Novel aspects of Soret and Dufour in entropy generation minimization for Williamson fluid flow

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Williamson fluid; Rotating disks; Activation energy; Viscous dissipation; Entropy generation; Soret and Dufour effects; Stratification.

**Abstract.** Soret and Dufour effects on MHD flow of Williamson fluid between two rotating disks are examined in this study. Impacts of stratification, viscous dissipation, and activation energy are also considered. Bejan number and entropy generation for the stratified flow are discussed. The governing Partial Differential Equations (PDEs) are converted into ODE using von Kármán transformations. The convergent solution of complicated ODE is found using a homotopic procedure. The results of physical quantities are discussed through plots and numerical values. It is noted that axial and radial velocities are higher in the case of greater Weissenberg number. Temperature and concentration profiles are the decreasing functions of thermal and solutal stratification parameters, respectively. Entropy and Bejan number show the opposite trends for higher Weissenberg number and Brinkman number.

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1. Introduction

Recent researchers have particularly focused on the flows of non-Newtonian fluids because these liquids have penetrated industrial and technological processes. In this regard, they have suggested a number of models for such liquids due to their diverse properties. In general, subclasses of differential-type liquids, known as second, third, and fourth grades, are extensively analyzed. However, Williamson fluid model has received insignificant attention. Williamson [1] studied the pseudoplastic material experimentally. Nadeem et al. [2] discussed Williamson fluid flow that results from the stretching velocity of a surface. This flow, which is characterized by pressure-dependent viscosity, was numerically analyzed by Zehra et al. [3]. Khan et al. [4] and Qayyum et al. [5] studied entropy generation during heat transfer of Williamson nanofluid due to stretching sheet and rotating disks, respectively. Further studies on non-Newtonian fluids can be seen in [6–10].

Non-Newtonian fluid flow with rotating disks has many applications in engineering, e.g., air cleaning machine, medical equipment and aerodynamical engineering processes, etc. Karman [11] firstly examined the steady flow using an infinite rotating disk. The flow generated by rotating disks with radiative heat flux and variable thickness was examined by Hayat et al. [12]. Khan et al. [13] presented MHD Eyring-Powell fluid flow with a rotating disk. Doh and Muthamilselvan discussed Micropolar fluid flow by a rotating disk with MHD and thermophoretic particle deposition effects [14]. Khan et al. examined magnetic field and double diffusion in the couple stress fluid flow by a rotating disk [15]. Griffiths et al. addressed the
stability analysis of flow using a rotating disk [16]. Qayyum et al. [17] discussed disorder in the system based on Williamson fluid motion.

It is a well-known fact that mass transfer exists in view of species concentration difference in a mixture. The species characterized by different concentrations transport themselves in a mixture from a region with higher concentration to a region with lower concentration. Moreover, activation energy is the minimum quantity of energy that must be possessed by reactants before any specified chemical reaction occurs. This process, which is followed by a chemical reaction with activation energy in mass transfer, is usually used in food processing, geothermal reservoirs, chemical engineering, etc. Bestman investigated convection of a binary mixture flowing in a porous space with activation energy [18]. Makinde et al. discussed unsteady radiative flow in the presence of chemical reaction [19]. Awad et al. described unsteady rotating flow with activation energy and chemical reaction [20]. Shafique et al. studied the rotating flow containing chemically reactive species and activation energy [21].

In recent years, to find optimal designs for engineering system, the second law of thermodynamics has been applied to the analysis of Entropy Generation Minimization (EGM). EGM helps determine the growing rate of irreversibility during a process. Reversibility of heat and mass transfer and irreversibility of viscous dissipation can be determined by entropy generation. During the convection process, irreversibility takes place inside the cavity. It is quite necessary to diminish the irreversibility process to ensure energy conservation. To study the effects of entropy generation inside a thermal system, the second law of thermodynamics is considered. Firstly, Bejan introduced entropy generation [22] and explained that entropy was generated due to viscous effects and thermal conductivity. Amani and Nobari investigated entropy generation in a curved pipe at a constant wall temperature [23]. Hayat et al. studied peristaltic rotating flow of nanoparticles with entropy generation [24]. Shit et al. found that nanofluid flow resulted from exponential stretching sheet and entropy generation [25]. Some of the recent works about entropy generation can be seen in [26–30].

Here, this study analyzes the effects of entropy generation on MHD Williamson fluid flow between two rotating disks. Further stratification, viscous dissipation, and activation energy effects are considered. To the best of the authors’ knowledge, such an attempt at studying Williamson fluid has not been considered yet. Homotopy technique [31–42] is used to develop convergent solutions. Behaviors of temperature, velocity, entropy generation, Nusselt number, Bejan number, and skin friction are discussed via graphs and tabulated values. At the end, concluding remarks are given.

2. Formulation

Three-dimensional steady flow of Williamson fluid between two rotating stretchable disks was considered. This study aims to scrutinize FGM in flow with effects of Dufour/Soret, stratification, and viscous dissipation. Chemical reaction with activation energy was also investigated. Flow takes place due to the stretching of disks. The lower \( z = 0 \) and upper \( z = h \) disks are characterized by respective angular velocities of \( \Omega_1 \) and \( \Omega_2 \). Flow is caused by the stretching of lower disk. Lower and upper disks correspond to temperatures \( (T_1, T_2) \) and concentration \( (\tilde{C}_1, \tilde{C}_2) \). A uniform magnetic field of strength \( (B_0) \) is exerted in the Z-direction (see Figure 1).

An extra stress tensor \( (\tilde{\tau}) \) of Williamson fluid is given below:

\[
\tilde{\tau} = \left[ \tilde{\mu}_\infty + \frac{(\tilde{\mu}_0 - \tilde{\mu}_\infty)}{1 - \Gamma \dot{\gamma}} \right] \mathbf{A},
\]

where \( \mathbf{A} \) denotes the first Rivlin-Ericksen tensor, \( \tilde{\mu}_0 \) is the zero shear rate viscosity, \( \tilde{\mu}_\infty \) is the infinite shear rate viscosity, and \( \Gamma > 0 \) is a time constant. Herein, \( \dot{\gamma} \) is defined below:

\[
\dot{\gamma} = \sqrt{\frac{1}{2} \text{tr} \mathbf{A}^2}.
\]

Through Eq. (1), one obtains the following:

\[
\tilde{\tau} = (\tilde{\mu}_0 + (\tilde{\mu}_0 - \tilde{\mu}_\infty) \Gamma \dot{\gamma}) \mathbf{A},
\]

which further yields:

\[
\tilde{\tau}_r = 2 ((\tilde{\mu}_0 - \tilde{\mu}_\infty) \Gamma \dot{\gamma} + \tilde{\mu}_0) \left( \frac{\partial \tilde{\omega}}{\partial r} \right).
\]

![Figure 1. Schematic diagram of the problem.](image-url)
\[
\begin{align*}
\tau_{\theta\theta} &= 2((\dot{\mu}_0 - \dot{\mu}_\infty) \Gamma_{\gamma} + \dot{\mu}_0) \left( \frac{1}{r} \frac{\partial \dot{\theta}}{\partial \theta} + \frac{\dot{\theta}}{r} \right) - k_1^2(\dot{C} - \dot{\hat{C}}_2) \left( \frac{T}{T_2} \right)^n \exp \left( - \frac{E_a}{kT} \right), \\
\tau_{zz} &= 2 \left( (\dot{\mu}_0 - \dot{\mu}_\infty) \Gamma_{\gamma} + \dot{\mu}_0 \right) \left( \frac{\partial \dot{\theta}}{\partial \theta} \right), \\
\tau_{\theta r} &= \tau_{r \theta} = (\dot{\mu}_0 - \dot{\mu}_\infty) \Gamma_{\gamma} + \dot{\mu}_0 \left( \frac{1}{r \partial \theta} \frac{\partial \dot{\theta}}{\partial r} + \frac{\dot{\theta}}{r} \right), \\
\tau_{rz} &= \tau_{z r} = (\dot{\mu}_0 - \dot{\mu}_\infty) \Gamma_{\gamma} + \dot{\mu}_0 \left( \frac{1}{r} \frac{\partial \dot{\theta}}{\partial r} + \frac{\dot{\theta}}{r} \right), \\
\tau_{r \theta} &= \tau_{z \theta} = (\dot{\mu}_0 - \dot{\mu}_\infty) \Gamma_{\gamma} + \dot{\mu}_0 \left( \frac{1}{r} \frac{\partial \dot{\theta}}{\partial \theta} + \frac{\dot{\theta}}{r} \right). \tag{4}
\end{align*}
\]

Eq. (5) is shown in Box I. Mathematical statements of the problem under consideration satisfy the following [5]:
\begin{alignat*}{2}
\frac{\partial \dot{\theta}}{\partial r} + \frac{\dot{\theta}}{r} + \frac{\partial \dot{\theta}}{\partial z} &= 0, \tag{6}
\end{alignat*}
\begin{alignat*}{2}
\rho \left( \frac{\partial \dot{\theta}}{\partial z} + \frac{\partial \dot{\theta}}{\partial r} - \frac{\dot{\theta}}{r} \right) &= - \frac{\partial p}{\partial r} + \frac{\partial \tau_{rr}}{\partial z} + \frac{\partial \tau_{rz}}{\partial z} - \frac{\tau_{\theta \theta}}{r} - \sigma B_0^2 \dot{\theta}, \tag{7}
\end{alignat*}
\begin{alignat*}{2}
\rho \left( \frac{\partial \dot{\theta}}{\partial r} + w \frac{\partial \dot{\theta}}{\partial z} + \frac{\dot{\theta}}{r} \right) &= \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{\theta \theta}}{r} - \sigma B_0^2 \dot{\theta}, \tag{8}
\end{alignat*}
\begin{alignat*}{2}
\rho \left( \frac{\partial \dot{w}}{\partial r} + w \frac{\partial \dot{w}}{\partial z} - \frac{\dot{w}}{r} \right) &= - \frac{\partial p}{\partial z} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{\theta \theta}}{r}. \tag{9}
\end{alignat*}

\begin{align*}
\frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} &= \frac{k}{\rho C_p} \left( \frac{\partial^2 \dot{T}}{\partial r^2} + \frac{\partial^2 \dot{T}}{\partial z^2} + \frac{1}{r} \frac{\partial \dot{T}}{\partial r} \right) + \tau L + \frac{D_m K_T}{C_p C_s} \left( \frac{\partial^2 \dot{C}}{\partial r^2} + \frac{\partial^2 \dot{C}}{\partial z^2} + \frac{1}{r} \frac{\partial \dot{C}}{\partial r} \right), \tag{10}
\end{align*}
\begin{align*}
\frac{\partial \dot{C}}{\partial r} + w \frac{\partial \dot{C}}{\partial z} &= D \left( \frac{\partial^2 \dot{C}}{\partial r^2} + \frac{\partial^2 \dot{C}}{\partial z^2} + \frac{1}{r} \frac{\partial \dot{C}}{\partial r} \right) + \frac{D_m K_T}{T_m} \left( \frac{1}{r} \frac{\partial \dot{T}}{\partial r} + \frac{\partial^2 \dot{T}}{\partial r^2} + \frac{\partial^2 \dot{T}}{\partial z^2} \right), \tag{11}
\end{align*}
\begin{align*}
\dot{\gamma} &= \sqrt{2 \left( \left( \frac{\partial \dot{\theta}}{\partial r} \right)^2 + \left( \frac{\partial \dot{\theta}}{\partial z} \right)^2 + \left( \frac{\dot{\theta}}{r} + \frac{1}{r} \frac{\partial \dot{\theta}}{\partial \theta} \right)^2 \right) + \left( \frac{1}{r} \frac{\partial \dot{\theta}}{\partial \theta} + \frac{\partial \dot{\theta}}{\partial r} - \frac{\dot{\theta}}{r} \right)^2 + \left( \frac{\partial \dot{w}}{\partial z} + \frac{1}{r} \frac{\partial \dot{w}}{\partial \theta} \right)^2 + \left( \frac{\partial \dot{w}}{\partial r} + \frac{\partial \dot{w}}{\partial z} \right)^2}, \tag{15}
\end{align*}

Box I
\[ (1 + \text{We} \gamma) \ddot{f}' + \text{Re}(\ddot{g}^2 - \ddot{f}' - M \ddot{f} - \ddot{h} \ddot{f}') + \frac{\text{We} A^2}{\gamma} \left( \dddot{f}'' + \dddot{f}'' + \dddot{h}' \dddot{f}' \right) + \frac{2 \text{We}}{\gamma} \left( \dddot{h}' \dddot{h}' + 3 \dddot{f}'' \dddot{f} + \dddot{g}'' \dddot{g}'' \right) = 0, \]

\[ (1 + \text{We} \gamma) \ddot{g}'' + \frac{\text{WeRe}}{\gamma} \left( \dddot{g}'' + \dddot{g}'' + \dddot{h}' \dddot{h}' + \dddot{g}' \dddot{g}' \right) = 0. \]  

(16)

(17)

(18)

(19)

\[ \text{Re} \theta'' + (\theta + S_1) - (\text{Re})(\text{Pr}) (\ddot{h}' + S_1 \ddot{f} + \dddot{f}) + D_f (\text{Re} \theta'' + \dddot{f} + S_2 \text{Ec})(\text{Pr})(1 + \text{We} \gamma) \left( 4 \dddot{f} + 4 \dddot{f} + 4 \dddot{g} + 2 \dddot{h} \right) = 0. \]  

(20)

with boundary conditions:

\[ \ddot{f}(0) = A_1, \quad \dddot{h}(0) = 0, \quad \dddot{g}(0) = 1, \]

\[ \dddot{\varphi}(0) = 1 - S_2, \quad \dddot{\theta}(0) = \Omega, \quad \dddot{\theta}(1) = 0, \quad P(1) = 0, \quad \dddot{\varphi}(1) = 0, \quad \dddot{\varphi}(1) = 1. \]  

(21)

\[ \dot{\gamma} = \sqrt{4 \dddot{f} + A^2 \left( \dddot{g}^2 + \dddot{g}^2 \right) + 2 \dddot{h}^2}, \]  

(22)

where:

\[ M = \frac{\sigma B_0^2}{\rho \Omega_1}, \quad \text{Re} = \frac{\Omega_1 \nu^2}{\nu}, \quad \sigma = \frac{k^2}{\Omega_1}, \]

\[ E = \frac{E_0}{\kappa T_2}, \quad S_\tau = \frac{D_m \kappa T_2}{D T_2}, \quad \text{Re} = \frac{\dot{T}_1 - \dot{T}_0}{C_1 - C_0}, \quad \dot{\epsilon} = \frac{\dot{T}_1 - \dot{T}_0}{\dot{T}_2}. \]  

(23)

Herein, Re represents the local Reynolds number, Ec Eckert number, Pr Prandtl number, We Weissenberg number, M magnetic field parameter, Sc Schmidt number, Df Dufour number, Sr Soret number, St thermal stratification parameter, S2 solutal stratification parameter, E dimensionless activation energy, \( \sigma \) dimensionless reaction rate, and \( \dot{\epsilon} \) temperature difference parameter.

Radial and tangential shear stresses and heat transfer rate at the lower disk are given below:

\[ C_{f,1} = \frac{r_{\tau,1}}{B_0 \Omega_1}, \quad C_{f,1} = \frac{r_{\tau,1}}{B_0 \Omega_1}. \]  

(24)

\[ \text{Nu}_{\tau,1} = \sqrt{\text{Re} \theta'(0)}. \]  

(25)

The definitions of Nusselt number and skin friction coefficients at the lower disk are:

\[ \text{Nu}_{\tau,1} = \sqrt{\text{Re} \theta'(0)}. \]  

\[ \text{Nu}_{\tau,2} = \frac{r_{\tau,1}}{B_0 \Omega_1}. \]  

(26)

For the upper disk (radial, tangential), shear stresses and Nusselt number are given below:

\[ C_{f,1} = \frac{r_{\tau,1}}{B_0 \Omega_1}, \quad C_{f,1} = \frac{r_{\tau,1}}{B_0 \Omega_1}. \]  

(27)
Thus, the Nusselt number and skin friction for the upper disk are obtained as:

\[
\frac{Nu_{22}}{\sqrt{Re}} = \frac{C_{l_2}}{\sqrt{Re}} = \left[1 + (We)f'(1) \sqrt{4\hat{h}^2(1)+2\hat{h}^{2}(1)+Re \left(\hat{g}^2(1)+\hat{f}^2(1)\right)} \right].
\]

\[
\frac{C_{l_{21}}}{\sqrt{Re}} = \left[1 + (We)g'(1) \right] \sqrt{4\hat{h}^2(1)+Re \left(\hat{g}^2(1)+\hat{f}^2(1)\right)+2\hat{h}^{2}(1)} .
\] (28)

For lower and upper disks, the Sherwood number can be written as follows:

\[
Sh = \frac{r_{jw}}{D(C_1-C_0)\left|_{1-h} \right.}, \quad Sh = \frac{r_{jw}}{D(C_1-C_0)\left|_{-\infty} \right.}, \quad (29)
\]

where \(J_{jw}\) is:

\[
J_{jw} = -\left[D \frac{\partial \tilde{C}}{\partial z}\right]. \quad (30)
\]

Thus, after applying transformations, the Sherwood numbers take the following form:

\[
\frac{Sh}{\sqrt{Re}} = -\varphi'(0), \quad \frac{Sh}{\sqrt{Re}} = -\varphi'(1). \quad (31)
\]

3. Solution technique

Auxiliary linear operators and initial approximations are:

\[
\begin{align*}
L_h & = \hat{h}', & L_f & = \hat{f}', & L_g & = \hat{g}', \\
L_\theta & = \hat{\theta}', & L_\phi & = \hat{\phi}', \\
\hat{h}_0 & = 0, & \hat{f}_0 & = A_1 - A_1 \exp(-\xi) + A_2(\xi), \\
\hat{g}_0 & = 1 + \xi(\Omega - 1), & \hat{\theta}_0 & = (1 - \xi)(1 - S_1), \\
\hat{\phi}_0 & = (1 - \xi)(1 - S_2).
\end{align*}
\] (32)

with

\[
\begin{align*}
L_h[Z_1] & = 0, \\
L_f[Z_2 + Z_3\xi] & = 0, \\
L_g[Z_4 + Z_5\xi] & = 0, \\
L_\theta[Z_6 + Z_7\xi] & = 0, \\
L_\phi[Z_8 + Z_9\xi] & = 0.
\end{align*}
\] (34)

where the constants are \(Z_i (i = 1 - 5)\).

4. Convergence

Auxiliary variables \((h, h_\ell, h_\ell, h_\ell, h_\ell, h_\ell)\), play a prominent role in the convergence analysis. Figure 2 shows \(h\)-curves for the \(m\)-th-order approximation. The solutions are found to be convergent for the regions \(-1.7 \leq h_\ell \leq -0.3, -1.2 \leq h_f \leq -0.3, -1.0 \leq h_g \leq -0.4, -0.9 \leq h_\theta \leq -0.6, -1.0 \leq h_\phi \leq -0.7\). Table 1 comprises numerical values of velocity, temperature, and concentration. Table 1 shows the numerical values of velocity concentration and temperature distribution. Clearly, a meaningful solution of \(\hat{h}(0), \hat{f}(0), \hat{g}(0), \hat{\theta}(0), \hat{\phi}(0)\) is derived from the 20th-order approximations. Table 2 shows good agreement between the obtained results and those in the previous literature.

5. Entropy

By considering the roles of thermal irreversibility, Joule heating irreversibility, viscous dissipation irreversibility, and mass transfer irreversibility, the formulation of entropy generation is presented. Its dimensional form is represented by Eq. (35) as shown in Box II, where \(\tilde{C}_m\) and \(\tilde{T}_m\) are mean concentration and temperature, respectively, and \((\nabla T), (\nabla C)\) and \((\Phi)\) are defined below:

\[
\nabla \tilde{T} = \left(\frac{\partial \tilde{T}}{\partial r}\right) \tilde{e}_r + \left(\frac{1}{r} \frac{\partial \tilde{T}}{\partial \theta}\right) \tilde{e}_\theta + \left(\frac{\partial \tilde{T}}{\partial z}\right) \tilde{e}_z, \quad (36)
\]

\[
\nabla \tilde{C} = \left(\frac{\partial \tilde{C}}{\partial r}\right) \tilde{e}_r + \left(\frac{1}{r} \frac{\partial \tilde{C}}{\partial \theta}\right) \tilde{e}_\theta + \left(\frac{\partial \tilde{C}}{\partial z}\right) \tilde{e}_z, \quad (37)
\]

\[
\Phi = \mu_0 \left[\left(\frac{\partial \tilde{\theta}}{\partial \tilde{y}}\right)^2 + \Gamma \left(\frac{\partial \tilde{\theta}}{\partial \tilde{y}}\right)^3\right]. \quad (38)
\]

After applying transformation, one obtains the

Figure 2. \(h\)-curves for axial, radial, tangential velocity, temperature, and concentration profiles.
Table 1. Series solutions convergence for \( \Omega = 0.2, n = 1, A_1 = 0.1, A_2 = 0.4, n = 0.1, E_1 = 0.5, \sigma = 0.5, \delta = 0.5, S_1 = S_2 = 0.02, D_f = 0.5, Ec = 0.1, M = 0.7, Sc = 1, Re = 0.7, We = 0.2, Pr = 0.6, Sr = 0.3, h_\beta = h_j = h_y = h_\theta = h_\xi = -0.8.

| Order of approximation | \(-\hat{h}(0)\) | \(\hat{f}'(0)\) | \(-\hat{g}'(0)\) | \(\hat{\theta}'(0)\) | \(-\nabla'\theta(0)\) |
|------------------------|----------------|----------------|----------------|----------------|----------------|
| 1                      | 0.20000        | 0.5272         | 1.0380         | -0.11008       | 0.6884         |
| 5                      | 0.20000        | 0.4890         | 1.192          | -0.09493       | 0.6863         |
| 11                     | 0.20000        | 0.4812         | 1.192          | 0.09682        | 0.5462         |
| 16                     | 0.20000        | 0.4812         | 1.192          | 0.2655         | 0.4127         |
| 20                     | 0.20000        | 0.4812         | 1.192          | 0.2655         | 0.3554         |
| 25                     | 0.20000        | 0.4812         | 1.192          | 0.2655         | 0.3554         |
| 30                     | 0.20000        | 0.4812         | 1.192          | 0.2655         | 0.3554         |
| 40                     | 0.20000        | 0.4812         | 1.192          | 0.2655         | 0.3554         |

\[
S_G^p = \frac{\hat{\mu}}{T_m} \Phi + \frac{k}{T_m^2} \left(\nabla^T\right)^2 + \frac{\sigma}{T_m} \left(\beta^2 + \hat{\omega}^2 \right) + \frac{R_g D}{C_m} \frac{\left(\nabla \Phi \right)^2}{T_m^2} + \frac{D R_g}{C_m} \left(\nabla \Phi, \nabla^T \right). \tag{35}
\]

**Viscous dissipation irreversibility**  **Thermal irreversibility**  **Joule dissipation irreversibility**  **Mass transfer irreversibility**

Box II

Table 2. Comparison table for the validation of the problem with Ref. [5] by varying \(\text{We}\).

| We    | \(C_{f0} [\text{Present}]\) | \(C_{f0} [\text{Ref. [5]}]\) | \(\hat{C}_{f0} [\text{Present}]\) | \(\hat{C}_{f0} [\text{Ref. [5]}]\) |
|-------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 0.1   | 0.550420                    | 1.12934                     | 0.50420                     | 1.12934                     |
| 0.2   | 0.590397                    | 1.20342                     | 0.569379                    | 1.20342                     |
| 0.3   | 0.630044                    | 1.27751                     | 0.630044                    | 1.27751                     |

following:

\[
N_G = \alpha_1 \left[ \text{Re} \hat{\theta}^2 + (S_1 + \theta)^2 \right] + M (\text{Re}) \left( \text{Br} \right) \left( \hat{f}^2 + \hat{g}^2 \right) + \left( \text{Br} \right) (1 + \text{We} \hat{\gamma}) \left[ 4 \hat{f}^2 + \text{Re} \hat{f}^2 + \text{Re} \hat{g}^2 + 2 \hat{h}^2 \right] + L \left[ \frac{\alpha_2}{\alpha_1} \right] \left[ \text{Re} \hat{\theta}^2 + (\hat{\varphi} + S_1)^2 \right] + L \left[ \left( \hat{\theta} + S_1 \right) \left( S_2 + \hat{\varphi} \right) + \text{Re} \hat{\theta}^2 \hat{\theta} \right]. \tag{39}
\]

Dimensionless parameters are given as follows:

\[
\alpha_1 = \frac{T_1 - T_0}{T_m}, \quad L = \frac{R_g D \left( \hat{C}_1 - \hat{C}_0 \right)}{k}.
\]

\[
\text{Br} = \frac{\hat{\mu} r^2 \Omega^2}{k (T_1 - T_0)}, \quad N_G = \frac{r^2 T_m S_{G1}^p}{k (T_1 - T_0)}.
\]

\[
\alpha_2 = \frac{\hat{C}_1 - \hat{C}_0}{C_m}. \tag{40}
\]

where \(\text{Br}\) and \(N_G\) are Brinkman number and local entropy generation, respectively, \(\alpha_1\) and \(\alpha_2\) are temperature and concentration ratio parameters, and \(L\) is the diffusion parameter.

Herein, the dimensionless form of Bejan number (Be) is given by Eqs. (41) and (42) as shown in Box III.

6. Discussion

This section discusses a graphical interpretation of various physical parameters for velocity, temperature, entropy generation, Bejan number, skin friction, Nusselt number, and Sherwood number.

6.1. Velocity profile

Figures 3–8 are designed to analyze the behavior of Hartmann number \((M)\) and Weissenberg number \((\text{We})\) on \(\hat{h}(\xi), \hat{f}(\xi),\) and \(\hat{\theta}(\xi)\) in axial, radial, and transverse directions. These quantities are discussed for both upper and lower disks. Figures 3–5 show the analysis of the behavior of \(M\) for axial \(\hat{h}(\xi),\) radial \(\hat{f}(\xi),\) and tangential \(\hat{\theta}(\xi)\) velocities at both disks. It is found that the magnitude of velocities \(\hat{h}(\xi), \hat{f}(\xi),\) and \(\hat{\theta}(\xi)\) is reduced as the Hartman number \((M)\)
$$\text{Be} = \frac{\text{Entropy generation due to heat and mass transfer}}{\text{Total entropy generation}}$$

$$\text{Be} = \frac{\alpha_1 \left[ \text{Re} \dot{\theta}^2 + (S_1 + \theta)^2 \right] + L \left[ \frac{\alpha_1}{\alpha_2} \right] \left[ \text{Re} \dot{\varphi}^2 + (S_1 + \varphi)^2 \right] + L \left[ \left( S_1 + \dot{\theta} \right) \left( S_2 + \dot{\varphi} \right) \right] + \left( S_1 + \dot{\theta} \right) \left( S_2 + \dot{\varphi} \right) + \text{Re} \dot{\varphi}^2 \dot{\theta}^2}{L \left[ \left( S_1 + \dot{\theta} \right) \left( S_2 + \dot{\varphi} \right) + \text{Re} \dot{\varphi}^2 \right] + M \left( \text{Re} \right) \left( 2 \ddot{\varphi} + \ddot{\varphi}^2 \right) + \alpha_1 \left( S_1 + \theta \right)^2 + \text{Re} \dot{\varphi}^2 \dot{\theta}^2}$$

Box III

Figure 3. Axial velocity \( \dot{h}(\xi) \) via \( M \).

Figure 4. Radial velocity \( \dot{f}(\xi) \) via \( M \).

Figure 5. Tangential velocity \( \dot{g}(\xi) \) via \( M \).

Figure 6. Axial velocity \( \dot{h}(\xi) \) via \( \text{We} \).

Figure 7. Radial velocity \( \dot{f}(\xi) \) for \( \text{We} \).

Figure 8. Tangential velocity \( \dot{g}(\xi) \) via \( \text{We} \).
increases. A resistive force is produced when the transverse magnetic field is in effect. A force acting as Lorentz force generates resistance and reduces velocity. The effect of Weissenberg number (We) on $\hat{h}(\xi)$, $\hat{f}(\xi)$, and $\hat{g}(\xi)$ is shown in Figures 6–8. The axial and radial velocities, $\hat{h}(\xi)$, and $\hat{f}(\xi)$, rise when Weissenberg number (We) increases and tangential velocity $\hat{g}(\xi)$, decreases. At a higher We number, the rotational velocity ($\Omega_1$) improves and, thus, $\hat{h}(\xi)$, and $\hat{f}(\xi)$ increase (see Figures 6 and 7).

### 6.2. Temperature

Figures 9–12 illustrate the trend of temperature $\left(\hat{\theta}(\xi)\right)$ against Dufour number ($D_1$), thermal stratification parameter ($S_1$), Weissenberg number (We), and Prandtl number (Pr). Figure 9 shows the impact of $D_1$ on $\hat{\theta}(\xi)$ at lower and upper disks. Fluid temperature via $D_1$ improves. Energy flux is enhanced due to increase in the concentration gradient for varying $D_1$ and it leads to the enhancement of the fluid temperature. Temperature close to the lower disk is higher than that to the upper disk because of the higher temperature of the lower disk than the upper disk, i.e., $\bar{T}_1 > \bar{T}_2$. Figure 10 shows the impact of thermal stratification parameter $S_1$ on $\hat{\theta}(\xi)$. To estimate the large number of $S_1$, the temperature decreases due to a potential drop between the ambient fluid temperature and the surface condition. Figure 11 shows the effect of Weissenberg number We, on $\hat{\theta}(\xi)$. Temperature is enhanced in the case of estimating higher $S_1$. The behavior of Prandtl number, Pr, with respect to $\hat{\theta}(\xi)$ is shown in Figure 12. For larger Pr, thermal diffusivity has a smaller value, which causes a reduction in temperature $\hat{\theta}(\xi)$.

### 6.3. Concentration

Figures 13–18 show the trend of concentration ($\hat{\phi}(\xi)$) for variations of Schmidt number (Sc), Soret number (Sr), dimensionless activation energy parameter ($E_1$).

![Figure 9. Temperature profile via $D_1$.](image9)

![Figure 10. Temperature profile via $S_1$.](image10)

![Figure 11. Temperature profile via We.](image11)

![Figure 12. Temperature profile via Pr.](image12)

![Figure 13. Temperature profile via Sc.](image13)
Generative chemical reaction is promoted due to a decrease in Arrhenius function \( \frac{T}{T_0} \exp \left( \frac{-E_a}{R T} \right) \). When \( E_a \) increases, concentration profile \( \phi (\xi) \) is enhanced. Figure 16 analyzes the effect of chemical reaction parameter \( \sigma \) on \( \phi (\xi) \). For larger \( \sigma \), a reduction in the concentration \( \phi (\xi) \) is noticed. Due to concentration \( \phi (\xi) \) gradient, this behavior shows weak buoyancy effect; thus, a reduction in \( \phi (\xi) \) occurs. Figure 17 shows that the concentration \( \phi (\xi) \) is reduced for the larger temperature difference parameter \( \delta \). According to Figure 18, the concentration \( \phi (\xi) \) is the decreasing function of the larger solutal stratification parameter \( S_2 \).

### 6.4. Entropy generation minimization

This section emphasizes the graphical interpretation of various physical parameters of entropy generation (\( N_g \)) and Bejan number (Be). Br effects on \( N_g \) and Be are shown in Figures 19 and 20. The opposite trend for Be is observed due to a increment in the disorderness of system for larger Be. For a larger estimation of Br due to its dissipation, a lower rate of conduction is produced and, thus, entropy generation (\( N_g \)) is enhanced. The impact of diffusion \( L \) on \( N_g \) and Be is shown in Figures 21 and 22. Both \( N_g \) and Be experience an increase for larger \( L \). For the
increasing $L$, the diffusion rate of nanoparticles is enhanced; therefore, Bejan number and total entropy of the system increase. The effect of $We$ on $N_g$ and Be is discussed in Figures 23 and 24. For larger $We$, the entropy generation increases, while Bejan number decreases. Figures 25–28 show the impact of solutal and thermal stratification parameters $S_2$ and $S_1$ on $N_g$ and Be. Both $N_g$ and Be are reduced for the larger estimation of $S_1$ and $S_2$. Figures 29 and 30 clearly reveal the effect of $M$ on $N_g$, and the enhancement of Be in $N_g$ parallel to increase in the magnetic parameter ($M$) is witnessed. For larger $M$, the fluid resistance grows due to rise in Lorentz force and, consequently, $N_g$ increases. Bejan number (Be) reduces for larger $M$. Herein, the irreversibility of fluid friction prevailed over the heat and mass transfer irreversibilities.

6.5. Skin friction and Nusselt and Sherwood numbers

Effects of Reynold number (Re) on skin friction coefficient at lower and upper disks are analyzed in Figures 29 and 30. There is an increase in skin friction
at both disks for the larger Re. From Figures 31 and 32, for the increasing Eckert number (Ec) the heat transfer rate near the surface of lower disk decreases, while reverse behavior is observed for the upper disk. Figures 33 and 34 are drawn to investigate the effect of stratification parameter ($S_2$) on the Sherwood number. For varying $S_2$, the Sherwood number near the surface of the lower disk is reduced. However, in proximity to the upper disk, the Sherwood number experiences a reverse behavior.
7. Conclusions

The following observations are worth mentioning:

- Velocities \( \hat{h}(\xi), \hat{f}(\xi) \) and \( \hat{g}(\xi) \) were reduced for \( M \) at both disks, while \( \tilde{h}(\xi) \) and \( \tilde{f}(\xi) \) were increasing functions of \( \text{We} \);
- For larger \( S_1 \) and \( \text{Pr} \), the temperature \( \tilde{\theta}(\xi) \) was reduced, while \( \text{We} \) and \( D_1 \) experienced an ascending trend;
- Opposite trends of concentration \( \tilde{c}(\xi) \) were found for \( S_2 \) and \( \text{Sc} \);
- Increasing behavior of \( \text{Re} \) for the skin friction was observed;
- Entropy rate through \( (\text{Br}), (\ell), \) and \( (\text{We}) \) was enhanced, while it decreased for thermal and solutal stratification parameters \( (S_1) \) and \( (S_2) \);
- Bejan number showed a decreasing behavior for larger \( (S_1), (S_2), (\text{We}), \) and \( (\text{Br}) \), while it increased for larger \( (\ell) \).

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