AN INDEPENDENCE SYSTEM AS KNOT INvariant

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ABSTRACT

An independence system (with respect to the unknotting number) is defined for a classical knot diagram. It is proved that the independence system is a knot invariant for alternating knots. The exchange property for minimal unknotting sets are also discussed. It is shown that there exists an infinite family of knot diagrams whose corresponding independence systems are matroids. In contrast, infinite families of knot diagrams exist whose independence systems are not matroids.

Keywords: Unknotting number; independence system; I-chromatic number; exchange property; matroid.

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1 Introduction

A knot $K_1$ is isotopic to another knot $K_2$ if, by the continuous deformation, $K_1$ is converted to $K_2$. A quantity defined for a knot $K$, that is same for all knots isotopic to $K$, is known as knot invariant. The knot invariants distinguish knots up to isotopy. A diagram of a knot $K$ is the presentation of the knot on a plane where each point on the diagram of a knot is the shadow of some point on the knot $K$. A point on the diagram which is shadow of two points on the knot, is termed as crossing. All the diagrams considered are regular, i.e, no point on the plane is a shadow of three or more points on the knot $K$. The crossing number of a diagram $D$ of a knot, denoted by $c(D)$, is the number of crossings in $D$. A diagram of a knot where no crossing can be removed just by twisting, is called a reduced knot diagram and such a crossing that vanishes merely by twisting is called nugatory crossing (reducible crossing). In the Fig. 1, $v_1$ and $v_2$ are reducible crossings.

![Figure 1: reducible crossing](image)

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The crossing number of a knot $K$, denoted by $c(K)$, is the minimum number of crossings, ranging over all possible reduced diagrams of the knot $K$. The unknotting number of a diagram $D$ of a knot $K$, denoted by $u(D)$, is the minimum number of switching (overcrossing is turned to undercrossing or vice versa) required to untangle the diagram. The unknotting number of a knot $K$, denoted by $u(K)$, is the minimum number of crossings required to switch to unknot the knot, ranging over all possible diagrams of the knot $K$ (readers are referred to [1] for more detailed definitions and examples of the notions discussed so far). The unknotting number $u(D) = u_{\text{min}}(K)$ only for a reduced diagram $D$, having minimum crossings (or $c(D) = c(K)$), of a knot $K$ is defined in [12]. For a knot $K$, the following inequality holds

$$u(K) \leq u_{\text{min}}(K).$$

For the knot $10_8$, $u(K) < u_{\text{min}}(K)$ and for the knot $6_1$, $u(K) = u_{\text{min}}(K)$. An alternating knot is a knot which possesses a diagram in which crossings alternate between under and over crossings. The unknotting set for a knot diagram is the set of those crossings which when switched, the knot is transformed to unknot. An unknotting set is minimal if it contains no unknotting set as its proper subset.

**Definition 1** The unknotting sets $W_1$ and $W_2$ for a diagram $D$ of a knot $K$ are said to have the exchange property if whenever $W_1$ and $W_2$ are any two minimal unknotting sets and for every crossing $u \in W_1$, there exists a crossing $v \in W_2$ such that $(W_1 \setminus \{u\}) \cup \{v\}$ is also a minimal unknotting set.

A property defined on a finite set that is possessed by all of its subsets, is called a hereditary property [13]. An independence family $I$ on a finite ground set $E$ is a non-empty collection of sets $X \subset E$, satisfying the hereditary property. An independence system $(E, I)$ on the set $E$, is a system consisting of an independence family $I$ of subsets of $E$:

$$I = \{ X \in E : A \in I \forall A \subseteq X \}.$$  

The maximal independent sets are called the bases of $(E, I)$. An independence system is called matroid if all its bases have the same cardinality [13].

All the ingredients are ready to give the key definition in the paper.

**Definition 2** A set $W$ of crossings of a given diagram $D$ of a knot $K$ is called a $U$-independent set if for every non-empty $S \subseteq W$, $W \setminus S$ is not an unknotting set.

The definition of $U$-independent set leads to the $U$-independence system $(E, I)$ for a knot diagram $D$ where $E$ is the set of all crossings of $D$ and $I$ is the independence family consisting of the $U$-independent sets for the diagram $D$.

$$I = \{ W \subset E : W \setminus X \text{ is not an unknotting set } \forall X \subseteq W \text{ with } X \neq \emptyset \}.$$  

A $U$-independent set is maximal if it is not contained in any other $U$-independent set. By the definition, every minimal unknotting set is a maximal $U$-independent set.
All the reduced alternating diagrams $D$ of a knot $K$ has the same number of crossings $\mathcal{K}$. Therefore, $u(D) = u_{\min}(K)$ for every reduced alternating diagram $D$ of $K$. The $U$-independence system for a reduced alternating diagram of a knot as a knot invariant is declared by the following theorem.

**Theorem 3** Let $(E_1, I_1)$ and $(E_2, I_2)$ be the $U$-independence systems of reduced alternating diagrams $D_1, D_2$ of a knot $K$. Then, there exists an isomorphism $\varphi$ between $(E_1, I_1)$ and $(E_2, I_2)$.

The knot given by $c_1, c_2, \ldots, c_j$ in Conway notation is denoted by $(c_1, c_2, \ldots, c_j)$. Whether the $U$-independence systems, for some diagrams of the families of knots $(2n + 1, 1, 2n); (2n + 1);$ and $(2n, 2)$, are matroids or otherwise are described in the following theorems.

**Theorem 4** The $U$-independence system for the diagram (Fig. 15) of a knot in the family $(2n + 1, 1, 2n)$, $n \geq 2$ is not a matroid.

**Theorem 5** The $U$-independence system for the diagram (Fig. 18) of a knot $K$ in the family with the Conway notation $(2n + 1)$, for $n \geq 1$ is a matroid.

**Theorem 6** For $n > 1$, the $U$-independence system for the diagram (Fig. 19) of a knot $K$ in the family of knots with the Conway notation $(2n, 2)$ is not a matroid.

There is a noteworthy advantage when a $U$-independence system for a knot diagram is a matroid. If it is a matroid, every maximal $U$-independent has the same cardinality. As a consequence, every minimal unknotting set has the same cardinality and hence minimum. In other words, one has to find a minimal unknotting set in order to determine the unknotting number of the diagram. This makes algorithmic methods to find the unknotting number more workable.

The rest of sections are organized as follows. In section 2, basic information and examples of independence systems and matroids are supplied; also, the exchange property for minimal unknotting sets is discussed here with examples. In section 3, properties of a $U$-independent system are discussed and the proof of Theorem 3 is given; the section also highlights how the invariants of the $U$-independent system, can be used as invariants of knots in combination with $u_{\min}(K)$ for a knot $K$; furthermore, two natural questions are posed. Section 4 is devoted to the proofs of Theorems 4, 5, and 6.

## 2 Information and Examples of Basic Notions

### 2.1 Independence System

An independence system is also called *hereditary system* \cite{13} and *abstract simplicial complex* \cite{5}. The set $X$ is called independent set if $X \subset I$ and called dependent otherwise. The empty set $\phi$ is independent and $E$ is dependent by definition. Based on the definition of the independence in different contexts,
there is a variety of independence systems, i.e., in linear algebra, the independence system is the usual linearly independence \[14\]. Similarly, for a simple undirected graph \(G(V, E)\), the property is edge-independence, i.e, a set \(S\) of edges in \(E\) is independent if its induced graph is acyclic \[13\]. The independent sets of each independence system \((E, I)\) form different partitions of the ground set \(E\). The partition of \(E\) into the smallest number of independent sets is a minimum partition. The number of independent sets in a minimum partition of \(E\) is called the \(I\)-chromatic number of \((E, I)\), denoted by \(\chi(E, I)\) (see \[15\] for details).

### 2.2 Matroids

A matroid is a generalization of the linear independence in linear Algebra. For later use, the formal definition of a matroid is given here.

**Definition 7** \[13\] The independence system \((E, I)\) consisting of a family \(I\) of subsets of a finite set \(E\), is a matroid if it satisfies the following property.

(Uniformity property): all the maximal independent subsets of \(E\) in \(I\) have the same cardinality.

The independence systems described in subsection 2.1 form matroids. The first independence system of linearly independent sets in a vector space, is known as *matric matroid* and the second whose independent sets are acyclic sets of edges in \(E\) for a simple undirected graph \(G = (V, E)\) is known as *graphic matroid* \[10\]. A set of vertices in a simple graph is called vertex independent set if no two vertices in the set are adjacent to each other. The vertex independence system of a simple graph is not a matroid in general.

### 2.3 The Exchange Property for Minimal Unknotting Sets

If the exchange property holds for all maximal independent sets (bases) of an independence system then bases have the same cardinality \[13\]. Since minimal unknotting sets are maximal \(U\)-independent sets for a knot diagram. Therefore, the following remark is worth mentioning.

**Remark 8** For a knot diagram having the exchange property for minimal unknotting sets, the unknotting number of the diagram can be determined by just finding a minimal unknotting set.

All the minimal unknotting sets for the diagram of figure eight knot (Fig. 2) have cardinality 1. Therefore, the exchange property holds here trivially.
To show that the exchange property does not hold, it is sufficient to show that there exist two minimal unknotting sets of different cardinalities. For example, the three twist knot diagram (Fig. 3) has minimal unknotting sets \( \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\} \) and \( \{v_2, v_3\} \).

The minimal unknotting sets do not have the same cardinality. Therefore, the exchange property does not hold. However, the condition is not necessary, i.e., the exchange property may not hold for the minimal unknotting sets of the same cardinality. For example, the reduced diagram 8 (Fig. 4) has two minimal unknotting sets \( \{v_1, v_2\} \) and \( \{v_5, v_6\} \) of the same cardinality.
All possible sets obtained by exchanging elements of these sets are \( \{v_1, v_5\} \), \( \{v_1, v_6\} \), \( \{v_2, v_5\} \), and \( \{v_2, v_6\} \) which are not unknotting sets. For example, when the crossings \( v_1 \) and \( v_5 \) are switched (Fig. 5), the knot 8_3 is not transformed to unknot.

Figure 5: \( v_1 \) and \( v_5 \) are switched.

The set \( \{v_1, v_6\} \) is also not an unknotting set (Fig. 6).

Figure 6: \( v_1 \) and \( v_6 \) are switched
Similarly, \{v_2, v_5\} and \{v_2, v_6\} not unknotting sets. The following table lists some reduced knot diagrams up to 8 crossings depicting their exchange property.

| Knot | Exchange Prop. holds |
|------|----------------------|
| 3_1  | yes                  |
| 4_1  | yes                  |
| 5_1  | yes                  |
| 5_2  | no                   |
| 6_1  | no                   |
| 6_2  | no                   |
| 6_3  | no                   |
| 7_1  | yes                  |
| 7_2  | no                   |
| 7_3  | no                   |
| 7_4  | no                   |
| 7_5  | no                   |
| 7_6  | no                   |
| 7_7  | no                   |
| 8_1  | no                   |
| 8_2  | no                   |
| 8_3  | no                   |
| 8_4  | no                   |
| 8_5  | no                   |
| 8_6  | no                   |

3 $U$-Independence System of a Knot Diagram

3.1 Basic Properties

The idea of converting a minimality in one sense to a maximality in another sense, was first introduced by Boutin in [3], where $det$-independent and $res$-independent sets were defined for determining and resolving sets in simple graphs. In this paper, the definition of $U$-independent set (Definition 2) is slightly different than the one in [3]. The definition is modified to suit our purpose. The diagram of the knot 7_3 (Fig. 7) given in the Rolfsen knot table [11] has the unknotting number 2.

![Figure 7: 7_3 knot](image)

Let $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ be the set of all crossings in the reduced diagram (Fig. 7). Some of the unknotting sets are $W_1 = \{v_1, v_2\}, W_2 = \{v_1, v_3\}, W_3 = \{v_1, v_4\}, W_4 = \{v_2, v_3\}, W_5 = \{v_2, v_4\}$ and $W_6 = \{v_3, v_4\}$. All these unknotting sets are $U$-independent. For example, $W_1 \setminus \{v_1\}$ and $W_1 \setminus \{v_2\}$
are not unknotting sets. There may be other $U$-independent sets not necessarily unknotting sets, e.g, $\{v_1, v_5\}$ is not unknotting set but $U$-independent because $\{v_1, v_5\} \setminus \{v_1\}$ and $\{v_1, v_5\} \setminus \{v_5\}$ are not unknotting sets.

A minimum unknotting set has the smallest cardinality among all the minimal unknotting sets of a knot diagram $D$. This smallest cardinality is actually $u(D)$ of the diagram $D$. Every minimum unknotting set is minimal but the converse may not be true in general, e.g., for the diagram (Fig. 7) of the knot $7_3$, all minimal unknotting sets are:

\begin{itemize}
  \item $\{v_1, v_2\}$,
  \item $\{v_1, v_3\}$,
  \item $\{v_1, v_4\}$,
  \item $\{v_2, v_3\}$,
  \item $\{v_2, v_4\}$,
  \item $\{v_3, v_4\}$,
  \item $\{v_1, v_5, v_6\}$,
  \item $\{v_1, v_6, v_7\}$,
  \item $\{v_1, v_7\}$,
  \item $\{v_2, v_5, v_6\}$,
  \item $\{v_2, v_5, v_7\}$,
  \item $\{v_2, v_6, v_7\}$,
  \item $\{v_3, v_5, v_6\}$,
  \item $\{v_3, v_5, v_7\}$,
  \item $\{v_3, v_6, v_7\}$,
  \item $\{v_4, v_5, v_6\}$,
  \item $\{v_4, v_5, v_7\}$ and $\{v_4, v_6, v_7\}$.
\end{itemize}

While the minimum unknotting sets are only $W_1, W_2, W_3, W_4, W_5,$ and $W_6$.

It is known that a knot $K$ has infinite many diagrams. It is not necessary that $u(K)$ is always obtained from a reduced diagram of $K$. There may be another diagram of $K$, not necessarily a reduced one, having the same unknotting number as $u(K)$. For many knots listed in the Rolfsen Table of knots [11], $u(K)$ is the same for the reduced and other diagrams of $K$. However, for the knot $10_8$ ((5, 1, 4) in Conway notation [4]), the reduced diagram (Fig. 8) is unknotted by switching, at least, 3 crossings with a minimum unknotting set $\{v_2, v_4, v_6\}$.

There is another diagram (Fig. 9) of $10_8$ which turns to unknot by switching only 2 crossings with a minimum unknotting set $\{v_6, v'_9\}$. 

![Figure 8: reduced diagram of (5, 1, 4)](image8)

![Figure 9: (5, 1, 4)](image9)
The unknotting number of this diagram is actually the unknotting number of 10_8 (see [2, 9]).

### 3.2 U-independence System as knot invariant

**Definition 9** Let \((E_k, I_k), k = 1, 2\), be two independence systems. Let there exist a bijection \(\varphi : E_1 \to E_2\) such that \(\varphi(X) \subset I_2\) if and only if \(X \subset I_1\). Then, \(E_1\) and \(E_2\) are isomorphic.

In order to prove that the \(U\)-independence system is a knot invariant for an alternating knot, the following well-known conjecture of Tait (proved in [8] by Menasco and Thistlethwaite) is needed.

**Theorem 10** [The Tait flyping conjecture] Given reduced alternating diagrams \(D_1, D_2\) of a knot (or link). Then, it is possible to transform \(D_1\) to \(D_2\) by a sequence of flypes (Fig. 10).

**Proof of Theorem 3** Let \(v_i\) be a crossing in the diagram \(D_1\). Apply the flype (Fig. 10) to \(D_1\) to remove the crossing \(v_i\) and create a new crossing with the same label \(v_i\). More precisely, the tangle (the shaded disc in Fig. 10) is turned upside-down to map the crossing (one to its left) to the crossing (one to its right). During the application of the flype, all the unknotting/not unknotting sets of the diagram \(D_1\) are preserved. Consequently, all the \(U\)-independent sets are preserved in the process. By Theorem 10, the diagram \(D_1\) can be converted to \(D_2\), through a sequence of the flypes, preserving the \(U\)-independent sets. As a result, an isomorphism \(\varphi\) between \((E_1, I_1)\) and \((E_2, I_2)\) is established.

![Figure 10: flype](image)

It is now established, by Theorem 3, that the \(U\)-independence system (defined for a reduced alternating diagram \(D\) of a knot \(K\)) itself and all its invariants are knot invariants. The number \(u_{\text{min}}(K)\) can also be defined as the cardinality of a \(U\)-independent set which is also a minimum unknotting set. The number is not a complete invariant, i.e., there are non-isotopic knots having the same \(u_{\text{min}}(K)\). However, other invariants of the \(U\)-independence systems of non-isotopic alternating knots may distinguish them where \(c(K)\) and \(u_{\text{min}}(K)\) fails to do so. Here are two such examples.

**The Number of \(U\)-independent Sets of a Fixed Cardinality**

The number of \(U\)-independent sets of a fixed cardinality is a knot invariant.

**Example 11** Consider the knots 6_1 and 6_2 with the same \(c(K)\) and \(u_{\text{min}}(K)\).
For the knot $6_1$, the set of all crossings $E = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ (Fig. 11) is divided into two disjoint sets $A$ and $B$: the set $A = \{v_1, v_2, v_3, v_4\}$ and $B = \{v_5, v_6\}$. In $A$, no single crossing switching turns the knot to unknot. In contrast, when each crossing in $B$ is switched, the knot is unknotted. All possible subsets of $\{v_1, v_2, v_3, v_4\}$ of cardinality 2, are minimal unknotting sets. Furthermore, every subset of cardinality 3, 4, or 5 contains an unknotting set. Thus, all the $U$-independent sets are: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_6\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_3, v_4\}$. There are 6 $U$-independent sets of cardinality 2.

For the knot $6_2$, the set $E = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ (Fig. 12) is divided into three disjoint sets $A$, $B$, and $C$: the set $A = \{v_1, v_2, v_3\}$, there is no crossing in $A$ which turns the knot to unknot; the set $B = \{v_4\}$ is an unknotting set; and the set $C = \{v_5, v_6\}$ contains no unknotting set. When any two crossings from $A \cup B$ are switched, the knot is unknotted. However, there is no unknotting set of cardinality 2 in $B \cup C$. Furthermore, every subset of cardinality 3, 4, or 5 contains an unknotting set. Thus, the $U$-independent sets are $\{v_1\}$,
{v_2}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_5, v_6\}. There are 10 U-independent sets of cardinality 2. The knots 6_1 and 6_2 are distinguished by the number of U-independent sets of cardinality 2.

**The I-chromatic number**

The U-independence system partitions the set of crossings of a reduced diagram D into U-independent sets and the minimum number of such U-independent sets gives a minimum partition of E. The number of U-independent sets in a minimum partition of E gives the I-chromatic number $\chi(E, I)$. The number $\chi(E, I)$ for the diagram D can be used as a knot invariant in combination with $u_{\text{min}}(K)$. In other words, two knots can be distinguished by the $\chi(E, I)$ if the knots have the same $c(K)$ and $u_{\text{min}}(K)$.

**Example 12** Consider the knots 7_2 and 7_7 with the same $c(K)$ and $u_{\text{min}}(K)$.

![Figure 13: reduced diagram of 7_2](image)

For the knots 7_2, the set $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ (Fig. 13) is divided into two disjoint subsets A and B: $A = \{v_1, v_2, v_3, v_4, v_5\}$ and $B = \{v_6, v_7\}$. The set A contains no unknotting set of cardinality 1 and 2. Each subset of A of cardinality 3 is a minimal unknotting set. Every crossing in B unknits the knot, but B itself is not an unknotting set. Every subset of E containing $\{v_6\}$ or $\{v_7\}$ is not a minimal unknotting set. Furthermore, every set of cardinality 4, 5, and 6 contains an unknotting set. Thus, the U-independent sets are:

$\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_7\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}.$

A minimum partition is $\{\{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_6\}, \{v_7\}\}$ and $\chi(E, I) = 4$. 
For the knot $7_7$, the set $E = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ (Fig. 14) is divided into three disjoint subsets $A$, $B$, and $C$. The set $A = \{v_1, v_2, v_3\}$; $B = \{v_4, v_5\}$; and $C = \{v_6, v_7\}$. In the set $A$, no unknotting set of cardinality 1 exists but every set of cardinality 2 is unknotting except $\{v_1, v_2\}$. In $B$, $\{v_4\}$ and $\{v_5\}$ are unknotting sets but $B$ itself is not unknotting. In $C$, neither a set of cardinality 1 nor $C$ itself is unknotting set. Every set of cardinality 3, 4, 5 and 6 contains an unknotting set. Thus, the $U$-independent sets are: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_6\}$, $\{v_7\}$, $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_6\}$, $\{v_1, v_7\}$, $\{v_2, v_3\}$, $\{v_2, v_6\}$, $\{v_2, v_7\}$, $\{v_3, v_6\}$, $\{v_3, v_7\}$, $\{v_6, v_7\}$.

A minimum partition is $\{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6, v_7\}\}$ and $\chi(E, I) = 5$. Hence, the knots $7_2$ and $7_7$ are distinguished by $\chi(E, I)$.

On the same lines, independence systems can also be defined for a knot diagram with respect to other invariants like forbidden unknotting number, bridge number, algebraic unknotting number etc. The corresponding invariants of these independence systems may also be used as knot invariants in combination with these invariants. Every independence system $(E, I)$ is an abstract simplicial complex [5]. Therefore, the homology of $(E, I)$ can be investigated for finer invariants of the corresponding knots. Similarly independence systems can be associated and studied for virtual knots [7].

One can show that the $U$-independence systems for reduced alternating diagrams of $6_1$ and $6_3$ are isomorphic. The following question is natural to be posed at this stage.

**Question 13** Does there exist two non-isotopic alternating knots (not the mirror image of each other) of the same $c(K)$ and $u_{\text{min}}(K) \geq 2$ with isomorphic $U$-independent systems defined for reduced alternating diagrams?

**Question 14** Is the $U$-independence system defined for a reduced diagram (with minimum crossings) of a non-alternating knot an invariant?

Figure 14: reduced diagram of $7_7$ knot
4  U-independence as a Matroid

4.1  Family \((2n + 1, 1, 2n)\) for \(n \geq 2\)

It is proved in [2] that a reduced diagram of \((2n + 1, 1, 2n)\) for \(n \geq 2\), has \(u(K) = n+1\). It is also proved in the same paper that a diagram of \((2n+1,1,2n)\) for \(n \geq 2\) (Fig. 15) has \(u(K) = n\).

Proof of Theorem \[\text{[4]}\] The diagram \(D\) (Fig. 15) of a knot diagram \(D\) with \(u(D) = n\) has two minimal unknotting sets \(\{w, u'_3, u'_5, \ldots, u'_{2n-1}\}\) and \(\{v_2, v_4, v_6, \ldots, v_{2n}, w\}\) of cardinalities \(n\) and \(n + 1\) respectively. Consequently, there are two maximal \(U\)-independent sets of different cardinalities and the \(U\)-independence system is not a matroid by Definition \[\text{[7]}\].

![Figure 15: \((2n + 1, 1, 2n)\)](image)

4.2  Family \((2n + 1)\)

The following result may be well known for an expert in knot theory. Anyhow, it is proved here for the sake of completion.

**Lemma 15** A knot \(K\) in the family with Conway notation \((2n + 1)\), for \(n \geq 1\) has \(u(K) = n\).

**Proof.** Apply induction on \(n\).

**Case I** For \(n = 1\), \((2n + 1) = (3)\) is the trefoil knot with \(u(K) = 1\).

**Case II** Suppose for \(n = m\), \((2m + 1)\) has \(u(K) = m\).
Case III For \( n = m + 1 \), \((2(m + 1) + 1) = (2m + 3)\) is a family of knots with 2m + 3 alternating crossings (Fig. 17).

When the crossing \( v_{2m+3} \) is switched, the crossing \( v_{2m+2} \) is also killed and the knot \((2m + 1)\) is obtained (Fig. 16). By case II, \( u(K) \leq m + 1 \). The knot \((2m + 1)\) can not be unknotted by fewer than \( m \) crossings because if \( m - 1 \) crossings are switched, \( 2(m - 1) \) alternating crossings are untangled and the trefoil knot \((2m + 1 - (2m - 2)) = (3)\) is obtained. Therefore, \((2m + 3)\) has the unknotted number \( m + 1 \).

Proof of Theorem 5 The diagram (Fig. 18) has the special property that every subset \( A \) of cardinality \( n \) in \( E = \{v_1, v_2, v_3, \ldots, v_{2n}, v_{2n+1}\} \) is an unknotted set. The set \( A \) must be minimal by Lemma 16. Hence, every subset of \( E \) of cardinality \( n \) is a \( U \)-independent set. Therefore, there is no maximal \( U \)-independent set of cardinality \( < n \). Also, there is no \( U \)-independent set \( B \) of cardinality \( > n \) because \( B \) contains an unknotted set of cardinality \( n \). Consequently, all maximal \( U \)-independent sets are of cardinality \( n \) and the system is a matroid by Definition 7.
4.3 Family $(2n, 2)$

For $n \geq 1$, the diagram (Fig. 19) for the family $(2n, 2)$ of knots is such that its each member has the unknotting number 1 and its $U$-independent system is not a matroid except for the figure eight knot (for $n = 1$).

Proof of Theorem For the diagram (Fig. 19) for $n > 1$, the sets \{w\} and \{v_1, v_2, v_3, v_4, \ldots, v_n\} are minimal unknotting sets of cardinality 1 and $n$ respectively. Thus, there are two maximal $U$-independent sets having different cardinalities and $U$-independent system is not a matroid by Definition.
The table given below shows whether the $U$-independent systems for the reduced knot diagrams up to 8 crossings are matroids or not.

| Knot | Matroid | Knot | Matroid |
|------|---------|------|---------|
| $3_1$ | yes     | $7_4$ | no      |
| $4_1$ | yes     | $7_5$ | no      |
| $5_1$ | yes     | $7_6$ | no      |
| $5_2$ | no      | $7_7$ | no      |
| $6_1$ | no      | $8_1$ | no      |
| $6_2$ | no      | $8_2$ | no      |
| $6_3$ | no      | $8_3$ | no      |
| $7_1$ | yes     | $8_4$ | no      |
| $7_2$ | no      | $8_5$ | no      |
| $7_3$ | no      | $8_6$ | no      |

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