LARGE CONE ANGLES ON A PUNCTURED SPHERE.

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Abstract. Do and Norbury found a so-called differential relation which relates the volume of the moduli space of singular surface with a cone point to that of a smooth surface obtained by forgetting the cone point. Their procedure is valid for cone angles less than \( \pi \) by work of Tan, Wong and Zhang. We study the moduli space of a surface with a single cone point of angle ranging from 0 to \( 2\pi \) using a coordinate system closely related to Penner’s \( \lambda \)-lengths. We compute the action of the mapping class group in these coordinates, give an explicit expression for Wolpert’s symplectic form and use this to justify Do and Norbury’s approach using just hyperbolic geometry.

1. Introduction

The aim of this paper is to extend the results of Nakanishi and Naatanen [6] to hyperbolic cone surfaces of signature \((0; 0, 0, 0, \theta)\) where \( \theta \) is a cone angle in the interval \([\pi, 2\pi]\). Our motivation for this is to understand geometrically the "differential recurrences" introduced by Do and Norbury [2]. In particular we want to see how the hyperbolic structure degenerates as the cone angle \( \theta \) approaches \( 2\pi \).

We use a different coordinate system to Nakanishi and Naatanen, who use traces of matrices, and Maloni, Palesi and Tan [4] also study the 4-holed sphere using similar ideas. Each of our coordinates is a cross ratio and so our results should generalise to representations into higher rank groups as in [11] and [3]. The price to pay for this (see Section 5.1) is that determining the action of the (extended) mapping class group is slightly more delicate in our coordinates. In the trace coordinates the generators are obtained via so-called root flipping or Vieta jumping. Finally we note that [8] have a different approach, based on generalising quadratic differentials, to the problems posed by large cone angles.

The main difficulty is that, although it is easy to construct a hyperbolic cone surfaces with this signature, it is not clear that given any other representation in the same component of the relative character variety that it is also the holonomy of a (unique) hyperbolic structure (Theorem 8.2.1).

1.1. Surfaces with geodesic boundary or cone points. As in [9, 2] we consider a cone point on a hyperbolic surface \( \Sigma^* \) to be a “generalized boundary component” having purely imaginary length.

(1) If the surface \( \Sigma^* \) has a single puncture and \([\delta] \subset \pi_1(\Sigma^*)\) represents a the boundary of a small embedded disc around the puncture then its Teichmuller space embeds as a connected subset of the \( SL_2(\mathbb{R}) \)-character variety

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of \(\pi_1\) containing representations such that \(|\text{tr} \rho(\delta)| = 2\). By convention \(\delta\) represents a boundary component of zero length.

(2) If \(\Sigma^*\) has a single totally geodesic boundary component represented by a conjugacy class of loops \([\delta] \subset \pi_1(\Sigma^*)\), the space of marked hyperbolic structures on \(\Sigma^*\) such that the boundary geodesic has length \(\ell_\delta > 0\) embeds in the \(SL_2(\mathbb{R})\)-character variety of its fundamental group. The image is a connected component of the relative \(SL_2(\mathbb{R})\)-character variety of \(\pi_1\), that is, the subvariety such that \(|\text{tr} \rho(\delta)| = 2 \cosh(\ell_\delta)\).

(3) If the puncture is “replaced” by a single cone point of angle \(\theta \leq \pi\), one identifies the space of hyperbolic structures on the surface \(\Sigma^*\) with a component of the \(SL_2(\mathbb{R})\)-character variety such that \(|\text{tr} \rho(\delta)| = 2 \cos(\theta/2)\).

It proves useful to adopt the convention that \(\delta\) represents a boundary component of purely imaginary length \(\ell_\delta = i\theta\).

In each of these three cases we say that \(\delta\) is a generalised boundary component.

We consider representations \(\rho\) obtained by deforming the holonomy of a metric on a surface with a single cone point of angle \(\theta \leq 2\pi\) through representations such that \(|\text{tr} \rho(\delta)| = |2 \cos(\theta/2)|\). The set of such representation forms a subset of the \(SL_2(\mathbb{R})\)-character variety which we call a geometric component of the relative character variety. Note that, if \(\theta > \pi\) then none of the representation \(\rho\) of \(\pi_1\) is discrete and faithful. Consequently, we may not assume that given an essential simple loop \(\gamma\) and a representation \(\rho\) the isometry \(\rho(\gamma) \in SL_2(\mathbb{R})\) will be hyperbolic.

With this in mind, our aim is to show that the geometric component of the relative character variety enjoys the following properties which should be familiar from the theory of Teichmüller space;

(1) every simple loop is hyperbolic.
(2) the action of the outer automorphism group of \(\pi_1\) is proper.
(3) the set of lengths of simple loops has Fibonacci growth.
(4) every representation is the holonomy of a hyperbolic structure with a single cone point on a surface.

1.2. Generalised four holed spheres. The fundamental group of the four holed sphere is freely generated by three peripheral loops \(\alpha, \beta, \gamma\). After possibly replacing some of these elements by their inverses, we may assume that \(\delta = \gamma \beta \alpha\) is also a peripheral element. Throughout each of \(\alpha, \beta, \gamma\) will be either hyperbolic or parabolic element of \(SL_2(\mathbb{R})\) with positive trace. On the other hand we impose no restriction on the type of \(\delta\) but its trace will always be strictly greater than \(-2\).

We make the following definition: a component (of the relative character variety \(|\text{tr} \rho(\delta)| = \text{constant}\)) is geometric iff it contains the holonomy of a metric of a cone surface. For example, if one takes \(a = b = c = 3\) then \(\kappa(3,3,3) = 2\), the matrices \(A, B, C\) are in the congruence subgroup \(\Gamma_3 < SL(2,\mathbb{Z})\) and \(\mathbb{H}/\langle A, B, C \rangle\) is the regular 4 punctured sphere.

**Theorem 1.2.1.** For the geometric component of the character variety of the generalised 4 holed sphere

(1) every simple loop is hyperbolic.
(2) the action of the outer automorphism group is proper.
(3) the set of lengths of simple loops has Fibonacci growth.
(4) every representation is the holonomy of a hyperbolic structure with a single cone point on a surface.
1.3. **An inequality for generalised four holed spheres.** If the four holed sphere is equipped with a complete hyperbolic metric, and the boundary \( \delta \) is totally geodesic of lengths \( \ell_\delta > 0 \) then the loops \( \alpha \beta \) and \( \beta \gamma \) can be represented by closed simple geodesics which meet in exactly two points and which satisfy

\[
\sinh\left(\frac{\ell_{\alpha \beta}}{4}\right) \sinh\left(\frac{\ell_{\beta \gamma}}{4}\right) \geq 1.
\]

This inequality follows almost immediately from the Collar Lemma but we will deduce it from a more general result which we explain now.

As discussed in the preceding paragraph, we view a three holed sphere with a single cone point of angle \( \theta < 2\pi \) as a four holed sphere with a generalized boundary component, represented by an element \( \delta \in \pi_1 \) such that \( \text{tr} \rho(\delta) = 2 \cos\left(\frac{\theta}{2}\right) \).

Under the hypothesis that \( \theta < 2\pi \), the elements \( \alpha \beta \) and \( \beta \gamma \) are always hyperbolic (Theorem 1.2.1) so that there are real numbers \( \ell_{\alpha \beta}, \ell_{\beta \gamma} \) such that

\[
\text{tr} \rho(\alpha \beta) = -2 \cosh\left(\frac{\ell_{\alpha \beta}}{2}\right), \quad \text{tr} \rho(\beta \gamma) = -2 \cosh\left(\frac{\ell_{\beta \gamma}}{2}\right)
\]

and we prove

**Theorem 1.3.1.** With the above notation we have the following inequality

\[
\sinh\left(\frac{\ell_{\alpha \beta}}{4}\right) \sinh\left(\frac{\ell_{\beta \gamma}}{4}\right) \geq \cos\left(\frac{\theta}{4}\right).
\]

It is worth taking a moment to consider the extremal cases \( \theta = 0, 2\pi \):

- when \( \theta = 0 \) this corresponds to a cone angle of zero, in other words a cusp, and we recover the inequality (1) above.
- when \( \theta = 2\pi \) one has \( \sinh\left(\frac{\ell_{\alpha \beta}}{4}\right) \sinh\left(\frac{\ell_{\beta \gamma}}{4}\right) \geq \cos(\pi/2) = 0 \), so that, as one expects, the inequality is redundant.

1.4. **Motivation: volume computations.** The essential motivation for this paper is understanding Do and Norbury’s approach to volume recurrences and Maloni, Palesi and Tan [4] study the mapping class group action on the four holed sphere.

Nakanishi and Naatanen determined the symplectic volume of the moduli spaces of the once holed torus \( V_1(l_1) \) and the 4 holed sphere \( V_0(l_1, l_2, l_3, l_4) \). The volumes are quadratic polynomials in the lengths of the boundary components:

\[
V_1(l_1) = \frac{1}{24}(4\pi^2 + l_1^2), \quad V_0(l_1, l_2, l_3, l_4) = \frac{1}{2}(4\pi^2 + \sum_i l_i^2).
\]

Their method consists of finding a fundamental region for the action of the mapping class group on the relative character variety when the peripheral loops are hyperbolic or possibly parabolic. They do this by imitating Wolpert’s [12] analysis of the mapping class group action on the Teichmüller space of the once punctured torus.

Do and Norbury found a so-called differential relation which, in the simplest case, relates the volume of the moduli space of a singular surface with a cone point to that of a smooth surface obtained by forgetting the cone point. For a 3 punctured sphere with a cone point of angle \( \theta \) the volume of the moduli space is

\[
V_0(0, 0, 0, i\theta) = \frac{1}{2}(4\pi^2 - \theta^2),
\]

and we see that for \( \theta = 2\pi \) the value of \( V_0(0, 0, 0, 2\pi i) \) is 0 which is the volume of the moduli space of the 3-punctured sphere.

A more interesting case is that of the punctured torus with a single cone point of angle \( \theta \), the volume, computed by Mirzakhani is:
\[ V_1(i\theta, l_2) = \frac{1}{192} (4\pi^2 - \theta^2 + l_2^2)(12\pi^2 - \theta^2 + l_2^2) \]

so for \( \theta = 2\pi \) the value of \( V_1(i\theta, 0) \) is a quartic in \( l_2 \) is not the volume of the moduli space of the one holed torus.

However, if we take the derivative at \( 2\pi i \) we will recover the volume of the one holed torus as a factor:

\[
\frac{d}{dl_1} V_1(l_1, l_2) = \frac{1}{192} (4\pi^2 + l_1^2 + l_2^2)(12\pi^2 + l_1^2 + l_2^2)
\]

\[
\frac{d}{dl_1} \bigg|_{2\pi i} V_1(l_1, l_2) = \frac{2\pi i}{96} (8\pi^2 + 2l_2^2) = \frac{2\pi i}{4} V_1(l_1)
\]

As mentioned above Maloni, Palesi and Tan [4] study the mapping class group action on the (relative) \( SL(2, \mathbb{C}) \) character varieties of the four-holed sphere. They describe a domain of discontinuity, and, in the case of real characters, show that this domain of discontinuity may be non-empty on the components where the relative euler class is non-maximal. This is to be expected, at least heuristically, as the volume polynomial \( \frac{1}{2} (4\pi^2 + \sum i l_i^2) \) does not vanish on these non-maximal components. They should conjecturally decompose as a union of wandering domains and a set where the mapping class action is ergodic.

2. MOEBIUS MAPS AND FIXED POINTS

Our methods are based on elementary algebra and geometry. It is important, however, to recall some basic facts and definitions.

Recall that if \( M = \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix} \in SL(2, \mathbb{R}) \) then it acts on \( \mathbb{H} \) by moebius transformation

\[ z \mapsto \frac{M_{1,1}z + M_{1,2}}{M_{2,1}z + M_{2,2}} \]

The fixed points of Moebius transformation are given by the following formula:

\[ z_{\pm} = \frac{(M_{1,1} - M_{2,2}) \pm \sqrt{\text{tr}^2 M - 4}}{M_{2,1}} = \frac{(\text{tr} M - 2M_{2,2}) \pm \sqrt{\text{tr}^2 M - 4}}{M_{2,1}} \]

Note further that, if \( M \) is elliptic, that is \( |\text{tr} M| < 2 \), then \( z_{\pm} \) are complex conjugate and

(3) \[ \text{Re}(z_{\pm}) = \frac{\text{tr} M - 2M_{2,2}}{M_{2,1}} \]

on the other hand if \( M \) is hyperbolic, \( |\text{tr} M| > 2 \), then the fixed points are points of the extended real line \( \mathbb{R} \cup \{\infty\} \).
3. The generalized four punctured sphere

Following the convention, one views a 3 punctured sphere with a single cone point as a generalized 4 holed sphere that is a four holed sphere with three boundary components of length zero and a single generalized boundary component $\delta$ of purely imaginary length.

The fundamental group $\pi_1$ of the 4 punctured sphere is freely generated by 3 peripheral loops $\alpha, \beta, \gamma$. After possibly replacing some of these elements by their inverses, we may assume that $\delta = \gamma\beta\alpha$ is also a peripheral element and we have the presentation

$$\pi_1 = \langle \alpha, \beta, \gamma, \delta | \delta = \gamma\beta\alpha \rangle.$$

3.1. Explicit matrices and a parametrization. We begin by studying the set of representations of $\pi_1 = \langle \alpha, \beta, \gamma, \delta | \delta = \gamma\beta\alpha \rangle$. The group is freely generated by $\alpha, \beta, \gamma$ and we may define a representation $\rho : \pi_1 \to SL(2, \mathbb{R})$ by setting

$$\rho(\alpha) = A, \rho(\beta) = B, \rho(\gamma) = C$$

with $A, B, C \in SL(2, \mathbb{R})$ parabolic with distinct fixed points. Recall that an element of $SL(2, \mathbb{R})$ is parabolic iff it is conjugate to

$$\pm \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \ p \neq 0$$

and that such a transformation has a single (ideal) fixed point in $\partial \mathbb{H}$. The action of $SL_2(\mathbb{R})$ is transitive on triples of distinct points in $\partial \mathbb{H}$ and we normalize so that the fixed points of $A, B, C$ are respectively fixes $0, 1, \infty$. So that, for some $a, b, c > 0$

$$C = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 + b & -b \\ b & 1 - b \end{pmatrix}, \ A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Since $\pi_1$ is freely generated by $\alpha, \beta, \gamma$, the triple $(A, B, C)$ determines a point of the $SL_2(\mathbb{R})$ representation variety of $\pi_1$ and we denote $[(A, B, C)]$ the corresponding point in the character variety. Thus we have a map from $\mathbb{R}^3$ to the representation variety

$$(a, b, c) \mapsto (A, B, C).$$

and from $\mathbb{R}^3$ to the character variety

$$(a, b, c) \mapsto [(A, B, C)].$$

The traces of elements of $\rho(\pi_1)$ form a natural class of functions on the representation variety and one can compute explicit expressions for these functions in terms of $a, b, c$. For example

$$BA = \begin{pmatrix} 1 + b - ab & -b \\ b + a - ab & 1 - b \end{pmatrix},$$

so that

$$\text{tr} \ AB = \text{tr} \ BA = 2 - ab.$$ 

Likewise, by explicit computation, one has

$$\text{tr} \ AC = 2 - ac, \ \text{tr} \ BC = 2 - bc.$$
Recall that an element of $SL_2(\mathbb{R})$ is hyperbolic iff $|\text{tr}| > 2$. For all our representations $\text{tr} A = \text{tr} B = 2 > 0$, so whenever $(A, B)$ is discrete, by the Triple Trace Theorem [5],

$$2 - ab = \text{tr} AB \leq -2.$$  

We shall consider non discrete representations $\rho$ but we will show (Lemma 4.1.1) that the inequality [4] still holds for all the representations considered provided $\text{tr} CBA > -2$.

Finally we check that our map gives a parameterization of the character variety:

**Lemma 3.1.1** (Smooth parameterization). *The restriction of the map*

$$(a, b, c) \mapsto [(A, B, C)]$$

*to $\{ab, bc, ac > 2\}$ is a diffeomorphism onto its image.*

**Proof.** It suffices to check that the map

$$(a, b, c) \mapsto (x, y, z) = (\text{tr} BC, \text{tr} AC, \text{tr} AB) = (2 - bc, 2 - ac, 2 - ab)$$

is a diffeomorphism. Both the jacobian and the inverse of this map are easy to compute and we leave this to the reader to check. $\square$

### 3.2. Geometric interpretation of $a, b, c$.

If the three matrices $A, B, C$ generate a fuchsian group $\Gamma$ then the ideal triangle $0, 1, \infty$ embeds in the quotient surface $\mathbb{H}/\Gamma$. The surface $\mathbb{H}/\Gamma$ has 3 (and possibly 4 if $CBA$ is parabolic) cusps, that is one for each of the ideal vertices of $0, 1, \infty$. Each of the the edges of the triangle $0, 1, \infty$ embeds as an arc joining distinct pairs of cusps and each of these cusps, by Shimura’s Lemma, lies in one of 3 pairwise disjoint cusp regions of area 1. In this paragraph we give an interpretation of $a, b, c$ in terms of this configuration.

The region $\{\text{Re} z \geq c\} \subset \mathbb{H}$ embeds as a cusp region of area $\int_0^c \frac{dx}{e^x} = 1$ and the portion of the triangle $0, 1, \infty$ contained in this region is $\int_0^1 \frac{dx}{e^x} = \frac{1}{e}$. Such a portion, that is the intersection of an ideal triangle with a horoball centered at one of its vertices and meeting exactly two of the sides of the triangle, is often called a *prong*. By a similar argument, one sees that the prong of the triangle $0, 1, \infty$ contained in the lift of a cusp region based at 0 (resp. 1) is exactly $\frac{1}{e}$ (resp. $\frac{1}{2}$).

On each side of an ideal triangle there is a well-defined midpoint. A simple calculation yields that the distance from the midpoint to the prong of area $h$ is $\log(h)$. It follows that the arc, that is the portion of the geodesic $0, \infty$, outside of lift of the cusp region of area $1/e$ based at $\infty$ and the cusp region of area $1/a$ based at 0, is $\log(ac)$. In this way one sees that the quantity $ac$ is one of Penner’s $\lambda$-length (see [2]).

### 3.3. Topology of the relative character variety.

A relative character variety is defined to be a level set of the function $\kappa$ where

$$\kappa(a, b, c) := \text{tr} CBA.$$  

In general this has more than one connected component (of codimension 0 and possibly infinitely many of codimension $> 1$). Here we are interested in the codimension 0 component parameterized by $(\mathbb{R}^+)^3$ under the map $(a, b, c) \mapsto [(A, B, C)]$ and for this parameterization

$$\kappa(a, b, c) = 2 + abc - ab - bc - ac.$$
Here we are interested a codimension 0 component which contains at least one representation which is the holonomy of metric on a cone surface.

**Definition** A component (of the character variety) is geometric if it contains the holonomy of a metric on a cone surface.

**Example 1:** with $a = b = c = 3$, $\kappa(3,3,3) = 2$, the matrices $A, B, C$ are in the congruence subgroup $\Gamma_3 < SL(2, \mathbb{Z})$ and $\mathbb{H}/\langle A, B, C \rangle$ is the regular 4 punctured sphere.

**Example 2:** with $a = b = c = 2$, $\kappa(2,2,2) = -2$, the matrices $A, B, C$ are in the congruence subgroup $\Gamma_2 < SL(2, \mathbb{Z})$ and $\mathbb{H}/\langle A, B, C \rangle$ is the 3 punctured sphere. Note that the image is not a free group since $CBA$ is a torsion of order 2.

**Lemma 3.3.1** (Subvariety). The level sets of $\kappa$ are smooth subvarieties of $\mathbb{R}^3$ except for $\kappa^{-1}(0)$ and $\kappa^{-1}(-2)$ each containing a unique singular point respectively $(0,0,0)$ and $(2,2,2)$.

**Proof.** By direct computation one sees that $\kappa$ is a submersion at $(a,b,c)$ unless

$$a = b = \frac{c}{c-1} \text{ and } c^2 - 2c = 0.$$ 

This system has exactly two solutions, namely $(0,0,0) \in \kappa^{-1}(0)$ and $(2,2,2) \in \kappa^{-1}(-2)$. The former corresponds to the trivial representation and the latter to representations such that $CBA = -I_2$. □

It follows that the geometric component is a smooth subvariety which we shall now describe more fully. Observe, that if $ab - a - b \neq 0$ then

$$c = \frac{\kappa - (2 - ab)}{ab - a - b}$$

and we use this formula to determine the topology of the components of the relative character variety.

**Lemma 3.3.2** (Connected components). If $t > -2$ then

$$\kappa^{-1}(t) = \{(a,b,c) \in \mathbb{R}^3 : 2 + abc - ab - bc - ac = t\}$$

consists of three components each of which is a graph over one of the components of $\{ab - a - b \neq 0\} \subset \mathbb{R}^2$.

**Proof.** If $ab - a - b = 0$ then $ab > 4$ so $2 - ab - \kappa \neq 0$. (otherwise there may be a point of indeterminacy in the formula for $c$ above and the map $(a,b,c) \mapsto c$ fails to be a diffeomorphism onto the components of $ab - a - b \neq 0$.)

The equation $ab - a - b = 0$ defines a right hyperbola with asymptotes $a = 1$ and $b = 1$ so that $\{ab - a - b \neq 0\}$ has three simply connected components, namely:

(1) $ab - a - b > 0, a, b > 1$

(2) $ab - a - b < 0$

(3) $ab - a - b > 0, a, b < -1$. □
4. HYPERBOLICITY AND INEQUALITIES

In this section we establish two inequalities for functions on the geometric component of \( \kappa^{-1}(t > -2) \). Firstly, we show that \( \text{tr} BA < -2 \) so that \( BA \) (and by symmetry \( CB, AC \)) is always a hyperbolic element. Secondly, we prove an inequality that shows that a weak verion of the Collar Lemma holds.

4.1. Hyperbolicity of the product \( BA \). We begin by showing that \( ab > 4 \) on the geometric component of \( \kappa > -2 \) which implies that \( BA \) is always a hyperbolic element. Another important consequence is that no two of \( a, b, c \) are simultaneous equal to 2 on the geometric component, it follows from this that the fixed points sets of the restrictions of the involutions \( I_a, I_b, I_c \) to the geometric component are disjoint.

**Lemma 4.1.1** (Product). If \( \kappa > -2 \) and \( a, b, c > 1 \) then

\[
ab > 4,
\]

so that \( \text{tr} \rho(\alpha\beta) = \text{tr} BA = 2 - ab < -2 \).

**Proof.** By the preceding lemma, we need to determine a lower bound for \( ab \) over the region \( X = \{(a, b) \in \mathbb{R}^2 : a, b > 1, ab - a - b > 0\} \), and this is equivalent to minimizing subject to the constraints

\[
a > 1, \quad b > \frac{a}{a - 1}.
\]

Thus

\[
ab > \left(a \times \frac{a}{a - 1}\right) = \left(\frac{a^2}{a - 1} - 4\right) + 4 = \frac{(a - 2)^2}{a - 1} + 4 \geq 4,
\]

since \( a - 1 > 0 \). \( \square \)

4.2. Proof of the inequality (1). From the preceding paragraph the elements \( \alpha\beta \) and \( \beta\gamma \) are hyperbolic so that there are positive real numbers \( \ell_{\alpha\beta}, \ell_{\beta\gamma} \) such that

\[
\text{tr} \rho(\alpha\beta) = 2 - ab = -2 \cosh(\ell_{\alpha\beta}/2), \quad \text{tr} \rho(\beta\gamma) = 2 - ab = -2 \cosh(\ell_{\beta\gamma}/2).
\]

It follows that

\[
(4 - ab)(4 - bc) = (2 + \text{tr}(\alpha\beta))(2 + \text{tr}(\beta\gamma)) = (2 - 2 \cosh(\ell_{\alpha\beta}/2))(2 - 2 \cosh(\ell_{\beta\gamma}/2)) = 16 \sinh^2(\ell_{\alpha\beta}/4) \sinh^2(\ell_{\beta\gamma}/4)
\]

We minimize \( (ab - 4)(bc - 4) \) over the relative character variety to obtain (2).

**Lemma 4.2.1.**

\[
(ab - 4)(bc - 4) \geq 4(\kappa + 2)
\]

**Proof.** Set \( ab = h, h > 4 \). The region \( ab - a - b > 0, a > 1, b > 1 \) is foliated by arcs of the hyperboloids \( ab = h \). Each leaf meets \( ab - a - b = 0 \) in exactly 2 points. To prove the inequality (7) it suffices, to minimize \( bc \) along each of the leaves of this foliation.

Begin by observing that

\[
hc - h - bc - \frac{hc}{b} = \kappa - 2
\]
and consequently,
\[ bc = b \left( \frac{\kappa - 2 + h}{h - b - h/b} \right) = \frac{\kappa - 2 + h}{h/b - 1 - h/b^2} \]

Minimising \( bc \) whilst keeping \( h \) constant is equivalent to maximising the function
\[ b \mapsto \frac{h}{b} - 1 - \frac{h}{b^2} = \left( \frac{h}{4} - 1 \right) - h \left( \frac{1}{b} - \frac{1}{2} \right)^2 \]

From the latter expression one sees that the function has a unique minimum when \( b = 2 \) and the corresponding value of \( bc - 4 \) is
\[ 4 \left( \frac{\kappa - 2 + h}{h - 4} \right) - 4 = 4 \left( \frac{\kappa + 2}{h - 4} + 1 \right) = 4 \left( \frac{\kappa + 2}{ab - 4} \right) \]

\[ \square \]

5. Dynamics

We define involutions of \( \pi_1 \) which are the analogues of the “diagonal exchanges” in the punctured torus \( \alpha, \beta \mapsto \alpha, \beta^{-1} \). These maps correspond to “topological reflections” (Section 3 [?]), and it is well known that they generate a subgroup of finite index in the mapping class group. Subsequently, we give explicit formulae for the induced maps on the character variety and use these to investigate the dynamics.

5.1. Induced automorphisms of the character variety. In this section we show how to calculate the induced map \((a, b, c) \mapsto (a', b', c')\) using cross ratios of fixed points of parabolics.

Let \( f : \pi_1 \to \pi_1 \) be an automorphism. The automorphism \( f \) induces a homeomorphism on the Gromov boundary \( \partial_\infty \pi_1 \) of the fundamental group which, for discrete, faithful representations of \( \pi_1 \), induces a \( \pi_1 \) equivariant homeomorphism \( f^\sim \) on the limit set. This simply means that if \( \gamma \in \pi_1 \) acts on \( \partial H \) with a fixed point \( \gamma^+ \) then \( f(\gamma) \) has a fixed point \( f(\gamma)^+ \) and
\[ f^\sim(\gamma^+) = f(\gamma)^+. \]

This relation that allows us to calculate \( a', b', c' \) explicitly using the cross ratio on \( \mathbb{R} \cup \{\infty\} \) as follows. Recall that \( \alpha, \beta, \gamma \) fix 0, 1, \( \infty \) respectively, so we set \( \alpha^+ = 0, \beta^+ = 1, \gamma^+ = \infty \).

We shall write each of \( a, b, c \) as (a function of) a cross ratio of these three points plus another point \( x \in \mathbb{R} \). For \( x \neq 0, 1, \infty \), one has the following identity
\[ x = [x, 1, 0, \infty] = \frac{x - 0}{x - \infty}, \frac{1 - \infty}{1 - 0} = \frac{x - \alpha^+}{x - \gamma^+}, \frac{\beta^+ - \gamma^+}{\beta^+ - \alpha^+}. \]

Now observe that
\[ a = \frac{1}{A(\infty)} = \frac{1}{\alpha(\gamma^+)} \]
\[ c = C(0) = -\gamma(\alpha^+) \]
\[ b = \frac{1}{B(\infty) - 1} = \frac{1}{\beta(\gamma^+) - 1}. \]

The following lemma is an immediate consequence of these three equations and [8], [9].
Lemma 5.1.1. Let $f : \pi_1 \to \pi_1$ be an automorphism. Then $f$ induces a map
\[
\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}
\]
on the set of $(a, b, c)$ such that the corresponding representation is discrete faithful. The values of $a', b', c'$ are given by:
\[
\begin{align*}
a' & = \frac{1}{F(f(\alpha)(f(\gamma)^+))} \\
b' & = \frac{1}{F(f(\beta)(f(\gamma)^+)) - 1} \\
c' & = -F(f(\gamma)(f(\alpha)^+))
\end{align*}
\]
where the function $F : \mathbb{R} \setminus \{0, 1\}$ is defined by
\[
F(x) = \frac{x - f(\alpha)^+ f(\beta)^+ - f(\gamma)^+}{x - f(\gamma)^+ f(\beta)^+ - f(\alpha)^+}.
\]
We apply the lemma to determine the map induced by the involution $I_b$.

Corollary 5.1.2. The action of the map $I_b$ induced by the automorphism $\phi_\beta$ is
\[
I_b : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a(b - 1) \\ b/(b - 1) \\ c(b - 1) \end{pmatrix}
\]
Proof. The morphism $\phi_\beta$ is defined by
\[
\begin{align*}
\alpha & \mapsto \alpha^{-1} \\
\beta & \mapsto \alpha^{-1} \beta^{-1} \alpha \\
\gamma & \mapsto (\beta \alpha)^{-1} \gamma^{-1} (\beta \alpha).
\end{align*}
\]
We use these equations to determine fixed points as follows
\[ f(\alpha)^+ = (\alpha^{-1})^+ = \alpha^+ = 0 \]
\[ f(\beta)^+ = \alpha^{-1}(\beta^+) = \alpha^{-1}(1) = \frac{1}{1-a} \]
\[ f(\gamma)^+ = (\beta\alpha)^{-1}(\gamma^+) = (\beta\alpha)^{-1}(\infty) = \frac{1-b}{ab-a-b} \]

One also has, by direct computation,
\[ f(\gamma)(f(\alpha)^+)(f(\gamma)(0)) = -b^2c + 2bc - c = \frac{ab^2c - 2abc + ac - b^2c + bc - 1}{ab^2c - 2abc + ac - b^2c + bc - 1} \]

It is then straightforward to check, either by hand or using a computer algebra package, that the morphisms induced is \( I_b \).

5.2. Fixed point sets of induced automorphisms. Each of the three involutions \( I_a, I_b, I_c \), has a non empty fixed point set. The dynamics of the group generated by these three involutions is determined by the geometric configuration of these fixed point sets

Lemma 5.2.1.

1. Viewed as a self map of \( \mathbb{R}^3 \setminus \{(b-1)\}, \) the fixed point set of the map \( I_b \) consists of the plane \( \Pi_b := \{b = 2\} \) and \((0,0,0)\).
2. The plane \( \Pi_a \) (resp. \( \Pi_b, \Pi_c \)) meets the level sets \( \kappa^{-1}(t > -2), a,b,c > 1 \) in a hyperbola. The three hyperbolae obtained in this way are disjoint.
3. The three planes meet the level set \( \kappa^{-1}(t = -2), a,b,c > 1 \) in the lines:
   \[ \{a = 2\}, \{b = 2\}, \{c = 2\} \]
   These lines are concurrent at the singular point of \( \kappa \), \((2,2,2)\).

Proof. For the first point, observe that if \((a,b,c)\) is a point of the fixed point set of \( I_b \) then one has
\[ b = \frac{b}{b-1}, \quad a = a(b-1), \quad c = c(b-1). \]
Clearly, if \( b \neq 0 \) then \( b - 1 = 1 \) and if \( b = 0 \) then \( a = -a, c = -c \) so that \( a = b = c = 0 \).

For the second point, it is convenient given to consider \( I_c \). If \( c = 2 \) then \( \kappa = ab - 2a - 2b + 2 = (a-2)(b-2) - 2 \) and this is the equation of a right hyperbola in the plane \( c = 2 \) provided \( \kappa + 2 > 0 \).

Finally, if \( \kappa + 2 > 0 \) the hyperbola \( c = 2 \) is asymptotic to the lines \( a = 2, b = 2 \) and so the hyperbolae are disjoint as required. If \( \kappa = -2 \) the intersection of \( c = 2 \) with the level set is the pair of lines \( a = 2, b = 2 \) and the third point of the lemma follows immediately.

5.2.1. The action is proper \( \kappa > -2 \). We apply the preceding lemma to show that the group of automorphisms acts properly discontinuously and to determine a fundamental domain for this action.

Theorem 5.2.1 (Proper). The three involutions generate a group \( \Gamma \) isomorphic to \( \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \) which acts properly on the geometric component. Furthermore a \( \Gamma \)-fundamental domain is
\[ \Delta = \{(a,b,c) : \min(a,b,c) > 2\} \]
Proof. To prove that $\Delta$ is a fundamental domain we have to show the following:

- $\Delta, I_a(\Delta), I_b(\Delta), I_c(\Delta)$ are disjoint.
- if $x \in \kappa^{-1}(t > -2)$ then there is a word in $w \in \Gamma$ such that $w(x) \in \overline{\Delta}$

For the first of these points, we note that

$$a > 2 \Rightarrow \frac{a}{a-1} < 2$$

so $\Delta \cap I_a(\Delta) = \emptyset$ (and by symmetry $\Delta \cap I_b(\Delta), \Delta \cap I_c(\Delta) = \emptyset$.)

For the second point suppose $(a, b, c) \notin \Delta$ and, imitating Wolpert [?] for the punctured torus, we choose an “energy” function $E$ on the level set. An energy function is a proper function such that for any $(x, y, z) \notin$ the closure of the fundamental domain $\overline{\Delta}$, there one of the involution $I_a, I_b, I_c$ which decreases its value, that is

$$\min(E \circ I_a(x, y, z), E \circ I_b(x, y, z), E \circ I_c(x, y, z)) < E(x, y, z)$$

Note that

$$(\Delta)^c = \{ (a, b, c) : \min(a, b, c) < 2 \}.$$

The function

$$E : (a, b, c) \mapsto abc,$$

satisfies this second condition since

$$E \circ I_a(a, b, c) = abc(a-1) < E(a, b, c)$$

if $1 < a < 2$. The function $E$ is clearly continuous so it is proper if the preimage of every bounded set is bounded. By Lemma 4.1.1 if $\kappa > -2$ then, $ab, bc, ac > 4$ it follows immediately that, for any $K > 0$,

$$E(a, b, c) \leq K \Rightarrow \max\{a, b, c\} \leq K/4.$$ 

\qed

6. Symplectic volume

Wolpert has shown that the symplectic form takes the form $d\ell_\alpha \wedge d\tau_\alpha$ where $\ell_\alpha$ is the length as before and $\tau_\alpha$ is a Fenchel-Nielsen twist parameter. We derive an expression for this in terms of $a, b, c$ and use it to calculate the volume.
6.0.2. *Fenchel Nielsen coordinates.* From the preceding discussion we have

\[ 2 \cosh\left(\frac{1}{2} \ell \right) = ab - 2 = -\operatorname{tr} \alpha \]

Then

\[
d(2 \cosh\left(\frac{1}{2} \ell \right)) = \sinh\left(\frac{1}{2} \ell \right) d\ell = \frac{1}{2} \left( (\operatorname{tr} \alpha)^2 - 4 \right)^{\frac{3}{2}} d\ell
\]

where \( \Delta(ab) := \sqrt{(ab - 2)^2 - 4} = 2 \sinh\left(\frac{1}{2} \ell \right) \).

So \( d\ell = \frac{2}{\Delta} d(ab) \) depends only on \( ab \) and we will exploit this by writing \( d\tau_\alpha \) as a sum where one term is \( F(ab) d(ab) \) for some function \( F : \mathbb{R} \rightarrow \mathbb{R} \).

The twist parameter is given by the signed distance between the image of a pair of reference points under the projection to the axis of \( \alpha \). We choose the fixed points of \( A, C \) (respectively 0 and \( \infty \)) as reference points and then

\[
\tau_\alpha = \log \left( \frac{\alpha + \infty}{\alpha - 0} \cdot \frac{\alpha - \infty}{\alpha - \infty} \right) = \log \left( \frac{\alpha^+}{\alpha} \right)
\]

Now, by direct calculation, we express this as a function of \( a, b \)

\[
\alpha^\pm = \frac{1 + b - ab - (1 - b) \pm \Delta}{b + a - ab}
\]

so that

\[
d\tau_\alpha = d \log(2b - ab + \Delta) - d \log(2b - ab - \Delta) = \frac{2\Delta(2d + F(ab)d(ab))}{(2b - ab)^2 - \Delta^2},
\]

for some function \( F \). Thus

\[
d\ell_\alpha \land d\tau_\alpha = \frac{db \land d(ab)}{b(ab - a - b)} = \frac{db \land da}{ab - b - a}.
\]

6.0.3. *Volume.* The expression for the volume form in these coordinate is just:

\[
\omega_{WP} = \frac{da \land db}{ab - b - a} = \frac{db \land dc}{bc - b - c} = \frac{dc \land da}{ac - a - c}
\]

\[
\int_2^\infty \int_2^{2 - 2b - k} \frac{da}{ab - a - b} db.
\]

For the particular case \( k = 2 \) this becomes the Fundamental domain is a region between two hyperboloids \((a - 2)(b - 2) = 0\) and \((a - 2)(b - 2) = 4\) so the integral becomes

\[
\int_2^\infty \int_2^{\frac{a = f(b)}{a - 1}(b - 1) - 1} \frac{da}{(a - 1)(b - 1) - 1} db,
\]
where \((f(b) - 2)(b - 2) = 4\). Using the change of variable \(u = a - 2, b = v - 2\)
\[
\int_0^\infty \int_0^{4/v} \frac{du}{(u+1)(v+1) - 1} dv = \int_0^\infty \frac{2}{v+1} \log(\frac{v+2}{v}) dv
\]
\[
= 2 \int_0^\infty \frac{\log(v+2)}{v+1} dv - 2 \int_0^\infty \frac{\log(v)}{v+1} dv
\]
\[
= 2[-L_i(2(-t-1) - L_i(-t) - \log(t)\log(t+1))]_0^\infty
\]
\[
= \frac{2\pi^2}{3} - \frac{\pi^2}{6} = \frac{\pi^2}{2}
\]

So the volume of the quotient of the component of the character variety of the 4 by
the group generated by the involutions is \(\pi^2/2\). By considering the action on the
abelianisation of the fundamental group one sees that the mapping class group is
a finite subgroup of order 4 in this latter group so that volume of the moduli space
is \(2\pi^2\) which agrees with the value of \(V_0(0,0,0,i\theta) = \frac{1}{4}(4\pi^2 - \theta^2)\) when \(\theta = 0\).

7. Hyperbolicty implies Fibonacci Growth

The free product \((\mathbb{Z}/2\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})\) has a Bass-Serre tree for which the edge
stabiliser are isomorphic to \(\mathbb{Z}/2\mathbb{Z}\) and the vertex stabiliser are isomorphic to \(\mathbb{Z}/3\mathbb{Z}\).
One identifies the Markoff tree with the Bass-Serre tree by identifying \(PSL(2,\mathbb{Z})\)
with the (smallest) group of automorphisms of the Markoff cubic that is transitive
on ordered triples of integers \((x,y,z)\) that are solutions. The vertices of the Markoff
tree are the unordered triples \(\{x,y,z\}\) and the edges unordered pairs \(\{x,y\}\). One
can embed the Markoff tree, viewed as the Bass-Serre tree of \(PSL(2,\mathbb{Z})\), in the
Poincaré disc. The complement consists of countably many complementary regions
three of which meet at every vertex \(\{x,y,z\}\) and one of these numbers can be
associated to each region in a consistent manner (see Bowditch [1] or [10] for details).

Using this construction, Bowditch defined Fibonacci growth for the \(PSL(2,\mathbb{Z})\)
orbit of solutions. Let \(z_0 \geq y_0 \geq x_0\) be a vertex and consider the binary subtree
which is union of the edge \(e = \{x_0,y_0\}\) and the component of \(T \setminus \text{int}(e)\) containing
the vertex \(\{x_0,y_0,z_0\}\). We think of this subtree as starting at the edge \(e = \{x_0,y_0\}\).
One introduces a comparison function defined recursively on (complementary regions
of an embedding in the disc of) such a binary subtree. If \(\{X,Y\}\) is the first edge one sets

\[
F_e(X) = \log(x_0), F_e(Y) = \log(y_0)
\]

If the function is defined at the edge \(\{X,Y\}\) and \(\{X,Y,Z\}\) is a vertex such that
\(F_e(Z)\) is as yet undefined then
\[
F_e(Z) = F_e(X) + F_e(Y).
\]

A function \(f\) on the vertices admits upper and lower Fibonacci bounds if there exists
\(K^-, K^+ > 0 > 0\) such that for every vertex \(v\) one has

\[
K^- F_e(v) \leq f(v) \leq K^+ F_e(v).
\]

An important observation of Bowditch is that if there exists a constant \(\delta > 0\) such
that a real valued function \(f\) defined on the vertices of the tree satisfies

\[
f(z) \geq f(x) + f(y) - \delta
\]

at each vertex \(z \geq y \geq x\) in a binary sub tree starting at \(z_0 \geq y_0 \geq x_0\) then

\[
f(z) \geq (m - \delta) F_e(z) + \delta,
\]
where \( m = \min(x_0, y_0, z_0) \). In particular it admits a lower Fibonacci bound.

**Lemma 7.0.2.** The function \( f : (a, b, c) \mapsto \log(ab) \) satisfies the inequality (12) above with \( \delta \leq \log(4) \).

**Proof.** It is convenient to adopt the following notation: for a function \( (a, b, c) \mapsto f(a, b, c) \), \( I^*_b(f(a, b, c)) := f \circ I_b(a, b, c) \). Under this operation the functions

\[
(a, b, c) \mapsto ab,
(a, b, c) \mapsto bc,
(a, b, c) \mapsto ac
\]

transform as follows:

- \( I^*_b(ab) = ab \), \( I^*_b(bc) = bc \) that is these are invariant functions
- \( I^*_b(ac) = ac(b-1)^2 \).

Consider the ratio

\[
\frac{I^*_b(ac)}{I^*_b(ab)I^*_b(bc)} = \frac{ac(b-1)^2}{ab^2c} = \frac{(b-1)^2}{b^2} = \left(1 - \frac{1}{b}\right)^2.
\]

Taking logs:

\[
\log(I^*_b(ac)) = \log(I^*_b(ab)) + \log(I^*_b(bc)) + 2\log(1 - 1/b)
\]

\[
= \log(ab) + \log(bc) - 2\log\left(\frac{b}{b-1}\right).
\]

By hypothesis \( b > 2 \) so that \( 2\log(b/(b-1)) < 2\log(2) = \log(4) \) and it follows that this system satisfies Bowditch’s condition (12) if

\[\delta = \log(4) \leq \min(\log(ab), \log(bc)).\]

This is equivalent to

\[4 \leq \min(ab, bc),\]

which is immediate from Lemma [4.1.1]. \( \square \)

8. A HYPERBOLIC METRIC WITH HOLONYM \( \rho \)

Finally we show that each representations is the holonomy of a singular hyperbolic metric on the 3 punctured sphere. This is done by constructing a convex polygon \( P_b \) in \( \mathbb{H} \) such that

1. each of \( A, B, C \) act as side pairing transformations
2. the quotient space obtained by identifying sides is a 3 punctured sphere.

The \( P_b \) has 3 ideal vertices, namely the fixed points of \( A, B, C \), and 3 finite vertices which are the fixed points of the elliptic elements \( CBA, BAC \) and \( ACB \) respectively.

8.1. The fixed point of \( CBA \).

**Lemma 8.1.1.** If \( a + b - ab < 0, b > 2, 2 > \kappa > -2 \) then \( CBA \) is elliptic and its unique fixed point \( z \in \mathbb{H} \) satisfies

\[
\text{Re}(z_{\pm}) = \frac{\kappa - 2 + 2b}{a + b - ab} < 0.
\]
Proof. One computes $CBA$
\[
CBA = \begin{pmatrix} 1 + b - ab - ca - cb + abc & -b - c + bc \\ a + b - ab & 1 - b \end{pmatrix}
\]
and applies the formula (3). □

8.2. Proof of existence.

Theorem 8.2.1. Provided $\kappa(a, b, c) > -2$ the representation $\rho$ associated to the triple $(a, b, c)$ is the holonomy of a hyperbolic metric with cone angle $\theta$ such that
\[
2 \cos \theta / 2 = \kappa.
\]

Proof. Let $z \in \mathbb{H}$ be the fixed point of $CBA$. Begin by observing that $C^{-1}$ conjugates $CBA$ to $C^{-1}CBA = BAC$ and $BA$ conjugates $CBA$ to $BA(CBA)(BA)^{-1} = BAC$. Thus we need to check that the “fundamental polygon” with the following 6 vertices $z, 0, C^{-1}(z), (BA)^{-1}(0), 1, 0$ is convex.

This follows from Lemma 8.1.1. □

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