Clustered Error Correction of Codeword-Stabilized Quantum Codes

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Codeword stabilized (CWS) codes are a general class of quantum codes that includes stabilizer codes and many families of non-additive codes with good parameters. For such a non-additive code correcting all $t$-qubit errors, we propose an algorithm that employs a single measurement to test all errors located on a given set of $t$ qubits. Compared with exhaustive error screening, this reduces the total number of measurements required for error recovery by a factor of about $3^t$.

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Quantum computation admits polynomial complexity for many classical algorithms believed to be hard [1–3]. To preserve coherence, quantum computations must be protected by quantum error correcting codes [3, 4]. Stabilizer codes [5, 6] represent an important class of quantum codes that can be encoded and decoded in polynomial time. Recent Refs. [7, 8] introduce a larger class of codeword-stabilized (CWS) codes. It includes important code families, such as the stabilizer codes and generally non-additive union stabilizer (UST) codes [9]. CWS codes have a broader range of code parameters which can be superior to those of any stabilizer code [3, 5, 6].

The most important advantage of the CWS codes is their close relation with the classical codes. In particular, a qubit CWS code $Q$ can be mapped onto a classical binary code $C$, with the quantum Pauli errors also mapped into some binary error patterns [3]. This way, within CWS framework, quantum code design can be reduced to classical codes and employ the wealth of different techniques developed for the latter.

On the other hand, quantum error correction must preserve the original quantum state in all intermediate measurements, and therefore is more restrictive than many classical algorithms. Thus, design of CWS codes must be complemented by an efficient non-damaging quantum error correction algorithm. In this paper, our main goal is to address this important unresolved problem.

We consider a general non-additive CWS code $(n, K, d)$ of distance $d$ which encodes $K$ quantum dimensions into a $K$-dimensional subspace of the Hilbert space of $n$ qubits. This code detects all errors that corrupt up to $(d-1)$ qubits, and corrects all errors corrupting $t \equiv \lfloor (d-1)/2 \rfloor$ or fewer qubits. As a benchmark for our study, we consider generic algorithms that project a corrupted code state into different subspaces. This brute-force technique is similar to the exhaustive error screening in nonlinear classical codes, and requires up to

\begin{equation}
B(n, t) = \sum_{i=0}^{t} \binom{n}{i} 3^i
\end{equation}

measurements to screen all errors of weight $t$ or less.

To reduce the number of such measurements, we first design an error detection algorithm for USt codes [3]. In CWS framework, the classical code $C$ associated with the USt code $Q = (\langle n, m 2^k, d \rangle)$ is decomposed as a group $C_0$ of $2^k$ codewords shifted by $m$ binary “translation” vectors. We prove the following

**Theorem 1** For a USt code of length $n$, with a group of size $2^k$ and dimension $K = m 2^k$, an error-detecting measurement requires no more than $2m(n - k)(n + 3)$ two-qubit gates.

Then, for a general CWS code $Q$, we propose an error-correcting method that simultaneously screens all $4^t$ different errors located on any given subset of $t$ qubits, by designing an auxiliary USt code which uses binary maps of these errors as generators of the group $C_0$, and the codewords of the associated classical code $C$ as translations. This requires only $\binom{n}{t} - 1$ measurements to screen all groups. Once the corrupted qubits are located, we need up to $2t$ extra measurements to find the actual error within the group. Overall, this reduces the number $B(n, t)$ of measurements about $3^t$ times to

\begin{equation}
N(n, t) = \binom{n}{t} + 2t - 1.
\end{equation}

Our main result is summarized as

**Theorem 2** Consider any $t$-error correcting CWS code of length $n$ and dimension $K$. Then this code can correct errors using at most $N(n, t)$ measurements, each of which requires at most $2K(\frac{n-1}{2})(n+3)$ two-qubit gates.

**Definitions.** Throughout the paper, we use the Hilbert space $\mathcal{H}^n_2 \equiv \mathbb{C}^n$ to represent any $n$-qubit state. Also, $\mathbb{P}_n = \{+1, -1\} \{I, X, Y, Z\}^\otimes n$ denotes the Pauli group, where the number of non-trivial terms in the tensor product is the weight of a given $E \in \mathbb{P}_n$. We say that a space $\mathcal{P}$ is stabilized by a measurement operator $M$ with all eigenvalues $\lambda = \pm 1$ (this includes all Hermitian operators in $\mathbb{P}_n$) if $M \ket{\psi} = \ket{\psi}$ for any state $\ket{\psi}$ in $\mathcal{P}$. We will also use the term anti-stabilized if
A general quantum code (\( (n, K, d) \)) is a subspace \( \mathcal{Q} \subseteq \mathcal{H}_2^\otimes n \) of dimension \( K \), such that any detectable error either takes any non-zero state \( |\psi\rangle \in \mathcal{Q} \) into a state outside of \( \mathcal{Q} \), \( E |\psi\rangle \not\in \mathcal{Q} \), or acts trivially on \( \mathcal{Q} \), \( E |\psi\rangle = C_E |\psi\rangle \) with \( C_E \) independent of \( |\psi\rangle \). A combination \( E_1^\dagger E_2 \) of any two errors from a set \( \mathcal{E} \) of correctable errors is detectable.

The errors are in the same degeneracy class iff \( E_1 E_2 \) acts trivially on \( \mathcal{Q} \). For a distance-\( d \) code, all Pauli errors of weight up to \((d-1)\) are detectable, and all Pauli errors of weight up to \( t = \lfloor (d-1)/2 \rfloor \) are correctable.

A stabilizer code \( \mathcal{Q} \) is defined as the stabilizer space of an Abelian group \( \mathcal{S} \) generated by Hermitian Pauli operators \( G_i, i = 1, \ldots, n-k \), and correctable errors in different degeneracy classes. For a single measure-

A union stabilizer (UST) code \( \mathcal{Q} \) can be defined as a CWS code \( \mathcal{Q} \) whose word operators contain a group,

\[
\mathcal{W} = \{ t_j \prod_{i=1}^k g_{i}^{\alpha_i} : j = 1, \ldots, m, \alpha_i \in \{0,1\} \}.
\] (5)

Here \( g_i \) are generators of the group \( \mathcal{W} = \langle g_1, \ldots, g_k \rangle \) forming an additive code \( \mathcal{Q}_0 = \text{span} \{ (W |s\rangle)_{W \in \mathcal{W}} \} \) with dimension \( K_0 = 2^k \). The operators \( t_j \) form a set \( T \) of \( m \) translations for the code \( \mathcal{Q}_0 \). The translated spaces \( t_j(\mathcal{Q}_0) \) are mutually orthogonal, which implies that the dimension of the code \( \mathcal{Q} = K = m 2^k \).

The standard form of a CWS code is defined in terms of a graph \( \mathcal{G} \) with \( n \) vertices and a classical code \( C \) containing \( K \) binary codewords \( c_i \) of length \( n \). The graph adjacency matrix \( R \in \{0,1\}^{n \times n} \) defines the generators of the stabilizer, \( S_i = X_{i_1} Z_{i_2} \cdots Z_{i_m} \), while the classical codewords define the codeword operators \( W_i = \{ |\psi\rangle : \mathcal{Q} \} \text{∽} Z_{1}^{c_1} \cdots Z_{n}^{c_n} \). Most importantly, the graph relates the error-correction properties \( \mathcal{Q} \) of the quantum CWS code \( \mathcal{Q} = (\mathcal{G}, C) \) and the classical code \( \mathcal{C} \). Indeed, the action of a single-qubit error \( \mathcal{E} \) on the code is equivalent (up to an overall phase) to that of \( \mathcal{X} S_i \mathcal{X} = Z_{1}^{c_1} \cdots Z_{n}^{c_n} \) for any Pauli operator \( \mathcal{E} = Z_{N} X_{n} \).

For a pair of correctable errors \( \mathcal{E}_1, \mathcal{E}_2 \) from different degeneracy classes, \( \mathcal{C}_G(\mathcal{E}_1) \neq \mathcal{C}_G(\mathcal{E}_2) \), the corrupted spaces are always orthogonal, \( \mathcal{E}_1(\mathcal{Q}) \perp \mathcal{E}_2(\mathcal{Q}) \).

Any CWS code \( \mathcal{Q} \) is locally Clifford-equivalent to a code in standard form \( \mathcal{Q} \). For CWS codes in standard form, we will denote the corresponding set of word operators and word stabilizer as \( \mathcal{W}_G \) and \( \mathcal{S}_G \), respectively.

**Exhaustive screening for CWS codes.** We can detect errors by measuring the operator \( M_0 = 2 P_\mathcal{Q} - \mathbb{1} \).

\[
P_\mathcal{Q} = \sum_{W \in \mathcal{W}} W |s\rangle \langle s| W^\dagger.
\] (7)

The corresponding ancilla measurement circuit which uses \( 2K[n^2 + O(n)] \) two-qubit gates can be constructed as the special case of Eq. (6) below. A different circuit which requires up to \( n^2 + K O(n) \) two-qubit gates is constructed in Ref. [12].

The operators \( E M_0 E^\dagger \) stabilize the spaces \( \mathcal{E}(\mathcal{Q}) \). For a CWS code \( \mathcal{Q} \), these spaces are orthogonal for mutually non-degenerate correctable errors \( \mathcal{E} \). This implies that an error can be located by measuring such operators for \( \mathcal{E} \) from different degeneracy classes. For a \( t \)-error correcting code we can exhaustively test all correctable errors using up to \( B(n, t) \) measurements [Eq. (6)]. This bound is tight.
Consider an additive CWS code. The code is a stabilizer code, it is the common stabilized space of the $n-k$ generators $G_i$ of the code stabilizer $S_0$, $Q_0 = \bigcap_{i=1}^{n-k} P(G_i)$. According to Eq. (8), we also have

$$M_0 \equiv M_{Q_0} = \bigwedge_{i=1}^{n-k} G_i,$$

and can construct the corresponding measurement circuit by analogy with Fig. 1 using associativity. This requires $2(n-k)$ controlled $n$-qubit Pauli operators and $(n-k-1)$ three-qubit Toffoli gates. Adding the corresponding complexities [14], we obtain the overall complexity of up to $2(n-k)(n+3)$ two-qubit gates.

This measurement can be done in the basis of the original CWS code. The $n$ generators $S_1 \in \mathbb{P}_n$ of the word stabilizer $S$ can be chosen [12] to satisfy the orthogonality condition $S_t g_j = (-1)^{\delta_{ij}} g_j$. Now, the $k$ logical operators of the code can be chosen as $X_j = g_j, Z_j = S_j$, and the remaining generators of the orthogonalized word stabilizer can serve as the generators $G_i = S_{i+k}, i = 1, \ldots, n-k$ of the code stabilizer $S_0$.

**DECOMPOSITION OF A UST CODE.** Now consider a Ust code $Q$ with the set $W$ of word operators in the form [5]. Given the generators $G_i$ of the stabilizer $S_0$ of the additive subcode $Q_0$, the generators of the translated code $t_j(Q_0)$ can be written as $t_j G_i t_j^\dagger$. Then, the corresponding measurement operators [cf. Eq. (13)]

$$M_j \equiv t_j M_0 t_j^\dagger = \bigwedge_{i=1}^{n-k} t_j G_i t_j^\dagger.$$

The code $Q$ is spanned by the orthogonal vector spaces

$$Q \equiv P(M_Q) = \bigcap_{j=1}^{m} P(M_j), \quad P(M_i) \perp P(M_j),$$

which is equivalent to the symmetric difference $Q = P(M_1) \bigtriangleup P(M_2) \bigtriangleup \ldots \bigtriangleup P(M_m)$. According to Eq. (4), this is also equivalent to the decomposition

$$M_Q = \bigoplus_{j=1}^{m} M_j = \bigoplus_{j=1}^{m} \bigwedge_{i=1}^{n-k} \left( t_j G_i t_j^\dagger \right).$$

Since the XOR (“⊕”) of several measurements is implemented as concatenation [Fig. 3], it requires no overhead; the resulting complexity is then given by Theorem 4.
Clustered measurements for CWS codes. For a \( t \)-error correcting CWS code \( \mathcal{C} \), consider any subset of correctable errors, \( \mathcal{E}' \subset \mathcal{E} \), and any correctable error \( E \) not degenerate with those in \( \mathcal{E}' \). Then, the space \( \mathcal{E}'(\mathcal{Q}) \equiv \text{span}_{E \in \mathcal{E}'} E(\mathcal{Q}) \) is orthogonal to \( E(\mathcal{Q}) \). Furthermore, errors located on any \( t \) qubits (specified by the set of qubit indices \( A = \{i_1, \ldots, i_t\} \)) form a group of correctable errors \( \mathcal{E}_A \equiv \langle X_{i_1} Z_{i_1}, \ldots, X_{i_t} Z_{i_t} \rangle \). Thanks to the group property of \( \mathcal{E}_A \), for the set \( \mathcal{E}' \equiv \mathcal{E}_A \), we also have \( \mathcal{E}'_{\mathcal{Q}} \) a more restrictive identity \( E(\mathcal{Q}_A) \perp \mathcal{E}_A \), where \( \mathcal{Q}_A \equiv \mathcal{E}_A(\mathcal{Q}) \). Thus, \( \mathcal{Q}_A \) is a quantum code which can detect errors \( E \in \mathcal{E} \) not degenerate with those in \( \mathcal{E}_A \).

Our clustered measurement technique is based on the observation that \( \mathcal{Q}_A \) is actually a USt code. Indeed, consider the original CWS code in standard form, \( \mathcal{Q} = (G, C) \). The set of operators \( \mathcal{D}_A \equiv \{ Z^{C} : E \in \mathcal{E}_A \} \) forms an Abelian group of size \( 2^k \geq |\mathcal{E}_A| = 2^t \) since the operators \( Z^{C} \) obey the same multiplication table as \( E \in \mathcal{E}_A \) but are not necessarily independent. By construction, different elements of \( \mathcal{D}_A \) are in different error degeneracy classes, therefore the spaces \( \mathcal{e}_i(\mathcal{Q}) \) are mutually orthogonal for different \( e_i \in \mathcal{D}_A \). The additional degenerate elements in \( \mathcal{E}_A \) do not add to the span, therefore \( \mathcal{Q}_A \equiv \mathcal{E}_A(\mathcal{Q}) = \mathcal{D}_A(\mathcal{Q}) \). Since \( \mathcal{D}_A \) and \( W_{\mathcal{Q}} \) are combinations of \( Z \)-operators only, \( \mathcal{Q}_A \) is a USt code in standard form which uses the same stabilizer state \( |s\rangle \) as \( \mathcal{Q} \), the Abelian group \( W = \mathcal{D}_A \), and the codeword operators \( W_{\mathcal{Q}} \) of the code \( \mathcal{Q} \) as the translation set \( T(\mathcal{Q}) \).

To form the measurement \( M_{A} \equiv M_{\mathcal{Q}_A} \) that stabilizes the USt code \( \mathcal{Q}_A \), we construct a set of \((n-k)\) orthogonal generators \( G_i \) for the additive code \( \mathcal{Q}_0 = \mathcal{D}_A(\text{span} |s\rangle) \), see Eq. (14). The actual measurement [cf. Eq. (13)],

\[
M_A = \bigoplus_{W \in W_{\mathcal{Q}}} \left[ \bigwedge_{i=1}^{n-k} (WG_i W^*) \right], \tag{14}
\]

satisfies the complexity bound of Theorem 1. The measurement \( M_A \) has eigenvalue 1 for all states in \( \mathcal{Q}_A \), and \(-1\) for all states in \( \mathcal{Q}_A^\perp \), which corresponds to all correctable errors not degenerate with those in \( \mathcal{E}_A \).

To determine the error, we first perform measurements \( M_A^{(j)} \) for all (but the last one) size-\( t \) index sets \( A^{(j)} \). After locating the covering cluster \( A \) with Abelian group \( \mathcal{D}_A \) of size \( |\mathcal{D}_A| = 2^t \leq 2^{t_0} \), we can find the error by going over all \( s \leq 2t \) subgroups of \( \mathcal{E}_A \) with \( s-1 \) generators. Each measurement determines whether or not the omitted generator is a part of the error. The error is identified as a product of the generators present in all auxiliary codes that detected no errors. Overall, this requires up to \( N(n, t) \) measurements as in Eq. (14). Thus, for any code length \( n \geq 3 \), the former number of \( B(n, t) \) measurements [see Eq. (1)] is reduced by a factor

\[
B(n, t)/N(n, t) \geq \begin{cases} \frac{3(n+1)}{n+4}, & \text{if } t = 1, \\ 3^t, & \text{if } t > 1. \end{cases} \tag{15}
\]

Some additional acceleration can be gained if the original CWS code is a USt code, with the set of codeword operators \( \mathcal{D}_A \). In this case, for a given index set \( A \), our scheme employs a bigger group \( \mathcal{D}' \) which includes the generators of both \( \mathcal{D}_A \) and the original group \( W \), and a smaller translation set \( T \) of size \( m < K \). The complexity of a single measurement would then be reduced to \( 2mn^2 \), compared to \( 2Kn^2 \) in Theorem 2. Screening of \( N(n, t) \) or fewer qubit clusters will locate the error.

Note also that in the special case of stabilizer codes, our error-grouping technique is equivalent to the syndrome-based recovery [2]. Indeed, for a stabilizer code \( \mathcal{Q} = [[n, k, d]] \), the degeneracy classes form an Abelian group \( \mathcal{E} = \langle e_1, \ldots, e_{n-k} \rangle \) whose \( 2^{n-k} \) elements are enumerated by different syndromes [2]. To locate the error, we can go over all \((n-k)\) USt codes \( \mathcal{E}_\alpha(\mathcal{Q}) \) generated by the subgroups of \( \mathcal{E} \) with one generator, \( e_\alpha \), missing. Then, the code \( \mathcal{E}_\alpha(\mathcal{Q}) \) is a stabilizer code that has to correct only one non-trivial error, \( \mathcal{E}_\alpha = \langle e_\alpha \rangle \). The corresponding stabilizer \( S_\alpha \) has only one generator. Thus, error can be located by independent measurements of \( n-k \) Pauli operators, as we do to measure the syndrome.

In conclusion, we constructed an accelerated clustered quantum error correction algorithm for a non-additive CWS code which uses a set of auxiliary USt codes associated with groups of correctable errors on size-\( t \) clusters. For a generic non-additive code, this reduces the number of error-correcting measurements approximately \( 3^t \) times, compared to exhaustive screening of all correctable errors of weight \( t \) and smaller.

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