UNIFORM STABILITY OF THE BALL WITH RESPECT TO THE FIRST DIRICHLET AND NEUMANN $\infty$–EIGENVALUES

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Abstract. In this note we analyze how perturbations of a ball $B_r \subset \mathbb{R}^n$ behaves in terms of their first (non-trivial) Neumann and Dirichlet $\infty$–eigenvalues when a volume constraint $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r)$ is imposed. Our main result states that $\Omega$ is uniformly close to a ball when it has first Neumann and Dirichlet eigenvalues close to the ones for the ball of the same volume $B_r$. In fact, we show that, if
\[ |\lambda_D^{\infty}(\Omega) - \lambda_D^{\infty}(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_N^{\infty}(\Omega) - \lambda_N^{\infty}(B_r)| = \delta_2, \]
then there are two balls such that
\[ B_r(\delta_1 r + 1) \subset \Omega \subset B_r(\delta_2 r - 1). \]

In addition, we also obtain a result concerning stability of the Dirichlet $\infty$–eigenfunctions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (connected open subset) with smooth boundary, $1 < p < \infty$ and $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$ (the standard $p$-Laplacian operator). Historically (cf. [13]), it well-known that the first eigenvalue (referred as the principal frequency in physical models) of the $p$–Laplacian Dirichlet eigenvalue problem
\[ \left\{ \begin{array}{ll} -\Delta_p u &= \lambda_{1,p}^{D}(\Omega)|u|^{p-2}u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{array} \right. \]
can be characterized variationally as the minimizer of the following (normalized) problem:
\[ (\text{p-Dirichlet}) \quad \lambda_{1,p}^{D}(\Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla u|^p \, dx : \int_{\Omega} |u|^p \, dx = 1 \right\}. \]

In the theory of shape optimization and nonlinear eigenvalue problems obtaining (sharp) estimates for the eigenvalues in terms of geometric quantities of the domain (e.g. measure, perimeter, diameter, among others) plays a fundamental role due to several applications of these problems in pure and applied sciences. We recall that the explicit value to (\text{p-Dirichlet}) is known only for some specific values of $p$ or for very particular domains $\Omega$. Notice that upper bounds for $\lambda_{1,p}^{D}(\Omega)$ are usually obtained by selecting particular test functions in (\text{p-Dirichlet}). Nevertheless, lower bounds are a more challenging task. In this direction we have the remarkable

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Faber-Krahn inequality: Among all domains of prescribed volume the ball minimizes \((p\text{-Dirichlet})\). More precisely,
\[
\lambda_{1,p}^{D}(\Omega) \geq \lambda_{1,p}^{D}(\mathfrak{B}),
\]
where \(\mathfrak{B}\) is the \(n\)-dimensional ball such that \(\mathcal{L}^{n}(\mathfrak{B})\) (along this paper \(\mathcal{L}^{n}(\Omega)\) will denote the Lebesgue measure of \(\Omega\) that is assumed to be fixed). Using isoperimetric or isodiametric inequality similar lower bounds for \((p\text{-Dirichlet})\) in terms of the perimeter (resp. diameter) of \(\Omega\) are also available (cf. \cite{1} and \cite{14} page 224, and the references therein). Recently, stability estimates for certain geometric inequalities were established in \cite{10}, thereby providing an improved version of (1.2) by adding a suitable remainder term, i.e.,
\[
\lambda_{1,p}^{D}(\Omega) \geq \lambda_{1,p}^{D}(\mathfrak{B}) \left(1 + \gamma_{p,n}(S(\Omega))^{2+p}\right),
\]
where \(S(\Omega)\) is the so-called Fraenkel asymmetry of \(\Omega\), which is precisely defined as
\[
S(\Omega) := \inf_{x_{0} \in \mathbb{R}^{n}} \left\{ \frac{\mathcal{L}^{n}(\Omega \triangle \mathfrak{B}_{r}(x_{0}))}{\mathcal{L}^{n}(\Omega)} : \mathcal{L}^{n}(\mathfrak{B}_{r}(x_{0})) = \mathcal{L}^{n}(\Omega) \right\},
\]
and \(\gamma_{p,n}\) is a constant. Observe that \(S\) measures the distance of a set \(\Omega\) from being a ball. For such quantitative estimates and further related topics we quote \cite{2}, \cite{4}, \cite{9} and references therein.

Our main goal here is to find stability results for the limit case \(p = \infty\).

First, we introduce what is known for the limit as \(p \to \infty\) in the eigenvalue problem for the \(p\)-Laplacian. When one takes the limit as \(p \to \infty\) in the minimization problem \((p\text{-Dirichlet})\), one obtains
\[
(\infty\text{-Dirichlet}) \quad \lambda_{1,\infty}^{D}(\Omega) := \lim_{p \to \infty} \sqrt[p]{\lambda_{1,p}^{D}(\Omega)} = \inf_{u \in W_{0}^{1,\infty}(\Omega) \setminus \{0\}} \|\nabla u\|_{L^{\infty}(\Omega)} > 0,
\]
see \cite{11}. Concerning the limit equation, also in \cite{11} it is proved that any family of normalized eigenfunctions \(\{u_{p}\}_{p \geq 1}\) to \((p\text{-Dirichlet})\) converges (up to a subsequence) locally uniformly to \(u_{\infty} \in W_{0}^{1,\infty}(\Omega)\), a minimizer for \((\infty\text{-Dirichlet})\) with \(\|u_{\infty}\|_{L^{\infty}(\Omega)} = 1\). Moreover, the pair \((u_{\infty}, \lambda_{1,\infty}^{D}(\Omega))\) is a nontrivial solution to
\[
\begin{cases}
\min \left\{ -\Delta_{\infty} v_{\infty}, |\nabla v_{\infty}| - \lambda_{1,\infty}^{D}(\Omega) v_{\infty} \right\} = 0 & \text{in } \Omega \\
v_{\infty} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Solutions to (1.3) must be understood in the viscosity sense (cf. \cite{6} for a survey) and \(\Delta_{\infty} u(x) := \nabla u(x)^{T} D^{2} u(x) \cdot \nabla u(x)\) is the well-known \(\infty\text{-Laplace operator}. In addition, also in \cite{11}, it is given an interesting and useful geometrical characterization for \((\infty\text{-Dirichlet})\):
\[
(1.4) \quad \lambda_{1,\infty}^{D}(\Omega) = \left( \max_{x \in \Omega} \text{dist}(x, \partial \Omega) \right)^{-1}.
\]
Such an information means that the “principal frequency” for the \(\infty\)-eigenvalue problem can be detected from the geometry of the domain: it is precisely the reciprocal of radius \(r_{\Omega} > 0\) of the largest ball inscribed in \(\Omega\). For more references concerning the first eigenvalue (1.3) we refer to \cite{12}, \cite{15} and \cite{18}.
Now, let us turn our attention to Neumann boundary conditions and consider the following eigenvalue problem:

\[
\begin{cases}
-\Delta_p u = \lambda N_1 p(\Omega)|u|^{p-2}u & \text{in } \Omega \\
|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

As before, we stress that the first non-zero eigenvalue of (1.5) can also be characterized variationally as the minimizer of the following normalized problem:

\[
(p\text{-Neumann}) \quad \lambda N_1 p(\Omega) := \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p dx : \int_\Omega |u|^p dx = 1 \text{ and } \int_\Omega |u|^{p-2}u dx = 0 \right\}.
\]

The celebrated Payne-Weinberger inequality provides a lower bound (on any convex domain \(\Omega \subset \mathbb{R}^n\)) for the first (non-trivial) Neumann \(p\)-eigenvalue (cf. \[8\] and \[17\])

\[
\lambda N_1 p(\Omega) \geq (p - 1) \left( \frac{2\pi}{p \cdot \text{diam}(\Omega) \cdot \sin(\frac{\pi}{p})} \right)^p.
\]

For a stability estimate for this problem with \(p = 2\) we refer to \[2\].

When \(p \to \infty\), the minimization problem \((p\text{-Neumann})\) becomes

\[
(\infty\text{-Neumann}) \quad \lambda N_1 \infty(\Omega) := \lim_{p \to \infty} \sqrt[p]{\lambda N_1 p(\Omega)} = \inf_{u \in W^{1,\infty}(\Omega)} \|\nabla u\|_{L^\infty(\Omega)},
\]

see \[7\] and \[16\]. Concerning the limit equation, also in \[7\] and \[16\], it is proved that any family of normalized eigenfunctions \(\{u_p\}_{p>1}\) to \((p\text{-Neumann})\) converges (up to subsequence) locally uniformly to \(u_\infty \in W^{1,\infty}(\Omega)\) with \(\|u_\infty\|_{L^\infty(\Omega)} = 1\). Moreover, the pair \((u_\infty, \lambda N_1 \infty(\Omega))\) is a nontrivial solution to

\[
\begin{cases}
\min \left\{ -\Delta_\infty v_\infty, |\nabla v_\infty| - \lambda N_1 \infty(\Omega) v_\infty \right\} = 0 & \text{in } \Omega \cap \{v > 0\} \\
\max \left\{ -\Delta_\infty v_\infty, -|\nabla v_\infty| - \lambda N_1 \infty(\Omega) v_\infty \right\} = 0 & \text{in } \Omega \cap \{v < 0\} \\
-\Delta_\infty v_\infty = 0 & \text{in } \Omega \cap \{v = 0\} \\
\frac{\partial v_\infty}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

In addition, we have the following geometrical characterization for \(\lambda N_1 \infty(\Omega)\):

\[
\lambda N_1 \infty(\Omega) = \frac{2}{\text{diam}(\Omega)},
\]

where the intrinsic diameter of \(\Omega\) is defined as

\[
\text{diam}(\Omega) := \max_{\Omega \times \Omega} d_\Omega(x, y) = \max_{\partial \Omega \times \partial \Omega} d_\Omega(x, y),
\]

being \(d_\Omega(x, y)\) the geodesic distance given by \(d_\Omega(x, y) = \inf_{\gamma} \text{Long}(\gamma)\), where the infimum is taken over all possible Lipschitz curves in \(\Omega\) connecting \(x\) and \(y\).

We remark that in the limit case \(p = \infty\), the geometrical characterization (1.8) of \((\infty\text{-Neumann})\) yields several interesting consequences:

✓ If \(\mathcal{L}^n(\Omega) = \mathcal{L}^n(\mathcal{B})\), \(\mathcal{B}\) being a ball, then \(\lambda N_1 \infty(\Omega) \leq \lambda N_1 \infty(\mathcal{B})\), which establishes a Szegö-Weinberger type inequality: among all domains of prescribed volume the ball maximizes \((\infty\text{-Neumann})\).
\( \lambda_{1,\infty}^N(\Omega) \leq \lambda_{1,\infty}^D(\Omega) \) for any convex \( \Omega \) with equality if and only if \( \Omega \) is a ball.

\( \lambda_{1,\infty}^N(\Omega) \leq \lambda_{1,\infty}^D(\Omega) \) for any convex \( \Omega \) with equality if and only if \( \Omega \) is a ball.

The Payne-Weinberger inequality, (1.6), becomes an equality when \( p = \infty \).

Taking account the previous historic overview, we arrive to our main result, which establishes the stability of the ball with respect to small perturbations of their first Dirichlet and Neumann \( \infty \)-eigenvalues. More precisely, if a domain \( \Omega \subset \mathbb{R}^n \) has Dirichlet and Neumann \( \infty \)-eigenvalues close enough to those of the ball \( B_r \) of the same Lebesgue measure, then \( \Omega \) is uniformly “almost” ball-shaped.

**Theorem 1.1.** Let \( \Omega \) be an open domain satisfying \( \mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r) \). If for some \( \delta_i > 0 \) \((i = 1, 2)\) small enough it holds that

\[
|\lambda_{1,\infty}^D(\Omega_k) - \lambda_{1,\infty}^D(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega_k) - \lambda_{1,\infty}^N(B_r)| = \delta_2,
\]

then there are two balls such that

\[
B_{\frac{r}{1+\delta_1r}} \subset \Omega \subset B_{\frac{r}{1+\delta_2r}}.
\]

The previous theorem implies the following convergence result.

**Theorem 1.2.** Let \( \{\Omega_k\}_{k \in \mathbb{N}} \) be a family of uniformly bounded domains satisfying \( \mathcal{L}^n(\Omega_k) = \mathcal{L}^n(B_r) \). If

\[
|\lambda_{1,\infty}^D(\Omega_k) - \lambda_{1,\infty}^D(B_r)| = o(1) \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega_k) - \lambda_{1,\infty}^N(B_r)| = o(1)
\]

as \( k \to \infty \), then

\[
\Omega_k \to B_r
\]

in the sense that the Hausdorff distance between \( \Omega \) and a ball \( B_r \) goes to zero, i.e.,

\[
d_H(\Omega_k, B_r) := \max \left\{ \sup_{x \in \Omega_k} \inf_{y \in B_r} d(x, y), \sup_{y \in B_r} \inf_{x \in \Omega_k} d(x, y) \right\} \to 0.
\]

Note that our results imply that

\[
\max \left\{ \mathcal{L}^n(\Omega \triangle B_{\frac{r}{1+\delta_1r}}), \mathcal{L}^n(\Omega \triangle B_{\frac{r}{1+\delta_2r}}) \right\} \leq C(n, \delta_1, r)r^n.
\]

where \( C(n, \delta_1, r) = \omega_n \max \{ (\delta_1 r + 1)^n - 1, (n-1)\delta_2 \} \to 0 \) as \( \delta_1 \to 0 \). Hence, we can control the Fraenkel asymmetry of the set, \( S(\Omega) \). But our results give much more since we have a sort of uniform control on how far the set is from being a ball (for instance, we have convergence in Hausdorff distance in Theorem 1.2).

Another important question in this theory consists on how the corresponding \( \infty \)-ground states (solutions to (1.3)) behave in relation to perturbations of the \( \infty \)-eigenvalues of the ball. The next result provides an answer for this issue, showing that Dirichlet \( \infty \)-eigenfunctions are uniformly close to a cone when the first Dirichlet and Neumann \( \infty \)-eigenvalues are close to those for the ball. Note that, in general, the \( \infty \)-eigenvalue problem (1.3) may have multiple solutions (the first eigenvalue may not be simple), see [3] and [18].

**Theorem 1.3.** Let \( \Omega \) be an open domain satisfying \( \mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r) \). Given \( \varepsilon > 0 \) there are \( \delta_i(\varepsilon) > 0 \) \((i = 1, 2)\) small enough such that:

\[
|\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| < \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| < \delta_2,
\]
then
\[ |u(x) - v_\infty(x)| < \varepsilon \text{ in } \Omega \cap B_r, \]
where
\[ v_\infty(x) = 1 - \frac{|x|}{r} \]
is the normalized \( \infty \)-ground state to (1.3) in \( B_r \).

Theorem 1.3 can be rewritten as follows:

**Corollary 1.4.** Let \( \{u_k\}_{k \in \mathbb{N}} \) be a family of normalized solutions to (1.3) in \( \Omega_k \) such that
\[ |\lambda_{D,1,\infty}^1(\Omega_k) - \lambda_{D,1,\infty}^1(B_r)| = o(1) \quad \text{and} \quad |\lambda_{N,1,\infty}^N(\Omega_k) - \lambda_{N,1,\infty}^N(B_r)| = o(1) \quad \text{as } k \to \infty. \]
Then,
\[ u_k \to v_\infty \text{ locally uniformly in } B_r, \]
where
\[ v_\infty(x) = 1 - \frac{|x|}{r} \]
is the normalized \( \infty \)-ground state to (1.3) in \( B_r \).

Our approach can be applied for other classes of operators with \( p \)-Laplacian type structure. We can deal with \( p \)-Laplacian type problems involving an anisotropic \( p \)-Laplacian operator
\[ -Q_p u := -\text{div}(F^{p-1}(\nabla u)F_\xi(\nabla u)), \]
where \( F \) is an appropriate (smooth) norm of \( \mathbb{R}^n \) and \( 1 < p < \infty \). The necessary tools for studying the anisotropic Dirichlet eigenvalue problem, as well as its limit as \( p \to \infty \) can be found in [3]. Here, to obtain results similar to ours, one has to replace Euclidean balls with balls in the norm \( F \).

The paper is organized as follows: in Section 2 we prove our main stability results including the behavior of the corresponding \( \infty \)-eigenfunctions and in Section 3 we collect several examples that illustrate our results.

2. **Proof of the Main Theorems**

Before proving our main result we introduce some notations which will be used throughout this section. Given a bounded domain \( \Omega \subset \mathbb{R}^n \) and a ball \( B_r \subset \mathbb{R}^n \) of radius \( r > 0 \) we denote \( \lambda_{D,1,\infty}^1(\Omega) \) and \( \lambda_{D,1,\infty}^1(B_r) \) the first Dirichlet eigenvalues (1.4) in \( \Omega \) and in \( B_r \), respectively; analogously, \( \lambda_{N,1,\infty}^N(\Omega) \) and \( \lambda_{N,1,\infty}^N(B_r) \) stand for the first nontrivial Neumann eigenvalues (1.8) in \( \Omega \) and in \( B_r \).

We introduce the following class of sets which will play an important role in our approach. For non-negative constants \( \delta_1 \) and \( \delta_2 \) we define the class:
\[ \Xi_{\delta_1,\delta_2}(B_r) := \left\{ \Omega \subset \mathbb{R}^n \mid \begin{array}{c}
\Omega \text{ bounded domain with } \quad |\lambda_{D,1,\infty}^1(\Omega) - \lambda_{D,1,\infty}^1(B_r)| = \delta_1 \\
\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r) \\
|\lambda_{N,1,\infty}^N(\Omega) - \lambda_{N,1,\infty}^N(B_r)| = \delta_2
\end{array} \right\}. \]
Notice that, \( \Xi_{0,0}(B_r) \) consists of the family of all balls with radius \( r > 0 \). Another important remark is that the elements of \( \Xi_{\delta_1,\delta_2}(B_r) \) are invariant by rigid movements (rotations, translations, etc).
Similarly, we can define the class $\Xi_{D,\delta_1}^D(B_r)$ (resp. $\Xi_{N,\delta_2}^N(B_r)$) as being $\Xi_{\delta_1,\delta_2}^D(B_r)$ with the restriction on the Dirichlet (resp. Neumann) eigenvalues only.

In the next lemma we show that a control on the difference of the first Dirichlet eigenvalue implies that $\Omega$ contains a large ball.

**Lemma 2.1.** If $\Omega \in \Xi_{D,\delta_1}^D(B_r)$ then there exists a ball such that

$$\mathfrak{B}_{\frac{r}{\delta_1 r + 1}} \subset \Omega.$$  

Moreover,

$$\mathcal{L}^n(\Omega \triangle \mathfrak{B}_{\frac{r}{\delta_1 r + 1}}) \leq c(n, \delta_1, r)r^n,$$

where $c = o(1)$ as $\delta_1 \to 0$.

**Proof.** According to (1.4) we have that

$$\delta_1 = \left| \lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(\mathfrak{B}_r) \right| = \frac{1}{r_{\Omega}} - \frac{1}{r}.$$  

It follows that

$$r_{\Omega} \geq \frac{r}{\delta_1 r + 1},$$

and then there is ball such that

$$\mathfrak{B}_{\frac{r}{\delta_1 r + 1}} \subset \Omega.$$  

Finally,

$$\mathcal{L}^n(\Omega \triangle \mathfrak{B}_{\frac{r}{\delta_1 r + 1}}) = \mathcal{L}^n(\Omega) - \mathcal{L}^n(\mathfrak{B}_{\frac{r}{\delta_1 r + 1}})$$

$$= \omega_n r^n \left( 1 - \frac{1}{(\delta r + 1)^n} \right)$$

$$\leq \omega_n r^n ((\delta r + 1)^n - 1)$$

$$= c(n, \delta, r)r^n$$

and the lemma follows. $\square$

Now, we show that a control on the difference of the first Neumann eigenvalue implies that $\Omega$ is contained in a small ball.

**Lemma 2.2.** If $\Omega \in \Xi_{N,\delta_2}^N(B_r)$ then there is a ball such that

$$\Omega \subset \mathfrak{B}_{\frac{r}{\delta_2 r + 1}}.$$  

Moreover,

$$\mathcal{L}^n(\Omega \triangle \mathfrak{B}_{\frac{r}{\delta_2 r + 1}}) \leq (n - 1)\omega_n r^n\delta_2.$$  

**Proof.** Using (1.8) we have that

$$\delta_2 = \left| \lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(\mathfrak{B}_r) \right| = \left| \frac{2}{\text{diam}(\Omega)} - \frac{1}{r} \right|.$$  

It follows that

$$\text{diam}(\Omega) \leq \frac{2r}{1 - \delta_2 r} = r + \frac{r(1 + \delta r)}{1 - \delta_2 r},$$

and then there exists a ball such that

$$\Omega \subset \mathfrak{B}_{\frac{\text{diam}(\Omega)}{2}} = \mathfrak{B}_{\frac{r}{\delta_2 r + 1}}.$$
Moreover,
\[
\mathcal{L}^n \left( \Omega \setminus \overline{B_{\text{diam}(\Omega)}} \right) = \mathcal{L}^n \left( \overline{B_{\text{diam}(\Omega)}} \right) - \mathcal{L}^n (\Omega) \\
= \omega_n r^n \left( \left( 1 + \frac{\delta_2}{1 - \delta_2 r} \right)^n - 1 \right) \\
= \omega_n r^n \sum_{k=2}^{n} \left( \frac{\delta_2}{1 - \delta_2 r} \right)^k \\
\leq (n-1) \omega_n \delta_2 r^n
\]
and the lemma follows. \(\Box\)

Proof of Theorem 1.1. The proof of Theorem 1.1 follows as an immediate consequence of Lemmas 2.1 and 2.2. \(\Box\)

Next, we will prove Theorem 1.2.

Proof of Theorem 1.2. The hypothesis implies that \(\Omega_k \in \Xi_{\delta_k, \varepsilon_k}(B_r)\) for \(\delta_k, \varepsilon_k = o(1)\) as \(k \to \infty\). For this reason, by Theorem 1.1 there are two balls such that 
\[
B_{\frac{r}{1+\varepsilon_k}} \subset \Omega_k \subset B_{\frac{r}{1-\varepsilon_k}}.
\]
Now, using that all these balls are centered at points that are bounded (since we assumed that the family \(\Omega_k\) is uniformly bounded), we can extract a subsequence such that the centers converge and therefore we conclude that there is a ball \(B_r\) such that \(\Omega_k \to B_r\) as \(k \to \infty\). \(\Box\)

Proof of Theorem 1.3. The proof follows by contradiction. Let us suppose that there exists an \(\varepsilon_0 > 0\) such that the thesis of Theorem fails to hold. This means that for each \(k \in \mathbb{N}\) we might find a domain \(\Omega_k\) and \(u_k\), a normalized \(\infty\)-ground state to (1.3) in \(\Omega_k\), such that \(\Omega_k \in \Xi_{\gamma_k, \zeta_k}(B_r)\) with \(\gamma_k, \zeta_k = o(1)\) as \(k \to \infty\), that is,
\[
|\lambda_{1, \infty}^P(\Omega_k) - \lambda_{1, \infty}^P(B_r)| < \gamma_k \quad \text{and} \quad |\lambda_{1, \infty}^N(\Omega_k) - \lambda_{1, \infty}^N(B_r)| < \zeta_k,
\]
with \(\gamma_k, \zeta_k = o(1)\) as \(k \to \infty\), together with
\[
|u_k(x) - v_{\infty}(x)| > \varepsilon_0 \quad \text{in} \quad \Omega_k \cap B_r,
\]
for every \(k \in \mathbb{N}\).

Using our previous results, we can suppose that every \(\Omega_k \subset B_{2r}\). Then, by extending \(u_k\) to zero outside of \(\Omega_k\), we may assume that \(\{u_k\}_{k \in \mathbb{N}} \subset W_0^{1, \infty}(B_{2r})\). In this context, standard arguments using viscosity theory show that, up to a subsequence, \(u_k \to u_{\infty}\) uniformly in \(B_{2r}\), being the limit \(u_{\infty}\) a normalized eigenfunction for some domain \(\Omega\) with \(\Omega \Subset B_{2r}\). Moreover, we have that \(\lambda_{1, \infty}^P(\Omega_k) \to \lambda_{1, \infty}^P(\Omega)\).

According to Theorem 1.2, \(\Omega_k \to B_r\) as \(k \to \infty\). By the previous sentences we conclude that \(\Omega = B_r\). Now, by uniqueness of solutions to (1.3) in \(B_r\) we conclude that \(u_{\infty} = v_{\infty}\). However, this contradicts (2.1) for \(k \gg 1\) (large enough). Such a contradiction proves the theorem. \(\Box\)
3. Examples

Given a fixed ball \( B \) and a domain \( \Omega \) having both of them the same volume, Theorem 1.1 says that if the \( \infty \)-eigenvalues are close each other then \( \Omega \) is almost ball-shaped uniformly. The following examples illustrate Theorem 1.1 and 1.2.

Example 3.1. The reciprocal in Theorem 1.1 (and Theorem 1.2) is not true: given a fixed ball \( B \), clearly, there are domains \( \Omega \) fulfilling (1.9) such that the difference between the Neumann (and Dirichlet) eigenvalues in \( \Omega \) and in \( B \) is not small. Let us present some illustrative examples.

1. A stadium. Let \( B \) be the unit ball in \( \mathbb{R}^2 \) and \( \Omega \) the stadium domain given in Figure 3 (a) with \( \ell = \frac{\pi (1 - \varepsilon^2)}{2 \varepsilon} \). In this case \( \mathcal{L}_n^N(B) = \mathcal{L}_n^N(\Omega) = \pi \) for any \( 0 < \varepsilon < 1 \). However,
   \[
   \lambda_{1,\infty}^N(B) = 1, \quad \lambda_{1,\infty}^N(\Omega) = \frac{2}{\text{diam}(\Omega)} = \frac{4\varepsilon}{\pi + \varepsilon^2(4 - \pi)} < \frac{1}{3} \quad \text{if} \quad \varepsilon < \frac{1}{4}. 
   \]

2. A ball with holes. If \( \Omega = B(0, \sqrt{1 + \varepsilon^2}) \setminus B(0, \varepsilon) \) is the domain given in Figure 3 (b), then \( \mathcal{L}_n^D(B) = \mathcal{L}_n^D(\Omega) = \pi \), however
   \[
   \lambda_{D,\infty}^D(B) = 1, \quad \lambda_{D,\infty}^D(\Omega) = \frac{1}{\sqrt{1 + \varepsilon^2}} > \frac{3}{2} \quad \text{if} \quad \frac{3}{4} < \varepsilon < 1. 
   \]

3. A ball with thin tubular branches. If \( \Omega \) is the domain given in Figure 3 (c), the condition \( \mathcal{L}_n^N(B) = \mathcal{L}_n^N(\Omega) \) gives the relation
   \[
   r(r + \varepsilon) + \varepsilon \left( \frac{1}{\pi} + \frac{\varepsilon}{2} \right) = 1, \quad \text{diam}(\Omega) = 1 + r + \pi(1 + r). 
   \]
   For instance, if we take \( \varepsilon = 10^{-3} \) it follows that \( r \approx 0.999465 \) and then
   \[
   \lambda_{1,\infty}^N(B) = \frac{2}{\text{diam}(B)} = 1, \quad \lambda_{1,\infty}^N(\Omega) = \frac{2}{\text{diam}(\Omega)} \approx 0.2415. 
   \]

Figure 1. Three examples of domains

Hence, in view of these examples we conclude that a domain that has Dirichlet and Neumann \( \infty \)-eigenvalues close to the ones for the ball is close to a ball not only in the sense that \( \mathcal{L}_n^N(\Omega \triangle B_r) \) is small but it can not contain holes deep inside (small holes near the boundary are allowed) and can not have thin tubular branches.
Example 3.2. The regular polygon $\mathbb{P}_k$ of $k$-sides ($k \geq 3$) centered at the origin such that $\mathbb{L}^{n}(\mathbb{P}_k) = \mathbb{L}^{n}(\mathbb{B}_r)$ satisfies

$$|\lambda_{1,\infty}^{D}(\mathbb{P}_k) - \lambda_{1,\infty}^{D}(\mathbb{B}_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^{N}(\mathbb{B}_r) - \lambda_{1,\infty}^{N}(\mathbb{P}_k)| = \delta_2,$$

where

$$\delta_1 = \frac{1}{r} \left( \frac{\pi}{k \tan\left(\frac{\pi}{k}\right)} \right) - \frac{1}{r} \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{r \sqrt{k \sin^2\left(\frac{\pi}{k}\right)}}.$$

Therefore, we can recover the well known convergence $\mathbb{P}_k \to \mathbb{B}_r$ as $k \to \infty$.

Example 3.3. Given $k \in \mathbb{N}$ and positive constants $a_1^k, \ldots, a_n^k$, the $n$-dimensional ellipsoid given by

$$\mathcal{E}_k := \left\{ (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} \left( \frac{x_i}{a_i^k} \right)^2 < 1 \right\}$$

such that $\mathbb{L}^{n}(\mathcal{E}_k) = \mathbb{L}^{n}(\mathbb{B}_r)$ satisfies

$$|\lambda_{1,\infty}^{D}(\mathcal{E}_k) - \lambda_{1,\infty}^{D}(\mathbb{B}_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^{N}(\mathbb{B}_r) - \lambda_{1,\infty}^{N}(\mathcal{E}_k)| = \delta_2,$$

where

$$\delta_1 = \frac{1}{\min_i \{a_i^k\}} - \frac{1}{r}, \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{\max_i \{a_i^k\}}.$$

Therefore, we recover the fact that if $\min_i a_i^k \to r$ and $\max_i a_i^k \to r$ as $k \to \infty$, then $\mathcal{E}_k \to \mathbb{B}_r$.

Example 3.4. Given $r > 0$ let $k_0 \in \mathbb{N}$ such that $\frac{1}{2\pi} \sqrt{\frac{1}{2} + 4\pi^2 r^2} > \frac{1}{k_0}$ for all $k \geq k_0$. For each $k \in \mathbb{N}$ let $\Omega_k$ be the planar stadium domain from Figure 1 (a) with $l_k = \frac{1}{k}$ and $\varepsilon_k = \frac{1}{2\pi} \sqrt{\frac{1}{2} + 4\pi^2 r^2} - \frac{1}{k}$. It is easy to check that $\Omega_k \equiv \frac{1}{\max_i \{a_i^k\}} - \frac{1}{r} \mathbb{B}_r$. Furthermore, in this case we have that the eigenfunctions are explicit and given by

$$u_k(x) = \frac{1}{\varepsilon_k} \text{dist}(x, \partial \Omega_k).$$

Finally, form Corollary 1.4

$$u_k(x) \to v_\infty(x) = \frac{1}{r} \text{dist}(x, \partial \mathbb{B}_r) \quad \text{locally uniformly in } \mathbb{B}_r \text{ as } k \to \infty.$$

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