A long memory time series with a periodic degree of fractional differencing

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Abstract

This article develops a periodic version of a time varying parameter fractional process in the stationary region. It is a partial extension of Hosking (1981)'s article which dealt with the case where the coefficients are invariant in time. We will describe the probabilistic theories of this periodic model. The results are followed by a graphical representation of the autocovariances functions.

Keywords: Periodic ARFIMA. Periodic autocovariance. Fractionally process. Simulation. Periodically stationary model. Periodic autocorrelation. Long memory.

1 Introduction and Notation

Non-stationary periodically correlated processes find their interesting application in many fields such as economics, hydrology as well as environmental studies. Much attention has been given to the periodic autoregressive moving average process. Most of the extant periodic models in the literature are concerned with the identification, estimation and testing problems. The periodic autoregressive moving average model is introduced by Hannan [10] to describe rainfall data. Gladyshev [7, 8] proposed a periodically correlated process and gave a necessary and sufficient condition for a series to be periodic. A discrete time process is a periodically correlated process if there is an integer $p > 0$ such that,
\( E(X_{t+p}) \equiv E(X_t) \) and \( \text{Cov}(X_{t+p}, X_{t'+p}) = \text{Cov}(X_t, X_{t'}) \). \hspace{1cm} (1.1)

The properties of a periodic autoregressive model are stated by Troutman [16] adopting the associated stationary multivariate autoregressive model. On the other hand, the invertibility property of the periodic moving average models with its elementary concern has been studied by Cipra [5], Ghysels and Hall [6] and Bentarzi and Hallin [3]. Lund et al [14] studied the techniques for fitting parsimonious periodic time series models to periodic data, considering the periodic autoregressive moving average time series model, see also Box et al [4] for more background, all of the articles cited above have taken only a short-memory case. From their description by Lawrance and Kottekoda [13], considerable attention was given to the long memory time series models in the literature, the introduction of this family model is developed by Hosking [11]. He generalized the well-known ARIMA(p,d,q) models of Box and Jenkins to the autoregressive fractionally integrated moving average ARFIMA(p,d,q), this generalization consists of permitting the degree of differencing \( d \) to take any real value, specifically \( 0 < d < \frac{1}{2} \), this fractionally differenced process became a standard model to study the behavior of long term persistence processes and is commonly detected in the analysis of real-life time series data in many areas; for example, in finance, in hydrology, meteorology. He also derived explicit expressions for autocovariances \( \gamma_h^X \), and autocorrelation \( \rho_h^X = \frac{\gamma_h^X}{\gamma_0^X} \) for the zero mean purely AFRIMA \((0,d,0)\) process \( (X_t, t \in \mathbb{Z}) \) represented by \((1 - B)^d X_t = \varepsilon_t\), and showed that the autocovariance function \( \gamma_h^X \) satisfies

\[
\gamma_h^X \sim C h^{-\alpha}, \hspace{0.5cm} C > 0, \hspace{0.5cm} 0 < \alpha < 1, \hspace{0.5cm} \text{si} \hspace{0.2cm} h \to \infty.
\] \hspace{1cm} (1.2)

An extension of this process in the sense of the periodic variation in the differencing parameter \( d \) is proposed in this work, for a know period \( p \), the PtvARFIMA idea is to drive an ARFIMA equation at a periodically time varying long memory parameter, this model is proposed by Amimour and Belaïde[1], they proved that it enjoys a local asymptotic normality property. In this article, we will provide some results for this new model. Stressing that this model is a generalization of the model proposed by Hosking [11] and can be exploited for expressing better the data which present both the periodic structure and the long term dependence patterns. It is more significative also to consider the short memory exponents in the model of the kind. However, Hui and Li [12] proposed a periodically correlated 2-process composed of two parameters of fractional integration based in two white noise independents,

\[
X_t^{(1)} = (1 - B)^{-d_1} \varepsilon_t^{(1)} \hspace{0.5cm} \text{and} \hspace{0.5cm} X_t^{(2)} = (1 - B)^{-d_2} \varepsilon_t^{(2)}, \hspace{1cm} (1.3)
\]
where $d_t$ is the 2-periodic fractional parameter, they are used this model for modeling the Hong Kong United Christian Hospital attendance series, the data concern seventy five weeks (approximately one and half years) on the average number of people entering the emergency unit observed on the weekday and the weekend, mentioning that the ARMA model is not considered because the autocorrelations indicate a long term dependence pattern in this periodic series. The conditional log likelihood function is determined for estimating the periodic memory parameter. This long memory model is a member of the non-stationary class of process (but periodically stationary).

The outline of our article is as follows. We describe the periodic fractional model and these characteristics in the second section. In the section three we calculate the periodic autocovariance and autocorrelation function and we study its asymptotic behavior. The section four contains the graphical representations for the autocovariances functions.

# Definitions and notations

## 2.1 Periodically time-varying long memory parameter

A purely fractionally differenced process $(X_t, t \in \mathbb{Z})$ with period $p$ (denoted by $\text{PtvARFIMA}_p (0, d_t, 0)$) has the following stochastic equation

\[
(1 - B)^{d_t} X_{i+pm} = \varepsilon_{i+pm},
\]

where for all $t \in \mathbb{Z}$, $\exists i = \{1, \ldots, p\}$, $m \in \mathbb{Z}$, such that $t = i + pm$ and $p$ represents the period in $\mathbb{N}$, $d_t$ is the periodic degree of fractional differencing whose values lie in $(0, \frac{1}{2})$, and $(\varepsilon_t, t \in \mathbb{Z})$ is a zero mean white noise with finite variance $\sigma^2_t$, the variance is periodic in $t$ such that $\sigma^2_{i+pm} = \sigma^2_t$.

If $d_t > 0$, the process (2.1) is invertible and has an infinite autoregressive representation is as follows

\[
\varepsilon_{i+pm} = (1 - B)^{d_t} X_{i+pm} = \sum_{j=0}^{\infty} \pi^i_j X_{i+pm-j},
\]

where

\[
\pi^i_j = \frac{\Gamma(j-d_t)}{\Gamma(j+1)\Gamma(-d_t)}.
\]

If $d_t < \frac{1}{2}$, the process (2.1) is causal and has an infinite moving-average representation is as follows

\[
\varepsilon_{i+pm} = \sum_{j=0}^{\infty} \pi^i_j X_{i+pm-j},
\]

where

\[
\pi^i_j = \frac{\Gamma(j-d_t)}{\Gamma(j+1)\Gamma(-d_t)}.
\]
2.1 Periodically time-varying long memory parameter

\[ X_{i+pm} = (1 - B)^{-d_i} \varepsilon_{i+pm} = \sum_{j=0}^{\infty} \psi_j^i \varepsilon_{i+pm-j}, \]  

(2.3)

where

\[ \psi_j^i = \frac{\Gamma(j+d_i)}{\Gamma(j+1)\Gamma(d_i)}, \]

\[ \psi_j^i \sim v_i j^{d_i - 1}, v_i > 0, \text{as } j \to \infty. \]  

(2.4)

**Proposition 2.1.** Let \((X_t)_{t \in \mathbb{Z}}\) a PtvarFIMA\((0, d_t, 0)\) where, for all \(t \in \mathbb{Z}\), there exists \(m \in \mathbb{Z}, p \in \mathbb{N}^*; p > 1\) is the period and \(d_{i+pm} = d_i\) with \(i = 1, ..., p\), such that \(t = i + pm\). The PtvarFIMA\((0, d_i, 0)\) is causal for all \(i = 1, ..., p\), then the series

\[ \sum_{j=0}^{\infty} \psi_j^i B^j(\varepsilon_{i+pm}) = \sum_{j=0}^{\infty} \psi_j^i \varepsilon_{i+pm-j}, \]  

(2.5)

converge in quadratic mean.

**Proof.** For all \(m \in \mathbb{Z}, i = 1, ..., p\), the coefficients \(\psi_j^i\) are given in (2.3), let \(t, s\) a positive integers, such that \(t < s\), we define \(S_t = \sum_{j=0}^{t} \psi_j^i \varepsilon_{i+pm-j}\), then for \(i = 1, ..., p\) we have

\[ ||S_s - S_t||^2 = E \left[ \sum_{v=0}^{s} \psi_v^i \varepsilon_{i+pm-v} - \sum_{j=0}^{t} \psi_j^i \varepsilon_{i+pm-j} \right]^2 \]

\[ = E \left[ \sum_{j=t+1}^{\infty} (\psi_j^i)^2 \varepsilon_{i+pm-j}^2 + \sum_{v=0}^{s} \sum_{j=t+1}^{\infty} \psi_v^i \psi_j^i \varepsilon_{i+pm-v} \varepsilon_{i+pm-j} \right] \]

\[ = \sigma_i^2 \sum_{j=t+1}^{\infty} (\psi_j^i)^2. \]

By the Cauchy criterion, one can easily verify that \(\sum_{j=t+1}^{\infty} (\psi_j^i)^2 < \infty\), for all \(i = 1, ..., p\). \(\square\)

**Proposition 2.2.** Let \((X_t)_{t \in \mathbb{Z}}\) a PtvarFIMA\((0, d_i, 0)\) where, for all \(t \in \mathbb{Z}\), there exists \(m \in \mathbb{Z}, p \in \mathbb{N}^*; p > 1\) is the period and \(d_{i+pm} = d_i\) with \(i = 1, ..., p\), such that \(t = i + pm\). The PtvarFIMA\((0, d_i, 0)\) is invertible for all \(i = 1, ..., p\), then the series

\[ \sum_{j=0}^{\infty} \pi_j^i B^j(X_{i+pm}) = \sum_{j=0}^{\infty} \pi_j^i X_{i+pm-j}, \]  

(2.6)

converge in quadratic mean.

**Proof.** The proof is similar to the first proof of the proposition (2.1). \(\square\)
2.2 Periodically correlated process

Proposition 2.3. Let \((X_t)_{t \in \mathbb{Z}}\) a PtvARFIMA\((0,d_i,0)\) where, for all \(t \in \mathbb{Z}\), there exists \(m \in \mathbb{Z}\), \(p \in \mathbb{N}^*\); \(p > 1\) is the period and \(d_{i+pm} = d_i\) with \(i = 1, ..., p\), such that \(t = i + pm\). The PtvARFIMA\((0,d_i,0)\) is a periodically correlated process.

Proof. From the equation (2.3) the proof can be obtained fairly simply, it is easy to show that \(\psi_j^{t+pm} = \psi_j^t\), is periodic with period \(p\), then

\[
\gamma_X(t + pm, t' + pm) = E(X_{t+pm}X_{t'+pm}) = E\left(\sum_{j=0}^{\infty} \psi_j^{t+pm}\varepsilon_{t+pm-j}\sum_{j=0}^{\infty} \psi_j^{t'+pm}\varepsilon_{t'+pm-j}\right) = \sum_{j=0}^{\infty} \psi_j^{t+pm}\psi_j^{t'+pm}E(\varepsilon_t^2) + \sum_{j \not= j}^{\infty} \psi_j^{t+pm}\psi_j^{t'+pm}E(\varepsilon_{t-j}\varepsilon_{t-i}) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^{t}\psi_j^{t'} = \gamma_X(t,t').
\]

\[\square\]

3 Periodic autocovariance function

3.1 Periodic autocovariance function with period two

Proposition 3.1. For \(p = 2\) then \(i = 1\) or 2, the autocovariance function of the model 2.1 is given by

\[
\gamma^i_X(h) = \begin{cases} 
\sigma_i^2 \frac{\Gamma(1-d_i-d_{i+1})\Gamma(d_{i+1}+h)}{\Gamma(d_i)\Gamma(1-d_i)\Gamma(1+h-d_i)} & \text{if } h \text{ odd} \\
\sigma_i^2 \frac{\Gamma(1-2d_i)\Gamma(d_i+h)}{\Gamma(d_i)\Gamma(1-d_i)\Gamma(1+h-d_i)} & \text{if } h \text{ even}
\end{cases}, \quad (3.1)
\]

Proof. When \(i = 1\) or 2, the autocovariance function can be calculated by

\[
\gamma^i_X(h) = \sigma_i^2 \sum_{j=0}^{\infty} \frac{\Gamma(j + d_i)\Gamma(j + d_{i+h} + h)}{\Gamma(d_i)\Gamma(1)\Gamma(d_{i+h})\Gamma(j + 1 + h)}.
\]

\(\gamma^i_X(h)\) depends to \(i\) the model \(X_{i+2m}\) is not stationary but periodically stationary.(proposition 2.3), by a simph calculus the autocovariance function may be written as

\[
\gamma^i_X(h) = \sigma_i^2 \frac{\Gamma(d_{i+h} + h)}{\Gamma(d_{i+h})\Gamma(h + 1)}F(d_i, d_{i+h} + h, 1 + h, 1), \quad (3.2)
\]
3.2 Periodic autocovariance function with period $p \in \mathbb{N}; p > 1$

where $F$ is the hypergeometric function, see Abramowitz and Stegun ([2], 556), next and using the property of hypergeometric function, we obtain

$$\gamma_i^X(h) = \sigma_i^2 \frac{\Gamma(1 - d_i - d_{i+h})\Gamma(d_{i+h} + h)}{\Gamma(d_{i+h})\Gamma(1 - d_{i+h})\Gamma(1 + h - d_i)}. \quad (3.3)$$

\[ \square \]

**Remark 1.** We will generalize this result later in the case where the period $p$ is in $\mathbb{N}$.

3.1.1 Asymptotic behavior of autocovariances with period two

**Proposition 3.2.**

$$\gamma_i^X(h) \simeq \begin{cases} \sigma_i^2 \frac{\Gamma(1-d_i-d_{i+1})}{\Gamma(d_{i+1})\Gamma(1-d_{i+1})} (h)^{d_i+d_{i+1}-1} & \text{if } h \text{ odd} \\ \sigma_i^2 \frac{\Gamma(1-2d_i)}{\Gamma(d_i)\Gamma(1-d_i)} (h)^{2d_i-1} & \text{if } h \text{ even} \end{cases} \quad (3.4)$$

For $h$ large, the components of the autocovariance function $(\gamma_i^X(h), \gamma_j^X(h))$ decay at a hyperbolic rate. (slowly decreases to 0 when $h$ tends to infinity).

**Proof.** According to (3.3) and using the standard approximation derived from Stirling’s formula, that for $h$ large, $\frac{\Gamma(d_{i+h}+h)}{\Gamma(1+h-d_i)}$ is well approximated by $(h)^{d_i+d_{i+h}-1}$. \[ \square \]

3.2 Periodic autocovariance function with period $p \in \mathbb{N}; p > 1$

Now we will generalize the results that we have facilitated in proposition 3.1 when $p \in \mathbb{N}$ i.e $i = 1, \ldots, p$ and $h \equiv k[p]$ , we take the formula 3.3, we have $p$ components

$$\gamma_i^X(h) = \sigma_i^2 \frac{\Gamma(1 - d_i - d_{i+h})\Gamma(d_{i+h} + h)}{\Gamma(d_{i+h})\Gamma(1 - d_{i+h})\Gamma(1 + h - d_i)}. \quad (3.5)$$

using the standard approximation derived from Stirling’s formula, that for large $h$. It follows that the approximation of $\gamma_i^X(h)$ is

$$\gamma_i^X(h) \simeq \sigma_i^2 \frac{\Gamma(1 - d_i - d_{i+k})}{\Gamma(d_{i+k})\Gamma(1 - d_{i+k})} (h)^{d_i+d_{i+k}-1}. \quad (3.6)$$

Then,

$$\gamma_i^X(h) \sim C_i h^{-\alpha_i}, \quad C_i > 0, \quad 0 < \alpha_i < 1, \text{ as } h \to \infty,$$

with

$$\alpha_i = -d_i - d_{i+k} + 1, \quad h \equiv k[p],$$

and

$$C_i = \sigma_i^2 \frac{\Gamma(1 - d_i - d_{i+k})}{\Gamma(d_{i+k})\Gamma(1 - d_{i+k})}. \quad (3.7)$$
4 Autocorrelation function for a PtvARFIMA model

Proposition 4.1. The periodic autocorrelation function with period $p \in \mathbb{N}$ (i.e $i = 1, \ldots, p$ and $h \equiv k[p]$) is given by

$$
\rho^X_i(h) = \frac{\Gamma(1-d_i)\Gamma(1-d_i-d_i+k)\Gamma(d_i+k+h)}{\Gamma(1-2d_i)\Gamma(d_i+k)\Gamma(1-d_i+k)\Gamma(1+h-d_i)}.
$$

(4.1)

Proof. For $p=2, i=1$ the autocorrelation function can be calculated by

$$
\rho^X_1(h) = \frac{\gamma^X_1(h)}{\gamma^X_1(0)} = \begin{cases}
\frac{\Gamma(1-d_1)\Gamma(1-d_1-d_2)\Gamma(d_2+k+h)}{\Gamma(1-2d_1)\Gamma(1-d_2)\Gamma(1-d_1+k)\Gamma(1+h-d_1)} & \text{if } h \text{ odd} \\
\frac{\Gamma(1-d_1)\Gamma(d_1+k+h)}{\Gamma(d_1)\Gamma(1+h-d_1)} & \text{if } h \text{ even}
\end{cases},
$$

similary for $i=2$, then we can deduce the (proposition 4.1) \hfill \square

From the structure of autocovariance or autocorrelation obtained, we deduce that the PtvARFIMA model is a persistent process, more precisely, the periodic autocorrelation function $\rho^X_i(h)$ does not decrease at a geometric rate, but exhibits an asymptotic behavior equivalent to $(h)^{d_i+d_i+k-1}$, with $h \equiv k[p]$ as $h$ tends to infinity, it means that these correlations are not absolutely summable. In addition, the autocorrelations are expressed as a function of the different long memory parameters, distributed periodically according to the lag $h$.

5 The graphical representations

In this section we represent graphically, the periodic autocovariance function for a period $p = 2$, with different values of $d_i$, for us we show the influence of the memory parameter $d$ on the behavior of the covariance function. We will take two cases, in the first case we take the two different values of $d_i : d_1 = 0.3, d_2 = 0.4$, in the second we take $d : d_1 = 0.09, d_2 = 0.49$.

![Figure 1: Periodic autocovariance function, with period 2 and lag $h = 0$ to 100, $d_1 = 0.3, d_2 = 0.4$.](image)

The figure(Fig 1) illustrates that the periodicity is caused by the fractional parameters $d = (0, 3; 0, 4)$, we see that, $\gamma^X_1(h) < \gamma^X_2(h)$ because the value of $d_1$ is less than the value of $d_2$,.
the periodic autocovariance $\gamma^i_X(h)$ decrease hyperbolically, for the values of $d_i$ close to 0, we find that the autocovariance function decreases rapidly towards zero.

![Graph of periodic autocovariance function](image)

Figure 2: Periodic autocovariance function, with period 2 and lag $h = 0$ to 100, $d_1 = 0.09$, $d_2 = 0.49$.

The figure (Fig 2) confirms the previous observation because for a value of d close to $\frac{1}{2}$ we notice that the autocovariance function decreases very slowly, whereas for the value of d very close to 0 we observe a strong decay towards 0 for the function of covariance. Between the two figures (Fig 1) and (Fig 2), in the second case a very important shift is observed between the functions in figure (Fig 2) for the values of d distant ($d_1 = 0.09$, $d_2 = 0.49$), while for the values of d close enough ($d_1 = 0.3$, $d_2 = 0.4$), there is less shift between the functions in Figure (Fig 1) (first case), which shows the influence of the parameters of differentiation on the values of the function as well on their behavior. Indicates a long term dependence and the presence of a cycle of different periodicity.

6 Conclusion

The article presents a model inspired by Hosking [11], considering the long memory parameter, d, a periodically time-varying function. We have described the asymptotic properties of the model, and studied the behavior of the autocovariance function, which was illustrated by the graphical representations. The results could be the opening wedge of the new subjects in the periodic ARFIMA models.

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