Thermodynamics of black holes with an infinite effective area of a horizon

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In some kinds of classical dilaton theory there exist black holes with (i) infinite horizon area $A$ or infinite $F$ (the coefficient at curvature in Lagrangian) and (ii) zero Hawking temperature $T_H$. For a generic static black hole, without an assumption about spherical symmetry, we show that infinite $A$ is compatible with a regularity of geometry in the case $T_H = 0$ only. We also point out that infinite $T_H$ is incompatible with the regularity of a horizon of a generic static black hole, both for finite or infinite $A$. Direct application of the standard Euclidean approach in the case of an infinite “effective” area of the horizon $A_{eff} = AF$ leads to inconsistencies in the variational principle and gives for a black hole entropy $S$ an indefinite expression, formally proportional to $T_H A_{eff}$. We show that treating a horizon as an additional boundary (that is, adding to the action some terms calculated on the horizon) may restore self-consistency of the variational procedure, if $F$ near the horizon grows not too rapidly. We apply this approach to Brans-Dicke black holes and obtain the same answer $S = 0$ as for ”usual” (for example, Reissner-Nordström) extreme classical black holes. We also consider the exact solution for a conformal coupling, when $A$ is finite but $F$ diverges and find that in the latter case both the standard and modified approach give rise to an infinite action. Thus, this solution represents a rare exception of a black hole without
nontrivial thermal properties.

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I. INTRODUCTION

At present, black hole thermodynamics is put on a firm basis due to the elaborated Euclidean action formalism. However, some gaps in important issues still persist here. In particular, it concerns gravity with dilaton (scalar) field. It was shown that some kinds of such a theory (in particular, Brans-Dicke theory \[1\]) predict quite unusual objects, having no analogs in general relativity - black holes with an infinite surface area of an event horizon \[-2\] \[-4\]. The natural question about thermodynamical interpretation (first of all, entropy) arises and an unusual character of such black holes makes it challenging.

Naive application of the Bekenstein - Hawking formula for the black hole entropy, proportional to the horizon area, would give here a physically meaningless answer. On the other hand, it turns out that the surface gravity \(\kappa\) for such black holes and the corresponding Hawking temperature \(T_H = \frac{\kappa}{2\pi}\) are exactly zero. This seemed to suggest another way of treating black hole thermodynamics, in the spirit of the already elaborated approach to extreme black holes \[5\] \[-7\]. According to it, an arbitrary finite temperature and zero entropy should be ascribed to classical extreme black holes. As the horizon area does not enter explicitly here, it would seem, on the first glance, that an unusual character of black holes with infinite horizon area does not change the approach. Nevertheless, thorough examination, suggested below, shows that this is not the case. First, naive calculation of the Euclidean action leads, as we will see, to the model-dependent contribution for a candidate on the entropy, that contradicts our expectations, based on knowledge of universality of black hole physics. Moreover, one cannot guarantee its finiteness and even non-negativity. Second, it turns out that the standard expressions for the Euclidean action has the following undesirable feature: it is not invariant in the case under discussion with respect to conformal transformation because of some additional horizon contribution, so its form depends
on the conformal frame. Third, if we trace back the variational principle, from which the solutions under discussion are obtained, it turns out that the variation of the action contains the terms (including the normal derivative of the metric) that vanished on the horizon for "usual" black holes but persist (and even may become infinite) for solutions admitting an infinite horizon area.

To resolve these difficulties, we suggest modification of the Euclidean action that resolves all three problems, inherent to the infinite area case, at once. In so doing, the third point is central since it concerns the foundation of the variation principle from which other properties stem. The modification consists in treating a horizon as an additional boundary that formally means adding to the action some terms calculated on the horizon. For "usual" extreme black holes these terms are automatically zero, so this modification agrees with previously obtained results [5] - [7]. Our analysis is valid for any static black holes and is not restricted to Brans-Dicke black holes, including them only as an example. We also examine the thermodynamic interpretation of black holes when the horizon area is finite, but the coupling between dilaton and curvature diverges.

It is worth noting that an attempt to examine thermodynamic interpretation of black holes in Brans-Dicke theory was made in [8]. In our view, the conclusion, made there, about complete failure of thermodynamic approach to black holes in Brans-Dicke theory, is incorrect. The range of parameters, considered in [8], corresponds to singular horizons, when it is obvious in advance that thermodynamics has no physical meaning. However, there exists another range (omitted in [8]), where the geometry on the horizon is regular and thermodynamics of Brans-Dicke black holes is well-defined (see details below).

II. REGULAR HORIZONS WITH AN INFINITE AREA

Before addressing thermodynamics issues, let us proof the following lemma.

*Static black holes with a regular horizon, having and infinite surface area $A$ can be extreme only (the surface gravity $\kappa = 0$).*
Consider an arbitrary static spacetime with an event horizon. It is convenient to use the coordinate system, exploited in \([9]\):

\[ds^2 = -V^2 dt^2 + \rho^2 dV^2 + \gamma_{ab} d\theta^a d\theta^b, \ a, b = 1, 2.\]

(1)

Then, it follows from the corresponding geometrical formulas that the Kretscmann invariant

\[I \equiv \frac{1}{4} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = G_i^{(3)} G_j^{(3)} + V^{-2} Y_{ij} Y^{ij}, \ Y_{ij} \equiv V_{;ij}\]

(2)

where \((3)\) refers to the three-geometry of slices \(t = const\), the covariant derivatives are also taken with respect to three-geometry. Simple calculations give us

\[Y_{ij} Y^{ij} = \rho^{-2} [K_{ab} K^{ab} + 2 \rho^{-2} \rho_{;a} \rho^{;a} + \rho^{-4} \left( \frac{\partial \rho}{\partial V} \right)^2],\]

(3)

where \(K_{ab}\) is two-dimensional extrinsic curvature tensor,

\[K_{ab} = \frac{1}{2\rho} \frac{\partial \gamma_{ab}}{\partial V}.\]

(4)

All indices in the right hand side of (2) and (3) are raised and lowed with respect to the three-dimensional metric of spacelike slices \(t = const\) which is positive-definite. Therefore, the first and the second terms in the right hand side of (2) and (3) are non-negative and, therefore, should be finite separately. In particular, it is true for terms with \(K_{ab} K^{ab}\). One can, for example, without the loss of generality, diagonalize \(K_{ab}\) and conclude that both \(K_1, K_2\) as well as the extrinsic curvature \(K^{(2)} = K_1 + K_2\) should be finite. In our context, below we will take advantage of the finiteness of the second term in (2).

Now observe that it follows from (4) that

\[\frac{\partial \eta}{\partial V} = \langle K^{(2)} \rho \rangle, \ \eta \equiv \ln A, \ \langle ... \rangle = A^{-1} \int d\theta^1 d\theta^2 \sqrt{\gamma}(...), \ \gamma = \det(\gamma_{ab}),\]

(5)

where \(A = \int d\theta^1 d\theta^2 \sqrt{\gamma}\) is the surface area of the slice with a constant \(V\). Actually, the quantity \(V\) enumerates equipotential surfaces \(-g_{00} = V^2 = V_0^2 = const\). If we take some surface \(f(V, y^1, y^2) = 0\), find the components \(n_\mu \sim \partial_\mu f\) of the vector, orthogonal to this surface and calculate \(n_\mu n^\mu\), one can easily see that, for the metric (3) and the choice
$f = V^2 - V_0^2$, such a quantity on the surface $V = V_0$ is proportional to $V^2$. Thus, when $V \to 0$, the vector $n_\mu$ becomes isotropic. This means that the surface under discussion represents a Killing horizon which, due to the staticity of the metric, coincides with the event horizon. Thus, the position of the horizon is determined by the condition $V = 0$ and does not involve other metric functions or coordinates. As $g_{00} = 0$ on the horizon and $g_{00} < 0$ everywhere in the static region, as usual (in particular, $g_{00} \to -1$ at infinity in asymptotically flat spacetimes), $g_{00}$ as well as the quantity $V$ cannot vanish in an intermediate region between a horizon and infinity, so one can choose $V > 0$ there (for example, $V = \sqrt{1 - \frac{2m}{r}}$ in the Schwarzschild case in curvature coordinates).

As one approaches the horizon, $V \to 0$ and it follows from the finiteness of $I$ that $\rho, a \to 0$. Thus, one obtains the constancy of the surface gravity $\kappa = \rho_0^{-1}$, where $\rho_0 = \lim_{V \to 0} \rho$ (the zero law of black hole thermodynamics (10)).

Let a black hole be non-extreme, $\kappa \neq 0$, $\rho_0$ is finite. Then we see from (2), (3), (5) that $K^{(2)} \sim a(\theta^1, \theta^2)V \to 0$ near the horizon, where $a(\theta^1, \theta^2)$ is bounded on the horizon. Taking into account that, by assumption, $\rho_0$ is finite, we obtain that (i) $\frac{\partial \eta}{\partial V} \sim V \to 0$. If we want to have a black hole with an infinite horizon area, (ii) $\eta \to \infty$ near the horizon. It is obvious that both properties (i) and (ii) are mutually inconsistent that forbids the existence of regular nonextreme black holes with an infinite horizon area. This completes the proof.

However, if a black hole is extreme, $\kappa = 0$, $\rho \to \infty$, $\frac{\partial \eta}{\partial V} \sim V(\rho)$ and we have a competition of two factors $V$ and $\rho$. This leaves the opportunity of infinite $A$ that is indeed realized in some scalar-gravitation theories, as is clear from corresponding exact solution in the spherically-symmetrical case (1) - (4).

Now we address briefly one more issue. There was made an observation in (11) for arbitrary spherically-symmetrical configurations that an infinite $T_H$ is inconsistent with a regularity of a horizon. Now we extend this observation to an arbitrary static (not necessarily spherically-symmetrical) configuration. Let $\kappa = \infty$. This means that $\rho \to 0$ near the horizon. It follows from (2), (3) that (a) $x \equiv \rho^{-2} \to \infty$, (b) $\frac{\partial x}{\partial V} \to 0$, when $V \to 0$. It is clear that (a) and (b) are mutually inconsistent. It is worth noting that this conclusion is
valid irrespective of whether the horizon area is finite or infinite.

Thus, an arbitrary static black hole with a regular horizon cannot have an infinite surface gravity (infinite Hawking temperature).

III. STANDARD EUCLIDEAN ACTION APPROACH TO DILATON BLACK HOLES

In this section we rederive basic formulas of black hole thermodynamics for the presence of a scalar field (dilaton). We use the generalization of the Hilbert action to the dilaton case. If the Hamiltonian constraint is taken into account, the action, as we will see, takes the thermodynamic form with the local temperature (or its inverse $\beta$) on the boundary as a relevant thermodynamic parameter. In other words, fixing $\beta$ on the boundary, we work with the canonical ensemble throughout the paper. By itself, such an extension to dilaton theories is quite direct. Meanwhile, we will need them for our purposes in general setting. In so doing, I follow almost the same line as in [12], where the general canonical approach to self-gravitating systems was suggested (having generalized previous observations for the spherically-symmetrical case [13]).

We restrict ourselves to static spacetimes only. Then the Euclidean metric can be written as

$$ds^2 = b^2 d\tau^2 + g_{ij} dx^i dx^j,$$

where all functions are independent of $\tau$. Consider the system governed by the Euclidean gravitation-dilaton action $I_{gd} = I_V + I_B$, where the bulk part

$$I_V = -\frac{1}{16\pi} \int_M d^4x \sqrt{-g} [RF(\phi) + V(\phi)(\nabla\phi)^2 + U(\phi)]$$

is taken over the manifold $M$, and the term over the boundary $\partial M$

$$I_B = \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{g_3} FK$$

(7)
is necessary to make the variational procedure self-consistent, $K$ is the second fundamental form of the boundary. If $F = 1$, we return to the case of general relativity. In the Euclidean action integration is performed over Euclidean time $\tau$, $0 \leq \tau \leq \beta_0 = T_0^{-1}$. In what follows it is assumed that the Euclidean manifold is regular and does not contain conical singularities. This implies that we consider either non-extreme black holes with $T_0 = T_H$ or extreme ones ($T_H = 0$).

One can derive from (7) the field equations

$$Z_{\nu}^{\mu} \equiv 2FG_{\nu}^{\mu} + 2(\delta_{\mu}^{\nu} \Box F - \nabla_{\mu} \nabla_{\nu} F) - U\delta_{\mu}^{\nu} + 2V\nabla_{\mu} \phi \nabla_{\nu} \phi - \delta_{\mu}^{\nu} V(\nabla \phi)^2 = 0, \quad (9)$$

$\nabla_{\mu}$ is the operator of four-dimensional covariant differentiation with respect to the metric $g_{\mu \nu}$.

Take into account that

$$R = -2G_{\tau}^{\tau} - 2\frac{\Delta_3 b}{b}, \quad (10)$$

where Laplacian should be calculated with respect to the three-dimensional metric $g_{ik}$. The extrinsic curvature $K = -\nabla_{\mu} n^\mu$, where $n^\mu$ is vector, orthogonal to the boundary surface and pointed outward. Then, writing $K = -\frac{(b\sqrt{g_{3\nu}})_{,\nu}}{b\sqrt{g_{3\mu}}}$, we easily obtain

$$K = K^{(2)} - \frac{b_i n^i}{b}, \quad (11)$$

comma denotes ordinary derivative. Here $K^{(2)}$ is the extrinsic curvature, measured with respect to a three-dimensional metric $g_{ij}$, $K^{(2)} = -\frac{(n^i \sqrt{g_{3}})_{,i}}{\sqrt{g_{3}}}$ . Applying the Gauss theorem, after some algebra we obtain

$$I_{gd} = \frac{\beta_0}{16\pi} \int d^3 x \sqrt{g_3} b Z_0^0 + \int_B d\sigma \beta \varepsilon + Y_1 - Y_2, \quad (12)$$

$$\varepsilon = -\frac{(n^i F)_{,i}}{8\pi} = -\frac{(n^i \sqrt{g_{3}} F)_{,i}}{\sqrt{g_{3}}}, \quad (13)$$

$$Y_1 = \frac{\beta_0}{8\pi} \int_H d\sigma b F_{,i} n^i, \quad (14)$$
\[ Y_2 = \frac{\beta_0}{8\pi} \int_H d\sigma \frac{\partial b}{\partial l} F, \]  

(15)

where \( d\sigma \) is the element of a two-dimensional surface, \( \frac{\partial b}{\partial l} = b_i n^i \) is the normal derivative, \( \beta = \beta_0 b \) is the inverse local Tolman temperature, \( \varepsilon \) has the meaning of quasilocal energy density for gravitation-dilaton system and directly generalizes the corresponding formula for pure gravitation case due to the factor \( F \). The index ”B” refers to a physical boundary (it is supposed that we have a black hole enclosed in a cavity), ”H” is related to the event horizon.

Now consider separately two cases. First, let on the horizon both \( F \) and the surface area \( A \) be finite (”normal” case). Then \( Y_1 = 0 \) due to the factor \( b \). On the horizon \( \frac{\partial b}{\partial l} \to \kappa \), where \( \kappa \) is a surface gravity, constant on it due to the zero law of black hole thermodynamics. As a result, we get the thermodynamic form for the action

\[ I_{gd} = \int_B d\sigma \beta \varepsilon - S, \]  

(16)

where the black hole entropy is identified with \( Y_2 \), its value is equal to

\[ S = \frac{A_{eff} T_H}{4 T_0}, \]  

(17)

\( T_H \equiv \frac{\kappa}{2\pi} \) is the Hawking temperature, \( T_0 = \beta_0^{-1} \), \( A_{eff} = AF_h \), \( F_h \) is the value of \( F \) on the horizon. The formula (17) embraces at once two kinds of regular topologies - nonextreme black holes with \( T_H = T_0 \) and extreme ones.

(i) In the first case we obtain for the black hole entropy the Bekenstein-Hawking value, generalized to the presence of a scalar field:

\[ S = \frac{A_{eff}}{4}. \]  

(18)

(ii) If a black hole is extreme \( (T_H = 0) \), and if we identified the Euclidean time with an arbitrary finite period \( T_0^{-1}(T_0 \neq 0) \), we get from (17) that \( S = 0 \) in accordance with prescription of [5] - [7]. The basic formulas remain practically intact if the terms, say, with electromagnetic field are included into the scheme.
However, what we will be interested in is not this slight generalization of the action formalism to the scalar field, but the special cases of infinite $A_{\text{eff}}$ which exist due to this field entirely and have no analogs in general relativity. We will see that direct application of the above formulas fails and we are led to some modification of the boundary terms in the action.

IV. CONFORMAL TRANSFORMATIONS

Consider the transformations

$$g_{\mu\nu} = e^{2\psi} g_{\mu\nu}, \sqrt{g} = \sqrt{g} e^{4\psi}, \psi = \psi(\phi).$$

(19)

Then

$$R = e^{-2\psi} \{ \tilde{R} - 6[(\nabla \psi)^2 + \Box \psi] \}. $$

(20)

Now we should take into account that $n^\nu = e^{-\psi} \tilde{n}^\nu$,

$$K = -\left( \tilde{\n}^\mu \sqrt{\tilde{g}} e^{3\psi} \right)^{,\mu} = e^{-\psi} \tilde{K} - 3 e^{-\psi} \psi_{,\mu} \tilde{n}^\mu. $$

(21)

We see that $I_{gd}(g_{\mu\nu}, F, U, V) = I_{gd}(\tilde{g}_{\mu\nu}, \tilde{F}, \tilde{U}, \tilde{V}) + \Delta I$, where

$$\tilde{F} = Fe^{2\psi}, \tilde{U} = U e^{4\psi}, \tilde{V} = e^{2\psi} \left[ V + 6\psi'(F\psi' + F') \right], $$

(22)

prime denotes here differentiation with respect to $\phi$,

$$\Delta I = -\frac{3\beta_0}{8\pi} \int_H d^2x \sqrt{g_2} \tilde{F} \psi' \phi_i \tilde{n}^i, $$

(23)

integration being performed over the horizon surface. For the "normal" case (with finite $A_{\text{eff}}$) this quantity vanishes due to the factor $b$. Thus, the total Euclidean action in the form (17), (18) is conformally invariant: $I_{gd}(g_{\mu\nu}, F, U, V) = I_{gd}(\tilde{g}_{\mu\nu}, \tilde{F}, \tilde{U}, \tilde{V})$. Apart from this, the entropy itself is also conformally invariant. For the nonextreme case (when $S = \frac{A_{\text{eff}}}{4}$) it follows from the fact that $A_{\text{eff}}$ is invariant due to (19), (22). For the extreme one $S = 0$ this invariance becomes obvious. The conformal invariance of the entropy is physically important.
because of intimate links between entropy and conformal structure associated with it (see for example, the recent work [14] and literature quoted there).

The formula (16) can be also rewritten as

\[ I_{\text{eff}} = \int_B d\sigma_{\text{eff}} \beta \varepsilon_{\text{eff}} - S, \]  

where \( d\sigma_{\text{eff}} = d\sigma F, \) \( S = \frac{A_{\text{eff}}}{4}. \)

\[ \varepsilon_{\text{eff}} = F^{-1} \varepsilon = -\frac{(n^i F)_{;i}}{8\pi F} \]  

The advantage of the form (24), (25) consists in that it manifests explicitly invariance of the entropy and action under conformal transformations (19). In so doing, the product \( \beta \varepsilon_{\text{eff}} \) by itself is invariant, in contrast to \( \beta \varepsilon. \)

For the ”anomalous” case (infinite \( A_{\text{eff}} \)) the situation is more complicated: we have a play of two factors \( b \) and \( A_{\text{eff}} \) that may result in unsatisfactory behavior of the action under conformal transformations. It is worth stressing that we always may perform the conformal transformation that makes \( F = 1 \) or \( F = -1 \) but this transformation leads to the system, non-equivalent to the original one in the anomalous case since the new metric may become singular because of infinite or zero \( F \) on the horizon.

V. SELF-CONSISTENCY OF VARIATIONAL PROCEDURE AND MODIFIED ACTION

The thermodynamic interpretation follows from (16) - (18), provided both \( F \) and \( A \) are finite. However, our goal is just to handle the situation when this is not the case. Moreover, ”standard” pure scalar (dilaton) black holes are ruled out by general no-hair theorems (see the review [18] and literature quoted there), so in the absence of electromagnetic (or other gauge fields) what remains is the anomalous case only. The existence of black holes in the Brans- Dicke theory, without contradiction to the aforementioned theorems, stems from the fact that for them \( A \to \infty \) and \( F \to 0 \) or \( F \to \infty \), so the arguments of [18] do not apply. As
a result, it turns out that (for example, for the Brans-Dicke theory) we get in (14), (15) the competition of three factors \((F, A \text{ and } b \text{ or } b_i)\). Therefore, one can identify in an obvious way the entropy neither with (18) nor with \(S = 0\), typical of ”normal” extreme black holes. For the same reasons, there is no guarantee that \(\Delta I = 0\), thus leaving very undesirable dependence of the form of the action on the conformal frame.

To elucidate the origin of the difficulties, let us return to the action principle as a starting point and trace back, how the Hamiltonian constraint appears as a result of variation with respect to \(\beta \) (or \(b \)). Preliminarily, take into account that \(2G^0_0 = -R_3\), where \(R_3\) is the curvature of the slice \(\tau = \text{const}\). It is easy to show from (9) that

\[
Z^0_0 = -FR_3 + 2\Delta_3F - U - V (\nabla \varphi)^2
\]

(26)
does not contain \(b\). Then, assuming that \(\beta_0 = \text{const}\) (for example, one may choose \(\beta_0 = 2\pi\)), we get immediately from (12) that

\[
\delta I_{gd} = \frac{\beta_0}{16\pi} \int d^3x \delta b \sqrt{g_3}Z^0_0 + \int_B d\sigma \delta \beta \varepsilon + \delta Y_1 - \delta Y_2.
\]

(27)

In the normal case \(\delta Y_1 = \frac{\beta_0}{8\pi} \int_H d\sigma \delta b F_{,i} n^i = 0\) since on the horizon \(b = 0\) is kept fixed, so \(\delta b = 0\) as well, and all other quantities in the integrand remain finite. Moreover, by a suitable conformal transformation one can achieve \(F = \pm 1\), so the term \(Y_1\) does not appear in the action at all. \(\delta Y_2 = 0\) since for extreme topologies \(Y_2 = 0\) and for nonextreme ones, according to (17), (18) \(Y_2 = \frac{A_{\text{eff}}}{4}\), the metric on the horizon being supposed to held fixed. Assuming that \(\delta b = 0\) on the boundary, one gets the Hamiltonian constraint from (27).

However, in the anomalous case these arguments do not work for the reasons explained above. For example, in spite of \(\delta b \to 0\) near the horizon, the surface element \(d\sigma\) or \(F_{,i}\) (or both) diverge. Therefore, to exclude undesirable terms on the horizon with \(\delta b\) and \(\delta \frac{\delta b}{\delta l}\), one is led to kill them by introducing corresponding counterparts. Namely, let us add on the horizon the term having the same form as (8) (formally, that means that we treat a horizon as an additional boundary). Then, after some rearrangement, we get the modified action

\[
\tilde{I}_{gd} = \int_B d\sigma \beta \varepsilon - \int_H d\sigma \beta \varepsilon = \int_B d\sigma_{\text{eff}} \beta \varepsilon_{\text{eff}} - \int_H d\sigma_{\text{eff}} \beta \varepsilon_{\text{eff}}
\]

(28)
(The minus sign in the second term in (28) arises due to the fact the outward normal is pointed now into the direction of the horizon.)

We would like to stress that the result $S = 0$ is obtained now for the anomalous case that required automatically some modification of the action principle. Thus, in combination with the conclusions made in [3] - [4], this completes the proof of the property $S = 0$ for classical extreme black holes, including here black holes of an infinite horizon area.

Now, with the modified action, one can easily check that the term (23) does not appear at all and the total modified action, including boundary terms (recall that now the horizon is an essential piece of boundary) obeys in the anomalous case the relation $\tilde{I}_{gd}(g_{\mu\nu}, F, U, V) = \tilde{I}_{gd}(\bar{g}_{\mu\nu}, \bar{F}, \bar{U}, \bar{V})$ similar to that in the normal one. In other words, its general form (but, of course, not the concrete values of the coefficients $F, U, V$) does not depend on the conformal frame, as it should be.

VI. COMPARISON TO NORMAL EXTREME AND NON-EXTREME BLACK HOLES

It is instructive to compare the modified action with that for extreme black holes in the normal case and with the non-extreme one.

Treatment of black holes topologies in [6] (that extended the approach of [15] to include extremal ones) revealed that there are terms on the horizon of nonextreme black holes which are absent in the extreme case. On the first glance, this exhibits close analogy with the role of an inner boundary on the horizon in our situation, because of which it would be tempting to interpret our approach simply as reformulation of the well-known results. Actually, however, there is a big difference here since not only the value of the action, but also its form should be modified from the outset in the "anomalous" case (we are speak about the Hilbert action or its generalization to the dilaton case). This can be explained as follows.

We can rewrite the action (12), (14) for the normal case as

$$I_{gd} = \frac{1}{16\pi} \int d^3x \sqrt{g_3} \beta Z_0^0 + B - S, \quad (29)$$
where \( B = \int_B d\sigma \beta \varepsilon \),
\[
S = \frac{A_{\text{eff}}}{8} \chi, \quad (30)
\]
\( \chi = \frac{2T_H T_0}{T_0} \). The value of \( \chi \) coincides with the Euler characteristic of the manifold, \( \chi = 2 \) for non-extreme black holes and \( \chi = 0 \) for extreme ones. In the case \( F = 1 \) eq. (29) coincides with eqs. (12), (18) of [6], if one identifies \( I_{\text{can}} \) with \( \frac{1}{16\pi} \int d^3 x \sqrt{g_3} \beta b \mathcal{Z}^0_0 \), \( B_{\infty} \) with \( B \) (if a boundary moves to infinity), \( I_H \) with \( I_{gd} \). Thus, for the normal case the conclusion \( S = 0 \) for the extreme black holes is made on the basis of the standard Euclidean Hilbert action (7), (8).

However, in our situation the action is modified:
\[
\tilde{I}_{gd} = I_{gd} + I_1, \quad I_1 \equiv \frac{1}{8\pi} \int_H d^3 x \sqrt{g_3} F K. \quad (31)
\]
Taking into account that \( \sqrt{g_3} = b \sqrt{g_2} \) and using again the relation (11), we see that for "normal" extreme black holes, when \( b, n_1 \to 0 \), \( I_1 \) vanishes. However, it does not vanish in the anomalous case and it is just this term that regularizes the action and gives the well-defined value for the entropy.

For non-extreme topologies one may also choose to place an inner boundary and, as a horizon is screened in this case for an external observer, he would not detect any entropy. However, there is a big difference here between such a system and our anomalous black hole. For the non-extreme case one can put an inner boundary in any place between a horizon and an outer boundary, an inner shell being physical in the sense it is built up from matter. Then the action has the general form (8) but its concrete value changes due to adding this shell.

Meanwhile, for anomalous extreme black holes introducing an inner boundary is mandatory. This boundary should be place on a horizon but the "shell" itself is fictitious. This is simply the way to express the fact that now the action has the modified form (31), which take into account the contribution from the horizon.

To summarize, there are three typical situations: (1) non-extreme black holes, (2) "normal" extreme black holes, (3) "anomalous" extreme black holes. The form of the Hilbert ac-
tion in the cases (1) and (2) is the same, but the value of the entropy is different \((S = A_{eff}/4 \text{ and } S = 0)\); in the case (3) the form of the Hilbert action (more exactly, its generalization to the dilaton case) differs from that in (1), (2) but the value of the entropy is the same as in (2). The papers [3], [4], [5] make accent on difference between (1) and (2), whereas the present article - on difference between (2) and (3).

VII. EXACT SOLUTIONS

In this section we consider, as examples, some exact solutions of self-consistent scalar-gravity theories. In all case the metric is spherically-symmetrical:

\[ ds^2 = d\tau^2 b^2 + \alpha^2 dy^2 + r^2(y)(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(32)

It follows from the standard relations for the Hawking temperature \(T_H = \frac{\kappa}{2\pi}, \kappa = \lim \frac{\partial b}{\partial l}\) (hererafter the sign "lim" refers to the horizon) that

\[ T_H = \frac{1}{2\pi} \lim \alpha^{-1} \frac{\partial b}{\partial y}. \]  

(33)

The formula for the energy takes a very simple form: \(E = 4\pi \varepsilon r^2 = A_{eff} \varepsilon_{eff}, \) where \(A_{eff} = 4\pi r^2 F\) and, according to (13), (25)

\[ \varepsilon_{eff} = -\frac{1}{8\pi} \frac{(Fr^2)'}{Fr^2 \alpha}, \]  

(34)

\[ \varepsilon = \varepsilon_{eff} F, \]  

(35)

\[ E = -\frac{1}{2} \frac{(Fr^2)'}{\alpha}, \]  

(36)

prime denotes differentiation with respect to \(y\). In a similar way,

\[ Y_1 = \lim \frac{\beta_b b^2 r^2 F'}{2\alpha}, \]  

(37)

\[ Y_2 = \lim \frac{\beta_b' b^2 r^2 F}{2\alpha}, \]  

(38)
\[ \Delta I = -\frac{3}{2} \beta_0 \psi'(\phi) Y_3, \quad Y_3 = \lim_{r^2 b F \phi'} \frac{r^2 b F \phi'}{\alpha}. \]  

(39)

It is assumed that the \( r \) coordinate runs from smaller to larger values from a horizon towards infinity.

The modified gravitation-dilaton Euclidean action is

\[ \tilde{I}_{gd} = \beta_B E_B - \beta^\text{loc}_H E_H, \]  

(40)

\( \beta^\text{loc} \) is the inverse local temperature that tends to zero on the horizon. We will see below that, as one approaches the horizon, \( E_H \) tends to infinity but, nevertheless, in some cases the product \( \beta^\text{loc}_H E_H \) remains finite.

A. Black holes in Brans-Dicke theory

Consider the Brans-Dicke theory, for which \( F = \phi, \ V = -\omega \phi^{-1}, \ U = 0 \ (\omega = \text{const}) \). There exist exact solutions within this theory, describing black holes [1] - [4]. They fall into two classes. Consider the first one, using notations of (32):

1. Case 1

\[ \begin{align*}
   b &= z^{(Q - \chi)/2}, \quad \alpha = z^{-Q/2}, \quad r^2 = y^2 z^{1 - Q}, \quad z = (1 - \frac{y_+}{y}).
   \phi &= z^{\chi/2}.
\end{align*} \]  

(41)

It is supposed that

\[ Q > \chi, \quad Q \geq 2. \]  

(43)

The first condition in (43) ensures that \( y = y_+ \) is a horizon, the second follows from its regularity (finiteness of the Kretschmann invariant (2), see [1] for details).

We get from (33) that
\[ T_H = \frac{Q - \chi}{8\pi y_+} \lim_{z \to 0} z^{\gamma/2}, \quad \gamma \equiv 2Q - \chi - 2. \] \hspace{1cm} (44)

It follows directly from (43) that \( \gamma > 0 \). Thus, \( T_H = 0 \) as a direct consequence of the regularity conditions.

Direct calculations give us that \( Y_3 \sim z^{1+\chi/2} \) on the horizon, so \( Y_3 \) diverges for \( \chi < -2 \). We also get

\[ Y_1 = \frac{\beta_0 y_+ \chi}{4}, \quad Y_2 = \frac{\beta_0 y_+}{4}(Q - \chi). \] \hspace{1cm} (45)

Apart from this, on the horizon

\[ A_{\text{eff}} \simeq 4\pi y_+^2 z^{1-Q+\chi/2} \to \infty, \] \hspace{1cm} (46)

where we took into account the inequality \( Q > 1 + \chi/2 \) that follows from (43). The energy density

\[ \varepsilon_{\text{eff}} = \frac{Q - 1 - \chi/2}{8\pi y_+} z^{Q/2-1}, \] \hspace{1cm} (47)

\( \varepsilon \sim z^{(Q+\chi)/2-1} \). Thus, on the horizon \( z = 0 \) \( \varepsilon_{\text{eff}} \) remains finite or even vanish due to the regularity condition (43), while \( \varepsilon \) may diverge for negative \( \chi \) large enough. It follows from eq. (48) that

\[ \beta_{H}^{\text{loc}} E_H = \frac{\beta_0 y_+}{2}(Q - \frac{\chi}{2} - 1). \] \hspace{1cm} (48)

2. Comparison with the results of [8]

Thermodynamics of black holes described by the exact solutions (41), (42) was discussed in [8]. In the space of parameters the authors considered three cases. Not counting the trivial case of the the Schwarzschild metric \( (Q = 1, \chi = 0, \phi = \text{const}) \), they considered two cases for which they obtained \( T_H = \infty \), whereas we get \( T_H = 0 \). As this value for the Hawking temperature contrasts sharply with what is obtained in our article, we will dwell
upon on the reason of this discrepancy. The parameters of the metric (11) obey the condition (eq.(4) of [8])

$$Q^2 + (1 + \omega \frac{3}{2})\chi^2 - Q\chi - 1 = 0. \quad (49)$$

Here $\omega$ is the Brans-Dicke parameter. Both cases I and II considered in [8] correspond to $\omega + \frac{3}{2} > 0$. It follows directly from (49) that in this case the parameter $\gamma$ that appears in (44) is negative and one gets formally $T_H = \infty$. However, this contradicts the regularity condition (43). This also conflicts with the incorrect statement made in [8] (in discussion after listing formulas with $T_H = \infty$) that black holes with $Q - \frac{\chi^2}{2} < 1$ are regular. (But the authors themselves point out rightly that in their case II the horizon is singular.) In our view, thermodynamics of black holes with a singular horizon has no physical meaning at all. For example, if the finiteness of the invariant (2) is relaxed, proof of the constancy of the surface gravity on the horizon (the condition $\rho_\alpha = 0$) loses its sense, the zero law of thermodynamics fails and one cannot even introduce the notion of a black hole temperature. Therefore, thermodynamic approach to such objects does not apply and not only the claim made in [8] about the value $S = 0$ but also the notion of the entropy itself is no longer valid under these circumstances. In other words, ”thermodynamics” is considered in [8] in the range of parameters, where there are no thermodynamic objects.

On the other hand, in the complimentary range of parameters ($\omega + \frac{3}{2} < 0$ or, equivalently, (43)), omitted in [8], thermodynamic properties of black holes are well defined, provided the Euclidean approach is properly modified.

3. Case 2

$$b = \exp[-(c + \frac{s}{2})y], \alpha = y^{-2} \exp[(c - \frac{s}{2})y], r^2 = y^{-2} \exp[(2c - s)y]. \quad (50)$$

$$\phi = e^{sy}. \quad (51)$$

Here $2c - s > 0$, $c > 0$, $-2c < s < 2c$ (see [11] for details). The horizon lies at $y \to \infty$. The formula (33) gives us
\[ T_H = \frac{s + 2c}{8\pi} \lim_{y \to \infty} y^2 \exp(-2cy) = 0. \] (52)

Again, direct calculations shows that \( Y_1 = -\frac{\beta_0 s}{2}, \) \( Y_2 = \frac{s + 2c}{2\pi}, \) \( Y_3 \) diverges as \( e^{sy}, \)

\[ A_{\text{eff}} = 4\pi y^{-2} \exp(2cy) \to \infty, \] (53)

on the horizon

\[ \varepsilon_{\text{eff}} = \frac{cy^2}{4\pi} \exp\left[\left(\frac{s - 2c}{2}\right) y\right] \to 0, \] (54)

\[ \varepsilon \sim y^2 \exp\left[\left(\frac{3s - 2c}{2}\right) y\right] \] (55)

may be finite or infinite dependent on the relation between \( s \) and \( c, \)

\[ \beta_{\text{loc}}^{E_H} = \beta_0 c. \] (56)

**B. BBMB solution**

This is exact solution for the coupling \( F = 1 - \xi \phi^2, \) where \( \xi \) corresponds to the conformal case \[16\]. It represents the rare exception, when a black hole with a finite area remains regular in spite of the presence of scalar hair. Its form coincides with the extremal Reissner-Nordström one:

\[ b = (1 - \frac{M}{r}), \alpha = b^{-1}, r(y) = y, T_H = 0, \] (57)

\[ \phi = q(r - M)^{-1}, M = \left(\frac{4\pi q^2}{3}\right)^{1/2}. \] (58)

In so doing, this hair is a discrete one, manifesting itself in a choice of the sign of \( \phi \) since for a given mass \( M \) there are two possible values of \( q. \) It was shown in \[17\] that, in spite of divergencies in \( F \) and \( \phi \) on the horizon, nothing pathological occurs with a particle, approaching the horizon, even if it is coupled to \( \phi. \) However, whereas mechanics remains
well-defined, the standard approach would fail for thermodynamics since in the present case \( Y_1 = \beta_0 \xi q^2 (r - M)^{-1} \), \( Y_2 = -\frac{1}{2} \beta_0 \xi q^2 (r - M)^{-1} \), \( Y_3 \) diverges as \((r - M)^{-2}\), \( A_{\text{eff}} = -4\pi M^2 \xi q^2 (r - M)^{-2} \to -\infty \).

The attempt to apply the standard formalism here leads to the meaningless result - entropy that not only diverges, but even is negative since \( Y_2 < 0 \). This drawback can be repaired according to the prescription, described above with the result \( S = 0 \). However, a new difficulty arises here. Simple calculations show that near the horizon

\[
\varepsilon_{\text{eff}} = (4\pi M)^{-1} \tag{59}
\]

and

\[
\beta_{H}^{\text{loc}} E_H = -\beta_0 q^2 \xi (r - M)^{-1}. \tag{60}
\]

In so doing, the total Euclidean action, according to (40), \( I \sim (r - M)^{-1} \to +\infty \).

### C. Common and distinct features of all three solutions

| \( Y_1 \) | \( \beta_0 y + \chi \) | \( \beta_0 y + \chi \) | \( \beta_0 y + \chi \) |
| --- | --- | --- | --- |
| \( Y_2 \) | \( \beta_0 y + (Q - \chi) > 0 \) | \( \beta_0 y + (Q - \chi) > 0 \) | \( \beta_0 y + (Q - \chi) > 0 \) |
| \( Y_3 \) | infinite or finite depending on parameters | infinite | infinite |
| \( A_{\text{eff}} \) | \( +\infty \) | \( +\infty \) | \( -\infty \) |
| \( T_H \) | 0 | 0 | 0 |
| \( S \) | 0 | 0 | 0 |
| \( I \) | finite | finite | +\( \infty \) |
| \( I \) | finite | finite | +\( \infty \) |

Here we did not display the sign of \( Y_3 \) since it is unimportant in the given context.

That the standard approach gives an unsatisfactory answer for the BBMB solution follows from divergencies in the action. It is also obvious that \( Y_2 \) cannot be considered as a candidat
on the BBMB entropy because $Y_2 \to -\infty$. For Brans-Dicke black holes the action is finite but the entropy, if identified with $Y_2$ according to (17), would also have given an unphysical result. Indeed, $Y_2$ is model dependent and would be incompatible with the relation between the Euler characteristics and entropy [3], [4], [5] in contrast to the universal form $S = 0$ for "normal" extreme black holes.

The fact that $Y_1 \neq 0$, explains the roots of these difficulties. Indeed, let us, according to the general formula (14), write down $Y_1$ in the spherically-symmetrical case, singling out the factor $b$, as $Y_1 = y_1 b$. Then the variation with respect to $b$ gives us $\delta Y_1 = y_1 \delta b$. If $Y_1$ is finite, near the horizon $y_1 \sim b^{-1}$. In the variation procedure the function $b$ and its variation $\delta b$ within the same class behave in a similar way, as one approaches the horizon. As a result, $\delta Y_1 \neq 0$ and, according to (27), the variational procedure fails and one cannot state that the Euclidean metrics under consideration were obtained in the self-consistent way (in the normal case we would have finite $y_1$, so the product $y_1 \delta b \to 0$ due to the behavior of $b$ near the horizon). In the case of BBMB solutions the situation even gets worse since $Y_1$ diverges. Thus, the standard Euclidean action formalism fails in all three cases.

As far as the modification (treating a horizon as an inner boundary with the corresponding terms in the action) is concerned, we see deep distinction between Brans-Dicke black holes and the BBMB solution. For both types of Brans-Dicke black holes black holes (i) the black hole entropy $S$ is well-defined, $S = 0$; (ii) the energy associated with a horizon is infinite but the Euclidean action itself is finite. The latter reveals itself in the finite nonzero product $\beta^{loc}_{EH}$, where the first term tends to zero, while the second one diverges.

However, for the BBMB solution, in spite of the fact that we found formally $S = 0$, the total action turns out to be infinite and positive both with or without a boundary on the horizon. This means that such solution cannot contribute into the partition function $Z \sim \exp(-I)$, so thermodynamics (if any), which can be assigned to such solutions, is very poor. It is seen directly from the table that some properties of the BBMB black holes are in a sense "more peculiar" than those of the Brans-Dicke ones. Indeed, both $A_{eff}$ and $Y_2$ (analogues of the quantities, proportional to the entropy of "normal" black holes) are negative, and these
oddities make it impossible to restore reasonable thermodynamic properties even after the "improvement" of the action. The failure of thermodynamic interpretation for the BBMB black holes can be ascribed to the fact that, with a finite qualilocal energy density on the horizon $\varepsilon_{\text{eff}}$, the effective area $A_{\text{eff}}$ as well as energy $\varepsilon_{\text{eff}}A_{\text{eff}}$ grows more rapidly than the local inverse Tolman temperature tends to zero.

VIII. SUMMARY AND CONCLUSION

It is shown, without using the assumption of the spherical symmetry, that any static black hole with an infinite horizon area but regular horizon should have a zero Hawking temperature. It follows from our consideration that black holes with an infinite effective horizon area occupy, formally speaking, an intermediate place between non-extreme and "normal" extreme black holes in what concerns thermodynamics. Formally, it is seen from (17): the factor $\frac{T_{H}}{T_{0}} \to 0$ (what is typical of extreme black holes), but the factor $\frac{A_{\text{eff}}}{\varepsilon}$ (typical of non-extreme ones) remains important and even tends to infinity. As a result, their product $Y_{2}$ may be finite (as is the case for Brans-Dicke black holes discussed above). However, the true situation is even more complicated than this rough analogy since, as we saw, one cannot identify (17) with the entropy. Moreover, the corresponding quantity can be negative and diverge as it happens to BBMB solutions.

We traced some subtleties in the action principle for such unusual geometries and demonstrated that the action formalism, provided it is modified properly, handles even rather exotic situations, when either the horizon area or the dilaton coefficient $F$ diverges. However, this does not guarantee in advance that black holes in any theory of this kind represent well-defined thermodynamic objects. Explicit examination of exact solutions showed that this is the case for Brans-Dicke black holes but not for the BBMB solution.

As far as Brans-Dicke theory is concerned, is just the unusual character of black holes under consideration (infinite $A_{\text{eff}}$) that makes the link between conformal properties of the action, self-consistency of the variational procedure and the zero value of the black hole en-
entropy nontrivial. The Euclidean action is finite in spite of the fact that the quasilocal energy $E_H$ associated with a horizon diverges. These divergencies receive a simple explanation: they appear due to an infinite effective area $A_{eff}$, while the effective energy $\varepsilon_{eff}$ per unit effective area turns out to be finite or even vanish.

The key point in our treatment consisted in placing an additional inner boundary on the horizon of extremal black holes with infinite $A_{eff}$. The reason, why this was necessary, consists in failure of the variational procedure without corresponding boundary terms. This failure would reveal itself in the appearance of superfluous terms which, in particular, would contain the normal derivatives of the local temperature. They are automatically equal to zero in a normal case (finite $A_{eff}$) due to properties of the horizon but, in general, persist in our case (infinite $A_{eff}$). The suggested approach removes all these undesirable terms and gives a quite definite answer for the black hole entropy.

On the other hand, failure of thermodynamic interpretation of the BBMB black holes is, in our view, especially interesting. We have already paid attention that thermodynamics interpretation of some models of two-dimensional dilaton gravity fails [19], [20]. This turns out to be possible due to quantum effects entirely and refer to nonextreme horizons. Now we see that, in an essence, BBMB black hole can be thought of as a classical extremal counterpart of such exceptional solutions.

On the other hand, it is also instructive to carry out some parallels bewteen semiclassical two-dimensional dilaton theories and pure classical Branse-Dicke black holes considered in the present paper. As is shown in [19], [20], [21], when the quantum-corrected quantity $F$ diverges on the horizon, in some special two-dimensional models infinite backreaction remains compatible with regularity of the geometry. As the quantity $F$ of two-dimensional models, obtained by sperically-symmetrical reduction, is similar to $r^2$, divergencies in $F$ resemble an infinite horizon area of four-dimensional ones. Thus, although the results of the present paper are restricted to the classical domain only, the aforementioned analogy seems to testify that at least for some models the value of the black hole entropy $S = 0$, inherent to classical extremal black holes, survives with quantum backreaction taken into account.
When $A_{\text{eff}}$ is infinite, the issue of quantum backreaction in so unusual situation deserves separate treatment. Anyway, it was necessary, as the first step, to elaborate self-closed and self-consistent approach to thermodynamics of objects with infinite $A_{\text{eff}}$ within pure classical framework, and this task is performed in the present article.

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