We report a new three and four coupled nonlinear partial differential-difference equations each admits Lax representation, possess infinitely many generalized (nonpoint) symmetries, conserved quantities and a recursion operator. Hence they are completely integrable both in the sense of Lax and Liouville.

Keywords: Integrable equations; nonlinear partial differential-difference equations; soliton equations.

1. Introduction

In recent years, searching for new integrable discrete systems governed by nonlinear partial differential-difference equations (PΔEs) is an important and interesting task in nonlinear systems[3, 4, 14, 16, 19, 22, 23, 28, 30, 31]. A variety of analytical techniques have been devised toward this goal both for nonlinear partial differential equations (PDEs) and PΔEs [1, 8, 15, 18, 20, 25, 27, 29]. As a result, considerable number of completely integrable nonlinear scalar PΔEs with polynomial forms with (1+1) dimensions have been identified. More often these integrable equations exhibit rich mathematical structures such as Lax representation [1, 5, 9, 16, 22, 23, 31], an infinitely many generalized symmetries, conserved densities [10–13, 15, 25] and master symmetries [8, 11, 20], etc. [7, 17, 21, 32] which are common properties of completely integrable systems. However only a limited number of integrable coupled nonlinear PΔEs with (1+1) dimensions exist in the literature. Also, if one introduces more components to a known scalar nonlinear PΔEs possessing mathematical structures related with integrability, the resulting equation may not preserve all the characteristics of original equation and hence it is important to investigate further towards their integrability. Thus it is interesting to identify integrable coupled nonlinear PΔEs. With this aim, in this article we report a new integrable three and four coupled nonlinear partial differential-difference equations (PΔEs).
More specifically we consider a 3- and 4-coupled PD\(\Delta\)Es, respectively given by

\[
\frac{\partial u_n}{\partial t} = \frac{1}{v_n} \frac{p_n}{u_{n+1}v_n},  \quad (1.1a)
\]

\[
\frac{\partial v_n}{\partial t} = \frac{v_n p_{n-1}}{u_n u_{n-1}v_{n-1}} - \frac{1}{u_n},  \quad (1.1b)
\]

\[
\frac{\partial p_n}{\partial t} = \frac{p_n}{u_n v_n} - \frac{p_n^2}{u_n v_n u_{n+1}},  \quad (1.1c)
\]

and

\[
\frac{\partial u_n}{\partial t} = \frac{1}{v_n} \frac{q_n}{u_{n+1}v_n} - \frac{v_n p_n}{u_{n+1}v_n},  \quad (1.2a)
\]

\[
\frac{\partial p_n}{\partial t} = \frac{v_n p_{n-1}}{u_n u_{n-1}v_{n-1}} - \frac{1}{u_n} + \frac{p_n}{u_{n+1}},  \quad (1.2b)
\]

\[
\frac{\partial q_n}{\partial t} = \frac{p_n q_{n-1}}{v_{n-1} u_{n-1} v_{n-1}} - \frac{p_n q_n}{v_{n-1} u_{n+1}},  \quad (1.2c)
\]

\[
\frac{\partial q_n}{\partial t} = \frac{q_n}{u_n v_n} - \frac{q_n q_{n+1}}{u_n v_{n+1} u_{n+1}} - \frac{p_n q_{n-1}}{u_{n-1} v_{n-1}},  \quad (1.2d)
\]

where \(u_n = u(n, t), v_n = v(n, t), p_n = p(n, t), q_n = q(n, t), u_{n-1} = u(n-1, t), u_{n+1} = u(n+1, t)\) and show that both of them are Hamiltonian ones and admit Lax representation with \((2\times2)\) Lax matrices and possess infinitely many generalized (nonpoint) symmetries, conserved quantities and recursion operator. Hence \((1.1)\) and \((1.2)\) are completely integrable in the sense of Lax and Liouville.

Different research groups have been engaged in this direction and reported a limited number of integrable coupled equations with both polynomial and rational terms with \((1+1)\) dimensions [26, 30, 31]. It is appropriate to mention that \((1.1)\) and \((1.2)\) reduce into the following integrable 2-coupled nonlinear PD\(\Delta\)Es [26]

\[
\frac{\partial u_n}{\partial t} = \frac{1}{v_n} \frac{u_n}{u_{n+1}v_n},  \quad (1.3a)
\]

\[
\frac{\partial v_n}{\partial t} = \frac{v_n}{u_n v_{n-1}} - \frac{1}{u_n},  \quad (1.3b)
\]

when \(p_n = u_n\) in \((1.1)\) and \(p_n = 0\) and \(q_n = u_n\) in \((1.2)\). We would like to mention that Blaszak and Marciniak [6] have reported three and four component nonlinear integrable PD\(\Delta\)Es. Furthermore the authors of reference [33, 34] have shown that the Blaszak and Marciniak system arises from the Lax representation with \((3\times3)\) and \((4\times4)\) Lax matrices. Also \((1.1)\) and \((1.2)\) cannot be deduced from the Blaszak and Marciniak systems [6] and hence to the best of our knowledge \((1.1)\) and \((1.2)\) are new multicomponent integrable equations.

The plan of the paper is as follows: In Sec. 2, we show that \((1.1)\) and \((1.2)\) admit Lax representation indicating that they are integrable in the sense of Lax. In Sec. 3, we establish that both \((1.1)\) and \((1.2)\) are Hamiltonian ones. In Sec. 4, we show explicitly that both 3-coupled and 4-coupled nonlinear PD\(\Delta\)Es possess an infinitely many generalized
(nonpoint) symmetries and conserved quantities and hence they are integrable in the sense of Liouville. In Sec. 5, we derive the recursion operator for coupled system (1.1) and (1.2) separately. In Sec. 6, we give summary of our results.

2. Lax Representation of Differential-Difference Equations

A nonlinear (autonomous) PDE with two independent variables (one continuous + one discrete) is an equation of the form

\[ \frac{\partial u_n}{\partial t} = F(u_n, u_{n-1}, u_{n+1}, \ldots), \tag{2.1} \]

where \( u_n \) and \( F \) are vector valued functions. Consider a linear system

\[ \Phi_{n+1}(t, \lambda) = L_n(t, \lambda) \Phi_n(t, \lambda), \quad \frac{d}{dt} \Phi_n(t, \lambda) = M_n(t, \lambda) \Phi_n(t, \lambda). \tag{2.2} \]

where \( L_n(t, \lambda) \) and \( M_n(t, \lambda) \) are nonsingular square matrices. When the Lax matrices \( L_n(= L_n(t, \lambda)) \) and \( M_n(= M_n(t, \lambda)) \) are \((2 \times 2)\) square matrices then (2.2) reads

\[
\begin{pmatrix}
\phi_{1n+1}(t, \lambda) \\
\phi_{2n+1}(t, \lambda)
\end{pmatrix} =
\begin{pmatrix}
L_{11n}(t, \lambda) & L_{12n}(t, \lambda) \\
L_{21n}(t, \lambda) & L_{22n}(t, \lambda)
\end{pmatrix}
\begin{pmatrix}
\phi_{1n}(t, \lambda) \\
\phi_{2n}(t, \lambda)
\end{pmatrix},
\tag{2.3a}
\]

\[
\begin{pmatrix}
\frac{d}{dt} \phi_{1n}(t, \lambda) \\
\frac{d}{dt} \phi_{2n}(t, \lambda)
\end{pmatrix} =
\begin{pmatrix}
A_n(t, \lambda) & B_n(t, \lambda) \\
C_n(t, \lambda) & D_n(t, \lambda)
\end{pmatrix}
\begin{pmatrix}
\phi_{1n}(t, \lambda) \\
\phi_{2n}(t, \lambda)
\end{pmatrix},
\tag{2.3b}
\]

where \( \lambda \) is the spectral parameter and \( L_{11n}(t, \lambda), A_n(t, \lambda), B_n(t, \lambda), C_n(t, \lambda), D_n(t, \lambda) \) and \( M_n(t, \lambda) \) are functions of \( u_n \) and their shifts. The compatibility of the linear system, (2.2) gives

\[ \frac{d}{dt} L_n + L_n M_n - M_{n+1} L_n = 0 \tag{2.4} \]

and the compatibility of (2.3) yields

\[
\frac{d}{dt} \begin{pmatrix}
L_{11n} + L_{12n} A_n + L_{21n} C_n - A_{n+1} L_{11n} - B_{n+1} L_{21n} = 0 \\
L_{22n} + L_{12n} B_n + L_{21n} D_n - A_{n+1} L_{22n} - B_{n+1} L_{22n} = 0
\end{pmatrix},
\tag{2.5a}
\]

\[
\frac{d}{dt} \begin{pmatrix}
L_{21n} + L_{22n} A_n + L_{21n} C_n - A_{n+1} L_{21n} - D_{n+1} L_{21n} = 0 \\
L_{22n} + L_{22n} B_n + L_{22n} D_n - C_{n+1} L_{22n} - D_{n+1} L_{22n} = 0
\end{pmatrix} = 0.
\tag{2.5b}
\]

\[
\frac{d}{dt} \begin{pmatrix}
L_{12n} + L_{11n} A_n + L_{12n} B_n + C_n + L_{11n} D_n - L_{11n} L_{21n} = 0 \\
L_{22n} + L_{21n} B_n + L_{22n} D_n - C_{n+1} L_{21n} - D_{n+1} L_{22n} = 0
\end{pmatrix} = 0. \tag{2.5c}
\]

The explicit form of the Lax matrices \( L_n \) and \( M_n \) can be derived by extending a well known procedure devised by Ablowitz, Kaup, Newell and Segur (AKNS) for nonlinear partial differential equations\cite{2}. More precisely for a given suitable matrix \( L_n \), the entries of the matrix \( M_n \) can be derived by expanding its entries as a polynomial in the spectral parameter \( \lambda \) satisfying the Lax Eq. (2.4).
2.1. Lax representation of (1.1) and (1.2)

To start with, we consider the discrete spectral problem (2.2) for (1.1) with \( L_n \) and \( M_n \) as

\[
L_n = \begin{bmatrix}
0 & \lambda p_n \\
-\lambda v_n & 1 + \lambda^2 u_n v_n
\end{bmatrix}, \quad \quad M_n = \begin{bmatrix}
A_n & B_n \\
C_n & D_n
\end{bmatrix}.
\]

Then the compatibility condition (2.4) gives the following

\[
C_n p_n + B_{n+1} v_n = 0, \quad \quad u_{nt} = u_n A_n - \lambda u_n^2 C_n - u_n D_n + \frac{B_n}{\lambda} \frac{C_n u_n}{\lambda v_n} + \frac{C_{n+1} p_n}{\lambda^2 v_n} + \frac{D_{n+1}}{\lambda^2 v_n} D_n
\]

(2.7)

\[
v_{nt} = D_{n+1} u_n - \lambda v_n A_n + \frac{C_n u_n}{\lambda} \lambda u_n v_n,
\]

\[
p_{nt} = A_{n+1} p_n - p_n D_n + \frac{B_{n+1}}{\lambda} + \lambda B_{n+1} u_n v_n.
\]

In order to find the entries of the associated matrix \( M_n \) we expand each of them as a quadratic polynomial in the spectral parameter \( \lambda \), that is

\[
A_n = \sum_{l=0}^{2} a^{(l)}_n \lambda^l, \quad B_n = \sum_{l=0}^{2} b^{(l)}_n \lambda^l, \quad C_n = \sum_{l=0}^{2} c^{(l)}_n \lambda^l, \quad D_n = \sum_{l=0}^{2} d^{(l)}_n \lambda^l,
\]

where \( a^{(l)}_n \), \( b^{(l)}_n \), \( c^{(l)}_n \) and \( d^{(l)}_n \) are unknown functions to be determined. Substituting the above expansions into (2.7) and then equating the like powers of \( \lambda \) to zero we obtain a system of equations along with evolution equations and solving them consistently yields the explicit form of \( A_n, B_n, C_n \) and \( D_n \). As a result the matrix \( M_n \) for (1.1) reads

\[
M_n = \begin{bmatrix}
-\frac{\lambda^2}{2} & \frac{p_{n-1}}{u_n u_{n-1} v_{n-1}} & \frac{\lambda p_{n-1}}{u_n u_{n-1} v_{n-1}} \\
\frac{1}{u_n} & \frac{\lambda}{u_n} & \frac{\lambda^2}{2}
\end{bmatrix}.
\]

(2.8)

Proceeding in a similar manner we find that (1.2) arises from the compatibility condition (2.4) with Lax matrices \( L_n \) and \( M_n \) as

\[
L_n = \begin{bmatrix}
p_n & \lambda q_n \\
-\lambda v_n & 1 + \lambda^2 u_n v_n
\end{bmatrix}, \quad \quad M_n = \begin{bmatrix}
\frac{\lambda^2}{2} & \frac{q_{n-1}}{u_n u_{n-1} v_{n-1}} & \frac{q_{n-1}}{u_n u_{n-1} v_{n-1}} \\
\frac{1}{u_n} & \frac{1}{u_n} & \lambda
\end{bmatrix}.
\]

satisfying (2.4). Thus 3- and 4-coupled systems given in (1.1) and (1.2) are integrable in the sense of Lax.

3. Hamiltonian Structure of (1.1) and (1.2)

Let us recall some of the basics related with Hamiltonian system governed by nonlinear partial differential and differential-difference equations [20]. Let \( H : L^0 \rightarrow L^0 \) be a linear
operator and $V_H$ be a formal evolutionary vector field with characteristic is the $q$-tuple,

$$ (H\theta)_n = \sum_{\beta=1}^{q} H_{n\beta} \theta^\beta $$

(3.1)

of vertical uni-vector. Then the prolongation of the vector field is given by

$$ PrV_{H\theta} = \sum_{\alpha,J} E_J \left( \sum_{\beta} H_{\alpha\beta} \theta^\beta \right) \frac{\partial}{\partial E_J u^\alpha_n}, $$

(3.2)

where $E$ is a shift operator defined by $Ef(n) = f(n + 1)$.

**Definition 3.1.** A linear operator $H$ is said to be a Hamiltonian operator of (2.1) if it is skew symmetric and satisfies Jacobi’s identity [21].

**Definition 3.2.** A system of coupled nonlinear PD\DeltaEs is said to be a Hamiltonian system if it can be written as

$$ \frac{\partial u_n}{\partial t} = H \left( \delta H \delta u_n \right), $$

(3.3)

where $H$ is a Hamiltonian operator and $\mathcal{H}$ is the appropriate Hamiltonian functional.

In order to prove that the skew symmetric operator $H$ is Hamiltonian it remains to prove that it satisfies the Jacobi’s identity. For clarity, we mention the following theorem for a system of nonlinear partial differential equations

$$ \frac{\partial u_n}{\partial t} = K(u) $$

due to Olver [21].

**Theorem.** Let $D$ be a skew-adjoint $q \times q$ matrix differential operator of the system of partial differential equations

$$ \frac{\partial u_n}{\partial t} = K(u) $$

and

$$ \Theta = \frac{1}{2} \int \left( \Theta \wedge D \Theta \right) dx, $$

the corresponding functional bi-vector. Then $D$ is Hamiltonian if and only if

$$ PrV_D\Theta = 0. $$

(3.4)

Here $\theta = \theta(x, t, u)$.

Recent investigations by Sanders and Wang [24] suggest that the above result holds for nonlinear PD\DeltaEs as well. For nonlinear PD\DeltaEs, the prolongation of the vector field takes the form given in (3.2).

We now establish the Hamiltonian structure of (1.1) and (1.2) through the definitions and theorems stated above. The 3-coupled system (1.1) can be written as

$$ \begin{pmatrix} u_{nt} \\ v_{nt} \\ p_{nt} \end{pmatrix} = H \begin{pmatrix} \frac{\delta H}{\delta u_n} \\ \frac{\delta H}{\delta v_n} \\ \frac{\delta H}{\delta p_n} \end{pmatrix} = \begin{pmatrix} 0 & -u_n v_n & 0 \\ u_n v_n & 0 & v_n p_n \\ 0 & -v_n p_n & 0 \end{pmatrix} \begin{pmatrix} p_n \\ u_n v_n v_{n+1} \\ \frac{p_n}{u_n^2 v_n v_{n+1}} - \frac{1}{u_n v_n} \end{pmatrix}. $$

(3.5)
Using the property of wedge product \( \theta \)

Next, define a bi-vector \( \Theta \) of \( H \) and further calculation shows that

**Theorem 3.1.** The operator \( H \) given in (3.5) is a Hamiltonian operator for the 3-coupled system (1.1).

**Proof.** Let \( \theta = [\theta_1, \theta_2, \theta_3]^T \). Then

\[
H\theta = H \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -u_n v_n \theta_2 \\ u_n v_n \theta_1 + v_n \theta_3 \\ -v_n \theta_2 \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix}.
\]

(3.7)

Next, define a bi-vector \( \Theta \) of \( H \) by

\[
\Theta_H = \frac{1}{2} \sum [-u_n v_n \theta_1 \wedge \theta_2 + u_n v_n \theta_2 \wedge \theta_1 + v_n \theta_3 \wedge \theta_1 - v_n \theta_1 \wedge \theta_3].
\]

(3.8)

And further calculation shows that

\[
P \Theta V_H (\Theta_H) = 0
\]

(3.9)

and hence the skew symmetric operator \( H \) is Hamiltonian of (1.1).

**Theorem 3.2.** The operator \( J \) given in (3.6) is a Hamiltonian operator for the 4-coupled system (1.2).
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Proof. The proof is similar to that of Theorem 3.1 and hence omitted. □

Thus the 3-coupled system (1.1) and 4-coupled system (1.2) are Hamiltonian systems.

4. Generalized Symmetries and Conserved Densities

4.1. Generalized symmetries: 3-coupled system (1.1)

In this subsection, we present the computational details of the derivation of generalized symmetries for the 3-coupled system (1.1). Obviously (1.1) is invariant under the scaling (dilation) symmetry

\[
(t, u_{n}, v_{n}, p_{n}) \rightarrow (s^{-2}t, s^{-1}u_{n}, s^{-1}v_{n}, s^{-1}p_{n}),
\]

where \(s\) is an arbitrary parameter. Let us assume that (1.1) is invariant under a continuous non-point transformations

\[
\begin{align*}
\eta &= n, \\
\eta' &= t, \\
\eta_{n} &= u_{n} + \epsilon G_{i}^{(1)}(n) + O(\epsilon^{2}), \\
\eta_{p} &= v_{n} + \epsilon G_{i}^{(2)}(n) + O(\epsilon^{2}), \\
\eta_{p} &= p_{n} + \epsilon G_{i}^{(3)}(n) + O(\epsilon^{2}), \\
& \quad i = 1, 2, \ldots,
\end{align*}
\]

where

\[
\begin{align*}
G_{i}^{(1)}(n) &= G_{i}^{(1)}(\ldots n_{p+1}, n_{p}, n_{p-1}, n_{p-2}, \ldots), \\
G_{i}^{(2)}(n) &= G_{i}^{(2)}(\ldots n_{p+1}, n_{p}, n_{p-1}, n_{p-2}, \ldots), \\
G_{i}^{(3)}(n) &= G_{i}^{(3)}(\ldots n_{p+1}, n_{p}, n_{p-1}, n_{p-2}, \ldots),
\end{align*}
\]

provided \(u_{n}, v_{n}\) and \(p_{n}\) satisfy (1.1). For clarity, we denote \(G_{i}(n) = (G_{i}^{(1)}(n), G_{i}^{(2)}(n), G_{i}^{(3)}(n))\) and the subscript \(i\) represents the \(i\)th order generalized symmetry. Consequently, we obtain the following invariant equations

\[
\begin{align*}
\frac{\partial G_{i}^{(1)}(n)}{\partial t} &= \frac{\nu_{n} \nu_{p+1} G_{i}^{(1)}(n + 1) - \nu_{n+1} \nu_{p} G_{i}^{(2)}(n) - \nu_{n+1} \nu_{p} G_{i}^{(3)}(n) - \nu_{n+1} \nu_{p} G_{i}^{(3)}(n)}{\nu_{n} \nu_{p+1}}, \\
\frac{\partial G_{i}^{(2)}(n)}{\partial t} &= \frac{G_{i}^{(1)}(n) + G_{i}^{(2)}(n) \nu_{n} + G_{i}^{(3)}(n) (n - 1)}{\nu_{n} \nu_{n-1} \nu_{p+1}} - \frac{\nu_{n} \nu_{p} \nu_{n-1} G_{i}^{(3)}(n) + \nu_{n} \nu_{n-1} G_{i}^{(3)}(n) (n - 1) + \nu_{n} \nu_{p} \nu_{n-1} G_{i}^{(3)}(n) (n - 1)}{\nu_{n} \nu_{n-1} \nu_{p+1}}, \\
\frac{\partial G_{i}^{(3)}(n)}{\partial t} &= \frac{\nu_{n} \nu_{p} G_{i}^{(3)}(n) - \nu_{n} \nu_{p} G_{i}^{(3)}(n) - \nu_{n} \nu_{p} G_{i}^{(3)}(n) - 2 \nu_{n} \nu_{p} G_{i}^{(3)}(n)}{\nu_{n} \nu_{n+1} \nu_{p+1}} \\
+ \frac{\nu_{n} \nu_{p} G_{i}^{(3)}(n) + \nu_{n} \nu_{p} G_{i}^{(3)}(n) (n + 1) + \nu_{n} \nu_{p} G_{i}^{(3)}(n) (n + 1)}{\nu_{n} \nu_{n+1} \nu_{p+1}}. \\
\end{align*}
\]

The invariant equations (4.3) can be solved for the generalized symmetry \(G_{i}(n) = (G_{i}^{(1)}(n), G_{i}^{(2)}(n), G_{i}^{(3)}(n))\) in more than one ways [7, 15, 18, 21, 25]. We show below how
to derive the generalized symmetries of (1.1) through the algorithmic procedure developed by Hereman and his co-workers\[15\]. Basically, Hereman’s algorithmic procedure is based on the concept of weights and ranks. To start with, we briefly explain the concept of weights and ranks. The weight, \( w \), of a variable is defined as the exponent in the scaling parameter \( s \) which multiplies the variable. Similarly the rank of a monomial is defined as the total weight of the monomial. An expression is uniform in rank if all its terms have the same rank.

We set \( w(\frac{d}{dt}) = 2 \). From (1.1) we see

\[
w\left( \frac{d}{dt} \right) + w(u_n) = -w(v_n) = w(p_n) - w(u_{n+1}) - w(v_n),
\]

\[
w\left( \frac{d}{dt} \right) + w(v_n) = w(r_n) + w(p_{n-1}) - w(u_n) - w(u_{n-1}) - w(v_{n-1}) = -w(u_n),
\]

\[
w\left( \frac{d}{dt} \right) + w(p_n) = w(p_n) - w(u_n) - w(v_n) = 2w(p_n) - w(u_n) - w(u_{n+1}) - w(v_n)
\]

and so

\[w(u_n) = -1, \quad w(v_n) = -1, \quad w(p_n) = -1,\]

and hence (1.1a)–(1.1c) are of rank 1, 1 and 1, respectively. We wish to mention that Hereman and his collaborators\[12, 13, 15\] have developed a Mathematical software package (known as INVARIANTSSYMMETRIES.M) in Mathematica for finding higher-order symmetries and conservation laws for nonlinear PDEs and nonlinear PDEΔEs with two independent variables provided the weight of the dependent variable is positive. Since the weights of the dependent variables associated with (1.1) are negative, the software package is not applicable. However we demonstrate that one can derive higher-order generalized symmetries for (1.1) and (1.2) by exploiting their ideas. In this article, we have computed the generalized symmetries and conserved densities manually. Hereafter, we use the more compact notation

\[u_n = u, \quad v_n = v, \quad p_n = p, \quad u_{n-1} = u, \quad v_{n-1} = v, \quad p_{n-1} = p,\]

\[u_{n-2} = u, \quad v_{n-2} = v, \quad p_{n-2} = p, \quad u_{n+1} = u, \quad v_{n+1} = v, \quad p_{n+1} = p,\]

\[p_{n+2} = p, \quad u_{n+2} = u, \quad v_{n+2} = v, \quad p_{n+2} = p, \quad \text{etc.}\]

Note that the trivial generalized symmetry is of rank \((1, 1, 1)\), then the next nontrivial generalized symmetry \(G_{2}(n) = (G^{(1)}_{2}(n), G^{(2)}_{2}(n), G^{(3)}_{2}(n))^{T}\) must have rank \((3, 3, 3)\). With this in mind, we first form monomial in \(u, v, p\) and \(r\) of rank \((3, 3, 3)\) that leads to a set \(L = \{u, v, p, \frac{du}{dt}, \frac{dv}{dt}, \frac{dp}{dt}, \frac{du}{dt}, \frac{dv}{dt}, \frac{dp}{dt}, \frac{du}{dt}, \frac{dv}{dt}, \frac{dp}{dt}, \frac{du}{dt}, \frac{dv}{dt}, \frac{dp}{dt}\}\). Then the necessary partial derivatives with respect to \(t\) in each monomial of \(L\) along with (1.1) and leads to another set \(M\) involving \(u, v, p\) and its backward and forward shifts. Note that each monomial in \(M\) is of rank 3. The linear combination of the monomials in \(M\) gives the most general form of the nontrivial generalized symmetry \(G_{2}(n) = (G^{(1)}_{2}(n), G^{(2)}_{2}(n), G^{(3)}_{2}(n))^{T}\). Substituting the above linear combination in the
invariant equations along with (1.1), leads to a system of linear equations and solving them consistently gives the first nontrivial generalized symmetry with rank (3,3,3) as

\[ G_2^{(1)}(n) = \frac{p}{u v w} - \frac{1}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} - \frac{p}{u v w} - \frac{1}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} \]  

\[ G_2^{(2)}(n) = \frac{2p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} - \frac{p}{u v w} \]  

\[ G_2^{(3)}(n) = \frac{p^2}{u v w} + \frac{p^2}{u v w} + \frac{p^2}{u v w} + \frac{p^2}{u v w} - \frac{p^2}{u v w} - \frac{p^2}{u v w} \]  

(4.4a)

(4.4b)

(4.4c)

Proceeding as above, we obtain the next generalized symmetry \( G_3(n) = (G_3^{(1)}(n), G_3^{(2)}(n), G_3^{(3)}(n))^2 \) with rank (5, 5, 5), where

\[ G_3^{(1)}(n) = \frac{1}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} - \frac{p}{u v w} - \frac{p}{u v w} \]  

\[ G_3^{(2)}(n) = \frac{v}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} \]  

\[ G_3^{(3)}(n) = \frac{u}{v w} + \frac{p}{u v w} - \frac{p}{u v w} + \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} + \frac{p}{u v w} - \frac{p}{u v w} \]  

(4.5a)

(4.5b)

(4.5c)
In a similar manner, we have checked that 3-coupled system (1.1) admits a sequence of higher-order generalized symmetries $G_i(n)$ with ranks $(2i - 1, 2i - 1, 2i - 1), i = 4, 5, \ldots$ which involve a huge number of terms and hence we refrain from presenting it here. We have checked that the obtained generalized symmetries are commutative, that is, the generalized symmetries satisfies the following relations

$$[G_l(n), G_k(n)] = G_l(n)[G_k(n)] - G_k(n)[G_l(n)] = 0, \ i, k = 1, 2, \ldots,$$

where $G_l(n)[G_k(n)] = \frac{d}{d\epsilon}G_k(nu_n + \epsilon G_l(n))|_{\epsilon=0}$ is the Frechet derivative of $G_k(n)$ along the direction of $G_l(n)$.

4.2. Generalized symmetries: 4-coupled system (1.2)

Proceeding as above, we have checked that 4-coupled system (1.2) admits a sequence of generalized symmetries $G_i(n)$ with rank $(2i - 1, 2i - 1, 2i - 1, 2i - 1), i = 1, 2, \ldots$. The first two members of the sequence of generalized symmetries are as follows:

$$G_1(n) = \begin{pmatrix}
G_1^{(1)}(n) \\
G_1^{(2)}(n) \\
G_1^{(3)}(n) \\
G_1^{(4)}(n)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{v} & q & \frac{u}{uv} & 0 \\
0 & \frac{1}{v} & q & \frac{u}{uv} \\
0 & 0 & \frac{1}{v} & q \\
0 & 0 & 0 & \frac{1}{v}
\end{pmatrix}, \quad G_2(n) = \begin{pmatrix}
G_2^{(1)}(n) \\
G_2^{(2)}(n) \\
G_2^{(3)}(n) \\
G_2^{(4)}(n)
\end{pmatrix}, \quad (4.6)$$

where

$$G_2^{(1)}(n) = \frac{p^2}{u v} + \frac{1}{u v} + \frac{2 q}{v} - \frac{u^2}{u v^2} - \frac{q^2}{u v^2} - \frac{q^2}{u v^2} - \frac{p q}{u v} + \frac{q}{u v^3},$$

$$G_2^{(2)}(n) = \frac{p^2}{u v} + \frac{1}{u v} + \frac{2 q}{v} - \frac{u^2}{u v^2} - \frac{q^2}{u v^2} - \frac{q^2}{u v^2} + \frac{p q}{u v} - \frac{q}{u v^3},$$

$$G_2^{(3)}(n) = \frac{p^2}{u v} + \frac{1}{u v} + \frac{2 q}{v} - \frac{u^2}{u v^2} - \frac{q^2}{u v^2} - \frac{q^2}{u v^2} + \frac{p q}{u v} + \frac{q}{u v^3},$$

$$G_2^{(4)}(n) = \frac{p^2}{u v} + \frac{1}{u v} + \frac{2 q}{v} - \frac{u^2}{u v^2} - \frac{q^2}{u v^2} - \frac{q^2}{u v^2} + \frac{p q}{u v} - \frac{q}{u v^3}.$$
Conserved densities and fluxes:

Here again we have checked that the obtained generalized symmetries $J_i$ and the associated fluxes $j_i$ to a system of linear equations and solving them consistently gives the conserved density $n^{(1)}$ with rank 2 as

\[ n^{(1)} = \frac{1}{u} \quad \frac{p}{u^2 v} \]

and the associated fluxes $j^{(1)}_n$ is

\[ j^{(1)}_n = \frac{p p}{u^2 v} + \frac{p}{u^2 v} \]

Proceeding as above, we obtain the next conserved densities $\rho^{(2)}_n$ with rank 4

\[ \rho^{(2)}_n = \frac{1}{2} \frac{p^2}{u^2 v} + \frac{p}{u^2 v} \frac{p}{u^2 v} + \frac{p p}{u^2 v} \frac{p}{u^2 v} \]
and the associated fluxes \( J_n^{(2)} \) are

\[
J_n^{(2)} = \frac{p^2}{u^2 v^2} + \frac{p}{u^2 v^2} + \frac{pp}{u^2 v^2} - \frac{p}{u^2 v^2} - \frac{q^2}{u^2 v^2} + \frac{q}{u^2 v^2} - \frac{pq}{u^2 v^2} + \frac{pq}{u^2 v^2} - \frac{pp}{u^2 v^2}.
\]

(4.11)

In a similar manner we have checked that the 3-coupled system (1.1) admits a sequence of conserved densities \( \rho_n^{(1)} \) with rank 2, \( i = 3, 4, 5, \ldots \) along with the flux \( J_n^{(1)} \) of rank 2(\( i + 1 \)), \( i = 3, 4, 5, \ldots \), respectively, which involves lengthy expressions and so the details are omitted here.

**4.4. Conserved densities and fluxes: 4-coupled system (1.2)**

Proceeding as above, we have checked that 4-coupled system (1.2) admits a sequence of conserved densities \( \rho_n^{(3)} \) with rank 2, \( i = 1, 2, \ldots \) and fluxes \( J_n^{(3)} \) with rank 2(\( i + 1 \)), \( i = 1, 2, \ldots \). The first two members of the sequence of conserved quantities are as follows:

\[
\rho_n^{(1)} = \frac{1}{u} - \frac{q}{u v},
\]

\[
J_n^{(1)} = \frac{q}{u v} + \frac{p}{u v} + \frac{q}{u v} + \frac{q}{u v}.
\]

and

\[
\rho_n^{(2)} = \frac{1}{2 u^2 v^2} + \frac{q^2}{u^2 v^2} + \frac{q}{u^2 v^2} + \frac{p q}{u^2 v^2} - \frac{q}{u^2 v^2} - \frac{q}{u^2 v^2} + \frac{q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} - \frac{q}{u^2 v^2}
\]

\[
J_n^{(2)} = \frac{q^2}{u^2 v^2} + \frac{q}{u^2 v^2} + \frac{q}{u^2 v^2} + \frac{q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} - \frac{q}{u^2 v^2} - \frac{q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} + \frac{p q}{u^2 v^2} - \frac{q}{u^2 v^2}.
\]

5. **Recursion Operator: PDAEs**

An operator valued function \( R \) is said to be a recursion operator of Eq. (2.1) if it connects symmetries into symmetries, that is,

\[
G_{k+1}(x) = R G_k(x) \quad \forall k,
\]

(5.1)

where \( G_k(n) \) and \( G_{k+1}(n) \) are consecutive generalized symmetries. Note that there exist different methods to construct recursion operator \( R \) for PDAEs [15, 20, 24, 25]. We show below how to derive \( R \) through the algorithmic procedure developed by Hereman and his
collaborators. For 3-component systems, (5.1) becomes
\[
\begin{bmatrix}
G_{k+1}^{(1)}(n) \\
G_{k+1}^{(2)}(n) \\
G_{k+1}^{(3)}(n)
\end{bmatrix} = R
\begin{bmatrix}
G_{k}^{(1)}(n) \\
G_{k}^{(2)}(n) \\
G_{k}^{(3)}(n)
\end{bmatrix},
\]
(5.2)
where \(G_k(n) = (G_k^{(1)}(n), G_k^{(2)}(n), G_k^{(3)}(n))^T\) and \(G_{k+1}(n) = (G_{k+1}^{(1)}(n), G_{k+1}^{(2)}(n), G_{k+1}^{(3)}(n))^T\) are the generalized symmetries. The entries \(R_{ij}\) of \(R\) involve dependent variables along with their shifts and its inverse, difference operators and inverse difference operators, that is,[15]
\[
R_{ij} = U_{ij}(u_n)\mathcal{O}(\triangle^{-1}, E^{-1}, I, E)V_{ij}(u_n),
\]
where \(u_n = (u_u, v_u, p_u)\), \(U_{ij}\) and \(V_{ij}\) are functions of the potentials \(u_u, v_u, p_u\) and their shifts, \(E^{-1}f(n) = f(n - 1)\), \(I_f(n) = f(n)\), \(Ef(n) = f(n + 1)\) and \(\triangle\) is difference operator defined by
\[
\triangle f(n) = (E - I)f(n) = f(n + 1) - f(n)
\]
and \(\triangle^{-1}\) is inverse difference operator defined as
\[
\triangle^{-1}f(n) = \frac{1}{2}\left[\sum_{k=-\infty}^{-1} [f(n + 1 + 2k) - f(n + 2k)] - \sum_{k=1}^{\infty} [f(n - 1 + 2k) - f(n - 2 + 2k)]\right].
\]

5.1. Recursion operator: 3-coupled system (1.1)

The construction of the recursion operator \(R\) for the 3-coupled system (1.1) is as follows: For \(k = 2\), (5.2) becomes
\[
\begin{bmatrix}
G_{k+1}^{(1)}(n) \\
G_{k+1}^{(2)}(n) \\
G_{k+1}^{(3)}(n)
\end{bmatrix} = R
\begin{bmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{bmatrix}
\begin{bmatrix}
G_{k}^{(1)}(n) \\
G_{k}^{(2)}(n) \\
G_{k}^{(3)}(n)
\end{bmatrix},
\]
(5.3)
where \((G_{2}^{(1)}(n), G_{2}^{(2)}(n), G_{2}^{(3)}(n))^T\) and \((G_{3}^{(1)}(n), G_{3}^{(2)}(n), G_{3}^{(3)}(n))^T\) are consecutive generalized symmetries of rank \((3, 3, 3)\) and \((5, 5, 5)\) given in (4.4) and (4.5), respectively. From (5.3) it is clear that the entries \(R_{11}, R_{12}, R_{13}, R_{21}, R_{22}, R_{23}, R_{31}, R_{32}\) and \(R_{33}\) of the matrix operator \(R\) must be of rank 2 which can be determined from the following relations,
\[
\begin{align*}
\text{rank } G_{1}^{(1)}(n) &= \text{rank } R_{11} + \text{rank } G_{2}^{(1)}(n) = \text{rank } R_{12} + \text{rank } G_{2}^{(2)}(n) \\
&= \text{rank } R_{13} + \text{rank } G_{2}^{(3)}(n), \\
\text{rank } G_{1}^{(2)}(n) &= \text{rank } R_{21} + \text{rank } G_{2}^{(1)}(n) = \text{rank } R_{22} + \text{rank } G_{2}^{(2)}(n) \\
&= \text{rank } R_{23} + \text{rank } G_{2}^{(3)}(n), \\
\text{rank } G_{1}^{(3)}(n) &= \text{rank } R_{31} + \text{rank } G_{2}^{(1)}(n) = \text{rank } R_{32} + \text{rank } G_{2}^{(2)}(n) \\
&= \text{rank } R_{33} + \text{rank } G_{2}^{(3)}(n).
\end{align*}
\]
(5.4)
With this goal, we expand $R_{ij}$, $i, j = 1, 2, 3$ as the functions of dependent variable along with their shifts, difference operator and inverse difference operator, with rank 2, that is,

$$R_{ij} = U_{ij}(u_n)O(\triangle^{-1}, E^{-1}, I, E) V_{ij}(u_n)$$  \( (5.5) \)

with the following relations

$$\text{rank } R_{ij} = \text{rank } U_{ij}(u_n) + \text{rank } V_{ij}(u_n).$$  \( (5.6) \)

After a tedious calculation, we find that (5.3) along with (5.4) satisfies for the following forms of $R_{ij}$, $i, j = 1, 2, 3$ as

$$R_{11} = \left( \frac{p}{uv} - \frac{p}{u^2 v^2} + \frac{2}{u v} \right) I + \frac{p}{uv} E + \frac{1}{u} \triangle^{-1} \frac{1}{u}$$

$$+ u \triangle^{-1} \left( \frac{p}{uv} + \frac{p}{u^2 v^2} - \frac{1}{u v} \right),$$  \( (5.7a) \)

$$R_{12} = \left( \frac{p}{uv} - \frac{1}{u} \right) I + \frac{p}{uv} - \frac{1}{u} \triangle^{-1} \frac{1}{u} + u \triangle^{-1} \left( \frac{p}{uv} - \frac{1}{u v^2} \right),$$  \( (5.7b) \)

$$R_{13} = -u \triangle^{-1} \frac{1}{u v}$$  \( (5.7c) \)

$$R_{21} = \left( \frac{1}{u} - \frac{v p}{uv} \right) I + \frac{v p}{uv} E^{-1} + \frac{1}{u v} - \frac{v p}{uv^2} \triangle^{-1} \frac{1}{u}$$

$$- v \triangle^{-1} \left( \frac{p}{uv} + \frac{p}{u^2 v} - \frac{1}{u v} \right),$$  \( (5.7d) \)

$$R_{22} = \frac{v p}{uv} E^{-1} + \frac{1}{u} \frac{v p}{uv} - \frac{1}{u v} + v \triangle^{-1} \left( \frac{1}{uv} - \frac{p}{uv^2} \right),$$  \( (5.7e) \)

$$R_{23} = -\frac{v}{uv} + v \triangle^{-1} \frac{1}{uv},$$  \( (5.7f) \)

$$R_{31} = \left( \frac{p^2}{u v^3} - \frac{v p}{u v^2} - \frac{p}{u v} \right) I + \frac{p^2}{u v^3} E + \frac{p^2}{uv^2} - \frac{p}{uv} \triangle^{-1} \frac{1}{u}$$

$$+ p \triangle^{-1} \left( \frac{p}{u v} + \frac{p}{u^2 v} - \frac{1}{u v} \right),$$  \( (5.7g) \)

$$R_{32} = \left( \frac{p^2}{u v^3} - \frac{p}{u v} \right) I + \left( \frac{p^2}{u v^3} - \frac{p}{u v} \right) \triangle^{-1} \frac{1}{u} + p \triangle^{-1} \left( \frac{p}{u v^2} - \frac{1}{u v^2} \right),$$  \( (5.7h) \)

$$R_{33} = -\frac{1}{u v} I - p \triangle^{-1} \frac{1}{u v},$$  \( (5.7i) \)

Proceeding as above, we have checked that (5.2) holds for $k = 3, 4, \ldots$ with the recursion operator given above. Thus we conclude that $R$ with the entries $R_{ij}$, $i, j = 1, 2, 3$ given in (5.7) is a recursion operator for 3-coupled system (1.1).
5.2. Recursion operator: 4-coupled system (1.2)

For 4-component systems, (5.1) becomes

\[
\begin{pmatrix}
G_{k+1}^{(1)}(n) \\
G_{k+1}^{(2)}(n) \\
G_{k+1}^{(3)}(n) \\
G_{k+1}^{(4)}(n)
\end{pmatrix} = R
\begin{pmatrix}
G_{k}^{(1)}(n) \\
G_{k}^{(2)}(n) \\
G_{k}^{(3)}(n) \\
G_{k}^{(4)}(n)
\end{pmatrix}
\]

(5.8)

where \( G_k(n) = (G_k^{(1)}(n), G_k^{(2)}(n), G_k^{(3)}(n), G_k^{(4)}(n))^T \) and \( G_{k+1}(n) = (G_{k+1}^{(1)}(n), G_{k+1}^{(2)}(n), G_{k+1}^{(3)}(n), G_{k+1}^{(4)}(n))^T \) are the generalized symmetries. For \( k = 2 \), (5.8) becomes

\[
\begin{pmatrix}
G_{3}^{(1)}(n) \\
G_{3}^{(2)}(n) \\
G_{3}^{(3)}(n) \\
G_{3}^{(4)}(n)
\end{pmatrix} = R
\begin{pmatrix}
R_{11} & R_{12} & R_{13} & R_{14} \\
R_{21} & R_{22} & R_{23} & R_{24} \\
R_{31} & R_{32} & R_{33} & R_{34} \\
R_{41} & R_{42} & R_{43} & R_{44}
\end{pmatrix}
\begin{pmatrix}
G_{2}^{(1)}(n) \\
G_{2}^{(2)}(n) \\
G_{2}^{(3)}(n) \\
G_{2}^{(4)}(n)
\end{pmatrix}
\]

(5.9)

Note that the rank of generalized symmetries \( (G_{2}^{(1)}(n), G_{2}^{(2)}(n), G_{2}^{(3)}(n), G_{2}^{(4)}(n))^T \) is (3,3,4,3) while the rank of \( (G_{3}^{(1)}(n), G_{3}^{(2)}(n), G_{3}^{(3)}(n), G_{3}^{(4)}(n))^T \) is (5,5,6,5). Thus it is clear that, for uniformity in rank the entries \( R_{ij} \), \( i,j = 1,2,3,4 \) must be of rank 2. Proceeding as above, after a tedious calculation, we find that (5.9) satisfies for the following forms of \( R_{ij} \):

\[
R_{11} = \left( \frac{q}{w} + \frac{q}{uw} - \frac{2}{w} + \frac{p}{w^2} \right) I + \left( \frac{up}{w^2} + \frac{q}{w^2} \right) E + \left( \frac{q}{w} + \frac{up}{w} - \frac{1}{w} \right) \Delta^{-1} \frac{1}{u}
\]

+ \( u \Delta^{-1} \left( \frac{q}{u^2 w^2} + \frac{q}{w^2} - \frac{1}{uw^2} \right) \),

(5.10a)

\[
R_{12} = \left( \frac{q}{w^2} + \frac{up}{w} - \frac{1}{w} \right) I + \left( \frac{q}{w^2} + \frac{up}{w} - \frac{1}{w} \right) \Delta^{-1} \frac{1}{w} + u \Delta^{-1} \left( \frac{q}{w^2} - \frac{1}{w^2} \right),
\]

(5.10b)

\[
R_{13} = 0,
\]

(5.10c)

\[
R_{14} = -u \Delta^{-1} \frac{1}{uw},
\]

(5.10d)

\[
R_{21} = \left( \frac{1}{w^2} + \frac{vq}{uw} - \frac{p}{wu} \right) I + \frac{vq}{uw} E^{-1} - \frac{p}{wu} E + \left( \frac{1}{w} - \frac{vq}{uw} - \frac{p}{wu} \right) \Delta^{-1} \frac{1}{w}
\]

- \( v \Delta^{-1} \left( \frac{q}{w^2} + \frac{1}{w} - \frac{1}{uw} \right) \),

(5.10e)

\[
R_{22} = \left( \frac{u}{w} + \frac{vq}{uw} - \frac{p}{wu} \right) E^{-1} + \left( \frac{1}{w} - \frac{vq}{uw} - \frac{p}{wu} \right) \Delta^{-1} \frac{1}{w} + v \Delta^{-1} \left( \frac{q}{w^2} - \frac{1}{w^2} \right),
\]

(5.10f)

\[
R_{23} = 0,
\]

(5.10g)
Also we have checked that (5.8) holds for k = 3, 4, . . . with the recursion operator given above. Thus we conclude that \( R \) with the entries \( R_{ij}, i, j = 1, 2, 3 \) given in (5.10) is a recursion operator for 4-coupled system (1.2).

6. Summary

In this article, we report a new multicomponent nonlinear P\( \Delta \)DEs which are Hamiltonian ones admitting Lax representation, possessing infinitely many generalized symmetries, conserved quantities and recursion operator. Hence both of them are integrable in the sense of Lax and Liouville. One of the characteristics of integrable nonlinear P\( \Delta \)DEs with two independent variables is the existence of recursion operator which connects the consecutive members of the sequence of generalized symmetries [10].

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