A certain minimization property implies a certain
integrability

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Abstract

The manifold $M$ being compact and connected and $H$ being a Tonelli Hamiltonian
such that $T^*M$ is equal to the dual tiered Mañé set, we prove that there is a partition
of $T^*M$ into invariant $C^0$ Lagrangian graphs. Moreover, among these graphs, those
that are $C^1$ cover a dense $G_δ$ subset of $T^*M$. The dynamic restricted to each of these
sets is non wandering.

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1 Introduction

In these articles, we go on with our study of the so-called tiered Mañé set. We began this study in \[1\]. Let us recall that the dual tiered Mañé set \(\mathcal{N}_t^T(H)\) of a Tonelli Hamiltonian \(H\) is the union of all the dual Mañé sets of \(H\) associated to all the cohomology classes of \(M\).

In \[1\], we proved that for a generic Tonelli Hamiltonian, the tiered Mañé set has no interior.

In our new article, we consider the following (non-generic) case: we assume that \(\mathcal{N}_t^T(H) = T^*M\). In other words, we assume that every orbit of the Hamiltonian flow of \(H\) is globally minimizing for \(L - \lambda\), where \(L\) is the Lagrangian associated to \(H\) and \(\lambda\) a closed 1-form (that depends on the considered orbit).

Such flows are part of a set of more general Tonelli Hamiltonian flows: those that have no conjugate points. For example, it is proved in \[18\] that any Anosov Hamiltonian level of a Tonelli Hamiltonian has no conjugate points. The same result for geodesic flows was proved in the 70's by W. Klingenberg in \[11\]. But the tiered Mañé set of an Anosov geodesic flow has no interior (see \[1\]) hence in this case, the dual tiered Mañé set is not equal to \(T^*M\). In fact, we prove:

**Theorem 1** Let \(M\) be a compact and connected manifold and let \(H : T^*M \to \mathbb{R}\) be a Tonelli Hamiltonian. Then the two following assertions are equivalent:

1. there exists a partition of \(T^*M\) into invariant Lipschitz Lagrangian graphs;
2. the dual tiered Mañé set of \(H\) is the whole cotangent bundle \(T^*M\).

Moreover, in this case:

- there exists an invariant dense \(G_\delta\) subset \(\mathcal{G}\) of \(T^*M\) such that all the graphs of the partition that meets \(\mathcal{G}\) are in fact \(C^1\).
- Mather’s \(\beta\) function is everywhere differentiable.

Let us emphasize why this result is surprising: we just ask that all the orbits are, in a certain way, minimizing, and we prove that they are well-distributed on invariant Lipschitz Lagrangian graphs.

An easy corollary is the following:

**Corollary 2** Let \(M\) be a compact and connected manifold and let \(H : T^*M \to \mathbb{R}\) be a Tonelli Hamiltonian. Then the two following assertions are equivalent:

1. there exists a partition of \(T^*M\) into invariant Lipschitz Lagrangian graphs;
2. \(T^*M\) is covered by the union of its invariant Lipschitz Lagrangian graphs.

\[1\] all these notions will be precisely defined in next section
The same statement is true if we replace everywhere “Lipschitz” by “smooth”.

In [3], we proved a Birkhoff multidimensional theorem for Tonelli Hamiltonians. We deduce:

**Corollary 3** Let $M$ be a closed and connected manifold and let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. Then the two following assertions are equivalent:

1. there exists a partition of $T^*M$ into Lagrangian invariant smooth graphs;
2. $T^*M$ is covered by the union of its Lagrangian invariant smooth submanifolds that are Hamiltonianly isotopic to some Lagrangian smooth graph.

These results give us a characterization of a weak form of integrability; following [2], we say that a Tonelli Hamiltonian is $C^0$-integrable if there is a partition of $T^*M$ into invariant $C^0$-Lagrangian graphs, one for each cohomology class in $H^1(M, \mathbb{R})$. We then prove that if all the orbits are in some Mañé set, then the Hamiltonian is $C^0$-integrable. A natural question is then:

**Question 1** : does there exist any Tonelli Hamiltonian that is $C^0$-integrable but not $C^1$-integrable (i.e. for which the invariant graphs are not all $C^1$)?

Let us notice that we finally prove that our hypotheses implies that the function $\beta$ is everywhere differentiable. An interesting question, well-known from specialists, is : when the function $\beta$ is everywhere differentiable, is the Hamiltonian $C^0$-integrable? In the case of closed surfaces, a positive answer to this question is given in [15].

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## 2 An overview of Mather-Mañé-Fathi theory of minimizing orbits

### 2.1 Tonelli Lagrangian and Hamiltonian functions

Let $M$ be a compact and connected manifold endowed with a Riemannian metric. We denote a point of the tangent bundle $TM$ by $(q,v)$ with $q \in M$ and $v$ a vector tangent to $M$ at $q$. The projection $\pi : TM \to M$ is then $(q,v) \to q$. The notation $(q,p)$ designates a point of the cotangent bundle $T^*M$ with $p \in T^*_qM$ and $\pi^* : T^*M \to M$ is the canonical projection $(q,p) \to q$.

We consider a Lagrangian function $L : TM \to \mathbb{R}$ which is $C^2$ and:

- uniformly superlinear: uniformly on $q \in M$, we have: $\lim_{\|v\| \to +\infty} \frac{L(q,v)}{\|v\|} = +\infty$;
strictly convex: for all \((q, v) \in TM\), \(\frac{\partial^2 L}{\partial v^2}(q, v)\) is positive definite.

Such a Lagrangian function will be called a Tonelli Lagrangian function.

We can associate to such a Lagrangian function the Legendre map \(L = L_L : TM \rightarrow T^*M\) defined by: \(L(q, v) = \frac{\partial L}{\partial v}(q, v)\) which is a fibered \(C^2\) diffeomorphism and the Hamiltonian function \(H : T^*M \rightarrow \mathbb{R}\) defined by: \(H(q, p) = p(L^{-1}(q, p)) - L(L^{-1}(q, p))\) (such a Hamiltonian function will be called a Tonelli Hamiltonian function). The Hamiltonian function \(H\) is then superlinear, strictly convex in the fiber and \(C^2\). We denote by \((f_t^L)\) or \((f_t)\) the Euler-Lagrange flow associated to \(L\) and \((\varphi_t^H)\) or \((\varphi_t)\) the Hamiltonian flow associated to \(H\); then we have: \(\varphi_t^H = L \circ f_t^L \circ L^{-1}\).

If \(\lambda\) is a \((C^\infty)\) closed 1-form of \(M\), then the map \(T_\lambda : T^*M \rightarrow T^*M\) defined by: \(T_\lambda(q, p) = (q, p + \lambda(q))\) is a symplectic \((C^\infty)\) diffeomorphism; therefore, we have: \((\varphi_t^{H \circ T_\lambda}) = (T_\lambda^{-1} \circ \varphi_t \circ T_\lambda)\), i.e. the Hamiltonian flow of \(H\) and \(H \circ T_\lambda\) are conjugated. Moreover, the Tonelli Hamiltonian function \(H \circ T_\lambda\) is associated to the Tonelli Lagrangian function \(L - \lambda\), and it is well-known that: \((f_t^L) = (f_t^{L - \lambda})\); the two Euler-Lagrange flows are equal. Let us emphasize that these flows are equal, but the Lagrangian functions, and then the Lagrangian actions differ and so the minimizing “objects” may be different.

2.2 Tiered sets: Mather, Aubry and Mañé

For a Tonelli Lagrangian function \((L\) or \(L - \lambda\)), J. Mather introduced in [17] (see [13] too) a particular subset \(\mathcal{A}(L - \lambda)\) of \(TM\) which he called the “static set” and which is now usually called the “Aubry set” (this name is due to A. Fathi[2]). There exist different but equivalent definitions of this set (see [8], [10], [13] and subsection 2.3) and it is known that two closed 1-forms which are in the same cohomological class define the same Aubry set:

\[ [\lambda_1] = [\lambda_2] \in H^1(M) \Rightarrow \mathcal{A}(L - \lambda_1) = \mathcal{A}(L - \lambda_2).\]

We can then introduce the following notation: if \(c \in H^1(M)\) is a cohomological class, \(\mathcal{A}_c(L) = \mathcal{A}(L - \lambda)\) where \(\lambda\) is any closed 1-form belonging to \(c\). \(\mathcal{A}_c(L)\) is compact, non-empty and invariant under \((f_t^L)\). Moreover, J. Mather proved in [17] that it is a Lipschitz graph above a part of the zero-section (see [10] or subsection 2.3 too).

As we are as interested in the Hamiltonian dynamics as well as in the Lagrangian ones, let us define the dual Aubry set:

- if \(H\) is the Hamiltonian function associated to the Tonelli Lagrangian function \(L\), its dual Aubry set is \(\mathcal{A}^\ast(H) = \mathcal{L}_L(\mathcal{A}(L))\);

---

[2] These sets extend the notion of “Aubry-Mather” sets for the twist maps.
These sets are invariant under the Hamiltonian flow \((\varphi^H_t)\).

Another important invariant subset in the theory of Tonelli Lagrangian functions is the so-called Mather set. For it, there exists one definition (which is in [10], [13], [16] and subsection 2.4) : it is the closure of the union of the supports of the minimizing measures for \(L\); it is denoted by \(\mathcal{M}(L)\) and the dual Mather set is \(\mathcal{M}^\ast(H) = \mathcal{L}_L(\mathcal{M}(L))\) which is compact, non empty and invariant under the flow \((\varphi^H_t)\). As for the Aubry set, if \(c \in H^1(M)\) is a cohomological class, we define : \(\mathcal{M}_c(L) = \mathcal{L}(L - \lambda)\) which is independent of the choice of the closed 1-form \(\lambda\) belonging to \(c\). Then \(\mathcal{M}_c^\ast(H) = \mathcal{L}_L(\mathcal{M}_c(L)) = T_\lambda(\mathcal{M}^\ast(H \circ T_\lambda))\) is invariant under \((\varphi^H_t)\); we name it the dual Mather set.

In a similar way, if \(\mathcal{N}(L)\) is the Mañé set, the dual Mañé set is \(\mathcal{N}^\ast(H) = \mathcal{L}_L(\mathcal{N}(L))\); we note that if \(c \in H^1(M)\) and \(\lambda \in c\), then \(\mathcal{N}_c(L) = \mathcal{N}(L - \lambda)\) is independent of the choice of \(\lambda \in c\) and then the c-dual Mañé set is \(\mathcal{N}_c^\ast(H) = \mathcal{L}_L(\mathcal{N}_c(L)) = T_\lambda(\mathcal{N}^\ast(H \circ T_\lambda))\); it is invariant under \((\varphi^H_t)\), compact and non empty but is not necessarily a graph.

For every cohomological class \(c \in H^1(M)\), we have the inclusion : \(\mathcal{M}_c^\ast(H) \subset \mathcal{N}_c^\ast(H)\). Moreover, there exists a real number denoted by \(\alpha_H(c)\) such that : \(\mathcal{N}_c^\ast(H) \subset H^{-1}(\alpha_H(c))\) (see [4] and [16]), i.e. each dual Mañé set is contained in an energy level. For \(c = 0\), the value \(\alpha_H(0)\) is named the “critical value” of \(L\).

**Definition.** If \(H : T^*M \rightarrow \mathbb{R}\) is a Tonelli Hamiltonian function, the tiered Aubry set, the tiered Mather set and the tiered Mañé set are :

\[
\mathcal{A}_c^T(L) = \bigcup_{c \in H^1(M)} \mathcal{A}_c(L); \quad \mathcal{M}_c^T(L) = \bigcup_{c \in H^1(M)} \mathcal{M}_c(L); \quad \mathcal{N}_c^T(L) = \bigcup_{c \in H^1(M)} \mathcal{N}_c(L).
\]

Their dual sets are :

\[
\mathcal{A}_c^T(H) = \bigcup_{c \in H^1(M)} \mathcal{A}_c^\ast(H); \quad \mathcal{M}_c^T(H) = \bigcup_{c \in H^1(M)} \mathcal{M}_c^\ast(H); \quad \mathcal{N}_c^T(H) = \bigcup_{c \in H^1(M)} \mathcal{N}_c^\ast(H).
\]

### 2.3 Mañé potential, Peierls barrier, Aubry and Mañé sets

We gather in this sections some well-known results; the ones concerning the Peierls barrier are essentially due to A. Fathi (see [10]), the others concerning Mañé potential are given in [12], [6] and [7].

In the whole section, \(L\) is a Tonelli Lagrangian function.

**Notations.**
• given two points $x$ and $y$ in $M$ and $T > 0$, we denote by $\mathcal{C}_T(x,y)$ the set of absolutely continuous curves $\gamma : [0,T] \to M$ with $\gamma(0) = x$ and $\gamma(T) = y$;

• the Lagrangian action along an absolutely continuous curve $\gamma : [a,b] \to M$ is defined by:

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))dt;$$

• for each $t > 0$, we define the function $h_t : M \times M \to \mathbb{R}$ by:

$$h_t(x,y) = \inf \{ A_L + \alpha H(0); \gamma \in C_T(x,y) \};$$

• the Peierls barrier is then the function $h : M \times M \to \mathbb{R}$ defined by:

$$h(x,y) = \liminf_{t \to +\infty} h_t(x,y);$$

• we define the (Mañé) potential $m : M \times M \to \mathbb{R}$ by:

$$m(x,y) = \inf \{ A_L + \alpha H(0); \gamma \in \bigcup_{T>0} \mathcal{C}_T(x,y) \} = \inf \{ h_t(x,y); t > 0 \}.$$  

Then, the Mañé potential verifies:

**Proposition 4** We have:

1. $m$ is finite and $m \leq h$;
2. $\forall x, y, z \in M, m(x, z) \leq m(x, y) + m(y, z)$;
3. $\forall x \in M, m(x, x) = 0$;
4. if $x, y \in M$, then $m(x, y) + m(y, x) \geq 0$;
5. if $M_1 = \sup \{ L(x, v); \|v\| \leq 1 \}$, then $\forall x, y \in M, |m(x, y)| \leq (M_1 + \alpha_H(0))d(x, y)$;
6. $m : M \times M \to \mathbb{R}$ is $(M_1 + \alpha_H(0))$-Lipschitz.

Now we can define:

**Definition.**

• a absolutely continuous curve $\gamma : I \to M$ defined on an interval $I$ is a ray if:

$$\forall [a,b] \subset I, A_{L+\alpha H(0)}(\gamma|_{[a,b]}) = h_{|b-a|}(\gamma(a), \gamma(b));$$

a ray is always a solution of the Euler-Lagrange equations;

• a absolutely continuous curve $\gamma : I \to M$ defined on an interval $I$ is semistatic if:

$$\forall [a,b] \subset I, m(\gamma(a), \gamma(b)) = A_{L+\alpha H(0)}(\gamma|_{[a,b]});$$

a semistatic curve is always a ray;
• the Mañé set is then: $\mathcal{N}(L) = \{v \in TM; \gamma_v \text{ is semistatic}\}$ where $\gamma_v$ designates the solution $\gamma_v : \mathbb{R} \to M$ of the Euler-Lagrange equations with initial condition $v$ for $t = 0$; $\mathcal{N}(L)$ is contained in the critical energy level;

• an absolutely continuous curve $\gamma : I \to M$ defined on an interval $I$ is static if:

$$\forall [a, b] \subset I, -m(\gamma(b), \gamma(a)) = A_{L + \alpha_H(0)}(\gamma|_{[a, b]});$$

a static curve is always a semistatic curve;

• the Aubry set is then: $\mathcal{A}(L) = \{v \in TM; \gamma_v \text{ is static}\}$.

The following result is proved in [7]:

**Proposition 5** If $v \in TM$ is such that $\gamma_v|_{[a, b]}$ is static for some $a < b$, then $\gamma_v : \mathbb{R} \to M$ is static, i.e. $v \in \mathcal{A}(L)$.

The Peierls barrier verifies (this proposition contains some results of [9], [10] and [5]):

**Proposition 6** (properties of the Peierls barrier $h$)

1. the values of the map $h$ are finite and $m \leq h$;
2. if $M_1 = \sup\{L(x, v); \|v\| \leq 1\}$, then:

$$\forall x, y, x', y' \in M, |h(x, y) - h(x', y')| \leq (M_1 + \alpha_H(0))(d(x, x') + d(y, y'));$$

therefore $h$ is Lipschitz;
3. if $x, y \in M$, then $h(x, y) + h(y, x) \geq 0$; we deduce: $\forall x \in M, h(x, x) \geq 0$;
4. $\forall x, y, z \in M, h(x, z) \leq h(x, y) + h(y, z)$;
5. $\forall x \in M, \forall y \in \pi(\mathcal{A}(L)), m(x, y) = h(x, y)$ and $m(y, x) = h(y, x)$;
6. $\forall x \in M, h(x, x) = 0 \iff x \in \pi(\mathcal{A}(L))$.

The last item of this proposition gives us a characterization of the projected Aubry set $\pi(\mathcal{A}(L))$. Moreover, we have:

**Proposition 7** (A. Fathi, [10], 6.3.3) When $t$ tends to $+\infty$, uniformly on $M \times M$, the function $h_t$ tends to the Peierls barrier $h$.

A corollary of this result is given in [7]:

**Corollary 8** ([7], 4-10.9) All the rays defined on $\mathbb{R}$ are semistatic.

Let us give some properties of the Aubry and Mañé sets (see [13] and [6]):
∀ all of whose minimizing for $L$ in $\mathcal{N}(L)$;

- the Aubry set is a Lipschitz graph above a part of the zero section;

- if $\gamma : \mathbb{R} \to M$ is semistatic, then $(\gamma, \dot{\gamma})$ is a Lipschitz graph above a part of the zero section;

- the $\omega$ and $\alpha$-limit sets of every point of the Mañé set are contained in the Aubry set.

Last item in proposition 6 gives us a criterion to some $q \in M$ belong to some projected Aubry set. We will need a little more than this: we will need to know what happens for its lift, the Aubry set.

**Proposition 10** Let $c \in H^1(M)$ and $\lambda \in c$, $\varepsilon > 0$ and let $L : TM \to \mathbb{R}$ be a Tonelli Lagrangian function. Then there exists $T_0 > 0$ such that:

$\forall T \geq T_0, \forall (q_0, v_0) \in \mathcal{A}(L), \forall \gamma : [0, T] \to M$ minimizing for $L - \lambda$ between $q_0$ and $q_0$, i.e.:

$\forall \eta : [0, T] \to M, \eta(0) = \eta(T) = q_0 \Rightarrow \int_0^T (L(\gamma, \dot{\gamma}) - \lambda(\dot{\gamma}) + \alpha_H(c)) \leq \int_0^T (L(\eta, \dot{\eta}) - \lambda(\dot{\eta}) + \alpha_H(c))$

then we have: $d((q_0, v_0), (q_0, \gamma'(0))) \leq \varepsilon$

**Proof** Let us assume that the result is not true; then we may find a sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+^*$ tending to $+\infty$, a sequence $\gamma_n : [0, T_n] \to M$ of absolutely continuous loops, all of whose minimizing for $L - \lambda$ from $q_n$ to $q_n$ where $(q_n, w_n) \in \mathcal{A}(L)$ such that the sequence $(q_n, v_n) = (\gamma_n(0), \dot{\gamma}_n(0))$ satisfies: $\forall n \in \mathbb{N}, d((q_n, v_n), (q_n, w_n)) \geq \varepsilon$.

The sequence $(q_n, v_n)$ is bounded (it is a consequence of the so-called “a priori compactness lemma” (see [10], corollary 4.3.2)); therefore we may extract a converging subsequence; we call it $(q_n, v_n)$ again and $(q_\infty, v_\infty)$ is its limit. Then $q_\infty \in \pi(\mathcal{A}(L))$ because the Aubry set is closed. We denote by $(q_\infty, w_\infty) \in \mathcal{A}(L)$ its lift. Then $w_\infty = \lim_{n \to \infty} w_n$ because $\mathcal{A}(L)$ is closed. Then: $d((q_\infty, v_\infty), (q_\infty, w_\infty)) \geq \varepsilon$.

Now we use proposition 7: we know that if we define $h^\lambda_t : M \times M \to \mathbb{R}$ by $h^\lambda_t(x, y) = \inf \{ A_{L-\lambda+\alpha_H(c)}(\gamma) ; \gamma \in \mathcal{C}(x, y) \}$ and $h^\lambda(x, y) = \lim_{t \to +\infty} h^\lambda_t(x, y)$, the functions $h^\lambda_t$ tend uniformly to $h^\lambda$ when $t$ tends to $+\infty$; we have then:

$h^\lambda_{T_n}(q_n, q_n) = A_{L-\lambda+\alpha_H(c)}(\gamma_n)$ tends to $h^\lambda(q_\infty, q_\infty) = 0$ when $n$ tends to the infinite.

Let $\gamma_\infty$ be the solution of the Euler-lagrange equations such that $(\gamma_\infty(0), \dot{\gamma}_\infty(0)) = (q_\infty, v_\infty)$. We want to prove that $\gamma_\infty$ is static: we shall obtain a contradiction. When $n$ is big enough, $\gamma_n(T_n) = \gamma_n(0)$ is close to $q_\infty$ and $\gamma_n(1)$ is close to $\gamma_\infty(1)$. Let us fix $\eta > 0$; then we define $\Gamma^\eta_n : [0, T_n + 2\eta] \to M$ by:

- $\Gamma^\eta_{n|[0,1]} = \gamma_\infty|_[0,1]$;
• $\Gamma^n_{n[1,1+\eta]}$ is a short geodesic joining $\gamma(1)\gamma_\infty(1)$ to $\gamma(1)$;
• $\forall t \in [1 + \eta, T_n + \eta], \Gamma^n(t) = \gamma_n(t - \eta)$;
• $\Gamma^n_{n[T_n+\eta,T_n+2\eta]}$ is a short geodesic joining $\gamma_n(T_n)$ to $\gamma_\infty(0)$.

If we choose carefully a sequence $(\eta_n)$ tending to 0, we have:

$$\lim_{n \to \infty} A_{L-\lambda+\alpha_H(c)}(\Gamma^n_n) = \lim_{n \to \infty} A_{L-\lambda+\alpha_H(c)}(\gamma_n) = 0.$$ 

Because the contribution to the action of the two small geodesic arcs tends to zero (if the $\eta_n$ are well chosen), this implies:

$$A_{L-\lambda+\alpha_H(c)}(\gamma_\infty(0), \gamma_\infty(1)) + m^\lambda(\gamma_\infty(1), \gamma_\infty(0)) \leq 0,$$

where $m^\lambda$ designates Mañe potential for the Lagrangian function $L-\lambda$. We deduce then from the definition of Mañe potential that $m^\lambda(\gamma_\infty(0), \gamma_\infty(1)) + m^\lambda(\gamma_\infty(1), \gamma_\infty(0)) = 0$ and that:

$$A_{L-\lambda+\alpha_H(c)}(\gamma_\infty(0), \gamma_\infty(1)) = m^\lambda(\gamma_\infty(0), \gamma_\infty(1)).$$

It implies then that $A_{L-\lambda+\alpha_H(c)}(\gamma_\infty(0), \gamma_\infty(1)) = -m^\lambda(\gamma_\infty(1), \gamma_\infty(0))$. Let us notice that, changing slightly $\Gamma^n_n$, we obtain too:

$$\forall [a, b] \subset [0, +\infty[, A_{L-\lambda+\alpha_H(c)}(\gamma_\infty(a,b)) = -m^\lambda(\gamma_\infty(b), \gamma_\infty(a));$$

therefore $\gamma_\infty(0, +\infty]$ is static. To conclude, we use proposition 5.

### 2.4 Minimizing measures, Mather $\alpha$ and $\beta$ functions

The general references for this section are [16] and [15]. Let $\mathcal{M}(L)$ be the space of compactly supported Borel probability measures invariant under the Euler-Lagrange flow $(f_t^L)$. To every $\mu \in \mathcal{M}(L)$ we may associate its average action $A_L(\mu) = \int_{TM} Ld\mu$. It is proved in [16] that for every $f \in C^1(M, \mathbb{R})$, we have:

$$\int df(q).vd\mu(q, v) = 0.$$ 

Therefore we can define on $H^1(M, \mathbb{R})$ a linear functional $\ell(\mu)$ by:

$$\ell(\mu)([\lambda]) = \int \lambda(q).vd\mu(q, v)$$

(Here $\lambda$ designates any closed 1-form). Then there exists a unique element $\rho(\mu) \in H_1(M, \mathbb{R})$ such that:

$$\forall \lambda, \int_{TM} \lambda(q).vd\mu(q, v) = [\lambda, \rho(\mu)].$$

The homology class $\rho(\mu)$ is called the rotation vector of $\mu$. Then the map $\mu \in \mathcal{M}(L) \to \rho(\mu) \in H^1(M, \mathbb{R})$ is onto. We can then define Mather $\beta$-function $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ that associates the minimal value of the average action $A_L$ over the set of measures of $\mathcal{M}(L)$ with rotation vector $h$ to each homology class $h \in H_1(M, \mathbb{R})$. We have:

$$\beta(h) = \min_{\mu \in \mathcal{M}(L); \rho(\mu) = h} A_L(\mu).$$

A measure $\mu \in \mathcal{M}(L)$ realizing such a minimum, i.e. such that $A_L(\mu) = \beta(\rho(\mu))$ is called a minimizing measure with rotation vector $\rho(\mu)$. The $\beta$ function is convex.
and superlinear, and we can define its conjugate function (given by Fenchel duality) $\alpha : H^1(M, \mathbb{R}) \to \mathbb{R}$ by:

$$\alpha(\lambda) = \max_{h \in H_1(M, \mathbb{R})} ([\lambda], h - \beta(h)) = -\min_{\mu \in \mathcal{M}(L)} A_{L-\lambda}(\mu).$$

A measure $\mu \in \mathcal{M}(L)$ realizing the minimum of $A_{L-\lambda}$ is called a $[\lambda]$-minimizing measure.

Being convex, Mather’s $\beta$ function has a subderivative at any point $h \in H_1(M, \mathbb{R})$; i.e. there exists $c \in H^1(M, \mathbb{R})$ such that:

$$\forall k \in H_1(M, \mathbb{R}), \beta(h) + c.(k - h) \leq \beta(k).$$

We denote by $\partial \beta(h)$ the set of all the subderivatives of $\beta$ at $h$. By Fenchel duality, we have:

$$c \in \partial \beta(h) \iff c.h = \alpha(c) + \beta(h).$$

Then we introduce the following notations:

- if $h \in H_1(M, \mathbb{R})$, the Mather set for the rotation vector $h$ is:
  $$\mathcal{M}^h(L) = \{\text{supp}\mu; \mu \text{ is minimizing with rotation vector } h\};$$

- if $c \in H^1(M, \mathbb{R})$, the Mather set for the cohomology class $c$ is:
  $$\mathcal{M}_c(L) = \{\text{supp}\mu; \mu \text{ is } c - \text{minimizing}\}.$$ 

The following equivalences are proved in [15] for any pair $(h, c) \in H_1(M, \mathbb{R}) \times H^1(M, \mathbb{R})$:

$$\mathcal{M}^h(L) \cap \mathcal{M}_c(L) \neq \emptyset \iff \mathcal{M}^h(L) \subset \mathcal{M}_c(L) \iff c \in \partial \beta(h).$$

As explained in subsection [2.2] the dual Mather set for the cohomology class $c$ is defined by:

$$\mathcal{M}^*_c(H) = L^*_L(\mathcal{M}_c(L)).$$

If $\mathcal{M}^*(H)$ designates the set of compactly supported Borel probability measures of $T^\ast M$ that are invariant by the Hamiltonian flow $(\varphi_t)$, then the map $L^\ast : \mathcal{M}(L) \to \mathcal{M}^*(H)$ that push forward the measures by $L$ is a bijection.

We denote $L^\ast(\mu)$ by $\mu^\ast$ and say that the measures are dual. We say too that $\mu^\ast$ is minimizing if $\mu$ is minimizing in the previous sense.

Moreover, the Mather set $\mathcal{M}^*_c(H)$ is a subset of the Mañé set $\mathcal{N}^*_c(H)$ and every invariant Borel probability measure the support of whose is in $\mathcal{N}^*_c(H)$ is $c$-minimizing.

### 2.5 The link with the weak KAM theory

If $\lambda$ is a closed 1-form on $M$, we can consider the Lax-Oleinik semi-groups of $L - \lambda$, defined on $C^0(M, \mathbb{R})$ by:

- the negative one: $T^\lambda_{-t} u = \min \left( u(\gamma(0)) + \int_0^t (L(\gamma(s), \dot{\gamma}(s)) - \lambda(\gamma(s))\dot{\gamma}(s)) ds \right)$;
  where the infimum is taken on the set of $C^1$ curves $\gamma : [0, t] \to M$ such that $\gamma(t) = q$.
• the positive one: $T^+_t u(q) = \max \left( u(\gamma(t)) - \int_{0}^{t} (L(\gamma(s), \dot{\gamma}(s)) - \lambda(\gamma(s)) \cdot \dot{\gamma}(s)) ds \right)$;

where the infimum is taken on the set of $C^1$ curves $\gamma: [0, t] \to M$ such that $\gamma(0) = q$.

A. Fathi proved in [10] that for each closed 1-form $\lambda$, there exists $k \in \mathbb{R}$ and $u \in C^0(M, \mathbb{R})$ such that: $\forall t > 0, T^+_t u = u - kt$ (resp. $\forall t > 0, T^-_t u = u + kt$). In this case, we have: $k = \alpha([\lambda])$. The function $u$ is called a negative (resp. positive) weak KAM solution if and only if it is a negative weak KAM solution if and only if it is a solution for the Hamilton-Jacobi equation: $H(q, \lambda(q) + du(q)) = \alpha([\lambda])$. It is equivalent too to the fact that the graph of $\lambda + du$ is invariant by the Hamiltonian flow $\varphi_t^H$.

Moreover, it is proved too that a function $u : M \to \mathbb{R}$ that is $C^1$ is a positive weak KAM solution if and only if it is a negative weak KAM solution if and only if it is a solution of the Hamilton-Jacobi equation: $H(q, \lambda(q) + du(q)) = \alpha([\lambda])$. It is equivalent too to the fact that the graph of $\lambda + du$ is invariant by the Hamiltonian flow $\varphi_t^H$.

But in general, the weak KAM solutions are not $C^1$ and the graph of $\lambda + du$ is not invariant by the Hamiltonian flow. There is an invariant subset contained in all these graphs: the dual Aubry set. Let us now recall which characterization of this set is given by A. Fathi in [10].

A pair $(u_-, u_+)$ of negative-positive weak KAM solution is called a pair of conjugate weak KAM solutions if $u_-|_{\pi(M(L-\lambda))} = u_+|_{\pi(M(L-\lambda))}$. Each negative weak KAM solution has an unique conjugate positive weak KAM solution, and we define for any pair $(u_-, u_+) \in S^-_\lambda \times S^+_\lambda$ of conjugate weak KAM solutions for $L-\lambda$:

- $\mathcal{I}(u_-, u_+) = \{ q \in M, u_-(q) = u_+(q) \}$;

- $\tilde{\mathcal{I}}(u_-, u_+) = \{ (q, du_+(q)) \colon q \in \mathcal{I}(u_-, u_+) \} = \{ (q, du_+(q)) \colon q \in \mathcal{I}(u_-, u_+) \}$.

Then: $\mathcal{A}^\ast_{[\lambda]}(H) = T_\lambda(\bigcap \tilde{\mathcal{I}}(u_-, u_+))$ where the intersection is taken on all the pairs of conjugate weak KAM solutions for $L-\lambda$. Moreover: $N^\ast_{[\lambda]}(H) = T_\lambda(\bigcup \tilde{\mathcal{I}}(u_-, u_+))$ where the union is taken on all the pairs of conjugate weak KAM solutions for $L-\lambda$.

An immediate corollary of all these results is the following: if $\pi^\ast(\mathcal{A}^\ast_{[\lambda]}(H)) = M$, then there is a unique negative weak KAM solution $u$ and a unique positive weak KAM solution for $L-\lambda$, they are equal and $C^{1,1}$ (i.e. $C^1$ with a Lipschitz derivative). In this case, we have: $\mathcal{A}^\ast_{[\lambda]}(H) = N^\ast_{[\lambda]}(H)$ is the graph of $\lambda + du$.

### 3 Proof of theorem [1]

We assume that $H$ is a Tonelli Hamiltonian such that $N^T(H) = T^*M$.

In order to prove theorem [1] we begin by proving that the periodic orbits are on some invariant totally periodic Lagrangian graphs:
Proposition 11  For every closed 1-form $\lambda$ of $M$, for every $(q_0,p_0) \in T^*M$ that is $T$-periodic for a certain $T > 0$ and whose orbit under the Hamiltonian flow is minimizing for $L - \lambda$, then $(q_0,p_0)$ belongs to a $C^1$ invariant Lagrangian graph $T$ such that the orbit of every element of $T$ is $T$-periodic, homotopic to the one of $(q_0,p_0)$ and has the same action for the Lagrangian $L - \lambda$ as the orbit of $(q,p)$. Moreover, $T$ is the graph of a closed 1-form that has the same cohomology class as $\lambda$.

Proof  Let us consider $(q_0,p_0)$ as in the statement. Then, if we denote the cohomology class of $\lambda$ by $[\lambda]$, we have : $(q_0,p_0) \in N^*_{[\lambda]}(H)$, i.e. $(q_0,p_0)$ belongs to the Mañé set associated to the cohomology class of $\lambda$. Let us use the notation : $\gamma_0(t) = \pi \circ \varphi_t(q_0,p_0)$.

Because of Tonelli theorem, we know that for every $q \in M$, there exists a piece of orbit $(\varphi_t(q,p))_{t \in [0,T]}$ such that, if we denote the projection of this piece of orbit by $\gamma_q$ (i.e. $\gamma_q(t) = \pi \circ \varphi_t(q,p)$), then we have :

- $\gamma_q(T) = \gamma_q(0) = q$;
- $\gamma_q$ is homotopic to $\gamma_0$;
- for every absolutely continuous arc $\eta : [0,T] \to M$ that is homotopic to $\gamma_0$ and such that : $\eta(0) = \eta(T) = q$, we have : $\int_0^T (L(\gamma_q, \dot{\gamma}_q) - \lambda(\dot{\gamma}_q)) \leq \int_0^T (L(\eta, \dot{\eta}) - \lambda(\dot{\eta}))$.

As every point on $T^*M$ is in some Mañé set, then the orbit of every point has to be a graph by proposition 9. We deduce that : $\varphi_T(q,p) = (q,p)$, hence $(q,p)$ is a $T$-periodic point. It defines an invariant probability measure $\mu_q$, the one equidistributed along this orbit, defined by :

$$\forall f \in C^0(T^*M, \mathbb{R}), \int f d\mu = \frac{1}{T} \int_0^T f \circ \varphi_t(q,p) dt.$$ 

As the support of this measure is in some Mañé set, this measure is minimizing for $L + \nu$ where $\nu$ is some closed 1-form. The rotation vector of this measure is $\frac{1}{T}[\gamma] = \frac{1}{T}[\gamma_0]$ where we denote the homology class of $\gamma$ by $[\gamma]$; hence, having the same rotation vector, the supports of the measures $\mu_q$ and $\mu_{q_0}$ belong to the same Mather set and the support of $\mu_q$ is in $N^*_{[\lambda]}(H)$. We deduce that :

$$\forall q \in M, -T \alpha([\lambda]) = \int_0^T (L(\gamma, \dot{\gamma}) - \lambda(\dot{\gamma})) = \int_0^T (L(\gamma_0, \dot{\gamma}_0) - \lambda(\dot{\gamma}_0))$$

because all these measures are minimizing for $L - \lambda$.

Finally, for all $q \in M$, we have found a point $(q,p)$ that is in the Mather set $\mathcal{M}^*_{[\lambda]}(H)$. As the Mather set is a Lipschitz graph, then the set of these points $(q,p)$ is a Lipschitz graph and coincides with the Mather set $\mathcal{M}^*_{[\lambda]}(H)$. Moreover, we know that the Aubry set is a graph that contains the Mather set. Hence $\mathcal{A}^*_{[\lambda]}(H) = \mathcal{M}^*_{[\lambda]}(H)$. 

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In this case, this set is the graph of a Lipschitz closed 1-form whose cohomology class is $[\lambda]$ (see subsection 2.5). As the dynamic restricted to this $C^0$-Lagrangian graph is totally periodic, i.e. as $\phi_T|A^c(\mathcal{H}) = \text{Id}|A^c(\mathcal{H})$, we know that this graph is in fact $C^1$ (this is proved in [2] by way of the so-called Green bundles).

We can apply this proposition to every periodic orbit. Indeed, such a periodic orbit is always contained in some Mañe set $\mathcal{N}^c_*(\mathcal{H})$. We deduce from the previous proposition that $A^c_*(\mathcal{H})$ is a $C^1$ Lagrangian graph, and that all the orbits contained in $A^c_*(\mathcal{H})$ are periodic with the same period and are homotopic to each other. Moreover, we have seen in subsection 2.5 that when the Aubry set is a graph above the whole zero section, then it coincides with the Mañe set. Hence, we have proved that $\mathcal{N}^c_*(\mathcal{H})$ is a $C^1$ Lagrangian graph, and that all the orbits contained in $\mathcal{N}^c_*(\mathcal{H})$ are periodic with the same period and are homotopic to each other.

Let us now explain what happens to the other Mañe sets, that correspond to the other cohomology classes.

**Proposition 12** For every cohomology class $c \in H^1(M, \mathbb{R})$, we have : $A^c_*(\mathcal{H}) = \mathcal{N}^c_*(\mathcal{H})$ is the graph $G_c$ of a Lipschitz closed 1-form.

**Proof** Let us assume that $(q,p) \in A^c_*(\mathcal{H})$. Let $\lambda$ be a closed 1-form such that $[\lambda] = c$. Then there exists a sequence $(T_n)$ tending to $+\infty$ and a sequence $(\gamma_n)$ of absolutely continuous arcs $\gamma_n : [0,T_n] \to M$ that are minimizing, such that $\gamma(0) = \gamma(T_n) = q$ and such that : $\lim_{n \to \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c)) dt = 0$ where $\alpha$ designates the $\alpha$ function of Mather. As every $\gamma_n$ is minimizing, it is the projection of a piece of orbit : $\gamma_n(t) = \pi \circ \varphi(t,q,p_n)$. The corresponding orbit, being in a certain Mañe set, has to be a graph, hence it is periodic : $\varphi_{t_n}(q,p_n) = (q,p_n)$. Moreover, we know (see proposition 10) that in this case : $\lim_{n \to \infty} (q,p_n) = (q,p).

We can use proposition 11. Let $c_n \in H^1(M, \mathbb{R})$ be the cohomology class such that $(q,p_n) \in \mathcal{N}^c_{c_n}(\mathcal{H})$. Then there exists a closed 1-form $\lambda_n$, whose cohomology class is $c_n$, so that $\mathcal{N}^c_{c_n}(\mathcal{H})$ is the graph of $\lambda_n$. We have in particular : $p_n = \lambda_n(q)$ and $p = \lim_{n \to \infty} \lambda_n(q)$. Let us now prove that for every $Q \in M$, the sequence $(Q, \lambda_n(Q))$ converges to some point $(Q,P)$ that belongs to $A^c_*(\mathcal{H})$. We will deduce that $A^c_*(\mathcal{H}) = \mathcal{N}^c_*(\mathcal{H})$ is the graph of a Lipschitz closed 1-form and then the proposition.

So let us consider $Q \in M$. For every $n \in \mathbb{N}$, we know by proposition 11 that $(Q, \lambda_n(Q))$ is $t_n$-periodic and that if we denote the projection of its orbit by $\Gamma_n(t) = \pi \circ \varphi_{t_n}(Q, \lambda_n(Q))$, then we have :

- $\Gamma_n$ is homotopic to $\gamma_n$;
\[ \int_0^{t_n} (L(\Gamma_n(t), \hat{\Gamma}_n(t)) - \lambda_n(\hat{\Gamma}_n(t)))dt = \int_0^{t_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \lambda_n(\hat{\gamma}_n(t)))dt. \]

We can then compute (the notation \([\lambda][\gamma]\) is just the usual product of a cohomology class with a homology class):

\[
\int_0^{t_n} (L(\Gamma_n(t), \hat{\Gamma}_n(t)) - \lambda(\hat{\Gamma}_n(t))) + \alpha(c)dt =
\]

\[
\int_0^{t_n} (L(\Gamma_n(t), \hat{\Gamma}_n(t)) - \lambda_n(\hat{\Gamma}_n(t)))dt - [\lambda - \lambda_n][\Gamma_n] + \alpha(c)t_n =
\]

\[
\int_0^{t_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \lambda_n(\hat{\gamma}_n(t)))dt - [\lambda - \lambda_n][\gamma_n] + \alpha(c)t_n =
\]

\[
\int_0^{t_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \lambda(\hat{\gamma}_n(t))) + \alpha(c)dt.
\]

Then : \( \lim_{n \to \infty} \int_0^{t_n} (L(\Gamma_n(t), \hat{\Gamma}_n(t)) - \lambda(\hat{\Gamma}_n(t))) + \alpha(c)dt = 0. \) By proposition 10, this implies that \( Q \) belongs to the projected Aubry set \( \pi(A^*_c(H)) \) and that the sequence \( (Q, \lambda_n(Q)) \) converges to the unique point of \( A^*_c(H) \) that is above \( Q \).

**Proposition 13** With the previous notations, the graphs \( G_c \) are disjoint:

\[ \forall c, d \in H^1(M, \mathbb{R}), c \neq d \Rightarrow G_c \cap G_d = \emptyset. \]

**Proof** We borrow the main elements of the proof to [14]. Let us assume that there exists \( c, d \in H^1(M, \mathbb{R}) \) such that \( G_c \cap G_d \neq \emptyset. \) Then \( G_c \cap G_d \) is a compact invariant subset and there exists an invariant Borel probability measure \( \mu^* \) (dual of \( \mu \)) whose support is contained in \( G_c \cap G_d. \) Hence \( \mu \) is minimizing for \( L - \lambda \) and \( L - \eta \) if \( [\lambda] = c \) and \( [\eta] = d \):

\[ \int (L - \lambda + \alpha(c))d\mu = 0 \quad \text{and} \quad \int (L - \eta + \alpha(d))d\mu = 0. \]

We deduce that for every \( t \in [0, 1] \), we have:

\[ \int (L - (t\lambda + (1-t)\eta) + t(\lambda + (1-t)\eta))d\mu = 0 \]

and then : \( \alpha(tc + (1-t)d) \geq \int (L - (t\lambda + (1-t)\eta))d\mu = \alpha(c) + (1-t)\alpha(d). \) As the function \( \alpha \) is convex, this implies : \( \alpha(tc + (1-t)d) = t\alpha(c) + (1-t)\alpha(d). \) Hence \( \mu \) is minimizing for \( L - (t\lambda + (1-t)\eta) \). This implies that the support of \( \mu^* \) is contained in \( \mathcal{A}^*_{tc+1-d}(H) \subset \mathcal{A}^*_{tc+1-d}(H) = \mathcal{A}^*_{tc+1-d}(H) = \mathcal{G}_{tc+1-d}(H). \) Let us now consider \( (q, p) \) belongs to \( \mathcal{A}^*_{2(c+d)}(H) \), there exists a sequence \( (T_n) \) tending to \( +\infty \) and a sequence of \( C^1 \) arcs \( \gamma_n : [0, T_n] \rightarrow M \) such that \( \gamma_n(0) = \gamma_n(T_n) = q \) and:

\[ \lim_{n \to \infty} (\int_0^{T_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \frac{1}{2}(\lambda + \eta)(\gamma_n(t)) + \alpha(\frac{1}{2}(c+d))dt) = 0. \]

The left term of the previous equality is the limit of the sum of two terms:

\[ \frac{1}{2} \int_0^{T_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \lambda(\hat{\gamma}_n(t)) + \alpha(n))dt \quad \text{and} \quad \frac{1}{2} \int_0^{T_n} (L(\gamma_n(t), \hat{\gamma}_n(t)) - \eta(\hat{\gamma}_n(t)) + \alpha(d))dt, \]

each of these terms being non negative. We deduce that:
\[
\begin{align*}
\lim_{n \to \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \lambda(\dot{\gamma}_n(t)) + \alpha(c)) \, dt = 0; \\
\lim_{n \to \infty} \int_0^{T_n} (L(\gamma_n(t), \dot{\gamma}_n(t)) - \eta(\dot{\gamma}_n(t)) + \alpha(d)) \, dt = 0;
\end{align*}
\]

and by proposition 10:

\[
\lim_{n \to \infty} (\gamma_n(0), \dot{\gamma}_n(0)) \in A_c(H) \cap A_d(H).
\]

We have finally proved that \(G_{1/2}(c+d) = A^*_c(H) \cap A^*_d(H) = G_c \cap G_d\), hence the two graphs \(G_c\) and \(G_d\) are equal, and their cohomology classes are also equal: \(c = d\).

Let us now finish the proof of theorem 1. We have found a partition of \(T^*M\) into Lipschitz Lagrangian graphs \((G_c)_{c \in H_1(M,\mathbb{R})}\), where \(G_c\) is the graph of a Lipschitz 1-form whose cohomology class is \(c\) and is equal to \(A^*_c(H) = N^*_c(H)\). Each Mañé set being chain recurrent, we deduce that the dynamic restricted to each \(G_c\) is chain recurrent. We are then exactly in the case of a \(C^0\)-integrable Hamiltonian that we described in [2]. We can apply the results of [2] and deduce that there exists a dense \(G_\delta\)-subset of \(T^*M\) filled by invariant \(C^1\) Lagrangian graphs. Finally, let us notice that it is proved in [15] that the \(\beta\) function of every \(C^0\) integrable Tonelli Hamiltonian is differentiable everywhere. This ends the proof of the implication: \(2 \Rightarrow 1\).

Let us now prove that \(1 \Rightarrow 2\). We assume that there is a partition of \(T^*M\) into invariant Lagrangian Lipschitz graph. Then to each of these Lipschitz graphs corresponds a \(C^{1,1}\) weak KAM solution and then the orbit of every point of this graph is in some Mañé set. This implies: \(T^*M = N^T_*(M)\).

## 4 Proof of the corollaries

### 4.1 Proof of corollary 2

We only have to prove that \(2 \Rightarrow 1\). We assume that \(T^*M\) is covered by the union of the invariant Lipschitz Lagrangian graphs (resp. smooth Lagrangian graphs). Then to each of these Lipschitz graphs correspond a \(C^{1,1}\) weak KAM solution and then the orbit of every point of this graph is in some Mañé set. This implies: \(T^*M = N^T_*(H)\). We can apply theorem 1 and proposition 12. Then there exists a partition of \(T^*M\) into Lipschitz Lagrangian graphs \((G_c)_{c \in H_1(M,\mathbb{R})}\), where \(G_c\) is the graph of a Lipschitz 1-form whose cohomology class is \(c\) and is equal to \(A^*_c(H) = N^*_c(H)\). Let us look at what happens in the smooth case: if \(N\) is one of the smooth invariant Lagrangian graphs, then it is contained in some Mañé set and then is equal to some \(G_c\). We obtain then
that there is a partition of $T^*M$ into some smooth $\mathcal{G}_c$. As $(\mathcal{G}_c)_{c \in H^1(M, \mathbb{R})}$ is a partition of $T^*M$, we deduce that all the $\mathcal{G}_c$ are smooth.

4.2 Proof of corollary 3

We just have to prove that $2 \Rightarrow 1$. We assume $T^*M$ is covered by the union of its Lagrangian invariant smooth submanifolds that are Hamiltonianly isotopic to some smooth Lagrangian graph. We have proved in [3] a multidimensional Birkhoff theorem: every Lagrangian invariant smooth submanifold that is Hamiltonianly isotopic to some smooth Lagrangian graph is a smooth graph. Then corollary 3 becomes a corollary of corollary 2.
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