WEIGHTED K-STABILITY OF $\mathbb{Q}$-FANO SPHERICAL VARIETIES

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Abstract. Let $G$ be a connected, complex reductive Lie group and $X$ a $\mathbb{Q}$-Fano $G$-spherical variety. In this paper we compute the weighed non-Archimedean functionals of a $G$-equivariant normal test configurations of $X$ via combinatory data. Also we define a modified Futaki invariant with respect to the weight $g$, and give an expression in terms of intersection numbers. Finally we show the equivalence of different notations of stability and gives a stability criterion on $\mathbb{Q}$-Fano spherical varieties, which is also a criterion of existence of Kähler-Ricci $g$-solitons.

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1. Introduction

Let $X$ be an $n$-dimensional projective variety, $D$ an effective divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $\Theta \in 2\pi c_1(B)$ be a closed positive $(1,1)$-current for a $\mathbb{Q}$-line bundle $B$. Assume that $L = -(K_X + D) - B$ is ample. Suppose that $T_R \cong (S^1)^r$ be a real torus of rank $r$ whose complexification $T \cong (\mathbb{C}^*)^r$, which acts effectively and holomorphically on $X$ and preserves the divisor $D$. We further assume that $B$ is also $T$-linearized so that $L = -(K_X + D) - B$ is also $T$-linearized. If the $T$-action is further Hamiltonian, then for any Kähler form $\omega_0 \in 2\pi c_1(L)$, we have the moment map

$$\mathbf{m}_{\omega_0} : X \rightarrow \Delta \subset \mathfrak{t}^*.$$ 

The image $\Delta$ of the moment map is a convex polytope, which is in fact independent with the choice of $\omega_0$. Let $g$ be any smooth positive function defined on $\Delta$. For any $x \in X$, set $g_{\omega_0}(x) := g(\mathbf{m}_{\omega_0}(x))$. For any Kähler potential $\phi \in \mathcal{E}_{T_R}(\omega_0)$, the space of $T_R$-invariant Kähler potentials with finite energy (cf. [11, Definition 2.30]), one can define $\mathbf{m}_{\omega_0}(\cdot)$ and $g_{\omega_0}(\cdot)$ for $\omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi$ in a same way. Following [11], we say that a metric $\omega_\phi$ with $\phi \in \mathcal{E}_{T_R}(\omega_0)$ is a generalized Kähler-Ricci soliton (KR $g$-soliton) if

$$(1.1) \quad \text{Ric}(\omega_\phi) = \omega_\phi + [D] + \Theta + \sqrt{-1} \partial \bar{\partial} \ln g_{\omega_\phi}.$$ 

Note that the existence of a KR $g$-soliton implies that the pair $(X, D + \Theta)$ is klt.

Denote by $\text{Aut}(X, D)$ the automorphism group of $(X, D)$ and $\text{Aut}_{T_R}(X, D, \Theta)$ its connected subgroup which preserves $\Theta$ and commutes with $T_R$. It is proved in [11, Theorem 1.7] that:

- If $(X, D + \Theta, T_R)$ is $\mathfrak{g}$-uniformly $g$-Ding-stable over $(T \times \mathfrak{g})$-equivariant test configurations for a connected reductive group $\mathfrak{g}$, then $X$ admits a KR $g$-soliton;
- If $X$ admits a KR $g$-soliton and $\mathfrak{g}$ a connected reductive subgroup of $\text{Aut}_{T_R}(X, D, \Theta)$ that contains a maximal torus of $\text{Aut}_{T_R}(X, D, \Theta)$. Then $(X, D + \Theta, T_R)$ is $\mathfrak{g}$-uniformly $g$-Ding-stable over $(T \times \mathfrak{g})$-equivariant test configurations.

The $\mathfrak{g}$-uniformly $g$-Ding stability of $(X, D + \Theta, T_R)$ is defined in terms of $g$-weighted non-Archimedean Ding functional and $g$-weighted non-Archimedean J-functional (cf. [11, Section 5]). This stability implies properness of certain modified Ding functional (cf. [11, Section 6]) and the existence can be derived via variational methods. When $g = 1$ and $\Theta = D = 0$, [11] reduces to the usual Kähler-Einstein problem, and the $\mathfrak{g}$-uniformly $g$-Ding stability reduces to the usual $\mathfrak{g}$-uniformly Ding stability (cf. [3,15]). In the following we recall a non-trivial example. Assume that $\xi$ is a holomorphic vector field on $M$ which generates a rank $r$ torus $T$-action on $M$. Denote by $\xi_1, \ldots, \xi_r$ the generators of $T$. Then by a suitable choice of the generators, $\xi = \sum_{A=1}^r c_A \xi_A$ for constants $c_1, \ldots, c_r \in \mathbb{R}$. In fact, the soliton vector field $\xi$ can be uniquely determined by [23]. Recall the Kähler metric $\omega_\phi \in 2\pi c_1(L)$. Let $\theta_A(\omega_\phi)$ be the Hamiltonian of $\xi_A$ with respect to $\omega_0$,

$$\iota_{\xi_A} \omega_\phi = \sqrt{-1} \partial \bar{\partial} \theta_A(\omega_\phi), \ A = 1, \ldots, r.$$ 

Then

$$\mathbf{m}_{\omega_\phi}(x) = (\theta_1(\omega_\phi), \ldots, \theta_r(\omega_\phi)).$$
Take \( D = \Theta = 0 \) and
\[
g_\omega = e^{\theta_\omega} = e^{\sum_{A=1}^c A \theta_A(\omega)}.
\]
Then (1.1) reduces to the Kähler-Ricci soliton equation. \cite[Theorem 1.7]{11} then gives an existence criterion of the Kähler-Ricci solitons. Note that if we choose \( \{ \lambda_A := \theta_A(\omega) \}_{A=1}^c \) as the coordinates on \( \Delta \), then the corresponding \( g(y) = e^{\sum_{A=1}^c c A \lambda_A} \) is exponential of an affine function on \( \Delta \). The Mabuchi metric problem can also be treated in this framework (cf. \cite{19, 27}).

On the other hand, Wang-Zhou-Zhu \cite{26} introduced the modified Futaki invariant and modified K-stability for the Kähler-Ricci soliton problem. They defined the modified Futaki invariant of a test configuration via weighted total weights (cf. \cite{26} Sections 1–2), which generalized the modified Futaki invariant of vector fields defined in \cite{23}. Moreover, they showed that this invariant has an integration-expression on the central fibre for any special test configuration. As an application, they showed that any toric Fano variety is modified K-stable, which then implies that the modified K-energy is proper.

Motivated by the works cited above, in this paper we consider the general KR g-soliton problem when \( D = \Theta = 0 \) and \( g \) a general continuous function. As in \cite{26} we define g-modified Futaki invariants of a \( \mathbb{Q} \)-Fano variety (see Section 2.2 below). Concerning the existence of KR \( g \)-solitons with \( D = \Theta = 0 \) on \( \mathbb{Q} \)-Fano varieties with at most klt singularities, we are now having various notions of \( (g-) \)stability: The \( g \)-modified K-stability which is defined according to the sign of the \( (g\text{-modified}) \) Futaki invariant (see Definition 2.7 below) introduced in the sense of \cite{7, 26}. The \( g \)-K-stability which is defined according to the sign of the \( (g\text{-weighted}) \) non-Archimedean Mabuchi functional (see Definition 2.7 below), which is introduced in the sense of \cite{3}. The \( g \)-Ding-stability which is defined according to the sign of the \( (g\text{-weighted}) \) non-Archimedean Ding functionals, which was introduced in \cite{11}. We showed the existence of \( g \)-modified Futaki invariant when \( g \) is a non-negative polynomial and compare the above notions of \( (g\text{-}) \)stability. We will show their equivalence on \( \mathbb{Q} \)-Fano varieties:

**Theorem 1.1.** Let \( X \) be a \( \mathbb{Q} \)-Fano variety and \( T \subset \text{Aut}(X) \) be complex torus. Assume that \( -K_X \) is \( T \)-linearized through some fixed lifting. Let \( \Delta \) be the moment polytope with respect to this lifting and \( g \) a polynomial function on \( \Delta \). Then for any \( T \)-equivariant normal test configuration of \( (X, -K_X) \), the \( g \)-modified Futaki invariant exists. Furthermore, if \( g \geq 0 \), then \( X \) is \( g \)-modified K-polystable in the sense of Definition 2.7 if and only if it is \( T \)-equivariantly \( g \)-K-polystable in the sense of Definition 2.8.

The existence part will be proved in Theorem 2.4. In fact we prove an intersection formula of the \( g \)-modified Futaki invariant for polynomial \( g \). Comparing with the intersection formula of the non-Archimedean Mabuchi functionals, Theorem 1.1 is then a consequence of Proposition 2.9 below.

On the other hand, let \( G \) be a connect, complex reductive group and \( X \) a \( \mathbb{Q} \)-Fano \( G \)-spherical varieties. We will show in Section 5 the existence of \( g \)-modified Futaki invariant for any smooth \( g \). With the help of Proposition 5.1 and Corollary 5.2 below, we can significantly strengthen Theorem 1.1 to the following:

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1. In fact, this assumption is independent with the choice of a lifting. See Section 2.1.3 below.
Theorem 1.2. Let $X$ be a $\mathbb{Q}$-Fano $G$-spherical variety and $T \subset \text{Aut}^0_G(X)$ be a complex torus that commutes with $G$. Let $\Delta$ be the moment polytope of the $T$-action and $g$ an arbitrary smooth function on $\Delta$. Then for any $G$-equivariant normal test configuration of $(X, -K_X)$, the $g$-modified Futaki invariant exists. Moreover, $X$ is $G$-equivariantly $g$-modified $K$-polystable in the sense of Definition 2.7 if and only if it is $G$-equivariantly $g$-K-polystable in the sense of Definition 2.8.

The existence part will be showed in Proposition 5.1, and the equivalence part in Corollary 5.2. Finally we get the following stability/existence criterion for $\mathbb{Q}$-Fano spherical varieties, which is a generalization of [5, Theorem A]:

Theorem 1.3. Let $X$ be a $\mathbb{Q}$-Fano $G$-spherical variety. Set $\mathbb{G} := G \times \text{Aut}^0_G(X)$. Then the following are equivalent:

1. The $g$-weighted barycenter

\[ b_g(\Delta_+) := \frac{1}{V_g} \int_{\Delta_+} \lambda g \pi(\lambda) d\lambda \in \kappa_P + \text{RelInt}(-\mathcal{V}(G/H))^\vee; \] 

2. $X$ is $\mathbb{G}$-uniformly $g$-Ding stable;
3. $X$ is $\mathbb{G}$-uniformly $g$-K-stable;
4. $X$ is $\mathbb{G}$-equivariantly $g$-K-polystable.

Remark 1.4. By [11] (cf. [11, Theorems 1.6 and 6.3]), one can conclude that (1.2) holds if and only if $X$ admits a Kähler-Ricci $g$-soliton. Consequently, (1.2) implies that $X$ is $g$-Ding/K-polystable (regardless group actions).

Our method is to direct computing the non-Archimedean functionals by using the intersection formula in [25, Section 18], and the Futaki invariant using the asymptotic expression of the total weights. In particular, we get an inequality of the non-Archimedean Mabuchi functional and the Futaki invariant for non-negative, smooth, compactly supported weight $g$ (see Corollary 5.2 below).

The paper is organized as follows: Section 2 studies $g$-weighted non-Archimedean functionals, $g$-weighted Futaki invariant on general $\mathbb{Q}$-Fano varieties. We introduce various notations of $g$-modified/weighted stabilities. Especially in Section 2.1 we give change of lifting formulas for Archimedean functionals, which will play an important role in computations. In Section 2.2 we define the $g$-modified Futaki invariant. We also study the change of lifting formula of the $g$-modified Futaki invariants and give a formula of the $g$-modified Futaki invariant for polynomial $g$ via some intersection numbers. In Section 2.3 we prove Theorem 1.1. Sections 3-6 are devoted to the KR $g$-soliton problem on $\mathbb{Q}$-Fano $G$-spherical varieties. In Section 3 we recall preliminaries of spherical varieties. In particular we study the fibre product construction introduced in [11, Section 2.1] on polarized spherical varieties. In Section 4 we compute the $g$-weighted non-Archimedean functionals. In Section 5 we study the $g$-modified Futaki invariant. In Section 6 we prove the stability criterion Theorem 1.3. In the Appendix we collect useful Lemmas.

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2. The notations of stability

The various notations of stability of a polarized variety are usually stated in terms of test configurations. Let \((X, L)\) be a polarized variety. A normal test configuration of \((X, L)\) consists of the following data:

- A normal variety \(\mathcal{X}\) with a \(\mathbb{C}^*\)-action;
- An ample line bundle \(\mathcal{L}\) on \(\mathcal{X}\);
- A \(\mathbb{C}^*\)-equivariant flat morphism \(\tilde{\pi} : (\mathcal{X}, \mathcal{L}) \to \mathbb{C}\) so that the fibre \((\mathcal{X}_t, \mathcal{L}_t)\) over \(t \in \mathbb{C}^*\) is isomorphic to \((X, m_0L)\) for some \(m_0 \in \mathbb{N}_+\).

For our later use, in the following we compactify \((\mathcal{X}, \mathcal{L})\) to a family over \(\mathbb{P}^1\) by adding a trivial fibre \((X, m_0L)\) at \(\infty \in \mathbb{P}^1\). Alternatively, we glue \((\mathcal{X}, \mathcal{L})\) with \((X, m_0L) \times \mathbb{C} \cong (X, m_0L) \times (\mathbb{C} \cup \{\infty\})\) along the common part \((X, m_0L) \times \mathbb{C}^*\).

From now on, by \((\mathcal{X}, \mathcal{L})\) we always refer to the compactified family. Also, by resolution of singularity, we can assume that \((\mathcal{X}, \mathcal{L})\) is dominating, that is, there is a \(\mathbb{C}^*\)-equivariant birational morphism \(p : \mathcal{X} \to X \times \mathbb{P}^1\).

A test configuration \((\mathcal{X}, \mathcal{L})\) is called product if \((\mathcal{X}, \mathcal{L}) \cong (X, m_0L) \times \mathbb{P}^1\). Let \(\mathfrak{G}\) be a group acts on \((X, L)\). A test configuration \((\mathcal{X}, \mathcal{L})\) is called \(\mathfrak{G}\)-equivariant if \(\mathfrak{G}\) acts on \((\mathcal{X}, \mathcal{L})\) and \(\tilde{\pi}\) is \(\mathfrak{G}\)-invariant. That is, \(\mathfrak{G}\) acts on each fibre.

In the remaining part of Section 2.1, we always assume that \(X\) is \(\mathbb{Q}\)-Fano and take \(L = -K_X\).

2.1. The \(g\)-weighted non-Archimedean functionals.

2.1.1. Definition under the canonical lifting. Suppose that \(X\) is an \(n\)-dimensional \(\mathbb{Q}\)-Fano variety with an \(r\)-dimensional torus \(T\)-action. Then the anticanonical line bundle \(L = -K_X\) is automatically \(T\)-linearized with a canonical lifting of \(T\)-action on it. Let \((\mathcal{X}', L)\) be a normal test configuration of \((X, -K_X)\) with central fibre \(\mathcal{X}_0\). Clearly, there is an induced lifting of the \(T\)-action on \(\mathcal{X}\). Denote by \(V = (-K_X)^n\) the volume of \(X\). Under the canonical lifting of the \(T\)-action, the usual non-Archimedean functionals \(E^{NA}(\cdot), I^{NA}(\cdot), J^{NA}(\cdot), H^{NA}(\cdot), M^{NA}(\cdot), L^{NA}(\cdot)\), and \(D^{NA}(\cdot)\) are defined by (cf. [3]):

\[
E^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{(n + 1)V(L^{n+1})},
\]

\[
J^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{V} \mathcal{L} \cdot L_{\mathbb{P}^1} - E^{NA}(\mathcal{X}, \mathcal{L}),
\]

\[
L^{NA}(\mathcal{X}, \mathcal{L}) = \text{let}_{(\mathcal{X}, -K_X)}(\mathcal{X}_0) - 1;
\]

\[
D^{NA}(\mathcal{X}, \mathcal{L}) = L^{NA}(\mathcal{X}, \mathcal{L}) - E^{NA}(\mathcal{X}, \mathcal{L}),
\]

\[
M^{NA}(\mathcal{X}, \mathcal{L}) = \frac{1}{V n!} \mathcal{L}^n \cdot K^{log}_{X/\mathbb{P}^1} + nE^{NA}(\mathcal{X}, \mathcal{L}),
\]

where

\[
K^{log}_{X/\mathbb{P}^1} := K_X + \mathcal{X}_0\text{red} - \tilde{\pi}^*\{\infty\},
\]

is understood as a Weil divisor. The Weil divisor \(K_X\) can be realized as following: Let \(\mathcal{X}_{\text{reg}}\) be the regular locus of \(\mathcal{X}\). Since \(\mathcal{X}\) is normal, the singular locus \(\mathcal{X} \setminus \mathcal{X}_{\text{reg}}\) as codimension at least 2. On \(\mathcal{X}_{\text{reg}}\) there is a section \(\delta\) of \(K_X|_{\mathcal{X}_{\text{reg}}}\) which defines a divisor \(\delta_0\) in \(\mathcal{X}_{\text{reg}}\) and \(K_X\) in (2.1) is the closure of \(\delta_0\) in \(\mathcal{X}\).
Then we recall the definition of $g$-weighted non-Archimedean functionals under this canonical lifting. This was first formulated in [11]. The functionals are defined by $g$-weighted intersection numbers (cf. [11 Sections 5, 10]).

**Step 1.** $g$ is a monomial. In this case, the $g$-weighted intersection numbers are defined as intersection numbers of line bundles over a fibre product variety $(X^{[k]}, L^{[k]})$. Let us recall this construction of [11, pp.7-8]. Let $X$ be a $\mathbb{Q}$-Fano variety which admits a rank $r$ torus $T$-action. Also assume that $T$ acts on $-K_X$ is $T$-linearized. Suppose that $\{\xi_A\}_{A=1}^r \subset \mathfrak{t}$ is a set of generators of $T$. Then $\{\xi_A\}_{A=1}^r$ is a basis of $\mathfrak{t}^*$. Let $\{\theta_A\}_{A=1}^r$ be the coordinates of $\mathfrak{t}^*$ under this basis. Suppose that

$$g(\theta_1, ..., \theta_r) = \prod_{A=1}^r \theta_A^{k_A} \quad (= \theta^k),$$

where $k = (k_1, ..., k_r) \in \mathbb{N}_r$. Denote $\mathbb{C}^{[k]+1} = \prod_{A=1}^r \mathbb{C}^{k_A+1}$. Consider the action of a torus $T \cong T$ on $X \times \mathbb{C}^{[k]+1}$,

$$\vartheta(x; z^{(A)}, \xi_A) := (\iota(\vartheta)x; \vartheta_A z^{(A)}, \xi_A), \vartheta \in T$$

where $\iota : \mathbb{T} \to T$ is the isomorphism between $\mathbb{T}$ and $T$. Let $\{z^{(A)}, \xi_A\}_{(A, \xi_A)=0}$ are the coordinates on $\mathbb{C}^{k_A+1}$, and we write $(x; z^{(A)}, \xi_A)$ in short of $(x; z^{(1)}, \xi_0, ..., z^{(k_1)}, \xi_1, ..., z^{(r)}, \xi_0, ..., z^{(r)}, \xi_r)$. Take $L = -K_X$, define

$$(X^{[k]}, L^{[k]}) := (X, L) \times (\mathbb{C}^{k_A+1} \setminus \{O\}) / \mathbb{T}.$$  

Then $X^{[k]}$, $L^{[k]}$ is a bundle over $\mathbb{P}^k = \mathbb{P}^{k_1} \times ... \times \mathbb{P}^{k_r}$,

$$\varpi^{[k]} : X^{[k]} \to \mathbb{P}^{[k]}.$$ 

where is the projection. Moreover, each fibre of $(X^{[k]}, L^{[k]})$ is isomorphic to $(X, L)$.

Set

$$V_g := \int_X g^n \omega^n / n!,$$

$k! := k_1!...k_r!$, and $|k| := k_1 + ... + k_r$. Also, for any $T$-equivariant normal test configuration $(\mathfrak{X}, \mathcal{L})$, define\footnote{Our convention differs from [11] by a factor $\frac{|k|!}{n!}$. Also, we normalized the Fubini-Study metric on the $m$-dimensional projective space as $\int_{\mathbb{P}^m} \omega^n = 1$ instead of $n!$.}

\begin{align}
(2.3) \quad & E_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = \frac{k!}{(n + |k| + 1) V_g} (L^{[k]})^{n+|k|+1}, \\
(2.4) \quad & I_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n} L \cdot L_{p}^{n} - \frac{k!}{(n + |k|)!} (L - \rho^* L_{p})^{[k]} (L^{[k]})^{n+|k|}, \\
(2.5) \quad & J_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = \frac{1}{L^n} L \cdot L_{p}^{n} - E_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}), \\
(2.6) \quad & H_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = \frac{k!}{V_g(n + |k|)!} (L^{[k]})^{n+k} (K_{X \times \mathbb{P}^1}^{\log} |k| - (\rho^* \cdot (K_{X \times \mathbb{P}^1}^{\log}) |k|), \\
(2.7) \quad & D_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = L_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) - E_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}), \\
(2.8) \quad & M_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) = H_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) - I_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L}) + J_{g}^{\mathfrak{N}A}(\mathcal{X}, \mathcal{L})
\end{align}
Here the Weil divisor \((\log K_X)_{\|g\|}\) is defined as following: Recall the section \(\tilde{\delta}\) of \(K_X\)|\(_{\tilde{\mathcal{X}}_{\|g\|}}\). Suppose that it has \(T\)-weight \(\tilde{\lambda}_0\) under the canonical lifting of the \(T\)-action. Let \(\delta_0\) be the closure of \(\tilde{\delta}\) in \(\mathcal{X}\). Then it is a \(T\)-invariant Weil divisor of \(\mathcal{X}\) and \(\delta_0|_T := \delta_0 \times (\mathbb{C}^{k+1} \setminus \{\lambda\}) / T\) is a Weil divisor of \(\mathcal{X}^{|k|}\). Then we define

\[
K^\log_{\mathcal{X}/\mathbb{P}^1} := \delta_0|_T + \sum_{A=1}^r \text{pr}_A^*O_{\mathcal{X}}(\lambda_{0A}) + \mathcal{X}^{|k|}_{\tilde{\mathcal{X}}_{\|g\|}} - (\tilde{\pi}^*\{\lambda\})|_T \]  

(2.9)

Note that since \(g(\lambda) = \prod_{A=1}^r \lambda_A^{|k|}\) is a monomial of degree \(|k|\), it holds

\[
g(\lambda) = \frac{1}{(n + |k| + 1)!} (L|_T)^{n + |k| + 1} = V_{g(\lambda)} E^{|\mathcal{X}}_{\mathcal{X}}(L),
\]

where \(y(\nabla g(\lambda)) = \sum_{A=1}^r \lambda_A \frac{\partial}{\partial \lambda_A} g(\lambda)\).

**Step-2.** \(g\) is a polynomial. Let

\[
g = \sum_{k} a_k \xi^k, \quad a_k \in \mathbb{C}
\]

be a polynomial. Also, for \(F \in \{E, J, M\}\), define

\[
F^{|\mathcal{X}}_{g}(\mathcal{X}, L) := \frac{1}{V_g} \sum_{k} a_k V_{g_k} \cdot F^{|\mathcal{X}}_{g_k}(\mathcal{X}, L).
\]

**Step-3.** \(g\) is a continuous function. Let \(g\) be a general \(C^0\)-function on \(\Delta\). Then there is a sequence of polynomials \(\{g_k\}_{k=1}^{+\infty}\) so that \(g_k\) converges to \(g\) uniformly on \(\Delta\). Define

\[
V_g = \lim_{k \to +\infty} V_{g_k},
\]

\[
F^{|\mathcal{X}}_{g}(\mathcal{X}, L) = \lim_{k \to +\infty} F^{|\mathcal{X}}_{g_k}(\mathcal{X}, L), \quad F \in \{E, I, J, H, D, M\}.
\]

It is proved by [11] Sections 5 and 10] that none of the above limits depends on the choice of \(\{g_k\}_{k=1}^{+\infty}\). Hence they are well-defined.

An important property proved in [11] is that the non-Archimedean functionals defined above satisfies the slope formula, which means that they are the slope of the corresponding Archimedean functionals at infinity (cf. [11] Propositions 5.8 and 10.8).

2.1.2. **Change of the lifting.** For our later use, we will consider the expression of the non-Archimedean functionals \(F^{|\mathcal{X}}_{g}(\cdot, L)\) under an arbitrary lifting of the \(T\)-action on \(L = -K_X\). Given a \(\mathbb{Q}\)-Fano variety \(X\) and a torus \(T \subset \text{Aut}(X)\). Let us fix a lifting \(\sigma\) of the \(T\)-action on \(L\). Denote by \(\Delta \subset \mathfrak{t}^*\) the corresponding polytope and choose a coordinate \(y_1, ..., y_n\) on it. For a \(T\)-equivariant normal test configuration \((\mathcal{X}, L)\) of \((X, L)\), we will denote by \((\mathcal{X}, L^\sigma)\) to emphasize the lifting \(\sigma\). Also we omit \(\sigma\) when we refer to the canonical lifting. In general, \(L^{|k|}\) and \(L^{|k|}\) (\(L^{|k|}\) and \(L^{|k|}\), resp.) are different line bundles on \(X^{|k|}\) (\(X^{|k|}\), resp.).

For a test configuration \((\mathcal{X}, L)\), by resolution of singularity, we can assume that it is dominating. That is, there is a \(\mathbb{C}^*\)-equivariant birational morphism \(\rho : \mathcal{X} \to X \times \mathbb{P}^1\). Then \(L = \rho^*L + D\) for some \(\mathbb{Q}\)-Cartier divisor \(D\), and \((\mathcal{X}, L)\) induces a canonical non-Archimedean metric on \(L^{|N|}\). The non-Archimedean functionals can be derived from the corresponding Archimedean ones by taking slope at infinity.

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3We would like to thank Professor Chi Li for pointing us this relation.
From the construction Steps-2, 3 in Section 2.1, it suffices to compute $F^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma)$ when $g$ is a monomial given by (2.2).

**Proposition 2.1.** Suppose that $g$ is a monomial given by (2.2). Then for a general lifting $\sigma$ of the $T$-action on $L-K_X$, it holds

\begin{equation}
E_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) = \frac{k!}{(n+|k|+1)!} (\mathcal{L}^\sigma[k])^{n+|k|+1},
\end{equation}

\begin{equation}
J_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) = \frac{k!}{(n+|k|)!} \mathcal{L}^\sigma[k] (-\rho[k] K^{\sigma[k]}_{X \times P^1})^{n+|k|} - E_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma),
\end{equation}

\begin{equation}
I_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) = \frac{k!}{(n+|k|)!} \mathcal{L}^\sigma[k] (-\rho[k] K^{\sigma[k]}_{X \times P^1})^{n+|k|} - \frac{k!}{(n+|k|)!} (\mathcal{L}^\sigma[k] + \rho[k] K^{\sigma[k]}_{X \times P^1})(\mathcal{L}^\sigma[k])^{n+|k|},
\end{equation}

\begin{equation}
H_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) = \frac{k!}{(n+|k|)!} (K^{\log[k]}_{\mathcal{X}/P^1} - \rho[k] K^{\sigma[k]}_{X \times P^1})(\mathcal{L}^\sigma[k])^{n+|k|},
\end{equation}

\begin{equation}
M_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) = H_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) - I_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma) + I_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma)
\end{equation}

\begin{equation}
= \frac{k!}{(n+|k|)!} (K^{\log[k]}_{\mathcal{X}/P^1})(\mathcal{L}^\sigma[k])^{n+|k|} + \frac{k!}{(n+|k|)!} (\rho[k] K^{\sigma[k]}_{X \times P^1})(\mathcal{L}^\sigma[k])^{n+|k|}
\end{equation}

\begin{equation}
- \frac{k!}{(n+|k|)!} (\mathcal{L}^\sigma[k] + \rho[k] K^{\sigma[k]}_{X \times P^1})(\mathcal{L}^\sigma[k])^{n+|k|} + (n+|k|)E_g^{\text{NA}}(\mathcal{X}, \mathcal{L}^\sigma).
\end{equation}

**Proof.** Following [11] Section 2.1, we have for any $\omega_0$-psh functions $\phi_0, \phi = \phi_1$ with finite Monge-Ampère energy and path $\{\phi_t\}_{t \in [0,1]}$ joining them, it holds

\begin{equation}
E_g(\phi) = \frac{1}{V_g} \int_X \phi g(m_\omega_{\phi_0}) \frac{\omega_{\phi_0}^n}{n!}
\end{equation}

\begin{equation}
= \frac{k!}{V_g} \int_X \phi_0 \frac{\omega_{\phi_0}^n}{n!} (\omega_{\phi_0} + \sum_{A=1}^r \omega_{FS,A})^{n+|k|}
\end{equation}

\begin{equation}
= \frac{k!}{(n+|k|+1)!V_g} \sum_{l=1}^{n+|k|} \int_X (\phi[k] - \phi_0[k]) (\omega_{\phi_0} + \sum_{A=1}^r \omega_{FS,A})^{l} \wedge
\end{equation}

\begin{equation}
\wedge (\omega_{\phi} + \sum_{A=1}^r \omega_{FS,A})^{n+|k|-l},
\end{equation}

where $\omega_{FS}$ denotes the Fubini-Study metric on $P^m$, and $\omega_{FS,A}$ denotes $\omega_{FS}$ on $P^{k_A}$. Note that $\mathcal{L}^\sigma[k]$ is a metric on $L^\sigma[k]$. Taking slope at infinity in the above relation we get (2.12).

For (2.13), we have

\begin{equation}
J_g(\phi) = \frac{1}{V_g} \int_X (\phi - \phi_0) g(m_\omega_{\phi_0}) \frac{\omega_{\phi_0}^n}{n!} - E_g(\phi)
\end{equation}

\begin{equation}
= \frac{k!}{V_g} \int_X (\phi[k] - \phi_0[k]) \frac{\omega_{\phi_0}^n}{(n+|k|)!} (\omega_{\phi_0} + \sum_{A=1}^r \omega_{FS,A})^{n+|k|} - E_g(\phi).
\end{equation}
Thus
\[ J_{g}^{N_{g}}(X', L_{\sigma}^{\ast}) = \frac{k!}{(n + |k|)!} \left( \mathcal{L}^{n} + \rho_{k}K_{X_{\mathbb{P}^{1}/\mathbb{P}^{1}}}(- \rho_{k}K_{X_{\mathbb{P}^{1}/\mathbb{P}^{1}}})^{n+|k|} \right) \]
(2.17)
and (2.13) follows from the fact that
\[ (- \rho_{k}K_{X_{\mathbb{P}^{1}/\mathbb{P}^{1}}})^{n+|k|+1} = 0. \]
Similarly, we get (2.14).

Finally we prove (2.15). Rewrite \( \omega_{0} = \sqrt{-1} \bar{\partial} \partial \phi_{0} \) so that it is the curvature of the Hermitian metric \( e^{-\phi_{0}} \) on \( -K_{X} \). Then \( d\mu_{0} = e^{-\phi_{0}} \) is a globally defined measure on \( X \). Recall [11, Section 10.1],
\[
H_{g}(\phi) = \frac{1}{V_{g}} \int_{X} \ln \left( \frac{g^{(m_{\omega_{0}})}k_{0}^{n}}{n!d\mu_{0}} \right) g(m_{\omega_{0}}) \frac{\omega_{0}^{n}}{n!} \]
(2.18)
\[
= \frac{k!}{V_{g}} \int_{X[k]} \ln \left( \frac{\omega_{0}^{n}}{n!d\mu_{0}} \right) \frac{\omega_{0} + \sum_{A=1}^{r} \omega_{FS;A}^{n+|k|}}{(n + |k|)!} \]
\[
+ \frac{k!}{V_{g}} \int_{X[k]} \ln g(m_{\omega_{0}}) \frac{(\omega_{0} + \sum_{A=1}^{r} \omega_{FS;A}^{n+|k|})}{(n + |k|)!}.
\]

On the other hand, as showed in [11, Section 10.1],
\[
\int_{X[k]} \ln \left( \frac{\omega_{0}^{n}}{n!d\mu_{0}} \right) \frac{\omega_{0} + \sum_{A=1}^{r} \omega_{FS;A}^{n+|k|}}{(n + |k|)!}
\]
\[
= \langle \ln(\omega_{0}^{n}) \rangle^{[k]}_{X[k]} \phi^{[k]}_{X[k]} - \langle \ln(e^{-\phi_{0}}) \rangle^{[k]}_{X[k]} \phi^{[k]}_{X[k]},
\]
where \( \langle \ldots \rangle_{X[k]} \) denotes the metric on the Deligne pair. Note that \( \ln(\omega_{0}^{n}) \rangle^{[k]} \) and \( \ln(e^{-\phi_{0}}) \rangle^{[k]} \) are always metrics on line bundles with respective to the canonical lifting. We get (2.15). The relation (2.16) then follows from (2.12)-(2.15). \( \square \)

Clearly, the second term in (2.16) vanishes if \( \sigma \) is the canonical lifting.

2.1.3. *Invariance of the non-Archimedean functionals.* Suppose that there are two different liftings \( \sigma_{1}, \sigma_{2} \) of the \( T \)-action on \( -K_{X} \) so that
\[
\sigma_{2} = \sigma_{1} + \chi
\]
(2.18)
for some \( T \)-character \( \chi \). Then the corresponding moment maps
\[
m_{2} = m_{1} + \chi.
\]
Clearly, the corresponding moment polytopes \( \Delta_{2} = \Delta_{1} + \chi \).

Consider the equation
\[
g^{(1)}_{\omega_{0}} \omega^{n}_{\phi} = n! e^{-\phi_{0}} \omega_{0}^{n},
\]
(2.19)
where \( g^{(i)}_{\omega_{0}} := g^{(i)} \circ m_{1} \) with \( g^{(i)} : \Delta_{1} \to \mathbb{R} \). The equation (2.19) can be changed into
\[
g^{(2)}_{\omega_{0}} \omega^{n}_{\phi} = n! e^{-\phi_{0}} \omega_{0}^{n},
\]
(2.20)
where \( g^{(2)}_{\omega_{0}} := g^{(2)} \circ m_{1} \) with
\[
g^{(2)} = g^{(1)}(m_{1} - \chi) : \Delta_{2} \to \mathbb{R}.
\]
(2.21)
The equations (2.19), (2.20) are associated to the weighted non-Archimedean Mabuchi functionals $M^{|\mathcal{O}|}_{g^{(1)}}(\cdot)$ and $M^{|\mathcal{O}|}_{g^{(1)}}(\cdot)$, respectively.

Suppose that $(\mathcal{X}, \mathcal{L})$ is a $T$-equivariant normal test configuration of $(\mathcal{X}, -K_{\mathcal{X}})$. We denote by $(\mathcal{X}, \mathcal{L}^\sigma_i)$, $i = 1, 2$ for $(\mathcal{X}, \mathcal{L})$ with the $T$-action via $\sigma_i$, $i = 1, 2$, respectively. In the following, we will show the invariance of the non-Archimedean functionals:

**Proposition 2.2.** Suppose that $g^{(1)} \in C^\infty(\Delta_1)$. Then

\[
F^{|\mathcal{O}|}_{g^{(1)}}(\mathcal{X}, \mathcal{L}^\sigma_1) = F^{|\mathcal{O}|}_{g^{(2)}}(\mathcal{X}, \mathcal{L}^\sigma_2), \quad F \in \{E, I, J, H, M, D\}.
\]

**Proof.** As in the previous section, it suffices to prove the Proposition when $g^{(1)}$ is given by (2.2). For convenience, we may choose $\sigma_1$ to be the canonical lifting.

We first show the Proposition for $F = E$. Denote by $k = (k_1, ..., k_r)$. For convenience, for another $i = (i_1, ..., i_r)$, we write $i \leq k$ if each $i_A \leq k_A$. Also, set $0 = (0, ..., 0)$. By (2.18),

(2.22) \hspace{2cm} (\mathcal{L}^\sigma_1)^{|k|} = (\mathcal{L}^\sigma_2)^{|k|} - \sum_{A=1}^r \text{pr}_A^* \mathcal{O}_{p^k_A}(\chi_A),

where $\text{pr}_A : \mathcal{X}^{|k|} \to \mathbb{P}^{|k|}$ is the projection to the $A$-th factor in $\mathbb{P}^{|k|}$. Thus

\[
((\mathcal{L}^\sigma_1)^{|k|})^{n+|k|+1} = ((\mathcal{L}^\sigma_2)^{|k|} - \sum_{A=1}^r \text{pr}_A^* \mathcal{O}_{p^k_A}(\chi_A))^{|k|}) = \sum_{0 \leq i \leq k} \frac{(n+|k|+1)!}{(n+|k|+1-|i|)!}(-1)^{|i|}(\mathcal{L}^\sigma_2)^{|k|}^{n+|k|+1-|i|} \times \prod_{A=1}^r (\text{pr}_A^* \mathcal{O}_{p^k_A}(\chi_A))^{i_A}.
\]

Note that

(2.23) \hspace{2cm} (\mathcal{L}^\sigma_2)^{|k|}^{n+|k|+1-|i|} \prod_{A=1}^r (\text{pr}_A^* \mathcal{O}_{p^k_A}(1))^{i_A} = (\mathcal{L}^\sigma_2)^{|k-i|}^{n+|k|+1-|i|}.

We get

\[
\prod_{A=1}^r \text{pr}_A^* \mathcal{O}_{p^k_A}(1) = \prod_{A=1}^r C^{|i|}_{k_A}^{i_A}.
\]

Here we write $C^{|i|}_{k_A}^{i_A} = \prod_{A=1}^r C^{i_A}_{k_A}$ for short.

On the other hand,

(2.24) \hspace{2cm} g^{(2)}(\lambda) = g^{(1)}(\lambda - \chi) = \prod_{A=1}^r (\lambda_A - \chi_A)^{k_A} = \sum_{0 \leq i \leq k} C^{|i|}_{k_A}^{i_A}(-1)^{|i|} \chi^i \lambda^{k-i}.

Combining with (2.3), we see that

\[
F^{|\mathcal{O}|}_{g^{(2)}}(\mathcal{X}, \mathcal{L}^\sigma_2) = E^{|\mathcal{O}|}_{g^{(1)}}(\mathcal{X}, \mathcal{L}^\sigma_1).
\]

The case $F = D$ can be checked by using the above relation and (2.18).

For the case $F = J$, recall the fact

(2.25) \hspace{2cm} \rho^{[k]}_A K^{\mathcal{L}^\sigma_1}_{\mathcal{X}_{\mathcal{O}_{p^1}}(\mathcal{Y})} = \rho^{[k]}_A K^{\mathcal{L}^\sigma_2}_{\mathcal{X}_{\mathcal{O}_{p^1}}(\mathcal{Y})} + \sum_{A=1}^r \text{pr}_A^* \mathcal{O}_{p^k_A}(\chi_A).
Combining with (2.22),

\[ (L^{a_1}|k| + \rho|k|*K^{\sigma_1}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}}) = (L^{a_2}|k| + \rho|k|*K^{\sigma_2}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}}). \]

By (2.17) we proved the Proposition for F = J. The case F = I can be showed in a same way.

Finally we consider the case F = M. Recall that we assume \( \sigma_1 \) is the canonical lifting. As in the case F = E, by (2.22), (2.23), we can show that

\[
k! (L^{a_1}|k|)^{n+|k|} (K^{\log}_{X\times\mathbb{P}^{1}}|k|) = ((L^{a_2}|k| - \sum_{A=1}^{r} \text{pr}_A^* O_{\mathbb{P}^A} (\chi_A))^{n+|k|} \cdot (K^{\log}_{X\times\mathbb{P}^{1}}|k|)
\]

(2.26) = \sum_{0 \leq i \leq k} \frac{(n + |k|)!}{(n + |k| - i)!} C^{k}_{k} (-\chi)^i (k - i)! (K^{\log}_{X\times\mathbb{P}^{1}})^{k-i} ((L^{a_2}|k|)^{k-i})^{(n+|k|-i)}.

Similarly,

\[
|k| E_{\gamma_1}^{\text{NA}}(X, L^{a_1}) = \sum_{0 \leq i \leq k} (-\chi)^i |k - i| C^{k}_{k} E_{\gamma_1}^{\text{NA}}(X, L^{a_2}) + \sum_{0 \leq i \leq k} (-\chi)^i |k - i| C^{k}_{k} E_{\gamma_1}^{\text{NA}}(X, L^{a_2})\]

(2.27) = \sum_{0 \leq i \leq k} (-\chi)^i |k - i| C^{k}_{k} E_{\gamma_1}^{\text{NA}}(X, L^{a_2}) = E_{\gamma_1}^{\text{NA}}(\chi(\nabla(\lambda - \chi)^k)(X, L^{a_2}),

where

\[
\chi(\nabla(\lambda - \chi)^k) = \sum_{A=1}^{r} \chi_A \frac{\partial}{\partial \lambda_A} (\lambda - \chi)^k.
\]

On the other hand, by (2.25),

\[
\frac{j!}{(n + |j|)!} (\rho|j|*K^{\sigma_2}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}} - \rho|j|*K^{\sigma_1}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}})(L^{a_2}|j|)^{n+|j|}
\]

= \frac{j!}{(n + |j|)!} \left( \sum_{A=1}^{r} \text{pr}_A^* O_{\mathbb{P}^A} (\chi_A)) (L^{\sigma_1}|j|)^{n+|j|}, \right.

= \frac{1}{(n + |j|)!} \sum_{A=1}^{r} \chi_A \frac{j!}{j_A^!} (L^{a_2}|j_A|)^{n+|j_A|+1},

= - \frac{\sum_{A=1}^{r} \chi_A j_A E_{\gamma_A}^{\text{NA}}(X, L^{a_2}) = -E_{\gamma_A}^{\text{NA}}(\sum_{A=1}^{r} \chi_A \frac{\partial}{\partial \lambda_A} (X, L^{a_2}),

where used the fact |j| = |j_A| + 1. Thus,

\[
\sum_{j \leq k} C^{k}_{k} (-\chi)^j \frac{j!}{(n + |j|)!} (\rho|j|*K^{\sigma_2}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}} - \rho|j|*K^{\sigma_1}_{X\times\mathbb{P}^{1}/\mathbb{P}^{1}})(L^{a_2}|j|)^{n+|j|} = -E_{\gamma_1}^{\text{NA}}(\chi(\nabla(\lambda - \chi)^k)(X, L^{a_2}),
\]
Plugging this equation into (2.27), we have

\[
|k|E_{g^1}^{NA}(X, \mathcal{L}^\sigma) = \sum_{0 \leq i \leq k_0} C_i \left( -\chi + \frac{1}{n+1} \right) (\rho[j] \cdot \frac{1}{n+1} - \rho[j] \cdot \frac{1}{n+1} \cdot (\mathcal{L}^\sigma)^{n+1} \cdot j)
+ \sum_{0 \leq i \leq k} (-\chi) |k| i^2 C_i E_{k-1}^{\mathcal{L}}(X, \mathcal{L}^\sigma).
\]

Combining with (2.26) and (2.24) we get the Proposition for F = M.

\[\square\]

2.2. The $g$-modified Futaki invariant. There is also a geometric way to construct test configuration of a $\mathbb{Q}$-Fano variety $X$ (cf. [12 Section 2.2] and [26 Section 1]). Suppose that there is a Kodaira embedding of $X$ by $-n_0K_X$ for some $n_0 \in \mathbb{N}_+$.

\[i : X \to \mathbb{P}(H^0(X, -n_0K_X)) =: \mathbb{P}^{N+1}.
\]

Choose a vector $\Lambda \in \mathfrak{psl}_N(\mathbb{C})$ so that $\Lambda$ generates a rank 1 torus of $\text{PSL}_N(\mathbb{C})$. Then it defines a test configuration $(X, \mathcal{L})$ via

\[\mathcal{X}_t := \exp(z\Lambda) \cdot i(X), \ t = e^z \in \mathbb{C}^*\]

and

\[\mathcal{L}|_{\mathcal{X}_t} := O_{\mathbb{P}^{N+1}}(1)|_{\mathcal{X}_t}.
\]

Also define $\mathcal{X}_0 := \lim_{t \to 0} \mathcal{X}_t$ as the limit of algebraic cycle, and $\mathcal{L}_0 := \mathcal{L}_{\mathcal{X}_0}$. Indeed, any test configuration can be realized in this way.

Without loss of generality we assume that $n_0 = 1$. Otherwise, we replace $-K_X$ by $-n_0K_X$. Suppose that $\Lambda$ is a (real) holomorphic vector field on $X$ so that $T = \exp(t\xi)$. We also assume that $\Lambda$ commutes with $\xi$ so that the test configuration $(X, \mathcal{L})$ is $T$-equivariant. We can also lift $\xi$ to an element of $\mathfrak{psl}_N(\mathbb{C})$.

Choose a basis $\{e_p\}_{p=1}^{h^0(X, -K_X)}$ of $H^0(X_0, -\mathcal{L}_0)$ so that each $e_p$ is a common eigenvector of both the exp($t\xi$)- and exp($t\lambda$)-actions. Denote by $\{e_p\}_{p=1}^{h^0(X, -K_X)}$ and $\{e_p\}_{p=1}^{h^0(X, -K\mathcal{L}_0)}$ the eigenvalues of the canonical lifting of the exp($t\xi$)- and exp($t\lambda$)-actions on $H^0(X_0, -\mathcal{L}_0)$, respectively. Here we use the fact that $h^0(X, -K\mathcal{L}_0) = h^0(X_0, -\mathcal{L}_0)$. Fix a background Kähler metric $\omega_0 \in 2\piC_1(X)$ and denote by $\theta_\xi$ a potential of $\xi$ with respect to $\omega_0$. Suppose that $T$ has generators $\{\xi_\Lambda\}_{\Lambda=1}^{r}$. As in [26 Section 1], for a $C^1$-function $g$ defined on some interval and $g_{\omega_0} = g(\theta_\xi, \ldots, \theta_\xi)$, define

\[
S_{1/k}^{(g)}(X, \mathcal{L}) := \sum_{p=1}^{h^0(X, -kK_X)} g\left(\frac{\xi_1}{k}, \ldots, \frac{\xi_r}{k}\right) \Lambda^k_p,
\]

\[
S_{2/k}^{(g)}(X, \mathcal{L}) := \frac{1}{2} \sum_{p=1}^{h^0(X, -kK_X)} \sum_{\Lambda=1}^{r} \frac{\partial g\left(\frac{\xi_1}{k}, \ldots, \frac{\xi_r}{k}\right)}{\partial \theta_{\xi_\Lambda}} \left(\frac{\xi_1}{k}, \ldots, \frac{\xi_r}{k}\right) \Lambda^k_p.
\]

Once it holds the formal asymptotic expression,

\[
\frac{S_{2/k}^{(g)} - S_{1/k}^{(g)}}{kh^0(X, -kK_X)}(X, \mathcal{L}) =: F_0(X, \mathcal{L}) + F_1(X, \mathcal{L})k^{-1} + O(k^{-2}), \ k \to +\infty,
\]

we can define the $g$-modified Futaki invariant in a similar way of [26 Section 1],
**Definition 2.3.** Suppose that (2.30) holds. Then the $g$-modified Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is defined as

$$\text{Fut}_g(\mathcal{X}, \mathcal{L}) := F_1(\mathcal{X}, \mathcal{L}).$$

It was showed by [25, Section 1] that (2.30) holds for $g = e^{\theta t}$ on a Fano manifold. That is, the modified Futaki invariant in Definition 2.3 for Kähler-Ricci solitons is well-defined. At the beginning of Section 5, we will show that (2.30) holds for an equivariant test configuration of a polarized spherical variety with $g \in C^\infty(\Delta)$.

Hence the $g$-modified Futaki invariant is defined. Indeed, we can show the $g$-modified Futaki invariant of an equivariant test configuration can be expressed as intersection numbers of some line bundle (see Lemma 6.2 below).

### 2.2.2. Change of lifting

As in Section 2.1.2, we consider the expression of the Futaki invariant under a general lifting of the $T$-action on $-K_X$. Denote by $\sigma_0$ the canonical lifting and $\sigma$ so that there is a $T$-character $\chi$ such that

\[
\sigma = \sigma_0 + \chi
\]

Then the corresponding weights

\[
\xi_{\Lambda_p}^k(\sigma) = \xi_{\Lambda_p}^k(\sigma_0) + \chi(\xi_{\Lambda_p}^k), \quad \forall p \in \mathbb{N}_+ \text{ and } k = 1, \ldots, h^0(X, -kK_X).
\]

Also denote by $g^{(\sigma)}(\cdot)$, $g^{(\sigma_0)}(\cdot)$ the weight functions on each moment polytope, respectively. Then define

\[
\xi_{1/k}^{(g^{(\sigma)})}(\mathcal{X}, \mathcal{L}) := \sum_{p=1}^{h^0(X, -kK_X)} g^{(\sigma)} \left( \frac{\xi_{1/k}^k(\sigma)}{k}, \ldots, \frac{\xi_{r/k}^k(\sigma)}{k} \right) \Lambda_p^k,
\]

\[
\xi_{2/k}^{(g^{(\sigma)})}(\mathcal{X}, \mathcal{L}) := \frac{1}{2} \sum_{p=1}^{h^0(X, -kK_X)} \sum_{A=1}^{r} \frac{\partial g^{(\sigma)}}{\partial \theta_{\xi_A}} \left( \frac{\xi_{1/k}^k(\sigma)}{k}, \ldots, \frac{\xi_{r/k}^k(\sigma)}{k} \right) \Lambda_p^k,
\]

and the $g$-weighted Futaki invariant is defined as Definition 2.3. In can be checked that $Fut_g^{(\sigma)}(\cdot)$ derived from (2.32) - (2.33) coincides with that of $Fut_{g^{(\sigma_0)}}(\cdot)$ derived from (2.28) - (2.29).

### 2.2.2. The case of polynomial $g$

Let $X$ be a $\mathbb{Q}$-Fano variety. Without loss of generality we can assume that $L = -K_X$ is very ample so that $X$ is embedded in a projective space by $| - K_X|$. Suppose that $T \subset \text{Aut}(X)$ is a complex torus of rank $r$, with $\{\xi_{\Lambda}^k\}_{A=1}$ the generators. Also suppose that $(\mathcal{X}, \mathcal{L})$ is a test configuration of $(X, -K_X)$ constructed at the beginning of Section 2.2, with $L$ the generator of the corresponding $\mathbb{C}^*$-action. In the following we will show (2.30) holds for polynomial $g$. We will adopt the argument of [3, Section 3].

**Theorem 2.4.** Let $X$ be a $\mathbb{Q}$-Fano variety and $T \subset \text{Aut}(X)$ acts on $-K_X$ through some fixed lifting $\sigma$. Suppose that $g$ is a polynomial function on the moment polytope $\Delta$ of the lifting $\sigma$. Then for any $T$-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$, (2.30) holds. Moreover, the $g$-modified Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is given as following:

1. When $g$ is a monomial given by (2.2),

\[
\text{Fut}_g(\mathcal{X}, \mathcal{L}) = \frac{k!}{(n + |k|)!W(K_{\mathcal{X}^{1/p_1}}^{|k|})(\mathcal{L}^{\sigma[|k|]}_{X^{1/p_1}})^{n+|k|} + \frac{k!}{(n + |k|)!W}(\rho^{|k|} K_{\mathcal{X}^{1/p_1}}^{|k|})(\mathcal{L}^{\sigma[|k|]}_{X^{1/p_1}})^{n+|k|} - \frac{\rho^{|k|} K_{\mathcal{X}^{1/p_1}}^{|k|}}{V(n + |k|)} E_{\mathcal{X}}^N(\mathcal{X}, \mathcal{L}^\sigma);
\]
(2) When $g$ is a monomial given by (2.10),

\[ \text{Fut}_g(\mathcal{X}, \mathcal{L}) := \sum_k a_k \text{Fut}_{\lambda^k}(\mathcal{X}, \mathcal{L}). \]

**Proof.** We will prove the Proposition using the argument of \[ Section 3. \] It suffices to prove it when $\mathcal{L}$ is the canonical lifting. In view of the change of lifting formula (2.32)--(2.33), we may fix a lifting $\sigma'$ of the $T$-action on $-K_X$ so that the corresponding moment polytope $\Delta'$ lies in the first quadrant. In fact, we may even assume that $\mathcal{L}^{\sigma'\{k\}}$ is ample. We first deal with the case when $g'$ is a monomial on $\Delta'$ given by (2.2). Note that if $\sigma' = \sigma_0 + \mu$, then $g'(\cdot + \mu)$ is the corresponding weight on the moment polytope $\Delta = \Delta' - \mu$ of the canonical lifting $\sigma_0$.

For a normal test configuration $pr : \mathcal{X} \to \mathbb{P}^1$, set $k := (k_1, ..., k_r)$ and consider $(\mathcal{X}^{[k]}, \mathcal{L}^{[k]})$. Denote by $pr^{[k]} : \mathcal{X}^{[k]} \to \mathbb{P}^{[k]}$ the projection. We also consider the variety $\mathbb{P}^1$ with trivial $T$-action. Then $(\mathbb{P}^1)^{[k]} = \mathbb{P}^1 \times \mathbb{P}^{[k]}$, and we have the following diagram:

\[
\begin{array}{ccc}
\mathcal{X} \times (\mathbb{C}^{k+1}\{O\}) & \xrightarrow{T} & \mathcal{X}^{[k]} \\
pr \times \text{Id} & \downarrow & \downarrow pr^{[k]} \\
\mathbb{P}^1 \times (\mathbb{C}^{k+1}\{O\}) & \xrightarrow{T} & \mathbb{P}^1 \times \mathbb{P}^{[k]} \xrightarrow{pr^{[k]}} \mathbb{P}^1
\end{array}
\]

The diagram commutes since the $T_0 := \exp(tA)$- and $T$-actions commute. For any $m \in \mathbb{N}_+$, denote by $H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{(\eta, \lambda)}$ the sections in $H^0(\mathcal{X}_0, -m\mathcal{L}_0)$ so that $T$ acts on it through character $\eta = (\eta_1, ..., \eta_r) \in \mathcal{X}(T) \cong \mathbb{Z}^r$ and $\exp(tA)$ acts through character $\lambda \in \mathbb{Z}$. Also we have $(m\mathcal{L})^{\sigma'\{k\}} = m\mathcal{L}^{\sigma'\{k\}}$. Let $\overline{pr} := pr^0 \circ pr^{[k]}$. Then using \[ Proposition 1.3, \]

\[
\overline{pr} \circ \mathcal{O}_{\mathcal{X}^{[k]}}(m\mathcal{L}^{\sigma'\{k\}}) = \oplus_{\lambda \in \mathbb{Z}} H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{\lambda}^{(T_0)} \otimes \mathcal{O}_{\mathbb{P}^1}(\lambda),
\]

recall that here $T_0$ is the $C^*$-action of the test configuration which acts trivially on $\mathbb{C}^{k+1}\{O\}$. Clearly,

\[
H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{\lambda}^{(T_0)} = \oplus_{\eta \in \mathcal{X}(T)} H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{(\eta, \lambda)}^{(T \times T_0)},
\]

since the actions of $T_0$ and $T$ commutes. By Lemma 7.1 in the Appendix,

\[
\dim H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{\lambda}^{(T_0)} = \sum_{\eta \in \mathcal{X}(T)} \dim H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{(\eta, \lambda)}^{(T \times T_0)} C_{k_0}^{m+k_0}.
\]

Thus,

\[
\chi(\mathbb{P}^1, \overline{pr} \circ \mathcal{O}_{\mathcal{X}^{[k]}}(m\mathcal{L}^{\sigma'\{k\}})) = \sum_{(\eta, \lambda) \in \mathcal{X}(T) \otimes \mathbb{Z}} \dim H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{(\eta, \lambda)}^{(T \times T_0)} (\lambda + 1) C_{k_0}^{m+k_0}.
\]

Hence, we get

\[
\chi(\mathbb{P}^1, \overline{pr} \circ \mathcal{O}_{\mathcal{X}^{[k]}}(m\mathcal{L}^{\sigma'\{k\}}))
\]

(2.36)

\[
= \sum_{(\eta, \lambda) \in \mathcal{X}(T) \otimes \mathbb{Z}} (\lambda + 1) \dim H^0(\mathcal{X}_0, -m\mathcal{L}_0)_{(\eta, \lambda)}^{(T \times T_0)} \prod_{A=1}^{r} \left( \frac{(\eta_A + 1) \cdot \ldots \cdot (\eta_A + k_A)}{k_A!} \right).
\]
Also, by the Riemann-Roch formula [3, Theorem A.1],
\[
\chi(X^{[k]}, mL^{[k]}) = \frac{(L^{[\sigma]}[k])^{n+|k|+1}}{(n + |k| + 1)!} m^{n+|k|+1} \cdot \frac{K_{X^{[k]}} \cdot (L^{[\sigma]}[k])^{n+|k|}}{2(n + |k|)!} m^{n+|k|} + O(m^{n+|k|-1}), \ m \to +\infty.
\]
(2.37)
On the other hand, since \(L\) is ample, by the Leray spectral sequence [28, Chapter 1, Section 4.2],
\[
\chi(\mathbb{P}^1, \mathbb{P}^1; O_{X^{[k]}}(mL^{[k]})) = \chi(X^{[k]}, mL^{[k]}).
\]
(2.38)
Combining with (2.36)-(2.37), we get the sum
\[
\sum_{(\eta; \lambda) \in X(T) \otimes \mathbb{Z}} \frac{\lambda^{\eta_k}}{k!} \dim H^0(X_0, mL_0)(T \times T_0)^{(T \times T_0)}
\]
is a polynomial function for large \(m \gg 1\),
\[
= \frac{(L^{[\sigma]}[k])^{(n+|k|+1)}}{(n + |k| + 1)!} m^{n+|k|+1} + O(m^{n+|k|-1}), \ m \to +\infty.
\]
(2.39)
Similarly, denote by \(k_A = (k_1, \ldots, k_A - 1, \ldots, k_r)\), it holds
\[
\sum_{(\eta; \lambda) \in X(T) \otimes \mathbb{Z}} \frac{\lambda^{\eta_{k_A}}}{k_A!} \dim H^0(X_0, mL_0)(T \times T_0)^{(T \times T_0)}
\]
\[
= \frac{(L^{[\sigma]}[k_A])^{(n+|k|)}}{(n + |k|)!} m^{n+|k|} + O(m^{n+|k|-1}), \ m \to +\infty.
\]
(2.40)
Also,
\[
= \frac{(L^{[\sigma]}[k_A])^{(n+|k|)}}{(n + |k|)!} m^{n+|k|} + O(m^{n+|k|-1}), \ m \to +\infty.
\]
(2.41)
Thus, plug (2.36)-(2.37) and (2.39)-(2.41) into (2.38), we get
\[
S_{1[m]}(X, \mathcal{L}) = \sum_{(\eta; \lambda) \in X(T) \otimes \mathbb{Z}} \lambda^{\eta} \left( \frac{\eta_A}{m} \right) \dim H^0(X_0, mL_0)(T \times T_0)^{(T \times T_0)}
\]
\[
= k! \left( \frac{(L^{[\sigma]}[k])^{n+|k|+1}}{(n + |k| + 1)!} m^{n+1} - \frac{K_{X^{[k]}} \cdot (L^{[\sigma]}[k])^{n+|k|}}{2(n + |k|)!} m^n \right)
\]
\[
- k! \sum_{A=1}^r (k_A + 1) \left( \frac{(L^{[\sigma]}[k_A])^{n+|k|}}{(n + |k|)!} m^n - \frac{K_{X^{[k]}} \cdot (L^{[\sigma]}[k])^{n+|k|}}{(n + |k|)!} m^n \right)
\]
\[
+ O(m^{n+|k|-2}), \ m \to +\infty.
\]
(2.42)
We want to simplify the above equation. The relation of \(-K_{X^{[k]}}\) and \((-K_X)[k]\) is derived in Lemma 7.2 in the Appendix. On the other hand, by definition, a
$T$-invariant divisor $D$ in $pr^*_A \mathcal{O}_{\mathbb{P}^1}(1)$ satisfies

$$
(D, \mathcal{L}^\sigma|D|) \cong (\mathcal{X}'|\mathcal{A}', \mathcal{L}'|\mathcal{A}').
$$

It then follows

$$
pr^*_A \mathcal{O}_{\mathbb{P}^1}(k_A + 1) \cdot (\mathcal{L}^\sigma|k|)^{n+|k|} = (k_A + 1)(\mathcal{L}'|\mathcal{A}'|)^{n+|k|}. \tag{2.43}
$$

Also, consider the projection $pr : \mathcal{X} \to \mathbb{P}^1$. Since $-K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$, it has a divisor $-K_{\mathbb{P}^1} = 2[0]$. We have

$$
-(pr^*[k])^* K_{\mathbb{P}^1} \cdot (\mathcal{L}^\sigma|k|)^{n+|k|} = 2(\mathcal{L}'|\mathcal{A}'|)^{n+|k|}. \tag{2.44}
$$

Plugging (2.43)+(2.44) into (2.42) and using Lemma 7.2, we get that when $g'$ is in form of (2.2) with respect to the lifting $\sigma'$,

$$
S_1^{(g')} (\mathcal{X}, \mathcal{L}) = k! \frac{\mathcal{L}^\sigma|k|^{n+|k|+1}}{(n+|k|+1)!} m^{n+1} - k! \frac{K_{\mathcal{X}/\mathbb{P}^1} \cdot (\mathcal{L}^\sigma|k|)^{n+|k|}}{2(n+|k|)!} m^n + O(m^{n-1}), \ m \to +\infty. \tag{2.45}
$$

Now suppose that $g$ is a monomial (2.2) on the canonical polytope $\Delta$. Then it corresponds to

$$
g' (\eta) = g(\eta - \mu) = \sum_{0 \leq i < k} c_i^k (-1)^{|i|} \prod_{A=1}^r \mu^i \eta^{k-i}
$$
on $\Delta'$. Apply (2.45) to each $\eta^{k-i}$ and using linearity, we see that when $g$ is given by (2.2) with respect to the canonical lifting $\sigma_0$,

$$
S_1^{(g)} (\mathcal{X}, \mathcal{L}) = k! \frac{\mathcal{L}^{\sigma_0}|k|^{n+|k|+1}}{(n+|k|+1)!} m^{n+1} - k! \frac{K_{\mathcal{X}/\mathbb{P}^1} \cdot (\mathcal{L}^{\sigma_0}|k|)^{n+|k|}}{2(n+|k|)!} m^n + O(m^{n-1}), \ m \to +\infty. \tag{2.46}
$$

Also, it is direct to check

$$
S_2^{(g)} (\mathcal{X}, \mathcal{L}) = \frac{1}{2} \sum_{(\eta, \lambda) \in \mathcal{X} \cap \mathbb{Z}^2} \lambda \sum_{A=1}^r \frac{\eta A}{m} \frac{\partial^r g}{\partial \eta_A} \frac{(\eta A)}{m} \dim H^0(\mathcal{X}_0, \mathcal{M}_0)_{(T \times T')} \left( m \mathcal{L}_0 \right)_{(T \times T')}
$$

$$
= \frac{|k|}{2} k! \frac{\mathcal{L}^{\sigma_0}|k|^{n+|k|+1}}{(n+|k|+1)!} m^n + O(m^{n-1}), \ m \to +\infty. \tag{2.47}
$$

Combining (2.46)+(2.47) with the fact that

$$
h^0(\mathcal{X}, -MK_X) = V \frac{m^n}{n!} \left( 1 + \frac{1}{2} nm^{-1} + O(m^{-2}) \right), \ m \to +\infty \tag{2.48}
$$

for $L = -K_X$, we see that (2.30) holds, and the $g$-modified Futaki invariant is given by (2.34).

Now we prove (2.35). Suppose that $g$ polynomial satisfying (2.10). From the linearity of (2.10), it is easy to check that (2.46)+(2.47) also holds. By a direct computation one gets (2.35).

Recall the fact that when $g$ is a monomial, $\sum_{A=1}^r \eta A \frac{\partial g}{\partial \eta_A} = k_A g$. By linearity of (2.35), we directly see (2.30) and holds for any polynomial $g$. \hfill \square
Recall (2.1), we have $K_{X/p^1} = K_{X/p^1}^{\log} + (\mathcal{X}_0 - \mathcal{X}_{0,\text{red}})$. By Theorem 2.4 and the definition of $M^\text{NA}_g(-)$, we have:

**Proposition 2.5.** Let $X$ be a $\mathbb{Q}$-Fano variety with effective $T$-action and $g \geq 0$ a polynomial on $\Delta$ given by (2.11). Suppose that $(\mathcal{X}, \mathcal{L})$ is a $T$-equivariant normal test configuration. Then $\frac{1}{V_g} \text{Fut}_g(\mathcal{X}, \mathcal{L}) \geq M^\text{NA}_g(\mathcal{X}, \mathcal{L})$. Moreover, the equality holds if and only if $(\mathcal{X}, \mathcal{L})$ has reduced central fibre.

**Proof.** Suppose that $\mathcal{X}_{0,\text{red}} = \sum_{a=1}^{n_0} \mathcal{X}_{0,a}$ and $\mathcal{X}_0 = \sum_{a=1}^{n_0} \mathcal{X}_{0,a}$ for reduced, irreducible varieties $\mathcal{X}_{0,1}, \ldots, \mathcal{X}_{0,n_0}$ and positive integers $m_1, \ldots, m_{n_0}$. It suffices to show that under certain lifting of the $T$-action,

$$
\sum_k a_k(\mathcal{L}^{[k]} \cdot | \mathcal{X}_0 - \mathcal{X}_{0,\text{red}}|) = \sum_{a=1}^{n_0} (m_a - 1) \sum_k a_k(\mathcal{L}^{[k]} \cdot | \mathcal{X}_{0,a} |) \geq 0.
$$

(2.49)

In fact, from the change of lifting formulas (2.21) and (2.22), it is direct to check that (2.49) does not depend on the choice of lifting of the $T$-action on $-K_X$.

On the other hand, since $T$ acts on each $(\mathcal{X}_{0,a}, \mathcal{L}|_{\mathcal{X}_{0,a}})$, by [12, Section 2] (see also Lemma 7.4 in the Appendix),

$$
\sum_k a_k(\mathcal{L}^{[k]} \cdot | \mathcal{X}_{0,a} |) = \sum_k a_k(\mathcal{L}^{[k]} |_{\mathcal{X}_{0,a}} \cdot | \mathcal{X}_{0,a} |) = \frac{(n + |k|)!}{k!} \int_{\Delta_k(\mathcal{L}|_{\mathcal{X}_{0,a}})} g(\lambda) \text{DH}_T(\mathcal{X}_{0,a}, \mathcal{L}|_{\mathcal{X}_{0,a}})(\lambda),
$$

and we get (2.49). \hfill \Box

**Remark 2.6.** By using the equivariant Riemann-Roch formula, it is showed in [26, Section 1] that the modified Futaki invariant exists when $g$ is exponential of the potential of the soliton vector field.

2.3. Variants of Stability.

2.3.1. $K$-polystability. In the sense of [12, 26] we have

**Definition 2.7.** We say that a $\mathbb{Q}$-Fano $\mathfrak{g}$-variety $X$ is ($\mathfrak{g}$-equivariantly) $g$-modified $K$-semistable if the $g$-modified Futaki invariant for any $\mathfrak{g} \times T$-equivariant test configuration is nonnegative, and is ($\mathfrak{g}$-equivariantly) $g$-modified $K$-polystable if in addition the $g$-modified Futaki invariant vanishes precisely on product $\mathfrak{g} \times T$-equivariant test configurations. When $X$ is not ($\mathfrak{g}$-equivariantly) $g$-modified $K$-semistable, we say it is $g$-modified $K$-unstable.

Also, in the sense of [11], one can define the $g$-$K$-stability using the $g$-weighted non-Archimedean Mabuchi functional:

**Definition 2.8.** We say that a $\mathbb{Q}$-Fano $\mathfrak{g}$-variety $X$ is ($\mathfrak{g}$-equivariantly) $g$-$K$-semistable if the $g$-weighed non-Archimedean Mabuchi functional for any $\mathfrak{g} \times T$-equivariant test configuration is nonnegative, and is ($\mathfrak{g}$-equivariantly) $g$-$K$-polystable if in addition the $g$-weighed non-Archimedean Mabuchi functional vanishes precisely on product $\mathfrak{g} \times T$-equivariant test configurations. When $X$ is not ($\mathfrak{g}$-equivariantly) $g$-$K$-semistable, we say it is $g$-$K$-unstable.
By Proposition 2.5, we can prove that the $\mathfrak{G}$-equivariantly $g$-modified K-polystability and $\mathfrak{G}$-equivariantly $g$-K-polystability coincide with each other, provided $g$ is a polynomial with non-negative coefficients.

**Proposition 2.9.** Let $X$ be a $\mathbb{Q}$-Fano variety with a reductive group $\mathfrak{G}$-action. Let $g \geq 0$ be a polynomial. Then to test the $\mathfrak{G}$-equivariantly $g$-modified K-polystability or $g$-K-polystability of $X$, it suffices to consider $\mathfrak{G}$-equivariant normal test configurations with reduced central fibre. Consequently, $X$ is $\mathfrak{G}$-equivariantly $g$-modified K-polystable in the sense of Definition 2.7 if and only if it is $\mathfrak{G}$-equivariantly $g$-K-polystable in the sense of Definition 2.8.

**Proof.** Suppose that there is a non-product $\mathfrak{G}$-equivariant normal test configuration $(X, L)$ so that $\text{Fut}_g(X, L) \leq 0$. Then by Proposition 2.5, $M^\text{NA}_g(X, L) \leq 0$. By [17, Section 5.1], there is a sufficiently divisible $d \in \mathbb{N}_+$ so that the test configuration $(X', L')$ is non-product and has reduced central fibre. Then by Proposition 2.5,

$$(2.50) \quad \text{Fut}_g(X', L') = M^\text{NA}_g(X', L') = dM^\text{NA}_g(X, L) \leq 0.$$  

On the other hand, suppose that there is a non-product $\mathfrak{G}$-equivariant normal test configuration $(X, L)$ so that $M^\text{NA}_g(X, L) \leq 0$. Take $(X', L')$ as above, it holds $$(2.50).$$ Hence we get the Proposition. □

We will see that Proposition (2.9) holds for arbitrary continuous $g$ when $X$ is a $\mathbb{Q}$-Fano spherical variety. See Section 5 below.

2.3.2. $\mathfrak{G}$-uniform $g$-stability. The following uniform stability are closely related to the existence of KR $g$-soliton:

**Definition 2.10.** Let $X$ be a $\mathbb{Q}$-Fano variety with a complex torus $T$-action. Let $\mathfrak{G}$ be a connected, complex reductive group. A closed subgroup $H \subset \mathfrak{G}$ is called a spherical subgroup of $\mathfrak{G}$ if there is a Borel subgroup $B$ of $\mathfrak{G}$ acts on $G/H$ with an open orbit. Then we say that $X$ is $\mathfrak{G}$-uniformly $g$-Ding-stable ($g$-K-stable, resp.) if there exists a constant $\epsilon_0 > 0$ such that for any $\mathfrak{G} \times T$-equivariant test configuration $(X, L)$ it holds

$$D^\text{NA}_g(X, L) \left( M^\text{NA}_g(X, L), \text{resp.} \right) \geq \epsilon_0 \cdot \inf_{\sigma \in T} J^\text{NA}_g(\sigma^*(X, L)).$$

Here by $\sigma^*(X, L)$ we mean the twist of $(X, L)$ by $\sigma$.

The precise relationship between $\mathfrak{G}$-uniformly $g$-Ding-stability and existence of KR $g$-solitons are studied in [11, Section 6].

3. Polarized $\mathfrak{G}$-spherical varieties

3.1. Preliminaries on spherical varieties. In the following we overview the theory of spherical varieties. The origin goes back to [22]. We use [24, 25] as main references.

**Definition 3.1.** Let $G$ be a connected, complex reductive group. A closed subgroup $H \subset G$ is called a spherical subgroup of $G$ if there is a Borel subgroup $B$ of $G$ acts on $G/H$ with an open orbit. In this case $G/H$ is called a spherical homogeneous space. A spherical embedding of $G/H$ (or simply a spherical variety) is a normal variety $X$ equipped with a $G$-action so that there is an open dense $G$-orbit isomorphic to $G/H$. 
3.1.1. Homogenous spherical datum. Let $H$ be a spherical subgroup of $G$ with respect to the Borel subgroup $B$. The action of $G$ on the function field $\mathbb{C}(G/H)$ of $G/H$ is given by

$$(g^*f)(x) := f(g^{-1} \cdot x), \forall g \in G, x \in G/H \text{ and } f \in \mathbb{C}(G/H).$$

A function $f(\neq 0) \in \mathbb{C}(G/H)$ is called $B$-semiinvariant if there is a character of $B$, denote by $\varpi$ so that $b^*f = \varpi(b)f$ for any $b \in B$. By [25, Section 25.1], $\mathbb{C}(G/H)^B = \mathbb{C}$. Two $B$-semiinvariant functions associated to a same character can differ from each other only by multiplying a non-zero constant.

Let $\mathfrak{M}(G/H)$ be the lattice of $B$-characters which admits a corresponding $B$-semi-invariant functions, and $\mathfrak{M}(G/H) = \text{Hom}_\mathbb{Z}(\mathfrak{M}(G/H), \mathbb{Z})$ its $\mathbb{Z}$-dual. The rank $r_0$ of $\mathfrak{M}(G/H)$ is called the rank of $G/H$. There is a map $\varrho$ which maps a valuation $D$ of $\mathbb{C}(G/H)$ to an element $\varrho_D$ in $\mathfrak{M}_\mathbb{Q}(G/H) = \mathfrak{M}(G/H) \otimes_\mathbb{Z} \mathbb{Q}$ by

$$\varrho_D(\varpi) = \nu(f),$$

where $f \in \mathbb{C}(G/H)_{\varpi}$. Again, this is well-defined since $\mathbb{C}(G/H)_{\varpi} = \mathbb{C}$. It is a fundamental result that $\varrho$ is injective on $G$-invariant valuations and the image forms a convex cone $\mathcal{V}(G/H)$ in $\mathfrak{M}_\mathbb{Q}(G/H)$, called the valuation cone of $G/H$ (cf. [25, Section 19]). Moreover, $\mathcal{V}(G/H)$ is a solid cosimplicial cone which is a (closed) fundamental chamber of a certain crystallographic reflection group, called the little Weyl group (denoted by $W_G^H$, cf. [25, Sections 22]). In fact, $W_G^H$ is the Weyl group of the spherical root system $\Phi^{G/H}$ of $G/H$ (cf. [25, Section 30]). The simple spherical roots are defined to be the primitive generators of edges of $(-\mathcal{V}(G/H))^\vee$.

The set of simple spherical roots is denoted by $\Pi_{G/H}$.

A $B$-stable prime divisors in $G/H$ is called a colour. Denote by $\mathcal{D}(G/H)$ the set of colours. A colour $D \in \mathcal{D}(G/H)$ also defines a valuation on $G/H$. However, the restriction of $\varrho$ on $\mathcal{D}(G/H)$ is usually non-injective.

Now we briefly introduce the homogeneous spherical datum, which by a deep result in [20] (cf. [25, Theorem 30.22]) characterizes the spherical homogeneous space. Let $P_\alpha$ be the minimal standard parabolic subgroup of $G$ containing $B$ corresponding to the simple root $\alpha \in \Pi_G$. Set

$$\mathcal{D}(\alpha) := \{D \in \mathcal{D}(G/H)|D \text{ is not } P_\alpha\text{-stable}\}.$$ 

Then $\mathcal{D}(G/H) = \cup_{\alpha \in \Pi_G} \mathcal{D}(\alpha)$. We see that a colour $D \in \mathcal{D}(G/H)$ is of

- type $a$ (denote the collection by $\mathcal{D}^a(G/H)$): if $D \in \mathcal{D}(G/H)$ for $\alpha \in \Phi^{G/H}$;
- type $a'$ (denoted by $\mathcal{D}^{a'}(G/H)$): if $D \in \mathcal{D}(G/H)$ for $2\alpha \in \Phi^{G/H}$;
- type $b$ (denoted by $\mathcal{D}^b(G/H)$): Otherwise.

Note that although a colour $D$ may belong to different $\mathcal{D}(\alpha)$, the type of $D$ is well-defined. Also, set

$$\Pi_{G/H}^a := \{\alpha \in \Pi_G|\mathcal{D}(\alpha) = 0\},$$

and $\mathcal{D}^a(G/H)$ the set of all colours of type $a$.

**Definition 3.2.** The quadruple $(\mathfrak{M}(G/H), \Phi^{G/H}, \Pi_{G/H}^a, \mathcal{D}^a(G/H))$ is called the homogeneous spherical datum of $G/H$.

The homogeneous spherical datum was introduced by [21]. It is proved by [20] that the homogeneous spherical datum uniquely determines $G/H$ up to $G$-equivariant isomorphism. The axioms that an abstract quadruple $(\mathfrak{M}, \Phi, \Pi^a, \mathcal{D}^a)$ forms a homogeneous spherical datum can be found in [25, Section 30].
3.1.2. **Line bundles and polytopes.** Let $X$ be a complete spherical variety, which is a spherical embedding of some $G/H$. Let $L$ be a $G$-linearized line bundle on $X$. In the following we will associate to $(X, L)$ several polytopes, which encode the geometric structure of $X$.

**Moment polytope of a line bundle.** Let $(X, L)$ be a polarized spherical variety. Then for any $k \in \mathbb{N}$ we can decompose $H^0(X, L^k)$ as direct sum of irreducible $G$-representations,

\[ H^0(X, L^k) = \bigoplus_{\lambda \in \Delta_{L,k}} V_\lambda, \tag{3.1} \]

where $\Delta_{L,k}$ is a finite set of $B$-weights and each $V_\lambda$ is called an *isotypic component*, which is isomorphic to the irreducible representation of $G$ with highest weight $\lambda$. Set

\[ \Delta_+(L) := \bigcup_{k \in \mathbb{N}} \left( \frac{1}{k} \Delta_{L,k} \right). \]

Denote by $X(B)$ the lattice of $B$-weights. Then $\Delta_+(L)$ is indeed a polytope in $X(B)$. Moreover, denote by $\Phi^+_G$ the positive roots of $G$ with respective to $B$, then $\Delta_+(L)$ lies in the dominant Weyl chamber determined by $\Phi^+_G$. We call $\Delta_+(L)$ the *moment polytope of $(X, L)$* (cf. [25, Section 17]). Clearly, the moment polytope of $(X, L^k)$ is $k$-times the moment polytope of $(X, L)$ for any $k \in \mathbb{N}_+$.

We also introduce here a useful weight function $\pi(\lambda)$ defined for $\lambda \in \Delta_+(L)$. Suppose that $\lambda \in \mathcal{M}(G/H)$. By Weyl character formula [29, Section 3.4.4],

\[
\dim(V_\lambda) = \frac{\prod_{\alpha \in \Phi^+_G, \alpha \not\perp \Delta_+(L)} \langle \alpha, \rho + k\lambda \rangle}{\prod_{\alpha \in \Phi^+_G, \alpha \not\perp \Delta_+(L)} \langle \alpha, \rho \rangle} = C_{G/H} (\pi(\lambda) + \rho(\nabla \pi(\lambda)) + \text{(lower order terms)}),
\]

where $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+_G} \alpha$, the constant

\[ C_{G/H} = \frac{1}{\prod_{\alpha \in \Phi^+_G, \alpha \not\perp \Delta_+(L)} \langle \alpha, \rho \rangle}, \]

and

\[ \pi(\lambda) = \prod_{\alpha \in \Phi^+_G, \alpha \not\perp \Delta_+(L)} \langle \alpha, \lambda \rangle. \tag{3.2} \]

**Polytope of a divisor.** Recall that the spherical embedding $X$ of $G/H$ is uniquely determined by its coloured fan $\mathfrak{F}_X$ (cf. [22, 24, 25]). Denote by $\mathcal{I}_G(X) = \{D_1, \ldots, D_{d_0}\}$ the set of $G$-invariant prime divisors in $X$. Then any $D \in \mathcal{I}_G(X)$ corresponds to a 1-dimensional cone $(C_D, \emptyset)$ in the coloured fan $\mathfrak{F}_X$ of $X$. Denote by $u_D$ the prime generator of $C_D$. Recall that $\mathcal{D}(G/H)$ is the set of colours, which are $B$-stable but not $G$-stable in $X$. Any $B$-stable $\mathbb{Q}$-Weil divisor can be written as

\[ \mathfrak{d} = \sum_{D \in \mathcal{I}_G(X)} c_D D + \sum_{D \in \mathcal{D}(G/H)} c_D D \tag{3.3} \]

for some $c_D \in \mathbb{Q}$. Set

\[ \mathcal{D}_X := \{ \mathfrak{d} \subset \mathcal{D}(G/H) | \exists (\mathfrak{c}, \mathfrak{d}) \in \mathfrak{F}_X \} . \]
we can choose $\chi_Q$.

**Corollary 6.5.** Let $(X, L)$ be a polarized spherical variety, then the Kodaira ring $M_X$ has a canonical choice of $m_D$ where the coefficients $m_D$'s are explicitly obtained in [9] according to the type of each colour $D$ (cf. [9, Theorem 1.5]). In fact, this divisor corresponds to a $B$-semiinvariant section of $-K_X$ (in case $X$ is Gorenstein Fano) with $B$-weight $\kappa_p = \sum_{\lambda \in \Phi^+_G, \lambda \not\perp \Delta_+(-K_X)} \lambda$. The polytope of $d_0$ is the $(\mathbb{Q})$-reflexible polytopes defined in [9].

**3.1.4. The $G$-equivariant automorphism group.** Let $X$ be a $G$-spherical variety, which is a spherical embedding of some $G/H$. It is known that the $G$-automorphism group $\text{Aut}_G(G/H) = N_G(H)/H$ and is a commutative group (cf. [24, Proposition 1.8]). The action of $G \times N_G(H)/H$ on $G/H$ is defined as

$$(g,p)g_0H := ggp^{-1}H, \forall g, g_0 \in G \text{ and } p \in N_G(H)/H.$$ 

This action is well-defined since $p^{-1}Hp = H$. Its neutral component $\text{Aut}_G^0(G/H)$ is isomorphic to the neutral compone $\text{Aut}_G^0(X)$ of $\text{Aut}_G(X)$ (cf. [20]).

It is known that for a spherical variety $X$, $\text{Aut}_G^0(X)$ is a complex torus (cf. [14, Corollary 6.5]). Let $(X, L)$ be a polarized spherical variety, then the Kodaira ring (homogeneous coordinate ring) of $X$ is

$$R(X, L) = \oplus_{k \in \mathbb{N}} R_k, \quad R_k = H^0(X, kL) = \oplus_{\lambda \in \Delta_{L,k}} V_\lambda.$$ 

The group $\text{Aut}_G^0(X)$ acts on $R(X, L)$ preserving each $R_k$. Let $\xi \in \text{aut}_G^0(X)$ be a rational element which generates a $1$-dimensional torus $T_\xi$-action. Then each $R_k$ can be decomposed into direct sums of irreducible $T_\xi$-representations. Since the...
\( T_\xi \)-action commutes with the \( G \)-action, each isotypic component \( V_\lambda, \lambda \in \Delta_{L,k} \) is \( T_\xi \)-invariant, and \( \xi \) acts on any \( s \in V_\lambda \) through a common weight \( \nu_\xi(\lambda) \).

Suppose that \( \lambda_i \in \Delta_{L,k}, i = 1, 2 \). Then for any \( s_i \in V_{\lambda_i} \), it holds
\[
\xi(s_1 \cdot s_2) = \xi(s_1) + \xi(s_2).
\]

On the other hand, by [17, Proposition 3.1],
\[
V_{\lambda_1} \cdot V_{\lambda_2} = \bigoplus_{\lambda_1 + \lambda_2 - \beta_i} V_{\lambda_1 + \lambda_2 - \beta_i},
\]
where each \( \beta_i \) is a non-negative \( \mathbb{Q} \)-linear combination of simple spherical roots. In particular there is a component with \( \beta = 0 \). Thus for each \( \beta_i \) appeared in (3.5),
\[
\nu_\xi(\lambda_1 + \lambda_2 - \beta_i) = \nu_\xi(\lambda_1 + \lambda_2) = \xi(\lambda_1) + \xi(\lambda_2).
\]

Combining with [17, Remark 3.3] we can conclude from the first equality of (3.6) that
\[
\nu_\xi(\alpha) = 0, \ \alpha \in \Phi^{G/H}.
\]
On the other hand, from the second equality of (3.6), it holds
\[
\nu_\xi(p\lambda) = p\nu_\xi(\lambda), \ \forall p \in \mathbb{N},
\]
and
\[
\nu_\xi\left(\frac{k_1}{k_1 + k_2} \lambda_1 + \frac{k_2}{k_1 + k_2} \lambda_2\right) = \frac{k_1}{k_1 + k_2} \nu_\xi(\lambda_1) + \frac{k_2}{k_1 + k_2} \nu_\xi(\lambda_2).
\]
Thus \( \nu_\xi(\cdot) \) descends to an affine function
\[
\nu_\xi(\lambda) = V_\xi(\lambda) + \chi_\xi, \ \lambda \in \Delta_+(L)
\]
on \( \Delta_+(L) \) so that for each \( \lambda \in \Delta_{L,k} \),
\[
\nu_\xi(\lambda) = kV_\xi\left(\frac{1}{k} \lambda\right) + k\chi_\xi.
\]
Moreover, \( V_\xi \in \mathcal{V}(G/H) := \mathcal{V}(G/H) \cap (\mathcal{V}(G/H)) \), the central part of \( \mathcal{V}(G/H) \), and different choices of the constant \( \chi_\xi \) correspond to different liftings of the \( T_\xi \)-action on \( L \). In the following, we will identify \( \xi \in \text{aut}_G^0(X) \) with \( V_\xi \in \mathcal{V}(G/H) \).

Suppose that \( T \subset \text{Aut}_G^0(X) \) is an \( r \)-dimensional torus. Then we can choose a set of generators \( \{\xi_A\}_{A=1}^r \subset \mathcal{V}(G/H) \). Let \( \xi^*_A \in t^* \) be the dual of \( \xi_A \). The \( T \)-weights on each \( R_k \) is given by (3.8) and each choice of the character \( \chi = \sum_{A=1}^r \chi_A \xi^*_A \) of \( T \) correspond to a lifting of the \( T \)-action on \( L \).

As showed in [13, Theorem 4.2], the character associated to the canonical lifting is
\[
\chi_0 = - \sum_{A=1}^r \kappa_{P,A} \xi^*_A,
\]
the restriction of \( -\kappa_P \) on \( t \).
3.2. The polarized variety \((X^{[k]}, L^{[k]})\). In this section we compute the combinatorial data of \((X^{[k]}, L^{[k]})\) for a general polarized \(G\)-spherical variety \((X, L)\). Let \((X, L)\) be a polarized spherical embedding of \(G/H\). We further assume that \(L\) is \(G \times N_G(H)\)-linearized. 

Suppose that \(\xi \in \text{aut}_G(X)\) which generates a rank \(r\) torus \(T \subset \text{Aut}_G^0(X)\). Denote by \(\mathfrak{t}\) the Lie algebra of \(T\). Fix a lifting of the \(T\)-action on \(L\) with corresponding \(T\)-character \(\chi\). From the previous section we have the embedding

\[
\mathfrak{t} \xrightarrow{\iota_t} \text{aut}_G(X) \cong \mathcal{V}_{\mathbb{R}}(G/H) \xrightarrow{\iota_G} \mathfrak{g},
\]

and \((e, p)H = p^{-1}H = (p^{-1}, e)H\) for any \(p \in N_G(H)/H\). Then \(\mathfrak{t}\) is identified with an \(r\)-dimensional rational linear subspace of \(\mathcal{V}_{\mathbb{R}}(G/H)\), and the moment map \(m_{\omega_\phi} : X \to \Delta \in \mathfrak{t}^*\) can be decomposed as \(m_{\omega_\phi}() = r \circ m_{\omega_\phi}() + \chi\), where

\[
m_{\omega_\phi} : X \to \Delta_+ (L) \subset \mathfrak{X}_{\mathbb{R}}(B)
\]

is the moment map with respect to the \(T_0\)-action for the maximal torus \(T_0 = B \cap B^-\) of \(G\), and \(r : \mathfrak{t}_0^* \to \mathfrak{t}^*\) is the restriction map defined by \(r(\lambda) = \lambda|_t\), \(\forall \lambda \in \mathfrak{X}_{\mathbb{R}}(B)(= \mathfrak{t}_0^*)\). 

Here we consider the restriction \(r(\lambda)\) of an element \(\lambda \in \mathfrak{t}_0^*\) on \(\mathfrak{t}\) as an element in \(\mathfrak{t}^*\). Consequently,

\[
\Delta = \{r(\lambda) + \chi \in \mathfrak{t}^*| \lambda \in \Delta_+ (L)\}.
\]

Recall the construction of \((X^{[k]}, L^{[k]})\) in Section 2.1, Step-1. Let \(\{\xi_A\}_{A=1}^{r}\) be a set of generators of \(T\) and \(\xi_A = i(\xi_A')\). Then \(\xi_A \in \mathfrak{t}, A = 1, \ldots, r\) generate \(T\). Also denote by \(T^{[k]+1} = \prod_{A=1}^r (\mathbb{C}^*)^{k_A+1}\) the \((|k| + r)\)-dimensional complex torus. We have:

**Proposition 3.4.** Let \((X, L)\) be a polarized spherical embedding of \(G/H\) with moment polytope \(\Delta_+ (L)\). Suppose that the lifting of \(T\) on \(L\) is chosen so that the moment polytope \(\Delta\) lies in the first quadrant of \(\mathfrak{t}^*\), that is, \(\xi_A(\lambda + \chi) \geq 0\) for any \(A \in \{1, \ldots, r\}\) and \(\lambda \in \Delta\). Then \((X^{[k]}, L^{[k]})\) is a polarized \(G \times T^{[k]+1}\)-spherical variety with polytope

\[
\Delta_+ (L^{[k]}) = \{(\lambda; \mu_{(A),i_A}) \in \Delta_+ (L) \oplus \oplus_{A=1}^r \mathbb{R}_{\geq 0}^{k_A+1}| \lambda_A + \chi_A = \sum_{i_A=0}^{k_A} \mu_{(A),i_A}\},
\]

where \(\chi\) is the \(T\)-character corresponding to the lifting and \(\lambda_A\) is the restriction of the character \(\lambda\) on the 1-dimensional torus \(\exp(t\xi_A)\).

**Proof.** For each \(A \in \{1, \ldots, r\}\) choose an \(i_A \in \{0, \ldots, k_A\}\). Set \([i] = (i_1, \ldots, i_r)\) and \(U_{[i]} = \mathbb{C}^{[k]+1} \cap \{z^{(A),i_A} \neq 0| A = 1, \ldots, r\}\). It is easy to see that the open set

\[
X_{[i]}^{[k]} := \{[(x; z^{(B),i_B})] \in X^{[k]}| z^{(A),i_A} \neq 0, A = 1, \ldots, r\} \cong X \times U_{[i]}.
\]
We may choose local coordinates \( x, \zeta_{i[B]}^{(B),i_B} := \frac{z(B, j_B)}{z(B, i_B)} \) on \( X^{[B]} \) for \( B = 1, \ldots, r \) and \( j_B = 0, \ldots, k_B \) with each \( j_B \neq i_B \). Suppose that \( \{(x, z^{(B), j_B})\} \in X^{[B]} \cap X^{[i']} \). Then

\[
\left\{ x, \zeta_{[1]}^{(1,0), i_1}, \ldots, \zeta_{[i]}^{(1,i_1), i_1}, \ldots, \zeta_{[i]}^{(B),1}, \ldots, \zeta_{[i]}^{(B), i_B-1}, 1, \zeta_{[i]}^{(B), i_B+1}, \ldots, \zeta_{[i]}^{(A), k_A} \right\}
\]

is normal.

Then \( (g; t^{(A), i_A}) \in G \times T^{[k]+1} \) so that

\[
(g; t^{(A), i_A})[eH, 1, \ldots, 1] = [eH, 1, \ldots, 1].
\]

Then

\[
\begin{align*}
gH &= \exp(- \sum_A c_A \xi_A)H, \\
t^{(A), i_A} &= e^{c_A},
\end{align*}
\]

where \( c_1, \ldots, c_r \) are \( r \) constants in \( \mathbb{C} \). Thus we see that

\[
\text{Stab}_{G \times T^{[k]+1}}([eH, 1, \ldots, 1]) = \{(\varrho (T^{-1}) h, \varrho) \mid h \in H, \varrho \in T\}.
\]

Since \( H \) is a spherical in \( G \), we see that \( \text{Stab}_{G \times T^{[k]+1}}([eH, 1, \ldots, 1]) \) is spherical in \( G \times T^{[k]+1} \).

We are going to compute the combinatorial data of \( X^{[k]} \). Write \( \hat{G} = G \times T^{[k]+1} \) and \( \hat{B} = B \times T^{[k]+1} \) for short. It is direct to determine \( \mathbb{C}(X^{[k]})(\hat{B}) \). Suppose that \( f \in \mathbb{C}(X^{[k]})(\hat{B}) \). Then the pull-back of \( f \) is a \( \hat{B} \)-semiinvariant and \( T \)-invariant function on \( X \times (\mathbb{C}[k]+1) \). Denote by \( (\varpi, \mu_1, \ldots, \mu_r) \), where each \( \mu_A = (\mu_A, 0, \ldots, \mu_A, k_A) \in \mathbb{Z}^{k_A+1} \), the \( \hat{B} \)-character of \( f \). Then \( \varpi \in \mathfrak{M}(G/H) \). Also, by \( T \)-invariance we get

\[ (3.11) \quad \varpi_A + \chi_A = \mu_A,0 + \ldots + \mu_A, k_A, \quad \forall 1 \leq A \leq r. \]

Conversely, for any given tuple \( (\varpi, \mu_1, \ldots, \mu_r) \) satisfying \( (3.11) \),

\[
\hat{f} = \hat{f} = \prod_{A=1}^r \prod_{j_A=0}^{k_A} (z^{(A), j_A})^{\mu_A, j_A},
\]

where \( \hat{f} \in \mathbb{C}(X^{[k]}(\hat{B})) \), descends to a function in \( \mathbb{C}(X^{[k]})(\varpi, \mu_1, \ldots, \mu_r) \). Hence the lattice of \( \mathbb{C}(X^{[k]})(\hat{B}) \) is

\[ (3.12) \quad \mathfrak{M}(X^{[k]}) = \{(\varpi, \mu_1, \ldots, \mu_r) \in \mathfrak{M}(G/H) \otimes \mathbb{Z}[k]+1 \mid (\varpi, \mu_1, \ldots, \mu_r) \text{ satisfies (3.11)}\}. \]
Then we compute the \( \hat{B} \)-invariant divisors of \( \mathfrak{f}(X^{[k]}) \) and there image in \( \mathfrak{M}(X^{[k]}) \). Fix a set of basis \( \{ \lambda_{\alpha} \}_{\alpha=1}^{r_0} \) of \( \mathfrak{M}(G/H) \). Then the vectors

\[
\hat{\lambda}_{\alpha} = (\lambda_{\alpha}; \lambda_{\alpha 1}, 0, ..., 0; (\mu_{\alpha, 0} =) \lambda_{\alpha A}, 0, ..., (\mu_{\alpha, k_A} =) 0; ..., \lambda_{\alpha r}, 0, ..., 0, 1 \leq \alpha \leq r_0
\]

\[
\hat{\mu}_{A,i} = (O; 0, ..., 0; (\mu_{A, 0} =) -1, 0, ..., (\mu_{A, i_A} =) 1, 0, ..., 0, 1 \leq i_A \leq k_A, 1 \leq A \leq r
\]

form a basis of \( \mathfrak{M}(X^{[k]}) \).

Recall the open covering \( \{ X^{[k]} \}_{[i]} \). There are three types of \( \hat{B} \)-invariant divisors on \( X^{[k]} \):

**Type-1.** For any \( D \in \mathcal{I}_G(X) \cup \mathcal{D}(G/H) \), \( D^{[k]} := D \times (\mathbb{C}^{[k]}+1 \setminus \{ O \}) / \mathbb{T} \) is a \( \hat{B} \)-invariant divisor of \( X^{[k]} \). The image of \( D^{[k]} \) in \( \mathfrak{M}(X^{[k]}) \) is characterized by

\[
D^{[k]}(\hat{\lambda}_{\alpha}) = D(\lambda_{\alpha}), \quad 1 \leq \alpha \leq r_0,
\]

\[
D^{[k]}(\hat{\mu}_{A,i}) = 0, \quad 1 \leq A \leq r, 1 \leq i_A \leq k_A.
\]

**Type-2.** \( D_{A,i'} := \{ z^{(A)}, i' = 0 \} / \mathbb{T}, 1 \leq A \leq r, 1 \leq i' \leq k_A \). Its image in \( \mathfrak{M}(X^{[k]}) \) is characterized by

\[
D_{A,i'}(\hat{\lambda}_{\alpha}) = 0, \quad 1 \leq \alpha \leq r_0,
\]

\[
D_{A,i'}(\hat{\mu}_{A,i''}) = \delta_{AB} \delta_{i_A i''}, \quad 1 \leq B \leq r, 1 \leq j_B \leq k_B.
\]

**Type-3.** \( D_{A,0} := \{ z^{(A)}, 0 \} / \mathbb{T}, 1 \leq A \leq r \). Its image in \( \mathfrak{M}(X^{[k]}) \) is characterized by

\[
D_{A,0}(\hat{\lambda}_{\alpha}) = \lambda_{\alpha A}, \quad 1 \leq \alpha \leq r_0,
\]

\[
D_{A,0}(\hat{\mu}_{A,i''}) = -\delta_{AB}, \quad 1 \leq B \leq r, 1 \leq j_B \leq k_B.
\]

Finally we determine all \( \hat{B} \)-semiinvariant sections of \( (X^{[k]}, (L^{[k]})^p) \) for any \( p \in \mathbb{N} \). Suppose that \( s \in H^0(X^{[k]}, (L^{[k]})^p) \). Then the pull-back \( \tilde{s} \) of \( s \) on \( X \times ((\mathbb{C}^{[k]}+1 \setminus \{ O \}) / \mathbb{T} \) is a \( \hat{B} \)-semiinvariant and \( \mathbb{T} \)-invariant section.

Suppose that the \( \hat{B} \)-character associated to \( s \) is \( (\varpi, \mu_1, ..., \mu_r) \). Then so is \( \tilde{s} \). Let \( \vartheta = (e^a_1, ..., e^a_r) \in \mathbb{T} \cong (\mathbb{C}^*)^r \) with each \( a_A \in \mathbb{C} \). Then for any \( (x; z^{(A)}) \in X \times (\mathbb{C}^{[k]}+1 \setminus \{ O \}) \), by \( \mathbb{T} \)-invariance,

\[
(\vartheta \cdot \tilde{s})(x, z^{(A)}) = \vartheta \cdot (\tilde{s}(\varpi(x); e^{-a_A}z^{(A)})) \]

\[
= \vartheta \cdot (\vartheta^{-1} \cdot (\varpi|_{\mathbb{C}^*}) (\vartheta(\varpi))) \prod_{A=1}^{r} \mu_A(e^{a_A}) \tilde{s}
\]

\[
= (\varpi|_{\mathbb{C}^*})(\vartheta(\varpi)) \prod_{A=1}^{r} \mu_A(e^{a_A}) \cdot \tilde{s}(x; z^{(A)}).
\]

Hence the tuple \( (\varpi, \mu_1, ..., \mu_r) \) satisfies (3.11).

By restricting \( s \) on each \( X^{[k]}_{[i]} \), it holds

\[
s(x; z^{(A)}) = \prod_{A=1}^{r} \prod_{j_A=0, j_A \neq i_A, z^{(A)} 0 }^{k_A} \mu_A(z^{(A)}, j_A) \cdot s (x; \text{sgn}(|z^{(A)}, j_A|)).
\]

Since \( s \) is holomorphic,

\[
D^{[k]}(s) \geq 0, \quad D \in \mathcal{I}_G(X) \cup \mathcal{D}(G/H),
\]

\[
D_{A,i_A}(s) \geq 0, \quad 1 \leq A \leq r, \quad 0 \leq i_A \leq k_A,
\]
The relation (3.13) implies that \( \varpi \in \Delta_{L,p} \). The relation (3.14) implies \( \mu_{A,t_A} \geq 0 \) for all \( 1 \leq A \leq r \), and \( 0 \leq i_A \leq k_A \). Note that by our assumption on the lifting of \( T \)-action on \( L \), each \( \lambda_A + \chi_A \geq 0 \) and (3.10) is a convex polytope. Combining with (3.11) we get (3.10).

\[
\square
\]

3.3. \( G \)-equivariant normal test configurations. Let \( (X, L) \) be a polarized \( G \)-spherical variety. The \( G \)-equivariant (ample) normal test configurations of \( (X, L) \) have been classified by [6, 17].

**Proposition 3.5.** Let \( (X, L) \) be a polarized \( G \)-spherical variety with \( \Delta_+(L) \) its moment polytope. Then any \( G \)-equivariant normal test configuration \( (\mathcal{X}, \mathcal{L}) \) of \( (X, L) \) is a polarized \( G \times \mathbb{C}^* \)-spherical variety. Moreover, there is a rational, concave, piecewise linear function \( f : \Delta_+(L) \rightarrow \mathbb{R}_+ \), with \( \nabla f \in \mathcal{V}(G/H) \) so that the moment polytope \( \Delta_+(\mathcal{L}) \) of \( (\mathcal{X}, \mathcal{L}) \) is given by

(3.15) \( \Delta_+(\mathcal{L}) = \{ (y, t) \in \Delta_+(L) \times \mathbb{R} \mid 0 \leq t \leq f(y) \} \).

Proposition 3.5 was first proved by [6] Theorem 4.1 based on [11, 2]. It can also be derived from a general classification result [17, Theorem 3.4] of \( \mathbb{R} \)-equivariant test configurations (cf. [17, Remark 3.6]).

Suppose that the function \( f \) corresponding to \( (\mathcal{X}, \mathcal{L}) \) is given by

(3.16) \( f(\lambda) = \min_{a=1,\ldots,N_f} \{ l_{a}(\lambda) := C_a + \Lambda_a(\lambda) \} : \Delta_+(L) \rightarrow \mathbb{R}_+ \),

where each \( C_a \in \mathbb{Q} \), \( \Lambda_a \in \mathfrak{N}_0(G/H) \). Here we assume that the set \( \{ l_{a}(y) \}_{a=1}^{N_f} \) is minimal so that deleting any \( l_{a} \) will change \( f \). Also we associate to each \( \Lambda_a \) a number \( m_a \in \mathbb{N}_+ \) which is the minimal positive integer so that \( m_a \Lambda_a \in \mathfrak{N}(G/H) \). Then \( (\mathcal{X}, \mathcal{L}) \) is a \( G \times \mathbb{C}^* \)-divisorial of \( \mathcal{X} \) are

- \( \Delta = D \times \mathbb{C}^* \), \( D \in \mathcal{I}_G(X) \);
- \( \mathcal{X}_a \), the primitive \( G \times \mathbb{C}^* \)-invariant divisor corresponding to the \( a \)-th piece of \( f \);
- \( \mathcal{X}_\infty \cong X \), the divisor corresponding to the \( \Delta_+(L) \times \{ 0 \} \), which is the fibre of \( \mathcal{X} \) at \( \infty \in \mathbb{P}^1 \).

The colours are

- \( \mathcal{X}_\infty \cong X \), the divisor corresponding to the \( \Delta_+(L) \times \{ 0 \} \), which is the fibre of \( \mathcal{X} \) at \( \infty \in \mathbb{P}^1 \).

Also, the central fibre

(3.17) \( \mathcal{X}_0 = \sum_{a=1}^{N_f} m_a \mathcal{X}_a \).

We directly conclude that \( \mathcal{X}_0 \) is reduced if and only if all \( m_a = 1 \), or equivalently, each \( \Lambda_a \) in (3.10) is integral; it has one irreducible component if and only if \( f \) is affine. Moreover, \( \mathcal{X}_0 \) is normal if and only if \( f \) is affine and has integral coefficients (cf. [17, Corollary 3.9]).

Assume that the exponent of \( (\mathcal{X}, \mathcal{L}) \) is \( m_0 \). Choose a \( B \)-semiinvariant section \( s \) of \( L^{m_0} \) with \( B \)-character \( \varpi_s \) so that its divisor

(3.18) \( \delta_s = m_0 \sum_{D \in \mathcal{I}_G(X) \cup \mathcal{D}(G/H)} C_D D \),

then it is direct to derive the following Lemma from [23, Section 17.4],
Lemma 3.6. There is a $B \times \mathbb{C}^*$-semiinvariant section $\hat{s}$ of $\mathcal{L}$ with $B \times \mathbb{C}^*$-character $(\varpi_s, 0)$ whose divisor

$$\hat{\delta}_s = m_0(\sum_{D \in I_G(X) \cup \mathcal{D}(G/H)} C_D \hat{D} + \sum_{a=1}^{N_f} m_a(C_a + \Lambda_a(\varpi_s)) \mathcal{X}_{0,a}).$$

When $X$ is $\mathbb{Q}$-Fano, take $L = -K_X$ and consider the $G$-equivariant normal test configuration $(X, \mathcal{L})$ of $(X, L)$. There is a $B$-stable Weil divisor

$$-K_X = \sum_{D \in I_G(X)} D + \sum_{D \in \mathcal{D}(G/H)} n_D D,$$

and $-k_0K_X$ is Cartier for sufficiently divisible $k_0 \in \mathbb{N}_+$. Also there is a canonical $B$-semiinvariant section $s_0$ of $-k_0K_X$ with weight $k_0\kappa_P$. It follows that the $B \times \mathbb{C}^*$-character of the $\mathbb{C}^*$-invariant rational section $s_0$ of $k_0\mathcal{L}$ induced by $s_0$ is $(k_0\kappa_P, 0)$. Also,

$$-K_X = \sum_{D \in I_G(X)} \hat{D} + \sum_{D \in \mathcal{D}(G/H)} n_D \hat{D} + \sum_{a=1}^{N_f} \mathcal{X}_{0,a} + \mathcal{X}_\infty,$$

and consequently, by Lemma 3.6

$$-K_X^{\log} |_{\pi_1} = \sum_{D \in I_G(X)} \hat{D} + \sum_{D \in \mathcal{D}(G/H)} n_D \hat{D} = \mathcal{L} - \sum_{a=1}^{N_f} m_a(C_a + \Lambda_a(\kappa_P)) \mathcal{X}_{0,a},$$

whose $B \times \mathbb{C}^*$-character is also $(\kappa_P, 0)$.

4. The $g$-weighted non-Archimedean functionals of $G$-equivariant normal test configurations

Let $X$ be a $\mathbb{Q}$-Fano $G$-spherical variety, which is a spherical embedding of some $G/H$. Let $(\mathcal{X}, \mathcal{L})$ be any $G$-equivariant normal test configuration of $(X, -K_X)$. Then it is a spherical embedding of $G \times \mathbb{C}^*/H \times \{e\}$. From Section 3.1.2 we know that $\text{Aut}_{G \times \mathbb{C}^*}(X) = N_G(H)/H \times \mathbb{C}^*$ and $(\mathcal{X}, \mathcal{L})$ is automatically $G \times \text{Aut}_{G}(X)$-equivariant.

Denote by $\Delta_+$ the moment polytope $\Delta_+(-K_X)$ with respect to the canonical lifting of the $G$-action for short. In the remaining, we will compute the $g$-weighted non-Archimedean functionals. Suppose that we have a lifting of $T$-action on $L$ with respect to the character $\chi$. Then $\Delta = \tau_X(\Delta_+)$ with $\tau_X(\cdot) = \tau(\cdot) + \chi$, and we may identify $g$ with its pull-back through

$$\tau_X^* g : \Delta_+(L) \to \mathbb{R}$$

so that $g$ can be identified with a function on $\Delta_+(L)$.

Denote by $\{\xi_A\}_{A=1}^r$ a basis of $\mathfrak{g}(T)$-the lattice of one-parameter subgroups of $T \cong (\mathbb{C}^*)^r$. From the above discussion we may choose suitable coordinates for $\lambda = (\lambda_1, ..., \lambda_r) \in \mathfrak{X}_g(B)$, where $r_G$ is the rank of $G$, so that the first $r$ coordinates $\lambda_A = \xi^T_A, A = 1, ..., r$. Then $g(\tau_X(\lambda)) = \hat{g}(\lambda_1, ..., \lambda_r)$ for some function $\hat{g}$ on $\Delta_+$ which depends only on the first $r$-arguments of $\lambda$. We have
Lemma 4.1. Let $X$ be a $\mathbb{Q}$-Fano $G$-spherical variety with moment polytope $\Delta_+$. For a positive continuous weight $g$ on $X$, it holds

\begin{equation}
V_g = n! \int_{\Delta_+} g(r_x(\lambda)) \pi(\lambda) d\lambda.
\end{equation}

Proof. Denote by $\theta$ the points in $\Delta$. As in [11, Section 2],

\begin{equation}
V_g = \int_{\Delta} g(\theta) \left( m_{\omega^g} \frac{\omega_0^n}{n!} \right) (\theta) = \int_{\tau_x(\Delta_+)} g(r_x(\lambda))(r_x \circ \mu_{\omega_0})_* \frac{\omega_0^n}{n!} = n! \int_{\Delta_+} g(r_x(\lambda)) \pi(\lambda) d\lambda,
\end{equation}

which gives (4.2). \qed

In the following we compute the $g$-weighted non-Archimedean functionals of $G$-equivariant normal test configurations of $(X, -K_X)$. Fix the canonical lifting $\sigma_0$ of the $T$-action on $L$ with $\chi_0$ the associated $T$-character, and $\Delta_0$ the corresponding moment polytope. Denote by $F_D$ the facet of $\Delta_+$ that corresponds to the $B$-invariant divisor $D$. Set

\begin{equation}
\tau_D = \begin{cases} 
\frac{1 - \kappa_P(u_D)}{|u_D|}, & \text{if } D \in \mathcal{I}_G(X), \\
\frac{n_D - \kappa_P(g_D)}{|g_D|}, & \text{if } D \in \mathcal{D}(G/H).
\end{cases}
\end{equation}

Denote by $d\sigma_0$ the Lebesgue measure of $\partial \Delta_+$. Define a measure $d\sigma$ of $\partial \Delta_+$ so that $d\sigma|_{F_D} = \tau_D d\sigma_0|_{F_D}$ on each facet $F_D$.

We have:

**Proposition 4.2.** Let $X$ be a $\mathbb{Q}$-Fano $G$-spherical variety with moment polytope $\Delta_+$. Let $\Delta_0$ be the moment polytope of the canonical lifting $\sigma_0$ of the $T$-action on $L$, and the weight $g \in C^0(\Delta_0)$. Then for the $G$-equivariant normal test configuration $(X, \mathcal{L})$ of $(X, -K_X)$ corresponding to the function $f$, it holds

\begin{equation}
E^N_g(X, \mathcal{L}) = \frac{1}{V_g} \int_{\Delta_+} f g(r_{\chi_0}(\lambda)) \pi(\lambda) d\lambda,
\end{equation}

\begin{equation}
J^N_g(X, \mathcal{L}) = \frac{1}{V_g} \int_{\Delta_+} (\max f - f) g(r_{\chi_0}(\lambda)) \pi(\lambda) d\lambda,
\end{equation}

\begin{equation}
D^N_g(X, \mathcal{L}) = f(\kappa_P) - \frac{1}{V_g} \int_{\Delta_+} f g(r_{\chi_0}(\lambda)) \pi(\lambda) d\lambda
\end{equation}

\begin{equation}
M^N_g(X, \mathcal{L}) = - \frac{1}{V_g} \int_{\Delta_+} (\nabla f, \lambda - \kappa_P) g(r_{\chi_0}(\lambda)) \pi(\lambda) d\lambda.
\end{equation}

Proof. Step-1. $g$ is a monomial (2.2) of $\theta = r_{\chi_0}(\lambda) \in \Delta_0$. We will mainly use [25, Theorem 18.8] to compute the intersection numbers. Denote by

\[ \Sigma_m(c) = \{ \mu_A \in \mathbb{R}^{m+1}_{\geq 0} | c = \mu_0 + ... + \mu_m \} \]

the $c$-dilation of the standard $m$-dimensional simplex. Then its normalized volume is $\frac{c^m}{m!}$.

The line bundle $\mathcal{L}^{\sigma_0[k]}$ may not be ample in general. In order to use the intersection formula [25, Theorem 18.8], we consider another lifting $\sigma$ of the $T$-action
on $L$ with a suitable character $\chi_0 + \chi$ so that $L^{|\mathbb{N}|}$ for each $0 \leq j \leq k$ is ample. Take $\sigma_1 = \sigma_0$ and $\sigma_2 = \sigma$ in (2.22), we can always choose such a $\chi$.

To prove (4.4), by Proposition (2.2),

$$E_g^{NA}(X, L) = E_g^{NA}(X, L^\sigma)$$

(4.8)

$$= \sum_{0 \leq i \leq k} \frac{(k - i)!C_k^{k-i}}{(n + |k - i| + 1)!V_g}(-\chi)^i(L^\sigma)^{n+|k-i|+1}.$$

Recall that $(X, L^\sigma)$ has moment polytope (3.15). Note that the moment polytope $\Delta_+(L)$ is determined by the lifting of the $G$-action on $L$ rather that the $T$-action, it leaves unchanged when replacing $\sigma_0$ by $\sigma$. But the map

$$t_{\chi_0 + \chi}() = t_{\chi_0}() + \chi : \Delta_+(L) \rightarrow \Delta_0,$$

translates the $T$-moment polytope of $L^\sigma_0$ by $\chi$. Applying Proposition 5.4 and [25, Theorem 18.8] to $L^\sigma$, for each $j := k - i$, we have

$$\frac{j!}{(n + j + 1)!V_g}(L^\sigma)^{n+j+1} = \frac{j!}{V_g} \int_{\Delta_+(L^\sigma)} \pi(\lambda)d\lambda \wedge dt \wedge d\mu$$

$$= \frac{j!}{V_g} \int_{\Delta_+(L)} g(t_{\chi_0} + \chi(\lambda))\pi(\lambda)d\lambda \wedge dt \cdot \prod_{A=1}^{r} \frac{1}{|A|}$$

$$= \frac{1}{V_g} \int_{\Delta_+} f g(t_{\chi_0}(\lambda) + \chi(\lambda))\pi(\lambda)d\lambda.$$

(4.9)

Here the factor $\frac{1}{j!}$ in the second line is the normalized volume of $\Sigma_{j=1}(1)$, and in the last line we used (5.13). Plugging the above relation into (4.8) we get (4.4). The relation (4.6) follows from (4.1),

$$D_g^{NA}(X, L) = L^{NA}(X, L) - E_g^{NA}(X, L)$$

and (cf. 17, Section 5),

$$L^{NA}(X, L) = f(\kappa_P).$$

Now we turn to (4.5). The $g$-weighted non-Archimedean J-functional

$$J_g^{NA}(F) := \frac{1}{(-K_X)^{n+1}}L^\sigma(-K_X)^{n+1} - E_g^{NA}(F).$$

By (4.4) it suffices to compute the first term on the right-hand-side. To compute the intersection number, we use the method of [25, Section 18]. Note that for any $\epsilon > 0$, the Newton polytope of the ample line bundle $L'_\epsilon := \epsilon L + (-K_X)^{n+1}$ is $P_\epsilon := \epsilon P_{L} + (\Delta_+ \times \{0\})$. By [25, Corollary 18.28],

$$\frac{1}{n+1!}L^{n+1}_\epsilon = \int_{P_\epsilon} \pi d\lambda \wedge dt = \epsilon \max f \cdot \int_{P_\epsilon} \pi dy + O(\epsilon^2), \epsilon \rightarrow 0^+.$$

Hence

$$L \cdot L^p_n = n! \frac{d}{d\epsilon} \bigg|_{\epsilon_0} L^{n+1}_\epsilon = n! \max f \cdot \int_{P_+} \pi dy = L^n \cdot \max f.$$

Combining with (4.4) we get (4.5).

Finally we prove (4.7). In view of (4.4) and

$$(4.10) \quad M_g^{NA}(X, L) = \frac{k!}{V_g(n + |k|)!} (K_{X/X}^{log})^{|k|} (L^\sigma)^{n+|k|} + (n + |k|)E_g^{NA}(X, L),$$

it suffices to compute the term $(K_{X/X}^{log})^{|k|}(L^\sigma)^{n+|k|}$.
By (2.9) and (3.19),
\[
\frac{k!(K^\log X_{[k]})_{[k]}(\mathcal{L}^{|\sigma_0[k]|}n + |k|)}{(n + |k|)V_g} = - \frac{k!}{(n + |k|)V_g} \left( \sum_{i=1}^{N_f} m_a(C_a + \Lambda_a(\kappa_P))\mathcal{L}^{|\sigma_0[k]|}(\mathcal{L}^{|\sigma_0[k]|}n + |k|) \right) + \frac{k!}{(n + |k|)V_g} \sum_{a=1}^{N_f} m_a(C_a + \Lambda_a(\kappa_P))\mathcal{L}^{|\sigma_0[k]|}(\mathcal{L}^{|\sigma_0[k]|}n + |k|)
\]
(4.11)

It remains to deal with the second term. Choose the lifting \( \sigma = \sigma_0 + \chi \) of the \( T \)-action as before, we have
\[
\frac{k!}{(n + |k|)!} \chi_{0,a}^{|k|}(\mathcal{L}^{|\sigma_0[k]|}n + |k|) = \sum_{0 \leq j : k - i \leq k} \frac{1}{(n + |k - i|)!} \left( \mathcal{L}^{|\sigma_0[k]|}n + |k - i| \right) \chi_{0,a}^{|k - i|}(\mathcal{L}^{|\sigma_0[k]|}n + |k - i|).
\]
(4.12)

For each \( 0 \leq j := k - i \leq k \),
\[
\chi_{0,a}^{|k|}(\mathcal{L}^{|\sigma_0[j]|}n + |j|) = (\mathcal{L}^{|\sigma_0[j]|}n + |j|) \chi_{0,a}^{|k|}(\mathcal{L}^{|\sigma_0[j]|}n + |j|).
\]
(4.13)

Note that each \( \chi_{0,a} \) is a \( G \times \mathbb{C}^* \)-semi-invariant rational function of \( \Delta \) that corresponds to the coloured cone \( \mathcal{C} = \mathbb{Q}_{\geq 0}(m_a \Lambda_a, -m_a) \) in \( \mathfrak{M}(G/H) \). Also, take any integral point \( \hat{\lambda} \) of \( \Delta \) that lies in the \( a \)-th piece of the graph of \( f \), it corresponds to a section \( \hat{s} \in H^0(X, \mathcal{L}|X_0,a) \) that does not vanish on \( X_0,a \). Hence \( \hat{s}|_{X_0,a} \) gives a section of the ample line bundle \( \mathcal{L}|X_0,a \). On the other hand, by [23] Theorem 15.14, the lattice of \( B \times \mathbb{C}^* \)-semi-invariant rational functions
\[
\mathfrak{M}(X_0,a) = \mathfrak{M}(G/H \times \mathbb{C}^*) \cap (m_a \Lambda_a, -m_a)^\perp,
\]
and each \( \hat{f}_\mu \in \mathcal{C}(X_0,a)^{(B \times \mathbb{C}^*)} \) with \( \mu \in \mathfrak{M}(X_0,a) \) is the restriction \( f_\mu|_{X_0,a} \) of some \( f_\mu \in \mathcal{C}(G/H \times \mathbb{C}^*)^{(B \times \mathbb{C}^*)} \). Thus any \( B \times \mathbb{C}^* \)-semi-invariant rational section of \( \mathcal{L}|X_0,a \) is the restriction of \( f_\mu \hat{s} \) for some \( f_\mu \in \mathcal{C}(G/H \times \mathbb{C}^*)^{(B \times \mathbb{C}^*)} \) with \( \mu \in \mathfrak{M}(G/H \times \mathbb{C}^*) \) perpendicular to \( (m_a \Lambda_a, -m_a) \). Using [3] Theorem 1.2 (or essentially, [8] Theorem 2.8 (d)), we see that \( (f_\mu \hat{s})|_{X_0,a} \) is holomorphic on \( X_0,a \) if and only if it is holomorphic on \( X \). Thus, denote by \( \Omega_a \) the domain in \( \Delta \) where \( f = l_a \) and \( F_a \) the graph of \( l_a \) over \( \Omega_a \), we conclude that
\[
H^0(X_0,a, \mathcal{L}|X_0,a)^k \cong \bigoplus_{\chi \in \mathfrak{M}(G/H \times \mathbb{C}^*)} V(\lambda, \Lambda_0, \chi, \lambda|X_0,a, \chi, \lambda),
\]
where each \( V_{\lambda,m} \) is considered as an irreducible \( G \times \mathbb{C}^* \)-representation with highest weight \( (\lambda, m) \). Clearly,
\[
\dim V_{\lambda,m} = V_{\lambda}, \quad \forall \lambda \in X_+(G) \text{ and } m \in \mathbb{Z},
\]
and by [25] Theorem 18.8,
\[
((\mathcal{L}|X_0,a)^{|\sigma_0[j]|}n + |j|) = \int_{F_a} \frac{1}{m_a \sqrt{1 + |\Lambda_a|^2}} g(\chi_{0,a}(\lambda) + \chi)\pi(\lambda)d\sigma_0
\]
\[
= \int_{\Omega_a} g(\chi_{0,a}(\lambda) + \chi)\pi(\lambda)d\lambda,
\]
where \( d\sigma_0 \) denotes the standard Lebesgue measure on \( F_a \). Plugging the above equality into (4.13), also note that
\[
C_a + \Lambda_a(\kappa_P) = f + (\kappa_P - \lambda, \nabla f), \quad \forall \lambda \in \Omega_a.
\]
we get
\[
\begin{align*}
\frac{j! \lambda^{[j]}_a (\mathcal{L}^{[j]}_\sigma)^{n+j}}{(n + [j])! V_g} &= \frac{1}{m_a V_g} \int_{\Delta_+} \langle \kappa_P - \lambda, \nabla_f \rangle g(t_{x_0}(\lambda) + \chi) \pi(\lambda) d\lambda \\
&+ \frac{j!}{m_a (n + [j] + 1)! V_g} (\mathcal{L}^{[j]}_\sigma)^{n+j+1},
\end{align*}
\]
where in the last line we used (4.9). Plugging this relation into (4.12) and combining with (4.11), we have
\[
\begin{align*}
\frac{k! (K^{log}_{\mathcal{X}/\mathcal{P}})[k](\mathcal{N}_{0,k})^{n+|k|}}{V_g(n + |k|)!} &= - \frac{k! (n + |k|)}{(n + |k| + 1)! V_g} (\mathcal{L}^{[k]}_\sigma)n+|k|+1 \\
&+ \frac{1}{V_g} \int_{\Delta_+} \langle \kappa_P - \lambda, \nabla_f \rangle g(t_{x_0}(\lambda)) \pi(\lambda) d\lambda,
\end{align*}
\]
and we get (4.7) by using (4.10) and (4.4).

Step-2. The case of a general $\mathcal{C}^0$-weight $g$. Now we turn to the case when $g$ is a general $\mathcal{C}^0$-function on $\Delta_+ (L)$. Suppose that $g$ is given by (2.10). Recall the construction in Section 2.1, Step-2. Since the functionals $V_g \cdot N^\mathcal{S}_{\mathcal{N}}(\mathcal{X}, \mathcal{L})$, $N \in \{ \mathcal{E}, \mathcal{J}, \mathcal{M} \}$ are all linear in $g$, clearly Proposition 4.2 holds for polynomial $g$. For general $\mathcal{C}^0$-function $g$, we can approximate it by polynomials in the $\mathcal{C}^0$-topology and then take limit. We conclude the Proposition.

Remark 4.3. When $X$ is $\mathcal{Q}$-Fano and $L = -K_X$, it is proved in [9] that $\Delta_+$ is a $\mathcal{Q}$-reflexive polytope. Denote by $\nu$ the unit outer normal vector of $\partial \Delta_+$. Then by (4.3) we have
\[
\pi d\sigma = \langle \lambda, \nu \rangle \pi d\sigma_0.
\]

Taking integration by parts and using homogeneity of $\pi(\lambda)$,
\[
\begin{align*}
\int_{\Delta_+} fg \pi d\sigma &= \int_{\Delta_+} \langle \nabla (fg \pi), \lambda \rangle d\lambda + r_0 \int_{\Delta_+} fg \pi d\lambda \\
&= \int_{\Delta_+} \langle \nabla f, \lambda \rangle g \pi d\lambda + \int_{\Delta_+} f \langle \nabla g, \lambda \rangle \pi d\lambda \\
&+ (n - r_0) \int_{\Delta_+} fg \pi d\lambda + r_0 \int_{\Delta_+} fg \pi d\lambda,
\end{align*}
\]
and we get another expression
\[
\begin{align*}
M^\mathcal{N}_{\mathcal{X}}(\mathcal{X}, \mathcal{L}) := - \frac{1}{V_g} \left( \int_{\partial \Delta_+} fg(t_{x_0}(\lambda)) \pi d\sigma - \int_{\Delta_+} \kappa_P (\nabla f) g(t_{x_0}(\lambda)) \pi d\lambda \right) \\
&- n \int_{\Delta_+} fg(t_{x_0}(\lambda)) \pi d\lambda - \int_{\Delta_+} f \langle \lambda, \nabla g(t_{x_0}(\lambda)) \rangle \pi d\lambda.
\end{align*}
\]

We have the following inequality:

Proposition 4.4. Let $X$ be a $\mathcal{Q}$-Fano $G$-spherical variety and $(\mathcal{X}, \mathcal{L})$ a $G$-equivariant normal test configuration of $(X, -K_X)$. Let $g > 0$ be a $\mathcal{C}^1$-function. Then
\[
\begin{align*}
M^\mathcal{N}_{\mathcal{X}}(\mathcal{X}, \mathcal{L}) &\geq D^\mathcal{N}_{\mathcal{X}}(\mathcal{X}, \mathcal{L}),
\end{align*}
\]

[4.9]
and the equality holds if and only if the central fibre $X_0$ of $(\mathcal{X}, \mathcal{L})$ has only one irreducible component.

Proof. Let $f$ be the piecewise linear concave function associated to $(\mathcal{X}, \mathcal{L})$. By concavity,

$$-f(\lambda) + f(\kappa P) \leq \langle \nabla(-f), \lambda - \kappa P \rangle,$$

with the equality holds if and only if $f$ is affine. Plugging this into (4.7) (in the sense of Proposition 4.2),

$$M^N_g(\mathcal{X}, \mathcal{L}) \geq \frac{1}{V_g} \int_{\Delta^+} (f(\kappa P) - f(\lambda)) g\pi d\lambda = D^N_g(\mathcal{X}, \mathcal{L}),$$

with the equality holds if and only if $f$ is affine. The Proposition then follows from (4.6) and Proposition 4.2. □

Remark 4.5. By [17, Section 5.1], one concludes that (4.15) holds if and only if $(\mathcal{X}, \mathcal{L})$ is special after a possible base change.

5. The $g$-modified Futaki invariant of equivariant test configurations

In this section we first compute the $g$-modified Futaki invariant of an equivariant test configuration when $g$ is smooth in a neighbourhood of $\Delta$. By Theorem 2.4, the $g$-modified Futaki invariant can be computed using a similar argument as in Section 4. However, as we have assumed that $g \in C^\infty_0(\mathbb{R}^r)$, we can apply the Euler-Maclaurin formula [10, Theorem 4.1] to give a direct computation of the $g$-modified Futaki invariant according to Definition 2.3. The advantage of the following argument is that we need not to do the computation step-by-step from the case of monomial $g$ to smooth $g$.

Proposition 5.1. Let $(X, L)$ be a $\mathbb{Q}$-Fano $G$-spherical variety and $T \subset Aut_G(X)$ be an $r$-dimensional torus. Suppose that $(\mathcal{X}, \mathcal{L})$ the $G \times T$-equivariant test configuration of $(X, L)$ that is associated to the concave function $f$. Let $\Delta \subset \mathbb{R}^r$ be the moment polytope of the $T$-action with respect to the canonical lifting and $g \in C^\infty_0(\mathbb{R}^r)$. Then the $g$-modified Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is well-defined and satisfies

$$2 \int_{\Delta^+} \pi d\lambda \text{Fut}_g(\mathcal{X}, \mathcal{L}) = -\int_{\partial \Delta^+} fg\pi d\sigma + n \int_{\Delta^+} fg\pi d\lambda + \int_{\Delta^+} \kappa P(\nabla f) g\pi d\lambda + \int_{\Delta^+} f(\lambda, \nabla g)\pi d\lambda + \sum_{a=1}^{N_f} (1 - \frac{1}{m_a}) \int_{\Omega_a} g\pi(\lambda) d\lambda,$$

(5.1)

where the measure $d\sigma$ is $d\sigma|_{F_D} = \tau_D d\sigma|_{F_D}$ given by (4.3).

Proof. Let $(\mathcal{X}, \mathcal{L})$ be the normal test configuration associated to $f$ defined in Proposition 3.3. Then up to a uniform translation, the eigenvalue of the $\exp(\xi f)$- and $\exp(\Lambda)$-actions on the isotypic factor $V_\lambda \subset H^0(X_0, -k\mathcal{L}_0)$ are $e^{f(\lambda)}$ and $e^{k[f(\lambda)/k]}$, respectively (cf. [17, Section 3]). On the other hand, each isotypic factor $V_\lambda$ corresponds to a unique isotypic factor of $H^0(X, -kK_X)$ of the same $\lambda$ (cf. [30] [5]).
Thus, we have

\[ S_{1/k}^{(g)}(X, \mathcal{L}) = \sum_{\lambda \in \Delta_+} g(\frac{\xi_k}{k})[k f(\lambda/k)] \dim(V_\lambda) \]

\[ = \left( \sum_{(\lambda, t) \in \Delta_+} - \sum_{(\lambda, o) \in (\Delta_+ \times \{O\})} \right) g(\frac{\xi_k}{k}) \dim(V_\lambda). \]

We want to apply the Euler-Maclaurin formula of [10, Section 4]. By simplicial division, we can divide \( \Delta_+ (\mathcal{L}) \) into a union of rational simplex. In fact, up to replace \( L \) by \( L^n \) for sufficiently divisible \( r_0 \in \mathbb{N}_+ \), we may assume all simplexes are integral. Then we apply [10, Theorem 4.2] on each simplexes and take sum. Before proceeding, let us fix some notations. Suppose that \( Q \) is a full dimensional integral convex polytope in some lattice \( \mathfrak{M} \). Denote by \( \{ F_A \}_{A=1}^{d_0} \) its facets. Suppose that \( u_A \) is the primitive outer normal vector of \( F_A \) and denote by \( d\sigma \) the measure on \( \partial Q \) so that \( d\sigma|_{F_A} = \frac{1}{u_A} d\sigma_0 \), where \( d\sigma_0 \) is the standard Lebesgue measure. Such a measure arises in counting the number of integral points in \( kQ \),

\[ \# \{ kQ \cap \mathfrak{M} \} = k^{\dim(Q)} \Vol(Q) + \frac{1}{2} k^{\dim(Q)-1} \Vol_{d\sigma}(\partial Q) + O(k^{\dim(Q)-2}), \; k \to +\infty. \]

We also need to deal with the \( \dim(V_\lambda) \)-terms. Recall [32]. We have

\[ \rho - \frac{1}{2} \kappa_P = \frac{1}{2} \sum_{\alpha \in \Phi^G_+, \alpha \perp \Delta_+} \alpha \perp \Delta_+, \]

and

\[ \langle \nabla \pi(\lambda), \rho - \frac{1}{2} \kappa_P \rangle = \sum_{\alpha \in \Phi^G_+} (\prod_{\beta \neq \alpha, \beta \in \Phi^G_+ \perp \Delta_+} \langle \beta, \lambda \rangle) (\alpha, \rho - \frac{1}{2} \kappa_P). \]

Consider \( \Phi^L := \Phi^G \cap \Delta_+ \). Then \( \Phi^L \) is a sub-root system of \( \Phi^G \) and the Weyl group \( W_L \) of \( \Phi^L \) permutes

\[ \Phi^G_+ \setminus \Phi^L = \{ \alpha \in \Phi^G_+ | \alpha \notin \Delta_+ \}. \]

Hence \( \nabla \pi(\lambda) \) is \( W_L \)-invariant. Choose \( w_0 \in W_L \) the longest element of \( W_L \). Then

\[ w_0(\rho - \frac{1}{2} \kappa_P) = -(\rho - \frac{1}{2} \kappa_P). \]

Hence

\[ \langle \nabla \pi(\lambda), \rho - \frac{1}{2} \kappa_P \rangle = \langle w_0(\nabla \pi(\lambda)), w_0(\rho - \frac{1}{2} \kappa_P) \rangle = 0, \]

and

\[ \dim(V_\lambda) = C_{G/H}(\pi(\lambda) + \frac{1}{2} \kappa_P(\nabla \pi(\lambda)) + (\text{lower order terms})). \]

Combining with the Euler-Maclaurin formula [10, Theorem 4.2] (see also [13]),

\[ S_{1/k}^{(g)}(X, \mathcal{L}) = k^{n+1} \int_{\Delta_+} f g \pi d\lambda + \frac{1}{2} k^n \int_{\partial \Delta_+} f g \pi d\sigma + \frac{1}{2} k^n \sum_{a=1}^{N_f} \int_{F_a} g \pi d\sigma \]

\[ + \frac{1}{2} k^n \int_{\Delta_+} f g \kappa_P(\nabla \pi) d\lambda - \frac{1}{2} k^n \int_{\Delta_+} g \pi d\lambda + O(k^{n-1}), \; k \to +\infty, \]

(5.2)
where \( \mathcal{F}_a = \{(\lambda, t) | t = f(\lambda), \lambda \in \Omega_a\} \) is the facet of \( \Delta_+(\mathcal{L}) \) that lies on the \( a \)-th piece of the graph of \( f \). Note that the primitive normal vector of \( \mathcal{F}_a \) is \( (m_a \Lambda_a, -1) \). We have

\[
\int_{\mathcal{F}_a} g \pi d\sigma = \int_{\mathcal{F}_a} g \pi \frac{1}{m_a(\Lambda_a - 1)} d\sigma_0 = \frac{1}{m_a} \int_{\Omega_a} g \pi d\lambda, \ a = 1, \ldots, N_f.
\]

Thus

\[
S_1^{(g)}(\mathcal{X}, \mathcal{L}) = k^{n+1} \int_{\Delta_+} f g \pi d\lambda + \frac{1}{2} k^n \int_{\partial \Delta_+} f g \pi d\sigma + \frac{1}{2} k^n \int_{\Delta_+} f g \kappa P(\nabla \pi) d\lambda
\]

\[
+ \frac{1}{2} k^n \sum_{a=1}^{N_f} \left( \frac{1}{m_a} - 1 \right) \int_{\Omega_a} g \pi d\lambda + O(k^{n-1}), \ k \to +\infty,
\]

Clearly,

\[
S_2^{(g)}(\mathcal{X}, \mathcal{L}) = \frac{1}{2} k^n \int_{\Delta_+} f \sum_{A=1}^r \xi_A(\lambda - \kappa P) \frac{\partial g}{\partial \xi_A}(\xi_A(\lambda)) \pi d\lambda + O(k^{n-1})
\]

\[
= \frac{1}{2} k^n \int_{\Delta_+} f(\lambda - \kappa P, \nabla g) \pi d\lambda + O(k^{n-1}), \ k \to +\infty.
\]

Also, as in (5.2)

\[
h^0(\mathcal{X}, -kK_X) = k^n \int_{\Delta_+} \pi d\lambda + \frac{1}{2} k^{n-1} \int_{\partial \Delta_+} \pi d\sigma
\]

\[
+ k^{n-1} \int_{\Delta_+} \langle \nabla \pi, \lambda \rangle d\lambda + O(k^{n-2}), \ k \to +\infty.
\]

Plugging (5.3)-(5.5) and the relation

\[
\int_{\partial \Delta_+} \pi d\sigma = \int_{\partial \Delta_+} \langle \lambda, \nu \rangle d\sigma_0 = n \int_{\Delta_+} \pi dy
\]

into (2.30), we get

\[
2 \int_{\Delta_+} \pi d\lambda \int_{\Delta_+} g \pi d\lambda \text{ Fut}_g(\mathcal{X}, \mathcal{L}) = - \int_{\partial \Delta_+} f g \pi d\sigma + n \int_{\Delta_+} f g \pi d\lambda
\]

\[
\quad - \int_{\Delta_+} f \kappa P(\nabla \pi) g d\lambda + \int_{\Delta_+} f(\lambda - \kappa P, \nabla g) \pi d\lambda
\]

\[
\quad + \sum_{a=1}^{N_f} \left( 1 - \frac{1}{m_a} \right) \int_{\Omega_a} g(\lambda) d\lambda,
\]

\[
(5.3)
\]

\[
(5.4)
\]

\[
(5.5)
\]

\[
4\text{Here we use the following identity in [10, Theorem 4.2] (essentially in [10, Eq. (3.15)]): Denote by } f(z) = z^n - 1 \text{ and } \omega = e^\frac{2\pi i}{n}, \text{ then } \sum_{k=1}^{n-1} \frac{1}{1 - \omega^k} = \frac{f''(1)}{2f'(1)} = \frac{n-1}{2}. \text{ Thus the Todd functions } \frac{1}{n}(\tau(s) + \sum_{k=1}^{n-1} \tau_\phi(s)) = 1 + \frac{1}{2}s + O(s^2), \ s \to 0.
\]
Note that for outer unit normal vector $\nu$ on each facet, $d\sigma = \langle \lambda, \nu \rangle d\sigma_0 = d\bar{\sigma} + \langle \kappa_P, \nu \rangle d\sigma_0$. Taking integration by parts to the third term on the right-hand side
\[
\int_{\Delta^+} \kappa_P f(\nabla \pi) g d\lambda = \int_{\partial \Delta^+} f(\kappa_P, \nu) \pi g d\sigma_0 - \int_{\Delta^+} \kappa_P(\nabla f) g \pi d\lambda \\
- \int_{\Delta^+} f \kappa_P(\nabla g) \pi d\lambda,
\]
we get the Proposition. \qed

Compare with \cite{[4,4]}, for $\mathbb{Q}$-Fano spherical varieties we can strengthen Theorem 2.9 to the following:

**Corollary 5.2.** Let $(X, L)$ be a $\mathbb{Q}$-Fano $G$-spherical variety which is locally $\mathbb{Q}$-factorial and $T \subset \text{Aut}_G(X)$ be an $r$-dimensional torus. Suppose that $(X, L)$ the $G \times T$-equivariant test configuration of $(X, L)$ that is associated to the concave function $f$. Let $\Delta \subset \mathbb{R}^r$ be the moment polytope of the $T$-action with respect to the canonical lifting and $g \in C^\infty(\mathbb{R}^r)$. Then
\[
\frac{\mathcal{V}}{V_g} \text{Fut}_g(X, L) = M\text{NA}_g(X, L) + 1
\]
\[
\sum_{a=1}^{N_f} (1 - \frac{1}{m_a}) \int_{\Omega_a} g \pi(\lambda) d\lambda \geq M\text{NA}_g(X, L).
\]

Consequently, $\frac{\mathcal{V}}{V_g} \text{Fut}_g(X, L) = M\text{NA}_g(X, L)$ if and only if $(X, L)$ has reduced central fibre.

**Proof.** The relation (5.6) can be proved in a same way as (4.14). Note that
\[
\int_{\Omega_a} g \pi(\lambda) d\lambda > 0, \quad a = 1, ..., N_f.
\]
The last point then follows from (3.17). \qed

6. **The stability criterion**

In this section we will prove a combinatorial criterion of $G$-uniformly $g$-modified stability.

To prove Theorem 1.3 we need a technical lemma. Set
\[
\mathcal{C}(\Delta^+) = \{ f : \Delta^+ \rightarrow \mathbb{R} | f \text{ is concave and } \nabla f \in \mathcal{V}(G/H) \}.
\]
Define two functionals
\[
\mathcal{D}_g(f) := f(\kappa_P) - \frac{1}{V_g} \int_{\Delta^+} f g \pi dy,
\]
\[
\mathcal{J}_g(f) := \max_{\Delta^+} f - \frac{1}{V_g} \int_{\Delta^+} f g \pi dy.
\]
By Proposition 4.2 for a $G$-equivariant normal test configuration $(X, L)$ of $(X, -K_X)$ is associated to $f$, it holds
\[
\mathcal{D}_g^{\text{NA}}(X, L) = \mathcal{D}_g(f), \quad \mathcal{J}_g^{\text{NA}}(X, L) = \mathcal{J}_g(f).
\]

**Lemma 6.1.** Suppose that $X$ is a $\mathbb{Q}$-Fano spherical embedding of $G/H$ with moment polytope $\Delta^+$. Suppose that the barycenter satisfies \cite{[1,2]} and (1.2). Then for any
\[
f \in \hat{\mathcal{C}}(\Delta^+) := \mathcal{C}(\Delta^+) \cap \{ \text{pr}_z(\nabla f(\kappa_P)) = 0, \max f = 0 \},
\]

it holds
\[(6.1) \quad \mathcal{D}_g(f) \geq \epsilon_0 \mathcal{J}_g(f) .\]

Here
\[
\operatorname{pr} : \mathfrak{N}_\mathbb{R}(G/H) \to \mathcal{V}_z\mathbb{R}(G/H) (= \cap_{\alpha \in \Phi G/H} \{ \Lambda | \alpha(\Lambda) = 0 \})
\]
is the projection with respect to the scalar product \(\langle \cdot, \cdot \rangle\).

**Proof.** Denote by \(\operatorname{pr}_z^\perp = \text{Id} - \operatorname{pr}_z\). Then
\[
V_g \cdot \mathcal{D}_g(f) = \int_{\Delta_+} (f(\kappa P) - f)(\nabla f(\kappa P))g\pi dy
\]
\[
= \int_{\Delta_+} (f(\kappa P) - f + \nabla f(\kappa P)(y - \kappa P))g\pi dy
\]
\[
- \int_{\Delta_+} (\operatorname{pr}_z(\nabla f(\kappa P))(y - \kappa P))g\pi dy
\]
\[
- \int_{\Delta_+} (\operatorname{pr}_z^\perp(\nabla f(\kappa P))(y - \kappa P))g\pi dy
\]

By (6.2), the second term
\[
\int_{\Delta_+} (\operatorname{pr}_z(\nabla f(\kappa P))(y - \kappa P))g\pi dy = V_g \cdot (\operatorname{pr}_z(\nabla f(\kappa P))(b(\Lambda_0) - \kappa P)) = 0,
\]
and the third term
\[(6.2) \quad - \int_{\Delta_+} (\operatorname{pr}_z^\perp(\nabla f(\kappa P))(y - \kappa P))g\pi dy \geq 0,
\]
since \(-\operatorname{pr}_z^\perp(\nabla f(\kappa P)) \in (-\mathcal{V}(G/H))\). By concavity
\[(6.3) \quad f(\kappa P) - f + \nabla f(\kappa P)(y - \kappa P) \geq 0.
\]
Thus
\[(6.4) \quad V_g \cdot \mathcal{D}_g(f) \geq 0, \quad \forall f \in \hat{\mathcal{C}}(\Delta_+).
\]

Suppose that (6.1) is not true. There is a sequence \(\{ f_p \}_{p=1}^{+\infty} \subset \hat{\mathcal{C}}(\Delta_+)\) so that
\[(6.5) \quad V_g \cdot \mathcal{J}_g(f_p) = \int_{\Delta_+} (-f_p)g\pi dy = 1,
\]
\[(6.6) \quad \lim_{p \to +\infty} \mathcal{D}_g(f_p) = 0.
\]
By (6.5) and \(f_p \leq 0\), up to passing to a subsequence, \(f_p\) converges locally uniformly to some \(f_\infty \in \hat{\mathcal{C}}(\Delta_+)\). Combining with (6.4), we have
\[
0 \leq \mathcal{D}_g(f_\infty) \leq \lim_{p \to +\infty} \mathcal{D}_g(f_p) = 0.
\]
By (6.2) and (6.3), we see that
\[
f_\infty(y) = \zeta(y) + C
\]
for some \(\zeta \in \mathcal{V}_z(G/H)\) and \(C \in \mathbb{R}\). But since \(f_\infty \in \hat{\mathcal{C}}(\Delta_+)\),
\[
\operatorname{pr}_z(\nabla f_\infty(\kappa P)) = \zeta = 0 \quad \text{and} \quad C = 0.
\]
Hence \(f_\infty \equiv 0\).
Since
\[ \lim_{p \to +\infty} f_p(\kappa_p) = f_\infty(\kappa_p) = 0, \]
by (6.5),
\[ V_g \cdot D_g(f_p) = V_g \cdot f_p(\kappa_p) - \int_{\Delta^+} f_p g \pi dy \to 1, \quad p \to +\infty. \]
A contradiction to (6.6). Hence (6.1) is true. \( \square \)

**Proof of Theorem 7.3.** The direction (1) \( \Rightarrow \) (2): Note that if a \( G \)-equivariant normal test configuration \( (\mathcal{X}, \mathcal{L}) \) of \( (X, -K_X) \) is associated to \( f \). Then the twist of \( (\mathcal{X}, \mathcal{L}) \) by an \( \exp(\Lambda) \)-action with \( \Lambda \in \mathcal{V}_{\mathbb{R}}(G/H) \cong \text{aut}_G(X) \) is associated to \( f_\lambda(\lambda) := f(\lambda) - \Lambda(\lambda) \). Also note that all the NA-functional is invariant if we add to \( f \) a constant. The direction is then a combination of (4.5) and Lemma 6.1.

The direction (2) \( \Rightarrow \) (3) follows directly from Proposition 4.4.

The direction (3) \( \Rightarrow \) (4) is trivial;

The direction (4) \( \Rightarrow \) (1): Suppose that (1.2) fails. Then as in [16, Lemma 2.4] one can construct a non-product \( G \)-equivariant normal test configuration \( (\mathcal{X}, \mathcal{L}) \) of \( (X, -K_X) \) so that
\[ D^\text{NA}(\mathcal{X}, \mathcal{L}) = M^\text{NA}(\mathcal{X}, \mathcal{L}) \leq 0. \]
A contradiction to (4). \( \square \)

7. **Appendix: Some Lemmas on \( (X^{[k]}, L^{[k]}) \)**

7.1. **The dimension of \( \text{H}^0(X^{[k]}, L^{[k]}) \).**

**Lemma 7.1.** Suppose that \( (X, L) \) is polarized variety. Let \( T \subset \text{Aut}(M) \) be a torus that acts on \( L \) through some lifting \( \sigma \). Then on \( (X^{[k]}, L^{[k]}) \) it holds
\[ \dim \text{H}^0(X^{[k]}, L^{[k]}) = \sum_{\lambda \in X(T)} \dim \text{H}^0(X, L)^{\langle T \rangle}_{\lambda} \prod_{A=1}^r \dim \mathcal{O}_{p^A}(\lambda_A), \]
where \( \lambda = (\lambda_1, \ldots, \lambda_r) \in X(T) \cong \mathbb{Z}^r \). In particular, if each \( \lambda \) with \( \text{H}^0(X, L)^{\langle T \rangle}_{\lambda} \neq 0 \) lies in \( \mathbb{N}^r \), then
\[ \dim \text{H}^0(X^{[k]}, L^{[k]}) = \sum_{\lambda \in X(T)} \dim \text{H}^0(X, L)^{\langle T \rangle}_{\lambda} \mathcal{O}_{k^{[k]}}^{\langle\lambda\rangle}. \]

**Proof.** Since \( T \) acts on \( L \),
\[ \text{H}^0(X, L) = \bigoplus_{\lambda \in X(T)} \text{H}^0(X, L)^{\langle T \rangle}_{\lambda}. \]

We fix a basis \( \{ s(\lambda)_i \}_{i=1}^{d_\lambda} \) of each \( \text{H}^0(X, L)^{\langle T \rangle}_{\lambda} \).

On the other hand, suppose that \( \tilde{s} \in \text{H}^0(X^{[k]}, L^{[k]}) \). Then the pull-back \( \tilde{s} \) of \( \tilde{s} \) is a \( T \)-invariant section of \( L \times (\mathbb{C}^{k+1} \setminus \{O\}) \). We can write
\[ \tilde{s} = \sum_{\lambda \in X(T)} \sum_{i=1}^{d_\lambda} s(\lambda)_i \tilde{f}(\lambda)_i, \]
where each \( \tilde{f}(\lambda)_i \in \mathbb{C}[\mathbb{C}^{k+1} \setminus \{O\}] \). Since \( \cup_{\lambda \in X(T)} \{ s(\lambda)_i \}_{i=1}^{d_\lambda} \) is a basis of \( \text{H}^0(X, L) \), each \( s(\lambda)_i \tilde{f}(\lambda)_i \) must be \( T \)-invariant. Consequently, \( \tilde{f}(\lambda)_i \) descenders to a section.
\[ \tilde{f}_{\lambda i} \in \bigotimes_{A=1}^{n} \mathcal{O}_{\tilde{p}^A}(\lambda_A). \] Hence we get \((7.1)\). \((7.2)\) follows from \((7.1)\) and the identity \((28, \text{Chapter 2, Section 2.5})\),
\[ \chi(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k} (\lambda)) = \frac{1}{k!} (\lambda + 1) \ldots (\lambda + k). \]

\[ \square \]

### 7.2. The canonical divisor of \(X^{[k]}\).

**Lemma 7.2.** Suppose that \(X\) is \(\mathbb{Q}\)-Fano variety. Let \(T \subset \text{Aut}(M)\) be a torus that acts on \(-K_X\) through the canonical lifting \(\sigma_0\). Then it holds
\[ -K_{X^{[k]}} = (-K_X)^{\sigma_0}[k] + \sum_{A=1}^{r} \text{pr}_A^* \mathcal{O}_{\mathbb{P}^A}(k_A + 1). \]
on the variety \((X^{[k]}, (-K_X)^{\sigma_0}[k])\).

**Proof.** Without loss of generality we may assume that \(-K_X\) is Cartier. Otherwise we replace \(-K_X\) by some \(-m_0K_X\), \(m_0 \in \mathbb{N}_+\). As \(T\) acts on \(H^0(X, -K_X)\), there is a section \(s_0 \in H^0(X, -K_X)^{(T)}\). That is,
\[ t \cdot s(t^{-1}x) = t^{m_0}s(x), \ \forall x \in X \text{ and } t \in T. \]

By definition, the pull back of any section \(\tilde{s} \in H^0(X^{[k]}, (-K_X)^{\sigma_0}[k])\) on \(X \times \mathbb{C}^{k+1}\) is a \(T\)-invariant section of \(-K_X \times \mathbb{C}^{k+1}\), and any such section descend to a section in \(\mathcal{H}^0(X^{[k]}, (-K_X)^{\sigma_0}[k])\). Set \(\tilde{s}_0 = s_0 \times \mathbb{C}^{k+1}\). Then any
\[ \tilde{s}_{0, \mu} = \tilde{s}_0 \prod_{A=1}^{r} \left(\zeta^{(A), 0} \mu_A, 0 \ldots \zeta^{(A), k_A} \mu_A, k_A\right) \]
with \(\mu_A, 0 + \ldots + \mu_A, k_A = \chi_0, A\) for each \(A = 1, \ldots, r\) is a \(T\)-invariant section of \(-K_X \times \mathbb{C}^{k+1}\), that descends to a section \(s_{0, \mu}\) of \((-K_X)^{\sigma_0}[k]\). Clearly, the divisor of \(\tilde{s}_{0, \mu}\) is \(T\)-invariant and descends to the divisor \(d_{s_{0, \mu}}\) of \(s_{0, \mu}\). Denote by \(d_{s_0} \subset X\) the divisor of \(s_0\) in \(X\), which is also \(T\)-invariant, we have
\[ (7.3) \]
\[ d_{s_{0, \mu}} = d_{s_0}^{[k]} + \sum_{A=1}^{r} \text{pr}_A^* \mathcal{O}_{\mathbb{P}^A}(\chi_{0, A}). \]

On the other hand, recall the local coordinate charts \(U_{[i]}\) with each \(i_A \in \{0, \ldots, k_A\}\) constructed in Section 3.2, and the local coordinates \((x_{[i]}, \zeta_{[i]})\) on it. It is direct to check that on each \(U_{[i]} \cap U_{[j]}\),
\[ x_{[i]} = (\zeta_{[i]}^{j}) \cdot x_{[j]}, \quad \zeta_{[j]} = \zeta_{[i]}^{j}(\zeta_{[i]}). \]
In particular, \(\zeta_{[j]}\) is a function that depends only on \(\zeta_{[i]}\). Thus under the local section
\[ e_{[i]} = (dx_{[i]} \wedge d\zeta_{[i]} = dx_{[i]}^1 \wedge \ldots \wedge dx_{[i]}^n) \bigwedge_{A=1}^{r} (d\zeta_{[i]}^{(A), 1} \wedge \ldots \wedge d\zeta_{[i]}^{(A), k_A}) \]
of \(-K_{X^{[k]}},\) the transition is given by
\[ e_{[i]} = \det \left( \frac{\partial x_{[i]}}{\partial x_{[j]}} \right) \det \left( \frac{\partial \zeta_{[i]}}{\partial \zeta_{[j]}} \right) e_{[j]}. \]
it follows from [11, Section 2] that choice of moment map. Thus by choosing a smooth \( \tilde{\omega} \),

It is well-known that both \( \Delta^+ \) and \( \Delta^- \) are related by:

\[
\Delta^+ \Delta^- \text{ is monomial} \quad (2.2)
\]

Proof. Consider a \( T \)-invariant \( \sigma \) on \( X \). Let \( \Delta^+ \) be the moment polytope and \( g \) a monomial \( g_{\mathfrak{X}} \) on it. Denote by \( \Delta^+ \) the corresponding Duistermaat-Heckman measure. Then

\[
\frac{k!}{(n + |k|)!} \Delta^+(X, L)^n = \int_{\Delta^+(X, L)} g(\lambda) \Delta^+(X, L)(\lambda).
\]

Remark 7.3. Let \( X \) be a \( \mathbb{Q} \)-Fano G-spherical variety. Then there always exists a \( B \)-semiinvariant section \( s \) of \( -K_X \) with divisor \( \sigma \) given by \([5, 3]\) and \( B \)-weight \( \kappa_F \) (cf. [9] Theorem 1.2). By \([3, 9]\), \( s \) is \( T \)-invariant under the canonical lifting of the \( T \)-action and we get a divisor

\[
\mathcal{F} = \sigma[k] + \sum_{A=1}^r \text{pr}_A^* O_{P^A}(k_A + 1)
\]

of \( -K_X[k] \). This is precisely the divisor we get when applying [9] Theorem 1.2 to the variety \( X[k] \).

7.3. An intersection formula.

Lemma 7.4. Let \( X \) be a projective variety with effective \( T \)-action and \( L \) a \( T \)-linearized ample line bundle on it. Let \( \Delta^+ \) be the moment polytope and \( g \) a monomial \( g_{\mathfrak{X}} \) on it. Denote by \( \Delta^+ \) the corresponding Duistermaat-Heckman measure. Then

\[
\frac{k!}{(n + |k|)!} \Delta^+(X, L)^n = \int_{\Delta^+(X, L)} g(\lambda) \Delta^+(X, L)(\lambda).
\]

Proof. Consider a \( T \)-invariant resolution \( \varphi : \tilde{X} \to X \) and \( \tilde{L} := \varphi^* L \) on it. Let \( \omega \in 2\pi c_1(\tilde{L}) \) be a Kähler form on \( X \). Then \( \tilde{\omega} := \varphi^* \omega \) is a semi-positive closed \((1,1)\)-form on \( \tilde{X} \). The moment maps of the \( T \)-actions on \((X, L, \omega)\) and \((\tilde{X}, \tilde{L}, \tilde{\omega})\) are related by:

\[
\tilde{m}_\omega = m_\omega \circ \varphi.
\]

As a consequence, \( \Delta^+(X, L) = \Delta^+(\tilde{X}, \tilde{L}) \) and

\[
\Delta^+(X, L) = \Delta^+(\tilde{X}, \tilde{L}) \quad \text{and} \quad \Delta^+(X, L) = \Delta^+(\tilde{X}, \tilde{L}).
\]

It is well-known that both \( \Delta^+(\tilde{X}, \tilde{L}) \) and \( \Delta^+(\tilde{X}, \tilde{L}) \) are independent with the choice of moment map. Thus by choosing a smooth \( \tilde{\omega} \) \in 2\pi c_1(\tilde{L}) \) and using \( \tilde{m}_\omega \), it follows from [11] Section 2 that

\[
\frac{k!}{(n + |k|)!} \Delta^+(\tilde{L})^n = \int_{\Delta^+(\tilde{X}, \tilde{L})} g(\lambda) \Delta^+(\tilde{X}, \tilde{L})(\lambda)
\]

On the other hand, \( \varphi \) is a generically finite surjective proper map and \( \deg(\varphi) = 1 \). Hence \( (L)^n = (\tilde{L})^n \) and we get the Lemma. \( \Box \)
References

[1] V. A. Alexeev and M. Brion, *Stable reductive varieties II: Projective case*, Adv. Math., 184 (2004), 382-408.
[2] V. A. Alexeev and L. V. Katzarkov, *On K-stability of reductive varieties*, Geom. Funct. Anal., 15 (2005), 297-310.
[3] S. Boucksom, T. Hisamoto and M. Jonsson, *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, Ann. Inst. Fourier (Grenoble), 67 (2017), 743-841.
[4] M. Brion, *Groupe de Picard et nombres caractéristiques des variétés sphériques*, Duke. Math. J., 58 (1989), 397-424.
[5] T. Delcroix, *K-Stability of Fano spherical varieties*, Ann Sci Éc Norm Supér., 53 (2020), 615-662.
[6] T. Delcroix, *Uniform K-stability of polarized spherical varieties*, arXiv:2009.06463.
[7] S. Boucksom, T. Hisamoto and M. Jonsson, *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, Ann. Inst. Fourier (Grenoble), 67 (2017), 743-841.
[8] S. Boucksom, T. Hisamoto and M. Jonsson, *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, Ann. Inst. Fourier (Grenoble), 67 (2017), 743-841.
[9] S. Boucksom, T. Hisamoto and M. Jonsson, *Uniform K-stability, Duistermaat-Heckman measures and singularities of pairs*, Ann. Inst. Fourier (Grenoble), 67 (2017), 743-841.
[10] V. Guillemin and S. Sternberg, *Riemann sums over polytopes*, Ann. Inst. Fourier (Grenoble), 57 (2007), 2183-2195.
[11] J.-Y. Han and Ch. Li, *On the Yau-Tian-Donaldson conjecture for generalized Kähler-Ricci soliton equations*, arXiv:2006.00903.
[12] J.-Y. Han and Ch. Li, *Algebraic uniqueness of Kähler-Ricci flow limits and optimal degenerations of Fano varieties*, arXiv:2009.01010.
[13] J.-M. Kantor and A. G. Khovanskii, *Une application du théorème de Riemann-Roch combinatoire au polynôme d’Ehrhart des polytopes entiers de $\mathbb{R}^d$*, C. R. Acad. Sci. Paris Sér. I, Math. 317 (1993), 501-507.
[14] F. Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. 9 (1996), 153-174.
[15] Ch. Li, *G-uniform stability and Kähler-Einstein metrics on Fano varieties*, Invent. Math., 227 (2022), 661-744.
[16] Y. Li and Zh.-Y. Li, *Finiteness of $Q$-Fano compactifications of semisimple group with Kähler-Einstein metrics*, arXiv:2006.01998. To appear in Int. Math. Res. Not. IMRN.
[17] Y. Li and Zh.-Y. Li, *Equivariant $R$-test configurations of polarized spherical varieties*, arXiv:2206.04880.
[18] Y. Li and F. Wang, *On the limit of Kähler-Ricci flow on $Q$-Fano spherical variety*, preprint, 2022.
[19] Y. Li and B. Zhou, *Mabuchi metrics and properness of modified Ding functional*, Pacific J. Math., 302 (2019), 659-692.
[20] I. V. Losev, *Uniqueness property for spherical homogeneous spaces*, Duke Math. J., 147 (2009), 315-343.
[21] D. Luna, *Variétés sphériques de type A*, Inst. Hautes Etudes Sci. Publ. Math., 94 (2001), 161-226.
[22] D. Luna and T. Vust, *Plongements d’espaces homogènes*, Comm. Math. Helv., 58 (1983), 186-245.
[23] G. Tian and X.-H. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comm. Math. Helv., 77 (2002), 297-325.
[24] D. A. Timashëv, *Equivariant embeddings of homogeneous spaces*, Surveys in geometry and number theory: reports on contemporary Russian mathematics, 226-278, London Math. Soc. Lecture Note Ser., 338, Cambridge Univ. Press, Cambridge, 2007.
[25] D. A. Timashëv, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, 138. Invariant Theory and Algebraic Transformation Groups, 8. Springer-Verlag, Berlin-Heidelberg, 2011.
[26] F. Wang, B. Zhou and X.-H. Zhu, *Modified Futaki invariant and equivariant Riemann-Roch formula*, Adv. Math., 286 (2016), 1205-1235.
[27] Y. Yao, *Mabuchi Solitons and Relative Ding Stability of Toric Fano Varieties*, arXiv:1701.04016. To appear in Int. Math. Res. Not. IMRN.
[28] V. I. Danilov, Kogomologii algebraiqeskih mnogoobrazi/ishort, Itogi nauki i mehni.
Ser. Sovrem. probl. mat. Fundam. направления, том 35 (1989), 5-130.
Eng.: V. I. Danilov, Cohomology of algebraic varieties (in Russian), Itogi Nauki i Tekhniki.
Ser. Sovrem. Probl. Mat. Fund. Napr., 35 (1989), 5-130.

[29] D. P. Желобенко и А. И. Штерн, Представления групп Ли, Издательство Наука, Москва, 1983 г.
Eng.: D. P. Zhelobenko i A. I. Shtern, Representations of Lie groups (in Russian), Press “Nauka”, Moscow, 1983.

[30] V. L. Popov, Стабилизация действий редуктивных алгебраических групп, Математический Сборник, 130(172) (1986), 310-334.
Eng.: V. L. Popov, Contractions of the actions of reductive algebraic groups, Mat. Sb., 130(172) (1986), 310-334.

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