REGULARITY RESULTS FOR FULLY NONLINEAR INTEGRO-DIFFERENTIAL OPERATORS WITH NONSYMMETRIC POSITIVE KERNELS

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Abstract. In this paper, we consider fully nonlinear integro-differential equations with possibly nonsymmetric kernels. We are able to find different versions of Alexandro-Backelman-Pucci estimate corresponding to the full class $S^0$ of uniformly elliptic nonlinear equations with $1 < \sigma < 2$ (subcritical case) and to their subclass $S^0_\eta$ with $0 < \sigma \leq 1$. We show that $S^0_\eta$ still includes a large number of nonlinear operators as well as linear operators. And we show a Harnack inequality, Hölder regularity, and $C^{1,\alpha}$-regularity of the solutions by obtaining decay estimates of their level sets in each cases.

1. Introduction

In this paper, we are going to consider the regularity of the viscosity solutions of integro-differential operators with possibly nonsymmetric kernel:

$$(1.0.1) \quad L u(x) = \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y)K(y)\,dy$$

where $\mu(u, x, y) = u(x+y) - u(x) - (V u(x) \cdot y)\chi_{B_1}(y)$, which describes the infinitesimal generator of given purely jump processes, i.e. processes without diffusion or drift part [CS]. We refer the detailed definitions of notations to [KL]. Then we see that $L u(x)$ is well-defined provided that $u \in C^{1,1}(x) \cap B(\mathbb{R}^n)$ where $B(\mathbb{R}^n)$ denotes the family of all real-valued bounded functions defined on $\mathbb{R}^n$. If $K$ is symmetric (i.e. $K(-y) = K(y)$), then an odd function $\left[ (V u(x) \cdot y)\chi_{B_1}(y) \right] K(y)$ will be canceled in the integral, and so we have that

$$(1.0.2) \quad L u(x) = \text{p.v.} \int_{\mathbb{R}^n} \left[ u(x+y) + u(x-y) - 2u(x) \right]\,K(y)\,dy.$$ 

On the other hand, if $K$ is not symmetric, the effect of $\left[ (V u(x) \cdot y)\chi_{B_1}(y) \right] K(y)$ persists and we can actually observe that the influence of this gradient term becomes stronger as we try to get an estimate in smaller regions.

Nonlinear integro-differential operators come from the stochastic control theory related with

$$I u(x) = \sup_{\alpha} L_\alpha u(x),$$

or game theory associated with

$$I u(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x),$$

when the stochastic process is of Lévy type allowing jumps; see [CS] [KL]. Also an operator like $I u(x) = \sup_{\alpha} \inf_{\beta} L_{\alpha\beta} u(x)$ can be considered. Characteristic properties of these operators can easily be derived as follows;

$$(1.0.3) \quad \inf_{\alpha\beta} L_{\alpha\beta} v(x) \leq I [u + v](x) - I u(x) \leq \sup_{\alpha\beta} L_{\alpha\beta} v(x).$$

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1.1. operators. In this section, we introduce a class of operators. All notations and the concepts of viscosity solution follows [KL] where a more general class of operators has been considered. Similar concepts can be found at [CS] for symmetric kernel.

For our purpose, we shall restrict our attention to the operators $L$ where the measure $\mu$ is given by a positive kernel $K$ which is not necessarily symmetric. That is to say, the operators $L$ are given by

\[ Lu(x) = \text{p.v.} \int_{\mathbb{R}^n} \mu(u, x, y)K(y) \, dy \]

where $\mu(u, x, y) = u(x + y) - u(x) - (Vu(x) \cdot y)\chi_E(y)$.

And we consider the class $\mathcal{L}_0$ of the operators $L$ associated with the measures $\mu$ given by positive kernels $K \in \mathcal{K}_0$ satisfying that

\[ (2 - \alpha) \frac{\lambda}{|y|^{n+\alpha}} \leq K(y) \leq (2 - \alpha) \frac{\lambda}{|y|^{n+\alpha}}, \quad 0 < \alpha < 2. \]

The maximal operator and the minimal operator with respect to a class $\mathcal{L}$ of linear integro-differential operators are defined by

\[ M_+ u(x) = \sup_{L \in \mathcal{L}} L u(x) \quad \text{and} \quad M_- u(x) = \inf_{L \in \mathcal{L}} L u(x). \]

We say that $P$ is a paraboloid of opening $M$ if $P(x) = \ell_0 + \ell(x) \pm \frac{M}{2} |x|^2$ where $M$ is a positive constant, $\ell_0$ is real constant and $\ell$ is a linear function. Then $P$ is called convex when we have $+$ in the above and concave when we have $-$ in the above. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given two semicontinuous functions $u, v$ defined on an open subset $U \subset \Omega$ and a point $x_0 \in U$, we say that $v$ touches $u$ by above at $x_0 \in U$ if $u(x_0) = v(x_0)$ and $u(x) \leq v(x)$ for any $x \in U$. Similarly, we say that $v$ touches $u$ by below at $x_0 \in U$ if $u(x_0) = v(x_0)$ and $u(x) \geq v(x)$ for any $x \in U$. For a semicontinuous function $u$ on $\Omega$ and an open subset $U \subset \Omega$, we define $\Theta^+(u, U)(x_0)$ to be the infimum of all positive constants $M$ for which there is a convex paraboloid of opening $M$ that touches $u$ by above at $x_0 \in U$. Also we define $\Theta^+(u, U)(x_0) = \infty$ if no such constant $M$ exists. Similarly, we define $\Theta^-(u, U)(x_0)$ to be the infimum of all positive constants $M$ for which there is a concave paraboloid of opening $M$ that touches $u$ by below at $x_0 \in U$, and also we define $\Theta^-(u, U)(x_0) = \infty$ if no such constant $M$ exists. Finally we set $\Theta(u, U)(x_0) = \max(\Theta^+(u, U)(x_0), \Theta^-(u, U)(x_0)) \leq \infty$. For these definitions, the readers can refer to [CC].

**Definition 1.1.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u : \Omega \rightarrow \mathbb{R}$ be a semicontinuous function. Given $x_0 \in \Omega$, we say that $u$ is $C^{1,1}_+$ at $x_0$ (resp. $C^{1,1}_-$ at $x_0$) if $\Theta^+(u, U)(x_0) < \infty$ (resp. $\Theta^-(u, U)(x_0) < \infty$) for an open neighborhood $U$ of $x_0$ and we write $u \in C^{1,1}_+[x_0]$ if $\Theta^+(u, U)(x_0) < \infty$. Given a fixed $\epsilon \in (0, 1)$, we set $\Theta(u, \epsilon)(x) = \Theta(u, \Omega \cap B(x))$ for $x \in \Omega$ and we write $u \in C^{1,1}_+[\Omega]$ if $\sup_{x \in \Omega} \Theta(u, \epsilon)(x) = \Theta[\Omega] < \infty$.

**Remark 1.1.2.** If $u \in C^{1,1}[x_0]$ then $u$ is differentiable at $x_0$ because $u$ lies between two tangent paraboloids in an open neighborhood of $x_0$.

**Definition 1.1.3.** A function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be $C^{1,1}$ at $x \in \mathbb{R}^n$ (we write $u \in C^{1,1}(x)$), if there are a vector $v \in \mathbb{R}^n$, $r_0 > 0$ and $M > 0$ such that

\[ |u(x + y) - u(x) - v \cdot y| \leq M|y|^2 \quad \text{for any } y \in B_{r_0}. \]

We write $u \in C^{1,1}(U)$ if $u \in C^{1,1}(x)$ for any $x \in U$ and the constant $M$ in (1.1.4) is independent of $x$, where $U$ is an open subset of $\mathbb{R}^n$.

**Remark 1.1.4.** (a) Such vector $v$ exists uniquely and moreover $v = Vu(x)$.

(b) If $u \in C^{1,1}[x]$ for $x \in \Omega$, then we see that $u \in C^{1,1}(x)$. Moreover, it is easy to show that the converse holds. Thus we have $C^{1,1}[\Omega] = C^{1,1}(\Omega)$. 
For $x \in \Omega$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ which is semicontinuous on $\overline{\Omega}$, we say that $\phi$ belongs to the function class $C^2_0(u;x)^*$ (resp. $C^1_0(u;x)^*$) and we write $\phi \in C^2_0(u;x)^*$ (resp. $\phi \in C^1_0(u;x)^*$) if there are an open neighborhood $U \subset \Omega$ of $x$ and $\phi \in C^2(U)$ such that $\phi(x) = u(x)$ and $\phi > u$ (resp. $\phi < u$) on $U \setminus \{x\}$. We note that geometrically $u - \phi$ having a local maximum at $x$ in $\Omega$ is equivalent to $\phi \in C^2_0(u;x)^*$ and $u - \phi$ having a local minimum at $x$ in $\Omega$ is equivalent to $\phi \in C^2_0(u;x)^*$. For $x \in \Omega$ and $\phi \in C^2_0(u;x)^*$, we write
\[
\mu(u, x, y; \nabla \phi) = u(x + y) - u(x) - (\nabla \phi(x) \cdot y) \chi_{\Omega}(y),
\]
and the expression for $L_{\phi} u(x; \nabla \phi)$ and $I u(x; \nabla \phi)$ may be written as
\[
L_{\phi} u(x; \nabla \phi) = \int_{\Omega^c} \mu(u, x, y; \nabla \phi) K_{\phi}(y) \, dy,
\]
\[
I u(x; \nabla \phi) = \inf \sup_{\phi} L_{\phi} u(x; \nabla \phi),
\]
where $K_{\phi} \in K_0$. Then we see that $M_{\phi}^+ u(x; \nabla \phi) \leq I u(x; \nabla \phi) \leq M_{\phi}^+ u(x; \nabla \phi)$, and $M_{\phi}^+ u(x; \nabla \phi)$ and $M_{\phi}^- u(x; \nabla \phi)$ have the following simple forms;
\[
M_{\phi}^+ u(x; \nabla \phi) = (2 - \alpha) \int_{\mathbb{R}^n} \Lambda \mu^+(u, x, y; \nabla \phi) - \lambda \mu^- u, x, y; \nabla \phi) \, dy,
\]
\[
(1.1.5)
\]
\[
M_{\phi}^- u(x; \nabla \phi) = (2 - \alpha) \int_{\mathbb{R}^n} \Lambda \mu^+(u, x, y; \nabla \phi) - \lambda \mu^- u, x, y; \nabla \phi) \, dy,
\]
where $\mu^+$ and $\mu^-$ are given by
\[
\mu^\pm(u, x, y; \nabla \phi) = \max(\pm \mu(u, x, y; \nabla \phi), 0).
\]
We note if $u \in C^{1,1}[\chi]$, then $I u(x; \nabla \phi) = I u(x)$ and $M_{\phi}^+ u(x; \nabla \phi) = M_{\phi}^+ u(x)$. We shall use these maximal and minimal operators to obtain regularity estimates.

Let $K(x) = \sup_{\phi} K_{\phi}(x)$ where $K_{\phi}$’s are all the kernels of all operators in a class $\mathcal{U}$. For any class $\mathcal{U}$, we shall assume that
\[
(1.1.6)
\]
\[
\int_{\mathbb{R}^n} (|y|^2 \wedge 1) K(y) \, dy < \infty.
\]
Using the extremal operators, we provide a general definition of ellipticity for nonlocal equations. The following is a kind of operators of which the regularity result shall be obtained in this paper.

**Definition 1.1.5.** Let $\mathcal{U}$ be a class of linear integro-differential operators. Assume that (1.1.6) holds for $\mathcal{U}$. Then we say that an operator $\mathcal{F}$ is elliptic with respect to $\mathcal{U}$, if it satisfies the following properties:
\[
(a) \quad \mathcal{F} u(x) \text{ is well-defined for any } u \in C^{1,1}[\chi] \cap B(\mathbb{R}^n).
\]
\[
(b) \quad \text{If } u \in C^{1,1}[\Omega] \cap B(\mathbb{R}^n) \text{ for an open } \Omega \subset \mathbb{R}^n, \text{ then } \mathcal{F} u \text{ is continuous on } \Omega.
\]
\[
(c) \quad \text{If } u, v \in C^{1,1}[\chi] \cap B(\mathbb{R}^n), \text{ then we have that}
\]
\[
(1.1.7)
\]
\[
M_{\phi}^+[u - v](x) \leq \mathcal{F} u(x) - \mathcal{F} v(x) \leq M_{\phi}^-[u - v](x).
\]
And We denote by $S^\mathcal{E}$ the class of integro-differential operators which is elliptic with respect to $\mathcal{E}$.

The concept of viscosity solutions, its comparison principle and stability properties can be found in $C^\mathcal{E}$ for symmetric kernels and in $K^\mathcal{E}$ for possibly nonsymmetric kernels. Kim and Lee $K^\mathcal{E}$ considered much larger class of operators but prove the regularity of viscosity solutions only for $1 < \alpha < 2$.

Now we are going to consider a subclass $S^\mathcal{E}_0$ of $S^\mathcal{E}$ where the drift effect created by the nonsymmetric kernel is controllable. For $x \in B_\delta$ and $\phi \in C^2_{B_\delta}(u;x)^*$, we set
\[
\mu_\phi(u, x, y; \nabla \phi) = u(x + y) - u(x) - (\nabla \phi(x) \cdot y) \chi_{B_\delta}(y).
\]
For \( u \in C^{1,1}[x] \), we write \( \mu_k(u, x, y) = \mu_k(u, x, y; Vu) \). Then we define \( \mu_k^{u_0} \) and \( M_{u_0, R}^L u(x; \nabla \varphi) \) by replacing \( \mu \) by \( \mu_k \) in the definition \( M_{u_0}^L u(x; \nabla \varphi) \). We note if \( u \in C^{1,1}[x] \), then \( M_{u_0, R}^L u(x; \nabla \varphi) = M_{u_0, R}^L u(x) \equiv \operatorname{p.v.} \int_{\mathbb{R}^n} \mu_k(u, x, y)K(y)dy \).

**Definition 1.1.6.** Let \( 0 < \eta \leq 1 \) and \( I \in S^E \), where \( E \) is a class of linear integro-differential operators. Then we say that \( I \in S^E_{\eta} \) if, for \( R \in (0, 1] \), there are \( B_k^E : \mathbb{R}^n \to \mathbb{R} \) such that

1. \( B_k^E(i) \) is homogeneous of degree one, i.e. \( B_k^E(0) = 0 \) and \( B_k^E(Vu) = B_k^E \left( \frac{\nabla u}{|\nabla u|} \right) |Vu| \) for \( |\nabla u| \neq 0 \).
2. \( ||v \in S^{n-1} : B_k^E(v) < 0 || \eta |S^{n-1}| > 0 \).
3. \( M_{i, \eta, R}^L u(x) \leq I u(x) - I 0(x) \leq M_{\eta, R}^L u(x) \) whenever \( u \in C^{1,1}[x] \cap B(\mathbb{R}^n) \) for \( x \in B_R \),

where \( M_{i, \eta, R}^L u(x) := M_{\eta, R}^L u(x) + B_k^E(Vu(x)) \pm (2-a)CR^{1-\eta} |Vu(x)| \).

**Definition 1.1.7.** Let \( L \in \mathcal{U} \) be a linear integro-differential operator with a kernel \( K \). For \( 0 < R < 1 \), the drift vector \( b_{L,R} \) of \( L \) at \( R \) is defined by

\[
b_{L,R} = (2-a) \int_{\mathbb{R}^n \setminus B_R} y K(y)dy.
\]

**Lemma 1.1.8.** Let \( 0 < a < 2 \) and \( 0 < R < 1 \). Let \( \mathcal{U} \) be a class of linear integro-differential operators. Then we have the following results:

1. If \( L \) is a linear integro-differential operator which is in \( \mathcal{U} \), then \( L \in S^E_{\eta} \) for \( 0 < \eta \leq \frac{1}{2} \).
2. Let \( L_i \) be a linear integro-differential operator with a kernel \( K_i \) for \( i = 1, \ldots, N \). Let \( b_{L_i,R} \) be the drift vectors of \( L_i \). Assume that there is a unit vector \( \alpha \) such that for any nonzero drift vector \( b_{L_i,R} \),

\[
\alpha \cdot \frac{b_{L_i,R}}{|b_{L_i,R}|} > 0.
\]

If \( I \in S^E \) satisfies that

\[
\min_{i=1, \ldots, N} L_i u(x) \leq I u(x) - I 0(x) \leq \max_{i=1, \ldots, N} L_i u(x)
\]

whenever \( u \in C^{1,1}[x] \cap B(\mathbb{R}^n) \) for \( x \in B_R \), then \( I \in S^E_{\eta} \) for some \( \eta > 0 \).
3. Let \( L_a \) be a linear integro-differential operator with a kernel \( K_a \), \( a \in I \). Let \( b_{L_a,R} \) be the drift vectors of \( L_a \). Assume that there is a vector \( \alpha \in S^{n-1} \) and \( \eta > 0 \) such that for any nonzero drift vectors \( b_{L_a,R} \),

\[
\alpha \cdot \frac{b_{L_a,R}}{|b_{L_a,R}|} \geq \eta \frac{|\alpha|}{|b_{L_a,R}|} \quad \text{for any } \alpha \in I.
\]

If \( I \in S^E \) satisfies that

\[
\min_{a \in I} L_a u(x) \leq I u(x) - I 0(x) \leq \max_{a \in I} L_a u(x)
\]

whenever \( u \in C^{1,1}[x] \cap B(\mathbb{R}^n) \) for \( x \in B_R \), then we have \( I \in S^E \).

**Proof.** (1) Take \( u \in C^{1,1}[x] \cap B(\mathbb{R}^n) \) for \( x \in B_R \). Then we have that

\[
Lu(x) = (2-a) \int_{\mathbb{R}^n} \mu_k(u, x, y)K(y)dy.
\]

\[
= (2-a) \int_{\mathbb{R}^n} \mu_k(u, x, y)K(y)dy + (2-a) \int_{\mathbb{R}^n \setminus B_R} (y \cdot \nabla u(x))K(y)dy.
\]

\[
\leq M_{\eta, R}^L u(x) + B_k^E(Vu(x))
\]

for \( B_k^E(a) = b_{L,R} \cdot a \). And if \( b_{L,R} \neq 0, ||a \in S^{n-1} : B_k^E(a) \leq 0 || = \frac{1}{2} |S^{n-1}| \), which means \( 0 < \eta \leq \frac{1}{2} \); otherwise, \( \eta = 1 \).
Lemma 1.1.9.

(1) If $\mathbf{x} \in B$ then we have that

$$\left\{ \frac{\psi - \psi_*}{|B_{\psi,R}|} \right\} \leq \eta_{i=1} \cdots n \text{ and } b_{\psi,R} \neq 0.$$  

Take $u \in C^{1,1}[x] \cap B(R^n)$ for $x \in B_R$. Then we have that

$$\max_{i=1,\cdots,N} L_i u(x) = (2-\sigma) \max_{i=1,\cdots,N} \int_{\mathbb{R}^n} \mu K_i(y) dy.$$  

$$\geq \max_{i=1,\cdots,N} \int_{\mathbb{R}^n} \mu K_i(y) dy + (2-\sigma) \max_{i=1,\cdots,N} \int_{\mathbb{R}^n} (y \cdot \nabla u(x) K_i(y) dy.$$  

$$\leq M_{\psi,R} u(x) + b_R^*(Vu(x))$$  

for $b_R^*(a) = \max_{i=1,\cdots,N}(b_{\psi,R} \cdot a)$. Also we have that $|[a : b_R^*(a) \leq 0] | \geq \eta |S^{n-1}|.$

(3) It can be obtained from the same argument as (2). $\square$

Lemma 1.1.9.

(1) Let $0 < \sigma < 2$ and let $\mathcal{L}$ be a class of linear integro-differential operators. If $\mathcal{L} \in S^0$ with a symmetric kernel $K$, then $\mathcal{L} \in S^0_{\eta}$ for some $\eta \in (0, 1]$. In addition, if $\mathcal{L}_{\eta}$ has symmetric kernel for all $\eta$, then $\sup_{\eta} \mathcal{L}_{\eta} \in S^0_{\eta}$ for $1 \geq \eta > 0$.

(2) If $0 < \sigma < 2$, then $S^{0}_{\eta} = S^{0}_{\eta} \cap B(R^n)$ for $x \in B_R$. We set

$$b_R(x) = (2-\sigma) \int_{\mathbb{R}^n} \frac{y(x, y) K(y) dy}{|y|^n \sigma}.$$  

Then we easily obtain that, for $1 < \sigma < 2$,  

$$|b_R(x)| \leq (2-\sigma) CR^{-\sigma}.$$  

Therefore we have that

$$I u(x) - 0(x) \leq M_{\psi,0}^* u(x) = M_{\psi,0}^* u(x) + (2-\sigma) \int_{\mathbb{R}^n} \frac{\lambda^* \mu - \lambda^* \mu^R}(|y|^{n+\sigma}) dy$$  

$$\leq M_{\psi,0}^* u(x) - b_R \cdot \nabla u(x)$$  

$$\leq M_{\psi,0}^* u(x) + (2-\sigma) \cdot CR^{1-\sigma} |\nabla u(x)|.$$  

The lower bound can be obtained similarly. Therefore $I \in S^0_{\eta}$ for $0 < \eta \leq 1$. $\square$

1.2. Main equation. The natural Dirichlet problem for such a nonlocal operator $I$. Let $\Omega$ be an open domain in $\mathbb{R}^n$. Given a function $g$ defined on $\mathbb{R}^n \setminus \Omega$, we want to find a function $u$ such that

$$I u(x) = \begin{cases} 0 & \text{for any } x \in \Omega, \\ g(x) & \text{for } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$  

Note that the boundary condition is given not only on $\partial \Omega$ but also on the whole complement of $\Omega$. This is because of the nonlocal character of the operator $I$. From the stochastic point of view, it corresponds to the fact that a discontinuous Lévy process can exit the domain $\Omega$ for the first time jumping to any point in $\mathbb{R}^n \setminus \Omega$.

In this paper, we shall concentrate mainly upon the regularity properties of viscosity solutions to an equation $I u(x) = 0$. We shall briefly give a very general comparison principle.
from which existence of the solutions can be obtained in smooth domains. Since kernels of integro-differential operators are comparable to the kernel of the fractional Laplace operator $(-\Delta)^{\sigma/2}$, the theory we want to develop can be understood as a theory of viscosity solutions for fully nonlinear operators of fractional order.

The differences between local and nonlocal operators have been discussed at [KL].

1.3. Known results and Key Observations. There are some known results about Harnack inequalities and Hölder estimates for integro-differential operators with positive symmetric kernels (see [J] for analytical proofs and BBC, BK1, BK2, BL, KS, SV for probabilistic proofs). The estimates in all these previous results blow up as the index $\sigma$ of the operator approaches 2. In this respect, they do not generalize to elliptic partial differential equations. However there is some known result on regularity results for fully nonlinear integro-differential operators with possibly nonsymmetric kernels, the authors introduced larger class of operators and proved Harnack inequalities and Hölder estimates when $1 < \sigma < 2$ (see [KL]).

In this paper, we are going to consider nonlinear integro-differential operators with possibly nonsymmetric kernels, when $0 < \sigma < 2$.

Throughout this paper we would like to briefly present the necessary definitions and prove some regularity estimates. Our results in this paper are

- A nonlocal version of the Alexandroff-Backelman-Pucci estimate for fully nonlinear integro-differential equations.
- A Harnack inequality, Hölder regularity and an interior $C^{1,\alpha}$-regularity result for certain fully nonlinear integro-differential equations.

Key observations are the following:

- For the nonsymmetric case, $K(y)$ and $K(-y)$ can be chosen any of $\lambda/|y|^{n+\sigma}$ or $\Lambda/|y|^{n+\sigma}$.

Therefore there could be an extra term $\int_{\mathbb{R}^n} \frac{[\nabla u(x) \cdot y] \chi_{B_1}(y)}{|y|^{n+\sigma}} dy$.

- The equation is not scaling invariant due to $|\chi_{B_1}(y)|$.
- Somehow the equation has a drift term, not only the diffusion term. The case $1 < \sigma < 2$ and the case $0 < \sigma \leq 1$ require different technique due to the difference of the blow rate as $|y|$ approaches to zero and the decay rate as $|y|$ approaches to infinity. When $1 < \sigma < 2$, a controllable decay rate of kernel allows Hölder regularities in a larger class, which is invariant under an one-sided scaling i.e. if $u$ is a solution of the homogeneous equation, then so is $u(x) = e^{-\alpha x} u(e x)$ for $0 < \alpha \leq 1$. Critical case ($\sigma = 1$) and supercritical case ($0 < \sigma < 1$) have been studied in BBC with different techniques due to the slow decay rate of the kernel as $|x| \to \infty$.

1.4. Outline of Paper. In Section 2 we show various nonlocal versions of the Alexandroff-Backelman-Pucci estimate to handle the difficulties caused by the gradient effect. It has different orders at subcritical, critical and supercritical cases. In Section 3 we construct a special function and apply A-B-P estimates to obtain the decay estimates of upper level sets which is essential in proving Hölder estimates in Section 4.

In Section 4 we prove a Harnack inequality which plays an important role in analysis. And then the Hölder estimates and an interior $C^{1,\alpha}$-estimates come from the arguments at [KL].

2. A nonlocal Alexandroff-Bakelman-Pucci estimate

The Alexandroff-Bakelman-Pucci (A-B-P) estimate plays an important role in Krylov and Sofonov theory [KS] on Harnack inequality for linear uniformly elliptic equations with measurable coefficients. The concept of viscosity solution is given pointwise through
touching test function; see [KL]. A-B-P estimate tells us that the maximum value is controlled by an integral quantity of the source term on the contact set, which will give us key lemma (Lemma 2.1.1) saying that the pointwise value of nonnegative function gives the lower bound of the measure of lower level set. We employ measure theoretical version of A-B-P estimate introduced at [CS] and extended to nonsymmetric case at [KL].

New A-B-P estimates below are two main differences from the arguments at [CS, KL].

- The operators considered at [CS, KL] are scaling invariant, but (2.1.2) doesn’t have such property due to $\chi_{B}(y)$ in the gradient term. So we keep the size of the domain $B_R$ at the following estimates.

- The control of bad set, Lemma 2.1.1 deteriorates as $R \to 0$ since $R^{n-2}J_{\sigma}(R)$ goes to $\infty$ as $R \to 0$. A-B-P estimate will be used to prove key Lemma 3.2.1 where we have an extra term $R$ to subdue the blow-up rate. But we have still $R \times R^{n-2}J_{\sigma}(R) \approx R^{n-1}$ (for $0 < \sigma < 1$) and $-\log(R)$ (for $\sigma = 1$) which blows up when $0 < \sigma \leq 1$. So we introduced a subclass $S_{\sigma}$ of $S$ and a different version of A-B-P estimate (Lemma 2.2.1) for $0 < \sigma \leq 1$ where we have better control of gradient term. Still $S_{\sigma}$ includes a large class of nonlinear operators, Lemma 1.1.8.1.9.

Let $R \in (0, R_0]$ for some $R_0 \in (0, 1)$ (in fact, the existence of $R_0$ was mentioned in [KL]) and let $u : \mathbb{R}^n \to \mathbb{R}$ be a function which is not positive outside the ball $B_{R_0/2}$ and is upper semicontinuous on $B_R$. We consider its concave envelope $\Gamma$ in $B_{2R}$ defined as

$$\Gamma(x) = \begin{cases} \inf\{p(x) : p \in \Pi, p > u^* \text{ in } B_{2R}\} & \text{in } B_{2R}, \\ 0 & \text{in } \mathbb{R}^n \setminus B_{2R}, \end{cases}$$

where $\Pi$ is the family of all the hyperplanes in $\mathbb{R}^n$. Also we denote the contact set of $u$ and $\Gamma$ in $B_R$ by $C(u, \Gamma, B_R) = \{y \in B_R : u(y) = \Gamma(y)\}$.

2.1. A-B-P estimate with blow-up rate.

**Lemma 2.1.1.** Let $0 < \sigma < 2$ and $0 < R \leq R_0$. Let $u \leq 0$ in $\mathbb{R}^n \setminus B_R$ and let $\Gamma$ be its concave envelope in $B_{2R}$. If $u \in B(\mathbb{R}^n)$ is a viscosity subsolution to $\mathcal{M}_{\nu}^+ u = -f$ on $B_R$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a function with $f > 0$ on $C(u, \Gamma, B_R)$, then there exists some constant $C > 0$ depending only on $n, \lambda$ and $\Lambda$ (but not on $\sigma$) such that for any $x \in C(u, \Gamma, B_R)$ and any $M > 0$ there is some $k \in \mathbb{N} \cup \{0\}$ such that

$$\left|\{y \in B_R : \mu^* (u, x, y; \nabla \Gamma) \geq M_0 \sigma^2\}\right| \leq C \frac{R^{n-2}(f(x) + J_{\sigma}(R)|\nabla \Gamma(x)|)}{M} |R_k|$$

where $R_k = B_{r_n} \setminus B_{r_n+1}$ for $r_n = (\sigma/2) \sigma^{-1/2} R$, $\theta_0 = 1/(16 \sqrt{n})$ and $J_{\sigma}(R)$ is $\frac{1}{\varphi(1 - R^{-\sigma})}$ for $\sigma \in (0, 1) \cup (1, 2)$ and $-\log(R)$ for $\sigma = 1$. Here, $\nabla \Gamma(x)$ denotes any element of the superdifferential $\partial \Gamma(x)$ of $\Gamma$ at $x$.

**Proof.** Let $0 < \sigma < 2$ and $0 < R \leq R_0$. Take any $x \in C(u, \Gamma, B_R)$. Since $u$ can be touched by a hyperplane from above at $x$, we see that $\nabla \wp(x) = \nabla \Gamma(x)$ for some $\wp \in C_{B_R}(u^*; x)^\circ$. Thus $\mathcal{M}_{\nu}^+ u(x; \nabla \Gamma)$ is well-defined and we have that

$$\mathcal{M}_{\nu}^+ u(x; \nabla \Gamma) = (2 - \sigma) \int_{\mathbb{R}^n} \frac{\Lambda \mu^* (u, x, y; \nabla \Gamma) - \lambda \mu^* (u, x, y; \nabla \Gamma)}{|y|^{n+\sigma}} dy.$$

We note that $\mu(u, x, y; \nabla \Gamma) = u(x + y) - u(x) - (\nabla \Gamma(x) \cdot y)\chi_{B}(y) \leq 0$ for any $y \in B_R$ by the definition of concave envelope of $u$ in $B_{2R}$. Since $\mu^* (u, x, y; \nabla \Gamma) \leq |\nabla \Gamma(x)||y|\chi_{B}(y)$ for any $y \in (\mathbb{R}^n \setminus B_R)$, we have that

$$\int_{\mathbb{R}^n} \frac{\Lambda \mu^* (u, x, y; \nabla \Gamma)}{|y|^{n+\sigma}} dy \leq \int_{B_R \setminus B_R} \frac{\Lambda |\nabla \Gamma(x)||y|}{|y|^{n+\sigma}} dy = \omega_n \Lambda J_{\sigma}(R)|\nabla \Gamma(x)|.$$
where \( \omega_n \) denotes the surface area of \( S^{n-1} \) and

\[
J_{\sigma}(R) = \begin{cases} 
\frac{1}{\sigma} (1 - R^{1-\sigma}) & \text{for } \sigma \in (0, 1) \cup (1, 2), \\
- \log(R) & \text{for } \sigma = 1. 
\end{cases}
\]

Here we see that \(|J_{\sigma}(R)|\) is finite for \( 0 < \sigma < 2 \). Thus it follows from simple calculation that

\[
-f(x) \leq M_{\omega_n} u(x; VT) = (2 - \sigma) \left( \int_{B_{r_0}} -\frac{\lambda \mu^+(u, x, y; VT)}{|y|^{n+\sigma}} dy + \int_{R^d} -\frac{\Lambda \mu^+(u, x, y; VT)}{|y|^{n+\sigma}} dy \right)
\]

\[
\leq (2 - \sigma) \int_{B_{r_0}} -\frac{\lambda \mu^+(u, x, y; VT)}{|y|^{n+\sigma}} dy + (2 - \sigma) \omega_n \Lambda J_{\sigma}(R) |VT(x)|
\]

for any \( x \in C(u, \Gamma, B_R) \), where \( r_0 = \varrho_0 R \). Splitting the above integral in the rings \( R_k \), we have that

\[
f(x) \geq (2 - \sigma) \lambda \sum_{k=0}^{\infty} \int_{R_k} \frac{\mu^+(u, x, y; VT)}{|y|^{n+\sigma}} dy - (2 - \sigma) \omega_n \Lambda J_{\sigma}(R) |VT(x)|. \tag{2.1.4}
\]

Assume that the conclusion (2.1.1) does not hold, i.e. for any \( C > 0 \) there are some \( x_0 \in C(u, \Gamma, B_R) \) and \( M_0 > 0 \) such that

\[
\left| \{ y \in R_k : \mu^+(u, x_0, y; VT) \geq M_0 r_k^2 \} \right| \geq C \frac{R^{d-2} \{ f(x_0) + J_{\sigma}(R) |VT(x)| \}}{M_0}
\]

for all \( k \in \mathbb{N} \cup \{0\} \). Since \((2 - \sigma) \lambda \) remains bounded below for \( \sigma \in (0, 1] \), it thus follows from (2.1.5) that

\[
f(x_0) \geq \frac{\lambda}{2 - \sigma} \sum_{k=0}^{\infty} \int_{R_k} \frac{\mu^+(u, x_0, y; VT(x))}{|y|^{n+\sigma}} dy - \omega_n \Lambda J_{\sigma}(R) |VT(x)|
\]

\[
\geq c \sum_{k=0}^{\infty} M_0 \frac{r_k^2}{R} CR^{d-2} \frac{f(x_0) + J_{\sigma}(R) |VT(x)|}{M_0} - \omega_n \Lambda J_{\sigma}(R) |VT(x)|.
\]

Thus this implies that

\[
f(x_0) + (2 - \sigma) \omega_n \Lambda J_{\sigma}(R) |VT(x)| \geq \frac{C \varrho_0^2}{1 - 2^{-1-\sigma}} C (f(x_0) + J_{\sigma}(R) |VT(x)|)
\]

\[
\geq C (f(x_0) + (2 - \sigma) J_{\sigma}(R) |VT(x)|)
\]

for any \( C > 0 \). Taking \( C \) large enough, we obtain a contradiction. Hence we are done. \( \square \)

Remark. Lemma 2.1.1 would hold for any particular choice of \( \varrho_0 \) by modifying \( C \) accordingly. The particular choice \( \varrho_0 = 1/(16 \sqrt{n}) \) is convenient for the proofs in Section 3.

Lemma 2.1.2. \( \text{[CS]} \) Let \( \Gamma \) be a concave function on \( B_r(x) \) where \( x \in \mathbb{R}^n \) and let \( h > 0 \). If \(|y \in S_r(x) : \Gamma(y) < \Gamma(x) + (y - x) \cdot VT(x) - h| \leq \varepsilon |S_r(x)| \) for any small \( \varepsilon > 0 \) \( S_r(x) = B_r(x) \setminus B_{r/2}(x) \), then we have \( \Gamma(x) \geq \Gamma(x) + (y - x) \cdot VT(x) - h \) for any \( y \in B_{r/2}(x) \).

Corollary 2.1.3. For any \( \varepsilon > 0 \), there is a constant \( C > 0 \) such that for any function \( u \) with the same hypothesis as Lemma 2.1.1 there is some \( r \in (0, \varrho_0 R \sqrt{n}) \) such that

\[
\left| \left\{ y \in S_r(x) : u(y) < u(x) + (y - x) \cdot VT(x) - C R^{n-2} (f(x) + J_{\sigma}(R) |VT(x)|)^2 \right\} \right| \leq \varepsilon,
\]

\[
\int_Q \xi_\varepsilon(VT(y)) \det(D^2 \Gamma(y)) \frac{d y}{|S_r(x)|} \leq C \varepsilon^{n-2} \sup_{y \in Q} \left( J_{\sigma}(R)^n + \varepsilon^{-n} |f(y)|^n \right) |Q|
\]

for any \( \varepsilon > 0 \) and any cube \( Q \in B_{r/2}(x) \) with diameter \( d \) such that \( x \in \overline{Q} \) and \( r/4 < d < r/2 \), where \( \varrho_0 = 1/(16 \sqrt{n}) \) and \( \xi_\varepsilon(z) = (|z|^{n(\varepsilon^{-1})} + \varepsilon^{n(\varepsilon^{-1})})^{-n} \).
Proof. The first part can be obtained by taking $M = CR^{-\varepsilon/2}(f(\hat{x}) + J_\alpha(R)|\nabla \Gamma(\hat{x}))$ in Lemma 2.1.1. Also the second part follows as a consequence of Lemma 2.1.2 and concavity. If $\Gamma$ is $C^2$, then we have that
\[
\det[D^2\Gamma(x)]^\alpha \leq C(R^{\alpha-2}f(x) + R^{\alpha-2}J_\alpha(R)|\nabla \Gamma(x)|^\alpha \]
\[
\leq 4^n C R^{\alpha} J_\alpha(R)^{\alpha} + \xi^{-\alpha} R^{\alpha-2} |f(x)|^\alpha.
\]
Then we have that $\int_{Q} g_\xi(\nabla \Gamma(x)) \det[D^2\Gamma(x)]^\alpha \, dx \leq 4^n C R^{\alpha} \int_{Q} (R^{\alpha-2} J_\alpha(R)^{\alpha} + \xi^{-\alpha} R^{\alpha-2} |f(x)|^\alpha) \, dy$ if $\Gamma$ is $C^2$, where $Q' = \mathcal{C}(u, \Gamma, B_\delta) \cap \overline{Q}$. It is also true for the general concave envelope $\Gamma$ through an approximation with $C^2$-concave functions.

Take any $y \in \mathcal{C}(u, \Gamma, B_\delta) \cap Q$ where $Q \subset B_{1/4}(x)$ is a cube with diameter $\delta$ such that $x \in \overline{Q}$ and $r/4 < d < r/2$. Similarly to the above, we can obtain that $\int_{Q} g_\xi(\nabla \Gamma(y)) \det[D^2\Gamma(y)]^\alpha \, dy \leq 4^n C \int_{Q} (R^{\alpha-2} J_\alpha(R)^{\alpha} + \xi^{-\alpha} R^{\alpha-2} |f(y)|^\alpha) \, dy$ because $\det[D^2\Gamma(\cdot)]^\alpha = 0$ a.e. on $\overline{Q} \setminus \mathcal{C}(u, \Gamma, B_\delta)'$ as in (CC).

Hence this implies the second part. \(\square\)

We obtain a nonlocal version of Alexandrov-Bakelman-Pucci estimate in the following theorem.

**Theorem 2.1.4.** Let $u$ and $\Gamma$ be functions as in Lemma 2.1.1. Then there exist a finite family $\{Q_i\}_{i=1}^m$ of open cubes $Q_i$ with diameters $d_i$, such that

(a) Any two cubes $Q_i$ and $Q_j$ do not intersect, (b) $Q'_i = \mathcal{C}(u, \Gamma, B_\delta) \cap \overline{Q}_i \neq \emptyset$ for $j$, (c) $\mathcal{C}(u, \Gamma, B_\delta) \subset \bigcup_{i=1}^m Q'_i$, (d) $d_i \leq \varepsilon_0 2^{\sigma} R$ where $\varepsilon_0 = 1/(16 \sqrt{n})$, (e) $\int_{Q_i} g_\xi(\nabla \Gamma(y)) \det[D^2\Gamma(y)]^\alpha \, dy \leq CR^{\alpha-2}(\sup_{Q_i} (J_\alpha(R)^{\alpha} + \xi^{-\alpha} |f(y)|^\alpha)) |Q_i|$, (f) $|y \in \mathcal{C}(u, \Gamma, B_\delta) : u(y) \geq \Gamma(y) - CR^{\alpha-2}(\sup_{Q_i} (J_\alpha(R)^{\alpha} + \xi^{-\alpha} |f(y)|^\alpha)) |Q_i| \geq \varepsilon_0 |Q_i|$, where the constants $C > 0$ and $\varepsilon_0 > 0$ depend on $n$, $\Lambda$ and $\lambda$ (but not on $\alpha$).

**Proof.** In order to obtain such a family, we start by covering $B_\delta$ with a tiling of cubes of diameter $\varepsilon_0 2^{\sigma} R$. Then discard all those that do not intersect $\mathcal{C}(u, \Gamma, B_\delta)$. Whenever a cube does not satisfy (e) and (f), we split it into $2^n$ cubes of half diameter and discard those whose closure does not intersect $\mathcal{C}(u, \Gamma, B_\delta)$.

Now our goal is to prove that eventually all cubes satisfy (e) and (f) and this process ends after a finite number of steps.

Assume that the process does not finish in a finite number of steps. Then we can have an infinite nested sequence of cubes. The intersection of their closures will be a point $\hat{x}$. So we may choose a sequence $\{x_k\} \subset \mathcal{C}(u, \Gamma, B_\delta)$ with $\lim_{k \to \infty} x_k = \hat{x}$. Since $u(x_k) = \Gamma(x_k)$ for all $k \in \mathbb{N}$, by the upper semicontinuity of $u$ on $\overline{B_\delta}$ we have that $\Gamma(\hat{x}) = \limsup_{k \to \infty} u(x_k) \leq u(\hat{x})$.

Also we have that $u(\hat{x}) \leq \Gamma(\hat{x})$ because $u \leq \Gamma$ on $B_{2\delta}$ by the definition of the concave envelope $\Gamma$ in $B_{2\delta}$. Thus we obtain that $u(\hat{x}) = \Gamma(\hat{x})$. We will now get a contradiction by showing that eventually one of these cubes containing $\hat{x}$ will not split.

Take any $\varepsilon > 0$. Then by Corollary 2.1.3 there is a radius $r \in (0, \varepsilon_0 2^{\sigma} R)$ such that

\[
\frac{\|y \in \mathcal{S}_r(\hat{x}) : u(y) < u(\hat{x}) + (y - \hat{x}) \cdot \nabla \Gamma(\hat{x}) - CR^{\alpha-2}(f(\hat{x}) + J_\alpha(R)|\nabla \Gamma(\hat{x})|)^2 \|^2}{|\mathcal{S}_r(\hat{x})|} \leq \varepsilon,
\]

\[
\int_{Q_i} g_\xi(\nabla \Gamma(y)) \det[D^2\Gamma(y)]^\alpha \, dy \leq CR^{\alpha-2}\sup_{y \in \overline{Q}_i} (J_\alpha(R)^{\alpha} + \xi^{-\alpha} |f(y)|^\alpha) |Q_i|
\]

for any $\eta > 0$ and a cube $Q_i \subset B_{1/4}(\hat{x})$ with diameter $d_i$ such that $x \in \overline{Q}_i$, and $r/4 < d_i < r/2$. So we easily see that $\overline{Q}_i \subset B_{1/4}(\hat{x})$ and $B_{1/4}(\hat{x}) \subset 4 \sqrt{n} Q_i$. We recall that $\Gamma(y) \leq u(\hat{x}) + (y - \hat{x}) \cdot \nabla \Gamma(\hat{x})$ for any $y \in B_{2\delta}$ because $\Gamma$ is concave on $B_{2\delta}$ and $\Gamma(\hat{x}) = u(\hat{x})$. Since $d_i$ is comparable to $r$, it
Thus we proved (f). Moreover, (e) holds for $Q_j$ because $Q_j \subset B_{\sqrt{2}}(\bar{x})$ and $B_j(\bar{x}) \subset 4\sqrt{n}Q_j$. Hence the cube $Q_j$ would not split and the process must stop there. \hfill \Box

2.2. A-B-P estimate for a class $S^0_\alpha$: $0 < \sigma < 2$. For $I \in S^0_\alpha$, there are $\mathcal{B}^\alpha_R$ as Definition 1.6. Set
\[
\mathcal{C}^\alpha(u, \Gamma, B_R) = \{x \in C(\Omega, \Gamma, B_R) : \mathcal{B}^\alpha_R(\nabla \chi(x)) \leq 0\}
\]
for a concave envelope $\Gamma$ of $u$ in $B_{2R}$.

Lemma 2.2.1. Let $0 < \sigma < 2$, $I \in S^0_\alpha$, and $0 < R \leq R_0$. Let $u \leq 0$ in $\mathbb{R}^n \setminus B_R$ and let $\Gamma$ be its concave envelope in $B_{2R}$. If $u \in B(\mathbb{R}^n)$ is a viscosity subsolution to $Iu = f$ on $B_R$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a function with $f > 0$ on $C(\Omega, \Gamma, B_R)$, then there exist constants $C > 0$ depending only on $n$, $\lambda$ and $\Lambda$ (but not on $\alpha$) such that for any $x \in C^\alpha(u, \Gamma, B_R)$ and $M > 0$ there is some $k \in \mathbb{N}$ such that
\[
\left| Iu(x; \nabla \chi) \right| \leq C \frac{R^{n-2}(f(x) + R^{1-\sigma}\|\nabla \chi(x)\|)}{M} |B|,
\]
where $B_k = B_{\sqrt{k}} \setminus B_{\sqrt{k+1}}$ for $n = 16\sqrt{n}$ and $\alpha = 1/(16\sqrt{n})$. Here, $\nabla \chi(x)$ denotes any element of the superdifferential $\partial \Gamma(x)$ of $\Gamma$ at $x$.

Proof. Take any $x \in C^\alpha(u, \Gamma, B_R)$. From the definition of $S^0_\alpha$, we have that
\[
Iu(x; \nabla \chi) \leq M^\alpha_{y, x, u}(x; \nabla \chi) + \mathcal{B}^\alpha_R(\nabla \chi(x)) + (2 - \alpha)CR^{1-\sigma}\|\nabla \chi(x)\|
\]
where $\mathcal{B}^\alpha_R(\nabla \chi(x)) \leq 0$ from the assumption and $\mu^\alpha_R(u, x, \cdot; \nabla \chi) = 0$ on $\mathbb{R}^n$. The conclusion comes from similar arguments as Lemma 2.1.1. \hfill \Box

Now we have the following Corollary as Section 2.1.

Corollary 2.2.2. Let $u$ and $\Gamma$ be functions as in Lemma 2.2.1. Then there exist a finite family $\mathcal{Q}^\alpha_{i=1}$ of open cubes $Q_j$ with diameters $d_j$, such that
(a) Any two cubes $Q_i$ and $Q_j$ do not intersect, (b) $\bar{Q}_j = C^\alpha_j(u, \Gamma, B_R) \cap \bar{Q}_i \neq \emptyset$ for any $j$, (c) $C^\alpha_j(u, \Gamma, B_R) \subset \bigcup_{i=1}^{\delta} Q_j$, (d) $d_j \leq \sqrt{x_0}2^{-n-\varepsilon}R$ where $\varepsilon = 1/(16\sqrt{n})$, (e) $\sum_{i=1}^{\delta} \chi_i(\nabla \chi(x)) \det(\nabla \chi(x)) \leq CR^{n-2}(\sup_{\Omega_j}(R^{1-\varepsilon}\|\nabla \chi(x)\|)\|Q_j\|)$, (f) $|4\sqrt{n}Q_j| : u(y) \geq \Gamma(y) - CR^{n-2}(\sup_{\Omega_j}(f + R^{1-\varepsilon}\|\nabla \chi(x)\|)\|Q_j\|)$, where the constants $C > 0$, $\lambda$ and $\Lambda$ (but not on $\sigma$).}

2.3. Discussion of A-B-P estimates. At this subsection, we are going to discuss motivations and differences of the A-B-P estimates at previous subsections.

Remark 2.3.1. (1) Key step in the A-B-P estimate is the control of the volume of the gradient image, $|\nabla \chi(B_R)|$, in terms of $\mathcal{B}^\alpha_R$ for $M_\alpha = \sup_{B_R} u$. From the concavity $\Gamma(x)$, we have $B_{\alpha M_\alpha} \subset \nabla \chi(B_R)$ where $\alpha M_\alpha \leq \varepsilon \|u\|_{\nabla \chi}$ (see Lemma 9.2, (11)). Similarly for $\mathcal{B}^\alpha_R \neq \emptyset$, we have $\{z \in B_{\alpha M_\alpha} : \mathcal{B}^\alpha_R(z) \leq 0\} \subset \nabla \chi(B_{\alpha M_\alpha}, B_R)$ and then $\eta \|u\|_{\nabla \chi} \leq |\nabla \chi(B_{\alpha M_\alpha}, B_R)|$. 

(2) The different A-B-P estimates have been considered to control the effect of the gradient term \( \nabla \Gamma(x) \) caused by the fact that the Kernel is not symmetric. And they will be used at Lemma 32.7 to prove the decay estimate of the upper level set of super-solutions. When we apply A-B-P estimate with blow-up rate, we have an extra term \( (R^{-2}J_n)|B_R| \approx 1 \) (for \( 0 < \sigma < 2 \)), \(- \log(R) \) (for \( \sigma = 1 \)), and \( R^{\sigma-1} \) (for \( 0 < \sigma < 1 \)) caused by \( \nabla \Gamma(x) \). It is bounded only at \( 1 < \sigma < 2 \). This is the main reason that we another A-B-P estimates.

(3) For the operators in \( S_{\nu} \), \( (R^{-2}R^{1-\nu})|B_R| \approx 1 \) (for \( 0 < \sigma < 2 \)) gives us the decay estimate, Lemma 3.3.1.

(4) For \( 1 < \sigma < 2 \), Lemma 2.1.1 and Lemma 2.2.1 are equal since \( f_s(R) \approx R^{1-\nu} \) for \( 0 < R \leq 1 \).

3. Decay Estimate of Upper Level Sets

In this section, we are going to show the geometric decay rate of the upper level set of nonnegative solution \( u \). The key Lemma 3.2.1 says that if a nonnegative function \( u \) has a value smaller than one in \( Q_k \) then the lower level set \( \{ x : u \leq M \} \) has uniformly positive amount of measure \( \nu(Q_k) \) which will be proven through ABP estimate. But the assumption of ABP estimate on a subsolution requires its special shape: it should be negative out side of \( Q_k \) and positive at some interior point. So we are going to construct a special function \( \Psi \) so that \( \Psi - u \) meets the requirement of ABP estimate.

3.1. Special functions. The construction of the special function is based on the idea in \( CS \) \( KL \). Nontrivial finer computation has been done to take care of the influence of the gradient term and the lack of scaling.

**Lemma 3.1.1.** There exist some \( \sigma^* \in (0, 2) \) and \( p > 0 \) such that the function

\[
\eta(x) = \min\{2/R^{1-\sigma}, |x|^p\}
\]

is a subsolution to \( M_{\eta} \eta(x) \geq 0 \) for any \( \sigma \in (\sigma^*, 2) \) and \( x \in B_R \).

**Proof.** It is enough to show that there is some \( \sigma^* \in (1, 2) \) so that

\[
M_{\eta} \eta(x) \geq 0
\]

for \( x = R_1e_n = (0, 0, \ldots, 0, R_1) \in \mathbb{R}^n \); for every other \( x \) with \( |x| = R \geq R_1 \), the above inequality follows by rotation. In order to prove (3.1.1), we use the following elementary inequality that holds for any \( a > b > 0 \) and \( p > 0 \);

\[
(a + b)^p \geq a^p \left(1 - \frac{p}{2!} \frac{b}{a} + \frac{p(p + 1)}{3!} \left(\frac{b}{a}\right)^2 - \frac{p(p + 1)(p + 2)}{4!} \left(\frac{b}{a}\right)^3\right).
\]

Using this inequality and \( \mu(f, R_1e_n, y) = R_1^{-\nu} \mu(f, e_n, \|y\|) \) for \( \|y\| = y/R_1 \), we have that

\[
\mu(f, e_n, \|y\|) = |e_n + \|y\|^p - 1 + p \|y\|^p - 1 + p \|y\| \geq -\left(\frac{p + 2}{2}\right) \|y\|^p \frac{\|y\|^2}{(1 + \|y\|^2)^{p+1}}
\]

\[
\quad + \frac{p(p + 2)}{2} \frac{\|y\|^2}{(1 + \|y\|^2)^{p+1}} - \frac{p(p + 2)(p + 4)}{6} \frac{\|y\|^3}{(1 + \|y\|^2)^{p+3}}
\]

for any \( y \in B_R \). We choose some sufficiently large \( p > 0 \) so that

\[
\frac{p(p + 2)}{2(1 + r^2)^{p/2+2}} \int_{S_{n-1}} \sigma_n^2 \delta_n d\sigma(\theta) + \frac{\alpha_n}{(1 + r^2)^{p/2+1}} - \left(\frac{p}{2} + 1\right) \omega_n = \delta_0(r) > 0
\]
for any sufficiently small \( r > 0 \). Since \( \int_{\mathbb{S}^{n-1}} \theta_n \, d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} \theta_n^2 \, d\sigma(\theta) = 0 \), it follows from (1.12), (1.15), (3.1.2) and (3.1.3) that
\[
\mathcal{M}_{\nu_j} f(e_0)
\geq (2 - \alpha) R_1^{-p} \left( \int_{\mathbb{R}^n} \frac{\lambda \mu^+(f, e_{0_j}, y/R_1)}{|y|^{n+s}} \, dy - \int_{\mathbb{R}^n} \frac{\Lambda \mu^-(f, e_{0_j}, y/R_1)}{|y|^{n+s}} \, dy \right)
\geq (2 - \alpha) R_1^{-p-\alpha} \left( \lambda \int_{\mathbb{R}^n} \frac{\mu(f, e_{0_j}, y)}{|y|^{n+s}} \, dy - \Lambda \int_{\mathbb{R}^n \setminus B_r} \frac{\mu(f, e_{0_j}, y)}{|y|^{n+s}} \, dy \right)
\geq (2 - \alpha) R_1^{-p-\alpha} \left( \frac{\lambda \delta_0(\rho)}{2 - \alpha} - (2^p + 1) \Lambda \int_{\mathbb{R}^n \setminus B_r} \frac{1}{|y|^{n+s}} \, dy - p \Lambda \int_{\mathbb{R}^n \setminus B_r} \frac{1}{|y|^{n+s}} \, dy \right)
= R_1^{-p-\alpha} \left( \lambda \delta_0(\rho) - (2^p + 1) \Lambda \omega_{n-1} \frac{2 - \alpha}{\alpha} r^{-\alpha} - p \Lambda \omega_{n-1} \frac{2 - \alpha}{\alpha} r^{-\alpha} \right)
\]
for \( r \in (0, \frac{1}{2} R) \), where \( \omega_{n} \) denotes the surface measure of \( S^{n-1} \). Thus we may take some sufficiently small \( r \in (0, 1/2) \) and take some \( \sigma^* \in (1, 2) \) close enough to \( 2 \) in the above so that
\[
\mathcal{M}_{\nu_j} f(e_0) \geq 0
\]
for any \( \sigma \in (\sigma^*, 2) \). Hence we complete the proof. \( \square \)

Now we have the following Corollary as Corollary 4.1.2, [KL]. The only difference is that the influence of non-symmetry of the kernel is \( p \mid g(x) \mid \chi_{\mathbb{S}^1}(y) \), not \( p \mid g \) in the proof since authors considered a larger class of operators for \( 1 < \sigma < 2 \) at [KL].

**Corollary 3.1.2.** Given any \( \alpha_0 \in (1, 2) \), there exist some \( \delta > 0 \) and \( p > 0 \) such that the function
\[
f(x) = \min[(4^\sigma \Lambda^{-1} p)^{1/\sigma}] \]
is a subsolution to \( \mathcal{M}_{\nu_j} f(x) \geq 0 \) for any \( \sigma \in (\alpha_0, 2) \) and \( x \in B_{R/4} \).

**Lemma 3.1.3.** Given any \( \sigma \in (0, 1] \), there exist \( 0 < R_0 < 1 \), \( \delta > 0 \), \( p > 0 \), and \( \epsilon_0 > 0 \) such that the function
\[
f(x) = \exp(-|x|^{\sigma / \delta})
\]
is a subsolution to \( \mathcal{M}_{\nu_j} f(x) \geq 0 \) for any \( 0 < R < R_0 \) and \( x \in B_{\sqrt{2} R_0} \setminus B_{R/4} \).

**Proof.** We consider \( f(x) = \exp(-|x|^{\sigma / \delta}) \) for \( \sigma > 0 \) to be selected later. For any \( x \in B_{\sqrt{2} R_0} \setminus B_{R/4} \), we may assume \( x = R_1 e_\alpha \) for \( R/4 \leq R_1 < 2 \sqrt{\Lambda} R \) without losing generality. Then we have
\[
\mu(f, R_1 e_\alpha, y) = e^{-\rho_1 R_1^{-1} |y|^{1/n} \chi_{\mathbb{S}^1}(y)} + ap R_1^{-n-1} e^{-\rho_1 R_1^{-1} |y|^{1/n} \chi_{\mathbb{S}^1}(y)}
\]
and
\[
\mathcal{M}_{\nu_j} f(R_1 e_\alpha) = (2 - \alpha) \lambda \int_{\mathbb{R}^n} \frac{\mu^+(f, R_1 e_\alpha, y)}{|y|^{n+s}} \, dy - (2 - \alpha) \Lambda \int_{\mathbb{R}^n} \frac{\mu^-(f, R_1 e_\alpha, y)}{|y|^{n+s}} \, dy.
\]
We note that \( \mu^-(f, R_1 e_\alpha, y) \leq 0 \) for any \( y \in B_{\rho_1(R_1)} \) and \( \rho(R_1) \geq c_1 R_1 \) for a uniform constant \( c_1 > 0 \) from simple geometric observation. It is easy to check that \( \mu^-(f, R_1 e_\alpha, y) \leq e^{-\rho_1 R_1^{-1} |y|^{1/n} \chi_{\mathbb{S}^1}(y)} + ap R_1^{-n-1} e^{-\rho_1 R_1^{-1} |y|^{1/n} \chi_{\mathbb{S}^1}(y)} \). So we see that
\[
-\mathcal{J}_2 f(R_1 e_\alpha) = (2 - \alpha) \Lambda \int_{\mathbb{R}^n \setminus \mathbb{S}^1} \frac{\mu^-(f, R_1 e_\alpha, y)}{|y|^{n+s}} \, dy
\leq (2 - \alpha) \Lambda e^{-\rho_1 R_1^{-1} |y|^{1/n} \chi_{\mathbb{S}^1}(y)} \int_{\mathbb{R}^n \setminus \mathbb{S}^1} \frac{1}{|y|^{n+s}} \, dy.
\leq (2 - \alpha) \Lambda e^{-\rho_1 R_1^{-1} \left( \frac{c_1}{R_1^2} + \frac{ap c_2}{R_1^2} R_1^{-1} \right)}.
\]
Also we have that
\[ \mathcal{J}_1 f(R_1 e_\alpha) \geq (2 - \alpha) \lambda \int_{B_{R_1}/2 < |y + R_1 e_\alpha| < R_1} e^{-\alpha p e^{p \sigma} R_1^p} \frac{\delta^2}{|y|^{n+2}} dy \]
\[ \geq \frac{c_1(2 - \alpha) \lambda}{R_1^p} e^{-\alpha p e^{p \sigma} R_1^p} \] for a uniformly small \( \delta > 0 \) and \( c_3 > 0 \).

Set \( \alpha = \sigma/4 \) for a uniformly small \( \epsilon_0 \). There are a large \( p > 0 \) and small \( R_0, \delta > 0 \) such that if \( 0 < R < R_0 \), then we have \( e^{-\alpha p e^{p \sigma} R_1^p} \ll e^{-\alpha p e^{p \sigma} R_1^p} \) and
\[ -\mathcal{J}_2 f(R_1 e_\alpha) \leq (2 - \alpha_0) e^{-\alpha p e^{p \sigma} R_1^p} \left( \frac{c_1}{R_1^p} + \frac{\alpha p e^{p \sigma} R_1^p}{R_1^p} \right) \ll \mathcal{J}_1 f(R_1 e_\alpha) \]
for a uniformly small \( \delta > 0 \) and \( R/4 \leq R_1 < \sqrt{n} R \). \( \square \)

**Lemma 3.1.4.** Given any \( \alpha_0 \in (1, 2) \) and \( 0 < R < R_0 \), there exists a function \( \Psi \in B(R^n) \) such that
(a) \( \Psi \) is continuous on \( R^n \), (b) \( \Psi = 0 \) on \( B_2 \sqrt{n} R \), (c) \( \Psi > 2 \) on \( Q_{3R} \),
(d) \( \Psi \leq M \) on \( R^n \) for some \( M > 1 \), (e) \( M_{\psi} \Psi \) is continuous on \( B_2 \sqrt{n} R \),
(f) \( M_{\psi} \Psi > -\psi/R^\alpha \) on \( R^n \) where \( \psi \) is a positive bounded function on \( R^n \) which is supported in \( \overline{B}_{R/4} \) for any \( \sigma \in (\alpha_0, 2) \).

**Proof.** Let \( f(x) = \min(|x|^\alpha, |x|^\beta) \). Then we consider the function \( \Psi \) given by
\[
\Psi = \begin{cases} 
0 & \text{in } R^n \setminus B_2 \sqrt{n} R, \\
(f(|x|) - f(2 \sqrt{n} R)) & \text{in } B_2 \sqrt{n} R \setminus B_R/4, \\
P & \text{in } B_R/4,
\end{cases}
\]
where \( P \) is a quadratic paraboloid chosen so that \( \Psi \) is \( C^{1,1} \) across \( \partial B_{R/4} \). We now choose the constants \( c \) and \( \delta \) so that \( \Psi(x) > 2 \) for \( x \in Q_{3R} \) and \( \Psi \leq M \) on \( R^n \) for some \( M > 1 \) (recall that \( Q_{3R} \subset B_2 \sqrt{n} R/2 \subset B_2 \sqrt{n} R \)). Since \( \Psi \in C^{1,1}(B_2 \sqrt{n} R) \), \( M_{\psi} \Psi \) is continuous on \( B_2 \sqrt{n} R \). Also by Lemma 3.1.1 we see that \( M_{\psi} \Psi \geq 0 \) on \( B_{R/4} \). Hence this completes the proof. \( \square \)

**Lemma 3.1.5.** Given any \( \sigma \in (0, 1] \) and \( 0 < R < R_0 \), there exists a function \( \Psi \in B(R^n) \) such that
(a) \( \Psi \) is continuous on \( R^n \), (b) \( \Psi = 0 \) on \( B_2 \sqrt{n} R \), (c) \( \Psi > 2 \) on \( Q_{3R} \),
(d) \( \Psi \leq M \) on \( R^n \) for some \( M > 1 \), (e) \( M_{\psi} \Psi \) is continuous on \( B_2 \sqrt{n} R \),
(f) \( M_{\psi} \Psi > -\psi/R^\alpha \) on \( R^n \) where \( \psi \) is a positive bounded function on \( R^n \) which is supported in \( \overline{B}_{R/4} \).

**Proof.** Let \( f(x) = c \exp(-|x|^\alpha) \) as in Lemma 3.1.3. From Lemma 3.1.3 we know that \( M_{\psi} \left( f(|x|) - f(2 \sqrt{n} R) \right) \geq 0 \) on \( B_2 \sqrt{n} R \setminus B_R/4 \). And the zero function is a solution of \( M_{\psi} u = 0 \) on \( B_{R/4} \). Since the supremum of two solutions is a subsolution, we have
\[ M_{\psi} \left[ \max(0, f(|x|) - f(2 \sqrt{n} R)) \right] \geq 0 \] on \( B_{R/4} \). The conclusion comes from the construction of \( \Psi \) as Lemma 3.1.4. \( \square \)

### 3.2. Estimates in measure

The main tool that shall be useful in proving Hölder estimates is a lemma that connects a pointwise estimate with an estimate in measure. The corresponding lemma in our context is the following.

**Lemma 3.2.1.** Let \( \alpha_0 \in (1, 2) \) and assume that \( \sigma \in (0, 1] \) or \( \sigma \in (\alpha_0, 2) \). If \( R \in (0, R_0) \), then there exist some constants \( \epsilon_0 > 0 \) and \( \nu \in (0, 1) \) and \( M > 1 \) for which if \( u \in B(R^n) \) is a viscosity supersolution to either \( M_{\psi} u \leq \epsilon_0 e^R \) with \( \sigma > 1 \) or \( M_{\psi} u \leq \epsilon_0 e^R \) with \( \sigma \leq 1 \), \( u \geq 0 \) on \( Q_{4\sqrt{n} R} \) and \( \inf Q_{3R} u \geq 1 \), then \( \|u - M\| \cap Q_3 \geq \nu |Q_3| \). For \( \sigma \in (\alpha_0, 2) \), \( \epsilon_0, \nu \) and \( M \) depend only on \( \lambda, \Lambda, n, \epsilon_0 \). And for \( \sigma \in (0, 1] \), \( \epsilon_0, \nu \) and \( M \) depend only on \( \lambda, \Lambda, n, \sigma, \alpha, \eta \).
Remark. We denote by $Q_r(x)$ an open cube $\{y \in \mathbb{R}^n : |y - x|_\infty \leq r/2\}$ and $Q_r = Q_r(0)$. If we set 
$Q = Q_r(x)$, then we denote by $sQ = Q_{sr}(x)$ for $s > 0$.

Proof of Lemma 3.2.1 (Case I: $1 < \sigma < 2$.) We consider the function $\psi := \Psi - u$ where $\Psi$ is the special function constructed in Lemma 3.1.4. Then we easily see that $\psi$ is upper semicontinuous on $B_{2\sqrt{\sigma}}$ and $\psi$ is not positive on $\mathbb{R}^n \setminus B_{2\sqrt{\sigma}}$. Moreover, $\psi$ is a viscosity subsolution to $\mathcal{M}_n^\sigma \psi \geq \mathcal{M}_n^\sigma \psi - \mathcal{M}_n^\sigma u \geq -(\psi + \epsilon_0)/R^\sigma$ on $B_{2\sqrt{\sigma}}$. So we want to apply Theorem 2.1.4 (rescaled) to $\psi$. Let $\Gamma$ be the concave envelope of $\psi$ in $B_{2\sqrt{\sigma}}$. Since inf$_{B_{\sqrt{\sigma}}}(\psi - u) \leq 1$, inf$_{B_{\sqrt{\sigma}}}(\psi) > 2$ and $Q_{2\sqrt{\sigma}} \subset B_{2\sqrt{\sigma}}$, we easily see that $M_0 := sup_{B_{2\sqrt{\sigma}}} \psi = \psi(x_0) > 1$ for some $x_0 \in B_{2\sqrt{\sigma}}$. We consider the function $g$ whose graph is the cone in $\mathbb{R}^n \times \mathbb{R}$ with vertex $(x_0, M_0)$ and base $\partial B_{\sqrt{\sigma}}(x_0) \times \{0\}$. For any $\xi \in \mathbb{R}^n$ with $|\xi| < M_0/6 \sqrt{n\sigma}$, the hyperplane

$$H = \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : x_{n+1} = L(x) := M_0 + \xi \cdot (x - x_0)\}$$

is a supporting hyperplane for $g$ at $x_0$ in $B_{2\sqrt{\sigma}}(x_0)$. Then $H$ has a parallel hyperplane $H'$ which is a supporting hyperplane for $\psi$ in $B_{2\sqrt{\sigma}}$ at some point $x_1 \in B_{2\sqrt{\sigma}}$. By the definition of concave envelope, we see that $H'$ is also the hyperplane tangent to the graph of $\Gamma$ at $x_1$, so that $\xi = V\Gamma(x_1)$. This implies that $B_{M_0/6 \sqrt{n\sigma}}(0) \subset \Gamma(B_{2\sqrt{\sigma}})$. Thus we have that

$$C(n) \ln \left(\frac{(M_0/R)^n}{\xi^n} + 1\right) \leq \int_{C(\xi, \Gamma, B_{2\sqrt{\sigma}})} g_\xi(V\Gamma(y)) \det[D^2\Gamma(y)]^{-\frac{1}{2}} dy,$$

where $g_\xi$ is the function given in Corollary 2.1.3. We also observe as shown in [CC] that

$$\left|\Gamma(B_{2\sqrt{\sigma}} \setminus C(\psi, \Gamma, B_{2\sqrt{\sigma}}))\right| = 0.$$

Let $(Q_i)$ be the finite cube family of cubes given by Theorem 2.1.4 (rescaled on $B_{2\sqrt{\sigma}}$). Then it follows from (3.2.6), (3.2.2) and Theorem 2.1.4 that, for $Q_i = Q_i \cap C(\psi, \Gamma, B_{2\sqrt{\sigma}})$,

$$\ln \left(\frac{(sup_{B_{2\sqrt{\sigma}}} \psi)/R^n}{\xi^n} + 1\right) \leq C \int_{C(\xi, \Gamma, B_{2\sqrt{\sigma}})} g_\xi(V\Gamma(y)) \det[D^2\Gamma(y)]^{-\frac{1}{2}} dy$$

$$\leq C \left(\sum_j \sup_{Q_j} (R^{n-2}J_0(R))^n + \xi^{-n}(R^{n-2}(\psi + \epsilon_0)/R^\sigma)^n \right) |Q_i|$$

$$\leq C \left(R^{n-2}J_0(R)^n \sum_j |Q_j| + \xi^{-n} \sum_j \sup_{Q_j} (\psi + \epsilon_0)/R^\sigma|Q_j|\right)$$

$$\leq C \left(R^{n-2}J_0(R)^n + \xi^{-n} \sum_j \sup_{Q_j} (\psi + \epsilon_0)/R^\sigma|Q_j|\right).$$

Here we note that $K_R := \exp((R^{n-2}J_0(R))^n) \leq C < \infty$ for $1 < \sigma < 2$ and $R < 1$. If we set

$$\xi = \left(\sum_j \sup_{Q_j} (\psi + \epsilon_0)/R^\sigma|Q_j|\right)^{1/n}$$

in (3.2.7), then we have that

$$\sup_{B_{2\sqrt{\sigma}}} \psi \leq CR \left(\sum_j \sup_{Q_j} (\psi + \epsilon_0)/R^\sigma|Q_j|\right)^{1/n}$$

$$\leq C\epsilon_0 + CR \left(\sum_j (sup \psi)^n|Q_j|\right)^{1/n}.$$

Since inf$_{Q_j} \psi \leq 1$ and inf$_{Q_j} \Psi > 2$, we see that sup$_{B_{2\sqrt{\sigma}}} \psi > 1$. If we choose $\epsilon_0$ small enough, the above inequality (3.2.4) implies that

$$\frac{1}{2^{|n|} R} \leq C \left(\sum_j (sup \psi)^n|Q_j|\right)^{1/n}.$$
We recall from the proof of Lemma 3.1.4 that $\psi$ is supported on $B_{R/4}$ and bounded on $\mathbb{R}^n$. Thus the above inequality becomes
\[
\frac{1}{2}|Q_k| \leq C\left(\sum_{Q \cap B_{R/4} \neq \emptyset} |Q_j|\right).
\]
which provides a lower bound for the sum of the volumes of the cubes $Q_j$ intersecting $B_{R/4}$ as follows;
\begin{equation}
(3.2.5) \sum_{Q_j \cap B_{R/4} \neq \emptyset} |Q_j| \geq \nu|Q_k|.
\end{equation}
Since $y \in C(\nu, \Gamma, B_{2\sqrt{\nu}})$ implies $\nu(y) \geq 0$, and thus $u(y) \leq \Psi(y) \leq M$ by Lemma 3.1.4 Hence it follows from (3.2.5) that
\[
||u - M|| \cap Q_k \geq ||u - \Psi|| \cap Q_k \geq |C(\nu, \Gamma, B_{2\sqrt{\nu}}) \cap Q_k| \geq \sum_{Q_j \cap B_{R/4} \neq \emptyset} |Q_j| \geq |Q_k|.
\]
Hence we complete the proof of (Case 1).

(Case 2: $0 < \sigma \leq 1$) We know $B_{\sigma R_{0}/(\sigma - \sqrt{2})Q}(0) \subset VT(B_{2\sqrt{\nu}})$ and $|VT(B_{2\sqrt{\nu}}) \cap C_{\epsilon}(u, \Gamma, B_{2\sqrt{\nu}})| \geq \eta|B_{\sigma R_{0}/(\sigma - \sqrt{2})Q}(0)|$

Thus we have that, $Q_j = \overline{Q_j} \cap C_{\epsilon}(u, \Gamma, B_{2\sqrt{\nu}})$

(3.2.6) $C(n) \ln \left(\frac{\eta(M_{0}/R)^{\nu}}{\epsilon^{n}} + 1\right) \leq \int_{C_{\epsilon}(u, \Gamma, B_{2\sqrt{\nu}})} \psi(y) \det[D^2 \Gamma(y)] - dy,$

where $\psi$ is the function given in Corollary 2.1.3. Let $|Q|$ be the finite family of cubes given by Theorem 2.1.4(rescaled on $B_{2\sqrt{\nu}}$). Then it follows from (3.2.6), (3.2.2) and Theorem 2.1.4 that
\begin{equation}
\ln \left(\frac{\eta[(\sup_{B_{2\sqrt{\nu}}} R)^{\nu}]/R^{\nu}}{\epsilon^{n}} + 1\right) \leq C \int_{C_{\epsilon}(u, \Gamma, B_{2\sqrt{\nu}})} \psi(y) \det[D^2 \Gamma(y)] - dy
\end{equation}
\[
\leq C \left(\sup_{j} \int_{Q_j} (R^{n-2} R^{1-\epsilon})^{\nu} \psi(y) / R^{\nu} dy \right)
\leq C \left(\sum_{j} |Q_j| \epsilon^{n} \sup_{Q_j} (\psi(y) / R^{\nu}) \right)
\leq C \left(1 + \epsilon^{n} \sum_{j} |Q_j| \sup_{Q_j} (\psi(y) / R^{\nu}) \right).
\end{equation}
The conclusion comes from the similar argument as (Case 1).

We split $Q_k$ into $2^n$ cubes of half side. We do the same splitting step with each one of these $2^n$ cubes and we continue this process. The cubes obtained in this way are called dyadic cubes. If $Q$ is a dyadic cube different from $Q_k$, then we say that $Q$ is the predecessor of $Q$ if $Q$ is one of $2^n$ cubes obtained from splitting $Q$.

Lemma 3.2.2. Let $A, B$ be measurable sets with $A \subset B \subset Q_k$. If $\delta \in (0, 1)$ is some number such that (a) $|A| \leq \delta$ and (b) $Q \subset B$ for any dyadic cube $Q$ with $|A \cap Q| > \delta |Q|$, then $|A| \leq \delta |B|$. The following lemma is a consequence of Lemma 3.2.1 and Lemma 3.2.2

Lemma 3.2.3. Under the same condition as Lemma 3.2.1 there are universal constants $C > 0$ and $\epsilon_{0} > 0$ such that
\[
||u - t|| \cap Q_k \leq C \epsilon^{\epsilon_{0}} |Q_k|, \forall \epsilon > 0.
\]

Remark 3.2.4. We note that $B_{R/2} \subset Q_k \subset Q_{3R} \subset B_{3\sqrt{2}R/2} \subset B_{2\sqrt{\nu}}$. 

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Proof. (Case 1: $1 < \sigma_0 < \sigma < 2$.)

First, we shall prove that

\[(3.2.8) \quad |u > M^k| \cap Q_k| \leq (1 - \nu)^k |Q_k|, \quad \forall k \in \mathbb{N},\]

where $\nu > 0$ is the constant as in Lemma 3.2.1 and $M > 1$ is the constant chosen in Lemma 3.2.1.

If $k = 1$, then it has been done in Lemma 3.2.1. Assume that the result (3.2.8) holds for $k - 1 (k \geq 2)$ and let

\[A = \{u > M^k\} \cap Q_k \quad \text{and} \quad B = \{u > M^{k-1}\} \cap Q_k.\]

If we can show that $|A| \leq (1 - \nu)|B|$, then (3.2.8) can be obtained for $k$. To show this, we apply Lemma 3.2.2. By Lemma 3.2.1 it is clear that $A \subset B \subset Q_k$ and $|A| \leq |u > M| \cap Q_k| \leq (1 - \nu)|Q_k|$. So it remains only to prove (b) of Lemma 3.2.2 that is, we need to show that if $Q = Q_{2^{-g}}(x_0)$ is a dyadic cube satisfying

\[(3.2.9) \quad |A \cap Q| > (1 - \nu)|Q|\]

then $\overline{Q} \subset B$. Indeed, we suppose that $\overline{Q} \not\subset B$ and take $x, \in \overline{Q}$ such that

\[(3.2.10) \quad u(x) \leq M^{k-1}.\]

We now consider the transformation $x = x_0 + y, y \in Q_{2^{-g}}$, $x \in Q = Q_{2^{-g}}(x_0)$ and the function $v(y) = u(x)/M^{k-1}$. If we can show that $v$ satisfies the hypothesis of Lemma 3.2.1, then we have that $v|Q < |u(x) \leq M^k| \cap Q)$, and thus $|Q \setminus A| > v|Q|$ which contradicts (3.2.9).

To complete the proof, we consider once again the transformation

\[x = x_0 + z, \quad z \in B_2 \sqrt{\nu} x \in B_2 \sqrt{\nu} \subset B_2 \sqrt{\nu} \]

and the function $v(z) = u(x)/M^{k-1}$. It now remains to show that $v$ satisfies the hypothesis of Lemma 3.2.1. We now take any $\varphi \in C^2_{2 \sqrt{\nu} - g}(i; z)^{-}$. If we set $\psi = M^{-1} \varphi(-, x_0)$, then we observe that

\[\varphi \in C^2_{2 \sqrt{\nu} - g}(i; z)^{-} \Rightarrow \psi \in C^2_{2 \sqrt{\nu} - g}(x_0)(u; x_0 + z)^{-}.\]

Since $B_2 \sqrt{\nu} - g(x_0) \subset B_2 \sqrt{\nu}$, we have that

\[M_{\nu} u(z; \nabla \psi) \leq \mathcal{L} v(z; \nabla \psi)\]

\[= \frac{1}{M^{k-1}} \int_{R^n} \mu(u, x_0 + z, y, \nabla \psi)K(y) \, dy \]

\[\leq \frac{1}{M^{k-1}} \mathcal{L} u(x_0 + z; \nabla \psi)\]

for any $\mathcal{L} \in \mathfrak{d}$. Taking the infimum of the right-hand side in the above inequality, we obtain that

\[M_{\nu} v(z; \nabla \psi) \leq \frac{1}{M^{k-1}} M_{\nu} u(x_0 + z; \nabla \psi).\]

Thus we have that

\[M_{\nu} v(z; \nabla \psi) \leq \frac{\varepsilon}{(2 - R)^{\nu}},\]

because $M_{\nu} u \leq \frac{\varepsilon}{(2 - R)^{\nu}}$ on $B_2 \sqrt{\nu}$. Also it is obvious that $\nu \geq 0$ on $\mathbb{R}^n$ and we see from (3.2.10) that $\inf_\nu v \leq 1$. Finally the result follows immediately from (3.2.8) by taking $C = (1 - \nu)^{-1}$ and $\varepsilon > 0$ so that $1 - \nu = M^{-\nu}$.

(Case 2: $0 < \sigma \leq 1$.) The argument above is based on Lemma 3.1.1 C-Z decomposition Lemma 3.2.2 and the invariance under translation that $M_{\nu, \mathbb{R}^n} v(z; \nabla \psi)$ also has. A similar argument gives us the conclusion.

By a standard covering argument we obtain the following theorem.
Theorem 3.2.5. Under the same condition as Lemma 3.2.1, then there are universal constants $C > 0$ and $\varepsilon > 0$ such that

$$\left| \{ u > t \} \cap B_\delta \right| \leq C \sqrt{1/\delta} |B_\delta|, \quad \forall \ t > 0.$$  

In contrast to symmetric cases, we note that we can not obtain the following theorem by rescaling the above theorem because our cases are not scaling invariant. We note that Theorem 3.2.6 on $r \in (0,1)$ shall be applied to obtain a Harnack inequality, and also Theorem 3.2.6 on $r \in [R,2R]$ will be used to prove Hölder estimates and an interior $C^{1,\alpha}$-regularity.

**Theorem 3.2.6.** Let $\alpha_0 \in (1,2)$ and assume that $\sigma \in (0,1]$ or $\sigma \in (\alpha_0,2]$ and that $R \in (0,R_0]$. For a constant $\varepsilon_0 > 0$ given at Theorem 3.2.5 if $u$ is a viscosity supersolution to either $\mathcal{M}_{\sigma_0}^+ u \leq \varepsilon_0/R^\sigma$ with $\sigma > 1$ or $\mathcal{M}_{\sigma_0,R_0}^- u \leq \varepsilon_0/R^\sigma$ with $\sigma \leq 1$ such that $u \geq 0$ on $\mathbb{R}^n$, then there are universal constants $\varepsilon_0 > 0$ and $C > 0$ such that

$$\left| \{ u > t \} \cap B_{\delta} \right| \leq C \left( u(x) + \varepsilon_0 R^\sigma \right) t^{\sigma - \varepsilon_0}, \quad \forall \ t > 0.$$  

**Proof.** (Case 1: $1 < \alpha_0 < \sigma < 2$) Let $x \in \mathbb{R}^n$ and set $v(z) = u(z+x)/q$ for $z \in B_{r_2}$ where $q = u(x) + \varepsilon_0 R^\sigma$. Take any $\varphi \in \mathcal{C}_c^\infty (B_{r_2}; \mathbb{R})$. If we set $\psi = q \varphi(x)$, then we see that $\psi \in \mathcal{C}_c^\infty (B_{r_2}; \mathbb{R})$. Thus by the change of variables we have that

$$\mathcal{M}_{\sigma_0}^- v(z; \nabla \varphi) \leq \mathcal{L} v(z; \nabla \psi)$$

$$= \frac{1}{q} \int_{\mathbb{R}^n} \mu(z+x,y; \nabla \psi) K(y) \, dy$$

$$:= \frac{1}{q} L u(z+x; \nabla \psi)$$

for any $\mathcal{L} \in \mathcal{L}_0$. Taking the infimum of the right-hand side in the above inequality, we get that

$$\mathcal{M}_{\sigma_0}^- v(z; \nabla \varphi) \leq \frac{1}{q} \mathcal{M}_{\sigma_0}^- u(rz + x; \nabla \psi) \leq \frac{\varepsilon_0}{r^\sigma}.$$  

Thus we have that $\mathcal{M}_{\sigma_0}^- v \leq \frac{\varepsilon_0}{r^\sigma}$ on $B_{r_2}$. Applying Theorem 3.2.5 to the function $v$, we complete the proof.

(Case 2: $0 < \sigma \leq 1$) A similar argument gives us the conclusion. \hfill $\square$

4. Regularity Theory

**Theorem 4.1.1.** Let $\alpha_0 \in (1,2)$ and assume that $\sigma \in (0,1]$ or $\sigma \in (\alpha_0,2]$ and that $R \in (0,R_0]$. If $u \in B(\mathbb{R}^n)$ is a positive function such that

$$\mathcal{M}_{\sigma_0}^- u \leq \frac{C_0}{R^\sigma_0} \quad \text{and} \quad \mathcal{M}_{\sigma_0}^+ u \geq -\frac{C_0}{R^\sigma_0} \quad \text{with} \quad \alpha_0 < \sigma < 2 \quad \text{on} \quad B_{2R}$$  

or

$$\mathcal{M}_{\sigma_0,R_0}^- u \leq \frac{C_0}{R^\sigma_0} \quad \text{and} \quad \mathcal{M}_{\sigma_0,R_0}^+ u \geq -\frac{C_0}{R^\sigma_0} \quad \text{with} \quad 0 < \sigma \leq 1 \quad \text{on} \quad B_{2R}$$

in the viscosity sense, then there is uniform constant $C > 0$ such that

$$\sup_{B_{r_2}} u \leq C \left( \inf_{B_{r_2}} u + C_0 \right).$$  

For $\sigma \in (\alpha_0,2)$, $C$ depends only on $\Lambda, \Lambda$, the dimension $n$, and $\alpha_0$. And for $\sigma \in (0,1)$, $C$ depends only on $\lambda, \Lambda$, $n$, $\sigma$, and $\eta$.

**Proof.** (Case 1: $1 < \alpha_0 < \sigma < 2$) Let $\hat{x} \in B_{R/2}$ be a point so that $\inf_{B_{R/2}} u = u(\hat{x})$. Then it is enough to show that

$$\sup_{B_{R/2}} u \leq C \left( u(\hat{x}) + C_0 \right).$$  

4.1. Harnack inequality.
Without loss of generality, we may assume that \( u(\hat{x}) \leq 1 \) and \( C_0 = 1 \) by dividing \( u \) by \( u(\hat{x}) + C_0 \). Let \( \varepsilon > 0 \) be the number given in Theorem 3.2.6 and let \( \beta = n/\varepsilon \). We now set \( s_0 = \inf\{ s > 0 : u(x) \leq s(1 - |x|/R)^{-\beta} \quad \forall x \in B_R \} \). Then we see that \( s_0 > 0 \) because \( u \) is positive on \( \mathbb{R}^n \). Also there is some \( x_0 \in B_R \) such that \( u(x_0) = s_0(1 - |x_0|/R)^{-\beta} = s_0 \left( \frac{d_0}{R} \right)^{-\beta} \) where \( d_0 = \delta(\mathcal{B}_0, \partial B_R) \leq R \).

To finish the proof, we have only to show that \( s_0 \) can not be too large because \( u(x) \leq C_1(1 - |x|/R)^{-\beta} \leq C \) for any \( x \in B_{2\delta}(x_0) \) if \( C_1 > 0 \) is some constant with \( s_0 \leq C_1 \). Assume that \( s_0 \) is very large. Then by Theorem 3.2.6 we have that

\[
\left| \left| u \geq u(x_0)/2 \right| \cap B_R \right| \leq \frac{2}{u(x_0)} \left| B_R \right| \leq C s_0^{-\varepsilon} d_0^n.
\]

Since \( |B_r| = C d_0^n \) for \( r = d_0/2 < R \), we easily obtain that

\[
\left| \left| u \geq u(x_0)/2 \right| \cap B_r(x_0) \right| \leq \frac{2}{u(x_0)} \left| B_r \right| \leq C s_0^{-\varepsilon} |B_r|.
\]

In order to get a contradiction, we estimate \( \left| \left| u \leq u(x_0)/2 \right| \cap B_{\delta}(x_0) \right| \) for some very small \( \delta > 0 \) (to be determined later). For any \( x \in B_{2\delta}(x_0) \), we have that \( u(x) \leq s_0(\delta_0 - \delta d_0/\delta) \leq u(x_0)(1 - \delta)^{-\beta} \) for \( \delta > 0 \) so that \((1 - \delta)^{-\beta} \) is close to 1. We consider the function

\[
v(x) = (1 - \delta)^{-\beta} u(x) - u(x).
\]

Then we see that \( v \geq 0 \) on \( B_{2\delta}(x_0) \), and also \( M_{\mu_v} v \leq \frac{1}{\delta} \) on \( B_{\delta}(x_0) \) because \( M_{\mu_v} u \geq \frac{1}{\delta} \) on \( B_{\delta}(x_0) \). We now want to apply Theorem 3.2.6 to \( v \), however \( v \) is not positive on \( \mathbb{R}^n \) but only on \( B_{\delta}(x_0) \). To apply Theorem 3.2.6, we consider \( w = v \) instead of \( v \). Since \( w = v + v^\ast \), we have that \( M_{\mu_w} w \leq M_{\mu_v} v + M_{\mu_v} v^\ast \leq \frac{1}{\delta} + M_{\mu_v} v^\ast \) on \( B_{\delta}(x_0) \). Since \( v^\ast \equiv 0 \) on \( B_{2\delta}(x_0) \), if \( x \in B_{\delta}(x_0) \) then we have that \( \mu(v^\ast, x, y; \nabla \delta) = v^\ast(x + y), \quad y \in B_{\delta}(x_0) \) and \( \delta \in C_{\delta_{\delta_0}(v^\ast; x)^\ast} \). Take any \( \varphi \in C_{\delta_{\delta_0}(v^\ast; x)^\ast}^2 \) and any \( x \in B_{\delta_0}(x_0) \). Since \( x + B_{\delta_0} \subset B_{2\delta}(x_0) \), we thus have that

\[
M_{\mu_v} w(x; \nabla \delta) < \frac{1}{R^2} + (2 - \alpha) \int_{R^2} \frac{\Lambda \mu^+ (v^\ast, x, y; \nabla \delta)}{|y|^{n+\alpha}} dy - \frac{1}{R^2} + (2 - \alpha) \int_{|y|^n + |x+y| < \delta} \frac{-\Lambda \mu (x+y)}{|y|^{n+\alpha}} dy 
\]

\[
\leq \frac{1}{R^2} + (2 - \alpha) \int_{R^2} \left( u(x + y) - (1 - \delta)^{-\beta} u(x_0) \right) \frac{dy}{|y|^{n+\alpha}}.
\]

We consider the function \( h_\delta(x) = c(1 - |x|^2/R^2) \) for \( c > 0 \) and we set

\[
c_1 = \sup\{ c > 0 : u(x) \geq h_\delta(x), \quad \forall x \in \mathbb{R}^n \}.
\]

Then there is some \( x_1 \in B_R \) such that \( u(x_1) = c_1(1 - |x_1|^2/R^2) \) and we see that \( c_1 \leq 4/3 \) because \( u(\hat{x}) \leq 1 \). Since \( \nabla h_\delta(x) = -\frac{2x}{R^2} \), we have that

\[
(2 - \alpha) \int_{R^2} \frac{\mu^+ (u, x_1, y; \nabla h_\delta)}{|y|^{n+\alpha}} dy 
\]

\[
\leq (2 - \alpha) \int_{R^2} \left( h_\delta(x_1 + y) - h_\delta(x_1) - y : \nabla h_\delta(x_1) \right) \frac{dy}{|y|^{n+\alpha}} 
\]

\[
\leq C(2 - \alpha) \int_{\mathcal{B}_0} \frac{|y|^2/R^2}{|y|^{n+\alpha}} dy + C(2 - \alpha) \int_{\mathcal{B}_1 \setminus \mathcal{B}_0} \frac{|y|/R}{|y|^{n+\alpha}} dy 
\]

\[
\leq C(2 - \alpha) \int_{\mathcal{B}_0} \frac{|y|^2/R^2}{|y|^{n+\alpha}} dy + C(2 - \alpha) \int_{\mathcal{B}_1 \setminus \mathcal{B}_0} \frac{|y|/R}{|y|^{n+\alpha}} dy 
\]
for some constant $C > 0$ which is independent of $\sigma$, and so we have that

$$
\Lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{\mu^-(u, x_i, y; \nabla h_i)}{|y|^{n+\sigma}} \, dy \leq \frac{C}{R^\sigma}.
$$

Since $M_{\sigma}^R u(x_i) \leq \frac{1}{R^\sigma}$ on $B_{2R}$, by (4.1.3) we have that

$$
\frac{1}{R^\sigma} \geq M_{\sigma}^R u(x_i; \nabla h_i)
$$

$$
\geq \lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{\mu^+(u, x_i, y; \nabla h_i)}{|y|^{n+\sigma}} \, dy
- \Lambda(2 - \sigma) \int_{\mathbb{R}^n} \frac{\mu^-(u, x_i, y; \nabla h_i)}{|y|^{n+\sigma}} \, dy.
$$

Thus we obtain that 

$$
(2 - \sigma) \Lambda \int_{\mathbb{R}_+^n} \frac{(u(x + y) - (1 - \delta)^{-\sigma}u(x_i))}{|y|^{n+\sigma}} \, dy
\leq C(2 - \sigma) \Lambda \int_{\mathbb{R}_+^n} \frac{(u(x_i + y) - 4/R)}{|y|^{n+\sigma}} \, dy
\leq (2 - \sigma) \int_{\mathbb{R}^n} \frac{\mu^+(u, x_i, y; \nabla h_i)}{|y|^{n+\sigma}} \, dy \leq \frac{C}{R^\sigma}
$$

for any $x \in B_{\sigma}(x_i)$. Thus by (4.1.2) we obtain that

$$
M_{\sigma_0}^R w(x) \leq \frac{C}{R^\sigma} \leq \frac{C}{(\delta R)^\sigma} \text{ on } B_{\sigma}(x_i).
$$

Since $u(x_i) = s_0(d_i/R)^{-\sigma} = 2^{-\sigma}s_0(r/R)^{-\sigma}$ and $\beta_\varepsilon = n$, applying Theorem 3.2.6 we have that

$$
\left|\{u \leq u(x_i)/2 \cap B_{3\sigma_0}(x_i)\} = \left[\{u \geq u(x_i)((1 - \delta)^{-\sigma} - 1/2) \cap B_{3\sigma_0}(x_i)\} \right.
\leq C(\delta R)^{\sigma}\left[(1 - \delta)^{-\sigma} - 1\right]u(x_i) + C(\delta R)^{-\sigma}\left[(1 - \delta)^{-\sigma} - 1\right]u(x_i)\right]^{\varepsilon\sigma}
\leq C(\delta R)^{\sigma}\left[(1 - \delta)^{-\sigma} - 1\right]^{\varepsilon\sigma} + \delta^{-\varepsilon\sigma}s_0^{-\varepsilon\sigma}.
$$

We now choose $\delta > 0$ so small enough that $C(\delta R)^{\sigma}(1 - \delta)^{-\sigma} - 1 \leq |B_{3\sigma_0}(x_i)|/4$. Since $\delta$ was chosen independently of $s_0$, if $s_0$ is large enough for such fixed $\delta$ then we get that $C(\delta R)^{\sigma}s_0^{-\varepsilon\sigma} \leq |B_{3\sigma_0}(x_i)|/4$. Therefore we obtain that $\left|\{u \leq u(x_i)/2 \cap B_{3\sigma_0}(x_i)\} \leq |B_{3\sigma_0}(x_i)|/2$. Thus we conclude that

$$
\left|\{u \geq u(x_i)/2 \cap B_{\sigma}(x_i)\} \geq \left|\{u \geq u(x_i)/2 \cap B_{3\sigma_0}(x_i)\} \right.
\geq \left|\{u > u(x_i)/2 \cap B_{3\sigma_0}(x_i)\} \right.
\geq \left|B_{3\sigma_0}(x_i)\right| - \left|B_{3\sigma_0}(x_i)/2\right|
= \left|B_{3\sigma_0}(x_i)/2\right| = C|B_1|
$$

which contradicts (4.1.1) if $s_0$ is large enough. Thus we complete the proof.

(Case 2: $0 < \sigma \leq 1$) The main step in the argument above is to control the error $M_{\sigma}^R u$ for $M_{\sigma_0}^{R\sigma} w(x; \nabla \psi)$. The conclusion comes from the same line of arguments by replacing $M_{\sigma}^R u$ by $M_{\sigma_0}^{R\sigma} u$. \qed
4.2. Hölder estimates. In this subsection, we obtain Hölder regularity result. The following technical lemma is very useful in proving it. As in [CS][KL], its proof can be derived from Theorem 3.2.6.

**Lemma 4.2.1.** Let \( \sigma_0 \in (1,2) \) and assume that \( \sigma \in (0,1] \) or \( \sigma \in (\sigma_0,2] \) and that \( R \in (0,R_0] \). If \( u \) is a bounded function with \( |u| \leq 1/2 \) on \( \mathbb{R}^n \) such that

\[
M_{\sigma_0} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad M_{\sigma_0} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}
\]

or

\[
M_{\sigma_0,R} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad M_{\sigma_0,R} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \leq 1 \text{ on } B_{2R}
\]

in the viscosity sense where \( \varepsilon_0 > 0 \) is some sufficiently small constant, then there is some universal constant \( \alpha > 0 \) (depending only on \( \lambda, \Lambda, n, \sigma_0 \)) such that \( u \in C^\alpha \) at the origin. More precisely,

\[
|u(x) - u(0)| \leq C \frac{|x|^\alpha}{R^\alpha}
\]

for some universal constant \( C > 0 \). For \( \sigma \in (\sigma_0,2) \), \( \alpha \) and \( C \) depend only on \( \lambda, \Lambda, n, \sigma_0 \). And for \( \sigma \in (0,1] \), \( \alpha \) and \( C \) depend only on \( \lambda, \Lambda, n, \sigma, \) and \( \eta \).

Lemma 4.2.1 and a simple rescaling argument give the following theorem as in [CS][KL].

**Theorem 4.2.2.** Let \( \sigma_0 \in (1,2) \) and assume that \( \sigma \in (0,1] \) or \( \sigma \in (\sigma_0,2] \) and that \( R \in (0,R_0] \). If \( u \) is a bounded function on \( \mathbb{R}^n \) such that

\[
M_{\sigma_0} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad M_{\sigma_0} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } \sigma_0 < \sigma < 2 \text{ on } B_{2R}
\]

or

\[
M_{\sigma_0,R} u \leq \frac{C_0}{R^\sigma} \quad \text{and} \quad M_{\sigma_0,R} u \geq -\frac{C_0}{R^\sigma} \quad \text{with } 0 < \sigma \leq 1 \text{ on } B_{2R}
\]

in the viscosity sense, then there is some constant \( \alpha > 0 \) such that

\[
\|u\|_{C^\alpha(B_{\varepsilon R/2}^n)} \leq \frac{C}{R^\alpha} \left( \|u\|_{L^\infty(K)} + C_0 \right)
\]

where \( C > 0 \) is some universal constant. For \( \sigma \in (\sigma_0,2) \), \( \alpha \) and \( C \) depend only on \( \lambda, \Lambda, n, \sigma_0 \). And for \( \sigma \in (0,1] \), \( \alpha \) and \( C \) depend only on \( \lambda, \Lambda, n, \sigma, \) and \( \eta \).

4.3. \( C^{\alpha,d} \)-estimates. We consider the class \( \mathcal{U}_0^d \) consisting of the operators \( \mathcal{L} \in \mathcal{U}_0 \) associated with kernels \( K \) for which (1.1.2) holds and there exists some \( q_1 > 0 \) such that

\[
\sup_{h \in B_{\sigma_1/2}^n} \int_{R^n \setminus B_{\sigma_1/2}} \frac{|K(y) - K(y - h)|}{|h|} \, dy \leq C.
\]

If \( K \) is a radial function satisfying (1.1.2), then it is interesting that the condition 4.3.1 is not required. Indeed, if \( K(y) = (2 - \sigma)A/|y|^{n+d} \) for \( \lambda \leq A \leq \Lambda \), then it follows from the mean value theorem and the Schwartz inequality that

\[
\sup_{h \in B_{\sigma_1/2}^n} \int_{R^n \setminus B_{\sigma_1/2}} \frac{|K(y) - K(y - h)|}{|h|} \, dy = (2 - \sigma)2^{n+d+1} \frac{(n + \sigma)\Lambda \omega_n}{(\sigma + 1)q_1^{n+d}} \leq C,
\]

because \( |y| \geq 2|h| \) for any \( h \in B_{\sigma_1/2} \) and \( y \in R^n \setminus B_{\sigma_1/2} \) and \( |y - vh| \geq |y| - |h| \geq |y| - |y|/2 = |y|/2 \) for \( \tau \in [0,1] \).

If we apply Theorem 4.2.2 on the Hölder difference quotients which satisfies the same class of operators as the solution, we will have the following interior \( C^{\alpha,d} \)-estimates as in [CS][CC]. For \( R \in (0,R_0] \), we set \( ||u||_{C^{\alpha,d}(B_R^n)} = ||u||_{L^\infty(B_R^n)} + ||D^\alpha u||_{L^\infty(B_R^n)} + ||R^{1+d}||_{L^\infty(B_R^n)} \).

**Theorem 4.3.1.** For \( \sigma_0 \in (1,2) \), then there is some \( q_1 > 0 \) so that if \( I \in S_{\sigma_0}^d \) for \( \sigma \in (\sigma_0,2] \) or \( I \in S_{\sigma_0}^d \) for \( \sigma \in (0,1) \) is a nonlocal elliptic operator with respect to \( \mathcal{U}_0^d \) in the sense of Definition 1.1.5.
and $u \in C(B(\mathbb{R}))$ is a viscosity solution to $Iu = 0$ on $B_1$, then there is a universal constant $\alpha > 0$ such that

$$||u||_{C^{\alpha}(B_{(1/2)})} \leq C \left( ||u||_{C^{\alpha}(B_1)} + R^{\alpha(I0)} \right)$$

for some constant $C > 0$ and the constant given in (4.3.1) (where we denote by $I0$ the value we obtain when we apply $I$ to the constant function that is equal to zero). For $\sigma \in (\sigma_0, 2)$, $\rho_1$, $\alpha$ and $C$ depend only on $\lambda$, $\Lambda$, the dimension $n$, and $\sigma_0$. And for $\sigma \in (0, 1]$, $\rho_1$, $\alpha$ and $C$ depend only on $\lambda$, $\Lambda$, $n$, $\sigma$ and $\eta$.

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References

[A] Sayah Awaﬁf, Équations d’Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. I. Unicité des solutions de viscosité. Comm. Partial Differential Equations 16(6-7), 1991, 1057–1074.

[BI] Guy Barles and Cyril Imbert, Second-order elliptic integro-diﬀerential equations: Viscosity solutions’ theory revisited, Preprint.

[BBC] Martin T. Barlow; Richard F. Bass; Zhen-Qing Chen; Moritz Kassmann, Non-local Dirichlet forms and symmetric jump processes, Trans. Amer. Math. Soc. 361, 2009, no. 4, 1963–1999.

[BK1] Richard F. Bass and Moritz Kassmann, Harnack inequalities for non-local operators of variable order, Trans. Amer. Math. Soc. 357(2), 2005, 837–850.

[BK2] Richard F. Bass and Moritz Kassmann, Hölder continuity of harmonic functions with respect to operators of variable order, Comm. Partial Differential Equations 30(7-9), 2005, 1249–1259.

[BL] Richard F. Bass and David A. Levin, Harnack inequalities for jump processes, Potential Anal. 17(4), 2002, 375–388.

[C] Luis A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. of Math. (2) 130(1), 1989, 189–213.

[CS] Luis A. Caffarelli and Luis Silvestre, Regularity theory for fully nonlinear integro-diﬀerential equations, Comm. Pure Appl. Math. 62(5), 2009, 597–638.

[CC] Luis A. Caffarelli and Xavier Cabré, Fully nonlinear elliptic equations, volume 43 of American Mathematical Society, Colloquium Publications. American Mathematical Society, Providence, RI, 1995.

[GT] David Gilbarg and Neil S. Trudinger, Elliptic partial differential equations of second order, Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 224. Springer-Verlag, Berlin, 1983. xxii+513 pp.

[GD] E. De Giorgi and G. Dal Maso, $\Gamma$-convergence and calculus of variations, Lecture notes in Mathematics 979, 1983, Springer-Verlag.

[I] Hitoshi Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs, Comm. Pure Appl. Math. 42(1), 1989, 15–45.

[J] Robert Jensen, The maximum principle for viscosity solutions of fully nonlinear second order partial diﬀerential equations, Arch. Rational Mech. Anal. 101(1), 1988, 1–27.

[KL] Yong-Cheol Kim and Ki-Ahm Lee, Regularity results for fully nonlinear integro-diﬀerential operators with nonsymmetric positive kernels: Subcritical Case, submitted

[KS] N. V. Krylov and M. V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure, Dokl. Akad. Nauk. USSR 245, 1979, 18–20.

[S] Luis Silvestre, Hölder estimates for solutions of integro-diﬀerential equations like the fractional laplace, Indiana Univ. Math. Jour. 55(3), 2006, 1155–1174.

[S0] Halil Mete Soner, Optimal control with state-space constraint. II, SIAM J. Control Optim. 24(6), 1986, 1110–1122.

[SV] Renming Song and Zoran Vondraček, Harnack inequality for some classes of Markov processes, Math. Z. 246(1-2), 2004, 177–202.

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