A STABILITY RESULT FOR SPARSE CONVOLUTIONS

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ABSTRACT
We will establish in this note a stability result for sparse convolutions on torsion-free additive (discrete) abelian groups. Sparse convolutions on torsion-free groups are free of cancellations and hence admit stability, i.e. injectivity with a universal lower bound \(\alpha = \alpha(s, f)\), only depending on the cardinality \(s\) and \(f\) of the supports of both input sequences. More precisely, we show that \(\alpha\) depends only on \(s\) and \(f\) and not on the ambient dimension. This statement follows from a reduction argument which involves a compression into a small set preserving the additive structure of the supports.

1. INTRODUCTION
In this work, we will prove a \(\ell^2\)-norm inequality for \((s, f)\)-sparse convolutions on \(\ell^2_s(G) \times \ell^2_f(G)\) for one–dimensional abelian torsion-free discrete groups \(G = (G, +, |\cdot|)\) equipped with the counting measure \(|\cdot|\). We define for a natural number \(k\) the set of \(k\)-sparse sequences in \(\ell^2(G)\):

\[
\ell^2_k(G) := \left\{ x : G \to \mathbb{C} \mid \|x\|^2 := \sum_{i \in G} |x_i|^2 < \infty, \text{supp} \ x \leq k \right\}.
\]

(1)

Then for two \((s, f)\)-sparse sequences \(x \in \ell^2_s(G)\) and \(y \in \ell^2_f(G)\) its convolution \(x \ast y\) is given by the sequence with elements:

\[
(x \ast y)_j = \sum_{i \in G} x_i y_{j-i} \text{ for all } j \in G
\]

(2)
each being a finite sum. Let us define the set of \(k\)-sparse vectors in \(\mathbb{C}^n\) by \(\Sigma^k_n\) and the set of all support sets with cardinality \(k\) by \([0, n-1]_k := \{ A \in \{0, 1, \ldots, n-1\} \mid |A| = k \}\). For any \(A \in [0, n-1]_k\), the projection operator \(P_A : \mathbb{C}^n \to \mathbb{C}^k\) cuts out from the \(n \times n\)–matrix \(B\) an \(k \times k\) principal submatrix \(B^A = P_A B P_A^*\). Further, we denote by \(B_A\) an \(n \times n\)–Hermitian Toeplitz matrix generated by the autocorrelation \(b_k(a) = \sum \text{w}_{\text{divis}\ a \text{all}} \text{w}_{\text{divis}\ k} a \in \Sigma^k_n\) with symbol

\[
b(a, \omega) = \sum_{k=-(n-1)}^{n-1} b_k(a) e^{ik\omega} = 1 + \sum_{k=1}^{n-1} (\mu_k \cos(k\omega) + \nu_k \sin(k\omega)), \quad \omega \in [0, 2\pi)
\]

(3)

with \(\mu_k := 2\Re(b_k(a))\) and \(\nu_k := -2\Im(b_k(a))\),

which defines a trigonometric polynomial of order not larger than \(n\).

2. MAIN RESULT AND PROOF

The following theorem is a generalization of a result in [3], (i) in the sense of the extension to infinite sequences on torsion-free abelian groups (note if one adds consecutive \(n-1\) zeros to sparse vectors in \(\Sigma^k_n\) the circular convolution in \(\mathbb{C}^{2n-1}\) can be written with \(G = (\mathbb{Z}/(2n-1)\mathbb{Z}, \oplus)\) by [2], since the addition modulo \(\oplus\) (modulo \(2n-1\)) equals then the regular addition + (ii) extension to the complex case, which actually only replaces SZEGÖ factorization with FEJER-RIESZ factorization in the proof and (iii) with a precise determination of the dimension parameter \(\bar{\ell}\).

\[\text{Actually, the estimate of the dimension } n = \tilde{\ell} \text{ of the constant } \alpha_\tilde{\ell} \text{ in [3], was quite too optimistic.}\]
Theorem. Let $s$ and $f$ be natural numbers and $G$ a one-dimensional torsion-free, discrete, additive abelian group. Then there exist constants $0 < \alpha(s, f) \leq \beta(s, f) = \sqrt{\min\{s, f\}} < \infty$ depending solely on $s$ and $f$, s.t. for all $x \in \ell^2_s(G)$ and $y \in \ell^2_f(G)$

$$\alpha(s, f) \|x\| \|y\| \leq \|x \ast y\| \leq \beta(s, f) \|x\| \|y\|$$

holds. Moreover, we have with

$$\alpha^2(s, f) = \min \left\{ \min_{y : \sum_{j=1}^{\infty} |y_j| = 1} \lambda(B^1), \min_{x : \sum_{j=1}^{\infty} |x_j| = 1} \lambda(B^1) \right\},$$

which is a decreasing sequence in $s$ and $f$. For $\beta(s, f) = 1$ we get equality with $\alpha(s, f) = 1$.

Proof. The upper bound is trivial and follows from the Cauchy-Schwartz inequality and the Young inequality for $p = 1, q = r = 2$. For $x = 0$ or $y = 0$ the inequality is trivial as well, hence we assume that $x$ and $y$ are non-zero. If $|\text{supp } x| = 1$ then there exist $i \in G$ such that $x_i \neq 0$ and $x_j = 0$ for all $j \neq i \in G$. The norm of the convolution equals then $\|x \ast y\| = |x_i y_i|$ and the inequality (4) becomes an equality.

We consider therefore the normalized version of the convolution for $s, f \geq 2$, i.e. the following problem:

$$\alpha(s, f) := \inf_{0 \neq x \in \ell^2_s(G)} \frac{\|x \ast y\|}{\|x\| \|y\|} = \inf_{0 \neq x, y \in \ell^2_s(G) \ast \ell^2_f(G)} \frac{\|x \ast y\|}{\|x\| \|y\|}. $$

This is a bi-quadratic optimization problem which is known to be NP-hard in the general case [5]. The squared norm of the convolution of two finitely supported sequences is given by (2) as:

$$\|x \ast y\|^2 = \sum_{j \in G} \left| \sum_{i \in G} x_i y_{j-i} \right|^2. $$

Let $I$ and $J$ be sets of $G$ such that $\text{supp } x \subseteq I$ and $\text{supp } x \subseteq J$ with $|I| = s, |J| = f$ for some $2 \leq s, f \in \mathbb{N}$. For such $I, J \subset G$ with $I = \{i_0, \ldots, i_{s-1}\}$ and $J = \{j_0, \ldots, j_{f-1}\}$ (ordered sets) we can represent $x$ and $y$ by complex vectors $u \in \mathbb{C}^s$ and $v \in \mathbb{C}^f$ component-wise given by:

$$x_i = \sum_{\theta=0}^{s-1} u_{\theta} \delta_{i,i_\theta}, \quad y_j = \sum_{\gamma=0}^{f-1} v_{\gamma} \delta_{j,j_\gamma} \quad \text{for all } \quad i, j \in G. $$

Inserting this representation in (7) yields:

$$\|x \ast y\|^2 = \sum_{j \in G} \left| \sum_{i \in G} \left( \sum_{\theta=0}^{s-1} u_{\theta} \delta_{i,i_\theta} \right) \left( \sum_{\gamma=0}^{f-1} v_{\gamma} \delta_{j,j_\gamma} \right) \right|^2$$

$$= \sum_{j \in G} \left| \sum_{\theta=0}^{s-1} \sum_{\gamma=0}^{f-1} \left( \sum_{i \in G} u_{\theta} \delta_{i,i_\theta} v_{\gamma} \delta_{j,j_\gamma+i_\theta} \right)^2 \right. \quad \text{(9)}$$

Since the inner $i$–sum is over $G$, we can shift $I$ by $i_0$ if we set $i \rightarrow i + i_0$ (note that $x \neq 0$), without changing the value of the sum:

$$= \sum_{j \in G} \left| \sum_{\theta=0}^{s-1} \sum_{\gamma=0}^{f-1} \left( \sum_{i \in G} u_{\theta} \delta_{i+i_0,j+i_0} \delta_{j,j_\gamma+i_\theta} \right) \right|^2 \quad \text{(11)}$$

By the same argument we can shift $J$ by setting $j \rightarrow j + j_0$ and get:

$$= \sum_{j \in G} \left| \sum_{\theta=0}^{s-1} \sum_{\gamma=0}^{f-1} \left( \sum_{i \in G} u_{\theta} \delta_{i-i_0,j-j_0} \delta_{j,j_\gamma+j_0+i_\theta} \right) \right|^2 \quad \text{(12)}$$
Therefore, we always can assume that the supports \( I, J \subset G \) have \( i_0 = j_0 = 0 \) in \([0,1,2,3]\). From \([10]\) we get:

\[
\begin{align*}
\sum_{j \in G} \left| \sum_{\theta} \sum_{\gamma} (u_0 v_{\gamma} \delta_{j,j+1}) \right|^2 \\
= \sum_{j \in G} \sum_{\theta} \sum_{\gamma} u_0 v_{\gamma} \delta_{j,j+1}
\end{align*}
\]

\[
\begin{align*}
\sum_{j \in G} \sum_{\theta} \sum_{\gamma} u_0 v_{\gamma} \delta_{j,j+1} = b_{1,j}(u,v).
\end{align*}
\]

Usually, fourth order tensors like \( \delta_{i_0+j_1,i_0'j_1'} \) make such bi-quadratic optimization problems over \( \mathbb{C}^s \times \mathbb{C}^f \) NP-hard, see \([5]\).

The interesting question is now: what is the smallest dimension to represent this tensor, i.e. preserving the additive structure? Let us consider a mapping \( \phi \) of the indices. For \( I, J \subset G \) with \( 0 \in I \cap J \) an injective map:

\[
\phi : I + J \rightarrow \mathbb{Z}
\]

which additional satisfies (preserves additive structure of the indices):

\[
\forall i, i' \in I, j, j' \in J : i + j = i' + j' \Rightarrow \phi(i) + \phi(j) = \phi(i') + \phi(j')
\]

is called a Freiman homomorphism on \( I, J \) and is a Freiman isomorphism if:

\[
\forall i, i' \in I, j, j' \in J : i + j = i' + j' \Leftrightarrow \phi(i) + \phi(j) = \phi(i') + \phi(j'),
\]

see e.g. \([3\text{, pp.}299]\)]. If we could show that \( \phi(I), \phi(J) \subset [0, n-1] = \{0, 1, \ldots, n-1\} \), where \( n = n(s, f) \), for any \( I, J \subset G \) with \( |I| = s, |J| = f \) the minimization problem reduces to an \( n \)-dimensional problem. Indeed, this was a conjecture by KONYAGIN and LEV \([4]\), which was proved very recently by GRYNIKIEWICZ in \([3\text{, Theorem}20.10]\) for Freiman dimension \( d = 1 \). He could even prove a more generalized compression argument of arbitrary sum sets with finite Freiman dimension \( d \) in torsion-free abelian groups. We will state here a restricted version of his result for additive abelian groups with two sets \( A_1 \) and \( A_2 \):

**Lemma \([3]\).** Let \( G \) be a torsion-free additive abelian group and \( A_1, A_2 \subset G \) be finite sets containing zero with \( m := |A_1 \cup A_2| \) and having finite Freiman dimension \( d = \dim^\perp(A_1 + A_2) \). Then there exists an injective Freiman homomorphism:

\[
\phi : A_1 + A_2 \rightarrow \mathbb{Z}
\]

such that

\[
diam(\phi(A_1)), \ diam(\phi(A_2)) \leq dl^2 \left( \frac{3}{2} \right)^d - 2m^2 - \frac{3^{d-1}-1}{2}.
\]

For simplicity we have restricted our statements here solely on discrete (countable) groups. Thus, for \( A_1 = A_2 = A := I \cup J \) we get \( |A_1 \cup A_2| = |A| \leq m \). By a simple upper bound for \( d \) in \([3\text{, Corollary}5.42]\) we get \( d \leq m-2 \) for \( m \geq 2 \) and by Gryniewicz a bijective Freiman homomorphism on \( A + A \), which is a Freiman isomorphism on \( A \) with

\[
diam(\phi(A)) \leq (m-2)^2 \left( \frac{3}{2} \right)^m - 2m^2 - \frac{3^{m-1}-1}{2} < 2^{2(m-1)\log_2(m-1)}.
\]

Since \( |I| = s, |J| = f \) with \( s, f \geq 2 \) and \( 0 \in I \cap J \) we always have \( 3 \leq m \leq s + f - 1 \). Then \( diam(\phi(A)) := max(\phi(A)) - min(\phi(A)) \) and hence \( diam(\phi(I) \cup \phi(J)) \leq n \) with

\[
n := 2^{2(s+f-2)\log_2(s+f-2)} + 1.
\]

Let us now define the minimum in the image:

\[
e^* := \min_{c \in I \cup J} \phi(c).
\]

Then we can translate the Freiman isomorphism by setting \( \phi' = \phi - e^* \) (still satisfy (16)) and define \( \tilde{I} := \phi'(I) \) and \( \tilde{J} := \phi'(J) \). With \( n \in \[21\] \) we have:

\[
0 \in \tilde{I} \cup \tilde{J} \subset \{0,1,2,\ldots,n-1\} = [0,n-1].
\]
Unfortunately, a Freiman isomorphism does not necessarily preserve the order. However, this is not a problem, since we only need to know, that indices not larger than \( n - 1 \) are needed to express the combinatorics of the convolution, i.e. \( (14) \) reads now as:

\[
b_{I,J}(u, v) = \sum_{\theta, \theta'} \sum_{i, j} u_{\theta i} v_{\theta' j} \delta_{i_\theta + j, j_i + i_{\theta'}}.
\]

(24)

and the norm of the convolution is indeed reduced to \( n \) dimensions. Next, we can define the embedding of \( u, v \) into \( \mathbb{C}^n \) by:

\[
\tilde{x}_i = \sum_{\theta} u_{\theta i} \delta_{i, i_\theta}, \quad \tilde{y}_j = \sum_{\gamma} v_{\gamma j} \delta_{j, j_\gamma},
\]

\[ \text{for all } i, j \in [0, n - 1]. \]

(25)

Since for all \( \theta \) and \( \gamma \) there exist unique \( i_\theta \in \tilde{I} \) resp. \( j_\gamma \in \tilde{J} \) (\( \phi' \) is bijective) we get:

\[
u_{\gamma} = \sum_{j=0}^{n-1} \tilde{y}_j \delta_{j, j_\gamma},
\]

(26)

and inserting this into \( (24) \) yields:

\[
b_{I,J}(u, v) = \sum_{\theta, \theta'} \sum_{i, j} u_{\theta i} \delta_{i, i_\theta} \delta_{\theta' j, j_\gamma} \sum_{\gamma, \gamma'} v_{\gamma j} \delta_{j, j_{\gamma'}} = \sum_{j} \tilde{y}_j \tilde{x}_j = \langle \tilde{x}, B_{\tilde{y}} \tilde{x} \rangle,
\]

(27)

(25)

(28)

(29)

where \( B_{\tilde{y}} \) is a \( n \times n \) Hermitian Toeplitz matrix with first row \( (B_{\tilde{y}})_{0,k} = \sum_{j=0}^{n-k} \bar{y}_j y_{j+k} = b_k(\tilde{y}) \) resp. first column \( (B_{\tilde{y}})_{k,0} = b_{-k} \) for \( k \in [0, n - 1] \) and symbol \( b(\tilde{y}, \omega) \) given by \( (3) \), see e.g. \[1\]. We call \( b(\tilde{y}, \omega) \) for each \( \tilde{y} \in \mathbb{C}^n \) with \( \|\tilde{y}\| = 1 \) a normalized trigonometric polynomial of order \( n - 1 \). Minimizing the scalar product in \( (29) \) over all \( \tilde{x} \in \mathbb{C}^n \) with \( \|\tilde{x}\| = 1 \) defines the smallest eigenvalue of \( B_{\tilde{y}} \):

\[
\lambda(B_{\tilde{y}}) := \min_{\tilde{x} \in \mathbb{C}^n, \|\tilde{x}\|=1} \langle \tilde{x}, B_{\tilde{y}} \tilde{x} \rangle.
\]

(30)

By the well-known FEJER-RIESZ Factorization, see e.g. \[2\], Thm.3], we know that the symbol of \( B_{\tilde{y}} \) is non-negative \[4\] for every \( \tilde{y} \in \mathbb{C}^n \), i.e. \( 0 \leq \min_{\omega} b(\tilde{y}, \omega) \). By \[1\] (10.2) we then have \( \lambda(B_{\tilde{y}}) > 0 \). Hence \( B_{\tilde{y}} \) is invertible and the determinant \( \det(B_{\tilde{y}}) \neq 0 \). Using:

\[
\frac{1}{\lambda(B_{\tilde{y}})} = \|B_{\tilde{y}}^{-1}\|
\]

(31)

in \[1\], p.59, we can estimate the smallest eigenvalue (singular value) with \[1\], Thm. 4.2 by the determinant as:

\[
\lambda(B_{\tilde{y}}) \geq \left| \det(B_{\tilde{y}}) \right| \frac{1}{\sqrt{n} \left( \sum_k |b_k(\tilde{y})|^2 \right)^{(n-1)/2}}.
\]

(32)

In the following we will not further explicity account for the sparsity of \( \tilde{y} \) which may improve the next steps. For our purpose it will be sufficient here to show a non-zero lower bound. The \( \ell^2 \)-norm of the sequence \( b_k(\tilde{y}) \) can be upper bounded for \( n > 1 \) by the CAUCHY-SCHWARTZ inequality (instead one may also utilize the upper bound of the theorem):

\[
\sum_{k} |b_k(\tilde{y})|^2 \leq 1 + 2 \sum_{k=1}^{n-1} |\sum_{j=0}^{n-1} \bar{y}_j y_{j+k}|^2 \leq 1 + 2 \sum_{k=1}^{n-1} |\tilde{y}|^4 = 1 + 2(n - 1) < 2n.
\]

(33)

\[2\] Note, there exist \( \tilde{y} \in \mathbb{C}^n \) with \( |\tilde{y}| = 1 \) and \( b(\tilde{y}, \omega) = 0 \) for some \( \omega \in [0, 2\pi) \). That’s the reason why things are more complicated here. Moreover, we want to find a universal lower bound over all \( \tilde{y} \), which is equivalent to a universal lower bound over all non-negative trigonometric polynomials of order \( n - 1 \). By the best knowledge of the authors, there exist no analytic lower bound for \( \alpha(s, f) \).
which is independent of \( \mathbf{y} \in \mathbb{C}^n \) with \( \|\mathbf{y}\| = 1 \)!

Since the determinant is a continuous function in \( \mathbf{y} \) over a compact set, the minimum is attained and is denoted by \( 0 < d_n := \min_{\mathbf{y}} |\det(\mathbf{B}_\mathbf{y})| \). Note, that \( d_n \) is a decreasing sequence, since we extend the minimum to a larger set by increasing \( n \). Hence we get:

\[
\min_{\mathbf{y} \in \mathbb{C}^n, \|\mathbf{y}\| = 1} \left( \frac{1}{\sqrt{n(2n)^{(n-1)/2}}} \right) = \frac{\sqrt{2}}{(2n)^{n/2}}d_n.
\]

(34)

This is a valid lower bound by (32) for the smallest eigenvalue of all \( \mathbf{B}_\mathbf{y} \). Hence we have shown

\[
\min_{\mathbf{y} \in \mathbb{C}^n, \|\mathbf{y}\| = 1} \lambda(\mathbf{B}_\mathbf{y}) > \sqrt{2}(2n)^{-n/2}d_n > 0.
\]

(35)

Now, bringing the support back into play, we see that \( \mathbf{x} \) and \( \mathbf{y} \) are fully realized by the Freiman isomorphism as \( \bar{I} = \phi'(I), \bar{J} = \phi'(J) \), where \( \mathbf{x} \) cuts out (in a symmetrical way) for a fixed \( \mathbf{y} \in \mathbb{C}^n \) an \( s \times s \) Hermitian matrix \( \mathbf{B}_\mathbf{y}^{\bar{I}} = \mathbf{P}_f \mathbf{B}_\mathbf{y} \mathbf{P}_f^* \) (principal submatrix, actually also Toeplitz) given by the green elements (here we have re-ordered \( I \) such that \( \bar{I} \) is ordered)

\[
\mathbf{B}_\mathbf{y} := \begin{pmatrix}
   b_0 & \ldots & b_{i_0} & \ldots & b_{i_1} & \ldots & b_{i_{n-1}} & \ldots & b_{n-1} \\
   \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   b_{i_0} & \ldots & b_{i_{n-1}} & \ldots & b_{i_{n-1}} & \ldots & b_{n-1} & \ldots & b_{n-1} \\
   \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   b_{i_1} & \ldots & b_{i_1} & \ldots & b_{i_1} & \ldots & b_{n-1} & \ldots & b_{n-1} \\
   \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
   b_{i_{n-1}} & \ldots & b_{i_{n-1}} & \ldots & b_{i_{n-1}} & \ldots & b_{n-1} & \ldots & b_{n-1} \\
   b_{n-1} & \ldots & b_{n-1} & \ldots & b_{n-1} & \ldots & b_{n-1} & \ldots & b_{n-1}
\end{pmatrix}.
\]

(36)

Minimizing over all \( \mathbf{u} \in \mathbb{C}^s \) we have by CAUCHY’S Interlacing Theorem, see e.g. [1, Prop.9.19], for all \( s \leq n \in \mathbb{N} \)

\[
\lambda(\mathbf{B}_\mathbf{y}^{\bar{I}}) \geq \lambda(\mathbf{B}_\mathbf{y}) > 0 \quad \bar{y} \in \mathbb{C}^n, \bar{I} \in [n].
\]

(37)

Hence, this also holds for \( \bar{y} \in \Sigma^s_j \) and we get for our problem in (6)

\[
\alpha^2(s, f) = \inf_{\|\mathbf{x} + \mathbf{y}\| = 1} \|\mathbf{x} + \mathbf{y}\| \geq \min_{\bar{I} \in [0, n-1]} \min_{\bar{y} \in \Sigma^s_j} \lambda(\mathbf{B}_\mathbf{y}^{\bar{I}}), \min_{\bar{I} \in [0, n-1]} \min_{\|\mathbf{y}\| = 1} \lambda(\mathbf{B}_\mathbf{y}) \geq \min_{\|\mathbf{z}\| = 1} \lambda(\mathbf{B}_\mathbf{z}) \geq \min_{\|\mathbf{n}\| = 1} \lambda(\mathbf{B}_\mathbf{n}) =: \alpha_n^2.
\]

(38)

Unfortunately, the combinatoric can only be removed by using the CAUCHY Interlacing theorem, which obtains only a lower bound \( \alpha_n \) for \( \alpha(s, f) \). Moreover, the lower bound attained in by the double minimum may still be too large: First, \( n \) may to large and even if \( n \) is the right dimension for the Freiman isomorphism, there are not all \( \bar{I} \in [0, n-1] \), resp. \( \bar{I} \in [0, n-1] \) needed to represent the convolution.

\[ \square \]

3. CONCLUSION

There are several applications for this inequality. For example, in [8] the authors have shown a statement for stable phase retrieval from magnitude of \( 4n - 3 \) symmetrized Fourier measurements. The stability result is independent of the ambient dimension in the regime of \( \mathcal{O}((s + f - 2) \log_2 (s + f - 2)) < \log_2 n \). Furthermore, tools from spectral theory of Toeplitz matrices may be used to obtain more precise estimates for lower bound of the smallest eigenvalue \( \alpha^2(s, f) \).
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