A Reflection Equivalence for Gorenstein-Projective Quiver Representations
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Abstract: For \( \Lambda \) a selfinjective algebra, and \( Q \) a finite quiver without oriented cycles, the algebra \( \Lambda Q \) is a Gorenstein algebra and the category \( \text{G-proj} \Lambda Q \) of Gorenstein-projective \( \Lambda Q \)-modules is a Frobenius category. For a sink \( v \) of \( Q \), we define a functor \( F(v) : \text{G-proj} \Lambda Q \to \text{G-proj} \Lambda Q(v) \) between the stable categories modulo projectives, where \( \hat{Q}(v) \) is obtained from \( Q \) by changing the direction of each arrow ending in \( v \). The functor is given by an explicit construction on the level of objects and homomorphisms. Our main result states that \( F(v) \) is an equivalence of categories. In the case where the underlying graph of \( Q \) is a tree, we deduce that the stable category \( \text{G-proj} \Lambda Q \) does not depend on the orientation of \( Q \). Moreover, if \( Q \) is a quiver of type \( A_3 \) and \( \Lambda = k[T]/(T^n) \) the bounded polynomial algebra, we use the symmetry of the octahedron in the octahedral axiom to verify that the composition of twelve reflections yields the identity on objects.

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1. Introduction

Throughout this paper, let \( k \) be a field, \( \Lambda \) be a finite-dimensional selfinjective \( k \)-algebra, and \( Q \) a finite quiver which is acyclic, that is, \( Q \) has no oriented cycles. The path algebra of the quiver \( Q \) with coefficients in the algebra \( \Lambda \) is denoted by \( \Lambda Q = \Lambda \otimes_k kQ \). Since \( \Lambda \) is selfinjective, and \( kQ \) hereditary, \( \Lambda Q \) is a Gorenstein algebra; then the category \( \text{G-proj} \Lambda Q \) of Gorenstein-projective modules is a Frobenius category.

Let \( Q \) be a quiver and let the vertex \( v \in Q_0 \) be a sink. The reflection of \( Q \) at \( v \) is the quiver \( Q(v) \) on the same set of vertices, but with each arrow ending in \( v \) replaced by its opposite.

In this paper we use a pull-back construction on the level of objects and homomorphisms in the category \( \text{G-proj} \Lambda Q \) to define the functor \( F(v) \) in the following main result.

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Theorem 1.1. For $\Lambda$ a selfinjective algebra and $Q$ a finite acyclic quiver, the functor

$$F(v) : \text{G-proj} \Lambda Q \to \text{G-proj} \Lambda Q(v)$$

is an equivalence of categories.

In [1], Corollaries 7.3–7.4, the authors proved that there is an equivalence of singularity categories $D_{sg}(\Lambda Q) \cong D_{sg}(\Lambda Q(v))$, where $\Lambda$ is a Gorenstein algebra. As a consequence, they obtain a different proof for the equivalence $\text{G-proj} \Lambda Q \cong \text{G-proj} \Lambda Q(v)$. Due to the approach via singularity categories, it is not immediately clear how the objects in $\text{G-proj} \Lambda Q$ and in $\text{G-proj} \Lambda Q(v)$ correspond to each other. By comparison, the functor $F(v)$ in our paper is given by an explicit construction.

The main tool in our approach is the characterization of Gorenstein-projective quiver representations by Luo and Zhang [15,16] as the separated monic representations, see Section 2.2 below. In particular for a quiver of type $A_n$ in linear operation, the Gorenstein-projective quiver representations are the chains of monomorphisms. Thus, in the special case where $\Lambda = k[T]/(T^n)$ is the bounded polynomial ring, we are dealing with systems consisting of a nilpotent linear operator and a chain of $n-1$ invariant subspaces, see [19] and [14]; such systems occur in applications, for example in control theory [17].

Gorenstein-projective modules feature prominently in relative homological algebra where they assume the role played by projectives modules in homological algebra [8,9]. Ringel and Zhang exhibit a link to derived categories: If $\Lambda = k[\varepsilon] = k[T]/(T^2)$ is the bounded polynomial ring, the authors show the equivalence $\text{G-proj} \Lambda Q \cong D^b(\text{mod} kQ)/[1]$ in [20].

In the second part of this paper we deal with examples where the quiver $Q$ has type $A_3$ and $\Lambda$ is the bounded polynomial ring $k[T]/(T^n)$. Up to quiver automorphism, there are three orientations for the quiver:

$$Q : 1 \to 2 \to 3, \quad Q(3) : 1 \to 2 \leftarrow 3, \quad Q(2,3) : 1 \leftarrow 2 \to 3.$$  

So our result yields three categorical equivalences:

$$F(3) : \text{G-proj} \Lambda Q \to \text{G-proj} \Lambda Q(3),$$
$$F(2,3) : \text{G-proj} \Lambda Q(3) \to \text{G-proj} \Lambda Q(2,3),$$
$$F(1,2,3) : \text{G-proj} \Lambda Q(2,3) \to \text{G-proj} \Lambda Q.$$

The composition of three reflections yields a selfequivalence $F^3$ on $\text{G-proj} \Lambda Q$. 
We show in Section 4 that the composition of twelve reflections, \((F^3)^4\), yields isomorphisms for objects:

**Theorem 1.2.** Suppose \(Q\) is a quiver of type \(A_3\) and \(\Lambda = k[T]/(T^n)\). Let \(M\) be a representation for \(Q\) with coefficients in \(\Lambda\) such that no branch \(M_i\) has a nonzero projective direct summand. Let \(A\) be a right approximation for \(M\) in \(\text{G-proj} \, \Lambda Q\). Let \(B\) be obtained from \(A\) by applying twelve reflections. Then

\[ B \cong A \quad \text{in} \quad \text{G-proj} \, \Lambda Q. \]

The twelve reflections can be visualized on the octahedron in the octahedral axiom. Recall that of the eight faces of the octahedron, four correspond to exact triangles while the other four represent commutative triangles, each given by a pair of composable maps and their product. The four products form the cyclically oriented square in the octahedron which we consider the “equator”.

As an oriented graph, the octahedron has a symmetry group which is cyclic of order four (Observation 4.3). A generator \(\rho\) is given by rotating the octahedron by 90° about the axis perpendicular to the equator, in the direction given by the four products, followed by a reflection on the plane of the equator. We show that \(F(1, 2, 3)\) maps a composable pair to the pair in the octahedron obtained by applying \(\rho\), up to the inverse of the suspension \([-1]\). Since \(\rho\) has order four, any composition of twelve reflections yields the identity in the stable category, up to applications of the suspension.

We conclude the paper with examples.

First we exhibit the Auslander-Reiten quivers for the Gorenstein-projective modules over \(\Lambda Q\), \(\Lambda Q(3)\) and \(\Lambda Q(2, 3)\) where \(\Lambda = k[T]/(T^2)\) and \(Q\) the above quiver of type \(A_3\) in linear orientation. While the three stable categories of Gorenstein-projective modules are equivalent, this result does not hold for the categories of all Gorenstein-projective modules, of all monic representations, or of all modules.

Next we compute the orbit of a simple representation under reflections and illustrate it in the Auslander-Reiten quivers of \(\Lambda Q\), \(\Lambda Q(3)\) and \(\Lambda Q(2, 3)\) for \(\Lambda = k[T]/(T^3)\). As a byproduct we obtain the orbit in Figure 1. The icons will be explained in the text. The modules in the top row are representations of \(\Lambda Q\), those in the second row of \(\Lambda Q(2, 3)\) and those in the bottom row of \(\Lambda Q(3)\).

We summarize the contents of the paper.
In Section 2 we study the interplay between the categories of monic representations of $Q$ over $\Lambda$ and of representations of $Q$ with objects and morphisms in the stable category $\text{mod} \Lambda$ modulo projectives.

The aim of Section 3 is to present the construction of objects and homomorphisms in $\text{G-proj} \Lambda Q(v)$ from those in $\text{G-proj} \Lambda Q$. Moreover, we sketch the proof of Theorem 1.1.

In Section 4 we revisit the octahedral axiom and use it to trace the composition of twelve reflections on objects in $\text{G-proj} \Lambda Q$ where the quiver $Q$ has type $A_3$, showing Theorem 1.2.

The last Section 5 gives examples which illustrate our construction and visualize the composition of the twelve reflections.

2. Monic Representations

2.1. Notation. We specify notation to define for $X \in \text{G-proj} \Lambda Q$ and $v$ a sink in $Q$ the corresponding module $X(v) \in \text{G-proj} \Lambda Q(v)$.

For a vertex $w \in Q_0$, let $w^- = \{s(\alpha) : \alpha \in Q_1$ and $t(\alpha) = w\}$ be the multiset of predecessors of $w$. Put

$$X_{w^-} = \bigoplus_{t(\alpha)=w} X_{s(\alpha)}.$$ 

We recall that a Gorenstein-projective module $X$ is separated monic (see definition and results in Subsection 2.2), so the map

$$u_{X,w} = (X_{\alpha})_{t(\alpha)=w} : X_{w^-} \to X_w$$

is a monomorphism, hence the short exact sequence

$$0 \to X_{w^-} \xrightarrow{u_{X,w}} X_w \to X_w^+ \to 0$$
defines a $\Lambda$-module $X$. We will sometimes consider $X_{w'}$ as a submodule of $X_w$. Regarding the map $u_{X,w}$, we sometimes omit one or both of the subscripts.

2.2. Separated monic representations. Given a finite quiver $Q$ without oriented cycles, in their papers [15,16], the authors define separated monic representations of $Q$ over an algebra $\Lambda$ as follows:

**Definition:** A representation $X = (X_i, X_\alpha)$ of $Q$ over $\Lambda$ is a separated monic representation, or a separated monic $\Lambda Q$-module, if for each vertex $i \in Q_0$ the map

$$u_{X,i} : X_i \rightarrow X_i$$

is a monomorphism. This means that $X$ satisfies the following two conditions:

1. For each $\alpha \in Q_1$, $X_\alpha : X_{s(\alpha)} \rightarrow X_{t(\alpha)}$ is an injective homomorphism, and
2. For each $i \in Q_0$, the sum $\sum_{t(\alpha) = i} \text{Im} X_\alpha$ is a direct sum.

If $\Lambda$ is a selfinjective algebra, then the Gorenstein-projective representations of $Q$ over $\Lambda$ are exactly the separated monic representations [15,16].

2.3. Minimal Monomorphisms. The algebra $\Lambda Q$ is Gorenstein, so every $\Lambda Q$-module has a right approximation in the category $\text{G-proj } \Lambda Q$.

Here we present a stepwise construction for such an approximation of a $\Lambda Q$-module $M$ in $\text{G-proj } \Lambda Q$ which we call Mono($M$) and define Mimo($M$) to be the minimal version of this right approximation.

For a vertex $i \in Q_0$, let $P(i)$ denote the $k$-linear projective representation of the quiver $Q$ which corresponds to the vertex $i$, it gives rise to a functor

$$P_i : \text{mod } \Lambda \rightarrow \text{mod } \Lambda Q, \ A \mapsto A \otimes_k P(i).$$

Suppose $M \in \text{mod } \Lambda Q$ is a module and $\alpha : v \rightarrow w$ an arrow. We construct a module $N = \text{Mono}_\alpha(M)$ which is such that the map $N_\alpha$ is monic and the sum $\text{Im} N_\alpha \oplus \sum_{t(\beta) = w, \beta \neq \alpha} \text{Im} N_\beta$ is direct.

Choose an injective envelope $e : M_v \rightarrow J = E(M_v)$. Then $\text{Mono}_\alpha(M)$ is given as the middle term of the short exact sequence

$$0 \rightarrow P_w(J) \rightarrow \text{Mono}_\alpha(M) \rightarrow M \rightarrow 0$$
which is split exact in each component. For \( \beta : i \to j \) an arrow, the map \( \text{Mono}_\alpha(M)_\beta \) is given by the vertical map in the middle of the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{p:w \to i} J & \overset{(0_1)}{\longrightarrow} & M_i & \bigoplus_{p:w \to i} J & \overset{(1_0)}{\longrightarrow} & M_i & \longrightarrow & 0 \\
\downarrow \text{incl} & & \downarrow \text{incl} & & \downarrow & & \downarrow \text{incl} & & \downarrow M_\beta & & \downarrow 0 \\
0 & \longrightarrow & \bigoplus_{p:w \to j} J & \overset{(0_1)}{\longrightarrow} & M_j & \bigoplus_{p:w \to j} J & \overset{(1_0)}{\longrightarrow} & M_j & \longrightarrow & 0 \\
\end{array}
\]

where \( E_\beta = 0 \) unless \( \beta = \alpha \) in which case

\[
E_\alpha = \begin{pmatrix} e_0 \\ 0 \end{pmatrix} : M_v \to J \bigoplus_{p:w \to j} J.
\]

where the first component of the module on the right is the summand corresponding to the path \( p = \alpha \).

The following two results are straightforward.

**Lemma 2.1.** Let \( M \in \text{mod } \Lambda Q \) be a module and \( \alpha, \beta \) arrows.

1. The \( \Lambda Q \)-module \( N = \text{Mono}_\alpha(M) \) satisfies that \( N_\alpha \) is monic and the sum \( \text{Im } N_\alpha \oplus \sum_{t(\beta) = w, \beta \neq \alpha} \text{Im } N_\beta \) is a direct sum.

2. Suppose the arrows \( \alpha, \beta \) are such that there is no path from \( t(\alpha) \) to \( s(\beta) \), or from \( t(\beta) \) to \( s(\alpha) \). Then

\[
\text{Mono}_\alpha(\text{Mono}_\beta(M)) \cong \text{Mono}_\beta(\text{Mono}_\alpha(M)).
\]

Note that if there are paths from \( t(\alpha) \) to \( s(\beta) \), then for each path, \( \text{Mono}_\beta(\text{Mono}_\alpha(M)) \) has an additional copy of \( E(M_{s(\beta)}) \) in the \( t(\beta) \)-branch and in each branch corresponding to a successor of \( t(\beta) \).

For a list of arrows, \( A = (\alpha_s, \ldots, \alpha_1) \) such that for \( i > j \) there is no path from \( t(\alpha_i) \) to \( s(\alpha_j) \), we define

\[
\text{Mono}_A(M) = \text{Mono}_{\alpha_1}(\ldots \text{Mono}_{\alpha_s}(M) \ldots).
\]

In particular, if \( v \) is a vertex in \( Q \) and \( (\alpha_s, \ldots, \alpha_1) \) the list of all arrows ending in \( v \), we put

\[
\text{Mono}_v(M) = \text{Mono}_{\alpha_1}(\ldots \text{Mono}_{\alpha_s}(M) \ldots).
\]

Since \( Q \) is an acyclic quiver, we can always arrange the set of all arrows such that for \( i > j \) there is no path from \( t(\alpha_i) \) to \( s(\alpha_j) \). For such an arrangement of arrows, we put

\[
\text{Mono}(M) = \text{Mono}_{\alpha_1}(\ldots \text{Mono}_{\alpha_s}(M) \ldots),
\]
where $s$ is the number of arrows in $Q$.

**Proposition 2.2.** Let $M \in \text{mod} \Lambda Q$ be a module.

1. Let $A = (\alpha_s, \ldots, \alpha_1)$ be a sequence of arrows such that for $i > j$ there is no path from $t(\alpha_i)$ to $s(\alpha_j)$. If $\text{Mono}_A(M)$ is separated monic, then the epimorphism in the short exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{s} P_{t(\alpha_i)}(E(M_{s(\alpha_i)})) \rightarrow \text{Mono}_A(M) \rightarrow M \rightarrow 0$$

is a right approximation for $M$ in $\text{G-proj} \Lambda Q$.

2. In particular, if $A$ is the sequence of arrows in the definition of $\text{Mono}(M)$, the epimorphism in the short exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{s} P_{t(\alpha_i)}(E(M_{s(\alpha_i)})) \rightarrow \text{Mono}(M) \rightarrow M \rightarrow 0$$

is a right approximation for $M$ in $\text{G-proj} \Lambda Q$.

**Proof.** See [9], Theorem 11.5.1. ■

**Definition:** For $M \in \text{mod} \Lambda Q$, the minimal monomorphism $\text{Mimo}(M)$ is given by the right minimal version $\text{Mimo}(M) \rightarrow M$ of the map $\text{Mono}(M) \rightarrow M$.

Since $\text{Mono}(M)$ is a finite length module (Lemma 2.2), the approximation $\text{Mono}(M) \rightarrow M$ has a right minimal version; this map $\text{Mimo}(M) \rightarrow M$ is the minimal right approximation for $M$ in $\text{G-proj} \Lambda Q$. To obtain the right minimal version of $\text{Mono}(M) \rightarrow M$, we need to split off some projective direct summand from $\text{Mono}(M)$.

We have

**Proposition 2.3.** Let $Q$ be a finite acyclic quiver, $n$ a source in $Q$, $\Lambda$ a selfinjective algebra and $M$ a separated monic representation of $Q$ over $\Lambda$. If $M_n = I_n \oplus M'_n$ with $I_n$ being a projective $\Lambda$-module, then $P_n(I_n)$ is a direct summand of $M$.

**Proof.** There is an embedding $\sigma : I_n \rightarrow M_n$ which induces a monomorphism $P_n(\sigma) : P_n(I_n) \rightarrow M$ since $M$ is separated monic (Lemma 2.3 in [13]). With $S = \text{Cok} P_n(\sigma)$, we have an exact sequence

$$0 \rightarrow P_n(I_n) \xrightarrow{P_n(\sigma)} M \xrightarrow{\pi} S \rightarrow 0.$$  

Since $P_n(I_n)$ is a projective-injective object in $\text{G-proj} \Lambda Q$, then to prove the assertion, we just need to prove that $S$ is separated monic.
For each $i \in Q_0 \setminus \{n\}$, the above short exact sequence gives the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & P_n(I_n)_{i-} & \rightarrow & M_{i-} & \rightarrow & S_{i-} & \rightarrow & 0 \\
& & \downarrow u_{P,i} & & \downarrow u_{M,i} & & \downarrow u_{S,i} & & \\
0 & \rightarrow & P_n(I_n)_{i} & \rightarrow & M_{i} & \rightarrow & S_{i} & \rightarrow & 0
\end{array}
\]

Recall that a module of the form $P_w(V)$ where $w \in Q_0$ and $V \in \text{mod} \Lambda$ has the property that $u_{P_w(V),i}$ is an isomorphism for every vertex $i \neq w$. Here, since $i \neq n$, the left hand map always is an isomorphism. Hence, by the Snake Lemma, for each $i \in Q_0$,

$u_{S,i} : S_i \rightarrow S_i$

is a monomorphism which follows from the facts that $M$ is separated monic and $\text{Cok}(u_{P,i}) = 0$.

\[\textbf{Proposition 2.4.} \text{ Let } Q \text{ be a finite acyclic quiver, } \Lambda \text{ a selfinjective algebra and } M \text{ be a separated monic representation of } Q \text{ over } \Lambda. \text{ There is an exact sequence}
\]

\[0 \rightarrow I \rightarrow M \rightarrow N \rightarrow 0\]

\[\text{such that } I \text{ is a projective object in } \text{G-proj} \Lambda Q \text{ and no branch of the } \Lambda Q\text{-module } N \text{ has a non-zero projective direct summand.}\]

\[\textbf{Proof.} \text{ Use induction on the number } n = |Q_0| \text{ of vertices in } Q. \text{ If } n = 1, \text{ then the assertion is clear. Suppose that the assertion is true for quivers with fewer vertices.}
\]

\[\text{Let } n \text{ be a source of } Q. \text{ Suppose that } M_n = I_n \oplus L_n \text{ such that } I_n \text{ is a projective } \Lambda\text{-module and } L_n \text{ has no non-zero projective direct summand. By Proposition 2.3, we have an exact sequence}
\]

\[0 \rightarrow P_n(I_n) \overset{P_n(\sigma)}{\rightarrow} M \overset{\pi}{\rightarrow} S \rightarrow 0,
\]

\[\text{with } S = \text{Cok} P_n(\sigma) \text{ being a separated monic representation of } Q \text{ over } \Lambda \text{ and } S_n = L_n. \text{ Restricting } S \text{ to the full subquiver } Q' \text{ with set of vertices } Q_0 \setminus \{n\}, \text{ we get a separated monic representation } S' \text{ of } Q' \text{ over } \Lambda. \text{ Since } |Q'_0| = n - 1, \text{ by the hypothesis, we have an exact sequence}
\]

\[0 \rightarrow I' \rightarrow S' \rightarrow N' \rightarrow 0,
\]

\[\text{such that } I' \text{ is a projective object in } \text{G-proj} \Lambda Q' \text{ and no branch of } N' \text{ has a non-zero projective direct summand. Consider } I' \text{ as a representation}
of $Q$; let $N$ be the target of the cokernel map in the diagram

$$
0 \to I' \to S' \to N' \to 0
$$

Thus, $N_i = N'_i$ for $i \neq n$ and $N_n = S_n$. In particular, no branch of $N$ contains a non-zero projective-injective direct summand. Take $I$ to be the pullback of $\pi$ and $I' \to S$, given by the following commutative diagram

$$
\begin{array}{ccc}
& & 0 & \to & I & \to & I' & \to & 0 \\
0 & \to & P_n(I_n) & \to & I & \to & I' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & P_n(I_n) & \to & M & \to & S & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
N & \to & N & \to & 0 & \to & 0 \\
\end{array}
$$

with exact rows and columns. Since $P_n(I_n)$ and $I'$ are projective objects in $\text{G-proj} \Lambda Q$, so is $I$.

If $M$ is a separated monic representation of $Q$ over $\Lambda$ and for each vertex $i \in Q_0$, $M_i$ is a projective $\Lambda$-module, then by Proposition 2.4, we obtain an exact sequence

$$
0 \to I \to M \to N \to 0
$$

such that $I$ is a projective object in $\text{G-proj} \Lambda Q$ and no branch of $N$ has a non-zero projective direct summand. It follows that $N = 0$ and $M \cong I$. We have shown the following result:

**Corollary 2.5.** Let $Q$ be a finite acyclic quiver, $\Lambda$ a selfinjective algebra and $M$ be a separated monic representation of $Q$ over $\Lambda$. If $M_i$ is a projective $\Lambda$-module for each vertex $i \in Q_0$, then $M$ is a projective representation $Q$ over $\Lambda$. ■
2.4. **Lifting isomorphisms from the stable category.** The following result adapts [IS Theorem 4.2] to our situation.

**Proposition 2.6.** Let $X, Y$ be representations of $Q$ over $\Lambda$ such that none of the components $X_i, Y_i$ has a nonzero injective direct summand. Then $\text{Mono}(X) \cong \text{Mono}(Y)$ if and only if for each vertex $i$, there is an isomorphism $f_i : X_i \to Y_i$ such that for each arrow $\alpha : i \to j$, the map $Y_\alpha f_i - f_j X_\alpha$ factors through an injective object.

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow x_\alpha & & \downarrow y_\alpha \\
X_j & \xrightarrow{f_j} & Y_j
\end{array}
\]

*Proof.* Assume first that $g : \text{Mono}(X) \to \text{Mono}(Y)$ is an isomorphism. Write $\text{Mono}(X)_i = X_i \oplus D_i$, $\text{Mono}(Y)_i = Y_i \oplus E_i$ with $D_i$ and $E_i$ injective $\Lambda$-modules, and let $g_i = (\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix})$. Since $X_i$ and $E_i$, and $Y_i$ and $D_i$ have no isomorphic indecomposable direct summands, the maps $a_i$ and $d_i$ must be isomorphisms.

For each arrow $\alpha : i \to j$, we have $\text{Mono}(Y)_\alpha \circ g_i = g_j \circ \text{Mono}(X)_\alpha$, hence, by restriction, there is the commutative diagram

\[
\begin{array}{ccc}
X_i & \xrightarrow{(a_i \ b_i)} & \text{Mono}(Y)_i \\
\downarrow (X_\alpha \ u) & & \downarrow (Y_\alpha \ v) \\
\text{Mono}(X)_j & \xrightarrow{(a_j \ b_j)} & Y_j
\end{array}
\]

and $Y_\alpha a_i - a_j X_\alpha = b_j u - v c_i$ factors through an injective module.

For the converse, assume isomorphisms $f_i : X_i \to Y_i$ are given such that for each arrow $\alpha : i \to j$, the map $Y_\alpha f_i - f_j X_\alpha$ factors through an injective object. We recall that $\text{Mono}(M) = \text{Mono}_{\alpha_1}(\ldots \text{Mono}_{\alpha_s}(M) \ldots)$ and proceed along the sequence of arrows $\alpha_s, \alpha_{s-1}, \ldots, \alpha_1$.

Let $k \in \{s, \ldots, 1\}$ and write $U = \text{Mono}_{\alpha_{k+1}}(\ldots \text{Mono}_{\alpha_s}(X) \ldots)$ and $V = \text{Mono}_{\alpha_{k+1}}(\ldots \text{Mono}_{\alpha_s}(Y) \ldots)$. Suppose we have already constructed a map (not necessarily a homomorphism) $g^{k+1} : U \to V$ which is a $\Lambda$-isomorphism in each branch and which is such that each square corresponding to $\alpha_{k+1}, \ldots, \alpha_s$ is commutative. Write $\alpha = \alpha_k$, say $\alpha : i \to j$, and consider the square corresponding to this arrow,
recall that it may not be commutative.

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow^{(X_\alpha)} & & \downarrow^{(Y_\alpha)} \\
X_j \oplus Q_j & \xrightarrow{(f_j \ a_j \ b_j)} & Y_j \oplus R_j \\
\end{array}
\]

(The sequence of arrows is chosen such that we have \( U_i = X_i, V_i = Y_i \) and \( g_i^{k+1} = f_i \), but in the \( j \)-component there may be additional projective summands \( Q_j, R_j \) with \( b_j : Q_j \to R_j \) an isomorphism.) In the construction \( \text{Mono}_\alpha \) for \( U \) and \( V \), we have added injective envelopes \( d_i : X_i \to I_i, e_i : Y_i \to J_i \). The given isomorphism \( f_i : X_i \to Y_i \) can be extended to an isomorphism \( u_i : I_i \to J_i \) such that \( u_id_i = e_if_i \). The map \( f_iX_\alpha - Y_\alpha f_i \) factors through an injective module, hence factors through the injective envelope \( d_i : X_i \to I_i \), so there is \( h_\alpha : I_i \to Y_j \) such that \( f_iX_\alpha - Y_\alpha f_i = -h_\alpha d_i \). We construct the map \( g^k : \text{Mono}_\alpha(U) \to \text{Mono}_\alpha(V) \). The square corresponding to \( \alpha \) is as follows.

\[
\begin{array}{ccc}
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow^{(X_\alpha \ d_i)} & & \downarrow^{(Y_\alpha \ e_i)} \\
X_j \oplus Q_j \oplus I_i & \xrightarrow{(f_j \ a_j \ h_\alpha \ 0 \ \ b_j \ 0 \ u_i)} & Y_j \oplus R_j \oplus J_i \\
\end{array}
\]

For each remaining vertex \( \ell \neq i, j \) we construct the map \( g^k_\ell : \text{Mono}_\alpha(U)_\ell \to \text{Mono}_\alpha(V)_\ell \) as follows. Note that \( \text{Mono}_\alpha(U)_\ell = U_\ell \oplus P_j(I_i)_\ell \) where \( P_j(I_i)_\ell = \bigoplus_{p,j \to \ell} I_i^{(p)} \), the sum being taken over all paths \( p \) from \( j \) to \( \ell \) and each summand \( I_i^{(p)} = I_i \) being marked by the path to which it corresponds. Define

\[
g^k_\ell = \begin{pmatrix} g^{k+1}_\ell & (V_p \circ (h_\alpha)_0)_{p} \\ 0 & \bigoplus_{u_i^{(p)}} \end{pmatrix} : U_\ell \oplus \bigoplus I_i^{(p)} \to V_\ell \oplus \bigoplus J_i^{(p)};
\]

here, \( (h_\alpha)_0 : I_i \to V_j \oplus R_j \) is part of the map \( g^k_j \) pictured above, \( V_p = V_{\beta_1} \circ \cdots \circ V_{\beta_1} \) is the composition of the structural maps in \( V \) if the path is given as \( p = \beta_1 \cdots \beta_1 \), and \( \bigoplus_{p,j \to \ell} u_i^{(p)} : \bigoplus I_i^{(p)} \to \bigoplus J_i^{(p)} \) is the diagonal map with entries \( u_i^{(p)} = u_i \).

The map \( g^k : \text{Mono}_\alpha(U) \to \text{Mono}_\alpha(V) \) thus constructed has the following properties: Like \( g^{k+1} \), for each arrow the corresponding square is commutative, up to a morphism which factors through an injective object. Like \( g^{k+1} \), for each arrow \( \alpha_{k+1}, \ldots, \alpha_s \), the corresponding square
is commutative. And in addition, the square for the arrow \( \alpha = \alpha_k \) is commutative. Hence we can continue our iteration.

With the iteration completed, it follows that the map \( g = g^1 : \text{Mono}(X) \to \text{Mono}(Y) \) is a homomorphism since for each arrow, the corresponding diagram is commutative, and hence an isomorphism since each branch map is.

We illustrate this map in the example of the quiver \( Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \). For representations \( X, Y \) of \( Q \) over \( \Lambda \) and isomorphisms \( f_i : X_i \to Y_i \) as in Proposition 2.6, the corresponding map \( g : \text{Mono}_\alpha(\text{Mono}_\beta(X)) \to \text{Mono}_\alpha(\text{Mono}_\beta(Y)) \) is as follows.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X_2 \oplus I_1 & \xrightarrow{(f_2 \ h_\alpha \ 0 \ u_1)} & Y_2 \oplus J_1 \\
\downarrow & & \downarrow \\
X_3 \oplus I_2 \oplus I_1 & \xrightarrow{\delta_\alpha \ h_\beta \ (0 \ u_2 \ 0 \ \alpha)} & Y_3 \oplus J_2 \oplus J_1
\end{array}
\]

Remark: From the above proof we can see that the condition that none of the components \( X_i, Y_i \) have a nonzero injective direct summand is not necessary for the sufficiency part of the proposition.

3. The equivalence of stable Gorenstein-projective categories

Let \( Q \) be a finite acyclic quiver with sink \( v \) and \( \Lambda \) a selfinjective algebra. The quiver \( Q(v) \) is obtained from \( Q \) by reversing all the arrows ending in \( v \). In this section, we construct the equivalence between \( \text{G-proj} \Lambda Q \) and \( \text{G-proj} \Lambda Q(v) \) from Theorem 1.1 on the level of objects and morphisms.

3.1. Reflection on objects. Let \( X \in \text{G-proj} \Lambda Q \). The \( \Lambda \)-homomorphism \( u_X : X_v^0 \to X_v^0 \) from Section 2.1 and a projective cover \( \delta_X : C_X \to X_\pi \) of the cokernel of \( u_X \) give rise to the following commutative diagram
with exact rows.

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega_X & \xrightarrow{\sigma_X} & C_X & \xrightarrow{\delta_X} & X_\pi & \rightarrow & 0 \\
& & k_X & \downarrow & t_X & & & \\
0 & \rightarrow & X_{v^-} & \xrightarrow{u_X} & X_v & \xrightarrow{\pi_X} & X_\pi & \rightarrow & 0 
\end{array}
\]

We index the components of the kernel map \( k_X \) using the arrows \( \beta \) in \( Q(v) \) starting at \( v \), thus \( k_X = (k_{X,\beta})_{s(\beta) = v} \) for \( \Lambda \)-homomorphisms \( k_{X,\beta} : \Omega_X \rightarrow X_{t(\beta)} \).

We define the \( \Lambda Q(v) \)-module \( X'(v) \) in terms of \( \Lambda \)-modules

\[
X'(v)_i = \begin{cases} 
X_i & \text{if } i \neq v \\
\Omega_X & \text{if } i = v 
\end{cases}
\]

and \( \Lambda \)-linear morphisms \( X'(v)_\beta \) where \( \beta \) is an arrow in \( Q(v) \)

\[
X'(v)_\beta = \begin{cases} 
X_\beta & \text{if } s(\beta) \neq v \\
k_{X,\beta} & \text{if } s(\beta) = v 
\end{cases}
\]

Note that the representation \( X'(v) \) may not be separated monic.
Hence we define

\[
X(v) = \text{Mono}_v(X'(v)).
\]

It follows that \( X(v) \) is separated monic, hence Gorenstein-projective, and the epimorphism in the short exact sequence

\[
0 \rightarrow \bigoplus_{s(\beta) = v} P_{t(\beta)}(C_X) \rightarrow X(v) \rightarrow X'(v) \rightarrow 0
\]

in Proposition 2.2 (1) is a right approximation for \( X'(v) \) in \( \text{G-proj} \Lambda Q \).
In particular, we have:

**Lemma 3.1.** *The module \( X(v) \) is determined uniquely, up to isomorphy.*

For later use we summarize the construction for \( X(v) \):

Let \( X \in \text{G-proj} \Lambda Q \) and \( v \in Q_0 \) a sink. Choose as above a projective cover \( C_X \rightarrow X_\pi \) with kernel \( \sigma_X : \Omega_X \rightarrow C_X \) and maps \( k_{X,\beta} : \Omega_X \rightarrow X_{t(\beta)} \), where \( \beta : v \rightarrow t(\beta) \) is an arrow in \( Q(v) \) starting at \( v \). Let

\[
P = \bigoplus_{s(\beta) = v} P_{t(\beta)}(C_X)
\]

be the direct sum, indexed by the arrows starting in \( v \), of the projective \( \Lambda Q(v) \)-modules \( P_{t(\beta)}(C_X) = C_X \otimes_k P(t(\beta)) \) where \( P(t(\beta)) \) is the
projective $kQ(v)$-module corresponding to the vertex $t(\beta)$. Then $X(v)$ is as follows:

$$X(v)_i = \begin{cases} (X \oplus P)_i & \text{if } i \neq v \\ \Omega_X & \text{if } i = v \end{cases}$$

and

$$X(v)_\beta = \begin{cases} (X \oplus P)_\beta & \text{if } s(\beta) \neq v \\ \left(\begin{array}{c} k_{X,\beta} \\ \sigma_X \end{array}\right) & \text{if } s(\beta) = v \end{cases}$$

where the components of the target of the map

$$\left(\begin{array}{c} k_{X,\beta} \\ \sigma_X \end{array}\right) : \Omega_X \longrightarrow X_{t(\beta)} \oplus C_{X,\beta} \bigoplus_{p: v \rightarrow t(\beta)} C_{X,p}$$

are indexed by the paths $p : v \rightarrow t(\beta)$ in the construction of $P$. Here, $C_{X,\beta} = C_X$ for the given arrow $\beta$ starting at $v$ and $C_{X,p} = C_X$ for each path $p \neq \beta$ from $v$ to $t(\beta)$, but the only nonzero map corresponds to the component of $P$ given by the arrow $\beta$.

### 3.2. The converse construction and density

Now, consider the converse of the above construction defined in Subsection 3.1. Let $Y$ be an object in $\text{G-proj } \Lambda Q(v)$. Then the injective envelope $\sigma_Y : Y_v \rightarrow C$ and the homomorphism $k : Y_v \rightarrow Y_{v^+} = \bigoplus_{s(\beta)=v} Y_{t(\beta)}$ give a pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y_v & \longrightarrow^{\sigma_Y} & C & \longrightarrow & \text{Cok } \sigma_Y & \longrightarrow & 0 \\
& & \downarrow^{k} & & \downarrow & & \parallel & & \\
0 & \longrightarrow & Y_{v^+} & \longrightarrow^{b} & B & \longrightarrow & \text{Cok } \sigma_Y & \longrightarrow & 0
\end{array}$$

Write $b = (b_\alpha)$ where $\alpha^{op}$ runs over all arrows in $Q(v)$ starting at $v$. Let $X = (X_i, X_\alpha)$ be the representation of $Q$ over $\Lambda$, where

$$X_i = \begin{cases} B, & \text{if } i = v; \\ Y_i, & \text{otherwise} \end{cases} \quad \text{and} \quad X_\alpha = \begin{cases} b_\alpha, & \text{if } t(\alpha) = v; \\ Y_\alpha, & \text{otherwise}. \end{cases}$$

Since $b$ is a monomorphism and $Y$ is separated monic, then $X$ is a separated monic representation of $Q$ over $\Lambda$. Moreover, $X_\tau = \text{Cok } \sigma_Y$. Note that $C \rightarrow \text{Cok } \sigma_Y$ is a projective cover. It follows that $X'(v) \cong Y$ in $\text{G-proj } \Lambda Q(v)$ (see Subsection 3.1). Since $X'(v)$ is in $\text{G-proj } \Lambda Q(v)$ and projective objects are injective in $\text{G-proj } \Lambda Q(v)$ (see Remark 10.2.2 in [9]), the exact short sequence in Subsection 3.1 splits. Hence $X(v) \cong X'(v) \oplus P$, so $X(v) \cong Y$ in $\text{G-proj } \Lambda Q(v)$ and we have shown that the construction is dense.
3.3. Reflection on homomorphisms. Let \( f : X \to Y \) be a homomorphism in \( \text{G-proj} \Lambda Q \). We define a corresponding homomorphism \( f(v) : X(v) \to Y(v) \) in \( \text{G-proj} \Lambda Q(v) \). Let \( u_X, u_Y \) be the \( \Lambda \)-homomorphisms used in the construction for \( X(v) \) and \( Y(v) \), they give rise to the commutative diagram

\[
\begin{array}{ccc}
X_{v^-} & \xrightarrow{u_X} & X_v \\
f_v \downarrow & & \downarrow f_v \\
Y_{v^-} & \xrightarrow{u_Y} & Y_v \\
\end{array}
\]

where \( f_{v^-} = \bigoplus_{w \in v^-} f_w \) is the componentwise map. Hence there is a homomorphism between the cokernels, \( f_{v^-} : X_{\pi} \to Y_{\pi} \) such that \( f_{v^-} \pi_X = \pi_Y f_v \) as in the diagram below. Fix projective covers \( \delta_X : C_X \to X_{\pi} \) and \( \delta_Y : C_Y \to Y_{\pi} \) of \( X_{\pi} \) and \( Y_{\pi} \), respectively. Let \( \Omega_X = \ker \delta_X \), \( \Omega_Y = \ker \delta_Y \), then there are homomorphisms \( t_X : C_X \to X_v \) and \( k_X : \Omega_X \to X_{v^-} \) such that \( \pi_X t_X = \delta_X, u_X k_X = t_X \sigma_X \) and \( t_Y : C_Y \to Y_v \) and \( k_Y : \Omega_Y \to Y_{v^-} \) such that \( \pi_Y t_Y = \delta_Y, u_Y k_Y = t_Y \sigma_Y \). Since \( \delta_Y \) is an epimorphism, there exists \( C_f : C_X \to C_Y \) such that \( \delta_Y C_f = f_{v^-} \delta_X \) and \( \Omega_f : \Omega_X \to \Omega_Y \) such that \( \sigma_Y \Omega_f = C_f \sigma_X \). So we have a diagram in which every square is commutative except for the left and the middle ones.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Omega_X & \xrightarrow{\sigma_X} & C_X & \xrightarrow{\delta_X} & X_\pi & \longrightarrow & 0 \\
0 & \longrightarrow & \Omega_Y & \xrightarrow{\sigma_Y} & C_Y & \xrightarrow{\delta_Y} & Y_\pi & \longrightarrow & 0 \\
0 & \longrightarrow & X_{v^-} & \xrightarrow{u_X} & X_v & \xrightarrow{t_X} & X_\pi & \longrightarrow & 0 \\
0 & \longrightarrow & Y_{v^-} & \xrightarrow{u_Y} & Y_v & \xrightarrow{t_Y} & Y_\pi & \longrightarrow & 0 \\
\end{array}
\]

Since \( \pi_Y t_Y C_f - f_{v^-} t_X = 0 \), then there is a morphism \( \omega_f : C_X \to Y_{v^-} \) such that \( u_Y \omega_f = t_Y C_f - f_{v^-} t_X \). And \( \omega_f \sigma_X = k_Y \Omega_f - f_{v^-} k_X \) follows from the injectivity of \( u_Y \). We index \( \omega_f = (\omega_f, \beta)_{\beta : \beta \in Q(v)} \). This is to say, for each \( \beta : v \to i \) in \( Q(v)_1 \), we have \( \omega_f, \beta \sigma_X = k_{Y, \beta} \Omega_f - f_{v^-} k_{X, \beta} \).

Remark: Note that the map \( f'(v) : X'(v) \to Y'(v) \) defined by

\[
f'(v)_i = \begin{cases} 
\Omega_f, & \text{if } i = v \\
f_{i, v}, & \text{if } i \neq v 
\end{cases}
\]

is not necessarily a homomorphism since the left square in the above diagram may not commute.
Let \( f(v) = (f(v)_i) \) where \( f(v)_v = \Omega_f \) and for each \( i \neq v \)

\[
f(v)_i = \left( f_i \left( Y_{p' \beta}^{\omega_f} \right) \right) : X_i \oplus \bigoplus_{p \neq p'} C_{X,p} \rightarrow Y_i \oplus \bigoplus_{p \neq p'} C_{Y,p},
\]

where \( p \) runs over all non-trival paths from \( v \) to \( i \) in \( Q(v) \) such that \( \beta \) is the starting arrow of \( p \), \( p' \) is the subpath of \( p \) such that \( p = p' \beta \) and \( E \) is the unit matrix of the size \(|\{p : v \rightarrow i\}|\). Note that when \( p \) is an arrow, then \( p' \) is a trivial path \( t(p) \). In this case \( Y_{p' \beta}^{\omega_f} \) is an identity morphism of \( Y_{t(p)} \). One can check that \( f(v) : X(v) \rightarrow Y(v) \) is a homomorphism in \( \mathcal{G}-\text{proj} \Lambda Q(v) \).

We illustrate this construction in an example.

**Example:** Let \( Q \) be the quiver \( 1 \rightarrow 2 \rightarrow 3 \), then \( Q(3) \) is the quiver \( 1 \rightarrow 2 \leftarrow 3 \). Let \( k \) be a field, \( A = k[x]/(x^2) \) the algebra of dual numbers, \( \sigma : k \rightarrow A \) an embedding and \( \pi : A \rightarrow k \) the canonical epimorphism. Consider the homomorphism \( f : X \rightarrow Y \) between two separated monic representations given by the diagram.

\[
\begin{align*}
X &: 0 \longrightarrow k \overset{\sigma}{\longrightarrow} A \\
& \quad \downarrow \quad \downarrow \sigma \quad \downarrow 1_A \\
Y &: k \overset{\sigma}{\longrightarrow} A \overset{1_A}{\longrightarrow} A
\end{align*}
\]

Since \( X_3 = \text{Cok} \sigma \cong k \), we have \( C_X = A, \Omega_X = k \). Then

\[
X'(3) = 0 \rightarrow k \overset{\text{id}_k}{\leftarrow} k, \quad P_2(C_X) = 0 \rightarrow A \leftarrow 0,
\]

so we have \( X(3) = 0 \rightarrow k \oplus A \overset{\text{id}_k}{\leftarrow} k \). Since \( Y_3 = \text{Cok} \text{id}_A = 0 \), we have \( C_Y = \Omega_Y = 0 \). Then \( Y(3) = k \rightarrow A \leftarrow 0 \).

Now, we need to build \( f(3) \). We illustrate the related homomorphisms with the following diagram

\[
\begin{align*}
0 & \longrightarrow k \overset{\sigma}{\longrightarrow} A \overset{\pi}{\longrightarrow} k \longrightarrow 0 \\
0 & \longrightarrow 0 \overset{\text{id}_k}{\longrightarrow} 0 \overset{\text{id}_X}{\longrightarrow} 0 \longrightarrow 0 \\
0 & \longrightarrow k \overset{\sigma}{\longrightarrow} A \overset{\omega_f}{\longrightarrow} A \overset{\pi}{\longrightarrow} k \longrightarrow 0 \\
0 & \longrightarrow 0 \longrightarrow 0 \overset{\text{id}_A}{\longrightarrow} 0 \longrightarrow 0 \\
0 & \longrightarrow A \overset{\sigma}{\longrightarrow} A \overset{\text{id}_A}{\longrightarrow} A \longrightarrow 0 \longrightarrow 0
\end{align*}
\]
where $\omega f \sigma = -f_2id_k = -\sigma$. Taking $\omega f = -id_A$, then $f(3) = (\sigma, -id_A)$. Then the homomorphism $f(3) : X(3) \to Y(3)$ is as follows:

\[
\begin{align*}
X(3) : & \quad 0 \longrightarrow k \oplus A \quad \overset{\sigma}{\longrightarrow} \quad k \\
Y(3) : & \quad k \quad \overset{\sigma}{\longrightarrow} \quad A \quad \overset{0}{\longrightarrow} \quad 0
\end{align*}
\]

In general, the construction of $f(v)$ depends on choices for $C_f, \omega_f$ and so on. It is straightforward to verify the following

**Lemma 3.2.** The class of $f(v)$ in the stable category $G\text{-proj} \Lambda Q(v)$ depends only on the class of $f$ in $G\text{-proj} \Lambda Q$. Moreover, the map

\[
\text{Hom}_{G\text{-proj} \Lambda Q}(X, Y) \to \text{Hom}_{G\text{-proj} \Lambda Q(v)}(X(v), Y(v)), \quad f \mapsto f(v)
\]

is a homomorphism of abelian groups.

### 3.4. Equivalence of stable categories.

The assignments $X \mapsto X(v), f \mapsto f(v)$ studied in the previous two subsections give rise to a functor $F(v) : G\text{-proj} \Lambda Q \to G\text{-proj} \Lambda Q(v)$. Actually, this functor $F(v)$ induces an equivalence of categories.

**Theorem 3.3.** Let $\Lambda$ be a selfinjective algebra, $Q$ be a finite acyclic quiver, and $v$ a sink in $Q$. Then the functor

\[
F(v) : G\text{-proj} \Lambda Q \to G\text{-proj} \Lambda Q(v)
\]

yields an equivalence between the two stable categories modulo projectives.

**Proof.** In Subsections 3.1 and 3.3 we defined $F(v)$ as a construction for objects and homomorphisms. The proof that $F(v)$ gives rise to a functor is lengthy and but straightforward and is omitted. By Subsection 3.2, we know that $F(v)$ is dense.

Claim 1: $F(v)$ is faithful.

Let $f : X \to Y$ be a morphism in $G\text{-proj} \Lambda Q$ such that the class of $f(v)$ is 0 in $G\text{-proj} \Lambda Q(v)$, that is to say, $f(v) = gh$ where $X(v) \to M \to Y(v)$ and $M$ is projective in $G\text{-proj} \Lambda Q(v)$. We decompose the branches of $X(v), Y(v)$ into direct sums as in the summary after Lemma 3.1. Denote by

\[
g_i = \begin{cases} 
g_i, & i = v; \\
(y^i_{(g_i)_p}), & i \neq v.
\end{cases} \quad h_i = \begin{cases} 
h_i, & i = v; \\
(h^v_{(h_i)_p}), & i \neq v.
\end{cases}
\]
where \( g_i' : M_i \rightarrow Y_i, h_i' : X_i \rightarrow M_i \), and \( g_p : M_i \rightarrow C_{Y_p}, h_p : C_{X_p} \rightarrow M_i \), for each path \( p \) from \( v \) to \( i \) where \( p \) runs over all non-trivial paths from \( v \) to \( i \).

We first construct a projective representation \( N \) over \( \Lambda Q \), and then define maps \( h' : X \rightarrow N \) and \( g' : N \rightarrow Y \) such that \( f = g'h' \).

Let \( N = (N_i, N_\alpha) \) be an object in \( \text{G-proj}\Lambda Q \) where \( N_i = M_i \) for each \( i \neq v \), \( N_v = (\bigoplus_{t(\alpha)=v} M_{s(\alpha)}) \oplus C_Y \); \( N_\alpha = M_\alpha \) for each \( \alpha \) with \( t(\alpha) \neq v \), \( N_\alpha : M_{s(\alpha)} \rightarrow N_v \) are natural embeddings for each arrow \( \alpha \) ending at \( v \). By construction, \( N_i \) is a separated monic representation and \( N_i \) is projective for each \( i \in Q_0 \). Then \( N \) is projective in \( \text{G-proj}\Lambda Q \) by Corollary 2.5.

Since \( \sigma_X : \Omega_X \rightarrow C_X \) is a monomorphism and \( M_v \) is an injective \( \Lambda \)-module, then there exists a map \( u : C_X \rightarrow M_v \) such that \( h_v = u\sigma_X \).

Since \( h : X(v) \rightarrow M \) is a homomorphism, for each \( \alpha \) ending at \( v \), \( h_{s(\alpha)}(\frac{k_X}{\sigma_X}) = M_{\alpha^p}h_v = M_{\alpha^p}u\sigma_X \), that is, \( h_{s(\alpha)}(\alpha) = (M_{\alpha^p}u - h_{\alpha^p})\sigma_X \).

Now we use the form of \( X(v) \) as described in Subsection 3.1. Hence \( (h_{s(\alpha)})(\alpha)k_X = (M_{\alpha^p}u - h_{\alpha^p})\sigma_X \) where \( \alpha \) runs over all arrows ending at \( v \) in \( Q_1 \). Since \( X_v \) is a pushout of \( (k_X, \sigma_X) \), there is an unique map \( (z_{\alpha^p})_\alpha : X_v \rightarrow \bigoplus_{e(\alpha)=v} M_{s(\alpha)} \) such that

\[
(h_{s(\alpha)})_\alpha = (z_{\alpha^p})_\alpha u_X, (M_{\alpha^p}u - h_{\alpha^p})_\alpha = (z_{\alpha^p})_\alpha t_X. \tag{3.1}
\]

By the equations \( f(v)_v = \Omega_f = g_vh_v \), \( \sigma_Y\Omega_f = C_f\sigma_X \) and \( h_v = u\sigma_X \), we have \( (\sigma_Yg_vu - C_f)\sigma_X = 0 \), hence there is a map \( \gamma : X_\pi \rightarrow C_Y \) such that \( \sigma_Yg_vu - C_f = \gamma\delta_X \). Since \( g \) is a homomorphism, \( f(v) = gh \) and the equation (3.1), we have that

\[
u_Y(g_{s(\alpha)})(\alpha)(z_{\alpha^p})_\alpha t_X = t_Y\gamma\delta_X + t_YC_f - u_Y\omega_f = t_Y\gamma\delta_X + f_v t_X.
\]

Hence

\[
[u_Y(g_{s(\alpha)})(\alpha)(z_{\alpha^p})_\alpha - t_Y\gamma\pi_X]u_X = u_Y(f_{s(\alpha)})(\alpha)
\]
\[
[u_Y(g_{s(\alpha)})(\alpha)(z_{\alpha^p})_\alpha - t_Y\gamma\pi_X]t_X = t_YC_f - u_Y\omega_f.
\]

Since \( X_v \) is the pushout of \( (k_X, \sigma_X) \), \( u_Y(f_{s(\alpha)})(\alpha)k_X = (t_YC_f - u_Y\omega_f)\sigma_X \), and \( f_vu_X = u_Y(f_{s(\alpha)})(\alpha), f_v t_X = t_YC_f - u_Y\omega_f \), then by the uniqueness of \( f_v \), we have \( f_v = u_Y(g_{Y\alpha}) \alpha (z_{\alpha^p})_\alpha - t_Y\gamma\pi_X \). Let \( h_v' = (\frac{(z_{\alpha^p})_\alpha}{\gamma\pi_X}) : X_v \rightarrow N_v, \ g'_v = (u_Y(g_{s(\alpha)})(\alpha), -tv) : N_v \rightarrow Y_v \), then \( f_v = g'_v h_v' \). Let \( g' = (g'_i) \) and
$h' = (h'_i)$, where

$$g'_i = \begin{cases} g'_{v_i}, & i = v_i; \\ g'_Y, & i \neq v_i, \end{cases}, \quad h'_i = \begin{cases} h'_{v_i}, & i = v_i; \\ h'_Y, & i \neq v_i. \end{cases}$$

then $g'$ and $h'$ are homomorphisms and $f = g'h'$ which means that $f$ factorizes through the projective object $N$ in $\text{G-proj} \Lambda Q$.

Claim 2: $F(v)$ is full.

In fact, let $g : X(v) \to Y(v)$ be a homomorphism in $\text{G-proj} \Lambda Q(v)$, we need to find a homomorphism $f : X \to Y$ in $\text{G-proj} \Lambda Q$, such that the class of $f(v)$ is equal to $g$ in $\text{G-proj} \Lambda Q(v)$. For each arrow $\beta : v \to i$ in $Q(v)$, we write $g_i : X(v)_i \to Y(v)_i$ as

$$\begin{pmatrix} g_{\beta Y} & (g_{\beta Y}_v) \\ g_{\beta} & (g_{\beta}_v) \end{pmatrix} : X_i \oplus C_{X,\beta} \oplus \bigoplus_{p \neq \beta} C_{X,p} \to Y_i \oplus C_{Y,\beta} \oplus \bigoplus_{q \neq \beta} C_{Y,q}$$

where $p$ and $q$ run over all non-trivial paths from $v$ to $i$ excluding $\beta$. Since $g_i X(v)_\beta = Y(v)_\beta g_v$ for each arrow $\beta : v \to i$ in $Q(v)$, we have $g_{i XY} k_{X,\beta} + g_{i \beta} \sigma_X = k_{Y,\beta} g_v, g_{i X,\beta} k_{X,\beta} + g_{i \beta} \sigma_X = \sigma_Y g_v, (g_{i X q}) q k_{X,\beta} + (g_{i q}) q \sigma_X = 0$. So

$$(g_{\beta XY}) \beta k_X + (g_{\beta y Y}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix} \sigma_X = k_Y g_v,$$

$$(g_{\beta X}) \beta k_X + (g_{\beta}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix} \sigma_X = \begin{pmatrix} 1 \\ i \end{pmatrix} \sigma_Y g_v, \quad (3.2)$$

where $\beta$ runs over all arrows starting at $v$. Since $u_Y k_Y = t_Y \sigma_Y$, we have

$$[\frac{1}{m} (t_Y, \cdots, t_Y) (g_{\beta XY}) \beta - u_Y (g_{\beta XY}) \beta ] k_X$$

$$= [u_Y (g_{\beta Y Y}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{1}{m} (t_Y, \cdots, t_Y) (g_{\beta YY}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix}] \sigma_X$$

where $m$ is the number of arrows starting at $v$. Since $X_v$ is the pushout of $(k_X, \sigma_X)$, then there is a unique map $s : X_v \to Y_v$ such that

$$su_X = \frac{1}{m} (t_Y, \cdots, t_Y) (g_{\beta XY}) \beta - u_Y (g_{\beta XY}) \beta,$$

$$st_X = u_Y (g_{\beta Y Y}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix} - \frac{1}{m} (t_Y, \cdots, t_Y) (g_{\beta YY}) \beta \begin{pmatrix} 1 \\ i \end{pmatrix}. $$

Since $u_X$ is a monomorphism and $C_{Y,\beta} = C_Y$ is an injective $\Lambda$-module, then there is a $\Lambda$-map $\gamma = (\gamma_\beta) \beta : X_v \to \bigoplus_{\beta} C_{Y,\beta}$ where $\beta$ runs over all
arrows starting at \( v \), such that \((g_{\ell(\beta)X\beta})_\beta = \gamma u_X\). Let

\[
f_i = \begin{cases} g_{\ell XY}, & i \neq v; \\ \frac{1}{m}(t_Y, \ldots, t_Y)\gamma - s, & i = v, \end{cases}
\]

then \( f_v u_X = u_Y(g_{\ell(\beta)XY})_\beta \). Moreover, \( f_\ell X \rightarrow Y \) is a homomorphism following from the fact that \( g : X(v) \rightarrow Y(v) \) is a homomorphism and \( f_v u_X = u_Y(g_{\ell(\beta)XY})_\beta \). By the construction of \( f(v) \), we know that \( f(v)_v = \Omega_f \) and

\[
f(v)_i = \left( \begin{array}{c} f_i \vspace{1em} \\ 0 \end{array} \right),
\]

for each \( i \neq v \) where \( p \) runs over all non-trivial paths from \( v \) to \( i \) and \( \beta \) is the starting arrow of \( p \) such that \( p = p'\beta \) and the size of \( E \) is determined by the number of non-trivial paths from \( v \) to \( i \).

Sub-claim: the class of \( f(v) \) is equal to \( g \) in \( G_{\text{proj}}AQ(v) \). That is to say, there is a projective object \( P \) and two homomorphisms \( h : X(v) \rightarrow P \) and \( k : P \rightarrow Y(v) \) such that \( g - f(v) = kh \) in \( G_{\text{proj}}AQ(v) \).

Let \( T = \frac{1}{m}(1, \ldots, 1) \gamma t_X + \frac{1}{m}(1, \ldots, 1)(g_{\ell(\beta)XY})_\beta \left( \begin{array}{c} 1 \\ \vdots \end{array} \right) \), where \( \beta \) runs over all arrows starting at \( v \). Since \( \delta_Y = \pi_Y v_Y, \pi_Y u_Y = 0 \) and \( u_Y \omega_f = t_Y C_f - f_v t_X \), then we have \( \delta_Y (C_f - T) = 0 \). Hence there is a homomorphism \( l : C_X \rightarrow \Omega_Y \) such that \( \sigma_Y l = C_f - T \). Since \( C_f \sigma_X = C_f \Omega_f, t_X \sigma_X = u_X k_X, \gamma u_X = (g_{\ell(\beta)XY})_\beta \) and the equation (3.2), then there holds \( \sigma_Y l \sigma_X = \sigma_Y (\Omega_f - g_v) \). So \( l \sigma_X = \Omega_f - g_v \) follows from the fact that \( \sigma_Y \) is a monomorphism.

Let \( P = P_\ell(C_X) \oplus \bigoplus_{s(\beta)=v} P_{\ell(\beta)}(C_Y), h = (h_i) \) and \( k = (k_i) \), where \( h_v = \sigma_X, k_v = -l \) and for each \( i \neq v \),

\[
h_i = \left( \begin{array}{c} 0 \\ (g_{\ell XY})_q \end{array} \right), \quad k_i = \left( \begin{array}{c} (g_{\ell XY} - Y_{\ell(\beta)XY})_p \end{array} \right), \]

where \( p \) and \( q \) runs over all non-trivial paths from \( v \) to \( i \), and \( p' \) is the subpath of \( p \) such that \( p = p'\beta \) with \( \beta \in Q(v)_1, E \) is the identity matrix whose size is determined by the number of non-trivial paths from \( v \) to \( i \). Hence \( k_v h_v = -l \sigma_X = g_v - \Omega_f = g_v - f(v)_v \). For each \( i \neq v \), by the multiplication of matrix, we have \( k_i h_i = g_i - f(v)_i \). So for each \( i \in Q(v)_0, k_i h_i = g_i - f(v)_i \). Moreover, we can prove that \( k \) and \( h \) are homomorphisms in \( G_{\text{proj}}AQ(v) \). Hence \( F \) is full.

\[\blacksquare\]

Iterated application of the previous result yields
Theorem 3.4. Suppose that $\Lambda$ is a selfinjective algebra, $Q$ and $\tilde{Q}$ are finite acyclic quivers, and that the quiver $\tilde{Q}$ can be obtained from the quiver $Q$ by a sequence of reflections. Then

$$\text{G-proj}\Lambda Q \simeq \text{G-proj}\Lambda\tilde{Q}.$$ 

4. The Quiver $A_3$ revisited

In this section we aim to show the following statement.

Proposition 4.1. Let $\Lambda$ be a selfinjective algebra and $Q$ be the quiver $Q : 1 \to 2 \to 3$. Suppose that $M$ is a representation for $Q$ with coefficients in $\Lambda$ such that none of the branches $M_i$, $i \in Q_0$, of $M$ has a nonzero injective direct summand.

Let $A = \text{Mimo}(M)$ be the corresponding Gorenstein-projective representation and let $B = A(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3)$ be obtained from $A$ by performing twelve reflections. Then

$$B \cong \Omega^2_\Lambda A.$$ 

The functor $\Omega_\Lambda$ is the inverse of the suspension functor [1] for the stable category $\text{mod}\Lambda$. The functor $\Omega^2_\Lambda$ is applied pointwise on the $\Lambda Q$-module $A$.

Corollary 4.2. Suppose that $\Lambda$ is a selfinjective algebra such that the functor $\Omega^2_\Lambda$ is naturally equivalent to the identity functor on the stable category, and $Q$ is the quiver $Q : 1 \to 2 \to 3$. If $M$, $A$, $B$ are as in Proposition 4.1, then

$$B \cong A.$$ 

4.1. On the geometry of the octahedron in the octahedral axiom. Consider a pair $(u, v)$ of composable maps in a triangulated category. The three maps $u, v, v \circ u$ give rise to three exact triangles. The octahedral axiom states that the mapping cones of these three triangles are the vertices of a fourth exact triangle such that "everything commutes".

The four exact triangles form four of the eight faces of an octahedron. The other four faces consist of pairs of composable maps which together with their composition form a non-oriented triangle. These four triangles are arranged such that the four products form an oriented cycle.

Note that an octahedron contains three squares such that any two are perpendicular to each other. One square is given by the oriented cycle of the four products just mentioned. The other two squares are perpendicular to it, and to each other, and are required by the axiom.
to be commutative squares. The starting vertex of one square is the end vertex of the other, and conversely. Those two vertices, \( Y' \) and \( Y \) in the picture below, define an axis which is perpendicular to the oriented cycle.

Consider the oriented cycle as the “equator” of the octahedron. We remark that the orientation given by the oriented cycle does not distinguish a “North pole” over a “South pole” (or conversely) as the axiom is symmetric with respect to reflection on the plane given by the equator. But the orientation of the oriented cycle does define a cyclic ordering of the four exact triangles involved in the octahedron.

**Observation 4.3.** Consider the octahedron in the octahedral axiom as an oriented graph \( G \). The symmetry group of \( G \) is cyclic of order four. It is generated by a rotation-reflection \( \rho \) given by rotating the octahedron about the axis mentioned above by \( 90^\circ \) in the direction of the orientation of the cyclically oriented square, and then reflecting the octahedron about the plane given by that square.

We present in Figure 2 the octahedron using a cylindrical projection from the axis which is perpendicular to the cyclically oriented square. Thus, the equator given by the square consists of the four products and is given by the horizontal line in the middle. Above and below the equator, exact triangles and triangles given by a pair of composable maps alternate. Under the cylindrical projection, the generator \( \rho \) in Observation 4.3 appears as a glide-reflection. Note that while \( \rho^4 \) acts as the identity map on the octahedron, it operates as the square [2] of the suspension on its exact triangles. (This is to be expected since each pole \( Y \) or \( Y' \) occurs both as starting term and as end term of two exact triangles.)

In the diagram, vertices and arrows are labelled as in [11, Page 3]. The four faces of the octahedron which are given by pairs of composable maps occur as rectangles which contain entries; the other four faces are exact triangles which appear as empty rectangles.

4.2. **The first reflection.** Let \( Q \) be the quiver of type \( \mathbb{A}_3 \) in linear orientation,

\[
Q : \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3
\]

and \( \Lambda \) a selfinjective algebra.

Consider \( M \), a representation of \( Q \) over \( \Lambda \), such that no component \( M_i \) has a non-zero injective direct summand. Thus, \( M \) consists of a pair of composable maps in the stable category \( \text{mod} \Lambda \):\\

\[
M : \quad X \xrightarrow{u} Y \xrightarrow{v} Z
\]
Figure 2. The octahedron of the axiom in cylindrical projection

We fix the octahedron given by this pair of composable maps and call it $\mathcal{O}$.

Then $A = \text{Mimo}(M) : A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3$ is an object in $\text{G-proj} \Lambda Q$, we apply reflections to $A$.

We first reflect at 3. Recall that $A(3) = \text{Mimo}(A'(3))$ where $A'(3)$ is a representation of $Q(3) : 1 \xrightarrow{\alpha} 2 \xleftarrow{3}$ given as

$$A'(3) : A_1 \xrightarrow{\alpha} A_2 \xleftarrow{k} \Omega_A$$

where the map $k$ representing $\gamma$ is as in the following diagram.

$$
\begin{array}{cccc}
0 & \longrightarrow & \Omega_A & \xrightarrow{\sigma_A} C_A & \xrightarrow{\delta_A} A_3 & \longrightarrow & 0 \\
\downarrow{k} & & \downarrow{t_A} & & & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\beta} A_3 & \xrightarrow{\pi} A_3 & \longrightarrow & 0
\end{array}
$$

We assume that $\Omega_A$ has no nonzero injective direct summand. In the stable category, the diagram represents the triangle

$$A_2 \xrightarrow{\beta} A_3 \xrightarrow{\pi} A_3 \xleftarrow{-k[1]}$$

which is equivalent to the triangle in the octahedron $\mathcal{O}$,

$$Y \xrightarrow{v} Z \xrightarrow{gok} X' \xleftarrow{j'}$$

since the maps $A_2 \rightarrow Y$, $A_3 \rightarrow Z$ are isomorphisms in $\text{mod} \Lambda$ which make the left square commutative; hence $\Omega_A \cong X'[-1]$ in $\text{mod} \Lambda$ and the square is commutative.

$$
\begin{array}{c}
\Omega_A \xrightarrow{k} A_2 \\
\downarrow \cong \\
X'[\cdot-1] \xrightarrow{j'[-1]} Y
\end{array}
$$
We obtain a representation of $Q(3)$ consisting of objects and morphisms in the octahedron $O$.

$$M(3) : \quad X \xrightarrow{u} Y \xrightarrow{j'[-1]} X'[-1].$$

4.3. **The second reflection.** Next, we reflect at vertex 2. The representation for $Q(3)$,

$$U = A(3) = \text{Mimo}(A'(3)) : \quad U_1 \xrightarrow{\alpha} U_2 \xleftarrow{\gamma} U_3$$

is separated monic, so we can compute $U'(2)$ using the following diagram. We assume that $\Omega_U$ has no nonzero injective direct summand.

\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega_U & \xrightarrow{\sigma_U} & C_U & \xrightarrow{\delta_U} & U_2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U_1 \oplus U_3 & \xrightarrow{(\alpha,\gamma)} & U_2 & \xrightarrow{\pi} & U_2 & \rightarrow & 0
\end{array}
\]

Hence, $U'(2)$ is the representation for the quiver $Q(2, 3) : 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$ given by

$$U'(2) : \quad U_1 \xleftarrow{k_1} \Omega_U \xrightarrow{k_3} U_3.$$  

The interesting step is to detect the maps $k_1 : \Omega_U \rightarrow U_1$ and $k_3 : \Omega_U \rightarrow U_3$ in the octahedron. We first deal with $k_3$. Using that $U$ is separated monic, we obtain the commutative diagram with exact rows and columns.

\[
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
U_1 & \xrightarrow{(1,0)} & U_1 & \xrightarrow{\alpha} & U_2 & \xrightarrow{\pi} & U_2 & \rightarrow & 0 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
0 & \rightarrow & U_1 \oplus U_3 & \xrightarrow{(\alpha,\gamma)} & U_2 & \xrightarrow{U_2/U_1 \oplus U_3} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & U_3 & \xrightarrow{\text{can}_{U_1} \circ \gamma} & U_2 & \xrightarrow{\text{can}_{U_1}} & U_2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Note that the middle row of this diagram is the bottom row of the diagram above. Combining the two diagrams yields the commutative diagram with exact rows.
By composing the vertical maps and omitting the middle row we obtain the following diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega_U & \xrightarrow{\sigma_U} & C_U & \xrightarrow{\delta_U} & U_T & \longrightarrow & 0 \\
\downarrow{k_1} & & \downarrow{t_U} & & \downarrow{\delta} & & \downarrow{\delta_U} & & \downarrow{0} \\
0 & \longrightarrow & U_1 \oplus U_3 & \xrightarrow{(\alpha, \gamma)} & U_2 & \xrightarrow{\pi} & U_T & \longrightarrow & 0 \\
\downarrow{(0,1)} & & \downarrow{\text{can}_{U_1}} & & \downarrow{\text{can}_{U_2}} & & \downarrow{\text{can}} & & \downarrow{0} \\
0 & \longrightarrow & U_3 & \xrightarrow{\text{can}_{U_1}\gamma} & U_2/_{U_1} & \xrightarrow{\text{can}} & U_T & \longrightarrow & 0
\end{array}
\]

Since \( C_U \) is a projective cover, this diagram represents the triangle in \( \text{mod} \Lambda \):

\[
\begin{array}{c}
U_3 \xrightarrow{\text{can}_{U_1}\gamma} U_2/_{U_1} \xrightarrow{\text{can}} U_T \xrightarrow{-k_3[1]} 0
\end{array}
\]

Consider the first morphism. The map \( \gamma : U_3 \rightarrow U_2 \) is equivalent to \( j'[-1] : X'[-1] \rightarrow Y \) (compare \( A(3) \) and \( M(3) \) above); the canonical map \( \text{can}_{U_1} : U_2 \rightarrow U_2/_{U_1} \), which is the cokernel of \( \alpha \), is equivalent to the map \( i \) in the octahedron since \( \alpha \) is equivalent to \( u \) and \( X \xrightarrow{u} Y \xrightarrow{i} Z' \rightarrow \) is a triangle in which \( i \) follows \( u \). Hence the above triangle is equivalent to the following triangle in the octahedron:

\[
\begin{array}{c}
X'[-1] \xrightarrow{-i\sigma j'[-1]} Z' \xrightarrow{f} Y' \xrightarrow{g} Z
\end{array}
\]

Hence the map \( k_3 : \Omega_U \rightarrow U_3 \) is equivalent to the map \( g[-1] : Y'[-1] \rightarrow X'[-1] \).

In a similar way we obtain \( k_1 \). Consider the first diagram in this subsection and the second diagram with the roles of \( U_1 \) and \( U_3 \) exchanged. Again, the bottom row of the first diagram is just the middle row of the second, so we obtain the following diagram by composing the vertical maps and omitting the middle row.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega_U & \xrightarrow{\sigma_U} & C_U & \xrightarrow{\delta_U} & U_T & \longrightarrow & 0 \\
\downarrow{k_1} & & \downarrow{t_U} & & \downarrow{\delta} & & \downarrow{\delta_U} & & \downarrow{0} \\
0 & \longrightarrow & U_1 & \xrightarrow{\text{can}_{U_2}\alpha} & U_2/_{U_1} & \xrightarrow{\text{can}} & U_T & \longrightarrow & 0
\end{array}
\]
We read off that
\[ U_1 \xrightarrow{\text{can}_{U_1} \alpha} U_2 \xrightarrow{\text{can}} U_3 \xrightarrow{k_1[1]} U_2 \]
is a triangle. By comparing first maps, we see that it is equivalent to the following triangle in the octahedron.
\[ X \xrightarrow{v \circ u} Z \xrightarrow{k} Y \xrightarrow{k'} \]
Hence \( k_1 \) is equivalent to the map \( k'[−1] \) in the octahedron.

In conclusion, the representation \( U'[2) \), and hence \( A(2, 3) = \text{Mimo}(U'(2)) \), is equivalent in \( \text{mod} \Lambda \) to the following representation which consists of objects and morphisms in \( \mathcal{O} \).
\[
\begin{align*}
M(2, 3) : & \quad X \xleftarrow{k'[−1]} Y'[-1] \xrightarrow{g[-1]} X'[−1].
\end{align*}
\]
Note that, together, the maps \( u, j'[-1] \) defining \( M(3) \) and the maps \( k'[-1] \) and \( g[-1] \) defining \( M(2, 3) \) form a square in the octahedron \( \mathcal{O} \)
\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{k'[−1]} & & \downarrow{j'[-1]} \\
\downarrow{g[-1]} & & \downarrow{}
\end{array}
\]
the commutativity of which is required by the axiom.

4.4. The third reflection. In order to return to a representation of the quiver \( Q \), it remains to reflect at vertex 1. We are given a separated monic representation for the quiver \( Q(2, 3) \):
\[
V = A(2, 3) : \quad V_1 \xleftarrow{δ} V_2 \xrightarrow{β} V_3
\]
We may assume that \( V_2 \) has no nonzero injective direct summand. For the reflection, we consider the diagram in which \( \Omega_V \), again, is supposed to have no nonzero injective direct summand.
\[
\begin{array}{ccc}
0 & \xrightarrow{σ_V} & C_V \\
\downarrow{\ell} & & \downarrow{τ_V} \\
0 & \xrightarrow{δ} & V_1 \\
\downarrow{π} & & \downarrow{V_2} \\
0 & & \xrightarrow{V_3} \xrightarrow{V'1}
\end{array}
\]
We obtain the representation \( V'(1) \) for the quiver \( Q \):
\[
V'(1) : \quad V_1 \xrightarrow{ℓ} V_2 \xrightarrow{β} V_3
\]
We locate a map equivalent to \( \ell \) in the octahedron. Recall that \( \delta \) is equivalent to the corresponding map \( k'[\{-1\} : Y'[\{-1\}] \to X \) in \( M(2, 3) \) which occurs in the triangle
\[
Z[-1] \xrightarrow{k[-1]} Y'[\{-1\}] \xrightarrow{\delta[-1]} X \xrightarrow{\nu_{0u}}.
\]
Here is the representation \( M(1, 2, 3) \) of \( Q \) with objects and morphisms in the octahedron \( O \). It is equivalent in \( \text{mod} \Lambda \) to \( A(1, 2, 3) = \text{Mimo}(V'(1)) \).
\[
M(1, 2, 3) : \quad Z[-1] \xrightarrow{k[-1]} Y'[\{-1\}] \xrightarrow{g[-1]} X'[-1]
\]

4.5. **Summary of the first three reflections.** In the octahedron, \( M(1, 2, 3) \) is represented by a pair of composable maps. We compare this pair \( (k[-1], g[-1]) \) to the pair \( (u, v) \) for \( M \). Up to a shift by \(-1\), the pair \( (k[-1], g[-1]) \) is adjacent to the original pair \( (u, v) \), but on the opposite side of the equator of the octahedron. Hence up to the shift \(-1\), three subsequent reflections are represented by the rotation-reflection \( \rho \) of the octahedron, see Observation 4.3.

4.6. **Twelve reflections.** We continue this process to determine the representation obtained from \( M \) after 12 reflections, this is \( M(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3) \). For this we can repeat the three above steps four times and trace the effect on the octahedron in Figure 2.

Recall that \( M \) is given by a pair of composable maps,
\[
M : \quad X \xrightarrow{u} Y \xrightarrow{v} Z.
\]
We have seen that \( M(1, 2, 3) \) is given by the following pair of composable maps,
\[
M(1, 2, 3) : \quad Z[-1] \xrightarrow{k[-1]} Y'[\{-1\}] \xrightarrow{g[-1]} X'[-1].
\]
Repeating this process shows that \( M(1, 2, 3, 1, 2, 3) \) is as follows.
\[
M(1, 2, 3, 1, 2, 3) : \quad X'[\{-2\}] \xrightarrow{\nu[-2]} Y[-1] \xrightarrow{\nu[-1]} Z'[\{-1\}]
\]
Similarly,
\[
M(1, 2, 3, 1, 2, 3, 1, 2, 3) : \quad Z'[\{-2\}] \xrightarrow{\nu[-2]} Y'[-2] \xrightarrow{k[-2]} X[-1].
\]
Finally, we obtain
\[
M(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3) : \quad X[-2] \xrightarrow{u[-2]} Y[-2] \xrightarrow{\nu[-2]} Z[-2].
\]
In conclusion, \( M(1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3) \) equals \( M[-2] \) is obtained by applying \( \rho \) four times to the composable pair \( (u, v) \) (resulting in \( (u[2], v[2]) \)) and by applying the inverse suspension \([-1\]) also four times. This finishes the proof of Proposition 4.1. We illustrate the construction in Section 5.2 in an example.
The examples in this section are Gorenstein-projective representations for \( \Lambda Q \) where \( Q \) is a quiver of type \( A_3 \) in various orientations and \( \Lambda = k[T]/(T^n) \) the bounded polynomial ring where \( n \) is 2 or 3.

5.1. Some Gorenstein-projective representations. In the case where \( n = 2 \) we present a table to show that the category of Gorenstein-projective quiver representations may behave nicer than either the monic representations, or all representations of the quiver in the sense that the Gorenstein-projective representations form a Frobenius category, and the stable part of this category does not depend on the orientation of the quiver, as we have seen in Theorem 1.1.

For \( Q \) a quiver of type \( A_3 \) in any orientation and \( \Lambda = k[T]/(T^n) \), the category of \( \Lambda Q \)-modules has finite representation type. We list in Table 5.1 for each orientation: the number of indecomposables, the shape of the stable part of the AR-quiver, the number of monic indecomposables, the number of Gorenstein-projective indecomposables, and the shape of the stable part of the AR-quiver for G-proj \( \Lambda Q \).

We picture the Auslander-Reiten quivers for the three categories of Gorenstein-projective representations. In each case we describe in detail how the objects give rise to the icons which occur in the Auslander-Reiten quiver. The Auslander-Reiten quivers are obtained using universal coverings as in [19, Part A].

Recall that a nilpotent linear operator \( V \) consists of a finite dimensional \( k \)-vector space, also denoted \( V \), together with a linear map \( T : V \rightarrow V \) such that \( T^n = 0 \) for some \( n \in \mathbb{N} \). Up to isomorphy, \( V \) is given uniquely by a partition recording the dimensions of the indecomposable direct summands, that is, the sizes of the Jordan blocks of \( T \). We
picture the parts of the partition as columns of empty squares; the $i$-th square under the top square in the $j$-th column corresponds to the basic element $T^i x_j$ where $x_j$ is a generator of the $j$-th direct summand of $V$. The two examples are for the partitions (2) and (3, 2).

\[
V_1 = k[T]/(T^2) : \quad V_2 = k[T]/(T^3) \oplus k[T]/(T^2) : \]

Suppose the quiver $Q$ has \textbf{linear orientation}, then a representation is a triple $(V, U, W)$ where the \textit{ambient space} $V$ is a nilpotent linear operator, the \textit{intermediate subspace} $U$ of $V$ is invariant under the action of $T$, and the \textit{small subspace} $W$ of $V$ is also $T$-invariant and contained in the intermediate subspace $U$. In each example in this paper, the generators of the intermediate space can be chosen as basic elements or sums of basic elements of the ambient space and will be pictured as circles or connected circles that are contained in the squares for the corresponding basic elements. Similarly, the generators for the small subspace are indicated by bullets or connected bullets.

\[
(V_1, kT, 0) : \quad (V_2, (T)/(T^3) \oplus (T)/(T^2), k(T^2, T)) : \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Gorenstein-projective_kQbar_modules_linear_orientation.png}
\caption{Gorenstein-projective $kQ[\varepsilon]$-modules (linear orientation)}
\end{figure}
For the quiver $Q$ in **V-orientation**, a representation is a triple $(V,U_{\downarrow},U_{\uparrow})$ consisting of the ambient space $V$ together with a *left subspace* $U_{\downarrow}$ and a *right subspace* $U_{\uparrow}$, both subspaces are $T$-invariant. Here is the example which occurs in the middle of the Auslander-Reiten quiver below.

\[
(k[T]/(T^2) \oplus k, k(0,1), k(T,1)) : \begin{array}{c}
\end{array}
\]

\begin{figure}[h]
\centering
\begin{tikzpicture}[scale=0.8]
\begin{scope}
\node (a) at (0,0) [fill=white] {$\blacklozenge$};
\node (b) at (1,1) [fill=white] {$\blacklozenge$};
\node (c) at (2,0) [fill=white] {$\blacklozenge$};
\node (d) at (1,-1) [fill=white] {$\blacklozenge$};
\node (e) at (0,-2) [fill=white] {$\blacklozenge$};
\node (f) at (-1,-1) [fill=white] {$\blacklozenge$};
\node (g) at (-2,0) [fill=white] {$\blacklozenge$};
\node (h) at (-1,1) [fill=white] {$\blacklozenge$};
\end{scope}
\begin{scope}[yshift=-2cm]
\node (i) at (0,0) [fill=white] {$\blacklozenge$};
\node (j) at (1,1) [fill=white] {$\blacklozenge$};
\node (k) at (2,0) [fill=white] {$\blacklozenge$};
\node (l) at (1,-1) [fill=white] {$\blacklozenge$};
\node (m) at (0,-2) [fill=white] {$\blacklozenge$};
\node (n) at (-1,-1) [fill=white] {$\blacklozenge$};
\node (o) at (-2,0) [fill=white] {$\blacklozenge$};
\node (p) at (-1,1) [fill=white] {$\blacklozenge$};
\end{scope}
\end{tikzpicture}
\caption{Gorenstein-projective $kQ[\varepsilon]$-modules (V-orientation)}
\end{figure}

For the quiver $Q$ in **A-orientation**, a representation $(V,W,U_{V},U_{W})$ consists of two nilpotent linear operators, the *left space* $V$ and the *right space* $W$, and also a subspace $U_{V} \cong U_{W}$ which is embedded as $U_{V}$ in the left space and as $U_{W}$ in the right space. In the icon, we separate the left space from the right space by a vertical line and indicate the subspace (abusing notation) as a subspace of the direct sum of the left space and the right space. The example occurs on the left end of the upper dotted line in the Auslander-Reiten quiver.

\[
(k, k[T]/(T^2), k, kT) : \begin{array}{c}
\end{array}
\]
Similarly, we obtain \( X \). Reflecting the identity map, and compute \( X \) and compute the cokernel as \( \text{Cok}((0 \rightarrow Y \rightarrow \Lambda \rightarrow k)) \).

Our example where we illustrate the process from Section 4 which leads to Theorem 1.2 in the Twelve reflections, starting with a simple object.

5.2. Twelve reflections, starting with a simple object. We illustrate the process from Section 4 which leads to Theorem 1.2 in the example where \( Q \) is the linear quiver of type \( A_3 \), \( Q = (1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) \), and \( \Lambda \) any local selfinjective algebra with unique maximal ideal \( m \) and radical factor field \( k \).

Take \( X^0 : (0 \rightarrow 0 \rightarrow k) \), and reflect at vertex 3: In the first step, we compute the cokernel as \( \text{Cok}(X^0_3) = \overline{X}^0 = k \); in the second step, we take the kernel of the projective cover, \( Y^0_3 = m \), and compute \( Y^0_{3\text{op}} : m \rightarrow 0 \); since this map is not monic, we take in the third step \( Y_2 = \Lambda \) and \( Y_{3\text{op}} : m \rightarrow \Lambda \) the inclusion. This yields the representation \( X^1 = (0 \rightarrow \Lambda \leftarrow m) \), for \( Q(3) = (1 \rightarrow 2 \leftarrow 3) \).

We reflect \( X^1 \) at vertex 2: The cokernel of the map \( (X^1_0, X^1_2) : X^1_1 \oplus X^1_3 \rightarrow X^1_2 \) is \( k \), so we obtain \( Y' = (0 \leftarrow m \rightarrow m) \), where the map \( Y'_{3\text{op}} \) is the identity map, and compute \( X^2 = (\Lambda \leftarrow m \rightarrow m) \).

We reflect \( X^2 \) at vertex 1 to obtain \( X^3 = (m \rightarrow m \rightarrow m) \).

Reflecting \( X^3 \) at vertex 3 yields \( X^4 = (m \rightarrow m \leftarrow 0) \).

Similarly, we obtain \( X^5 = (m \leftarrow 0 \rightarrow 0) \); \( X^6 = (k \rightarrow \Lambda \rightarrow \Lambda) \), \( X^7 = (k \rightarrow \Lambda \leftarrow 0) \), \( X^8 = (k \leftarrow k \rightarrow \Lambda) \), \( X^9 = (0 \rightarrow k \rightarrow \Lambda) \).
$X^{10} = (0 \to k \leftarrow k)$, $X^{11} = (0 \leftarrow 0 \to k)$, and finally, $X^{12} = X^0 = (0 \to 0 \to k)$, so after 12 steps we are back where we started.

\[ \text{Figure 6. Gorenstein-projective } kQ[\zeta]\text{-modules (linear orientation)} \]

5.3. **Examples where $T$ has nilpotency index three.** Finally, we consider the case where $\Lambda = k[\zeta] = k[T]/(T^3)$ and $Q$ is a quiver of type $A_3$ in either linear orientation, V-orientation or \( \Lambda \)-orientation.

For each of the three orientations, we picture in Figures 6.8 the Auslander-Reiten quiver $\Gamma$ for G-proj $\Lambda Q$. Each has 27 indecomposable Gorenstein-projective modules of which 24 are stable and 3 are projective-injective. The shape of the stable part of $\Gamma$ is $\overline{\Gamma} = \mathbb{Z}E_6/(\tau^4)$. Note that the positions of the three projective-injective objects depend on the orientation of $Q$. 

\[ \text{Figure 6. Gorenstein-projective } kQ[\zeta]\text{-modules (linear orientation)} \]
We label each of the above modules $X^i$ in Section 5.2 by the symbol \( \diamond \) in the Auslander-Reiten quiver for the category $\text{G-proj} \Lambda Q$ where the quiver $Q$ is oriented such that the module $X^i$ is defined.

We have seen that each reflection gives rise to a categorical equivalence of the stable category of Gorenstein-projective modules, hence gives rise to an isomorphism of translation quivers between the stable parts of the AR-quivers. It turns out that in this case, this isomorphism is determined uniquely by each pair $X^i, X^{i+1}$ of successive objects. Thus we can read off from the AR-quivers how the reflection functors act on each object.
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Figure 8. Gorenstein-projective $kQ[\zeta]$-modules (A-orientation)

In particular the orbit which is pictured in Figure 8 is obtained by tracing the objects on the central $\tau$-orbits in Figures II under the action of those translation quiver isomorphisms.

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