QUANTUM GROUPS AND BOUNDED
SYMMETRIC DOMAINS

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Abstract

Recent results of the authors on quantum bounded symmetric domains and quantum
Harish-Chandra modules are expounded.

1. Here is a general outline of problems in non-compact quantum group theory to be
discussed in this talk.

The geometrical methods of studying Harish-Chandra modules are very important in
representation theory of real reductive groups. Our obstacle that arises in passage from
ordinary to quantum groups is that necessary methods of non-commutative geometry
are not at a hand. This forces us to combine the process of searching for geometric
realizations of quantum Harish-Chandra modules with looking for suitable concepts and
results of non-commutative geometry.

For such approach to be successful, one has to choose properly the class of modules to
be considered. Our choice is based on the notion of quantum bounded symmetric domain
\[ \Xi \].

It is well known that every irreducible bounded symmetric domain admits a standard
realization in a complex prehomogeneous vector space. We begin with a result on \( \ast \)-
algebras of ‘polynomials on quantum prehomogeneous vector spaces’.

Turn to precise formulations.

2. We assume \( \mathbb{C} \) as a ground field. The parameter \( q \) is supposed to be in the interval
\( (0, 1) \). All the algebras are unital, unless the contrary is stated explicitly.

Consider the \( \ast \)-algebra \( \text{Pol}(\mathbb{C})_q \) given by a single generator \( z \) and the defining relation
\[
z^\ast z = q^2zz^\ast + 1 - q^2.
\]

\( \text{Pol}(\mathbb{C})_q \) is a quantum analogue of the polynomial algebra on a complex plane treated as
a two dimensional real vector space. We are interested in representations of \( \ast \)-algebras by
bounded linear operators.

Let \( T_F \) be a representation of \( \text{Pol}(\mathbb{C})_q \) in \( l^2(\mathbb{Z}_+) \) given by
\[
T_F(z)e_n = (1 - q^{2(n+1)})e_{n+1}, \quad T_F(z^\ast)e_n = \begin{cases} (1 - q^{2n})e_{n-1}, & n \in \mathbb{N}, \\ 0, & n = 0, \end{cases}
\]
with \( \{e_n\} \) being the standard basis of \( l^2(\mathbb{Z}_+) \). This is the so called Fock representation.
One can readily use the spectral theory of operators and the commutation relations
\[ zy = q^{-2}yz, \quad z^*y = q^2yz^*, \] with \( y = 1 - zz^* \), to derive the following easy and well known

**Proposition 2.1**

1. \( T_F \) is a faithful irreducible \(*\)-representation of \( \text{Pol}(\mathbb{C})_q \) by bounded linear operators in a Hilbert space.

2. Every representation with these properties is unitarily equivalent to \( T_F \).

This Proposition could be treated as a q-analogue of the Stone-von Neumann theorem since the commutation relation between the elements
\[ a^+ = (1 - q^2)^{-\frac{1}{2}}z, \quad a = (1 - q^2)^{-\frac{1}{2}}z^* \]
is a q-analogue of the canonical commutation relation \( aa^+ - a^+a = 1 \).

During 90’s q-analogues were found for polynomial algebras on some special prehomogeneous vector spaces. This became a background for producing an advanced extension of Proposition 2.1 and lead to the notion of quantum bounded symmetric domain. Describe a construction of those q-analogues for polynomial algebras.

Consider a simple complex Lie algebra \( g \) of rank \( l \) whose Cartan matrix is \( a = (a_{ij}) \).

Up to an isomorphism this Lie algebra can be described in terms of the generators \( \{ H_i, E_i, F_i \}_{i=1,2,...,l} \) and the well known defining relations.

The linear span \( \mathfrak{h} \) of \( H_1, H_2, \ldots, H_l \) is a Cartan subalgebra, and the linear functionals \( \alpha_1, \alpha_2, \ldots, \alpha_l \) on \( \mathfrak{h} \) defined by
\[ \alpha_j(H_i) = a_{ij}, \quad i, j = 1, 2, \ldots, l, \]
form a system of simple roots for the Lie algebra \( g \).

Choose a simple root \( \alpha_{l_0} \) which appears in the decomposition \( \delta = \sum_{i=1}^{l} n_i \alpha_i \) of the maximal root \( \delta \) with coefficient 1.\(^1\) Let \( H \in \mathfrak{h} \) be the element given by
\[ \alpha_{l_0}(H) = 2, \quad \alpha_j(H) = 0, \quad j \neq l_0, \]
and
\[ \mathfrak{e} = \{ \xi \in g \mid [H, \xi] = 0 \}, \quad \mathfrak{p}^\pm = \{ \xi \in g \mid [H, \xi] = \pm 2\xi \}. \]

Then one has \( g = \mathfrak{p}^- \oplus \mathfrak{e} \oplus \mathfrak{p}^+ \), which constitutes a \( \mathbb{Z} \)-gradation:
\[ [\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{e}, \quad [\mathfrak{e}, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm, \quad [\mathfrak{p}^+, \mathfrak{p}^-] \subset \mathfrak{e}, \quad [\mathfrak{p}^+, \mathfrak{p}^+] = [\mathfrak{p}^-, \mathfrak{p}^-] = 0. \]

Consider the complex simply connected Lie group with the Lie algebra \( g \) and its subgroup \( K \) of those elements which preserve the gradation
\[ K = \{ g \in G \mid \text{Ad}_g \mathfrak{e} = \mathfrak{e}, \ \text{Ad}_g \mathfrak{p}^\pm = \mathfrak{p}^\pm \}. \]

We follow H. Rubenthaler in calling \( \mathfrak{p}^\pm \) the prehomogeneous vector spaces of commutative parabolic type. The prehomogeneity means existence of an open \( K \)-orbit.

A construction of the \(*\)-algebra \( \text{Pol}(\mathfrak{p}^-)_q \), a quantum analogue of the polynomial algebra on \( \mathfrak{p}^- \), can be found in [8]. Describe the outline of this construction and one of its steps.

\(^1\) Such simple roots exist for all simple complex Lie algebras except \( E_8, F_4, G_2 \).
For that, we need a Hopf \ast\text{-algebra} \((U_q\mathfrak{g}, \ast)\), with \(U_q\mathfrak{g}\) being the quantum universal enveloping algebra. This Hopf \ast\text{-algebra} is given by its generators

\[K_i, K_i^{-1}, E_i, F_i, \quad i = 1, 2, \ldots, l,\]

the well known Drinfeld-Jimbo relations [3]. The comultiplication \(\Delta\), the counit \(\varepsilon\), the antipode \(S\), and the involution \(\ast\) are given by

\[\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i,\]
\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1,\]
\[S(E_i) = -K_i^{-1}E_i, \quad S(F_i) = -F_iK_i, \quad S(K_i) = K_i^{-1},\]
\[(K_j)^\ast = K_j^{\pm 1}, \quad j = 1, 2, \ldots, l;\]
\[E_j^\ast = \begin{cases} K_jF_j, & j \neq l_0; \\ -K_jF_j, & j = l_0. \end{cases} \quad F_j^\ast = \begin{cases} E_jK_j^{-1}, & j \neq l_0; \\ -E_jK_j^{-1}, & j = l_0. \end{cases}\]

In the special case \(l = 1\) one has the Hopf \ast\text{-algebra} \(U_q\mathfrak{sl}_{1,1} \overset{\text{def}}{=} (U_q\mathfrak{sl}_2, \ast)\).

In what follows all the \(U_q\mathfrak{g}\)-modules are assumed to be weight:

\[V = \bigoplus_{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{Z}^l} V_{\lambda}, \quad V_{\lambda} = \{v \in V | K_i^{\pm 1}v = q_i^{\pm \lambda_i}v, \quad i = 1, 2, \ldots, l\},\]

with \(q_i = q^{d_i}\) and \(d_i\) being the coprime numbers that symmetrize the Cartan matrix: \(d_ia_{ij} = d_ja_{ji}\). Define the linear operators \(H_1, H_2, \ldots, H_l\) in \(V\) by \(H_i|V_{\lambda} = \lambda_i\). Obviously, in \(\text{End} V\) one has the relations

\[K_i^{\pm 1} = q_i^{\pm H_i}, \quad i = 1, 2, \ldots, l.\]

Consider the Hopf subalgebra \(U_q\mathfrak{k}\) generated by

\[K_j^{\pm 1}, \quad j = 1, 2, \ldots, l; \quad E_i, F_i, \quad i \neq l_0.\]

The weight finitely generated \(U_q\mathfrak{g}\)-module \(V\) is called a Harish-Chandra module if it splits as a sum of simple finite dimensional \(U_q\mathfrak{k}\)-modules, each of those having a finite multiplicity in \(V\).

The construction and classification of simple quantum Harish-Chandra modules are important open problems.

Turn back to a construction of the \ast\text{-algebra} \(\text{Pol}(\mathfrak{p}^-)_q\). Its relation to quantum bounded symmetric domains is to be based on the fact that \(\text{Pol}(\mathfrak{p}^-)_q\) is a \((U_q\mathfrak{g}, \ast)\)-module algebra. Recall the latter notion.

Consider an algebra \(F\) which is also a module over a Hopf algebra \(A\). \(F\) is called an \(A\)-module algebra if the multiplication \(m : F \otimes F \to F\) is a morphism of \(A\)-modules. In the case of a unital algebra \(F\) one has to require additionally an \(A\)-invariance of its unit. In the presence of involutions in \(A\) and in \(F\) they have to be compatible:

\[(af)\ast = (S(a))\ast f\ast, \quad a \in A, \quad f \in F.\]

The initial step towards \(\text{Pol}(\mathfrak{p}^-)_q\) is in producing a \(U_q\mathfrak{g}\)-module algebra \(\mathbb{C}\mathfrak{p}^-_q\), which is a \(q\)-analogue for the algebra \(\mathbb{C}\mathfrak{p}^-\) of holomorphic polynomials on \(\mathfrak{p}^-\). We follow the
idea of V. Drinfeld for producing by duality function algebras on quantum groups. To
construct the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[p^-]_q$, we start with considering the ‘dual’ coalgebra. This coalgebra is going to be a generalized Verma module, specifically the $U_q\mathfrak{g}$-module with a generator $v^{(0)}$ and defining relations

$$E_i v^{(0)} = (K_i^\pm - 1) v^{(0)} = 0, \quad i = 1, 2, \ldots, l; \quad F_j v^{(0)} = 0, \quad j \neq l_0.$$ 

The comultiplication is essentially determined by

$$\Delta v^{(0)} = v^{(0)} \otimes v^{(0)}$$

(see [8] for details). It is worthwhile to note that the multiplicities of simple finite dimensional $U_q\mathfrak{k}$-modules in $\mathbb{C}[p^-]_q$ are the same as the multiplicities of corresponding simple finite dimensional $U\mathfrak{k}$-modules in $\mathbb{C}[p^-]$.

In the special case $l = 1$ one has the polynomial algebra in one variable $z$ and with the following $U_q\mathfrak{sl}_2$-action:

$$K^\pm f(z) = f(q^\pm z), \quad F f(z) = q^\frac{1}{2} \frac{f(q^{-2}z) - f(z)}{q^{-2}z - z}, \quad (2.3)$$

$$E f(z) = -q^\frac{1}{2} z^2 \frac{f(z) - f(q^2z)}{z - q^2z} \quad (2.4)$$

The next step is in producing the $U_q\mathfrak{g}$-module algebra $\mathbb{C}[p^+]_q$ of ‘antiholomorphic polynomials on the quantum vector space $p^-$’. It is a very easy step:

- $\mathbb{C}[p^+]_q$ is just $\mathbb{C}[p^-]_q$ as an Abelian group,
- the identity map
  $$\ast : \mathbb{C}[p^-]_q \rightarrow \mathbb{C}[p^+]_q, \quad \ast : f \mapsto f \quad (2.5)$$
  is antilinear,
- the action of $U_q\mathfrak{g}$ on $\mathbb{C}[p^+]_q$ is given by
  $$(\xi f)^\ast = (S(\xi))^{\ast} f^\ast, \quad \xi \in U_q\mathfrak{g}, \; f \in \mathbb{C}[p^-]_q.$$ 

The algebras $\mathbb{C}[p^\pm]_q$ are equipped with gradations as follows:

$$\mathbb{C}[p^\pm]_q = \bigoplus_{j=0}^{\infty} \mathbb{C}[p^\pm]_{q,j}, \quad \mathbb{C}[p^\pm]_{q,j} = \{ f \in \mathbb{C}[p^\pm]_q | Hf = 2jf \}.$$ 

One can demonstrate that those algebras are generated by their homogeneous components $\mathbb{C}[p^\pm]_{q,\pm 1}$ and they are quadratic algebras.

At the final step the vector space

$$\text{Pol}(p^-)_q \overset{\text{def}}{=} \mathbb{C}[p^-]_q \otimes \mathbb{C}[p^+]_q$$

is equipped with a structure of $(U_q\mathfrak{g}, \ast)$-module algebra. An involution $\ast$ is defined via the antilinear map (2.5) in an obvious way. A multiplication is imposed via the Drinfeld universal R-matrix. Specifically, introduce the notation

$$m^\pm : \mathbb{C}[p^\pm]_q \otimes \mathbb{C}[p^\pm]_q \rightarrow \mathbb{C}[p^\pm]_q, \quad m^\pm : f_1 \otimes f_2 \mapsto f_1 f_2$$

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for multiplications in $\mathbb{C}[p^+]_q$ and a $q$-analogue for the permutation of tensor multiples

$$\tilde{R} : \mathbb{C}[p^+]_q \otimes \mathbb{C}[p^-]_q \rightarrow \mathbb{C}[p^-]_q \otimes \mathbb{C}[p^+]_q,$$

determined by the action of the universal R-matrix in $\mathbb{C}[p^+]_q \otimes \mathbb{C}[p^-]_q$.

$\tilde{R}$ is well defined, as one can demonstrate that the weights of the $U_q g$-module $\mathbb{C}[p^-]_q$ are non-negative while the weights of the $U_q g$-module $\mathbb{C}[p^+]_q$ are non-positive (the zero weight determines one-dimensional weight spaces $\mathbb{C} \cdot 1$).

A multiplication $m : (\mathbb{C}[p^-]_q \otimes \mathbb{C}[p^+]_q)^{\otimes 2} \rightarrow \mathbb{C}[p^-]_q \otimes \mathbb{C}[p^+]_q$ is defined via $m^\pm$ and $\tilde{R}$:

$$m \overset{\text{def}}{=} (m^- \otimes m^+) \left( \text{id}_{\mathbb{C}[p^-]_q} \otimes \tilde{R} \otimes \text{id}_{\mathbb{C}[p^+]_q} \right).$$

Such multiplication equips $\text{Pol}(p^-)_q$ with a structure of $(U_q g, *)$-module algebra [8].

**Example 2.2** Let $l = 1$. One has $Ez^* = q^{-\frac{3}{2}}$, $q^{-\frac{\frac{\partial H}{\partial H}}{2}} z^* \otimes z = q^2 z^* \otimes z$,

$$R(z^* \otimes z) = (1 + (q^{-1} - q) E \otimes F) q^{-\frac{\frac{\partial H}{\partial H}}{2}} z^* \otimes z = q^2 z^* \otimes z + 1 - q^2;$$

$$\tilde{R}(z^* \otimes z) = q^2 z \otimes z^* + 1 - q^2, \quad z^* z = q^2 z z^* + 1 - q^2.$$ 

A description of $\text{Pol}(p^-)_q$ in terms of generators and relations in a more general case $\mathfrak{g} = \mathfrak{sl}_{l+1}$ is given in [7]. A complete list of irreducible *-representations is known for some of those *-algebras [6, 9].

Note that $\mathbb{C}[p^+]_{q,-1} \hookrightarrow \mathbb{C}[p^+]_q \hookrightarrow \text{Pol}(p^-)_q$. A vector $\nu_0 \neq 0$ from a space of *-representation $T$ of $\text{Pol}(p^-)_q$ is called a vacuum vector if $T(f) \nu_0 = 0$ for all $f \in \mathbb{C}[p^+]_{q,-1}$.

**Theorem 2.3**

1. There exists a unique (up to a unitary equivalence) faithful irreducible *-representation $T_F$ of $\text{Pol}(p^-)_q$ by bounded linear operators in a Hilbert space $H_F$.

2. There exists a unique (up to a scalar multiple) vacuum vector for $T_F$ as above.

To match correspondence with the special case $\mathfrak{g} = \mathfrak{sl}_2$ considered above, we keep the term ‘Fock representation’ for $T_F$.

3. Let $dv$ be an invariant measure on $\mathbb{D}$. Our goal is to obtain a $q$-analogue for the algebra $\mathcal{D}(\mathbb{D})$ of smooth functions on $\mathbb{D}$ with compact supports and a $q$-analogue for the invariant integral

$$\nu : \mathcal{D}(\mathbb{D}) \rightarrow \mathbb{C}, \quad \nu : f \mapsto \int_{\mathbb{D}} f \, dv.$$ 

These are to be used to introduce a $q$-analogue for the space $L^2(dv)$ which is involved into formulating the principal problem of harmonic analysis in $\mathbb{D}$. (Note that polynomials are not in $L^2(dv)$: $\int_{\mathbb{D}} 1 \, dv = \infty$.)

Consider the space of the Fock representation $T_F$ and the one-dimensional orthogonal projection $P_0$ onto the vacuum subspace. To produce $\mathcal{D}(\mathbb{D})_q$, it would be well to have an element $f_0 \in \text{Pol}(p^-)_q$ such that

$$T_F f_0 = P_0,$$ (3.1)

but it does not exist. This is a motivation to attach such $f_0$ to $\text{Pol}(p^-)_q$.

Let us act formally. Extend the $(U_q g, *)$-module algebra $\text{Pol}(p^-)_q$ by attaching an element $f_0$ which satisfies the following relations that are motivated by (3.1) and a continuity argument:
- $f_0^2 = f_0$, $f_0^* = f_0$,
- $\psi^* f_0 = f_0 \psi = 0$ for all weight $\psi \in \mathbb{C}[\mathfrak{p}^-]_q$, $\psi \notin \mathbb{C} \cdot 1$,
- $K_j^{\pm 1} f_0 = f_0$, $j = 1, 2, \ldots, l$,

\[
F_j f_0 = \begin{cases} 
\frac{q_j^{1/2}}{q_0^{1/2} - 1} f_0 z_{\text{low}}^*, & j = l_0, \\
0, & j \neq l_0.
\end{cases} \quad E_j f_0 = \begin{cases} 
\frac{q_j^{1/2}}{1 - q_0^{1/2}} z_{\text{low}} f_0, & j = l_0, \\
0, & j \neq l_0.
\end{cases}
\]

Here $z_{\text{low}}$ is the unique element of $\mathbb{C}[\mathfrak{p}^-]_q$ with the properties

\[
K_i^{\pm 1} z_{\text{low}} = q_i^{\pm h_{di}} z_{\text{low}}, \quad F_i z_{\text{low}} = \begin{cases} 
\frac{1}{q_{l_0}}, & i = l_0, \\
0, & i \neq l_0.
\end{cases}
\]

We keep the notation $T_F$ for the natural extension of the Fock representation onto the above algebra. The two-sided ideal $\mathcal{D}(\mathbb{D})_q$ of this algebra generated by $f_0$ will be called the algebra of finite functions on the quantum bounded symmetric domain $\mathbb{D}$. $\mathcal{D}(\mathbb{D})_q$ is a $(U_q \mathfrak{g}, \ast)$-module algebra.

Recall that a linear functional $\nu$ on an $A$-module algebra $F$ is called an invariant integral if $\nu(af) = \epsilon(a)\nu(f)$ for all $a \in A, f \in F$.

**Proposition 3.1** There exists a non-zero $U_q \mathfrak{g}$-invariant integral on the $U_q \mathfrak{g}$-module algebra $\mathcal{D}(\mathbb{D})_q$. It is unique up to a constant multiple and can be chosen to be positive: $\int f^* f \, d\nu > 0$, with $f \neq 0$.

The integral looks like a q-trace. We formulate this result. Since $\mathcal{D}(\mathbb{D})_q = \mathbb{C}[\mathfrak{p}^-]_q f_0 \mathbb{C}[\mathfrak{p}^+]_q$, one has $\mathcal{H}_F \overset{\text{def}}{=} \mathcal{D}(\mathbb{D})_q f_0 = \mathbb{C}[\mathfrak{p}^-]_q f_0 = \text{Pol}(\mathfrak{p}^-)_q f_0$.

Obviously, $\mathcal{H}_F$ is a $\mathcal{D}(\mathbb{D})_q$-module, a $\text{Pol}(\mathfrak{p}^-)_q$-module, and a weight $U_q \mathfrak{g}$-module. Introduce the notation $\mathcal{F}_F$ for the corresponding representations of $\mathcal{D}(\mathbb{D})_q$ and $\text{Pol}(\mathfrak{p}^-)_q$ in the vector space $\mathcal{H}_F$ (the Hilbert space $H_F$ is a completion of the pre-Hilbert space $\mathcal{H}_F$).

**Proposition 3.2** Let $\rho$ be the half sum of positive roots, $\rho = \frac{1}{2} \sum_{i=1}^{l} n_i \alpha_i$ and $\check{\rho} = \frac{1}{2} \sum_{i=1}^{l} n_i d_i H_i$. The linear functional on $\mathcal{D}(\mathbb{D})_q$

\[
\int_{\mathbb{D}_q} f \, d\nu = \text{const} \cdot \text{tr} \left( \mathcal{F}_F(f) q^{-2\check{\rho}} \right), \quad \text{const} > 0, \quad (3.2)
\]

is a positive $U_q \mathfrak{g}$-invariant integral.

Now (3.2) can be readily used to obtain a q-analogue of weighted Bergman spaces and the well known geometric realizations for the so called holomorphic discrete series in those spaces.

4. In the classical case $q = 1$ one has a well known geometric realization for the non-degenerate principal series of Harish-Chandra modules on the open $K$-orbit $\Omega$ of the flag variety.
It is easy to obtain a quantum analogue for such a geometric realization, and then to use it as a background for producing a quantum analogue for the principal series. We restrict ourselves to producing a $U_q\mathfrak{g}$-module algebra $\mathbb{C}[[\Omega]]_q$, which is a quantum analogue for the algebra $\mathbb{C}[[\Omega]]$ of regular functions on the affine algebraic variety $\Omega$.

Recall a general background on spherical weights. Every $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in \mathbb{Z}_+^l$ determines a simple finite dimensional weight $U_q\mathfrak{g}$-module $L(\lambda)$ with the highest weight $\lambda$:

$$E_j v(\lambda) = 0, \quad K_j^\pm v(\lambda) = q_j^{\pm \lambda_j} v(\lambda), \quad E_j^{\lambda_j+1} v(\lambda) = 0.$$  

Such a weight $\lambda$ is said to be spherical if $L(\lambda)$ contains a non-zero $U_q\mathfrak{k}$-invariant vector. This vector is unique up to a constant multiple. The set $\Lambda$ of spherical weights is of the form $\Lambda = \mathbb{Z}_+^l$, with $\mu_1, \mu_2, \ldots, \mu_l$ being the so-called fundamental spherical weights. Here $r$ is the rank of the bounded symmetric domain $\mathbb{D}$. In the simplest case $l = 1$ the fundamental spherical weight $\mu$ is $2\omega$.

Equip the $U_q\mathfrak{g}$-module $\mathbb{C}[X]_q^{\text{spher}} = \bigoplus_{\lambda \in \Lambda} L(\lambda)$ with a structure of $U_q\mathfrak{g}$-module algebra in a way which is normally used in producing quantum flag varieties. For any $\lambda', \lambda'' \in \Lambda$, $L(\lambda' + \lambda'')$ occurs in $L(\lambda') \otimes L(\lambda'')$ with multiplicity 1. This allows one to introduce the morphisms of $U_q\mathfrak{g}$-modules

$$m_{\lambda', \lambda''} : L(\lambda') \otimes L(\lambda'') \to L(\lambda' + \lambda''), \quad m_{\lambda', \lambda''} : v(\lambda') \otimes v(\lambda'') \mapsto v(\lambda' + \lambda''),$$

and a structure of $U_q\mathfrak{g}$-module algebra in $\mathbb{C}[X]_q^{\text{spher}}$

$$f' \cdot f'' \overset{\text{def}}{=} m_{\lambda', \lambda''}(f' \otimes f''), \quad f' \in L(\lambda'), \quad f'' \in L(\lambda'').$$

Apply the Peter-Weyl decomposition $\mathbb{C}[G]_q = \bigoplus_{\lambda \in \mathbb{Z}_+^l} (L(\lambda) \otimes L(\lambda)^*)$ to obtain an embedding of $U_q\mathfrak{g}$-module algebras

$$i : \mathbb{C}[X]_q^{\text{spher}} \hookrightarrow \mathbb{C}[G]_q, \quad i : v(\lambda) \mapsto c^\lambda_{\lambda', \lambda}, \quad \lambda \in \Lambda.$$

Here $c^\lambda_{\lambda', \lambda}$ are the matrix elements of representation $\pi_\lambda$ associated to the $U_q\mathfrak{g}$-modules $L(\lambda)$. This embedding allows one to treat the elements of $\mathbb{C}[X]_q^{\text{spher}}$ as $q$-analogues for sections of bundles on the flag variety. Of course, $\mathbb{C}[X]_q^{\text{spher}}$ has no zero divisors, as these are absent in $\mathbb{C}[G]_q$.

Choose non-zero $U_q\mathfrak{k}$-invariants

$$\psi_j \in L(\mu_j), \quad j = 1, 2, \ldots, r.$$  

Of course, $\psi_1, \psi_2, \ldots, \psi_r \in \mathbb{C}[X]_q^{\text{spher}}$.

**Proposition 4.1** The elements $\psi_1, \psi_2, \ldots, \psi_r$ pairwise commute and the multiplicative subset

$$\Psi = \{ \psi_1^{j_1} \psi_2^{j_2} \cdots \psi_r^{j_r} \mid j_1, j_2, \ldots, j_r \in \mathbb{Z}_+ \} \subset \mathbb{C}[X]_q^{\text{spher}}$$

is both right and left Ore set.

Let $\mathbb{C}[X]_q^{\text{spher}}, \Psi$ be the localisation of $\mathbb{C}[X]_q^{\text{spher}}$ with respect to the multiplicative set $\Psi$.

**Proposition 4.2** There exists a unique extension of the structure of $U_q\mathfrak{g}$-module algebra from $\mathbb{C}[X]_q^{\text{spher}}$ onto $\mathbb{C}[X]_q^{\text{spher}}, \Psi$. 

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Our proof uses more elementary observations than those of a work by V. Luntz and A. Rosenberg [5].

Equip the algebra $\mathbb{C}[X_{\text{spher}}]_{q,\Psi}$ with a $\mathbb{Z}^r$-gradation so that

$$\text{deg } f = (j_1, j_2, \ldots, j_r) \iff f \in L(j_1\mu_1 + j_2\mu_2 + \ldots + j_r\mu_r).$$

The subalgebra of zero degree elements

$$\mathbb{C}[\Omega]_q \overset{\text{def}}{=} \{ f \in \mathbb{C}[X_{\text{spher}}]_{q,\Psi} \mid \text{deg } f = 0 \}$$

is a $U_q\mathfrak{g}$-module algebra. This is just the $q$-analogue for the algebra of regular functions on the open $K$-orbit $\Omega$.

In the simplest case $l = 1$

$$i : \mathbb{C}[X_{\text{spher}}]_q \hookrightarrow \mathbb{C}[SL_2]_q,$$

and the subalgebra $i(\mathbb{C}[X_{\text{spher}}]_q)$ is generated by

$$t_{11}^2, \quad t_{11}t_{12}, \quad t_{12}^2. \quad (4.1)$$

In this case $r = 1$, $i\psi_1 = \text{const} \cdot t_{11}t_{12}$, and $\Psi$ is an Ore set because $t_{11}t_{12}$ quasi-commutes with each of the generators (4.1). In this special case $\mathbb{C}[\Omega]_q$ is isomorphic to the algebra of Laurent polynomials in the indeterminate $z = t_{12}^{-1}t_{11}$ just as in the classical case $q = 1$. The action of the generators $K^{\pm 1}, F, E$ of $U_q\mathfrak{sl}_2$ on Laurent polynomials $f(z)$ is given by (2.3), (2.4).

5. Many results related to the subjects of this talk remain intact. For example, the works by H. Jakobsen and T. Tanisaki’s team [2, 4] study the $U_q\mathfrak{t}$-module algebras isomorphic to $\mathbb{C}[p^-]_q$. The latter of those works suggests $q$-analogues of Sato-Bernstein polynomials for quantum prehomogeneous vector spaces in question. Note that such quantum prehomogeneous vector spaces were found independently in [4, 2, 8]. $q$-Analogues for the Shilov boundaries of the bounded symmetric domains can be produced. They can be used to obtain geometric realizations for unitary principal degenerate series of $U_q\mathfrak{g}$-modules (see [1] for application of this geometric realizations).

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