Optimal control problem for linear fractional-order systems, described by equations with Hadamard-type derivative

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Abstract. Two kinds of optimal control problem are investigated for linear time-invariant fractional-order systems with lumped parameters which dynamics described by equations with Hadamard-type derivative: the problem of control with minimal norm and the problem of control with minimal time at given restriction on control norm. The problem setting with nonlocal initial conditions studied. Admissible controls allowed to be the $p$-integrable functions ($p > 1$) at half-interval. The optimal control problem studied by moment method. The correctness and solvability conditions for the corresponding moment problem are derived. For several special cases the optimal control problems stated are solved analytically. Some analogies pointed for results obtained with the results which are known for integer-order systems and fractional-order systems describing by equations with Caputo- and Riemann-Liouville-type derivatives.

1. Introduction

Dynamics and optimal control are the actively developing topics for fractional-order systems now. There are many interesting results obtained already in this field. Note that character of these results depend on, in addition, from the type of fractional derivative definition chosen for system description [1]. So, the maximum principle, which is analogous to L.S. Pontryagin maximum principle, were proved recently for general type system described by equations with Riemann-Liouville derivative [2]. In case of systems described by equations with Caputo derivative such principle were proved for linear systems only [3]. The other effective and powerful approach for optimal control search is the moment method [4]. This method was successfully applied to investigation of optimal control problem for linear dynamic systems described by equations with Caputo derivative [5, 6, 7] and Riemann-Liouville derivative [8].

Caputo and Riemann-Liouville derivatives are most popular types of fractional derivative. In accordance with their definitions the fractional differentiation operator specified as convolution with fractional-power kernel. When the differentiation index is equal to 1, both types of the derivative reduce to ordinary first derivative. The principal difference between these two derivatives is the kind of initial conditions [9, 10, 11]. In case of equations with Caputo derivative the initial conditions are local and define the values of function and its integer-order derivatives at initial time. In case of equations with Riemann-Liouville derivative the nonlocal initial conditions used which define the value of some integral functional at initial time. Let us illustrate it by the following example from [11] (fractional derivative and integral implied here in the sense of Riemann-Liouville).
In the fractional Voight model describing the behaviour of some viscoelastic bodies under loading, the stress $\sigma(t)$ and the strain $\varepsilon(t)$ coupled by the following constitutive equation:

$$\sigma(t) = E\varepsilon(t) + K_0 D_t^\alpha \varepsilon(t),$$

where $K$ and $E$ — real coefficients. The relaxation after stress pulse of type $B\delta(t)$ impact will be described by the following equation:

$$\sigma(t) = E\varepsilon(t) + K_0 D_t^\alpha \varepsilon(t), t > 0.$$

For this equation the initial condition needed which define the value $0 D_t^{\alpha-1} \varepsilon(t) = 0 I_t^{1-\alpha} \varepsilon(t)$ at $t \to 0+$. If we integrate the constitutive equation then we obtain:

$$0 D_t^{-1} \sigma(t) - E_0 D_t^{\alpha-1} \varepsilon(t) = K_0 D_t^{\alpha-1} \varepsilon(t).$$

The first term here is equal to $B$ when $t \to 0+$ [11]. Taking into account physical considerations (i.e., impossibility of instantaneous deformation) or the requirement of boundedness of function $0 D_t^{\alpha-1} \varepsilon(t)$, which define the initial condition, one can demonstrate that the second term vanishes when $t \to 0+$ [11]. So, we obtain:

$$0 D_t^{\alpha-1} \varepsilon(t) = 0 I_t^{1-\alpha} \varepsilon(t) = B/K.$$

The example considered above shown that nonlocal initial conditions for the strain $\varepsilon(t)$ can be re-written in the form of local initial conditions for stress $\sigma(t)$, which have more clear physical sense.

One of the first articles concerning the optimal control problem for fractional-order systems with nonlocal initial conditions is [10]. Later such problem statement studied by many researchers, particularly, in [2].

In this paper the optimal control problem investigated for the systems with lumped parameters, which described in terms of Hadamard fractional derivative, defined as convolution with kernel containing the logarithmic function in fractional power. This derivative type, analogously to the Riemann-Liouville derivative, need to set the nonlocal initial conditions but didn’t reduce already to the first derivative when the differentiation index is equal to 1. Optimal control problems investigated by the moment method. The optimal control problem for one-dimensional linear time-invariant system studied in general setting. It’s demonstrated that this problem can be reduced to the moment problem. The multi-dimensional systems of special type (the chain of integrators) considered analogously. The comparison evaluated with integer-order systems and analogous fractional-order systems, which described by equations with Riemann-Liouville derivative.

2. Problem setting

We will consider the following dynamical systems:

$$0 D_t^{\alpha_i} q_i(t) = a_{ij} q_j(t) + b_{ij} u_j(t) + f_i(t), t \in (t_0, T], i, j = 1, ..., N,$$

where functions $q_i(t)$, $u_i(t)$ and $f_i(t)$ are the phase coordinates, controls and perturbations correspondingly, $T > t_0 > 0$; $a_{ij}$ and $b_{ij}$ are known real coefficients. The repeated indices suppose summation. Here $0 D_t^{\alpha_i}$ is operator of the left-side fractional Hadamard derivative (see [9, §2.7]), $\alpha_i \in (0, 1]$:

$$0 D_t^{\alpha_i} q_i(t) \equiv \frac{1}{\Gamma(1 - \alpha_i)} \frac{d}{dt} \int_0^t \left( \ln \frac{t}{\tau} \right)^{-\alpha_i} \frac{q_i(\tau) d\tau}{\tau}.$$
Functions \( q_i(t), u_i(t) \) and \( f_i(t) \) can be considered as components of vector functions \( \vec{q}(t) = (q_1(t), \ldots, q_N(t)), \vec{u}(t) = (u_1(t), \ldots, u_N(t)) \) and \( \vec{f}(t) = (f_1(t), \ldots, f_N(t)). \)

The nonlocal initial conditions are determined [9, §4.1]:

\[
\lim_{t \to t_0^+} \left[ t_0 t_1^{\alpha_i} q_i(t) \right] = s_i^0, \quad i = 1, \ldots, N,
\]

where \( t_0 t_1^{\alpha_i} q_i(t) = \frac{1}{\Gamma(\alpha_i)} \int_{t_0}^{t} \left( \ln \frac{t}{s} \right)^{\alpha_i - 1} \frac{q_i(s)ds}{s} \) is the left-side Hadamard integral of order \( \alpha_i \) [9, §2.7].

The final conditions are determined in ordinary manner:

\[
q_i(T) = q_i^T, \quad i = 1, \ldots, N.
\]

Let us consider control \( \vec{u}(t) \in L_p(t_0, T], \ p > 1 \). The control norm will be defined by the following expressions [4, 13]:

\[
\|u(t)\| = \left( \int_{t_0}^{T} \left( \sum_{i=1}^{N} |u_i(t)|^p \right) dt \right)^{1/p}, \quad 1 < p < \infty,
\]

\[
\|u(t)\| = \text{vrai max}_{t \in (t_0, T]} \max_{i} |u_i(t)|, \ p \to \infty.
\]

Let us take that functions \( q_i(t) \) and \( f_i(t), \ i = 1, \ldots, N \) (or \( \vec{q}(t) \) and \( \vec{f}(t) \)) possess all properties required for existence of solutions of equations studied further on, are, in particular, differentiable at least once.

Let us set the following optimal control problem: determine such control \( \vec{u}(t) \in L_p(t_0, T] \) at which the system (1) will pass from the given initial state (2) to the assigned final state (3) and herewith: 1) norm of control will be minimal with the assigned control time \( T \) (OCP A), control time \( T \) will be minimal provided \( \|\vec{u}\| \leq l, \ l > 0 \), where \( l \) is the assigned constant (OCP B).

3. The moment problem

3.1. Preliminaries

The classical \( l \)-problem of moments can be formulated in a following way [4, 14, 15].

Let we have a system of functions \( g_i(t) \in L^{p'}(t_0, T], \ i = 1, \ldots, N, \ p' > 1 \). Let we also have the assigned numbers \( c_i \) (calling by moments), \( i = 1, \ldots, N \) and \( l > 0 \). We should find function \( u(t) \in L^p(t_0, T], \ p > 1 \) that satisfies the following conditions:

\[
\int_{t_0}^{T} g_i(\tau)u(\tau)d\tau = c_i,
\]

\[
\|u(t)\| \leq l.
\]

and The spaces \( L^p(t_0, T] \) and \( L^{p'}(t_0, T] \) are adjoint:

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

In order to the moment problem (4)-(5) will be solvable it’s necessary and sufficient the existence of number \( \lambda_N > 0 \) and numbers \( \xi_1^*, \ldots, \xi_N^* \) that give a solution to the following equivalent conditional minimum problem [4, 14, 15].
To find
\[
\min_{\xi_1, \ldots, \xi_N} \left( \frac{1}{\lambda_N} \right) \left[ \int_0^T \left( \sum_{i=1}^N \xi_i g_i(t) \right)^{p'/(p'-1)} dt \right]^{1/(p'-1)} = \left( \frac{1}{\lambda_N} \right) \left( \int_0^T \left( \sum_{i=1}^N \xi_i^* g_i(t) \right)^{p'/p} dt \right)^{1/p'}
\]
with additional condition
\[
\sum_{i=1}^N \xi_i c_i = \sum_{i=1}^N \xi_i^* c_i = 1.
\]

In case of problem (7-8) is solvable the optimal control for OCP A will be given by the following expression [4]:
\[
u(t) = \lambda_N^{p'} \left[ \int_0^T \xi_i^* g_i(t) dt \right]^{p'-1} \text{sign} \left[ \sum_{i=1}^N \xi_i^* g_i(t) \right], \quad t \in (t_0, T].
\]
OCP B may be resolved by the following formula:
\[
u(t) = l^{p'} \left[ \int_0^T \xi_i^* g_i(t) dt \right]^{p'-1} \text{sign} \left[ \sum_{i=1}^N \xi_i^* g_i(t) \right], \quad t \in (t_0, T],
\]
where \(T^*\) is the minimal non-negative real root of equation
\[
\lambda_N (T^*) = l.
\]

For solvability of moment problem (7-8) it is necessary and sufficient to satisfy one of two equivalent conditions [4]: (1) \(\lambda_N > 0\); (2) functions \(g_i(t)\) are linearly independent. The single question is the possibility of correct posing of the moment problem, which depend on the existence of norm of functions \(g_i(t)\) in space \(L^{p'}(t_0, T]\) and the existence of at least one nonzero component in number set \(c_i\).

**Definition 1.** The moment problem (4)-(5) is calling correct if the norm of functions \(g_i(t)\) is determined in space \(L^{p'}(t_0, T]\), \(p' > 1\) and there exist at least one nonzero component in number set \(c_i\).

System (1) with fractional derivative operator in sense of Hadamard (Riemann-Liouville, Caputo) we will name below as Hadamard (Riemann-Liouville, Caputo) system or H-system (RL-system, C-system).

It’s known that optimal control problems in form of OCP A and OCP B can be reduced to the moment problem (4)-(5) in case of integer-order systems with lumped parameters [4]. It was shown also that the same is valid for Caputo systems [5, 6, 7] and for Riemann-Liouville systems [8]. Below we will demonstrate that for Hadamard systems the optimal control problem in form of OCP A and OCP B also can be reduced to the moment problem (4)-(5). We will consider the one-dimensional system of general view and special multi-dimensional system — \(N\)-fold integrator, corresponding to the system (1) at \(a_{ij} = \delta_{i,j-1}, b_{ij} = \delta_{i,N}; f_i(t) = 0.\)

### 3.2. The moment problem for one-dimensional system

The general solution of (1) in case of H-system at \(N = 1\) can be represented by the following expression [9, p. 234] (subscripts are omitted):
\[
q(t) = s^0 \left( \ln \frac{t}{t_0} \right)^{a-1} E_{a, a} \left[ a \left( \ln \frac{t}{t_0} \right)^a \right] +
\]
\[ + \int_{t_0}^{t} \left( \ln \frac{t}{\tau} \right)^{\alpha - 1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{t}{\tau} \right) \right] \frac{bu(\tau) + f(\tau)}{\tau} \, d\tau, \]

where \( E_{\alpha,\alpha}(t) \) is two-parameter Mittag-Leffler function [9, \S 1.8]. Solution (12) at \( t = T \) subject to the final condition (3) can be written in the form of expression (4), where

\[ g(\tau) = \frac{b}{\tau} \left( \ln \frac{T}{\tau} \right)^{\alpha - 1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{\tau} \right) \right], \quad (13) \]

\[ c(T) = q^T - s^0 \left( \ln \frac{T}{t_0} \right)^{\alpha - 1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t_0} \right) \right] - \int_{t_0}^{T} \left( \ln \frac{T}{\tau} \right)^{\alpha - 1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{\tau} \right) \right] \frac{f(\tau)}{\tau} \, d\tau. \quad (14) \]

**Theorem 1.** Let us suppose \( b \neq 0 \) and \( c \neq 0 \). The moment problem (4) subject to (13) and (14) will be correct and solvable if and only if the following condition is valid:

\[ \alpha > \frac{p'}{p' - 1}. \quad (15) \]

**Proof.** Let us prove the correctness of moment problem correspondingly to the Def. 1. The moment \( c \) is non-zero by the data. For the norm of function \( g(\tau) \in L_{p'}(t_0, T) \) subject to the (13) the following estimation will be valid [13]:

\[ \| g(\tau) \| \leq |b| \cdot \left\| \frac{1}{\tau} \right\| \cdot \left\| \left( \ln \frac{T}{\tau} \right)^{\alpha - 1} \right\| \cdot \left\| E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{\tau} \right) \right] \right\|. \quad (16) \]

We have \( b \neq 0 \) by the data. The norm of Mittag-Leffler function defined on half-interval \( t \in (t_0, T) \) at \( \alpha > 0 \) [9, p. 42]. The second multiplier in (16) can be represented by the following formulas:

\[ \left\| \frac{1}{\tau} \right\| = \ln \frac{T}{t_0} \]

at \( p' = 1 \),

\[ \left\| \frac{1}{\tau} \right\| = \left[ \frac{T^{1-p'}}{1-p'} \left( 1 - \left( \frac{t_0}{T} \right)^{1-p'} \right) \right]^{1/p'} \]

at \( p' > 1 \). Consequently, the norm of function \( \frac{1}{\tau} \) at \( 0 < t_0 < T \) also is fully defined (bounded and positive). Finally, the norm of third multiplier in (16) will be represented by the following expression:

\[ \left\| \left( \ln \frac{T}{\tau} \right)^{\alpha - 1} \right\| = \left( \int_{t_0}^{T} \left( \ln \frac{T}{\tau} \right)^{p'(\alpha - 1)} \, d\tau \right)^{1/p'}. \]

Integral in the right side of the expression by substitution \( \ln \frac{T}{\tau} = x \) reduces to the following view:

\[ \int_{t_0}^{T} \left( \ln \frac{T}{\tau} \right)^{p'(\alpha - 1)} \, d\tau = T \int_{0}^{\sigma} x^{p'(\alpha - 1)} e^{-x} \, dx, \quad (17) \]
where \(\sigma = \ln \left( \frac{T}{t_0} \right) > 0\). Convergence condition for integral in right side of (17) can be obtained from comparison criterion [20, Ch. 9]. Integrant in right side of (17) majorized, obviously, by function \(x^{p'(\alpha-1)}\), which order of greatness equal to \(p'(1-\alpha)\). Correspondingly to comparison criterion integral in right side of (17) will be converge if this order is less than 1. Consequently, we obtain the condition (15). The correctness in sense of Def. 1 is proven.

Now we will prove the solvability of the moment problem (4) subject to (13) and (14). Let us consider interpolation problem (7)-(8) for \(N = 1\). One can express \(\xi\) from (8) and obtain the following unconditional minimization problem:

\[
\left( \int_{t_0}^{T} \left| \frac{g(\tau)}{c} \right|^{p'} d\tau \right)^{1/p'} = \frac{1}{\lambda_N}. \tag{18}
\]

Function \(g(\tau)\) (in accordance with (13)) at \(b \neq 0\) is positive definite function on half-interval \(t \in (t_0, T]\), \(0 < t_0 < T\). And \(c \neq 0\) by the data. Consequently, left side in this expression is positively defined and \(\lambda_N > 0\). So, the solvability condition is satisfied. Thus, theorem 1 is proven.

**Remark 1.** Condition (15) match with analogous conditions obtained for Caputo [5, 6, 7] and Riemann-Liouville [8] systems.

Let us consider now the special one-dimensional system — single integrator (it correspond to (1) at \(a = 0, b = 1, f(t) = 0\)). In this case solution of equation (1) can be obtained explicitly by its fractional integration:

\[
q(t) = s^0 t^\alpha \Gamma(\alpha) \left( \ln t \right)^{\alpha - 1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \left( \ln \frac{t}{\tau} \right)^{\alpha - 1} \frac{u(\tau)}{\tau} d\tau. \tag{19}
\]

When \(t = T\) the expression (19) subject to final condition (3) represents the moment problem (4), where:

\[
g(\tau) = \frac{1}{\Gamma(\alpha)} \left( \ln \frac{T}{\tau} \right)^{\alpha - 1} \frac{1}{\tau}, \tag{20}
\]

\[
c = q^T - s^0 \frac{1}{\Gamma(\alpha)} \left( \ln \frac{T}{t_0} \right)^{\alpha - 1}. \tag{21}
\]

Note that expressions (20) and (21) can be derived from (13) and (14) correspondingly by passage to the limit at \(a \to 0\).

### 3.3. The moment problem for \(N\)-fold integrator of fractional order

Let us consider system (1) at \(a_{ij} = \delta_{i,j-1}, b_{ij} = \delta_{i,N}, f_i(t) = 0\):

\[
\begin{aligned}
& \left\{ \begin{array}{l}
t_0 D_t^{\alpha_1} q_1(t) = q_2(t), \\
t_0 D_t^{\alpha_2} q_2(t) = q_3(t), \\
\cdots \\
t_0 D_t^{\alpha_N} q_N(t) = u(t).
\end{array} \right.
\end{aligned} \tag{22}
\]
If we integrate each of equations in (22) in sense of Hadamard with corresponding index \( (\alpha_i \text{ for } i\text{-th equation}) \) then we will obtain the following expressions for phase coordinates:

\[
\begin{align*}
q_1(t) &= \frac{s_0^0}{\Gamma(\alpha_1)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_1 - 1} = t_0^\alpha_1 q_2(t), \\
q_2(t) &= \frac{s_0^2}{\Gamma(\alpha_2)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_2 - 1} = t_0^\alpha_2 q_3(t), \\
& \quad \ldots \\
q_N(t) &= \frac{s_0^N}{\Gamma(\alpha_N)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_N - 1} = t_0^\alpha_N u(t).
\end{align*}
\]

Substituting sequentially phase coordinates from the last equation to penultimate one etc. and using the semigroup property of Hadamard fractional integration operators we will obtain:

\[
\begin{align*}
q_1(t) - \sum_{k=1}^{N} \frac{s_0^k}{\Gamma(\alpha_1 + \ldots + \alpha_k)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_1 + \ldots + \alpha_k - 1} &= t_0^\alpha_1 q_2(t), \\
q_2(t) - \sum_{k=2}^{N} \frac{s_0^k}{\Gamma(\alpha_2 + \ldots + \alpha_k)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_2 + \ldots + \alpha_k - 1} &= t_0^\alpha_2 q_3(t), \\
& \quad \ldots \\
q_N(t) &= \frac{s_0^N}{\Gamma(\alpha_N)} \left( \ln \frac{T - t}{t_0} \right)^{\alpha_N - 1} = t_0^\alpha_N u(t).
\end{align*}
\]

System (23) at \( t = T \) subject to final conditions (3) represent the \( N \)-dimensional moment problem (4), where the following expressions are valid:

\[
\begin{align*}
g_k(\tau) &= \frac{1}{\Gamma(\alpha_k + \ldots + \alpha_N)} \left( \ln \frac{T - \tau}{\tau} \right)^{\alpha_k + \ldots + \alpha_N - 1} \frac{1}{\tau}, \\
c_k &= q_k^T \sum_{l=k}^{N} \frac{s_0^l}{\Gamma(\alpha_l + \ldots + \alpha_N)} \left( \ln \frac{T - t_0}{t_0} \right)^{\alpha_l + \ldots + \alpha_N - 1}.
\end{align*}
\]

**Theorem 2.** Let us suppose that at least one of moments \( c_i \) is non-zero. Then the moment problem (4) subject to (24), (25) will be correct and solvable if and only if the following condition is valid:

\[
\alpha_N > \frac{p' - 1}{p'}. \tag{26}
\]

**Proof.** Correctness of moment problem (4) for system (23) can be proven analogously to the Theorem 1. And condition (26) will be caused by existence of norm of function \( q_N(\tau) \). Existence of norm for other functions \( q_i(\tau), i = 1, \ldots, N - 1 \) caused by the following conditions:

\[
\alpha_k + \ldots + \alpha_N > \frac{p' - 1}{p'}.
\]

These conditions will be satisfied if condition (26) is valid since \( \alpha_i > 0 \) by the data.

Solvability of the moment problem (4) subject to (24), (25) caused by linear independency of functions \( q_i(\tau) \). Thus, the theorem is proven.

**Remark 2.** Condition (26) match with analogous conditions obtained later for Caputo [5, 7] and Riemann-Liouville [8] systems.

### 4. Moment problem solution and optimal control properties

#### 4.1. One-dimensional system

In this case we can solve the problem (7)-(8) explicitly and construct the optimal control for OCP A and OCP B.

**Theorem 3.** Let the one-dimensional system (1) given with initial and final conditions (2) and (3). Let \( b \neq 0, c \neq 0 \) (where moment \( c \) defined by (14)) and condition (15) satisfied (i.e., moment problem (4)-(5) subject to (13)-(14) is correct and solvable). Then:
(i) OCP A solution at \( p \to \infty \) is given by the following expression:

\[
u(t) = \frac{c}{bK_1^\alpha(t_0,T)} \left[ \frac{1}{t} \left( \ln \frac{T}{t} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right] \right]^{p'-1}, \tag{28}\]

where \( \alpha > 0, \ t \in (t_0,T), \ E_{\alpha}(t) = E_{\alpha,1}(t) \) — one-parameter Mittag-Leffler function;

(ii) OCP A solution at \( 1 < p < \infty \) is given by the following expression:

\[
u(t) = \frac{c}{bK_1^\alpha(t_0,T)} \left[ \frac{1}{t} \left( \ln \frac{T}{t} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right] \right]^{p'-1}, \tag{28}\]

where \( \alpha > 1/p, \ t \in (t_0,T), \ K_1(t_0,T) = \left( \int_{t_0}^T \left[ \frac{E_{\alpha,\alpha}[a(\ln \frac{T}{t})^\beta]}{\tau(\ln \frac{T}{t})^{1-\alpha}} \right]^{p'}/d\tau \right)^{1/p'}; \)

(iii) OCP B solution at \( p \to \infty \) is given by the following expression:

\[
u(t) = l \operatorname{sign} \left( \frac{b}{c} \right), \tag{29}\]

where \( \alpha > 0, \ t \in (t_0,T^*) \) and \( T^* \) can be found from equation \( |\tfrac{d}{d\tau} E_{\alpha}(\tau)\| \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right]^{-1} = 1; \)

(iv) OCP B solution at \( 1 < p < \infty \) is given by the following expression:

\[
u(t) = l \left| \frac{b}{ct} \left( \ln \frac{T}{t} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right] \right|^{p'-1} \operatorname{sign} \left( \frac{b}{c} \right), \tag{30}\]

where \( \alpha > 1/p, \ t \in (t_0,T^*) \) and \( T^* \) can be found from equation \( |\tfrac{d}{d\tau} E_{\alpha}(\tau)\| \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right]^{-1} = l. \)

**Proof.** Let us consider the case \( p \to \infty \) and solve the minimization problem (7)-(8) subject to (13)-(14). As we have shown above the problem (7)-(8) reduces to unconditional minimization problem (18). Taking into account (13)-(14) and \( p' = 1 \) we will obtain:

\[rac{1}{\lambda} = \int_{t_0}^T \frac{bE_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right]}{ct \left( \ln \frac{T}{t} \right)^{1-\alpha}} d\tau. \tag{31}\]

In order to calculate integral in (31) we will use Mittag-Leffler function representation by power series [9, p. 42]:

\[
E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{32}\]

This series converge absolutely on number axis at \( \alpha > 0 \) and \( \beta > 0 \) [9, p. 42]. Then we can substitute representation (32) into (31), perform the termwise integration and summarize resulting series:

\[rac{1}{\lambda} = \int_{t_0}^T \frac{bE_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t} \right)^{\alpha} \right]}{ct \left( \ln \frac{T}{t} \right)^{1-\alpha}} d\tau = \frac{b}{c} \sum_{k=0}^{\infty} \int_{t_0}^T \frac{a^k \left( \ln \frac{T}{t_0} \right)^{k+\alpha-1}}{\Gamma(\alpha k + \alpha)} \frac{d\tau}{\tau} =
\]

\[
= \frac{1}{\alpha} \left[ a \left( \ln \frac{T}{t_0} \right)^{\alpha-1} \right] = \frac{1}{\alpha} \left[ \frac{a}{c} \right] \left( E_{\alpha} \left[ a \left( \ln \frac{T}{t_0} \right)^{\alpha} \right] - 1 \right). \tag{33}\]
Expressing $\lambda$ from the last formula we will obtain:

$$\lambda = \left| \frac{c}{b} \right| \frac{1}{E_{\alpha} \left[ a \left( \ln \frac{T}{t_0} \right)^{\alpha} \right]} - 1.$$  \hfill (33)

Substituting expression (33) to (9) we will obtain formula (27). OCP B solution (29) in this case result from (10) and (11) subject to (33).

Let us consider now the case $1 < p < \infty$. Obviously, the expression (18) subject to (13)-(14) could be rewritten as the following formula:

$$\lambda = \left| \frac{c}{b} \right| \frac{1}{K_1(t_0, T)}.$$  \hfill (34)

Substituting formula (34) into expression (9) we will obtain OCP A solution (28). From (10) and (11) subject to (34) one can obtain OCP B solution (30). Thus, the theorem is proven.

**Remark 3.** Optimal controls (27) and (29) have no switching points on considered time half-interval, which corresponds to corollary from Feldbaum’s interval theorem [16] for integer-order system of order $[\alpha] + 1$.

For single integrator analogous solutions for OCP A and OCP B can be obtained by direct solution of problem (7)-(8) subject to (20), (21) or by passage of expressions (27), (28), (29) and (30) to the limit at $a \to 0$. So, in case of $p \to \infty$ we can obtain the following expression for $\lambda$:

$$\lambda = \left| \frac{c}{b} \right| \frac{\Gamma(\alpha + 1)}{(\ln \frac{T}{t_0})^{\alpha}}.$$  \hfill (35)

OCP A solution for studied system will be written by formula:

$$u(t) = \frac{c\Gamma(\alpha + 1)}{(\ln \frac{T}{t_0})^{\alpha}}.$$  \hfill (36)

Optimal control in case of OCP B will be described by formula (10), where minimal time $T^*$ can be found (in general case numerically) from (11) subject to (35). In case of $q^T = 0$ this equation can be solved explicitly and the following formula will take place:

$$T^* = t_0 \exp \left( \frac{|s_0|^\alpha}{l} \right).$$  \hfill (37)

Analogously, we can find for single integrator at $1 < p < \infty$:

$$\lambda = \left| \frac{c}{b} \right| \frac{\Gamma(\alpha)}{K_2(t_0, T)}.$$  \hfill (38)

where $K_2(t_0, T) = \left( \int_{t_0}^{T} \left(\frac{1}{\tau^{(\ln \frac{T}{t_0})^{1-\alpha}}} \right)^{1/p'} \right)^{1/p'}$. Solutions of OCP A and OCP B can be found from (9), (10) and (11) subject to (38).

Now we will illustrate the results obtained above. Calculations and visualization of the results carried out in MatLab 7.9. A special module was used for Mittag-Leffler function calculation [17]. Numerical integration was performed by Gauss-Kronrod method [18].

Firstly we will illustrate the case $p \to \infty$ ($p' = 1$ correspondingly). At fig. 1 and 2 the dependencies represented of control norm from differentiation index $\alpha$, which calculated using...
Figure 1. The dependency of control norm from differentiation index $\alpha$ at $q^T = 0$. Logarithmic scale used for ordinates.

Figure 2. The dependency of control norm from differentiation index $\alpha$ at $q^T = 2$. Logarithmic scale used for ordinates.

Figure 3. The dependency of control norm from control time $T$ at $p' = 1$, $q^T = 0$ and several values of index $\alpha$. Logarithmic scale used for ordinates.

Figure 4. The dependency of control norm from control time $T$ at $p' = 1$, $q^T = 2$ and several values of index $\alpha$. Logarithmic scale used for ordinates.

Figure 5. The dependency of control norm from differentiation index $\alpha$ at $p = p' = 2$ and different $q^T$: $q^T = 0$ (solid lines), $q^T = 2$ (dashed lines).

Formulas (33) and (35) for one-dimensional system (1) at different $a$ and for single integrator ($a = 0$ correspondingly). The dependencies obtained for different final coordinates $q^T$: $q^T = 0$ (fig. 1) and $q^T = 2$ (fig. 2). Other parameters were fixed as follows: $q^0 = 1$, $t_0 = 1$, $T = 100$, $b = 1$. As it seen on fig. 1 the curves differs qualitatively at $a \leq 1$ and at $a > 1$: in first case the monotonic growth obtained (which expressed more slightly in case of $a = 1$, than in case of $a < 1$), and in second case the curve monotonically decreasing. This tendency take place at $a \geq 1$ for $q^T = 2$ (fig. 2), but in this case behaviour of curves changes dramatically at $a < 1$: its demonstrate extremal character at $0.1 < a < 0.9$ and decrease monotonically at $a \leq 0.1$, and monotonic growth present only at $a \geq 0.9$. Figs. 3 and 4 represent the dependencies of control norm from control time $T$ for single integrator at severla fixed values of $\alpha$. It’s seen that,
in general, the curves decreasing but in case of \( q^T = 2 \) at small \( T \) the maximum appear and norm behaviour changes in dependence from \( \alpha \): at \( q^T = 0 \) the norm increases and at \( q^T = 2 \) it decreases.

At fig. 5 analogical results for control norm dependency from index \( \alpha \) represented for \( 1 < p < \infty \). It’s seen that the majority of curves monotonically increasing although at \( q^T = 2 \) for \( a = 0.1 \) slight extremum appears at small \( \alpha \).

### 4.2. Double integrator

Double integrator is represented by system (22) at \( N = 2 \). Let us consider the case \( p \to \infty \) and suppose that at least one of the moments \( c_1, c_2 \) (defined by (25)) is non-zero. Then the moment problem is correct and solvable for \( \alpha_1 > 0, \alpha_2 > 0 \) in accordance with Theorem 2. Using expressions (24) at \( k = 1, 2 \) one can study the problem (7)-(8) and reduce it to the problem of unconditional minimization of the following one-variable function:

\[
F(\xi_2) = \frac{1 - \xi_2 c_2}{c_1 \Gamma(\alpha_1 + \alpha_2 + 1)} \left[ (\ln \frac{T}{t_0})^{\alpha_1 + \alpha_2} - 2 \left( \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_2)} \frac{\xi_2 c_1}{\xi_2 c_2 - 1} \right)^{\alpha_1 + \alpha_2} \right] + \frac{\xi_2}{\Gamma(\alpha_2 + 1)} \left[ (\ln \frac{T}{t_0})^{\alpha_2} - 2 \left( \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_2)} \frac{\xi_2 c_1}{\xi_2 c_2 - 1} \right)^{\alpha_2} \right].
\]

We can calculate the first derivative of this function and demonstrate that its extrema are roots of the following equation:

\[
\left( \ln \frac{T}{t_0} \right)^{\alpha_2} \left[ \frac{1}{\Gamma(\alpha_2 + 1)} - \frac{c_2}{c_1 \Gamma(\alpha_1 + \alpha_2 + 1)} \right] - \frac{2}{\Gamma(\alpha_2)} \left( \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_2)} \frac{\xi_2 c_1}{\xi_2 c_2 - 1} \right)^{\alpha_2} \left[ \frac{1}{\alpha_2} - \frac{1}{\alpha_1 + \alpha_2} \frac{\xi_2 c_2}{\xi_2 c_2 - 1} \right] = 0. \tag{39}
\]

Equation (39) at arbitrary values of indices \( \alpha_1 \) and \( \alpha_2 \) have no explicit solution analogously to the same cases for Caputo \([5, 7]\) and Riemann-Liouville \([8]\) double integrators. But explicit solution of this equation can be obtained in some particular cases, for example, in case of \( c_2 = 0 \). Then we can solve equation (39) analytically and to find \( \xi_2^* \):

\[
\xi_2^* = -2 \frac{\alpha}{\alpha_2} \frac{\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} \left( \ln \frac{T}{t_0} \right)^{\alpha_1}. \tag{40}
\]

From (8) at \( c_2 = 0 \) we can find:

\[
\xi_1 = \frac{1}{c_1}. \tag{41}
\]

Using (41), (40) and (24) at \( k = 1, 2 \) one can obtain:

\[
\xi_1^* g_1(t) + \xi_2^* g_2(t) = \frac{1}{c_1 t \Gamma(\alpha_1 + \alpha_2)} \left[ (\ln \frac{t}{T})^{\alpha_2 - 1} - 2 \frac{\alpha}{\alpha_2} \left( \ln \frac{T}{t_0} \right)^{\alpha_2} \right]. \tag{42}
\]

Function (42) reverse sign at the following point:

\[
t' = T \left( \frac{T}{t_0} \right)^{-2} \frac{\alpha_2}{\alpha_1}. \tag{43}
\]
Substituting (42) into (7) and using (43) one can calculate:

$$
\lambda_2 = \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{1 - 2^{-\frac{\alpha_1}{\alpha_2}}} \frac{|c_1|}{(\ln \frac{T}{t_0})^{\alpha_1 + \alpha_2}}.
$$

(44)

OCP A solution result from (9) subject to (42) and (44):

$$
u(t) = \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{1 - 2^{-\frac{\alpha_1}{\alpha_2}}} \frac{c_1}{(\ln \frac{T}{t_0})^{\alpha_1 + \alpha_2}} \text{sign} \left[ \left( \ln \frac{T}{t} \right)^{\alpha_1} - 2^{-\frac{\alpha_1}{\alpha_2}} \left( \ln \frac{T}{t_0} \right)^{\alpha_1} \right].
$$

(45)

OCP B solution result from (10) subject to (42):

$$
u(t) = \text{sign} \left[ \left( \ln \frac{T^*}{t} \right)^{\alpha_1} - 2^{-\frac{\alpha_1}{\alpha_2}} \left( \ln \frac{T^*}{t_0} \right)^{\alpha_1} \right],
$$

(46)

where $T^*$ can be found numerically from (11) subject to (44). In some special cases one can obtain an explicit solution, for example, in case of $s_0^1 = q^T = 0$ we will have:

$$
T^* = t_0 \exp \left( \frac{|s_0^1| \Gamma(\alpha_1 + \alpha_2 + 1)}{2^{-\frac{\alpha_1}{\alpha_2}} \Gamma(\alpha_1) (1 - 2^{-\frac{\alpha_1}{\alpha_2}})} \right).
$$

(47)

**Remark 4.** Optimal controls (45) and (46) have one switching point, which corresponds to the corollary from Feldbaum’s interval theorem [16] for integer-order system of order $[\alpha_1]+1+[\alpha_2]+1$.

---

![Figure 6](image1.png)  
**Figure 6.** The dependency of control norm from index $\alpha_1$ at $\alpha_2$ fixed for double integrator at $q^T = 0$ (three upper curves) and at $q^T = 2$ (three lower curves). Dependencies calculated for several values of index $\alpha_2$: $\alpha_2 = 0.2$ (solid lines), $\alpha_2 = 0.5$ (dashed lines) and $\alpha_2 = 0.9$ (dash-dotted lines). Logarithmic scale used for ordinates.

![Figure 7](image2.png)  
**Figure 7.** The dependency of control norm from index $\alpha_2$ at $\alpha_1$ fixed for double integrator at $q^T = 0$ (three upper curves) and at $q^T = 2$ (three lower curves). Dependencies calculated for several values of index $\alpha_1$: $\alpha_1 = 0.2$ (solid lines), $\alpha_1 = 0.5$ (dashed lines) and $\alpha_1 = 0.9$ (dash-dotted lines). Logarithmic scale used for ordinates.

Now we will illustrate results obtained in this subsection. At fig. (6) and (7) represented dependencies of control norm (44) from each of indices $\alpha_1$ and $\alpha_2$ at other index fixed. It’s seen that dependency from $\alpha_1$ is more expressive. It’s also demonstrated that different values $q^T$ lead to qualitatively different dependencies. At fig. 8 shown dependencies of minimal control time $T^*$ from index $\alpha_1$ at $\alpha_2$ fixed, which demonstrate increasing character. Dependencies of $T^*$ from index $\alpha_2$ at $\alpha_1$ fixed are analogous to the curves at fig. 8.
5. Some features of phase trajectories of Hadamard double integrator

In this section we continue to study Hadamard double integrator with control $u(t) \in L_\infty(t_0, T]$ and focus on two basic aspects: boundary trajectories calculation and calculation of system phase trajectories in optimal control mode.

**Definition 2.** We will calling boundary trajectories the system phase trajectories, which correspond to boundary values of control $u(t) = \pm l$.

**Remark 5.** In case of $\alpha_1 = \alpha_2 = 1$ the boundary trajectories are identical to the boundaries of integral vortex of differential inclusion [19], which correspond to considered system. Boundary trajectories bound the phase plane region containing the all admissible trajectories of considered system.

To find boundary trajectories we suppose $u(t) = \pm l$ in the system (23) at $N = 2$. After calculations of corresponding integrals we obtain:

\[
\begin{align*}
q_1^{\pm l}(t) &= \frac{s_0^0}{\Gamma(\alpha_1)} \left( \ln \frac{t}{t_0} \right)^{\alpha_1-1} \pm \frac{s_2^0}{\Gamma(\alpha_1+\alpha_2)} \left( \ln \frac{t}{t_0} \right)^{\alpha_1+\alpha_2-1} \pm \frac{\ln t}{\Gamma(\alpha_1+\alpha_2+1)} \left( \ln \frac{t}{t_0} \right)^{\alpha_1+\alpha_2} \\
q_2^{\pm l}(t) &= \frac{s_0^0}{\Gamma(\alpha_2)} \left( \ln \frac{t}{t_0} \right)^{\alpha_2-1} \pm \frac{\ln t}{\Gamma(\alpha_2+1)} \left( \ln \frac{t}{t_0} \right)^{\alpha_2}.
\end{align*}
\]

(48)

In general case we can’t eliminate time from equations (48) and obtain obvious dependence $q_2^{\pm l}(q_1^{\pm l})$, although it’s possible to make in some special cases. For example, at $s_2^0 = 0$ we can express time from the second equation in (48) and , substituting it to the first equation, obtain finally:

\[
q_1^{\pm l} = \frac{s_0^0}{\Gamma(\alpha_1)} \left( \frac{\Gamma(\alpha_2+1)}{\pm l} q_2^{\pm l} \right)^{\frac{\alpha_1-1}{\alpha_2}} \pm \frac{l}{\Gamma(\alpha_1+\alpha_2+1)} \left( \frac{\Gamma(\alpha_2+1)}{\pm l} q_2^{\pm l} \right)^{\frac{\alpha_1}{\alpha_2+1}}.
\]

In case of $\alpha_2 = 1$ one can analogously obtain the following expression:

\[
q_1^{\pm l} = \frac{s_0^0}{\Gamma(\alpha_1)} \left( \frac{\pm l - s_2^0}{\pm l} \right)^{\alpha_1-1} \pm \frac{s_2^0}{\Gamma(\alpha_1+1)} \left( \frac{\pm l - s_2^0}{\pm l} \right)^{\alpha_1} \pm \frac{l}{\Gamma(\alpha_1+2)} \left( \frac{\pm l - s_2^0}{\pm l} \right)^{\alpha_1+1}.
\]

At fig. 9 and 10 some examples of boundary trajectories shown, which correspond to the following initial states: $s_1^0 = 1, s_2^0 = 0$. Other parameters were fixed as follows: $t_0 = 1, T = 100, l = 1$. 

\[\text{Figure 8. The dependency of minimal control time from index } \alpha_1 \text{ at } \alpha_2 \text{ fixed for double integrator at } q_1^T = 0. \text{ Dependencies calculated for several values of index } \alpha_2: \alpha_2 = 0.2 \text{ (solid lines), } \alpha_2 = 0.5 \text{ (dashed lines) and } \alpha_2 = 0.9 \text{ (dash-dotted lines).} \]
It’s seen from fig. 9 that in case of \( \alpha_1 = 1 \) boundary trajectories are similar to the same for double integrator of integer order [19], Caputo double integrator [5, 7] and Riemann-Liouville double integrator [8]. At \( \alpha_2 = 1 \) these trajectories bound maximal area on phase plane which decrease with \( \alpha_2 \) decreasing. Boundary trajectories at \( \alpha_1 < 1 \) (fig. 10) differs qualitatively from the same for integer-order double integrator, Riemann-Liouville and Caputo double integrators. The regions between solid and dashed curves in upper and lower half-planes at fig. 9 and 10 are inaccessible for the system. And decreasing of \( \alpha_2 \) corresponds to decreasing of area, bounded by boundary trajectories. Note also that at fig. 10 in the region bounded by dashed lines phase trajectories localized either in upper half-plane, or in lower half-plane.

Let us calculate now phase trajectories of Hadamard double integrator in optimal control mode. Substituting control (45) in system (23) at \( N = 2 \) we can derive:

\[
\begin{aligned}
q_1(t) &= \frac{s_1^0}{\Gamma(\alpha_1)} \left( \ln \frac{t}{t_0} \right)^{\alpha_1-1} + \frac{s_2^0}{\Gamma(\alpha_1+\alpha_2)} \left( \ln \frac{t}{t_0} \right)^{\alpha_1+\alpha_2-1} + \\
&\quad \times \left[ \left( \ln \frac{t}{t_0} \right)^{\alpha_1+\alpha_2} - 2 \left( \ln \frac{t}{t_0} \right)^{\alpha_1+\alpha_2} \Theta(t-t') \right], \\
q_2(t) &= \frac{s_1^0}{\Gamma(\alpha_2)} \left( \ln \frac{t}{t_0} \right)^{\alpha_2-1} + \\
&\quad \times \left[ \left( \ln \frac{t}{t_0} \right)^{\alpha_2} - 2 \left( \ln \frac{t}{t_0} \right)^{\alpha_2} \Theta(t-t') \right],
\end{aligned}
\]

(49)

where \( \Theta(t-t') \) is the Heaviside function, \( t' \) — control switching point (43). It can be verified directly that formulas (49) satisfy final conditions (3) (taking into account \( c_2 = 0 \)).

At fig. 11 and 12 represented some examples of phase trajectories, calculated using formulas (49). Initial state values and other parameters fixed identically to trajectories calculation for fig. 9 and 10. Final states fixed as follows: \( q_1^f = q_2^f = 0 \). It’s seen that at \( \alpha_1 = 1 \) (fig. 11) decreasing of index \( \alpha_2 \) lead to switching point shift to final state. In case of \( \alpha_1 < 1 \) (fig. 12) the overcontrol effect take place: phase trajectory go beyond final point, switches and return to final point. This behaviour is similar to the same for Caputo [5, 7] and Riemann-Liouville [8] double integrators.

6. Discussion

In this section we will compare obtained results with the same for integer-order systems and Caputo and Riemann-Liouville systems.
Figure 11. Phase trajectories of Hadamard double integrator in optimal control mode at $\alpha_1 = 1$ and several fixed values of $\alpha_2$: $\alpha_2 = 0.2$ (dot-dashed line), $\alpha_2 = 0.5$ (dotted line), $\alpha_2 = 0.8$ (dashed line) and $\alpha_2 = 1$ (solid line).

Figure 12. Phase trajectories of Hadamard double integrator in optimal control mode at some fixed values of indices $\alpha_1$ and $\alpha_2$.

6.1. Optimal control behaviour at integer values of differentiation indices

Note that for Hadamard systems the functions $g_i(t)$, obtained in this paper and defined by (13), (20) and (24), at $\alpha_1 = 1$, $\alpha_2 = 1$ doesn’t reduce to analogical functions for integer-order systems. Analogous situation take place for moments, defined by (14), (21) and (25) (saving moments $c_N$ in (25)). This fact is qualitative difference between Hadamard systems and Caputo and Riemann-Liouville systems, illustrated below by example.

Let us consider single integrator. If in this case the left side of equation (1) contains only classic differentiation operator of first order, then its solution subject to initial condition (2) can be written as follows:

$$q(t) = s^0 + \int_{t_0}^{t} u(\tau)d\tau. \quad (50)$$

Solution (19) for Hadamard single integrator reduces at $\alpha = 1$ to the following expression:

$$q(t) = s^0 + \int_{t_0}^{t} \frac{u(\tau)}{\tau}d\tau. \quad (51)$$

At $t = T$ both of solutions (50) and (51) can be rewritten as a moment problem (4) with the same moments but different functions $g(\tau)$. So, motion laws for ordinary single integrator and Hadamard single integrator of first order will differ from each other. It’s shown at fig. 13 for boundary values of control $u(t) = \pm l$.

It’s known [4, Ch. 2] that in case of $u(t) \in L_\infty(t_0, T)$ OCP A solution for ordinary single integrator can be expressed by the following formula (here and below it’s taking into account that for considered in this paper systems initial conditions established not at $t = 0$ but at $t = t_0 > 0$):

$$u^1(t) = \frac{q^T - s^0}{T - t_0}, t \in (t_0, T]. \quad (52)$$

OCP B solution for this case represented by expressions [4, Ch. 2]:

$$u^1(t) = l\text{sign}(q^T - s^0), t \in (t_0, T^*], \quad (53)$$

$$T^*_1 = t_0 + \frac{|q^T - q^0|}{l}. \quad (54)$$
Figure 13. Motion laws for ordinary single integrator (solid lines) and for Hadamard single integrator of first order (dashed lines) at $u(t) = \pm l$. Other parameters were fixed as follows: $t_0 = 1$, $T = 10$, $l = 1$, $s^0 = 1$.

If we fix $\alpha = 1$ at expression (36) subject (21) then we can obtain the following formula, which obviously differs from (52):

$$u(t) = \frac{q^T - s^0}{\ln \frac{T}{t_0}}.$$  

Expression (10) subject to (21) at $\alpha = 1$ is identical to expression (53) but expression (37) reduces to the following formula, which didn’t match with (54):

$$T^* = t_0 \exp \left( \frac{|s^0|}{l} \right).$$  

Let us consider the classical double integrator of first order. At arbitrary initial and final conditions solution of conditional minimization problem (7)-(8) lead to the following quadratic equation for $\xi_2$ [4, Ch. 2]:

$$(c_1 + Ac_2)\xi_2^2 - 2(c_1 + Ac_2)\xi_2 + A = 0,$$  

where $A = A^c = \left( \frac{c_2}{c_1} \frac{T - t_0}{2} - 1 \right) (T - t_0)$. Equation (39) at $\alpha_1 = \alpha_2 = 1$ reduces to equation (57) with another $A$: $A = A^H = \left( \frac{c_2}{c_1} \ln \frac{T}{t_0} - 1 \right) \ln \frac{T}{t_0}$.

OCP A solution for double integrator of first order at $s_2^0 = q_2^T = 0$ is given by the following formula [4, Ch. 2]:

$$u^2(t) = \frac{4(q_1^T - s_1^0)}{(T - t_0)^2} \text{sign} \left( \frac{T - t_0}{2} - \frac{t - t_0}{2} \right).$$  

OCP A solution (45) for Hadamard double integrator subject to $s_2^0 = q_2^T = 0$, $\alpha_1 = \alpha_2 = 1$ and (25) at $k = 1, 2$ reduces to the following expression:

$$u^2(t) = \frac{4(q_1^T - s_1^0)}{\left(\ln \frac{T}{t_0}\right)^2} \text{sign} \left( \ln \frac{T}{t_0} - \frac{1}{2} \ln \frac{T}{t_0} \right).$$  

OCP B solution for classical double integrator of first order at $q_1^T = 0$ and $s_1^0 > 0$ given by the following formulas [4, Ch. 2]:

$$u^2(t) = l \text{sign} \left( \frac{t - t_0}{s_1^0} - \frac{1}{\sqrt{ts_1^0}} \right), t \in (t_0, T_2^w].$$
\[ T_2^* = t_0 + 2\sqrt{\frac{s_0^0}{l}}. \] (61)

At the same assumptions formulas (46) and (47) for Hadamard double integrator at \( \alpha_1 = \alpha_2 = 1 \) reduces to the following expressions:

\[ u(t) = l\text{sign} \left( \frac{1}{s_0^0} \ln \frac{t}{t_0} - \frac{1}{\sqrt{ls_0^0}} \right), \] (62)

\[ T^* = t_0 \exp \left( \frac{4|s_0^0|}{l} \right)^{\frac{1}{2}}. \] (63)

Thus, we have demonstrated in detail that for Hadamard systems solutions of optimal control problem and motion laws doesn’t reduce to the same for integer-order systems when differentiation indices fixed to be equal to 1. It differs qualitatively Hadamard systems from analogous Caputo and Riemann-Liouville systems. And there is one interesting feature take place: all of the formulas obtained for Hadamard systems reduce to the same for corresponding integer-order systems when we change the value \( T - t_0 \) or function \( t - \tau \) by the value \( \ln \frac{T}{t_0} \) or function \( \ln \frac{t}{\tau} \).

On the other hand, expressions (48) for boundary trajectories of Hadamard double integrator at \( \alpha_1 = \alpha_2 = 1 \) reduced to the same equation, which can be obtained for classic double integrator of first order [19, §35]:

\[ q_1^{\pm t} = s_0^0 \pm s_0^2 q_2^{\pm t} - s_0^0 \pm \frac{1}{2l} \left( q_2^{\pm t} - s_0^2 \right)^2. \]

To the same equation reduced expresses for boundary trajectories of Caputo [5, 7] and Riemann-Liouville [8] double integrator at \( \alpha_1 = \alpha_2 = 1 \).

### 6.2. Comparison of optimal control features for Hadamard, Caputo and Riemann-Liouville systems

As noted above, today optimal control problems are studied most carefully for Caputo and Riemann-Liouville systems. Hadamard systems, investigated in this paper, are more similar to the systems of second type: both of Hadamard and Riemann-Liouville differentiation operators represented by first derivative from the convolution of considering function with some kernel; the statement of Cauchy problem need the nonlocal initial conditions for both types of systems. These facts allow to make not only qualitative but also quantitative comparison for results concerning Hadamard and Riemann-Liouville systems. Comparison of results obtained for Hadamard systems with the same for Caputo systems will be more qualitative than quantitative and will be made more briefly.

As for integer-order systems we can derive all the expressions for Riemann-Liouville systems from the same expressions for Hadamard systems by performing the change of value \( \ln \frac{T}{t_0} \) by value \( T - t_0 \) and of function \( \ln \frac{t}{\tau} \) by function \( t - \tau \) correspondingly.

Let us consider firstly the single integrator in case of \( u(t) \in L_{\infty}(t_0, T] \). For the Hadamard system at \( q^T = 0 \) expression (35) subject to (21) reduces to the following formula:

\[ \lambda^H = \frac{|s_0^0|\alpha}{\ln \frac{t}{t_0}}. \] (64)

The same formula for Riemann-Liouville single integrator obtained in [8] (formula (32) subject to (31)) and can be written here as follows:

\[ \lambda^{RL} = \frac{|s_0^0|\alpha}{T - t_0}. \] (65)
Thus, the dependency of control norm from differentiation index is linear for both types of single integrator and differ from each other only by scale multiplier. The value of control norm can vary significantly for different types of integrator, for example, as follows from (64) and (65), at \( T >> t_0 \geq 1 \) and identical initial conditions the control norm for Hadamard integrator is distinctly more than for Riemann-Liouville integrator. Additionally, it results from (64)-(65) that ratio of control norms corresponding to Hadamard and Riemann-Liouville integrators doesn’t depend on differentiation index and defines only by time interval value:

\[
\frac{\lambda^H}{\lambda^{RL}} = \frac{T - t_0}{\ln \frac{T}{t_0}}.
\]

It can be obtained analogously that dependencies for minimal control time from differentiation index also differ qualitatively for Hadamard and Riemann-Liouville integrators: in the first case exponential dependency take place and linear — in the second case.

In more general case of linear one-dimensional system at arbitrary final condition control norm for Hadamard system defines by formula (33), which can be written subject to (14) as follows:

\[
\lambda^H = \frac{a}{|b|} \left| q^T - s^0 \left( \ln \frac{T}{t_0} \right)^{\alpha-1} E_{\alpha,\alpha} \left[ a \left( \ln \frac{T}{t_0} \right) \alpha \right] \right| E_{\alpha} \left[ a \left( \ln \frac{T}{t_0} \right) \alpha \right] - 1.
\] (66)

For the same Riemann-Liouville system the following expression take place (it result from formula (24) subject to expression (23) in [8]):

\[
\lambda^{RL} = \frac{a}{|b|} \left| q^T - s^0 (T - t_0)^{\alpha-1} E_{\alpha,\alpha} \left[ a(T - t_0)^{\alpha} \right] \right| E_{\alpha} \left[ a(T - t_0)^{\alpha} \right] - 1.
\] (67)

**Figure 14.** The dependency of control norm from index \( \alpha \) for one-dimensional Hadamard (solid lines) and Riemann-Liouville (dashed lines) systems at \( q^T = 0 \) and different values of parameter \( a \). Logarithmic scale used for ordinates.

**Figure 15.** The dependency of control norm from index \( \alpha \) for one-dimensional Hadamard (solid lines) and Riemann-Liouville (dashed lines) systems at \( q^T = 2 \) and different values of parameter \( a \). Logarithmic scale used for ordinates.

At the same initial conditions and other parameters and at \( q^T = 0, T >> t_0 \geq 1 \) it result from (66)-(67) that control norm for Hadamard system is more than for Riemann-Liouville one (see fig. 14). And difference between control norms decreasing with parameter \( a \) growth. In case of \( q^T \neq 0 \) this difference reveal non-monotonic behaviour for \( a < 1 \) and eliminates for \( a = 1 \) (see fig. 15).
In case of double integrator expressions for control norm for Hadamard and Riemann-Liouville systems also differ qualitatively from each other. In first case control norm can be calculated using (44) subject to (25) at \( k = 1 \):

\[
\lambda^H_2 = \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{1 - \frac{\alpha_1}{\alpha_2}} \frac{|q_1^T - \frac{s_1^0}{\Gamma(\alpha_1)} (\ln \frac{T}{t_0})^{\alpha_1-1}|}{(\ln \frac{T}{t_0})^{\alpha_1+\alpha_2}}.
\]

(68)

For Riemann-Liouville double integrator control norm expressed by formula (which result from formula (40) subject to (37), obtained in [8]):

\[
\lambda^{RL}_2 = \frac{\Gamma(\alpha_1 + \alpha_2 + 1)}{1 - \frac{\alpha_1}{\alpha_2}} \frac{|q_1^T - \frac{s_1^0}{\Gamma(\alpha_1)} (T - t_0)^{\alpha_1-1}|}{(T - t_0)^{\alpha_1+\alpha_2}}.
\]

(69)

It’s interesting that ratio of control norms (68) and (69) at \( q_1^T = 0 \) and the same initial conditions and other parameters depend on \( \alpha_2 \) and time interval value only:

\[
\frac{\lambda^H_2}{\lambda^{RL}_2} = \left( \frac{T - t_0}{\ln \frac{T}{t_0}} \right)^{\alpha_2+1}.
\]

Let us compare now the motion laws and phase trajectories for Hadamard and Riemann-Liouville double integrators. Boundary trajectories of the Hadamard system in general case are expressed by formulas (48). Analogous formulas for the Riemann-Liouville system obtained in [8] (formulas (52)):

\[
\begin{align*}
q_1^{\pm l}(t) &= \frac{s_1^0}{\Gamma(\alpha_1)} (t - t_0)^{\alpha_1-1} + \frac{s_1^0}{\Gamma(\alpha_1+\alpha_2)} (t - t_0)^{\alpha_1+\alpha_2-1} \\
&\quad \pm \frac{1}{\Gamma(\alpha_1+\alpha_2+1)} (t - t_0)^{\alpha_1+\alpha_2},
\end{align*}
\]

(70)

Expressions (70) differ from (48) but in special cases, when we can eliminate time and obtain an explicit expression for \( q_2^{\pm l}(q_1^{\pm l}) \), these expression is the same for both of Hadamard and Riemann-Liouville systems. Particularly, formulas obtained above from (48) at \( s_1^0 = 0 \) and \( \alpha_2 = 1 \) are identical to formulas obtained from (70) at the same conditions (see formulas (53) and (54) in [8]). Comparison of expressions (48) and (70) shown that boundary trajectories for Riemann-Liouville system envelop more area at phase plane that boundary trajectories for Hadamard system at the same time interval and other parameters (fig. 16).

In optimal control mode phase trajectories of the Hadamard double integrator at \( c_2 = 0 \) described by formulas (49) and analogous expressions for the Riemann-Liouville system were obtained in [8] (formulas (55)) and can be written as follows:

\[
\begin{align*}
q_1(t) &= \frac{s_1^0}{\Gamma(\alpha_1)} (t - t_0)^{\alpha_1-1} + \frac{s_1^0}{\Gamma(\alpha_1+\alpha_2)} (t - t_0)^{\alpha_1+\alpha_2-1} + \\
&\quad + \frac{c^{RL}}{1 - \frac{\alpha_1}{\alpha_2}} \frac{T - t_0}{(T - t_0)^{\alpha_1+\alpha_2}} \times \\
&\quad \times [(t - t_0)^{\alpha_1+\alpha_2} - 2(t - t_0')^{\alpha_1+\alpha_2} \Theta(t - t_0')],
\end{align*}
\]

(71)

\[
\begin{align*}
q_2(t) &= \frac{s_2^0}{\Gamma(\alpha_2)} (t - t_0)^{\alpha_2-1} + \\
&\quad + \frac{c^{RL}}{1 - \frac{\alpha_1}{\alpha_2}} \frac{T - t_0}{\Gamma(\alpha_1+\alpha_2+1)} \times \\
&\quad \times [(t - t_0)^{\alpha_2} - 2(t - t_0')^{\alpha_2} \Theta(t - t_0')] ;
\end{align*}
\]
Figure 16. Three-dimensional representation of boundary trajectories for Hadamard (heavy lines) and Riemann-Liouville (thin lines) double integrators at $\alpha_1 = \alpha_2 = 0.5$, $s_1^0 = 1$, $s_2^0 = 0$, $t_0 = 1$, $T = 100$, $l = 1$.

Figure 17. Phase trajectories for Hadamard (solid lines) and Riemann-Liouville (dashed lines) double integrators at some values of indices $\alpha_1$ and $\alpha_2$ and $s_1^0 = 1$, $s_2^0 = 0$, $t_0 = 1$, $T = 100$.

where $c_1^{RL} = q_1^T - \frac{s_2^0(T-t_0)^{\alpha_1 + \alpha_2 - 1}}{\Gamma(\alpha_1 + \alpha_2)} - \frac{s_1^0(T-t_0)^{\alpha_1 - 1}}{\Gamma(\alpha_1)}$. For switching point of optimal control for Riemann-Liouville double integrator in this case the following formula valid:

$$t_{RL}^\prime = T \left( 1 - 2^{-\frac{1}{\alpha_2}} \right) + 2^{-\frac{1}{\alpha_2}} t_0. \quad (72)$$

Comparison of calculations by formulas (49) and (71) demonstrates that in case of Hadamard double integrator overcontrolling effect expressed more significantly (fig. 17).

At fig. 18 the dependencies of switching point of optimal control from index $\alpha_2$ represented for both types of double integrators studied here. It’s seen that for the Hadamard system switching of control happens earlier than for the Riemann-Liouville system. And for both of systems dependencies decreasing with $\alpha_2$ growth which correspond to switching point shifting to the left boundary of time interval (it also take place for Caputo systems [5, 7]).

7. Conclusion
In this paper investigation of the optimal control problem carried out for the linear fractional-order systems, which described by equations with Hadamard derivative. One-dimensional system
Figure 18. The dependence of switching point of optimal control from index α_2 for Hadamard (solid lines) and Riemann-Liouville (dashed lines) double integrators at t_0 = 1, T = 100.

of general view and N-fold integrator studied. The optimal control problem reduced to the l-problem of moments. For the last problem conditions of correctness and solvability obtained. Analytical solutions of the studied problem obtained and its features analyzed. Comparison of results performed with the same results for integer-order systems and fractional-order systems, described by equations with Riemann-Liouville and Caputo derivative.

Obtained results can be used for optimal control investigation concerning fractional-order systems and for model and parameter choosing of particular control systems.

Acknowledgements.

This paper dedicated to the memory of my teacher and co-author V.A. Kubyshkin.

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