A novel hierarchy of integrable lattices.

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Abstract

In the framework of the reduction technique for Poisson-Nijenhuis structures, we derive a new hierarchy of integrable lattice, whose continuum limit is the AKNS hierarchy.

In contrast with other differential-difference versions of the AKNS system, our hierarchy is endowed with a canonical Poisson structure and, moreover, it admits a vector generalisation. We also solve the associated spectral problem and explicitily construct action-angle variables through the r-matrix approach.

1 Introduction

The search for discrete integrable systems got a novel impetus in recent years: the quantisation of integrable PDE.s \cite{1 - 4}, the recent findings on integrable discrete-time systems \cite{5 - 7}, including the “extreme” case of cellular automata \cite{8}, and even some results in string theory and $2 - D$ quantum gravity (related to the so-called “discrete string equations”) are perhaps the main motivations for such a growing interest in that
field. The results reported in this paper can be ascribed to the above line of research, although they belong in a sense to a more traditional approach, aiming at deriving integrable systems with discrete space and continuous time.

In fact, we present and discuss an integrable differential-difference version of so-called AKNS hierarchy [8], already mentioned in [11]. In comparison with other discretisations [11] [3], it has certain advantages and some drawbacks. The latters are essentially given by the non-existence of one-field reductions, unlike the model derived by Ablowitz and Ladik [11]: such reductions are in fact admissible only in the continuum limit. However, this seems the price to be paid (i) to preserve the first hamiltonian structure of the AKNS hierarchy, namely the canonical one, and (ii) to allow for a vector generalisation, both such features being exhibited by our model. We have to mention that one equation belonging to our hierarchy can be interpreted as a Backlund tranformations for the continuous AKNS system, and, as such, it has been recently derived by Yamilov and Svinolupov [12].

The paper is organised as follows.

In Section 2, the hierarchy under scrutiny is derived by using a nowadays standard geometrical reduction technique for Poisson-Nijenhuis structures.

In Section 3, the underlying direct and inverse problem is solved: for the sake of simplicity, only the $2 \times 2$ matrix case is considered.

In Section 4, the $r$-matrix structure of the system is revealed, and, through the $r$-matrix, the action-angle variables are explicitly given in terms of the spectral data.

In section 5, it is shown that the whole hierarchy goes into the AKNS one in a suitable simple continuum limit.

In Section 6 we make a few concluding remarks and mention some interesting open problems.

2 Poisson-Nijenhuis structure

In a recent paper, P.M.Santini and one of the authors [13], have considered the following abstract linear problem:

$$E\psi = (Q + \lambda A)\psi$$

In formula (1), $\lambda \in \mathbb{C}$ is a spectral parameter, $Q, A, \psi$ take values in an associative algebra $\mathcal{A}$ with unit element $I$, endowed with a trace-form $< \cdot, \cdot >$, $E$ is an algebra
automorphism; \( A \) is assumed to be a constant (a fixed point of \( E \)) element in the algebra.

It has been shown in [13] that to the linear problem (1) one can naturally associate the following two compatible Poisson tensors:

\[
\theta_1 \doteq L_Q R_A - R_Q L_A \\
\theta_2 \doteq N \theta_1
\]  

where the symbol \( L_x \) (resp. \( R_x \)) denote left (resp. right) multiplication by \( x \), and \( N \) is the hereditary recursion operator, or Nijenhuis tensor [14], defined as:

\[
N \doteq (L_Q E - R_Q)(L_A E - R_A)^{-1}
\]  

Furthermore, one has proved there that the family of vector fields:

\[
K_B \doteq (R_B - L_B)Q \doteq [Q, B]; \quad [A, B] = 0
\]  

is an invariant family for \( N \), namely the Lie-derivative of \( N \) along \( K_B \) vanishes:

\[
\mathcal{L}_{K_B} N = 0
\]

We can then assert that to the linear problem (1) it is naturally associated the following hierarchy of (commuting) bi-hamiltonian systems:

\[
\frac{\partial Q}{\partial t_s} = N^s K_B
\]

Out of the abstract hierarchy (7), one can construct bi-hamiltonian lattices through a convenient realization of the algebra \( A \), chosen to be the algebra of matrix valued sequences, approaching an arbitrary constant value as \(|n| \to \infty \), and of the automorphism \( E \), taken to be the “shift operator” on sequences:

\[
E X_n = X_{n+1} \quad n \in Z
\]

In particular, throughout this paper, we shall consider \((N+1) \times (N+1)\) real matrices. Accordingly, the “field variable” \( Q \) will be parametrized as follows:

\[
Q = \begin{pmatrix}
\frac{p}{r} & <q| \\
|r> & \hat{Q}
\end{pmatrix}
\]
where $p$ is a scalar, $< q \mid$ is the $1 \times N$ matrix $(q_1, \ldots, q_N)$ (“row vector”), $\mid r >$ is the $N \times 1$ matrix $(r_1, \ldots, r_N)$ (“column vector”) and $\hat{Q}$ is an $N \times N$ matrix.

The constant matrix $A$ will be chosen as:

$$A = \begin{pmatrix} 1 & < 0| \\ |0 > & \hat{0} \end{pmatrix}$$

(10)

where $< 0| (|0 >)$ is the null row (column) vector, and $\hat{0}$ is the null $N \times N$ matrix.

The sequence $\{Q_n\}_{n \in \mathbb{Z}}$ of matrices of type (9) is assumed to fulfil the boundary condition:

$$\lim_{|n| \to \infty} Q_n = \begin{pmatrix} 0 & < 0| \\ |0 > & \hat{I} \end{pmatrix}$$

(11)

where $\hat{I}$ denotes the $N \times N$ identity matrix.

Let $\mathcal{I} \subset \mathcal{A}$ be the subalgebra of $\mathcal{A}$, consisting of matrix-valued sequences obeying homogeneous boundary conditions. Once equipped with the Lie-product given by the point-wise commutator:

$$[X, Y]_n = X_nY_n - Y_nX_n$$

$\mathcal{I}$ becomes a Lie-algebra and moreover an ideal of $\mathcal{A}$, and can be identified with its dual through the bilinear form:

$$(X, Y) = \sum_{n=-\infty}^{+\infty} < X_n, Y_n > \overset{=} \text{Tr} \sum_{n=-\infty}^{+\infty} X_nY_n$$

(12)

Our configuration space $M$ will be the affine hyperplane to $\mathcal{I}$ of matrix-valued sequences obeying (11); accordingly, its generic point will be again denoted by $Q$.

Its tangent bundle $T(M)$ will then be the set $\{(Q, K_Q) : Q \in M, K_Q \in \mathcal{I}\}$ while the points of its cotangent bundle $T^*(M)$ will be parametrized as $\{(Q, \gamma_Q) : Q \in M, \gamma_Q \in \mathcal{I}^* \simeq \mathcal{I}\}$.

In the following, for the sake of simplicity, elements of $T(M)$ (resp. $T^*(M)$) will be shortly denoted by $K_Q$ (resp. $\gamma_Q$). In the present concrete realisation of the abstract setting introduced in [13], the Poisson tensors $\theta_1$, $\theta_2$ defined by (23), as linear maps form $T(M)$ and $T^*(M)$, are in fact linear operators on $\mathcal{I}$, and the same is true for the Nijenhuis tensor $\mathcal{N}$, given by (4), which is a linear map form $T(M)$ into itself.

The main result of present Section is contained in the following:

**Theorem 1** The Nijenhuis tensor $\mathcal{N}$ reduces by restriction on the submanifold $\text{Im} \theta_1$.  

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As a consequence, the abstract bi-hamiltonian hierarchy (7) restricts to a bi-hamiltonian hierarchy of evolution equations on a one-dimensional lattice.

The theorem (1) will be proved in a constructive way, which will then also yield the concrete form of the restricted Poisson and Nijenhuis tensors.

Let us start by evaluating the image of $\theta_1$:

$$Im\theta_1 = \{K_Q = \theta_1 \gamma_Q; \gamma_Q \in \mathcal{I}\}$$  (13)

Let us choose for $K_Q$ and $\gamma_Q$ the parametrisation induced by (9):

$$K_Q = \begin{pmatrix} K_p & <K_q| \\ |K_r> & K_\hat{Q} \end{pmatrix}; \quad \gamma_Q = \begin{pmatrix} \gamma_p & <\gamma_r| \\ |\gamma_q> & \Gamma_{\hat{Q}} \end{pmatrix}$$  (14)

One immediately gets, for $K_Q \in Im\theta_1$, i.e. for $K$ of the form:

$$K_Q = Q\gamma_Q A - A\gamma_Q Q \quad \text{(for some $\gamma_Q$)}$$  (15)

the necessary condition:

$$K_{\hat{Q}} = \hat{0}$$  (16)

Hence, $\hat{Q}$ must be a constant $N \times N$ matrix. In view of the boundary condition (11), we have:

$$\hat{Q} = \hat{I}$$  (17)

In turn, formula (17) implies:

$$K_p = <K_q|r> + <q|K_r>$$

which entails:

$$p = qr + c \quad (c = \text{const.})$$  (18)

Summarizing, we can assert that:

1. under conditions (10), (11), $Im\theta_1$ is the submanifold of $\mathcal{I}$ consisting of vector fields $K_Q$ such that:

$$K_{\hat{Q}} = \hat{0} \quad ; \quad K_p = <K_q|r> + <q|K_r>$$  (19)

2. the set $S \subset M$, such that $T(S) = Im\theta_1$, given by:

$$S = \{Q : p = <q|r>, \hat{Q} = \hat{I}\}$$  (20)

is a characteristic leaf of $\theta_1$. 

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We are now ready to show that $\mathcal{N}$ reduces by restriction on $S$, namely that $T(S) = \text{Im} \theta_1$ is an invariant submanifold for $\mathcal{N}$. To this aim, let us introduce the auxiliary vector field $\xi_Q$ through the formula:

$$K_Q = (E\xi_Q)A - A\xi_Q$$

whence it follows, on $T(M)$:

$$K'_Q = N K_Q = (E\xi_q)Q - Q\xi_Q$$

Parametrizing $\xi_Q$ as

$$\xi_Q = \begin{pmatrix} \xi_0 & <\xi_q| \\ |\xi_r| & \xi \end{pmatrix}$$

we get from (21)

$$\xi_0 = (E - 1)^{-1}K_p$$
$$<\xi_q| = -<K_q|$$
$$|\xi_r| = E^{-1}|K_r|$$

while the matrix $\dot{\xi}$ stays undetermined.

On the other hand, eq. (22) yields, for $Q \in S$:

$$K'_p = <q|r>(E-1)K_p + <E\xi_q|r> - <q|\xi_r>$$
$$<K'_q| = <q|(E\xi_0) + (E-<q|r>)<\xi_q|-<q|\dot{\xi}$$
$$|K'_r| = (<q|r|E-1)|\xi_r| - \xi_0|r| + (E\dot{\xi})|r|$$

By imposing the reduction condition: $\dot{K}'_Q = 0$, one gets:

$$\dot{\xi} = (E - 1)^{-1}(-|E\xi_r|<q| + |r|<\xi_q|)$$

or, using (25, 26):

$$\dot{\xi} = -(E - 1)^{-1}(+|K_r|<q| + |r|<K_q|)$$

Then, to establish that $S$ is an invariant submanifold for $\mathcal{N}$, we have just to show that:

$$K_p = <K_q|r| + <q|K_r|$$

entails:

$$K'_p = <K'_q|r| + <q|K'_r|$$

and this can be seen by a direct straightforward (although tedious) calculation. $\square$
Hence, the theorem (1) is proved and moreover we can give the explicit form of \( N|_S \), more precisely we have:

\[
\begin{bmatrix}
  |K_q^*> \\
  |K_r^*>
\end{bmatrix}
= \begin{bmatrix}
  (\frac{<q|r>-E|K_q>}{E}\left(\frac{<q|K_q>+<q|K_r>}{E}\right)q>
  \\
  (\frac{<q|r>-E^{-1}|K_r>}{E}\left(\frac{<q|K_q>+<q|K_r>}{E}\right)q>
\end{bmatrix} \tag{33}
\]

The explicit form of \( \theta_1|_S \) can be easily obtained by noting that the points of \( S \) (resp. \( T(S) \)) are completely determined once the \( 2N \) vector \( (<q|, <r|)_t \) (resp. \( (<K_q|, <K_r|)_t \)) are given.

Hence points of \( T^*(S) \) will be fully characterized by the value of the \( 2N \) vector \( (<\beta_q|, <\beta_r|)_t \) defined by the duality condition:

\[
Tr \sum_{n=-\infty}^{+\infty} \gamma_{Q,n} K_{Q,n} = \sum_{n=-\infty}^{+\infty} <\beta_{q,n}|K_{q,n}> + <\beta_{r,n}|K_{r,n}> \tag{34}
\]

which entails:

\[
<\beta_q| = <\gamma_r|-\gamma_p<r| <\beta_r| = <\gamma_q|-\gamma_p<q|
\]

By direct calculation, one can check that the relation:

\[
K_Q = \theta_1 \gamma_Q
\]

once restricted to \( S \), implies:

\[
\begin{bmatrix}
  |K_q> \\
  |K_r>
\end{bmatrix}
= \begin{bmatrix}
  0 & -\hat{I} \\
  \hat{I} & 0
\end{bmatrix}
\begin{bmatrix}
  |\beta_q> \\
  |\beta_r>
\end{bmatrix}
\doteq J \begin{bmatrix}
  |\beta_q> \\
  |\beta_r>
\end{bmatrix} = \begin{bmatrix}
  |\beta_q> \\
  |\beta_r>
\end{bmatrix} \tag{36}
\]

whence it follows that \( \theta_1|_S \) is the canonical Poisson tensor or, in other words, that the fields variables \( |q>, |r> \) are endowed with canonical Poisson brackets:

\[
\{q_j(n), q_k(m)\}_1 = \{r_j(n), r_k(m)\}_1 = 0
\]

\[
\{q_j(n), r_k(m)\}_1 = \delta_{j,k}\delta_{n,m} \tag{37}
\]

It is worthwhile to notice that the restricted Poisson tensor \( \theta_2|_S \doteq N|_S \theta_1|_S \) is on the other hand nonlocal and has a rather cumbersome form: its non-locality is intimately related with the non degeneracy of \( \theta_1|_S \).

According with the general theory of bi-hamiltonian system, we have still to show that the restricted vector fields:

\[
\tilde{K}_B = [Q|_S, B] \quad ( [B, A] = 0 )
\]

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are invariant vector fields for \( \mathcal{N} |_S \); but this follows by the invariant nature of the Lie derivative, provided that \( K_B \) belongs to \( T(S) \).

Actually, the commutativity condition \([B,A] = 0\) implies for matrix \( B \) the form:

\[
B = \begin{pmatrix}
b & < 0 \\
0 & \hat{B}
\end{pmatrix}
\]

so that, for any \( Q \in S \), we have:

\[
\tilde{K}_B = \begin{pmatrix}
0 & < q | (b - \hat{B}) \\
(B - b) | r > & 0
\end{pmatrix}
\]

(38)

So, the family of starting commuting symmetries can be written in terms of the \( 2N \)-vectors:

\[
\begin{bmatrix}
|K^c_q > \\
|K^c_r >
\end{bmatrix} = \left[ \begin{array}{c}
\dot{C}^t |q > \\
-\hat{C} |r >
\end{array} \right]
\]

(39)

\( \hat{C} = b - \hat{B} \) being an arbitrary \( N \times N \) matrix.

However, we should notice that the resulting evolution equations:

\[
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
|_{t_0} = [\mathcal{N} |_S]^t
\begin{pmatrix}
|K^c_q > \\
|K^c_r >
\end{pmatrix}
\]

(40)

will be in general non-local. Local equations will be get whenever \( \hat{C} = c\hat{I} \).

Let us now give some concrete examples of local evolution equations associated with the linear problem (1), corresponding to the choice \( \hat{C} = \hat{I} \):

1. \( (l = 0) \)

\[
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
|_{t_0} = \begin{pmatrix}
|q > \\
-|r >
\end{pmatrix} = \theta_1 |_S
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
\]

or, in components:

\[
\frac{\partial q_j}{\partial t_0} = q_j; \quad \frac{\partial r_j}{\partial t_0} = -r_j \quad (j = 1, ..., N)
\]

(42)

with hamiltonian density:

\[
h_0 = - < q | r > = - \sum_{j=1}^{N} q_j r_j
\]

(43)
2. \( (l = 1) \)

\[
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
_{t_1} = \begin{pmatrix}
(<q|r> - E)|q > \\
(E^{-1} - <q|r>)|r >
\end{pmatrix}
\]  

(44)

with hamiltonian density:

\[
h_1 = -\frac{1}{2} <q|r>^2 + <Eq|r>
\]  

(45)

3. \( (l = 2) \)

\[
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
_{t_2} = - \begin{pmatrix}
-E^2|q> + E(<q|r>|q>) + (<Eq|r> + <q|E^{-1}r> - <q|r>^2)|q>
\end{pmatrix}
\]

(46)

with hamiltonian density:

\[
h_2 = -\frac{1}{3} <q|r>^3 + <q|r>(<q|E^{-1}r> + <Eq|r> - <Eq|E^{-1}r>)
\]  

(47)

4. \( (l = -1) \)

\[
\begin{pmatrix}
|q > \\
|r >
\end{pmatrix}
_{t_1} = [\mathcal{N}|s]^{-1}
\begin{pmatrix}
|q > \\
-|r >
\end{pmatrix}
\]

(48)

with hamiltonian density:

\[
h_{-1} = \ln(1 + <q|Er>)
\]  

(49)

We will now show that the above geometric reduction procedure is perfectly equivalent to the perhaps more familiar technique, based on the so-called discrete zero-curvature condition:

\[
U_t + UV - (EV)U = 0
\]  

(50)

obtained by enforcing compatibility of linear problem \([\mathcal{L}]\) with the auxiliary linear problem:

\[
\frac{\partial \psi}{\partial t} = V(\lambda)\psi
\]  

(51)

In our case:

\[
U = Q + \lambda A
\]  

(52)
where $A$ is given by (10) and $Q$ belongs to manifold $S$ (20), i.e.:

$$ Q = \left( \begin{array}{cc} <q|r> & <q| \n \end{array} \right) $$

To extract from (5) the hierarchy (40) we start by considering the “stationary equation” associated with (50), namely:

$$ UW - (EW)U = 0 \quad (53) $$

Parametrizing $W$ as:

$$ W = \left( \begin{array}{c} w <u| \\
|v > \hat{W} \n \end{array} \right) \quad (54) $$

one gets

$$ \hat{W} = (E - 1)^{-1}[|r > < u| - |Ev > < q|] + \hat{W}(\lambda) \quad (55) $$

$$ w = -tr \hat{W} = (E - 1)^{-1}[|r > < u| - |Ev > < q|] - tr \hat{W}(\lambda) \quad (56) $$

$\hat{W}(\lambda)$ being an arbitrary constant (i.e., field independent) matrix, and the following “eigenvalue equations” for $<u|, |v>$:

$$ \lambda < u| = (E - <q|r>) < u| + < q|(Ew - \hat{W}) \quad (57) $$

$$ \lambda E|v > = (E - 1 - <q|r>)(Ev > + (w - E\hat{W})|r > \quad (58) $$

with $w, \hat{W}$ given by (55),(56).

Eqs. (57, 58) can be solved recursively assuming $W$, and thus $<u|, |v>$, to be given by the following Laurent series in $\lambda$:

$$ \sum_{k=0}^{\infty} W^{(k)}\lambda^{-k} \quad (59) $$

and requiring $\hat{W}(\lambda)$ in (55),(56), to be in fact $\lambda$-independent.

One gets:

$$ < u^{(0)}| = |v^{(0)} >= 0 \quad (60) $$

$$ < u^{(k+1)}| = (E - <q|r>) < u^{(k)}| + < q|(Ew^{(k)} - \hat{W}^{(k)}) \quad (60) $$

$$ E|v^{(k+1)} > = (E - 1 - <q|r>)(Ev^{(k)} > + (w^{(k)} - E\hat{W}^{(k)})|r > \quad (61) $$

On the other hand, if $W$ is a solutions of (53), the same is true of course for $\lambda^nW$, where $k$ is any positive integer.
Given a two-sided Laurent series on the unit circle:

\[ f(\lambda) = \sum_{s=-\infty}^{+\infty} f_s \lambda^s , \]

we shall denote, as usual, by \( f_+(\lambda) \) (resp. \( f_-(\lambda) \)) the part containing only non negative (resp. strictly negative) powers of \( \lambda \). Therefore, eq. (53) can be rewritten as:

\[ U(\lambda^k W)_+ - (\lambda^k E W)_+ U = -U(\lambda^k W)_- + (\lambda^k E W)_- U \tag{62} \]

But now, as \( U \) is linear in \( \lambda \), the l.h.s. of (62) cannot contain any negative power of \( \lambda \), while its r.h.s. cannot contain any strictly positive one. Hence both sides of (62) are \( \lambda \)-independent (order “zero” in \( \lambda \)), and we have:

\[ \text{l.h.s.} = QW^{(k)} - (EW^{(k)})Q \tag{63} \]
\[ \text{r.h.s.} = -AW^{(k+1)} - (EW^{(k+1)})A \tag{64} \]

As far as its \( \lambda \)-dependence is concerned, (63) (and of course 64) is then compatible with \( U_t \). On the other hand it may be readily checked that it belongs indeed to the manifold \( T(S) \), defined in eqs. (19-20). Hence, we can assert that the hierarchy of evolution equations associated to (1), (51) is given by:

\[ \frac{\partial U}{\partial t_k} = -QW^{(k)} + (EW^{(k)})Q = AW^{(k+1)} - (EW^{(k+1)})A \tag{65} \]

They correspond to the following choice for the matrix \( V \) appearing in formulas (51), (50):

\[ V = V^{(k)} = (\lambda^k W)_+ \tag{66} \]

and clearly coincide with (10), by choosing \( \hat{C} = (tr \hat{W}) \hat{I} + \hat{W} \) (see rqs. (39), (53), (54)). In particular, \( \hat{C} = \hat{I} \leftrightarrow \hat{W} = -\frac{1}{N+1} \hat{I} \). Indeed, looking at (55), (23), we see that \( W \) is the generating function of the fields \( \xi_Q \).

### 3 Direct and inverse problem

In this Section, we outline the solution of the direct and inverse problem associated to (1); for simplicity, we restrict considerations to the \( 2 \times 2 \) matrix case, when we have:

\[ U_n(\lambda) = Q_n + \lambda A = \begin{pmatrix} \lambda + q_n r_n & q_n \\ r_n & 1 \end{pmatrix} \tag{67} \]
As explained in section 2, \( \{q_n\}_{n \in \mathbb{Z}}, \{r_n\}_{n \in \mathbb{Z}} \) are real valued sequences vanishing at \( \pm \infty \)

\[
\lim_{|n| \to \infty} q_n = \lim_{|n| \to \infty} r_n = 0 \quad (68)
\]

For the linear problem:

\[
\psi_{n+1} = U_n \psi_n \quad (69)
\]

we can naturally define the transfer matrix:

\[
T_{n,m}(\lambda) = U_{n-1} \ldots U_m \quad (n \geq m) \quad (70)
\]

\[
T_{n,m}(\lambda) = U_{m-1}^{-1} \ldots U_{n+1}^{-1} \quad (n < m) \quad (71)
\]

such that:

\[
\psi_n = T_{n,m} \psi_m \quad (72)
\]

We can then introduce the “Jost matrices”:

\[
T^-_n(\lambda) \doteq \lim_{m \to -\infty} T_{n,m}(\lambda) E_m(\lambda) = (\phi_n(\lambda), \tilde{\phi}_n(\lambda)) \quad (73)
\]

\[
T^+_n(\lambda) \doteq \lim_{m \to +\infty} T_{n,m}(\lambda) E_m(\lambda) = (\tilde{\phi}_n(\lambda), \psi_n(\lambda)) \quad (74)
\]

where:

\[
E_n(\lambda) \doteq \begin{pmatrix} \lambda^n & 0 \\ 0 & 1 \end{pmatrix} \quad (75)
\]

is a fundamental matrix solution of the asymptotic (or “undressed”) problem:

\[
E_{n+1}(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} E_n(\lambda) \quad (76)
\]

and \( \varphi_n, \tilde{\varphi}_n, \psi_n, \tilde{\psi}_n \) are 2-column vector solutions of (1), the “Jost solutions”.

Clearly, the asymptotic solution \( E_n(\lambda) \) \( (75) \) is bounded on the unit circle \( |\lambda| = 1 \), which will then be the continuous spectrum of (1), (68).

On the unit circle, the monodromy matrix is then defined as:

\[
T(\lambda) = T^+_n(\lambda)[T_n(\lambda)]^{-1} \quad (77)
\]

In the following, we shall call “spectral parameters” the elements of monodromy matrix:

\[
T(\lambda) = \begin{pmatrix} a(\lambda) & \tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix} \quad (78)
\]
As \( \det U_n = \lambda \), we have:

\[
\det T_{n,m}^{\pm}(\lambda) = \lambda^{n-m}, \quad \det T_n^{\pm}(\lambda) = \lambda^n
\]  

so that:

\[
\det T(\lambda) = 1
\]  

Formulas (79) implies that \( \varphi_n, \tilde{\varphi}_n \) and \( \psi_n, \tilde{\psi}_n \) are two pairs of independent vector solutions of (1) on the unit circle \( (|\lambda| = 1) \), while formulas (78), (80) entail, on the unit circle:

\[
a(\lambda)a(\lambda) = 1 + b(\lambda)b(\lambda)
\]  

**Direct problem**

The direct problem amounts to determine the monodromy matrix (78) once the fields \( \{q_n\}, \{r_n\} \) are given.

To this aim, it is convenient to introduce the normalized vector sequences:

\[
\tilde{\chi} = \lambda^{-n}\tilde{\psi}_n; \quad \varphi = \lambda^{-n}\phi_n
\]  

such that:

\[
\lim_{n \to -\infty} \tilde{\chi}_n(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lim_{n \to +\infty} \varphi_n(\lambda)
\]  

Somewhat loosely, in the following \( \tilde{\chi}_n, \varphi_n \) will be denoted as “Jost solution” as well.

The spectral parameters are naturally defined as Wronskians of independent vector solutions; namely we have:

\[
a(\lambda) = W(\varphi, \psi)
\]

\[
\tilde{a}(\lambda) = W(\tilde{\varphi}, \tilde{\chi})
\]

\[
b(\lambda) = \lambda^n W(\tilde{\chi}, \varphi)
\]

\[
\tilde{b}(\lambda) = \lambda^{-n} W(\tilde{\phi}, \psi)
\]

where \( W(a, b) \) is the determinant of the matrix whose columns are the 2-vectors \( a \) and \( b \).

It easily seen that the vector sequences \( \tilde{\chi}_n, \varphi_n, \tilde{\varphi}_n \) satisfy the following “discrete integral” equations:

\[
\tilde{\chi}_n(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=n}^{\infty} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{k-n} \end{pmatrix} \begin{pmatrix} q_k & q_k \\ r_k & 0 \end{pmatrix} \tilde{\chi}_k(\lambda)
\]
\[ \psi_n(\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k=n}^{\infty} \begin{pmatrix} \lambda^{n-(k+1)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_kr_k & q_k \\ r_k & 0 \end{pmatrix} \psi_k(\lambda) \] (86)

\[ \varphi_n(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{k=-\infty}^{n-1} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{k-n} \end{pmatrix} \begin{pmatrix} q_kr_k & q_k \\ r_k & 0 \end{pmatrix} \varphi_k(\lambda) \] (87)

\[ \tilde{\varphi}_n(\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sum_{k=-\infty}^{n-1} \begin{pmatrix} \lambda^{n-k-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_kr_k & q_k \\ r_k & 0 \end{pmatrix} \tilde{\varphi}_k(\lambda) \] (88)

The analyticity properties of the Jost solutions are summarized by the following:

**Theorem 2** If the sequences \( \{q_n\}, \{r_n\} \) are such that:

\[ \lim_{|n| \to \infty} n^2 q_n = \lim_{|n| \to \infty} n^2 r_n = 0 \] (89)

then \( \psi_n, \varphi_n \) are analytic functions of \( \lambda \) in the domain \(|\lambda| > 1\) and are continuously differentiable for \(|\lambda| \geq 1\); analogously, \( \tilde{\chi}_n, \tilde{\varphi}_n \) are analytic functions of \( \lambda \) for \(|\lambda| < 1\) and continuously differentiable for \(|\lambda| \leq 1\).

We outline the proof of theorem (2) for \( \varphi_n \).

First of all, we equip \( \mathbb{C}^N \) with the norm:

\[ \|x\| = \max_{k=1\ldots N} |x_k| \]

so that linear transformations in \( \mathbb{C}^N \) are naturally equipped with the norm:

\[ \|X\| = \max_{j=1\ldots N} \sum_{j=1}^{N} |x_{ij}| \]

Then, the following inequality holds for (87):

\[ \left\| \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{k-n} \end{pmatrix} U_{1,k} \right\| \leq \|U_{1,k}\| \quad (|\lambda| \geq 1) \] (90)

where

\[ U_{1,k} = \begin{pmatrix} q_k r_k & q_k \\ r_k & 0 \end{pmatrix} \] (91)

So, writing the Neumann series solutions of (85):

\[ \varphi_n(\lambda) = \sum_{l=0}^{\infty} (-1)^l F_n^{(l)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] (92)

where:

\[ F_n^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
\[ F_{n}^{(l)}(\lambda) = \sum_{k_1 > n} G(n, k_1) \sum_{k_2 > k_2} G(k_1, k_2) \cdots \sum_{k_l > k_2} G(k_{l-1}, k_l) \quad (l > 0) \quad (93) \]

\[ G(n, k) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda^{k-n} \end{pmatrix} U_{1,k} \quad (94) \]

The inequality (90) implies, by iteration:

\[ \| F_{n}^{(l)}(\lambda) \| \leq \frac{1}{l!} \left( \sum_{k > n} \| U_{1,k} \| \right)^l \quad l \geq 0 \quad (95) \]

Therefore:

\[ \| \varphi(\lambda) \| = \sup_{n \in \mathbb{Z}} \| \varphi_n(\lambda) \| \leq \exp \gamma \quad (96) \]

where:

\[ \gamma = \sum_{n \in \mathbb{Z}} \| U_{1,n} \| \quad (97) \]

Hence, if \( \gamma < \infty \), \( \varphi_n(\lambda) \) is analytic for \( |\lambda| > 1 \). On the other hand:

\[ \| U_{1,n} \| = \max \{ |q_n| [1 + |r_n|], |r_n| \} \leq |q_n| + |r_n| + 1/2 [||q_n|^2 + |r_n|^2] \]

so that the existence of \( \gamma \) is guaranteed whenever the sequences \( \{q_n\}, \{r_n\} \) belong to \( l_1 \).

A similar procedure leads to the following result for \( \frac{\partial \varphi_n}{\partial \lambda} \):

\[ \left\| \frac{\partial \varphi_n}{\partial \lambda} \right\| \leq a + |n|b \quad |\lambda| \geq 1 \quad (98) \]

which holds, with suitable coefficients \( a \) and \( b \), whenever:

\[ \sum_{n \in \mathbb{Z}} (1 + |n|)\| U_{1,n} \| < \infty \quad (99) \]

Hence, provided \( q_n \) and \( r_n \) vanish faster than \( n^{-2} \) as \( |n| \to \infty \), \( \varphi_n(\lambda) \) is continuously differentiable with respect to \( \lambda \) for \( |\lambda| \geq 1 \). □

We can thus assert that, whenever \( \{q_n\}, \{r_n\} \) belong to \( l_1 \), the diagonal entries of the monodromy matrix \( a(\lambda) \) and \( \bar{a}(\lambda) \) (3.14) are analytic respectively outside and inside the unit circle. Moreover, due to their asymptotic behaviour in \( \lambda \):

\[ \lim_{|\lambda| \to \infty} a(\lambda) = \lim_{|\lambda| \to \infty} \text{det}(\varphi, \psi) = \text{det} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \quad (100) \]

\[ \lim_{|\lambda| \to 0} \bar{a}(\lambda) = \lim_{|\lambda| \to \infty} \text{det}(\bar{\varphi}, \bar{\psi}) = \prod_{j \neq n}^{+\infty} (1 + r_j q_j) \text{det} \begin{pmatrix} q_{n-1} & 1 \\ 1 & -r_n \end{pmatrix} \]

\[ = -\prod_{j = -\infty}^{-1} (1 + r_j q_j) \quad (102) \]
both \(a\) and \(\tilde{a}\) have at most a finite number of zeros, say \(N\) and \(\tilde{N}\) respectively, in their analyticity domains. These zeroes will be denoted as \(\{\lambda_j\}_{j=1}^{N}\) and \(\{\tilde{\lambda}_j\}_{j=1}^{\tilde{N}}\) and will be assumed to be simple.

If in addition the stronger condition (99) is satisfied, then the entries of the monodromy matrix (78) are Hölder continuous on the unit circle (see again (84)). Moreover, the analyticity properties of \(a(\lambda)\) and \(\tilde{a}(\lambda)\) imply that the scalar Riemann problem on the unit circle (81) can be solved through the formulas:

\[
a(\lambda) = \frac{\Pi_{j=1}^{N}(\lambda - \lambda_j)}{\Pi_{j=1}^{\tilde{N}}(\lambda - \tilde{\lambda}_j)} \lambda^\sigma \exp \left[ \frac{1}{2\pi i} \oint_{|z|=1} \ln \left(\frac{1 + b(\lambda)\tilde{b}(z)}{z - \lambda}\right) - \sigma \ln z \, dz \right]
\]

for \(|\lambda| > 1\)

\[
\tilde{a}(\lambda) = \frac{\Pi_{j=1}^{\tilde{N}}(\lambda - \tilde{\lambda}_j)}{\Pi_{j=1}^{N}(\lambda - \lambda_j)} \exp \left[ \frac{1}{2\pi i} \oint_{|z|=1} \ln \left(\frac{1 + b(\lambda)\tilde{b}(z)}{z - \lambda}\right) - \sigma \ln z \, dz \right]
\]

for \(|\lambda| < 1\)

where \(\sigma\) is the index of the Riemann problem, i.e. the variation of \(\arg(1 + b(\lambda)\tilde{b}(\lambda))\) after a cycle; condition (100) clearly implies \(\sigma = N - \tilde{N}\).

In the following, we shall assume the index \(\sigma\) to be zero so that \(N = \tilde{N}\): this is certainly true in the reflectionless case \((b = \tilde{b} = 0)\).

**Inverse problem**

The inverse problem amounts to reconstruct the sequences \(\{q_n\}, \{r_n\}\), once the monodromy matrix is given.

To solve it, we rewrite eqs. (77) in terms of the appropriate column vector solutions:

\[
\frac{\psi_n}{a} = \psi_n + \frac{b}{a} \lambda^n \tilde{\chi}_n 
\]

(105)

\[
\frac{\varphi_n}{a} = \tilde{\chi}_n + \frac{b}{a} \lambda^{-n} \psi_n 
\]

(106)

Under the above analyticity conditions on \(\phi, \psi, \varphi, \tilde{\chi}\), eqs. (105, 107) are two vector Riemann-Hilbert problems, which can be solved (i.e. reduced to singular integral
equations) through the well-known Plemelj formulas. One gets:

\[
\psi_n(\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \oint \tilde{\rho}(\zeta) \frac{\zeta^n \tilde{\chi}_n(\zeta)}{\zeta - \lambda} d\zeta - \sum_{k=1}^{N} \frac{\tilde{\lambda}_k^n \tilde{\gamma}_k \tilde{\chi}(\tilde{\lambda}_k)}{\lambda_k - \lambda} 
\]

(108)

\[
\tilde{\chi}_n(\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{2\pi i} \oint \rho(\zeta) \frac{\zeta^{-n} \psi_n(\zeta)}{\zeta - \lambda} d\zeta - \sum_{k=1}^{N} \frac{\lambda_k^{-n} \gamma_k \psi(\lambda_k)}{\lambda_k - \lambda} 
\]

(109)

where we have set:

\[
\rho(\lambda) = b(\lambda)/a(\lambda); \quad \tilde{\rho}(\lambda) = \tilde{b}(\lambda)/\tilde{a}(\lambda)
\]

(110)

\[
\gamma_k = b_k/a'(\lambda_k); \quad \tilde{\gamma}_k = \tilde{b}_k/\tilde{a}'(\tilde{\lambda}_k)
\]

(111)

In formulas (111) \( b_k \) (resp. \( \tilde{b}_k \)) are the ratios between \( \phi \) and \( \psi \) (resp. \( \tilde{\phi} \) and \( \tilde{\psi} \)) at \( \lambda = \lambda_k \) (resp. \( \lambda = \tilde{\lambda}_k \)).

Of course, in the reflectionless case formulas (108, 109) yield a system of 2\( N \) linear algebraic equations for \( \psi_n(\lambda_k), \tilde{\chi}_n(\tilde{\lambda}_k) \).

Finally, as usual, the sequences \( \{q_n\}, \{r_n\} \) are given in terms of the leading terms of the asymptotic behaviour of the vector solutions \( \psi_n, \tilde{\chi}_n \). For instance, taking into account that:

\[
\psi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda^{-1} \begin{pmatrix} \psi_{1,n}^{(1)} \\ \psi_{2,n}^{(1)} \end{pmatrix} + O(\lambda^{-2})
\]

we have:

\[
q_n = -\psi_{1,n}^{(1)}
\]

\[
r_n = \frac{\psi_{2,n+1}^{(1)} - \psi_{2,n}^{(1)}}{\psi_{1,n}^{(1)}}
\]

To find the time-evolution of the spectral data corresponding to eq. (50), (51), we notice that, if the matrix \( V \) appearing in the auxiliary linear problem (51) is given by (66), the monodromy matrix undergoes the time evolution:

\[
\frac{\partial T(\lambda)}{\partial t_k} = [\bar{V}(k)(\lambda), T(\lambda)]
\]

(112)

where:

\[
\bar{V}(k)(\lambda) = \lim_{|n|\to\infty} V(k)_n(\lambda) = \lambda^k \begin{pmatrix} -tr \bar{W} & 0 \\ 0 & \bar{W} \end{pmatrix}
\]

(113)
Restricting considerations to the $2 \times 2$ case, we can thus write:

$$\bar{V}_{(k)}(\lambda) = c\lambda^k \sigma_3$$  \hspace{1cm} (114)

where $c$ is an arbitrary scalar constant and $\sigma_3$ is the usual Pauli matrix.

Consequently we have:

$$\frac{\partial a}{\partial t_k} = \frac{\partial \bar{a}}{\partial t_k} = 0 \hspace{1cm} (115)$$

$$\frac{\partial b}{\partial t_k} = -2c\lambda^k b \hspace{1cm} ; \hspace{1cm} \frac{\partial \bar{b}}{\partial t_k} = 2c\lambda^k \bar{b} \hspace{1cm} (116)$$

of course, eqs. (115) imply that $\lambda_r$, $\bar{\lambda}_r$ are constant in time for any equation of the hierarchy ("isospectral deformation"), while the normalization coefficient $\gamma_r$, $\bar{\gamma}_r$ evolve according to equations:

$$\frac{\partial \gamma_r}{\partial t_k} = -2(\lambda_r)^k \gamma_r \hspace{1cm} ; \hspace{1cm} \frac{\partial \bar{\gamma}_r}{\partial t_k} = 2(\bar{\lambda}_r)^k \bar{\gamma}_r \hspace{1cm} (117)$$

4 \hspace{1cm} r\text{-matrix and action-angle variables}

We have seen in sec. 2 (formulas (36),(37)) that our hierarchy of discrete evolution equations are hamiltonian with respect to the canonical Poisson tensor or, in other words, that the fields variables $|q>$, $|r>$ are endowed with the canonical Poisson bracket.

As consequence, we have the "ultra-local" Poisson bracket relation [15]:

$$\{U_n(\lambda) \otimes U_m(\mu)\} = [r(\lambda, \mu), U_n(\lambda) \otimes U_m(\mu)]\delta_{n,m} \hspace{1cm} (118)$$

Restricting again considerations to $2 \times 2$ matrices $U_n$, we have for $4 \times 4$ r-matrix $r(\lambda, \mu)$ the formula:

$$r(\lambda, \mu) = r(\lambda - \mu) = \frac{1}{\lambda - \mu} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \hspace{1cm} (119)$$

i.e. the same expression as for the NLS case.

From (118) one easily gets the analogous relation for the transfer matrix, and finally the Poisson-bracket relation for the monodromy matrix, that reads:

$$\{T(\lambda) \otimes T(\mu)\} = r^+(\lambda, \mu)[T(\lambda) \otimes T(\mu)] - [T(\lambda) \otimes T(\mu)]r^-(\lambda, \mu) \hspace{1cm} (120)$$
where:
\[
r^\pm(\lambda, \mu) = \lim_{n \to \infty} (E^{-1}(n, \lambda) \otimes E^{-1}(n, \mu)r(\lambda, \mu)E(n, \lambda) \otimes E(n, \mu))
\]
that is:
\[
r^\pm = \begin{pmatrix}
\frac{1}{\lambda - \mu} & 0 & 0 & 0 \\
0 & 0 & \mp i\pi \delta(\lambda - \mu) & 0 \\
0 & \pm i\pi \delta(\lambda - \mu) & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\lambda - \mu}
\end{pmatrix}
\]
(121)

In formula (121), \(\frac{1}{\lambda - \mu}\) denotes the principal-value distribution, and we have used the distribution formula:
\[
\lim_{n \to \pm \infty} \left(\mu/\lambda \right)^n = \mp i\pi \delta(\lambda - \mu)
\]
(122)

Eq. (120) implies the following Poisson-brackets for spectral parameters:
\[
\{a(\lambda), a(\mu)\} = \{a(\lambda), \tilde{a}(\mu)\} = 0
\]
(123)
\[
\{b(\lambda), b(\mu)\} = \{b(\lambda), \tilde{b}(\mu)\} = 0
\]
(124)
\[
\{b(\lambda), \tilde{b}(\mu)\} = 2\pi i\delta(\lambda - \mu) \tilde{a}(\mu)a(\mu)
\]
(125)
\[
\{a(\lambda), b(\mu)\} = -\{b(\mu), a(\lambda)\} = -\frac{1}{\lambda - \mu}a(\lambda)b(\mu) - i\pi \delta(\lambda - \mu) b(\mu)a(\mu)
\]
(126)

We can thus construct canonical spectral variables, given by:
\[
\alpha(\lambda) \doteq \ln a(\lambda)\tilde{a}(\lambda) \quad \beta(\lambda) \doteq \frac{1}{4\pi i} \ln \frac{\tilde{b}(\lambda)}{a(\lambda)}
\]
\[
\lambda_j \quad \nu_j \doteq \ln \gamma_j \quad \tilde{\lambda}_j \quad \tilde{\nu}_j \doteq \ln \tilde{\gamma}_j
\]
(127)

obeying the Poisson bracket realtions:
\[
\{\alpha(\lambda), \beta(\mu)\} = \delta(\lambda - \mu)
\]
\[
\{\lambda_j, \nu_k\} = \{\lambda_{j'}, \nu_{k'}\} = \delta_{jk}
\]
(128)

(all other P.B. being identically zero).

As usual, in terms of the spectral variables, the continuous and the discrete spectrum contributions are separated. Moreover, \(\alpha(\lambda), \{\lambda_j\}, \{\tilde{\lambda}_j\}\) are constant along any flow of the hierarchy, while \(\beta(\lambda), \{\nu_j\}, \{\tilde{\nu}_j\}\) evolve linearly.

The “action” variable \(\alpha(\lambda)\) is defined only on the unit circle. However, it can be expressed as a sum of two functions having a single-valued analytic branch for \(|\lambda| > \max_j |\lambda_j|\) and \(|\lambda| < \min_j |\tilde{\lambda}_j|\) respectively, uniquely defined by the asymptotics.
Such branches will be denoted as $\ln a(\lambda)$ and $\ln \tilde{a}(\lambda)$ respectively. Then formulas (103), (104) imply for such branches the following power series expansions:

$$\ln a(\lambda) = \sum_{k=1}^{\infty} \lambda^{-k} J_k$$  \hspace{1cm} (129)

$$\ln \tilde{a}(\lambda) = \sum_{k=0}^{\infty} \lambda^k \tilde{J}_k$$ \hspace{1cm} (130)

where:

$$J_k = -\frac{1}{2\pi i} \oint_{|z|=1} z^{k-1} \alpha(z)dz + \sum_{j=1}^{N} \frac{1}{k}(\tilde{\lambda}_j^k - \lambda_j^k)$$ \hspace{1cm} (131)

$$\begin{cases}
\tilde{J}_k = \frac{1}{2\pi i} \oint_{|z|=1} z^{k+1} \alpha(z)dz + \sum_{j=1}^{N} \frac{1}{k}(\lambda_j^{-k} - \tilde{\lambda}_j^{-k}) & (k \neq 0) \\
\tilde{J}_0 = \frac{1}{2\pi i} \oint_{|z|=1} z \alpha(z)dz + \sum_{j=1}^{N} (\ln \tilde{\lambda}_j - \ln \lambda_j) & (k = 0)
\end{cases}$$ \hspace{1cm} (132)

In the following, we will show that the evolution equations (65) are generated by the hamiltonians $J_k$, i.e.:

$$\frac{\partial U_n(\lambda)}{\partial t_k} = \{J_k, U_n(\lambda)\}$$ \hspace{1cm} (133)

Indeed, from the Poisson-bracket relation (118), we obtain:

$$\{trT(\mu), U_n(\lambda)\} = \frac{1}{\lambda - \mu}(V_{n+1}(\mu)(\lambda A + Q) - (\lambda A + Q)V_n(\mu)) =$$

$$= V_{n+1}(\mu)A - AV_n(\mu) + \frac{1}{\lambda - \mu}(V_{n+1}(\mu)U(n, \mu) - U(n, \mu)V_n(\mu))$$ \hspace{1cm} (134)

where $V_n$ is defined in terms of the Jost solutions:

$$V_n(\mu) = \begin{pmatrix}
\psi_{2,n}(\mu)\varphi_{1,n}(\mu) & -\psi_{1,n}(\mu)\varphi_{1,n}(\mu) \\
\psi_{3,n}(\mu)\varphi_{2,n}(\mu) & -\psi_{1,n}(\mu)\varphi_{2,n}(\mu)
\end{pmatrix} + \begin{pmatrix}
-\tilde{\psi}_{2,n}(\mu)\tilde{\varphi}_{1,n}(\mu) & \tilde{\psi}_{1,n}(\mu)\tilde{\varphi}_{1,n}(\mu) \\
-\tilde{\psi}_{3,n}(\mu)\tilde{\varphi}_{2,n}(\mu) & \tilde{\psi}_{1,n}(\mu)\tilde{\varphi}_{2,n}(\mu)
\end{pmatrix}$$ \hspace{1cm} (135)

It can be immediately seen that

$$(V_{n+1}(\mu)U(n, \mu) - U(n, \mu)V_n(\mu)) = 0$$

Then formula (134) entails:

$$\{a(\mu), U_n(\lambda)\} = V_{n+1}^{(+)}(A - AV_n^{(+)})$$
\{\tilde{a}(\mu), U_n(\lambda)\} = V_{n+1}^{(-)} A - AV_n^{(-)}

where \(V^\pm\) are of course the projections of \(V\) outside and inside the unit circle. Hence:

\[
\{\ln a(\mu), U_n\} = \bar{W}^{(+)}(n+1) A - AW_n^{(+)}
\]

(136)

\[
\{\ln \tilde{a}(\mu), U_n\} = \bar{W}^{(-)}(n+1) A - AW_n^{(-)}
\]

(137)

where

\[
\bar{W}^{(\pm)} = \begin{cases}
a^{-1}V^{(+)} \\
\tilde{a}^{-1}V^{(-)}
\end{cases}
\]

(138)

are again analytic functions of \(\lambda\) for \(|\lambda| > \max_j |\lambda_j|\) and \(|\lambda| < \min_j |\tilde{\lambda}_j|\) respectively; they obey the asymptotic conditions:

\[
\lim_{|\lambda| \to \infty} \bar{W}^{(+)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(139)

\[
\lim_{|\lambda| \to 0} \bar{W}^{(-)} = \frac{1}{1 + r_nq_{n-1}} \begin{pmatrix} r_nq_{n-1} & q_{n-1} \\ r_n & 1 \end{pmatrix}
\]

(140)

\[
\lim_{n \to \infty} \bar{W}^{(+)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(141)

\[
\lim_{n \to -\infty} \bar{W}^{(-)} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}
\]

(142)

Formulas (136), (137) yield:

\[
\{J_k, U_n\} = \bar{W}^{(+)(k)}_{n+1} A - AW_n^{(+)(k)}
\]

(143)

\[
\{\tilde{J}_k, U_n\} = \bar{W}^{(-)(k)}_{n+1} A - AW_n^{(-)(k)}
\]

(144)

On the other hand, \(W^{(+)}\), defined by (138) and \(W\), defined by (53) (59), obey the same stationary equation (138) and have the same analyticity properties with respect to \(\lambda\). Moreover, their asymptotic behaviours differ essentially just for a constant multiple of the identity matrix, which plays no role in the recursion relation: hence formula (143) coincides with (53).
We end this section by noting that formula (118) does hold even in the \((N + 1) \times (N + 1)\) matrix case. For this general spectral problem, the \(r\)-matrix is given by:

\[
\begin{pmatrix}
    R^1_1 & \ldots & R^N_{1+1} \\
    \vdots & \ddots & \vdots \\
    R^1_{N+1} & \ldots & R^N_{N+1}
\end{pmatrix}
\]

where

\[
(R^l_m)_{ij} = \delta_{il} \delta_{mj} \quad (i, j = 1, \ldots, N + 1)
\]  

(146)

\[\lambda - \mu
\]

(145)

5 Continuum limit

**Theorem 3** In the continuum limit: \(h \to 0, n \to 0, x = nh \) finite, with the rescalings:

\[
q \to h q \quad , \quad r \to hr
\]

(147)

we have:

\[
\lim_{h \to 0} \frac{\mathcal{N} + 1}{h} = \Lambda
\]

(148)

where \(\Lambda\) is the recursion operator of the vector AKNS hierarchy.

In fact, from eq.(33):

\[
K'_q = (q - E) K_q + \left( \frac{E}{E-1} (q K_q + K_q q) + \frac{1}{E-1} (r q K_q + K_q q r) \right) q
\]

(149)

\[
K' r = (q r - E^{-1}) K_r - \left( \frac{1}{E-1} (q K_q + K_q q) + \frac{E}{E-1} (r q K_q + K_q q r) \right) r
\]

by noting that:

\[
E = e^{h \partial_x} = 1 + h \partial_x + O(h^2)
\]

(150)

\[
E^{-1} = e^{-h \partial_x} = 1 - h \partial_x + O(h^2)
\]

(151)

and thus

\[
E^E = \frac{1}{h} \partial_x^{-1} + O(1) \quad ; \quad \frac{1}{E-1} = \frac{1}{h} \partial_x^{-1} + O(1)
\]

(152)
we obtain:

\[
\begin{align*}
K'q &= -Kq + h[\partial_x Kq + \partial_x^{-1}(\langle Kq|r\rangle + \langle q|Kr\rangle + \langle r|q\rangle)]q + O(h^2) \\
K'r &= -Kq + h[\partial_x Kr - \partial_x^{-1}(\langle Kq|r\rangle + \langle q|Kr\rangle + \langle r|q\rangle)]r + O(h^2)
\end{align*}
\]

(153)

so that:

\[
\lim_{h \to 0} \frac{\mathcal{N} + I}{h} = \begin{bmatrix}
-\partial_x + \partial_x^{-1}(\langle r|+|r\rangle)q & -\partial_x^{-1}(\langle q|+|q\rangle)q \\
-\partial_x^{-1}(\langle r|+|r\rangle)q & \partial_x - \partial_x^{-1}(\langle q|+|q\rangle)q
\end{bmatrix}
\]

(154)

which is the recursion operator for the vector AKNS case \([16]\). In particular, in the 2×2 matrix case, we get the familiar recursion operator of the standard AKNS hierarchy:

\[
\lim_{h \to 0} \frac{\mathcal{N} + I}{h} = \begin{bmatrix}
-\partial_x + 2q \int_{-\infty}^{\infty} r \cdot q & 2q \int_{-\infty}^{\infty} q \cdot r \\
-2r \int_{-\infty}^{\infty} r \cdot q & \partial_x - 2r \int_{-\infty}^{\infty} q \cdot r
\end{bmatrix}
\]

(155)

For instance, by taking \((\frac{\mathcal{N} + I}{h})^2\) we get the equations:

\[
\begin{align*}
q_t &= q_{xx} - 2q^2r \\
r_t &= -r_{xx} + 2r^2q
\end{align*}
\]

(156)

It is worth noting that the continuous equation (156) can be so obtained by taking the continuum limit of a suitable linear combination of the \(l = -1, l = 0, l = 1\) flows (eqs. (42), (44), (48)).

6 Concluding remarks

We would like to stress here that the system presented in this paper is, to our Knowledge, the only integrable discrete version of AKNS hierarchy that keeps the canonical Poisson structure of the continuum model and admits a natural vector generalisation.

As further developments of the research reported in this paper, we mention the derivation of Backlund tranformations for our lattice equations, to be considered as integrable fully discrete two-dimensional lattices, and the formulation of a proper quantum version in the framework of the Quantum Inverse Method.

Research has already started on both the above issues, with encouraging preliminary results.
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