Canonical description of ideal magnetohydrodynamics and integrals of motion.

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Abstract

In the framework of the variational principle there are introduced canonical variables describing magnetohydrodynamic (MHD) flows of general type without any restrictions for invariants of the motion. It is shown that the velocity representation of the Clebsch type introduced by means of the variational principle with constraints is equivalent to the representation following from the generalization of the Weber transformation for the case of arbitrary MHD flows. The integrals of motion and local invariants for MHD are under examination. It is proved that there exists generalization of the Ertel invariant. It is expressed in terms of generalized vorticity field (discussed earlier by Vladimirov and Moffatt (V. A. Vladimirov, H. K. Moffatt, J. Fl. Mech., 283, pp. 125–139, 1995) for the incompressible case). The generalized vorticity presents the frozen-in field for the barotropic and isentropic flows and therefore for these flows there exists generalized helicity invariant. This result generalizes one obtained by Vladimirov and Moffatt in the cited work for the incompressible fluid. It is shown that to each invariant of the conventional hydrodynamics corresponds MHD invariant and therefore our approach allows correct limit transition to the conventional hydrodynamic case. The additional advantage of the approach proposed enables one to deal with discontinuous flows, including all types of possible breaks.

1 Introduction.

It is well-known that description of the solid media flows in terms of the canonical (hamiltonian) variables is very useful and effective, see for instance [1, 2]. In terms of the hamiltonian variables it is possible to deal with all nonlinear processes in unified terms not depending on the specific problem related to the media under investigation. For instance, all variants of the perturbation theory are expressed in terms of different order nonlinear vertices, which along with the linear dispersion relation contain the specific information relating to the concrete system under investigation, cf. Refs. [3, 4]. In the problems of the nonlinear stability investigations the conventional Hamiltonian approach based upon the corresponding variational principle allows...
one to use the Hamiltonian along with other integrals of motion (momentum, number of quasiparticles, topological invariants) in order to construct the relevant Lyapunov functional, cf. Refs. 9, 8, 7, 6, 5. Therefore, it makes important the problem of introducing the canonical variables and corresponding variational principle for the general type MHD flows (i.e., non-barotropic and including all types of the breaks possible for MHD) and obtaining the complete set of the local invariants, see definition and discussions in original papers 10, 11, 12, 13 and in the recent review 1. As for the first item, the example of the variational principle describing all possible breaks is presented in the recent work 14.

Here in the framework of some modification of the variational principle of the cited work we examine the problem of the MHD invariants. Note that the set of invariants for MHD discussed in the literature till now is incomplete. It becomes evident if one takes into account that for the vanishing magnetic field this set has to go over to the set of the conventional hydrodynamic invariants. But this limit transition does not reproduce Ertel, vorticity and helicity invariants existing for the hydrodynamic flows. For the particular case of incompressible MHD flows generalized vorticity and helicity invariants were obtained in the paper 8. Below we show that the generalized vorticity and helicity invariants exist also for compressible barotropic flows, and derive MHD generalization for the Ertel invariant.

The plan of the paper is as follows. In section 2 we briefly discuss appropriate variational principle, introducing the Clebsch type velocity representation by means of constraints and defining the canonical variables. In the following section 3 we develop generalization of the Weber transformation and show that it leads to the velocity representation, which is equivalent to the one following from the variational principle under discussion. In section 4 we examine MHD integrals of motion, introducing ‘missing’ MHD invariants, and discuss their transformation properties relating to change of gauge. In section 5 we make some conclusions and formulate problems to be solved later.

2 Variational principle and canonical variables.

Let us briefly describe the variational principle and subsidiary variables describing dissipation-free MHD. Starting with the standard Lagrangian density

\[ \mathcal{L} = \rho \frac{v^2}{2} - \rho \varepsilon (\rho, s) - \frac{H^2}{8 \pi} , \]

where \( \rho, s \) and \( \varepsilon (\rho, s) \) present the fluid density, entropy and internal energy, respectively, \( H \) denotes magnetic field, we have to include to the action \( A \) the constraint terms. Then the action can be presented in the form

\[ A = \int dt L', \quad L' = \int dx L', \quad L' = L + L_c , \]

where \( L_c \) is the part of the Lagrangian density respective for constraints,

\[ L_c = \rho D \varphi + \lambda D \mu + \sigma D s + M \cdot \left( \frac{\partial A}{\partial t} - [v, \text{curl} A] + \nabla \Lambda \right) + \frac{H \text{curl} A}{4 \pi} . \]
Here $D = \partial_t + (\mathbf{v} \nabla)$ is substantial (material) derivative and $\mathbf{A}$ is the vector potential.\[^1\] Including the last two terms into $L_c$ allows us to introduce relation $\mathbf{H} = \text{curl}\mathbf{A}$ strictly into the variational principle (it follows after variation with respect to $\mathbf{H}$).

Supposing first that all variables introduced (including velocity) are independent we obtain the set of variational equations of the form

\[
\delta \varphi \implies \partial_t \rho + \text{div}(\rho \mathbf{v}) = 0, \quad (4)
\]

\[
\delta \rho \implies D \varphi = w - v^2/2, \quad (5)
\]

\[
\delta \mathbf{A} \implies D \mathbf{\mu} = 0, \quad (6)
\]

\[
\delta \mu_m \implies \partial_t \lambda_m + \text{div}(\lambda_m \mathbf{v}) = 0, \quad (7)
\]

\[
\delta \sigma \implies D s = 0, \quad (8)
\]

\[
\delta \mathbf{M} \implies \partial_t \mathbf{A} = [\mathbf{v}, \text{curl}\mathbf{A}] - \nabla \Lambda, \quad (10)
\]

\[
\delta \mathbf{A} \implies \partial_t \mathbf{M} = \frac{\text{curl}\mathbf{H}}{4\pi} + \text{curl}[\mathbf{v}, \mathbf{M}]. \quad (11)
\]

\[
\delta \mathbf{H} \implies \mathbf{H} = \text{curl}\mathbf{A}, \quad (12)
\]

\[
\delta \Lambda \implies \text{div}\mathbf{M} = 0, \quad (13)
\]

where $w$ and $T$ are the enthalpy density and temperature.

Note that in this section we suppose the velocity field to be independent on other variables. Therefore, variation with respect to $\mathbf{v}$ gives us the velocity representation:

\[
\delta \mathbf{v} \implies \rho \mathbf{v} = -\rho \nabla \varphi - \lambda_m \nabla \mu_m - \sigma \nabla s - [\mathbf{H}, \mathbf{M}]. \quad (14)
\]

It is convenient to rewrite it in a shortened form that emphasizes it’s structure. Bearing in mind that the velocity potential $\varphi$, vector Lagrange markers $\mathbf{\mu}$, entropy $s$ and the vector potential $\mathbf{A}$ can be treated as generalized coordinates, one can see that $\rho$, $\lambda$, $\sigma$ and subsidiary field $\mathbf{M}$ are conjugated momenta, respectively. Let

\[
\mathbf{Q} = (\varphi, \mathbf{A}), \quad \mathbf{Q} = (\varphi, \mathbf{\mu}, s), \quad \mathbf{P} = \delta \mathbf{A}/\delta \partial_t \mathbf{Q}, \quad \mathbf{P} = (P, \mathbf{M}). \quad (15)
\]

\[^1\] This form of the action slightly differs from the one proposed in Ref. [14]. The main difference consists in introducing the vector potential for the magnetic field. Therefore, here the canonical pair is $\mathbf{A}, \mathbf{M}$ instead of $\mathbf{H}, \mathbf{S}$, where $\mathbf{S} = \text{curl}\mathbf{M}$. We do not deal here with the discontinuous flows and thus we omit the surface term in the action. But adding corresponding surface term we can easily take the breaks into account.
Then the velocity representation takes the transparent form

\[ \mathbf{v} = \mathbf{v}_0(\mathcal{P}, \nabla \mathcal{Q}), \quad \mathbf{v}_0 = \mathbf{v}_h + \mathbf{v}_M, \quad \mathbf{v}_h = -\frac{\mathcal{P}}{\rho} \nabla \mathcal{Q}, \quad \mathbf{v}_M = -\frac{1}{\rho} [\mathbf{H}, \mathbf{M}]. \]  

(16)

Here subindexes \( h \) and \( M \) correspond to the "hydrodynamic" and "magnetic" parts of the velocity field. The hydrodynamic part, \( \mathbf{v}_h \), corresponds to the generalized Clebsch representation, compare papers [13, 14, 15], and magnetic part, \( \mathbf{v}_M \), coincides with the conventional term if we replace the divergence-free field \( \mathbf{M} \) by \( \text{curl}\mathbf{S} \). The latter was first introduced by Zakharov and Kuznetsov, cf. Ref. [17].

From the velocity representation Eq. (16) and the equations of motion (4) – (11) it strictly follows that the velocity field \( \mathbf{v} = \mathbf{v}_0 \) satisfies Euler equation with the magnetic force taken into account. Namely, providing differentiation we obtain

\[ \rho D \mathbf{v}_0 = -\nabla p + \frac{1}{4\pi} [\text{curl}\mathbf{H}, \mathbf{H}], \]  

(17)

where \( p \) is the fluid pressure.

### 2.1 Canonical variables.

The variational principle can be easily reformulated in the Hamiltonian form. Excluding the magnetic and velocity fields by means of Eqs. (12), (16) we arrive to the following Hamiltonian density

\[ \mathcal{H} = \mathcal{H}(\mathcal{P}, \nabla \mathcal{Q}) = \mathcal{P} \partial_t \mathcal{Q} - \mathcal{L}' = \rho \frac{\mathbf{v}_0^2}{2} + \rho \varepsilon(\rho, s) + \frac{(\text{rot}\mathbf{A})^2}{8\pi} - (\mathbf{M}, \nabla \Lambda). \]  

(18)

Equations of motion (4) – (11) can be expressed now in the canonical form

\[ \partial_t \mathcal{Q} = \delta \mathcal{H}_V / \delta \mathcal{P}, \quad \partial_t \mathcal{P} = -\delta \mathcal{H}_V / \delta \mathcal{Q}, \quad \mathcal{Q} = (\varphi, \mu, s; \mathbf{A}), \quad \mathcal{P} = (\rho, \lambda, \sigma; \mathbf{M}); \]  

(19)

Eq. (12) serves as a definition of magnetic field, and divergence-free condition for the subsidiary field \( \mathbf{M} \), Eq. (13), follows from variation of the action

\[ \mathcal{A} = \int dt \int d\mathbf{r} (\mathcal{P} \partial_t \mathcal{Q} - \mathcal{H}) \]  

(20)

with respect to \( \Lambda \). Note that it is possible to put \( \Lambda = 0 \). Under this assumption the divergence-free condition for the field \( \mathbf{M} \) vanishes, but from Eq. (14) follows that \( \text{div}\mathbf{M} \) is conserved quantity, and supposing that \( \text{div}\mathbf{M} = 0 \) holds for some initial moment we arrive to the conclusion that this is valid for arbitrary moment. Nevertheless, it proves convenient to deal with \( \Lambda \neq 0 \) that makes it possible to use different gauge conditions for the vector potential.

The variational principle presented gives us the set of dynamic equations from which follow conventional MHD equations, (4), (9), (17) and equation for the frozen-in magnetic field, which follows from Eq. (14) after taking curl operation,

\[ \partial_t \mathbf{H} = \text{curl}[\mathbf{v}, \mathbf{H}]. \]  

(21)
On the contrary, if at some initial moment, \( t = \bar{t} \), we have the conventional MHD fields \( \bar{\rho}, \bar{s}, \bar{v} \) and \( \bar{H} \), then we can find the initial subsidiary fields \( \bar{\varphi}, \bar{\mu}, \bar{\lambda}, \bar{\sigma}, \bar{A}, \bar{M} \) and \( \bar{\Lambda} \), satisfying Eqs. (12) – (14). This can be done up to the gauge transformations (do not changing the velocity and magnetic field) due to the fact that the subsidiary fields play a role of generalized potentials. Then, if the uniqueness conditions hold both for the conventional MHD equations and for the set of variational equations, we arrive to the conclusion that corresponding solutions coincide for all moments. In this sense we can say that these sets of equations are equivalent, cf. Ref. [4].

The complete representation of the velocity field in the form of the generalized Clebsch representation, Eq. (16) allows, first, to deal with the MHD flows of general type, including all types of breaks, cf. Ref. [14]; second, for the zero magnetic field it gives correct limit transition to the conventional hydrodynamics, cf. Refs. [16], [15]; third, it allows to obtain the additional to the known ones integrals and invariants of motion for the MHD flows: for instance, generalized Ertel invariant, generalized vorticity and generalized helicity, see below. The two last integrals were deduced for the particular case of incompressible flows in the paper [8].

Moreover, it is possible to show that representation (16) is equivalent to the one following from the Weber transformation, cf. Refs. [18, 19] and the recent review [1]. The generalization of the Weber transformation for the ideal MHD incompressible flows was obtained by Vladimirov and Moffatt, cf. Ref. [8].

3 Generalized Weber transformation.

Suppose here that the fluid particles are labelled by Lagrange markers \( a = (a_1, a_2, a_3) \). The label of the particle passing through point \( r = (x_1, x_2, x_3) \) at time \( t \) is then

\[
a = a(r, t),
\]

and

\[
D_a = \frac{\partial a}{\partial t} + (v \cdot \nabla)a = 0.
\]

The particle paths and velocities are given by the inverse function

\[
r = r(a, t), \quad v = Dr(a, t) = \left. \frac{\partial r}{\partial t} \right|_{a=\text{const}}.
\]

Let the initial position of the particle labelled \( a \) is \( X \), i.e.,

\[
r(a, 0) = X(a).
\]

A natural choice of label would be \( X(a) = a \); however it is convenient to retain the extra freedom represented by the “rearrangement function” \( X(a) \).

We now seek to transform the equation of motion (17) to integrable form, by generalization of the argument of Weber [18] (see, for example, Refs. [20], [1], and [8]. It is convenient to represent here the equation of motion in the following form

\[
Dv = -\nabla w + T \nabla s + [J, h],
\]

3. Generalized Weber transformation.
where \( \mathbf{h} = \mathbf{H}/\rho \) and the vector \( \mathbf{J} \) is defined according to

\[
\mathbf{J} = \frac{\text{curl}\mathbf{H}}{4\pi},
\]

being proportional to the current density. Multiplying Eq. (26) by \( \partial x_k/\partial a_i \) we have

\[
(Dv_k)\frac{\partial x_k}{\partial a_i} = -\frac{\partial w}{\partial x_k} \frac{\partial x_k}{\partial a_i} + T \frac{\partial s}{\partial x_k} \frac{\partial x_k}{\partial a_i} + [\mathbf{J}, \mathbf{h}]_k \frac{\partial x_k}{\partial a_i}.
\]

The l.h.s. can be represented in the form

\[
(Dv_k)\frac{\partial x_k}{\partial a_i} = D\left( v_k \frac{\partial x_k}{\partial a_i} \right) - \frac{\partial}{\partial a_i} \left( \frac{v^2}{2} \right),
\]

where we have taken into account that operator \( D \equiv \partial/\partial t |_{a=\text{const}} \) and therefore \( Dx_k = v_k \) and \( D \) commute with derivative \( \partial/\partial a_i \). Eq. (28) takes now the form

\[
D\left( v_k \frac{\partial x_k}{\partial a_i} \right) = \frac{\partial}{\partial a_i} \left( \frac{v^2}{2} - w \right) + T \frac{\partial s}{\partial a_i} + [\mathbf{J}, \mathbf{h}]_k \frac{\partial x_k}{\partial a_i}.
\]

It is convenient to transform the last term by means of the dynamical equation for the subsidiary field \( \mathbf{m} = \mathbf{M}/\rho \) (compare Eq. (11))

\[
D\mathbf{m} = (\mathbf{m}, \nabla)\mathbf{v} + \mathbf{J}/\rho.
\]

Then we can transform the last term in the r.h.s. of Eq. (30) to the form of substantial derivative, see Appendix

\[
[J, h]_k \frac{\partial x_k}{\partial a_i} = D\left( [\mathbf{m}, \mathbf{H}]_k \frac{\partial x_k}{\partial a_i} \right).
\]

Analogously, the first two terms in the r.h.s. of Eq. (30) can be presented as substantial derivatives by means of introducing subsidiary functions \( \varphi \) and \( \sigma \), which satisfy equations (compare Eqs. (9), (5))

\[
D\left( \frac{\sigma}{\rho} \right) = -T,
\]

\[
D\varphi = w - \frac{v^2}{2}.
\]

Then

\[
T \frac{\partial s}{\partial a_i} = -\frac{\partial s}{\partial a_i} D\left( \frac{\sigma}{\rho} \right) = -D\left( \frac{\partial s}{\partial a_i} \frac{\sigma}{\rho} \right), \quad \frac{\partial}{\partial a_i} (v^2/2 - w) = -D\left( \frac{\partial \varphi}{\partial a_i} \right),
\]

where we have taken into account that \( Ds = 0 \) along with \( D(\partial s/\partial a_i) = 0 \). Therefore, we can present the Euler equation (30) in the integrable form

\[
D\left( v_k \frac{\partial x_k}{\partial a_i} \right) = -D\left( \frac{\partial \varphi}{\partial a_i} \right) - D\left( \frac{\partial s}{\partial a_i} \frac{\sigma}{\rho} \right) + D\left( [\mathbf{m}, \mathbf{H}]_k \frac{\partial x_k}{\partial a_i} \right).
\]
Integration leads to the relation

$$v_k \frac{\partial x_k}{\partial a_i} = - \frac{\partial \varphi}{\partial a_i} - \frac{\partial s}{\partial a_i} \sigma - [H, m]_k \frac{\partial x_k}{\partial a_i} + b_i,$$

(37)

Here \( b = b(a) \) does not depend on time explicitly, \( D b = 0 \), presenting the vector constant of integration. Multiplying this relation by \( \frac{\partial a_i}{\partial x_j} \) allows reverting from Lagrangian variables \((a, t)\), to the Eulerian ones, \((r, t)\),

$$v = -\nabla \varphi + b_k \nabla a_k - \frac{\sigma}{\rho} \nabla s - [h, M].$$

(38)

This representation obviously coincides with the Clebsch representation obtained above from the variational principle with constraints if one identifies \( b \) with \(-\lambda/\rho\) and \( a \) with \( \mu \). Moreover, this proves equivalence of description of the general type magnetohydrodynamic flows in terms of canonical variables introduced and the conventional description in Lagrange or Euler variables. The equations of motion for the generalized coordinates and momenta follow now from definitions of the subsidiary variables \( a, m = M/\rho, \sigma, \varphi \) and \( b \).

Emphasize here that the vector field \( M = \rho m \) introduced by Eq. (31) satisfies integral relation

$$\frac{\partial}{\partial t} \int_{\Sigma} (M, d\Sigma) = \int_{\Sigma} (J, d\Sigma),$$

(39)

where \( \Sigma \) presents some oriented area moving with the fluid. The proof of this statement see in Appendix. Expressing \( M = \text{curl} S \) and making use of the Stocks theorem we arrive to a conclusion that time derivative of the vector \( S \) circulation over the closed frozen-in contour \( \partial \Sigma \) is proportional to the current (remind, \( J = (4\pi)^{-1}\text{curl} H \) and differs from the current density by constant multiplier) intersecting the surface defined by this contour,

$$\frac{\partial}{\partial \Sigma} \int_{\partial \Sigma} (S, dl) = \int_{\Sigma} (J, d\Sigma) = (4\pi)^{-1} \int_{\partial \Sigma} (H, dl)$$

(40)

that highlights the physical meaning of the subsidiary field \( S \) usually introduced for the canonical description of MHD flows. This fact was first indicated in Ref. [8] for the incompressible flows. Now we see that it holds true for the general case.

The vector constant of integration, \( b \), may be expressed in terms of the initial conditions,

$$b_i = \nabla_i (a) \frac{\partial X_k}{\partial a_i} + \frac{\partial \varphi_0}{\partial a_i} + c_0 \frac{\partial s}{\partial a_i},$$

$$\varphi_0 = \varphi(a, 0), \quad c_0 = \left( \frac{\sigma}{\rho} \right) \bigg|_{t=0}, \quad \nabla_i (a) = V_k (a) + [h_0, M_0]_k, \quad V_k (a) = v_k (a, 0),$$

$$h_0 \equiv h_0(a) = h(x(a, 0), 0) = h(X(a), 0), \quad M_0 \equiv M_0(a) = M(x(a, 0), 0) = M(X(a), 0).$$

(41)

Under special conditions, namely, for

$$X(a) = a, \quad r(a, 0) = a, \quad a(r, 0) = r,$$

(42)

from Eq. (41) follows

$$b_i = \nabla_i (a) + \frac{\partial \varphi_0}{\partial a_i} + c_0 \frac{\partial s}{\partial a_i}.$$

(43)
Adopting zero initial conditions,
\[ M_0 = 0, \quad \varphi_0 = 0, \quad \sigma_0 = 0, \] (44)
we obtain
\[ b = \nabla(a) = \tilde{v}(a, 0) \equiv \tilde{v}_0(a) = v(a, 0), \] (45)
where tilde indicates that we are dealing with the velocity field in the Lagrange description, i.e., \( \tilde{v}(a, t) \) denotes velocity of the fluid particle with label \( a \) at time \( t \). Evidently, \( \tilde{v}(a, t) = v(r, t) \), where \( a \) and \( r \) are linked by relations (22) and (24) for the specific choice given by Eqs. (42), (44). Then the velocity representation takes the particular form
\[ v = v_h - [h, M], \quad v_h \equiv -\nabla \varphi + \tilde{v}_0 \nabla a_k - \frac{\sigma}{\rho} \nabla s, \quad \tilde{v}_0(r) = v(r, 0), \quad a(r, 0) = r. \] (46)
It differs from the one presented in Ref. [4] by involving the entropy term. Emphasize here that existence of this term allows to describe general type MHD and hydrodynamic flows with arbitrary possible discontinuities, including shocks, slides and rotational breaks, cf. Ref. [15, 16, 14]. One can omit this term for continuous barotropic and isentropic flows.

4 Integrals of motion.

The conservation laws, as it is well-known, follow from the specific symmetries of the action. Existence of the relabelling transformations group (first discussed by Salmon in Ref. [21]) of the Lagrange markers, \( \mu \), leads to the additional to the energy, the fluid momentum and mass integrals of motion. These additional integrals are expressed in terms of the Lagrange description of the motion, i.e., in terms of the Lagrange markers, etc. Therefore, as a rule, they are gauge dependent. The frozen-in character of the magnetic field leads to the specific topological integrals of motion, namely, magnetic helicity and, cross-helicity, first discussed by Moffatt in Ref. [22], see also review [1]. Corresponding densities are respectively
\[ h_M = (A, H), \] (47)
and
\[ h_C = (v, H). \] (48)
As it strictly follows from the dynamic equations, the local conservation law for the magnetic helicity holds true for general type MHD flows
\[ \partial_t h_M + div q_M = 0, \quad q_M = vh_M - H \cdot ((A, v) + \Lambda). \] (49)
On the contrary, the cross-helicity in general case is governed by equation
\[ \partial_t h_C / \partial t = -div \left[ vh_C + (w - v^2/2)H \right] + Tdiv(sH) \]
and is not conserved. But for barotropic and isentropic flows the pressure \( p = p(\rho) \) and \( h_C \) is conserved:
\[ \partial_t h_C + div q_C = 0, \quad q_C = vh_C + (\chi - v^2/2)H, \] (50)
where $\chi = \int dp/\rho$.

For the general case there is known one more conserved quantity first discovered by Gordin and Petviashvili, cf. Ref. [23]. Corresponding density is

$$h_P = (H, \nabla s),$$

(51)

and

$$\partial_t h_P + \text{div} q_P = 0, \quad q_P = v h_P .$$

(52)

With this local conserved quantity there is linked integral conservation law. Namely, integrating $h_P$ over arbitrary substantial volume $\tilde{V}$ we obtain conserved quantity $I_P$,

$$I_P = \int_{\tilde{V}} d\tau h_P, \quad \partial_t I_P = 0 .$$

(53)

Note here that the latter quantity gives us example of the so called local Lagrange invariants, cf. Refs. [10, 11, 12, 13] and [2, 1]. By definition they obey the following equations

$$\partial_t \alpha + (v, \nabla)\alpha = 0 , \quad \partial_t I + (v, \nabla)I = 0 ,$$

(54)

$$\partial_t J + (v, \nabla)J - (J, \nabla)v = 0 ,$$

(55)

$$\partial_t L + (v, \nabla)L + (L, \nabla)v + [L, \text{curl}v] = 0 , \text{ or, equivalently, } \partial_t L + \nabla(v, L) - [v, \text{curl}L] = 0 .$$

(56)

Here $\alpha$ and $I$ present the scalar and vector Lagrange invariants, $J$ is frozen-in field, and $L$ presents $S$-type invariant in terminology of Ref. [12], related to frozen-in surface. To these invariants it is necessary to add the density $\rho$. Evidently, the quantity $h_P/\rho$ presents $\alpha$-type invariant. The Lagrange markers $\mu$ and quantities $\lambda/\rho$ give us examples of the vector Lagrange invariants, magnetic field $H$ divided by $\rho$, $h = H/\rho$ is invariant of the $J$- type, gradient of any scalar Lagrange invariant is $S$-type invariant,

$$L' = \nabla \alpha .$$

(57)

There exist also another relations between different type invariants, see Refs. [2, 1], allowing to produce new invariants. For instance, scalar product of the $J$ and $L$ invariants presents some scalar Lagrange invariant, symbolically

$$\alpha' = (J, L).$$

(58)

The presented above invariant $h_P/\rho$ can be obtained by means of this relation if we put $J = h$ and $L = \nabla s$. Another examples present relations generating $J$- (L)- type invariants by means of two $L$- (J-) type invariants,

$$J' = [L, L']/\rho ,$$

(59)

$$L' = \rho[J, J'] .$$

(60)
Note here that integrating of the density $h_M$ over arbitrary substantial volume does not lead to the conserved integral. It is easy to check that

$$ I_M = \int \tilde{V} \, d\mathbf{r} h_M $$

satisfies relation

$$ \partial_t I_M = \int_{\partial \tilde{V}} d\Sigma \left( (\mathbf{A}, \mathbf{v}) + \Lambda \right) H_n, \quad H_n = (\mathbf{H}, \mathbf{n}), $$

where integration in the right-hand side is performed over the boundary $\partial \tilde{V}$ of the volume $\tilde{V}$, $\mathbf{n}$ is outward normal and $d\Sigma$ presents infinitesimal area of the surface $\partial \tilde{V}$. It is obvious that $I_M$ will be integral of motion if $H_n$ equals zero. This fact leads to the conclusion that $I_M$ presents integral of motion if we choose the substantial volume in such a way that initial volume $\tilde{V} |_{t=t_0}$ is such that $H_n |_{t=t_0} = 0$ because this condition is invariant of the motion: if equality $H_n = 0$ holds for the initial moment then it holds true in the future.

Another way to make $I_M$ invariant consists in fixing the gauge of the vector potential $\mathbf{A}$ in such a way that $(\mathbf{A}, \mathbf{v}) + \Lambda = 0$. Then the dynamic equation for $\mathbf{A}$, (10), takes the form

$$ \partial_t \mathbf{A} + \nabla (\mathbf{v}, \mathbf{A}) - [\mathbf{v}, \text{curl} \mathbf{A}] = 0, $$

i.e., $\mathbf{A}$ presents invariant of the $L$-type. Under this gauge condition the quantity $h_M / \rho$ presents the scalar Lagrange invariant, $D(h_M / \rho) = 0$.

As for the local conservation law for the cross-helicity, Eq. (50), it obviously leads to the integral conserved quantity $I_C$ for the barotropic flows but with following restriction: integration have to be performed over the specific substantial volume such one that condition $H_n |_{\partial \tilde{V}} = 0$ (this condition is invariant of the motion) holds,

$$ \partial_t I_C = 0, \quad I_C \equiv \int \tilde{V} \, d\mathbf{r} h_C, \quad H_n |_{\partial \tilde{V}} = 0. $$

Existence of the recursive procedure allowing one to construct new invariants on the basis of the starting set of invariants, see Refs. [2, 1], accentuates the role of the local invariants among other conserved quantities. Although in terms of the Lagrangian variables (such as the markers $\mu$) there exist a wide set of invariants, see, for instance, Ref. [1], the most interesting invariants are such that can be expressed in Eulerian (physical) variables and are gauge invariant. Emphasize here that in the conventional hydrodynamics there exists Ertel invariant $\alpha_E$,

$$ \alpha_E = h_E / \rho, \quad h_E = (\omega, \nabla s), $$

where $\omega = \text{curl} \mathbf{v}$ is vorticity,

$$ \partial_t h_E + \text{div} \mathbf{q}_E = 0, \quad \mathbf{q}_E = h_E \mathbf{v}, \quad D \alpha_E = 0. $$

Corresponding integral of motion reads

$$ \partial_t I_E = 0, \quad I_E \equiv \int \tilde{V} \, d\mathbf{r} h_E. $$

Note here that $I_E = 0$ holds true for arbitrary substantial volume $\tilde{V}$. 

The Ertel invariant density has the structure of the Eq. (58) with $L = \nabla s$, $J = \omega/\rho$, where $\omega$ is vorticity, $\omega = \text{curl}\, \mathbf{v}$ (remind that $\omega$ is a frozen-in field for the barotropic hydrodynamic flows). In the hydrodynamic case there exists also the helicity invariant

$$h_H = (\omega, \mathbf{v}),$$

which has topological meaning, defining knottness of the flow. It satisfies equation

$$\partial_t h_H + \text{div} q_H = 0, \quad q_H = h_H \mathbf{v} + (\chi - v^2/2)\omega,$$

and evidently leads to the corresponding integral conservation law

$$\partial_t I_H = 0, \quad \text{for} \quad \omega_{n|\nabla \tilde{V}} = 0, \quad I_H \equiv \int \tilde{V} \, dr \, h_E.$$

For the MHD case the vector $\omega/\rho$ does not present frozen-in field due to the fact that magnetic force is not potential. It seems rather evident that for the MHD case there have to exist integrals of motion generalizing the conventional helicity and Ertel invariant along with vorticity one, which have to pass into conventional ones for vanishing magnetic field. The generalization for the vorticity and helicity invariants was obtained by authors of the paper [8] for the particular case of the incompressible flows. In the following section it is shown that there exists MHD generalization for the Ertel invariant, and results of the paper [8] relating to the vorticity and helicity can be extended for incompressible barotropic MHD flows.

4.1 Generalized vorticity.

Let us prove that the quantity $\omega_h/\rho$, where

$$\omega_h \equiv \text{curl}\, \mathbf{v}_h = - \left[ \nabla \left( \frac{P}{\rho} \right), \nabla Q \right] = - \left[ \nabla \left( \frac{\lambda_m}{\rho} \right), \nabla \mu_m \right] - \left[ \nabla \left( \frac{\sigma}{\rho} \right), \nabla s \right],$$

presents frozen-in field (‘hydrodynamic’ part of the vorticity) for the barotropic MHD flows. It would be trivial consequence of the fact that $[L, L']/\rho$, where $L, L'$ are Lamb type invariants, is local invariant of the frozen-in type if all quantities $Q$ and $P/\rho$ satisfy homogeneous transport equations being $\alpha$- or $\mathbf{I}$ type invariants (remember, that $\nabla \alpha$ and $\nabla I_m$ are $L$ type invariants). But $\varphi$ and $\sigma/\rho$ satisfy inhomogeneous equations of motion. Therefore, let us start with equation of motion for the ‘hydrodynamic’ part of the velocity. Differentiating representation (16) and making use of relations

$$D(\nabla X) = \nabla (DX) - (\nabla v_m) \cdot \partial_m X$$

we have

$$D\mathbf{v}_h = -D \left( \frac{P}{\rho} \right) \cdot \nabla Q - \frac{P}{\rho} \cdot \nabla (DQ) + \frac{P}{\rho} (\nabla v_m) \cdot \partial_m Q = T \nabla s - \nabla (w - v^2/2) - v_{hm}(\nabla v_m),$$

or, after simple rearrangements,

$$D\mathbf{v}_h = -\nabla p/\rho + (v_m - v_{hm}) \cdot \nabla v_m.$$
Taking the curl of this equation leads to
\[
\partial_t \omega_h = -\operatorname{curl}((v_m \partial_m) \nabla v) + [\nabla \rho, \nabla p]/\rho^2 - \operatorname{curl}(v_h \nabla v_m) = \\
= [\nabla \rho, \nabla p]/\rho^2 + \operatorname{curl}[(v_m \nabla v_h) - (v_h \nabla v_m)].
\]
The term in the square brackets is equal to \([v_m \nabla v_h] - (v_h \nabla v_m)] = [\nu, \omega_h]\) and we obtain
\[
\partial_t \omega_h = [\nabla \rho, \nabla p]/\rho^2 + \operatorname{curl}[\nu, \omega_h].
\] (71)

For barotropic flows the first term in the r.h.s. becomes zero and we can see that \(\omega_h/\rho\) is frozen-in field,
\[
D \left( \frac{\omega_h}{\rho} \right) = \left( \frac{\omega_h}{\rho}, \nabla \right) \nu.
\] (72)

For \(\mathbf{H} = 0\) \(\omega_h\) corresponds to the conventional hydrodynamic vorticity.

In spite of the gauge dependence of the generalized vorticity, it frozeness gives us possibility to introduce the generalized helicity integral of motion.

### 4.2 Generalized helicity.

Now we can prove that generalized helicity, \(h_H\), defined in terms of the ‘hydrodynamic’ part of the velocity,
\[
h_H = (\omega_h, \nu_h),
\] (73)
is integral of motion for barotropic flows. Namely, differentiating Eq. (73) and taking for account Eqs. (70), (71) we arrive for the barotropic flows to the local conservation law of the form (rather cumbersome calculations are presented in Appendix):
\[
\partial_t h_H + \operatorname{div} q_H = 0, \quad q_H = h_H \nu + (\chi - v^2/2) \omega_h.
\] (74)

In analogy with the hydrodynamic case we arrive to the conclusion that the integral helicity \(I_H\) (defined by means of Eq. (68)) is integral invariant, moving together with the fluid if the normal component of the vorticity tends zero, \(\omega_{hn} = 0\), on the surface of the corresponding substantial volume \(\tilde{V}\). Note that the condition \(\omega_{hn} = 0\) is invariant of the flow (due to the frozen–in character of \(\omega_h/\rho\)) and therefore it can be related to the initial surface only.

### 4.3 Generalized Ertel invariant.

Let us show here that there exists strict generalization of the Ertel invariant for the MHD case. For this purpose let us prove that without any restrictions related to the character of the flow the quantity
\[
h_E = (\omega_h, \nabla s)
\] (75)

satisfies conservation law of the form
\[
\partial_t h_E + \operatorname{div} q_E = 0, \quad q_E = h_E \nu.
\] (76)
Equivalently, the quantity $\alpha_E = h_E/\rho$ is transported by the fluid

$$D\alpha_E = 0, \quad \alpha_E = h_E/\rho,$$

presenting $\alpha$-type invariant. For the barotropic flows it immediately follows from the fact that $\omega_h/\rho$ presents frozen-in field if one takes for account the composition rules given by Eqs. (58) and (57). In order to make the proof for the non barotropic flows more transparent let us consider something more general situation. Let $\tilde{J}$ satisfy equation of motion of the form

$$D\tilde{J} = (\tilde{J}, \nabla) v + Z,$$

differing from the frozen field equation (55) by existence of the term $Z$ that violates homogeneity. Then, if $\alpha$ represents any scalar Lagrange invariant, we have

$$D(\tilde{J}, \nabla\alpha) = (D\tilde{J}, \nabla\alpha) + ((\tilde{J}, \nabla) v, \nabla\alpha) - \left( \tilde{J}, (\nabla v_m) \cdot \partial_m \alpha \right).$$

Here the two last terms cancel and we get

$$D(\tilde{J}, \nabla\alpha) = (Z, \nabla\alpha) \quad \text{if} \quad D\tilde{J} = (\tilde{J}, \nabla) v + Z, \quad \text{and} \quad D\alpha = 0.$$ 

For $Z = 0$ this relations prove the generating rule of Eq. (58). But we can see that $(\tilde{J}, \nabla\alpha)$ will present the local Lagrange invariant under more restrictive condition $(Z, \nabla\alpha) = 0$. That is the case for the Ertel invariant: $Z = [\nabla \rho, \nabla p]/\rho^2$ is orthogonal to $\nabla s$ due to the fact that the scalar product of any three thermodynamic quantities is equal zero (because any thermodynamic variable in the equilibrium state can be presented as function of two basic variables). This ends the proof.

The conserved integral quantity associated with $\alpha_E$ is

$$I_E = \int_{\tilde{V}} dV h_E, \quad \partial_t I_E = 0.$$ 

Note here that by the structure $I_E$ is not gauge invariant in contrast to the hydrodynamic case. Let us examine it change under gauge transformation changing $v_h \Rightarrow v'_h, \ v_M \Rightarrow v'_M$ with

$$v'_h + v'_M = v_h + v_M.$$ 

Then

$$I'_E - I_E = \int_{\tilde{V}} dV (\nabla s, \omega'_h - \omega_h) = \int_{\tilde{V}} dV (\nabla s, \omega_M - \omega'_M).$$

But $(\nabla s, \omega_M - \omega'_M) = -\text{div}[\nabla s, (v'_M - v_M)]$ and therefore we can proceed as follows

$$I'_E - I_E = -\int_{\partial\tilde{V}} d\Sigma(n, [\nabla s, (v'_M - v_M)]).$$

Taking into account that $v'_M - v_M = -[h, M' - M]$ we obtain

$$I'_E - I_E = \int_{\partial\tilde{V}} d\Sigma (n, [\nabla s, [h, M' - M]]).$$
Inasmuch as both $M'$ and $M$ satisfy Eq. (11), their difference is governed by homogeneous equation

$$\partial_t \overline{M} = \text{curl} [\mathbf{v}, \overline{M}],$$

i.e. $\overline{m} = \overline{M}/\rho$ is frozen-in field. Then we arrive to the conclusion that the vector $[\nabla s, [h, \overline{m}]]$ entering the integrand presents frozen-in field, as it follows from recursive relations Eqs. (57) – (59). Therefore, if we adopt relation $(n, [\nabla s, [h, \overline{m}]]))_{\partial \tilde{V}} = 0$ as initial condition, then it holds true for all moments. But we cannot choose the (initial) substantial volume in such a way that relation

$$(n, [\nabla s, [h, m']])_{\partial \tilde{V}} = (n, [\nabla s, [h, m]])_{\partial \tilde{V}}$$

holds true for any change of the gauge. Thus integral Ertel invariant is gauge dependent. Nevertheless, we can point out some subset of the gauge transformations under which $I_E$ is invariant. Namely, let $M|_{t=t_0} = fH$ for some initial moment, $t = t_0$, where $f$ have to satisfy condition $(H, \nabla f) = 0$, following from the divergence–free character of $M$. Then relation (82) fulfills for the initial moment and therefore it holds true at all moments also. The specific choice $f = 0$ leads to additional restriction for the gauge transformations but it is convenient due to its simplicity. Summarizing, we can say that the Ertel invariant is partly gauge independent.
5 Conclusions.

The results obtained can be summarized as follows. First, there is presented variant of introducing the canonical description of the MHD flows by means of the variational principle with constraints. It is shown that in order to describe general type MHD flows it is necessary to use in the generalized Clebsch type representation of the fluid velocity field vector Clebsch variables (the Lagrange markers and conjugate momenta) along with the entropy term (compare papers [15, 16] describing hydrodynamic case) and the conventional magnetic term introduced first in the paper [17]. Such complete representation allows one to deal with general type MHD flows, including all type of breaks, see Ref. [14]. Second, it is proved that introduced in the paper generalized Weber transformation leads to the velocity representation, which equivalent to the one introduced by means of the variational principle. Third, there is proved existence of the generalized Ertel invariant for MHD flows. Forth, there are generalized the vorticity and helicity invariants for the compressible barotropic MHD flows (first discussed for the incompressible case in cf. [8]). Fifth, the relations between the local and integral invariants are discussed along with the gauge dependence of the latter.

As a consequence of the completeness of the representation proposed we arrive to the correct limit transition from the MHD to conventional hydrodynamic flows. The results obtained allow one to deal with the complicated MHD problems by means of the Hamiltonian variables. The use of such approach was demonstrated for the specific case of incompressible flows in the series of papers [8, 9] devoted to the nonlinear stability criteria. Emphasize, that existence of the additional invariants proved in our paper is of very importance for the stability problems.

Note here that existing of the additional basic invariants of the motion makes it actual to examine the problem of the complete set of independent invariants, cf. [1]. This problem needs special discussion together with related problem of their gauge invariance. One more open problem is connected with the great number of the generalized coordinates and momenta involved in the approach discussed. Here the question arises if it is possible to reduce this number without loosing the generality.

Appendix A

In order to prove Eq. (32) let us substitute $J$ from Eq. (31) into expression $[J, h]_k \frac{\partial x_k}{\partial a_i}$. Then

$$[J, h]_k \frac{\partial x_k}{\partial a_i} = [Dm, H]_k \frac{\partial x_k}{\partial a_i} - [(m, \nabla)v, H]_k \frac{\partial x_k}{\partial a_i} =$$

$$= \frac{\partial x_k}{\partial a_i} D (\frac{\partial (m, H)}{\partial k}) - [(m, \nabla)v, H]_k \frac{\partial x_k}{\partial a_i}. \tag{83}$$

Proceeding with the terms in the second brackets we obtain

$$[m, D(\rho h)]_k + [(m, \nabla)v, H]_k = [m, h]_k \cdot D\rho + [\rho m, Dh]_k + [(m, \nabla)v, H]_k =$$

$$= -[M, h]_k \cdot divv + [M, (h, \nabla)v]_k + [(M, \nabla)v, h]_k = -[M, h]_k \partial_k v_s, \tag{84}$$

15
where \( \mathbf{M} = \rho \mathbf{m} \) and there is taken for account dynamic equation \( D \mathbf{h} = (\mathbf{h}, \nabla) \mathbf{v} \) and identity

\[
[M, (h, \nabla)v]_k + [(M, \nabla)v, h]_k = [M, h]_k \partial_s v_s - [M, h]_s \partial_k v_s.
\]

Introducing for brevity notation

\[
\mathbf{Y} = [\mathbf{m}, \mathbf{H}] \equiv [\mathbf{M}, \mathbf{h}],
\]

we can represent the r.h.s. of Eq. (83) as

\[
\frac{\partial x_k}{\partial a_i} \cdot DY_k + Y_s \frac{\partial x_k}{\partial a_i} \partial_k v_s = \frac{\partial x_k}{\partial a_i} \cdot DY_k + Y_s \frac{\partial v_s}{\partial a_i} = \frac{\partial x_k}{\partial a_i} \cdot DY_k + Y_s \frac{\partial}{\partial a_i} (D x_s) = D \left( Y_s \frac{\partial x_k}{\partial a_i} \right).
\]

This proves Eq. (32).

Let us check up now the integral relation (39). It is sufficient to prove the differential form, namely

\[
D(\mathbf{M}, d\Sigma) = (\mathbf{J}, d\Sigma),
\]

where \( d\Sigma \) presents some infinitesimal oriented area moving with the fluid. It can be presented in the form

\[
d\Sigma = [d l_1, d l_2],
\]

where \( d l_1, d l_2 \) are frozen-in linear elements. Thus, \( d l_a, a = 1, 2, \) are invariants of the \( \mathbf{J} \) type and satisfy equations

\[
D(d l_a) = (d l_a, \nabla) \mathbf{v}.
\]

Consequently, from the recursion relation Eq. (60) it follows that \( \rho d\Sigma \) is \( \mathbf{L} \)-type invariant and therefore is governed by dynamic equation of the form:

\[
D(\rho d\Sigma) = -\nabla (\rho \mathbf{v} d\Sigma) + [\mathbf{v}, \text{curl}(\rho d\Sigma)],
\]

or in the coordinates,

\[
D(\rho d\Sigma_i) = - (\rho d\Sigma_k) \partial_i v_k.
\]

Now it is easy to prove relation (33) without any restrictions for the type of flow. Namely,

\[
D(\mathbf{M}, d\Sigma) = D(\mathbf{m}, \rho d\Sigma) = (D \mathbf{m}, \rho d\Sigma) + m_i D(\rho d\Sigma_i) =
\]

\[
= (\rho d\Sigma, (\mathbf{m}, \nabla) \mathbf{v}) + (\mathbf{J}, d\Sigma) - m_i \rho d\Sigma_k \partial_i v_k = (\mathbf{J}, d\Sigma).
\]

In order to prove the helicity conservation, Eq. (74), let us consider some scalar quantity of the form

\[
\mathbf{Y} = (\mathbf{v}_h \mathbf{J}),
\]

where \( \mathbf{J} \) is frozen-in field. Then, taking for account that Eq. (70) for the barotropic flows can be presented in the form

\[
D \mathbf{v}_h = -\nabla (\chi - v^2/2) - v_{hm} \cdot \nabla v_m, \quad \chi \equiv \int dp/\rho,
\]

we obtain

\[
D \mathbf{Y} = (D \mathbf{v}_h, \mathbf{J}) + (\mathbf{v}_h, D \mathbf{J}) = - (\nabla (\chi - v^2/2), \mathbf{J}).
\]
For \( \mathbf{J} = \omega_h/\rho \) we proceed
\[
D(\mathbf{v}_h, \omega_h/\rho) = -\rho^{-1} \left( \nabla (\chi - v^2/2), \omega_h \right) = -\rho^{-1} \text{div} \left( (\chi - v^2/2)\omega_h \right).
\]
Then
\[
D(\mathbf{v}_h, \omega_h) = \rho D(\mathbf{v}_h, \omega_h/\rho) + (\mathbf{v}_h, \omega_h/\rho) D\rho = -\text{div} \left( (\chi - v^2/2)\omega_h \right) - (\mathbf{v}_h, \omega_h) \text{div} \mathbf{v},
\]
or
\[
\partial_t (\mathbf{v}_h, \omega_h) = -\text{div} \mathbf{q}_h, \quad \mathbf{q}_h = (\chi - v^2/2)\omega_h + \mathbf{v} (\mathbf{v}_h, \omega_h)
\] (89)
that evidently coincides with Eq. (74).

It is noteworthy that the proof is valid for any \( \mathbf{J} \)-type invariant if the field \( \rho \mathbf{J} \) is divergence–free:
\[
\partial_t (\rho \mathbf{J}, \mathbf{v}_h) = -\text{div} \mathbf{q}, \quad \mathbf{q} = (\chi - v^2/2)\rho \mathbf{J} + \mathbf{v} (\rho \mathbf{J}, \mathbf{v}_h) \quad \text{for} \quad \text{div}(\rho \mathbf{J}) = 0.
\] (90)
For instance, choosing \( \mathbf{J} = \mathbf{h} \) immediately leads to cross–helicity invariant if one takes for account that \( (\mathbf{H}, \mathbf{v}_h) = (\mathbf{H}, \mathbf{v}) \).

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