*hynet*: An Optimal Power Flow Framework for Hybrid AC/DC Power Systems

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**Abstract**—High-voltage direct current (HVDC) systems are increasingly incorporated into today’s AC power grids, necessitating optimal power flow (OPF) tools for the analysis, planning, and operation of such hybrid systems. To this end, we introduce *hynet*, a Python-based open-source OPF framework for hybrid AC/DC grids with point-to-point and radial multi-terminal HVDC systems. *hynet*’s software design promotes ease of use, extensibility, and a manifold of solving options, which range from interior-point methods to relaxation-based solution techniques. This paper introduces the underlying mathematical framework, including the system model and OPF formulation. To support large-scale hybrid grids, the presented model balances modeling depth and complexity to offer both adequate accuracy and computational tractability. Additionally, the OPF formulation is simplified by a proposed state space relaxation that unifies the representation of AC and DC subgrids. Furthermore, two supported convex relaxations of the OPF problem and some related results are discussed and generalized. Finally, *hynet*’s software design is illustrated and related to the presented mathematical framework, while highlighting its amenability to extensions.

**Index Terms**—Power system modeling, hybrid power systems, HVDC transmission, power system simulation, optimal power flow, power system economics, optimization, convex relaxation.

**I. INTRODUCTION**

To counteract the climate change, many countries consider a decarbonization of the energy sector, especially via a transition of electricity generation based on fossil fuels toward renewable energy sources (RES) [1], [2]. This transition introduces an increasingly distributed and fluctuating energy production, which generally necessitates additional transmission capacity as well as stronger interconnections of regional and national grids to balance and smooth the variability of RES-based generation [2], [3]. In this regard, high-voltage direct current (HVDC) systems are considered as a key technology due to their advantages in long-distance, underground, and submarine transmission as well as their ability to connect asynchronous grids and, in case of voltage source converter (VSC) HVDC systems, to provide flexible power flow control and reactive power compensation [2], [3]. Already today, a large number of point-to-point HVDC (P2P-HVDC) systems and several multi-terminal HVDC (MT-HVDC) systems are installed and many are planned [2], [4]. With the above developments, this trend is destined to continue, leading to large-scale hybrid AC/DC power systems.

Due to the importance of the optimal power flow (OPF) in operational and grid expansion planning as well as techno-economic studies, these structural changes necessitate an OPF framework for large-scale hybrid AC/DC power systems. OPF denotes the optimization problem of identifying the cost-optimal allocation of generation resources and the corresponding system state to serve a given load, while satisfying all boundary conditions of the grid. It involves a large number of optimization variables, system constraints, and, to accurately capture the physics, the power flow equations based on Kirchhoff’s laws, which render the problem inherently nonconvex and challenging to solve. As the OPF problem specification and solution is rather involved, a software framework for this task is desired. Furthermore, for transparency, reproducibility, and flexible adoption in research, it should be available as open-source software. Several open-source software packages for OPF computation have already been published, including the established toolboxes MATPOWER [5], [6] (and its Python-port PYPower [7]) and PSAT [8], [9] as well as the recently released PowerModels [10], [11] and pandapower [12], [13]. While PSAT is targeted at small to medium-sized systems, MATPOWER, PowerModels, and pandapower also support large-scale systems, but they are limited to a simple model of P2P-HVDC systems and do not support MT-HVDC systems.

To close this gap, we developed *hynet* [14], an open-source OPF framework for hybrid AC/DC grids with P2P- and radial MT-HVDC systems. In the process, particular care was taken that *hynet* is (a) freely accessible, (b) very easy to use, and (c) extensible, while featuring a (d) solid and rigorous mathematical model and inherent support of (e) convex relaxations. To address (a), *hynet* was written in Python [15], a popular high-level open-source programming language that is freely available for all major platforms. For (b), the interface to the system and result data was designed similar to MATPOWER, which has been widely successful due to its intuitive accessibility. For (c), we introduced an object-oriented software design, which renders the framework’s structure clear and intuitive while offering inherent support for extensions. For (d), we developed a mathematical model formulation with a substantial modeling depth, while providing several layers of notational abstraction to support its adoption in future works and OPF extensions. Finally, for (e), we devised a state space relaxation that unifies the representation of AC and DC subgrids in the OPF formulation, which simplifies the study of convex relaxations for hybrid AC/DC grids substantially.

**A. Contributions and Outline**

In the following, Section II and III presents the system model for hybrid AC/DC power systems as implemented in *hynet*. This model aims at a balance of modeling depth and
complexity to offer adequate accuracy while being tractable for large-scale hybrid grids. To this end, a novel converter model as well as a unified representation of AC and DC subgrids is introduced. The latter simplifies the mathematical modeling significantly and enables the straightforward generalization of previously derived AC models to DC subgrids. Subsequently, Section IV formulates the corresponding OPF problem, which is then simplified by a proposed state space relaxation to unify the voltage representation. Besides streamlining the implementation, which is utilized in hynet, this unified OPF formulation further enables straightforward convex relaxation. This is illustrated in Section V, which discusses two popular convex relaxations of the OPF problem, i.e., the semidefinite and second-order cone relaxation, which are both included in hynet. Additionally, a novel approach to the bus voltage recovery for those relaxations is presented, which comprises a least-squares rank-1 approximation on a particular sparsity pattern, where the latter is defined by the system’s network topology and, therewith, focuses the approximation to system-relevant parts. Furthermore, to support the use of relaxations in hynet, some results on conditions for exactness and locational marginal prices are generalized to hybrid AC/DC grids with MT-HVDC systems. In the former, exactness requires the absence of certain pathological price profiles. This result is complemented by a proposed price profile deformation, which potentially establishes exactness in case of a pathological price profile. Finally, Section VI highlights hynet’s fundamental software design and relates it to the presented system model and OPF formulations. Section VII concludes the paper.

B. Notation

The set of natural numbers is denoted by \( \mathbb{N} \), the set of integers by \( \mathbb{Z} \), the set of real numbers by \( \mathbb{R} \), the set of nonnegative real numbers by \( \mathbb{R}_+ \), the set of positive real numbers by \( \mathbb{R}_{++} \), the set of complex numbers by \( \mathbb{C} \), and the set of Hermitian matrices in \( \mathbb{C}^{N \times N} \) by \( \mathbb{S}^N \). The imaginary unit is denoted by \( i = \sqrt{-1} \). For \( x \in \mathbb{C} \), its real part is \( \text{Re}(x) \), its imaginary part is \( \text{Im}(x) \), its absolute value is \( |x| \), its principal value of its argument is \( \arg(x) \in (-\pi, \pi] \), and its complex conjugate is \( x^* \). For a matrix \( A \), its transpose is \( A^T \), its conjugate (Hermitian) transpose is \( A^H \), its trace is \( \text{tr}(A) \), its rank is \( \text{rank}(A) \), its Frobenius norm is \( \| A \|_F \), and its element in row \( i \) and column \( j \) is \( [A]_{ij} \). For two matrices \( A, B \in \mathbb{S}^N \), \( A \succeq B \) denotes that \( A - B \) is positive semidefinite. For real-valued vectors, inequalities are component-wise. The vector \( e_n \) denotes the \( n \)th standard basis vector of appropriate dimension. For a countable set \( C \), its cardinality is \( |C| \). For a set \( N \subseteq \mathbb{N} \) and vectors or matrices \( x_n \in S \), with \( n \in N \), \( x_N \) denotes the \( |N| \)-tuple \( x_N = (x_n)_{n \in N} \); and \( S_N \) the \( |N| \)-fold Cartesian product \( S_N = \prod_{n \in N \backslash S} \).

II. SYSTEM MODEL

In the literature, several models of hybrid AC/DC grids with MT-HVDC systems were proposed for OPF studies, cf. [17–24]. For hynet, we developed a model that balances modeling depth and complexity to offer adequate accuracy while being tractable for large-scale hybrid grids and accessible to rigorous mathematical studies. To this end, the model for hybrid AC/DC grids with P2P-HVDC systems in [16], with the refinements in [25], is extended with DC subgrids and converters. A unified representation of AC and DC subgrids is introduced, which enables the generalization of the electrical model in [16] to DC subgrids while simplifying the implementation and mathematical exposition. Hereafter, AC lines, cables, transformers, and phase shifters are referred to as AC branches, DC lines and cables as DC branches, inverters, rectifiers, VSCs, and back-to-back converters as converters, and points of interconnection, generation injection, and load connection are called buses.

A. Network Topology

The network topology of the hybrid AC/DC power system, which consists of the interconnection of an arbitrary number of AC and DC subgrids via converters, is described by the directed multigraph \( \tilde{G} = (\mathcal{V}, \mathcal{E}, \mathcal{C}, \mathcal{E}, \mathcal{C}, \mathcal{E}, \mathcal{C}) \). where

1) \( \mathcal{V} = \{1, \ldots, |\mathcal{V}|\} \) is the set of buses,
2) \( \mathcal{E} = \{1, \ldots, |\mathcal{E}|\} \) is the set of branches,
3) \( \mathcal{C} = \{1, \ldots, |\mathcal{C}|\} \) is the set of converters,
4) \( \mathcal{E} : \mathcal{E} \to \mathcal{V} \) maps a branch to its source bus,
5) \( \mathcal{E} : \mathcal{E} \to \mathcal{V} \) maps a branch to its destination bus,
6) \( \mathcal{C} : \mathcal{C} \to \mathcal{V} \) maps a converter to its source bus, and
7) \( \mathcal{C} : \mathcal{C} \to \mathcal{V} \) maps a converter to its destination bus.

The directionality of branches and converters is not related to the direction of power flow and can be chosen arbitrarily. The buses \( \mathcal{V} \) are partitioned into a set \( \tilde{\mathcal{V}} \) of AC buses and a set \( \mathcal{V} \) of DC buses, i.e., \( \mathcal{V} = \tilde{\mathcal{V}} \cup \mathcal{V} \) and \( \tilde{\mathcal{V}} \cap \mathcal{V} = \emptyset \). AC and DC buses must not be connected by a branch, i.e., the branches \( \mathcal{E} \) are partitioned into AC and DC branches: \( \mathcal{E} = \tilde{\mathcal{E}} \cup \mathcal{E} \) and \( \tilde{\mathcal{E}} \cap \mathcal{E} = \emptyset \). Accordingly, the set \( \mathcal{E} \) of AC branches and the set \( \mathcal{E} \) of DC branches is given by

\[
\tilde{\mathcal{E}} = \{ k \in \mathcal{E} : \tilde{\ell}(k), \tilde{\varepsilon}(k) \in \tilde{\mathcal{V}} \} \quad \text{(1a)}
\]
\[
\mathcal{E} = \{ k \in \mathcal{E} : \ell(k), \varepsilon(k) \in \mathcal{V} \} . \quad \text{(1b)}
\]

Note that the terminal buses of converters are not restricted, i.e., the model supports AC/DC and DC/AC as well as AC and DC back-to-back converters. To support the mathematical exposition later on, some terms and expressions are defined.

Definition 1: Consider the directed subgraph \( \mathcal{G}' = (\mathcal{V}, \mathcal{E}, \hat{\ell}, \hat{\varepsilon}) \) with all buses and branches. A connected component [26] in the underlying [26] undirected graph of \( \mathcal{G}' \) is called subgrid. A subgrid comprising buses in \( \tilde{\mathcal{V}} \) is called AC subgrid. A subgrid comprising buses in \( \mathcal{V} \) is called DC subgrid.

Definition 2: The set \( \mathcal{B}_C(n) \subseteq \mathcal{C} \) and \( \mathcal{B}_E(n) \subseteq \mathcal{E} \) of branches outgoing and incoming at bus \( n \in \mathcal{V} \), respectively, is

\[
\mathcal{B}_C(n) = \{ k \in \mathcal{C} : \ell(k) = n \} \quad \text{(2a)}
\]
\[
\mathcal{B}_E(n) = \{ k \in \mathcal{E} : \ell(k) = n \} . \quad \text{(2b)}
\]

Definition 3: The set \( \mathcal{B}_C(n) \subseteq \mathcal{C} \) and \( \mathcal{B}_E(n) \subseteq \mathcal{C} \) of converters outgoing and incoming at bus \( n \in \mathcal{V} \), respectively, is

\[
\mathcal{B}_C(n) = \{ l \in \mathcal{C} : \hat{\gamma}(l) = n \} \quad \text{(3a)}
\]
\[
\mathcal{B}_E(n) = \{ l \in \mathcal{E} : \hat{\gamma}(l) = n \} . \quad \text{(3b)}
\]

Finally, a generally valid property of the network topology is established, which is utilized later on.
Definition 4 (Self-Loop Free Network Graph): The multigraph $\mathcal{G}$ does not comprise any self-loops, i.e.,
\[ \exists k \in \mathcal{E} : \hat{e}(k) = \hat{e}(k) \quad \text{and} \quad \exists l \in \mathcal{C} : \hat{\gamma}(l) = \hat{\gamma}(l). \] (4)

B. Electrical Model

The electrical model for branches and buses is adopted from [16] and extended with DC subgrids and a proposed converter model. The characterization of generation and load is adopted from [25] and generalized to the concept of injectors.

1) Branch Model: Branches are represented via the common branch model in Fig. 1b. For branch $k \in \mathcal{E}$, it comprises two shunt admittances $\hat{y}_k, \bar{y}_k \in \mathbb{C}$, a series admittance $\bar{y}_k \in \mathbb{C}$, and two complex voltage ratios $\hat{\rho}_k, \bar{\rho}_k \in \mathbb{C} \setminus \{0\}$. In the latter, $|\hat{\rho}_k|$ and $|\bar{\rho}_k|$ is the tap ratio and $\arg(\hat{\rho}_k)$ and $\arg(\bar{\rho}_k)$ the phase shift of the respective transformer, while
\[ \rho_k = \hat{\rho}_k \bar{\rho}_k \] (5)
denotes the total voltage ratio. Let the bus voltage vector $\mathbf{v}$, source current vector $\mathbf{i}$, and destination current vector $\hat{\mathbf{i}}$ be
\[ \mathbf{v} = [V_1, \ldots, V_\mathcal{V}]^T \in \mathbb{C}^{\mathcal{V}} \] (6)
\[ \mathbf{i} = [I_1, \ldots, I_\mathcal{V}]^T \in \mathbb{C}^{\mathcal{V}} \] (7)
\[ \hat{\mathbf{i}} = [\hat{I}_1, \ldots, \hat{I}_\mathcal{V}]^T \in \mathbb{C}^{\mathcal{V}} \] (8)
They are related by Kirchhoff’s and Ohm’s law, which renders
\[ \hat{\mathbf{i}} = \hat{\mathbf{Y}} \mathbf{v} \quad \text{and} \quad \mathbf{i} = \mathbf{Y} \mathbf{v} \] (9)
where $\hat{\mathbf{Y}}, \mathbf{Y} \in \mathbb{C}^{\mathcal{E} \times \mathcal{V}}$ are given in [16, Eq. (6) and (7)]. In the following, bus voltages are used as state variables. While AC subgrids exhibit complex-valued effective (rms) voltage phasors, DC subgrids exhibit real-valued voltages. This is considered by restricting $\mathbf{v}$ to $\mathcal{U}$, where
\[ \mathcal{U} = \{ \mathbf{v} \in \mathbb{C}^{\mathcal{V}} : V_n \in \mathbb{R}_+, n \in \hat{\mathcal{V}} \}. \] (10)
Furthermore, DC lines and cables are modeled via their series resistance, which is captured as follows.

Definition 5: DC branches equal a series conductance, i.e.,
\[ \forall k \in \hat{\mathcal{E}} : \hat{\rho}_k = \bar{\rho}_k = 1, \quad \hat{y}_k = \bar{y}_k = 0, \quad \Im(\bar{y}_k) = 0. \] (11)
Finally, a generally valid physical property of DC branches is observed, which is utilized later on.

Definition 6 (Lossy DC Branches): The series conductance of all DC branches is positive, i.e., $\Re(\bar{y}_k) > 0, \forall k \in \hat{\mathcal{E}}$.

2) Bus Model: Buses are modeled as depicted in Fig. 1a. For bus $n \in \mathcal{V}$, it comprises a shunt admittance $\bar{y}_n \in \mathbb{C}$, connections to the outgoing branches $k \in \mathcal{E}_n$, and connections to the incoming branches $k \in \mathcal{E}_n$ and an injection port. Let the injection current vector $\hat{\mathbf{i}}$ be
\[ \hat{\mathbf{i}} = [I_1, \ldots, I_\mathcal{V}]^T \in \mathbb{C}^{\mathcal{V}}. \] (12)
It is related to $\mathbf{v}$ by Kirchhoff’s and Ohm’s law, i.e.,
\[ \mathbf{i} = \mathbf{Y} \mathbf{v} \] (13)
where the bus admittance matrix $\mathbf{Y} \in \mathbb{C}^{\mathcal{V} \times \mathcal{V}}$ is given in [16, Eq. (10)]. Finally, note that the shunt $\bar{y}_n$ usually models reactive power compensation and is irrelevant in DC subgrids.

Definition 7: DC buses exhibit a zero shunt admittance, i.e.,
\[ \bar{y}_n = 0, \forall n \in \mathcal{V}. \]

3) Converter Model: For hynet, the converter model illustrated in Fig. 1c is introduced, which balances modeling depth and complexity.\(^1\) For converter $l \in \mathcal{C}$, it considers the source and destination apparent power flow $\hat{S}_l, \bar{S}_l \in \mathbb{C}$, respectively, where active power is converted with a forward and backward conversion loss factor $\hat{\eta}_l, \bar{\eta}_l \in (0, 1)$, respectively, while reactive power may be provided, i.e.,
\[ \hat{S}_l = \hat{p}_l - (1 - \hat{\eta}_l)\bar{p}_l - i\hat{q}_l \] (14a)
\[ \bar{S}_l = \bar{p}_l - (1 - \bar{\eta}_l)\hat{p}_l - i\bar{q}_l. \] (14b)
Therein, $\hat{p}_l, \bar{p}_l \in \mathbb{R}_+$ and $\hat{q}_l, \bar{q}_l \in \mathbb{R}_+$ is the nonnegative active power flow from bus $\hat{\gamma}(l)$ to $\hat{\gamma}(l)$ and vice versa, respectively, while $\hat{q}_l, \bar{q}_l \in \mathbb{C}$ is the reactive power support. The P/Q-capability of the converter at the source and destination bus is approximated by the polyhedral set $\mathcal{F}_l$ and $\hat{\mathcal{F}}_l$, respectively, cf. Fig. 2a. In this model, this is captured by
\[ \mathbf{f}_l = [\hat{p}_l, \bar{p}_l, \hat{q}_l, \bar{q}_l]^T \in \mathcal{F}_l \subset \mathbb{R}_+^2 \times \mathbb{R}^2 \] (15)
\(^1\)Complementary, the transformer, filter, and phase reactor of a VSC can be modeled using two AC branches and a capacitive shunt, cf. [22], [24].
\(^2\)In the analysis of VSC loss models in [24], the difference of the “Avg” model [24, Eq. (2)] and the “Complete” model [24, Eq. (1)] implies that an individual loss parametrization of the rectifier and inverter mode is essential, while the similar performance of the “Avg” model and the “Prop” model [24, Eq. (3)] suggests that individual proportional loss models for both modes potentially offer adequate accuracy, which motivates this formulation.
Its valid operating points are specified by the capability region which injects active power, i.e., DC-side of a converter is established. Inequality constraints that describe the capability regions of reactive power, which is associated with a certain cost. The real-valued function over the P/Q-plane. This is utilized for an abstraction of these entities to injectors. An injection vector \( H \) are collected in the injection vector \( S_j \). Moreover, the nature of DC subgrids is respected.

Definition 8: The DC-side of all converters exclusively inject active power, i.e., \( S_j \subset \mathbb{R}^2 \), \( \forall j \in \mathbb{V} \), where \( \mathbb{V} \) are a fixed load, shift, while \( S_j \) is singleton and \( C_j \) is associated with an operating point. The injector's terminal bus \( S_j \) is specified by an intersection of up to \( 8 \) half-spaces, which offers adequate accuracy at a moderate number of constraints and parametrization complexity.

In the latter, \( H \in \mathbb{R}^{2\times 4\mid I\mid} \) and \( h \in \mathbb{R}^F \) capture the \( F \in \mathbb{N} \) inequality constraints that describe the capability regions of all converters. Finally, the absence of reactive power on the DC-side of a converter is established.

Definition 10: Injectors connected to DC buses exclusively inject active power, i.e., \( S_j \subset \mathbb{R} \times \{0\} \), \( \forall j \in \bigcup_{n \in \mathbb{V}} B_S(n) \). For example, for a generator, \( S_j \) is a convex approximation of its P/Q-capability, while \( C_j \) reflects the generation cost. For a fixed load, \( S_j \) is singleton and \( C_j \) maps to a constant value. For a flexible load, \( S_j \) characterizes the implementable load shift, while \( C_j \) reflects the cost for load dispatching.

In the software framework, fixed loads are modeled explicitly for ease of use. Furthermore, the P/Q-capability \( S_j \) is specified as a polyhedral set defined by the intersection of up to \( 8 \) half-spaces, which can approximate the physical capabilities and restrict the power factor, cf. Fig. 2b. The cost function \( C_j : S_j \subset \mathbb{R}^2 \rightarrow \mathbb{R} \) is considered linearly separable in the active and reactive power costs, i.e.,

\[
C_j(s) = C_j^p(e_1^T s) + C_j^q(e_2^T s) \tag{20}
\]

where \( C_j^p, C_j^q : \mathbb{R} \rightarrow \mathbb{R} \) are convex and piecewise linear.

5) Power Balance: The flow conservation arising from Kirchhoff's current law balances the nodal injections with the power balance flow into branches and converters. This is captured by the power balance equations

\[
v^H P_n v + p_n^T f = e_1^T \sum_{j \in B_S(n)} s_j, \quad \forall n \in \mathbb{V} \tag{21a}
\]

\[
v^H Q_n v + q_n^T f = e_2^T \sum_{j \in B_S(n)} s_j, \quad \forall n \in \mathbb{V} \tag{21b}
\]

Therein, the left hand side describes the flow of active and reactive power into the branches and converters, respectively, while the right hand side accumulates the nodal active and reactive power injection. The matrices \( P_n, Q_n \in \mathbb{S}^{\lvert \mathbb{V} \rvert} \) are a function of the bus admittance matrix and given in [16, Eq. (14)]. The vectors \( p_n, q_n \in \mathbb{R}^{2\mid I\mid} \), which include (14) and characterize the flow into converters, can be derived as

\[
p_n = \sum_{l \in B_C(n)} (e_{4l-3} - (1 - \eta_l)e_{4l-2}) + \sum_{l \in B_C(n)} (e_{4l-2} - (1 - \eta_l)e_{4l-3}) \tag{22}
\]

\[
q_n = - \sum_{l \in B_C(n)} e_{4l-1} + \sum_{l \in B_C(n)} e_{4l} \tag{23}
\]

6) Electrical Losses: The total electrical losses amount to the difference of total active power generation and load, i.e., the sum of the right-hand side in (21a) over all \( n \in \mathbb{V} \) or, equivalently, the respective sum of the left-hand side. With
the latter and \( v^T A v = \text{tr}(A v v^T) \), the \textit{total electrical losses} can be derived as \( L(v v^H, f) \), where \( L : \mathbb{S}^{[V]} \times \mathbb{R}^{[I]} \rightarrow \mathbb{R} \) is

\[
L(V, f) = \text{tr}(L V) + l^T f
\]

with \( L \in \mathbb{S}^{[V]} \) and \( l \in \mathbb{R}^{[I]} \) given by

\[
L = \frac{1}{2}(Y + Y^H) \quad \text{and} \quad l = \sum_{i \in c} [\bar{\eta}_i e_{4i-3} + \bar{\eta}_i e_{4i-2}].
\]

III. SYSTEM CONSTRAINTS

In the following, the formulation of physical and stability-related limits in hyenet is presented. This formulation is based on [16], where the common apparent power flow limit ("MVA rating") is substituted by its underlying ampacity, voltage drop, and angle difference constraint (cf. [28, Ch. 6.1.12]) to improve expressiveness and mathematical structure.

1) Voltage: Due to physical and operational requirements, the voltage at bus \( n \in V \) must satisfy \( |V_n| \in [V_n, \bar{V}_n] \subset \mathbb{R}_+ \), where \( \bar{V}_n > V_n > 0 \). With \( M_n = e_n e_n^T \in \mathbb{S}^{[V]} \), this reads

\[
V_n^2 \leq v^H M_n v \leq \bar{V}_n^2, \quad \forall n \in V.
\]

2) Ampacity: The thermal flow limit on branch \( k \in E \) can be expressed as \( |I_k| \leq \bar{I}_k \) and \( |\bar{I}_k| \leq \bar{I}_k \), where \( \bar{I}_k, \bar{I}_k \in \mathbb{R}_+^+ \). In quadratic form, this renders

\[
v^H \bar{I}_k v \leq \bar{I}_k^2, \quad v^H \bar{I}_k v \leq \bar{I}_k^2, \quad \forall k \in E
\]

where \( \bar{I}_k, \bar{I}_k \in \mathbb{S}^{[V]} \) are given in [16, Eq. (18)].

3) Voltage Drop: The stability-related limit on the voltage drop \( \nu_k \in \mathbb{R} \) along AC branch \( k \in \hat{E} \), i.e.,

\[
\nu_k = |V_{\hat{k}}(k)|/|V_{\hat{k}}(k)| - 1
\]

is \( \nu_k \in [\nu_k, \bar{\nu}_k] \subset [-1, \infty), \) with \( \nu_k < \bar{\nu}_k \). This is captured by

\[
v^H M_k v \leq 0, \quad v^H \bar{M}_k v \leq 0, \quad \forall k \in \hat{E}
\]

in which \( M_k, \bar{M}_k \in \mathbb{S}^{[V]} \) are given in [16, Eq. (24) and (25)].

4) Angle Difference: The stability-related limit on the voltage angle difference \( \delta_k \in \mathbb{R} \) along AC branch \( k \in \hat{E} \), i.e.,

\[
\delta_k = \arg(V_{\hat{k}}^* V_{\hat{k}}(k))
\]

reads \( \delta_k \in [\delta_k, \bar{\delta}_k] \subset (-\pi/2, \pi/2) \), with \( \delta_k < \bar{\delta}_k \). Note that

\[
\text{Re}(V_{\hat{k}}^* V_{\hat{k}}(k)) \geq 0 \quad \text{and} \quad \text{Im}(V_{\hat{k}}^* V_{\hat{k}}(k)) = \text{tan}(\delta_k) \leq \text{tan}(\bar{\delta}_k)
\]

is an equivalent formulation of this constraint that can be put as

\[
v^H A_k v \leq 0, \quad v^H \bar{A}_k v \leq 0, \quad \forall k \in \hat{E}
\]

where

\[
A_k = -\bar{M}_k - \bar{M}_k^T
\]

with \( \bar{M}_k = e_{\hat{k}} e_{\hat{k}}^T \in \mathbb{R}^{[V] \times [V]} \) and \( \bar{A}_k, \bar{A}_k \in \mathbb{S}^{[V]} \) as given in [16, Eq. (30) and (31)]

IV. OPTIMAL POWER FLOW

The OPF problem identifies the optimal utilization of the grid infrastructure and generation resources to satisfy the load, where optimality is typically considered with respect to minimum injection costs or minimum electrical losses. With the system model and system constraints above, the OPF problem can be cast as the following optimization problem.

\[
\begin{align*}
\text{minimize} & \quad \sum_{j \in \mathcal{I}} C_j(s_j) + \tau L(v v^H, f) \\
\text{subject to} & \quad (21), (26), (27), (29), (33), (35b)
\end{align*}
\]

The objective consists of the injection costs and a penalty term comprising the electrical losses weighted by an (artificial) \textit{loss price} \( \tau \geq 0 \), which enables injection cost minimization, electrical loss minimization, and a combination of both.

In (35), it can be observed that the objective and constraints consider AC and DC subgrids analogously, while their bus voltages are treated differently by restricting \( v \) to \( \mathcal{U} \). With respect to the Lagrangian dual domain and convex relaxations, this restriction complicates further mathematical studies. To avoid these issues, it is observed that all currently installed and almost all planned HVDC systems are P2P-HVDC or \textit{radial MT-HVDC systems} [4]. This observation is formalized by the following definition and, in the next section, it is utilized to unify the representation of AC and DC voltages.

\textit{Definition 11 (Radial DC Subgrids):} The underlying undirected graph of the directed subgraph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \bar{\varphi}, \bar{\epsilon}) \) is \textit{acyclic} [26, i.e., its connected components are trees [26].}

A. Unified Voltage Representation

To eliminate the restriction of \( v \) to \( \mathcal{U} \) and, therewith, unify the representation of AC and DC voltages in the OPF problem, let (31) also be imposed on DC subgrids, i.e.,

\[
v^H A_k v \leq 0, \quad \forall k \in \hat{E}
\]

with \( A_k, \bar{A}_k \in \mathbb{S}^{[V]} \) in (34). Jointly, Def. 11 and the complementary constraints in (36) enable the following result.

\textit{Theorem 1:} Consider any \( f \in \mathcal{F} \) and \( s_j \in \mathcal{S}_j \), for \( j \in \mathcal{I} \). Let \( v \in \mathbb{C}^{[V]} \) satisfy the constraints in (35b) and (36) for the given \( f \) and \( s_j \). Then, the bus voltage vector \( v \in \mathcal{U} \) given by

\[
[v]_n = \begin{cases} [v]_n & \text{if } n \in \mathcal{V} \\ |[v]_n| & \text{if } n \in \mathcal{V} \end{cases}
\]

satisfies (35b) and (36) for the given \( f \) and \( s_j \) with equivalent constraint function values and \( L(v v^H, f) = L(v v^H, f) \).

\textit{Proof:} See Appendix A. \hfill \blacksquare

Considering that \( \mathcal{U} \subset \mathbb{C}^{[V]} \), Theorem 1 enables an \textit{exact state space relaxation} of the OPF problem (35) to \( v \in \mathbb{C}^{[V]} \) under Def. 11 and the complementary constraints (36). Thus, for radial DC subgrids the representation of AC and DC voltages can be \textit{unified}, where the corresponding OPF formulation reads

\[
\begin{align*}
\text{minimize} & \quad \sum_{j \in \mathcal{I}} C_j(s_j) + \tau L(v v^H, f) \\
\text{subject to} & \quad (21), (26), (27), (29), (33), (36)
\end{align*}
\]
**B. Optimal Power Flow in hynet**

Concluding, to define hynet’s OPF formulation, the constraints in (38b) and the restriction of $f$ to $F$ are expressed as $(v v^T, f, s_z) \in \mathcal{X}$, where

\[
\mathcal{X} = \left\{ (V, f, s_z) \in \mathbb{S}^{|V|} \times \mathbb{R}^{|E|} \times \mathbb{R}_+^2 : \right. \\
\left. \begin{align*}
\text{tr}(P_n V) + p_n^T f &= e_i^T \sum_{j \in B_z(n)} s_j, \quad \forall n \in V, \\
\text{tr}(Q_n V) + q_n^T f &= e_i^T \sum_{j \in B_z(n)} s_j, \quad \forall n \in V, \\
\text{tr}(C_m V) + c_m^T f &\leq b_m, \quad \forall m \in M \right\}. \quad (39d)
\]

Therein, (39d) captures (17), (26), (29), (33), and (36), where $C_m$, $c_m$, and $b_m$, $m \in M = \{1, \ldots, |M|\}$, reproduce the $|M| = F + 2|V| + 3|E| + 4|\bar{E}|$ inequality constraints. With $X$ above, hynet’s OPF formulation is defined as

\[
p^* = \min_{s_j \in S_j, v \in \mathbb{S}^{|V|}} \sum_{j \in J} C_j(s_j) + \tau L(v v^T, f) \quad (40a)
\]

subject to $(v v^T, f, s_z) \in \mathcal{X}. \quad (40b)$

**V. Convex Relaxation**

Due to the power balance equations and indefiniteness of some constraint matrices (cf. [16]), the OPF problem (40) is a nonconvex optimization problem and, in general, is solved to local optimality as its globally optimal solution is NP-hard. A recent approach to improve computational tractability of a globally optimal solution of OPF problems is convex relaxation, see e.g. [29]–[33] and the references therein. In hynet, the following two popular relaxations are included, which are very simple to derive by virtue of the unified representation of AC and DC subgrids.

**A. Semidefinite Relaxation**

By definition, the objective in (40) and $S_j$ are convex, while $\mathcal{X}$ is a polyhedral set and, thus, also convex. Consequently, the nonconvexity of (40) arises from the quadratic dependence on the bus voltage vector $v$. The outer product $v v^T$ may be expressed equivalently by a Hermitian matrix $V \in \mathbb{S}^{|V|}$, which is positive semidefinite (psd) and of rank 1. As the set of psd matrices is a convex cone, the nonconvexity is then lumped to the rank constraint. In semidefinite relaxation (SDR), the rank constraint is omitted to obtain a convex optimization problem, i.e., the SDR of the OPF problem (40) reads

\[
\tilde{p}^* = \min_{s_j \in S_j, v \in \mathbb{S}^{|V|}} \sum_{j \in J} C_j(s_j) + \tau L(V, f) \quad (41a)
\]

subject to $(V, f, s_z) \in \mathcal{X} \quad (41b)$

\[
V \succeq 0. \quad (41c)
\]

Therewith, the OPF problem gains access to the powerful theory of convex analysis as well as solution algorithms with polynomial-time convergence to a globally optimal solution. However, SDR is in general only suitable if the optimizer $V^* \in \mathbb{S}^{|V|}$ obtained from (41) has rank 1, in which case the relaxation is called exact.

**B. Second-Order Cone Relaxation**

Besides exactness, another issue with SDR is the quadratic increase in dimensionality by $|V|^2 - 2|V|$ (real-valued) variables, which impedes computational tractability for large-scale grids. This uplift can be mitigated by an additional second-order cone relaxation (SOCR), see e.g. [25], [29], [30] and the references therein. To this end, it is observed that the constraint matrices $P_n$, $Q_n$, and $C_m$ in (39) may only exhibit a nonzero element in row $i$ and column $j$ if $(i, j) \in J$, where

\[
J = \{(i, j) \in V \times V : i, j \in \{\hat{e}(k), \check{e}(k)\}, k \in E\}, \quad (42)
\]

see [16, Appendix B]. If the psd constraint (41c) is relaxed to psd constraints on $2 \times 2$ principal submatrices, then only those elements of $V$ that are involved in (39) need to be retained. Correspondingly, the SOC of (41) can be stated as

\[
\tilde{p}^* = \min_{s_j \in S_j, v \in \mathbb{S}^{|V|}} \sum_{j \in J} C_j(s_j) + \tau L(V, f) \quad (43a)
\]

subject to $(V, f, s_z) \in \mathcal{X} \quad (43b)$

\[
S_k^H S_k \succeq 0, \quad k \in E \quad (43c)
\]

in which $S_k = [e_{\hat{e}(k)} e_{\check{e}(k)}] \in \mathbb{R}^{|V| \times 2}$ and $\mathbb{S}^{|V|}$ is the set of Hermitian partial matrices on the graph $G = (V, E, \hat{e}, \check{e})$, i.e.,

\[
\mathbb{S}^{|V|} = \left\{ P(V) : V \in \mathbb{S}^{|V|} \right\} \subseteq \mathbb{S}^{|V|} \quad (44)
\]

with the projection $P$ onto the sparsity pattern given by

\[
P(V) = \sum_{(i, j) \in J} |V|_{i,j} e_{i} e_{j}^T. \quad (45)
\]

In the SOC, (43c) can be implemented as second-order cone constraints, cf. e.g. [34]. By virtue of the partial matrix, (43) introduces only $2|E| - |V|$ additional (real-valued) variables compared to (40). The SOC is exact if the optimizer $V^*$ obtained from (43) permits a rank-1 completion.

**C. Bus Voltage Recovery**

Considering the potential inexactness of a relaxation and the finite precision of solvers, the bus voltages $\hat{v}^*$ associated with an optimizer $V^*$ of the SDR or SOC must be recovered via a rank-1 approximation of $V^*$ on its sparsity pattern.\(^4\) In the least-squares sense, this reconstruction problem reads

\[
\hat{v}^* = \arg \min_{v \in \mathbb{S}^{|V|}} \| P(V v^H - V^*) \|_F^2. \quad (46)
\]

The related (mean) reconstruction error $\hat{\kappa}(V^*)$ is\(^5\)

\[
\hat{\kappa}(V^*) = \| P(V^*) \|_F^2 / |J|. \quad (47)
\]

In the software framework, the (nonconvex) problem (46) is solved with a Wirtinger calculus gradient descent method [35] and Armijo’s rule for step size control [36, Sec. 8.3] using an initial point obtained with the method in [37, Sec. III-B-3].

\(^4\)Due to voltage-decoupled subgrids, an optimizer $V^*$ of the SDR (41) still allows exact recovery if it permits a decomposition $V^* = \sum_{k=1}^N v_k v_k^H$, where $N$ is the number of subgrids and $v_k$ comprises only nonzero elements for bus voltages of subgrid $k$.

\(^5\)The bus voltage error can be difficult to assess. Due to this, the software also reports the induced power balance error, which is intuitively accessible.
D. Locational Marginal Prices

In electricity markets, nodal pricing can be applied to account for system constraints and losses and, under perfect competition (price takers), the optimal nodal prices are the locational marginal prices (LMPs) [38]. The LMP at a bus is the cost of serving an increment of load by the cheapest possible injection [38]. Thus, it quantifies the sensitivity of the optimal objective value of a cost-minimizing OPF problem with respect to a perturbation of the nodal power balance [25]. Due to the nonconvexity of the OPF problem, accurate LMPs are in general hard to obtain [39], [40], but in an exact SDR and SOCR the LMPs for active and reactive power are given by the optimal Lagrangian dual variables of the power balance constraints [25]. By virtue of the unified AC/DC representation, the result in [25] generalizes\(^6\) to the SDR in (41) and the SOCR in (43) for \(\tau = 0\). In hynet, primal-dual interior-point solvers are applied to the SDR and SOCR. Thus, in case of exactness, the LMPs are obtained as a byproduct of the OPF computation.

E. Exactness of the Relaxations

While exactness of a relaxation can be verified a posteriori via the reconstruction error in (47), a priori information about the applicability of a relaxation is desired, i.e., on a system’s tendency toward exactness. In the literature, several results were presented for AC grids [41]–[46], DC grids [44], [47], [48], and hybrid AC/DC grids with P2P-HVDC systems [16], [25]. By virtue of the unified AC/DC representation, the result in [25, Sec. VII] generalizes\(^7\) under Def. 12 to the SOCR (43) for \(\tau = 0\) and, thus, to the SDR (41) for \(\tau = 0\).

Definition 12 (Hybrid Architecture): The underlying undirected graph of the directed subgraph \(G' = \{V, E, \bar{e}, \bar{\epsilon}\}\) is acyclic, i.e., its connected components are trees.

The result states that for the hybrid architecture the SDR and SOCR may only be inexact if the LMPs form a pathological price profile, i.e., if they combine to a point in a union of linear subspaces – a set of measure zero [25, Sec. VII-B]. This suggests that inexactness is unlikely and, indeed, we only observed it in cases where the LMP for active power is (close to) zero at some buses – a pathological case.\(^8\) Thus, if the SDR or SOCR in hynet is applied to a system with radial (acyclic) subgrids, e.g., radial distribution grids [49], exactness may be expected under normal operating conditions. Yet, if inexactness occurs, the following result may be utilized.

Proposition 1: Consider the OPF problem (40). Including a loss penalty with \(\tau > 0\) is equivalent to increasing the marginal cost of active power by \(\tau\) for all injectors \(j \in I\).

Proof: Considering Section II-B6, it follows that

\[
\sum_{j \in I} C_j(s_j) + \tau L(vv^H, f) = \sum_{j \in I} C_j(s_j) + \tau \sum_{w \in V} v^T e_j s_j = \sum_{j \in I} [C_j(s_j) + \tau e_j^T s_j].
\]

Thus, \(\tau > 0\) deforms and uplifts the “price profile”. If inexactness occurs with the hybrid architecture, we observed that this alteration often circumvents pathology and establishes exactness. In case that zero marginal costs cause inexactness, the loss penalty is also motivated from an engineering point of view to avoid wasteful use of power offered free of charge.

VI. SOFTWARE FRAMEWORK

The documentation of the hynet software framework and its application is provided separately on the project website [14]. This section briefly describes the fundamental design of hynet and relates it to the presented system model and OPF formulations, see Fig. 3. An OPF study is initiated by loading a scenario from a grid database into a scenario object, where the latter organizes the data using pandas \([50]\) data frames. After optional adjustments, the scenario is used to create a system model object, which implements the mathematical model in Section II and III. This object serves as a builder \([51]\) for the associated OPF problem, which is represented by an object of a quadratically-constrained quadratic problem (QCQP). This QCQP is solved via a solver interface, where the underlying implementation may solve the nonconvex QCQP (40), its SDR (41), or its SOCR (43). The solver returns an object containing the result and status information, which is routed through a factory function \([51]\) of the system model object to obtain an appropriate representation of the OPF result data.

\(^6\) The proof follows along the lines of [25, Sec. VI-B], while considering the change to minimization in the primal domain.

\(^7\) The proof follows along the lines of [25, Sec. VII], while considering that the sufficient condition for existence of a psd rank-1 completion in [25, Eq. (25)] generalizes to voltage-decoupled radial subgrids. The latter is evident if the rank-1 completion method in [37, Sec. III-B-3] is considered.

\(^8\) This may occur, e.g., in presence of renewables with zero marginal costs.
This object-oriented design offers a transparent structure and data flow, while rendering *hynet* amenable to extensions. For example, extensions to additional solvers and other relaxations can be implemented via corresponding solver classes. Furthermore, the system model may be extended by subclassing the system model class and, by overriding the respective factory function, the OPF result representation can be customized.

VII. CONCLUSION

This paper introduced the open-source OPF software *hynet* as well as its underlying mathematical framework. The latter was developed with a focus on a unified representation of AC and DC subgrids to simplify the study of hybrid AC/DC power systems, from mathematical notation and convex relaxation to the software implementation and data handling. Furthermore, it was introduced with several layers of notational abstraction to facilitate its utilization in future works and model extensions. The software was developed with a focus on ease of use, while its object-oriented design was kept transparent and flexible to encourage its utilization in derivative works. We hope that *hynet* can be valuable to the study of hybrid AC/DC power systems and may support new findings in research.

APPENDIX A

PROOF OF THEOREM 1

To begin with, consider the following lemma.

**Lemma 1**: Consider any \( f \in F \) and \( s_j \in S_j \), for \( j \in I \). Let \( v \in \mathbb{C}^{\mathbf{v}|\mathbf{V}|} \) satisfy (21) for the given \( f \) and \( s_x \). Then, \( \text{Im}(V_{\hat{e}(k)} V_{\hat{e}(k)}) = 0 \), \( \forall k \in \mathcal{E} \).

**Proof**: See Appendix B.

Thus, for all \( k \in \mathcal{E} \), it holds that

\[
\text{arg}(V_{\hat{e}(k)}) - \text{arg}(V_{\hat{e}(k)}) = r \pi, \quad \text{for some } r \in \mathbb{Z}.
\]

(48)

Furthermore, it follows from (36), which implements (31), that

\[
-\pi/2 \leq \text{arg}(V_{\hat{e}(k)}) - \text{arg}(V_{\hat{e}(k)}) \leq \pi/2.
\]

(49)

Jointly, (48) and (49) imply

\[
\text{arg}(V_{\hat{e}(k)}) = \text{arg}(V_{\hat{e}(k)}), \quad \forall k \in \mathcal{E}.
\]

(50)

Therefore, the phase of all elements in \( v \) associated with a certain DC subgrid is equal. As the constraints in (35b) and (36) are quadratic\(^9\) in \( v \) and the individual constraints only involve elements of \( v \) that are associated with the same subgrid,\(^10\) it follows that these constraints are invariant with respect to a common phase shift of all elements in \( v \) for a certain subgrid. Thus, if \( v \) satisfies (35b) and (36), then \( v \) in (37) satisfies (35b) and (36) with equivalent constraint function values, as its construction only involves a common phase shift in the individual DC subgrids. Furthermore, this implies that \( L(v v^\mathsf{H}, f) = L(v v^\mathsf{H}, f) \), as \( L \) equals the summation of the left-hand side of (21a).

\(^9\)For \( a \) and \( a \), it follows \( a^\mathsf{H} A a = a^\mathsf{H} A a \), where each block relates to a subgrid. Footnote 9 applies per block, i.e., a blockwise phase shift does not affect the constraints.

\(^10\)Note that converters decouple the voltages of different subgrids. W.l.o.g., the buses can be numbered such that the constraint matrices are block-diagonal (see also [16, App. B]), where each block relates to a subgrid. Footnote 9 applies per block, i.e., a blockwise phase shift does not affect the constraints.

APPENDIX B

PROOF OF LEMMA 1

For \( n \in \mathcal{V} \), it follows from (21b), Def. 8, and Def. 10 that \( v^\mathsf{H} Q \mathbf{v} = 0 \) and, with Def. 5 and Def. 7, that\(^11\)

\[
\sum_{k \in \mathcal{E}(n)} \bar{y}_k \text{Im}(V_{\hat{e}(k)}^* V_n) = \sum_{k \in \mathcal{E}(n)} \bar{y}_k \text{Im}(V_{\hat{e}(k)}^* V_n).
\]

(51)

Thus, if \( v \) is not restricted to \( \mathcal{U} \), there may emerge a circulation of “artificial reactive power” in DC subgrids.

Now, consider an individual DC subgrid, which is radial by Def. 11. Without loss of generality, assume that it comprises no parallel branches (consider their single branch equivalent), consider one of its buses as the reference bus, and let all its branches point toward this reference bus. At all leaf nodes \( n \) of this directed tree graph, (51) reduces to

\[
\bar{y}_k \text{Im}(V_{\hat{e}(k)}^* V_n) = 0
\]

(52)

where \( k \) is the only DC branch connected to DC bus \( n = \hat{e}(k) \) due to Def. 11, Def. 4, and the absence of parallel branches. It follows from Def. 6 that \( \bar{y}_k \neq 0 \), thus (52) implies

\[
\text{Im}(V_{\hat{e}(k)}^* V_{\hat{e}(k)}) = 0 \implies \text{Im}(V_{\hat{e}(k)}^* V_{\hat{e}(k)}) = 0
\]

(53)

i.e., the inflow of artificial reactive power arising from DC branch \( k \) at its destination bus \( \hat{e}(k) \) is zero. Therefore, all DC branches that are connected to leaf nodes do not contribute to the circulation of artificial reactive power and, thus, can be excluded for this analysis. Considering the resulting reduced directed tree graph, the above argument can be repeated until the reduced directed tree graph equals the reference node.

This proves by induction that \( \text{Im}(V_{\hat{e}(k)}^* V_{\hat{e}(k)}) = 0 \) for all DC branches \( k \) of the considered DC subgrid.

Repition of this inductive argument for all DC subgrids implies that \( \text{Im}(V_{\hat{e}(k)}^* V_{\hat{e}(k)}) = 0 \), \( \forall k \in \mathcal{E} \). Note that the proof can be adapted to an arbitrary directionality of DC branches.

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\(^11\)Alternatively, (51) can be derived via the reactive power balance equation

\[
\text{Im}(V_n^* I_n) = 0 \quad \text{at DC bus } n \in \mathcal{V}
\]

using the electrical model in Section II-B.
