Large scale behavior of the energy spectra of the quantum random antiferromagnetic Ising chain with mixed transverse and longitudinal fields

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In recent years it became clear that the metallic regime of systems that exhibit a many body localization (MBL) behavior show properties which are quite different than the vanilla metallic region of the single particle Anderson regime. Here we show that the large scale energy spectrum of a canonical microscopical model featuring MBL, displays a non-universal behavior at intermediate scales, which is distinct from the deviation from universality seen in the single particle Anderson regime. The crucial step in revealing this behavior is a global unfolding of the spectrum performed using the singular value decomposition (SVD) which takes into account the sample to sample fluctuations of the spectra. The spectrum properties may be observed directly in the singular value amplitudes via the scree plot, or by using the SVD to unfold the spectra and then perform a number of states variance calculation. Both methods reveal an intermediate scale of energies which follow super Poissonian statistics.

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I. INTRODUCTION

Many-body localization (MBL)\cite{1} has captured the imagination of researchers since its inception more than a decade and a half ago. Once interactions are introduced to a many-particle system for which all single particle states are localized, a parameter region where the many-particle states are extended should appear, as expected from the many-body thermalization hypothesis\cite{2}. Other regions of the parameter space remain localized even in the presence of interactions. Almost immediately, an effort to identify the transition point by analyzing the spectra of microscopic models of disordered interacting many-particle systems began. The spectra of 1D spin chains and electronic models\cite{3-8} were probed in order to identify a signature of a transition (or crossover) between the two regions. Nevertheless, despite much effort a definitive answer remains elusive.

For microscopic models of MBL one runs into an insurmountable obstacle in analyzing the energy spectra. The Hilbert space grows exponentially and for conventional computers it is hard to imagine that one will reach large enough systems for which the analysis of the spectra will give an indisputable finite size scaling. Nevertheless, there is still a point in looking into the spectral properties of small microscopic models, for two main reasons. The first, is that although the systems studied are small, there are nevertheless some behaviors which emerge in a robust form even for these sizes. Although it might not be possible to prove that these behaviors survive in the thermodynamical limit, it is still worthwhile to understand them\cite{9}. Second, many current experimental studies searching for a signature of the MBL\cite{10-12} are performed on systems of similar small size.

Here we would like to examine a particular microscopical model of a quantum random antiferromagnetic Ising chain with mixed transverse and longitudinal fields, sometimes referred in the MBL literature as the Imbrie model. The ground state of this model has been known to exhibit a rich phase diagram\cite{13,14} and recently the model has garnered considerable interest in the context of MBL\cite{15,16}. This interest stems from the assertion that under some assumptions, it is possible to rigorously show that it undergoes a MBL transition from metallic to localized behavior as disorder increase\cite{17}.

What is the nature of the extended region of the Imbrie model? Is this region analogous to the single particle Anderson metallic phase? These are the questions we would like to address in this paper. For single particle Anderson metallic regime, the energy spectrum follows the random matrix predictions\cite{18,19,20} up to an energy scale known as the Thouless energy\cite{21} above which a different behavior is observed, where the Thouless energy $E_{Th} = hD/L^2 = g\delta$ ($D$ is the diffusion constant, $L$, is the linear dimension, $g$, is the dimensionless conductance, and $\delta$ the average level spacing). The physical origin of the Thouless energy is the time needed for a wave packet to cover the whole sample known as the Thouless time $t_{Th} = h/E_{Th} = L^2/D$. At shorter times (larger energy scales) the system is not ergodic, hence the different energy spectrum behavior at this scale.

The generalized Rosenzweig-Porter random matrix model (GRP)\cite{22,23} is probably the simplest random matrix model which shows three distinct phases: Localized at strong disorder, non-ergodic extended (NEE) phase for intermediate disorder, and a fully ergodic extended phase at weak disorder. The NEE phase exhibits unusual features such as fractality of the wave functions\cite{24,25} and super Poissonian behavior of the energy spectrum at intermediate energy scales\cite{26,27}. Focusing on the energy spectrum, one discovers that the nearest neighbor statistics (small energy scale, corresponding to long times) is indistinguishable from the extended metallic phase, while for intermediate energy scales a super-Poissonian behavior of the n-th level spacing distribution has been observed\cite{28}. Examining the singular value decomposition (SVD) of the spectrum of an ensemble of realizations supports this conclusion. Moreover, the SVD amplitude
scree plot, which in the NEE phase show three different regimes as function of the mode number (essentially inverse energy, where low modes correspond to large energy scales). High modes (small energy scales) show a Wigner behavior, then it crosses to a super-Poissonian behavior for the intermediate range of modes, finally switching at low modes (large energy scales) to a Poisson form. Thus two transition energies in the spectrum emerge. The lower transition energy corresponds to a transition from a universal Wigner behavior typical to the metallic regime, to a non-universal NEE regime. Following previous work, the energy at which this transition occurs will be termed the Thouless energy. It is important though to note that although the same terminology for the Thouless energy as for the single particle Anderson transition is used, it does not necessarily mean that the same physics is behind it. This will be discussed further on. The second transition between the NEE regime and Poisson like behavior behavior has no direct analogue in the single particle metallic systems. One must keep in mind that for any system the energy spectrum on large scales is determined by the global band structure, which is captured in the first few modes in the scree plot. The NEE behavior has an additional time scale which is the onset of the extended behavior. Thus, a region of small modes of the SVD will be needed to capture the pre-extended region. The energy for which the extended behavior is manifested is the second transition energy and will be termed, $E_{Ex}$, the extended energy. Much of the interest in the GRP model stems from the proposal that it might capture properties relevant to MBL systems. Indeed we will demonstrate that these two transition energies emerge also for the Imbrie model, and the meaning of this large energy scale will be discussed further on.

One of the most interesting questions investigated by the MBL community is the nature of the metallic regime close to the localized regime. This region exhibits at intermediate times (intermediate energy scales) different behavior than expected from the canonical Anderson transition. For example, the time evolution of the system is sub-diffusive and relaxation toward equilibrium is anomalously slow, a behavior that might have been seen also in experiment. The same region also exhibits fractal behavior of the eigenfunctions.

The fractal behavior of the eigenfunctions as well as the sub-diffusive time evolution have several complementary explanations. One of the main routes to an explanation of the NEE behavior is via the picture localization in the Fock space, which has originally motivated the study of MBL. Essentially the coupling of states in the Fock space creates a quantum random graph which leads to non-ergodic behavior and fractal structure of the states in Fock space. This behavior has also been associated to rare regions in the 1D systems known as the Griffiths region, which could explain the sub-diffusive behavior. Nevertheless, this cannot be the whole picture since sub-diffusive behavior is seen also in systems of higher dimensionality, both numerically and experimentally, while the Griffiths regions influence is limited to 1D systems. A complementary view suggests that the MBL transition is a Kosterlitz Thouless (KT) transition. In this picture, rare regions of extended states may appear which will lead to an avalanche delocalizing the whole sample if disorder is not too strong. This will lead naturally to fractal structure of the Hilbert space and to a NEE behavior.

Here we would explore whether there are signatures of super Poissonic statistics at intermediate energies in the Imbrie model, similar to the behavior seen in GRP. The usual way of examining the behavior at large energy scales is the variance of the number of levels as function of the size of an energy window after a local unfolding of the energy spectrum, i.e., $\langle \delta^2 n(E) \rangle = \langle (n(E) - \langle n(E) \rangle)^2 \rangle$, where $\langle \cdots \rangle$ denotes an average over an ensemble of different realizations of disorder and $n(E)$ is the number of levels within an energy window $E$. In the Poisson regime $\langle \delta^2 n(E) \rangle \sim \langle n(E) \rangle$, while in the Wigner regime it grows logarithmic. Deviation from the logarithmic behavior to a stronger than linear behavior at large energies have been seen in metallic system beyond the Thouless energy and the Sachdev-Ye-Kitaev (SYK) model and many body localization systems. As we have shown in Refs. 41,89 there are some problems in the application of the local unfolding in systems where the local density shows strong sample to sample fluctuations or a non-smooth band structure which may skew the results. In order to circumvent these problems we will use a different method to study the properties of the spectra, known as singular value decomposition (SVD). This method has been successfully applied to analyze the transition from Wigner to Poisson statistics in the Anderson transition and to characterizing the NEE in the GRP model. To study the large energy scale spectrum behavior beyond the Thouless energy in metallic systems and very recently to the MBL transition in the Heisenberg chain.

As will be discussed in detail in the appendix, SVD essentially returns a set of modes which can be used to construct the energy spectra of the different realizations in the ensemble. Arranging the modes according to the size of their amplitude squared, $\lambda_k$ (where $k = 1$ is the largest), the first few $\lambda_k$ (O(1)) correspond to global features of the spectrum. Thus one can globally unfold the spectra by filtering out these modes when reconstructing the spectrum. Then the unfolded spectrum can be used to obtain the number variance. A different way to obtain a comprehensive picture of the behavior of the energy spectrum is to plot $\lambda_k$ vs. $k$, also known as a scree plot. Usually, a power law behavior $\lambda \sim k^{-\alpha}$, is detected for certain ranges of $k$. The power law exponent corresponds to the statistics of the energy spectrum with $\alpha = 2$ for the Poisson behavior, $\alpha = 1$ for the Wigner regime. For energies larger than $E_{Th}$ (small values of $k$) in the metallic regime of a single particle Anderson model $\alpha = 1 + d/2$ (where $d$ is the dimensionality).

In Ref. 41 we have shown that for GRP model in the
NEE phase shows for intermediate values of $k$ a power-law behavior with $\alpha > 2$. This behavior is consistent with super-Poissonian statistics. At large values of $k$ (short energy scales) the singular value curve returns to the $\alpha = 2$ exponent, i.e., Poissonian statistics. Thus the super Poisson behavior starts at the Thouless energy, and terminates at $E_{Ex}$. Moreover, as has been discussed in Ref. [29], the power law of the SVD amplitude scree plot is connected to the power law behavior of the number variance, $\langle \delta n(E) \rangle \propto \langle n(E) \rangle^\beta$, with $\beta = \alpha - 1$, where in the Poisson regime $\alpha = 2$ and $\beta = 1$, while in the Wigner regime $\alpha = 1$ and $\beta = 0$ (actually logarithmic). In the NEE regime one expects a super Poissonian behavior of the number variance $\beta > 1$ and therefore $\alpha > 2$.

In this paper we shall introduce the random antiferromagnetic Ising chain (Imbrie model) in Sec. [II]. The dependence of the density of states and ratio statistics (nearest neighbor level statistics) on the strength of disorder is presented in Sec. [II]. As expected, above a certain strength of disorder finite size scaling indicates a localized regime, while for weaker disorder an extended regime emerges. Then (Sec. [IV]) the locally unfolded spectra is used to study the level number variance. The results deviate from RMT predictions (whether Wigner in the extended regime or Poissonian the localized) for higher energy scales, and in the extended regime seem to follow a super Poissonian behavior. Nevertheless, due to the structure of density of states as well as to strong sample to sample fluctuations, one must question the validity of the local unfolding. Therefore we turn to the SVD scree plot, in order to get a better picture of the larger energy scale behavior of the spectrum in Sec.V. The scree plot suggests that for small energies, the system follows the expectations garnered from the ratio statistics. Then at a particular mode (corresponding to the Thouless energy) the power law change to a super Poissonian behavior. Nevertheless, due to the structure of the level number variance, the results deviate from RMT predictions (whether Wigner in the extended regime or Poissonian the localized) for higher energy scales, and in the extended regime seem to follow a super Poissonian behavior. Nevertheless, due to the structure of density of states as well as to strong sample to sample fluctuations, one must question the validity of the local unfolding.

III. SMALL ENERGY SCALES

As a first step we would like to calculate the nearest neighbor level spacing statistics in order to establish for which values of disorder $W$ we see extended states. Since small energy scales for the extended regime follow the Wigner statistics while for the localized regime they follow Poisson statistics, a finite size scaling of a measure probing the nearest neighbor level statistics should reveal in what regime the system is. As a measure we shall use the ratio statistic [27], defined as:

$$r_n = \frac{E_n - E_{n-1}}{E_{n+1} - E_n},$$

where $E_n$ is the $n$-th eigenvalue of the Hamiltonian and $\langle \ldots \rangle$ is an average over different realizations of disorder and a range of eigenvalues around the middle of the energy spectrum. This measure has the advantage of avoiding the unfolding procedure. For the Poisson statistics $r_n = 2 \ln 2 - 1 \approx 0.3863$, while for the GOE Wigner distribution $r_n \approx 0.5307^{[21]}

In Fig. [II] $r_s$, for sample sizes, $L = 12, 13, 14, 15$ (corresponding to Hilbert space sizes of $2^L$), is presented. The matrices were exactly diagonalized and all eigenvalues $E_n$ were obtained. $r_s$ was averaged over $M$ realizations, where $M = 3000$ for $L = 12, 13, 14$, while for $L = 15, M = 1000$ realizations, using $P = 2L/2$ eigenvalues around the center of the band. For finite $W$ a typical transition pattern is seen: Above $W \sim 5$ the larger is $L$ the lower the value of $r_s$ and the closer to the Poisson value it becomes. Below the value of $W \sim 5$ (except for in the vicinity of $W = 0$) the order is the opposite, the larger the $L$, the higher its $r_s$ value and the closer its value is to the GOE statistics. All the curves seem to cross at the same value of $W \sim 5$. Thus, roughly speaking, the behavior of $r_s$ shows the finite size scaling features of a second order localization transition. At $W = 0$ the system again coalesces at the Poisson value.

The Hamiltonian for the random antiferromagnetic Ising chain of length $L$ with mixed transverse and longitudinal fields (Imbrie model) is given by [20][23]

$$\hat{H} = \sum_{i=1}^{L} h_i \hat{S}_i^z + \sum_{i=1}^{L} \gamma_i \hat{S}_i^x + \sum_{i=1}^{L-1} J_i \hat{S}_i^z \hat{S}_{i+1}^z.$$ (1)

where $S_i^\alpha$ is the spin on site $i$ in direction $\alpha$, $h_i$ is a random magnetic field in the $\hat{z}$ direction on site $i$ drawn from a box distribution between $-W/2$ and $W/2$, and $\gamma_i = 1$. $J_i$ are the nearest-neighbor spin-spin antiferromagnetic interactions which following Abanin et al.[23] are drawn from a box distribution in the range 0.8 and 1.2. $L$ is the length of the chain.

The corresponding to Hilbert space has size of $2^L$, and we calculate the eigenvalues $E_i$ using exact diagonalization for $M$ realizations for a given disorder.

II. IMBRIE MODEL

The Hamiltonian for the random antiferromagnetic Ising chain of length $L$ with mixed transverse and longitudinal fields (Imbrie model) is given by [20][23].
while for realization. Circles correspond to the numerical results, while \( W \) values of continuous and dashed lines. On the left side the whole range transition to the localized regime, the values of tended regime and not to determine the nature of the \( W = 0 \).

Thouless like behavior indicates that the crossover occurs very close to \( W = 0 \).

Since our aim in this study was to investigate the extended regime and not to determine the nature of the transition to the localized regime, the values of \( W \) around the intersection of the curves was not calculated with enough points around it and the averaging was not performed on a sufficient large number of realization to establish that the crossing corresponds to a second order transition. At this point we can not be sure that the crossing does not drift with size or show Kosterlitz-Thouless like behavior.

**IV. NUMBER VARIANCE WITH LOCAL UNFOLDING**

We start by plotting the average density of states \( \langle \nu(\varepsilon) \rangle \) for different values of disorder. As can be seen in Fig. 2 the level density widens as expected when the disorder increases. Moreover, it is also apparent that the density becomes more smooth as \( W \) increases. For \( W < 2 \) some additional (quasi-) regular structure of the density is seen.

Even for stronger disorder where the average density of states seems smooth, significant realization dependent structure remain. This can be seen in Fig. 3 where the averaged level spacing over a range of eigenvalues \( l \) around the \( p \)-th level \( \delta_p(l) = (E_{p+l/2} - E_{p-1/2})/l \) is plotted. Two typical realizations are presented for \( W = 2 \), where \( p \) ranges over the the middle half of the eigenvalues, while three different values of \( l = 100, 200, 400 \) are presented. In addition to the expected smooth global increase of \( \delta \) as \( p \) moves from the center of the band, \( \delta_p(l) \) shows long range sample specific fluctuations on scales of hundreds of levels.

Such a behavior hints towards the existence of a large scale structure of the energy spectrum and sample to sample fluctuations. This poses a challenge since when one studies the number variance one would like to filter out global or sample dependence regular behavior. This can be problematic since one has to separate global behavior from sample to sample fluctuations. Let us start by a naive application of the local unfolding. In order to calculate \( \langle \delta^2 n(E) \rangle \), we unfold the spectrum by \( \varepsilon_i = \varepsilon_{i-1} + 2m(E_i - E_{i-1})/(E_{i+m} - E_{i-m}) \) where \( m = 6 \) (other values were used with no significant change). We place the window at the center of the band then the averages \( \langle n(E) \rangle \) and \( \langle n^2(E) \rangle \) are calculated over all \( M \) realizations. The results are shown in Fig. 4.
The variance $\langle \delta^2 n(E) \rangle$ as function of $\langle n(E) \rangle$ for $L = 14$ with disorder $W = 0$ and $W = 2$. The Poisson behavior, $\langle \delta^2 n(E) \rangle = \langle n(E) \rangle$, and Wigner behavior, $\langle \delta^2 n(E) \rangle = (2/\pi^2) \ln(\langle n(E) \rangle)$, are indicated by dashed curves. The inset zooms into the small $n$ region, where the expected Poisson (for $W = 0$) and Wigner ($W = 2$) behavior is seen. For larger energy scales depicted in the main figure, a completely different behavior is seen. For $W = 0$ a very non-monotonous behavior is observed, while for $W = 2$, $\langle \delta^2 n(E) \rangle \sim \langle n(E) \rangle^{\beta}$ with $\beta = 2.02$ fits reasonably well.

The variance $\langle \delta^2 n(E) \rangle$ as function of $\langle n(E) \rangle$ is depicted in Fig. 4 for $L = 14$ (matrix linear size $2^{14} = 16384$) and two values of disorder $W = 0$ and $W = 2$. As we have seen from the ratio statistics $r_n$ (Fig. 3), $W = 0$ follows Poisson statistics for small energy scales while $W = 2$ follows Wigner at these scales. Indeed, as can be seen in the inset of Fig. 4 for small values of $\langle n \rangle$ the expected behavior of the number variance is followed, i.e., $\langle \delta^2 n(E) \rangle = \langle n \rangle$ for Poisson and $\langle \delta^2 n(E) \rangle = (2/\pi^2) \ln(\langle n(E) \rangle) + 0.44$ for Wigner (GOE). Nevertheless, as larger energies are examined, strong deviations from the Poisson or Wigner behavior are observed.

The large scale behavior is very different between these two values of disorder. For $W = 0$ the linear behavior quickly saturates, but a very non-monotonous behavior is apparent. One cannot escape the feeling that a large scale structure with strong sample to sample fluctuation that lurks in the spectra is not correctly addressed by the local unfolding. For $W = 2$ the large scale behavior is quite monotonous, shows a strong super Poissonian behavior where the number variance shows a power law dependence on the average number of states, $\langle \delta^2 n(E) \rangle \sim \langle n \rangle^{\beta}$ with $\beta = 2.02 \gg 1$. Although the fit seems rather decent, one must wonder how reliable is it and whether we are seeing an artifact of the local unfolding.

A possible cure to the sample to sample fluctuations is averaging also over the center of the energy window. In Fig. 5 the number variance is also averaged over 21 positions of the center of the energy window, $\tilde{E}$, equally spaced around the band center, where the furthest point is no more than 1/15 of the bandwidth from the center.

The number of states, $n(E, \tilde{E})$, in a window of width $E$ centered at $\tilde{E}$, is calculated, then the averages $\langle n(E) \rangle$ and $\langle n^2(E) \rangle$ are taken over all positions of the center $\tilde{E}$ and all $M$ realizations. As can be seen, the large scale non-monotonous behavior for $W = 0$ is somewhat dampened, while the behavior for $W = 2$ remains essentially the same. Nevertheless, the question remains how much of these results are an artifact of the unfolding and sample to sample fluctuations. We shall address global methods of unfolding in the next sections.

V. SINGULAR VALUE DECOMPOSITION SCREE PLOT

As result of these difficulties with the local unfolding, we change tack and use a different method to study the spectrum, i.e., the SVD method. In this method no local unfolding is performed, and is replaced by global unfolding. Essentially the spectrum of $M$ realizations of disorder each with $P$ eigenvalues is arranged as a matrix $X$ of size $M \times P$ where $X_{mn}$ is the $p$ level of the $m$-th realization. As detailed in the appendix after carrying out SVD on $X$, we can write the matrix as a sum of amplitudes, $\sigma_k$, multiplied by matrices, $X^{(k)}$, i.e., $X = \sum_k \sigma_k X^{(k)}$.

One may rank the amplitudes from the largest to the smallest, and thus the lower values of $k$ represent modes with higher contributions to reconstructing the matrix. Moreover, the lower modes tend to code the global behavior of the matrix. Plotting the singular values squared $\lambda_k = \sigma_k^2$ according to their rank is known as the singular value scree plot[12-13] and much information can be gleaned from it. This approach has been applied to the spectrum of disordered systems in several studies[11,20,29] the first few $\lambda_k$ ($k \leq O(1)$) correspond to global features.

FIG. 4: The variance $\langle \delta^2 n(E) \rangle$ as function of $\langle n(E) \rangle$ for $L = 14$ with disorder $W = 0$ and $W = 2$. The Poisson behavior, $\langle \delta^2 n(E) \rangle = \langle n(E) \rangle$, and Wigner behavior, $\langle \delta^2 n(E) \rangle = (2/\pi^2) \ln(\langle n(E) \rangle) + 0.44$, are indicated by dashed curves. The inset zooms into the small $n$ region, where the expected Poisson (for $W = 0$) and Wigner ($W = 2$) behavior is seen. For larger energy scales depicted in the main figure, a completely different behavior is seen. For $W = 0$ a very non-monotonous behavior is observed, while for $W = 2$, $\langle \delta^2 n(E) \rangle \sim \langle n(E) \rangle^{\beta}$ with $\beta = 2.02$ fits reasonably well.

FIG. 5: As for Fig. 4 with an additional average over different positions of the center of the energy window. The averaging does not change much. For the large energy scales in the localized case ($W = 0$), the non-monotonous behavior is somewhat dampened, while for $W = 2$, $\beta = 1.98$ is similar to the previous result.
of the spectra. Higher SV (λ_k) show a power law behavior \( k^{-\alpha} \). In the Poisson regime \( \alpha = 2 \) for high modes, while \( \alpha = 1 \) in the Wigner regime.

For the GRP model\(^{22}\), the same behavior was seen for weakly disordered (extended) and strongly disordered (localized) regime. For the intermediate disorder NEE regime, one expects small energy scales to show Wigner properties. Indeed, large \( k \)'s follow \( \alpha \sim 1 \). Intermediate values of \( k \), corresponding to intermediate energy scales show unconventional behavior. They follow a power law, but with \( \alpha > 2 \). This super Poissonian behavior was interpreted as the signature of the NEE phase. At small \( k \) corresponding to larger energy scales (small times) a return to an exponent of \( \alpha = 2 \) is observed.

A somewhat similar picture emerges for the Imbrie model. Increasing the disorder results in a change of the dependence of the SV amplitudes on the mode number \( k \).

For \( W = 0 \) the high \( k \) values follow a power law \( \lambda_k \sim k^{-\alpha} \) with \( \alpha = 2 \), as expected from a localized system, matching with ratio statistics results (Fig.5). A sudden switch in the exponent to \( \alpha = 1 \) occurs at \( k \sim 200 \). This exponent is equal to the exponent exhibited by Wigner statistics.

As can be seen in Fig.6 for weak disorder (\( W = 1,2 \)) the behavior of the SV is quite different. For \( k > 200 \) and \( W = 1,2 \) an exponent of \( \alpha = 1 \) is evident, as expected from systems in the Wigner (extended) regime. This changes to an exponent larger than two (for \( W = 1, \alpha = 2.3 \) for \( 95 < k < 150 \); for \( W = 2, \alpha = 2.3 \) for \( 40 < k < 105 \)) for intermediate values of \( k \). Then, similarly to \( W = 0 \), the exponent switches back to \( \alpha = 1 \). Thus, in the regime of extended behavior, the SV amplitudes have three distinct behaviors for different ranges of \( k \). Wigner for large values of \( k \) (small energy scales, long times), super-Poissonian (\( \alpha > 2 \)) for an intermediate range of \( k \), and back to \( \alpha = 1 \). This indicates that the extended regime in the Imbrie model is far from trivial and signatures of different physics show up at intermediate energy scales. This behavior is somewhat similar to the behavior seen for the SV in the GRP model\(^{22}\). Both the GRP and Imbrie models in the extended regime exhibit Wigner behavior at large times (small energy scales, large \( k \)), super Poissonian behavior associated with non ergodicity at intermediate times and energy scales. For short times (large energy scales, small \( k \)) the Imbrie and the GRP models show a different behavior expressed by different exponents (\( \alpha = 2 \) for GRP, \( \alpha = 1 \) for Imbrie).

That is the result of the large scale structure of the density of states seen in Fig. 2 very clearly for \( W = 0 \), but still hinted for somewhat stronger disorder. We shall elaborate on it in the following section. Thus, SVD provides support for the existence of a NEE regime for the Imbrie model in the weakly disordered extended regime.

For \( W = 3 \) a crossover behavior is seen. For \( 700 < k < 2000, \alpha = 1.5 \) while for \( 50 < k < 700, \alpha = 2.1 \), and then for \( 10 < k < 30, \alpha = 1 \). Clearly, even for large \( k \) we do not see a clear GOE behavior expected on the basis of the ratio statistics behavior (Fig. 1). The SV scree plot behavior seems as a crossover between Poisson and Wigner. Thus, although finite size behavior of nearest neighbor ratio statistics unequivocally puts the \( W = 3 \) disorder in the Wigner regime, the larger energy scales do not show it. This indicates that the larger energy scales which correspond to short times are crossing over to a closer to Poisson behavior earlier than the short energy scales. The Wigner regime transits to higher \( \alpha \) values, while the NEE regime moves towards smaller values of \( \alpha \) closer to two. Indeed, The \( W = 4 \) shows an almost pure Poisson behavior although the disorder is smaller than the critical disorder associated with the ratio statistics. A similar difference between the ratio statistics corresponding to level spacing scale and scree plot behavior was very recently noted in Ref. \(^{80}\) for the Heisenberg chain.

In the localized regime (\( W = 5,6 \)) the expected Poisson exponent, \( \alpha = 2 \), is seen for \( k > 100 \). For smaller values of \( k \) the exponent tappers, and it is hard to determine whether \( \lambda_k \) even follows a power law at all. Nevertheless, it looks that this region becomes smaller as \( W \) increases.

Returning to the weakly disordered metallic regime, we would like to examine more carefully the intermediate energy scale for which the super Poissonian behavior is observed. The first issue to address is the dependence of the scree plot on the number of realizations \( M \). In Fig. 7 \( \lambda_k \) as function of \( k \) for \( L = 14 \) and \( W = 2 \) with a range of \( P = 2^L/2 = 8192 \) eigenvalues around the center of the band are shown. Four different numbers of realizations \( M = 1000, 2000, 4000, 6000 \) have been calculated and are presented in the inset of Fig. 7. As the number of SV modes \( r = \min(M, P) \) (see appendix), and here \( P > M \) in all cases, the number of modes is \( r = M \). It is clear...
that for small $k$'s the curves are very similar. Recalling $\lambda_k$ to $\lambda_k/M$ results in all the curves falling on top of each other for $k < 100$ (Fig. 7). In the intermediate regime $40 < k < 100$ for which the super Poissonian regime with an exponent of $\alpha = 2.3$ is observed the scaled curves coalesce almost perfectly. For higher modes ($k > 100$) although the curves do not coalesce (which is natural since they terminate at different values of $k = M$), nevertheless, the exponents are all $\alpha = 1$ for a significant range of $k$. Thus, for a reasonable number of realizations one gets a decent representation of large and intermediate scale behavior of the energy spectrum.

When one increases the range of eigenvalues, $P$, while keeping the number of realization $M$ fixed it is possible to track the two energies determining the crossover from GOE to super Poissonian behavior, $E_{TH}$, and the transition from the super Poissonian regime to a Poissonian regime, $E_{EX}$. One might expect that since SVD modes describe the energy spectrum of width $P\delta$, resulting in the $k$th mode corresponding to a $P\delta/k$ energy range. Thus the position of $k_{TH}$, the mode for which the exponent changes should depend linearly on $P$. Indeed, from Fig. 8 which presents a scree plot of the SV of $L = 14$ ($2^L = 16384$) and $L = 15$ ($2^L = 32768$) deep in the weak disorder regime ($W = 2$) for $M = 4096$ ($L = 14$) or $M = 2048$ ($L = 15$) realizations, and different ranges of eigenvalues $P$ centered around the middle of the band, one can see that the SV amplitudes, $\lambda_k$, scale as $1/P$. As can be seen in the insets, all curves coincide after rescaling. Estimating the energy scales from the scree plots leads to: $E_{TH} \sim P\delta/k_{TH} \sim 80\delta$ ($L = 14$) and $E_{TH} \sim 160\delta$ ($L = 15$). Similarly, $E_{EX} \sim M\delta/k_{EX} \sim 200\delta$ ($L = 14$) and $E_{EX} \sim 400\delta$ ($L = 15$). Since, roughly speaking, $\delta \sim B/2^L$, (where $B$ is the band width which depends only weakly on $L$), one may postulate that $E_{TH}$ and $E_{EX}$ correspond to a fixed fraction of the band width for the same disorder. The values of $E_{TH}$ and $E_{EX}$ are within the same ranges for which we observed the large scale structure in Fig. 3.

In Fig. 9 we probe the influence of size, $L$, on the intermediate region. Here $L = 12, 13, 14, 15$, $W = 2$ and $M = 2048$ realizations are considered for all sizes. In all cases $P = 2^L/2$ (half of the eigenvalues around the middle of the band). The exponent in the intermediate energy range increases as the size becomes larger. For $L = 12, 13$ the exponent $\alpha = 2.2$, for $L = 14$ its $\alpha = 2.3$, and for $L = 15$, the largest size considered here, $\alpha = 2.65$. One may conclude that the intermediate super Poissonian behavior is enhanced by the increase of the system
FIG. 9: The scree plot of $\lambda_k$, as function of $k$ for $W = 2$ and four sizes of $L = 12, 13, 14, 15$ with $M = 2048$ realizations. In all cases $P = 2^h/2$. Fits to power laws $\lambda_k \sim k^{-\alpha}$, are depicted by the lines, where the blue line correspond to $\alpha = 2.2$ (fit $L = 12, 13$), the cyan to $\alpha = 2.3$ (L = 14), and the dark green to $\alpha = 2.65$ (L = 15). The transition between the universal and super Poissonian behavior, $k_{Th}$, and between the super Poissonian and large scale behavior, $k_{Ex}$, is similar for all sizes.

size. Moreover, the crossover regions between the regions becomes sharper and $k_{Th}$ and $k_{Ex}$ easier to pinpoint as $L$ increase. It can be also seen that for all sizes the Thouless and large scale energy scales do not vary much, in line with our previous conclusion that they depend on the band width.

Although $E_{Th}$ moves to lower energies as $W$ increases (at least when $W$ approaches the transition value) similar to the Thouless energy for the single particle Anderson model there are nevertheless important differences for the larger energy scales. Indeed, for both Anderson localization and MBL (or GRP) the behavior on larger scales is super Poissonian ($\alpha > 2$), but there are two main differences. The first is that while for the Anderson case $\alpha$ depends mainly on dimension and only weakly on $W$ and not at all on $L$, for the MBL model the power law has a very strong dependence on disorder and system size. The second difference is that for the MBL case an additional energy scale ($E_{Ex}$) is evident, while for the Anderson model it is absent.

Thus, the super Poissonian regime seems robust and not a fluke of the range of eigenvalues considered or small size. Nevertheless, from the available data it is not possible to extrapolate what is the $\alpha$ value at infinite size.

VI. SINGULAR VALUE DECOMPOSITION
GLOBAL UNFOLDING

Another way that SVD can be used, is to apply its results for unfolding the spectra and then perform a standard number variance calculation. The unfolding is based on reconstructing the matrix $X$ where the first (or few) contributions of the SV decomposition are dropped since they encode the global behavior. Specifically, capturing the global behavior of the energy spectrum by $X_{ip} = \sum_{k=m}^{r} \sigma_k X_k^{(i)}$ (see appendix) with $m$ determined by examining the scree plot and identifying the point where the first few modes change the behavior. For example in the scree plot for the $L = 14$, $W = 2$ case seen in the inset of Fig. [6] one chooses $m = 4$.

Defining the global unfolded $l$-th eigenvalue of the $p$-th realization as:

$$\tilde{\varepsilon}_l = \tilde{\varepsilon}_{l-1} + \frac{X_{ip} - X_{i(p-1)}}{X_{ip} - X_{i(p-1)}} + 1,$$

and calculating the number variance centered on the middle of the unfolded spectra, results in the number variance presented in Fig. [10]. Here we focus on the weak disorder regime. First, lets examine the behavior of the number variance for small average numbers, corresponding to small energy scales shown in the main panel. For $W = 0$ we see a close to linear behavior with $\langle \delta^2 n(E) \rangle = \langle n \rangle$. For $W = 1, 2$ we see see in the inset a Wigner (GOE) behavior, $\langle \delta^2 n(E) \rangle = (2/\pi^2) \ln(\langle n(E) \rangle) + 0.44$, which holds up to $\langle n(E) \rangle \sim 10$ for $W = 1$ and $\langle n(E) \rangle \sim 20$ for $W = 2$.

For large energy scales, a different dependence emerges. The variance saturates with quasi-periodic oscillations which are very pronounced for small disorder and dampened at higher $W$. This behavior conforms to the low modes (small $k$) power law seen for $\lambda_k$ (Fig. [6]). In this region $\alpha = 1$, and assuming $\langle \delta^2 n(E) \rangle \sim \langle n(E) \rangle^{\beta - 1}$, leading to the expectation that the power law behavior of the number variance for large energy scale will correspond to $\beta = 0$, i.e., saturation. Nevertheless, on top of the saturation a quasi-periodic oscillations is observed. This is the result of the finite range of $k$ for which the exponent is equal to 1.

For the weakly disordered regime ($W = 1, 2$) the long time (small energy scales) Wigner behavior is followed by an intermediate time and energy scale for which $\langle \delta^2 n(E) \rangle \sim \langle n(E) \rangle^\beta$, and $\beta = 1.3$ (for $W = 1$ fit to the range $10 < \langle n(E) \rangle < 50$, and for $W = 2$ to $30 < \langle n(E) \rangle < 100$). This corresponds to powers larger than 2 we have seen in the scree plot for the SV amplitudes. Thus, this regime corresponds to $E_{Th} < \langle n(E) \rangle \delta < E_{Ex}$. The estimation in the previous section (assuming a factor two) for $L = 14$ and $W = 1$ of $E_{Th} \sim 20\delta$ and $E_{Ex} \sim 40\delta$, while for $W = 2$, $E_{Th} \sim 40\delta$ and $E_{Ex} \sim 100\delta$. These estimations fits reasonably well the range of the super Poissonian behavior seen for the globally unfolded number variation. Moreover, $\beta = 0 = 1.3$ in line with the behavior of the exponent observed for the SV in the region of $k_{Th} < k < k_{Ex}$.

VII. DISCUSSION

In the previous sections it has been shown that the energy spectra of the quantum random antiferromagnetic
the range of eigenvalues were considered there, similar to the deviation from the universal behavior of the energy fluctuations are taken into account using the SVD method, both by scrutinizing the scree plot of the SV amplitudes as well as studying the number variance of the spectra after unfolding the spectra by SVD.

Thus, for a model which is one of the canonical microscopical model for studying the MBL transition, the metallic phase is far from trivial. The small energy scales has also been very recently seen for an other canonical microscopical model for MBL, the Heisenberg chain. Nevertheless, $E_{E_{3}}$ is not observed there since only a small range of eigenvalues were considered there, similar to the $P = 2048$ case depicted in Fig. 5. As in itself the deviation from universal behavior of the spectrum at larger energy scales is not surprising, since a somewhat similar deviation from the universal behavior of the energy spectra is seen in the single particle energy spectrum and associated with the Thouless energy. There the reason for the termination of the universal behavior is very clear. At short times (corresponding to large energy scales) diffusive behavior has not had time to evolved and experience the whole sample and therefore the behavior is not yet universal. For the MBL model the crossover from the short time behavior does not occur directly to the diffusive (universal) behavior, but there is an intermediate times for which the motion of a wave packet is extended, but nevertheless it does not cover the whole phase space and only on longer times it crosses over to the diffusive regime. Both crossovers leave a distinct signature in the energy spectrum and establishes energy scales ($E_{E_{3}}$ and $E_{Th}$) which can be extracted using SVD. This regime exists only in the metallic regime, while in the localized regime there is only a transition from non-universal short times behavior to a localized behavior.

The origin of this intermediate energy (or time) regime is not clarified by this study. Whether is stems from the structure of the coupling of states in the Fock space resulting in a quantum random graph, or other explanations which hinge on static or dynamical rare regions in the system such as Griffiths regions which may drive KT transitions, needs more study. Of course clarifying the finite size scaling of the intermediate regime is highly desirable, but unfortunately seems beyond current and reasonable future numerical capabilities. A possible continuation to this study would be the study of the energy spectrum of other models models with a different geometry than the 1D chains, such as the a random network or a modified SYK model. Although one will continue to suffer from the constrains of small systems, one will have freedom of tweaking geometry which may help understanding the physics behind this intermediate region.

**Appendix: Singular Value Decomposition**

The singular value decomposition (SVD) is a method to decompose a matrix $X$ of size $M \times P$ (X is not necessarily Hermitian nor square) into a sum of matrices. The matrix $X$ represents data arranged by rows and columns, where the arrangement depends on the application. For the SVD analysis of the spectrum one writes the $M$ realizations of disorder and the $P$ eigenvalues each, as a matrix $X$ of size $M \times P$ where $X_{mp}$ is the $p$ level of the $m$-th realization. The matrix $X$ is decomposed to $X = U \Sigma V^{T}$, where $U$ and $V$ are $M \times M$ and $P \times P$ matrices correspondingly, and $\Sigma$ is a diagonal matrix of size $M \times P$ and rank $r = \min(M,P)$. The $r$ diagonal elements of $\Sigma$, denoted as $\sigma_{k}$ are the singular values (SV) of the matrix which are positive and could be ordered by their size such that $\sigma_{1} \geq \sigma_{2} \geq \ldots \sigma_{r}$. The Hilbert-Schmidt norm of the matrix $\|X\|_{HS} = \sqrt{Tr(X^{T}X)} = \sum_{k} \lambda_{k}$ (where $\lambda_{k} = \sigma_{k}^{2}$). Therefore, using the SVD the matrix $X$ could be written as a series composed of matrices $X^{(k)}$, where $X_{ij}^{(k)} = U_{ik} V_{jk}^{T}$ and $X_{ij} = \sum_{k} \sigma_{k} X_{ij}^{(k)}$. Thus, this se-
ries in an approximation of matrix $X$, where the sum of the first $m$ modes gives a matrix \( \tilde{X} = \sum_{k=1}^{m} \sigma_k X^{(k)} \), for which \( ||X||_{HS} - ||\tilde{X}||_{HS} \) is minimal.

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