SOME COMBINATORICS IN THE CANCELLATION OF POLES OF EISENSTEIN SERIES FOR $GL(n, \mathbb{A}_Q)$

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Abstract. The cancellations of poles of degenerate Eisenstein series were studied by Hanzer and Muić. This paper generalized the result to Eisenstein series constructed from inducing two Speh representations $\Delta(\tau, m) |^*1 \otimes \Delta(\tau, n) |^*2$ for the group $GL(m + n, \mathbb{A}_Q)$ for self-dual cuspidal automorphic representation $\tau$ by describing the combinatorics of the relevant Weyl group coset.

1. Introduction

In [MW89], Mœglin and Waldspurger described the residual automorphic spectrum of $GL(N)$. A residual automorphic representation can always be realized as a generalized Speh representation $\Delta(\tau, n)$, where $\tau$ is an irreducible unitary cuspidal automorphic representation of the group $GL(a)$ with $a$ satisfying $N = an$. For convenience, we will assume the cuspidal automorphic representation $\tau = \otimes_v \tau_v$ has a unitary central character, and $\tau_v$ is tempered at each local place $v$. The automorphic representation $\Delta(\tau, n)$ is a global Langlands quotient of a principal series representation, and can be realized as the automorphic representation generated by the residue of an Eisenstein series. For any partition $N = (N_1, \ldots, N_r)$ of $N$, denoting the standard parabolic subgroup with Levi subgroup $M_N$ isomorphic to $GL(N_1) \times \ldots GL(N_r)$ by $P_N = M_N U_N$, we can choose cuspidal automorphic representations $\tau_1, \ldots, \tau_r$ of $GL(a_i)$, where $a_i n_i = N_i$, and construct the principal series representation

$$I(\tau, \underline{s}) = \text{Ind}_{P_N}^{G_N} (\Delta(\tau_1, n_1) |^*1 \otimes \ldots \otimes \Delta(\tau_r, n_r) |^*r).$$

This paper investigates the poles of the Eisenstein series constructed from a section in $I(\tau, \underline{s})$ with isomorphic cuspidal data $\tau_1 = \tau_2 = \tau$ in the case $r = 2$.

Theorem A. Fixing an irreducible unitary self-dual cuspidal automorphic representation $\tau$, we construct the principal series

$$I(\tau, \underline{s}) = \text{Ind}_{P_N}^{G_N} (\Delta(\tau, m) |^*1 \otimes \Delta(\tau, n) |^*2)$$

where $N = (m, n)$. In the region $s_1 - s_2 \geq 0$, the Eisenstein series

$$E(\cdot, \underline{s}) : I(\tau, \underline{s}) \rightarrow \mathcal{A}_{G_N}$$

has poles of maximal order 1 at $s_1 - s_2 \in \frac{m+n}{2} - \{0, 1, \ldots, \min\{m, n\} - 1\}$.

The proof of this theorem is a generalization of the combinatorial method in [HM15] to cuspidal automorphic induction data. The residues of this Eisenstein series are described in the following corollary:

Corollary B. The residue of Eisenstein series $\text{Res}_{s = \frac{m+n}{2}} E(\cdot, \underline{s})$ defines an intertwining operator sending $I(\tau, \underline{s})$ to the irreducible representation $\text{Ind}_{P_{m+n}}^{G_{m+n}} (\Delta(\tau, m + n - \sigma) |^*| \otimes \Delta(\tau, \sigma) |^*).$

2. Principal Series and Eisenstein Series

Throughout this section, we denote the space of automorphic functions on $G_n = GL(na)$ by $\mathcal{A}_{G_n}$. The standard parabolic subgroup with Levi subgroup isomorphic to $GL(a) \times n$ is denoted by $P_n = M_n \cup n$. 

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2.1. Induction from Cusp Forms. The cuspidal automorphic representation $\tau$ on the group $GL(a)$ can be decomposed as a restricted tensor product $\tau = \bigotimes_{i=1}^{a} \tau_n$ of irreducible unitary representations $\tau_n$ over local fields $\mathbb{Q}_v$. One can choose a pure tensor vector $f = \otimes_v f_v$ which realizes the completed automorphic $L$-function as the normalization factor of the intertwining operator (for the construction of such a vector c.f. [Sha10]). Thus, if we consider the tensor product $\bigotimes_{k=1}^{n} \tau_k^{|\nu_k|}$ as a representation on the group $GL(a)^{\times n}$, we can still choose a pure tensor $\bigotimes_{k=1}^{n} (\otimes_v f_{k,v}) \in \tau$ satisfying the same property. This vector can be realized as an automorphic function $\hat{f}_\nu \in \bigotimes_{k=1}^{n} \mathcal{A}_{G_n}$. By the Iwasawa decomposition of $G_n = GL_{na}$, we can choose a section $\hat{f}_\nu \in \text{Ind}_{\rho_n}^{G_n} (\bigotimes_{k=1}^{n} \tau_k^{|\nu_k|})$ of the principal series satisfying
\[
\hat{f}_\nu(umk) = f_\nu(m)|m|^{|\nu_k|} \hat{f}_\nu(k),
\]
where $|m|^{|\nu_k|} = |\det m_1|^{|\nu_1|} \ldots |\det m_n|^{|\nu_n|}$ for any $m \in M_{[a,n]}$. The Eisenstein series
\[
E_\nu(f_\nu, \nu)(g) = \sum_{\gamma \in P_n(k) \backslash G(k)} \hat{f}_\nu(\gamma g)
\]
defines an intertwining operator between $G_n$ representations:
\[
E_\nu(\cdot, \nu) : I(\tau, \nu) := \text{Ind}_{\rho_1}^{G_1} \bigotimes_{k=1}^{n} \tau_k^{|\nu_k|} \longrightarrow \mathcal{A}_{G_n},
\]
and can be meromorphically continued to the whole $\mathbb{C}^n$ from the domain of convergence $\Re(\nu_i - \nu_j) > 0$ for $i < j$.

2.2. The Rankin-Selberg $L$-Function. The Langlands-Shahidi method can be applied to define the Rankin-Selberg $L$-function $L(s, \tau \times \tau)$ described in [JPSSS3] for any cuspidal automorphic representation $\tau$ as normalization factors of intertwining operators. An exposition of such method can be found in [Sha10, Chapter 5]. We will be mostly following the setup in [MW99, Appendice], but will also point out other properties of the Rankin-Selberg $L$-function $L(s, \tau \times \tau)$ which we will be using in this paper.

We consider two irreducible unitary cuspidal automorphic representations $\tau_1, \tau_2$ on $GL(n_1, \mathbb{A})$ and $GL(n_2, \mathbb{A})$, respectively. The constant term of the Eisenstein series
\[
E_{\nu_1, \nu_2}(s, g) : \text{Ind}_{\rho_{n_1+n_2}}^{G_{n_1+n_2}}(\tau_1^{|\nu_1|} \otimes \tau_2^{|\nu_2|}) \longrightarrow \mathcal{A}_{G_{n_1+n_2}}
\]
will have only one summand unless $n_1 = n_2$. In any case, following [Sha83], the normalization factor $r(s, w_0)$ of the intertwining operator $M(s, w_0)$ can be expressed as
\[
r(s, w_0) = \frac{L(s, \tau_1 \times \tau_2)}{L(1+s, \tau_1 \times \tau_2)} L(s, \psi, \tau_1 \times \tau_2)
\]
where $L(s, \tau_1 \times \tau_2)$ is the completed Rankin-Selberg $L$-function with its local factors described in [JPSSS3]. In [MW99, Appendice, Corollaire], it is shown that the function $L(s, \tau_1 \times \tau_2)$ is entire unless $n_1 = n_2$ and there exists a complex number $t$ such that $\tau_1 = \tau_2$: $|t|$. The poles of the function $L(s, \tau \times \tau)$ are simple and are located at $s = 0, 1$. A zero-free region of $L(s, \tau \times \tau)$ is given in [GL18] by
\[
\Re(s) > 1 - \frac{c}{(\log(|\Im(s)| + 2))^{5}}.
\]
A consequence of the zero-free region given above is that there are no zeros in the region $\Re(s) \geq 1$ and $\Re(s) \leq 0$. Therefore, one expects no cancellations of poles between the numerators and the denominators apart from the cancellations from the zeros on the critical strip.

2.3. Residues of Eisenstein Series and Speh Representation. In this section, we summarize the key facts concerning the poles and residues of the Eisenstein series $E_\nu(\cdot, \nu)$ defined in [11] from the previous section. By the Langlands constant term formula [MW95, II.1.7] and the cuspidality of $\tau$, the constant term along the unipotent radical of $U_n$ of the parabolic subgroup $P_n$ can be expressed as
\[
e_{U_n} E_\nu(f_\nu)(g) = \sum_{w \in W_n \setminus W_{na} = S_n} M(w, \nu) \hat{f}(g).
\]
Each intertwining operator \( M(w, \nu) \) is defined in the dominant cone \( \text{Re}(\nu_i - \nu_j) \gg 0 \), where \( i < j \), as the following formal integral
\[
M(w, \nu) \tilde{f}(g) = \int_{U_n \cap w^{-1} U_n w \backslash U_n} \tilde{f}(wng) \, dn.
\]
The operator \( M(w, \nu) \) can be meromorphically continued to the whole \( \mathbb{C}^n \). By [MW89] Appendix, each summand \( M(w, \nu) \tilde{f}(g) \) has the same set of poles as the function \( r(w, \nu) \) known as the normalization factor:
\[
r(w, \nu) = \prod_{i < j, w(i) > w(j)} \frac{L(\nu_i - \nu_j, \tau \times \overline{\tau})}{L(1 + \nu_i - \nu_j, \tau \times \overline{\tau}) \epsilon(\nu_i - \nu_j, \tau \times \overline{\tau}, \psi)}.
\]
The normalized intertwining operator \( N(w, \nu) = r(w, \nu)^{-1} M(w, \nu) \) is holomorphic in the region \( \text{Re}(\nu_i - \nu_j) \geq 0 \) with \( i < j \).

The pole of the highest possible codimension \( r(w, \nu) \) occurs when \( w(i) > w(j) \) for all \( i < j \). In this case, the normalization factor \( r(w, \nu) \) has a simple pole at the intersection of the hyperplanes \( \nu_i - \nu_{i+1} = 1 \), i.e., at the point \( \nu = -\Delta_n = (n^{-1}, \frac{n^{-1}}{2}, \ldots, \frac{n^{-1}}{n}) \) (pay attention to the minus sign). We can take the residue
\[
\text{Res}_n E_n(f, \nu) = \lim_{\nu \to -\Delta_n} \prod_{i=1}^{n-1} (\nu_i - \nu_{i+1} - 1) E_n(f, \nu).
\]
of the Eisenstein series \( E_n(f, \nu) \) at the point \( -\Delta_n \). The irreducible automorphic representation \( \Delta(\tau, n) \) generated by \( \text{Res}_n E_n(f, \nu) \) is usually referred to as the Speh representation, which constitutes the whole non-cuspidal discrete spectrum if we let \( \tau \) run through all possible choices of cuspidal automorphic representations for all block sizes \( a \).

The composition \( \text{Res}_n E_n(f, \nu) \) of the Eisenstein series operator and the residue operator is a quotient operator from the principal series \( \text{Ind}_{F_n}(\otimes_{i=1}^n \tau) \cdot \nu \) to \( A_G \). It factors through the embedding \( \Delta(\tau, n) \hookrightarrow A_G \). By the Langlands classification ([Lan89] and [Lan96]), on the pole \( \nu \) where a residue exists, the representation \( \Delta(\tau, n) \) is also the image of the intertwining operator
\[
M(w_0, \nu) : I(\tau, \nu) \longrightarrow I(\tau, w_0 \nu)
\]
for the longest element \( w_0 \) of the Weyl group \( S_n \). On each finite place, the local component of the residual representation \( \Delta(\tau, n) \) is isomorphic to the Langlands quotient of the principal series \( I(\tau, \nu) \).

2.4. Induction from Speh Representations. As in Theorem [A] for \( m \leq n \) and an irreducible cuspidal unitary automorphic representation \( \tau \), we construct the induced representation
\[
I(\tau, \xi) := \text{Ind}_{G_m}^{G_{m+n}} (\Delta(\tau, m) \cdot [\xi^1 \boxtimes \Delta(\tau, n)] \cdot [\xi^2])
\]
from two Speh representations \( \Delta(\tau, m) \) and \( \Delta(\tau, n) \) for the groups \( G_m \) and \( G_n \), respectively. We would like to develop a combinatorial method to understand the poles of the Eisenstein series
\[
E^{m,n}(\cdot, \xi) : I(\tau, \xi) \longrightarrow A_G
\]
on a section \( f(\tau, \xi) \). Since the Speh representation \( \Delta(\tau, m) \cdot [\xi^1 \boxtimes \Delta(\tau, n)] \cdot [\xi^2] \) can be realized as a subrepresentation of \( I(m, \lambda_m) \cdot [\xi^1 \boxtimes I(n, \lambda_n)] \cdot [\xi^2] \), any vector of \( I(\tau, \xi) \) can be realized as a vector in the principal series
\[
I(m, n, \xi) = \text{Ind}_{G_m}^{G_{m+n}} (I(m, \lambda_m) \cdot [\xi^1 \boxtimes I(n, \lambda_n)] \cdot [\xi^2]).
\]
For any such vector \( f \), the constant term along the unipotent radical \( U_{m+n} \) can be expressed as the following sum of formal integrals over Weyl group cosets:
\[
(c_{U_{m+n}} E^{m,n}(f, \xi))(g) = \sum_{\gamma \in P_m(k) \backslash G_{m+n}(k) \backslash U_{m+n}(k)} \int_{U_{m+n}(k) \backslash U_{m+n}(A)} f(\gamma ng) \, dg.
\]
By the Bruhat decomposition, $\gamma$ can be represented by product $w(1, 0, 1)$, and we can decompose $n$ into the product $n = n_1n_2$, such that $n_1 \in U_{[m, n]}$ and $n_2 \in U_m \times U_n$. Writing $n_1 = (1, 0, 1)$ and $n_2 = (u_1, 0, u_2)$, we have

$$\gamma n = w(1, 0, 1) (u_1, 0, u_2) = w(u_1, 0, u_2) w(1, 0, 1) (u_1, 0, 0) = w(u_1, 0, u_2) w(1, 0, 1).$$

For any section $f \in I(m, n, S)$, since

$$f(\gamma g) = f \left( w \left( \begin{array}{cc} u_1 & 0 \\ 0 & u_2 \end{array} \right) w^{-1} w \left( \begin{array}{cc} 1 & u_1^{-1}(a+\beta)u_2 \\ 0 & 1 \end{array} \right) g \right),$$

and $f$ is invariant under the left regular action of $U_{m+n}$, we have

$$f(\gamma g) = f \left( \begin{array}{cc} 1 & u_1^{-1}(a+\beta)u_2 \\ 0 & 1 \end{array} \right) g \right).$$

Therefore, for the section $f \in I(m, n, S)$, the constant term $c_{U_{m+n}} E_{m,n}(f, S)$ can simply be written as the sum

$$c_{U_{m+n}} E_{m,n}(f, S) = \sum_{w \in S_m \times S_n \setminus S_{m+n}} \int_{U_{[m,n]}(A)wU_{[m,n]}(A)wU_{[m,n]}(A)} f(wg) dw = \sum_{w \in S_m \times S_n \setminus S_{m+n}} M(w, S) f.$$

We can summarize this result into the following lemma:

**Lemma 1.** For any section $f \in I(m, n, S) \hookrightarrow I(\tau, S)$, the Eisenstein series $E_{m,n}(f, S)$ has the same set of poles with the constant term

$$c_{U_{m+n}} E_{m,n}(f, S) = \sum_{w \in S_m \times S_n \setminus S_{m+n}} M(w, S) f.$$

We can further organize the sum in Lemma 1 by grouping $M(w, S) f$ for different $w$’s based on which space $M(w, S) f$ lives in. Denoting

$$\rho_{m,n}(\tau, S) = \left( \bigotimes_{i=1}^{m} \tau \cdot \left| \frac{1}{2} - \frac{i-1-s_1}{m} \right| \right) \otimes \left( \bigotimes_{j=1}^{n} \tau \cdot \left| \frac{1}{2} - \frac{j-1+s_2}{n} \right| \right),$$

the intertwining operator $M(w, S) f$ is a vector in the conjugation $(\rho_{m,n}(\tau, S))^w$. We will denote by $O_{\tau}(w)$ the set of all Weyl group representatives $u' \in S_m \times S_n \setminus S_{m+n}$ such that $(\rho_{m,n}(\tau, S))^w \cong (\rho_{m,n}(\tau, S))^{u'}$. Thus the sum in Lemma 1 can be grouped into sums on orbits $O_{\tau}(w)$:

$$c_{U_{m+n}} E_{m,n}(f, S) = \sum_{O_{\tau}(w)} \left( \sum_{w \in O_{\tau}(w)} M(w, S) f \right).$$

3. **Segments and Weyl Group Elements**

In this section, we classify the orbits of elements in $S_m \times S_n \setminus S_{m+n}$ and describe the representatives of each orbit.

3.1. **A Double Coset.** We define $W_{m,n}$ as the collection of representatives of the double coset

$$(W_m \times W_n) \setminus W_{m+n} / \{e\} = S_m \times S_n \setminus S_{m+n},$$

which has a one-to-one correspondence with the set of elements $w \in S_{m+n}$ preserving the order in the strings $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$, respectively. Such an element satisfies the property

$$w(1) < \ldots < w(m), \ w(m+1) < \ldots < w(m+n).$$

The action of $w$ on the string $\{1, \ldots, m+n\}$ shuffles between the two strings $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$. We denote the interlacing intervals originated from each one of these two strings by $v_i$ and $u_i$, respectively, as shown in the following diagram:
The "light gray" \( v_i \)-intervals and the "dark gray" \( u_i \)-intervals assemble back to the two original strings:

\[
12 \ldots (m - 1)m = \begin{array}{cccc}
v_0 & v_1 & v_2 & v_3 & \ldots \\
u_1 & u_2 & u_3 & u_4 & \ldots \\
\end{array}
\]

\[
(m + 1) \ldots (m + n) = \begin{array}{cccc}
\end{array}
\]

It is possible that \( v_0 \) is empty, in which case the sequence \( w(1 \ldots (m + n)) \) will start with the interval \( u_1 \):

\[
\begin{array}{cccc}
u_1 & v_1 & u_2 & \ldots & u_i & v_i & \ldots \\
\end{array}
\]

For any interval \( I \), we denote by \( l(I) \) the index of its starting point on the left and \( r(I) \) the index of its ending point on the right. In the string corresponding to \( w \), if an interval \( I_1 \) appears to the left of \( I_2 \), then we say \( I_1 \) precedes \( I_2 \), and we denote this situation by \( I_1 \prec I_2 \).

3.2. Segments. The full principal series induced from the tensor product \( \rho_{m,n}(\tau, s) \) can be perceived as the automorphic analogue of the induction from Bernstein-Zelevinsky segments \( \Delta + s_1 \) and \( \Delta' + s_2 \) (c.f. \cite{Zel80}): \[
\begin{array}{cccc}
\frac{1-m}{2} & \frac{3-m}{2} & \frac{5-m}{2} & \ldots & \frac{1-m}{2} + \sigma & \ldots & \frac{m-5}{2} & \frac{m-3}{2} & \frac{m-1}{2} \\
\frac{1-n}{2} & \frac{3-n}{2} & \frac{5-n}{2} & \ldots & \frac{n-5}{2} & \frac{n-3}{2} & \frac{n-1}{2} \\
\end{array}
+ s_1 + s_2
\]

The integer \( \sigma \geq 0 \) equals to the length of the overlapping part of the two segments. The smallest possible \( \sigma \) is equal to 0, in which case the two segments are juxtaposed, as shown in the following diagram:

\[
\begin{array}{cccc}
\frac{1-m}{2} & \frac{3-m}{2} & \frac{5-m}{2} & \ldots & \frac{1-m}{2} + \sigma & \ldots & \frac{m-5}{2} & \frac{m-3}{2} & \frac{m-1}{2} \\
\frac{1-n}{2} & \frac{3-n}{2} & \frac{5-n}{2} & \ldots & \frac{n-5}{2} & \frac{n-3}{2} & \frac{n-1}{2} \\
\end{array}
+ s_1
\]

In general, if the length of the overlapping part of the two segments is \( \sigma \), we have:

\[
\frac{n - 1}{2} + s_2 = \frac{1 - m}{2} + s_1 - 1 + \sigma,
\]

and thus \( \sigma \) determines \( s = s_1 - s_2 = \frac{m+n}{2} - \sigma \), where \( \sigma \) is allowed to be any integer between 0 and \( \lfloor \frac{m+n}{2} \rfloor \), so that the segments \( \Delta + s_1 \) and \( \Delta' + s_2 \) intersect in the following three ways:

- **Case I**: \( \sigma \leq \min\{m, n\} \):
  \[
  \Delta + s_1
  \]
  \[
  \Delta' + s_2
  \]

- **Case II**: \( m \leq \sigma \leq n \):
  \[
  \Delta + s_1
  \]
  \[
  \Delta' + s_2
  \]

- **Case III**: \( n \leq \sigma \leq m \):
  \[
  \Delta + s_1
  \]
  \[
  \Delta' + s_2
  \]

The case when \( \sigma > \max\{m, n\} \) is called \( \Delta + s_1 \) precedes \( \Delta' + s_2 \), which will result in a negative \( s = s_1 - s_2 \) and will not be considered.

Since the representation \( \rho_{m,n}(\tau, s) \) depends on the choice of \( \sigma \), the orbits \( O_{\Sigma}(w) \) of Weyl group cosets \( S_m \times S_n \backslash S_{m+n} \) also depend on the choice of \( \sigma \).
3.3. Orbits. This section generalizes the method developed in [HM15]. The condition for two Weyl group representatives \( w, w' \in W_{m,n} \) belonging to the same orbit is \((\rho_{m,n}(\tau, 1))w = (\rho_{m,n}(\tau, 1))w'\), which is equivalent to requiring \(w(i) = w(i)\) or \(w(m+n-\sigma + i)\).

for \(i \in \{1, \ldots, m\}\) and \(m+n-\sigma + i \in \{m+1, \ldots, m+n\}\). Therefore, there is a transitive action by a group \(G_\sigma = \mathbb{Z}_2^n\) (which is related to the \(R\)-group, and will be described in the following sections) of permissible moves for some integer \(r\) on each orbit \(O_\sigma\) permuting the pairs \(\{w(i), w(m+n-\sigma + i)\}\). For any choice of \(\sigma\), we can choose the base point \(w_0 \in O_\sigma\) to be the unique element in the orbit satisfying
\[
\forall i \text{ such that } i \in \{1, \ldots, m\} \text{ and } m+n-\sigma + i \in \{m+1, \ldots, m+n\}.
\]

3.4. Permissible Moves on Basepoints. In this section, we describe the aforementioned group \(G_\sigma\) of permissible moves.

\[
\ldots \quad v_i \quad u_{i+1} \quad \ldots
\]

\textbf{Definition 2.} Fixing an \(s = s_1 - s_2 = \frac{m+n}{2} - \sigma\), a permissible move \(g \in G_\sigma\) is a permutation of the string \(\{1, 2, \ldots, (m+n)\}\), such that

1. For any element \(w\) satisfying (3), \(gw\) also satisfies (3).
2. The two elements \(w\) and \(gw\) satisfy \(\rho_{m,n}(\tau, 1)w = \rho_{m,n}(\tau, 1)gw\), i.e.
\[
gw(i) \in \{w(i), w(m+n-\sigma + i)\}.
\]

Any element that can be affected by a permissible move is said to be alive, otherwise it is said to be dead. If there is a continuous segment of living elements, we refer to that segment as a living segment, and vice versa for dead segments.

For a basepoint \(w_0\), we will color the living elements of the string \(w_0(12 \ldots (m+n))\) either green or red based on whether they originate from the string \(12 \ldots m\) or \((m+1) \ldots (m+n)\), respectively. Recall that we require the basepoint element \(w_0\) to satisfy the property \(w_0(i) < w_0(m+n-\sigma + i)\), on the basepoint element, we will color \(w_0(i)\) green and \(w_0(m+n-\sigma + i)\) red. We will call them green elements (intervals, resp.) and red elements (intervals, resp.) for convenience. Note that any green element lies on a “light gray” \(v_i\)-interval while any red element lies on a “dark gray” \(u_i\)-interval. We will use this diagram to prove properties of all permissible moves from a basepoint \(w_0\).

\textbf{Lemma 3.} For a basepoint \(w_0\), the following properties are true for green elements:

1. All elements between two green elements on the same “light gray” \(v_i\)-interval are green.
\[
\ldots \quad v_i \quad u_{i+1} \quad \ldots
\]
\[
I_i
\]

2. The right endpoint of \(v_i\) and the right endpoint of the corresponding \(I_i\) are aligned: \(w(r(v_i)) = w(r(I_i))\).
\[
\ldots \quad v_i \quad u_{i+1} \quad \ldots
\]
\[
I_i
\]

Similarly, for red elements, they form a gap-free interval on any \(u_i\), and their left endpoints are aligned with the left endpoint of \(u_i\) they lie in:

\[
\ldots \quad v_i \quad u_{i+1} \quad \ldots
\]
\[
J_i
\]


**Lemma 5.** That there is actually no gap between two consecutive living intervals.

In the Part (1) of the lemma above, and the right endpoint of a continuous interval satisfies the same property as between continuous green/red intervals. One green interval is dead, the left endpoint of a continuous interval satisfies the same property as adjacent green interval $I_{i+1}$ in $v_{i+1}$, with their left endpoints aligned.

**Remark 1.** There is no specification about how on any basepoint element $w_0$, the elements are moved between continuous green/red intervals. One green interval $I_i$ can be split across two $v$-intervals, and one red interval $J_i$ can be split across two $v$-intervals by a permissible move. The only fact we know for sure is that the left endpoint of a continuous interval satisfies the same property as $I_{i+1}$ and $J_{i+1}$ in the Part (2) of the lemma above, and the right endpoint of a continuous interval satisfies the same property as $I_i$ and $J_i$ in the Part (1) of the lemma above.

The following lemma describes the permissible moves near the dead elements:

**Lemma 4.** On the string representing the basepoint $w_0$, the following statements are true:

1. If the element with index $w(r(I_i)) + 1$ lying to the right of a green interval $I_i$ is dead, the permissible moves can only move $I_i$ into the adjacent red interval $v_{i+1}$, with their right endpoints aligned:

$$\ldots v_i \quad u_{i+1} \quad v_{i+1} \quad u_{i+2} \quad \ldots$$

Similarly, if the element with index $w(l(I_{i+1})) - 1$ to the left of $J_{i+1} \subset u_{i+2}$ is dead, the permissible moves can only move $J_{i+1}$ leftwards to its adjacent green interval $I_{i+1}$ in $v_{i+1}$, with their left endpoints aligned.

2. If the intervals $I_i \subset v_i$ and $I_{i+1} \subset v_{i+1}$ are connected (i.e. without any gap in between), then the whole $u_{i+1}$ is a red interval $J_i$.

**Proof.** The first part is a consequence of (1) in Definition 2 and Lemma 3, since there is no green living interval adjacent to $I_i$ on the right. The second part is a consequence of the first part and (2) in Definition 2.

**Corollary A.** On the same $v_i$, the green elements form a gap-free interval, and on each “light gray” interval $v_i$ there is only one interval of green elements. We denote this interval by $I_i$. Similarly, on the same $u_i$, the red elements form a unique gap-free interval, and we denote this interval by $J_i$.

The following lemma describes the gaps between two continuous intervals. In fact, this lemma implies that there is actually no gap between two consecutive living intervals.

**Lemma 5.** (1) For the following configuration on four consecutive intervals $v_i, u_{i+1}, v_{i+1}, u_{i+2}$:

$$\ldots v_i \quad u_{i+1} \quad v_{i+1} \quad u_{i+2} \quad \ldots$$

Setting $J'_i = u_{i+1} \setminus J_i$ and $I'_{i+1} = v_{i+1} \setminus I_{i+1}$, swapping $I'_{i+1}$ and $J'_i$ is a permissible move.

(2) For the following configuration,

$$\ldots v_i \quad u_{i+1} \quad v_{i+1} \quad u_{i+2} \quad \ldots$$

one can construct a permissible move for the whole interval between $r(I_i) + 1$ and $l(I_{i+k}) - 1$.

**Proof.** (1) Since $J'_i$ has the same length as $I'_{i+1}$, swapping $J'_i$ with $I'_{i+1}$ is in fact a permissible move.
(2) We can list the intervals between \(J_i\) and \(I_{i+k}\) as
\[
J_i', v_{i+1}, u_{i+2}, \ldots, u_{i+k}, I_{i+k}'.
\]
Since the right endpoints of \(J_i\) and \(J_i\) correspond, and the left endpoints of \(I_{i+k}\) and \(I_{i+k}\) correspond, denoting the lengths of the “dark grey” intervals \(J_i', u_{i+2}, \ldots, u_{i+k}\) by \(p_1, \ldots, p_k\), and the lengths of “light grey” intervals \(v_{i+1}, \ldots, I_{i+k}'\) by \(q_1, \ldots, q_k\). We have \(\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i\). From the partitions \((p_1, \ldots, p_k)\) and \((q_1, \ldots, q_k)\), we can obtain a refined partition of the dark grey and light grey intervals. The move between corresponding intervals is a permissible move.

\[\square\]

From the previous lemmas, we can find that on the segment \(12, \ldots, m\), the green elements only exist along \((\max\{\sigma - n + 1, 1\}, \ldots, \min\{\sigma, m\})\), and can be organized into mutually connected intervals \(K_1, \ldots, K_r\), possibly straddling across different \(v\)-intervals. Each \(K_i\) moves as a whole under the action of \(G_\sigma\). The order of \(G_\sigma\) is equal to \(2^r\).

3.5. **General Description of the Permissible Moves.** Recall from Section 3.1 that the string \(w(12, \ldots, m + n)\) can be expressed as the interlacing \(v, u\) intervals:
\[
\begin{array}{cccccccc}
  v_0 & u_1 & v_1 & u_2 & \ldots & u_i & v_i & \ldots \\
\end{array}
\]

We would like to use this diagram, combined with the lemmas we stated in the previous section to describe the green and red living intervals and the corresponding permissible moves.

**Proposition 6.** On the string representing a basepoint element \(w_0\),

(1) The first green element is the first green element on the left such that there is no dead element from \(\{m + 1, \ldots, m + n\}\) to its right.

(2) The last red element is the last red element on the right such that there is no dead element from \(\{1, \ldots, m\}\) to its left.

(3) An element \(i \in \{1, \ldots, \sigma\}\) is dead if between \(w(i)\) and \(w(m + n - \sigma + i)\) there is a dead element coming from \(\{m + n - \sigma + 1, \ldots, m + n\}\).

**Proof.** Part (1) and (2) follow from the definition of living elements and the proof of Lemma 3. Part (3) is true simply because permissible moves do not allow any move across dead elements. \(\square\)

In the proof of Lemma 5, we have already seen that there is no dead element between the first and the last living elements. Denoting the first living green interval by \(v_k' = v_k\) and the last living red interval by \(u_i' = u_i\). The positions of these living elements are displayed as in the following diagram:
\[
\begin{array}{cccccccc}
  v_k' & u_k & v_{k+1} & u_{k+1} & \ldots & u_i & v_i & \ldots & u_i' \\
\end{array}
\]

The permissible moves can be constructed with the following algorithm:

(1) Denoting the lengths of the intervals \(v_k', \ldots, v_l\) and \(u_k, \ldots, u_l\) by \(p_k, \ldots, p_l\) and \(q_k, \ldots, q_l\), for any \(1 \leq r \leq \sum_{i=k}^{l} p_i = \sum_{i=k}^{l} q_i\), the collection of \(S\) of “end points of refined intervals” \(\{t_1, \ldots, t_N\}\) is defined as all the \(r\)'s such that \(r\) corresponds to an end point of either a green or a red interval.
\[
\begin{array}{cccccccc}
  v_k' & u_k & v_{k+1} & u_{k+1} & \ldots & u_i & v_i & \ldots & u_i' \\
\end{array}
\]

(2) Between these end points, we can obtain a refinement of the living green and red intervals. We label each interval of each color from left to right with an index:
\[
\begin{array}{cccccccc}
  v_k' & u_k & v_{k+1} & u_{k+1} & \ldots & u_i & v_i & \ldots & u_i' \\
\end{array}
\]

Following the convention of step 2, the labels of each color are \(\{1, \ldots, N\}\). Each labeled interval with index \(i\) is denoted by \(\delta_i\).
3.6.1. Head intervals for Cases I and II.

We describe the head and tail intervals in all the three cases described in Section 3.2. The Proposition 8. Dead Intervals. 3.6. be done for red intervals, and we can define $L_i$ to a living element. Therefore, all

This theorem is a consequence of Proposition 6. There cannot be any dead element on the right of $m$. Proof. Then the set of all dead “light gray” elements in the head is the union of all

If (3) happens before (1) or (2) could happen, one will be able to construct a permissible move on the living intervals to the right of the element $\sum_{j=1}^{k-1} |R_j| + 1$.

3.6. Dead Intervals. In this and the following sections, we will call the dead intervals to the left of the living intervals the head, and those dead intervals to the right of the living intervals the tail. In this section, we describe the head and tail intervals in all the three cases described in Section 3.2.

3.6.1. Head intervals for Cases I and II. In the cases I and II, we have $n \geq \sigma$, thus:

Proposition 8. The head can be constructed through the following procedure.

Denote

- the set of elements $i$ in $\{1, \ldots, m\}$ such that $w(i) < w(m + n - \sigma)$ by $R_1$,
- the set of elements $i$ in $\{1, \ldots, m\}$ such that $w(|R_1|) < w(i) < w(m + n - \sigma + |R_1|)$ by $R_2$,
- the set of elements $i$ in $\{1, \ldots, m\}$ such that $w(|R_1| + |R_2|) < w(i) < w(m + n - \sigma + |R_1| + |R_2|)$ by $R_3$,
- \ldots
- the set of elements $i$ in $\{1, \ldots, m\}$ such that $w(\sum_{j=1}^{k-1} |R_j|) < w(i) < w(m + n - \sigma + \sum_{j=1}^{k-1} |R_j|)$ by $R_k$,
- The head ends where $R_{k+1} = \emptyset$.

Then the set of all dead “light gray” elements in the head is the union of all $R_j$'s.

Proof. This theorem is a consequence of Proposition 8. There cannot be any dead element on the right of a living element. Therefore, all $i$'s such that $w(i) < w(m + n - \sigma)$ are dead, and so are all the elements $m + n - \sigma + i$ such that $i \in R_1$. As a consequence, all elements $i$ such that $w(i) < w(m + n - \sigma + |R_1|)$ are also dead. We can continue with this process, until any of the following three situations occur:

1. All “light gray” elements from $\{1, \ldots, m\}$ are finished.
2. All “dark gray” elements from $\{m + 1, \ldots, n\}$ are finished.
3. $R_{k+1} = \emptyset$ for some $k$.

If (3) happens before (1) or (2) could happen, one will be able to construct a permissible move on the living intervals to the right of the element $\sum_{j=1}^{k} |R_j| + 1$.

Lemma 9. If for any $1 \leq j \leq k$ such that $R_j \neq \emptyset$, we have

- $i \leq w(i) < n - \sigma + i$ for $i \in R_1$,
- $n - \sigma + i \leq w(i) < n - \sigma + |R_1| + i$ for $i \in R_2$,
- \ldots
- $n - \sigma + \sum_{j=1}^{k-1} |R_j| + i \leq w(i) < n - \sigma + \sum_{j=1}^{k-1} |R_j| + i$ for $i \in R_k$.

Proof. The inequalities follows simply from the enumeration of intervals. The strict inequality is due to the nonemptyness of $R_j$. □
3.6.2. **Head intervals for Case III.** In Case III, we can construct the head following a similar process. In this case, we have $\sigma > n$, and the dead intervals can be obtained via the process described in the following proposition:

**Proposition 10.** The head can be constructed through the following procedure in Case III:

![Diagram](image)

Denote

- the set of elements $i$ in $\{1, \ldots, n\}$ such that $w(m + i) < w(\sigma - n)$ by $R'_1$,
- the set of elements $i$ in $\{1, \ldots, n\}$ such that $w(m + |R'_1|) < w(m + i) < w(\sigma - n + |R'_1|)$ by $R'_2$,
- the set of elements $i$ in $\{1, \ldots, n\}$ such that $w(m + |R'_1| + |R'_2|) < w(m + i) < w(\sigma - n + |R'_1| + |R'_2|)$ by $R'_3$,
- \ldots
- the set of elements $i$ in $\{1, \ldots, n\}$ such that $w(\sum_{j=1}^{k-1} |R'_j|) < w(m + i) < w(\sigma - n + \sum_{j=1}^{k-1} |R'_j|)$ by $R'_k$,
- The head ends where \(R'_{k+1} = \emptyset\).

Then the set of all dead “dark gray” elements in the head is the union of all $R'_j$’s.

**Proof.** In Case III, since $\sigma > n$, all the elements between 1 and $\sigma - n$ are dead, as well as the elements from $\{m+1, \ldots, m+n\}$ interlacing the elements $1, \ldots, \sigma - n$. Therefore, we can imitate the process in Proposition 8 to exhaust the dead elements in the head for Case III. The process finishes when one of the following three situations occur:

1. All “light gray” elements from $\{1, \ldots, m\}$ are finished.
2. All “dark gray” elements from $\{m + 1, \ldots, n\}$ are finished.
3. $R'_{k+1} = \emptyset$ for some $k$.

Similar to Proposition 8 if (3) occurs before (1) or (2), we can construct a permissible move involving the element $\sum_{j=1}^k |R'_j| + 1$.

The lengths of the $R'_j$ intervals satisfy the following lemma:

**Lemma 11.** If for any $1 \leq j \leq k$, $R'_j \neq \emptyset$, we have

- $i \leq w(i) \leq i + |R'_1|$ for $i \in \{1, \ldots, \sigma - n\}$
- $i + |R'_1| \leq w(i) \leq i + |R'_1| + |R'_2|$ for $i \in \sigma - n + R'_1$,
- $i + |R'_1| + |R'_2| \leq w(i) \leq i + |R'_1| + |R'_2| + |R'_3|$ for $i \in \sigma - n + R'_2$,
- \ldots
- $i + \sum_{j=1}^{k-1} |R'_j| \leq w(i) \leq i + \sum_{j=1}^k |R'_j|$ for $i \in \sigma - n + R'_{k-1}$,
- $w(i) = i + \sum_{j=1}^k |R'_j|$ for $i \in \sigma - n + R'_k$.

At the right endpoint of each interval, the left part of the inequalities are strict.

**Proof.** The lemma follows easily from counting. The strict inequality at the right endpoint of each interval follows from the nonzeroness of $|R'_j|$ for $j \leq k$.

3.6.3. **Tail Intervals for Case I, III.** We can construct the tail of the dead elements symmetric to the situation described in Lemma 10.

![Diagram](image)

**Proposition 12.** The tail can be constructed through the following algorithm: Denote

- the elements $j$ in $\{m+1, \ldots, m+n\}$ such that $w(\sigma + 1) \prec w(j)$ by $S_1$,
- the elements $j$ in $\{m+1, \ldots, m+n\}$ such that $w(\sigma - |S_1| + 1) \prec w(j) \prec w(\sigma)$ by $S_2$,
The elements $j$ in $\{m+1, \ldots, m+n\}$ such that $w(\sigma - |S_1| - |S_2| + 1) < w(j) < w(\sigma - |S_1|)$ by $S_3$, 

... 

The elements $j$ in $\{m+1, \ldots, m+n\}$ such that $w(\sigma - \sum_{j=1}^{k-1} |S_j| + 1) < w(j) < w(\sigma - \sum_{j=1}^{k-2} |S_j|)$ by $S_k$ 

There exists a minimal $k$ such that $S_{k+1} = \emptyset$, and the set of all dead light gray elements in the tail is the union of all nonempty $S_j$.

Proof. In this situation, all the elements between $\sigma + 1$ and $m$ are dead. Therefore, the dead elements coming from $\{m+1, \ldots, m+n\}$ are those which interlace with the elements $\sigma + 1, \ldots, m$. Following the same procedure as in the proof of Proposition 10 from right to left along the tail, we can follow the process described in the statement of the proposition until one of the following three situations occur:

1. All “light gray” elements from $\{1, \ldots, m\}$ are finished.
2. All “dark gray” elements from $\{m+1, \ldots, m+n\}$ are finished.
3. $S_{k+1} = \emptyset$ for some $k$.

If (3) happens before (1) and (2), the adjacent first element to the left of $S_k$ is the last living element. □

Denoting the index of the right-most living element from $(m+1) \ldots (m+n)$ by $q$, we have the following lemma

Lemma 13. For any $1 \leq j \leq k$ such that $S_j \neq \emptyset$, we have

- $w(i) = q - m + i$ for $i \in S_k$,
- $w(i) \geq q - m + i$ for $i \in S_{k-1}$,
- $w(i) \geq q - m + |S_k| + i$ for $i \in S_{k-2}$,
- ...
- $w(i) \geq q - m + \sum_{j=1}^{k} |S_j| + i$ for $i \in S_1$.

Proof. The proof follows from the similar enumeration technique as in Lemma 11 □

3.6.4. Tail Intervals for Case II. We can construct the tail of the dead elements symmetric to the situation in Proposition 8.

![Diagram of tail intervals](image)

Proposition 14. The tail can be constructed through the following algorithm: Denote

- the elements $j$ in $\{1, \ldots, m\}$ such that $w(\sigma - m + 1) < w(j) < w(\sigma - m)$ by $S'_1$,
- the elements $j$ in $\{1, \ldots, m\}$ such that $w(\sigma - m - |S'_1| + 1) < w(j) < w(\sigma - m)$ by $S'_2$,
- the elements $j$ in $\{1, \ldots, m\}$ such that $w(\sigma - m - |S'_1| - |S'_2| + 1) < w(j) < w(\sigma - m - |S'_1|)$ by $S'_3$,
- ...
- the elements $j$ in $\{1, \ldots, m\}$ such that $w(\sigma - m - \sum_{j=1}^{k-1} |S'_j| + 1) < w(j) < w(\sigma - m - \sum_{j=1}^{k-2} |S'_j|)$ by $S'_k$.

There exists a minimal $k$ such that $S'_{k+1} = \emptyset$, and the set of all dead light gray elements in the tail is the union of all nonempty $S'_j$.

Proof. The proof follows the same procedure as in the proof of Proposition 8 from right to left along the tail. Similarly, the algorithm terminates when one of the following three situations occur:

1. All “light gray” elements from $\{1, \ldots, m\}$ are finished.
2. All “dark gray” elements from $\{m+1, \ldots, m+n\}$ are finished.
3. $S'_{k+1} = \emptyset$ for some $k$.

The adjacent element to the left of $S'_k$ does not interlace with any dead interval, and thus is an living element. □

Denoting the index of the right-most living element from $1 \ldots m$ by $q'$, we have the following lemma

Lemma 15. For any $1 \leq i \leq n$ such that $S'_j \neq \emptyset$, we have
\( n - \sigma + q' + |S'_n| + i < w(i) \leq n - \sigma + q' + |S'_n| + |S'_{n-1}| + i \) for \( i \in S'_{n-1} \),
\( n - \sigma + q' + |S'_n| + |S'_{n-1}| + i < w(i) \leq n - \sigma + q' + \sum_{j=n-2}^{n} |S'_j| + i \) for \( i \in S'_{n-1} \),
\[ \ldots \]
\( w(i) > n - \sigma + q' + \sum_{j=1}^{n} |S'_j| + i \) for \( i \in S'_{n} \).

**Proof.** The proof follows from the similar enumeration technique as in Lemma 9. \( \Box \)

### 4. L-Function Calculations

#### 4.1. L-functions

For self-dual cuspidal automorphic representations \( \tau \), the Rankin-Selberg \( L \)-functions \( L(s, \tau \times \hat{\tau}) \) satisfies the following functional equation

\[
L(s, \tau \times \hat{\tau}) = \epsilon(s, \tau \times \hat{\tau}) L(1-s, \tau \times \hat{\tau})
\]

with the epsilon factor \( \epsilon = e^{c \frac{\tau}{2}} \) for some rational number \( c \). Without incurring any confusion, we will simply denote \( L(s, \tau \times \hat{\tau}) \) and \( \epsilon(s, \tau \times \hat{\tau}) \) by \( L(s) \) and \( \epsilon(s) \), respectively. By [MW89], the normalizing factor defined in the previous section is equal to the product

\[
r(w, s) = \prod_{i<j} \prod_{w(i) > w(j)} \frac{L(\nu_i - \nu_j)}{L(1 + \nu_i - \nu_j) \epsilon(\nu_i - \nu_j, \tau \times \hat{\tau})}.
\]

In the situation we are interested in, we specialize the parameters to the following values:

\[
\begin{align*}
\nu_i &= \frac{1 - m}{2} + i - 1 + s_1, \\
\nu_j &= \frac{1 - n}{2} + j - m - 1 + s_2, \\
&= s = s_1 - s_2.
\end{align*}
\]

For any element \( w \) representable by the diagram

\[
\begin{array}{cccccccc}
v_0 & u_1 & v_1 & u_2 & \ldots & u_i & v_i & \ldots
\end{array}
\]

the collection of inverted pairs \( \text{inv}(w) = \{ (i, j) \mid i < j, w(i) > w(j) \} \) is the set

\[
\text{inv}(w) = (u_1 \times u_1) \bigcup (u_2 \times (u_1 \cup u_2)) \bigcup (u_3 \times (u_1 \cup u_2 \cup u_3)) \bigcup \ldots.
\]

Since \( \nu_i - \nu_j = s + \frac{m+n}{2} + i - j \). For any \( w \in S_m \times S_n \backslash S_{m+n} \), we denote by \( i_w \) the smallest \( i \) such that there exists a \( j > i \) with \( w(j) < w(i) \). For each fixed \( i \geq i_w \), denoting the largest \( j \) such that \( (i, j) \in \text{inv}(w) \) by \( j_i \), the product corresponding to all \( j \)'s such that \( (i, j) \in \text{inv}(w) \) is

\[
\prod_{w(j) < w(i)} \frac{L(\nu_i - \nu_j)}{L(1 + \nu_i - \nu_j) \epsilon(\nu_i - \nu_j)}
\]

\[
= \frac{L(s + \frac{m+n}{2} + i - (m + 1)) \times \ldots \times L(s + \frac{m+n}{2} + i - j_i)}{L(s - \frac{m+n}{2} + i) \times \prod \text{products of } \epsilon \text{'s}}.
\]

Replacing \( s \) by \( s = \frac{m+n}{2} - \sigma + t \) and taking into account that

\[
w(i) = i + j_i - m,
\]

after plugging in all the parameters, the product of \( \epsilon \)-factors can be expressed as

\[
\rho_i(w, t) = e^{\sum_{i=1}^{j_i} \left( s + \frac{m+n}{2} + i - k \right)} = e^{(w(i)-i)(n-i-1)+t} = \left( w(i)-i \right)\rho_i(w, t)^{-1},
\]

and the normalization factor of the intertwining operator is thus equal to

\[
r(w, t) = \prod_{i=i_w} L(s + 2i - w(i) + t) L(s - i + t)^{-1} \rho_i(w, t)^{-1}.
\]
For any $\sigma$, we will be only interested in the analytic property of $r(w,t)$ in the region $t \geq 0$, in which there will be no cancellation of the poles of the denominator by the zeros of the denominator in the critical strip.

The following lemma is a generalization of [HMT15] Lemma 8.5:

**Lemma 16.** For any section $f \in I(\tau, \underline{s})$, the images of the normalized intertwining operator $N(w, \underline{s})f$ are equal as long as $w$ lies in the same orbit $O_{\underline{s}}$.

**Proof.** We would like to compare two $N(w_1, \underline{s})$ and $N(w_2, \underline{s})$ for different $w_1, w_2$ with isomorphic twists $\rho_{m,n}(\tau, \underline{s})^{w_1} \cong \rho_{m,n}(\tau, \underline{s})^{w_2}$.

The normalized intertwining operators are holomorphic in the dominant chamber, and each normalized intertwining operator

$$N(w, \underline{s}) : \text{Ind}^G_P \rho_{m,n}(\tau, \underline{s}) \rightarrow \text{Ind}^G_P \rho_{m,n}(\tau, \underline{s})^{w_i},$$

is invertible. Since the cuspidal representations $\tau$ are unitary, and by [KS71] Proposition 38] the normalized intertwining operators satisfy

(1) $N(w_1w_2, \underline{s}) = N(w_1, \underline{s})N(w_2, \underline{s})$
(2) $N(w, \underline{s})^* = N(w^{-1}, -\underline{s})$.
(3) $N(w, \underline{s})$ is unitary for $s = -\underline{s}$.

Since we can factorize the elements $w_2 = uw_1$ such that $u$ is a permissible move, which is known to be an involution. The intertwining operator

$$N(u, w_2 \underline{s}) : \text{Ind}^G_P \rho_{m,n}(\tau, \underline{s})^{w_2} \rightarrow \text{Ind}^G_P \rho_{m,n}(\tau, \underline{s})^{w_1},$$

is thus a self-adjoint, unitary involution. The operator $N(u, w_2 \underline{s})$ can be factorized further into:

$$N(u, w_2 \underline{s}) = N(v^{-1}, vw_2 \underline{s})N(u, v \underline{s})N(v, w_2 \underline{s})$$

where $v$ is the Weyl group element sending each pairs of corresponding blocks in the permissible move to adjacent blocks, and the operator $N(u, v \underline{s})$ is the transposition of these adjacent blocks. In fact, the intertwining operator $N(\iota, v \underline{s})$ is a product of self-intertwining operators $N(0)$ of $\text{Ind}^G_P \rho_{m,n}(\tau \boxtimes \iota)$ corresponding to the Weyl group element swapping the two Levi blocks. By [KSS88 Proposition 6.3], such an operator $N(0) = 1$, and thus $N(u, w_2 \underline{s}) = 1$.

As a result of this lemma, we can collect the normalization factors in the constant term of $E^{m,n}(\otimes_v f_v, \underline{s})$ into sums over orbits, as in

$$c_{\iota, m,n} E^{m,n}(\otimes_v f_v, \underline{s}) = \sum_{\mathcal{O}_{\underline{s}}(w)} \left( \sum_{w' \in \mathcal{O}_{\underline{s}}(w)} r(w', \underline{s}) \right) \bigotimes_v N(v, w', \underline{s})f_v.$$

Each sum over an orbit is denoted by $R(w, \underline{s}) = \sum_{w' \in \mathcal{O}_{\underline{s}}(w)} r(w', \underline{s})$. We will discuss the cancellations of poles in the sum $R(w, \underline{s})$.

**4.2. Calculation on a Single Living Interval.** Now we calculate the sums on a single living interval. In this section, following Proposition [7] for each living interval $K_i$ with a starting interval $\Xi_{k_i-1} + 1$ and ending interval $\Xi_{k_i}$, we denote these individual green intervals by

$$\Xi_{k_i-1} + 1, \Xi_{k_i-1} + 2, \ldots, \Xi_{k_i}$$

with $|\Xi_{k_i-1} + 1| + \ldots + |\Xi_{k_i}| = |K_i|$. Their corresponding red intervals are denoted by

$$\Sigma_{k_i-1} + 1, \Sigma_{k_i-1} + 2, \ldots, \Sigma_{k_i}.$$

Denoting by $A_p(t)$ the product of $L(n - \sigma + 2i - w(i) + t)\rho_1(w, t)^{-1}$ along each $K_p$, and $B_p(t)$ the product of $L(n - \sigma + 2i - w(n + n - \sigma + i) + t)\rho_1(w, t)^{-1}$ along each $L_p$. Since $i = s(\Xi_{k_p-1} + 1) + r$ and $w(i) = n - \sigma + 2s(\Xi_{k_p-1} + 1) + \sum_{k_p-1}^{\Xi_{k_p-1} + 1} \Sigma_{k_i} w(i) |\Sigma_{k_p-1} + s| + r$, then

$$n - \sigma + 2i - w(i) = r - \sum_{k_p-1}^{\Xi_{k_p-1} + 1} \Sigma_{k_i} w(i) |\Sigma_{k_p-1} + s|.$$
After swapping the corresponding green and red intervals, we obtain
\[ w(m + n - \sigma + i) = n - \sigma + 2s(\Xi_{p-1}+1) + \sum_{\Xi_{p-1}+s < w(m+n-\sigma+i)} |\Xi_{p-1} + s| + r \]
and thus
\[ n - \sigma + 2i - w(m + n - \sigma + i) = r - \sum_{\Xi_{p-1}+s < w(m+n-\sigma+i)} |\Xi_{p-1} + s|. \]

4.3. **Product of \( L \) Factors.** In this section we will calculate \( A_p(t) \) and \( B_p(t) \). We understand that some of the \( \Xi_i, \Sigma_i \) intervals group together on the same \( v, u \) interval. If we denote the endpoints of these intervals by
\[ s(\Xi_{p-1}+1) + \{a_1, \ldots, a_r\}, \]
then the corresponding product of \( L \)-functions in \( A_p(t) \) and \( B_p(t) \) are given by
\[ A'_p(t) = L(1 + t) \cdots L(a_1 + t) L(a_1 - b_1 + 1 + t) \cdots L(a_2 - b_1 + t) \cdots L(a_{r-1} - b_r - 1 + t) \cdots L(a_r - b_r + t) \]
and
\[ B'_p(t) = L(-a_1 + 1 + t) \cdots L(b_1 - a_1 + t) L(b_1 - a_2 + 1 + t) \cdots L(b_2 - a_2 + t) \cdots L(b_{r-1} - a_r + 1 + t) \cdots L(b_r - a_r + t). \]

We can compare these two products, and it turns out that:

**Lemma 17.** The two products above satisfy
\[ A'_p(t) = c \sum \left( \sum_{\Xi_{p-1}+s < w(m+n-\sigma+i)} |\Xi_{p-1} + s| - \sum_{\Xi_{p-1}+s < w(i)} |\Xi_{p-1} + s| \right) B_p'(-t). \]

**Proof.** Since each \( b_i < a_i \), we can reorder the factors in \( A'_p(t) \) and \( B_p'(-t) \). If we represent these factors in \( A'_p(t) \) as intervals, and they can be illustrated in the following diagram:

\[
\begin{align*}
L(1 + t) \cdots L(a_1 + t) & \quad \text{[Red]} \\
L(a_1 - b_1 + 1 + t) \cdots L(a_2 - b_1 + t) & \quad \text{[Green]} \\
\vdots & \\
L(a_{r-1} - b_r - 1 + t) \cdots L(a_r - b_r + t) & \\
\end{align*}
\]

while the factors in \( B_p'(t) \) correspond to the intervals

\[
\begin{align*}
L(-a_1 + 1 + t) \cdots L(b_1 - a_1 + t) & \quad \text{[Red]} \\
L(b_1 - a_2 + 1 + t) \cdots L(b_2 - a_2 + t) & \quad \text{[Green]} \\
\vdots & \\
L(b_{r-1} - a_r + 1 + t) \cdots L(b_r - a_r + t) & \\
\end{align*}
\]

between which a bijection can be established after sending each index \( p \) to \( 1 - p \). By the functional equation of the Rankin-Selberg \( L \)-function \( L(s, r \times \hat{r}) \), we have
\[ A'_p(t) = c \sum \left( r - \sum_{\Xi_{p-1}+s < w(i)} |\Xi_{p-1} + s| - 1/2 + t \right) B_p'(-t). \]

Applying the functional equation again, we can see that
\[ 1 = c \sum \left( r - \sum_{\Xi_{p-1}+s < w(i)} |\Xi_{p-1} + s| - 1/2 + t \right) c \sum \left( r - \sum_{\Xi_{p-1}+s < w(m+n-\sigma+i)} |\Xi_{p-1} + s| - 1/2 - t \right). \]

Therefore,
\[
\sum_r \left( r - \sum_{\Xi_{p-1}+s < w(i)} |\Xi_{p-1} + s| + \sum_{\Xi_{p-1}+s < w(m+n-\sigma+i)} |\Xi_{p-1} + s| \right) \frac{1}{2} = 0
\]
and the original factor becomes
\[ A'_p(t) = c \left( \sum_{r} \left( \sum_{\xi_k} \frac{\xi_k}{2} \right) \right) B'_p(-t). \]

4.4. **Product of \( \rho \)-factors.** This subsection calculates the cancellations of poles in \( R(w, s) \) from the living intervals.

**Proposition 18.** Along each living interval, \( A_p(t) + B_p(t) \) is holomorphic.

**Proof.** Each individual \( \rho \)-factor can be calculated with the following procedure:

For any \( i \), we have
\[
\rho_i(w, t) = c^{(w(i) - i)(n - \sigma + i - 1 + t) - \frac{2}{3}(w(i) - i)^2} \\
= c^{(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i) - \frac{1}{2}(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i)^2} \\
\]
and
\[
\rho_i'(w, t) = c^{(w(m + n - \sigma + i - 1 + t) - \frac{2}{3}(w(m + n - \sigma + i) - i)^2} \\
= c^{(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i) - \frac{1}{2}(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i)^2}.
\]

The differences of their products is
\[
\sum \xi_i(\xi_{j} - \xi_{j})(n - \sigma + s + r - 1 + t) - \left( \sum \xi_i(\xi_{j} - \xi_{j}) \right) = c \sum \xi_i(\xi_{j} - \xi_{j})(r - 1 + t - \frac{2}{3}(w(i) - i)^2)
\]
Therefore,
\[
A_p(t) = A'_p(t)\rho_i(w, t) = c^{(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i) - \frac{1}{2}(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i)^2} \\
B_p(t) = B'_p(t)\rho_i'(w, t) = c^{(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i) - \frac{1}{2}(n - \sigma + s(\xi_{k_{p-1}}) + \sum \xi_i)^2}.
\]

The sum can be reduced to
\[
A_p(t) + B_p(t) \\
= \left( \prod \rho_i(w, t)^{-1} \right) \left( B'_p(t)c^{\sum \xi_i(\xi_{j} - \xi_{j})(r - 1 + t - \frac{2}{3}(w(i) - i)^2)} + B'_p(t) \right) \\
= \left( \prod \rho_i'(w, t)^{-1} \right) \left( B'_p(t)c^{\sum \xi_i(\xi_{j} - \xi_{j})(r - 1 + t - \frac{2}{3}(w(i) - i)^2)} + B'_p(t) \right)
\]
The exponent summing over all \( \sum \xi_i(\xi_{j} - \xi_{j})(r - 1 + t - \frac{2}{3}(w(i) - i)^2) \) is in fact equal to
\[
\sum \xi_i(\xi_{j} - \xi_{j})(r - 1 + t - \frac{2}{3}(w(i) - i)^2) = \sum_{i \in \text{living elements}} w(i) - w(m + n - \sigma + i) - \frac{n - \sigma + 2i - w(i) + w(m + n - \sigma + i) + 1}{2} + t.
\]
Noting that the individual terms in the summation above can be rearranged into the form:
\[
- \sum_{i} \left( n - \sigma + 2i - w(i) \right)^2 - \sum_{i} \left( n - \sigma + 2i - w(m + n - \sigma + i) \right)^2 \\
+ \sum_{i} \left( n - \sigma + 2i - w(i) - (n - \sigma + 2i - w(m + n - \sigma + i)) \right)^2 \\
- t \sum_{i} ((n - \sigma + 2i - w(i)) - (n - \sigma + 2i - w(m + n - \sigma + i)).
\]
The first sum can be further rearranged into the form of \(k^2 - (1-k)^2 = 2k - 1\), which cancels out the second sum. Therefore,

\[
A_p(t) + B_p(t) = \left( \prod_i \rho_i'(w,t)^{-1} \right) \left( e^{-\alpha} B_p'(t) + B_p''(t) \right).
\]

Since both \(A_p'(t)\) and \(B_p'(t)\) has a simple pole at \(t = 0\), and since \(e^{-\alpha}\) is holomorphic with zeroth order coefficient 1, we conclude that \(A_p(t) + B_p(t)\) is holomorphic.

\[\square\]

4.5. Other Factors and Proof of the Theorem. Now we finish the calculation of the sum

\[
R(w, s) = \sum_{w' \in \mathcal{O}_s(w)} r(w', s)
\]

outside of the contributions of the living intervals. Recall that \(i_w\) is the first index \(i\) such that there exist a \(j > i\) such that \(w(i) > w(j)\). The location of the first living interval may affect \(i_w\) when 1 is a living element. When that happens, we are forced to require \(m + n - \sigma + 1 = m + 1\), in which case \(n = \sigma\).

In this case, if 1 is a living element for \(w\), \(R(w, s)\) can be separated into two terms \(R'(t)\) and \(R''(t)\) which consist of the products of \(L\)-function factors with \(i_w'\) for \(w' \in \mathcal{O}_s(w)\) the same as the \(i_w\) of the basepoint element \(w_0\) or not, respectively.

\[
R'(t) = \sum_{w' \in \mathcal{O}_s(w), \ i_{w'} = i_{w_0}} r(w', s) = \frac{\prod_{i \in \text{head \& tail}} L(n - \sigma + 2i - w_0(i) + t) \rho_i^{-1}(w_0, t)}{\prod_{i=0}^{\sigma} L(n - \sigma + i + t)} \prod_{i=1}^{r}(A_i(t) + B_i(t)).
\]

and the sum of the terms \(r(w', s)\) satisfying \(i_{w'} > 1\) by

\[
R''(t) = \sum_{w' \in \mathcal{O}_s(w), \ i_{w'} \neq i_{w_0}} r(w', s) = \frac{\prod_{i \in \text{head \& tail}} L(n - \sigma + 2i - w_0(i) + t) \rho_i^{-1}(w_0, t)}{\prod_{i=1}^{\sigma} L(n - \sigma + i + t)} B_0(t) \prod_{i=1}^{r}(A_i(t) + B_i(t)),
\]

then

\[
R(w, s) = R'(t) + R''(t).
\]

In this case, there are no indices to the left of all the movable indices. The denominator is the product \(L(1+t) \ldots L(m+t)\), and the tail does not contribute to any poles since \(w_0(i) < 2i\) on tails.

On the other hand, if 1 is not a movable element, then \(R(w, s)\) simply takes the form

\[
R(w, s) = \sum_{w' \in \mathcal{O}_s(w)} r(w', s) = \frac{\prod_{i \in \text{head \& tail}} L(n - \sigma + 2i - w(i) + t) \rho_i^{-1}(w, t)}{\prod_{i=1}^{\sigma} L(n - \sigma + i + t)} \prod_{i=1}^{r}(A_i(t) + B_i(t)).
\]

We will have to calculate the contribution from the head and the tails separately.

4.5.1. Head Factors. Now we calculate the contribution to the poles from the head terms in the numerator. In Case I and II, by Lemma \[\text{I}1\] we have

- \(n - \sigma + 2i - w(i) > i\) for \(i \in R_1\),
- \(n - \sigma + 2i - w(i) > i - |R_1|\) for \(i \in R_2\),
- \(\ldots\)
- \(n - \sigma + 2i - w(i) > i - \sum_{j=1}^{k-1} |R_j|\) for \(i \in R_k\).

Therefore, since the head terms can be expressed as

\[
\prod_{j=1}^{k} \prod_{i \in R_j} L(n - \sigma + 2i - w(i) + t),
\]

the point \(t = 0\) is not a pole because all the \(n - \sigma + 2i - w(i) > 1\) for \(i \in \bigcup_{j=1}^{k} R_j\). For Case III, by Lemma \[\text{III}\] we have
n - \sigma + 2i - w(i) \leq i - (\sigma - n) \text{ for } i \in \{1, \ldots, \sigma - n\}, \text{ and when } i = \sigma - n \text{ the inequality is strict,}

n - \sigma + 2i - w(i) \leq i - (\sigma - n) - |R_k'| \text{ for } i \in \sigma - n + R_k', \text{ and when } i = \sigma - n + r(R_k') \text{ the inequality is strict,}

\ldots
\ldots
n - \sigma + 2i - w(i) = i - (\sigma - n) - \sum_{j=1}^{k} |R'_j| \text{ for } i \in \sigma - n + R'_k, \text{ and when } i = \sigma - n + r(R'_k) \text{ we have } n - \sigma + 2i - w(i) = 0.

Therefore, in Case III when \sigma > n, the only possible contribution to the pole on the numerator occurs when \( i = \sigma - n + r(R'_k) \), and when \( \sigma = n \) there will be no head factors. However, this pole will be cancelled by the denominator when \( \sigma \geq n \) as in Case III since we allow \( i = \sigma - n \) and \( i = \sigma - n + 1 \) in the factors of the denominator: the denominator \( \prod_{l=1}^{m} L(n - \sigma + i + t) \) has a pole of order 2 at \( t = 0 \) when \( \sigma > n \), and a pole of order 1 at \( t = 0 \) when \( \sigma = n \). In the case when there are head factors, the contribution to the pole \( t = 0 \) on the numerator will be cancelled by the pole on the denominator.

4.5.2. Tail Factors. In the Cases I and III, the contribution to the poles from the tail factors can be calculated with Lemma [13] Since by Lemma [13] on each interval \( S_k \), we have

\begin{align*}
n - \sigma + 2i - w(i) &< i - (q' + |S_n'|) < 0 \text{ for } i \in S_n', \\
n - \sigma + 2i - w(i) &< i - (q' + |S_n'| + |S_{n-1}'|) < 0 \text{ for } i \in S_{n-1}', \\
\ldots
n - \sigma + 2i - w(i) &< i - (q' + \sum_{j=1}^{n} |S_j'|) < 0 \text{ for } i \in S_1'.
\end{align*}

Therefore, there will be no contribution to the pole from the numerator of \( R(w, s) \). Thus, the tail factors will contribute to the poles of \( R(w, s) \) if and only if \( 0 \leq \sigma < \min\{m, n\} \).

Thus, we have concluded the proof of Theorem A that the possible simple poles occur only when \( \sigma \in \{0, 1, \ldots, \min\{m, n\} - 1\} \).

4.6. The Residue of \( E(\cdot, s) \). Fixing any \( \sigma \in \{0, 1, \ldots, \min\{m, n\} - 1\} \), now we will prove Corollary [13] of Theorem A which describes the residue of \( E(\cdot, s) \) at \( s = s_1 - s_2 = \frac{m+n}{2} - \sigma \).

Proof. For the character

\( \Delta = (s_1, \ldots, s_m; t_1, \ldots, t_n) \),

denote the full principal series induced from \( P_{[1, \ldots, 1]} \) by

\( I(\Delta) = \text{Ind}_{P_{[m+n]}}^{G_{m+n}} (\tau) \cdot |s_1| \otimes \ldots \otimes |\tau| \cdot |s_m| \otimes |\tau| \cdot |t_1| \otimes \ldots \otimes |\tau| \cdot |t_n| \).

Assuming \( \sigma \in \{0, 1, \ldots, \min\{n_1, n_2\} - 1\} \), we define the following four Weyl group elements:

\begin{align*}
w_1 : & (1, 2, \ldots, m; m+1, m+2, \ldots, m+n) \mapsto (m, m-1, \ldots, 1; m+n, m+n-1, \ldots, m+1) \\
w_2 : & (1, 2, \ldots, m; m+1, m+2, \ldots, m+n) \mapsto (m+1, \ldots, m+n-\sigma; 1, \ldots, m; m+n-\sigma+1, \ldots, m+n) \\
w'_1 : & (1, 2, \ldots, m; m+1, m+2, \ldots, m+n) \mapsto (1, 2, \ldots, m; m+\sigma+1, \ldots, m+n; m+1, \ldots, m+\sigma) \\
w'_2 : & (1, 2, \ldots, m; m+1, m+2, \ldots, m+n) \mapsto (m+n-\sigma, \ldots, 1; m+n-\sigma+1, \ldots, m+n).
\end{align*}
These four elements satisfy the relation $w_\sigma w_1 = w'_1 w'_\sigma$. Consider following four intertwining operators corresponding to these Weyl group elements:

$$
I(\Delta) \xrightarrow{N(w_1 \Delta)} I(w_1 \Delta) \\
N(w'_\sigma \Delta) \xrightarrow{I(w'_\sigma \Delta)} I(w_\sigma w_1 \Delta),
$$

The image of the normalized intertwining operator $N(w_\sigma w_1, \xi)$ in $I(w_\sigma w_1 \Delta)$ is the representation

$$
I_\sigma = \text{Ind}^{G_{m+n}}_{P_{[m,n]}} \left( \Delta(\tau, m + n - \sigma)| \cdot | \cdot \frac{\sigma - n}{\sigma} \otimes \Delta(\tau, \sigma)| \cdot | \cdot \frac{\sigma + n}{\sigma} \right),
$$

which is irreducible when $\sigma \in \{0, 1, \ldots, \min\{m, n\} - 1\}$ by MW89 I.7, I.11 for archimedean places and [Tad13 Theorem 1.1] for the nonarchimedean places. Since $w_\sigma w_1 = w'_1 w'_\sigma$, by the properties of the normalized intertwining operators

$$
N(w_1, \Delta) = N(w_1^{-1}, w'_1 w'_\sigma \Delta) N(w'_1, w'_\sigma \Delta) N(w'_\sigma \Delta),
$$

and since $N(w_1, \Delta) \neq 0$, we have $N(w_\sigma w_1, \Delta) \neq 0$. The image of the intertwining operator $N(w_1, \Delta)$ is isomorphic to the image $I(\tau, \xi)$ of the residue operator

$$
\text{Res}_{(-\lambda_m + s_1, -\lambda_n + s_2)} = \prod_{1 \leq i \leq m-1} (s_i - s_{i+1} - 1) \prod_{1 \leq j \leq n-1} (t_j - t_{j+1} - 1) E(\xi, \Delta) \bigg|_{\Delta \to (-\lambda_m + s_1, -\lambda_n + s_2)}: I(\Delta) \to I(w_1 \Delta)
$$

as a submodule in $I(w_1 \Delta)$. By calculating the poles of the normalized intertwining operator $N(w_\sigma, w_1 \Delta)$, the residue operator $\text{Res}_{s_1 - s_2 = \frac{m+n}{2} - \sigma} E(\xi, \Delta)$ kills all the $M(w, \Delta)f$ terms in the constant term formula of $E(\xi, \Delta)$ except for those corresponding to the Weyl group element $w = w_2 w_1$ with $w_2$ satisfying

$$
w_2 (m + 1) < w_2 (m + 2) < \ldots < w_2 (m + n - \sigma) < w_2 (1) < w_2 (2) < \ldots < w_2 (m).
$$

as well as (3). For any such $w_2$, $N(w_2 w_1^{-1}, w_\sigma w_1 \Delta)$ maps $I_\sigma$ either isomorphically onto its image or to 0, and thus, the constant term of $\text{Res}_{s_1 - s_2 = \frac{m+n}{2} - \sigma} \text{Res}_{(-\lambda_m + s_1, -\lambda_n + s_2)} E(\xi, \Delta)$ is equal to the constant term of the Eisenstein series constructed from $M(w_\sigma w_1, w_\sigma w_1 \Delta)f$ as a vector in $I_\sigma$ up to a nonzero constant scalar. Thus, the automorphic representation generated by $\text{Res}_{s_1 - s_2 = \frac{m+n}{2} - \sigma} \text{Res}_{(-\lambda_m + s_1, -\lambda_n + s_2)} E(\xi, \Delta)$ is isomorphic to $I_\sigma$.

**EXAMPLES**

1. **Case $m = n = 2$**. For $\sigma = 0$, the two segments $\Delta_1 = [\frac{1}{2}, \frac{3}{2}]$ and $\Delta_2 = [-\frac{3}{2}, -\frac{1}{2}]$ are **juxtaposed**. The orbits and the corresponding $R(w, \Delta)$ are listed in the following table:

| Orbit | $\rho_{m,n}$ | $R(w, \frac{m+n}{2} - \sigma + t)$ |
|-------|---------------|----------------------------------|
| $e$   | $\left\{ \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{5}{2} \right\}$ | $\frac{1}{c^{t+2}L(t+2)}$ |
| $(1324)$ | $\left\{ \frac{1}{2}, -\frac{3}{2}, \frac{7}{2}, -\frac{1}{2} \right\}$ | $\frac{L(t+2)}{c^{2t+1}L(t+2)}$ |
| $(1342)$ | $\left\{ \frac{1}{2}, -\frac{7}{2}, -\frac{1}{2}, \frac{9}{2} \right\}$ | $\frac{L(t+4)}{c^{4t+2}L(t+2)(t+4)}$ |
| $(3124)$ | $\left\{ -\frac{1}{2}, \frac{7}{2}, -\frac{1}{2}, \frac{5}{2} \right\}$ | $\frac{L(t+4)}{c^{2t+1}L(t+2)^2}$ |
| $(3142)$ | $\left\{ -\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{7}{2} \right\}$ | $\frac{L(t+4)}{L(t+2)L(t+4)}$ |
| $(3412)$ | $\left\{ -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \right\}$ | $\frac{L(t+4)}{L(t+3)L(t+4)}$ |

For $\sigma = 1$, the poles of $r(w, \Delta)$ in one of the orbits $\{(3142), (3412)\}$ cancel:
Orbit \( \rho_{m,n} \) \( R(w, \frac{m+n}{2} - \sigma + t) \)

\( e \) \[ \{0, 1, -1, 0\} \] \( \frac{1}{c^{-t} L(t+2)} + \frac{1}{c^{-2t} L(t+1)} \)

(1324) \[ \{0, -1, 1, 0\} \]

(1342) \[ \{0, -1, 0, 1\} \]

(3124) \[ \{-1, 0, 1, 0\} \]

(3142), (3412) \[ \{-1, 0, 0, 1\} \]

\[ \frac{
 e^{-\frac{5t}{2}} L(t+1)(L(1-t)+L(1+t))}{L(t+2)L(t+3)} \]

For \( \sigma = 2 \), more cancellations occur:

Orbit \( \rho_{m,n} \) \( R(w, \frac{m+n}{2} - \sigma + t) \)

\( e, (3142) \) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(1324), (1342), (3124), (3142) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

\[ \frac{e^{-2t}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+1)L(t+2)} \]

\[ \frac{e^{-t-\frac{5t}{2}} L(t+3)}{L(t+2)L(t+3)} \]

\[ \frac{e^{-2t-2}(L(1-t)+L(t+1))}{L(t+3)} \]

\[ \frac{e^{-4t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

2. Case \( m = 2, n = 3 \). For \( \sigma = 2 \), the \( R(w, s) \)'s are displayed in the following table:

Orbit \( \rho_{m,n} \) \( R(w, \frac{m+n}{2} - \sigma + t) \)

\( e \)

(13245) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(13425), (13452) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(31245), (34512) \[ \{-\frac{5}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(31425), (34152), (34125), (34152) \[ \{-\frac{5}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

\[ \frac{e^{-t-\frac{5t}{2}} L(t+3)}{L(t+2)L(t+3)} \]

\[ \frac{e^{-2t-4}(L(1-t)+L(t+1))}{L(t+3)} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

\[ \frac{e^{-3t-2}(c^{2t} L(t+1)L(t+2)+L(1-t)L(2-t))}{L(t+2)L(t+3)} \]

3. Case \( m = 3, n = 2 \). Similar to the previous subsection, for \( \sigma = 1 \),

Orbit \( \rho_{m,n} \) \( R(w, \frac{m+n}{2} - \sigma + t) \)

\( e \)

(12435) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(12453) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(14235) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(14253) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(14523) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(41235) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(41253) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

(41523), (45123) \[ \{-\frac{1}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{-4t-2}(L(1-t)+L(t+1))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{-4t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

4. Case \( m = 3, n = 3 \). For \( \sigma = 3 \), we have:

Orbit \( \rho_{m,n} \) \( R(w, \frac{m+n}{2} - \sigma + t) \)

\( e, (456123) \)

(124356), (451623) \[ \{-0, 1, 0, 1\} \]

(124536), (124636), (451236), (455263) \[ \{-0, 1, 0, 1\} \]

(142356), (145623), (412356), (415623) \[ \{-0, 1, -1, 0, 1\} \]

(142536), (145263), (412536), (414523) \[ \{-1, -1, 0, 1, 0\} \]

(145236), (145263), (415236), (415263) \[ \{-1, -1, 0, 0, 1\} \]

\[ \frac{e^{5t}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{5t-2}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{5t-2}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{5t-2}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

\[ \frac{e^{5t-2}(c^{2t} L(t+1)L(t+2)L(3-t))}{L(t+1)L(t+2)L(t+3)} \]

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