Semigroup generation properties of streaming operators with non–contractive boundary conditions.

Bertrand Lods
Politecnico di Torino, Dipartimento di Matematica,
Corso Duca degli Abruzzi, 24,
10129 Torino, Italy.
lods@calvino.polito.it

Abstract. We present $c_0$–semigroup generation results for the free streaming operator with abstract boundary conditions. We recall some known results on the matter and establish a general theorem (already announced in \[1\]). We motivate our study with a lot of examples and show that our result applies to the physical cases of Maxwell boundary conditions in the kinetic theory of gases as well as to the non–local boundary conditions involved in transport–like equations from population dynamics.

Key words. Generation theorem, boundary operators, transport theory, population dynamics.

1 Introduction

In this paper, we investigate the well-posedness of the following initial–boundary–value problem in $L^p$–spaces ($1 \leq p < \infty$)

\[
\frac{\partial \phi}{\partial t}(x, v, t) + v \cdot \nabla_x \phi(x, v, t) = Q(\phi)(x, v, t) \quad (x, v) \in \Omega \times V, \; t > 0 \tag{1.1a}
\]

\[
\phi(x, v, 0) = \phi_0(x, v) \quad (x, v) \in \Omega \times V \tag{1.1b}
\]

\[
\phi_{|\Gamma_-}(x, v, t) = H(\phi_{|\Gamma_+})(x, v, t) \quad (x, v) \in \Gamma_-, \; t > 0. \tag{1.1c}
\]

where $\Omega$ is a smooth open subset of $\mathbb{R}^N$ ($N \geq 1$), $V$ is the support of a positive Radon measure $d\mu$ on $\mathbb{R}^N$ and $\phi_0 \in L^p(\Omega \times V, dxd\mu(v))$ ($1 \leq p < \infty$). The operator $Q$ at the right–hand side of (1.1a) is a suitable linear operator on $L^p(\Omega \times V, dxd\mu(v))$. In
Γ− (respectively Γ+) denotes the incoming (resp. outgoing) part of the boundary of the phase space Ω × V

\[ \Gamma_{\pm} = \{(x,v) \in \partial \Omega \times V ; \pm v \cdot n(x) > 0\} \]

where \( n(x) \) stands for the outward normal unit at \( x \in \partial \Omega \). The boundary condition \( (1.1c) \) expresses that the incoming flux \( \phi|_{\Gamma_{-}}(\cdot,\cdot, t) \) is related to the outgoing one \( \phi|_{\Gamma_{+}} \) through a linear operator \( H \) that we shall assume to be bounded on some suitable trace spaces.

The kinetic model \( (1.1) \) arises in different fields of applied sciences:

- **Mathematical physics.** In the kinetic theory of gases or in neutron transport theory, the unknown \( \phi(x,v,t) \) represents the density of particles (neutrons, molecules of gas, etc) having the position \( x \in \Omega \) and the velocity \( v \in V \) at time \( t \geq 0 \). In this case, \( Q(\phi) \) represents the interaction between particles and the host medium due to collisions [2, 3, 4].

- **Mathematical biology.** In population dynamics, the variables \( (x,v) \) do no longer represent the position and velocity but any other state variables of a given cell populations. In this case \( \phi(x,v,t) \) is the distribution function of cells having the state \( (x,v) \) at time \( t \geq 0 \), \( Q(\phi) \) represents then the transition from one state to another. We refer to [5] for such transport–like equations in the context of population dynamics and, more generally, to [6] for generalized kinetic models in the applied sciences.

In the present paper, we will focus our attention on the influence of the boundary operator \( H \) on the well–posedness of \( (1.1) \). We will only consider the so–called collisionless form of \( (1.1) \), i. e. we will assume that

\[ Q = 0. \]

We adopt here the semigroup framework and the main purpose of this paper is to identify the right class of boundary operators \( H \) for which the free–streaming operator (whose domain includes the boundary condition \( (1.1c) \))

\[ T_H \phi(x,v) = -v \cdot \nabla_x \phi(x,v) \quad (x,v) \in \Omega \times V \]

generates a \( c_0 \)–semigroup in \( L^p(\Omega \times V, dxd\mu(v)) (1 \leq p < \infty) \). Note that, despite the simple aspect of the transport equation \( (1.1) \), this question is far from being trivial whenever \( H \) is not a contraction. Actually, it is well–known that for contractive boundary conditions \( \|H\| < 1 \), \( T_H \) is a generator of a \( c_0 \)–semigroup of contractions in \( L^p(\Omega \times V, dxd\mu(v)) (1 \leq p < \infty) \) [4]. The case of non–contractive boundary conditions is much more involved because of the difficulty to control the growth of the flux \( \phi(\cdot,\cdot,t) \). We point out that such boundary conditions arise naturally in population dynamics. Indeed, in this case the boundary operator \( H \) models the birth–law of the cell population so that \( H \) is multiplicative. Typically, for a proliferating population of cells, during the mitosis, mother
cells undergo fission to give birth to two daughter cells, i.e. \( \| Hu \| = 2 \| u \| \forall u \geq 0 \). The question of the well–posedness of (1.1) has been already addressed in several recent papers, see for instance [9, 10, 11] and the references therein. We present in this paper various approaches to answer this question and give also some new results. More precisely, our aim is to determine sufficient condition on the boundary operator \( H \) for which \( T_H \) generates a \( c_0 \)-semigroup in \( L^p(\Omega \times V, dx d\mu(v)) \) \( (1 \leq p < \infty) \). Our main result (Theorem 5.3) (already announced in [1]) answer this question in general \( L^p \)–spaces with arbitrary \( 1 \leq p < \infty \) by a constructive approach. Actually, our proof consists in deriving, by an appropriate change of unknown, an evolution problem equivalent to (1.1) and involving contractive boundary conditions. Note that the afore–mentioned result on contractive boundary conditions turns out to be a direct consequence of our main result. Moreover, known results referring to the so–called phase space approach (see Section 4 for more details) [12, 13] are also simple corollaries of our main theorem. We apply our results successfully to the following boundary conditions arising in practical situations:

- **Local boundary conditions** of Maxwell–type which are known to be well–suited to the kinetic theory of gases [2] and to neutron transport theory [1].

- **Non–local** boundary conditions as the ones used in population dynamics. Note that this type of boundary conditions may be handle thanks to compactness arguments.

The outline of the paper is as follows. In the following section, we present some of the boundary conditions commonly adopted in the kinetic theory of gases and in population dynamics. These are the motivating examples we had in mind to apply our main result. In section 3, we introduce the functional setting and prove the classical generation theorem for contractive boundary conditions. In section 4 we present the so–called phase space approach. We begin with the particular case of slab geometry (section 4.1) and recall then the general result [12] which identify the class of phase spaces in which (1.1) is well-posed without any assumption on the boundary operator. After some examples showing that, out of this class of phase spaces, assumptions on the boundary conditions are needed, we present our main result Theorem 5.3 and show that all the afore–mentioned results are simple consequences of it. Finally, in section 5 we show that our result applies to the physical boundary conditions afore–mentioned. In an Appendix, we propose a brief discussion on the use of Batty and Robinson Theorem [14, 15] in the context of kinetic theory and we end this paper by some concluding remarks and open problems.

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2 Examples of boundary conditions

We present in this section some examples of boundary conditions arising in applications. These examples are coming from the kinetic theory of gases or from population dynamics. The main feature of these latter is their non–local character whereas the boundary conditions are local in the kinetic theory of gases.

2.1 Local boundary conditions

Let us consider in this section the case of Maxwell–type boundary conditions which plays a fundamental role in the kinetic theory of gases (see for instance [3]) and in neutron transport theory [4]. For simplicity, we assume throughout this section that \( \mu(\cdot) \) is the Lebesgue measure with support \( V \subset \mathbb{R}^N (N \geq 1) \). The natural class of boundary operators arising in the kinetic theory of gases is the one of boundary operators local with respect to \( x \in \partial \Omega \). Typically, such a boundary operator reads

\[
H(\psi|\Gamma_\pm)(x,v) = \int_{\{v' \in V ; v' \cdot n(x) > 0\}} \psi|\Gamma_\pm(x,v') \, d\Pi(x,v)(v') \quad (x,v) \in \Gamma_-,
\]

where, for a.e. \((x,v) \in \Gamma_-\), \(d\Pi(x,v)(\cdot)\) is a non–negative and bounded Radon measure on \( \{v' \in V ; v' \cdot n(x) > 0\} \). Precisely, \(d\Pi(x,v)(v')\) is the probability that a particle (molecule of gas, neutron...) striking the wall \( \partial \Omega \) at the point \( x \) with velocity between \( v' \) and \( v' + dv' \) will re–emerge at (practically) the same point with velocity between \( v \) and \( v + dv \) (see [2, 3, 4] for details). A particularly interesting model is the following.

**Example 2.1.** Let us assume that a fraction \( \alpha \) \((0 < \alpha < 1)\) of particles undergoes a specular reflection while the remaining fraction \( 1 - \alpha \) is diffused with the Maxwellian distribution of the wall \( M_\omega \):

\[
M_\omega(v) = \frac{1}{(2\pi \theta_0)^{N/2}} \exp\left(-\frac{v^2}{2\theta_0}\right) \quad v \in V, \tag{2.1}
\]

\( \theta_0 \) being the temperature of the surface \( \partial \Omega \) (which is assumed to be constant). Then

\[
d\Pi(x,v)(\cdot) = \alpha \, d\delta(v' - v + 2(v \cdot n(x)n(x)) + (1 - \alpha)M_\omega(v)|v' \cdot n(x)|dv', \quad (x,v) \in \Gamma_-
\]

where \(d\delta(\cdot)\) is the usual Dirac mass centered in 0. This corresponds to the classical Maxwell model, commonly adopted in the kinetic theory of gases [2].

More generally, let us introduce the following definition of regular reflection boundary conditions, due to A. Palczewski [16].
Definition 2.2. Let $R \in \mathcal{L}(L^p_+, L^p_-)$. One say that $R$ is a regular reflection boundary operator if there exists a $C^1$–piecewise mapping $V : \Gamma_- \to \mathbb{R}^N$ such that

i) For any $(x, v) \in \Gamma_-, (x, V(x, v)) \in \Gamma_+.$

ii) $|V(x, v)| = |v|$ for any $(x, v) \in \Gamma_-.$

iii) $|n(x) \cdot v| = |n(x) \cdot \nabla \det \frac{\partial V}{\partial v}(x, v)|, (x, v) \in \Gamma_-.$

iv) $V(x, \lambda v) = \lambda V(x, v)$ for any $(x, v) \in \Gamma_-$ and $\lambda > 0.$

v) $R(\varphi)(x, v) = \varphi(x, V(x, v)) \quad \forall (x, v) \in \Gamma_-, \varphi \in L^p_+.$

Example 2.3. In practical situations, the most frequently used regular reflection conditions are

(a) the \textit{specular reflection boundary conditions} which corresponds to

\[ V(x, v) = v - 2(v \cdot n(x)) n(x) \quad (x, v) \in \Gamma_- . \]

(b) The \textit{bounce–back reflection conditions} for which $V(x, v) = -v,$ $(x, v) \in \Gamma_-$ and $V$ has to be symmetric with respect to 0.

The main important feature of such boundary operators is that they are \textit{conservative}, i. e., for any regular reflection operator $R$:

\[ \| R \varphi \| = \| \varphi \| \quad \forall \varphi \in L^p_+ . \tag{2.2} \]

Definition 2.4. We shall say that a boundary operator $H \in \mathcal{L}(L^p_+, L^p_-)$ is of \textit{Maxwell–type} if

\[ H(\psi|_{\Gamma_+})(x, v) = K(\psi|_{\Gamma_+})(x, v) + C(\psi|_{\Gamma_+})(x, v) \quad (x, v) \in \Gamma_- , \]

with $C \in \mathcal{L}(L^p_+, L^p_-)$ given by

\[ C(\psi|_{\Gamma_+})(x, v) = \alpha(x) R(\psi|_{\Gamma_+})(x, v) \]

where $\alpha(\cdot) \in L^\infty(\partial \Omega)$ is non–negative, $R$ is a regular reflection operator, and

\[ K(\psi|_{\Gamma_+})(x, v) = \int_{\{v \cdot n(x) > 0\}} h(x, v, v') \psi|_{\Gamma_+}(x, v') |v' \cdot n(x)| dv', \quad (x, v) \in \Gamma_- , \]

where $h(\cdot, \cdot, \cdot) \geq 0$ is measurable.

Remark 2.5. If $C = 0$, the boundary operator is said to be diffusive. More generally, the operator $K$ is called the diffusive–part of $H$. 
2.2 Non–local boundary conditions

For transport–like equations arising in population dynamics, the boundary conditions are no longer assumed to be local with respect to $x \in \partial \Omega$ (see for instance [8, 17, 18] and the monograph [5]) as illustrated by the following example:

**Example 2.6.** In [19], the author, together with M. Mokhtar–Kharroubi, studied a model of growing cell population proposed by Lebowitz and Rubinow [8]:

$$
\begin{aligned}
\frac{\partial \varphi(a, \ell, t)}{\partial t} + \frac{\partial \varphi(a, \ell, t)}{\partial a} + \mu(a, \ell) \varphi(a, \ell, t) &= 0 \\
\varphi(0, \ell) &= \int_{\ell_1}^{\ell_2} k(\ell, \ell') \varphi(\ell', \ell') \, d\ell' + c \varphi(\ell, \ell) \\
\varphi(a, \ell, 0) &= \varphi_0(a, \ell) \in X_p
\end{aligned}
$$

(2.3)

where

$$
\varphi_0, \mu(\cdot, \cdot) \in L^\infty(\Omega).$$

This is a model of a proliferating cell population with inherited properties. The variable $\ell$ is the cycle length of cells, that is the time between cell birth and cell division. It is assumed to be determined at birth. The variable $a$ represents the age of the individual cell. At birth, the age is obviously null whereas, at division, $a = \ell$. The constant $\ell_1$ (respectively $\ell_2$) denotes the minimum (resp. maximum) cycle length. The unknown $\varphi(a, \ell, t)$ denotes the density of the cell population with age $a$ and cycle length $\ell$ at time $t \geq 0$. The function $\mu(\cdot, \cdot)$ is the rate of cell mortality which is assumed to be bounded and non–negative. The boundary condition describes the birth–law (i.e. the transition from mother cycle length to daughter cycle length). For this model, the velocity space $V$ reduces to the singleton

$$
V = \{(1, 0)\},
$$

endowed with the Dirac mass centered in $(1, 0)$. One has $X_p = L^p(\Omega, da \, d\ell)$ ($1 \leq p < \infty$) and $\Gamma_- = \{(0, \ell); \ell_1 < \ell < \ell_2\}$ and $\Gamma_+ = \{(\ell, \ell); \ell_1 < \ell < \ell_2\}$. Let us consider the biological case

$$
\ell_1 = 0.
$$

The free–streaming operator $T_H$ is given then by

$$
T_H \varphi(a, \ell) := -\frac{\partial \varphi(a, \ell)}{\partial a},
$$

with its usual domain and, in Eq. (2.3), the boundary operator $H \in \mathcal{L}(L^p((0, \ell_2), d\ell))$ ($1 \leq p < \infty$) is non–local with respect to $x = (a, \ell) \in \Omega$:

$$
H(\psi|_{\Gamma_+})(\ell) = \int_0^{\ell_2} k(\ell, \ell') \psi|_{\Gamma_+}(\ell') \, d\ell' + c \psi|_{\Gamma_+}(\ell) \quad 0 < \ell < \ell_2.
$$

\[\diamond\]
As suggested by the above example, we can introduce *non–local Maxwell–type boundary operators*.

**Definition 2.7.** Let \( H \in \mathcal{L}(L^p_+, L^p_-) \), \( H \) is said to be a *non–local Maxwell–type boundary operators* if \( H \) writes
\[
H = K + \mathcal{C},
\]
where \( \mathcal{C} \in \mathcal{L}(L^p_+, L^p_-) \) is a contractive boundary operator and \( K \in \mathcal{L}(L^p_+, L^p_-) \) is a *non–local* integral operator.

### 3 Setting of the problem and the classical case of contractive boundary conditions

Let us first introduce the functional setting we shall use in the sequel. Let
\[
X_p = L^p(\Omega \times V, dxd\mu(v)) \quad 1 \leq p < \infty,
\]
where \( \Omega \) is a smooth interior (respectively exterior) domain of \( \mathbb{R}^N \) (\( N \geq 1 \)), i.e., \( \Omega \) is bounded (resp. \( \mathbb{R}^N \setminus \Omega \) is bounded). The boundary of the phase space \( \partial \Omega \times V \) splits as
\[
\partial \Omega \times V = \Gamma_- \cup \Gamma_+ \cup \Gamma_0
\]
where \( \Gamma_\pm = \{(x, v) \in \partial \Omega \times V; \pm v \cdot n(x) > 0\} \) and \( \Gamma_0 = \{(x, v) \in \partial \Omega \times V; \pm v \cdot n(x) = 0\} \).

We will assume throughout this paper that \( \Gamma_0 \) is of zero measure with respect to \( d\gamma(\cdot)d\mu(\cdot) \), \( d\gamma(\cdot) \) being the Lebesgue measure on \( \partial \Omega \). We define the partial Sobolev space
\[
W_p = \{\psi \in X_p; v \cdot \nabla_x \psi \in X_p\}.
\]
Suitable \( L^p \)-spaces for the traces on \( \Gamma_\pm \) are defined as
\[
L^p_\pm = L^p(\Gamma_\pm; |v \cdot n(x)| d\gamma(x)d\mu(v)).
\]
For any \( \psi \in W_p \), one can define the traces \( \psi|_{\Gamma_\pm} \) on \( \Gamma_\pm \), however these traces do not belong to \( L^p_\pm \) but to a certain weighted space \([20, 21]\). Let us define
\[
\tilde{W}_p = \{\psi \in W_p; \psi|_{\Gamma_\pm} \in L^p_\pm\}.
\]
Let \( H \) be a bounded linear operator from \( L^p_+ \) to \( L^p_- \)
\[
H \in \mathcal{L}(L^p_+, L^p_-) \quad 1 \leq p < \infty.
\]
The free–streaming operator associated with the boundary condition \( H \) is
\[
\begin{align*}
T_H : D(T_H) &\subset X_p \rightarrow X_p \\
\varphi &\mapsto T_H \varphi(x,v) := -v \cdot \nabla_x \varphi(x,v),
\end{align*}
\]
with domain
\[ D(T_H) := \{ \psi \in \widetilde{W}_p \text{ such that } \psi|_{\Gamma_-} = H(\psi|_{\Gamma_+}) \}. \]
A crucial role will be played in the sequel by the so-called time of sojourn in \( \Omega \).

**Definition 3.1.** For any \((x,v) \in \overline{\Omega} \times V\), define
\[
\tau(x,v) := \sup \{ t > 0 ; x - sv \in \Omega, \ \forall 0 < s < t \} = \inf \{ s > 0 ; x - sv \notin \Omega \}.
\]
For the sake of convenience, we will set
\[
\tau(x,v) := t(x,v) \quad \text{if} \ (x,v) \in \partial \Omega \times V.
\]
From a heuristic point of view, \( t(x,v) \) is the time needed by a particle having the position \( x \in \Omega \) and the velocity \(-v \in V\) to go out \( \Omega \). One notes [22] that \( \tau(x,v) = 0 \) for any \((x,v) \in \Gamma_-\) whereas, if \( v \cdot n(x) > 0, \tau(x,v) > 0 \). In particular,
\[
\{(x,v) \in \Gamma_+ ; \tau(x,v) = 0 \} = \{(x,v) \in \Gamma_+ ; v \cdot n(x) = 0 \}.
\]
Moreover, for any \((x,v) \in \overline{\Omega} \times V\)
\[
(x - t(x,v)v,v) \in \Gamma_-.
\]
Let us now derive the resolvent of \( T_H \). For any \( \lambda \in \mathbb{C} \) such that \( \text{Re} \ \lambda > 0 \), define
\[
\begin{align*}
M_\lambda : \ & L^p_- \to L^p_+ \\
& u \mapsto M_\lambda u(x,v) = u(x - \tau(x,v)v,v) e^{-\lambda \tau(x,v)}, \ (x,v) \in \Gamma_+ ;
\end{align*}
\]
\[
\begin{align*}
B_\lambda : \ & L^p_- \to X_p \\
& u \mapsto B_\lambda u(x,v) = u(x - t(x,v)v,v) e^{-\lambda t(x,v)}, \ (x,v) \in \Omega ;
\end{align*}
\]
\[
\begin{align*}
G_\lambda : \ & X_p \to L^p_+ \\
& \varphi \mapsto G_\lambda \varphi(x,v) = \int_0^{\tau(x,v)} \varphi(x - sv,v) e^{-\lambda s} ds, \ (x,v) \in \Gamma_+ ;
\end{align*}
\]
and
\[
\begin{align*}
C_\lambda : \ & X_p \to X_p \\
& \varphi \mapsto C_\lambda \varphi(x,v) = \int_0^{t(x,v)} \varphi(x - tv,v) e^{-\lambda t} dt, \ (x,v) \in \Omega .
\end{align*}
\]
Thanks to Hölder’s inequality, all these operators are bounded on their respective spaces. More precisely, for any \( \text{Re} \lambda > 0 \)
\[
\begin{align*}
\|M_\lambda\| & \leq 1, & \|B_\lambda\| & \leq (p \text{Re} \lambda)^{-1/p}, \\
\|G_\lambda\| & \leq (q \text{Re} \lambda)^{-1/q}, & \|C_\lambda\| & \leq (\text{Re} \lambda)^{-1}, \quad 1/p + 1/q = 1.
\end{align*}
\]
The resolvent of \( T_H \) is given by the following (see for instance [23]).
Proposition 3.2. Let \( H \in \mathcal{L}(L^p_+,L^p_-) \) be such that there exists \( \lambda_0 \) such that
\[
\sigma_r(M_\lambda H) < 1 \quad \forall \Re \lambda > \lambda_0.
\]
Then, for any \( \Re \lambda > \lambda_0 \),
\[
(\lambda - T_H)^{-1} = B_\lambda (I - M_\lambda H)^{-1} G_\lambda + C_\lambda.
\] (3.1)

We recall now the well–known generation result concerning contractive boundary conditions. It can be found in [7, Theorem 2.2, Chapter XII] (see also [24]). We recall here the proof of this (now classical) result since it will play a fundamental role in the sequel.

Theorem 3.3. Let \( H \in \mathcal{L}(L^p_+,L^p_-) \) \((1 \leq p < \infty)\) be such that \( \| H \| < 1 \). Then, \( T_H \) generates a contraction \( c_0 \)–semigroup in \( X_p \).

Proof: The proof consists in showing that \( T_H \) is dissipative. From Proposition 3.2 one sees first that \( \{ \lambda \in \mathbb{C} ; \Re \lambda > 0 \} \subset \rho(T_H) \), where \( \rho(T_H) \) stands for the resolvent set of \( T_H \) (in particular \( T_H \) is closed). Let us now consider the case \( 1 < p < \infty \) and let \( \psi \in D(T_H) \). Since
\[
v \cdot \nabla_x (|\psi|^p)(x,v) = p|\psi|^{p-2}(x,v)\psi(x,v)(v \cdot \nabla_x \psi(x,v)),
\]
one gets
\[
\langle T_H \psi, |\psi|^{p-2}\psi \rangle := \int_{\Omega \times V} |\psi|^{p-2}(x,v) \psi(x,v)(-v \cdot \nabla_x \psi(x,v))dxd\mu(v)
= -\frac{1}{p} \int_{\Omega \times V} v \cdot \nabla_x |\psi|^p(x,v)dxd\mu(v).
\]
Green’s identity yields
\[
\langle T_H \psi, |\psi|^{p-2}\psi \rangle = -\frac{1}{p} \int_{\partial \Omega \times V} |\psi|^p(x,v) v \cdot n(x) d\gamma(x)d\mu(v)
= \frac{1}{p} \int_{\Gamma_-} |\psi|_\Gamma^{-}(x,v)|^p|v \cdot n(x)|d\gamma(x)d\mu(v)
- \frac{1}{p} \int_{\Gamma_+} |\psi|^p_\Gamma^+(x,v)|v \cdot n(x)|d\gamma(x)d\mu(v)
= \frac{1}{p} (\|\psi|_{\Gamma_-}^p_{L^p_-} - \|\psi|_{\Gamma_+}^p_{L^p_+}).
\]
Since \( H \) is a contraction and \( \psi|_{\Gamma_-} = H(\psi|_{\Gamma_+}) \), one deduces that
\[
\langle T_H \psi, |\psi|^{p-2}\psi \rangle < 0.
\]
For \( p = 1 \), one shows in the same way that
\[
\langle T_H \psi, \text{sign}\psi \rangle < 0 \quad \forall \psi \in D(T_H).
\]
Now, let $\psi \in D(T_H)$ and $\text{Re}\lambda > 0$ be fixed. Set $\varphi = (\lambda - T_H)\psi$ and denotes

$$\psi^* = \begin{cases} |\psi|^{p-2}\psi & \text{if } 1 < p < \infty \\ \text{sign}\psi & \text{if } p = 1. \end{cases}$$

One has $\text{Re}\lambda \|\psi\|^p = \text{Re}\langle \lambda \psi, \psi^* \rangle$. Consequently

$$\text{Re}\lambda \|\psi\|^p \leq \text{Re}\langle \lambda \psi, \psi^* \rangle - \langle T_H \psi, \psi^* \rangle \leq \|\varphi\| \|\psi\|^{p-1}.$$ 

Therefore, for any $\text{Re}\lambda > 0$, $\|\psi\| \leq \|\varphi\|/\text{Re}\lambda$, i.e.

$$\|\lambda - T_H\|^{-1} \leq \frac{1}{\text{Re}\lambda} \quad (\text{Re}\lambda > 0). \quad (3.2)$$

The proof follows then from Lumer–Phillips Theorem.

Remark 3.4. Note that, resuming the above arguments, one can easily check that estimate $\text{3.2}$ remains true if one assumes $\|H\psi\| = \|\psi\| \forall \psi \in L^p_+$. Indeed, with the notations of the above proof, $\langle T_H \psi, \psi^* \rangle = 0$ for any $\psi \in D(T_H)$. Unfortunately, this is not sufficient to prove that $T_H$ generates a $c_0$–semigroup in $X_p$ as illustrated by the following example due to J. Voigt [22].

Example 3.5. Let us consider a 1D transport model in $L^1$. Define

$$\Omega = [0,1[ \quad \text{and} \quad V = [0, +\infty[$$

and assume that $d\mu(\cdot) = \text{Lebesgue measure on } V$. One sees that $\Gamma_+ = \{1\} \times V$ and $\Gamma_- = \{0\} \times V$, so that $L^1_+ = L^1([0, +\infty[, \text{d}v)$. Let us consider the identical boundary operator

$$H(\psi(1, \cdot)) = \psi(0, \cdot) \quad \forall \psi \in W_1.$$

Let us prove that $T_H$ is not a closed operator in $X_1$. Let $h \in L^1([0, +\infty[, \text{d}v)$ be such that

$$\int_0^\infty |h(v)| \text{d}v = \infty. \quad (3.3)$$

For any $n \in \mathbb{N}$, denote

$$\varphi_n(x,v) = \begin{cases} h(v) & \text{if } 0 < v < n \\ 0 & \text{else.} \end{cases}$$

Clearly, $\varphi_n \in W_1$ for any $n \in \mathbb{N}$ and, since

$$\int_0^n |h(v)| \text{d}v < \infty \quad \forall n \in \mathbb{N},$$

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one has $\varphi_n|_{\Gamma_\pm} \in L^1_\pm$ and $\varphi_n \in D(T_H)$ for any $n \in \mathbb{N}$. Now, one can easily show that
\[
\varphi_n \rightarrow \varphi \quad \text{and} \quad T_H \varphi_n \rightarrow 0 \quad (n \rightarrow \infty)
\]
with $\varphi(x,v) = h(v)$ for almost every $(x,v) \in \Omega \times V$, $\varphi \in X_1$. Now, according to (3.3)
\[
\varphi_{\Gamma_-} = h \notin L^1_-
\]
This proves that $\varphi \notin D(T_H)$ and $T_H$ is not a closed operator in $X_1$.

**Remark 3.6.** The above example shows that, for $\|H\| = 1$, $T_H$ may not be closed and consequently may not be the generator of a $c_0$-semigroup in $X_p$. Nevertheless, under the additional assumption $H \geq 0$, it is possible to show, by a monotone convergence argument, that there exists an extension of $T_H$ that generates a $c_0$–semigroup in $X_p$ [24], [7 Theorem 2.3, Chapter XII]. For more considerations on non–negative conservative boundary conditions, we refer the reader to [25].

**Remark 3.7.** If $\|H\| < 1$, Theorem 3.3 implies that the type $\omega(T_H)$ of the $c_0$-semigroup generated by $T_H$ is non–positive. Actually, it is possible to derive finer estimates of $\omega(T_H)$. We refer for instance to [26] in the case when $0 \notin V$ (see also Remark 4.2 thereafter in the case of the slab).

## 4 The phase space approach

### 4.1 The particular case of a slab

We begin this section by dealing with the study of the free streaming operator in slab geometry. This particular case has its own historical importance and received a peculiar interest during the last decade (see for instance [10], [27]). Precisely, let $\Omega \times V = [a,a] \times [-1,1]$ ($a > 0$) and
\[
X_p = L^p([-a,a],[1,1],dx,\xi) \quad (1 \leq p < \infty).
\]
In this case, the incoming and the outgoing part of $\Omega \times V$ are
\[
\Gamma_- := \{-a\} \times [0,1] \cup \{a\} \times [-1,0] \quad \text{and} \quad \Gamma_+ := \{-a\} \times [-1,0] \cup \{a\} \times [0,1]. \quad (4.1)
\]
For any $H \in \mathcal{L}(L^p_+,L^p_-)$, the free streaming operator is given then by
\[
T_H f(x,\xi) = -\xi \frac{\partial f(x,\xi)}{\partial x} \quad f \in D(T_H)
\]
with
\[
D(T_H) = \{\psi \in W_p : H(\psi|_{\Gamma_+}) = \psi|_{\Gamma_-}\}, \quad R(T_H) \subset X_p.
\]
It is possible to prove the following.
Theorem 4.1. For any \( H \in \mathcal{L}(L^p_+, L^p_-) \), the free streaming operator \( T_H \) is a generator of a \( c_0 \)-semigroup \( \{U_H(t); t \geq 0\} \) in \( X_p \) \((1 \leq p < \infty)\). Moreover,

\[
\|U_H(t)\| \leq \max\{1, \|H\|\} \exp\left( t \max\{\frac{1}{2a} \ln \|H\|, 0\}\right), \quad t \geq 0. \tag{4.2}
\]

Remark 4.2. Note that, because of the definition of \( \Gamma_\pm \) (4.1), any boundary operator \( H \in \mathcal{L}(L^p_+, L^p_-) \) admits a matrix representation \( [28] \) which allows to improve the estimate (4.2) (see \( [27] \)).

This theorem has been proved independently by several authors. Let us mention here the seminal works of G. Borgioli and S. Totaro \( [9] \) and S. Totaro \( [27] \) who proved the result in the particular case \( p = 1 \) using a general theorem of Batty and Robinson \( [15] \) (for more details on the result of Batty and Robinson, see also the Appendix). More recently, M. Boulouanour proved Theorem 4.1 using a renormalization process similar to that used in Section 4.2 \( [10] \).

The above result calls for comments. Surprisingly, Theorem 4.1 asserts that, whatever the boundary operator \( H \) is, the free–streaming operator \( T_H \) generates a \( c_0 \)-semigroup in \( X_p \) \((1 \leq p < \infty)\). Actually, as we will see hereafter, this result follows from the particular nature of the slab geometry. The drawback of this result is that it does not give any information of what may occur in other kind of geometry and leaves in the darkness the real mathematical difficulty. In fact, Theorem 4.1 is a simple consequence of the more general case studied in the following section.

4.2 The general phase space approach

The following illustrates the fact that the geometry of the phase space plays a crucial role for the well–posedness of kinetic equations \( [29], [12] \).

Theorem 4.3. Let us assume that the phase space \( \Omega \times V \) is such that

\[
\tau_0 := \operatorname{ess} \inf_{(x,v) \in \Gamma_+} \tau(x,v) > 0. \tag{4.3}
\]

Then, for any \( H \in \mathcal{L}(L^p_+, L^p_-) \), \( T_H \) is a generator of a \( c_0 \)-semigroup \( \{U_H(t); t \geq 0\} \) in \( X_p \) \((1 \leq p < \infty)\) such that

\[
\|U_H(t)\| \leq \max\{1, \|H\|\} \exp\left( t \max\{0, \ln \|H\|/\tau_0\}\right) \quad (t \geq 0). \tag{4.4}
\]

Remark 4.4. Using the terminology of \( [12] \), any phase space \( \Omega \times V \) satisfying (4.3) is said to be regular.
Transport equations in slab geometry are governed by the above Theorem since the phase space \([-a,a] \times [-1,1]\) is regular. Indeed, for any \((x,\xi) \in [-a,a] \times [-1,1]\)

\[
t(x,\xi) = \begin{cases} 
\inf\{s > 0 ; x - \xi s \leq -a\} & \text{if } \xi > 0 \\
\inf\{s > 0 ; x - \xi s \geq a\} & \text{if } \xi < 0,
\end{cases}
\]

i. e.

\[
t(x,\xi) = \frac{x - \text{sign}(\xi)a}{|\xi|} \quad (x,\xi) \in [-a,a] \times [-1,1], \; \xi \neq 0.
\]

Therefore

\[
t_0 = \inf_{(x,\xi) \in D} \tau(x,\xi) = 2a > 0
\]

which proves that the phase space is regular.

**Remark 4.5.** One notes that estimate (4.2) follows from (4.5) and (4.4). Consequently, Theorem 4.1 turns out to be a simple consequence of Theorem 4.3.

Theorem 4.3 has been proved by M. Boulanouar [12] and his proof is based upon a suitable renormalization argument. More precisely, it consists in studying the problem

\[
\begin{cases} 
\frac{d\varphi}{dt}(t) = T_H \varphi(t) \\
\varphi(0) = \phi_0 \in X_p,
\end{cases}
\]

in a weighted space \(L^p_\omega := L^p(\Omega \times V, \omega(x,v)dx \mu(v)) \; (1 \leq p < \infty)\) where \(\omega(\cdot,\cdot)\) is a suitable nonnegative function such that \(\omega|_{\Gamma^+} = \|H\|^p, \; \omega|_{\Gamma^-} = 1\) and, because of (4.3),

\[
\text{ess sup}_{(x,v) \in \Omega \times V} \omega(x,v) \leq \|H\|^p.
\]

This last inequality implies that the norms on \(X_p\) and on \(L^p_\omega\) are equivalent. The end of the proof is based on Hille–Yosida theorem applied in \(L^p_\omega\) and consists in resuming the arguments of the proof of Theorem 3.3.

Note that the proof of Theorem 4.3 in [29] is carried out by the method of characteristics, using the fact that, because of (4.3), the lengths of the characteristics curves have a positive lower bound. The proof of [29] also uses the above renormalization argument.

Theorem 4.3 illustrates the important fact that the time of sojourn is the quantity to handle for who wants to deal with the well–posedness of linear kinetic equations. Unfortunately, in practical situations, this theorem only applies in the case of slab geometry (see Remark 4.3 above) and in some particular cases from population dynamics (such like the Rotenberg model with maturation velocity bounded from below [30]). Indeed, for a bounded convex domain \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\), if \(V\) is such that

\[
\{v/|v|, \; v \in V, v \neq 0\} = S^{N-1} \; (\text{the unit sphere of } \mathbb{R}^N)
\]
then, one can easily check that
\[ \inf \{ \tau(x, v), (x, v) \in \Gamma_+ \} = 0, \]
i.e. \( \Omega \times V \) is a non–regular phase space.

5 The influence of the boundary operator

The results of Section 4.2 illustrate the fact that, to prove the well–posedness of kinetic equations associated to a non–contractive boundary operator \( H \), the main difficulty relies on the fact that, for a convex domain \( \Omega \subset \mathbb{R}^N \) with \( N > 1 \), the time of sojourn of particles in \( \Omega \) may be arbitrary small. Recall that Theorem 4.3 asserts that, for a regular phase space (for which this time of sojourn is bounded away from zero), no assumption on the boundary operator is needed. This is no more the case in full generality as it is illustrated by the following example:

Example 5.1 (Bounce–back reflections). Let \( \Omega \) be a smooth open and convex subset of \( \mathbb{R}^N \) \((N \geq 1)\) and let \( V = \mathbb{R}^N \) be endowed with the Lebesgue measure. Let us consider the boundary operator:

\[ H(\psi)(x, v) = \alpha \psi(x, -v) \quad (x, v) \in \Gamma_-, \psi \in L^p_+ \]

with \( \alpha > 1 \). Clearly \( H \in \mathcal{L}(L^p_+, L^p_-) \) and \( \|H\| = \alpha > 1 \). In [31], the spectrum of the associated free–streaming operator \( T_H \) is investigated and one can show that

\[ \sigma(T_H) \supseteq \bigcup_{k \in \mathbb{Z}} \text{R}_{\text{ess}}(F_k) \]

where \( \text{R}_{\text{ess}}(F_k) \) is the essential range of the measurable mapping:

\[ F_k : (x, v) \in \Omega \times V \mapsto F_k(x, v) = \frac{\ln \alpha - 2ik\pi}{t(x, v) + t(x, -v)} \quad (k \in \mathbb{Z}). \]

Consequently,

\[ s(T_H) := \sup \{ \text{Re} \lambda ; \lambda \in \sigma(T_H) \} = \text{ess sup} \frac{\ln \alpha}{t(x, v) + t(x, -v)} = +\infty. \]

This proves that the spectrum of \( T_H \) is not confined in any left half–plane. In particular, \( T_H \) is not a generator of a \( c_0 \)–semigroup in \( X_p \) \((1 \leq p < \infty)\).

Remark 5.2. It is possible to exhibit similar examples from neutron transport models with specular reflection conditions [32] and for transport–like equations from population dynamics [19] (see also Example 2.6 below).
The previous example shows that, for a non-regular phase space, some assumption on the boundary operator is needed to prove that the associated streaming operator generates a $c_0$-semigroup in $X_p$. Moreover, Theorem 4.3 indicates intuitively that $T_H$ will be the generator of a $c_0$-semigroup in $X_p$ provided $H$ "does not take too much into account" the set $\{(x,v) \in \Gamma_+ ; \tau(x,v) = 0\}$.

Let us make more precise what we mean by this. For any $\varepsilon > 0$, denotes $\chi_\varepsilon$ the multiplication operator in $L^p_+$ by the characteristic function of the set $\{(x,v) \in \Gamma_+ ; \tau(x,v) \leq \varepsilon\}$, i.e. $\chi_\varepsilon \in \mathcal{L}(L^p_+)$ is given by

$$\chi_\varepsilon u(x,v) = \begin{cases} u(x,v) & \text{if } \tau(x,v) \leq \varepsilon \\ 0 & \text{else}, \end{cases}$$

for any $u \in L^p_+$. Our main result is the following.

**Theorem 5.3.** Let $H \in \mathcal{L}(L^p_+,L^p_-)$. If

$$\limsup_{\varepsilon \to 0} \|H\chi_\varepsilon\|_{\mathcal{L}(L^p_+,L^p_-)} < 1,$$

then $T_H$ generates a $c_0$-semigroup $\{U_H(t) ; t \geq 0\}$ in $X_p$ ($1 \leq p < \infty$). Moreover, there exists $C \geq 1$ such that

$$\|U_H(t)\| \leq C \exp(t \max\{\frac{1}{\varepsilon_0} \ln \|H\|, 0\}), \quad \forall t \geq 0,$$

where $\varepsilon_0 = \sup\{\varepsilon > 0 ; \|H\chi_\varepsilon\| < 1\}$.

**Remark 5.4.** Roughly speaking, assumption (5.3) is a smallness assumption of $H$ in the neighborhood of $\{(x,v) \in \Gamma_+ ; \tau(x,v) = 0\} = \{(x,v) \in \Gamma_+ ; v \cdot n(x) = 0\}$. This means that the tangential velocities are weakly taken into account by $H$ regardless of its norm.

**Remark 5.5.** A particular version of Theorem 5.3 has been first proved in [11] in the case $p = 1$ thanks to Batty–Robinson’s theorem. Nevertheless, it appears that the result of [11] only apply to regular phase-spaces (see Appendix for details).

**Remark 5.6.** Note that it is possible to show, in the spirit of [19] Theorem 4.4], that $\{U_H(t) ; t \geq 0\}$ depends continuously on $H \in \mathcal{L}(L^p_+,L^p_-)$ (see [23] for details).

Let us explain the strategy we follow to prove this result. This strategy is inspired by a model from population dynamics (see Example 2.6) studied together with M. Mokhtar–Kharroubi [19]. Our aim is to prove that the following evolution problem

$$\begin{cases} \frac{\partial \psi}{\partial t}(x,v,t) + v \cdot \nabla_x \psi(x,v,t) = 0 \\ \psi|_{\Gamma_-} = H(\psi|_{\Gamma_+}) \\ \psi(x,v,0) = \psi_0(x,v), \end{cases}$$

\hspace{10cm} (5.3)
where \( \psi_0 \in X_p \) \((1 \leq p < \infty)\), is governed by a \(c_0\)-semigroup in \(X_p\). We make use of a suitable change of unknown in the spirit of the one used in [19] (see also [7, Chapter XIII]). This new unknown satisfies then an equivalent evolution problem (see below (5.5)) which, under assumption (5.1), involves a contractive boundary operator.

Let us introduce some useful definitions. For any \(0 < q < 1\), define the multiplication operator in \(L^p_+\) \((1 \leq p < \infty)\):

\[
M_q : u \in L^p_+ \mapsto M_q u(x,v) = q^{\tau_k(x,v)}u(x,v) \in L^p_+,
\]

where \(\tau_k(x,v) = \min\{\tau(x,v); k\}\), \((x,v) \in \Gamma_+, k\) being any fixed positive real number. Let \(B_q\) be defined by

\[
B_q : \varphi \in X_p \mapsto B_q \varphi(x,v) = q^{t_k(x,v)}\varphi(x,v) \in X_p,
\]

with \(t_k(x,v) = \min\{t(x,v); k\}\), \((x,v) \in \overline{\Omega} \times V\). Since \(M_q \in \mathcal{L}(L^p_+, L^p_-)\), it is possible to define the absorption operator associated to \(HM_q \in \mathcal{L}(L^p_+, L^p_-)\)

\[
T_{H_q} : D(T_{H_q}) \subset X_p \rightarrow X_p
\]

\[
\varphi \mapsto T_{H_q} \varphi(x,v) := -v \cdot \nabla_x \varphi(x,v) - q \varphi(x,v)
\]

where

\[
D(T_{H_q}) = \{ \psi \in \dot{W}_p; \psi|_{\Gamma_-} = HM_q(\psi|_{\Gamma_+}) \}.
\]

The unbounded operators \(T_H\) and \(T_{H_q}\) are related by the following.

**Lemma 5.7.** For any \(0 < q < 1\), \(B_q^{-1}D(T_H) = D(T_{H_q})\) and \(T_H = B_q T_{H_q} B_q^{-1}\).

**Proof:** Let \(0 < q < 1\) be fixed. One sees easily that \(B_q\) is a continuous bijection from \(X_p\) onto itself. Its inverse is given by

\[
B_q^{-1} : \varphi \in X_p \mapsto B_q^{-1} \varphi(x,v) = e^{-t_k(x,v)\ln q} \varphi(x,v) \in X_p.
\]

Note that \(B_q^{-1} \in \mathcal{L}(X_p)\) because \(\sup\{t_k(x,v); (x,v) \in \Omega \times V\} \leq k\). Now, let \(\varphi \in D(T_H)\) and \(\psi = B_q^{-1} \varphi\). Let us first show that \(\psi \in W_p\). Indeed, for almost every \((x,v) \in \Omega \times V\)

\[
v \cdot \nabla_x \psi(x,v) = \lim_{s \rightarrow 0} \frac{\psi(x + sv,v) - \psi(x,v)}{s}
\]

\[
= \lim_{s \rightarrow 0} \frac{e^{-t_k(x+sv,v)\ln q} \varphi(x + sv,v) - e^{-t_k(x,v)\ln q} \varphi(x,v)}{s}.
\]

Since, for a. e. \((x,v) \in \Omega \times V\),

\[
t(x + sv,v) = s + t(x,v) \quad \forall 0 \leq s < t(x,v),
\]

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one gets $t_k(x + sv, v) = s + t_k(x, v)$ for any $0 < s < k - t_k(x, v)$ and
\[
v \cdot \nabla_x \psi(x, v) = e^{-t_k(x, v) \ln q} \lim_{s \to 0} \frac{e^{-s \ln q} \varphi(x + sv, v) - \varphi(x, v)}{s}.
\]
Using that $\varphi \in W_p$ one gets
\[
v \cdot \nabla_x \psi(x, v) = e^{-t_k(x, v) \ln q} (-\ln q \varphi(x, v) + v \cdot \nabla_x \varphi(x, v))
\]
so that $\psi \in W_p$. Moreover, since $t_k(x, v) = 0$ for any $(x, v) \in \Gamma_-$, it is clear that
\[
\varphi|_{\Gamma_-} = \psi|_{\Gamma_-},
\]
and
\[
\psi|_{\Gamma_+}(x, v) = e^{-\tau_k(x, v) \ln q} \varphi|_{\Gamma_+}(x, v), \quad (x, v) \in \Gamma_+.
\]
Thus $\psi|_{\Gamma_\pm} \in L^p_\pm$ and
\[
\psi|_{\Gamma_-} = H M_q(\psi|_{\Gamma_+}).
\]
This proves that $\psi \in D(T_{H_q})$ i. e.
\[
B_q^{-1} D(T_H) \subset D(T_{H_q}).
\]
The converse inclusion is proved similarly. Finally, for $\varphi \in D(T_H)$, according to (5.4)
\[
T_{H_q} B_q^{-1} \varphi(x, v) = -v \cdot \nabla_x (e^{-t_k(x, v) \ln q} \varphi(x, v)) - \ln q e^{-t_k(x, v) \ln q} \varphi(x, v)
\]
\[
= e^{-t_k(x, v) \ln q} (-\ln q \varphi(x, v) - v \cdot \nabla_x \varphi(x, v)).
\]
Consequently
\[
B_q T_{H_q} B_q^{-1} \varphi(x, v) = -v \cdot \nabla_x \varphi(x, v) = T_H \varphi(x, v)
\]
which achieves the proof. ■

As a consequence, one has the following.

**Proposition 5.8.** For any $0 < q < 1$, $T_{H_q}$ generates a $c_0$–semigroup $\{V_{H_q}(t) ; t \geq 0\}$ in $X_p$ if and only if $T_H$ is a generator of a $c_0$–semigroup $\{U_H(t) ; t \geq 0\}$ in $X_p$ ($1 \leq p < \infty$).

Moreover,
\[
U_H(t) = B_q V_{H_q}(t) B_q^{-1} \quad (t \geq 0).
\]

In other words, Proposition 5.8 indicates that the following evolution problem
\[
\begin{align*}
\frac{\partial \varphi}{\partial t}(x, v, t) + v \cdot \nabla_x \varphi(x, v, t) + \ln q \varphi(x, v, t) &= 0, \\
\varphi|_{\Gamma_-} &= H M_q(\varphi|_{\Gamma_+}), \\
\varphi(x, v, 0) &= e^{-t_k(x, v) \ln q} \psi_0(x, v),
\end{align*}
\]
is equivalent to problem (5.3) thanks to the change of variables
\[ \varphi(x,v,t) = e^{-k(x,v)\ln q} \psi(x,v,t). \]

We are now in position to prove Theorem 5.3.

**Proof of Theorem 5.3:** According to Theorem 3.3, it is enough to prove the result when \( \|H\| \geq 1 \). Define \( Q = \{ 0 < q < 1 ; \| HM_q \| < 1 \} \). Proposition 5.8 together with Theorem 3.3 assert that if \( Q \neq \emptyset \) then \( T_H \) generates a \( c_0 \)-semigroup \( \{ U_H(t) ; t \geq 0 \} \) such that
\[ U_H(t) = B_q V_H B_q^{-1} \quad \forall t \geq 0, q \in Q, \tag{5.6} \]
where \( \{ V_H(t) ; t \geq 0 \} \) is the \( c_0 \)-semigroup in \( X_p \) with generator \( T_{H_q} (q \in Q) \).

Thanks to assumption (5.1), let us fix \( 0 < \varepsilon < k \) so that \( \| H \chi_\varepsilon \| < 1 \). Then, for any \( 0 < q < 1 \),
\[ \| HM_q \| \leq \| H \chi_\varepsilon M_q \| + \| H(I - \chi_\varepsilon) M_q \| \]
\[ \leq \| H \chi_\varepsilon \| + \| H \| \| (I - \chi_\varepsilon) M_q \|. \]

Moreover
\[ \| (I - \chi_\varepsilon) M_q \| = \sup \{ e^{\tau(x,v)\ln q} ; (x,v) \in \Gamma_+ \text{ and } \tau_k(x,v) \geq \varepsilon \} \leq e^{\varepsilon \ln q}. \]

Consequently,
\[ \| HM_q \| \leq \| H \chi_\varepsilon \| + \| H \| e^{\varepsilon \ln q} \]
and, if
\[ e^{\varepsilon \ln q} < \frac{1 - \| H \chi_\varepsilon \|}{\| H \|} \tag{5.7} \]
then \( q \in Q \). One has then \( Q \neq \emptyset \) and \( T_H \) is a generator of a \( c_0 \)-semigroup \( \{ U_H(t) ; t \geq 0 \} \) in \( X_p \). On the other hand, it is clear that
\[ \| V_{H_q}(t) \| \leq e^{-\ln q t} \quad \forall t \geq 0, q \in Q, \]
and one checks that
\[ \| B_q \| \leq 1 \quad \text{and} \quad \| B_q^{-1} \| \leq e^{-k \ln q} \leq e^{-\varepsilon \ln q}, \quad q \in Q. \]

Then, (5.6) implies
\[ \| U_H(t) \| \leq e^{-\varepsilon \ln q e^{-\ln q t}} \quad \forall t \geq 0, q \in Q. \]
One deduces from (5.7) the following estimate
\[ \| U_H(t) \| \leq \| H \| e^{\ln (1 - \| H \chi_\varepsilon \|) e^{\frac{1}{\varepsilon} \ln \| H \|}} \quad t \geq 0 \]
for any \( 0 < \varepsilon < k \) such that \( \| H \chi_\varepsilon \| < 1 \), which achieves the proof. ■
Remark 5.9. It has been shown above that, provided $H$ fulfills (5.1),
\[
\limsup_{q \to 0} \|HM_q\| < 1.
\]
Therefore, setting $\lambda = -\ln q$, with the notations of Section 3 one gets $\tau_\sigma(M_\lambda H) < 1$ for sufficiently large $\lambda$.

The results of the previous section are now simple corollaries of Theorem 5.3. Indeed, let us assume that
\[
\tau_0 := \operatorname{ess inf} \tau(x,v) > 0.
\]
Then, for any bounded operator $H \in \mathcal{L}(L^p_+, L^p_-)$, one has
\[
\|H\chi_\varepsilon\| = \begin{cases} 0 & \text{if } 0 < \varepsilon < \tau_0 \\ \|H\| & \text{if } \varepsilon \geq \tau_0. \end{cases} \tag{5.8}
\]
Therefore, Theorem 4.3 follows directly from Theorem and assumption (5.1) is met by any bounded boundary operator $H$. Note also that the estimate (4.4) follows from (5.8) and (5.2).

6 Application to Maxwell–type boundary conditions

We briefly show in this section how the results of the previous section apply to the boundary conditions described in Section 2.

We begin by the local boundary conditions introduced in Definition 2.4. For $p = 1$, we have the following.

Proposition 6.1. Assume $p = 1$ and let $H \in \mathcal{L}(L^1_+, L^1_-)$ be a Maxwell–type boundary operator given by Definition 2.4. If
\[
\lim_{\varepsilon \to 0} \operatorname{ess sup}_{\tau(x,v') \leq \varepsilon} \int_{\{v' \cdot n(x) > 0\}} h(x,v,v') |v' \cdot n(x)| dv' < 1 - \operatorname{ess sup}_{x \in \partial \Omega} \alpha(x),
\]
then $T_H$ generates a $c_0$–semigroup in $X_1$.

Proof : It is easy to check that
\[
\|H\chi_\varepsilon\|_{\mathcal{L}(L^1_+, L^1_-)} \leq \operatorname{ess sup}_{\tau(x,v') \leq \varepsilon} \int_{\{v' \cdot n(x) > 0\}} h(x,v,v') |v' \cdot n(x)| dv' + \operatorname{ess sup}_{x \in \partial \Omega} \alpha(x), \quad \varepsilon > 0.
\]
Then, Theorem 5.3 leads to the conclusion. \[ \square \]

When $1 < p < \infty$, we have the following.
Proposition 6.2. Let $1 < p < \infty$. Assume $H$ is a Maxwell-type boundary operator given by Definition 2.4. Moreover, let us assume that

$$\lim_{\varepsilon \to 0} \sup_{x \in \partial \Omega} \int_{\{v \cdot n(x) < 0\}} |v \cdot n(x)| dv \times \left( \int_{\{v \cdot n(x) > 0\} \cap \{\tau(x,v') \leq \varepsilon\}} |h(x,v,v')|^q |v' \cdot n(x)| dv' \right)^{\frac{2}{q}} < 1 - \sup_{x \in \partial \Omega} \alpha(x) \quad (1/p + 1/q = 1). \quad (6.1)$$

Then $T_H$ is a generator of a $c_0$-semigroup in $X_p$.

Proof: The proof is a direct application of Theorem 5.3 and follows from straightforward calculations (for the details see [23]). \hfill \blacksquare

Remark 6.3. It is possible to replace assumption (6.1) by

$$\lim_{\varepsilon \to 0} \left( \sup_{(x,v) \in \Gamma_-} \int_{\{v \cdot n(x) > 0\} \cap \{\tau(x,v') \leq \varepsilon\}} h(x,v,v') |v' \cdot n(x)| dv' \right)^{\frac{1}{q}} \times \left( \sup_{(x,v') \leq \varepsilon} \int_{\{v \cdot n(x) < 0\}} h(x,v,v') |v \cdot n(x)| dv \right)^{\frac{1}{p}} < 1 - \sup_{x \in \partial \Omega} \alpha(x) \quad (1/p + 1/q = 1).$$

For practical situations (see Example 2.1), it is useful to state the following.

Proposition 6.4. Assume $H = K + C$ with $C$ given by Def. 2.4 and

$$K(\psi|_{\Gamma_+})(x,v) = \beta(x) \int_{\{v' \cdot n(x) > 0\}} k(v,v') \psi|_{\Gamma_+}(x,v') |v' \cdot n(x)| dv',$$

for any $(x,v) \in \Gamma_-$, where $\beta(\cdot) \in L^\infty(\partial \Omega)$ is non-negative. Moreover, if $1 < p < \infty$, assume that

$$\sup_{x \in \partial \Omega} \int_{\{v \cdot n(x) < 0\}} |v \cdot n(x)| dv \left( \int_{\{v \cdot n(x) > 0\}} |k(v,v')|^q |v' \cdot n(x)| dv' \right)^{p/q} < \infty, \quad (6.2)$$

where $1/p + 1/q = 1$. Then $T_H$ generates a $c_0$-semigroup in $X_p$ ($1 \leq p < \infty$) provided

$$\sup_{x \in \partial \Omega} \alpha(x) < 1.$$
Proof : The proof will consist in showing that the diffusive–part $K$ is such that

$$\lim_{\varepsilon \to 0} \| K \chi_\varepsilon \| = 0. \quad (6.3)$$

We will restrict ourselves with the case $1 < p < \infty$, the case $p = 1$ being much simple.

For any $\varepsilon > 0$, define

$$f_\varepsilon(x) = \int_{\{v \cdot n(x) \leq 0\}} |v \cdot n(x)| \, dv \times \left( \int_{\{v' \cdot n(x) \geq 0\} \cap \{\tau(x, v') \leq \varepsilon\}} |k(v, v')|^q |v' \cdot n(x)| \, dv' \right)^{p/q} \quad (x \in \partial \Omega).$$

Clearly, for any $0 \leq \varepsilon < \varepsilon'$,

$$0 \leq f_\varepsilon(x) \leq f_{\varepsilon'}(x) \leq f_0(x) \quad (x \in \partial \Omega), \quad (6.4)$$

where

$$f_0(x) = \int_{\{v \cdot n(x) \leq 0\}} |v \cdot n(x)| \, dv \left( \int_{\{v' \cdot n(x) \geq 0\}} |k(v, v')|^q |v' \cdot n(x)| \, dv' \right)^{p/q}.$$

Note that $f_0 \in L^\infty(\Omega)$ according to (6.2). Moreover, using the continuity of $n(\cdot)$ and $\tau(\cdot, \cdot)$ (see [22]) it is possible to show [23, p. 194–195] that $f_\varepsilon(\cdot)$ is continuous on $\partial \Omega (\varepsilon \geq 0)$.

Now, for a.e. $(x, v) \in \Gamma_-$

$$\lim_{\varepsilon \to 0} \int_{\{v' \cdot n(x) \geq 0\} \cap \{\tau(x, v') \leq \varepsilon\}} |k(v, v')|^q |v' \cdot n(x)| \, dv'$$

$$= \int_{\{v' \cdot n(x) \geq 0\} \cap \{\tau(x, v') = 0\}} |k(v, v')|^q |v' \cdot n(x)| \, dv'$$

$$= \int_{\{v' \cdot n(x) = 0\}} |k(v, v')|^q |v' \cdot n(x)| \, dv' = 0.$$

Thus, using (6.2) together with the dominated convergence theorem,

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = 0 \text{ a.e. } x \in \partial \Omega.$$

Using (6.4) and the continuity of $f_\varepsilon(\cdot)$, Dini's Theorem yields

$$\lim_{\varepsilon \to 0} \sup_{x \in \partial \Omega} f_\varepsilon(x) = 0.$$

Now, since

$$\|H \chi_\varepsilon\|_{L^p(L^p, L^p)} \leq \|\beta\|_{\infty} \|f_\varepsilon\|^{1/p}_{\infty},$$

one gets (6.3). Finally, since $\|C\| \leq \text{ess sup}_{x \in \partial \Omega} \alpha(x) < 1$, Theorem 5.3 leads to the conclusion. \[\square\]
Remark 6.5. The main notable fact of Proposition 6.4 is that generation occurs for arbitrarily large $\beta(\cdot)$. This comes from the fact that $\beta(\cdot)$ is only space–dependent and does not care about the tangential velocities (see Remark 5.4).

Example 6.6. Let us consider the Maxwell model described previously. Precisely, assume that, for any $\psi \in L^p$, 

$$H(\psi|_{\Gamma_+})(x, v) = \alpha(x) \psi|_{\Gamma_+}(x, v - 2(v \cdot n(x))n(x)) + (1 - \alpha(x))M_\omega(v) \int_{\{v' \cdot n(x) \geq 0\}} \psi|_{\Gamma_+}(x, v')|v' \cdot n(x)|dv',$$

where $\alpha \in L^\infty(\partial \Omega)$ is non–negative and $M_\omega$ is the Maxwellian of the wall given by (2.1). One easily derive from Proposition 6.4 that, if

$$\sup_{x \in \partial \Omega} \alpha(x) < 1,$$

then $T_H$ is a generator of a $c_0$–semigroup in $X_p$ ($1 < p < \infty$).

The case of non–local boundary operators as described in by Definition 2.7 is covered by the following result when $p = 1$.

Theorem 6.7. Let $p = 1$. Assume that $H = K + C$ where $\|C\| < 1$ and $K \in \mathcal{L}(L^1_+, L^1_-)$ is given by

$$K(\psi)(x, v) = \int_{\Gamma_+} \kappa(x, v, y, v')\psi(y, v')|v' \cdot n(y)|d\gamma(y)d\mu(v'),$$

where the kernel $\kappa(\cdot, \cdot, \cdot, \cdot) \geq 0$ is measurable and $d\gamma(\cdot)$ is the Lebesgue measure on the surface $\partial \Omega$. If

$$\limsup_{\varepsilon \to 0} \sup_{\{\tau(y, v') \leq \varepsilon\}} \int_{\Gamma_-} \kappa(x, v, y, v')|v \cdot n(x)|d\gamma(x)d\mu(v') < 1 - \|C\|,$$

then $T_H$ generates a $c_0$–semigroup in $X_p$.

Proof: The proof follows from Theorem 5.3 and from the fact that

$$\|K\chi_\varepsilon\|_{\mathcal{L}(L^1_+, L^1_-)} = \sup_{\{\tau(y, v') \leq \varepsilon\}} \int_{\Gamma_-} \kappa(x, v, y, v')|v \cdot n(x)|d\gamma(x)d\mu(v'),$$

since $\kappa(\cdot, \cdot, \cdot, \cdot)$ is non–negative.

For $1 < p < \infty$, one has the following result, based on compactness arguments.
Theorem 6.8. Let $1 < p < \infty$. Assume that $H = K + C$ where $K : L^p_+ \to L^p_-$ is compact and $\|C\| < 1$, then $T_H$ generates a $c_0$--semigroup in $X_p$.

Proof : Note that

\[ \|H\chi\| \leq \|K\chi\| + \|C\| = \|\chi K^*\| + \|C\| \quad \forall \varepsilon > 0 \]

where $K^* \in L(L^q_-, L^p_+)$ denotes the dual operator of $K$ $(1/p + 1/q = 1)$. Since the truncation operator $\chi\varepsilon$ goes to zero as $\varepsilon \to 0$ in the strong operator topology (and consequently uniformly on any compact subset of $L^q_-$) it follows from the compactness of $K^*$ that

\[ \lim_{\varepsilon \to 0} \|\chi\varepsilon K^*\|_{L(L^q_-, L^p_+)} = 0. \]

Hence $\limsup_{\varepsilon \to 0} \|H\chi\| \leq \|C\| < 1$ which ends the proof thanks to Theorem 5.3. \[
\]

Example 2.6 (revisited). Let us go back to Example 2.6. Let the boundary operator $H \in L(L^p((0, \ell_2), d\ell))$ $(1 \leq p < \infty)$ by given by

\[ H(\psi|\Gamma_+)(\ell) = \int_0^{\ell_2} k(\ell, \ell')\psi|\Gamma_+ (\ell') d\ell' + c\psi|\Gamma_+ (\ell) \quad 0 < \ell < \ell_2. \]

If $p = 1$, one deduces from Theorem 6.7 that, provided

\[ \lim_{\varepsilon \to 0} (\text{ess sup}_{\ell' \in (0, \varepsilon)} \int_0^{\ell_2} k(\ell, \ell') d\ell) < 1 - c \]

then $T_H$ generates a $c_0$--semigroup in $X_1$ (see also [19, Corollary 3.2]). For $1 < p < \infty$, it is also possible to prove the well--posedness of (2.3) thanks to Theorem 6.8 under some (natural) assumption on the transition kernel $k(\cdot, \cdot)$ (see [19, Corollary 3.1] for details). 

Remark 6.9. Note that, if $\ell_1 > 0$, the phase space $\Omega \times V$ is regular so that, thanks to Theorem 6.3, $T_H$ generates a $c_0$--semigroup in $X_p$ for any $H \in L(L^p((\ell_1, \ell_2), d\ell))$ [19].

7 Concluding remarks

We gave in this paper an overview of $c_0$--semigroup generation results for free--streaming operators with abstract boundary conditions. Actually, we emphasize here that, to our mind, the right approach is the one explained in Section 5 which consists in dealing with the boundary operators rather than with the phase space. Indeed, for applications, the phase space is given a priori and it appears to us that the interesting question is to determine, for a given phase space, the class of boundary operators $H$ such that $T_H$ generates a
A \(c_0\)-semigroup in some suitable \(L^p\)-space. One saw that this occurs under some suitable smallness assumption on \(H\) in the vicinity of the tangential velocities. The important feature of such a result (Theorem 5.3) is that no global assumption on \(H\) is needed. Moreover, already known generation results for regular phase space turn out to be simple consequence of our main result. This comes from the fact that, for this kind of geometry, the set of tangential velocities is empty. We also emphasize the fact that Theorem 5.3 is well-suited to the study of transport-like equations with practical boundary conditions arising in the field of mathematical physics (neutron transport equations, linear kinetic of gases...) or from population dynamics.

We point out that, by standard perturbation arguments, the results of this paper imply the well-posedness (in the semigroup sense) of the initial-boundary value problem given in Introduction

\[
Q(f)(x,v) = \int_V \kappa(x,v,w)f(x,w)d\mu(w) - \sigma(x,v)f(x,v)
\]

Precisely, at least for \(\sigma(\cdot,\cdot) \in L^\infty(\Omega \times V)\) and for a measurable kernel \(\kappa(\cdot,\cdot,\cdot)\) such that the operator

\[
K : \psi(x,v) \in X_p \mapsto \int_V \kappa(x,v,w)f(x,w,t)d\mu(w) \in X_p
\]

is bounded, then \(T_H + Q\) generates a \(c_0\)-semigroup in \(X_p\) provided \(T_H\) is. It is an open question to know whether such a result is still valid for unbounded cross-sections \(\sigma\) and \(K\).

A question is of relevant interest in the study of the linearized Boltzmann equation (see [3]). Hopefully, one should generalize the generation result proposed in [33] (dealing with the absorbing case \(H = 0\)) to more general boundary conditions. Results in this direction are already known in the peculiar case of slab geometry [34, 35] and, more generally, for regular phase space [29].

We conclude this section with an interesting conjecture. To our knowledge, all the existing examples of free-streaming operator \(T_H\) that does not generate a \(c_0\)-semigroup in \(X_p\) (\(1 \leq p < \infty\)) are such that the spectrum of \(T_H\) does not lie in a left half-space or that \(T_H\) is not closed (see Examples 2.5 or 4.1 for instance). Moreover, one saw that the smallness assumption on \(H\) (5.1) can be seen as an existence assumption of the resolvent of \(T_H\) for large \(\lambda\) (see Remark 5.9). This suggests the following conjecture.

\textbf{Conjecture 1.} Let \(H \in \mathcal{L}(L^p_+,L^p_-)\) (\(1 \leq p < \infty\)) be a bounded boundary operator. Then, \(T_H\) generates a \(c_0\)-semigroup in \(X_p\) if and only if there exists \(\lambda_0 \in \mathbb{R}\) such that \([\lambda_0, +\infty[ \subseteq \rho(T_H)\).

Actually, the use of Batty–Robinson Theorem in \(L^1\)-space (see the following Appendix) supports us in the belief that the main difficulty to prove that \(T_H\) is a generator is not to find a suitable estimate on the resolvent of \(T_H\) but rather to prove that this resolvent does exist. Work is in progress in this direction.
Appendix: The Batty–Robinson Theorem

In this section, we say a few words about a useful tool used in kinetic theory to derive generation theorem in $L^1$–space. The following abstract result is due to J. K. Batty and D. W. Robinson [15] (see also [14] for a very elegant proof of this theorem).

Let $X$ be an ordered Banach space whose positive cone is generating and normal, i.e., $X = X_+ - X_-$ and $X^* = X_+^* - X_-^*$ where $X_\pm$ (respectively $X_\pm^*$) denote the positive and negative cone in $X$ (resp. in $X^*$).

An operator $A$ on $X$ is said to be resolvent positive if there exists $\omega \in \mathbb{R}$ such that $]\omega, +\infty[ \subset \rho(A)$ (the resolvent set of $A$) and $(\lambda - A)^{-1} \geq 0$ for any $\lambda > \omega$.

**Theorem (Batty–Robinson).** Let $A$ be a densely defined resolvent positive operator in $X$. If there exists $\lambda_0 > s(A)$ and $c > 0$ such that
\[
\| (\lambda_0 - A)^{-1} \varphi \| \geq c \| \varphi \| \quad \forall \varphi \in X_+,
\]
then $A$ is a generator of a (positive) $c_0$–semigroup in $X$.

Note that the hypothesis (A.1) requires an inverse estimate with respect to the Hille–Yosida theorem. Note also that, in practical situations, the Banach space $X$ is a $L^1$–space.

The use of Batty–Robinson’s Theorem in kinetic theory is due to our knowledge to G. Borgioli and S. Totaro [9] in order to prove Theorem 4.1 in a $L^1$–setting. More recently, this result has been used successfully by several authors [36, 11]. In particular, K. Latrach and M. Mokhtar–Kharroubi [11] proved a particular version of Theorem 5.3 for $p = 1$:

**Theorem (Latrach–Mokhtar-Kharroubi).** Let us assume that $H$ satisfies (5.1) and the following additional assumptions:
\[
H \geq 0,
\]
and
\[
\| H \psi \| \geq \| \psi \| \quad \forall \psi \in L^1_+.
\]
Then, $T_H$ generates a $c_0$–semigroup in $L^1(\Omega \times V)$.

Actually, we already saw that according to Remark 5.9 there exists $\lambda_0 > 0$ such that
\[
r_\sigma(M_\lambda H) < 1 \quad \forall \lambda > \lambda_0.
\]
Now, it suffices to appeal to Proposition 3.2 together with (A.2) which ensure that, for any $\lambda > \lambda_0$, $(\lambda - T_H)^{-1}$ exists and is nonnegative. Let us show how to derive Estimate (A.1). We follow the strategy of [11, Theorem 5.2]. Let $\lambda > \lambda_0$ and let $\varphi \in X_1$, $\varphi \geq 0$. Set $\psi = (\lambda - T_H)^{-1} \varphi$ the nonnegative solution of
\[
\lambda \psi(x,v) + v \cdot \nabla_x \psi(x,v) = \varphi(x,v) \quad (x,v) \in \Omega \times V.
\]
Integrating with respect to $x$ and $v$ together with Green’s identity leads to
\[
\lambda \|\psi\| + \int_{\Gamma^+} \psi(x,v) |v \cdot n(x)| d\gamma(x) d\mu(v) - \\
- \int_{\Gamma^-} \psi(x,v) |v \cdot n(x)| d\gamma(x) d\mu(v) = \|\varphi\|
\]
which is noting else but $\lambda \|\psi\| + (\|\psi|_{\Gamma^+} - \|H\psi|_{\Gamma^+}) = \|\varphi\|$. Therefore, thanks to (A.3),
\[
\|(\lambda - T_H)^{-1}\varphi\| \geq \frac{1}{\lambda} \|\varphi\|
\]
which gives the estimate (A.1).

**Remark 7.1.** The above result of [11] calls for comments. Actually, it turns out that the assumptions (5.1) and (A.3) are compatible only for regular phase–space. Indeed, let us assume that $\inf \{\tau(x,v); (x,v) \in \Gamma^+_+\} = 0$ and define, for any $\varepsilon > 0$,
\[
\Gamma_\varepsilon = \{(x,v) \in \Gamma^+_+; \tau(x,v) \leq \varepsilon\}
\]
and
\[
u_\varepsilon(x,v) = \chi_{\Gamma_\varepsilon}(x,v) \quad (x,v) \in \Gamma^+_+.
\]
According to Assumption (A.3),
\[
\|Hu_\varepsilon\| \geq \|u_\varepsilon\|
\]
and, since $Hu_\varepsilon = H_\varepsilon u_\varepsilon$ (where we used the notations of Section 5), this shows that
\[
\|H_\varepsilon\| \geq 1
\]
and contradicts Assumption (5.1). This fact has not been noticed by the authors of [11] and suggests that the Batty–Robinson’s Theorem applies in the kinetic theory only to regular phase–spaces.

**References**

[1] B. Lods, A generation theorem for kinetic equations with non–contractive boundary operators. *C. R. Acad. Sci. Paris*, Ser. I 335 655–660 (2002).

[2] C. Cercignani, *The Boltzmann equation and its applications*, Springer–Verlag, New York (1988).

[3] C. Cercignani, R. Illner and M. Pulvirenti, *The mathematical theory of dilute gases*, Springer–Verlag, New York (1994).

[4] M. M. R. Williams, *Mathematical Methods in Particle Transport Theory*, Butterworth, London (1971).
[5] G. F. Webb, *Theory of nonlinear age–dependent population dynamics*, Marcel Dekker, New York (1985).

[6] N. Bellomo and M. Pulvirenti Eds., *Modeling in applied sciences: A kinetic theory approach*, Birkhäuser, Boston (2000).

[7] W. Greenberg, C. Van der Mee and V. Protopopescu, *Boundary Value Problems in Abstract Kinetic theory*, Birkhäuser Verlag, Basel (1987).

[8] J. L. Lebowitz and S. I. Rubinow, A theory for the age and generation time distribution of a microbial population. *J. Math. Biol.* 117–36 (1974).

[9] G. Borgioli and S. Totaro, 3D–streaming operator with multiplying boundary conditions: semigroup generation properties. *Semigroup Forum* 55 110–117 (1997).

[10] M. Boulanouar, Le transport neutronique avec des conditions aux limites générales. *C. R. Acad. Sci. Paris.*, Ser. I 329 121–124 (1999).

[11] K. Latrach and M. Mokhtar–Kharroubi, Spectral analysis and generation results for streaming operators with multiplying boundary conditions. *Positivity* 3 273–296 (1999).

[12] M. Boulanouar, Opérateur d’advection: Existence d’un semi–groupe (I). *Transp. Theory Stat. Phys.* 31 169–176 (2002).

[13] M. Boulanouar, Opérateur d’advection: Existence d’un semi–groupe (II). *Transp. Theory Stat. Phys.* 32 185–197 (2003).

[14] W. Arendt, Resolvent Positive Operators. *Proc. London Math. Soc.* 54 321–349 (1987).

[15] J. K. Batty and D. W. Robinson, Positive one parameter semigroups on ordered spaces. *Acta Appl. Math.* 1 221–296 (1984).

[16] A. Palczewski, Velocity averaging for boundary value problems, In *Nonlinear kinetic theory and mathematical aspects of hyperbolic systems*, (Edited by V. Boffi, F. Bampi, G. Toscani), World Scientific. Series Adv. Math. Sci. *Vol. 9* (1992).

[17] G. F. Webb, A model of proliferating cell population with inherited cycle length. *J. Math. Biol.* 23 269–282 (1986).

[18] K. Latrach and A. Zeghal, Existence results for a boundary value problem arising in growing cell populations. *Math. Models Methods Appl. Sci.* 13 1–17 (2003).

[19] B. Lods and M. Mokhtar–Kharroubi, On the theory of a growing cell population with zero minimum cycle length. *J. Math. Anal. Appl.* 266 70–99 (2001).

[20] M. Cessenat, Théorèmes de traces Lp pour les espaces de fonctions de la neutronique. *C. R. Acad. Sci. Paris.*, Ser I 299 831–834 (1984).

[21] M. Cessenat, Théorèmes de traces pour les espaces de fonctions de la neutronique. *C. R. Acad. Sci. Paris.*, Ser. I 300 89–92 (1985).

[22] J. Voigt, *Functional analytic treatment of the initial boundary value problem for collisionless gases*, München, Habilitationsschrift (1981).

[23] B. Lods, Théorie spectrale des équations cinétiques, Thèse de doctorat. Université de Franche–Comté (2002).
[24] R. Beals and V. Protopopescu, Abstract time–dependent transport equations. *J. Math. Anal. Appl.* **121** 370–405 (1987).

[25] L. Arlotti and B. Lods, work in progress.

[26] F. Ammar–Khodja and M. Mokhtar–Kharroubi, On the exponential stability of advection semigroups with boundary operators. *Math. Mod. Meth. Appl. Sci.* **8** 95–106 (1996).

[27] S. Totaro, Study of the free streaming operator in slab geometry in dependence of the boundary conditions. *Math. Meth. Appl. Sci.* **20** 717–736 (1997).

[28] K. Latrach, Théorie spectrales d’équations cinétiques. Thèse de doctorat. Université de Franche–Comté (1992).

[29] C. Van der Mee, Time dependent kinetic equations with collision terms relatively bounded with respect to collision frequency. *Transp. Theory Stat. Phys.* **30** 63–90 (2001).

[30] M. Boulanouar and H. Emamirad, A transport equation in cell population dynamics. *Differential Integral Equations* **13** 125–144 (2000).

[31] B. Lods, On the spectrum of transport operator with specular and bounce-back reflections conditions. *work in progress*.

[32] Chen Jun and Yang Ming–Zhu, Linear transport equation with specular reflection boundary conditions. *Transp. Theory Stat. Phys.* **20** 281–306 (1991).

[33] B. Lods, On linear kinetic equations involving unbounded cross–sections. *Math. Methods Appl. Sci.* **27** 1049–1075 (2004).

[34] M. Chabi and K. Latrach, On singular mono-energetic transport equations in slab geometry. *Math. Methods Appl. Sci.* **25** 1121–1147 (2002).

[35] M. Chabi and K. Latrach, Singular one-dimensional transport equations on $L_p$-spaces. *J. Math. Anal. Appl.* **283** 319–336 (2003).

[36] S. Mancini and S. Totaro, Solutions of the Vlasov equation in a slab with source terms on the boundaries. *Riv. Math. Univ. Parma* **2** 33–47 (1999).