Conformal fields: a class of representations of Vect(N)

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Abstract
Vect(N), the algebra of vector fields in N dimensions, is studied. Some aspects of local
differential geometry are formulated as Vect(N) representation theory. There is a new class
of modules, conformal fields, whose restrictions to the subalgebra sl(N + 1) \subset Vect(N)
are finite-dimensional sl(N + 1) representations. In this regard they are simpler than
tensor fields. Fock modules are also constructed. Infinities, which are unremovable even
by normal ordering, arise unless bosonic and fermionic degrees of freedom match.

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1. Introduction

It seems to be a reasonable assumption that the physical content of a theory is independent of the choice of coordinate system. This leads by definition to the conclusion that any physical quantity must be an intrinsic object in the sense of differential geometry, i.e. that it must transform as a representation of the group $Diff(M)$ of all coordinate transformations, or diffeomorphisms, on the base manifold $M$. Something which is not a representation of $Diff(M)$ is clearly an artefact of the choice of coordinate system and thus unsuitable for physics.

Diffeomorphisms on arbitrary manifolds have been studied by mathematicians from various points of view for a long time\textsuperscript{1−6}, and they have recently attracted some interest by physicists as natural generalizations of the one-dimensional case\textsuperscript{7−12}. Nevertheless, the representation theory of $Diff(M)$ is not very well understood when $\dim M > 1$. In particular, no irreducible or lowest-weight representations are known to us\textsuperscript{13}. In order to study the diffeomorphism group it is reasonable to start with its Lie algebra of vector fields on $M$. Moreover, we only consider vector fields locally; we expect problems of homological nature to appear in a global approach, but an understanding of the local properties is certainly a prerequisite for global analysis. Such a program has of course been carried out in great detail on the circle, where it leads to the Virasoro algebra, i.e. the universal central extension of $Vect(1)^{13−15}$. The same algebra is also at the core of conformal field theory which is important to the theory of critical phenomena in two dimensions\textsuperscript{16}.

In section 2 we define $Vect(N)$, the algebra of vector fields in $N$-dimensional space. The algebra is given in a plane wave basis; this is always possible to do locally. We also show that it does not admit any central extension except when $N = 1$. Section 3 is the main part of this paper. We begin by formulating local differential geometry (tensor fields, forms and exterior derivatives) as $Vect(N)$ representation theory. Tensor fields are intimately related to tensors of $gl(N) \subset Vect(N)$. However, the largest finite-dimensional subalgebra is not $gl(N)$ but $cl(N) \cong sl(N+1) \subset Vect(N)$, which is obtained from $gl(N)$ by adding translations and conformal transformations. Any $Vect(N)$ module yields of course an $sl(N+1)$ module by restriction, but tensor fields do not correspond to finite-dimensional $sl(N+1)$ representations. We construct conformal fields, which are $Vect(N)$ modules having this desirable property, and initiate their study. Since conformal fields are more natural than tensor fields, at least regarding their conformal properties, they could be a good tool for interesting physics. Section 4 contains a brief discussion of Fock modules. It is found that infinities cannot be removed by normal ordering, but they can be cancelled between bosons and fermions. In the final section it is noted that any representation of $Vect(N)$ gives rise to a representation of the Poisson algebra in $N$ dimensions. Using the results of section 3, a host of Poisson modules can thus be written down.

2. Definition of $Vect(N)$

Select a point on a manifold $M$ such that $\dim M = N$. In some neighborhood of this point we introduce local coordinates $x = (x_1, \ldots, x_N)$, where the origin is the selected point. For simplicity we only consider coordinate patches that are hypercubic in the
sense that any function can be expanded in a plane wave basis. A basis for the function algebra is thus given by \( \{e^{im \cdot x}\}_{m \in \Lambda} \), where \( m = (m^1, \ldots, m^N) \in \Lambda \subset \mathbb{R}^N \) is a point of an \( N \)-dimensional lattice and \( m \cdot x \equiv m^\mu x_\mu \). The summation convention is implied unless otherwise stated.

A vector field locally has the form \( f(x)\partial^\mu \), which in this region is a linear combination of the basis elements

\[
L^\mu(m) = -i \exp(im \cdot x)\partial^\mu.
\] (2.1)

It is immediate to check that these generators obey the Lie algebra \( Vect(N) \), with basis \( \{L^\mu(m)\}_{m \in \Lambda} \) and brackets

\[
[L^\mu(m), L^\nu(n)] = n^\mu L^\nu(m + n) - m^\nu L^\mu(m + n),
\] (2.2)

where \( \mu, \nu = 1, \ldots, N = \dim \Lambda \), and \( m, n \in \Lambda \). This algebra is sometimes referred to as a generalized Witt algebra [6]. By means of the dual lattice,

\[
\Lambda^* = \{\alpha = (\alpha_1, \ldots, \alpha_N) : m \cdot \alpha \in 2\pi \mathbb{Z}\},
\] (2.3)

the neighborhood can be described as the torus \( \mathbb{R}^N/\Lambda^* \). All considerations in this paper are local, but on this torus the results hold globally.

By a complex rescaling of the lattice \( \Lambda \), (2.1) can be brought to the form

\[
L^\mu(m) = e^{m \cdot x}\partial^\mu,
\] (2.4)

which also satisfies (2.2). Since only algebraic properties are of interest in this paper we take this as the defining representation of \( Vect(N) \) and model other representations on it. This formalism saves many factors of \( i \), but more conventional expressions can be recovered by restricting \( \Lambda \) to a purely imaginary lattice.

\( Vect(N) \) has also another natural realization, as the algebra of holomorphic vector fields in \( N \) dimensions.

\[
L^\mu(m) = (x_1)^{m^1}(x_2)^{m^2} \cdots (x_N)^{m^N}x_\mu \partial^\mu \quad \text{(no sum on } \mu)\). \] (2.5)

In this case the basis should be restricted to \( \{L^\mu(m) | m^\mu \geq -1 \text{ and } m^\nu \geq 0 \text{ for } \nu \neq \mu\} \), which ensures that the vector fields do not have any singularities at the origin. Thus, this is a basis for vector fields which can be expanded in a Taylor series around the origin. A point worth noting is that the \( N = 1 \) version of this “amputated” algebra is a Virasoro subalgebra for any value of the central charge, because the central extension only enters in the brackets \([L(m), L(-m)]\) with \( m \geq 2 \). Hence any Virasoro module, for arbitrary \( c \), yields a representation of amputated \( Vect(1) \) by restriction. This is of course true only on the level of linear representations; once unitarity is considered, involution brings back the central charge.

We will henceforth focus on representations modelled on (2.4), but most results can be translated to the second kind (2.5) by means of the following dictionary.

\[
e^{m \cdot x} \rightarrow \prod_\mu (x_\mu)^{m^\mu}, \quad \partial^\mu \rightarrow x_\mu \partial^\mu \quad \text{(no sum on } \mu),
\]
\[m \cdot x \rightarrow \sum_{\mu} m^\mu \log(x_\mu), \quad x_\mu \rightarrow \log(x_\mu).\]  

(2.6)

At this point, there is an obvious question for anyone acquainted to the Virasoro algebra. The answer to this question is negative; \(\text{Vect}(N)\) does not admit any non-trivial central extension except when \(N = 1\). A proof of this was given already in Ref. 9, but we give the argument here for convenience. Look for a central extension of the form \(f^{\mu\nu}(m)\delta(m+n)\), where \(f^{\mu\nu}(-m) = -f^{\mu\nu}(m)\) and \(\delta(m)\) is the multi-dimensional Kronecker symbol. Let \(L_k = \alpha \cdot L(km)/\alpha \cdot m, k,l \in \mathbb{Z}, m \in \Lambda, \) and \(\alpha_\mu\) is a fixed vector. Then

\[[L_k, L_l] = (l-k) L_{k+l} + \frac{\alpha \cdot f(km)}{(\alpha \cdot m)^2} \delta((k+l)m),\]

(2.7)

so \(L_k\) satisfies a central extension of the Witt algebra. As is well known, any non-trivial such extension must be cubic. Since this relation holds for arbitrary choices of \(m\) and \(\alpha_\mu\), \(f^{\mu\nu}(m)\) must in fact be cubic itself. Make the most general ansatz possible, \(f^{\mu\nu}(m) = c^{\mu\nu}_{\rho \sigma \tau} m^\rho m^\sigma m^\tau \equiv m \cdot m \cdot m \cdot c^{\mu\nu}\), where the coefficient is separately symmetric in its three lower and two upper indices. The Jacobi identities yield the conditions

\[n^\nu m \cdot m \cdot m \cdot c^{\mu \sigma} + n^\sigma m \cdot m \cdot m \cdot c^{\mu \nu} + n^\mu m \cdot m \cdot m \cdot c^{\sigma \nu} = 3m^\nu m \cdot m \cdot m \cdot c^{\sigma \mu}\]  

(2.8)

\[m^\nu m \cdot n \cdot m \cdot c^{\sigma \mu} = n^\mu m \cdot m \cdot n \cdot c^{\sigma \nu},\]  

(2.9)

for all \(1 \leq \mu, \nu, \rho, \sigma, \tau \leq N\). These equations hold identically when \(N = 1\), but do not have a solution otherwise. E.g., in the second equation the LHS is symmetric in \(\mu\) and \(\sigma\), whereas the RHS is symmetric in \(\nu\) and \(\sigma\). Hence both sides must be symmetric in all three indices, which clearly is impossible.

We have earlier looked for non-central extensions of \(\text{Vect}(N)\) which reduce to the usual Virasoro term when \(\text{dim} M = 1^{10,11}\), but we now believe this to be an uninteresting problem. The reason for this is that it seems to be no natural way to generate non-central extensions in Fock modules, and therefore we doubt that the non-central terms have anything to do with representation theory. This point will be elaborated in section 4, where we show that normal ordering gives rise to infinite central extensions rather than to non-central ones.

Ragoucy and Sorba\textsuperscript{12} have recently discussed central extensions of current algebras in \(N\) dimensions, i.e. higher-dimensional Kac-Moody algebras. \(\text{Vect}(N)\) arises in this context as derivations of these algebras. However, not all vector fields are compatible with their extensions, which was stressed in Ref. 11. Our point of view is therefore complimentary to theirs: they look for central extensions and reject vector fields which are not compatible with these, whereas we consider arbitrary vector fields and hence reject central extensions.
3. Conformal fields

It is straightforward to formulate most aspects of differential geometry as representation theory of \( \text{Vect}(N) \). For example, a tensor field is a \( \text{Vect}(N) \) module, constructed as follows. If \( T_\mu^\nu \) is a \( gl(N) \) generator, i.e.

\[
[T_\nu^\sigma, T_\tau^\mu] = \delta_\nu^\mu T_\tau^\sigma - \delta_\nu^\sigma T_\tau^\mu,
\]

then

\[
L^\mu(m) = e^{m x} (\partial^\mu + m T^\mu)
\]
satisfies \( \text{Vect}(N) \). This is proved by direct computation:

\[
[L^\mu(m), L^\nu(n)] = \left[ e^{m x} (\partial^\mu + m T^\mu), e^{n x} (\partial^\nu + n T^\nu) \right]
\]

\[
= e^{(m+n) x} \left( n^\mu (\partial^\nu + n T^\nu) + n^\nu m T^\mu - m \leftrightarrow n \right)
\]

\[
= n^\mu e^{(m+n) x} (\partial^\nu + (m + n) T^\nu) - m \leftrightarrow n
\]

\[
= n^\mu L^\nu(m + n) - m \leftrightarrow n,
\]

where \( m \leftrightarrow n \) stands for the analogous terms with \( m \) and \( n \) (and \( \mu \) and \( \nu \)) interchanged.

This observation provides us with a host of \( \text{Vect}(N) \) representations, one for each finite-dimensional \( gl(N) \) representation. As is well known (and easy to verify), there are \( gl(N) \) modules \( T^\rho_q(\lambda) \) with bases \( v_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} \) and action

\[
T^\mu_{\nu} v_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} = \lambda \delta^\mu_{\nu} v_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} - \sum_{i=1}^p \delta^\mu_{\nu i} v_{\tau_1...\tau_q}^{\sigma_1...\sigma_i...\sigma_p} + \sum_{j=1}^q \delta^\mu_{\tau_j} v_{\nu_{\tau_1...\nu_{\tau_q}}}^{\sigma_1...\sigma_p}.
\]

The action of (3.2) on \( \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_p}(x) = v_{\tau_1...\tau_q}^{\sigma_1...\sigma_p} \otimes f(x) \) yields the corresponding \( \text{Vect}(N) \) modules.

\[
L^\mu(m) \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_p}(x) = e^{m x} \left( (\lambda m^\mu + \partial^\mu) \phi_{\tau_1...\tau_q}^{\sigma_1...\sigma_p}(x) \right.
\]

\[
- \sum_{i=1}^p m^\sigma_i \phi_{\tau_1...\tau_q}^{\sigma_i...\sigma_p}(x) + \sum_{j=1}^q \delta^\mu_{\tau_j} m^\nu \phi_{\nu_{\tau_1...\nu_{\tau_q}}}^{\sigma_1...\sigma_p}(x) \bigg). \tag{3.5}
\]

These modules, which we also denote by \( T^\rho_q(\lambda) \), are called tensor fields. There are submodules consisting of symmetric, skew-symmetric and traceless tensors, etc.

Moreover, the only class of module homomorphisms connects anti-symmetric tensor fields (forms); this is the exterior derivative. The simplest non-tensorial field is the connection, whose transformation law reads

\[
L^\mu(m) \Gamma^\rho_\sigma_\tau (x) = e^{m x} \left( \partial^\mu \Gamma^\rho_\sigma_\tau (x) - m^\rho \Gamma^\mu_\tau (x) - m^\sigma \Gamma^\rho_\tau (x) \right.
\]

\[
+ \delta^\mu_{\nu} m^\nu \Gamma^\rho_\sigma_\tau (x) + m^\rho m^\sigma \delta^\mu_\tau \bigg). \tag{3.6}
\]
By means of the connection we can define the simplest kind of binary homomorphism, the covariant derivative.

Despite their apparent simplicity, tensor fields are in a sense quite complicated objects. To see this we must consider their restriction to the largest finite-dimensional subalgebra. Morally speaking, \( Vect(N) \) has a \( gl(N) \) subalgebra consisting of rigid general linear transformations. This is not strictly true, since a vector field is an everywhere small diffeomorphism while a small linear transformation is not small sufficiently far from the origin, but it is true on the group level: \( GL(N) \subset Diff(\mathbb{R}^N) \). Therefore there is a close relationship between representations of \( Vect(N) \) and \( gl(N) \), which is evident in the case of tensor fields.

From (3.2) it follows by formal manipulations that

\[
J^\mu_\nu = \frac{\partial L^\mu(m)}{\partial m^\nu} \bigg|_{m=0} = x_\nu \partial^\mu + T^\mu_\nu
\]

satisfies \( gl(N) \). Note that we treat \( m \) as a continuous variable, which means that the manifold under consideration is really flat space \( \mathbb{R}^N \). The first term in (3.7) can be thought of as orbital angular momentum (this is what it would be if \( gl(N) \) were replaced by \( so(N) \)), and the last term is the intrinsic “spin”. It is now clear that the restriction of the \( Vect(N) \) module \( T^\mu_\nu(\lambda) \) to \( gl(N) \) is \( \Omega \oplus T^\mu_\nu(\lambda) \), where the orbital representation \( \Omega \) of \( gl(N) \) is defined by \( J^\mu_\nu = x_\nu \partial^\mu \).

However, \( gl(N) \) is not the largest finite-dimensional \( Vect(N) \) subalgebra (in the same moral sense as above). We can add translations and a kind of conformal transformations to \( gl(N) \) to obtain a new finite-dimensional subalgebra. The result is related to \( gl(N) \) in the same way as the ordinary conformal algebra is related to \( so(N) \), and therefore we call it the conformal linear algebra. The special conformal generators do not quite have the standard form, which would require introduction of additional structure in the form of a metric, but the kinship with the usual conformal algebra is obvious.

The conformal linear algebra \( cl(N) \) is the \( N(N+2) \)-dimensional Lie algebra with basis \( \{ P^\mu, T^\mu_\nu, K_\nu \}^N_{\mu,\nu=1} \) and brackets

\[
\begin{align*}
[P^\mu, P^\nu] &= 0 \quad [J^\mu_\nu, P^\sigma] = -\delta^\sigma_\nu P^\mu \\
[K_\mu, K_\nu] &= 0 \quad [J^\mu_\nu, K_\tau] = \delta^\mu_\tau K_\nu \\
[J^\mu_\nu, J^\tau_\sigma] &= \delta^\tau_\nu J^\mu_\sigma - \delta^\sigma_\nu J^\mu_\tau \quad [P^\mu, K_\nu] = \delta^\mu_\nu J^\sigma_\nu + J^\mu_\nu.
\end{align*}
\]

It is straightforward to check that \( cl(N) \) is a Lie algebra by direct verification of the Jacobi identities. It is even simpler to note that it is isomorphic to \( sl(N+1) \), which is the Lie algebra with basis \( \{ J^A_B \}^N_{A,B=1} \), subject to the conditions

\[
[J^A_B, J^C_D] = \delta^A_D J^C_B - \delta^C_D J^A_B, \quad J^A_A = 0.
\]

The isomorphism is given by the following identifications:

\[
J^A_B \equiv \begin{pmatrix} J^0_0 & J^0_\nu \\ J^\mu_0 & J^\mu_\nu \end{pmatrix} = \begin{pmatrix} -J^\sigma_\nu & -K_\nu \\ P^\mu & J^\mu_\nu \end{pmatrix},
\]
where \( A = (0, \mu), B = (0, \nu) \) are \( N + 1 \)-dimensional indices. Here and henceforth \( N + 1 \)-dimensional indices are denoted by capital Latin letters from the beginning of the alphabet. To prove the claimed isomorphism, we must e.g. verify that

\[
[J_\mu^\nu, J_\sigma^\rho] = [J_\mu^\nu, P_\sigma^\rho] = -\delta^\rho_\sigma P_\nu - \delta_\nu^\rho J_\mu^\sigma - \delta_\nu^\sigma J_\mu^\rho,
\]

(3.11)

because \( \mu \neq 0 \). The other five brackets are checked similarly.

If \( L^\mu(m) \) satisfies \( V e c t(N) \), the following generators satisfy \( c l(N) \).

\[
P^\mu = L^\mu(0), \quad J_\nu^\mu = \frac{\partial L^\mu(m)}{\partial m^\nu} \bigg|_{m=0}, \quad K_\nu = \frac{\partial^2 L^\mu(m)}{\partial m^\mu \partial m^\nu} \bigg|_{m=0}.
\]

(3.12)

This observation is the reason why \( c l(N) \) is important to understand \( V e c t(N) \), because it means that every \( V e c t(N) \) module gives rise to a \( c l(N) \) module by restriction. In particular, from the scalar representation of \( V e c t(N) \) we obtain a \( c l(N) \) representation by differentiating \( L^\mu(m) = e^{m \cdot x} \partial^\mu \) with respect to \( m \) at \( m = 0 \).

\[
P^\mu = \partial^\mu, \quad J_\nu^\mu = x_\nu \partial^\mu, \quad K_\nu = x_\nu x \cdot \partial.
\]

(3.13)

In analogy with the corresponding \( g l(N) \) representation, this deserves to be called the orbital representation of \( c l(N) \) and denoted by \( \Omega \).

The restriction of the tensor field \( T_\lambda^p(\lambda) \) to \( c l(N) \) reads

\[
P^\mu = \partial^\mu, \quad J_\nu^\mu = x_\nu \partial^\mu + T_\nu^\mu, \quad K_\nu = x_\nu x \cdot \partial + x \cdot T_\nu + x_\nu T_\sigma^\nu.
\]

(3.14)

whereas the connection (3.6) gives upon restriction

\[
P^\mu \Gamma_\tau^\rho = \partial^\mu \Gamma_\tau^\rho \\quad J_\nu^\mu \Gamma_\tau^\rho = x_\nu \partial^\mu \Gamma_\tau^\rho - \delta_\nu^\rho \Gamma_\mu^\sigma \Gamma_\tau^\rho + \delta_\nu^\sigma \Gamma_\tau^\rho - \delta_\nu^\rho \Gamma_\mu^\sigma \Gamma_\tau^\rho - \delta_\nu^\sigma \Gamma_\tau^\rho + \delta_\nu^\rho \delta_\tau^\sigma + \delta_\nu^\sigma \delta_\tau^\rho.
\]

(3.15)

Note that the \( g l(N) \) subalgebra is not able to distinguish the connection from a tensor field of type \( T_1^q(0) \); only the last two terms in the action of the special conformal generator achieve this.

Eq. (3.14) points at a fundamental incompleteness of tensor fields, which motivated us to search for a new class of representations. Since \( c l(N) \cong s l(N + 1) \) we know much about its representations; in particular the irreducible finite-dimensional representations \( T_\lambda^p((p - q)/(N + 1)) \) are \( g l(N + 1) \) tensors with \( p \) upper and \( q \) lower indices \( (\lambda = (p - q)/(N + 1) \) by tracelessness). However, restriction of the \( V e c t(N) \) module \( T_\lambda^p(\lambda) \) does not yield any of the non-trivial finite-dimensional \( s l(N + 1) \) representations, although it does yield all \( g l(N) \) modules according to (3.7). This is not surprising since tensor fields by definition are \( V e c t(N) \) modules induced from \( g l(N) \) tensors. It is thus natural to ask if there are \( V e c t(N) \) modules whose \( c l(N) \) restriction contain finite-dimensional \( s l(N + 1) \) modules. The positive answer is the main result of this paper.
Theorem 3.1

The following expression satisfies $\text{Vect}(N)$.

\[
L^\mu(m) = e^{m\cdot x} \left( \partial^\mu + m\cdot T^\mu + (1 - m\cdot x)T^\mu_0 \\
+ cm^\mu (m\cdot T\cdot x + m\cdot T^0 - m\cdot x T^0_0 - m\cdot x T_0^0) \right),
\]

where

\[
T^A_B = \begin{pmatrix} T^0_0 & T^0_\nu \\ T^\mu_0 & T^\mu_\nu \end{pmatrix}
\]
satisfies $gl(N + 1)$.

Proof: The proof straight-forward, but since it is our main result we give the details.

\[
[L^\mu(m),L^\nu(n)] = e^{(m+n)\cdot x} \left( \frac{n\mu (\partial^\nu + n\cdot T^\nu + (1 - n\cdot x)T^\nu_0)}{n\mu (\partial^\nu + n\cdot T^\nu + (1 - n\cdot x)T^\nu_0)} \\
+ cn^\nu (n\cdot T\cdot x + n\cdot T^0 - n\cdot x T^0_0 - n\cdot x T_0^0) - n\mu T^\nu_0 \\
+ cn^\nu (n\cdot T^\mu - n\mu T^\mu_0 - n\mu T^0_0 - n\cdot x T^\mu_0) + n\mu \cdot T^\nu - (1 - n\cdot x)\mu T^\mu_0 \\
+ cn^\nu (n\mu m\cdot T\cdot x - m\cdot x n\cdot T^\mu + n\mu m\cdot T^\mu_0 + n\cdot x m\cdot x T^\mu_0) \\
+ (1 - m\cdot x)cn^\nu (n\mu T^0_0 - n\cdot T^\mu_0 + n\cdot x T^\mu_0) \\
+ e^2m\mu n\nu n\cdot x (m\cdot T\cdot x + m\cdot T^0 + m\cdot x T^0_0 \\
- m\cdot T^0_0 - (m\cdot T\cdot x - m\cdot x T^0_0 + m\cdot x T_0^0) - m \leftrightarrow n
\right) - m \leftrightarrow n
\]

\[
= n\mu e^{(m+n)\cdot x} \left( \partial^\nu + (m + n)\cdot T^\nu + (1 - (m + n)\cdot x)T^\nu_0 + cn^\nu ((m + n)\cdot T\cdot x \\
+ (m + n)\cdot T^0 - (m + n)\cdot x T^0_0 - (m + n)\cdot x T_0^0) \right) - m \leftrightarrow n
\]

\[
= n\mu L^\mu(m+n) - m \leftrightarrow n.
\]  \hspace{8cm} (3.16)

We used that $m^\nu n^\mu f(m+n) - m \leftrightarrow n = n^\mu (m^\nu + n^\nu) f(m+n) - m \leftrightarrow n$ for any function that depends on $m + n$ only. \hfill \blacksquare

We claim that theorem 3.1 is the most general expression satisfying $\text{Vect}(N)$ from the following class. $L^\mu(m)$ is $\Lambda$-graded, depends on the derivative only through $e^{m\cdot x} \partial^\mu$, and it depends otherwise only on $m$, $x$ and the generators of $sl(N + 1)$. The most general ansatz in this class is

\[
L^\mu(m) = e^{m\cdot x} \left( \partial^\mu + m\cdot T^\mu + \alpha(m\cdot x)T^\mu_0 + m\mu (\beta(m\cdot x)m\cdot T\cdot x \\
+ \gamma(m\cdot x)m\cdot T^0 + \epsilon(m\cdot x)T^0_0 + \phi(m\cdot x)T_0^0) \right),
\]  \hspace{8cm} (3.17)
where \( \alpha, \ldots, \phi \) are functions of \( m \cdot x \) and \( T_B^A \in gl(N + 1) \). Note that no term proportional to \( T_0^0 \) is included because it equals \(-T_0^0\) in \( sl(N + 1) \). When this ansatz is inserted into the brackets, a slightly more general expression than that in the theorem turns out to be consistent. However, if \( T_B^A \) satisfies \( gl(N + 1) \), so does

\[
T'^A_B = \left( \frac{T_0^0}{\alpha T_0^0} \right) \left( \frac{T_0^0}{T_0^0} / \alpha \right).
\] (3.18)

Once this freedom is eliminated the expression above results.

It should be emphasized that although our motivation for the inadequacy of tensor fields depends on differentiation with respect to \( m \), which is a quite formal manipulation, the result in theorem 3.1 does not. To better understand the nature this result it is useful to introduce an \( N + 1 \)-dimensional formalism, with momenta \( m^A \) and coordinates \( x_B \), by the following definitions

\[
m^A \equiv (m^0, m^\mu) = (-m \cdot x, m^\mu), \quad x_B \equiv (x_0, x_\nu) = (1, x_\nu).
\] (3.19)

It is clear that \( m^A x_A \equiv 0 \). Moreover, we indicate contraction of \( (N + 1) \)-dimensional indices by double dots: \( m:x \equiv m^A x_A \). A single dot indicates \( N \)-dimensional contraction, as before: \( m \cdot x \equiv m^\mu x_\mu \). Thus theorem 3.1 acquires the form

**Theorem 3.2**

The following expression satisfies \( Vect(N) \).

\[
L^\mu(m) = e^{m \cdot x} \left( \partial^\mu + T_0^\mu + m:T^\mu + cm^\mu m:T:x \right),
\]

where \( T_B^A \in gl(N + 1) \) and

\[
m:x \equiv 0, \quad [\partial^\mu, n^A] = -\delta^A_0 n^\mu, \quad [\partial^\mu, x_B] = \delta^\mu_B, \quad x_0 = 1.
\]

**Proof:**

\[
[L^\mu(m), L^\nu(n)] = \left[ e^{m \cdot x} \left( \partial^\mu + T_0^\mu + m:T^\mu + cm^\mu m:T:x \right), \right. \\
e^{n \cdot x} \left( \partial^\nu + T_0^\nu + n:T^\nu + cn^\nu n:T:x \right) \left. \right]
\]

\[
= e^{(m+n) \cdot x} \left( n^\mu (\partial^\nu + T_0^\nu + n:T^\nu + cn^\nu n:T:x) - n^\mu T_0^\nu \right.
\]

\[
+ cn^\nu (-n^\mu T_0^0:x + n:T^\mu) + n^\mu T_0^\nu + cn^\nu (n^\mu T_0^0:x - n:T^\mu x_0)
\]

\[
+ n^\mu m:T^\nu + cn^\nu (n^\mu m:T:x - m:x n:T^\mu)
\]

\[
+ c^2 m^\mu n^\nu n:x m:T:x \right) - m \leftrightarrow n
\]

\[
= n^\mu e^{(m+n) \cdot x} \left( \partial^\nu + T_0^\nu + (m + n):T^\nu + cn^\nu (m + n):T:x \right) - m \leftrightarrow n
\]

\[
= n^\mu L^\nu(m + n) - m \leftrightarrow n.
\]

We used that \( m:x = n:x = 0 \).
It is now clear that we can build new Vect(N) modules by considering the action given by theorem 3.1 on elements of the form
\[ \phi(x) = \nu \otimes f(x). \]  
(3.21)
where \( \nu \) is a \( T^p_q(\lambda) \) \( gl(N+1) \) tensor and \( f(x) \) is a scalar function. We denote this \( Vect(N) \) module by \( C^\nu_q(\lambda, c) \). As examples we write down the action of \( Vect(N) \) on \( C^1_0(0, c), C^0_1(0, c) \) and \( C^0_0(\lambda, c) \).

\[
L^\mu(m)\phi^A = e^{m\cdot x}(\delta^A_0\phi^\mu - m^A\phi^\mu - cm^\mu m^A x: \phi) \\
L^\mu(m)\phi^B = e^{m\cdot x}(\delta^\mu_0 \phi^B + \delta^\mu_0 \phi^0 + \delta^\mu_0 m: \phi + cm^\mu x_B m: \phi) \\
L^\mu(m)\phi = e^{m\cdot x}(\partial^\mu \phi + \lambda m^\mu \phi + cm^\mu (\lambda m: x) \phi)
\]
(3.22)

It is clear that \( C^\nu_q(\lambda, c) \cong T^\nu_0(\lambda) \).

By differentiation of the expression in theorem 3.1 with respect to \( m \), it is found that the expressions in theorems 3.1-2 correspond to the following \( cl(N) \) generators.

\[
P^\mu = \partial^\mu + T^\mu_0 \\
J^\mu_\nu = x_\nu \partial^\mu + T^\mu_\nu \\
K_\nu = x_\nu x: \partial + x_\nu T^\sigma_\sigma - x_\nu T^\sigma_0 x + T_\nu: x + c(N + 1)(T_\nu: x - T_0: x x_\nu).
\]
(3.23)
The expression for \( K_\nu \) reads in \( N \)-dimensional notation
\[
K_\nu = x_\nu x: \partial + c(N + 1)T^0_\nu + x_\nu (T^\sigma_\sigma - c(N + 1)T^0_0) \\
- (1 + c(N + 1))x_\nu T^0_0 x + (1 + c(N + 1))T_\nu: x,
\]
(3.24)
and particularly when \( c = -1/(N + 1) \) and \( T^A_0 = 0 \),
\[
K_\nu = x_\nu x: \partial - T^0_\nu.
\]
(3.25)

From the last expression, it is clear that the restriction of \( C^p_q((p - q)/(N + 1), -1/(N + 1)) \) to \( sl(N + 1) \) is \( \Omega \oplus T^p_q((p - q)/(N + 1)) \). This means that these modules give rise to all finite-dimensional representations of the conformal algebra \( sl(N + 1) \) upon restriction, which would motivate to name them conformal fields.

Let us finally discuss on the physical meaning of some of the parameters characterizing conformal fields. To this end we make some simple observations. \( P^\mu \) is the generator of rigid translations, i.e. the momentum operator, and we can therefore identify its eigenvalue in a \( P^\mu \) eigenstate with the momentum of this state. Since \( P^\mu = \partial^\mu + T^\mu_0, T^\mu_0 \) takes the role of a characteristic momentum, and it is tempting to identify an eigenvalue of \( T^0_N \) as a mass, \( N \) being the time direction. The eigenstates of the dilatation operator \( J^\mu_\mu = x: \partial + T^\mu_\mu \) are scale invariant, wherefore the eigenvalues of \( T^\mu_\mu \) should give critical exponents, which could be relevant to critical systems in \( N \) dimensions.
4. Fock modules

In this section we discuss the construction of Fock modules for \( \text{Vect}(N) \), and substantiate the claim in section 2 that infinite central extensions arise. Any Fock module, or more generally any lowest-weight module, is characterized by a \( \mathbb{Z} \)-gradation and a lowest-weight state of minimal degree. We focus our attention to modules whose gradation is by one component of the momentum. If this is the time component, the gradation is by energy and the lowest weight can be thought of as a mass. Other \( \mathbb{Z} \)-gradation are clearly possible, e.g. according to the value of the dilatation operator.

In a Fourier transformed basis, a tensor field is a \( \text{Vect}(N) \) module with basis \( \{ \phi(n) \}_{n \in \Lambda} \) and action

\[
L^\mu(m) \phi(n) = (n^\mu + m \cdot T^\mu) \phi(m + n) \quad (4.1)
\]

A slightly more general representation is found by considering \( \psi(n) = \phi(n + h) \), \( h \) a constant vector. The action follows from (4.1) by replacing \( n^\mu \) by \( n^\mu + h^\mu \). Of course, \( \psi \) and \( \phi \) are related by a change of basis if \( h \in \Lambda \), so \( h \) is only defined modulo \( \Lambda \). It is tempting to interpret \( h \) as a “mass”, or rather as a characteristic momentum which might point in the time direction. Note that this “mass” is related to conformal weights of primary Virasoro fields.

Let \( a(m) \) and \( \bar{a}(n) \) be bosonic oscillators, satisfying the canonical commutation relations

\[
[a(m), \bar{a}(n)] = \delta(m + n), \quad [a(m), a(n)] = [\bar{a}(m), \bar{a}(n)] = 0. \quad (4.2)
\]

Then the following expression satisfies \( \text{Vect}(N) \).

\[
L^\mu(m) = -\sum_{s \in \Lambda} \bar{a}(m-s) (s^\mu + h^\mu + m \cdot T^\mu) a(s), \quad (4.3)
\]

and the action of (4.3) on \( a(n) \) is the shifted variant of (4.1). This representation extends naturally to arbitrary polynomials in \( a(n) \), i.e. symmetrized tensor powers of (4.1). Because (4.3) commutes with the bosonic number operator

\[
\sum_{s \in \Lambda} \bar{a}(m-s) a(s), \quad (4.4)
\]

each monomial in \( a \) is closed under the action of \( \text{Vect}(N) \).

To make this into a Fock module, we introduce a division of the lattice \( \Lambda \),

\[
\Lambda = \Lambda_- \cup \{0\} \cup \Lambda_+, \quad (4.5)
\]

and write \( m > 0 \) \( (m < 0) \) if \( m \in \Lambda_+ \) \( (m \in \Lambda_-) \). The decomposition must be such that \( m, n > 0 \) implies that \( m + n > 0 \) and \( -m < 0 \). A division of this kind can e.g. be defined by introducing a constant vector \( k_\mu: m > 0 \) iff \( k \cdot m > 0 \). If there are non-zero points satisfying \( k \cdot m = 0 \) some extra effort has to be taken to divide these points equally between the two halves. We can now define a vacuum state \( |0\rangle \) by \( \bar{a}(0)|0\rangle = 0 \) and \( a(m)|0\rangle = \bar{a}(m)|0\rangle = 0 \) for all \( m < 0 \). This means that \( L^\mu(m)|0\rangle \) also vanishes for all \( m < 0 \), where \( L^\mu(m) \) given
by (4.3), because either $a(s) < 0$ or $\bar{a}(m - s) < 0$ and the two commute. However, the action of $L^\mu(0)$ diverges.

$$L^\mu(0) |0\rangle = - \sum_{s \in \Lambda} \bar{a}(-s) (s^\mu + h^\mu) a(s) |0\rangle = \sum_{s > 0} (s^\mu + h^\mu) |0\rangle. \quad (4.6)$$

When $N = 1$, the standard approach to avoid this infinity is normal ordering, but that idea does not work for $N > 1$. To see what goes wrong, consider the case $h^\mu = T^\mu_\nu = 0$. The only normal ordered generators differing from (4.6) are

$$L^\mu(0) = - \sum_{s < 0} \bar{a}(-s) s^\mu a(s) - \sum_{s > 0} a(s) s^\mu \bar{a}(-s), \quad (4.7)$$

When computing $[L^\mu(m), L^\nu(n)]$ we pick up a central term proportional to $\delta(m + n)$, and the proportionality constant is

$$\sum_{0 < s < m} s^\mu (m^\nu - s^\nu). \quad (4.8)$$

In one dimension this sum can be readily performed, yielding $(m^3 - m)/6$ ($c = -2$), but when $N \geq 2$ the set of points between 0 and $m$ is infinite. More precisely, the sum is proportional to $\infty^{N-1}$, where $\infty$ is the number of integers. Of course, this is a signal that the normal ordering prescription breaks down, which is in accordance with the result of section 2 that there is no central extension.

The same divergence has been noted by Figueirido and Ramos$^9$, who draw the bold conclusion that the Jacobi identities have to be abandoned. We propose a less drastic way out. Note that the same steps can be repeated with fermionic oscillators, satisfying canonical anti-commutation relations

$$\{b(m), \bar{b}(n)\} = \delta(m + n), \quad \{b(m), b(n)\} = \{\bar{b}(m), \bar{b}(n)\} = 0, \quad (4.9)$$

and the $Vect(N)$ generators given by (4.3) with all $a$’s replaced by $b$’s. We find that

$$L^\mu(0)|0\rangle = - \sum_{s > 0} (s^\mu + h^\mu) |0\rangle. \quad (4.10)$$

If we now consider a theory with $N_B$ bosonic and $N_F$ fermionic species, with “masses” $h_i$ and $h_j$, respectively, the total vacuum eigenvalue becomes

$$L^\mu(0)|0\rangle = \sum_{s > 0} \left((N_B - N_F)s^\mu + \left(\sum_{i=1}^{N_B} h_i^\mu - \sum_{j=1}^{N_F} h_j^\mu\right)\right)|0\rangle. \quad (4.11)$$

This expression vanishes, in spite of the divergent sum over $s$, provided that $N_F = N_B$ and $\sum h_i^\mu = \sum h_j^\mu$. 

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Eq. (4.11) is an expression of type $\infty \times 0$, which in general could be anything, but we argue that it must vanish for the following reason. If the sum over $s$ is cut off at large momenta, we obtain generators which obey some approximation to $\text{Vect}(N)$. In the limit that the cutoff approaches infinity, this approximation should be increasingly good. However, (4.11) is identically zero for every finite cutoff, and thus the limit is also zero. Note that the result was formulated for tensor fields, but it also holds for conformal fields, because $L^\mu(0)$ is of the form (4.6) with $h^\mu = T^\mu_0$.

The impossibility of normal ordering is not a peculiarity of the plane wave basis. We can write down an expression for the generators using a position space basis,

$$L^\mu(m) = -\int d^N x e^{\mu \cdot x} \bar{a}(x) (\partial^\mu + h^\mu + mT^\mu) a(x), \quad (4.12)$$

where

$$[a(x), \bar{a}(y)] = \delta(x-y), \quad (4.13)$$

The oscillators can be expanded as

$$a(x) = \sum_k a_k \varphi_k(x), \quad \bar{a}(x) = \sum_k \bar{a}_k \varphi_k(x), \quad (4.14)$$

where $\{\varphi_k\}_{k \in I}$ is a complete orthogonal function basis and $I$ is an index set. We can now define a total order based upon the first component of $k$ and a corresponding Fock module. Divergences arise because the other components of $k$ can take infinitely many values, except in one dimension where a single index suffices to label a complete set of functions. In particular we can use the spherical basis $x = (r, \Omega)$, $k = (n, l)$ and $\varphi_k(x) = r^n Y_l(\Omega)$, where $Y_l$ is the $N$-dimensional spherical harmonics. In this way we obtain Fock modules graded according to the dilatation eigenvalue.

To summarize, the vacuum eigenvalue does not diverge provided that the number of bosonic and fermionic degrees of freedom are the same, as well as the total bosonic and fermionic “masses”. This condition is slightly reminiscent of supersymmetry, but it is not equivalent. The $\text{Vect}(N)$ generators do not have any fermionic partners, and the bosonic and fermionic number operators still commute with $L^\mu(m)$, wherefore Fock modules decompose into sectors with a fixed number of particles. The situation is thus somewhat paradoxical; bosons and fermions do not transform into each other, but they interact in a subtle way through the vacuum to remove infinities.

The Fock space construction is completely general and can be applied to other multi-graded Lie algebras, e.g. $\text{Map}(N, g)$, the algebra of maps from $N$-dimensional space to a finite-dimensional Lie algebra $g$. If $M^a$ are matrices in a finite-dimensional representation of $g$,

$$T^a(m) = \sum_{s \in \Lambda} \bar{a}(m - s) M^a a(s), \quad (4.15)$$

satisfies $\text{Map}(N, g)$ (representation indices are suppressed). If this expression is normal ordered, one picks up a central extension proportional to the number of points between 0 and $m$, i.e. infinity. Again, this infinity could be cancelled against a fermionic contribution.
Eq. (4.3) defines a quadratic embedding of Vect\((N)\) in an infinite Heisenberg algebra. There are two other important quadratic embeddings of Vect\((1)\): in Kac-Moody algebras (Sugawara construction) and in the algebra of bosonic string oscillators\(^{14}\). However, it is easy to see that neither of these constructions have any higher-dimensional counterpart, even on the classical level, because the index structure would be wrong. The Sugawara construction would be something like

\[
L^\mu(m) = \sum_{s \in \Lambda} T^a(m - s) T^a(s),
\]

which does not make sense because there is a vector to the left and a scalar to the right. String oscillators should satisfy

\[
[a(m), a(n)] = m^\mu \delta(m + n),
\]

but since \(m^\mu\) has a vector index \(a(m)\) would have to be “half-vector”, which we do not know how to treat.

5. Discussion

The results in section 3 can be used to construct representations of the algebra of Poisson brackets in a \(N\)-dimensional phase space (\(N\) even). It is given by the brackets

\[
[f, g] = \omega_{\mu \nu} \partial^\mu f \partial^\nu g,
\]

where \(\omega_{\mu \nu}\) is the constant, anti-symmetric, non-degenerate symplectic form. By expanding the functions in the plane-wave basis \(\{E(m) \equiv e^{m \cdot x}\}_{m \in \Lambda}\), we obtain

\[
[E(m), E(n)] = \omega_{\mu \nu} m^\mu n^\nu E(m + n).
\]

The adjoint representation of (5.2) is given by

\[
E(m) = e^{m \cdot x} \omega_{\mu \nu} m^\mu \partial^\nu.
\]

We now note that the defining Vect\((N)\) representation (2.4) is given by \(L^\mu(n) = e^{m \cdot x} \partial^\mu\), and thus

\[
E(m) = \omega_{\mu \nu} m^\mu L^\nu(m)
\]

satisfies (5.2), provided that \(L^\mu(n)\) is in the defining representation. However, it is easy to check that a sufficient condition for the expression (5.4) to satisfy (5.2) is that \(L^\mu(n)\) obeys (2.2) \((L^\mu(m)\) commutes with \(\omega_{\sigma \tau}\)), and hence we can insert any Vect\((N)\) representation in (5.4) to obtain a new representation of the Poisson algebra. Together with the results of section 3 this gives many new representations.
The Poisson algebra admits a Lie algebra deformation, the Moyal algebra, which takes the form
\[ [E(m), E(n)] = (e^{i\hbar \omega_{\mu\nu} m^\mu n^\nu} - e^{-i\hbar \omega_{\mu\nu} m^\mu n^\nu}) E(m + n). \] (5.5)

After a trivial rescaling of the generators, the \( \hbar \to 0 \) limit of (5.5) is clearly (5.2). Replacing the Poisson algebra by the Moyal algebra is one route to quantization, advocated by Bayen et al.\(^{17}\) (see also Ref. 18). It is now natural to ask if (5.4) also can be deformed, i.e. if one can make the substitution \( \partial^\mu \to e^{-m \cdot x} L^\mu(m) \) in the adjoint representation
\[ E(m) = (e^{i\hbar \omega_{\mu\nu} m^\mu \partial^\nu} - e^{-i\hbar \omega_{\mu\nu} m^\mu \partial^\nu}). \] (5.6)

The answer is negative; it seems impossible to generalize (5.4) to the Moyal algebra.

We hope that the new representations of \( Vect(N) \) discovered in this paper could eventually have some applications to physics. A rather obvious field is quantum gravity, which almost by definition is intimately related to action of the diffeomorphism group. It is safe to say that quantized gravity theories based on tensor fields have not been very successful; conformal fields may fare better. Another idea could be to look for a classification of \( N \)-dimensional phase transitions, similar to conformal field theory in two dimensions. Of course, the diffeomorphism group is much bigger than the conformal group, even in two dimensions, so this would require much more than a direct generalization of conformal field theory to \( N \) dimensions. On the other hand, unless we wish to consider artefacts of the choice of coordinate system, it is hard to see how arbitrary diffeomorphisms can fail to be a symmetry of any sensible theory. However, such a theory will presumably include gravity.

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