Weighted Supermembrane Toy Model

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Abstract

A weighted Hilbert space approach to the study of zero-energy states of supersymmetric matrix models is introduced. Applied to a related but technically simpler model, it is shown that the spectrum of the corresponding weighted Hamiltonian simplifies to become purely discrete for sufficient weights. This follows from a bound for the number of negative eigenvalues of an associated matrix-valued Schrödinger operator.

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1 Introduction

There are many difficulties in the study of zero-energy states of supersymmetric matrix models. Some arise due to the fact that the spectrum of the associated Hamiltonian is continuous\(^1\), starting at zero. Accordingly, we expect it to be useful to shift to a weighted Hilbert space on which the corresponding operator has a discrete spectrum.

In this work we illustrate the applicability of the technique to a simplified model which, despite its technical simplicity, still shares many of the features (and difficulties) of the original matrix models. The spectral properties of this so-called supermembrane toy model, and its purely bosonic counterpart, has been previously studied in [1, 2, 3, 4, 5, 6, 7, 8], while the underlying geometry of the model was emphasized in [9]. Our results on the technically

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\(^1\)or purely essential, to be precise
much more complicated matrix models will be presented in a forthcoming paper.

In Section 2 we recall the formulation of the original toy model and introduce the weighted Hilbert space approach. In Sections 3 and 4 we investigate the spectral properties of the weighted model, and show that, under a certain condition on the parameter of the weight, the spectrum of the weighted model is in fact discrete. This is accomplished using a Cwikel-Lieb-Rozenblum-type bound for operator-valued potentials which is derived in Section 5 from a result in [10]. We also note that the technique we use could provide a geometric understanding of the eigenvalue asymptotics of the purely bosonic model.

2 The original and weighted models

The supermembrane toy model, also called the supersymmetric $x^2y^2$ potential, is defined by the Hamiltonian operator

$$H = -\Delta + V + H_F = -\partial_x^2 - \partial_y^2 + x^2y^2 + x\gamma_1 - y\gamma_2$$

(1)

(where $\gamma_k$ are Pauli matrices), acting on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^2, dx dy) \otimes \mathbb{C}^2.$$ 

A corresponding hermitian supercharge operator is given by

$$Q = -i(\partial_x \gamma_1 + \partial_y \gamma_2) + xy\gamma_3,$$

such that $Q^2 = H \geq 0$.

The matrix-valued Schrödinger operator $H$ can be formally defined as a self-adjoint operator through the closure of the quadratic form corresponding to the expression (1) on $C_0^\infty(\mathbb{R}^2) \otimes \mathbb{C}^2$. The spectrum of $H$ is $\sigma(H) = [0, \infty)$ due to potential valleys along the coordinate axes, where the lower eigenvalue of the matrix potential,

$$V + (H_F)_- = x^2y^2 - \sqrt{x^2 + y^2},$$

tends to negative infinity – precisely cancelling the localization energy due to the narrowing of the valley.

We would like to make the spectrum of the model discrete by introducing a weighted Hilbert space. We define

$$\mathcal{H}_w := L^2(\mathbb{R}^2, \rho(x, y) dx dy) \otimes \mathbb{C}^2, \quad \rho(x, y) := (1 + x^2 + y^2)^{-\frac{3}{2}},$$
with \( \alpha \geq 0 \). The inner product on \( \mathcal{H}_w \) is then given by

\[
\langle \Phi, \Psi \rangle_w = \langle \Phi, \rho \Psi \rangle = \int_{\mathbb{R}^2} \frac{\langle \Phi(x, y), \Psi(x, y) \rangle C_2}{(1 + x^2 + y^2)^{\frac{\alpha}{2}}} \, dx \, dy.
\]

The corresponding self-adjoint operator \( \tilde{H} \) on \( \mathcal{H}_w \) is defined through the same quadratic form on \( C_0^\infty(\mathbb{R}^2) \otimes \mathbb{C}^2 \);

\[
\langle \Psi, \tilde{H} \Psi \rangle_w := \langle \Psi, H \Psi \rangle = \| Q \Psi \|_2^2 \geq 0.
\]

It follows that, if we define \( \tilde{Q} := \rho^{-\frac{1}{2}} Q \) (with adjoint w.r.t \( \mathcal{H}_w \) given by \( \tilde{Q}^* = \rho^{-\frac{1}{2}} Q \rho^\frac{1}{2} \)) we have

\[
\langle \Psi, \tilde{H} \Psi \rangle_w = \| \tilde{Q} \Psi \|_w^2.
\]

We observe in general that any solution of \( H \Psi = 0 \) in \( \mathcal{H} \) is also a solution of \( \tilde{H} \Psi = 0 \) in the weighted Hilbert space. On the other hand, finding a solution of \( \tilde{H} \Psi = 0 \) in \( \mathcal{H}_w \) does yield a (smooth\( ^3 \)) solution to the differential equation \( H \Psi = 0 \), but its decay rate may be insufficient for square-integrability. For this particular model it is known \cite{7} that there is no solution in \( \mathcal{H} \).

### 3 Spectrum of \( \tilde{H} \) for \( \alpha < 2 \)

Continuity of the spectrum of \( H \) can be proved (see \cite{4}) by, for any \( \mu \geq 0 \), finding a Weyl sequence \( (\Psi_t) \) in \( \mathcal{H} \) such that \( \| \Psi_t \| = 1 \) \( \forall t \) and

\[
\langle \Psi_t, (H - \mu)^2 \Psi_t \rangle \to 0, \quad t \to \infty.
\]

Explicitly (and for \( \mu = 0 \) for simplicity), we take

\[
\Psi_t(x, y) := \chi_t(x) \phi_x(y) \xi,
\]

where \( \chi_t \) is a cut-off function s.t. \( \chi_t(x) := t^{-\frac{1}{2}} \chi(x/t), \chi \in C_0^\infty[1, 2], \int_\mathbb{R} \chi^2 = 1 \), \( \phi_x \) is the normalized groundstate of the harmonic oscillator \(-\partial_y^2 + x^2 y^2 \),

\[
\phi_x(y) := \left( \frac{x}{\pi} \right)^{\frac{1}{4}} e^{-\frac{1}{2} y^2},
\]

and \( \xi \in \mathbb{C}^2 \) is a unit eigenvector, \( \gamma_1 \xi = -\xi \). One finds that \( \| \Psi_t \|^2 = 1 \) and

\[
\langle \Psi_t, H \Psi_t \rangle \leq \int |\chi_t \chi''_t| \, dx + c_1 \int x^{-1} |\chi_t \chi'_t| \, dx + c_2 \int x^{-2} |\chi_t|^2 \, dx \leq ct^{-2} \]

\( ^3 \)by elliptic regularity
(here, and in the following, $c, c_1, \ldots$ denote some positive constants).

Taking the same sequence for the weighted case, we also find

$$\langle \Psi_t, \tilde{H} \Psi_t \rangle_w = \langle \Psi_t, H \Psi_t \rangle \leq ct^{-2}.$$  

However, the norm is now

$$\|\Psi_t\|_w^2 = \int_x |\chi_t(x)|^2 \int_y \frac{|\phi_x(y)|^2}{(1 + x^2 + y^2)\tilde{\alpha}^2} dy dx$$

$$\geq c_1 \int_x \frac{|\chi_t(x)|^2}{(1 + 4t^2 + c_2t^{-1})\tilde{\alpha}^2} dx \geq c_3(1 + 4t^2)^{-\frac{\tilde{\alpha}}{2}},$$

so that, for $\alpha < 2$,

$$\frac{\langle \Psi_t, \tilde{H} \Psi_t \rangle_w}{\|\Psi_t\|_w^2} \leq c(1 + 4t^2)^{\frac{2}{\tilde{\alpha}}}, \quad t \to \infty.$$  

Thus, $\Psi_t$ still approximates a zero-energy eigenfunction, but since its support moves out to infinity, this indicates that the spectrum of $\tilde{H}$ is still continuous for $0 \leq \alpha < 2$.

### 4 Spectrum of $\tilde{H}$ for $\alpha > 2$

The spectrum of $\tilde{H}$ is discrete if and only if, for all $\lambda > 0$, the rank of the spectral projection of $\tilde{H}$ on $(-\infty, \lambda)$ is finite. Equivalently, if and only if

$$\sup_{W_\lambda} \dim W_\lambda < \infty,$$

where $W_\lambda$ are subspaces of $C_\infty^0(\mathbb{R}^2) \otimes \mathbb{C}^2$ such that, for all $\Psi \in W_\lambda$,

$$\langle \Psi, \tilde{H} \Psi \rangle_w < \lambda \|\Psi\|_w^2.$$  

Note that (2) is equivalent to

$$\langle \Psi, (H - \lambda \rho) \Psi \rangle < 0.$$  

It follows that the spectrum of $\tilde{H}$ is discrete if and only if the operator $H - \lambda \rho$, on the original Hilbert space $\mathcal{H}$, has finitely many negative eigenvalues for any $\lambda > 0$, or more precisely, $N(H - \lambda \rho) < \infty$, where we denote by $N(A)$ the (possibly infinite) rank of the spectral projection on $(-\infty, 0)$ of a self-adjoint operator $A$. We will prove the following theorem.
Theorem 1. For all $\lambda > 0$ and $\alpha > 2$, the operator

$$H_\lambda := H - \lambda \rho = -\partial_x^2 - \partial_y^2 + x^2 y^2 + x \gamma_1 - y \gamma_2 - \frac{\lambda}{(1 + x^2 + y^2)^\frac{\alpha}{2}}$$  \hspace{1cm} (3)$$

has finitely many negative eigenvalues. Furthermore, the number of negative eigenvalues is bounded by

$$N(H_\lambda) \leq C(\alpha) + \frac{2^{12} \pi C_3}{27(\alpha - 2)^3} \lambda^\frac{\alpha}{2} - \epsilon(\alpha),$$  \hspace{1cm} (4)$$

where $C(\alpha)$ and $C_3$ are positive constants, and $0 < \epsilon(\alpha) < \frac{1}{2}(\alpha - 2)$.

Our strategy is to prove this by splitting the domain of the operator into different regions – based on the geometry of the potential valleys – and introducing Dirichlet boundary conditions between these regions by means of a partition of unity. The unbounded region along the potential valley is then shown to admit only finitely many negative eigenvalues using a Cwikel-Lieb-Rozenblum bound for operator-valued potentials. In order to illustrate the latter part of this procedure, let us first prove that $H_\lambda$ defined on the region $x > 1$ and with Dirichlet boundary condition at $x = 1$ has finitely many eigenvalues below zero when $\alpha > 2$. However, despite the fact that there is a reflection symmetry between $x$ and $y$, this result cannot be directly applied to prove Theorem 1 because of inconvenient intersections between regions of this form. Instead, we will introduce a different set of coordinates and define the regions with respect to those.

4.1 Cartesian coordinates

In the cartesian coordinates $(x, y)$ we consider the region $\Omega := (1, \infty) \times \mathbb{R}$ and the semi-bounded operator $H_{xy}^{\lambda}$ defined by closure of the quadratic form corresponding to (3) on $C_0^\infty(\Omega) \otimes \mathbb{C}^2$. Note that, for $\Psi \in C_0^\infty(\Omega) \otimes \mathbb{C}^2$,

$$\int_{\Omega} \langle \Psi, H_{xy}^{\lambda} \Psi \rangle_{\mathbb{C}^2} dxdy \geq \int_{\Omega} \left\langle \Psi, \left(-\partial_x^2 - \partial_y^2 + x^2 \left(y - \frac{1}{2x^2} \gamma_2\right)^2 - \frac{1}{4x^2} - x - \frac{\lambda}{x^\alpha}\right) \Psi \right\rangle_{\mathbb{C}^2} dxdy.$$

Choosing the representation $\gamma_2 = \text{diag}(1, -1)$, and making separate coordinate transformations $\tilde{y} := y \pm \frac{1}{2x^2}$ in the integrals over the corresponding components of $\Psi$, we find

$$H_{xy}^{\lambda} \geq -\partial_x^2 - \frac{1}{4x^2} - \partial_y^2 + x^2 \tilde{y}^2 - x - \frac{\lambda}{x^\alpha}.$$
where the two-dimensional scalar Schrödinger operator on the r.h.s. can be considered as a Schrödinger operator on the interval \((1, \infty)\) with an operator-valued potential \(V(x) = -\partial_x^2 + x^2 \tilde{y}^2 - x - \lambda x^{-\alpha}\) acting on \(L^2(\mathbb{R}, d\tilde{y})\). This shifted harmonic oscillator, with the projection onto its \(k\):th eigenvector denoted \(P_k\), is bounded below by its negative part

\[
V(x)_- = \sum_{k=0}^{\infty} (2kx - \lambda x^{-\alpha})_+ P_k \geq -\lambda x^{-\alpha} \sum_{0 \leq k \leq \lambda/2} P_k.
\]

Applying Lemma 7 below (with a factor 2 coming from the trace over \(\mathbb{C}^2\)), we find

\[
N(H^{xy}_\lambda) \leq N \left( \left( -\partial_x^2 - \frac{1}{4x^2} \right) - \frac{\lambda}{x^\alpha} \sum_{0 \leq k \leq \lambda/2} P_k \right) \leq 8\pi C_3 \int_1^\infty (1 + \lambda/2) \left( \lambda x^{-\alpha} \right)^{3/2} x^2 (\ln x)^2 dx,
\]

which is finite for \(\alpha > 2\).

### 4.2 Parabolic coordinates

Consider the coordinate transformation (cp. [9])

\[
(x, y) \mapsto (u, v) := \left( \frac{1}{2}(x^2 - y^2), xy \right),
\]

which is conformal everywhere except at the origin, and maps e.g. the open right half-plane bijectively onto the whole plane with the negative real line removed. We introduce the regions (see Figure 1)

\[
\mathcal{A} : -M < u < M,
\]

\[
\mathcal{B}_1 : u > M, x > 0,
\]

and the corresponding reflections \(\mathcal{B}_{2,3,4}\) of \(\mathcal{B}_1\) in the symmetry lines \(x = 0\) and \(x = y\), together with their union \(\mathcal{B} := \bigcup_{j=1}^4 \mathcal{B}_j = \mathbb{R}^2 \setminus \mathcal{A}\). We will also make use of rescaled versions of these regions, e.g. \(\kappa \mathcal{A}\), with a fixed \(\kappa > 1\).

Take a partition of unity, \(1 = \chi^2_\mathcal{A} + \chi^2_\mathcal{B}\), such that \(\chi_{\mathcal{A},\mathcal{B}} \in C^\infty(\mathbb{R}^2; [0, 1])\), \(\chi_{\mathcal{A}} = 1\) on \(\mathcal{A}\), and \(\chi_{\mathcal{B}} = 1\) on \(\kappa \mathcal{B}\). It follows that, for any \(\Psi \in C^\infty_0(\mathbb{R}^2) \otimes \mathbb{C}^2\),

\[
\langle \Psi, H_\lambda \Psi \rangle = \langle \Psi, H_\lambda (\chi^2_\mathcal{A} + \chi^2_\mathcal{B}) \Psi \rangle = \langle \chi_\mathcal{A} \Psi, H_{\lambda A}^4 \chi_\mathcal{A} \Psi \rangle + \langle \chi_\mathcal{B} \Psi, H_{\lambda B}^B \chi_\mathcal{B} \Psi \rangle,
\]

where

\[
H_{\lambda A}^4 := H_\lambda - |\nabla \chi_{\mathcal{A}}|^2 - |\nabla \chi_{\mathcal{B}}|^2
\]
denotes the corresponding operator restricted to the domain $\kappa A$ resp. $B$ with Dirichlet boundary condition at the boundary $|u| = \kappa^2 M$ resp. $|u| = M$. As we will see, the additional negative potential terms in (6), denoted $- V \chi$, will not cause any problems because they are supported on a region $\kappa A \cap B$ where the potential tends rapidly to infinity. Using that $N(A + B) \leq N(A) + N(B)$ for any two self-adjoint operators $A, B$, we obtain from the quadratic form expression (5) that

$$N(H_\lambda) \leq N(H_\lambda^A) + N(H_\lambda^B) = N(H_\lambda^A) + \sum_{j=1}^{4} N(H_\lambda^{B_j}).$$

Consider first the region $B_1$. Under the coordinate transformation, we find (cp. [9]) \( \Delta_{xy} = h^{-2} \Delta_{uv} \) and \( dxdy = h^2 dudv \), with scale factor \( h = (x^2 + y^2)^{-\frac{1}{2}} = 2^{-\frac{1}{4}}(u^2 + v^2)^{-\frac{1}{4}} \), so that for any $\Psi \in C^\infty_0(B_1) \otimes \mathbb{C}^2$

$$\int_{B_1} \langle \Psi, (-\Delta_{xy} + x^2 y^2 - x\gamma_1 - y\gamma_2 - \lambda \rho - V_\lambda) \Psi \rangle_{C^2} dxdy$$

$$= \int_{B_1} \langle \Psi, (-\Delta_{uv} + h^2 \rho^2 + h^2 \gamma_u - \lambda h^2 \rho - V^{uv}_\chi) \Psi \rangle_{C^2} dudv$$

$$\geq \int_{u=M}^{\infty} \int_{v=-\infty}^{\infty} \left\langle \Psi, \left( -\partial_u^2 - \partial_v^2 + \frac{v^2}{2u^2 + v^2} - \frac{1}{\sqrt{2(u^2 + v^2)}} \right)\lambda \right. \left. - \frac{1}{2\sqrt{u^2 + v^2}(1 + \sqrt{u^2 + v^2})^{\frac{3}{2}}} - V^{uv}_\chi \right\rangle_{C^2} dvdu,$$

where $\mu := h(x\gamma_1 - y\gamma_2)$, so that $\mu^2 = 1$. We have also used that

$$h^2 V_\chi = (h|\nabla_{xy} \chi A|)^2 + (h|\nabla_{xy} \chi B|)^2 = |\nabla_{uv} \chi A|^2 + |\nabla_{uv} \chi B|^2 = V^{uv}_\chi,$$

which (with a suitably chosen $\chi A$) is independent of $v$, bounded by $c_1/M^2$, and supported on $M \leq |u| \leq \kappa^2 M$. 

7
Let us think of the resulting scalar Schrödinger operator in the r.h.s. of (7), call it \( H_{uv} \), as acting on \( L^2([M, \infty), du) \otimes \mathcal{H} \) with fiber \( \mathcal{H} = L^2(\mathbb{R}, dv) \).

Let

\[
H_u := -\frac{\partial^2}{\partial v^2} + \frac{v^2}{2(u^2 + v^2)^{1/2}} - \frac{1}{\sqrt{2(u^2 + v^2)^{1/2}}},
\]

(8)
denote part of the one-dimensional Schrödinger operator acting on \( \mathcal{H} \), and observe that

\[
H_{uv}^\lambda \geq -\frac{\partial^2}{\partial v^2} + \left( H_u - \frac{\lambda}{2u(1 + 2u)^{1/2}} - V_{\chi_{uv}} \right)_- \geq -\frac{\partial^2}{\partial v^2} + \left( H_u - \frac{\lambda}{u^{11/2}} - V_{\chi_{uv}} \right)_-,
\]

(9)

where \((A)_-\) denotes the spectral projection on the negative part of the spectrum of \( A \).

First, let us make a rough estimate of the spectrum of \( H_u \) to prove that this spectral projection is one-dimensional when \( u \) is sufficiently large. Splitting \( H_u \) into three regions (again using a partition of unity), \( v < -\delta u \), \( -u < v < u \), resp. \( v > \delta u \), with a fixed \( 0 < \delta < 1 \) and Dirichlet boundary conditions at \( v = \pm u \) resp. \( v = \pm \delta u \), we find (with some constant \( c_2 \geq (1 - \delta)^2 \))

\[
H_u|_{|v| < u} \geq -\frac{\partial^2}{\partial v^2} + \frac{v^2}{2(u^2 + u^2)^2} - \frac{1}{\sqrt{2u}} - \frac{c_2}{u^2} = -\frac{\partial^2}{\partial v^2} + \frac{1}{2\pi u} v^2 - \frac{1}{\sqrt{2u}} - \frac{c_2}{u^2},
\]

whose spectrum is bounded below by \( \left\{ \frac{1}{\sqrt{2\pi u}} (2k + 1) - \frac{1}{\sqrt{2u}} - \frac{c_2}{u^2} \right\}_{k=0,1,2,...} \), while

\[
H_u|_{v > \delta u} \geq -\frac{\partial^2}{\partial v^2} + \frac{v^2}{2(\delta^2 - v^2 + v^2)^2} - \frac{1}{\sqrt{2u}} - \frac{c_2}{u^2} \geq \frac{\delta^2 u}{2\sqrt{2}} - \frac{1}{\sqrt{2u}} - \frac{c_2}{u^2},
\]

and similarly for \( H_u|_{v < -\delta u} \), so that

\[
N \left( H_u - \frac{\lambda}{u^{1+\frac{1}{2}}} \right) \leq N \left( H_u|_{v < -\delta u} - \frac{\lambda}{u^{1+\frac{1}{2}}} \right) + N \left( H_u|_{|v| < u} - \frac{\lambda}{u^{1+\frac{1}{2}}} \right) + N \left( H_u|_{v > \delta u} - \frac{\lambda}{u^{1+\frac{1}{2}}} \right) \leq 0 + 1 + 0
\]

for \( \alpha \geq 2 \) and \( u \) sufficiently large. Hence, only the ground state energy of \( H_u \) contributes to (9) when \( M \) is taken sufficiently large, e.g. \( M \geq (c_1 + c_2 + \lambda)^{\frac{1}{2}} \) with \( \delta = 0.8 \).

A sufficient bound for the ground state energy is provided by Proposition [2] below, showing that \( H_u \geq -\frac{1}{4u^2} \) for all \( u > 0 \), so that (9) becomes

\[
H_{uv}^\lambda \geq \left( -\frac{\partial^2}{\partial u^2} - \frac{1}{4u^2} \right) \otimes 1_\mathcal{H} - \left( \frac{\lambda}{u^{11/2}} + V_{\chi_{uv}} \right) \otimes P_0,
\]

8
where \( P_0 \) denotes the projection onto the ground state of \( H_u \). Applying Lemma 7, we find (after extending trivially to \([1, \infty)\))

\[
N(H_{uv}^\lambda) \leq 8\pi C_3 \int_M^{\infty} \left( \frac{\lambda}{u^{1+\frac{\alpha}{2}}} + V_{uv}^\lambda \right)^{\frac{3}{2}} u^2 \ln u^2 \, du
\leq c_3 \left( \lambda M^{-\frac{\alpha}{2}} + c_1 \right)^{\frac{3}{2}} (\ln \kappa^2 M)^2 + 8\pi C_3 \lambda^{\frac{3}{2}} \int_{\kappa M}^{\infty} u^{\frac{1}{2} - \frac{\alpha}{2}} \ln u^2 \, du,
\]

which is finite for \( \alpha > 2 \). This implies that \( N(H_{B1}^\lambda) < \infty \) and, by reflection symmetry, also \( N(H_{B1}^\lambda) < \infty \) for all \( \lambda > 0 \).

It remains to prove that \( N(H_{A2}^\lambda) < \infty \). Taking the scalar lower bound for the potential of \( H_{A2}^\lambda \),

\[
V_A^\lambda := x^2 y^2 - \sqrt{x^2 + y^2} - \lambda (1 + x^2 + y^2)^{-\frac{\alpha}{2}} - V_x,
\]

and using that \( V_x = (x^2 + y^2)V_{uv}^\lambda \leq 2\sqrt{u^2 + v^2}c_1 M^{-2} \), we have on the region \( \kappa A \)

\[
H_A^\lambda \geq -\Delta_{xy} + v^2 - \sqrt{2(\kappa^4 M^2 + v^2)^{\frac{1}{2}} - \frac{\lambda}{(1 + 2\sqrt{u^2 + v^2})^{\frac{\alpha}{2}}} - V_x
\geq -\Delta_{xy} + v^2 - \sqrt{2(\kappa^4 M^2 + v^2)^{\frac{1}{2}} - \lambda - 2\sqrt{\kappa^4 M^2 + v^2}c_1 M^{-2}},
\]

and since the potential of the Schrödinger operator on the right hand side tends to infinity as \( |x| \to \infty \Rightarrow |v| \to \infty \), it follows that the spectrum of \( H_A^\lambda \) is purely discrete, and \( N(H_A^\lambda) < \infty \).

We have proved the first statement of Theorem 1. The second statement follows from (10) with \( M = (c_1 + c_2 + \lambda)^{\frac{\alpha}{2}} \), together with the following bound for scalar Schrödinger operators in two dimensions (see e.g. Theorem 20, Chapter 8.4 in [11]):

\[
N(-\Delta + V) \leq 1 + C_q \int_{\mathbb{R}^2} |V(x)|^q \left( 1 + \ln |x||^{2q-1} |x|^{2(q-1)} \, dx,
\]

with \( q > 1 \) and \( C_q \) a positive constant. Extending \( V_A^\lambda \) by zero outside \( \kappa A \), it follows that

\[
N(H_A^\lambda) \leq 2 + 2C_q \int_{\kappa A} |V_A(x)|^q \left( 1 + \ln |x||^{2q-1} |x|^{2(q-1)} \, dx.
\]

\[3\text{Whether the bound } (12) \text{ extends to } q = 1 \text{ is currently unknown (we note that there is an error in [12]).}\]
Figure 2: The bounded region where $V^A < 0$.

For large $\lambda$, we have on the unbounded region $|v| \geq \kappa^2 M$ (similarly to (11)) that

$$V^A \geq v^2 - \sqrt{2}(2v^2)^{\frac{1}{4}} - \lambda(1 + 2|v|)^{-\frac{5}{2}} - 2(2v^2)^{\frac{1}{2}}c_1M^{-2} \geq 0.$$ 

Hence, the integral reduces to the bounded region $|u|, |v| < \kappa^2 M$, i.e.

$$N(H^A_{\lambda}) \lesssim \int_{x^2 + y^2 < \frac{1}{2} \kappa^2 M} (-V^A)^q \left( 1 + \ln \sqrt{x^2 + y^2} \right)^{2q-1} (x^2 + y^2)^{q-1} dxdy$$

$$\leq \int_0^{c_1 \lambda^\frac{1}{4}} \int_{-\pi}^\pi \left( \frac{r^4}{4} \sin^2 2\varphi + r + \lambda(1 + r^2)^{-\frac{5}{2}} + r^2c_1M^{-2} \right)^{q} \cdot (1 + \ln r)^{2q-1} r^{2q-1} drd\varphi,$$

where we switched to polar coordinates $(r, \varphi)$. Furthermore, this region increases in size with $\lambda$ at a faster rate than the geometry of the potential valleys, so we can split the integral into a central part and four narrowing regions along the valleys (see Figure 2). We obtain the bound

$$2\pi \int_0^{r_{\lambda}} \left( r + \lambda(1 + r^2)^{-\frac{5}{2}} + r^2c_1\lambda^{-\frac{1}{2}} \right)^q (1 + \ln r)^{2q-1} r^{2q-1} dr$$

$$+ 8 \int_{r_{\lambda}}^{c_1 \lambda^\frac{1}{4}} \int_{-\varphi}^{\varphi} \left( r + \lambda(1 + r^2)^{-\frac{5}{2}} + r^2c_1\lambda^{-\frac{1}{2}} \right)^q (1 + \ln r)^{2q-1} r^{2q-1} drd\varphi,$$

where $r_{\lambda}$ is the solution to

$$-\frac{1}{4}r^4 + r + \lambda(1 + r^2)^{-\frac{5}{2}} + r^2c_1M^{-2} = 0.$$
i.e. \( r_\lambda \sim \lambda^{\tfrac{4}{4+\alpha}} \), and \( \varphi_r \sim \tfrac{1}{r^4} (r + \lambda r^{-\alpha})^{\tfrac{3}{2}} \). The first integral is bounded by

\[
\int r_\lambda^2 \left( r_\lambda + \lambda + r_\lambda^2 c_1 \lambda^{-\tfrac{3}{2}} \right)^q (1 + \ln r_\lambda)^{2q-1} \leq c_6 \lambda^{\tfrac{4}{4+\alpha}} \ln^q \lambda^{2q-1},
\]

and the second by

\[
c_6 \int_{c_7 \lambda^{1+\alpha}}^{c_7 \lambda^{1+\alpha}} \frac{1}{r^2} \left( r + \lambda r^{-\alpha} + r^2 c_1 \lambda^{-\tfrac{3}{2}} \right)^q (r \ln r)^{2q-1} dr \leq \frac{c_8}{q-1} \lambda^{\tfrac{3}{4} q - \tfrac{1}{2}} \ln^q \lambda^{2q-1}.
\]

(14)

Now, for \( \alpha > 2 \) we can choose \( q \) sufficiently close to 1 to make these expressions dominated by \( o(\lambda^{\tfrac{4}{3}}) \). On the other hand, the first term of the r.h.s. of (10) is asymptotically bounded by

\[
c_9 \left( \lambda \cdot (\lambda^{2/3})^{1-\frac{4}{3q}} \right)^{\frac{3}{2}} \ln^2 \lambda = c_9 \lambda^{\tfrac{3}{4} - \tfrac{4}{3q} (\alpha - 2)} \ln^2 \lambda^2
\]

for \( 2 < \alpha \leq 5 \) and by \( c_9 \ln^2 \lambda^2 \) otherwise, and for the second term we have, with \( \alpha = 2 + \frac{4}{3} a \) and any \( 0 \leq \epsilon < 1 \),

\[
\int_{M}^\infty u^{-1-a}(\ln u)^2 du \leq M^{-\epsilon a} \int_{1}^\infty u^{-1-(1-\epsilon)a}(\ln u)^2 du \leq \lambda^{-\epsilon a} \cdot \frac{2}{((1-\epsilon)a)^3}.
\]

Summing up, we obtain

\[
N(H_\lambda) \leq C(\alpha) + 32\pi C_3 \lambda^{\tfrac{4}{2} - \epsilon(\alpha)} \cdot \frac{128}{27(\alpha - 2)^3} \forall \lambda > 0,
\]

for some constant \( C(\alpha) \) and sufficiently small \( \epsilon(\alpha) > 0 \).

It follows from Theorem 1 that the asymptotic eigenvalue distribution of the weighted Hamiltonian \( \tilde{H} \) is given by

\[
N(\tilde{H} - \lambda) \sim o(\lambda^{\tfrac{4}{3}}), \quad \lambda \to \infty,
\]

regardless of \( \alpha > 2 \). We note that the same approach can be applied to the purely bosonic model, i.e. the scalar Schrödinger operator \( H_B = -\Delta + x^2 y^2 \), with \( \alpha \geq 0 \). In this case there will be no contribution from the region \( B \) when \( M \sim \lambda^2 \), and the correct leading order eigenvalue asymptotics for \( \alpha = 0 \) (see [5]),

\[
N(H_B - \lambda) \sim \lambda^{\tfrac{4}{3}} \ln \lambda, \quad \lambda \to \infty,
\]

would be matched by the corresponding bound (13) for the central region with \( q = 1 \), while for the cut off valleys there is a bound analogous to (14).
with
\[
\int_{\lambda^{1/\alpha}}^{\lambda} \left( r^{-\alpha} + r^2 e_1 \lambda^{-4} \right)^{q+\frac{1}{2}} r^2 r^{-3} (\ln r)^{2q-1} \, dr \leq \frac{c_{10}}{q - 1} \lambda^{4q+2} + 2(q-1)(\ln \lambda)^{2q-1}.
\]

One could try to improve this by instead letting \( M \) be fixed and reconsidering the bound on the region \( B_1 \). In any case, we have for a nonzero weight that \( N(\tilde{H}_B - \lambda) \sim o(\lambda^{\frac{q}{2}}) \), \( \lambda \to \infty \).

4.2.1 Asymptotics of \( H_u \)

We conclude this section with some useful properties of the operator \( H_u \) in the limit \( u \to \infty \). By the change of variable \( v = u^{\frac{1}{4}} t \), we write
\[
\hat{H}(\epsilon) := -\partial_t^2 + \frac{t^2}{2(1 + \epsilon t^2)^{\frac{3}{4}}} - \frac{1}{\sqrt{(2(1 + \epsilon t^2)^{\frac{3}{4}}}}.
\]

**Proposition 2.** \( \hat{H}(\epsilon) \geq -\frac{\epsilon t^2}{4}, \) for all \( \epsilon > 0 \).

**Proof.** We use that for any \( f = f(t) \)
\[
(-i \partial_t + i f)(-i \partial_t - i f) \geq 0,
\]
i.e. \( -\partial_t^2 + f^2 - f' \geq 0 \). As a first attempt, let
\[
f_0 := \frac{t}{\sqrt{2(1 + \epsilon t^2)^{\frac{3}{4}}}},
\]
resulting in
\[
\hat{H}(\epsilon) \geq -\frac{\epsilon t^2}{2\sqrt{2}(1 + \epsilon t^2)^{\frac{3}{4}}}. \tag{16}
\]

While the r.h.s. is bounded and vanishes as \( \epsilon \to 0 \) pointwise, it does not so uniformly. Consider instead \( f = f_0 + \epsilon f_1 \), with
\[
f_1 := -\frac{t}{4(1 + \epsilon t^2)}.
\]

We so get the bound \((16)\) pushed to \( O(\epsilon) \):
\[
\hat{H}(\epsilon) \geq -\frac{\epsilon t^2}{2\sqrt{2}(1 + \epsilon t^2)^{\frac{3}{4}}} - 2\epsilon f_0 f_1 - \epsilon^2 f_1^2 + \epsilon f_1'
\]
\[
= -\epsilon^2 f_1^2 + \epsilon f_1' = -\epsilon \cdot \frac{1 - \frac{3\epsilon t^2}{4(1 + \epsilon t^2)^2}}{4} \geq -\frac{\epsilon}{4}.
\]
\[\square\]
Let $\hat{P}_0$ denote the projection onto the ground state of $\hat{H}_0 := \hat{H}(0)$, i.e.

$$(P_0\psi)(t) = \varphi_0(t) \int \overline{\varphi_0(\tau)}\psi(\tau) d\tau,$$

where $\varphi_0(t) = (\sqrt{2\pi})^{-\frac{1}{4}}e^{-t^2/(2\sqrt{2})}$ is its normalized wave function, and let $\hat{P}_0^\perp := 1 - \hat{P}_0$. Note that

$$\hat{H}(\epsilon) - \hat{H}_0 = -\frac{t^2}{2} \left( 1 - (1 + \epsilon t^2)^{-\frac{1}{2}} \right) + \frac{1}{\sqrt{2}} \left( 1 - (1 + \epsilon t^2)^{-\frac{1}{2}} \right)$$

and

$$\langle \varphi_0, \hat{H}(\epsilon)\varphi_0 \rangle = -\frac{\epsilon}{4} + o(\epsilon). \tag{17}$$

**Proposition 3.** $\hat{H}(\epsilon) \geq \left( -\frac{\epsilon}{4} + o(\epsilon) \right) \hat{P}_0 + c\hat{P}_0^\perp$, as $\epsilon \to 0$, where $c > 0$.

We start with

**Lemma 4.** For small $\epsilon > 0$,

$$\hat{P}_0^\perp \hat{H}(\epsilon)\hat{P}_0^\perp \geq \frac{\sqrt{2}}{2} \hat{P}_0^\perp.$$

*(Note: $\sqrt{2}$ is the excitation energy of $\hat{H}_0$.)

**Proof.** We again use a partition of unity and let $\tilde{f}_i = \tilde{f}_i(s)$, $(i = 1, 2)$ be smooth functions with $\tilde{f}_1^2 + \tilde{f}_2^2 = 1$, $\tilde{f}_2(s) = 0$ for $|s| \leq 1$, and $\tilde{f}_1(s) = 0$ for $|s| \geq 2$. Set $f_i(t) = \tilde{f}_i(t/R)$. Then

$$\hat{H}(\epsilon) = f_1\hat{H}(\epsilon)f_1 + f_2\hat{H}(\epsilon)f_2 + O(R^{-2}).$$

For large $R$, that error is $\leq \sqrt{2}/10$ and

$$\|[f_1, \partial_t^2]\hat{P}_0\|, \quad \|(1 - f_1)\hat{P}_0\|$$

have the same bound (for later use). The potential of $\hat{H}(\epsilon)$ in (15), denote it $V(\epsilon, t)$, satisfies

$$V(\epsilon, t) \geq \frac{t^2}{2(1 + t^2)^{\frac{1}{2}}} - \frac{1}{\sqrt{2}}$$

for $0 < \epsilon < 1$. This is $\geq \sqrt{2}$ for $t \in \text{supp } f_2$ if $R$ is large enough. Hence, we obtain $f_2\hat{H}(\epsilon)f_2 \geq \sqrt{2}f_2^2$. Now, for fixed $R$,

$$f_1\hat{H}(\epsilon)f_1 = f_1\hat{H}_0f_1 + O(\epsilon),$$

13
and we take \( \epsilon \) small enough that \( |O(\epsilon)| \leq \sqrt{2}/10 \). We consider
\[
\hat{P}_0^+ f_1 \hat{H}_0 f_1 \hat{P}_0^+ = f_1 \hat{P}_0^+ \hat{H}_0 \hat{P}_0^+ f_1 + (\hat{P}_0^+ f_1 - f_1 \hat{P}_0^+) \hat{H}_0 f_1 \hat{P}_0^+ + f_1 \hat{P}_0^+ \hat{H}_0 (f_1 \hat{P}_0^+ - \hat{P}_0^+ f_1).
\]
(18)

Using \( \hat{H}_0 \hat{P}_0 = 0 \) we have
\[
\hat{H}_0 (f_1 \hat{P}_0^+ - \hat{P}_0^+ f_1) = \hat{H}_0 (\hat{P}_0 f_1 - f_1 \hat{P}_0) = (f_1 \hat{H}_0 - \hat{H}_0 f_1) \hat{P}_0 = [f_1, -\partial^2_{\varphi}] \hat{P}_0
\]
for the last term of (18), and similarly for the second. Together with the bound \( \hat{P}_0^+ \hat{H}_0 \hat{P}_0^+ \geq \sqrt{2} \hat{P}_0^+ \) we conclude
\[
\hat{P}_0^+ \hat{H}(\epsilon) \hat{P}_0^+ \geq \sqrt{2}(f_1 \hat{P}_0^+ f_1 + f_2^2) - \frac{\sqrt{2}}{10} (1 + 1 + 2).
\]

Multiplying again with \( \hat{P}_0^+ \) and using
\[
\hat{P}_0^+ f_1 \hat{P}_0^+ f_1 \hat{P}_0^+ = \hat{P}_0^+ f_1^2 \hat{P}_0^+ - \hat{P}_0^+ f_1 \hat{P}_0 f_1 \hat{P}_0^+, \quad \text{and} \quad -\hat{P}_0^+ f_1 \hat{P}_0 = \hat{P}_0^+ (1 - f_1) \hat{P}_0,
\]
we obtain
\[
\hat{P}_0^+ \hat{H}(\epsilon) \hat{P}_0^+ \geq \sqrt{2} \hat{P}_0^+ (f_1^2 + f_2^2) \hat{P}_0^+ - \frac{\sqrt{2}}{2} \hat{P}_0^+.
\]

\[ \square \]

**Proof of Proposition**

We decompose
\[
\hat{H}(\epsilon) = \hat{P}_0 \hat{H}(\epsilon) \hat{P}_0 + \hat{P}_0^+ \hat{H}(\epsilon) \hat{P}_0^+ + \hat{P}_0^+ (\hat{H}(\epsilon) - \hat{H}_0) \hat{P}_0 + \hat{P}_0 (\hat{H}(\epsilon) - \hat{H}_0) \hat{P}_0^+,
\]

since \( \hat{P}_0^+ \hat{H}_0 \hat{P}_0 = 0 \). The first two terms are greater than \( -\frac{\sqrt{2}}{2} \hat{P}_0 + o(\epsilon) \) by (17), resp. \( \frac{\sqrt{2}}{2} \hat{P}_0^+ \) by Lemma 5. Expectations of the third one are bounded as
\[
|\langle \psi, \hat{P}_0^+ (\hat{H}(\epsilon) - \hat{H}_0) \hat{P}_0 \psi \rangle| \leq \| \hat{P}_0^+ \psi \| \| (\hat{H}(\epsilon) - \hat{H}_0) \hat{P}_0 \psi \| \leq c\| \hat{P}_0^+ \psi \| \| \hat{P}_0 \psi \| \\
\leq c\epsilon \left( \epsilon^{-\frac{1}{2}} \| \hat{P}_0^+ \psi \|^2 + \epsilon^\frac{1}{2} \| \hat{P}_0 \psi \|^2 \right),
\]
and so for the fourth one. Therefore,
\[
\hat{H}(\epsilon) \geq \left( -\frac{\epsilon}{4} + o(\epsilon) \right) \hat{P}_0 + \left( \frac{1}{\sqrt{2}} - c\epsilon^\frac{3}{2} \right) \hat{P}_0^+,
\]
where the second bracket is positive for \( \epsilon \) small enough. \[ \square \]
5 CLR bound for operator-valued potentials

Given a separable Hilbert space $\mathfrak{h}$, we denote by $\mathcal{S}^p(\mathfrak{h})$ the set of compact symmetric operators $A$ on $\mathfrak{h}$ s.t. $\text{tr}_\mathfrak{h} |A|^p = \text{tr}_\mathfrak{h} (A^* A)^{p/2} < \infty$. The following theorem is given as Corollary 2.4 in [10] (see also [14]):

**Theorem 5.** Let $\mathfrak{h}$ be some auxiliary Hilbert space and $V$ a potential in $L^{d/2}(\mathbb{R}^d, S^{d/2}(\mathfrak{h}))$, $d \geq 3$. Then

$$N(-\Delta \otimes 1_\mathfrak{h} + V) \leq C_d \int_{\mathbb{R}^d} \text{tr}_\mathfrak{h} |V(x)|^\frac{d}{2} \, dx$$

for some positive constant $C_d$.

It is also noted in [10] that the operator $H_d := -\Delta \otimes 1_\mathfrak{h} + V$ is self-adjoint and semi-bounded from below on the corresponding Sobolev space $H_1(\mathbb{R}^d; \mathfrak{h})$, and that $\sigma_{\text{ess}}(H_d) \subseteq [0, \infty)$.

From the above theorem we can derive the following [15]:

**Lemma 6.** Assume $\mathfrak{h}$ is an auxiliary Hilbert space and $V : [0, \infty) \to S^{\frac{d}{2}}(\mathfrak{h})$ a smooth operator-valued potential. Let $H_1 := -\partial^2 \otimes 1_\mathfrak{h} + V$ be self-adjoint and defined by Friedrichs extension on $C_0^\infty(\mathbb{R}^d; \mathfrak{h})$. Then

$$N(H_1) \leq 4\pi C_3 \int_0^\infty \text{tr}_\mathfrak{h} |V(x)| \frac{3}{2} x^2 \, dx.$$

**Proof.** Consider $N(H_1) = \sup_{W \in \mathcal{W}_1} \dim W$, where $\mathcal{W}_1$ denotes the set of linear subspaces $W \subseteq C_0^\infty(\mathbb{R}^d; \mathfrak{h}) \subseteq L^2(\mathbb{R}^d; \mathfrak{h})$ s.t. $\langle u, H_1 u \rangle < 0 \forall u \in W$. For $u \in W \in \mathcal{W}_1$ and $x \in \mathbb{R}^3$, $r := |x|$, we let $\psi(x) := \frac{1}{r} u(r)$. Then $\psi \in C_0^\infty(\mathbb{R}^3; \mathfrak{h})$ and

$$\langle \psi, H_3 \psi \rangle = \int_{\mathbb{R}^3} \langle \psi, (-\Delta_{\mathbb{R}^3} \otimes 1_\mathfrak{h} + V(|x|)) \psi \rangle_\mathfrak{h} \, dx$$

$$= |S^2| \int_0^\infty \left\langle \frac{1}{r} u(r), \left( -\frac{1}{r} \frac{\partial^2}{\partial r^2} r \otimes 1_\mathfrak{h} + V(r) \right) \frac{1}{r} u(r) \right\rangle_\mathfrak{h} \, r^2 \, dr$$

$$= 4\pi \int_0^\infty \langle u(r), \left( -\partial^2_r \otimes 1_\mathfrak{h} + V(r) \right) u(r) \rangle_\mathfrak{h} \, dr$$

$$= 4\pi \langle u, H_1 u \rangle < 0.$$

Hence, $\psi \in W' \in \mathcal{W}_3$ for some $W'$, where $\mathcal{W}_3$ denotes the corresponding set of linear subspaces $W' \subseteq C_0^\infty(\mathbb{R}^3; \mathfrak{h}) \subseteq L^2(\mathbb{R}^3; \mathfrak{h})$ s.t. $\langle \psi, H_3 \psi \rangle < 0 \forall \psi \in W'$. Also, if $u_1, u_2$ in $W \in \mathcal{W}_1$ are orthogonal, then so are the associated $\psi_1, \psi_2$.
in $W' \in \mathcal{W}_3$, so that to each $W \in \mathcal{W}_1$ there corresponds a $W' \in \mathcal{W}_3$ with
$\dim W' \geq \dim W$. Hence,

$$N(H_1) = \sup_{W \in \mathcal{W}_1} \dim W \leq \sup_{W' \in \mathcal{W}_3} \dim W' = N(H_3)$$

$$\leq C_3 \int_{\mathbb{R}^3} \text{tr}_x |V(x) - \frac{3}{2} x^2| dx = 4\pi C_3 \int_0^\infty \text{tr}_r |V(r) - \frac{3}{2} r^2| r^2 dr,$$

by Theorem 5.

Lemma 7. With $\kappa$ and $V$ as above, let $H_2 := (-\partial_x^2 - \frac{1}{4x^2}) \otimes 1_x + V$ be self-adjoint and defined by Friedrichs extension on $C_0^\infty((1, \infty); \kappa)$. Then

$$N(H_2) \leq 4\pi C_3 \int_1^\infty \text{tr}_x |V(x) - \frac{3}{2} x^2| dx.$$

Proof. We have for $u \in C_0^\infty((1, \infty); \kappa)$ that

$$\langle u, H_2 u \rangle = \int_1^\infty \left( \|u'(x)\|_\kappa^2 - \frac{1}{4x^2}\|u(x)\|_\kappa^2 + \langle u(x), V(x)u(x) \rangle_\kappa \right) dx.$$

Note that

$$\| (\partial_x - \frac{1}{2x}) u(x) \|_\kappa^2 = \|u'(x)\|_\kappa^2 - \frac{1}{2x} (\langle u'(x), u(x) \rangle_\kappa + \langle u(x), u'(x) \rangle_\kappa) + \frac{1}{4x^2}\|u(x)\|_\kappa^2,$$

so that after integrating by parts,

$$\langle u, H_2 u \rangle = \int_1^\infty \left( \| (\partial_x - \frac{1}{2x}) u(x) \|_\kappa^2 + \langle u(x), V(x)u(x) \rangle_\kappa \right) dx.$$

We can write $u(x) = x^{1/2} v(x)$, with $v \in C_0^\infty((1, \infty); \kappa)$, implying

$$\langle u, H_2 u \rangle = \int_1^\infty \left( \|x^{1/2} v'(x)\|_\kappa^2 + \langle v(x), xV(x)v(x) \rangle_\kappa \right) dx.$$

Put $t := \ln x$ and $w(t) := v(e^t)$. Then $w \in C_0^\infty((0, \infty); \kappa)$ and

$$\langle u, H_2 u \rangle = \int_0^\infty \left( \|w'(t)\|_\kappa^2 + \langle w(t), e^{2t}V(e^t)w(t) \rangle_\kappa \right) dt = \int_0^\infty \langle w(t), (-\partial_t^2 \otimes 1_x + e^{2t}V(e^t)) w(t) \rangle_\kappa dt.$$
We also note that there is a 1-to-1 correspondence between linearly independent sets of such \( u \in C_0^\infty((1, \infty); \hat{h}) \) and \( w \in C_0^\infty((0, \infty); \hat{h}) \). Applying Lemma [8] with the potential \( W(x) = e^{2x}V(e^x) \) we find

\[
N(H_2) = N \left( -\partial_x^2 \otimes 1 + e^{2x}V(e^x) \right)
\leq 4\pi C_3 \int_0^\infty \text{tr}\left| (e^{2x}V(e^x))_+ \right|^\frac{3}{2} x^2 \, dx
= 4\pi C_3 \int_1^\infty \text{tr}\left| V(s)_- \right|^\frac{3}{2} s^2 (\ln s)^2 \, ds,
\]

where we substituted \( s := e^x \).

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