Multiparticle entanglement and its applications to cryptography

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Entanglement between three or more parties exhibits a realm of properties unknown to two-party states. Bipartite states are easily classified using the Schmidt decomposition. The Schmidt coefficients of a bipartite pure state encompass all the non-local properties of the state and can be "seen" by looking at one party's density matrix only. Pure states of three and more parties however lack such a simple form. They have more invariants under local unitary transformations than any one party can "see" on their sub-system. These "hidden non-localities" will allow us to exhibit a class of multipartite states that cannot be distinguished from each other by any party. Generalizing a result of BPRST and using a recent result by Nielsen we will show that these states cannot be transformed into each other by local actions and classical communication. Furthermore we will use an orthogonal subset of such states to hint at applications to cryptography and illustrate an extension to quantum secret sharing (using recently suggested ((n,k))-threshold schemes).

I. INTRODUCTION

The entanglement properties of bipartite pure states have already been treated extensively. The analysis of entanglement and its properties for these states is much easier than for 3 or more party shared states due to a particularly convenient form that captures all non-local parameters: the (unique) Schmidt decomposition. An interesting question which arises in attempts to classify entanglement is which states can be obtained from a given state if we allow local actions and classical communication of the parties. By classical communication we mean an a priori unlimited amount of two-way classical communications. We will call these transformations of a k-party state k-LOCC (k-party local operations and classical communication). The crucial difference between pure local unitary action and LOCC is that each party may perform (generalized) measurements on its subsystem and broadcast the outcomes via classical channels between the parties. The other parties may choose their subsequent actions conditional on the outcomes of these measurements.

For bipartite pure states Nielsen has recently found necessary and sufficient conditions for the process of entanglement transformation via 2-LOCC to be possible. A key tool in this result is the Schmidt-decomposition of bipartite states and the conditions involve the Schmidt-coefficients of the states only. So once we are given the density matrix of one party a bipartite pure state contains no more secret to us: The eigenvalues of one party’s density matrix completely characterize the state (up to equivalence under local unitary operations) and give us complete knowledge about its entanglement transformation properties under local operations and classical communication between the parties. In other words given a sufficient supply of copies of a certain state shared by two parties each of the parties is able to determine (up to a certain precision) its equivalence class under local unitaries and which other states it can be transformed into via 2-LOCC.

The situation is drastically different for multipartite states involving more than two parties. No convenient (locally invariant) form–analogous to the Schmidt-decomposition–can be given. The number of invariants of a state under local unitaries grows exponentially with the number of parties (see Section II). Attempts to find canonical points on the orbits of multipartite states have been made but yield unwieldy outcomes. We will say that two multiparty-states are unitarily equivalent (|Ψ⟩ ∼ |Φ⟩) if they can be transformed into each other by local (single-party) unitary operations only (without classical communication). Linden and Popescu have given a lower bound on the number of parameters needed to describe equivalence classes of multipartite states. To parameterize inequivalent states they also exhibited an explicit polynomial form for invariants of a multipartite state under local unitaries (see Section III B). Some of these invariants are functions of the eigenvalues of the local density matrices of all parties. For three (and more) parties however the number of independent invariants under local unitaries is bigger than the number of independent eigenvalues of all local density matrices. This means that if we get all possible information from each party’s subsystem there will be invariants under local unitaries that we cannot determine. We will call these parameters hidden non-localities of our quantum-state.

Complete knowledge of each local system thus does not

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give us complete information on the equivalence class of the multipartite state under local unitary operations. Let’s review Nielsen’s result to make the difference between bipartite and multipartite states more precise: For bipartite pure states and 2-LOCC there is a partial ordering on the states that characterizes their mutual entanglement transformation properties [1]:

\[
|\Psi\rangle \xrightarrow{2\text{-LOCC}} |\Phi\rangle \text{ iff } \rho^\Psi_A \prec \rho^\Phi_A
\]  

(1)

where \( \rho_A \) is the density matrix of one party and \( \rho^\Psi_A \prec \rho^\Phi_A \) means that the eigenvalues \( \lambda_1^\Psi, \ldots, \lambda_k^\Psi \) of \( \rho^\Psi_A \) are majorized by the eigenvalues \( \lambda_1^\Phi, \ldots, \lambda_k^\Phi \) of \( \rho^\Phi_A \), i.e.

\[
\sum_{i=1}^k \lambda_i^\Psi \leq \sum_{i=1}^k \lambda_i^\Phi \quad \forall k
\]  

(2)

The arrow indicates that the eigenvalues have to be put into decreasing order.

This gives a partial ordering in the space of all non-local parameters of bipartite states (remember that the non-local parameters are the independent eigenvalues of the density matrix of one party). In the case of just two qubits shared by two parties this even gives a total ordering on the states meaning that given two states either the first can be transformed into the second or vice versa (there is only one independent Schmidt-coefficient). Among higher dimensional bipartite states however we also find sets of states that cannot be transformed into each other either way by LOCC. These states have been termed incommensurate. The smallest system to provide us with two bipartite incommensurate pure states is the 9-dimensional space of two qutrits [2]. Also note that the commensurateness or incommensurateness of two bipartite states can be immediately identified by looking at the density matrix of one subsystem.

Two bipartite pure states whose one-party density matrices have the same eigenvalues are always mutually obtainable from each other via 2-LOCC. Also

\[
\frac{1}{n} I \prec \rho_A^\Psi \quad \forall |\Phi\rangle
\]  

(3)

implies that starting with an EPR-type bipartite state (unique up to local unitaries with the property that its density matrix obtained by tracing out one party is proportional to the identity matrix) we can extract every given bipartite state \(|\Phi\rangle\) with local operations and classical communication. The partially ordered set of states under 2-LOCC has just one maximal state (up to unitary equivalence)!

We will show that this structure is very different for multipartite states. BPRST\(^1\) [1, 2] have found two 3-party states—each party having 2 qubits—of dimension \(2^6\) that are incommensurate although all of their sub-density matrices are identical. Following their argument we will use Nielsen’s result [1] to generalize their proof and show that even for the smallest 3-party state (of dimension 8) there are incommensurate states that have identical or similar local density matrices. Their incommensurateness can not be “seen” by looking at subsystems of the state (it is hidden). We connect hidden non-localities to hidden incommensurateness to see that two multipartite states with similar density matrices on each party are incommensurate if and only if they are not unitarily equivalent.

We give some examples of locally equivalent k-LOCC incommensurate states. We will suggest how to “encode into hidden non-localities” with the help of an orthogonal subset of such states. These states have the property that they are totally indistinguishable from each other for each party alone and cannot be transformed into each other by local operations and classical communication between the parties. Furthermore we can find a set of such states that are maximal in the sense that they cannot be obtained from any other (unitarily not equivalent) state by k-LOCC. Only if the parties perform a collective (orthogonal) measurement they will be able to (perfectly) distinguish these states. This area needs further exploration.

We will analyze a recently suggested cryptographic protocol for quantum secret sharing to identify a class of incommensurate and locally equivalent states in them. An \(((n, k))\) threshold scheme \((k < n)\) is a method to encode and divide a secret quantum state between \(n\) parties such that from any \(k\) shares the state can be perfectly recovered and from any \(k - 1\) or fewer shares no information whatsoever about the state can be inferred. The scheme as introduced in [6] assumes that all parties are honest when they participate in reconstructing the secret. Allowing for the possibility of some parties being dishonest in order to retrieve the secret alone we will show how the scheme can be “misused” for cheating by one party if it is used to encode a “classical” bit and how this cheating can be prevented by using incommensurate locally identical states.

II. COUNTING HIDDEN NON-LOCALITIES

Linden and Popescu [3] have classified the orbits of multipartite states under local unitary operations and determined the dimension of generic orbits and the number of parameters needed to describe the location of such an orbit in Hilbert space. In the case of \(k\) parties each having one spin 1/2 particle (qubit) there at least \(2^{k+1} - 2 - 3k\) real parameters that characterize non-local properties. (Initially each \(2^k\)-dimensional state has \(2^k\) complex parameters and the requirement of unit norm leaves \(2^{k+1} - 1\) real parameters. The group of equivalence transformations is \(U(1) \times SU(2) \times SU(2) \times \ldots \times SU(2)\) each local unitary \(U(2) \simeq U(1) \times SU(2)\) but each local phase can be fac-
tored out to one single global phase. The group (and the
generic orbit) then has dimension $3k+1$ or less.)
Furthermore an explicit form for polynomial invariants
of an orbit has been given [3] (see Section IIIB). To get
a picture of how the number of hidden non-localities
grows let’s analyze the three, four and five-party cases.
We can easily count parameters in the three party spin
1/2-case: We have 5 (independent) non-local parameters
but only three (in general) different density matrices each
characterized by one eigenvalue. So there are 2 non-local
parameters that we cannot “see” by only looking at vari-
sous subsystems of our entangled state. Now let’s look
at the four-party spin-1/2 case. Here we have at least
18 non-local parameters but at most 7 independent sub-
density matrices where two density matrices are inde-
pendent if the eigenvalues of the first do not completely
determine the eigenvalues of the second. The four one-
party matrices each have one non-local parameter and
the 3 independent density matrices of 2 joint parties have
at most 3 parameters each thus leaving a total of at least
$18 - 4 \times 1 - 3 \times 3 = 5$ hidden non-localities. Of the 18
non-local parameters 14 cannot be seen by only one party
alone, they are hidden if we look at one-party subsystems
only. For the 3-party case the number of non-local pa-
rarneters that cannot be seen by looking at each party
locally only is $\leq 58$. There are at least 18 non-local pa-
rarneters that cannot be accessed by looking at any one
and two-party sub-density matrices of the system.
In general the number of hidden nonlocalities grows ex-
ponentially with the number of parties.

III. A CLASS OF 3-LOCC INCOMMENSURATE
STATES

We now want to show that there are 3-LOCC incom-
mensurate states for the 3-spin-1/2 system. There are 5
independent invariants under local unitaries, 3 of them
are of the form $tr \rho_p^2$ ($p = A, B, C$) and completely
characterize the eigenvalues of $\rho_p$ (see Section IIIB). Suppose
that we have two states $|\Psi\rangle$ and $|\Phi\rangle$ that differ only in
the last two hidden invariants, i.e. the three one-party
density-matrices of $|\Psi\rangle$ and $|\Phi\rangle$ have the same eigenval-
es. For the ease of argument choose them such that $\rho_A$,
$\rho_B$ and $\rho_C$ have full rank 2.
Claim: $|\Psi\rangle$ and $|\Phi\rangle$ are 3-LOCC incommensurate.
Proof: How does a general 3-LOCC protocol look like?
First one party, say Alice, will perform a generalized
measurement and broadcast her outcome. Then Bob
and Charlie will continue with generalized measurements
on their subsystems conditional on Alice’s outcome and
broadcast their outcomes. At a certain point Alice will
continue and so on. Let’s for a moment (mentally) merge
Bob’s and Charlie’s systems and look at the 3-LOCC pro-
tocol as a protocol between the systems A and BC. Ev-
erything that Bob and Charlie do after receiving Alice’s
outcome and before Alice’s next action can be viewed as
a generalized measurement on the BC subsystem. So the
whole 3-LOCC protocol can be viewed as a specific case
of a 2-LOCC protocol between A and BC. In particular
this means that if $|\Psi\rangle$ could be transformed into $|\Phi\rangle$ via
3-LOCC it certainly could be transformed into $|\Phi\rangle$ via
2-LOCC on A and BC.
Assume there were a 3-LOCC protocol that transforms
$|\Phi\rangle$ to $|\Psi\rangle$. We have chosen the states such that
$$\rho_A^\Psi \sim \rho_A^\Phi \sim \begin{pmatrix} \cos^2 \alpha & 0 \\ 0 & \sin^2 \alpha \end{pmatrix}.$$  (4)

Let $|v_A\rangle$ and $|v_A^+\rangle$ be the two eigenvectors of $\rho_A^\Psi$ and rewrite
$$|\Psi\rangle = \cos \alpha |v_A\rangle |v_{BC}\rangle + \sin \alpha |v_A^+\rangle |v_{BC}^+\rangle.$$  (5)

This is the Schmidt decomposition of $|\Psi\rangle$ as a bipartite
A-BC state. In particular $|v_{BC}\rangle$ and $|v_{BC}^+\rangle$ are orthogonal
in the joint BC-Hilbert space.
Alice performs the first generalized measurement $M = \{M_1, M_2, \ldots\}$ (with $\sum_i M_i^\dagger M_i = I$) and obtains the out-
come $i$. Hereby she transforms the state $|\Psi\rangle$ to a state $|\Psi_i\rangle$ with
$$|\Psi_i\rangle = \frac{1}{N_i}(\cos \alpha M_i |v_{A}\rangle |v_{BC}\rangle + \sin \alpha M_i |v_{A}^+\rangle |v_{BC}^+\rangle).$$  (6)

$(N_i$ is the normalization factor.)
From here Bob and Charlie will continue $|\Psi_i\rangle \xrightarrow{3-LOCC} |\Phi_i\rangle$
so in particular we know that $|\Psi_i\rangle \xrightarrow{2-LOCC_{A,BC}} |\Phi_i\rangle$. From
$$|\Psi\rangle \xrightarrow{2-LOCC} |\Psi_i\rangle \xrightarrow{2-LOCC_{A,BC}} |\Phi_i\rangle.$$  (7)

Nielsen’s criterion [3] tells us
$$\rho_A^\Psi \prec \rho_A^\Phi \prec \rho_A^\Psi \sim \rho_A^\Psi.$$  (8)
so $\rho_A^\Psi = \cos^2 \alpha |v\rangle \langle v| + \sin^2 \alpha |v^\perp\rangle \langle v^\perp|$ has to have the same
eigenvalues as
$$\rho_A^\Psi = \frac{1}{N_2^2}(\cos^2 \alpha M_i |v\rangle \langle v| M_i^\dagger + \sin^2 \alpha M_i |v^\perp\rangle \langle v^\perp| M_i^\dagger) = \frac{M_i}{N} \rho_A^\Psi \frac{M_i}{N}.$$  (9)

This implies that there is a unitary transformation $U$ s.t.
$$U^\dagger \rho_A^\Psi U = \frac{M_i}{N} \rho_A^\Psi \frac{M_i}{N}.$$  (10)

It follows that $U \frac{M_i}{N}$ has to be unitary and diagonal in the
same basis as $\rho_A^\Psi$ (just pick a basis where $\rho_A^\Psi$ is diagonal
and write out the matrix-elements in their most general
form using that $\rho_A^\Psi$ has full rank). We see that $M_i |v_A\rangle$
and $M_i|\psi_A^1\rangle$ have to be orthogonal and that Alice’s generalized measurement reduces to a local unitary operation on her qubit.

Continuing this argument for each subsequent step of the 3-LOCC protocol it follows that the whole protocol ends up to be a succession of local unitaries. But we have chosen our states to be non-equivalent under local unitaries. This completes the proof.

Note that the constraint to full-rank local density matrices can be lifted if we only look at the restriction of $M_i$ onto the support of $\rho_A, \rho_B, \rho_C$.

We have thus shown that even in the simplest 3-party case there are states that—having the same eigenvalues of all sub-density-matrices—are 3-LOCC-incommensurate. Furthermore once we fix the eigenvalues of $\rho_A, \rho_B, \rho_C$ we have two additional parameters to specify different classes of 3-LOCC-incommensurate states. In the 5-dimensional space of unitarily non-equivalent states we have found a 2-dimensional subspace of mutually incommensurate states.

This proof generalizes trivially to more than 2 dimensions of each party’s Hilbert space and to $k \geq 3$ parties. To see the latter we note that at each step of a $k$-LOCC protocol we can divide the system into two parts—one party that performs a local operation and the other $k-1$ parties—and apply Nielsen’s criterion as in the 3-party case.

It follows from our proof that throughout each step of a $k$-LOCC transformation protocol each party’s density matrices of the state obtained at a particular step have to have rank 3 at least. But even for the smallest three-party system of dimension 8 there are incommensurate states. We can find states with identical local states—gives separable density matrices when tracing out any one party—whereas 3EPR is not tri-separable. (Tracing out $A$ in 3EPR gives $\frac{1}{4}I \otimes |EPR_{B2C2}\rangle\langle EPR_{B2C2}|$ which is obviously not separable.)

Note that this proof generalizes trivially to $k$-partite states and $k$-LOCC:

$\textit{(k-1)GHZ and } (\frac{k}{2}) \textit{ EPR are } k\textit{-LOCC incommensurate}.$

B. Two locally non-distinguishable 3-LOCC incommensurate states of dimension 8

Note that the smallest bipartite system that contains two incommensurate states has to have dimension 9 at least with each party possessing a qutrit. This is because two density matrices that are not majorized either way have to have rank 3 at least. But even for the smallest three-party system of dimension 8 there are incommensurate states. We can find states with identical local density matrices that cannot be transformed into each other via 3-LOCC. To keep calculations easier we looked for particularly simple states of the following form:

$$|\Psi\rangle = \alpha_+|000\rangle + \alpha_-|vvv\rangle$$

where $|v\rangle$ is a normalized state. The equivalence classes of these states are characterized by two parameters—say $\alpha_+\alpha_-^*$ and $|\langle 0|v\rangle|$—and have equivalent density matrices on all three sub-parties. So from the 5 independent invariants of (generic) states under local unitary transforms in this case only (at most) 2 are algebraically independent.

Let’s look at the invariants in the general case for states of the form $|\Psi\rangle = \sum_{i,j,k} \alpha_{ijk}|e_i e_j e_k\rangle$. From the coefficients $\alpha_{ijk}$ we can form polynomials that are manifestly incommensurate states.

3EPR = $\frac{(|0_{A10_{B1}}\rangle + |1_{A11_{B1}}\rangle)}{\sqrt{2}} \otimes \frac{(|0_{A20_{B2}}\rangle + |1_{A21_{B2}}\rangle)}{\sqrt{2}}$ (12)

In the 3EPR-state the three parties Alice, Bob and Charlie share three EPR-pairs, one between A and B, one between A and C and one between B and C. In the 2GHZ state they share just share two GHZ-states. In both cases the density matrices of Alice, Bob and Charlie are identical:

$$\rho = \frac{1}{4}I$$ (13)

So in any LOCC transformation protocol from 2GHZ to 3EPR and vice versa Alice, Bob and Charlie are restricted to local unitaries. It is however impossible to transform 2GHZ to 3EPR via local unitaries. One simple way to see this is to observe that 2GHZ is a tri-separable state—gives separable density matrices when tracing out any one party—whereas 3EPR is not tri-separable. (Tracing out $A$ in 3EPR gives $\frac{1}{4}I \otimes |EPR_{B2C2}\rangle\langle EPR_{B2C2}|$ which is obviously not separable.)
invariant under local unitaries, like the degree 2 polynomial:

\[ I_1 = \sum_{ijk} \alpha_{ijk} \alpha^*_{ijk} \]  

which is the norm of the state. To fourth degree we get three polynomials:

\[ I_2 = \sum_{ijkmpq} \alpha_{kij} \alpha^*_{mij} \alpha_{mpq} \alpha^*_{kpq} = tr \rho_A^2 \]

\[ I_3 = \sum_{ijkmpq} \alpha_{kij} \alpha^*_{imj} \alpha_{pmq} \alpha^*_{pkq} = tr \rho_B^2 \]

\[ I_4 = \sum_{ijkmpq} \alpha_{ijk} \alpha^*_{ijm} \alpha_{pqm} \alpha^*_{pkq} = tr \rho_C^2 \]

which in general are algebraically independent. One of the higher degree invariants is

\[ I_5 = \sum_{ijklmnopq} \alpha_{ijk} \alpha^*_{ilm} \alpha_{nlo} \alpha^*_{pjo} \alpha_{pqm} \alpha^*_{nqk} \]  

which in general is not algebraically dependent of \( I_2, I_3, I_4 \). For the simple state above \( (14) \) we have \( I_2 = I_3 = I_4 \) and a symbolical calculation (Groebner Basis) shows that \( I_5 \) and \( I_2 \) are algebraically independent. We now exhibit two states of the above simple form \( (14) \) which have similar one-party density matrices (and the same \( I_2 \)) but different \( I_5 \) thus being 3-LOCC incommensurate:

\[ |\Psi\rangle = 2 \frac{\sqrt{3}}{37} |000\rangle - \frac{5}{\sqrt{37}} |111\rangle \]  

and

\[ |\Phi\rangle = 4 \frac{\sqrt{2}}{37} |000\rangle - \frac{5}{\sqrt{37}} |\nu\rangle \]  

where \( |\nu\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \) is the state \( |1\rangle \) rotated by 45 degrees.

\[ I_2^\Psi = I_2^\Phi = \frac{769}{1369} \]  

and

\[ I_5^\Psi \approx 0.343 \neq I_5^\Phi \approx 0.242 \]

So these two states are 3-LOCC incommensurate. We can apply a local unitary transformation on each subsystem to one of the states to make their density-matrices diagonal in the same basis so that they are completely indistinguishable for each party.

C. The \((3, 2)\)-threshold states

In a recent paper an encoding of a qudit into a tripartite state has been given (see Section VA). The encoded state is of the following form:

\[ |\Phi(\alpha, \beta, \gamma)\rangle = \alpha(|000\rangle + |111\rangle + |222\rangle) + \beta(|012\rangle + |120\rangle + |201\rangle) + \gamma(|021\rangle + |102\rangle + |210\rangle) \]  

The density matrix of any one party is proportional to the identity matrix. So all of these states have the same one-party density matrices. Most of these states will differ in the hidden nonlocalities and be 3-LOCC incommensurate. Here we will give a set of three locally indistinguishable orthogonal states of the above form:

\[ |\Phi_1\rangle = |\Phi(1, 0, 0)\rangle = (\langle 000 | + |111 | + |222 |) \]

\[ |\Phi_2\rangle = |\Phi(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\rangle = \frac{1}{\sqrt{2}} (\langle 012 | + |120 | + |201 |) \]

\[ + \frac{1}{\sqrt{2}} (|021 | + |102 | + |210 |) \]

\[ |\Phi_3\rangle = |\Phi(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})\rangle = \frac{1}{\sqrt{2}} (\langle 012 | + |120 | + |201 |) \]

\[ - \frac{1}{\sqrt{2}} (|021 | + |102 | + |210 |) \]

These states differ in the value of \( I_5 \) \((17)\) which takes on 1/9, 1/18 and 0 for the three states respectively. They are thus 3-LOCC incommensurate.

IV. CRYPTOGRAPHY–ENCODING INTO HIDDEN NONLOCALITIES

We have exhibited states that cannot be transformed into each other by local operations and classical communication involving three parties and shown that there is a large number of them. We think that these states can have a fruitful application in (quantum)-cryptographic protocols like 3-party quantum bit commitment schemes. We can produce states that for each subsystem are indistinguishable and yet have some hidden non-local property that makes them different.

How could we \textbf{encode information into the hidden nonlocalities} and how can we access them? One possibility is to find orthogonal states that have the same respective sub-party density matrices and differ only in these hidden parameters. Encode a bit-string into each of those and give a part of the corresponding state to the three parties, A, B and C without telling them which specific state they share. While the parties are locally separated and only allowed to perform local actions and classical communication they have no way to transform the states into each other. Only when they get together (or send their share of the state through a quantum channel) they can perform an orthogonal measurement and determine the encoded bitstring. To ensure that there is no common state \( \Omega \) from which two different states can be obtained via \( k \)-LOCC we can choose states with local density matrices proportional to the identity. Since the identity matrix on a subsystem majorizes every other density matrix,
The singlet state \( \rho_A' \) would have to be proportional to the identity as well. We have shown that in this case each party is restricted to local unitaries in their attempt to change the state via LOCC.

Another example of 2 tripartite states (apart from (23)) that have identical one-party density matrices, are 3-LOCC-incommensurate and orthogonal is the actual 2GHZ-3EPR example from Section II.A if we use the singlet state

\[ \text{EPR}' = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \quad (24) \]

instead of the EPR states. 3EPR' and 2GHZ are 3-LOCC incommensurate and orthogonal.

A detailed analysis of some cryptographic schemes for the potential use of incommensurate states should be done. Here we will restrict ourselves to a rather illustrative example involving quantum secret sharing.

A. How Bob can cheat using the ((3, 2)) threshold scheme and how to prevent that

The ((3, 2)) threshold scheme in 3 encodes a qutrit

\[ |\Psi\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle \quad (25) \]

into the state \( |\Phi(\alpha, \beta, \gamma)\rangle \) (22). Each party obtains one qutrit of the encoded state: Alice the first, Bob the second and Charlie the third. This scheme allows any two parties together to completely extract the secret state \( |\Psi\rangle \) (23). But no party alone can infer any information about the secret state \( |\Psi\rangle \): each party’s local density matrix is proportional to the identity.

The procedure to retrieve \( |\Psi\rangle \) from say the first two qutrits is the following 3: First the first register is added to the second (modulo 3) and then the (resulting) second qutrit is added to the first (modulo 3). These operations can be performed without any measurement. This changes an encoded state \( |\Phi(\alpha, \beta, \gamma)\rangle \) to

\[
\begin{align*}
\alpha(|000\rangle + |111\rangle + |222\rangle) &+ \\
\beta(|012\rangle + |120\rangle + |201\rangle) &+ \\
\gamma(|021\rangle + |102\rangle + |210\rangle) &\rightarrow \\
\frac{\alpha(|000\rangle + |021\rangle + |012\rangle)}{\sqrt{3}} &+ \\
\frac{\beta(|112\rangle + |100\rangle + |121\rangle)}{\sqrt{3}} &+ \\
\frac{\gamma(|221\rangle + |212\rangle + |200\rangle)}{\sqrt{3}} &
\end{align*}
\]

\[ = (\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \otimes (|00\rangle + |21\rangle + |12\rangle) \quad (26) \]

The secret state is completely restored in the first register. Analogous decoding procedures apply for AC and BC.

In the original scheme 3 it is assumed that the parties are honest when they participate in reconstructing the secret quantum state.

Now assume the president of the bank uses this procedure to encode one of three (classical) “trits” \( b = 0, 1 \) or 2). He may want to use three orthogonal states \( |\Phi_0\rangle \), \( |\Phi_1\rangle \), \( |\Phi_2\rangle \) that can be completely distinguished by an orthogonal measurement. So he distributes one of three known orthogonal states to his three vice-presidents. He does not want any of them alone to get knowledge about the encoded trit, only two of them together should be able to find out what the secret was. For illustration let’s suppose that he uses the states

\[ |\Psi_0\rangle = |0\rangle \quad |\Psi_1\rangle = |1\rangle \quad |\Psi_2\rangle = |2\rangle \quad (27) \]

to encode \( b = 0, 1 \) and 2 respectively and creates and distributes one of the three encoded states \( |\Phi(1,0,0)\rangle \), \( |\Phi(0,1,0)\rangle \) resp. \( |\Phi(0,0,1)\rangle \). The three parties know the set of encoded states but not the actual state they are sharing.

Now let’s assume Bob decides to obtain the secret on his own without having to share his knowledge with Alice or Charlie. He thinks of the following strategy:

He applies a unitary transformation \( U \) to his share of the encoded secret state

\[ U : |0\rangle \rightarrow |1\rangle \quad |1\rangle \rightarrow |2\rangle \quad |2\rangle \rightarrow |0\rangle \quad (28) \]

The set of encoded states after this transformation will have changed to

\[
\begin{align*}
|\Phi(1,0,0)\rangle &\rightarrow (|001\rangle + |121\rangle + |202\rangle) \\
|\Phi(0,1,0)\rangle &\rightarrow (|022\rangle + |100\rangle + |211\rangle) \\
|\Phi(0,0,1)\rangle &\rightarrow (|001\rangle + |112\rangle + |220\rangle)
\end{align*}
\]

(29)

Alice and Charlie have no way of detecting Bob’s dishonest action. Suppose now that at the time for two parties to find out what the secret was, Alice and Bob were the two to jointly retrieve the state. If they apply (26) to the changed state they obtain

\[
\begin{align*}
b = 0 : & \quad |1\rangle \\
b = 1 : & \quad |2\rangle \otimes (|10\rangle + |01\rangle + |22\rangle) \\
b = 2 : & \quad |0\rangle
\end{align*}
\]

(30)

At the end of this procedure Alice and Bob are supposed to know the value of \( b \). Assume \( b = 0 \). Alice will think that \( b = 1 \). Bob, however, having changed the state, knows that if he jointly with Alice gets the outcome “\( b = 1 \)”, the actual trit \( b \) is 0! So he has obtained the actual secret alone and misled Alice! Bob can apply \( U^{-1} \) afterwards to erase the traces of his cheating completely. Similar misleading happens if Bob and Charlie retrieve the secret. Of course if Alice and Charlie were the two to recover the secret trit Bob’s action would not help and they will obtain \( b = 0 \).

This type of cheating is possible, because the set of orthogonal states chosen to encode \( b \) is equivalent under local unitaries. Bob has applied a local unitary transformation \( U \) to change \( \rho_{AB}^b \) to \( \rho_{AB}^b \) etc. Hereby he has changed the state corresponding to \( b = 0 \) to a state \( |\Psi\rangle \) with \( \rho_{AB}^\Psi = \rho_{AB}^b \cdot |\Psi\rangle \) and the actual state corresponding to \( b = 1 \) are related by a local unitary on Charlie’s
system ($\Psi$) is a purification of $\rho_{AB}^1$ and so is the state corresponding to $b = 1$). This can only be possible if the states encoding $b = 0$ and $b = 1$ are unitarily equivalent. To prevent this type of cheating by a dishonest misleading party, the president of the bank has to select a set of incommensurate orthogonal states like (23). They are not transformable into each other by any local action (and classical communication). The class of incommensurate states has helped us to choose a quantum secret.

V. CONCLUSION

We have exhibited a class of locally equivalent multipartite states that belong to an essential different class of entanglement. Actually almost all locally similar multi-particle states cannot be transformed into each other either way by local operations and classical communication: they are incommensurate. The partial order induced on multipartite states by transformation via $k$-LOCC is different from the bipartite case: There is a multidimensional manifold of unitarily nonequivalent states that are maximal in the sense that there is no other state from which they can be obtained by $k$-LOCC. The number of parameters to characterize different classes of entanglement grows exponentially with the number of parties involved. This space of locally indistinguishable and yet incommensurate states suggests itself for cryptographic applications involving several parties. We have shown that a set of incommensurate orthogonal and locally indistinguishable states can improve an ($n, k$)-threshold scheme against a form of cheating by a party. Other possible applications in cryptography should be investigated. For instance, it is conceivable to find states $|\Omega\rangle$ shared between $k$ parties such that any of them by choosing a local action could transform the whole state into either $|\Phi\rangle$ or $|\Psi\rangle$ where the last two states have the same local density matrices for each party. This shared state can then be used to share a secret between multiple users that none of them can reveal to an outsider. Only in getting together they can find out what the secret was. The partial order of multipartite states should be investigated beyond classes of locally equivalent states. Another way to follow would be to suggest “multipartite” quantum bit commitment schemes involving sets of incommensurate states. Note that all proofs of the “no-go” theorem [3], [4] for two-party quantum bit commitment schemes (like [6]) use the Schmidt-decomposition of a bipartite state (or in other words the non-existence of hidden parameters for two-party entanglement!). Multi-party protocols do not obey their line of argument.

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