Dynamical Reflection Maps

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Abstract

In this paper, by making use of category theory, we construct dynamical reflection maps, solutions to a version of the reflection equation associated with suitable dynamical Yang-Baxter maps, set-theoretic solutions to the braid relation that is equivalent to a version of the quantum Yang-Baxter equation. Quiver-theoretic solutions to the reflection equation are also discussed.

1 Introduction

The quantum Yang-Baxter equation \[2, 3, 34\] has been studied intensively in mathematics and physics. Much research for finding solutions to this equation gave birth to the quantum groups \[6, 10\], examples of non-commutative and non-cocommutative Hopf algebras.

The quantum Yang-Baxter equation is defined on the tensor product \(V \otimes V \otimes V\) of a vector space \(V\). Instead of the tensor products, Drinfel’d \[7, \text{Section 9}\] proposed to study the quantum Yang-Baxter equation on the Cartesian product \(S \times S \times S\) of a set \(S\), whose solution is called a Yang-Baxter map \[8, 19, 32, 33\]. This Yang-Baxter map plays an important role in discrete integrable systems \[1\].

The quantum Yang-Baxter equation is exactly the braid relation \[3.15\] in the tensor category consisting of \(\mathbb{C}\)-vector spaces, and, from this viewpoint,
the Yang-Baxter map is regarded as a solution to the braid relation in the
tensor category consisting of sets. We can consider the braid relation in
another tensor category $\text{Set}_H$ [28, 29], which is embedded into the tensor
category $\text{Quiv}_H$ consisting of quivers over a nonempty set $H$ (For $\text{Set}_H$ and
$\text{Quiv}_H$, see Sections 2 and 9 respectively).

Suitable ternary operations [12, 13, 27], dynamical braces [20], and braided
semigroups [21] produce dynamical Yang-Baxter maps [20, 27], solutions to
the braid relation in $\text{Set}_H$. This dynamical Yang-Baxter map is useful in
discrete integrable systems [17]. Moreover, by means of the dynamical Yang-
Baxter map, we can construct quiver-theoretic solutions to the braid relation
[22] and Hopf algebroids [23, 30], which give birth to rigid tensor categories
consisting of finite-dimensional dynamical representations [30].

We will try to investigate so-called dynamical reflection equation algebras
[18] associated with the dynamical Yang-Baxter maps. As a first step, in
order to confirm that this dynamical reflection equation algebra is rich in
representations, this paper focuses on the reflection equation in [5, Eq.(10)].
A set-theoretic version of this reflection equation is considered in [4], which
clarifies a way to construct reflection maps, (set-theoretic) solutions to this
reflection equation, by means of actions of skew left braces (For skew left
braces, see Definition 8.12).

The aim of this paper is to construct dynamical reflection maps, solutions
to the reflection equation (3.26) in the tensor category $\text{Set}_H$, associated with
the dynamical Yang-Baxter map.

Let $L$ be a left quasigroup with a unit (Definition 6.2), and let $G$ be a
group isomorphic to $L$ as sets. We set $H = L$. This left quasigroup $L$ with
a unit can give birth to a monoid $L = (L, m, \eta)$ in $\text{Set}_H$ (See Definition 4.1).
Let $\sigma : L \otimes L \to L \otimes L$ be a dynamical Yang-Baxter map (6.1) defined by
the ternary operation $\mu^G_1$ (6.1) related to the group $G$. By means of a left
$(L, m, \eta)$-module $(X, m_X)$ in $\text{Set}_H$ and a family of homomorphisms of the
group $G$, we can produce a dynamical reflection map $k : L \otimes X \to L \otimes X$
associated with the dynamical Yang-Baxter map $\sigma : L \otimes L \to L \otimes L$ (See
Theorem 6.10 and the end of Section 6). As an application, we construct
quiver-theoretic solutions to the reflection equation from the dynamical re-
flection maps (Proposition 9.2).

The organization of this paper is as follows.

Section 2 contains a brief summary of the tensor category $\text{Set}_H$, in which
we will consider the braid relation and the reflection equation. In Section 3
we present a way to construct dynamical reflection maps, by generalizing the
method in [4] from the viewpoint of category theory.

We introduce the notions of monoids in Section 4 and their left modules
in Section 5 which play a key role in constructing dynamical reflection maps.
in this paper. In particular, braided monoids in Definition 4.2 and braid-commuting pairs of left modules in Definition 5.4 are essential. The relations in Section 3 imply these notions (See Remarks 5.5 and 5.6 also). The tensor category $\text{Set}_H$ has an advantage in constructing desirable braided monoids (See Proposition 6.9).

Our main result, Theorem 6.10, is stated in Section 6 and proved in Section 7. By means of this theorem, suitable left modules and group homomorphisms can produce the dynamical reflection maps.

In Section 8 we provide several examples. The end of this section deals with reflection maps also. In Section 9 the final section, we prove that the dynamical reflection maps give birth to quiver-theoretic solutions to the reflection equation, following [22].

2 Tensor category $\text{Set}_H$

This section contains a brief summary of the tensor category $\text{Set}_H$ [28, 29], which plays an important role in this paper. About the tensor categories, we follow the definition and the notation of [15, Chapter XI] throughout the paper.

Definition 2.1. A tensor category $(\mathcal{C}, \otimes, I, a, l, r)$ is a category $\mathcal{C}$, together with:

1. a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called a tensor product;

2. a natural isomorphism $a : \otimes(\otimes \times \text{id}) \to \otimes(\text{id} \times \otimes)$ called an associativity constraint;

3. an object $I$ called a unit;

4. a natural isomorphism $l : \otimes(I \times \text{id}) \to \text{id}$ called a left unit constraint with respect to the unit $I$;

5. $r : \otimes(\text{id} \times I) \to \text{id}$ called a right unit constraint with respect to the unit $I$,

satisfying the pentagon axiom and the triangle axiom (See [15, Definition XI.2.1]): for $U, V, W, X \in \mathcal{C}$;

\[(1) \quad (1_W \otimes a_{VW}) \circ a_{UV \otimes W} \circ (a_{UV} \otimes 1_X) = a_{UVW} \circ a_{V \otimes VW} X;\]

\[(2) \quad (1_V \otimes l_W) \circ a_{VW} = r_V \otimes 1_W.\]
We now recall the definition of \( \mathcal{Set}_H \). Let \( H \) be a nonempty set. \((X, \cdot_X)\) is an object of the category \( \mathcal{Set}_H \), iff \( X \) is a set and \( \cdot_X \) is a map from \( H \times X \) to \( H \). For \((X, \cdot_X), (Y, \cdot_Y) \in \mathcal{Set}_H \), we call \( f : (X, \cdot_X) \to (Y, \cdot_Y) \) a morphism of \( \mathcal{Set}_H \), iff \( f \) is a map from \( H \) to \( \text{Map}(X,Y) \) satisfying

\[
\lambda \cdot_Y f(\lambda)(x) = \lambda \cdot_X x \quad (\forall \lambda \in H, \forall x \in X).
\]

Here, we denote by \( \text{Map}(X,Y) \) the set of all maps from \( X \) to \( Y \). For morphisms \( f : (X, \cdot_X) \to (Y, \cdot_Y) \) and \( g : (Y, \cdot_Y) \to (Z, \cdot_Z) \) of \( \mathcal{Set}_H \), we define the composition \( g \circ f : (X, \cdot_X) \to (Z, \cdot_Z) \) by \((g \circ f)(\lambda) = g(\lambda) \circ f(\lambda) \) \( \lambda \in H \).

Then \( \mathcal{Set}_H \) is a category with the following identity \( 1_{(X,X)} : (X, \cdot_X) \to (X, \cdot_X) \): For \((X, \cdot_X) \in \mathcal{Set}_H \), the map \( 1_{(X,X)} : H \to \text{Map}(X,X) \) is defined by \( 1_{(X,X)}(\lambda)(x) = x \) \( \lambda \in H, x \in X \).

The next task is to show that \( \mathcal{Set}_H \) is a tensor category. Let \( X = (X, \cdot_X) \) and \( Y = (Y, \cdot_Y) \) be objects in \( \mathcal{Set}_H \). We define the tensor product \((X \otimes Y, \cdot_{X \otimes Y})\) by:

\[
X \otimes Y = (X \times Y, \cdot_{X \otimes Y}); \lambda \cdot_{X \otimes Y} (x,y) = (\lambda \cdot_X x, \cdot_Y y) \quad (\forall \lambda \in H, \forall (x,y) \in X \times Y).
\]

This \( X \otimes Y \) is an object of \( \mathcal{Set}_H \).

The tensor product \( f \otimes g : (X, \cdot_X) \otimes (X', \cdot_{X'}) \to (Y, \cdot_Y) \otimes (Y', \cdot_{Y'}) \) of two morphisms \( f : (X, \cdot_X) \to (Y, \cdot_Y) \) and \( g : (X', \cdot_{X'}) \to (Y', \cdot_{Y'}) \) is defined by

\[
(f \otimes g)(\lambda)(x,y) = (f(\lambda)(x), g(\lambda \cdot_X x)(y)) \quad (\lambda \in H, (x,y) \in X \times Y).
\]

It is easily seen that \( f \otimes g \) is a morphism of \( \mathcal{Set}_H \).

The associativity constraint \( a : \otimes(\otimes \times \text{id}) \to \otimes(\text{id} \times \otimes) \) is given as follows: for \( X, Y, Z \in \mathcal{Set}_H \),

\[
a_{X,Y,Z}(\lambda)((x,y),z) = (x,(y,z)) \quad (\lambda \in H, ((x,y),z) \in ((X \times Y) \times Z)).
\]

Let \( I = \{\bullet\} \) denote a set of one element \( \bullet \), together with the map \( \cdot_I : H \times I \to H \) defined by \( \lambda \cdot_I \bullet = \lambda \) \( \lambda \in H \). This \( I = (I, \cdot_I) \) is an object in \( \mathcal{Set}_H \), which is called a unit of \( \mathcal{Set}_H \).

The left and right unit constraints \( l : \otimes(I \times \text{id}) \to \text{id} \) and \( r : \otimes(\text{id} \times I) \to \text{id} \) with respect to the unit \( I \) are given by:

\[
l_X(\lambda)(\bullet, x) = x, \quad r_X(\lambda)(x, \bullet) = x \quad (\lambda \in H, x \in X)
\]

for every \( X \in \mathcal{Set}_H \).

We can check that \((\mathcal{Set}_H, \otimes, I, a, l, r)\) is a tensor category.
3 Reflection equation

In this section, we establish a way to construct solutions to the reflection equation in tensor categories with suitable properties. Finally we will present a sufficient condition for one of these properties in the case of the tensor category \( \text{Set} \).

Let \( C \) be a tensor category (Definition 2.1).

**Proposition 3.1.** Let \( A, X \in C \), and write \( Y = A \otimes X \). Suppose that morphisms \( m : A \otimes A \to A, \eta : I \to A, m_X : A \otimes X \to X, m_Y : A \otimes Y \to Y, \) and \( \sigma : A \otimes A \to A \otimes A \) of \( C \) satisfy

(3.1) \( m(1_A \otimes \eta)r_A^{-1} = 1_A \),
(3.2) \( m_X(\eta \otimes 1_X) = l_X \),
(3.3) \( m_Y(m \otimes 1_Y) = m_Y(1_A \otimes m_Y)a_{AAY} \),
(3.4) \( m_Y(1_A \otimes m_Y^{triv}) = m_Y^{triv}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)a_{AAY}^{-1} \),
(3.5) \( m_Xm_Y = m_X(1_A \otimes m_X) \).

Here,

(3.6) \( m_Y^{triv} := (m \otimes 1_X)a_{AAX}^{-1} : A \otimes Y \to Y \).

Then the morphism

(3.7) \( k = m_Y(1_A \otimes ((\eta \otimes 1_X)l_X^{-1})) : A \otimes X \to A \otimes X \)

of \( C \) enjoys:

(3.8) \( m_Xk = m_X \);
(3.9) \( k(m \otimes 1_X) = (m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX} \).

**Proof.** We can easily prove (3.8) by taking account of (3.2), (3.5), and (3.7). It follows from (3.1), (3.4), (3.6), and (3.7) that

(3.10) \( m_Y = (m \otimes 1_X)a_{AAX}^{-1}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1} \).

In fact, the right-hand-side of (3.10) is

\[ m_Y^{triv}(1_A \otimes m_Y)a_{AAY}(\sigma \otimes 1_Y)(1_A \otimes (\eta \otimes 1_X))(1_A \otimes l_X^{-1})a_{AAX}^{-1} \]

by (3.6) and (3.7). (3.4) yields that the right-hand-side of the above equation is \( m_Y(1_A \otimes m_Y^{triv})a_{AAY}(1_A \otimes (\eta \otimes 1_X))(1_A \otimes l_X^{-1})a_{AAX}^{-1} \), which is exactly \( m_Y \) on account of (2.2), (3.1), and (3.6).
Proposition 3.2. Let $A, X \in \mathcal{C}$, and set $Y = A \otimes X$. We assume that morphisms $m : A \otimes A \to A, \eta : I \to A, m_\chi : A \otimes X \to X, m_\nu : A \otimes Y \to Y$, and $\sigma : A \otimes A \to A \otimes A$ of $\mathcal{C}$ satisfy (3.3) and

\[(3.10) \quad m(\eta \otimes 1_A)l_A^{-1} = 1_A = m(1_A \otimes \eta)r_A^{-1},
\]

\[(3.12) \quad m(\eta \otimes 1_A) = m(1_A \otimes \eta),
\]

\[(3.13) \quad (m \otimes 1_A)a_{AA}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A) = \sigma(1_A \otimes m)a_{AAA},
\]

\[(3.14) \quad \sigma(1_A \otimes \eta) = (\eta \otimes 1_A)l_A^{-1}r_A,
\]

\[(3.15) \quad a_{AAA}(\sigma \otimes 1_A)a_{A}\overrightarrow{\lambda}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)
\]

\[= (1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)a_{A}\overrightarrow{\lambda}(1_A \otimes \sigma)a_{AAA},
\]

\[(3.16) \quad m_Y(1_A \otimes m_\nu^x) = m_Y^x(1_A \otimes m_\nu)a_{AA}(\sigma \otimes 1_Y)a_{A\overrightarrow{\lambda}}.
\]

Here,

\[(3.17) \quad m_Y^x = (1_A \otimes m_\chi)a_{AAA}(\sigma \otimes 1_X)a_{A\overrightarrow{\lambda}} : A \otimes Y \to Y.
\]

If $\sigma : A \otimes A \to A \otimes A$ is an isomorphism, then the morphism $k : A \otimes X \to A \otimes X$ (3.7) enjoys

\[(3.18) \quad k(1_A \otimes m_\chi)
\]

\[= (1_A \otimes m_\chi)a_{AA}(\sigma \otimes 1_X)a_{A\overrightarrow{\lambda}}(1_A \otimes k)a_{AA}(\sigma \otimes 1_X)a_{A\overrightarrow{\lambda}}.
\]

Remark 3.3. We will prove Proposition 3.2 by refering to the proof of [4, Lemma 7.4], which was discussed with the help of graphical calculation. From the viewpoint of category theory, we proceed with a proof of Proposition 3.2.

Proof. In view of (3.12), the right-hand-side of (3.18) is

\[(3.19) \quad ((m(\eta \otimes 1_X)l_X^{-1}) \otimes 1_X)(1_A \otimes m_\chi)a_{AA}(\sigma \otimes 1_X)a_{A\overrightarrow{\lambda}}(1_A \otimes k)a_{AA}(\sigma \otimes 1_X)a_{A\overrightarrow{\lambda}}
\]

\[= (m \otimes m_\chi)a_{A\otimes AA}\overrightarrow{\lambda}(1_A \otimes \sigma)a_{AAA}((\sigma \otimes 1_A)a_{A\overrightarrow{\lambda}}(1_A \otimes k)\sigma a_{A\otimes AA}\overrightarrow{\lambda}(1_A \otimes \sigma)a_{AAA}((\sigma \otimes 1_A)a_{A\overrightarrow{\lambda}}(1_A \otimes k)) \otimes 1_A)a_{A\overrightarrow{\lambda}}.
\]

In fact, we can prove it by using (((\eta \otimes 1_A)l_A^{-1}) \otimes 1_A)\sigma = a_{A\otimes AA}(1_A \otimes \sigma)a_{AAA}((\eta \otimes 1_A)l_A^{-1}) \otimes 1_A), which comes from the fact that $l_{A\otimes A} = (l_A \otimes 1_A)a_{A\overrightarrow{\lambda}}$ [15, Lemma XI.2.2].
Because \( \sigma \) is an isomorphism and satisfies (3.14), the right-hand-side of (3.19) is

\[
(m \otimes m_X) a_{A \otimes AAX}((a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}) \otimes 1_X) a^{-1}_{A \otimes AAX} (1_A \otimes k) a_{A \otimes AAX}
\]

\[
((\sigma \otimes 1_A) \otimes 1_X)(((\sigma^{-1} \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)) \otimes 1_X)
\]

\[
(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX},
\]

which is

\[
(3.20) \quad (1_A \otimes m_X) a_{AAX}(((m \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)) \otimes 1_X)
\]

\[
a^{-1}_{A \otimes AAX} (1_A \otimes k) a_{A \otimes AAX}
\]

\[
((a_{AAA}^{-1}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma^{-1})a_{AAA}) \otimes 1_X)
\]

\[
(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX}
\]

owing to (3.17). On account of (3.13) and (3.17), (3.20) is

\[
(3.21) \quad m^\sigma_Y a_{AAX}(((1_A \otimes m)a_{AAA}) \otimes 1_X) a^{-1}_{A \otimes AAX} (1_A \otimes k) a_{A \otimes AAX}
\]

\[
((a^{-1}_{AAA}(1_A \otimes \sigma)a_{AAA}(\sigma \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma^{-1})a_{AAA}) \otimes 1_X)
\]

\[
(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX}
\]

\[
=m^\sigma_Y (1_A \otimes ((m \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X))) a_{AAX}\]

\[
((a_{AAA}(\sigma \otimes 1_A)a_{AAA}^{-1}(1_A \otimes \sigma^{-1})a_{AAA}) \otimes 1_X)
\]

\[
(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX}.
\]

We note that (3.10) holds because of (3.4), (3.7), and (3.12). It follows from (3.10) that the right-hand-side of (3.21) is

\[
m^\sigma_Y (1_A \otimes m_Y)a_{AAX}((\sigma \otimes 1_Y)a_{AAX}) a^{-1}_{AAA}(1_A \otimes \sigma^{-1})a_{AAA} \otimes 1_X)(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX},
\]

which is exactly the same as

\[
(3.22) \quad (m \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a_{AAX}^{-1}
\]

\[
(1_A \otimes ((1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X))) a_{AAX} \otimes 1_X)
\]

\[
(((1_A \otimes \sigma^{-1})a_{AAA}^{-1}(1_A \otimes \sigma^{-1})) \otimes 1_X) \otimes 1_X)a^{-1}_{AAX}
\]

\[
=m(\otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}((\sigma \otimes 1_X)a_{AAX}^{-1}
\]

\[
(1_A \otimes ((1_A \otimes m_X)a_{AAX})) a_{AAX} \otimes 1_X)
\]

\[
(((1_A \otimes \eta)r^{-1}_A) \otimes 1_A) \otimes 1_X)a^{-1}_{AAX}
\]

\[
=m(\otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}((\sigma(1_A \otimes \eta)) \otimes 1_X)a_{AAX}^{-1}
\]

\[
(1_A \otimes ((1_A \otimes m_X)a_{AAX})) a_{AAX} \otimes 1_X)((r^{-1}_A \otimes 1_A) \otimes 1_X)
\]

\[
a^{-1}_{AAX}
\]

in view of (3.7), (3.16), and (3.17).
Then satisfy (3.23) in much the same way. By using (3.12), together with the triangle axiom (2.2) and $l_{A\otimes X} = (l_A \otimes 1_X)a_{IAX}^{-1}$ [15, Lemma XI.2.2], we deduce that the above morphism is the left-hand-side of (3.18). This proves the proposition. □

The equation (3.15) is exactly the braid relation in $C$.

**Proposition 3.4.** For $A, X \in C$, we suppose that morphisms $m : A \otimes A \to A, m_X : A \otimes X \to X, \sigma : A \otimes A \to A \otimes A$, and $k : A \otimes X \to A \otimes X$ of $C$ satisfy (3.8), (3.9), (3.18), and

$$m\sigma = m.$$ (3.23)

Then

$$m \otimes 1_X)A_{AAX}(1_A \otimes k)A_{AAX}(\sigma \otimes 1_X)A_{AAX}(1_A \otimes k)A_{AAX}$$ (3.24)

$$= (m \otimes 1_X)A_{AAX}(1_A \otimes k)A_{AAX}(\sigma \otimes 1_X)A_{AAX}(1_A \otimes k),$$

(3.25)

$$= (1_A \otimes m_X)A_{AAX}(\sigma \otimes 1_X)A_{AAX}(1_A \otimes k)A_{AAX}(\sigma \otimes 1_X)A_{AAX}(1_A \otimes k).$$

**Proof.** It follows from (3.9) that the left-hand-side of (3.24) is $k(m \otimes 1_X)(\sigma \otimes 1_X)A_{AAX}$, which is $k(m \otimes 1_X)A_{AAX}$ because of (3.23). By using (3.9) again, we conclude that $k(m \otimes 1_X)A_{AAX}$ is the right-hand-side of (3.24).

By making use of (3.8) and (3.18) instead of (3.9) and (3.23), we can show (3.25) in much the same way. □

A pair of morphisms $(f : X \to Y, g : X \to Y)$ of $C$ is called monic, iff this pair satisfies the following condition:

$$fh = fh' \text{ and } gh = gh' \text{ (h, h' : A \to X) induce h = h'}. $$

This notion is a generalization of the usual monomorphism.

**Corollary 3.5.** For $A, X \in C$, we suppose that morphisms $m : A \otimes A \to A, m_X : A \otimes X \to X, \sigma : A \otimes A \to A \otimes A$, and $k : A \otimes X \to A \otimes X$ of $C$ satisfy (3.8), (3.9), (3.18), and (3.23). If the pair $(m \otimes m, (1_A \otimes m_X)m_{AAX})$ of
morphisms is monic, then \( k : A \otimes X \to A \otimes X \) is a solution to the reflection equation associated with \( \sigma \):

\[
(3.26) \quad a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k) = (\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k).
\]

**Proof.** From Proposition 3.4, (3.8), (3.23), (3.24), and (3.25),

\[
\begin{align*}
(m \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX} \quad & = (m \otimes 1_X)(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k), \\
(1_A \otimes m_X)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X) \quad & = (1_A \otimes m_X)(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}(1_A \otimes k)a_{AAX}(\sigma \otimes 1_X)a^{-1}_{AAX}.
\end{align*}
\]

Because the pair \((m \otimes 1_X, (1_A \otimes m_X)a_{AAX})\) is monic, (3.26) follows. \(\square\)

The dynamical reflection map is a solution to the reflection equation (3.26) in the tensor category \(\text{Set}_H\).

We now present a sufficient condition for the pair \((m \otimes 1_X, (1_A \otimes m_X)a_{AAX})\) of the tensor category \(\text{Set}_H\) to be monic.

**Proposition 3.6.** For \( A, X \in \text{Set}_H \), let \( m : A \otimes A \to A \) and \( m_X : A \otimes X \to X \) be morphisms of \(\text{Set}_H\). If the maps \( A \ni b \mapsto m(\lambda)(a, b) \in A \) are injective for any \( \lambda \in H \) and \( a \in A \), then the pair \((m \otimes 1_X, (1_A \otimes m_X)a_{AAX})\) is monic.

**Proof.** Let \( \lambda \in H, a_1, a_2, b_1, b_2 \in A, \) and \( x_1, x_2 \in X \). It suffices to prove that \((m \otimes 1_X)(\lambda)(a_1, b_1, x_1) = (m \otimes 1_X)(\lambda)(a_2, b_2, x_2)\) and \((1_A \otimes m_X)a_{AAX}(\lambda)(a_1, b_1, x_1) = ((1_A \otimes m_X)a_{AAX})(\lambda)(a_2, b_2, x_2)\) induce \((a_1, b_1, x_1) = (a_2, b_2, x_2)\).

From these equations

\[
\begin{align*}
(m(\lambda)(a_1, b_1), x_1) = (m(\lambda)(a_2, b_2), x_2), \\
(a_1, m_X(\lambda a_1)(b_1, x_1)) = (a_2, m_X(\lambda a_2)(b_2, x_2)).
\end{align*}
\]

Thus \( a_1 = a_2, x_1 = x_2, \) and \( m(\lambda)(a_1, b_1) = m(\lambda)(a_2, b_2) = m(\lambda)(a_1, b_2) \). Because the map \( A \ni b \mapsto m(\lambda)(a, b) \in A \) is injective, \( b_1 = b_2 \). This completes the proof. \(\square\)

The above proposition is useful for the construction of dynamical Yang-Baxter maps in Section 5 (See Proposition 6.9).
4 Braided monoids

In this section, we focus on the relations (3.1) and (3.12)–(3.14), which can produce monoids in the tensor category (Cf. [4, Section 5]).

Let $C$ be a tensor category.

**Definition 4.1.** An object $A$ of $C$, together with morphisms $m : A \otimes A \to A$ and $\eta : I \to A$, is a monoid, iff the morphisms satisfy (3.12) and

\[ m(m \otimes 1_A) = m(1_A \otimes m)a_{AAA}. \]

This monoid is also called a ring in [14, Definition 4.3.1].

**Definition 4.2.** A monoid $(A, m, \eta)$ with a morphism $\sigma : A \otimes A \to A \otimes A$ is braided, iff the morphisms $m$, $\eta$, and $\sigma$ satisfy (3.13), (3.14), and

\[ \sigma(\eta \otimes 1_A) = (1_A \otimes \eta)r^{-1}_A, \]

\[ (1_A \otimes m)a_{AAA}(\sigma \otimes 1_A) = (\sigma \otimes 1_A)a^{-1}_{AAA}; \]

\[ \sigma(1_A \otimes \eta) = (1_A \otimes \eta)r^{-1}_A. \]

For braided semigroups in the tensor category, see [21].

**Proposition 4.3.** If $(A, m, \eta, \sigma)$ is a braided monoid in $C$, then $A \otimes A$ is a monoid in $C$ with morphisms:

\[ m_{A \otimes A} = (m \otimes m)a_{A \otimes A \otimes A}(a^{-1}_{A\otimes A \otimes A}(1_A \otimes \sigma) \otimes 1_A) \]

\[ (a_{AAA} \otimes 1_A)a^{-1}_{A \otimes A \otimes A}; \]

\[ \eta_{A \otimes A} = (\eta \otimes \eta)r^{-1}_A. \]

**Proof.** For the proof, it suffices to show:

\[ m_{A \otimes A}(m_{A \otimes A} \otimes 1_{A \otimes A}) = m_{A \otimes A}(1_{A \otimes A} \otimes m_{A \otimes A})a_{A \otimes A \otimes A \otimes A}; \]

\[ m_{A \otimes A}(\eta_{A \otimes A} \otimes 1_{A \otimes A})l^{-1}_{A \otimes A} = 1_{A \otimes A} = m_{A \otimes A}(1_{A \otimes A} \otimes \eta_{A \otimes A})r^{-1}_{A \otimes A}. \]

On account of (3.14) and (3.3), it is a simple matter to show (4.7). We prove (4.6) only, when the tensor category $C$ is strict [15, Definition XI.2.1]; hence, we assume in this proof that the associativity constraint $a$ and the unit constraints $l, r$ are all identities.

From (4.4), the left-hand-side of (4.6) is

\[ (m \otimes m)(1_A \otimes \sigma \otimes 1_A)((m \otimes m)(1_A \otimes \sigma \otimes 1_A)) \otimes 1_{A \otimes A} \]

\[ (m \otimes m)(1_A \otimes \sigma(m \otimes 1_A) \otimes 1_A)(m \otimes 1_{A \otimes A \otimes A})(1_A \otimes \sigma \otimes 1_{A \otimes A \otimes A}). \]
By virtue of (4.2), the right-hand-side of the above equation is

\[
(m \otimes m)(1_A \otimes ((1_A \otimes m)(\sigma \otimes 1_A)(1_A \otimes \sigma)) \otimes 1_A)(m \otimes 1_{A \otimes A \otimes A \otimes A})
\]

\[
(1_A \otimes \sigma \otimes 1_{A \otimes A \otimes A})
\]

\[
= (m(m \otimes 1_A) \otimes 1_A)(1_{A \otimes A \otimes A} \otimes m(m \otimes 1_A))
\]

\[
(1_{A \otimes A} \otimes ((\sigma \otimes 1_A)(1_A \otimes \sigma)) \otimes 1_A)(1_A \otimes \sigma \otimes 1_{A \otimes A \otimes A}),
\]

which coincides with

\[
(4.8) \quad (m(1_A \otimes m) \otimes 1_A)(1_{A \otimes A \otimes A} \otimes m(1_A \otimes m))
\]

\[
(1_{A \otimes A} \otimes ((\sigma \otimes 1_A)(1_A \otimes \sigma)) \otimes 1_A)(1_A \otimes \sigma \otimes 1_{A \otimes A \otimes A})
\]

\[
= (m \otimes 1_A)(1_{A \otimes A} \otimes m(1_A \otimes m))
\]

\[
(1_A \otimes ((m \otimes 1_A)(1_A \otimes \sigma)(\sigma \otimes 1_A)) \otimes 1_{A \otimes A})(1_{A \otimes A \otimes A} \otimes \sigma \otimes 1_A)
\]

in view of (4.1). Because of (3.13), the right-hand-side of (4.8) is

\[
(m \otimes 1_A)(1_{A \otimes A} \otimes m(1_A \otimes m))(1_A \otimes (\sigma(1_A \otimes m)) \otimes 1_{A \otimes A})(1_{A \otimes A \otimes A} \otimes \sigma \otimes 1_A)
\]

\[
= (m \otimes m)(1_A \otimes \sigma \otimes 1_A)(1_{A \otimes A} \otimes ((m \otimes m)(1_A \otimes \sigma \otimes 1_A))),
\]

which is exactly the right-hand-side of (4.6) with the aid of (4.4). This completes the proof.

The monoid in the above proposition is called a twisted monoid and denoted by $A_{tw}$.

## 5 Left modules of monoids

In this section, we introduce the notion of left modules of monoids in an arbitrary tensor category $\mathcal{C}$ (Cf. [4, Sections 7 and 8]).

Let $(A, m, \eta)$ be a monoid in $\mathcal{C}$ (Definition 4.1).

**Definition 5.1.** An object $X$ of $\mathcal{C}$ with a morphism $m_X : A \otimes X \to X$ is a left $A$-module in $\mathcal{C}$, iff the morphisms satisfy (3.2) and

\[
m_X(m \otimes 1_X) = m_X(1_A \otimes m_X)a_{AAX}.
\]

We note that (3.3) holds, if $(Y, m_Y)$ is a left $A$-module in $\mathcal{C}$.

**Proposition 5.2.** For a monoid $(A, m, \eta)$ and its left module $(X, m_X)$ in $\mathcal{C}$, we set $Y = A \otimes X$. Then $(Y, m_Y^{\text{triv}})$ is a left $A$-module. Here, $m_Y^{\text{triv}} : A \otimes Y \to Y \in \text{Set}_H$ is defined by (3.6).
Proof. Combining (3.6) and (4.1) with the pentagon axiom (2.1) yields $m_Y^{triv}(m \otimes 1_Y) = m_Y^{triv}(1_A \otimes m_Y^{triv})a_{AY}$. By using (3.6) and (3.12), together with $l_Y = (l_A \otimes 1_X)a_{AX}^{-1}$ [15, Lemma XI.2.2], we see at once that $m_Y^{triv}(\eta \otimes 1_Y) = l_Y$. □

**Proposition 5.3.** For a braided monoid $(A, m, \eta, \sigma)$ (Definition 4.2) and its left module $(X, m_X)$ in $\mathcal{C}$, we set $Y = A \otimes X$. Then $(Y, m_Y')$ is a left $A$-module. Here, $m_Y' : A \otimes Y \to Y \in \text{Set}_H$ is defined by (3.17).

**Proof.** We prove the relations below:

(5.2) \[ m_Y'(m \otimes 1_Y) = m_Y'(1_A \otimes m_Y')a_{AY}; \]

(5.3) \[ m_Y'(\eta \otimes 1_Y) = l_Y. \]

We only prove (5.2). On account of (3.17) and (5.1), the right-hand-side of (5.2) is

(5.4) \[ (1_A \otimes (m_X(1_A \otimes m_X)))a_{AY}(\sigma \otimes 1_Y)a_{AY}^{-1} \]

\[ = (1_A \otimes (a_{AX}(\sigma \otimes 1_X)a_{AX}^{-1})))a_{AY} \]

\[ = (1_A \otimes m_X)a_{AX}((1_A \otimes m)a_{AAA}(\sigma \otimes 1_A)a_{AX}^{-1}(1_A \otimes \sigma)) \otimes 1_X) \]

\[ a_{AX}^{-1}(1_A \otimes a_{AX})a_{AY}. \]

By using (1.2), we see that the right-hand-side of (5.4) is

(5.5) \[ (1_A \otimes m_X)a_{AX}((\sigma(m \otimes 1_A)a_{AX}^{-1}) \otimes 1_X)a_{AX}^{-1}(1_A \otimes a_{AX})a_{AY}, \]

which is exactly the left-hand-side of (5.2) because of the pentagon axiom (2.1) and $Y = A \otimes X$. □

**Definition 5.4.** Let $(V, m_V)$ and $(V, m'_V)$ be left modules of a monoid $(A, m, \eta)$ in $\mathcal{C}$, and let $\sigma : A \otimes A \to A \otimes A$ be a morphism of $\mathcal{C}$. A pair $(m_V, m'_V)$ braid-commutes, iff $m_V(1_A \otimes m'_V) = m'_V(1_A \otimes m_V)a_{AV}(\sigma \otimes 1_Y)a_{AV}^{-1}$.

**Remark 5.5.** Let $(X, m_X)$ and $(Y, m_Y)$ be left modules of a monoid $(A, m, \eta)$ in $\mathcal{C}$, and let $\sigma : A \otimes A \to A \otimes A$ be a morphism of $\mathcal{C}$. We note that (3.4) holds, if and only if the pair $(m_Y, m_Y^{triv})$ braid-commutes.

**Remark 5.6.** Let $(X, m_X)$ and $(Y, m_Y)$ be left modules of a braided monoid $(A, m, \eta, \sigma)$ in $\mathcal{C}$. Then (3.16) is a necessary and sufficient condition for $(m_Y, m_Y')$ to braid-commute.

Let $(A, m, \eta, \sigma)$ be a braided monoid in $\mathcal{C}$ and let $(X, m_X)$ be a left $(A, m, \eta)$-module in $\mathcal{C}$. Write $Y = A \otimes X$.

We will use Theorems 5.7, 5.8 and Corollary 5.9 for the proof of Corollary 7.12.
**Theorem 5.7.** Let \((Y, m_Y)\) be a left \((A, m, \eta)\)-module in \(C\) satisfying \([3.5]\) and that the pair \((m_Y, m_Y^{\text{triv}})\) braid-commutes (For \(m_Y^{\text{triv}}\), see \([3.6]\)). We set
\[
(5.5) \quad \theta_Y = m_Y^{\text{triv}}(1_A \otimes m_Y) a_{AA Y} : (A \otimes A) \otimes Y \to Y.
\]

Then \((Y, \theta_Y)\) is a left module of the twisted monoid \(A \otimes A\) (See the proof of Proposition \([4.3]\) below) in \(C\) satisfying:
\[
(5.6) \quad m_X \theta_Y = m_X(1_A \otimes m_X) a_{AA X}(1_{A \otimes A} \otimes m_X);
(5.7) \quad \theta_Y((1_A \otimes \eta) \otimes 1_Y) = (m \otimes 1_X) a_{AA X}^{-1}(r_A \otimes 1_Y).
\]

**Proof.** For simplicity we assume that the tensor category \(C\) is strict \([15]\), Definition XI.2.1]. In order to prove that \((Y, \theta_Y)\) is a left module of \(A \otimes A\), we show:
\[
(5.8) \quad \theta_Y(m_{A \otimes A} \otimes 1_Y) = \theta_Y(1_{A \otimes A} \otimes \theta_Y);
(5.9) \quad \theta_Y(\eta_{A \otimes A} \otimes 1_Y) = 1_Y.
\]

For the morphisms \(m_{A \otimes A}\) and \(\eta_{A \otimes A}\), see \([14.4]\) and \([14.5]\).

We can easily show \((5.9)\) by means of \((5.5)\), Proposition \([5.2]\), and the fact that \((Y, m_Y)\) is a left \((A, m, \eta)\)-module.

We next prove \((5.8)\). On account of Remark \([5.5]\) and \((5.5)\), the right-hand-side of \((5.8)\) is
\[
(5.10) \quad m_Y^{\text{triv}}(1_A \otimes (m_Y(1_A \otimes m_Y^{\text{triv}})))(1_{A \otimes A} \otimes 1_A \otimes m_Y)
\]

\[
= m_Y^{\text{triv}}(1_A \otimes (m_Y^{\text{triv}}(1_A \otimes m_Y)(\sigma \otimes 1_Y)))(1_{A \otimes A} \otimes 1_A \otimes m_Y)
\]

\[
= m_Y^{\text{triv}}(1_A \otimes m_Y^{\text{triv}})(1_A \otimes 1_A \otimes (m_Y(1_A \otimes m_Y)))(1_A \otimes \sigma \otimes 1_{A \otimes Y}).
\]

Because \((Y, m_Y)\) and \((Y, m_Y^{\text{triv}})\) are left \((A, m, \eta)\)-modules (See Proposition \([5.2]\)), the right-hand-side of \((5.10)\) is
\[
m_Y^{\text{triv}}(m \otimes 1_Y)(1_A \otimes 1_A \otimes (m_Y(m \otimes 1_Y)))(1_A \otimes \sigma \otimes 1_{A \otimes Y})
\]

\[
= m_Y^{\text{triv}}(1_A \otimes m_Y)(m \otimes 1_{A \otimes Y})(1_A \otimes 1_A \otimes m \otimes 1_Y)(1_A \otimes \sigma \otimes 1_{A \otimes Y}),
\]

which is exactly
\[
\theta_Y(m \otimes 1_{A \otimes Y})(1_A \otimes 1_A \otimes m \otimes 1_Y)(1_A \otimes \sigma \otimes 1_{A \otimes Y})
\]

\[
= \theta_Y(m \otimes m \otimes 1_Y)(1_A \otimes \sigma \otimes 1_{A \otimes Y})
\]

in view of \((5.5)\), \((5.8)\) thus holds.

From \((5.5)\), \((5.6)\), \((5.7)\), and \((5.5)\), we can show \((5.6)\). By means of \((3.6)\), \((5.5)\), and the fact that \((Y, m_Y)\) is a left \((A, m, \eta)\)-module, we see that \((5.7)\) holds. The proof is therefore complete. \(\square\)
Theorem 5.8. Let $(Y, \theta_Y)$ be a left $A \otimes A$-module satisfying (5.6) and (5.7). Write

\begin{equation}
(5.11) \quad m_Y = \theta_Y((\eta \otimes 1_A)(\xi^{-1}) \otimes 1_Y) : A \otimes Y \to Y.
\end{equation}

Then $(Y, m_Y)$ is a left $A$-module in $C$ satisfying (3.5) and that the pair $(m_Y, m_Y^{\text{triv}})$ braid-commutes.

Proof. We assume that the tensor category $C$ is strict. On account of (3.2) and (5.6), it is a simple matter to show (3.5).

We next prove that $(Y, m_Y)$ is a left $A$-module. For this purpose, we show the relation below only.

\begin{equation}
(5.12) \quad m_Y(m \otimes 1_Y) = m_Y(1_A \otimes m_Y).
\end{equation}

On account of (1.4) and the fact that $(Y, \theta_Y)$ is a left $A \otimes A$-module, the right-hand-side of (5.12) is

\begin{align}
(5.13) \quad \theta_Y(1_{A \otimes A} \otimes \theta_Y)((\eta \otimes 1_A) \otimes 1_A \otimes 1_{A \otimes Y}) & = \theta_Y(m_{A \otimes A} \otimes 1_Y)((\eta \otimes 1_A) \otimes 1_A \otimes 1_{A \otimes Y})

& = \theta_Y(1_A \otimes m \otimes 1_Y)((\eta \otimes 1_A) \otimes 1_A \otimes 1_Y)

& = \theta_Y((1_A \otimes \sigma)(1_{A \otimes A} \otimes \eta)) \otimes 1_{A \otimes Y}((\eta \otimes 1_A) \otimes 1_{A \otimes Y}).
\end{align}

From (5.12) and (5.14),

\begin{align}
(5.14) \quad (m \otimes 1_A)(1_A \otimes \sigma)(1_{A \otimes A} \otimes \eta) = 1_{A \otimes A}.
\end{align}

The right-hand-side of (5.13) is hence $\theta_Y(1_A \otimes m \otimes 1_Y)((\eta \otimes 1_A) \otimes 1_A \otimes 1_Y)$, which is exactly the same as the left-hand-side of (5.12).

The task is now to prove that the pair $(m_Y, m_Y^{\text{triv}})$ braid-commutes. For the proof, we show that either side of (3.4) is $\theta_Y(\sigma \otimes 1_Y)$.

It follows from (5.7) that

\begin{equation}
(5.15) \quad m_Y^{\text{triv}} = \theta_Y((1_A \otimes \eta) \otimes 1_Y),
\end{equation}

and the right-hand-side of (3.4) is consequently

\begin{align}
(5.16) \quad \theta_Y((1_A \otimes \eta) \otimes 1_Y) & = \theta_Y(1_{A \otimes A} \otimes \theta_Y)((1_A \otimes (\eta \otimes 1_A) \otimes 1_Y))((\sigma \otimes 1_Y)

& = \theta_Y(m_{A \otimes A} \otimes 1_Y)((1_A \otimes \eta) \otimes 1_{A \otimes A \otimes Y})(1_A \otimes (\eta \otimes 1_A) \otimes 1_Y)((\sigma \otimes 1_Y)

& = \theta_Y(m \otimes m \otimes 1_Y)(((1_A \otimes \sigma)(1_A \otimes (1_A \otimes \eta))) \otimes 1_A \otimes 1_Y)

& = \theta_Y((1_A \otimes \eta) \otimes 1_A \otimes 1_Y)((\sigma \otimes 1_Y).
\end{align}
because of (4.4) and the fact that $\theta_Y$ is a left $A \otimes tw A$-module.

From (3.12) and (5.14), the right-hand-side of (5.16) is
\[
\theta_Y(1_A \otimes m \otimes 1_Y)((1_A \otimes \eta) \otimes 1_A \otimes 1_Y)(\sigma \otimes 1_Y)
= \theta_Y(1_A \otimes (m(\eta \otimes 1_A)) \otimes 1_Y)(\sigma \otimes 1_Y)
= \theta_Y(\sigma \otimes 1_Y).
\]

On account of (5.15) and the fact that $m_Y = \theta_Y((\eta \otimes 1_A) \otimes 1_Y)$, the left-hand-side of (3.4) is $\theta_Y(1_{A \otimes A} \otimes \theta_Y)((\eta \otimes 1_A) \otimes ((1_A \otimes \eta) \otimes 1_Y))$, which is
\[
(5.17) \quad \theta_Y(m_{A \otimes A} \otimes 1_Y)((\eta \otimes 1_A) \otimes ((1_A \otimes \eta) \otimes 1_Y))
= \theta_Y(((m(\eta \otimes 1_A)) \otimes (m(1_A \otimes \eta))) \otimes 1_Y)(\sigma \otimes 1_Y)
= \theta_Y(\sigma \otimes 1_Y)
\]
owing to (3.12), (4.4), and the fact that $\theta_Y$ is a left $tw A -$module. This completes the proof.

Corollary 5.9. The correspondence in Theorem 5.7 is the inverse of that in Theorem 5.8 and vice versa.

The proof of this corollary is straightforward.

6 Construction

This section deals with the construction of dynamical reflection maps by means of results in the preceding sections. In this paper, we focus on the following solutions to the braid relation (3.15) in the tensor category $Set_H$, called dynamical Yang-Baxter maps [26, 27].

Remark 6.1. The dynamical Yang-Baxter map in this paper is called a dynamical braiding map in [26, 27].

Let $L$ be a set with a binary operation $\cdot : L \times L \ni (a, b) \mapsto ab \in L$ and an element $e_L \in L$.

Definition 6.2. The triplet $(L, \cdot, e_L)$ is a left quasigroup with a unit, iff there uniquely exists the element $b \in L$ such that $ab = c$ for all $a, c \in L$ and $ae_L = e_La = a$ for every $a \in L$.

For $a, c \in L$, let $a \backslash c$ denote the unique element $b \in L$ satisfying $ab = c$. Hence, $a(a \backslash c) = c$ and $a \backslash (ac) = c$.

Every group is a left quasigroup with a unit. However, there exists a left quasigroup with a unit that is not associative.
Example 6.3. Let $L = \{e_L, l_1, l_2, l_3, l_4, l_5\}$ denote the set of six elements with the binary operation $\cdot : L \times L \to L$ defined by Table 1. Here, $l_2 \cdot l_3 = l_1$ and $l_3 \cdot l_2 = e_L$. This $(L, \cdot, e_L)$ is a left quasigroup with a unit $e_L$. Because $(l_1 \cdot l_2) \cdot l_3 = l_2$ and $l_1 \cdot (l_2 \cdot l_3) = l_5$, it is not associative.

We write $H = L$.

Proposition 6.4. $(L, \cdot) \in \mathsf{Set}_H$.

Let $G$ be a group such that $G \cong L$ as sets and the map $\pi : L \to G$ means a set-theoretic bijection. Let $\mu^G_1 : G \times G \times G \to G$ denote the ternary operation of $G$ defined by

$$
\mu^G_1(a, b, c) = ab^{-1}c \quad (a, b, c \in G).
$$

Remark 6.5. This $\mu^G_1 : G \times G \times G \to G$ is a (classical) torsor \cite{28} Remark 5.2 [For torsors, see \cite{16} Section 4.2 and \cite{31} Section 1].

The maps $\xi_\lambda : L \times L \to L$ and $\eta_\lambda : L \times L \to L$ ($\lambda \in H$) are given by

$$
\xi_\lambda(a, b) = \lambda \backslash \pi^{-1}(\mu^G_1(\pi(\lambda), \pi(\lambda a), \pi((\lambda a)b))); \\
\eta_\lambda(a, b) = (\lambda \xi_\lambda(a, b))(\lambda a)b \quad (a, b \in L).
$$

For $\lambda \in H, a, b \in L$, we define the map $\sigma(\lambda) : L \times L \to L \times L$ by

$$
\sigma(\lambda)(a, b) = (\xi_\lambda(a, b), \eta_\lambda(a, b)).
$$

This $\sigma$ is a dynamical Yang-Baxter map; that is to say,

Proposition 6.6. $\sigma : L \otimes L \to L \otimes L$ is a solution to the braid relation \cite{5,15} in $\mathsf{Set}_H$.

|     | $e_L$ | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ |
|-----|------|------|------|------|------|------|
| $e_L$ | $e_L$ | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ |
| $l_1$ | $l_1$ | $l_5$ | $l_3$ | $l_4$ | $l_2$ | $e_L$ |
| $l_2$ | $l_2$ | $l_3$ | $l_5$ | $l_1$ | $e_L$ | $l_4$ |
| $l_3$ | $l_3$ | $l_4$ | $e_L$ | $l_2$ | $l_5$ | $l_1$ |
| $l_4$ | $l_4$ | $e_L$ | $l_1$ | $l_5$ | $l_3$ | $l_2$ |
| $l_5$ | $l_5$ | $l_2$ | $l_4$ | $e_L$ | $l_1$ | $l_3$ |

Table 1: The binary operation on $L$
Because the ternary operation \( \mu_1^G \) satisfies
\[
\mu_1^G(a, \mu_1^G(a, b, c), \mu_1^G(\mu_1^G(a, b, c), c, d)) = \mu_1^G(a, b, \mu_1^G(b, c, d)),
\]
\[
\mu_1^G(\mu_1^G(a, b, c), c, d) = \mu_1^G(\mu_1^G(a, b, \mu_1^G(b, c, d)), \mu_1^G(b, c, d)),
\]
for all \( a, b, c, d \in G \), the morphism \( \sigma \) \eqref{6.4} satisfies the braid relation \eqref{3.15} in \( \text{Set}_H \) (See \cite[Theorem 3.3]{22} and \cite[Theorem 3.2]{27}).

**Remark 6.7.** This solution is a dynamical braiding map constructed by \((G, \mu_1^G)\) in \cite[Section 6]{27} for the group \( G \) (See also \cite[Remark 6.7]{27}).

**Proposition 6.8.** \( \sigma : L \otimes L \to L \otimes L \) is an isomorphism in \( \text{Set}_H \).

The inverse \( \sigma^{-1} : L \otimes L \to L \otimes L \) is defined by \( \sigma^{-1}(\lambda)(a, b) = (\lambda \backslash c, c \backslash ((\lambda)a)b) \), where \( c = \pi^{-1}(\pi((\lambda)a)b)\pi(\lambda)a^{-1}\pi(\lambda) \) (Cf. \cite[Proposition 5.1]{26}).

We set
\[
(6.5) \quad m(\lambda)(a, b) = \lambda \backslash ((\lambda)a)b; \eta(\lambda)(\bullet) = e_L \quad (\lambda \in H, a, b \in L, I = \{\bullet\}).
\]

**Proposition 6.9.** \((L, m, \eta, \sigma)\) is a braided monoid in \( \text{Set}_H \) (For the definition of the braided monoid, see Definition \[4.2\]). Moreover, \( m \sigma = m \) and the map \( L \ni b \mapsto m(\lambda)(a, b) \in L \) is injective for all \( \lambda \in H \) and \( a \in L \).

**Proof.** We only prove \eqref{3.13}. We show
\[
(6.6) \quad (m \otimes 1)_L(\lambda)a_{LLL}^{-1}(\lambda)(1_L \otimes \sigma)(\lambda)a_{LLL}(\lambda)(\sigma \otimes 1)_L(\lambda)((a, b), c)
\]
\[
= \sigma(\lambda)(1_L \otimes m)(\lambda)a_{LLL}(\lambda)((a, b), c)
\]
for all \( \lambda \in H(= L) \) and \( a, b, c \in L \).

Because of \eqref{6.4}, the left-hand-side of \eqref{6.6} is
\[
(\lambda \backslash ((\lambda \xi_\lambda(a, b))\xi_\lambda(a, b)(\eta_\lambda(a, b), c)), \eta_\lambda(a, b, c)),
\]
which is exactly the same as
\[
(\lambda \backslash \pi^{-1}(\pi(\lambda\xi_\lambda(a, b))\pi((\lambda\xi_\lambda(a, b))\eta_\lambda(a, b), c)),
\]
\[
((\lambda\xi_\lambda(a, b))\xi_\lambda(a, b)(\eta_\lambda(a, b), c)) \backslash (((\lambda\xi_\lambda(a, b))\eta_\lambda(a, b), c))
\]
\[
= (\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda - 1)\pi((\lambda\lambda)b)c), \pi^{-1}(\pi(\lambda)\pi(\lambda - 1)\pi((\lambda\lambda)b)c) \backslash (((\lambda\lambda)b)c))
\]
in view of \eqref{6.2} and \eqref{6.3}.

Taking account of \eqref{6.1} again, we see that the right-hand-side of \eqref{6.6} is
\[
(\xi_\lambda(a, (\lambda)a) \backslash (((\lambda\lambda)b)c)), \eta_\lambda(a, (\lambda)a) \backslash (((\lambda\lambda)b)c)),
\]
which coincides with
\[
(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda - 1)\pi((\lambda\lambda)b)c), \pi^{-1}(\pi(\lambda)\pi(\lambda - 1)\pi((\lambda\lambda)b)c) \backslash (((\lambda\lambda)b)c))
\]
by virtue of \eqref{6.2} and \eqref{6.3}. Hence \eqref{6.6} holds. \( \blacksquare \)
For $\lambda \in H$, the map $\iota^\lambda : L \to L \times L$ is given by
\[(6.7)\]
$$\iota^\lambda(a) = (a, (\lambda a)\lambda) \quad (a \in L).$$

Let $(X, m_X)$ be a left $(L, m, \eta)$-module in $\text{Set}_H$. We set $Y = L \otimes X \in \text{Set}_H$. Let $\lambda_0(\in L)$ denote the unique element satisfying that the element $\pi(\lambda_0)$ is the unit of the group $G$.

We can now formulate our main result whose proof will be given in the next section.

**Theorem 6.10.** (1) Let $(Y, m_Y)$ be a left $(L, m, \eta)$-module in $\text{Set}_H$ satisfying (3.5) and that two pairs $(m_Y, m_Y^{\text{triv}})$ and $(m_Y, m_Y^\pi)$ braid-commute respectively. We define the morphism $\theta_Y : (L \otimes L) \otimes Y \to Y(\in \text{Set}_H)$ by (5.3).

For $\lambda \in H, a, b \in L, x \in X$, let $\lambda \frac{a}{x}b \in L$ denote the first component of $\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\lambda, x)))$. The map $f^\lambda_x : G \to G \ (x \in X)$ is given by
$$f^\lambda_x(a) = \pi(\lambda_0)\pi(\lambda_0((\lambda_0\lambda l^{-1}(a))\lambda_0_0(\lambda_x))^{-1}a \quad (a \in G).$$

Then $\{f^\lambda_x : G \to G \ | \ x \in X\}$ is a family of group homomorphisms.

(2) Let $\{f^\lambda_x : G \to G \ | \ x \in X\}$ be a family of group homomorphisms. For $\lambda \in H, a, b \in L, x \in X$, we define:
\[(6.8)\]
$$\frac{\lambda}{x}b = \lambda l^{-1}(\pi(\lambda a)\pi(\lambda)l^{-1}(\lambda b)f^\lambda_0_{m_X(\lambda_0)(\lambda_0\lambda_x)}(\pi(\lambda))$$

$$f^\lambda_{m_X(\lambda_0)(\lambda_0\lambda_x)}(\pi(\lambda)^{-1});$$
$$m_Y(\lambda)(a, (b, x)) = (\lambda\lambda c, m_X(c)(\lambda c), m_X(\lambda a)(b, x)).$$

Here, $c = (\lambda a)((\lambda a)\lambda l_{m_X(\lambda_0)(b, x)} b) \in L$. Then $m_Y : L \otimes Y \to Y$ is a morphism of $\text{Set}_H$. In addition, $(Y, m_Y)$ is a left $(L, m, \eta)$-module in $\text{Set}_H$ satisfying (3.5) and that two pairs $(m_Y, m_Y^{\text{triv}})$ and $(m_Y, m_Y^\pi)$ braid-commute respectively.

(3) The correspondence in (1) is the inverse of that in (2) and vice versa.

It follows from the above theorem, together with Propositions 3.1, 3.2, 6.6, 6.8 and Remarks 5.5, 5.6 that Corollary 3.5 and Proposition 3.6 can produce dynamical reflection maps $k : L \otimes X \to L \otimes X \in \text{Set}_H$ (3.7).

7 Proof of Theorem 6.10

This section is devoted to giving a proof of Theorem 6.10 (Cf. [4, Section 8]).
Let $L = (L, \cdot, e_L)$ be a left quasigroup with a unit (Definition 6.2). We set $H = L$. Let $G$ be a group isomorphic to $L$ as sets. By Proposition 6.9 $(L, m, \eta, \sigma)$ is a braided monoid in $\text{Set}_H$ (see Definition 6.2, 6.4, and (6.5)).

Let $(X, m_X)$ be a left $(L, m, \eta)$-module in $\text{Set}_H$. We write $Y = L \otimes X \in \text{Set}_H$. For $\lambda \in H, a, b, c \in L$, elements $a \cdot b$ and $\rho^\lambda_b(a) \in L$ are defined by:

\begin{align*}
(7.1) & \quad a \cdot b = \lambda \backslash \pi^{-1}(\mu_1^G(\pi(\lambda a), \pi(\lambda), \pi(\lambda b))); \\
(7.2) & \quad \rho^\lambda_b(a) = \lambda \backslash \pi^{-1}(\mu_1^G(\pi((\lambda b)a), \pi(\lambda b), \pi(\lambda)))).
\end{align*}

Here, $\mu_1^G : G \times G \times G \ni (a, b, c) \mapsto ab^{-1}c \in G$ is the ternary operation in (6.1).

**Remark 7.1.** $L$ with the binary operation $(7.1)$ is a group (Cf. [26, Proposition 4.11]). Its unit element is $e_L$ and the inverse of $a \in L$ is $\lambda \backslash \pi^{-1}(\mu_1^G(\pi(\lambda), \pi(\lambda a), \pi(\lambda)))$ (See [11] Section 1 and [28, Remark 5.3]).

**Proposition 7.2.** Let $(Y, \theta_Y)$ be a left module of the twisted monoid $L \otimes L$ satisfying $(6.6)$ and $(6.7)$ (For the twisted monoid, see below Proposition 4.3). We define $a^\lambda_x b \in L \ (\lambda \in H, a, b \in L, x \in X)$ by the first component of $\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash x))) \ (\text{For } \iota^\lambda(a), \text{ see (6.1)). Then they enjoy the following for all } \lambda \in H, a, b, c \in L, \text{ and } x \in X:$

\begin{align*}
(7.3) & \quad (a \cdot b)^\lambda_x c = a^\lambda_x (b \square_x c); \\
(7.4) & \quad e_L^\lambda_x b = b; \\
(7.5) & \quad \lambda \backslash((\lambda b)(a^\lambda_x c)) = \rho^\lambda_b(a) \cdot m_X(\lambda(b, x)) (\lambda \backslash((\lambda b)c)).
\end{align*}

**Proof.** We first prove (7.3) and (7.4) by means of the following claim.

**Claim 7.3.** For $\lambda \in H, a, b \in L, \text{ and } x \in X,$

\[ \theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b)\backslash x))) = (a^\lambda_x b, m_X(\lambda(a^\lambda_x b))((\lambda(a^\lambda_x b)\backslash x))). \]

Assuming this claim for the moment, we complete the proof of (7.3) and (7.4).

It follows from the above claim and the definition of $(a \cdot b)^\lambda_x c$ that, for $\lambda \in H, a, b, c \in L, \text{ and } x \in X,$

\[ ((a \cdot b)^\lambda_x c, m_X(\lambda((a \cdot b)^\lambda_x c))((\lambda((a \cdot b)^\lambda_x c)\backslash x))) = \theta_Y(\lambda)(\iota^\lambda(a \cdot b), (c, m_X(\lambda c)((\lambda c)\backslash x))). \]

19
which is

\[(7.6) \quad \theta_Y(\lambda)(m_{L \otimes L}(\lambda)(\iota^\lambda(a), \iota^\lambda(b))(c, m_X(\lambda(c))((\lambda c) \setminus \lambda, x))),\]

since (4.4), (6.2), (6.3), (6.5), (6.7), and (7.1) induce (7.6)

\[(7.7) \quad \iota^\lambda(a \cdot b) = m_{L \otimes L}(\lambda)(\iota^\lambda(a), \iota^\lambda(b)).\]

Because \((Y, \theta_Y)\) is a left module of the twisted monoid \(L \otimes L\) and \(\lambda \iota^\lambda(a) = \lambda, (7.6)\) is

\[(7.8) \quad \theta_Y(\lambda)(\iota^\lambda(a), \theta_Y(\lambda)(\iota^\lambda(b), (c, m_X(\lambda(c))((\lambda(c) \setminus \lambda, x))))).

By using Claim 7.3 again, we can show that (7.8) coincides with \((a \underline{\circ} (b \underline{\circ} c)), m_X(\lambda(a \underline{\circ} (b \underline{\circ} c))((\lambda(a \underline{\circ} (b \underline{\circ} c)) \setminus \lambda, x))), and we have thus proved (7.3), comparing the first components.

The next task is to show (7.4) with the aid of Claim 7.3. From this claim and (7.5),

\[(7.9) \quad (e_L, \lambda x, b, m_X(\lambda(e_L, \lambda x))((\lambda(e_L, \lambda x)) \setminus \lambda, x))\]

\[= \theta_Y(\lambda)((e_L, \lambda x), (b, m_X(\lambda(b))((\lambda(b) \setminus \lambda, x)))\]

\[= \theta_Y(\eta_{L \otimes L} \otimes 1_Y)(\lambda)((\lambda(b)a), ((\lambda(b)a) \setminus \lambda(b)), (\lambda(b) \setminus \lambda, x))).\]

Since \((Y, \theta_Y)\) is a left module of the twisted monoid \(L \otimes L\), the right-hand-side of (7.9) coincides with \(l_{L \otimes L}(\lambda)((\lambda(b)a), ((\lambda(b)a) \setminus \lambda(b)), (\lambda(b) \setminus \lambda, x))) = (b, m_X(\lambda(b)((\lambda(b) \setminus \lambda, x))), which establishes (7.4).

The task is now to prove (7.5). For the proof, we need the following.

**Claim 7.4.** For \(\lambda \in H, a, b, c \in L,\) and \(x \in X,

\[(7.10) \quad m_{L \otimes L}(\lambda)((b, e_L), \iota^\lambda(a)) = m_{L \otimes L}(\lambda)(\iota^\lambda(\rho^\lambda(a)), (b, e_L))\]

\[= (\lambda((\lambda(b)a), ((\lambda(b)a) \setminus \lambda(b)), (\lambda(b) \setminus \lambda, x)).\]

\[(7.11) \quad \theta_Y(\lambda)((b, e_L), (c, m_X((\lambda(b)c))((\lambda(b)c) \setminus \lambda(b), x)))\]

\[= (\lambda((\lambda(b)c), m_X((\lambda(b)c))((\lambda(b)c) \setminus \lambda, m_X(\lambda(b), (b, x)))).\]

Assuming this claim for the moment, we complete the proof of (7.5). From (7.10),

\[(7.12) \quad \theta_Y(\lambda)(m_{L \otimes L}(\lambda)(\iota^\lambda(\rho^\lambda(a)), (b, e_L)), (c, m_X((\lambda(b)c))((\lambda(b)c) \setminus \lambda(b), x)))\]

\[= \theta_Y(\lambda)(m_{L \otimes L}(\lambda)((b, e_L), \iota^\lambda(a)), (c, m_X((\lambda(b)c))((\lambda(b)c) \setminus \lambda(b), x))).\]
and we compute the both sides of (7.12).

Because of (7.11), \( \lambda (\nu^\lambda (\rho_b^\lambda (a))) = \lambda \), and the fact that \((Y, \theta_Y)\) is a left module of the twisted monoid \( L \otimes L \), the left-hand-side of (7.12) is

\[
\theta_Y(\lambda)(\nu^\lambda (\rho_b^\lambda (a))), \theta_Y(\lambda \nu^\lambda (\rho_b^\lambda (a)))(b, e_L), (c, m_X((\lambda b)c)(((\lambda b)c)(\lambda b), x)))
\]

\[
= \theta_Y(\lambda)(\nu^\lambda (\rho_b^\lambda (a))), (\lambda \nu^\lambda ((\lambda b)c), m_X((\lambda b)c)(((\lambda b)c)\lambda, m_X(\lambda)(b, x))).
\]

By the definition, the first component of the right-hand-side of the above equation is exactly \( \rho_b^\lambda (a) \)

A slightly change in the proof actually shows that the first component of the right-hand-side of (7.12) is \( \lambda \nu^\lambda ((\lambda b)(a b)) \), and the proof of (7.5) is therefore complete.

\[ \blacksquare \]

Proof of Claim 7.3. By the definition, \( a b \nu^\lambda (b)(\in L) \) \( (\lambda \in H, a, b \in L, x \in X) \) is the first component of \( \theta_Y(\lambda)(\nu^\lambda (a), (b, m_X(\lambda)(b\lambda, x))) \), and we define \( y \in X \) by

\[ (7.13) \quad \theta_Y(\lambda)(\nu^\lambda (a), (b, m_X(\lambda)(b\lambda, x))) = (a b \nu^\lambda (b), y). \]

Our goal is to show

\[ (7.14) \quad y = m_X(\lambda)(a b \nu^\lambda (b))((\lambda a b \nu^\lambda (b))\lambda, x). \]

From (5.6),

\[ (7.15) \quad (m_X \theta_Y)(\lambda)(\nu^\lambda (a), (b, m_X(\lambda)(b\lambda, x))) = m_X(\lambda)(1L \otimes m_X)(\lambda)a_{LLX}(\lambda)(\nu^\lambda (a), m_X(\lambda)(b, m_X(\lambda)(b\lambda, x))). \]

On account of (6.5) and the fact that \((X, m_X)\) is a left \((L, m, \eta)\)-module,

\[ (7.16) \quad m_X(\lambda)(b, m_X(\lambda)(b\lambda, x)) = (m_X(m \otimes 1_X)a_{LLX}^{-1}(\lambda)(b,(b\lambda)\lambda, x)) = m_X(\lambda)(e_L, x) = (m_X(\eta \otimes 1_X))(\lambda)(\bullet, x) = l_X(\lambda)(\bullet, x) = x, \]

21
and the right-hand-side of (7.15) is consequently \(m_X(\lambda)(a, m_X(\lambda a)((\lambda a)\backslash x, x))\), which coincides with \(x\) due to (7.16). In view of (7.13), the left-hand-side of (7.15) is \(m_X(\lambda)(a\square b, y)\), and \(x = m_X(\lambda)(a\square b, y)\) as a result.

The right-hand-side of (7.14) is hence

\[
m_X(\lambda(a\square b))((\lambda(a\square b))\backslash, m_X(\lambda)(a\square b, y)) = (m_X(1_L \otimes m_X))((\lambda(a\square b))\backslash, (a\square b, y)),
\]

which coincides with \(y\), because of (6.5) and the fact that \((X, m_X)\) is a left \((L, m, \eta)\)-module. Therefore, the claim follows. \(\square\)

**Proof of Claim 7.4.** On account of (5.7), we can prove (7.11) immediately. Let us show (7.10). Because of (4.4), (6.2), (6.3), (6.5), and (6.7), the right-hand-side of (7.10) is

\[
m_L\otimes L(\lambda)((\rho_b^\lambda(a), (\lambda\rho_b^\lambda(a))\backslash), (b, e_L)) = (\lambda, \pi^{-1}(\pi(\lambda\rho_b^\lambda(a))\pi(\lambda)^{-1}\pi(\lambda b)))((\lambda b)).
\]

By (7.2), \(\pi^{-1}(\pi(\lambda\rho_b^\lambda(a))\pi(\lambda)^{-1}\pi(\lambda b)) = (\lambda b)a\), and the right-hand-side of (7.10) is \((\lambda\backslash((\lambda b)a), ((\lambda b)a)\backslash(\lambda b))\) as a result.

In the same manner, we can see that the left-hand-side of (7.10) is

\[
(m(\lambda)(b, \xi_b(e_L, a)), m((\lambda b)\xi_b(e_L, a))(((\lambda b)\xi_b(e_L, a))\backslash(((\lambda b)\xi_b(e_L, a)), (\lambda b)\xi_b(e_L, a))) = (\lambda((\lambda b)a), ((\lambda b)a)\backslash(\lambda b)),
\]

and (7.10) is thus proved. \(\square\)

**Proposition 7.5.** Let \(\lambda b, \lambda c \in H, a, b, c \in L, x \in X\) be elements of \(L\) satisfying (7.3), (7.4), and (7.5) for all \(\lambda \in H, a, b, c \in L, x \in X\). We define \(\theta_Y(\lambda)((a, b), (c, x)) \in Y \backslash H, a, b, c \in L, x \in X\) by:

\[
\begin{align*}
\theta_Y(\lambda)((a, b), (c, x)) &= (\lambda d, m_X(d)((\lambda a)b, m_X((\lambda a)b)(c, x))) (7.17) \\
(\lambda d, m_X(d)((\lambda a)b), (\lambda a)b(c, x))) &= (\lambda d, m_X(d)((\lambda a)b(c, x))). (7.18)
\end{align*}
\]

Here, \(d = ((\lambda a)b)(((\lambda a)b)\backslash(\lambda a))_m X((\lambda a)b)(c, x)). Then \((Y, \theta_Y)\) is a left module of the twisted monoid \(L \otimes L\) satisfying (5.6) and (5.7).
Proof. An easy computation shows that $\lambda \theta_Y(\lambda)((a, b), (c, x)) = \lambda((a, b), (c, x))$ for $\lambda \in H$, $(a, b) \in L \times L$, and $(c, x) \in L \times X$, and $\theta_Y : (L \otimes L) \otimes Y \to Y$ is consequently a morphism in $\text{Set}_H$.

Because of (6.5), (7.17), and the fact that $\eta \in L$ for every $a \in L$, we check at once that $\theta_Y : (L \otimes L) \otimes Y \to Y$ satisfies (5.7). From the definition of the $\text{Set}_H$ and the fact that $(X, m_X)$ is a left $(L, m, \eta)$-module, we can show (5.7).

On account of (3.2), (4.5), and (7.4), it is immediate that $\theta_Y(\eta_L \otimes 1_Y) = 1_Y$.

Now we show $\theta_Y(1_L \otimes \theta_Y) = \theta_Y(m_L \otimes 1_Y)a^{-1}_{(L \otimes L) \otimes LY}$. For the proof, we need:

**Claim 7.6.** For $\lambda \in H (= L)$, $a, b, c, d, f \in L$, and $x \in X$,

(7.19) $\lambda \setminus ((\lambda a)(b \cdot \lambda a \sqcup (\lambda \setminus ((\lambda a)c)(d \cdot \lambda a \sqcup f))))$

$= (\rho_a^\lambda(b) \cdot \rho_a^\lambda(c))_{m_X(\lambda)((\lambda a)c,x)} (\lambda \setminus ((\lambda a)c)(f))$,

(7.20) $\theta_Y(\lambda)(m_L \otimes L)(\xi^\lambda(a), (b, e_L)), (f, m_X((\lambda b)\setminus ((\lambda b))\setminus ((\lambda b), x)))$

$= (a \cdot \xi^\lambda(b), \lambda \setminus ((\lambda b)f), m_X(\lambda(a \cdot (\lambda b)f)))$

For every $\lambda \in H (= L)$, the following map $F^\lambda : (L \times L) \times ((L \times L) \times Y) \to (L \times L) \times ((L \times L) \times Y)$ is bijective:

$$F^\lambda((a, b), ((c, d), (f, x))) = (m_L \otimes L)(\xi^\lambda(a), (b, e_L), (c, e_L), (\xi^\lambda)(d), (f, m_X((((\lambda a)c(f))((((\lambda a)c(f))\setminus ((\lambda a)c(f)))(x))))).

Here, $((a, b), ((c, d), (f, x))) \in (L \times L) \times ((L \times L) \times Y)$.

Assuming these claims for the moment, we complete the proof.

On account of Claim 7.4, it suffices to show

(7.21) $(\theta_Y(1_L \otimes \theta_Y))(\lambda)F^\lambda((a, b), ((c, d), (f, x)))$

$= (\theta_Y(m_L \otimes 1_Y)a^{-1}_{(L \otimes L) \otimes LY})(\lambda)F^\lambda((a, b), ((c, d), (f, x)))$

for $\lambda \in H, a, b, c, d, f \in L$, and $x \in X$.

Because of (6.5) and the fact that $(X, m_X)$ is a left $(L, m, \eta)$-module,

$$m_X((\lambda a)c(f), m_X((((\lambda a)c(f))((((\lambda a)c(f))\setminus ((\lambda a)c(f))))(x)) = x.$$
and consequently the left-hand-side of (7.21) is
\[
\theta_Y(\lambda)((\lambda\setminus((\lambda a)b), (\lambda a)) - (\lambda a)c), (f, m_X(((\lambda a)c)(d x f))),
\]
(7.22)
\[
=\theta_Y(\lambda)((\lambda\setminus((\lambda a)b), (\lambda a)) - (\lambda a)c), (f, m_X(((\lambda a)c)(d x f))),
\]
(7.23)
\[
m_X(((\lambda a)c)(d x f))((((\lambda a)c)(d x f)) - (\lambda a)c),
\]
in view of (7.10) and (7.17).

By using (4.6) for \(A = L, \mu, \eta\), and the fact that \((X, m_X)\) is a left \((L, m, \eta)\)-module, we deduce that the right-hand-side of the above equation is
(7.22)
\[
(\lambda\setminus p, m_X(p)(((\lambda a)c), (\lambda a)c, x)).
\]
Here, \(p = (\lambda a)(b, m_X(\lambda a)(c, x))\).

Because of (7.10) and the fact that \(\lambda^\lambda(p_\alpha^\lambda(b)) = (\lambda p_\alpha^\lambda(b))((\lambda p_\alpha^\lambda(b)) - \lambda) = \lambda\)
and \(\lambda(a, e_L) = \lambda a,\) the right-hand-side of (7.21) is
(7.23)
\[
\theta_Y(\lambda)(m_{L\square L}(\lambda(\lambda a), (a, e_L), \lambda a, d)) = \theta_Y(\lambda)(m_{L\square L}(\lambda(\lambda a), (a, e_L), \lambda a, d)).
\]
(7.24)
\[
m_{L\square L}(\lambda)((a, e_L), x) = (\lambda\setminus((\lambda a)c), (\lambda a)c, x)).
\]
By taking account of (4.4) for \(A = L, \mu, \eta\), and the fact that \(\lambda(p_\alpha^\lambda(d)) = \lambda^\lambda(p_\alpha^\lambda(d)) = \lambda,\) we obtain
(7.24)
\[
\lambda(p_\alpha^\lambda(d)) = \lambda^\lambda(p_\alpha^\lambda(d)) = \lambda,
\]
and (7.20), the right-hand-side of (7.23) is exactly the same as (7.22), and the proposition follows. \(\square\)
Proof of Claims 7.6 and 7.7. We first prove (7.19) in Claim 7.6. Because of (6.5), (7.5), and the fact that \((X, m_X)\) is a left \((L, m, \eta)\)-module, the left-hand-side of (7.19) is

\[
\rho^\lambda_a(b) \overset{\lambda}{\boxdot} m_X(\lambda)(\lambda\\{((\lambda a)c)x\},x) (\lambda\\{(\lambda a)c\}(f)),
\]

which coincides with

\[
(\rho^\lambda_a(b), \rho^\lambda_{\lambda\{((\lambda a)c)x\}}(d)) \overset{\lambda}{\boxdot} m_X(\lambda\\{((\lambda a)c)x\},x) (\lambda\\{(\lambda a)c\}(f)))
\]

in view of (7.3). From (7.2), \(\rho^\lambda_a(\rho^\lambda_{\{((\lambda a)c)x\}}(d)) = \rho^\lambda_{\lambda\{((\lambda a)c)x\}}(d)\), and the above element is exactly the right-hand-side of (7.19) as a result.

The next task is to show (7.20). From (4.4) and (6.2)–(6.4), the left-hand-side of (7.20) is

\[
\theta_Y(\lambda)((\lambda\\{(\lambda a)\xi_{\lambda a}((\lambda a)\\{\lambda, b\})m_X((\lambda a)\\{(\lambda a)\xi_{\lambda a}((\lambda a)\\{\lambda, b\})\eta_{\lambda a}((\lambda a)\\{\lambda, b\})e_L)),
(f, m_X((\lambda b)f)(((\lambda b)f)\\{(\lambda b)f\}(\lambda b), x))),
\]

which coincides with

\[
(\lambda\\{g\}, m_X(g)(g\\{(\lambda b)f\}, m_X((\lambda b)f)(((\lambda b)f)\\{(\lambda b)f\}(\lambda b), x)))
\]

because of (7.17) and the fact that \(m_X(\lambda b)(f, m_X((\lambda b)f)(((\lambda b)f)\\{(\lambda b)f\}(\lambda b), x)) = x\). Here, \(g = (\lambda b)(((\lambda b)\{((\lambda a)\xi_{\lambda a}((\lambda a)\{\lambda, b\}))))\overset{\lambda}{\boxdot} f\).

On account of (7.2) and (7.5), \(\lambda\\{g\} = a \overset{\lambda}{\boxdot} m_X(\lambda)(\lambda b, x) (\lambda\\{(\lambda b)f\})\), and the element (7.25) is exactly the right-hand-side of (7.20) with the aid of (6.5) and the fact that \((X, m_X)\) is a left \((L, m, \eta)\)-module. This gives (7.20), and the proof of Claim 7.6 is complete.

The map \(F^\lambda\) has its inverse \(G^\lambda\) defined by

\[
G^\lambda((a, b), ((c, d), (f, x)))
= ((\lambda\\{(\lambda a)b\}, ((\lambda a)b)\\{(\lambda a)b\}d, (\lambda a)\{((\lambda a)b)c\}d), (f, m_X(((\lambda a)b)c)(f, x))",

which proves Claim 7.7.

\[\square\]

Proposition 7.8. The correspondence in Proposition 7.2 is the inverse of that in Proposition 7.5 and vice versa.
**Proof.** We only show the correspondence in Proposition 7.5 is a left inverse of that in Proposition 7.2. Let \((Y, \theta_Y)\) be a left module of the twisted monoid \(L \otimes L\) satisfying (5.6) and (5.7), and let \(\theta_Y^L : (L \otimes L) \otimes Y \to Y\) denote the morphism of \(\text{Set}_H\) defined by the right-hand-side of (7.17), in which \(a, b \in L\) is the first component of \(\theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b) \lambda, x)))\) as was explained in Proposition 7.2. We show \(\theta_Y^L = \theta_Y\).

For the proof, we make use of the following claim.

**Claim 7.9.** For \(\lambda \in H(= L), a, b \in L, and x \in X,\)

\[
\tag{7.26} \theta_Y^L(\lambda)((a, e_L), (b, x)) = \theta_Y(\lambda)((a, e_L), (b, x)),
\]

\[
\tag{7.27} \theta_Y^L(\lambda)(\iota^\lambda(a), (b, x)) = \theta_Y(\lambda)(\iota^\lambda(a), (b, x)).
\]

Assuming this claim for the moment, we complete the proof.

By (7.10), \(m_{L \otimes L}(\lambda)\)((\lambda)((\lambda a)b, e_L), \iota^{(\lambda a)b}((\lambda a)b)\lambda)) = (a, b)\) for \(\lambda \in H, a, b, c \in L,\) and \(x \in X,\) and, as a result,

\[
\theta_Y^L(\lambda)((a, b), (c, x)) = (\theta_Y^L(m_{L \otimes L} \otimes 1_Y))(\lambda)(((\lambda a)b, e_L), \iota^{(\lambda a)b}((\lambda a)b)\lambda), (c, x)).
\]

From Proposition 7.5, \((Y, \theta_Y)\) is a left module of the twisted monoid \(L \otimes L\), and, the right-hand-side of the above equation is consequently

\[
(\theta_Y^L(1_{L \otimes L} \otimes \theta_Y^L)a_{L \otimes LL \otimes LY})(\lambda)(((\lambda a)b, e_L), \iota^{(\lambda a)b}((\lambda a)b)\lambda), (c, x)) = \theta_Y^L(\lambda)(((\lambda a)b, e_L), \theta_Y((\lambda a)b)(\iota^{(\lambda a)b}((\lambda a)b)\lambda), (c, x))).
\]

We now apply this argument again, with \(\theta_Y^L\) replaced by \(\theta_Y\), to obtain

\[
\theta_Y(\lambda)((a, b), (c, x)) = \theta_Y((\lambda a)b)(\iota^{(\lambda a)b}((\lambda a)b)\lambda), (c, x))).
\]

Claim 7.9 implies that \(\theta_Y^L(\lambda)((a, b), (c, x)) = \theta_Y(\lambda)((a, b), (c, x))\), which is the desired conclusion. \(\square\)

**Proof of Claim 7.9.** We first prove (7.26). From (7.24) and (7.17),

\[
\theta_Y^L(\lambda)((a, e_L), (c, x)) = (\lambda)((\lambda a)c, m_X((\lambda a)c)(e_L, x))
\]

for \(\lambda \in H, a, c \in L,\) and \(x \in X.\) Since \((X, m_X)\) is a left \((L, m, \eta)\)-module, \(m_X((\lambda a)c)(e_L, x) = x,\) and consequently \(\theta_Y^L(\lambda)((a, e_L), (c, x)) = (\lambda((\lambda a)c), x).\)
From (5.7),
\[
\theta_Y(\lambda)((a, e_L), (c, x)) = (\theta_Y((1_L \otimes \eta) \otimes 1_Y))(\lambda)((a, \bullet), (c, x)) \\
= ((m \otimes 1_X) a_{LX}^{-1}(r_L \otimes 1_Y))(\lambda)((a, \bullet), (c, x)) \\
= (\lambda \cdot ((\lambda a)c), x),
\]
which gives (7.26).

Let \( \lambda \in H, a, b \in L, \) and \( x \in X. \) Because of (6.5) and the fact that \( (X, m_X) \) is a left \( (L, m, \eta) \)-module,
\[
\theta_Y(\lambda)(\iota^\lambda(a), (b, x)) = \theta_Y(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b) \setminus \lambda, m_X(\lambda)(b, x)))),
\]
which is exactly
\[
(7.28) \; (a \begin{array}{c} \lambda \\ m_X(\lambda)(b, x) \end{array}, b, m_X(\lambda(a \begin{array}{c} \lambda \\ m_X(\lambda)(b, x) \end{array}) b))((\lambda(a \begin{array}{c} \lambda \\ m_X(\lambda)(b, x) \end{array}) b) \setminus \lambda, m_X(\lambda)(b, x)))
\]
by Claim 7.3.

It follows from the definition of \( \theta_Y' \) (7.28) that \( \theta_Y'(\lambda)(\iota^\lambda(a), (b, x)) \) coincides with (7.28), and (7.27) follows. \( \square \)

We also rephrase the condition that the pair \((m_Y, m_Y')\) defined by (5.11) and (3.17) braid-commutes (Definition 5.4), by means of the elements \( a_{x}^{\lambda}b \in L. \)

**Proposition 7.10.** Let \((Y, \theta_Y)\) be a left module of the twisted monoid \( L \otimes L \)
satisfying (5.6) and (5.7) (For the twisted monoid, see below Proposition 4.3). We define \( a_{x}^{\lambda}b = (\lambda \in H, a, b \in L, x \in X) \) by the first component of \( \theta_Y(\lambda)(\iota^\lambda(a), (b, m_X(\lambda b)((\lambda b) \setminus \lambda, x))) \) (For \( \iota^\lambda(a) \), see (6.1)). In addition, \( m_Y : L \otimes Y \rightarrow Y \) and \( m_Y' : L \otimes Y \rightarrow Y \) are defined by (5.11) and (3.17) for \( A = (L, m, \eta) \), respectively. Then the following two conditions are equivalent:

1. \((m_Y, m_Y')\) braid-commutes (See Definition 5.4);
2. For all \( \lambda \in H, a, b, c \in L, x \in X, \)
\[
(7.29) \; a_{x}^{\lambda}(b \cdot c) = \lambda \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1} \pi(\lambda b)\pi(\lambda)^{-1} \pi(\lambda(a_{x}^{\lambda}c))).
\]

**Proof.** For \( \lambda \in H(= L), a, b, c \in L, \) and \( x \in X, \) we write
\[
(f_1, y_1) = (m_Y(1_L \otimes m_Y'))(\lambda)(a, (b, (c, m_X(((\lambda a)b)c)(((\lambda a)b)c) \setminus ((\lambda a)b), x))),
\]
\[
(f_2, y_2) = (m_Y'(1_L \otimes m_Y)a_{LY}^{-1}(\sigma \otimes 1_Y)a_{LY}^{-1})(\lambda) \\
\quad (a, (b, (c, m_X(((\lambda a)b)c)(((\lambda a)b)c) \setminus ((\lambda a)b), x))).
\]

27
Lemma 7.11. If $f_1 = f_2$, then $y_1 = y_2$.

Proof. We note that Propositions 7.2, 7.5, and 7.8 induce (7.17). On account of (5.11) and (7.17),

\begin{equation}
(7.30) \quad m_Y(\lambda)(a, (b, x)) = (\lambda \backslash p, m_X(p)(p \backslash (\lambda a), m_X(\lambda a)(b, x))) \\
= (\lambda \backslash p, m_X(p)(p \backslash ((\lambda a)b), x)).
\end{equation}

Here, $p = (\lambda a)((\lambda a) \backslash (\lambda a) \backslash b)$. It follows from (3.17) and (7.30) that

$$(f_1, y_1) = (\lambda \backslash p', m_X(p')(p' \backslash ((\lambda a)b), x)).$$

Here, $p' = (\lambda a)((\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash q, m_X(\lambda a)(b, c))$.

We can see in a similar way that

$$(f_2, y_2) = m_X^*(\lambda)(\xi_\lambda(a, b), ((\lambda \xi_\lambda(a, b)) \backslash q, m_X(q)(q \backslash ((\lambda a)b), x)))$$

$$(\lambda \xi_\lambda(a, b), (\lambda \xi_\lambda(a, b)) \backslash q, ((\lambda a)b), x)).$$

Here, $q = ((\lambda a)b)(((\lambda a)b) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash (\lambda a) \backslash q, (\lambda a)b, c)$. Hence, $f_1 = f_2$, if and only if

\begin{equation}
(7.31) \quad \lambda \backslash p' = \xi_\lambda(\xi_\lambda(a, b), (\lambda \xi_\lambda(a, b)) \backslash q),
\end{equation}

which immediately induces $y_1 = y_2$. This is our assertion.

From this lemma and the fact that

\begin{equation}
\begin{aligned}
m_X(((\lambda a)b)c)(((\lambda a)b)c \backslash ((\lambda a)b), m_X((\lambda a)b)(c, x')) \\
= m_X(((\lambda a)b)c)(e_L, x') = x'
\end{aligned}
\end{equation}

for every $x' \in X$, the condition (1) in Proposition 7.10 is equivalent to the following condition (3):

(3) $f_1 = f_2$ for all $\lambda \in H, a, b, c \in L$, and $x \in X$. 

28
Because of Proposition 7.2, (7.34) holds, and we can rewrite the both sides of (7.31) equivalent to the condition (3) by means of (6.2), (7.2), and (7.5).

\[
\lambda \lfloor (\lambda a)((\lambda a)\backslash \lambda) \underbrace{\lambda}_{m_X((\lambda a)(b,x))} \xi_{\lambda a}(b, c))
\]

\[
= (\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underbrace{\lambda}_{m_X((\lambda a)^{(\lambda a)b}, x)} \xi_{\lambda a}(b, c))
\]

(7.33) \[
\xi_{\lambda}(\xi_{\lambda a}(a, b), (\lambda \xi_{\lambda a}(a, b)) \backslash ((\lambda a)b)((\lambda a)b \backslash (\lambda \xi_{\lambda a}(a, b))) \underbrace{\lambda}_{m_X((\lambda a)b), x})
\]

and, as a result, the condition (3) is equivalent to the condition (4):

(4) The right-hand-side of (7.32) coincides with that of (7.33) for all \( \lambda \in H, a, b, c \in L \), and \( x \in X \).

This condition (4) is equivalent to the condition (2). We prove (2) from (4). First we respectively substitute \(((\lambda a)b \backslash (\lambda c')\) for \( c \) and \( m_X((\lambda a)b)((\lambda a)b )\backslash \lambda, x'\) for \( x \) in the equation in (4). Because of (6.5) and the fact that \((X, m_X)\) is a left \((L, m, \eta, η)\)-module,

\[
m_X(\lambda)\backslash ((\lambda a)b), m_X((\lambda a)b)((\lambda a)b \backslash \lambda, x') = m_X(\lambda)(e_L, x') = x',
\]

and consequently

\[
(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underbrace{\lambda}_{x'}(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda c')))
\]

\[
= \lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda a)\pi(\lambda c')\pi(\lambda c'))
\]

(7.34) \[
(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underbrace{\lambda}_{x'}(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda c')))
\]

for all \( \lambda \in H, a, b, c' \in L \), and \( x' \in X \). By substituting \((\lambda a)^{-1}(\pi(\lambda)\pi(\lambda b')^{-1}\pi(\lambda a))\) for \( b \) in (7.34), we deduce that

(7.35) \[
(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))) \underbrace{\lambda}_{x'}(\lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda c')))
\]

\[
= \lambda \backslash \pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda a)\pi(\lambda c')\pi(\lambda c'))
\]

\[
\pi(\lambda a)^{-1}(\pi(\lambda a)^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))(\lambda \underbrace{\lambda}_{x'})}}
\]

29
for all $\lambda \in H, a, b', c \in L$, and $x \in X$. Substituting $\lambda \pi^{-1}(\pi(\lambda)\pi(\lambda a')^{-1}\pi(\lambda))$ for $a$ in (7.35) yields the condition (2).

We can follow the steps above in reverse, and the condition (2) implies (4). This proves the proposition. □

Proposition (7.10) immediately induces the following corollary on account of Theorems 5.7, 5.8, Corollary 5.9, and Propositions 7.2, 7.5, 7.8.

**Corollary 7.12.** (1) Let $(Y, m_Y)$ be a left $(L, m, \eta)$-module satisfying (3.5) and that two pairs $(m_Y, m_Y^{\text{triv}})$ and $(m_Y, m_Y^{\pi})$ braid-commute respectively.

For $\lambda \in H, a, b \in L, x \in X$, let $a \boxed{b} \in L$ denote the first component of $\theta_Y(\lambda)(\pi^X(a), (b, m_X(\lambda b)(\lambda b), x))$ (For $\theta_Y : (L \otimes L) \otimes Y \rightarrow Y \in \text{Set}_H$, see (5.5)). Then they enjoy (7.3), (7.4), (7.5), and (7.29) for all $\lambda \in H, a, b, c \in L, x \in X$.

(2) Let $a \overset{\lambda}{\boxed{b}} (\lambda \in H, a, b \in L, x \in X)$ be elements of $L$ satisfying (7.3), (7.4), (7.5), and (7.29) for all $\lambda \in H, a, b, c \in L, x \in X$. Then $(Y, m_Y)$ defined by (6.8) is a left $(L, m, \eta)$-module satisfying (3.5) and that two pairs $(m_Y, m_Y^{\text{triv}})$ and $(m_Y, m_Y^{\pi})$ braid-commute respectively.

(3) The correspondence in (1) is the inverse of that in (2) and vice versa.

The following proposition states that $a \overset{\lambda}{\boxed{b}} \in L$ are recovered by the elements $a \overset{\lambda}{\boxed{e_L}}$.

**Proposition 7.13.** (1) If elements $a \overset{\lambda}{\boxed{b}} \in L (\lambda \in H, a, b \in L, x \in X)$ satisfy (7.3), (7.4), (7.5), and (7.29) for any $\lambda \in H, a, b, c \in L, x \in X$, then the elements $\beta^\lambda_x(a) = a \overset{\lambda}{\boxed{e_L}} \in L (\lambda \in H, a \in L, x \in X)$ enjoy:

\[
\beta^\lambda_x(a \cdot b) = \lambda \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda \beta^\lambda_x(b))\pi(\lambda a)^{-1}\pi(\lambda \beta^\lambda_x(a))),
\]

\[
(\lambda b)\beta^\lambda_{\pi^{-1}}(b_{\pi^{-1}}(\lambda b)(\lambda a))(\rho_{(\lambda b)}(\lambda a)(a)) = \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda \beta^\lambda_x(a)))
\]

for all $\lambda \in H, a, b \in L, x \in X$. Here, $e_L$ is the unit element of the left quasigroup $(L, \cdot)$.

(2) We assume that elements $\beta^\lambda_x(a) \in L$ satisfy (7.36) and (7.37) for all $\lambda \in H, a, b \in L, x \in X$. We write

\[
a \overset{\lambda}{\boxed{b}} = \lambda \pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda a)^{-1}\pi(\lambda \beta^\lambda_x(a)))
\]

\[
(= \lambda ((\lambda b)\beta^\lambda_{\pi^{-1}}(b_{\pi^{-1}}(\lambda b)(\lambda a))(\rho_{(\lambda b)}(\lambda a)(a))))
\]

30
for \( \lambda \in H, a, b \in L, x \in X \). Then the elements \( \lambda \cdot x \) satisfy (7.38), (7.39), and (7.37) for all \( \lambda \in H, a, b, c \in L, x \in X \).

(3) The correspondence in (1) is the inverse of that in (2) and vice versa.

Proof. We only prove (1) and (2).

(1) Let us first show (7.36). Substituting \( e_L \) for \( \lambda \cdot a \) in (7.29) yields

\[
(7.40) \quad \lambda \cdot x = \lambda \cdot x = \lambda \cdot x \]

for \( \lambda \in H(= L), a, b \in L, x \in X \), because \( a \cdot x = a \) by the definition (7.3). It follows from (7.1) and (7.40) that

\[
(7.41) \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c
\]

for all \( \lambda \in H, a, b, c \in L, x \in X \). On account of (7.3), we get (7.36).

The next task is to show (7.37). For the proof, we use the fact that (7.5) is equivalent to the following: for \( \lambda \in H, a, b, c \in L, x \in X \),

\[
(7.41) \quad \lambda \cdot x = \lambda \cdot x = \lambda \cdot x
\]

We show that (7.3) induces (7.41) only. Because of (6.5) and the fact that \( X, m_X \) is a left \( (L, m, \eta) \)-module, \( m_X(\lambda)(b, m_X(\lambda)(\lambda \cdot a, x)) = x \), and the right-hand-side of (7.41) is consequently \( \rho_\lambda^\lambda(b)(\rho_\lambda^\lambda(a)) \cdot (\lambda \cdot (\lambda \cdot c)) \) owing to (7.5). This element is exactly the left-hand-side of (7.41), since \( \rho_\lambda^\lambda(b)(\rho_\lambda^\lambda(a)) = a \) by (7.2), and this is our claim.

We now proceed the proof of (7.37). By substituting \( e_L \) for \( \lambda \cdot a \) in (7.41), we obtain

\[
(7.42) \quad \lambda \cdot x = \lambda \cdot x = \lambda \cdot x
\]

and combining (7.40) and (7.42) yields (7.37).
(2) A straightforward computation with the aid of (7.1) and (7.38) shows (7.4) easily.

Substituting $e_L$ for $a$ and $b$ in (7.36) yields $\beta_{x}^{\lambda}(e_L) = e_L$ for all $\lambda \in H$ and $x \in X$, and $\rho_{(\lambda b)c}\lambda(e_L) = e_L$ by the definition (7.2). Combining these, we obtain (7.4).

Since $(X, m_X)$ is a left $(L, m, \eta)$-module, $m_X((\lambda b)c)((\lambda b)c)(\lambda b), x) = m_X((\lambda b)c)((\lambda b)c)(\lambda m_X(\lambda)(b, x))$ for $\lambda \in H, b, c \in L$, and $x \in X$, in view of (7.36). In addition, $\rho_{(\lambda b)c}\lambda(\rho_{(\lambda b)c}\lambda_a(a)) = \rho_{(\lambda b)c}(\lambda_a\rho_{(\lambda b)c}\lambda_a(a))$ by the definition (7.2). Combining these with (7.42) yields (7.5).

The task is now to show (7.29). From (7.1) and (7.38),

$$a_{\lambda x}^\lambda(b \cdot c) = \lambda \Pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda)^{-1}\pi(\lambda c)\pi(\lambda a)^{-1}(\lambda b_x^\lambda(a)))$$

for all $\lambda \in H, a, b, c \in L$, and $x \in X$. The right-hand-side of the above equation coincides with that of (7.29), since

$$\pi(\lambda c)\pi(\lambda a)^{-1}(\lambda b_x^\lambda(a)) = \pi(\lambda)\pi(\lambda a)^{-1}(\lambda b_x^\lambda(\Pi_{\lambda x}^\lambda(a)))$$

on account of (7.38), and (2) is therefore proved.

We now introduce $\Pi_{\lambda x}^\lambda(a) \in L$ instead of the elements $\beta_{x}^{\lambda}(a) \in L$.

**Proposition 7.14.** (1) If elements $\beta_{x}^{\lambda}(a) \in L$ ($\lambda \in H, a \in L, x \in X$) satisfy (7.36) and (7.37) for any $\lambda \in H, a, b \in L, x \in X$, then the elements $\Pi_{\lambda x}^\lambda(a) \in L$ ($\lambda \in H, a \in L, x \in X$) defined by

$$(7.43) \quad \Pi_{\lambda x}^\lambda(a) = \lambda \Pi^{-1}(\pi(\lambda)\pi(\lambda b_x^\lambda(a))^{-1}(\lambda a)) \in L$$

enjoy

$$(7.44) \quad \Pi_{\lambda x}^\lambda(a \cdot b) = \Pi_{\lambda x}^\lambda(a) \cdot \Pi_{\lambda x}^\lambda(b),$$

$$(7.45) \quad \Pi_{\lambda x}^\lambda((\lambda a)b) = \lambda \Pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}(\pi((\lambda a)\Pi_{m_X(\lambda a)\lambda x}^\lambda(a))\pi(\lambda)^{-1}(\lambda b_x^\lambda(a))))$$

for all $\lambda \in H, a, b \in L, x \in X$.

(2) If elements $\Pi_{\lambda x}^\lambda(a) \in L$ satisfy (7.44) and (7.45) for any $\lambda \in H, a, b \in L, x \in X$, then the elements

$$(7.46) \quad \beta_{x}^{\lambda}(a) = \lambda \Pi^{-1}(\pi(\lambda a)\pi(\lambda \Pi_{\lambda x}^\lambda(a))^{-1}(\lambda)) \in L$$

enjoy (7.36) and (7.37) for all $\lambda \in H, a, b \in L, x \in X$.

(3) The correspondence in (1) is the inverse of that in (2) and vice versa.
Obviously, \((7.33)\) induces \((7.46)\) and vice versa.

Remark 7.15. On account of Remark 7.1, \((7.44)\) means that every map \(\Pi_\lambda^L : L \to L\) is a homomorphism of the group \((L, \cdot)\).

Proof of Proposition 7.14. We prove (1) and (2) only. The proof will be divided into a sequence of lemmas and a corollary.

Lemma 7.16. If elements \(\beta_\lambda^L(a) \in L (\lambda \in H, a, b \in L, x \in X)\) satisfy \((7.36)\), then the elements \(\Pi_\lambda^L(a)\) defined by \((7.43)\) enjoy \((7.44)\). Conversely, if elements \(\Pi_\lambda^L(a) \in L (\lambda \in H, a, b \in L, x \in X)\) satisfy \((7.44)\), then the elements \(\beta_\lambda^L(a)\) defined by \((7.46)\) enjoy \((7.36)\).

Proof. Let \(\beta_\lambda^L(a) (\lambda \in H, a, b \in L, x \in X)\) be elements of \(L\) satisfying \((7.36)\). By \((7.1)\), \((7.36)\), \((7.43)\), and \((7.46)\), the left-hand-side of \((7.44)\) is

\[
\lambda\pi^{-1}(\pi(\lambda)\pi(\lambda^L_x(a))^{-1}\pi(\lambda a)\pi(\lambda^L_x(b))^{-1}\pi(\lambda b)),
\]

which is exactly the right-hand-side of \((7.44)\) because of \((7.1)\) and \((7.46)\).

Conversely, let \(\Pi_\lambda^L(a) (\lambda \in H, a, b \in L, x \in X)\) be elements of \(L\) satisfying \((7.44)\). From \((7.1)\), \((7.43)\), \((7.44)\), and \((7.46)\), the left-hand-side of \((7.36)\) is

\[
\lambda\pi^{-1}(\pi(\lambda a)\pi(\lambda)^{-1}\pi(\lambda b)\pi(\lambda\Pi_\lambda^L(b))^{-1}\pi(\lambda)\pi(\lambda\Pi_\lambda^L(a))^{-1}\pi(\lambda)),
\]

which coincides with the right-hand-side of \((7.36)\) due to \((7.46)\).



By taking account of this lemma, we are left with the task of clarifying a relation between \((7.37)\) and \((7.45)\).

Lemma 7.17. If elements \(\Pi_\lambda^L(a) \in L (\lambda \in H, a, b \in L, x \in X)\) satisfy \((7.44)\), then \(\Pi_\lambda^L(e_L) = e_L\) and

\[
(7.47) \quad \pi(\lambda)\pi(\lambda\Pi_\lambda^L(a))^{-1}\pi(\lambda) = \pi(\lambda\Pi_\lambda^L(\lambda\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))))
\]

for all \(\lambda \in H, a \in L, x \in X\).

Proof. From \((7.1)\), \(e_L \cdot e_L = e_L\) for every \(\lambda \in H (= L)\); and substituting \(e_L\) for \(a\) and \(b\) in \((7.44)\), together with \((7.1)\), yields \(\Pi_\lambda^L(e_L) = e_L\) for all \(\lambda \in H\) and \(x \in X\) as a result.

Moreover, by substituting \(\lambda\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda))\) for \(b\) in \((7.44)\), we can show: the left-hand-side is \(e_L\) because of \((7.1)\) and the fact that \(\Pi_\lambda^L(e_L) = e_L\); and the right-hand-side is

\[
\lambda\pi^{-1}(\pi(\lambda\Pi_\lambda^L(a))\pi(\lambda)^{-1}\pi(\lambda\Pi_\lambda^L(\lambda\pi^{-1}(\pi(\lambda)\pi(\lambda a)^{-1}\pi(\lambda)))))
\]

due to \((7.1)\). This gives \((7.47)\) immediately.
Lemma 7.18. Let $\beta_\lambda^\lambda(a) \ (\lambda \in \mathcal{H}, a, b \in L, x \in X)$ be elements of $L$ satisfying (7.36), and we define $\Pi_\lambda^\lambda(a) \ (\lambda \in \mathcal{H}, a \in L, x \in X)$ by (7.43). Then the following three conditions are equivalent:

1. (7.37) for all $\lambda \in L, a, b \in L$, and $x \in X$;

2. $\Pi_\lambda^\lambda(\lambda(a,x))(b) = \lambda^\lambda \pi^\lambda((\lambda b)\Pi_\lambda^\lambda(\lambda(b)))$ for all $\lambda \in L, a, b \in L$, and $x \in X$;

3. (7.45) for all $\lambda \in L, a, b \in L$, and $x \in X$.

Proof. As we have already pointed out, (7.43) induces (7.43). From (7.2), (7.46), and (7.47), the left-hand-side of (7.37) is

$$\pi^{-1}(\pi((\lambda b)\Pi_\lambda^\lambda(\lambda(b)))((\lambda b)\pi^{-1}(\pi((\lambda a)\pi^{-1}(\lambda b))))).$$

From Lemmas 7.16 and 7.17, (7.47) holds, and, by (7.46) and (7.47), the right-hand-side of (7.37) is

$$\pi^{-1}(\pi((\lambda a)\pi^{-1}(\lambda b))\pi^{-1}(\lambda a)(\lambda b))\pi^{-1}(\lambda a)\pi^{-1}(\lambda b)) = \pi^{-1}(\pi((\lambda a)\pi^{-1}(\lambda b)).$$

The condition (1) is consequently equivalent to:

$$\pi((\lambda b)\Pi_\lambda^\lambda(\lambda(b)))((\lambda b)\pi^{-1}(\pi((\lambda a)\pi^{-1}(\lambda b)))) = \pi((\lambda a)\Pi_\lambda^\lambda(\lambda(b))$$

for all $\lambda \in L, a, b \in L$, and $x \in X$.

Replacing $a$ in (7.48) by $\lambda^\lambda \pi^{-1}(\pi((\lambda b)\pi^{-1}(\lambda b)))$, we can assert that the above condition is equivalent to:

$$\Pi_\lambda^\lambda(a)$$

for all $\lambda \in L, a, b \in L$, and $x \in X$. In view of (6.5), (7.2), and the fact that $(X, m_X)$ is a left $(L, m, \eta)$-module, the above condition is exactly the same as the condition (2), which coincides with the condition

$$\pi((\lambda a)\Pi_\lambda^\lambda(\lambda(b))) = \pi((\lambda a)\Pi_\lambda^\lambda(\lambda(b))$$

for all $\lambda \in L, a, b \in L$, and $x \in X$.

This condition is equivalent to that

$$\pi((\lambda a)(\rho_{(\lambda a)}^\lambda(b))) = \pi((\lambda a)(\rho_{(\lambda a)}^\lambda(b)))$$

for all $\lambda \in L, a, b \in L$, and $x \in X$.

34
for all $\lambda \in H$, $a, b \in L$, and $x \in X$. In fact, substituting respectively $\lambda a$ and $(\lambda a) \lambda$ for $\lambda$ and $a$ in \textcolor{red}{(7.49)} gives \textcolor{red}{(7.50)} and vice versa.

From \textcolor{red}{(7.1)} and \textcolor{red}{(7.44)},

\begin{align*}
\lambda \pi^{-1}(\pi(\lambda \Pi^1_x(\rho^\lambda(a)) \pi(\lambda a))) &= \Pi^1_x(\rho^\lambda(b)) \cdot \Pi^1_x(a) \\
&= \Pi^1_x(\rho^\lambda(b) \lambda a),
\end{align*}

which is exactly the left-hand-side of \textcolor{red}{(7.45)} in view of \textcolor{red}{(7.1)} and \textcolor{red}{(7.2)}.

Therefore, \textcolor{red}{(7.45)} holds, if and only if the right-hand-side of \textcolor{red}{(7.45)} is equal to the left-hand-side of \textcolor{red}{(7.51)}. Because this equation is equivalent to \textcolor{red}{(7.50)}, the conditions (2) and (3) are equivalent. This is our assertion. \hfill \square

Lemma 7.16 immediately implies the following corollary.

**Corollary 7.19.** Let $\Pi^\lambda_x(a)$ ($\lambda \in H$, $a, b \in L$, $x \in X$) be elements of $L$ satisfying \textcolor{red}{(7.44)}, and we define $\beta^\lambda_x(a)$ ($\lambda \in H$, $a \in L$, $x \in X$) by \textcolor{red}{(7.43)}. Then the three conditions in the previous lemma are equivalent.

The above sequence of lemmas and a corollary completes the proof of Proposition 7.14 (1) and (2).

Let $\lambda_0(\in L)$ denote the unique element satisfying $\pi(\lambda_0) = e_G$. Here, $e_G$ is the unit element of the group $G$.

The following proposition is our final step in the proof of Theorem 6.10.

**Proposition 7.20.** (1) We assume that elements $\Pi^\lambda_x(a)$ ($\lambda \in H$, $a, b \in L$, $x \in X$) satisfy \textcolor{red}{(7.44)} and \textcolor{red}{(7.45)} for all $\lambda \in H$, $a, b \in L$, $x \in X$. Then the maps $f_{x}^{\lambda_0} : G \to G$ defined by

\begin{equation}
(7.52) \quad f_{x}^{\lambda_0}(a) = \pi(\lambda_0 \Pi^\lambda_x(\lambda_0 \pi^{-1}(a)))
\end{equation}

are group homomorphisms and enjoy

\begin{equation}
(7.53) \quad \Pi^\lambda_x(a) = \lambda \pi^{-1}(\pi(\lambda f_{mX}^{\lambda_0(\lambda_0 \pi^{-1}(a))) \pi(\lambda a)) f_{mX}^{\lambda_0(\lambda_0 \pi^{-1}(a))) \pi(\lambda a)})^{-1})
\end{equation}

for all $\lambda \in H$, $a \in L$, $x \in X$.

(2) Let $\{f_{x}^{\lambda_0} : G \to G \mid x \in X\}$ be a family of group homomorphisms. If we define elements $\Pi^\lambda_x(a)$ ($\lambda \in H$, $a \in L$, $x \in X$) by \textcolor{red}{(7.53)}, then they satisfy \textcolor{red}{(7.44)} and \textcolor{red}{(7.45)} for all $\lambda \in H$, $a, b \in L$, $x \in X$.

(3) The correspondence in (1) is the inverse of that in (2), and vice versa.
Proof. We only prove (1) and (2).
(1) We begin by showing that \( f^x_{\lambda_0}(ab) = f^x_\lambda(a)f^x_\lambda(b) \) for all \( a, b \in G \). Because of (7.1) and the fact that \( \pi(\lambda_0) = e_G \),
\[
(7.54) \quad f^x_{\lambda_0}(ab) = \pi(\lambda_0 \Pi_x^0(\lambda_0 \\pi^{-1}(a\pi(\lambda_0)^{-1}b))) = \pi(\lambda_0 \Pi_x^0((\lambda_0 \\pi^{-1}(a)) \cdot (\lambda_0 \\pi^{-1}(b)))).
\]
From (7.44), the right-hand-side of (7.54) is
\[
\pi(\lambda_0(\Pi_x^0(\lambda_0 \\pi^{-1}(a))) \cdot \Pi_x^0(\lambda_0 \\pi^{-1}(b))) = \pi(\lambda_0 \Pi_x^0(\lambda_0 \\pi^{-1}(a)))\pi(\lambda_0 \Pi_x^0(\lambda_0 \\pi^{-1}(b)))
\]
which coincides with \( f^x_\lambda(a)f^x_\lambda(b) \), and the map \( f^x_{\lambda_0} : G \to G \) is consequently a group homomorphism.

The next task is to show (7.53). From (7.45),
\[
(7.55) \quad \Pi_{m_X(\lambda_0)(\lambda_0 \\lambda)}(b) = (\lambda a) \pi^{-1}(\lambda a \pi(a))^{-1}(\lambda \Pi_x^0((\lambda a) \pi(b))) = (\lambda \Pi_x^0(a))^{-1}(\pi(\lambda)).
\]
Since \( (X, m_X) \) is a left \( (L, m, \eta) \)-module, \( m_X(\lambda)(\lambda \\lambda, m_X(\lambda_0)(\lambda_0 \\lambda, x)) = x \) on account of (6.5). As a result, we deduce (7.53) by means of (7.52), substituting \( \lambda, \lambda_0 \\lambda, a, m_X(\lambda_0)(\lambda_0 \\lambda, x) \) for \( \lambda, a, b, x \) in (7.55). This is our claim. (2) Since \( (X, m_X) \) is a left \( (L, m, \eta) \)-module, \( m_X(\lambda_0)(\lambda_0 \\lambda, m_X(\lambda_0)(\lambda_0 \\lambda, x)) = m_X(\lambda_0)(\lambda_0 \\lambda, x) \) in view of (6.5), and we can immediately prove (7.45) by using (7.53).

On account of (7.1), (7.53), and the fact that every \( f^x_{\lambda_0} : G \to G \) is a group homomorphism, we can show (7.44), and (2) is proved. 

Combining Corollary 7.12 and Propositions 7.13, 7.14, 7.20 we can prove Theorem 6.10.

We are now in a position to describe the dynamical reflection map \( k : L \otimes X \to L \otimes X \) in Set_H in (5.67) by means of the elements \( \Pi_x^0(a) \): for \( \lambda \in H, a \in L, x \in X \),
\[
(7.56) \quad k(\lambda)(a, x) = (\Pi_{m_X(\lambda)(a, x)}^0)(a), m_X(\lambda \Pi_{m_X(\lambda)(a, x)}^0(a)), (\lambda \Pi_{m_X(\lambda)(a, x)}^0(a))(\lambda x, x)).
\]

8 Examples

From Proposition 6.9 the left quasigroup with a unit \( (L, \cdot, e_L) \) (Definition 6.2) and the group \( G \) isomorphic to \( L \) as sets give birth to a braided monoid.
\((L, m, \eta, \sigma)\) in \(\text{Set}_H\) (Here, \(H = L\)). With the aid of Propositions 3.1, 3.2, 6.6, 6.8, Remarks 5.5, 5.6, Corollary 3.5, and Proposition 3.6, Theorem 6.10 may be summarized by saying that every \((L, m, \eta)\)-module \((X, m_X)\) in \(\text{Set}_H\), together with a family (indexed by \(X\)) of homomorphisms of the group \(G\), can produce a dynamical reflection map \(k : L \otimes X \to L \otimes X \in \text{Set}_H\) (3.7).

In this section, we construct examples of dynamical reflection maps and reflection maps from this viewpoint (Cf. [4, Section 3]).

We first present examples of \((L, m, \eta)\)-module \((X, m_X)\).

**Example 8.1.** \((L, m)\) is a left \((L, m, \eta)\)-module, as is easy to check. Here, \(m : L \otimes L \to L\) is defined by (6.5). This module is called left regular.

The set \(X\) of one element is a left \((L, m, \eta)\)-module.

**Example 8.2.** Let \(X\) be a set of one point. Write \(X = \{x\}\). We fix any \(\lambda_1 \in H\), and define \(\lambda \cdot_X x = \lambda_1\). Then \((X, \cdot_X)\) is an object of \(\text{Set}_H\); moreover, \((X, m_X)\) is a left \((L, m, \eta)\)-module. Obviously, \(m_X(\lambda)(a, x) = x\) (\(\lambda \in H, a \in L\)).

Every nonempty set \(X\) has a left \((L, m, \eta)\)-module structure.

**Example 8.3.** Let \(L \times X \ni (a, x) \mapsto ax \in X\) be a map satisfying that, for all \(a \in L\) and \(y \in X\), there uniquely exists \(x \in X\) such that \(ax = y\). We will use the symbol \(a\backslash y\) to denote this unique element \(x \in X\). Thus, \(a(a\backslash y) = y\) and \(a\backslash(ay) = y\).

For any map \(f : X \to H(= L)\), we define \(\lambda \cdot_X x \in H\) (\(\lambda \in H, x \in X\)) by

\[
\lambda \cdot_X x = f(e_L \backslash(\lambda x)) \quad (\lambda \in H(= L), x \in X).
\]

Then

\[
\lambda \cdot_X (\lambda \backslash((\lambda a)x)) = (\lambda a) \cdot_X x
\]

for all \(\lambda \in H(= L), a \in L, \) and \(x \in X\). The proof is immediate from the definition of \(\lambda \cdot_X x \in H\). \((X, \cdot_X)\) is hence an object of \(\text{Set}_H\).

**Remark 8.4.** If a map \(\cdot_X : L \times X \to X\) satisfies (8.2), then the set \(\{e_L \cdot_X y \in L \mid y \in X\}\) completely determines \(\lambda \cdot_X x\) for all \(\lambda \in H(= L)\) and \(x \in X\). In fact, \(\lambda \cdot_X x = e_L \cdot_X (e_L \backslash(\lambda x))\) from (8.2).

Let \(m_X(\lambda)\) \((\lambda \in H)\) denote the map from \(L \times X\) to \(X\) defined by

\[
m_X(\lambda)(a, x) = \lambda \backslash((\lambda a)x) \quad (a \in L, x \in X).
\]

It follows from (8.2) that \(m_X : L \otimes X \to X\) is a morphism of \(\text{Set}_H\).
Moreover, \((X, m_X)\) is a left \((L, m, \eta)\)-module. In fact, for \(\lambda \in H(= L), a, b \in L\), and \(x \in X\),
\[
(m_X(m \otimes 1_X))(\lambda)((a, b), x) = \lambda \setminus ((\lambda_a b)x) \\
= \lambda \setminus ((\lambda a) \setminus ((\lambda a) b)x) \\
= (m_X(1_L \otimes m_X) a_{LX})(\lambda)((a, b), x),
\]
and \((m_X(\eta \otimes 1_X))(\lambda)(\bullet, x) = \lambda \setminus ((\lambda e_L)x) = \lambda \setminus (\lambda x) = x = l_X(\lambda)(\bullet, x)\). The morphism \(m_X : L \otimes X \rightarrow X\) thus satisfies \((3.2)\) and \((5.1)\).

Let \((X, m_X)\) be a left \((L, m, \eta)\)-module, and let \(\lambda_0 \in L\) denote the unique element satisfying \(\pi(\lambda_0) = e_G\). Here, \(e_G\) is the unit element of the group \(G\). Next we introduce a family \(\{f^\lambda_x : G \rightarrow G \mid x \in X\}\) of homomorphisms of the group \(G\).

**Example 8.5.** Let \(f^\lambda_x : G \rightarrow G \mid x \in X\) denote the group homomorphisms of \(G\) defined by \(f^\lambda_x(a) = e_G\ (a \in G)\). By means of \((7.53)\), the family \(\{f^\lambda_x : G \rightarrow G \mid x \in X\}\) produces \(\Pi^\lambda_x(a) = e_L\ (\lambda \in H(= L), a \in L, x \in X)\), and the dynamical reflection map \(k : L \otimes X \rightarrow L \otimes X\) in \((7.56)\) is consequently \(k(\lambda)(a, x) = (e_L, m_X(\lambda)(a, x))\) (Cf. \([4, \text{Example 3.1}]\)).

**Example 8.6.** For every \(x \in X\), we set \(f^\lambda_x = 1_G\), which is a group homomorphism. By means of \((7.53)\), the family \(\{f^\lambda_x : G \rightarrow G \mid x \in X\}\) yields
\[
(8.3) \quad \Pi^\lambda_x(a) = \lambda \setminus \pi^{-1}(\pi(\lambda) \pi(\lambda a) \pi(\lambda)^{-1}) \quad (\lambda \in H(= L), a \in L, x \in X),
\]
and the dynamical reflection map \(k : L \otimes X \rightarrow L \otimes X\) in \((7.56)\) is
\[
k(\lambda)(a, x) \\
= (\lambda \setminus \pi^{-1}(\pi(\lambda) \pi(\lambda a) \pi(\lambda)^{-1}), \\
m_X(\pi^{-1}(\pi(\lambda) \pi(\lambda a) \pi(\lambda)^{-1}))(\pi^{-1}(\pi(\lambda) \pi(\lambda a) \pi(\lambda)^{-1} \setminus (\lambda a), x)).
\]
If the group \(G\) is abelian, then \(\Pi^\lambda_x(a) = a\ (\lambda \in H(= L), a \in L, x \in X)\) from \((8.3)\) (Cf. \([4, \text{Example 3.4}]\)), and the dynamical reflection map is trivial: \(k(\lambda)(a, x) = (a, x)\).

If the group \(G\) is abelian, then the map \(G \ni a \mapsto a^{-1} \in G\) is a group homomorphism.

**Example 8.7.** We assume that the group \(G\) is abelian. For every \(x \in X\), the map \(f^\lambda_x : G \rightarrow G\) is defined by \(f^\lambda_x(a) = a^{-1}\ (a \in G)\). From \((7.53)\), the family \(\{f^\lambda_x : G \rightarrow G \mid x \in X\}\) gives birth to \(\Pi^\lambda_x(a) = \lambda \setminus \pi^{-1}(\pi(\lambda)^2 \pi(\lambda a)^{-1})\)
Table 2: The map $L \times X \ni (a, x) \mapsto ax \in X$

|     | $x_1$ | $x_2$ | $x_3$ |
|-----|-------|-------|-------|
| $e_L$ | $x_1$ | $x_2$ | $x_3$ |
| $l_1$ | $x_2$ | $x_1$ | $x_3$ |
| $l_2$ | $x_3$ | $x_2$ | $x_1$ |
| $l_3$ | $x_1$ | $x_3$ | $x_2$ |
| $l_4$ | $x_2$ | $x_3$ | $x_1$ |
| $l_5$ | $x_3$ | $x_1$ | $x_2$ |

$(\lambda \in H(= L), a \in L, x \in X)$, and the dynamical reflection map $k : L \otimes X \to L \otimes X$ in (7.56) is

$$k(\lambda)(a, x) = (\lambda \pi^{-1}(\pi(\lambda)^2 \pi(\lambda a)^{-1}),$$

$$m_X(\pi^{-1}(\pi(\lambda)^2 \pi(\lambda a)^{-1}))(\pi^{-1}(\pi(\lambda)^2 \pi(\lambda a)^{-1}))(\lambda a, x)).$$

A family of the inner automorphisms of the group $G$ can produce a dynamical reflection map.

**Example 8.8.** Let $g_x (x \in X)$ be elements of $G$, and let $f^\lambda_x : G \to G$ ($x \in X$) denote the inner automorphisms of the group $G$: $f^\lambda_x(a) = g_x^{-1}ag_x$ ($a \in G$). Then the dynamical reflection map $k : L \otimes X \to L \otimes X$ in (7.56) is $k(\lambda)(a, x) = (\lambda \pi^c, m_X(b)(\pi^c(\lambda a), x))$ for $\lambda \in H, a \in L$, and $x \in X$ (Cf. [4, Examples 3.6 and 3.7]). Here,

$$b = \pi^{-1}(\pi(\lambda)(g_{m_X}(\lambda_0)(\lambda_0, x)))^{-1}\pi(\lambda a)\pi(\lambda)^{-1}g_{m_X}(\lambda_0)(\lambda_0, x)).$$

If a left $(L, \cdot, e_L)$-module $(X, m_X)$ is left regular (Example 8.1) and $g_c = \pi(\lambda_0c)$ for $c \in L(= X)$, then the dynamical reflection map $k : L \otimes L \to L \otimes L$ in the above example is exactly the morphism $\sigma^2 : L \otimes L \to L \otimes L$ (Cf. [4, Examples 4.3 and 8.6]).

The following example shows that there exists a dynamical reflection map $k : L \otimes X \to L \otimes X$ that is dependent on $\lambda \in H$.

**Example 8.9.** Let $(L, \cdot, e_L)$ denote the left quasigroup with a unit in Example 6.3. We set $H = L$ and $X = \{x_1, x_2, x_3\}$. The map $L \times X \ni (a, x) \mapsto ax \in X$ is defined by Table 2. We note that, for any $a \in L$ and $y \in X$, there uniquely exists $x \in X$ such that $ax = y$.

On account of Example 8.3, we can define $\lambda \cdot_X x \in L$ ($\lambda \in H, x \in X$) by (8.1), in which the following map $f : X \to H$ is used: $f(x_1) = l_2; f(x_2) = l_4$. 

39
Table 3: The map \( \cdot : H \times X \to H \)

| \( e_L \) | \( l_2 \) | \( l_4 \) | \( l_3 \) |
|---|---|---|---|
| \( l_1 \) | \( l_4 \) | \( l_2 \) | \( l_3 \) |
| \( l_2 \) | \( l_3 \) | \( l_4 \) | \( l_2 \) |
| \( l_3 \) | \( l_2 \) | \( l_3 \) | \( l_4 \) |
| \( l_4 \) | \( l_4 \) | \( l_3 \) | \( l_2 \) |
| \( l_5 \) | \( l_3 \) | \( l_2 \) | \( l_4 \) |

and \( f(x_3) = l_3 \). We present the map \( \cdot : H \times X \to H \) in Table 3. Example 8.3 can produce a left \( (L, m, \eta) \)-module \( (X, m_X) \).

Let \( G \) denote the symmetric group \( S_3 \) of \( \{1, 2, 3\} \). We next define a bijection \( \pi : L \to G \) by:

\[
\begin{align*}
\pi(e_L) &= \text{id}; \\
\pi(l_1) &= (123); \\
\pi(l_2) &= (132); \\
\pi(l_3) &= (12); \\
\pi(l_4) &= (13); \\
\pi(l_5) &= (23).
\end{align*}
\]

We write \( g_{x_1} = (132), g_{x_2} = (13), \) and \( g_{x_3} = (12) \).

Remark 8.10. Each family of the group homomorphisms \( f^\lambda_x : G \to G \) (\( x \in X \)) in Examples 8.5–8.8 is defined regularly; however, we can choose them more variously, because we do not need any condition among the group homomorphisms \( f^\lambda_x : G \to G \) (\( x \in X \)). For example, we can produce a dynamical reflection map by means of the group homomorphisms \( f^\lambda_x : G \to G \) (\( x \in X \)) defined by

\[
f^\lambda_x(a) = \begin{cases} 
  e_G, & \text{for some } x \in X; \\
  a, & \text{otherwise.}
\end{cases}
\]

At the end of this section, we deal with the reflection map, a dynamical reflection map \( k(\lambda) : L \times X \to L \times X \) that is independent on the dynamical parameter \( \lambda \in H \). For this purpose, we first introduce a necessary and sufficient condition for the dynamical Yang-Baxter map \( \sigma(\lambda) : L \times L \to L \times L \) to be independent on the dynamical parameter \( \lambda \).

Let \( * : L \times L \to L \) denote the binary operation defined by

\[
a * b = \pi^{-1}(\pi(a)\pi(b)) \quad (a, b \in L).
\]

That is to say, \( (L, *) \) is a group isomorphic to the group \( G \) via the bijection \( \pi : L \to G \).
Proposition 8.11. If \((L, \cdot, e_L)\) is a group, then the following two conditions are equivalent:

(1) \(\pi(e_L) = e_G\), and the map \(\sigma(\lambda)\) defined by (6.3) is independent on the dynamical parameter \(\lambda \in H\). Here, \(e_G\) is the unit element of the group \(G\);

(2) \(a(b*c) = (ab)\overline{a-1}(ac)\) for all \(a, b, c \in L\). Here, the element \(a^{-1}(\in L)\) is the inverse of \(a \in L\) with respect to the group structure \((L, \cdot)\).

Under the above conditions,

\[
\sigma(\lambda)(a, b) = (\overline{a^{-1}}(ab), \overline{a^{-1}}(ab)ab) \quad (a, b \in L).
\]

Here, \(\overline{c}(\in L)\) is the inverse of \(c \in L\) with respect to the group structure \((L, \cdot)\).

Proof. Let us first prove the condition (2) from (1). Since \(\sigma(\lambda)\) is independent on the dynamical parameter \(\lambda\), \(\xi_\lambda(a, b) = \xi_{e_L}(a, b)\) for all \(\lambda \in H(= L)\) and \(a, b \in L\). From (6.1), (6.2), and \(\pi(e_L) = e_G\),

\[
\lambda a (\lambda a)^{-1} \pi((\lambda a)b)) = \pi^{-1}(\pi(a)\pi^{-1}(ab))
\]

for all \(\lambda \in H\) and \(a, b \in L\).

Because of (8.6), \(\pi(e_L) = e_G\), and the fact that

\[
\begin{align*}
\overline{a^{-1}} &= \pi^{-1}(\pi(a))^{-1}, \\
(ab)\overline{a^{-1}}(ac) &= \pi^{-1}(\pi(ab)\pi(a)\pi^{-1}(ac)) \\
&= ab\pi^{-1}(\overline{b})\pi^{-1}(\overline{bc}) \\
&= ab(\overline{b}\pi^{-1}(\pi(b)\pi(c))) \\
&= a(b*c)
\end{align*}
\]

for all \(a, b, c \in L\), which is the desired conclusion.

Next we show the condition (1) from (2). It follows from the condition (2) that \(\pi(e_L) = e_G\). In fact, by substituting \(e_L\) for \(a\) and \(c\) in the condition (2), we obtain \(b*e_L = b*e_L^{-1}*e_L = b\) for every \(b \in L\), which means that \(e_L\) is the unit element of the group \((L, \cdot)\). Hence, \(\pi(e_L) = e_G\).

From the condition (2), \(a\pi^{-1}(\pi(b)\pi(c)) = \pi^{-1}(\pi(ab)\pi(a)\pi^{-1}(ac))\) for all \(a, b, c \in L\). On account of this equation, (6.2), and \(\pi(e_L) = e_G\),

\[
\xi_\lambda(a, b) = \lambda a \pi^{-1}(\pi(\lambda)\pi^{-1}(\pi((\lambda a)b))) \\
= \lambda a \pi^{-1}(\pi(\pi(\overline{\lambda})\pi(b))) \\
= \pi^{-1}(\pi(a)\pi^{-1}(ab)),
\]

41
and $\xi_\lambda(a, b)$ is thus independent on the dynamical parameter $\lambda$.

In view of (6.3) and the fact that $(L, \cdot)$ is a group,

\begin{equation}
\eta_\lambda(a, b) = (\lambda \xi_\lambda(a, b)) \lambda ab = \xi_\lambda(a, b) ab,
\end{equation}

and $\eta_\lambda(a, b)$ is also independent on $\lambda$. Therefore (1) holds.

(6.4), (8.4), and (8.7)–(8.9) immediately induce (8.5), and the proof is complete.

The set $L$ in Proposition 8.11 is a skew left brace $[4, 9]$.

**Definition 8.12.** $(L, \cdot, \ast)$ is a skew left brace, iff $(L, \cdot)$ and $(L, \ast)$ are groups satisfying the condition (2) in Proposition 8.11.

From this definition, Proposition 8.11 may be summarized by saying that the structure of the skew left brace is a necessary and sufficient condition for the dynamical Yang-Baxter map $\sigma$ (6.4) to be a Yang-Baxter map (Cf. [4, Theorem 5.3]).

In the reminder of this section we assume that $(L, \cdot, \ast)$ is a skew left brace. Then the map $m(\lambda) : L \times L \to L$ defined by (6.5) is independent on $\lambda$, since $m(\lambda)(a, b) = \lambda ab$ for all $a, b \in L$. As a result, the map $m(\lambda)$ is exactly a binary operation of the group $(L, \cdot)$; that is to say, the monoid $(L, m, \eta)$ is the group $(L, \cdot)$.

Let $X$ be a left $(L, \cdot)$-action. That is, for any $a \in L$ and $x \in X$, there exists $ax \in X$ satisfying $(ab)x = a(bx)$ and $e_L x = x$ for all $a, b \in L$ and $x \in X$.

**Proposition 8.13.** For any map $f : X \to H(= L)$, we define $\lambda \cdot_x x \in H$ $(\lambda \in H, x \in X)$ by $\lambda \cdot_x x = f(\lambda x)$. Then $(X, \cdot_x)$ is an object of $\text{Set}_H$.

We set $m_X(\lambda)(a, x) = ax$ for all $\lambda \in H, a \in L$, and $x \in X$. Because $\lambda \cdot_x m_X(\lambda)(a, x) = (\lambda a) \cdot_x x$ $(\lambda \in H, a \in L, x \in X)$, $m_X : L \otimes X \to X$ is a morphism of $\text{Set}_H$, and moreover

**Proposition 8.14.** $(X, m_X)$ is a left $(L, m, \eta)$-module in $\text{Set}_H$.

That is, every $(L, \cdot)$-action $X$, together with a suitable $\cdot_X : H \times X \to H$, can be regarded as a left module of $(L, \cdot)$ in $\text{Set}_H$.

From the above proposition and $\lambda_0 = e_L$, a left $(L, \cdot)$-action $X$, together with a family $\{f^x_x \mid x \in X\}$ of homomorphisms of the group $G$ isomorphic to $(L, \ast)$, can produce a dynamical reflection map $k(\lambda) : L \times X \to L \times X$ $(\lambda \in H)$ associated with the Yang-Baxter map $\sigma : L \times L \to L \times L$ (8.5).

We are now in a position to show a necessary and sufficient condition for the dynamical reflection map $k(\lambda)$ to be independent on the dynamical parameter $\lambda \in H$. 

42
Proposition 8.15. The following two conditions are equivalent:

1. The dynamical reflection map $k(\lambda)$ is independent on $\lambda$;
2. $\pi(\lambda)f_{\lambda(ax)}^e(\pi(\lambda a)) = \pi(\lambda)\pi^{-1}(f_{ax}^e(\pi(a)))f_{\lambda(ax)}^e(\pi(\lambda))$ for all $\lambda \in H(= L), a \in L,$ and $x \in X$.

Under the above conditions,

\[ (8.10) \quad k(\lambda)(a, x) = (\pi^{-1}(f_{ax}^e(\pi(a))), (\pi^{-1}(f_{ax}^e(\pi(a))a)x) \quad (a \in L, x \in X). \]

Proof. From (7.56), $k(\lambda)(a, x) = (\Pi_{ax}^\lambda(a), (\Pi_{ax}^\lambda(a)a)x),$ and the condition (1) is hence equivalent to the following condition:

3. $\Pi_{ax}^\lambda(a) = \Pi_{ax}^e(a)$ for all $a \in L$ and $x \in X.$

Since $f_{ax}^e : G \to G$ is a group homomorphism, $f_{ax}^e(e_G) = e_G$. Because of (7.53) and $\pi(e_L) = e_G$ (See Proposition 8.11), the condition (3) is exactly the same as

\[ \lambda \pi^{-1}(\pi(\lambda)f_{\lambda(ax)}^e(\pi(\lambda a))f_{\lambda(ax)}^e(\pi(\lambda))^{-1}) = \pi^{-1}(f_{ax}^e(\pi(a))) \]

for all $\lambda \in H(= L), a \in L,$ and $x \in X$ (We note that $\lambda_0 = e_L$, which is equivalent to the condition (2)).

The above condition (3), (7.53), and (7.56) immediately induce (8.10), and the proof is complete. \(\square\)

Remark 8.16. On account of (8.10), this dynamical reflection map $k(\lambda) : L \times X \to L \times X$ is exactly the same as the reflection map in [4, Theorem 8.5].

By means of the family of group homomorphisms in Example 8.7, we can produce a reflection map associated with the Yang-Baxter map (8.5).

Example 8.17. We assume that $(L, \ast)$ is abelian, and then so is the group $G$.

The family of group homomorphisms in Example 8.7 satisfies the condition (2) of Proposition 8.15.

In fact, by substituting $\lambda, a,$ and $a^{-1}$ for $a, b,$ and $c$ in the condition (2) of Proposition 8.11 respectively, we obtain $\lambda \ast (\lambda a)^{-1} = (\lambda a^{-1}) \ast \lambda^{-1}$ for all $\lambda \in H(= L)$ and $a \in L$. Here we note that $e_L(\in L)$ is the unit element of the group $(L, \ast)$, since $\pi(e_L) = e_G$. Because $a \ast b = \pi^{-1}(\pi(a)\pi(b))$ and $\lambda^{-1} = \pi^{-1}(\pi(\lambda)^{-1})$ (8.7), it follows from the above condition that $\pi(\lambda)\pi(\lambda a)^{-1} = \pi(\lambda\pi^{-1}(\pi(a)^{-1}))\pi(\lambda)^{-1}$ for all $\lambda \in H(= L)$ and $a \in L$, which is exactly the same as the condition (2) of Proposition 8.15.

Therefore, for every left $(L, \cdot)$-action $X$, the dynamical reflection map constructed here is independent on the dynamical parameter $\lambda$, and, from (8.10), it is a reflection map $k : L \times X \ni (a, x) \mapsto (a^{-1}, ((a^{-1})a)x) \in L \times X$ associated with the Yang-Baxter map (8.5) as a result.
Remark 8.18. If the group $(L, \ast)$ is abelian, then $(L, \cdot, \ast)$ is a brace \cite{9, 25}.

9 Quiver-theoretic solutions to reflection equation

The dynamical Yang-Baxter maps can produce quiver-theoretic solutions to the (quantum) Yang-Baxter equation, which is due to \cite{22}. This section demonstrates that the dynamical reflection maps give birth to quiver-theoretic solutions to the reflection equation (9.2).

We first introduce the category of quivers following \cite{22}. Let $H$ be a nonempty set. A quiver over $H$ is, by definition, a set $Q$ with two maps $s_Q : Q \to H$ and $t_Q : Q \to H$, called the source map and the target map respectively. An object of the category Quiv$_H$ is a quiver over $H$. For two quivers $Q$ and $Q'$ over $H$, a morphism $f : Q \to Q'$ is a (set-theoretic) map $f : Q \to Q'$ satisfying $s_{Q'}(f(a)) = s_Q(a)$ and $t_{Q'}(f(a)) = t_Q(a)$ for every $a \in Q$. The identity $1_Q : Q \to Q$ is defined by the identity map $1_Q : Q \to Q$, and the composition of two morphisms $f : Q \to Q'$, $g : Q' \to Q''$ is the composition of maps $f$ and $g$: $gf : Q \to Q''$.

This category Quiv$_H$ of quivers over $H$ is a tensor category (Definition \ref{2.1}) with the following fiber product.

For $Q, R \in$ Quiv$_H$, we set $Q \times_H R = \{(a, b) \in Q \times R \mid t_Q(a) = s_R(b)\}$. This fiber product $Q \times_H R$ is an object of Quiv$_H$, together with the source map $s_{Q \times_H R}(a, b) = s_Q(a)$ and the target map $t_{Q \times_H R}(a, b) = t_R(b)$.

For two morphisms $f : Q \to Q'$ and $g : R \to R'$ of Quiv$_H$, we define the fiber product $f \times_H g : Q \times_H R \to Q' \times_H R'$ by $(f \times g)(a, b) = (f(a), g(b))$. In fact, $(f \times g)(a, b) \in Q' \times_H R'$ for every $(a, b) \in Q \times_H R$.

The ordinary associativity of sets is the associativity constraint of Quiv$_H$: $a_{QRS} : (Q \times_H R) \times_H S \ni ((a, b), c) \mapsto (a, (b, c)) \in Q \times_H (R \times_H S)$ for $Q, R, S \in$ Quiv$_H$. Obviously, this associativity constraint satisfies the pentagon axiom \ref{2.1}.

The set $H$ with the source and the target maps $s_H = t_H = 1_H$ is a unit of Quiv$_H$, and the left and the right unit constraints with respect to the unit $H$ are respectively defined by: $l_Q : H \times_H Q \ni (\lambda, a) \mapsto a \in Q$ and $r_Q : Q \times_H H \ni (a, \lambda) \mapsto a \in Q$ for every $Q \in$ Quiv$_H$.

This $(\text{Quiv}_H, \times_H, H, a, l, r)$ is a tensor category (Definition \ref{2.1}), because the triangle axiom \ref{2.2} holds.

A useful functor $Q : \text{Set}_H \to$ Quiv$_H$ is constructed in \cite{22}. For $X \in \text{Set}_H$, we set $Q(X) = H \times X$. This $Q(X)$, together with the source map $s_{Q(X)}(\lambda, x) = \lambda$ and the target map $t_{Q(X)}(\lambda, x) = \lambda x$, is an object of Quiv$_H$.
For a morphism \( f : X \to Y \) of \( \text{Set}_H \), we define the map \( Q(f) : Q(X) \to Q(Y) \) by \( Q(f)(\lambda, x) = (\lambda, f(\lambda)(x)) \) (\( (\lambda, x) \in Q(X) \)). This \( Q(f) : Q(X) \to Q(Y) \) is a morphism of \( \text{Quiv}_H \), and \( Q : \text{Set}_H \to \text{Quiv}_H \) is hence a functor.

Moreover, this \( Q : \text{Set}_H \to \text{Quiv}_H \) is a tensor functor [15, Definition XI.4.1]. In fact, \( \varphi_0 : H \ni \lambda \mapsto (\lambda, \bullet) \in Q(I) \) is an isomorphism of \( \text{Quiv}_H \); and \( \varphi_2(X, Y) : Q(X) \times_H Q(Y) \ni ((\lambda, x), (\kappa, y)) \mapsto (\lambda, (x, y)) \in Q(X \otimes Y) \) is a natural isomorphism for all \( X, Y \in \text{Set}_H \). Since they satisfy

\[
\varphi_2(X, Y \otimes Z)(1_{Q(X)} \times_H \varphi_2(Y, Z))a_{Q(X)Q(Y)Q(Z)} = Q(a_{XYZ})\varphi_2(X \otimes Y, Z)(\varphi_2(X, Y) \times_H 1_{Q(Z)}),
\]

\[
l_{Q(X)} = Q(l_X)\varphi_2(I, X)(\varphi_0 \times_H 1_{Q(X)}),
\]

\[
r_{Q(X)} = Q(r_X)\varphi_2(X, I)(1_{Q(X)} \times_H \varphi_0)
\]

for all \( X, Y, Z \in \text{Set}_H \), \( Q : \text{Set}_H \to \text{Quiv}_H \) is a tensor functor.

Remark 9.1. The above tensor functor \( Q : \text{Set}_H \to \text{Quiv}_H \) is fully faithful [22, Theorem 2.7].

Through this tensor functor \( Q : \text{Set}_H \to \text{Quiv}_H \), every dynamical Yang-Baxter map \( \sigma : L \otimes L \to L \otimes L \) gives birth to a solution

\[
\tilde{\sigma} = \varphi_2(L, L)^{-1}Q(\sigma)\varphi_2(L, L) : Q(L) \times_H Q(L) \to Q(L) \times_H Q(L)
\]

to the braid relation (3.15) in \( \text{Quiv}_H \) [22, Theorem 2.9].

Let \( k : L \otimes X \to L \otimes X \) be a dynamical reflection map, a solution to the reflection equation (3.20) in \( \text{Set}_H \), associated with the dynamical Yang-Baxter map \( \sigma : L \otimes L \to L \otimes L \). The proof of the following proposition is straightforward.

**Proposition 9.2.** The morphism \( \tilde{k} = \varphi_2(L, X)^{-1}Q(k)\varphi_2(L, X) : Q(L) \times_H Q(X) \to Q(L) \times_H Q(X) \) satisfies the following reflection equation in \( \text{Quiv}_H \) associated with the solution \( \tilde{\sigma} : Q(L) \times_H Q(L) \to Q(L) \times_H Q(L) \) (3.11).

\[
a_{Q(L)Q(L)Q(X)}^{-1}(1_{Q(L)} \times_H \tilde{k})a_{Q(L)Q(L)Q(X)}(\tilde{\sigma} \times_H 1_{Q(X)})a_{Q(L)Q(L)Q(X)}^{-1} = (\tilde{\sigma} \times_H 1_{Q(X)})a_{Q(L)Q(L)Q(X)}^{-1}(1_{Q(L)} \times_H \tilde{k})a_{Q(L)Q(L)Q(X)}(\tilde{\sigma} \times_H 1_{Q(X)})a_{Q(L)Q(L)Q(X)}^{-1}.
\]

Therefore, our method in this paper can produce the above \( \tilde{k} : Q(L) \times_H Q(X) \to Q(L) \times_H Q(X) \) called quiver-theoretic solutions to the reflection equation.
Remark 9.3. The reflection equation is the defining relation of the Artin monoid \([21]\) associated with the Coxeter matrix \(
abla \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \). Hence, our results in this paper may be summarized by saying that we can construct morphisms satisfying this defining relation in the tensor categories \(Set_H\) and \(Quiv_H\).

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