Finite-Time Amplitudes In Matrix Theory

Øyvind Tafjord and Vipul Periwal *

Department of Physics
Princeton University
Princeton, NJ 08544

We evaluate one-loop finite-time amplitudes for graviton scattering in Matrix theory and compare to the corresponding amplitudes in supergravity. We find agreement for arbitrary time intervals at leading order in distance, providing a functional agreement between supergravity and Matrix theory. At subleading order, we find corrections to the effective potential found from previous phase shift calculations in Matrix theory.
1. Introduction

Matrix theory \([1]\) is a remarkable proposal for a non-perturbative definition of M-theory, a supposedly consistent 11-dimensional quantum theory having supergravity as its low energy limit. Matrix theory is defined as the maximally supersymmetric quantum mechanics of \(U(N)\) matrices, describing the lowest states of open strings connecting \(N\) D0-branes. Supergravitons, for instance, appear as bound states of these D0-branes. The original conjecture of Banks, Fischler, Shenker and Susskind \([1]\) relates the large \(N\) limit of Matrix theory to M-theory in the Infinite Momentum Frame. Later, Susskind \([2]\) expanded the conjecture to a relation between finite \(N\) Matrix theory and M-theory with a compact null-direction.

There have been several successful tests of these conjectures so far, see, e.g., \([3]\) for an overview. Part of the original evidence involved comparing graviton scattering phase shifts computed in Matrix theory and supergravity in the limit of large impact parameter. The fact that these agree originates from the work of Douglas, Kabat, Pouliot, and Shenker \([4]\), where it was shown how supersymmetry leads to a cancellation of the contribution from the massive modes of the strings connecting the D0-branes, implying that the long distance behavior can be reproduced by keeping only the lowest open string modes. These phase shifts corresponds to an eikonal approximation where the gravitons move along infinite straight lines at large impact parameter. Similar calculations have later successfully extended this to processes involving higher branes (as in \([4,5,6]\)) and spin effects \([8]\), as well as processes involving longitudinal momentum transfer \([9]\). The Matrix theory calculation of the graviton scattering process has lately been pushed to two loops \([10,11]\), giving further evidence for the finite \(N\) conjecture. Recently Seiberg \([12]\) has given arguments for why the Matrix theory conjecture is correct, and this was further examined in the context of graviton scattering in \([13]\).

From the infinite time phase shift calculation in Matrix theory one derives an effective potential between gravitons of relative velocity \(v\) and relative distance \(r\). This potential is a double expansion in \(v^2/r^4\) and \(1/r^3\) (the latter is the loop expansion), starting out as \([11]\)

\begin{align}
-V_{\text{eff}} &= \frac{15}{16} \kappa^2 \frac{v^4}{r^7} + 0 \cdot \kappa^{8/3} \frac{v^6}{r^{11}} + \frac{9009}{4096} \kappa^{10/3} \frac{v^8}{r^{15}} + \mathcal{O}(\kappa^4 \frac{v^{10}}{r^{17}}) \\
&+ 0 \cdot \kappa^{10/3} \frac{v^4}{r^{10}} + \frac{225}{32} \kappa^4 \frac{v^6}{r^{14}} + \mathcal{O}(\kappa^{14/3} \frac{v^8}{r^{17}}) \\
&+ \mathcal{O}(\kappa^{14/3} \frac{v^4}{r^{13}}),
\end{align}

(1.1)
where we have indicated how each term scales with the 11-dimensional gravitational coupling, otherwise (when setting $\kappa = 1$ above) the units are as in [10]. So far, the two first terms along the diagonal of this double expansion, with integer powers of $\kappa$, have been found to coincide with supergravity [11]. It has been argued that all the terms along the diagonal will agree with classical supergravity and that all terms to the left of the diagonal should vanish [14,15,13], while other non-zero terms in this expansion should come from higher-derivative terms of the supergravity effective actions [13].

These infinite time straight line phase shift calculations remain one of very few quantities one can actually calculate directly in Matrix theory. It is important to try to extend this repertoire as much as possible. In this paper we will consider a finite time version of this calculation, and from the resulting “phase shift” we can read off an effective potential locally, rather than integrated along an infinite path. We will only work to one loop, and there we find again the $\frac{15}{16} v^4 r^7$ potential of supergravity to leading order, for arbitrary time intervals. This provides a stronger equivalence between Matrix theory and supergravity than is demonstrated from the infinite-time case, as the finite time amplitude contains more information. Of course, the main conjecture of [1] concerns the $S$-matrix, so this result suggests that their conjecture could perhaps be strengthened. However, higher order terms in the potential will be modified compared to the infinite-time case, by interesting terms that integrate to zero along the infinite line, thus showing that the complete matching of the leading term is non-trivial. Working on a finite time interval means that we have to be careful about what boundary conditions are put on the high energy modes that are integrated out. We will consider the most natural boundary condition for comparing to supergravity, and also investigate how the answer varies with other choices. We find that the leading term is robust towards changing the boundary conditions, while the subleading terms are more sensitive. In the large $N$ limit proposed in [1], these terms are also subleading in powers of $N$, but they may be interesting in Susskind’s finite $N$ conjecture [2]. Note that if the Planck scale $l_P$ is the only relevant scale in supergravity, then since $\kappa_{11}^{2/3} \sim l_P^3$, these terms seem to indicate corrections to the supergravity action dimensionally going like $R + R^{5/2} + \ldots$ rather than $R + R^2 + \ldots$.

This paper is organized as follows: In sect. 2, we set up the basic problem. Sect. 3 contains our calculations, and sect. 4 has some concluding remarks.
2. Comparing Matrix theory and supergravity

Graviton scattering in Matrix theory is described by $U(N_1 + N_2)$ supersymmetric quantum mechanics, where $N_1/R$ and $N_2/R$ are the longitudinal lightlike momenta of the gravitons, $R$ being the radius of the compact null-direction. We will take $N_1 = N_2 = 1$ here, as usual the leading order factors of $N_i$ can be easily reinstated afterwards. The bosonic part of the $U(2)$ action is given by

$$S = \text{Tr} \int dt \left( \frac{1}{2} (D_t X^i)^2 + \frac{1}{4} [X^i, X^j]^2 \right),$$

(2.1)

where for simplicity we suppress dependences on $R$ and the 11-dimensional Planck scale (see [11]). We defined $D_t X^i = \partial_t X^i + [A, X^i]$, with $A$, $X^i$ being $U(2)$ matrices which we can decompose as

$$X^i = \frac{i}{2} (X^i_0 I + X^i_a \sigma^a).$$

(2.2)

The $\sigma^a$, $a = 1 \ldots 3$, are the Pauli matrices. There are also corresponding fermionic fields $\psi$. The $X^i_0$ fields describe the center of mass motion, and they (together with $A_0$ and $\psi_0$) decouple from the rest and will be ignored from now on. We interpret $X^i_3$ as the relative coordinate of the gravitons, while the corresponding gauge field $A_3$ can in principle be gauged away (in background field gauge, which we employ here, it is set to a constant). The fermionic $\psi_3$ fields incorporate the spin and polarization of the (super-)gravitons. The off-diagonal 1,2 modes do not have such an intuitive description in the long distance supergravity, in fact they can be thought of as arising from open string considerations valid only at short distances, and they then represent the unexcited modes of these open strings.

When the gravitons are far apart, the off-diagonal modes are all very heavy, with masses roughly proportional to the distance. One can then imagine integrating these modes out in a Born-Oppenheimer type of approach, as (non-local) internal degrees of freedom. This leaves a theory only involving the diagonal modes that are interpreted as the positions of supergravitons. This separation of light and heavy modes is only exact in the limit of large $<X^i_3>$ though, and a more systematic understanding of this is needed, especially at higher loop orders.

When integrating out the off-diagonal modes, we need to supply boundary conditions on the fields. In previous infinite time phase shift calculations, the fields are taken to vanish at $\pm \infty$, but for finite time intervals we should be more careful. The off-diagonal modes are at one loop level described by harmonic oscillators, with frequency proportional
to their mass, and specifying boundary conditions can be done by specifying the initial and final states of the harmonic oscillators. These harmonic oscillator modes do not appear at all in supergravity, so exciting them out of their ground state would seem to take us out of the realm of supergravity. For comparing finite time amplitudes, the most natural assumption therefore appears to be to put these modes in their ground state for both the initial and final states. We will also consider other choices of boundary conditions.

If we also specify boundary conditions for \( X^3 \) and integrate it out, we will find the amplitude for a finite-time graviton propagation. We can write this schematically as (we now work in Euclidean time)

\[
\langle X_{3,s}^i, T_s | X_{3,f}^i, T_f \rangle = \mathcal{N} \int_{X_{3,s}^i}^{X_{3,f}^i} \mathcal{D}X a e^{-S(X_a)} \equiv \mathcal{N} \int_{X_{3,s}^i}^{X_{3,f}^i} \mathcal{D}X_3 e^{-S_{\text{eff}}(X_3)} \equiv e^{-S_0 + \delta}.
\]

Here \( S_0 \) is the action of the classical trajectory implementing the boundary conditions, and in an abuse of language we call \( \delta \) the finite time phase shift and try to relate it to an effective potential through \( \delta = -\int V_{\text{eff}} \). Since we are not considering any sort of spin effects, \( \psi_3 \) (as well as \( A_3 \), the ghost \( C_3 \) and \( X_3 \) once the boundary conditions have been implemented) just go along for the ride as massless free fields here at one-loop level.

In supergravity the most convenient way to do a similar computation, is to give one of the gravitons a large longitudinal momentum such that it can be treated as a classical source for the other graviton. One can then derive an action for the probe graviton moving in the field of this source, \( S_{\text{SG}}(X_3^i) \). There are then two ways to view the comparison between Matrix theory and supergravity. On one hand, we can directly compare the effective actions of the light modes \( S_{\text{eff}}(X_3^i) \) and \( S_{\text{SG}}(X_3^i) \) for various paths \( X_3^i(t) \) and see how they match—our computations in this paper are equivalent to doing this for a finite line segment. This seems to be the cleanest way of interpreting our results. Alternatively we can evaluate the finite time amplitude in supergravity, which, at the semiclassical level, is given by integrating the action along the geodesic connecting the two space time points. At large separation of the gravitons, the leading order amplitude can be found using an eikonal approximation, replacing the geodesic by a constant velocity straight line.

We will choose coordinates such that the straight line connecting the two space time points is given by

\[
\begin{align*}
X_{3,0}^1(t) &= vt, \\
X_{3,0}^2(t) &= b, \\
X_{3,0}^{i>2}(t) &= 0.
\end{align*}
\]

(2.4)
The supergravity action for the probe graviton is given in [11], and we find that the supergravity action evaluated for a finite time line segment (alternatively, an eikonal approximation to the finite time phase shift) is given by

$$\delta_{SG} = \int_{T_s}^{T_f} \left[ \frac{15}{16} \frac{v^4}{r^7(t)} + \mathcal{O}(\frac{v^6}{r^{14}}) \right] dt,$$  

(2.5)

where \( r(t) = \sqrt{v^2 t^2 + b^2} \). The \( \mathcal{O}(\frac{v^6}{r^{14}}) \) term we don’t expect to see until two loops in Matrix theory and will not concern us here. Note that, as argued in [18], the corrections to the eikonal approximation are of order \( v^2/r^7 \) and thus fall along the diagonal in (1.1) and will not be of concern to us here, either.

3. Finite-time calculation in Matrix theory

On the Matrix theory side, to evaluate (2.3) we implement the boundary conditions on \( X^i_{\bar{3}} \) by a suitable solution to the equations of motion. These solutions are straight lines, so we expand about \( X^i_{\bar{3},0}(t) \) in (2.4). The path integral can then be evaluated as in [4], at one loop it yields a product of determinants. The phase shift is given by

$$\delta = \ln[\det^{-\frac{1}{2}}(-\partial^2_t + \omega^2(t)) \det^{-1}(-\partial^2_t + r^2 + 2v) \det^{-1}(-\partial^2_t + r^2 - 2v)] \times \det^4(-\partial^2_t + r^2 + v) \det^4(-\partial^2_t + r^2 - v)].$$  

(3.1)

Now these determinants represent the evolution of a time dependent harmonic superoscillator from a state \( |\psi_s\rangle \) at time \( T_s \) to a state \( |\psi_f\rangle \) at time \( T_f \). The boundary conditions we put on the fermionic part should be determined by supersymmetry from the boundary conditions on the bosonic part. We here choose to represent each determinant by a separate bosonic harmonic oscillator of the appropriate frequency, evolving from corresponding initial states to final states. This prescription preserves a symmetry between all the modes, and we believe it gives equivalent results to treating the full superoscillator. We write this as

$$\det^{-\frac{1}{2}}(-\partial^2_t + \omega^2(t)) = \langle \psi_f | \exp \left[ - \int_{T_s}^{T_f} H(t) dt \right] |\psi_s\rangle$$

$$= \mathcal{N} \int d\varphi_f d\varphi_s \psi^*_f(\varphi_f) \psi_s(\varphi_s) \int_{\varphi_s}^{\varphi_f} \mathcal{L}(\varphi) \exp \left[ - \int_{T_s}^{T_f} L(t) dt \right]$$

$$= \mathcal{N} \int d\varphi_f d\varphi_s \psi^*_f(\varphi_f) \psi_s(\varphi_s) \exp \left[ - \frac{1}{2} (\varphi_f \dot{\varphi}_0(T_f) - \varphi_s \dot{\varphi}_0(T_s)) \right] \det^{-\frac{1}{2}}(-\partial^2_t + \omega^2(t)), $$

(3.2)
where
\[ H = \frac{1}{2} \left( p^2 + \omega^2 \varphi^2 \right), \quad p = -i \partial \varphi, \]
\[ L = \frac{1}{2} \left( \dot{\varphi}^2 + \omega^2 \varphi^2 \right), \]
\[ \left[ -\dot{\varphi}^2 + \omega^2(t) \right] \varphi_0(t) = 0, \quad \varphi_0(T_s) = \varphi_s, \quad \varphi_0(T_f) = \varphi_f. \tag{3.3} \]

By \( \det_0 \) we denote the determinant evaluated on the space of functions that vanish at both endpoints.

In order to extract the leading order supergravity potential, we actually need to do very little work, the simplest adiabatic approximation is sufficient. In this approximation, we assume the harmonic oscillator to stay in its ground state throughout, and we find the estimate
\[ \det^{-\frac{1}{2}}(-\partial^2 + \omega^2(t)) \approx \exp \left[ -\int_{T_s}^{T_f} \frac{1}{2} \omega(t) dt \right]. \tag{3.4} \]

Using \( \omega^2(t) = v^2 t^2 + b^2 + \alpha v \), inserting this into (3.1), and expanding in powers of \( v \), we get the phase shift
\[ \delta_{ad} = \int_{T_s}^{T_f} dt \left[ \frac{15}{16} v^4 \frac{1}{r^7} + \frac{315}{128} v^6 \frac{1}{r^{11}} + \frac{27027}{4096} v^8 \frac{1}{r^{15}} + \ldots \right]. \tag{3.5} \]

We see that the leading term at large distance agrees with the supergravity result (2.7), while the higher terms do not even agree with the infinite time phase shift calculation. This way of deriving the potential energy between the gravitons is tantamount to summing up the zero point energy of the harmonic oscillators representing the off-diagonal modes, which is how various potential energies were computed for instance in [5]. We see that at subleading order there is a distinction between these two methods of extracting an effective potential.

We will now embark on a more systematic evaluation of the finite time amplitude. We will use two approaches, valid, roughly speaking, at long and short time intervals respectively.

For long time intervals, we use the last line of (3.2). The classical trajectory \( \varphi_0(t) \) is a parabolic cylinder function [19]. We can systematically solve for it using a WKB type expansion (equivalent to the Darwin expansion in [19]). To this end, write
\[ \varphi_0(t) = \exp \left[ \int_{T_s}^{t} \sum_{n=0}^{\infty} h^{n-1} f_n(t') dt' \right], \tag{3.6} \]
and consider \( \omega(t) \) to be of order \( 1/h \). Solving the equation for \( \varphi_0 \) in (3.3) order by order in \( \hbar \) (and then setting \( \hbar = 1 \)), we find the first few \( f \)’s

\[
\begin{align*}
    f_0 &= \omega, \\
    f_1 &= -\frac{v^2 t}{2 \omega^2}, \\
    f_2 &= \frac{v^2}{8 \omega^5} (2 \omega^2 - 5v^2 t^2), \\
    f_3 &= \frac{3v^4 t}{8 \omega^8} (3 \omega^2 - 5v^2 t^2), \\
    f_4 &= \frac{v^4}{128 \omega^{11}} (-76 \omega^4 + 884 \omega^2 v^2 t^2 - 1105v^4 t^4).
\end{align*}
\] (3.7)

There is also another solution obtained by \( f_{2n} \rightarrow -f_{2n} \) for all \( n \). We see that

\[
f_n \sim \omega \left( \frac{v}{\omega^2} \right)^n g_n(\nu t/\omega), \tag{3.8}
\]

where \( g_n \) only contains non-negative powers of its argument. Thus this is an expansion in \( v/\omega^2 \) which is what we want. If we define

\[
A(t) = \sum f_{2n}(t), \quad B(t) = \sum f_{2n+1}(t), \tag{3.9}
\]

the general solution can be written

\[
\varphi_0(t) = c_1 e^{\int_{T_s}^t (B + A)} + c_2 e^{\int_{T_s}^t (B - A)}. \tag{3.10}
\]

Solving the boundary conditions, we can evaluate

\[
\begin{align*}
    \varphi_0(T_s) &= \varphi_s B_s - \varphi_s A_s \coth \int A + \frac{\varphi_f A_s e^{-\int B}}{\sinh \int A}, \\
    \dot{\varphi}_0(T_f) &= \varphi_f B_f + \varphi_f A_f \coth \int A - \frac{\varphi_s A_f e^{\int B}}{\sinh \int A}. \tag{3.11}
\end{align*}
\]

Here \( A_s \equiv A(T_s) \) and so on. It is now straightforward to evaluate the integral over \( \varphi_{s,f} \) in (3.2). The ground state wave function is

\[
\psi(\varphi) = \left( \frac{\omega}{\pi} \right)^{1/4} e^{-\frac{1}{2} \omega \varphi^2}, \tag{3.12}
\]

and the integral is gaussian. The remaining determinant \( \text{det}_0 (-\partial_t^2 + \omega^2(t)) \) can be evaluated using the method explained in [16], p. 340. The prescription is to solve the equation

\[
(-\partial_t^2 + \omega^2(t)) \chi(t) = 0; \quad \chi(T_s) = 0, \dot{\chi}(T_s) = 1, \tag{3.13}
\]
then
\[ \det_0(-\partial_t^2 + \omega^2(t)) = N'\chi(T_f). \] (3.14)

We already solved this equation above. Solving for the boundary conditions, we find
\[ \chi(T_f) = \frac{e^\frac{B}{A_s}}{\sinh \int A - \frac{1}{\sqrt{A_sA_f}}} \sinh \int A, \] (3.15)
using \( B = -\dot{A}/(2A) \) which is easily derived from the equation.

We now restrict to the regime where \( \int \omega \gg 1 \), we can then replace all hyperbolic functions by exponentials up to errors of order \( \exp(-2 \int \omega) \). Doing the gaussian integral, we find
\[ \det(-\partial_t^2 + \omega^2(t)) = N'' \left[ 1 + \frac{A' - B_s}{2\omega_s} \right] \left[ 1 + \frac{A' + B_f}{2\omega_f} \right] \exp \left[ \int (\omega + A') \left[ 1 + O(e^{-2\int \omega}) \right] \right] \]
\[ = N'' \exp \left[ \int_{T_s}^{T_f} \omega dt + \frac{1}{8} \int_{T_s}^{T_f} \frac{v^4 t^2}{\omega^5} dt - \frac{v^4}{32} \left( \frac{T_f^2}{\omega_f^6} + \frac{T_s^2}{\omega_s^6} \right) + O(v^3/\omega^6) + O(e^{-2\int \omega}) \right]. \] (3.16)

Here we defined \( A' = A - f_0 \) and \( B' = B - f_1 \). The result takes a simple symmetric form, every second term in the expansion can be written as an integral, while the others have equal contributions from each endpoint (the first of these actually cancels completely). This latter type of term could in principle be removed by a suitable normalization of the initial and final wave functions we use, however as far as we can see, there is no rationale for doing that. There are no ambiguous time-dependent normalization factors floating around here, as we see from comparison with the short-time calculation below. When we write \( O(v^3/\omega^6) \), it is meant up to positive powers of \( vt/\omega \). We can now plug this into (3.11) and expand in powers of \( v \), this gives the phase shift
\[ \delta = \int_{T_s}^{T_f} \frac{15 v^4}{16 r^7} dt + \int_{T_s}^{T_f} \frac{315 v^6}{128 r^{11}} (1 - 11(\hat{r} \cdot \hat{v})^2) dt + \frac{45}{4} v^8 \left( \frac{T_f^2}{r_f^{14}} + \frac{T_s^2}{r_s^{14}} \right) \]
\[ + \int_{T_s}^{T_f} \frac{429 v^8}{4096 r_s^{15}} [63 + 1500(\hat{r} \cdot \hat{v})^2 - 12070(\hat{r} \cdot \hat{v})^4] dt + O(v^8/r^{16}) + O(e^{-2\int r}). \] (3.17)

Here \( \hat{r} = \vec{r}/r \). This result reduces to the first line of (1.11) in the limit \( T_s \to -\infty, T_f \to \infty \) as it should, but we see the interesting feature that it differs for finite time intervals. The leading order potential still matches supergravity perfectly. The subleading potential is
different from both the instantaneous zero-point energy potential and the potential derived from the infinite time phase shift—in a sense it interpolates between these two. There are also terms depending on the end points additively and thus cannot be written as an integrated potential. The interpretation of these terms is not clear. Note that the first subleading term is a simple total derivative,

\[
\frac{v^6}{r^{11}}(1 - 11(\hat{r} \cdot \hat{v})^2) = \frac{d}{dt} \frac{v^5}{r^{10}}\hat{r} \cdot \hat{v}
\]  

(3.18)

for the straight-line trajectory. It seems unlikely that the corresponding term for a curved trajectory would be a total derivative \[20\].

It is interesting to see how the result changes for different choices of boundary conditions on the off-diagonal fields. If we choose boundary conditions such that the fields vanish at the end points, the determinants are just given by \(\det_0\), and we find the phase shift

\[
\delta_{\nu\nu} = \int_{T_s}^{T_f} \frac{15}{16} \frac{v^4}{r^7} dt - 3v^4 \left( \frac{1}{r^{8}_{s}} + \frac{1}{r^{8}_{f}} \right) = \int_{T_s}^{T_f} \frac{1575}{128} \frac{v^6}{r^{11}}(1 - 11(\hat{r} \cdot \hat{v})^2) dt + \ldots. 
\]  

(3.19)

This also reduces to (1.1) in the infinite time limit, as expected, and the leading potential is unchanged, but the subleading terms have changed. We can also imagine picking boundary conditions such that we have the ground state initially while summing over all final states, this corresponds to putting \(\psi_f(\varphi_f) = 1\) in (3.2). Also in this case we find the same leading order potential, while the higher terms are now asymmetric in \(T_f\) and \(T_s\) and cannot be written in a particularly illuminating form.

The approach above breaks down for \(\Delta T = T_f - T_s\) very small. We want to check further what the Matrix theory predicts in this regime. In the infinite-momentum frame, it is not clear to us whether physics at short time scales as described by Matrix theory should agree with physics at short time scales as described by supergravity. Since supergravity is an effective low-energy description of M-theory, one might expect Matrix theory to behave differently from supergravity in this regime. To deal with short time-intervals, we use the first representation of the determinant in (3.2). We write

\[
H = H_s + H_p,
\]

\[
H_s = \frac{1}{2}(p^2 + \omega_s^2 \varphi^2),
\]

\[
H_p = \frac{1}{2}v^2(t^2 - T_s^2)\varphi^2,
\]

(3.20)
and treat $H_p$ as a time-dependent perturbation. This expansion should be useful for $v^2(T_f^2 - T_s^2)$ small, in particular $\omega_s\Delta T$ can be allowed large thus giving an overlap in the region of validity with the expansion considered above. We expand the time ordered product as

$$\det\left(-\frac{1}{2}\partial_t^2 + \omega^2(t)\right) = \langle 0_f|e^{-H_s\Delta T}|0_s\rangle - \int_{T_s}^{T_f}\langle 0_f|e^{-H_s(T_f-t)}H_p(t)e^{-H_s(t-T_s)}|0_s\rangle + \ldots.$$  

(3.21)

Here $|n_s\rangle$ denotes the energy eigenstates of the harmonic oscillator at time $T_s$. To evaluate this expansion, it is convenient to expand $|0_f\rangle$ in terms of the $|n_s\rangle$. Representing $\varphi^2$ in terms of creation and annihilation operators it is then a matter of straightforward, but tedious algebra to evaluate the determinant. The result for short time intervals is

$$\det(-\partial_t^2+\omega^2(t)) = \exp\left[\int_{T_s}^{T_f}\omega dt + \frac{v^4T_s^2}{8\omega_s^4}\Delta T^2 + \frac{v^4T_s}{24\omega_s^4}\left(-2\omega_s T_s + 3 - 6\frac{v^2T_s^2}{\omega_s^2}\right)\Delta T^3 + O(\Delta T^4)\right].$$  

(3.22)

We have checked the consistency between this expansion and the WKB expansion above in the region where $v^2T_s\Delta T \ll 1$, $\omega\Delta T \gg 1$ (we checked all terms involving $v^4$). From the determinant we calculate the short time expansion of the phase shift,

$$\delta = \Delta T \left[\frac{15}{16} v^4 r_s^5 + \frac{315}{128} v^6 r_s^{11} + \ldots\right] + \Delta T^2 \left[\left(-\frac{105}{32} v^5 r_s^8 - \frac{3465}{256} v^7 r_s^{12} + \ldots\right)(\hat{r}_s \cdot \hat{v}) + \left(-\frac{15}{r_s^{10}} v^6 + \frac{105}{r_s^{14}} v^8 + \ldots\right)(\hat{r}_s \cdot \hat{v})^2\right] + O(\Delta T^3).$$  

(3.23)

We see that even at short time we reproduce the supergravity result to leading order, the piece linear in $\Delta T$ is nothing but the simple adiabatic result from above, while we see deviations from this approximation at order $\Delta T^2$. These deviations cannot be easily written in form of a potential, which would have required $\delta$ to take the form

$$\delta = -\int_{T_s}^{T_s+\Delta T} V_{\text{eff}}(t) dt = -V_{\text{eff}}(T_s)\Delta T - \frac{1}{2} \dot{V}_{\text{eff}}(T_s)\Delta T^2 + \ldots$$  

(3.24)

The new terms appearing at order $\Delta T^2$ are indicative of terms in the effective Lagrangian that are not expressible as a single local time integral.

If we consider the short time behavior for the case of vanishing boundary conditions on the off-diagonal fields, we find a very different result. We can evaluate the short-time determinant by expanding $\chi(t)$ in a Taylor series, and solve (3.13) order by order in $\Delta T$.  

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The first terms in the determinant will then be proportional to $\omega^2$, $\omega^4$ and so on, and these terms all cancel when multiplying the determinants for the phase shift. In fact the phase shift vanishes all the way up to $\Delta T^8$ and we find

$$\delta_{vv} = \frac{v^4}{1575} \Delta T^8 + O(\Delta T^{10}).$$

This strange result probably shows that this is a particularly unwise choice of boundary conditions at short time intervals.

4. Discussion

When considering finite time amplitudes in Matrix theory, one must be careful in specifying boundary conditions on the off-diagonal modes that have no counterpart in supergravity. The most natural choice seems to be to demand these modes to be in their ground state. In this case we derive an effective potential between the gravitons that differs from the one obtained from the infinite time calculations. The leading order term, which is the one reproduced by supergravity, still matches perfectly, providing a functional agreement between supergravity and Matrix theory. In comparing these Matrix theory and supergravity calculations, one expects the low energy considerations to be valid only away from very short time intervals. We find that for long, but finite time intervals, there is a correction to the effective potential, depending on the angle between $\vec{r}$ and $\vec{v}$, compared to the one read off from the infinite time phase shifts. There are also new terms that are not expressible in terms of a potential, whose role is not so clear. In particular the first subleading term in the potential, of order $v^6/r^{11}$, is non-zero. Note that this term dominates the $v^6/r^{14}$-term (which has been matched between two loop Matrix theory and supergravity) at long distances. Since supergravity should be corrected by M-theory effects, it would be interesting to interpret the term we found as such an M-theory correction.

This type of calculation should be straightforward to extend to include spin effects and other objects than gravitons. We expect the same features to show up also then, namely that the leading term matches supergravity as in the infinite time case, with corrections to the subleading terms. It would also be interesting to consider extensions to higher loop calculations.

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References

[1] T. Banks, W. Fischler, S. Shenker, and L. Susskind, Phys. Rev. D55 (1997) 5112, hep-th/9610043
[2] L. Susskind, Another conjecture about M(atrix) theory, hep-th/9704080
[3] T. Banks, The state of Matrix theory, hep-th/9706168
[4] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, Nucl. Phys. B485 (1997) 85, hep-th/9608024
[5] O. Aharony and M. Berkooz, Nucl. Phys. B491 (1997) 184, hep-th/9611215
[6] G. Lifschytz and S.D. Mathur, Supersymmetry and Membrane Interactions in M(atrix) Theory, hep-th/9612087
[7] I. Chepelev and A.A. Tseytlin, Long-distance Interactions of D-brane Bound States and Longitudinal 5-brane in M(atrix) Theory, hep-th/9704127
[8] P. Kraus, Spin-Orbit Interaction from Matrix Theory, hep-th/9709199
[9] J. Polchinski and P. Pouliot, Membrane Scattering with M-Momentum Transfer, hep-th/9704028
[10] K. Becker and M. Becker, A two-loop test of M(atrix) theory, hep-th/9705091
[11] K. Becker, M. Becker, J. Polchinski and A.A. Tseytlin, Phys. Rev. D56 (1997) 3174
[12] N. Seiberg, Why is the Matrix model correct?, hep-th/9710009
[13] S.P. de Alwis, Matrix Models and String World Sheet Duality, hep-th/9710219
[14] I. Chepelev and A.A. Tseytlin, Long-distance interactions of branes: correspondence between supergravity and super Yang-Mills descriptions, hep-th/9709087
[15] J.M. Maldacena, Branes Probing Black Holes, hep-th/9709099
[16] S. Coleman, Aspects of Symmetry, Cambridge University Press (Cambridge, U.K., 1985)
[17] M. Claudson and M.B. Halpern, Nucl. Phys. B250 (1985) 689
[18] V. Balasubramanian and F. Larsen, Relativistic brane scattering, hep-th/9704143
[19] M. Abramowitz and I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1965
[20] Work in progress