LOG CANONICAL THRESHOLDS ON VARIETIES WITH BOUNDED SINGULARITIES

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Abstract. We consider pairs $(X, A)$, where $X$ is a variety with klt singularities and $A$ is a formal product of ideals on $X$ with exponents in a fixed set that satisfies the Descending Chain Condition. We also assume that $X$ has (formally) bounded singularities, in the sense that it is, formally locally, a subvariety in a fixed affine space defined by equations of bounded degree. We prove in this context a conjecture of Shokurov, predicting that the set of log canonical thresholds for such pairs satisfies the Ascending Chain Condition.

1. Introduction

The log canonical threshold is a fundamental invariant in birational geometry. It is attached to a divisor with real coefficients on a variety with mild singularities. An outstanding conjecture due to Shokurov [Sho] predicts that in any fixed dimension, if the coefficients of the divisors are taken in any given set of positive real numbers satisfying the descending chain condition (DCC), then the set of all possible log canonical thresholds satisfies the ascending chain condition (ACC).

This conjecture has attracted considerable attention due to its implications to the Termination of Flips Conjecture. More precisely, Birkar showed in [Bir] the following: if Shokurov’s conjecture is known in dimension $n$, and if the log Minimal Model Program is known in dimension $(n-1)$, then there are no infinite sequences of flips in dimension $n$ for pairs of non-negative log Kodaira dimension. We note that due to the results in [BCHM], Termination of Flips is the remaining piece in order to establish the log Minimal Model Program in arbitrary dimension. There is another outstanding open problem in the area, the Abundance Conjecture, but the circle of ideas we are discussing does not have anything to say in that direction.

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1A set of real numbers satisfies DCC (respectively, ACC) if it does not contain any infinite sequence that is strictly decreasing (respectively, strictly increasing). For short, such a set will be called a DCC set (respectively, ACC set).
Shokurov’s conjecture was proved in the case of smooth (and, more generally, locally complete intersection) ambient varieties in [dFEM], building on work from [dFM] and [Kol1]. In this note we deal with the more general case of varieties that have bounded singularities, in a sense to be explained below.

Let \( k \) be an algebraically closed field of characteristic zero. We assume that our ambient varieties are defined over \( k \), and are normal and \( \mathbb{Q} \)-Gorenstein. Let \( X \) be any such variety. Instead of dealing with \( \mathbb{R} \)-divisors, we work in the more general setting of \( \mathbb{R} \)-ideals on \( X \): these are formal products \( A = a_1^{q_1} \cdots a_r^{q_r} \), where the \( a_i \) are nonzero ideal sheaves and the \( q_i \) are positive real numbers. If the \( q_i \) lie in a subset \( \Gamma \) of \( \mathbb{R}^+ \), we say that \( A \) is a \( \Gamma \)-ideal. Given two \( \mathbb{R} \)-ideals \( A \) and \( B \) on \( X \) and a point \( x \in \text{Supp}(A) \), if \( (X, B) \) is log canonical, then one defines the mixed log canonical threshold \( \text{lct}_{(X, B),x}(A) \) to be the largest \( c \) such that the pair \( (X, A^c B) \) is log canonical at \( x \). One reduces to the more familiar setting of log canonical thresholds when \( B = \mathcal{O}_X \).

In order to study limits of log canonical thresholds, the basic ingredient in the methods used in [dFM, Kol1, dFEM] is the construction of generic limit ideals. The main obstruction in proving Shokurov’s Conjecture in its general form comes from the problem of constructing a “generic limit ambient space” where the generic limit ideal should live. From this point of view, the advantage in the smooth and locally complete intersection cases is that one can easily reduce to work with one fixed polynomial ring, so that in the end, in order to construct generic limit ideals, it suffices to take a field extension and complete the ring at the origin.

In this paper we consider the case of bounded singularities. We say that a collection of germs of algebraic varieties \( (X_i, x_i) \) has (formally) bounded singularities if there are integers \( m \) and \( N \) such that for every \( i \) there is a subscheme \( Y_i \) in \( \mathbb{A}^N \) whose ideal is defined by equations of degree \( \leq m \), and a point \( y_i \in Y_i \) such that \( \mathcal{O}_{X_i, x_i} \simeq \mathcal{O}_{Y_i, y_i} \). Equivalently, this means that there exists a morphism \( \pi: \mathcal{Y} \to T \), such that for every \( i \) there is a closed point \( t_i \in T \) and a point \( y_i \) in the fiber \( \mathcal{Y}_{t_i} \) over \( t_i \) such that \( \mathcal{O}_{X_i, x_i} \simeq \mathcal{O}_{Y_{t_i}, y_i} \). At the moment this appears to be the most general context where the approach through generic limits can be put to work, and it seems likely that new methods will be needed to attack the conjecture in its general form.

We can now state our main result.

**Theorem 1.1.** If \( \Gamma \subset \mathbb{R}^+ \) is a DCC set, then there is no infinite strictly increasing sequence of log canonical thresholds

\[
\text{lct}_{(X_1, B_1),x_1}(A_1) < \text{lct}_{(X_2, B_2),x_2}(A_2) < \ldots,
\]

where the \( (X_i, x_i) \) form a collection of klt varieties with bounded singularities and \( A_i, B_i \) are \( \Gamma \)-ideals such that all pairs \( (X_i, B_i) \) are log canonical.

The result in [dFEM] covers the case when the \( X_i \) are assumed to be nonsingular (or more generally, locally complete intersection), and \( \Gamma = \mathbb{Z}^+ \). In the nonsingular setting,
the first result in this direction was obtained in [dFM], where it was shown using ultrafilter constructions that every limit of invariants of the form \(\text{let}(X_i, a_i)\), with \(\dim(X_i) = n\) for all \(i\), is again an invariant of the same form. Kollár replaced in [Kol1] the ultrafilter approach by a generic limit construction, using more traditional algebro-geometric methods. In addition, using the results in [BCHM] he proved a semicontinuity property for log canonical thresholds that allowed him to treat a special case of the conjecture, namely when the log canonical threshold of the limit is computed by a divisor with center at one point. In [dFEM] we gave a more elementary proof of Kollár’s semicontinuity result, and showed that this can be used in fact to deduce the full statement of the above theorem when all \(X_i\) are nonsingular (and \(\Gamma = \mathbb{Z}_{>0}\)). More general cases, such as when the \(X_i\) are locally complete intersection or have quotient singularities, were deduced from the nonsingular case in a direct fashion.

Regarding the statement of Theorem 1.1, we emphasize that while the category of varieties with bounded singularities is quite large, it is not large enough for the applications to the Minimal Model Program. More precisely, it is not the case that, for example, terminal \(\mathbb{Q}\)-factorial singularities in a fixed dimension have bounded singularities. This is simply because one can construct quotient singularities that satisfy these properties, and of arbitrary embedding dimension. Furthermore, as Miles Reid pointed out to us, starting from dimension five there are families of terminal singularities of arbitrary high embedding dimension that are not nontrivial quotients by finite group actions.

The proof of Theorem 1.1 is based on the generic limit construction from [Kol1], suitably adapted to our setting. The main novelty in the proof of the above theorem is the simultaneous construction of a generic limit of the ambient spaces and of the ideal sheaves involved. The fact that the embedded dimension of the varieties is bounded is necessary in order to have the “generic limit variety” being defined by an ideal in a power series ring with finitely many variables. We use the fact that the singularities themselves are bounded to guarantee that such limit variety is normal, \(\mathbb{Q}\)-Gorenstein, and klt.

There are however several technical difficulties that arise when working in this general setting. Some of these technical points are of a more general nature, not necessarily related to the main topic of the paper, and therefore their treatment will be deferred to the end of the paper. This will result in two appendices.

The first technical difficulty comes from the fact that we work with singular varieties in the formal setting. It has became evident since [dFM] that the formal setting is very natural when dealing with this kind of problems. However, while in the previous papers [dFM, Kol1, dFEM] the formal setting always occurred at regular points, in the present paper we need to work with possibly singular schemes of finite type over complete local rings.

The generic limit construction that is essential for proving Theorem 1.1 requires us to develop the theory of log canonical pairs in a slightly more general framework than usual: working with \(\mathbb{R}\)-ideals on schemes of finite type over a complete Noetherian ring (of
characteristic zero). This is the case since starting with a sequence as in the theorem, the generic limit construction provides an ambient space that is the spectrum of a complete local ring. While this ring is the completion at a closed point of a scheme of finite type over a field, we need to consider ideals in this ring that do not come via completion from the finite type level. This will require us to extend the basic results on log canonical thresholds to this setting.

In particular, in order to have the notion of relative canonical class in this setting, we will need to develop a theory of sheaves of differentials that is adapted to this context. This part is extracted from the main body of the paper and forms the first appendix.

The second appendix is devoted to another technical complication arising in the proof of Theorem 1.1. The problem comes from the fact that we need to be able to bound the Gorenstein index of the varieties appearing in the statement in order to conclude that the limit variety is \(\mathbb{Q}\)-Gorenstein. To this end, we prove a general result on the behavior of the Gorenstein index in bounded families (see Theorem B.1). This result is of independent interest, and a slightly simplified version of it can be stated as follows.

**Theorem 1.2.** Let \(f : X \to T\) be a morphism of normal complex varieties such that every fiber of \(f\) is a normal variety with rational singularities. Then there is a nonempty Zariski open subset \(T^o \subseteq T\) and a positive integer \(s\) such that for every point \(x \in X\) with \(t = f(x) \in T^o\), the fiber \(X_t\) is \(\mathbb{Q}\)-Gorenstein at \(x\) if and only if the total space \(X\) is \(\mathbb{Q}\)-Gorenstein at \(x\); furthermore, in this case both \(sK_X\) and \(sK_{X_t}\) are Cartier at \(x\).

A variant of the result holding over arbitrary algebraically closed fields of characteristic zero is given in Theorem B.8. This will be the version of the result applied in the proof of Theorem 1.1.

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2. **Log canonical pairs on schemes of finite type over a complete local ring**

Throughout this section, let \(k\) be a field of characteristic zero, and let \(R = k[x_1, \ldots, x_n]\). Our goal is to define and prove the basic properties of log canonical and log terminal pairs when the ambient space is a scheme of finite type over \(R\). Of course, the definitions parallel to the ones in the case of schemes of finite type over fields. The main difference is that in order to define the relative canonical class, we need to work with sheaves of special differentials as defined in Appendix A. The theory of special differentials enables us to define the notion of relative canonical divisor in this setting (see in particular Lemma A.11). Once we have the notion of relative canonical class, the theory of singularities of pairs can be built in the same way as in the case of schemes of finite type over a field, for which we
refer to [Kol2]. However, we will need to work with $\mathbb{R}$-ideals (as opposed to $\mathbb{R}$-divisors), hence we give all definitions in this setting.

In the following, let $X$ be a scheme of finite type over $R$. An $\mathbb{R}$-ideal on $X$ is a formal product $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$, where $r$ is a positive integer, each $a_i$ is a nonzero (coherent) ideal sheaf on $X$, and the $p_i$ are positive real numbers. We call $\mathfrak{A}$ a proper $\mathbb{R}$-ideal if there is $i$ with $a_i \neq \mathcal{O}_X$. If the $p_i$ are required to lie in some subset $\Gamma \subseteq \mathbb{R}_{>0}$, then $\mathfrak{A}$ is called a $\Gamma$-ideal.

The notions we are interested in are invariant with respect to the equivalence relation that identifies two $\mathbb{R}$-ideals if they have the same order of vanishing along all divisorial valuations. More precisely, we consider all proper birational morphisms $\pi : Y \to X$, with $Y$ normal, and all prime divisors $E$ on $Y$. Every such $E$ defines a valuation $\text{ord}_E$ of the function field of $X$. The image of $E$ on $X$ is the center of $E$ on $X$, and it is denoted by $c_X(E)$. If $a$ is an ideal sheaf on $X$, then $\text{ord}_E(a)$ is the minimum of $\text{ord}_E(w)$, where $w$ varies over the sections of $a$ defined at the generic point of $c_X(E)$. If $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$ is an $\mathbb{R}$-ideal on $X$, then

$$\text{ord}_E(\mathfrak{A}) := \sum_{i=1}^{r} p_i \cdot \text{ord}_E(a_i).$$

The equivalence relation identifies $\mathfrak{A}$ and $\mathfrak{A}'$ whenever $\text{ord}_E(\mathfrak{A}) = \text{ord}_E(\mathfrak{A}')$ for every $E$ as above.

**Remark 2.1.** By Theorem 2.3 below, whenever we consider a valuation $\text{ord}_E$ as above, we may assume that the model $Y$ on which $E$ lies is nonsingular.

**Example 2.2.** If $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$, where all $p_i \in \mathbb{Q}$, then $\mathfrak{A}$ is identified with $b^{1/m}$, where $m$ is a positive integer such that $mp_i \in \mathbb{Z}$ for all $i$, and $b = \prod_{i=1}^{r} a_i^{mp_i}$. Furthermore, two such $\mathbb{Q}$-ideals $b_1^{1/m}$ and $b_2^{1/m}$ are identified if and only if for some positive integer $q$, the ideals $b_1^q$ and $b_2^q$ have the same integral closure.

The product of $\mathbb{R}$-ideals is defined in the obvious way, by concatenating the factors. Similarly, if $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$ is as above, and $q \in \mathbb{R}_{>0}$, then $\mathfrak{A}^q := \prod_{i=1}^{r} a_i^{qp_i}$. Note that these operations preserve the above equivalence classes.

Suppose now that $X$ is normal, and let $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$ be an $\mathbb{R}$-ideal on $X$. A log resolution of $(X, \mathfrak{A})$ is a log resolution for the pair $(X, \prod_{i=1}^{r} a_i)$. Recall that this is a proper birational morphism $\pi : Y \to X$, with $Y$ nonsingular, such that the exceptional locus of $\pi$ and the inverse images of the subschemes $V(a_i)$ are Cartier divisors, and all these divisors have simple normal crossings. Since we are in characteristic zero, the existence of log resolutions in our setting is guaranteed by the results in [Tem]. For completeness, we explain how to get log resolutions from the results in loc. cit. The two theorems below
hold for arbitrary quasi-excellent schemes\(^2\), so they hold in particular in our setting, for schemes of finite type over \(R\).

**Theorem 2.3.** ([Tem]) For every integral scheme \(X\) of finite type over \(R\), there is a proper birational morphism \(\pi: Y \to X\) with \(Y\) nonsingular. Furthermore, we may construct \(\pi\) such that it is an isomorphism over \(X_{\text{reg}}\).

**Theorem 2.4.** ([Tem]) If \(X\) is a nonsingular scheme as above, and \(D\) is an effective divisor on \(X\), then there is a proper birational morphism \(\pi: Y \to X\) such that \(Y\) is nonsingular and \(\pi^*(D)\) has simple normal crossings. Furthermore, we may assume that \(\pi\) is an isomorphism over \(X \setminus \text{Supp}(D)\).

Let us explain how to combine these two theorems in order to get log resolutions. Suppose that \((X, \mathfrak{A})\) is a pair as above, with \(\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}\), and \(X\) normal. We first apply Theorem 2.3 to construct \(\pi_1: Y_1 \to X\) proper and birational, and such that \(Y_1\) is nonsingular. Since \(X\) is normal, there is an open subset \(U \subseteq X\) such that \(\pi_1\) is an isomorphism over \(U\), and \(Z := X \setminus U\) has \(\text{codim}(Z, X) \geq 2\). We note that if \(\varphi: Y \to Y_1\) is proper and birational, with \(Y\) nonsingular, and if \(\varphi^{-1}(\pi_1^{-1}(Z))\) is a divisor, then

\[
\text{Exc}(\pi_1 \circ \varphi) = \text{Exc}(\varphi) \cup \varphi^{-1}(\pi_1^{-1}(Z)).
\]

In particular, this exceptional locus is a divisor (recall that \(\text{Exc}(\varphi)\) is a divisor; this follows for instance from Lemma A.11).

We blow-up successively along \(\prod_{i=1}^{r} a_i\), and along the inverse image of \(Z\), to get \(\pi_2: Y_2 \to Y_1\). We now apply one more time Theorem 2.3 to get a proper and birational morphism \(\pi_3: Y_3 \to Y_2\) with \(Y_3\) nonsingular. Furthermore, we do this so that \(\pi_3\) is an isomorphism over \((Y_2)_{\text{reg}}\). It follows from (1) that \(E := \text{Exc}(\pi_1 \circ \pi_2 \circ \pi_3)\) is an effective divisor on \(Y_3\), and we have effective divisors \(E_i\) on \(Y_3\) such that \(a_i \cdot \mathcal{O}_{Y_3} = \mathcal{O}_{Y_3}(-E_i)\). Furthermore, if \(Z' = (\pi_1 \circ \pi_2 \circ \pi_3)^{-1}(Z)\), then by construction

\[
\text{Supp}(Z') \subseteq \text{Supp}(E) \subseteq \text{Supp}(Z') \cup \text{Supp}(E_1 + \cdots + E_r).
\]

We apply Theorem 2.4 to get a proper birational morphism \(\pi_4: Y \to Y_3\) with \(Y\) nonsingular, and such that \(\pi_4^*(Z' + E_1 + \cdots + E_r)\) has simple normal crossings. Furthermore, we may and will assume that this is an isomorphism over \(Y_3 \setminus \text{Supp}(Z' + E_1 + \cdots + E_r)\). We let \(\pi: Y \to X\) be the composition. Using (1) and (2) we see that \(\text{Exc}(\pi)\) is a divisor, and that it is contained in the support of \(\pi_4^*(Z' + E_1 + \cdots + E_r)\). Therefore \(\pi\) is a log resolution of \((X, \mathfrak{A})\). A similar argument can be used to show that any two log resolutions of \((X, \mathfrak{A})\) are dominated by a third one.

Suppose now that \(X\) is \(\mathbb{Q}\)-Gorenstein, and let \(\pi: Y \to X\) be a log resolution of a pair \((X, \mathfrak{A})\), where \(\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}\). Let \(K_{Y/X}\) be the relative canonical divisor as defined

\[^2\text{A scheme is quasi-excellent if it is covered by affine open subsets of the form \text{Spec}(A_i), with each } A_i \text{ a quasi-excellent ring; the definition of quasi-excellent ring is similar to that of excellent ring, but one does not require the ring to be universally catenary, see [Mat1, p. 260].}\]
in Appendix A (cf. Lemma A.11). Since $K_{Y/X}$ is supported on the exceptional locus, it follows that there is a simple normal crossings divisor $\sum_{j=1}^{\ell} E_j$ on $Y$ with

$$K_{Y/X} = \sum_{j=1}^{\ell} \kappa_j E_j, \quad a_i \cdot \mathcal{O}_Y = \mathcal{O}_Y \left( - \sum_{j=1}^{\ell} \alpha_{i,j} E_j \right) \text{ for } 1 \leq i \leq r.$$  

The pair $(X, \mathfrak{A})$ is called log canonical if

$$\kappa_j + 1 \geq \sum_{i=1}^{r} \alpha_{i,j} p_i = \text{ord}_E(\mathfrak{A})$$

for all $j$. If all inequalities in (4) are strict, the pair is Kawamata log terminal (or klt, for short). If $\mathfrak{A} = \mathcal{O}_X$, we simply say that $X$ is log canonical or klt, respectively. Note that the definitions are independent of the representative for $\mathfrak{A}$ is our equivalence class. The fact that the definition is independent of the log resolution follows in the same way as in the case of schemes of finite type over a field. The key ingredients are given by Lemma A.11 iii), and the fact that any two log resolutions can be dominated by a third one.

**Remark 2.5.** It follows from Remark A.12 that if $X$ is nonsingular, then the log canonicity of a pair $(X, \mathfrak{A})$ is independent of the $R$-scheme structure of $X$. Again, it is not clear to us whether the same remains true if $X$ is singular (however, see Remark 2.12 below for one case when this holds, which is the one that concerns us most).

Let $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$ and $\mathfrak{B} = \prod_{i=1}^{s} b_i^{q_i}$ be $R$-ideals on $X$, with $\mathfrak{A}$ a proper ideal. If the pair $(X, \mathfrak{B})$ is log canonical, then we define the *mixed log canonical threshold* $\text{lct}_{(X, \mathfrak{B})}(\mathfrak{A})$ (written also as $\text{lct}_{\mathfrak{B}}(\mathfrak{A})$ when there is no ambiguity about the ambient scheme) as the largest $c \geq 0$ such that $(X, \mathfrak{A} + \mathfrak{B})$ is log canonical. If $\mathfrak{B} = \mathcal{O}_X$, then we simply write $\text{lct}(X, \mathfrak{A})$ or $\text{lct}(\mathfrak{A})$, and we call it the *log canonical threshold* of $\mathfrak{A}$. If $\pi$ as above is a log resolution of $(X, \mathfrak{A} \cdot \mathfrak{B})$, and if $b_i \cdot \mathcal{O}_Y = \mathcal{O}_Y (-\sum_{j=1}^{s} \beta_{i,j} E_j)$ for $1 \leq i \leq s$, then

$$\text{lct}_{(X, \mathfrak{B})}(\mathfrak{A}) = \min_j \frac{\kappa_j + 1 - \sum_{i=1}^{s} \beta_{i,j} q_i}{\sum_{i=1}^{r} \alpha_{i,j} p_i}.$$  

Note that since $\mathfrak{A}$ is assumed to be proper, there are $i$ and $j$ such that $\alpha_{i,j} > 0$, hence the above minimum is finite. If $E_j$ is such that the minimum in (5) is achieved, we say that $E_j$ computes $\text{lct}_{(X, \mathfrak{B})}(\mathfrak{A})$.

We also consider a local version of the above invariant. If $\mathfrak{A} = \prod_{i=1}^{r} a_i^{p_i}$ is an $R$-ideal on $X$, we denote by $\text{Supp}(\mathfrak{A})$ the union of the closed subsets of $X$ defined by the ideals $a_i$. If $x \in \text{Supp}(\mathfrak{A})$, and $(X, \mathfrak{B})$ is log canonical in some open neighborhood of $x$, then $\text{lct}_{(X, \mathfrak{B}),x}(\mathfrak{A})$ is the largest $c \geq 0$ such that $(X, \mathfrak{A} + \mathfrak{B})$ is log canonical in some neighborhood of $x$. If $\mathfrak{B} = \mathcal{O}_X$, we write $\text{lct}_x(X, \mathfrak{A})$ or $\text{lct}_x(\mathfrak{A})$. Of course, $\text{lct}_{(X, \mathfrak{B}),x}(\mathfrak{A})$ can be described by a formula analogous to (5), in which the minimum is over those $j$ such that $x \in c_X(E_j)$.

For simplicity, we will state most of the basic properties of log canonical thresholds only in the unmixed setting, since we will only need these versions. The following lemma
Lemma 2.6. Suppose that $X$ is log canonical, and let $\mathfrak{A} = \prod_{i=1}^{r} a_i^{b_i}$ and $\mathfrak{B} = \prod_{i=1}^{s} b_i^{q_i}$ be proper $\mathbb{R}$-ideals on $X$. If $s \leq r$, and $a_i \subseteq b_i$ and $p_i \geq q_i$ for all $i \leq s$, then $\text{lct}(\mathfrak{A}) \leq \text{lct}(\mathfrak{B})$. A similar assertion holds for the local version of log canonical thresholds.

It is sometimes convenient to reduce the study of log canonical thresholds of $\mathbb{R}$-ideals to that of $\mathbb{Q}$-ideals (hence to that of usual ideals). This can be done using the following two lemmas (the first one deals with the log canonical threshold, while the second one treats the divisors computing the log canonical threshold).

Lemma 2.7. Assume that $X$ is log canonical, and let $\mathfrak{A} = \prod_{i=1}^{r} a_i^{b_i}$ be a proper $\mathbb{R}$-ideal on $X$. If $(p_{i,m})_{m \geq 1}$ are sequences of positive real numbers with $\lim_{m \to \infty} p_{i,m} = p_i$ for every $i \leq r$, and if $\mathfrak{A}_m = \prod_{i=1}^{r} a_i^{p_{i,m}}$, then $\lim_{m \to \infty} \text{lct}(\mathfrak{A}_m) = \text{lct}(\mathfrak{A})$. A similar assertion holds for the local version of log canonical threshold $\text{lct}_x(\mathfrak{A})$.

Proof. The assertion follows immediately from formula (5). \qed

Lemma 2.8. Suppose that $X$ is log canonical, and let $E$ be a divisor computing $\text{lct}_x(\mathfrak{A}) = \lambda$, for some $\mathbb{R}$-ideal $\mathfrak{A} = \prod_{i=1}^{r} a_i^{b_i}$ on $X$, containing $x$ in its support. Then one can find sequences of rational numbers $(p_{i,m})_{m \geq 1}$ with $\lim_{m \to \infty} p_{i,m} = \lambda p_i$, and such that if we put $\mathfrak{A}_m = \prod_{i=1}^{r} a_i^{p_{i,m}}$, then $\text{lct}_x(\mathfrak{A}_m) = 1$ and $E$ computes $\text{lct}_x(\mathfrak{A}_m)$ for every $m$.

Proof. Let $\pi : Y \to X$ be a log resolution of $\mathfrak{A}$ such that $E$ is a divisor on $Y$. With the notation in (3), after restricting to a suitable open neighborhood of $x$, we may assume that $x \in c_X(E_j)$ for all $j$. Consider the rational polyhedron

$$P = \{(u_1, \ldots, u_r) \mid \kappa_j + 1 \geq \sum_{i=1}^{r} \alpha_{i,j} u_i \text{ for all } j\}.$$

If $E = E_{j_0}$, then we see that $(\lambda p_1, \ldots, \lambda p_r)$ lies on the face $P_E$ of $P$ defined by $\kappa_{j_0} + 1 = \sum_i \alpha_{i,j_0} u_i$. Since $P_E$ is itself a rational polyhedron, it follows that $(\lambda p_1, \ldots, \lambda p_r)$ can be written as the limit of a sequence $(p_{1,m}, \ldots, p_{r,m}) \in P_E \cap \mathbb{Q}^r$. It is clear that this sequence satisfies our requirements. \qed

The following lemma allows one to reduce the study of the log canonical threshold to the case when this invariant is computed by a divisor with center equal to a closed point. The proof is the same as that of [dFEM, Lemma 5.2], so we omit it.

Lemma 2.9. Suppose that $X$ is log canonical, $\mathfrak{A}$ is a proper $\mathbb{R}$-ideal on $X$, and $x \in X$ is a closed point defined by the ideal $m_x$. If $c = \text{lct}_x(\mathfrak{A})$, then there is a nonnegative real number $t$ such that $c = \text{lct}_x(m_x^t \cdot \mathfrak{A})$, and this log canonical threshold is computed by a divisor $E$ over $X$ having center equal to $x$. 

We will mainly be interested in the case when the ambient variety is either a scheme of finite type over a field, or the spectrum of the completion of the local ring of such a scheme at a closed point. The following proposition gives the compatibility of the log canonical threshold with respect to taking such a completion. Suppose that \( X \) is a scheme of finite type over \( k \) and \( x \in X \) is a closed point, and consider \( g : Z = \text{Spec}(\widehat{O}_{X,x}) \to X \). If \( \mathfrak{A} = \prod_{i=1}^r a_i^{n_i} \) is an \( R \)-ideal on \( X \), we denote by \( \widehat{\mathfrak{A}} \) the \( R \)-ideal \( \prod_{i=1}^r (a_i O_Z)^{n_i} \). We consider the Cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & Y \\
\downarrow f & & \downarrow \pi \\
Z = \text{Spec}(\widehat{O}_{X,x}) & \xrightarrow{g} & X
\end{array}
\]

where \( \pi : Y \to X \) is a proper birational morphism with \( Y \) nonsingular.

**Remark 2.10.** Since \( g \) is a regular morphism (see [Mat1, Chapter 32]), it follows that so is \( h \). Recall that a morphism of Noetherian schemes is regular if it is flat, and has geometrically regular fibers. Since \( g \) is regular, we see that \( X \) is normal around \( x \) if and only if \( Z \) is normal, and since \( h \) is regular, we see that \( W \) is nonsingular.

**Proposition 2.11.** With the above notation, the following hold:

i) The pair \((X, \mathfrak{B})\) is log canonical (respectively, klt) in a neighborhood of \( x \) if and only if \((Z, \widehat{\mathfrak{B}})\) is log canonical (respectively, klt).

ii) If \((X, \mathfrak{B})\) is log canonical in a neighborhood of \( x \in \text{Supp}(\mathfrak{A}) \), then \( \widehat{\mathfrak{A}} \) is a proper \( R \)-ideal and

\[
lct_{(X, \mathfrak{B}), x}(\mathfrak{A}) = lct_{(Z, \widehat{\mathfrak{B}})}(\widehat{\mathfrak{A}}).
\]

iii) Under the assumptions in ii), if \( E_i \) and \( F \) are as in Remark A.15, then \( F \) computes \( lct_{(X, \mathfrak{B}), x}(\mathfrak{A}) \) if and only if \( E_i \) computes \( lct_{(Z, \widehat{\mathfrak{B}})}(\widehat{\mathfrak{A}}) \).

**Proof.** Note that if \( \pi \) is a log resolution of \((X, \mathfrak{A} \cdot \mathfrak{B})\), then \( f \) is a log resolution of \((Z, \widehat{\mathfrak{A}} \cdot \widehat{\mathfrak{B}})\) because \( h \) is a regular morphism. Furthermore, if \( F \) is a nonsingular prime divisor on \( Y \) such that \( x \in c_X(F) \), and \( E \) is a component of \( h^*(F) \), then \( E \) is a nonsingular prime divisor on \( W \). It is clear that the coefficient of \( F \) in a simple normal crossings divisor \( D \) on \( Y \) is equal to the coefficient of \( E \) in \( h^*(D) \). We now deduce the assertions in the proposition from Proposition A.14. \(\square\)

**Remark 2.12.** In the setting of the proposition, it follows from Proposition A.14 that the divisor \( K_{W/Z} \) does not depend on the presentation of \( \widehat{O}_{X,x} \) via Cohen’s Structure Theorem. Furthermore, if \( \mathfrak{A}' \) is an arbitrary \( R \)-ideal on \( Z \) (not necessarily coming from \( X \)), and if we consider a log resolution \( W' \to Z \) of \((Z, \mathfrak{A}')\), then it follows from Lemma A.13 and Remark A.12 that \( K_{W'/Z} \) is independent of the presentation of \( \widehat{O}_{X,x} \). Therefore, the (mixed) log canonical thresholds on \( Z \) are independent of this presentation.
Remark 2.13. Note that in the setting of the proposition, if $X$ is nonsingular, then the conclusion of the proposition also holds if the localization is at a non-closed point. Indeed, when we deal with nonsingular schemes, then we do not need to consider $\mathcal{O}(K_X)$, as the divisors $K_{Y/X}$ and $K_{W/Z}$ can be computed using the 0th Fitting ideals of the corresponding sheaves of relative differentials, and $h^*\Omega_{Y/X} \cong \Omega_{W/Z}$.

The following lemma concerns the behavior of singularities of pairs for schemes of finite type over a field under the extension of the ground field. In particular, it allows us to reduce the study of singularities of such pairs to the case when the ground field is algebraically closed.

Lemma 2.14. Let $X$ be a normal scheme of finite type over a field $k$. If $K$ is a field extension of $k$, and if $\varphi : \overline{X} = X \times_{\text{Spec}(k)} \text{Spec}(K) \to X$ is the projection, then for every $R$-ideal $\mathfrak{A} = \prod_{i=1}^r a_i^{p_i}$ on $X$, and every $\overline{x} \in \overline{X}$ and $x = \varphi(\overline{x})$, we have:

i) $rK_X$ is Cartier at $x$ if and only if $rK_{\overline{X}}$ is Cartier at $\overline{x}$.

ii) The pair $(X, \mathfrak{A})$ is log canonical (respectively, klt) in some neighborhood of $x$ if and only if $(\overline{X}, \overline{\mathfrak{A}})$ is log canonical (respectively, klt) in some neighborhood of $\overline{x}$, where

$$\overline{\mathfrak{A}} = \prod_{i=1}^r (a_i \mathcal{O}_{\overline{X}})^{p_i}.$$ 

iii) If $X$ is log canonical at $x$, then $\text{lct}_x(X, \mathfrak{A}) = \text{lct}_{\overline{x}}(\overline{X}, \overline{\mathfrak{A}})$.

iv) If $F$ is a divisor that computes $\text{lct}_x(X, \mathfrak{A})$, and if $E$ is a component of the divisor $\overline{F} = F \times_{\text{Spec}(k)} \text{Spec}(K)$ on $\overline{X}$ whose center contains $\overline{x}$, then $E$ computes $\text{lct}_{\overline{x}}(\overline{X}, \overline{\mathfrak{A}})$.

Proof. Note that the fibers of $\varphi$ are disjoint unions of zero-dimensional, reduced schemes. Since we are in characteristic zero and $\varphi$ is flat, we deduce from this fact that $\varphi$ is regular. In particular, $\overline{X}$ is normal (though it might be disconnected), and $\varphi^{-1}(X_{\text{reg}}) = \overline{X}_{\text{reg}}$. It is also easy to deduce that if $F$ is a reflexive sheaf on $X$, then $\varphi^*(F)$ is reflexive, and $F$ is generated by one element around $x$ if and only if $\varphi^*(F)$ is generated by one element around $\overline{x}$. Since $\Omega_{\overline{X}/\overline{K}} \cong \varphi^*(\Omega_{X/K})$, this implies that we can take $\varphi^*(K_X) = K_{\overline{X}}$, and $rK_X$ is Cartier at $x$ if and only if $rK_{\overline{X}}$ is Cartier at $\overline{x}$.

Suppose now that $X$ is $\mathbb{Q}$-Gorenstein, and let $f : Y \to X$ be a log resolution of $(X, \mathfrak{A})$. If $\overline{Y} = Y \times_{\text{Spec}(k)} \text{Spec}(K)$, then we have a Cartesian diagram

$$\begin{array}{ccc}
\overline{Y} & \xrightarrow{\psi} & Y \\
g \downarrow & & \downarrow f \\
\overline{X} & \xrightarrow{\varphi} & X.
\end{array}$$

If $f$ is an isomorphism over $U \subseteq X$, then $g$ is an isomorphism over the dense open subset $\varphi^{-1}(U)$ of $\overline{X}$. Since $\psi$ is regular, arguing as in the proofs of Propositions A.14 and 2.11, we see that $g$ is a log resolution of $(\overline{X}, \overline{\mathfrak{A}})$, and we have $K_{\overline{Y}/\overline{X}} = \psi^*(K_{Y/X})$. Note also that
if $E$ is a prime nonsingular divisor on $Y$ such that $x \in c_X(E)$, then there is a component $F$ of $E = \psi^*(E)$ such that $x \in c_X(F)$. For every such $F$, the valuation $\text{ord}_F$ restricts to the valuation $\text{ord}_E$ on the function field of $X$. The remaining assertions in the proposition are easy consequences of these observations.

We now give some further properties of log canonical thresholds that will be used in the proof of our main result. These generalize corresponding results for schemes of finite type over a field, and for usual ideals. Let us fix the notation. In what follows $X$ is a log canonical scheme of finite type over a field $k$. Let $x \in X$ be a closed point $\dim(\mathcal{O}_{X,x}) = n$, and let $g: Z = \text{Spec}(\widehat{\mathcal{O}_{X,x}}) \to X$ be the canonical morphism. We have seen in Proposition 2.11 that if $A$ is a $R$-ideal on $X$ such that $x \in \text{Supp}(A)$, then $\text{lct}(Z, \widehat{A}) = \text{lct}_x(X, A)$. The following lemma allows us to approximate every log canonical threshold on $Z$ by log canonical thresholds of pull-backs of $R$-ideals on $X$.

**Proposition 2.15.** Let $B = \prod_{i=1}^r b_i^q_i$ be an $R$-ideal on $Z$. If $m$ is the ideal defining the closed point on $Z$, and if we put $B_d = \prod_{i=1}^r (b_i + m^d)^q_i$, then

$$\lim_{d \to \infty} \text{lct}(Z, B_d) = \text{lct}(Z, B).$$

Furthermore, if there is a divisor $E$ over $Z$ computing $\text{lct}(Z, B)$ and with center equal to the closed point, then $\text{lct}(Z, B_d) = \text{lct}(Z, B)$ for $d \gg 0$.

**Proof.** The proof follows verbatim the proof of [dFM, Proposition 2.5] (the hypothesis in loc. cit. that the ambient scheme is nonsingular does not play any role). The key step is to show that

$$\text{lct}(Z, B) = \inf_F \frac{\text{ord}_F(K_{W/Z}) + 1}{\text{ord}_F(B)},$$

where the infimum is over the divisors $F$ over $Z$ (lying on some $W$) having center equal to the closed point. We refer to loc. cit. for details.

**Remark 2.16.** With the notation in the proposition, if $n$ denotes the maximal ideal in $\mathcal{O}_{X,x}$, then $m = n \cdot \widehat{\mathcal{O}_{X,x}}$, and $\mathcal{O}_{X,x}/n^d \cong \widehat{\mathcal{O}_{X,x}}/m^d$ for every $d$. It follows that, after possibly replacing $X$ by an affine open neighborhood of $x$, every $R$-ideal $B_d$ in the proposition can be written as $\widehat{A}_d$ for some $R$-ideal $A_d$ on $X$.

**Lemma 2.17.** Suppose that $X$ is klt. If $n$ is the ideal defining $x \in X$, then $\text{lct}(X, n) \leq n$.

**Proof.** We apply Lemma 2.14, with $K = \overline{k}$, the algebraic closure of $k$. We see that $\text{lct}(X, n) = \text{lct}_{\overline{x}}(\overline{X}, \overline{n})$ for any point $\overline{x} \in \varphi^{-1}(x)$. Since in some neighborhood of $\overline{x}$, $\overline{n}$ is equal to the ideal defining $\overline{x}$, we see that we may replace $X$ by $\overline{X}$. Therefore we may assume that $k$ is algebraically closed.

As pointed out by Kawakita, the bound now follows from the proof of [Kaw, Theorem 2.2]. For the sake of the reader, we briefly recall the argument. After replacing $X$
by its index one cover corresponding to $\mathcal{O}(K_X)$, we may assume that $K_X$ is Cartier. Let $f: Y \rightarrow X$ be a log resolution of $(X, n)$, and write $n \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$, with $E = \sum_i m_i E_i$. We can choose $F = E_{i_0}$ such that $\mathcal{O}(-E)|_F$ is big and nef. For every $m \geq 1$, we have an exact sequence

$$0 \rightarrow \mathcal{O}(K_{Y/X} - mE) \rightarrow \mathcal{O}(K_{Y/X} - mE + F) \rightarrow \mathcal{O}(K_F) \otimes \mathcal{O}(-mE)|_F \rightarrow 0.$$ 

By the Kawamata-Viehweg Vanishing Theorem, it follows that

$$P(m) := h^0(F, \mathcal{O}(K_F) \otimes \mathcal{O}(-mE)|_F) = \chi(F, \mathcal{O}(K_F) \otimes \mathcal{O}(-mE)|_F),$$

and this is a polynomial of degree $(n-1)$ since $((-E)^{n-1} \cdot F) > 0$. It follows that there is an integer $s$, with $1 \leq s \leq n$ such that $P(s) \neq 0$.

On the other hand, another application of the Kawamata-Viehweg Vanishing Theorem implies that $R^1 f_* \mathcal{O}(K_{Y/X} - sE) = 0$, hence the sequence

$$0 \rightarrow f_* \mathcal{O}(K_{Y/X} - sE) \rightarrow f_* \mathcal{O}(K_{Y/X} - sE + F) \rightarrow f_* (\mathcal{O}(K_F) \otimes \mathcal{O}(-sE)|_F) \rightarrow 0$$

is exact. It follows that $f_* \mathcal{O}(K_{Y/X} - sE) \neq f_* \mathcal{O}(K_{Y/X} - sE + F)$, hence there is a rational function $\varphi$ on $X$ such that $D := \text{div}_Y(\varphi) + K_{Y/X} - sE + F \geq 0$, and the coefficient of $F$ in $D$ is zero. Since $K_{Y/X} - sE + F$ is $f$-exceptional, it follows that $\varphi \in \mathcal{O}(X)$. Therefore $\text{ord}_F(\varphi) \geq 0$, and we conclude that $\text{ord}_F(K_{Y/X}) + 1 \leq s \cdot \text{ord}_F(n)$, and hence $\text{lct}(X, n) \leq s \leq n$.

**Lemma 2.18.** If $a$ and $b$ are ideals on $X$ such that their supports contain $x \in X$, then $\text{lct}_x(a + b) \leq \text{lct}_x(a) + \text{lct}_x(b)$.

**Proof.** Arguing as in the proof of the previous lemma, we may assume that $k$ is algebraically closed. It is now convenient to use the language of multiplier ideals, for which we refer to [Laz, Chapter 9]. The version of the Summation Theorem from [JM, Corollary 2] implies that for every $\lambda \geq 0$ we have the following description for the multiplier ideals of exponent $\lambda$ of a sum of ideals:

$$\mathcal{J}(X, (a + b)^\lambda) = \sum_{\alpha + \beta = \lambda} \mathcal{J}(X, a^\alpha b^\beta).$$

Recall that $\text{lct}_x(a)$ is the smallest $\alpha$ such that $x$ lies in the support of $\mathcal{J}(X, a^\alpha)$. Let $c_1 = \text{lct}_x(a)$ and $c_2 = \text{lct}_x(b)$. It is enough to show that $x$ lies in the support of $\mathcal{J}(X, (a+b)^{c_1+c_2})$. This follows from (7), since given $\alpha$, $\beta$ such that $\alpha + \beta = c_1 + c_2$, then either $\alpha \geq c_1$, or $\beta \geq c_2$. In the first case we have

$$\mathcal{J}(X, a^\alpha b^\beta) \subseteq \mathcal{J}(X, a^\alpha) \subseteq \mathcal{J}(X, a^{c_1}),$$

whose support contains $x$. The case $\beta \geq c_2$ is similar.

**Proposition 2.19.** Suppose that $X$ is klt in a neighborhood of $x$. Let $\mathfrak{B} = \prod_{i=1}^r b_i^q_i$ be a proper $\mathbb{R}$-ideal on $Z$, and let $m$ be the ideal defining the closed point of $Z$.

i) If $b_i \subseteq m^{s_i}$ for every $i$, then $\text{lct}(Z, \mathfrak{B}) \leq \frac{n}{s_1 q_1 + \cdots + s_r q_r}$. 


ii) Suppose that \( \mathfrak{A} = \prod_{i=1}^r a_i^{q_i} \) is another \( \mathbb{R} \)-ideal on \( Z \), and let \( \varepsilon > 0 \) be a real number. If \( d \) is a positive integer such that \( d \geq \frac{n}{\varepsilon q_i} \) and \( a_i + m^d = b_i + m^d \) for all \( i \), then

\[
| \text{lct}(\mathfrak{A}) - \text{lct}(\mathfrak{B}) | \leq \varepsilon.
\]

Proof. For i), it is clear that if \( s = s_1 q_1 + \cdots + s_r q_r \), then

\[
\text{lct}(Z, \mathfrak{A}) \leq \text{lct}(Z, \mathfrak{B}) = \frac{\text{lct}(Z, \mathfrak{m})}{s}.
\]

By Proposition 2.11 we have \( \text{lct}(Z, \mathfrak{m}) = \text{lct}(X, n) \), where \( n \) is the ideal defining \( x \in X \), and we conclude by Lemma 2.17.

For ii), we first show that if \( a \) and \( b \) are proper ideals on \( Z \) such that \( a + m^\ell = b + m^\ell \), then \( | \text{lct}(a) - \text{lct}(b) | \leq n/\ell \). Note that by Proposition 2.15, it is enough to show that \( | \text{lct}(a + m^N) - \text{lct}(b + m^N) | \leq n/\ell \) for all \( N \geq \ell \). Therefore we may assume that there are ideals \( a' \) and \( b' \) on \( X \) such that \( a = \hat{a}' \) and \( b = \hat{b}' \). By Proposition 2.11, it is enough to show that if \( a' + n^\ell = b' + n^\ell \), then \( | \text{lct}_x(a') - \text{lct}_x(b') | \leq n/\ell \). We deduce from Lemmas 2.17 and 2.18 that

\[
\text{lct}_x(a') \leq \text{lct}_x(a' + n^\ell) = \text{lct}_x(b' + n^\ell) \leq \text{lct}_x(b') + \frac{\text{lct}(n)}{\ell} \leq \text{lct}_x(b') + \frac{n}{\ell}.
\]

The inequality \( \text{lct}_x(b') \leq \text{lct}_x(a') + \frac{n}{\ell} \) follows by symmetry.

We now prove ii). After writing each \( q_i \) as a decreasing limit of rational numbers, we see using Lemma 2.7 that it is enough to prove ii) when all \( q_i \in \mathbb{Q} \). Let us choose a positive integer \( p \) such that all \( pq_i \) are integers.

It is enough to show that \( \text{lct}(\mathfrak{B}) \geq \text{lct}(\mathfrak{A}) - \varepsilon \), as the other inequality will follow by symmetry. After replacing each \( a_i \) by \( a_i + m^d \), we may assume that \( a_i = b_i + m^d \). Therefore \( a_i^{pq_i} = (b_i + m^d)^{pq_i} \), and this ideal has the same integral closure as \( b_i^{pq_i} + m^{dpq_i} \). Since \( \prod_i (b_i^{pq_i} + m^{dpq_i}) \) and \( \prod_i b_i^{pq_i} \) have the same image mod \( m^\ell \), where \( \ell = dp \cdot \min_i q_i \), we deduce

\[
\text{lct}(\mathfrak{A}) = p \cdot \text{lct} \left( \prod_i a_i^{pq_i} \right) = p \cdot \text{lct} \left( \prod_i (b_i^{pq_i} + m^{dpq_i}) \right)
\leq p \cdot \left( \text{lct} \left( \prod_i b_i^{pq_i} \right) + \frac{n}{\ell} \right) \leq \text{lct}(\mathfrak{B}) + \varepsilon,
\]

where the last inequality follows from the assumption that \( \ell \geq np/\varepsilon \). \qed

The following result is a key ingredient in the proof of our main result. In the case of schemes of finite type over a field and usual ideals, it was proved in [Kol1] and [dFEM].

**Proposition 2.20.** Consider the proper \( \mathbb{R} \)-ideals \( \mathfrak{A} = \prod_{i=1}^r a_i^{q_i} \) and \( \mathfrak{B} = \prod_{i=1}^r b_i^{q_i} \) on \( Z \), and suppose that \( E \) is a divisor that computes \( \text{lct}(\mathfrak{A}) \), having center equal to the closed point of \( Z \). If \( a_i + p_i = b_i + p_i \) for all \( i \), where \( p_i = \{ u \in \overline{O_{X,x}} \mid \text{ord}_E(u) > \text{ord}_E(a_i) \} \), then \( \text{lct}(\mathfrak{A}) = \text{lct}(\mathfrak{B}) \).

We will need the following lemma.
Lemma 2.21. If $E$ is a divisor over $Z = \text{Spec}(\mathcal{O}_{X,x})$ with center equal to the closed point of $Z$, then the restriction of $\text{ord}_E$ to the function field of $X$ is of the form $\text{ord}_F$ for some divisor $F$ over $X$ with center $x$. Moreover, one can find a Cartesian diagram as in (6) such that $F$ appears as a prime nonsingular divisor on $Y$ and $E$ appears as $h^*(F)$.

Proof. We first note that the valuation ring $\mathcal{O}_v$ of $v = \text{ord}_E$ is essentially of finite type over $\mathcal{O}_{X,x}$, and its residue field $k(E)$ has transcendence degree $(n - 1)$ over $k$ (recall that $n = \dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X,x})$). Indeed, let us realize $E$ as a prime divisor on some nonsingular $T$, with $\varphi: T \to Z$ birational. By assumption, $E$ lies in the fiber $T_0$ of $T$ over the closed point in $Z$. Note that $T_0$ is a scheme of finite type over $K$, where $K = k(x)$, hence over $k$ (recall that $K$ is finite over $k$). Let $y \in E$ be a closed point. It follows from the Dimension Formula (see [Mat1, Theorem 15.6]) that $\dim(\mathcal{O}_{T,y}) = n$. Since $E$ is a prime divisor on $T$, we deduce that $\mathcal{O}_v = \mathcal{O}_{T,E}$ is essentially of finite type over $k$. Its residue field $k(E)$ is the fraction field of $\mathcal{O}_{E,y}$. This has dimension $(n - 1)$, hence $k(E)$ has transcendence degree $(n - 1)$ over $K$ (equivalently, over $k$).

We now show that we can find a sequence of schemes $Z_0, \ldots, Z_m$, with $Z_0 = Z$ and each $Z_i$ being the blow-up of $Z_{i-1}$ at the center $C_{i-1}$ of $\text{ord}_E$ on $Z_{i-1}$, such that the center of $\text{ord}_E$ on $Z_m$ has codimension one. This is well-known in the case of schemes of finite type over a field, and for example the proof of [KM1, Lemma 2.45] can be easily adapted to our setting. Indeed, arguing as in loc. cit. one first shows that if the $Z_i$ are constructed as above, then $\mathcal{O}_v = \bigcup_{i \geq 1} \mathcal{O}_{Z_i,C_i}$. If $y_1, \ldots, y_{n-1} \in \mathcal{O}_v$ are such that their residues give a transcendence basis of $k(E)$ over $K$, let $i$ be large enough such that all the $y_j$ lie in $\mathcal{O}_{Z_i,C_i}$. Therefore the residue field of $\mathcal{O}_{Z_i,C_i}$ has transcendence degree $(n - 1)$ over $K$. Another use of the Dimension Formula implies that $\dim(\mathcal{O}_{Z_i,C_i}) = 1$. Since $\mathcal{O}_{X,x}$ is a Nagata ring, so is $\mathcal{O}_{Z_i,C_i}$, hence the normalization $S$ of $\mathcal{O}_{Z_i,C_i}$ is finite over $\mathcal{O}_{Z_i,C_i}$, $S$ is a Dedekind ring, and $\mathcal{O}_v$ is the localization of $S$ at a maximal ideal. If $m$ is such that $\mathcal{O}_{Z_m,C_m}$ contains $S$, then we see that $\text{codim}(C_m, Z_m) = 1$.

We similarly construct a sequence of varieties $X_i$, where $X_0 = X$ and $X_i$ is the blow-up of $X_{i-1}$ along the center $C_i^{X_i}$ of the restriction $w$ of $\text{ord}_E$ to the function field of $X$. It follows by induction on $i$ that we have Cartesian diagrams

$$
\begin{array}{c}
\begin{array}{c}
Z_i \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
X_i
\end{array}
\end{array}
\quad
\begin{array}{c}
\begin{array}{c}
Z_{i-1}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
X_{i-1}
\end{array}
\end{array}
$$

such that $C_i = (g_i)^{-1}(C_i^{X_i})$ (this follows since $C_i^{X_i}$ is clearly the closure of $g_i(C_i)$, and since $C_i$ lies over the closed point in $Z$). Since $g_m$ induces an isomorphism between $C_m$ and $C_m^{X_m}$, it follows that $\text{codim}(C_m^{X_m}, X_m) = 1$. Therefore $w$ is a divisorial valuation. The last assertion in the lemma follows by taking a model $Y$ over $X$ on which the proper transform of $C_m^{X}$ is nonsingular. \qed
Proof of Proposition 2.20. If \( k \) is algebraically closed, then the assertion for usual ideals on \( \text{Spec}(O_{X,x}) \) follows from [dFEM, Theorem 1.4] (see also [Kol1]). The assertion extends to the case when \( k \) is not algebraically closed via Lemma 2.14.

We now extend this to the case of (usual) ideals \( \mathfrak{A} \) and \( \mathfrak{B} \) on \( Z \) (that is, we assume \( r = 1 \) and \( q_1 = 1 \)). By Lemma 2.21 below, there is a divisor \( F \) over \( X \), with center \( x \), such that \( \text{ord}_F \) is equal to the restriction of \( \text{ord}_E \) to the function field of \( X \). Furthermore, it follows from the lemma and Proposition 2.11 that given an ideal \( \mathfrak{A}' \) on \( X \) with \( x \in \text{Supp}(\mathfrak{A}') \), \( F \) computes \( \text{lc}_x(\mathfrak{A}') \) if and only if \( E \) computes \( \text{lc}(\mathfrak{A}') \).

Recall that \( n \) is the ideal defining \( x \in X \), and \( m \) is the ideal defining the closed point in \( Z \). For every \( d \geq 1 \), after replacing \( X \) by any affine neighborhood of \( x \), we can find ideals \( \mathfrak{A}'_d \) and \( \mathfrak{B}'_d \) on \( X \) such that \( \mathfrak{A}'_d = \mathfrak{A} + m^d \) and \( \mathfrak{B}'_d = \mathfrak{B} + m^d \). Since \( \text{ord}_E(m) \geq 1 \), it follows that \( \text{ord}_E(\mathfrak{A} + m^d) = \text{ord}_E(\mathfrak{A}) \) for \( d > \text{ord}_E(\mathfrak{A}) \). Let us fix such \( d \). Since \( E \) computes \( \text{lc}(\mathfrak{A}) \), we deduce that \( \text{lc}(\mathfrak{A} + m^d) \leq \text{lc}(\mathfrak{A}) \), and if equality holds, then \( E \) computes \( \text{lc}(\mathfrak{A} + m^d) \). On the other hand, the inclusion \( \mathfrak{A} \subseteq \mathfrak{A} + m^d \) implies \( \text{lc}(\mathfrak{A}) \leq \text{lc}(\mathfrak{A} + m^d) \). We conclude that \( \text{lc}(\mathfrak{A}) = \text{lc}(\mathfrak{A} + m^d) \), and \( E \) computes \( \text{lc}(\mathfrak{A} + m^d) \). Therefore \( F \) computes \( \text{lc}(\mathfrak{A}'_d) = \text{lc}(\mathfrak{A}) \). Since in a neighborhood of \( x \), the ideals \( \mathfrak{B}'_d \) and \( \mathfrak{A}'_d \) are congruent modulo \( \{ u \mid \text{ord}_F(u) > \text{ord}_F(\mathfrak{A}'_d) \} \), we deduce from the case we already know that \( \text{lc}(\mathfrak{B} + m^d) = \text{lc}(\mathfrak{B}'_d) = \text{lc}(\mathfrak{A}) \). This holds for all \( d > \text{ord}_E(\mathfrak{A}) \), hence it follows from Proposition 2.15 that \( \text{lc}(\mathfrak{A}) = \text{lc}(\mathfrak{A}') \).

Let us deduce now the statement of the proposition in the case of \( \mathbb{R} \)-ideals. We first note that the hypothesis implies that \( \text{ord}_E(a_i) = \text{ord}_E(b_i) \) for all \( i \).

Claim. For all positive integers \( \ell_1, \ldots, \ell_r \), we have

\[
\prod_{i=1}^r a_i^{\ell_i} + J_{\ell_1, \ldots, \ell_r} = \prod_{i=1}^r a_i^{\ell_i} + J_{\ell_1, \ldots, \ell_r},
\]

where \( J_{\ell_1, \ldots, \ell_r} = \{ f \in \mathcal{O}_{X,x} \mid \text{ord}_E(f) > \text{ord}_E(a_1^{\ell_1} \cdots a_r^{\ell_r}) \} \).

After replacing each \( a_i \) and \( b_i \) by \( \ell_i \) copies of itself, we see that it is enough to prove the claim when all \( \ell_i = 1 \). If \( u_i \in b_i \), let us write \( u_i = w_i + h_i \), with \( w_i \in a_i \), and \( h_i \in p_i \). In this case,

\[
\prod_{i=1}^r u_i - \prod_{i=1}^r w_i = \sum_{m=1}^{r} \left( h_m \cdot \prod_{i<m} w_i \cdot \prod_{j>m} u_j \right).
\]

Since each of the terms on the right-hand side of (9) has order \( > \sum m \text{ord}_E(a_m) \), we deduce the inclusion \( \subseteq \) in (8), and the opposite inclusion follows by symmetry. This proves the claim.

We now apply Lemma 2.8 to get sequences of positive rational numbers \( (q_{m,j})_j \) for \( 1 \leq j \leq r \), with \( \lim_{m \to \infty} q_{m,j} = \text{lc}(\mathfrak{A}) \cdot q_j \) and such that, for all \( m \), \( \text{lc}(\prod_{i=1}^r a_i^{q_{m,i}}) = 1 \) and this log canonical threshold is computed by \( E \). We choose for each \( m \) a positive integer \( N_m \).
such that $N_{m,q_{m,j}} \in \mathbb{Z}$ for all $j$. If we put $a^{(m)} := \prod_{i=1}^{r} a_{i}^{N_{m,q_{m,i}}}$ and $b^{(m)} := \prod_{i=1}^{r} b_{i}^{N_{m,q_{m,i}}}$, then $\text{lct}(a^{(m)}) = 1/N_{m}$ is computed by $E$. The above claim, together with the case we already know (for usual ideals) gives $\text{lct}(b^{(m)}) = 1/N_{m}$ for every $m$. We now deduce $\text{lct}(\mathfrak{B}) = \text{lct}(\mathfrak{A})$ from Lemma 2.7, which completes the proof of the proposition. \hfill \square

3. Generic limit constructions

In this section we give a variant of the generic limit construction from [Kol1] (see also [dFEM]), that allows us to deal with the fact that in Theorem 1.1 the ambient space is not fixed. Let us fix an algebraically closed field $k$ of characteristic zero. Let $R = k[x_1, \ldots, x_N]$, let $m$ be the maximal ideal in $R$, and for every field extension $K/k$ let $R_K = K[x_1, \ldots, x_N]$ and $m_K = mR_K$.

For every $d \geq 1$, we consider the quotient homomorphism $R \to R/m^d$. We identify the ideals in $R/m^d$ with the ideals in $R$ containing $m^d$. Let $H_d$ be the Hilbert scheme parametrizing the ideals in $R/m^d$, with the reduced scheme structure. Since $\dim_k(R/m^d) < \infty$, $H_d$ is an algebraic variety. For every field extension $K/k$, the $K$-valued points of $H_d$ correspond to ideals in $R_K/m_K^d$. Mapping an ideal in $R/m^d$ to its image in $R/m^d$ gives a surjective map $\tau_d: H_d \to H_{d-1}$. This is not a morphism. However, by generic flatness we can cover $H_d$ by finitely many disjoint locally closed subsets such that the restriction of $\tau_d$ to each of these subsets is a morphism.

Let us fix a positive integer $m$. We also consider a parameter space $G$ for ideals in $k[x_1, \ldots, x_N]$ generated by polynomials of degree $\leq m$, and vanishing at the origin (we consider on $G$ the reduced structure). Each such ideal is determined by its intersection with the vector space of polynomials of degree $\leq m$, hence $G$ is an algebraic variety. Given a field extension $K/k$, the $K$-valued points of $G$ are in bijection with the ideals in $K[x_1, \ldots, x_N]$ vanishing at the origin and generated in degree $\leq m$.

We now fix also a positive integer $r$. Consider the product $Z_d := G \times (H_d)^r$ and the map $t_d: Z_d \to Z_{d-1}$ that is given by the identity on the first component, and by $\tau_d$ on the other components. As in the case of $\tau_d$, even though $t_d$ is not a morphism, we can cover $Z_d$ by disjoint locally closed subsets such that the restriction of $t_d$ to each of these subsets is a morphism. In particular, for every irreducible closed subset $Z \subseteq Z_d$, the map $t_d$ induces a rational map $Z \dashrightarrow Z_{d-1}$.

We now describe the generic limit construction. The main difference with the treatment in [dFEM] is coming from the first factor in $Z_d$. Suppose that we have $(r+1)$ sequences $(p_i)_{i \in I_0}, (a^{(1)}_i)_{i \in I_0}, \ldots, (a^{(r)}_i)_{i \in I_0}$ indexed by $I_0 = \mathbb{Z}_+$. Each $p_i$ is an ideal in $k[x_1, \ldots, x_N]$ generated in degree $\leq m$ and vanishing at the origin, and each $a^{(j)}_i$ is an ideal in $k[x_1, \ldots, x_N]$.

We consider sequences of irreducible closed subsets $Z_d \subseteq Z_d$ for $d \geq 1$ with the following properties:
(⋆) For every $d \geq 1$, the projection $t_{d+1}$ induces a dominant rational map $\varphi_{d+1} : Z_{d+1} \to Z_d$.

(⋆⋆) For every $d \geq 1$, there are infinitely many $i$ with $(p_i, a_i^{(1)} + m^d, \ldots, a_i^{(r)} + m^d) \in Z_d$, and the set of such $(r+1)$-tuples is dense in $Z_d$.

Given such a sequence $(Z_d)_{d \geq 1}$, we define inductively nonempty open subsets $Z_d^0 \subseteq Z_d$, and a nested sequence of infinite subsets

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots,$$

as follows. We put $Z_1^0 = Z_1$ and $I_1 = \{i \in I_0 \mid (p_i, a_i^{(1)} + m, \ldots, a_i^{(r)} + m) \in Z_1^0\}$. For $d \geq 2$, let $Z_d^0 = \varphi_d^{-1}(Z_{d-1}^0) \subseteq \text{Domain}(\varphi_d)$ and $I_d = \{i \in I_0 \mid (p_i, a_i^{(1)} + m^d, \ldots, a_i^{(r)} + m^d) \in Z_d^0\}$. It follows by induction on $d$ that $Z_d^0$ is open in $Z_d$, and condition (⋆⋆) implies that each $I_d$ is infinite. Furthermore, it is clear that $I_d \supseteq I_{d+1}$.

Sequences $(Z_d)_{d \geq 1}$ satisfying (⋆) and (⋆⋆) can be constructed as in [dFEM, Section 4]. Suppose now that we have a sequence $(Z_d)_{d \geq 1}$ with these two properties. The rational maps $\varphi_d$ induce a nested sequence of function fields $k(Z_d)$. Let $K := \bigcup_{d \geq 1} k(Z_d)$. Each morphism $\text{Spec}(K) \to Z_d \subseteq Z_d$ corresponds to an $(r+1)$-tuple $(\tilde{p}_d, \tilde{a}_d^{(1)}, \ldots, \tilde{a}_d^{(r)})$. All $\tilde{p}_d$ are equal, and we denote this ideal in $K[x_1, \ldots, x_N]$ by $p$. This is generated in degree $\leq m$ and vanishes at the origin. Furthermore, the compatibility between the morphisms $\text{Spec}(K) \to Z_d$ implies that there are (unique) ideals $a_j^{(i)}$ in $R_K$ (for $1 \leq j \leq r$) such that $\tilde{a}_d^{(j)} = a_j^{(i)} + m_j^d$ for every $d$. We call the $(r+1)$-tuple $(p, a^{(1)}, \ldots, a^{(r)})$ the generic limit of the given $(r+1)$ sequences of ideals.

We record in the next lemma some easy properties of this construction. The proof is straightforward, so we omit it.

**Lemma 3.1.** With the above notation, for every $j$ with $1 \leq j \leq r$, the following hold.

i) If $a_j^{(i)} = b$ for some ideal $b \subseteq R$ and every $i$, then $a_j^{(i)} = bR_K$.

ii) If $a_j^{(i)} \subseteq m$ for every $i$, then $a_j^{(1)} \subseteq m_K$.

iii) If $a_j^{(1)}, \ldots, a_j^{(s)} = (0)$, then for every $q \geq 1$ there are infinitely many $d$ such that $a_d^{(i)} \subseteq m^q$ for $1 \leq \alpha \leq s$.

Let $I_d$ be the universal ideal on $H_d \times A_k^N$, whose restriction to the fiber over a point in $H_d$ corresponding to the ideal $b$ containing $m^d$ is equal to $b$. Similarly, let $J$ be the universal ideal on $G \times A_k^N$. Let $\beta_j$ be the composition of the embedding $Z_d \times A_k^N \to Z_d \times A_k^N$ with the projection $Z_d \times A_k^N \to H_d \times A_k^N$ if $j \neq 0$ (or $Z_d \times A_k^N \to G \times A_k^N$ when $j = 0$) induced by the projection $H_d' \to H_d$ onto the $j$th factor (respectively by the projection $G \times H_d' \to G$). We consider the ideals on $Z_d \times A_k^N$ defined as follows: $I_d^{(0)} = (\beta_0)^{-1}(J)$ and $I_d^{(j)} = (\beta_j)^{-1}(I_d)$ for $1 \leq j \leq r$. For a not necessarily closed point $z \in Z_d$, we denote by $(I_d^{(j)})_z$ the restriction of $I_d^{(j)}$ to the fiber over $z$. Each tuple $(p_i, a_i^{(1)} + m^d, \ldots, a_i^{(r)} + m^d)$, with $i \in I_d$, corresponds to a closed point $t_{d,i} \in Z_d$ such that $(I_d^{(j)})_{t_{d,i}} \cdot R = a_i^{(j)} + m^d$ for
1 \leq j \leq r$, and $(\mathcal{I}_d^{(0)})_{t_d,i} = p_i$. By construction, the set $T_d := \{t_{d,i} | i \in I_d\}$ is dense in $Z_d$. Similarly, if $\eta_d \in Z_d$ is the generic point, we have $(\mathcal{I}_d^{(j)})_{\eta_d} \cdot R_K = a^{(j)} + m^d_K$ for $j \geq 1$, and $(\mathcal{I}_d^{(0)})_{\eta_d} \cdot K[x_1, \ldots, x_N] = p$. We denote by $\sigma_d: Z_d \to Z_d \times A^N_k$ the morphism given by $\sigma_d(u) = (u, 0)$.

**Lemma 3.2.** With the above notation, suppose that $p_i R \subseteq a^{(j)}$ for every $i \in I_0$ and every $j$ with $1 \leq j \leq r$. For every $d \geq 1$ there is a nonempty open subset $U_d$ of $Z_d$ such that $\mathcal{I}_d^{(0)} \subseteq \mathcal{I}_d^{(j)}$ over $U_d$ for $1 \leq j \leq r$. In particular, $p R_K \subseteq a^{(j)}$.

**Proof.** Since the support of $\mathcal{I}_d^{(j)}$ lies in $\sigma_d(Z_d)$, it follows that the open subset of $Z_d \times A^N_k$ where $\mathcal{I}_d^{(0)}$ is contained in $\mathcal{I}_d^{(j)}$ is the inverse image of an open subset $U_d$ in $Z_d$. This is nonempty, since by assumption all $t_{d,i}$, with $i \in T_d$, lie in $U_d$. Since $\eta_d \in U_d$, we deduce that $p \subseteq (a^{(j)} + m^d) \cap K[x_1, \ldots, x_N]$. This holds for every $d$, hence $p \cdot R_K \subseteq a^{(j)}$. \hfill $\square$

Suppose now that the sequences $(p_i)_{i \in I_0}, (a^{(1)}_{i})_{i \in I_0}, \ldots, (a^{(r)}_{i})_{i \in I_0}$ satisfy the hypothesis in Lemma 3.2. In this case we put $W_i = \text{Spec}(R/p_i R)$ and $W = \text{Spec}(R_K/p R_K)$. We denote by $\bar{a}^{(j)}_i = a^{(j)}_i/p_i R$ the ideal on $W_i$ corresponding to $a^{(j)}_i$, for $1 \leq j \leq r$. It follows from Lemma 3.2 that we may consider the ideals $\bar{a}^{(j)} = a^{(j)}/p R_K$ on $W$, for $1 \leq j \leq r$. We denote by $\bar{m}_i$ and $\bar{m}_K$ the ideals defining the closed points in $W_i$ and, respectively, in $W$.

The following is the main technical result about generic limits in our setting.

**Proposition 3.3.** With the above notation, if all $W_i$ are klt, and all $a^{(j)}_i$ are proper ideals, then the following hold.

i) $W$ is klt.

ii) For every $d$ there is an infinite subset $I^o_d \subseteq I_d$ such that for all nonnegative real numbers $p_1, \ldots, p_r$, and for every $i \in I^o_d$

$$\lct(W, \prod_{j=1}^r (\bar{a}^{(j)} + \bar{m}^d_K)^{p_j}) = \lct(W_i, \prod_{j=1}^r (\bar{a}^{(j)}_i + \bar{m}^d_i)^{p_j}).$$

iii) If $E$ is a divisor over $W$ computing $\lct(W, \prod_{j=1}^r (\bar{a}^{(j)}_i)^{p_j})$ for some nonnegative real numbers $p_1, \ldots, p_r$, and having center equal to the closed point, then there is an integer $d_E$ such that for every $d \geq d_E$ the following holds: there is an infinite subset $I^E_d \subseteq I_d$, and for every $i \in I^E_d$ a divisor $E_i$ over $W_i$ computing $\lct(W_i, \prod_{j=1}^r (\bar{a}^{(j)}_i + \bar{m}^d_i)^{p_j})$, such that $\text{ord}_E(\bar{m}_K) = \text{ord}_{E_i}(\bar{m}_i)$ and $\text{ord}_E(\bar{a}^{(j)} + \bar{m}^d_K) = \text{ord}_{E_i}(\bar{a}^{(j)}_i + \bar{m}^d_i)$ for every $j$. In particular, $E_i$ has center the closed point of $W_i$.

We emphasize that in part iii) of the proposition, both $d_E$ and the sets $I^E_d$ depend on $p_1, \ldots, p_r$, and $E_i$ depends on $d$. 
Proof. With the notation in Lemma 3.2, let $X_d$ be the closed subscheme of $U_d \times \mathbb{A}^N_k$ defined by $I_d^{(0)}$, and let $f : X_d \to U_d$ be the morphism induced by projection. It follows from the Lemma 3.2 that we can define ideal sheaves $b_d^{(j)}$ on $X_d$ as the quotient of (the restrictions to $X_d$ of) $I_d^{(j)}$ by $I_d^{(0)}$. We denote by $(X_d)_{\xi}$ the fiber of $X_d$ over a (not necessarily closed) point $\xi \in U_d$, and by $(b_d^{(j)})_{\xi}$ the restriction of $b_d^{(j)}$ to $(X_d)_{\xi}$. We will apply Corollary B.9 to $f$ and to the ideals $b_d^{(j)}$, with $1 \leq j \leq r$.

Note that each $W_i$ is obtained by completing at $\sigma_d(t_{d,i})$ the fiber $(X_d)_{t_{d,i}}$. Since $W_i$ is klt, it follows from Proposition 2.11 that $(X_d)_{t_{d,i}}$ is klt by another application of Proposition 2.11. This proves $i$).

The assertion in $ii$) follows directly from Proposition 2.11 and Corollary B.9 (see also Remark B.10). In order to prove $iii$), note first that if $d_E > \operatorname{ord}_E(\tilde{a}^{(j)})$ for all $j$ with $1 \leq j \leq r$, then $E$ also computes

$$\lct(W, \prod_j(\tilde{a}^{(j)} + \tilde{m}_K^d)^{p_j}) = \lct(W, \prod_j(\tilde{a}^{(j)})^{p_j})$$

for all $d \geq d_E$ (the inequality “$\leq$” follows since the assumption on $d$ implies $\operatorname{ord}_E(\tilde{a}^{(j)} + \tilde{m}_K^d) = \operatorname{ord}_E(\tilde{a}^{(j)})$ for all $j$, while the reverse inequality follows from the inclusions $\tilde{a}^{(j)} \subseteq \tilde{a}^{(j)} + \tilde{m}_K^d$). By Lemma 2.21, the restriction of the valuation $\operatorname{ord}_E$ to the fraction field of $K[x_1, \ldots, x_N]$ is equal to $\operatorname{ord}_F$, for some divisor $F$ over $\mathbb{A}^N_k$. We can choose $d_E$ large enough, so that for $d \geq d_E$ the divisor $F$ comes by base extension from a divisor $F_d$ defined over $k(Z_d)$. It follows from Proposition 2.11 and Lemma 2.14 that $F_d$ computes $\lct_{\sigma_d(\eta_d)}((X_d)_{\eta_d}, (\prod_j b_d^{(j)})_{\eta_d})$. The assertion in iii) now follows from Corollary B.9 $ii$). This completes the proof of the proposition. $\square$

Corollary 3.4. With the notation and assumption in Proposition 3.3, for every sequence $(i_d)_{d \geq 1}$ with $i_d \in I^0_d$, and for all nonnegative real numbers $p_1, \ldots, p_r$ we have

$$\lct(W, \prod_j(\tilde{a}^{(j)})^{p_j}) = \lim_{d \to \infty} \lct(W_{i_d}, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j}).$$

In particular, if the sequence $(\lct(W_{i_d}, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j})_{i_d \geq 1}$ is convergent, then it converges to $\lct(W, \prod_j(\tilde{a}^{(j)})^{p_j})$.

Proof. Since $W$ and all $W_i$ are klt, it follows from Proposition 2.19 $ii$) that given any $\varepsilon > 0$, if $d \geq \frac{N}{\varepsilon p_j}$ for every $j$ such that $p_j > 0$, then we have

$$|\lct(W, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j}) - \lct(W, \prod_j(\tilde{a}^{(j)} + \tilde{m}_K^d)^{p_j})| \leq \varepsilon,$$

$$|\lct(W_{i_d}, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j}) - \lct(W_{i_d}, \prod_j(\tilde{a}^{(j)} + \tilde{m}_{i_d}^d)^{p_j})| \leq \varepsilon.$$

It follows from the choice of $I^0_d$ in Proposition 3.3 that

$$|\lct(W, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j}) - \lct(W_{i_d}, \prod_j(\tilde{a}^{(j)}_{i_d})^{p_j})| \leq 2\varepsilon.$$
This gives the assertion in the corollary.

\[ \square \]

Remark 3.5. The argument in the proof of the above corollary can also be carried out if some \( \bar{a}^{(j)} \) is zero; in this case, one has to apply part i) in Proposition 2.19, instead of part ii). In particular, we see that if the sequence \( \left( \text{lct} \left( W_i, \prod_j (\bar{a}^{(j)}_i)_{p_j} \right) \right)_{i\geq 1} \) converges to a positive real number, then all \( \bar{a}^{(j)} \), with \( 1 \leq j \leq r \) are nonzero.

4. Proof of the main result

Our goal is to give a proof of Theorem 1.1. Let us fix an algebraically closed ground field \( k \) of characteristic zero. Consider a collection \( (X_i, x_i) \) of schemes of finite type over \( k \), with \( x_i \in X_i \) closed points. We say that the family has bounded singularities if there are positive integers \( m \) and \( N \) such that for every \( i \) there is a closed subscheme \( Y_i \) of \( A_k^N \) whose ideal is defined by polynomials of degree \( \leq m \), and a closed point \( y_i \in Y_i \) such that \( \overline{O_{X_i,x_i}} \simeq \overline{O_{Y_i,y_i}} \).

Remark 4.1. The above condition is equivalent with the existence of a morphism \( \pi : \mathcal{Y} \to T \) of schemes of finite type over \( k \) such that for every \( i \) there is a closed point \( t_i \in T \) and a closed point \( y_i \) in the fiber \( \mathcal{Y}_{t_i} \) over \( t_i \) such that \( \overline{O_{X_i,x_i}} \simeq \overline{O_{Y_i,y_i}} \). Indeed, if the collection of varieties has bounded singularities, then it is enough to take \( T \) to be a parameter space parametrizing closed subschemes of \( A_k^N \) defined by ideals generated in degree \( \leq m \), and let \( \mathcal{Y} \to A_k^N \times T \) be the universal subscheme. Conversely, given \( \pi \) we can find finite affine open covers \( T = \bigcup_j V_j \) and \( \mathcal{Y} = \bigcup_j U_j \) such that \( \pi(U_j) \subseteq V_j \) for every \( j \). It is enough to take \( N \) and \( m \) such that each \( U_j \) can be embedded as a closed subscheme of \( A_k^N \), with the ideal generated in degree \( \leq m \).

We now set the notation for the rest of this section. Let us fix \( N \) and \( m \), and let \( \mathcal{X}_{N,m} \) be the set of all klt schemes of the form \( \text{Spec}(\overline{O_{X,x}}) \), where \( X \) is a closed subscheme of \( A_k^N \) defined by an ideal generated by polynomials of degree \( \leq m \) and \( x \in X \) is a closed point. After a suitable translation we may always assume that \( x \) is the origin in \( A_k^N \). We will freely use the notation introduced in the previous section in the construction of generic limits. The following is our main result.

Theorem 4.2. With the above notation, if \( \Gamma \subset \mathbb{R}_+ \) is a DCC subset, then there is no infinite strictly increasing sequence of log canonical thresholds

\[ \text{lct}_{(W_1, \mathfrak{A}_1)}(\mathfrak{A}_1) < \text{lct}_{(W_2, \mathfrak{B}_2)}(\mathfrak{B}_2) < \cdots, \]

where all \( W_i \in \mathcal{X}_{N,m} \), and \( \mathfrak{A}_i, \mathfrak{B}_i \) are \( \Gamma \)-ideals on \( W_i \), with \( (W_i, \mathfrak{B}_i) \) log canonical.

Proof. Let us assume that there is a strictly increasing sequence as in the statement, with \( W_i = \text{Spec}(\overline{O_{X,i}}) \) klt, where \( X_i \) is a closed subscheme of \( A_k^N \) defined by an ideal \( \mathfrak{p}_i \subset k[x_1, \ldots, x_N] \) generated in degree \( \leq m \). Let \( c_i = \text{lct}_{(W_i, \mathfrak{B}_i)}(\mathfrak{A}_i) \). 
Let us write \( \mathfrak{A}_i = \prod_{j=1}^{r_i} (\mathfrak{a}_i^{(j)})^{p_{i,j}} \) and \( \mathfrak{B}_i = \prod_{j=1}^{s_i} (\mathfrak{b}_i^{(j)})^{q_{i,j}} \) where all \( p_{i,j}, q_{i,j} \in \Gamma \). We may and will assume that all \( \mathfrak{a}_i^{(j)} \) and \( \mathfrak{b}_i^{(j)} \) vanish at 0 (though it may happen that \( \mathfrak{B}_i = \mathcal{O}_{W_i} \), in which case \( s_i = 0 \)). Let \( \mathfrak{a}_i^{(j)} \) and \( \mathfrak{b}_i^{(j)} \) be the ideals in \( R = k[x_1, \ldots, x_n] \) such that \( \mathfrak{a}_i^{(j)} = \mathfrak{a}_i^{(j)}/p_i R \) and \( \mathfrak{b}_i^{(j)} = \mathfrak{b}_i^{(j)}/p_i R \).

Since \( \Gamma \) satisfies DCC, it follows that there is \( \varepsilon > 0 \) such that \( p_{i,j}, q_{i,j} > \varepsilon \) for every \( i \) and \( j \). Since \( W_i \) is klt around 0, it follows from Proposition 2.19 that

\[
    c_i \leq \dim(W_i, \mathfrak{A}_i) \leq \frac{\dim(W_i)}{\varepsilon r_i} \leq \frac{N}{\varepsilon r_i}.
\]

This clearly implies that the sequence \( (c_i)_{i \geq 1} \) is bounded, and therefore it converges to some \( c \in \mathbb{R}_{>0} \). It also implies that the sequence \( (r_i)_{i \geq 1} \) is bounded. Therefore, after passing to a subsequence we may assume that \( r_i = r \) for all \( i \). Similarly, since \( \dim(W_i, \mathfrak{B}_i) \geq 1 \) for every \( i \), it follows that we may assume that \( s_i = s \) for every \( i \).

Using again that \( \Gamma \) is a DCC set, we see that after passing to a subsequence \( r + s \) times, we may assume that each of the sequences \( (p_{i,j})_{i \geq 1} \) and \( (q_{i,j})_{i \geq 1} \) is non-decreasing. Recall that \( c_i \leq N/p_{i,j} \) and \( 1 \leq N/q_{i,j} \) for every \( i \) and \( j \), hence the sequences \( (p_{i,j})_{i \geq 1} \) and \( (q_{i,j})_{i \geq 1} \) are bounded. We put \( p_j = \lim_{i \to \infty} p_{i,j} \) and \( q_j = \lim_{i \to \infty} q_{i,j} \).

Let \( c'_i := \dim(W_i, \mathfrak{A}_i) \), where \( \mathfrak{A}_i = \prod_{j=1}^{r_i} (\mathfrak{a}_i^{(j)})^{p_j} \) and \( \mathfrak{B}_i = \prod_{j=1}^{s_i} (\mathfrak{b}_i^{(j)})^{q_j} \). Since \( p_{i,j} \leq p_j \) and \( q_{i,j} \leq q_j \) for every \( i \) and \( j \), it follows that \( c'_i \leq c_i \) for every \( i \). On the other hand, for every \( \eta \in (0,1) \), we have \( p_{i,j}/p_j, q_{i,j}/q_j \geq 1 - \eta \) for all \( j \) and all \( i \gg 0 \). This implies \( c_i \leq c'_i/(1-\eta) \) for all \( i \gg 0 \). We deduce that the sequence \( (c'_i)_{i \geq 1} \) contains a strictly increasing subsequence converging to \( c \). Therefore, in order to derive a contradiction, we may assume that \( p_{i,j} = p_j \) and \( q_{i,j} = q_j \) for every \( i \) and \( j \).

**Case 1.** We first treat the case when \( \mathfrak{B}_i = \mathcal{O}_X \) for every \( i \). The argument for this case now follows closely the proof of [dFEM, Theorem 5.1], using though the version of generic limit construction introduced in the previous section. We consider the sequences of ideals

\[
    (\mathfrak{p}_i)_{i \in I_0}, (\mathfrak{a}_i^{(1)})_{i \in I_0}, \ldots, (\mathfrak{a}_i^{(r)})_{i \in I_0}, (\mathfrak{m}_i)_{i \in I_0},
\]

where \( \mathfrak{m} \) is the maximal ideal in \( R \). We choose a generic limit \( (\mathfrak{p}, \mathfrak{a}^{(1)}, \ldots, \mathfrak{a}^{(r)}, \mathfrak{m}_K) \) constructed as in §4, with \( \mathfrak{a}^{(j)} \) proper ideals in \( R_K = K[x_1, \ldots, x_N] \).

As in Proposition 3.3, we consider the scheme \( W = \text{Spec}(R_K/\mathfrak{p} R_K) \), and the ideals \( \overline{\mathfrak{a}}^{(j)} = \mathfrak{a}^{(j)}/(\mathfrak{p} R_K) \). Note that by Remark 3.5 all \( \overline{\mathfrak{a}}^{(j)} \) are nonzero. Let \( \mathfrak{A} \) be the \( \mathbb{R} \)-ideal \( \prod_{j=1}^{r} (\overline{\mathfrak{a}}^{(j)})^{p_j} \). It follows from Corollary 3.4 that \( c = \dim(W, \mathfrak{A}) \).

By Lemma 2.9, we can find a nonnegative real number \( t \) such that \( \text{lct}(W, \overline{\mathfrak{m}}_K \cdot \mathfrak{A}) = c \), and this is computed by a divisor \( E \) over \( W \) having center equal to the closed point (recall that \( \overline{\mathfrak{m}}_K \) defines the closed point of \( W \), and \( \overline{\mathfrak{m}}_t \) defines the closed point on \( W_t \)). It follows from Proposition 3.3 iii) that we can find \( d_E \) such that for every \( d \geq d_E \) the following holds: there is an infinite subset \( I^*_d \subseteq I_0 \) such that for each \( i \in I^*_d \) there is a divisor \( E_i \)
over $W$, having center equal to the closed point, computing the log canonical threshold 
\[ \text{lct}(W, \mathfrak{m}^t_i \cdot \prod_{j=1}^{t} (\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j)^{p_j}) , \]
and such that \[ \text{ord}_E(\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j) = \text{ord}_{E_i}(\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j) \] for every $j$.

Fix $d \geq d_E$ such that, in addition, $d > \text{ord}_E(\mathfrak{a}^{(j)}_i)$ for all $j \geq 1$. Since \[ \text{lct}(W, \mathfrak{m}^t_i \cdot \mathfrak{a}) = \text{lct}(W, \mathfrak{m}^t_i \cdot \prod_j (\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j)^{p_j}) , \]
and the right-hand side is computed by $E$. On the other hand, it follows from Proposition 3.3 ii) that we may assume that for all $i \in I_d^E$
\[ \text{lct} \left( W, \mathfrak{m}^t_i \cdot \prod_j (\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j)^{p_j} \right) = \text{lct} \left( W, \mathfrak{m}^t_i \cdot \prod_j (\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j)^{p_j} \right) . \]

By combining equations (10), (11) and (13), we conclude that
\[ c = \text{lct} \left( W, \mathfrak{m}^t_i \cdot \prod_j (\mathfrak{a}^{(j)}_i + \mathfrak{m}^d_j)^{p_j} \right) \leq \text{lct}(W, \mathfrak{a}) < c , \]
a contradiction. This completes the proof of this case.

**Case 2.** We now treat the general case. Consider the sequences of ideals
\[ (p_i)_{i \in I_0}, (a_i^{(1)})_{i \in I_0}, \ldots, (a_i^{(r)})_{i \in I_0}, (b_i^{(1)})_{i \in I_0}, \ldots, (b_i^{(s)})_{i \in I_0} . \]
Again, we construct a generic limit $(p, a^{(1)}, \ldots, a^{(r)}, b^{(1)}, \ldots, b^{(s)})$ as in §3. Let \[ W = \text{Spec}(R_K/pR_K) , \]
and \[ \mathfrak{a}^{(j)} = a^{(j)}/pR_K \] and \[ \mathfrak{b}^{(j)} = b^{(j)}/pR_K . \] We consider the $\mathbb{R}$-ideals \[ \mathfrak{a} = \prod_{j=1}^{r} (\mathfrak{a}^{(j)})^{p_j} \] and \[ \mathfrak{b} = \prod_{j=1}^{s} (\mathfrak{b}^{(j)})^{q_j} . \]

For every $c' < c$, we have $c_i > c'$ for $i \gg 0$. Therefore \[ \text{lct}(W, \mathfrak{a}^i \cdot \mathfrak{a}^c) \geq 1 \] for such $i$. By Proposition 3.3 ii), \[ \text{lct}(W, \mathfrak{b} \cdot \mathfrak{a}^c) \] is a limit point of the sequence \[ \left( \text{lct}(W, \mathfrak{a}^i \cdot \mathfrak{a}^c) \right)_{i \geq 1} , \]
hence \[ \text{lct}(W, \mathfrak{b} \cdot \mathfrak{a}^c) \geq 1 . \] Since this holds for every $c' < c$, we have \[ \text{lct}(W, \mathfrak{b} \cdot \mathfrak{a}^c) \geq 1 . \]

Another application of Proposition 3.3 ii) gives that \[ \text{lct}(W, \mathfrak{b} \cdot \mathfrak{a}^c) \] is a limit point of the sequence \[ \left( \text{lct}(W, \mathfrak{a}^i \cdot \mathfrak{a}^c) \right)_{i \geq 1} . \] On the other hand, it follows from Case 1 that the set \[ \{ \text{lct}(W, \mathfrak{a}^i \cdot \mathfrak{a}^c) \mid i \geq 1 \} \] contains no strictly increasing sequences. We deduce that there are infinitely many $i$ such that \[ \text{lct}(W, \mathfrak{a}^i \cdot \mathfrak{a}^c) \geq \text{lct}(W, \mathfrak{b} \cdot \mathfrak{a}^c) \geq 1 . \] For every such $i$ we have $c_i \geq c$, a contradiction. This completes the proof of the theorem. \hfill $\Box$

**Remark 4.3.** It follows from Proposition 2.11 that the statement in the above theorem implies the version stated in the Introduction in Theorem 1.1 in terms of bounded families of singularities.
Appendix A. Sheaves of differentials for schemes of finite type over a formal power series ring

In this appendix we work in the following setting. Let \( k \) be a field of characteristic zero, and \( R = k[x_1, \ldots, x_n] \). All our schemes will be of finite type over such a formal power series ring. We note that since \( R \) is an excellent ring (see [Mat1, p. 260]), it follows that the nonsingular locus of such a scheme is open. Furthermore, \( R \) is universally catenary. The usual sheaves of differentials over \( k \) are not the right objects in our setting (in particular, they are not coherent). Our aim in this section is to introduce an appropriate version of sheaves of differentials that is better behaved.

For every \( R \)-module \( M \), the special \( k \)-derivations \( D: R \to M \) are those \( k \)-derivations with the property that \( D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i) \) for every \( f \in R \). Note that this is automatically true for a \( k \)-derivation if \( M \) is separated in the \((x_1, \ldots, x_n)\)-adic topology, but not in general.

For an \( R \)-algebra \( A \) and an \( A \)-module \( M \), the module \( \text{Der}'_k(A, M) \) of special \( k \)-derivations consists of all \( k \)-derivations \( D: A \to M \) such that the restriction of \( D \) to \( R \) is a special \( k \)-derivation \( R \to M \). It is clear that \( \text{Der}'_k(A, M) \) is an \( A \)-submodule of \( \text{Der}_k(A, M) \). Note that the definition depends on the fixed ring \( R \).

If \( w: M \to N \) is a morphism of \( A \)-modules, composing with \( w \) induces a morphism of \( A \)-modules \( \text{Der}'_k(A, M) \to \text{Der}'_k(A, N) \). We say that \( A \) has a module of special differentials if there is an \( A \)-module \( \Omega'_{A/k} \) with a special \( k \)-derivation \( d'_{A/k}: A \to \Omega'_{A/k} \) that induces an isomorphism of \( A \)-modules

\[
\text{Hom}_A(\Omega'_{A/k}, M) \to \text{Der}'_k(A, M)
\]

for every \( A \)-module \( M \). Of course, in this case \( \Omega'_{A/k} \) is called the module of special differentials (note that it is unique, up to a canonical isomorphism commuting with \( d'_{A/k} \)).

In order to avoid cluttering the notation we do not include \( R \) in the notation for \( \Omega'_{A/k} \). However, the reader should keep in mind that the definition was made in reference to a fixed \( R \).

Lemma A.1. If \( A \to B = A/I \) is a surjective morphism of \( R \)-algebras, and if \( \Omega'_{A/k} \) exists, then \( \Omega'_{B/k} \) exists, and we have an exact sequence

\[
\frac{I/I^2}{I/I^2} \to \Omega'_{A/k} \otimes_A B \xrightarrow{u} \Omega'_{B/k} \to 0,
\]

where \( \delta(f) = d'_{A/k}(f) \otimes 1 \) and \( u(d'_{A/k}(f) \otimes 1) = d'_{B/k}(f) \).

Proof. The assertion follows as in the case of usual differentials from the fact that the corresponding sequence of \( B \)-modules

\[
0 \to \text{Der}'_k(B, M) \to \text{Der}'_k(A, M) \to \text{Hom}_A(I/I^2, M)
\]

is exact for every \( A \)-module \( M \). \( \square \)
Lemma A.2. The module of special differentials $\Omega'_{R/k}$ exists, and it is a free $R$-module with basis $d'_{R/k}(x_1), \ldots, d'_{R/k}(x_n)$.

Proof. The assertion follows from the fact that by definition, every $D \in \text{Der}'_k(R, M)$ is uniquely determined by the $D(x_i)$, which can be chosen arbitrarily. \hfill \Box

Lemma A.3. Let $S$ be an $R$-algebra, and $A = S[y_i | i \in I]$ a polynomial ring over $S$. If $\Omega'_{S/k}$ exists, then $\Omega'_{A/k}$ exists and it is isomorphic to the direct sum of $\Omega'_{S/k} \otimes_S A$ and a free $A$-module with basis $\{d'_{A/k}(y_i) | i \in I\}$.

Proof. The assertion follows from the fact that every $D \in \text{Der}'_k(A, M)$ is uniquely determined by $D|_S$ and by the $D(y_i)$, which can be chosen arbitrarily. \hfill \Box

By combining the above three lemmas we deduce the following existence result.

Corollary A.4. For every $R$-algebra $A$, the module $\Omega'_{A/k}$ exists. Furthermore, if $A$ is of finite type over $R$, then $\Omega'_{A/k}$ is finitely generated over $A$.

Remark A.5. Since $\text{Der}'_k(A, M) \subseteq \text{Der}_k(A, M)$ for every $A$-module $M$, it follows that we have a surjective morphism $\Omega_{A/k} \rightarrow \Omega'_{A/k}$. In particular, $\Omega'_{A/k}$ is generated as an $A$-module by $\{d'_{A/k}(a) | a \in A\}$.

Lemma A.6. If $\varphi: A \rightarrow B$ is a morphism of $R$-algebras, then we have an exact sequence

$$\Omega'_{A/k} \otimes_A B \xrightarrow{\bar{u}} \Omega'_{B/k} \xrightarrow{\bar{v}} \Omega_{B/A} \rightarrow 0,$$

where $u(d'_{A/k}(f) \otimes 1) = d'_{B/k}(f)$ and $v(d'_{B/k}(f)) = d_{B/A}(f)$.

Proof. The assertion follows as in the case of usual derivations from the fact that for every $B$-module $M$, the corresponding sequence

$$0 \rightarrow \text{Der}_A(B, M) \rightarrow \text{Der}'_k(B, M) \rightarrow \text{Der}'_k(A, M)$$

is exact (note that if $D \in \text{Der}_A(B, M)$, then $D$ is trivially a special $k$-derivation, since its restriction to $R$ is zero). \hfill \Box

Lemma A.7. Let $A$ be an $R$-algebra. If $S$ is a multiplicative system in $A$, then we have a canonical isomorphism $S^{-1}\Omega'_{A/k} \simeq \Omega'_{S^{-1}A/k}$.

Proof. It is enough to note that for every $S^{-1}A$-module $M$, we have canonical isomorphisms

$$\text{Hom}_{S^{-1}A}(S^{-1}\Omega'_{A/k}, M) \simeq \text{Hom}_A(\Omega'_{A/k}, M) \simeq \text{Der}'_k(A, M) \simeq \text{Der}'_k(S^{-1}A, M).$$

The last isomorphism follows from the fact that every $k$-derivation $D: A \rightarrow M$ admits a unique extension $\tilde{D}: S^{-1}A \rightarrow M$, and it is clear that $D$ is special if and only if $\tilde{D}$ is. The assertion in the lemma follows from (14). \hfill \Box
The case when $A$ is regular will play an important role. We show that in this case $\Omega'_{A/k}$ is locally free. In fact, we have the following more precise result.

**Proposition A.8.** If $A$ is an algebra of finite type over $R$, and if $q \in \text{Spec}(A)$ is such that $A_q$ is a regular ring of dimension $r$, then $\Omega'_{A_q/k}$ is a free $A_q$-module, of rank equal to $r + \dim_{k(q)}(\Omega'_{k(q)}/k)$, where $k(q) = A_q/qA_q$. Furthermore, if $u_1, \ldots, u_r \in A$ induce a regular system of parameters in $A_q$, then the images of $d'_{A/k}(u_1), \ldots, d'_{A/k}(u_r)$ in $\Omega'_{A_q/k}$ are part of a basis.

**Proof.** Our argument is based on the results in [Mat2]. Since the regular locus of $A$ is open, we may replace $A$ by a localization $A_f$ in order to assume that $A$ is a regular ring, and further, that it is a domain. Let us choose an isomorphism $A \simeq S/P$, with $S = R[y_1, \ldots, y_N]$ and $P \in \text{Spec}(S)$. Let $Q \in \text{Spec}(S)$ be such that $q = Q/P$. We put $s = \text{codim}(P)$, so that $\text{codim}(Q) = r + s$.

A key property of $S$ is that it satisfies the strong Jacobian condition over $k$. At the prime $Q$, this means that there are $D_1, \ldots, D_{r+s} \in \text{Der}_k(S) \cdot S_Q$, and $w_1, \ldots, w_{r+s} \in Q$, such that $\det(D_i(w_j)) \notin QS_Q$ and $[D_i, D_j] \in \sum_{\ell+t=1} S_Q \cdot D_t$. The fact that $S$ satisfies this condition follows from [Mat2, Theorem 6], which says that rings with the strong Jacobian condition are closed under taking polynomial or formal power series rings (the statement therein is in terms of absolute derivations, but the proof goes through if one works with $k$-derivations). In this case [Mat2, Theorem 5] implies that $S_Q/PS_Q$ is regular if and only if there are $w'_1, \ldots, w'_s \in P$ such that some $s$-minor of the matrix $(D_i(w'_j))_{i,j}$ does not lie in $QS_Q$.

Lemma A.1 gives (after localizing and using Lemma A.7) an exact sequence of $A_q$-modules

$$PS_Q/P^2S_Q \xrightarrow{\phi} \Omega_{S/k} \otimes_S A_q \rightarrow \Omega'_{A_q/k} \rightarrow 0.$$  
Since $S_Q/PS_Q$ is regular, $PS_Q/P^2S_Q$ is a free $A_q$-module of rank $s$. Note that since $S$ is separated with respect to the $(x_1, \ldots, x_n)$-adic topology, we have Der$_k(S) = \text{Der}_k(S)$. Therefore $D_1, \ldots, D_{r+s}$ define an $A_q$-linear map $\psi: \Omega_{S/k} \otimes_S A_q \rightarrow A_q^{s+s}$ such that $\psi \circ \varphi$ is split injective. It follows from the above exact sequence that $\Omega'_{A_q/k}$ is a free $A_q$-module of rank $n + N - s$, and we also see that $w'_1, \ldots, w'_s$ generate $PS_Q$. Running the same argument with $P$ replaced by $Q$, we see that $\dim_{k(q)}(\Omega'_{k(q)}/k) = n + N - (r + s)$, which gives the assertion about the rank of $\Omega'_{A_q/k}$.

For the last assertion in the proposition, note that if the $\tilde{u}_i \in S$ are lifts of the $u_i$, then it follows that $\tilde{u}_1, \ldots, \tilde{u}_r, w'_1, \ldots, w'_s$ give a minimal system of generators of $QS_Q$. By writing $w_1, \ldots, w_{r+s}$ in terms of this system of generators, we see that

$$d'_{S/k}(\tilde{u}_1) \otimes 1, \ldots, d'_{S/k}(\tilde{u}_r) \otimes 1, d_{S/k}(w'_1) \otimes 1, \ldots, d_{S/k}(w'_s) \otimes 1$$

are part of a basis of $\Omega'_{S/k} \otimes_S A_q$. We deduce from this the last assertion in the proposition. \qed
For future reference, we include here the following lemma, describing the behavior of the module of special differentials with respect to field extensions.

**Lemma A.9.** Let $K \hookrightarrow L$ be a field extension, where $K$ is an $R$-algebra. In this case we have an exact sequence

$$0 \to \Omega'_{K/k} \otimes_K L \to \Omega'_{L/k} \to \Omega_{L/K} \to 0.$$  

**Proof.** By Lemma A.6, it is enough to show that the morphism $\Omega'_{K/k} \otimes_K L \to \Omega'_{L/k}$ is injective; equivalently, for every $L$-module $M$, the map $\text{Der}_k(L, M) \to \text{Der}_k(K, M)$ is surjective. This follows from the fact that the map $\text{Der}_k(L, M) \to \text{Der}_k(K, M)$ is surjective (recall that $\text{char}(k) = 0$), by noticing that a $k$-derivation $D: L \to M$ is special if and only if $D|_K$ is special. \hfill $\square$

We will need a comparison between the usual module of differentials for schemes of finite type over a field, and the module of special differentials for the completion at a closed point. We consider the following setting. Suppose that $\varphi: A \to B$ is a morphism of finitely generated $k$-algebras, and $m$ is a maximal ideal in $A$. The field $K = A/m$ is a finite extension of $k$. We put $A' = \widehat{A}_m$ and $B' = B \otimes_A \widehat{A}_m$. By Cohen’s Structure Theorem, we can find a surjective local morphism of $k$-algebras $\psi: S = K[x_1, \ldots, x_N] \to A'$. In this case we take $R = k[x_1, \ldots, x_N]$ and $m_R = (x_1, \ldots, x_N)$. Note that $A'$ becomes naturally an $R$-algebra via the inclusion $R \hookrightarrow S$. We use this structure when considering the modules of special derivations $\Omega'_{A'/k}$ and $\Omega'_{B'/k}$. Since $K/k$ is finite, $A'$ is finite as an $R$-algebra, hence $B'$ is a finitely generated $R$-algebra.

**Proposition A.10.** With the above notation, we have a canonical isomorphism

$$\Omega_{B/k} \otimes_B B' \simeq \Omega'_{B'/k}. \tag{15}$$

**Proof.** We first show that $\Omega_{A/k} \otimes_A A' \simeq \Omega'_{A'/k}$. Note that if $M$ is a finitely generated $A'$-module, then $M$ is separated with respect to the $m_{A'}$-adic topology (where $m_{A'}$ is the maximal ideal of $A'$), hence it is separated with respect to the $m_R$-adic topology. Therefore $\text{Der}'_k(A', M) = \text{Der}_k(A', M)$. Let $d_{A/k}: A \to \Omega_{A/k}$ be the universal $k$-derivation on $A$, and let $j: \Omega_{A/k} \to \Omega_{A/k} \otimes_A A'$ be the canonical map. Note that $\Omega_{A/k} \otimes_A A'$ is complete in the $m_{A'}$-adic topology, hence we get a unique $\widehat{d}_{A/k} \in \text{Der}_k(A', \Omega_{A/k} \otimes_A A')$ such that $\widehat{d}_{A/k} \circ \iota = j \circ d_{A/k}$, where $\iota: A \to A'$ is the completion map. Since $\Omega_{A/k} \otimes_A A'$ is a finitely generated $A'$-module, it follows that $\widehat{d}_{A/k}$ is a special derivation, and we have a unique morphism of $A$-modules $f: \Omega'_{A'/k} \to \Omega_{A/k} \otimes_A A'$ such that $f \circ d'_{A'/k} = \widehat{d}_{A/k}$.

On the other hand, since $d'_{A'/k}$ is a derivation, there is a unique morphism of $A$-modules $g: \Omega_{A/k} \to \Omega'_{A'/k}$ such that $g \circ d_{A/k} = d'_{A'/k} \circ \iota$. Since $\Omega'_{A'/k}$ is finitely generated
over $A'$ by Corollary A.4, it is complete, hence $g$ induces a (unique) morphism of $A'$-modules $\hat{g}: \Omega_{A/k} = \Omega_{A} \otimes_A A' \to \Omega'_{A'/k}$ such that $\hat{g} \circ j = g$. It is now easy to check that $f$ and $\hat{g}$ are inverse isomorphisms.

In order to prove the general statement for $B$, let us write $B \cong A[y_1, \ldots, y_m]/I$. In this case we have

\[(16) \quad \Omega_{B/k} \cong \left( (\Omega_{A/k} \otimes_A B) \oplus \bigoplus_{i=1}^{m} B \cdot d_{B/k}(y_i) \right)/B \cdot \{d_{B/k}(u) \mid u \in I\},\]

which after tensoring with $B'$ gives a description of $\Omega_{B'/k} \otimes_B B'$. Since we have a corresponding isomorphism $B' \cong A'[y_1, \ldots, y_m]/(\iota(I))$, using Lemmas A.1 and A.3 we get an analogous formula for $\Omega'_{B'/k}$. The isomorphism (15) now follows from the corresponding isomorphism in the case $B = A$. □

It is standard to deduce from Lemma A.7 that for every scheme $X$ over $R$, there is a quasicoherent sheaf $\Omega'_{X/k}$ such that for every affine open subset $U$ of $X$, the restriction of $\Omega'_{X/k}$ to $U$ is canonically isomorphic to the sheaf associated to $\Omega'_{\mathcal{O}(U)/k}$. It follows from Corollary A.4 that if $X$ is of finite type over $R$, then $\Omega'_{X/k}$ is coherent. Furthermore, Proposition A.8 implies that if $X$ is nonsingular, then $\Omega'_{X/k}$ is locally free. The exact sequences in Lemmas A.1 and A.6 globalize in a straightforward way.

We now use the sheaves of special differentials to introduce the notion of relative canonical class in this setting. Let $X$ be a normal scheme of finite type over $R$. Since the discussion that follows can be done separately on each connected component of $X$, we may and will assume that $X$ is irreducible. Recall that since $R$ is excellent, the nonsingular locus $X_{\text{reg}}$ of $X$ is an open subset of $X$. Since $X$ is normal, the complement $X \setminus X_{\text{reg}}$ has codimension $\geq 2$ in $X$. In particular, restriction induces an isomorphism of class groups $\text{Cl}(X) \cong \text{Cl}(X_{\text{reg}})$.

The restriction $\Omega'_{X/k}|_{X_{\text{reg}}}$ is locally free, and let $M$ be its rank. On $X$ we have a Weil divisor $K_X$, uniquely defined up to rational equivalence, such that $\mathcal{O}(K_X)|_{X_{\text{reg}}} \cong \wedge^M \Omega'_{X_{\text{reg}}/k}$. As in the case of schemes of finite type over a field, we say that $X$ is $Q$-Gorenstein if there is a positive integer $r$ such that $rK_X$ is a Cartier divisor (the smallest such $r$ is the index of $X$; any other $r$ with this property is a multiple of the index).

Suppose now that $\pi: Y \to X$ is a proper birational morphism of schemes over $R$, with $Y$ nonsingular. The following lemma shows that the relative canonical class $K_{Y/X}$ can be defined in the same way as in the case of schemes of finite type over a field (see [Kol2]).

**Lemma A.11.** With the above notation, the following hold:

i) We may take $K_X = \pi_*(K_Y)$. 


ii) If $rK_X$ is Cartier, then there is a unique $\mathbb{Q}$-divisor $K_{Y/X}$ supported on the exceptional locus of $\pi$ such that $rK_Y$ and $\pi^*(rK_X) + rK_{Y/X}$ are linearly equivalent. If $X$ is nonsingular, then $K_{Y/X}$ is effective and its support is the exceptional locus $\text{Exc}(\pi)$.

iii) Suppose that $X$ is nonsingular, and that $E_1 + \cdots + E_q$ is a divisor on $X$ having simple normal crossings. If $F$ is a prime nonsingular divisor on $Y$ with corresponding valuation $\text{ord}_F$, and if $\text{ord}_F(E_i) = a_i$ for every $i$, then $\text{ord}_F(K_{Y/X}) \geq a_1 + \cdots + a_q - 1$.

Proof. In order to prove i), we may restrict to $X_{\text{reg}}$, and therefore assume that $X$ is nonsingular. If $y \in Y$ and $x = \pi(y)$, then Lemma A.6 gives an exact sequence

$$U := \Omega'_{X/k,x} \otimes \mathcal{O}_{Y,y} \rightarrow V := \Omega'_{Y/k,y} \rightarrow \Omega_{Y/X,y} \rightarrow 0.$$ 

Since $\pi$ is birational, it follows from the Dimension Formula (see [Mat1, Theorem 15.6]) that

$$\dim(\mathcal{O}_{Y,y}) = \dim(\mathcal{O}_{X,x}) + \text{trdeg}(k(y)/k(x)).$$

Since $\dim_k(\Omega_{k(y)/k(x)}) = \text{trdeg}(k(y)/k(x))$, we deduce from Lemma A.9 and Proposition A.8 that $U$ and $V$ are free $\mathcal{O}_{Y,y}$-modules of the same rank $M$. It follows that $\wedge^M w$ is given by the equation of an effective divisor $K_{Y/X}$. The support of this divisor is the locus where $\pi$ is not étale, which in this case is precisely the exceptional locus of $\pi$. The assertions in i) and ii) now easily follow. Due to the last assertion in Proposition A.8, we can deduce iii) via the same computation as in the usual case of schemes of finite type over a field. 

Remark A.12. It follows from the above proof that if $X$ is nonsingular, then $K_{Y/X}$ is independent of the structure of $X$ as an $R$-scheme. Indeed, $K_{Y/X}$ is the effective divisor defined by the $0^{\text{th}}$ Fitting ideal of $\Omega_{Y/X}$. It is not clear to us whether the same remains true if $X$ is singular.

Lemma A.13. If $Y' \xrightarrow{\varphi} Y \xrightarrow{\pi} X$ are proper birational morphisms, with both $Y$ and $Y'$ nonsingular, and if $X$ is $\mathbb{Q}$-Gorenstein, then

$$K_{Y'/X} = K_{Y'/Y} + \varphi^*(K_{Y/X}).$$

Proof. It is enough to observe that if $rK_X$ is Cartier, then $r(K_{Y'/Y} + \varphi^*(K_{Y/X}))$ is $\pi \circ \varphi$-exceptional, and it is linearly equivalent to $rK_{Y'} - (\pi \circ \varphi)^*(rK_X)$. 

In the next proposition we consider an integral scheme $X$, of finite type over a field $k$ (assumed, as always, to have characteristic zero). Suppose that $\pi: Y \rightarrow X$ is a proper birational morphism, with $Y$ nonsingular. Let $x \in X$ be a closed point, and consider the
Cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & Y \\
\downarrow f & & \downarrow \pi \\
Z = \text{Spec}(\hat{O}_{X,x}) & \xrightarrow{g} & X
\end{array}
\]

(see Remark 2.10 for general properties of such a diagram).

From now on, let us assume that \( X \) is normal. In this case \( W \) is connected: otherwise the fiber over the unique closed point of \( Z \) would be disconnected, but this is the same as the fiber \( \pi^{-1}(x) \). Since \( \pi \) is proper and birational, we deduce that \( f \) as well has these two properties. We consider both \( Z \) and \( W \) as schemes over a formal power series ring over \( k \), as in Proposition A.10. Since \( g \) and \( h \) are flat, we may pull-back Weil divisors via both \( g \) and \( h \).

**Proposition A.14.** With the above notation, we may take \( K_Z = g^*(K_X) \). In particular, \( rK_X \) is Cartier in a neigborhood of \( x \) if and only if \( rK_Z \) is Cartier, and in this case \( h^*(K_{Y/X}) = K_{W/Z} \).

**Proof.** Since \( g \) is a regular morphism, it follows that \( g^{-1}(X_{\text{reg}}) = Z_{\text{reg}} \). The first assertion in the proposition follows from the fact that if \( g_0 : Z_{\text{reg}} \to X_{\text{reg}} \) is the restriction of \( g \), then \( g_0^*(\Omega_{X_{\text{reg}}}/k) \simeq \Omega_{Z_{\text{reg}}}^\nu \) by Proposition A.10. Furthermore, the same proposition implies that we may take \( K_W = h^*(K_Y) \).

Note now that if \( D \) is a divisor on \( X \), then \( D \) is Cartier in a neighborhood of \( x \) if and only if \( g^*(D) \) is Cartier. Indeed, for this we may assume that \( D \) is effective, and let \( I = \mathcal{O}(-D) \cdot \hat{O}_{X,x} \). In this case \( \mathcal{O}(-g^*(D)) = I \cdot \hat{O}_{X,x} \), since for every prime ideal \( P \) in \( \mathcal{O}_{X,x} \) and every minimal prime ideal \( Q \) in \( \hat{O}_{X,x} \) containing \( P \), we have \( P \cdot (\hat{O}_{X,x}/Q) = Q \cdot (\hat{O}_{X,x}/Q) \) (this follows from the fact that the fiber over \( P \) is nonsingular). It is now enough to note that \( I \cdot \hat{O}_{X,x} \) is principal if and only if \( I \) is principal (more generally, \( I \) and \( I \cdot \hat{O}_{X,x} \) have the same minimal number of generators).

In particular, we see that \( rK_X \) is Cartier in a neighborhood of \( x \) if and only if \( rK_Z \) is Cartier. The last assertion in the proposition now follows from the fact that \( h^*(K_{Y/X}) \) is supported on the inverse image via \( h \) of the exceptional locus of \( \pi \), hence on the exceptional locus of \( f \).

**Remark A.15.** Suppose that \( F \) is a prime nonsingular divisor on \( Y \). The pull-back \( h^*(F) \) is a nonsingular divisor on \( W \). If we consider the irreducible components \( E_1, \ldots, E_m \) of \( h^*(F) \), then the restriction of each \( \text{ord}_{E_i} \to \text{field of } X \) is equal to \( \text{ord}_F \). We note that if the center of \( F \) is \( x \), then \( h^*(F) \) is abstractly isomorphic to \( F \). In particular, \( h^*(F) \) is a prime divisor.
APPENDIX B. RATIONAL $\mathbb{Q}$-GORENSTEIN SINGULARITIES IN FAMILIES

Throughout this appendix, all varieties and schemes are of finite type over a field $k$ of characteristic zero. At one point, we will need to assume that $k = \mathbb{C}$. Our goal is to prove Theorem B.8 on the behavior of the canonical class and Gorenstein index in families. This implies the corollary about the generic behavior of the log canonical threshold in families that is used in the proof of Proposition 3.3.

In fact, Theorem B.8 follows from the following more precise result, for which we need to assume that $k$ is the field of complex numbers. Given a positive integer $r$, we say that a normal variety $X$ is $r$-Gorenstein at a point $x$ if $rK_X$ is Cartier at $x$. For a scheme $X \to T$ over $T$ we denote by $X_\xi$ the fiber over the not necessarily closed point $\xi \in T$.

**Theorem B.1.** Let $f : X \to T$ be a morphism of normal varieties over $\mathbb{C}$ such that every fiber of $f$ is normal. Then there are a positive integer $s$ and a nonempty Zariski open set $T^o \subseteq T$ such that for every closed point $t \in T^o$, if $X_t$ has rational singularities at a closed point $x$, then the following conditions are equivalent:

- (a) $X_t$ is $\mathbb{Q}$-Gorenstein at $x$;
- (b) $X_t$ is $s$-Gorenstein at $x$;
- (c) $X$ is $\mathbb{Q}$-Gorenstein at $x$;
- (d) $X$ is $s$-Gorenstein at $x$.

Before giving the proof of the theorem, we start with some general considerations. Recall first Grothendieck’s Generic Freeness Theorem (see, for example, [Eis, Theorem 14.4]).

**Theorem B.2** (Generic Freeness Theorem). Let $\varphi : A \to B$ be a ring homomorphism of finite type, with $A$ a Noetherian integral domain. If $M$ is a finitely generated $B$-module, then there is a nonzero $a \in A$ such that $M_a$ is a free $A_a$-module.

**Corollary B.3.** If $f : X \to T$ is a scheme morphism of finite type, with $T$ a Noetherian integral scheme, then there is a nonempty open subset $W$ in $T$ such that $f^{-1}(W) \to W$ is flat. Furthermore, given a complex of coherent sheaves on $X$

$$\mathcal{C} : \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'',$$

with homology sheaf $\mathcal{H}(\mathcal{C})$, we can choose $W$ such that for every $\xi \in W$ the canonical morphism $\mathcal{H}(\mathcal{C}) \otimes \mathcal{O}_{X_\xi} \to \mathcal{H}(\mathcal{C} \otimes \mathcal{O}_{X_\xi})$ is an isomorphism.

**Proof.** The first assertion follows easily from the theorem. For the second one, note that by the theorem, we may choose $W$ such that the images and the cokernels of the arrows in $\mathcal{C}$ are all flat over $W$. It is then easy to see that $W$ has the required property. $\square$

The following lemmas will be used in the proof of Theorems B.1 and B.8.
Lemma B.4. Let \( f : X \to T \) be a morphism of normal schemes such that all fibers of \( f \) are normal. For every positive integer \( m \), there is an open subset \( W_m \subseteq T \) such that for every \( \xi \in W_m \) we have a canonical isomorphism
\[
\mathcal{O}(mK_X)|_{X_{\xi}} \cong \mathcal{O}(mK_{X_{\xi}}).
\]
In particular, for every \( \xi \in W_m \), the divisor \( mK_X \) is Cartier at a point \( x \in X_{\xi} \) if and only if \( mK_{X_{\xi}} \) is Cartier at \( x \).

Proof. By Corollary B.3, after replacing \( T \) by an open subset we may assume that \( f \) is flat. We may clearly also assume that \( T \) is nonsingular. In particular, if \( x \) is a nonsingular point of \( X_{\xi} \), then both \( f \) and \( X \) are smooth at \( x \). In this case we clearly have a canonical isomorphism \( \mathcal{O}(mK_X)|_{X_{\xi}} \cong \mathcal{O}(mK_{X_{\xi}}) \) in a neighborhood of \( x \) (where \( X_{\xi} \) is considered as a scheme over \( \text{Spec}(k(\xi)) \)). Since the complement of \( (X_{\xi})_{\text{reg}} \) in \( X_{\xi} \) has codimension \( \geq 2 \), it is enough to find \( W_m \) such that for every \( \xi \in W_m \), the restriction \( \mathcal{O}(mK_X)|_{X_{\xi}} \) is reflexive.

After covering \( X \) by affine open subsets, we may assume that \( X \) is affine. Since \( \mathcal{O}(mK_X) \) is reflexive, we may write it as the kernel of a morphism \( \varphi : \mathcal{E}_1 \to \mathcal{E}_0 \) of free coherent sheaves on \( X \). By Corollary B.3, there is an open subset \( W_m \subseteq T \) such that for every \( \xi \in W_m \) the restricted sheaf \( \mathcal{O}(mK_X)|_{X_{\xi}} \) is isomorphic to the kernel of the restriction of \( \varphi \) to \( X_{\xi} \), which is a reflexive sheaf. This completes the proof.

Lemma B.5. If \( X \) is a normal scheme, and
\[
U = \{ x \in X \mid X \text{ is } \mathbb{Q}-\text{Gorenstein at } x \}
\]
is the \( \mathbb{Q} \)-Gorenstein locus of \( X \), then \( U \) is open in \( X \), and there is a positive integer \( s \) such that \( sK_X \) is Cartier on \( U \).

Proof. For every positive integer \( m \), the set
\[
U_m = \{ x \in X \mid X \text{ is } m\text{-Gorenstein at } x \}
\]
is open in \( X \) (it is nonempty, since it contains \( X_{\text{reg}} \)). Note that \( U = \bigcup_{m \geq 1} U_m \). Furthermore, we have \( U_k \subseteq U_m \) if \( k \) divides \( m \). It follows by the Noetherian property that there is a unique maximal set among all these open sets. In other words, there is a positive integer \( s \) such that \( U = U_s \).

Lemma B.6. Let \( X \) be a normal scheme, and let \( g : Y \to X \) be a resolution of singularities. For a positive integer \( m \), the divisor \( mK_X \) is Cartier at a closed point \( x \in X \) if and only if there is an open neighborhood \( V \) of \( x \) and a \( g \)-exceptional divisor \( E \) on \( Y \) such that \( \mathcal{O}(mK_Y) \cong \mathcal{O}(E) \) on \( g^{-1}(V) \). Furthermore, if the ground field is \( \mathbb{C} \), then it is enough to find an open neighborhood \( V \) of \( x \) in the analytic topology such that \( \mathcal{O}(mK_Y)^{\text{an}} \cong \mathcal{O}(E)^{\text{an}} \) on \( g^{-1}(V) \).

Proof. Note that after fixing the Cartier divisor \( K_Y \) on \( Y \), we may take \( K_X = g_\ast K_Y \). If \( mK_X \) is Cartier, then \( mK_Y - g^\ast(mK_X) \) is an integral exceptional divisor. Thus, given \( x \in X \) such that \( mK_X \) is Cartier at \( x \), it is enough to take an open neighborhood \( V \) of
where $mK_X$ is principal. Conversely, if there is $E$ as in the statement, then taking the push-forward and observing that $g_*(mK_Y) = mK_X$ and $g_*E = 0$, we see that $mK_X$ is linearly equivalent to zero in a neighborhood of $x$.

Suppose now that $X$ is a complex variety, and assume that $\mathcal{O}(mK_Y)^{an} \cong \mathcal{O}(E)^{an}$ on $g^{-1}(V)$, where $V$ is an open neighborhood of $x$ in the analytic topology. It follows that there is a meromorphic function $\varphi$ on $Y$ such that $\text{div}_Y(\varphi) = mK_Y - E$ on $g^{-1}(V)$. In this case $\text{div}_X(\varphi) = mK_X$ on $V$. Therefore $\mathcal{O}(mK_X)^{an}$ is locally free of rank one at $x$. Since $\mathcal{O}(mK_X)$ and $\mathcal{O}(mK_X)^{an}$ have isomorphic completions at $x$, it follows by Nakayama’s Lemma that $\mathcal{O}(mK_X)$ is locally free of rank one at $x$, hence $mK_X$ is Cartier at this point. □

We are now ready to prove the key result of this appendix.

**Proof of Theorem B.1.** In this proof we only consider the closed points of the schemes involved. Let $g: Y \to X$ be a resolution of singularities whose exceptional locus is a divisor with simple normal crossings. By a theorem of Verdier [Ver], we can write $X$ as a finite disjoint union $X = \bigsqcup X^\alpha$, with each $X^\alpha$ an irreducible locally closed subset of $X$, such that the restriction $g^\alpha: Y^\alpha \to X^\alpha$ of $g$ to $Y^\alpha = g^{-1}(X^\alpha)$ is topologically locally trivial. Let $Z^\alpha := \overline{X^\alpha} \setminus X^\alpha$ (the closure being taken inside $X$). Note that each $Z^\alpha$ is a closed subset of $X$.

By Lemma B.5, there is a positive integer $s$ such that $X$ is $Q$-Gorenstein at a point $x$ if and only if $X$ is $s$-Gorenstein at $x$. By generic smoothness, generic flatness, and Lemma B.4, after possibly replacing $T$ by a nonempty open subset, we can assume that the following properties hold:

1. $T$ is smooth;
2. $Y \to T$ is smooth, the exceptional locus of $g$ has relative simple normal crossings over $T$, and for every point $t \in T$, the induced morphism $g_t: Y_t \to X_t$ is a resolution of singularities and every $g_t$-exceptional divisor is the restriction to $Y_t$ of a $g$-exceptional divisor;
3. $X$ is flat over $T$, and both $\overline{X^\alpha}$ and $Z^\alpha$ are flat over $T$ for every $\alpha$;
4. For every $t \in T$, there is a canonical isomorphism $\mathcal{O}(sK_X)|_{X_t} \cong \mathcal{O}(sK_{X_t})$.

We will show that after this reduction the conclusion of the theorem holds for every $t \in T$.

Fix any $t \in T$, and suppose that $x$ is a point where $X_t$ has rational singularities. Since $f$ is flat and $T$ is smooth, this implies that $X$ has rational singularities at $x$, and hence in a neighborhood of $x$ (cf. [Elk, Théorème 2 and Théorème 4]).

By Lemma B.5, we see that the conditions (c) and (d) are equivalent. Furthermore, condition (4) implies that (b) and (d) are equivalent, and clearly (b) implies (a). Therefore, in order to conclude it suffices to show that (a) implies (c).
We thus assume that (a) holds, that is, that there is a positive integer \( m \) such that \( mK_{X_t} \) is Cartier at \( x \). We denote by \( X_t^\alpha, (\overline{X}_\alpha)_t, \) and \( Z_t^\alpha \) the fibers of \( X^\alpha, \overline{X}_\alpha, \) and \( Z^\alpha \) over \( t \). Let \( \mathcal{A} := \{ \alpha \mid x \in \overline{X}_t^\alpha \} \). Note that

\[
x \in \text{Int} \left( \bigcup_{\alpha \in \mathcal{A}} X^\alpha \right).
\]

Indeed, if this were false, then for every open neighborhood \( V \) of \( x \) in \( X_t \) we could find an \( \alpha \notin \mathcal{A} \) such that \( V \cap X_t^\alpha \neq \emptyset \). By considering a nested sequence of open neighborhoods of \( x \), we see that we can pick \( \alpha \) independent of \( V \). As this holds for every \( V \), we conclude that \( x \in \overline{X}_t^\alpha \), which contradicts the definition of \( \mathcal{A} \).

**Claim.** We have

\[
x \in \text{Int} \left( \bigcup_{\alpha \in \mathcal{A}} X^\alpha \right).
\]

**Proof of claim.** We argue by contradiction. Let us assume that \( x \) is not in the interior of \( \bigcup_{\alpha \in \mathcal{A}} X^\alpha \). Arguing as above, we conclude that there is an \( \alpha \notin \mathcal{A} \) such that \( x \in \overline{X}_t^\alpha \). Consider the morphism \( \overline{X}_t^\alpha \to T \). We have \( x \in (\overline{X}_t^\alpha)_t \), and since \( x \notin \overline{X}_t^\alpha \), there is an open neighborhood \( V \) of \( x \) in \( (\overline{X}_t^\alpha)_t \) that is disjoint from \( X_t^\alpha \), and hence from \( X^\alpha \). Therefore \( V \) is contained in \( Z^\alpha \), hence in \( Z_t^\alpha \). The closure of \( V \) in \( (\overline{X}_t^\alpha)_t \), and hence the closure of \( Z_t^\alpha \) in \( (\overline{X}_t^\alpha)_t \), contains some irreducible component \( W \) of \( (\overline{X}_t^\alpha)_t \). Since both \( Z^\alpha \) and \( \overline{X}_t^\alpha \) are flat over \( T \), it follows that if \( x' \in W \) is a general (closed) point, then

\[
\dim(W) = \dim(O_{Z^\alpha,x'}) - \dim(T) = \dim(O_{\overline{X}_t^\alpha,x}) - \dim(T)
\]

(see [Har, Proposition III.9.5]). The fact that \( \dim(O_{Z^\alpha,x'}) = \dim(O_{\overline{X}_t^\alpha,x}) \) contradicts the fact that \( Z^\alpha \) is a proper closed subset of the irreducible set \( \overline{X}_t^\alpha \), and thus completes the proof of the claim. \( \square \)

We now fix a small contractible analytic open neighborhood \( V \subseteq X \) of \( x \) fully contained in \( \bigcup_{\alpha \in \mathcal{A}} X^\alpha \), and such that \( H^1(V, O_X^an) = 0 \). Let \( V_t = V \cap X_t \), we may and will assume that \( V_t \) has rational singularities. Furthermore, we may assume that \( V_t \) is contained in any given neighborhood of \( x \), hence by Lemma B.6 and the fact that \( mK_{X_t} \) is Cartier at \( x \) we may assume that there is a \( g_t \)-exceptional divisor \( E_t \) on \( Y_t \) such that

\[
(18) \quad O(mK_{Y_t})^{an} \cong O(E_t)^{an} \quad \text{on} \quad g_t^{-1}(V_t).
\]

It follows from condition (2) that \( mK_{Y_t} \) is the restriction of \( mK_Y \) to \( Y_t \), and \( E_t \) is the restriction of a \( g \)-exceptional divisor \( E \) on \( Y \). By Lemma B.6, in order to conclude the proof of the theorem, it suffices to show that there is \( \ell \geq 1 \) such that \( L^\ell \) is trivial, where \( L = O(mK_Y - E)^{an}|_{g^{-1}(V)} \). Let \( \gamma = c^1(L) \in H^2(g^{-1}(V), \mathbb{Z}) \). For every \( x \in V \), we denote by \( \gamma_x \) the image of \( \gamma \) in \( H^2(Y_x, \mathbb{Z}) \) via the map induced by \( g^{-1}(x) = Y_x \hookrightarrow g^{-1}(V) \).

Arguing by contradiction, let us assume that \( L^\ell \) is nontrivial for all \( \ell \geq 1 \). It follows from [KM2, (12.1.4)] and the proof therein that in this case we can find a \( g \)-exceptional curve \( C \subseteq Y \), with image \( p := g(C) \) in \( V \), such that \( (L \cdot C) \neq 0 \). In particular, \( \gamma_p \neq 0 \).
By our choice of $V$, we have $p \in X^\alpha$ for some $\alpha \in \mathcal{A}$. Note that $V_t \cap X^\alpha_t \neq \emptyset$ by the definition of $\mathcal{A}$ (recall that by [Ser, Proposition 5], the analytic closure of $X^\alpha_t$ in $X_t$ agrees with the Zariski closure $\overline{X^\alpha_t}$). Pick any point $q \in V_t \cap X^\alpha_t$. Since $X^\alpha$ is connected, and hence path connected, we can fix a path $w: [0, 1] \to X^\alpha$ joining $p$ to $q$. As $g^\alpha$ is topologically locally trivial, moving along the path $w$ induces an isomorphism $H^2(Y_p, \mathbb{Z}) \cong H^2(Y_q, \mathbb{Z})$. Note that $\gamma_p$ is mapped to $\gamma_q$ via this isomorphism, hence $\gamma_q \neq 0$.

On the other hand, (18) implies that $L|_{g^{-1}(V) \cap Y_t}$ is trivial, hence so is $L|_{Y_q}$. Therefore $\gamma_q = 0$, a contradiction. This completes the proof of the theorem.

Remark B.7. It follows from the above proof that in Theorem B.1 one can take any $s$ as given by Lemma B.5, that is, such that $sK_X$ is Cartier on the largest open subset of $X$ that is normal and $\mathbb{Q}$-Cartier.

We will need the following version of the result, which holds over any algebraically closed field $k$ of characteristic zero.

Theorem B.8. Let $f: X \to T$ be a morphism of schemes of finite type over $k$, with $T$ integral, and let $\sigma: T \to X$ be a section of $f$, i.e. $f \circ \sigma = 1_T$. Suppose that we have a countable dense subset $T_0 \subseteq T$ of closed points such that for all $t \in T_0$, at $\sigma(t)$ the fiber $X_t := f^{-1}(t)$ is $\mathbb{Q}$-Gorenstein and has rational singularities. Then there is a nonempty open subset $U$ of $T$, and a positive integer $s$ such that

i) $X$ is normal and $sK_X$ is Cartier in a neighborhood of $\sigma(U)$.

ii) For every (not necessarily closed) point $\xi \in U$, the fiber $X_\xi$ is normal at $\sigma(\xi)$, and we have a canonical isomorphism

$$\mathcal{O}(sK_X)|_{X_\xi} \cong \mathcal{O}(sK_{X_\xi})$$

in a neighborhood of $\sigma(\xi)$. In particular, $sK_{X_\xi}$ is Cartier at $\sigma(\xi)$.

Proof. It is clear that we may replace $T$ by any nonempty open subset $V$ (note that the set $T_0 \cap V$ is countable and dense in $V$). Furthermore, if $W \subseteq X$ is an open subset such that $\sigma^{-1}(W)$ is nonempty, it is enough to prove the theorem for $W \cap f^{-1}(\sigma^{-1}(W)) \to \sigma^{-1}(W)$ (note that $\sigma$ induces a section of this morphism).

After replacing $T$ by an open smooth subset, we may assume that $T$ is smooth, and $f$ is flat (we again use generic flatness). By [Elk, Théorème 4], there is an open subset $W_1$ of $X$ whose closed points $x \in W_1$ are precisely those such that $X_{f(x)}$ has rational singularities at $x$. Since $\sigma(t) \in W_1$ for every $t \in T_0$, we see that $\sigma^{-1}(W_1)$ is nonempty. After replacing $X$ by $W_1 \cap f^{-1}(\sigma^{-1}(W_1))$, we may assume that every closed fiber $X_t$ has rational singularities. In this case, by [Elk, Théorème 2] $X$ has rational singularities. Furthermore, all fibers of $f$ are normal.

We apply Lemma B.5 to get the open subset $W \subseteq X$ consisting of the points in $X$ where $K_X$ is $\mathbb{Q}$-Cartier. Let $s$ be such that $sK_X$ is Cartier on $W$. In order to prove
the theorem it is enough to show that $\sigma^{-1}(W)$ is nonempty. Indeed, if this is the case we may replace $X$ by $W \cap f^{-1}(\sigma^{-1}(W))$, in which case $sK_X$ is Cartier. After replacing $T$ by a nonempty open subset, we may assume by Lemma B.4 that $\mathcal{O}(sK_X)|_{X_\xi} \simeq \mathcal{O}(sK_{X_\xi})$ for every $\xi \in T$. This would prove the theorem.

If the ground field is $\mathbb{C}$, then by Theorem B.1 (see also Remark B.7) we may replace $T$ by an open subset and assume that for every closed point $t \in T$, the divisor $sK_{X_t}$ is Cartier at a closed point $x$ if and only if $x \in W$. In this case we see that $T_0 \subseteq \sigma^{-1}(W)$, hence $\sigma^{-1}(W)$ is nonempty.

For an arbitrary $k$, we can find a subfield $k'$ of $k$ of countable transcendence degree over $\mathbb{Q}$, such that there are morphisms $f': X' \to T'$ and $\sigma': T' \to X'$ of schemes over $k'$, with $f$ and $\sigma$ obtained from $f'$, respectively $\sigma'$, by base-extension via $\text{Spec}(k) \to \text{Spec}(k')$, and such that the points in $T_0$ are defined over $k'$. If $\varphi: T \to T'$ is the natural morphism, it follows that $T'_0 := \varphi(T_0)$ consists of $k'$-rational closed points.

There is an embedding $k' \to \mathbb{C}$. Let $\tilde{f}: \tilde{X} \to \tilde{T}$ and $\tilde{\sigma}: \tilde{T} \to \tilde{X}$ be the morphisms obtained from $f'$, respectively $\sigma'$, by base-extension to $\mathbb{C}$. If $\psi: \tilde{T} \to T'$ is the natural morphism, we choose (closed) points $\tilde{t} \in \psi^{-1}(t')$ for all $t' \in T'_0$. Let $\tilde{T}_0$ be the set consisting of these closed points. It follows from Lemma 2.14 i) that for every $\tilde{t} \in \tilde{T}_0$, the fiber $\tilde{X}_{\tilde{t}}$ is $\mathbb{Q}$-Gorenstein and with rational singularities at $\tilde{\sigma}(\tilde{t})$ (the assertion about rational singularities follows easily from definition by considering base-extensions of resolutions of singularities).

On the other hand, if $W' \subseteq X'$ and $\tilde{W} \subseteq \tilde{X}$ are the subsets where $X'$ and $\tilde{X}$, respectively, are normal and $\mathbb{Q}$-Gorenstein, then by Lemma 2.14 i) we see that $\tilde{W} = W' \times_{\text{Spec}(k')} \text{Spec}(\mathbb{C})$ and $W = W' \times_{\text{Spec}(k')} \text{Spec}(k)$. Furthermore, the divisors $sK_{X}$ and $sK_{\tilde{X}}$ are Cartier on $W'$ and $\tilde{W}$, respectively. We have already seen that $\tilde{\sigma}^{-1}(\tilde{W})$ is a nonempty open subset of $\tilde{T}$. The closure of $\psi(\tilde{T} \setminus \tilde{\sigma}^{-1}(\tilde{W}))$ is a proper closed subset of $T'$. If $t'$ is a closed point in the complement of this closed set, and if $t \in \varphi^{-1}(t')$, then $\sigma(t) \in W$. Therefore $\sigma^{-1}(W)$ is nonempty, and this completes the proof of the theorem.

In order to state the next corollary, we introduce some notation. Let $f: X \to T$ be a morphism of schemes of finite type over $k$, with $T$ integral, and $\sigma: T \to X$ a section of $f$. Suppose that $(t_m)_{m \geq 1}$ is a dense sequence of closed points in $T$ such that each $X_{t_m}$ is klt around $\sigma(t_m)$. Suppose that $\mathfrak{a} = \prod_{j=1}^r a_j^{p_j}$ is an $\mathbb{R}$-ideal on $X$, such that each $a_j$ vanishes along $\sigma(T)$, but it does not vanish along any fiber of $f$. For every (not necessarily closed) point $\xi \in T$, we put $a_{j,\xi} = a_j \cdot \mathcal{O}_{X_\xi}$ and $\mathfrak{a}_\xi = \prod_j a_{j,\xi}^{p_j}$. We also denote by $m_\xi$ the ideal defining $\sigma(\xi)$ in $X_\xi$.

**Corollary B.9.** With the above notation and assumptions, the following hold:
Proof. It is clear that we are allowed to replace $T$ by any open subset $V$, in which case we need to replace $\mathbb{Z}_{>0}$ by $J = \{ i \mid t_i \in V \}$. Since klt varieties have rational singularities (see [Kol2, Corollary 11.14]), we may apply Theorem B.8 to $f$. Let $U \subseteq T$ and $s$ be given by this theorem. After replacing $T$ by $U$, we may assume $U = T$. Since $sK_X$ is Cartier around $\sigma(T)$, and since we are only interested in the behavior around $\sigma(T)$, we may replace $X$ by the open subset where $sK_X$ is Cartier, and therefore assume $sK_X$ is Cartier.

Consider now a log resolution $h: Y \to X$ of $(X, \mathcal{A})$. Let $E$ be the simple normal crossings divisor on $Y$ given by the sum of the $h$-exceptional divisors and of the divisors appearing in the supports of the ideals $a_j \mathcal{O}_Y$. By generic smoothness, after possibly replacing $T$ by an open subset, we may assume the following properties:

1. The composition $f \circ h$ is smooth, and $E$ has relative simple normal crossings over $T$.
2. For every prime divisor $F_j$ in $E$, its image $Z_j$ in $X$ is flat over $T$ and maps onto $T$.
3. Furthermore, we may and will assume that each $Z_j$ contains $\sigma(T)$ (otherwise we simply replace $T$ by $T \setminus \sigma^{-1}(Z_j)$).

It follows from (1) that for every (not necessarily closed) point $\xi \in U$, the morphism $h_\xi: Y_\xi \to X_\xi$ is a log resolution of $(X_\xi, \mathcal{A}_\xi)$. By (2), if $F$ is a component of $E$ that is $h$-exceptional, then $F_\xi$ is $h_\xi$-exceptional (note that $F_\xi$ is smooth, but might not be connected). Since $\mathcal{O}(sK_X)|_{X_\xi} \simeq \mathcal{O}(sK_\xi)$ in a neighborhood of $\sigma(\xi)$, we deduce that $K_{Y_\xi/X_\xi} = K_{Y_\xi/X_\xi}$ over the inverse image of an open neighborhood of $\sigma(\xi)$. Since $\sigma(\xi) \in h_\xi(F_\xi)$ for every prime divisor $F$ in $E$, it follows that each $X_\xi$ is klt and $\text{lct}(X_\xi, \mathcal{A}_\xi) = \text{lct}(X, \mathcal{A})$. Applying this with $\xi = t_i$ and $\xi = \eta$ gives the assertions in i).

Suppose now that $E$ is as in ii). $E$ appears as a prime divisor on some log resolution of $(X_\eta, \mathcal{A}_\eta \cdot m_\eta)$. Since this log resolution is defined over $k(\eta)$, it follows that after replacing $T$ by a suitable open subset, we may assume that this log resolution is equal to $h_\eta$ for some log resolution $h: Y \to X$ as above. In fact, we may assume that $h$ is a log resolution of $(X, \mathcal{A} \cdot a_{\sigma(T)})$, where $a_{\sigma(T)}$ is the ideal defining $\sigma(T)$ in $X$. We may again assume that $h$ satisfies (1)-(3) above. There is a prime divisor $F$ in $E$ such that $E = F_\eta$. It is then clear that, for every $i$, we may take $E_i$ to be any connected component of $E_{t_i}$, and that the divisors $E_i$ satisfy ii). 

\qed
Remark B.10. It follows from the proof of the corollary that the set $J$ in i) can be chosen independently of the exponents $p_1, \ldots, p_r$. In fact, while the convention for $R$-ideals is that all exponents are positive, it is clear that the result in the corollary still holds if some (but not all) of the $p_i$ are allowed to be zero.

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