UNIQUENESS FOR AN INVERSE PROBLEM IN ELECTROMAGNETISM WITH PARTIAL DATA

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ABSTRACT. A determination result for the recovery of the electric and magnetic coefficients in the time-harmonic Maxwell equations from local boundary measurements is proven. No special geometrical condition is imposed on the inaccessible part of the boundary of the domain, apart from imposing that the boundary of the domain is $C^{1,1}$. The coefficients are assumed to coincide on a neighbourhood of the boundary, a natural property in applications.

1. Introduction

Let $\mu$, $\varepsilon$, $\sigma$ be positive functions on a nonempty, bounded, open set $\Omega$ in $\mathbb{R}^3$, describing the permeability, permittivity and conductivity, respectively, of an inhomogeneous, isotropic medium modelled by $\Omega$. Let $\partial \Omega$ denote the boundary of $\Omega$ and $N$ the outward unit vector field normal to the boundary. Consider the electric and magnetic fields, $E$, $H$, verifying the so-called time-harmonic Maxwell equations at a frequency $\omega > 0$, namely

\[
\begin{aligned}
\nabla \times H + i\omega \gamma E &= 0, \\
\nabla \times E - i\omega \mu H &= 0,
\end{aligned}
\]

in $\Omega$, where $\gamma = \varepsilon + i\sigma/\omega$, $i$ denotes the imaginary unit, and $\nabla \times$ denotes the curl operator.

There exist positive values of $\omega$ for which the equations (1.1), posed in proper spaces and domains, assuming non-negative coefficients $\varepsilon$, $\sigma$, $\mu$ for $\varepsilon$, $\mu$ bounded from below, with the tangential boundary condition either $N \times H|_{\partial \Omega} = 0$ or $N \times E|_{\partial \Omega} = 0$, have non-trivial solutions (see [61], [43]). Such values are called resonant frequencies.

The boundary data corresponding to the inverse boundary value problem (IBVP) for the system (1.1) only can be given by a boundary mapping (the
impedance or admittance map) if \( \omega \) is not a resonant frequency. The fact that the position of resonant frequencies depends on the unknown coefficients (as it is stated in \[53\]) motivated Pedro Caro in \[16\] to consider a Cauchy data set instead of a boundary map as boundary data. Cauchy data sets have been used in \[14, 56, 57, 16, 17, 20\].

This work is aimed at the IBVP for the system \((1.1)\) with local boundary measurements established by a Cauchy data set taken just on a part of \(\partial \Omega\). More precisely, Definition \[1.1\] describes the conditions for the domain and the part of its boundary where the measurements are taken and Definition \[1.2\] (used in \[17\]) introduces the boundary data for the IBVP studied in this article.

**Definition 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a non-empty, bounded domain in \( \mathbb{R}^3 \) with \( C^{1,1} \) boundary \( \partial \Omega \). Assume \( \Gamma \) is a smooth proper non-empty open subset of \( \partial \Omega \). We call \( \Gamma \) the accessible part of the boundary \( \partial \Omega \) and \( \Gamma_c := \partial \Omega \setminus \Gamma \) the inaccessible part of the boundary.

**Definition 1.2.** Let \( \Omega \) and \( \Gamma \) be as in Definition \[1.1\]. For a pair of smooth coefficients \( \mu, \gamma \) on \( \Omega \) according to Definition \[1.4\] define the Cauchy data set restricted to \( \Gamma \), write \( C(\mu, \gamma; \Gamma) \), at frequency \( \omega > 0 \) by the set of couples \((T, S) \in TH_0(\Gamma) \times TH(\Gamma)\) such that there exists a solution \((E, H) \in (H(\Omega; \text{curl}))^2\) of \((1.1)\) in \( \Omega \) satisfying \( N \times E|_{\partial \Omega} = T \) and \( N \times H|_{\Gamma} = S \), where the spaces \( H(\Omega; \text{curl}), TH(\Gamma), TH_0(\Gamma) \) are defined in Definition \[1.3\].

It is known that if the domain \( \Omega \) is not convex and its boundary is not \( C^{1,1} \), Maxwell equations may not admit solutions in \( H^1(\Omega) \) even for boundary data in \( H^{1/2}(\partial \Omega) \) (see \[3, 9, 18, 59, 23\]). Thus, for a less regular domain (e.g., Lipschitz), some non-standard Sobolev spaces are necessary. Some of them, which will be used in these notes, appear in the following

**Definition 1.3.** Let \( \Omega \) and \( \Gamma \) be as in Definition \[1.1\]. Let \( B^{1/2}(\partial \Omega) \) denote the Besov space \( B^{s,p}_q(\partial \Omega) \) with \( s = 1/2 \), \( p = q = 2 \). These Besov spaces on nonsmooth boundaries of domains are defined, e.g., in \[36\]. Define \( B^{1/2}(\Gamma) = \{ f|_{\Gamma} : f \in B^{1/2}(\partial \Omega) \} \), with norm \( \| g \|_{B^{1/2}(\Gamma)} = \inf \{ \| f \|_{B^{1/2}(\partial \Omega)} : f|_{\Gamma} = g \} \), and \( B^{1/2}_0(\Gamma) = \{ f \in B^{1/2}(\partial \Omega) : \text{supp } f \subset \overline{\Gamma} \} \), with norm \( \| f \|_{B^{1/2}_0(\Gamma)} = \| f \|_{B^{1/2}(\partial \Omega)} \). Write \( B^{-1/2}(\partial \Omega) \) for the dual space of \( B^{1/2}(\partial \Omega) \).

Consider the space \( H(\Omega; \text{curl}) = \{ u \in L^2(\Omega; \mathbb{C}^3) : \nabla \times u \in L^2(\Omega; \mathbb{C}^3) \} \) with the usual graph norm, and the following

\[ TH(\partial \Omega) = \{ u \in B^{-1/2}(\partial \Omega; \mathbb{C}^3) : N \times v|_{\partial \Omega} = u, \text{ for some } v \in H(\Omega; \text{curl}) \}, \]

\[ TH(\Gamma) = \{ u|_{\Gamma} : u \in TH(\partial \Omega) \}, \]

\[ TH_0(\Gamma) = \{ u \in TH(\partial \Omega) : \text{supp } u \subset \overline{\Gamma} \}, \]

with norms \( \| u \|_{TH(\partial \Omega)} = \inf \{ \| v \|_{H(\Omega; \text{curl})} : v \in H(\Omega; \text{curl}), N \times v|_{\partial \Omega} = u \}, \)

\( \| u \|_{TH(\Gamma)} = \inf \{ \| v \|_{TH(\partial \Omega)} : v|_{\Gamma} = u \}, \)

\( \| u \|_{TH_0(\Gamma)} = \| u \|_{TH(\partial \Omega)} \). Some properties of these spaces can be found, e.g., in \[18, 16, 17\].
Next, the class of admissible coefficients for the uniqueness result proven in this article is set.

**Definition 1.4.** Let $\Omega$ be as in Definition 1.1. Let $M > 0$. The pair of coefficients $\mu, \gamma$ is admissible if $\mu, \gamma \in C^{1,1}(\Omega) \cap W^{2,\infty}(\Omega)$, and the following conditions are satisfied:

- $\text{Re} \gamma \geq M^{-1}, \mu \geq M^{-1}$ in $\Omega$,
- $\|\gamma\|_{W^{2,\infty}(\Omega)} + \|\mu\|_{W^{2,\infty}(\Omega)} \leq M$.

The main result of this work reads as follows.

**Theorem 1.1.** Let $\Omega$ and $\Gamma$ be as in Definition 1.1 and $\omega > 0$ the time-harmonic frequency. Assume $\mu_j, \gamma_j$ (with $j = 1, 2$) are two pairs of admissible coefficients such that $\text{supp}(\mu_1 - \mu_2), \text{supp}(\gamma_1 - \gamma_2) \subset \Omega$. Then, if $C(\mu_1, \gamma_1; \Gamma) = C(\mu_2, \gamma_2; \Gamma)$ then $\gamma_1 = \gamma_2$ and $\mu_1 = \mu_2$ in $\Omega$.

The IBVP for Maxwell equations can be seen as a vector generalization of the inverse conductivity problem of Calderón. In his seminal paper [15], Calderón posed two questions as follows: Firstly, is it possible to uniquely determine the conductivity of an unknown object from boundary measurements? Secondly, in affirmative case, can this conductivity be reconstructed? Here the boundary measurements are set by the Dirichlet-to-Neumann map $\Lambda_\sigma$, which for a conductivity $\sigma \in L^\infty(\Omega)$ defined on a bounded domain $\Omega$ modelling the object, is defined by $\Lambda_\sigma f = \sigma|_{\partial \Omega}(\partial u/\partial N)|_{\partial \Omega}$, where $f \in H^{1/2}(\partial \Omega)$, $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega, \quad u = f \text{ on } \partial \Omega,$$

and $(\partial u/\partial N)|_{\partial \Omega}$ denotes the normal derivative of $u$ on $\partial \Omega$.

Matti Lassas in [42] proved that the boundary measurements of the Calderón problem are a low-frequency limit of the boundary data (impedance map) of the IBVP for time-harmonic Maxwell equations under some restrictions.

Concerning the Calderón problem in the plane, there are three main global uniqueness proofs giving reconstruction D-bar methods based on complex geometrical optics (CGO) solutions: the Schrödinger equation approach for twice differentiable $\sigma$ by Nachmann [51], the first-order system approach for once differentiable $\sigma$ by Brown and Uhlmann [12], and the Beltrami equation approach assuming no smoothness ($\sigma \in L^\infty(\Omega)$) by Astala and Päivärinta [4]. The assumption $\sigma \in L^\infty(\Omega)$ was the one originally used by Calderón in [15]. Several stability estimates have been proven: [6, 7, 21, 27].

In dimension $n \geq 3$ the best known uniqueness result for the Calderón problem is due to Haberman and Tataru [30] for continuously differentiable conductivities. A novel argument of decay in average using Bourgain-type spaces is introduced there. We cite some previous uniqueness results: the foundational [64] for smooth conductivities by Sylvester and Uhlmann, [50] where Nachman presents a reconstruction algorithm, and [10, 11, 55]. Concerning conditional stability, the best result is by the third author et al. in [18] for $C^{1,\varepsilon}$ conductivities on Lipschitz
domains using the method in [30]. A previous stability result was given by Heck in [31]. Roughly speaking, the method introduced by Alessandrini in [1] gives the main guidelines followed by most stability methods for both the scalar and vector problems.

To deal with inverse problems from partial data for scalar elliptic equations, two main approaches are found in the literature in dimension $n > 2$ (see [3] for $n = 2$): using Carleman estimates [12, 39, 32] and using reflection arguments [35, 33]. This work applies to the vector case the density argument shown in [2] for the scalar Schrödinger equation by Gunther Uhlmann and Habib Ammari.

The IBVP for stationary Maxwell equations with global data was originally proposed by Somersalo et al. in [61], where the coefficients are supposed to deviate only slightly from constant values. The same year the unique recovery of the parameters from the scattering amplitude for $\mu$ constant was presented in [22], and a local uniqueness result for the IBVP from global data was proven in [63]. The first global determination result for the IBVP with general coefficients $\mu, \gamma$ from global boundary measurements was proven by Lassi Päivärinta et al. in [52], assuming $C^3$ smoothness on $\mu, \gamma$ on $C^{1,1}$ domains. The proof is constructive. Later on, the proof was simplified via a relation between Maxwell equations and a matrix Helmholtz equation with a potential in [54]. Boundary determination results appeared in [46] and [37] for smooth boundaries. Chiral media were studied in [47]. Stability from global boundary data was obtained in [16]. Other inverse problems in electromagnetism in settings different to the ones in this paper have been considered in [60, 34, 41, 45, 38, 20].

The uniqueness and stability issue of the IBVP for Maxwell equations with local boundary data has been little studied. The only works in this direction the authors are aware of are [19, 17], where an extension of Isakov’s method in [35] to Maxwell system is performed. Another extension of methods used in the scalar case for partial data to vector systems is in [57], where uniqueness for a Dirac-type system is proven following the ideas of [39].

Our proof uses the CGO solutions given in [16, 17] to a matrix Schrödinger-type equation related to Maxwell system and a Dirac-type system not related to Maxwell equations. As in Lemma 3.2 of [38] we give an integral identity (Proposition 4.2) involving a solution $Z_1$ to the Schrödinger-type equation and a solution $Y_2$ to the Dirac system. In order that such an identity holds for boundary data restricted to $\Gamma$, the solutions have to satisfy certain local homogeneous boundary conditions on $\Gamma_c$. In [17] the solutions with such properties are constructed from the CGO solutions following the reflection principle in [35], arising this way the strong geometrical constraint on $\Gamma_c$ of being plane or part of a sphere.

We manage to avoid this annoying restriction by the density argument given by Lemma 2 in [2] for the scalar Schrödinger equation adapted to a vector Helmholtz equation satisfied by the electric (and magnetic with different coefficients) field related to $Z_1$ and another matrix Schrödinger-type equation verified by $Y_2$. Here,
unique continuation principles for the aforementioned vector equations and a stability estimate for the inverse of the Dirac-type operator are required.

The density argument makes the assumption that the boundary is $C^{1,1}$ and the coefficients coincide on a neighbourhood of the boundary, the latter being a natural property in applications. The rest of the proof is valid with just Lipschitz boundary.

This article is organized as follows. In Section 2 some key matrix equations are introduced with the novelty with respect to previous works of a Schrödinger-type equation verified by the solutions to the Dirac-type equation $(P + W^*)\hat{Y} = 0$ not related to the Maxwell system. The CGO solutions to some of such equations used in this article are recalled in Section 3. Section 4 is devoted to showing an orthogonality identity involving the potentials and solutions to matrix equations corresponding to two couples of admissible coefficients. Two density results which are essential in the proof are presented in Section 5. Finally, our proof of uniqueness is expounded in Section 6 which contains a demonstration of the bounded invertibility of a Dirac-type operator with certain boundary conditions.

Throughout this work the following notation is used.

**Notation.** Given a couple of coefficients $(\mu_j, \gamma_j)$ for $j = 1, 2$, denote $C^j_{\Gamma} := C(\mu_j, \gamma_j; \Gamma)$, where $C(\mu_j, \gamma_j; \Gamma)$ is defined in Definition 1.2. For an expression like $U \cdot V$, with $U, V \in C^m$-valued vector fields and $m$ a natural number, $\cdot$ denotes the analytic extension to $C^m$ of the Euclidean real-inner product on $\mathbb{R}^m$. $I_m$ stands for the $m \times m$ identity matrix. For a matrix of complex entries $U$, the expressions $U^t$ and $U^*$ stand for its transpose and conjugate transpose, $U^t$, respectively. For the domain $\Omega$ and complex vector fields $U, V \in H^{1/2} (\Omega; C^m)$, denote

$$(U|V)_\Omega := \int_{\Omega} V^* U \, dx, \quad (U|V)_{\partial\Omega} := \int_{\partial\Omega} V^* U \, ds,$$

where $ds$ denotes the restriction of the Lebesgue measure of $\mathbb{R}^3$ to $\partial\Omega$.

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2. The equations

Here some differential systems related to Maxwell equations are presented.

Let us start with the classical Schrödinger-type equation approach by Petri Ola and Erkki Somersalo in [54]. Fix a frequency $\omega > 0$. Assume the coefficients $\mu, \gamma$ to be in $C^{1,1}(\overline{\Omega})$ on a bounded domain with boundary locally described by
the graph of a Lipschitz function. Following the notation in [16], write
\[
\alpha := \log \gamma, \quad \beta := \log \mu, \quad \kappa := \omega^{1/2} \gamma^{1/2}.
\]

\(E, H \in H(\Omega; \text{curl})\) solve Maxwell equations (1.1) with coefficients \(\mu, \gamma\) if and only if \(X = (h, H^t | e, E^t)^t\) solves the so-called augmented system \((P + V)X = 0\) and the scalar fields \(e, h\) vanish, where

\[
(2.1) \quad P := \begin{pmatrix} D_\alpha & D_\beta \\ D_\beta & -D_x \end{pmatrix}, \quad V := \begin{pmatrix} \omega \mu I_3 \\ \omega \gamma I_3 \end{pmatrix},
\]

with \(D := 1/i\nabla\), \(\nabla\) denoting the gradient operator, and \(\nabla \cdot\) the divergence operator. Further, \(X\) solves \((P + V)X = 0\) if and only if \(Y = \text{diag}(\mu^{1/2} I_4, \gamma^{1/2} I_4) X\) solves the rescaled system \((P + W)Y = 0\) with

\[
(2.2) \quad W := \kappa I_8 + \frac{1}{2} \begin{pmatrix} D_\beta & D_\alpha \\ D_\alpha & -D_\beta \end{pmatrix},
\]

where \(\times\) denotes the cross product. Define the terms

\[
Q := WP - PW^t - WW^t, \quad \tilde{Q} := W^*P - P\tilde{W} - W^*\tilde{W}.
\]

It can be checked that the expressions \(WP - PW^t\) and \(W^*P - P\tilde{W}\) are zeroth-order. Therefore, the second-order operators

\[
-\Delta I_8 + Q = (P + W)(P - W^t), \quad -\Delta I_8 + \tilde{Q} = (P + W^*)(P - \tilde{W}),
\]

which do not contain first-order terms, are Schrödinger-type. In [16, 17] a further zeroth-order operator \(Q'\) is considered which it is not used in this work.

Note that if \((-\Delta I_8 + Q)Z = 0\) and \(X := \text{diag}(\mu^{-1/2} I_4, \gamma^{-1/2} I_4) (P - W^t)Z\) then \((P + V)X = 0\). If additionally the scalar fields in \(X\) are identically zero, the vector fields in \(X\) give the electromagnetic fields verifying Maxwell equations. Moreover, if \(\tilde{Z}\) solves \((-\Delta I_8 + \tilde{Q})\tilde{Z} = 0\) then \(\tilde{Y} := (P - \tilde{W})\tilde{Z}\) is solution to \((P + W^*)\tilde{Y} = 0\).

Finally, a new matrix Schrödinger potential is introduced, namely \(\tilde{Q}\) in (2.4), satisfying Lemma 2.1 below.

Specify explicitly the \((\kappa, \alpha, \beta)\)-dependence of \(W, W^t\) and \(W^*\) by writing

\[
W = W(\kappa, \alpha, \beta), \quad W^t = W^t(\kappa, \alpha, \beta), \quad W^* = W^*(\kappa, \alpha, \beta).
\]

A straightforward computation gives \(W^*(\kappa, \alpha, \beta) = -W^t(-\pi, \alpha, \beta)\). Since the relation among \(\kappa, \alpha\) and \(\beta\) is not involved in the proof of the fact that \(-PW^tWP\) is zeroth-order, we deduce that \(-PW^t(-\pi, \alpha, \beta) + W(-\pi, \alpha, \beta)P\) is zeroth-order. Thus, the matrix operator

\[
(P + W(-\pi, \alpha, \beta))(P - W^t(-\pi, \alpha, \beta)) = -\Delta I_8 + \tilde{Q}
\]
is Schrödinger-type, where
\[
(2.4) \quad \tilde{Q} := -PW^t(-\kappa, \alpha) + W(-\kappa, \alpha) P - W(-\kappa, \alpha) W^t(-\kappa, \alpha)
\]
is zeroth-order. We deduce the following

**Lemma 2.1.** Assume \( \tilde{Y} \in H^1(\Omega; \mathbb{C}^8) \). If \((P + W^*)\tilde{Y} = 0\) then \((-\Delta I_8 + \tilde{Q}) \tilde{Y} = 0\).

**Notation.** In the rest of the manuscript, for two pairs of coefficients \( \mu_j, \gamma_j \) with \( j = 1, 2 \), we will write \( Q_j, \tilde{Q}_j, W_j \) to refer to the zeroth-order matrix operators \( Q, \tilde{Q}, W \) defined in \((2.3), (2.4), (2.2)\), respectively, for the case \( \mu = \mu_j, \gamma = \gamma_j \).

3. **The special solutions**

In this section we recall the almost exponentially growing solutions \( Z, Y \) constructed in \([16]\) for the systems \((-\Delta I_8 + Q) Z = 0, (P + W^*) Y = 0\) based on ideas of the papers \([64], [10], [54], [38]\). Here the coefficients \( \mu_j, \gamma_j \) \( (j = 1, 2) \) under Theorem \([11]\)'s conditions have to be considered extended to the whole Euclidean space \( \mathbb{R}^3 \). We denote the extended coefficients in the same manner \( \mu_j, \gamma_j \). The extensions fulfill the properties as follows:

1. They are Whitney type (see \([62]\) for their construction).
2. The extensions preserve the regularity and a priori conditions stated in Definition \([14]\).
3. The extended functions \( \mu_j, \gamma_j \) satisfy that \( \mu_j = \mu_0, \gamma_j = \varepsilon_0 \) outside a ball \( B(O, \rho) \) centered at origin \( O \) with radius \( \rho > 0 \) such that \( \overline{\Omega} \subset B(O, \rho) \), where \( \mu_0, \varepsilon_0 > 0 \) are constants.
4. Denoting likewise the matrices \( Q, \tilde{Q} \) obtained by replacing the coefficients by their extensions, the matrices \( \omega^2 \varepsilon_0 \mu_0 I_8 + Q, \omega^2 \varepsilon_0 \mu_0 I_8 + \tilde{Q} \) are compactly supported in \( B(O, \rho) \).

These properties allow Caro to prove Proposition 9 and Proposition 11 in \([16]\), which we present here for the reader’s convenience.

**Notation:** \( ||f||_{L^3_\delta} = \int_{\mathbb{R}^3} (1 + |x|^2)^{\delta} |f(x)|^2 \, dx \), for \( \delta \in \mathbb{R} \).

**Proposition 3.1.** Let \(-1 < \delta < 0\) and \( \zeta \in \mathbb{C}^3 \) with \( \zeta \cdot \zeta = \omega^2 \varepsilon_0 \mu_0 \). Assume

\[
|\zeta| > C(\delta, \rho) \left( \sum_{j=1,2} ||\omega^2 \varepsilon_0 \mu_0 + q_j||_{L^\infty(B_\rho)} + ||\omega^2 \varepsilon_0 \mu_0 I_8 + Q||_{L^\infty(B_{\rho};\mathcal{M}_{8\times 8})} \right),
\]

where \( \mathcal{M}_{8\times 8} \) denotes the space of \( 8 \times 8 \) matrices with complex entries, and

\[
q_1 = -\frac{1}{2} \Delta \beta - \kappa^2 - \frac{1}{4} (D\beta \cdot D\beta), \quad q_2 = -\frac{1}{2} \Delta \alpha - \kappa^2 - \frac{1}{4} (D\alpha \cdot D\alpha).
\]

Then there exists a solution
\[
Z(x, \zeta) = e^{i\zeta \cdot x} (L(\zeta) + R(x, \zeta)),
\]
to $(-\Delta I_8 + Q)Z = 0$ in $\mathbb{R}^3$ with $Z|_\Omega \in H^2(\Omega; \mathbb{C}^8)$, where

$$L(\zeta) = \frac{1}{|\zeta|} \begin{pmatrix} \zeta \cdot A_1 \\ \omega \varepsilon_0 \mu_0^{1/2} B_1 \\ \zeta \cdot B_1 \\ \omega \varepsilon_0 \mu_0^{1/2} A_1 \end{pmatrix}$$

with $A_1, B_1$ constant complex vector fields, and

$$\|R\|_{L^2_\Omega(\mathbb{R}^3; \mathbb{C}^8)} \leq \frac{C(\delta, \rho)}{|\zeta|} \|L\| \|\omega^2 \varepsilon_0 \mu_0 I_8 + Q\|_{L^\infty(B_\rho; M_8 \times M_8)}.$$

Furthermore, $Y := (P - W^*)Z$ solves $(P + W)Y = 0$ in $\mathbb{R}^3$ and has the form

$Y = (0 \mu^{1/2} H^t | 0 \gamma^{1/2} E^t)^t,$

with $E, H$ solutions of (1.1) in $\mathbb{R}^3$.

**Proposition 3.2.** Let $\zeta \in \mathbb{C}^3$ with $\zeta \cdot \zeta = \omega^2 \varepsilon_0 \mu_0$ and

$$|\zeta| > C(\rho) \left( \|\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q}\|_{L^\infty(B_\rho; M_8 \times M_8)} \right)$$

Then there exists a solution

$$\hat{Y}(x, \zeta) = e^{i\zeta \cdot x} (M(\zeta) + S(x, \zeta))$$

to $(P + W^*)\hat{Y} = 0$ in $\mathbb{R}^3$ with $\hat{Y}|_\Omega \in H^1(\Omega; \mathbb{C}^8)$,

$$M(\zeta) = \frac{1}{|\zeta|} \begin{pmatrix} \zeta \cdot A_2 \\ -\zeta \times A_2 \\ \zeta \cdot B_2 \\ \zeta \times B_2 \end{pmatrix},$$

where $A_2, B_2$ are constant complex vector fields, and

$$\|S\|_{L^2(\Omega; \mathbb{C}^8)} \leq \frac{C(\rho, \Omega)}{|\zeta|} \left( \|\omega^2 \varepsilon_0 \mu_0 I_8 + \hat{Q}\|_{L^\infty(B_\rho; M_8 \times M_8)} + \|W\|_{L^\infty(\Omega; M_8 \times M_8)} \right).$$

### 4. An orthogonality identity

This section is aimed at proving an orthogonality identity given by Proposition 4.2 involving solutions on the open set $\Omega$ to certain matrix partial differential equations whose traces contain information supported on $\bar{\Omega}$.

**Lemma 4.1.** Let $\mu_j, \gamma_j$ belong to $C^{0,1}(\bar{\Omega})$ for $j = 1, 2$. Let

$Y_1 = (0 \mu^{1/2} H_1^t | 0 \gamma^{1/2} E_1^t)^t,$

with $E_1, H_1 \in H(\Omega; \text{curl})$ solutions of

$$\nabla \times H_1 + i\omega \gamma_1 E_1 = 0, \tag{4.1}$$

$$\nabla \times E_1 - i\omega \mu_1 H_1 = 0, \tag{4.2}$$
in $\Omega$ such that $N \times E_1 = 0$ on $\Gamma_c$. In addition, suppose that
\[ Y_2 = (f^1 (u^1)^t | f^2 (u^2)^t)^t \]
is a solution to $(P + W^*_2)Y_2 = 0$ in $\Omega$ with $f^j \in H^1(\Omega)$, $w^j \in H(\Omega; \text{curl})$ and $f^1 = N \times u^2 = 0$ on $\Gamma_c$. Hence, for any pair $E_2, H_2$ in $H(\Omega; \text{curl})$ of solutions to
\[
\begin{cases}
\nabla \times H_2 + i \omega \gamma_2 E_2 = 0, \\
\nabla \times E_2 - i \omega \mu_2 H_2 = 0,
\end{cases}
\]
in $\Omega$ such that $N \times E_2|_{\partial \Omega} = 0$ on $\Gamma_c$, the following estimate holds:
\[
\|(Y_1|PY_2)_{\Omega} - (PY_1|Y_2)_{\Omega}\|
\leq C \left( \|N \times (E_1 - E_2)|_{\partial \Omega}\|_{TH_0(\Gamma)} + \|N \times (H_1 - H_2)|_{\Gamma}\|_{TH(\Gamma)} \right)
\times \left( \left\| \mu_2^{-1/2} \right\|_{C^{0,1}(\Gamma)} \|f^2\|_{B^{1/2}(\Gamma)} + \left\| \gamma_2^{1/2} \right\|_{C^{0,1}(\Gamma)} \|N \times u^1\|_{TH(\Gamma)} + \left\| \mu_2^{-1/2} \right\|_{C^{0,1}(\Gamma)} \|N \times u^2\|_{TH_0(\Gamma)} \right)
+ C \left( \|N \times E_1\|_{TH_0(\Gamma)} + \|N \times H_1\|_{TH(\Gamma)} \right)
\times \left( \left\| \mu_1^{-1/2} - \mu_2^{-1/2} \right\|_{C^{0,1}(\Gamma)} \|f^2\|_{B^{1/2}(\Gamma)} + \left\| \gamma_1^{1/2} - \gamma_2^{1/2} \right\|_{C^{0,1}(\Gamma)} \|N \times u^1\|_{TH(\Gamma)} + \left\| \mu_1^{-1/2} - \mu_2^{-1/2} \right\|_{C^{0,1}(\Gamma)} \|N \times u^2\|_{TH_0(\Gamma)} \right).
\]

Lemma 4.1 follows from the proof of Lemma 3.3 in [17] by making the solutions $Y_1, Y_2$ on $\Omega$ play the role of $\mathcal{Y}_1, \mathcal{Y}_2$ on $U$ in Lemma 3.3 from [17]. This is achieved imposing directly to $Y_1, Y_2$ the appropriate boundary conditions on $\partial \Omega$, namely the tangential component of the electric field appearing in the structure of $Y_1$ vanishes on the inaccessible part of the boundary, and concerning $Y_2$, the trace of the first component and the tangential component of the second vector field also vanish on the inaccessible part of the boundary. In Lemma 3.3 from [17] such boundary conditions for $\mathcal{Y}_1, \mathcal{Y}_2$ come from a reflection argument using the special geometric conditions assumed to $\partial U \setminus \overline{\Gamma}$ there, which can not be used here.

The proof of Lemma 4.1 uses Lemma 2.2, Lemma 2.4, Lemma 2.5 and Lemma 2.6 in [17]. Lemma 4.1’s proof is omitted since, up to these comments, is identical to Lemma 3.3’s proof in [17].

**Proposition 4.2.** Let $\mu_j, \gamma_j$ be an admissible pair of coefficients ($j = 1, 2$) such that $C^1_\Gamma = C^2_\Gamma$. Additionally, suppose $\mu_1 = \mu_2$, $\gamma_1 = \gamma_2$, $\partial_{x_l} \mu_1 = \partial_{x_l} \mu_2$ and $\partial_{x_l} \gamma_1 = \partial_{x_l} \gamma_2$ on $\overline{\Gamma}$ for $l = 1, 2, 3$. Then
\[
((Q_1 - Q_2)Z_1|Y_2)_{\Omega} = 0
\]
holds for any \( Z_1 \in H^1(\Omega; \mathbb{C}^8) \) solving \( (P - W'_1)Z_1 = Y_1 \) in \( \Omega \) with
\[
Y_1 = (0 \quad \mu_1^{1/2} H_1^t | 0 \quad \gamma^{1/2} E_1^t)^t,
\]
where \( E_1, H_1 \in H(\Omega; \text{curl}) \) are solutions of
\[
\begin{align*}
\nabla \times H_1 + i \omega \gamma_1 E_1 &= 0, \\
\nabla \times E_1 - i \omega \mu_1 H_1 &= 0,
\end{align*}
\]
in \( \Omega \) and \( N \times E_1|_{\partial \Omega} = 0 \) on \( \Gamma_c \), and for any \( Y_2 \in H^1(\Omega; \mathbb{C}^8) \) verifying \( (P + W'_2)Y_2 = 0 \) in \( \Omega \) such that \( Y_2|_{\partial \Omega} = 0 \) on \( \Gamma_c \).

**Proof of Proposition 4.2** Following the proof of Proposition 1 in [17], we obtain the identity
\[
((Q_1 - Q_2)Z_1|Y_2)_\Omega = ((W'_1 - W'_2)Z_1|P_N Y_2)_{\partial \Omega} + (Y_1|P Y_2)_{\Omega} - (P Y_1|Y_2)_{\Omega},
\]
whose proof involves integration by parts and the relations \( (P + W'_2)Y_2 = 0, \ Y_1 = (P - W'_1)Z_1, (P + W'_1)Y_1 = 0 \). Here, denote
\[
P_N := \frac{1}{i} \begin{pmatrix} \begin{array}{cc} N \cdot & N \times \\ N \times & -N \times \end{array} \end{pmatrix}.
\]
Since \( \mu_1 = \mu_2, \gamma_1 = \gamma_2, \partial_{x_l} \mu_1 = \partial_{x_l} \mu_2 \) and \( \partial_{x_l} \gamma_1 = \partial_{x_l} \gamma_2 \) on \( \Gamma \) (for \( l = 1, 2, 3 \)), it follows that
\[
W'_1 - W'_2 = 0 \quad \text{on} \ \Gamma.
\]
From the fact that \( Y_2|_{\partial \Omega} \) is supported on \( \Gamma \), we have
\[
P_N Y_2|_{\Gamma_c} \equiv 0.
\]
By \((4.6), (4.7), \)
\[
((W'_1 - W'_2)Z_1|P_N Y_2)_{\partial \Omega} = 0.
\]
Since \( \mu_1 = \mu_2 \) and \( \gamma_1 = \gamma_2 \) on \( \Gamma \),
\[
\begin{align*}
\| \mu_1^{1/2} - \mu_2^{1/2} \|_{C^{0,1}(\Gamma)} &= \| \mu_1^{-1/2} - \mu_2^{-1/2} \|_{C^{0,1}(\Gamma)} = \| \gamma_1^{1/2} - \gamma_2^{1/2} \|_{C^{0,1}(\Gamma)} \\
&= \| \gamma_1^{-1/2} - \gamma_2^{-1/2} \|_{C^{0,1}(\Gamma)} = 0.
\end{align*}
\]
Since \( (N \times E_1|_{\partial \Omega}, N \times H_1|_{\Gamma}) \in C^1_\Gamma = C^2_\Gamma \), there exist solutions \( E_2, H_2 \) in \( H(\Omega; \text{curl}) \) to the Maxwell system \((4.3)\) such that \( N \times E_2|_{\partial \Omega} \) is supported on \( \Gamma \) and
\[
(N \times E_1|_{\partial \Omega}, N \times H_1|_{\Gamma}) = (N \times E_2|_{\partial \Omega}, N \times H_2|_{\Gamma}.
\]
Applying Lemma 4.1 to \( Y_1, Y_2, E_2, H_2 \) and by \((4.9) - (4.10), (4.11), \)
\[
(Y_1|P Y_2)_{\Omega} - (P Y_1|Y_2)_{\Omega} = 0.
\]
By identities (1.8), (4.12), we conclude \(((Q_1 - Q_2)Z_1|Y_2)\Omega = 0.\]

\[\]

5. Density and unique continuation results

In the remainder of the paper, let \(\Omega\) and \(\Gamma\) be as in Definition 1.1. If \(E, H \in H(\Omega; \text{curl})\) solve system (1.1) in \(\Omega\) for certain coefficients \(\mu, \gamma\), then \(E, H\) are also solutions to the following second order system:

\[
\nabla \times (\mu^{-1}\nabla \times E) - \omega^2 \gamma E = 0, \\
\nabla \times (\gamma^{-1}\nabla \times H) - \omega^2 \mu H = 0.
\]

Notation. For known coefficients \(\mu, \gamma\) and frequency \(\omega > 0\), notation \(L\) will refer to the following Helmholtz-type vector second order differential operator defined in the sense of distributions for \(U \in (C_0^\infty)'(\Omega; \mathbb{C}^3)\) by

\[
LU = \nabla \times (\mu^{-1}\nabla \times U) - \omega^2 \gamma U.
\]

For Lipschitz continuous functions \(\mu, \gamma\) on \(\Omega\), assuming \(\mu\) to be bounded from below, \(LU\) is an \(L^2\) vector field if \(U \in H(\Omega; \text{curl})\) and \(\nabla \times (\nabla \times U) \in L^2(\Omega; \mathbb{C}^3)\), since \(\nabla \times (\mu^{-1}\nabla \times U) = (\nabla \mu^{-1}) \times (\nabla \times U) + \mu^{-1}\nabla \times (\nabla \times U)\).

Note the following integration by parts formula for any \(E, F \in C^\infty(\Omega; \mathbb{C}^3)\):

\[
\int_\Omega (LE) \cdot F \, dx = \int_\Omega E \cdot (LF) \, dx \\
- \int_{\partial \Omega} \mu^{-1} (E \cdot (N \times (\nabla \times F))) + (\nabla \times E) \cdot (N \times F)) \, ds.
\]

Next, the density result for the scalar Schrödinger equation given by Lemma 2 in [2] is adapted to Schrödinger-type matrix equations (Proposition 5.1) and the second order operator \(L\) (Proposition 5.2).

**Proposition 5.1.** Let \(\Omega'\) be an open subset of \(\mathbb{R}^3\) with \(C^2\) boundary. Assume \(\Omega' \subset \subset \Omega\) and \(\Omega \setminus \Omega'\) is connected. Let \(\tilde{Q}\) be the zeroth-order \(8 \times 8\) matrix operator defined in (2.4) for an admissible pair of coefficients \(\mu, \gamma\). Then the set

\[
\tilde{K}(\Omega) := \{g \in H^2(\Omega; \mathbb{C}^8) : (-\Delta f + \tilde{Q})g = 0 \text{ in } \Omega, \ g|_{\partial \Omega} = 0 \text{ on } \Gamma_c\}
\]

is dense in the space \(K(\Omega) := \{v \in H^2(\Omega; \mathbb{C}^8) : (-\Delta f + \tilde{Q})v = 0 \text{ in } \Omega\}\) with respect to the topology in \(L^2(\Omega'; \mathbb{C}^8)\). Here, \(g, v\) denote \(8 \times 1\) vector fields.

**Proposition 5.2.** Let \(\Omega'\) be an open subset of \(\mathbb{R}^3\) with \(C^2\) boundary. Assume \(\Omega' \subset \subset \Omega\) and \(\Omega \setminus \Omega'\) is connected. Let \(\tilde{L}\) be the differential operator defined in (5.1) for an admissible pair of coefficients \(\mu, \gamma\). Then the set \(\tilde{N}(\Omega)\) of vector functions \(\tilde{E} \in H(\Omega; \text{curl})\) such that \(\nabla \times (\nabla \times \tilde{E}) \in L^2(\Omega; \mathbb{C}^3)\), \(\tilde{L} \tilde{E} = 0 \text{ in } \Omega\), \(N \times \tilde{E}|_{\partial \Omega} = 0 \text{ on } \Gamma_c\), is dense in the space

\[
N(\Omega) := \{E \in H(\Omega; \text{curl}) : \nabla \times (\nabla \times E) \in L^2(\Omega; \mathbb{C}^3), \ L E = 0 \text{ in } \Omega\}
\]

with respect to the topology in \(L^2(\Omega'; \mathbb{C}^3)\).
The following unique continuation principles, Lemma 5.3 and Lemma 5.4, are used to prove Proposition 5.1 and Proposition 5.2.

**Lemma 5.3. (Unique continuation principle for matrix Schrödinger-type equations)** Let \( \bar{Q} \) be the zeroth-order \( 8 \times 8 \) matrix operator defined in (2.3) for a pair \( \mu, \gamma \) of admissible coefficients. Assume \( \Omega' \subset \subset \Omega \) and \( \Omega \setminus \Omega' \) is connected with \( \partial \Omega' \subset C^2 \). Hence,

- i) If \( u \in H^2(\Omega; \mathbb{C}^8) \) satisfies \((-\Delta I_8 + \bar{Q})u = 0 \) in \( \Omega \) and \( u = 0 \) on \( B \) for some open ball \( B \) such that \( \overline{B} \subset \Omega \), then \( u = 0 \) in \( \Omega \).
- ii) Suppose \( u \in H^2(\Omega \setminus \overline{\Omega'}; \mathbb{C}^8) \) verifies \((-\Delta I_8 + \bar{Q})u = 0 \) in \( \Omega \setminus \overline{\Omega'} \), \( u = 0 \) on \( \partial \Omega \), \( (\frac{\partial}{\partial \nu} I_8)u \big|_{\partial \Omega} = 0 \) on \( \Gamma \), where \( \Gamma \) is a smooth proper non-empty open subset of \( \partial \Omega \). Then \( u = 0 \) on \( \Omega \setminus \overline{\Omega'} \).

**Lemma 5.4. (Unique continuation principle for \( L \))** Let \( \mathcal{G} \) be a nonempty, open, bounded, connected subset of \( \mathbb{R}^3 \) with Lipschitz boundary \( \partial \mathcal{G} \). Let \( L \) denote the operator (5.1) for scalar functions \( \mu, \gamma \in C^1(\mathcal{G}) \) with \( \mu \geq C \), \( \Re \gamma \geq C \) in \( \mathcal{G} \) for some constant \( C > 0 \).

Further, assume \( U \in H(\mathcal{G}; \mathrm{curl}) \) and \( \nabla \times (\nabla \times U) \in L^2(\mathcal{G}; \mathbb{C}^3) \). Therefore:

- i) If \( LU = 0 \) in \( \mathcal{G} \) and \( U = 0 \) in \( B \) for some open ball \( B \) such that \( \overline{B} \subset \mathcal{G} \), then \( U = 0 \) in \( \mathcal{G} \).
- ii) Let \( \Gamma' \) denote a nonempty, smooth, open subset of \( \partial \mathcal{G} \). If \( LU = 0 \) in \( \mathcal{G} \), \( N \times U \big|_{\partial \mathcal{G}} = 0 \) on \( \Gamma' \) and \( N \times (\nabla \times U) \big|_{\partial \mathcal{G}} = 0 \) on \( \Gamma' \), then \( U = 0 \) in \( \mathcal{G} \). Here \( N \) also denotes the outward unit vector field normal to \( \partial \mathcal{G} \).

**Proof of Lemma 5.3** Part i) of Lemma 5.3 can be proven by trivially rewriting Theorem 6.5.1’s proof in [25] for the vector case taking \( W_1 = \left\| \bar{Q} \right\|_{L^\infty(\Omega; \mathbb{M}_{8 \times 8})} \), \( W_2 = 0 \), \( C = \emptyset \). Part ii) follows from part i), remarking that the boundary conditions on \( \Gamma \) guarantee that the extension of the solution by zero on a neighbourhoood \( \Gamma_x \) in \( \mathbb{R}^3 \) such that \( \Gamma_x \cap \Omega = \emptyset \) and \( \overline{\Gamma_x} \cap \overline{\Omega'} \) is an open subset of \( \Gamma \) with respect to the relative topology on \( \partial \Omega \) induced by the Euclidean topology of \( \mathbb{R}^3 \), satisfies the same equation and maintains the \( H^2 \)-regularity on \( \text{int}(\overline{\Omega \setminus \overline{\Omega'}}) \). Indeed, the kernel of the trace operator \( (u \big|_{\partial \mathcal{G}}, (\partial u / \partial N) \big|_{\partial \mathcal{G}}) \) defined for \( u \in H^2(\mathcal{G}) \), is the closure of \( C^1(\mathcal{G}) \) in \( H^2(\mathcal{G}) \) (usually denoted by \( H^2_0(\mathcal{G}) \)), for any domain \( \mathcal{G} \) with \( C^{1,1} \) boundary \( \partial \mathcal{G} \) (see [44] or e.g. [29] Theorem 1.5.1.5). For clarity Figure 1 illustrates the sets \( \Gamma_x, \Omega, \Omega' \) in the plane (although they must be considered in \( \mathbb{R}^3 \)).

**Proof of Lemma 5.4** Under the conditions of part i), define \( V := (i \omega \mu)^{-1} \nabla \times U \) and check that \( (U, V) \) solves in \( \mathcal{G} \) the Maxwell equations \( \nabla \times V + i \omega \gamma U = 0 \), \( \nabla \times U - i \omega \mu V = 0 \), and \( U, V \in H(\mathcal{G}; \mathrm{curl}) \) with \( \nabla \cdot U, \nabla \cdot V \in L^2(\mathcal{G}) \). By [23] Chapter I, Corollary 2.10, \( U, V \in H^1_{\text{loc}}(\mathcal{G}; \mathbb{C}^3) \). Consider another open ball \( B' \) with \( B \cap B' \neq \emptyset \) and \( B' \subset \mathcal{G} \). Since the restrictions of \( U, V \) to \( B' \) are in \( H^1(B'; \mathbb{C}^3) \), from the unique continuation result across \( C^2 \)-surfaces by Eller and
Yamamoto [26, Corollary 1.2] for the Maxwell system with $C^1$ coefficients, we deduce that $U = V = 0$ on $B'$. Propagating this argument we conclude that $U$ and $V$ vanish on any neighbourhood in $\mathcal{G}$. This proves part i).

Let $\Gamma_\varepsilon$ be a nonempty, open, connected subset of $\mathbb{R}^3$ with Lipschitz boundary such that $\Gamma_\varepsilon \cap \mathcal{G} = \emptyset$, $\Gamma_\varepsilon \cap \overline{\mathcal{G}}$ is an open subset of $\Gamma'$ with respect to the relative topology on $\partial \mathcal{G}$ induced by the Euclidean topology of $\mathbb{R}^3$. Figure 1 with $\Omega = \mathcal{G}$, $\Gamma = \Gamma'$ illustrates the choice of $\Gamma_\varepsilon$ in the plane. The conditions of part ii) guarantee that the extension $\tilde{U}$ of $U$ by zero on $\Gamma_\varepsilon$ verifies $\tilde{U} \in H(\mathcal{G}'; \text{curl})$, $\nabla \times (\nabla \times \tilde{U}) \in L^2(\mathcal{G}' ; \mathbb{C}^3)$ and $L \tilde{U} = 0$ in $\mathcal{G}'$, where $\mathcal{G}' := \text{int}(\mathcal{G} \cup \overline{\Gamma_\varepsilon})$. This property follows from the fact that the $H(\text{curl})$-vector functions on a bounded, Lipschitz domain that can be approximated by smooth compactly supported functions in $H(\text{curl})$-norm are exactly those ones with zero tangential trace (see e.g. [49, Theorem 3.33] for details). By part i) of this Lemma 5.4, $\tilde{U} = 0$ in $\mathcal{G}'$. In particular, $U = 0$ in $\mathcal{G}$.

\[ \square \]

Regarding the aforementioned result in [26], note that a counterexample for the stationary Maxwell system with coefficients in the Hölder class $C^\alpha$ for every $\alpha < 1$ is provided in [24] by Demchenko.

---

**Proof of Proposition 5.1**

Following the lines of Lemma 2’s proof in [2], suppose $v \in K(\Omega)$ satisfies $(g|v)_{\Omega'} = \int_{\Omega'} v^* g \, dx = 0$ for any $g \in \tilde{K}(\Omega)$. We are going to prove that $v = 0$ in $\Omega$.

Consider the Dirichlet Green’s function $G$ in $\Omega$ verifying for $x \in \Omega$,
\[ (-\Delta_y I_8)G(x, y) + G(x, y)\widetilde{Q}(y) = \delta(y - x)I_8, \text{ for } y \in \Omega, \ G(x, y) = 0, \text{ for } y \in \partial\Omega, \]

where \( \delta \) denotes the Dirac delta function with pole at the origin and \( I_8 \) the \( 8 \times 8 \) identity matrix. For \( g \in \tilde{K}(\Omega) \) and \( x \in \Omega \) we have, by Green’s formula,

\[
g(x) = \int_{\Omega} (\delta(y - x)I_8)g(y)dy = \int_{\Omega} (-(-\Delta_y I_8)G(x, y) + G(x, y)\widetilde{Q}(y))g(y)dy
\]

\[
= \int_{\Omega} G(x, y)(-\Delta_y I_8 + \widetilde{Q}(y))g(y)dy - \int_{\partial\Omega} \left( \left( \frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) g(y) ds(y)
\]

\[
= -\int_{\Gamma} \left( \left( \frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) g(y) ds(y).
\]

In particular note that for \( x \in \Omega \),

\[
U(x) = \int_{\Omega} G(x, y)F(y)dy \text{ provided that } (-\Delta I_8 + \widetilde{Q})U = F \text{ in } \Omega \text{ and } U = 0 \text{ on } \partial\Omega.
\]

By Fubini’s theorem,

\[
\int_{\Gamma} \int_{\Omega'} v(x)^* \left( \left( \frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) dx g(y) ds(y)
\]

\[
= \int_{\Omega'} v(x)^* \int_{\Gamma} \left( \left( \frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) g(y) ds(y) dx
\]

\[
= -\int_{\Omega'} v(x)^* g(x) dx = 0.
\]

Thus, for \( y \in \Gamma \),

\[
\int_{\Omega'} v(x)^* \left( \left( \frac{\partial}{\partial \nu(y)} I_8 \right) G(x, y) \right) dx = 0.
\]

Define the vector field \( u \) by

\[
u(y)^* := \int_{\Omega'} v(x)^* G(x, y) dx.
\]

Since \( G(x, y) = 0 \) for \( y \in \partial\Omega \) and \( x \in \Omega \), \( u(y) = 0 \) for \( y \in \partial\Omega \). By (5.3),

\[
(\frac{\partial}{\partial \nu(y)} I_8) u(y) = 0 \quad \text{for } y \in \Gamma.
\]

Since \( (-\Delta I_8 + \widetilde{Q})^* u = 0 \) in \( \Omega \setminus \Omega' \), \( u |_{\partial\Omega} = 0 \) and by (5.4), it follows that \( u = 0 \) in \( \Omega \setminus \Omega' \) by the unique continuation principle (Lemma 5.3). In particular,

\[
u(y)^* := \int_{\Omega'} v(x)^* G(x, y) dx = 0 \quad \text{on } \partial\Omega'.
\]
UNIQUENESS FOR AN INVERSE PROBLEM WITH PARTIAL DATA

Note that \((-\Delta I_8 + (\tilde{Q})^*)u = v\) in \(\Omega'\). Now, we can write

\[
\int_{\Omega'} v(y)^*v(y) \, dy = \int_{\Omega'} ((-\Delta y I_8)u(y)^* + u(y)^*\tilde{Q}(y))v(y) \, dy
\]

\[= -\int_{\Omega'} u(y)^*(\Delta I_8) v(y) \, dy \]

\[+ \int_{\partial\Omega'} \left( u(y)^*\left( \frac{\partial}{\partial \nu(y)} I_8 \right) v(y) - \left( \frac{\partial}{\partial \nu(y)} I_8 \right) u(y)^* v(y) \right) ds(y)\]

\[+ \int_{\Omega'} u(y)^*\tilde{Q}(y) v(y) \, dy\]

\[= \int_{\Omega'} u(y)^*(-\Delta I_8 + \tilde{Q}(y)) v(y) \, dy = 0,
\]

where identity (5.5) follows from Green’s formula. Hence \(v = 0\) in \(\Omega'\). Since, \((-\Delta I_8 + \tilde{Q})v = 0\) in \(\Omega\), by unique continuation (Lemma 5.3), \(v = 0\) in \(\Omega\).

**Proof of Proposition 5.2** Consider the 3 \(\times\) 3 matrix valued Green’s function \(\tilde{G}\) given by

\[
L_y \tilde{G}(x, y)^t = \delta(y - x) I_3, \quad y \in \Omega,
\]

\[
N_y \times \tilde{G}(x, y)^t = 0, \quad y \in \partial\Omega,
\]

where \(x \in \Omega\). Here, \(L_y\) stands for the operator \(L\) acting in the variable \(y\), and \(N_y\) stands for the outward unit normal vector to the boundary \(\partial\Omega\) at the point \(y \in \partial\Omega\).

Using the integration by parts formula (5.2) one can check that any \(\tilde{E} \in \tilde{N}(\Omega)\) can be represented as follows for \(x \in \Omega\):

\[
\tilde{E}(x) = \int_{\Omega} \delta(y - x) I_3 \tilde{E}(y) \, dy = \int_{\Omega} (L_y \tilde{G}(x, y)^t)^t \tilde{E}(y) \, dy
\]

\[= \int_{\Omega} \tilde{G}(x, y) L_y \tilde{E}(y) \, dy\]

\[+ \mu(y)^{-1} \left( \left( N_y \times \tilde{G}(x, y)^t \right)^t \left( N_y \times (\nabla \times \tilde{E}) \right) \right) \, ds(y)\]

\[+(\nabla_y \times \tilde{G}(x, y)^t)^t \left( N_y \times \tilde{E} \right) ds(y)\]

\[= -\int_{\Gamma} \mu(y)^{-1} (\nabla_y \times \tilde{G}(x, y)^t)^t \left( N_y \times \tilde{E} \right) ds(y).
\]

**Remark:** We deduce that if \(LU = F\) in \(\Omega\) and \(N \times U = 0\) on \(\partial\Omega\) then \(U(x) = \int_{\Omega} \tilde{G}(x, y) F(y) \, dy\), for \(x \in \Omega\).
Fix $E \in N(\Omega)$ such that $\int_{\Omega'} \overline{E} \cdot \tilde{E} \, dx = 0$ for any $\tilde{E} \in \tilde{N}(\Omega)$. By (5.6)-(5.7), we have for every $\tilde{E} \in \tilde{N}(\Omega)$,

\begin{equation}
0 = \int_{\Omega'} E^*(x) \overline{\tilde{E}}(x) \, dx
\end{equation}

\begin{equation}
= - \int_{\Omega'} E^*(x) \mu(y)^{-1} (\nabla_y \times \tilde{G}(x, y))_t^t (N_y \times \tilde{E}(y)) \, ds(y) \, dx
\end{equation}

\begin{equation}
= - \int_{\Gamma} \mu(y)^{-1} \int_{\Omega'} E^*(x) (\nabla_y \times \tilde{G}(x, y))_t^t \, dx (N_y \times \tilde{E}(y)) \, ds(y)
\end{equation}

Since in (5.8)-(5.9) the factor $N \times \tilde{E}$ can be chosen arbitrarily in the space $TH_0(\Gamma)$, we deduce that the tangential component of the term $\int_{\Omega'} (\nabla_y \times \tilde{G}(x, y)) \overline{E}(x) \, dx$ vanishes on $\Gamma$. In particular,

\begin{equation}
N_y \times \int_{\Omega'} (\nabla_y \times \tilde{G}(x, y)_t) \overline{E}(x) \, dx = 0 \quad \text{for a.e. } y \in \Gamma.
\end{equation}

Define $E'(y) := \int_{\Omega'} \tilde{G}(x, y)_t \overline{E}(x) \, dx$, for $y \in \Omega$. We have

\begin{equation}
L E' = \begin{cases}
\overline{E} & \text{in } \Omega', \\
0 & \text{in } \Omega \setminus \Omega'.
\end{cases}
\end{equation}

The boundary condition of the Green’s function implies that $N \times E'|_{\partial\Omega} = 0$ and by (5.10), $N \times (\nabla \times E')|_{\partial\Omega} = 0$ on $\Gamma$. By the uniqueness result stated in part ii) of Lemma 5.4 for $G = \Omega \setminus \Omega'$ and $\Gamma' = \Gamma$, it follows that $E' = 0$ in $\Omega \setminus \Omega'$. In particular,

\begin{equation}
E'|_{\partial\Omega'} = 0, \quad \nabla \times E'|_{\partial\Omega'} = 0.
\end{equation}

Now, we write

\[
\int_{\Omega'} E^*(y) E(y) \, dy = \int_{\Omega'} (L E')(y) \, (\nabla \times E)(y) \, dy
\]

\[
= \int_{\Omega'} (E'(y)) \, (L E)(y) \, dy
\]

\[
- \int_{\partial\Omega'} \mu(y)^{-1} ((E'(y))_t (N_y \times (\nabla \times E(y))) + (\nabla \times E'(y))_t (N_y \times E(y))) \, ds(y),
\]

where last identity follows from formula (5.2). Hence,

\[
\int_{\Omega'} E^*(y) E(y) \, dy = 0
\]

and $E = 0$ in $\Omega'$. Since $L E = 0$ in $\Omega$, we deduce by the unique continuation principle stated in part i) of Lemma 5.4 with $G = \Omega$, that $E = 0$ in $\Omega$. 

\hfill \Box
6. Proof of uniqueness

Here, the outline of [1], Section 3.2 in [16], Section 3.4 in [17] is adapted to prove Theorem 1.1.

Let $\omega > 0$ be the time-harmonic frequency. Assume $\mu_j, \gamma_j$ is an admissible pair of coefficients for each $j = 1, 2$, according to Definition 1.1 such that $\text{supp}(\mu_1 - \mu_2), \text{supp}(\gamma_1 - \gamma_2) \subseteq \Omega$. Let $\Omega'$ be an open subset of $\mathbb{R}^3$ with $C^2$ boundary such that $\Omega' \subset \subset \Omega, \Omega \setminus \overline{\Omega'}$ is connected and

\begin{equation}
\mu_1 = \mu_2 \quad \text{and} \quad \gamma_1 = \gamma_2 \quad \text{in} \quad \overline{\Omega} \setminus \Omega'.
\end{equation}

Suppose $C_1 = C_2$. The extended coefficients to $\mathbb{R}^3$ according to the extensions described in Section 3 will be written likewise, $\mu_j, \gamma_j$. Remember that $\mu_j = \mu_0, \gamma_j = \varepsilon_0$ outside the ball $B(O, \rho)$, where $\overline{\Omega} \subset B(O, \rho)$, and $\mu_0, \varepsilon_0$ are constants.

Let $j \in \{1, 2\}$. Define

\[ \alpha_j := \log \gamma_j, \quad \beta_j := \log \mu_j, \quad \kappa_j := \omega \mu_j^{1/2} \gamma_j^{1/2}, \]

\[ f := \chi_\Omega \cdot \left( \frac{1}{2} \Delta (\alpha_1 - \alpha_2) + \frac{1}{4} (\nabla \alpha_1 \cdot \nabla \alpha_1 - \nabla \alpha_2 \cdot \nabla \alpha_2) + (\kappa_2^2 - \kappa_1^2) \right), \]

\[ g := \chi_\Omega \cdot \left( \frac{1}{2} \Delta (\beta_1 - \beta_2) + \frac{1}{4} (\nabla \beta_1 \cdot \nabla \beta_1 - \nabla \beta_2 \cdot \nabla \beta_2) + (\kappa_2^2 - \kappa_1^2) \right), \]

where $\chi_\Omega$ denotes the characteristic function of $\Omega$.

Fix $\xi \in \mathbb{R}^3 \setminus 0$. Let $S^2$ denote the unit sphere in $\mathbb{R}^3$. Assume $\tau \geq 1$ and take $\eta_1, \eta_2 \in S^2$ with $\eta_2 \cdot \eta_1 = \eta_1 \cdot \xi = \eta_2 \cdot \xi = 0$, and

\[ \zeta_1 = -\frac{1}{2} \xi + i \left( \tau^2 + \frac{1}{4} \right)^{1/2} \eta_1 + (\tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2} \eta_2, \]

\[ \zeta_2 = \frac{1}{2} \xi - i \left( \tau^2 + \frac{1}{4} \right)^{1/2} \eta_1 + (\tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2} \eta_2. \]

Note that $\zeta_j \in \mathbb{C}^3$ satisfies $\zeta_j \cdot \zeta_j = \omega^2 \varepsilon_0 \mu_0$, and

\[ |\zeta_j| = (|\text{Re}(\zeta_j)|^2 + |\text{Im}(\zeta_j)|^2)^{1/2} = (|\xi|^2/2 + 2 \tau^2 + \omega^2 \varepsilon_0 \mu_0)^{1/2}. \]

Further, $\zeta_1 - \overline{\zeta_2} = -\xi$, and as $\tau \to \infty$,

\[ \frac{\zeta_1}{|\zeta_1|} = \frac{1}{\sqrt{2}} (i \eta_1 + \eta_2) + O(\tau^{-1}), \quad \frac{\zeta_2}{|\zeta_2|} = \frac{1}{\sqrt{2}} (-i \eta_1 + \eta_2) + O(\tau^{-1}), \]

where the implicit constants depend on $|\xi|$ (and $\omega, \varepsilon_0, \mu_0$).

Consider the special solutions

\[ Z_1(x, \zeta_1) = e^{i \zeta_1 \cdot x} (L_1(\zeta_1) + R_1(x, \zeta_1)), \quad Y_2(x, \zeta_2) = e^{i \zeta_2 \cdot x} (M_2(\zeta_2) + S_2(x, \zeta_2)) \]

from Proposition 3.1 and Proposition 3.2 applied to the case $\mu = \mu_1, \gamma = \gamma_1, \zeta = \zeta_1$ and $\mu = \mu_2, \gamma = \gamma_2, \zeta = \zeta_2$, respectively, so that $Z_1, Y_2$ solve $(-\Delta L_8 + Q_1) Z_1 = 0$,
\((P + W^*_2)Y_2 = 0\) in \(\mathbb{R}^3\). Choosing such solutions with \(B_j = 0\) and \(A_j\) such that
\[
\left(\frac{i \eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot A_1 = \left(\frac{i \eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot \overline{A_2} = 1,
\]
one obtains
\[(6.2) \quad ((Q_1 - Q_2)Z_1|Y_2)_\Omega = \hat{f}(\xi) + O(\tau^{-1}),\]
as \(\tau \to \infty\). Analogously, choosing \(Z_1, Y_2\) with \(A_j = 0\) and \(B_j\) such that
\[
\left(\frac{i \eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot B_1 = \left(\frac{i \eta_1}{\sqrt{2}} + \frac{\eta_2}{\sqrt{2}}\right) \cdot \overline{B_2} = 1,
\]
one can prove
\[(6.3) \quad ((Q_1 - Q_2)Z_1|Y_2)_\Omega = \hat{g}(\xi) + O(\tau^{-1}),\]
as \(\tau \to \infty\). In (6.2), (6.3) the implicit constant depends on \(M, \xi, \gamma, \rho, \omega, \varepsilon_0, \mu_0\).

Fix \(\varepsilon > 0\). For each choice of \(Z_1\), define \(Y_1 := (P - W^*_1)Z_1\). Hence, \((P + W^*_1)Y_1 = 0\) in \(\mathbb{R}^3\) and by Proposition 3.1 \(Y_1\) reads \(Y_1 = (0 \ \mu_1^{1/2}H_1 \ 0 \ \gamma_1^{1/2}E_1^*)^t\), with \(E_1, H_1\) solutions of
\[(6.4) \quad \nabla \times H_1 + i \omega \gamma_1 E_1 = 0,
\]
\[\nabla \times E_1 - i \omega \mu_1 H_1 = 0,
\]
in \(\mathbb{R}^3\). In particular, \(L_1E_1 = 0\) in \(\Omega\), where \(L_1\) denotes the second order operator \(L\) defined in 5.1 for \(\mu_1, \gamma_1\). By Proposition 3.1 \(Z_1|_\Omega \in H^2(\Omega; \mathbb{C}^8)\). Thus, by the Lipschitz regularity and the a priori bounds from below for \(\mu_1, \gamma_1\), and from equation (6.4) we deduce that \(E_1|_\Omega \in H(\Omega; \text{curl})\) and \(\nabla \times (\nabla \times E_1)|_\Omega \in L^2(\Omega; \mathbb{C}^3)\).

By Proposition 5.2, there exists \(\widetilde{E}_1 \in H(\Omega; \text{curl})\) such that \(\nabla \times (\nabla \times \widetilde{E}_1) \in L^2(\Omega; \mathbb{C}^3), \ L_1 \widetilde{E}_1 = 0\) in \(\Omega, N \times \widetilde{E}_1|_{\partial \Omega} = 0\) on \(\Gamma_c\), and \(\left\| E_1 - \widetilde{E}_1 \right\|_{L^2(\Omega; \mathbb{C}^3)} < \varepsilon\).

Due to the a priori condition \(\mu_1 \geq M^{-1}, \mu_1\) does not vanish. Define \(\widetilde{H}_1 := (1/i \omega \mu_1) \nabla \times \widetilde{E}_1\). Therefore \(\widetilde{E}_1, \widetilde{H}_1 \in H(\Omega; \text{curl})\) solve
\[
\nabla \times \widetilde{H}_1 + i \omega \gamma_1 \widetilde{E}_1 = 0,
\]
\[\nabla \times \widetilde{E}_1 - i \omega \mu_1 \widetilde{H}_1 = 0,
\]
in \(\Omega\). Define \(\widetilde{Y}_1 := (0 \ \mu_1^{1/2} \widetilde{H}_1^t \ 0 \ \gamma_1^{1/2} \widetilde{E}_1^*)^t\).

6.1. Invertibility of the Dirac-type operator \(P\).

Along this subsection the letters \(M, N, E\) refer to mathematical entities which are different from their meanings in the rest of the paper.

Jochen Br"uning and Matthias Lesch in [13] generalize the analysis of Dirac-type operators considered in the well-known paper by Atiyah, Patodi and Singer [4]. Concerning the general Dirac-type operators studied there on compact manifolds with boundary, in [13] Section 1.B] an operator \(D\) is introduced acting on sections
of a hermitian vector bundle $E$ over an open subset $M$ of a compact oriented Riemannian manifold $\tilde{M}$ such that its boundary $N = \partial M$ is a compact hypersurface in $\tilde{M}$. The authors call $\tilde{E}$ the vector bundle over $\tilde{M}$, and $E_N := \tilde{E} \upharpoonright N$. The differential operator $D$ is said to be of Dirac type if it is first order, symmetric and elliptic in $L^2(E)$ with domain $C_0^\infty(E)$ verifying that $D^2$ has scalar principal symbol given by the metric tensor.

Taking $M = \Omega'$ and $E = M \times \mathbb{C}^8$ the trivial bundle over $M$, each fiber equipped with the standard hermitian inner product of $\mathbb{C}^8$, the operator $P$ on $\Omega'$ defined in (2.1) falls into the category of these Dirac type operators, since $P^2 = -\Delta I_8$, $\langle PU, V \rangle_{\Omega'} = \langle U, PV \rangle_{\Omega'}$ for any $U, V \in C_0^\infty(\Omega'; \mathbb{C}^8)$ and the characteristic form of $P$, namely $Q(\lambda) = \det(\Lambda(\lambda)) = -i|\lambda|^8$, does not vanish for any $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \setminus 0$. Here, $\Lambda(\lambda)$ denotes the symbol of $P$ given by the matrix form

$$\Lambda(\lambda) = \frac{1}{i} \begin{pmatrix} 0 & A(\lambda) & 0 \\ B(\lambda) & 0 \end{pmatrix},$$

with

$$A(\lambda) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & \lambda_3 & -\lambda_2 \\ \lambda_2 & -\lambda_3 & 0 & \lambda_1 \\ \lambda_3 & \lambda_2 & -\lambda_1 & 0 \end{pmatrix}, \quad B(\lambda) = \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & 0 \end{pmatrix}.$$

In [13] it is proved that $D$ admits self-adjoint extensions by imposing non-local boundary conditions given by an orthogonal projection $\pi$ in $L^2(E_N)$, which is a classical pseudodifferential operator on $E_N$ satisfying a certain symmetry property (condition (1.13) in [13]) related to the structure of the operator (see [13] Lemma 1.1 for details). For such $\pi$ and by [13] Theorem 1.5 and the interpretation by Y. Kurylev and M. Lassas [40, Theorem 2.1], it turns out that $P \upharpoonright \mathcal{D}$ is self-adjoint with empty essential spectrum and finite-dimensional eigenspaces, where $\mathcal{D} := \{U \in H^1(\Omega'; \mathbb{C}^8) : \pi(U|_{\partial\Omega'}) = 0\}$.

The domain $\mathcal{D}$ in $H^1(\Omega'; \mathbb{C}^8)$ with the graph norm associated with $P$ is continuously embedded into $H^1(\Omega'; \mathbb{C}^8)$. The space $H^1(\Omega'; \mathbb{C}^8)$ is compactly embedded into $L^2(\Omega'; \mathbb{C}^8)$. Since $W \in L^\infty(\Omega'; \mathcal{M}_{8 \times 8})$ for admissible $\mu, \gamma$, the operator of multiplication by $W^t$, which we write $\mathcal{M}_{W^t}$, is bounded and linear in $L^2(\Omega'; \mathbb{C}^8)$. Therefore, $\mathcal{M}_{W^t}$ is $(P \upharpoonright \mathcal{D})$-compact.

Thus, $P - W^t$ has also empty essential spectrum and finite-dimensional eigenspaces. If $0$ is in the spectrum of $P - W^t$, then $0$ must be an eigenvalue with finitely many linearly independent eigen- and associated functions. We can make $0$ no longer be an eigenvalue by choosing a new set of boundary conditions which are not satisfied by any of the finitely many linearly independent eigen- and associated functions in the root spaces associated with $0$. Let us keep denoting the resultant boundary operator by $\pi$ so that the condition $\pi(Z|_{\partial\Omega'}) = 0$ guarantees
the existence of a constant $C_{\text{stblty}}$ independent of $Z$ such that
\begin{equation}
\|Z\|_{L^2(\Omega'; \mathbb{C}^8)} \leq C_{\text{stblty}} \| (P - W^t)Z \|_{L^2(\Omega'; \mathbb{C}^8)},
\end{equation}
provided that $Z, (P - W^t)Z \in L^2(\Omega'; \mathbb{C}^8)$.

The argument presented in Subsection 6.1 together with a trick based on an auxiliary system which improves the regularity of $(P - W^t)Z$ when Maxwell equations are satisfied, leads to the following

**Lemma 6.1.** For admissible coefficients $\mu, \gamma$, assume that $(P - W^t)Z = Y$ in $\Omega'$, where $Y$ reads $Y = (0 \quad \mu^{1/2}H^t | 0 \quad \gamma^{1/2}E^t)^t$, with $E, H$ verifying (1.1) in $\Omega'$. Additionally, suppose $\pi(Z|_{\partial\Omega'}) = 0$ for the boundary operator $\pi$ introduced in Subsection 6.1. Then there exists a constant $C$ only depending on $C_{\text{stblty}}, M, \omega$, such that $\|Z\|_{L^2(\Omega'; \mathbb{C}^8)} \leq C \|E\|_{L^2(\Omega'; \mathbb{C}^3)}$.

**Proof of Lemma 6.1.** Writing $Z = (Z_1 \quad Z_H^t | Z_2 \quad Z_E^t)^t$, where $Z_j$ ($j = 1, 2$) are scalar fields and $Z_H$, $Z_E$ vector fields, and defining $Z_{\text{aux}} := (Z_1 \quad Z_H^t | Z_2 \quad (Z_E^t)^t)^t$ with $Z_E' := Z_E + \mu^{-1/2}\omega^{-1}E$, the dependence on the electric field $E$ of the vector function $(P - W^t)Z_{\text{aux}}$ is zeroth-order. Indeed, it is straightforward to check that
\[
(P - W^t)Z_{\text{aux}} = \begin{pmatrix}
(-i/\omega)(-2\nabla\mu^{-1/2} + \mu^{-1/2}\nabla\alpha) \cdot E \\
0 \\
0 \\
(\gamma^{1/2} - \kappa\mu^{-1/2}\omega^{-1})E
\end{pmatrix}.
\]
By (6.5),
\[
\|Z\|_{L^2(\Omega'; \mathbb{C}^8)} \leq \|Z_{\text{aux}}\|_{L^2(\Omega'; \mathbb{C}^8)} + M^{1/2}\omega^{-1}\|E\|_{L^2(\Omega'; \mathbb{C}^3)} \\
\leq C_{\text{stblty}} \|(P - W^t)Z_{\text{aux}}\|_{L^2(\Omega'; \mathbb{C}^8)} + M^{1/2}\omega^{-1}\|E\|_{L^2(\Omega'; \mathbb{C}^3)} \\
\leq (C_{\text{stblty}}C(M, \omega) + M^{1/2}\omega^{-1})\|E\|_{L^2(\Omega'; \mathbb{C}^3)}.
\]

For $Y_1$ defined above, let $\tilde{Z}_1$ be a solution to the system $(P - W^t_1)\tilde{Z}_1 = Y_1$ in $\Omega'$ such that $\pi((Z_1 - \tilde{Z}_1)|_{\partial\Omega'}) = 0$. By Lemma 6.1
\begin{equation}
\|Z_1 - \tilde{Z}_1\|_{L^2(\Omega'; \mathbb{C}^8)} \leq C \|E_1 - \tilde{E}_1\|_{L^2(\Omega'; \mathbb{C}^3)} \leq \epsilon C.
\end{equation}

For each choice of $Y_2$ (with $Y_2|_{\Omega} \in H^1(\Omega; \mathbb{C}^8)$), since $(-\Delta I_8 + \tilde{Q}_2)Y_2 = 0$ in $\Omega$ by Lemma 2.1, where $\tilde{Q}_2$ denotes the zeroth order matrix operator $Q$ defined in (2.4) for $\mu_2, \gamma_2$, by elliptic regularity and Proposition 5.1 there exists $\tilde{Y}_2 \in H^2(\Omega; \mathbb{C}^8)$ verifying $(-\Delta I_8 + \tilde{Q}_2)\tilde{Y}_2 = 0$ in $\Omega$ with $\tilde{Y}_2|_{\partial\Omega} = 0$ on $\Gamma_c$, and
\begin{equation}
\|Y_2 - \tilde{Y}_2\|_{L^2(\Omega'; \mathbb{C}^8)} < \epsilon.
\end{equation}
Condition (6.1) implies that \( \mu_1 = \mu_2, \gamma_1 = \gamma_2, \partial_{x_l} \mu_1 = \partial_{x_l} \mu_2 \) and \( \partial_{x_l} \gamma_1 = \partial_{x_l} \gamma_2 \) on \( \Gamma \) (for \( l = 1, 2, 3 \)). By Proposition 1.2

\[
(Q_1 - Q_2) \frac{Z_1}{Y_2} = 0.
\]

Applying (6.6) and (6.7) write

\[
\left| \left( (Q_1 - Q_2)Z_1 \right) \Omega - \left( (Q_1 - Q_2) \frac{Z_1}{Y_2} \right) \Omega^c \right|
\]

\[
= \left| \int_{\Omega} (Q_1 - Q_2) Z_1 \cdot \left( \frac{Y_2}{Z_2} - \frac{Z_1}{Y_2} \right) dx - \int_{\Omega} (Q_1 - Q_2)(Z_1 - Z_2) \cdot \frac{Y_2}{Z_2} dx \right|
\]

\[
\leq \| Q_1 - Q_2 \|_{L^\infty(\Omega'; M; S^2)} \left( \| Z_1 \|_{L^2(\Omega'; C^2)} \left\| \frac{Y_2}{Z_2} - \frac{Z_1}{Y_2} \right\|_{L^2(\Omega'; C^2)} \right)
\]

\[
+ \| \frac{Y_2}{Z_2} \|_{L^2(\Omega'; C^2)} \left( \| Z_1 \|_{L^2(\Omega'; C^2)} \right)
\]

\[
\leq C(C_{stbly}, M, \omega) \epsilon \left( \| Z_1 \|_{L^2(\Omega'; C^2)} + \left\| \frac{Y_2}{Z_2} \right\|_{L^2(\Omega'; C^2)} \right)
\]

\[
\leq \epsilon C(M, \Omega, \rho, \xi, \omega, \varepsilon_0, \mu_0, C_{stbly}) e^{c(\tau + |\xi|)}
\]

where \( c = c(\Omega) \) and last inequality follows from the fact

\[
\| \frac{Y_2}{Z_2} \|_{L^2(\Omega'; C^2)} \leq \| Y_2 - \frac{Y_2}{Z_2} \|_{L^2(\Omega'; C^2)} + \| Y_2 \|_{L^2(\Omega'; C^2)} \leq 1 + \| Y_2 \|_{L^2(\Omega'; C^2)}
\]

the exponential behavior of \( Z_1, Y_2 \) and estimates (3.2), (3.4). Therefore, denoting

\[
e' \epsilon := \epsilon C(\Omega, \rho, \xi, \omega, \varepsilon_0, \mu_0, C_{stbly}),
\]

since \( Q_1 = Q_2 \) in \( \Omega \setminus \Omega' \) and by (6.8), we have

\[
|((Q_1 - Q_2)Z_1) \Omega| = |((Q_1 - Q_2)Z_1) \Omega'|
\]

\[
\leq |((Q_1 - Q_2) Z_1) \Omega^c| + e' \epsilon e^{c(\tau + |\xi|)} = e' \epsilon e^{c(\tau + |\xi|)}.
\]

Thus, for fixed \( \tau \) and \( \xi \), letting \( \epsilon \to 0 \) in (6.9)-(6.10), we get

\[
(Q_1 - Q_2)Z_1 \Omega = 0,
\]

for both choices of \( Z_1, Y_2 \). By (6.2), (6.3) and (6.11), we have for large enough \( \tau \),

\[
|f(\xi)| + |g(\xi)| \leq \frac{C}{\tau},
\]

where \( C = C(M, \xi, |\Omega|, \rho, \omega, \mu_0, \varepsilon_0) \). For any fixed \( \xi \in \mathbb{R}^3 \), by letting \( \tau \to \infty \) deduce that \( f(\xi) = g(\xi) = 0 \). Hence, \( f = g = 0 \).

Using a Carleman estimate, Pedro Caro in [17] proves the following inequality

\[
e^{d_1/h} \sum_{j=1,2} \left( h \| \phi_j \|^2_{L^2(\Omega)} + h^3 \| \nabla \phi_j \|^2_{L^2(\Omega)} \right) \leq C e^{d_2/h}
\]

\[
x \left( h^4 \left( \| f \|^2_{L^2(\Omega)} + \| g \|^2_{L^2(\Omega)} \right) + \sum_{j=1,2} \left( h \| \phi_j \|^2_{L^2(\partial \Omega)} + h^3 \| \nabla \phi_j \|^2_{L^2(\partial \Omega)} \right) \right),
\]
where \( \phi_1 := \gamma_1^{1/2} - \gamma_2^{1/2} \), \( \phi_2 := \mu_1^{1/2} - \mu_2^{1/2} \), \( C = C(\Omega, M) \), \( 0 < h < C^{-1/3} \leq 1 \), and

\[
d_1 := \inf \{|x - x_0|^2 : x \in \Omega\}, \quad d_2 := \sup \{|x - x_0|^2 : x \in \Omega\},
\]

for certain point \( x_0 \notin \partial \Omega \). Under Theorem 1.1’s conditions the summation term of norms on \( \partial \Omega \) in (6.13) vanishes. From this fact together with \( f = g = 0 \) in \( \Omega \), we conclude \( \mu_1 = \mu_2 \) and \( \gamma_1 = \gamma_2 \) in \( \Omega \).

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