Geographical effects on cascading breakdowns of scale-free networks

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Cascading breakdowns of real networks are severe accidents in recent years, such as the blackouts of the power transportation networks in North America. In this paper, we study the effects of geographical structure on the cascading phenomena of load-carried scale-free networks, find that more geographically constrained networks tend to have larger cascading breakdowns. Explanations by the effects of circles and large betweenness of small degree nodes are provided.

Recently dynamical processes on networks has been highly concerned and widely investigated \cite{1, 2, 3}. Among many of the dynamical features of networks, robustness attracts much attention \cite{4, 5, 6, 7}, much of which focus on scale-free (SF) networks, i.e., the degrees of nodes satisfy a power law distribution: \( P(k) \sim k^{-\lambda} \), for their ubiquity in real systems \cite{8}. The heterogeneity of the degrees often makes the scale-free networks sensitive to intentional attack \cite{9, 10, 11}, while it is resilience to random breakdowns \cite{6, 12, 13, 14}, and also resilience under avalanche phenomena by the role of the hubs that sustain large amounts of grain, playing the role of reservoirs \cite{15}. Furthermore, for cascading failures, the load-carried SF network is fragile even when one attacks only one node or very few nodes with the largest degrees \cite{16}. Since many real networks exist in two or three dimensional physical spaces, it is helpful to study the geographical complex networks and it has attracted much attention recently \cite{17, 18, 19, 20, 21}. It has been shown that geographical structure has great influence on percolation thresholds \cite{22}. Since many real systems bear cascading failures, such as power grid networks, traffic lines, Internet, etc., and also lay on the two dimensional global surface, the influence of geographical structures on cascading breakdowns is of highly importance and up to now is rarely studied.

In this paper, we study the effects of geographical structure on the cascading phenomena of load-carried scale-free networks, in which each node carries a certain type of load, such as power, traffic, etc., and if the node is broken down, its load will be redistributed to its neighbors. We investigate the Bak-Tang-Wiesenfeld (BTW) sandpile model \cite{23, 24} as a prototypical model on a weighted lattice embedded SF (WLESF) network \cite{25}; and further study the betweenness distribution. Both validate that the more spatially loosely connected network is more robust under cascading failures, i.e., they have less huge avalanche events. The network is generated as follows \cite{26}. It begins with an \( L \times L \) lattice, with periodical boundary conditions, and for each node assigned a degree \( k \) drawn from the prescribed SF degree distribution \( P(k) \sim k^{-\lambda} \), \( k \geq m \). Then a node \( i \) is picked out randomly, according to a Gaussian weight function \( f_j(r) = De^{-\left(\frac{r}{\sqrt{\lambda}}\right)^2} \), it selects other nodes and establishes connections until its degree quota \( k_i \) is filled or until it has tried many enough times, avoiding duplicate connections. The process is carried out for all the nodes in the lattice. The clustering parameter \( A \) controls the spatial denseness of the connections. For large \( A \) limits, e.g. \( A \sqrt{m} > L \), the weight function will be trivial, and the network becomes a SF random (SFR) network, i.e., random otherwise than SF degree distribution \cite{27}. To compare, we also investigated lattice embedded SF (LESF) networks with nearest neighbor connections \cite{28}. Here, we assume that the time scales governing the dynamics are much smaller than that characterizing the network evolvement, thus the static geographical network models are suitable for discussing the problems.

The rules we adopted for sandpile dynamics are as follows: (i) At each time step, a grain is added at a randomly chosen node \( i \). (ii) If the height at the node \( i \) reaches or exceeds a prescribed threshold \( z_i - k_i \), the degree of the node \( i \), then it becomes unstable and all the grains at the node topple to its adjacent nodes: \( h_i = h_i - k_i \); and for each \( i \)’s neighbor \( j \): \( h_j = h_j + 1 \); during the transfer, there is a small fraction \( f \) of grains being lost, which plays the role of sinks without which the system becomes overloaded in the end. (iii) If this toppling causes any of the adjacent nodes to be unstable, subsequent topplings follow on those nodes in parallel until there is no unstable node left, forming an avalanche event. (iv) Repeat (i) –(iii).

The main feature of the BTW sandpile model on Euclidean space is the emergence of a power law with an exponential cutoff in the avalanche size distribution, \( p(s) \sim s^{-\gamma} \exp(-s/s_c) \), where \( s \) is the avalanche size, i.e., the number of toppling nodes in an avalanche event, and \( s_c \) is its characteristic size. In our studies, nodes toppled more than once in an avalanche event is seldom \cite{29}, unless for the very large avalanches, which have already exceeded the exponential cutoffs. Thus we study the avalanche area, which is the number of distinct nodes that toppled in an avalanche event, instead of avalanche size. The avalanche area distribution follows the same form as that
FIG. 1: Number of avalanches with size $j$ or area $j$, for LESF networks out of $10^6$ avalanche events on one network configuration. $m = 4$, $N = 10^5$.

of avalanche sizes

$$p(a) \sim a^{-\tau} e^{-a/a_c},$$  \hspace{1cm} (1)

where $a$ is the avalanche area, and $a_c$ its characteristic size. A typical example is shown in Fig. 1.

For BTW sandpile model on SFR networks, K. S. Goh et al. [9] have shown that the avalanche area exponent $\tau$ increases as $\lambda$ decreases, caused by the increasing number of hubs playing the role of reservoirs. Here, we will demonstrate that for the densely connected scale-free geographical networks, the reservoir effect is weakened, and the network has a smaller $\tau$.

Figure 2 represents the avalanche area distribution for different $\lambda$ of LESF networks and WLESF networks with $A = 1$. It shows that as $\lambda$ decreases, the curve of avalanche area distribution is steeper, corresponds to larger $\tau$. These are the same as the results in Ref. [9].

The avalanche area exponent $\tau$ for these data are fitted by formula (1) and is presented in Fig. 3 together with that of SFR networks for comparison. The data for SFR networks we obtained is consistent with that of [9]. For large $\lambda$ large $N$ limits, the SFR network tends to ER random graphs, for which $\tau \approx 1.5$ [8, 13]; while LESF network tends to a super lattice, with each node has $m$ neighbors; since in our studies $m = 4$, the network limits to a normal 2D lattice, which has a value of 1.01(2) for $\tau$, consistent with the previous results [16, 19].

The avalanche area exponent for different $A$ of WLESF network is shown in Fig. 4. As $A$ goes larger, avalanche area exponent $\tau$ increases, the curves of avalanche area distribution become sharper in the double-log plot (see inset of Fig. 4), which corresponds to fewer large avalanche events. This transition in $\tau$ illuminates that when the network is geographically more loosely connected, it will be harder for large cascading events to occur.

FIG. 2: Avalanche area distribution for LESF (left panel) and WLESF $A = 1$ (right panel) networks. For both panels, from up to down $\lambda = 10.0$, 5.0, 4.0, 3.5, 3.0, 2.8, 2.6, 2.4. The loosing probability is $f = 0.001$, and $m = 4$, $N = 10^5$. 10 network realizations are carried out and for each $10^6$ avalanche events are recorded for statistics.

FIG. 3: Avalanche area exponent $\tau$ vs the SF degree exponent $\lambda$. The data are fitted by formula (1) from the data presented in Fig. 2 and that of SFR networks. The network parameters and the statistics for SFR network are the same as that in Fig. 2.

The range of an edge is the length of the shortest paths between the nodes it connected in the absence of itself [2, 20]. If an edge’s range is $l$, it will probably belong to an $l + 1$ circle. Thus the distribution of range in a network sketches the distribution of circles. The inset of Fig. 5 shows that when the spatial constrains is slighter, as $A$ goes larger, the range distribution drifts to larger ranges. It means that spatially loosely connected networks have fewer small order circles but more higher order circles. If there are many small order circles, the toppling grains are easier to meet, and the nodes with much less grains, i.e., fewer than $z - 1$, especially those
with \( z - 2 \) or \( z - 3 \) grains, could also reach the toppling threshold \( z \) and topple. Larger order circles contribute less to this effect. The main frame of Fig. 4 shows the fraction of nodes toppled in avalanches that have precisely \( z - 1 \) grains. As the network is less geographically constrained and has fewer small order circles, the fraction of toppling nodes with \( z - 1 \) grains increases, justifies our reasoning. This effect contributes to the large avalanche events of the densely connected networks, and explains the decrease of avalanche area exponent \( \tau \) as the network is more geographically constrained.

In the following section, we studied the betweenness distribution of these geographical networks. The betweenness, or betweenness centrality, of node \( i \) is defined as the total number of shortest paths between pairs of nodes that pass through \( i \). If a pair of nodes has two shortest paths, the nodes along those paths are given a betweenness of \( 1/2 \) each. The betweenness distribution for SF networks is reported to follow a power law \( P_b \sim b^{-\delta} \), and for \( 2 < \lambda < 3 \), the exponent is \( \delta \approx 2.2(1) \). We find that the betweenness distribution of LESF network decays much slower than that of SFR networks, as Fig. 6 demonstrates for a particular case. The distributions for WLESF networks lay between them, but do not appear in the graph for clearness. The same holds for other \( \lambda \) and \( m \) values. Thus there are more large betweenness nodes in LESF networks than in SFR networks. To comprehend this, we plot the betweenness vs node’s degree in Fig. 7. For LESF networks the betweenness of the same degree is distributed much more diffusively, and on average are larger. It could be seen that even nodes with small degree \( k \) could have unusually large betweenness.

When an avalanche occurs, the front of toppling nodes spread along geodesics, i.e., along the shortest paths between nodes. Since the betweenness of a node is the number of shortest paths passing through it, larger betweenness means that it will have higher possibility to receive grains in avalanching processes. In the above sandpile model, the toppling threshold is the node’s degree, thus the node that has large betweenness but small degree will be easier to topple. As Fig. 7 shows, LESF network have more such nodes than SFR networks, and the situation changes continuously for WLESF network with...
increasing $A$. This could also account for the decreasing avalanche area component $\tau$ as the network is more geographically constrained.

In conclusion, by studying avalanching processes on geographical SF networks, we find that besides the reservoir effects of the hubs in SF networks, geography has great influences on the critical exponents of these systems. The decreasing avalanche area component $\tau$ for the more geographically constrained network hints high risks for such network to breakdown through cascading failures, since they have a much higher possibility to experience huge avalanche events, due to the denser connections and huge number of smaller order circles and larger betweenness of small degree nodes. Since many real networks that carried some kinds of loads, i.e., power, traffic, data packets, etc., are imbedded in the 2D global surface and highly clustered, our results indicate that they will suffer more severe risks under node failures.

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