A Simple Method for Computing Soliton Statistics

Donald Spector†

Department of Physics, Eaton Hall
Hobart and William Smith Colleges
Geneva, NY 14456 USA

I propose a simple method for identifying the statistics of kink-type solitons in a large class of theories. The method is based on the Witten index, but can in fact be used to determine soliton statistics in non-supersymmetric theories as well.
The spectra of field theories with multiple classical vacua include kinks, solitons corresponding to field configurations that interpolate between vacua. Even though such solitons are field configurations of a bosonic field, the kinks themselves can in fact be fermionic \[1\]. In this letter, I will propose an extremely simple method for identifying the statistics of certain kinks. The method is so simple because it takes advantage of the Witten index, and the Witten index can be computed reliably in very simple ways \[2\]. Importantly, however, the applicability of this method is not restricted to supersymmetric theories. Because a discrete quantity, such as the statistics, cannot change under continuous deformations of the parameters of a theory, if we know the statistics of the kink in a supersymmetric theory, we know too the statistics of the kink in any non-supersymmetric theory which can be reached by a smooth change in the parameters of the original theory. In this way, then, we can use results on statistics obtained in supersymmetric theories (where the Witten index turns out to make the relevant calculations simple) to obtain results about statistics in non-supersymmetric theories.

This paper is meant to present the method and to highlight its essential features, and so I will proceed in the way that seems most effective, which is by means of an example, followed by more general discussion. The model I consider is in 2+1 dimensions, but there is nothing special about this case; as will be readily apparent, the method I use is valid in any number of dimensions (although, of course, results in specific models will vary) \[1\].

The outline of the paper is as follows. First, we see how in supersymmetric theories the Witten index leads to an easy identification of the statistics of the states corresponding to classical vacua. I then argue that the determination of vacuum statistics leads to a determination of the statistics of the kink solitons that interpolate between those vacua. Next, I discuss the extension of these results to non-supersymmetric theories by means of continuity arguments, and thus find a large class of non-supersymmetric theories in which our arguments easily give the kink statistics. I then discuss some aspects of the general application of this method, and present briefly some of the results and implications of this method in other theories. I close with mention of some open problems.

To begin the argument, then, consider a 2+1-dimensional model with \(N = 1\) supersymmetry. (See \[3\] for a review of 2+1 dimensional supersymmetry.) The simplest such theories may be described in terms of a single real scalar superfield \(\Phi(x^\mu, \theta^\alpha)\), where

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1 The potential role of anyons, which are special to 2+1 dimensions, is discussed in the final section of this paper.
\( \theta^\alpha \) is a real, two-component anticommuting coordinate. The superfield \( \Phi \) contains as its physical components a real scalar field \( \phi \) and a real, two-component fermion \( \psi^\alpha \). Defining the covariant derivative \( D_\alpha = \partial/\partial \theta^\alpha + i \theta^\beta \gamma_\alpha^\beta \partial_\mu \), we can write a supersymmetric Lagrangian

\[
\mathcal{L} = \int d^2 \theta (\frac{1}{2} D_\alpha \Phi D^\alpha \Phi + W(\Phi)) ,
\]

(1)

where \( W(\Phi) \) is the superpotential. In terms of the physical components, this theory has Lagrangian

\[
\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{i}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi + \left( \frac{\partial W}{\partial \phi} \right)^2 + \frac{\partial^2 W}{\partial \phi^2} \bar{\psi} \psi .
\]

(2)

The Witten index of a supersymmetric theory is the number of bosonic zero energy states minus the number of fermionic zero energy states, written formally as \( \text{tr}(-1)^F \). Because states of non-zero energy in a supersymmetric theory must appear in bose-fermi pairs, the index may be calculated exactly using an approximation scheme that respects supersymmetry; for example, both perturbative and semiclassical arguments give exact values for the Witten index. Similarly, the index is unaffected by supersymmetry-preserving changes in the parameters of a theory (as long as these do not change the behavior of the potential at infinity); states can only be lifted from or lowered to zero energy in bose-fermi pairs under such variations, and so such variations cannot change the Witten index.

We now consider evaluating the index for theories of the type described above in (1). For the purposes of this paper, it is sufficient to restrict our attention to superpotentials of the form

\[
W(\Phi) = \frac{g}{3} \Phi^3 - \alpha \Phi ,
\]

(3)

where \( g \) and \( \alpha \) are real. Without loss of generality, we take \( g > 0 \). Note that the Witten index in this theory is unaffected by changes in \( \alpha \). We will be interested in comparing the Witten index at negative and positive values of \( \alpha \).

We will calculate the Witten index by identifying the minima of the classical potential which have zero energy, and then examining the perturbative spectrum about each such minimum. Supersymmetric classical minima are given by the condition

\[
\frac{\partial W}{\partial \phi} = 0 ,
\]

(4)

which in our case yields

\[
\phi^2 = -\frac{\alpha}{g} ,
\]

(5)
In the case that $\alpha$ is negative, this equation (5) has no solutions, as $\phi$ is a real field. Consequently, at the classical level already, there are no zero energy states, and so the Witten index is zero.

In the case that $\alpha$ is positive, however, this equation (5) has two solutions,

$$\phi = \pm \sqrt{\frac{\alpha}{g}}.$$  

(6)

It is easy to check that all the perturbative excitations about each of these minima are massive. Consequently, the Witten index is given simply by counting up the classical vacua. Here there are two such vacua, and so, initially, one might expect the Witten index to have value two.

However, the Witten index cannot be affected by changes in $\alpha$, and we saw unambiguously that, for $\alpha < 0$, the Witten index is zero. Therefore, the Witten index must be zero even for $\alpha > 0$. In order for this to be so, one of the vacua in (6) must be bosonic and the other vacuum must be fermionic. Then these two classical vacua enter the Witten index sum with opposite signs, producing a Witten index of zero.

Thus we have established quite easily that the two vacua have opposite fermion number. Let us now consider the theory with $\alpha > 0$ a little bit further. As we know from (6), the scalar potential

$$V(\phi) = (g\phi^2 - \alpha)^2,$$  

(7)

has two minima. There are kink-type solitons which interpolate between these two classical vacua. What can we say regarding the statistics of such solitons? In the next paragraphs, I would like to argue that the only physically sensible picture is that these solitons are themselves fermions.

The physical idea underlying our argument is that since we can move from one minimum of the potential to the other by, loosely speaking, acting on the initial vacuum state with a soliton operator, and since the vacua have opposite fermion number, then the soliton itself must be a fermion, not a boson. We will now refine this informal argument into a more fully developed physical picture.

Let the two minima of the potential occur at $\phi(x^\mu) = \phi_1$ and $\phi(x^\mu) = \phi_2$. Suppose we have an operator that converts the first vacuum into the second one; clearly such an operator must be fermionic. Suppose, then, that a soliton is created, passes through space, and continues on far past some observer. What does such an observer see? Initially, the observer sees the first vacuum state, $\phi(x^\mu) = \phi_1$. The soliton is created (i.e., a
soliton creation operator acts on this vacuum), so that the field configuration \( \phi(x^\mu) \) is an interpolation from \( \phi_2 \) to \( \phi_1 \). This soliton then goes past the observer. After a sufficiently long time, the observer would see the state of the system to be, to arbitrarily good accuracy, indistinguishable from the second vacuum, for which \( \phi(x^\mu) = \phi_2 \) everywhere. If no soliton had appeared, this transition would not have occurred. Thus, the action of the soliton on the first vacuum has been, in effect, to convert it into the second vacuum. Evidently, then, the kink soliton must be fermionic.

We can refine this argument further by placing space on a tube with the topology of \( R \times S^1 \). To have an easy way to refer to the ends of this tube, let the left end be the end of the tube at \( -\infty \) on the \( R \)-axis, and let the right end be the end of the tube at \( +\infty \) on the \( R \)-axis. With \( \phi_1 \) and \( \phi_2 \) the field values corresponding to the two classical vacua, one ordinarily formulates this theory with four sectors, corresponding respectively to the following boundary conditions: (1) \( \phi = \phi_1 \) at both ends; (2) \( \phi = \phi_2 \) at both ends; (3) \( \phi = \phi_2 \) on the left and \( \phi = \phi_1 \) on the right; and (4) \( \phi = \phi_1 \) on the left and \( \phi = \phi_2 \) on the right. The lowest energy state in each of these sectors corresponds respectively to the first classical vacuum; the second classical vacuum; a soliton; and an anti-soliton.

However, for our purposes, it is advantageous to use an alternative formulation of the theory in which there are only two sectors. These sectors are given, respectively, by the following boundary conditions: (1) \( \phi = \phi_1 \) on the left; and (2) \( \phi = \phi_2 \) on the left. The lowest energy states in these two sectors are, respectively, the first and the second classical vacua, since in each sector, the energy is minimized by \( \phi(x^\mu) \) having a uniform value, with the particular value fixed by the boundary condition on the left. Thus we still have the same vacuum structure as with the more familiar formulation.\(^2\) The kink field configurations still exist with the boundary conditions of the two-sector formulation; they simply are not stable. The instability of the soliton is not a problem here; it is, in fact, an advantage, as we can determine some of the properties of the soliton by studying what it can evolve into, just as we routinely determine the properties of particles produced at accelerators by studying their decays.

Given a kink field configuration, which interpolates between the two classical vacua, it can be deformed continuously into a vacuum configuration, while staying within the space of field configurations which have finite energy and respect the boundary condition, through a continuous change in the value of the field at the right end of the tube. (The

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\(^2\) Note, too, that for an infinitely long tube, there is no tunneling between these vacua.
end of the tube is a circle of finite radius.) Consequently, to an observer near the right end of the tube, the soliton will not appear as a sensible approximate notion. However, to an observer near the left end of the tube, as long as the tube is extremely long, the soliton will appear as a sensible approximate notion. Thus, near the left end of the tube, it is physically sensible to speak of creating a kink, and then to watch its evolution in time. We can use the quantum numbers of the final state to help us determine the quantum numbers of the soliton.

So now let us consider this two-sector version of the theory, and let us imagine that initially the system is in the first vacuum state. Now imagine a soliton is created at the left end of the tube, so that $\phi = \phi_2$ on the left, but $\phi = \phi_1$ still on the right, with the velocity of the soliton pointing toward the right end of the tube. As the kink moves rightward, it leaves behind a longer and longer stretch over which the field $\phi$ has the value $\phi_2$. In the far future, when the disturbance reaches the right end of the tube, the value of the field at the right end will change smoothly from $\phi_1$ to $\phi_2$ (the finite radius of the circle $S^1$ makes this possible), leaving the system finally in the second vacuum state.

In other words, the state created by acting on the first vacuum with a soliton creation operator evolves into the second vacuum state. Since the first and second vacuum states have opposite fermion number, then, the soliton must itself be fermionic.

Obviously, this all works, as it must, in the other direction, too. The state created by acting on the second vacuum with an anti-soliton creation operator can evolve into the first vacuum, verifying that the anti-soliton, like the soliton, has fermion number $+1$.

Thus we see that we have used a Witten index calculation to identify the fermion number of a soliton. It is striking to have such a simple way to determine that a soliton is a fermion. In the remainder of this paper, I examine some of the details associated with this method, and explore some of its consequences. These remarks are meant to cover the essentials, not to be exhaustive.

First, there is a large class of non-supersymmetric theories in which the above type of argument still determines the statistics of kink solitons. This is because the statistics of a soliton will not be changed by any deformation of the parameters of the theory, including deformations that violate supersymmetry, as discrete quantum numbers cannot change under continuous changes in the parameters. Thus in any model which can be obtained from a supersymmetric theory by continuous (non-supersymmetric) changes in the parameters of that theory, and which has essentially the same structure of classical
vacua (e.g., under the change of parameters, classical vacua should not appear, disappear, or coalesce), the statistics of the soliton will be the same as in the supersymmetric theory.

Some clarification of these remarks is in order in the case of two spatial dimensions, where anyon statistics can occur. In this paper, we should take the following as the definition of \((-1)^F\) in the 2+1 dimensional example we study. The operator \((-1)^F\) is the operator that squares to unity; anticommutes with the supercharges; and (anti)commutes with those fundamental fields which are quantized according to (anti)commutation relations. Since the theories I am considering can always be formulated in terms of such quantum fields, this provides a complete definition of \((-1)^F\). Statistics then refers to the eigenvalue of this operator, which must be \(\pm 1\). In the 2+1 dimensional model in question, we have an extended object whose \((-1)^F\) eigenvalue is opposite to that of the local field out of which it is built.

In three or more spatial dimensions, such a definition of statistics is identical to the notions of statistics more conventionally defined. In two spatial dimensions, however, there is an alternative definition of statistics which is more familiar and conventional, namely statistics defined in terms of the relative phases associated with multiparticle configurations in the functional integral, which gives rise to the possibility of fractional statistics realized by anyons. This definition of statistics is distinct from the one considered above, and allows, in 2+1 dimensions, for the possibility of statistics which change continuously as the parameters of the theory change. In higher dimensions, where anyons are not possible, this possibility does not arise. In order to preserve a discrete notion of statistics in 2+1 dimensions, therefore, we use the less conventional definition of statistics and \((-1)^F\) given above. It is worth commenting here that in the example model in this paper, one can argue that the soliton must have half-integral spin, by using a Witten index argument based on the conventional definition of statistics, along with the results of [4]. Indeed, one can combine the method here with the results in [5] (which connects these different notions of statistics) to obtain further results on kinks in 2+1 dimensional supersymmetric theories.

Second, one might worry that non-zero fermion number might be emitted in the evolution of a soliton into a vacuum state. We can use the invariance of the kink statistics under changes in the parameters of a theory to argue that this problem does not arise. Let us imagine deforming our original theory so that the scalar vacua remain unchanged, but so that the perturbative fermionic particles are more massive than the soliton. This deformation cannot change a discrete quantum number such as the kink statistics. It also
does not alter our argument connecting vacuum statistics to kink statistics. Consequently, in this modification of the original theory, clearly the only particles that can carry off the released energy when the soliton decays are bosons, and so no statistics are lost to particle emission. Thus, we can safely conclude, as we did above, that the kink is fermionic in the deformed theory. By continuity, the soliton must be fermionic in the original theory as well, and so even if the perturbative fermions are light enough to be emitted in the transition from a soliton to a vacuum background, they must be admitted in pairs to maintain consistency with the results we obtained at large fermion mass.

Third, note that the basic structure of the argument presented here does not depend on dimension (despite the special issues associated with anyons in two spatial dimensions). However, the argument does distinguish between the cases of real and complex scalar fields. Now in some dimensions and some situations, supersymmetry only requires real fields, whereas in other cases complex fields are necessary. Thus, while the methods described above may be applied in any number of dimensions, the conclusions one draws will be different depending upon whether the supersymmetric theory involves real or complex fields. In fact, the distinctions between real and complex fields may actually be used to relate the computation of the Witten index to the fundamental theorem of algebra (and to the failure of such a theorem to hold for polynomials over the reals) [6].

Fourth, it is worth noting that in such a familiar example as the sine-Gordon model, our argument reproduces in simple fashion the familiar result that the solitons in that model are fermions [6].

An important extension of this argument would be to generalize it to determine the statistics of other extended objects which do not have the same statistics as their constituent field(s). Topological solitons such as the skyrmion provide a prime example of this transmutation of statistics, yet are not covered by the argument presented above.

Finally, it would be instructive to connect the arguments of this paper with more conventional treatments of soliton statistics. For example, one can directly determine the statistics of classical vacua; indeed, performing such a calculation in supersymmetric quantum mechanics explicitly yields a structure of the kind inferred via Witten index arguments in the above model [6]. One can already see from the results presented that there is a kind of “Berry’s sign” that arises in mapping from one of the vacua to the other in the model theory I have discussed in this paper.
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