On the successive passage times of certain one-dimensional diffusions.

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Abstract

We show in detail some results, outlined in a previous paper regarding the case of Brownian motion (BM), about the distribution of the $n$th-passage time of a one-dimensional diffusion obtained by a space or time transformation of BM, through a constant barrier $a$. Some explicit examples are reported.

Keywords: First-passage time, Second-passage time, one-dimensional diffusion

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1 Introduction

This is a continuation of [1], where we studied excursions of Brownian motion with drift, and we found explicitly the distribution of the $n$th-passage time of Brownian motion (BM) through a linear boundary. In the present paper, by using the results of [1], we study in depth the $n$th-passage time of a one-dimensional diffusion process $X(t)$, which is obtained from BM by a space transformation or a time change. Particularly, focusing on the second passage-time, we find explicit formulae for the density of the second-passage time, and we calculate its Laplace transform. We use numerical computation and/or simulation, whenever analytical expressions cannot be achieved, and we show graphically the results in a number of figures.

The successive-passage times of a diffusion $X(t)$ through a boundary $S(t)$ are related to the excursions of $Y(t) := X(t) - S(t)$; indeed, when $Y(t)$ is entirely positive or entirely negative on the time interval $(s,u)$, it is said that it is an excursion of $Y(t)$. Excursions have interesting applications in Biology, Economics, and other applied sciences (see e.g. [1], [14]). They are also related to the last passage time of a diffusion through a boundary; actually, last passage times of continuous martingales play an important role in Finance, for instance, in models of default risk (see e.g. [6], [8]). In the special case when $X(t)$ is BM, its excursions follow the arcsine law, namely the probability that BM has no zeros in the time interval $(s,u)$ is $\frac{2}{\pi} \arcsin \sqrt{u/s}$ (see e.g. [10]). Really, by using Salminen’s formula for the last passage time of BM through a linear boundary (see [13]), in [1] we found the law of the excursions of drifted BM, and we derived the distribution of the $n$th passage time of BM through a linear boundary; in this article, as we said, our aim is to obtain analogous results for transformed BM through a constant boundary.

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The paper is organized as follows: in Section 2, we recall the results of [1], concerning the nth-passage time of BM through a linear boundary, in Section 3 we extend these results to certain diffusions \( X(t) \), studying also the Laplace transform of the second-passage time of \( X(t) \). Section 4 reports some explicit examples, while Section 5 is devote to conclusions and final remarks.

## 2 Preliminary results on the nth-passage time of Brownian motion

The first-passage time of BM through the linear boundary \( S(t) = a + bt \), when starting from \( x \), is defined by \( \tau_i^B(S|x) = \inf\{t > 0 : x + B_t = a + bt\} \), that will be denoted in this section by \( \tau_1(x) \), dropping, for simplicity, the superscript \( B \) which refers to BM and the dependence on \( S \); we recall the Bachelier-Levy formula for the distribution of \( \tau_1(x) \):

\[
P(\tau_1(x) \leq t) = 1 - \Phi((a - x)/\sqrt{t} + b\sqrt{t}) + \exp(-2b(a - x))\Phi(b\sqrt{t} - (a - x)/\sqrt{t}), \ t > 0
\]

where \( \Phi(y) = \int_y^{\infty} \phi(z) \, dz \), with \( \phi(z) = e^{-z^2/2}/\sqrt{2\pi} \), is the cumulative distribution function of the standard Gaussian variable. If \( (a - x)b > 0 \), then \( P(\tau_1(x) < \infty) = e^{-2b(a-x)} \), whereas, if \( (a - x)b \leq 0 \), \( \tau_1(x) \) is finite with probability one and it has the Inverse Gaussian density, which is non-defective (see e.g. [5], [9]):

\[
f_{\tau_1}(t) = f_{\tau_1}(t|x) = \frac{d}{dt} P(\tau_1(x) \leq t) = \frac{|a - x|}{t^{3/2}} \phi \left( \frac{a + bt - x}{\sqrt{t}} \right), \ t > 0; \quad (2.1)
\]

moreover, if \( b \neq 0 \), the expectation of \( \tau_1(x) \) is finite, being \( E(\tau_1(x)) = \frac{|a-x|}{|b|} \).

The second-passage time of BM through \( S(t) \), when starting from \( x \), is defined by \( \tau_2(x) := \tau_2^B(S|x) = \inf\{t > \tau_1(x) : x + B_t = a + bt\} \), and generally, for \( n \geq 2 \), \( \tau_n(x) = \inf\{t > \tau_{n-1}(x) : x + B_t = a + bt\} \) denotes the nth-passage time of BM through \( S(t) \).

Now, we assume that \( b \leq 0 \) and \( x < a \), or \( b \geq 0 \) and \( x > a \), so that \( P(\tau_1(x) < \infty) = 1 \). For fixed \( t > 0 \), we consider the last-passage time prior to \( t \) of BM, starting from \( x \), through the boundary \( S(t) = a + bt \), that is:

\[
\chi_S^t = \begin{cases} 
\sup\{0 \leq u \leq t : x + B_u = S(u)\} & \text{if } \tau_1(x) \leq t \\
0 & \text{if } \tau_1(x) > t.
\end{cases} \quad (2.2)
\]

Notice that [1] contains a slight imprecision in the definition of \( \chi_S^t \), and the function there denoted by \( \psi_t(u) \), has to be thought indeed as the density of \( \chi_S^t \), conditional to the event \( \{\tau_1(x) \leq t\} \); in other words, \( \chi_S^t \) is zero with probability \( P(\tau_1(x) > t) \), and it takes values spread on the interval \( (0, t) \) with conditional density \( \psi_t(u) = P(\chi_S^t \in du | \tau_1(x) \leq t) \). However, this does not affect the correctness of the results of [1]. In fact, it holds (see [13], [1]):

\[
\psi_t(u) = \frac{d}{du} P(\chi_S^t \leq u | \tau_1(x) \leq t) = \frac{1}{\sqrt{2\pi u}} \exp(-b^2 u/2) \int_{-\infty}^{+\infty} \nu_{x-u}(t-u, \hat{S}) \, dx, \ u \leq t \quad (2.3)
\]

where \( \hat{S}(t) = S(t-u), \ u \leq t \), and

\[
\nu_{x}(v, \hat{S}) = \exp\{-b(x-bt) - b^2v/2\}/\sqrt{2\pi v^3} \exp\{-b^2v^2/2\} \quad (2.4)
\]
Then, the following explicit formula is obtained (see [1]):

$$\psi_t(u) = \frac{e^{-\frac{u^2}{2}}}{\pi \sqrt{u(t-u)}} \left[ e^{-\frac{u^2}{2(t-u)}} + \frac{b}{2} \sqrt{2\pi(t-u)} \left( 2\Phi(b\sqrt{t-u}) - 1 \right) \right], \quad 0 < u < t \quad (2.5)$$

Notice that $\psi_t$ is independent of $a$; if $b = 0$, one gets $\psi_t(u) = \frac{1}{\pi \sqrt{u(t-u)}}$, $0 < u < t$, that is, the arc-sine law with support in $(0,t)$.

**Remark 2.1** For $z < t$, the event $\{\lambda^t_z \leq z|\tau_1(x) \leq t\}$ is nothing but the event $\{x + B_u - S(u) \text{ has no zeros in the interval } (z,t)\}$. In order to study the second-passage time, $\tau_2(x)$, of $x + B_t$ through the linear boundary $S(t) = a + bt$, when $x < a$ and $b \leq 0$ (the case when $b \geq 0$ and $x > a$ can be studied in a similar way), we set $T_1(x) = \tau_1(x)$ and $T_2(x) = \tau_2(x) - \tau_1(x)$. It can be proved (see [1]) that $T_2(x)$ is finite with probability one, only if $b = 0$. Conditionally to $\tau_1(x) = s$, the event $\{\tau_2(x) > s + t\}$ ($t > 0$), is nothing but the event $\{x + B_u - S(u) \text{ has no zeros in the interval } (s,s + t)\}$. Therefore, by using the above expression of $\psi_t(u)$ one obtains

$$P(\tau_2(x) > \tau_1(x) + t|\tau_1(x) = s) = P(\lambda^t_s \leq s|\tau_1(x) \leq s + t) = \int_0^s \psi_{s+t}(y)dy$$

and so

$$P(T_2(x) \leq t|\tau_1(x) = s) = 1 - \int_0^s \psi_{s+t}(y)dy. \quad (2.6)$$

Then:

$$P(T_2(x) \leq t) = 1 - \int_0^{+\infty} f_{\tau_1}(s)ds \int_0^s \psi_{s+t}(y)dy. \quad (2.7)$$

By taking the derivative with respect to $t$, one gets the density of $T_2(x)$:

$$f_{T_2}(t) = \int_0^{+\infty} f_{\tau_1}(s) \left[ e^{-\frac{b^2(s+t)}{2}} \frac{\sqrt{s}}{\pi \sqrt{t(s+t)}} \right] ds. \quad (2.8)$$

Notice that $\int_0^s \psi_{s+t}(y)dy$ is decreasing in $t$ and $f_{T_2}(t) \sim \text{const}/\sqrt{t}$, as $t \to 0^+$. Moreover, from (2.7) it follows that, if $b \neq 0$, the distribution of $T_2(x)$ is defective. In fact (see [1]):

$$P(T_2(x) = +\infty) = 2\text{sgn}(b)E \left[ \Phi(b\sqrt{\tau_1(x)}) - \frac{1}{2} \right] > 0, \quad (2.9)$$

where $\text{sgn}(b) = \begin{cases} 0 & \text{if } b = 0 \\ \frac{|b|}{b} & \text{if } b \neq 0 \end{cases}$.

Since the function $g(s) = 2\text{sgn}(b) \left( \Phi(b\sqrt{s}) - \frac{1}{2} \right)$ is concave, by Jensen’s inequality one gets

$$P(T_2(x) = +\infty) \leq \gamma(b), \quad \text{where } \gamma(b) = 2\text{sgn}(b) \left( \Phi(b\sqrt{[(a-x)/b]} \right) - \frac{1}{2} \right).$$

Notice that $\gamma$ is an even function of $b$.

On the contrary, if $b = 0$, from (2.9) we get that $P(T_2(x) = +\infty) = 0$, that is, $T_2(x)$ is a proper random variable, and also $\tau_2(x) = T_2(x) + \tau_1(x)$ is finite with probability one; precisely, by calculating the integral in (2.7) we have:

$$P(T_2(x) \leq t) = \int_0^{+\infty} \frac{2}{\pi} \arccos \sqrt{\frac{s}{s + t}} \frac{|a-x|}{\sqrt{2\pi s^{3/2}}} e^{-(a-x)^2/2s} ds \quad (2.10)$$
then, taking the derivative with respect to \( t \), the density of \( T_2(x) \) turns out to be:

\[
 f_{T_2}(t) = \int_{0}^{+\infty} \frac{1}{\pi(s+t)\sqrt{t}} \frac{|a-x|}{\sqrt{2}\pi s} e^{-\frac{(a-x)^2}{2s}} ds. \quad (2.11)
\]

In the Figure 1, taken from \[1\], the plot of \( P(\tau_2(x) = +\infty) = P(T_2(x) = +\infty) \), as a function of \( b \leq 0 \), for \( a = 1 \) and \( x = 0 \).

In the Figure 2, taken from \[1\], it is reported the probability density of \( T_2(x) \), obtained from (2.8), by calculating numerically the integral, for \( x = 0 \), \( a = 1 \), and various values of the parameter \( b \leq 0 \).

As far as the expectation of \( \tau_2(x) \) is concerned, it is obviously infinite for \( b \neq 0 \), while \( E(\tau_1(x)) \) is finite. If \( b = 0 \), \( E(\tau_1(x)) \) and \( E(\tau_2(x)) \) are both infinite. As for \( \tau_1(x) \) this is well-known, as for \( \tau_2(x) \) it derives from the fact that (see \[1\]):

\[
 E(T_2(x)) = \int_{0}^{+\infty} P(T_2(x) > t)dt = 
\int_{0}^{+\infty} ds \frac{2}{\pi} \frac{|a-x|}{\sqrt{2}\pi s^{3/2}} e^{-\frac{(a-x)^2}{2s}} \int_{0}^{+\infty} \arcsin \sqrt{\frac{s}{s+t}} dt = +\infty.
\]

By taking the derivative with respect to \( t \) in (2.6), one obtains the density of \( T_2(x) \) conditional to \( \tau_1(x) = s \), that is:

\[
f_{T_2|\tau_1}(t|s) = -\frac{d}{dt} \int_{0}^{s} \psi_{s+t}(y)dy = e^{-\frac{b^2(s+t)^2}{2}} \frac{\sqrt{s}}{\pi(s+t)\sqrt{t}}, \quad (2.12)
\]

and, for \( b = 0 \):

\[
f_{T_2|\tau_1}(t|s) = \frac{\sqrt{s}}{\pi(s+t)\sqrt{t}}. \quad (2.13)
\]
Since $\tau_2(x) = \tau_1(x) + T_2(x)$, by the convolution formula, the density of $\tau_2(x)$ follows:

$$f_{\tau_2}(t) = \int_0^t f_{T_2|\tau_1}(t - s)f_{\tau_1}(s)ds = \frac{e^{-b^2t/2}}{\pi t} \int_0^t \frac{|a - x|}{\sqrt{2\pi s\sqrt{t-s}}} e^{-\frac{(a+bs-x)^2}{2s}}ds. \quad (2.14)$$

Of course, the distribution of $\tau_2(x)$ is defective for $b \neq 0$, namely $\int_0^{+\infty} f_{\tau_2}(t)dt = 1 - P(\tau_2(x) = +\infty) < 1$, since $P(\tau_2(x) = +\infty) = P(T_2(x) = +\infty) > 0$.

If $b = 0$, we obtain:

$$f_{\tau_2}(t) = \frac{1}{\pi t} \int_0^t \frac{|a - x|}{\sqrt{2\pi s\sqrt{t-s}}} e^{-\frac{(a-x)^2}{2s}}ds, \quad (2.15)$$

which is non-defective.

In the Figure 3, we report the probability density of $\tau_2(x)$, obtained from (2.14) by calculating numerically the integral, for $x = 0$, $a = 1$, and various values of the parameter $b \leq 0$. Although the shapes appear to be similar to that of the inverse Gaussian density (2.1), the density of $\tau_2(x)$ is more concentrated around its maximum. In the Figure 4, we report from [1] the comparison between the probability density of $\tau_2(x)$ and the inverse Gaussian density, for $a = 1$, $b = 0$ and $x = 0$.

The following holds (see [1]):

**Proposition 2.2** Let be $T_1(x) = \tau_1(x)$, $T_n(x) = \tau_n(x) - \tau_{n-1}(x)$, $n = 2, \ldots$

Then:

$$P(T_1(x) \leq t) = 2(1 - \Phi(a - x/\sqrt{t})), \quad (2.16)$$

$$P(T_n(x) \leq t) = 1 - \int_0^{+\infty} f_{\tau_{n-1}}(s)ds \int_0^s \psi_{s+1}(y)dy, \quad n = 2, \ldots \quad (2.17)$$

Moreover, the density of $\tau_n(x)$ is:

$$f_{\tau_n}(t) = \int_0^t f_{T_n|\tau_{n-1}}(t - s)f_{\tau_{n-1}}(s)ds, \quad (2.18)$$
Figure 3: Approximate density of $\tau_2(x)$ for $x = 0$, $a = 1$, and various values of the parameter $b$; from top to the bottom, with respect to the peak of the curve: $b = -2$, $b = -1$, $b = -0.5$, $b = 0$.

Figure 4: Comparison between the probability density of $\tau_2(x)$ (upper peak) and the inverse Gaussian density (lower peak), for $a = 1$, $b = 0$ and $x = 0$. 
Figure 5: Comparison between the LT of the inverse Gaussian density (upper curve) and the LT of the second-passage time $\tau_2(x)$ of BM through the boundary $S(t) = a + bt$ (lower curve), for $x = 0$ and $a = 1, b = 0$.

where $f_{\tau_{n-1}}$ and $f_{T_{n|\tau_{n-1}}}$ can be calculated inductively, in a similar way, as done for $f_{\tau_2}$ and $f_{T_2|\tau_1}$.

If $b = 0$, $T_1(x), T_2(x), ...$ are finite with probability one.

We conclude this section, by summarizing the Laplace transforms of the first and second passage-time of $B_t$ through the boundary $S(t) = a + bt$:

- the Laplace transform (LT) of $\tau_1(x)$ is the LT of the inverse Gaussian density:
  $$\psi(\lambda) = E\left[e^{-\lambda \tau_1(x)}\right] = e^{-(a-x)(\sqrt{b^2+2\lambda} - b)}, \ \lambda > 0; \quad (2.19)$$

- the Laplace transform (LT) of $\tau_2(x)$ is:
  $$\tilde{\psi}(\lambda) = E\left[e^{-\lambda \tau_2(x)}\right] = \int_0^{+\infty} f_{\tau_2}(t)e^{-\lambda t} \ dt = \int_0^{+\infty} e^{-\lambda t} \left[ \frac{e^{-\frac{b^2}{2t}}}{\sqrt{2\pi}} \int_0^t \frac{|a-x|e^{-\frac{(a+bs-x)^2}{2s}}}{\sqrt{2\pi}s \sqrt{t-s}} \ ds \right] dt, \quad (2.20)$$

In the Figure 5, we compare the shape of the LT of the inverse Gaussian density with the LT of the second-passage time $\tau_2(x)$ of BM through the boundary $S(t) = a + bt$, for $a = 1, b = 0$ and $x = 0$.

3 The nth-passage time of a transformed Brownian motion

In this section we will extend the previous results to a one-dimensional, time homogeneous diffusion $X(t)$, obtained from BM by a space transformation or a time change. We start with diffusions conjugated to BM.
3.1 The nth-passage time of a diffusion conjugated to BM

Let be $X(t)$ a one-dimensional, time-homogeneous diffusion process, which is driven by the SDE

$$dX(t) = \mu(X(t)) dt + \sigma(X(t)) dB_t, \quad X(0) = x,$$

where the coefficients $\mu(\cdot)$ and $\sigma(\cdot)$ satisfy the usual conditions for the existence and uniqueness of the solution (see e.g. [7]).

We recall that $X(t)$ is said to be conjugated to BM (see [4]), if there exists an increasing function $v$ with $v(0) = 0$, such that:

$$X(t) = v^{-1}(B_t + v(x)), \quad t \geq 0.$$

Examples of diffusions conjugated to BM are (see also [4]):

(i) the Feller process or Cox-Ingersoll-Ross (CIR) model, which is driven by the SDE

$$dX(t) = \frac{1}{4} dt + \sqrt{X(t)} dB_t, \quad X(0) = x,$$

and is conjugated to BM via the function $v(x) = 2\sqrt{x}$, i.e. $X(t) = \frac{1}{4}(B_t + 2\sqrt{x})^2$;

(ii) the Wright and Fisher-like process, which is driven by the SDE

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2}X(t)\right) dt + \sqrt{X(t)(1 - X(t))} dB_t, \quad X(0) = x \in [0, 1],$$

and is conjugated to BM via the function $v(x) = 2 \arcsin \sqrt{x}$, i.e.

$$X(t) = \sin^2\left(B_t/2 + \arcsin \sqrt{x}\right).$$

Suppose now that $X(t)$ is conjugated to BM via the function $v$; let $\tau_1(a|x) = \inf\{t > 0 : X(t) = a\}$ be the FPT of $X(t)$ through the level $a > x$; as easily seen, one has $\tau_1(a|x) = \tau^B_1(v(a)|v(x))$, where $\tau^B_1(\alpha|y)$, denotes the FPT of BM, starting from $y < \alpha$, through the level $\alpha$. Then, from (2.1), one gets the density of $\tau_1(a|x)$:

$$f_{\tau_1}(t) = \frac{v(a) - v(x)}{t^{3/2}} \phi\left(\frac{v(x) - v(a)}{\sqrt{t}}\right), \quad (3.1)$$

implying that $E[\tau_1(a|x)] = +\infty$. For $x < a$, one has:

$$\tau_2(a|x) = \inf\{t > \tau_1(a|x) : X(t) = a\} = \inf\{t > \tau_1(a|x) : B_t = v(a) - v(x)\} = \tau^B_2(v(a)|v(x)),$$

and

$$\tau_n(a|x) = \inf\{t > \tau_{n-1}(a|x) : B_t = v(a) - v(x)\} = \tau^B_n(v(a)|v(x)),$$

where the superscript $B$ refers to BM, namely $\tau^B_n(a|x), \quad n = 2, \ldots$ is the $n$th-passage time of BM through the level $a$, when starting from $x < a$. By using the same notation of the previous section, setting $T_1(x) = \tau_1(a|x) = \tau^B_1(v(a)|v(x))$, $T_2(x) = \tau_2(a|x) - \tau_1(a|x) = \tau^B_2(v(a)|v(x)) - \tau^B_1(v(a)|v(x))$, and using (2.8) with $b = 0$, we get

$$P(T_2(x) \leq t) = \frac{2}{\pi} \int_0^{+\infty} f_{\tau_1}(s) \arccos\sqrt{\frac{s}{s + t}} ds. \quad (3.2)$$
Thus, the density of $T_2$ is

$$f_{T_2}(t) = \int_0^{+\infty} f_{\tau_1}(s) \frac{\sqrt{s}}{\pi \sqrt{t(s + t)}} \, ds = \int_0^{+\infty} \frac{v(a) - v(x)}{s^{3/2}} e^{-(v(x) - v(a))^2 / 4s} \frac{\sqrt{s}}{\pi \sqrt{t(s + t)}} \, ds$$

$$= \frac{v(a) - v(x)}{\pi \sqrt{2\pi}} \int_0^{+\infty} \frac{1}{(s + t) \sqrt{t}} \frac{1}{s} e^{-\frac{(v(x) - v(a))^2}{4s}} \, ds. \quad (3.3)$$

The expectation of $\tau_2(a|x)$ is infinite, because $E[T_2(x)] = +\infty$. From (2.12) with $b = 0$, we get

$$f_{T_2|\tau_1}(t|s) = -\frac{d}{dt} \int_0^{s} \psi_{s+t}(y) \, dy = \frac{\sqrt{s}}{\pi(s + t) \sqrt{t}}. \quad (3.4)$$

Moreover, from (2.15) $(b = 0)$:

$$f_{T_2}(t) = \int_0^{t} \frac{v(a) - v(x)}{\pi ts \sqrt{t - s} \sqrt{2\pi}} \cdot e^{-\frac{(v(x) - v(a))^2}{2s}} \, ds. \quad (3.5)$$

Setting $T_n(x) = \tau_n(a|x) - \tau_{n-1}(a|x)$, and using Proposition 2.2, we get

$$P(T_1(x) \leq t) = 2 \left( 1 - \Phi \left( \frac{v(a) - v(x)}{\sqrt{t}} \right) \right), \quad (3.6)$$

$$P(T_n(x) \leq t) = 1 - \int_0^{+\infty} f_{\tau_{n-1}}(s) \, ds \int_0^{s} \psi_{s+t}(y) \, dy, \quad n = 2, \ldots. \quad (3.7)$$

The density of $\tau_n(a|x)$ is:

$$f_{\tau_n}(t) = \int_0^{t} f_{T_n|\tau_{n-1}}(t - s|s) \cdot f_{\tau_{n-1}}(s) \, ds, \quad (3.8)$$

where $f_{\tau_{n-1}}$ and $f_{T_n|\tau_{n-1}}$ can be calculated inductively, in a similar way, as done for $f_{T_2}$ and $f_{T_2|\tau_1}$. In conclusion, $T_1(x), T_2(x), \ldots$ are finite with probability one.

### 3.2 The nth-passage time of time-changed Brownian motion

The previous arguments can be applied also to time-changed BM, namely

$$X(t) = x + B(\rho(t)), \quad (3.9)$$

where $\rho(t) \geq 0$ is an increasing, differentiable function of $t > 0$, with $\rho(0) = 0$. Such kind of diffusion process $X(t)$ is a special case of Gauss-Markov process; in particular the form (3.9) is taken by certain integrated Gauss-Markov processes (see [2], [3]). Let us consider the constant barrier $S = a$, and $x < a$; then, the FPT of $X(t)$ through $a$ is $\tau_1(a|x) = \rho^{-1}(\tau^B_1(a|x))$, where $\tau^B_1(a|x)$ denotes the first-passage time of BM, starting from $x$, through the barrier $a$. Thus, the density of $\tau_1(a|x)$ is:

$$f_{\tau_1}(t) = f_{\tau_1^B}(\rho(t)) \rho'(t) = \frac{|a - x|}{\rho(t)^{3/2}} \phi \left( \frac{a - x}{\sqrt{\rho(t)}} \right) \rho'(t). \quad (3.10)$$
The expectation is:
\[
E[\tau_1(a|x)] = \int_0^{+\infty} t f_{\tau_1}(t) \, dt = \int_0^{+\infty} t |a - x| \cdot \rho'(t) e^{-\frac{(x-a)^2}{2\rho^2(t)}} \, dt
\]  \tag{3.11}

Suppose that \( \rho(t) \sim c \cdot t^\alpha \), as \( t \to +\infty \); then, as easily seen, for \( \alpha > 2 \) the integral converges, namely \( E[\tau_1(a|x)] < +\infty \), unlike the case of BM, for which the expectation of FPT is infinite. One has:
\[
\tau_2(a|x) = \rho^{-1}(\tau_2^B(a|x)) \quad \text{with} \quad \tau_2^B(a|x) = \inf\{s > \tau_1^B(a|x) : x + B_s = a\},
\]
\[
\tau_n(a|x) = \rho^{-1}(\tau_n^B(a|x)) \quad \text{with} \quad \tau_n^B(a|x) = \inf\{s > \tau_{n-1}^B(a|x) : X + B_s = a\};
\]
\[
T_1(x) = \tau_1(a|x) = \rho^{-1}(\tau_1^B(a|x)), \quad T_2(x) = \tau_2(a|x) - \tau_1(a|x) = \rho^{-1}(\tau_2^B(a|x)) - \rho^{-1}(\tau_1^B(a|x)).
\]

It results
\[
P(T_2(x) \leq t) = \int_0^{+\infty} P\left(\rho^{-1}(\tau_2^B(a|x)) \leq t + s\right) f_{\rho^{-1}(\tau_2^B(a|x))}(s) \, ds,
\]  \tag{3.12}
and so the density of \( f_{T_2} \) is
\[
f_{T_2}(t) = \int_0^{+\infty} f_{\tau_2^B}(\rho(t+s)) \rho'(t+s) f_{\tau_1}(s) \, ds = \int_0^{+\infty} f_{\tau_2^B}(\rho(t+s)) \rho'(t+s) f_{\tau_2^B}(\rho(s)) \rho'(s) \, ds.
\]  \tag{3.13}

Moreover,
\[
P(T_2(x) = +\infty) = \lim_{t \to +\infty} P(T_2(x) > t)
= 1 - \int_0^{+\infty} f_{\tau_1}(s) \left[ \lim_{t \to +\infty} P(\tau_2^B(a|x) \leq \rho(t+s)) \right] \, ds.
\]

Therefore, by using (2.15):
\[
P(T_2(x) = +\infty) = 1 - \int_0^{+\infty} f_{\tau_1}(s) \left[ \lim_{t \to +\infty} \int_0^{\rho(t+s)} f_{\tau_2^B}(u) \, du \right] \, ds
= 1 - \int_0^{+\infty} f_{\tau_1}(s) \left[ \lim_{t \to +\infty} \int_0^{\rho(t+s)} \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2\pi y}} \frac{|a - x| e^{-\frac{(a-x)^2}{2y}}}{\sqrt{u-y}} \, dy \, du \right] \, ds.
\]

Moreover
\[
E[T_2(x)] = \int_0^{+\infty} P(T_2(x) > t) \, dt
= \int_0^{+\infty} \left[ 1 - \int_0^{+\infty} P(\tau_2^B(a|x) \leq \rho(t+s)) f_{\tau_1}(s) \, ds \right],
\]  \tag{3.14}
which, unfortunately, it is impossible to calculate, due to the complexity of the integral.

The density of \( \tau_2(a|x) \) is
\[
f_{\tau_2}(t) = \frac{d}{dt} P(\tau_2^B(a|x) \leq \rho(t)) = f_{\tau_2^B}(\rho(t)) \cdot \rho'(t)
\]
Covariance process with mean zero and covariance function the integrated BM, which has many important application (see e.g. [2], [3]). It is a Gaussian zero, we consider successive-passage time of $B$

$$\tau = \inf \{ t > 0 : X(t) = a \} = \inf \{ t > 0 : B(t^3/3) = a \},$$

and, as before,

$$T_1(x) = \tau_1(a|0), \ T_2(x) = \tau_2(a|0) - \tau_1(a|0).$$

Then, by using (3.10), we get the density of $\tau_1(a|0)$:

$$f_{\tau_1}(t) = \frac{|a - x|}{(t^3)^{3/2}} \cdot \phi \left( \frac{a - x}{(t^3)^{1/2}} \right) = 3\sqrt{3} \frac{|a - x|}{t^{5/2}} e^{\frac{3(a-x)^2}{2t^3}}.$$ (4.2)
Figure 6: Comparison between the shapes of the FPT density of $B(t^3/3)$ (4.2), for different values of the barrier $a$; from top to the bottom, with respect to the peak of the curve: $a = 1, a = 2, a = 3$.

In the Figure 6 we report the FPT density of $B(t^3/3)$ obtained from (4.2), for various values of the barrier: $a = 1, a = 2, a = 3$. The expectation of $\tau_1(a|0)$ is obtained by (3.11):

$$E[\tau_1(a|0)] = 3\sqrt{3} |a - x| \int_0^{+\infty} e^{-\frac{3(a-x)^2}{2s}} \frac{1}{t^{3/2}} dt. \quad (4.3)$$

Notice that, unlike the case of BM, it is finite (in fact $\rho(t) = t^3/3 \sim c \cdot t^3$ with $\alpha = 3 > 2$ (see the observation after equation (3.11)). Calculating numerically the integral e.g. for $a = 1$, we have obtained $E[\tau_1(a|0)] \approx 3.594$ , which is not far from the estimate $\hat{E}[\tau_1(a|0)] = 3.704$ , obtained by Monte Carlo simulation. By using (3.13), one has:

$$f_{T_2}(t) = \int_0^{+\infty} \frac{3|a - x|}{\pi (t + s)^3} \left[ \int_0^{(t+s)^3/3} e^{-\frac{(a-x)^2}{2u}} \frac{du}{u \sqrt{(t+s)^3 - u}} \right] (t + s)^2 s^2 \sqrt{\frac{3|a - x|}{s^{5/2}}} e^{-\frac{3(a-x)^2}{2s^3}} ds$$

$$= \frac{9\sqrt{3}(a - x)^2}{2\pi^2} \int_0^{+\infty} e^{-\frac{3(a-x)^2}{2s^3}} \frac{du}{(t + s)\sqrt{s}} \left[ \int_0^{(t+s)^3/3} e^{-\frac{(a-x)^2}{du}} \frac{du}{u \sqrt{(s+t)^3 - u}} \right] ds. \quad (4.4)$$

In the Figure 7 we report the probability density of $T_2(x)$, obtained from (4.4) by numerical computation, for various values of the parameter $a$.

By using (3.15), one gets the density of $\tau_2(a|x)$ :

$$f_{\tau_2}(t) = \frac{3}{\pi t} \int_0^{\frac{3}{2}} \frac{|a - x|}{\sqrt{2\pi s\sqrt{\frac{1}{3} - s}}} e^{-\frac{(a-x)^2}{2s^3}} ds. \quad (4.5)$$

Calculating numerically the integral, we obtained the shape of the density of $\tau_2(a|x)$, for different values of the parameter $a$; the results are shown in the Figure 8.
Figure 7: Approximate density of $T_2(x)$ \([4.4]\), in the case of $B(t^3/3)$, for various values of the parameter $a$; from top to bottom, with respect to the value at $t = 0$: $a = 3$, $a = 2$, $a = 1$.

Figure 8: Approximate density of $\tau_2(a|x)$ in the case of $B(t^3/3)$, for various values of the parameter $a$; from top to the bottom: $a = 1$, $a = 2$, $a = 3$.
Figure 9: Comparison between the density of $\tau_2(a|0)$ (lower peak) and the density of $\tau_1(a|0)$ (upper peak), in the case of $B(t^3/3)$, for $a = 1$.

As for the expectation of $\tau_2(a|0)$, one has $E[\tau_2(a|0)] = -\frac{d}{d\lambda}\tilde{\psi}(\lambda) \big|_{\lambda=0}$, where $\tilde{\psi}(\lambda)$ is the Laplace transform of $\tau_2(a|x)$, i.e.

$$
\tilde{\psi}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f_{\tau_2}(t) dt = \frac{3|a - x|}{\pi \sqrt{2\pi}} \int_0^{+\infty} e^{-\lambda t} \left[ \int_0^{t^3/3} e^{-\frac{(a-x)^2}{2s}} ds \right] dt. \quad (4.6)
$$

Thus:

$$
E[\tau_2(a|x)] = \frac{3|a - x|}{\pi \sqrt{2\pi}} \int_0^{+\infty} \left[ \int_0^{t^3/3} e^{-\frac{(a-x)^2}{2s}} ds \right] dt. \quad (4.7)
$$

For not too large values of $a$, Monte Carlo simulation provides the estimate $\hat{E}[\tau_2(a|0)]$, not far from the exact value, obtained by using (4.7).

In the Figure 9, we report the comparison between the density, (4.5), of $\tau_2(a|0)$ and the density, (4.2), of $\tau_1(a|0)$, for $a = 1$. Notice that in both the Figures 8 and 9 the curves appear to be somewhat jagged, due to approximation in numerical computation. This also happens in the next Figure 13 and Figure 15.
Figure 10: Comparison between LT of $\tau_1(a|x)$ (upper curve) and LT of $\tau_2(a|x)$ (lower curve), in the case of $B(t^3/3)$, for $a = 1$.

We conclude the example of $B(t^3/3)$, reporting the LTs of $\tau_1(a|0)$ and $\tau_2(a|0)$.

- **LT of $\tau_1(a|x)$**:

$$\psi(\lambda) = E \left[ e^{-\lambda \tau_1(a|x)} \right] = \int_0^{+\infty} f_{\tau_1}(t) e^{-\lambda t} dt = \int_0^{+\infty} \frac{3\sqrt{3}|a-x|}{t^{5/2}} e^{-\lambda t} \cdot \frac{e^{-\frac{3(a-x)^2}{2t^3}}}{\sqrt{2\pi}} dt; \quad (4.8)$$

- **LT of $\tau_2(a|x)$**:

$$\tilde{\psi}(\lambda) = E \left[ e^{-\lambda \tau_2(a|x)} \right] = \int_0^{+\infty} f_{\tau_2}(t) e^{-\lambda t} dt = \frac{3|a-x|}{\pi \sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-\lambda t}}{t} \left[ \int_0^{\frac{3}{2}} \frac{e^{-\frac{(a-x)^2}{2s}}}{s \sqrt{\frac{3}{2} - s}} ds \right] dt. \quad (4.9)$$

2) **The Ornstein-Uhlenbeck (OU) process**

It is the solution of:

$$\begin{cases} 
    dX(t) = -\mu X(t) dt + \sigma dB(t) \\
    X(0) = x 
\end{cases} \quad (4.10)$$

with $\mu, \sigma$ positive constants. Explicitly, one has

$$X(t) = e^{-\mu t} \left( x + \int_0^t \sigma e^{\mu s} dB(s) \right). \quad (4.11)$$

Moreover, (see e.g. [4]) $X(t)$ can be written as $e^{-\mu t} (x + B(\rho(t)))$, where $\rho(t) = \frac{\sigma^2}{2\mu}(1-e^{-2\mu t})$, namely in terms of time-changed BM. Therefore, $X(t)$ has normal distribution with mean $xe^{-\mu t}$ and variance $\frac{\sigma^2}{2\mu}(1-e^{-2\mu t})$. 

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We choose the time-varying barrier $S(t) = ae^{-\mu t}$, with $x < a$, and we reduce to the passage-times of BM through the constant barrier $a$. In fact:

$$\tau_1(S|x) = \inf \{ t > 0 : X(t) = S(t) \} = \inf \{ t > 0 : x + B(\rho(t)) = a \} \quad (4.12)$$

and

$$\tau_1(S|x) = \rho^{-1} (\tau_1^B(a|x)), \quad (4.13)$$

where $\tau_1^B(a|x)$ is the first passage time of BM, through the barrier $a$. Moreover,

$$\tau_2(S|x) = \rho^{-1} (\tau_2^B(a|x)), \ldots, \tau_n(S|x) = \rho^{-1} (\tau_n^B(a|x)),$$

which are the successive-passage times of BM through $a$. Thus, by using (3.10), we get the density of $\tau_1(S|x)$:

$$f_{\tau_1}(t) = \frac{2\sqrt{2\mu} \cdot |a - x|}{\sigma (e^{2\mu} - 1)^{3/2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(a - x)^2}{(e^{2\mu} - 1)^2} + 2\mu t}. \quad (4.14)$$

In the Figure 11 we report the density of $\tau_1(S|x)$, for $x = 0$, $\mu = 2$, $\sigma = 0.2$ and various values of $a$.

As for the expectation of $\tau_1(a|x)$, we have:

$$E [\tau_1(S|x)] = \frac{|a - x| \cdot 2\mu \sqrt{\mu}}{\sigma \sqrt{\pi}} \cdot \int_0^{+\infty} te^{-\frac{(x-a)^2}{(e^{2\mu} - 1)^2} + 2\mu t} dt; \quad (4.15)$$

calculating numerically the integral e.g. for $x = 0$, $a = 1$, $\mu = \sigma = 1$, one obtains $E [\tau_1(S|x)] = 1.025$, while, Monte Carlo simulation provides the estimate $\hat{E} [\tau_1(S|x)] = 1.048$. 

Figure 12: Approximate density of $T_2(x)$ for $x = 0, \mu = 2, \sigma = 0.2$ and various values of the parameter $a$; from top to the bottom: $a = 1, a = 2, a = 3$. It refers to the OU process through the boundary $S(t) = ae^{-\mu t}$.

By using (3.13), one has:

$$f_{T_2}(t) = \frac{(2\mu)^{5/2}\sigma(a - x)^2}{2\pi} \cdot \int_0^{+\infty} \frac{e^{-\frac{\mu(a-x)^2}{2\sigma^2(e^{2\mu t} - 1)}} + 2\mu(t+3s)}{(e^{2\mu t} - 1)(e^{2\mu s} - 1)^{3/2}} \cdot \int_0^{\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)} \frac{e^{-\frac{(a-x)^2}{2u}}}{u^{3/2}} du \cdot ds.$$  \hspace{1cm} (4.16)

In the Figure 12 we report the probability density of $T_2(x)$, obtained from (3.13), by calculating numerically the integral, for $x = 0, \sigma = 0.2, \mu = 2$ and various values of the parameter $a$. Notice that, for $a = 3$ the curve appears to be overlapped with the time axis. By using (3.15), we obtain

$$f_{\tau_2}(t) = \frac{2\mu|a - x| e^{2\mu t}}{\pi \sqrt{2\pi} \cdot (e^{2\mu t} - 1)} \cdot \int_0^{\frac{\sigma^2}{2\mu}(e^{2\mu t} - 1)} \frac{e^{-\frac{(a-x)^2}{2s}}}{s^{3/2} \sqrt{e^{2\mu t} - 1} - s} ds.$$  \hspace{1cm} (4.17)

Calculating numerically the integral, one gets the density of $\tau_2(S|x)$; for $x = 0, \sigma = 0.2, \mu = 2$ and different values of the parameter $a$, the results are shown in the Figure 13.

We summarize the LTs of $\tau_1(S|x)$ and $\tau_2(S|x)$ in the case of OU process and $S(t) = ae^{-\mu t}$.

- LT of $\tau_1(S|x)$:

$$\psi(\lambda) = E \left[ e^{-\lambda \tau_1(S|x)} \right] = \int_0^{+\infty} f_{\tau_1}(t)e^{-\lambda t} dt = \frac{(2\mu)^{3/2} \cdot |a - x|}{\sigma \sqrt{2\pi}} \int_0^{+\infty} \frac{e^{2\mu t - \frac{\mu(a-x)^2}{\sigma^2(e^{2\mu t} - 1)}} - \lambda}{(e^{2\mu t} - 1)^{3/2}} dt.$$  \hspace{1cm} (4.18)
Figure 13: Approximate density of $\tau_2(S|x)$ for $x = 0, \sigma = 0.2, \mu = 2$ and various values of the parameter $a$; from top to the bottom: $a = 1, a = 2, a = 3$. It refers to the OU process through the boundary $S(t) = ae^{-\mu t}$.

- LT of $\tau_2(S|x)$:

$$
\bar{\psi}(\lambda) = E\left[e^{-\lambda \tau_2(S|x)}\right] = \int_0^{+\infty} f_{\tau_2}(t)e^{-\lambda t} dt
$$

$$
= \frac{2\mu|a - x|}{\pi \sqrt{2\pi}} \int_0^{+\infty} \frac{e^{2\mu t - \lambda}}{e^{2\mu t} - 1} \left[ \int_0^{\frac{a^2}{2\mu}(e^{2\mu t} - 1)} \frac{e^{-\frac{(a-x)^2}{4s}}}{s \sqrt{\frac{a^2}{2\mu}(e^{2\mu t} - 1) - s}} ds \right] dt. \quad (4.19)
$$

As far the expectation of $\tau_2(S|x)$ is concerned, it is not possible to obtain the exact value, by taking minus the derivative with respect to $\lambda$ in (4.19), that is:

$$
- \frac{d}{d\lambda} \bar{\psi}(\lambda) \big|_{\lambda=0} = \frac{2\mu|a - x|}{\pi \sqrt{2\pi}} \int_0^{+\infty} \frac{te^{2\mu t}}{e^{2\mu t} - 1} \left[ \int_0^{\frac{a^2}{2\mu}(e^{2\mu t} - 1)} \frac{e^{-\frac{(a-x)^2}{4s}}}{s \sqrt{\frac{a^2}{2\mu}(e^{2\mu t} - 1) - s}} ds \right] dt, \quad (4.20)
$$

because of the complexity of the integrals involved in the calculation, so it has to be estimated by Monte Carlo simulation.

In the Figure 14, with regard to OU through the boundary $S(t) = ae^{-\mu t}, a = 1$, we report the comparison between the LT of $\tau_1(S|x)$ and that of $\tau_2(S|x)$, for $x = 0, \sigma = 0.2, \mu = 2$; in the Figure 15, we report the comparison between the density of $\tau_1(S|x)$ and that of $\tau_2(S|x)$, for the same set of parameters.

5 Conclusions and final remarks

By using the results of [1] on the $n$th-passage time of Brownian motion through a straight line, we studied the distribution of the $n$th-passage time through a barrier $a$, of a diffusion process $X(t)$, given by a space or time transformation of Brownian motion $B_t$; precisely, we
Figure 14: Comparison between Laplace transforms of $\tau_1(S|x)$ (upper curve) and $\tau_2(S|x)$ (lower curve), in the case of OU process and the boundary $S(t) = ae^{-\mu t}$, for $\sigma = 0.2$, $\mu = 2$, $x = 0$ and $a = 1$.

Figure 15: Comparison between the density (4.17) of $\tau_2(S|x)$ (lower peak) and the density (4.14) of $\tau_1(S|x)$ (upper peak), in the case of OU process and the boundary $S(t) = ae^{\mu t}$, for $\sigma = 0.2$, $\mu = 2$, $x = 0$ and $a = 1$.  


have considered the cases: \( X(t) = v^{-1}(B_t + v(x)) \), where \( v(x) \) is an increasing function with \( v(0) = 0 \), and \( X(t) = B(\rho(t)) \), where \( \rho(t) \) is an increasing function with \( \rho(0) = 0 \). We also found the Laplace transform of the first and second-passage time of such processes, and we have reported some explicit examples.

Notice that the results of this paper can be extended to diffusions which are more general than the process \( X(t) \) here considered, for instance to a process of the form

\[
X(t) = v^{-1}(\hat{B}(\rho(t)) + v(x)),
\]

where \( \hat{B}(t) \) is BM, \( v(x) \) and \( \rho(t) \) are regular enough, increasing functions with \( \rho(0) = v(0) = 0 \); such a process \( X(t) \) is obtained from BM combining a space transformation and a time-change (see e.g. the discussion in [4]). When \( \rho(t) = t \), the process \( X(t) \) is conjugated to BM, according to the definition given in Section 3.1 (see [4]). The process \( X(t) \) given by (5.1) is indeed a weak solution of the SDE:

\[
dX(t) = -\frac{\rho'(t)v''(X(t))}{2(v'(X(t)))^3} dt + \frac{\sqrt{\rho'(t)}v'(X(t))}{v'(X(t))} dB_t,
\]

where \( v'(x) \) and \( v''(x) \) denote first and second derivative of \( v(x) \).

Provided that the deterministic function \( \rho(t) \) is replaced with a random function, the representation (5.1) is also valid for a time homogeneous one-dimensional diffusion driven by the SDE

\[
dX(t) = \mu(X(t))dt + \sigma(X(t))dB_t, \quad X(0) = x,
\]

where the drift \( \mu \) and diffusion coefficients \( \sigma^2 \) satisfy the usual conditions (see e.g. [7]) for existence and uniqueness of the solution of (5.3). In fact, let \( w(x) \) be the scale function associated to the diffusion \( X(t) \) driven by the SDE (5.3), that is, the solution of \( Lw(x) = 0, \ w(0) = 0, \ w'(0) = 1 \), where \( L \) is the infinitesimal generator of \( X \) given by \( Lh = \frac{1}{2}\sigma^2(x)\frac{d^2h}{dx^2} + \mu(x)\frac{dh}{dx} \). As easily seen, if the integral \( \int_0^t \frac{2\mu(z)}{\sigma^2(z)} \ dz \) converges, the scale function is explicitly given by

\[
w(x) = \int_0^x \exp \left( -\int_0^t \frac{2\mu(z)}{\sigma^2(z)} \ dz \right) dt.
\]

If \( \zeta(t) := w(X(t)) \), by Itô’s formula one obtains

\[
\zeta(t) = w(x) + \int_0^t w'(w^{-1}(\zeta(s)))\sigma(w^{-1}(\zeta(s)))dB_s,
\]

that is, the process \( \zeta(t) \) is a local martingale, whose quadratic variation is

\[
\rho(t) \triangleq \langle \zeta \rangle_t = \int_0^t [w'(X(s))\sigma(X(s))]^2 \, ds, \quad t \geq 0.
\]

The (random) function \( \rho(t) \) is differentiable, increasing, and \( \rho(0) = 0 \). If \( \rho(+\infty) = +\infty \), by the Dambis, Dubins-Schwarz theorem (see e.g. [11]) one gets that there exists a BM \( \hat{B} \) such that \( \zeta(t) = \hat{B}(\rho(t)) + w(\eta) \). Thus, since \( w \) is invertible, one obtains the representation (5.1) with \( w \) in place of \( u \).

Notice, however, that the successive-passage time problem for the process \( X(t) \) given by (5.1) can be addressed as in the case when \( \rho(t) \) is a deterministic function, only if the distribution of the random process \( \rho(t) \) is explicitly known.
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