Arithmetical chaos and quantum cosmology

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Abstract

In this paper, we present the formalism to start a quantum analysis for the recent billiard representation introduced by Damour, Henneaux and Nicolai in the study of the cosmological singularity. In particular we use the theory of Maass automorphic forms and recent mathematical results about arithmetical dynamical systems. The predictions of the billiard model give precise automorphic properties for the wavefunction (Maass–Hecke eigenform), the asymptotic number of quantum states (Selberg asymptotics for \( \text{PSL}(2, \mathbb{Z}) \)), the distribution for the level spacing statistics (the Poissonian one) and the absence of scarred states. The most interesting implication of this model is perhaps that the discrete spectrum is fully embedded in the continuous one.

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1. Introduction

The beautiful singularity theorems of Hawking and Penrose [11] establish the presence of a singularity in the generic solution to Einstein field equations under very reasonable assumptions of the kind of matter. These theorems use tools from differential topologies (Morse theory for Lorentzian manifolds) but do not say anything about the behavior of the singularity, being existence results. Thus the works by Belinskii et al (BKL in the following) [2] regarding the general behavior of a homogeneous cosmological singularity are very important in understanding the (spacelike) singularity in general relativity. Their approach is based on the assumption that close to the cosmological singularity the spatial gradients can be neglected with respect to the temporal ones in the field equations, this being equivalent to a spatial decoupling of the points on the singular hypersurface (BKL limit). Thus each point (i.e., the space part of the metric) evolves according to an infinite succession of Kasner eras, leading to an approximate discretized dynamics which turns out to be isomorphic to the Gauss map (the continued fraction expansion of a real number), hence chaotic. The analysis of BKL has
been recently improved and generalized by Damour et al (DHN in the following) \cite{6} to the case of a general inhomogeneous cosmological singularity and supergravity/string theories (extra dimensions, exotic matter, etc). Their result is that, in the BKL limit, the dynamics of pure gravity in four dimensions can be reformulated as a billiard motion in a suitable domain embedded in a flat three-dimensional pseudo-Riemannian manifold (we comment on the chaoticity of this billiard too). This billiard table has a basis on the hyperbolic plane corresponding to the fundamental domain of the extended modular group \( \text{PGL}(2, \mathbb{Z}) \), which is a so-called arithmetic group. The arithmetic nature of the asymptotic billiard allows us to derive a precise quantum analysis and make some predictions for this model using powerful and recent theorems about the quantum behavior of arithmetical dynamical systems. This quantum analysis is the main point of this paper. In particular, we can give the precise form of the wavefunction, count the asymptotic number of quantum states and comment on the so-called scarred states. Finally, the role of the Selberg trace formula for \( \text{PSL}(2, \mathbb{Z}) \) as a convergent semi-classical quantization rule for gravity in this regime is briefly pointed out.

Two appendices, on the Kac–Moody algebra \( \text{HA}_1^{(1)} \) and the theory of Maass waveforms, are added at the end and should be read in parallel with the relevant sections.

This paper is based on \cite{7}, which also contains more details, references and background material. For the BKL limit, the original references and many other related questions we suggest the recent reviews \cite{13, 18}; the latter deals especially with the cosmological billiards in relation to other theories (supergravity, etc) too.

We want to stress that, usually, with the term chaotic cosmology one refers to the fact that the asymptotic (i.e., in the limit toward the singularity) classical evolution exhibits some chaotic features, but one has to consider what the implications of that are in the quantum behavior of the system. As we will see the corresponding quantum system is not ‘chaotic’, since the quantum manifestations of classical chaos for arithmetical systems are closer to a classically integrable system, and this is due to the arithmetic nature of the billiard. In fact, the existence of Hecke operators, which form at the quantum level an infinite family of commuting self-adjoint operators which mimic an integrable system, allows for an anomalous spectral statistics (a Poissonian law for the high energy spectrum of the quantum Hamiltonian instead of a GOE/GUE ensemble) and allows us to derive the quantum unique ergodicity theorem.

2. The BKL oscillatory behavior and the billiard representation

Let us briefly review the BKL result, restricting to the case of a homogeneous Bianchi IX universe, or ‘mixmaster’ as renamed by C Misner for the chaotic features it exhibits. Moreover, we consider only the case of pure gravity in four dimensions. Then, the spacetime metric can be written in a synchronous reference system as

\[
\begin{aligned}
\text{d}s^2 &= -\text{d}t^2 + \text{d}l^2 \\
&= -\text{d}t^2 + h_{ij} \text{d}x^i \text{d}x^j
\end{aligned}
\]

with \( h_{ij} \) diagonal

\[
h_{ij} = a^2(t)l_il_j + b^2(t)m_im_j + c^2(t)n_in_j
\]

and where the frame vectors \( l, m, n \) depend on the spatial coordinates \( x^i \). The behavior of the three scale factors \( a, b, c \) has been studied by BKL in the limit \( t \to 0 \).

Their evolution is represented in figure \( 1 \). The asymptotic dynamics is such that there exists an infinite number of Kasner eras, each one formed by a certain number of Kasner epochs. Precisely, during each

1 Regarding the validity of the BKL regime, we have to say that it should be settled down in the pre-inflationary universe, thus unaccessible to present observation. Thus it concerns the very early stages of the universe evolution, matching the Planckian era with the inflationary behavior. Of course, one should always consider the problem of the falsification of a theory of quantum gravity, since it deals with Planck scale physics.

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Kasner era, two of the scale factors oscillate and the third one decreases monotonically. On passing from one era to the next one, the monotonic decrease is transferred to another of the three scale factors. The time interval in which each scale factor is represented as a straight segment is called a Kasner epoch, i.e., it is a regime of the solution to the field equation where one can approximate the exact solution with a Kasner solution. Given an irrational number $u > 1$, the number of Kasner epochs contained in a Kasner era corresponds to the partial denominators in the continued fraction expansion for $u$.

In other words, the result is that the asymptotic evolution can be described as infinite succession of Kasner epochs with a certain law of replacement of the Kasner exponents when passing over from one epoch to the next one. This law corresponds to the Gauss map

$$ T(x) = \frac{1}{x} - \left[ \frac{1}{x} \right] = \left\{ \frac{1}{x} \right\} $$

whose metric entropy can be computed

$$ h(T\text{Gauss}) = \frac{\pi^2}{6 \ln 2}. $$

Note that the Gauss map accounts only for the transitions between successive Kasner eras (i.e., in the picture between the times $\Omega_0$ and $\Omega_1$ and so on). One can do better and consider also the oscillations in two of the scale factors inside a single Kasner era and include them in a more complete map which describes the discrete evolution (still approximated). This can be realized through the Farey map, whose topological entropy is given by

$$ \tau(T\text{Farey}) = 2 \ln 2. $$

This map accounts for oscillations (one pair of axes oscillates while the third one decreases monotonically) and bounces (when the role of the three axes are interchanged and a different axis decreases monotonically), according to a chaotic Farey tale [5]. This paper uses fractal techniques (which are observer independent) to claim that the mixmaster universe is indeed chaotic.

Finally, note that in the BKL approach (although fascinating) it is not easy to write down explicitly a quantization procedure.
BKL also considered the case of an inhomogeneous universe and showed that there exists a generalized Kasner solution near the singularity
\[ dl^2 = h_{ij} dx^i \, dx^j, \]
with
\[ h_{ij} = a^2 \delta_{ij} + b^2 m_i m_j + c^2 n_i n_j, \]
and
\[ a \sim t^p, \quad b \sim t^m, \quad c \sim t^n \]
but now, different from the homogeneous case, the Kasner exponents are functions of the spatial coordinates. One expresses this fact by saying that inhomogeneous cosmology can be modelled on a Bianchi IX universe, since there exists an analog infinite succession of Kasner epochs/eras with local behavior, i.e. each point on the singular hypersurface evolves with the same laws (Bianchi IX) but with different parameters. In this sense, one says that the behavior of a generic singularity is \textit{local and oscillatory}.

The positivity of the metric entropy for the Gauss map implies its ergodicity. This result was derived for the first time by Artin (much before the introduction of entropy in dynamical systems) relating the properties of the Gauss map to the geodesic flow on \( \text{PSL}(2, \mathbb{Z}) \) \( \setminus \mathbb{H} \), which was known to be ergodic (see [21]). Thus, one can ask if there exists a precise physical system associated with this geodesic flow in the same way as the asymptotic evolution of a Bianchi IX universe can be approximately described by the Gauss map. The answer to this question is positive and the physical system is pure gravity in four dimensions near the singularity \textit{and} without any symmetry assumption for the metric. This leads us to the billiard representation introduced by DHN.

The result by DHN [6] is that the BKL oscillatory behavior of pure gravity in four dimensions can be reformulated as a billiard motion in an auxiliary spacetime\(^2\). More precisely, the dynamics of the generic solution to Einstein equations close to the cosmological singularity (and in the BKL limit, i.e. assuming BKL’s conjecture) is equivalent to a \textit{null geodesic motion} (i.e., on a light ray) inside a billiard table given by a Coxeter polytope in a three-dimensional Minkowski space\(^3\). The remarkable fact is that this polytope corresponds to the Weyl chamber

2 One can speculate whether this representation is photographic or holographic (since we go from a four-dimensional spacetime to a three-dimensional auxiliary Minkowski space), according to the words of T Damour at the last 11th Marcel Grossmann meeting.

3 Generic means without any symmetry assumption for the metric. If accidental symmetries are present in the metric, the analysis is not valid. For example, it is known that the Schwarzschild solution
\[ ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\Omega^2 \]
has a spacelike singularity at \( r = 0 \); moreover, inside the horizon \( (r < 2m) \) the \( r \) coordinate is timelike (there is a minus sign in front of \( dr^2 \)). If we take the limit \( r \to 0 \), we obtain
\[ \lim_{r \to 0} ds^2 = \frac{2m}{r} dt^2 - \frac{r}{2m} dr^2 + r^2 d\Omega^2, \]
which is a Kasner metric
\[ -dr^2 + t^{-2/3} ds^2 + t^{2/3}(\sin(\theta)) \, d\theta^2, \]
and for curved spacetimes
\[ \sigma = (4m/3)^{1/3}, \quad \bar{\sigma} = (9m/2)^{1/3} \theta \quad \text{and} \quad \bar{\Phi} = (9m/2)^{1/3} \phi, \]
where \( \bar{\sigma} \) and \( \bar{\Phi} \) are the Schwarzschild solution (in the neighborhood of the singularity) to a single Kasner epoch, not to a never-ending succession of Kasner eras.

Finally, the analysis does not apply to timelike (like that in the Reissner–Nordström solution, the charged black hole) or null singularities where a causal decoupling of spatial points does not occur. The question about the general behavior of not-spacelike singularities is still open.
of the hyperbolic Kac–Moody algebra $\text{HA}_1$, the canonical hyperbolic extension of $\mathfrak{A}_1$ ($\text{su}(2)$).

The walls are the hyperplanes pointwise fixed by Weyl reflections in the three simple roots which define the algebra and the reflections at the walls are elastic. In this language, each Kasner epoch is represented by a null geodesic segment between two successive reflections. In particular, given a Kasner epoch and the wall where this epoch crashes/ends, the following one is obtained by Weyl reflection with respect to the simple root orthogonal to that face of the Weyl chamber. In other words, Weyl reflections with respect to simple roots send null geodesic segments into null geodesic segments, i.e. transform Kasner solutions into Kasner solutions (with different values of Kasner exponents). Because of this, we have a hint that the Weyl group of the algebra acts on the space of solutions of Einstein equations, transforming Kasner solutions into Kasner solutions which can then be associated with the simple roots (and more generally to the real roots of the algebra). Of course, this would be a technique to generate solutions (also approximate) to Einstein equations starting from very simple solutions. This was already done by R Geroch, who found that gravity reduced to 1+1 dimensions has a hidden $\mathfrak{A}_1$ symmetry (affine $\text{su}(2)$), which is a sub-algebra of $\text{HA}_1$. Thus in this case, the affine algebra (or better its corresponding infinite-dimensional Lie group, known as the Geroch group) can be considered a symmetry of the theory. One can go further and check if the whole hyperbolic algebra (not only its Weyl group, on which we focus in this work) is a symmetry of gravity and in which regime, see [13].

In this formalism only null geodesics are physical and correspond to Kasner solutions. Spacelike and timelike geodesics seem to play no role. Note that the walls of the Weyl chamber are timelike, that is their orthogonal vectors are spacelike, since for the simple roots one has $(\alpha_i, \alpha_i) = 2 > 0$. Because of this, every reflection preserves the null character of the velocity vector.

The (flat) metric in this Minkowski space is given by the Cartan matrix of the algebra

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

whose signature is $(- + +)$. For this reason, one has three kinds of geodesics, but only null geodesics appear.

Note that in the DHN approach, the three-dimensional billiard is peculiar, in fact the walls move together with the trajectories. During this motion the geometry of the billiard remains the same, i.e. the structure of the algebra is preserved and the Weyl group is uniquely determined. This means that the walls of the (reduced) billiard on the hyperbolic plane are motionless and the two-dimensional analysis is much easier than studying the three-dimensional billiard with (say) co-moving walls. As we will say many times in this work, the Weyl group of the algebra is an arithmetic group (being isomorphic to $\text{PGL}(2, \mathbb{Z})$), thus we are coping with a physical system represented by an arithmetical dynamical system.

The very essence of arithmetical chaos (see [4]) must be researched in its quantum aspects, since from the classical point of view no difference exists between generic chaotic dynamical systems and arithmetical dynamical systems, except for the degeneracy of lengths of periodic orbits. In fact, for generic systems, one does not expect a degeneracy of this kind but those which come from the symmetry of the model. For example, for time-invariant systems, this multiplicity, on average, is 2, which corresponds to the two possible ways to run a geodesic. Arithmetical systems are very exceptional, since the degeneracy of lengths of periodic orbits grows exponentially

$$\frac{1}{C} e^{L/2},$$

\[ (14) \]
where \( L \) is the length of such a geodesic and \( C \) is a constant depending on the particular group \((C = 1 \text{ for } \text{PSL}(2, \mathbb{Z}))\). To this, one should add the Horowitz–Randol theorem which states that these degeneracies are unbounded. The large degeneracy of lengths of periodic orbits seems to have no importance in classical mechanics. These systems are ergodic as any other model on negatively curved surfaces, but the quantum aspects are anomalous. Thus, it is very interesting to start a quantum analysis of the cosmological billiard using its arithmetic nature. Before doing that, in the following section we comment on the (classical) chaoticity of the cosmological billiard.

### 3. General considerations on the cosmological billiard

Let us stress again that the billiard motion occurs in a pseudo-Riemannian three-dimensional space, not in a Euclidean space. This space is \( h^*_\mathbb{R} \), in the notion of Kac [14] (see also appendix A). Thus the billiard motion is an interrupted null flow on a pseudo-Riemannian manifold. This situation is different from the typical billiards which are embedded in Riemannian manifolds. In fact, in pseudo-Riemannian manifolds, one has three kinds of geodesics, and consequently three kinds of geodesic flows (indeed one first needs to define in a rigorous way flows for manifolds with an indefinite metric). The second step is the definition of the billiard flow, which is even more involved. In fact, because of the three kinds of directions (null-like, timelike, spacelike), each reflection on a wall depends both on the character of the incoming trajectory and on the character of the wall.

To clarify this important point, let us consider the usual billiard motion in a convex polyhedron \( Q \) in a Euclidean space, that is a Riemannian manifold with the usual (diagonal) flat metric (see [21]). The boundary of this polyhedron is given by the union of its faces \( \Gamma_i \). Each elastic reflection in the wall \( \Gamma_i \) is given by the geometric reflection \( \sigma_i \)

\[
\sigma_i(v) = v - 2(n_i, v)n_i, \tag{15}
\]

where \( v \) is the velocity vector of the incoming trajectory, \( n_i \) is the unit vector orthogonal to each face and \((, )\) is the standard Euclidean scalar product (compare with the expression of the Weyl reflections in appendix A). To understand the properties of the billiard flow inside \( Q \), one has to consider the group \( G_Q \) generated by the reflections \( \{\sigma_i\} \); this is a subgroup of all isometries of the Riemannian manifold. The ergodicity of the billiard depends on \( G_Q \), precisely: if \( G_Q \) is a finite group, then the billiard flow inside \( Q \) is not ergodic (thus it is not chaotic). In two dimensions, the finiteness of the group is equivalent to the commensurability of all angles on the polygon \( Q \).

The situation for generic polyhedra is still open. Indeed, one knows that the metric entropy of a billiard inside an arbitrary, not necessarily convex, polyhedron is zero. All these statements are valid for Riemannian manifolds.

In our case, the analogy is the following. The polyhedron \( Q \) is replaced by the Weyl chamber of \( \text{HA}^{(1)} \), the Riemannian manifold is replaced by the pseudo-Riemannian space \( h^*_\mathbb{R} \), and the billiard motion occurs on null segments. The faces of the billiard are timelike walls. Finally, the reflection group \( G_Q \) is replaced by the Weyl group of the algebra \( W \), which is an infinite Coxeter group. At the moment of writing this work (and modulo over-present

4 We prefer the term pseudo-Riemannian instead of Lorentzian because for people who study billiards as dynamical systems, the word Lorentzian has a completely different meaning.

5 The problem is the definition of the billiard law, i.e. how to use the rule ‘angle of incidence = angle of reflection’ in a pseudo-Riemannian setting. When one speaks of billiards, one must be careful, because together with the billiard table, one has also to specify the billiard law. In Euclidean spaces, one usually considers the law of geometrical optics, but in principle one can study different billiard laws (in the same table), as it happens for outer billiards which have nowadays surprising applications.
Figure 2. Schematic representation for the cosmological billiard in the Weyl chamber of $\text{HA}_1^{(1)}$ and projection on the Poincaré disc.

ignorance), there are no mathematical results which imply the ergodicity (or the integrability) of our billiard motion, because the difficulty is that the motion occurs in pseudo-Riemannian manifolds, thus we cannot invoke the previous theorems for billiard flows in Riemannian manifolds. In particular, we know that $W$ is isomorphic to $\text{PGL}(2, \mathbb{Z})$, and that the free motion on the hyperbolic plane inside a region with finite (hyperbolic) area is a chaotic flow, being an Anosov flow [21]. Thus we can state that there exists a chaotic motion on the ‘basis’ of the billiard, but this does not imply the chaoticity of the null billiard flow in the full three-dimensional Weyl chamber. In fact, the full motion could be less chaotic or even integrable. For example, the Bunimovich stadium is a well-known example of a two-dimensional Euclidean chaotic billiard, but if we consider a three-dimensional Euclidean billiard raising the stadium as a basis in the $z$-direction, then the billiard is integrable, because of a translation symmetry in the $z$-direction. Thus, a billiard with a chaotic basis (i.e., which has the property of being ergodic or mixing or hyperbolic, etc) is not necessarily chaotic. Our case is even more complicated (see figure 2), because the full three-dimensional billiard is a Coxeter polytope in a pseudo-Riemannian space, and the projected billiard corresponds to the fundamental domain of $\text{PGL}(2, \mathbb{Z})$ on the hyperbolic plane (which is a Riemannian manifold)\(^6\).

To conclude this section, one cannot state that the Kac–Moody billiard is chaotic only from the property that the billiard table is contained inside the light-cone, but more powerful (and at the moment missing) theorems are needed. The study of billiard flows on pseudo-Riemannian manifolds is a completely new field and should be an interesting topic of future research. It seems that the only mathematical papers on this subject are very recent and due to Khesin and Tabachnikov [16], where they define a billiard law which is area-preserving and does not change the type of a geodesic.

Let us also observe that the metric entropy of the geodesic flow inside the fundamental domain of $\text{PGL}(2, \mathbb{Z})$ can be explicitly calculated\(^7\)

\[
h((S^1) \text{ on } \text{PGL}(2, \mathbb{Z}) \setminus H) = 1
\]

thus the billiard representation on the hyperbolic plane and the BKL approach based on the Gauss map are not equivalent, as they are not isomorphic as dynamical systems. In fact, the first describes the behavior of a generic inhomogeneous singularity, the second describes

\(^6\) With the projection of the three-dimensional billiard on the hyperbolic plane, we mean that we consider the geodesic segment on the hyperbolic plane corresponding to the intersection in three dimensions of a null segment with a plane through the origin.

\(^7\) In fact, for the geodesic flow $\{S^1\}$ on a surface of negative constant curvature $K$, the following formula holds:

\[
h((S^1)) = \sqrt{-K}
\]

and one usually puts the Gaussian curvature $K = -1$. 

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the fate of a Bianchi IX homogeneous universe. Yet, as we mentioned above, a famous result due to Artin relates the ergodicity of the geodesic flow on \( \text{PGL}(2, \mathbb{Z}) \setminus \mathbb{H} \) to the ergodicity of the Gauss map. Artin’s theorem supports the conjecture that the behavior of a generic spacelike singularity is somehow well described by a Bianchi IX homogeneous cosmological model, a conjecture for which nowadays there is a lot of numerical evidence \[3\]. At this point, we should add that some theoretical progress has been made in understanding the general behavior of the cosmological singularity thanks to Uggla et al (see \[22\] for a review), who have shown that asymptotic silence holds, i.e. particle horizons along all timelines shrink to zero for generic solutions. With Uggla’s words at the last 11th Marcel Grossmann Meeting, ‘everybody dies alone’.

We can make an interesting consideration when we pass from the three-dimensional Minkowskian billiard to the two-dimensional Euclidean one. The light rays which move in the auxiliary Minkowskian space have no mass of course. When we study the wave equation on the reduced domain (i.e. the Schrödinger equation for \( \text{PGL}(2, \mathbb{Z}) \)), there is a subtlety. Indeed, this equation allows for a mass different from zero because it corresponds to the quantization of a classically chaotic particle on the hyperbolic plane (we always put \( m = 1, \hbar = 1 \)). But this particle ‘arises’ from light-like rays in three dimensions. Thus, we have an example of a mechanism to generate mass on a Euclidean manifold from a null motion on a Minkowskian manifold.

4. The wavefunction of the universe

Let us now come to the quantization of the cosmological billiard. In the language of \[6\], we can decompose the motion in the three-dimensional billiard in an angular part inside a fundamental domain for \( \text{PGL}(2, \mathbb{Z}) \) plus a radial part. The latter gives a temporal contribution to the full wavefunction, because the vertical direction in the Weyl chamber is a temporal one. Thus, with the term wavefunction in this section we mean the space part of the full wavefunction\[8\] which is solution of the Schrödinger equation on the hyperbolic plane. How to arrive at this equation after some changes of variables is already described in many works (see \[18\] and references therein), we briefly recall the main steps below. But let us stress that the wave equation on the hyperbolic plane derives from the Wheeler–DeWitt equation for the three-dimensional problem. We believe that the time dependence of the solution to the full wave equation does not change the physical spectrum, which is thus contained in the two-dimensional billiard. In \[18\] (following previous works by Misner), the authors transform the domain for Bianchi IX using an approximation which makes the domain integrable (they modify the lower arc of the billiard with a horizontal segment, but note that the latter is not a geodesic segment on the hyperbolic plane) and with the same area. The same approach was used for Bianchi IX in \[8\], where the authors identify the fundamental domain on the hyperbolic plane specific\[9\] for this situation and try to resolve the corresponding wave equation (see figure 3).

Here we keep the exact domain. The novelty with respect to previous works is our widespread use of the arithmetic properties of the billiard, which, although mentioned in some

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8 To be more precise, one should also say that the full wavefunction of the system is, heuristically, the product of the wavefunctions in each space point on the hypersurface. But since the classical behavior is local, we need only to study the quantum behavior in a generic point, where we approximate the physical evolution by the billiard representation.

9 They restrict to Bianchi IX without rotation of axes. If one considers also the rotation of axes, the fundamental domain is the one for \( \Gamma_0(2) \) \[17\], which is a subgroup of \( \text{PSL}(2, \mathbb{Z}) \). It is reasonable for the generic/inhomogeneous case to expect a generic discrete group, and in fact \( \text{PSL}(2, \mathbb{Z}) \) occurs.
To be more specific, using the notation of [8], the three-dimensional quantum equation reads as follows:

\[-\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \log \rho \right) \frac{\partial \Psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \log \rho \right) \frac{\partial \Psi}{\partial \rho} + \frac{1}{\sinh^2 \zeta} \Delta_{LB} - U(\rho, \zeta, \phi) \right] \Psi = 0 \tag{18}\]

where \(\Delta_{LB} = \frac{1}{\sinh \zeta} \frac{\partial}{\partial \zeta} \left( \sinh \zeta \frac{\partial}{\partial \zeta} \right) + \frac{1}{\sinh \zeta} \frac{\partial^2}{\partial \phi^2}, U \) is a potential term

\[U(\rho, \zeta, \phi) = (\ln \rho)^2 \rho^4 \cosh^4 \zeta (V - 1), \tag{19}\]

\(\rho\) is the radial (time) variable, \(\zeta\) and \(\phi\) are the angular (space) variables. This equation is exact, in the sense that it is derived from the Bianchi IX metric without any other assumptions\(^{10}\).

If we now take the limit \(\rho \to 0^+\) (which corresponds to the case \(\Omega \to +\infty\) in the BKL variables), then the potential term vanishes inside and is plus infinite outside the triangular domain pictured in the figure. This makes it possible to factorize the solution as follows:

\[\Psi = \Phi(\rho) \psi(\zeta, \phi) \tag{21}\]

and we consider only the equation

\[-\Delta_{LB} \psi_i = E_i \psi_i \tag{22}\]

\(^{10}\) In particular, in the original Misner variables the Bianchi IX metric (1) can be written as

\[\text{d} s^2 = -d\tau^2 + \tilde{a}^2(t) (e^{2\beta} \omega_i \omega^i), \tag{20}\]

where \(\beta = \text{diag}(\beta_+, \sqrt{3} \beta_-, \beta_-, -\sqrt{3} \beta_+)\) is a 3 \times 3 traceless matrix and the \(\omega_i\)'s are a basis of 1-forms for \(\text{SO}(3)\). In terms of these variables, the term \(V\) in the potential \(U\) is \(V = \frac{1}{4} \text{Tr} (1 - 2 e^{-2\beta} + e^{4\beta})\). Finally, the variables \(\beta, \tilde{a}\) are transformed to hyperbolic coordinates \(\rho, \zeta, \phi\) [8].
which after some redefinitions of variables corresponds to the spectral problem for the hyperbolic Laplacian (see [18, 8], in the latter the approximate temporal evolution for the three-dimensional wavefunction is computed too).

To go beyond, as we have seen, the general inhomogeneous case is modelled on a billiard problem inside the fundamental Weyl chamber of the hyperbolic algebra \( HA^{1(1)} \) whose projection on the hyperbolic plane is \( \text{PGL}(2, \mathbb{Z}) \) and not \( \text{PSL}(2, \mathbb{Z}) \), thus we have to consider the spectral problem with Neumann boundary conditions

\[
-\Delta \psi = E \psi \\
\psi \in L^2(\mathcal{F}_3, \mu) \\
\partial_n \psi|_{\partial \mathcal{F}_3} = 0,
\]

where \( \mathcal{F}_3 \) is the halved standard modular domain, \( \Delta \) is the hyperbolic Laplacian and \( \mu \) is the usual measure on the hyperbolic plane (the notation of this section is explained in appendix B with more details). Thus solutions to the Neumann problem for \( \text{PGL}(2, \mathbb{Z}) \) are given by even Maass cusp forms for \( \text{PSL}(2, \mathbb{Z}) \). In fact, the spectrum of \( \Delta \) on \( \text{PSL}(2, \mathbb{Z}) \) is purely discrete on the space of odd functions, but on the even space there is a continuous spectrum given by the interval \( \left[ \frac{1}{4}, \infty \right) \) and, what is less obvious, a discrete spectrum. Thus the space of Maass waveforms (which for \( \text{PSL}(2, \mathbb{Z}) \) are also cusp forms, i.e. they are zero at the cusps) splits into odd/even forms under the symmetry \( R_1 \psi(z) = \pm \psi(z) \).

Since the translation \( T(z) = z + 1 \) belongs to the modular group, an even/odd Maass form can be written as \( \psi(x) = \sum_n a_\psi(n) e^{2\pi i n x} \). Considering the eigenfunction condition \( \Delta \psi + \left( \frac{1}{4} + t^2 \right) \psi = 0 \) and the square-integrability condition, one arrives at a more explicit form for the Maass cusp forms for \( \text{PSL}(2, \mathbb{Z}) \), that is

\[
\psi(z) = \sum_{n \neq 0} a_\psi(n) y^{1/2} K_{it}(2\pi |n| y) e^{2\pi i n x},
\]

where \( K_{it}(2\pi |n| y) \) are modified Bessel functions such that \( K_{it}(y) \ll e^{-2\pi y} \) for \( y \to \infty \). The coefficients \( a_\psi(n) \) are the Fourier coefficients of the \( \psi(z) \). For even forms, \( a_\psi(-n) = a_\psi(n) \), while for odd ones \( a_\psi(-n) = -a_\psi(n) \). Finally, since for \( \text{PSL}(2, \mathbb{Z}) \) (and for all so-called Hecke triangle groups) there is an obvious symmetry with respect to the imaginary axis, one can write

\[
\psi(z) = \sum_{n=1}^{+\infty} a_\psi(n) y^{1/2} K_{it}(2\pi n y) \frac{\cos(2\pi i n x)}{\sin(2\pi i n x)},
\]

depending on whether \( \psi \) is even or odd. The precise statement, then, is that the wavefunction of the universe is an even Maass cusp form for the modular group \( \text{PSL}(2, \mathbb{Z}) \). Apart from the trivial eigenvalue \( E_0 = 0 \) and the corresponding constant eigenfunction \( \psi_0 \), no other eigenvalues/eigenfunctions are known analytically. Some eigenvalues, calculated via numerical investigations, are given at the end of section 5.

Actually, more is true. In fact, since \( \text{PSL}(2, \mathbb{Z}) \) is arithmetic and the cuspidal spectrum is very likely simple, one can diagonalize the hyperbolic Laplacian and the Hecke operators simultaneously. Thus it turns out that the wavefunction is a Maass–Hecke eigenform. This is an even more interesting statement. In fact, Hecke eigenvalues are multiplicative, i.e. \( \lambda(mn) = \lambda(m) \lambda(n) \), and this should put some conditions on the physical interpretation of them.

In recent years, in string theory, a lot of work has been done on counting the entropy of black holes (see for example [19] for a review). It turns out that in some cases this
entropy is counted by the Fourier coefficients of certain automorphic functions. Thus, it is remarkable that the quantization of gravity in the BKL limit leads to a wavefunction which has automorphic properties. In the present case, the question would be the following: is the gravitational entropy computable from the Fourier coefficients of Maass–Hecke eigenforms?

Another remark is the following. In the semi-classical limit, the expected statistics for the energy levels spacing distributions for $X = \text{PSL}(2, \mathbb{Z}) \backslash \mathcal{H}$ is the Poisson distribution. Indeed, from general considerations regarding chaotic systems, one would expect that the system follows predictions from random matrix theory (GOE/GUE ensemble). These ensembles present level repulsion at short distance and rigidity at long distance. But, again, due to the arithmetic property, we have an anomalous statistics, the Poissonian one, which exhibits the clustering property. The knowledge of the expected statistics could, in principle, allow us to compare with the observations.

The inclusion of matter changes the shape of the billiard and also increases the number of dimensions of the billiard. The most interesting case is perhaps supergravity theory in 11 dimensions, which is believed to be a kind of ultimate theory unifying all fundamental interactions. The supergravity billiard is the fundamental Weyl chamber for $E_{10}$, thus in this case the statement is that the wavefunction is a Maass wavefunction with respect to the (discrete) Weyl group of $E_{10}$ (which is not known). In this case, we cannot speak safely of Maass cusp forms, and we must use only the general term Maass waveform, because we do not know if the residual spectrum is empty as in the cases of each $\Gamma(N)$ (the congruence subgroups of $\text{PSL}(2, \mathbb{Z})$). Besides, the discrete spectrum could also be degenerated. The existence of Maass forms with respect to $W(E_{10})$ should be guaranteed by the fact $W(E_{10})$ is arithmetic.

Usually in quantum mechanics, one has a discrete and a continuous spectrum for a self-adjoint Hamiltonian and these are separated. Bound states are the proper eigenfunctions of the discrete spectrum, whereas the continuous part is interpreted as a free motion. Our model of quantum cosmology is different from this usual situation. In fact, from the Selberg theory for the automorphic Laplacian inside a fundamental domain of some $\Gamma(N)$, we know that the discrete spectrum is not separated from the continuous one $\left[\tfrac{1}{4}, \infty\right)$ (whose improper eigenfunctions are given by the Eisenstein series), but it is embedded in the continuous part\[^{11}\].

We believe this says something about the nature of quantum gravity, i.e. this suggests that quantum gravity/cosmology is a non-trivial mixing of discrete and continuous concepts.

It would be interesting to understand if other approaches to the problem of quantum gravity like the formalism of loop quantum gravity or string theory say something regarding this point.

5. Scarring and asymptotic number of quantum states

In [1], Barrow and Levin have analyzed the case of a finite universe emerging from a compact octagon on the hyperbolic plane. The octagon is a compact domain (but the associated discrete group is not arithmetic), thus the classical motion inside is chaotic. They found a fractal structure, which, in the classical to quantum transition, can persist in forms of scars, ridges of enhanced amplitude in the semi-classical wavefunction. They conclude that if the universe is finite and negatively curved, the cobweb of luminous matter might be a residue of these primordial quantum scars.

Of course we can never know if our universe is finite or not. Yet, in the case of a generic singularity, we have seen that the interesting domain is the one for $\text{PGL}(2, \mathbb{Z})$, which is an arithmetic group. Thus, there is no scarring (see appendix B). The conclusions of Barrow

\[^{11}\text{There are cases in scattering problems where points of the discrete spectrum lie in the continuous spectrum, but our situation is distinguished, because the whole discrete spectrum is embedded in the continuous one.}\]
and Levin do not apply in this case and it remains to understand the physical (cosmological) meaning for the absence of these scarred states. Results in this direction by loop quantum gravity/cosmology or string theory would be interesting, in order to confirm or discard the roles of scarred states in quantum cosmology.

Note one more thing. Our analysis is limited to the case of pure gravity in four dimensions, thus the conclusion about the absence of scarred states in quantum cosmology is valid only in this context. Thus one may think that the inclusion of matter or extra dimensions could change the situation. If we consider the case of supergravity theory in 11 dimensions (or a not well-defined quantum version of it, like M-theory or whatever), then, as we mentioned, the cosmological billiard is the Weyl chamber of $E_{10}$ [6] (the role of the fermions is still matter of debate) whose Weyl group is still arithmetic. Thus for the hyperbolic manifold $W(E_{10}) \setminus \mathbb{H}^9$ the quantum unique ergodicity theorem should be true and, again, there is no scarring effect. Eleven-dimensional supergravity is a candidate theory to describe the universe and all of its interactions. The message is that, with the knowledge we have today, it does seem that in the early universe scarred states are absent. And, in our opinion, quantum gravity has to cope with quantum chaos. The absence of scarred states is deduced from the arithmetic quantum unique ergodicity theorem, which says that the only invariant measure at the quantum level is the one given by the Lebesgue measure on the hyperbolic manifold (which means uniqueness of the invariant measure for this model of quantum cosmology). In other words, all the features of classical chaos disappear at the quantum level, giving a system which is closer to a classically integrable system.

Finally, regarding the asymptotic number of quantum states, we can count them using the Selberg result for $\text{PSL}(2, \mathbb{Z})$

$$N_{\text{PSL}(2, \mathbb{Z})}(R) = \sum_{0 < E_i \leq R} 1 \sim \frac{\mu(X)}{4\pi} R$$

for $R \to \infty$. This gives the asymptotic number of eigenvalues/eigenfunctions, thus it counts the asymptotic number of discrete states in quantum cosmology. Actually, we have to consider only the eigenvalues whose eigenfunctions are even, so only a part (on average, half) of this asymptotics is physical. Again, a confirmation of this from other approaches to the problem to quantum cosmology would be extremely interesting. For $E_{10}$, there is no such result.

The first eigenvalues of $\Delta$ on $X$ are $E_1 = 91.12$, $E_2 = 148.43$, $E_3 = 190.13$, $E_4 = 206.16$. They are respectively even, odd, odd, even and odd, with respect to the symmetry about $x = 0$. See [12, 20] for the description of the numerical methods used to determine eigenvalues and an approximate form for the eigenfunctions. It is interesting also to compare the energy levels computed in [18] using the approximate domain with those computed from the ‘exact’ domain (but always via numerical methods).

6. Conclusions, some speculations and points to be developed in the future

Assuming the billiard representation introduced by DHN in the study of the cosmological singularity in the BKL limit, we have commented on the question of integrability/chaoticity of the null billiard flow inside the Weyl chamber of $\text{HA}_{10}$: one must be careful about statements which derive the chaoticity of cosmological billiard from the finite-volume argument (typical of the Euclidean case), as we are in a pseudo-Riemannian setting. In fact, from the Anosov property of the flow on the basis of the billiard (the fundamental domain for $\text{PGL}(2, \mathbb{Z})$), it does not follow the chaoticity for the dynamics in the full three-dimensional Weyl chamber, since from the ergodicity of a dynamical system in a proper subset of the phase space one cannot infer the ergodicity of the system in the full phase space. We do hope to study billiards and
flows in pseudo-Riemannian manifolds in the near future, since these are the natural evolution of classical billiards toward considering the problem of ergodicity in special/general relativity. The next step, in fact, should be the study of relativistic billiards confined in some polyhedron in a Minkowski spacetime.

We have also carried on the quantum analysis for the cosmological billiard using the arithmetic properties of the modular group. The result is that, for pure gravity in four dimensions, the wavefunction of the universe (or better, the angular part projected on the hyperbolic plane, i.e. the space part of the full wavefunction) is an automorphic form for $\text{PGL}(2, \mathbb{Z})$, precisely an even Maass cusp form for $\text{PSL}(2, \mathbb{Z})$. Indeed, since $\text{PSL}(2, \mathbb{Z})$ is arithmetic, the wavefunction is a Maass–Hecke eigenform, being also eigenfunction of the Hecke operators (which commute with the Hamiltonian). In our opinion, the most important prediction/hint of the model is that the discrete spectrum is fully embedded in the continuous spectrum, which is a very distinguished situation. In fact, for every point in the discrete spectrum and every normalizable even Maass cusp form, there exists a corresponding not-normalizable Eisenstein series with the same value of the generalized eigenvalue. This is an odd situation, thus we would like to understand if this phenomenon hides some deep truth which is an intrinsic property of quantum gravity/cosmology or if it is due to the billiard model which is too simplified.

The arithmetic nature of $\text{PGL}(2, \mathbb{Z})$ allows us to state that in the early universe scarred states are absent, thanks to the quantum unique ergodicity theorem by Lindenstrauss. The model allows also to count the asymptotic number of quantum states, whose number is half of the full Selberg asymptotics for $\text{PSL}(2, \mathbb{Z})$, and to identify a distribution for the level spacing statistics. This distribution is the Poissonian one and is characterized by the clustering property. One should understand the meaning of this as opposite to the level repulsion predicted from random matrix theory for non-arithmetic chaotic systems.

These conclusions, true for pure gravity in four dimensions if we believe the billiard representation, are very likely valid also in the case of eleven-dimensional supergravity close to the singularity, whose billiard is modelled on the $E_{10}$ hyperbolic Kac–Moody algebra. Very briefly, one can say that the model predicts specific automorphic properties for the wavefunction, and gives the asymptotic number of quantum states and the statistics for the level spacing. Arithmetic systems at the quantum level seem to be closer to integrable systems than to generic chaotic systems, thus, for this model, any manifestation of classical chaos disappears at the quantum level.

In [7], we have pointed out that the Selberg trace formula for $\text{PSL}(2, \mathbb{Z})$ gives a semiclassical quantization rule for pure gravity in four dimensions, as typically occurs for the Gutzwiller trace formula in quantum chaos. But the difference is that the Gutzwiller trace formula is divergent, whereas the Selberg trace formula is convergent. Thus, in the BKL limit to the singularity and in the billiard representation, a semiclassical quantization of gravity is well defined. Indeed, the Selberg trace formula is a kind of path integral. This point of view is emphasized in the book by Grosche [9]. But remember that indeed we have an infinite number of semi-classical quantization rules, since the class of test functions entering the trace formula is very large [20]. This is the best one can do in a rigorous/convergent way for the quantum systems whose semi-classical limit is a Hamiltonian flow for which the Selberg trace formula is valid. Nevertheless, the trace formula contains a lot of information, in fact Selberg derived his asymptotics using a particular Gaussian test function.

It may be that this quantization helps in finding a correct theory of quantum gravity, as happened in the early days of quantum mechanics after the introduction of Bohr–Sommerfeld quantization rules. The Selberg trace formula, in a sense, is an identity between quantum mechanics (because it contains the spectrum of the quantum Hamiltonian) and classical
mechanics (because it contains a sum over all classical periodic orbits). In [7], we have made an attempt to interpret the imaginary roots of the hyperbolic algebra in terms of the periodic orbits which enter the Selberg trace formula. These periodic orbits live inside $\text{PSL}(2, \mathbb{Z})$, which is the projected billiard. The idea is to check if these periodic orbits on the hyperbolic plane can be lifted up to periodic orbits (which escape the singularity) in the three-dimensional billiard such that they correspond to solutions (also approximate) of Einstein equations in the asymptotic regime as it happens for the Kasner solutions. In fact, to the simple roots of the algebra and their Weyl images (i.e., the real roots) one can associate Kasner solutions, because every reflection is a Weyl reflection from a Kasner epoch to another one. We would like to understand if something similar is possible for the imaginary roots, in particular relating the Selberg trace formula to the structure of the hyperbolic algebra. A future paper of more mathematical taste is in progress.

The billiard representation is very interesting from many points of view. It is evident that in this formalism one has another example of a relation between general relativity and geometrical optics. One can wonder if this reformulation can be applied to other models too. We would like to understand if it is possible to reformulate problems of general relativity as billiard-like problems (although billiard problems are generally very difficult to solve). This would help in studying the integrability or ergodicity of general relativity in particular situations.

For example, the variables used to describe Choptuik’s solution to the collapse of a spherically symmetric scalar field show a profile with spikes which resemble (billiard) bounces, see for example figure 3 in [10]. Moreover, this system exhibits discrete self-similarity, a property which is very likely shared by the BKL limit too (let us remember that the collapse studied numerically by Choptuik implies the formation of a black hole, that is a singularity, thus possible analogies with the cosmological singularity are reasonable). However, let us remember that Choptuik’s solution is physical, while the BKL behavior, as far as we know, is theoretical.

We have limited our analysis to the case of pure gravity in 3+1 dimensions, where, in the limit to the singularity, spatial points decouple and the dynamics involves only a single (time) variable. The question now is: what could be the analog for a quantum theory when all four (time and space) variables must be considered? Is it possible to describe gravity with the tools of higher-dimensional modular groups in particular regimes where one does not have spatial decoupling? Of course, this is a statement motivated more by beautiful mathematics than physics, thus it can be wrong. However, let us remember that higher-dimensional groups share a lot of nice properties with $\text{PSL}(2, \mathbb{Z})$, like existence of waveforms, trace formulae (the so-called Arthur trace formula), etc. In other words, the construction of a quantum theory would be viable under the aegis of what we would call an ‘automorphic principle’. An interesting mathematical mechanism which could do the job is the lifting of automorphic forms to higher dimensions, but again the connection with the physics is all to prove. Everything is mathematically well defined and beautiful, and leads, unavoidably, to the

12 The author thanks Professor T Damour for explaining Choptuik’s discovery to him.
13 Note that this proposal is different from the one contained for example in [6] (and references therein). DHN’s approach is based on the assumption (for which there is indeed evidence) that supergravity theory in 11 dimensions (or a quantum extension of it) exhibits a hidden $E_{10}$ symmetry at the level of the Lagrangian, i.e. there should exist a formulation of this theory invariant under the infinite-dimensional Lie group $E_{10}\backslash K(E_{10})$, where $K(E_{10})$ is the formal maximal compact subgroup of the $E_{10}$ Lie group. In this setting, the discrete group one looks for to build the quantum theory is a discretized version of $E_{10}$, $E_{10}(\mathbb{Z})$. Our proposal is different: we suggest looking at discrete groups like $\text{GL}(n, \mathbb{Z})$ and subgroups, not to some $\text{HA}_{1}^{1}(\mathbb{Z})$. Building automorphic forms for $\text{GL}(n, \mathbb{Z})$ groups (and subgroups) is a well-established technique, while automorphic forms for $E_{10}(\mathbb{Z})$ or $\text{HA}_{1}^{1}(\mathbb{Z})$, at the moment, are out of reach. Anyway, all these considerations can have nothing to do with true physics.
classical Langlands program. Very recently, Witten et al have given a physical interpretation of the geometric Langlands program in terms of gauge theory. This construction uses a lot Hecke eigensheaves, whose classical counterpart is Hecke operators and Hecke eigenfunctions. It is remarkable, in my opinion, that these objects appear in general relativity in a different context and with a different language. It could be another subtle indication of the so-called gauge-gravity correspondence, realized via primitive (arithmetic) objects.

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Appendix A. The hyperbolic Kac–Moody algebra $HA_1^{(1)}$

In this section we do not review the theory of Kac–Moody algebras (see [14] for details), but just fix the notation.

The algebra $HA_1^{(1)}$ is an infinite-dimensional Lie algebra, being a hyperbolic Kac–Moody algebra. This means that it is identified by the following (generalized) Cartan matrix:

$$A = a_{ij} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

or equivalently by its Dynkin diagram.

The precise definition of the algebra is standard and goes as follows. Let $\mathfrak{h}$ be a complex vector space whose dimension is 3, and $\mathfrak{h}^*$ its dual. Then there exist linearly independent indexed sets $\Pi := \{\alpha_i\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{h_i\} \subset \mathfrak{h}$, such that $\alpha_j(h_i) = a_{ij}$ ($i, j = 1, 2, 3$). The $\alpha_i$ ($h_i$) are called simple roots (dual simple roots). The sets $\Pi$ and $\Pi^\vee$ are uniquely determined by $A$ up to isomorphism.

Then the Kac–Moody algebra $HA_1^{(1)}$ is the complex Lie algebra generated by $\mathfrak{h} \cup \{e_i, f_i\}$ with the following defining relations:

$$[e_i, f_j] = \delta_{ij}h_i, \quad [h, h'] = 0 \quad \text{for} \quad h, h' \in \mathfrak{h}$$

$$[h_i, e_j] = a_{ij}e_i, \quad [h_i, f_j] = -a_{ij}f_i$$

$$(\text{ad } e_i)^{a_{ij}}e_j = 0, \quad (\text{ad } f_j)^{a_{ij}}f_j = 0 \quad \text{for} \quad i \neq j.$$
rank 3, being the canonical hyperbolic extension of $A_1$ (su(2)) through the affine node $\alpha_2$ and the hyperbolic node $\alpha_3$. The scalar products between the simple roots $(\alpha_i, \alpha_j) = a_{ij}$ are

$$
(\alpha_1, \alpha_2) = -2, \quad (\alpha_1, \alpha_3) = 0, \quad (\alpha_2, \alpha_3) = -1.
$$

(A.4)

$A$ is a flat metric on a certain pseudo-Riemannian vector space $h_\mathbb{R}^*$ (see below). Given the simple roots, one has the notion of real roots, which are $W$-equivalent to the simple roots, i.e. they are the image through the Weyl group of the simple roots, and imaginary roots. It turns out that the real roots $\alpha$ have positive-squared norm, while for imaginary roots $\alpha^2 \leq 0$. Indeed, for hyperbolic (symmetrizable) Kac–Moody algebras, the imaginary roots are precisely all the vectors in the root lattice $\bigoplus_i \mathbb{Z} \alpha_i$ which have zero or negative-squared norm (Moody theorem).

The Weyl group $W$ is defined by the fundamental reflections in the simple roots

$$
R_i(\beta) = \beta - 2\frac{(\alpha_i, \beta)}{(\alpha_i, \alpha_i)}\alpha_i.
$$

(A.5)

It is a Coxeter group, i.e. it is a discrete group generated by the fundamental reflections with the following relations:

$$
R_i^2 = 1, \quad (R_i R_j)^{m_{ij}} = 1,
$$

(A.6)

where the Coxeter exponents $m_{ij}$ are positive integers or $\infty$ (in this case one puts $x_\infty = 1$). For Kac–Moody algebras, one has $m_{ij} = 2, 3, 4, 6$ or $\infty$ according to $a_{ij} a_{ji} = 0, 1, 2, 3$ or $\geq 4$. Thus, different Kac–Moody algebras with the same Coxeter exponents has isomorphic Weyl groups\(^{14}\). For $\text{HA}_1^{(1)}$

$$
R_i^2 = (R_1 R_3)^2 = (R_2 R_3)^3 = 1
$$

(A.8)

and the Weyl group can be easily identified with $\text{PGL}(2, \mathbb{Z})$ putting $T = R_2 R_1$ and $S = R_1 R_3$, where $T$, $S$ are the standard generators for $\text{PSL}(2, \mathbb{Z})$. The even (positive) Weyl group $W^+$ is thus isomorphic to $\text{PSL}(2, \mathbb{Z})$. The properties of $\text{PSL}(2, \mathbb{Z})$, the modular group, are recalled in the following section. Every Coxeter group defines a Coxeter polytope, which is the polyhedron whose faces are pointwise fixed by the fundamental reflections. When the Coxeter group is the Weyl group of a Kac–Moody algebra, this polyhedron is called the Weyl chamber\(^{15}\) of the algebra and is contained in $h_\mathbb{R}^*$. Thus the faces of the Weyl chamber are orthogonal (with respect to the metric given by the Cartan matrix) to the simple roots and are hyperplanes in the pseudo-Riemannian space $h_\mathbb{R}^*$.

Appendix B. Maass automorphic forms for $\text{PSL}(2, \mathbb{Z})$

Let us recall the main properties of the modular group $\text{PSL}(2, \mathbb{Z})$. First we consider $\text{PGL}(2, \mathbb{Z})$, sometimes called the extended modular group, whose fundamental domain we call $\mathcal{F}_3$

\(^{14}\) For example, the two rank-3 algebras

$$
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -4 \\
0 & -1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -4 & 2
\end{pmatrix}
$$

(A.7)

have the same Weyl group as $\text{HA}_1^{(1)}$.

\(^{15}\) To be honest, the Weyl chamber is the domain in $\mathfrak{h}$ defined by

$$
|h|_{\mathfrak{h}}(h) \geq 0,
$$

(A.9)

where $\mathfrak{h} = \{ h \in \mathfrak{h} | a_i(h) \in \mathbb{R} \}$ is a $W$-stable real subspace of $\mathfrak{h}$. In this notation, what we call Weyl chamber should more correctly called the dual Weyl chamber, since we are interested in the polyhedron in $h_\mathbb{R}^*$.\(^{16}\)
Figure 4. The modular domain (the domain for PGL\(^{(2, \mathbb{Z})}\)) is the desymmetrized one) and the modular surface (from [20]).

(remember that one of the angles is \(\pi/3\)). It is a discrete group acting on the hyperbolic plane and is generated by the hyperbolic reflections in the three sides

\[
R_1(z) = -\bar{z}, \quad R_2(z) = -\bar{z} + 1, \quad R_3(z) = \frac{1}{z}.
\]  

(B.1)

PSL\((2, \mathbb{Z})\) is its most important subgroup, its domain \(\mathcal{D}\) (see figure 4) is twice the fundamental domain of PGL\((2, \mathbb{Z})\)

\[
\mathcal{D} = \{z \in \mathbb{H} : |z| > 1, |x| \leq \frac{1}{2}\}
\]  

(B.2)

and its hyperbolic area is

\[
\mu(\mathcal{D}) = \frac{\pi}{3},
\]

(B.3)

where \(\mu\) is the usual hyperbolic measure on the hyperbolic plane, \(d\mu = dx \, dy/y^2\). The standard generators for PSL\((2, \mathbb{Z})\) are

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(B.4)

considered as fractional linear transformations, i.e.

\[
T(z) = z + 1, \quad S(z) = -\frac{1}{z}.
\]

(B.5)

\(S\) is such that \(S^2 = 1 (S = S^{-1})\); moreover \(\text{Tr } S = 0 < 2\), \(\text{Tr } T = 2\) thus \(S\) is an elliptic element, \(T\) is a parabolic element (for details on hyperbolic geometry and Fuchsian groups see [15]).

\(T\) and \(S\) can be expressed of course in terms of the reflections \(R_i\) belonging to PGL\((2, \mathbb{Z})\)

\[
T = R_2 R_1 \neq R_1 R_2 = T^{-1}, \quad S = R_1 R_3 = R_3 R_1.
\]

(B.6)

Note that PGL\((2, \mathbb{Z})\) contains hyperbolic reflections, that is negative isometries with determinant \(-1\), while PSL\((2, \mathbb{Z})\) contains only positive isometries. Thus we can consider the hyperbolic surface \(X := \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H}\), called the modular surface (see figure 4), which is an oriented, finite area, non-compact two-dimensional Riemannian manifold. It is also a Riemann surface with the complex structure inherited by \(\mathbb{H}\). It has a cusp at infinity, which corresponds to the fixed point \(i\infty\) of the parabolic transformation \(T\). We are interested in the spectral problem for \(X\). See the excellent survey [20] for more details and references.

The so-called fundamental spectral problem of quantum chaos is the following:

\[
\begin{cases}
\Delta \psi + E \psi = 0, & E = \frac{1}{4} + t^2 > 0 \\
\psi(\gamma z) = \psi(z) & \forall \gamma \in \text{PSL}(2, \mathbb{Z}) \\
\int_X |\psi(z)|^2 \, d\mu(z) < +\infty.
\end{cases}
\]  

(B.7)
Here $\Delta = y^2 (\partial_x^2 + \partial_y^2)$ is the hyperbolic Laplacian. The numbers $0 = E_0 < E_1 \leq E_2 \leq \cdots$ for which the spectral problem has solutions from the discrete spectrum (eigenvalues) for $X$. The only known eigenvalue is $E_0 = 0$, to which it corresponds the constant eigenfunctions $\psi_0(z)$ (remember that $\mathcal{F}$ has finite area). Solutions to the previous spectral problem are called Maass automorphic forms (or Maass waveforms), after the mathematician H Maass, who first studied them. The term automorphic refers to the periodicity under $\text{PSL}(2, \mathbb{Z})$, i.e. $\psi(\gamma z) = \psi(z)$. It is not evident that solutions to this problem exist, since the modular surface is not compact. Moreover, no explicit eigenvalues or eigenfunctions are known or expected for $X$, although Maass himself produced an explicit subsequence of eigenvalues for other discrete groups (e.g. $\Gamma(4)$).

One would like to formulate for this case the analog of the Weyl law for the Dirichlet problem for the Euclidean Laplacian on a compact domain in $\mathbb{R}^2$ on the asymptotic number of eigenvalues. This is possibly thanks to Selberg’s theory, in particular his famous trace formula. Selberg found that there exists a continuous spectrum too, and the latter is the whole interval $[\frac{1}{4}, \infty)$ with multiplicity one. The corresponding not-normalizable eigenfunctions are given by the Eisenstein series, which, for $X$, read as follows:

$$E(z, s) = \sum_{g \in \Gamma_\infty \setminus \text{PSL}(2, \mathbb{Z})} \frac{\psi}{|cz + d|^s} \quad \text{for } \Re s > 1$$

(B.8)

where $\Gamma_\infty$ is the stabilizer of $\infty$, which is the only cusp, $z = x + iy$ and the transformation $g$ is represented as usual by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The Eisenstein series extend meromorphically to the whole $\mathbb{C}$-plane and are analytic on $\Re s = \frac{1}{2}$. The continuous spectrum is thus furnished by these generalized eigenfunctions $E(z, \frac{1}{2} + it), t \geq 0$,

$$\Delta E(z, \frac{1}{2} + it) + \left(\frac{1}{4} + t^2\right) E(z, \frac{1}{2} + it) = 0$$

(B.9)

and these are of course $\text{PSL}(2, \mathbb{Z})$-periodic

$$E(gz, s) = E(z, s) \quad \text{for any } g \in \text{PSL}(2, \mathbb{Z})$$

(B.10)

that is they are generalized Maass automorphic forms. Let us call $\overline{\psi}(s)$ the constant term in the Fourier expansion of the Eisenstein series: this is meromorphic in $\mathbb{C}$ and for $\Re s \geq 1/2$ its poles are in $\left\{ \frac{1}{2}, 1 \right\}$. The residues at these poles are solutions to the spectral problem and form the residual spectrum of $X$. If we now take the orthogonal complement in $L^2(X, \mu)$ of the continuous and residual spectrum, we obtain the cuspidal space $L^2_{\text{cusp}}(X)$. It is invariant under the hyperbolic Laplacian and the resolvent $(\Delta - \lambda)^{-1}$ is compact when restricted $L^2_{\text{cusp}}$. A Maass form which also lies in $L^2_{\text{cusp}}$ is called a Maass cusp form. These cusp forms are a kind of building blocks for the theory of automorphic forms. Their existence is of course tied to the size of $L^2_{\text{cusp}}$, since $L^2_{\text{cusp}} \neq \{0\}$ is not obvious.

However, it turns out that for $\text{PSL}(2, \mathbb{Z}), \overline{\psi}(s)$ has no poles in $\left(\frac{1}{2}, 1\right)$, thus the residual spectrum is empty and any Maass form is automatically a cusp form. Using his trace formula and the fact that for $\text{PSL}(2, \mathbb{Z}) \overline{\psi}(s)$ can be expressed in terms of a ratio involving the (completed) Riemann zeta-function, Selberg was able to show that the contribution of the continuous spectrum to the Weyl law is negligible, and that

$$N^\text{cusp}_{\text{PSL}(2, \mathbb{Z})}(R) := \sum_{0 < E_j \leq R} 1 \sim \frac{\mu(\mathcal{F})}{4\pi} R \quad (R \to \infty)$$

(B.11)

that is solutions to the spectral problem exist and in abundance (each eigenvalue is counted with its multiplicity). Let us stress that no eigenvalues/eigenfunctions are explicitly known.
The existence of Maass cusp forms for $\text{PSL}(2, \mathbb{Z})$ is deeply related to its arithmetic nature\(^{16}\). One can also consider the spectral problem for $\text{PGL}(2, \mathbb{Z})$; this latter contains negative isometries, thus one cannot form a hyperbolic manifold by taking the quotient, but one can consider the spectral problem with Neumann boundary conditions

\[
-\Delta \psi = E \psi \\
\psi \in L^2(F_3, \mu) \\
\partial_n \psi \mid_{\partial F_3} = 0.
\]

Solutions to this problem are given by the even Maass cusp forms for $\text{PSL}(2, \mathbb{Z})$.

In full generality, let $X(N)$ be the surfaces built out of the congruence subgroups $\Gamma(N)$. The spectral problem applies exactly as above. Regarding the low-energy spectrum of $X(N)$, let us call $E_1(N)$ the closest eigenvalue to $E_0 = 0$. A deep conjecture (still open) of Selberg states that

\[
E_1(N) \geq \frac{1}{4}.
\]

For $N = 1$ ($X \equiv X(1)$), the case of the modular group, the numerical calculations show that the first eigenvalue is $E_1 = 91.12 \ldots$ Since there is no residual spectrum\(^{17}\), we can take for $E_1$ the smallest eigenvalue of a Maass cusp form on $X$. One can understand the number $1/4$ by recalling that a result due to McKean shows that the spectrum of $\Delta$ on the universal covering $L^2(H)$ is $[\frac{1}{4}, \infty)$. Moreover, P Cartier conjectured that the cuspidal spectrum of $\Delta$ on $X$ is simple (the situation may indeed be very different for the other surfaces $X(N)$), a result confirmed by many numerical experiments. For the high-energy behavior of the spectrum, one would expect, using standard arguments from random matrix theory, that it would fit the GOE-ensemble predictions (the system is time invariant), as the geodesic flow on $X$ is an Anosov flow (thus chaotic). Yet, all the numerical experiments show that the high-energy spectrum follows a Poissonian distribution (which is instead typical of systems whose semi-classical limit is integrable). The reason for that is, again, the arithmetic nature of $\text{PSL}(2, \mathbb{Z})$, in particular the existence of additional symmetries, the so-called Hecke operators, which for $n > 0$ read as follows:

\[
T_n \psi(z) := \frac{1}{\sqrt{n}} \sum_{ad=n, b \mod n} \psi \left( \frac{az + b}{d} \right)
\]

the sum going over all positive integers $a$, $d$ with $ad = n$ and $0 \leq b < d$. These operators are a commutative family\(^{18}\) of self-adjoint operators on $L^2(X)$ and commute with $\Delta$ and the reflection $R_1$. Thus they preserve the even/odd eigenspaces of $\Delta$ and each eigenspace has a basis consisting of simultaneous eigenfunctions of all Hecke operators. Such eigenfunctions are called Maass–Hecke eigenforms. Given such an eigenfunction $\psi$, $T_n \psi = \lambda_n \psi$, its Fourier coefficients are given by

\[
a_\psi(n) = a_\psi(1) \lambda_n
\]

and we can normalize the first Fourier coefficient $a_\psi(1) = 1$; in this way the $n$th Fourier coefficient is the Hecke eigenvalue $\lambda_n$. Hecke eigenvalues enjoy a lot of nice properties, the

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\(^{16}\) Phillips and Sarnak have more generally shown that the arithmetic groups are the only ones which allow for solutions. Many numerical experiments support the absence of eigenvalues and Maass forms for non-arithmetic cases. For the precise definition of an arithmetic Fuchsian group see [15].

\(^{17}\) Besides $E = 0$, which thus should not be considered in the discrete spectrum.

\(^{18}\) Mathematicians have no doubts that the anomalous statistics for arithmetical dynamical systems is due to these Hecke operators, which at the quantum level commute with the Laplacian $\Delta$ and somehow mimic an integrable system. A rigorous proof of that is lacking.
most important is that they are multiplicative

\[ \lambda_{mn} = \lambda_m \lambda_n \quad m, n \text{ co-prime} \]  \hspace{1cm} (B.16)

\[ \lambda_p \cdot \lambda_p = \lambda_{p^2} + \lambda_{p^2 - 1} \quad p \text{ prime}. \]  \hspace{1cm} (B.17)

Let us conclude this section by recalling a deep and recent mathematical result about the quantum unique ergodicity for arithmetic hyperbolic manifolds. Let us first observe that, from the classical point of view, arithmetical dynamical systems are chaotic as any other model on compact negatively curved manifolds. But at the quantum level the arithmetic property is peculiar. If one defines the probability measures \( \mu_j \)

\[ d\mu_j = |\psi_j|^2 \, d\text{vol}, \]  \hspace{1cm} (B.18)

where \( \psi_j \) is an eigenfunction of the Laplacian and \( d\text{vol} \) is the Riemannian volume element, then the quantum unique ergodicity conjecture by Rudnick and Sarnak states that for compact manifolds of negative curvature, the measures \( \mu_j \) converge to \( d\text{vol} \) (in the weak * topology). The conjecture by Rudnick and Sarnak, if true, is remarkable, because it asserts that at quantum level and in the semi-classical limit, there is no manifestation of chaos. In particular, one would have quantum unique ergodicity, that is only one possible quantum limit, whereas classical unique ergodicity, i.e. uniqueness of the invariant measure for the Hamiltonian flow, is never satisfied for chaotic systems.

This conjecture is motivated by a theorem due to Shnirelman, which says that for an ergodic system there exists a subsequence \( j_k \) for which the measures \( \mu_{j_k} \) converge to the standard normalized Lebesgue measure. Shnirelman’s theorem is a statement of equidistribution, that is one often says that the eigenfunctions equidistribute because the probability densities \( |\psi_{j_k}| \) tend to a constant independent on any point on the manifold. Thus, for non-exceptional \( E_n, |\psi_n|^2 \) can never localize to just a finite number of closed geodesics. This theorem has been improved by Colin de Verdiere and Zelditch, who showed that eigenfunctions still equidistribute on non-compact manifolds like \( \text{PSL}(2, \mathbb{Z}) \backslash \mathbb{H} \). The presence of an exceptional set is of course troubling. In stadium-like domains, the so-called scarring effect has been observed. For numerous \( n \), the topography of \( \psi_n \) is found to contain clear ridges of mass of scars, situated roughly along what would appear of closed geodesics. The location of these scars changes with \( n \). In practice, scars are what is left of periodic orbits [12].

But for negatively curved manifolds, the QUE conjecture by Rudnick and Sarnak denies the existence of these scars. The proof of this conjecture is still lacking, yet progress has been made for arithmetic hyperbolic surfaces thanks to E. Lindenstrauss. He has shown that for a compact arithmetic quotient the quantum unique ergodicity conjecture is true and that, moreover, the same statement holds for the modular surface (which is non-compact). Thus, in the latter case, there is no scarring, i.e. there is no manifestation of classical chaos at the quantum level.

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