On some properties of transitions operators

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Abstract. We study a general transition operator, generated by a random walk on a graph $X$; in particular we give necessary and sufficient condition on the matrix coefficient (1-step transition probabilities) to be a bounded operator from $l^\infty(X)$ into itself. Moreover we characterize compact operators and we relate this property to the behaviour of the associated random walk. We give a necessary and sufficient condition for the pre-adjoint of the discrete Laplace operator to be an injective map.

Keywords: random walk, Laplace operator, transition probabilities, stationary measures.

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1. Introduction

In this paper we consider the transition operator $P$ associated with a general irreducible Markov chain $\{Z_n\}_{n \in \mathbb{N}}$ defined on a probability space $(X, \Omega, \mathbb{P})$ with a state space $X$ which is finite or countable. It is well known that one of the consequences of the time-independence property of Markov chains is that $\mathbb{P}[Z_{n+1} = y | Z_n = x]$ is independent of $n$, hence the transition operator $P$ is uniquely determined (according to equation (2)) by the coefficients $\{p(x, y)\}_{x, y \in X}$ (called 1-step transition probabilities) defined by

$$p(x, y) := \mathbb{P}[Z_1 = y | Z_0 = x].$$ 

This map gives rise to an amount of interesting concepts (such as harmonic and superharmonic functions, stationary measures and so on) which allow us to get information about the behaviour of the Markov chains. Take for instance the characterization of recurrent random walk in terms of non-negative superharmonic functions (see Theorem 1.16 of [1] or Theorem 5.3 of [2]) or in terms of excessive measures (see [1] Theorem 1.18). Other applications may be found in the study of the asymptotic behaviour of the $n$-step transition probabilities $p^{(n)}(x, y)$ (see for instance [3]).

One of the most interesting topics is given by the discrete harmonic analysis on graphs (see [4] and [5]) and the corresponding Dirichlet problem (see for instance [1], Chapter 4).
More recently we started to study mean value properties for finite variation measures on graphs (see [6]) with respect to suitable families of harmonic functions; in that paper it is shown how these properties are related with the range of the preadjoint of the discrete Laplace operator (see Section 3). These are some of the reasons which justify the present paper. For those who are not familiar with some terminology we suggest to look at [7] for functional analysis reference and at [1] for random walk theory reference.

We begin (Section 2) dealing with “generalized” transition operator (with more general kernels \( p(x, y) \)) defined by the equation (2): we give a complete characterization of continuous transition maps (Theorem 2.1) and of compact transition maps with non-negative kernel (Theorem 2.2); moreover we show that compactness implies recurrence (Proposition 2.3), meanwhile the converse it is not true (Example 2.4).

In Section 3 we consider the stochastic kernel defined by equation (1) and we deal with the corresponding (stochastic) transition operator which is a continuous map from \( l^\infty(X) \) into itself. We construct the preadjoint of the discrete analogous of the Laplace operator and we turn our attention to its null space and its range. In particular we give a necessary and sufficient condition for this operator to be an injective map (Theorem 3.2); this result generalizes Theorem 1.18 of [1]. We finally make some remarks about the topological properties of its range.

We fix now the basic notation: let \( \Phi \) represent the real field \( \mathbb{R} \) or the complex field \( \mathbb{C} \) and \( p : X \times X \to \Phi \) be a function. We consider the domain \( D \) and the linear operator \( P \) depending by \( p \) as follows

\[
D(P) := \left\{ f : X \to \Phi : \sum_{y \in X} |p(x, y)||f(y)| < +\infty, \forall x \in X \right\},
\]

\[
(Pf)(x) := \sum_{y \in X} p(x, y)f(y), \quad \forall f \in D(P), \forall x \in X.
\]

The properties of the linear map \( P \) are strictly related to the functional space where it is restricted: for instance if the coefficients \( p \) satisfy equation (1) (which is equivalent to \( p(x, y) \geq 0 \) for all \( x, y \in X \) and \( \sum_{y \in X} p(x, y) = 1 \) for every \( x \in X \) then the transition operator \( P \) is called stochastic; in this case it is easy to show that \( P \) is a bounded linear map from \( l^\infty(X, \mu) \) into itself (for any real or complex measure \( \mu \) on \( X \)) and \( \|P\| = 1 \); furthermore given any excessive, positive measure \( \nu \) on \( X \) (see Section 3), any stochastic map \( P \) is bounded, with \( \|P\| \leq 1 \), from \( L^p(X, \nu) \) into itself (\( p \in [1, +\infty) \)).

If \( P \) is generated by a reversible random walk (see for instance [8], Paragraph 2.A) and if \( \nu \) is the reversibility measure then \( P \) is a linear, bounded, selfadjoint operator from \( L^2(X, \nu) \) into itself, satisfying \( \|P\|_{L^2(X, \nu)} = \rho(P) \), where \( \rho(P) \) is the spectral radius of the random walk \((X, P)\) (see [1] Chapter 1, Paragraph B). More precisely it is possible to show that an operator \( K \) defined as in eq. (2) with kernel \( k(x, y) \) (instead of \( p(x, y) \)) is selfadjoint if and only if \( \nu(x)k(x, y) = \nu(y)k(y, x) \) for all \( x, y \in X \).
2. Compactness of the transition operator

In this section we give a necessary and sufficient condition for the general linear map $P$ (defined by eq. (2)) to be a bounded map from from $l^\infty(X)$ into itself. Then we characterize all the maps with non-negative kernels which are compact; in the case of stochastic maps, this conditions is related to the recurrence property. The interest in the space $l^\infty(X)$ will be justified in the next section. We start with boundeness conditions.

**Theorem 2.1.** Let $P$ the transition operator defined by eq. (2) (where $p(x,y)$ are real (complex) numbers for any $x, y \in X$); then the following assertions are equivalent:

(i) $P$ is a continuous linear operator from $l^\infty(X)$ into itself;

(ii) $\sup_{x \in X} \sum_{y \in X} |p(x, y)| < \infty$.

(iii) $D(P) \supseteq l^\infty(X)$ and $P(l^\infty(X)) \subseteq l^\infty(X)$.

If one of the previous condition holds, then $\|P\| = \sup_{x \in X} \sum_{y \in X} |p(x, y)|$.

**Proof.** Let us discuss the complex case. Since $X$ is at most countable, we may suppose $X = \{x_i\}_{i \in J}$, where $J \subseteq \mathbb{N}$ has the same cardinality of $X$. If $X$ is finite then the theorem is trivial, we suppose that $X$ is countable (and $J = \mathbb{N}$). Let $x \in X$ and define

$$f^n_x(x_i) := \begin{cases} \frac{p(x, x_i)}{|p(x, x_i)|} & \text{if } p(x, x_i) \neq 0, i \leq n, \\ 1 & \text{if } p(x, x_i) = 0, i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

then $\|f^n_x\|_\infty = 1$ for every $n \in \mathbb{N}$ and for every $x \in X$.

(i) $\implies$ (ii). We easily note that, $f^n_x \in D(P)$ for every $n \in \mathbb{N}, x \in X$ and $\|P f^n_x\|_\infty \geq |(P f^n_x)(x)| = \sum_{i \leq n} |p(x, x_i)|$, then if $n$ tends to infinity and $P$ is bounded, we have

$$\sum_{y \in X} |p(x, y)| = \sum_{i \in \mathbb{N}} |p(x, x_i)| \leq \|P\|$$

and the arbitrary choice of $x$ leads to the conclusion.

(ii) $\implies$ (i). If $\alpha := \sup_{x \in X} \sum_{y \in X} |p(x, y)| < \infty$ then for every $f \in l^\infty(X)$ we obtain

$$\|Pf\|_\infty = \sup_{x \in X} |(Pf)(x)| \leq \sup_{x \in X} \sum_{y \in X} |p(x, y)||f(y)| = \alpha \|f\|_\infty;$$

using the last equation and eq. (3) we obtain that $D(P) \supseteq l^\infty(X)$, $P(l^\infty(X)) \subseteq l^\infty(X)$ and $\alpha = \|P\|$.

(iii) $\implies$ (ii). Let us note that $l^\infty(X) \subseteq D(P)$ implies that for every $f \in l^\infty(X)$ and for every $x \in X$ we have $\sum_{y \in X} |p(x, y)||f(y)| < +\infty$ which is equivalent to the condition $\{p(x, y)\}_{y \in X} \in l^1(X)$ for every $x \in X$. Let $\lambda_x \in l^\infty(X)^*$ defined by $\lambda_x(f) :=$
\[ \sum_{y \in X} p(x, y)f(y), \text{ then } \|\lambda_x\|_{l^\infty(X)^*} = \sum_{y \in X} |p(x, y)|. \] Now the condition \( P(l^\infty(X)) \subseteq l^\infty(X) \) implies \( \sup_{x \in X} |\lambda_x(f)| < +\infty \), for every \( f \in l^\infty(X) \), then, using the principle of Uniform Boundeness, we have \( \sup_{x \in X} \|\lambda_x\|_{l^\infty(X)^*} < +\infty \) which is equivalent to (ii).

(i) \( \implies \) (iii). It is trivial.

We turned our attention now to the compactness property for a transition operator with non negative kernel.

**Theorem 2.2.** Let \( X \) a countable graph and let us choose an enumeration \( \{x_i\} \) for \( X \). Let \( P \) a transition operator on \( X \) with non negative elements, satisfying the condition
\[ \sup_{x \in X} \sum_{y \in X} p(x, y) < +\infty. \] Then \( P \) is a bounded, linear operator from \( l^\infty(X) \) into itself; moreover \( P \) is compact if and only if
\[ \lim_{n \to +\infty} \sup_{x \in X} \sum_{i>n} p(x, x_i) = 0. \] (4)

The last condition is independent from the chosen enumeration.

**Proof.** Theorem 2.1 implies the boundedness of \( P \). If \( P \) is compact, then, for every bounded sequence \( \{f_i\} \) in \( l^\infty(X) \), \( \{Pf_i\} \) is relatively compact, hence there exists a subsequence \( \{n_j\} \) such that \( \{Pf_{n_j}\} \) is a Cauchy sequence. Let \( f_n(x_i) \) equal to 1 if \( i > n \) and 0 otherwise, then
\[ (Pf_n)(\cdot) = \sum_{i>n} p(\cdot, x_i) \]
and if \( m > n \), since \( p(x, y) \geq 0 \) for every \( x, y \in X \),
\[ \|Pf_n - Pf_m\|_\infty = \sup_{x \in X} \sum_{i=n+1}^m p(x, x_i). \]
By the Cauchy property, for every \( \varepsilon > 0 \) there exists \( j_\varepsilon \) such that for all \( j_2 > j_1 \geq j_\varepsilon \) we have
\[ \sup_{x \in X} \sum_{i=n_{j_1}+1}^{n_{j_2}} p(x, x_i) < \varepsilon/2; \]
we note that, for every fixed \( x \in X \), \( m \mapsto \sum_{i=n+1}^m p(x, x_i) \) (resp. \( n \mapsto \sum_{i=n+1}^m p(x, x_i) \)) is not decreasing (resp. not increasing), hence
\[ \sup_{x \in X} \sum_{i=n_{j_\varepsilon}+1}^\infty p(x, x_i) \leq \varepsilon/2 < \varepsilon, \]
which implies \( \lim_{n \to +\infty} \sup_{x \in X} \sum_{i>n} p(x, x_i) = 0. \)

Vice versa if we consider the finite range (compact) projections on \( l^\infty(X) \) defined by
\[ V_i(f)(x_n) := \begin{cases} f(x_n) & \text{if } n \leq i \\ 0 & \text{if } n > i, \end{cases} \]

\[ \sum_{y \in X} p(x, y)f(y), \text{ then } \|\lambda_x\|_{l^\infty(X)^*} = \sum_{y \in X} |p(x, y)|. \] Now the condition \( P(l^\infty(X)) \subseteq l^\infty(X) \) implies \( \sup_{x \in X} |\lambda_x(f)| < +\infty \), for every \( f \in l^\infty(X) \), then, using the principle of Uniform Boundeness, we have \( \sup_{x \in X} \|\lambda_x\|_{l^\infty(X)^*} < +\infty \) which is equivalent to (ii).

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\[ \sup_{x \in X} \sum_{y \in X} p(x, y) < +\infty. \] Then \( P \) is a bounded, linear operator from \( l^\infty(X) \) into itself; moreover \( P \) is compact if and only if
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\[ (Pf_n)(\cdot) = \sum_{i>n} p(\cdot, x_i) \]
and if \( m > n \), since \( p(x, y) \geq 0 \) for every \( x, y \in X \),
\[ \|Pf_n - Pf_m\|_\infty = \sup_{x \in X} \sum_{i=n+1}^m p(x, x_i). \]
By the Cauchy property, for every \( \varepsilon > 0 \) there exists \( j_\varepsilon \) such that for all \( j_2 > j_1 \geq j_\varepsilon \) we have
\[ \sup_{x \in X} \sum_{i=n_{j_1}+1}^{n_{j_2}} p(x, x_i) < \varepsilon/2; \]
we note that, for every fixed \( x \in X \), \( m \mapsto \sum_{i=n+1}^m p(x, x_i) \) (resp. \( n \mapsto \sum_{i=n+1}^m p(x, x_i) \)) is not decreasing (resp. not increasing), hence
\[ \sup_{x \in X} \sum_{i=n_{j_\varepsilon}+1}^\infty p(x, x_i) \leq \varepsilon/2 < \varepsilon, \]
which implies \( \lim_{n \to +\infty} \sup_{x \in X} \sum_{i>n} p(x, x_i) = 0. \)

Vice versa if we consider the finite range (compact) projections on \( l^\infty(X) \) defined by
\[ V_i(f)(x_n) := \begin{cases} f(x_n) & \text{if } n \leq i \\ 0 & \text{if } n > i, \end{cases} \]
for all \( i \in \mathbb{N} \), then \((PV_i f)(x) = \sum_{i \leq n} p(x, x_i) f(x_i)\). By Theorem 2.1 \( \|P - PV_i\| = \sup_{x \in X} \sum_{i > n} p(x, x_i) \) then \( \|P - PV_i\| \) tends to 0 if \( n \) tends to infinity. By Theorem 4.18(f) of Rudin [9], \( PV_i \) is compact for every \( i \in \mathbb{N} \), hence \( P \) is compact since Theorem 4.18(c) of Rudin [9] holds.

The condition (4) does not depend on the choice of the enumeration, since compactness is defined “a priori”.

We note that the previous theorem says that \( P \) is compact if and only if for any \( \varepsilon > 0 \) there exists a finite subset \( A_\varepsilon \subset X \) such that \( \sup_{x \in X} \sum_{y \in A_\varepsilon} p(x, y) < \varepsilon \); this means, for instance, that a necessary condition for the compactness property is that \( \lim_{y \to \infty} p(x, y) = 0 \) holds uniformly with respect to \( x \in X \) (where the limit is taken in the Alexandroff compactification of \( X \) with the discrete topology).

We immediately note that if \( X \) is locally finite, then \( P \) is not compact. In fact in this case, for every \( n \in \mathbb{N} \), \( \sum_{i \leq n} \text{deg}(x_i) < +\infty \); this means that there exists \( n_1 > n \) such that \( x_i \) is not a neighbour of \( x_{n_1} \) for any \( i \leq n \), hence \( \sum_{i > n} p(x_{n_1}, x_i) = 1 \) and eq. (4) cannot be satisfied. Moreover one can show that if \( P \) is compact then it is a recurrent transition operator.

**Proposition 2.3.** Let \( X \) be a graph and \( P \) a random walk on \( X \) which is a compact, transition operator from \( l^\infty(X) \) into itself; then \( P \) is recurrent.

**Proof.** Let \( A := \{x_0, x_1, \ldots, x_n\} \) and \( Z_n \) the Markov chain associated to \( P \); when \( P \) is compact then, by Theorem 2.2, if \( x \in X \)

\[
\mathbb{P}[Z_m \notin A | Z_0 = x] = \sum_{y \in X} \mathbb{P}[Z_{m-1} = y | z_0 = x] p(y, x_i) = \\
\sum_{y \in X} \mathbb{P}[Z_{m-1} = y | z_0 = x] \sum_{i > n} p(y, x_i) \leq \\
\leq \sum_{y \in X} \mathbb{P}[Z_{m-1} = y | z_0 = x] \sup_{w \in X} \sum_{i > n} p(w, x_i) \leq \sup_{w \in X} \sum_{i > n} p(w, x_i) \xrightarrow{n \to +\infty} 0
\]

If \( B := \{\exists k \in \mathbb{N}; Z_n \notin A, \forall n \geq k\} \equiv \bigcup_{k \in \mathbb{N}} \cap_{n \geq k} \{Z_n \notin A\} \) it is clear that

\[\mathbb{P}(B) \leq \sum_{k \in \mathbb{N}} \mathbb{P}(\cap_{n \geq k} \{Z_n \notin A\}),\]

but \( \cap_{n \geq k} \{Z_n \notin A\} \subseteq \{Z_m \notin A\} \) for every \( m \geq k \) which implies \( \mathbb{P}(\cap_{n \geq k} \{Z_n \notin A\}) = 0 \) and \( \mathbb{P}(B) = 0 \). Thus by Woess [2] Theorem 3.5, \( P \) is recurrent.

The necessary condition given in the previous proposition is not sufficient; it is not difficult to find random walks which are recurrent and the associated transition operator is not compact.
Example 2.4. Let us give an example of random walk giving rise to a compact transition operator. To this aim we consider any sequence of real number \( \{ p_i \}_{i \in \mathbb{N}} \) such that \( p_0 = 1 \) and \( p_i \in (0, 1] \) for every \( i \geq 1 \). Let us take \( X = \mathbb{N} \) and

\[
p(x, y) := \begin{cases} 
p_x & \text{if } x \in \mathbb{N} \text{ and } y = x + 1 \\
1 - p_x & \text{if } y = 0 \text{ and } x \neq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The condition (4) is easily equivalent to \( \lim_{n \to +\infty} p_n = 0 \); moreover, using Theorem 3.11 of \([2]\), it is not difficult to show that \((X, P)\) is recurrent (resp. positive recurrent) if and only if \( \lim_{n \to +\infty} \prod_{i=0}^{n} p_i = 0 \) (resp. \( \sum_{n=0}^{\infty} \prod_{i=0}^{n} p_i < +\infty \)). This proves that \((X, P)\) positive recurrent (and hence \((X, P)\) recurrent) does not imply the compactness of the transition operator \( P \).

3. The null space of the pre-adjoint of the Laplace operator and finite variation stationary measures.

The discrete analogous of the Laplace operator is given by \((P - 1_{\infty} l_{\infty}) : l^\infty(X) \to l^\infty(X)\) where \( 1_{\infty} \) id the identity operator on \( l^\infty(X) \). It is straightforward to show that the preadjoint map \((P - 1_{\infty})^*\) is given by \((Q - 1_1) : l^1(X) \to l^1(X)\) where

\[
(Q\nu)(y) := \sum_{x \in X} p(x, y)\nu(x), \quad \forall y \in X,
\]

and \( 1_1 \) is the identity map on \( l^1(X) \); from now on, if \( A : D \to Y \) is a map, we denote by \( \text{Rg}(A) \) the image (or range) of \( D \) through \( A \) (that is \( A(D) \)). A bounded function is said to be harmonic if is an element of the null space of the discrete laplacian (we denote the set of all bounded harmonic functions by \( \mathcal{H}^\infty(X, P) \)).

In \([6]\) it was shown that a finite variation measure \( \nu \) on \( X \) (i.e. \( \nu \in l^1(X) \)) has the weak mean value property with respect to \( o \in X \) (that is, \( \sum_{x \in X} f(x)\nu(X) \text{d}\nu = f(o) \sum_{x \in X} \nu(x) \), for every \( f \in \mathcal{H}^\infty(X, P) \)) if and only if \( (\nu - \delta_o \sum_{x \in X} \nu(x)) \in \text{Rg}(Q - 1_1) \) (where \( \delta_o \) is the (finite variation) Dirac measure with support in \( \{o\} \)). From this point of view, it is important to know when \( Q - 1_1 \) is injective and when it has a closed range. In this section we give a complete answer to the first question and we make some remarks related to the second.

To this aim, we characterize in particular all the stationary measures with finite variation. We recall that a signed measure is called stationary (resp. excessive) if

\[
(Q\nu)(y) = \nu(y), \quad \forall y \in X \quad (\text{resp. } (Q\nu)(y) \leq \nu(y), \quad \forall y \in X)
\]

provided that \( (Q\nu)(y) \) exists for every \( y \in X \). We note that a finite variation measure \( \nu \) is stationary if and only if \( \nu \in \ker(Q - 1_1) \).

**Lemma 3.1.** Let \( \nu \) be a signed stationary measure on \( X \), then \(-|\nu|\) is a negative excessive measure which is stationary if and only if \( \nu = |\nu| \) or \( \nu = -|\nu| \).
Proof. It is well known that if \( f \) is a complex integrable function on a measure space \((Y, \mu)\) then
\[
\left| \int_Y f \, d\nu \right| \leq \int_Y |f| \, d\nu
\]
and the equality holds if and only if there exists \( \alpha \in [0, 2\pi) \) such that \( f = |f| \exp(i\alpha) \) \( \mu \)-a.e.
If we consider the measure space \( X \) with the counting measure and \( f_y(x) := \nu(x)p(x, y) \) then by hypothesis \( f_y \in L^1(X) \) for every \( y \in X \) and
\[
|\nu|(y) = |\nu(y)| = \sum_{x \in X} \nu(x)p(x, y) \leq \sum_{x \in X} |\nu|(x)p(x, y) = (Q|\nu|)(y)
\]
and the equality holds if and only if \( \nu(x)p(x, y) = |\nu|(x)p(x, y) \exp(i\alpha) \) (where \( \alpha \in \{0, \pi\} \), since \( f_y \) is a real function) which leads to the conclusion. \( \square \)

We are ready to prove the following theorem which characterizes all the stationary measures (i.e. the null space of \( Q - I_1 \)). Before state the Theorem we recall that for any irreducible random walk it is possible to define a natural number called the period of the random walk (see [8], Section 5.A).

**Theorem 3.2.** Let \((X, P)\) be an irreducible random walk, then there exists a finite variation, stationary measure \( \nu \neq 0 \) if and only if \((X, P)\) is positive recurrent. In this case there exists \( \alpha \in \mathbb{R} \setminus \{0\} \) such that \( \nu = \alpha \mu \) where \( \mu \) satisfies
\[
\mu(y) = \limsup_{n \to \infty} p^{(nd+j-i)}(x, y)/d
\]
(the right hand side is seen to be independent of \( x \) and \( d \) is the period of the random walk).

*Proof.* If we suppose that there exists a stationary measure \( \nu \) with finite variation and \( C_0, C_1, \ldots, C_{d-1}, C_d \equiv C_0 \) is the partition of \( X \) given by the periodicity classes, then, by Lemma 3.1, \(-|\nu|\) is an excessive measure; Lemmas 2.4, 2.5 and Theorem 2.2 of Woess [2] and Tonelli-Fubini’s Theorem imply
\[
|\nu|(C_{i+1}) \sum_{y \in C_{i+1}} |\nu(y)| \leq \sum_{y \in C_{i+1}} \sum_{x \in C_i} |\nu(x)|p(x, y) = \\
= \sum_{x \in C_i} \sum_{y \in C_{i+1}} |\nu(x)|p(x, y) = |\nu|(C_i)
\]
then \( |\nu|(C_0) \leq |\nu|(C_1) \leq \cdots \leq |\nu|(C_{d-1}) \leq |\nu|(C_d) \equiv |\nu|(C_0) \), hence \( |\nu|(C_i) = |\nu|(X)/d \) for every \( i = 0, 1, \ldots, d-1 \).

Using the “Renewal Theorem” by Erdös-Feller-Pollard (see [10] or Theorems 3.6 and 3.7 of [2]) and Lebesgue bounded convergence Theorem,
\[
|\nu|(y) \leq \sum_{x \in C_i} |\nu|(x)p^{(nd)}(x, y) \xrightarrow{n \to +\infty} d \cdot \mu(y) \sum_{x \in C_i} |\nu|(x);
\]  \( \square \)
since \( \nu \neq 0 \) then there exists \( i \) and \( y \in C_i \) such that \( |\nu|(y) > 0 \), thus eq. (5) implies that \( \mu(y) > 0 \), hence \((X, P)\) is positive recurrent.

On the other hand, if \((X, P)\) is positive recurrent, Theorem 3.9 of Woess [2] implies that \( \mu \) is a stationary, probability measure.

If \( \nu \) is another stationary, finite variation measure on \( X \) \((\nu \neq 0)\) then by eq. (5), \(|\nu|(y) \leq |\nu|(X)\mu(y)\); if we suppose, by contradiction, that there exist \( y \in X \) such that \(|\nu|(y) < |\nu|(X)\mu(y)\), then we have

\[
1 = \sum_{y \in X} |\nu|(y)/|\nu|(X) < \sum_{y \in X} \mu(y) = 1
\]

which is a contradiction; hence \(|\nu|(.)/|\nu|(X) \equiv \mu(\cdot)\). If we define \( \varphi(y) := \nu(y)/|\nu|(X) \) then \( |\varphi| \equiv \mu \) and \((2\mu - \varphi)/(2\mu(X) - \varphi(X))\) is a stationary, probability measure. By Theorem 3.9 of Woess [2], \((2\mu - \varphi)/(2\mu(X) - \varphi(X)) \equiv \mu\) which is equivalent to \( \varphi = \varphi(X)\mu \), that is, \( \nu = \nu(X)\mu \).

As a consequence of this theorem we obtain that the bounded, linear map \( Q - \mathbb{1}_1 \) is injective if and only if \((X, P)\) is not positive recurrent.

We try now to answer to the second question: when \( \text{Rg}(Q - \mathbb{1}_1) \) is closed?

By Schauder’s Theorem (see Brezis [7] Theorem VI.4), since \( P = Q^* \), we have that the operator \( Q \) from \( l^1(X) \) into itself, is compact if and only if eq. (4) holds. Now it is well known (see for instance [11], Theorem 4.23) that if \( Q \) is compact operator from a Banach space into itself then \( Q - \mathbb{1} \) has closed range. Hence if equation (4) holds we have that a finite variation measure \( \nu \) on \( X \) has the weak mean value property with respect to \( o \in X \) if and only if \((\nu - \delta_o \sum_{x \in X} \nu(x)) \in \text{Rg}(Q - \mathbb{1}_1)\).

This is obviously only a partial answer to our question; a way to reach a complete a satisfactory answer, which we don’t undertake here is given by the following remarks.

We recall that if \((Z, \| \cdot \|_Z), (Y, \| \cdot \|_Y)\) are Banach spaces, \( D \) is a linear subspace of \( Z \) and \( A : D \to Y \) is a linear map such that \( \sup_{x \in D} \|Ax\|_Y =: \beta \), then there exists a unique bounded, linear map \( \overline{A} : D \to Y \) which extends \( A \); moreover \( \overline{A} \) is bounded by the same constant \( \beta \) and

\[
\inf_{x \in D} \|Ax\|_Y = \inf_{x \in \overline{D}} \|\overline{A}x\|_Y.
\]

Therefore, if \( A : D \to Y \) is a linear and injective map, then

\[
\sup_{y \in \text{Rg}(A): \|y\|_Y = 1} \|A^{-1}y\|_Z = 1/ \inf_{x \in D: \|x\|_Z = 1} \|Ax\|_Y
\]

(where, by definition, \( 1/0 := +\infty \)).

Now using the Open Mapping Theorem it is simple to show that if \( A : Z \to Y \) is a linear, bounded, injective map then

\[
\text{Rg}(A) = \overline{\text{Rg}(A)} \iff \inf_{x \in D: \|x\|_Z = 1} \|Ax\|_Y > 0.
\]

In our case, if the Markov chain is not positive recurrent, then \( \text{Rg}(Q - \mathbb{1}_1) \) is closed if and only if

\[
\inf_{\nu \in l^1(X): \|\nu\|_1 = 1} \|Q\nu - \nu\|_1 > 0.
\]
Bibliography

[1] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, 138, Cambridge Univ. Press, 2000.

[2] W. Woess, *Catene di Markov e Teoria del Potenziale nel Discreto*, Quaderno U.M.I. 41, Ed. Pitagora, Bologna (1996).

[3] T. Coulhon, A. Grigoryan, *Random walks on graphs with regular volume growth*, GAFA 8 (1998), 656-701.

[4] P. Cartier, *Harmonic analysis on trees*, Proc. Sympos. Pure Math., vol. 26, Amer. Math. Soc., Providence, R.I., 1972, 419-424.

[5] A. Figà-Talamanca, M. A. Picardello, *Harmonic analysis on free groups*, Lecture Notes in Pure and Appl. Math., vol. 87, Dekker, New York and Basel, 1987.

[6] F. Zucca, *The mean value property for harmonic functions on graphs and trees*, appearing on Ann. Mat. Pura. App.

[7] H. Brezis, *Analyse Fonctionnelle-Théorie et Applications*, Masson, Paris, (1983).

[8] W. Woess, *Random walks on infinite graphs and groups - A survey on selected topics*, Bull. London Math. Soc. 26 (1994), 1-60.

[9] W. Rudin, *Real and Complex Analysis*, Mc Graw-Hill, (1987).

[10] P. Erdős, W. Feller, H. Pollard, *A theorem on power series*, Bull. Amer. Soc. 55, (1949) 201-204.

[11] W. Rudin, *Functional Analysis*, Mc Graw-Hill, (1991).