THE CONE OF BETTI DIAGRAMS OF BIGRADED
ARTINIAN MODULES OF CODIMENSION TWO

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Abstract. We describe the positive cone generated by bigraded
Betti diagrams of artinian modules of codimension two, whose res-
solutions become pure of a given type when taking total degrees. If
the differences of these total degrees, \( p \) and \( q \), are relatively prime,
the extremal rays are parametrised by order ideals in \( \mathbb{N}^2 \) contained
in the region \( px + qy < (p - 1)(q - 1) \). We also consider some
examples concerning artinian modules of codimension three.

Introduction

In \cite{2}, D.Eisenbud, J.Weyman, and the second author gave for every
sequence of integers \( d : d_0 < d_1 < \cdots < d_n \) a construction of pure
resolutions of graded artinian modules over a polynomial ring \( S = \mathbb{k}[x_1, \ldots, x_n] \) (char \( \mathbb{k} = 0 \))
\[ S(-d_0)^{\beta_0} \leftarrow S(-d_1)^{\beta_1} \leftarrow \cdots \leftarrow S(-d_n)^{\beta_n}. \]
Moreover these resolutions were \( GL(n) \)-equivariant, and so in particu-
lar invariant for the diagonal matrices and hence \( \mathbb{Z}^n \)-graded.

In the case when \( S = \mathbb{k}[x_1, x_2] \), the first author and J.Söderberg
in \cite{1} Remark 3.2 gave a different construction of pure resolutions of
artinian bigraded modules. It had a bigraded Betti diagram distinct
from that of the equivariant resolution.

Example 0.1. Suppose \( d_1 - d_0 = 2 \) and \( d_2 - d_1 = 3 \). The equivariant
resolution has the following form where we have written the bidegrees
of the generators below the terms.

\[
\begin{align*}
(1) &\quad S^3 \leftarrow S^5 \leftarrow S^2. \\
&\quad (2,0) \quad (4,0) \quad (4,3) \\
&\quad (1,1) \quad (3,1) \quad (3,4) \\
&\quad (0,2) \quad (2,2) \\
&\quad (1,3) \\
&\quad (0,4)
\end{align*}
\]

Let \( \beta_1 \) be its bigraded Betti table. The resolution in \cite{1} is of a quotient
of a pair of monomial ideals. For the type above the resolution has the
following bidegrees.\[\begin{array}{c|c|c}
(2) & S^3 & S^5 \\
(4,0) & (6,0) & (6,3) \\
(2,2) & (4,2) & (3,6) \\
(0,4) & (3,3) & (2,4) \\
& (0,6) & \\
\end{array}\]

Denote by $\beta_2$ be its Betti diagram.

This indicated that there may be many types of multigraded Betti diagrams of $\mathbb{Z}^n$-graded artinian modules of codimension $n$ whose resolutions become pure of a given type when taking total degrees. In [4] the second author showed that the multigraded Betti diagram of the equivariant resolution has a fundamental position. This diagram and its twists with $a \in \mathbb{Z}^2$ form a basis for the linear space generated by multigraded Betti diagrams of artinian $\mathbb{Z}^n$-graded modules whose resolutions become pure of the given type when taking total degrees. Even more natural it is to describe the positive cone generated by the multigraded Betti diagrams.

In this paper we to this in the case when $S = \mathbb{k}[x_1, x_2]$. Let $e_1 = d_1 - d_0$ and $e_2 = d_2 - d_1$. We describe all the extremal rays of the positive cone $P(e_1, e_2)$ generated by bigraded Betti diagrams of artinian bigraded modules of codimension two whose resolutions become pure when taking total degrees, and where the differences of these total degrees are $e_1$ and $e_2$. In the example above the two resolutions, or rather their Betti diagrams, are essentially the full story in the sense that the extremal rays in $P(2,3)$ are exactly the rays generated by $\beta_1(a)$ and $\beta_2(a)$ for $a \in \mathbb{Z}^2$. To explain the general situation assume here for simplicity that $e_1$ and $e_2$ are relatively prime. Let $R(e_1, e_2)$ be the integer coordinate points in the region of the first quadrant of the coordinate plane bounded by the line $e_1x + e_2y < (e_1 - 1)(e_2 - 1)$. There is a partial order on $\mathbb{N}^2$ given by $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$, and the region $R(e_1, e_2)$ inherits this. An order ideal in $R(e_1, e_2)$ corresponds to a partition $\lambda$. We give a construction which to each partition $\lambda$ in $R(e_1, e_2)$ associates a bigraded resolution

$$S^{e_2} \leftarrow S^{e_2+e_1} \leftarrow S^{e_1}.$$ 

Let $\beta_{\lambda}$ be the bigraded Betti diagram of this complex. The following is our main result in the case that $e_1$ and $e_2$ are relatively prime.

**Theorem.** The extremal rays in the cone $P(e_1, e_2)$ are the $\beta_{\lambda}(a)$ where $a$ varies over $\mathbb{Z}^2$ and $\lambda$ ranges over partitions contained in the region $R(e_1, e_2)$.

The general case is formulated in Theorems 2.8 and 3.2. In the region $R(e_1, e_2)$ there are two distinguished partitions, the maximal one and the empty one. It turns out that the maximal one corresponds to the
equivariant complex and the empty one corresponds to the bigraded resolution of a quotient of monomial ideals constructed in [1].

The organisation of the paper is as follows. Section 1 contains preliminaries. First we give the multigraded Herzog-Kühl equations which give strong restrictions on Betti diagrams of multigraded artinian modules. We recall the equivariant resolution, and the result of [4] that its twists generate the linear space of multigraded Betti diagrams of artinian \( \mathbb{Z}^n \)-graded modules of codimension \( n \) whose resolution becomes pure when taking total degrees. This gives us a very simple alternative description of the positive cone \( P(e_1, e_2) \). This is used in Section 2 where we show that the extremal rays of the positive cone \( P(e_1, e_2) \) are generated by the Betti diagrams \( \beta_\lambda(a) \) for \( a \in \mathbb{Z}^2 \), provided these diagrams really come from resolutions. And that such resolutions really exist is established in Section 3. In Section 4 we briefly discuss the positive cone in the case of three variables, providing an example.

1. Preliminaries

Let \( S = k[x_1, \ldots, x_n] \) be the polynomial ring over a field \( k \). We shall study \( \mathbb{Z}^n \)-graded free resolutions of artinian \( \mathbb{Z}^n \)-graded \( S \)-modules

\[
F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n.
\]

For a multidegree \( a = (a_1, a_2, \ldots, a_n) \) in \( \mathbb{Z}^n \) let \( |a| = \sum a_i \) be its total degree. We shall be interested in the case that these resolutions become pure resolutions if we make them singly graded by taking total degrees. That is there is a sequence \( d_0 < d_1 < \cdots < d_n \) such that

\[
 F_i = \oplus_{|a| = d_i} S(-a)^{\beta_i, a}.
\]

1.1. Betti diagrams and the multigraded Herzog-Kühl equations. The multigraded Betti diagram of such a resolution is the element

\[
\{\beta_i, a\}_{i = 0, \ldots, n} \in \oplus_{Z^n} \mathbb{Q}^{n+1}.
\]

A way of representing a multigraded Betti table which will be very convenient for us is to represent \( \beta = \{\beta_i, a\} \) where \( i = 0, \ldots, n \) and \( a \in \mathbb{Z}^n \) by Laurent polynomials

\[
B_i(t) = \sum_{a \in \mathbb{Z}^n} \beta_i, a \cdot t^a.
\]

We thus get an \((n + 1)\)-tuple of Laurent polynomials

\[
B = (B_0, B_1, \ldots, B_n).
\]

Also the module \( \oplus_{a \in \mathbb{Z}^n} S(-a)^{\beta_i, a} \) may be conveniently denoted as \( S.B_i \).

Let \( e_i = d_i - d_{i-1} \), so we get the differences \( e = (e_1, \ldots, e_n) \). Now let \( L(e) \) be the linear subspace of \( \oplus_{a \in \mathbb{Z}^n} \mathbb{Q}^{n+1} \) generated by multigraded
Betti diagrams of \( \mathbb{Z}^n \)-graded artinian \( S \)-modules whose resolutions become pure when taking total degrees, and where the difference sequence of these total degrees is \( e \). Similarly let \( P(e) \) be the positive cone in \( \oplus_{a \in \mathbb{Z}^n} \mathbb{Q}^{n+1} \) generated by such Betti diagrams.

There are some natural restrictions on \( L(e) \) coming from the multigraded Herzog-Kühl equations. If the resolution resolves the artinian module \( M \), the multigraded Hilbert series of \( M \) is the polynomial

\[
    h_M(t) = \sum_{i,a} (-1)^i \beta_{i,a} t^a \prod_{k=1}^n (1 - t_i),
\]

which gives

\[
    \sum_{i,a} (-1)^i \beta_{i,a} t^a = h_M(t) \cdot \prod_{k=1}^n (1 - t_i). \tag{3}
\]

For each multigrade \( a \in \mathbb{Z}^n \) and integer \( k = 1, \ldots, n \), let the projection \( \pi_k(a) \) be \((a_1, \ldots, \hat{a}_k, \ldots, a_n)\), the \( n-1 \)-tuple where we omit \( a_k \).

Now we have the multigraded analogs of the Herzog-Kühl (HK) equations. We obtain these by setting \( t_k = 1 \) in (3) for each \( k \). This gives for every \( \hat{a} \) in \( \mathbb{Z}^{n-1} \) and \( k = 1, \ldots, n \) an equation

\[
    \sum_{i, \pi_k(a) = \hat{a}} (-1)^i \beta_{i,a} = 0. \tag{4}
\]

Let \( L'(e) \) be the linear space of elements in \( \oplus_{a \in \mathbb{Z}^n} \mathbb{Q}^{n+1} \) which fulfil the multigraded HK-equations above, and which become pure diagrams when taking total degrees with the difference sequence of these total degrees equal to \( e \). Also let \( P'(e) \) be the cone in \( L'(e) \) consisting of the elements with nonnegative coordinates. There are natural injections \( L(e) \to L'(e) \) and \( P(e) \to P'(e) \). In [4] the second author showed that the first injection is an isomorphism and moreover gave an explicit basis for \( L(e) \) which we now describe.

1.2. The equivariant resolution. In [2] the second author together with D.Eisenbud and J.Weyman constructed \( GL(n) \)-equivariant pure resolutions of artinian modules. For a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) let \( S_{\lambda} \) be the associated Schur module, it is an irreducible representation of \( GL(n) \) (see for instance [5]). The action of the diagonal matrices in \( GL(n) \) gives a decomposition of \( S_{\lambda} \) as a \( \mathbb{Z}^n \)-graded vector space. The basis elements are given by semi-standard Young tableau of shape \( \lambda \) with entries from \( 1, 2, \ldots, n \). All the nonzero graded pieces in this decomposition have total degree \( |\lambda| = \sum_{i=1}^{n} \lambda_i \). The free module \( S \otimes_k S_{\lambda} \) then becomes a free multigraded module where the generators all have total degree \( |\lambda| \).

Now given the difference vector \( e \), let

\[
    \lambda_i = \Sigma_{j=i+1}^{n} e_j - 1
\]
and define a sequence of partitions for $i = 0, \ldots, n$ by
\[ \alpha(e, i) = (\lambda_1 + e_1, \lambda_2 + e_2, \ldots, \lambda_i + e_i, \lambda_{i+1}, \ldots, \lambda_n). \]
The construction in [2] then gives a $GL(n)$-equivariant resolution
\[ E(e) : S \otimes_k S_{\alpha(e, 0)} \leftarrow S \otimes_k S_{\alpha(e, 1)} \leftarrow \cdots \leftarrow S \otimes_k S_{\alpha(e, n)} \]
of an artinian $S$-module.

In the case of two variables $S = \mathbb{k}[x_1, x_2]$ the resolution takes the form
\[ E(e_1, e_2) : S \otimes_k S_{e_2-1,0} \leftarrow S \otimes_k S_{e_1+e_2-1,0} \leftarrow S \otimes_k S_{e_1+e_2-2,1}. \]

1.3. The linear space of Betti diagrams of multigraded artinian modules. For a multigraded Betti diagram $\beta = \{\beta_i,a\}$ and a multidegree $t$ in $\mathbb{Z}^n$, we get the twisted Betti diagram $\beta(-t)$ which in homological degree $i$ and multidegree $a$ is given by $\beta_{i,a-t}$. If $F$ is a resolution with Betti diagram $\beta$, then $F(-t)$ is a resolution with Betti diagram $\beta(-t)$.

Also let $F_r : S \rightarrow S$ be the map sending $x_i \mapsto x_i^r$. Denote by $S^{(r)}$ the ring $S$ with the $S$-module structure given by $F_r$. Given any complex $F$, we may tensor it with $- \otimes_S S^{(r)}$ and get a complex we denote by $F^{(r)}$. Note that if $F$ is pure with degrees $d$, then $F^{(r)}$ is pure with degrees $r \cdot d$.

In [4] we showed the following.

Theorem 1.1. Let $m = \gcd(e_1, \ldots, e_n)$ and let $e = m \cdot e'$. The space $L(e)$ is equal to the space $L'(e')$ of diagrams fulfilling the HK-equations, and the $\beta_{E(e')}^{(m)}(a)$ where $a$ varies over $\mathbb{Z}^n$, form a basis for $L(e)$.

Moreover if $E'$ is another resolution such that the $\beta_{E'}(a)$ form a basis, then $\beta_{E'}$ is an integer multiple of $\beta_{E(e')}^{(m)}(a)$ for some $a$.

This may also be formulated in terms of the associated $(n+1)$-tuple of Betti polynomials.

Corollary 1.2. Let $s = (s_0, \ldots, s_n)$ be the $(n+1)$-tuple of Betti polynomials of $E(e')^{(m)}$. If $B = (B_0, \ldots, B_n)$ is any $(n+1)$-tuple of Betti polynomials of an artinian $\mathbb{Z}^n$-graded module whose resolution becomes pure when taking total degrees and with difference vector $e$ of the total degrees, then $B = p \cdot s$ for some homogeneous Laurent polynomial $p$.

1.4. The linear space in the case of two variables. Now assume $S = \mathbb{k}[x_1, x_2]$. Let $\xi_d(t,u) = t^{d-1} + t^{d-2}u + \cdots + u^{d-1}$ be the cyclotomic polynomial. The first and last Betti polynomials of the equivariant resolution (6) are then respectively
\[ \xi_{e_2}(t,u), \quad (tu)^{e_2}\xi_{e_1}(t,u) \]
and the middle Betti polynomial is
\[ \xi_{e_1+e_2} = t^{e_2}\xi_{e_1}(t,u) + u^{e_1}\xi_{e_2}(t,u) = u^{e_2}\xi_{e_1}(t,u) + t^{e_1}\xi_{e_2}(t,u). \]

By Corollary 1.2 the space $L(e_1, e_2)$ may now be described as follows.
Lemma 1.3. Let $e_1 = mq$ and $e_2 = mp$ where $m$ is the greatest common divisor of $e_1$ and $e_2$. A triple of homogeneous Laurent polynomials $B_0, B_1, B_2$ whose degrees have $e_1$ and $e_2$ as differences, is in $L(e_1, e_2)$ if and only if the following two equations hold:

\[
B_2(t, u) \cdot \xi_p(t^m, u^m) = (tu)^{mp}B_0(t, u) \cdot \xi_q(t^m, u^m),
\]
\[
B_1(t, u) = u^{-pm}B_2(t, u) + u^{-qm}B_0(t, u)
\]

(8) \hspace{1cm} (9)

Proof. By Corollary 1.2 we have

\[
(B_0, B_1, B_2) = f(t, u) \cdot (\xi_q(t^m, u^m), \xi_{p+q}(t^m, u^m), (tu)^{p+q}\xi_q(t^m, u^m))
\]

This gives (8). Also (9) follows by (7). Conversely, if (8) and (9) hold, we may deduce that the equation above holds, so $(B_0, B_1, B_2)$ is in $L(e_1, e_2)$. □

For a homogeneous Laurent polynomial $f(t, u)$ denote by $f^{dh}(t)$ its dehomogenisation with respect to $u$. If we now dehomogenise equation (8) we get an equation

\[
B_2^{dh}(t) \cdot \xi_p(t^m) = B_0^{dh} \cdot \xi_q(t^m).
\]

Each of the first factors are uniquely determined by the other, and if the triple comes from an actual complex, the coefficients are non-negative.

With some abuse of notation we also identify the cone $P' = P'(e_1, e_2)$ with the positive cone of pairs of Laurent polynomials $(A(t), B(t))$ in one variable $t$ and with non-negative coefficients, such that

\[
B(t)\xi_p(t^m) = A(t)\xi_q(t^m).
\]

We shall in the next section describe the cone $P'$ completely. Recall that we have an injective map $P(e_1, e_2) \rightarrow P'(e_1, e_2)$. In Section 3 we show that this map is an isomorphism.

2. The positive cone of bigraded Betti diagrams

In this section we describe completely the positive cone $P'(e_1, e_2)$ of diagrams fulfilling the HK-equations (4). We shall show that there is a finite number of diagrams $\beta_\lambda$ parametrised by certain partitions $\lambda$ such that extremal rays in the positive cone are the one-dimensional rays generated by $\beta_\lambda(a)$ for $a \in \mathbb{Z}$.

Note. In the following we let $e_1 = mq$ and $e_2 = mp$ where $m$ is the greatest common divisor of $e_1$ and $e_2$.

2.1. Partitions. Let $\mathbb{N}^2$ have the partial ordering where $(a_1, a_2) \leq (b_1, b_2)$ if $a_1 \leq b_1$ and $a_2 \leq b_2$. An order ideal $T$ in $\mathbb{N}^2$ (a set closed under taking smaller elements) gives rise to two partitions. The first is given by

\[
\lambda_j = 1 + \max\{i \mid (i, j) \in T\}, \ j \geq 0.
\]
The second is the dual partition
\[ \mu_i = 1 + \max\{j \mid (i, j) \in T\}, \ i \geq 0. \]
(If for a given \( j \) no \((i, j)\) is in \( T \), we set \( \lambda_j = 0 \) and correspondingly for \( \mu_i \).) Note that \( \lambda \) and \( \mu \) are dual partitions. So \( \mu_i \) is the cardinality of \( \{j \mid \lambda_j > i\} \).

We shall be interested in order ideals \( T \) which are contained in the region \( R(p, q) \) in the first quadrant bounded by the following strict inequality
\[ px + qy < (p - 1)(q - 1). \]

**Lemma 2.1.** Let the order ideal \( T \) correspond to the partition \( \lambda \). Then \( T \) is contained in the region above if and only if every \( aq - p\lambda_{p-1-a} \) is nonnegative for \( 0 \leq a < p \). Correspondingly for the dual partition \( \mu \).

**Proof.** First note that \( aq - p\lambda_{p-1-a} \geq 0 \) if and only if
\[ (p - 1 - a)q + (\lambda_{p-1-a} - 1)p \leq pq - p - q. \]
Assume \( 0 \leq a < p \). If \( T \) is contained in \( R(p, q) \) then if \( \lambda_{p-1-a} \geq 1 \) it fulfills the second equation above and therefore the first. If \( \lambda_{p-1-a} = 0 \) the first equation is also fulfilled. Suppose now that \( T \) fulfills the first equation. Then when \( \lambda_{p-1-a} \geq 1 \) the point \((p - 1 - a, \lambda_{p-1-a} - 1)\) is in \( R(p, q) \), so \( T \) is contained in \( R(p, q) \). \( \square \)

The following easy lemma will be useful.

**Lemma 2.2.** Let \( P(t) = \sum c_it^i \) be a polynomial with positive coefficients. Write \( P(t)\xi_d(t) = \sum_{j \in \mathbb{Z}} \alpha_j t^j \). Then \( \alpha_j - \alpha_{j-1} = c_j - c_{j-d} \).

**Proof.** This is clear from \( \alpha_j = \sum_{i=j-d+1}^j c_i \). \( \square \)

The following result will essentially describe the extremal rays.

**Proposition 2.3.** Suppose \( p \) and \( q \) are relatively prime and let \( T \) be an order ideal in \( R(p, q) \). Write
\[ A_T(t) = \sum_{a=0}^{p-1} t^{aq-p\lambda_{p-1-a}}, \quad B_T(t) = \sum_{a=0}^{q-1} t^{ap-qu_{q-1-a}}. \]
Then
\[ A_T(t)\xi_q(t) = B_T(t)\xi_p(t). \]

**Proof.** Note that since \( p \) and \( q \) are relatively prime, the coefficient of each power \( t^j \) in \( A_T \) or \( B_T \) is 0 or 1. Writing \( \sum \alpha_j t^j \) for the product \( A_T(t)\xi_q(t) \) we see that when \( \alpha_j > \alpha_{j-1} \) we have \( \alpha_j = \alpha_{j-1} + 1 \). We shall show that the indices \( j \) for which this happens are exactly when \( j = 0 \) or \( j = pq - p - q - qu - pw \) where \((u, v)\) is a maximal element in the poset \( T \), i.e. \((u, v)\) is in \( T \), but neither \((u + 1, v)\) nor \((u, v + 1)\) is in \( T \). Since the analog holds for the product \( B_T(t)\xi_p(t) \), these products must increase exactly at the same indices. An analog argument also
show that they decrease at exactly the same indices, namely $\alpha_j < \alpha_{j-1}$ iff $j = pq - qu - pv$ where $(u, v)$ is not in $T$ but $(u - 1, v)$ and $(u, v - 1)$ are either in $T$ or have $-1$ as a coordinate. Hence the products are equal.

Now $\alpha_j > \alpha_{j-1}$ when $j = aq - p\lambda_{p-1-a}$ for some $a$ but $(a - 1)q - p\lambda_{p-1-a}$ is not a power in $A(t)$. Thus either $a = 0$ or $\lambda_{p-a} < \lambda_{p-1-a}$. But this means that $j = 0$ or $(u, v) = (\lambda_{p-1-a} - 1, p - 1 - a)$ is a maximal element in $T$. We easily compute that

$$j = aq - p\lambda_{p-1-a} = pq - p - q - qu - pv.$$  

$\Box$

**Remark 2.4.** The empty poset $T = \emptyset$ corresponds to the polynomials

$$A_{\emptyset}(t) = \xi_p(t^p)$$

$$B_{\emptyset}(t) = \xi_q(t^q).$$

Via the correspondence at the end of Subsection 1.4 these corresponds to the Betti diagram of a resolution of an artin module. This is the module described in [1, Remark 3.2] which is the quotient $I/J$ of two monomial ideals in $k[x, y]$: the ideal $I = (x^{(p-1)q}, x^{(p-2)q}y^q, \ldots, y^{(p-1)q})$ and the ideal $J = (x^{pq}, x^{p(q-1)}y^p, \ldots, y^{pq})$.

**Remark 2.5.** There is also a maximal order ideal $\hat{T}$ in the region $R(p,q)$ and this corresponds to the polynomials

$$A_{\hat{T}}(t) = \xi_p(t)$$

$$B_{\hat{T}}(t) = \xi_q(t)$$

which again via the correspondence at the end of Subsection 1.4 corresponds to the Betti diagrams of the $GL(2)$-equivariant resolutions $E(p,q)$ constructed in [2].

2.2. **Decomposing.** Now any polynomial $A(t)$ may be written

$$A(t) = \sum_{a=0}^{p-1} \sum_{b \in \mathbb{Z}} \alpha_{a,b} t^{aq-bp}.$$  

For each $a$ let $\lambda_{p-1-a}$ be the maximum of the set $\{b | \alpha_{a,b} \neq 0\}$. We may then write

$$A(t) = A_{\min}(t) + A_{\perp}(t)$$

where

$$A_{\min}(t) = \sum_{a=0}^{p-1} \alpha_{a,\lambda_{p-1-a}} t^{aq-\lambda_{p-1-a}p}.$$  

Correspondingly we may write

$$B(t) = \sum_{a=0}^{p-1} \sum_{b \in \mathbb{Z}} \beta_{a,b} t^{ap-bq}.$$
For each a let $\mu_{q-1-a}$ be the maximum of the set $\{b \mid \beta_{a,b} \neq 0\}$. We may then write

$$B(t) = B_{\min}(t) + B_+(t)$$

where

$$B_{\min}(t) = \sum_{a=0}^{q-1} \beta_{a,\mu_{q-1-a}} t^{ap-\mu_{q-1-a}p}.$$

**Proposition 2.6.** Let $p$ and $q$ be relatively prime. Assume $A(t)$ and $B(t)$ are polynomials with nonnegative coefficients and nonzero constant terms. Suppose

$$A(t)\xi_q(t) = B(t)\xi_p(t).$$

Let $\lambda$ and $\mu$ be the sequences corresponding to $A_{\min}(t)$ and $B_{\min}(t)$. Then these sequences are partitions which are dual.

**Proof.** Write the product above as $\sum \alpha_j t^j$. Let $0 \leq b < p - 1$ and assume $bq - p\lambda$ occurs as a power in $A_{\min}(t)$, so $\lambda = \lambda_{p-1-b}$. We want to show that $\lambda_{p-b-2} \geq \lambda_{p-1-b}$. If $(b + 1)q - p\lambda$ occurs as a power in $A(t)$ then clearly $\lambda_{p-2-b} \geq \lambda = \lambda_{p-1-b}$. So assume $(b + 1)q - p\lambda$ does not occur in $A(t)$. By Lemma 2.2 applied to $A(t)\xi_q(t)$:

$$\alpha_{(b+1)q-p\lambda} < \alpha_{(b+1)q-p\lambda-1}$$

so $(b + 1)q - p(\lambda + 1)$ is a power in $B(t)$. We may now write

$$(b + 1)q - p(\lambda + 1) = (q - \lambda - 1)p - q(p - b - 1).$$

There will then be an $a' \leq q - \lambda - 1$ such that $a'p - q(p - b - 1)$ is in $B(t)$ but $(a' - 1)p - q(p - b - 1)$ is not. By Lemma 2.2 applied to $B(t)\xi_p(t)$:

$$\alpha_{a'p-q(p-b-1)} > \alpha_{a'p-q(p-b-1)-1}.$$ 

Now we may write

$$a'p - q(p - b - 1) = (b + 1)q - p(a' - 1)$$

and recall that $q - a' \geq \lambda + 1$. Again by Lemma 2.2 we get that the number in this equation will occur as a power in $A(t)$. But this means that

$$\lambda_{p-2-b} \geq q - a' = \lambda_{p-1-b}.$$ 

Since $A(t)$ and equivalently $B(t)$ has nonzero constant term, we have $\lambda_{p-1} = 0$ so we get a partition $\lambda$.

An analog argument gives that the sequence of $\mu_i$’s also form a partition.

Now let $T$ be the order ideal corresponding to $\lambda$ and $T'$ the order ideal corresponding to $\mu$. We show that they are equal and so $\lambda$ and $\mu$ will be dual partitions. Suppose $\lambda_{p-b-1} < \lambda_{p-b-2}$. Then $bq - p\lambda_{p-b-2}$ is not in $A(t)$. By Lemma 2.2

$$\alpha_{(b+1)q-p\lambda_{p-b-2}} > \alpha_{(b+1)q-p\lambda_{p-b-2}-1}.$$
And this implies again by Lemma 2.2 that \((b + 1)q - p\lambda_{p-b-2}\) occurs as a power in \(B(t)\). Rewriting, this is \((q - 1 - (\lambda_{p-b-2} - 1))p - q(p - b - 1)\). And this means that \((\lambda_{p-b-2} - 1, r)\) is in \(T'\) for some \(r \geq p - b - 2\). The upshot is that \(T'\) contains \(T\). Analogously we could show the opposite inclusion so these are in fact equal. □

Corollary 2.7. The polynomials \(A(t) = \sum_{T,i} \gamma_{T,i} t^{c_{T,i}} A_T(t)\) and \(B(t) = \sum_{T,i} \gamma_{T,i} t^{c_{T,i}} B_T(t)\) where the sum is over order ideals \(T\) in \(R(p, q)\) and a running index \(i\) for each \(T\).

\[ A'(t) \xi_q(t) = B'(t) \xi_p(t) \]

and we may proceed inductively, since then new polynomials have no more terms than the original ones, and one of them strictly less. □

From this we obtain our goal of describing the extremal rays of the cone \(P'\) described at the end of Subsection 1.4.

Theorem 2.8. Let \(e_1 = mq\) and \(e_2 = mp\) where \(p\) and \(q\) are relatively prime. The rays generated by \((t^{e_i} A_T(t^m), t^{e_i} B_T(t^m))\) where \(T\) is an order ideal in \(R(p, q)\) and \(c \in \mathbb{Z}\), are the extremal rays in the cone \(P'(e_1, e_2)\). In particular any element in this cone may be written as a positive linear combination of these elements.

\[ A_i(t^m) \xi_q(t^m) = B_i(t^m) \xi_p(t^m) \]

for each \(i\). By Corollary 2.7 we may then conclude. □

Remark 2.9. Such a positive linear combination is in general not unique.

Remark 2.10. We see that the extremal rays fall into classes, one for each order ideal \(T\) in \(R(p, q)\). These form a poset with a minimal element \(T = \emptyset\) and a maximal element \(\hat{T}\). In Remarks 2.4 and 2.5 we showed that these correspond to Betti diagrams of well known resolutions.
3. Existence of resolutions

We will now show that for any extremal ray in $P'(e_1, e_2)$ there is a resolution whose Betti diagram is in this extremal ray. This will show that $P'(e_1, e_2) = P(e_1, e_2)$.

Given an order ideal $T$ in $R(p, q)$ where $p$ and $q$ are relatively prime. If $e_1 = mp$ and $e_2 = mq$, we have the two polynomials $A_T(t^m)$ and $B_T(t^m)$. Homogenising these we may construct an associated triple $B_0, B_1, B_2$ fulfilling the equations of Lemma 1.3, with positive integer coefficients. These lie on an extremal ray in $P'(e_1, e_1)$. Note that in $B_0$ and $B_2$ each monomial occurs with coefficient 0 or 1 and similarly for $B_2$. We may therefore apply the following proposition whose proof will occupy this section.

**Proposition 3.1.** Let $(B_0, B_1, B_2)$ be a triple of homogeneous Laurent polynomials of increasing degrees, fulfilling the HK-equations (7). If the coefficients of each monomial of $B_0$ and $B_2$ is 0 or 1, there is a resolution

$$S.B_0(t, u) \xleftarrow{\alpha} S.B_1(t, u) \xleftarrow{\beta} S.B_2(t, u)$$

of an artinian $S$-module.

As a consequence we get the following.

**Theorem 3.2.** Let $e_1 = mq$ and $e_2 = mp$ where $p$ and $q$ are relatively prime. Let $(B_0, B_1, B_2)$ be the triple of homogeneous Laurent polynomials associated to an order ideal $T$ in $R(p, q)$, with $t^m$ as argument. Then this is a triple of Betti polynomials associated to a bigraded artinian module. Hence the cone $P(e_1, e_2) = P'(e_1, e_2)$.

**Remark 3.3.** Proposition 3.1 holds for any $B_0$ and $B_2$ with nonnegative integer coefficients. But for ease of demonstration we make the above assumptions.

**Remark 3.4.** In the case of three variables it is not true that $P(e_1, e_2, e_3)$ is equal to $P'(e_1, e_2, e_3)$. We provide an example where this is not so in the last section.

We shall prove the above proposition towards the end of this section. But the following outlines what we need to show. Since ker $\alpha$ is a free module, ker $\alpha/\text{im} \beta$ will be either 0 or nonzero of codimension one or zero. But the latter is equivalent to coker $\beta'$ being of codimension one or zero. Hence we need to show the following.

- coker $\alpha$ is of codimension two.
- coker $\beta'$ is of codimension two.
- The composition $\alpha \circ \beta = 0$.

First we have the following.
Lemma 3.5. Given a bidegree $(i, j)$ with $i + j \geq \deg B_2(t, u) - 1$. Then the dimension of the bigraded part $S.B_1(t, u)_{i,j}$ is the sum of the dimensions of $S.B_0(t, u)_{i,j}$ and $S.B_2(t, u)_{i,j}$.

Proof. The bigraded Hilbert function is

$$h(t, u) = \frac{\sum_{i,j} (-1)^i \beta_{i,j} \cdot t^i t^j}{(1-t)(1-u)}$$

for some polynomial $h$. Writing $h(t, u)$ as $\sum \alpha_{i,j} t^i u^j$, the coefficient $\alpha_{i,j}$ will be the alternating sum of the dimensions of the $S.B_\nu(t, u)_{i,j}$. We will show that $\alpha_{i,j} = 0$ for $i + j \geq \deg B_2(t, u) - 1$. But if such a coefficient is nonzero, the pair $(i + 1, j + 1)$ must occur as a power in the numerator in the fraction above. But this implies in turn that $i + j + 2$ is less or equal to the degree of $B_2(t, u)$. \qed

To facilitate the discussion we now introduce some notation. Let $s, e : [1, \ldots, n] \to [1, \ldots, m]$ be two weakly increasing functions such that $s(i) \leq e(i)$. The subset $D = \{(i, j) \mid s(i) \leq j \leq e(i)\}$ of $[1, \ldots, n] \times [1, \ldots, m]$ is a thick diagonal. We then write $s = s_D$ and $e = e_D$. If $s(1) = 1$ and $e(n) = m$ and $s$ and $e$ are strictly increasing we call $D$ a strict thick diagonal. If $s$ is only strictly increasing as soon as $s(i) > 1$ and $e$ is only strictly increasing as long as $e(i) < m$, we call $D$ semi-strict.

Let $B_0(1,1) = p$ and $B_2(1,1) = q$ and write $B_0(t, u) = \sum_{i=1}^p t^{a_1^i} u^{a_2^i}$ where $\{a_1^i\}$ is strictly increasing and $\{a_2^i\}$ is strictly decreasing. Similarly for $B_2(t, u)$ with pairs $(c^1_k, c^2_k)$ and for $B_1(t, u)$ with pairs $(b^1_j, b^2_j)$ but now with the $\{b^1_j\}$ only weakly increasing and the $\{b^2_j\}$ only weakly decreasing.

We may now note that the positions where $\alpha$ may have nonzero entries, i.e. those pairs $(i, j)$ such that $(a_1^i, a_2^i) \leq (b^1_j, b^2_j)$, form a thick diagonal $D_\alpha$ of $[1, \ldots, p] \times [1, \ldots, p + q]$. It has no zero rows because of the HK-equations (4): for each $(a_1^k, a_2^k)$ there is a $(b^1_j, b^2_j)$ with $a_1^k = b^1_j$. Similarly we have a thick diagonal $D_{\beta\nu}$ in $[1, \ldots, q] \times [1, \ldots, p + q]$.

Lemma 3.6. a. $s_{D_\alpha}(i) = j$ if and only if $j$ is the smallest index such that $a_1^i = b^1_j$.

b. $e_{D_\alpha}(i) = j$ if and only if $j$ is the largest index for which $a_2^i = b^2_j$.

The analog result holds for $D_{\beta\nu}$.

Proof. Let $s_{D_\alpha}(i) = j$ and let $\tilde{j}$ be the smallest index such that $a_1^i = b^1_{\tilde{j}}$. Such an index exists by the HK-equations. Clearly $j \leq \tilde{j}$. But if $j < \tilde{j}$ then $b^1_j < b^1_{\tilde{j}}$ and so we could not have $(a_1^i, a_2^i) \leq (b^1_j, b^2_j)$ The other arguments are analogous. \qed

Corollary 3.7. The thick diagonal $D_\alpha$ is strict. Similarly $D_{\beta\nu}$ is strict.

Proof. Since the $a_1^i$ are strictly increasing, we get that $s_{D_\alpha}$ is strictly increasing. We thus need to show that $s_{D_\alpha}(1) = 1$. Let $s_{D_\alpha}(1) = j$. 

Suppose $j$ is not 1. Then $b_1^1 < a_1^1$. By the HK-equations there will be $(c_k^1, c_k^2)$ with $c_k^1 = b_1^1$. But then again there will be a $(b_j^j, b_j^{j'})$ with $b_j^{j'} = c_k^2$ and this would have $b_j^j < c_k^1 = b_1^1$ which is impossible. Thus $s_{D_\alpha}(1) = j = 1$. 

**Lemma 3.8.** Let $D$ be a semi-strict diagonal of $[1, \ldots, n] \times [1, \ldots, n+1]$ with $e_D(1) > 1$ and $s_D(n) < n+1$. Let $A$ be a general matrix of type $D$. Then there is a vector in the null space of $A$ with nonzero first and last coordinates.

**Proof.** If we omit the first column we get an $n \times n$-matrix of semi-strict diagonal type. But a general such matrix is easily seen to be non-singular. Hence a null vector must have nonzero first coordinate. Similarly for the last coordinate. 

**Lemma 3.9.** If $\alpha$ is nonzero in positions $(i, s_{D_\alpha}(i))$ and $(i, \varepsilon_{D_\alpha}(i))$ for $i = 1, \ldots, p$, then $\text{coker} \alpha$ has codimension two. Similarly for the map $\beta^\vee$.

**Proof.** By Lemma 3.6, in the first position of each row there is a power of $y$. Hence for the matrix to degenerate we must have $y = 0$. Similarly there is a power of $x$ in the last position, and so $x = 0$ when the matrix degenerates. 

Now when $\alpha$ and $\beta$ are composed, columns in $\beta$ are multiplied with the rows of $\alpha$. Motivated by this we have the following.

**Lemma 3.10.** Let $k$ be a column in $D_\beta$ which starts in position $(j_0, k)$ and ends in $(j_1, k)$. Then $D_\alpha$ restricted to $[1, \ldots, p] \times [j_0, j_1]$ has $j_1 - j_0$ nonzero rows, say the interval $[i_0, i_1]$ where $j_1 - j_0 = i_1 - i_0 + 1$, and $D_\alpha$ restricted to $[i_0, i_1] \times [j_0, j_1]$ is semi-strict with $e_{D_\alpha}(i_0) > j_0$ and $s_{D_\alpha}(i_1) < j_1$.

**Proof.** 1. That $j_1 - j_0 = i_1 - i_0 + 1$ follows from Lemma 3.5 by restricting to the bidegree $(c_k^1, c_k^2)$.

2. Now we show $e_{D_\alpha}(i_0) > j_0$. By the HK-equations there is a $(b_j^j, b_j^{j'})$ with $b_j^j = c_k^2$. Since the $b_j^j$ are decreasing, this must happen for $j = j_0$. (This is the analog of Lemma 3.6 for $\beta^\vee$.) Clearly $e_{D_\alpha}(i_0) \geq j_0$. If we have equality, by Lemma 3.6 $a_{i_0}^2 = b_{j_0}^2$. But then $a_{i_0}^2 = c_k^2$ and by the HK-equations there must then be two $(b_j^j, b_j^{j'})$ with $b_j^{j'} = a_{i_0}^2 = c_k^2$. But this would again give $s_D(i_0) > j_0$. Similarly we can argue that $s_{D_\alpha}(i_1) < j_1$.

3. That the restriction is semi-strict follows from i) $s_{D_\alpha}$ and $e_{D_\alpha}$ are strictly increasing, ii) $s_{D_\alpha}(i_0) \leq j_0$, and iii) $e_{D_\alpha}(i_1) \geq j_1$. To show ii) note that if $s_{D_\alpha}(i_0) > j_0$ then clearly $s_{D_\alpha}(i_1) > j_0 + i_1 - i_0 = j_1 - 1$. But this is not possible since $e_{D_\alpha}(i_1) \leq j_1 - 1$. Similarly we can show iii).
Proof of Proposition 3.1. We choose $\alpha$ to be a general matrix, homogeneous with respect to the multidegrees. It will be of type $D_{\alpha}$ and it degenerates in codimension two by Lemma 3.9.

By Lemma 3.10 we get for each column $k$ in $D_{\beta}$ a vector in the kernel of $\alpha$ which is nonzero in positions $s_{D_{\beta}}(k)$ and $e_{D_{\beta}}(k)$. Hence these kernel vectors make up the columns of a map $\beta$ such that $\beta^\vee$ degenerates in codimension two by Lemma 3.9. Also the composition $\alpha \circ \beta = 0$, and this is what we needed to show.

4. Resolutions of Trigraded Artinian Modules of Codimension Three

In the case of trigraded artinian modules over the polynomial ring $\mathbb{k}[x, y, z]$ where the resolution has pure total degrees, we do not know much. The following are natural questions.

- For Betti diagrams with given total degrees, are there, up to translation, only a finite number of extremal rays in the positive cone of such Betti diagrams?
- Suppose the above property 2. holds. From Section 2 we know that the translation classes of extremal rays form a poset with a unique minimal member and a unique maximal member. Is there a maximal member in the translation classes in the three variable case also?

We do not know the answer to these questions. A general fact we do know is that $L(e) = L'(e)$. However in three variables it is not the case that the injection $P(e) \to P'(e)$ is an isomorphism. Let us consider as example the case of resolutions of type 0, 1, 2, 1. The equivariant resolution of this type has the form (we have listed the tridegrees of the generators below each free module)

\[
(11) \quad \begin{array}{cccc}
S^3 & \leftarrow & S^6 & \leftarrow & S^6 & \leftarrow & S^4, \\
& 100 & 200 & 211 & 221 & \\
010 & 020 & 121 & 212 & \\
001 & 002 & 112 & 122 & \\
& 110 & 220 & \\
& 101 & 202 & \\
& 011 & 022 & \\
\end{array}
\]

To facilitate notation write \( \sum_i k_i \beta(a_i, b_i, c_i) \) as \( \sum_i [k_i(a_i, b_i, c_i)] \beta \). Let $\beta$ be the Betti diagram of the complex (11). One may check that

\[
[(2, 1, 0) + (0, 2, 1) + (1, 0, 2) - (1, 1, 1)] \beta
\]

gives a diagram with no negative entries (and it fulfills the HK-equations). But no multiple of this is the Betti diagram of a module. If $F_\bullet$ is a complex with this diagram, then $S(-3, -1, 0)$ is a term in $F_0$. But there is no term $S(-3, -1, *)$ in $F_1$ (but there is one in $F_3$), and so the cokernel of $F_1 \to F_0$ cannot have codimension three. In particular
this diagram is in $P'(1,2,1)$ but not in $P(1,2,1)$. However let $\alpha$ be the diagram

$$[(2, 1, 0) + (2, 0, 1) + (1, 2, 0) + (0, 2, 1) + (1, 0, 2) + (0, 1, 2) - (1, 1, 1)] \beta.$$ 

**Claim 1.** The diagrams $\beta$ and $\alpha$ are Betti diagrams of resolutions of indecomposable artinian trigraded modules of codimension three, and they generate rays which are extremal rays in the cone $P(1, 2, 1)$.

**Proof.** That $\beta$ is a Betti diagram is clear and that it resolves an indecomposable module is also immediate to see from the resolution. That $\alpha$ is a Betti diagram of a resolution of an indecomposable module, may be checked on Macaulay 2 by filling in general monomial matrices with the tridegrees of $\alpha$. Now the only way $\alpha$ can decompose into nonnegative diagrams which are not on its ray, may be worked out to be as follows.

$$[c_1((2, 1, 0) + (0, 2, 1) + (1, 0, 2) - (1, 1, 1))$$

$$+ c_2((1, 2, 0) + (0, 1, 2) + (2, 0, 1) - (1, 1, 1)) + c_3(1, 1, 1)] \beta$$

where $c_1 = c_2 = c_3$. But the same argument used to show that the diagram corresponding to the first term is not a resolution may be used to show that a linear combination as above is the diagram of a resolution only if $c_1 = c_2 = c_3$ is a positive integer. \hfill $\square$

It would be interesting to know if there are other extremal rays in the cone $P$ apart from the translates of $\alpha$ and $\beta$.

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