Super Yang-Mills Theory on Lattice and the Transformation

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We present a new lattice super Yang-Mills theory and its SUSY transformation. After our formulation of the model in a fundamental lattice, it is extended to the whole lattice with a substructure of modulo 2.

1. INTRODUCTION

A lattice formulation is one of the most powerful tools for a nonperturbative study. We present here a completely new formulation of the super Yang-Mills theory on lattice with an exact fermionic symmetry.

There are several pioneering attempts for constructing super Yang-Mills theories on lattice. The first approach, the Wilson fermion and mass fine-tuning procedure, is a well-simulated method \cite{1,2}. It is based on an observation that, in the perturbative expansion, a gaugino mass is the unique relevant operator which breaks supersymmetry around the SUSY fixed point \cite{1,3}. Nevertheless, nobody knows whether this observation is nonperturbatively correct and there exists the SUSY fixed point in the Wilson fermion theory.

Second is an approach with the staggered fermion \cite{4,5}. This formulation may allow us to have a correspondence between fermion and boson. However, the counting of the degrees of freedom is still difficult for Yang-Mills theory and the full theory has not been constructed until now.

Third approach is based on the domain wall fermion \cite{6}. This approach naturally has a vanishing mechanism of the gaugino mass, although, in order to impose a Majorana condition, one must introduce a nonlocal condition along the fifth direction. A simulation has also been done \cite{7}.

Our new formulation may look like the second approach in some aspects. However, the crucial difference is the presence of an exact fermionic symmetry in our formulation. No fine-tuning is required for the symmetry and we show that it is not BRS-like.

We start our consideration with a model defined on a fundamental lattice. The model will be called as the one-cell model. Then we show that the model may be extended to the whole space while keeping the fermionic symmetry found for the one-cell model. The extended model has a slightly unusual lattice structure, which will be explained later.

2. LATTICE SUSY TRANSFORMATION FOR GAUGE AND FERMION

In this section, we present the lattice action and SUSY transformations for the one-cell model. Each coordinate of a site takes only 0 or 1 in the unit of \(a\) (lattice constant).

Our action of gauge and fermion system is a normal one except the facts that it is restricted on a fundamental lattice and fermi fields, \(\psi\), are real.

\begin{align*}
S_g &= -\beta \sum_{n,\mu\nu} \text{tr} (U_{n(\mu\nu),\mu\nu} + U_{n(\mu\nu),\nu\mu}) \\
S_f &= \sum_{n,\rho} b_{n(\rho)} \text{tr} (\psi_{n(\rho)} U_{n(\rho),\rho} \psi_{n(\rho)} + U_{n(\rho),\rho}^\dagger)
\end{align*}
\(-\psi_{n(\rho)} + b_{\rho} U_{n(\rho), \rho}^\dagger \psi_{n(\rho)} U_{n(\rho), \rho}\)

where \(n(\mu \nu)\) is a base point for the plaquette \((n, \mu \nu)\) and a symbol \(n(\rho)\) implies \(n_\rho = 0\) for the \(\rho\) coordinate. The lattice has one plaquette in \(D=2\) case and six plaquettes in \(D=3\). The coefficients \(b_{\rho}\) of the fermion action appear similarly to the usual staggered fermion. However the fermion here is the real fermion.

We also use the symbol \(\rho(n)\) to indicate either \(\hat{\rho}\) or \(-\hat{\rho}\), allowed inside the cell when we start from the site \(n\). The SUSY transformation of gauge fields is defined as

\[
\delta U_{n(\mu), \mu} = \sum_\rho (\alpha^\rho_{n(\mu), \mu} \xi^\rho_{n(\mu)}) U_{n(\mu), \mu} + U_{n(\mu), \mu} \alpha^\rho_{n\mu + \hat{\rho}, \mu} \xi^\rho_{n(\mu) + \hat{\rho}}
\]

where \(\xi^\rho_n\) is a gauge-covariantly translated fermion, \(U_{\nu, \mu} \psi_{\mu, \mu} U_{\mu, \mu}^\dagger\) and \(\alpha\) is a grassmannian parameter.

For fermion fields, we write the transformation as

\[
\delta \psi_n = \sum_{0 < \mu < \nu} C^\rho_{\mu \nu}(n) (U_{n(\mu \nu)\rho}(n) - U_{n(\nu \mu)\rho}(n))
\]

where \((\mu \nu)(n)\) implies \((-)^{n_\mu} \hat{\mu}(-)^{n_\nu} \hat{\nu}\) and \(C^\rho_{\mu \nu}\) is a grassmannian parameter.

3. SUSY INVARIANCE OF OUR THEORY

Now, we derive relations between the introduced parameters, which are required for the SUSY invariance. They come from four important conditions.

**Condition (1) \(\delta U, S_f = 0\)**

This is the invariance of the fermion action under gauge field variation. It is nothing but the vanishing condition of fermion cubed terms:

\[
b_{\rho}(n) \alpha^\rho_{n, \rho} + b_{\mu}(n) \alpha^\mu_{n, \mu} = 0.
\]

Note that this condition is consistent with the staggered Majorana fermion;

\[
b_{\mu}(n) = (-)^{|n|} \eta_{\mu}(n) b.
\]

Here \(b\) is a constant and the constraint on \(\alpha\) is

\[
\alpha^\mu_{n, \nu} = - \frac{b_{\mu}(n)}{b_{\nu}(n)} \alpha^\nu_{n, \mu}.
\]

In particular, we find that any diagonal element vanishes

\[
\alpha^\rho_{n, \rho} = 0.
\]

**Condition (2) \(\delta S = \delta U, S_b + \delta \psi, S_f = 0\)**

By satisfying the first and second conditions altogether, the total action becomes invariant under the SUSY transformation. The second condition produces two types of graphs, i.e. those with three different indices and those with two different indices.

![Figure 1. Two kinds of graphs](image)

Corresponding to two types of graphs, we find the relations,

\[
b_{\rho}(n) C^\rho_{\mu \nu}(n) = \beta((-)^n \alpha^\rho_{n, \mu} - (-)^n \alpha^\rho_{n, \nu}),
\]

\[
b_{\mu}(n) C^\rho_{\nu \mu}(n) + b_{\nu}(n) C^\rho_{\nu \mu} = -\beta(\alpha_{n, \mu} - \alpha_{n, \nu}).
\]

**Condition (3) \(\delta U, S_b \neq 0\)**

Third one is to confirm that this symmetry is not BRS-like. We see that this condition is trivially satisfied because the left hand sides of the relations obtained from the second condition are nonzero.

**Condition (4) Invariance of Path Integral Measure**

The condition is not so trivial. We consider in two steps.

(A) In the first step, only fermion variables change.

In particular, we find that any diagonal element vanishes

\[
\alpha^\rho_{n, \rho} = 0.
\]

By satisfying the first and second conditions altogether, the total action becomes invariant under the SUSY transformation. The second condition produces two types of graphs, i.e. those with three different indices and those with two different indices.
\[ U_{n,\mu}' = U_{n,\mu} \]
\[ \psi_n' = \psi_n + C_{n}^{\mu\nu}(U_{n,\mu\nu} - U_{n,\nu\mu}) \]

(B) In the second step only gauge field variables change.

\[ U_{n,\mu}'' = e^{\alpha_{n,\mu}} \xi_{n\mu}' \psi_n' \]
\[ \psi_n'' = \psi_n \]

where \( \xi_{n\rho}' = U_{n,\rho}'(\psi_{n+\hat{\rho}}' - C_{n+\hat{\rho},\mu\nu}(U_{n+\hat{\rho},\mu\nu} - U_{n+\hat{\rho},\nu\mu}))U_{n,\rho}' \). It is easy to show that Jacobian \( J_A = 1 \). But \( J_B = 1 \) implies \( \alpha_{n,\mu} = 0 \). This is consistent with the solution for the first condition. We have completed the check of the SUSY invariance of our one-cell model.

4. MULTI-CELL MODEL

Now, we extend one-cell model to the whole space. If we do it naively, we encounter \( O(a) \) reduced SUSY not \( O(1) \). So we have to consider a new type lattice theory. Let us decompose the whole lattice into three units.

(B) In a blue unit, a one-cell model lives.
(R) In a red unit, another one-cell model lives.
(Bl) In blank, there are no cell models.

Our total action is obtained by summing over blue and red cells.

![Multi-cell pattern](image)

Figure 2. A multi-cell pattern

From a point of view in fields,

(A) Link Variables (Gauge Fields) are in either a blue or a red cell.

(B) Plaquettes are put only in a blue or red cell but not blank unit.

(C) Site Variables (Fermi Fields) are on a boundary site of a blue or red cell. So fermi fields have dual property.

Now we have to check not only the in-Cell condition but also inter-Cell conditions. There are many constraints but we can find a solution.

5. SUMMARY AND DISCUSSIONS

Let us summarize our work and give some discussions. We presented a new model for lattice super Yang-Mills theory. After finding the SUSY invariant one-cell model, it is extended to the whole space. Note that the representation of a link variable, \( U \), must be real as the SUSY transformation of a fermi field.

What is the indication of our symmetry? For example, Ward-Takahashi identities can be written down. A nontrivial example relates the field strength and two-body fermion correlation.

We leave the investigation of the SUSY algebra including the actual spinor structure as a future problem.

Finally, the embedding pattern of the one-cell model allows the modulo 2 translational invariance. The pattern may imply an embedding of spinor structure to lattice. In our forthcoming paper [8], some discussions will be given on this point.

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