BELTRAMI EQUATION FOR THE HARMONIC
DIFFEOMORPHISMS BETWEEN SURFACES

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Abstract. In this article it is shown that the study of harmonic
diffeomorphisms, with nonvanishing Hopf differential, reduces to
the study of the Beltrami equation of a certain type: the imaginary
part of the logarithm of the Beltrami function coincides with the
imaginary part of the logarithm of the Hopf differential, therefore
is a harmonic function. The real part of the logarithm of the Bel-
trami function satisfies an elliptic nonlinear differential equation,
which in the case of constant curvature is an elliptic sinh-Gordon
equation. Solutions are calculated for the constant curvature case
in a unified way. The harmonic maps are therefore classified by
the classification of the solutions of the sinh-Gordon equation.

1. Introduction and Statement of the Results

The aim of this article is to develop a method to construct har-
monic diffeomorphisms, with nonvanishing Hopf differential, between
Riemann surfaces $M$ and $N$.

The case when $N$ is of constant curvature is studied in more detail:
The method to find a harmonic map is to first find a solution of the
elliptic sinh-Gordon equation, next solve the Beltrami equation and
finally describe the metric on $N$ of constant curvature.

There are only a few examples of harmonic diffeomorphisms that
are not conformal, see for example [2, 9, 16, 19]. Using the proposed
method and the elliptic functions, we can find a family of harmonic
maps to constant curvature spaces, that includes the above examples
and generalizes them.

It is central in the theory of harmonic maps the study of harmonic
diffeomorphisms between two Riemann surfaces (see for example the
classical article [17]). The case that has been studied the most is when
the surfaces are of constant curvature (see for example [11, 7, 9, 10, 13,
16] and the references therein). The preparation of this article was

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prompted by the work in [11] that proves a conjecture of R. Schoen on quasiconformal harmonic diffeomorphisms between hyperbolic spaces.

In this paper, it is shown that the study of harmonic maps reduces to the study of the Beltrami equation of a certain type: the imaginary part of the logarithm of the Beltrami function coincides with the imaginary part of the logarithm of the Hopf differential, therefore is a harmonic function. The real part of the logarithm of the Beltrami function satisfies an elliptic nonlinear differential equation, which in the case of constant curvature is an elliptic sinh-Gordon equation. This result reduces the harmonic map equations to a special case of the Beltrami equation that has been already extensively studied, see for example [15].

Furthermore, one could apply any property or calculation on a harmonic map to the associated Beltrami equation and vice versa. This result connects two previously extensively studied aspects of mathematics, i.e. the harmonic maps and the Beltrami equation.

The harmonic maps to surfaces of constant curvature are closely related to the elliptic sinh-Gordon equation. The sinh-Gordon and sine-Gordon equations have many applications and they have also been studied extensively (see for example [5, 6]). A close relation to the theory of constant mean curvature surfaces has been already known, see for example [3, 8].

Using one-soliton solutions of the elliptic sinh-Gordon equation one can construct examples of harmonic maps to constant curvature spaces. Elliptic functions arise in course of this construction. With this approach one can study positive, negative and zero constant curvature surfaces in a unified way and find concise formulas that provide new examples of harmonic maps. As an application, we can recover the examples of harmonic maps in the articles [16] and [9], by proving that they correspond to the one-soliton solution of the sinh-Gordon equation. There are several model solutions of the sinh-Gordon equations as the soliton solutions, solutions by using seperation of variables etc. All these models imply models of solutions of the associated Beltrami equation and finally of the corresponding harmonic map problem.

A Bäcklund transform arises and it provides a connection between the solutions of an elliptic sinh-Gordon and an elliptic sine-Gordon equation. This result provides solutions of an elliptic sinh-Gordon equation by the known solutions of a sine-Gordon equation and vice versa.

The main result in this article could be summarized in the following theorem.
Theorem 1. A harmonic diffeomorphism $u : M \to N$ between Riemann surfaces, with nonvanishing Hopf differential $e^{-\lambda(z)}dz^2$, is a solution of the Beltrami equation,

$$\frac{\partial u}{\partial \bar{z}} \mu(\bar{z}, z) = e^{-2\omega + i \text{Im } \lambda(z)},$$

where

$$\omega_{z\bar{z}} = \frac{K_N}{2} e^{-\text{Re } \lambda} \sinh 2\omega,$$

and $K_N$ is the curvature of the surface $N$.

Corollary 2. Let $u : M \to N$ be a harmonic diffeomorphism between Riemann surfaces, with nonvanishing Hopf differential $e^{-\lambda(z)}dz^2$. If $N$ is a surface of constant curvature $K_N$, then in the specific coordinate system

$$\zeta = \int e^{-\lambda(z)/2} dz,$$

$u$ is a solution of the Beltrami equation,

$$\frac{\partial u}{\partial \bar{\zeta}} \mu(\zeta, \bar{\zeta}) = e^{-2\omega},$$

where

$$\omega_{\zeta\bar{\zeta}} = -\frac{K_N}{2} \sinh 2\omega.$$ (1.1)

Then, the solutions of the harmonic map equation such that

$$\frac{\partial U}{\partial \zeta} = e^{-2\omega(\zeta, \bar{\zeta})},$$

can be written as

$$U = f(u(\zeta, \bar{\zeta})),$$

where $f(z)$ and $\lambda(z)$ are holomorphic functions and the metric on $N$ is of constant curvature $K_N$.

Note that (1.1) is the elliptic sinh-Gordon equation which has been already extensively studied.

There are only a few explicit formulas for harmonic diffeomorphisms that are not conformal. Using a model as the one-soliton solution of the elliptic sinh-Gordon equation, one can construct examples of harmonic maps to constant curvature spaces in a unified way. In the special case $K_N = -1$ these examples include, as a special case, the examples in [9] and [16] that were previously unrelated.

In Section 2 the necessary formulas are introduced. Next, in Section 3 we prove Theorem 1. In Section 4 the constant curvature case is
discussed and in Section 5 analytic calculations of the constant curvature case is studied by using one-soliton solutions of the sinh-Gordon equation. In Section 6 it is shown that the solutions given in [9, 16] correspond to the one-soliton solutions of the sinh-Gordon equation and analytic formulas are given. The results of this paper can be extended to the positive curvature case by using the analytic formulas of Section 5. These results are given in unified formulas for positive, negative and zero curvature. Also the Bäcklund transform is discussed in Section 7. The Bäcklund transform could be used in order to generate new solutions of the sinh-Gordon equation and therefore new solutions of the Beltrami equation and the harmonic map problem. Next, Section 8 contains some perspectives for future research. Finally, there is an Appendix about elliptic functions.

2. Preliminaries

2.1. Isothermal Coordinates. Let \( u : M \to N \) be a map between Riemann surfaces \( (M, g), (N, h) \). The map \( u \) is locally represented by \( u = u(z) = R + iS \). The standard notation is that

\[
\partial_z = \frac{1}{2} (\partial_x - i \partial_y), \quad \partial_{\bar{z}} = \frac{1}{2} (\partial_x + i \partial_y).
\]

It is a known fact the existence of isothermal coordinates on an arbitrary surface with a real analytic metric (see [4, Section 8, p. 396]). Consider an isothermal coordinate system \( (x, y) \) on \( M \) such that \( g = e^{f(x,y)}(dx^2 + dy^2) = e^{f(z,\bar{z})}d\bar{z}dz = e^{f(z,\bar{z})}|dz|^2, \)
where \( z = x + iy \). Consider isothermal coordinate system \( (R, S) \) on \( N \) such that

\[
h = e^{F(R,S)}(dR^2 + dS^2) = e^{F(u,\bar{u})}dud\bar{u} = e^{F(u,\bar{u})}|du|^2,
\]
where \( u = R + iS \).

Note that

\[
K_N = K_N(u, \bar{u}) = -\frac{1}{2} e^{-F}\Delta F = -2\partial_{u\bar{u}}F e^{-F}
\]
is the Gauss curvature of the metric

\[
h = e^{F(u,\bar{u})}|du|^2.
\]

2.2. Harmonic Maps and the Beltrami Equation. In the case of isothermal coordinates (see [4] Section 8, p. 397), the map \( u \) is harmonic if it satisfies

\[
\partial_{\bar{z}} \partial_z u + \partial_u F(u, \bar{u}) \partial_z u \partial_{\bar{z}} u = 0. \tag{2.1}
\]
Notice that this equation only depends on the complex structure of $M$ and not on the metric $g$ of $M$.

Denote by
\[
\|\partial_z u\|^2 = e^{F(u, \bar{u})}e^{-f(z, \bar{z})}|\partial_z u|^2,
\]
\[
\|\partial_{\bar{z}} u\|^2 = e^{F(u, \bar{u})}e^{-f(z, \bar{z})}|\partial_{\bar{z}} u|^2.
\]
These are the norms of the $(1,0)$-part and $(0,1)$-part of $du$ respectively.

The Jacobian of $u$ is defined by
\[
J(u) = \|\partial_z u\|^2 - \|\partial_{\bar{z}} u\|^2
= e^{F(u, \bar{u})}e^{-f(z, \bar{z})}(|\partial_z u|^2 - |\partial_{\bar{z}} u|^2). \tag{2.2}
\]

Let $u: M \to N$ be a diffeomorphism. Then the Jacobian $J(u) = e^{F(R, S)}e^{-f(x, y)}(\partial_x R\partial_y S - \partial_y R\partial_x S)$ is nowhere vanishing.

In order to prove Theorem 1, the following observation is required, (see [4, Section 8, p. 399]).

**Proposition 2.1.** A necessary and sufficient condition for $u$ to be a harmonic map, it is the Hopf differential to be holomorphic, i.e.
\[
e^{F(u, \bar{u})}\partial_z u\partial_{\bar{z}} \bar{u} = e^{-\lambda(z)},
\]
where $\lambda(z)$ is a holomorphic function.

Consider the Beltrami coefficient
\[
\mu(z, \bar{z}) = \frac{\partial_z u}{\partial_{\bar{z}} u}.
\]

The following relations are valid:
\[
du = \partial_z u dz + \partial_{\bar{z}} u d\bar{z} = \partial_z u (dz + \mu(z, \bar{z})d\bar{z}),
\]
where
\[
\frac{\partial_z u}{\partial_{\bar{z}} u} = \mu(z, \bar{z}) = e^{-2\omega(z, \bar{z})+i\phi(z)}.
\]
This is the well known Beltrami Equation. Note that in general the Beltrami coefficient $\mu(z, \bar{z})$ is a complex function.

### 3. Proof of the Theorems

We shall study harmonic maps in local isothermal coordinates. The harmonic map from a Riemann surface $M$ to a Riemann surface $N$ can be defined by (2.1). Thus, the following equations are valid:
\[
\partial_{zz} u + \partial_u F(u, \bar{u})\partial_z u\partial_{\bar{z}} u = 0 \quad \text{and} \quad \partial_{zz} \bar{u} + \partial_{\bar{u}} F(u, \bar{u})\partial_{\bar{z}} u\partial_{\bar{z}} \bar{u} = 0. \tag{3.1}
\]

The Hopf differential of $u$ is defined by
\[
\Lambda(z)dz^2 = (e^{F(u, \bar{u})}\partial_z u\partial_{\bar{z}} \bar{u})dz^2. \tag{3.2}
\]
It is a well known result [4, Section 8, p. 399] that \( u \) is harmonic if and only if the Hopf differential is holomorphic. The following formulation will be used in the text. If \( u \) is a harmonic map then
\[
e^{F(u, \bar{u})} \partial_z u \partial_{\bar{z}} \bar{u} = \Lambda(z) = e^{-\lambda(z)} \quad \text{and} \quad e^{F(u, \bar{u})} \partial_{\bar{z}} u \partial_z \bar{u} = \overline{\Lambda(z)} = e^{-\overline{\lambda(z)}},
\]
where \( \lambda(z) \) is a holomorphic function.

Motivated by the above relations (3.3), we set
\[
e^v = e^{\frac{F + \lambda}{2}} u_z \text{ and } v = \omega + i\theta.
\]

Then
\[
e^{\bar{v}} = e^{\frac{F}{2}} e^{\frac{-\lambda}{2}} u_z = \frac{1}{e^{\frac{F}{2}} e^{\frac{\lambda}{2}} u_z}, \quad e^{-v} = e^{\frac{F}{2}} e^{\frac{-\lambda}{2}} \bar{u}_{\bar{z}} = \frac{1}{e^{\frac{F}{2}} e^{\frac{\lambda}{2}} \bar{u}_{\bar{z}}}
\]
\[
e^\theta = e^{\frac{F}{2}} e^{\frac{\lambda}{2}} \bar{u}_{\bar{z}} = \frac{1}{e^{\frac{F}{2}} e^{\frac{-\lambda}{2}} u_z}, \quad e^{-\theta} = e^{\frac{F}{2}} e^{\frac{-\lambda}{2}} \bar{u}_{\bar{z}} = \frac{1}{e^{\frac{F}{2}} e^{\frac{-\lambda}{2}} u_z}
\]
and
\[
e^{F + \lambda} u_z \bar{u}_{\bar{z}} = 1, \quad e^{F + \overline{\lambda}} \bar{u}_z u_{\bar{z}} = 1.
\]
The above equations imply
\[
e^{-(v + \bar{v})} = e^{-2\omega} = \frac{e^{\frac{\lambda}{2}} u_z}{e^{\frac{\lambda}{2}} u_z} = \frac{e^{\frac{-\lambda}{2}} \bar{u}_{\bar{z}}}{e^{\frac{-\lambda}{2}} \bar{u}_{\bar{z}}},
\]
therefore the harmonic map satisfies the Beltrami equation:
\[
\frac{u_z}{u_{\bar{z}}} = e^{-2\omega + i \text{Im} \lambda(z)} = \mu(z, \bar{z}).
\]
The imaginary part of the logarithm of the Beltrami function \( \mu \) is a harmonic function, since it is the imaginary part of a conformal function. Thus the following proposition has been proved:

**Proposition 3.1.** Let \( u : M \to N \) be a harmonic map defined by (2.1). Then the function \( u \) is a solution of the Beltrami equation (3.8) and the imaginary part of the logarithm of the Beltrami coefficient \( \mu \) is a harmonic function.

Let us now prove the inverse of the above Proposition. We suppose that \( u \) is a solution of this Beltrami equation, such that

\[
\text{Re } \log(\mu(z, \bar{z})) = -2\omega(z, \bar{z}), \quad \text{Im } \log(\mu(z, \bar{z})) = \phi(z, \bar{z}) \quad \text{and} \quad \phi_{z\bar{z}} = 0.
\]
Then, there exists a holomorphic function \( \lambda(z) \) such that \( \phi(z, \bar{z}) = \text{Im} \lambda(z) \). In this case (3.8) is true and it implies (3.7), so it follows that
\[
e^{\lambda} u_z \bar{u}_{\bar{z}} = e^{\lambda} \bar{u}_{\bar{z}} u_z = e^{-\sigma},
\]
where $\sigma$ is a real function of $z$ and $\bar{z}$. The Jacobian (see \eqref{2.2}) is nonvanishing i.e.
\[ |u_z|^2 - |u_{\bar{z}}|^2 \neq 0. \]
Since $u$ is a diffeomorphism, we can consider that $\sigma = \sigma(u, \bar{u})$. Thus,
\[ e^{\sigma} u_z u_{\bar{z}} = e^{-\lambda(z)}, \]
where $e^{-\lambda(z)}$ is a holomorphic function. The above relation is the Hopf differential and it follows that the function $u$ satisfies the equation
\[ u_{z\bar{z}} + \sigma(u, \bar{u}) u_z u_{\bar{z}} = 0. \]
Thus, there is a harmonic map $u$ from a surface $M$ to the surface $N$ equipped with the metric
\[ ds^2 = e^{\sigma} dud\bar{u}. \]
Therefore the following result is valid.

**Proposition 3.2.** Let $u$ be a diffeomorphism that is a solution to the Beltrami equation, and the imaginary part of the logarithm of the Beltrami coefficient is a harmonic function. Then $u$ is a harmonic map.

Both the above Propositions 3.1 and 3.2 could be summarized to the following result.

**Proposition 3.3.** A diffeomorphism between Riemann surfaces is harmonic if and only if it is the solution of the Beltrami equation, where the imaginary part of the logarithm of the Beltrami coefficient is a harmonic function.

The above theorem implies that both problems are equivalent, therefore one could apply any property or calculation on a harmonic map to the associated Beltrami equation and vice versa. Equations \eqref{3.4}, \eqref{3.5} and \eqref{3.6} imply that
\[ e^{2\omega} = e^{v + \bar{v}} = e^{F + \text{Re}\lambda}|u_z|^2, \quad e^{-2\omega} = e^{-(v + \bar{v})} = e^{F + \text{Re}\lambda}|u_{\bar{z}}|^2 \]
and
\[ \sinh(2\omega) = \frac{e^{F + \text{Re}\lambda}}{2} \left(|u_z|^2 - |u_{\bar{z}}|^2\right), \quad \cosh(2\omega) = \frac{e^{F + \text{Re}\lambda}}{2} \left(|u_z|^2 + |u_{\bar{z}}|^2\right). \]
As a consequence of \eqref{3.5}, it follows that
\[ e^{2v} = \frac{u_z}{u_{\bar{z}}}, \quad 2v = 2\omega + 2i\theta = \log u_z - \log u_{\bar{z}}. \]
After some elementary calculations and taking into consideration \eqref{3.1}, it follows that
\[ 2v_{z\bar{z}} = 2\omega_{z\bar{z}} + 2i\theta_{z\bar{z}} = F_{u\bar{u}} \left(|u_z|^2 - |u_{\bar{z}}|^2\right) + (F_u^2 - F_{u\bar{u}}) u_z u_{\bar{z}} - (F_{\bar{u}}^2 - F_{u\bar{u}}) \bar{u}_z \bar{u}_{\bar{z}}. \]
Therefore, 
\[ 2\omega \bar{z} = F_{u\bar{u}} \left( |u_z|^2 - |u_{\bar{z}}|^2 \right) = 2F_{u\bar{u}} e^{-F - \Re \lambda} \sinh 2\omega = -K_N e^{-\Re \lambda} \sinh 2\omega, \]
where \( K_N \) the curvature of the surface \( N \). Notice that when \( \lambda = 0 \) and \( N \) is a surface with constant curvature, the equation
\[ \omega \bar{z} = -\frac{K_N}{2} \sinh 2\omega \]
is the well known elliptic sinh-Gordon differential equation, with many applications in physics. There is extensive bibliography on the solutions of this equation, see for example [6] and the references therein. Next section we shall discuss the calculation of the harmonic map between surfaces by using the solution of the elliptic sinh-Gordon equation. The following theorem holds true.

**Theorem 1.** A harmonic diffeomorphism \( u : M \rightarrow N \) between Riemann surfaces, with nonvanishing Hopf differential \( e^{-\lambda(z)} dz^2 \), is a solution of the Beltrami equation,
\[ \frac{\partial \bar{u}}{\partial z} = \mu(z, \bar{z}) = e^{-2\omega + i \Im \lambda(z)}, \]
where
\[ \omega \bar{z} = -\frac{K_N}{2} e^{-\Re \lambda} \sinh 2\omega, \]
and \( K_N \) is the curvature of the surface \( N \).

In the domain surface \( M \) of a harmonic map, one can choose a specific coordinate system. This choice will facilitate considerably the calculations. This specific system is defined by the conformal transformation
\[ \zeta = \xi + i \eta = \int e^{-\lambda(z)/2} \, dz, \quad (3.10) \]
where \( \lambda(z) \) is the holomorphic function given by the Hopf differential \( (3.2) \) and it is related to the imaginary part of the logarithm of the Beltrami function \( \mu \), see equation (3.8). In this specific system the equations in Section 3 could be simplified by substituting \( \lambda(z) = 0 \).

Then,
\[ \partial_{\bar{\zeta}} u + \partial_u F(u, \bar{u}) \partial_{\bar{\zeta}} u \partial_{\bar{\zeta}} u = 0, \quad e^{F(u, \bar{u})} \partial_{\bar{\zeta}} \partial_{\bar{\zeta}} u = 1. \quad (3.11) \]
The corresponding Beltrami equation is given by
\[ \frac{\partial \bar{u}}{\partial \zeta} u = e^{-2\omega(\zeta, \bar{\zeta})}, \quad (3.12) \]
where the function \( \omega(\zeta, \bar{\zeta}) \) satisfies the elliptic sinh-Gordon equation
\[ \omega_{\zeta \bar{\zeta}} = -\frac{K_N}{2} \sinh 2\omega. \quad (3.13) \]
The above equations are interesting in the case of the constant mean curvature of the target manifold $N$. In this case, given a solution of the elliptic sinh-Gordon equation (3.13), one has to calculate a solution of the Beltrami equation (3.12). By the conformal change of coordinates (3.10) of this solution, we can calculate the solution in the original coordinates $z = x + iy$ and the general solution of the problem is a holomorphic function of this solution. The above is a strategy to solve the harmonic map problem from a domain surface $M$ to an image surface $N$ of constant curvature. The following corollary holds true.

**Corollary 2.** Let $\omega$ be a solution of the elliptic sinh-Gordon equation

$$\omega_{\zeta\zeta} = -\frac{K_N}{2}\sinh 2\omega,$$

and let $u$ be one solution of the Beltrami equation

$$\frac{\partial \bar{\zeta}}{\partial \zeta}u - e^{-2\omega(\zeta, \bar{\zeta})} = 0.$$ 

(4.1)

Then, the solutions of the harmonic map equation such that

$$\frac{\partial U}{\partial \zeta} = e^{-2\omega(\zeta, \bar{\zeta})},$$

can be written as

$$U = f(u(\zeta, \bar{\zeta})), \quad \zeta = \xi + i\eta = \int e^{-\lambda(z)/2} \, dz,$$

where $f(z)$ and $\lambda(z)$ are holomorphic functions and the metric on $N$ is of constant curvature $K_N$.

4. **Solution of the Beltrami Equation**

From the above discussion one could notice that when a solution of the elliptic sinh-Gordon equation (3.13) is known, then the solution of the Beltrami equation (3.12) is a solution of the harmonic map problem.

Let us consider one solution of the elliptic sinh-Gordon equation (3.13), then the Beltrami equation (3.12) can be written as a first order P.D.E.

$$e^{\omega(\zeta, \bar{\zeta})}u_\bar{\zeta} - e^{-\omega(\zeta, \bar{\zeta})}u_\zeta = 0.$$ 

(4.1)

The relations $\zeta = \xi + i\eta$ and $u = R(\xi, \eta) + iS(\xi, \eta)$ and the fact that $\omega(\zeta, \bar{\zeta})$ is a real function, imply that the following system of first order PDE’s holds true:

$$\sinh \omega R_\xi - \cosh \omega S_\eta = 0$$

(4.2)

$$\cosh \omega R_\eta + \sinh \omega S_\xi = 0.$$ 

(4.3)
By multiplying the equation (4.2) by \( R_\eta \) and equation (4.3) by \( S_\eta \), the following Proposition holds true.

**Proposition 4.1.** The solution of the Beltrami equation with a real Beltrami coefficient \( \mu(\zeta, \bar{\zeta}) \) corresponds to a harmonic mapping which preserves the orthogonality of the local coordinate systems, i.e.

\[
R_\xi R_\eta + S_\xi S_\eta = 0.
\]

The system (4.2, 4.3) can be separated as two second order O.D.E.’s

\[
(tanh \omega R_\xi)_\xi + (coth \omega R_\eta)_\eta = 0 \quad (4.4)
\]

\[
(tanh \omega S_\xi)_\xi + (cot \omega S_\eta)_\eta = 0. \quad (4.5)
\]

Equation (4.4) can be written as

\[
\tan \omega R_\xi + \cot \omega R_\eta + \frac{\omega_\xi R_\xi}{\sinh^2 \omega} - \frac{\omega_\eta R_\eta}{\cosh^2 \omega} = 0. \quad (4.6)
\]

This is an elliptic second order P.D.E and when a solution \( \omega(\xi, \eta) \) of the elliptic sinh-Gordon equation

\[
\omega_{\xi\xi} + \omega_{\eta\eta} = -2K_N(\xi, \eta) \sinh 2\omega
\]

is known, then one can solve equation (4.3), by using the standard methods of solution of such an equation, see [14, p. 72]. In other words, one has to solve the equation of characteristics

\[
\frac{d\eta}{d\xi} = i \coth \omega(\xi, \eta). \quad (4.7)
\]

The solution of the above equation is of the form

\[
\Phi(\xi, \eta) = \text{constant}.
\]

Then, we observe that the solutions of (4.1) are given by

\[
u = \text{Re}\Phi + i\text{Im}\Phi = R(\xi, \eta) + iS(\xi, \eta).
\]

Thus, the strategy to find solutions of the harmonic map problem is the following. It is enough to take a solution of the elliptic sinh-Gordon equation and find a solution of the Beltrami equation by solving (4.7). Then, the solutions \( U \) of the harmonic map equation such that

\[
\frac{\partial U}{\partial \xi} = e^{-2\omega(\zeta, \bar{\zeta})},
\]

can be written as

\[
U = f(u(\zeta, \bar{\zeta})), \quad \zeta = \xi + i\eta = \int e^{-\lambda(z)/2} dz, \quad (4.8)
\]

where \( f(z) \) is a holomorphic function. It is interesting to apply this strategy in the case of constant curvature.
5. Constant curvature spaces

In this section we consider the case when $N$ is of constant curvature $K_N$.

In the specific coordinates (3.10), the elliptic sinh-Gordon equation is

$$\omega_{\zeta\bar{\zeta}} = -\frac{K_N}{2} \sinh 2\omega, \text{ where } K_N = \pm 1, 0.$$ 

Consider next a one-soliton solution of the above equation

$$\omega = \omega(\gamma \eta - \delta \xi), \gamma = \rho \cos \tau \text{ and } \delta = \rho \sin \tau.$$ 

In order to simplify the calculations, a new system of coordinates $Z = X + iY$ is introduced:

$$X = \xi \rho \cos \tau + \eta \rho \sin \tau \text{ or } Z = \rho \epsilon^{-i\tau} \zeta, \bar{Z} = \rho \epsilon^{i\tau} \bar{\zeta}.$$ 

In these coordinates $\omega = \omega(Y)$ and the elliptic sinh-Gordon equation is written

$$\frac{d^2\omega}{dY^2} = -\frac{2K_N}{\rho^2} \sinh 2\omega \text{ or } \left(\frac{d\omega}{dY}\right)^2 = C \left(1 - \frac{4K_N}{C\rho^2} \sinh^2 \omega\right), (5.1)$$

or

$$\left(\frac{\omega'(Y)}{\sqrt{C}}\right)^2 + (m - 1) \sinh^2 \omega(Y) = 1, (5.2)$$

where

$$C = (\omega'_0)^2 + \frac{4K_N}{\rho^2} \sinh^2 \omega_0, \omega'_0 = \omega'(Y_0), \omega_0 = \omega(Y_0), m = 1 + \frac{4K_N}{C\rho^2}.$$ 

The parameter $Y_0$ corresponds to the choice of the initial conditions.

The solution of the equation (5.2) can be calculated by using the Jacobi elliptic functions, discussed in the Appendix.

More precisely, we find that

$$\frac{\omega'(Y)}{\sqrt{C}} = cd(\epsilon \sqrt{Cm}(Y - Y_0) + v_0 |\frac{1}{m}|),$$

where

$$v_0 = sd^{-1}(\sqrt{m} \sinh \omega_0 |\frac{1}{m}|), \epsilon = \pm 1.$$ 

Moreover,

$$\tanh \omega = \frac{1}{\sqrt{m}} sn(\epsilon \sqrt{Cm}(Y - Y_0) + v_0 |\frac{1}{m}|).$$ 

In the coordinates $Z = X + iY$, the equation (4.1) is written as

$$e^{\omega(Y) + i\tau} u_{\bar{Z}} - e^{-\omega(Y) - i\tau} u_Z = 0, (5.3)$$
and the equation of the characteristics \( 4.7 \) takes the form
\[
\frac{dY}{dX} = \coth(i\omega(Y) + \tau).
\] (5.4)

Then, a solution of \( 5.3 \) is given by
\[
u = R + iS = R_0 + iS_0 + \alpha \left( X - X_0 - \int_{Y_0}^{Y} \tanh(i\omega(t) + \tau) dt \right),
\]
where
\[
R - R_0 = \alpha \left( X - X_0 - \text{Re} \left( \int_{Y_0}^{Y} \tanh(i\omega(t) + \tau) dt \right) \right),
\]
\[
S - S_0 = -\alpha \left( \text{Im} \left( \int_{Y_0}^{Y} \tanh(i\omega(t) + \tau) dt \right) \right).
\]

Elementary calculations give that
\[
\frac{\partial R}{\partial Y} = -\alpha \text{Re} \tanh(i\omega + \tau) = -\frac{\alpha \tan \tau (1 - \tanh^2 \omega)}{1 + \tan^2 \tau \tanh^2 \omega} = -\frac{(m - 1) \sin \tau \cos \tau}{M} \frac{\omega'(Y)}{\sqrt{CM}},
\] (5.5)

where
\[
M = 1 + \frac{4K_N}{C \rho^2} \cos^2 \tau = \frac{m + \tan^2 \tau}{1 + \tan^2 \tau}.
\]

Also,
\[
\frac{\partial S}{\partial Y} = -\alpha \text{Im} \tanh(i\omega + \tau) = -\frac{\alpha \tanh \omega (1 + \tan^2 \tau)}{1 + \tan^2 \tau \tanh^2 \omega} = \alpha C \frac{\omega''(Y)}{M} \frac{\omega'(Y)}{\sqrt{CM}},
\] (5.6)

and
\[
\Phi = e^F = \frac{1}{u_\xi \bar{u}_\xi} = \frac{4}{\alpha^2 \rho^2} \left( \cos^2 \tau \cosh^2 \omega + \sin^2 \tau \sinh^2 \omega \right).
\]

One should notice an interesting relation:
\[
\frac{\partial R}{\partial Y} = -\frac{4 \sin \tau \cos \tau}{\alpha \rho^2} \frac{1}{\Phi}.
\] (5.7)
Another interesting relation, which can be used in Section 6, is the following one:

\[
\left( \frac{\partial S}{\partial Y} \right)^2 = -\frac{16\tan^2 \tau}{(\tan^2 \tau + 1)^2} \frac{1}{\alpha^2 \rho^4 \Phi^2} + \frac{4(\tan^2 \tau - 1)}{(\tan^2 \tau + 1)^2 \rho^2 \Phi} + \alpha^2. \tag{5.8}
\]

Therefore (see the detailed explanations and calculations in the Appendix),

\[
R - R_0 = \alpha (X - X_0) + \frac{\alpha (m - 1) \tan \tau}{m + \tan^2 \tau} \left( \Pi \left( \frac{m(1 + \tan^2 \tau)}{m + \tan^2 \tau}, \frac{\omega'(Y)}{\sqrt{CM}} \right) - \Pi \left( \frac{m(1 + \tan^2 \tau)}{m + \tan^2 \tau}, \frac{\omega'_0}{\sqrt{CM}} m \right) \right) \tag{5.9}
\]

where \( \Pi(n, x|m) \) is the elliptic integral of the third kind. Also,

\[
S - S_0 = \frac{\alpha}{\sqrt{CM}} \left( \text{arctanh} \frac{\omega'(Y)}{\sqrt{CM}} - \text{arctanh} \frac{\omega'_0}{\sqrt{CM}} \right) \tag{5.10}
\]

and

\[e^F = \frac{4M}{(m - 1)\alpha^2 \rho^2 \cosh^2 \Sigma},\]

where

\[
\Sigma = \frac{\sqrt{CM}}{\alpha} (S - S_0) + \text{arctanh} \frac{\omega'_0}{\sqrt{CM}}. \tag{5.11}
\]

Note that the metric on \( N \) is of constant curvature and that the results in Section 6 cover all the cases of positive, negative and zero constant curvature in a unified formulation.

6. Explicit solutions

6.1. The strip model of Shi, Tam and Wan [16]. This Section focuses on the explicit solution of the harmonic map problem in the influential work [16] that generalizes the solutions in [2, 18]. The solution is a quasi-conformal harmonic diffeomorphism between hyperbolic planes. This result can be recovered by the results in Section 5. In this section we prove that the calculations of the paper [16] correspond to the one soliton solution of the sinh-Gordon equation. Here only a short description of the methods of this paper are given, the detailed calculations will appear in the future as a selfcontained note.

Consider the strip model for hyperbolic plane. In [16] the authors find a harmonic map which takes the form \( R(x, y) = \alpha x + h(y) \) and \( S(x, y) = g(y) \). Let \( a = h'(\frac{\pi}{2}) \) and \( b = g'(\frac{\pi}{2}) \). They show that \( \frac{\partial R}{\partial y} = a^2 \sin^2 g \) and \( \cot g = z \), where

\[
\int_0^{\pi(y)} \frac{dz}{\sqrt{\alpha^2 z^4 + c^2 z^2 + b^2}} = \frac{\pi}{2} - y,
\]
and \( c^2 = \alpha^2 + b^2 + a^4 \). They extend \( g, h \) to \([0, \pi]\) such that
\[
h(y) = h(\pi) - h(\pi - y), \quad g(y) = \pi - g(\pi - y),
\]
and they prove that there are appropriate constants \( a, b \) such that the harmonic map is a quasi-conformal harmonic diffeomorphism between the hyperbolic strips.

The same harmonic map can be recovered by the method presented earlier. More precisely, let \( x = X, y = Y, X_0 = 0, Y_0 = \pi/2 \),
\[
w_1 = \frac{1}{\alpha\sqrt{2}} \sqrt{c^2 - \sqrt{c^4 - 4\alpha^2b^2}}, \quad w_2 = \frac{1}{\alpha\sqrt{2}} \sqrt{c^2 + \sqrt{c^4 - 4\alpha^2b^2}}.
\]
Consider
\[
\rho = \frac{2}{\alpha\sqrt{w_2^2 - w_1^2}}, \quad \tan \tau = -\sqrt{\frac{w_2^2 - 1}{1 - w_1^2}}, \quad \omega'_0 = 0,
\]
\[
C = -\alpha^2 w_1^2, \quad M = \frac{1}{w_1^2}, \quad m = \frac{w_2^2}{w_1^2}, \quad \Sigma = i(S - \frac{\pi}{2}).
\]
We observe that
\[
\sqrt{(w_2^2 - 1)(1 - w_1^2)} = \frac{a^2}{\alpha}.
\]
Considering the choice of the parameters in [16], we find that
\[
K' = \alpha w_2 \frac{\pi}{2} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 - (1 - \frac{w_1^2}{w_2^2}) \sin^2 \theta}
\]
where \( K' \) is the quarter period of the elliptic Jacobi functions, see equations (16.1.1) and (16.1.2) of [12]. In this case we can see that
\[
\tanh \omega = \frac{1}{\sqrt{m}} sn(\sqrt{Cm}(Y - Y_0) + v_0) \frac{1}{m} = \frac{w_1}{w_2} sn(i\alpha w_2 (Y - \frac{\pi}{2}) + v_0) \frac{w_2^2}{w_1^2}.
\]
We find that
\[
\tanh \omega = \frac{w_1}{w_2} cd(i\alpha w_2 (Y - \frac{\pi}{2}) \frac{w_2^2}{w_1^2}) = \frac{w_1}{w_2} nd(\alpha w_2 (Y - \frac{\pi}{2}) 1 - \frac{w_2^2}{w_1^2}) = dn(\alpha w_2 Y|1 - \frac{w_2^2}{w_1^2}).
\]
Given that, we find that
\[
\frac{\partial S}{\partial y} = \frac{\alpha w_2^2 dn(\alpha w_2 y|1 - \frac{w_2^2}{w_1^2})}{w_2^2 + (1 - w_2^2) \sin^2(\alpha w_2 y|1 - \frac{w_2^2}{w_1^2})}.
\]
and
\[
\frac{\partial R}{\partial y} = \frac{a^2 \ sn^2(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})}{w_2^2 + (1 - w_2^2) \ sn^2(\alpha w_2 y | 1 - \frac{w_1^2}{w_2^2})}.
\]

A lengthy but standard computation, that will also be contained in a future self-contained note, can show that this result is identical with the result in [16]. Then, (5.7) and (5.8) coincide with the corresponding equations (4.3) in [16]. More precisely we find that
\[
\frac{\partial R}{\partial Y} = -\frac{4 \sin \tau \cos \tau \ 1}{\alpha \rho^2 \Phi} = a^2 \sin^2 S
\]
and
\[
\left( \frac{\partial S}{\partial Y} \right)^2 = -\frac{16 \tan^2 \tau}{(\tan^2 \tau + 1)^2} \frac{1}{\alpha^2 \rho^4 \Phi^2} + \frac{4 (\tan^2 \tau - 1)}{(\tan^2 \tau + 1) \rho^2 \Phi} + \alpha^2
\]
\[
= \alpha^2 + (b^2 + a^4 - \alpha^2) \sin^2 S - a^4 \sin^4 S,
\]
where \( \Pi(n, x|m) \) is the elliptic integral of the third kind. Also,
\[
S - \frac{\pi}{2} = i \arctanh \left( w_1 \frac{\omega'(Y)}{\sqrt{C}} \right), \tag{6.1}
\]
\[
e^F = \frac{1}{\sin^2 S},
\]
and
\[
\frac{\omega'(Y)}{\sqrt{C}} = cd(\alpha w_2 i(Y - \frac{\pi}{2}) + v_0 \frac{w_1^2}{w_2^2}) = -sn(\alpha w_2 i(Y - \frac{\pi}{2}) | \frac{w_1^2}{w_2^2})
\]
\[
= -isc(\alpha w_2 (Y - \frac{\pi}{2}) | 1 - \frac{w_1^2}{w_2^2}) = i \frac{w_2}{w_1} \cs(\alpha w_2 Y | 1 - \frac{w_1^2}{w_2^2}).
\]

Thus, we find that
\[
S = \cot^{-1} \left( \frac{w_2 \cs(\alpha w_2 Y | 1 - \frac{w_1^2}{w_2^2})}{w_1} \right),
\]
and it is a lengthy computation that this result coincides with the result in [16].

6.2. The result of Li and Tam. This section focuses on the explicit solution of the harmonic map problem in the influential work [9]. This result can also be recovered by the results in Section 5 as shown below.
Consider the solution
\[
\omega(\zeta, \bar{\zeta}) = -\log \tanh \xi
\]
of the equation
\[
\Delta \omega = 2 \sinh 2\omega.
\]
Let $a > 0$. Then, the equation
\[ \frac{\partial u}{\partial \bar{\zeta}} = e^{-2\omega} \]
admits the solutions
\[ u(\zeta, \bar{\zeta}) = \left( \frac{2\eta}{a}, -2 \sinh \frac{2\xi}{a} \right). \]
If $z = -\frac{2i}{a} \zeta$, then
\[ u(z, \bar{z}) = (x, \frac{1}{a} \sinh ay). \]

The hyperbolic metric that corresponds to the target $N$ is $e^{F(R,S)} = \frac{1}{\sqrt{s}}$, see Section 5 for more details. This is a family of harmonic maps between hyperbolic spaces, that has been studied in [9].

Note that this result can also be obtained by the general result in Section 5 by considering the initial conditions $u(x_0, y_0) = (x_0, \frac{1}{a} \sinh ay_0)$, $\omega_0 = -\log \tanh y_0$, $\omega'_0 = \frac{1}{\cosh 2y_0}$. In this case $K_N = -1$, $\rho = 1$, $C = 0$, $\tau = -\frac{\pi}{2}$ and one can observe that the limiting case of the solution in Section 5 taking $C \to 0$, $\tau \to -\frac{\pi}{2}$ provides the explicit solution obtained in [9].

7. Bäcklund Transform of the sinh-Gordon equation

In this section we discuss a variation of the Bäcklund transform of the sinh-Gordon equation, by applying the methods discussed in the previous sections.

Recall that $e^v = e^{\frac{F_{uu}}{2}} u_z$ and $v = \omega + i\theta$. In what follows we shall prove the following proposition.

**Proposition 7.1.** The equation
\[ 2i\theta_\zeta \zeta = e^{-F} \left( (F^2_u - F_{uu}) e^{2i\theta} - (F^2_{\bar{u}} - F_{\bar{u}\bar{u}}) e^{-2i\theta} \right) \]
is the Bäcklund transform of the sinh-Gordon equation
\[ \omega_{\zeta \zeta} = -\frac{K_N}{2} \sinh 2\omega, \quad (7.1) \]
where
\[ K_N = K_N(u, \bar{u}) = -\frac{1}{2} e^{-F} \Delta F \]
is the Gauss curvature of the metric
\[ h = e^{F(u, \bar{u})} du^2. \]
Taking into consideration (3.5) and (3.6) one can calculate the following relations,

\[ e^{2i\theta} = e^{v - \bar{v}} = e^{F + Re\lambda u_\zeta u_\zeta}, \quad e^{-2i\theta} = e^{-(v - \bar{v})} = e^{F + Re\lambda u_\zeta \bar{u}_\zeta}, \]

and

\[
\sin(2\theta) = \frac{e^{F + Re\lambda}}{2i} (u_\zeta u_\bar{\zeta} - u_\zeta u_\bar{\zeta}), \quad \cosh(2\theta) = \frac{e^{F + Re\lambda}}{2} (u_\zeta u_\bar{\zeta} + u_\zeta u_\bar{\zeta}).
\]

From equation (3.9) it follows that

\[
2i\theta_{\zeta\bar{\zeta}} = \left( F_\zeta^2 - F_{\zeta\zeta} \right) u_\zeta u_\bar{\zeta} - \left( F_{\bar{\zeta}\zeta} - F_{\bar{\zeta}\bar{\zeta}} \right) \bar{u}_\zeta \bar{u}_\bar{\zeta} = e^{-(F + Re\lambda)} \left( (F_\zeta^2 - F_{\zeta\zeta}) e^{2i\theta} - (F_{\bar{\zeta}\zeta} - F_{\bar{\zeta}\bar{\zeta}}) e^{-2i\theta} \right),
\]

or in the specific coordinate system \( \zeta = \xi + i\eta \) given by (3.10):

\[
2i\theta_{\zeta\bar{\zeta}} = e^{-F} \left( (F_\zeta^2 - F_{\zeta\zeta}) e^{2i\theta} - (F_{\bar{\zeta}\zeta} - F_{\bar{\zeta}\bar{\zeta}}) e^{-2i\theta} \right). \tag{7.2}
\]

Using the specific coordinate system (3.10), equations (3.5) and (3.6) can be written as follows:

\[
e^{v - \frac{F}{2}} = u_\zeta, \quad e^{-v - \frac{F}{2}} = \bar{u}_\zeta, \quad e^{v - \frac{F}{2}} = \bar{u}_\zeta, \quad e^{-v - \frac{F}{2}} = u_\zeta, \quad v = \omega + i\theta. \tag{7.3}
\]

Consider the relation

\[ u_\zeta \bar{\zeta} = u_\zeta \bar{\zeta}. \]

Then \( v \) satisfies the linear partial differential equation

\[
\left( e^{v - \frac{F}{2}} \right)_\zeta = \left( e^{-v - \frac{F}{2}} \right)_{\bar{\zeta}}.
\]

Observe that the above equations can be written as a system of first order non linear differential equations:

\[
\omega_{\xi} - \theta_{\eta} = \frac{1}{2} \tanh \omega \frac{\partial F}{\partial \xi},
\]

\[
\omega_{\eta} + \theta_{\xi} = \frac{1}{2} \coth \omega \frac{\partial F}{\partial \eta}.
\]

Therefore, by the chain rule we have

\[
\frac{\partial F}{\partial \xi} = F_{u\xi} \omega_{\xi} + F_{u\bar{\xi}} \theta_{\xi}, \quad \frac{\partial F}{\partial \eta} = F_{u\eta} \omega_{\eta} + F_{u\bar{\eta}} \theta_{\eta}.
\]

In these equations we can replace the partial derivatives of the function \( u \) by the relations given in equation (7.3).

After trivial lengthy calculations, we can check that the function \( \theta(\xi, \eta) \) satisfies equation (7.2) and \( \omega(\xi, \eta) \) satisfies the sinh-Gordon type equation (7.1). Therefore the function \( \theta \) is indeed the Bäcklund Transform of the function \( \omega \).
One interesting remark arising directly from equation (7.2) is the following.

**Remark 1.** In the case of the hyperbolic upper half-plane metric, 
\[ F = \log \left( \frac{1}{S^2} \right), \quad S = \text{Im} \, u, \]
the function \(\omega\) satisfies the sinh-Gordon equation
\[ \omega_{\zeta \zeta} = \frac{1}{2} \sinh(2\omega) \]
and the Bäcklund transform \(\theta\), satisfies the sine-Gordon equation
\[ \theta_{\zeta \zeta} = -\frac{1}{2} \sin(2\theta). \]
These two equations are related by the Bäcklund transform
\[ \omega_\xi - \theta_\eta = 2 \sinh \omega \sin \theta \]
\[ \omega_\eta + \theta_\xi = 2 \cosh \omega \cos \theta. \]
This is a Bäcklund transform and it provides a connection between the solutions of an elliptic sinh-Gordon and an elliptic sine-Gordon equations. Thus, one can obtain solutions of the elliptic sinh-Gordon equation by the known solutions of the sine-Gordon equation and vice versa.

8. Perspectives for future investigation

An evident application of the methods introduced in this paper is to classify the known solutions of the sinh-Gordon equation and to generate new solutions of the harmonic map problem. This is under investigation.

It would be interesting to generalize the above results in higher dimensions. Furthermore, theoretical results can be obtained by applying the theory of Beltrami equation in order to study harmonic maps between surfaces.

On the other hand, one can apply maximum principle arguments and deduce theoretical results on harmonic diffeomorphisms. There is also a connection with the theory of constant mean curvature surfaces, implied by the sinh-Gordon equation. The case of the flat complex plane is still of great interest. More generally, it is interesting to find solutions (for example group invariant solutions) of the sinh-Gordon equation and then obtain families of harmonic maps. On the other hand, the known formulas for harmonic maps can provide examples of solutions to the elliptic sinh-Gordon and elliptic sine-Gordon equation.
Note that the Bäcklund transform obtained can relate known solutions of the one equation with solutions to the other. The general problem of solving the sinh-Gordon equation and then the corresponding Beltrami equation, is still open.

**Appendix A. Elliptic Functions summary**

In this Appendix the definitions and the properties of the elliptic integrals and Jacobi functions, used in this paper, are presented for clarity reasons. The formulation, used in this paper, is taken from [12].

The first kind elliptic integral \( F(\phi|n) \) and the Jacobi elliptic function \( sn(v|n) \) are defined by the formula

\[
F(\phi|n) = v = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - nt^2)}} = sn^{-1}(x|n) \tag{A.1}
\]

where \( x = \sin \phi = sn(v|n) \).

Equation (5.1) can be written as the sinh-Gordon equation

\[
\left( \frac{\omega'(Y)}{\sqrt{C}} \right)^2 + (m-1) \sinh^2 \omega(Y) = 1 \quad \text{or} \quad \frac{\omega'(Y)}{\sqrt{1 - (m-1) \sinh^2 \omega}} = \epsilon \sqrt{C}, \quad \epsilon = \pm 1.
\tag{A.2}
\]

If we put

\[
v(Y) = \sqrt{m \tanh \omega(Y)}
\]

then (A.2) can be rewritten as

\[
\frac{v'(Y)}{\sqrt{(1 - v^2)(1 - \frac{1}{m} v^2)}} = \epsilon \sqrt{Cm}.
\]

Therefore equation (A.2) gives after integration

\[
\int_{\sqrt{m \tanh \omega(0)}}^{\sqrt{m \tanh \omega}} \frac{dv}{\sqrt{(1 - v^2)(1 - \frac{1}{m} v^2)}} = \epsilon \sqrt{Cm} (Y - Y_0).
\]

The left hand part of the equation can be replaced by using the definition of the Jacobi elliptic functions (A.1) and we can find \( \omega \) explicitly. More precisely we find that

\[
v - v_0 = sn^{-1}\left(\sqrt{m \tanh \omega} \left| \frac{1}{m} \right. \right) - sn^{-1}\left(\sqrt{m \tanh \omega_0} \left| \frac{1}{m} \right. \right) = \epsilon \sqrt{Cm} (Y - Y_0)
\]

or

\[
\tanh \omega = \frac{1}{\sqrt{m}} sn(\epsilon \sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m}). \tag{A.3}
\]
From the well known relations between the Jacobi elliptic functions
\[ 1 - n \, sn^2(v|n) = dn^2(v|n) \] and \[ nd^2(v|n) - 1 = n \, sd^2(v|n) \]
we find
\[ \cosh \omega = nd(\sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m}), \] (A.4)
and hence
\[ \sinh \omega = \frac{1}{\sqrt{m}}sd(\epsilon \sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m}), \] (A.5)
where
\[ v_0 = sn^{-1}(\sqrt{m} \tanh \omega_0|\frac{1}{m}) = sd^{-1}(\sqrt{m} \sinh \omega_0|\frac{1}{m}). \] (A.6)
Thus
\[ \omega = \log \frac{sd(\epsilon \sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m}) + \sqrt{m} nd(\epsilon \sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m})}{2\sqrt{m}}. \] (A.7)
Furthermore, we find that
\[ \frac{\omega'(Y)}{\sqrt{C}} = cd(\epsilon \sqrt{Cm}(Y - Y_0) + v_0|\frac{1}{m}). \] (A.8)

From equation (5.6) we have after some elementary calculations
\[ S_Y = -\alpha \frac{\cosh \omega \sinh \omega}{\cos^2 \tau + \sinh^2 \omega} = \frac{\alpha}{CM} \frac{\omega'}{1 - \left(\frac{\omega'(Y)}{\sqrt{C}}\right)^2}, \] (A.9)
using the formulas (A.4) and (A.5) we find
\[ S_Y = -\frac{\alpha}{\sqrt{m}} \frac{nd(v|\frac{1}{m})sd(v|\frac{1}{m})}{\cos^2 \tau + \frac{1}{m} nd^2(v|\frac{1}{m})}, \] (A.10)
where
\[ v = \epsilon \sqrt{Cm}(Y - Y_0) + v_0, \]
and we can verify the implicit formula (5.1). Formula (5.10) can be calculated by integration of the right hand side of equation (A.9).

From equation (5.5) by similar algebraic calculations we find
\[ \frac{\partial R}{\partial Y} = -\frac{\alpha \tan \tau}{\cosh^2 \omega(1 + \tan^2 \tau \tan^2 \omega)} = -\frac{\alpha \tan \tau \, dn^2(v|\frac{1}{m})}{(1 + \frac{1}{m} \tan^2 \tau \, sn^2(v|\frac{1}{m}))}. \]
Therefore we obtain formula (5.9).
Using the following expression
\[ R_Y = -\frac{(m-1)\sin \tau \cos \tau}{M} \frac{\alpha}{1 - \left(\frac{\omega'(Y)}{\sqrt{CM}}\right)^2}, \tag{A.11} \]
the sinh-Gordon equation can be written as follows:
\[ \frac{\omega''}{\sqrt{Cm}} = -\sqrt{C} \sqrt{\left(1 - m \left(\frac{\omega'(Y)}{\sqrt{Cm}}\right)^2\right) \left(1 - \left(\frac{\omega'(Y)}{\sqrt{Cm}}\right)^2\right)}. \]

Then (A.11) can be written as
\[ R_Y = \frac{\alpha (m-1) \tan \tau}{m + \tan^2 \tau}. \]

From the definition of the elliptic integral of the third kind, by integration we can obtain formula (5.9). In this formula \( \omega' \) can be replaced by the corresponding elliptic function given by (A.8).

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