Abstract. We construct Morse-Smale-Witten complex for an effective orientable orbifold. For a global quotient orbifold, we also construct a Morse-Bott complex. We show that certain type of critical points of a Morse function has to be discarded to construct such a complex, and gradient flows should be counted with suitable weights. The homology of these complexes are shown to be isomorphic to the singular homology of the quotient spaces under the self-indexing assumptions.

1. Introduction

Morse theory is one of the most important tools to understand the topology of manifolds. A modern approach to the Morse theory, Morse-Smale-Witten complex has been very popular as its infinite dimensional application, Floer homology theory, has proven to be a very powerful tool in the area of symplectic and differential geometry. Morse-Smale-Witten complex is a free module generated by critical points of a Morse function graded by their indices, and the differential on this complex is given by counting (signed) number of gradient lines between critical points of index difference one.

The notion of a differentiable orbifold was introduced by Satake in the fifties under the name V-manifold, as a natural generalization of the notion of differentiable manifold. An effective orbifold is a space which is locally the quotient space of a smooth manifold by the effective action of a finite group.

In this paper, we develop Morse-Smale-Witten complex for effective orbifolds. Morse functions are given by invariant functions whose local lift is Morse. Morse theory on orbifolds, such as Morse inequalities or informations about local Morse data has been known for a while since the work of Lerman-Tolman. But the construction of Morse-Smale-Witten complex has not been available.

We construct such a complex, and show that its homology is isomorphic to the singular homology of the quotient space (with certain assumptions). There are a few interesting differences from the case of manifolds. The first one is that a broken trajectory (assume only one breaking) may be a limit of several different families of smooth trajectories, whereas for manifolds, there exist only one family of smooth trajectories converging to such a broken trajectory. This is because there can be several different lifts of broken trajectories, which are not equivalent via local group action and we analyze them carefully in section 4.

The second one is that one should count with suitable weights to define a chain complex depending on the order of related isotropy groups. Namely, each gradient trajectory, which was counted as one in the case of manifolds, should be counted...
with weights depending on the stabilizer of the end points and that of trajectory itself.

Most interesting of all is probably that one has to discard certain critical points (namely unorientable critical points) and consider a subcomplex generated by only orientable critical points, to define Morse-Smale-Witten complex. This is related to the observation, already in [LT], that when the local group action does not preserve the orientation of the unstable directions, then there exists no change of topology when passing through such a critical point.

One drawback is that it is known to be very difficult to study Morse-Smale condition (transversality between any unstable and stable manifolds of critical points) together with the local invariance condition. For example, Morse functions on orbifolds are dense among smooth functions (by [Wa], [H]) but they may not be Morse-Smale. Hence throughout the paper, we only consider the case when the given function is indeed Morse-Smale. It is also difficult to carry out an analogue of the usual invariance proof of the homology of the chain complex due to such issues. (We hope to return to this issues in the future research.) Instead, we will prove that the homology is isomorphic to the singular homology of the quotient space.

The work of Dixon, Harvey, Vafa and Witten on string theory on orbifolds [DHVW], and the discovery of new ring structure on cohomology of inertia orbifolds by Chen and Ruan [CR], and orbifold Gromov-Witten invariants [CR2], has prompted many exciting new developments on the study of orbifolds in the last decades. But the Fukaya category theory on orbifolds has not been developed yet. This paper lays a foundation to define orbifold Fukaya category theory. Namely, we expect that the new phenomenons which appeared in this paper on Morse theory should also be present in orbifold Fukaya category theory, and they should be dealt in a similar way as in this paper. We hope to explore this elsewhere jointly with Mainak Poddar. We remark that for toric orbifolds, Lagrangian Floer theory for smooth Lagrangian torus fibers has been developed in [CP].

We also remark that the equivariant cohomology version of Morse-Bott theory has been defined by Austin and Braam [AB] for compact (connected) Lie group $G$, but their construction does not immediately generalize for finite $G$ unless the group action preserves the orientation of critical submanifolds and orientations of unstable directions.

We consider local coefficient system on each critical submanifolds from the determinant bundle of normal directions, and take an invariant subcomplex (where invariance involves local coefficients) to define a Morse-Smale-Witten complex in such a case. This gives a correct analogue of their construction of equivariant cohomology Morse-Bott complex in the case of finite group $G$.

Here is the outline of the paper. In section 2, the case of global quotient is developed with careful examination of orientation issues. In section 3, we reformulated the construction of section 2 in a more intrinsic form. In section 4, we define a Morse-Smale-Witten complex for effective orbifolds and show that $\partial^2 = 0$. In section 5, we compare the homology of Morse-Smale-Witten complex with singular homology of the quotient space. In section 6, we consider the case of non-orientable manifolds and also Morse-Bott case.
2. MORSE-SMALE-WITTEN COMPLEX FOR GLOBAL QUOTIENTS

Let $M$ be a closed oriented connected manifold and suppose the finite group $G$ acts on $M$ effectively in an orientation preserving way. Set $X$ to be the global quotient orbifold $M/G$ and denote by $X$ the quotient space and $\pi : M \to X$ the natural projection to the orbit space. By [Wa], we can take a $G$-invariant Morse function $f : M \to \mathbb{R}$. Suppose further that $f$ is Morse-Smale with respect to a $G$-invariant metric. In this section, we construct a Morse-Smale-Witten complex for $\bar{f} : X \to \mathbb{R}$, where $f = \bar{f} \circ \pi$. A more general Morse-Bott case will be discussed later, but we explain this case here for the introduction and to present a clearer picture on orientation issues.

We begin by recalling that Morse homology of orbifolds has been studied by Lerman and Tolman [LT], where they analyzed the local Morse data near a critical point in orbifold setting and proved Morse inequalities for orbifolds. The equivariant cohomology complex of $M$ with respect to $G$-action in terms of Morse theory has been defined by Austin and Braam in [AB] using Cartan model together with Morse-Bott complex, when $G$ is compact connected Lie group. (See remark 6.2 for more discussions on this).

Although it is not stated in [AB], they implicitly assume that $G$ is connected compact Lie group, because first, they are using Cartan model, and second they do not discuss orientation issues of the group action, which becomes trivial for connected Lie group actions.

In fact, as we will see, the orientation issues are very important even to set up Morse-Witten-Smale complex for finite group actions. Even though the group action is assumed to preserve the given orientation of manifold, it may not preserve the orientation of unstable directions.

The critical points of $\bar{f}$ will be divided into two types, orientable and non-orientable critical points (see Definition 2.2 below). We will discard the non-orientable critical points and show that invariant chains in the complex made of orientable critical points define Morse homology of the orbifold $X$.

We recall that a smooth function on orbifold by definition has smooth invariant lift on each uniformizing chart.

**Definition 2.1.** A smooth function $\bar{f} : X \to \mathbb{R}$ is called Morse if every point $\bar{x}$ in the orbifold has a uniformizing chart $(\bar{U}_\bar{x}, G_\bar{x}, \pi_{\bar{x}})$ such that $\bar{f} \circ \pi_{\bar{x}}$ is Morse on $\bar{U}_\bar{x}$.

From now on, we consider global quotient orbifold $X = M/G$ as above. Let $f$ be a Morse function on $M$ which is in addition $G$-invariant, then $f$ induces a Morse function $\bar{f}$ on $X$. Denote the set of critical points of $f$ and $\bar{f}$ by $\text{crit}(f)$ and $\text{crit}(\bar{f})$, respectively. i.e. $\bar{p} \in \text{crit}(\bar{f})$ if there exists $p \in \text{crit}(f)$ such that $\pi(p) = \bar{p}$. As usual, we define $Cf_*(M)$ as a complex of $\mathbb{R}$-vector spaces freely generated by each $p \in \text{crit}(f)$. We write $W^+(p)$ and $W^-(p)$ to denote stable and unstable manifolds at $p$, respectively (see for example [Ni]). Also, denote the set of all critical points of $f$ with index $i$ by $\text{crit}_i(f)$, for simplicity.

We construct a Morse-Smale-Witten complex associated to $\bar{f}$ as a certain subcomplex of $Cf_*(X)$ as follows: First, orient $W^-(p)$ for each $p \in \text{Crit}(f)$.

**Definition 2.2.** We define the type of a critical point $\bar{p} \in \text{crit}(\bar{f})$ as follows. If $G_{\bar{p}}$-action on the unstable manifold $W^-(p)$ at $p \in \pi^{-1}(\bar{p})$ is orientation preserving, then $p$ is called orientable critical point, and non-orientable otherwise. Denote by...
\( \text{crit}^+(f) \) (resp. \( \text{crit}^-(f) \)) the set of all orientable (resp. non-orientable) critical points of \( f \).

We use the similar notation for critical points of \( f \).

**Remark 2.3.** The idea of non-orientable critical points was considered already in [LT] and in several subsequent works such as [H]. As observed in [LT], this is very natural in terms of local Morse data. Indeed, we will see later that attaching cells which arise at non-orientable critical points do not contain any topological information for the quotient space (see Corollary 5.4).

**Remark 2.4.** If \( G_p \) is orientation preserving for one of \( p \in \pi^{-1}(\bar{p}) \), then it is so for other \( p' \in \pi^{-1}(\bar{p}) \).

Let \( \text{crit}_i^+(f) = \text{crit}_i(f) \cap \text{crit}_i^k(f) \), which induces the decomposition \( Cf_i(M) = Cf_i^+(M) \oplus Cf_i^-(M) \). \( G \)-action not only preserves \( \text{crit}(f) \) but also preserves a index and the type (orientable or not) of a critical point and hence \( G \) naturally acts on \( Cf_i^+(M) \). Define
\[
Cf_i^+(X) := Cf_i^+(M)^G,
\]
\( G \)-invariant chains in \( Cf_i(X) \) consisting of orientable critical points of degree \( i \). Let \( \bar{p} \in \text{crit}(f) \), we formally write
\[
(2.1) \quad [\bar{p}] := \sum_{p \in \pi^{-1}(\bar{p})} p.
\]
\( Cf^+(X)(= \oplus_i Cf_i^+(X)) \) is freely generated by such \([\bar{p}]\)'s for \( \bar{p} \in \text{crit}^+(\bar{f}) \).

We will define a boundary map \( \partial_i : Cf_i^+(X) \to Cf_{i-1}^+(X) \) to make it a chain complex. For each orientable critical point \( \bar{p} \) of \( \bar{f} \), take a \( G \)-invariant orientation on \( W^-(p) \), for all \( p \in \pi^{-1}(\bar{p}) \). For non-orientable \( \bar{p} \), and take arbitrary orientation for \( W^-(p) \).

As \( f \) is a Morse function on the manifold \( M \), we have Morse-Smale-Witten differential \( \partial_i : Cf_i(M) \to Cf_{i-1}(M) \) associated to \( f \) (defined by using the above choice of orientations), defined as follows:

**Definition 2.5.** For \( p, q \in \text{crit}(f) \), define \( \widehat{M}(p, q) \) to be the set of all negative gradient flow lines from \( p \) to \( q \) and by taking quotient under time translation,
\[
M(p, q) = \widehat{M}(p, q)/\mathbb{R}.
\]
Then, we define
\[
\partial_{\mu(p)}p = \sum_{q, \mu(q) = \mu(p)-1} \#M(p, q) q.
\]
Here, \( \mu(p) \) is the Morse index of \( p \) and \( \#M(p, q) \) is the signed number of oriented moduli space which is a finite set. (i.e. \( \#M(p, q) \) is the ‘signed’ number of negative gradient flow lines from \( p \) to \( q \). For the sign rule, see below.)

Now, we define a differential \( \partial \) for \( \bar{f} : X \to \mathbb{R} \) on \([\bar{p}]\) from the formula \(2.1\) and using the differential for \( f : M \to \mathbb{R} \). We claim that this defines a differential for \( Cf^+(X) \). To show this, we need the following two crucial lemmas:

**Lemma 2.6.** If \( \bar{p} \in \text{crit}^+(\bar{f}) \), then
\[
(2.2) \quad \partial[\bar{p}] = \partial \left( \sum_{\bar{p}' \in \pi^{-1}(\bar{p})} \bar{p} \right) \in Cf^+(X),
\]
Proof. Let \( \bar{p} \) be of index \( \bar{i} \). Then, every \( p \) in the sum has index \( i \). We will show that the coefficient of an arbitrary non-orientable critical point \( q \in \text{crit}_{-1}(f) \) is zero. Set \( \mathcal{M}(\bar{p}, q) \) be the set of negative gradient flow lines from any \( p \in \pi^{-1}(\bar{p}) \) to \( q \). Then, \( \mathcal{M}(\bar{p}, q) \) is a signed set. i.e. there is a prescribed sign for each element \( \gamma \in \mathcal{M}(\bar{p}, q) \) according to the standard sign rule in Morse homology theory.

We briefly recall the sign rule for reader’s convenience. For an oriented manifold \( M \) and a Morse function \( f \), fix an orientation of each unstable manifold, it will orient every stable manifolds so that for each critical point \( M \) where \([ ]\) means the oriented frames of the tangent spaces at \( p \). Hence, \( W^{-}(p) \cap W^{+}(q) \) has an induced orientation (we follow the orientation conventions of [GP]).

Let \( \gamma \) be a negative gradient flow line connecting \( p \) and \( q \). Then, \( \text{im} \gamma \subset W^{-}(p) \cap W^{+}(q) \). If the negative gradient flow orientation of \( \gamma \) matches the induced orientation, then it is counted as +1 and otherwise as -1.

The following convention also gives the same sign. Fix \( s \) so that \( f(q) < s < f(p) \). Orient the set \( f^{-1}(s) \) so that \( |\nabla f| f^{-1}(s) = |M| \). Consider \( S^{-}(p) := W^{-}(p) \cap f^{-1}(s) \) which is oriented as a boundary of \( D^{-}(p) := W^{-}(p) \cap f^{-1}([s, \infty)) \). Similarly, consider \( S^{+}(q) \) which is oriented as a boundary of \( D^{+}(q) \). In fact, they are diffeomorphic to \( S^{-1} \) and \( S^{n^{-1}} \), respectively. Because \( S^{-}(p) \) and \( S^{+}(q) \) are of complementary dimensions to each other in the slice \( f^{-1}(s) \), we can count their signed intersection number. One can check that this sign agrees with the above convention (following sign rules of [GP]).

Suppose that \( q \) is a non-orientable critical point. We split \( \mathcal{M}(\bar{p}, q) = \cup_{x} \mathcal{M}(p, q) \) into a disjoint union \( \mathcal{M}(\bar{p}, q) = \mathcal{M}(\bar{p}, q)^{+} \cup \mathcal{M}(\bar{p}, q)^{-} \) with respect to their signs. Clearly, the sum of these signs will be the coefficient of \( q \) in \((4.7)\) and hence we have to show that \( |\mathcal{M}(\bar{p}, q)^{+}| = |\mathcal{M}(\bar{p}, q)^{-}| \).

Pick any \( g \in G_{q} \) which reverses the orientation of \( W^{-}(q) \). Then \( g \) will give a permutation of \( \mathcal{M}(\bar{p}, q) \), which also written as \( g \), since \( g \) preserves \( \pi^{-1}(\bar{p}) \). We claim that \( g \) sends \( \mathcal{M}(\bar{p}, q)^{+} \) to \( \mathcal{M}(\bar{p}, q)^{-} \). To see this, we consider the action of \( g \) on \( f^{-1}(s) \). Note that the \( G \)-action preserves each slice of \( f \). If \( g \cdot p = p' \), \( g \) sends \( S^{-}(p) \) to \( S^{-}(p') \) and preserves \( S^{+}(q) \) (note that \( g \in G_{q} \) is an automorphism of \( f^{-1}(s) \)). Let \( x \in S^{-}(p) \cap S^{+}(q) \) represent \( \gamma \in \mathcal{M}(\bar{p}, q)^{+} \). By the sign rule given above, \( S^{-}(p) \) and \( S^{+}(q) \) intersect positively at \( x \) in \( f^{-1}(s) \), meaning that
\[
[S^{-}(p)]_{x} [S^{+}(q)]_{x} = [f^{-1}(s)]_{x}.
\]
Here the subscript \( x \) means they are oriented frames of tangent spaces at \( x \). Now, let’s pass through the automorphism of \( f^{-1}(s) \) induced by \( g \). \( G \)-invariant orientations on unstable manifolds at \( p \)‘s implies that:
\[
g \cdot [S^{-}(p)]_{x} = [S^{-}(p')]_{g_{x}}.
\]
and, by the assumption that \( g \) reverses the orientation of unstable manifold at \( q \),
\[
g \cdot [S^{+}(q)]_{x} = -[S^{+}(q)]_{g_{x}}.
\]
Since \( g \) preserves the orientation of \( M \) and \( f \) is \( g \)-invariant, hence \( g \) preserves the orientation of \( f^{-1}(s) \), or \( g \cdot [f^{-1}(s)]_{x} = [f^{-1}(s)]_{g_{x}} \). In conclusion, considering the oriented frames at \( g \cdot x \), we have
\[
[S^{-}(p')]_{g_{x}} [S^{+}(q)]_{g_{x}} = (g \cdot [S^{-}(p)]_{x}) (-g \cdot [S^{+}(q)]_{x}) = -g \cdot [f^{-1}(s)]_{x} = -[f^{-1}(s)]_{g_{x}}.
\]

i.e., \( \partial[\bar{p}] \) has nonzero coefficients only at orientable critical points of \( f \).
This means $S^-(p')$ and $S^+(q)$ intersect negatively at $g \cdot x$. Because the point $g \cdot x$ represent $g \cdot \gamma$, the sign of $g \cdot \gamma$ should be minus. This proves our claim. By the same argument $g^{-1}$ sends $\mathcal{M}(\bar{p}, q)^+$ to $\mathcal{M}(\bar{p}, q)^+$. But since $g$ and $g^{-1}$ are inverse to each other, we get a bijection $g$ from $\mathcal{M}(\bar{p}, q)^+$ to $\mathcal{M}(\bar{p}, q)^-$. In particular $|\mathcal{M}(\bar{p}, q)^+| = |\mathcal{M}(\bar{p}, q)^-|$. □

Lemma 2.7. The following expression \[ \partial \left( \sum_{p \in \pi^{-1}(\bar{p})} P \right) \] is $G$-invariant if $\bar{p}$ is orientable.

Proof. By the previous lemma, \[ \partial \] only consists of orientable critical points. Consider two orientable points $q$ and $q' := g \cdot q$ appearing non trivially in \[ \partial \]. We need to show that coefficients of $q$ and $q'$ are equal. However, this is obvious since $g$ and $g^{-1}$ give the sign preserving isomorphisms between $M(\bar{p}, q)$ and $M(\bar{p}, q')$. This is because we chose orientations on unstable manifolds at each orientable critical points of $f$ so that they are $G$-invariant. □

We have shown that the Morse boundary map $\partial$ preserves $Cf^+(M)^G \subset Cf_*(M)$. Thus, we conclude that $Cf^+_*(X) = Cf^+_*(M)^G$ is a subcomplex of $Cf_*(M)$. We denote $Cf^+_*(X)$ by $CM_*(X, \bar{f})$ for simplicity and use the same notation $\partial$ for the restriction of $\partial : Cf_*(M) \to Cf_*(M)$ to $CM_*(X, \bar{f})$. Note that $\partial^2 = 0$ automatically follows from that of $M$.

In fact, the resulting homology group $HM_*(X, \bar{f})$ is isomorphic to the singular homology of the orbit space (quotient space). We postpone its proof to section 4 proposition 5.6 where we prove it in more general case.

Theorem 2.8. $HM_*(X, \bar{f}) \cong H_*(M/G) = H_*(X)$.

Example 2.9. Consider the famous heart $S^2$ with the Morse function $h$ given by the height function $h$ with two maximum $p, q$, minimum $s$ and one saddle point $r$. We assume that the heart $S^2$ admits an $\mathbb{Z}/2\mathbb{Z}$-action given by 180 degree rotation which interchanges $p$ and $q$ and fixes $r$ and $s$ and assume that this $\mathbb{Z}/2\mathbb{Z}$-action preserves $h$. Then, the quotient space $S^2/(\mathbb{Z}/2\mathbb{Z})$ is again topologically $S^2$. A naive $G$-invariant Morse chain complex is

$$0 \to < (p + q) > \to < r > \to < s > \to 0.$$ 

where $< (p + q) >$ denotes the one dimensional vector space generated by $p + q$. Here, the differential all vanishes, and the resulting homology does not equal to the singular homology of $S^2$.

A correct Morse chain complex for the quotient orbifold is

$$0 \to < (p + q) > \to 0 \to < s > \to 0.$$
Here, critical points $p, q, s$ is orientable whereas the critical point $r$ is non-orientable as the half-rotation reverses the orientation of the unstable manifold at $r$. Hence, we discard $<r>$ and do not use it as a generator in the above complex. In this way, we obtain a Morse-homology of $S^2/(\mathbb{Z}/2\mathbb{Z})$ isomorphic to the singular homology of $S^2$.

In fact, this is $G$-invariant part of the Morse chain complex of $M$ when the critical points of $M$ are decorated with orientation sheaf of unstable manifolds. Such a point of view will be explained in more detail in the last section in the Bott-Morse setting.

But this approach of taking invariant subcomplex obviously does not work for general orbifolds which are not global quotients.

3. INTRINSIC FORMULA

In order to extend the result to general orbifolds, we will reformulate the formula of $\partial$ for the global quotient orbifold $X$ to be more intrinsic form. To do this, consider two orientable critical points $\bar{p}$ and $\bar{q}$ of indices $k$ and $k-1$, respectively and suppose there exists a negative gradient flow line $\gamma$ from $\bar{p}$ to $\bar{q}$ in $X$. We want to find the contribution of $\gamma$ to the coefficient at $[\bar{q}]$ in $\partial[\bar{p}]$. Recall that $[\bar{p}] = \sum_{p \in \pi^{-1}(\bar{p})} p$. Let $\gamma$ be one of its liftings of $\gamma$.

**Lemma 3.1.** Given a negative gradient flow line $\gamma$ in $M$, the isotropy groups $G_x$ are all isomorphic for any point $x \in \gamma$.

**Proof.** Indeed, they are all the same because the diffeomorphism $\Phi_t$ of $M$ induced by the negative gradient vector field of $f$ is $G$-equivariant. More precisely, let $y$ be another point in $\gamma$. Since both $x$ and $y$ are not critical points of $f$, there exists $t$ such that $\Phi_t(x) = y$. Then, for any $g \in G$, $G$-equivariance of $\Phi_t$ implies $\Phi_t(g \cdot x) = g \cdot \Phi_t(x) = g \cdot y$ and hence $G_x = G_y$. □

By the above lemma, we may denote $G_x$ by $G_{\gamma}$ for $x \in \text{im} \gamma$. It is natural to define $G_{\gamma}$ as the conjugacy class of $G_{\gamma}$. Then, $|G_{\gamma}|$ is well-defined.

Note that the number of lifts of $\gamma$ in $M$ is $|G|/|G_{\gamma}|$. Hence, there exist $\left( \sum_{\bar{q} \in \pi^{-1}(\bar{q})} [G]/|G_{\gamma}| \right)$-negative flow lines connecting critical points of $\bar{p}$ and $\bar{q}$. We want the coefficient of $[\bar{q}] = \sum_{q \in \pi^{-1}(\bar{q})} q$ instead of one of single $q$, and therefore we divide $\sum_{\bar{q} \in \pi^{-1}(\bar{q})} [G]/|G_{\gamma}|$ by the number of $q'$s as $|G|^{|G_{\gamma}|}$. Note that all coefficients of $q'$s in the sum are equal because of $G$-action. Therefore:

$$
\partial[\bar{p}] = \sum_{\bar{q} \in \text{crit}_{G}^{}(f) \backslash \gamma} \sum_{\bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{1}{|\pi^{-1}(\bar{q})|} \frac{|G_{\bar{q}}|}{|G_{\gamma}|} [\bar{q}]
$$

$$
= \sum_{\bar{q} \in \text{crit}_{G}^{}(f) \backslash \gamma} \sum_{\bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{q}}|}{|G|} \frac{|G|}{|G_{\gamma}|} [\bar{q}]
$$

$$
= \sum_{\bar{q} \in \text{crit}_{G}^{}(f) \backslash \gamma} \sum_{\bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{q}}|}{|G_{\gamma}|} [\bar{q}]
$$

Here, by $G_{\bar{q}}$, we denote the conjugacy class of $G_q$, $q \in \pi^{-1}(\bar{q})$. The sum is taken over all orientable critical points $\bar{q}$ of index $k-1$. Also $\epsilon(\gamma) = \pm 1$ assigned to $\gamma$. The sign $\epsilon(\gamma)$ is
from the sign convention explained before. From now on, we use $\bar{p}$ itself instead of $[\bar{p}]$ for simplicity.

We denote

$$n(p, \bar{q}) := \sum_{\gamma : \bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}, \quad \nu_q(\bar{\gamma}) := \epsilon(\gamma) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|}.$$  

On a minimal chart around $\bar{q}$, the preimage of $\bar{\gamma}$ is $|\nu_q(\bar{\gamma})|$ copies of gradient flow lines which can be obtained by $G_{\bar{q}}$-action to a single lifting $\gamma$. (namely, the chart $(\bar{U}_{\bar{q}}, G_{\bar{q}}, \pi_{\bar{q}})$ in which $\bar{U}_{\bar{q}}$ is an open subset of an Euclidean space equipped with linear $G_{\bar{q}}$-action and we assume that there is unique lifting $q$ of $\bar{q}$ which is the origin).

So, $\nu_q(\bar{\gamma})$ can be regarded as a multiplicity of $\bar{\gamma}$ at $\bar{q}$ and $n(p, \bar{q})$ can be seen as the number of negative gradient flow lines from $\bar{p}$ to $\bar{q}$ counted with multiplicity or weight. We also denote $\nu_p(\bar{\gamma}) = \epsilon(\bar{\gamma}) \frac{|G_p|}{|G_{\bar{\gamma}}|}$, the number of liftings of $\bar{\gamma}$ in an uniformizing chart around $\bar{p}$ counting with signs.

Hence we obtain:

$$\partial \bar{p} = \sum_{\bar{q}} \left( \sum_{\gamma : \bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{q}}|}{|G_{\bar{\gamma}}|} \right) \bar{q} = \sum_{\bar{q}} \nu_{\bar{q}}(\bar{\gamma}) \bar{q} = \sum_{\bar{q}} n(\bar{p}, \bar{q}) \bar{q}. \quad (3.1)$$

We emphasize that $\nu_q(\bar{\gamma})$ and $n(p, \bar{q})$ make sense for an arbitrary orbifold $X$ with a given Morse-Smale function. Namely, coefficients of $(3.1)$ are intrinsic, only considering the critical points of $f$, gradient flow lines in the orbit space, and the local groups at each critical points. Also note that if the group action is trivial we get the usual formula of the Morse boundary operator. In the next section, we define a Morse-Smale-Witten complex of a general orbifold using the above formula.

We would like to introduce an alternative formula of the Morse boundary operator which is also intrinsic in the above sense. We simply use

$$\langle \bar{p} \rangle := \frac{|G_{\bar{p}}|}{|G|} \sum_{p \in \pi^{-1}(\bar{p})} \bar{p}$$

instead of $[\bar{p}]$. Note that $\langle \bar{p} \rangle$ can be seen as the average of $p$ with respect to $G$-action since $\frac{|G_{\bar{p}}|}{|G|}$ is the cardinality of the orbit containing $p$. With this slight change of generators, the boundary operator is computed as follows.

$$\partial \langle \bar{p} \rangle = \sum_{\bar{q} \in crit_{-1}^+(f)} \sum_{\gamma : \bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{p}}|}{|G_{\bar{\gamma}}|} \cdot \langle \bar{q} \rangle ,$$

which has a nice shape as much as the old one. To avoid the confusion, we will use $\partial \bar{q}$ for general orbifolds to denote the operator coming arise from the above choice of generators. i.e.

$$\partial \bar{p} := \sum_{\bar{q} \in crit_{-1}^+(f)} \sum_{\gamma : \bar{p} \rightarrow \bar{q}} \epsilon(\gamma) \frac{|G_{\bar{p}}|}{|G_{\bar{\gamma}}|} \cdot \bar{q}.$$  

Note that two homology groups are obviously isomorphic by the scaling

$$\psi : \bar{p} \mapsto |G_{\bar{p}}| \cdot \bar{p},$$
which is a chain map with respect to \((\partial, \partial)\) since we are using \(\mathbb{R}\)-coefficients for both sides.

4. \textbf{Morse-Smale-Witten complex of General Orbifolds}

From now on, let \(X\) be a compact oriented connected \(n\)-dimensional effective orbifold, which may not be a global quotient orbifold. It is known that we can still choose a Morse function on \(X\) (see [H]) in the sense of definition 2.1. Unfortunately, it is not always possible to choose it to be a Morse-Smale function. We assume that we already have a Morse-Smale function \(\bar{f}\) on \(X\). We write \(X\) the underlying orbit space of \(X\) and the same \(\bar{f}\) for the underlying continuous map on the orbit space \(X \to \mathbb{R}\).

We define the orientability of critical points as in Definition 2.2. The index of critical points can be defined, and these two notions are well-defined independent of choice of local uniformizing charts.

Again, we denote by \(\text{crit}_k^+ (\bar{f})\) by the set of all orientable critical points of \(\bar{f}\) of index \(k\). Regardless of \(\bar{p} \in \text{crit}(\bar{f})\) being orientable or not, we fix an orientation of \(W^-(\bar{p})\) for future reason. It can be happen that \(g \in G_{\bar{p}}\) reverses the given orientation on \(W^-(\bar{p})\). Let \(CM_k(X, \bar{f})\) be the \(\mathbb{R}\)-vector space generated by \(\text{crit}_k^+ (\bar{f})\).

By using the notation of (3.1), we define

\[
\partial \bar{p} = \sum_{\bar{q} \in \text{crit}_{k-1}^+ (\bar{f})} n(\bar{p}, \bar{q}) \bar{q} = \sum_{\bar{q} \in \text{crit}_{k-1}^+ (\bar{f})} \sum_{\bar{\gamma} \in M(\bar{p}, \bar{q})} \nu_\bar{q}(\bar{\gamma}) \bar{q},
\]

for \(\bar{p} \in \text{crit}_k^+ (\bar{f})\).

The main theorem of this section is

**Theorem 4.1.** \((CM_\ast (X, \bar{f}), \partial)\) defines a chain complex. i.e \(\partial^2 = 0\)

The proof of this theorem will occupy the rest of the section. Before we proceed for its proof, we first explain the main difference from the case of manifolds.

\[\text{Figure 1. limits of gradient flows for (a) manifolds and (b) orbifolds}\]

Recall that the standard Morse homology argument to show \(\partial^2 = 0\) uses the compactifications of moduli spaces of negative gradient flow lines between critical points whose index difference is 2. The same analysis still works on the uniformizing covers, but there is a crucial difference. As illustrated in the figure a given
broken trajectory (representing $\partial^2$) on $X$ can become limits of several smooth flow trajectories (different even after the local group action). This can be easily seen as in the following example.

**Example 4.2.** Let $\gamma$ be the negative gradient flow lines from $\bar{p}$ to $\bar{q}$ and $\delta$ be the one from $\bar{q}$ to $\bar{r}$. Assume $G_\gamma = G_\delta = 1$, for simplicity. Then, on an uniformizing neighborhood $(\bar{U}_\gamma, G_\gamma, \pi_\gamma)$, there are $|G_\gamma|$-flow lines which lift $\gamma$ and also $|G_\gamma|$-flow lines which lift $\delta$.

Choose $\gamma, \delta$ to be one of the flow trajectories covering $\gamma$ and $\delta$ in the cover $\bar{U}_\gamma$. For each $g_1, g_2 \in G_\gamma$, the choice of lifts $g_1 \cdot \gamma$ together with $g_2 \cdot \delta$ gives another broken trajectory in the cover.

In this way, we find $|G_\gamma|^2$-broken trajectories of the lifting $\bar{f}$ of $\bar{f}$ in $\bar{U}_\gamma$ and hence, we have $|G_\gamma|^2$-families of smooth gradient flow lines converging to each of $|G_\gamma|^2$-broken trajectories in $\bar{U}_\gamma$. Since $G_\gamma$-action on the set of broken trajectories in $\bar{U}_\gamma$ is free and so is on the set of (local) gluings in $\bar{U}_\gamma$, we get $|G_\gamma|(|G_\gamma|^2/|G_\gamma|)$-families after quotient by $G_\gamma$-action. We see that there are $|G_\gamma|$ different families of smooth trajectories converging to a single broken trajectory $(\gamma, \delta)$ near $\bar{q}$ in this case.

Recall that the convergence of 1-parameter family of negative gradient flow lines to a broken trajectory is given as uniform convergence in locally compact subsets. Also, the standard gluing lemma which guarantees that there exists an 1-parameter family of flow lines which converges to the given broken trajectory is also carried out locally. So, these facts are still true for general orbifolds. We are going to use these facts freely.

For effective orbifolds, gradient vector fields can be studied in the following way, which we learned from E. Lerman. For any orbifold $X$, its Frame bundle $Fr(X)$ is a smooth manifold with a smooth, effective, and almost free $O(n)$-action. Then, $X$ is naturally isomorphic to the quotient orbifold $Fr(X)/O(n)$ ( Theorem 1.23 of [ALR]). Or consider any manifold $M$ and compact Lie group $G$ with $X = M/G$ with smooth, effective, almost free $G$-action. Then, we can lift our Morse function $f : X \to \mathbb{R}$ to a function $\bar{f} : M \to \mathbb{R}$ simply by setting $\bar{f} := f \circ \pi$. Note that the critical submanifolds of $\bar{f}$ typically look like $\pi^{-1}(\bar{p}) \cong G/\bar{p}$ for $\bar{p} \in \pi^{-1}(\bar{p})$, $\bar{p} \in \text{crit}(\bar{f})$. The hessian of $\bar{f}$ along the normal direction to the critical submanifold $\pi^{-1}(\bar{p})$ precisely equals to that of $f$ at $\bar{p}$. Therefore, we can conclude that the lift $\bar{f} : M \to \mathbb{R}$ is Morse-Bott. Thus, the properties of gradient trajectories in $M$ are already well-known (see [AB]) including analytic properties of gradient trajectories of $\bar{f}$ such as convergence, gluing and so on. This implies the desired properties for gradient trajectories of $\bar{f}$.

**Remark 4.3.** Another way to define Morse-Smale-Witten complex for an orbifold $X$ would be to use $G$-equivariant Morse-Bott complex for $\bar{f} : M \to \mathbb{R}$. As discussed in remark [6.2] the construction in [AB] provides such a complex if the assumptions of [AB] are met, such as critical submanifolds are orientable, $G$-action preserves orientations of unstable and stable manifolds, and the submersiveness of evaluation maps from the trajectories. Even when all these conditions are met, the complex in [AB] uses Cartan model of $BG$, and it is not clear what is the relation of the differential there and geometric counting of gradient flow lines. For $\mathbb{R}$ or $\mathbb{Q}$-coefficients, the $G$-equivariant cohomology of $M$ in this setting is isomorphic to the singular homology of $M/G$ (Proposition 2.12 [ALR]). Hence it would be interesting to find...
a relation between the construction of $[AB]$ and the construction in this paper in the above setting.

Consider the moduli space of gradient flow trajectories between critical points of index difference two, which is of dimension one. The above example illustrates that the compactified moduli space in this case topologically, near each broken trajectory, is given by a join of several copies of interval $[0, 1)$ at 0's equipped with a $G_{\bar{q}}$ action. Also note that before compactification, the moduli space should be understood as an orbifold as the trajectories lies in the uniformizing covers with group actions on them. We also remark that the orbifold structures of each limiting trajectories to a given broken trajectory may not be isomorphic to each other in non-abelian cases.

To prove the main theorem, we first prove a couple of lemmas on stabilizers of the gradient flow trajectories.

First, we set the notations as follows: Consider $\bar{p} \in \text{crit}_+^k(\bar{f})$, $\bar{q}, \bar{q}' \in \text{crit}_{k-1}(\bar{f})$, $\bar{r} \in \text{crit}_+^{k-2}(\bar{f})$. Note that $\bar{q}$ and $\bar{q}'$ are not assumed to be orientable. Let $\bar{\gamma}$ (resp. $\bar{\gamma}'$) be negative gradient flow lines from $\bar{p}$ to $\bar{q}$ (resp. $\bar{q}'$) and let $\bar{\delta}$ (resp. $\bar{\delta}'$) be flow lines from $\bar{q}$ (resp. $\bar{q}'$) to $\bar{r}$. Suppose that two broken trajectories $(\bar{\gamma}, \bar{\delta})$ and $(\bar{\gamma}', \bar{\delta}')$ are connected by 1-parameter family of negative gradient flow lines from $\bar{p}$ to $\bar{r}$. Take the set of flows lines in the above 1-parameter family and call it $\mathcal{P}$.

**Remark 4.4.** Even if $\mathcal{P}$ flows between two orientable critical point (and hence will be oriented as we shall see below), breaking points (either $\bar{q}$ or $\bar{q}'$) of limiting broken trajectories are not necessarily orientable. This is the reason why we didn’t impose any condition on the orientability of $\bar{q}$ and $\bar{q}'$. Indeed, example 2.9 already shows this phenomenon. We shall see that, however, each of limiting broken trajectories has a well defined sign as a boundary of $\mathcal{P}$.

**Lemma 4.5.** $\mathcal{P}$ is an one dimensional oriented orbifold whose stabilizers $G_{\bar{\gamma}}$ are all isomorphic for each $\bar{\gamma} \in \mathcal{P}$. (This is an ineffective orbifold for nontrivial $G_{\bar{\gamma}}$.)

**Proof.** We first explain how to obtain a natural orbifold structure on $\mathcal{P}$. Fix $\bar{\gamma} \in \mathcal{P}$ and $t \in \mathbb{R}$. Let $(\bar{U}, G)$ be a uniformizing chart of some neighborhood $U \ni \bar{\gamma}(t)$ with the quotient map $\pi : \bar{U} \to U$ and $f$ be the lifting of $\bar{f}$ on $\bar{U}$. Consider the level set of $f = \bar{f} \circ \pi$, given as $f^{-1}(\bar{f}(\bar{\gamma}(t)))$. Consider the intersection of the level set $f^{-1}(\bar{f}(\bar{\gamma}(t)))$ with the corresponding gradient flow trajectories (from $\mathcal{P}$) in $\bar{U}$ and denote the intersection by $\mathcal{T}$. From the standard Morse theory, $\mathcal{T}$ is an open one dimensional manifold. Then, as we consider the moduli space (before compactification, after $\mathbb{R}$-quotient), $\mathcal{P}$ is given by $\mathcal{T}/G$ in the neighborhood $U$. Hence, we can give $\mathcal{P}$ an orbifold structure locally as a suborbifold of $\mathcal{X}$. This does not depend on the choice of $t \in \mathbb{R}$ by lemma 3.1. And the resulting orbifold is oriented as we consider flows between orientable critical points as explained in section 2.

Now we show that stabilizers are isomorphic to each other. But this is clear, since any connected one dimensional orientable orbifold satisfies such a property since a finite group action on an interval say $(-1, 1)$ is either identity or $x \mapsto -x$ up to diffeomorphism. But the latter cannot be orientation preserving. Hence local groups act trivially and hence the stabilizers are isomorphic to each other. \qed
Next, we consider compactification $\overline{\mathcal{P}}$ of each component $\mathcal{P}$ by adding limit broken trajectories $(\bar{\gamma}, \bar{\delta}), (\bar{\gamma}', \bar{\delta}')$ to $\mathcal{P}$. To consider the orbifold structure of $\overline{\mathcal{P}}$, we compare the stabilizers of the limiting trajectories and that of its limit.

Consider the uniformizing chart $(\bar{U}_{\bar{q}}, G_{\bar{q}}, \pi_{\bar{q}})$ around $\bar{q}$ with $U_{\bar{q}} = \pi_{\bar{q}}(\bar{U}_{\bar{q}})$. Let $\Gamma$ be the set of all liftings of $\bar{\gamma} \cap U_{\bar{q}}$ and $\Delta$ be that of $\bar{\delta} \cap U_{\bar{q}}$. Then $G_{\bar{q}}$ naturally acts on $\Gamma \times \Delta$ by the diagonal action. Recall that there is the unique gluing for a given broken trajectory in the uniformizing cover, and hence by diagonal action, the quotient set $\Gamma \times \Delta / G_{\bar{q}}$ can be seen as the set of all possible smooth trajectories converging to $(\bar{\gamma}, \bar{\delta})$ in $X$. Here, we can observe that there may be several different gluings for the single broken trajectory $(\bar{\gamma}, \bar{\delta})$. Hence,

**Lemma 4.6.** $\mathcal{P}$ determines an element of $\Gamma \times \Delta / G_{\bar{q}}$, say $[\gamma, \delta] \in \Gamma \times \Delta / G_{\bar{q}}$ and this correspondence is one to one locally around $\bar{q}$.

Now, consider $G_{\gamma}, G_{\delta}$ the isotropy groups of $\gamma$ and $\delta$ respectively. Their intersection $G_{\gamma} \cap G_{\delta} \subset G_{\bar{q}}$ is regarded as the isotropy group at the boundary point $(\bar{\gamma}, \bar{\delta})$ of $\mathcal{P}$. We denote its conjugacy class as $G_{[\gamma, \delta]}$. In general, the limit of isotropy groups are always a subgroup of the isotropy group at the limit point. For the moduli space of gradient flow trajectories, we also have the converse, which is crucial in proving $\partial^2 = 0$.

**Lemma 4.7.** $G_{[\gamma, \delta]} \cong G_{\bar{x}}$ for any $\bar{x} \in \mathcal{P}$.

*Proof.* We prove that $G_{[\gamma, \delta]} \cong G_{\bar{x}}$, for sufficiently close $\bar{x}$ to the boundary point $(\bar{\gamma}, \bar{\delta})$. This will be enough by lemma 4.5. Take a uniformizing neighborhood around $\bar{q}$, $(\bar{U}_{\bar{q}}, G_{\bar{q}}, \pi_{\bar{q}})$ and consider the lifting of one parameter family $\tilde{\mathcal{P}}$ converging to one of liftings $(\gamma, \delta)$ of $(\bar{\gamma}, \bar{\delta})$. Taking two different slice of $f$ one of which meets $\gamma$ and $\delta$ respectively, the usual continuity argument shows that $G_{\bar{x}} \subset G_{\gamma}$ and $G_{\bar{x}} \subset G_{\delta}$ for $x$ in the slice of $f$ project down to $\bar{x}$. Therefore $G_{\bar{x}} \subset G_{[\gamma, \delta]}$. Conversely, assume there exists $g \in G_{\bar{q}}$ which fixes $(\gamma, \delta)$ but does not fix $\tilde{\mathcal{P}}$. Then, $g \cdot \tilde{\mathcal{P}}$ would be a different family from $\tilde{\mathcal{P}}$ converging to the same limit $(\gamma, \delta)$. This is impossible from the standard Morse theory on the uniformizing chart. \hfill $\square$

**Remark 4.8.** Note that $G_{\gamma}$'s are conjugate to each other for liftings $\gamma$ of $\bar{\gamma}$ but the intersection $G_{\gamma} \cap G_{\delta}$ depends on each choice of lift and its cardinality may depend on the choice of the lift.

Therefore, the set $\overline{\mathcal{P}}$ maybe considered as ineffective orbifolds, where the same isotopy group acts on every points trivially. Also, it carries a natural orientation. As in the standard Morse theory, this can be used to prove that the natural orientation at the boundary broken trajectories of $\overline{\mathcal{P}}$ are opposite to each other. To be more precise, we introduce a sign rule for “boundary” of $\overline{\mathcal{P}}$. We will in fact show that $\Gamma \times \Delta$ inherits a sign rule from $\Gamma \times \Delta$. Namely, we have:

**Lemma 4.9.** For $(\gamma, \delta) \in \Gamma \times \Delta$, define $\epsilon(\gamma, \delta)$ as the product of $\epsilon(\gamma)$ and $\epsilon(\delta)$. Whether $\bar{q}$ is orientable or not, $G_{\bar{q}}$-action on $\Gamma \times \Delta$ preserves $\epsilon(\gamma, \delta)$, i.e.

$$\epsilon(g \cdot \gamma, g \cdot \delta) = \epsilon(\gamma, \delta)$$

for all $g \in G_{\bar{q}}$.

*Proof.* Suppose $g \in G_{\bar{q}}$ reverses the orientation of $W^-(q)$. (Otherwise, there’s nothing to prove.) Since $\bar{p}$ and $\bar{q}$ are both orientable, exactly the same argument
in lemma 2.6 shows that \(\epsilon(g \cdot \gamma) = -\epsilon(\gamma)\) and \(\epsilon(g \cdot \delta) = -\epsilon(\delta)\). This proves the lemma.

From the lemma, the following sign rule makes sense.

**Definition 4.10.** For \([\gamma, \delta] \in \Gamma \times \Delta/G\bar{q}\), we assign a sign to it as
\[
\epsilon[\gamma, \delta] := \epsilon(\gamma, \delta) = \epsilon(\gamma) \cdot \epsilon(\delta).
\]

If \(\bar{q}\) is orientable, we can give \(\bar{\gamma}\) and \(\bar{\delta}\) well-defined signs. Clearly, \(\epsilon[\gamma, \delta] = \epsilon(\bar{\gamma}) \cdot \epsilon(\bar{\delta})\) for all \([\gamma, \delta] \in \Gamma \times \Delta/G\bar{q}\) since \(G\bar{q}\)-action preserves all signs in concern. So, the orientation of broken trajectories \((\bar{\gamma}, \bar{\delta})\) as a boundary of any component of \(\mathcal{P}\) converging to it is given by \(\epsilon(\bar{\gamma}) \cdot \epsilon(\bar{\delta})\).

Consequently, the orientation issue of \(\mathcal{P}\) can be rephrased as follows:

**Lemma 4.11.** If there is an one parameter family \(\mathcal{P}\) which corresponds to \([\gamma, \delta]\) and \([\gamma', \delta']\) in the sense of lemma 4.6, \(\epsilon[\gamma, \delta]\) and \(\epsilon[\gamma', \delta']\) should be opposite.

As we said, the proof is not different from the classical one at all. (see [AB] for example). This sign cancellation is the base of proving \(\partial^2 = 0\) in smooth case. But, to count gradient flow trajectories and to describe cancellation phenomenon in orbifold case correctly, we should take a weighted sum to take into account the orbifold structure.

**Definition 4.12.** For the compactification \(\overline{\mathcal{P}}\) as above, the following expression will be called the weighted boundary of \(\overline{\mathcal{P}}\):
\[
\partial \overline{\mathcal{P}} = \left[\epsilon[\gamma, \delta]/[G[\gamma, \delta]]\right](\bar{\gamma}, \bar{\delta}) + \left[\epsilon[\gamma', \delta']/[G[\gamma', \delta']]\right](\bar{\gamma}', \bar{\delta}').
\]

We call the numbers \(\epsilon[\gamma, \delta]/[G[\gamma, \delta]]\), \(\epsilon[\gamma', \delta']/[G[\gamma', \delta']]\) the weights and write them as \(\omega_\mathcal{P}(\bar{\gamma}, \bar{\delta})\), \(\omega_\mathcal{P}(\bar{\gamma}', \bar{\delta}')\), respectively.

Now, the standard arguments of proving \(\partial^2 = 0\) in the smooth case together with the above choice of weights gives the following equation. (lemma 4.7)
\[
\sum_{(\zeta, \eta) \in \partial \overline{\mathcal{P}}} \omega_\mathcal{P}(\zeta, \eta) = 0 \tag{4.1}
\]

Denote by \(\overline{\mathcal{M}}(\bar{p}, \bar{r})\) the compactified moduli space of negative gradient flow lines from \(\bar{p}\) to \(\bar{r}\). Geometrically, as explained in the beginning of the section, this is given by several copies of compact intervals (equipped with trivial actions of corresponding isotropy groups) which are joined at boundary points if they define families of flow lines whose limits at that boundary coincide. Also note that the limiting flows to a fixed broken trajectory might have non-isomorphic stabilizers by lemma 4.7. So we cannot really think of \(\overline{\mathcal{M}}(\bar{p}, \bar{r})\) as orbifold with boundary. This is somewhat different from the smooth case where compactified moduli spaces are manifolds with corners. Denote \(\partial \overline{\mathcal{M}}(\bar{p}, \bar{r}) := \overline{\mathcal{M}}(\bar{p}, \bar{r}) - \mathcal{M}(\bar{p}, \bar{r})\).

**Definition 4.13.** If \((\zeta, \eta) \in \partial \overline{\mathcal{M}}(\bar{p}, \bar{r})\), we define
\[
\omega(\zeta, \eta) := \sum_{\mathcal{P} \text{ with } (\zeta, \eta) \in \partial \mathcal{P}} \omega_\mathcal{P}(\zeta, \eta), \tag{4.2}
\]
where the sum is taken over all 1-parameter family $\mathcal{P}$ one of whose boundary is $(\bar{\zeta}, \bar{\eta})$. Finally, we denote the sum of all weight associated to the gluings converging to one of broken trajectories through $\bar{\nu}$, $\bar{\nu}'$ and $\bar{\nu}''$ as

$$\omega(\bar{\nu}, \bar{\nu}', \bar{\nu}'') := \sum_{(\bar{\zeta}, \bar{\eta}) \in \partial M(\bar{\nu}, \bar{\nu}') \times M(\bar{\nu}', \bar{\nu}'')} \omega(\bar{\zeta}, \bar{\eta}).$$

For $\partial \overline{M}(\bar{\nu}, \bar{\nu}'')$ (which is the set of all broken trajectories from $\bar{\nu}$ to $\bar{\nu}'$), note that we have $M(\bar{\nu}, \bar{\nu}) \times M(\bar{\nu}, \bar{\nu}'') \subset \partial \overline{M}(\bar{\nu}, \bar{\nu}'')$ for any $\bar{\nu}$.

Since all intervals contained in $\overline{M}(\bar{\nu}, \bar{\nu}'')$ are oriented so that they are compatible with their boundary orientations, all terms in the sum of (4.2) have the same signs.

From (4.1), we get the following equality.

$$\sum_{\bar{\nu} \in \text{crit}_{k-1}(\bar{\nu})} \omega(\bar{\nu}, \bar{\nu}') = \sum_{(\bar{\zeta}, \bar{\eta}) \in \partial \overline{M}(\bar{\nu}, \bar{\nu}'')} \omega(\bar{\zeta}, \bar{\eta}) = \sum_{\mathcal{P}} \sum_{(\bar{\zeta}, \bar{\eta}) \in \partial \overline{P}} \omega_{\mathcal{P}}(\bar{\zeta}, \bar{\eta}) = 0$$

The following figure explains how we sum up weighted contributions near orientable critical points. The lines in the figure represent (oriented)1-dimensional moduli spaces, and these converge to broken trajectories, which are drawn as $\circ$’s. In the figure (b), $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$ contributes to $\omega(\bar{\nu}, \bar{\nu}'', \bar{\nu}'')$, and summation on a neighboring dotted circle contributes to $\omega(\bar{\nu}, \bar{\nu}'', \bar{\nu}'')$.

Near an unorientable critical points, additional cancellation phenomenon as in lemma 2.6 occurs, and this will be explained below in (4.5).

Now, we are ready to prove our main theorem 4.1.
Proof. Observe that
\[ \nu_{\tilde{\rho}}(\tilde{\delta}) = \epsilon(\tilde{\delta}) \frac{|G_{\tilde{\delta}}|}{|G_{\tilde{\delta}}|} = \frac{|G_{\tilde{\delta}}|}{|G_{\tilde{\delta}}|} \cdot \nu_{\tilde{\rho}}(\tilde{\delta}). \]

Therefore,
\[
\partial^2 \tilde{p} = \partial \left( \sum_{\tilde{\rho} \in \text{crit}_{k-1}(f) \cap \mathcal{M}(\tilde{p}, \tilde{q})} \sum_{\tilde{\gamma}} \nu_{\tilde{\rho}}(\tilde{\gamma}) \right) \\
= \sum_{\tilde{\rho} \in \text{crit}_{k-1}(f)} \left( \sum_{\tilde{\gamma}, \tilde{\delta}} \nu_{\tilde{\rho}}(\tilde{\gamma}) \nu_{\tilde{\rho}}(\tilde{\delta}) \right) \tilde{r} \\
= \sum_{\tilde{\rho} \in \text{crit}_{k-1}(f)} |G_{\tilde{\rho}}| \left( \sum_{\tilde{\gamma}, \tilde{\delta}} \nu_{\tilde{\rho}}(\tilde{\gamma}) \nu_{\tilde{\rho}}(\tilde{\delta}) \right) \tilde{r} \\
= \sum_{\tilde{\rho} \in \text{crit}_{k-1}(f)} |G_{\tilde{\rho}}| \left( \sum_{\tilde{\gamma}, \tilde{\delta}} \omega(\tilde{\rho}, \tilde{\gamma}, \tilde{\delta}) \right) \tilde{r}
\]
where the last sum is taken over all broken trajectories \((\tilde{\gamma}, \tilde{\delta})\) through \(\tilde{p}, \tilde{q}\) and \(\tilde{r}\). The last equality follows from the lemma below, which directly implies the theorem.

Lemma 4.14. If \(\tilde{q}\) is orientable, then
\[ \sum_{(\tilde{\gamma}, \tilde{\delta})} \frac{\nu_{\tilde{\rho}}(\tilde{\gamma}) \nu_{\tilde{\rho}}(\tilde{\delta})}{|G_{\tilde{\rho}}|} = \omega(\tilde{p}, \tilde{q}, \tilde{r}), \]
and if \(\tilde{q}\) is unorientable, then \(\omega(\tilde{p}, \tilde{q}, \tilde{r}) = 0\). Therefore,
\[ \sum_{\tilde{\gamma}, \tilde{\delta}} \omega(\tilde{p}, \tilde{q}, \tilde{r}) = \sum_{\tilde{\gamma}, \tilde{\delta}} \omega(\tilde{p}, \tilde{q}, \tilde{r}) = 0. \]

Proof. The first identity is nothing but a direct application of the weighted version of the orbit counting lemma 4.15. As before, in the uniformizing chart \((\tilde{U}_{\tilde{q}}, G_{\tilde{q}}, \pi_{\tilde{q}})\) around \(\tilde{q}\) let \(\Gamma\) be the set of all liftings of \(\tilde{\gamma} \cap \pi(\tilde{U}_{\tilde{q}})\) and \(\Delta\) be that of \(\tilde{\delta} \cap \pi(\tilde{U}_{\tilde{q}})\). The quotient space is the set of all gluings of \((\tilde{\gamma}, \tilde{\delta})\) in \(X\). From 4.3 \(\omega(\tilde{p}, \tilde{q}, \tilde{r})\) is equivalent to the “weighted” number of elements of the space \(\bigcup_{\Gamma} \Gamma \times \Delta / G_{\tilde{q}}\) for all broken trajectories \((\tilde{\gamma}, \tilde{\delta})\) through \(\tilde{q}\), where the weight \([\gamma, \delta] \in \Gamma \times \Delta / G_{\tilde{q}}\) is given by \(\frac{\epsilon(\gamma)}{\omega(\gamma, \delta)}\). Thus, the weighted number of elements in \(\Gamma \times \Delta / G_{\tilde{q}}\) is
\[ \frac{1}{|G_{\tilde{q}}|} \sum_{(\gamma, \delta) \in \Gamma \times \Delta} \epsilon(\gamma, \delta) \cdot |\{ g \in G_{\tilde{q}} | g \cdot \gamma = \gamma, g \cdot \delta = \delta \}| \]
by lemma 4.15 and since \( |\{ g \in G_{\tilde{q}} | g \cdot \gamma = \gamma, g \cdot \delta = \delta \}| = |G_{\gamma} \cap G_{\delta}| = |G_{[\gamma, \delta]}| \), it equals to
\[ \frac{1}{|G_{\tilde{q}}|} \sum_{(\gamma, \delta) \in \Gamma \times \Delta} \epsilon(\gamma, \delta) = \frac{1}{|G_{\tilde{q}}|} \sum_{(\gamma, \delta) \in \Gamma \times \Delta} \epsilon(\gamma) \cdot \epsilon(\delta). \]
If \(\tilde{q}\) is orientable, \(\epsilon(\tilde{\gamma}) \cdot \epsilon(\tilde{\delta})\) is constant for all \((\gamma, \delta) \in \Gamma \times \Delta\) so that
\[
\frac{1}{|G_{\bar{q}}|} \sum_{(\gamma,\delta) \in \Gamma \times \Delta} \epsilon[\gamma,\delta] = \frac{\epsilon(\bar{\gamma})\epsilon(\bar{\delta})}{|G_{\bar{q}}|} \sum_{(\gamma,\delta) \in \Gamma \times \Delta} 1
\]
\[
= \epsilon(\bar{\gamma})\epsilon(\bar{\delta}) \cdot |\Gamma \times \Delta|
\]
\[
= \epsilon(\bar{\gamma})\epsilon(\bar{\delta}) \cdot |G_{\bar{q}}| \cdot |G_{\bar{q}}|
\]
\[
= \frac{\nu_{\bar{q}}(\bar{\gamma})\nu_{\bar{q}}(\bar{\delta})}{|G_{\bar{q}}|}.
\]

Therefore, \(\omega(\bar{p}, \bar{q}, \bar{r}) = \sum_{(\bar{\gamma},\bar{\delta})} \nu_{\bar{q}}(\bar{\gamma})\nu_{\bar{q}}(\bar{\delta}) \cdot |G_{\bar{q}}|\), if \(\bar{q}\) is orientable.

On the other hand, suppose \(\bar{q}\) is unorientable. Pick any \(g \in G_{\bar{q}}\) which reverses the orientation of \(W^{-}(p)\). Then, \(g\) gives a permutation \(\Gamma \times \Delta\) by \(g \cdot (\gamma, \delta) := (g \cdot \gamma, \delta)\).

Note that \(\epsilon(g \cdot \gamma) \cdot \epsilon(\delta) = -\epsilon(\gamma) \cdot \epsilon(\delta)\).

By the same argument in the case of global quotients (lemma 2.6), the number of elements in \(\Gamma \times \Delta\) which have positive signs should agree with the number of elements with negative signs. Thus, \(\sum_{(\gamma,\delta) \in \Gamma \times \Delta} \epsilon(\gamma) \cdot \epsilon(\delta) = 0\) and \(\omega(\bar{p}, \bar{q}, \bar{r}) = 0\) when \(\bar{q}\) is not orientable.

**Lemma 4.15.** Let \(X\) be a finite set on which a finite group \(G\) acts and suppose \(X/G\) is a weighted set so that each element \(\bar{x} \in X/G\) has the weight \(\lambda_{\bar{x}}\). Then,
\[
\sum_{\bar{x} \in X/G} \lambda_{\bar{x}} = \frac{1}{|G|} \sum_{x \in X} \lambda_{x} |G_{x}|.
\]

**Proof.** Just follow the standard proof of the Burnside’s lemma. \(\square\)

**Remark 4.16.** The proof of \(\partial^{2} = 0\) is similar. Indeed, this is automatic since we have a \((\partial, \partial^{2})\)-chain map \(\psi: \bar{p} \mapsto |G_{\bar{p}}| \cdot \bar{p}\) which is an \((\mathbb{R}\text{-vector space})\) isomorphism.

5. Comparison with the Homology of the Orbit Space

In this section, we show that the homology of the Morse-Smale-Witten complex of general orbifolds \((CM_{*}(X, \bar{f}), \partial)\) equals the singular homology of the orbit space.

We assume in this section that \(\bar{f}\) is self-indexing, meaning that \(\bar{f}(\bar{p}_{i}) = \lambda_{i}\), \(\lambda_{i}\) the Morse index of \(\bar{p}_{i}\).

**Remark 5.1.** For the general case without self-indexing assumption, one may use the filtration
\[
X_{k} := \bigcup_{\text{ind}(\bar{p}) \leq k} W^{-}(\bar{p}),
\]
\[
\phi = X_{-1} \subset X_{0} \subset X_{1} \subset \cdots \subset X_{n-1} \subset X_{n} = X
\]
instead of the one described below and proceed as in [S], where \(W^{-}(\bar{p})\) is given in [5.2]. Note that \(X_{k}\) is compact.

We will apply the topological method of [Ni] which uses the cell structure of \(X\) induced by Morse data of \(f\). This kind of cell structure was already revealed by several authors, for example [LT] and [H].
Theorem 5.2. (Theorem 7.6 of [H]) Let $\tilde{p} \in \text{crit}_k(f)$ and $f(\tilde{p}) = c$. Suppose $\tilde{p}$ is the only critical point in $f^{-1}[c-\epsilon, c+\epsilon]$ for small $\epsilon < \frac{1}{2}$. Then, $f^{-1}(-\infty, c-\epsilon]$ is homotopic to $f^{-1}(-\infty, c+\epsilon]$ along $D^k/G_{\tilde{p}}$ attached with $\partial D^k/G_{\tilde{p}}$. Here, $D^k$ is a small invariant disc in the unstable manifold in a uniformizing chart around $\tilde{p}$ and hence endowed with $G_{\tilde{p}}$-action.

Proof. See theorem 7.6 of [H] and compare it with 3.2 of [M].

We need an elementary fact of equivariant topology to compute homological information of attaching cells.

Theorem 5.3. (Theorem 2.4 of [BR]) Let $K$ be a (regular) $G$-simplicial complex with $G$ finite and $L$ be a subcomplex. Then,

$$H_* (K, L; \mathbb{R})^G \cong H_* (K/G, L/G; \mathbb{R}),$$

where the left hand side means the subset of $H_* (K, L; \mathbb{R})$ fixed by $G$.

Corollary 5.4. Let $D^n$ be the $n$-dimensional disc and the finite group $\Gamma$ act on $(D^n, \partial D^n)$. Then, the homology group $H_* (D^n/\Gamma, \partial D^n/\Gamma; \mathbb{R})$ are given as follows.

$$H_* (D^n/\Gamma, \partial D^n/\Gamma; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } i = n \text{ and } \Gamma \text{ preserves the orientation of } D^n \\ 0 & \text{otherwise} \end{cases}$$

Proof. It suffices to note that there exists a $\Gamma$-invariant triangulation of $(D^n, \partial D^n)$ by [H] and that we can achieve the regularity condition in the theorem by subdivisions.

Recall that in smooth case, the coefficient of $q$ in $\partial p$ is defined from the (relative) intersection number between the unstable of $p$ and the stable manifold of $q$ (see for example [Ni]). In what follows, we will use instead the integration of Thom forms as they are more suitable in the orbifold setting. Recall from [CR] (or [ALR]) that Thom form of suborbifold $N$ of $X$ is defined locally as an invariant Thom forms of the preimage of $N$ in each uniformizing chart. Let $N$ denote the underlying space of $N$.

Remark 5.5. On an uniformizing chart, integration of Thom form (which defined by that of the preimage $\tilde{N}$ of $N$) along a normal fiber of $\tilde{N}$ at $p \in \tilde{N}$ is 1 where $\pi(p) = \tilde{p}$, according to the usual definition of Thom forms on Euclidean spaces. Hence, the orbifold integration of the Thom form along a normal fiber of $N$ in $X$ at $p$ is $1/G_{\tilde{p}}$. See [ALR] for more details about Thom forms and Poincare duals of suborbifolds.

We will also need the Stokes theorem for orbifolds, which goes back to [Sa]. We recall it here for readers convenience. A $C^\infty$ singular simplex $s$ of dimension $k$ in $X$ is defined by a smooth map $\tilde{s}$ from a $k$-dimensional simplex $\Delta_k$ to $X$. Suppose the image of $\tilde{s}$ lies in a single uniformizing chart $(\tilde{U}, G, \pi)$ so that it admits a lifting $s : \Delta_k \to \tilde{U}$ with $\pi \circ s = \tilde{s}$. Consider a $k$-form $\tilde{\omega}$ on $\pi(\tilde{U})$, which is given by an invariant $k$-form $\omega$ on $\tilde{U}$. We define

$$\int_{\tilde{s}} \tilde{\omega} = \int_{\Delta_k} s^* \omega.$$

For general $\tilde{s}$, use a partition of unity to define $\int_{\tilde{s}} \tilde{\omega}$. One can prove Stokes formula:

$$\int_{\tilde{s}} d\omega = \int_{\partial \tilde{s}} \tilde{\omega}.$$
Proposition 5.6. The homology of \((CM_\ast(X, \bar{f}), \partial)\), which is constructed in section 4, equals the singular homology of the underlying space \(X\).

Proof. We begin with the filtration of singular homology of \(X\). Let \(X_k = \bar{f}^{-1}(-\infty, k + 1 - \epsilon), 0 < \epsilon \ll 1\), and \(Y_k = \bar{f}^{-1}[k - \epsilon, k + 1 - \epsilon]\). Then,

\[
\tag{5.1} C_\ast(X_0; \mathbb{R}) \subset C_\ast(X_1; \mathbb{R}) \subset \cdots \subset C_\ast(X_n; \mathbb{R}) = C_\ast(X; \mathbb{R})
\]
gives a filtration on the singular chain complex \(C_\ast(X; \mathbb{R})\). For a critical point \(\bar{p}\) with \(\bar{f}(\bar{p}) = k\), let \(W^\pm(\bar{p})\) the stable and unstable manifolds at \(\bar{p}\), respectively. i.e.

\[
\tag{5.2} W^\pm(\bar{p}) = \{x \in X : \lim_{t \to \pm \infty} \Phi_t(x) = \bar{p}\}.
\]

Set \(D^\pm(\bar{p}) = W^\pm(\bar{p}) \cap Y_k\). Then, topologically (since \(\epsilon\) is small enough)

\[
D^\pm(\bar{p}) \cong D^\pm(p)/G_{\bar{p}},
\]

where \(D^\pm(p)\) are small invariant neighborhoods of \(p \in \pi^{-1}(\bar{p})\) in stable and unstable manifolds of \(p\) with respect to the lift \(f\) of \(\bar{f}\). By \(\partial D^\pm(\bar{p})\), we mean the image of \(\{\partial D^\pm(p)\}/G_{\bar{p}}\), equivalently

\[
\partial D^+(\bar{p}) = D^+(\bar{p}) \cap \{\bar{f} = k + 1 - \epsilon\},
\]

\[
\partial D^-(\bar{p}) = D^-(\bar{p}) \cap \{\bar{f} = k - \epsilon\}.
\]

By the excision, we have (see [III])

\[
H_\ast(X_k, X_{k-1}; \mathbb{R}) = \begin{cases} \bigoplus_{\bar{p} \in \text{crit}_k(\bar{f})} H_k(D^-(\bar{p}), \partial D^-(\bar{p}); \mathbb{R}) & * = k \\ 0 & \text{otherwise} \end{cases}
\]

From [5.4], \(H_k(D^-(\bar{p}), \partial D^-(\bar{p}); \mathbb{R}) \cong H_k(D^-(p), \partial D^-(p))^{G_{\bar{p}}}\) is isomorphic to \(\mathbb{R}\) if \(\bar{p}\) is orientable and vanishes otherwise.

The \(E^1\)-terms of (5.1) are the following chain complex.

\[
\cdots \rightarrow H_{k+1}(X_{k+1}, X_k; \mathbb{R}) \rightarrow H_k(X_k, X_{k-1}; \mathbb{R}) \rightarrow H_{k-1}(X_{k-1}, X_{k-2}; \mathbb{R}) \rightarrow \cdots
\]

where the boundary map is given by the composition

\[
H_k(X_k, X_{k-1}; \mathbb{R}) \rightarrow H_{k-1}(X_{k-1}; \mathbb{R}) \rightarrow H_{k-1}(X_{k-1}, X_{k-2}; \mathbb{R})
\]

Let \(st(W^-(\bar{p}))\) be the conjugacy class represented by the subgroup

\[
\{g \in G_{\bar{p}} : g \cdot x = x, \forall x \in D^-(p)\}
\]
of \(G_{\bar{p}}\). We now choose a generator of \(H_k(X_k, X_{k-1}; \mathbb{R})\) to be a multiple of

\[
(D^-(\bar{p}), \partial D^-(\bar{p})) \subset (X_k, X_{k-1})
\]

for \(\bar{p} \in \text{crit}^+_k(\bar{f})\). More precisely, we denote \(\frac{1}{|st(W^-(\bar{p}))|}\)-times this generator by \(|\bar{p}|\) to get rid of the influence of the ineffective action of \(G_{\bar{p}}\) on \(D^-(p)\). i.e.

\[
|\bar{p}| := \frac{1}{|st(W^-(\bar{p}))|} \cdot [(D^-(\bar{p}), \partial D^-(\bar{p}))] \in H_k(X_k, X_{k-1}; \mathbb{R}).
\]

Similarly, we set \(|q| = \frac{1}{|st(W^-(\bar{q}))|} \cdot [(D^-(\bar{q}), \partial D^-(\bar{q}))]|\) for \(\bar{q} \in \text{crit}^+_k(\bar{f})\), generating \(H_k(X_{k-1}, X_{k-2}; \mathbb{R})\). Then, there exist real numbers \(a_\bar{q}\) for each \(\bar{q}\) such that

\[
\tag{5.3} \partial |\bar{p}| = \sum_{\bar{q} \in \text{crit}^+_k(\bar{f})} a_\bar{q} |\bar{q}|,
\]
or, equivalently, there exist \( a_\bar{q} \) for each \( \bar{q} \) satisfying

\[
(5.4) \quad \frac{1}{|st(W^- (\bar{p}))|} \cdot [\partial D^- (\bar{p})] = \sum_{\bar{q} \in crit_{k-1}^-(f)} \frac{a_\bar{q}}{|st(W^- (\bar{q}))|} \cdot [(D^- (\bar{q}), \partial D^- (\bar{q})].
\]

We remark that the following proof is rather complicated than that of \([\text{Ni}]\) for the case of manifolds, as we are working with quotient spaces in the above filtration.

Now, it is enough to show that

\[
a_\bar{q} = n(\bar{p}, \bar{q}) = \sum_{\gamma \in \mathcal{M}(\bar{p}, \bar{q})} \epsilon(\gamma) \frac{|G_\gamma|}{|G_{\gamma}|}.
\]

For this, we will consider chains \( \partial |\bar{p}| \), and \( |\bar{q}| \) in the subspace \( Y_{k-1} \) of \( X \) and use Thom form of \( D^+ (\bar{q}) \) to identify the constant \( a_\bar{q} = n(\bar{p}, \bar{q}) \). More precisely, consider the following two sets,

\[
\hat{Y}_{k-1} := Y_{k-1} \setminus \{ \bar{f} = k - 1 - \epsilon \}
\]

and

\[
\hat{D}^+ (\bar{q}) := D^+ (\bar{q}) \setminus \partial D^+ (\bar{q}) \subset \hat{Y}_{k-1}.
\]

Since \( \bar{f} \) is invariant, \( \hat{Y}_{k-1} \) carries a natural orbifold structure inherited from \( X \) and \( \hat{D}^+ (\bar{q}) \) can be considered as a suborbifold of \( \hat{Y}_{k-1} \). So one can take \( \eta_{\bar{q}} \), a Thom form of \( D^+ (\bar{q}) \) on \( \hat{Y}_{k-1} \). On the other hand, setting \( D^- (\bar{p}) := W^- (\bar{p}) \cap \bar{f}^{-1}[k - \epsilon, \infty) \) and \( \partial D^- (\bar{p}) := D^- (\bar{p}) \cap \bar{f}^{-1}(k - \epsilon', \infty) \) for \( \epsilon < \epsilon' \), we may identify

\[
\partial |\bar{p}| = \frac{1}{|st(W^- (\bar{p}))|} \partial D^- (\bar{p}),
\]

by flowing \( \partial |\bar{p}| \) along negative gradient flows from \( \bar{f}^{-1}(\epsilon) \) to \( \bar{f}^{-1}(\epsilon') \). This is to consider possible intersections of \( \partial D^- (\bar{p}) \) and \( \hat{D}^+ (\bar{q}) \).

For simplicity, we denote

\[
b_\bar{p} := |st(W^- (\bar{p}))|, \quad b_\bar{q} := |st(W^- (\bar{q}))|.
\]

Now, by the definition of \( a_\bar{q} \) in the identity \((5.4)\), we have a formal sum of simplicial complexes \( K \) which maps (say, via \( \tau \)) to \( \hat{Y}_{k-1} \), whose boundary \( \tau : \partial K \to Y_{k-1} \) is given by

\[
\frac{1}{b_\bar{p}} \partial D^- (\bar{p}) \cup \bigcup_\bar{q} \frac{a_\bar{q}}{b_\bar{q}} D^- (\bar{q}),
\]

(with the opposite orientation on the first component) union some subset of \( X_{k-2} \) as its image.

Here, we consider \( \tau : K \to \hat{Y}_{k-1} \), \( \partial D^- (\bar{p}) \) and \( D^- (\bar{q}) \) as singular chains on \( \hat{Y}_{k-1} \). The rational coefficients can appear as we work with \( \mathbb{R} \)-coefficients in singular homology. By subdividing simplices repeatedly if necessary, we may assume that the map \( \tau \) when restricted to each simplex in \( K \) has a lift in some uniformizing charts of \( X \).

Then, Stokes theorem of \([\text{Sa}]\) tells us that

\[
\int_{\partial K} \tau^* \eta_{\bar{q}} = \int_K d(\tau^* \eta_{\bar{q}}) = \int_K \tau^* (d\eta_{\bar{q}}) = 0
\]
We will compute the integral on the left hand side to get $a_{\tilde{q}}$. Since the support of $\eta_{\tilde{q}}$ can be shrunken so that it lies in an arbitrary small open neighborhood of $\tilde{D}^+(\tilde{q})$, (and since for each $\tilde{q}$, $\tilde{D}^+(\tilde{q})$’s are disjoint,) 

\begin{equation}
\int_{\partial K} \tau^* \eta_{\tilde{q}} = -\int_I \tau^* \eta_{\tilde{q}} + \int_J \tau^* \eta_{\tilde{q}} = 0,
\end{equation}

where $I$ and $J$ are sub-complexes of $\partial K$ mapping to $|\bar{p}| = \frac{1}{b_\tilde{p}} \partial D^-(\bar{p})$ and $a_{\tilde{q}} \eta_{\bar{q}} = \frac{a_{\tilde{q}}}{b_{\tilde{q}}} \tilde{D}^-(\tilde{q})$, respectively. The numbers $1/b_{\tilde{q}}$ and $a_{\tilde{q}}/b_{\tilde{q}}$ here are considered to be coefficients of singular chains.

To compute the integral over $I$, we may assume $\epsilon$ is so small that there exists an uniformizing chart $(\tilde{U}_\tilde{p}, G_\tilde{p}, \pi_\tilde{p})$ around $\tilde{p}$ with $\partial D^-(\tilde{p}) \subset \pi_\tilde{p}(\tilde{U}_\tilde{p})$. Let $\partial D^-(p)$ be the preimage of $\partial D^-(\tilde{p})$ and $\bar{\eta}_{\tilde{q}}$ represent $\eta_{\tilde{q}}$ on $\tilde{U}_\tilde{p}$. Then, the preimage of $\tilde{D}^+(\tilde{q})$, say $\tilde{D}^+(q)$, on $U$ will meet $\partial D^-(p)$, \[ \left( \sum_{\gamma: \tilde{p} \to \tilde{q}} \frac{\epsilon(\gamma)|G_{\tilde{p}}|}{|G_{\gamma}|} \right) \times \text{times}. \]

Obviously, $\int_I \tau^* \eta_{\tilde{q}}$ equals to $1/b_{\tilde{p}} \times \text{times}$ the integral of $\eta_{\bar{q}}$ over $\partial D^-(\bar{p})$, where we mean by $\int_{\partial D^-(p)} \eta_{\bar{q}}$ the integration of $\bar{\eta}_{\tilde{q}}$ over $\partial D^-(p)$ divided by $|G_{\tilde{p}}|/b_{\tilde{p}}$. Here, the division is due to the degree of the projection map $\pi_{\tilde{p}}: \partial D^-(\bar{p}) \to \partial D^-(p)$ since $G_{\tilde{p}}/st(W^-_{\tilde{p}})$ acts on $\partial D^-(p)$ effectively. By the definition of $\eta_{\tilde{q}}$, we get

\begin{align*}
\int_I \tau^* \eta_{\tilde{q}} &= \frac{1}{b_{\tilde{p}}} \cdot \frac{b_{\bar{p}}}{|G_{\tilde{p}}|} \int_{\partial D^-(p)} \bar{\eta}_{\tilde{q}} \\
&= \frac{1}{|G_{\tilde{p}}|} \cdot \left( \sum_{\gamma: \tilde{p} \to \tilde{q}} \frac{\epsilon(\gamma)|G_{\tilde{p}}|}{|G_{\gamma}|} \right) \cdot \int_F \bar{\eta}_{\tilde{q}} \\
&= \sum_{\gamma: \tilde{p} \to \tilde{q}} \frac{\epsilon(\gamma)|G_{\tilde{p}}|}{|G_{\gamma}|},
\end{align*}

where $F$ denotes a general fiber of the normal bundle of $\tilde{D}^+(\tilde{q})$ in $\tilde{U}_\tilde{p}$. Here, we use the transversality at intersection points of $\partial D^-(\bar{p})$ and $\tilde{D}^+(\tilde{q})$.

On the other hand, using an uniformizing chart $\tilde{V}_{\tilde{q}}$ around $\tilde{q}$ where the preimage $D^-_{\tilde{q}}$ of $D^-_{\tilde{q}}$ is used to calculate $\int_J \tau^* \eta_{\tilde{q}}$,

\begin{align*}
\int_J \tau^* \eta_{\tilde{q}} &= \frac{a_{\tilde{q}}}{b_{\tilde{q}}} \cdot \frac{b_{\bar{q}}}{|G_{\tilde{q}}|} \int_{D^-_{\tilde{q}}} \bar{\eta}_{\tilde{q}} \\
&= \frac{a_{\tilde{q}}}{|G_{\tilde{q}}|} \int_F \bar{\eta}_{\tilde{q}} \\
&= \frac{a_{\tilde{q}}}{|G_{\tilde{q}}|},
\end{align*}

(We abbreviate $\bar{\eta}_{\tilde{q}}$ to denote the representative of $\eta_{\tilde{q}}$ on $\tilde{V}_{\tilde{q}}$.)

By comparing both integrals and \[(5.5),\] we conclude that

\[ a_{\tilde{q}} = \sum_{\gamma: \bar{p} \to \bar{q}} \frac{\epsilon(\gamma)|G_{\bar{q}}|}{|G_{\gamma}|}. \]

\[ \square \]

**Remark 5.7.** Note that there are several points in the proof where we use the fact that $CM_*(X, \hat{f})$ is defined on the field coefficient, although it would be still a chain complex using $\mathbb{Z}$-coefficients.
Therefore, under the existence of a Morse-Smale function $\bar{f}$ on $X$, we can prove the Poincare duality of the singular homology of the orbit space $X$ by considering $-\bar{f}$. However, the inner product which gives the Poincare pairing between $HM_\ast(X, \bar{f})$ and $HM_\ast(X, -\bar{f})$ induced by slight different pairing

$$<,>: CM_\ast(X, \bar{f}) \otimes CM_\ast(X, -\bar{f}) \rightarrow \mathbb{R},$$

where $<\bar{p}, \bar{q}> = 1/|G_{\bar{p}}|$ and $<\bar{p}, \bar{q}> = 0$ if $\bar{p} \neq \bar{q}$. Let $\partial_+$ and $\partial_-$ be boundary operators of $CM_\ast(X, \bar{f})$ and $CM_\ast(X, -\bar{f})$, respectively. Then,

$$<\partial_+ \bar{p}, \bar{q}> = n(\bar{p}, \bar{q}) <\bar{q}, \bar{q}> = \frac{1}{|G_{\bar{q}}|} \sum_{\bar{\gamma} \in M(\bar{p}, \bar{q})} \epsilon(\bar{\gamma}) |G_{\bar{q}}|$$

$$= \sum_{\bar{\gamma} \in M(\bar{p}, \bar{q})} \epsilon(\bar{\gamma}) |G_{\bar{\gamma}}|,$$

and similarly,

$$<\bar{p}, \partial_- \bar{q}> = n(\bar{q}, \bar{p}) <\bar{p}, \bar{p}> = \sum_{\gamma \in M(\bar{p}, \bar{q})} \epsilon(\gamma) |G_{\gamma}|$$

so that

$$<\partial_+ \bar{p}, \bar{q}> = <\partial_- \bar{q}, \bar{p}>$$

and $<,>$ induces a pairing on homologies.

**Remark 5.8.** To get a similar pairing between $(CM_\ast(X, \bar{f}), \partial)$ and $(CM_\ast(X, -\bar{f}), \partial)$, one should modify $<,>$ such that $<\bar{p}, \bar{p}> = |G_{\bar{p}}|$.

### 6. Morse-Bott case for global quotient orbifolds

We generalize the construction of the section 2 of Morse functions on global quotients to the Morse-Bott case. Let a finite group $G$ act effectively on the compact smooth connected oriented manifold $M$ and assume that there is a $G$-invariant Morse-Bott function $f: M \rightarrow \mathbb{R}$.

**Definition 6.1.** $f$ is called Morse-Bott if $\text{crit}(f)$ is a finite union of compact connected submanifold of $M$ and the Hessian of $f$ is nondegenerate at each critical submanifold in normal direction. In addition, we require the Morse-Smale condition that each unstable manifold and stable manifold (for each critical submanifold) intersect transversally.

**Remark 6.2.** The construction in this section is related to the construction of equivariant Morse-Smale-Witten complex by Austin and Braam in [AB].

(i) The construction of Austin and Braam is for equivariant cohomology for manifold $M$ with $G$-action. We believe that authors [AB] implicitly assume that $G$ is connected Lie group, as they have used the Cartan model, and have not discussed orientation issues.

(ii) For the case of finite $G$, the associated Lie-algebra is trivial, and the construction of [AB] would provide a complex, built on invariant differential forms of critical submanifolds. But in general, $G$-action may not preserve the orientations of unstable directions (of critical submanifolds) or even orientations of critical submanifolds, and in such cases, one cannot obtain the singular cohomology of the quotient space (as discussed so far). Hence,
the construction in this subsection may be considered as an extension of \[AB\] for the case of finite \(G\) with orientation issues considered.

(iii) We also consider slightly generalized setting without assuming that the end point map from the moduli space of the connecting trajectories (i.e. flow lines at \(\pm \infty\)) is submersive.

**Remark 6.3.** There are several approaches for Morse-Bott homology such as \[F\], \[L\] etc. Banyaga and Hurtubise assumed different transversality condition (called the Morse-Bott-Smale transversality condition) which enables them to go through issues about fiber product without perturbation. See \[BH\].

If \(S\) is a critical submanifold, the unstable manifold and the stable manifold over \(S\) is defined as follows.

\[
W^-(S) := \{ x \in M : \lim_{t \to -\infty} \Phi_t(x) \in S \} \\
W^+(S) := \{ x \in M : \lim_{t \to \infty} \Phi_t(x) \in S \}
\]

By now, there exist several versions of Morse-Bott theory, and we will follow the one using currents as in \[Hu\] (which is based on the construction by Fukaya \[F\]). We do not use the de Rham version as in the setting of \[AB\] because non-trivial assumptions about relative orientability of critical submanifolds and unstable manifolds over them in order to define fiberwise integration of differential forms. The currents were introduced to deal with the following problems: the end point map \(e_+ : M(S_1, S_2) \to S_2\) is not a submersion in general and hence, we cannot integrate differential forms along the fiber of this map. Furthermore, the moduli space \(\mathcal{M}(S_1, S_2) := W^-(S_1) \cap W^+(S_2) / \mathbb{R}\) might not be (invariantly) orientable, even when \(S_1, S_2\) and \(M\) are all orientable.

We will introduce the Morse-Bott chain complex of \(M\) (following that of Hutchings \[Hu\]) and then restrict to \(G\)-invariant subcomplex.

Let \(\sigma\) be a simplex in \(S_1\) and define the following fiber product \(\overline{\mathcal{M}}(\sigma, S_2) := \sigma \times_{S_2} M(S_1, S_2)\), the set of flow lines from \(\sigma\) to \(S_2\). Moreover, we have the following diagram of fiber product.

\[
\begin{array}{ccc}
\overline{\mathcal{M}}(\sigma, S_2) & \xrightarrow{i} & M(S_1, S_2) \\
\downarrow & & \downarrow e_+ \\
\Delta^k & \xrightarrow{\sigma} & S_1 \\
\end{array}
\]

(6.1)

We require that \(\sigma\) is transversal to \(e_+\) because we want \(\overline{\mathcal{M}}(\sigma, S_2)\) to be a manifold with boundary (or with corners). Indeed we will think of \(\overline{\mathcal{M}}(\sigma, S_2)\) as a current on \(S_2\) given the genericity assumption on \(\sigma\) which will be explained later. We will write this current as \((i \circ e_+) \left[ \overline{\mathcal{M}}(\sigma, S_2) \right]\).

**Remark 6.4.** This is always possible by generic perturbation (when \(G\)-action is not involved). Note that if \(e_+\) is a submersion, the desired transversality follows automatically.
For \( \gamma \in \mathcal{M}(\sigma, S_2) \), orientations of \( \sigma \), \( W^-(p_1) \) and \( W^-(p_2) \) determine a local orientation of \( \mathcal{M}(\sigma, S_2) \) where \( \gamma \) runs from \( p_1 \) to \( p_2 \). (see \([6.4]\)).

We introduce the fiberwise orientation sheaf of the pair \((W^-(S), S)\) for a critical submanifold \( S \) of \( M \). It is a locally constant sheaf \( O \) whose stalk is defined by

\[
O_p := H_{\mu(S)-1}(W^-(p) \setminus p) \simeq \mathbb{R},
\]

where \( \mu(S) \) is the number of negative eigenvalues of the Hessian of \( f \) at a point in \( S \), or simply, the Morse-Bott index of \( S \). Let \( C^*_{\text{sing}}(S, O) \) be the space of singular chains with coefficients in \( O \).

Definition 6.5. To define the fiber products, we consider a subspace of currents spanned by pairs \((\sigma, o)\) where \( \sigma \) satisfies conditions, below.

(i) \( \sigma \) is smooth,
(ii) each face of \( \sigma \) is transverse to \( e_+ \) of all moduli spaces of flow lines between critical submanifolds and all iterated fiber products thereof.

\( \sigma \) satisfying the above conditions is generic. We will call such a simplex as a generic simplex.

Let \( C_*(S, O) \) denote the resulting chain complex equipped with the differential defined in a standard way. We now define a chain complex as follows. The \( k \)-th chain group is

\[
C^B_{k} := \bigoplus_S C_{k-\mu(S)}(S, O).
\]

If \( \sigma \in C_*(S, O) \) is a generic simplex and for \( S' \neq S \), we have a well-defined current

\[
(i \circ e_+) \left[ \mathcal{M}(\sigma, S') \right] \in C_*(S', O).
\]

For simplicity, from now on, we write \( e_+ \), instead of \( i \circ e_+ \) and also sometimes write \( \sigma \) instead of \( (\sigma, o) \) to denote elements of \( C_*(S, O) \).

Define

\[
D \sigma := \partial \sigma + \sum_{S' \neq S} e_+ \left[ \mathcal{M}(\sigma, S') \right] .
\]

One can easily check that \( D : C^B_k \to C^B_{k-1} \) and the standard argument using the convergence of gradient flows shows \( D^2 = 0 \).

Now, we consider finite group \( G \) action on them. Since \( G \) acts on singular chains of \( S \) and on local sections of \( O \), we have an well-defined action of \( G \) on \( C^B_k \). Note that \( G \) obviously preserves transversality condition of singular chains in Definition 6.5 as the moduli spaces of gradient flows also admit \( G \)-action on them (\( \mathcal{M}(S_1, S_2) \to S_1 \) is \( G \)-equivariant). We check whether the \( G \)-action is compatible with the differential \( D \) or not.

Proposition 6.6. The boundary operator \( D \) is \( G \)-equivariant so that

\[
C^B_k^{G} := \{ \alpha \in C^B_k : g \cdot \alpha = \alpha \ \forall g \in G \}
\]

forms a sub-complex of \( C^B_k \). Similarly, we define invariant subcomplex \( C^*_G(S, O) \) of \( C_*(S, O) \).
As mentioned earlier, $\sigma$ in (6.2) should be thought of as $(\sigma, o)$, where $o$ determines continuously varying orientations on fibers of $W^- (S) \to S$ over $\sigma$.

**Proof.** It suffices to show that $D(g \cdot (\sigma, o)) = g \cdot D(\sigma, o)$ for $g \in G$ and $(\sigma, o) \in C_k^{Bott}$, which implies that the boundary operator $D$ preserves $C_k^{Bott,G}$. It is easy to see that the only non-trivial part of the above is the sign. We explain the related signs more explicitly.

$o$ will induce a fiberwise orientation of $e_+ [\mathcal{M}(\sigma, S')]$, which is denoted as $D(o)$ (see (6.4)). Then we need to show

$$(e_+ [\mathcal{M}(g \cdot (\sigma, o), S')], D(g \cdot o)) = (g \cdot e_+ [\mathcal{M}(\sigma, o), S'], g \cdot D(o))$$

As sets without concerning orientations, the above identity holds because $G$-action is compatible with gradient flows of $f$. To see the equivalence of signs, we recall the sign rule in [Hu] (see [F] also). The sign or orientation of $e_+ [\mathcal{M}(\sigma, S')]$ is determined by the following equality of oriented vector spaces:

$$T_{g \cdot \sigma} \oplus T_{g \cdot \sigma} W^-(p_1) \cong T_{\mathcal{M}}(\mathcal{M}(\sigma, S')) \oplus T_{\gamma} \mathcal{M}(\sigma, S') \oplus T_{p_2} W^-(p_2),$$

where $p_1 \in \sigma$ and the flow line $\gamma$ runs from $p_1$ to $p_2 \in S'$. The sign of $(\sigma, o)$ is exactly described as the orientation of the left hand side. Since $T_{\gamma}$ has a canonical orientation (in negative gradient direction), the orientation on $T_{\mathcal{M}}(\mathcal{M}(\sigma, S')) \oplus T_{p_2} W^-(p_2)$ is determined by $(\sigma, o)$, which corresponds to the orientation of $e_+ [\mathcal{M}(\sigma, S')]$ itself together with the fiberwise orientation of unstable manifold over $e_+ [\mathcal{M}(\sigma, S')]$ at $p_2$. Now apply $g \in G$ to this equation (6.4). Then, we obtain:

$$T_{g \cdot \sigma} \oplus T_{g \cdot \sigma} W^-(g \cdot p_1) \cong g \cdot (T_{\mathcal{M}}(\mathcal{M}(\sigma, S')) \oplus T_{p_2} W^-(p_2)) \oplus T(g \cdot \gamma).$$

The left hand side determines $D(g \cdot o)$ whereas the orientation of the right hand side corresponds to $g \cdot D(o)$.

\[\square\]

**Remark 6.7.** Suppose $f$ is not only Morse-Bott, but Morse. Then, for a non-orientable critical points $p \in \text{crit}^{-}(f)$, consider the element with orientation, $(p, o) \in C_k^{Bott}$. Non-orientability of $p$ implies that some $g \in G_p$ (in fact half of $G_p$) reverses the orientation $o$, and hence, the average of $(p, o)$ via $G_p$ in fact vanishes. Thus $(p, o)$ is not an element of $C_k^{Bott,G}$. Hence, $C_k^{Bott,G}$ agrees with our earlier invariant Morse complex when $f$ is Morse. i.e. we have

$$\sum_{g \in G} g \cdot (p, o) = \sum_{g \in G^+} g \cdot (p, o) + \sum_{h \in G^-} h \cdot (p, o)$$

$$= \sum_{G^+} (p, o) - \sum_{G^-} (p, o) = 0,$$

where $G^\pm$ denote the set of elements preserving or reversing orientation, respectively.

Now, we show that the homology of $(C_{\ast}^{Bott,G}, \partial)$ is isomorphic to the singular homology of $M/G$ with the assumption that $f$ is furthermore self-indexing. We first recall the compactification of unstable manifolds of $f$ in [AB]. For a critical submanifold $S_i$ of index $i$ ($\mu(S_i) = i$) and a sequence $i_0 < i_1 < \cdots < i_m = i$, define

$$Y_{i_0, i_1, \ldots, i_m} = M(S_{i_0}, S_{i_0-1}) \times S_{i_0-1} \cdots \times S_{i_m} W^-(S_{i_0}),$$
where the index of points in $S_i$ is $i_k$. Na"ively, $Y_{i_0,i_1,\ldots,i_m}$ is a product of $W^-(S_i)$ with the moduli space of broken trajectories passing through $S_{i_{m-1}},\ldots,S_i$ to $S_{i_0}$.

**Lemma 6.8.** (lemma 3.3 of [AB]) The unstable manifold $W^-(S_i)$ can be compactified so that

$$\partial W^-(S_i) = \bigcup_{i_0 < i_1 < \cdots < i_m = i} Y_{i_0,i_1,\ldots,i_m}.$$  

Here, the codimension of the stratum $Y_{i_0,i_1,\ldots,i_m}$ in $W^-(S_i)$ is exactly $m$. The compactified space has the structure of a manifold with corners.

Moreover, there is a natural immersion $\iota_{S_i}$ of the $W^-(S_i)$ into $M$, which is not injective nor proper in general. For example, the immersion of $\partial W^-(S_i)$ restricted on the boundary component $Y_{i_0,i_1,\ldots,i_m}$ is just the projection from $Y_{i_0,i_1,\ldots,i_m}$ to the last factor $W^-(S_{i_0})$. Also we have

$$\partial W^-(S_i) = \bigcup_{S'} M(S,S') \times_{S'} W^-(S').$$

For $\tau \in C_k(S_i) \in C_{k+i}^{Bott}$, let

$$W^-(\tau) := \tau \times_{S_i} W^-(S_i).$$

From our genericity assumption, this define a current on $M$. Note that

$$\dim W^-(\tau) = \dim \tau + \dim W^-(S_i) - \dim S_i = k + i + \dim S_i - \dim S_i = k + i.$$

Since the element of $C_k(S_i)$ also contains the information of the local orientation of fiber of $W^-(S_i) \to S_i$ around the image of simplex, say $o$, $(\tau,o)$ defines an orientation on $W^-(\tau)$. Therefore, we get a map $\Phi : C_k^{Bott} = \oplus_i C_{k-i}(S_i) \to C_k(M)$ sending $\tau \mapsto W^-(\tau)$. Here, we regard $C_k(M)$ as a singular chain complex of $M$ generated by currents.

**Remark 6.9.** As a current on $M$, $W^-(S_i)$ is equivalent to its closure in $M$ or embedding of its compactification since they only differ from measure zero subsets of $M$.

We can show that $\Phi$ becomes isomorphism of homology groups when restricted to $C_{k+i}^{Bott, G}$. First, we prove

**Proposition 6.10.** $\Phi : C_k^{Bott} \to C_k(M)$ is a $G$-equivariant chain map.

**Proof.** Consider $S \subset \text{crit}(f)$ and $\sigma$, one of generators of $C_k(S,O)$. Then, we need to show:

$$\Phi(D\sigma) = \partial \Phi(\sigma),$$

where $\partial$ is the usual boundary operator of the singular chain complex of $M$. We will only check the identify as a set, and the check on orientation can be done without difficulty. So we will simply write $\sigma$ instead of $(\sigma,o)$. The left hand side can be written as

$$(6.5) \quad \Phi(D\sigma) = \partial \sigma \times_{S} W^-(S) + \sum_{S' \neq S} e_+ X(M(\sigma,S') \times_{S'} W^-(S')).$$
On the other hand,
\[
\partial \Phi(\sigma) = \partial \left( \sigma \times_S W^-(S) \right) \\
= \partial \sigma \times_S W^-(S) + \sigma \times_S \partial W^-(S) \\
= \partial \sigma \times_S W^-(S) + \sum_{S' \neq S} \sigma \times_S \left( \mathcal{M}(S, S') \times_{S'} W^-(S') \right)
\]

Here, only codimension 1 boundary components of \( W^-(S) \) appear since we are considering them as currents. The map \( \mathcal{M}(S, S') \times_{S'} W^-(S') \to S \) is induced by \( e_- : \mathcal{M}(S, S') \to S \). Now, the identification easily follows by considering the diagram (6.1). □

Since \( \Phi \) is \( G \)-equivariance, we get a map \( \Phi : C_{Bott,G}^* \to C_p^G(M) \) by restricting original \( \Phi \), where \( C_p^G(M) \) denotes \( G \)-invariant singular chains in \( M \). From now on, assume further that the Morse-Bott function \( f \) satisfies the self-indexing condition, i.e \( f(S_i) = i \). Then, we get a filtration of complex \( C_p^G(M) \) as follows. Let \( M_k = f^{-1}(-\infty, k + \frac{1}{2}) \). Each \( M_k \) is \( G \)-invariant and gives a filtration on \( C_p^G(M) \),
\[
0 \subset C_p^G(M_0) \subset C_p^G(M_1) \subset \cdots \subset C_p^G(M_n) = C_p^G(M), \text{ i.e. } (C_p^G(M))^k = C_p^G(M_k)
\]
where \( n = \dim M \). For \( C_{Bott,G}^* \), we take the following filtration:
\[
(C_{Bott,G}^p)^k = \bigoplus_{i \leq k} C_{Bott,G}^p(S_i, \mathcal{O}_j),
\]
where \( \mathcal{O}_j \) means the fiberwise orientation sheaf on \( (W^-(S_j), S_j) \).

We prove that \( \Phi \) induces an isomorphism between cohomology groups. As the homology of \( C_p^G(M) \) is the homology of \( M/G \) (see Theorem 5.3), we have

**Proposition 6.11.** \( \Phi_* : H_{*Bott,G}^*(M) \xrightarrow{\sim} H_*(M/G) \) is an isomorphism.

**Proof.** We are going to show that \( \Phi \) induces an isomorphism between the homology groups of associated graded complexes of \( C_{Bott,G}^* \) and \( C_p^G(M) \). Since our complexes are bounded, spectral sequences of \( C_{Bott,G}^* \) and \( C_p^G(M) \) (with respect to the filtrations given above) will converge to associated graded groups of homologies. Thus if we can prove \( \Phi \) induces an isomorphism between \( E^1 \)-terms of spectral sequences which are homology groups of associated graded complexes of \( C_{Bott,G}^* \) and \( C_p^G(M) \), then we will get the desired result. Let
\[
GC_{p}^{k} = C_p^G(M_k)/C_p^G(M_{k-1})
\]
and
\[
GC_p^k = C_{p-k}(S_k, \mathcal{O}_k).
\]
They are precisely the \( E^1 \)-terms of our complexes. The homology of the first complex is simply the \( G \)-invariant relative homology group of the pair \( (M_k, M_{k-1}) \). i.e.
\[
H_p(GC_{p}^{k}) = H_p^G(M_k, M_{k-1}).
\]
For \( S_k = \text{crit}(f) \cap (M_k \setminus M_{k-1}) \), define \( F_k = W^-(S_k) \cap \{ x : k - \frac{1}{2} \leq f(x) \leq k + \frac{1}{2} \} \). \( F_k \)'s are \( G \)-invariant, obviously and hence there is natural \( G \)-action on \( F_k \).
Combining theorem 5.3 and excision, we get:

\[ H^G_p(M_k, M_{k-1}) \cong H_p(M_k/G, M_{k-1}/G) \cong H_p(F_k/G, \partial F_k/G) \cong H^G_p(F_k, \partial F_k). \]

(The gradient flow defines a deformation retract of the pair \((M_k, M_{k-1})\) onto \((F_k \cup \partial F_k, M_{k-1})\) and hence, a retract between pairs of \(G\)-quotients.)

Thus, what we are left with is to show the following morphism of chain complexes induces the isomorphism on homology groups of them:

\[ \Phi' : C^G_p(S_k, O_k) \to C^G_p(F_k, \partial F_k). \]

Keeping track of our isomorphisms used above, one can find that this map is defined as

\[ \Phi'(\sigma) = F_k(\sigma) := \sigma \times_{S_k} F_k. \]

We again emphasize that chain \(\sigma\) contains information of the coherent orientation of the fiber of \(e_-| : F_k \to S_k\) and that the genericity condition on \(\sigma\) guarantees the well-definedness of this map. So, \(F_k(\sigma)\) carries a natural orientation from this information.

Observe that \(\Phi'\) induces the homology Thom isomorphism, or dual to well-known cohomology Thom isomorphism defined by the integration along fiber. Even if neither a bundle nor a base is not orientable, the Thom isomorphism still holds true. However, we have to use a relative orientation sheaf as a coefficient ring (as we introduced \(O_k\)). See theorem 7.10 of [BT] or [SE] for more details.

\[ \square \]

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