On a Hyperbolic-Parabolic Parasitoid-Parasite System: 
Well Posedness and Control

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In other words, (1.1) provides a possible structure for the search for an optimal strategy in biological pest control. Preliminary general numerical results are provided in [6].

A specific situation that fits the present framework is the current attempt to limit the spreading of *Drosophila suzukii* (a pest damaging fruits’ cultivation) by means of *ad hoc* deployments of *Trichopria drosophilæ* (a parasitoid laying its eggs in the larvae of the *Drosophila suzukii*), see [7, 13, 15]. An obvious question risen by the adoption of these biological strategies is the search for the optimal time and space choices for the release of parasitoids in the environment. The present paper offers a framework to test and compare different strategies, see Section 3.

From the analytic point of view, besides the introduction of the control, the mixed system (1.1) comprehends the one studied in [4] also by taking into account general source terms that may depend on the unknown variables, as well as on both *t* and *x*. Moreover, the flow \(uv(t,w)\) in the first equation in (1.1) accounts for the velocity chosen by predators in response to the prey density distribution \(w\). A key feature of the mixed system (1.1) is the non locality and nonlinearity of the function \(v\) with respect to the prey density. For instance, the choice

\[
(v(t,w)) (x) = \kappa(t,x) \frac{\nabla(w \ast \eta)(x)}{\sqrt{1 + \|\nabla(w \ast \eta)(x)\|^2}},
\]

(1.2)

means that predators are directed towards regions where the concentration of prey is greater. Above, the positive function \(\kappa\) is the maximal speed of predators and may depend on time and space. For any fixed positive smooth mollifier \(\eta\), the space-convolution product \((w(t) \ast \eta)(x)\) is an average of the prey density at time \(t\) around position \(x\). The denominator in (1.2) acts as a smooth normalisation factor.

The next section is devoted to the well posedness and stability of the Cauchy Problem for (1.1). Then, we also deal with the optimal control of the solutions to (1.1) by means of the control \(q\) and aiming at the minimization of a given integral functional. A specific application of these theoretical results is in Section 3. All analytic proofs are deferred to Section 4.

2 Main Results

Below, we fix \(T > t_o \geq 0\), possibly allowing the case \(T = +\infty\), and correspondingly we set

\[
I = [t_o, T] \text{ or } I = [t_o, +\infty[ \text{ and } \quad J = \left\{ (t_1, t_2) \in I^2 : t_1 < t_2 \right\}. \tag{2.1}
\]

The space dimension \(n\) is fixed throughout, as well as the parameter \(\mu > 0\). For the heat kernel we use the notation \(H_\mu(t,x) = (4\pi \mu t)^{-n/2} \exp\left(-\|x\|^2/(4\mu t)\right)\), where \(t \in I, x \in \mathbb{R}^n\). As it is well known, \(\|H_\mu(t)\|_{L^1(\mathbb{R}^n;\mathbb{R})} = 1\).

We recall below the definition of solution to (1.1), slightly extending that in [4], and adapting it to the present setting of time and space dependent coefficients.

**Definition 2.1.** A pair \((u,w) \in C^0(I;L^1(\mathbb{R}^n;\mathbb{R}^2))\) is a solution to problem (1.1) on \(I\) if

- setting \(a(t,x) = g(t,x,u(t,x),w(t,x))\), \(w\) is a weak solution to \(\partial_tw - \mu \Delta w = aw\);
- setting \(b(t,x) = f(t,x,w(t,x))\) and \(c(t,x) = (v(t,w(t)))(x)\), \(u\) is a weak solution to \(\partial_u + \nabla \cdot (u c) = bu + q\).
Then, for any initial datum in the sense of Definition 2.1 and, moreover, and the norm
\[ \| (u, w) \|_{X} = \| u \|_{L^1(I; \mathbb{R}^n; \mathbb{R})} + \| w \|_{L^1(I; \mathbb{R}^n; \mathbb{R})}. \]

We are now ready to state the key well posedness and stability result of this paper.

**Theorem 2.2.** Consider problem (1.1) under the following assumptions:

(v) \( v : I \times (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}) \to (C^2 \cap W^{1,\infty})(\mathbb{R}^n; \mathbb{R}^n) \) admits two maps \( K_v \in L^\infty_{loc}(I; \mathbb{R}_+) \) and \( C_v \in L^\infty_{loc}(I \times \mathbb{R}_+; \mathbb{R}_+) \) weakly increasing in each argument and such that, for all \( t \in I \) and \( w, w_1, w_2 \in (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}), \)

\[
\begin{align*}
&\| v(t, w) \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq K_v(t) \| w \|_{L^1(\mathbb{R}^n; \mathbb{R})}, \\
&\| \nabla v(t, w) \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \leq K_v(t) \| w \|_{L^1(\mathbb{R}^n; \mathbb{R})}, \\
&\| v(t, w_1) - v(t, w_2) \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq K_v(t) \| w_1 - w_2 \|_{L^1(\mathbb{R}^n; \mathbb{R})}, \\
&\| \nabla \cdot (v(t, w)) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \leq C_v(t) \| w \|_{L^1(\mathbb{R}^n; \mathbb{R})}, \\
&\| \nabla \cdot (v(t, w_1) - v(t, w_2)) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \leq C_v(t) \| w_1 - w_2 \|_{L^1(\mathbb{R}^n; \mathbb{R})}.
\end{align*}
\]

(f) \( f : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) admits a weakly increasing map \( K_f \in L^\infty_{loc}(I; \mathbb{R}_+) \) such that, for a.e. \( t \in I, \) all \( w_1, w_2 \in \mathbb{R}_+ \) and all \( w \in BV(\mathbb{R}^n; \mathbb{R}), \)

\[
\begin{align*}
&\sup_{x \in \mathbb{R}^n} | f(t, x, w_1) - f(t, x, w_2) | \leq K_f(t) | w_1 - w_2 |, \\
&\sup_{x \in \mathbb{R}^n} f(t, x, w_1) \leq K_f(t) (1 + w_1), \\
&TV f(t, \cdot, w(\cdot)) \leq K_f(t) \left( 1 + \| w \|_{L^\infty(\mathbb{R}^n; \mathbb{R})} + TV(w) \right).
\end{align*}
\]

(g) \( g : I \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) admits a weakly increasing map \( K_g \in L^\infty_{loc}(I; \mathbb{R}_+) \) such that, for a.e. \( t \in I \) and all \( u_1, u_2, w_1, w_2 \in \mathbb{R}_+, \)

\[
\begin{align*}
&\sup_{x \in \mathbb{R}^n} | g(t, x, u_1, w_1) - g(t, x, u_2, w_2) | \leq K_g(t) \left( | w_1 - w_2 | + | w_1 - w_2 | \right), \\
&\sup_{(x, u, w) \in \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+} g(t, x, u, w) \leq K_g(t).
\end{align*}
\]

(q) \( q \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}_+) \cap L^\infty(I; L^1(\mathbb{R}^n; \mathbb{R}_+)) \) and \( q(t) \in BV(\mathbb{R}^n; \mathbb{R}_+), \) for \( t \in I. \)

Then, for any initial datum \( (u_0, w_0) \in X^+, \) problem (1.1) admits a unique solution \( (u, w) \in C^0(I, L^1(\mathbb{R}^n; \mathbb{R}_+)) \)
in the sense of Definition 2.7 and, moreover,
Remark 2.3. Note the different behaviors of the total variation in space. 

To prove Theorem 2.2, following the general lines of [4], we study separately, but symmetrically, the parabolic and the hyperbolic problems that constitute (1.1), namely
\[ \frac{\partial_t w}{w} - \mu \Delta w = a(t, x, w) \quad \text{and} \quad \partial_t u + \nabla \cdot (c(t, x, u) w) = b(t, x, u) + q(t, x), \]
with \( a, b \) and \( c \) as in Definition 2.1. All estimates use exclusively the \( L^1 \) or \( L^\infty \) norms and the total variation in space.

Remark 2.3. Note the different behaviors of \( f \) and \( g \) allowed by conditions (f) and (g), namely \( \sup_{x \in \mathbb{R}^n} f(t, x, w) \leq K_f(t) (1 + w) \) and \( \sup_{(x, u, w) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+} g(t, x, u, w) \leq K_g(t) \). For instance, \( f \) may well increase in \( w \), while \( g \) may decrease in both \( u \) and \( w \). Thus, the classical Lotka-Volterra source terms \( f(w) = \alpha w - \beta \) and \( g(u) = \gamma - \delta u \) (for \( \alpha, \beta, \gamma, \delta \) positive and constant) are compatible with (f) and (g) comprising the problem studied in [4] when \( q \equiv 0 \).

Theorem 2.2 allows to consider optimal control problems based on (1.1). To this aim, introduce a cost functional measuring the relevance of the presence of the pest, for instance quantifying its effect on cultivation. Inspired by [6] § 4, we propose a cost of the general form
\[ I = \int_I \int_{\mathbb{R}^n} \Phi(t, x, u(t, x), w(t, x)) \, dx \, dt. \] (2.6)

It is clear that various assumptions on the function \( \Phi \) ensure that the integral on the right hand side of (2.6) is a continuous function of \((u, w)\) in \( \mathcal{A} \). Therefore, (3) in Theorem 2.2 ensures that \( I \) is a continuous function of the control \( q \) in \( L^1 \).
In practice, the choice of a real strategy depends on a finite set of parameters, say \( p \in \mathbb{R}^m \), defining, for instance, the (time/space) support of \( q \), or the maximal value of \( q \), or its (time/space) integral. We are thus lead to minimize a compositions of maps of the type

\[
\mathbb{R}^m \to L^\infty (I; L^1(\mathbb{R}^n; \mathbb{R})) \to X^+ \to \mathbb{R}
\]

\( p \to q \to (u, w) \to \mathcal{I} \)

to which, thanks to Theorem 2.2, Weierstraß Theorem can be applied, ensuring the existence of an optimal strategy \( p^* \). The actual computation of \( p^* \) can be achieved through standard numerical procedures dedicated to the optimization of Lipschitz continuous functions. The next section is devoted to specific examples.

3 Optimized Timing of Parasitoids’ Releases

We present below a sample of the possible behaviors of solutions to (1.1). Further examples can be found in [6].

Inspired by [13, 15], we address the problem of optimizing the timing and the location of parasitoids’ (=predators’) releases in the case of a parasite (=prey) whose reproduction is seasonal and geographically localized. To this aim, we consider the following instance of (1.1) in the case of \( n = 2 \) space dimensions

\[
\begin{aligned}
\partial_t u + \nabla \cdot (u \, v(w)) &= (\alpha \, w - \beta)u + q(t, x) \\
\partial_t w - \mu \Delta w &= \left( \gamma (1 - \sin t) \chi_B(x) \left( \frac{1 - w}{C} \right) - \delta u \right)w,
\end{aligned}
\]

Here, as usual, \( t \) is time and \( x \) is the space coordinate in \( \mathbb{R}^2 \). Moreover, \( \alpha \, w \) is the predator natality due to predation, \( \beta \) is the predators’ mortality, \( \delta \) is the prey mortality due to predation and \( C \) is the prey carrying capacity. The prey natality \( \gamma (1 - \sin t) \chi_B(x) \) is seasonal, i.e. it is \( 2\pi \)-periodic in time, and localized, i.e. it is supported in the ball \( B \) centered at the origin with radius 2. The speed \( v \) is chosen as in (1.2), with \( \kappa \) constant. The parasitoids predate hunting for parasites in the direction of the highest average prey density gradient within a radius \( \ell \), which hence measures the predator horizon. We summarize here the choices of functions and parameters in (3.1)–(1.2), apart from \( q \) to be chosen below:

| \( \alpha \) | \( \beta \) | \( \gamma \) | \( \delta \) | \( C \) | \( \ell \) | \( \kappa \) |
|---|---|---|---|---|---|---|
| 0.25 | 2.00 | 9.00 | 0.50 | 10.0 | 0.80 | 2.00 |

\( \eta(x) = \begin{cases} 
\frac{4}{\pi \ell^2} \left( 1 - \frac{\|x\|^2}{\ell^2} \right)^{3} & \|x\| \leq \ell, \\
0 & \|x\| > \ell. 
\end{cases} \) (3.2)

We now seek strategies \( q = q(t, x) \) to release parasitoids so that the parasite population is kept small in the rectangle \( R = [1, 3] \times [-3, 3] \), which we assume is the region where the presence of the parasites is most harmful. The regions \( B \) and \( R \) are chosen so that they are different but overlapping. Thus, for simplicity, we aim at the minimization of

\[
\mathcal{I} = \int_{4\pi}^{12\pi} \int_{R} w(t, x) \, dx \, dt ,
\]

\( \chi_B \) is the characteristic function of the set \( B \): \( \chi_B(x) = 1 \iff x \in B \) and \( \chi_B(x) = 0 \iff x \in \mathbb{R}^n \setminus B \).
although within the present framework (2.6) more complex costs can be considered. Another natural choice, for instance, might be the minimization of the pest population $w$ only in specific periods, e.g. when fruits are ripening on the trees, as in the case of the Drosophila suzukii. As initial datum we choose

$$u_0(x) \equiv 0, \quad w_0(x) = 2 \chi_B(x). \quad (3.4)$$

Clearly, Theorem 2.2 applies to (3.1)–(1.2)–(3.2)–(3.4) and the cost (3.3) fits into (2.6).

In the examples below, we use the Lax–Friedrichs scheme [12, § 12.5] to integrate the hyperbolic convective term and an explicit finite difference algorithm to deal with the parabolic equation. Furthermore, we exploit dimensional splitting [12, § 19.5] and a further splitting to take care of the source terms [12, § 17.1], which are computed through a second order Runge–Kutta method (corresponding to $\alpha = 1/2$ in [16, § 12.5, p. 327]). Refer to [1, 2, 14] for alternative algorithms. The numerical domain is the rectangle $[-4 - \ell, 4 + \ell] \times [-4 - \ell, 4 + \ell]$ and we let the parameters $\alpha, \beta$ and $\gamma$ vanish outside the physical domain $[-4, 4] \times [-4, 4]$.

The computations below were obtained with a uniform mesh consisting of $2^{10} \times 2^{10}$ points.

First, as a reference case, we integrate (3.1)–(1.2)–(3.2)–(3.4) with $q \equiv 0$. The results are displayed in Figure 1. Since parasitoids are absent, parasites evolve with a logistic growth with capacity $C$ and a $2\pi$–periodic natality. After two periods, the total number of parasites is approximately time periodic, with a high mean value.

We now assume that at time $4\pi$ measures need to be taken to reduce the presence of parasites. This is achieved through the release in the environment of the parasitoid $u$, which is described by the function $q$ in (3.1). Different strategies correspond to different choices of $q$. The ones we consider below differ both in the space and time dependence: they may take place in the ball $B$ where the parasites are born, or on the rectangle $R$ where parasites are harmful. Moreover, they can take place uniformly in time (on $I_0 = [4\pi, 12\pi]$) or in the time intervals where parasites are more ($I_1 = \sin^{-1}([-1, -1/\sqrt{2}] \cap I_0)$, middle ($I_2 =$

![Figure 1: Left, the initial datum (3.4) for $w$ in the $x$–plane and, right, the total amount of parasites $\int_{[-4,4]^2} w(t,x) \, dx$ on the whole physical domain as a function of time.](image-url)
\[ \cos^{-1}([-1, -1/\sqrt{2}] \cap I_0) \text{ or less (} I_3 = \sin^{-1}([1/\sqrt{2}, 1]) \cap I_0 \text{)} \text{ prolific, see Figure 2. These} \]

![Figure 2: Characteristic functions of the time intervals, from left to right, I_0, I_1, I_2 and I_3 used in the definitions of the controls (3.5), plotted together with the map } t \to 1 - \sin t \text{ appearing in the natality of the parasite in (3.1).} \]

strategies correspond to the following choices of \( q \):

\[
\begin{align*}
q_B^0(t) &= 3.166287 \chi_{I_0}(t) \chi_B(x) \\
q_R^0(t) &= 3.315728 \chi_{I_0}(t) \chi_R(x) \\
q_B^1(t) &= 12.66515 \chi_{I_1}(t) \chi_B(x) \\
q_R^1(t) &= 13.26291 \chi_{I_1}(t) \chi_R(x) \\
q_B^2(t) &= 12.66515 \chi_{I_2}(t) \chi_B(x) \\
q_R^2(t) &= 13.26291 \chi_{I_2}(t) \chi_R(x) \\
q_B^3(t) &= 12.66515 \chi_{I_3}(t) \chi_B(x) \\
q_R^3(t) &= 13.26291 \chi_{I_3}(t) \chi_R(x) .
\end{align*}
\] (3.5)

The above values are chosen so that the amount of parasitoids inserted in the environment is constant, i.e.

\[
\int_0^{12\pi} \int_{R^2} q_A^i(t, x) \, dx \, dt = 1000 \quad \text{for } i = 0, 1, 2, 3 \text{ and } A = B, R .
\]

The numerical integrations of (3.1)–(3.2)–(3.4) with the controls (3.5) yield the following values for the cost (3.3):

| \( I \) | 0    | 1    | 2    | 3    |
|------|------|------|------|------|
| B    | 1179.05 | 1318.74 | 1332.75 | 1232.41 |
| R    | 874.420 | 1068.13 | 1098.85 | 1080.19 |

when \( q \equiv 0, I = 1866.98 .\)

In the different cases of the controls in (3.5), the instantaneous costs \( t \to \int_R w(t, x) \, dx \) are displayed in Figure 3. All solutions to (3.1)–(3.2)–(3.4) show a somewhat periodic behavior for \( t > 4\pi \).

With respect to the cost (3.3), where the rectangle \( R \) obviously plays a key role, the most effective strategy consists in a constant release of parasitoids over the rectangle \( R \), corresponding to the control \( q_R^0 \) in (3.5). This solution is somewhat periodic and displays a maximum, respectively a minimum, of the running cost at the time \( t \approx 33.30 \), respectively \( t \approx 30.79 \); level plots of the corresponding solutions computed at these times are in Figure 4.

It is evident that the convective term in the first equation in (3.1) allows the parasitoids to move towards the region with the highest parasite concentration. On the other hand, the Laplace operator in the second equation makes the parasites diffuse everywhere.
Figure 3: Graphs of the instantaneous cost \( t \to \int_R w(t, x) \, dx \) corresponding to the controls (3.5) on the time interval \([4\pi, 12\pi]\). Figure 3a corresponds to the diffusion of parasites with no control. The most effective strategy, in the sense it minimizes (3.3), is in Figure 3c.

Figure 4: Contour plots of the solution to corresponding to the best strategy \( q_0^R \) in (3.5). Left, at time \( t = 30.79 \) approximately corresponding to a maximum of the running cost and, right, at time \( t = 33.30 \) approximately corresponding to a minimum.
Let Lemma 4.2.

The following lemmas will be of use below. The proofs, where immediate, are omitted.

4 Analytic Proofs

Boundary conditions deserve a specific treatment on their own. At the modeling level, the immigration of parasites is neglected in the present work. At the analytic level, general well posedness and stability results are currently apparently still missing, see [5] for recent preliminary results. The numerical algorithm to deal with boundary conditions would then be necessarily adapted.

We expect that a precise simulation of a real scenario requires a model more complex than (3.1)– (3.3), as well as the obvious tuning of the various parameters. For instance, also $\alpha, \beta$ and $\delta$ are likely to be better substituted by “seasonal” (i.e., time periodic) functions. While such an experimental fitting is out of the scopes of the present work, we remark that the generality of the framework presented here, and in particular Theorem 2.2 allows to comprehend it.

Let Lemma 4.3.

\begin{align*}
\text{Lemma 4.1} \quad \text{([4], Formula (1.8) and Remark 1.16]. Let } \psi \in (L^1 \cap L^\infty \capBV) (\mathbb{R}^n; \mathbb{R}). \text{ Then, there exists a sequence } \psi_h \in \mathcal{C}^\infty (\mathbb{R}^n; \mathbb{R}) \text{ such that for } h \in \mathbb{N} \setminus \{0\} \\
\psi_h \to \psi \text{ in } L^1 (\mathbb{R}^n; \mathbb{R}), \quad \| \psi_h \|_{L^\infty (\mathbb{R}^n)} \leq \| \psi \|_{L^\infty (\mathbb{R}^n)}, \quad \TV (\psi_h) \to \TV (\psi). \tag{4.1}
\end{align*}

\begin{align*}
\text{Lemma 4.2. Let } \psi \in (L^\infty \capBV) (\mathbb{R}^n; \mathbb{R}). \text{ Then, there exists a sequence } \psi_h \in \mathcal{C}^\infty (\mathbb{R}^n; \mathbb{R}) \text{ such that for } h \in \mathbb{N} \setminus \{0\}, \psi_h \to \psi \text{ in } L^\infty (\mathbb{R}^n; \mathbb{R}), \text{ so that also } \psi_h \to \psi \text{ in } L^1_{\text{loc}} (\mathbb{R}^n; \mathbb{R}), \text{ and} \\
\| \psi_h \|_{L^\infty (\mathbb{R}^n)} \leq \| \psi \|_{L^\infty (\mathbb{R}^n)}, \quad \TV (\psi_h) \leq \TV (\psi).
\end{align*}

\begin{proof}
Let $\rho$ be a mollifier: $\rho \in \mathcal{C}^\infty_c (\mathbb{R}^n, \mathbb{R}), \rho \geq 0,$ spt $\rho \subseteq \{ x \in \mathbb{R}^2 : \| x \| \leq 1 \}$ and $\int_{\mathbb{R}^n} \rho = 1.$ Define $\rho_h (x) = h^n \rho (h x)$ for $h \in \mathbb{N} \setminus \{0\}$ and set $\psi_h = \rho_h * \psi.$ The $L^1_{\text{loc}}$ convergence follows from $\| \psi_h - \psi \|_{L^\infty (\mathbb{R}^n)} \to 0,$ ensured by [8] Theorem 8.14. The $L^\infty$ estimate is a consequence of [8] Proposition 8.7. Finally, [8] Proposition 8.68 implies the latter bound.
\end{proof}

\begin{align*}
\text{Lemma 4.3. Let } \psi \in L^\infty (I \times \mathbb{R}^n; \mathbb{R}) \text{ be such that for all } t \in I, \psi(t) \in BV (\mathbb{R}^n; \mathbb{R}). \text{ Then, there exists a sequence } \psi_h \in L^\infty (I \times \mathbb{R}^n; \mathbb{R}) \text{ such that for all } h \in \mathbb{N} \setminus \{0\} \text{ and for a.e. } t \in I, \psi_h (t) \in (\mathcal{C}^\infty \capBV) (\mathbb{R}^n; \mathbb{R}), \psi_h (t) \to \psi (t) \text{ in } L^1_{\text{loc}} (\mathbb{R}^n; \mathbb{R}) \text{ and} \\
\| \psi_h (t) \|_{L^\infty (\mathbb{R}^n)} \leq \| \psi (t) \|_{L^\infty (\mathbb{R}^n)}, \quad \TV (\psi_h (t)) \leq \TV (\psi (t)).
\end{align*}

4.1 About the Parabolic Equation $\partial_t w - \mu \Delta w = a(t, x) w$

We focus on the parabolic problem:

\[
\begin{cases}
\partial_t w - \mu \Delta w = a(t, x) w \\
w(t_0, x) = w_0(x)
\end{cases} \quad (t, x) \in I \times \mathbb{R}^n. \tag{4.2}
\]

Similarly to [4], solutions to (4.2) are sought as $L^1$ function defined on $\mathbb{R}^n$ and all estimates refer to the $L^1$ or $L^\infty$ norms, see (2.3), which is somewhat unusual in relation to (4.2).
Definition 4.4. Let $a \in L_\infty(I \times \mathbb{R}^n; \mathbb{R})$ and $w_o \in L_1^1(\mathbb{R}^n; \mathbb{R})$. A solution to problem (4.2) is a function $w \in C^0(I; L_1^1(\mathbb{R}^n; \mathbb{R}))$ such that

$$w(t, x) = (H_\mu(t) * w_o)(x) + \int_0^t \left( H_\mu(t - \tau) * (a(\tau) w(\tau)) \right)(x) \, d\tau.$$  

(4.3)

Lemma 4.5 ([H Lemma 2.4]). Let $a \in L_\infty(I \times \mathbb{R}^n; \mathbb{R})$. Assume that $w_o \in L_1^1(\mathbb{R}^n; \mathbb{R})$ and $w \in C^0(I; L_1^1(\mathbb{R}^n; \mathbb{R}))$. Then, the following statements are equivalent:

1. The function $w$ solves (4.2) in the sense of Definition 4.4.

2. The function $w$ is a weak solution to (4.2), i.e., for all test functions $\varphi \in C^2_0(I \times \mathbb{R}^n; \mathbb{R})$

$$\int_{t_o}^T \int_{\mathbb{R}^n} (w \partial_t \varphi + \mu w \Delta \varphi + a w \varphi) \, dx \, dt = 0$$

(4.4)

and $w(t_o, x) = w_o(x)$.

Proposition 4.6 ([H Proposition 2.5]). Fix $a \in L_\infty(I \times \mathbb{R}^n; \mathbb{R})$. Then, (4.2) generates the process

$$\mathcal{P} : J \times L_1^1(\mathbb{R}^n; \mathbb{R}) \to L_1^1(\mathbb{R}^n; \mathbb{R})$$

$$t \mapsto w_o \to w$$

with $w$ defined as in (4.3), with the following properties, for a suitable $\mathcal{O} \in L_\infty(I; \mathbb{R})$ that depends only on norms of the map $a$ on $I \times \mathbb{R}^n$.

(P1) $\mathcal{P}$ is a Process: $\mathcal{P}_{t,t} = \text{Id}$ for all $t \in I$ and $\mathcal{P}_{t_1,t_2} \circ \mathcal{P}_{t_2,t_3} = \mathcal{P}_{t_1,t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.

(P2) Regularity in time: for all $w_o \in L_1^1(\mathbb{R}^n; \mathbb{R})$, the map $t \mapsto \mathcal{P}_{t_0,t} w_o$ is in $C^0(I; L_1^1(\mathbb{R}^n; \mathbb{R}))$, and, moreover, for every $\vartheta \in [0, 1]$ and for all $\tau, t_1, t_2 \in I$, with $t_2 \geq t_1 \geq \tau > 0$,

$$\|\mathcal{P}_{t_0,t_2} w_o - \mathcal{P}_{t_0,t_1} w_o\|_{L_1^1(\mathbb{R}^n)} \leq \|w_o\|_{L_1^1(\mathbb{R}^n)} \left[ \frac{n}{\tau - t_o} + \mathcal{O}(t_2) \right] |t_2 - t_1|^\vartheta.$$  

(P3) Regularity in space: for all $t > t_o$, $w(t) \in C^\infty(\mathbb{R}^n; \mathbb{R})$.

(P4) Regularity in $(t, x)$: if $w_o \in (L_1^1 \cap C^1)(\mathbb{R}^n; \mathbb{R})$, then $(t, x) \to (\mathcal{P}_{t_0,t} w_o)(x)$ in $C^1(I \times \mathbb{R}^n; \mathbb{R})$.

(P5) $L_1^1$ continuous dependence on $w_o$: for all $t \in I$, the map $\mathcal{P}_{t_0,t} : L_1^1(\mathbb{R}^n; \mathbb{R}) \to L_1^1(\mathbb{R}^n; \mathbb{R})$ is linear and continuous, with $\|\mathcal{P}_{t_0,t} w_o\|_{L_1^1(\mathbb{R}^n)} \leq \mathcal{O}(t) \|w_o\|_{L_1^1(\mathbb{R}^n)}$.

(P6) Stability with respect to $a$: let $a_1, a_2 \in L_\infty(I \times \mathbb{R}^n; \mathbb{R})$ with $a_1 - a_2 \in L_1^1(I \times \mathbb{R}^n; \mathbb{R})$ and call $\mathcal{P}_1^1, \mathcal{P}_2^1$ the corresponding processes. Then, for all $t \in I$ and for all $w_o \in (L_1^1 \cap L_\infty)(\mathbb{R}^n; \mathbb{R})$,

$$\|\mathcal{P}_{t_0,t}^1 w_o - \mathcal{P}_{t_0,t}^2 w_o\|_{L_1^1(\mathbb{R}^n)} \leq \mathcal{O}(t) \|w_o\|_{L_\infty(\mathbb{R}^n)} \|a_1 - a_2\|_{L_1^1(I \times \mathbb{R}^n)}.$$  

(P7) $L_\infty$-estimate: for all $w_o \in (L_1^1 \cap L_\infty)(\mathbb{R}^n; \mathbb{R})$, for all $t \in I$, $\|\mathcal{P}_{t_o,t} w_o\|_{L_\infty(\mathbb{R}^n)} \leq \mathcal{O}(t) \|w_o\|_{L_\infty(\mathbb{R}^n)}$. 

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(P8) **W**^{1,1}-estimate: for all \( w_0 \in L^1(\mathbb{R}^n; \mathbb{R}) \), for all \( t \in I \) with \( t > t_0 \),

\[
\left\| \nabla (P_{t_0,t}w_0) \right\|_{L^1(\mathbb{R}^n; \mathbb{R})} \leq \frac{J_n}{\sqrt{\mu (t - t_0)}} \left\| w_0 \right\|_{L^1(\mathbb{R}^n)} \\
\times \left( 1 + 2 (t - t_0) \left\| a \right\|_{L^\infty(t_0,t] \times \mathbb{R}^n)} e^{\int_{t_0}^t \left\| a(\tau) \right\|_{L^\infty(\mathbb{R}^n)} d\tau} \right),
\]

where \( J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \) and \( \Gamma \) is the Gamma function.

The latter estimate above and (P3) provide a BV bound on the solution \( P_{t_0,t}w_0 \) for \( t > t_0 \).

In the sequel, we need the following strengthened version of (P8).

**Proposition 4.7.** Let \( a \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \) and assume \( w_0 \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^n; \mathbb{R}) \). Call \( w(t) \) the solution to (4.2). Then, for all \( t \in I \), \( w(t) \in BV(\mathbb{R}^n; \mathbb{R}) \) and the following estimate holds:

\[
TV(w(t)) \leq TV(w_0) + \frac{2 J_n}{\sqrt{\mu}} \mathcal{O}(t) \left\| a \right\|_{L^\infty([t_0,t] \times \mathbb{R}^n)} \left\| w_0 \right\|_{L^1(\mathbb{R}^n)},
\]

where \( J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \) and \( \Gamma \) is the Gamma function.

**Proof.** Approximate \( w_0 \) by means of a sequence \( w^h_0 \) as defined in Lemma 4.1. Define \( w_h \) through (4.3) by

\[
w_h(t,x) = \left( H_\mu(t) * w^h_0 \right)(x) + \int_{t_0}^t \left( H_\mu(t - \tau) * (a(\tau) w_h(\tau)) \right)(x) d\tau.
\]

Let \( w \) be defined by (4.3) and compute

\[
\left\| w_h(t) - w(t) \right\|_{L^1(\mathbb{R}^n)} \leq \left\| w^h_0 - w_0 \right\|_{L^1(\mathbb{R}^n)} + \int_{t_0}^t \left\| a(\tau) \right\|_{L^\infty(\mathbb{R}^n)} \left\| w_h(\tau) - w(\tau) \right\|_{L^1(\mathbb{R}^n)} d\tau.
\]

An application of Gronwall Lemma [17] Chapter I, 1.III yields

\[
\left\| w_h(t) - w(t) \right\|_{L^1(\mathbb{R}^n)} \leq \left\| w^h_0 - w_0 \right\|_{L^1(\mathbb{R}^n)} \exp \left( \int_{t_0}^t \left\| a(\tau) \right\|_{L^\infty(\mathbb{R}^n)} d\tau \right).
\]

Thus, as \( h \) goes to \( +\infty \), \( w_h(t) \) converges to \( w(t) \) in \( L^1(\mathbb{R}^n; \mathbb{R}) \) for a.e. \( t \in I \).

It follows immediately from (4.6) and from the regularity of the heat kernel \( H_\mu \) that \( w_h(t) \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) for a.e. \( t \in I \). Moreover,

\[
\nabla w_h(t,x) = \left( H_\mu(t) * \nabla w^h_0 \right)(x) + \int_{t_0}^t \nabla H_\mu(t - \tau) * (a(\tau) w_h(\tau)) (x) d\tau,
\]

so that, using the properties of the heat kernel \( H_\mu \) and (P5) in Proposition 4.6, we obtain

\[
\left\| \nabla w_h(t) \right\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \leq \left\| \nabla w^h_0 \right\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} + \int_{t_0}^t \left\| \nabla H_\mu(t - \tau) \right\|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \left\| a(\tau) \right\|_{L^\infty(\mathbb{R}^n)} \left\| w_h(\tau) \right\|_{L^1(\mathbb{R}^n)} d\tau
\]
Stability with respect to estimate:

Proof. Let now \( h \to +\infty \): Lemma 4.11 and the lower semicontinuity of the total variation imply that:

\[
\text{TV}(w(t)) \leq \lim_{h \to +\infty} \text{TV}(w_h(t)) = \lim_{h \to +\infty} \left\| \nabla w_h(t) \right\|_{L^1([0, t] \times \mathbb{R}^n)} \leq \text{TV}(w_o) + \mathcal{O}(t) \left\| w_o \right\|_{L^1([0, t] \times \mathbb{R}^n)} \left\| a \right\|_{L^\infty([0, t] \times \mathbb{R}^n)} \frac{2J_n}{\sqrt{\mu}} \sqrt{t - t_o},
\]

completing the proof. \( \square \)

We need the following improvements of the estimates in propositions 4.6 and 4.7 that hold in the case of positive initial data.

Corollary 4.8. Let \( a \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \), \( w_o \in L^1(\mathbb{R}^n; \mathbb{R}) \) with \( w_o \geq 0 \). Then,

(P9) \textbf{Positivity:} \( P_{t_o, t} w_o \geq 0 \) for all \( t \in I \).

(P10) \textbf{A priori estimates:} assume that \( w_o \in (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}) \) and set, for all \( t \in I \), \( A(t) = \sup_{\xi \in \mathbb{R}^n} a(t, \xi) \). Then,

\[
\left\| P_{t_o, t} w_o \right\|_{L^1(\mathbb{R}^n)} \leq \left\| w_o \right\|_{L^1(\mathbb{R}^n)} \exp \int_{t_o}^t A(\tau) \, d\tau, \tag{4.7}
\]

\[
\left\| P_{t_o, t} w_o \right\|_{L^\infty(\mathbb{R}^n)} \leq \left\| w_o \right\|_{L^\infty(\mathbb{R}^n)} \exp \int_{t_o}^t A(\tau) \, d\tau.
\]

(P11) \textbf{Stability with respect to} \( a \): let \( a_1, a_2 \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \) with \( a_1 - a_2 \in L^1(I \times \mathbb{R}^n; \mathbb{R}) \) and call \( P^1, P^2 \) the corresponding processes. Then, for all \( t \in I \) and for all \( w_o \in (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}) \),

\[
\left\| P^1_{t_o, t} w_o - P^2_{t_o, t} w_o \right\|_{L^1(\mathbb{R}^n)} \leq \left\| w_o \right\|_{L^\infty(\mathbb{R}^n)} e^{\int_{t_o}^t \left[ \left\| a_1(\tau) \right\|_{L^\infty(\mathbb{R}^n)} + \left\| a_2(\tau) \right\|_{L^\infty(\mathbb{R}^n)} \right] \, d\tau} \left\| a_1 - a_2 \right\|_{L^1([0, t] \times \mathbb{R}^n)}. \tag{4.8}
\]

(P12) \textbf{BV estimate:} if \( w_o \in (L^1 \cap L^\infty \cap BV)(\mathbb{R}^n; \mathbb{R}) \), define \( A(t) = \sup_{x \in \mathbb{R}^n} a(t, x) \), then

\[
\text{TV}(P_{t_o, t} w_o) \leq \text{TV}(w_o) + \frac{2J_n}{\sqrt{\mu}} \sqrt{t - t_o} \left\| a \right\|_{L^\infty([0, t] \times \mathbb{R}^n)} \left\| w_o \right\|_{L^1(\mathbb{R}^n)} e^{\int_{t_o}^t A(\tau) \, d\tau}, \tag{4.9}
\]

where \( J_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \) and \( \Gamma \) is the Gamma function.

**Proof.** The positivity \((\text{P9})\) follows from \[\text{Point 6. in Proposition 2.5}\], based on \[\text{Chapter 2, Section 4, Theorem 9}\].

Starting now from \( (4.3) \), we have

\[
w(t, x) = (H\mu(t) * w_o)(x) + \int_{t_o}^t \int_{\mathbb{R}^n} H\mu(t - \tau, x - \xi) a(\tau, \xi) w(\tau, \xi) d\xi \, d\tau
\]

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\[
\leq (H_\mu(t) * w_\alpha)(x) + \int_{t_0}^{t} A(\tau) \int_{\mathbb{R}^n} H_\mu(t - \tau, x - \xi) w(\tau, \xi) \, d\xi \, d\tau.
\]

In both cases of the \( \mathbf{L}^1 \) and \( \mathbf{L}^\infty \) estimate, an application of Gronwall Lemma [17, Chapter I, 1.11] completes the proof of (P10).

Concerning the stability with respect to \( a \), denote \( w_i(t) = \mathcal{P}_{t_0,t}^i w_\alpha \), for \( i = 1, 2 \) and \( t \in I \), and using (4.3), compute

\[
w_1(t, x) - w_2(t, x) = \int_{t_0}^{t} \int_{\mathbb{R}^n} H_\mu(t - \tau, x - \xi) (a_1(\tau, \xi) w_1(\tau, \xi) - a_2(\tau, \xi) w_2(\tau, \xi)) \, d\xi \, d\tau
\]

so that

\[
\| w_1(t) - w_2(t) \|_{\mathbf{L}^i(\mathbb{R}^n)} \leq \int_{t_0}^{t} \| a_1(\tau) - a_2(\tau) \|_{\mathbf{L}^1(\mathbb{R}^n)} \| w_1(\tau) \|_{\mathbf{L}^\infty(\mathbb{R}^n)} \, d\tau + \int_{t_0}^{t} \| a_2(\tau) \|_{\mathbf{L}^\infty(\mathbb{R}^n)} \| w_1(\tau) - w_2(\tau) \|_{\mathbf{L}^i(\mathbb{R}^n)} \, d\tau.
\]

By Gronwall Lemma,

\[
\| w_1(t) - w_2(t) \|_{\mathbf{L}^i(\mathbb{R}^n)} \leq \int_{t_0}^{t} \| a_1(\tau) - a_2(\tau) \|_{\mathbf{L}^1(\mathbb{R}^n)} \| w_1(\tau) \|_{\mathbf{L}^\infty(\mathbb{R}^n)} \, d\tau \exp \left( \int_{t_0}^{t} \| a_2(\tau) \|_{\mathbf{L}^\infty(\mathbb{R}^n)} \, d\tau \right)
\]

completing the proof of (P11).

Finally, (P12) follows from Proposition 4.7 from (P9) and from the \( \mathbf{L}^\infty \) bound (4.7).

\( \square \)

4.2 About the Balance Law \( \partial_t u + \nabla \cdot (c(t, x) u) = b(t, x) u + q(t, x) \)

We focus on the following Cauchy problem for a linear balance law

\[
\begin{cases}
\partial_t u + \nabla \cdot (c(t, x) u) = b(t, x) u + q(t, x) \\
 u(t_0, x) = u_0(x).
\end{cases}
\]

(4.10)

Recall the following conditions on the functions defining problem (4.10):

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(b) \( b \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \).

(b+) \( b \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \) and \( b(t) \in BV(\mathbb{R}^n; \mathbb{R}) \) for \( t \in I \).

(c1) The map \( c \) satisfies \( c \in (C^0 \cap L^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n) \), \( c(t) \in C^1(\mathbb{R}^n; \mathbb{R}^n) \) for all \( t \in I \) and \( \nabla c \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n \times n}) \).

(c2) The map \( c \) satisfies \( c \in (C^0 \cap L^\infty)(I \times \mathbb{R}^n; \mathbb{R}^n) \), \( c(t) \in C^2(\mathbb{R}^n; \mathbb{R}^n) \) for all \( t \in I \), \( \nabla c \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}^{n \times n}) \) and \( \nabla \cdot c \in L^1(I \times \mathbb{R}^n; \mathbb{R}^n) \).

(q-) \( q \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \) and \( L^\infty(I; L^1(\mathbb{R}^n; \mathbb{R})) \).

Definition 4.9. Let \((b) (c1)\) and \((q-)\) hold and choose \( u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}) \). A solution to \((4.10)\) is a function \( u \in C^0(I; L^1(\mathbb{R}^n; \mathbb{R})) \) such that

\[
\begin{align*}
\int_0^t u(s, X(s; t, x_0)) \frac{d}{dt} (b(\tau, X(\tau; t, x))) &- \nabla \cdot c(\tau, X(\tau; t, x)) \, d\tau \\
&+ \int_0^t q(s, X(s; t, x)) \frac{d}{dt} (\nabla \cdot (b(\tau, X(\tau; t, x)) - \nabla \cdot c(\tau, X(\tau; t, x)))) \\
&= -\nabla \cdot (b(\tau, X(\tau; t, x)) - \nabla \cdot c(\tau, X(\tau; t, x))) \\
&+ q(\tau, X(\tau; t, x)) \, d\tau,
\end{align*}
\]

where

\[
(4.11)
\]

Lemma 4.10 ([3, Lemma 2.7] and [3, Lemma 5.1]). Let \((b) (c1) (q-)\) hold. Fix \( u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^n; \mathbb{R}) \) and \( u \in C^0(I; L^1(\mathbb{R}^n; \mathbb{R})) \). Then, the following three statements are equivalent:

1. \( u \) is a Kružkov solution to \((4.10)\), i.e. \( u(t_0) = u_0 \) and for all \( k \in \mathbb{R} \) and \( \varphi \in C^1(I \times \mathbb{R}^n; \mathbb{R}^n) \),

\[
\int_I \int_{\mathbb{R}^n} \left[ (u - k)(\partial_t \varphi + c \cdot \nabla \varphi) + (bu + q - k \nabla \cdot c) \varphi \right] \operatorname{sgn}(u - k) \, dx \, dt \geq 0.
\]

(4.13)

2. \( u \) is a weak solution to \((4.10)\), i.e. \( u(t_0) = u_0 \) and for all \( \varphi \in C^1_c(I \times \mathbb{R}^n; \mathbb{R}) \),

\[
\int_I \int_{\mathbb{R}^n} (u \partial_t \varphi + uc \cdot \nabla \varphi + (bu + q) \varphi) \, dx \, dt = 0.
\]

(4.14)

3. \( u \) solves \((4.10)\) in the sense of Definition 4.9.

The proof amounts to mix the techniques used in [3, Lemma 2.7] and [3, Lemma 5.1].

We recall a different approach to the study of linear balance laws of type \((4.10)\), which is adopted in [11, Lemma 3.4]. That Lemma guarantees the existence of a weak solution, in the sense of \((4.14)\) in Lemma 4.10 and provides an explicit formula for the solution in terms of characteristics, corresponding exactly to \((4.11)\). The regularity requirements in [11] on the functions defining problem \((4.10)\) are the following: for \( T \in \mathbb{R}, \, T > 0 \),

\[
u_0 \in L^1(\mathbb{R}^n; \mathbb{R}), \quad b \in L^1((0, T); L^\infty(\mathbb{R}^n; \mathbb{R})), \quad q \in L^1((0, T); L^1(\mathbb{R}^n; \mathbb{R})),
\]

and \( c \in C^0((0, T); C^1(\mathbb{R}^n; \mathbb{R})) \) is globally Lipschitz continuous in space. Notice that, for \( T \in \mathbb{R}, \, T > 0 \), our assumptions \((b) (c1) (q-)\) are stronger than those required in [11, Lemma 3.4], allowing to apply that result in the present setting.

The next proposition is not only an extension of [3, Proposition 2.8] to the present setting, but it also improves it sharply.
Proposition 4.11. Under the assumptions (b) and (c1) the Cauchy Problem (4.10) generates the map

$$\mathcal{H} : J \times U \to U$$

$$(t_0, t), \ u_0 \to u$$

where $u$ is defined by (4.11), with the following properties:

(H1) $\mathcal{H}$ is a process: $\mathcal{H}_{t,t} = \text{Id}$ for all $t \in I$ and $\mathcal{H}_{t_1,t_2} \circ \mathcal{H}_{t_2,t_3} = \mathcal{H}_{t_1,t_3}$ for all $t_1, t_2, t_3 \in I$, with $t_1 \leq t_2 \leq t_3$.

(H2) Positivity: if $q \geq 0$ and $u_0 \in U^+$, then $\mathcal{H}_{t_0,t} u_0 \in U^+$ for all $t \in I$.

(H3) $L^1$ continuous dependence on $u_0$: for all $t \in I$ the map $\mathcal{H}_{t_0,t} : U \to U$ is linear, continuous and

$$\| \mathcal{H}_{t_0,t} u_0 \|_{L^1(\mathbb{R}^n)} \leq \left( \| u_0 \|_{L^1(\mathbb{R}^n)} + \| q \|_{L^1([t_0,t] \times \mathbb{R}^n)} \right) \exp \int_{t_0}^t \| b(\tau) \|_{L^\infty(\mathbb{R}^n)} d\tau .$$

Moreover, if $u_0 \geq 0$ and $q \geq 0$, then

$$\| \mathcal{H}_{t_0,t} u_0 \|_{L^1(\mathbb{R}^n)} \leq \left( \| u_0 \|_{L^1(\mathbb{R}^n)} + \| q \|_{L^1([t_0,t] \times \mathbb{R}^n)} \right) \exp \int_{t_0}^t \left( \sup_{x \in \mathbb{R}^n} b(\tau, x) \right) d\tau .$$

(H4) $L^\infty$–estimate: for all $u_0 \in U$, for all $t \in I$,

$$\| \mathcal{H}_{t_0,t} u_0 \|_{L^\infty(\mathbb{R}^n)} \leq \left( \| u_0 \|_{L^\infty(\mathbb{R}^n)} + \| q \|_{L^1([t_0,t] ; L^\infty(\mathbb{R}^n))} \right) \exp \int_{t_0}^t \left( \| b(\tau) \|_{L^\infty(\mathbb{R}^n)} + \| \nabla \cdot c(\tau) \|_{L^\infty(\mathbb{R}^n)} \right) d\tau .$$

Moreover, if $u_0 \geq 0$ and $q \geq 0$, then

$$\| \mathcal{H}_{t_0,t} u_0 \|_{L^\infty(\mathbb{R}^n)} \leq \left( \| u_0 \|_{L^\infty(\mathbb{R}^n)} + \| q \|_{L^1([t_0,t] ; L^\infty(\mathbb{R}^n))} \right) \exp \int_{t_0}^t \left( \sup_{x \in \mathbb{R}^n} b(\tau, x) \right) + \| \nabla \cdot c(\tau) \|_{L^\infty(\mathbb{R}^n)} d\tau .$$

(H5) Stability with respect to $b, c, q$: if $b, \tilde{b}$ satisfy (b+) with $b - \tilde{b} \in L^1(I \times \mathbb{R}^n; \mathbb{R})$; $c, \tilde{c}$ satisfy (c2) with $\nabla \cdot (c - \tilde{c}) \in L^1(I \times \mathbb{R}^n; \mathbb{R})$ and $q, \tilde{q}$ satisfy (q). Call $\mathcal{H}, \tilde{\mathcal{H}}$ the corresponding processes. Then, for all $t \in I$ and for all $u_0 \in U$,

$$\| \mathcal{H}_{t_0,t} u_0 - \tilde{\mathcal{H}}_{t_0,t} u_0 \|_{L^1(\mathbb{R}^n)} \leq O(t) \| c - \tilde{c} \|_{L^1([t_0,t] ; L^\infty(\mathbb{R}^n; \mathbb{R})]} \left[ \| u_0 \|_{L^\infty(\mathbb{R}^n)} + \text{TV} (u_0) \right. \left. + \int_{t_0}^t \max \left\{ \| q(\tau) \|_{L^\infty(\mathbb{R}^n)}, \| \tilde{q}(\tau) \|_{L^\infty(\mathbb{R}^n)} \right\} \bigg] \right) d\tau \right) \right) + O(2) \| q - \tilde{q} \|_{L^1([t_0,t] \times \mathbb{R}^n)}$$

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\[ + O_2(t) \left( \|u_0\|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \max \left\{ \|q(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{q}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau \right) \times \left( \|b - \tilde{b}\|_{L^1([t_0,t] \times \mathbb{R}^n)} + \|\nabla \cdot (c - \tilde{c})\|_{L^1([t_0,t] \times \mathbb{R}^n)} \right), \]

where
\[
O_1(t) = \exp \int_{t_0}^t \max \left\{ \|b(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{b}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau
\times \exp \int_{t_0}^t \max \left\{ \|\nabla c(\tau)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})}, \|\nabla \tilde{c}(\tau)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \right\} \, d\tau
\times \left[ 1 + \int_{t_0}^t \max \left\{ TV \left( b(s) \right) + \|\nabla \cdot (c(s))\|_{L^1(\mathbb{R}^n)}, \right\} \, ds \right],
\]

\[
O_2(t) = \exp \int_{t_0}^t \max \left\{ \|b(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{b}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau .
\]

**H6** Total variation bound: let \((b+)\) \((c2)\) and \((q)\) hold. If \(u_o \in \mathcal{U}\), then, for all \(t \in I\),
\[
TV \left( \mathcal{H}_{t_o,t} u_o \right) \leq O(t) \left( \|u_0\|_{L^\infty(\mathbb{R}^n)} + TV (u_o) + \int_{t_0}^t \left( \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} + TV (q(\tau)) \right) \, d\tau \right),
\]

where
\[
O(t) = \exp \left( \int_{t_0}^t \left( \|b(\tau)\|_{L^\infty(\mathbb{R}^n)} + \|\nabla c(\tau)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})} \right) \, d\tau \right)
\times \left( 1 + \int_{t_0}^t \left( TV \left( b(\tau) \right) + \|\nabla \cdot c(\tau)\|_{L^1(\mathbb{R}^n; \mathbb{R}^{n \times n})} \right) \, d\tau \right).
\]

**H7** Regularity in time: let \((b+)\) \((c2)\) and \((q)\) hold. For all \(u_o \in \mathcal{U}\), the map \(t \to \mathcal{H}_{t_o,t} u_o\) is in \(C^{0,1}(I; L^1(\mathbb{R}^n; \mathbb{R}))\), moreover for all \(t_1, t_2 \in I\), with \(O(t)\) as above,
\[
\|\mathcal{H}_{t_1,t} u_o - \mathcal{H}_{t_0,t} u_o\|_{L^1(\mathbb{R}^n)}
\leq O(t_1 \vee t_2) \left( \|u_0\|_{L^\infty(\mathbb{R}^n)} + TV (u_o) + \int_{t_0}^{t_1 \vee t_2} \left( \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} + TV (q(\tau)) \right) \, d\tau \right) |t_2 - t_1|.
\]

**H8** Finite propagation speed: if, for all \(t \in I\), the map \(x \to q(t,x)\) is compactly supported and \(u_o \in \mathcal{U}\) has compact support, then, for \(t \in I\) also, \(\text{supp} \mathcal{H}_{t_o,t} u_o\) is compact.

**Proof.** Statement \([H1]\) directly follows from Definition 4.9, Lemma 4.10 and \([H1] \text{ Lemma 3.4}\), thanks to \((b)\) \((c1)\) and \((q)\). Using \((4.11)\), points \([H2]\) \([H4]\) and \([H8]\) are ensured.

To get the \(L^1\) bound \((H3)\), exploit the change of variable \(y = X(s,t,x)\), see also \([3] \text{ § 5.1}\). Denoting the Jacobian of this change of variable by \(J(t,y) = \det (\nabla x X(t,s,y))\), \(J\) solves
\[
\frac{dJ(t,y)}{dt} = \nabla \cdot c(t, X(t;s,y)) J(t,y) \quad \text{with} \quad J(s,y) = 1.
\]
Thus, \( J(t, y) = \exp \left( \int_s^t \nabla \cdot c(\tau, X(\tau; s, y)) \, d\tau \right) \), so that \( J(t, y) > 0 \) for \( t \in I \) and \([H3]\) follows.

To prove the remaining points, we exploit the techniques used in the proof of \([5, \text{Lemma 4.4} \) and \( \text{Lemma 4.6} \) for an initial boundary value problem for a conservation law, thus without source term. To this aim, we approximate \( b \), respectively \( q \), by a sequence \( b_h \), respectively \( q_h \), as in \( \text{Lemma 4.3} \). Regularize also the initial datum \( u_o \) and call \( u^h_0 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) the sequence defined by \( \text{Lemma 4.1} \). Using (4.11), define the corresponding sequence \( u_h \) of solutions to

\[
\begin{aligned}
\partial_t u_h + \nabla \cdot (c(t, x) u_h) &= b_h(t, x) u_h + q_h(t, x) \\
u_h(t_o, x) &= u^0_h(x),
\end{aligned}
\]

so that

\[
u_h(t, x) = u^0_h(X(t_o; t, x)) \exp \left( \int_{t_o}^t \left( b_h(\tau, X(\tau; t, x)) - \nabla \cdot c(\tau, X(\tau; t, x)) \right) \, d\tau \right) + \int_{t_o}^t q_h(s, X(s; t, x)) \exp \left( \int_s^t \left( b_h(\tau, X(\tau; t, x)) - \nabla \cdot c(\tau, X(\tau; t, x)) \right) \, d\tau \right) \, ds,
\]

where \( X \) is defined in (4.12). Observe that for a.e. \( t \in I \), the map \( x \to u_h(t, x) \) is of class \( C^1 \), due to \( \text{Lemma 4.3} \) applied to both \( b \) and \( q \), and to \([c2]\).

Pass now to \([H6] \). Differentiate the solution to (4.12) with respect to the initial point, that is, for \( \tau \in [t_o, t] \),

\[
\nabla_x X(\tau; t, x) = \text{Id} + \int_{\tau}^t \nabla_x c(s, X(s; t, x)) \nabla_x X(s; t, x) \, ds,
\]

so that, by Gronwall Lemma,

\[
\|\nabla_x X(\tau; t, x)\| \leq 1 + \int_{\tau}^t \|\nabla_x c(s, X(s; t, x))\| \|\nabla_x X(s; t, x)\| \, ds.
\]

By (4.15) and the properties of \( u^h_0 \), the gradient \( \nabla u_h(t) \) is well defined and continuous:

\[
\nabla u_h(t, x) = \exp \left( \int_{t_o}^t (b_h - \nabla \cdot c)(\tau, X(\tau; t, x)) \, d\tau \right) \left( \nabla u^0_h(X(t_o; t, x)) \nabla X(t_o; t, x) \right)
\]

\[
+ u^0_h(X(t_o; t, x)) \int_{t_o}^t \nabla (b_h - \nabla \cdot c)(\tau, X(\tau; t, x)) \nabla X(\tau; t, x) \, d\tau
\]

\[
+ \int_{t_o}^t \exp \left( \int_s^t (b_h - \nabla \cdot c)(\tau, X(\tau; t, x)) \, d\tau \right) \left( \nabla q_h(s, X(s; t, x)) \nabla X(s; t, x) \right)
\]

\[
+ q_h(s, X(s; t, x)) \int_s^t \nabla (b_h - \nabla \cdot c)(\tau, X(\tau; t, x)) \nabla X(\tau; t, x) \, d\tau \right) \, ds.
\]
Therefore, for every \( t \in I \), we use the change of variable described at the beginning of the proof together with (4.16) to get
\[
\| \nabla u_h(t) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \\
\leq \exp \left( \int_{t_0}^t \| b_h(\tau) \|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \exp \left( \int_{t_0}^t \| \nabla c(\tau) \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n)} \, d\tau \right) \\
\times \left[ \| \nabla u_0^h \|_{L^1(\mathbb{R}^n)} + \int_{t_0}^t \| q_h(\tau) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \, d\tau \\
+ \left( \| u_0 \|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \| q(\tau) \|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \int_{t_0}^t \| \nabla(\nabla \cdot c)(\tau) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \, d\tau \right] ,
\]
(4.17)

Let \( u \) be defined as in (4.11); Lemma 4.3 and Lemma 4.4 imply that \( u_h \to u \) in \( L^1(\mathbb{R}^n; \mathbb{R}) \). By the lower semicontinuity of the total variation, by (4.17) and (4.1), for \( t \in I \) we obtain
\[
TV \left( u(t) \right) \leq \lim_{h \to 0} TV \left( u_h(t) \right) = \lim_{h \to 0} \| \nabla u_h(t) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \\
\leq \exp \left( \int_{t_0}^t \| b(\tau) \|_{L^\infty(\mathbb{R}^n)} + \| \nabla c(\tau) \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n)} \, d\tau \right) \left[ TV \left( u_0 \right) + \int_{t_0}^t TV \left( q(\tau) \right) \, d\tau \\
+ \left( \| u_0 \|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \| q(\tau) \|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \int_{t_0}^t \left( TV \left( b(\tau) \right) + \| \nabla \cdot \nabla \cdot c(\tau) \|_{L^1(\mathbb{R}^n; \mathbb{R}^n)} \right) \, d\tau \right] ,
\]
(4.18)

concluding the proof of (H6).

The proof of (H7) is entirely analogous, leading to
\[
\| u(t_2) - u(t_1) \|_{L^1(\mathbb{R}^n)} \leq TV \left( u \left( \max\{t_1, t_2\} \right) \right) |t_2 - t_1|.
\]

To prove (H5), we follow the idea of the proof of [5 Lemma 4.6], adapting it to the present setting. With obvious notation, we denote by \( b_h \) and \( \tilde{b}_h \) sequences of functions converging to \( b \) and \( \tilde{b} \), with the properties in Lemma 4.3. Similarly, we denote by \( q_h \) and \( \tilde{q}_h \) sequences of functions converging to \( q \) and \( \tilde{q} \), with the properties in Lemma 4.3. Consider also the regularization of the initial datum \( u_0^h \in C^\infty(\mathbb{R}^n; \mathbb{R}) \) provided by Lemma 4.1. For \( \vartheta \in [0, 1] \) set
\[
b^\vartheta_h(t, x) = \vartheta \tilde{b}_h(t, x) + (1 - \vartheta) \tilde{b}_h(t, x), \\
c^\vartheta(t, x) = \vartheta c(t, x) + (1 - \vartheta) \tilde{c}(t, x), \\
q^\vartheta_h(t, x) = \vartheta q_h(t, x) + (1 - \vartheta) \tilde{q}_h(t, x).
\]

Let \( u^\vartheta_h \) be the solution to
\[
\begin{align*}
\partial_t u^\vartheta_h + \nabla \cdot \left( c^\vartheta(t, x) u^\vartheta_h \right) &= b^\vartheta_h(t, x), \\
u^\vartheta_h(t_0, x) &= u_0^\vartheta(x),
\end{align*}
\]
where
\[
\begin{align*}
X^\vartheta &= c^\vartheta(t, X^\vartheta), \\
X^\vartheta(t_0) &= x_0,
\end{align*}
\]

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that is

\[ u^b_h(t, x) = u_0^b(X^\vartheta(t_0; t, x)) \exp \left( \int_{t_0}^t \left( \frac{\partial}{\partial \vartheta} \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right) \right) (\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \]

\[ + \int_{t_0}^t q_h^\vartheta(s, X^\vartheta(s; t, x)) \exp \left( \int_{s_0}^s \left( \frac{\partial}{\partial \vartheta} \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right) \right) (\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \, ds. \] (4.19)

Compute the derivative of \( X^\vartheta \) with respect to \( \vartheta \), recalling that \( X^\vartheta(t; t, x) = x \) for all \( \vartheta \):

\[
\begin{align*}
\partial_\vartheta X^\vartheta(\tau; t, x) &= \partial_\vartheta \left( c(\tau, X^\vartheta(\tau; t, x)) - \tilde{c}(\tau, X^\vartheta(\tau; t, x)) + \nabla c^\vartheta(\tau, X^\vartheta(\tau; t, x)) \partial_\vartheta X^\vartheta(\tau; t, x) \right) \\
\partial_\vartheta X^\vartheta(t; t, x) &= 0.
\end{align*}
\]

The solution to the above problem satisfies

\[
\partial_\vartheta X^\vartheta(\tau; t, x) = \int_{\tau}^t \exp \left( \int_{s}^{\tau} \nabla c^\vartheta(s, X^\vartheta(s; t, x)) \, ds \right) (c - \tilde{c})(s, X^\vartheta(s; t, x)) \, ds \\
= \int_{\tau}^t \exp \left( \int_{s}^{\tau} -\nabla c^\vartheta(s, X^\vartheta(s; t, x)) \, ds \right) (\tilde{c} - c)(s, X^\vartheta(s; t, x)) \, ds. \] (4.20)

Derive (4.19) with respect to \( \vartheta \):

\[
\begin{align*}
\partial_\vartheta u^b_h(t, x) &= \exp \left( \int_{t_0}^t \left( \frac{\partial}{\partial \vartheta} \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right) \right) (\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \left\{ \nabla u_0^b(X^\vartheta(t_0; t, x)) \partial_\vartheta X^\vartheta(t_0; t, x) \\
& \quad + u_0^b(X^\vartheta(t_0; t, x)) \int_{t_0}^t \left( b_h - \tilde{b}_h - \nabla \cdot (c - \tilde{c}) \right) (\tau, X^\vartheta(\tau; t, x)) \, d\tau \\
& \quad + u_0^b(X^\vartheta(t_0; t, x)) \int_{t_0}^t \nabla \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right)(\tau, X^\vartheta(\tau; t, x)) \partial_\vartheta X^\vartheta(\tau; t, x) \, d\tau \right\} \\
& \quad + \int_{t_0}^t \exp \left( \int_{s}^{\tau} \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right)(\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \\
& \times \left\{ (q_h - \tilde{q}_h)(t, X^\vartheta(s; t, x)) + \nabla q_h^\vartheta(s, X^\vartheta(s; t, x)) \partial_\vartheta X^\vartheta(s; t, x) \\
& \quad + q_h^\vartheta(s, X^\vartheta(s; t, x)) \int_{s_0}^s \left( b_h - \tilde{b}_h - \nabla \cdot (c - \tilde{c}) \right)(\tau, X^\vartheta(\tau; t, x)) \, d\tau \\
& \quad + q_h^\vartheta(s, X^\vartheta(s; t, x)) \int_{s_0}^s \nabla \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right)(\tau, X^\vartheta(\tau; t, x)) \partial_\vartheta X^\vartheta(\tau; t, x) \, d\tau \right\} \, ds \\
& \leq \exp \left( \int_{t_0}^t \left( b_h^\vartheta - \nabla \cdot c^\vartheta \right)(\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \left\{ \int_{t_0}^t (q_h - \tilde{q}_h)(t, X^\vartheta(s; t, x)) \, ds \\
& \quad + \left( \nabla u_0^b(X^\vartheta(t_0; t, x)) + \int_{t_0}^t \nabla q_h^\vartheta(\tau, X^\vartheta(\tau; t, x)) \, d\tau \right) \right\}
\end{align*}
\]
\[ \times \int_0^t \exp \left( \int_0^s -\nabla c^\vartheta (\sigma, X^\vartheta(t,x)) \, d\sigma \right) \left( \tilde{c} - c \right) (s, X^\vartheta(s,t,x)) \, ds \]

+ \left( u^\vartheta_h(X^\vartheta(t_0;t,x)) + \int_{t_0}^t q^\vartheta_h(\tau, X^\vartheta(\tau;t,x)) \, d\tau \right) \]

\times \left[ \int_{t_0}^t \left( b_h - \tilde{b}_h - \nabla \cdot (c - \tilde{c}) \right) (\tau, X^\vartheta(\tau;t,x)) \, d\tau \right.

\left. + \int_{t_0}^t \nabla (b^\vartheta_h - \nabla \cdot c^\vartheta)(\tau, X^\vartheta(\tau;t,x)) \right]

\times \left[ \int_{t_0}^t \exp \left( \int_\tau^s -\nabla c^\vartheta (\sigma, X^\vartheta(\sigma,t,x)) \, d\sigma \right) \left( \tilde{c} - c \right) (s, X^\vartheta(s,t,x)) \, ds \right] \, d\tau \right) \]

where we made use of (4.20). Call \( u_h \) and \( \tilde{u}_h \) the functions defined by (4.19) for \( \vartheta = 0 \) and \( \vartheta = 1 \), that is \( u_h = u^0_{\vartheta=0} \) and \( \tilde{u}_h = u^1_{\vartheta=1} \). Compute

\[ \| u_h(t) - \tilde{u}_h(t) \|_{L^1(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \left| \frac{1}{\int_0^t \partial \vartheta u^\vartheta_h(t, x) \, dx} \right| \, dx \leq \int_0^1 \int_{\mathbb{R}^n} |\partial \vartheta u^\vartheta_h(t, x)| \, dx \, d\vartheta. \quad (4.21) \]

Exploiting the change of variable introduced at the beginning of the proof, compute

\[ \int_{\mathbb{R}^n} \left| \partial \vartheta u^\vartheta_h(t, x) \right| \, dx \]

\[ \leq \exp \left( \int_0^t \left\| \tilde{b}^\vartheta_h(\tau) \right\|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \left\{ \int_0^t \left( \left| q^\vartheta_h(\tau) \right|_{L^1(\mathbb{R}^n)} \right) \, d\tau \right. \]

\[ + \left( \int_{\mathbb{R}^n} \left| \nabla u^\vartheta_h(y) \right| \, dy + \int_{t_0}^t \int_{\mathbb{R}^n} \left| \nabla q^\vartheta_h(\tau, y) \right| \, dy \, d\tau \right) \]

\[ \times \exp \left( \int_0^t \left| \nabla c^\vartheta(\sigma) \right|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \, d\sigma \right) \int_{t_0}^t \left| (c - \tilde{c})(\tau) \right|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \, d\tau \]

\[ + \left( \left| u^\vartheta_h \right|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \left| q^\vartheta_h(\tau) \right|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \int_{t_0}^t \left| (b_h - \tilde{b}_h - \nabla \cdot (c - \tilde{c}))(\tau) \right|_{L^1(\mathbb{R}^n)} \, d\tau \]

\[ + \left( \left| u^\vartheta_h \right|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \left| q^\vartheta_h(\tau) \right|_{L^\infty(\mathbb{R}^n)} \, d\tau \right) \int_{t_0}^t \left| \nabla (b^\vartheta_h - \nabla \cdot c^\vartheta)(\tau) \right|_{L^1(\mathbb{R}^n;\mathbb{R}^n)} \, d\tau \]

\[ \times \exp \left( \int_0^t \left| \nabla c^\vartheta(\sigma) \right|_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \, d\sigma \right) \int_{t_0}^t \left( c - \tilde{c}(s) \right)_{L^\infty(\mathbb{R}^n;\mathbb{R}^n)} \, ds \right) \]

Inserting the result above in (4.21), by the definitions of \( b^\vartheta_h, q^\vartheta_h \) and their properties as stated in Lemma 1.3, we have

\[ \| u_h(t) - \tilde{u}_h(t) \|_{L^1(\mathbb{R}^n)} \]
\[
\begin{align*}
\leq & \exp \left( \int_{t_0}^t \max \left\{ \|b(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\bar{b}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau \right) \left\{ \int_{t_0}^t \| (q_h - \bar{q}_h)(\tau) \|_{L^1(\mathbb{R}^n)} \, d\tau \right. \\
& \quad + \left( \|u_0^b\|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \max \left\{ \|q(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\bar{q}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau \right) \\
& \quad \times \int_{t_0}^t \| (b_h - \bar{b}_h - \nabla \cdot (c - \bar{c}))(\tau) \|_{L^1(\mathbb{R}^n)} \, d\tau \\
& \quad + \exp \left( \int_{t_0}^t \max \left\{ \|\nabla c(s)\|_{L^\infty(\mathbb{R}^{n\times n})}, \|\nabla \bar{c}(s)\|_{L^\infty(\mathbb{R}^{n\times n})} \right\} \, ds \right) \\
& \quad \times \int_{t_0}^t \| (c - \bar{c})(s) \|_{L^\infty(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \, ds \\
& \quad \times \left[ \int_{\mathbb{R}^n} \| \nabla u_0^b(y) \| \, dy \right. \\
& \quad \left. + \int_{t_0}^t \max \left\{ \| \nabla q_h(s)\|_{L^1(\mathbb{R}^n)}, \| \nabla \bar{q}_h(s)\|_{L^1(\mathbb{R}^n)} \right\} \, ds \right] \\
& \quad \times \left. \int_{t_0}^t \max \left\{ \| \nabla (b_h - \nabla \cdot c)(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)}, \| \nabla (\bar{b}_h - \nabla \cdot \bar{c})(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \right\} \, ds \right) \right) .
\end{align*}
\]

Let now \( h \) tend to \( +\infty \). We have:

\[
\begin{align*}
& \|u_h(t) - \bar{u}_h(t)\|_{L^1(\mathbb{R}^n)} \to \|u(t) - \bar{u}(t)\|_{L^1(\mathbb{R}^n)} \\
& \|u_0^b\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \quad \text{by (4.1)} \\
& \| (q_h - \bar{q}_h)(\tau) \|_{L^1(\mathbb{R}^n)} \to \| (q - \bar{q})(\tau) \|_{L^1(\mathbb{R}^n)} \quad \text{by Lemma 4.3} \\
& \| (b_h - \bar{b}_h - \nabla \cdot (c - \bar{c}))(\tau) \|_{L^1(\mathbb{R}^n)} \to \| (b - \bar{b} - \nabla \cdot (c - \bar{c}))(\tau) \|_{L^1(\mathbb{R}^n)} \quad \text{by Lemma 4.3} \\
& \| \nabla u_0^b\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \to \text{TV} \ (u_0) \quad \text{by (4.1)} \\
& \| \nabla b_h(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \leq \text{TV} \ (b(s)) \quad \text{by Lemma 4.3} \\
& \| \nabla \bar{b}_h(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \leq \text{TV} \ (\bar{b}(s)) \quad \text{by Lemma 4.3} \\
& \| \nabla q_h(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \leq \text{TV} \ (q(s)) \quad \text{by Lemma 4.3} \\
& \| \nabla \bar{q}_h(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \leq \text{TV} \ (\bar{q}(s)) \quad \text{by Lemma 4.3}
\end{align*}
\]

Therefore,

\[
\|u(t) - \bar{u}(t)\|_{L^1(\mathbb{R}^n)} \quad (4.22)
\]

\[
\begin{align*}
\leq & \exp \left( \int_{t_0}^t \max \left\{ \|b(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\bar{b}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau \right) \left\{ \int_{t_0}^t \| (q - \bar{q})(\tau) \|_{L^1(\mathbb{R}^n)} \, d\tau \right. \\
& \quad + \left( \|u_0\|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \max \left\{ \|q(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\bar{q}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} \, d\tau \right) \\
& \quad \times \int_{t_0}^t \| (c - \bar{c})(s) \|_{L^\infty(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \, ds \\
& \quad \times \left[ \int_{\mathbb{R}^n} \| \nabla u_0^b(y) \| \, dy \right. \\
& \quad \left. + \int_{t_0}^t \max \left\{ \| \nabla q_h(s)\|_{L^1(\mathbb{R}^n)}, \| \nabla \bar{q}_h(s)\|_{L^1(\mathbb{R}^n)} \right\} \, ds \right] \\
& \quad \times \left. \int_{t_0}^t \max \left\{ \| \nabla (b_h - \nabla \cdot c)(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)}, \| \nabla (\bar{b}_h - \nabla \cdot \bar{c})(s)\|_{L^1(\mathbb{R}^{n\times n}; \mathbb{R}^n)} \right\} \, ds \right) \right) .
\end{align*}
\]

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Proof of Theorem 2.2.

This completes the proof. □

Step 0:

For all complete metric space and that its limit solves (1.1). We divide the proof into several steps. The existence part of the proof amount to verify that \( w \) and

\[
\begin{align*}
\text{Step 0:} & \quad \text{For all } t \in I, u_i(t, x) = w_\circ(x) \text{ for } t, x \in I \times \mathbb{R}^n. \\
& \text{Then, construct recursively for } i = 1, 2, \ldots \text{ the following sequences of functions:}
\end{align*}
\]

\[
\begin{align*}
a_i(t, x) &= g(t, x, u_{i-1}(t, x), w_{i-1}(t, x)); \\
b_i(t, x) &= f(t, x, w_{i-1}(t, x)); \\
c_i(t, x) &= \left( v \left( t, w_{i-1}(t) \right) \right)(x);
\end{align*}
\]

\[
\begin{align*}
u_i(t, x) &\text{ solves } \partial_t u_i + \nabla \cdot (c_i(t, x)u_i) = b_i(t, x)u_i + q(t, x) \quad \text{with } u_i(t_0, x) = u_\circ(x); \\
\end{align*}
\]

\[
\begin{align*}
v_i(t, x) &\text{ solves } \partial_t v_i - \mu \Delta v_i = a_i(t, x)w_i \quad \text{with } v_i(t_0, x) = v_\circ(x).
\end{align*}
\]

The existence part of the proof amount to verify that \( (u_i, w_i) \) is a Cauchy sequence in a suitable complete metric space and that its limit solves (1.1). We divide the proof into several steps.

Step 0: For all \( i \in \mathbb{N}, (u_i, w_i) \) is well defined and

\[
\begin{align*}
\text{for all } t \in I, u_i(t) &\in \mathcal{U}^+ \quad \text{and} \quad u_i \in C^{0,1}(I; L^1(\mathbb{R}^n; \mathbb{R}^+)), \\
\text{for all } t \in I, w_i(t) &\in \mathcal{U}^+ \quad \text{and} \quad w_i \in C^0(I; L^1(\mathbb{R}^n; \mathbb{R}_+)).
\end{align*}
\]

Proof of Step 0: For \( i = 0 \), the thesis holds true due to the choice of the initial data and the definition of \( u_0 \) and \( w_0 \). We proceed by induction.

Assume now that the claim holds for \( i - 1 \), with \( i \geq 1 \). Then, \( a_i \in L^\infty(I \times \mathbb{R}^n; \mathbb{R}) \) for all \( t \in I \), by (g) and by the inductive hypothesis. Proposition 4.6, Proposition 4.7 and Corollary 4.8 hence ensure that \( w_i \) is well defined, with \( w_i(t) \in \mathcal{U}^+ \) for all \( t \in I \). Similarly, \( b_i \) satisfies (b+) by (f) and \( c_i \) satisfies (c2) by (v). An application of Proposition 4.11 ensures...
the existence of \(u_i\), with \(u_i(t) \in \mathcal{U}^+\) for all \(t \in I\). The time regularity of \(w_i\) follows from \(\text{(P2)}\) in Proposition 4.6 and, for \(u_i\), from \(\text{(H7)}\) in Proposition 4.11.

**Step 1:** For all \(i \in \mathbb{N}\), for all \(t \geq t_o\)

\[
\|w_i(t)\|_{L^1(\mathbb{R}^n)} \leq \|w_o\|_{L^1(\mathbb{R}^n)} e^{K_g(t)(t-t_o)}, \quad \|w_i(t)\|_{L^\infty(\mathbb{R}^n)} \leq \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)},
\]

\[ (4.25) \]

\[
\|u_i(t)\|_{L^1(\mathbb{R}^n)} \leq \left(\|u_o\|_{L^1(\mathbb{R}^n)} + \|q\|_{L^1([t_o, t] \times \mathbb{R}^n)} \right) \times \exp \left[ K_f(t) (t-t_o) \left( 1 + \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)} \right) \right],
\]

\[ (4.26) \]

\[
\|u_i(t)\|_{L^\infty(\mathbb{R}^n)} \leq \left(\|u_o\|_{L^\infty(\mathbb{R}^n)} + \|q\|_{L^1([t_o, t] \times \mathbb{R}^n)} \right) \times \exp \left[ (K_f(t) + K_v(t)) (t-t_o) \left( 1 + \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)} \right) \right].
\]

\[ (4.27) \]

(The \(L^1\) and \(L^\infty\) estimates on \(w\) are independent of \(u\). This fact plays a key role throughout, in particular in Step 6 below.)

**Proof of Step 1:** By \((g)\) and \((4.23)\), with obvious notation, for all \(\tau \in [t_o, t], \)

\[
A_i(\tau) := \sup_{\xi \in \mathbb{R}^n} a_i(\tau, \xi) = \sup_{\xi \in \mathbb{R}^n} g(\tau, \xi, u_{i-1}(\tau, \xi), w_{i-1}(\tau, \xi)) \leq K_g(\tau) \leq K_g(t).
\]

Hence, \((4.25)\) follows by \((\text{P10})\) in Corollary 4.8.

Proceeding now similarly, using \((4.23)\), \((f)\) and \((4.25)\), compute for \(\tau \in [t_o, t], \)

\[
\sup_{x \in \mathbb{R}^n} b_i(\tau, x) = \sup_{x \in \mathbb{R}^n} f(\tau, x, w_{i-1}(\tau, x)) \leq \sup_{x \in \mathbb{R}^n} K_f(\tau) \left( 1 + w_{i-1}(\tau, x) \right)
\]

\[
\leq K_f(\tau) \left( 1 + \|w_i(\tau)\|_{L^\infty(\mathbb{R}^n)} \right) \leq K_f(t) \left( 1 + \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)} \right).
\]

Estimate \((4.26)\) now follows from \((\text{H3})\) in Proposition 4.11 and \((4.23)\). Moreover, by \((v)\)

\[
\|\nabla \cdot c(\tau)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq K_v(\tau) \|w_{i-1}(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq K_v(t) \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)}.
\]

Using now \((\text{H4})\) in Proposition 4.11 and \((4.23)\), the bound \((4.27)\) follows.

**Step 2:** There exists \(G \in C^0(I; \mathbb{R}_+)\) such that for all \(t \in I\) and \(i \in \mathbb{N}\), \(TV(w_i(t)) \leq G(t).\)

**Proof of Step 2:** By the definition of \(a_i\) given in \((4.23)\), by \((g)\) and by \((\text{P12})\) in Corollary 4.8 we obtain \(TV(w_i(t)) \leq G(t)\) where

\[
G(t) = TV(w_o) + \frac{2J_n \sqrt{t-t_o}}{\sqrt{\mu}} K_g(t) \|w_o\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_o)}.
\]

**Step 3:** There exists \(F \in C^0(I; \mathbb{R}_+)\) such that, for all \(t \in I\) and all \(i \in \mathbb{N}\), \(TV(u_i(t)) \leq F(t).\)
Proof of Step 3: Exploiting the definitions of $b_i$ and $c_i$ given in (4.23), by (6) for $\tau \in [t_0, t]$,
\[
\|\nabla c_i(\tau)\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \leq K_v(\tau) \|w_{i-1}(\tau)\|_{L^\infty(\mathbb{R}^n)} \leq K_v(\tau) \|w_{i}^{o}\|_{L^\infty(\mathbb{R}^n)} e^{K_g(\tau) (\tau-t_0)} ,
\]
\[
\|\nabla \nabla \cdot c_i(\tau)\|_{L^1(\mathbb{R}^n)} \leq C_v \left( \tau, \|w_{i-1}(\tau)\|_{L^1(\mathbb{R}^n)} \right) \|w_{i-1}(\tau)\|_{L^1(\mathbb{R}^n)} \\
\leq C_v \left( \tau, \|w_{o}\|_{L^1(\mathbb{R}^n)} e^{K_g(\tau) (\tau-t_0)} \right) \|w_{i}^o\|_{L^1(\mathbb{R}^n)} e^{K_g(\tau) (\tau-t_0)} ,
\]
and by (f), (4.25) and Step 2,
\[
TV (b_i(\tau)) = TV \left( f(\tau, \cdot, w_{i-1}(\cdot)) \right) \\
\leq K_f(\tau) \left( 1 + \|w_{i-1}(\tau)\|_{L^\infty(\mathbb{R}^n)} + TV (w_{i-1}(\tau)) \right) \\
\leq K_f(\tau) \left( 1 + TV (w_{o}) + \|w_{o}\|_{L^\infty(\mathbb{R}^n)} \left( 1 + \frac{2J_n \sqrt{\tau - t_0}}{\sqrt{\mu}} K_g(\tau) \right) e^{K_g(\tau) (\tau-t_0)} \right) .
\]
Insert the latter estimates above in (H6) of Proposition 4.11 to get $TV (u_i(t)) \leq F(t)$, where
\[
F(t) = \left( \|u_{o}\|_{L^\infty(\mathbb{R}^n)} + TV (u_{o}) + \int_{t_0}^{t} \left( \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} + TV (q(\tau)) \right) d\tau \right) \\
\times \exp \left( \int_{t_0}^{t} \left( K_f(\tau) + \|w_{o}\|_{L^\infty(\mathbb{R}^n)} (K_f(\tau) + K_v(\tau)) e^{K_g(\tau)(\tau-t_0)} \right) d\tau \right) \\
\times \left( 1 + \int_{t_0}^{t} C_v \left( \tau, \|w_{o}\|_{L^1(\mathbb{R}^n)} e^{K_g(\tau)(\tau-t_0)} \right) \|w_{o}\|_{L^1(\mathbb{R}^n)} e^{K_g(\tau)(\tau-t_0)} d\tau \right) \\
+ K_f(t)(t-t_0) \left( 1 + TV (w_{o}) + \frac{4J_n}{3 \sqrt{\mu}} \sqrt{t-t_0} K_g(t) \|w_{o}\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_0)} \right) \\
\]
concluding the proof of Step 3.

Observe for later use that, due to (f), (g) and (v) on a bounded time interval $[t_0, T]
\]
an\{a_i+1 - a_i, b_{i+1} - b_i, \nabla \cdot (c_{i+1} - c_i) \} \in L^1([t_0, T] \times \mathbb{R}^n; \mathbb{R}) . \quad (4.28)

Step 4: Referring to (2.2), (2.3), Step 2 and Step 3, consider the complete metric space
\[
\mathcal{Y}_T = \left\{(u, w) \in C^0([t_0, T]; \mathcal{X}^+) : TV (u(t)) \leq F(t) \text{ and } TV (w(t)) \leq G(t) \text{ for all } t \in [t_0, T] \right\},
\]
\[
d ((u_1, w_1), (u_2, w_2)) = \sup_{t \in [t_0, T]} \| (u_1(t) - u_2(t), w_1(t) - w_2(t)) \|_{\mathcal{X}} . \quad (4.29)
\]
Moreover, for $r > 0$ introduce the following subset of $\mathcal{X}^+$:
\[
\mathcal{X}^+_r = \left\{(u, w) \in \mathcal{X}^+ : \|u\|_{L^\infty(\mathbb{R}^n)} \leq r, TV (u) \leq r, \|w\|_{L^\infty(\mathbb{R}^n)} \leq r, TV (w) \leq r \right\} . \quad (4.30)
\]
Then, given $(u_o, w_o) \in \mathcal{X}^+_r$, there exists a continuous function $K_r : [t_0, T] \to \mathbb{R}_+$, for a suitable $T \in I$ with $T > t_0$, such that for all $i \in \mathbb{N}
\]
d \left( (u_{i+1}, w_{i+1}), (u_{i}, w_{i}) \right) \leq K_r(T)(T-t_0) d \left( (u_{i}, w_{i}), (u_{i-1}, w_{i-1}) \right) . \quad (4.31)
**Proof of Step 4:** In the following, we make use of the bounds (4.25)–(4.27). Start from (P11) in Corollary 4.8 for all $t \in [t_0, T]$, using (4.23) and (g) we obtain

$$
\left\| w_{i+1}(t) - u_i(t) \right\|_{L^1(\mathbb{R}^n)} \\
\leq K_g(t)(t - t_0) \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} \sup_{\tau \in [t_0, t]} \left\| (u_i(\tau) - u_{i-1}(\tau), w_i(\tau) - w_{i-1}(\tau)) \right\|_{X} \\
\leq K^u_r(T) (T - t_0) d((u_i, w_i), (u_{i-1}, w_{i-1}))
$$

(4.32)

with

$$
K^u_r(T) = r K_g(T) e^{2(T-t_0)K_g(T)}.
$$

(4.33)

Now consider (H5) in Proposition 4.11 by (v) and (f) setting

$$
\tilde{O}_1(t) = \exp \left( K_f(t)(t - t_0) + \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} (t - t_0) \left( K_f(t) + K_v(t) \right) e^{K_g(t)(t-t_0)} \right) \\
\times \left[ 1 + (t - t_0) C_v(t) \left\| w_0 \right\|_{L^1(\mathbb{R}^n)} e^{K_g(t)(t-t_0)} \left\| w_0 \right\|_{L^1(\mathbb{R}^n)} e^{K_g(t)(t-t_0)} \right. \\
+ K_f(t)(t - t_0) \left. \left( 1 + TV(w_0) + \frac{4J_n}{3\sqrt{\mu}} \sqrt{t-t_0} \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} K_g(t) e^{K_g(t)(t-t_0)} \right) \right],
$$

(4.34)

and

$$
\tilde{O}_2(t) = \exp \left( K_f(t)(t - t_0) \left( 1 + \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} e^{K_g(t)(t-t_0)} \right) \right),
$$

(4.35)

we get

$$
\left\| w_{i+1}(t) - u_i(t) \right\|_{L^1(\mathbb{R}^n)} \\
\leq \left[ \tilde{O}_1(t)(t - t_0) \left( \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} + TV(w_0) + \int_{t_0}^t \left( \left\| q(\tau) \right\|_{L^\infty(\mathbb{R}^n)} + TV(q(\tau)) \right) d\tau \right) \right] K_v(t) \\
+ \tilde{O}_2(t)(t - t_0) \left( \left\| w_0 \right\|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^t \left\| q(\tau) \right\|_{L^\infty(\mathbb{R}^n)} d\tau \right) \left( K_f(t) + C_v \left( t, \left\| w_0 \right\|_{L^\infty} e^{K_g(t)(t-t_0)} \right) \right) \right] \\
\times \sup_{\tau \in [t_0, t]} \left\| w_1(\tau) - w_{i-1}(\tau) \right\|_{L^1(\mathbb{R}^n)} \\
\leq K^v_r(T) (T - t_0) \sup_{\tau \in [t_0, T]} \left\| w_1(\tau) - w_{i-1}(\tau) \right\|_{L^1(\mathbb{R}^n)},
$$

(4.36)

with

$$
K^v_r(T) = \left( r + \int_{t_0}^T \left\| q(\tau) \right\|_{L^\infty(\mathbb{R}^n)} d\tau \right) \exp \left( K_f(T)(T - t_0) \left( 1 + re^{K_g(T)(T-t_0)} \right) \right) \\
\times \left[ K_f(T) + C_v \left( T, re^{K_g(T)(T-t_0)} \right) \right] \\
+ \left( 2r + \int_{t_0}^T \left( \left\| q(\tau) \right\|_{L^\infty(\mathbb{R}^n)} + TV(q(\tau)) \right) d\tau \right) \\
\times \exp \left( K_f(T)(T - t_0) \left( 1 + re^{K_g(T)(T-t_0)} \right) \right)
$$

(4.37)
× \K_v(T) \exp \left( \K_v(T) (T - t_o) r e^{\K_g(T)(T-t_o)} \right)
\times \left( 1 + r (T - t_o) C_v(T, r e^{\K_g(T)(T-t_o)}) e^{\K_g(T)(T-t_o)} \right) + (T - t_o) K_f(T) \left( 1 + r + \frac{4 J_n}{3 \sqrt{\mu}} K_g(T) \sqrt{T - t_o} r e^{\K_g(T)(T-t_o)} \right).

Thus, collecting together (4.32) and (4.36),
\[ d \left( (u_{i+1}, w_{i+1}), (u_i, w_i) \right) = \sup_{t \in [0, T]} \left( \|u_{i+1}(t) - u_i(t)\|_{L^1(\mathbb{R}^n)} + \|w_{i+1}(t) - w_i(t)\|_{L^1(\mathbb{R}^n)} \right) \leq (K^n_v(T) + K^w_v(T)) (T - t_o) d \left( (u_i, w_i), (u_{i-1}, w_{i-1}) \right). \]

This proves (4.31), with \( K_r(T) = K^n_v(T) + K^w_v(T) \) and \textbf{Step 4} is completed.

\textbf{Step 5:} For any \( r > 0 \), there exists a \( T_r > t_o \) such that for all \((u_o, w_o) \in X_r^+\), the sequence \((u_i, w_i)\) converges in \( \mathcal{Y}_{T_r} \) to a \((u_s, w_s)\) solving (1.1) in the sense of Definition 2.1.

\textbf{Proof of Step 5:} Choose \( T_r > t_o \) such that \( K_r(T_r) (T_r - t_o) < 1 \). Thanks to (4.31), the sequence \((u_i, w_i)\) defined through (4.23) is a Cauchy sequence and converges in the complete metric space \( (\mathcal{Y}_{T_r}, d) \) defined in (4.29). Call \((u_s, w_s)\) the limit. Clearly, \( u_s \in C^0([t_o, T_r]; U^+) \) and \( w_s \in C^0([t_o, T_r]; U^+) \). It remains to prove that \((u_s, w_s)\) is a solution to (1.1) in the sense of Definition 2.1. By Lemma 4.5 and Lemma 4.10, it is sufficient to prove that \( u_s \) is a weak solution to (4.10) and \( w_s \) is a weak solution to (4.2) with
\begin{align*}
a(t, x) &= g(t, x, u_s(t, x), w_s(t, x)) , \\
b(t, x) &= f(t, x, w_s(t, x)) , \\
c(t, x) &= \left( v(t, w_s(t)) \right)(x). \end{align*}

The initial condition is satisfied: \((u_s, w_s)(0) = (u_o, w_o)\). Using the weak formulations (4.14) and (4.4), applying the Dominated Convergence Theorem, thanks to \([f]\) and \([g]\), we obtain that \((u_s, w_s)\) solves (1.1) on \([t_o, T_r]\), with initial datum \((u_o, w_o)\), in the sense of Definition 2.1

\textbf{Step 6:} The solution constructed above can be uniquely extended to all \( I \).

\textbf{Proof of Step 6:} The uniform continuity in time of \((u_s, w_s)\) on \([t_o, T_r]\) ensures that \((u_s(T_r), w_s(T_r)) = \lim_{t \to T_r-} (u_s(t), w_s(t))\) is in \( X_r^+ \). The above results can be iteratively applied, proving that \((u_s, w_s)\) can be uniquely extended to a maximal time interval \([t_o, T_s]\).

The \( L^1 \) and \( L^\infty \) bounds in (4.25), together with the \( BV \) bound in \textbf{Step 2}, ensure that the limit \( \lim_{t \to T_s-} w_s(t) \) exists and is in \( U^+ \), so that we can define \( w_s(T_s) = \lim_{t \to T_s-} w_s(t) \). Similarly, Proposition 4.11 allows to uniquely extend \( u_s \) in \( T_s \), setting \( u_s(T_s) = \lim_{t \to T_s-} u_s(t) \) with \( u_s(T_s) \in U^+ \). A further application of the steps above then allows to further prolong \((u_s, w_s)\) beyond time \( T_s \), unless \( T_s = \sup I \), completing the proof of this step.

\textbf{Step 7:} Let \( r > 0 \). Given \((u_o, w_o), (\tilde{u}_o, \tilde{w}_o) \in X_r^+\), call \((u, w)\) and \((\tilde{u}, \tilde{w})\) the corresponding solutions to (1.1). Then, for all \( t \in I \), (2.4) holds, with \( C_o \) defined in (4.48).
Proof of Step 7: Define for \((t, x) \in I \times \mathbb{R}^n\) the following functions
\begin{align*}
    a(t, x) &= g(t, x, u(t, x), w(t, x)), \\
    b(t, x) &= f(t, x, w(t, x)), \\
    c(t, x) &= (v(t, w(t)))(x), \\
    \alpha(t, x) &= g(t, x, \tilde{u}(t, x), \tilde{w}(t, x)), \\
    \beta(t, x) &= f(t, x, \tilde{w}(t, x)), \\
    \epsilon(t, x) &= (v(t, \tilde{w}(t)))(x).
\end{align*}
\label{eq:4.38}

Let \(\tilde{w}\) be the solution to (4.2) with \(a\) in the source term and initial datum \(\tilde{w}_o\), and let \(\hat{u}\) be the solution to (4.10) with coefficients \(b, c\) and initial datum \(\tilde{u}_o\). More precisely,
\begin{align*}
    \begin{cases}
    \partial_t \hat{w} - \mu \Delta \hat{w} = a(t, x) \hat{w}, \\
    \hat{w}(t_o, x) = \tilde{w}_o(x) \\
    \end{cases}
    \quad \text{and} \quad
    \begin{cases}
    \partial_t \hat{u} + \nabla \cdot (c(t, x) \hat{u}) = b(t, x) \hat{u} + q(t, x) \\
    \hat{u}(t_o, x) = \tilde{u}_o(x).
    \end{cases}
\end{align*}
\label{eq:4.39}

By (2.3), we need to compute
\begin{align*}
    \|(u(t), w(t)) - (\tilde{u}(t), \tilde{w}(t))\|_\mathcal{X} &= \|u(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} + \|w(t) - \tilde{w}(t)\|_{L^1(\mathbb{R}^n)} \\
    &\leq \|u(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} + \|\hat{u}(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} + \|w(t) - \tilde{w}(t)\|_{L^1(\mathbb{R}^n)} + \|\hat{w}(t) - \tilde{w}(t)\|_{L^1(\mathbb{R}^n)}.
\end{align*}
\label{eq:4.40}

Compute each term in (4.40) separately. The first one is the \(L^1\)-distance between solutions to balance laws of the type (4.10) with different initial data. Exploiting (4.11) for the solution to these balance laws and the bounds obtained in the proof of Step 1, we get
\begin{equation}
    \|u(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} \leq \|u_o - \tilde{u}_o\|_{L^1(\mathbb{R}^n)} \exp \left[ K_f(t) (t - t_o) \left( 1 + \|w_o\|_{L^\infty(\mathbb{R}^n)} \epsilon K_g(t)(t-t_o) \right) \right].
\end{equation}
\label{eq:4.42}

The second term in (4.40) is the \(L^1\)-distance between solutions to balance laws of the type (4.10) with different coefficients \(b, c\) and same initial datum. Exploiting the computations in the proof of Step 4, as well as (H5) in Proposition 4.11, we get
\begin{align*}
    &\|\hat{u}(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} \\
    \leq & \left\{ \hat{O}_1(t, r) \left( \|\tilde{w}_o\|_{L^\infty(\mathbb{R}^n)} + \text{TV} (\tilde{w}_o) + \int_{t_o}^t \left( \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} + \text{TV} (q(\tau)) \right) d\tau \right) K_v(t) \\
    &+ \hat{O}_2(t, r) \left( \|\tilde{w}_o\|_{L^\infty(\mathbb{R}^n)} + \int_{t_o}^t \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} d\tau \right) \left( K_f(t) + C_v \left( t_o, \|\tilde{w}_o\|_{L^\infty(\mathbb{R}^n)} \epsilon K_g(t)(t-t_o) \right) \right) \right\} \times \int_{t_o}^t \|w(\tau) - \tilde{w}(\tau)\|_{L^1(\mathbb{R}^n)} d\tau,
\end{align*}
\label{eq:4.43}
with
\begin{align*}
    \hat{O}_1(t, r) &= \exp \left( K_f(t) (t - t_o) + r (t - t_o) \epsilon K_g(t)(t-t_o) (K_f(t) + K_v(t)) \right) \times \left[ 1 + (t - t_o) C_v(t, r \epsilon K_g(t)(t-t_o)) R \epsilon K_g(t)(t-t_o) \right],
\end{align*}
\label{eq:4.44}
An application of Gronwall Lemma yields:

\[ +K_f(t) (t - t_o) \left( 1 + r + \frac{4J_n}{3\sqrt{\mu}} r K_g(t) \sqrt{t - t_o} e^{K_g(t)(t-t_o)} \right) \]

\[ \hat{O}_2(t, r) = \exp \left( K_f(t) (t - t_o) \left( 1 + r e^{K_g(t)(t-t_o)} \right) \right). \quad (4.45) \]

The first term in (4.41) is the \( L^1 \)-distance between solutions to equations of the type (4.2) with different initial data. Since \( P \) as defined in Proposition 4.6 is linear, by Step 1 we obtain

\[ \|w(t) - \hat{w}(t)\|_{L^1(\mathbb{R}^n)} \leq \|w_o - \hat{w}_o\|_{L^1(\mathbb{R}^n)} \exp \left( K_g(t)(t - t_o) \right). \quad (4.46) \]

The second term in (4.41) is the \( L^1 \)-distance between solutions to the parabolic equation (4.2) with different coefficients in the source term and the same initial datum. Exploiting the computations in the proof of Step 4, as well as (P11) in Corollary 4.8, we get

\[ \|\hat{w}(t) - \tilde{w}(t)\|_{L^1(\mathbb{R}^n)} \leq \|\tilde{w}_o\|_{L^\infty(\mathbb{R}^n)} K_g(t) e^{2K_g(t)(t-t_o)} \int_{t_o}^t \left\| \left( u(\tau) - \tilde{u}(\tau), w(\tau) - \tilde{w}(\tau) \right) \right\|_{\mathcal{X}} d\tau. \quad (4.47) \]

Hence, (4.42), (4.43), (4.46) and (4.47) yield

\[ \|(u(t), w(t)) - (\tilde{u}(t), \tilde{w}(t))\|_{\mathcal{X}} \leq \mathcal{K}_1(t, r) \|\det(\mathcal{K}), \det(\mathcal{K}) - (\tilde{u}_o, \tilde{w}_o)\|_{\mathcal{X}} + \mathcal{K}_2(t, r) \int_{t_o}^t \left\| \left( u(\tau) - \tilde{u}(\tau), w(\tau) - \tilde{w}(\tau) \right) \right\|_{\mathcal{X}} d\tau, \]

where we set

\[ \mathcal{K}_1(t, r) = \exp \left( \max \left\{ K_f(t) (t - t_o) \left( 1 + r e^{K_g(t)(t-t_o)} \right), e^{K_g(t)(t-t_o)} \right\} \right), \]

\[ \mathcal{K}_2(t, r) = \hat{O}_1(t, r) \left( 2r + \int_{t_o}^t \left( \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} + \text{TV} (q(\tau)) \right) d\tau \right) K_g(t) \]

\[ + \hat{O}_2(t, r) \left( r + \int_{t_o}^t \|q(\tau)\|_{L^\infty(\mathbb{R}^n)} d\tau \right) \left( K_f(t) + C_o \left( t, r e^{K_g(t)(t-t_o)} \right) \right) \]

\[ + r K_g(t) e^{2K_g(t)(t-t_o)}. \]

An application of Gronwall Lemma yields:

\[ \|(u(t), w(t)) - (\tilde{u}(t), \tilde{w}(t))\|_{\mathcal{X}} \leq \|\det(\mathcal{K}), \det(\mathcal{K}) - (\tilde{u}_o, \tilde{w}_o)\|_{\mathcal{X}} \int_{t_o}^t \mathcal{K}_1(s, r) \exp \left( \int_s^t \mathcal{K}_2(\tau, r) d\tau \right) \right\|_{\mathcal{X}} ds, \]

proving Step 7 with

\[ C_o(t, r) = \int_{t_o}^t \mathcal{K}_1(s, r) \exp \left( \int_s^t \mathcal{K}_2(\tau, r) d\tau \right) ds. \quad (4.48) \]

Step 8: Given \( q, \tilde{q} \) satisfying (q) call \((u, w)\) and \((\tilde{u}, \tilde{w})\) the solutions to (1.1) with the same initial datum \((u_o, w_o) \in \mathcal{X}_0^+\). Then, for all \( t \in I, (2.5) \) holds with \( C_o \) defined in (4.51).
Proof of Step 8: Define for \((t, x) \in I \times \mathbb{R}^n\) the functions \(a, \tilde{a}, b, \tilde{b}, c, \tilde{c}\) as in (4.38).

The \(L^1\) distance between \(w(t)\) and \(\tilde{w}(t)\) can be computed as in (4.47), leading to

\[
\|w(t) - \tilde{w}(t)\|_{L^1(\mathbb{R}^n)} \leq \|w_0\|_{L^\infty(\mathbb{R}^n)} K_g(t) e^{2K_g(t)(t-t_0)} \int_{t_0}^{t} \left\| (u - \tilde{u}, w - \tilde{w})(\tau) \right\|_{\mathcal{X}} \, d\tau.
\]

(4.49)

To compute the \(L^1\) distance between \(u(t)\) and \(\tilde{u}(t)\), we exploit (H5) in Proposition 4.11 and the computations in the proofs of Step 4 and Step 7, to get

\[
\|u(t) - \tilde{u}(t)\|_{L^1(\mathbb{R}^n)} \leq \hat{O}_1(t, r) \left[ \|u_0\|_{L^\infty(\mathbb{R}^n)} + TV(u_0) \right. \\
+ \int_{t_0}^{t} \left( \max \left\{ \|q(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{q}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} + \max \left\{ TV(q(\tau)), TV(\tilde{q}(\tau)) \right\} \right) d\tau \\
\times K_v(t) \int_{t_0}^{t} \|w(\tau) - \tilde{w}(\tau)\|_{L^1(\mathbb{R}^n)} \, d\tau \\
+ \hat{O}_2(t, r) \left( \|u_0\|_{L^\infty(\mathbb{R}^n)} + \int_{t_0}^{t} \max \left\{ \|q(\tau)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{q}(\tau)\|_{L^\infty(\mathbb{R}^n)} \right\} d\tau \right) \\
\times \left( K_f(t) + C_v \left( t, \|w_0\|_{L^\infty} e^{2K_g(t)(t-t_0)} \right) \right) \\
\times \int_{t_0}^{t} \|w_i(\tau) - w_i^{-1}(\tau)\|_{L^1(\mathbb{R}^n)} \, d\tau + \hat{O}_2(t, r) \|q - \tilde{q}\|_{L^1([t_0,t] \times \mathbb{R}^n)},
\]

(4.50)

where \(\hat{O}_1(t, r)\) and \(\hat{O}_2(t, r)\) are as in (4.44)–(4.45). Collecting together (4.49) and (4.50) and an application of Gronwall Lemma completes the proof of Step 8 with

\[
\mathcal{C}_q(t, r) = \hat{O}_2(t, r) \int_{t_0}^{t} \exp \int_{s}^{t} \left\{ rK_g(\tau) e^{2K_g(\tau)(\tau-t_0)} + K_v(\tau) \hat{O}_1(\tau, r) \left[ 2r \\
+ \int_{t_0}^{\tau} \left[ \max \left\{ \|q(\sigma)\|_{L^\infty(\mathbb{R}^n)}, \|\tilde{q}(\sigma)\|_{L^\infty(\mathbb{R}^n)} \right\} + \max \left\{ TV(q(\sigma)), TV(\tilde{q}(\sigma)) \right\} \right] \, d\sigma \right. \\
\times \left( K_f(\tau) + C_v \left( \tau, re^{K_g(\tau)(\tau-t_0)} \right) \right) \right\} d\tau \, ds.
\]

(4.51)

□

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References

[1] R. Bürger, G. Chowell, E. Gavilán, P. Mulet, and L. M. Villada. Numerical solution of a spatio-temporal gender-structured model for hantavirus infection in rodents. *Math. Biosci. Eng.*, 15(1):95–123, 2018.

[2] R. Bürger, G. Chowell, E. Gavilán, P. Mulet, and L. M. Villada. Numerical solution of a spatio-temporal predator-prey model with infected prey. *Math. Biosci. Eng.*, 16(1):438–473, 2019.

[3] R. M. Colombo, M. Herty, and M. Mercier. Control of the continuity equation with a non local flow. *ESAIM Control Optim. Calc. Var.*, 17(2):353–379, 2011.

[4] R. M. Colombo and E. Rossi. Hyperbolic predators vs. parabolic prey. *Commun. Math. Sci.*, 13(2):369–400, 2015.

[5] R. M. Colombo and E. Rossi. Nonlocal conservation laws in bounded domains. *SIAM J. Math. Anal.*, 50(4):4041–4065, 2018.

[6] R. M. Colombo and E. Rossi. A modeling framework for biological pest control. *Math. Biosci. Eng.*, 17(2):1713–1427, 2020.

[7] D. Crowder. Impact of release rates on the effectiveness of augmentative biological control agents. *J. Insect Sci.*, 7(15), 2007.

[8] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication.

[9] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[10] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[11] A. Keimer, L. Pflug, and M. Spinola. Existence, uniqueness and regularity of multi-dimensional nonlocal balance laws with damping. *J. Math. Anal. Appl.*, 466(1):18–55, 2018.

[12] R. J. LeVeque. *Finite volume methods for hyperbolic problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[13] F. Pfab, M. Stacconi, G. Anfora, A. Grassi, V. Walton, and A. Pugliese. Optimized timing of parasitoid release: a mathematical model for biological control of Drosophila suzukii. *Theoretical Ecology*, 11(4):489–501, 2018.

[14] E. Rossi and V. Schleper. Convergence of numerical scheme for a mixed hyperbolic-parabolic system in two space dimensions. *ESAIM Math. Model. Numer. Anal.*, 50(2):475–497, 2016.

[15] M. Rossi Stacconi, A. Grassi, C. Ioriatti, and G. Anfora. Augmentative releases of Trichopria drosophilae for the suppression of early season Drosophila suzukii populations. *BioControl*, 64(1):9–19, 2019.

[16] E. Suli and D. F. Mayers. *An introduction to numerical analysis*. Cambridge University Press, Cambridge, 2003.

[17] W. Walter. *Differential and integral inequalities*. Translated from the German by Lisa Rosenblatt and Lawrence Shampine. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 55. Springer-Verlag, New York-Berlin, 1970.