THE CLOSED STEINHAUS PROPERTIES
OF \( \sigma \)-IDEALS ON TOPOLOGICAL GROUPS

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Dedicated to the 130th birthday of Hugo Steinhaus (1887–1972)

ABSTRACT. We prove that any meager quasi-analytic subgroup of a topological group \( G \) belongs to every \( \sigma \)-ideal \( \mathcal{I} \) on \( G \) possessing the closed \( \pm n \)-Steinhaus property for some \( n \in \mathbb{N} \). An ideal \( \mathcal{I} \) on a topological group \( G \) is defined to have the closed \( \pm n \)-Steinhaus property if for any closed subsets \( A_1, \ldots, A_n \not\in \mathcal{I} \) of \( G \) the product \((A_1 \cup A_1^{-1}) \cdots (A_n \cup A_n^{-1})\) is not nowhere dense in \( G \). Since the \( \sigma \)-ideal \( \mathcal{E} \) generated by closed Haar null sets in a locally compact group \( G \) has the closed \( \pm 2 \)-Steinhaus property, we conclude that each meager quasi-analytic subgroup \( H \subset G \) belongs to the ideal \( \mathcal{E} \). For analytic subgroups of the real line this result was proved by Laczkovich in 1998. We shall discuss possible generalizations of the Laczkovich Theorem to non-locally compact groups and construct an example of a meager Borel subgroup in \( \mathbb{Z}^\omega \) which cannot be covered by countably many closed Haar-null (or even closed Haar-meager) sets. On the other hand, assuming that \( \text{cof}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \text{cov}(\mathcal{N}) \) we construct a subgroup \( H \subset \mathbb{Z}_2^\omega \) which is meager and Haar null but does not belong to the \( \sigma \)-ideal \( \mathcal{E} \). The construction uses a new cardinal characteristic \( \text{voc}^*(\mathcal{I}, \mathcal{J}) \) which seems to be interesting by its own.

1. INTRODUCTION

By a classical result of S. Banach [1], any subgroup \( H \) with the Baire property in a topological group \( G \) is either open or belongs to the \( \sigma \)-ideal \( \mathcal{M} \) of meager sets in \( G \). On the other hand, the classical result of Steinhaus [28] implies that every Haar measurable subgroup \( H \) of a locally compact \( G \) is either open or belong to the \( \sigma \)-ideal \( \mathcal{N} \) of Haar null sets in \( G \). Much later, in 1998 Laczkovich [20] unified these two results and proved that each analytic subgroup \( H \) of a Polish locally compact group \( G \) is either open or belongs to the \( \sigma \)-ideal \( \mathcal{E} \) generated by closed Haar-null sets in \( G \). We recall that a topological space \( A \) is analytic if it is a continuous image of a Polish space. A non-empty family \( \mathcal{I} \) of subsets of a set \( X \) is called an ideal on \( X \) if \( \mathcal{I} \) is closed under taking subsets and finite unions. If \( \mathcal{I} \) is closed under countable unions, then \( \mathcal{I} \) is called a \( \sigma \)-ideal.

Motivated by these classical results, in this paper we shall consider the following problem.

**Problem 1.1.** Detect \( \sigma \)-ideals \( \mathcal{I} \) on Polish groups \( G \) containing all meager analytic subgroups of \( G \).

To answer this problem in Section 3, we shall define ideals with the closed \( \pm n \)-Steinhaus property on topological groups and shall prove that such ideals contain all non-open analytic subgroups and all meager quasi-analytic subgroups of the group. Quasi-analytic spaces are introduced and studied in Section 2. In Section 4, we shall construct examples of \( \sigma \)-ideals distinguishing the closed \( \pm n \)-Steinhaus properties for various \( n \). In Section 5, we discuss possible extensions of the Laczkovich theorem to non-locally compact Polish groups and in Section 6 we shall prove that such ideals contain all non-open analytic subgroups and all meager quasi-analytic subgroups of the group. Quasi-analytic spaces are introduced and studied in Section 2. In Section 4, we shall construct examples of \( \sigma \)-Polish subgroup \( G \) in \( \mathbb{Z}^\omega \) which cannot be covered by countably many Haar-null sets in \( \mathbb{Z}^\omega \). In the final Section 7, given two ideal \( \mathcal{I}, \mathcal{J} \) on a group \( G \) we study the problem of the existence of a subgroup \( H \subset G \) such that \( H \in \mathcal{I} \setminus \mathcal{J} \). In particular, assuming that \( \text{cof}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \text{cov}(\mathcal{N}) \) we construct a subgroup \( H \) in the Cantor cube \( 2^\omega \) which belongs to the family \( (\mathcal{M} \cap \mathcal{N}) \setminus \mathcal{E} \) (by the Laczkovich Theorem [20] such subgroup \( H \) cannot be analytic). In the construction of \( H \) an important role belongs to a new cardinal characteristic \( \text{voc}^*(\mathcal{I}, \mathcal{J}) \), for which many natural questions still remain open.

2. QUASI-ANALYTIC AND UNIVERSALLY \( \mathcal{I} \)-MEAGER SETS

A topological space \( X \) is defined to be

- **Baire** if for any countable family \( \mathcal{U} \) of open dense subsets in \( X \) the intersection \( \bigcap \mathcal{U} \) is dense in \( X \);

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• **hereditarily Baire** if every closed subspace of \( X \) is Baire;
• **quasi-analytic** if \( X \) is a continuous image of a hereditarily Baire metrizable separable space.

Since Polish spaces are hereditarily Baire, every analytic space is quasi-analytic. Examples of spaces which are not quasi-analytic are uncountable universally meager spaces.

Following [34], we call a subset \( A \) of a Polish space \( Y \) **universally meager** if for any continuous nowhere constant map \( f: X \to Y \) defined on a Polish space \( X \) the preimage \( f^{-1}(A) \) is meager in \( X \). A map \( f: X \to Y \) is called **nowhere constant** if for each non-empty open set \( U \subset X \) the image \( f(U) \) contains more than one point.

This notion is a partial case of the notion of a universally \( I \)-meager set defined for any ideal \( I \) as follows.

**Definition 2.1.** Let \( I \) be an ideal on a topological space \( X \). A subset \( A \subset X \) is called **universally \( I \)-meager** if for any continuous function \( f: Z \to A \) defined on a Baire separable metrizable space \( Z \) there exists a non-empty open set \( U \subset Z \) with \( f(U) \in I \).

Definition 2.1 implies the following simple proposition.

**Proposition 2.2.** Let \( I \subset J \) be two ideals on a topological space \( X \).

1. Each set \( A \in I \) is universally \( I \)-meager.
2. If a set \( A \subset X \) is universally \( I \)-meager, then \( A \) is universally \( J \)-meager.

**Proposition 2.3.** A subset \( A \) of a separable metrizable space \( X \) belongs to the ideal \( M \) of meager sets in \( X \) if and only if \( A \) is universally \( M \)-meager in \( X \).

**Proof.** The “only if” part follows from Proposition 2.2. To prove the “if” part, assume that the set \( A \) is not meager in \( X \). Let \( U \) be the largest open subset of \( A \) which belongs to the \( \sigma \)-ideal \( M \). The set \( U \) is well-defined and is equal to the union of all open subsets of \( A \), which are meager in \( X \). Let \( U \) be the closure of \( U \) in \( A \). Since \( U \) is nowhere dense in \( A \) and in \( X \), the set \( U \) is meager in \( X \). Then the set \( A \setminus U \) is non-meager in \( X \). Moreover, by the maximality of \( U \), each non-empty open subset \( V \) of \( A \setminus U \) is not meager in \( X \) and hence non-meager in \( A \). This implies that the space \( A \setminus U \) is Baire.

Observe that for the identity map \( f: A \setminus U \to A \subset X \) and any non-empty open set \( V \) of \( A \setminus U \) the set \( f(V) = V \) is not meager in \( X \). Consequently, \( A \) fails to be universally \( M \)-meager.

**Proposition 2.4.** For a subset \( A \) of a Polish space \( X \) the following conditions are equivalent:

1. \( A \) is universally meager;
2. \( A \) is universally \([X]^{<\omega}\)-meager for the ideal \([X]^{<\omega}\) of finite subsets of \( X \);
3. \( A \) is universally \([X]^{\leq \omega}\)-meager for the ideal \([X]^{\leq \omega}\) of countable subsets of \( X \).

**Proof.** (1) \( \Rightarrow \) (2). Assume that \( A \) is universally meager in \( X \). To prove that \( A \) is universally \([X]^{<\omega}\)-meager, fix any continuous map \( f: Z \to A \) defined on a Baire separable metrizable space \( Z \). By Kuratowski Theorem [21, 3.8], the map \( f \) can be extended to a continuous map \( \tilde{f}: P \to X \) defined on some Polish space \( P \) containing \( Z \) as a dense subset. Since \( A \) is universally meager and the preimage \( \tilde{f}^{-1}(A) \supset Z \) is not meager in \( P \), the map \( \tilde{f} \) is constant on some non-empty open set \( U \subset P \). Then \( U \cap Z \) is a non-empty open set in \( Z \) such that \( f(U \cap Z) \subset f(U) \in [X]^{<\omega} \).

The implication (2) \( \Rightarrow \) (3) is trivial. (3) \( \Rightarrow \) (1). Assume that \( A \) is universally \([X]^{\leq \omega}\)-meager in \( X \). To prove that \( A \) is universally meager in \( X \), we should show that for any nowhere constant continuous map \( f: P \to X \) defined on a Polish space \( P \) the preimage \( f^{-1}(A) \) is meager in \( P \). To derive a contradiction, assume that \( f^{-1}(A) \) is not meager in \( P \). Then there is a non-empty open set \( W \subset P \) such that the space \( Z = W \cap f^{-1}(A) \) is Baire and dense in \( W \). Consider the map \( g = f|Z: Z \to A \). Since the set \( A \) is universally \([X]^{\leq \omega}\)-meager in \( X \), there exists a non-empty open set \( U \subset W \) such that the set \( C = g(U \cap Z) \) is countable. Since the space \( U \cap Z \subset \bigcup_{y \in C} U \cap g^{-1}(y) \) is Baire, for some \( y \in C \) the preimage \( g^{-1}(y) \) contains some non-empty open set \( V \subset U \). The density of \( V \cap Z \) in \( Z \) guarantees that \( f(V) = f(V \cap Z) = g(V \cap Z) = \{y\} \) is a singleton, which means that \( f \) is not nowhere constant. But this contradicts the choice of \( f \).

**Proposition 2.5.** A quasi-analytic set \( A \) is a topological space \( X \) belongs to a \( \sigma \)-ideal \( I \) on \( X \) if and only if \( A \) is universally \( I \)-meager in \( X \).
Proposition 3.3. Let $\varepsilon$ sequence $\mathbb{M}$-meager in $G$. The classical Steinhaus theorem [28] (see also [29]) implies that for a locally compact group $G$ the ideal $\mathbb{N}_G$ of Haar null sets and the $\sigma$-ideal $\mathcal{E}_G$ generated by closed Haar null subsets in $G$ both have the closed $2$-Steinhaus property.

Theorem 3.4. Assume that a $\sigma$-ideal $\mathcal{I}$ on a topological group $G$ has the closed $\pm n$-Steinhaus property for some $n \in \mathbb{N}$. If subsets $A_1, \ldots, A_n \subset G$ are not universally $\mathcal{I}$-meager in $G$, then the set $A_1^\pm \cdots A_n^\pm$ is not meager in $G$.

Proof. The “only if” part follows from Proposition 2.2. To prove the “if” part, assume that $A$ is universally $\mathcal{I}$-meager in $G$. The space $A$, being quasi-analytic, is the image of a hereditarily Baire metrizable separable space $B$ under a continuous surjective map $f : B \to A$. Let $U$ be the family of all open sets $U \subset B$ such that $f(U) \in \mathcal{I}$. Since the space $B$ is hereditarily Lindelöf, the union $U = \bigcup U$ equals to the union $\bigcup V$ of some countable subfamily $V \subset U$, which implies that the set $f(U) \in \mathcal{I}$ (we recall that $\mathcal{I}$ is a $\sigma$-ideal).

If $A \notin \mathcal{I}$, then $U \neq A$ and the closed subset $B' = B \setminus U$ of $B$ is not empty. By the maximality of $U$, for any non-empty open set $V \subset B'$ the set $f(V)$ does not belong to the ideal $\mathcal{I}$. Since the space $B$ is hereditarily Baire, the closed subspace $B'$ is Baire. Then the map $f|B' : B' \to A$ witnesses that $A$ is not universally $\mathcal{I}$-meager. This contradiction shows that $A \in \mathcal{I}$. 

By [23], uncountable universally meager sets exist in ZFC. On the other hand, Miller [22] constructed a model of ZFC in which every perfectly (and universally) meager subset $A \subset \mathbb{R}$ has cardinality $|A| \leq \omega_1 < \omega_1 = \mathfrak{c}$. We recall [23] that a set $A \subset \mathbb{R}$ is called perfectly meager if for any closed subset $P \subset \mathbb{R}$ without isolated points the intersection $A \cap P$ is meager in $P$.

By Proposition 2.3 each universally $\mathcal{M}$-meager subset of a Polish space $X$ is meager.

Problem 2.6. Is it consistent that each $\mathcal{E}$-meager subset of a compact topological group $G$ belongs to the $\sigma$-ideal $\mathcal{E}$ generated by closed Haar null sets in $G$?

Problem 2.7. Is it consistent that each subset $A \subset \mathbb{R}$ of cardinality $|A| = \mathfrak{c}$ is quasi-analytic?

3. The closed Steinhaus properties of ideals on topological groups

In this section we shall introduce various closed Steinhaus properties of ideals on topological groups and shall prove that such ideals contains all meager quasi-analytic subgroups.

An ideal $\mathcal{I}$ on a group $G$ is called invariant if for any $A \in \mathcal{I}$ and $x, y \in G$ we get $xAy \in \mathcal{I}$; $\mathcal{I}$ is symmetric if for any $A \in \mathcal{I}$ the set $A^{-1} = \{ a^{-1} : a \in A \}$ belongs to $\mathcal{I}$.

For subsets $A_1, \ldots, A_n$ of a semigroup $G$ let $A_1 \cdots A_n = \{ a_1 \cdots a_n : a_1 \in A_1, \ldots, a_n \in A_n \}$ be the product of the sets $A_1, \ldots, A_n$ in $G$. If all sets $A_i$ are equal to a fixed subset $A \subset G$, then the product $A_1 \cdots A_n$ will be denoted by $A^n$. For a subset $A$ of a group $G$ we put $A^\pm = A \cup A^{-1}$.

In the following definition we introduce two central notions considered in this paper.

Definition 3.1. An ideal $\mathcal{I}$ on a topological group $G$ is defined to have the closed $n$-Steinhaus property (resp. the closed $\pm n$-Steinhaus property) if for any closed subsets $A_1, \ldots, A_n \notin \mathcal{I}$ of $G$ their product $A_1 \cdots A_n$ (resp. $A_1^\pm \cdots A_n^\pm$) is not nowhere dense in $G$.

Observe that each ideal with the closed $n$-Steinhaus property has the closed $\pm n$-Steinhaus property.

The closed $n$-Steinhaus property is a partial case of the closed $\varepsilon$-Steinhaus property defined as follows.

Definition 3.2. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ for some $n \in \mathbb{N}$. An ideal $\mathcal{I}$ on a topological group $G$ is defined to have the closed $\varepsilon$-Steinhaus property if for any closed subsets $A_1, \ldots, A_n \notin \mathcal{I}$ of $G$ their $\varepsilon$-product $A_1^{\varepsilon_1} \cdots A_n^{\varepsilon_n} = \{ x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} : x_1 \in A_1, \ldots, x_n \in A_n \}$ is not nowhere dense in $G$.

Observe that the closed $n$-Steinhaus property is equal to the closed $\varepsilon$-Steinhaus property for the unique sequence $\varepsilon \in \{1\}^n$ consisting of $n$ units.

The following simple proposition follows immediately from the definitions.

Proposition 3.3. Let $\mathcal{I}$ be an ideal on a topological group and $\varepsilon \in \{-1, 1\}^n$ for $n \in \mathbb{N}$.

1. If $\mathcal{I}$ has the closed $\varepsilon$-Steinhaus property, then $\mathcal{I}$ has the closed $\pm n$-Steinhaus property;

2. If $\mathcal{I}$ is symmetric, then $\mathcal{I}$ has the closed $\varepsilon$-Steinhaus property if and only if $\mathcal{I}$ has the closed $n$-Steinhaus property.

It is clear that the ideal $\mathcal{M}_G$ of meager subsets in a topological group $G$ has the closed $1$-Steinhaus property. The classical Steinhaus theorem [28] (see also [29]) implies that for a locally compact group $G$ the ideal $\mathcal{N}_G$ of Haar null sets and the $\sigma$-ideal $\mathcal{E}_G$ generated by closed Haar null subsets in $G$ both have the closed $2$-Steinhaus property.
Corollary 3.5. If a σ-ideal $\mathcal{I}$ on a topological group $X$ has the closed $\pm n$-Steinhaus property for some $n \in \omega$, then each meager subgroup $H \subset G$ is universally $\mathcal{I}$-meager.

By analogy we can prove the following modification of Theorem 3.4:

Theorem 3.6. Assume that a σ-ideal $\mathcal{I}$ on a topological group $G$ has the $\varepsilon$-Steinhaus property for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$, $n \in \omega$. If subsets $A_1, \ldots, A_n \subset G$ are not universally $\mathcal{I}$-meager in $G$, then the set $A_1^{\varepsilon_1} \cdots A_n^{\varepsilon_n}$ is not meager in $G$.

Corollary 3.7. If a σ-ideal $\mathcal{I}$ on a topological group $X$ has the closed $n$-Steinhaus property for some $n \in \omega$, then each meager subsemigroup $H \subset G$ is universally $\mathcal{I}$-meager.

Corollaries 3.5 and 3.7 will be combined with the following classical result of Piccard [25] and Pettis [26] on sums of non-meager sets with the Baire property in topological groups. We recall that a subset $A$ of a topological space $X$ has the Baire property if for every non-empty open set $U \subset X$ the symmetric difference $(U \setminus A) \cup (A \setminus U)$ is meager in $X$. By Nikodým Theorem [31 4.9], each analytic set in a regular topological space has the Baire property.

Theorem 3.8 (Piccard-Pettis). For any non-meager sets $A, B$ with the Baire property in a topological group $G$, the set $AB$ has non-empty interior in $G$ and the set $AA^{-1}$ is a neighborhood of the unit in $G$.

Corollary 3.5 will be used to give the following answer to Problem 1.1:

Theorem 3.9. If a σ-ideal $\mathcal{I}$ on a topological group $G$ has the closed $\pm n$-Steinhaus property for some $n \in \mathbb{N}$, then $\mathcal{I}$ contains every meager quasi-analytic subsemigroup of $G$ and every non-open analytic subsemigroup of $G$.

Proof. Assume that some quasi-analytic subgroup $H$ of $G$ does not belong to the ideal $\mathcal{I}$. By Proposition 2.3 $H$ is not universally $\mathcal{I}$-meager and by Corollary 3.5 $H$ is not meager. If the subgroup $H$ is analytic, then $H$ has the Baire property by Nikodým Theorem [31 4.9] and the set $H \cdot H \subset H$ has non-empty interior in $G$ by Piccard-Pettis Theorem 3.8.

By analogy we can derive the following theorem from Corollary 3.7:

Theorem 3.10. If a σ-ideal $\mathcal{I}$ on a topological group $G$ has the closed $n$-Steinhaus property for some $n \in \mathbb{N}$, then $\mathcal{I}$ contains every meager quasi-analytic subsemigroup of $G$ and every analytic subsemigroup with empty interior in $G$.

We shall say that a σ-ideal $\mathcal{I}$ on a topological group $G$ has the closed $\pm \omega$-Steinhaus property if for any closed sets $A_n \not\in \mathcal{I}$, $n \in \mathbb{N}$, in $G$ the set $A_1^{\pm_1} \cdots A_m^{\pm_m}$ is not nowhere dense in $G$ for some $m \in \mathbb{N}$. It is clear that each ideal $\mathcal{I}$ with the closed $\pm n$-Steinhaus property for some $n \in \omega$ has the closed $\pm \omega$-Steinhaus property.

Problem 3.11. Does each non-open analytic subgroup $H$ of a Polish group $G$ belong to every invariant σ-ideal with the closed $\pm \omega$-Steinhaus property?
4. Constructing ideals possessing various closed Steinhaus properties

The results of the preceding section motivate a problem of finding natural examples of $\sigma$-ideals possessing the closed ±-Steinhaus property. Many examples of such ideals can be constructed as follows.

Given a subset $K$ of a topological group $G$, consider the family $B^\perp_K$ of all closed subsets $B \subset G$ such that for every $x, y \in G$ the set $K \cap (xBy^{-1})$ is nowhere dense in $K$. Let $I^\perp_K$ be the $\sigma$-ideal generated by the family $B^\perp_K$. The $\sigma$-ideal $I^\perp_K$ consists of subsets of countable unions of sets of the family $B^\perp_K$. It is clear that for any set $K$ in a topological group $G$ the $\sigma$-ideal $I^\perp_K$ is invariant. If $K = K^{-1}$, then the ideal $I^\perp_K$ is symmetric.

For a Baire subspace $K$ of a topological group $G$ the closed ±-Steinhaus property of the ideal $I^\perp_K$ can be characterized as follows.

**Proposition 4.1.** Let $K$ be a Baire subspace of a topological group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed ±-Steinhaus property for some $n \in \mathbb{N}$ if and only if for any non-open sets $U_1, \ldots, U_n \subset K$ and points $x_1, y_1, \ldots, x_n, y_n \in G$ the set $(x_1U_1y_1)^\pm \cdots (x_nU_ny_n)^\pm$ is nowhere dense in $G$.

**Proof.** If $I^\perp_K$ fails to have the closed ±-Steinhaus property, then there are closed sets $A_1, \ldots, A_n \notin I^\perp_K$ in $G$ such that the product $A_1^\pm \cdots A_n^\pm$ is nowhere dense in $G$. It follows from $A_1, \ldots, A_n \notin B^\perp_K$ that for every $i \leq n$ there are points $x_i, y_i \in G$ such that the set $K \cap (x_i^{-1}A_iy_i^{-1})$ contains a non-empty open subset $U_i$ of $K$. It follows that $U_i \subset x_i^{-1}A_iy_i^{-1}$ and hence $x_iU_iy_i \subset A_i$. Then the set $\left( x_1U_1y_1 \right)^\pm \cdots \left( x_nU_ny_n \right)^\pm \subset A_1^\pm \cdots A_n^\pm$ is nowhere dense in $G$. This completes the proof of the "if" part.

To prove the "only if", assume that for some non-open sets $U_1, \ldots, U_n \subset K$ and some points $x_1, y_1, \ldots, x_n, y_n \in G$ the set $x_1U_1y_1^\pm \cdots (x_nU_ny_n)^\pm$ is nowhere dense in $G$. We claim that for every $i \leq n$ the closed set $A_i = x_iU_iy_i$ does not belong to the ideal $I^\perp_K$. Assuming the opposite, we can find a countable subfamily $F_i \subset B^\perp_K$ such that $A_i \subset \bigcup F_i$. Since the space $K$ is Baire, so is its open subspace $U_i$ and its shift $x_iU_iy_i \subset A_i \subset \bigcup F_i$. Since $x_iU_iy_i$ is Baire, there is a set $F \in F_i$ such that the set $x_iU_iy_i \cap F$ contains some open subset $V_i$ of $x_iU_iy_i$. Thus $x_i^{-1}V_i \subset U_i \subset K \cap x_i^{-1}Fy_i^{-1}$ is a non-empty open subset of $K$, which contradicts the inclusion $F \subset B^\perp_K$. This contradiction shows that for every $i \leq n$ the closed set $A_i = x_iU_iy_i$ does not belong to the ideal $I$. Since the product $A_1^\pm \cdots A_n^\pm = (x_1U_1y_1)^\pm \cdots (x_nU_ny_n)^\pm$ is nowhere dense in $G$, the ideal $I$ fails to have the closed ±-Steinhaus property.

By analogy we can prove a characterization of $\sigma$-ideals $I^\perp_K$ possessing the closed $\epsilon$-Steinhaus property.

**Proposition 4.2.** Let $K$ be a Baire subspace of a topological group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed $\epsilon$-Steinhaus property for some $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, $n \in \mathbb{N}$, if and only if for any non-open sets $U_1, \ldots, U_n \subset K$ and points $x_1, y_1, \ldots, x_n, y_n \in G$ the set $(x_1U_1y_1)^{\epsilon_1} \cdots (x_nU_ny_n)^{\epsilon_n}$ is nowhere dense in $G$.

For symmetric subsets $K$ the characterization given in Proposition 4.2 can be simplified. A subset $K$ of a group $G$ is called symmetric if $K^{-1} = c^{-1}Kc^{-1}$ for some point $c \in G$ (called the center of symmetry of $K$).

**Proposition 4.3.** Let $K$ be a Baire symmetric subspace of a topological group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed $\epsilon$-Steinhaus property for some $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, $n \in \mathbb{N}$, if and only if for any non-open sets $U_1, \ldots, U_n \subset K$ and points $x_1, y_1, \ldots, x_n, y_n \in G$ the set $(x_1U_1y_1)^{\epsilon_1} \cdots (x_nU_ny_n)^{\epsilon_n}$ is nowhere dense in $G$.

For abelian groups the characterizations given in Propositions 4.4 and 4.5 can be simplified.

**Proposition 4.4.** Let $K$ be a Baire subspace of a topological abelian group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed ±-Steinhaus property for some $n \in \mathbb{N}$ if and only if for any non-empty open sets $U_1, \ldots, U_n \subset K$ there is $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ such that $U_1^{\epsilon_1} \cdots U_n^{\epsilon_n}$ is not nowhere dense in $G$.

**Proposition 4.5.** Let $K$ be a Baire subspace of a topological abelian group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed $\epsilon$-Steinhaus property for some $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, $n \in \mathbb{N}$, if and only if for any non-empty open sets $U_1, \ldots, U_n \subset K$ the product $U_1^{\epsilon_1} \cdots U_n^{\epsilon_n}$ is not nowhere dense in $G$.

**Proposition 4.6.** Let $K$ be a Baire symmetric subspace of a topological abelian group $G$. The $\sigma$-ideal $I^\perp_K$ has the closed $\epsilon$-Steinhaus property for some $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$, $n \in \mathbb{N}$, if and only if for any non-empty open sets $U_1, \ldots, U_n \subset K$ the product $U_1 \cdots U_n$ is not nowhere dense in $G$.

Now we apply Propositions 4.4 and 4.6 to sets $K$ having product structure.
Proposition 4.7. Let $A_k \subset G_k$ be non-empty sets in discrete topological abelian groups $G_k$, $k \in \omega$. For the closed set $K = \prod_{k \in \omega} A_k$ in the topological group $G = \prod_{k \in \omega} G_k$ the $\sigma$-ideal $I_K^\perp$ has the closed $\pm n$-Steinhaus property for some $n \in \mathbb{N}$ if and only if for some sequence $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and some $k_0 \in \omega$ the set $A_{k_0}^{\varepsilon_1} \cdots A_{k_0}^{\varepsilon_n}$ equals $G_k$ for all $k \geq k_0$.

Proof. If for every sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ the set $A_k^{\varepsilon_1} \cdots A_k^{\varepsilon_n}$ is not equal to $G_k$ for infinitely many numbers $k$, then the countable product $\prod_{k \in \omega} A_k^{\varepsilon}$ is nowhere dense in the topological group $G = \prod_{k \in \omega} G_k$ and the closed set $K^{\pm n} = \bigcup_{\varepsilon \in \{-1, 1\}^n} K^{\varepsilon_1} \cdots K^{\varepsilon_n}$ is nowhere dense in $G$. Since $K \notin I_K^\perp$, the ideal $I_K^\perp$ fails to have the closed $\pm n$-Steinhaus property.

Now assume that for some sequence $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and some $k_0 \in \omega$ the set $A_{k_0}^{\varepsilon_1} \cdots A_{k_0}^{\varepsilon_n} = G_k$ for all $k \geq k_0$. By Proposition 4.4 to prove that the ideal $I_K^\perp$ has the closed $\pm n$-Steinhaus property, it suffices to check that for any non-empty subsets $U_1, \ldots, U_n \subset G$ the product $U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$ is not nowhere dense in $G$. By the definition of the Tychonoff product topology on $K = \prod_{k \in \omega} A_k$, there are a number $m \geq k_0$ and points $u_0, \ldots, u_n \in \prod_{k < m} G_k$ such that $\{u_i\} \times \prod_{k \geq m} A_k \subset U_i$ for all $i \leq n$. Taking into account that $A_k^{\varepsilon_1} \cdots A_k^{\varepsilon_n} = G_k$ for all $k \geq m \geq k_0$, we conclude that $\{u_0 \cdots u_n\} \times \prod_{k \geq m} G_k \subset U_1^{\varepsilon_1} \cdots U_n^{\varepsilon_n}$, which means that the latter set has non-empty interior and is not nowhere dense in the topological group $G$. □

By analogy we can prove the following characterization.

Proposition 4.8. Let $A_k \subset G_k$ be non-empty sets in discrete topological abelian groups $G_k$, $k \in \omega$. For the closed set $K = \prod_{k \in \omega} A_k$ in the topological group $G = \prod_{k \in \omega} G_k$ the $\sigma$-ideal $I_K^\perp$ has the closed $\varepsilon$-Steinhaus property for some $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, $n \in \mathbb{N}$, if and only if for some sequence $k_0 \in \omega$ the set $A_{k_0}^{\varepsilon_1} \cdots A_{k_0}^{\varepsilon_n} = G_k$ for all $k \geq k_0$.

For symmetric sets $A_k$ in abelian groups $G_k$ the characterizations given in Propositions 4.7 and 4.8 can be unified.

Proposition 4.9. Let $n \in \mathbb{N}$ and $A_k \subset G_k$ be non-empty symmetric sets in discrete topological abelian groups $G_k$, $k \in \omega$. For the closed set $K = \prod_{k \in \omega} A_k$ in the topological group $G = \prod_{k \in \omega} G_k$ and the $\sigma$-ideal $I_K^\perp$ the following conditions are equivalent:

1. $I_K^\perp$ has the closed $n$-Steinhaus property;
2. $I_K^\perp$ has the closed $\pm n$-Steinhaus property;
3. the set $A_k^n = \{a_1 \cdots a_n : a_1, \ldots, a_n \in A_k\}$ equals $G_k$ for all but finitely many $k$.

Proof. The equivalence (1) $\iff$ (3) follows from Proposition 4.8 and (1) $\implies$ (2) is trivial. So, it remains to check that (2) $\implies$ (3). Assuming that $I_K^\perp$ has the closed $\pm n$-Steinhaus property and applying Proposition 4.7 we can find numbers $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $k_0 \in \omega$ such that $A_{k_0}^{\varepsilon_1} \cdots A_{k_0}^{\varepsilon_n} = G_k$ for all $k \geq k_0$. Since each set $A_k \subset G_k$ is symmetric, we can find an element $c_k \in G_k$ such that $A_k^{\varepsilon_1} \cdots A_k^{\varepsilon_n} = G_k = c_k^{-1} A_k c_k^{-1}$. Observe that $G = A_k^{\varepsilon_1} \cdots A_k^{\varepsilon_n} = A_k^n c_k^{\varepsilon_1-1} \cdots c_k^{\varepsilon_n-1}$ and hence $A_k = G_k$ for all $k \geq k_0$. □

Now we are ready construct $\sigma$-ideals distinguishing the closed $\pm n$-Steinhaus properties and the closed $m$-Steinhaus properties for various $n$ and $m$. For $n \in \mathbb{N}$ by

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

we denote the cyclic group of order $n$.

Example 4.10. For every $n \geq 2$ there exists an $F_\sigma$-supported invariant symmetric ideal $I_n$ on the compact group $C_{2n+1}$ which has the closed $n$-Steinhaus property but fails to have the closed $(n-1)$-Steinhaus property.

Proof. Apply Proposition 4.9 to the symmetric sets $A_k = \{e^{i\varphi} : \varphi \in \{-1, 0, 1\} \times \pi \mathbb{Z}ceil}$ in the cyclic groups $G_k = C_{2n+1}$ for $k \in \omega$. □

Example 4.11. For some number sequence $(n_k)_{k \in \omega}$ the compact group $G = \prod_{k \in \omega} C_{n_k}$ contains a closed subset $K$ such that the $\sigma$-ideal $I_K^\perp$ has the closed $\pm 2$-Steinhaus property but fails to have the closed $n$-Steinhaus property for every $n \in \mathbb{N}$. Moreover, the symmetric $\sigma$-ideal $I_K^\perp$ has the closed $\pm 2$-Steinhaus property but fails to have the closed $\varepsilon$-Steinhaus property for every $\varepsilon \in \bigcup_{n=1}^\infty \{-1, 1\}^n$. □
Proof. By \[17\] or \[18\], for every \( k \in \mathbb{N} \) there is a number \( n_k \) and a subset \( A_k \) in the cyclic group \( C_{n_k} \) such that \( A_k A_k^{-1} = C_{n_k} \), but the set \( A_k = \{ a_1 \cdots a_k : a_1, \ldots, a_k \in A_k \} \) is not equal to \( C_{n_k} \). By Proposition \[17\] for the closed set \( K = \prod_{k=1}^{\infty} A_k \) in the compact topological group \( G = \prod_{k=1}^{\infty} C_{n_k} \), the \( \sigma \)-ideal \( \mathcal{I}_K \) has the strong \( \pm 2 \)-Steinhaus property but fails to have the strong \( n \)-Steinhaus property for all \( n \in \mathbb{N} \).

Now consider the symmetric \( \sigma \)-ideal \( \mathcal{I}_{K_{\pm}} \). To show that it has the closed \( \pm 2 \)-Steinhaus property, fix any closed sets \( A_1 \), \( A_2 \notin \mathcal{I}_{K_{\pm}} \) in \( G \) and find points \( a_1, a_2 \in G \) such that \( a_1 A_1 \cap K_{\pm} \) and \( a_1 A_2 \cap K_{\pm} \) are not nowhere dense in \( K_{\pm} = K \cup K^{-1} \). Since \( A_k \cap A_k^{-1} \neq A_k \) for all \( k \geq 2 \), the intersection \( K \cap K^{-1} \) is nowhere dense in \( K \) and \( K^{-1} \). Consequently, we can find numbers \( \varepsilon_1, \varepsilon_2 \in \{-1, 1\} \) and non-empty open sets \( U_1, U_2 \subset K \) such that \( U_1 \cap U_2 \subset A_1 \cap K_{\pm} \) for every \( i \in \{1, 2\} \), and hence the set \( A_1^{\varepsilon_1} \cdot A_2^{\varepsilon_2} \) is not Haar null in \( G \). This completes the proof of the closed \( \pm 2 \)-Steinhaus property of the ideal \( \mathcal{I}_{K_{\pm}} \).

Now we prove that \( \mathcal{I}_{K_{\pm}} \) fails to have the closed \( \varepsilon \)-Steinhaus property for every \( \varepsilon \in \{-1, 1\}^n \), \( n \in \mathbb{N} \). Since the ideal \( \mathcal{I}_{K_{\pm}} \) is symmetric, it suffices to check that it fails to have the closed \( n \)-Steinhaus property (see Proposition \[5\]). The choice of the sets \( A_k \subset G \) guarantees that for every \( n \in \mathbb{N} \) the set \( K^n = \prod_{k=1}^{n} A_k^n \) is closed and nowhere dense in \( G \). Observing that \( K \) is not nowhere dense in \( K_{\pm} = K \cup K^{-1} \), we conclude that \( K \notin \mathcal{I}_{K_{\pm}} \), witnessing that \( \mathcal{I}_{K_{\pm}} \) fails to have the closed \( n \)-Steinhaus property.

**Remark 4.12.** By \[15\], \[16\] \( \S 12.4 \) the real line contains a Borel subgroup \( H \) of arbitrary Hausdorff dimension \( \alpha \in (0, 1) \). This result combined with Theorem \[5\] implies that for every \( \alpha \in (0, 1) \) and \( n \in \mathbb{N} \) the \( \sigma \)-ideal \( \mathcal{I}_d \) generated by closed subsets of Hausdorff dimension \( \leq \alpha \) in \( \mathbb{R} \) does not have the closed \( n \)-Steinhaus property.

## 5. Ideals of Haar-null and Haar-meager sets in Polish groups

The classical Steinhaus Theorem \[28\] (see also \[29\]) implies that for every locally compact topological group \( G \) the \( \sigma \)-ideal \( \mathcal{E}_G \) generated by closed sets of Haar measure zero has the closed \( 2 \)-Steinhaus property. This allows us to apply Theorem \[5\] and deduce the following result proved for analytic subgroups of the real line by Laczkovich \[20\].

**Theorem 5.1.** Every meager quasi-analytic subsemigroup \( H \) in a locally compact group \( G \) belongs to the \( \sigma \)-ideal \( \mathcal{E}_G \) and hence can be covered by countably many closed sets of Haar measure zero.

It would be interesting to find versions of Theorem 5.1 for non-locally compact Polish groups. For this we should find a counterpart of the ideal \( \mathcal{E}_G \). This can be done in six different ways using the notions of \{generically\} (left, right) Haar null set. For a Polish space \( X \) by \( \text{exp}(X) \) we shall denote the space of all non-empty compact sets in \( X \) endowed with the Vietoris topology, and by \( P(X) \) the space of all Borel \( \sigma \)-additive probability measures on \( X \), endowed with the weak-star topology.

**Definition 5.2.** An analytic subset \( A \) of a Polish group \( G \) is called

- **left-Haar null** if there exists a measure \( \mu \in P(G) \) such that \( \mu(xA) = 0 \) for all \( x \in G \);
- **right-Haar null** if there exists a measure \( \mu \in P(G) \) such that \( \mu(AY) = 0 \) for all \( y \in G \);
- **Haar null** if there exists a measure \( \mu \in P(G) \) such that \( \mu(xA) = 0 \) and \( \mu(AY) = 0 \) for all \( x, y \in G \);
- **generically left-Haar null** if the set \( \{ \mu \in P(G) : \forall x \in G \mu(xA) = 0 \} \) is comeager in \( P(G) \);
- **generically right-Haar null** if the set \( \{ \mu \in P(G) : \forall y \in G \mu(AY) = 0 \} \) is comeager in \( P(G) \);
- **generically Haar null** if the set \( \{ \mu \in P(G) : \forall x, y \in G \mu(xAy) = 0 \} \) is comeager in \( P(G) \).

Haar null sets were introduced by Christensen \[3\] who proved the following important fact:

**Theorem 5.3** (Christensen). If an analytic subset \( A \) of a Polish abelian group \( G \) is not Haar null, then the set \( A^{-1}A \) is a neighborhood of the unit in \( G \).

Generically Haar null sets were introduced by Dodos \[10\]. In \[10\] Dodos proved that each analytic generically Haar null subset of a Polish abelian group is meager. By \[11\], a Polish abelian group \( G \) is locally compact if and only if each closed Haar null set in \( G \) is generically Haar null. The following fact was proved by Dodos in \[12\].
Theorem 5.4 (Dodos). If an analytic subset $A$ of a Polish group $G$ is not generically left-Haar null, then the set $A^{-1}A$ is not meager in $G$ and the set $A^{-1}AA^{-1}$ is a neighborhood of the unit in $G$.

Corollary 5.5. Each non-open analytic subgroup of a Polish group is generically left-Haar null and generically right-Haar null.

Next, we consider “Haar” modifications of the notion of a meager sets, suggested by Darji [8]. By [8], each Borel (left or right) Haar null set in a Polish group is (left, right) Haar-meager.

Definition 5.6. We shall say that a subset $A$ of a topological group $G$ is

- **Haar-meager** if there is a continuous map $f : K \to X$ defined on a metrizable compact space $K$ such that for every $x, y \in G$ the set $f^{-1}(xAy)$ is meager in $K$;
- **left-Haar-meager** if there is a continuous map $f : K \to X$ defined on a metrizable compact space $K$ such that for every $x \in G$ the set $f^{-1}(xA)$ is meager in $K$;
- **right-Haar-meager** if there is a continuous map $f : K \to X$ defined on a metrizable compact space $K$ such that for every $y \in G$ the set $f^{-1}(A)$ is meager in $K$;
- **generically Haar-meager** if the subset $\{ K \in \exp(X) : \forall x, y \in G \ K \cap (xAy) \in M_K \}$ is comeager in $\exp(X)$;
- **generically left-Haar-meager** if the subset $\{ K \in \exp(X) : \forall x \in G \ K \cap (xA) \in M_K \}$ is comeager in $\exp(X)$;
- **generically right Haar-meager** if the subset $\{ K \in \exp(X) : \forall y \in G \ K \cap (Ay) \in M_K \}$ is comeager in $\exp(X)$.

Haar-meager sets were introduce by Darji [8]. The notion of a generically (left, right) Haar-meager set seems to be new (we intend to devote a paper [2] to studying this new notion). By [3], each Borel (left or right) Haar-meager set in a Polish group $G$ is meager in $G$.

Proposition 5.7. Each closed (left, right) Haar null set in a Polish group is (left, right) Haar-meager.

Proof. Assuming that a closed subset $A$ is a Polish group $G$ is Haar null, find a measure $\mu \in P(G)$ such that $\mu(xAy) = 0$ for all $x, y \in G$. Since each Borel $\sigma$-additive measure on a Polish space is Radon (see [4, 7.1.7]), there exists a compact set $K \subset G$ with $\mu(K) > 0$. Let $U \subset K$ be the largest open subset of measure $\mu(U) = 0$ in $K$. Then $\mu(K \setminus U) = \mu(K) > 0$ and each non-empty relatively open subset $V$ of $K \setminus U$ has positive measure $\mu(V)$. Replacing $K$ by $K \setminus U$, we can assume that each non-empty open subset of $K$ has positive $\mu$-measure. In this case the compact set $K$ witnesses that the set $A$ is Haar-meager. Indeed, assuming that for some $x, y \in G$ the closed subset $K \cap xAy$ is non-meager in $K$, we conclude that it contains a non-empty open subset $V$ of $K$, which implies that $\mu(xAy) \geq \mu(K \cap xAy) \geq \mu(V) > 0$. But this contradicts the choice of the measure $\mu$. By analogy we can prove the left and right versions of this proposition. \qed

Closed (left, right) Haar-meager sets in Polish groups can be characterized as closed sets which are not (left, right) prethick.

Definition 5.8. A subset $T$ of a topological group $G$ is called

- **thick** if for every compact set $K \subset G$ there are points $x, y \in G$ such that $K \subset xTy$;
- **prethick** if for every compact set $K \subset G$ there is a finite set $F \subset G$ such that $K \subset FTF$;
- **left thick** if for every compact set $K \subset G$ there is a point $x \in G$ such that $K \subset xT$;
- **left prethick** if for every compact set $K \subset G$ there is a finite set $F \subset G$ such that $K \subset FT$;
- **right thick** if for every compact set $K \subset G$ there is a point $y \in G$ such that $K \subset Ty$;
- **right prethick** if for every compact set $K \subset G$ there is a finite set $F \subset G$ such that $K \subset TF$.

Proposition 5.9. For a closed subset $A$ of a Polish group $G$ the following conditions are equivalent:

1. $A$ is (left, right) Haar-meager;
2. there is a compact subset $K \subset G$ such that for every $x, y \in G$ the set $K \cap xAy$ (resp. $K \cap xA, K \cap Ay$) is nowhere dense in $K$;
3. $A$ is not (left, right) prethick.

Proof. The implication (2) $\Rightarrow$ (1) is trivial. To see that (1) $\Rightarrow$ (3), assume that $A$ is (left, right) Haar meager and find a continuous map $f : K \to G$ defined on a compact metrizable space $K$ such that for every $x, y \in G$ the preimage $f^{-1}(xAy)$ (resp. $f^{-1}(xA), f^{-1}(Ay)$) is meager in $K$. Then for every finite subset $F \subset G$ the
preimage $f^{-1}(FAF) = \bigcup_{x,y \in F} f^{-1}(xAy)$ (resp. $f^{-1}(FA)$, $f^{-1}(AF)$) is meager in $K$, which implies that the compact subset $f(K)$ of $G$ is not contained in the set $FAF$ (resp. $FA$, $AF$) and hence witnesses that $A$ is not prethick.

To prove that $(3) \Rightarrow (2)$ we shall prove that the negation of $(2)$ implies that $A$ is (left, right) thick. We shall prove the two-sided version of this implication (the left and right versions can be proved by analogy). Fix a compact set $K \subset G$ and a complete metric $d$ generating the topology of the Polish space $G$. For a point $x \in G$, set $A \subset G$, and positive number $\varepsilon > 0$, put $B(x; \varepsilon) = \{y \in G : d(y, x) < \varepsilon\}$ and $B(A; \varepsilon) = \bigcup_{a \in A} B(a; \varepsilon)$.

For every $n \in \omega$ choose a finite cover $\mathcal{C}_n$ of $K$ by non-empty closed sets of $d$-diameter $\leq 2^{-n}$. In each set $C \in \mathcal{C}_n$ pick up a point $p_C$. For every $n \in \omega$ consider the compact set $\sigma_n = \bigcup_{m \geq n} \bigcup_{C \in \mathcal{C}_m} p_C^{-1}C$ and observe that the sequence $(\sigma_n)_{n \in \omega}$ converges to the unit $1_G$ of $G$ in the sense that each neighborhood $U \subset G$ of $1_G$ contains all but finitely many sets $\sigma_n, n \in \omega$. By induction we shall construct a sequence $(\Sigma_n)_{n \in \omega}$ of non-empty compact sets in $K$ satisfying the following conditions for every $n \in \omega$:

\begin{itemize}
  \item $\bigcup_{l \in \omega} \Sigma_{n+1} \subset B(\Sigma_n; 2^{-n})$;
  \item any open set $U \subset \Sigma_{n+1}$ meeting the set $\Sigma_n$ contains the set $x \cdot \sigma_l$ for some $x \in \Sigma_n$ and some $l \in \omega$.
\end{itemize}

We start the inductive construction letting $\Sigma_1 = \emptyset$ and $\Sigma_0 = \{1_G\}$. Assume that for some $n \in \omega$ the set $\Sigma_n$ has been constructed. Fix a countable dense set $\{x_m\}_{m \in \omega}$ in $\Sigma_n$. For every $m \in \omega$ find a number $l(m) \in \omega$ such that $x_m \cdot \sigma_{l(m)} \subset B(x_m; 2^{-n-\varepsilon})$. Such number $l(m)$ exists since the sequence $(\sigma_l)_{l \in \omega}$ converges to $1_G$.

Then $\Sigma_{n+1} = \Sigma_n \cup \bigcup_{m \in \omega} x_m \sigma_{l(m)}$ is compact and satisfies the conditions (\textcircled{1}) and (\textcircled{2}).

After completing the inductive construction, consider the closure $\overline{\Sigma}$ of the union $\Sigma = \bigcup_{n \in \omega} \Sigma_n$. The conditions (\textcircled{1}), $n \in \omega$, guarantee that the set $\overline{\Sigma}$ is totally bounded in the complete metric space $(G, d)$ and hence $\overline{\Sigma}$ is compact. By the negation of (2), there exist points $x, y \in G$ such that the set $\overline{\Sigma} \cap xAy$ is not nowhere dense in $\overline{\Sigma}$ and hence contains a non-empty open subset $U$ of $\overline{\Sigma}$. By the conditions (\textcircled{1}), $n \in \omega$, the set $U$ contains a shift $z\sigma_l$ of the set $\sigma_l$ for some $l \in \omega$. Then

$$
  xAy \supset U \supset z\sigma_l \supset \bigcup_{C \in C_l} z p_C^{-1} C
$$

and $K = \bigcup_{C \in C_l} \bigcup_{C \in C_l} p_C z^{-1} xAy \subset FAF$ for the finite set $F = \{y\} \cup \{p_C z^{-1} x : C \in C_l\}$, witnessing that the set $A$ is prethick.

The following Steinhaus-type property of Haar-meager sets was proved by Jabłońska [20].

**Theorem 5.10** (Jabłońska). If a Borel subset $A$ of a Polish abelian group $G$ is not Haar-meager, then $A^{-1}A$ is a neighborhood of zero.

Generically left Haar-meager sets have a weaker property.

**Theorem 5.11.** If an (analytic) subset $A$ of a Polish group $X$ is not generically left Haar-meager, then $A^{-1}A$ is not meager in $X$ (and hence $A^{-1}AA^{-1}A$ is a neighborhood of the unit in $X$).

**Proof.** To derive a contradiction, assume that the set $A^{-1}A$ is meager in $X$. Then we can find a meager $F_\sigma$-set $F \subset X$ containing $A^{-1}A$. Consider the continuous map $\mu : X \times X \to X, \mu : (x, y) \mapsto x^{-1}y$, and observe that it is open. This implies that $\mu^{-1}(F)$ is a meager $F_\sigma$-set in $X \times X$ and its complement $G = (X \times X) \setminus \mu^{-1}(F)$ is a dense $G_\delta$-set in $X \times X$. For a subset $K \subset X$ put $(K)^2 = \{(x, y) \in K \times K : x \neq y\} \subset X \times X$. By Mycielski-Kuratowski Theorem [21, 19.1], the set $H = \{K \in \exp(X) : (K)^2 \subset G\}$ is a dense $G_\delta$-set in the hyperspace $\exp(X)$. Since the Polish group $G$ contains a non-empty meager subset, it is not discrete and hence contains no isolated points. By [21, 4.31], the subset $P \subset \exp(X)$ consisting of compact subsets without isolated points is a dense $G_\delta$-set in $X$. Then $H \cap P$ is a dense $G_\delta$-set in $\exp(X)$. We claim that for each compact subset $K \subset H \cap P$ and every $x \in A$ the set $K \cap xA$ contains at most one point and hence is meager in $K$. Thus, $K \cap xA$ contains two distinct points $xa, xb$, we conclude that $\mu(xa, xb) = a^{-1}b \in A^{-1}A$ and hence $(xa, xb) \in (K)^2 \cap \mu^{-1}(F)$, which contradicts the choice of $K \in H$. Therefore $H \cap P \subset \{K \in \exp(X) : \forall x \in G. K \cap xA \in \mathcal{M}_K\}$, which implies that $A$ is generically left Haar-meager. But this contradicts the choice of $A$.

Now assume that $A$ is analytic. Then the set $A^{-1}A$ is analytic and non-meager in the Polish group $G$. By Picard-Pettis Theorem [21, 9.9], $A^{-1}AA^{-1}A$ is a neighborhood of the unit in $G$. 

Theorems 5.4 and 5.11 imply the following partial answer to Problem 1.1.
Corollary 5.12. Every analytic non-open subgroup $H$ of a Polish group $G$ is generically left-Haar-meager and generically right Haar-meager in $G$.

6. Three counterexamples

In this section we present three counterexamples to possible extensions of Theorem 5.1. First of them will show that in contrast to Theorem 5.1, Corollaries 5.3 and 5.12 (treating analytic subgroups) are not true for subsemigroups of Polish groups.

Example 6.1. The closed nowhere dense subsemigroup $\mathbb{R}_+^\omega$ is thick in the Polish abelian group $\mathbb{R}^\omega$ and hence $H$ is neither Haar-null nor Haar-meager in $\mathbb{R}^\omega$.

The second example shows that “two-sided” versions Corollaries 5.3 and 5.12 are false.

Example 6.2. There exists a Polish group $G$ containing a closed nowhere dense thick subgroup $H$. This subgroup is neither Haar-null nor Haar-meager. On the other hand, $H$ is generically left (and right) Haar-null and generically left (and right) Haar-meager in $G$.

Proof. Let $X$ be a countable set and $Z \subset X$ be a proper countable subset of $X$. Let $FS_X$ be the discrete group of finitely supported permutations of $X$ and $FS_Z$ be the subgroup of $FS_X$ consisting of permutations $f : X \to X$ with finite support supp$(f) = \{x \in X : f(x) \neq x\} \subset Z$. Consider the Polish group $G = FS_X$ and closed nowhere dense subgroup $H = FS_Z$ of $G$. It is easy to show that for any finite subset $A \subset FS_X$ there is a permutation $g \in FS_X$ such that $gAg^{-1} \subset FS_Z$. This means that the subgroup $FS_Z$ is thick in $FS_X$. This property implies that the countable product $H = FS_Z$ is thick in the Polish group $G = FS_Z$. By Propositions 5.9 and 5.12 the closed subgroup $H$ is neither Haar-meager nor Haar-null in $G$.

By Corollaries 5.3 and 5.12 the nowhere dense subgroup $H$ is generically left (and right) Haar-null and generically left (and right) Haar-meager in $G$. □

Our final example shows that Theorem 5.1 is not true for non-locally compact Polish groups. A topological space $X$ is defined to be $\sigma$-Polish if $X$ can be written as a countable union $X = \bigcup_{i \in \omega} X_i$ of closed Polish subspaces of $X$. It is easy to see that each $\sigma$-Polish subspace $X$ of a Polish space $Y$ can be written as the difference $A \setminus B$ of two $F_\sigma$-sets in $Y$. Consequently, $X$ is of Borel classes $F_\sigma$ and $G_\delta$.

Example 6.3. The Polish group $G = Z^\omega$ contains a meager $\sigma$-Polish subgroup $H \subset \{0\} \cup \{z \in Z^\omega : |z^{-1}(0)| < \omega\}$ which cannot be covered by countably many closed Haar-meager sets in $G$. The subgroup $H$ is generated by a $G_\delta$-subset $P \subset Z^\omega$ such that for any non-empty relatively open set $U \subset P$ its closure $\bar{U}$ is prethick in $G$.

Proof. In the construction of the $G_\delta$-set $P$ we shall use the following elementary lemma.

Lemma 6.4. There exists an infinite family $T$ of thick subsets of $Z$ and an increasing number sequence $(\Xi_m)_{m \in \omega}$ such that for any positive numbers $m \leq n$, non-zero integer numbers $\lambda_1, \ldots, \lambda_n \subset [-m, m]$, pairwise distinct sets $T_1, \ldots, T_n \in T$ and points $x_i \in T_i$, $i \leq n$, such that $\{x_1, \ldots, x_n\} \not\subset [-\Xi_m, \Xi_m]$ the sum $\lambda_1x_1 + \cdots + \lambda_nx_n$ is not equal to zero.

Proof. For every $m \in \omega$ let $\xi_m \in \omega$ be the smallest number such that $2^{2^m - x} > m^2(2^{2^m - x} - 1) + x$ for all $x \geq \xi_m$, and put $\Xi_m = 2^{2^m + \xi_m}$. Choose an infinite family $A$ of pairwise disjoint infinite subsets of $\mathbb{N}$ and for every $A \in A$ consider the thick subset $T_A = \bigcup_{a \in A}[2^{2^a} - a, 2^{2^a} + a]$ of $Z$. Here by $[a, b]$ we denote the segment $\{a, \ldots, b\}$ of integers. We claim that the family $T = \{T_A\}_{A \in A}$ and the sequence $(\Xi_m)_{m \in \omega}$ have the required property.

Take any positive numbers $m \leq n$, non-zero integer numbers $\lambda_1, \ldots, \lambda_n \subset [-m, m]$, pairwise distinct sets $T_1, \ldots, T_n \in T$ and points $x_1 \in T_1, \ldots, x_n \in T_n$ such that $\{x_1, \ldots, x_n\} \not\subset [-\Xi_m, \Xi_m]$. For every $i \leq n$ find an integer number $a_i$ such that $x_i = 2^{2^{2a_i}} + \xi_i$ for some $\xi_i \in [-a_i, a_i]$. Since the family $A$ is disjoint and the sets $T_1, \ldots, T_n$ are pairwise distinct, the points $a_1, \ldots, a_n$ are pairwise distinct, too. Let $j$ be the unique number such that $a_j = \max\{a_i : 1 \leq i \leq n\}$. Taking into account that $\{x_1, \ldots, x_n\} \not\subset [0, \Xi_m] = [0, 2^{2\Xi_m} + \xi_m]$, we conclude that $2^{2^{2a_i}} + a_j \geq 2^{2^{2a_i}} + \xi_j = x_j > 2^{2\Xi_m} + \xi_m$ and hence $a_j > \xi_m$. The definition of the number $\xi_m$ guarantees that $2^{2^{2a_j}} - a_j > m^2(2^{2^{2a_j}} - a_j)$, hence $|\lambda_jx_j| \geq x_j = 2^{2^{2a_j}} + \xi_j \geq 2^{2^{2a_j}} - a_j > m^2(2^{2^{2a_j}} - a_j) + a_j > \sum_{i \neq j} |\lambda_i(2^{2^{2a_i}} + a_i)| + \sum_{i \neq j} |\lambda_i x_i|$, i.e., $|\lambda_jx_j| > \sum_{i \neq j} |\lambda_i x_i|$. 

and hence $\sum_{i=1}^{n} \lambda_i x_i \neq 0$.

Now we are ready to start the construction of the $G_{\delta}$-set $P \subset \mathbb{Z}^{\omega}$. This construction will be done by induction on the tree $\omega^{<\omega} = \bigcup_{n \in \mathbb{N}} \omega^n$ consisting of finite sequences $s = (s_0, \ldots, s_{n-1}) \in \omega^n$ of finite ordinals. For a sequence $s = (s_0, \ldots, s_{n-1}) \in \mathbb{N}$ and a number $m \in \omega$ by $s' m = (s_0, \ldots, s_{n-1}, m) \in \omega^{n+1}$ we denote the concatenation of $s$ and $m$.

For an infinite sequence $s = (s_n)_{n \in \omega} \in \mathbb{Z}^{\omega}$ and a natural number $l \in \omega$ let $s[l] = (s_0, \ldots, s_{l-1})$ be the restriction of the function $s : \omega \to \mathbb{Z}$ to the subset $l = \{0, \ldots, l-1\}$. Observe that the topology of the Polish group $\mathbb{Z}^{\omega}$ is generated by the ultrametric

$$d(x, y) = \inf \{2^{-n} : n \in \omega, x|n = y|n\}, \quad x, y \in \mathbb{Z}^{\omega}.$$ 

Observe also that for every $z \in \mathbb{Z}^{\omega}$ and $n \in \omega$ the set $U(z|n) = \{z \in \mathbb{Z}^{\omega} : z|n = z|n\}$ coincides with the closed ball $B(z; 2^{-n}) = \{x \in \mathbb{Z}^{\omega} : d(x, z) \leq 2^{-n}\}$ centered at $z$.

Using Lemma 6.4 choose a number sequence $(\Xi_m)_{m \in \mathbb{N}}$ and a sequence $(T_m)_{m \in \omega^{<\omega}}$ of thick sets in the discrete group $\mathbb{Z}$ such that for every positive integer numbers $n \leq m$, finite set $F \subset \omega^{<\omega}$ of cardinality $|F| \leq n$, function $\lambda : F \to [-m, m] \setminus \{0\}$, and numbers $z_s \in T_s$, $s \in F$, such that $\{z_s\}_{s \in F} \not\subset [-\Xi_m, \Xi_m]$ the sum $\sum_{s \in F} \lambda(s) \cdot z_s$ is not equal to zero.

For every $s \in \omega^{<\omega}$ we shall construct a sequence $(z_s)_{s \in \omega^{<\omega}}$ of points of $\mathbb{Z}^{\omega}$ and a sequence $(l_s)_{s \in \omega^{<\omega}}$ of ordinals satisfying the following conditions for every $s \in \omega^{<\omega}$:

1. $U(z_s\langle i \rangle | l_s) \cap U(z_{s\langle j \rangle} | l_{s\langle j \rangle}) = \emptyset$ for any distinct numbers $i, j \in \omega$;
2. $l_s > l_s + i$ for every $i \in \omega$;
3. the closure of the set $(z_s\langle i \rangle)_{i \in \omega}$ contains the prethick set $T_s = \{z_s | l_s\} \times \prod_{n \geq l_s} T_{s, n}$ and is contained in the set $T^*_{s} = \{z_s | l_s\} \times \prod_{n \geq l_s} (T_{s, n} \cup \{n, n\}) \subset U(z_s | l_s)$.

We start the inductive construction letting $z_0 = 0$ and $l_0 = 0$. Assume that for some $s \in \omega^{<\omega}$ a point $z_s \in \mathbb{Z}^{\omega}$ and a number $l_s \in \omega$ have been constructed.

Consider the prethick sets $T_s$ and $T^*_s$ defined in the conditions (2) and (3). Since $T_s$ is nowhere dense in $T^*_s$, we can find a sequence $(z_s\langle i \rangle)_{i \in \omega}$ of pairwise distinct points of $T^*_s$ such that the space $D_s = \{z_s\langle i \rangle | i \in \omega\}$ is discrete and contains $T_s$ in its closure. Since $D_s$ is discrete, for every $i \in \omega$ we can choose a number $l_s > l_s + i$ such that the open sets $U(z_s | l_s, i)$, $i \in \omega$, are pairwise disjoint. Observing that the sequences $(z_s)_{s \in \omega^{<\omega}}$ and $(l_s)_{s \in \omega^{<\omega}}$ satisfy the conditions (1) and (3), we complete the inductive step.

We claim that the $G_{\delta}$-subset $P = \bigcap_{n \in \omega} \bigcup_{i \in \omega^n} U(z_s | l_s)$ of $\mathbb{Z}^{\omega}$ has the required properties. First observe that the map $h : \omega^{\omega} \to P$ assigning to each infinite sequence $s \in \omega^{\omega}$ the unique point $z_s$ of the intersection $\bigcap_{n \in \omega} U(z_s | l_s)$ is a homeomorphism of $\omega^{\omega}$ onto $P$. Then the inverse map $h^{-1} : P \to \omega^{\omega}$ is a homeomorphism too.

Claim 6.5. For every non-empty open set $U \subset P$ its closure $\bar{U}$ in $\mathbb{Z}^{\omega}$ is prethick in $\mathbb{Z}^{\omega}$.

Proof. Given any non-empty open set $U \subset P$, pick any point $p \in U$ and find a unique infinite sequence $t \in \omega^{\omega}$ such that $\{p\} = \bigcap_{n \in \mathbb{N}} U(z_t | l_t | m)$. Since the family $\{U(z_t | l_t | m)\}_{m \in \mathbb{N}}$ is a neighborhood base at $p$, there is $m \in \mathbb{N}$ such that $U(z_t | l_t | m) \subset U$. Consider the finite sequence $s = t | m$. The Baire Theorem guarantees that for every $i \in \omega$ the intersection $P \cap U(z_s | l_s, i)$ contains some point $y_s | i$. Taking into account that

$$d(z_s, y_s | i) \leq \text{diam} U(z_s | l_s, i) \leq 2^{-l_s, i} \leq 2^{-i}$$

and $T_s$ is contained in the closure of the set $(z_s | i)_{i \in \omega}$, we conclude that the prethick set $T_s$ is contained in the closure of the set $(y_s | i)_{i \in \omega} \subset P \cap U(z_s | l_s, i) \subset U$, which implies that $\bar{U}$ is prethick.

Claim 6.6. The subgroup $H \subset \mathbb{Z}^{\omega}$ generated by $P$ cannot be covered by countably many closed Haar-meager sets in $G$.

Proof. To derive a contradiction, assume that $H \subset \bigcup_{n \in \omega} H_n$ where each set $H_n$ is closed and Haar-meager in $\mathbb{Z}^{\omega}$. Since the Polish space $P = \bigcup_{n \in \omega} P \cap H_n$ is Baire, for some $n \in \omega$ the set $P \cap H_n$ contains a non-empty open subset $U$ of $P$. Taking into account that closure $\bar{U} \subset H_n$ is prethick in $G$, we conclude that the set $H_n$ is prethick and not Haar-meager (according to Proposition 5.9).
It remains to prove that the subgroup $H \subset \mathbb{Z}^\omega$ generated by the $G_\delta$-set $P$ is $\sigma$-Polish and meager in $\mathbb{Z}^\omega$. This is the most difficult (in technical respect) part of the proof.

We recall that by $h : \omega^\omega \to P$ we denote the homeomorphism assigning to each infinite sequence $s \in \omega^\omega$ the unique point $z_s$ of the intersection $\bigcap_{n \in \omega} U(z_s|_m)$. For every $n \in \omega$ denote by $R_n : \omega^\omega \to \omega^\omega$, $R_n : s \mapsto s|n$, the restriction operator. For any finite subset $F \subset \omega^\omega$ denote by $\delta_F \in \omega$ the smallest number such that the restriction $R_n|F$ is injective. Let also $\Lambda_F = \max\{|l_{s\delta_F} : s \in F\}$. Taking into account that $l_s|_m \geq m$ for every $s \in \omega^\omega$ and $m \in \omega$, we conclude that $\Lambda_F \geq \delta_F$ for every finite subset $F$ of $P$.

Claim 6.7. For every $m \in \mathbb{N}$, non-empty finite set $F \subset \omega^\omega$ of cardinality $|F| \leq m$ and function $\lambda : F \to [-m, m] \setminus \{0\}$ we get

$$\sum_{s \in F} \lambda(s) \cdot z_s \in \{y \in \mathbb{Z}^\omega : \forall k > \max\{\Lambda_F, \Xi_m\} \ y(k) \neq 0\}.$$ 

Proof. Given any number $k > \max\{\Lambda_F, \Xi_m\}$, consider the projection $\forall_k : \mathbb{Z}^\omega \to \mathbb{Z}$, $\forall_k : (x_i)_{i \in \omega} \mapsto x_k$, to the $k$-th coordinate. The claim will be proved as soon as we check that the element $y = \sum_{s \in F} \lambda(s) \cdot z_s$ has non-zero projection $\forall_k(y)$. Observe that $\forall_k(y) = \sum_{s \in F} \lambda(s) \cdot \forall_k(z_s)$. For every $s \in F$ the equality $\{z_s\} = \bigcap_{n \in \omega} U(z_s|_m)l_{s|_m}$ implies that $\forall_k(z_s) = \forall_k(z_{s|_m})$ for any $m \in \omega$ such that $l_{s|m} > k$.

Taking into account that $k > \Lambda_F = \max_{s \in F} l_{s\delta_F}$, for every $s \in F$ we can find a number $m_s \geq \delta_F$ such that $k \in [l_{s|m_s}, l_{s|(m_s+1)}]$. Then

$$\forall_k(z_s) = \forall_k(z_{s|((m_s+1))}l_{s|((m_s+1))}) \in T_{s|m_s,k} \cup \{s|m_s,k\} \subset T_{s|m_s} \setminus [-k, k] \subset T_{s|m_s} \setminus [-\Xi_m, \Xi_m]$$

by the condition $(3_{s|m_s})$ of the inductive construction.

The definition of the number $\delta_F$ guarantees that the sequences $s|\delta_F$, $s \in F$, are pairwise distinct and so are the sequences $s|m_s$, $s \in F$. Then $\forall_k(y) = \sum_{s \in F} \lambda(s)\forall_k(z_s) \neq 0$ by the choice of the family $(T_s)_{s \in \omega^\omega}$. \qed

For every $n \in \omega$ and a finite sequence $s \in \omega^n$ consider the basic closed-and-open subset $V_s = \{v \in \omega^\omega : v|n = s\}$ in $\omega^\omega$ and observe that $h(V_s) = P \cap U(z_s)$. For any non-empty finite set $F \subset \omega^\omega$ put

$$V_F = \prod_{s \in F} V_s|_{\delta_F}.$$ 

Given any function $\lambda : F \to \mathbb{Z} \setminus \{0\}$ we shall prove that the function

$$\Sigma_{\lambda} : V_F \to H, \quad \Sigma_{\lambda} : v \mapsto \sum_{s \in F} \lambda(s) \cdot z_{v(s)}$$

is a closed topological embedding. This will be done in three steps. Let $||\lambda|| = \max_{s \in F} |\lambda(s)|$.

Claim 6.8. The function $\Sigma_{\lambda} : V_F \to H$ is injective.

Proof. Choose any distinct sequences $u, v \in V_F$ and consider the symmetric difference $D$ of the sets $\{u(s)\}_{s \in F}$ and $\{v(s)\}_{s \in F}$. Put $m = \max\{|F|, ||\lambda||\}$ and $k = 1 + \max\{\Lambda_{k(D)}, \Xi_m\}$. Claim 6.7 guarantees that the element

$$y = \Sigma_{\lambda}(u) - \Sigma_{\lambda}(v) = \sum_{s \in F} \lambda(s)z_{u(s)} - \sum_{s \in F} \lambda(s)z_{v(s)}$$

has $y(k) \neq 0$, which implies that $y \neq 0$ and $\Sigma_{\lambda}(u) \neq \Sigma_{\lambda}(v)$. \qed

Claim 6.9. The function $\Sigma_{\lambda} : V_F \to H$ is a topological embedding.

Proof. It suffices to show that a sequence $(v_n)_{n \in \omega} \in V_F'$ contains a subsequence convergent to a point $v_\infty \in V_F$ if the sequence $(\Sigma_{\lambda}(v_n))_{n \in \omega}$ converges to the point $\Sigma_{\lambda}(v_\infty)$ in $\mathbb{Z}^\omega$. Replacing $(v_n)_{n \in \omega}$ by a suitable subsequence, we can assume that for every $s \in F$ either the sequence $(v_n(s))_{n \in \omega}$ converges to $v_\infty(s)$ or the point $v_\infty(s)$ has a neighborhood $W(v_\infty(s)) \subset \omega^\omega$ containing no point of the set $\{v_n(s)\}_{n \in \omega}$. If the set $F_c = \{s \in F : \lim_{n \to \infty} v_n(s) = v_\infty(s)\}$ coincides with $F$, then the sequence $(v_n)_{n \in \omega}$ converges to $v_\infty$ and we are done.

So, we assume that $F_c \neq F$ and hence the set $F_c' = F \setminus F_c$ is not empty. For every $s \in F_c'$ choose a number $m_s \geq \delta_F$ such that the basic neighborhood $V_{s|m_s}$ of $s$ is disjoint with the set $\{v_n(s)\}_{n \in \omega}$. Then for every $n \in \omega$ the set $D_n = \bigcup_{s \in F_c'} \{v_n(s), v_\infty(s)\}$ has $\delta_{D_n} \leq \max_{s \in F_c} m_s$. Put $m = \max\{|F|, ||\lambda||\}$ and $k = 1 + \max\{\Xi_m, \max_{s \in F_c'} m_s\}$. 


Consider the sequence \((v'_n)_{n \in \omega} \in \Pi_{\omega}^\kappa\) defined by
\[
v_n(s) = \begin{cases} v_{\infty}(s) & \text{if } s \in F_c \\ v_n(s) & \text{if } s \in F'_c \end{cases}
\]
for every \(n \in \omega\). Claim 6.7 guarantees that for every \(n \in \omega\) the element \(y_n = \Sigma_{\lambda}(v'_n) - \Sigma_{\lambda}(v_n)\) has \(y_n(k) \neq 0\) (here we also use the inequality \(k \geq \delta_{D_n}\)), which implies that \(d(\Sigma_{\lambda}(v'_n), \Sigma_{\lambda}(v_n)) \geq 2^{-k}\). On the other hand, the convergence of the sequence \((v_n(s))_{n \in \omega}\) to \(v_{\infty}(s)\) for \(s \in F_c\) implies that \(\lim_{n \to \infty} d(\Sigma_{\lambda}(v'_n), \Sigma(v_n)) = 0\). Taking into account the triangle inequality we conclude that the sequence \(d(\Sigma_{\lambda}(v_n)), \Sigma_{\lambda}(v_{\infty})))_{n \in \omega}\) does not converge to zero, which is a desired contradiction completing the proof of the claim.

**Claim 6.10.** The function \(\Sigma_{\lambda} : V_F \to H\) is a closed topological embedding.

**Proof.** It suffices to prove that a sequence \((v_n)_{n \in \omega} \in V_F^\kappa\) contains a convergent subsequence if the sequence \((\Sigma_{\lambda}(v_n))_{n \in \omega}\) converges to some point \(g \in H\). Write \(g = \sum_{u \in E} \mu(u)z_u\) for some finite set \(E \subset \omega^\omega\) and some function \(\gamma : E \to \mathbb{Z} \setminus \{0\}\).

Replacing \((v_n)_{n \in \omega}\) by a suitable subsequence, we can assume that for every \(u \in E\) either for some \(s \in F\) the sequence \((v_n(s))_{n \in \omega}\) converges to \(u\) or the point \(u\) is a point of a neighborhood \(W_u \subset \omega^\omega\) containing no point of the set \(\{v_n(s) : n \in \omega, s \in F\}\). Consider the sets \(E_c = \{u \in E : \exists s \in F \lim_{n \to \infty} v_n(s) = u\}, E'_c = E \setminus E_c\) and for every \(u \in E'_c\) choose a point \(s_u \in F\) such that \(u = \lim_{n \to \infty} v_n(s_u)\). Since the family \((V_{d(\gamma)}n \in F)\) is disjoint, the point \(s_u\) is unique. Let \(F_c = \{s \in F : \exists u \in E \ s = s_u\} = \{s \in F : \exists u \in E \ u = \lim_{n \to \infty} v_n(s)\}\) and for every \(s \in F_c\) put \(u_s = \lim_{n \to \infty} v_n(s)\) in \(V_{d(\gamma)}\).

If \(F_c = F\), then the sequence \((v_n)_{n \in \omega}\) is convergent in \(V_F\) and we are done. So, we assume that \(F_c \neq F\) and hence the set \(F'_c = F \setminus F_c\) is not empty. For every \(u \in E'_c\) choose a number \(m_u \geq \delta_{F_c}F\) such that the neighborhood \(V_{d(\gamma)}m_u\) is disjoint with the set \(\{v_n(s) : s \in F, n \in \omega\}\).

Consider the sequence \((v'_n)_{n \in \omega} \in \Pi^\kappa_{\omega}\) defined by
\[
v'_n(s) = \begin{cases} u_s & \text{if } s \in F_c \\ v_n(s) & \text{if } s \in F \setminus F_c \end{cases}
\]
for every \(n \in \omega\).

Let \(m = \max\{|F| + |E|, \|\lambda\| + \|\mu\|\}\) and \(k = 1 + \max\{\Xi_{m,n}, \max_{q \in E'_c} m_q\}\). Claim 6.7 ensures that the point \(y = \Sigma_{\lambda}(v'_n) - \Sigma_{\mu}(u)\) has \(y(k) \neq 0\), which implies \(d(\Sigma_{\lambda}(v'_n), g) = d(\Sigma_{\lambda}(v'_n), \Sigma_{\lambda}(v_n)) \geq 2^{-k}\). On the other hand, the convergence of the sequences \((v_n(s))_{n \in \omega}\), \(s \in F_c\) implies that \(d(\Sigma_{\lambda}(v'_n), \Sigma_{\lambda}(v'_n)) \to 0\). Then the triangle inequality ensures that the sequence \((\Sigma_{\lambda}(v_n), g)_{n \in \omega}\) does not converge to zero, contradicting with the choice of \(g\).

**Claim 6.11.** The set \(\Sigma_{\lambda}(V_F) \cap \{y \in \mathbb{Z}^\omega : \exists m \in \omega \ \forall k \geq m \ y(k) \neq 0\}\) is meager.

Denote by \([\omega^{<\omega}]^{<\omega}\) the family of finite subsets of \(\omega^{<\omega}\). For the empty set \(F = \emptyset\) and the unique map \(\lambda \in \mathbb{Z}^F_{=0}\) we put \(\Sigma_{\lambda}(V_F) = \{0\}\). Claims 6.10, 6.11 and the obvious equality
\[
H = \bigcup_{F \in [\omega^{<\omega}]^{<\omega}} \bigcup_{\lambda \in \mathbb{Z}^F_{=0}} \Sigma_{\lambda}(V_F)
\]
imply that the subgroup \(H\) is \(\sigma\)-Polish and is contained in the meager subset \(\{0\} \cup \bigcup_{m \in \omega} \{z \in \mathbb{Z}^\omega : \forall k \geq m \ z(k) = 0\}\) of \(\mathbb{Z}^\omega\).

7. **Set-Theoretic Constructions of Subgroups and the Cardinal Characteristic \(\text{voc}^\ast\)**

In Example 5.3 we constructed a subgroup \(H \subset \mathbb{Z}^\omega\) which is meager and Haar null but does not belong to the \(\sigma\)-ideal \(E\) generated by closed Haar null subsets of \(\mathbb{Z}^\omega\). It is natural to ask if a subgroup with these properties exists in any Polish group, in particular in the Cantor cube \(2^\omega\), considered as the countable power of the two-element group \((0,1)\) endowed with the operation of addition modulo 2.

**Problem 7.1.** Is it true that the Cantor cube \(2^\omega\) contain a meager Haar null subgroup \(G\) which cannot be covered by countably many closed Haar null sets? (By Theorem 5.1 such subgroup \(G\) cannot be quasi-analytic.)
In this section we shall prove that under Martin’s Axiom the answer to this problem is affirmative. More precisely, it is affirmative under the assumption \( \text{cov}(\mathcal{N}) = \text{cof}(\mathcal{M}) = \text{cof}(\mathcal{N}) \).

Problem 7.3 is a partial case of the following more general problem.

**Problem 7.2.** Find conditions on two \( \sigma \)-ideals \( \mathcal{I}, \mathcal{J} \) on a group \( G \) under which the family \( \mathcal{I} \setminus \mathcal{J} \) contains an subgroup of \( G \).

It turns out that even for the standard \( \sigma \)-ideals \( \mathcal{M} \) and \( \mathcal{N} \) on the Cantor cube \( 2^\omega \) the answer to Problem 7.2 is not trivial (and not symmetric). On one hand, for the family \( \mathcal{N} \setminus \mathcal{M} \) we have the following ZFC-result due to Talagrand [30].

**Theorem 7.3 (Talagrand).** The compact topological group \( 2^\omega \) contains a subgroup \( H \subset 2^\omega \) (which is an ideal on \( \omega \)) that belongs to the family \( \mathcal{N} \setminus \mathcal{M} \).

On the other hand, for the family \( \mathcal{M} \setminus \mathcal{N} \) a counterpart of Talagrand’s result is independent of ZFC, see [14] or [27].

**Theorem 7.4 (Burke).** It is consistent that each meager subgroup of \( 2^\omega \) is Haar null. On the other hand, Martin’s Axiom implies the existence of a subgroup \( H \in \mathcal{M} \setminus \mathcal{N} \) in \( 2^\omega \).

We shall give consistent answers to Problems 7.1 and 7.2 using some cardinal characteristics of ideals \( \mathcal{I}, \mathcal{J} \) on a group \( X \). In the sequel, we shall consider only ideals \( \mathcal{I} \) on a set \( X \) with the property: \( \bigcup \mathcal{I} = X \notin \mathcal{I} \). In this case the following four cardinal characteristics are well-defined:

- \( \text{add}(\mathcal{I}) = \min\{|A| : A \subset I \, \cup A \notin \mathcal{I}\}; \)
- \( \text{non}(\mathcal{I}) = \min\{|A| : A \subset X \, A \notin \mathcal{I}\}; \)
- \( \text{cov}(\mathcal{I}) = \min\{|A| : A \subset I \, \bigcup A = X\}; \)
- \( \text{cof}(\mathcal{I}) = \min\{|A| : A \subset I \, \forall I \in \mathcal{I} \exists A \in A (I \subset A)\}. \)

In fact, these four cardinal characteristics are partial cases of the following two cardinal characteristics defined for ideals \( \mathcal{I} \subset \mathcal{J} \) on \( X \):

- \( \text{add}(\mathcal{I}, \mathcal{J}) = \min\{|A| : A \subset I \, \bigcup A \notin \mathcal{J}\}; \)
- \( \text{cof}(\mathcal{I}, \mathcal{J}) = \min\{|A| : A \subset \mathcal{J} \, \forall I \in \mathcal{I} \exists J \in \mathcal{A} (I \subset J)\}; \)
- \( \text{cov}(\mathcal{I}, \mathcal{J}) = \min\{|A| : A \subset I \, G \setminus \bigcup A \notin \mathcal{J}\}; \)

It is clear that \( \text{add}(\mathcal{I}) = \text{add}(\mathcal{I}, \mathcal{I}) \), \( \text{cof}(\mathcal{I}) = \text{cof}(\mathcal{I}, \mathcal{I}) \), \( \text{cov}(\mathcal{I}) = \text{cov}(\mathcal{F}, \mathcal{I}) = \text{cov}(\mathcal{F}, \mathcal{I}) \) and \( \text{non}(\mathcal{I}) = \text{add}(\mathcal{F}, \mathcal{I}) \), where \( \mathcal{F} \) is the ideal of finite subsets of \( X \).

The following trivial proposition gives a simple answer to Problem 7.2.

**Proposition 7.5.** Let \( \mathcal{I} \subset \mathcal{J} \) be two ideals on a group \( G \). If \( \text{non}(\mathcal{I}) < \text{non}(\mathcal{J}) \), then there exists a subgroup \( H \in \mathcal{J} \setminus \mathcal{I} \).

**Proof.** By the definition of the cardinal \( \text{non}(\mathcal{I}) \), there is a subset \( A \subset G \) of cardinality \( |A| = \text{non}(\mathcal{I}) \) such that \( A \notin \mathcal{I} \). Then the subgroup \( H \) generated by the set \( A \) has cardinality \( |H| \leq \max\{\aleph_0, \text{non}(\mathcal{I})\} = \text{non}(\mathcal{I}) < \text{non}(\mathcal{J}) \) and hence \( H \) belongs to the ideal \( \mathcal{J} \).

A more complicated answer to Problem 7.2 will be given with help of the cardinal characteristic \( \text{voc}^*(\mathcal{I}, \mathcal{J}) \) defined for two ideals \( \mathcal{I}, \mathcal{J} \) on a group \( G \) as follows:

- \( \text{cov}^*(\mathcal{I}, \mathcal{J}) = \min\{|X| : X \subset G \, \text{and} \exists I \in \mathcal{I} \, \forall X \subset G \, (X \cdot I \notin \mathcal{J})\}; \)
- \( \text{voc}^*(\mathcal{I}, \mathcal{J}) = \sup\{\kappa : \exists I \in \mathcal{I} \forall X \subset G \, (|X| < \kappa \Rightarrow \exists x \in X (xI \notin \mathcal{J}))\}; \)
- \( \text{non}^*(\mathcal{I}) = \min\{|X| : X \subset G \, \text{and} \forall g \in G \, g \cdot X \notin \mathcal{I}\}; \)
- \( \text{cov}^*(\mathcal{I}) = \text{cov}^*(\mathcal{I}, \mathcal{I}); \)
- \( \text{voc}^*(\mathcal{I}) = \text{voc}^*(\mathcal{I}, \emptyset) \).

This definition implies that \( \text{voc}^*(\mathcal{I}, \mathcal{J}) = 1 \) for any ideals \( \mathcal{I} \subset \mathcal{J} \) on \( X \).

An ideal \( \mathcal{I} \) on a group \( G \) is called \textit{left-invariant} if for every \( A \in \mathcal{I} \) and \( g \in G \) the set \( gA = \{ga : a \in A\} \) belongs to \( \mathcal{I} \). It is easy to see that for any left-invariant ideals \( \mathcal{I} \subset \mathcal{J} \) on a group \( G \) we get \( \text{cov}(\mathcal{I}, \mathcal{J}) \leq \text{cov}^*(\mathcal{I}, \mathcal{J}) \).

Given two ideals \( \mathcal{I}, \mathcal{J} \) by \( \mathcal{I} \vee \mathcal{J} \) we denote the ideal generated by the union \( \mathcal{I} \cup \mathcal{J} \). For an ideal \( \mathcal{I} \) on a group \( G \) we put \( \mathcal{I}^{-1} = \{A^{-1} : A \in \mathcal{I}\} \).
Proposition 7.6. Let \( I, I' \) and \( J \) be two ideals on a group \( G \) such that \( G \in (I \cup I') \setminus J \). Then
\[
\text{cov}^*(I', J) \leq \text{voc}^*(I, J) \leq \text{non}^*(I^{-1}).
\]

Proof. By the definition of the cardinal \( \kappa = \text{cov}^*(I', J) \), there exists a set \( I' \in I' \) such that \( G \setminus \{X \cdot I'\} \in J \) for some set \( X \in G \) of cardinality \( |X| = \text{cov}^*(I', J) \). Since \( G \in I \cup I' \), we can enlarge the set \( I' \) and assume that \( I := G \setminus I' \in I \). To show that \( \kappa \leq \text{voc}^*(I, J) \) it suffices to show that for any subset \( Y \in G \) of cardinality \( |Y| < \kappa \) we get \( \bigcap_{y \in Y} yI \notin J \). To derive a contradiction, assume that for some set \( Y \subseteq G \) of cardinality \( |Y| < \kappa \) the set \( J = \bigcap_{y \in Y} yI \) belongs to the ideal \( J \). Then \( G \setminus J = \bigcup_{y \in Y} y(G \setminus I) = \bigcup_{y \in Y} yI' \) and hence \( G \setminus (Y \cdot I') \in J \), which contradicts the definition of the cardinal \( \text{cov}^*(I', J) > |Y| \).

Next, we prove that \( \text{voc}^*(I, J) \leq \text{non}^*(I^{-1}) \). Assuming that \( \text{voc}^*(I, J) > \text{non}^*(I^{-1}) \), we could find a set \( I \in I \) such that for any subset \( X \subseteq G \) with \( |X| \leq \text{non}^*(I^{-1}) \) we get \( \bigcap_{x \in X} xI \notin J \). By the definition of the cardinal \( \text{non}^*(I^{-1}) \), there exists a set \( X \subseteq G \) of cardinality \( |X| = \text{non}^*(I^{-1}) \) such that \( gX \notin I \) for all \( g \in G \). Then \( X^{-1}g \notin I \) and hence \( X^{-1}g \notin I \) for all \( g \in G \). We claim that \( \bigcap_{x \in X} x \notin \) and hence \( gX \notin I \). Indeed, for any point \( g \in G \) we use \( X^{-1}g \notin I \) to find a point \( x \in G \) such that \( x^{-1}g \notin I \) and hence \( gX \notin I \). □

A group \( G \) is defined to be Boolean if \( xxx = x \) for all \( x \in G \). A typical example of a Boolean group is the Cantor cube \( 2^{\omega} \). The following proposition (partially) answers Problem 7.2.

Proposition 7.7. Let \( I \subseteq J \) be two \( \sigma \)-ideals on a Boolean group \( G \). If there exists an ideal \( I' \) on \( G \) such that \( I \subseteq I' \), \( J \setminus I' \) and \( \text{cof}(I, J') \leq \text{voc}(J, I') \), then the group \( G \) contains a subgroup \( H \in J \setminus I \).

Proof. By the definition of \( \text{voc}^*(J, I') \geq \text{cof}(I, I') \), there exists a set \( A \subseteq J \) such that for any subset \( X \subseteq A \) of cardinality \( |X| < \text{cof}(I, I') \) we get \( \bigcap_{x \in X} xA \notin I' \). By the definition of the cardinal \( \kappa = \text{cof}(I, J') \), there is a subfamily \( \{X_\alpha\}_{\alpha < \kappa} \subseteq I' \) such that each set \( I \in I \) is contained in some set \( X_\alpha \), \( \alpha < \kappa \). By transfinite induction we shall construct a transfinite sequence of points \( (x_\alpha)_{\alpha < \kappa} \) in \( A \) such that for every \( \alpha < \kappa \) the point \( x_\alpha \) belongs to the set \( \bigcap_{x \in X_\alpha} xA \setminus X_\alpha \) for all \( x \in H_\alpha \subseteq G \) generated by the set \( \{x_\beta\}_{\beta < \alpha} \). It follows that for every \( x \in H_\alpha \) we get \( x_\alpha \in A \) and hence \( x^{-1}x_\alpha = x_\alpha x_\alpha \in A \). Consequently, \( H_\alpha = H_\alpha \cup x_\alpha H_\alpha \subseteq A \). After completing the inductive construction, consider the subgroup \( H \) generated by the set \( \{x_\alpha\}_{\alpha < \kappa} \). It follows that \( H \subseteq A \) and \( H \not\subseteq X_\alpha \) for all \( \alpha < \kappa \), which implies that \( H \subset J \setminus I \). □

For a locally compact topological group \( G \) by \( \mathcal{M}_G \) we shall denote the \( \sigma \)-ideal of meager sets in \( G \), by \( \mathcal{N}_G \) the \( \sigma \)-ideal of Haar null sets in \( G \) and by \( \mathcal{E}_G \) the \( \sigma \)-ideal generated by compact sets of Haar measure zero in \( G \). If the group \( G \) is clear from the context, they we shall omit the subscripts and shall denote these three \( \sigma \)-ideals by \( \mathcal{M}, \mathcal{N} \) and \( \mathcal{E} \). On the Cantor cube \( 2^\omega \) the \( \sigma \)-ideal \( \mathcal{M}, \mathcal{N}, \mathcal{E} \) were thoroughly studied in [5]. The cardinal characteristics of the ideals \( \mathcal{M}, \mathcal{N}, \mathcal{E} \) on the Cantor cube \( 2^\omega \) (or the real line) fit in the following extension of the famous Cichoń diagram (in which an arrow \( \kappa \rightarrow \lambda \) between two cardinals indicated that \( \kappa \leq \lambda \) in ZFC).

```
| cov(N) | non(M) | cof(E,N) | cof(E) | cof(M) | cof(N) | \( \mathfrak{c} \) |
|--------|--------|----------|--------|--------|--------|---------|
| add(E,M) | \( \omega_1 \) | add(M) | add(M) | add(M) | add(E) | add(M) |
|                     |        |        |        | add(E,N) | add(M) | add(M) |
|                     |        |        |        | non(N) | add(M) | non(N) |
```

All inequalities between the cardinal characteristics presented in this diagram are proved in Chapter 2 of [6] (more precisely, in Theorems 2.1.7, 2.2.9, 2.2.11, 2.3.7, 2.6.10, 2.6.14, 2.6.17 of [6]). It is well-known (see e.g. [32] or [8]) that Martin’s Axiom collapses all cardinal characteristics of the ideals \( \mathcal{M}, \mathcal{N}, \mathcal{E} \) appearing in this diagram to the continuum.

Applying Proposition 7.4 to the ideals \( \mathcal{J}, \mathcal{I}, \mathcal{I}' \in \{\mathcal{M}, \mathcal{N}, \mathcal{M} \cap \mathcal{N}, \mathcal{E}\} \), we get:

Corollary 7.8. For the \( \sigma \)-ideals \( \mathcal{M}, \mathcal{N}, \mathcal{E} \) on a Boolean locally compact topological group \( G \) the following statements hold.

1. If \( \text{cov}(\mathcal{N}) \leq \text{voc}^*(\mathcal{M}, \mathcal{N}) \), then there exists a subgroup \( H \in \mathcal{M} \setminus \mathcal{N} \).
2. If \( \text{cof}(\mathcal{M}) \leq \text{voc}^*(\mathcal{N}, \mathcal{M}) \), then there exists a subgroup \( H \in \mathcal{N} \setminus \mathcal{M} \).
(3) If \( \text{cof}(\mathcal{E}, \mathcal{N}) \leq \text{voc}^*(\mathcal{M}, \mathcal{N}) \), then there exists a subgroup \( H \in \mathcal{M} \setminus \mathcal{E} \).

(4) If \( \text{cof}(\mathcal{E}) \leq \text{voc}^*(\mathcal{M} \cap \mathcal{N}, \mathcal{E}) \), then there exists a subgroup \( H \in (\mathcal{M} \cap \mathcal{N}) \setminus \mathcal{E} \).

Corollary 7.8 motivates the problem of calculating the cardinals \( \text{voc}^*(\mathcal{I}, \mathcal{J}) \) for the standard \( \sigma \)-ideals \( \mathcal{I}, \mathcal{J} \in \{\mathcal{M}, \mathcal{N}, \mathcal{M} \cap \mathcal{N}, \mathcal{E}\} \). Proposition 7.6 implies the following lower and upper bounds.

**Corollary 7.9.** For the ideals \( \mathcal{M}, \mathcal{N}, \mathcal{E} \) on a locally compact topological group \( G \) we get

1. \( \text{cov}(\mathcal{N}) \leq \text{cov}^*(\mathcal{M}, \mathcal{N}) \leq \text{voc}^*(\mathcal{M} \cap \mathcal{N}, \mathcal{E}) \leq \text{voc}^*(\mathcal{M}, \mathcal{E}) \leq \text{voc}^*(\mathcal{M}) \leq \text{non}^*(\mathcal{M}) = \text{non}(\mathcal{M}) \);

2. \( \text{cov}(\mathcal{M}) \leq \text{voc}^*(\mathcal{N}, \mathcal{M}) \leq \text{voc}^*(\mathcal{N}, \mathcal{N} \setminus \mathcal{E}) \leq \text{voc}^*(\mathcal{N}) \leq \text{non}^*(\mathcal{N}) = \text{non}(\mathcal{N}) \).

Now we are going to evaluate the cardinal \( \text{voc}^*(\mathcal{M} \cap \mathcal{N}, \mathcal{E}) \).

**Lemma 7.10.** For locally compact topological groups \( G, H \) we get

\[
\text{voc}^*(\mathcal{M}_{G \times H} \cap N_{G \times H}, E_{G \times H}) \geq \text{min}\{\text{voc}^*(\mathcal{M}_G, E_G), \text{voc}^*(\mathcal{N}_H, E_H)\}.
\]

**Proof.** To derive a contradiction, assume that the cardinal \( \kappa = \text{voc}^*(\mathcal{M}_{G \times H} \cap N_{G \times H}, E_{G \times H}) \) is strictly smaller than \( \text{min}\{\text{voc}^*(\mathcal{M}_G, E_G), \text{voc}^*(\mathcal{N}_H, E_H)\} \). Let \( \lambda_G \) and \( \lambda_H \) be the Haar measures on the locally compact groups \( G, H \), respectively. By a *Haar measure* on a locally compact group we understand any left-invariant Borel regular \( \sigma \)-additive measure that takes positive finite values on compact sets with non-empty interior. It is well-known that each locally compact topological group has a Haar measure and such measure is unique up to a positive multiplicative constant. This uniqueness result implies that the tensor product \( \lambda = \lambda_G \otimes \lambda_H \) of the measures \( \lambda_G, \lambda_H \) is a Haar measure on the group \( G \times H \).

By the definition of the cardinals \( \text{voc}^*(\mathcal{M}_G, E_G), \text{voc}^*(\mathcal{N}_H, E_H) > \kappa \), there are sets \( M \in \mathcal{M}_G \) and \( N \in \mathcal{N}_H \) such that for any subsets \( X \subset G \) and \( Y \subset H \) with \( \max\{|X|, |Y|\} \leq \kappa \), we get \( \bigcap_{x \in X} xM \notin E_G \) and \( \bigcap_{y \in Y} yN \notin E_H \). We claim that for any subset \( Z \subset G \times H \) of cardinality \( |Z| \leq \kappa \) we get \( \bigcap_{z \in Z} z(M \times N) \notin E_{G \times H} \). To derive a contradiction, assume that the intersection \( \bigcap_{z \in Z} z(M \times N) \) belongs to the \( \sigma \)-ideal \( E_{G \times H} \) and hence can be covered by the union \( \bigcup_{n \in \omega} K_n \) of an increasing sequence \( (K_n)_{n \in \omega} \) of compact sets of Haar measure \( \lambda(K_n) = 0 \) in \( G \times H \). For every \( n \in \omega \) and point \( x \in G \) consider the compact set \( K_n^x = \{y \in H : (x, y) \in K_n\} \). The regularity of the measure \( \lambda_H \) implies that for every \( m \in \omega \) the set \( \{x \in G : \lambda_H(K_n^x) \geq 2^{-m}\} \) is compact, being a closed subset of the projection of \( K_n \) on \( G \). Let \( X \) and \( Y \) be the projections of the set \( Z \) on \( G \) and \( H \), respectively.

We claim that \( \bigcap_{x \in X} xM \subset \bigcup_{n, m \in \omega} P_{n, m} \). Indeed, assuming that some point \( g \in \bigcap_{x \in X} xM \) does not belong to \( \bigcup_{n, m \in \omega} P_{n, m} \), we conclude that \( \lambda_H(K_n^g) = 0 \) for all \( n \in \omega \) and then

\[
\{g\} \times \bigcap_{y \in Y} yN \subset ((\{g\} \times H) \cap \bigcap_{z \in Z} z(M \times N) \cap \bigcup_{n \in \omega} K_n = \{g\} \times \bigcup_{n \in \omega} K_n^g,
\]

which implies that the set \( \bigcap_{y \in Y} yN \subset \bigcup_{n \in \omega} K_n^g \) belongs to the \( \sigma \)-ideal \( E_H \). But this contradicts the choice of the set \( N \). This contradiction shows that the set \( \bigcap_{x \in X} xM \notin E_G \) belongs to the \( \sigma \)-compact set \( \bigcup_{n, m \in \omega} P_{n, m} \), which implies that \( \lambda_G(P_{n, m}) > 0 \) for some \( n, m \in \omega \). Since for every \( x \in P_{n, m} \) the set \( K_n^x \) has measure \( \lambda_H(K_n^x) \geq 2^{-m} \), the Fubini Theorem [4, 3.4.4] guarantees that \( \lambda(K_n) \geq 2^{-m} \cdot \lambda_G(P_{n, m}) > 0 \). But this contradicts the choice of the set \( K_n \).

This contradiction shows that the set \( M \times N \in \mathcal{M}_{G \times H} \cap N_{G \times H} \) witnesses that \( \text{voc}^*(\mathcal{M}_{G \times H} \cap N_{G \times H}) > \kappa \), which contradicts the definition of the cardinal \( \kappa \).

**Lemma 7.10** will help us to evaluate the cardinal \( \text{voc}^*(\mathcal{M} \cap \mathcal{N}, \mathcal{E}) \) for any locally compact group \( G \), isomorphic to its square \( G \times G \).

**Corollary 7.11.** If a locally compact group \( G \) is topologically isomorphic to its square \( G \times G \), then

\[
\text{voc}^*(\mathcal{M} \cap \mathcal{N}, \mathcal{E}) = \text{min}\{\text{voc}^*(\mathcal{M}, \mathcal{E}), \text{voc}^*(\mathcal{N}, \mathcal{E})\}.
\]

Combining Corollaries 7.8, 7.9, 7.10, and 7.11 with the equalities \( \text{cof}^*(\mathcal{E}, \mathcal{N}) = \text{non}(\mathcal{M}) \) and \( \text{cof}(\mathcal{E}) = \text{cof}(\mathcal{M}) \) (proved in Theorems 2.6.14 and 2.6.17 of [6]) we get another corollary.

**Corollary 7.12.** The compact Boolean group \( 2^\omega \) contains a subgroup

1. \( H_1 \in \mathcal{M} \setminus \mathcal{N} \) if \( \text{cof}(\mathcal{N}) = \text{cov}(\mathcal{N}) \).
2. \( H_2 \in \mathcal{M} \setminus \mathcal{E} \) if \( \text{non}(\mathcal{M}) = \text{cof}(\mathcal{M}) \).
3. \( H_3 \in (\mathcal{M} \cap \mathcal{N}) \setminus \mathcal{E} \) if \( \text{cof}(\mathcal{M}) = \text{cov}(\mathcal{M}) = \text{cov}(\mathcal{N}) \).
Remark 7.13. By Theorem 7.4 it is consistent that each meager subgroup of $2^\omega$ belongs to the ideal $\mathcal{N}$. So, the first statement of Corollary 7.12 cannot be proved in ZFC. We do not know if the same is true for the other two statements of Corollary 7.12.

Problem 7.14. Calculate the values of the cardinal characteristics $\text{voc}^*(\mathcal{M}, \mathcal{I})$ and $\text{voc}^*(\mathcal{N}, \mathcal{I})$ for the $\sigma$-ideals $\mathcal{I} \in \{\mathcal{F}, \mathcal{E}, \mathcal{M} \cap \mathcal{N}, \mathcal{M}, \mathcal{N}\}$ on the group $2^\omega$. Are they equal to some known cardinal characteristics of the continuum?

Corollary 7.9 combined with known information [6] about the cardinal characteristics of the $\sigma$-ideals $\mathcal{M}, \mathcal{N}, \mathcal{E}$ of the Cantor cube $2^\omega$ allow us to draw the following diagram.

\[
\begin{array}{cccccccc}
\text{cov}^*(\mathcal{M}) & \xrightarrow{} & \text{voc}^*(\mathcal{N}, \mathcal{M}) & \xrightarrow{} & \text{voc}^*(\mathcal{N}, \mathcal{E}) & \xrightarrow{} & \text{voc}^*(\mathcal{N}) & \xrightarrow{} & \text{non}^*(\mathcal{N}) \\
\text{cov}(\mathcal{M}) & \xrightarrow{} & \text{add}(\mathcal{E}, \mathcal{N}) & \xrightarrow{} & \text{add}(\mathcal{M}) & \xrightarrow{} & \text{voc}^*(\mathcal{M}, \mathcal{N}) & \xrightarrow{} & \text{voc}^*(\mathcal{M}) \\
\text{add}(\mathcal{M}) & \xrightarrow{} & \text{cov}(\mathcal{N}) & \xrightarrow{} & \text{cov}^*(\mathcal{N}) & \xrightarrow{} & \text{voc}(\mathcal{M}, \mathcal{E}) & \xrightarrow{} & \text{voc}^*(\mathcal{M}) \\
\end{array}
\]

Remark 7.15. The cardinal characteristic $\text{cov}^*(\mathcal{I})$ of various $\sigma$-ideals $\mathcal{I}$ on Polish groups have been studied in [6] §2.7, [24], [9], [13], [14]. By [9], for a non-locally compact Polish group $G$ possessing an invariant metric we get $\text{cov}^*(\mathcal{M}) = \text{cov}(\mathcal{M})$. On the other hand, it is consistent [24] that for some locally compact groups $G$ (like $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$ or $2^\omega$) we get $\text{cov}^*(\mathcal{M}) > \text{cov}(\mathcal{M})$. This implies the consistency of a strict inequality $\text{cov}(\mathcal{M}) < \text{voc}^*(\mathcal{M}, \mathcal{N})$. By [14], each Polish locally compact abelian group $G$ has $\text{cov}^*(\mathcal{E}) \leq \text{cof}(\mathcal{N})$ and it is consistent that $\text{cov}^*(\mathcal{E}) < \text{cof}(\mathcal{N})$.

For the Polish group $\mathbb{Z}^\omega$ the cardinal $\text{voc}^*(\mathcal{M})$ can be easily calculated using the combinatorial characterization of the cardinal $\text{non}(\mathcal{M})$.

Proposition 7.16. For the Polish group $G = \mathbb{Z}^\omega$ we get $\text{voc}^*(\mathcal{M}) = \text{non}(\mathcal{M})$.

Proof. In the Polish group $G$ consider the meager subset $F = \{x \in \mathbb{Z}^\omega : |\{n \in \omega : x(n) = 0\}| < \omega\}$. We claim that for any subset $X \subseteq G$ of cardinality $|X| < \text{non}(\mathcal{M})$ we get $\bigcap_{x \in X} xF \neq \emptyset$. Indeed, by Theorem 2.4.7 [6], there exists a function $y \in \mathbb{Z}^\omega$ such that for every $x \in X$ the set $\{n \in \omega : x(n) = y(n)\}$ is finite and hence $y \in xF$. This means that the intersection $\bigcap_{x \in X} xF$ is not empty and hence $\text{voc}^*(\mathcal{M}) \geq \text{non}(\mathcal{M})$. The inequality $\text{voc}^*(\mathcal{M}) \leq \text{non}(\mathcal{M})$ follows from Proposition 7.6.

Problem 7.17. Is $\text{voc}^*(\mathcal{M}) = \text{non}(\mathcal{M})$ for the ideal $\mathcal{M}$ of meager sets on the Cantor cube $2^\omega$?
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