Integration of Singular Subalgebroids

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Abstract

We establish a Lie theory for singular subalgebroids, objects which generalize singular foliations to the setting of Lie algebroids. First we carry out the longitudinal version of the theory. For the global one, a guiding example is provided by the holonomy groupoid, which carries a natural diffeological structure in the sense of Souriau. We single out a class of diffeological groupoids satisfying specific properties and introduce a differentiation-integration process under which they correspond to singular subalgebroids. In the regular case, we compare our procedure to the usual integration by Lie groupoids. We also specify the diffeological properties which distinguish the holonomy groupoid from the graph.

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Introduction

Subject of the paper

A coarse description of the subject of this paper is the Lie Theory of some very singular dynamical systems. They were introduced in [27], as singular subalgebroids, and were shown to encompass an abundance of examples. Our primary purpose here is to develop a theory of integration\footnote{Here “integration” is taken in same sense as a Lie group is an integration of a Lie algebra.} for such objects. To this end, first let us recall the definition of singular subalgebroid, as well as the constructions given in [27].

Definition. Let $A$ be a Lie algebroid over the manifold $M$. A singular subalgebroid is a $C^\infty(M)$-submodule $B$ of (compactly supported) sections of $A$, which is locally finitely generated and involutive with respect to the Lie algebroid bracket.

- Note that $B$ is quite generic as a $C^\infty(M)$-module; in particular it may contain sections which vanish to a prescribed order at certain points of $M$. To explain this, take $A = TM$ and a singular foliation $F$ in the sense of [2]. Vectors in $TM$ represent only first order differential equations, but sections may represent higher order ones. For instance, in the case $M = \mathbb{R}$; the module $F$ generated by the vector field $x^2 \partial_x$ vanishes to order 2 at zero. More generally, vectors of any vector bundle $A$ over $M$ (thanks to the Serre-Swan theorem these are equivalent to localizations of elements of a projective $C^\infty(M)$-module) represent only first order differential equations, but sections may represent higher order differential equations. In other words, the definition of $B$ is designed for Analytical purposes much more than Algebraic ones. In particular, it is designed so that singularities of any order fit in (hence the justification of the term “very” singular for $B$). In general, the $C^\infty(M)$-submodules $B$ of $\Gamma(A)$ we are looking at are not necessarily projective; they may admit a projective resolution though, possibly of infinite length.

- Also note that the notion of singular subalgebroid is a relative one, since it depends on the ambient Lie algebroid $A$.

- In fact, the notion of singular foliation $(M, F)$ from [2] is our guiding example of singular subalgebroid, where the ambient Lie algebroid is $TM$. Recall that $TM$ can be integrated by more than one Lie groupoid, for instance the fundamental groupoid and the pair groupoid. In [27] it was noticed that the holonomy groupoid $H(F)$ constructed in [2] really takes into account one more choice; that is, the choice of Lie groupoid integrating $TM$. Indeed, to every triple $(B, A, G)$ consisting of a singular subalgebroid $B$ of a Lie algebroid $A$ and a choice of Lie groupoid $G$ integrating the latter, [27] attached a topological groupoid $H^G(B)$. In the case of a singular foliation, applying this construction to $(F, TM, M \times M)$ recovers $H(F)$. In fact, $H^G(B)$ ought to be thought of as a holonomy groupoid, because it is a kind of pullback of a holonomy groupoid for a singular foliation on $G$.

- One understands that the integration problem for singular subalgebroids is even more relative than their definition: One not only needs to prescribe the ambient Lie algebroid $A$, but also a Lie groupoid $G$ integrating $A$. In fact, $H^G(B)$ comes together with a morphism of topological groupoids $\Phi: H^G(B) \to G$, and in the special case of wide Lie subalgebroids, the same is true for the integrations in the sense of [22].
As we said in the beginning, the purpose of this paper is to provide a definition of integration in this very general and very singular context, and show that our notion is an adequate one. Such a task is not easy, because in order to make sure that the definition produces reasonable descriptions (theorems), one first ought to envisage the traits that this definition should have.

Let us make some general considerations in a different context. First, the established notion of integration (by Lie groupoids) concerns Lie algebras and Lie algebroids, which do not present the kind of singularity we have in the case of singular subalgebroids. Even though the context is not identical – those classical objects do not have the relative nature that singular subalgebroids have – it is clear that the notion of integration we are looking for ought to include more general objects than Lie groupoids. On the other hand, we explained that the nature of singular subalgebroids is quite parallel with singular foliations. Elaborating further on these two observations, we also envisage the following about the integration problem:

- Even though we will not address it at all in this paper, consider again the classical integration problem for a Lie algebroid \( A \) over \( M \). Recall that in [8] a source-simply connected topological “monodromy” groupoid \( Mon(A) \) was attached to \( A \) (analogous to the fundamental groupoid), and the topological obstructions for it to be smooth were given. If we know a priori that a Lie groupoid \( G \) exists which differentiates to \( A \), then these obstructions vanish. However, even when these obstructions vanish, there may be quotients of \( Mon(A) \) which are not Lie groupoids, but which as quotients of a Lie groupoid carry a nice diffeological structure. Recall [24] that diffeology generalises the notion of smoothness. Although we will not address the following question in this paper, it is natural to wonder if the diffeology can be used to differentiate (in a suitable sense) these quotients to \( A \) as well. This would enlarge the integration problem beyond the smooth category. This question is closely related to the integration question for Lie-Rinehart algebras endowed with suitable extra structure (what is left of a singular subalgebroid after one forgets the ambient Lie algebroid), which is addressed in a special case in [12].

- From the point of view of foliation theory, let us recall the regular case, namely a constant rank involutive distribution \( F \) on \( M \). The “largest” (source simply-connected) integration of this Lie subalgebroid of \( TM \) is the monodromy groupoid. The holonomy groupoid \( H(F) \) is a quotient of the monodromy groupoid. In terms of category theory, \( H(F) \) is a terminal object among all integrations of \( F \), namely the adjoint integration, which means that it is a minimal integration in some sense. On the other hand, \( H(F) \) also quotients to the graph \( R(F) \) of the foliation. Note that both the holonomy groupoid and the graph can be constructed in the case of singular subalgebroids. In general, as far as adjoints are concerned, the integration problem for general Lie algebroids is very different from the integration problem for the special class of regular foliations. We will discuss this in §3.3.

This leads us to the following specific questions, which we address in this paper:

a) Is the notion of diffeology appropriate for the definition of integration we are seeking for singular subalgebroids?

b) If so, does the notion of diffeology provide a way to distinguish the holonomy groupoid of a singular subalgebroid from its graph, as far as integration is concerned?

We will show that for both questions the answer is positive.
Results

Since we wish to address the integration problem, let us start with a triple \((B, A, G)\) consisting of a singular subalgebroid \(B\) of a Lie algebroid \(A\) and a Lie groupoid \(G\) integrating the latter. It turns out that integration can be understood in two ways:

- **Longitudinally:** Applying the anchor map of \(A\) to \(B\) we obtain a (singular) foliation \(\mathcal{F}_B\) on the underlying manifold \(M\). Even though \(B\) is a singular object, over every leaf \(L\) of this foliation it induces a transitive Lie algebroid \(B_L\). We show in \([2]\) that this Lie algebroid is always integrable. Further, the restriction of the holonomy groupoid \(H^G(B)\) to the leaf \(L\) is a Lie groupoid integrating it. Whence, we can differentiate \(H^G(B)\) “leaf by leaf” to obtain Lie algebroids that are “restrictions” of \(B\) to each leaf. Notice that there is a loss of information passing from \(B\) to the collection of transitive Lie algebroids over the leaves, hence the longitudinal integration we just described cannot be expected to provide a global integration of \(B\).

- **Globally:** For the global integration a new approach is needed. It turns out that the center of focus is the specification of the correct notion of smoothness for the integrating groupoids. In the approach we propose, the integrals of singular subalgebroids are groupoids endowed with a specific diffeology. This was already announced in \([27]\), and diffeology in the context of the holonomy groupoid was later used by other authors \([12]\) \([19]\) \([20]\). We elaborate on this below, in the rest of this introduction.

A special case occurs when the singular subalgebroid \(B\) is projective, i.e. it consists of sections of a Lie algebroid \(B\) over \(M\), which comes with an almost injective morphism \(\tau: B \to A\). We show in \([3]\) building on work of Debord \([9]\), that the Lie algebroid \(B\) is always an integrable Lie algebroid. Further, the holonomy groupoid \(H^G(B)\) is a Lie groupoid integrating \(B\), and the canonical morphism \(\Phi: H^G(B) \to G\) integrates \(\tau: B \to A\). The global integrals we associate to \(B\) encompass all the Lie groupoids integrating \(B\), as we explain in the Theorem below, and it is an open question whether they also include non-smooth integrals. This applies in particular when \(B\) corresponds to a wide Lie subalgebroid.

We explain our approach to the global integration question. The first step is to get a grasp on differentiation. The holonomy groupoid \(H^G(B)\) provides a guide, since one certainly expects to be able to recover \(B\) from it. Hence we ask:

1) In what sense does \(H^G(B)\) differentiate to \(B\)?

The “link” between the holonomy groupoid and the Lie algebroid \(A\) is provided by the canonical morphism \(\Phi: H^G(B) \to G\). The special case of wide Lie subalgebroids suggests the following, which is indeed a true fact: \(B\) is recovered as the (compactly supported) elements of

\[
\left\{ \frac{d}{d\lambda}\big|_{\lambda=0}(\Phi \circ b_\lambda) : \{b_\lambda\}_{\lambda \in I} \text{ family of smooth global bisections for } H^G(B) \text{ s.t. } b_0 = \text{Id}_M \right\}.
\]

An important point is to make sense of “smooth” above. This is done using the natural diffeological structure on \(H^G(B)\) (recall that \(H^G(B)\) is a quotient of a disjoint union of manifolds).

We point out that here we are using two well-known and fundamental ideas about integration. First, a global integration question was considered by Souriau: Given a compact manifold \(M\), in what
sense can we apply the Lie functor to the group of diffeomorphisms $Diff(M)$ in order to obtain the Lie algebra $\mathfrak{X}(M)$ of vector fields? Souriau achieved this differentiation by introducing the notion of diffeology in [24].

The second key to the differentiation of $H^G(B)$ comes from H. Abels [11, Appendix], who showed that in order to recover the infinitesimal information (the Lie algebra) one just needs to make good sense of 1-parameter families, and this can be done even when we work with just a topological group. In fact, this idea can be traced back to Palais.

In §5, guided by the above example of the holonomy groupoid, we single out a class of objects that give rise to singular subalgebroids by a differentiation procedure. They are given by pairs $(H, \Psi)$ where $H$ is a diffeological groupoid [15, §8.3] and $\Psi: H \to G$ a morphism covering $Id_M$, subject to certain conditions modelled on the properties of the holonomy groupoid. Our approach is minimalistic, in the sense that the properties we require are the minimal assumptions under which we were able to extract a well-defined singular subalgebroid from our data. This leads to the definition of differentiation (Definition 5.3). The above groupoids do carry bisections, and their diffeological nature allows us to speak about smooth, 1-parameter families of bisections; in the context of groupoids, this is the appropriate analogue of 1-parameter groups of diffeomorphisms (cf. [21]). As expected, $H^G(B)$ differentiates to $B$ in this sense.

Diffeological groupoids occur in several contexts in the literature, including general relativity [6] and higher Lie theory [7]. We emphasize that in this paper we only consider diffeological groupoids whose space of objects is a smooth manifold and which come together with a morphism to a Lie groupoid. This simplifies many of our considerations and allows us to avoid some technical aspects of the theory of diffeological spaces (for instance, we never have to address their tangent spaces).

We can finally address the global integration question:

II) What is the correct notion of integration for a singular subalgebroid $B$?

Our answer is more refined than just pairs $(H, \Psi)$ with the expected requirement that they differentiate to $B$: we impose two more conditions that make the notion of integration as well-behaved as possible, while retaining all the desired classes of examples. We paraphrase Definition 6.1.

**Definition.** Fix a triple $(B, A, G)$ consisting of a singular subalgebroid $B$ of a Lie algebroid $A$ and a Lie groupoid $G$ for $A$. Let $H$ be a diffeological groupoid over $M$, and $\Psi: H \to G$ a smooth morphism of diffeological groupoids covering $Id_M$. We say that $(H, \Psi)$ is an integral of $B$ over $G$ if:

a) $(H, \Psi)$ differentiates to $B$ (Def. 5.3).

b) The diffeology of $H$ is generated by open maps (Def. 4.24).

c) The morphism $\Psi$ is almost injective.

Notice that we adopt the terminology “integral”, to distinguish this notion from the usual integration by Lie groupoids. To test the validity of our notion of integral, we apply it to the main classes of examples. We summarize the results as follows (Thm. 6.19, Prop. 6.6, Prop. 6.17, Rem. 6.18; notice that items ii), iii), iv) are in increasing order of generality. Here we use the term “smooth integral” to denote integrals that are Lie groupoids.

**Theorem.** i) For any singular subalgebroid $B$, the holonomy groupoid is an integral.
ii) When $A$ is a Lie algebra (thus $B = A$ is a Lie subalgebra), the integrals of $B$ are all smooth. The integrals of $B$ over $G$ are exactly the Lie groups covering the connected Lie subgroup of $G$ integrating the Lie subalgebra $B$.

iii) When $B = \Gamma_c(B)$ for a wide Lie subalgebroid $B$, the smooth integrals coincide with the integrations of $B$ in the sense of Moerdijk-Mrčun [22].

iv) A singular subalgebroid $B$ admits a smooth integral iff it is a projective, i.e. $B \cong \Gamma_c(B)$ for a Lie algebroid $B$. The pair $(H^G(B), \Phi)$ is a smooth integral of $B$ over $G$, and further it is the minimal smooth integral: the smooth integrals are exactly the coverings of the holonomy groupoid.

To put item iii) into context, we recall that the theory of integration of wide Lie subalgebroids $B$ of $A$ was developed in [22]. Fix a Lie groupoid $G$ integrating $A$. Moerdijk-Mrčun show that there exists a family of Lie groupoids $H$, whose Lie algebroid is $B$, which come together with Lie groupoid morphisms $\Psi : H \to G$ integrating the inclusion $B \hookrightarrow A$. Further they show that there is a minimal such Lie groupoid $H$, which they call “minimal integral of $B$ over $G$” (it agrees with the holonomy groupoid).

We finish with a few comments about the holonomy groupoid.

- For the holonomy groupoid, the openness property in the above Definition is implied by a kind of submersive property for the plots of the diffeological structure, which is called “local subduction” and we recall in §4.4.3. The investigation of a submersive property was suggested to us by Georges Skandalis. It has independent interest, in view of the construction of operator algebras from diffeological groupoids, which are interesting in Noncommutative Geometry. However we do not address this here.

- One can also consider the graph of $B$, a set-theoretic subgroupoid of $G$. It is defined analogously to the graph of a foliation, which is the collection of pairs of points belonging to the same leaf. The graph is quite a coarse object, but as a quotient of the holonomy groupoid $H^G(B)$ (actually also of every integral of $B$) it carries a diffeological groupoid structure, which turns out to differentiate to $B$. We explain in §7 that the openness property is not satisfied by the graph, which hence is not an integral. Clearly, the graph cannot be quotiented to any integral $Q$ of $B$, because the inclusion of the graph in $G$ is an injective map and as such it can not induce a well-defined map $Q \to G$.

Structure of the paper: The article starts §1 by recalling singular subalgebroids and their associated holonomy groupoids from [27]. In §2 we discuss longitudinal integration. §3 gives the integration of singular subalgebroids $B$ which are projective as $C^\infty(M)$-modules. In §4 we define diffeological groupoids and discuss the properties which are relevant to the integration of singular subalgebroids. §5 gives the differentiation process for diffeological groupoids as such. Global integration is discussed in §6. Finally, the graph of a singular subalgebroid is discussed in §7. In the appendix we collect two proofs and an interpretation of a certain assumption used in the main text.

Conventions and notation: All Lie groupoids are assumed to be source connected. Given a Lie groupoid $G \rightrightarrows M$, we denote by $t$ and $s$ its target and source maps, and by $i : G \to G$ the inversion map. We denote by $1_x \in G$ the identity element corresponding to a point $x \in M$, and by $1_M \subset G$ the submanifold of identity elements. Two elements $g, h \in G$ are composable if $s(g) = t(h)$. We use the term bisection to denote a right-inverse to $s$ defined on an open subset of $M$. We identify the Lie algebroid of $G$, which we denote by $AG$, with ker$(ds)|_M$. 7
Acknowledgements: I.A. thanks Georges Skandalis, Robert Yuncken and Omar Mohsen for various discussions and suggestions. M.Z. thanks Dan Christensen, Claire Debord, Alfonso Garmendia and Joel Villatoro for explanations related to this work. This work was partially supported by a Marie Curie Career Integration Grant PCLI09-GA-2011-290823 (Athens), by grants MTM2011-22612 and ICMAT Severo Ochoa SEV-2011-0087 (Spain), Pesquisador Visitante Especial grant 88881.030367/2013-01 (CAPES/Brazil), IAP Dygest, the long term structural funding – Methusalem grant of the Flemish Government, the FWO under EOS project G0H4518N, the FWO research project G083118N (Belgium), and SCHI 525/12-1 (DFG).

1 Singular Subalgebroids and holonomy groupoids

In this section we recall the notion of a singular subalgebroid (§1.1 §1.2 §1.3) and the construction of its holonomy groupoid (§1.4 §1.5). Later on, in §5 we will give the description of the holonomy groupoid as a diffeological space.

We follow closely [27], except for the material in §1.2-§1.3 which was not spelled out there.

1.1 Definition and main examples

Let $A$ be a Lie algebroid over a smooth manifold $M$. Throughout the sequel we assume $A$ is integrable and choose a (source-connected) Lie groupoid $G\rightarrow M$ integrating $A$. We will assume that $G$ is Hausdorff.

Definition 1.1. [27, Def. 1.1] A singular subalgebroid of $A$ is an involutive, locally finitely generated $C^\infty_c(M)$-submodule $B$ of $\Gamma_c(A)$, the compactly supported sections of $A$.

We briefly discuss the main examples of singular subalgebroids.

Example 1.2. A singular foliation on a manifold $M$ is an involutive, locally finitely generated $C^\infty(M)$-module of vector fields with compact support $\mathfrak{x}_c(M)$ [2]. The singular subalgebroids of $A = TM$ are exactly the singular foliations on $M$.

Example 1.3. 

a) Let $\psi: E \rightarrow A$ be a morphism of Lie algebroids covering the identity on the base manifolds. Then the image of the induced map of compactly supported sections,

$$B := \psi(\Gamma_c(E)),$$

is a singular subalgebroid of $A$. We say that $B$ arises from the Lie algebroid morphism $\psi$.

b) A special case of item a) is provided by projective singular subalgebroid $B$ of $A$, namely those for which there exists a vector bundle $B$ over $M$ such that $\Gamma_c(B) \cong B$ as $C^\infty(M)$-modules. There is a Lie algebroid structure on $B$ and almost injective Lie algebroid morphism $\tau: B \rightarrow A$ inducing the isomorphism $\Gamma_c(B) \cong B$, and these data are unique, as we shall see in §3.1. In particular, $B$ arises from the Lie algebroid morphism $\tau$.

c) Specializing further, we can consider $B = \Gamma_c(B)$ where $B$ is a wide Lie subalgebroid of $A$ (that is, a vector subbundle $B \rightarrow M$ whose sections are closed with respect to the Lie bracket $[21$, Def. 3.3.21]).
Example 1.4. Let $N$ be a closed embedded submanifold of $M$ and $B \to N$ a Lie subalgebroid of $A$ ([21] Def. 4.3.14]). Then
\[ \mathcal{B} := \{ \alpha \in \Gamma_c(A) : \alpha|_N \subset B \} \]
is a singular subalgebroid of $A$.

Let $n = \text{dim}(N)$ and $b = \text{rank}(B)$. To describe $\mathcal{B}$ near a point $p$ of $N$, choose coordinates $\{x_i\}_i \odot n$ vanish on $N$ and the restrictions of $\{x_i\}_{i \odot n}$ to $N$ are coordinates around $p$ in $N$. Let $\{\alpha_j\}$ be a frame of compactly supported sections of $A$ adapted to $B$, i.e. $\{\alpha_j|_N\}_{j \odot b} \subset B$. Then $\mathcal{B}$, locally near $p$, is generated by
\[ \{\alpha_j\}_{j \odot b} \cup \{x_i \cdot \alpha_j\}_{i \odot n, j \odot b}. \]

On open sets disjoint from $N$, $\mathcal{B}$ is given by restrictions of compactly supported sections of $A$.

When $N$ is a hypersurface (i.e. has codimension 1 in $M$), the $C^\infty(M)$-module $\mathcal{B}$ is projective. On the other hand, if $\text{codim}(N) \geq 2$ and $B \neq A|_N$, then $\mathcal{B}$ is not projective, because the number of generators above is strictly larger than the rank of $A$.

Last, let us recall from [27] §1 two structures associated with a singular subalgebroid $\mathcal{B}$:

a) The foliation $\overrightarrow{\mathcal{B}}$ on $G$: Since the Lie algebroid $A$ integrates to the Lie groupoid $G$, every section $\alpha$ of $A$ corresponds to a right-invariant vector field $\overrightarrow{\alpha}$ of $G$ (see [21] §3.5]). Hence the $C^\infty(M)$-module $\mathcal{B}$ gives rise to the $C^\infty(G)$-module $\overrightarrow{\mathcal{B}}$ generated by $\{\overrightarrow{\alpha} : \alpha \in \mathcal{B}\}$. Notice that $\overrightarrow{\mathcal{B}}$ consists of compactly supported vector fields[2], indeed it is a singular foliation on $G$ in the sense of [2]. Likewise, every $\alpha \in \mathcal{B}$ defines a left-invariant vector field $\overleftarrow{\alpha}$ of $G$ and a foliation $\overleftarrow{\mathcal{B}}$.

b) The foliation $F_\mathcal{B}$ on $M$: Let $\rho : A \to TM$ be the anchor map of $A$. Since $\mathcal{B}$ is locally finitely generated and involutive, so is the $C^\infty(M)$-module $F_\mathcal{B}$ generated by $\{\rho(\alpha) : \alpha \in \mathcal{B}\}$; whence $F_\mathcal{B}$ is a singular foliation of $M$. Note that $\rho(\alpha)$ has the same support as $\alpha$ whence it is compactly supported as well. Recall that $\overrightarrow{\alpha}$ is $t$-related with $\rho(\alpha)$, therefore the vector field $\overrightarrow{\alpha}$ is complete.

1.2 Associated vector spaces

We explain how, at every point, a singular subalgebroid induces a Lie algebra and a short exact sequence of vector spaces.

Pick a point $x \in M$ and consider the evaluation map as a morphism of $\mathbb{R}$-vector spaces
\[ \text{ev}_x : \mathcal{B} \to A_x, \quad \alpha \mapsto \alpha(x). \]

We denote the image of this map $B_x$ (it is a vector subspace of $A_x$), and its kernel by $B(x) := \{ \alpha \in \mathcal{B} : \alpha(x) = 0 \}$. Let $I_x^M$ denote the ideal of functions on $M$ vanishing at $x$. Put $\mathfrak{b}_x = \frac{B(x)}{I_x^M \mathcal{B}}$ and

---

[2]This is the reason why we assume the Lie groupoid $G$ to be Hausdorff: the compact support condition on non-Hausdorff manifolds leads to somewhat strange behaviours. Making this assumption we avoid complicating the exposition. (Allowing $G$ to be non-Hausdorff, one can consider sheaves of compactly supported sections on open subsets of $G$).
We obtain a short exact sequence of finite dimensional vector spaces
\[ 0 \to \mathfrak{b}_x \to \mathcal{B}_x \to \mathcal{B}_x \to 0. \] (1.1)

The kernel \( \mathfrak{b}_x \) has an obvious Lie algebra structure induced by the Lie bracket on \( \Gamma_x(A) \).

The next lemma is proved exactly as in [2, Prop. 1.5a].

**Lemma 1.5.** If \( \alpha_1, \ldots, \alpha_N \in \mathcal{B} \) are such that their images \([\alpha_1], \ldots, [\alpha_N]\) in \( \mathcal{B}_x \) form a basis, then the \( \alpha_i \)'s are generators of \( \mathcal{B} \) in a neighborhood of \( x \).

We also recall the next result from [27, §1]:

**Lemma 1.6.** Let \( \alpha_1, \ldots, \alpha_n \) be a finite subset of \( \mathcal{B} \). Then \([\alpha_1], \ldots, [\alpha_n]\) is a basis of \( \mathcal{B}/I^M_x \mathcal{B} \) iff \([\alpha_1], \ldots, [\alpha_n]\) is a basis of \( \overrightarrow{\mathcal{B}}/I^G_x \overrightarrow{\mathcal{B}} \). (And likewise for \( \mathcal{B} \).)

As a consequence, the sequence (1.1) is isomorphic to the short exact sequence associated to the foliation \( \overrightarrow{\mathcal{B}} \), which is
\[ 0 \to \overrightarrow{\mathcal{B}}(x)/I^G_x \overrightarrow{\mathcal{B}} \to \overrightarrow{\mathcal{B}}/I^G_x \overrightarrow{\mathcal{B}} \to \text{ev}_x(\overrightarrow{\mathcal{B}}) \to 0. \]

The isomorphism is essentially given by the map \( \overrightarrow{\mathcal{B}} \to \mathcal{B} \), obtained restricting sections of \( \text{ker}(s_x) \) to sections of \( \text{ker}(s_x)|_M \cong A \).

### 1.3 Associated Lie algebroids over the leaves

Let \( L \) be a leaf of \( \mathcal{F}_B \). For simplicity, we assume \( L \) is embedded in \( M \). Note that it is possible to formulate everything for immersed leaves as well, but we omit this so as not to blow up the length of the article.

**Lemma 1.7.** Put
\[ B_L := \bigcup_{x \in L} B_x. \]
Then \( B_L \) is a transitive Lie subalgebroid of \( A \) supported on \( L \). Its compactly supported sections are given by \( \mathcal{B}|_L = \{ \alpha|_L : \alpha \in \mathcal{B} \} \).

**Proof.** To show that \( B_L \) has constant rank, pick \( x, y \in L \). Assume there is \( \alpha \in \mathcal{B} \) such that the flow of \( \rho(\alpha) \) takes \( x \) to \( y \) (in general it is necessary to take a finite number of sections). Use that \([\alpha, \cdot]\) is a covariant differential operator on \( A \) preserving \( \mathcal{B} \), and whose induced Lie algebroid automorphism (the flow) takes \( A_x \) to \( A_y \).

Further \( B_L \) is a Lie subalgebroid of \( A \): Indeed, any two elements of \( \Gamma_c(B_L) \) can be extended to elements \( \alpha_1, \alpha_2 \in \mathcal{B} \), and \([\alpha_1, \alpha_2]_L \in \Gamma_c(B_L) \) since \( \mathcal{B} \) is involutive. The restriction of the bracket to \( L \) is independent of the choice of extensions, as a consequence of the Leibniz rule for the Lie algebroid \( A \). The Lie subalgebroid \( B_L \) over \( L \) is transitive by construction.

Now consider the ideal \( I_L = \{ f \in C^\infty(M) : f|_L = 0 \} \).

**Lemma 1.8.** Put
\[ B_L := \bigcup_{x \in L} B_x. \]
Then \( \mathcal{B}_L \) is a transitive Lie algebroid over \( L \). Its compactly supported sections are given by \( \mathcal{B}/I_L \mathcal{B} \).
Proof. We provide a proof along the lines of [3, Lemma 1.6]. The vector spaces $B_x$ have the same dimension for all $x \in L$ (use the same argument as in Lemma 1.7 above). Fix $x \in L$. Lifting the elements of a basis of $B_x$, we obtain generators $\alpha_1, \ldots, \alpha_N$ of $B$ in a neighborhood $U_x$ of $x$, by Lemma 1.5. Hence the $[\alpha_i] := (\alpha_i \mod I_L B)$ are a set of generators of the $C^\infty(L)$-module $B/I_L B$ on $V_x := U_x \cap L$. Further we observe that the $[\alpha_i]_y := (\alpha_i \mod I_y B)$ form a basis of $B_y$ for all $y \in V_y$.

We claim that the following holds on $V_x$, where the $g_i$ are smooth functions there:

$$\sum_i g_i[\alpha_i]_L = 0 \implies g_i = 0 \text{ for all } i.$$ 

To see this, extend the $g_i$ to functions $\hat{g}_i$ on $U_x \subset M$. Then $\sum_i \hat{g}_i \alpha_i \in I_L B$. For all $y \in V_x$ we have $I_L \subset I_y$, hence $\sum_i g_i(y)[\alpha_i]_y = 0 \in B_y$. The above observation implies that $g_i(y)$ for all $i$. Hence we conclude that $g_i = 0$, proving the claim.

Now we can cover $L$ by open subsets $V$ as above, and for each $V$ consider the trivial rank $N$ vector bundle over $V$ with canonical frame given by a choice of local generators $[\alpha^i]_L$ of $B/I_L B$ as above. Thanks to the claim, on overlaps $V \cap V'$, we can express each $[\alpha^i]_L$ uniquely in terms of the $[\alpha^i]_L, \ldots, [\alpha^N]_L$. This allows to glue the trivial vector bundles into a vector bundle over $L$, which becomes a Lie algebroid with the evaluation map as anchor and the Lie bracket on sections induced by the one on $B$.

The Lie algebroids introduced in Lemmas 1.7 and 1.8 fit in a short exact sequence of Lie algebroids over $L$, induced by the evaluation of sections on $L$:

$$\{0\} \rightarrow \bigcup_{x \in L} B_x \rightarrow B_L \xrightarrow{\text{ev}} B_L \rightarrow \{0\}. \quad (1.2)$$

Evaluating at a point $x \in L$, we obtain the short exact sequence of vector spaces $\{1.1\}$. 

Remark 1.9 (Unions of leaves). Notice that the above holds if we replace $L$ with any submanifold $N$ of $M$ such that $\mathcal{F}|_N \subset TN$ (i.e. $N$ is a union of leaves), provided $B/I_N B$ and $B|_N$ define constant rank bundles. Namely, we have a short exact sequence of $C^\infty(N)$ modules,

$$\{0\} \rightarrow B(N)/I_N B \rightarrow B/I_N B \xrightarrow{\text{ev}} B|_N \rightarrow \{0\}$$

(where $B(N) := \{\alpha \in B : \alpha|_N = 0\}$). It is induced by a short exact sequence of Lie algebroids over $N$, which generalizes $\{1.2\}$:

$$\{0\} \rightarrow \ker(\pi) \rightarrow B_N \xrightarrow{\text{ev}} B_N \rightarrow \{0\}.$$

1.4 The holonomy groupoid of a singular subalgebroid

Here we briefly recall the construction of the holonomy groupoid associated with a singular subalgebroid. The construction was carried out in [27, §2, §3] as an extension of the construction of Androulidakis-Skandalis for singular foliations [2].

Throughout this subsection we let $\mathcal{B}$ be a singular subalgebroid of an integrable Lie algebroid $A$, and fix a Lie groupoid $G$ integrating $A$. 

11
1.4.1 Bisubmersions

The main tool is the notion of submersion of a singular subalgebroid. Since an integrable Lie algebroid $A$ can be integrated by more than one Lie groupoid, we are forced to allow submersions to depend on the choice of Lie groupoid $G$ integrating $A$. Let us recall this notion, as well as some results that we’ll need in this sequel.

Let $U, V$ be manifolds and $\varphi : U \to V$ a smooth map. Let $E$ be a $C^\infty(V)$-submodule of $\mathfrak{X}_e(V)$. Denote $\varphi^1 TV$ the vector bundle on $U$ obtained as the the pullback of the tangent bundle. Put:

a) $\varphi^*(E)$ the $C^\infty(U)$-submodule of $\Gamma_c(U, \varphi^1 TV)$ generated by $f(\xi \circ \varphi)$ with $f \in C^\infty_c(U)$ and $\xi \in E$.

b) $\varphi^{-1}(E)$ the $C^\infty(U)$-submodule $\{X \in \mathfrak{X}_c(U) : d\varphi(X) \in \varphi^*(E)\}$ of $\mathfrak{X}_c(U)$.

Definition 1.10. [27, Prop. 2.4] A bisubmersion for $B$ is a triple $(U, \varphi, G)$ where $U$ is a manifold and $\varphi : U \to G$ a smooth map such that

i) $s_U := s \circ \varphi$ and $t_U := t \circ \varphi : U \to M$ are submersions,

ii) for every $\alpha \in B$, there is $Z \in \mathfrak{X}(U)$ which is $\varphi$-related to $\alpha$ and $W \in \mathfrak{X}(U)$ $\varphi$-related to $\alpha$,

iii) $\varphi^{-1}(B) = \Gamma_c(U, \ker ds_U)$ and $\varphi^{-1}(B) = \Gamma_c(U, \ker dt_U)$.

Now recall that, given any $\alpha \in B$ the vector field $\alpha$ on $G$ is complete. This allows us to show in the next proposition that bisubmersions exist:

Proposition-Definition 1.11. [27, Def. 2.16, Prop. 2.18] Let $x \in M$ and $\alpha_1, \ldots, \alpha_n \in B$ such that $[\alpha_1], \ldots, [\alpha_n]$ span $B/I_xB$. The associated path-holonomy bisubmersion is the map

$$\varphi : U_0 \to G, (\lambda, x) \mapsto \exp_x \sum \lambda_i \alpha_i,$$

where $U_0$ is a neighborhood of $(0, x)$ in $\mathbb{R}^n \times M$ in which the map $t \circ \varphi$ is a submersion, and $\exp$ denotes the time-1 flow.

We say that $(U_0, \varphi, G)$ is minimal if $[\alpha_1], \ldots, [\alpha_n]$ are a basis of $B/I_xB$.

Proposition-Definition 1.12. [27, Def. 2.21, 2.24, 2.27] Let $(U_i, \varphi_i, G)$, $i = 1, 2$ two bisubmersions for $B$. Put $s_i = s \circ \varphi$ and $t_i = t \circ \varphi$.

a) A morphism of bisubmersions is a smooth map $f : U_1 \to U_2$ such that $\varphi_2 \circ f = \varphi_1$.

b) The inverse of $(U_i, \varphi_i, G)$ is the bisubmersion $\iota \circ \varphi : U \to G$, where $\iota : G \to G$ is the inversion map.

c) The composition of $(U_1, \varphi_1, G)$ and $(U_2, \varphi_2, G)$ is the bisubmersion

$$m \circ (\varphi_1, \varphi_2) : U_1 \times_{s_1, t_2} U_2 \to G$$

where $m$ is the multiplication map of $G$. This bisubmersion is denoted $U_1 \circ U_2$.

We’ll use the next result in this sequel:
Lemma 1.13. ([27] Lemma 2.0) Let \((U, \varphi, G)\) a path-holonomy bisubmersion. The map \(\kappa : U \rightarrow U, \kappa(\lambda, x) = (-\lambda, t_U(\lambda, x))\) is a diffeomorphism and the following diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{\kappa} & U \\
\downarrow{\varphi} & & \downarrow{\kappa \circ \varphi} \\
G & & G
\end{array}
\] (1.3)

In particular, \(s_U \circ \kappa = t_U\) and \(t_U \circ \kappa = s_U\).

1.4.2 Bisections

We’ll also need the notion of bisection of a bisubmersion.

Definition 1.14. ([27] Def. 2.30) Let \((U, \varphi, G)\) be a bisubmersion for \(B\).

a) A bisection of \((U, \varphi, G)\) is a locally closed submanifold \(u\) of \(U\) such that the restrictions of both \(s_U\) and \(t_U\) to \(u\) are diffeomorphisms from \(u\) onto open subsets of \(M\). Then \(\varphi(u)\) is a bisection of \(G\) and the map \(\varphi : u \rightarrow \varphi(u)\) is a diffeomorphism.

b) Let \(u \in U\) and \(c\) a bisection of \(G\). We say that \(c\) is carried by \((U, \varphi, G)\) at \(u\) if there exists a bisection \(u\) of \(U\) such that \(u \in u\) and \(\varphi(u)\) is an open subset of \(c\).

Let \((U, \varphi, G)\) and \((U_i, \varphi_i, G)\), \(i = 1, 2\) be bisubmersions. The next properties were proven in ([27] §2.5) and ([27] §3.1):

a) Let \(u \in U\) and \(c\) a bisection of \(G\) carried by \((U, \varphi, G)\) at \(u\). Then \(c^{-1}\) is carried by the inverse bisubmersion \((U, \iota \circ \varphi, G)\) at \(u\).

b) Let \(u_i \in U_i\), \(i = 1, 2\) be such that \(s_{U_1}(u_1) = t_{U_2}(u_2)\) and let \(c_i\) be bisections of \(G\) carried by \((U_i, \varphi_i, G)\) at \(u_i\) respectively. Then \(c_1 \cdot c_2\) is carried by the composition \((U_1 \circ U_2, \varphi_1 \cdot \varphi_2, G)\) at \((u_1, u_2)\).

c) Let \((U_0, \varphi_0, G)\) be a minimal path-holonomy bisubmersion. Let \((U, \varphi, G)\) be any bisubmersion carrying the identity bisection at \(u \in U\). Then there exists an open neighborhood \(U'\) of \(u\) in \(U\) and a submersion \(g : U' \rightarrow U_0\) which is a morphism of bisubmersions.

Also, the following proposition is crucial for the construction of the holonomy groupoid. It is a corollary of item (c) above.

Proposition 1.15. ([27] Cor. 3.3) Let \((U_i, \varphi_i, G)\), \(i = 1, 2\) be bisubmersions of \(B\) and \(u_i \in U_i\) such that \(\varphi_1(u_1) = \varphi_2(u_2)\).

a) If the identity bisection \(1_M\) is carried by \(U_i\) at \(u_i\), for \(i = 1, 2\), there exists an open neighborhood \(U'_i\) of \(u_i\) in \(U_i\) and a morphism of bisubmersions \(f : U'_1 \rightarrow U_2\) such that \(f(u_1) = u_2\).

b) If there is a bisection of \(G\) carried by both \(U_1\) at \(u_1\) and by \(U_2\) at \(u_2\), there exists an open neighborhood \(U'_1\) of \(u_1\) in \(U_1\) and a morphism of bisubmersions \(f : U'_1 \rightarrow U_2\) such that \(f(u_1) = u_2\).

c) If there is a morphism of bisubmersions \(g : U_1 \rightarrow U_2\) such that \(g(u_1) = u_2\), then there exists an open neighborhood \(U'_2\) of \(u_2\) and a morphism of bisubmersions \(f : U'_2 \rightarrow U_1\) such that \(f(u_2) = u_1\).
1.4.3 Construction of the holonomy groupoid

Analogously to the case of singular foliations [2], the holonomy groupoid of a singular subalgebroid $B$ is a quotient. We recall briefly its construction.

**Definition 1.16.** [27 Def. 3.4, Def. C.4]

a) Consider a family $(U_i, \varphi_i, G)_{i \in I}$ of minimal path-holonomy bisubmersions defined as in Def. 1.11 such that $M = \cup_{i \in I} s_i(U_i)$. Let $U^G$ be the collection of all such bisubmersions, together with their inverses and finite compositions. We call $U^G$ a **path-holonomy atlas** of $B$ associated with $G$.

b) Let $(U, \varphi, G)$ a bisubmersion and $u \in U$. We say that $(U, \varphi, G)$ is adapted to $U^G$ at $u$ if there exists $i \in I$, an open subset $U' \subseteq U$ containing $u$ and a morphism of bisubmersions $U' \to U_i$.

c) We say that $(U, \varphi, G)$ is adapted to $U^G$ if every element $u \in U$ is adapted to $U^G$.

Proposition 1.15 c) shows that for $u_1 \in (U_1, \varphi_2, G), u_2 \in (U_2, \varphi_2, G)$ the relation

$$u_1 \sim u_2 \Leftrightarrow \text{there is an open neighborhood } U'_i \text{ of } u_1, \text{there is a morphism of bisubmersions } f : U'_i \to U_2 \text{ such that } f(u_1) = u_2$$

is an equivalence relation. This allows us to give the following definition:

**Definition 1.17.** [27 Def. 3.5] Let $B$ a singular subalgebroid of $A = AG$ and $U^G$ the associated path-holonomy atlas. The **holonomy groupoid of $B$ over $G$** is

$$H^G(B) := \coprod_{U \in U^G} U / \sim$$

The holonomy groupoid $H^G(B)$ has the following structure. Denote by $\natural : \coprod_{U \in U^G} U \to H^G(B)$ the quotient map, and $q_i := \natural|_{U_i}$.

a) Endowed with the quotient topology, $H^G(B)$ carries a topological groupoid structure with space of objects $M$. The source and target maps $s_H, t_H : H^G(B) \to M$ are determined by $s_H \circ q_i = s_i$ and $t_H \circ q_i = t_i$ for all $i \in I$. The multiplication is determined using the composition of bisubmersions, as follows: $q_i(u)q_j(v) = q_{i \circ j}(u, v)$ for all $i, j \in I$.

b) There is a canonical morphism of topological groupoids

$$\Phi : H^G(B) \to G, \quad \Phi([u]) = \varphi(u),$$

where $u$ is any point in a bisubmersion $(U, \varphi, G)$ belonging to the path-holonomy atlas $U^G$.

c) For every bisubmersion $(U, \varphi, G)$ adapted to $U^G$ there is a map $g_U : U \to H^G(B)$ such that, for every local morphism of bisubmersions $f : U' \subseteq U \to U_i$ and every $u \in U'$ we have $g_U(u) = q_i(f(u))$.

**Remark 1.18.** The equivalence relation introduced after Def. 1.16 can be also expressed as follows [27 Remark 3.6]:

$$u_1 \sim u_2 \Leftrightarrow \varphi_1(u_1) = \varphi_2(u_2), \exists \text{ bisections } u_i \text{ through } u_i \text{ s.t. } \varphi_1(u_1) = \varphi_2(u_2).$$
Usually the topology of the holonomy groupoid is quite bad, however in §2.3 we will show that it always is longitudinally smooth. In §5 we will show that the holonomy groupoid is also a diffeological groupoid. Here we show the following property of the topology of the holonomy groupoid.

**Proposition 1.19.** The quotient map \( \pi : \bigsqcup_{U \in \text{Int}^G} U \to H^G(B) \) an open map.

**Proof.** Fix an open subset \( A \subseteq \bigsqcup_{U \in \text{Int}^G} U \). We need to prove that the saturation \( \pi^{-1}(\pi(A)) \subseteq \bigsqcup_{U \in \text{Int}^G} U \) is open as well. Recall that

\[
\pi^{-1}(\pi(A)) = \{ u \in \bigsqcup_{U \in \text{Int}^G} U : \pi(u) = \pi(a) \text{ for some } a \in A \}
\]

Pick an element \( u \in \pi^{-1}(\pi(A)) \).

We need to show that there is an open neighborhood \( B \) of \( u \) in \( \bigsqcup_{U \in \text{Int}^G} U \) such that \( B \subseteq \pi^{-1}(\pi(A)) \). Pick \( a \in A \) such that \( \pi(u) = \pi(a) \) and assume that \( u, a \) belong to path-holonomy bisubmersions \( U_u, U_a \) respectively. Since \( \pi(u) = \pi(a) \), by definition of the equivalence relation in the path-holonomy atlas, there is an open subset \( \bar{U}_u \subseteq U_u \) containing the element \( u \), and a morphism of bisubmersions \( f : \bar{U}_u \to U_a \) such that \( f(u) = a \). Since \( A \) is open, we can arrange that \( f(\bar{U}_u) \subset A \) by shrinking the domain if necessary. Then, every other element \( u' \in \bar{U}_u \) satisfies \( \pi(u') = \pi(f(u')) \), and since \( f(u') \in A \) it follows that \( u' \in \pi^{-1}(\pi(A)) \). So the open subset \( \bar{U}_u \) of \( \bigsqcup_{U \in \text{Int}^G} U \) is contained in \( \pi^{-1}(\pi(A)) \). \( \square \)

### 1.5 Examples of holonomy groupoids

Here we put a few constructions from [27] that we’ll use in this sequel, and which provide the holonomy groupoids for the singular subalgebroids of Ex. 1.3. Again, we fix a source-connected Lie groupoid \( G \) integrating a Lie algebroid \( A \) (so \( A = AG \)). As a preparation, we need the following result.

**Proposition 1.20.** [27, Prop 2.13] Let \( \phi : K \to G \) be a morphism of (source-connected) Lie groupoids covering the identity on \( M \). Then \( \phi : K \to G \) is a bisubmersion for the singular subalgebroid \( B := \phi_*(\Gamma_c(AK)) \) of \( AG \).

Notice that Prop. 1.20 applies in particular to the singular subalgebroids that arises from Lie algebroid morphism \( \psi : E \to A \) (Ex. 1.3), provided the Lie algebroid \( E \) integrable: just take \( \phi \) to be a Lie groupoid morphism integrating \( \psi \).

The following proposition provides the holonomy groupoids for a class of singular subalgebroids that includes those appearing in Prop. 1.20 just above.

**Proposition 1.21** (Holonomy groupoids as quotients). [27, Prop 3.12] Let \( G \) be a Lie groupoid over \( M \) and \( B \) a singular subalgebroid of \( AG \). Let \( K \) be a Lie groupoid over \( M \). Let \( \phi : K \to G \) be a morphism of Lie groupoids covering \( \text{Id}_M \) which is also a bisubmersion for \( B \). Then:

i) \( H^G(B) = K/I \) as topological groupoids, where

\[
I := \{ k \in K : \exists \text{ a (local) bisection } u \text{ through } k \text{ such that } \phi(u) \subset 1_M \}
\]
ii) the canonical map $\Phi: H^G(B) \to G$ coincides with the map $K/I \to G$ induced by $\phi$.

A special case of the above proposition is the following. Let $B$ a wide Lie subalgebroid of $A$, and $B := \Gamma_c(B)$. Then $H^G(B)$ is the minimal integral of $B$ over $G$ in the sense of \cite[Thm. 2.3]{22}. This is shown in \cite[Prop 3.20]{27}, and we will recover this result in Ex \ref{ex:example}.

Example 1.22 (Lie subalgebras). Let $\mathfrak{g}$ a Lie algebra, $\mathfrak{k}$ a Lie subalgebra, and fix a connected Lie group $G$ integrating $\mathfrak{g}$. Let $\phi: K \to G$ be any morphism of Lie groups integrating the inclusion $\iota: \mathfrak{k} \hookrightarrow \mathfrak{g}$, where $K$ is assumed to be connected. (For instance, take $K$ to be the connected and simply connected integration of $\mathfrak{k}$.) Then

$$H(\mathfrak{k}) = K/\text{ker}(\phi).$$

Indeed, since the space of objects of $K$ is just a point, $k_1 \sim k_2$ iff $\phi(k_1) = \phi(k_2)$. Hence $H(\mathfrak{k})$ is a Lie group integrating $\mathfrak{k}$, and the map $\Phi: H(\mathfrak{k}) \to G$ induced by $\phi$ is an injective immersion and group homomorphism. In other words, $(H(\mathfrak{k}), \Phi)$ is the Lie subgroup of $G$ integrating $\mathfrak{k}$ (see \cite[pp. 92-93]{25}).

Remarks 1.23. a) Local bisections such as the ones mentioned in the definition of the equivalence relation in Prop. \ref{prop:local_bisections} can be obtained in the following computable way: Put $\mathcal{J}$ the kernel of the $C^\infty(M)$-linear map $d\phi: \Gamma_c(\mathcal{A}K) \to \mathcal{B}$. It is an involutive $C^\infty(M)$-submodule of $\Gamma_c(\mathcal{A}K)$, whence it corresponds to an involutive $C^\infty(M)$-submodule $\mathcal{J}$ of right-invariant vector fields of $K$. Their associated 1-parameter groups provide bisections which lie in the isotropy of $K$.

b) In the setting of Prop. \ref{prop:local_bisections}, we also note the following application of a classical algebraic result. Suppose $\mathcal{B}$ is locally finitely presented as a $C^\infty(M)$-module, that is to say for every open $W \subset M$ the $C^\infty(W)$-module $\mathcal{B}_W$ is a finitely presented module. Consider the $C^\infty(W)$-module $(\Gamma_c(\mathcal{A}K))_W$ and the exact sequence $0 \to \mathcal{J}_W \to (\Gamma_c(\mathcal{A}K))_W \xrightarrow{d\phi} \mathcal{B}_W \to 0$. Since $(\Gamma_c(\mathcal{A}K))_W$ is a free finite-rank module and $\mathcal{B}_W$ is finitely presented, a classical result from algebra implies that $\mathcal{J}_W$ is a finitely generated module. Whence $\mathcal{J}$ is a singular subalgebroid of $\mathcal{A}K$.

Notice that in general its holonomy groupoid $H^K(\mathcal{J})$ does not agree with the groupoid (actually a bundle of groups) $\mathcal{I}$ appearing in Prop. \ref{prop:local_bisections} not even up to covers. Indeed, as for any holonomy groupoid, the dimension of the source fibers of $H^K(\mathcal{J})$ is upper semicontinuous, while the source fibers of $\mathcal{I}$ behave in the opposite way.

2 Leafwise Integration

In this section we begin our study of the integration of a singular subalgebroid $\mathcal{B}$, by examining its restriction to a leaf $L$ of the associated foliation $(M, \mathcal{F}_B)$. Our conclusion is that this restriction is always integrable. More precisely: the restriction of the holonomy groupoid $H^G(\mathcal{B})$ to $L$ is always a Lie groupoid, and it integrates the transitive Lie algebroid $\mathcal{B}_L$ introduced in Lemma \ref{lem:transitive_algebroid}. Further, the restriction of the canonical morphism $\Phi: H^G(\mathcal{B}) \to G$ to $L$ integrates the evaluation map $\mathcal{B}_L \to A$. See Cor. \ref{cor:holonomy_integration} Prop. \ref{prop:transitive_integration} and Thm. \ref{thm:holonomy_groupoid}.

To this end, the crucial ingredient is Theorem \ref{thm:holonomy_equivalence} in §2.1 which is a result of independent interest. It provides the exact relation between the holonomy groupoid $H^G(\mathcal{B}) \to M$ of the singular subalgebroid
and holonomy groupoid \( H(\widetilde{\mathcal{B}}) \rightrightarrows G \) of the associated singular foliation on \( G \). (In particular, the former is a quotient of the latter.) This theorem allows us in an explicit manner to reduce the statements we are after to the analogous statements for singular foliations, which hold by Debord’s work \[9\].

### 2.1 An alternative construction of the Holonomy Groupoid

In this subsection we provide a different construction for the holonomy groupoid we introduced in \[\S 1.4\]. Let \( \mathcal{B} \) be a singular subalgebroid of a Lie algebroid \( A \) and \( (G, t, s) \) a Lie groupoid integrating the latter. The associated \( C^\infty_c(G) \)-module of right-invariant vector fields \( \widetilde{\mathcal{B}} \) is a (singular) foliation of the manifold \( G \). Let \( H(\widetilde{\mathcal{B}}) \) be its holonomy groupoid, as constructed in \[2\].

**Theorem 2.1.** The topological groupoid \( H(\widetilde{\mathcal{B}}) \rightrightarrows G \) is canonically isomorphic to the transformation groupoid

\[ H^G(\mathcal{B}) \times_{s_H, t} G \]

of the action of \( H^G(\mathcal{B}) \) on the map \( t : G \to M \), induced by the canonical groupoid morphism \( \Phi : H^G(\mathcal{B}) \to G \).

In order not to interrupt the flow of ideas here, we give the proof of Theorem 2.1 in Appendix A. Here we just mention that the canonical isomorphism appearing in Thm. 2.1 is given in eq. (A.2).

**Remark 2.2.** More precisely, the action appearing in Thm. 2.1 is \( (h, g) \mapsto \Phi(h)g \). Hence this is also the target map of the transformation groupoid, whereas the source map is \( (h, g) \mapsto g \) and the multiplication is given by \( (h, g)(h', g') = (hh', g') \).

Using the above isomorphism we can state the following corollaries.

**Corollary 2.3.** There is a canonical principal right action\(^3\) of \( G \) on \( H(\widetilde{\mathcal{B}}) \rightrightarrows G \) and the quotient by this action is a topological groupoid canonically isomorphic to \( H^G(\mathcal{B}) \rightrightarrows M \), i.e.

\[ H(\widetilde{\mathcal{B}})/G \cong H^G(\mathcal{B}). \]

**Proof.** Thanks to the isomorphism of Thm. 2.1 it is sufficient to prove the statement for \( H^G(\mathcal{B}) \times_{s_H, t} G \) in place of \( H(\widetilde{\mathcal{B}}) \). There is a canonical \( G \)-action on \( s : G \to M \) by right multiplication, and similarly a canonical \( G \)-action on the composition of the source map of \( H^G(\mathcal{B}) \times_{s_H, t} G \) with \( s \). One sees easily that the projection to the quotient is given by

\[
\begin{array}{c}
H^G(\mathcal{B}) \times_{s_H, t} G \\
\downarrow \\
G \\
\downarrow \\
M
\end{array} \xrightarrow{s_H, t} \begin{array}{c}
H^G(\mathcal{B}) \\
\downarrow \\
\end{array}
\]

(2.1)

where the upper horizontal map is \( (h, g) \mapsto h \).

\[^3\] More precisely: there are canonical right actions of \( G \rightrightarrows M \) on the maps \( s \circ s_{H(\widetilde{\mathcal{B}})} : H(\widetilde{\mathcal{B}}) \to M \) and on \( s : G \to M \).
Example 2.4 (Lie subalgebroids). Let $B \to M$ be a wide Lie subalgebroid of $A \to M$. With $B = \Gamma_c(B)$, the foliation $\mathcal{B}$ on the Lie groupoid $G$ is regular, hence the holonomy groupoid $H(\mathcal{B})$ is given by classes of paths in its leaves modulo holonomy. By Cor. 2.3 $H^G(\mathcal{B})$ is isomorphic to $H(\mathcal{B})/G$. The latter is called the minimal integral of $\mathcal{B}$ over $G$, and is the “smallest” Lie groupoid admitting an immersion to $G$ integrating the inclusion of $B \to A$. [22, Thm. 2.3]. In particular, when $B = \Gamma_c(A)$, we have $H^G(\mathcal{B}) \cong G$.

Corollary 2.5. Put $\Phi_x$ the restriction of $\Phi : H^G(\mathcal{B}) \to G$ to the $s_H$-fiber $H^G(\mathcal{B})_x$. Then $\Phi_x$ is injective if and only if the foliation $\mathcal{B}$ has trivial holonomy at $1_x \in G$.

Proof. Saying that the foliation $\mathcal{B}$ has trivial holonomy at $1_x$ is saying that the isotropy group of $H(\mathcal{B})$ at $1_x$ is trivial. Under the isomorphism (covering $Id_G$) of Thm. 2.1, the target-source map $\Psi : H^G(\mathcal{B}) \times_{s_H,G} G \to G \times G, (h, g) \mapsto (\Phi(h)g, g)$. The isotropy group is thus $\ker \Psi_x = \ker \Phi_x \times \{1_x\}$ and the result follows.

2.2 The restriction of the holonomy groupoid to a leaf

In this subsection we describe the restriction of $H^G(\mathcal{B})$ to leaves using Theorem 2.1. First, let us clarify which leaves we are referring to.

2.2.1 The relation between leaves of $(G, \mathcal{B})$ and leaves of $(M, \mathcal{F}_B)$

Given $g \in G$ put $x = s(g), y = t(g) \in M$. Let $\vec{L}_g \subset G$ be the leaf of the foliation $\mathcal{B}$ through $g$. Let $L_y \subset M$ be the leaf of $\mathcal{B}$ through $y$.

From Theorem 2.1 we get

$$\vec{L}_g = \{ \Phi(h)g : h \in H^G(\mathcal{B})_y \}$$

so the leaf $\vec{L}_g$ is the right-translation by $g$ of the leaf $L_y := \vec{L}_1$. Therefore, it suffices to consider only leaves $\vec{L}_x$ and $L_x$ for $x \in M$. We observe the following:

Observation 2.6. The restriction $\mid_{\vec{L}_x} : \vec{L}_x \to L_x$ is a surjective submersion, since for all $g' \in \vec{L}_x$ and $\alpha \in B$ we have $t_* (\vec{\alpha}_{|g'}) = t_* (\vec{\alpha}_{|t(g')}) = \rho(\alpha)_{t(g')}$. More generally, $\mid_{\vec{L}_g} : \vec{L}_g \to L_g$ is a surjective submersion for any $g \in G$ such that $y = t(g)$.

2.2.2 Description of $H^G(\mathcal{B})_{L_y}$

Let $g \in G$, put $x = s(g), y = t(g)$. Theorem 2.1 allows us to describe the groupoid $H^G(\mathcal{B})_{L_y}$ in this way:

Observations. a) Under the isomorphism of Thm. 2.1, we have

$$H(\mathcal{B})_{\vec{L}_y} \cong H^G(\mathcal{B})_{L_y} \times_{s_H,G} \vec{L}_g$$

That is because the right-hand side of (2.2) is the preimage of $\vec{L}_g$ under the source map of $H^G(\mathcal{B}) \times_{s_H,G} G$. 

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b) Thus $H^G(B)_{L_y} \times s_{H,t} L_g$ is the transformation groupoid associated with the action of the groupoid $H^G(B)_{L_y} \rightarrow L_y$ on the fibration $t : L_g \rightarrow L_y$, defined by the restriction of the map $\Phi$ to $H^G(B)_{L_y}$.

We recall that Debord [9, Prop. 2.2] proved that the holonomy groupoid of any singular foliation is longitudinally smooth, i.e. its restriction to a leaf is a Lie groupoid (see also [3, Thm. 4.16]). This applies in particular to the right hand side of eq. (2.2).

2.3 Smoothness of the holonomy groupoid over a leaf

We show that restricting the holonomy groupoid to leaves yields a Lie groupoid. Let $\mathcal{U}_G$ be a path-holonomy atlas associated with $B$ and $L_x$ the leaf of $\mathcal{F}_B$ through $x \in M$.

**Proposition 2.8.** There is a smooth manifold structure on $H^G(B)_{L_x} := s^{-1}_H(L_x)$ such that the quotient maps $q_U|_{L_x} : U_{L_x} \rightarrow H^G(B)_{L_x}$ are submersions for every $(U, \varphi, G) \in \mathcal{U}_G$, where $U_{L_x} := s^{-1}_U(L_x)$.

**Remark 2.9.**

a) The above smooth structure is unique (this follows from general properties of submersions).

b) If the leaf $L$ is immersed, the topology on $H^G(B)_{L_x}$ is the one obtained from the natural bijection with the fibered product $L \times_{(\iota, s)} H^G(B)$, where $\iota : L \rightarrow M$ is the inclusion map.

**Proof.** Every $(U, \varphi, G) \in \mathcal{U}_G$ gives rise to a path-holonomy bisubmersion $\vec{U} = U \times s_{U,t} G$ for $\vec{B}$ by [27, Prop. B.1], with source map $\vec{s}$ the second projection. Put $\vec{U}_{L_x} = \vec{s}^{-1}(L_x)$, thus $\vec{U}_{L_x} = U_{L_x} \times s_{U,t} L_x$. Recall that $H^G(B)_{L_x} \cong H^G(B)_{L_x} \times s_{U,t} L_x$ by eq. (2.2), using the canonical isomorphism of Thm. 2.1.

Thanks to a result by Debord [9, Prop. 2.2] for singular foliations we know that $H^G(B)_{L_x}$ is smooth, meaning [3, Def. 3.8] that it has a differentiable structure such that the quotient map

$$q_U|_{L_x} : \vec{U}_{L_x} \rightarrow H^G(B)_{L_x}$$

is a submersion for every $U \in \mathcal{U}_G$. Under the canonical isomorphism of Thm. 2.1 this submersion becomes (see eq. (A.2)) the top horizontal map in the commutative diagram

$$\begin{array}{ccc}
U_{L_x} \times s_{U,t} L_x & \xrightarrow{q_U \times \text{id}} & H^G(B)_{L_x} \times s_{H,t} L_x \\
\downarrow \pi & & \downarrow p \\
U_{L_x} & \xrightarrow{q_U} & H^G(B)_{L_x}
\end{array} \quad (2.3)
$$

Here $\pi$ and $p$ denote the projection onto the first factor.

We have to show that there is a smooth structure on $H^G(B)_{L_x}$ for which the bottom horizontal map in diagram (2.3) is a submersion, for every $U \in \mathcal{U}_G$. We know that the image of $q_U(U_{L_x})$ is an open
subset, by Prop. [1.19]. Let σ be a local section of the target map \( t : \overrightarrow{L}_x \rightarrow L_x \), which for simplicity we assume to be defined on \( s_{U}(U_{L_x}) \). Then

\[
T := \{(h, \sigma(s_H(h))) : h \in q_U(U_{L_x})\}
\]

is a smooth submanifold of \( H^G(B)_{L_x} \times_{s_U, t} \overrightarrow{L}_x \), since its preimage by the submersion is a smooth submanifold of \( U_{L_x} \times_{s_U, t} \overrightarrow{L}_x \), namely \( S := \{(u, \sigma(s_U(u))) : u \in U_{L_x}\} \).

The restriction of the projection \( p \) to \( T \) is a homeomorphism onto its image, hence can be used to transport the smooth structure of \( T \) to the open subset \( q_U(U_{L_x}) \) of \( H^G(B)_{L_x} \).

With this smooth structure, \( p \circ (q_U \times \text{id})|_S = q_U \circ (\pi|_S) \) is a submersion, and since \( p|_S \) is a diffeomorphism it follows that \( q_U|_{L_x} : U_{L_x} \rightarrow q_U(U_{L_x}) \subset H^G(B)_{L_x} \) is a submersion. Repeating the procedure with another submersion \( V \in \mathcal{U}^G \) such that \( q_U(U_{L_x}) \cap q_V(V_{L_x}) \) is non-empty defines the same smooth structure on the intersection: if \( q_U(u) = q_V(v) \), there exists a morphism of submersions \( f : U \rightarrow V \) mapping \( u \) to \( v \) (shrinking \( U \) if necessary), hence \( q_U = q_V \circ f \) is a smooth map for the smooth structure defined by \( V \) too.

\[ \square \]

Remarks 2.10. For a path-holonomy bisubmersion \( U \) of \( B \) which is minimal at \( x \), the quotient map \( q_U|_{L_x} : U_{L_x} \rightarrow H^G(B)_{L_x} \) is actually a diffeomorphism in a neighborhood of \((x, 0)\). Indeed, \( q_U|_{L_x} : \overrightarrow{U}_{L_x} \rightarrow H(B)_{L_x} \) is a diffeomorphism in a neighborhood of \((x, 0)\), as follows from [3] Prop. 2.2] and the proof of [3] Thm. 4.16]. Therefore the proof of Prop. [2.8] leads to the above claim.

It follows by the same arguments as in the proof of [3] Lemma 3.9] that:

Corollary 2.11. The restriction \( H^G(B)_{L_x} \) is a Lie groupoid, when endowed with the smooth structure of Prop. [2.8]

Thanks to [11] Lemma 7.1.4], Corollary 2.11 implies:

Corollary 2.12. Every \( s_{H} \)-fiber \( H^G(B)_{L_x} \) is a smooth Hausdorff manifold. It is a closed submanifold of \( H^G(B)_{L_x} \).

### 2.4 Integration of \( B_{L_x} \)

Here we discuss the integration of the Lie algebroid \( B_{L_x} \) defined in Lemma 1.8. Let us first make the following observations.

Observations 2.13. a) There is an infinitesimal action of the Lie algebroid \( B_{L_x} \) on the map \( \overrightarrow{t} : \overrightarrow{L}_x \rightarrow L_x \), given by

\[
\Gamma(B_{L_x}) \rightarrow \chi(\overrightarrow{L}_x) \quad (\alpha \mod I_{L_x} \mathcal{B}) \mapsto \overrightarrow{\alpha}|_{\overrightarrow{L}_x}.
\]

b) The associated transformation algebroid is denoted by \( t^* B_{L_x} \), since as a vector bundle it is the pullback of the vector bundle \( B_{L_x} \) along \( t \). The bracket on basic sections is given by the bracket of \( B_{L_x} \), and the anchor by the infinitesimal action.

c) We have an isomorphism of Lie algebroids \( t^* B_{L_x} \cong \overrightarrow{B}_{L_x} \). For all \( \alpha \in \mathcal{B} \), it sends the basic section \( (\alpha \mod I_{L_x} \mathcal{B}) \) to \( (\overrightarrow{\alpha} \mod I_{\overrightarrow{L}_x} \overrightarrow{\mathcal{B}}) \).
d) The latter is the Lie algebroid of the Lie groupoid \( H(\mathcal{B})_{L_x} \) by Debord’s result [9, Cor. 2.2] (see also [3, Thm. 5.1]).

**Proposition 2.14.** The Lie groupoid \( H^G(\mathcal{B})_{L_x} \) integrates \( \mathcal{B}_{L_x} \).

**Proof.** Let \( Z_{L_x} \) be the Lie algebroid of \( H^G(\mathcal{B})_{L_x} \). Then the Lie algebroid of the transformation Lie groupoid \( H^G(\mathcal{B})_{L_x} \times_{s_{H^G} L_x} \mathcal{B}_{L_x} \) is the transformation Lie algebroid of the infinitesimal action of \( Z_{L_x} \) on the map \( t : L_x \to L_x \) obtained differentiating the action of \( H^G(\mathcal{B})_{L_x} \). We denote this transformation Lie algebroid by \( \mathfrak{t}^* Z_{L_x} \). Composing we obtain an isomorphism over \( \text{Id}_{L_x} \) of Lie algebroids

\[
\mathfrak{t}^* B_{L_x} \cong \mathcal{B}_{L_x} \cong A \left( H(\mathcal{B})_{L_x} \right) \cong A \left( H^G(\mathcal{B})_{L_x} \times_{s_{H^G} L_x} \mathcal{B}_{L_x} \right) = \mathfrak{t}^* Z_{L_x},
\]

where the third isomorphism is induced by the isomorphism of Lie groupoids given by eq. (2.2). We denote this composition by \( \Theta \).

**Claim:** \( \Theta \) maps basic sections of \( \mathfrak{t}^* B_{L_x} \) to basic sections of \( \mathfrak{t}^* Z_{L_x} \).

Hence \( \Theta \) induces an isomorphism of Lie algebroids \( \theta : \mathcal{B}_{L_x} \to Z_{L_x} \) over \( \text{Id}_{L_x} \), implying the desired result.

We are left with proving the claim. To do this, take local generators \( \alpha_1, \ldots, \alpha_n \) of \( \mathcal{B} \), giving rise to a path-holonomy bisubmersion \( \mathcal{U} \) for \( \mathcal{B} \), and to a path-holonomy bisubmersion \( \mathcal{U} \to \mathbb{R}^n \to G \) for \( \mathcal{B} \).

- The first isomorphism in eq. (2.4) was described in Observations 2.13 c).

- The second isomorphism in eq. (2.4) is done so that it commutes with the map \([\mathcal{T}(U)_{L_x}] \to \mathcal{B}_{L_x} \), which sends the \( i \)-th canonical vertical basis vector \( e_i \) to \( (\alpha_i \mod L_x) \mathcal{B} \), and with the derivative of the quotient map \( U_{L_x} \to H(\mathcal{B})_{L_x} \). (See the proof of [3, Thm. 4.1]). Notice that both maps are surjective over the open subset of \( L_x \) where the generators are defined.

- The latter map and the derivative of \( q \times \text{id} : U_{L_x} \to H^G(\mathcal{B})_{L_x} \times_{s_{H^G} L_x} \mathcal{B}_{L_x} \), which sends the \( i \)-th canonical vertical basis vector \( e_i \) to a basic section of \( \mathfrak{t}^* Z_{L_x} \), commute with the third isomorphism in eq. (2.4). (See the proof of Prop. 2.8).

This proves the claim. \( \square \)

### 2.5 Integrating the evaluation map over a leaf

Here we discuss the integration of the morphism of Lie algebroids \( \mathcal{B}_{L_x} \to A \) defined by the evaluation map (see the short exact sequence (1.2)). To this end, we will use the following, to reduce the problem to the case of singular foliations.

1. Put \( L_x \) be the leaf of the foliation \( \mathcal{B} \) at \( 1_x \). Recall that \( t : L_x \to L_x \) is a submersion. We denote \( \iota : L_x \to G \) and \( \iota : L_x \to M \) the associated immersions (inclusion maps).

---

4 We consider these maps only at points of \( L_x \).
b) Recall that $\tilde{H} (B)_{L_x} -$ the restriction to $\tilde{L}_x$ of the holonomy groupoid of $\tilde{B}$ – integrates the transitive Lie algebroid $\tilde{B}_{L_x},$ see Observations 2.13 d). Since $\tilde{B}$ consists of $s$-vertical vector fields on $G,$ the map $(\tilde{t}_{H}^{s}, \tilde{s}_{H}): \tilde{H} (B) \to G \times G$ takes values in the subgroupoid $G \times_s G \to G.$ Whence, we obtain the following morphism of Lie groupoids:

$$\begin{array}{ccc}
\tilde{H} (B)_{L_x} & \xrightarrow{(\tilde{t}_{H}^{s}, \tilde{s}_{H})} & G \times_s G \\
\downarrow & \downarrow & \downarrow \\
\tilde{L}_x & \xrightarrow{\tilde{t}} & G \\
\end{array}$$

(2.5)

c) The image of this morphism is the pair groupoid $\tilde{L}_x \times \tilde{L}_x.$ On the other hand, the anchor map of the Lie algebroid $\tilde{B}_{L_x}$ over $\tilde{L}_x$ is just the evaluation map $\tilde{ev}: \tilde{B}_{L_x} \to T \tilde{L}_x.$ In fact, $\tilde{ev}$ is a morphism of Lie algebroids

$$\begin{array}{ccc}
\tilde{B} & \xrightarrow{\tilde{ev}} & \ker (ds) \\
\downarrow & \downarrow & \downarrow \\
\tilde{L}_x & \xrightarrow{\tilde{t}} & G \\
\end{array}$$

(2.6)

d) Diagram (2.5) clearly differentiates to diagram (2.6).

e) The Lie algebroid $\tilde{B}_{L_x}$ is isomorphic the action Lie algebroid $t^*(B_{L_x}),$ see Observations 2.13. Further it is well known that $\ker (ds) \to G$ is isomorphic to $t^*(A) = A \times_{pr_M,t} G.$ With this, diagram (2.6) becomes:

$$\begin{array}{ccc}
t^* B_{L_x} & \xrightarrow{ev \times t} & t^* A \\
\downarrow & \downarrow & \downarrow \\
L_x & \xrightarrow{\tilde{t}} & G \\
\end{array}$$

(2.7)

**Theorem 2.15.** Given $x \in M,$ let $L_x$ be the leaf of $\mathcal{F}_B$ through $x.$ Put $\iota: L_x \to M$ the natural immersion. Then the map $\Phi_{L_x}: H^G (B)_{L_x} \to G$ over $\iota$ obtained restricting $\Phi: H^G (B) \to G$ to $L_x$ integrates the evaluation map $ev: B_{L_x} \to A.$

**Proof.** Since the construction in Prop. 2.14 is functorial, we can quotient the immersion $\tilde{\iota}$ out of diagram (2.7) – i.e. pull back by the inclusion $L_x \to M$ – to obtain the evaluation map $ev: B_{L_x} \to A$ (which is a morphism of Lie algebroids over $\iota: L_x \to M$).

On the other hand, we know that $\tilde{H} (B)_{L_x}$ is diffeomorphic to $H^G (B)_{L_x} \times_{s_H,t} L_x$ by Observations 2.14 a). It is a long and tedious task to check that applying this diffeomorphism, the groupoid morphism
in diagram (2.5) becomes

\[
H^G_B L_x \times_{s_H, A} L_x \xrightarrow{\Phi_{L_x} \times _t} G \times_s G
\]

We conclude by pulling back by the inclusion \(L_x \to M\) again. \(\square\)

The next corollary follows from theorem 2.15 and corollary 2.5

**Corollary 2.16.** Given \(x \in M\), let \(L\) be the leaf of \(\mathcal{F}_B\) through \(x\). Then the following are equivalent:

a) The holonomy group \(H^\rightarrow_{\mathcal{F}_B} \mid_L^x\) is trivial.

b) The evaluation map \(\mathcal{F}_L \xrightarrow{ev} A\) is integrated by an injective morphism \(\Phi_{L_x} : H^G_B(L_x) \to G\).

**Example 2.17.** Let \(\mathcal{F}\) be a singular foliation of a manifold \(M\) (in the sense of \([2]\)). We can view \(\mathcal{F}\) as a singular subalgebroid of \(TM\), which is the Lie algebroid of the pair groupoid \(M \times M\). Recall that the \(s\)-simply connected cover of the pair groupoid is the fundamental groupoid \(\Pi(M)\). For every leaf \(L\) of \(\mathcal{F}\), we have:

- \(\mathcal{F}_L\) can be integrated by an injectively immersed groupoid \(H_L \to L \times L\) iff \(\mathcal{F}\) has trivial holonomy at \(L\).

- \(\mathcal{F}_L\) can be integrated by an injectively immersed groupoid \(H'_L \to \Pi(L)\) iff the pull-back of \(\mathcal{F}\) to the universal cover \(\tilde{M} \to M\) has trivial holonomy at \(\tilde{L}\). Here \(\tilde{L}\) is the leaf of the pull-back foliation at a point \(y \in \tilde{M}\) whose image under the covering map lies in \(L\).

## 3 Smooth holonomy groupoids

The main result of this section is that the holonomy groupoid of a singular subalgebroid \(B\) is a Lie groupoid if and only if \(B\) is a projective module, see Prop. 3.3.

### 3.1 Projective singular subalgebroids

Here we review projective singular subalgebroids. Recall first that a morphism of vector bundles \(E \to F\) over \(Id_M\) is called *almost injective* if it is fiber-wise injective on an open dense subset of \(M\), or equivalently, if the induced map at the level of sections is injective.

**Definition 3.1.** ([27], Ex 1.5) Let \(A \to M\) be a Lie algebroid. A singular subalgebroid \(B\) of \(A\) is *projective* if \(B\) is a locally projective \(C^\infty(M)\)-module. An equivalent characterization, due to the Serre-Swan theorem, is the existence of a vector bundle \(B \to M\) such that \(B \cong \Gamma_c(B)\) as \(C^\infty(M)\)-modules. We define \(\text{rank}(B)\) to be the rank of the vector bundle \(B\).

Let us point out the following easy facts. Item c) shows in particular that the vector bundle \(B\) is unique up to isomorphism, thus \(\text{rank}(B)\) is well defined.

---

5This means that every \(p \in M\) has a neighborhood \(V\) such that \(B\mid_V\) is a direct summand of a free \(C^\infty(V)\)-module.
a) A further equivalent characterization of $\mathcal{B}$ being projective is that $\dim(\mathcal{B} / I_x \mathcal{B})$ is a constant function of $x \in M$. This is a well-known fact, see for instance [5, Lemma 1.6] for a proof.

b) The vector bundle $\mathcal{B}$ acquires a Lie algebroid structure, by means of the $C^\infty(M)$-module isomorphism $\Gamma_c(\mathcal{B}) \cong \mathcal{B}$. The inclusion $\mathcal{B} \hookrightarrow \Gamma_c(A)$ induces an almost injective morphism of Lie algebroids $\tau: \mathcal{B} \to A$.

c) Put $\mathcal{B}_1$, $\tau_1$ another such pair. The $C^\infty(M)$-module isomorphism $\Gamma_c(\mathcal{B}) \cong \mathcal{B}$ induces an isomorphism of Lie algebroids $\zeta: \mathcal{B} \to \mathcal{B}_1$. The inclusion $\mathcal{B}_1 \hookrightarrow \Gamma_c(A)$ induces an almost injective morphism of Lie algebroids $\tau_1$. Hence the pair $(\mathcal{B}, \tau)$ is unique up to isomorphism. Further, the only Lie algebroid automorphism $\xi$ of $\mathcal{B}$ covering $\text{Id}_M$ and such that $\tau \circ \xi = \tau$ is the identity $\text{Id}_B$, as shown in [13, Prop. 1.22] in a special case.

d) Conversely, suppose there is a Lie algebroid $\mathcal{B}$ and an almost injective morphism of Lie algebroids $\psi: \mathcal{B} \to A$ such that $\tau(\Gamma_c(\mathcal{B})) = \mathcal{B}$. Then clearly $\mathcal{B}$ is a projective singular subalgebroid with $\text{rank}(\mathcal{B}) = \text{rank}(\mathcal{B})$.

The following lemma is immediate, since $\mathcal{B}$ is isomorphic to the (compactly supported) sections of a vector bundle:

**Lemma 3.2.** Let $A \to M$ be a Lie algebroid and $\mathcal{B}$ a projective singular subalgebroid of $A$. Let $x \in M$ and $\{\alpha_i\}_{1 \leq i \leq k} \subset \mathcal{B}$ whose image forms a basis of $\mathcal{B} / I_x \mathcal{B}$. Let $V$ be a neighborhood of $x$ on which the $\{\alpha_i\}_{1 \leq i \leq k}$ generate $\mathcal{B}$ (it exists by Rem. [3]). Then for every $y \in V$, the images of $\{\alpha_i\}_{1 \leq i \leq k}$ form a basis of $\mathcal{B} / I_y \mathcal{B}$.

### 3.2 The holonomy groupoid of a projective singular subalgebroid

Here we prove that the holonomy groupoid $H^G(\mathcal{B})$ is a Lie groupoid iff $\mathcal{B}$ is projective.

In the following, when we say “$H^G(\mathcal{B})$ is a Lie groupoid” we mean that there exists a (necessarily unique) smooth structure on $H^G(\mathcal{B})$ such that the quotient map $\bar{\tau}: \bigcup_{U \in \mathcal{U}^G} U \to H^G(\mathcal{B})$ (see [1, 3]) is a surjective submersion, where $\mathcal{U}^G$ is the path-holonomy atlas. With this smooth structure, $H^G(\mathcal{B})$ is a Lie groupoid, as can be seen using the arguments given in [3] for the case that $\mathcal{B}$ is a singular foliation.

**Proposition 3.3** (Projectivity and smooth holonomy groupoids). Let $A \to M$ be a Lie algebroid, $G$ a source-connected Lie groupoid integrating it, and $\mathcal{B}$ a singular subalgebroid of $A$. Then:

a) $H^G(\mathcal{B})$ is a Lie groupoid whose source fibers have dimension $k$ if and only if $\mathcal{B}$ is a projective $C^\infty(M)$-module of rank $k$.

b) In this case, denote by $\mathcal{B}$ the Lie algebroid such that $\mathcal{B} \cong \Gamma_c(\mathcal{B})$, and by $\tau: \mathcal{B} \to A$ the almost injective Lie algebroid morphism inducing this isomorphism. Then for any choice of Lie groupoid $G$ integrating $A$:

i) $H^G(\mathcal{B})$ is a Lie groupoid integrating $\mathcal{B}$.

ii) the canonical morphism $\Phi: H^G(\mathcal{B}) \to G$ is a Lie groupoid morphism integrating $\tau: \mathcal{B} \to A$.

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Proof. a) “⇒” For every $x \in M$, $\mathcal{B}/I_x\mathcal{B}$ has dimension equal to the source fiber $H^G(\mathcal{B})_x$, as a consequence of Prop. 2.14. If $H^G(\mathcal{B})$ is a Lie groupoid whose source fibers have dimension $k$, then $\mathcal{B}/I_x\mathcal{B}$ has dimension $k$ for all $x \in M$. This means exactly that $\mathcal{B}$ is a projective $C^\infty(M)$-module of rank $k$, as recalled just after Def. 3.1.

“⇐” Conversely, suppose that $\mathcal{B}$ is locally projective, of rank $k$. Let $x \in M$, $\{\alpha_i\}_{1 \leq i \leq k} \subset \mathcal{B}$ whose image forms a basis of $\mathcal{B}/I_x\mathcal{B}$, and $(U, \varphi, G)$ the corresponding path-holonomy bisubmersion. By Lemma 3.2 the minimality assumption of Rem. 2.10 is satisfied at every point $y$ lying in a neighborhood $V$ of $x$. We deduce that (shrinking $U$ if necessary) the quotient map

$$q_U: U \rightarrow H^G(\mathcal{B})$$

is injective. Hence there is a unique smooth structure in a neighborhood of $1_x$ in $H^G(\mathcal{B})$ such that $q_U$ is a diffeomorphism onto its image. Repeating for an open cover of $M$, we see that $H^G(\mathcal{B})$ is a Lie groupoid.

b) Suppose that $\mathcal{B}$ is projective.

**Claim 1.** The canonical map $\Phi: H^G(\mathcal{B}) \rightarrow G$ is a Lie groupoid morphism.

Let $(U, \varphi, G)$ be a minimal path holonomy bisubmersion of $\mathcal{B}$ at $x \in M$. We may assume that the bisubmersion $U$ is an open subset of $V \times \mathbb{R}^k$ about the point $(x, 0)$, where $V := s_U(U)$ is an open neighborhood of $x$ in $M$ and $k = \text{dim}(\mathcal{B}_x)$. As we explained in the proof of the implication “⇐” in item a), the quotient map $q_U$ is injective, so we can identify $U$ with the open subset $q_U(U)$ of $H^G(\mathcal{B})$ about the identity $1_x$. By the definition of the map $\Phi$ we have $\varphi = \Phi \circ q_U$, whence $\Phi|_{q_U(U)}$ corresponds to $\varphi$ under this identification, proving that $\Phi: H^G(\mathcal{B}) \rightarrow G$ is smooth and therefore a Lie groupoid morphism.

**Claim 2.** The Lie algebroid morphism associated to $\Phi$ is almost injective and induces an isomorphism $\Gamma_c(A(H^G(\mathcal{B}))) \cong \mathcal{B}$.

Using the identification of $U$ with the open subset $q_U(U)$ of $H^G(\mathcal{B})$, this Lie algebroid morphism is given as follows: it is the restriction of $d\varphi$ to $\text{Vert}_{\vert V}$, where the vertical bundle $\text{Vert}$ consists of the tangent spaces to the fibers of $s_U = pr_1: U \subset V \times \mathbb{R}^k \rightarrow V$. Using the definition of $\varphi$ (see Prop.-Def. 1.11) one computes that

$$(d_y\varphi)(y, \lambda) = \sum_i \lambda_i \alpha_i|_y,$$

where $y \in V$ and $\alpha_1, \ldots, \alpha_k$ are the local generators of $\mathcal{B}$ used to construct the path holonomy bisubmersion $U$. Hence the map of sections

$$d\varphi: \Gamma(\text{Vert}_{\vert V}) \rightarrow \{\alpha|_V: \alpha \in \mathcal{B}\}$$

is surjective. Its injectivity is a consequence of the fact that the singular subalgebroid $\mathcal{B}$ is projective. Indeed, assume that $\lambda: V \rightarrow \mathbb{R}^k$ is a smooth function such that $\sum_i \lambda_i \alpha_i = 0$. Then at every $y \in V$ we have $\sum_i \lambda_i(y)[\alpha_i] = 0 \in \mathcal{B}_y$, and we know by Lemma 3.2 that the $[\alpha_i]$ form a basis of $\mathcal{B}_y$. We conclude that $\lambda$ is the zero function, proving that the map in (3.1) is injective and therefore an isomorphism. By a partition of unity argument on $M$, it follows that the Lie algebroid morphism associated to $\Phi$ maps $\Gamma_c(A(H^G(\mathcal{B})))$ isomorphically onto $\mathcal{B}$. This implies in particular that it is almost injective.

Now we can proceed to prove the two items.
i) Consider the following composition of isomorphism of singular subalgebroids (the first
induced by $\Phi$ as in Claim 2, the second induced by $\tau$):

$$\Gamma_c\left(A(H^G(B))\right) \cong B \cong \Gamma_c(B).$$

\text{(3.2)}

It shows that the Lie algebroid of $H^G(B)$ is isomorphic to $B$, as desired.

ii) Again by Claim 2, the Lie algebroid morphism associated to $\Phi$, at the level of sections,

is an isomorphism $\Gamma_c\left(A(H^G(B))\right) \rightarrow B$. Applying the isomorphism \text{(3.2)} to the domain,

it becomes the isomorphism $\Gamma_c(B) \rightarrow B$ induced by $\tau$.

\[\square\]

An alternative proof of Prop. 3.3 b) is given in Appendix \[\[\]\]. Unlike the one above, it does not rely

of the smoothness results of \[\[\].

\subsection*{3.3 Adjoint and source-simply connected Lie groupoids}

Let $A$ be an integrable Lie algebroid. Let $\mathcal{B}$ be a projective singular subalgebroid, and $B$ the Lie
algebroid with $B \cong \Gamma_c(B)$. We just saw in \[\[\]\] that for any choice of Lie groupoid $G$

integrating $A$, the holonomy groupoid $H^G(B)$ is a Lie groupoid integrating $B$. We address the question of whether,

for some choice of $G$, the holonomy group is the “largest” or the “smallest” among the Lie groupoids

integrating $B$. In this sense, this subsection complements \[\[\]\] \[\[\]\].

\textbf{Definition 3.4.} ([13] Def. 1.19) Let $A$ be an integrable Lie algebroid over $M$. An \textit{adjoint} groupoid is a terminal object in the following category:

- Objects are pairs $(G, \phi)$ consisting of an $s$-connected Lie groupoid $G$ and a Lie algebroid

isomorphism $\phi: AG \rightarrow A$ covering $Id_M$. (Recall that $AG = \ker(d_s)|_M$.)

- Morphisms from $(G, \phi)$ to $(G', \phi')$ are Lie groupoid morphisms $\Psi: G \rightarrow G'$ whose associated

Lie algebroid morphism intertwines $\phi$ and $\phi'$, i.e. $\phi' \circ \Psi_\ast = \phi$.

Assume the adjoint groupoid $(G, \phi)$ of $A$ exists. Consider the holonomy groupoid $H^G(\mathcal{B})$

obtained using $G$. It is tempting to hope that $H^G(\mathcal{B})$ is the adjoint groupoid of $B$, when taken together with the

canonical isomorphism $A(H^G(\mathcal{B})) \cong B$ of eq. \text{(3.2)}. When $H^G(\mathcal{B})$ is the holonomy groupoid

$H(\mathcal{F})$ of a singular foliation, this is the case. (See [2] Prop. 3.8], together with the fact that when

$B$ is an almost injective Lie algebroid, the only Lie algebroid automorphism of $B$ covering $Id_M$ is the identity $\text{[13] Prop. 1.22}$.)

However in general it is not true that $H^G(\mathcal{B})$ is the adjoint groupoid of $B$, as Example 3.5 below shows. Further, given a Lie groupoid $K$ integrating $B$, there might not be any Lie groupoid morphism $K \rightarrow G$ integrating the almost injective morphism $\tau: B \rightarrow A$. (Among Lie groupoids $K$ that
do admit such a morphism, the holonomy groupoid is the minimal one, see Prop. \[\[\]\] later on.)

\textbf{Example 3.5.} Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{k}$ a Lie subalgebra, and $G$ a connected Lie group integrating

$\mathfrak{g}$. Then $H^G(\mathfrak{g})$ is the Lie subgroup of $G$ integrating $\mathfrak{k}$, by Ex. \[\[\].

Now choose $\mathfrak{g}$ for which the adjoint group $G$ exists (for instance, take $\mathfrak{g} = so(3)$, for which the

adjoint group is $G = SO(3)$.) Let $\mathfrak{k}$ be any one-dimensional Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{k}$, being an abelian Lie algebra, does not admit an adjoint Lie group. (In particular, the “circle” is not the

adjoint group.) In particular the Lie subgroup of $G$ integrating $\mathfrak{k}$ is not an adjoint group.
Similarly, taking \( G \) to be the source simply connected Lie groupoid integrating \( A \), the Lie groupoid \( H^G(B) \) integrating \( B \) is not source simply connected in general. This can be seen from the fact that given a simply connected Lie group \( G \), not every Lie subgroup is simply connected. Even when \( B \) is a foliation, \( H^G(B) \) can fail to be source simply connected, see [27, Ex. 3.31].

4 The holonomy groupoid as a diffeological groupoid

The notion of diffeology was introduced by Souriau in [24] in order to extend differential geometry to topologically pathological situations. In particular, Souriau used it to show that the group of diffeomorphisms differentiates to the (infinite-dimensional) Lie algebra of vector fields. Diffeology works very well with quotient spaces, like the holonomy groupoid we discuss in this article. On the other hand, analogously to the Lie algebra of vector fields studied by Souriau, singular subalgebroids are infinite dimensional Lie algebras as well. So it is natural to use Souriau’s ideas in order to provide a framework for the integration of singular subalgebroids.

In this section we introduce diffeological groupoids and show that the our holonomy groupoid is an example (Prop. 4.12). We also single out in §4.4 some properties that the holonomy groupoid – viewed as a diffeological groupoid – always satisfies: the “holonomy-like” property, the “source-submersive” property, and the “open map/local subduction” property.

4.1 Diffeological spaces

To make our account self-contained we start with an overview of diffeological spaces [24], following in part the monograph [15]. Let \( X \) be a set. We also consider the sets of maps:

- \( \chi : O \rightarrow X \), where \( O \subseteq \mathbb{R}^k \) for some \( k \in \mathbb{N} \). A map \( \chi \) as such is called a \( k \)-plot.
- \( h : O_h \rightarrow O'_h \), where \( O_h, O'_h \subseteq \mathbb{R}^n \) are open subsets.

We say that a pair \((h, \chi)\) as above is composable if \( h(O_h) \subseteq O_\chi \). Also, a family \( \{\chi_i : O_i \rightarrow X\}_{i \in I} \), where \( O_i \subseteq \mathbb{R}^k \) for every \( i \in I \), is called compatible if for every \( x \in O_i \cap O_j \) we have \( \chi_i(x) = \chi_j(x) \). In this case there is a unique smallest extension \( \chi : \bigcup_{i \in I} O_i \rightarrow X \).

**Definition 4.1.** Consider a set \( X \) and for every \( k \in \mathbb{N} \) let \( \mathcal{P}^k(X) \) be a collection of \( k \)-plots \( \chi \). The collection \( \mathcal{P}(X) = \bigcup_{k \in \mathbb{N}} \mathcal{P}^k(X) \) is called a **diffeology** if:

(D1) Every constant map \( x : O \rightarrow X \) is in \( \mathcal{P}(X) \).

(D2) Given a compatible family \( \{\chi_i\}_{i \in I} \) in \( \mathcal{P}^k(X) \), its smallest extension is in \( \mathcal{P}^k(X) \).

(D3) If \( h \) as above and \( \chi \in \mathcal{P}^k(X) \) are composable then \( \chi \circ h \) is in \( \mathcal{P}^n(X) \).

If \( F \) is a set of plots, the intersection of all diffeologies containing \( F \) is a diffeology denoted \( \{F\} \). It is called the diffeology generated by \( F \). It is the smallest diffeology containing \( F \). In [15 §1.68] it is characterized as follows:

\[For \text{ instance, take } G = SU(2) \text{ and the Lie subgroup consisting of diagonal matrices, which is a one-dimensional torus.}\]
Proposition 4.2. Let $F$ be a set of plots. A $k$-plot $\chi : \mathcal{O}_\chi \to X$ belongs to the diffeology $\langle F \rangle$ iff for every point $x$ in $\mathcal{O}_\chi$ there is a neighborhood $\mathcal{O}_x$ of $x$ such that either $\chi|_{\mathcal{O}_x}$ is constant or $\chi|_{\mathcal{O}_x} = \chi' \circ h$, where $\chi' \in F$ and $h$ is such that the pair $(\chi', h)$ is composable.

A diffeology on a set $X$ induces a canonical topology on $X$:

**Definition 4.3.** If $(X, \mathcal{P}(X))$ is a diffeological space, the $D$-topology of $X$ is defined as follows: A subset $W$ of $X$ is open iff for each plot $\chi$ in $\mathcal{P}(X)$ the inverse image $\chi^{-1}(W) \subset \mathcal{O}_\chi$ is open.

We also need to specify the notion of a smooth map in this context.

**Proposition-Definition 4.4.**

a) Let $(X, \mathcal{P}(X))$ and $(Y, \mathcal{P}(Y))$ two diffeological spaces. A map $f : X \to Y$ is called smooth if for every $\chi \in \mathcal{P}(X)$ the composition $f \circ \chi$ is in $\mathcal{P}(Y)$ (It suffices to check this condition for a generating set of $\mathcal{P}(X)$). The collection of smooth maps as such is denoted $[X, Y]$.

b) Let $(X, \mathcal{P}(X))$ a diffeological space and $X'$ a set. For any map $f : X \to X'$ there exists a finest diffeology $f_*(\mathcal{P}(X))$ on $X'$ which makes $f$ smooth. It is called pushforward diffeology and is characterized as follows: A map $\chi' : \mathcal{O}' \to X'$ is a plot in $f_*(\mathcal{P}(X))$ iff for every $r \in \mathcal{O}'$ there exists an open neighborhood $\mathcal{O}'_r$ of $r$ such that

i) either $\chi'|_{\mathcal{O}_r}$ is constant, or

ii) there exists a plot $\chi : \mathcal{O} \to X$ in $\mathcal{P}(X)$ such that $\chi'|_{\mathcal{O}_r} = f \circ \chi$.

When $f$ is surjective, this is called quotient diffeology, and in the above characterization item i) becomes superfluous.

**Remark 4.5** (Smooth maps on manifolds). In the special case that $X$ is a smooth manifold and $(Y, \mathcal{P}(Y))$ a diffeological space, $f : X \to Y$ is smooth iff for any coordinate neighbourhood $\mathcal{O} \subset X$ we have that $f|_{\mathcal{O}} : \mathcal{O} \to Y$ is a plot in $\mathcal{P}(Y)$. Here and in the following we abuse slightly notation by identifying the coordinate neighbourhood $\mathcal{O}$ with an open subset of $\mathbb{R}^{\dim(M)}$ using the chart given by the coordinates. The above is a consequence of the fact that $\mathcal{P}(X)$ is generated by $\{\phi^{-1} : \phi \in \mathfrak{a}\}$ where $\mathfrak{a}$ is an atlas for $M$, see Ex. [1.6]a) below.

The following examples show that the category of diffeological spaces defined above includes manifolds and quotient spaces. Our holonomy groupoid $H^G(B)$ is also an example of a diffeological space, but we will examine it more thoroughly in the next section [4.2].

**Examples 4.6.**

a) Let $M$ be an finite dimensional manifold. For every $n \in \mathbb{N}$, we consider $\mathcal{P}^n(M)$ to be the smooth maps $\mathcal{O} \to M$ in the usual sense of differential geometry. If $\mathfrak{a}$ is an atlas of $M$ then the diffeology $\mathcal{P}(M)$ is the one generated by $\mathfrak{a}^{-1} = \{\phi^{-1} : \phi \in \mathfrak{a}\}$. The $D$-topology of $M$ coincides with the topology induced on $M$ by the atlas $\mathfrak{a}$.

If $N$ is another manifold, we have $C^\infty(M, N) = [M, N]$. Notice that the diffeology $\mathcal{P}(M)$ determines uniquely\(^7\) the manifold structure on $M$.

b) Let $\{(X_i, \mathcal{P}(X_i))\}_{i \in I}$ be a family of diffeological spaces, $Y$ a set and $f_i : X_i \to Y$ maps. The final diffeology on $Y$ is the one generated by $f_i \circ \chi$ for all $i \in I$ and all plots $\chi \in \mathcal{P}(X_i)$. It is the smallest diffeology making all the $f_i$ smooth.

---

\(^7\)This is due to the fact that, on a smooth manifold $M$, the diffeomorphisms from open subsets of $M$ to open subsets of Euclidean space are exactly the charts of $M$. 

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c) If \( \{(X_i, \mathcal{P}(X_i))\}_{i \in I} \) is a family of diffeological spaces, we endow the disjoint union \( \bigsqcup_{i \in I} X_i \) with the final diffeology induced by the inclusion maps \( \iota_i : X_i \to \bigsqcup_{i \in I} X_i \).

d) Given a diffeological space \( (X, \mathcal{P}(X)) \) and an equivalence relation \( \sim \) on \( X \), we endow the quotient \( X/\sim \) with the final diffeology induced by the projection \( \pi : X \to X/\sim \). It agrees with the quotient diffeology introduced in Prop-Def. \[\text{1.14}\] The \( D \)-topology of the diffeology on \( X/\sim \) coincides with the quotient topology on \( X/\sim \) (induced by the projection \( \pi \) and the \( D \)-topology on \( X \)).

The following maps \[\text{15} \text{ §2.17}\] play the role of surjective submersions:

**Definition 4.7.** Let \( X, Y \) be diffeological spaces. A smooth surjective map \( f : X \to Y \) is a **local subduction** if the following holds for every \( x \in X \): for any plot \( \chi : O_\chi \to Y \) and any point \( r \in O_\chi \) with \( \chi(r) = f(x) \), there is an open neighborhood \( V \) of \( r \) and a plot \( \chi' : V \to X \) so that \( \chi|_V = f \circ \chi' \) and \( \chi'(r) = x \).

Local subductions are in particular **subductions** \[\text{15} \text{ Ch. 1.46}\]; the latter are defined as surjective maps such that the diffeology on the codomain is the quotient diffeology.

**Remark 4.8.** One can regard local subductions as being the analogue – in the diffeological category – of surjective submersions in the category of manifolds. Indeed, when \( Y \) is a manifold, the definition of local subduction can be interpreted as saying that every point of \( X \) lies on a local section of \( f \) (i.e. in the image of a smooth map \( \chi' \) such that \( f \circ \chi' = Id \)).

Further, local subductions are open maps with respect to the \( D \)-topologies, see \[\text{15} \text{ §2.20}\]. Open maps are often regarded as the analogues of submersions in the topological category. As an aside, this fact provides another proof of Prop. \[\text{1.19}\].

We finish with a lemma that is not essential for this paper, and will be used in Remark \[\text{6.3}\]

**Lemma 4.9.** Let \( (X, \mathcal{P}(X)) \) be a diffeological space and \( \{\chi^i_{op} : O^i_{op} \to X\}_{i \in I} \) be a generating set for \( \mathcal{P}(X) \) consisting of open maps. Let \( \{\chi^j : O^j \to X\}_{j \in J} \) be another generating set for \( \mathcal{P}(X) \). Then for every \( j \in J \) there is an open subset \( \tilde{O}^j \subset O^j \) such that \( \{\chi^j|_{\tilde{O}^j}\}_{j \in J} \) is a generating set for \( \mathcal{P}(X) \) consisting of maps such that the images \( \chi^j(\tilde{O}^j) \) are open in \( X \).

**Proof.** We provide a sketch. Fix \( i \in I \) and \( x \in O^i_{op} \). Since the \( \{\chi^j\}_{j \in J} \) form a generating set, there exists a neighborhood \( O^i_{op}(x) \) of \( x \), and index \( j \in J \) and a smooth map \( \tau^x \) making this diagram commute:

\[
\begin{array}{ccc}
O^j & \xrightarrow{\chi^j} & X \\
\downarrow{\tau^x} & & \uparrow{\chi^j_{op}} \\
O^i_{op}(x) & \xrightarrow{\chi^i_{op}} & X
\end{array}
\]

Notice that \( V^{i,x} := \chi^i_{op}(O^i_{op}(x)) \) is an open subset of \( X \). Now for every \( j \in J \) define \( \tilde{O}^j := \cup_{i \in I} \cup_{x \in O^i_{op}} (\chi^j)^{-1}(V^{i,x}) \).

\( \square \)
4.2 Diffeological groupoids

Roughly, a diffeological groupoid is a groupoid internal to the category of diffeological spaces, much like a Lie groupoid is essentially a groupoid internal to the category of smooth manifolds. Let us recall the definition from [15, §8.3] in more detail, but imposing the requirement that the set of objects is a manifold:

**Definition 4.10.** A **diffeological groupoid** is a groupoid $H ightrightarrows M$, where $(H, P(H))$ is a diffeological space and $M$ a manifold, such that

a) The source and target maps $s, t : H \to M$ are smooth.

b) The composition map $H \times_{s,t} H \to H$ is smooth, when $H \times_{s,t} H$ is equipped with the subset diffeology of $H \times H$.

c) The inversion map $\iota : H \to H$ is smooth.

d) The unit inclusion map $1 : M \to H$ is smooth.

**Remark 4.11.** It is obvious from definition 4.10 that diffeological groupoids with a single object $(M, pt)$ are diffeological groups in the sense of [14].

An example of diffeological groupoid is our holonomy groupoid: Let $B$ be a singular subalgebroid and $U^G$ its path-holonomy atlas (definition 1.16). Recall that $U^G$ consists of all minimal path-holonomy bisubmersions $(U, \varphi, G)$ together with their inverses and finite compositions. Using $U^G$ we attach a diffeology $P(U^G)$ to $H^G(B)$ as follows:

- Each $U \in U^G$ is a smooth manifold, whence a diffeological space with the diffeology described in item (a) of ex. 4.6. Its diffeology is generated by the usual manifold charts $\{((\phi_i^U)^{-1} : O_i \to U\}$, where $O_i$ are open subsets of $\mathbb{R}^{\dim U}$.

- Put $X = \bigsqcup_{U \in U^G} U$ and for every $U \in U^G$ consider the inclusion map $\iota_U : U \to X$. The set $X$ is a diffeological space with the diffeology defined in item (c) of ex. 4.6. Namely its diffeology is generated by $\iota_U \circ (\phi_i^U)^{-1} : O_i \to X$ for every $U \in U^G$.

- The **path-holonomy diffeology** $P(U^G)$ is the final diffeology induced by the projection map $\pi : X \to X/\sim$ (see item (d) of ex. 4.6). Its diffeology is generated by $\pi \circ \iota_U \circ (\phi_i^U)^{-1} : O_i \to H^G(B)$. Notice that $\pi \circ \iota_U$ is the quotient map $q_U : U \to H^G(B)$. Thanks to idem (d) of ex. 4.6, the associated $D$-topology of $H^G(B)$ is the quotient topology.

**Proposition 4.12** (The holonomy groupoid as diffeological groupoid). Let $B$ be a singular subalgebroid of a Lie algebroid $AG$. Then:

a) The path-holonomy diffeology is canonical.

b) The path-holonomy diffeology makes the holonomy groupoid $H^G(B)$ into a diffeological groupoid.

**Proof.** a) We first check that the path-holonomy diffeology is canonical. Recall that any two path-holonomy atlases are equivalent (see [27] Appendix C], also for the definitions). Let $U$ and $U'$ be two equivalent atlases of bisubmersions for the singular subalgebroid $B$. Denote
by $\pi : X \to X/\sim$ and $\pi' : X' \to X'/\sim$ the projection maps, whose final diffeologies on $H^G(B) = X/\sim = X'/\sim$ we denote by $\mathcal{P}(U)$ and $\mathcal{P}(U')$ respectively.

If $x \in U \in X$, there is a (smooth) morphism of bisubmersions $f : U_0 \to U'$ for some neighborhood $U_0$ of $x$ in $U$ and a bisubmersion $U' \in X'$, since the atlas $\mathcal{U}$ is adapted to $\mathcal{U}'$. Hence $\pi_{|U_0} = \pi' \circ f$ is a smooth map with respect to the diffeology $\mathcal{P}(U')$. This shows that $\mathcal{P}(U) \subset \mathcal{P}(U')$, and by symmetry we obtain the equality.

b) We have to show that the structure maps in Def. 4.10 are smooth. We do so only for b) (the composition map $m$), as the other cases are similar. Let $U_1, U_2 \in \mathcal{U}^G$, and let $\mathcal{O}_1, \mathcal{O}_2$ be coordinate charts there. Then $(q_{U_1}|_{\mathcal{O}_1}, q_{U_2}|_{\mathcal{O}_2})$ is a plot in the diffeology of the Cartesian product $H^G(B) \times H^G(B)$, and we use the same notation for plot obtained by restriction to the fiber products over $M$. We have to show that $m \circ ((q_{U_1}|_{\mathcal{O}_1}, q_{U_2}|_{\mathcal{O}_2})$ is a plot in the path-holonomy diffeology of $H^G(B)$. This holds because $q_U := m \circ (q_{U_1} \times q_{U_2})$, being the quotient map associated to the composition of two bisubmersions in $\mathcal{U}^G$, lies in $\mathcal{U}^G$.

\[
\begin{array}{c}
U := U_1 \times_{s_1,t_2} U_2 \\
\downarrow q_{U_1} \times q_{U_2} \\
H^G(B) \times_{s,t} H^G(B) \xrightarrow{m} H^G(B)
\end{array}
\]

□

Remarks 4.13. a) (The path-holonomy diffeology) Let us explain more precisely the diffeological structure of $H^G(B)$ near the identity: Consider a minimal path-holonomy bisubmersion $(U, \varphi, G)$. Shrinking $U$ if necessary, we may use a manifold chart to identify $U$ with an open subset of $\mathbb{R}^{\dim U}$. On the other hand, $q_U(U)$ is an open subset of $H^G(B)$ near the identity. Whence the set $H^G(B)_{op} = \bigcup_U q_U(U)$ is an open subset of $H^G(B)$ which contains the identity $M$. (Note that the above union is given only by single path-holonomy bisubmersions, and not by finite compositions of bisubmersions as such.) So the diffeology $Q$ of $H^G(B)_{op}$ is the one generated by the quotient maps $q_U : U \to H^G(B)_{op}$.

b) (Smoothness of $\Phi$) The groupoid $H^G(B)$ comes together with the map $\Phi : H^G(B) \to G$. Let us explain the smoothness of $\Phi$ in the diffeological setting: By construction we have that the following diagram commutes, for every minimal path-holonomy bisubmersion $U$:

\[
\begin{array}{ccc}
U & \xrightarrow{\varphi} & G \\
\downarrow q_U & & \downarrow \Phi \\
H^G(B)_{op} & & 
\end{array}
\]

This makes the restriction of $\Phi$ to $H^G(B)_{op}$ an element of $[H^G(B)_{op}, G]$. Allowing finite compositions and inverses in the above discussion, we see that $\Phi \in [H^G(B), G]$.

c) (The underlying topology) Recall that $H^G(B)$ is a topological groupoid, with the quotient topology induced by $\pi : X \to X/\sim = H^G(B)$. This topology agrees with the $D$-topology underlying the path-holonomy diffeology. This is an immediate consequence of Ex. 4.16 d).
d) (The projective case: diffeology) When $\mathcal{B}$ is projective, the diffeology of $H^G(\mathcal{B})$ defined by its smooth structure (cf. Prop. 3.3) coincides with the path-holonomy diffeology. To see this, recall from the proof of Prop. 3.3 a) (and remark 2.10) that in this case the quotient map $q_U : U \to H^G(\mathcal{B})$ is a diffeomorphism onto its image, for minimal path-holonomy bisubmersions.

e) (The projective case: topology) When $H^G(\mathcal{B})$ is a Lie groupoid – in the sense explained just before Prop. 3.3 – the topology induced by its (necessarily unique) smooth structure is exactly the $D$-topology (cf. Def. 1.3) induced by the path-holonomy diffeology $\mathcal{P}(U^G)$. This follows from item d) and the fact that the $D$-topology on a smooth manifold with the associated diffeology agrees with the usual topology on the manifold, see Ex. 4.6 a).

### 4.3 Bisections

Let $H \longrightarrow M$ be a diffeological groupoid. The key to the differentiation we carry out in §5 is the notion of a bisection, which we introduce and discuss here.

**Definition 4.14.** a) A **global bisection** of $H$ is a smooth map $b : M \to H$ which is right-inverse to $s_H$. We use the term **bisection** when the domain is any open subset of $M$.

b) Let $I = (-\delta, \delta)$ be an interval. Consider a family $\{b_\lambda : V \to H\}_{\lambda \in I}$ of bisections. This family is called **smooth** iff the map $V \times I \to H, (x, \lambda) \mapsto b_\lambda(x)$ is smooth as a map between diffeological spaces.

**Remarks 4.15.** a) Fix a generating set of plots $\mathcal{X} = \{\chi : O_\chi \to H\}$ in $\mathcal{P}(H)$. If $b : V_b \to H$ is a bisection and $V_b$ is small enough, there exists a plot $\chi : O_\chi \to H$ in $\mathcal{X}$ and a smooth map $u : V_b \to O_\chi$ satisfying $\chi \circ u = b$. This follows from Rem. 4.3 and Prop. 4.2.

b) An example of a global bisection is the identity bisection $1_M$ of $H$, namely the unit inclusion map, which is smooth by the definition of diffeological groupoid.

c) For the holonomy groupoid $H^G(\mathcal{B})$, the fact mentioned in the previous item can be also seen as follows: Let $(U, t, s)$ be a minimal path-holonomy bisubmersion and put $V = s(U)$. Since $V \to U, x \mapsto (x, 0)$ is a bisection of the bisubmersion, it follows that $1_V : V \to H^G(\mathcal{B})$ defined by $1_V(x) = q(x, 0)$ is a bisection of the holonomy groupoid.

Global bisections can be multiplied: for $i = 1, 2$ consider global bisections $b_i : M \to H$. We define $b_2 \ast b_1 : M \to H$ by

$$(b_2 \ast b_1)(x) = b_2((t_H \circ b_1)(x)) \cdot b_1(x)$$

where the product on the right-hand side is the groupoid multiplication of $H$. Similarly, the inverse to a global bisection $b$ is the global bisection $b^{-1}$ determined by $b^{-1}((t \circ b)(x)) = (b(x))^{-1}$ for all $x \in M$ (see [21, Prop. 1.42]).

**Definition 4.16.** A smooth family of global bisections $\{b_\lambda\}_{\lambda \in I}$ of $H$ is called a **1-parameter group** iff

a) $b_0 = 1_M$,

b) $b_{\lambda+\mu} = b_\lambda \ast b_\mu$ whenever $\lambda, \mu, \lambda + \mu \in I$.

The next result proves the existence of many 1-parameter groups of global bisections for $H^G(\mathcal{B})$. 


Proposition 4.17. Let $\mathcal{B}$ be a singular subalgebroid. Then for every $x \in M$ there is a neighborhood $V$ with this property: every $\alpha \in \mathcal{B}$ with support in $V$ gives rise to an 1-parameter group of compactly supported\footnote{More precisely this means: the bisections agree with the identity bisection of $V$ outside a compact subset of $V$.} global bisections $b_\lambda$ of $H^G(\mathcal{B})$ defined on $V$, satisfying $\frac{d}{d\lambda}|_{\lambda=0}(\Phi \circ b_\lambda) = \alpha$.

Proof. Given $x \in M$, consider a minimal path-holonomy bisubmersion $(U, \varphi, G)$ at $x$, and define $V := (s \circ \varphi)(U)$. Pick a section $\alpha \in \mathcal{B}$ with support in $V$, and consider the corresponding right-invariant vector field $\tilde{\alpha}$ of $G$. From definition 1.10 we have $\varphi^{-1}(\mathcal{B}) = \Gamma(U; \ker d\varphi)$, whence there is a vector field $\xi \in \Gamma(U; \ker d\varphi)$ which is $\varphi$-related to $\tilde{\alpha}$. Looking at [21 §3.6] we may assume that there is an interval $I = (-\epsilon, \epsilon)$ such that the flow $\{A_\lambda\}_{\lambda \in I}$ of $\tilde{\alpha}$ is defined on an open subset $W$ of $G$ which contains an open neighborhood of $V$ in the base manifold $M$. Since $\xi$ is $\varphi$-related to $\tilde{\alpha}$, we can assume that the flow $\{\Xi_\lambda\}_{\lambda \in I}$ of $\xi$ is defined in an open subset $\tilde{W}$ of $U$ which contains $V \times \{0\}$.

The situation is summarized in the following diagram:

$$
\begin{array}{ccc}
U & \xrightarrow{\varphi} & G \\
\xi, \{\Xi_\lambda\} & \xrightarrow{\tilde{\alpha}, \{A_\lambda\}} & \\
\end{array}
$$

For every $\lambda$ define $u_\lambda : V \to U$ by $u_\lambda(x) = \Xi_\lambda(x, 0)$. (4.2)

The family $\{u_\lambda\}_{\lambda \in I}$ is obviously smooth. Composing each $u_\lambda$ with the quotient map $q_U : U \to H^G(\mathcal{B})$ we obtain a 1-parameter family of bisections $\{b_\lambda\}_{\lambda \in I}$ of $H^G(\mathcal{B})$, with the following properties:

- It satisfies $\frac{d}{d\lambda}|_{\lambda=0}(\Phi \circ b_\lambda) = \alpha$ by construction, using that $\Phi \circ q_U = \varphi$.

- The bisections $b_\lambda$ are compactly supported. Indeed, for all $y \in V$ with $y \notin \text{Supp}(\alpha)$ we have $b_\lambda(y) = 1_y \in H^G(\mathcal{B})$, for every $\lambda$. This follows from Rem. 1.18 and the fact that in a neighborhood of $y$, $u_\lambda$ is a bisection of $U$ which carries the identity bisection of $G$ (i.e. $\varphi(u_\lambda) \subset I_M$ in such a neighborhood).

- It satisfies the properties of definition 4.16. Indeed:

  a) $b_0 \subset I_M$, as a consequence of $u_0(x) = (x, 0)$.

  b) We have to prove that the bisections $b_\lambda * b_\mu$ and $b_{\lambda+\mu}$ of $H^G(\mathcal{B})$ coincide. They both carry the bisection

  $$A_\lambda \, A_\mu = A_{\lambda+\mu}$$

  of $G$, since $\Phi$ is a groupoid morphism, since $\Phi \circ q_U = \varphi : U \to G$ and $\xi$ is $\varphi$-related to $\tilde{\alpha}$. Here the equality holds because $\{A_\lambda\}$ is the flow of a right invariant vector field on $G$ (namely, $\tilde{\alpha}$). Since both $b_\lambda * b_\mu$ and $b_{\lambda+\mu}$ carry the same bisection of $G$, they coincide (see the proof of the later Thm. 6.19 for a detailed explanation).

$\square$
4.4 Desirable properties of diffeological groupoids

For diffeological groupoids, we introduce three properties, which are always satisfied by the holonomy groupoid. The first of them, the “holonomy-like” property, plays an important role in the differentiation of diffeological groupoids, explained in §5.1. The second property is implied by the third one, which comes in two flavours (the “open map” and the “local subduction” property) and which will be important for the notion of integral in §6.

We first present an important lemma, that applies to all diffeological groupoids.

Lemma 4.18. Let \( H \to M \) be a diffeological groupoid. Let \( \{ \chi : O_\chi \to H \} \) be any generating set of plots in \( \mathcal{P}(H) \). Then for every \( x \in M \) there is a neighbourhood \( V \subset M \), a plot \( \chi \) in the generating set, and a submanifold \( e \subset O_\chi \), such that

i) \( \chi|_e : e \to 1(V) \) is a diffeomorphism,

ii) \( s_H \circ \chi \) and \( t_H \circ \chi \) are submersions at points of \( e \).

Proof. Fix \( x \in M \). The inclusion of the identity elements \( 1 : M \to H \) is smooth by Def. 4.10. There is an open neighbourhood \( V \) of \( x \) in \( M \) so that \( 1|_V \) is a plot for \( H \) in \( \mathcal{P}(H) \), since \( M \) is a manifold, see Rem. 4.5 (after identifying \( V \) with an open subset of \( \mathbb{R}^{\dim(M)} \) by means of a chart). There exists a generating plot \( \chi : O_\chi \to H \) as in Def. 4.19 and a smooth map \( \tilde{1} : V \to O_\chi \) with \( \chi \circ \tilde{1} = 1|_V \), by Prop. 4.2 after shrinking \( V \) if necessary.

Notice that, since \( s_H \circ 1|_V = Id_V \), the map \( \tilde{1} \) is a smooth section of \( s_H \circ \chi : O_\chi \to V \). This has two consequences.

The first is that \( e := \tilde{1}(V) \subset O_\chi \) is a submanifold of \( O_\chi \), shrinking \( V \) if necessary, because \( \tilde{1} \) is an injective immersion. Further \( \chi(e) = 1(V) \) is an open neighbourhood of \( 1_x \) in \( 1_M \). Also, \( \chi|_e : e \to \chi(e) = 1(V) \) is a diffeomorphism, because it is the inverse of the diffeomorphism \( \tilde{1} : V \to e \).

The second consequence is item ii) in the statement.

4.4.1 The “holonomy-like” property

We describe a specific property of diffeological groupoids, which will be used in §5.1. This property is reasonable in view of Lemma 4.18 and loosely speaking it implies that there is a generating set for \( H \) nearby the identity section consisting of “few plots” (see Prop. 4.2). The plots of this generating set can heuristically described as being “semi-local subductions over points of \( 1_M \)”, a variation of the notion of local subduction where a condition is imposed on submanifolds rather than points.

Definition 4.19. A diffeological groupoid \( H \to M \) is called holonomy-like if there exists an open neighborhood \( \tilde{H} \) of \( M \) in \( H \) and a generating set of plots \( \mathcal{X} = \{ \chi : O_\chi \to \tilde{H} \} \) in \( \mathcal{P}(\tilde{H}) \) such that the following holds for all \( \chi, \chi' \in \mathcal{X} \):

\[
\begin{array}{ccc}
e & & H \\
\uparrow & & \downarrow \\
\tilde{1} & & 1_V \\
\chi & & V \\
\end{array}
\]
Let \( e \subset \mathcal{O}_\chi, e' \subset \mathcal{O}_{\chi'} \) be submanifolds such that \( \chi(e) = \chi'(e') \) is an open subset of \( 1_M \subset H \) and \( \chi|_e: e \to \chi(e) \) and \( \chi'|_{e'}: e' \to \chi'(e') \) are diffeomorphisms. Let \( e' \in e' \). Then there exists a smooth map \( k: \mathcal{O}_{\chi'} \to \mathcal{O}_\chi \) with \( k(e') = e \) and \( \chi \circ k = \chi' \), after shrinking \( \mathcal{O}_{\chi'} \) to a smaller neighborhood of \( x' \) if necessary.

\[
\begin{array}{c}
\chi \\
\uparrow \\
H
\end{array}
\Rightarrow
\begin{array}{c}
\chi' \\
\downarrow \\
\mathcal{O}_{\chi'}
\end{array}
\Rightarrow
\begin{array}{c}
\mathcal{O}_\chi \\
\leftarrow \\
k
\end{array}
\]

**(Proposition 4.20)** The holonomy groupoid \( H^G(B) \) is holonomy-like.

**Proof.** Take \( \overset{\circ}{H} = H \) and \( \mathcal{X} \) equal to the path holonomy atlas (see definition 1.16). Let \( \chi, \chi', e, e' \) be as in Def. 4.19. In particular, \( e \) is a bisection of the bisubmersion \( \mathcal{O}_\chi \), and similarly for \( e' \) and \( \mathcal{O}_{\chi'} \). It follows that \( (\Phi \circ \chi)(e) \) and \( (\Phi \circ \chi')(e') \) are both mapped into the identity bisection of \( G \). Hence there is a morphism of bisubmersions \( k: \mathcal{O}_{\chi'} \to \mathcal{O}_\chi \) with \( k(e') = e \), as follows from the proof of Prop. 1.15 in [27, Prop. 3.2, Cor. 3.3] (that proof is analogous to [2, Prop. 2.10 b), Cor. 2.11 a]). We have \( \chi \circ k = \chi' \) by the definition of the equivalence relation used in the construction of the holonomy groupoid (see §1.4.3), since \( k \) is a morphism of bisubmersions.

4.4.2 The “source-submersive” property

Recall that the source map of a Lie groupoid is a surjective submersion. We show here that for diffeological groupoids the analogous property holds under certain assumptions.

**Definition 4.21.** A diffeological groupoid \( H \xrightarrow{\overset{\circ}{H}} M \) is **source-submersive** if through every point of \( H \) there is a (local) bisection. (Notice that in this case the maps \( s: H \to M \) and \( t: H \to M \) are local subductions, thus in particular open maps [9].)

The assumptions of the next lemma are clearly satisfied by the holonomy groupoid \( H^G(B) \) (thanks to Prop. 1.19) and by Lie groupoids. More generally, Prop. 1.23 shows that these assumptions are satisfied by diffeological groupoids generated by open maps, as introduced in §4.4.3.

**Lemma 4.22.** Let \( H \xrightarrow{\overset{\circ}{H}} M \) be a (source-connected) diffeological groupoid. Assume there exist plots \( \{ \chi_i: \mathcal{O}_{\chi_i} \to H \} \) in \( \mathcal{P}(H) \) such that

\begin{itemize}
  \item[i)] \( s \circ \chi_i: \mathcal{O}_{\chi_i} \to M \) and \( t \circ \chi_i \) are submersions, for all \( i \),
  \item[ii)] \( \overset{\circ}{H} := \bigcup_{i} \text{Image}(\chi_i) \) is an open neighborhood of \( M \) in \( H \).
\end{itemize}

Then \( H \) is source-submersive.

**Proof.** **Claim:** Through every point of \( \overset{\circ}{H} \) there is a bisection.

Let \( h \in \overset{\circ}{H} \). Then \( h = \chi_i(x) \) for some \( x \in \mathcal{O}_{\chi_i} \). By assumption i), there is a smooth submanifold of \( \mathcal{O}_{\chi_i} \) through \( x \) which is transverse to both the \( s \circ \chi_i \)-fibers and the \( t \circ \chi_i \)-fibers. It is the image of

---

[9] This follows immediately from Def. 4.17 and Remark 4.8.
a unique local section \( \sigma \) of \( s \circ \chi_i \). Hence \( \chi_i \circ \sigma \) is a smooth bisection of \( H \) through \( h \), proving the claim.

\[
\begin{array}{c}
\sigma \\
\xrightarrow{\chi_i} H \\
\downarrow s \\
M
\end{array}
\] (4.5)

Every source-connected topological groupoid is generated by any symmetric open neighborhood of the identity section (this can be proven as in [21, Prop. 1.5.8]). Hence, by assumption ii), any point \( h \in H \) is the product \( h_1 \cdots h_k \) of points in \( \mathcal{H} \). By the claim, there is a smooth bisection \( b_j \) through \( h_j \) for all \( j \). Their product \( b_1 \ast \cdots \ast b_k \) is a smooth bisection through \( h \). This proves the statement.

The following proposition provides a sufficient criterium for being source-submersive, and an additional property.

**Proposition 4.23.** Let \( H \to M \) be a diffeological groupoid. Assume there is an open neighborhood \( \mathcal{H} \) of \( 1_M \) in \( H \) whose diffeology is generated by open maps, i.e.: there exists a generating set of plots \( \{ \chi_i : \mathcal{O}_{\chi_i} \to \mathcal{H} \} \) of \( \mathcal{P}(\mathcal{H}) \) such that each \( \chi_i \) is an open map. Then:

i) the diffeological groupoid \( H \) is source-submersive,

ii) there is a neighborhood \( \mathcal{H} \) of the identity \( 1_M \) in \( H \) such that for all \( h \in \mathcal{H} \) there is a 1-parameter family of global bisections \( \{ b_\lambda \}_{\lambda \in [0,1]} \) with \( h \in b_1 \).

**Proof.** For every \( x \in 1_M \), by Lemma 4.18 we know that there is a neighbourhood \( V \) of \( x \) in \( M \), a plot \( \chi \) in the above generating set, and a submanifold \( e \) of \( \mathcal{O}_\chi \), such that

- \( \chi|_e : e \to 1(V) \subset 1_M \) is a diffeomorphism,
- \( s_H \circ \chi \) and \( t_H \circ \chi \) are a submersion at points of \( e \).

Therefore there is a neighborhood \( \mathcal{O}_\chi \) of \( e \) in \( \mathcal{O}_\chi \) so that

- the image of \( \mathcal{O}_\chi \) under \( \chi \) is a neighborhood of \( 1(V) \) in \( H \) (here we use the assumption that \( \chi \) is an open map),
- \( s_H \circ \chi \) and \( t_H \circ \chi \) are submersions when restricted to \( \mathcal{O}_\chi \).

We now prove the two items of the lemma.

i) The set of plots \( \{ \chi_i|_{\mathcal{O}_{\chi_i}} : \mathcal{O}_{\chi_i} \to \mathcal{H} \} \) satisfies the assumptions of Lemma 4.22.
ii) **Claim:** Fix an index $i$. Shrinking $O_{\chi_i}$ to a smaller neighborhood of $e_i$ if necessary, every point of $p \in O_{\chi_i}$ lies on $u_1$ where $u_\lambda$ are sections of $s_H \circ \chi_i: O_{\chi_i} \to V_i$ depending smoothly on $\lambda \in [0, 1]$, transverse to the fibers of $t_H \circ \chi_i$, with compact support, and with $u_0 = (\chi|_{e_i})^{-1}$.

Given the claim, we have $\chi_i(p) \in b_1$ where

$$b_\lambda := \chi_i \circ u_\lambda, \quad \lambda \in [0, 1],$$

is a 1-parameter family of bisections in $H$ with support in $V_i$, which hence can be extended trivially to global bisections. Taking $\hat{H} = \cup_i \chi_i(O_{\chi_i})$, this proves the desired statement.

To show the claim, since we are working locally, we may assume that $s_H \circ \chi_i: O_{\chi_i} \to V_i$ is a neighborhood of the zero section of a metric vector bundle with zero section $e_i$. Clearly $e_i$ is transverse to the fibers of $t_H \circ \chi_i$. For every unit length vector $v \in O_{\chi_i}$, extending $v$ to a compactly supported section of this vector bundle, we obtain a vertical fiberwise constant vector field, and there is $\epsilon > 0$ so that the image of $e_i$ under its flow is transverse to the fibers of $t_H \circ \chi_i$ for all times in $[0, \epsilon)$. Hence each of the points $tv$, for $t \in [0, \epsilon)$, satisfies the statement of the claim.

\[\square\]

### 4.4.3 The “open map” and “local subduction” property

Here we look at diffeological spaces and groupoids admitting a generating set of plots which define either an open map, or a local subduction. Recall these two kinds of maps are the analogues of (surjective) submersions respectively in the topological and the diffeological category.

**Definition 4.24.** Let $(X, P(X))$ a diffeological space. We say that the diffeology is **generated by open maps** if there exists a generating set $\{\chi_i: O_i \to X\}_{i \in I}$ of plots in $P(X)$ such that $\chi_i: O_i \to X$ is an open map for all $i$. Equivalently, so that the combined map $\sqcup \chi_i: \sqcup O_i \to X$ is an open map.

We already applied this notion to a diffeological groupoid $H$ in Lemma 4.23, where we showed that the diffeology of $H$ is generated by open maps, the $H$ is source-submersive.

**Remark 4.25.** Let $H$ be a diffeological groupoid. The following are equivalent:

- the diffeology of $H$ itself is generated by open maps,
- there is a open neighborhood $\hat{H}$ of $1_M$ whose diffeology is generated by open maps.

The implication from top to bottom is easy. For the other implication, notice that the existence of such $\hat{H}$ implies that a generating set for $P(H)$ is obtained composing the plots $\{\chi_i: O_{\chi_i} \to \hat{H}\}$ as in Prop. 4.23 with the right-translations $R_b$, where the $b$ range over all bisections of $H$. Each $R_b \circ \chi_i$ is an open map, being a composition of such.

**Definition 4.26.** Let $(X, P(X))$ a diffeological space. We say that the diffeology is **generated by local subductions** if there exists a generating set $\{\chi_i: O_i \to X\}_{i \in I}$ of plots in $P(X)$ such that $\sqcup \chi_i: \sqcup O_i \to X$ is local subduction.
Recall that local subductions were defined in Def. 4.24 are surjective by definition, and are always open maps. In the following remark we rephrase Def. 4.26 in terms of the individual plots $\chi_i$.

**Remark 4.27.** Given a diffeological space $X$ and a generating set of plots $\{\chi_i: O_i \to X\}_{i \in I}$ in $\mathcal{P}(X)$, the following is equivalent:

- the map $\sqcup \chi_i$ is a local subduction,
- i) for each $i \in I$, the image $\chi_i(O_i)$ is an open subset of $X$ (endowed with the D-topology),
- ii) for each $i \in I$, the map $\chi_i: O_i \to \chi_i(O_i)$ is a local subduction, where $\chi_i(O_i)$ is endowed with the subspace diffeology,
- iii) the map $\sqcup \chi_i: \sqcup O_i \to X$ is surjective.

We argue as follows. Assuming the first condition, the second is clear. For the other implication, let $\chi: O \to X$ be any plot in $\mathcal{P}(X)$, $r \in O$, and $x \in O_i$ (for some $i \in I$) such that $\chi(r) = \chi_i(x)$. Since $\chi_i(O_i)$ is an open subset of $X$, there is a neighborhood $V \subset O$ of $r$ such that $\chi|_V$ takes values in $\chi_i(O_i)$. The fact that $\chi_i: O_i \to \chi_i(O_i)$ is a local subduction finishes the argument.

Notice that conditions ii) and iii) above do not imply condition i): take the diffeology on $X = \mathbb{R}$ generated by the plots $\chi_1 = x^2: \mathbb{R} \to \mathbb{R}$ and $\chi_2 = -x^2: \mathbb{R} \to \mathbb{R}$. Each $\chi_i$ is a local subduction, but $\chi_1(\mathbb{R}) = \mathbb{R}_{\geq 0}$ is not open in the D-topology, because its preimage under $\chi_2$ is the singleton $\{0\}$.

The relation between Def. 4.24 and Def. 4.26 is manifest, because local subductions are always open maps:

**Lemma 4.28.** Let $(X, \mathcal{P}(X))$ a diffeological space. If the diffeology is generated by local subductions, then it is generated by open maps.

The following proposition states that the quotient map from the path-holonomy atlas to the holonomy groupoid plays the role of a surjective submersion in the diffeological category.

**Proposition 4.29.** Let $\mathcal{B}$ be a singular subalgebroid of a Lie algebroid $AG$. The quotient map $\xi: \bigsqcup_{U \in \mathcal{U}^{AG}} U \to H^G(\mathcal{B})$ is a local subduction.

**Proof.** The map $\xi$ is surjective, and it is smooth because the path-holonomy diffeology is exactly the quotient diffeology induced by $\xi$. Fix a point $u$ in a bisubmersion $U \in \mathcal{U}^{AG}$, denote $h := \xi(u)$ the element of the holonomy groupoid it represents. Take any plot $\chi: O_{\chi} \to H^G(\mathcal{B})$ and any point $r \in O_{\chi}$ with $\chi(r) = h$. There is a connected open neighborhood $V$ of $r$ and a plot $\chi': V \to \bigsqcup_{U \in \mathcal{U}^{AG}} U$ so that $\chi|_V = \xi \circ \chi'$, by definition of quotient diffeology (see Def.-Prop. 1.1). The image $u' := \chi'(r)$ lies in some bisubmersion $U'$. The point $u'$ might differ from the point $u$ we fixed, but we know that $\xi(u') = h = \xi(u)$. By the definition of holonomy groupoid, this means that there is a morphism of bisubmersions $\phi: U' \to U$ mapping $u'$ to $u$, shrinking $U'$ if necessary. As for every morphism of bisubmersions, we have $\xi \circ \phi = \xi$. Thus the composition $\phi \circ \chi': V \to U$ (which is well-defined after shrinking $V$ if necessary) is a plot mapping $r$ to $u$ and lifting the plot $\chi|_V$. 

![Diagram](image)

\[\square\]
5 Global differentiation

We show that a diffeological groupoid together with a morphism to the Lie groupoid \( G \) – both satisfying certain properties – gives rise to a singular subalgebroid (Thm. 5.2). We formalize this in the definition of differentiation (Def. 5.3). We provide examples, showing in particular that the holonomy groupoid of a singular subalgebroid \( B \) differentiates to \( B \) (Thm. 5.11). Finally we address the functoriality of differentiation.

5.1 Differentiation of global bisections to singular subalgebroids

In this subsection, in view of the holonomy groupoid, we fix the following data:

- a diffeological groupoid \( H \to M \), which is holonomy-like (see Def. 4.19),
- a Lie groupoid \( G \) and a smooth morphism of diffeological groupoids \( \Psi : H \to G \) covering \( \text{Id}_M \).

We use the apparatus developed in §4.2 and §4.3 to specify a differentiation process for families of global bisections which gives rise to a singular subalgebroid of \( AG \) (Thm. 5.2 and Def. 5.3 below).

To this end, first note that if \( \{ b_\lambda \}_{\lambda \in I} \) is a smooth family (resp. 1-parameter group) of global bisections of \( H \) then \( \{ \Psi \circ b_\lambda \}_{\lambda \in I} \) is a smooth family (resp. 1-parameter group) of global bisections of \( G \). Now let us consider the following set of velocity vectors (at time 0):

\[
S := \left\{ \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda) : \{ b_\lambda \}_{\lambda \in I} \text{ family of global bisections for } H \text{ s.t. } b_0 = \text{Id}_M \right\} \cap \Gamma_c(AG).
\]

We will argue that \( S \) “contains many elements” in Prop. 5.9 below, by constructing elements out of 1-parameter families of local bisections.

Let us make the following assumption, involving 1-parameter \( g \)roups. In §5.3 we show that the holonomy groupoid satisfies it.

\textbf{Assumption 5.1.} For every \( x \in M \) there is a neighborhood \( V \) with this property:

\[
\{ \alpha \in S : \text{Supp}(\alpha) \subset V \} \subset \left\{ \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda) : \{ b_\lambda \}_{\lambda \in I} \text{ 1-parameter group of global bisections for } H \right\}.
\]

The main statement of this section is:

\textbf{Theorem 5.2.} Let \( H \to M \) and \( \Psi : H \to G \) be as above (in particular \( H \) is holonomy-like), satisfying Assumption 5.1. Then the set \( S \subseteq \Gamma_c(AG) \) is a singular subalgebroid of \( AG \).

Now the next definition seems in order, in view of theorem 5.2:

\textbf{Definition 5.3 (Differentiation).} Let \( H \to M \) a diffeological groupoid, \( G \to M \) a Lie groupoid and \( \Psi : H \to G \) a smooth morphism of diffeological groupoids covering \( \text{Id}_M \). If

- \( H \) is holonomy-like (Def. 4.19), and
- this data satisfies Assumption 5.1,
then we say that \((H, \Psi)\) differentiates to the singular subalgebroid \(S\) of \(\Gamma_c(AG)\) (introduced at the beginning of §5.1).

Remarks 5.4.  

a) In the differentiation process of a diffeological groupoid, the property of being holonomy-like (cf. definition 4.19) is used only to prove that the module \(S\) is locally finitely generated, in Lemma 5.6. However, there do exist interesting examples of involutive \(C^\infty\)-submodules of vector fields which are not finitely generated. For instance, consider the partition to \(\mathbb{R}\) to \((0, +\infty), \{0\}\) and \(\{x\}\) for every \(x < 0\) and take the \(C^\infty_c(\mathbb{R})\)-module generated by all vector fields which are tangent to these submanifolds. This is an involutive \(C^\infty(\mathbb{R})\)-submodule of \(\mathfrak{X}(\mathbb{R})\), but it is not locally finitely generated. We do not know if diffeological groupoids can be used to treat the differentiation/integration process of non locally finitely generated modules of vector fields. Of course, Lemma 5.6 already excludes the holonomy-like groupoids as such.

b) Assumption 5.1 is used only to prove that \(S\) is involutive, in Lemma 5.7. The interested reader may check that a variation of the proof of lemma 5.7 gives the following result. Instead of making Assumption 5.1, suppose that \(S\) is holonomy-like is used to prove that the module \(\{\alpha\}_\lambda \subseteq \Gamma_c(A)\) such that \(\alpha\_\lambda \in S\) for all \(\lambda\) and \(\alpha_0 = 0\), the time derivative \(\frac{d}{d\lambda}\big|_{\lambda=0}\alpha\_\lambda\) lies in \(S\). Then \(S\) is involutive.

5.1.1  The proof of theorem 5.2

The proof of theorem 5.2 is provided by the following three lemmas.

Lemma 5.5. The set \(S \subseteq \Gamma_c(AG)\) is a \(C^\infty(M)\)-module.

Proof. Given \(f \in C^\infty(M)\) and an element \(\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ b\_\lambda)\) of \(S\), we show that their product lies in \(S\). Since elements of \(S\) are compactly supported, we may assume that \(f\) is compactly supported. Consider the family of global bisections defined by \(b\_f^\lambda(x) = b\_f(x)\). (Since \(f\) is a bounded function, \(\lambda\) is defined in an open subinterval of \(I\) containing zero, and passing to a smaller subinterval if necessary we ensure that \(b\_f^\lambda\) is a bisection.) The chain rule implies

\[
\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ b\_f^\lambda) = f \frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ b\_\lambda).
\]  

(5.1)

Now fix two elements \(\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ b\_\lambda)\) and \(\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ c\_\lambda)\) of \(S\), where \(b\_\lambda\) and \(c\_\lambda\) are 1-parameter families with \(b_0 = c_0 = Id_M\). We show that their sum lies in \(S\), and more concretely that it comes from the 1-parameter family \(b\_\lambda \ast c\_\lambda\) of global bisections of \(H\). Indeed, using the product of global bisections introduced in §4.3 we have

\[
\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ (b\_\lambda \ast c\_\lambda)) = \frac{d}{d\lambda}\big|_{\lambda=0}((\Psi \circ b\_\lambda) \ast (\Psi \circ c\_\lambda)) = \frac{d}{d\lambda}\big|_{\lambda=0}((\Psi \circ b\_\lambda) + \frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ c\_\lambda)),
\]

where in the first equality we used that \(\Psi\) is a groupoid morphism and in the second that \(b_0 = c_0\) is the identity bisection.

The assumption that the \(H\) is holonomy-like is used to prove that \(S\) is locally finitely generated (see also Rem. 5.4).

40
Lemma 5.6. The module $\mathcal{S} \subseteq \Gamma_c(AG)$ is locally finitely generated.

Proof. For every $x \in 1_M$, by Lemma 4.18 we know that there is a neighbourhood $V$ of $x$ in $M$, a plot $\chi$ in the above generating set, and a submanifold $e$ of $O_{\chi}$, such that

- $\chi|_e: e \to 1(V) \subseteq 1_M$ is a diffeomorphism,
- $s_H \circ \chi$ and $t_H \circ \chi$ are a submersion at points of $e$.

Further, $e$ is realized as the image of a smooth map $\tilde{1}: V \to O_{\chi}$.

This implies that

$$Vert|_e := \text{Ker}(s_H \circ \chi)|_e$$

is a vector bundle over $e$.

Claim:

$$\mathcal{S}|_V = (\Psi \circ \chi)_* \Gamma(Vert|_e).$$

Since $\Gamma(Vert|_e)$ is finitely generated as a $C^\infty(e)$-module, the claim implies that the same holds for $\mathcal{S}|_V$, as we wanted to show. Thus we are left with proving the claim.

Proof of claim:

""" Every element of $\mathcal{S}|_V$ is of the form $\frac{d}{dx}|_{0}\psi \circ b_{\lambda}$ where $\{b_{\lambda}\}_{\lambda \in I}$ is a 1-parameter family of bisections of $H$ defined on $V$, with $b_{0} = 1|_V$. By Def. 4.14, $b: I \times V \to H$ is smooth.

Fix $y \in V$. There is an open neighborhood $V_y$ in $V$ such that the map $b: I \times V \to \hat{\mathcal{H}}$ is well-defined, where $\hat{\mathcal{H}}$ is as in Def. 4.19. Shrinking $I$ to a smaller open interval about zero if necessary. Shrinking $V_y$ if necessary, $b|_{I \times V_y}$ is a plot in $\mathcal{P}(\hat{\mathcal{H}})$, by Rem. 4.3 and by Prop. 4.2, and there exists a generating plot $\chi': O_{\chi'} \to \hat{\mathcal{H}}$ (as in Def. 4.19) and a smooth map $h': I \times V_y \to O_{\chi'}$ such that $b|_{I \times V_y} = \chi' \circ h'$. For the sake of readability, let us temporarily assume that $V_y = V$.

We just “lifted” $b$ to $O_{\chi'}$, and now we argue that $b$ can be lifted to the fixed generating plot $O_{\chi}$ introduced at the beginning of the proof, in such a way that $\{0\} \times V$ is mapped to the submanifold $e$ fixed there. Notice that $\chi'$ maps the submanifold $h'_0(V)$ diffeomorphically onto $b_0(V) = 1(V)$, so since $H$ is holonomy-like (Def. 4.19) there exists a smooth map $k: O_{\chi'} \to O_{\chi}$
such that $\chi \circ k = \chi'$ mapping $h'_0(V)$ to $e$, shrinking $\mathcal{O}_{\chi'}$ to a smaller neighborhood of $h'_0(y)$ if necessary. Therefore

$$h := k \circ h' : I \times V \to \mathcal{O}_{\chi}$$

satisfies $\chi \circ h = \chi \circ k \circ h' = \chi' \circ h' = b$ (i.e., the diagram below commutes) and $h_0(V) = k(h'_0(V)) = e$.

$$\mathcal{O}_{\chi} \xrightarrow{\chi} H \xleftarrow{b} I \times V$$

(5.4)

Even more, $h_0$ coincides with $\bar{1} : V \to e$, by the commutativity of diagrams (5.2) and (5.4) together with the facts that $1|_V$ and $\chi|_e$ are diffeomorphisms onto their images.

For every $\lambda \in I$, $h_{\lambda}$ is a section of $s_H \circ \chi$, since $b_{\lambda} = \chi \circ h_{\lambda}$ is a section of $s_H$. Consider $\Psi \circ b_{\lambda} = (\Psi \circ \chi) \circ h_{\lambda}$. Taking derivatives (notice that $\Psi \circ \chi : \mathcal{O}_{\chi} \to G$ is smooth) we obtain

$$\frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_{\lambda}) = (\Psi \circ \chi)_{*} \frac{d}{d\lambda}|_{\lambda=0}(h_{\lambda}).$$

(5.5)

Notice that $\frac{d}{d\lambda}|_{\lambda=0}(h_{\lambda})$ is a section of $\text{Vert}_{|e}$, proving the desired inclusion under the assumption $V_y = V$.

For the general case, we argue as follows. There are open subsets $V^i$ covering $V$ and maps $h^i : I \times V^i \to \mathcal{O}_{\chi}$ so that $(h^i)_0 : V^i \to \mathcal{O}_{\chi}$ equals $1|_{V^i}$. Our reasoning above shows that eq. (5.5) holds on $V^i$, replacing $h$ by $h^i$. Let $\{\varphi^i\}$ be a partition of unity subordinate to the open cover $\{\bar{1}(V^i)\}$ of $e$. Then eq. (5.5) holds, on the whole of $V$, replacing the section $\frac{d}{d\lambda}|_{\lambda=0}(h_{\lambda})$ of $\text{Vert}_{|e}$ with the section $\sum \{\bar{1}^* \varphi^i\} \frac{d}{d\lambda}|_{\lambda=0}(h_{\lambda}^i)$.

"⇒ Any element of $\text{Vert}_{|e}$ can be written as $\frac{d}{d\lambda}|_{\lambda=0}(h_{\lambda})$ for a smooth family of sections $h_{\lambda} : V \to \mathcal{O}_{\chi}$ of $s_H \circ \chi$ through $e$. (This uses the fact that since $s_H \circ \chi$ is a submersion at points of $e$, it is a submersion also in a tubular neighbourhood of it.) Further $\chi \circ h_{\lambda}$ is a family of bisections of $H$ which is smooth by construction and goes through $1_V$. Hence $\frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ \chi \circ h_{\lambda})$ is an element of $S|_V$.

Assumption 5.1 is used to prove that $S$ is involutive. We will make use of the following fact about 1-parameter groups of global bisections of Lie groupoids. Let $\{\sigma_{\lambda} : M \to G\}_{\lambda \in I}$ be such a 1-parameter group. For any fixed $\lambda \in I$, denote by

$$L_{\sigma_{\lambda}} : G \to G, g \mapsto \sigma_{\lambda}(t_{G}(g)) \cdot g$$

the left translation by $\sigma_{\lambda}$. Then $L_{\sigma_{\lambda} \cdot \nu} = L_{\sigma_{\lambda} \cdot \sigma_{\nu}} = L_{\sigma_{\lambda}} \circ L_{\sigma_{\nu}}$, therefore $\{L_{\sigma_{\lambda}}\}_{\lambda \in I}$ is the flow of a vector field on $G$. By differentiation it is easily seen that this vector field is the right-invariant vector field $\frac{d}{d\lambda}|_{0}\sigma_{\lambda}$.

**Lemma 5.7.** The module $S \subseteq \Gamma_c(AG)$ is involutive.
Thus generated by its element such that for some Lemma 5.8. Let \( \{ V_i \} \) be a cover of \( M \) by open subsets, on which the inclusion in Assumption 5.1 holds; we may assume that finitely many of these open sets cover \( \text{Supp}(\alpha_1) \cap \text{Supp}(\alpha_2) \). Let \( \{ \varphi_i \} \) a partition of unity subordinate to \( \{ V_i \} \). Then \( [\alpha_1, \alpha_2] = \sum_{i,j} [\varphi_i \alpha_1, \varphi_j \alpha_2] \).

Thanks to Lemma 5.8, we only need to show that each summand lies in \( S \).

Fix indices \( i, j \). Thanks to Assumption 5.1 we can write \( \varphi_i \alpha_1 = \frac{d}{d\lambda} |_{0} (\Psi \circ b_{\lambda}) \) and \( \varphi_j \alpha_2 = \frac{d}{d\lambda} |_{0} (\Psi \circ b'_{\lambda}) \), where \( \{ b_{\lambda} \}_{\lambda \in I} \), \( \{ b'_{\lambda} \}_{\lambda \in I} \) are 1-parameter groups of global bisections of \( H \). Their Lie bracket is the restriction to \( M \) of the Lie bracket of the associated right-invariant vector fields on \( G \)

\[
\left[ \frac{d}{d\lambda} |_{0} (\Psi \circ b_{\lambda}) , \frac{d}{d\lambda} |_{0} (\Psi \circ b'_{\lambda}) \right],
\]

whose flows are given by the left translations \( \{ L_{\Psi \circ b_{\lambda}} \}_{\lambda \in I} \) and \( \{ L_{\Psi \circ b'_{\lambda}} \}_{\lambda \in I} \) respectively. Using the characterization of the Lie bracket of two vector fields on any manifold in terms of their flows, we find that the value of (5.6) at some \( g \in G \) is

\[
\frac{d}{d\lambda} |_{0} \left( L_{\Psi \circ b'_{\lambda}} \circ L_{\Psi \circ b_{\lambda}} \circ L_{\Psi \circ b'_{\lambda}} \circ L_{\Psi \circ b_{\lambda}} \right)(g) = \frac{d}{d\lambda} |_{0} \left( L_{\Psi \circ (b'_{\lambda} \circ b_{\lambda})} \circ L_{\Psi \circ (b_{\lambda} \circ b'_{\lambda})} \right)(g).
\]

Evaluating at all points of \( M \) this yields an element of \( S \), because \( b'_{\lambda} \circ b_{\lambda} \) is a 1-parameter family of global bisections of \( H \). Above we use the fact that \( \Psi \), being a groupoid morphism covering \( \text{Id}_M \), respects the multiplication \( \ast \) of 1-parameter groups of global bisections Thus \( [\varphi_i \alpha_1, \varphi_j \alpha_2] \in S \), concluding the proof.

Theorem 5.2 is thus proven, thanks to Lemmas 5.3, 5.6, 5.7.

### 5.1.2 About the module \( S \)

We end this subsection arguing that \( S \) “contains many elements”. First notice that any \( C^\infty(M) \)-module of compactly supported sections is generated by its elements supported on arbitrarily small open subsets, as one can prove using a partition of unity argument:

**Lemma 5.8.** Let \( \mathcal{M} \subseteq \Gamma_c(AG) \) be a \( C^\infty(M) \)-module and \( \{ V_i \} \) be open cover of \( M \). Then \( \mathcal{M} \) is generated by its element such that for some \( i \) their support is contained in \( V_i \).

The following proposition provides elements of \( S \) supported on arbitrarily small open subsets. It shows that if a compactly supported section \( \alpha \) of \( AG \) arises from a family of not necessarily compactly supported bisections of \( H \), then \( \alpha \) lies in \( S \).

**Proposition 5.9.** Let \( V \subset M \) be an open subset, let \( \alpha = \frac{d}{d\lambda} |_{\lambda=0} (\Psi \circ b_{\lambda}) \in \Gamma_c(AG|_V) \) where \( \{ b_{\lambda} \}_{\lambda \in I} \) is family of bisections of \( H \) defined on \( V \) s.t. \( b_0 = \text{Id}_V \). Then one can choose the \( b_{\lambda} \)'s so that their support is contained in a compact subset \( K \) of \( V \), shrinking \( I \) if necessary. Therefore \( \alpha \), extended trivially to the rest of \( M \), lies in \( S \).

**Proof.** Take \( f \in C^\infty_c(V) \) which is identically equal to 1 on \( \text{supp}(\alpha) \). Then

\[
\alpha = f \alpha = \frac{d}{d\lambda} |_{\lambda=0} (\Psi \circ b'_{\lambda})
\]
by eq. \ref{eq1}. Notice that the bisections $b^f = b_f$ have support contained in $K := \text{supp}(f)$ (meaning that they are the identity bisection outside of there). Extending these bisections of $H$ to global bisections, the last statement of the proposition follows. \hfill $\Box$

### 5.2 Differentiation and Lie groupoids

A first class of examples for differentiation (Def. 5.3) is provided by Lie groupoids:

**Proposition 5.10.** Let $H$ be a source-connected Lie groupoid and $\Psi : H \rightarrow G$ a Lie groupoid morphism covering $\text{Id}_M$. Define $\mathcal{B} := \Psi_*(\Gamma_c(AH))$. Then $H$ differentiates to $\mathcal{B}$.

**Proof.** Being a Lie groupoid, the diffeology of $H$ is generated by inverses of charts (see Ex. 4.6 a)), hence $H$ is holonomy-like (see Def. 4.19). Since $H$ is a Lie groupoid, we have

$$\Gamma(AH) = \left\{ \frac{d}{dx}|_{x=0}b_\lambda : \{b_\lambda\}_{\lambda \in I} \text{ 1-parameter group of global bisections of } H \right\}$$

$$= \left\{ \frac{d}{dx}|_{x=0}b_\lambda : \{b_\lambda\}_{\lambda \in I} \text{ 1-parameter family of global bisections of } H \text{ with } b_0 = \text{Id}_M \right\}.$$ 

Hence $\mathcal{S}$ equal $\Psi_*(\Gamma_c(AH)) = \mathcal{B}$. Further, the first equality implies that Assumption 5.1 is satisfied. \hfill $\Box$ 

### 5.3 Differentiation of the holonomy groupoid

A further class of examples for differentiation is given by holonomy groupoids. Let $\mathcal{B}$ be a singular subalgebroid of a Lie algebroid $\mathcal{A}$ with source-connected integration $G$.

**Theorem 5.11.** The holonomy groupoid $H^G(\mathcal{B})$ differentiates to $\mathcal{B}$ (in the sense of definition 5.3).

**Proof.** Since the holonomy-like property holds by Prop. 4.20 we only have to show that $\Phi : H^G(\mathcal{B}) \rightarrow G$ satisfies $\mathcal{S} = \mathcal{B}$ and Assumption 5.1. We will use the fact that, since both $\mathcal{S}$ and $\mathcal{B}$ are $C^\infty(M)$-submodules of $\Gamma_c(AG)$, they are generated by their sections which are supported on arbitrarily small open subsets (see Lemma 5.3). The inclusion $\mathcal{B} \subset \mathcal{S}$ follows from Proposition 4.17.

Now we prove $\mathcal{S} \subset \mathcal{B}$. To this end, let $x \in M$, and let $V$ be a sufficiently small neighborhood of $V$ of $x$. We start recalling a few facts:

- By Given any smooth bisection $b$ of $H^G(\mathcal{B})$ defined on $V$ and close enough (in the $C^0$-sense) to the identity bisection, there is a minimal path-holonomy bisubmersion $(U, \varphi, G)$ and a bisection $u$ of $U$ such that $q_U \circ u = b$, by Rem. 4.13 a). Notice that $\varphi \circ u = \Phi \circ b$.

- Denote by $\alpha_1, \ldots, \alpha_n \in \mathcal{B}$ the local generators of $\mathcal{B}$ (inducing a basis of $\mathcal{B}/\mathcal{I}_G$) used to construct the path-holonomy bisubmersion $U \subset \mathbb{R}^n \times M$. For any bisection $u = (Id, f)$ of $U$ (where $f : M \rightarrow \mathbb{R}^n$), we have $(\varphi \circ u)(y) = \exp(exp(0, y) \sum f_i(y) \alpha_i)$, see Def-Prop. 4.12.

Now let $\{b_\lambda\}$ be a smooth 1-parameter family of bisections of $H^G(\mathcal{B})$ defined on $V$, such that $b_0 = \text{Id}_M$. Denote by $u_\lambda$ a family of bisections of $U$ such that $q_U \circ u_\lambda = b_\lambda$. One can always arrange\footnote{Indeed, since $q_U \circ u_0 = b_0$ is the identity bisection of $H^G(\mathcal{B})$, composing with $\Phi$ we see that $\varphi \circ u_0$ is the identity section of $G$. Hence there is a (locally defined) morphism of bisubmersions $k : U \rightarrow U$ mapping $u_0(V)$ to $(0, V)$, by Prop. 4.13 and its proof. Since $q_U \circ k = q_U$, it follows that $k \circ u$ is also a family of bisections of $U$ lifting $b$, with the additional property that at time zero it is the zero section of $U$.} that $u_0$ is the zero section of $U$. 

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Write \( u_\lambda = (Id, f_\lambda) \). Then for all \( y \in V \) we have

\[
\frac{d}{d\lambda}|_{\lambda=0}(\varphi \circ u_\lambda)(y) = d_{(0,y)}\varphi\left(\frac{d}{d\lambda}|_{\lambda=0}u_\lambda(y)\right) = \sum_i g_i(y)\alpha_i|_y, \quad \text{for} \quad g(y) = \frac{d}{d\lambda}|_{\lambda=0}(f_\lambda)(y),
\]

where in the last equation we used that \( d_{(0,y)}\varphi \) maps the \( i \)-th canonical basis vector of \( T_y\mathbb{R}^n \subset T_yU \) to \( \alpha_i|_y \). Hence

\[
\frac{d}{d\lambda}|_{\lambda=0}(\varphi \circ b_\lambda) = \frac{d}{d\lambda}|_{\lambda=0}(\varphi \circ u_\lambda) = \sum_i g_i\alpha_i
\]

lies in \( \mathcal{B}|_V \), since \( \alpha_i \in \mathcal{B}|_V \) for all \( i \). Any element of \( \mathcal{S} \) with support contained in \( V \) is of the form \( \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda) \), for a 1-parameter family of global bisections \( b_\lambda \). We thus showed that \( \mathcal{S} \subset \mathcal{B} \).

Having showed that \( \mathcal{S} = \mathcal{B} \), we apply again Proposition 4.17 to conclude that Assumption 5.1 is satisfied.

### 5.4 The functoriality of differentiation

In this subsection we cast differentiation as a functor. We fix a manifold \( M \) and consider two categories. They both differ slightly from those introduced in [27, §4].

The category \( \text{SingSub}_M \) has:

- objects:
  \[(A, \mathcal{B}) \mid A \text{ a Lie algebroid over } M, \mathcal{B} \text{ a singular subalgebroid of } A\]

- arrows from \((A_1, \mathcal{B}_1)\) to \((A_2, \mathcal{B}_2)\):
  \[
  \{\psi: A_1 \to A_2 \text{ a morphism of Lie algebroids covering } Id_M, \text{ such that } \psi(\mathcal{B}_1) \subset \mathcal{B}_2\}
  \]

The category \( \text{DiffeoGrd}_M \) has:

- objects:
  \[
  \{\Phi: H \to G \text{ a morphism of diffeological groupoids covering } Id_M, \text{ where } H \text{ is diffeological groupoid which is holonomy-like, } G \text{ is a source-connected Lie groupoid, satisfying Assumption 5.1}\}
  \]

- arrows from \((\Phi_1: H_1 \to G_1)\) to \((\Phi_2: H_2 \to G_2)\):
  \[
  \{(\Xi, F) \mid \Xi: H_1 \to H_2 \text{ a morphism of diffeological groupoids over } Id_M, \text{ s.t. the diagram below commutes}\}
  \]

\[
\begin{array}{ccc}
H_1 & \xrightarrow{\Xi} & H_2 \\
\downarrow{\Phi_1} \quad & & \quad \downarrow{\Phi_2} \\
G_1 & \xrightarrow{F} & G_2
\end{array}
\]

(5.7)
Definition 5.12. We define the differentiation functor

\[ D : \text{DiffeoGrd}_M \to \text{SingSub}_M \]

\[ (\Phi: H \to G) \mapsto (A, B) := (AG, S) \]

\[ (\Xi, F) \mapsto F_* : AG_1 \to AG_2 \]

where \( S \) was defined in §5.1 and \((\Xi, F)\) is an arrow from \((\Phi_1 : H_1 \to G_1)\) to \((\Phi_2 : H_2 \to G_2)\).

The functor \( D \) is well-defined on objects it is because of §5.1. We now check that \( D \) is well-defined on morphisms, i.e. that \( F_* B_1 \subset B_2 \). Every element of \( B_1 \) is of the form \( \frac{d}{d\lambda}\big|_{\lambda=0}(\Phi_1 \circ b_\lambda) \) where \( \{b_\lambda\}_{\lambda \in I} \) is a 1-parameter family of global bisections for \( H^{G_1}(B_1) \). Using the commutativity of diagram (5.7) we have

\[ F_* \left( \frac{d}{d\lambda}\big|_{\lambda=0}(\Phi_1 \circ b_\lambda) \right) = \frac{d}{d\lambda}\big|_{\lambda=0}(F \circ \Phi_1 \circ b_\lambda) = \frac{d}{d\lambda}\big|_{\lambda=0}(\Phi_2 \circ \Xi \circ b_\lambda) \]

and the latter lies in \( B_2 \), since \( \Xi \circ b_\lambda \) is a global bisection for \( H^{G_2}(B_2) \).

We now present some remarks and examples. At first sight the differentiation functor at the level of arrows looks strange: it maps \((\Xi, F)\) to \( F_* \), hence \( \Xi \) seems to “be lost”. The following lemma shows that – under reasonable assumptions – this is not the case, since \( \Xi \) is determined by \( F \).

Lemma 5.13. Let \( \Phi_i : H_i \to G_i \) be an object in \( \text{DiffeoGrd}_M \), for \( i = 1,2 \). Assume that through every point of \( H_1 \) there passes a bisection. Assume that \( \Phi_2 \) is almost injective (see Def. 6.1 later on). Let \((\Xi, F)\) be an arrow in \( \text{DiffeoGrd}_M \) from \( \Phi_1 \) to \( \Phi_2 \), as in diagram 5.7. Then \( \Xi \) is determined uniquely by \( F \).

Proof. Assume that \((\Xi, F)\) and \((\tilde{\Xi}, F)\) are both morphisms in \( \text{DiffeoGrd}_M \) from \( \Phi_1 \) to \( \Phi_2 \). We have to show that \( \Xi = \tilde{\Xi} \).

Take \( \tilde{H}_2 \) to be a symmetric neighborhood of the identity in \( H_2 \) such that \( \tilde{H}_1 \cdot \tilde{H}_2 \subset \tilde{H}_2 \), where the latter is the neighborhood of the identity in \( H_2 \) satisfying the almost injective condition in Def. 6.1. Then \( H_1 := \Xi^{-1}(\tilde{H}_2) \cap \tilde{\Xi}^{-1}(\tilde{H}_2) \) is a neighborhood of the identity in \( H_1 \).

Let \( b \) be a bisection in \( H_1 \). Then

\[ (\Phi_2 \circ \Xi)b = (F \circ \Phi_1)b = (\Phi_2 \circ \tilde{\Xi})b \]

due to the commutativity of diagram 5.7. This means that \( \Xi \ b \ast (\tilde{\Xi} \ b)^{-1} \) is a bisection in \( \tilde{H}_2 \) that maps under \( \Phi_2 \) to the identity bisection of \( G_2 \). The assumption on \( H_2 \) implies that \( \Xi \ b = \tilde{\Xi} \ b \).

Hence \( \Xi \) and \( \tilde{\Xi} \) agree on \( \tilde{H}_1 \). Finally, by the source-connectedness of \( H_1 \), they agree on the whole of \( H_1 \).

Remark 5.14. As we saw in Thm. 5.11, given an object \((A, B)\) in \( \text{SingSub}_M \) such that \( A \) is an integrable Lie algebroid, the holonomy groupoid \( H^G(B) \) (for any choice of Lie groupoid \( G \) of \( A \)) differentiates to \((A, B)\). We now extend this statement to arrows.

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11 This holds in particular if \( H_1 \) satisfies the assumptions of Lemma 4.22 or Prop. 4.23.
Consider objects \( (A_1, B_1) \) and \( (A_2, B_2) \) in \( \text{SingSub}_M \) together with an arrow between them, i.e. a Lie algebroid morphism \( \psi: A_1 \to A_2 \) such that \( \psi_*(B_1) \subset B_2 \). Make a choice of Lie groupoid morphism \( F: G_1 \to G_2 \) integrating \( \psi \). (These data comprise exactly the arrows of the category \( \text{SingSub}^\text{Gpd}_M \) we defined in [27, §4].) We saw in [27, Thm. 4.6] that these data give rise canonically to an arrow \((\Xi, F)\) in \( \text{DiffeoGrd}_M \), where \( \Xi: H^G(B_1) \to H^G(B_2) \). The arrow \( \Xi, F \) differentiates to the original arrow \( \psi: A_1 \to A_2 \), by the definition of the functor \( D \).

Loosely speaking, given a singular subalgebroid \( B \) of \( A \), the canonical morphism \( \Phi: H^G(B) \to G \) differentiates to the inclusion of \( B \) in \( \Gamma_c(A) \). We make this precise as follows.

**Example 5.15.** Let \( B \) be a singular subalgebroid of \( A \), and \( G \) a Lie groupoid integrating \( A \). Consider the arrow \( \Phi, Id_G \) in \( \text{DiffeoGrd}_M \) from \( \Phi: H^G(B) \to G \) to \( Id_G \). It clearly differentiates to \( Id_A: A \to A \), which at the level of sections restricts to the inclusion \( B \to \Gamma_c(A) \).

Notice that the arrow \( Id_A \) in \( \text{SingSub}_M \) from \( (A, B) \) to \( (A, \Gamma_c(A)) \), by the procedure recalled in Remark 5.14, gives rise [27, Ex. 4.12] exactly to the arrow \( \Phi, Id_G \) in \( \text{DiffeoGrd}_M \) considered in Ex. 5.15.

### 6 Global Integration of a singular subalgebroid

In this section we introduce a global notion of integration for a singular subalgebroid \( B \) (Def. 6.1), which we call integral, and prove that the holonomy groupoid \( H^G(B) \) is an integral (Thm. 6.19). We also show that \( B \) admits a smooth integral iff it is projective (Cor. 6.15). In that case \( H^G(B) \) is the minimal smooth integral, and all other smooth integrals are coverings of \( H^G(B) \) (Prop. 6.16 and 6.17). When \( B \) is a Lie subalgebra of a Lie algebra, all integrals are smooth (Prop. 6.6).

#### 6.1 Definition of integral

Definition 6.1 below is inspired by [1, Appendix A]. It relies on the notion of differentiation introduced in Def. 5.3.

**Definition 6.1 (Integrals).** Let \( G \) be a Lie groupoid over \( M \), \( H \) be a diffeological groupoid over \( M \), and \( \Psi: H \to G \) a smooth morphism of diffeological groupoids covering \( Id_M \).

Let \( B \) be a singular subalgebroid of \( A := AG \). We say that \( (H, \Psi) \) is an integral of \( B \) over \( G \) if:

a) \( (H, \Psi) \) differentiates to \( B \) (Def. 5.3). (In particular \( H \) is holonomy-like, see Def. 4.19 and Assumption 5.1 is satisfied.)

b) The diffeology of \( H \) is generated by open maps (Def. 4.24).

c) The morphism \( \Psi \) is almost injective, in the following sense: There exists a neighborhood \( \tilde{H} \) of the identity \( 1_M \) in \( H \) such that if \( b \subset \tilde{H} \) is a (local) bisection carrying the identity bisection of \( G \), then \( b \subset 1_M \).

Several remarks are in order.
Remark 6.2 (The three conditions). We comment on the three conditions appearing in Def. 6.1.

a) Condition a) is certainly expected.

b) Condition b) implies that $H$ is source-submersive – i.e. through every point there passes a bisection –, thanks to Prop. 4.23 i). This makes condition c) a meaningful one.

c) The almost injective requirement on $\Psi$ is imposed to ensure that $H$ is “not larger than necessary”. For example, when $M$ is a point, so that $H$ and $G$ are groups, $\Psi$ must be an injection in a neighborhood of the identity element of $H$.

Remark 6.3 (Conditions on $H$). In Def. 6.1 on integrals, there are two requirements on the diffeological groupoid $H$: that it be holonomy-like (Def. 4.19), and the openness condition b). Each of these two requirements consist of the existence of a generating set of plots with certain properties, in some neighborhood $\tilde{H}$ of $1_M$ in $H$ (see Remark 4.25).

i) If the generating set of plots $\{\chi_i : O_{\chi_i} \to \tilde{H}\}$ required by the holonomy-like property consists of open maps, then both requirements are satisfied. A special case is when the diffeology of $H$ is generated by local subductions (Def. 4.26), according to Lemma 4.28. Spelled out, this means that for each $i$ the image $\chi_i(O_{\chi_i})$ is an open subset of $H$ and $\chi_i : O_{\chi_i} \to \chi_i(O_{\chi_i})$ is a local subduction, by Remark 4.27. This is exactly what occurs for the examples of integrals we display in §6.3 and §6.4, namely smooth integrals and the holonomy groupoid.

ii) The above two requirements together, by Lemma 4.9, imply that the generating set $\{\chi_i : O_{\chi_i} \to \tilde{H}\}$ for the holonomy-like property can be chosen so that the image of each map $\chi_i$ is an open subset (but not necessarily that the $\chi_i$ are open maps). This observation supports the scenario outlined in item i).

Remark 6.4 (Restatement of Assumption 5.1). In Def. 6.1 the differentiation condition a) contains in particular Assumption 5.1. For integrals, one can give an equivalent characterization of this assumption in terms of the liftability of 1-parameter groups of global bisections of $G$, see Corollary C.2. This equivalent characterization will be used only in the proof of Prop. 7.7, which states that the image $\Psi(H)$ is the same for all integrals $(H, \Psi)$.

Remark 6.5 (Functoriality). The differentiation functor introduced in §5.4 is well-behaved when restricted to integrals. Indeed Lemma 5.13 applies automatically to integrals, thanks to conditions b) and c) above.

6.2 The integrals of Lie subalgebras

Here we consider the special case of a Lie subalgebra of a (finite dimensional real) Lie algebra. We show that all integrals are actually Lie groups.

Proposition 6.6. Let $\mathfrak{g}$ a Lie algebra, $\mathfrak{k}$ a Lie subalgebra, fix a Lie group $G$ integrating $\mathfrak{g}$. Let $(H, \Psi)$ be an integral of $\mathfrak{k}$.

Then there exists a Lie group structure whose underlying diffeological group is $H$.

Remark 6.7. The above Lie group structure, which we denote by $H_{\text{Lie}}$, is unique by Ex. 4.6 a). Further $H_{\text{Lie}}$ integrates the Lie algebra $\mathfrak{k}$, and $\Psi$ – viewed as a map on $H_{\text{Lie}}$ – integrates the inclusion $\mathfrak{k} \hookrightarrow \mathfrak{g}$, see the proof of Prop. 6.6 below. Thus $H_{\text{Lie}}$ is a covering of the (unique) connected Lie subgroup of $G$ integrating the Lie subalgebra $\mathfrak{k}$.
Remark 6.8. We recall what it means that \((H, \Psi)\) is an integral of \(\mathfrak{t}\) (Def. 6.1):

- \(H\) is a diffeological group which is holonomy-like\(^{12}\) (we will not use the latter condition).
- \(\Psi : H \to G\) is a smooth morphism of diffeological groups such that

\[
\mathfrak{t} = \left\{ \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda) : \{b_\lambda\}_{\lambda \in I} \text{ 1-parameter group of elements of } H \right\}
\]

\[
= \left\{ \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda) : \{b_\lambda\}_{\lambda \in I} \text{ 1-parameter family of elements of } H \right\}
\]

and \(\Psi\) is injective on a neighborhood of the unit in \(H\).
- The diffeology of \(H\) is generated by open maps (we will not use this condition).

\[\text{Proof.}\] First observe that if \(\{b_\lambda\}\) is a 1-parameter group in \(H\), then \(\{\Psi \circ b_\lambda\}\) is a 1-parameter group in the Lie group \(G\), hence

\[
\Psi \circ b_\lambda = \exp(\lambda v)
\]

(6.1)

where \(v := \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda)\) and \(\exp\) is the Lie group exponential map.

Denote by \(K\) the unique connected Lie subgroup of \(G\) integrating the Lie subalgebra \(\mathfrak{t}\).

\[\text{Claim: } \Psi(H) = K.\]

\[\text{Proof of claim:}\]

- “⊃” There is a neighborhood of the unit in \(K\) consisting of elements of the form \(\exp(v)\) where \(v \in \mathfrak{t}\). By assumption, there is a 1-parameter group \(\{b_\lambda\}\) in \(H\) such that \(v = \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda)\).

By the above observation, taking \(\lambda = 1\), we get \(\Psi \circ b_1 = \exp(v)\). Conclude using that \(K\), being connected, is generated by any neighborhood of the unit.

- “⊂” For any diffeological space, the set of points that can be connected to a given point by a smooth path is open in the \(D\)-topology. (This can be proved exactly as \[^{17}\] Lemma 1.8, which states that the \(D\)-topology is locally arc-connected. Notice that while the composition of two smooth paths is not smooth in general, it is if we first apply a suitable time reparametrization.) In particular\[^{13}\], the set of points in \(H\) that can be connected to the unit \(e\) by a smooth path is an open subset \(\tilde{H}\) of \(H\).

Let \(h \in \tilde{H}\) and \(\{b_\lambda\}_{\lambda \in [0,1]}\) be a smooth path in \(H\) from the identity element \(e\) to \(h\) (in other words, a 1-parameter family). For every fixed \(\lambda_0 \in [0,1]\) consider the smooth path \(\{b_{\mu+\lambda_0} \cdot b_{\lambda_0}^{-1}\}_{\mu \in I_\mu}\) where \(I_\mu\) is an open interval containing zero. Since \(H\) differentiates to \(\mathfrak{t}\), we have

\[
\frac{d}{d\mu}|_{\mu=0} \left( \Psi(b_{\mu+\lambda_0} \cdot b_{\lambda_0}^{-1}) \right) \in \mathfrak{t}.
\]

\(^{12}\)By Def. \[^{10}\] this means that there exists an open neighborhood \(\tilde{H}\) of the unit 1 in \(H\) and a generating set of plots \(\mathcal{X} = \{\chi : \mathcal{O} \to \tilde{H}\}\) in \(\mathcal{P} (\tilde{H})\) such that: given any plots \(\chi, \chi'\) in \(\mathcal{X}\) and points \(e \in \mathcal{O}, e' \in \mathcal{O}\) such that \(\chi(e) = \chi'(e') = 1\), there exists a smooth map \(k : \mathcal{O} \to \mathcal{O}\) with \(k(e') = e\) and \(\chi \circ k = \chi'\), after shrinking \(\mathcal{O}\) to a smaller neighborhood of \(e'\) if necessary.

\(^{13}\)Alternatively, this also follows from Condition b) in Def. \[^{6}\].
Since $\Psi$ is a groupoid morphism, the left hand side equals
\[
\frac{d}{d\mu}\big|_{\mu=0} R_{\Psi(b_{\lambda_0})}^{-1}(\Psi(b_{\mu+\lambda_0})) = (R_{\Psi(b_{\lambda_0})})_* \frac{d}{d\lambda}|_{\lambda=\lambda_0}(\Psi(b_{\lambda}))
\]
where $R$ denotes right-translation on $G$. Hence $\frac{d}{d\lambda}|_{\lambda=\lambda_0}(\Psi(b_{\lambda}))$ lies in the right-invariant distribution $\mathfrak{f}$. Repeating for all $\lambda_0 \in [0, 1]$ we see that the curve $[0, 1] \ni \lambda \mapsto \Psi(b_{\lambda})$ lies in the leaf of $\mathfrak{f}$ through $e$, which is exactly the Lie subgroup $K$. We conclude that $\Psi(h) \in K$.

The morphism of topological groups $\Psi : H \to G$ is injective in a neighborhood of the unit, and by the claim its image is the Lie group $K$. Hence $K$, via $\Psi$, induces a Lie group structure on $H$, which we denote by $H_{\text{Lie}}$, and whose Lie algebra is isomorphic to $\mathfrak{f}$. We are left with showing that the diffeological structure underlying the manifold structure of $H_{\text{Lie}}$ coincides with the diffeological structure of $H$.

**Claim:** The identity map is an isomorphism of diffeological groups between $H$ and $H_{\text{Lie}}$.

**Proof of claim:** Since $\Psi : H \to G$ is smooth, the induced surjective map to the Lie group $K$ is a smooth map: for any plot $\chi$ into $H$, the composition $\Psi \circ \chi$ is a smooth plot into $K$, since $K$ is an initial submanifold of $G$ [15] Thm. 19.25]. This implies that the identity $H \to H_{\text{Lie}}$ is a smooth.

We now show that the identity $H_{\text{Lie}} \to H$ is a smooth. Thanks to Lemma 6.10 below, it suffices to show that all plots for the strong diffeology of $H_{\text{Lie}}$ are also plots for the diffeology on $H$. That is, we have to show that all rays in $H_{\text{Lie}}$—which as is well-known are of the form $\lambda \mapsto \exp_{H_{\text{Lie}}}(\lambda v)$ for elements $v$ in the Lie algebra $\mathfrak{h}$—are plots for $H$. To show this, fix $v \in \mathfrak{h}$, and consider $\Psi_*v \in \mathfrak{f}$. By assumption, there is a (smooth) 1-parameter group $\{b_\lambda\}_{\lambda \in I}$ in $H$ such that $\Psi_*v = \frac{d}{d\lambda}|_{\lambda=0}(\Psi \circ b_\lambda)$, and by the above observation $\Psi \circ b_\lambda = \exp_K(\lambda \Psi_*v)$. By the injectivity of $\Psi$ near the unit, this implies that $b_\lambda = \exp_{H_{\text{Lie}}}(\lambda v)$ for values of $\lambda$ close enough to zero, showing that $\lambda \mapsto \exp_{H_{\text{Lie}}}(\lambda v)$ is smooth for the diffeology of $H$.

\[\square\]

**Remark 6.9.** In the setting of Prop. 6.6 recall that $K$ is the holonomy groupoid $H^G(\mathfrak{f})$ of $\mathfrak{f}$, by Ex. 1.22. The integral $H$ quotients to $H^G(\mathfrak{f})$ (see the proof of Prop. 6.6). This fact will be generalized in Prop. 6.17.

We recall a few notions from [24] §6. Given a diffeological group $H$, a ray is a smooth group homomorphism $\mathbb{R} \to H$. The strong diffeology on $H$ is the smallest diffeology on $H$ containing all rays and so that the group multiplication and inversion are smooth. All plots for the strong diffeology are plots for the original diffeology on $H$.

For Lie groups the converse holds. This is the content of the following result, which was used in the proof of Prop. 6.6 above and was stated without proof by Souriau [24] §6.13]. We sketch a proof for the sake of completeness.

**Lemma 6.10.** Let $H$ be a Lie group. Then the strong diffeology of $H$ coincides with the Lie group diffeology.

**Proof.** It is well-known that the Lie group exponential map $\exp : \mathfrak{h} \to H$, once restricted to a suitable open neighborhood of the origin, provides a chart for the manifold structure on the Lie group $H$.  

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In view of the text just before this lemma, it suffices to show that \( \exp \) is smooth with respect to the strong diffeology.

To this aim, let \( n := \dim(H) \), and fix a basis \( v_1, \ldots, v_n \) of the Lie algebra \( \mathfrak{h} \). For a small enough neighborhood \( U \subset \mathbb{R}^n \) of the origin, the map

\[
S: U \to H, \quad (\lambda_1, \ldots, \lambda_n) \mapsto \exp(\lambda_1 v_1) \cdot \exp(\lambda_2 v_2) \cdots \cdot \exp(\lambda_n v_n)
\]

is a smooth map to the Lie group \( H \) (actually, a diffeomorphism onto its image), and it is also smooth for the strong diffeology, since it is obtained multiplying rays. Consider the map

\[
B: U \to \mathfrak{h}, \quad (\lambda_1, \ldots, \lambda_n) \mapsto BCH\left(\lambda_1 v_1, BCH\left(\ldots, BCH(\lambda_{n-1} v_{n-1}, \lambda_n v_n)\right)\right),
\]

where “BCH” denotes the Baker-Campbell-Hausdorff formula. The relation between these two maps is \( S = \exp \circ B \). Thus \( B = \exp^{-1} \circ S: U \to \mathfrak{h} \) is a diffeomorphism onto its image. Restricting to this image we have

\[
\exp = S \circ B^{-1},
\]

which is a plot for the strong diffeology as the composition the smooth map \( B^{-1} \) between open subsets of vector spaces and the plot \( S \) for the strong diffeology.

6.3 Smooth integrals

In this subsection we determine which singular subalgebroids admit integrals (in the sense of definition 6.1) which are smooth. For any such singular subalgebroid, we exhibit all the smooth integrals.

Let \( H \rightarrowtail M \) be a diffeological groupoid, in the sense of Def. 4.10. If there exists a manifold structure on \( H \) making the above diffeological groupoid a Lie groupoid, this manifold structure is unique, by Ex. 4.6 a).

**Definition 6.11.** An integral \((H, \Psi)\) of a singular subalgebroid \( B \) is said to be a smooth integral if the diffeological groupoid \( H \) admits a (necessarily unique) manifold structure making it a Lie groupoid.

**Remark 6.12.** If \((H, \Psi)\) is a smooth integral of \( B \), then \( \Psi \) is automatically a Lie groupoid morphism, by Ex. 4.6 a). Further, \( B = \Psi_* (\Gamma_c(AH)) \) by Def. 6.1 a).

**Lemma 6.13.** Assume \( H \) is a Lie groupoid and \( \Psi: H \to G \) a Lie groupoid morphism covering \( \text{Id}_M \). Define \( B := \Psi_* (\Gamma_c(AH)) \).

Then \( \Psi: H \to G \) is an integral for \( B \) iff the Lie algebroid morphism \( \Psi_*: AH \to AG \) is almost injective.\(^{14}\)

**Proof.** “\( \Rightarrow \)” Let \( \alpha \in \Gamma_c(AH) \) such that \( \Psi_* \alpha = 0 \). It suffices to show that \( \alpha = 0 \). Define a smooth family of bisections \( b_\lambda: M \to H \) by \( b_\lambda(x) = \exp_x(\lambda \overrightarrow{\alpha}) \). For all \( x \in M \),

\[
\Psi \circ \exp_x(\lambda \overrightarrow{\alpha}) = \exp_x(\lambda \Psi_* \overrightarrow{\alpha}) = x,
\]

where in the first equality we used that \( \Psi \) is a Lie groupoid morphism and in the second that \( \Psi_* \overrightarrow{\alpha} = \Psi_* \overrightarrow{\alpha} \) is the zero vector field on \( G \). This shows that \( \Psi \circ b_\lambda \) is the identity bisection of \( G \), for

\(^{14}\)Recall that this means that \( \Psi_* \) fiber-wise injective on a dense subset of \( M \).
all \( \lambda \) sufficiently close to zero, hence by a) in Def. 6.1 we obtain that \( b_\lambda \) is the identity section of \( H \) for all such \( \lambda \), and therefore that \( \alpha = 0 \).

\( \leftarrow \) Condition c) in Def. 6.1 was checked in Prop. 5.10. Condition b) holds for Lie groupoids, see Examples 4.6 a). Now we check condition a). Since \( \Psi_* \) is almost injective, for all \( x \) lying in a dense subset of \( M \) we have that the restriction of \( \Psi \) to the fiber \( s^{-1}_H(x) \) is injective nearby \( x \). This implies that \( \Psi \) is injective for bisections nearby the identity.

By Lemma 6.13, if \( \Psi : H \to G \) a smooth integral of a singular subalgebroid \( B \), then \( B \) is necessarily a projective singular subalgebroid (see §3.1), whose underlying Lie algebroid is isomorphic to \( AH \).

Given a projective singular subalgebroid, we present examples of smooth integrals for it.

**Example 6.14.** Let \( B \) be a projective singular subalgebroid of \( A = AG \), hence there exists a Lie algebroid \( B \) with an almost injective morphism

\[ \psi : B \to A \]

inducing an isomorphism \( \Gamma_c(B) \cong B \). Let \( H \) be a Lie groupoid integrating \( B \) with a morphism of Lie groupoids \( \Psi : H \to G \) integrating \( \psi \). Then \( (H, \Psi) \) is a smooth integral of \( B \) over \( G \), as follows immediately from Lemma 6.13. By Prop. 3.3 as \( H \) we can always take the holonomy groupoid \( H^G(B) \), which therefore is a smooth integral.

As an immediate corollary of the discussion above, we determine the singular subalgebroids for which there exists a smooth integral.

**Corollary 6.15.** Let \( B \) be a singular subalgebroid of \( AG \). Then \( B \) admits a smooth integral iff \( B \) is projective.

**Proof.** Let \( \Psi : H \to G \) be an integral for \( B \), where \( H \) is a Lie groupoid. Then \( \Psi_*(\Gamma_c(AH)) = S = B \), where the first equality holds by the definition of \( S \) (see §5.1) and the second because \( H \) is an integral for \( B \). Hence we can apply Lemma 6.13 which implies that \( B \) is projective.

Conversely, when \( B \) is projective, smooth integrals of \( B \) were constructed in Ex. 6.14.

Given a projective singular subalgebroid, in general we do not know whether all its integrals are smooth.

When \( B \) is projective, the holonomy groupoid \( H^G(B) \) is minimal among all of its possible smooth integrals.

**Proposition 6.16** (Minimality among smooth integrals). Let \( B \) be a singular subalgebroid of \( AG \). Let \( (H, \Psi) \) be a smooth integral of \( B \) over \( G \). Then there is a surjective morphism of Lie groupoids \( \phi : H \to H^G(B) \) such that this diagram commutes:

\[ \begin{array}{ccc}
H & \to & H^G(B) \\
\downarrow \phi & & \downarrow \\
\Psi & \to & G
\end{array} \]

**Proof.** As a consequence of Lemma 6.13 \( B = \Psi_*(\Gamma_c(AH)) \) is projective. Since \( \Psi : H \to G \) is a Lie groupoid morphism, it is a bisubmersion for \( B \) by Prop. 1.20. Prop. 1.21 then implies the statement. (The smoothness of the quotient map \( H \to H^G(B) \) follows from \( \Psi \) being a bisubmersion for \( B \)).
The following proposition characterizes the smooth integrals. It is the analogue, for projective singular subalgebroids, of a theorem of Moerdijk-Mrčun on the integration of wide Lie subalgebroids [22 Thm. 2.3].

**Proposition 6.17** (Characterization of smooth integrals). Let $B$ be a projective singular subalgebroid of $AG$. Denote by $B$ the corresponding Lie algebroid, satisfying $\Gamma_c(B) \cong B$. The smooth integrals of $B$ over $G$ are precisely the pairs

$$(H, \Phi \circ \pi),$$

where $\pi : H \to H^G(B)$ is a surjective Lie groupoid morphism integrating $\text{Id}_B$ (in particular $H$ is a Lie groupoid integrating $B$).

**Proof.** The pairs $(H, \Phi \circ \pi)$ in the statement are smooth integrals of $B$ over $G$, as explained in Ex. 6.14. All smooth integrals are of this kind, by Prop. 6.16.

**Remark 6.18.** When $B = \Gamma_c(B)$ for a wide Lie subalgebroid $B \subset AG$, its smooth integrals recover exactly the Lie groupoid morphisms of [22, Thm. 2.3].

### 6.4 The holonomy groupoid is an integral

Here we prove that $H^G(B)$ satisfies definition 6.1.

**Theorem 6.19.** The holonomy groupoid $(H^G(B), \Phi)$ is an integral of $B$ over $G$.

**Proof.** We already proved in §5.3 that $(H^G(B), \Phi)$ differentiates to $B$. We now prove that it satisfies the injectivity condition c) of definition 6.1. Let $b$ be a bisection of $H^G(B)$ carrying the identity bisection of $G$. Then locally there exists a bisubmersion $(U, \varphi, G)$ with a bisection $u$ such that $b$ is the image of $u$ by the quotient map $q_U : U \to H^G(B)$. So $u$ also carries the identity bisection of $G$, which means by Rem. 1.18 that $q_U(u) \subset 1_M$. Whence $b \subset 1_M$.

Finally, $H^G(B)$ satisfies Condition b) of definition 6.1 by Prop. 1.19 and Examples 4.6 d).

**Remark 6.20.** In view of Theorem 6.19, the question arises of whether the groupoid $H^G(B)$ is a minimal integral in some sense. In proposition 6.16 we gave a partial answer for projective singular subalgebroids, in the spirit of [22].

### 7 The graph of a singular subalgebroid

It is well known that, given a regular foliation, one can attach to it the holonomy groupoid as well as the graph of the foliation, namely the groupoid defined by the equivalence relation of “belonging to the same leaf”. The graph generally fails to be a smooth manifold, and this is one of the *raison d'être* of the holonomy groupoid: the holonomy groupoid is the minimal Lie groupoid with Lie algebroid isomorphic to the tangent distribution of the foliation.

In §7.1 for any singular subalgebroid $B$, we show that there is a natural diffeology on the graph, called path holonomy diffeology, making it into a diffeological groupoid. We remark that the construction of the path holonomy diffeology makes essential use of the holonomy groupoid. We

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15Although the holonomy groupoid is not always a Hausdorff manifold.
show that the graph differentiates to $B$ (Prop. 7.5). In particular, the path holonomy diffeology captures enough information to recover $B$. However the graph is not an integral of $B$ in general, since it generally violates Condition b) in the definition of integral (Prop. 7.6). This raises the question of whether the holonomy groupoid $H^G(B)$ is “minimal” – in a sense to make precise – among all integrals of $B$ over $G$. In §7.2 we relate the graph and the path holonomy diffeology to arbitrary integrals of $B$. In particular, we show that the image of any integral of $B$ under its canonical map to $G$ is exactly the graph (Prop. 7.7).

Let us start giving the definitions.

**Definition 7.1.** Let $B$ be a singular subalgebroid of $AG$. The graph of $B$ over $G$ is the subgroupoid $R^G(B)$ of $G$ defined as the image of the groupoid morphism $\Phi : H^G(B) \to G$.

In the case of a singular foliation $(M,F)$, the morphism $\Phi$ is the target-source map, so the graph $R(F) := R^{M \times M}(F)$ is the subgroupoid of the pair groupoid $M \times M$ given by the equivalence relation of “belonging to the same leaf of $F$”.

We now give an explicit description of the graph of an arbitrary singular subalgebroid $B$ of a Lie algebroid $AG$. This description makes clear that the graph does not depend on $B$ but just on its image under the evaluation map.

**Remark 7.2** (Description of the graph). Recall that for every leaf $L$ of (the singular foliation induced by) $B$, there is transitive Lie subalgebroid $B_L \to L$ of $AG$, obtained evaluating point-wise the elements of $B$ (see Lemma 1.7). By right-translation we obtain a regular foliation $\overrightarrow{B_L}$ on $G_L = \iota^{-1}(L)$, see also [10, Prop 5.1] for an different description. (When the leaf $L$ is immersed but not embedded, $G_L$ is endowed with the differentiable structure coming from the canonical bijection $G_L \cong L \times_{\{\iota\}} G$, where $\iota : L \to M$ is the inclusion.) The union of the leaves of $\overrightarrow{B_L}$ which intersect the manifold of unit elements is a subgroupoid of $G_L$, which we denote by $R^G(B)_L$. The graph agrees with the union

$$R^G(B) := \bigcup_L R^G(B)_L,$$

where $L$ ranges through all leaves of $B$.

To see this, for every leaf $L$, apply Thm. 2.15 $\Phi_L : H^G(B)_L \to G_L$ integrates the evaluation map $\text{ev}_L : B_L \to A_L$, whose image is $B_L$ (see the short exact sequence of Lie algebroids [1.2]).

### 7.1 The graph as diffeological groupoid

The holonomy groupoid, via the canonical morphism $\Phi : H^G(B) \to G$, induces a diffology on the graph, which we call path-holonomy diffeology. The graph $R^G(B)$ is thus naturally a diffeological groupoid, and in Prop. 7.5 we show that it differentiates to $B$. However it is not an integral of $B$ in general, since it generally violates Condition b) in the definition of integral (Prop. 7.6).

#### 7.1.1 The subspace diffeology on the graph

The subspace diffeology on the graph is the one obtained restricting the diffeology of $G$ arising from its smooth structure. The plots in this diffeology are all the smooth maps $f : U \subseteq \mathbb{R}^n \to G$ such that $f(U) \subseteq \Phi(H^G(B))$. Due to the following reasons, the subspace diffeology captures little information and is thus not useful for us:
• This diffeology does not depend on the map \( \Phi \) but rather on its image.

• This diffeology is not holonomy-like in general (see Example 7.3 b) below).

• \( R^G(\mathcal{B}) \), with the subspace diffeology, may not differentiate\(^{16}\) to \( \mathcal{B} \). This follows from Examples 7.3 below.

Example 7.3.  

a) Take \( M = \mathbb{R} \) with the foliation \( \mathcal{F}_i = \text{span}_{C^\infty(\mathbb{R})} \langle f^i \partial_x \rangle \), where \( i = 1, 2 \). Then \( R(\mathcal{F}_1) = R(\mathcal{F}_2) = (\mathbb{R}^+ \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{R}^-) \cup \{(0,0)\} \). But the differentiation of \( R(\mathcal{F}_i) \) with respect to the \((\mathbb{R} \times \mathbb{R})\)-diffeology makes the module generated by all vector fields on \( \mathbb{R} \) which vanish at 0.

b) Take \( M = \mathbb{R} \) with the foliation \( \mathcal{F} = \text{span}_{C^\infty(\mathbb{R})} \langle f \partial_x \rangle \), where \( f \) is a smooth function vanishing on \( \mathbb{R}^- \) and which is no-where vanishing on \( \mathbb{R}^+ \). Then \( R(\mathcal{F}) = \bigcup_{x \in 0} \{(x,x)\} \cup (\mathbb{R}^+ \times \mathbb{R}^+) \).

Its differentiation with respect to the \((\mathbb{R} \times \mathbb{R})\)-diffeology never returns \( \mathcal{F} \), indeed it gives the module generated by all vector fields on \( \mathbb{R} \) which vanish on \( \mathbb{R}^- \). But this module is not locally finitely generated, so this diffeology cannot be holonomy-like.

It is not surprising that \( R^G(\mathcal{B}) \) with the subspace diffeology does not differentiate to \( \mathcal{B} \). In the case of singular foliations, there may be more than one module of vector fields which induce the same partition into leaves. However, for a singular foliation \((M, \mathcal{F})\), the diffeological groupoid \( R^{M \times M}(\mathcal{F}) \) with the \( M \times M \)-diffeology depends only on the partition into leaves it induces on \( M \), hence it does not contain enough information to recover \( \mathcal{F} \) by differentiation.

7.1.2 The path holonomy diffeology on the graph

Much more useful is the path-holonomy diffeology, i.e. the quotient diffeology induced by the surjective map \( \Phi: H^G(\mathcal{B}) \rightarrow \Phi(H^G(\mathcal{B})) \) (see Prop-Def. 4.19 b)). This diffeology is generated by the plots \( \Phi \circ q_U \), where \((U, \varphi, G)\) is a bisubmersion in a path-holonomy atlas of \( \mathcal{B} \). With this diffeology, the inclusion \( \iota: R^G(\mathcal{B}) \hookrightarrow G \) is a smooth morphism, since \( \Phi \circ q_U: U \rightarrow G \) is smooth. The path-holonomy diffeology makes \( R^G(\mathcal{B}) \) into a diffeological groupoid.

The following two lemmas show that the path-holonomy diffeology is well-behaved.

Lemma 7.4. The path holonomy diffeology is holonomy-like (see Def. 4.19).

Proof. Let \( \chi: \mathcal{O} \rightarrow R^G(\mathcal{B}) \) be a plot, and \( e \subset \mathcal{O} \) a submanifold such that \( \chi|_e: e \rightarrow \chi(e) \) is a diffeomorphisms onto an open subset of \( 1_M \subset R^G(\mathcal{B}) \). We can lift \( \chi \) to a plot \( \tau \) for \( H^G(\mathcal{B}) \) (i.e. \( \chi = \Phi \circ \tau \)) after shrinking \( \mathcal{O} \) if necessary, by the definition of path holonomy diffeology. Notice that \( \tau(e) \) is a bisection of \( H^G(\mathcal{B}) \) that carries the identity of \( G \). Thus \( \tau(e) \) is contained in the set of units of \( H^G(\mathcal{B}) \), by the definition of holonomy groupoid (see the proof of Thm. 6.19 for more details).

\[ \begin{array}{ccc}
H^G(\mathcal{B}) & \xrightarrow{\Phi} & R^G(\mathcal{B}) \\
\tau \downarrow & & \chi' \downarrow \\
\mathcal{O} & \xrightarrow{\chi} & \mathcal{O}'
\end{array} \]

\(^{16}\)Here we are applying Def. 5.3 despite the fact that \( R^G(\mathcal{B}) \) with the \( G \)-diffeology might fail to be holonomy-like.
Repeat this procedure for another such plot $\chi': \mathcal{O}' \to R^G(\mathcal{B})$ satisfying $\chi(e) = \chi'(e')$. Since $\tau|_e: e \to \tau(e)$ and $\tau'|_{e'}: e' \to \tau'(e')$ are diffeomorphisms onto the same open subset of $1_M \subset H^G(\mathcal{B})$, the fact that $H^G(\mathcal{B})$ is holonomy-like (Prop. 4.20) implies that there exists $k: \mathcal{O}' \to \mathcal{O}$ with $k(e') = e$ and $\tau \circ k = \tau'$, after shrinking the domain if necessary. Composing both sides with $\Phi$ we find $\chi \circ k = \chi'$.

\textbf{Proposition 7.5.} Endowed with the path holonomy diffeology, the graph $R^G(\mathcal{B})$ differentiates to $\mathcal{B}$ (Def. 5.3).

\textit{Proof.} We will show that $\mathcal{B}$ equals

$$\left\{ \frac{d}{d\lambda}|_{\lambda=0} c_\lambda : \{c_\lambda\}_{\lambda \in I} \text{ smooth family of global bisections for } R^G(\mathcal{B}) \text{ s.t. } b_0 = Id_M \right\} \cap \Gamma_c(AG).$$

Using this, it is clear that $R^G(\mathcal{B})$ satisfies Assumption 5.1 since $H^G(\mathcal{B})$ satisfies it and $\Phi$ is a smooth groupoid morphism. Together with Lemma 7.4 this implies the claim.

\textit{“$\subset$”} Any $\alpha \in \mathcal{B}$ is of the form $\frac{d}{d\lambda}|_{\lambda=0}(\Phi \circ b_\lambda)$ for a smooth family of global bisections $\{b_\lambda\}$ of $H^G(\mathcal{B})$, since the holonomy groupoid differentiates to $\mathcal{B}$. Since $c_\lambda := \Phi \circ b_\lambda$ is a smooth family of global bisections of $R^G(\mathcal{B})$, this inclusion is proven.

\textit{“$\supset$”} let $\{c_\lambda\}_{\lambda \in I}$ be smooth family of global bisections of $R^G(\mathcal{B})$, such that $\alpha := \frac{d}{d\lambda}|_{\lambda=0} c_\lambda$ lies in $\Gamma_c(AG)$. We have to show that $\alpha \in \mathcal{B}$. To do this we use the fact that plots for $R^G(\mathcal{B})$ locally lift to plots for $H^G(\mathcal{B})$, by the definition of quotient diffeology. Take a finite open cover $\{U^i\}$ of $\text{Supp}(\alpha)$ so that the restriction of $\{c_\lambda\}$ to $U^i$ lifts for every $i$, i.e. there is a smooth family of bisections $\{b_\lambda^i\}$ for $H^G(\mathcal{B})$ such that $\Phi \circ b_\lambda^i = c_\lambda|_{U^i}$ for all $\lambda$. Take a partition of unity $\rho_i$ subordinate to this cover. Since $\rho_i \alpha = \frac{d}{d\lambda}|_{\lambda=0} c_{\rho_i \lambda}$ by the chain rule (exactly as in eq. (7.1)), it suffices to show that for all $i$

$$\frac{d}{d\lambda}|_{\lambda=0} c_{\rho_i \lambda} \in \mathcal{B}. \hspace{1cm} (7.1)$$

Notice that $b_{\rho_i \lambda}^i$ is a smooth family of bisections of $H^G(\mathcal{B})$ (in particular for $\lambda = 0$ its image is contained in $1_M$, by the definition of holonomy groupoid). Since $\Phi \circ b_{\rho_i \lambda}^i = c_{\rho_i \lambda}$ and the holonomy groupoid differentiates to $\mathcal{B}$, we deduce that eq. (7.1) holds. \hfill \Box

By the following proposition, the graph $R^G(\mathcal{B})$ is generally fails to be an integral of $\mathcal{B}$. Indeed, it generally does not admit a generating set of plots consisting of open maps.

\textbf{Proposition 7.6.} The graph $R^G(\mathcal{B})$, endowed with the path holonomy diffeology, in general does not satisfy the openness condition b) in Def. 6.1. In particular, the graph $R^G(\mathcal{B})$ is generally not an integral.

\textit{Proof.} We prove the proposition by providing a counter-example. Let $\mathcal{B} = \mathcal{F}$ be the singular foliation induced by the action of $G = S^1$ on $M = \mathbb{R}^2$ by rotations. We show that the graph $R(\mathcal{F}) \subset M \times M$ does not have any plot which is an open map with the point $(0,0) \in R(\mathcal{F})$ in its image.

Let $\chi: \mathcal{O} \to R(\mathcal{F})$ be a plot in the path holonomy diffeology and $x \in \mathcal{O}$ a point mapping to $(0,0)$. We want to show that $\chi$ is not an open map. There is a (open) neighborhood $\mathcal{O}^{(1)}$ of $x$ and a
plot \( \tilde{\chi} \) for the holonomy groupoid making this diagram commute, by definition of path holonomy diffeology:

\[
\begin{array}{ccc}
\mathcal{O}^{(1)} & \xrightarrow{\tilde{\chi}} & H(\mathcal{F}) \\
\downarrow & & \downarrow \Phi \\
\mathcal{O} & \xrightarrow{\chi} & R(\mathcal{F})
\end{array}
\]

Notice that the holonomy groupoid \( H(\mathcal{F}) \) is the transformation Lie groupoid \( G \times M \), and the map \( \Phi \) is the target-source map \((g,v) \mapsto (gv,v)\).

Take a connected open subset \( G^{(1)} \subset G \) such that \( G^{(1)} \times M \) contains \( \tilde{\chi}(x) \). Define \( \mathcal{O}^{(2)} \) to be the preimage of \( G^{(1)} \times M \) under \( \tilde{\chi}|_{\mathcal{O}(0)} \). It is an open neighborhood of \( x \) in \( \mathcal{O} \), since \( \tilde{\chi} \) is continuous. The following claim immediately implies that \( \chi \) is not an open map, as desired.

**Claim:** \( \chi(\mathcal{O}^{(2)}) \) is not an open subset of \( R(\mathcal{F}) \).

**Proof of claim:** The claim is equivalent to the following, since \( R(\mathcal{F}) \) is endowed with the quotient topology: \( \Phi^{-1}\left(\Phi(\tilde{\chi}(\mathcal{O}^{(2)}))\right) \) is not an open subset of \( H(\mathcal{F}) \). Since by construction \( \tilde{\chi}(\mathcal{O}^{(2)}) \subset G^{(1)} \times M \), we have

\[
\Phi^{-1}\left(\Phi(\tilde{\chi}(\mathcal{O}^{(2)}))\right) \subset \Phi^{-1}\left(\Phi(G^{(1)} \times M)\right).
\]

The right hand side equals \( (G^{(1)} \times M \setminus \{0\}) \coprod G \times \{0\} \), since the \( G \)-action is free on \( M \setminus \{0\} \) and fixes \( \{0\} \). Notice that the left hand side contains \( G \times \{0\} \). Pick \( g \in G \setminus G^{(1)} \). Then \((g,0)\) lies in the left hand side but due to (7.3) does not admit any (open) neighborhood in \( H(\mathcal{F}) \) lying in the left hand side.

\[
\begin{array}{c}
\Phi^{-1}\left(\Phi(\tilde{\chi}(\mathcal{O}^{(2)}))\right) \\
\Phi^{-1}\left(\Phi(G^{(1)} \times M)\right)
\end{array}
\]

We finish highlighting properties of the diffeological groupoid \( H^G(\mathcal{B}) \) that are not shared by \( R^G(\mathcal{B}) \).

- **Integrals:** We saw that \( H^G(\mathcal{B}) \) is an integral of \( \mathcal{B} \), while \( R^G(\mathcal{B}) \) is not in general (Thm. 6.19 and Prop. 7.6).
- **Smoothness:** In the regular case, i.e. when \( \mathcal{B} \) comes from a wide Lie subalgebroid \( B \subset AG \), the holonomy groupoid \( H^G(\mathcal{B}) \) is smooth, while in general \( R^G(\mathcal{B}) \) fails to be a Lie groupoid integrating \( B \).
- **Dimension of source-fibers:** The dimension of the source-fibers of \( H(\mathcal{B}) \) is upper semicontinuous, i.e. locally it can not increase. In contrast to this, the dimension of the source-fibers of \( R(\mathcal{B}) \) is lower semicontinuous, as is apparent in the case of singular foliations.
• Holonomy transformations: It was shown in [1] that the holonomy groupoid $H(F)$ of a singular foliation has a canonical injective morphism into the groupoid of holonomy transformations. The latter are the germs of diffeomorphisms between (fixed) slices transverse to $F$, modulo a certain equivalence relation. The injectivity implies that this morphism does not descend to any (non-trivial) quotient of $H(F)$, in particular not to $R(F)$.

These statements extend to any singular subalgebroid of a Lie algebroid $AG$, by taking the slices as follows: the slice through a point $x \in M$ is contained in the source-fiber $s^{-1}(x) \subset G$ and is transverse to the evaluation of $B$ at $x$. For more details see [20].

7.2 The graph as image of arbitrary integrals

Let $B$ be a singular subalgebroid of a Lie algebroid $AG$. We show that for all integrals of $B$, the image of the natural map to $G$ is always the same, namely the graph $R^G(B)$. We provide an assumption under which the diffeology on the image induced by the quotient integral coincides with the path-holonomy diffeology introduced in §7.1.2. We do now know if the assumption is always satisfied or not; this question is related to the question of whether the holonomy groupoid is a minimal integral.

**Proposition 7.7.** Let $(H, \Psi)$ be an integral of $B$ over $G$. Then $\Psi(H) = R^G(B)$.

In the proof of this proposition, the inclusion “$\supset$” makes essential use of the almost injectivity condition c) in Def. 6.1 and the inclusion “$\subset$” of the openness condition b) there.

**Proof.** “$\supset$”: Since $(H, \Psi)$ is an integral, by Cor. C.2 it satisfies Assumption C.3. Recall that Assumption C.3 requires for every $x \in M$ to choose a neighborhood, which we denote $V_x$. Endow $R^G(B)$ with the quotient topology induced by $\Phi: H^G(B) \to R^G(B)$.

**Claim:** There is a neighborhood $U$ of $1_M$ in $R^G(B)$ such that for every $g \in U$ there is $\alpha \in B$ with $g \in \exp_{1_M}(\hat{\alpha})$ and $\text{Supp}(\alpha) \subset V_{s(g)}$.

**Proof of claim:** Let $x \in M$, and denote by $L \subset M$ the leaf of $B$ through $x$. Assume first that $L$ is embedded. Consider the transitive Lie algebra $B_L$ over $L$ introduced in Lemma 1.7. It integrates the (transitive) Lie groupoid $R^G(B)_L$. It is a general fact about Lie groupoids that there is a neighborhood $V_x$ of $1_L$ in $R^G(B)_L$ such that for every $g$ there lies on $\exp_{1_L}(a)$ for some $a \in \Gamma_c(B_L)$ [10], §4, page 16. We will use the following stronger statement [28, Lemma 5.2]: for every neighborhood $V_x$ of $x$ in $L$ there is an open neighborhood $U_x$ of $1_L$ in $s_{R^G(B)_L}^{-1}(x)$ such that any point $g \in U_x$ lies on $\exp_{1_L}(a)$ for some $a \in \Gamma_c(B_L)$ with $\text{Supp}(a) \subset V_x$. We apply this to $V_x = V_x \cap L$. There exists $\alpha \in B$ with $\alpha|_L = a$, which can be chosen so that $\text{Supp}(\alpha) \subset V_x$. By construction, $g \in \exp_{1_M}(\hat{\alpha})$.

If the leaf $L$ is not embedded in $M$, we can apply the argument above to a neighborhood $L_x \subset L$ which is an embedded submanifold of $M$. To finish the argument, take $U$ to be a neighborhood of $1_M$ contained in $\cup_{x \in M} U_x \subset R^G(B)$.

Now let $g \in U \subset R^G(B)$ be as in the claim. By Assumption C.3 there are $N \in \mathbb{N}$ and a 1-parameter family $\{h_\lambda\}_{\lambda \in [0, \frac{1}{N}]}$ of global bisections of $H$ mapping to $\{\exp_{1_M}(\lambda \hat{\alpha})\}_{\lambda \in [0, \frac{1}{N}]}$ under $\Psi$.

In particular, for $x := s(g)$, we have $\Psi \left(h_\frac{1}{N}(x)\right) = \exp_x \left(\frac{1}{N} \hat{\alpha}\right)$, and taking the $N$-th power we see that $\exp_x(\alpha) = g \in \Psi(H)$. Since the groupoid $R^G(B)$ is source-connected, the desired inclusion $\Psi(H) \supset R^G(B)$ follows.
“⊂” We proceed as in the proof of Prop. 6.6: By the openness condition b) in Def. 6.1 we can apply Prop. 4.23 ii), hence there is a neighborhood \( \bar{H} \) of the identity \( 1_M \) in \( H \) such that for all \( h \in \bar{H} \) there is a 1-parameter family \({b_\lambda}\) of global bisections \({b_\lambda}\) with \( h \in b_1 \). Let \( h \in \bar{H} \) and \({b_\lambda}\) as above. For every fixed \( \lambda_0 \in [0,1] \) consider \({b_{\mu+\lambda_0} \ast b_{\lambda_0}^{-1}}\) where \( I_\mu \) is an open interval containing zero. It is a 1-parameter family of bisections of \( H \). Since \( H \) differentiates to \( \mathcal{B} \), we therefore have

\[
\frac{d}{d\mu}\bigg|_{\mu=0} \left( \Psi \circ (b_{\mu+\lambda_0} \ast b_{\lambda_0}^{-1}) \right) \in \mathcal{B}.
\]

Since \( \Psi \) is a groupoid morphism, the left hand side equals

\[
\frac{d}{d\mu}\bigg|_{\mu=0} R_{\Psi \circ b_{\lambda_0}^{-1}}(\Psi \circ b_{\mu+\lambda_0}) = (R_{\Psi \circ b_{\lambda_0}^{-1}})_* \frac{d}{d\lambda}\bigg|_{\lambda=0} (\Psi \circ b_\lambda)
\]

where \( R \) denotes right-translation on \( G \). Since \( \mathcal{B} \) is right-invariant, from this we conclude that \( \frac{d}{d\lambda}\bigg|_{\lambda=0} (\Psi \circ b_\lambda) \) is the restriction of an element of \( \mathcal{B} \) to \( \Psi \circ b_{\lambda_0} \). In particular, for \( x := s_H(h) \), the curve \([0,1] \ni \lambda \mapsto (\Psi \circ b_\lambda)(x)\) is tangent to \( \mathcal{B} \), so its endpoint \( \Psi(h) \) lies in \( R^G(\mathcal{B}) \). \(\Box\)

We now address diffeological structures. Thanks to Prop. 7.7 we know that \( \Psi(H) \) agrees \( R^G(\mathcal{B}) = \Phi(H^G(\mathcal{B})) \) as a set, so the assumption of the following statement is reasonable.

**Proposition 7.8.** Let \((H, \Psi)\) be an integral of \( \mathcal{B} \) over \( G \). Assume that there is a subduction \( \pi \) to the holonomy groupoid such that the following diagram commutes:

\[
\begin{array}{ccc}
H & \xrightarrow{\psi} & \mathcal{B} \\
\pi \downarrow & & \downarrow \phi \\
H^G(\mathcal{B}) & \xrightarrow{} & G
\end{array}
\]

Then the quotient diffeology on \( \Psi(H) \) (induced by \( \Psi \): \( H \to \Psi(H) \)) agrees with the path-holonomy diffeology.

We recall from [15] Ch. 1.46 what it means that \( \pi: H \to H^G(\mathcal{B}) \) is a subduction: it means that \( \pi \) is a surjective map and that the quotient diffeology induced by it coincides with the diffeology on \( H^G(\mathcal{B}) \). The latter property can be phrased as follows, see Prop-Def. 4.3 for any plot \( \chi \) for \( H^G(\mathcal{B}) \) and any point \( r \in O_\chi \) in its domain, there is an open neighborhood \( V \) and a plot \( \chi': V \to H \) so that \( \chi|_V = \pi \circ \chi' \).

**Proof.** By definition, the path-holonomy diffeology is the quotient diffeology obtained from the surjective map \( H^G(\mathcal{B}) \to \Phi(H^G(\mathcal{B})) = R^G(\mathcal{B}) \). The quotient diffeology is well-behaved under composition [15] §1.45, so the commutativity of the above diagram implies the statement. \(\Box\)

---

[15] We clarify what we mean by 1-parameter family parametrized by the closed interval \([0,1]\): by definition, this means that there is an open interval \( I \) containing \([0,1]\) and a (smooth) 1-parameter family defined in \( I \) extending \( \{b_\lambda\}_{\lambda \in [0,1]} \).
A Proof of Theorem 2.1

We prove Theorem 2.1. We first need some preliminary work. Let \((U, \varphi, G)\) be a bisubmersion for \(\mathcal{B}\). We consider \(\overrightarrow{U} := U \times_{s_U, t} G\) and maps

\[
\begin{align*}
\alpha : \overrightarrow{U} &\to G, \quad (u, g) \mapsto g \\
\beta : \overrightarrow{U} &\to G, \quad (u, g) \mapsto \varphi(u)g
\end{align*}
\]

It was shown in [27, Prop. B.1] that \((\overrightarrow{U}, \overrightarrow{t}, \overrightarrow{s})\) is a bisubmersion for the foliation \(\overrightarrow{\mathcal{B}}\).

**Lemma A.1.** If \((U, \varphi, G)\) is a path-holonomy bisubmersion for \(\mathcal{B}\), then \((\overrightarrow{U}, \overrightarrow{t}, \overrightarrow{s})\) is a path-holonomy bisubmersion for \(\overrightarrow{\mathcal{B}}\).

**Proof.** Let \(x \in M\) and \(\alpha_1, \ldots, \alpha_n \in \mathcal{B}\) such that \([\alpha_1], \ldots, [\alpha_n]\) span \(\mathcal{B}/I_x\mathcal{B}\). Then \(\alpha_1, \ldots, \alpha_n\) are generators in an open neighborhood \(V\), and we denote by \((U, \varphi, G)\) the corresponding bisubmersion (see Def. [1.11] in particular \(U \subset \mathbb{R}^n \times M\) is open). Then

\[U \times_{s_U, t} G = \{(\lambda, x, g) : t(g) = x \text{ where } (\lambda, x) \in U, g \in G\} \cong \{(\lambda, g) : \lambda \in \mathbb{R}, g \in G, (\lambda, t(g)) \in U\}\]

is an open subset of \(\mathbb{R}^n \times G\). Under this identification, we have

\[
\overrightarrow{s}(\lambda, g) = g \quad \text{and} \quad \overrightarrow{t}(\lambda, g) = exp_t(g)(\sum \lambda_i \alpha_i)g = exp(g)(\sum \lambda_i \alpha_i),
\]

using in the last equality the right invariance of the vector fields \(\alpha_i\). In other words, \((\overrightarrow{U}, \overrightarrow{t}, \overrightarrow{s})\) is the path-holonomy bisubmersion induced by the generators \(\alpha_1, \ldots, \alpha_n\) of \(\overrightarrow{\mathcal{B}}\) on the open \(t^{-1}(V) \subset G\) (they are really generators there by Lemma [1.6]). \(\square\)

**Proof of theorem 2.1.** For any \(x \in M\), choose a basis \([\alpha_i]\}_{i \in \mathbb{N}} of \(\mathcal{B}/I_x^M\mathcal{B}\), and let \(V\) a neighborhood in \(M\) such that \(\{\alpha_i\}_{i \in \mathbb{N}}\) generates \(\mathcal{B}/V\). Covering \(M\) by such open subsets \(V_j\) we obtain path-holonomy bisubmersions \((U_j, \varphi_j, G)\), and the atlas (for \(\mathcal{B}\)) they generate, which we denote by \(U^G\), consists of all the finite compositions of the \(U_j\)'s. (Notice that the inverse to each \(U_j\) is isomorphic to \(U_j\), by Lemma [1.3].) So \(H^G(\mathcal{B}) = \big( \prod_{U \in U^G} U \big) / \sim\).

Due to Lemma A.1, for every path-holonomy bisubmersion \((U, \varphi, G)\) of \(\mathcal{B}\), the triple \((\overrightarrow{U}, \overrightarrow{t}, \overrightarrow{s})\) where \(\overrightarrow{U} := U \times_{s_U, t} G\), is a path-holonomy bisubmersion for the foliation \(\overrightarrow{\mathcal{B}}\). The \(U_j\)'s generate an atlas \(\overrightarrow{U}\) for the foliation \(\overrightarrow{\mathcal{B}}\). So \(H(\overrightarrow{\mathcal{B}}) = \big( \prod_{\overrightarrow{U} \in \overrightarrow{U}} \overrightarrow{U} \big) / \sim\).

We make more explicit the multiplication in \(H(\overrightarrow{\mathcal{B}})\). Let \(U, U' \in U^G\). There is a canonical isomorphism of bisubmersions

\[
\begin{align*}
\overrightarrow{U} \circ \overrightarrow{U'} &= (U \times_{s_U, t} G) \circ (U' \times_{s_{U'}, t} G) \cong (U \circ U') \times_{s_{U \circ U'}, t} G = \overrightarrow{U} \circ \overrightarrow{U'}
\end{align*}
\]  

(A.1)
given by \((u, g), (u', g') \mapsto ((u, u'), g')\). This shows that for any element \(V\) of \(U^G\) (so \(V\) is a composition of path-holonomy bisubmersions), the corresponding \(\overrightarrow{V}\) lies in the atlas \(\overrightarrow{U}\).
We define the map
\[ \tau: H^G(B) \times_{s_{t,H}} G \to H(\bar{B}), \quad ([u], g) \mapsto [(u, g)], \] (A.2)
where \([u] \in H^G(B)\) is the class of the element \(u\) of some bisubmersion \(U \in \mathcal{U}^G\). Notice that \((u, g) \in U\).

**Claim:** The map \(\tau\) is a well-defined.

Let \((U, \varphi, G)\) and \((U', \varphi', G)\) be in \(\mathcal{U}^G\), and \(u_0 \in U\) and \(u'_0 \in U'\) be equivalent points. This means that there exists a morphism\(^{18}\) of bisubmersions \(f: U \to U'\) with \(f(u_0) = u'_0\). Then we obtain a map
\[ U = (U \times_{s_{t,H}} G) \to U' = (U' \times_{s_{t,H}} G), \quad (u, g) \mapsto (f(u), g), \]
which is a morphism of bisubmersions for \(\bar{B}\). Hence \([u, g] = [u', g]\). \(\Box\)

**Claim:** The map \(\tau\) is a homeomorphism.

The map \(\tau\) is surjective since every bisubmersion in \(U\) arises from one in \(\mathcal{U}^G\), by eq. (A.1). To show that \(\tau\) is injective, take equivalent points in \(U\) and \(U'\). They are necessarily of the form \((u_0, g_0)\) and \((u'_0, g_0)\) where \(u_0 \in U, u'_0 \in U', g_0 \in G\). We have to show that \([u_0] = [u'_0]\). There is a morphism of bisubmersions \(U \to U'\) mapping \((u_0, g_0)\) to \((u'_0, g_0)\). It is necessarily of the form \((u, g) \mapsto (F(u, g), g)\) where \(F: U \to U'\) satisfies \(\varphi'(F(u, g)) = \varphi(u)\). Now define \(f: U \to U'\) by \(f(u) = F(u, c(s_U(u)))\) where \(c: M \to G\) is a section of the target map \(t\) through \(g_0\). Then \(\varphi' \circ f = \varphi\), i.e. \(f\) is a morphism of bisubmersions for \(\bar{B}\), and \(f(u_0) = F(u_0, g_0) = u'_0\). Hence \([u_0] = [u'_0]\).

Finally, \(\tau\) is a homeomorphism, since it is induced by the identity maps on \(U \times_{s_{t,H}} G = U\) for all \(U \in \mathcal{U}^G\). \(\Box\)

**Claim:** The map \(\tau\) is groupoid morphism over \(Id_M\).

Let \((U, \varphi, G) \in \mathcal{U}^G\) and \(u \in U\) and take \(([u], g) \in H^G(B) \times_{s_{t,H}} G\). Its source and target are the same of those of \(\tau([u], g) = ([u], g) \in H(\bar{B})\), namely \(g\) and \(\varphi(u)g = Phi([u])g\) respectively. To show that \(\tau\) preserves the product, we notice that
\[ ([u], g) ([u'], g') = ([u][u'], g') = ([u \circ u'], g') \in H^G(B) \times_{s_{t,H}} G, \]
while eq. (A.1) shows that
\[ ([u], g) ([u'], g') = [(u \circ u'), g'] \in H(\bar{B}). \]

\(\Box\)

**B  An alternative proof of Proposition 3.3 b)**

In the main text, Prop. 3.3 b) was proven relying on the smoothness results of §2.3. Here we sketch a proof of Prop. 3.3 b) which instead uses that the holonomy groupoid is a quotient of a Lie groupoid, as in Prop. 1.21.

\(^{18}\)Actually, \(f\) is defined only on an open neighborhood of \(u_0\) in \(U\). Here and below, we will be not write this explicitly in order not to overburden the notation.
Let \( B \) be a projective singular subalgebroid of \( A \), and denote by \( B \) the associated almost injective Lie algebroid. The proof we sketch here is exactly the proof exhibited in [27, Prop. 3.18] for the special case that \( B \) is a wide Lie subalgebroid, upon the following modifications (using the notation of [27, Prop. 3.18]):

- \( B \) is an integrable Lie algebroid.

Indeed, the isomorphism \( \Gamma_c(B) \cong B \) is induced by an almost injective Lie algebroid morphism \( \tau: B \to A \). Recall [8, Def. 3.1] that for every \( x \in M \), the monodromy group \( N_x(B) \) consists of elements \( v \) in the center of the isotropy Lie algebra such that the constant \( B \)-path \( v \) is equivalent to the constant \( B \)-path \( 0 \), and similarly for \( A \). The Lie algebroid morphism \( \tau \) maps \( N_x(B) \) into \( N_x(A) \), since the equivalence of \( B \)-paths is defined in terms of a Lie algebroid morphism \( T \) on \( B \), and similarly for \( A \). Since \( A \) is an integrable Lie algebroid, by [8, Thm. 4.1] for every \( x \in M \) there is a neighborhood \( U \subset M \) such that \( U \times N_x(A) \) is contained in the zero section, where \( N_x(A) := \cup_{x \in M} N_x(A) \). We claim that there is an open neighborhood of \( x \) in \( \tau^{-1}(U) \subset B \) which has the same property. This follows from two facts. First, from [8, Thm. 5.10], which states that nearby \( x \) every element of \( N_x(B) \) can be extended to a smooth section of \( B \) lying in \( N_x(B) \). Second, from the fact that \( \tau: B \to A \) is fiberwise injective on a dense subset of \( M \).

- Let \( K \) be the s.s.c. Lie groupoid integrating \( B \), and \( \Psi: K \to G \) the Lie groupoid morphism integrating \( \tau: B \to A \). Let

\[
\mathcal{I} := \{ k \in K : \exists \text{ a local bisection } u \text{ through } k \text{ such that } \Psi(u) \subset 1_M \}.
\]

Then there exists a neighborhood \( V \subset K \) of \( 1_M \) such that \( \mathcal{I} \cap V = 1_M \).

Indeed, for all \( x \) belonging to a dense subset of \( M \), \( \tau_x: B_x \to A_x \) is injective and therefore the integrating Lie groupoid morphism \( \Psi \) satisfies that the restriction \( s_{1_M}^x \to s_{G}^x \) is an immersion at \( x \) and therefore injective in a neighborhood of \( x \).

- The Lie groupoid \( K/\mathcal{I} \) integrates \( B \).

Indeed, \( K \) does and \( \mathcal{I} \) is an s-discrete subgroupoid.

- The canonical map \( \Phi: H^G(B) \to G \) agrees with the map \( K/\mathcal{I} \to G \) induced by \( \Psi \), by Prop. [1.21] and the latter integrates the almost injective Lie algebroid morphism \( \tau: B \to A \) because \( \Psi \) does.

C Rephrasing Assumption 5.1 as a liftability condition

In Def. 6.1 about integrals of singular subalgebroids, the differentiation condition a) contains in particular Assumption 5.1. Here, for integrals, we give an equivalent characterization of this assumption in terms of the liftability of 1-parameter groups of global bisections of \( G \), see Corollary C.2.

The following lemma addresses when bisections of \( G \) can be “lifted” via \( \Psi \).

**Lemma C.1.** Let \( G \) be a Lie groupoid, \( H \) be a diffeological groupoid, and \( \Psi: H \to G \) a smooth morphism of diffeological groupoids covering \( Id_M \), satisfying the almost injectivity condition c) in Def. 6.1.
i) There exists a neighborhood $\tilde{H}$ of the identity in $H$ with the following property: for all bisections $c$ of $G$, if there exists a bisection of $\tilde{H}$ mapping to $c$ under $\Psi$, then it is unique.

ii) Let $\alpha \in \Gamma_c(AG)$, consider the corresponding 1-parameter group $c_\lambda := \exp_{1_M}((\lambda\alpha))$ of global bisections of $G$. Assume that there is a 1-parameter family $\{b_\lambda\}_{\lambda \in I}$ of global bisections of $H$ mapping to $\{c_\lambda\}_{\lambda \in I}$ under $\Psi$. Then $\{b_\lambda\}_{\lambda \in I}$ is a 1-parameter group, shrinking $I$ to a smaller open interval if necessary.

iii) For all $\alpha \in \Gamma_c(AG)$, the following two conditions are equivalent:

a) There is a 1-parameter group $\{b_\lambda\}_{\lambda \in I'}$ of global bisections of $H$ such that $\frac{d}{d\lambda}\big|_{\lambda=0}(\Psi \circ b_\lambda) = \alpha$.

b) There is a 1-parameter family $\{b_\lambda\}_{\lambda \in I}$ of global bisections of $H$ mapping to $\{\exp_{1_M}(\lambda\alpha)\}_{\lambda \in I}$ under $\Psi$.

Proof. 

i) Take $\tilde{H}$ to be a symmetric neighborhood of the identity such that $\tilde{H} \cdot \tilde{H} \subset \tilde{H}$. Let $b$ and $b'$ be bisections of $\tilde{H}$ with $\Psi(b) = c = \Psi(b')$. Then $b \ast (b')^{-1}$ is contained in $\tilde{H}$, and maps to the identity bisection of $G$ under $\Psi$. Thus property c) in Def. 6.1 implies that $b = b'$.

ii) When $\lambda, \mu, \lambda + \mu \in I$ we compute

$$\Psi(b_\lambda \ast b_\mu) = \Psi(b_\lambda) \ast \Psi(b_\mu) = c_\lambda \ast c_\mu = c_{\lambda+\mu} = \Psi(b_{\lambda+\mu}).$$

Shrinking $I$ if necessary, we can assure that $b_\lambda \ast b_\mu$ and $b_{\lambda+\mu}$ lie in $\tilde{H}$. Item i) then implies that these two global bisections agree. Together with $b_0 = 1_M$, this shows that $\{b_\lambda\}_{\lambda \in I}$ is a 1-parameter group.

iii) Assume a). Since $\{b_\lambda\}_{\lambda \in I}$ is a 1-parameter group and $\Psi$ is a groupoid morphism, $\{\Psi \circ b_\lambda\}_{\lambda \in I}$ is also a 1-parameter group. Hence it agrees with $\{\exp_{1_M}(\lambda\alpha)\}_{\lambda \in I}$, by the uniqueness of 1-parameter groups of bisections with initial velocity on a Lie groupoid.

Conversely, assume b). Using item ii), condition a) follows, for some open subinterval $I' \subset I$.

The following corollary is an immediate consequence of Lemma C.1 iii).

Corollary C.2. In the Def. 6.1 of integral, the requirement that Assumption 5.1 is satisfied can be equivalently replaced by Assumption C.3 below.

Assumption C.3. For every $x \in M$ there is a neighborhood $V$ with this property:

For all $\alpha \in B$ with $\text{Supp}(\alpha) \subset V$,

there is a 1-parameter family $\{b_\lambda\}_{\lambda \in I}$ of global bisections of $H$ mapping to $\{\exp_{1_M}(\lambda\alpha)\}_{\lambda \in I}$ under $\Psi$.

(Notice that the openness condition b) in Def. 6.1 is not used to obtain this corollary.)
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