Self-avoiding walk representation for the Green’s function and localization properties

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Abstract. The Green’s function contains much information about physical systems. Mathematically, the fractional moment method (FMM) developed by Aizenman and Molchanov connects the Green’s function and the transport of electrons in the Anderson model. Recently, it has been discovered that the Green’s function on a graph can be represented using self-avoiding walks on a graph, which allows us to connect localization properties in the system and graph properties. We discuss FMM in terms of the self-avoiding walks on a general graph with a small number of assumptions.

1. Introduction
We introduce the Anderson model which describes the motion of an electron in a disordered system. There exist many good reviews of the Anderson model such as [3,5] and we follow their notations in this article. In physics, it is common to model the system by a lattice $\mathbb{Z}^n$ or a graph $G$. In this article, we deal with the Anderson model on a graph $G$ with the following assumptions.

Let $G = (V, E)$ be a graph with vertices $V$ and edges $E$. We assume $G$ is a connected graph where the number of edges between any pair of vertices is either one or zero. We write $v \sim w$ if an edge connects the vertices $v$ and $w$. Let $N(v)$ be the degree of a vertex $v \in V$. i.e., $N(v) := \#\{w \in V : w \sim v\}$. We assume that the degree of a vertex is bounded above by some constant $N$, $N(v) \leq N < \infty$ for all $v \in V$. $d(v, w)$ is the graph distance from $v$ to $w$, which is the minimum number of edges from $v$ to $w$ on $G$.

$W(v, w)$ is the set of self-avoiding walks (sequences of vertices) $[v, v_1, \ldots, v_{d(v, w)}]$ with $d(v, w)$ steps starting at $v_0 = v$. The walks need not end at $w$.

Furthermore, $W'(v, w)$ is the set of self-avoiding walks $[v, v_1, \ldots, v_{d(v, w)}]$ with $d(v, w)$ steps starting at $v_0 = v$ and with $v_{d(v, w)}$ connected to $w$ in the graph obtained by deleting all edges attached to $[v, v_1, \ldots, v_{d(v, w)}]$.

We define a function $W$ which measures the maximum number of self-avoiding walks $W'(v, w)$ with $d$ steps that can happen in $G$.

$$W(d) = \max\{|W'(v, w)| : d(v, w) = d\},$$

thus $|W'(v, w)| \leq W(d(v, w))$.

We write the set of vertices on a sphere (or shell) with radius $d$ from some origin $v$ as $S(v, d)$. We write the set of vertices in a ball with radius $d$ from some origin $v$ as $B(v, d)$. $|S(v, d)|$ and $|B(v, d)|$ are the number of vertices in $S(v, d)$ and $B(v, d)$ respectively. Then, we define $S(d)$...
to be the largest possible value of $|S(v, d)|$ as $v$ ranges over the graph, $\mathcal{S}(d) = \max_v |S(v, d)|$. $\mathcal{S}(d)$ is bounded by the biggest possible value $N(N - 1)^d$.

Disordered matter can be described by a random Schrödinger operator acting on the Hilbert space $l^2(V)$:

$$H = -\sum_{\{v, w\}: v \sim w} (|v\rangle\langle w| + |w\rangle\langle v|) + \lambda \sum_{v \in V} \omega_v |v\rangle\langle v|$$

where the first term called next neighbour hopping operator describes the kinetic energy, the random potential $\lambda \omega_v$ is a multiplication operator on $l^2(V)$ with a coupling constant $\lambda > 0$. We assume the simplest case where $(\lambda \omega_v)_{v \in V}$ is a set of independent, identically distributed (i.i.d.) real-valued random variables with single-site distribution $\rho$. We assume $\rho$ is bounded and has compact support.

Large coupling constant $\lambda >> 1$ indicates large disorder (randomness) and small coupling constant $\lambda << 1$ indicates small disorder. As $\lambda$ increases, the distribution is spread out over larger supports and the random potential can take a wider range of possible random values.

Now we introduce dynamical localization, which implies the expectation of the amplitudes that an initial state of the system $\delta_y$ evolves into state $\delta_y$ after time $t$ is exponentially localized with the distance $d$ for all times.

**Definition 1.1. (Dynamical localization):** $H$ exhibits dynamical localization in $I$ if there exist constants $C < \infty$ and $\mu > 0$ such that

$$\sum_{y \in S(x, d)} \mathbb{E} \left( \sup_{t \in \mathbb{R}} |\langle \delta_y, e^{-itH}\chi_I(H)\delta_x \rangle| \right) \leq Ce^{-\mu d}$$

(1)

for all $x \in V$.

$\mathbb{E}$ is the expectation with respect to the probability measure for random variables $\lambda \omega_v$. $\chi_I$ is the characteristic function of $I$ and so $\chi_I(H)$ is the orthogonal projection onto the spectral subspace of $H$ corresponding to energies in $I$. i.e., we only deal with the initial states with energy in $I$. $\mu$ is a constant which expresses disorder and often can be written in terms of $\lambda$. Larger $\lambda$ implies larger $\mu$.

2. Dynamical localization and graph properties

The Green’s function can be represented using self-avoiding walks on a graph. There are several studies discussing it in different ways [3,6]. In this article, we introduce the self-avoiding walk (SAW) representation for the Green’s function in the way which gives us physical intuition.

We define a depleted random Schrödinger operator which can be made from $H$ by self-avoiding walk process as follows (Figure 1).

$$H_{v_0} = H + \sum_{v_1 \sim v_0} (|v_0\rangle\langle v_1| + |v_1\rangle\langle v_0|) \text{ (i.e., } H_{v_0} \text{ is } H \text{ without edges connected to } v_0.)$$

$$H_{v_0, \ldots, v_k} = H_{v_0, \ldots, v_k} + \sum_{v_i \sim v_{i-1}, v_i \sim v_{i+1}} (|v_{i-1}\rangle\langle v_i| + |v_i\rangle\langle v_{i-1}|).$$

where $\sum_{v_i \sim v_{i-1}}$ is summing over every possible vertex $v_i$ which can be reached by taking the
next step after the self-avoiding walk \([v_0, \ldots, v_{i-1}]\). Then we obtain the following proposition.\(^1\)

**Proposition 2.1. (SAW representation)** Let \(x = v_0, y \in V\) and \(z \in \mathbb{C} \setminus \mathbb{R}\). Then the Green’s function can be written as

\[
G(x, y; z) = \langle x| (H - z)^{-1} |y \rangle = \sum_{[v_0, \ldots, v_{d(x,y)-1}]} \prod_{i=0}^{d(x,y)-1} \langle v_i| (H_{[v_0,\ldots,v_{i-1}]-z})^{-1} |v_i \rangle \langle v_{d(x,y)}|(H_{[v_0,\ldots,v_{d(x,y)-1}]-z})^{-1} |y \rangle
\]

(2)

where \(\sum_{[v_0,\ldots,v_{d(x,y)}]}\) is summing over all self-avoiding walks starting at \(v_0\) with length \(d(x, y)\).

When \(i = 0\), \(H_{[v_0,v_0]} = H\). When \(i = 1\), \(H_{[v_0,v_0]} = H_{[v_0]}\).

Here, self-avoiding walks are sequences of vertices \([v_0, \ldots, v_{d(x,y)}]\). When moving from a vertex to one that is joined by multiple edges, we must delete all the edges joining those vertices (Figure 1). Note that if a walker can not find any edge to walk from \(v_0\) to \(v_1\) where he deleted all edges connected to \(v_i\), then the contribution of that walk to the Green’s function is 0 since \(\langle v_i|(H_{[v_0,\ldots,v_{i-1}]-z})^{-1} |y \rangle = 0\).

Now, we discuss the fractional moment bounds for the Green’s function represented by self-avoiding walks above.

Firstly, let \(W(x, y)\) be the set of self-avoiding walks \([x, v_1, \ldots, v_{d(x,y)}]\) with \(d(x, y)\) steps starting at \(v_0 = x\). Then, we can divide \(W(x, y)\) into three subsets:

\(\mathcal{Y}(x, y)\) : Self-avoiding walks in \(W(x, y)\) with \(v_{d(x,y)} = y\).
\(\mathcal{X}(x, y)\) : Self-avoiding walks in \(W(x, y)\) with \(v_{d(x,y)} \neq y\) where \(v_{d(x,y)}\) is connected to \(y\) in the graph obtained by deleting all edges attached to \([x, v_1, \ldots, v_{d(x,y)-1}]\).
\(\overline{\mathcal{X}}(x, y)\) : Self-avoiding walks in \(W(x, y)\) with \(v_{d(x,y)} \neq y\) where \(v_{d(x,y)}\) is not connected to \(y\) in the graph obtained by deleting all edges attached to \([x, v_1, \ldots, v_{d(x,y)-1}]\).

Only \(\mathcal{Y}(x, y)\) and \(\mathcal{X}(x, y)\) contribute to the Green’s function since

\(\langle v_{d(x,y)}|(H_{[v_0,\ldots,v_{d(x,y)-1}]-z})^{-1} |y \rangle = 0\) if \([x, v_1, \ldots, v_{d(x,y)}] \in \overline{\mathcal{X}}\).

Now we can define \(W'(x, y) = \mathcal{Y}(x, y) \cup \mathcal{X}(x, y)\) be the set of self-avoiding walks \([x, v_1, \ldots, v_{d(x,y)}]\) with \(d(x, y)\) steps starting at \(v_0 = x\) and \(v_{d(x,y)}\) is connected to \(y\) in the graph obtained by deleting all edges attached to \([x, v_1, \ldots, v_{d(x,y)-1}]\). \(y\) is connected to itself in the case \(v_{d(x,y)} = y\). Then, we obtain the following two theorems using the fractional moment

\(^1\) Proofs of propositions and theorems in this article are given in [5].
method.

**Theorem 2.2. (Fractional moment bounds)** Let us write the number of $W'(x, y)$ as $|W'(x, y)|$. Let $0 < s < 1$. Then, the fractional moment bounds for the Green’s function can be written as follows.

$$
E(|G(x, y; z)|^s) \leq |W'(x, y)| \left( \frac{C}{\lambda^s} \right)^{d(x, y)+1}
$$

**Theorem 2.3.** Let $I \subset \mathbb{R}$, $0 < s < 1$ and $\epsilon \in (0, 1/2)$. Then dynamical localization (1) holds for disorder $\lambda$ which satisfies the following condition.

$$
se \ln \lambda > \epsilon \ln C + \sup_{x,d} \left( \frac{\ln |S(x, d)|}{d} \right) \quad \text{and} \quad \sum_{d'=0}^{\infty} \mathcal{H}(d') \mathcal{W}(d') \left( \frac{C}{\lambda^s} \right)^{(1-2\epsilon)d'} < \infty
$$

where $C = \|\rho\|_\infty 2^{s+1} 2^{s^2-s}$

3. Conclusions and discussion

One of the most important open problems in random operator theory is to understand the transition between the localized regime and the extended states regime. There is an attempt to understand the transition using the *level statistics conjecture* and *random matrix theory* (RMT). This method allows physicists to distinguish the two regimes numerically.

We use the statistical distribution of the eigenvalues of finite volume restrictions of the Anderson model to distinguish two regimes. It is expected that the localized regime and the extended states regime are corresponding to Poisson statistics and Gaussian orthogonal ensemble (GOE) statistics of the eigenvalues respectively.

Some studies in physics communities make use of such method to test the transition. Giraud et al. [2] studied the model of a circular graph with on-site disorder where each vertex is linked with its two nearest-neighbours and also they added shortcut edges between random pairs of vertices. Therefore, this is the one-dimensional Anderson model with extra off-diagonal random elements.

They studied level spacing statistics for Hamiltonian and obtained GOE distribution for small on-site disorder $\lambda$ and Poisson distribution as they made on-site disorder $\lambda$ larger.

It might be possible to make a relation between this transition and our theorem 2.3. In our work, the connection between distant regions is reflected in the size of $\mathcal{W}(d)$. When $\mathcal{W}(d)$ is large, the system needs larger disorder to obtain dynamical localization.

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