Rigorous numerics for nonlinear heat equations in the complex plane of time

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Abstract
In this paper, we introduce a method for computing rigorous local inclusions of solutions of Cauchy problems for nonlinear heat equations for complex time values. The proof is constructive and provides explicit bounds for the inclusion of the solution of the Cauchy problem, which is rewritten as a zero-finding problem on a certain Banach space. Using a solution map operator, we construct a simplified Newton operator and show that it has a unique fixed point. The fixed point together with its rigorous bounds provides the local inclusion of the solution of the Cauchy problem. The local inclusion technique is then applied iteratively to compute solutions over long time intervals. This technique is used to prove the existence of a branching singularity in the nonlinear heat equation. Finally, we introduce an approach based on the Lyapunov–Perron method for calculating part of a center-stable manifold and prove that an open set of solutions of the Cauchy problem converge to zero, hence yielding the global existence of the solutions in the complex plane of time.

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1 Introduction

In this paper, we consider solutions of a nonlinear heat equation with the following setting:

\[
\begin{aligned}
    u_t &= u_{xx} + u^2, & x \in (0, 1), & t > 0, \\
    u(0, x) &= u_0(x), & x \in (0, 1)
\end{aligned}
\]  (1.1)

with the periodic boundary condition in \( x \) variable. Here the subscripts denote the derivatives in the respective variables and \( u_0(x) \) is a given initial data. It is well-known that solutions of the nonlinear heat equation (1.1) may blow up in finite time, that is, the \( L^\infty \)-norm of the solution tends to \( \infty \) as \( t \) tends to a certain \( B < \infty \). Such a \( B \) is called the blow-up time of (1.1). There are plenty of studies for blow-up problems of nonlinear heat equations. One can consult previous studies by, e.g., [7, 21] and references therein.

We extend the time variable of (1.1) into the complex plane. Changing the \( t \) variable into \( z \), the aim of this paper is to study dynamics of the complex-valued nonlinear heat equation

\[
\begin{aligned}
    u_z &= u_{xx} + u^2, & x \in (0, 1), & \text{Re}(z) \geq 0,
\end{aligned}
\]  (1.2)

under the periodic boundary condition in \( x \) with initial data \( u(0, x) = u_0(x) \). Here, the subscript \( z \) denotes the complex-derivative with respect to \( z \) and \( \text{Re}/\text{Im} \) denote the real/imaginary part of complex values, respectively.

In his pioneering works [14, 15] for this problem, Masuda considered the solution of (1.2) under the Neumann boundary condition. If the initial data \( u_0(x) \) is sufficiently close to a real constant, he proved the existence of two analytic solutions, one of which is defined on the shaded domain of Fig. 1a and the other on its mirror image with respect to the real axis in Fig. 1b. Masuda also proved that the initial data is a constant if the solution agrees in the intersection of the two domains with \( \text{Re}(z) > z_B \), where \( z_B \equiv B \) denotes the blow-up time (see Fig. 1b). It follows that, if \( u \) is the solution in the shaded domain and \( v \) is the one in the mirror image of the shaded domain, then \( v \) is the complex conjugate of \( u \). Therefore, one can conclude that there exists multiple solutions if the imaginary part of \( u \) and \( v \) are non-trivial. It indicates that the blow-up time is expected to be a branch point\(^1\), but it is not shown that the blowup happens at an isolated point in the complex plane.

Following Masuda, Cho et al. [4] numerically tested dynamics of the solution of (1.2) under the periodic boundary condition. They have shown that the solution may converge to the zero function on a straight path \( \Gamma_\theta \equiv \{ z \in \mathbb{C} : z = te^{i\theta}, \ t \geq 0 \} \) where \( \theta \in (-\pi/2, \pi/2) \) and \( i = \sqrt{-1} \). They have also shown that the solution of (1.2) may have only one singularity on the real axis, which branches the Riemann surface of complex-valued solutions. As conclusions of their study, two conjectures have been proposed.

\(^1\) A point is called a branch point if the multi-valued function is discontinuous on traversing a small circuit around this point (cf. [1,p.46]). We call such a singularity branching singularity in this paper.
The analytic function defined by the nonlinear heat equation (1.2) has branching singularities and only branching singularities, unless it is constant in $x$.

- The nonlinear Schrödinger equation, which is the case of $\theta = \pm \pi/2$, is globally well-posed for any real initial data, small or large.

The contribution of the current paper is to give two computer-assisted proofs for the complex-valued nonlinear heat equation (1.2). Although these results do not fully solve the above conjectures, they do provide a mathematical proof of the dynamical behavior of the solution numerically obtained by Cho et al. [4]. Additionally they also extend the results by Masuda [14, 15] in terms of treating the periodic boundary conditions.

Our first result shows the existence of a branching singularity, which appears at the blow-up point Cho et al. [4] calculated to be approximately $z_B \approx 0.0119$.

**Theorem 1.1** (Existence of branching singularity) For the complex-valued nonlinear heat equation (1.2) under the periodic boundary condition with the specific initial data $u_0(x) = 50(1 - \cos(2\pi x))$, there exists a branching singularity inside the region $\{z \in \mathbb{C} : \text{Re}(z) \leq 0.0145, \ 0 \leq \text{Im}(z) \leq 0.00725\sqrt{3} \} \setminus [0, 0.0116]$.

This theorem partially answers the first conjecture, that is it proves that the analytic function defined by (1.2) has at least one branching point. If we assume the second conjecture, then we would be able to conclude that there is a real blow-up point $z_B$ in the interval $(0.0116, 0.0145)$. However even then it is not clear that the set of branching singularities is a single point, and not a branch cut along an interval, cf. [24].

Our second result agrees with Masuda’s work for the case of periodic boundary conditions without assumption of closeness to a constant for the initial data.

**Theorem 1.2** (Global existence) For $\theta = \pi/3, \pi/4, \pi/6$ and $\pi/12$, setting a straight path $\Gamma_\theta : z = te^{i\theta}$ ($t \geq 0$) in the complex plane of time, the solution of the complex-valued nonlinear heat equation (1.2) under the periodic boundary condition with the specific initial data $u_0(x) = 50(1 - \cos(2\pi x))$ exists globally in $t$ and converges to the zero function as $t \to \infty$. 

---

**Fig. 1** a The shaded domain in which global existence of the solution is proved by Masuda [14, 15]. Here, $0 < \theta < \pi/2$ and $z_B$ denotes the blow-up time. b The shaded domain and its mirror-image about the real axis: intersection of two domains is plotted in dark gray.
The proofs of these results are obtained in Sect. 7 by using rigorous numerics, via a careful blend of functional analysis, semi-group theory, numerical analysis, fixed point theory, the Lyapunov–Perron method and interval arithmetic. It is worth mentioning that the study of finite time blow-up in ODEs [6, 16, 26] and global existence of solutions in PDEs [2, 18, 29] are beginning to be studied with the tools of rigorous numerics. Furthermore, several approaches of rigorous integration for evolution PDEs have been introduced in, e.g., [5, 10, 17, 30, 31]. We expect that these rigorous integrators will be applied to understand dynamics of various evolution PDEs soon. In recent works, we have applied the rigorous integrator developed in this paper to study the dynamics of (1.2) in further detail, proving the existence of heteroclinic orbits and other global solutions [8, 9].

It is also important to note that the techniques of rigorous numerics presented in this paper are by no means applicable only to nonlinear heat equations with the quadratic nonlinear term. These techniques can be easily applied to more general class of nonlinearities, such as $u^p$ when $p \geq 2$ is an integer. Moreover, we believe that our approach should be extended in principle to more general PDE models involving $e^u$ or even a derivative type nonlinear term $\partial_x (u^p)$.

The present paper is organized as follows. In Sect. 2 we set up the fixed point operator to solve the Cauchy problem (1.2) on a straight path in the complex plane. Using the Fourier series, we derive an infinite-dimensional system of differential equations and define a formulation of the fixed point operator to rigorously provide an enclosure of the solution on a short time interval. Section 3 is devoted to defining a solution map operator, which provides a solution of the linearized Cauchy problem with arbitrary forcing term. Such a solution map operator is defined by what is called the evolution operator. We show how we verify the existence of such an evolution operator by using rigorous numerics and give a computable estimate for a uniform bound of the evolution operator. Using the solution map operator, we derive in Sect. 4 a sufficient condition to have a fixed point for the fixed point operator. Theorem 4.1 provides rigorous enclosure of the solution of the Cauchy problem. In Sect. 5, we introduce a method for applying iteratively the local inclusion approach to compute solutions over long time intervals. This technique is then used to prove Theorem 1.1. In Sect. 6, we introduce an approach based on the Lyapunov–Perron method in order to calculate part of a center-stable manifold, which is used to prove that an explicit open set of solutions of the given Cauchy problem converge to zero, hence yielding the global existence in time of the solutions. This proves our second main result, namely Theorem 1.2. The numerical results are presented in details in Sect. 7.

2 Setting-up the fixed point operator to solve the Cauchy problem

In this section, we derive the formulation of the fixed point operator which is used to provide a rigorous enclosure for the solution of the Cauchy problem on a short time interval. Taking the straight path $\Gamma_\theta = \{ z \in \mathbb{C} : z = te^{i\theta}, \ t \geq 0 \}$, the complex-
valued nonlinear heat equation \((1.2)\) is transformed into the equation
\[
  u_t = e^{i\theta} (u_{xx} + u^2), \quad x \in (0, 1), \quad t > 0
\] (2.1)
with periodic boundary conditions. Here, we set \(\theta \in (-\pi/2, \pi/2)\) and the initial data \(u_0(x) = 50(1 - \cos(2\pi x))\). We expand the unknown function into the Fourier series:
\[
  u(t, x) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ik\omega x},
\] (2.2)
where we have set \(\omega = 2\pi\). Plugging (2.2) in the initial-boundary value problem (2.1), we have the following infinite-dimensional system of ODEs:
\[
  \frac{d}{dt} a_k(t) = e^{i\theta} \left[ -k^2 \omega^2 a_k(t) + (a(t) \ast a(t))_k \right] \quad (k \in \mathbb{Z}), \quad a(0) = \varphi,
\] (2.3)
where \(\ast\) denotes the discrete convolution product defined by
\[
  (b \ast c)_k \overset{\text{def}}{=} \sum_{m \in \mathbb{Z}} b_{k-m} c_m \quad (k \in \mathbb{Z})
\]
for bi-infinite sequences \(b = (b_k)_{k \in \mathbb{Z}}\) and \(c = (c_k)_{k \in \mathbb{Z}}\), and \(\varphi\) is defined by
\[
  \varphi_k = \begin{cases} 
    -25, & k = \pm 1 \\
    50, & k = 0 \\
    0, & \text{otherwise}
  \end{cases}
\] (2.4)
For a fixed time \(h > 0\), let us define \(J \overset{\text{def}}{=} [0, h]\) and the Banach space
\[
  X \overset{\text{def}}{=} C(J; \ell^1), \quad \|a\|_X \overset{\text{def}}{=} \sup_{t \in J} \|a(t)\|_{\ell^1},
\] (2.5)
where \(\ell^1 \overset{\text{def}}{=} \{a = (a_k)_{k \in \mathbb{Z}} : \sum_{k \in \mathbb{Z}} |a_k| < \infty, \ a_k \in \mathbb{C}\}\) with norm \(\|a\|_{\ell^1} \overset{\text{def}}{=} \sum_{k \in \mathbb{Z}} |a_k|\).
An important and useful feature of \(\ell^1\) is that it is a Banach algebra under discrete convolution, namely
\[
  \|a \ast b\|_{\ell^1} \leq \|a\|_{\ell^1} \|b\|_{\ell^1}, \quad \text{for all} \ a, b \in \ell^1.
\] (2.6)
To determine the Fourier coefficients \((a_k(t))_{k \in \mathbb{Z}}\) of the solution of the Cauchy problem (2.3), we use a numerically computed approximate solution with the maximal wave number \(N\), say \((\bar{a}_k(t))_{|k| \leq N}\). Let
\[
  \bar{a}(t) \overset{\text{def}}{=} (\ldots, 0, 0, \bar{a}_{-N}(t), \ldots, \bar{a}_N(t), 0, 0, \ldots)
\]
be an approximation of $a(t)$. We will rigorously include the Fourier coefficients in the neighborhood of numerical solution defined by

$$B_J(\bar{a}, \varrho) \overset{\text{def}}{=} \{ a \in X : \| a - \bar{a} \|_X \leq \varrho, \ a(0) = \varphi \}.$$  \hfill (2.7)

Define the Laplacian operator $L$ acting on a sequence of Fourier coefficients as

$$Lb \overset{\text{def}}{=} \left( -k^2\omega^2 b_k \right)_{k \in \mathbb{Z}}, \ b = (b_k)_{k \in \mathbb{Z}}.$$ \hfill (2.8)

The domain of the operator $L$ is denoted by $D(L) \overset{\text{def}}{=} \{ a = (a_k)_{k \in \mathbb{Z}} : \sum_{k \in \mathbb{Z}} k^2 |a_k| < \infty \} \subset \ell^1$. We define an operator acting on $a \in D \overset{\text{def}}{=} C((0, h]; D(L)) \cap C^1((0, h]; \ell^1)$ by

$$(F(a))(t) \overset{\text{def}}{=} \frac{d}{dt} a(t) - e^{i\theta} \left( La(t) + a(t) * a(t) \right).$$ \hfill (2.9)

Then we consider the Cauchy problem (2.3) as the zero-finding problem $F(a) = 0$ with the initial condition $a(0) = \varphi$. It is obvious that $a$ solves (2.3) if $F(a) = 0$ with $a(0) = \varphi$ holds. Let us define the following operator:

$$(T(a))(t) \overset{\text{def}}{=} \mathcal{A}_\varphi \left[ e^{i\theta} \left( a(t) * a(t) - 2\bar{a}(t) * a(t) \right) \right], \ T : X \to X,$$ \hfill (2.10)

where $\mathcal{A}_\varphi : X \to D \cap X \subset X$ is a solution map operator corresponding to the linearized problem of (2.3) around the approximate solution $\bar{a}$, which will be explicitly defined in the next section. As shown in Remark 3.1, this operator is alternative form of the simplified Newton operator. We expect that the simplified Newton operator (2.10) has a fixed point $\tilde{a} \in B_J(\bar{a}, \varrho)$ such that $\tilde{a} = T(\tilde{a})$.

### 3 The solution map operator

The simplified Newton operator (2.10) is characterized by the solution map operator. We define the solution map operator $\mathcal{A}_\varphi$ using an evolution operator $U(t, s)$ defined by $U(t, s)\varphi \overset{\text{def}}{=} b(t)$ for $0 \leq s \leq t \leq h$, where $b$ is the solution of the following homogeneous initial value problem (IVP):

$$\frac{d}{dt} b_k(t) + e^{i\theta} \left[ k^2\omega^2 b_k(t) - 2 \bar{a}(t) * b(t) \right] = 0 \quad (k \in \mathbb{Z})$$ \hfill (3.1)

for any initial sequence $b(s) = \varphi \in \ell^1$ $(0 \leq s \leq t)$. Here, $s$ is also a parameter since the evolution operator $U(t, s)$ is a two parameter family of bounded linear operators. We also note that (3.1) is equivalent to $DF[\bar{a}]b = 0$, where $DF[\bar{a}]$ is the Fréchet derivative of $F : D \to C((0, h]; \ell^1)$ at $\bar{a}$. Consider (3.1) with an arbitrary forcing
term $g(t) = (g_k(t))_{k \in \mathbb{Z}}$:

$$
\frac{d}{dt} b_k(t) + e^{i\theta} \left[ k^2 \omega^2 b_k(t) - 2 \left( \tilde{a}(t) * b(t) \right)_k \right] = g_k(t) \quad (k \in \mathbb{Z}).
$$

(3.2)

Setting the initial sequence $b(0) = \phi$ of (3.2), this leads to the definition of the solution map operator $\mathcal{A}_\phi : X \rightarrow X$ given by

$$(\mathcal{A}_\phi g)(t) \overset{def}{=} U(t, 0)\phi + \int_0^t U(t, s)g(s)ds. \quad (3.3)$$

In particular, $\mathcal{A}_\phi 0 = U(t, 0)\phi$.

**Remark 3.1** Suppose $\tilde{a} \in \mathcal{D}$, we take $\mathcal{A}^\dagger$ as the Fréchet derivative of $F : \mathcal{D} \rightarrow C((0, h]; \ell^1)$ at $\tilde{a}$, say $\mathcal{A}^\dagger = DF[\tilde{a}]$ defined by

$$
DF[\tilde{a}]b \overset{def}{=} \frac{d}{dt}b - e^{i\theta} (Lb + 2\tilde{a} * b).
$$

This is a linearized problem of (2.3). Setting the initial data (e.g., $b(0) = \phi$), we will validate that the problem has a unique solution that is continuous on $J$ satisfying $\mathcal{A}^\dagger b = 0$. If it succeeds, then we denote by $\mathcal{A}_\phi : X \rightarrow X$ the solution map operator that represents such a correspondence $b = \mathcal{A}_\phi 0$. This fact implies that the operator $\mathcal{A}_\phi$ satisfies $\mathcal{A}_\phi \mathcal{A}^\dagger a = a$ if $a \in \mathcal{D} \cap X$ with the initial data $a(0) = \varphi$. Furthermore, from the definition (3.3), $\mathcal{A}_{\varphi_1} g_1 + \mathcal{A}_{\varphi_2} g_2 = \mathcal{A}_{\varphi_1 + \varphi_2} (g_1 + g_2)$ holds for $\varphi_1, \varphi_2 \in \ell^1, g_1, g_2 \in X$. Then, the simplified Newton operator defined in (2.10) satisfies

$$
T(a) = \mathcal{A}_\varphi \left[ e^{i\theta} (a * a - 2\tilde{a} * a) \right]
$$

$$
= \mathcal{A}_\varphi \left[ \frac{d}{dt} a - e^{i\theta} (La + 2\tilde{a} * a) - \frac{d}{dt} a + e^{i\theta} (La + a * a) \right]
$$

$$
= \mathcal{A}_\varphi \left( \mathcal{A}^\dagger a - F(a) \right)
$$

$$
= \mathcal{A}_\varphi \mathcal{A}^\dagger a - \mathcal{A}_0 F(a)
$$

$$
a - \mathcal{A}_0 F(a)
$$

for $a \in \mathcal{D} \cap X$ with $a(0) = \varphi$. We note that this form $a - \mathcal{A}_0 F(a)$ is defined only on $\mathcal{D} \cap X$ but the simplified Newton operator (2.10) can be defined on $X$. Moreover, from the property of the operator $\mathcal{A}_\varphi : X \rightarrow \mathcal{D} \cap X \subset X$, which will be shown in Remark 3.6, the fixed point satisfies $\tilde{a} \in \mathcal{D} \cap X$ if such a fixed point is obtained.

The remaining tasks of this section consist of constructing the solution map operator (3.3). This requires showing the existence of the evolution operator $\{U(t, s)\}_{0 \leq s \leq t \leq h}$ obtaining a *computable* constant $W_h > 0$ satisfying

$$
\sup_{0 \leq s \leq t \leq h} \|U(t, s)\phi\|_{\ell^1} \leq W_h \|\phi\|_{\ell^1}, \quad \forall \phi \in \ell^1.
$$
More precisely, we will derive a hypothesis of existence of the evolution operator in Theorem 3.3, which can be verified via rigorous numerics. If such a hypothesis holds, the explicit value of $W_h > 0$ is also obtained. The constant $W_h > 0$ provides a uniform bound for the bounded linear operator norm of $U(\cdot, \cdot)$ over the simplex $S_h \overset{\text{def}}{=} \{(t, s) : 0 \leq s \leq t \leq h\}$, that is
\[
\|U(t, s)\|_{B(\ell^1)} \leq W_h, \quad \forall (t, s) \in S_h. \tag{3.4}
\]
Here, we use the notation $B(\ell^1)$ to denote the set of all bounded linear operators on $\ell^1$ with corresponding bounded operator norm $\|\cdot\|_{B(\ell^1)}$.

**Remark 3.2** It is important to notice that for a given $\bar{a}(t) \in \ell^1$ and $h$ small enough, there exists the solution of the linearized problem (3.1) (e.g. see [19]). This in turns yields the existence of the evolution operator $U(t, s)$ for all $(t, s) \in S_h$ for a small enough step size $h > 0$. However, once a fixed $h > 0$ has been chosen, it is in general difficult to show the existence of the evolution operator $U(t, s)$ for $(t, s) \in S_h$. In the next section, we derive a hypothesis of existence of the evolution operator. We also give an explicit and computable formula for the constant $W_h$.

### 3.1 Deriving a formulation for $W_h$ and existence of the evolution operator

We begin this section with some notation. Given a finite dimensional (Fourier) projection number $m \in \mathbb{N}$ and a vector $\phi = (\phi_k)_{k \in \mathbb{Z}} \in \ell^1$, define the projection $\pi^{(m)} : \ell^1 \rightarrow \ell^1$ as follows:
\[
\big(\pi^{(m)} \phi\big)_k = \begin{cases} 
\phi_k, & |k| \leq m \\
0, & |k| > m,
\end{cases}
\]
for $k \in \mathbb{Z}$. Given $\phi \in \ell^1$, denote $\phi^{(m)} \overset{\text{def}}{=} \pi^{(m)} \phi \in \ell^1$ and $\phi^{(\infty)} \overset{\text{def}}{=} (\text{Id} - \pi^{(m)}) \phi \in \ell^1$. Thus, $\phi$ is represented by $\phi = \phi^{(m)} + \phi^{(\infty)}$. For example, with this notation, the discrete convolution in (3.1) can be expanded as
\[
(\bar{a}(t) * b(t))_k = \left(\bar{a}(t) * b^{(m)}(t)\right)_k + \left(\bar{a}(t) * b^{(\infty)}(t)\right)_k
= \sum_{|k_1| \leq N, \ k_2 \leq m} \bar{a}_{k_1}(t)b_{k_2}(t) + \sum_{|k_1| \leq N, \ k_2 > m} \bar{a}_{k_1}(t)b_{k_2}(t) \quad (k \in \mathbb{Z}).
\]

As previously mentioned, given a step size $h > 0$, our goal in this section is to show the existence of the evolution operator $U(t, s)$ of (3.1) by computing a constant $W_h$ satisfying (3.4). However, from a computational point of view, the formulation of the homogeneous non-autonomous linear IVP (3.1) is daunting due to the fact that for each fixed $k \in \mathbb{Z}$ the convolution term $(\bar{a}(t) * b(t))_k$ involves all Fourier modes $(b_j(t))_{j \in \mathbb{Z}}$. In order to address this issue, we separate the equations by considering
yet another homogeneous IVP with respect to the sequence $c(t) = (c_k(t))_{k \in \mathbb{Z}}$,

$$
\frac{d}{dt} c_k(t) + \bar{e}^{i\theta} \left[ k^2 \omega^2 c_k(t) - 2 \left( \bar{\alpha} (t) * c^{(m)}(t) \right)_k \right] = 0 \quad (|k| \leq m)
$$

(3.5)

$$
\frac{d}{dt} c_k(t) + \bar{e}^{i\theta} \left[ k^2 \omega^2 c_k(t) - 2 \left( \bar{\alpha} (t) * c^{(\infty)}(t) \right)_k \right] = 0 \quad (|k| > m).
$$

(3.6)

This new decoupled formulation, while not being equivalent to (3.1), will be used to control the evolution operator associated to (3.1). Setting the initial sequence $c(s) = \phi (0 \leq s \leq t)$, let $(c_k(t))_{|k| \leq m}$ and $(c_k(t))_{|k| > m}$ be the solution of the $(2m + 1)$-dimensional equation (3.5) and the infinite dimensional equation (3.6), respectively. We define the solution operator of the IVP of (3.5) by $C^{(m)}(t, s)(c_k)_{|k| \leq m} \overset{\text{def}}{=} (c_k(t))_{|k| \leq m}$ for $0 \leq s \leq t \leq h$. Similarly, the solution operator corresponding to (3.6) is defined by $C^{(\infty)}(t, s)(c_k)_{|k| > m} \overset{\text{def}}{=} (c_k(t))_{|k| > m}$ for $0 \leq s \leq t \leq h$. We call $C^{(m)}(t, s)$ and $C^{(\infty)}(t, s)$ the evolution operators of (3.5) and (3.6), respectively. At this point, the existence of the operator $C^{(\infty)}(t, s)$ is not guaranteed, but will be verified in Sect. 3.3. We extend the action of the operator $C^{(m)}(t, s)$ (resp. $C^{(\infty)}(t, s)$) on $\ell^1$ by introducing the operator $\bar{U}^{(m)}(t, s)$ (resp. $\bar{U}^{(\infty)}(t, s)$) as follows. Given $\phi \in \ell^1$, define $\bar{U}^{(m)}(t, s) : \ell^1 \to \ell^1$ and $\bar{U}^{(\infty)}(t, s) : \ell^1 \to \ell^1$ by

$$
\left( \bar{U}^{(m)}(t, s) \phi \right)_k = \begin{cases} 
(C^{(m)}(t, s)(c_k)_{|k| \leq m})_k, & |k| \leq m \\
0, & |k| > m
\end{cases}
$$

(3.7)

$$
\left( \bar{U}^{(\infty)}(t, s) \phi \right)_k = \begin{cases} 
(C^{(\infty)}(t, s)(c_k)_{|k| > m})_k, & |k| \leq m \\
0, & |k| > m
\end{cases}
$$

(3.8)

The proof of existence of the evolution operator $U(t, s)$ of the original linearized problem (3.1) and the explicit bound $W_h$ satisfying (3.4) is presented in the following theorem.

**Theorem 3.3** Let $m \in \mathbb{N}$. Let $s, t \in J$ satisfying $0 \leq s \leq t \leq h$ and let $\bar{\alpha}$ be fixed. Assume that there exists a constant $W_m > 0$ such that

$$
\sup_{0 \leq s \leq t \leq h} \left\| \bar{U}^{(m)}(t, s) \right\|_{B(\ell^1)} \leq W_m.
$$

(3.9)

Assume that $C^{(\infty)}(t, s)$ exists and that $\bar{U}^{(\infty)}(t, s)$ defined in (3.8) satisfies

$$
\left\| \bar{U}^{(\infty)}(t, s) \right\|_{B(\ell^1)} \leq W^{(\infty)}(t, s) \overset{\text{def}}{=} e^{\mu_{m+1}(t-s) + 2 \int_s^t \|\bar{\alpha}(\tau)\|_1 d\tau},
$$

(3.10)
where $\mu_{m+1} \overset{\text{def}}{=} (m + 1)^2 \omega^2 \cos \theta$. Define the constants $W_\infty \geq 0$, $\bar{W}_\infty \geq 0$, $W_{\sup}^\infty > 0$ by

\begin{align*}
W_\infty & \overset{\text{def}}{=} e^{(2\|\bar{a}\|_X - \mu_{m+1})h} - 1, \\
\bar{W}_\infty & \overset{\text{def}}{=} \frac{W_\infty - h}{2\|\bar{a}\|_X - \mu_{m+1}}, \\
W_{\sup}^\infty & \overset{\text{def}}{=} \begin{cases} 
1, & \mu_{m+1} \geq 2\|\bar{a}\|_X \\
 e^{(2\|\bar{a}\|_X - \mu_{m+1})h}, & \mu_{m+1} < 2\|\bar{a}\|_X
\end{cases}
\end{align*}

(3.11) (3.12) (3.13)

respectively. Define $\bar{a}^{(s)}(t) \in \ell^1$ component-wise by

\[ \bar{a}_k^{(s)}(t) = \begin{cases} 
0, & k = 0 \\
\bar{a}_k(t), & |k| \leq N \text{ and } k \neq 0 \\
0, & |k| > N
\end{cases} \]

If

\[ \kappa \overset{\text{def}}{=} 1 - 4W_m\bar{W}_\infty\|\bar{a}^{(s)}\|_X^2 > 0, \]

(3.14)

then the evolution operator $U(t, s)$ exists and the following estimate holds

\[ \sup_{0 \leq s \leq t \leq h} \|U(t, s)\phi\|_{\ell^1} \leq W_h \|\phi\|_{\ell^1}, \quad \forall \phi \in \ell^1, \]

where

\[ W_h \overset{\text{def}}{=} \left\| \begin{bmatrix} W_m \kappa^{-1} & 2W_mW_\infty\|\bar{a}^{(s)}\|_X \kappa^{-1} \\
2W_mW_\infty\|\bar{a}^{(s)}\|_X \kappa^{-1} & W_{\sup}^\infty + 4W_mW_{\sup}^\infty\|\bar{a}^{(s)}\|_X^2 \kappa^{-1} \end{bmatrix} \right\|_1. \]

(3.15)

Remark 3.4 Equations (3.11) and (3.12) above implicitly assume that the denominator is non-zero. However, if the denominator is zero, these are $W_\infty = \bar{W}_\infty = h^2/2$. It easily follows from the proof of Lemma 3.5.

After the proof of Theorem 3.3, we introduce in Sect. 3.2 a rigorous computational method based on Chebyshev series to obtain the finite dimensional evolution operator $C^{(m)}(t, s)$, which will directly yield $U^{(m)}(t, s)$. We show how this helps computing $W_m > 0$ satisfying (3.9). Section 3.3 is devoted to show the existence of $C^{(\infty)}(t, s)$ (and hence the existence of $U^{(\infty)}(t, s)$) and to show that the hypothesis (3.10) holds. The proof of Theorem 3.3 uses the following elementary result.

Lemma 3.5 Consider the constants $W_\infty \geq 0$, $\bar{W}_\infty \geq 0$ and $W_{\sup}^\infty > 0$ as defined in (3.11), (3.12) and (3.13), respectively. Then $W^{(\infty)}$, defined in (3.10), obeys the following inequalities:

\[ \sup_{0 \leq s \leq t \leq h} W^{(\infty)}(t, s) \leq W_{\sup}^\infty \]

(3.16)
\[
\sup_{0 \leq s \leq t \leq h} \int_s^t W(\infty, \tau, s) d\tau, \quad \sup_{0 \leq s \leq t \leq h} \int_s^t W(\infty, t, \tau) d\tau \leq W_\infty \\
\sup_{0 \leq s \leq t \leq h} \int_s^t \int_s^\tau W(\infty, \tau, \sigma) d\sigma d\tau \leq \tilde{W}_\infty.
\]

**Proof** First, note that from (3.10)

\[
\sup_{0 \leq s \leq t \leq h} W(\infty, t, s) = \sup_{0 \leq s \leq t \leq h} e^{-\mu_{m+1}(t-s)+2 \int_s^t \|\bar{a}(\tau)\|_{L_1} d\tau}
\]

\[
\leq e^{-(\mu_{m+1}-2\|\bar{a}\|_{X}) (t-s)}
\]

\[
\leq \begin{cases}
1, & \mu_{m+1} \geq 2\|\bar{a}\|_{X} \\
\left(1 + \frac{1}{2\|\bar{a}\|_{X} - \mu_{m+1}}\right)^{h} \leq 2\|\bar{a}\|_{X} \bar{a}; & \mu_{m+1} < 2\|\bar{a}\|_{X}.
\end{cases}
\]

\[
= W_\infty^\text{sup}.
\]

Second, note that

\[
\sup_{0 \leq s \leq t \leq h} \int_s^t W(\infty, \tau, s) d\tau = \sup_{0 \leq s \leq t \leq h} \left(\int_s^t e^{-\mu_{m+1}(\tau-s)+2 \int_s^\tau \|\bar{a}(\sigma)\|_{L_1} d\sigma} d\tau\right)
\]

\[
\leq \sup_{0 \leq s \leq t \leq h} \left(\int_s^t e^{(2\|\bar{a}\|_{X} - \mu_{m+1})(\tau-s)} d\tau\right)
\]

\[
= \sup_{0 \leq s \leq t \leq h} \left(\frac{e^{(2\|\bar{a}\|_{X} - \mu_{m+1})(s-t)} - 1}{2\|\bar{a}\|_{X} - \mu_{m+1}}\right)
\]

\[
\leq \frac{e^{(2\|\bar{a}\|_{X} - \mu_{m+1})h} - 1}{2\|\bar{a}\|_{X} - \mu_{m+1}} = W_\infty
\]

and that

\[
\sup_{0 \leq s \leq t \leq h} \int_s^t W(\infty, t, \tau) d\tau = \sup_{0 \leq s \leq t \leq h} \left(\int_s^t e^{-\mu_{m+1}(t-\tau)+2 \int_s^\tau \|\bar{a}(\sigma)\|_{L_1} d\sigma} d\tau\right)
\]

\[
= \sup_{0 \leq s \leq t \leq h} \left(\int_s^t e^{(2\|\bar{a}\|_{X} - \mu_{m+1})(t-\tau)} d\tau\right)
\]

\[
= \sup_{0 \leq s \leq t \leq h} \left(\frac{e^{(2\|\bar{a}\|_{X} - \mu_{m+1})(t-s)} - 1}{2\|\bar{a}\|_{X} - \mu_{m+1}}\right)
\]

\[
\leq \frac{e^{(2\|\bar{a}\|_{X} - \mu_{m+1})h} - 1}{2\|\bar{a}\|_{X} - \mu_{m+1}} = W_\infty.
\]

Third,

\[
\int_s^t \int_s^\tau W(\infty, \tau, \sigma) d\sigma d\tau = \int_s^t \frac{e^{(2\|\bar{a}\|_{X} - \mu_{m+1})(\tau-s)} - 1}{2\|\bar{a}\|_{X} - \mu_{m+1}} d\tau
\]
and hence, it follows that

$$
\sup_{0 \leq s \leq t \leq h} \int_{s}^{t} W^{(\infty)}(\tau, \sigma) d\sigma d\tau \leq \left( \frac{1}{2\|\tilde{a}\| X - \mu_{m+1}} \right) \left( e^{(2\|\tilde{a}\| X - \mu_{m+1})h - 1} - h \right)
$$

$$
= \tilde{W}_{\infty}.
$$

**Proof of Theorem 3.3** First note that for $|k| \leq m$, the system of differential equations (3.1) is described by the non-homogeneous equation

$$
d/dt b_k(t) + e^{i\theta} \left[ k^2 \omega^2 b_k(t) - 2 \left( \tilde{a}(t) * b^{(m)}(t) \right)_k \right] = 2e^{i\theta} \left( \tilde{a}(t) * b^{(\infty)} \right)_k \quad (|k| \leq m)
$$

(3.19)

with the initial condition $b_k(s) = \phi_k$ for $|k| \leq m$. Consider the homogeneous equation (3.5) and denote by $C^{(m)}(t, s)$ the solution of the variational problem. We show in Sect. 3.2 how to rigorously compute $C^{(m)}(t, s)$. As before, let $\tilde{U}^{(m)}(t, s)$ be the extension of the action of the operator $C^{(m)}(t, s)$ on $\ell^1$. Using the evolution operator $\tilde{U}^{(m)}(t, s) : \ell^1 \to \ell^1$, we can integrate the system (3.19) to obtain the integral equation

$$
b^{(m)}(t) = \tilde{U}^{(m)}(t, s)\phi^{(m)} + 2e^{i\theta} \int_{s}^{t} \tilde{U}^{(m)}(t, \tau) X^{(m)} \left( \tilde{a}(\tau) * b^{(\infty)}(\tau) \right) d\tau.
$$

(3.20)

Here, for $|k| \leq m$,

$$
\left( \tilde{a} * b^{(\infty)} \right)_k = \sum_{k_1+k_2=k} \tilde{a}_{k_1} b_{k_2} = \sum_{k_1 \neq 0} \tilde{a}_{k_1} b_{k-k_1}
$$

$$
= \sum_{|k_1| \leq N, |k-k_1| > m} \tilde{a}_{k_1} p_{k-k_1},
$$

where the last equality holds because the index $k_1$ is never equal to zero. Combining (3.9) and (3.20), and using the property (2.6), it follows that

$$
\left\| b^{(m)}(t) \right\|_{\ell^1} \leq W_m \left\| \phi^{(m)} \right\|_{\ell^1} + 2W_m \int_{s}^{t} \left\| \tilde{a}^{(s)}(\tau) \right\|_{\ell^1} \left\| b^{(\infty)}(\tau) \right\|_{\ell^1} d\tau.
$$

(3.21)
Next, for the case of $|k| > m$, we rewrite the system of differential equations (3.1) as

$$\frac{d}{dt} b_k(t) + e^{i\theta} \left[ k^2 \omega^2 b_k(t) - 2 \left( \tilde{a}(t) \ast b^{(\infty)} \right)_k \right] = 2e^{i\theta} \left( \tilde{a}(t) \ast b^{(m)} \right)_k \quad (|k| > m)$$

(3.22)

with the initial condition $b_k(s) = \phi_k$ for $|k| > m$. Note that by assumption the evolution operator $C^{(\infty)}(t, s)$ associated to (3.6) exists. Define $\tilde{U}^{(\infty)}(t, s)$ as in (3.8). Using that operator, the system (3.22) can be re-written as

$$b^{(\infty)}(t) = \tilde{U}^{(\infty)}(t, s) \phi^{(\infty)} + 2e^{i\theta} \int_s^t \tilde{U}^{(\infty)}(t, \tau)(\text{Id} - \pi^{(m)}) \left( \tilde{a}(\tau) \ast b^{(m)}(\tau) \right) d\tau.$$  

(3.23)

For $|k| > m$, the following holds

$$\left( \tilde{a} \ast b^{(m)} \right)_k = \sum_{k_1 + k_2 = k \atop |k_1| \leq N, |k_2| \leq m} \tilde{a}_{k_1} b_{k_2} = \sum_{k_1 \neq 0 \atop |k_1| \leq N, |k-k_1| \leq m} \tilde{a}_{k_1} b_{k-k_1} = \sum_{|k_1| \leq N, |k-k_1| \leq m} \tilde{a}_{k_1}^{(s)} b_{k-k_1}.$$  

Combining (3.10), (3.23) and using the property (2.6), it follows that

$$\left\| b^{(\infty)}(t) \right\|_{\ell^1} \leq W^{(\infty)}(t, s) \left\| \phi^{(\infty)} \right\|_{\ell^1} + 2 \int_s^t \left\| \tilde{U}^{(\infty)}(t, \tau) \right\|_{\ell^1} \left\| \tilde{a}^{(s)}(\tau) \right\|_{\ell^1} \left\| b^{(m)}(\tau) \right\|_{\ell^1} d\tau.$$ 

(3.24)

Plugging (3.24) into (3.21), and using the inequalities (3.17) and (3.18) from Lemma 3.5, we have

$$\left\| b^{(m)}(t) \right\|_{\ell^1} \leq W_m \left\| \phi^{(m)} \right\|_{\ell^1} + 2W_m \int_s^t \left\| \tilde{a}^{(s)}(\tau) \right\|_{\ell^1} \left\{ W^{(\infty)}(\tau, s) \left\| \phi^{(\infty)} \right\|_{\ell^1} + 2 \int_s^\tau \left\| W^{(\infty)}(\tau, \sigma) \left\| \tilde{a}^{(s)}(\sigma) \right\|_{\ell^1} \left\| b^{(m)}(\sigma) \right\|_{\ell^1} d\sigma \right\} d\tau \right. 

= W_m \left\| \phi^{(m)} \right\|_{\ell^1} + \left( 2W_m \int_s^t \left\| \tilde{a}^{(s)}(\tau) \right\|_{\ell^1} \left\{ W^{(\infty)}(\tau, s) \left\| \phi^{(\infty)} \right\|_{\ell^1} + 4W_m \int_s^\tau \left\| \tilde{a}^{(s)}(\sigma) \right\|_{\ell^1} \left\| W^{(\infty)}(\tau, \sigma) \right\|_{\ell^1} \left\| b^{(m)}(\sigma) \right\|_{\ell^1} d\sigma \right\} d\tau \right. 

\leq W_m \left\| \phi^{(m)} \right\|_{\ell^1} + 2W_m W^{(\infty)} \left\| \tilde{a}^{(s)} \right\|_X \left\| \phi^{(\infty)} \right\|_{\ell^1} + 4W_m W^{(\infty)} \left\| \tilde{a}^{(s)} \right\|_X^2 \left\| b^{(m)} \right\|_X.$$ 

(3.25)

By assumption (3.14), $\kappa = 1 - 4W_m W^{(\infty)} \left\| \tilde{a}^{(s)} \right\|_X^2 > 0$ and using (3.25) yields that

$$\left\| b^{(m)} \right\|_X \leq \frac{W_m \left\| \phi^{(m)} \right\|_{\ell^1} + 2W_m W^{(\infty)} \left\| \tilde{a}^{(s)} \right\|_X \left\| \phi^{(\infty)} \right\|_{\ell^1}}{\kappa}.$$ 

(3.26)
Note that if the hypothesis \((3.14)\) holds, it guarantees the existence of the solution of the finite part of \((3.1)\). Using the inequalities \((3.16)\) and \((3.17)\) in Lemma 3.5 and \((3.26)\), the tail mode \((3.24)\) is bounded by

\[
\|b^{(\infty)}\|_X \leq W^\sup \|\phi^{(\infty)}\|_\ell^1 + 2W^\infty \|\tilde{a}^{(s)}\|_X \|b^{(m)}\|_X \\
\leq 2W^m W^\infty \|\tilde{a}^{(s)}\|_X \kappa^{-1} \|\phi^{(m)}\|_\ell^1 \\
+ \left(W^\sup + 4W^m W^2_\infty \|\tilde{a}^{(s)}\|_X \kappa^{-1}\right) \|\phi^{(\infty)}\|_\ell^1.
\]  

(3.27)

Finally, \((3.26)\) and \((3.27)\) yield

\[
\|b\|_X \leq \|b^{(m)}\|_X + \|b^{(\infty)}\|_X \\
= \left[\|b^{(m)}\|_X \|b^{(\infty)}\|_X\right]_1 \\
\leq \left[\frac{W^m \kappa^{-1}}{2W^m W^\infty \|\tilde{a}^{(s)}\|_X \kappa^{-1} W^\sup + 4W^m W^2_\infty \|\tilde{a}^{(s)}\|_X \kappa^{-1}}\right] \|\phi^{(m)}\|_\ell^1 \\
\leq W_h \|\phi\|_\ell^1.
\]

\(\Box\)

**Remark 3.6** For arbitrary forcing term \(g \in X\) with the initial value \(\phi \in \ell^1\), \(b = \mathscr{A}_\phi g\) defined in \((3.3)\) solves \((3.2)\) in the sense of classical solution (see, e.g., [19]). This implies \(b \in D \cap X\). Then, the solution map operator \(\mathscr{A}\) has a smoothing property from \(X\) to \(D \cap X\).

Having derived a computable formulation in \((3.15)\) for the constant \(W_h > 0\) which provides a uniform bound for the norm of evolution operator \(U(\cdot, \cdot)\) over \(S_h\), we now turn to the question of computing the bound \(W_m\), which is required to defined \(W_h\).

### 3.2 Computation of the bound \(W_m\)

The goal of this section is to present a rigorous computational approach to obtain the constant \(W_m\) satisfying \((3.9)\), that is a uniform bound for the operator norm of \(\bar{U}^{(m)}\) over the simplex \(S_h\). From \((3.7)\), defining \(\bar{U}^{(m)}(t, s)\) boils down to the computation of \(C^{(m)}(t, s)\), that is the evolution operators of the \((2m + 1)\)-dimensional equation \((3.5)\), which can be written component-wise as

\[
\frac{d}{dt} c_{k,j}(t) = e^{i\theta} \left[-k^2 \omega^2 c_{k,j}(t) + 2 \left(\tilde{a}(t) \ast e_j^{(m)}(t)\right)_{k}\right],
\]

\[
c_{k,j}(s) = \delta_{k,j} (|k|, |j| \leq m),
\]

(3.28)

for \(0 \leq s \leq t \leq h\), and where \(\delta_{k,j}\) denotes the Kronecker’s delta. Denote by \(A(t)\) the matrix representation of the right-hand side of \((3.28)\) acting on the coefficients \((c_{k,j}(t, s))_{|k|, |j| \leq m}\).
The evolution matrix\( C^{(m)}(t, s) \) can be decomposed as
\[
C^{(m)}(t, s) = \Phi(t) \Psi(s) \overset{\text{def}}{=} \Phi(t) \Phi(s)^{-1},
\]
where \( \Phi(t) \in M_{2m+1}(\mathbb{C}) \) is the principal fundamental solution and solves
\[
\frac{d}{dt} \Phi(t) = A(t) \Phi(t), \quad \Phi(0) = Id, \quad t \in [0, h],
\]
and where \( \Psi(s) = \Phi(s)^{-1} \) is the solution of the adjoint problem
\[
\frac{d}{ds} \Psi(s) = -\Psi(s) A(s), \quad \Psi(0) = Id, \quad s \in [0, h].
\]

Having introduce the set-up, the computation of \( W_m \) is twofold. First, using the approach of [12], we use the tools of rigorous numerics and Chebyshev series expansion to compute the fundamental matrix solutions \( \Phi_1(t) \) and \( \Psi_1(t) \) satisfying (3.30) and (3.31), respectively. Second, using interval arithmetic and the Chebyshev series representations of \( \Phi(t) \) and \( \Psi(t) \), we obtain \( W_m \) such that
\[
\sup_{t \in [0,h]} \| \Phi(t) \|_1 \cdot \sup_{s \in [0,h]} \| \Psi(s) \|_1 \leq W_m,
\]
where \( \| \cdot \|_1 \) represents the matrix norm induced by the \( \ell^1 \) norm on \( \mathbb{C}^{2m+1} \). By construction, the constant \( W_m \) obtained computationally in (3.32) satisfies (3.9). The remainder of this section is devoted to the computational method for obtaining the matrix solutions \( \Phi(t) \) and \( \Psi(t) \).

### 3.2.1 Rigorously computing \( \Phi(t) \) and \( \Psi(t) \) via Chebyshev series

The Chebyshev polynomials \( T_k : [-1, 1] \to \mathbb{R} \) \((k \geq 0) \) are orthogonal polynomials defined by \( T_0 = 1, T_1(t) = t \) and \( T_{k+1}(t) = 2t T_k(t) - T_{k-1}(t) \) for \( k \geq 1 \). Every Lipschitz continuous function \( v : [-1, 1] \to \mathbb{R} \) has a unique representation as an absolutely and uniformly convergent series \( v(t) = \sum_{k=0}^{\infty} a_k T_k(t) \) (e.g. see [27]). The more regular the function \( v \) is, the faster the decay rate of its Chebyshev coefficients.

Fixing a Fourier projection number \( N \) and a Chebyshev projection number \( n \), let us assume that we have numerically computed an approximate solution of the initial-boundary value problem (2.1) in the form
\[
\bar{u}(t, x) = \sum_{|k| \leq N} \bar{a}_k(t)e^{ik\omega x}, \quad \bar{a}_k(t) = \bar{a}_{0,k} + 2 \sum_{\ell=1}^{n-1} \bar{a}_{\ell,k} T_{\ell}(t),
\]
with \( \omega = 2\pi \). In practice, we perform this task by considering a Galerkin approximation of (2.3) of size \( 2N + 1 \) and using MATLAB software system Chebfun (e.g. see [20]) to compute a solution of the IVP on the time interval \([0, h]\). Given this approximate solution \( \bar{u}(t) = (\bar{u}_k(t))_{k=-N}^N \), the variational problem (3.30) consists of solving
the homogeneous initial value problem (IVP) with respect to sequence \((c_k(t))_{k=−m}^{m}\) satisfying (after rescaling the time interval \([0, h]\) to \([-1, 1]\))

\[
\dot{c}_k = G_k(c) \overset{\text{def}}{=} - \frac{he^{i\theta}}{2} \left[ k^2 \omega^2 c_k - 2 \left( \bar{a} \ast c^{(m)} \right)_k \right], \quad |k| \leq m. \tag{3.34}
\]

Assume that the initial condition of (3.34) is given by \(c_k(−1) = b_k\). In practice, we will solve rigorously \(2m + 1\) IVPs with initial conditions \((b_k)_{k=−m}^{m} = e_j \in \mathbb{C}^{2m+1}\) the canonical basis vectors. Rewriting the system (3.34) as an integral equation results in

\[
c_k(t) = b_k + \int_{−1}^{t} G_k(c(s)) \, ds, \quad |k| \leq m, \quad t \in [-1, 1]. \tag{3.35}
\]

For each \(k\), we expand \(c_k(t)\) using a Chebyshev series, that is

\[
c_k(t) = c_{0,k} + 2 \sum_{\ell \geq 1} c_{\ell,k} T_\ell(t) = c_{0,k} + 2 \sum_{\ell \geq 1} c_{\ell,k} \cos(\ell \vartheta)
= \sum_{\ell \in \mathbb{Z}} c_{\ell,k} e^{i\ell \theta} = \sum_{\ell \in \mathbb{Z}} c_{\ell,k} T_\ell(t), \tag{3.36}
\]

where \(c_{-\ell,k} = c_{\ell,k}, t = \cos(\vartheta), \vartheta = \cos^{-1}(t)\) and \(T_{-\ell}(t) \overset{\text{def}}{=} T_\ell(t)\). For each \(|k| \leq m\), we expand \(G_k(c(t))\) using a Chebyshev series, that is

\[
G_k(c(t)) = \psi_{0,k}(c) + 2 \sum_{\ell \geq 1} \psi_{\ell,k}(c) \cos(\ell \vartheta) = \sum_{\ell \in \mathbb{Z}} \psi_{\ell,k}(c) e^{i\ell \theta} = \sum_{\ell \in \mathbb{Z}} \psi_{\ell,k}(c) T_\ell(t).
\]

where

\[
\psi_{\ell,k}(c) = \lambda_k c_{\ell,k} + N_{\ell,k}(c).
\]

Letting \(N_k(c) \overset{\text{def}}{=} (N_{\ell,k}(c))_{\ell \geq 0}, \psi_k(c) \overset{\text{def}}{=} (\psi_{\ell,k}(c))_{\ell \geq 0}\) and noting that \((\lambda_k c_k)_{\ell} = \lambda_k c_{\ell,k}\), we get that

\[
\psi_k(c) = \lambda_k c_k + N_k(c). \tag{3.38}
\]

**Remark 3.7** Note that for problem (3.34),

\[
\lambda_k = - \frac{he^{i\theta}}{2} \omega^2 k^2
\]

and that

\[
N_{\ell,k}(c) = he^{i\theta} \left( \bar{a} \ast c^{(m)} \right)_{\ell,k} = he^{i\theta} \sum_{\substack{\ell_1+\ell_2=\ell \kbar_1+k_2=k \kbar_2<\kbar_1, |k_2| \leq m}} \bar{a}_{\ell_1,k_1} c_{\ell_2,k_2}. \tag{3.39}
\]
Combining expansions (3.36) and (3.37) leads to

\[
\sum_{\ell \in \mathbb{Z}} c_{\ell,k} T_\ell(t) = c_k(t) = b_k + \int_{-1}^{t} G_k(c(s)) \, ds = b_k + \int_{-1}^{t} \sum_{\ell \in \mathbb{Z}} \psi_{\ell,k}(c) T_\ell(s) \, ds
\]

and this results (e.g. see in [12]) in solving \( f = 0 \), where \( f = (f_{\ell,k})_{\ell,k} \) is given component-wise by

\[
f_{\ell,k}(c) = \begin{cases} 
  c_{0,k} + 2 \sum_{j=1}^{\infty} (-1)^j c_{j,k} - b_k, & \ell = 0, |k| \leq m \\
  2\ell c_{\ell,k} + (\psi_{\ell+1,k}(c) - \psi_{\ell-1,k}(c)), & \ell > 0, |k| \leq m.
\end{cases}
\]

Hence, for \( \ell > 0 \) and \( |k| \leq m \), we aim at solving

\[
f_{\ell,k}(c) = 2\ell c_{\ell,k} + \lambda_k(c_{\ell+1,k} - c_{\ell-1,k}) + (N_{\ell+1,k}(c) - N_{\ell-1,k}(c)) = 0.
\]

Finally, the problem that we solve is \( f = 0 \), where \( f = (f_{\ell,k})_{\ell,k} \) is given component-wise by

\[
f_{\ell,k}(c) = \begin{cases} 
  c_{0,k} + 2 \sum_{j=1}^{\infty} (-1)^j c_{j,k} - b_k, & \ell = 0, |k| \leq m \\
  -\lambda_k c_{\ell-1,k} + 2\ell c_{\ell,k} + \lambda_k c_{\ell+1,k} + (N_{\ell+1,k}(c) - N_{\ell-1,k}(c)), & \ell > 0, |k| \leq m.
\end{cases}
\]

Define the operators (acting on Chebyshev sequences) by

\[
T \overset{\text{def}}{=} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
-1 & 0 & 1 & 0 & \cdots \\
0 & -1 & 0 & 1 & \cdots \\
& & & & \ddots & \ddots & \ddots \\
0 & & & & 0 & -1 & 0 & 1 \\
& & & & & & \ddots & \ddots & \ddots \\
& & & & & & & \ddots & \ddots & \ddots \\
\end{pmatrix},
\]

and

\[
\Lambda \overset{\text{def}}{=} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 2 & 0 & 0 & \cdots \\
0 & 0 & 4 & 0 & \cdots \\
& & & & \ddots & \ddots & \ddots \\
0 & & & & 0 & 0 & 2\ell & 0 \\
& & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots & \ddots \\
\end{pmatrix}.
\]
Using the operators $T$ and $\Lambda$, we may write more densely for the cases $\ell > 0$ and $|k| \leq m$

$$f_k(c) = \Lambda c_k + T(\lambda_k c_k + N_k(c)).$$

Hence,

$$f_{\ell,k}(c) = \begin{cases} c_{0,k} + 2 \sum_{j=1}^{\infty} (-1)^j c_{j,k} - b_k, & \ell = 0, |k| \leq m \\ (\Lambda c_k + T(\lambda_k c_k + N_k(c)))_{\ell}, & \ell > 0, |k| \leq m. \end{cases} \quad (3.42)$$

Denoting the set of indices $I = \{ (\ell, k) \in \mathbb{Z}^2 : \ell \geq 0$ and $|k| \leq m \}$ and denote $f = (f_j)_{j \in I}$. Assume that using Newton’s method, we computed $\bar{c} = (\bar{c}_{\ell,k})_{\ell=0,\ldots,n-1 \atop k=-m,\ldots,m}$ such that $f(\bar{c}) \approx 0$. Fix $\nu \geq 1$ the Chebyshev decay rate and define the weights $\omega_{\ell,k} = \nu^\ell$. Given a sequence $c = (c_j)_{j \in I} = (c_{\ell,k})_{k=-m,\ldots,m, \ell \geq 0}$, denote the Chebyshev-weighted $\ell^1$ norm by

$$\|c\|_{X^m_v} \overset{\text{def}}{=} \sum_{j \in I} |c_j| \omega_j.$$

The Banach space in which we prove the existence of the solutions of $f = 0$ is given by

$$X^m_v \overset{\text{def}}{=} \{ c = (c_j)_{j \in I} : \|c\|_{X^m_v} < \infty \}.$$

The following result is useful to perform the nonlinear analysis when solving $f = 0$ in $X^m_v$. We omit the elementary proof which can be mimicked from the one of Lemma 3 in [28].

**Lemma 3.8** For all $a, b \in X^m_v$, $\|a \ast b\|_{X^m_v} \leq 4\|a\|_{X^m_v}\|b\|_{X^m_v}$.

The computer-assisted proof of existence of a solution of a function $f = 0$ relies on showing that a certain Newton-like operator $c \mapsto c - Af(c)$ has a unique fixed point in the closed ball $B_r(\bar{c}) \subset X^m_v$, where $r$ is a radius to be determined. Let us now define the operator $A$. Given $n$, a finite number of Chebyshev coefficients used for the computation of $c$, denote by $f^{(n,m)} : \mathbb{C}^{n(2m+1)} \to \mathbb{C}^{n(2m+1)}$ the finite dimensional projection used to compute $\bar{c} \in \mathbb{C}^{n(2m+1)}$, that is

$$f^{(n,m)}_{\ell,k}(c^{(n,m)}) \overset{\text{def}}{=} \begin{cases} c_{0,k} + 2 \sum_{j=1}^{n-1} (-1)^j c_{j,k} - b_k, & \ell = 0, |k| \leq m \\ (\Lambda c_k + T(\lambda_k c_k + N_k(c^{(n,m)})))_{\ell}, & 0 < \ell < n, |k| \leq m. \end{cases}$$
First consider $A^\dagger$ an approximation for the Fréchet derivative $Df(\tilde{c})$:

$$(A^\dagger c)_{\ell,k} = \begin{cases} (Df^{(n,m)}(\tilde{c})c^{(n,m)})_{\ell,k}, & 0 \leq \ell < n, |k| \leq m \\ 2\ell c_{\ell,k}, & \ell \geq n, |k| \leq m, \end{cases}$$

where $Df^{(n,m)}(\tilde{c}) \in M_{n(2m+1)}(\mathbb{C})$ denotes the Jacobian matrix. Consider now a numerical inverse $A^{(n,m)}$ of $Df^{(n,m)}(\tilde{c})$. We define the action of $A$ on a vector $c \in X^m$ as

$$(Ac)_{\ell,k} = \begin{cases} A^{(n,m)}c^{(n,m)}_{\ell,k}, & 0 \leq \ell < n, |k| \leq m \\ \frac{1}{2\ell}c_{\ell,k}, & \ell \geq n, |k| \leq m. \end{cases}$$

The following Newton-Kantorovich type theorem (for linear problems posed on Banach spaces) is useful to show the existence of zeros of $f$.

**Theorem 3.9** Assume that there are constants $Y_0, Z_0, Z_1 \geq 0$ having that

$$\|Af(\tilde{c})\|_{X^m} \leq Y_0, \tag{3.43}$$
$$\|\text{Id} - AA^\dagger\|_{B(X^m)} \leq Z_0, \tag{3.44}$$
$$\|A(Df(\tilde{c}) - A^\dagger)\|_{B(X^m)} \leq Z_1. \tag{3.45}$$

If

$$Z_0 + Z_1 < 1, \tag{3.46}$$

then for all

$$r \in \left(\frac{Y_0}{1 - Z_0 - Z_1}, \infty\right),$$

there exists a unique $\tilde{c} \in B_r(\tilde{c})$ such that $f(\tilde{c}) = 0$.

**Proof** We omit the details of this standard proof. Denote $\kappa \overset{\text{def}}{=} Z_0 + Z_1 < 1$. The idea is show that $T(c) \overset{\text{def}}{=} c - Af(c)$ satisfies $T(B_r(\tilde{c})) \subset B_r(\tilde{c})$ and then that $\|T(c_1) - T(c_2)\|_{X^m} \leq \kappa \|c_1 - c_2\|_{X^m}$ for all $c_1, c_2 \in B_r(\tilde{c})$. From the Banach fixed point theorem, there exists a unique $\tilde{c} \in B_r(\tilde{c})$ such that $T(\tilde{c}) = \tilde{c}$. The condition (3.46) implies that $\|\text{Id} - AA^\dagger\|_{B(X^m)} < 1$, and by construction of the operators $A$ and $A^\dagger$, it can be shown that $A$ is an injective operator. By injectivity of $A$, we conclude that there exists a unique $\tilde{c} \in B_r(\tilde{c})$ such that $f(\tilde{c}) = 0$.

Given a Fourier projection dimension $m$, we apply Theorem 3.9 to compute the solution of $2m + 1$ problems of the form $f = 0$ given in (3.42) with initial conditions

$$b = (b_k)_{k=1}^m \in \{e_1, e_2, \ldots, e_{2m+1}\}$$
yielding a sequence of solutions \( \tilde{c}^{(j)} : [-1, 1] \to \mathbb{C}^{2m+1} (j = -m, \ldots, m) \) with Chebyshev series representation

\[
\tilde{c}^{(j)}(t) = \tilde{c}^{(j)}_{0,k} + 2 \sum_{\ell \geq 1} \tilde{c}^{(j)}_{\ell,k} T_\ell(t).
\]

Finally we can define the fundamental matrix solution \( \Phi(t) \in M_{2m+1}(\mathbb{C}) \) satisfying (3.30) as

\[
\Phi(t) \overset{\text{def}}{=} \begin{pmatrix}
\vdots & \vdots & \vdots \\
\tilde{c}^{(1)}(t) & \tilde{c}^{(2)}(t) & \cdots & \tilde{c}^{(2m+1)}(t) \\
\vdots & \vdots & \vdots 
\end{pmatrix}.
\] (3.47)

Using a similar construction, we can construct \( \Psi(s) \in M_{2m+1}(\mathbb{C}) \) satisfying (3.31).

The rest of this section is dedicated to the explicit construction of the bounds \( Y_0, Z_0 \) and \( Z_1 \) of Theorem 3.9.

**The bound \( Y_0 \)** Recalling the definition of the quadratic term \( N_k \) of \( f \) in (3.39), given the numerical solution \( \tilde{c} = (\tilde{c}_{\ell,k})_{\ell=0,\ldots,n-1, k=-m,\ldots,m} \), one gets that \( N_k,j(\tilde{c}) = 0 \) for all \( j \geq 2n-1 \). This implies that the term \( f_j(\tilde{c}) \) has only finitely many nonzero terms. Hence, the computation of \( Y_0 \) satisfying \( \|Af(\tilde{c})\|_{\mathcal{X}_m} \leq Y_0 \) is only a finite computation with interval arithmetic.

**The bound \( Z_0 \)** The computation of the bound \( Z_0 \) satisfying (3.44) requires defining the operator

\[
B \overset{\text{def}}{=} \text{Id} - AA^\dagger,
\]

which action is given by

\[
(Bc)_{\ell,k} = \begin{cases}
(\text{Id} - A^{(n,m)}Df^{(n,m)}(\tilde{c}))c^{(n,m)} & 0 \leq \ell < n, \ |k| \leq m \\
0 & \ell \geq n, \ |k| \leq m
\end{cases}
\]

Using interval arithmetic, compute \( Z_0 \) such that

\[
\|B\|_{\mathcal{X}_m} = \sup_{j \in \mathcal{I}} \frac{1}{\omega_j} \sum_{i \in \mathcal{I}} |B_{i,j}| \omega_i
\]

\[
= \max_{\ell_2=0,\ldots,n-1} \frac{1}{v_{\ell_2}} \sum_{\ell_1=0,\ldots,n-1} \sum_{|k_1| \leq m} |B_{(\ell_1,k_1) ,(\ell_2,k_2)}| v_{\ell_1} \leq Z_0.
\]

**The bound \( Z_1 \).** For any \( c \in B_1(0) \), let

\[
z \overset{\text{def}}{=} [Df(\tilde{c}) - A^\dagger]c
\]
which is given component-wise by

\[ z_{\ell,k} = z_{\ell,k}(\tilde{a},c) \equiv \begin{cases} 
2 \sum_{j \geq n} (-1)^j c_{j,k}, & \ell = 0, \ |k| \leq m \\
he^{i\theta} \left( T(\tilde{a} \ast c^{(\infty,m)})_\ell \right) \ , & 0 < \ell < n, \ |k| \leq m \\
\lambda_k(T\tilde{c}_k) + he^{i\theta} \left( T(\tilde{a} \ast c^{(m)})_\ell \right) , & \ell \geq n, \ |k| \leq m, 
\end{cases} \]

where \( c^{(\infty,m)} = (c_{\ell,k})_{\ell \geq m, \ldots, m} \). Next, for the cases \( 0 \leq \ell < n \) and \( |k| \leq m \), we present component-wise uniform bounds \( \hat{z}_{\ell,k} \geq 0 \) such that \( z_{\ell,k}(\tilde{a},c) \leq \hat{z}_{\ell,k} \) for all \( c \in X_v^m \) with \( \|c\|_{X_v^m} \leq 1 \). These bounds will then be used to define \( Z_1 \). First, given \( c \in B_1(0) \) and \( |k| \leq m \),

\[
|z_{0,k}(c)| = \left| 2 \sum_{j \geq n} (-1)^j c_{j,k} \right| \leq 2 \sum_{j \geq n} |c_{j,k}| \frac{v_j}{\nu_j} \leq 2 \nu^n \sum_{j \geq n} |c_{j,k}| \nu_j \leq 2 \nu^n \|c\|_{X_v^m} \leq \hat{z}_{0,k} \equiv \frac{2}{\nu^n}. (3.48)
\]

The next step is to obtain \( \hat{z}_{\ell,k} \) for \( \ell = 1, \ldots, n-1 \). This task involves understanding \( \tilde{a} \ast c^{(\infty,m)} \) whose components can interpreted as

\[
(\tilde{a} \ast c^{(\infty,m)})_{\ell,k} = \sum_{\substack{\ell_1 + \ell_2 = \ell \\
k_1 + k_2 = k \\
|\ell_1| < n \leq |\ell_2| \\
|k_1| \leq m, |k_2| \leq m}} \tilde{a}_{\ell_1,k_1} c_{\ell_2,k_2} = \sum_{\ell_2 \geq 0} \sum_{\substack{|k_2| \leq m \\\n|k_1| \leq N, |k_2| \leq m}} \alpha^{(\ell,k)}_{\ell_2,k_2} c_{\ell_2,k_2},
\]

where

\[
\alpha^{(\ell,k)}_{\ell_2,k_2} \equiv \begin{cases} 
0, & \ell_2 = 0, \ldots, n - 1 \\
\tilde{a}_{\ell_2 - \ell,k_2}, & \ell_2 = n, \ldots, \ell + n - 1.
\end{cases}
\]

For \( 0 < \ell < n \) and \( |k| \leq m \), the term \((\tilde{a} \ast c^{(\infty,m)})_{\ell,k} \in \mathbb{C}\) can then be thought of as a linear functional acting on \( c = (c_j)_{j \in \mathcal{I}} \in X_v^m \). Using that representation, we get that for all \( c \in X_v^m \) with \( \|c\|_{X_v^m} \leq 1 \),

\[
|(\tilde{a} \ast c^{(\infty,m)})_{\ell,k}| \leq \sum_{j \in \mathcal{I}} |\alpha_j^{(\ell,k)}| |c_j| = \sum_{j \in \mathcal{I}} \left| \frac{\alpha_j^{(\ell,k)}}{\omega_j} \right| |c_j| \omega_j \leq \Psi^{(\ell,k)}(\tilde{a}) \|c\|_{X_v^m} \leq \Psi^{(\ell,k)}(\tilde{a})
\]

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where
\[
\psi^{(\ell, k)}(\bar{a}) \overset{\text{def}}{=} \sup_{j \in I} \left| \alpha_j^{(\ell, k)} \right| = \max_{|k| \leq m} \left\{ \frac{\left| \bar{a}_{\ell_2} - \ell, k - k_2 \right|}{\nu_2} \right\},
\]
which can easily be computed using interval arithmetic. For the cases \(0 \leq \ell < n\) and \(|k| \leq m\), this leads to the bound
\[
|z^{\ell, k}(\bar{a}, c)| \leq h \left| T(\bar{a} * c^{(\ell, m)}) \right| \overset{\text{def}}{=} \hat{z}^{\ell, k} \overset{\text{def}}{=} h \left| T\psi^{(\ell, k)}(\bar{a}) \right|, \quad (3.50)
\]
for all \(c \in B_1(0)\), where \(|T|\) denotes the operator with component-wise absolute values. We are ready to obtain the bound \(Z_1\). Given \(c \in B_1(0)\), we use Lemma 3.8 to conclude that
\[
\|A[Df(\bar{c}) - A^\dagger c]\|_{\chi^m_v} = \|Az\|_{\chi^m_v}
\]
where
\[
\|Az\|_{\chi^m_v} = \sum_{j \in I} \left| (Az)_j \right| \omega_j
\]
and
\[
\|A[Df(\bar{c}) - A^\dagger c]\|_{\chi^m_v} \leq \sum_{\ell = 0, \ldots, n-1} \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \sum_{\ell \geq n} \frac{1}{2n} \lambda_m |T c_k| + \lambda_m \left( \nu_2 - c_{\ell-1,k} + c_{\ell+1,k} \right) v^\ell
\]
\[
+ \sum_{\ell \geq n} \frac{1}{2n} \lambda_m \left( |T(\bar{a} * c^{(m)})_k| \right) v^\ell
\]
\[
+ \frac{1}{2n} \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \frac{1}{2n} \left( \lambda_m \right) \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \frac{1}{2n} \left( \lambda_m \right) \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \frac{1}{2n} \left( \lambda_m \right) \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \frac{1}{2n} \left( \lambda_m \right) \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
\[
+ \frac{1}{2n} \left( \lambda_m \right) \left( |A^{(n,m)}| \hat{z}^{(n,m)} \right)_{\ell, k} v^\ell
\]
Hence, by construction

\[ Z_1 \overset{\text{def}}{=} \sum_{\ell=0, \ldots, n-1} |A(n,m)_{\ell}^2(n,m)| \nu_{\ell,k} + \frac{1}{2n} (\nu + \frac{1}{\nu}) \left(|\lambda_m| + 4h\|\bar{a}\|_{\chi^m}\right), \]

satisfies (3.45).

### 3.3 Generation of the evolution operator $\bar{U}^{(\infty)}(t,s)$ on $\ell^1$

Finally, in this section, we verify the existence of the evolution operator $C^{(\infty)}(t,s)$ of the infinite dimensional equation (3.6). This in turns will verify the existence of the evolution operator $\bar{U}^{(\infty)}(t,s)$ generated on $\ell^1$. Moreover, we derive an estimate of $\bar{U}^{(\infty)}(t,s)$ defined in (3.8) using the bounded operator norm on $\ell^1$, which is the hypothesis (3.10) of Theorem 3.3.

Let $\ell_1^\infty \overset{\text{def}}{=} (\text{Id} - \pi^{(m)})\ell_1 = \{ (a_k)_{k\in\mathbb{Z}} \in \ell_1 : a_k = 0 (|k| \leq m) \} \subset \ell_1$ endowed with the norm $\|a\|_{\ell_1^\infty} \overset{\text{def}}{=} \sum_{|k|>m} |a_k|$. Consider the Laplace operator $L$ defined in (2.8) whose action is restricted on $\ell_1^\infty$, that is

\[ (La)_k = \begin{cases} 0, & |k| \leq m \\ -k^2 \omega^2 a_k, & |k| > m, \end{cases} \quad \forall a \in D(L), \]

where the domain of the operator $L$ is $D(L) = \{ a \in \ell_1^\infty : La \in \ell_1^\infty \}$ in this case. Denoting $\mathcal{L} = e^{i\theta} L$, it is easy to see that $\mathcal{L} : D(L) \subset \ell_1^\infty \rightarrow \ell_1^\infty$ is a densely defined closed operator on $\ell_1^\infty$. The operator $\mathcal{L}$ then generates the semigroup on $\ell_1^\infty$ (e.g. see [19]), which is denoted by $\{ e^{\mathcal{L}t} \}_{t \geq 0}$. Furthermore, the action of such a semigroup $\{ e^{\mathcal{L}t} \}_{t \geq 0}$ can be naturally extended to $\ell^1$ by

\[ (e^{\mathcal{L}t} \phi)_k = \begin{cases} 0, & |k| \leq m \\ (e^{2\mathcal{L}t} (\phi_k)_{|k|>m})_k, & |k| > m, \end{cases} \quad \forall \phi \in \ell^1. \]

In the following, unless otherwise noted, we consider the semigroup $\{ e^{\mathcal{L}t} \}_{t \geq 0}$ on $\ell^1$ as defined in (3.52). We also have the following estimate:

\[ \| e^{\mathcal{L}t} \|_{B(\ell^1)} \leq e^{-\mu_{m+1}t}, \quad \mu_{m+1} = (m+1)^2 \omega^2 \cos \theta. \]

We re-write (3.6) as the Cauchy problem in $\ell^1$

\[ \frac{d}{dt} c^{(\infty)}(t) - \mathcal{L} c^{(\infty)}(t) = 2e^{i\theta} (\text{Id} - \pi^{(m)}) \left( \bar{a} (t) * c^{(\infty)}(t) \right) \]
for any initial sequence \( c^{(\infty)}(s) = \phi^{(\infty)} \). Showing the existence of the solution of (3.54), the proof of existence of the evolution operator \( \bar{U}^{(\infty)}(t, s) \) is given by the following theorem.

**Theorem 3.10** For the infinite-dimensional system of differential equations (3.54) with a fixed initial time \( s \), there exists a unique solution that solves the integral equation in \( \ell^1 \)

\[
c^{(\infty)}(t) = e^{\Sigma(t-s)}\phi^{(\infty)} + 2e^{i\theta} \int_s^t e^{\Sigma(t-\tau)}(\text{Id} - \pi^{(m)})(\bar{a}(\tau) \ast c^{(\infty)}(\tau))d\tau.
\]

Furthermore, the evolution operator \( \bar{U}^{(\infty)}(t, s) \) exists and the following estimate holds

\[
\|\bar{U}^{(\infty)}(t, s)\|_{\ell^1} \leq e^{-\mu_{m+1}(t-s)} + 2 \int_s^t \|\bar{a}(\tau)\|_{\ell^1} d\tau \|\phi^{(\infty)}\|_{\ell^1}, \quad \forall \phi^{(\infty)} \in \ell^1.
\]

**Proof** For a fixed \( s \geq 0 \), let us define a map \( \mathcal{P} : X \to X \) acting on the \( c^{(\infty)}(t) \) as

\[
\mathcal{P}c^{(\infty)}(t) \overset{\text{def}}{=} e^{\Sigma(t-s)}\phi^{(\infty)} + 2e^{i\theta} \int_s^t e^{\Sigma(t-\tau)}(\text{Id} - \pi^{(m)})(\bar{a}(\tau) \ast c^{(\infty)}(\tau))d\tau
\]

and let a function space \( \mathcal{X}_\infty \) be defined by

\[
\mathcal{X}_\infty \overset{\text{def}}{=} \left\{ c^{(\infty)}(t) \in \ell^1 : \sup_{0 \leq s \leq t \leq h} e^{\mu_{m+1}(t-s)} \|c^{(\infty)}(t)\|_{\ell^1} < \infty \right\}
\]

with the distance\(^2\)

\[
d\left( c_1^{(\infty)}, c_2^{(\infty)} \right) \overset{\text{def}}{=} \sup_{0 \leq s \leq t \leq h} \left( e^{\mu_{m+1}(t-s)} - 2e^{i\theta} \int_s^t \|\bar{a}(\tau)\|_{\ell^1} d\tau \|c_1^{(\infty)}(t) - c_2^{(\infty)}(t)\|_{\ell^1} \right), \quad \beta > 1.
\]

We prove that the map \( \mathcal{P} \) becomes a contraction mapping under the distance \( d \) on \( \mathcal{X}_\infty \). For \( c_1^{(\infty)}, c_2^{(\infty)} \in \mathcal{X}_\infty \), we have using (3.53) and the property (2.6)

\[
e^{\mu_{m+1}(t-s)} - 2e^{i\theta} \int_s^t \|\bar{a}(\tau)\|_{\ell^1} d\tau \left\| \mathcal{P}c_1^{(\infty)}(t) - \mathcal{P}c_2^{(\infty)}(t) \right\|_{\ell^1}
\]

\[
\leq e^{\mu_{m+1}(t-s)} - 2e^{i\theta} \int_s^t e^{-\mu_{m+1}(t-\tau)} \|\bar{a}(\tau)\|_{\ell^1} \left\| c_1^{(\infty)}(\tau) - c_2^{(\infty)}(\tau) \right\|_{\ell^1} d\tau
\]

\[
\leq e^{\mu_{m+1}(t-s)} - 2e^{i\theta} \int_s^t e^{-\mu_{m+1}(t-\tau)} \|\bar{a}(\tau)\|_{\ell^1} d\tau \|c_1^{(\infty)}(t) - c_2^{(\infty)}(t)\|_{\ell^1} d\tau
\]

\[
\times 2 \int_s^t e^{-\mu_{m+1}(t-\tau)} \|\bar{a}(\tau)\|_{\ell^1} e^{\mu_{m+1}(t-s)} + 2e^{i\theta} \int_s^t \|\bar{a}(\sigma)\|_{\ell^1} d\sigma d\tau
\]

\[
= e^{-2\beta} \int_s^t \|\bar{a}(\tau)\|_{\ell^1} d\tau d\mathcal{P}\left( c_1^{(\infty)}, c_2^{(\infty)} \right) 2 \int_s^t \|\bar{a}(\tau)\|_{\ell^1} e^{2\beta} \int_s^t \|\bar{a}(\sigma)\|_{\ell^1} d\sigma d\tau
\]

\(\overset{\text{def}}{=} \sup_{0 \leq s \leq t \leq h} e^{\mu_{m+1}(t-s)} - 2e^{i\theta} \int_s^t \|\bar{a}(\tau)\|_{\ell^1} d\tau \|c_1^{(\infty)}(t) - c_2^{(\infty)}(t)\|_{\ell^1} < \infty\)

\[\beta > 1\]

The parameter \( \beta > 1 \) is used to prove that the map \( \mathcal{P} \) is a contraction mapping.
Since $\beta > 1$, $P$ becomes a contraction mapping on $X_{\infty}$. This yields that the solution of (3.54) uniquely exists in $X_{\infty}$, which satisfies

$$c(\infty)(t) = e^{\Sigma(t-s)}\phi(\infty) + 2e^{i\theta} \int_{s}^{t} e^{\Sigma(t-\tau)} (\text{Id} - \pi(m)) (\bar{a}(\tau) \ast c(\infty)(\tau)) d\tau.$$  

(3.55)

Moreover, letting $y(t) \overset{\text{def}}{=} e^{\mu_{m+1}(t-s)} \|c(\infty)(t)\|_{\ell^1}$, it follows from (3.55) using (3.53) and the property (2.6), that

$$y(t) \leq \|\phi(\infty)\|_{\ell^1} + 2 \int_{s}^{t} e^{\mu_{m+1}(\tau-s)} \|\bar{a}(\tau)\|_{\ell^1} \|c(\infty)(\tau)\|_{\ell^1} d\tau = \|\phi(\infty)\|_{\ell^1} + 2 \int_{s}^{t} \|\bar{a}(\tau)\|_{\ell^1} y(\tau) d\tau.$$  

Grönwall’s inequality yields

$$y(t) \leq \|\phi(\infty)\|_{\ell^1} e^{2 \int_{s}^{t} \|\bar{a}(\tau)\|_{\ell^1} d\tau}.$$  

Then, we conclude that the following inequality holds

$$\|\hat{U}(\infty)(t, s)\phi(\infty)\|_{\ell^1} \leq \|\phi(\infty)\|_{\ell^1} e^{-\mu_{m+1}(t-s) + \int_{s}^{t} \|\bar{a}(\tau)\|_{\ell^1} d\tau} = W(\infty)(t, s) \|\phi(\infty)\|_{\ell^1}$$

(3.56)

for any $\phi(\infty) \in \ell^1$, where $W(\infty)(t, s)$ is defined in (3.10).

\[\square\]

### 3.4 Rigorous construction of the solution map operator

Recall that the definition of the solution map operator $A_{\phi}$ in (3.3) requires proving the existence of the evolution operator $U(t, s)$, which is done by verifying the main three hypotheses of Theorem 3.3, namely (3.9), (3.10) and (3.14). Hypothesis (3.9) is verified in practice by using the theory of Sect. 3.2 to compute rigorously (with Chebyshev series) the fundamental matrix solutions $\Phi(t)$ and $\Psi(s)$ of problems (3.30) and (3.31), respectively, and then using (3.32). Hypothesis (3.10) has been verified in full generality in Sect. 3.3 in Theorem 3.10. Hypothesis (3.14) is computational and requires computing the constant $\kappa$ with interval arithmetic. Once all the hypotheses are verified successfully, the solution map operator $A_{\phi}$ exists, is defined as in (3.3) and can be used to study the fixed point of the simplified Newton operator $T$ defined in (2.10). The fixed point of $T$ is obtained using the approach of Sect. 4. The fixed
point so obtained yields the unique solution of $F = 0$ (with $F$ defined in (2.9)) and by construction, this yields the local rigorous inclusion of the solution of the Cauchy problem on the time interval $[0, h]$.

4 Local inclusion in a time interval

In this section, we present a sufficient condition whether a fixed point of the simplified Newton operator (2.10) exists and is unique in $B_J(\tilde{a}, \varrho)$ defined by (2.7). Such a sufficient condition can be rigorously checked by numerical computations based on interval arithmetic.

**Theorem 4.1** Consider the Cauchy problem (2.3). For a given initial sequence $\varphi$ and its approximation $\tilde{a}(0)$, assume that there exists $\varepsilon \geq 0$ such that $\|\varphi - \tilde{a}(0)\|_{\ell^1} \leq \varepsilon$. Assume also that $\tilde{a} \in X$ and any $a \in B_J(\tilde{a}, \varrho)$ satisfies

$$\sup_{t \in J} \sum_{k \in \mathbb{Z}} |[T(a)(t) - \tilde{a}(t)]_k| \leq f_\varepsilon (\varrho),$$

where $f_\varepsilon (\varrho)$ is defined by

$$f_\varepsilon (\varrho) \overset{\text{def}}{=} W_h \left[ \varepsilon + h \left( 2\varrho^2 + \delta \right) \right]. \quad (4.1)$$

Here, $W_h > 0$ and $\delta > 0$ satisfy $\sup_{(t,s) \in S_h} \|U(t,s)\|_{B(\ell^1)} \leq W_h$ and $\|F(\tilde{a})\|_X \leq \delta$, respectively. If

$$f_\varepsilon (\varrho) \leq \varrho$$

holds, then the Fourier coefficients $\tilde{a}$ of the solution of (2.1) are rigorously included in $B_J(\tilde{a}, \varrho)$ and are unique in $B_J(\tilde{a}, \varrho)$.

The proof of this theorem is based on Banach fixed point theorem.

**Proof** On the basis of Banach fixed point theorem, we prove that the simplified Newton operator $T$ defined by (2.10) becomes a contraction mapping on $B_J(\tilde{a}, \varrho)$. It is sufficient to show that the following two conditions hold:

1. $T(a) \in B_J(\tilde{a}, \varrho)$ for any $a \in B_J(\tilde{a}, \varrho)$,
2. there exists $\kappa \in [0, 1)$ such that $d(T(a_1), T(a_2)) \leq \kappa d(a_1, a_2)$ for $a_1, a_2 \in B_J(\tilde{a}, \varrho)$ with a distance $d$ in $B_J(\tilde{a}, \varrho)$.

Firstly, for a sequence $a \in B_J(\tilde{a}, \varrho)$, we have using (2.10)

$$T(a) - \tilde{a} = T(a) - T(\tilde{a}) + T(\tilde{a}) - \tilde{a}$$

$$= \mathcal{A}_{\varphi - \tilde{a}(0)} \left[ e^{i\theta} (a * a - \tilde{a} * \tilde{a} - 2\tilde{a} * (a - \tilde{a})) \right] - \mathcal{A}_0 F(\tilde{a}). \quad (4.2)$$
Let \( z \overset{\text{def}}{=} a - \bar{a} \). The first term of (4.2) follows

\[
\mathcal{A}_{z(0)} \left[ e^{i\theta} (a * a - \bar{a} * \bar{a} - 2\bar{a} * (a - \bar{a})) \right] = \mathcal{A}_{z(0)} \left[ e^{i\theta} (z * z) \right].
\]

Thus, (4.2) is represented by

\[
T(a) - \bar{a} = \mathcal{A}_{z(0)} \left( e^{i\theta} (z * z)_k - F_k(\bar{a}) \right)_{k \in \mathbb{Z}},
\]

where \( \mathcal{A}_{z(0)} \) is the solution map operator defined in Sect. 3 and

\[
F_k(\bar{a}) = \begin{cases} \frac{d}{dt} \bar{a}_k - e^{i\theta} ((L \bar{a})_k + (\bar{a} \ast \bar{a})_k), & |k| \leq N \\ -e^{i\theta} (\bar{a} \ast \bar{a})_k, & |k| > N. \end{cases}
\]  

Taking \( \ell^1 \) norm of \( T(a) - \bar{a} \), we have

\[
\| T(a) - \bar{a} \|_{\ell^1} = \sum_{k \in \mathbb{Z}} |(T(a) - \bar{a})_k|
\]

\[
= \left\| U(t, 0)z(0) + \int_0^t U(t, s)g(s) ds \right\|_{\ell^1}
\]

\[
\leq \| U(t, 0)z(0) \|_{\ell^1} + \int_0^t \| U(t, s)g(s) \|_{\ell^1} ds,
\]

where

\[
g(s) \overset{\text{def}}{=} e^{i\theta} (z(s) * z(s)) - (F(\bar{a}))(s).
\]

Taking \( \ell^1 \)-norm of \( g \), we have using the property (2.6)

\[
\| g(s) \|_{\ell^1} \leq \sum_{k \in \mathbb{Z}} \left| e^{i\theta} (z(s) * z(s))_k \right| + \| (F(\bar{a}))(s) \|_{\ell^1}
\]

\[
\leq 2\| z(s) \|_{\ell^1}^2 + \delta,
\]

where \( \delta \) satisfies \( \sup_{s \in J} \| (F(\bar{a}))(s) \|_{\ell^1} \leq \delta \). We remark that the last inequality in (4.6) is obviously overestimated, but we estimate it in this way to simplify the later notations in the second part of this proof. Since \( a \in B_J(\bar{a}, \varrho) \), \( \| z \|_X \leq \varrho \) holds. Finally, (4.4) is bounded by using the uniform bound \( W_h (3.4) \) discussed in the previous section as

\[
\sup_{t \in J} \sum_{k \in \mathbb{Z}} \left| ((T(a))(t) - \bar{a}(t))_k \right| \leq \sup_{t \in J} \| U(t, 0)z(0) \|_{\ell^1} + \sup_{t \in J} \int_0^t \| U(t, s)g(s) \|_{\ell^1} ds
\]

\[
\leq Wh \left[ \varepsilon + h \left( 2\varrho^2 + \delta \right) \right] = f_\varepsilon(\varrho),
\]
where $\varepsilon$ is the upper bound of the initial error such that $\|z(0)\|_{\ell^1} \leq \varepsilon$. From the assumption $f_{\varepsilon}(\bar{\varrho}) \leq \varrho$, $T(a) \in B_{J}(\bar{a}, \varrho)$ holds for any $a \in B_{J}(\bar{a}, \varrho)$.

Secondly, we will show the contraction property of $T$. For sequences $a_1, a_2 \in B_{J}(\bar{a}, \varrho)$, we define the distance in $B_{J}(\bar{a}, \varrho)$ as

$$d(a_1, a_2) \overset{\text{def}}{=} \|a_1 - a_2\|_\chi.$$  

The analogous discussion above yields

$$T(a_1) - T(a_2) = A_0 \left[ e^{i\theta} (a_1 \ast a_1 - a_2 \ast a_2 - 2\bar{a} \ast (a_1 - a_2)) \right].$$  

(4.7)

Let $\zeta \overset{\text{def}}{=} a_1 - a_2$ with $\zeta(0) = 0$. It follows

$$\|T(a_1) - T(a_2)\|_{\ell^1} = \sum_{k \in \mathbb{Z}} \left| (T(a_1) - T(a_2))_k \right| \leq \int_0^t \|U(t, s)\tilde{g}(s)\|_{\ell^1} ds,$$  

(4.8)

where $\tilde{g}$ is defined by

$$\tilde{g} \overset{\text{def}}{=} e^{i\theta} [(a_1 - \bar{a}) \ast (a_1 - a_2) + (a_2 - \bar{a}) \ast (a_1 - a_2)].$$

Then, from (4.7) and (4.8), the distance is estimated by

$$d(T(a_1), T(a_2)) \leq (2W_h \varrho) d(a_1, a_2).$$

Taking $\kappa = 2W_h \varrho$, it follows $\kappa < f_{\varepsilon}(\varrho)/\varrho \leq 1$ from the assumption of theorem. It is proved that the simplified Newton operator $T$ becomes the contraction mapping on $B_{J}(\bar{a}, \varrho)$. $\Box$

**Remark 4.2** Our task in the practical implementation is to rigorously compute the minimum values $\varrho$ such that $f_{\varepsilon}(\varrho) \leq \varrho$ by using interval arithmetic.

### 4.1 The bound $\varepsilon$

We show how we get the $\varepsilon$ bound such that $\|\varphi - \bar{a}(0)\|_{\ell^1} \leq \varepsilon$. From (3.33)

$$\bar{a}_k(0) = \bar{a}_{0,k} + 2 \sum_{\ell=1}^{n-1} \bar{a}_{\ell,k} T_\ell(0)$$

$$\quad = \bar{a}_{0,k} - 2\bar{a}_{1,k} + 2\bar{a}_{2,k} - \cdots + (-1)^{n-1} 2\bar{a}_{n-1,k} \quad (|k| \leq N),$$

where we used the fact $T_\ell(0) = (-1)^\ell$. Then, using interval arithmetic, $\varepsilon$ is given by

$$\varepsilon \overset{\text{def}}{=} \sum_{|k| \leq N} \left| \varphi_k - \left( \bar{a}_{0,k} - 2\bar{a}_{1,k} + 2\bar{a}_{2,k} - \cdots + (-1)^{n-1} 2\bar{a}_{n-1,k} \right) \right|.$$
We remark that $\varphi_k = 0$ for $|k| > N$ from the definition (2.4). If these are non-zero, $\varepsilon$ should include the tail term $\sum_{|k| > N} |\varphi_k|$.

4.2 The bound $\delta$

We also show how we get the defect bound of $F$ at the approximate solution $\bar{a}$. From (4.3), we recall

$$F_k(\bar{a}) = \begin{cases} \frac{d}{dt} \bar{a}_k - e^{i\theta} \left( -k^2 \omega^2 \bar{a}_k + (\bar{a} \ast \bar{a})_k \right), & |k| \leq N, \\ -e^{i\theta} (\bar{a} \ast \bar{a})_k, & |k| > N. \end{cases}$$

Here, we suppose that the first derivative of $\bar{a}$ is expressed by

$$\frac{d}{dt} \bar{a}_k(t) = \sum_{\ell=0}^{n-2} \bar{a}^{(1)}_{\ell,k} T_\ell(t),$$

where $\bar{a}^{(1)}_{\ell,k} \in \mathbb{C}$ can be computed by an recursive algorithm (see, e.g., [13, page 34]). For $|k| \leq N$, we have

$$F_k(\bar{a}) = \frac{d}{dt} \bar{a}_k - e^{i\theta} \left( -k^2 \omega^2 \bar{a}_k + (\bar{a} \ast \bar{a})_k \right)$$

$$= \sum_{\ell=0}^{n-2} \bar{a}^{(1)}_{\ell,k} T_\ell(t) + \sum_{\ell=0}^{n-1} e^{i\theta} k^2 \omega^2 w_\ell \bar{a}_{\ell,k} - \sum_{\ell_1 + \ell_2 = \ell, \ k_1 + k_2 = k} \bar{a}_{|\ell_1|, k_1} \bar{a}_{|\ell_2|, k_2} T_\ell(t)$$

$$- \sum_{\ell \geq n} \left( \sum_{\ell_1 + \ell_2 = \ell, \ k_1 + k_2 = k} \bar{a}_{|\ell_1|, k_1} \bar{a}_{|\ell_2|, k_2} \right) T_\ell(t),$$

where $w_\ell$ denotes a weight for the Chebyshev coefficients such that

$$w_\ell = \begin{cases} 1, & \ell = 0 \\ 2, & \ell > 0. \end{cases}$$

It then follows

$$\left| \frac{d}{dt} \bar{a}_k - e^{i\theta} \left( -k^2 \omega^2 \bar{a}_k + (\bar{a} \ast \bar{a})_k \right) \right|$$
\[ \begin{aligned} &\leq \sum_{\ell=0}^{n-2} \tilde{a}_{\ell,k}^{(1)} + e^{i\theta} k^2 \omega^2 \mathbf{w}_\ell \tilde{a}_{\ell,k} - \sum_{\ell_1+\ell_2=\pm\ell \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2} \\
&\quad + 2e^{i\theta} k^2 \omega^2 \tilde{a}_{n-1,k} - \sum_{\ell_1+\ell_2=\pm(n-1) \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2} \\
&\quad + \sum_{\ell \geq n} \sum_{\ell_1+\ell_2=\pm\ell \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2}. 
\end{aligned} \]

Furthermore, for \(|k| > N\), the tail part is given by

\[ |(\tilde{a} \ast \tilde{a})_k| \leq \sum_{\ell \geq 0} \sum_{\ell_1+\ell_2=\pm\ell \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2}. \]

Finally, the defect bound \(\delta\) is given by

\[ \sup_{t \in J} \| (F(\tilde{a}))(t) \|_{\mathcal{E}_1} \leq \sup_{t \in J} \left( \sum_{|k| \leq N} |F_k(\tilde{a})| + \sum_{|k| > N} |F_k(\tilde{a})| \right) \]

\[ \leq \sum_{|k| \leq N} \left( \sum_{\ell=0}^{n-2} \tilde{a}_{\ell,k}^{(1)} + e^{i\theta} k^2 \omega^2 \mathbf{w}_\ell \tilde{a}_{\ell,k} - \sum_{\ell_1+\ell_2=\pm\ell \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2} \\
+ 2e^{i\theta} k^2 \omega^2 \tilde{a}_{n-1,k} - \sum_{\ell_1+\ell_2=\pm(n-1) \atop k_1+k_2=k \atop |\ell_1|<n, |k_1| \leq N} \tilde{a}_{|\ell_1|,k_1} \tilde{a}_{|\ell_2|,k_2} \right). \]
\begin{align*}
+ \sum_{\ell \geq n} \sum_{|\ell_1| < n, |k_1| \leq N} \tilde{a}_{|\ell_1|, k_1} \tilde{a}_{|\ell_2|, k_2} \\
+ \sum_{|k| > N} \sum_{\ell \geq 0} \sum_{|\ell_1| < n, |k_1| \leq N} \tilde{a}_{|\ell_1|, k_1} \tilde{a}_{|\ell_2|, k_2} \quad \text{def} = \delta.
\end{align*}

This seems to be an infinite sum, but thanks to the finite number of nonzero elements in \( \tilde{a}_{\ell, k} \), it becomes a finite sum. Then the defect bound can be rigorously computed by interval arithmetic. For the rigorous computing of the discrete convolution, one can consult a FFT based algorithm by, e.g., [11].

5 Time stepping scheme

Once the rigorous inclusion of Fourier coefficients is obtained, we consider extending the time interval, say time step, in which the existence of the solution is verified. For this purpose the initial sequence in the next time step is replaced by a sequence at the endpoint of the current time step (e.g., \( a(h) \) for the first time step). Replacing \( J = (h, 2h) \), we apply Theorem 4.1 for the initial-boundary value problem on the next time step and recursively repeat this process several times. We introduce such a time stepping scheme in this section.

For \( K \in \mathbb{N} \), let \( 0 = t_0 < t_1 < \cdots < t_K < \infty \) be grid points of the time variable. We call \( J_i = (t_{i-1}, t_i) \) the \( i \)th time step and let \( h_i = t_i - t_{i-1} \) (\( i = 1, 2, \ldots, K \)) be the step size. Now we assume that the solution \( a(t) = (a_k(t))_{k \in \mathbb{Z}} \) of (2.3) is rigorously included until \( J_K \), i.e.,

\[ a(t) \in B_{J_i} (\tilde{a}, \varrho_i) \quad \text{def} \quad \{ a : \| a - \tilde{a} \|_{C(J_i; \ell^1)} \leq \varrho_i \text{ with initial data } a(t_{i-1}) \} \quad (t \in J_i) \]

holds for some \( \varrho_i > 0 \). In the following, we derive the error bound \( \varepsilon_i \) (\( i = 1, 2, \ldots, K \)) at each endpoint of time interval, which satisfies \( \| a(t_i) - \tilde{a}(t_i) \|_{\ell^1} \leq \varepsilon_i \). We call such an error estimate the point-wise error estimate.

Let us show the case of first time step. Recall that \( z(t) = a(t) - \tilde{a}(t) \) (\( t \in J_1 \)). At the endpoint of the time step \( t = t_1 \), it follows from (4.4)

\[ \| z(t_1) \|_{\ell^1} \leq \| U(t_1, t_0) z(t_0) \|_{\ell^1} + \int_{t_0}^{t_1} \| U(t_1, s) g(s) \|_{\ell^1} ds, \]

where \( g(s) \) is defined in (4.5) and \( \| g(s) \|_{\ell^1} \leq 2\varrho_1^2 + \delta_1 \) holds from (4.6). Here, the positive constant \( \delta_1 \) satisfies \( \sup_{t \in J_1} \| (F(\tilde{a}))'(s) \|_{\ell^1} \leq \delta_1 \), which is given in Sect. 4.2. We assume that the initial error is bounded by \( \| z(t_0) \|_{\ell^1} \leq \varepsilon_0 \), which is the same as
that in the hypothesis of Theorem 4.1 in Sect. 4. Then, to derive the bound \( \varepsilon_1 \), we need two positive constants \( W_{J_1} > 0 \) and \( W_{t_1} > 0 \) such that \( \sup_{s \in J_1} \| U(t_1, s) \|_{B(\ell^1)} \leq W_{J_1} \) and \( \| U(t_1, t_0) \|_{B(\ell^1)} \leq W_{t_1} \), respectively.

We reconsider the linearized problem (3.1) and represent the solution of (3.1) as \( b_s(t) \equiv b(t) \). As the analogous discussion in Sect. 3, we split the solution \( b_s(t) \) into the finite mode \( b_s^{(m)} \) and the tail mode \( b_s^{(\infty)} \). The finite mode is given by plugging \( t = t_1 \) in (3.20)

\[
b_s^{(m)}(t_1) = \tilde{U}^{(m)}(t_1, s)\phi^{(m)} + 2e^{i\theta} \int_{t}^{t_1} \tilde{U}^{(m)}(t_1, \tau)\pi^{(m)}(\tilde{a}(\tau) * b_s^{(\infty)}(\tau))d\tau.
\]

By the definition of \( \tilde{U}^{(m)}(t, s) \) given in (3.7), \( \| \tilde{U}^{(m)}(t, s) \|_{B(\ell^1)} = \| C^{(m)}(t, s) \|_1 \) holds for \( (t, s) \in S_h \). Furthermore, using the form \( C^{(m)}(t_1, s) = \Phi(t_1)\Psi(s) \) defined in (3.29), the property (2.6) and (3.27), we have

\[
\| b_s^{(m)}(t_1) \|_{\ell^1} \leq \| \Phi(t_1) \|_1 \left( \| \Psi(s) \|_1 \| \phi^{(m)} \|_{\ell^1} + 2 \int_{t}^{t_1} \| \Psi(\tau) \|_1 \| \tilde{a}(\tau) \|_{\ell^1} \| b_s^{(\infty)}(\tau) \|_{\ell^1} d\tau \right)
\]

\[
\leq \| \Phi(t_1) \|_1 \left( \| \Psi(s) \|_1 \| \phi^{(m)} \|_{\ell^1} + 2h_1 \sup_{s \in J_1} \| \Psi(\tau) \|_1 \| \tilde{a}(\tau) \|_{\ell^1} \right)
\]

\[
\leq \| \Phi(t_1) \|_1 \left( \| \Psi(s) \|_1 \| \phi^{(m)} \|_{\ell^1} + 2h_1 \sup_{s \in J_1} \| \Psi(\tau) \|_1 \| \tilde{a}(\tau) \|_{\ell^1} \right)
\]

\[
\times \left[ 2W_{m}W_{\infty}\| \tilde{a}(s) \|_{X}\kappa^{-1} \| \phi^{(m)} \|_{\ell^1} + \left( W_{\infty}^{\sup} + 4W_{m}W_{\infty}^{2}\| \tilde{a}(s) \|_{X}\kappa^{-1} \right) \| \phi^{(\infty)} \|_{\ell^1} \right].
\]

Taking the supremum norm with respect to \( s \), it follows

\[
\sup_{s \in J_1} \left( b_s^{(m)}(t_1) \right)_{\ell^1} \leq \| \Phi(t_1) \|_1 \sup_{s \in J_1} \| \Psi(s) \|_1 \left( 1 + 4h_1W_{m}W_{\infty}\| \tilde{a}(s) \|_{X}\kappa^{-1} \right) \| \phi^{(m)} \|_{\ell^1}
\]

\[
+ 2 \| \Phi(t_1) \|_1 h_1 \sup_{s \in J_1} \| \Psi(s) \|_1 \| \tilde{a}(s) \|_{X} \left( W_{\infty}^{\sup} + 4W_{m}W_{\infty}^{2}\| \tilde{a}(s) \|_{X}\kappa^{-1} \right) \| \phi^{(\infty)} \|_{\ell^1}.
\]

(5.2)

Let

\[
W_{J_1}^{(1,1)} \overset{\text{def}}{=} \| \Phi(t_1) \|_1 \sup_{s \in J_1} \| \Psi(s) \|_1 \left( 1 + 4h_1W_{m}W_{\infty}\| \tilde{a}(s) \|_{X}\kappa^{-1} \right)
\]

and

\[
W_{J_1}^{(1,2)} \overset{\text{def}}{=} 2 \| \Phi(t_1) \|_1 h_1 \sup_{s \in J_1} \| \Psi(s) \|_1 \| \tilde{a}(s) \|_{X} \left( W_{\infty}^{\sup} + 4W_{m}W_{\infty}^{2}\| \tilde{a}(s) \|_{X}\kappa^{-1} \right).
\]

\( \square \) Springer
Next, the tail mode is given by plugging $t = t_1$ in (3.23)

$$b_s(\infty)(t_1) = \tilde{U}(\infty)(t_1, s)\phi(\infty) + 2ei\theta \int_s^{t_1} \tilde{U}(\infty)(t_1, \tau)(\text{Id} - \pi^{(m)}) (\bar{a}(\tau) * b_s^{(m)}(\tau)) \, d\tau.$$ 

Using the bound (3.56), the property (2.6) and (3.26), we also have

$$\left\| b_s(\infty)(t_1) \right\|_{\ell^1} \leq W^{(\infty)}(t_1, s) \left\| \phi(\infty) \right\|_{\ell^1} + 2 \int_s^{t_1} W^{(\infty)}(t_1, \tau) \left\| \bar{a}(s)(\tau) \right\|_{\ell^1} \left\| b_s^{(m)}(\tau) \right\|_{\ell^1} \, d\tau \leq W^{(\infty)}(t_1, s) \left\| \phi(\infty) \right\|_{\ell^1} + 2 W_{\infty} \left\| \bar{a}(s) \right\|_{X} \left\| b_s^{(m)} \right\|_{X} \leq W^{(\infty)}(t_1, s) \left\| \phi(\infty) \right\|_{\ell^1} + 2 W_{\infty} \left\| \bar{a}(s) \right\|_{X} \left\| \phi(\infty) \right\|_{\ell^1}.$$

Taking the supremum norm with respect to $s$, it follows

$$\sup_{s \in J_1} \left\| b_s(\infty)(t_1) \right\|_{\ell^1} \leq \left( 2 W_{\infty} \left\| \bar{a}(s) \right\|_{X} \kappa^{-1} \right) \left\| \phi(\infty) \right\|_{\ell^1} + \left( W^{\sup} + 4 W_{\infty} \left\| \bar{a}(s) \right\|_{X} \kappa^{-1} \right) \left\| \phi(\infty) \right\|_{\ell^1}.$$  \hfill (5.3)

Moreover, let us define

$$W_{J_1}^{(2,1)} \overset{\text{def}}{=} 2 W_{m} W_{\infty} \left\| \bar{a}(s) \right\|_{X} \kappa^{-1} \quad \text{and} \quad W_{J_1}^{(2,2)} \overset{\text{def}}{=} W^{\sup} + 4 W_{m} W_{\infty} \left\| \bar{a}(s) \right\|_{X} \kappa^{-1}.$$ 

Summing up (5.2) and (5.3), we get the $W_{J_1}$ bound

$$\sup_{s \in J_1} \left\| U(t_1, s)\phi \right\|_{\ell^1} \leq \sup_{s \in J_1} \left\| b_s^{(m)}(t_1) \right\|_{\ell^1} + \sup_{s \in J_1} \left\| b_s(\infty)(t_1) \right\|_{\ell^1} \leq \left\| W_{J_1}^{(1,1)} \right\|_{1} \left\| \phi^{(m)} \right\|_{\ell^1} + \left\| W_{J_1}^{(2,1)} \right\|_{1} \left\| \phi(\infty) \right\|_{\ell^1} \leq W_{J_1} \left\| \phi \right\|_{\ell^1}, \quad \forall \phi \in \ell^1,$$

where

$$W_{J_1} \overset{\text{def}}{=} \left\| W_{J_1}^{(1,1)} \right\|_{1} \left\| W_{J_1}^{(2,1)} \right\|_{1}.$$
Setting $s = t_1$, (5.2) and (5.3) are changed by

$$
\left\| b_{h_0}^{(m)} (t_1) \right\|_{\ell^1} \leq \left\| \Phi(t_1) \right\|_1 \left( 1 + 4 h_1 \sup_{s \in J_1} \left\| \Psi(s) \right\|_1 W_m W_{\infty} \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right) \left\| \phi^{(m)} \right\|_{\ell^1}
+ 2 \left\| \Phi(t_1) \right\|_1 h_1 \left\| \bar{a}^{(s)} \right\|_X \left( W_{\infty}^{\sup} + 4 W_m W_{\infty}^2 \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right) \left\| \phi^{(\infty)} \right\|_{\ell^1}.
$$

and

$$
\left\| b_{h_0}^{(\infty)} (t_1) \right\|_{\ell^1} \leq \left( 2 W_m W_{\infty} \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right) \left\| \phi^{(m)} \right\|_{\ell^1}
+ \left( W^{(\infty)}(t_1, t_0) + 4 W_m W_{\infty}^2 \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right) \left\| \phi^{(\infty)} \right\|_{\ell^1},
$$

respectively. The analogous discussion yields the $W_{t_1}$ bound by

$$
W_{t_1} \overset{\text{def}}{=} \left\| \begin{bmatrix}
W_{t_1}^{(1,1)} & W_{t_1}^{(1,2)} \\
W_{t_1}^{(2,1)} & W_{t_1}^{(2,2)}
\end{bmatrix} \right\|_1,
$$

where

$$
W_{t_1}^{(1,1)} \overset{\text{def}}{=} \left\| \Phi(t_1) \right\|_1 \left( 1 + 4 h_1 \sup_{s \in J_1} \left\| \Psi(s) \right\|_1 W_m W_{\infty} \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right),
$$

$$
W_{t_1}^{(1,2)} \overset{\text{def}}{=} 2 \left\| \Phi(t_1) \right\|_1 h_1 \left\| \bar{a}^{(s)} \right\|_X \left( W_{\infty}^{\sup} + 4 W_m W_{\infty}^2 \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1} \right),
$$

$$
W_{t_1}^{(2,1)} \overset{\text{def}}{=} 2 W_m W_{\infty} \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1}, \quad \text{and} \quad W_{t_1}^{(2,2)} \overset{\text{def}}{=} W^{(\infty)}(t_1, t_0) + 4 W_m W_{\infty}^2 \left\| \bar{a}^{(s)} \right\|_X \kappa^{-1}.
$$

Consequently, the point-wise error is estimated by

$$
\left\| z(t_1) \right\|_{\ell^1} \leq \left\| U(t_1, t_0) z(t_0) \right\|_{\ell^1} + \int_{t_0}^{t_1} \left\| U(t_1, s) g(s) \right\|_{\ell^1} d\tau
\leq W_{t_1} \varepsilon_0 + W_{J_1} h_1 \left( 2 \delta_1^2 + \delta_1 \right) = \varepsilon_1.
$$

For the next time step, updating $\varepsilon \equiv \varepsilon_1$, we apply Theorem 4.1 on $J_2$ and derive the point-wise error $\varepsilon_2$ at the endpoint recursively. By repeating the time stepping, we have $\varepsilon_i = W_i \varepsilon_{i-1} + W_{J_i} h_i \left( 2 \delta_i^2 + \delta_i \right)$ ($i = 1, 2, \ldots, K$).

**Remark 5.1** The point-wise error $\varepsilon_i$ can be smaller than the previous one, i.e., $\varepsilon_i \leq \varepsilon_{i-1}$ holds when $W_{J_i} < 1$ with sufficiently small $\delta_i$, $i = 1, 2, \ldots, K$.

Furthermore, we note that the tiny numerical error may include at the end point. That is, the value of approximate solution at $t = t_1$ in $J_1$, say $\bar{a}^{J_1}(t_1)$, and that in $J_2$, say $\bar{a}^{J_2}(t_1)$, may be different. It is because the Chebyshev polynomial approximates...
a function globally in each time interval. In such a case, we should take care of the numerical error by using the following form:

$$
\epsilon_1 \defeq \left\| \bar{a}^{J_1}(t_1) - \bar{a}^{J_2}(t_1) \right\|_{\ell^1} = \sum_{|k| \leq N} \left| \bar{a}^1_{0,k} + 2\bar{a}^1_{1,k} + 2\bar{a}^1_{2,k} + \cdots + 2\bar{a}^1_{n-1,k} 
- (\bar{a}^2_{0,k} - 2\bar{a}^2_{1,k} + 2\bar{a}^2_{2,k} - \cdots + (-1)^{n-1}2\bar{a}^2_{n-1,k}) \right|.
$$

Hence, we should add $\epsilon_1$ in the point-wise error, i.e., $\epsilon_1 = W_{t_1}\epsilon_0 + W_{J_1}h_1 (2\varrho_1 + \delta_1) + \epsilon_1$.

6 Global existence in time

After several time steppings, we prove global existence of the solution. Our approach is based on a calculation of a center-stable manifold arising from the Cauchy problem (2.3).

6.1 Calculating part of a center-stable manifold

Starting from the PDE (2.1) we consider the system of differential equations in $\ell^1$ given by (2.3) In this section, we use the Lyapunov–Perron method for computing a foliation of portion of the center-stable manifold of the equilibrium at $a \equiv 0$. A good reference for this method in ODEs is [3], and for PDEs see [23]. In [29] this method is applied to give computer-assisted proofs of the stable manifold theorem in the Swift-Hohenberg PDE.

Let us define subspaces

$$
X_c \defeq \{ a \in \ell^1 | a_k = 0 \ \forall k \neq 0 \} \\
X_s \defeq \{ a \in \ell^1 | a_0 = 0 \}.
$$

We may then rewrite (2.3) as the following system:

$$
\dot{x}_c = N_c(x_c, x_s) \tag{6.1} \\
\dot{x}_s = \mathcal{L}x_s + N_s(x_c, x_s), \tag{6.2}
$$

where for $a = (x_c, x_s)$ we define:

$$
(\Sigma a)_k \defeq -e^{i\theta}k^2\omega^2a_k, \quad N_c(a_c, a_s) \defeq e^{i\theta} \sum_{k=-\infty}^{\infty} a_k a_{-k},
$$

$$
(N_s(a_c, a_s))_k \defeq e^{i\theta} \sum_{k_1+k_2=k, k_1, k_2 \in \mathbb{Z}} a_{k_1} a_{k_2}.
$$
Using the convolution product, the nonlinearities may be written as:

\[
\mathcal{N}_c(x_c, x_s) = e^{i\theta}(x_c^2 + (x_s * x_s)_0), \quad \mathcal{N}_s(x_c, x_s)_k = e^{i\theta}(2x_c x_s + x_s * x_s)_k.
\]

Note \(\|e^{t\mathcal{L}}\|_{B(\ell^1)} \leq e^{-\omega^2 \cos \theta t}, |\mathcal{N}_c(x_c, x_s)| \leq |x_c|^2 + \|x_s\|^2_{\ell^1}\) and

\[
\|\mathcal{N}_s(x_c, x_s)\|_{\ell^1} \leq 2|x_c|\|x_s\|_{\ell^1} + \|x_s\|^2_{\ell^1}.
\] (6.3)

To abbreviate, let us define \(\mu = \omega^2 \cos \theta\).

One may observe that the center subspace \(X_c\) is in fact invariant with respect to the nonlinear dynamics; \(X_c\) is a center manifold of the zero equilibrium. We can solve (6.1) restricted to this subspace:

\[
\dot{x}_c = e^{i\theta}x_c^2,
\] (6.4)

using the fact that the differential equation is separable. For an initial condition \(\phi \in \mathbb{C}\) then the solution of (6.4) is given by:

\[
\Phi(t, \phi) \overset{\text{def}}{=} \frac{\phi}{1 - \phi e^{i\theta}}.
\]

Most solutions limit to zero in forward and backwards time however if \(\text{Re}(\phi e^{i\theta}) > 0\) and \(\text{Im}(\phi e^{i\theta}) = 0\), then the solution \(\Phi(t, \phi)\) blows up in finite time.

For \(r_c, r_s > 0\) let us define the following sets

\[
B_c(r_c) = \{x_c \in X_c : |x_c| \leq r_c, \text{Re}(e^{i\theta} x_c) \leq 0\}
\]

\[
B_s(r_s) = \{x_s \in X_s : \|x_s\|_{\ell^1} \leq r_s\}.
\]

Note that if \(\phi \in B_c(r_c)\) then \(|\phi| \leq r_c\) and \(\text{Re}(e^{i\theta} \Phi(t, \phi)) \leq 0\) for \(t \geq 0\). Moreover \(|\Phi(t, \phi)| \leq r_c\) and \(\text{Re}(e^{i\theta} \Phi(t, \phi)) \leq 0\) for \(t \geq 0\), hence \(B_c(r_c)\) is a forward invariant set (see Fig. 2). We aim to locally characterize the center-stable manifold at 0 as a Lipschitz graph over \(B_c\). To that end we make the following definition.

**Definition 6.1** Fix \(\rho, \kappa, r_c, r_s > 0\). Write \(B_c = B_c(r_c)\) and \(B_s = B_s(r_s)\), and let us consider functions \(\tilde{\alpha} \in \text{Lip}(B_c \times B_s, X_c)\) satisfying the inequalities

\[
|\tilde{\alpha}(\phi_1, \xi_1) - \tilde{\alpha}(\phi_2, \xi_2)| \leq \rho \|\xi_1 - \xi_2\| \quad (6.5)
\]

\[
|\tilde{\alpha}(\phi_1, \xi_1) - \tilde{\alpha}(\phi_2, \xi_1)| \leq \kappa \|\phi_1 - \phi_2\| \quad (6.6)
\]

for all \(\phi_1, \phi_2 \in B_c\) and \(\xi_1, \xi_2 \in B_s\). Define the following sets of functions:

\[
\tilde{\mathcal{B}}_{\rho, \kappa} \overset{\text{def}}{=} \{\tilde{\alpha} \in \text{Lip}(B_c \times B_s, X_c) : (6.5) \text{ and } (6.6) \text{ hold, and } \tilde{\alpha}(\phi, 0) = 0 \ \forall \phi \in B_c\},
\]

\[
\mathcal{B}_{\rho, \kappa} \overset{\text{def}}{=} \{\alpha \in \text{Lip}(B_c \times B_s, X_c) : \alpha(\phi, \xi) = \phi + \tilde{\alpha}(\phi, \xi), \ \text{for some } \tilde{\alpha} \in \tilde{\mathcal{B}}\}.
\]
For short, we will often write $\tilde{B} \equiv \tilde{B}_{\rho, \kappa}$ and $B \equiv B_{\rho, \kappa}$. Also note that if $\alpha \in B$, then $\alpha$ satisfies (6.5) for all $\phi \in B_c$ and $\xi_1, \xi_2 \in B_s$.

Continuing with the Lyapunov–Perron method, for a fixed $\alpha \in B, \phi \in B_c(r_c), \xi \in B_s(r_s)$, we define $x(t, \phi, \xi, \alpha)$ as a solution with initial condition $x(0, \phi, \xi, \alpha) = \xi$ of the differential equation on $X_s$ below:

$$\dot{x}_s = \mathcal{L}x_s + \mathcal{N}_s(\alpha(\Phi(t, \phi), x_s), x_s).$$

(6.7)

One may see that (6.7) is no more than (6.2) with $x_c$ replaced by the nonautonomous term $\alpha(\Phi(t, \phi), x_s)$. If $\mathcal{L}$ sufficiently dominates the nonlinearity $\mathcal{N}_s$, then $\|x(t, \phi, \xi, \alpha)\|$ decreases exponentially as $t \to +\infty$, and consequently $\|\Phi(t, \phi) - \alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha))\|$ approaches zero exponentially fast, see Fig. 2.

Now we define the Lyapunov–Perron Operator for $\alpha \in B$ as follows:

$$\Psi[\alpha](\phi, \xi) = -\int_{0}^{\infty} \mathcal{N}_c(\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha)), x(t, \phi, \xi, \alpha)) \, dt. \quad (6.8)$$

In Proposition 6.12, we show that $\Psi : B \to B$ is a well defined operator. As is the case for Lyapunov–Perron operators, from the variation of constants formula, if $\Psi[\alpha] = \alpha$, then for all $(\phi, \xi) \in B_c \times B_s$, the trajectory $\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha))$ satisfies (6.1). Hence $(\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha)), x(t, \phi, \xi, \alpha))$ satisfies our original equation (2.3).

Since $\Phi(t, \phi)$ limits to zero, such a fixed point $\alpha = \Psi[\alpha]$ gives us a foliation of the center stable manifold over $B_c(r)$. By Corollary 6.4 we obtain an explicit neighborhood within which all points limit to the zero equilibrium. To prove the existence of a fixed point to our Lyapunov–Perron operator, and obtain explicit bounds, we prove the following theorem.

---

Fig. 2 (Left) A phase diagram of the center dynamics in (6.4), for $\theta = \frac{\pi}{4}$, and the forward invariant half disk $B_c(r_c)$. (Right) A schematic of an enclosure of $\alpha(\Phi(t, \phi), x(t, \phi, \xi, \alpha))$ for all $\alpha \in B$ and several initial conditions $(\phi, \xi) \in B_c(r_c) \times B_s(r_s)$.
Theorem 6.2  Fix $r_c, r_s, \rho > 0$ and define the following constants

\[
\begin{align*}
\delta_1 & \overset{\text{def}}{=} 2r_c + (1 + 2\rho)r_s, \\
\delta_2 & \overset{\text{def}}{=} 2r_c + 2(1 + 2\rho)r_s, \\
\delta_3 & \overset{\text{def}}{=} 2(\rho(r_c + r_s) + r_c), \\
\epsilon_1 & \overset{\text{def}}{=} \frac{2}{\mu - \delta_1} \left( r_c + \frac{1}{2} \rho r_s + \frac{2r_c r_s \rho}{\mu - \delta_2} + \frac{r_s^2 (1 + \rho^2)}{2\mu - \delta_1 - \delta_2} \right), \\
\lambda & \overset{\text{def}}{=} \frac{2(\rho(r_c + r_s) + r_c) r_s}{(\mu - \delta_1)(\mu - \delta_2)} + \frac{2(2r_c + \rho r_s)}{\mu - \delta_1}.
\end{align*}
\]

If the following inequalities are satisfied

\[
\delta_1, \delta_2 < \mu, \quad \frac{\delta_3}{\mu - \delta_2} < \rho, \quad \epsilon_1 < 1, \quad \lambda < 1, \quad (6.9)
\]

then there is a unique map $\alpha \in B_{\rho, \kappa_0}$ (for some $\kappa_0 > 0$) such that $\Psi[\alpha] = \alpha$.

The proof is given in Sect. 6.2.

Corollary 6.3  Suppose the hypothesis of Theorem 6.2 is satisfied and $\hat{\alpha} \in B$ is such that $\Psi(\hat{\alpha}) = \hat{\alpha}$. Fix $(\phi, \xi) \in B_c(r_c) \times B_s(r_s)$. Let $X(t)$ denote the solution to (2.3) with initial conditions $(\hat{\alpha}(\phi, \xi), \xi)$. Then $\lim_{t \to \infty} X(t) = 0$ and moreover

\[
\|X(t) - (\Phi(t, \phi), 0)\| \leq e^{-(\mu - \delta_1)t} (1 + \rho)|\xi|.
\]

The proof largely follows from Proposition 6.8.

Corollary 6.4  Fix $\rho, r_c, r_s > 0$ and write $B_c = B_c(r_c)$ and $B_s = B_s(r_s)$. Define the set $U \subseteq B_c \times B_s$ as follows:

\[
U \overset{\text{def}}{=} \{(x_c, x_s) \in B_c \times B_s : \rho \|x_s\|_{\ell_1} \leq \text{dist}(x_c, \partial B_c)\},
\]

where $\partial B_c$ denotes the boundary of $B_c$. Suppose the hypothesis of Theorem 6.2 is satisfied and there exists some $\alpha \in B$ such that $\Psi[\alpha] = \alpha$. Then $U$ is contained in the $\alpha$-skew image of $B_c \times B_s$; that is to say $U \subseteq \{(\alpha(x_c, x_s), x_s) : (x_c, x_s) \in B_c \times B_s\}$. Furthermore, $\lim_{t \to \infty} a(t) = 0$ under the differential equation (2.3) for all $a(0) \in U$.

Proof  Fix $(\phi_0, \xi) \in U \subseteq B_c \times B_s$. We wish to show there exists some $\phi \in B_c$ such that $\alpha(\phi, \xi) = \phi_0$. Let us define

\[
\tau_0 = \sup\{\tau \in [0, 1] : \emptyset \neq \alpha^{-1}(\phi_0, \tau \xi) \in \text{int}(B_c)\}.
\]

To show $U$ is contained in the $\alpha$-skew image of $B_c \times B_s$, it suffices to show that $\tau_0 = 1$. Since $\alpha(\phi, 0) = \phi$ for all $\phi \in B_c$, then $\tau_0 > 0$.

From Corollary 6.3 it follows that for each fixed $\xi \in B_s$ the map $\alpha(\cdot, \xi) : B_c \to B_c$ is injective. By the invariance of domain theorem, it follows that $\alpha(\cdot, \xi)$
is a homeomorphism from \( \text{int}(B_c) \) onto its image. We may then define a homotopy \( H : \text{int}(B_c) \times [0, 1] \rightarrow \mathbb{C} \times [0, 1] \) by \( H(\phi, \tau) = (\alpha(\phi, \tau \xi), \tau) \), which is also a homeomorphism from \( \text{int}(B_c) \times [0, 1] \) onto its image. Hence \( H^{-1}(\phi_0, (0, \tau_0)) = \{(\alpha^{-1}(\phi_0, \tau \xi), \tau) \in \text{int}(B_c) \times [0, 1] : \tau \in (0, \tau_0) \} \) is a continuous curve, and we may define a function \( \Gamma : (0, \tau_0) \rightarrow \text{int}(B_c) \) by \( \Gamma(\tau) \stackrel{\text{def}}{=} H^{-1}(\phi_0, \tau \xi) \). Fix \( \phi_1 \stackrel{\text{def}}{=} \lim_{\tau \rightarrow \tau_0} \Gamma(\tau) \in B_c \).

By way of contradiction, let us suppose that \( \tau_0 < 1 \). If \( \phi_1 \in \text{int}(B_c) \), then \( H \) maps an open set about \( (\phi_1, \tau_0) \) onto an open set about \( (\phi_0, \tau_0) \). Hence the initial choice of \( \tau_0 \) as the supremum above was incorrect, and \( \tau_0 \) could have been made even larger. Otherwise suppose \( \phi_1 \in \partial B_c \). But this is contradicted by our definition of \( U \), as seen by the following calculation:

\[
0 = |\phi_0 - \alpha(\phi_1, \tau_0 \xi)| \geq \inf_{\phi \in \partial B_c} |\phi_0 - \alpha(\phi, \tau_0 \xi)| \geq \text{dist}(\phi_0, \partial B_c) - \rho \tau_0 \|\xi\|_1 > 0.
\]

Hence \( \tau_0 = 1 \), and there is some \( \phi \in B_c \) such that \( \alpha(\phi, \xi) = \phi_0 \). Thereby \( U \) is contained in the \( \alpha \)-skew image of \( B_c \times B_s \).

Furthermore, if \( a(0) \in U \), then \( a(0) \) is in the \( \alpha \)-skew image of \( B_c \times B_s \). That is to say \( a(0) \) is in the center stable manifold of the equilibrium \( 0 \); \( \lim_{t \rightarrow \infty} a(t) = 0 \).

**Remark 6.5** By isolating the \( \rho \) terms in the inequality \( \frac{\delta_1}{\mu - \rho^2} < \rho \), we obtain the following:

\[
(6r_s)\rho^2 - (\mu - 4r_c - 2r_s) \rho + 2r_s < 0.
\]

Hence, we need at a minimum \( \mu > 4r_c + 2r_s \) in order to satisfy the hypotheses of Theorem 6.2. Using the quadratic formula, we obtain a necessary inequality,

\[
0 < \frac{\mu - 4r_c - 2r_s}{12r_s} - \frac{1}{2} \sqrt{\left(\frac{\mu - 4r_c - 2r_s}{6r_s}\right)^2 - \frac{4}{3}} < \rho.
\]

We can use this to define a nearly optimal \( \rho \) in terms of \( r_c, r_s \) and \( \mu \).

**Remark 6.6** For a given \( \theta \), there is something to be said about how to select \( r_c \) and \( r_s \), so that the hypothesis of Theorem 6.2 is satisfied. For the best bounds, \( \rho \) should be taken as small as possible, and nearly optimal bounds are explicitly given in Remark 6.5. Furthermore, we necessarily need to take \( r_c < \frac{1}{4} \omega^2 \cos \theta \). Below are some constants which will satisfy Theorem 6.2 and try to maximize \( r_c \).

| \( \theta \) | \( \frac{1}{4} \omega^2 \cos \theta \) | \( r_c \) | \( r_s \) | \( \rho \) | \( \epsilon_1 \) | \( \lambda \) |
|---|---|---|---|---|---|---|
| 0 | 9.87 | 9.770.010.060.990.99 | | | | |
| \( \pi/8 \) | 9.12 | 9.020.010.060.980.98 | | | | |
| \( \pi/4 \) | 6.98 | 6.880.010.060.980.98 | | | | |
| 3\( \pi/8 \) | 3.78 | 3.680.010.060.960.96 | | | | |

\( \odot \) Springer
6.2 Proof of Theorem 6.2

The proof of Theorem 6.2 is organized as follows. First in Propositions 6.8–6.10 we derive bounds on the norm of solutions to (6.7) and their dependence on initial conditions. In Proposition 6.12, we show that $\Psi : B \to B$ is a well defined endomorphism. In Proposition 6.14 we introduce a norm $\| \cdot \|_{\tilde{E}}$ on $\tilde{B}$. In Proposition 6.15, we show that if two maps $\alpha, \beta \in B$ are such that $\| \alpha - \beta \|_{\tilde{E}}$ is small, then their solutions to (6.7), taken with the same initial conditions but different maps $\alpha$ and $\beta$, will also remain close. Finally, we finish the proof of Theorem 6.2 by bounding the Lipschitz constant of $\Psi$ using this new norm. If this Lipschitz constant is less than 1, then $\Psi$ is a contraction mapping, and hence it will have a unique fixed point. Throughout, we will often make use of the following Grönwall type lemma below [29].

Lemma 6.7 Fix constants $c_0, c_1, c_2, c_3 \in \mathbb{R}$ with $c_1, c_2, c_3 \geq 0$. Define $\mu_0 = c_0 + c_2$ and fix $\mu_1 \in \mathbb{R}$ such that $\mu_0 \neq \mu_1$. If we have the inequality:

$$e^{-c_0 t} u_0(t) \leq \left( c_1 + \int_0^t e^{-c_0 \tau} c_3 \, d\tau \right) + \int_0^t c_2 e^{-c_0 s} u_0(s) \, ds,$$

then:

$$u_0(t) \leq c_1 e^{\mu_0 t} + c_3 \frac{e^{\mu_1 t} - e^{\mu_0 t}}{\mu_1 - \mu_0}. \quad (6.10)$$

Proposition 6.8 Fix $\alpha \in B$, $\phi \in B_c(r_c)$, $\xi \in B_s(r_s)$, and define

$$\delta_1 = 2r_c + (1 + 2\rho)r_s,$$

where $\rho > 0$ is in the definition of $B$. If $\delta_1 < \mu$ then:

$$\| x(t, \phi, \xi, \alpha) \|_{\ell^1} \leq e^{-\mu (\mu - \delta_1) t} \| \xi \|_{\ell^1} \quad \text{for all } t \geq 0.$$

Furthermore, $x(t, \phi, \xi, \alpha) \in B_s(r_s)$ for all $t \geq 0$.

Proof Let us abbreviate $x_x = x_x(t) = x(t, \phi, \xi, \alpha)$ and let us write $\alpha(\phi, \xi) = \phi + \tilde{\alpha}(\phi, \xi)$ for $\tilde{\alpha} \in \tilde{B}$. By variation of constants we have

$$\| x_x(t) \|_{\ell^1} \leq e^{-\mu t} \| \xi \| + \int_0^t e^{-\mu (t - \tau)} \| N_x(\alpha(\Phi(\tau, \phi), x_x(\tau)), x_x(\tau)) \|_{\ell^1} \, d\tau.$$

Since $|\Phi(t, \phi)| \leq r_c$ for $t \geq 0$, then by (6.5) we obtain:

$$|\alpha(\Phi(t, \phi), x_x)| = |\Phi(t, \phi) + (\tilde{\alpha}(\Phi(t, \phi), x_x) - \tilde{\alpha}(\Phi(t, \phi), 0))|$$

$$\leq r_c + \rho \| x_x \|. \quad (6.11)$$
Let us now assume that \( \|\xi\| < r_s \), by which there exists some positive \( T \equiv \sup\{t > 0 : \|x_s(t)\| \leq r_s\} \). Hence \( |\alpha(\Phi(t, \phi), x_s)| \leq r_c + \rho r_s \) for \( t \in (0, T) \). Using the estimate in (6.3) we obtain

\[
\|N_s(\alpha(\Phi(t, \phi), x_s), x_s)\|_{\ell^1} \leq 2(r_c + \rho r_s)\|x_s\|_{\ell^1} + \|x_s\|^2_{\ell^1}
\]

\[
\leq (2r_c + (1 + 2\rho)r_s)\|x_s\|_{\ell^1}
\]

\[
= \delta_1\|x_s\|_{\ell^1},
\]

for all \( t \in [0, T) \). Returning to our variation of constants formula, we have:

\[
e^{\mu t}x_s(t) \leq \|\xi\|_{\ell^1} + \int_0^t e^{\mu(\tau)}\delta_1\|x_s(\tau)\|_{\ell^1}d\tau.
\]

From Grönwall’s inequality, we obtain \( \|x_s(t)\| \leq e^{-(\mu - \delta_1)^t}\|\xi\| \) for all \( t \in (0, T) \). From this we may conclude that in fact \( T = +\infty \). By continuity the inequality also extends to the case \( \|\xi\| = r_s \).  

\( \square \)

**Proposition 6.9** Fix \( \alpha \in \mathcal{B}, \phi \in B_c(r_c) \), and \( \xi, \zeta \in B_s(r_s) \), and define

\( \delta_2 = 2r_c + 2(1 + 2\rho)r_s \).

If \( \delta_2 < \mu \) then

\[
\|x(t, \phi, \xi, \alpha) - x(t, \phi, \zeta, \alpha)\|_{\ell^1} \leq e^{-(\mu - \delta_2)t}\|\xi - \zeta\|_{\ell^1}.
\]

**Proof** Let us abbreviate \( x_s = x_s(t) = x(t, \phi, \xi, \alpha) \) and \( z_s = z_s(t) = x(t, \phi, \zeta, \alpha) \). Additionally, let us abbreviate \( x_c = \alpha(\Phi(t, \phi), x_s(t)) \) and \( z_c = \alpha(\Phi(t, \phi), z_s(t)) \). By variation of constants we have

\[
\|x_s(t) - z_s(t)\|_{\ell^1} \leq e^{-\mu t}\|\xi - \zeta\|_{\ell^1} + \int_0^t e^{-\mu(t-\tau)}\|N_s(x_c, x_s) - N_s(z_c, z_s)\|_{\ell^1}d\tau.
\]

Using the notational abbreviation \( x_s^2 = x_s * x_s \) for \( x_s \in X_s \), we bound the norm of the difference in the integrand below

\[
\|N_s(x_c, x_s) - N_s(z_c, z_s)\|_{\ell^1} \leq \|N_s(x_c, x_s) - N_s(x_c, z_s)\|_{\ell^1}
\]

\[
+ \|N_s(x_c, z_s) - N_s(z_c, z_s)\|_{\ell^1}
\]

\[
\leq 2\|x_c(x_s - z_s) + x_s^2 - z_s^2\|_{\ell^1} + \|2(x_c - z_c)z_s\|_{\ell^1}
\]

\[
\leq 2|x_c|\|x_s - z_s\|_{\ell^1} + \|x_s^2 - z_s^2\|_{\ell^1}
\]

\[
+ 2|x_c - z_c|\|z_s\|_{\ell^1}.
\]

(6.12)

By Proposition 6.8 then \( \|x_s\|, \|z_s\| \leq r_s \) for all \( t \geq 0 \). So from (6.11) we obtain \( |x_c| \leq r_c + \rho r_s \). By the Banach algebra property of \( * \) we have

\[
\|x_s^2 - z_s^2\|_{\ell^1} = \|(x_s + z_s) * (x_s - z_s)\|_{\ell^1} \leq \|x_s + z_s\|_{\ell^1}\|x_s - z_s\|_{\ell^1},
\]
and thereby we have \( \|x_s^2 - z_s^2\|_{\ell^1} \leq 2r_s\|x_s - z_s\|_{\ell^1} \) for all \( t \geq 0 \). Also note that by (6.5) then
\[
|x_c - z_c| = |\alpha(\Phi(t, \phi), x_s) - \alpha(\Phi(t, \phi), z_s)| \leq \rho\|x_s - z_s\|.
\]

Plugging all of this into (6.12) we obtain
\[
\|\mathcal{N}_s(x_c, x_s) - \mathcal{N}_s(z_c, z_s)\|_{\ell^1} \leq 2(r_c + \rho r_s)\|x_s - z_s\|_{\ell^1}
+ 2r_s\|x_s - z_s\|_{\ell^1} + 2r_s\rho\|x_s - z_s\|_{\ell^1}
= \delta_2\|x_s - z_s\|_{\ell^1}.
\]

Returning to our variation of constants formula, we have:
\[
e^{\mu t}\|x_s(t) - z_s(t)\|_{\ell^1} \leq \|\xi - \xi\|_{\ell^1} + \int_0^t e^{\mu \tau}\delta_2\|x_s(\tau) - z_s(\tau)\|_{\ell^1}d\tau.
\]

The proposition follows from Grönwall's inequality. □

**Proposition 6.10** Fix \( \rho, \kappa > 0 \) and \( \alpha \in \mathcal{B}_{\rho, \kappa} \). Then for all \( \phi_1, \phi_2 \in B_{c}(r_c) \) and \( \xi \in B_{s}(r_s) \) we have that
\[
\|x(t, \phi_1, \xi, \alpha) - x(t, \phi_2, \xi, \alpha)\|_{\ell^1} \leq 2(1 + \kappa r_s)\|\xi\|\|\phi_1 - \phi_2\|e^{-\mu_1 t} - e^{-\mu_2 t}
- (\mu - \delta_1) + (\mu - \delta_2).
\]

**Proof** Let us define \( x_s = x_s(t) = x(t, \phi_1, \xi, \alpha) \) and \( w_s = w_s(t) = x(t, \phi_2, \xi, \alpha) \).

Let us define \( x_c = x_c(t) = \alpha(\Phi(t, \phi_1), x_s(t)) \) and define \( w_c = w_c(t) = \alpha(\Phi(t, \phi_2), w_s(t)) \). Also let us write \( \tilde{\alpha}(\phi, \xi) = \phi + \tilde{\alpha}(\phi, \xi) \) for \( \tilde{\alpha} \in \tilde{\mathcal{B}}_{\rho, \kappa} \).

Analogous to the computation in (6.12), we compute and find
\[
\|\mathcal{N}_s(x_c, x_s) - \mathcal{N}_s(w_c, w_s)\| \leq 2|x_c|\|x_s - w_s\|_{\ell^1} + \|x_s^2 - w_s^2\|_{\ell^1} + 2|c_s - w_s|\|w_s\|_{\ell^1}
\leq \|x_s - w_s\|_{\ell^1}(2r_c + 2pr_s + 2r_s) + 2|w_s|\|w_s\|_{\ell^1}|x_c - w_c|.
\]

Using (6.5), (6.6) and the triangle inequality, we estimate \( |x_c - w_c| \) below:
\[
|x_c - w_c| \leq |\alpha(\Phi(t, \phi_1), x_s(t)) - \alpha(\Phi(t, \phi_1), w_s(t))|
+ |\alpha(\Phi(t, \phi_1), w_s(t)) - \alpha(\Phi(t, \phi_2), w_s(t))|
\leq \rho|x_s - w_s| + |\Phi(t, \phi_1) - \Phi(t, \phi_2)|
+ |\tilde{\alpha}(\Phi(t, \phi_1), w_s(t)) - \tilde{\alpha}(\Phi(t, \phi_2), w_s(t))|
\leq \rho|x_s - w_s| + (1 + \kappa\|w_s\|)|\Phi(t, \phi_1) - \Phi(t, \phi_2)|.
\]
By our assumption $\text{Re}(e^{i\theta} \phi_1) \leq 0$ it follows that $|1 - \phi_1 t e^{i\theta}|^2 \geq 1 + |\phi_1 t|^2$ for $t \geq 0$. We bound $|\Phi(t, \phi_1) - \Phi(t, \phi_2)|$ below:

$$
|\Phi(t, \phi_1) - \Phi(t, \phi_2)| = \frac{|\phi_1 - \phi_2|}{|1 - \phi_1 t e^{i\theta}| \cdot |1 - \phi_2 t e^{i\theta}|} \leq \frac{|\phi_1 - \phi_2|}{\sqrt{1 + |\phi_1 t|^2} \sqrt{1 + |\phi_2 t|^2}} \leq |\phi_1 - \phi_2|.
$$

Combining these estimate we obtain:

$$
\|w_s\| \|x_c - w_c\| \leq r_s \rho |x_s - w_s| + \|w_s\|(1 + \kappa r_s) |\phi_1 - \phi_2|.
$$

So by Proposition 6.8 and the definition of $\delta_2$, we obtain the bound:

$$
\|N_s(x_c, x_s) - N_s(w_c, w_s)\|_{\ell^1} \leq 2(1 + \kappa r_s) \|w_s\|_{\ell^1} |\phi_1 - \phi_2|
+ 2(r_c + r_s + 2\rho r_s) \|x_s - w_s\|_{\ell^1} \leq 2(1 + \kappa r_s) \|\xi\|_{\ell^1} e^{-(\mu - \delta_1) t} |\phi_1 - \phi_2| + \delta_2 \|x_s - w_s\|_{\ell^1}.
$$

By variation of constants we obtain the following:

$$
\|x_s(t) - w_s(t)\|_{\ell^1} = \int_0^t e^{-\mu(t - \tau)} \|N_s(x_c, x_s) - N_s(w_c, w_s)\|_{\ell^1} d\tau
\leq \int_0^t e^{-\mu(t - \tau)} 2(1 + \kappa r_s) \|\xi\|_{\ell^1} e^{-(\mu - \delta_1) t} |\phi_1 - \phi_2| d\tau
+ \int_0^t e^{-\mu(t - \tau)} \delta_2 \|x_s(\tau) - w_s(\tau)\|_{\ell^1} d\tau.
$$

$$
e^{\mu t} \|x_s(t) - w_s(t)\|_{\ell^1} \leq \int_0^t e^{\mu t(1 + \kappa r_s)} \|\xi\|_{\ell^1} e^{-(\mu - \delta_1) t} |\phi_1 - \phi_2| d\tau
+ \int_0^t e^{\mu t} \delta_2 \|x_s(\tau) - w_s(\tau)\|_{\ell^1} d\tau.
$$

By Lemma 6.7 we have:

$$
\|x_s(t) - w_s(t)\|_{\ell^1} \leq 2(1 + \kappa r_s) \|\xi\|_{\ell^1} |\phi_1 - \phi_2| \frac{e^{-(\mu - \delta_1) t} - e^{-(\mu - \delta_2) t}}{-(\mu - \delta_1) + (\mu - \delta_2)}.
$$

\[\square\]

**Proposition 6.11** Define:

$$
\delta_3 = 2(\rho (r_c + \rho r_s) + r_s).
$$

(6.15)
If $\delta_2 < \mu$ and $\frac{\delta_1}{\mu - \delta_2} < \rho$, then

$$|\Psi[\alpha](\phi, \xi) - \Psi[\alpha](\phi, \zeta)| \leq \rho \|\xi - \zeta\|_{\ell^1}$$

for all $\phi \in B_c$ and all $\xi, \zeta \in B_s$.

**Proof** Fix some $\alpha \in B$ and $\phi \in B_c(r_c)$ and $\xi, \zeta \in B_s(r_s)$. We show that $|\Psi[\alpha](\phi, \xi) - \Psi[\alpha](\phi, \zeta)| \leq \rho \|\xi - \zeta\|_{\ell^1}$. Let us abbreviate $x_s = x_s(t) = x(t, \phi, \xi, \alpha)$ and $z_s = z_s(t) = x(t, \phi, \xi, \alpha)$. Additionally, let us abbreviate $x_c = \alpha(\Phi(t, \phi), x_s(t))$ and $z_c = \alpha(\Phi(t, \phi), z_s(t))$. We calculate:

$$|N_c(x_c, x_s) - N_c(z_c, z_s)| \leq |x_c^2 - z_c^2| + |(x_s + z_s) \ast (x_s - z_s)|_0$$

$$\leq 2(r_c + \rho r_s)|x_c - z_c| + 2r_s \|x_s - z_s\|_{\ell^1}$$

$$\leq \delta_3 \|x_s - z_s\|_{\ell^1}.$$

We calculate:

$$|\Psi[\alpha](\phi, \xi) - \Psi[\alpha](\phi, \zeta)| \leq \int_0^\infty |N_c(x_c, x_s) - N_c(z_c, z_s)| dt$$

$$\leq \int_0^\infty \delta_3 \|x_s - z_s\|_{\ell^1} dt$$

$$\leq \int_0^\infty \delta_3 e^{-(\mu - \delta_2)t} \|\xi - \zeta\|_{\ell^1} dt$$

$$= \frac{\delta_3}{\mu - \delta_2} \|\xi - \zeta\|_{\ell^1}.$$

By our assumption that $\frac{\delta_1}{\mu - \delta_2} < \rho$, it follows that $|\Psi[\alpha](\phi, \xi) - \Psi[\alpha](\phi, \zeta)| \leq \rho \|\xi - \zeta\|_{\ell^1}$. \boxed{}

**Proposition 6.12** Define:

$$\epsilon_1 = \frac{2}{\mu - \delta_1} \left( r_c + \frac{1}{2^3} \rho r_s + \frac{2r_c r_s \rho}{\mu - \delta_2} + \frac{r_s^2 (1 + \rho^2)}{2 \mu - \delta_1 - \delta_2} \right). \quad (6.16)$$

If $\delta_2 < \mu$ and $\frac{\delta_1}{\mu - \delta_2} < \rho$ and $\epsilon_1 < 1$, then there exists a $\kappa_0 > 0$ such that the map $\Psi : B_{\rho, \kappa_0} \to B_{\rho, \kappa_0}$ is a well defined endomorphism.

**Proof** Fix $\kappa, \rho > 0$ and $\alpha \in B_{\rho, \kappa}$. Fix $\tilde{\alpha} \in B_{\rho, \kappa}$ for which $\alpha(\phi, \xi) = \phi + \tilde{\alpha}(\phi, \xi)$. By the previous proposition we know that

$$|\Psi[\alpha](\phi, \xi) - \Psi[\alpha](\phi, \zeta)| \leq \rho \|\xi - \zeta\|_{\ell^1}$$

for all $\phi \in B_c$ and all $\xi, \zeta \in B_s$. We may also see that $\Psi[\alpha](\phi, 0) = \phi$; by direct calculation we obtain

$$\Psi[\alpha](\phi, 0) = -\int_0^\infty N_c(\alpha(\Phi(t, \phi), 0), 0) dt$$

$\Box$
Thus the difference in (6.17) can be expressed by the integral below, which we estimate

\[ \int_0^\infty \Phi(t, \phi)^2 dt \]

having used the fact that \( \Phi(t, \phi) \) is a solution to (6.4).

To prove the proposition we will show that there exist \( \epsilon_0, \epsilon_1 > 0 \), given explicitly in (6.24) and (6.16) respectively, such that

\[ |(\Psi[\alpha](\phi_1, \xi) - \phi_1) - (\Psi[\alpha](\phi_2, \xi) - \phi_2)| \leq (\epsilon_0 + \epsilon_1\kappa)|\xi||\phi_1 - \phi_2| \quad (6.17) \]

for all \( \phi_1, \phi_2 \in B_c(r_c) \) and \( \xi \in B_s(r_s) \). That is to say \( \Psi[\alpha] \in \tilde{B}_{\rho, \epsilon_0 + \epsilon_1\kappa} \), where we define \( \Psi[\alpha](\phi, \xi) = \Psi[\alpha](\phi, \xi) - \phi \). Moreover \( \Psi[\alpha] \in B_{\rho, \epsilon_0 + \epsilon_1\kappa} \), and if we let \( \kappa_0 \geq \frac{\epsilon_0}{1-\epsilon_1} \), then \( \Psi : B_{\rho, \kappa_0} \rightarrow B_{\rho, \kappa_0} \) is a well defined endomorphism.

Fix \( \phi_1, \phi_2 \in B_c(r_c) \) and \( \xi \in B_s(r_s) \). Let us define \( x_s = x_s(t) = x(t, \phi_1, \xi, \alpha) \) and \( w_s = w_s(t) = x(t, \phi_2, \xi, \alpha) \). Let us define \( x_c = x_c(t) = \alpha(\Phi(t, \phi_1), x_s(t)) \) and define \( w_c = w_c(t) = \alpha(\Phi(t, \phi_2), x_s(t)) \). Since \( \Phi(t, \phi) \) is a solution to (6.4) then \( \phi = - \int_0^\infty N_c(\Phi(t, \phi), 0) dt \). Moreover for all \( \phi \in B_c \) and \( \xi \in B_s \), we have

\[ \Psi[\alpha](\phi, \xi) - \phi = - \int_0^\infty N_c(\Phi(t, \phi) + \tilde{\alpha}(\Phi(t, \phi), x_s), x_s) - N_c(\Phi(t, \phi), 0) dt \]

\[ = -e^{i\theta} \int_0^\infty 2\Phi(t, \phi)\tilde{\alpha}(\Phi(t, \phi), x_s) + \tilde{\alpha}(\Phi(t, \phi), x_s)^2 + (x_s * x_s)_0 dt. \]

Thus the difference in (6.17) can be expressed by the integral below, which we estimate in three parts:

\[ (\Psi[\alpha](\phi_1, \xi) - \phi_1) - (\Psi[\alpha](\phi_2, \xi) - \phi_2) \]

\[ = -e^{i\theta} \int_0^\infty \overbrace{2\Phi(t, \phi_1)\tilde{\alpha}(\Phi(t, \phi_1), x_s) - 2\Phi(t, \phi_2)\tilde{\alpha}(\Phi(t, \phi_2), w_s)}^{III} \]

\[ + \tilde{\alpha}(\Phi(t, \phi_1), x_s)^2 - \tilde{\alpha}(\Phi(t, \phi_2), w_s)^2 + (x_s * x_s)_0 - (w_s * w_s)_0 dt. \]

**Part I** By the triangle inequality, the integrand in Part I can be bounded as follows

\[ |\Phi(t, \phi_1)\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \Phi(t, \phi_2)\tilde{\alpha}(\Phi(t, \phi_2), w_s)| \]

\[ \leq |\Phi(t, \phi_1)| |\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), w_s)| \]

\[ + |\Phi(t, \phi_1) - \Phi(t, \phi_2)| |\tilde{\alpha}(\Phi(t, \phi_2), w_s)|. \quad (6.18) \]

Since \( \Phi(t, \phi) \in B_c \) for all \( t \geq 0 \) and \( \phi \in B_c \), then \( |\Phi(t, \phi_1)| \leq r_c \) for all \( t \geq 0 \). To continue bounding the first summand we apply inequalities (6.5) and (6.6) as below:

\[ |\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), w_s)| \]

\[ \leq |\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), x_s)| + |\tilde{\alpha}(\Phi(t, \phi_2), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), w_s)| \]
\[ \leq \kappa |x_s| |\Phi(t, \phi_1) - \Phi(t, \phi_2)| + \rho |x_s - w_s| \]

To bound these terms, we use Proposition 6.8, inequality (6.14), and Proposition 6.10 as below:

\[
|\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), w_s)| \\
\leq \kappa e^{-(\mu - \delta_1)t} |\xi| |\phi_1 - \phi_2| + 2\rho (1 + \kappa r_s) |\xi| |\phi_1 - \phi_2| \frac{e^{-(\mu - \delta_1)t} - e^{-(\mu - \delta_2)t}}{-(\mu - \delta_1) + (\mu - \delta_2)} \\
= |\xi| |\phi_1 - \phi_2| \left( \kappa e^{-(\mu - \delta_1)t} + 2\rho (1 + \kappa r_s) \frac{e^{-(\mu - \delta_1)t} - e^{-(\mu - \delta_2)t}}{-(\mu - \delta_1) + (\mu - \delta_2)} \right). \tag{6.19}
\]

From the estimate in (6.14) we have \(|\Phi(t, \phi_1) - \Phi(t, \phi_2)| \leq |\phi_1 - \phi_2|\). We finish bounding the second summand in Part I using (6.5) and Proposition 6.8 as below:

\[
|\tilde{\alpha}(\Phi(t, \phi_2), w_s)| \leq \rho \|w_s\| \leq \rho e^{-(\mu - \delta_1)t} |\xi|. \tag{6.20}
\]

Combining all of these estimates into inequality (6.18), we obtain a bound for the integrand of Part I:

\[
|\Phi(t, \phi_1)\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \Phi(t, \phi_2)\tilde{\alpha}(\Phi(t, \phi_2), w_s)| \\
\leq |\xi| |\phi_1 - \phi_2| \left( r_c \kappa + \rho \right) e^{-(\mu - \delta_1)t} + 2r_c \rho (1 + \kappa r_s) \frac{e^{-(\mu - \delta_1)t} - e^{-(\mu - \delta_2)t}}{-(\mu - \delta_1) + (\mu - \delta_2)}
\]

Integrating we obtain our bound on Part I,

\[
\left| \int_0^\infty 2\Phi(t, \phi_1)\tilde{\alpha}(\Phi(t, \phi_1), x_s) - 2\Phi(t, \phi_2)\tilde{\alpha}(\Phi(t, \phi_2), w_s) dt \right| \\
\leq 2|\xi| |\phi_1 - \phi_2| \int_0^\infty \left( r_c \kappa + \rho \right) e^{-(\mu - \delta_1)t} + 2r_c \rho (1 + \kappa r_s) \frac{e^{-(\mu - \delta_1)t} - e^{-(\mu - \delta_2)t}}{-(\mu - \delta_1) + (\mu - \delta_2)} dt \\
= 2|\xi| |\phi_1 - \phi_2| \left( r_c \kappa + \rho \right) \frac{1}{\mu - \delta_1} + \frac{2r_c \rho (1 + \kappa r_s)}{(\mu - \delta_1)(\mu - \delta_2)}. \tag{6.21}
\]

**Part II** To estimate the integrand in Part II, we begin by writing:

\[
\tilde{\alpha}(\Phi(t, \phi_1), x_s)^2 - \tilde{\alpha}(\Phi(t, \phi_2), w_s)^2 \\
= (\tilde{\alpha}(\Phi(t, \phi_1), x_s) + \tilde{\alpha}(\Phi(t, \phi_2), w_s)) (\tilde{\alpha}(\Phi(t, \phi_1), x_s) - \tilde{\alpha}(\Phi(t, \phi_2), w_s)).
\]

In equations (6.19) and (6.20) we calculated similar bounds for these terms, by which we obtain

\[
|\tilde{\alpha}(\Phi(t, \phi_1), x_s)^2 - \tilde{\alpha}(\Phi(t, \phi_2), w_s)^2|
\]
\[ \leq 2\rho e^{-(\mu-\delta_1)t} |\xi|^2 |\phi_1 - \phi_2| \left( \kappa e^{-(\mu-\delta_1)t} + 2\rho (1 + \kappa r_s) e^{-(\mu-\delta_2)t} \right) - \frac{e^{-(\mu-\delta_1)t} - e^{-(\mu-\delta_2)t}}{-(\mu - \delta_1) + (\mu - \delta_2)} \].

Integrating, we obtain a bound for Part II.

\[
\left| \int_{0}^{\infty} \bar{\alpha}(\Phi(t, \phi_1), x_\delta)^2 - \bar{\alpha}(\Phi(t, \phi_2), w_\delta)^2 \, dt \right|
\leq 2\rho |\xi|^2 |\phi_1 - \phi_2| \int_{0}^{\infty} e^{-(\mu-\delta_1)t} \left( \kappa e^{-(\mu-\delta_1)t} + 2\rho (1 + \kappa r_s) e^{-(\mu-\delta_2)t} \right) dt
\leq 2\rho |\xi|^2 |\phi_1 - \phi_2| \left( \frac{\kappa}{2(\mu - \delta_1)} + \frac{2\rho (1 + \kappa r_s)}{2(\mu - \delta_1)(2\mu - \delta_1 - \delta_2)} \right). \tag{6.22}
\]

**Part III** Using the Banach algebra property of $\ast$, the integrand in Part III may be re-expressed as follows:

\[
(x_s * x_s)_0 - (w_s * w_s)_0 = ((x_s + w_s) * (x_s - w_s))_0.
\]

Using the estimates from Propositions 6.8 and 6.10 we obtain the following bound

\[
| (x_s * x_s)_0 - (w_s * w_s)_0 | \leq 2|\xi| e^{-(\mu-\delta_1)t} 2(1 + \kappa r_s) |\xi| |\phi_1 - \phi_2| e^{-(\mu-\delta_1)t} - e^{-(\mu-\delta_2)t}
\leq 4(1 + \kappa r_s) |\xi|^2 |\phi_1 - \phi_2| e^{-(\mu-\delta_1)t} - e^{-(\mu-\delta_2)t}
\leq 4(1 + \kappa r_s) |\xi|^2 |\phi_1 - \phi_2| e^{-(\mu-\delta_1)t} - e^{-(\mu-\delta_2)t}
\leq 4(1 + \kappa r_s) \frac{2\rho (1 + \kappa r_s)}{2(\mu - \delta_1)(2\mu - \delta_1 - \delta_2)}. \tag{6.23}
\]

Thus, we have obtained bounds on Parts I, II and III. Putting together the results from (6.21) (6.22) and (6.23), we obtain

\[
| (\Psi[\alpha](\phi_1, \xi) - \phi_1) - (\Psi[\alpha](\phi_2, \xi) - \phi_2) |
\leq \int_{0}^{\infty} \left| 2\Phi(t, \phi_1)\bar{\alpha}(\Phi(t, \phi_1), x_\delta) - 2\Phi(t, \phi_2)\bar{\alpha}(\Phi(t, \phi_2), w_\delta) \right|
+ \left| \bar{\alpha}(\Phi(t, \phi_1), x_\delta)^2 - \bar{\alpha}(\Phi(t, \phi_2), w_\delta)^2 \right|
+ \left| (x_s * x_s)_0 - (w_s * w_s)_0 \right| \, dt
\leq 2|\xi| |\phi_1 - \phi_2| \left( \frac{r_\kappa + \rho}{\mu - \delta_1} + \frac{2r_\kappa (1 + \kappa r_s)}{(\mu - \delta_1)(\mu - \delta_2)} \right).
\]
Proposition 6.14
The space \( L^\infty \) has the following properties:

\[
+ 2\rho |\xi| r_s |\phi_1 - \phi_2| \left( \frac{\kappa}{2(\mu - \delta_1)} + \frac{2\rho (1 + kr_s)}{2(\mu - \delta_1)(2\mu - \delta_1 - \delta_2)} \right) \\
+ \frac{4(1 + kr_s) |\xi| r_s |\phi_1 - \phi_2|}{2(\mu - \delta_1)(2\mu - \delta_1 - \delta_2)} \\
= \frac{2|\xi||\phi_1 - \phi_2|}{\mu - \delta_1} \left( r_c \kappa + \rho + \frac{2r_c \rho (1 + kr_s)}{\mu - \delta_2} + \frac{r_s (1 + kr_s)(1 + \rho^2)}{2\mu - \delta_1 - \delta_2} \right) \\
= (\epsilon_0 + \epsilon_1|\kappa|)|\xi| |\phi_1 - \phi_2|.
\]

for \( \epsilon_0 \) and \( \epsilon_1 \) defined below

\[
\epsilon_0 \overset{\text{def}}{=} \frac{2}{\mu - \delta_1} \left( \rho + \frac{2r_c \rho}{\mu - \delta_2} + \frac{r_s (1 + \rho^2)}{2\mu - \delta_1 - \delta_2} \right) \\
\epsilon_1 \overset{\text{def}}{=} \frac{2}{\mu - \delta_1} \left( r_c + \frac{\rho r_s}{\mu - \delta_2} + \frac{r_s^2 (1 + \rho^2)}{2\mu - \delta_1 - \delta_2} \right).
\] (6.24)

Thus we have shown that \( \Psi[\alpha] \in \mathcal{B}_{\rho, \epsilon_0 + \epsilon_1|\kappa|} \). Thereby if \( \epsilon_1 < 1 \) and \( \kappa_0 \geq \frac{\epsilon_0}{1 - \epsilon_1} \), then \( \Psi : \mathcal{B}_{\rho, \kappa_0} \to \mathcal{B}_{\rho, \kappa_0} \) will be an endomorphism.

To show that \( \Psi \) is a contraction mapping, we will use a somewhat weaker norm.

Definition 6.13 Define the vector space below

\[
\mathcal{E} \overset{\text{def}}{=} \left\{ \beta \in \mathcal{C}^0(\mathcal{B}_c, X_c) : \beta(0) = 0 \text{ and } \|\beta\|_{\mathcal{E}} < \infty \right\}, \quad \|\beta\|_{\mathcal{E}} \overset{\text{def}}{=} \sup_{\xi \in \mathcal{B}_c(\tau_s)} \frac{|\beta(\xi)|}{\|\xi\|_{\ell^1}}.
\]

This can be seen to be a Banach space by the same argument as in [3]. Moreover \( L^\infty(\mathcal{B}_c, \mathcal{E}) \) is a Banach space and \( \tilde{\mathcal{B}}_{\rho, \kappa} \subseteq L^\infty(\mathcal{B}_c, \mathcal{E}) \). Furthermore the intersection \( L^\infty(\mathcal{B}_c, \mathcal{E}) \cap \tilde{\mathcal{B}}_{\rho, \kappa} \) is complete.

Proposition 6.14 The space \( L^\infty(\mathcal{B}_c, \mathcal{E}) \cap \tilde{\mathcal{B}}_{\rho, \kappa} \) with norm \( \| \cdot \|_{\tilde{\mathcal{E}}} \) is complete, where we define

\[
\|\tilde{\alpha}\|_{\tilde{\mathcal{E}}} \overset{\text{def}}{=} \sup_{\phi \in \mathcal{B}_c(\tau_s)} \|\tilde{\alpha}(\phi, \cdot)\|_{\mathcal{E}} = \sup_{\phi \in \mathcal{B}_c(\tau_s), \xi \in \mathcal{B}_c(\tau_s)} \frac{|\tilde{\alpha}(\phi, \xi)|}{\|\xi\|_{\ell^1}}.
\]

Proof Consider a Cauchy sequence \( \{\tilde{\alpha}_n\}_{n=1}^{\infty} \subseteq L^\infty(\mathcal{B}_c, \mathcal{E}) \cap \tilde{\mathcal{B}}_{\rho, \kappa} \) with limit point \( \tilde{\alpha} \in L^\infty(\mathcal{B}_c, \mathcal{E}) \). Note that all functions in \( \tilde{\mathcal{B}}_{\rho, \kappa} \) have a uniform bound on their Lipschitz constant. Hence the sequence is uniformly equicontinuous, and converges uniformly to a continuous function \( \tilde{\alpha} \in \mathcal{C}^0(\mathcal{B}_c \times \mathcal{B}_c, X_c) \). Moreover, as \( \tilde{\alpha}_n(\phi, 0) = 0 \) for all \( \phi \in \mathcal{B}_c \) and \( n \in \mathbb{N} \), then \( \tilde{\alpha}(\phi, 0) = 0 \). Furthermore, since there is a fixed bound on the Lipschitz constant for the entire sequence, then \( \tilde{\alpha} \) is Lipschitz continuous. By the same argument \( \tilde{\alpha} \) also satisfies the Lipschitz bounds in (6.5) and (6.6), hence \( \tilde{\alpha} \in \tilde{\mathcal{B}}_{\rho, \kappa} \).

We will show that \( \Psi \) is a contraction on a ball of functions using the \( \| \cdot \|_{\tilde{\mathcal{E}}} \) norm. But first we prove a helpful estimate.
Proposition 6.15  Fix $\alpha, \beta \in \mathcal{B}$, $\phi \in B_c(r_c)$, $\xi \in B_s(r_s)$. Fix $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{B}}$ such that $\alpha(\phi, \xi) = \phi + \tilde{\alpha}(\phi, \xi)$ and $\beta(\phi, \xi) = \phi + \tilde{\beta}(\phi, \xi)$. Then for all $t \geq 0$ we have:

$$\|x(t, \phi, \xi, \alpha) - x(t, \phi, \xi, \beta)\|_{\ell^1} \leq 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1} \left(\frac{e^{-(\mu - \delta_1) t} - e^{-(\mu - \delta_2) t}}{-(\mu - \delta_1) + (\mu - \delta_2)}\right).$$

Proof  Let us abbreviate $x_s = x_s(t) = x(t, \phi, \xi, \alpha)$ and $y_s = y_s(t) = x(t, \phi, \xi, \beta)$. Additionally, let us abbreviate $x_c = \alpha(\Phi(t, \phi), x_s(t))$ and $y_c = \beta(\Phi(t, \phi), y_s(t))$.

By variations of constants we have the following:

$$x_s(t) - y_s(t) = \int_0^t e^{L(t-t')} \left(\mathcal{N}_s(x_c, x_s) - \mathcal{N}_s(y_c, y_s)\right) d\tau.$$

In the same manner as in (6.12), we bound the difference of the nonlinearities:

$$\|\mathcal{N}_s(x_c, x_s) - \mathcal{N}_s(y_c, y_s)\|_{\ell^1} \leq 2|x_c||x_s - y_s|_{\ell^1} + \|x_s + y_s\|_{\ell^1} \|x_s - y_s\|_{\ell^1} + 2\|x_c - y_c\|y_s\|_{\ell^1}$$

where we used Proposition 6.8 to obtain $\|x_c\|, \|y_s\| \leq r_s$, and (6.11) to obtain $|x_c| \leq r_c + \rho r_s$. To bound $|x_c - y_c|$ we calculate further, using (6.5) and the definition of $\|\cdot\|_{\tilde{\ell}^1}$ to obtain the following:

$$|x_c - y_c| \leq |\alpha(\Phi(t, \phi), x_s) - \alpha(\Phi(t, \phi), y_s)| + |\alpha(\Phi(t, \phi), y_s) - \beta(\Phi(t, \phi), y_s)|$$

$$\leq \rho \|x_s - y_s\|_{\ell^1} + \|\tilde{\alpha}(\Phi(t, \phi), y_s) - \tilde{\beta}(\Phi(t, \phi), y_s)|$$

$$\leq \rho \|x_s - y_s\|_{\ell^1} + \|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1}.$$ (6.25)

Combining these estimates we obtain:

$$\|\mathcal{N}_s(x_c, x_s) - \mathcal{N}_s(y_c, y_s)\|_{\ell^1} \leq 2(r_c + 2\rho r_s + r_s)|x_s - y_s|_{\ell^1} + 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1}y_s\|_{\ell^1}$$

$$= \delta_2 \|x_s - y_s\|_{\ell^1} + 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1}y_s\|_{\ell^1}.$$

We may continue calculating using variation of constants:

$$e^{\mu t} \|x_s - y_s\|_{\ell^1} \leq \int_0^t e^{\mu \tau} \left(\delta_2 \|x_s - y_s\|_{\ell^1} + 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1}y_s\|_{\ell^1}\right) d\tau$$

$$\leq \int_0^t e^{\mu \tau} 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1}e^{-(\mu - \delta_1)\tau}\|\tilde{\xi}\|_{\tilde{\ell}^1} d\tau + \int_0^t e^{\mu \tau} \delta_2 \|x_s - y_s\|_{\ell^1} d\tau.$$

By Lemma 6.7, we have that:

$$\|x_s - y_s\|_{\ell^1} \leq 2r_s\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\ell}^1} \left(\frac{e^{-(\mu - \delta_1) t} - e^{-(\mu - \delta_2) t}}{-(\mu - \delta_1) + (\mu - \delta_2)}\right).$$
The proposition follows. \(\square\)

We can now show that \(\Psi\) is a contraction map.

**Proof of Theorem 6.2** By the Proposition 6.12 there exists a \(\kappa > 0\) such that \(\Psi : B_{\rho,\kappa} \to B_{\rho,\kappa}\) is an endomorphism. Define the function \(i_c : B_c \times B_s \to X_c\) by \(i_c(\phi, \xi) = \phi\), and define \(\tilde{\Psi}\) by

\[
\tilde{\Psi}[\tilde{\alpha}] \overset{\text{def}}{=} \Psi[i_c + \tilde{\alpha}] - i_c.
\]

It follows that \(\tilde{\Psi} : \tilde{B}_{\rho,\kappa} \to \tilde{B}_{\rho,\kappa}\) is also an endomorphism. For the constant

\[
\lambda \overset{\text{def}}{=} \frac{2(\rho(r_c + \rho r_s) + r_s)2r_s}{(\mu - \delta_1)(\mu - \delta_2)} + \frac{2(r_c + \rho r_s)}{\mu - \delta_1},
\]

we will show that:

\[
\|\tilde{\Psi}[\tilde{\alpha}] - \tilde{\Psi}[(\tilde{\beta})]\|_{\tilde{\xi}} \leq \lambda\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\xi}} \quad (6.26)
\]

for all \(\tilde{\alpha}, \tilde{\beta} \in \tilde{B}_{\rho,\kappa}\).

Fix \(\alpha, \beta \in \mathcal{B}\) and fix \(\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{B}}\) such that \(\alpha = i_c + \tilde{\alpha}\) and \(\beta = i_c + \tilde{\beta}\). Also fix \(\phi \in B_c(r_c), \xi \in B_s(r_s)\). Let us abbreviate \(x_s = x_s(t) = x(t, \phi, \xi, \alpha)\) and \(y_s = y_s(t) = x(t, \phi, \xi, \beta)\). Additionally, let us abbreviate \(x_c = \alpha(\Phi(t, \phi), x_s(t))\) and \(y_c = \beta(\Phi(t, \phi), x_s(t))\). We wish to bound the following:

\[
\left|\tilde{\Psi}[\tilde{\alpha}](\phi, \xi) - \tilde{\Psi}[\tilde{\beta}](\phi, \xi)\right| = |\Psi[\alpha](\phi, \xi) - \Psi[\beta](\phi, \xi)|
\]

\[
\leq \int_0^\infty |\mathcal{N}_c(x_c, x_s) - \mathcal{N}_c(y_c, y_s)| dt.
\]

By Proposition 6.8 we have \(\|x_s\|, \|y_s\| \leq r_s\) and by (6.11) we have \(\|x_c\|, \|y_c\| \leq r_c + \rho r_s\), hence:

\[
|\mathcal{N}_c(x_c, x_s) - \mathcal{N}_c(y_c, y_s)| \leq |x_c^2 - y_c^2| + \|x_s^2 - y_s^2\|_{\ell^1}
\]

\[
\leq 2(r_c + \rho r_s)|x_c - y_c| + 2r_s\|x_s - y_s\|_{\ell^1}.
\]

By (6.25) we have \(\|x_c - y_c\| \leq \rho\|x_s - y_s\|_{\ell^1} + \|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\xi}}\|y_s\|_{\ell^1}\), and thereby

\[
|\mathcal{N}_c(x_c, x_s) - \mathcal{N}_c(y_c, y_s)| \leq 2(\rho(r_c + \rho r_s) + r_s)\|x_s - y_s\|_{\ell^1} + 2(r_c + \rho r_s)\|\tilde{\alpha} - \tilde{\beta}\|_{\tilde{\xi}}\|y_s\|_{\ell^1}.
\]

Plugging this into the integral, and using Proposition 6.8 and Proposition 6.15 to bound \(\|x_s\|_{\ell^1}\) and \(\|x_s - y_s\|_{\ell^1}\) respectively, we obtain

\[
|\Psi[\alpha](\phi, \xi) - \Psi[\beta](\phi, \xi)| \leq 2(\rho(r_c + \rho r_s) + r_s)\int_0^\infty \|x_s - y_s\|_{\ell^1} dt
\]

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\[
+ 2(r_c + \rho r_s) \|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \int_0^\infty \|y_s\|_{\ell^1} dt
\leq 2(\rho(r_c + \rho r_s) + r_s) \int_0^\infty 2r_s \|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \|\xi\|_{\ell^1} dt \times \left( e^{-(\mu - \delta_2)t} - e^{-(\mu - \delta_1)t} \right) dt
+ 2(r_c + \rho r_s) \|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \int_0^\infty e^{-(\mu - \delta_1)t} \|\xi\|_{\ell^1} dt.
\]

Integrating, we obtain our final bound:

\[
\left| \tilde{\Psi}[\tilde{\alpha}](\phi, \xi) - \tilde{\Psi}[\tilde{\beta}](\phi, \xi) \right| \leq \frac{2(\rho(r_c + \rho r_s) + r_s)2r_s \|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \|\xi\|_{\ell^1}}{\mu - \delta_1(\mu - \delta_2)} + \frac{2(r_c + \rho r_s) \|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \|\xi\|_{\ell^1}}{\mu - \delta_1}.
\]

After factoring out \(\|\tilde{\alpha} - \tilde{\beta}\| \tilde{\varepsilon} \|\xi\|_{\ell^1}\), we have proved inequality (6.26) for our definition of \(\lambda\).

If \(\lambda < 1\), as assumed in the theorem, then \(\tilde{\Psi}\) is a contraction mapping on \(L^\infty(B_c, \mathcal{E}) \cap \tilde{B}_{\rho,\kappa}\) with norm \(\| \cdot \|_{\tilde{\varepsilon}}\). So by the Banach fixed point theorem, there exists a unique map \(\tilde{\alpha}_* \in \tilde{B}\) such that \(\tilde{\Psi}[\tilde{\alpha}_*] = \tilde{\alpha}_*\). If we then define \(\alpha_* = i_c + \tilde{\alpha}_*\) it follows that \(\Psi[\alpha_*] = \alpha_*\). Thereby \(\alpha_*\) is a chart for the center-stable manifold over \(B_c(r_c)\). \(\square\)

7 Computer-assisted proofs of main theorems

In this section, we show computer-assisted proofs of our main results for the initial-boundary value problem (2.1). All computations are carried out on Windows 10, Intel(R) Core(TM) i7-6700K CPU @ 4.00GHz, and MATLAB 2019a with INTLAB - INTerval LABoratory [22] version 11 and Chebfun - numerical computing with functions [20] version 5.7.0. All codes used to produce the results in this section are freely available from [25].

7.1 Proof of Theorem 1.1

It is obvious that the solution of (1.2) has a symmetry \(u(te^{-i\theta}, x) = \overline{u(te^{i\theta}, x)}\) for a real \(t\). Then, to prove Theorem 1.1, it is sufficient to show that the imaginary part of \(u(z, x)\) is a non-zero function at a certain point \(z \in \mathbb{R}\) satisfying \(z_B < z\), where \(z_B\) denotes the blow-up time of (1.1). We take a path \(\tilde{\Gamma}_\theta\) which bypasses the blow-up point \(z_B\) as shown in Fig. 3. Here, \(z_B \approx 0.0119\) holds [4] under the periodic boundary condition with the initial data \(u_0(x) = 50(1 - \cos(2\pi x))\).
Fig. 3 The path\[ \tilde{\Gamma}_\theta \overset{\text{def}}{=} \{ z : z = te^{i\theta} (O \to z_A), z = z_A + te^{-i\theta} (z_A \to z_C), \theta = \pi/3 \} \]is plotted, where\[ z_A = 0.00725(1 + \sqrt{3}) \text{i} \]and\[ z_C = 0.0145. \]We divide each segment (\( O \to z_A \) and \( z_A \to z_C \)) into 64 steps.

Fig. 4 Profiles of numerically computed solutions on each segment (a) \( O \to z_A \) and (b) \( z_A \to z_C \) are plotted. At the point \( z_C \), the imaginary part of \( \bar{u}(z_C, x) \) becomes a non-zero function.

We divide each segment into 64 steps and, by using our rigorous integrator introduced in Sects. 2–5, analytically continue to the \( z_C \) point in Fig. 3. From \( O \) to \( z_A \), we set \( \theta = \pi/3 \) in (2.1). After that, we change \( \theta = -\pi/3 \) from \( z_A \) to \( z_C \). For each time step \( J_i \) \( (i = 1, 2, \ldots, 128) \), we get an approximate solution by using Chebfun [20] as

\[
\hat{u}(t, x) = \sum_{|k| \leq N} \left( \tilde{a}_{0,k} + 2 \sum_{\ell=1}^{n-1} \tilde{a}_{\ell,k} T_{\ell}(t) \right) e^{ik\omega x}, \quad \omega = 2\pi, \quad x \in (0, 1), \quad t \in J_i \quad (7.1)
\]

with \( N = 25 \) and \( n = 13 \). We also set \( m = 0 \) in Theorem 3.3, which determines the size of variational problem. The profiles of numerically computed \( \Re(\hat{u}) \) and \( \Im(\hat{u}) \) are plotted in Fig. 4.

After 128 time stepping (at \( z_C \)), we show that the imaginary part of \( u(z_C, x) \) is a non-zero function. To prove this, we use the \( \ell^1 \) norm of the Fourier coefficients, say

\[
\|v\| \overset{\text{def}}{=} \|c\|_{\ell^1} = \sum_{k \in \mathbb{Z}} |c_k| \quad \text{for} \quad v(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\omega x}.
\]

Let the solution \( u(z_C, x) \) of (1.2) at \( z_C \) and its numerically computed solution be denoted by

\[
u(z_C, x) = \sum_{k \in \mathbb{Z}} a^z_{C,k} e^{ik\omega x} \quad \text{and} \quad \hat{u}(z_C, x) = \sum_{|k| \leq N} \bar{a}^z_{C,k} e^{ik\omega x},
\]
respectively. We also denote two bi-infinite complex-valued sequences \(a^{z_C} = (a_k^{z_C})_{k \in \mathbb{Z}}\) and \(\bar{a}^{z_C} = (\ldots, 0, \bar{a}_{-N}^{z_C}, \ldots, \bar{a}_N^{z_C}, 0 \ldots)\). Then, the imaginary part of the solution at \(z_C\) follows

\[
\| \text{Im} (u(z_C, \cdot)) \| = \| \text{Im} (\bar{u}(z_C, \cdot)) + \text{Im} (u(z_C, \cdot)) - \text{Im} (\bar{u}(z_C, \cdot)) \| \\
\geq \| \text{Im} (\bar{u}(z_C, \cdot)) \| - \| \text{Im} (u(z_C, \cdot) - \bar{u}(z_C, \cdot)) \| \\
\geq \| \text{Im} (\bar{u}(z_C, \cdot)) \| - \| a^{z_C} - \bar{a}^{z_C} \|_{\ell^1} \\
\geq \| \text{Im} (\bar{u}(z_C, \cdot)) \| - \varepsilon^{z_C},
\]

where \(\varepsilon^{z_C}\) is the point-wise error at \(z_C\) point and our rigorous computation yields \(\varepsilon^{z_C} = 0.5765\). Similarily, we have an upper bound of the imaginary part given by \(\| \text{Im} (u(z_C, \cdot)) \| \leq \| \text{Im} (\bar{u}(z_C, \cdot)) \| + \varepsilon^{z_C}\). Furthermore, the imaginary part of \(\bar{u}(z_C, x)\) is presented by

\[
\text{Im} (\bar{u}(z_C, x)) = \frac{1}{2} \sum_{|k| \leq N} \left[ \text{Im} (\bar{a}_k^{z_C}) + \text{Im} (\bar{a}_{-k}^{z_C}) - i \left( \text{Re} (\bar{a}_k^{z_C}) - \text{Re} (\bar{a}_{-k}^{z_C}) \right) \right] e^{ik_\omega x}.
\]

(7.2)

This is followed by the following fact:

\[
\text{Im} \left( \bar{a}_k^{z_C} e^{ik_\omega x} \right) = \text{Im} (\bar{a}_k^{z_C}) \cos(k_\omega x) + \text{Re} (\bar{a}_k^{z_C}) \sin(k_\omega x) \\
= \text{Im} (\bar{a}_k^{z_C}) \left( \frac{e^{ik_\omega x} + e^{-ik_\omega x}}{2} \right) + \text{Re} (\bar{a}_k^{z_C}) \left( \frac{e^{ik_\omega x} - e^{-ik_\omega x}}{2i} \right)
\]

for each \(|k| \leq N\). Summing up for \(k\), we have

\[
\sum_{|k| \leq N} \text{Im} \left( \bar{a}_k^{z_C} e^{ik_\omega x} \right) = \frac{1}{2} \sum_{|k| \leq N} \left[ \text{Im} (\bar{a}_k^{z_C}) \left( \frac{e^{ik_\omega x} + e^{-ik_\omega x}}{2} \right) \\
- i \cdot \text{Re} (\bar{a}_k^{z_C}) \left( \frac{e^{ik_\omega x} - e^{-ik_\omega x}}{2i} \right) \right] \\
= \frac{1}{2} \sum_{|k| \leq N} \left[ \text{Im} (\bar{a}_k^{z_C}) + \text{Im} (\bar{a}_{-k}^{z_C}) - i \left( \text{Re} (\bar{a}_k^{z_C}) - \text{Re} (\bar{a}_{-k}^{z_C}) \right) \right] e^{ik_\omega x}.
\]

We rigorously compute (7.2) based on interval arithmetic and the following is given

\[
\| \text{Im} (u(z_C, \cdot)) \| \in [660.4935, 661.6465].
\]

This implies \(\| \text{Im} (u(z_C, \cdot)) \| > 0\).

Consequently, we prove that the imaginary part of the solution \(u(z_C, x)\) is the non-zero function. Then, there exists a branching singularity inside the contour integral in Fig. 3, i.e., \(\{ z \in \mathbb{C} : \text{Re}(z) \leq z_C, 0 \leq \text{Im}(z) \leq 0.00725\sqrt{3} \}\) with \(z_C = 0.0145\). The execute time to prove the branching singularity was almost 105.95 s. The results of analytical continuation is shown in Fig. 5. As shown in this figure, the error bounds
For each time step $J_i$ ($i = 1, 2, \ldots, 128$), results of analytical continuation are plotted. a The point-wise error $\varepsilon_i$ for $i = 0, 1, \ldots, 128$. b The radius of the neighborhood $B_{J_i}(\bar{a}, \varrho_i)$ in which the exact solution is included for $i = 1, 2, \ldots, 128$.

Lastly, to obtain a lower bound on the location of the blow-up point $z_B$, it suffices to rigorously integrate our initial data $u_0(x)$ in purely real time with $\theta = 0$ in (2.1) for as long as possible. Again using Chebfun [20] we obtain an approximate solution as in (7.1) using computational parameters $N = 20$, $n = 15$, $m = 0$, and adaptive time stepping. By using our rigorous integrator introduced in Sects. 2–5, we are able to validate our solution until $t = 0.0116$, hence the blow-up point must satisfy $z_B \geq 0.0116$. The execute time to give the above lower bound was almost 69.361 s. □

Remark 7.1 Generally speaking, the propagation of error estimates makes rigorous integration difficult. As such, both the length of a contour and how close it approaches the blow-up point will affect the results of our rigorous integrator. We believe the longer contour is a principal reason for explaining why we are able to get 0.0003 close to the blowup point from below, but only 0.0031 close from above. The upper bound could also be improved by adjusting the step size more carefully.

7.2 Proof of Theorem 1.2

We show the proof of global existence on the straight path $\Gamma_\theta$. To prove the global existence of the solution of (1.2), we check the hypothesis of Theorem 6.2 by using rigorous numerics. More precisely, at $t = t_i$ (after $i$th time stepping), we rigorously have

$$\vert a_0(t_i) \vert \leq \vert \tilde{a}_0(t_i) \vert + \varepsilon_i, \quad \| a^{(s)}(t_i) \|_{\ell^1} \leq \| \tilde{a}^{(s)}(t_i) \|_{\ell^1} + \varepsilon_i,$$

where $a^{(s)}(t) = (a_k(t))_{k \in \mathbb{Z}, \; k \neq 0}$ and $\tilde{a}^{(s)}(t) = (\tilde{a}_k(t))_{|k| \leq N, \; k \neq 0}$. To construct a trapping region $U$ as in Corollary 6.4 which might contain $a(t_i)$, we fix mildly inflated radii constants

$$r_c \overset{\text{def}}{=} \left( \vert \tilde{a}_0(t_i) \vert + \varepsilon_i \right) + 0.02 \left( \| \tilde{a}^{(s)}(t_i) \|_{\ell^1} + \varepsilon_i \right) \quad r_s \overset{\text{def}}{=} \| \tilde{a}^{(s)}(t_i) \|_{\ell^1} + \varepsilon_i.$$

Then, for $\mu = \omega^2 \cos \theta$, we compute the nearly optimal bounds of $\rho$ explicitly given in Remark 6.5. If such a (positive) $\rho$ is not obtained, then we are unable validate the
Fig. 6 Results of rigorous integration for $a \theta = \pi/3$, $b \pi/4$, $c \pi/6$ and $d \pi/12$

(a) $\theta = \pi/3$: The step size is equidistantly taken as $h = 2.5 \times 10^{-3}$. We set $N = 14$ (maximum wave number of Fourier), $n = 13$ (number of Chebyshev basis), and $m = 2$. After 85 steps of the rigorous integration (at $t = 0.2125$), the hypothesis of Theorem 6.2 holds for $r_c = 4.741$, $r_s = 6.545 \times 10^{-3}$, $\rho = 1.719 \times 10^{-2}$, $\epsilon_1 = 0.9251$, and $\lambda = 0.9251$, and $a(t)$ is contained in the region $U$ described in Corollary 6.4. We succeed in proving the global existence of solution on $\Gamma_{\pi/3}$. The execute time was almost 263.2 sec.

(b) $\theta = \pi/4$: The step size is equidistantly taken as $h = 2.5 \times 10^{-3}$. We set $N = 14$, $n = 13$, and $m = 2$. At $t = 0.1575$, the hypothesis of Theorem 6.2 holds for $r_c = 6.588$, $r_s = 1.033 \times 10^{-2}$, $\rho = 1.341 \times 10^{-2}$, $\epsilon_1 = 0.8947$, and $\lambda = 0.8947$, and $a(t)$ is contained in the region $U$.

(c) $\theta = \pi/6$: The step size is equidistantly taken as $h = 2.5 \times 10^{-3}$. We set $N = 15$ and $n = 14$. We adapt $m = 0$ ($0 \leq t \leq 0.02$) and $m = 2$ ($t \geq 0.02$) for the success of our proof.

(d) $\theta = \pi/12$: The step size is adapted as $h = 1.25 \times 10^{-3}$ ($0 \leq t \leq 0.01$), $h = 6.25 \times 10^{-4}$ ($0.01 \leq t \leq 0.02$), and $h = 2.5 \times 10^{-3}$ ($t \geq 0.02$). We set $N = 19$ and $n = 15$. We also adapt $m = 0$ ($0 \leq t \leq 0.02$) and $m = 2$ ($t \geq 0.02$).

center stable manifold in a region large enough to contain $a(t_i)$. In such a case, we continue rigorous integration to the next time step, i.e., $J_{i+1}$. Figure 6 shows verified results of our rigorous integration for $\theta = \pi/3$, $\pi/4$, $\pi/6$ and $\pi/12$.
described in Corollary 6.4. Number of time stepping is 63. We also succeed in proving the global existence of solution on $\Gamma_{\pi/4}$. It took almost 195.3 s to complete the proof.

In the case of $\theta = \pi/6$, our rigorous integrator with the same setting above fails to prove local inclusion because the peak of solution becomes large ($\|\tilde{a}\|_X \approx 240$). So we change the value of $m$ as $m = 0$ ($0 \leq t \leq 0.02$) and $m = 2$ ($t \geq 0.02$). Then, as shown in Fig. 6c, our rigorous integrator succeeds in including the solution until $t = 0.1325$ (53 steps). At that time, the hypothesis of Theorem 6.2 holds for $r_c = 8.138$, $r_s = 1.382 \times 10^{-2}$, $\rho = 1.722 \times 10^{-2}$, $\epsilon_1 = 0.9095$, and $\lambda = 0.9095$, and $a(t)$ is contained in the region $U$ described in Corollary 6.4. Proof of the global existence on $\Gamma_{\pi/6}$ is complete. The execute time was almost 163.2 sec.

Finally, the case $\theta = \pi/12$ is slightly difficult to complete. The value of $\|\tilde{a}\|_X$ is almost 450 because this path is close to the blow-up point. We adapt both the step size of time stepping and the value of $m$ as

$$h = \begin{cases} 
1.25 \times 10^{-3} & (0 \leq t \leq 0.01), \\
6.25 \times 10^{-4} & (0.01 \leq t \leq 0.02), \\
2.5 \times 10^{-3} & (t \geq 0.02), 
\end{cases} \quad m = \begin{cases} 
0 & (0 \leq t \leq 0.02), \\
2 & (t \geq 0.02). 
\end{cases}$$

Then, after 66 time stepping, the hypothesis of Theorem 6.2 holds at $t = 0.125$ for $r_c = 8.939$, $r_s = 2.226 \times 10^{-2}$, $\rho = 1.914 \times 10^{-2}$, $\epsilon_1 = 0.8838$, and $\lambda = 0.8838$, and $a(t)$ is contained in the region $U$ described in Corollary 6.4. On $\Gamma_{\pi/12}$, we also prove the global existence of solution of (2.1). The results of rigorous integrator is shown in Fig. 6d. It took almost 206.9 s to complete the proof. □

**Conclusion**

In this paper, we introduced a computational method for computing rigorous local inclusions of solutions of the Cauchy problems for the nonlinear heat equation (1.2) for complex time values. The proof is constructive and provides explicit bounds for the inclusion of the solutions of the Cauchy problem rewritten as $F(a) = 0$ on the Banach space $X = C(J; \ell^1)$. The idea is to show that a simplified Newton operator $T : X \rightarrow X$ of the form $T(a) = a - \mathcal{A}_0 F(a)$ has a unique fixed point by the Banach fixed point theorem. The rigorous enclosure of the fixed point yields the solution of the Cauchy problem. The construction of the solution map operator $\mathcal{A}_0$ is guaranteed with the theory of Sect. 3 by verifying the hypotheses of Theorem 3.3. More explicitly, this requires first computing rigorously with Chebyshev series the solutions of the linearized problems (3.30) and (3.31), and second proving the existence of the evolution operator on the tail. Once the hypotheses are verified successfully, the solution map operator $\mathcal{A}_0$ exists, is defined as in (3.3), and the theory of Sect. 4 is used to enclose rigorously the fixed point of $T$, and therefore the local inclusion in time of the Cauchy problem. Section 5 then introduces a method for applying iteratively the approach to compute solutions over long time intervals. This technique is directly used to prove Theorem 1.1 about the existence of a branching singularity in the nonlinear
heat equation (1.2). Afterwards, in Sect. 6, we introduced an approach based on the Lyapunov–Perron method for calculating part of a center-stable manifold, which is then used to prove that an open set of solutions of the given Cauchy problem converge to zero, hence yielding the global existence in time of the solutions. This method is used to prove our second main result, namely Theorem 1.2. The results are presented in details in Sect. 7.

Finally, we conclude this paper by discussing some potential extensions and remained problems for the complex-valued nonlinear heat equation (1.2). We believe that our methodology of rigorous numerics could be extended to more general time-dependent PDEs. It is interesting to rigorously integrate the solution orbit of high (space) dimensional PDEs. Furthermore, the case where the time is pure imaginary offers us an interesting problem. The computation in [4] suggests strongly that the solution exists globally in time and decays slowly toward the zero solution. Since the semigroup generator becomes conservative in that case, our method in this paper, which depends on the existence of dissipation, cannot be used.

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