The Fuglede theorem and some intertwining relations

Bensaid, Ikram Fatima Zohra; Dehimi, Souheyb; Fuglede, Bent; Mortad, Mohammed Hichem

Published in:
Advances in Operator Theory

DOI:
10.1007/s43036-020-00110-5

Publication date:
2021

Document version
Early version, also known as pre-print

Document license:
CC BY

Citation for published version (APA):
Bensaid, I. F. Z., Dehimi, S., Fuglede, B., & Mortad, M. H. (2021). The Fuglede theorem and some intertwining relations. Advances in Operator Theory, 6(1), 1-8. [9]. https://doi.org/10.1007/s43036-020-00110-5
THE FUGLEDE THEOREM AND SOME INTERTWINNING RELATIONS

IKRAM FATIMA ZOHRA BENSADA, SOUHEYB DEHIMI, BENT FUGLEDE AND MOHAMMED HICHEM MORTAD*

Abstract. In this paper, we show a new and classic version of the celebrated Fuglede Theorem in an unbounded setting. A related counterexample is equally presented. In the second strand of the paper, we give a pair of a closed and self-adjoint (unbounded) operators which is not intertwined by any (bounded or closed) operator except the zero operator.

1. Introduction

Undoubtedly, the Fuglede Theorem is the second salient result in Operator Theory, at least, as far as normal operators are concerned. It has many applications. The most tremendous one is the fact that it improves the statement of the Spectral Theorem of normal operators. To cite only a little amount of applications of this powerful tool, we refer readers to [1], [5], [15], [16], [18], [21], [27], [29], [31], [38], [42], [43], [46] and [54].

Recall that this theorem states that if $T \in B(H)$ and $A$ is normal (not necessarily bounded), then

$TA \subseteq AT \iff TA^* \subseteq A^*T.$

The problem leading to this theorem was first raised by von Neumann in [36] who had already established it in a finite dimensional setting (since this is seemingly not well documented, readers may find it in e.g. Exercise 11.3.29 in [33]). Fuglede was the first one to answer this problem affirmatively in [11] (a quite different proof popped up shortly afterwards and it is due to Halmos [17]). Then Putnam [42] generalized the result to:

$TA \subseteq BT \iff TA^* \subseteq B^*T$

where $A$ and $B$ are normal (not necessarily bounded) and $T \in B(H)$.

There are different proofs of the Fuglede-Putnam Theorem besides the first two due to Fuglede and Putnam (e.g. the one in [17]). Perhaps the most elegant proof is due to Rosenblum (see [48]). Then came Berberian [3], who noted that the Fuglede’s version is actually equivalent to the Putnam’s version. Other proofs which are not well-spread may be consulted in [7] or [45]. See also [39].

There is a particular terminology to the transformation which occurs in the Fuglede-Putnam Theorem.

Definition. We say that $T \in B(H)$ intertwines two operators $A, B$ if $TA \subseteq BT$.

2010 Mathematics Subject Classification. Primary 47A05, Secondary 47B25, 47A62.

Key words and phrases. The Fuglede theorem. Intertwining relations. Closed and self-adjoint operators.

* Corresponding author.
Accordingly, we may restate the Fuglede-Putnam theorem as follows: *If an operator intertwines two normal operators, then it intertwines their adjoints.*

There have been many generalizations of the Fuglede-Putnam Theorem since Fuglede’s paper. We note a generalization to the so-called "spectral operators" by Dunford [10] (and another proof of the latter by Radjavi-Rosenthal [44]). See also [13], [14], [23], [24], [28], [35], [47], and [50] (among others). See also [30]. For new versions of the Fuglede-Putnam Theorem involving unbounded operators only, readers may wish to consult [26], [32], [40] and [41]. An interesting and related paper is [19].

Most of these generalizations seem to go into one direction only, that is, towards relaxing the normality hypothesis whilst there are still some unexplored territories as regards the very first version. To get to one main problem of this paper, observe that if $A$ is self-adjoint (and unbounded), then obviously $BA \subset AB$ implies that $B^*A \subset AB^*$ for any $B \in B(H)$. In [22], it was asked whether the assumption of the self-adjointness of $A$ can be relaxed to requiring only the closedness of $A$ and imposing the normality of $B$? The referee of the same reference informed Jorgensen of Fuglede’s example (which we will be recalling below). In the same reference it was shown that $B^*A \subset AB^*$ if for instance the complement of $\sigma(B)$ is connected and the interior is empty. Readers might also be interested in [8].

Closely related to what has just been alluded at, the following conjecture was proposed in [25] (it has resisted solutions for about three years). See Theorem 2.1 and Proposition 2.6.

**Conjecture 1.1.** Let $T$ be an operator (densely defined and closed if necessary) and let $B \in B(H)$ be normal. Then

$$BT \subset TB^* \implies B^*T \subset TB.$$  

What is interesting about this conjecture is the fact that it holds when $T \in B(H)$ (as we recover the bounded version of the Fuglede-Putnam Theorem), and as it stands, it is covered by none of the known (unbounded) generalizations of Fuglede-Putnam Theorem (see e.g. [32], [41] and [51]).

In this paper, we show that this conjecture is true in case $B$ has a finite pure point spectrum (Theorem 2.1). It is, however, not true even if we assume that $A$ is self-adjoint and $B$ is unitary. In the second part of this paper, we provide a pair of a closed and self-adjoint (unbounded) operators which is not intertwined by any (bounded or closed) operator except the zero operator.

Finally, we refer readers to [32] for properties and results about matrices of unbounded operators which will be helpful in the sequel. For the general theory of unbounded operators, readers may consult [4] or [49] or [53].

2. **Main Results**

**Theorem 2.1.** Let $B$ be a bounded normal operator with a finite pure point spectrum and let $A$ be a closed (possibly unbounded) operator on a separable complex Hilbert space $H$. Let $f, g : \mathbb{C} \to \mathbb{C}$ be two continuous functions. Then

$$BA \subset Af(B) \implies g(B)A \subset A(g \circ f)(B).$$

**Proof.** The hypothesis $BA \subset Af(B)$ clearly gives

$$D(A) = D(BA) \subset D(Af(B))$$
where $D$ stands for domains. Hence
\[
D(B^2A) = D[(BA)f(B)] = D[(BA)f(B)] = D[(Af(B))f(B)] = D[f(B)^2],
\]
and next successively, for any $x \in D(A)$,
\[
B^2Ax = B(BA)x = B(Af(B))x = (BA)f(B)x = (Af(B))f(B)x = Af(B)^2x.
\]
Hence $B^2A \subset A(f(B))^2$ and by iteration
\[
B^mA \subset A(f(B))^m
\]
for any $m \in \mathbb{N}$. Therefore,
\[
p(B)A \subset A(p \circ f)(B)
\]
for any polynomial $p \in \mathbb{C}(X)$.

By assumption, $B$ has a point spectrum with finitely many distinct eigenvalues $\lambda_j, j \in \{1, \cdots, n\}$, and corresponding eigenprojectors $E_j$ adding up to the identity operator $I$, so $B = \sum_{j=1}^n \lambda_j E_j$ is the spectral representation of $B$. For the given continuous function $g : \mathbb{C} \to \mathbb{C}$, there exists a polynomial $p$ such that $p(f(\lambda_j)) = g(f(\lambda_j))$. In fact, for any $k \in \{1, \cdots, n\}$, there is a polynomial $p_k$ with roots $f(\lambda_j)$, $j \neq k$, and with the value $p(f(\lambda_k)) = (g \circ f)(\lambda_k)$ at $f(\lambda_k)$. Then the polynomial $p := \sum_{k=1}^n p_k$ has the asserted property. From the hypothesis $BA \subset A(f(B))$, we obtain
\[
g(B)A = p(B)A \subset A(p \circ f)(B) = A(g \circ f)(B).
\]

\[\square\]

**Corollary 2.2.** With $A$ and $B$ as in Theorem 2.1, we have
\[
BA \subset AB^* \implies B^*A \subset AB.
\]

**Proof.** Just apply Theorem 2.1 to the functions $f, g : z \mapsto z$ (so that $g \circ f$ becomes the identity map on $\mathbb{C}$). \[\square\]

A similar reasoning applies to establish the following consequence:

**Corollary 2.3.** With $A$ and $B$ as in Theorem 2.1, we likewise have
\[
BA \subset AB \implies B^*A \subset AB^*.
\]

Using an idea by Berberian, we may generalize this result to the case of two normal operators whereby we obtain a Fuglede-Putnam style theorem.

**Proposition 2.4.** Let $B$ and $C$ be bounded normal operators with a finite pure point spectrum and let $A$ be a closed (possibly unbounded) operator on a separable complex Hilbert space $H$. Then
\[
BA \subset AC \implies B^*A \subset AC^*.
\]

**Proof.** Define $\tilde{B}$ on $H \oplus H$ by:
\[
\tilde{B} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}
\]
and let $\tilde{A} = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with $D(\tilde{A}) = H \oplus D(A)$. Since $BA \subset AC$, it follows that
\[
\tilde{B}\tilde{A} = \begin{pmatrix} 0 & BA \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} 0 & AC \\ 0 & 0 \end{pmatrix} = \tilde{A}\tilde{B}
\]
for \( D(\hat{B}\hat{A}) = H \oplus D(A) \subset H \oplus D(AC) = \hat{A}\hat{B} \).

Now, since \( B \) and \( C \) are normal, so is \( \hat{B} \). Finally, apply Corollary 2.3 to the pair \((\hat{B}, \hat{A})\) to get
\[
\hat{B}^* \hat{A} \subset \hat{A}\hat{B}^*
\]
which, upon examining their entries, yields the required result. \( \square \)

**Corollary 2.5.** Let \( B \) and \( C \) be bounded normal operators with a finite pure point spectrum and let \( A \) be a densely defined operator on a separable complex Hilbert space \( H \). Then
\[
BA \subset AC \implies CA^* \subset A^*B.
\]

**Proof.** Merely use the foregoing result, then take adjoints. \( \square \)

One may wonder whether \( BT \subset TB^* \) implies \( B^*T \subset TB \) in the events of the self-adjointness of \( T \) and the normality of \( B \in B(H) \)? The next example says that this is untrue, thus providing a counterexample to Conjecture 1.1.

**Proposition 2.6.** There is a unitary \( B \in B(H) \) and a self-adjoint \( T \) with domain \( D(T) \subset H \) such that \( BT \subset TB \) but \( B^*T \not\subset TB \).

First, we recall the following example (which appeared in [12]):

**Example 2.7.** There exists a unitary \( B \in B(H) \) and a closed and symmetric \( T \) with domain \( D(T) \subset H \) such that \( BT \subset TB \) but \( B^*T \not\subset TB \).

Now, we prove Proposition 2.6

**Proof.** Consider a unitary \( U \in B(H) \) and a closed \( A \) such that \( UA \subset AU \) and \( U^*A \not\subset AU^* \). Consider
\[
B = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.
\]

Then \( B \) is unitary and \( T \) is self-adjoint on \( D(A^*) \oplus D(A) \) (thanks to the closedness of \( A \)). Besides,
\[
BT = \begin{pmatrix} 0 & UA \\ U^*A & 0 \end{pmatrix} \quad \text{and} \quad TB^* = \begin{pmatrix} 0 & AU \\ A^*U & 0 \end{pmatrix}.
\]

Since \( UA \subset AU \), it follows by taking adjoints that \( U^*A \subset A^*U \). Therefore, \( BT \subset TB^* \). Since \( U^*A \not\subset AU^* \) is equivalent to \( U^*A^* \not\subset A^*U \), we may thereby get that
\[
B^*T \not\subset TB
\]
as \( D(B^*T) \not\subset D(TB) \). \( \square \)

Now, we pass to the second topic of the paper. Fuglede found in [11] a closed operator which did not commute with any bounded operator except scalar ones (i.e. \( \alpha I \) where \( \alpha \in \mathbb{C} \)). The next two results lie within the same scope. In addition, they allow us to establish the uniqueness of the solution of some particular equations.

**Proposition 2.8.** On some Hilbert space \( H \), there is a self-adjoint operator \( A \) and a densely defined closed operator \( B \) such that \( TA \subset BT \) (whenever \( T \in B(H) \)) implies \( T = 0 \). Also (for the same pair \( A \) and \( B \)), \( SB \subset AS \) for any \( S \in B(H) \) forces \( S = 0 \).
Proof. Let \( H = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) and let \( A \) be any unbounded self-adjoint operator with domain \( D(A) \subset H \) and let \( B \) be a closed operator such that

\[
D(B^2) = D\left(B^{*2}\right) = \{0\}
\]

(as in [9], cf. [34]). Let \( T \in B(H) \). Then, clearly

\[
TA \subset BT \implies TA^2 \subset BTA \subset B^2T.
\]

Hence

\[
D(TA^2) = D(A^2) \subset D(B^2T) = \{x \in H : Tx \in D(B^2) = \{0\}\} = \ker T.
\]

Since \( A^2 \) is densely defined, it follows that \( H = \overline{D(A^2)} \subset \ker T = \ker T \subset H \).

Hence \( \ker T = H \), that is, \( T = 0 \), as required.

Now, we pass to the second part of the question. Plainly,

\[
SB \subset AS \implies S^*A \subset B^*S^*.
\]

As before, we obtain

\[
S^*A^2 \subset B^*S^*.
\]

Similar arguments as above then yield \( S^* = 0 \) or simply \( S = 0 \), as needed. \( \square \)

Remark. In fact, the first case of the foregoing counterexample may be beefed up by even allowing \( B \) to be also symmetric and semi-bounded (see e.g. [53] for the definition of semi-boundedness). This is based on the famous counterexample by Chernoff in [9].

**Proposition 2.9.** On some Hilbert space \( H \), there are two densely defined closed operators \( A \) and \( B \) such that \( TA \subset BT \) implies \( T = 0 \) whenever \( T \) is closed.

**Proof.** Let \( H = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) and let \( A \) be a densely defined closed operator with domain \( D(A) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) such that \( A^2 = 0 \) on \( D(A^2) = D(A) \) (cf. [37]). An explicit and adapted example to our case is to consider

\[
A = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}
\]

where \( T \) is say an unbounded self-adjoint operator with domain \( D(T) \subset L^2(\mathbb{R}) \). By definition, \( D(A) = L^2(\mathbb{R}) \oplus D(T) \).

Then as may easily be seen

\[
A^2 = \begin{pmatrix} 0 & 0_{D(T)} \\ 0 & 0_{D(T)} \end{pmatrix}
\]

with \( D(A^2) = D(A) = L^2(\mathbb{R}) \oplus D(T) \). Now, let \( B \) be a closed operator defined on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) satisfying \( D(B^2) = \{0\} \) (as in [9]).

Now, clearly

\[
TA \subset BT \implies TA^2 \subset B^2T.
\]

But

\[
D(TA^2) = \{x \in D(A^2) : 0 \in D(T)\} = D(A) \quad \text{and} \quad D(B^2T) = \ker T.
\]

Hence

\[
D(A) \subset \ker T \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})
\]
and so upon passing to the closure (w.r.t. \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \))

\[ \ker T = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \]

because \( \ker T \) is closed for \( T \) is closed. Therefore, \(Tx = 0\) for all \( x \in D(T)\), i.e. \( T \subset 0 \). Accordingly, as \( T \) is bounded on \( D(T) \) and also closed, then \( D(T) \) becomes closed and so \( D(T) = H \), that is, \( T = 0 \) everywhere, as coveted. \( \square \)

3. Concluding Remarks and an Open Problem

It seems noteworthy that easy arguments allow us to show that \( BT = TB^* \) does imply that \( B^*T = TB \) when \( B \) is unitary even if \( T \) is any (unbounded) operator. In other words, the self-adjointness of \( BT \) entails that of \( B^*T \) if we further assume that \( T \) is self-adjoint. One may therefore wonder what happens if one assumes that \( B \) is only normal? The problem thus becomes: If \( B \in B(H) \) is normal and if \( T \) is (unbounded) self-adjoint, then

\[ BT \text{ is self-adjoint} \iff B^*T \text{ is self-adjoint?} \]

Recall that if \( T \) is bounded, then the self-adjointness of \( BT \) gives the self-adjointness of \( B^*T \) and vice versa. The analogous question in the case of normality of \( BT \) has already a negative answer as a famous counterexample by Kaplansky shows (see [22]. Cf. [2]). Going back to the main question, observe that a naive counterexample is not available either. In other words, if \( BT \) is closed, then \( B^*T \) is necessarily closed (and conversely). Indeed, the normality of \( B \) gives

\[ \|B^*Tx\| = \|B Tx\|, \forall x \in D(B^*T) = D(BT) = D(T). \]

Hence, the graph norms of \( B^*T \) and \( BT \) coincide and hence the closedness of one implies the closedness of the other. With the closedness of \( B^*T \) at hand, we may try to show that \( B^*T \) is normal and having a real spectrum. But honestly, we just do not know whether this would lead anywhere or one has to look for counterexamples?

References

[1] E. Albrecht, P. G. Spain, When products of self-adjoints are normal, Proc. Amer. Math. Soc., 128/8 (2000) 2509-2511.
[2] A. Benali, M. H. Mortad, Generalizations of Kaplansky's theorem involving unbounded linear operators, Bull. Pol. Acad. Sci. Math., 62/2 (2014) 181-186.
[3] S. K. Berberian, Note on a Theorem of Fuglede and Putnam, Proc. Amer. Math. Soc., 10 (1959) 175-182.
[4] M. Sh. Birman, M. Z. Solomjak, Spectral theory of selfadjoint operators in Hilbert space. Translated from the 1980 Russian original by S. Khrushčëv and V. Peller. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.
[5] Ch. Chellali and M. H. Mortad, Commutativity up to a factor for bounded and unbounded operators, J. Math. Anal. Appl., Elsevier, 419/1 (2014), 114-122.
[6] P. R. Chernoff, A Semibounded Closed Symmetric Operator Whose Square Has Trivial Domain, Proc. Amer. Math. Soc., 89/2 (1983) 289-290.
[7] J. B. Deeds, A proof of Fuglede's theorem, J. Math. Anal. Appl., 27 (1969) 101-102
[8] S. Dehimi and M. H. Mortad, Bounded and Unbounded Operators Similar to Their Adjoints, Bull. Korean Math. Soc., 54/1 (2017) 215-223.
[9] S. Dehimi, M. H. Mortad, Chernoff Like Counterexamples Related to Unbounded Operators, submitted, 
[arXiv:1808.09523v2].
[10] N. Dunford, Spectral operators. Pacific J. Math., 4 (1954) 321-354.
[11] B. Fuglede, A Commutativity Theorem for Normal Operators, Proc. Nati. Acad. Sci., 36 (1950) 35-40.
[12] B. Fuglede, Solution to Problem 3, Math. Scand., 2 (1954) 346-347.
[13] T. Furuta, On Relaxation of Normality in the Fuglede-Putnam Theorem, *Proc. Amer. Math. Soc.*, **77** (1979) 324-328.
[14] T. Furuta, Extensions of the Fuglede-Putnam-type theorems to subnormal operators, *Bull. Austral. Math. Soc.*, **31/2** (1985) 161-169.
[15] K. Gustafson, M. H. Mortad, Unbounded products of operators and connections to Dirac-type operators, *Bull. Sci. Math.*, **138/5** (2014), 626-642.
[16] K. Gustafson, M. H. Mortad, Conditions implying commutativity of unbounded self-adjoint operators and related topics, *J. Operator Theory*, **76/1** (2015) 159-169.
[17] P. R. Halmos, Commutativity and spectral properties of normal operators. *Acta Sci. Math. Szeged*, **12**, (1950). Leopoldo Fejer Frederico Riesz LXX annos natis dedicatus, Pars B, 153-156.
[18] P. R. Halmos, *A Hilbert Space Problem Book*, Springer, 1982 (2nd edition).
[19] Z. J. Jabłoński, Il B. Jung, J. Stochel, Unbounded quasi normal operators revisited, *Integral Equations Operator Theory*, **79/1** (2014) 135-149.
[20] P. E. T. Jørgensen, Unbounded operators: perturbations and commutativity problems, *J. Funct. Anal.*, **39/3** (1982) 281-307.
[21] Il Bong Jung, M. H. Mortad, J. Stochel, On normal products of selfadjoint operators, *Kyungpook Math. J.*, **57** (2017) 457-471.
[22] I. Kaplansky, Products of normal operators, *Duke Math. J.*, **20/2** (1953) 257-260.
[23] S. Mecheri, A generalization of Fuglede-Putnam theorem, *J. Pure Math.*, **21** (2004) 31-38.
[24] S. Mecheri, K. Tanahashi, A. Uchiyama, Fuglede-Putnam theorem for $p$-hyponormal or class $\mathcal{Y}$ operators, *Bull. Korean Math. Soc.*, **43/4** (2006) 747-753.
[25] M. Meziane, M. H. Mortad, Maximality of Linear Operators, *Rend. Circ. Mat. Palermo* series 2 (to appear). DOI: 10.1007/s12215-018-0370-x.
[26] M. H. Mortad, An Application of the Putnam-Fuglede Theorem to Normal Products of Self-adjoint Operators, *Proc. Amer. Math. Soc.*, **131/10** (2003) 3135-3141.
[27] M. H. Mortad, On some product of two unbounded self-adjoint operators, *Integral Equations Operator Theory*, **64** (2009) 399-408.
[28] M. H. Mortad, Yet More Versions of The Fuglede-Putnam Theorem, *Glasg. Math. J.*, **51/3**, (2009) 473-480.
[29] M. H. Mortad, Commutativity up to a Factor: More Results and the Unbounded Case, *Z. Anal. Anwendungen: Journal for Analysis and its Applications*, **29/3** (2010), 303-307.
[30] M. H. Mortad, Products and Sums of Bounded and Unbounded Normal Operators: Fuglede-Putnam Versus Embry, *Rev. Roumaine Math. Pures Appl.*, **56/3** (2011), 195-205.
[31] M. H. Mortad, On the Normality of the Sum of Two Normal Operators, *Complex Anal. Oper. Theory*, **6/1** (2012), 105-112.
[32] M. H. Mortad, An all-unbounded-operator version of the Fuglede-Putnam theorem, *Complex Anal. Oper. Theory*, **6/6** (2012) 1269-1273.
[33] M. H. Mortad, *An Operator Theory Problem Book*, World Scientific Publishing Co., (2018). https://doi.org/10.1142/10884. ISBN: 978-981-3236-25-7 (hardcover).
[34] M. H. Mortad, On the triviality of domains of powers and adjoints of closed operators (submitted). [arXiv:1811.08994]
[35] M. H. Mortad, S. M. S. Nabavi Sales, Fuglede-Putnam type theorems via the Aluthge transform, *Positivity*, **17/1** (2013) 151-162.
[36] J. von Neumann, Approximative properties of matrices of high finite order, *Portugaliae Math.*, **3** (1942) 1-62.
[37] S. Ota, Unbounded nilpotents and idempotents, *J. Math. Anal. Appl.*, **132/1** (1988) 300-308.
[38] S. Ota, K. Schmüdgen, On Some Classes of Unbounded Operators, *Integral Equations Operator Theory*, **12/2** (1989) 211-226.
[39] F. C. Palagiannis, On Fuglede’s theorem for unbounded normal operators, *Ricerche Mat.*, **51/2** (2002) 261-264.
[40] F. C. Palagiannis, A note on the Fuglede-Putnam theorem, *Proc. Indian Acad. Sci. Math. Sci.*, **123/2** (2013) 253-256.
[41] F. C. Palagiannis, A Generalization of the Fuglede-Putnam Theorem to Unbounded Operators, *J. Oper.*, 2015, Art. ID 804353, 3 pp.
[42] C. R. Putnam, On Normal Operators in Hilbert Space, *Amer. J. Math.*, **73** (1951) 357-362.
[43] H. Radjavi, P. Rosenthal, On roots of normal operators, *J. Math. Anal. Appl.*, **34** (1971) 653-664.
[44] H. Radjavi, P. Rosenthal, Hyperinvariant subspaces for spectral and \( n \)-normal operators, *Acta Sci. Math. (Szeged)*, 32 (1971) 121-126.

[45] W. Rehder, On the Adjoints of Normal Operators, *Arch. Math. (Basel)*, 37/2 (1981) 169-172.

[46] W. Rehder, On the Product of Self-adjoint Operators, *Internat. J. Math. and Math. Sci.*, 5/4 (1982) 813-816.

[47] M. Radjabalipour, An extension of Putnam-Fuglede theorem for hyponormal operators, *Math. Zeit.*, 194/1, (1987) 117-120.

[48] M. Rosenblum, On a theorem of Fuglede and Putnam, *J. London Math. Soc.*, 33, (1958) 376-377.

[49] K. Schmüdgen, *Unbounded Self-adjoint Operators on Hilbert Space*, Springer. GTM 265 (2012).

[50] J. G. Stampfli, B. L. Wadhwa, An asymmetric Putnam-Fuglede theorem for dominant operators, *Indiana Univ. Math. J.*, 25/4, (1976) 359-365.

[51] J. Stochel, An Asymmetric Putnam-Fuglede Theorem for Unbounded Operators, *Proc. Amer. Math. Soc.*, 129/8 (2001) 2261-2271.

[52] Ch. Tretter, *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, London, 2008.

[53] J. Weidmann, *Linear Operators in Hilbert Spaces*, Springer, 1980.

[54] J. Yang, Hong-Ke Du, A Note on Commutativity up to a Factor of Bounded Operators, *Proc. Amer. Math. Soc.*, 132/6 (2004) 1713-1720.

(The first author): Department of Mathematics, University of Cadiz. Avenida de la Universidad s/n E-11405. Jerez de la Frontera. Spain.

E-mail address: ikram.bensaid@alum.uca.es

(The second author): University of Mohamed El Bachir El Ibrahim, Bordj Bou Arreridj. Algeria.

E-mail address: sohayb20091@gmail.com

(The third author): Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen, Danmark.

E-mail address: fuglede@math.ku.dk

(The corresponding author) Department of Mathematics, University of Oran 1, Ahmed Ben Bella, B.P. 1524, El Menouar, Oran 31000, Algeria.

E-mail address: mhmortad@gmail.com, mortad.hichem@univ-oran1.dz.