Constraints on the primordial gravitational waves with variable sound speed from current CMB data

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Abstract: We make a comprehensive investigation of the observational effect of the inflation consistency relation. We focus on the general single-field inflation model with the consistency relation \( r = -8c_s n_t \), and investigate the observational constraints of sound speed \( c_s \) by using the Seven-Year WMAP data, the BICEP tensor power spectrum data, and the constraints on \( f_{\text{NL}}^{\text{equil}} \) and \( f_{\text{NL}}^{\text{orth}} \) from the Five-Year WMAP observations. We find that the constraints on the tensor-to-scalar ratio \( r \) is much tighter if \( c_s \) is small, since a large tilt \( n_t \) is strongly constrained by the observations. We obtain \( r < 0.37, 0.27 \) and 0.09 (\( dn_s/d\ln k = 0 \)) for \( c_s = 1, 0.1 \) and 0.01 models at 95.4\% confidence level. When taking smaller values of \( c_s \), the positive correlation between \( r \) and \( n_s \) also leads to slightly tighter constraint on the upper bound of \( n_s \), while the running of scalar spectral index \( dn_s/d\ln k \) is generally unaffected. For the sound speed \( c_s \), it is not well constrained if only the CMB power spectrum data is used, while the constraints are obtainable by taking \( f_{\text{NL}}^{\text{equil}} \) and \( f_{\text{NL}}^{\text{orth}} \) priors into account. With the constraining data of \( f_{\text{NL}}^{\text{equil}} \) and \( f_{\text{NL}}^{\text{orth}} \), we find that, \( c_s \lesssim 0.01 \) region is excluded at 99.7\% CL, and the \( c_s = 1 \) case (the single-field slow-roll inflation) is slightly disfavored at 68.3\% CL. In addition, the inclusion of \( f_{\text{NL}}^{\text{equil}} \) and \( f_{\text{NL}}^{\text{orth}} \) into the analysis can improve the constraints on \( r \) and \( n_s \). We further discuss the implications of our constraints on the test of inflation models.

Keywords: tensor perturbation, CMB, inflation, consistency relation.

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1. Introduction

An important task of modern cosmology is to understand the expansion history of the Universe. The standard hot big-bang model is successful in explaining various observations, including Hubble expansion, Big-bang Nucleosynthesis and microwave background radiation [1], yet still suffers from the flatness, horizon and monopole problems, etc. The inflation model, in which the vacuum energy drives the Universe exponentially expanding in the very early Universe [2], was proposed under such concerns. Besides the successful explanation of the above problem, inflationary cosmology can provide a viable mechanism for the origin of the cosmic structures.

There have been numerous inflation models proposed in the last several decades. In the face of so many competing candidates, it is necessary to find an effective way to figure out which one is realistic, or at least, which one is most favored by the cosmological observations. Especially, it is important to confirm or rule out the canonical single-field slow-roll (SFSR) inflation model.

It has been proved that the SFSR inflation can generate observable primordial scalar and tensor perturbations, which encode themselves in the cosmic microwave background (CMB) anisotropies. Thus, it is possible to test inflationary models from the current CMB observations, e.g., the Wilkinson Microwave Anisotropy Probe (WMAP) satellite [3], QUaD experiment [4], BICEP experiment [5] and other probes [6].
There have been a number of investigations made on testing the inflation models from the current and future CMB observations [7], mainly on the issues of constraining the SFSR model with the scalar spectral index $n_s$, running of the spectral index $dn_s/d\ln k$, and tensor-to-scalar ratio $r$ as free parameters. Besides the determination of the parameters in inflation models, it has also been proposed [8] that the consistency relations, which features various types of inflation models, can be used as a test to classify and distinguish different models of inflation. The possibility of the observational test of the consistency relations has been discussed in [8] in detail.

In this paper we make further investigations on the observational effect of the consistency relation $r = -8c_s n_t$. We focus on the general single-field inflation model, and discuss the current constraints on the sound speed $c_s$ from the CMB data, including the Seven-Year WMAP (WMAP7) power spectrum data [9], the BICEP data [5], and the constraints on $f_{NL}^{\text{equil}}$ and $f_{NL}^{\text{forth}}$ from the Five-Year WMAP (WMAP5) observations [10, 11]. We then discuss the results of the constraints on $n_s$, $r$, $dn_s/d\ln k$ parameters when $c_s \neq 1$.

This paper is organized as follows. In Sec. 2, we introduce the inflationary consistency relation in the general single-field inflation model. In Sec. 3, we briefly introduce the CMB data and data analysis methodology used in this paper. The results of constraints on cosmological parameters are presented in Sec. 4 and Sec. 5. We summarize our results in Sec. 6.

2. Single-field Inflation Model

Let’s start with the general single-field inflation model

$$S = \int d^4x\sqrt{-g} \left[ \frac{M_p^2}{2} R + P(X, \phi) \right],$$

where $M_p = 1/\sqrt{8\pi G}$ is the reduced Plack mass, $R$ is the Ricci scalar, $g$ is the determinant of the metric, and $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. $P(X, \phi)$ is an arbitrary function of $X$ and $\phi$. This action is the most general Lorentz invariant action for inflaton $\phi$ minimally coupled to Einstein gravity. The primordial scalar power spectrum of curvature perturbation is

$$\Delta^2_R = \frac{H^2/M_p^2}{8\pi^2 c_s \epsilon},$$

where

$$\epsilon = -\frac{\dot{H}}{H^2},$$

is the slow-roll parameter, and

$$c_s = \frac{P_{,X}}{P_{,X} + 2XP_{,XX}}$$

is the speed of sound. The spectral index of scalar curvature perturbation power spectra becomes

$$n_s - 1 = \frac{d \ln \Delta^2_R}{d \ln k} = -2\epsilon - \eta - s,$$
where

\[ \eta = \frac{\dot{\epsilon}}{H \epsilon}, \quad s = \frac{\dot{c}_s}{H c_s}, \quad (2.6) \]

are another two slow-roll parameters. Due to the dynamics of inflation, the spectral index \( n_s \) can be scale-dependent as well. Its scale-dependence is measured by the running of the spectral index \( dn_s/d \ln k \). The primordial power spectrum of scalar curvature perturbation then takes the form,

\[ \Delta^2_R(k) = \Delta^2_R(k_0) \left( \frac{k}{k_0} \right)^{n_s(k_0) - 1 + \frac{1}{2} dn_s/d \ln k}, \quad (2.7) \]

where \( k_0 \) is the pivot scale. The primordial power spectrum of gravitational waves perturbation generated during inflation only depends on the Hubble parameter during inflation

\[ \Delta^2_T = \frac{H^2/M_p^2}{\pi^2/2}, \quad (2.8) \]

with tilt

\[ n_t = \frac{d \ln \Delta^2_T}{d \ln k} = -2 \epsilon. \quad (2.9) \]

The tensor-to-scalar ratio is defined as

\[ r = \Delta^2_T/\Delta^2_R = 16 c_s \epsilon, \quad (2.10) \]

and then by combing with Eq. (2.9), we obtain the consistency relation

\[ r = -8 c_s n_t. \quad (2.11) \]

Here we should note that since the acceleration of scale factor takes the form \( \ddot{a} = H^2 a (1 - \epsilon) \), inflation only happens if \( \epsilon < 1 \), therefore from Eqs. (2.9) and (2.11), we know the valid ranges of values for \( n_t \) and \( r \) as

\[ -2 < n_t \leq 0, \quad \text{and} \quad r < 16 c_s. \quad (2.12) \]

If \( c_s = 1 \), the general single-field inflation model reduces to the single-field slow-roll inflation. But if \( c_s \ll 1 \), the non-trivial sound speed of inflation can generate non-Gaussian modes of perturbation, which results in a large non-local form of bispectrum \( f^\text{equil}_{\text{NL}} \). Although the non-local form of bispectrum has not been well classified, the two most general types, equilateral type with shape size \( f^\text{equil}_{\text{NL}} \), and orthogonal type measured by \( f^\text{orth}_{\text{NL}} \), have been widely discussed in literatures \[1, 11, 14\]. In \[14\], the observational constraint on the \( c_s \) from the bispectrum has been discussed and the requirement from the stability of the field theory \( (c_s^2 \geq 0) \) implies \( f^\text{orth}_{\text{NL}} \leq -0.054 f^\text{equil}_{\text{NL}} \).
3. Data Analysis Methodology

In the following data analysis, we will combine WMAP7 power spectrum \([9]\), with BICEP tensor power spectrum data \([5]\) and bispectrum constraints on \(f_{NL}^{\text{equil}}\) and \(f_{NL}^{\text{orth}}\) \([10, 11]\), to constrain inflation consistency relation. The WMAP \(TT\) power spectrum at \(2 \leq l \leq 1200\) is powerful to constrain the cosmological parameters, e.g. \(n_s\) and \(dn_s/d\ln k\). We also use the WMAP \(TE/EE\) data at \(2 \leq l \leq 800\), and the \(BB\) data mainly on large scales \(2 \leq l \leq 23\). To be consistent with the WMAP results \([9]\), we choose our pivot scale to be \(k_0 = 0.002\ \text{Mpc}^{-1}\).

We also use the BICEP tensor power spectrum data which mainly covers the region \(21 \leq l \leq 335\). Following the pipelines of \([5]\), we construct the expected bandpowers for the inflation models, and use the lognormal approximation to calculate the \(\chi^2\) function,

\[
\chi^2(p) = \left[\tilde{Z}^{BB} - Z(p)^{BB}\right]^T \left[D^{BB}(p)\right]^{-1} \left[\tilde{Z}^{BB} - Z(p)^{BB}\right],
\]

where \(p\) is the model parameters, and \(\hat{Z}^{BB}\) and \(Z(p)^{BB}\) are the observational and theoretical bandpowers. \(D^{BB}(p)\) is the covariance matrix which is dependent on the model parameters. The likelihood function then takes the form

\[
\mathcal{L} \propto \frac{1}{\sqrt{\det[D^{BB}(p)]}} e^{-\chi^2(p)/2}.
\]

In addition, the observational results of \(f_{NL}^{\text{equil}}\) and \(f_{NL}^{\text{orth}}\) can also constrain the value of \(c_s\). We use \(f_{NL}^{\text{equil}}\) and \(f_{NL}^{\text{orth}}\) priors obtained from the WMAP5 observations \([10, 11]\), and construct the \(\chi^2\) function as

\[
\chi^2(p) = v(p)^T_{\text{WMAP}} C^{-1} v(p)_{\text{WMAP}},
\]

where \(C\) is the covariance matrix given in \([11]\), and \(v(p)_{\text{WMAP}}\) is the difference between the observed and model values of \(f_{NL}^{\text{equil}}\) and \(f_{NL}^{\text{orth}}\) \([11]\),

\[
v(p)_{\text{WMAP}} = \begin{pmatrix} \langle \hat{f}_{NL}^{\text{equil}}(p) \rangle - \langle f_{NL}^{\text{equil}} \rangle_{\text{WMAP}} \\ \langle \hat{f}_{NL}^{\text{orth}}(p) \rangle - \langle f_{NL}^{\text{orth}} \rangle_{\text{WMAP}} \end{pmatrix}.
\]

The WMAP5 data yields to \([11]\)\(^1\)

\[
f_{NL}^{\text{equil}} = 155 \pm 140, \quad f_{NL}^{\text{orth}} = -149 \pm 110,
\]

where the errors given are the 1\(\sigma\) confidence level.

We will determine the best-fit parameters and the 68.3% and 95.4% confidence level (CL) ranges by using the Monte Carlo Markov chain (MCMC) technique. The whole set of our free parameters is

\[
P = \{\Omega_b h^2, \Omega_c h^2, \theta, \tau, n_s, dn_s/d\ln k, r, c_s, A_s, A_{SZ}\}^2.
\]

\(^1\)The covariance matrix \(C\) is dependent on the data \([11]\). Since the WMAP7 covariance matrix \(C\) has not yet been published, we will adopt the WMAP5 covariance matrix \(C\) in the following discussion.

\(^2\)\(\theta\) is the ratio of the sound horizon to the angular diameter distance; \(\tau\) is the the reionization optical depth; \(A_s\) is the primordial superhorizon power in the curvature perturbation on the pivot scale \(k_0 = 0.002\ \text{Mpc}^{-1}\); \(A_{SZ}\) is an SZ template normalization.
Figure 1: Power spectra for inflation models with different $c_s$. Models with $c_s=1$, 0.1 and 0.01 are plotted in blue solid, red dashed and green dotted lines, respectively. In all figures we fix $r=0.15$. The primordial tensor power spectrum and the $BB$, $TT$, $EE$, $TE$ power spectra are plotted. The WMAP data are plotted in black points. The $c_s=0.1$ model with $n_t=-0.1875$ leads to slightly larger values of power spectrum at the large scale (not very evident), while the $c_s=0.01$ model leads to significantly larger $P_T(k)/C_l$ in small-$k$/low-$l$ region.

We modify the publicly available CAMB and COSMOMC packages to include models with $c_s$ as a free parameter, and generate $O(10^5)$ samples for each set of results presented in this paper.

4. Cosmological Constraints of Fixed $c_s$ Models

In the following sections we will discuss the cosmological interpretations of the consistency relation. In this section we focus on three models with $c_s$ fixed as 1, 0.1 and 0.01. The free $c_s$ model will be discussed in the next section.

4.1 Effects of $c_s$ on the Power Spectrum

We firstly clarify how the different values of $c_s$ affect the shape of the angular power spectra of CMB.

In the upper-left panel of Fig. 1 we plot the primordial tensor power spectrum for $c_s=1$, 0.1 and 0.01 respectively. We see that $c_s$ has significant influence on the tilt of the
power spectrum through the consistency relation \( n_t = -r/(8c_s) \). In particular, at the large scale (small \( k \)), the \( c_s=0.01 \) and \( c_s=0.1 \) models have much larger values of \( P_t(k) \) than that in the model with \( c_s = 1 \).

This effect is also visible in \( BB, TT, EE \) and \( TE \) power spectra, which is shown in the upper-right and lower panels of Fig. 2 (with lensing). In all figures we take \( r = 0.15 \) and fix other parameters at their WMAP7 best-fit values. It is shown that the amplitude of the power spectrum for \( c_s=0.1 \) is slightly larger than the \( c_s=1 \) case (not evident), while for the \( c_s=0.01 \) case the low-\( l \) \( C_l \)'s are significantly larger. The panels indicate that the set of parameters \( r = 0.15, c_s = 0.01 \) is inconsistent with the WMAP data. Thus, we expect a tight constraint on \( r \) when \( c_s \) is small.

### 4.2 Results of Fitting

Our results of fitting for different models with fixed \( c_s \) are shown in Table 1.

| Data & Model          | \( r \) (95.4% CL) | \( n_t \) (95.4% CL) | \( n_s \) | \( dn_s/d\ln k \) |
|----------------------|-------------------|---------------------|---------|-----------------|
| \( WMAP \)           |                   |                     |         |                 |
| \( c_s = 1 \)        | < 0.37            | > -0.05             | 0.967\(_+0.026\)\(_-0.010\) | -         |
| \( c_s = 0.1 \)      | < 0.26            | > -0.33             | 0.972\(_+0.016\)\(_-0.014\) | -         |
| \( c_s = 0.01 \)     | < 0.09            | > -1.16             | 0.966\(_+0.017\)\(_-0.019\) | -         |
| \( WMAP+BICEP \)     |                   |                     |         |                 |
| \( c_s = 1 \)        | < 0.32            | > -0.04             | 0.966\(_+0.024\)\(_-0.008\) | -         |
| \( c_s = 0.1 \)      | < 0.26            | > -0.33             | 0.971\(_+0.019\)\(_-0.014\) | -         |
| \( c_s = 0.01 \)     | < 0.09            | > -1.16             | 0.969\(_+0.014\)\(_-0.014\) | -         |
| \( WMAP+BICEP \)     |                   |                     |         |                 |
| \( c_s = 1 \)        | < 0.36            | > -0.05             | 1.011\(_+0.030\)\(_-0.035\) | -0.023\(_+0.018\)\(_-0.024\) |
| \( c_s = 0.1 \)      | < 0.35            | > -0.44             | 1.014\(_+0.070\)\(_-0.046\) | -0.022\(_+0.021\)\(_-0.034\) |
| \( (+dn_s/d\ln k) \) | \( c_s = 0.01 \)  | < 0.10              | > -1.21 | 1.005\(_+0.068\)\(_-0.014\) | -0.019\(_+0.005\)\(_-0.030\) |

Here we list the results for \( c_s=1, 0.1 \) and 0.01. In the 4-9 rows, the constraints on \( r, n_t, n_s \) and \( dn_s/d\ln k \) by using the WMAP+BICEP data are listed, divided into the \( dn_s/d\ln k = 0 \) and \( dn_s/d\ln k \neq 0 \) cases. For comparison, in the 1-3 rows we also list the results obtained by using the WMAP data alone with \( dn_s/d\ln k = 0 \). We discuss the results of constraints in Sec. 4.2.1 and 4.2.2 in detail.

#### 4.2.1 The \( dn_s/d\ln k = 0 \) case

In this subsection we briefly discuss the fitting results of models without including \( dn_s/d\ln k \) as a free parameter.

Let us first see the constraints on the tensor-to-scalar ratio \( r \) which determines the amplitude of the tensor power spectrum. The fitting results of \( r \) are listed in the second column of Table 1 and the likelihood functions are plotted in the left panel of Fig. 2. Using the WMAP data alone, we find a 95.4% CL constraint \( r < 0.37 \), which is well consistent with the result obtained by the WMAP 7-yr constraint (\( r < 0.36 \)). As expected, we find the constraint on \( r \) becomes much tighter if the value of \( c_s \) becomes smaller (see the left panel of Fig. 2). Using the WMAP+BICEP data, we find \( r < 0.32 \), 0.26 and 0.09 (95.4% CL) for \( c_s = 1, 0.1 \) and 0.01, respectively. This result is quite reasonable. According to the consistency relation \( n_t = -r/(8c_s) \), if \( c_s \) is small, a large \( r \) would lead to a large \( n_t \), leading
Figure 2: Fitting results for the fixed $c_s$ models with $d\sigma_s/d\ln k = 0$. In the left and middle panels we plot the likelihoods of $r$ and $n_s$ for the $c_s = 1$, 0.1, 0.01 models in blue, red, green lines, and the results obtained by using the WMAP and WMAP+BICEP are shown in dotted and solid lines, respectively. In the right panel we also plot the marginalized 68.3% and 95.4% CL contours in the $r - n_s$ plane. We see that smaller values of $c_s$ lead to tighter constraints on $r$, and thus tighter upper bound constraints on $n_s$, due to their correlation. The inclusion of BICEP data slightly improves the constraints on $r$ and $n_s$ for the $c_s = 1$ case.

...to a large tensor mode on superhorizon scale, which is strongly constrained by low-$l$ CMB data. The necessary condition for inflation Eq. (2.12) is automatically satisfied by the constraint. The $r = 0.15$, $c_s = 0.01$ case shown in the Fig. 1 is excluded.

Except for $r$, another interesting issue is the fitting results of the scalar spectral index $n_s$. Using the WMAP+BICEP data, we find $n_s = 0.966_{-0.008}^{+0.024}$, $0.971_{-0.014}^{+0.019}$ and $0.969_{-0.013}^{+0.014}$ (68.3% CL) for $c_s = 1$, 0.1 and 0.01. All the results are consistent with Harrison-Zeldovich spectrum ($n_s = 1$) at 68.3% CL. The likelihood of $n_s$ in different cases are plotted in the middle panel of Fig. 2. Similar to $r$, we find that the upper bound constraint on $n_s$ also becomes tighter when $c_s$ is smaller. This effect is caused by the positive correlation between $r$ and $n_s$. In the right panel of Fig. 2, we plot the marginalized contours in the $r - n_s$ plane, which shows that the smaller $c_s$ is, the tighter constraints on $n_s$ and $r$.

The results of constraints from BICEP data are as follows. For the $c_s = 1$ model, we get $r < 0.37(0.32)$ (95.4% CL) by using the WMAP(WMAP+BICEP) data. The inclusion of BICEP data slightly improves the constraint by $\sim 14\%$. However, for the $c_s = 0.1$ and 0.01 cases, since the constraints on $r$ mainly come from constraints on $n_t$ by the low-$l$ WMAP data, the inclusion of BICEP data does not lead to significant improvement in the result.

The BICEP data can only affect $n_s$ through its correlation between $r$, so it does not have significant effect on the results of $n_s$.

### 4.2.2 The $d\sigma_s/d\ln k \neq 0$ case

We now discuss the results of constraints with running of spectral index $d\sigma_s/d\ln k$ as a free parameter.

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3The Two-Year BICEP data, which maps only $\sim 2\%$ of the sky, measures limited modes of perturbations, therefore does not contribute too much on the total constraining.
The fitting results are shown in the last three rows of Table 1 and Fig. 3. For the three models we find similar constraints on $dn_s/d\ln k$, thus $c_s$ does not have too much influence on $dn_s/d\ln k$.

The most striking effect of the inclusion of $dn_s/d\ln k$ is the significant amplification of the allowed region of $n_s$. In the middle panel of Fig. 3 we plot the likelihood functions of $n_s$ for $dn_s/d\ln k = 0$ (solid) and $dn_s/d\ln k \neq 0$ (dotted). We see that the width of $n_s$ likelihood is increased by nearly a factor of two. The inclusion of $dn_s/d\ln k$ also changes the central values of $n_s$ from below 1 to above 1. These phenomena are caused by the strong anti-correlation between $dn_s/d\ln k$ and $n_s$ (see the right panel of Fig. 3).

The inclusion of $dn_s/d\ln k$ slightly releases the upper bound constraints of $r$. For the $c_s = 1, 0.1, 0.01$ models, the 95.4% CL upper bounds on $r$ are 0.32, 0.26, 0.09 for $dn_s/d\ln k = 0$ and 0.36, 0.35, 0.10 for $dn_s/d\ln k \neq 0$.

5. Cosmological Constraints of Free $c_s$ Models

In this section we consider the more general case, i.e., treating $c_s$ as a free parameter. We not only use the WMAP and BICEP power spectrum data, but also take $f_{NL}^{\text{equil}}$ and $f_{NL}^{\text{orth}}$ into consideration in order to constrain $c_s$. A summary of the fitting results, including $c_s$, $r$, $n_t$, $n_s$ and $dn_s/d\ln k$ are given in Table 2. Notice that the word ‘UCON’ represents for ‘unconstrained’.

5.1 Results of fitting Without $f_{NL}$ Prior

In this subsection we discuss the results obtained by WMAP+BICEP data, without adding
Figure 4: Marginalized likelihoods of $c_s$, $r$, $dn_s/d\ln k$ and $n_s$ for the free $c_s$ case, obtained by using the WMAP and WMAP+BICEP data. Upper-left: The current CMB power spectrum data alone cannot constrain $c_s$. Upper-right: The inclusion of $dn_s/d\ln k$ slightly widens the distribution of $r$. Lower-left: The inclusion of BICEP data does not affect the distribution of $dn_s/d\ln k$ too much. Lower-right: The inclusion of $dn_s/d\ln k$ greatly broadens the width of $n_s$ likelihood, and shifts its central value to be greater than unity.

Table 2: Results of fitting with $c_s$ as a free parameter.

| Data & Model            | $c_s$   | $r$(95.4% CL) | $n_s$(95.4% CL) | $n_s$ | $dn_s/d\ln k$ |
|-------------------------|---------|---------------|----------------|-------|---------------|
| WMAP                    | UCON    | $< 0.36$      | $> -1.33$      | $0.966^{+0.024}_{-0.011}$ | $-0.020^{+0.023}_{-0.030}$ |
| WMAP + $dn_s/d\ln k$   | UCON    | $< 0.46$      | $> -1.44$      | $1.027^{+0.005}_{-0.031}$ | $-0.026^{+0.020}_{-0.027}$ |
| WMAP + BICEP + fNL     | UCON    | $< 0.32$      | $> -1.30$      | $0.967^{+0.012}_{-0.012}$ | $-0.026^{+0.020}_{-0.027}$ |
| WMAP + BICEP + fNL     | $0.016^{+0.021}_{-0.006}$ | $< 0.21$      | $> -0.91$      | $0.973^{+0.011}_{-0.016}$ | $-0.026^{+0.020}_{-0.027}$ |
|                         | $0.016^{+0.017}_{-0.003}$ | $< 0.29$      | $> -1.15$      | $1.016^{+0.064}_{-0.045}$ | $-0.024^{+0.022}_{-0.033}$ |

Let us first have a look at the constraint on $c_s$, and its likelihood is plotted in the upper-left panel of Fig. 4. The current CMB power spectrum data is not able to constrain on $c_s$, and the likelihood function shows that it can take any possible value given the current constraints.
Figure 5: Marginalized 68.3\% and 95.4\% CL contours in the $r - n_t$ (left) and $c_s - n_t$ (right) planes which are obtained by using the WMAP+BICEP data. In the left panel, the $n_t = -\frac{r}{8}$ line is plotted in the black dashed line. We see that $n_t > -2$ automatically satisfied, and the inclusion of $dn_s/d\ln k$ evidently amplifies the parameter space.

Secondly, the likelihoods of $r$ are shown in the upper-right panel of Fig. 4. We see the green and blue lines are close to each other, which means that the constraint on $r$ with free $c_s$ is similar to the result for $c_s=1$. \footnote{A similar conclusion was obtained in \cite{7}.} We get $r < 0.32$ (95.4\% CL) for both $c_s = 1$ and $c_s$ free models (WMAP+BICEP, $dn_s/d\ln k = 0$). The inclusion of BICEP data slightly improves constraint of $r$ from 0.36 to 0.32 ($dn_s/d\ln k = 0$). We see the inclusion of $dn_s/d\ln k$ as a free parameter boardens the width of distribution of $r$. At 95.4\% CL, the constraint is widened from $r < 0.32$ to $r < 0.41$ (WMAP+BICEP).

Thirdly, in the lower panels of Fig. 4 we plot the marginalized likelihoods of $dn_s/d\ln k$ (left) and $n_s$ (right). We find that the results are similar to the fixed $c_s$ models. It implies that the BICEP data almost does not affect the constraint of $dn_s/d\ln k$, and the inclusion of $dn_s/d\ln k$ greatly amplifies the distribution of $n_s$ and shifts its central value to above 1.

In addition, in Fig. 5 we plot the marginalized 2D-contours in the $r - n_t$ (left) and $c_s - n_t$ (right) planes. The slow-roll model with $n_t = -r/8$ is plotted in the black dashed line in the left panel, and the $dn_s/d\ln k = 0$ and $dn_s/d\ln k \neq 0$ cases are shown in orange dotted and black solid lines, respectively. By releasing $c_s$ as a free parameter, $n_t$ ranges to much smaller values ($\sim -1$ to -1.5 at 95.4\% CL), especially in the small $c_s$ region. We find $n_t > -2$ is still satisfied by the constraints. In both panels we see the inclusion of $dn_s/d\ln k$ boardens the ranges of the parameter space.

5.2 Results of fitting With $f_{NL}$ Prior

In this subsection let us take the $f_{NL}^{\text{equil.}}$ and $f_{NL}^{\text{orth.}}$ priors into account. The fitting results are shown in the last two rows of Table 2, and the likelihoods of $c_s$, $r$, $n_s$ and $dn_s/d\ln k$ are plotted in Fig. 6.
Figure 6: Marginalized likelihoods of $c_s$, $r$, $n_s$ and $dn_s/d\ln k$ for the $c_s$ free case from WMAP+BICEP data. The inclusion of $f_{NL}$ prior evidently tightens the constraints on $c_s$ and $r$, and slightly improves the upper-bound constraint on $n_s$. The running of the spectral index $dn_s/d\ln k$ remains unchanged.

Figure 7: Marginalized 68.3% and 95.4% CL contours in the $r-n_s$ (left) and $n_s-dn_s/d\ln k$ (right) planes, obtained by using the WMAP+BICEP data. In all figures we let $dn_s/d\ln k$ and $c_s$ as free parameters. The constraints with and without $f_{NL}$ cases are shown in dark cyan solid and orange dotted colors respectively.

The most striking effect is that the inclusion of $f_{NL}^{\text{equil}}$ and $f_{NL}^{\text{orth}}$ significantly improves the constraint on $c_s$. At 68.3% CL, we obtain $0.013 < c_s < 0.031$ and $0.013 < c_s < 0.033$
for the cases of without and with $dn_s/d\ln k$ as a free parameter. We find the large $c_s$ region, including the SFSR inflation with $c_s = 1$, is slightly disfavored at around 68.3\% CL, while $c_s \lesssim 0.01$ is excluded at 99.7\% CL. Thus, the bispectrum data is much more powerful than the power spectrum data for constraining $c_s$.

By narrowing the allowed range of $c_s$, the addition of $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ also has interesting effect on constraining the other parameters. The likelihood of $r$ is shown in the upper-right panel of Fig. 4. We see that, once $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ priors are considered, the constraint becomes much tighter. This can be also seen through the contours in the left panel of Fig. 4. Again, due to the correlation between $r$ and $n_s$, where $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ priors are included a slightly tighter constraint on the upper bound of $n_s$ is also obtained (see the lower-left panel of Fig. 4). But the effect of $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ priors on $dn_s/d\ln k$ is negligible (see the lower-right panels of Fig. 4).

Finally, the correlations between $r$, $n_s$ and $dn_s/d\ln k$ are shown in Fig. 4. The left panel shows the $r - n_s$ contours and the right panel shows the $n_s - dn_s/d\ln k$ contours. In all figures we set $c_s$, $dn_s/d\ln k$ as free parameters and use both WMAP and BICEP data. The cases of without and with $f_{NL}$ as a free parameter are shown in orange and dark cyan colors. One can see the strong correlation between $r$ and $n_s$, which suggests that if the distribution of $r$ is tighten, $n_s$ distribution is also constrained. However, the distribution of running spectral index $dn_s/d\ln k$, is not much affected by this correlation, because the change of $n_s$ is much smaller comparing with $r$.

6. Conclusion

In this paper we make a detailed investigation of the cosmological interpretation of the consistency relation from CMB data. We focus on the general single-field inflation model in which the spectral index $n_t$ of tensor perturbation power spectrum is related to the tensor-to-scalar ratio $r$ by $n_t = -r/(8c_s)$, and further investigate the effect of the sound speed $c_s$. The datasets used in this paper include the WMAP power spectrum data, the BICEP B-mode polarization data, and $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ priors obtained from the WMAP5 bispectrum data.

We discuss three models with fixed $c_s = 1$, 0.1 and 0.01. We find that when $c_s$ is small, the tilt of the tensor power spectrum $n_t$ becomes very large if $r$ is not too small, and then a tight constraint on $r$ is obtained for $c_s \ll 1$. Using the WMAP+BICEP data, we obtain the 95.4\% CL constraints of $r < 0.37$, 0.26, 0.09 for the $c_s = 1$, 0.1, 0.01 cases ($dn_s/d\ln k = 0$). Due to the positive correlation between $r$ and $n_s$, smaller values of $c_s$ lead to slightly tighter constraint on the upper bound of $n_s$. The effect of $c_s$ on the running of scalar spectral index $dn_s/d\ln k$ is not obvious. The inclusion of $dn_s/d\ln k$ significantly alters the constraints of $n_s$, and slightly amplifies the upper bound constraint of $r$.

For more general cases in which $c_s$ is taken as a free parameter, we find that $c_s$ is unconstrained if we only use the current CMB power spectrum data in the analysis, and the marginalized distribution of $r$, $n_s$ and $dn_s/d\ln k$ are all similar to the $c_s = 1$ case. However, after taking $f^\text{equil}_{NL}$ and $f^\text{orth}_{NL}$ priors into consideration, we find the sound speed $c_s$ is effectively constrained. The $c_s \lesssim 0.01$ region is ruled out, and the $c_s \gtrsim 0.03$ region
is disfavored at the 68.3% CL. From the constraints on $c_s$, the inclusion of $f_{\text{NL}}$ leads to tighter constraint on the $r$ and $n_s$. In the $dn_s/d\ln k = 0$ case, we find $r < 0.21/0.32$ (95.4% CL) with/without $f_{\text{NL}}$ prior ($dn_s/d\ln k = 0$), and the results for the $dn_s/d\ln k \neq 0$ case is $r < 0.29/0.41$. The running of spectral index $dn_s/d\ln k$ is almost unaffected.

To summarize, we find that the consistency relation has significant effect in the constraints on cosmological parameters $r$ and $n_s$ when $c_s$ is small, while the parameter $dn_s/d\ln k$ remains unaffected. Although the sound speed $c_s$ is unconstrained by the CMB power spectrum data, it can be effectively constrained by the CMB bispectrum data. Using the $f_{\text{NL}}^\text{equl}$ and $f_{\text{NL}}^\text{orth}$ priors obtained from the WMAP5 data, we find the SFSR model with $c_s=1$ is slightly disfavored at the 68.3% CL. Thus, we are expecting that, the on-going and upcoming CMB observations, such as Planck [18] and CMBPol [19], with much lower instrumental noise and better foreground clean, will provide stronger constraints on inflation models.

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