Exact solutions of the Klein-Gordon equation in the Kerr-Newman background and Hawking radiation

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Abstract. This work consider the influence of the gravitational field produced by a charged and rotating black hole (Kerr-Newman spacetime) on a charged massive scalar fields. We obtain exacts solutions of both angular and radial parts of Klein-Gordon equation in this spacetime, which are given in terms of the confluent Heun functions. From the radial solution, we obtain the exact waves solutions near the exterior horizon of the black hole, and discuss the Hawking radiation of scalar charged particles.

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1. Introduction

The study of the interaction of quantum systems with a gravitational fields goes back to the beginning of the last century, when the generalization of quantum mechanics to curved spaces was discussed, motivated by the idea of constructing a theory combining quantum physics and general relativity. Along this line of research the investigation of solutions of the Klein-Gordon equation in some gravitational fields as well as their consequences have been discussed in the literature \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\]. It is worth emphasize that the study of the behavior of scalar fields in black hole backgrounds could be used, in principle, to understand the physics of these objects. Therefore, it is important to find solutions of Klein-Gordon equation for real as well for complex fields and analyse the phenomena related to them, as for example, the radiation of scalar particles.

In a recent paper \[22\], we obtained the exact solutions of the Klein-Gordon equation for a massive real scalar field in the Kerr-Newman spacetime, valid in the whole space that corresponds to the black hole exterior, which means between the exterior event horizon and infinity. They are given in terms of the confluent Heun functions \[23\]. Thus, we extended the range in which the solutions are valid as compared with the ones obtained by Rowan and Stephenson \[8\], which is valid near the exterior horizon and at infinity.

Hawking radiation \[24\] is an interesting phenomenon concerning the emission of any types of particles by black holes. In particular, the emission of scalar particles has also been discussed in the literature \[25, 26, 27, 28\]. In order to study this phenomenon, many different methods has been proposed \[29, 30, 31, 32, 33, 34, 35, 36, 37\].

Recently, Zhang and Zhao \[38\] studied the Hawking radiation of a Kerr-Newman black hole by introducing a very particular coordinate system. The coordinates have some attractive properties: the time direction is a Killing vector, the metric is smooth at the horizons, among others. Huaifan et al. \[39\] introduced the tortoise coordinates and extended the classical Damour-Ruffini method \[31\] to discuss the general radiation spectrum. And Umetsu \[25\] presented the Hawking radiation derivation using the tunneling mechanism process and the dimensional reduction near the horizon.

In our present paper we obtain the exact solutions of the Klein-Gordon equation for a charged massive scalar field in the background under consideration, valid in the whole space that corresponds to the black hole exterior, which means between the exterior event horizon and infinity. Using the radial solution and making use of confluent Heun functions properties, we study the Hawking radiation of charged massive scalar particles.

This paper is organized as follows. In the section II, we present some features of the Kerr-Newman spacetime and some elements relevant to study the Hawking radiation. In the section III we introduce the Klein-Gordon equation for a charged massive scalar field in a curved background and write down this equation in the Kerr-Newman spacetime, separate the angular and radial parts, and we present the exact solution of both angular and radial equations. In section IV, we obtain the solutions for ingoing and outgoing
waves, near to the exterior horizon of the black hole. In section V, we extend the waves solutions from the outside of the black hole into the inside of the black hole. In the section VI, under the condition that the spacetime total energy, total charge and total angular momentum are conserved, we derive the black hole radiation spectra taking into account the reaction of the radiation to the spacetime. Finally, in section VII we present our conclusions.

2. Kerr-Newman spacetime

The metric generated by a black hole with angular momentum per mass unit \( a = J/M \), elctric charge \( Q \), and mass (energy) \( M \) is the Kerr-Newman metric [40], whose line element, in the Boyer-Lindquist coordinates [41], is given by

\[
ds^2 = \frac{\Delta}{\rho^2} \left( dt - a \sin^2 \theta \ d\phi \right)^2 - \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) \ d\phi - a \ dt \right]^2 - \frac{\rho^2}{\Delta} \ dr^2 - \rho^2 \ d\theta^2
\]

where

\[
\Delta = r^2 - 2Mr + a^2 + Q^2,
\]

and

\[
\rho^2 = r^2 + a^2 \cos^2 \theta.
\]

We can write the metric tensor of the Kerr-Newman spacetime as

\[
(g_{\sigma\tau}) = \begin{pmatrix}
\frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta) & 0 & 0 & \frac{a \sin^2 \theta}{\rho^2} [r^2 + a^2 - \Delta] \\
0 & \frac{\rho^2}{\Delta} & 0 & 0 \\
0 & 0 & -\rho^2 & 0 \\
\frac{a \sin^2 \theta}{\rho^2} [r^2 + a^2 - \Delta] & 0 & 0 & -\frac{\sin^2 \theta}{\rho^2} [r^2 + a^2 - \Delta] - a^2 \sin^2 \theta
\end{pmatrix}
\]

from which we obtain

\[
g \equiv \det(g_{\sigma\tau}) = -\rho^4 \sin^2 \theta
\]

Thus, the contravariant components of \( g_{\sigma\tau} \) are given by

\[
(g^{\sigma\tau}) = \begin{pmatrix}
\frac{1}{\rho^2 \Delta} (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta & 0 & 0 & \frac{a}{\rho^2 \Delta} [r^2 + a^2 - \Delta] \\
0 & \frac{\Delta}{\rho^2} & 0 & 0 \\
0 & 0 & \frac{1}{\rho^2} & 0 \\
\frac{a}{\rho^2 \Delta} [r^2 + a^2 - \Delta] & 0 & 0 & \frac{-1}{\rho^2 \Delta \sin^2 \theta} (\Delta - a^2 \sin^2 \theta)
\end{pmatrix}
\]

From Eq. (2), we have that the horizon surface equation of the Kerr-Newman spacetime is obtained from the condition

\[
\Delta = (r - r_+)(r - r_-) = 0.
\]
The solutions of Eq. (7) are
\begin{align}
r_+ &= M + \left[ M^2 - (a^2 + Q^2) \right]^{1/2}, \\
r_- &= M - \left[ M^2 - (a^2 + Q^2) \right]^{1/2},
\end{align}
and correspond to the event and Cauchy horizons of the Kerr-Newman black hole.

The gravitational acceleration on the black hole horizon surface \( r_+ \), and the Hawking radiation temperature are [39], respectively,
\begin{equation}
\kappa_+ \equiv \Delta'(r_+) = \frac{r_+ - r_-}{2(r_+^2 + a^2)},
\end{equation}
and
\begin{equation}
T_+ = \frac{\kappa_+}{2\pi}.
\end{equation}
The thermodynamic quantities associated with the black hole, such as the entropy at event horizon, \( S_+ \), the dragging angular velocity of the exterior horizon, \( \Omega_+ \), the angular momentum, \( J \), and the electric potential, \( \Phi_+ \), are given by
\begin{align}
S_+ &= \pi \left( r_+^2 + a^2 \right), \\
\Omega_+ &= \frac{a}{r_+^2 + a^2}, \\
J &= Ma, \\
\Phi_+ &= \frac{Qr_+}{r_+^2 + a^2}.
\end{align}
These quantities obtained above for the black hole event horizon satisfy the first law of thermodynamics
\begin{equation}
dE = T_+ dS_+ + \Omega_+ dJ + \Phi_+ dQ.
\end{equation}

3. The Klein-Gordon equation in a Kerr-Newman spacetime

Now, let us consider the covariant Klein-Gordon equation for a charged massive scalar field in the background generated by a charged and rotating black hole. In this case, we can write the Klein-Gordon equation as
\begin{equation}
\left[ \frac{1}{\sqrt{-g}} \partial_\sigma \left( g^{\sigma\tau} \sqrt{-g} \partial_\tau \right) - \frac{ie}{\sqrt{-g}} A^\sigma (\partial_\sigma \sqrt{-g}) - ie (\partial_\sigma A^\sigma) - 2ieA^\sigma \partial_\sigma - e^2 A^\sigma A_\sigma + \mu_0^2 \right] \Psi = 0,
\end{equation}
where \( \mu_0 \) is the mass of the scalar particle, and \( e \) is the charge of the particle. The units \( G \equiv c \equiv \hbar \equiv 1 \) were chosen. The 4-vector electromagnetic potential is given by [42]
\begin{equation}
A_\sigma = \frac{Qr}{\rho^2} (1, 0, 0, -a \sin^2 \theta).
\end{equation}
Susbtituting Eqs. (5), (6), and (17) into Eq. (16), we obtain the following equation

\[
\left\{ \frac{1}{\Delta} \left[ (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right] \frac{\partial^2}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( \Delta \frac{\partial}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)
- \frac{1}{\Delta \sin^2 \theta} \left( \Delta - a^2 \sin^2 \theta \right) \frac{\partial^2}{\partial \phi^2} + \frac{2a}{\Delta} \left[(r^2 + a^2) - \Delta\right] \frac{\partial^2}{\partial t \partial \phi}
+ 2ieQ \frac{r}{\Delta} \left[(r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi}\right] + \mu_0 r^2 - e^2 Q^2 r^2 \frac{1}{\Delta} \right\} \Psi = 0 .
\]

(18)

In order to solve the Eq. (18), we assume that its solution can be separated as follows

\[
\Psi = \Psi(r, t) = R(r) S(\theta) e^{im \phi} e^{-i\omega t} .
\]

(19)

Substituting Eq. (19) into (18), we find that

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) + \left[ - \left( \omega a \sin \theta - \frac{m}{\sin \theta} \right)^2 - \mu_0^2 a^2 \cos^2 \theta + \lambda \right] S = 0 ,
\]

(20)

\[
\frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + \left\{ - (\lambda + \mu_0^2 r^2) + \frac{1}{\Delta} \left[ \omega (r^2 + a^2) - am - eQ r \right]^2 \right\} R = 0 ,
\]

(21)

where \( \lambda \) is the separation constant, \( \omega \) is the energy of the particle, and \( m \) is the azimuthal quantum number. In what follows we will solve the angular and radial parts of Klein-Gordon equation, whose solutions are given in terms of confluent Heun functions.

### 3.1. Angular equation

Now, let us obtain the exact and general solution for the angular part of Klein-Gordon equation given by Eq. (20), which can be rewritten as

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{dS}{d\theta} \right) + \left( \Lambda_{lm} + \mu^2_0 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} \right) S = 0 ,
\]

(22)

where \( \mu_0^2 = a^2(\omega^2 - \mu_0^2) \), and the constant \( \Lambda_{lm} \) is defined by

\[
\Lambda_{lm} \equiv \lambda - a^2 \omega^2 + 2a \omega m .
\]

(23)

Note that the presence of the 4-vector external potential, \( A_\sigma \), given by Eq. (17), modifies slightly the angular part of the Klein-Gordon equation in a Kerr-Newman spacetime relative to the case of uncharged particle, discussed in our previous work [22]. This modification is reflected in the constant given by Eq. (23), which turns to be a function of kind \( \Lambda_{lm} = \Lambda_{lm}(a, \omega, m) \). In the literature, the solutions of Eq. (22) are the oblate spheroidal harmonic functions \( S_{lm}(ic_0, \cos \theta) \) with eigenvalues \( \Lambda_{lm} \), where \( l, m \) are integers such that \( |m| \leq l \) [43, 44]. We will show that the solutions of Eq. (22) can be expressed in terms of confluent Heun functions.
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To do this, let us rewrite Eq. (22) in the form which resembles a Heun equation by defining a new angular coordinate, \( x \), such that

\[
x = \cos^2 \theta .
\]

(24)

Thus, we can write Eq. (22) as

\[
d^2 S \left/ dx^2 \right. + \left( 1/2 \frac{1}{x} + \frac{1}{x - 1} \right) \frac{dS}{dx} + \frac{1}{x(x - 1)} \left[ -\frac{c_0^2 x}{4} - \frac{\Lambda_{lm}}{4} - \frac{m^2}{4(x - 1)} \right] S = 0 ,
\]

(25)

which has singularities at \( x = 0, 1 \) and \( x = \infty \). Equation (25) can also be written as

\[
d^2 S \left/ dx^2 \right. + \left( 1/2 \frac{1}{x} + \frac{1}{x - 1} \right) \frac{dS}{dx} + \left[ \frac{A_1}{x} + \frac{A_2}{x - 1} + \frac{A_3}{(x - 1)^2} \right] S = 0 ,
\]

(26)

where the coefficients \( A_1, A_2, \) and \( A_3 \) are given by:

\[
A_1 = \frac{\Lambda_{lm} - m^2}{4} ;
\]

(27)

\[
A_2 = -\left( \frac{\Lambda_{lm} - m^2 + c_0^2}{4} \right) ;
\]

(28)

\[
A_3 = -\frac{m^2}{4} .
\]

(29)

Defining a new function, \( S(x) \), by \( S(x) = Z(x)x^{-1/4}(x - 1)^{-1/2} \), we can write Eq. (26) in the following form

\[
d^2 Z \left/ dx^2 \right. + \left[ \frac{B_2}{x} + \frac{B_3}{x - 1} + \frac{B_4}{x^2} + \frac{B_5}{(x - 1)^2} \right] Z = 0 ,
\]

(30)

where the coefficients \( B_2, B_3, B_4, \) and \( B_5 \) are given by:

\[
B_2 = \frac{1 + 4A_1}{4} ;
\]

(31)

\[
B_3 = -\frac{1 + 4A_2}{4} ;
\]

(32)

\[
B_4 = \frac{3}{16} ;
\]

(33)

\[
B_5 = \frac{1 + 4A_3}{4} .
\]

(34)

Now, consider an equation in the standard form

\[
\frac{d^2 U}{dz^2} + p(z) \frac{dU}{dz} + q(z)U = 0 .
\]

(35)

Changing the function \( U(z) \), using the relation

\[
U(z) = Z(z)e^{-1/2 \int p(z)dz} ,
\]

(36)

Eq. (35) turns into the normal form

\[
\frac{d^2 Z}{dz^2} + I(z)Z = 0 ,
\]

(37)
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where

\[ I(z) = q(z) - \frac{1}{2} dp(z) \frac{d}{dz} - \frac{1}{4} [p(z)]^2 \].

(38)

Now, let us consider the confluent Heun equation [45]

\[
\frac{d^2 U}{dz^2} + \left( \alpha + \beta + 1 \frac{1}{z} + \gamma + 1 \frac{1}{z-1} \right) \frac{dU}{dz} + \left( \frac{\mu}{z} + \frac{\nu}{z-1} \right) U = 0 ,
\]

(39)

where \( U(z) = \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z) \) are the confluent Heun functions, with the parameters \( \alpha, \beta, \gamma, \delta \) and \( \eta \), related to \( \mu \) and \( \nu \) by

\[
\mu = \frac{1}{2}(\alpha - \beta - \gamma + \alpha \beta - \beta \gamma - \eta) ,
\]

(40a)

\[
\nu = \frac{1}{2}(\alpha + \beta + \gamma + \alpha \gamma + \beta \gamma + \delta + \eta) ,
\]

(40b)

according to the standard package of the Maple\textsuperscript{TM}17. Using Eqs. (35)-(38), we can write Eq. (39) in the normal form as [23]

\[
\frac{d^2 Z}{dz^2} + \left[ D_1 + \frac{D_2}{z} + \frac{D_3}{z-1} + \frac{D_4}{z^2} + \frac{D_5}{(z-1)^2} \right] Z = 0
\]

(41)

where the coefficients \( D_1, D_2, D_3, D_4, \) and \( D_5 \) are given by:

\[
D_1 \equiv -\frac{1}{4}\alpha^2 ;
\]

(42)

\[
D_2 \equiv \frac{1}{2}(1 - 2\eta) ;
\]

(43)

\[
D_3 \equiv \frac{1}{2}(-1 + 2\delta + 2\eta) ;
\]

(44)

\[
D_4 \equiv \frac{1}{4}(1 - \beta^2) ;
\]

(45)

\[
D_5 \equiv \frac{1}{4}(1 - \gamma^2).
\]

(46)

The angular part of the Klein-Gordon equation for a charged massive scalar particle in the Kerr-Newman spacetime in the exterior region to the event horizon, given by (30), can be written as (41), and therefore, its solution is given by

\[
Z(z) = U(z)e^{\frac{i}{2} \int (\alpha + \beta_{\pm 1} + \gamma_{\pm 1}) dz} ,
\]

(47)

where \( U(z) \) is a solution of the confluent Heun equation (39), and the parameters \( \alpha, \beta, \gamma, \delta, \) and \( \eta \) are obtained from the following relations:

\[
-\frac{1}{4}\alpha^2 = 0 ;
\]

(48a)

\[
\frac{1}{2}(1 - 2\eta) = \frac{1 + 4A_1}{4} ;
\]

(48b)

\[
\frac{1}{2}(-1 + 2\delta + 2\eta) = \frac{-1 + 4A_2}{4} ;
\]

(48c)
\[
\frac{1}{4}(1 - \beta^2) = \frac{3}{16} ; \quad (48d)
\]
\[
\frac{1}{4}(1 - \gamma^2) = \frac{1 + 4A_3}{4} . \quad (48e)
\]

Thus, from the above relations, we find that:

\[
\alpha = 0 ; \quad (49a)
\]
\[
\beta = \frac{1}{2} ; \quad (49b)
\]
\[
\gamma = m ; \quad (49c)
\]
\[
\delta = -\frac{\nu_0^2}{4} = \frac{1}{4} \left( a^2 \mu_0^2 - a^2 \omega^2 \right) ; \quad (49d)
\]
\[
\eta = \frac{1}{4} \left( 1 + m^2 - \Lambda_{lm} \right) = \frac{1}{4} \left( 1 + m^2 - \lambda + a^2 \omega^2 - 2a\omega m \right) . \quad (49e)
\]

The general solution of Eq. (20) over the entire range \(0 \leq x < \infty\) is obtained with the use of Eq. (47). It is given by

\[
S(x) = (x - 1)^{\frac{1}{4}} x^{\frac{1}{2} \left( \frac{3}{4} + \beta \right)}
\times \{ C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta; x) \\
+ C_2 x^\beta \text{HeunC}(\alpha, -\beta, \gamma, \delta, \eta; x) \} , \quad (50)
\]

where \(\text{HeunC}(\alpha, \pm \beta, \gamma, \delta, \eta, z)\) are the confluent Heun functions, \(C_1\) and \(C_2\) are constants and \(\alpha, \beta, \gamma, \delta, \) and \(\eta\) are fixed by relations \((49a)-(49e)\). These two functions form linearly independent solutions of the confluent Heun differential equation provided \(\beta\) is not integer.

### 3.2. Radial equation

Also, we can see that the presence of 4-vector external potential, \(A_\sigma\), given by Eq. (17), modifies slightly the radial part of Klein-Gordon equation in a Kerr-Newman spacetime relative to the case of an uncharged particle, discussed in our previous work [22]. Indeed, we can generalize the results obtained in our previous work [22].

Therefore, to solve the radial part of Klein-Gordon equation, we use Eq. (7) and write down Eq. (21) as

\[
\frac{d^2R}{dr^2} + \left( \frac{1}{r - r_+} + \frac{1}{r - r_-} \right) \frac{dR}{dr} + \frac{1}{(r - r_+)(r - r_-)} \left[ r^2 \left( \omega^2 - \mu^2 \right) + r \left[ \omega^2(r_+ + r_-) - 2eQ \omega \right] + \left[ e^2Q^2 - \lambda - 2am\omega - 2eQ\omega(r_+ + r_-) + 2a^2\omega^2 + \omega^2 \left( r_+^2 + r_+r_- + r_-^2 \right) \right] \right.

+ \left. \frac{(-am - eQr_+ + a^2\omega + r_+^2\omega^2)^2}{(r - r_+)(r_+ - r_-)} \right) R = 0 . \quad (51)
\]
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This equation has singularities at \( r = (a_1, a_2) = (r_+, r_-) \), and at \( r = \infty \). The transformation of (51) to a Heun-type equation is achieved by setting

\[
x = \frac{r - a_1}{a_2 - a_1} = \frac{r - r_+}{r_- - r_+}.
\]

(52)

Defining a new function, \( R(x) \), by \( R(x) = Z(x)[x(x - 1)]^{-1/2} \), we can write Eq. (51) in the following form

\[
\frac{d^2 Z}{dx^2} + \left[ D_1 + \frac{D_2}{x} + \frac{D_3}{x - 1} + \frac{D_4}{x^2} + \frac{D_5}{(x - 1)^2} \right] Z = 0,
\]

(53)

with the coefficients \( D_1, D_2, D_3, D_4, \) and \( D_5 \) given by:

\[
D_1 = (\omega^2 - \mu^2)(r_+ - r_-)^2;
\]

(54)

\[
D_2 = \frac{(4e^2Q^2 - 8am\omega + 8a^2\omega^2)r_+r_- - 4eQr_-\omega(a^2 + 3r_+^2)}{2(r_+ - r_-)^2}
+ \frac{-4\omega^2r_1^3(r_+ - 2r_-) + 4a^2(m - a\omega)^2 + 4aemQ(r_+ + r_-)}{2(r_+ - r_-)^2}
+ \frac{(2\lambda + 1 + 2r_1^2\mu_1)(r_+ - r_-)^2 + 4eQr_\omega(r_+^2 - a^2)}{2(r_+ - r_-)^2};
\]

(55)

\[
D_3 = \frac{-(4e^2Q^2 - 8am\omega + 8a^2\omega^2)r_+r_- + 4eQr_\omega(a^2 + 3r_-^2)}{2(r_+ - r_-)^2}
+ \frac{4\omega^2r_1^3(r_- - 2r_+) - 4a^2(m - a\omega)^2 - 4aemQ(r_+ + r_-)}{2(r_+ - r_-)^2}
- \frac{(2\lambda + 1 + 2r_1^2\mu_1)(r_+ - r_-)^2 + 4eQr_\omega(r_-^2 - a^2)}{2(r_+ - r_-)^2};
\]

(56)

\[
D_4 = \frac{(4e^2Q^2 - 8am\omega + 8a^2\omega^2)r_+^2 - 8eQr_\omega(a^2 + r_+^2) + 4r_+^4\omega^2}{4(r_+ - r_-)^2}
+ \frac{4a^2(m - a\omega)^2 + 8aemQr_\omega + (r_+ - r_-)^2}{4(r_+ - r_-)^2};
\]

(57)

\[
D_5 = \frac{(4e^2Q^2 - 8am\omega + 8a^2\omega^2)r_-^2 - 8eQr_\omega(a^2 + r_-^2) + 4r_-^4\omega^2}{4(r_+ - r_-)^2}
+ \frac{4a^2(m - a\omega)^2 + 8aemQr_\omega + (r_+ - r_-)^2}{4(r_+ - r_-)^2}.
\]

(58)

The radial part of the Klein-Gordon equation for a charged massive scalar particle in the Kerr-Newman spacetime in the exterior region of event horizon, given by Eq. (21), can be written as (11). Therefore, the general solution of Eq. (21) over the entire range \( 0 \leq x < \infty \) is obtained with the use of Eq. (17). It is given by

\[
R(x) = e^{2\alpha x}(x - 1)^{1/2}x^{1/2}
\times \{ C_1 \text{ HeunC}(\alpha, \beta, \gamma, \delta, \eta; x)
+ C_2 x^{-\beta} \text{ HeunC}(\alpha, -\beta, \gamma, \delta, \eta; x) \},
\]

(59)
where $C_1$ and $C_2$ are constants, and the parameters $\alpha$, $\beta$, $\gamma$, $\delta$, and $\eta$ are given by:

\begin{align}
\alpha &= 2(r_+ - r_-) \left( \mu^2 - \omega^2 \right)^{1/2} ; \\
\beta &= \frac{2i}{r_+ - r_-} \left[ \omega(r_+^2 + a^2) - am - eQr_+ \right] ; \\
\gamma &= \frac{2i}{r_+ - r_-} \left[ \omega(r_-^2 + a^2) - am - eQr_- \right] ; \\
\delta &= (r_+ - r_-) \left[ 2eQ\omega + (r_+ + r_-) \left( \mu^2 - 2\omega^2 \right) \right] ; \\
\eta &= \frac{-2a^2(m - \omega)^2 - 2aemQ(r_+ + r_-) - (\lambda + r_+^2\mu^2)(r_+ - r_-)^2}{(r_+ - r_-)^2} \\
&\quad \quad + \frac{-2e^2Q^2 - 4am\omega + 4a^2\omega^2)r_+r_- - 2eQr_+\omega \left( a^2 - 3r_+^2 \right)}{(r_+ - r_-)^2} \\
&\quad \quad + \frac{2\omega^2r_+^3(r_+ - 2r_-) - 2eQr_+\omega \left( r_+^2 - a^2 \right)}{(r_+ - r_-)^2} .
\end{align}

These two functions form linearly independent solutions of the confluent Heun differential equation provided $\beta$ is not integer. However, there is no any specific physical reason to impose that $\beta$ should be integer. If we consider the expansion in power series of the confluent Heun functions with respect to the independent variable $x$ in a vicinity of the regular singular point $x = 0$ \cite{Heun}, we can write

\begin{align}
\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, x) &= 1 + \frac{1}{2} \frac{(-\alpha\beta + \beta\gamma + 2\eta - \alpha + \beta + \gamma)}{(\beta + 1)} x \\
&\quad \quad + \frac{1}{8(\beta + 1)(\beta + 2)} \left( \alpha^2\beta^2 - 2\alpha\beta^2\gamma + \beta^2\gamma^2 \\
&\quad \quad - 4\eta\alpha\beta + 4\eta\beta\gamma + 4\alpha^2\beta - 2\alpha\beta^2 - 6\alpha\beta\gamma \\
&\quad \quad + 4\beta^2\gamma + 4\beta\gamma^2 + 4\eta^2 - 8\eta\alpha + 8\eta\beta + 8\eta\gamma \\
&\quad \quad + 3\alpha^2 - 4\alpha\beta - 4\alpha\gamma + 3\beta^2 + 4\beta\delta \\
&\quad \quad + 10\beta\gamma + 3\gamma^2 + 8\eta + 4\beta + 4\delta + 4\gamma \right) x^2 + ... ,
\end{align}

which is a useful form to be used in the discussion of Hawking radiation.

4. Hawking radiation

We will consider the charged massive scalar field near the horizon in order to discuss the Hawking radiation. From Eqs. (52) and (61) we can see that the radial solutions given by Eq. (59), near the exterior event horizon, that is, when $r \to r_+ \Rightarrow x \to 0$, behave asymptotically as

\begin{equation}
R(r) \sim C_1 (r - r_+)^{\beta/2} + C_2 (r - r_+)^{-\beta/2} ,
\end{equation}
where we are considering contributions only of the first term in the expansion, and all constants are included in $C_1$ and $C_2$. Thus, considering the time factor, near the black hole event horizon $r_+$, this solution is

$$
\Psi = e^{-i\omega t} (r - r_+)^{\pm \beta/2} .
$$

From Eq. (60b), for the parameter $\beta$, we have

$$
\frac{\beta}{2} = i \left[ \omega \frac{r_+^2 + a^2}{r_+ - r_-} - \left( m \frac{a}{r_+ - r_-} + e \frac{Qr_+}{r_+ - r_-} \right) \right] .
$$

From Eq. (9), we have

$$
\frac{1}{2\kappa_+} = \frac{r_+^2 + a^2}{r_+ - r_-} \Leftrightarrow r_+ - r_- = 2\kappa_+ \left( r_+^2 + a^2 \right) .
$$

Then, substituting Eq. (65) into Eq. (64), we obtain

$$
\frac{\beta}{2} = \frac{i}{2\kappa_+} \left[ \omega - \left( m\Omega_+ + e\Phi_+ \right) \right] = \frac{i}{2\kappa_+} (\omega - \omega_0) ,
$$

where $\omega_0 = m\Omega_+ + e\Phi_+$.

Therefore, on the black hole exterior horizon surface the ingoing wave solution is

$$
\Psi_{in} = e^{-i\omega t} (r - r_+)^{\frac{i}{2\kappa_+} (\omega - \omega_0)} ,
$$

while the outgoing wave solution is

$$
\Psi_{out}(r > r_+) = e^{-i\omega t} (r - r_+)^{\frac{i}{2\kappa_+} (\omega - \omega_0)} .
$$

Thus, we obtained the solutions for ingoing and outgoing waves near to the exterior horizon $r_+$ of a Kerr-Newman black hole. These solutions for the scalar fields near the horizon will be useful to investigate Hawking radiation of charged massive scalar particles. It is worth calling attention to the fact that we are using the analytical solution of the radial part of Klein-Gordon in the spacetime under consideration, differently from the calculations in the literature [38, 39].

For consistency and completeness, we will show that our solution is exactly the same one obtained by other methods [39, 38], unless a change of variable. Using the definitions of the tortoise and Eddington-Finkelstein coordinates, given by [31]

$$
dr_* = \frac{1}{\Delta r} (r^2 + a^2) \, dr \Rightarrow
$$

$$
\ln(r - r_+) = \frac{1}{r_+^2 + a^2} \frac{d\Delta r}{dr} \bigg|_{r=r_+} = 2\kappa_* r_* ,
$$

$$
\hat{r} = \frac{\omega - \omega_0}{\omega} r_* ,
$$

$$
v = t + \hat{r} ,
$$
we have the following ingoing wave solution:
\[
\Psi_{\text{in}} = e^{-i\omega v} e^{i\omega \hat{r}} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v} e^{i(\omega - \omega_0) r_+} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v}.
\]  
(73)

The outgoing wave solution is given by:
\[
\Psi_{\text{out}}(r > r_+) = e^{-i\omega v} e^{i\omega \hat{r}} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v} e^{i(\omega - \omega_0) r_+} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)} (r - r_+) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)}
= e^{-i\omega v}.
\]  
(74)

The solutions (73) and (74) are exactly the solutions obtained by Zhang & Zhao [38] and Huaifan et al. [39].

5. Analytic extension

In this section we obtain by analytic continuation a real damped part of the outgoing solution of the scalar field which will be used to construct an explicit expression for the decay rate $\Gamma$. This real damped part corresponds (at least in part) to the temporal contribution to the decay rate [46, 47] found in the tunneling method of Hawking radiation.

From Eq. (74), we see that this solution is not analytical in the exterior event horizon $r = r_+$. By analytic continuation, rotating $−\pi$ through the lower-half complex $r$ plane, we obtain
\[
(r - r_+) \rightarrow |r - r_+| e^{-i\pi} = (r_+ - r) e^{-i\pi}.
\]  
(75)

Thus, the outgoing wave solution on the horizon surface $r_+$ is
\[
\Psi_{\text{out}}(r < r_+) = e^{-i\omega v} (r_+ - r) \frac{1}{\tilde{x}^+_+(\omega - \omega_0)} e^{\frac{\pi}{2} \tilde{x}^+_+(\omega - \omega_0)}.
\]  
(76)

Now, using Eq. (70), the solution given by Eq. (76) can also be written in the following form
\[
\Psi_{\text{out}}(r < r_+) = e^{-i\omega v} e^{2i(\omega - \omega_0) r_+} e^{\frac{\pi}{2} \tilde{x}^+_+(\omega - \omega_0)}.
\]  
(77)

Equations (74) and (77) describe the outging wave outside and inside of the black hole, respectively. Therefore, for an outgoing wave of a particle with energy $\omega$, charge $e$ and angular momentum $m$, the outgoing decay rate or the relative scattering probability of the scalar wave at the event horizon surface $r = r_+$ is given by
\[
\Gamma_+ = \left| \frac{\Psi_{\text{out}}(r > r_+)}{\Psi_{\text{out}}(r < r_+)} \right|^2 = e^{-\frac{\pi}{2} \tilde{x}^+_+(\omega - \omega_0)}.
\]  
(78)

This result was formally obtained in the literature [38, 46, 47, 39] in different contexts.
6. Radiation spectrum

After the black hole event horizon radiates particles with energy $\omega$, charge $e$ and angular momentum $m$, in order to consider the reaction of the radiation of the particle to the spacetime, we must replace $M, Q, J$ by $M - \omega, Q - e, J - m$, respectively, in the line element of Kerr-Newman spacetime \cite{11}. Doing these changes, we must guarantee that the total energy, angular momentum and charge of spacetime are all conserved, that is,

\begin{equation}
\begin{aligned}
-\omega &= \Delta E, \\
-e &= \Delta Q, \\
-m &= \Delta J, \\
\end{aligned}
\end{equation}

where $\Delta E$, $\Delta Q$, and $\Delta J$ are the energy, charge, and angular momentum variations of the black hole event horizon, before and after the emission of radiation, respectively. Substituting Eqs. (15) and (79) into Eq.(78), we obtain the outgoing decay rate at the event horizon surface $r = r_+$:

\begin{equation}
\Gamma_+ = e^{-\frac{2\pi}{\kappa}(-\Delta E - m\Omega + e\Phi)} \\
= e^{-\frac{2\pi}{\kappa}(-\Delta E + \Delta J + \Phi + \Delta Q)} \\
= e^{-\frac{2\pi}{\kappa}(-T_+ \Delta S_+)} \\
= e^{\Delta S_+},
\end{equation}

where we have used Eq. (10). $\Delta S_+$ is the change of the Bekenstein-Hawking entropy, compared before and after the emission of radiation, and obtained from the expressions for the entropy \cite{11} and for the exterior event horizon \cite{8a}, as following:

\begin{equation}
\Delta S_+ = S_+(M - \omega, Q - e, J - m) - S_+(M, Q, J) \\
= \pi \left[ \frac{r_+^2(M - \omega, Q - e, J - m) + a^2(M - \omega, J - m)}{r_+^2(M, Q, J) + a^2(M, J)} \right] \\
- \pi \left[ \frac{r_+^2(M, Q, J) + a^2(M, J)}{2(M - \omega)^2 - (Q - e)^2} \right] \\
+ 2(M - \omega)\sqrt{(M - \omega)^2 - a_+^2 - (Q - e)^2} \\
\quad + Q^2 - 2M^2 - 2M\sqrt{M^2 - a^2 - Q^2},
\end{equation}

where $a = a(M, J) = J/M$ and

\begin{equation}
a_+^2 = \left( \frac{J - m}{M - \omega} \right)^2.
\end{equation}

According to the Damour-Ruffini-Sannan method \cite{31, 34}, a correct wave describing a particle flying off of the black hole is given by

\begin{equation}
\Psi_\omega(r) = N_\omega \left[ H(r - r_+) \Psi_\omega^{\text{out}}(r - r_+) \\
+ H(r_+ - r) \Psi_\omega^{\text{out}}(r_+ - r) \right. \left. e^{\frac{2\pi}{\kappa}((\omega - \omega_0)} \right],
\end{equation}

where $N_\omega$ is the normalization constant, such that

\begin{equation}
\langle \Psi_{\omega_1}(r) | \Psi_{\omega_2}(r) \rangle = -\delta(\omega_1 - \omega_2),
\end{equation}

where $\delta$ is the Dirac delta function.
where $H(x)$ is the Heaviside function and $\Psi_{\omega}^{\text{out}}(x)$ are the normalized wave function given, from Eq. (74), by

$$
\Psi_{\omega}^{\text{out}}(x) = e^{-i\omega v_x} x^{\frac{1}{\kappa_+}(\omega - \omega_0)}.
$$

Thus, from the normalization condition

$$
\langle \Psi_{\omega}(r) | \Psi_{\omega}(r) \rangle = 1 = |N_{\omega}|^2 \left[ e^{2\pi(\omega - \omega_0)} - 1 \right],
$$

we get the resulting Hawking radiation spectrum of charged scalar particles

$$
|N_{\omega}|^2 = \frac{1}{e^{2\pi(\omega - \omega_0)} - 1} = \frac{1}{e^{\frac{\kappa_+}{k_B T} - 1}}.
$$

Therefore, we can see that the resulting Hawking radiation spectrum of scalar particles has a thermal character, analogous to the blackbody spectrum, where $k_B T = \kappa_+/2\pi$, with $k_B$ being the Boltzmann constant – note that the SI units were restored.

7. Conclusions

In this paper, we presented analytic solutions for both angular and radial parts of the Klein-Gordon equation for a charged massive scalar field in the Kerr-Newman spacetime.

These solutions are analytic in whole spacetime, namely, in the region between the event horizon and infinity. The radial part of obtained solutions generalizes a previous result [22] in the sense that now we are considering a charged massive scalar field coupled to the electromagnetic field associated with the gravitational source. The solution is given in terms of the confluent Heun functions, and is valid over the range $0 \leq x < \infty$.

From these analytic solutions, we obtained the exact solutions for ingoing and outgoing waves near the exterior horizon of a Kerr-Newman black hole, and used those results to discuss the Hawking effect [24]. We considered the properties of the confluent Heun functions to obtain the results. This approach has the advantage that it is not necessary the introduction of any coordinate system, as for example, the particular one [38] or tortoise and Eddington-Finkelstein coordinates [39].

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