result follows from Lemmas 3.15 and 3.16.

Now, we are ready to prove the main result of this section.

**Proof.** (Proof of Theorem 3.11) We have \( |\rho_{f_n} - \rho_{f^*}| \leq |\rho_{f_n} - \tilde{\rho}_{f_n}^{f_n}| + |\tilde{\rho}_{f_n}^{f_n} - \hat{\rho}_{f_n}^{f_n}| + |\hat{\rho}_{f_n}^{f_n} - \rho_{f^*}| \). The result now follows from Lemmas 3.13, 3.17 and 3.14.

### 4. Finite State Approximations of MDPs with Non-Compact State Space

In this section we consider (Q1) for noncompact state MDPs with unbounded one-stage cost. To solve (Q1), we use the following strategy: (i) first, we define a sequence of compact-state MDPs to approximate the original MDP, (ii) we use Theorems 3.2 and 3.11 to approximate the compact-state MDPs by finite-state models, and (iii) we prove the convergence of the finite-state models to the original model. In fact, steps (ii) and (iii) will be accomplished simultaneously.

We impose the assumptions below on the components of the Markov decision process; additional assumptions will be imposed for the average cost problem. With the exception of the local compactness of the state space, these are the usual assumptions used in the literature for studying Markov decision processes with unbounded cost.

**Assumption 3.**

(a) The one-stage cost function \( c \) is nonnegative and continuous.

(b) The stochastic kernel \( p(\cdot|x,a) \) is weakly continuous in \((x,a)\) and setwise continuous in \( a \).

(c) \( X \) is locally compact and \( A \) is compact.

(d) There exist nonnegative real numbers \( M \) and \( \alpha \in [1,\frac{1}{2}) \), and a continuous weight function \( w: X \to [1,\infty) \) such that for each \( x \in X \), we have

\[
\sup_{a \in A} c(x,a) \leq Mw(x),
\]

where
\[ \sup_{a \in K} \int_X w(y)p(dy|x, a) \leq \alpha w(x), \quad (4.2) \]

and \( \int_X w(y)p(dy|x, a) \) is continuous in \((x, a)\).

Since \( X \) is locally compact separable metric space, there exists a nested sequence of compact sets \( \{K_n\} \) such that \( K_n \subset \text{int } K_{n+1} \) and \( X = \bigcup_{n=1}^{\infty} K_n \) [1, Lemma 2.76].

**Lemma 4.1.** For any compact subset \( K \) of \( X \) and for any \( \varepsilon > 0 \), there exists a compact subset \( K_\varepsilon \) of \( X \) such that

\[ \sup_{(x, a) \in K \times A} \int_{K_\varepsilon} w(y)p(dy|x, a) < \varepsilon, \]

where \( D^c \) denotes the complement of the set \( D \).

**Proof.** We prove the lemma by contradiction. Assume the claim is wrong. Since every compact subset \( K \) of \( X \) is a subset of \( K_n \) for some \( n \), the negation of the above lemma is equivalent to the following statement: there exists a compact set \( K \subset X \) and \( \varepsilon > 0 \) such that for all \( n \geq 1 \) we have

\[ \sup_{(x, a) \in K \times A} \int_{K_n} w(y)p(dy|x, a) \geq \varepsilon. \]

Note that \( w \) is integrable with respect to the probability measures in the set \( \{p(\cdot|x, a) : (x, a) \in K \times A\} \) since

\[ \sup_{(x, a) \in K \times A} \int_X w(y)p(dy|x, a) \leq \alpha \sup_{x \in K} w(x) < \infty. \]

For each \( n \), we prove that \( \int_{(\text{int } K_n)^c} w(y)p(dy|x, a) \) is an upper semi-continuous function on \( K \times A \). Recall that \( \int_X w(y)p(dy|x, a) \) is a continuous function of \((x, a)\). Let \((x_k, a_k) \to (x, a) \) in \( K \times A \). Then \( p(\cdot | x_k, a_k) \to p(\cdot | x, a) \) weakly and \( \int_X w(y)p(dy|x_k, a_k) \to \int_X w(y)p(dy|x, a) \) by our assumption. If we take \( f_k = g_k = f = g = w \) in [26, Theorem 3.3], this result implies that \( \nu_k(\cdot) \to \nu(\cdot) \) weakly, where

\[ \nu_k(D) = \int_D w(y)p(\cdot|x_k, a_k), \]

\[ \nu(D) = \int_D w(y)p(\cdot|x, a), \]

for all \( D \in \mathcal{B}(X) \). Then, by [2, Theorem A] we have

\[ \int_{(\text{int } K_n)^c} w(y)p(dy|x, a) : = \nu((\text{int } K_n)^c) \]

\[ \geq \limsup_{k \to \infty} \nu_k((\text{int } K_n)^c) : = \limsup_{k \to \infty} \int_{(\text{int } K_n)^c} w(y)p(dy|x_k, a_k). \]

Hence, \( \int_{(\text{int } K_n)^c} w(y)p(dy|x, a) \) is upper semi-continuous. Since \( K \times A \) is compact, there exists \((x_n, a_n) \in K \times A\) such that

\[ \sup_{(x, a) \in K \times A} \int_{(\text{int } K_n)^c} w(y)p(dy|x, a) = \int_{(\text{int } K_n)^c} w(y)p(dy|x_n, a_n). \]
The sequence \( \{(x_n, a_n)\} \) (being a sequence in a compact set \( K \times A \)) has an converging subsequence \( \{(x_{n_k}, a_{n_k})\} \) with the limit \( (x, a) \in K \times A \). Then, for all \( m \geq 2 \), we have

\[
\int_{K_{m-1}} w(y)p(dy|x, a) \geq \int_{(\text{int } K_m)^c} w(y)p(dy|x, a)
\]

\[
\geq \limsup_{k \to \infty} \int_{(\text{int } K_m)^c} w(y)p(dy|x_{n_k}, a_{n_k})
\]

\[
\geq \limsup_{k \to \infty} \int_{(\text{int } K_m)^c} w(y)p(dy|x_{n_k}, a_{n_k}) \geq \varepsilon,
\]

where the third inequality follows from the fact that \( (\text{int } K_m)^c \supset (\text{int } K_m)^c \) for \( k \) sufficiently large. But this is a contradiction because \( w \) is \( p(\cdot | x, a) \) integrable. \( \square \)

Let \( \{\nu_n\} \) be a sequence of probability measures such that for each \( n \geq 1 \), \( \nu_n \in \mathcal{P}(K_n^c) \) and

\[
\gamma_n := \int_{K_n^c} w(x)\nu_n(dx) < \infty,
\]

\[
\gamma := \sup_n \tau_n := \sup_n \max \left\{ 0, \sup_{(x, a) \in K_n^c} \int_{K_n^c} (\gamma_n - w(y)) p(dy|x, a) \right\} < \infty.
\]

For example, such probability measures can be constructed by choosing \( x_n \in K_n^c \) such that \( w(x_n) < \inf_{y \in K_n^c} w(y) + \frac{1}{n} \) and letting \( \nu_n(\cdot) = \delta_{x_n}(\cdot) \).

Similar to the finite-state MDP construction in Section 3, we define a sequence of compact-state MDPs, denoted as c-MDP, to approximate the original model. To this end, for each \( n \) let \( X_n = K_n \cup \{\Delta_n\} \), where \( \Delta_n \in K_n^c \) is a so-called pseudo-state.

We define the transition probability \( p_n \) on \( X_n \) given \( X_n \times A \) and the one-stage cost function \( c_n : X_n \times A \to [0, \infty) \) by

\[
p_n(\cdot | x, a) = \begin{cases} p(\cdot \cap K_n | x, a) + p(K_n^c | x, a)\delta_{\Delta_n}, & \text{if } x \in K_n \\ \int_{K_n^c} p(\cdot \cap K_n | z, a) + p(K_n^c | z, a)\delta_{\Delta_n} \nu_n(dz), & \text{if } x = \Delta_n, \end{cases}
\]

\[
c_n(x, a) = \begin{cases} c(x, a), & \text{if } x \in K_n \\ \int_{K_n^c} c(z, a)\nu_n(dz), & \text{if } x = \Delta_n. \end{cases}
\]

With these definitions, c-MDP is defined as a Markov decision process with the components \( (X_n, A, p_n, c_n) \). History spaces, policies, and cost functions are defined in a similar way as in the original model. Let \( \Pi_n \), \( \Phi_n \), and \( \mathcal{F}_n \) denote the set of all policies, randomized stationary policies and deterministic stationary policies of c-MDP, respectively. For each policy \( \pi \in \Pi_n \) and initial distribution \( \mu \in \mathcal{P}(X_n) \), we denote the cost functions for c-MDP by \( J_n(\pi, \mu) \) and \( V_n(\pi, \mu) \).

To obtain the main result of this section, we introduce, for each \( n \), another MDP, denoted by \( \text{MDP}_n \), with the components \( (X, A, q_n, b_n) \) where

\[
q_n(\cdot | x, a) = \begin{cases} p(\cdot | x, a), & \text{if } x \in K_n \\ \int_{K_n^c} p(\cdot | z, a)\nu_n(dz), & \text{if } x \in K_n^c, \end{cases}
\]

\[
b_n(x, a) = \begin{cases} c(x, a), & \text{if } x \in K_n \\ \int_{K_n^c} c(z, a)\nu_n(dz), & \text{if } x \in K_n^c. \end{cases}
\]

For each policy \( \pi \in \Pi \) and initial distribution \( \mu \in \mathcal{P}(X) \), we denote the cost functions for \( \text{MDP}_n \) by \( J_n(\pi, \mu) \) and \( V_n(\pi, \mu) \).
4.1. Discounted Cost. In this section we consider (Q1) for the discounted cost criterion with a discount factor \( \beta \in (0,1) \). The following result states that \( c\text{-MDP}_n \) and \( \overline{\text{MDP}}_n \) are equivalent for the discounted cost.

**Lemma 4.2.** We have

\[
J^*_n(x) = \begin{cases} 
J^*_n(x), & \text{if } x \in K^c_n \\
J^*_n(\Delta_n), & \text{if } x \in K_n,
\end{cases}
\] (4.5)

where \( J^*_n \) is the discounted value function of \( \overline{\text{MDP}}_n \) and \( J^*_n \) is the discounted value function of \( c\text{-MDP}_n \), provided that there exist optimal deterministic stationary policies for \( \overline{\text{MDP}}_n \) and \( c\text{-MDP}_n \). Furthermore, if, for any deterministic stationary policy \( f \in \mathbb{F}_n \), we define \( \bar{f}(x) = f(x) \) on \( K_n \) and \( \bar{f}(x) = f(\Delta_n) \) on \( K^c_n \), then

\[
\bar{J}_n(\bar{f}, x) = \begin{cases} 
J_n(f, x), & \text{if } x \in K_n \\
J_n(f, \Delta_n), & \text{if } x \in K^c_n.
\end{cases}
\] (4.6)

In particular, if the deterministic stationary policy \( f^*_n \in \mathbb{F}_n \) is optimal for \( c\text{-MDP}_n \), then its extension \( \bar{f}^*_n \) to \( X \) is also optimal for \( \overline{\text{MDP}}_n \).

**Proof.** The proof of (4.6) is a consequence of the following facts: \( b_n(x, a) = b_n(y, a) \) and \( q_n(\cdot | x, a) = q_n(\cdot | y, a) \) for all \( x, y \in K^c_n \) and \( a \in A \). In other words, \( K^c_n \) in \( \overline{\text{MDP}}_n \) behaves like the pseudo state \( \Delta_n \) in \( c\text{-MDP}_n \) when \( \bar{f} \) is applied to \( \overline{\text{MDP}}_n \).

Let \( \overline{\mathbb{F}}_n \) denote the set of all deterministic stationary policies in \( \mathbb{F} \) which are obtained by extending policies in \( \mathbb{F}_n \) to \( X \). If we can prove that \( \min_{f \in \overline{\mathbb{F}}_n} J_n(f, x) = \min_{f \in \mathbb{F}_n \setminus \overline{\mathbb{F}}_n} J_n(f, x) \) for all \( x \in X \), then (4.5) follows from (4.6). Let \( f \in \mathbb{F} \setminus \overline{\mathbb{F}}_n \). We have two cases: (i) \( \bar{J}_n(\bar{f}, z) = J_n(f, y) \) for all \( z, y \in K^c_n \) or (ii) there exists \( z, y \in K^c_n \) such that \( \bar{J}_n(\bar{f}, z) < J_n(f, y) \).

For the case (i), if we define the deterministic Markov policy \( \pi^0 = \{f_0, f, \ldots\} \), where \( f_0(x) = f(z) \) on \( K^c_n \) for some fixed \( z \in K^c_n \) and \( f_0(x) = f(x) \) on \( K_n \), then using the expression

\[
\bar{J}_n(\pi^0, x) = b_n(x, f_0(x)) + \beta \int_X \bar{J}_n(\bar{f}, x')q_n(dx' | x, f_0(x)),
\] (4.7)

it is straightforward to show that \( \bar{J}_n(\pi^0, x) = \bar{J}_n(\bar{f}, x) \) on \( K_n \) and \( \bar{J}_n(\pi^0, x) = \bar{J}_n(\bar{f}, z) \) on \( K^c_n \). Therefore, \( J_n(\pi^0, x) = \bar{J}_n(\bar{f}, x) \) for all \( x \in X \) since \( J_n(f, x) = \bar{J}_n(\bar{f}, f(z)) \) for all \( x \in K^c_n \). For all \( t \geq 1 \) define the deterministic Markov policy \( \pi^t \) as \( \pi^t = \{f_0, \pi^{t-1}\} \). Analogously, one can prove that \( \bar{J}_n(\pi^t, x) = J_n(\pi^{t+1}, x) \) for all \( x \in X \). Since \( \bar{J}_n(\pi^t, x) \to \bar{J}_n(\bar{f}, x) \) as \( t \to \infty \), we have \( \bar{J}_n(f_0, x) = J_n(f, x) \) for all \( x \in X \), where \( f_0 \in \mathbb{F}_n \).

For the second case, if we again consider the deterministic Markov policy \( \pi^0 = \{f_0, f, \ldots\} \), then by (4.7) we have \( \bar{J}_n(\pi^0, y) = \bar{J}_n(\bar{f}, f(z)) < \bar{J}_n(\bar{f}, y) \). Since \( \min_{f \in \mathbb{F}_n} J_n(f, y) \leq \bar{J}_n(\pi^0, y) \), this completes the proof. \( \blacksquare \)

For each \( n \), let us define \( w_n \) by letting \( w_n(x) = w(x) \) on \( K_n \) and \( w_n(x) = \int_{K^c_n} w(z) q_n(dz) =: \gamma_n \) on \( K^c_n \). Hence, \( w_n \in B(X) \) by (4.3).

**Lemma 4.3.** For all \( n \) and \( x \in X \), the components of \( \overline{\text{MDP}}_n \) satisfy the following:

\[
\sup_{a \in A} b_n(x, a) \leq M w_n(x)
\] (4.8)

\[
\sup_{a \in A} \int_X w_n(y)q_n(dy|x, a) \leq \alpha w_n(x) + \gamma,
\] (4.9)
where \( \gamma \) is the constant in (4.4).

**Proof.** It is straightforward to prove (4.8) by using the definitions of \( b_n \) and \( w_n \), and the equation (4.1). To prove (4.9), we have to consider two cases: \( x \in K_n \) and \( x \in K_n^c \). For the first case, \( q_n( \cdot | x, a ) = p( \cdot | x, a ) \), and therefore, we have

\[
\sup_{a \in A} \int_X w_n(y) p(dy|x, a) = \sup_{a \in A} \left\{ \int_X w(y) p(dy|x, a) + \int_K^{K_n^c} (\gamma_n - w(y)) p(dy|x, a) \right\}
\leq \sup_{a \in A} \int_X w(y) p(dy|x, a) + \gamma \quad (\text{by (4.4)})
\leq \alpha w(x) + \gamma = \alpha w_n(x) + \gamma \quad (\text{as } w_n = w \text{ on } K_n).
\]

For \( x \in K_n^c \), we have

\[
\sup_{a \in A} \int_X w_n(y) q_n(dy|x, a) = \sup_{a \in A} \int_{K_n^c} \left( \int_X w_n(y) p(dy|z, a) \right) \nu_n(dz)
\leq \int_{K_n^c} \sup_{a \in A} \left( \int_X w_n(y) p(dy|z, a) \right) \nu_n(dz)
\leq \int_{K_n^c} (\alpha w(z) + \gamma) \nu_n(dz)
(4.10)
\]

\[
= \alpha w_n(x) + \gamma,
\]

where (4.10) can be proved following the same arguments as for the case \( x \in K_n \).

This completes the proof. \( \square \)

Note that if we define \( c_{n,0}(x) = 1 + \sup_{a \in A} b_n(x, a) \) and \( c_{n,t}(x) = \sup_{a \in A} \int_X c_{n,t-1}(y) q_n(dy|x, a) \), by (4.8) and (4.9), and an induction argument, we obtain (see [16, p. 46])

\[
c_{n,t}(x) \leq L w_n(x) \alpha^t + L \gamma \sum_{j=0}^{t-1} \alpha^j \quad \text{for all } x \in X,
(4.11)
\]

where \( L = 1 + M \). Let \( \beta_0 > \beta \) be such that \( \alpha \beta_0 < 1 \) and let \( C_n : X \to [1, \infty) \) be defined by

\[
C_n(x) = \sum_{t=0}^{\infty} \beta_0^t c_{n,t}(x).
\]

Then, for all \( x \in X \), by (4.11) we have

\[
C_n(x) := \sum_{t=0}^{\infty} \beta_0^t c_{n,t}(x) \leq \frac{L}{1 - \beta_0 \alpha} w_n(x) + \frac{L \beta_0}{(1 - \beta_0)(1 - \beta_0 \alpha)} \gamma
\leq L_1 w_n(x) + L_2.
(4.12)
\]

Hence \( C_n \in B(X) \) as \( w_n \in B(X) \). Moreover, for all \( (x, a) \in X \times A, C_n \) satisfies (see [16, p. 45])

\[
\int_X C_n(y) q_n(dy|x, a) = \sum_{t=0}^{\infty} \beta_0^t \int_X c_{n,t}(y) q_n(dy|x, a)
\]
where $\alpha_0 := \frac{1}{\beta_0}$ and $\alpha_0 \beta < 1$ since $\beta_0 > \beta$. Therefore, for all $x \in X$, components of $\overline{\text{MDP}}_n$ satisfy

$$\sup_{a \in A} b_n(x, a) \leq C_n(x) \quad (4.13)$$

$$\sup_{a \in A} \int_X C_n(y) q_n(dy|x, a) \leq \alpha_0 C_n(x). \quad (4.14)$$

Since (i) $b_n(x, a)$ is continuous in $a$ for all $x \in X$, (ii) $q_n(\cdot|x, a)$ is setwise continuous in $a$ for all $x \in X$, (iii) $C_n \in B(X)$, and (iv) $\alpha_0 \beta < 1$, $\overline{\text{MDP}}_n$ satisfies the assumptions in [16, Theorem 8.3.6, p. 47].

Let us define the Bellman optimality operator $\overline{T}_n : B(X) \to B(X)$ (note that $B_n(X) = B(X)$) for $\overline{\text{MDP}}_n$ by

$$\overline{T}_n u(x) = \min_{a \in A} \left[ b_n(x, a) + \beta \int_X u(y) q_n(dy|x, a) \right]$$

$$= \begin{cases} \min_{a \in A} \left[ c(x, a) + \beta \int_X u(y) p(dy|x, a) \right], & \text{if } x \in K_n \\ \min_{a \in A} \int_{K_n^c} \left[ c(z, a) + \beta \int_X u(y) p(dy|z, a) \right] \nu_n(dz), & \text{if } x \in K_n^c. \end{cases}$$

Then successive approximations to the discounted value function of $\overline{\text{MDP}}_n$ are given by $v^0_n = 0$ and $v^{t+1}_n = \overline{T}_n v^t_n$ ($t \geq 1$). By [16, Theorem 8.3.6, p. 47] and [16, (8.3.34), p. 52] we have

$$v^t_n(x) \leq \overline{J}^*_n(x) \leq \frac{C_n(x)}{1 - \sigma_0} \quad \text{for all } x, \quad (4.15)$$

$$\|v^t_n - \overline{J}^*_n\|_{c_n} \leq \frac{\sigma_0^t}{1 - \sigma_0^t}. \quad (4.16)$$

where $\sigma_0 = \beta \alpha_0 < 1$.

Similar to $v^t_n$, let us define $v^0 = 0$ and $v^{t+1} = Tu^t$, where $T : B_w(X) \to B_w(X)$, the Bellman optimality operator for the original MDP, is given by

$$Tu(x) = \min_{a \in A} \left[ c(x, a) + \beta \int_X u(y) p(dy|x, a) \right].$$

Then, again by [16, Theorem 8.3.6, p. 47] and [16, (8.3.34), p. 52] we have

$$v^t(x) \leq J^*(x) \leq M \frac{w(x)}{1 - \sigma} \quad \text{for all } x, \quad (4.17)$$

$$\|v^t - J^*\|_w \leq M \frac{\sigma^t}{1 - \sigma}, \quad (4.18)$$

where $\sigma = \beta \alpha < 1$.

**Lemma 4.4.** For any compact set $K \subset X$, we have

$$\lim_{n \to \infty} \sup_{x \in K} |v^t_n(x) - v^t(x)| = 0. \quad (4.19)$$
Proof. We prove (4.19) by induction on $t$. For $t = 1$, the claim trivially holds since any compact set $K \subset X$ is inside $K_n$ for sufficiently large $n$, and therefore, $b_n = c$ on $K$ for sufficiently large $n$ (recall $v_n^1 = v^0 = 0$). Assume the claim is true for $t \geq 1$. Fix any compact set $K$. Recall the definition of compact subsets $K_n$ of $X$ in Lemma 4.1. By definition of $q_n$, $b_n$, and $w_n$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $q_n = p$, $b_n = c$, and $w_n = w$ on $K$. With these observations, for each $n \geq n_0$ we have

\[
\sup_{x \in K} |v_{n+1}^t(x) - v^t(x)| = \sup_{x \in \mathbb{R}} \left[ c(x, a) + \beta \int_X v_n^t(y) \, p(dy|x, a) \right] - \min_{A} \left[ c(x, a) + \beta \int_X v^t(y) \, p(dy|x, a) \right]
\]

\[
\leq \beta \sup_{(x, a) \in K \times A} \left| \int_X v_n^t(y) \, p(dy|x, a) - \int_X v^t(y) \, p(dy|x, a) \right|
\]

\[
= \beta \sup_{(x, a) \in K} \int_{K} (v_n^t(y) - v^t(y)) \, p(dy|x, a) + \int_{K \times A} (v_n^t(y) - v^t(y)) \, p(dy|x, a)
\]

\[
\leq \beta \left( \sup_{x \in K_n} |v_n^t(x) - v^t(x)| + \sup_{(x, a) \in K \times A} \left| \int_{K} (v_n^t(y) - v^t(y)) \, p(dy|x, a) \right| \right)
\]

Note that we have $v^t \leq M \frac{w}{1-\sigma}$ by (4.17). Since $w_n \leq \gamma_{\max} w$, where $\gamma_{\max} := \max\{1, \gamma\}$, we also have $v_n^t \leq \frac{L_1 \gamma_{\max} w + L_2}{1-\sigma_0} \leq \frac{(L_1 \gamma_{\max} w + L_2)w}{1-\sigma_0} \leq (4.12)$ and (4.15) (as $w \geq 1$). Let us define

\[
R := \frac{L_1 \gamma_{\max} + L_2}{1-\sigma_0} + \frac{M}{1-\sigma_0}
\]

Then by Lemma 4.1 we have

\[
\sup_{x \in K} |v_n^t(x) - v^t(x)| \leq \beta \sup_{x \in K} |v_n^t(x) - v^t(x)| + \beta R \varepsilon.
\]

Since the first term converges to zero as $n \to \infty$ by the induction hypothesis, and $\varepsilon$ is arbitrary, the claim is true for $t + 1$. This completes the proof. \( \Box \)

The following theorem states that the discounted value function of $\text{MDP}_n$ converges to the discounted value function of the original MDP uniformly on each compact set $K \subset X$.

**Theorem 4.5.** For any compact set $K \subset X$ we have

\[
\lim_{n \to \infty} \sup_{x \in K} |J_n^*(x) - J^*(x)| = 0.
\] (4.20)

**Proof.** Fix any compact set $K \subset X$. Since $w$ is continuous and therefore bounded on $K$, it is sufficient to prove $\lim_{n \to \infty} \sup_{x \in K} \frac{|J_n^*(x) - J^*(x)|}{w(x)}$. Let $n$ be chosen such that $K \subset K_n$, and so, $w_n = w$ on $K$. Then we have

\[
\sup_{x \in K} \frac{|J_n^*(x) - J^*(x)|}{w(x)} \leq \sup_{x \in K} \frac{|J_n^*(x) - v_n^t(x)|}{w(x)} + \sup_{x \in K} \frac{|v_n^t(x) - v^t(x)|}{w(x)} + \sup_{x \in K} \frac{|v^t(x) - J^*(x)|}{w(x)}
\]
By \[\text{Remark 8.3.10, p. 54}\], \(R\) is a sequence of positive real numbers such that \(\lim_{n \to \infty} \epsilon_n \leq \frac{\sigma^t}{1 - \sigma^t}\) (by (4.18)).

\[
\begin{align*}
&\leq \sup_{x \in K} \frac{|J^*_n(x) - u^t_n(x)|}{w(x)} + \sup_{x \in K} \frac{|v^t_n(x) - v^t(x)|}{w(x)} + M \frac{\sigma^t}{1 - \sigma^t} \quad \text{(by (4.18))}
&\leq \sup_{x \in K} \frac{|J^*_n(x) - u^t_n(x)|}{C_n(x)} \cdot (L_1 w_n(x) + L_2) + \sup_{x \in K} \frac{|v^t_n(x) - v^t(x)|}{w(x)} + M \frac{\sigma^t}{1 - \sigma^t} \quad \text{(by (4.12))}
&\leq (L_1 + L_2) \sup_{x \in K} \frac{|J^*_n(x) - u^t_n(x)|}{C_n(x)} + \sup_{x \in K} \frac{|v^t_n(x) - v^t(x)|}{w(x)} + M \frac{\sigma^t}{1 - \sigma^t} \quad \text{(by (4.16)).}
\end{align*}
\]

Since \(w \geq 1\) on \(X\), \(\sup_{x \in K} \frac{|v^t_n(x) - v^t(x)|}{w(x)} \to 0\) as \(n \to \infty\) for all \(t\) by Lemma 4.4. Hence, the last expression can be made arbitrarily small. This completes the proof. \(\square\)

In the remainder of this section, we use the above results and Theorem 3.2 to compute a near optimal policy for the original MDP. It is straightforward to check that for each \(n\), c-MDP\(_n\) satisfies the assumptions in Theorem 3.2. Let \(\{\epsilon_n\}\) be a sequence of positive real numbers such that \(\lim_{n \to \infty} \epsilon_n = 0\).

By Theorem 3.2, for each \(n \geq 1\), there exists a deterministic stationary policy \(f_\infty \in F\), obtained from the finite state approximations of c-MDP\(_n\), such that

\[
\sup_{x \in X} |J_n(f_\infty, x) - J^*_n(x)| \leq \epsilon_n,
\]

where for each \(n\), finite-state models are constructed replacing \((Z, A, p, c)\) with the components \((X_n, A, p_n, c_n)\) of c-MDP\(_n\) in Section 3. By Lemma 4.2, for each \(n \geq 1\) we also have

\[
\sup_{x \in X} |\tilde{J}_n(f_\infty, x) - \tilde{J}^*_n(x)| \leq \epsilon_n,
\]

where, with an abuse of notation, we also denote the extended (to \(X\)) policy by \(f_\infty\). Let us define operators \(\tilde{R}_n : B_{C_n}(X) \to B_{C_n}(X)\) and \(R_n : B_w(X) \to B_w(X)\) by

\[
\tilde{R}_n u(x) = \begin{cases} 
\frac{c(x, f_n(x)) + \beta \int_X u(y)p(dy|x, f_n(x))}{w(x)}, & \text{if } x \in K_n \\
\int_{K_n^c} \left[ c(z, f_n(z)) + \beta \int_X u(y)p(dy|z, f_n(z)) \right] \nu_n(dz), & \text{if } x \in K_n^c,
\end{cases}
\]

\[R_n u(x) = c(x, f_n(x)) + \beta \int_X u(y)p(dy|x, f_n(x)).\]

By [16, Remark 8.3.10, p. 54], \(\tilde{R}_n\) is a contraction operator with modulus \(\sigma_0\) and \(R_n\) is a contraction operator with modulus \(\sigma\). Furthermore, the fixed point of \(\tilde{R}_n\) is \(\tilde{J}_n(f_\infty, x)\) and the fixed point of \(R_n\) is \(J(f_\infty, x)\). For each \(n \geq 1\), let us define \(\tilde{u}_n^0 = u_n^0 = 0\) and \(\tilde{u}_n^{t+1} = \tilde{R}_n \tilde{u}_n^t, u_n^{t+1} = R_n u_n^t\) \((t \geq 1)\). One can prove that (see the proof of [16, Theorem 8.3.6])

\[
\tilde{u}_n^t(x) \leq \tilde{J}_n(f_\infty, x) \leq \frac{C_n(x)}{1 - \sigma_0}
\]

\[
\|\tilde{u}_n^t - \tilde{J}_n(f_\infty, \cdot)\|_{C_n} \leq \frac{\sigma^t}{1 - \sigma_0}
\]

\[
u_u^t(x) \leq J(f_\infty, x) \leq M \frac{w(x)}{1 - \sigma}
\]

\[
\|u_n^t - J(f_\infty, \cdot)\|_w \leq M \frac{\sigma^t}{1 - \sigma},
\]
Lemma 4.6. For any compact set $K \subset X$, we have
\[
\lim_{n \to \infty} \sup_{x \in K} |\bar{u}^n_t(x) - u^n_t(x)| = 0.
\]

Proof. The lemma can be proved using the same arguments as in the proof of Lemma 4.4 and so we omit the details.

Lemma 4.7. For any compact set $K \subset X$, we have
\[
\lim_{n \to \infty} \sup_{x \in K} |\bar{J}_n(f_n, x) - J(f_n, x)| = 0.
\](4.22)

Indeed, this is true for all sequences of policies in $\mathbb{P}$.

Proof. The lemma can be proved using the same arguments as in the proof of Theorem 4.5.

The following theorem is the main result of this section which states that the true cost functions of the policies obtained from finite state models converge to the value function of the original MDP. Hence, to obtain a near optimal policy for the original MDP, it is sufficient to compute the optimal policy for the finite state model that has sufficiently large number of grid points.

Theorem 4.8. For any compact set $K \subset X$, we have
\[
\lim_{n \to \infty} \sup_{x \in K} |J(f_n, x) - J^*(x)| = 0.
\]

Therefore,
\[
\lim_{n \to \infty} |J(f_n, x) - J^*(x)| = 0 \text{ for all } x \in X.
\]

Proof. The result follows from (4.20), (4.21), and (4.22).

4.2. Average Cost. In this section we obtain approximation results, analogous to Theorems 4.5 and 4.8, for the average cost criterion. To do this, we impose some new assumptions on the components of the original MDP in addition to Assumption 3. These assumptions are the unbounded counterpart of Assumption 2. With the exception of Assumption 4-(j), versions of these assumptions were imposed in [27] and [12] to study the existence of the solution to the Average Cost Optimality Equality (ACOE) and Inequality (ACOI). In what follows, for any finite signed measure $\vartheta$ and measurable function $h$ on $X$, we let $\vartheta(h) := \int_X h(x) \vartheta(dx)$ and
\[
\|\vartheta\|_w := \sup_{\|g\|_w \leq 1} \left| \int_X g(x) \vartheta(dx) \right|.
\]
Here $\|\vartheta\|_w$ is called the $w$-norm of $\vartheta$.

Assumption 4. Suppose Assumption 3 holds with item (b) and (4.2) replaced by conditions (j) and (e) below, respectively. In addition, there exist a probability measure $\eta$ on $X$ and a positive measurable function $\phi : X \times A \to (0, \infty)$ such that for all $(x, a) \in X \times A$
\begin{enumerate}
\item[(e)] $\int_X w(y)p(dy|x, a) \leq \alpha w(x) + \eta(w)\phi(x, a)$, where $\alpha \in (0, 1)$.
\item[(f)] $p(D|x, a) \geq \eta(D)\phi(x, a)$ for all $D \in \mathcal{B}(X)$.
\item[(g)] The weight function $w$ is $\eta$-integrable, i.e., $\eta(w) < \infty$.
\item[(h)] For each $n \geq 1$, $\inf_{(x,a) \in K_n \times A} \phi(x,a) > 0$.
\end{enumerate}
(j) The stochastic kernel $p(\cdot | x, a)$ is continuous in $(x, a)$ with respect to the $w$-norm.

Analogous with Theorem 3.9, the following theorem is a consequence of [27, Theorems 3.3] and [12, Lemma 3.5 and Theorem 2.6] (see also [16, Proposition 10.2.5]), which also holds with Assumption 4-(j) replaced by Assumption 3-(b).

**Theorem 4.9.** Under Assumption 4 the following hold.

(i) For each $f \in \mathcal{F}$, the stochastic kernel $p(\cdot | x, f)\) is positive Harris recurrent with unique invariant probability measure $\mu_f$. Furthermore, $w$ is $\mu$-integrable, and therefore, $\rho_f := \int_X c(x, f) \mu_f(dx) < \infty$.

(ii) There exist positive real numbers $R$ and $\kappa < 1$ such that

$$\sup_{f \in \mathcal{F}} \|p^t(\cdot | x, f(x)) - \mu_f\|_w \leq R\kappa^t$$

for all $x \in X$, where $R$ and $\kappa$ continuously depend on $\alpha$, $\eta(w)$, and $\inf_{f \in \mathcal{F}} \eta(\phi(y, f(y)))$.

(iii) There exist $f^* \in \mathcal{F}$ and $h^* \in B_w(X)$ such that the triplet $(h^*, f^*, \rho_{f^*})$ satisfies the average cost optimality inequality (ACOI), and therefore,

$$\inf_{\pi \in \Pi} V(\pi, x) := V^*(x) = \rho_{f^*},$$

for all $x \in X$.

Note that this theorem implies that for each $f \in \mathcal{F}$, the average cost is given by $V(f, x) = \int_X c(y, f(y)) \mu_f(dy)$ for all $x \in X$ (instead of $\mu$-a.e.).

**Remark.** We note that if we further assume the continuity of $\phi(x, \cdot)$ on $A$ for each $x \in X$ in Assumption 4, then by [27, Theorem 3.6] the ACOE holds in Theorem 4.9-(ii) instead of the ACOI.

Recall that $V_n$ and $\bar{V}_n$ denote the average costs of $c$-MDP$_n$ and $\bar{\text{MDP}}_n$, respectively. The value functions for average cost are denoted analogously to the discounted cost case. Similar to Lemma 4.2, the following result states that $\text{MDP}_n$ and $\bar{\text{MDP}}_n$ are not too different for the average cost.

**Lemma 4.10.** Suppose Theorem 4.9 holds for $\bar{\text{MDP}}_n$ and Theorem 3.9 holds for $\text{MDP}_n$. Then we have

$$\bar{V}_n^*(x) = \begin{cases} V_n^*(x), & \text{if } x \in K_n \\ V_n^*(\Delta_n), & \text{if } x \in K_n^c. \end{cases}$$

(4.24)

Furthermore, if, for any deterministic stationary policy $f \in \mathcal{F}_n$, we define $\bar{f}(x) = f(x)$ on $K_n$ and $\bar{f}(x) = f(\Delta_n)$ on $K_n^c$, then

$$\bar{V}_n(\bar{f}, x) = \begin{cases} V_n(f, x), & \text{if } x \in K_n \\ V_n(f, \Delta_n), & \text{if } x \in K_n^c. \end{cases}$$

(4.25)

In particular, if the deterministic stationary policy $f^*_n \in \mathcal{F}_n$ is optimal for $\text{MDP}_n$, then its extension $\bar{f}^*_n$ to $X$ is also optimal for $\bar{\text{MDP}}_n$.

**Proof.** Let the triplet $(h^*_n, f^*_n, \rho^*_n)$ satisfy the ACOI for $c$-MDP$_n$, so that $f^*_n$ is an optimal policy and $\rho^*_n$ is the average value function for $c$-MDP$_n$. It is straightforward to show that the triplet $(\bar{h}^*_n, \bar{f}^*_n, \bar{\rho}^*_n)$ satisfies the ACOI for $\bar{\text{MDP}}_n$, where

$$\bar{h}^*_n(x) = \begin{cases} h^*_n(x), & \text{if } x \in K_n \\ h^*_n(\Delta_n), & \text{if } x \in K_n^c. \end{cases}$$
and
\[ \tilde{f}^*_n(x) = \begin{cases} f^*_n(x), & \text{if } x \in K_n \\ f^*_n(\Delta_n), & \text{if } x \in K_n^c. \end{cases} \]

By [12, Theorem 2.6 and Lemma 5.2], this implies that \( \tilde{f}^*_n \) is an optimal stationary policy for \( \text{MDP}_n \) with cost function \( \rho^*_n \). This completes the proof of the first part.

For the second part, let \( f \in F_n \) with an unique invariant probability measure \( \mu_f \in \mathcal{P}(X_n) \) and let \( \tilde{f} \in \mathcal{F} \) denote its extension to \( X \) with an unique invariant probability measure \( \mu_{\tilde{f}} \). It can be proved that
\[ \mu_f(\cdot) = \mu_f(\cdot \cap K_n) + \mu_f(K_n^c)\delta_{\Delta_n}(\cdot). \]

Then we have
\[ \bar{V}_n(f, x) = \int_{X} b_n(x, \tilde{f}(x))\mu_f(dx) \]
\[ = \int_{K_n} c_n(x, f(x))\mu_f(dx) + \mu_f(K_n^c)c_n(\Delta_n, \tilde{f}(\Delta_n)) \]
\[ = \int_{X_n} c_n(x, f(x))\mu_f(dx) \]
\[ = V_n(f, x). \]

This completes the proof. \( \Box \)

By Lemma 4.10, in the remainder of this section we need only consider \( \text{MDP}_n \) in place of \( \text{MDP}_n \). Later we will show that Theorem 4.9 holds for \( \text{MDP}_n \) for \( n \) sufficiently large and that Theorem 3.9 holds for \( \text{c-MDP}_n \) for all \( n \).

Recall the definition of constants \( \gamma_n \) and \( \tau_n \) from (4.3) and (4.4). For each \( n \geq 1 \), we define \( \phi_n : X \times A \to (0, \infty) \) and \( \varsigma_n \in \mathbb{R} \) as
\[ \phi_n(x, a) := \begin{cases} \phi(x, a), & \text{if } x \in K_n \\ \int_{K_n^c} \phi(y, a)\nu_n(dy), & \text{if } x \in K_n^c, \end{cases} \]
\[ \varsigma_n := \int_{K_n^c} w(y)\eta(dy). \]

Since \( \eta(w) < \infty \) and \( \tau_n \) can be made arbitrarily small by properly choosing \( \nu_n \), we assume, without loss of generality, the following.

**Assumption 5.** The sequence of probability measures \( \{\nu_n\} \) is chosen such that the following holds
\[ \lim_{n \to \infty} (\tau_n + \varsigma_n) = 0. \tag{4.26} \]

Let \( \alpha_n := \alpha + \varsigma_n + \tau_n \).

**Lemma 4.11.** For all \( n \) and \( (x, a) \in X \times A \), the components of \( \text{MDP}_n \) satisfy the following:
\[ \sup_{a \in A} b_n(x, a) \leq Mw_n(x) \]
\[ \int_X w_n(y)q_n(dy|x, a) \leq \alpha_n w_n(x) + \eta(w_n)\phi_n(x, a), \tag{4.27} \]
\[ q_n(D|x,a) \geq \eta(D)\phi_n(x,a) \quad \text{for all } D \in \mathcal{B}(X). \]

**Proof.** The proof of the first inequality follows from Assumption 4 and definitions of \( b_n \) and \( w_n \). To prove the remaining two inequalities, we have to consider the cases \( x \in K_n \) and \( x \in K_n^c \) separately.

Let \( x \in K_n \), and therefore, \( q_n(\cdot|x,a) = p(\cdot|x,a) \). The second inequality holds since

\[
\int_X w_n(y)p(dy|x,a) = \int_X w(y)p(dy|x,a) + \int_{K_n^c} (\gamma_n - w(y)) p(dy|x,a) \\
\leq \int_X w(y)p(dy|x,a) + \tau_n \\
\leq \alpha w(x) + \eta(w)\phi(x,a) + \tau_n \\
\leq \alpha w_n(x) + \eta(w_n)\phi_n(x,a) + \zeta \phi_n(x,a) + \tau_n \quad \text{(as } w_n = w \text{ and } \phi_n = \phi \text{ on } K_n) \\
\leq \alpha_n w_n(x) + \eta(w_n)\phi_n(x,a), \quad \text{(as } \phi_n \leq 1 \text{ and } w_n \geq 1). 
\]

For the last inequality, for all \( D \in \mathcal{B}(X) \), we have

\[ q_n(D|x,a) = p(D|x,a) \geq \eta(D)\phi(x,a) = \eta(D)\phi_n(x,a) \quad \text{(as } \phi_n = \phi \text{ on } K_n). \]

Hence, inequalities hold for \( x \in K_n \).

For \( x \in K_n^c \), we have

\[
\int_X w_n(y)q_n(dy|x,a) = \int_{K_n^c} \left( \int_X w_n(y)p(dy|z,a) \right) \nu_n(dz) \\
\leq \int_{K_n^c} \left( \alpha w(z) + \eta(w_n)\phi(x,a) + \zeta \phi(x,a) + \tau_n \right) \nu_n(dz) \tag{4.28} \\
= \alpha w_n(x) + \eta(w_n)\phi_n(x,a) + \zeta \phi_n(x,a) + \tau_n \\
\leq \alpha_n w_n(x) + \eta(w_n)\phi_n(x,a), \quad \text{(since } \phi_n \leq 1 \text{ and } w_n \geq 1 \text{)}
\]

where (4.28) can be obtained following the same arguments as for the case \( x \in K_n \).

The last inequality holds for \( x \in K_n^c \) since

\[
q_n(D|x,a) = \int_{K_n^c} p(D|z,a)\nu_n(dz) \\
\geq \int_{K_n^c} \eta(D)\phi(z,a)\nu_n(dz) \\
= \eta(D)\phi_n(x,a).
\]

This completes the proof. \( \square \)

We note that by (4.26), there exists \( n_0 \geq 1 \) such that \( \alpha_n < 1 \) for \( n \geq n_0 \). Hence, for each \( n \geq n_0 \), Theorem 4.9 holds for \( \text{MDP}_n \) with \( w \) replaced by \( w_n \) for some \( R_n > 0 \), and \( \kappa_n \in (0,1) \), and we have \( R_{\max} := \sup_{n \geq n_0} R_n < \infty \) and \( \kappa_{\max} := \sup_{n \geq n_0} \kappa_n < 1 \).

In the remainder of this section, it is assumed that \( n \geq n_0 \).

**Lemma 4.12.** Let \( g : X \times A \to \mathbb{R} \) be any measurable function such that \( \sup_{x \in X} |g(x,a)| \leq M_g w(x) \) for some \( M_g \in \mathbb{R} \). Then, for all \( t \geq 1 \) and any compact set \( K \subset X \) we have

\[
\sup_{(y,f) \in K \times \mathcal{F}} \left| \int_X q_n(x,f(x))g_n^1(dy|y,f(y)) - \int_X g(x,f(x))p^t(dy|y,f(y)) \right| \to 0
\]
as \( n \to \infty \), where \( g_n(x,a) = g(x,a) \) on \( K_n \times A \) and \( g_n(x,a) = \int_{K_n} g(z,a) \nu_n (dz) \) on \( K_n \times A \).

\[ \text{Proof.} \] We will prove the lemma by induction. Fix any compact set \( K \subset X \). We note that in the inequalities below, we repeatedly use the fact \( \phi, \phi_n \leq 1 \) without explicitly referring to this fact. Recall the definition of the compact subsets \( K_\varepsilon \) of \( X \) in Lemma 4.1 and the constant \( \gamma_{\text{max}} = \max \{1, \gamma\} \). Note that \( \sup_{x \in A} |g_n(x,a)| \leq M_g w_n(x) \leq M_g \gamma_{\text{max}} w(x) \) for all \( x \in X \).

The claim holds for \( t = 1 \) by the following argument:

\[
\sup_{(y,f) \in K \times F} \left| \int_X g_n(x,f(x))q_n(dx|y,f(y)) - \int_X g(x,f(x))p(dx|y,f(y)) \right|
= \sup_{(y,f) \in K \times F} \left| \int_{K_\varepsilon} g_n(x,f(x))p(dx|y,f(y)) - \int_{K_\varepsilon} g(x,f(x))p(dx|y,f(y)) \right|
\]

(for \( n \) sufficiently large)

\[
\leq M_g(1 + \gamma_{\text{max}})\varepsilon,
\]

where the last inequality follows from Lemma 4.1. Since \( \varepsilon \) is arbitrary, the result follows.

Assume the claim is true for \( t \geq 1 \). Let us define \( l_f(z) := \int_X g_n(x,f(x))q_n^{t}(dx|z,f(z)) \) and \( l_f^t(z) := \int_X g_n(x,f(x))q_n^{t}(dx|z,f(z)) \). By recursively applying the inequalities in Assumption 4-(e) and in (4.27) we obtain

\[
\sup_{f \in F} |l_f(z)| \leq M_g \alpha^t w(z) + M_g \eta(w) \sum_{j=0}^{t-1} \alpha^j
\]

and

\[
\sup_{f \in F} |l_f^t(z)| \leq M_g \alpha^t w_n(z) + M_g \eta(w_n) \sum_{j=0}^{t-1} \alpha^j
\]

\[
\leq M_g \alpha_{\text{max}}^t \gamma_{\text{max}} w(z) + M_g \eta(w) \gamma_{\text{max}} \sum_{j=0}^{t-1} \alpha_{\text{max}}^j,
\]

where \( \alpha_{\text{max}} := \sup_{n \geq n_0} \alpha_n < 1 \). Then we have

\[
\sup_{(y,f) \in K \times F} \left| \int_X g_n(x,f(x))q_n^{t+1}(dx|y,f(y)) - \int_X g(x,f(x))p^{t+1}(dx|y,f(y)) \right|
= \sup_{(y,f) \in K \times F} \left| \int_{K_\varepsilon} l_f^t(z)q_n^{t+1}(dz|y,f(y)) - \int_{K_\varepsilon} l_f(z)p^{t+1}(dz|y,f(y)) \right|
\]

(for \( n \) sufficiently large)

\[
\leq \sup_{(y,f) \in K \times F} \left| \int_{K_\varepsilon} l_f^t(z)p(dz|y,f(y)) - \int_{K_\varepsilon} l_f(z)p(dz|y,f(y)) \right| + \sup_{(z,f) \in K_\varepsilon \times F} |l_f^t(z) - l_f(z)|
\]
\[

R \leq \sup_{(z,f) \in K_x \times F} |l_f^y(z) - l_f(z)|, \quad (4.29)
\]

where \( R \) is given by
\[
R := M_y \left( \alpha^t + \alpha_{\text{max}}^t \gamma_{\text{max}} + \eta(w) \sum_{j=0}^{t-1} \alpha_j^t + \eta(w) \gamma_{\text{max}} \sum_{j=0}^{t-1} \alpha_{\text{max}}^j \right)
\]
and the last inequality follows from Lemma 4.1. Since the claim holds for \( t \) and \( K_x \), the second term in (4.29) goes to zero as \( n \to \infty \). Since \( \varepsilon \) is arbitrary, the result follows. \( \square \)

In the remainder of this section the above results are used to compute a near optimal policy for the original MDP. Let \( \{\varepsilon_n\} \) be a sequence of positive real numbers converging to zero.

For each \( f \in F \), let \( \mu^n_f \) denote the unique invariant probability measure of the transition kernel \( q_n(\cdot | x, f(x)) \) and let \( \rho^n_f \) denote the associated average cost; that is, 
\[
\rho^n_f := \int_{K_x} b_n(y, f(y)) \mu^n_f(dy)
\]
for all initial points \( x \in X \). Therefore, the value function of \( \text{MDP}_n \), denoted by \( V_n^* \), is given by 
\[
V_n^*(x) = \inf_{f \in F} \rho^n_f, \quad \text{i.e., it is constant on } X.
\]

Before making the connection with Theorem 3.11, we prove the following result.

**Lemma 4.13.** The transition probability \( p_n \) of \( c\text{-MDP}_n \) is continuous in \((x, a)\) with respect to the total variation distance.

**Proof.** To ease the notation, we define \( M(X_n) \), \( M(X) \), and \( M_u(X) \) as the subsets of \( B(X_n) \), \( B(X) \), and \( B_u(X) \), respectively, whose elements have (corresponding) norm less than one. Let \((x_k, a_k) \to (x, a)\) in \( X_n \times \mathcal{A} \). Since the state \( \Delta_n \) is isolated and \( K_n \) is compact, we have two cases: (i) \( x_k = \Delta_n \) for all \( k \) large enough, or (ii) \( x_k \to x \) in \( K_n \).

For the first case we have
\[
\|p_n(\cdot | \Delta_n, a_k) - p_n(\cdot | \Delta_n, a)\|_{TV}
\]
where (4.30) follows since if for any \( g \in M(X_n) \) we define \( \bar{g} = g \) on \( K_n \) and \( \bar{g} = g(\Delta_n) \) on \( K_n^c \), then we have \( \bar{g} \in M(X) \) and 
\[
\int_{X_n} g(y)p_n(dy|x, a) = \int_{X} \bar{g}(y)q_n(dy|x, a)
\]
for all \((x, a) \in X_n \times \mathcal{A} \). Note that we have
\[
\sup_{g \in M_u(X)} \left| \int_X g(y)p(dy|z, a_k) - \int_X g(y)p(dy|z, a) \right|
\]
\[
\leq \int_X w(y)p(dy|z, a_k) + \int_X w(y)p(dy|z, a)
\leq 2(\alpha + \eta(w))w(z) \quad \text{(by Assumption 4-(e), } \phi \leq 1, \text{ and } w \geq 1).
\]

Since \( w \) (restricted to \( K_n^c \)) is \( \nu_n \)-integrable, by the dominated convergence theorem (4.31) goes to zero as \( k \to \infty \).

For the second case we have
\[
\|p_n(\cdot | x_k, a_k) - p_n(\cdot | x, a)\|_{TV} = \sup_{g \in M(X_n)} \left| \int_{X_n} g(y)p_n(dy|x_k, a_k) - \int_{X_n} g(y)p_n(dy|x, a) \right|
\leq \sup_{g \in M(X)} \left| \int_X g(y)q_n(dy|x_k, a_k) - \int_X g(y)q_n(dy|x, a) \right|
= \sup_{g \in M(X)} \left| \int_X g(y)p(dy|x_k, a_k) - \int_X g(y)p(dy|x, a) \right| \quad \text{(since } x_k, x \in K_n \text{)}
\leq \sup_{g \in M(X)} \left| \int_X g(y)p(dy|x_k, a_k) - \int_X g(y)p(dy|x, a) \right|
= \|p(\cdot | x_k, a_k) - p(\cdot | x, a)\|_w.
\]

By Assumption 4-(j) the last term goes to zero as \( k \to \infty \). \( \square \)

Thus we obtain that for each \( n \geq 1 \), c-MDP\(_n\) satisfies the assumption in Theorem 3.11 for
\[
\zeta(\cdot) = \eta(\cdot \cap K_n) + \eta(K_n^c)\delta_n(\cdot),
\]
\[
\theta(x, a) = \begin{cases} 
\phi(x, a), & \text{if } x \in K_n \\
_{K_n^c} \phi(y, a)\nu_n(dy), & \text{if } x = \Delta_n,
\end{cases}
\]
and some \( \lambda \in (0, 1) \), where the existence of \( \lambda \) follows from Assumption 4-(h) and the fact that \( \phi > 0 \).

Consequently, there exists a deterministic stationary policy \( f_n \in F_n \), obtained from the finite state approximations of c-MDP\(_n\), such that
\[
\sup_{x \in X_n} |V_n(f_n, x) - V_n^*(x)| \leq \varepsilon_n, \quad (4.32)
\]
where finite-state models are constructed replacing \((Z, A, p, c)\) with the components \((X_n, A, p_n, c_n)\) of c-MDP\(_n\) in Section 3. By Lemma 4.10, we also have
\[
|\rho_n^f - V_n^*| \leq \varepsilon_n, \quad (4.33)
\]
where, by an abuse of notation, we also denote the policy extended to \( X \) by \( f_n \).

**Lemma 4.14.** We have
\[
\sup_{f \in F} |\rho_n^f - \rho_f| \to 0 \quad (4.34)
\]
as \( n \to \infty \).

**Proof.** Fix any compact set \( K \subset X \). For any \( t \geq 1 \) and \( y \in K \), we have
\[
\sup_{f \in F} |\rho_n^f - \rho_f| = \sup_{f \in F} \left| \int_X b_n(x, f(x))\mu_f^t(dx) - \int_X c(x, f(x))\mu_f(dx) \right|
\leq \sup_{f \in F} \left| \int_X b_n(x, f(x))\mu_f^t(dx) - \int_X b_n(x, f(x))q_n(dy)f(y) \right|
\]
\[
20 X
The result follows from Lemma 4.12. □

**Theorem 4.15.** The value function of \( \text{MDP}_n \) converges to the value function of the original MDP, i.e.,

\[ |\bar{V}_n^* - V^*| \to 0 \]

as \( n \to \infty \).

**Proof.** Since

\[ |\bar{V}_n^* - V^*| = \left| \inf_{f \in \mathcal{F}} \rho_f^n - \inf_{f \in \mathcal{F}} \rho_f \right| \leq \sup_{f \in \mathcal{F}} |\rho_f^n - \rho_f|, \]

the result follows from Lemma 4.14 □

The following is the main result of this section which states that the true average cost of the policies \( f_n \) obtained from finite state approximations of \( \text{c-MDP}_n \) converges to the average value function \( V^* \) of the original MDP.

**Theorem 4.16.** We have

\[ |\rho_{f_n} - V^*| \to 0 \]

as \( n \to \infty \).

**Proof.** We have

\[ |\rho_{f_n} - V^*| \leq |\rho_{f_n} - \rho_{f_n}^n| + |\rho_{f_n}^n - \bar{V}_n^*| + |\bar{V}_n^* - V^*| \]

\[ \leq \sup_{f \in \mathcal{F}} |\rho_f - \rho_f^n| + \varepsilon_n + |\bar{V}_n^* - V^*| \quad \text{(by (4.33))} \]

The result follows from Lemma 4.14 and Theorem 4.15. □

**5. Discretization of the Action Space.** For computing near optimal policies using well known algorithms, such as value iteration, policy iteration, and Q-learning, the action space must be finite. In this section, we show that, as a pre-processing step, the action space can be taken to be finite if it has sufficiently large number of points for accurate approximation.

It was shown in [25] that any MDP with (infinite) compact action space and with bounded one-stage cost function can be well approximated by an MDP with finite action space under assumptions that are satisfied by \( \text{c-MDP}_n \) for each \( n \), for both the discounted cost and the average cost cases. Specifically, let \( d_A \) denote the metric on \( A \). Since \( A \) is compact, one can find a sequence of finite subsets \( \{A_k\} \) of \( A \) such that for all \( k \)

\[ \min_{\hat{a} \in A_k} d_A(a, \hat{a}) < 1/k \text{ for all } a \in A. \]
We define $c$-MDP$_{n,k}$ as the Markov decision process having the components $\{X_n, \Lambda_k, p_n, c_n\}$ and we let $F_n(\Lambda_k)$ denote the set of all deterministic stationary policies for $c$-MDP$_{n,k}$. Note that $F_n(\Lambda_k)$ is the set of policies in $F_n$ taking values only in $\Lambda_k$. Therefore, in a sense, $c$-MDP$_{n,k}$ and $c$-MDP$_n$ can be viewed as the same MDP, where the former has constraints on the set of policies. For each $n$ and $k$, by an abuse of notation, let $f_n^*$ and $f_{n,k}^*$ denote the optimal stationary policies of $c$-MDP$_n$ and $c$-MDP$_{n,k}$, respectively, for both the discounted and average costs. Then [25, Theorems 3.1 and 3.2] show that for all $n$, we have

$$
\lim_{k \to \infty} J_n(f_{n,k}^*, x) = J_n(f_n^*, x) := J_n^*(x)
$$

$$
\lim_{k \to \infty} V_n(f_{n,k}^*, x) = V_n(f_n^*, x) := V_n^*(x)
$$

for all $x \in X_n$. In other words, the discounted and average value functions of $c$-MDP$_{n,k}$ converge to the discounted and average value functions of $c$-MDP$_n$ as $k \to \infty$.

**Theorem 5.1.** Suppose that Assumption 3 holds for the discounted cost and Assumption 4 holds for the average cost. Then for any $x \in X$, there exists a subsequence $\{k_n\}$ such that

$$
\lim_{n \to \infty} J(f_{n,k_n}^*, x) = J^*(x)
$$

$$
\lim_{n \to \infty} V(f_{n,k_n}^*, x) = V^*(x),
$$

where $f_{n,k_n}^* \in F(\Lambda_{k_n})$ is the optimal stationary policy of $c$-MDP$_{n,k_n}$.

**Proof.** Let us fix $x \in X$. For $n$ sufficiently large (so $x \in K_n$), we choose $k_n$ such that $|J_n(f_{n,k_n}^*, x) - J_n(f_n^*, x)| < 1/n$ (or $|V_n(f_{n,k_n}^*, x) - V_n(f_n^*, x)| < 1/n$ for the average cost). We note that if $A$ is a compact subset of a finite dimensional Euclidean space, then by using [25, Theorems 4.1 and 4.2] one can obtain an explicit expression for $k_n$ in terms of $n$ under further continuity conditions on $c$ and $p$. By Lemmas 4.7 and 4.14, we have $|J_n(f_{n,k_n}^*, x) - J_n(f_n^*, x)| \to 0$ and $|V_n(f_{n,k_n}^*, x) - V_n(f_n^*, x)| \to 0$ as $n \to \infty$, where again by an abuse of notation, the policies extended to $X$ are also denoted by $f_{n,k_n}^*$. Since $J_n(f_{n,k_n}^*, x) = J_n(f_n^*, x)$ and $V_n(f_{n,k_n}^*, x) = V_n(f_n^*, x)$, using Theorems 4.5 and 4.15 one can immediately obtain

$$
\lim_{n \to \infty} J(f_{n,k_n}^*, x) = J^*(x)
$$

$$
\lim_{n \to \infty} V(f_{n,k_n}^*, x) = V^*(x).
$$

Theorem 5.1 implies that before discretizing the state space to compute the near optimal policies, one can discretize, without loss of generality, the action space $A$ in advance on a finite grid using sufficiently large number of grid points.

6. Rate of Convergence Analysis for Compact-State MDPs. In this section we consider (Q2) for MDPs with compact state space; that is, we derive an upper bound on the performance loss due to discretization in terms of the cardinality of the set $Z_n$ (i.e., number of grid points). To do this, we will impose some new assumptions on the components of the MDP in addition to Assumptions 1 and 2. First, we present some definitions that are needed in the development.

For each $g \in C_b(Z)$, let

$$
\|g\|_{\text{Lip}} := \sup_{(z,y) \in Z \times Z} \frac{|g(z) - g(y)|}{d_Z(z, y)}
$$
If \( \|g\|_{\text{Lip}} \) is finite, then \( g \) is called Lipschitz continuous with Lipschitz constant \( \|g\|_{\text{Lip}} \). \( \text{Lip}(Z) \) denotes the set of all Lipschitz continuous functions on \( Z \), i.e.,

\[
\text{Lip}(Z) := \{ g \in C_b(Z) : \|g\|_{\text{Lip}} < \infty \}
\]

and \( \text{Lip}(Z, K) \) denotes the set of all \( g \in \text{Lip}(Z) \) with \( \|g\|_{\text{Lip}} \leq K \). The Wasserstein distance of order 1 [28, p. 95] between two probability measures \( \zeta \) and \( \xi \) over \( Z \) is defined as

\[
W_1(\zeta, \xi) := \sup \left\{ \left| \int_Z gd\zeta - \int_Z gd\xi \right| : g \in \text{Lip}(Z, 1) \right\}.
\]

\( W_1 \) is also called the Kantorovich-Rubinstein distance. It is known that if \( Z \) is compact, then \( W_1(\zeta, \xi) \leq \text{diam}(Z) \|\zeta - \xi\|_{TV} \) [28, Theorem 6.13]. For compact \( Z \), the Wasserstein distance of order 1 is weaker than total variation distance. Furthermore, for compact \( Z \), the Wasserstein distance of order 1 metrizes the weak topology on the set of probability measures \( P(Z) \) [28, Corollary 6.11] which also implies that convergence in this sense is weaker than setwise convergence.

In this section we impose the following supplementary assumptions in addition to Assumption 1 and Assumption 2.

**Assumption 6.**

- (g) The one-stage cost function \( c \) satisfies \( c(\cdot, a) \in \text{Lip}(Z, K_1) \) for all \( a \in A \) for some \( K_1 \).
- (h) The stochastic kernel \( p \) satisfies \( W_1(p(\cdot | z, a), p(\cdot | y, a)) \leq K_2d_Z(z, y) \) for all \( a \in A \) for some \( K_2 \).
- (h') The stochastic kernel \( p \) satisfies: \( \|p(\cdot | z, a) - p(\cdot | y, a)\|_{TV} \leq K_2d_Z(z, y) \) for all \( a \in A \) and for some \( K_2 \).
- (j) \( Z \) is an infinite compact subset of \( \mathbb{R}^d \) for some \( d \geq 1 \), equipped with the Euclidean norm.

We note that Assumption 6-(j) implies the existence of a constant \( \alpha > 0 \) and finite subsets \( Z_n \subset Z \) with cardinality \( n \) such that

\[
\max_{z \in Z} \min_{y \in Z_n} d_Z(z, y) \leq \alpha(1/n)^{1/d}
\]

for all \( n \), where \( d_Z \) is the Euclidean distance on \( Z \). In the remainder of this section, we replace \( Z_n \) defined in Section 3 with \( Z_n \) satisfying (6.1) in order to derive explicit bounds on the approximation error in terms of the cardinality of \( Z_n \).

**6.1. Discounted Cost.** Assumptions 1 and 6 (without Assumption 6-(h')) are imposed throughout this section. Additionally, we assume that \( K_2\beta < 1 \). The last assumption is the key to prove the next result which states that the value function \( J^* \) of the original MDP for the discounted cost is in \( \text{Lip}(Z) \). Although this result is known in the literature [18], we give a short proof for the sake of completeness using a simple application of the value iteration algorithm.

**Theorem 6.1.** Suppose Assumptions 1, 6 (without Assumption 6-(h')) and \( K_2\beta < 1 \) hold. Then the value function \( J^* \) for the discounted cost is in \( \text{Lip}(Z, K) \), where \( K = K_1 \frac{1}{1 - \beta K_2} \).

**Proof.** Let \( u \in \text{Lip}(Z, K) \) for some \( K > 1 \). Then \( g = \frac{u}{K} \in \text{Lip}(Z, 1) \) and therefore, for all \( a \in A \) and \( z, y \in Z \) we have

\[
\left| \int_Z u(x)p(dx|z, a) - \int_Z u(x)p(dx|y, a) \right| = K \left| \int_Z g(x)p(dx|z, a) - \int_Z g(x)p(dx|y, a) \right|
\]
by Assumption 6-(h). Hence, the contraction operator $T$ defined in (3.2) maps $u \in \text{Lip}(Z, K)$ to $Tu \in \text{Lip}(Z, K_1 + \beta KK_2)$, since, for all $z, y \in Z$

$$|Tu(z) - Tu(y)| \leq \max_{a \in A} \left\{ |c(z, a) - c(y, a)| + \beta \left| \int_Z u(x) p(dx|z, a) - \int_Z u(x) p(dx|y, a) \right| \right\} \leq K_1 d_Z(z, y) + \beta KK_2 d_Z(z, y) = (K_1 + \beta KK_2) d_Z(z, y).$$

Now we apply $T$ recursively to obtain the sequence $\{T^n u\}$ by letting $T^n u = T(T^{n-1} u)$, which converges to the value function $J^*$ by the Banach fixed point theorem. Clearly, by induction we have for all $n \geq 1$

$$T^n u \in \text{Lip}(Z, K_n),$$

where $K_n = K_1 \sum_{i=0}^{n-1} (\beta K_2)^i + K(\beta K_2)^n$. If we choose $K < K_1$, then $K_n \leq K_{n+1}$ for all $n$ and therefore, $K_n \uparrow K_1 \frac{1}{1-\beta K_2}$ since $K_2 \beta < 1$. Hence, $T^n u \in \text{Lip}(Z, K_1 \frac{1}{1-\beta K_2})$ for all $n$, and therefore, $J^* \in \text{Lip}(Z, K_1 \frac{1}{1-\beta K_2})$ since Lip$(Z, K_1 \frac{1}{1-\beta K_2})$ is closed with respect to the sup-norm $\| \cdot \|$.

The following theorem is the main result of this section. Recall that the policy $\hat{\pi}_n \in \mathcal{P}$ is obtained by extending the optimal policy $\hat{f}_n^*$ of MDP to $Z$.

**Theorem 6.2.** We have

$$\|J(\hat{f}_n, \cdot) - J^*\| \leq \frac{\tau(\beta, K_2) K_1 \frac{1}{1-\beta K_2} + 2K_1}{1-\beta} \alpha(1/n)^{1/d},$$

where $\tau(\beta, K_2) = (2+\beta)\beta K_2 + \frac{\beta^2 + \beta + 2}{(1-\beta)^2}$ and $\alpha$ is the coefficient in (6.1).

**Proof.** To prove the theorem, we obtain upper bounds on the expressions derived in Section 3.1 in terms of the cardinality $n$ of $Z_n$. The proof of Theorem 3.2 gives

$$\|J(\hat{f}_n, \cdot) - J^*\| \leq \frac{\|T_{f_n} J^* - \hat{T}_{f_n} J^*\| + (1+\beta)\|\hat{f}_n^* - J^*\|}{1-\beta}.$$

To prove the theorem we upper bound $\|T_{f_n} J^* - \hat{T}_{f_n} J^*\|$ and $\|\hat{f}_n^* - J^*\|$ in terms $n$. For the first term we have

$$\|T_{f_n} J^* - \hat{T}_{f_n} J^*\| = \sup_{z \in Z} |T_{f_n} J^*(z) - \hat{T}_{f_n} J^*(z)| \leq \sup_{z \in Z} \int_Z \left| c(z, \hat{f}_n(z)) + \beta \int_Z J^*(y)p(dy|z, \hat{f}_n(z)) - c(x, \hat{f}_n(x)) \right| \nu_{n,i_n(z)}(dx)$$

$$- \beta \int_Z J^*(y)p(dy|x, \hat{f}_n(x)) \nu_{n,i_n(z)}(dx) \leq \sup_{z \in Z} \left( K_1 d_Z(x, z) + \beta \left| \int_Z J^*(y)p(dy|z, \hat{f}_n(z)) - \int_Z J^*(y)p(dy|x, \hat{f}_n(z)) \right| \nu_{n,i_n(z)}(dx) \right)$$

(since $\hat{f}_n(x) = \hat{f}_n(z)$ for all $x \in S_{n,i_n(z)}$)

$$\leq \sup_{z \in Z} \int_Z (K_1 + \beta \|J^*\|_{\text{Lip}} K_2) d_Z(x, z) \nu_{n,i_n(z)}(dx)$$
\[ \leq (K_1 + \beta \| J^* \|_{\text{Lip}}) \max_{i \in \{1, \ldots, n\}} \text{diam}(S_{n,i}) \]
\[ \leq (K_1 + \beta \| J^* \|_{\text{Lip}}) 2\alpha (1/n)^{1/d}. \]

For the second term, the proof of Theorem 3.7 gives
\[ \| J^* - u_{n}^* \| \leq \| \hat{T}_n J^* - F_n J^* \| + (1 + \beta) \| J^* - u_{n}^* \|. \]

First consider \( \| \hat{T}_n J^* - F_n J^* \| \). Define
\[ l(z, a) := c(z, a) + \beta \int X J^*(y)p(dy|z, a), \]
so that
\[ J^*(z) = \min_{a \in A} l(z, a). \]

It is straightforward to show that \( l(\cdot, a) \in \text{Lip}(Z, K_i) \) for all \( a \in A \), where \( K_1 = K_1 + \beta \| J^* \|_{\text{Lip}} K_2 \). By adapting the proof of Lemma 3.6 to the value function \( J^* \), we obtain
\[ \| \hat{T}_n J^* - F_n J^* \| = \sup_{z \in Z} \left| \min_{a \in A} \int l(x, a)\nu_{n,i,n}(z)(dx) - \int \min_{a \in A} l(x, a)\nu_{n,i,n}(z)(dx) \right| \]
\[ \leq \sup_{z \in Z} \int \sup_{y \in S_{n,i}} \left| l(y, a_i) - J^*(y) \right| \nu_{n,i,n}(z)(dy) \]
\[ \leq \max_{i \in \{1, \ldots, n\}} \int \sup_{y \in S_{n,i}} \{ |l(y, a_i) - l(z_i, a_i)| + |J^*(z_i) - J^*(y)| \} \nu_{n,i}(dy) \]
\[ \leq (K_1 + \| J^* \|_{\text{Lip}}) \max_{i \in \{1, \ldots, n\}} \text{diam}(S_{n,i}) \]
\[ \leq (K_1 + \| J^* \|_{\text{Lip}}) 2\alpha (1/n)^{1/d}. \]

For the expression \( \| J^* - u_{n}^* \| \), by Lemma 3.5 we have
\[ \| u_{n}^* - J^* \| \leq \frac{2}{1 - \beta} \inf_{r \in Z^n} \| J^* - \Phi_r \|, \]
where \( \Phi_r(z) = \Sigma_{i=1}^{k} r_i S_{n,i}(z), r = (r_1, \ldots, r_{k}). \)

Since \( \| J^* \|_{\text{Lip}} < \infty \), we have \( \inf_{r \in Z^n} \| J^* - \Phi_r \| \leq \| J^* \|_{\text{Lip}} \max_{i \in \{1, \ldots, n\}} \text{diam}(S_{n,i}) \leq \| J^* \|_{\text{Lip}} 2\alpha (1/n)^{1/d}. \) Hence
\[ \| u_{n}^* - J^* \| \leq \frac{2}{1 - \beta} \| J^* \|_{\text{Lip}} 2\alpha (1/n)^{1/d}. \]

Hence, by (6.3) and (6.4) we obtain
\[ \| \hat{J}_n^* - J^* \| \leq \left( (\beta K_2 + \frac{\beta + 3}{(1 - \beta)^2}) \| J^* \|_{\text{Lip}} + \frac{K_1}{1 - \beta} \right) 2\alpha (1/n)^{1/d}. \]

Then, the result follows from (6.2) and (6.5), and the fact \( \| J^* \|_{\text{Lip}} \leq K_1 \frac{1}{1 - \beta K_2}. \)
Remark. It is important to point out that if we replace Assumption 6-(h) with (h'), then Theorem 6.2 remains valid (with possibly different constants in front of the term \((1/n)^{1/d}\)). However, in this case, we do not need the assumptions \(K_2\beta < 1\).

6.2. Average Cost. In this section, we suppose that Assumptions 2 and 6 (without Assumption 6-(h)) hold. To obtain convergence rates for the average cost, we first prove a rate of convergence result for Lemma 3.12.

**Lemma 6.3.** For all \(t \geq 1\), we have

\[
\sup_{(y,f) \in \mathbb{Z} \times \mathbb{F}} \|p^t(\cdot | y, f(y)) - q^t_n(\cdot | y, f(y))\|_{TV} \leq K_2 \alpha (1/n)^{1/d} (2^{t+1} - 2),
\]

where \(\alpha\) is the coefficient in (6.1).

**Proof.** Similar to the proof of Lemma 3.12, we use induction. For \(t = 1\), recalling the proof of Lemma 3.12, the claim holds by the following argument:

\[
\sup_{(y,f) \in \mathbb{Z} \times \mathbb{F}} \|p(\cdot | y, f(y)) - q_n(\cdot | y, f(y))\|_{TV} \leq \sup_{y \in \mathbb{Z}} \sup_{x \in S_{n,i}(y)} \|p(\cdot | x, a) - p(\cdot | x, a)\|_{TV}
\]

\[
\leq K_2 \sup_{y \in \mathbb{Z}} \sup_{x \in S_{n,i}(y)} d_Z(x, y)
\]

\[
\leq K_2 \max_{i \in \{1, \ldots, n\}} \text{diam}(S_{n,i})
\]

\[
\leq 2K_2 \alpha (1/n)^{1/d}.
\]

Now, assume the claim is true for \(t \geq 1\). Again recalling the proof of Lemma 3.12, we have

\[
\sup_{(y,f) \in \mathbb{Z} \times \mathbb{F}} \|p^{t+1}(\cdot | y, f(y)) - q_n^{t+1}(\cdot | y, f(y))\|_{TV}
\]

\[
\leq 2 \sup_{(y,f) \in \mathbb{Z} \times \mathbb{F}} \|p^t(\cdot | y, f(y)) - q_n^t(\cdot | y, f(y))\|_{TV}
\]

\[
+ \sup_{(z,f) \in \mathbb{Z} \times \mathbb{F}} \|p(\cdot | z, f(z)) - q_n(\cdot | z, f(z))\|_{TV}
\]

\[
\leq 2K_2 \alpha (1/n)^{1/d} \sum_{i=1}^t 2^i + 2K_2 \alpha (1/n)^{1/d} = 2K_2 \alpha (1/n)^{1/d} \sum_{i=1}^{t+1} 2^i.
\]

This completes the proof. \(\square\)

The following theorem is the main result of this section. A somewhat similar result was obtained in [14, Section 3.5], where identical assumptions are imposed for both the original model and the approximating model (i.e. [14, Assumption 5.1]). Moreover, the approximating transition probability and one-stage cost function are assumed to converge to the original transition probability and one-stage cost function with respect to some rate; that is, \(\rho(n) := \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |b_n(x, a) - c(x, a)|\) and \(\pi(n) := \sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} \|q_n(\cdot | x, a) - p(\cdot | x, a)\|_{TV}\), where \(\rho(n), \pi(n) \to \infty\) as \(n \to \infty\). Although our result can be seen as a special case of the results in [14, Section 3.5], there are several differences: (i) we have an exact expression for the functions \(\rho(n)\) and \(\pi(n)\), so our rate of convergence result gives an explicit bounds on the approximation error, (ii) our assumptions are only imposed for the the original model, and (iii) in [14, Section 3.5] approximating models do not have finite state space while our approximating models are obtained by extending finite state models to the original state space, thereby, allowing for a constructive numerical method to calculate near optimal policies.
Recall that the policy \( \hat{f}_n \), the optimal policy of \( \hat{MDP}_n \), is obtained by extending the optimal policy \( f_n^* \) of MDP\(_n\) to \( \mathbb{Z} \).

**THEOREM 6.4.** For all \( t \geq 1 \), we have

\[
|\rho_{f_n} - \rho_f| \leq 4\|c\|R\kappa^t + 4K_1\alpha(1/n)^{1/d} + 2\|c\|K_2\alpha(1/n)^{1/d}(2^{t+1} - 2).
\]

**Proof.** The proof of Theorem 3.11 gives

\[
|\rho_{f_n} - \rho_f| \leq |\rho_{f_n} - \hat{\rho}_{f_n}^n| + |\hat{\rho}_{f_n}^n - \hat{\rho}_{f_n}^n| + |\hat{\rho}_{f_n}^n - \rho_f|.
\]

Hence, to prove the theorem we obtain an upper bounds for \( |\rho_{f_n} - \hat{\rho}_{f_n}^n| \), \( |\hat{\rho}_{f_n}^n - \hat{\rho}_{f_n}^n| \) and \( |\hat{\rho}_{f_n}^n - \rho_f| \) in terms of \( n \). Consider the first term (recall the proof of Lemma 3.13)

\[
|\rho_{f_n} - \hat{\rho}_{f_n}^n| \leq \sup_{f \in \mathcal{F}} |\hat{\rho}_f^n - \rho_f| \leq 2R\kappa\|c\| + \|c\| \sup_{(y, f) \in \mathcal{Z} \times \mathcal{F}} \|\nu_n^t(\cdot | y, f(y)) - \nu^t(\cdot | y, f(y))\|_{TV}
\]

\[
\leq 2R\kappa\|c\| + \|c\|K_2\alpha(1/n)^{1/d} \sum_{i=1}^t 2^i \text{ (by Lemma 6.3).} \quad (6.6)
\]

For the second term, the proof of Lemma 3.17 gives

\[
|\hat{\rho}_{f_n}^n - \hat{\rho}_{f_n}^n| \leq \sup_{f \in \mathcal{F}} |\hat{\rho}_f^n - \rho_f^n| \leq 2\|b_n - c\| \text{ (see the proof of Lemma 3.15)}
\]

\[
\leq 2 \sup_{(x, a) \in \mathcal{Z} \times \mathcal{A}} \int |c(x, a) - c(z, a)|\nu_{n,i_n}(z)(dx)
\]

\[
\leq 2\sup_{z \in \mathcal{Z}} \int K_1d_2(x, z)\nu_{n,i_n}(z)(dx)
\]

\[
\leq 2K_1 \max_{i \in \{1, \ldots, n\}} \text{diam}(\mathcal{S}_{n,i})
\]

\[
\leq 4K_1\alpha(1/n)^{1/d}. \quad (6.7)
\]

For the last term, we have

\[
|\hat{\rho}_{f_n}^n - \rho_f| = |\inf_{f \in \mathcal{F}} \hat{\rho}_f^n - \inf_{f \in \mathcal{F}} \rho_f|
\]

\[
\leq \sup_{f \in \mathcal{F}} |\hat{\rho}_f^n - \rho_f|
\]

\[
\leq 2R\kappa\|c\| + \|c\|K_2\alpha(1/n)^{1/d} \sum_{i=1}^t 2^i \text{ (by (6.6)).} \quad (6.8)
\]

Hence, for any \( t \geq 1 \), by combining (6.6), (6.7), and (6.8) we obtain the result. \( \square \)

To obtain a proper rate of convergence result (i.e., an upper bound that only depends on \( n \)) the \( t \) term in the upper bound in Theorem 6.4 has to be written as a
function of \( n \). This can be done by (approximately) minimizing the upper bound in Theorem 6.4 with respect to \( t \) for each \( n \). Let us define the constants \( I_1 := 4\|c\|R \), \( I_2 := 4K_1\alpha \), and \( I_3 := 2\|c\|K_2\alpha \), and therefore, the upper bound in Theorem 6.4 becomes

\[
I_1\kappa^t + I_2(1/n)^{1/d} + I_3(1/n)^{1/d}(2^{t+1} - 2). \tag{6.9}
\]

For each \( n \), it is straightforward to compute that the real number

\[
t'(n) := \ln\left(\frac{n^{1/d}}{I_4}\right) - \frac{1}{\ln(z)}
\]

makes the derivative of (6.9) zero, where \( I_4 := \left(\frac{t}{2I_3\ln(\frac{z}{4})}\right)^{-1} \). Taking \([t'(n)]\) for \( t \) in (6.9), we obtain the following result.

**Corollary 6.5.** We have

\[
|\rho_{f^n} - \rho_{f^t}| \leq (I_1I_4^q + 4I_3I_4^q)^{-} (1/n)^{q/t/d} + (I_2 - I_3)(1/n)^{1/d},
\]

where \( q_1 := -\frac{\ln(z)}{\ln(2/\kappa)} \).

### 7. Order Optimality for Approximation Errors in the Rate of Quantization

The following example demonstrates that the order of the performance losses in Theorems 6.2 and 6.4 cannot be better than \( O\left(\frac{1}{n}^{2}\right) \). More precisely, we exhibit a simple standard example where we can lower bound the performance loss by \( L(1/n)^{1/d} \), for some positive constant \( L \). A similar result was obtained in [25, Section IV] for the case of quantization of action space, where the action space was a compact subset of \( \mathbb{R}^m \) for some \( m \geq 1 \). Therefore, when both state and action spaces are quantized, then the resulting construction is order optimal in the above sense as the approximation error, in this case, is bounded by the sum of the approximation errors in quantization of state space and quantization of action space.

In what follows \( h(\cdot) \) and \( h(\cdot;\cdot) \) denote differential and conditional differential entropies, respectively [7].

Consider the additive-noise system:

\[
z_{t+1} = F(z_t, a_t) + v_t, \quad t = 0, 1, 2, \ldots,
\]

where \( z_t, a_t, v_t \in \mathbb{R}^d \). We assume that \( \sup_{(z,a)\in\mathbb{R}^d \times \mathbb{R}^d} \|F(z,a)\| < 1 \). The noise process \( \{v_t\} \) is a sequence of i.i.d. random vectors whose common distribution has density \( g \) supported on some compact subset \( V \) of \( \mathbb{R}^d \). We choose \( V \) such that \( Z = A \) can be taken to be compact subsets of \( \mathbb{R}^d \). For simplicity suppose that the initial distribution \( \mu \) has the same density \( g \). It is assumed that the differential entropy \( h(g) := -\int g(z) \log g(z) dz \) is finite. Let the one stage cost function be \( c(z, a) := \|z - a\| \). Clearly, the optimal stationary policy \( f^* \) is induced by the identity \( f^*(z) = z \), having the optimal cost \( J(f^*, \mu) = 0 \) and \( V(f^*, \mu) = 0 \). Let \( f_n \) be the piece-wise constant extension of the optimal policy \( f_n^* \) of the MDP \( n \) to the set \( Z \). Fix \( n \geq 1 \) and define \( D_t := E_\mu^n[c(z_t, a_t)] \) for all \( t \). Then, since \( a_t = f_n(z_t) \) can take at most \( n \) values in \( A \), by the Shannon lower bound (SLB) [30, p. 12] we have for \( t \geq 1 \)

\[
\log n \geq R(D_t) \geq h(z_t) + \theta(D_t) \\
= h(F(z_{t-1}, a_{t-1}) + v_{t-1}) + \theta(D_t)
\]

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$\geq h(F(z_t-1, a_t-1) + v_t-1 | z_t-1, a_t-1) + \theta(D_t) \tag{7.1}$

$= h(v_t-1) + \theta(D_t) \tag{7.2}$

where $\theta(D_t) = -d + \log \left( \frac{1}{\sqrt{V_d(d)(dD_t)^d}} \right)$, $R(D_t)$ is the rate-distortion function of $z_t$, $V_d$ is the volume of the unit sphere $S_d = \{ z : \|z\| \leq 1 \}$, and $\Gamma$ is the gamma function. Here, (7.1) follows from the fact that conditioning reduces the entropy $[7, \text{Theorem 2.6.5}]$ and (7.2) follows from the independence of $v_{t-1}$ and the pair $(z_{t-1}, a_{t-1})$. Note that $h(v_{t-1}) = h(g)$ for all $t$. Thus, $D_t \geq L(1/n)^{1/d}$, where $L := \frac{d}{2} \left( \frac{2^{h(g)}}{\sqrt{V_d(d)}} \right)^{1/d}$. Since we have obtained stage-wise error bounds, these give $|J(f^*, \mu) - J(\hat{f}_n, \mu)| \geq \frac{1}{1-\beta}(1/n)^{1/d}$ and $|V(f^*, \mu) - V(\hat{f}_n, \mu)| \geq L(1/n)^{1/d}$.

8. Numerical Examples. In this section, we consider two examples, the additive noise model and fisheries management problem, in order to illustrate our results numerically. Since computing true costs of the policies obtained from the finite models is intractable, we only compute the value functions of the finite models and illustrate their convergence to the value function of the original MDP as $n \to \infty$.

Before proceeding to the examples, we note that all results in this paper apply with straightforward modifications for the case of maximizing reward instead of minimizing cost.

8.1. Additive Noise System. In this example, the additive noise system is given by

$x_{t+1} = F(x_t, a_t) + v_t, \quad t = 0, 1, 2, \ldots$

where $x_t, a_t, v_t \in \mathbb{R}$ and $X = \mathbb{R}$. The noise process $\{v_t\}$ is a sequence of $\mathbb{R}$-valued i.i.d. random variables with common density $g$. Hence, the transition probability $p(\cdot | x, a)$ is given by

$p(D|x, a) = \int_D g(v - F(x,a))m(dv) \quad \text{for all } D \in \mathcal{B}(\mathbb{R}),$

where $m$ is the Lebesgue measure. The one-stage cost function is $c(x, a) = (x - a)^2$, the action space is $A = [-L, L]$ for some $L > 0$, and the cost function to be minimized is the discounted cost.

We assume that (i) $g$ is a Gaussian probability density function with zero mean and variance $\sigma^2$, (ii) $\sup_{a \in A} |F(x, a)|^2 \leq k_1 x^2 + k_2$ for some $k_1, k_2 \in \mathbb{R}_+$, (ii) $\beta < 1/\alpha$ for some $\alpha \geq k_1$, and (iv) $F$ is continuous. Hence, Assumption 3 holds for this model with $w(x) = k + x^2$ and $M = 4 \left( \frac{L^2}{\beta} + x^2 \right)$, for some $k \in \mathbb{R}_+$.

For the numerical results, we use the following parameters: $F(x, a) = x + a$, $\beta = 0.3$, $L = 0.5$, and $\sigma = 0.1$.

We selected a sequence $\{[-l_n, l_n]\}_{n=1}^{15}$ of nested closed intervals, where $l_n = 0.5 + 0.25n$, to approximate $\mathbb{R}$. Each interval is uniformly discretized using $[2k_l \mathbb{Z}]$ grid points, where $k_m = 5m$ for $m = 1, \ldots, 5$ and $[q]$ denotes the smallest integer greater than or equal to $q \in \mathbb{R}$. Therefore, the discretization is gradually refined. For each $n$, the finite state space is given by $\{x_{n,i}\}_{i=1}^{k_n} \cup \{\Delta_n\}$, where $\{x_{n,i}\}_{i=1}^{k_n}$ are the representation points in the uniform quantization of the closed interval $[-l_n, l_n]$ and $\Delta_n$ is a pseudo state. We also uniformly discretize the action space $A = [-0.5, 0.5]$ by using $2[\frac{L}{\beta}]$ grid points. For each $n$, the finite state models are constructed as
in Section 3 by replacing $Z$ with $[-l_n, l_n]$ and by setting $\nu(\cdot) = \frac{1}{2}m_n(\cdot) + \frac{1}{2}\delta_{\Delta_n}(\cdot)$, where $m_n$ is the Lebesgue measure normalized over $[-l_n, l_n]$.

We use the value iteration algorithm to compute the value functions of the finite models. Figure 8.1 displays the graph of these value functions corresponding to the different values for the number of grid points, when the initial state is $x = 0.7$. The figure illustrates that the value functions of the finite models converge to the value function of the original model.

![Figure 8.1. Optimal costs of the finite models when the initial state is $x = 0.7$](image)

### 8.2. Fisheries Management Problem.

In this example we consider the following population growth model, called a Ricker model, see [15, Section 1.3]:

$$x_{t+1} = \theta_1 a_t \exp\{-\theta_2 a_t + v_t\}, \quad t = 0, 1, 2, \ldots$$  \hspace{1cm} (8.1)

where $\theta_1, \theta_2 \in \mathbb{R}_+$, $x_t$ is the population size in season $t$, and $a_t$ is the population to be left for spawning for the next season, or in other words, $x_t - a_t$ is the amount of fish captured in the season $t$. The one-stage ‘reward’ function is $u(x_t - a_t)$, where $u$ is some utility function. In this model, the goal is to maximize the average reward.

The state and action spaces are $X = A = [\kappa_{\text{min}}, \kappa_{\text{max}}]$, for some $\kappa_{\text{min}}, \kappa_{\text{max}} \in \mathbb{R}_+$. Since the population left for spawning cannot be greater than the total population, for each $x \in X$, the set of admissible actions is $A(x) = [\kappa_{\text{min}}, x]$ which is not consistent with our assumptions. However, we can (equivalently) reformulate above problem so that the admissible actions $A(x)$ will become $A$ for all $x \in X$. In this case, instead of dynamics in equation (8.1) we have

$$x_{t+1} = \theta_1 \min(a_t, x_t) \exp\{-\theta_2 \min(a_t, x_t) + v_t\}, \quad t = 0, 1, 2, \ldots$$

and $A(x) = [\kappa_{\text{min}}, \kappa_{\text{max}}]$ for all $x \in X$. The one-stage reward function is $u(x_t - a_t)1\{x_t \geq a_t\}$.

Since $X$ is already compact, it is sufficient to discretize $[\kappa_{\text{min}}, \kappa_{\text{max}}]$. The noise process $\{v_t\}$ is a sequence of independent and identically distributed (i.i.d.) random variables which have common density $g$ supported on $[0, \lambda]$. Therefore, the transition probability $p(\cdot | x, a)$ is given by

$$p(D| x, a) = \Pr\left\{x_{t+1} \in D \left| x_t = x, a_t = a \right. \right\}$$
\[
= \Pr \left\{ \theta_1 \min(a, x) \exp(-\theta_2 \min(a, x) + v) \in D \right\}
= \int_D g \left( \log(v) - \log(\theta_1 \min(a, x)) + \theta_2 \min(a, x) \right) \frac{1}{\nu} m(dv),
\]
for all \( D \in \mathcal{B}(\mathbb{R}) \). To make the model consistent, we must have \( \theta_1 y \exp(-\theta_2 y + v) \in [\kappa_{\text{min}}, \kappa_{\text{max}}] \) for all \((y, v) \in [\kappa_{\text{min}}, \kappa_{\text{max}}] \times [0, \lambda] \).

We assume that (i) \( g > \epsilon \) for some \( \epsilon \in \mathbb{R}_+ \) on \([0, \lambda] \), (ii) \( g \) is continuous on \([0, \lambda] \), and (iii) the utility function \( u \) is continuous. Define \( h(v, x, a) := g(\log(v) - \log(\theta_1 \min(a, x)) + \theta_2 \min(a, x)) \frac{1}{\nu} \), and for each \((x, a) \in X \times A\), let \( S_{x,a} \) denote the support of \( h(\cdot, x, a) \). Then, Assumption 2 holds for this model with \( \theta(x, a) = \inf_{v \in S_{x,a}} h(v, x, a) \) (provided that it is measurable), \( \zeta = m_\kappa \) (Lebesgue measure restricted on \([\kappa_{\text{min}}, \kappa_{\text{max}}] \)), and for some \( \lambda \in (0, 1) \).

For the numerical results, we use the following values of the parameters:
\[
\theta_1 = 1.1, \quad \theta_2 = 0.1, \quad \kappa_{\text{max}} = 7, \quad \kappa_{\text{min}} = 0.005, \quad \lambda = 0.5.
\]
We assume that the noise process is distributed uniformly over \([0, 0.5] \). Hence, \( g \equiv 1 \) on \([0, 0.5] \) and otherwise zero. The utility function \( u \) is taken to be the shifted isoelastic utility function \([8, \text{Section 4.1}]\)
\[
u(z) = 3((z + 0.5)^{1/3} - (0.5)^{1/3}).
\]
We selected 25 different values for the number \( n \) of grid points to discretize the state space: \( n = 10, 20, 30, \ldots, 250 \). The grid points are chosen uniformly over the interval \([\kappa_{\text{min}}, \kappa_{\text{max}}] \). We also uniformly discretize the action space \( A \) by using the following number of grid points: \( 5n = 50, 100, 150, \ldots, 1250 \).

We use the relative value iteration algorithm \([4, \text{Chapter 4.3.1}]\) to compute the value functions of the finite models. For each \( n \), the finite state models are constructed as in Section 3 by replacing \( Z \) with \([\kappa_{\text{min}}, \kappa_{\text{max}}] \) and by setting \( \nu(\cdot) = m_\kappa(\cdot) \).

Figure 8.2 shows the graph of the value functions of the finite models corresponding to the different values of \( n \) (number of grid points), when the initial state is \( x = 2 \). It can be seen that the value functions converge (to the value function of the original model).

9. Conclusion. The approximation of a discrete time MDP by finite-state MDPs was considered for discounted and average costs for both compact and non-compact state spaces. Under usual conditions imposed for studying Markov decision processes, it was shown that if one uses a sufficiently large number of grid points to discretize the state space, then the resulting finite-state MDP yields a near optimal policy. Under the Lipschitz continuity of the transition probability and the one-stage cost function, explicit bounds were derived on the performance loss due to discretization in terms of the number of grid points for the compact state case. These results were then illustrated numerically by considering two different MDP models.

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Figure 8.2. Optimal rewards of the finite models when the initial state is $x = 2$

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