Minimal kinetic theory: a mathematical framework for non-equilibrium flowing matter

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Abstract. We discuss the intriguing ability of minimal kinetic theory to describe a broad variety of complex non-equilibrium flows across scales of motion. It is argued that, besides major computational progress, minimal kinetic theory also provides a new conceptual framework to investigate the complexities of flowing matter far from equilibrium.

1. Introduction

Statistical physics is the branch of theoretical physics in charge of unraveling the connections between the macroscopic world, as we perceive it, and its microscopic constituents (atoms, molecules and below). Kinetic theory, in turn, is the sector of statistical physics which deals with the transport phenomena by which macroscopic systems attain their thermodynamic equilibrium [1, 2]. The undisputed hero is Ludwig Boltzmann (1844-1906), who first laid down the mathematical bridge between the micro and macroworlds. The Boltzmann equation, the mathematical cornerstone of kinetic theory, is however very difficult to solve, on account of its high-dimensionality (six-dimensional phase-space plus time), non-linearity and non-locality in velocity space. Moreover, it was derived for dilute gases only, and the extension to dense fluids and liquids has proven pretty difficult beyond expectation [3].

In the last two decades, however, several forms of minimal kinetic theory (MKT), have been developed, which provide a very fertile background not only for the numerical simulation of complex non-equilibrium phenomena, but also for their conceptual study. In this paper, we wish to provide a brief survey of the leading themes behind these developments.

In the recent years, MKT, and most notably the Lattice Boltzmann method (LB), have made proof of an amazing and largely unanticipated capability of straddling across a broad range of fluid scales, from fully developed turbulence, all the way down to nanoscopic flows of biological interest, and lately even quantum-relativistic subnuclear fluids, spanning over ten spatial decades in the process [4].

This multi-scale capability often takes MKT into a theoretically 'unprotected' territory, where the conditions for a rigorous analysis of the continuum limit (Chapman-Enskog asymptotics) are no longer available [5]. For simplicity, we shall call this the non-hydrodynamic regime, which is typically but not exclusively, associated with the molecular mean free-path becoming comparable with the smallest macroscopic length-scale.

This non-hydrodynamic capability is still only partially understood and represents one the most interesting aspects of modern MKT. In this paper, starting from the by now largely
documented versatility of LB, we shall argue that, besides its practical impact as a computational
tool, the most far-reaching contribution of MKT is to provide a conceptually elegant and
computationally powerful mathematical framework for the study of non-equilibrium phenomena
in flowing matter across many scales of motion, including the quantum/classical border [6].

2. Boltzmann’s kinetic theory
Let us begin by writing the Boltzmann equation (BE) in its standard form (vector notation
relaxed for simplicity):
\[
\frac{df}{dt} = \frac{1}{\tau} Q(f, f)
\]
where \( df/dt \equiv \partial_t f + v \nabla_x f + a \nabla_v f \) is the streaming term in phase-space and \( C \equiv Q(f, f)/\tau \) is
the collision term, where the characteristic collisional time scale \( \tau \) is exposed for dimensional
clarity.

The central lengthscale of Boltzmann’s kinetic theory is the molecular mean-free-path, i.e.
the averaged distance traveled by a representative molecule before hitting into another molecule.
The mean-free path is given by
\[
\lambda \sim v_{th} \tau
\]
where \( v_{th} = \sqrt{Dk_B T/m} \) is the thermal speed in \( D \) spatial dimensions, \( m \) the mass of the molecule,
\( T \) the temperature and \( k_B \) Boltzmann’s constant. The collisional time \( \tau \), in turn, is given by
\[
\tau = n v_{th} \sigma(v_{th})
\]
where \( n \) is the number density of the gas and \( \sigma \) its cross section (length square), namely the
effective "size" of the molecule as dynamically perceived by its partner molecules in a collisional
event. For short-range potentials such as those pertaining to the Boltzmann scenario, \( \sigma \sim r_0^2 \),
\( r_0 \) being the range of the interaction potential.

Regardless of the complexity of the interaction potential, the collision operator obeys mass-
momentum-energy conservation laws, namely:
\[
\int C(f, f) \{1, mv_a, \frac{mv^2}{2} \} d^3v = 0
\]
where the latin subscript \( a = x, y, z \) labels the three spatial coordinates. In other words, mass,
momentum and energy, are microscopic invariant, i.e. they are left unchanged by molecular
collisions, no matter how complicated. This insensitivity to molecular details (Universality)
stands at the very roots of macroscopic hydrodynamics. Another central property of the collision
operator is the fact of vanishing at local equilibrium:
\[
C(f^{eq}, f^{eq}) = 0
\]
This condition implies a dynamical balance between direct and inverse collisions, sometimes also
called Loss and Gain terms (a direct collision takes a molecule away from the phase-space element
\( dxdv \), while an inverse one puts it back). Due to the existence of the microscopic invariants,
the local equilibrium takes the universal Maxwell-Boltzmann distribution (for classical=non
quantum and non-relativistic particles):
\[
f^{eq}(x, v; t) = n \frac{e^{-c^2/2}}{(2\pi v_{th}^2)^{D/2}}
\]
where \( c \equiv \frac{c-u}{v_{th}} \), \( u \) being the local flow speed, is the so-called peculiar speed, in units of the thermal
speed. Note that the five macroscopic quantities, \( (n, u_a, T) \) associated with the microscopic
invariants can be arbitrary functions of space and time, as long as their scale of variation
remains well above the mean-free path, (slow modes, at the molecular scale).

Finally, Boltzmann’s collision operator obeys the famous H-theorem, namely:

\[ \dot{s} \equiv - \int C(f,f) \log f \, dv \geq 0 \] (5)

where \( \dot{s} \) is the local rate of entropy production. The H-theorem stands as possibly the deepest
outcome of Boltzmann’s equation, offering the first mathematical bridge between the hitherto
disjoint fields of mechanics and thermodynamics. A bridge over ”troubled waters” indeed, as
history of science informs us, and yet a monumental one [7].

Conceptually, the BE is limited to dilute gases with binary short-range collisions, so that
triple collisions of the form \( Q(f,f,f) \) and higher can safely be neglected at the right-hand side.
This formally rules out dense gases and liquids, or, in a modern language, strongly interacting
fluids, in which the mean-free path is comparable with the range of molecular interactions.

Even so, the BE is computationally highly demanding, for it lives in \( 6 + 1 \) dimensions and
collisions come in the form of a quadratic integral operator in velocity space. Because of the
above, the use of the Boltzmann equation has been traditionally confined to the study of weakly
interacting fluids, i.e. dilute gases and rarefied states of matter in which kinetic energy dominates
over interparticle interactions. The computational tool of the trade is Bird’s Direct Simulation
Monte Carlo [8], a stochastic particle technique whereby computational molecules are advanced
along single particle trajectories and undergo binary collisions according to probabilistic rules
controlled by the cross section and the resulting molecular mean free path.

3. Minimal kinetic theory

The idea of minimal kinetic theory is not new: model equations aiming at relinquishing the
mathematical complexity of the BE without surrendering the main physics, have been available
for over half a century. Perhaps, the most popular one is the celebrated Bhatnagar-Gross-Krook
version, in which the quadratic integral \( Q(f,f) \) is replaced by the departure of the distribution
from its local equilibrium [9], namely:

\[ C_{BGK} = (f^{eq} - f)/\tau \]

What is new though, is the realization of the breadth of complex non-equilibrium problems that
can be handled by suitable generalizations of such minimal models. In this respect, the lattice
has proven an invaluable ally in affording computational efficiency, even though the conceptual
picture disclosed by MKT is not restricted to lattice realizations.

Lattice versions have shown that the hydrodynamic complexity of macroscopic equations,
typically Navier-Stokes, can be captured by a very stylized representation of velocity space. This
is a gift of collisional invariants, which secure lasting order out of molecular chaos (microscopic
conservation laws). In the following we shall refer to the Lattice Boltzmann equation (LBE)
for the sake of concreteness, but other forms of minimal kinetic theory, say Stochastic Rotation
methods [10] and Dissipative Particle Dynamics [11], inscribe to the same conceptual framework.

The LBE, in BGK form for simplicity, reads as follows [12, 13, 14, 15]

\[ f_i(x + c_i \Delta t, c_i; t + \Delta t) - f_i(x, c_i; t) = \omega (f_i^{eq} - f_i) + \gamma S_i \] (6)

where the index \( i \) labels the set of discrete velocities. In the above \( \omega = \Delta t/\tau \) is the ratio
of free-streaming version collision timescales, \( S_i \) is a source term associated with external/internal
forces and \( \gamma = \Delta/\tau_S \) is a measure of the strength of the external versus collisional relaxation.

The lattice local equilibrium comes in the form of a truncated polynomial expansion of the
local Maxwell-Boltzmann distribution in the local Mach number. This expansion is typically, but
Figure 1. The D2Q9 lattice, i.e. nine discrete velocities in two spatial dimensions. The lattice affords fourth order isotropy and permits to recover the isothermal, quasi-incompressible two-dimensional Navier-Stokes equations.

Figure 2. The D3Q19 lattice, i.e. nineteen discrete velocities in three spatial dimensions. The lattice provides fourth order isotropy and permits to recover the isothermal, quasi-incompressible three-dimensional Navier-Stokes equations.

not necessarily, truncated to second order since the Navier-Stokes equations are quadratically non linear. The lattice local equilibrium reads as follows:

\[ f_i^{eq}(x; t) = \sum_{k=0}^{2} H_{ik} M_k^{eq}(x; t) \]  

(7)

where \( M_0^{eq} = \rho \) is the fluid mass density, \( M_1^{eq} = J = \rho u \) is the flow current density and \( M_2^{eq} = \rho u^2 + P \) is the momentum flux, \( P \) being the fluid pressure. In \( D \) spatial dimensions, the current \( J_a \) is a \( D \)-dimensional vector and the momentum flux \( P_{ab} \) is a second-rank tensor. In the above \( H_{ik} \) is a set of lattice basis functions in velocity space, the analogue of Hermite polynomials \( H_k(v) \), evaluated at \( v = c_i \).

Once the set of discrete velocities \( c_i \) (typically of order 10 in two dimensions and 20 in three), provides enough symmetry to recover the isotropy of fourth order lattice tensors, and the lattice local equilibria ensure local mass and momentum conservation, the lattice BGK equation can be shown to reproduce the (weakly-compressible) Navier-Stokes equations of macroscopic hydrodynamics in the limit of small Knudsen numbers. This is usually defined in terms of the collisional relaxation time, i.e \( \lambda = c_s \tau \), where \( c_s \) is the lattice sound speed, typically \( O(1) \) in lattice units. In the strongly coupled (hydrodynamic) regime, the mean free path is typically much smaller than the lattice spacing, i.e. \( \lambda << \Delta x \). As a result, the effective Knudsen number is most appropriately defined as \( Kn = \Delta x / L \), \( L \) being the smallest macroscale of the problem.

This means that near-grid scales \( L \sim \Delta x \) are not guaranteed to obey macroscopic hydrodynamics, a point to which we shall return shortly. For now, let’s stay with the bona-fide statement that for lengthscales such that \( Kn << 1 \), the LBE provides a quantitative description of Navier-Stokes hydrodynamics in kinetic vests.

The LB has made proof of a series of remarkable properties, both conceptual and computational. Without delving into specialistic details, in the following we shall try to convey the essence of the story.

4. The benefits of extra-dimensions

By a mere counting of degrees of freedom, the fact that just ten or twenty discrete distributions \( f_i \) are sufficient to recover hydrodynamics is hardly a surprise. Indeed, it is well known from
continuum kinetic theory that hydrodynamics is just the emergent tip of a kinetic iceberg involving a virtually infinite hierarchy of kinetic moments:

\[ M_n(x; t) = \int f(x, v; t) H_n(v) dv \]  

where \( H_n(v) \) is a suitable basis in kinetic (Hilbert) space, typically Hermite’s polynomials. Tensor notation is again relaxed for simplicity, but it should be borne in mind that subscript \( n \) labels \( n \)-th order tensors.

The point, however, is not the number of degrees of freedom, but rather their structural organization! Kinetic theory exhibits a very elegant organization, directly subtended by the Stream-Collide structure of the Boltzmann equation, which contrasts with the Advection-Diffusion form of macroscopic transport equations.

In a catchy motto, in the stream-collide kinetic paradigm: Non-linearity is local and non-locality is linear, a property which is inevitably lost by any hydrodynamic representation.

Differently restated, Stream-Collide in (minimal) phase space is more transparent (and efficient) than the emergent Advection-Diffusion picture of macroscopic transport.

Let us spell this important point out.

4.1. Streaming versus Advection

The kinetic moments obey recursive PDE’s of the form:

\[ \partial_t M_n + \nabla_x N_{n+1} = C_n \]  

where \( M_n(x; t) = \int f(x, v; t) H_n(v) dv \) is the kinetic moment of order \( n \), \( N_{n+1} = \int f(x, v; t) v H_n(v) dv \) is the corresponding current and \( C_n \) is the collisional contribution to the rate of change of \( M_n \). By setting \( v = u + c \), where \( u \) is the local fluid velocity (space-time dependence omitted for simplicity)

\[ J_{n+1} = M_n u + J'_{n+1} \]  

where

\[ J'_{n+1} = \int f(x, u + c; t) c H_n(u + c) dc \]  

is the non-equilibrium component of the generalized current. The key point is the following; leaving aside the non-equilibrium current, information travels along the material streamlines defined by the flow velocity \( u(x; t) \). In a complex flow, say turbulence, or multiphase flows with complex interfaces, the material streamlines are overly complex, as they carry the complexity of macroscopic hydrodynamics, as expressed by the non-linear inertial acceleration:

\[ A_a = u_b \nabla_b u_a \]

where repeated indices are summed upon. Now, contrast this with the peaceful form of the (force-free) kinetic streaming operator:

\[ \hat{S} f \equiv v_a \nabla_a f \]

All the complexity is now embedded in \( f(x, v; t) \) and the streaming is happily unaware of the macroscopic complexity in configuration spacetime \( (x, t) \): information always travels along the straight streamlines \( dx_a = v dt \) as opposed to the material line \( dx_a = u(x; t) dt \).

The price to pay are the extra degrees of freedom: there are always more discrete velocities than hydrodynamic moments, but the reward is major: streaming is linear and can be solved exactly. Thus instead of \( u \nabla u \) which is non-linear and non-local at a time, in kinetic theory, non-linearity is local (collision operator) and non-locality is linear (streaming operator).
4.2. Relaxation versus Diffusion

It is well known that hydrodynamics results from kinetic theory in the limit where the momentum flux tensor \( P_{ab} = m \int f v_a v_b d^3v \) is enslaved to its hydrodynamic equilibrium form, \( P_{eq,ab} = \rho u_a u_b + p \delta_{ab} \), where \( p = p(\rho,T) \) is the pressure. As a result of this enslaving, dissipative forces take the form of a second-order spatial derivatives, coupled to the fluid viscosity:

\[
F_D^a \sim \tau \partial_b \partial_c Q_{eq}^{abc} = \nu \partial_b \partial_c (J_a \delta_{bc})
\]

In the above, \( Q_{eq}^{abc} = \int f^{eq} v_a v_b v_c d^3v \) is the third order energy-flux tensor, evaluated at local equilibrium. We have further taken \( \tau \) and \( \nu = c_s^2 \) as constant.

Second order spatial derivatives in combination with first order time derivatives imply irreversibility, hence dissipation.

In kinetic theory, irreversibility and dissipation are built-in in the collision operator via the H-theorem, securing monotonic entropy growth to equilibrium and compliance with the Second Principle of thermodynamics. To be noted that the H-theorem does not impose any non-local structure, such as the Laplacian. This is another major advantage of the kinetic formulation: dissipation is local, as it results from local enslaving of the momentum flux tensor, with no need of second order spatial derivatives. This is computationally convenient, especially near solid boundaries, as it permits to keep the boundary procedure first-order in space. It is also appealing from the analytical viewpoint, as it permits to work with discrete propagators of the form

\[
P_i(k,\omega) = \left[ e^{j(k_\gamma - \omega)} - 1 \right]^{-1},
\]

which are Galilean invariant at all orders. In the above, \( j \) stands for the imaginary unit, to avoid confusion with the discrete velocity index \( i \).

5. Beyond hydrodynamics

The discussion so far has been confined to macroscopic hydrodynamics. While it is very pleasing to observe that MKT has shown major competitiveness on a subject, computational fluid dynamics, which was considered a "no fly zone" for the Boltzmann equation, one may still argue that this is "just" a computational achievement.

This author would hardly subscribe to the "just" above, but this is another paper. So, let us proceed to show how MKT can eventually be taken into the theoretically unprotected non-hydrodynamic regime.

Hydrodynamics emerges from Boltzmann’s kinetic theory in the limit of vanishing Knudsen number, i.e. weak non-equilibrium, weak inhomogeneities. The theoretical backup is known as Chapman-Enskog asymptotics. Given that LB was originally targeted to hydrodynamics only, some authors have concluded that LB cannot and actually -should- not be used outside this specific realm. Yet, for more than a decade, a wide body of evidence has been accumulated showing that suitable LB extensions do carry valuable non-equilibrium information beyond the hydrodynamic level.

These extensions involve larger sets of discrete velocities [16], ensuring isotropy to six or eighth order, in combination with boundary conditions catering for momentum exchange between the fluid molecules and solid walls [17]. This is again not so surprising after all, since higher-order lattices contain many more discrete distributions than the thirteen moments of the standard Grad’s generalized hydrodynamics [18].

However, as discussed earlier on, the mere counting of the degrees of freedom alone does not say much: the right structural organization matters way more. Indeed the discrete-velocity organization gives rise to realizable boundary conditions, a property which is not shared by generalized hydrodynamics.

This is basically the Streaming versus Advection item discussed earlier on.

One further point regards the so called Regularization procedure [19, 20].

This consists in filtering out non-hydrodynamic moments after the streaming step, the one fueling non-equilibrium.
In equations:

(i) Stream: \( f_i(x,t) = f'_i(x - c_i \Delta t, t - \Delta t) \)

(ii) Regularize: \( h_i(x,t) = \mathcal{P} f_i(x,t) \)

(iii) Collide: \( f'_i(x,t) = (1 - \omega) h_i(x,t) + \omega h^q_i(x,t) \)

In the above, \( \mathcal{P} \), denotes a projector in kinetic space, filtering out the non-hydrodynamic (ghost) modes, i.e. \( \mathcal{P} f = h \) and \( (1 - \mathcal{P}) f = g \), \( f = h + g \). By hydrodynamic modes, we imply here density, momentum and momentum-flux, while all remaining moments classify under the ghost rubric (the kinetic moments lacking a direct macroscopic interpretation). In this respect, the hydro-ghost representation follows in the steps of Grad’s 13-moment generalized hydrodynamics [18], with two crucial twists: i) it contains a larger set of kinetic moments, ii) these moments are organized by the symmetry of the discrete lattice and not by the Hermite expansion in Hilbert space.

While still under exploration, the Regularized LB shows encouraging promise to reproduce at least the global features of finite-Knudsen flows, such as the transition from collective hydrodynamics to single-particle molecular regimes in finite Knudsen flows between parallel plates.

\[ \text{Figure 3. A high-order lattice featuring 93 discrete velocities ensuring 8th order isotropy. Courtesy of A. Montessori.} \]

\[ \text{Figure 4. The Knudsen paradox: Flow across a plate channel as a function of the Knudsen number for various LB Models. DnQm stands for LB in d-spatial dimensions with m discrete velocities. BB and KBC stand for bounce (no-slip) and Kinetic boundary conditions, while R stands for Regularized. Some Regularized LB’s prove capable of reproducing the Knudsen paradox, i.e. an increasing flow rate beyond a given Knudsen number, in close agreement with the Direct Simulation Monte Carlo (DSMC) solution of the Boltzmann equation. CEL and CEH refer to Cercignani’s asymptotic solutions in the low (high) Knudsen regimes, respectively. Finally, NSE denotes the Navier-Stokes solution. Courtesy of A. Montessori.} \]

Does this mean that LB can reproduce the BE?
The answer is a most certain NO!
What it means though, is that non-hydro kinetic models can be formulated which provide useful quantitative information, without resorting to the full Boltzmann equation.

This is a recurrent theme in MKT: filling the (many) gaps where hydrodynamics is no longer adequate, and yet full molecular details are not needed either. In this case, solving the full BE would be a computational overkill.

MKT has sometimes been heralded under the catchy motto "simple models for complex flows". This is nice, but only works as long as Universality stands by our side; out of equilibrium Universality recedes, by definition. The good news is that it often does it in graceful way, i.e. Universality is only weakly broken: *weakly broken universality* is possibly the richest hunting ground for MKT. In this sense, MKT really seems to offer a new computational and *conceptual* path in non-equilibrium statistical mechanics.

By "new path" we imply discrete features which can not be reconduced to direct discretizations of the Boltzmann equation.

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**Figure 5.** Sketch of the conceptual scenario by which minimal kinetic theory would generate new kinetic equations in the continuum. Take a LBE descending directly from BE by discretization (not all LBE’s do!), generalize it in the minimal (lattice in this figure) framework (GLBE), and export it back to the continuum (GBE). This gives rise to new kinetic equations, which may or may not be derivable as an extension of the BE in the continuum (empty horizontal arrow). Some GLBE’s may not admit a well-defined continuum limit, in which case they should be regarded as discrete dynamical models of complex moving matter, in their own right.

A very elegant example in this direction is offered by recent developments of the entropic version of LB, see [22]. The authors show that the LBE is a direct implication of kinetic theory when relaxation terms are retained to all orders, a step which is usually not taken in the context of standard continuum kinetic theory.

The Regularization procedure provides another example of discrete benefits which are not (at least, not easily) returned to the continuum. Indeed regularized versions of Grad’s 13-moment generalized hydrodynamic have been developed in the last decades [18, 21]. Even though such R13 regularizations do indeed mend Grad’s original weaknesses, i.e. bulk instability and hardly realizable boundary conditions, they still face with a number of difficulties. Arguably, much of these difficulties might stem from the fact of sticking to the advection-diffusion paradigm, rather than the stream-collide kinetic formalism. For exact LB solutions beyond Navier-Stokes, see [23].

**6. Minimal kinetic theory for dense fluids**

A major limitation of the Boltzmann’s equation is its (alleged) lack of applicability beyond the context of dilute gases. Indeed, extensions of Boltzmann equations to dense fluids and liquids
have proven exceedingly difficult, to the point that, historically, the research community in the mid 50’s turned to Molecular Dynamics simulation.

To a large extent, this picture is still with us: use Navier-Stokes as much as you can, typically by resort to generalized boundary conditions, and stick to the macroscopic PDE’s as much as possible.

The point is that dense fluids exhibit non-perturbative effects on both equilibrium (equation of state) and non-equilibrium (transport coefficients) properties. Capturing these corrections would, in principle, require the solution of the full BBGKY hierarchy, or at least the two-body Liouville equation. Helas, this is a 12+1 dimensional problem...

Some authors have argued that, being in fact a cartoon/caricature of Boltzmann kinetic theory, MKT inherits all of its limitations, at best. This intuitive view proves however overly simplistic and misses a great deal of interesting physics.

Indeed, somehow paradoxically, suitably enriched "caricatures" of the Boltzmann equation prove more general than the original Boltzmann equation itself!

This should by no means sound as a detrimental statement towards the BE: quite the opposite, it speaks even more assertively for the greatness of the Boltzmann equation! Indeed, it uncovers a whole world of mathematical richness which allows the BE equation to spread its wings across an amazingly broad spectrum of applications, far beyond the original framework of dilute gases!

So, how does one incorporate many (two)-body physics in MKT?

Here, one turns to another chapter of kinetic theory where Boltzmann is flanked by a towering figure of modern kinetic theory, Anatoly Vlasov (1908-1975).

The Boltzmann-Vlasov equation is characterized by the appearance of a mean-field force in the streaming operator:

\[ \partial_t f + v \nabla_x f + a \nabla_v f = C \]

where \( a = F/m \) is the acceleration experienced by a representative molecule. In the simplest instance, this is just an electric or gravitational field, but most interesting physics is opened up by generalizations in the spirit of density functional theory.

Perhaps, the most popular one is the so-called pseudo-potential approach, which takes the form [24]:

\[ a(x) = G \psi(x) \sum_i w_i \psi(x + r_i) \]

In the above, \( \psi(x) = \psi[\rho(x)] \) is a local functional of the fluid density, sometimes also known as generalized density, and \( G \) is the coupling strength associated with the pseudo-potential. This expression, with suitable choices of the generalized density, has proven immensely valuable for the study of a broad variety complex multiphase and multicomponent flows. Once again, a subject where one would have not even dreamt of using Boltzmann-like equations until two decades ago!

As an example, out of a huge class of applications, we mention here recent study of soft glassy materials, such as foams and emulsions [26], a rich and fascinating subject where the complexity of interfacial dynamics stands in the way of most analytical and numerical methods.

Unlike Regularized LB, here one can apply a "reverse MKT" argument, i.e return the lattice benefits back to the continuum, but in the form of new continuum equations.

More precisely, we argue that a Vlasov term of the form:

\[ a(x) = \psi(x) \int G(x, y) \psi(y) dy \]

may prove extremely useful for the study of complex flows with dynamic phase-transitions, no need to solve it on the lattice! To the best of our knowledge, the study of such kinetic equations in the continuum is largely (maybe totally) unexplored, both analytically and numerically.
Figure 6. LB simulation of a dry foam. The blue region represents the gas phase, with the liquid flowing along the sides of the polygon (Plateau borders). The simulation was performed via a two-component Shan-Chen LB with competing short-range attraction and mid-range repulsion [25]. Courtesy of M. Sbragaglia.

Figure 7. LB simulation of a wet foam. Here the fraction of liquid is 0.5, with the result that the Plateau borders invade the bulk fluid and take the shape of liquid droplets within gas droplets. Courtesy of M. Sbragaglia.

Of course, the approach remains heuristic, in that the "Green function" $G(x, y)$ is not derived ab-initio from the two-body Liouville equation, and most notably from the two-body radial correlation function. Nevertheless, a top-down approach, whereby one would experiment with a whole class of plausible model Green functions, might deliver a wealth of new knowledge in non-equilibrium thermodynamics.

And again, the progress is conceptual in the first place, no compelling need to solve the equations on the lattice.

7. Down to the atomistic scale
Here, we start by stating the obvious: MKT is not Molecular Dynamics!
For one, it does not contain molecular fluctuations. However, it is liable to generalizations whereby fluctuations can be inserted back, essentially through a stochastic source term. Again nothing new under the sky, fluctuating kinetic theory was proposed half a century ago by a number of authors, particularly Fox and Uhlenbeck [27]. Such fluctuating kinetic equations surely played a major role in advancing our basic knowledge of fluctuating nano-hydrodynamics. No less goes for fluctuating LB, as pioneered by A.J.C Ladd [28] and subsequently improved by several other authors, which have opened entirely new perspectives in the study of colloidal suspensions and nanofluids in general.

8. Summary
It is sometimes said that the scientific import of an equation is proportional to the conceptual distance between its left and right hand side. Very true for Boltzmann: non-equilibrium versus equilibrium, respectively, the "Evershifting battle" in Boltzmann’s own words. But it is also true that the actual physics is not contained in the equations themselves, but in their solutions
instead. And solutions for truly complex problems are preciously rare. Thus, it appears plausible
to argue that the scientific import of an equation rests largely with its amenability to workable
approximations. Possibly, one of the most profound merits of minimal kinetic theory is to have
unveiled the greatness of the Boltzmann equation in this respect too. Given how appreciative
Boltzmann himself was of practical aspects of science, I cherish the idea that he would be
delighted at the unsuspected advances spawned by suitable ”caricatures” of his own equation in
the last two decades.

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