BLACK BOX ALGEBRA AND HOMOMORPHIC ENCRYPTION

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Abstract. In the present paper, we study homomorphic encryption as an area where principal ideas of black box algebra are particularly transparent. This paper is a compressed summary of some principal definitions and concepts in the approach to the black box algebra being developed by the authors. We suggest that black box algebra could be useful in cryptanalysis of homomorphic encryption schemes, and that homomorphic encryption is an area of research where cryptography and black box algebra may benefit from exchange of ideas.

1. Homomorphic encryption

1.1. Homomorphic encryption and black box algebra. “Cloud computing” appears to be a hot topic in information technology; in a nutshell, this is the ability of small and computationally weak devices to delegate hard resource-intensive computations to third party (and therefore untrusted) computers. To ensure the privacy of the data, the untrusted computer should receive data in an encrypted form but still being able to process it. It means that encryption should preserve algebraic structural properties of the data.

This is one of the reasons for popularity of the idea of homomorphic encryption which we describe here with some simplifications aimed at clarifying connections with black box algebra, that is, part of computational algebra which deals with algebraic structures (fields, rings, groups, semigroups, etc.) represented by black boxes as defined in Section 2.1. Alternatively, black box algebra can be described as a study of categories where objects are algebraic structures with algebraic operations on them computable in probabilistic time polynomial in the input length, and morphisms are also computable in probabilistic polynomial time. The key aspect of black box algebra is that bijective morphisms can not be automatically assumed to be invertible.

1.2. Homomorphic encryption: basic definitions. Let $A$ and $X$ denote the sets of plaintexts and ciphertexts, respectively, and assume that we have some (say, binary) operators $\Box_A$ on $A$ needed for processing data and corresponding operators $\Box_X$ on $X$. An encryption function $E$ is homomorphic if

$$E(a_1 \; \Box_A \; a_2) = E(a_1) \; \Box_X \; E(a_2)$$

for all plaintexts $a_1$, $a_2$ and all operations on $A$.

Suppose Alice is the owner of data represented by plaintexts in $A$ which she would like to process using operators $\Box_A$, but has insufficient computation resources,
while Bob has computational facilities for processing ciphertexts using operators \( \otimes_X \). Alice may wish to enter into a contract with Bob; in a realistic scenario, Alice is one of the many customers of the encrypted data processing service run by Bob, and all customers use the same formats of data and operators which are for that reason are likely to be known to Bob. This is what is known in cryptology as Kerckhoff’s Principle: *obscurity is no security*, the security of encryption should not rely on details of the protocol being held secret; see [37] for historic details.

Alice encrypts plaintexts \( a_1 \) and \( a_2 \) and sends ciphertexts \( E(a_1) \) and \( E(a_2) \) to Bob.

Bob computes

\[
b = E(a_1) \otimes_X E(a_2)
\]

without having access to the content of plaintexts \( a_1 \) and \( a_2 \), then return the output \( b \) to Alice who decrypts it using the decryption function \( E^{-1} \):

\[
E^{-1}(b) = a_1 \otimes_A a_2
\]

In this set-up, we shall say the homomorphic encryption scheme is *based* on the algebraic structure \( A \) or that the homomorphism \( E \) is a *homomorphic encryption of* the algebraic structure \( A \).

To simplify exposition, we assume that the encryption function \( E \) is deterministic, that is, \( E \) establishes a one-to-one correspondence between \( A \) and \( X \). Of course, this is a strong assumption in the cryptographic context; it is largely unnecessary for our analysis, but, for the purposes of this paper, allows us to avoid technical details and makes it easier to explain links with the black box algebra.

1.3. **Back to algebra.** In algebraic terms, \( A \) and \( X \) as introduced above are algebraic structures (non-empty sets with operations on them which we refer to as *algebraic operations*) and

\[
E : A \rightarrow X
\]

is a homomorphism.

In this paper we assume that the algebraic structure \( A \) is finite. This is not really essential for our analysis, many observations are relevant for the infinite case as well, but handling probability distributions (that is, random elements) on infinite sets is beyond the scope of the present paper.

We discuss a class of potential attacks on homomorphic encryption which are based on a simple but fundamental fact of algebra: a map

\[
E : A \rightarrow B
\]

of algebraic structures of the same type is a homomorphism if and only if its graph

\[
\Gamma(E) = \{(a, E(a)) \mid a \in A\}
\]

is a substructure \( A \times B \), that is, closed under all algebraic operations on \( A \times B \).

Obviously, \( \Gamma(E) \) is isomorphic to \( A \). If \( A \) has rich internal configuration (has many substructures with complex interaction between them, automorphisms, etc.) so is \( \Gamma(E) \).

As it will be demonstrated in this paper using a few examples from black box algebra,
If an algebraic structure $A$ has a rich internal configuration, the graph $\Gamma(E)$ of a homomorphic encryption function $E : A \rightarrow X$ also has a rich (admittedly hidden) internal configuration, and this could make it vulnerable to a potential attack from Bob.

We suggest that,

before attempting to develop a homomorphic encryption scheme based on a particular algebraic structure $A$, the latter needs to be examined by black box theory methods -- as examples in this paper show, it could happen that all homomorphic encryption schemes on $A$ are insecure.

2. A black box attack on homomorphic encryption

As explained in Section 1, we assume that the algebraic structure $A$ of plaintexts is represented in some standard form known to Bob. In agreement with the standard language of algebra -- and with our terminology in [15] -- we shall use the words *plain element* or just *element* in place of ‘plaintext’ and *cryptoelement* in place of ‘ciphertext’.

We assume that Bob can accumulate a big dataset of cryptoelements sent from/to Alice, or intermediate results from running Alice’s programme, and that he can feed, without knowledge of Alice, cryptoelements into a computer system (the *black box*) which performs operations on them, and retain the outputs for peruse -- again without Alice’s knowledge. Bob’s aim is to compute the decryption function $E^{-1}$ in an efficient way, that is, in probabilistic time polynomial in terms of the lengths of plain elements and cryptoelements involved.

This setup can be described as a *black box algebraic structure* and is defined axiomatically in Section 2.1.

2.1. Axiomatic description of black box algebraic structures. A *black box algebraic structure* $X$ is a black box (or an oracle, or a device, or an algorithm) operating with 0–1 strings of uniform length which encrypt (not necessarily in a unique way) elements of some algebraic structure $A$.

In algorithms described in this paper, we have to build new black boxes (frequently for the same algebraic structure) from existing ones and work with several black box structures at once. For that reason we specify the functionality of a family $\mathcal{X}$ of black boxes $X$ by the following axioms.

**BB1** Each $X$ produces cryptoelements as strings of fixed length $l(X)$ (which depends on $X$) encrypting random (almost) uniformly distributed elements from some algebraic structure $A$; this is done in probabilistic time polynomial in $l(X)$.

**BB2** Each $X$ computes, in probabilistic time polynomial in $l(X)$, cryptoelements encrypting the outputs of algebraic operations applied to given cryptoelements.

**BB3** Each $X$ decides, in probabilistic time polynomial in $l(X)$, whether two cryptoelements encrypt the same element in $A$.

We shall say in this situation that a black box $X$ *encrypts* the algebraic structure $A$. 

So far, it appears that only finite groups, fields, rings, and, very recently, projective planes (in our paper [15]) got a black box treatment.

Black box algebraic structures had been introduced by Babai and Szemerédi [9] in the special case of groups as an idealized setting for randomized algorithms for solving permutation and matrix group problems in computational group theory. Our Axioms BB1–BB3 is a slight modification – and generalisation to arbitrary algebraic structures – of their original axioms. Black box groups have proved to be very useful in computational group theory.

In case of finite fields, the concept of a black box field can be traced back to Lenstra Jr [53], Boneh [13], and Boneh and Lipton [12], and in case of rings – to Arvind [4].

2.2. Construction of new black boxes. An example of a new black box algebraic structure built from old ones is a direct product $X \times Y$ of two black box structures $X$ and $Y$ encrypting algebraic structures $A$ and $B$, correspondingly; its cryptoelements are pairs $(x, y)$ with components being cryptoelements independently produced by $X$ and $Y$, correspondingly, and operations performed componentwise by the two black boxes.

A homomorphic image $Y$ of the black box $X$ under a homomorphism $\alpha : X \rightarrow Y$ can be seen as having the same strings as $X$ and the same algebraic operations but a redefined equality relation (a ‘congruence” in terminology of abstract algebra) $y_1 \equiv_Y y_2$ with the property that $y_1 \equiv_Y y_2 \iff \alpha(y_1) = \alpha(y_2)$;

this explains why Axiom BB3 is useful.

The reader will find in this paper a variety of other black box constructions.

2.3. Black boxes associated with homomorphic encryption. Returning to the homomorphic encryption setup, we see that $A$ can be seen as the set of plain elements, and $X$ as the set of cryptoelements, and the encryption function $E$ as an isomorphism $E : A \rightarrow X$.

Supply of random cryptoelements from $X$ postulated in Axiom BB1 can be achieved by sampling a big dataset of cryptoelements provided by Alice, or computed on request from Alice. The computer system controlled by Bob performs algebraic operations referred to in Axiom BB2.

Axiom BB3 is redundant under the assumption that $E : A \rightarrow X$ is a bijection, but gives us more freedom to construct new black boxes, for example, homomorphic images of $X$. It could also be useful for handling another quite possible scenario: For Alice, the cost of computing homomorphisms $E$ and $E^{-1}$ could be higher than the price charged by Bob for processing cryptoelements. In that case, it could be cheaper to transfer initial data to Bob (in encrypted form) and ask Bob to run a computer programme which uses the black box but does not send intermediate values back to Alice and returns to Alice only the final result. In that case, it becomes almost inevitable to have, among operation done by the black box, comparing the values of two cryptoelements – thus making Axiom BB3 necessary for the description of the black box.

2.4. Bob’s attack. As we have already explained, we can assume that Bob knows the algebraic structure $A$. 
Bob’s aim is to find an algorithm which works in probabilistic time polynomial in $l(X)$ and maps cryptoelements from $X$ to elements in $A$ and vice versa while preserving the algebraic operations on $X$ and $A$. In our terminology (see Section 4.1), this means solving the **constructive recognition problem** for $X$, that is, finding efficient isomorphisms

\[
\alpha : X \rightarrow A \\
\beta : A \rightarrow X
\]

such that $\alpha \circ \beta$ is the identity map on $A$.

Here and in the rest of the paper, “efficient” means “computable in probabilistic time polynomial in the input length”.

Assume that Bob solved the constructive recognition problem and can efficiently compute $\alpha$ and $\beta$.

Alice’s encryption function is a map $E : A \rightarrow X$; the composition $\delta = \alpha \circ E$ is an automorphism of $A$. Therefore Bob reads not Alice’s plaintexts $a \in A$, but their images $\delta(a) = \alpha(E(a))$ under an automorphism $\delta$ of $A$ still unknown to him.

This means that

**solving the constructive recognition problem for $X$ reduces the problem of inverting the encryption homomorphism $E : A \rightarrow X$ to a much simpler problem of inverting the automorphism $\delta : A \rightarrow A$.**

We are again in the situation of homomorphic encryption, but this time the sets of plaintexts and ciphertexts are the same. One would expect that this encryption is easier to break. For example, if Bob can guess the plaintexts of a few cryptoelements, and if the automorphism group Aut $A$ of $A$ is well understood, computation of $\delta$ and $\delta^{-1}$ could be a more accessible problem than the constructive recognition for $X$. For example, automorphism groups of finite fields are very small, and in that case $\delta^{-1}$ can be found by direct inspection.

As soon as $\delta^{-1}$ is known, Bob knows

\[
E^{-1} = \delta^{-1} \circ \alpha
\]

and can decrypt everything.

Moreover, the map $E$ is also known:

\[
E = \beta \circ \delta,
\]

and allows Bob to return to Alice cryptoelements which encrypt plaintexts of Bob’s choice.

We suggest that this approach to analysis of homomorphic encryption is useful because it opens up connections to black box algebra. Indeed the theory of black box structures is reasonably well developed for groups and fields, and its methods could provide insight into assessment of security of other algebraic structures if any are proposed for use in homomorphic encryption.

In Sections 3 and 5 (the latter is based on our papers [15, 21]), we give an example of how this perhaps could be done.
3. Black box fields

We define black box fields as a special case of black box algebraic structure, and invite the reader to compare our exposition with [13]. Notice that we slightly generalize the definition of a black box field given in [13, 56] by removing the assumption that the characteristic of the field is known.

A black box (finite) field $K$ is an oracle or an algorithm operating on 0-1 strings of uniform length (input length) which encrypts some finite field $F$. The oracle can compute $x + y$, $xy$, and $x^{-1}$ (the latter for $x \neq 0$ and decide whether $x = y$ for any strings $x, y \in K$. If the characteristic $p$ of $K$ is known then we say that $K$ is a black box field of known characteristic $p$. We refer the reader to [13, 56] for more details on black box fields of known characteristic and their applications to cryptography.

Maurer and Raub [56] proved that a construction of an isomorphism and its inverse between a black box field $K$ of known characteristic $p$ and an explicitly given field $F_p^n$ is reducible in polynomial time to the same problem for the prime subfield in $K$ and the field $F_p$.

Using our terminology, their proof can be reformulated to yield the following result.

**Theorem 3.1.** Let $K$ be a black box field of known characteristic $p$ encrypting an explicitly given finite field $F_p^n$ and $K_0$ the prime subfield of $K$. Then the problem of finding efficient two way isomorphisms between $K$ and $F_p^n$ can be efficiently reduced to the same problem for $K_0$ and $F_p$. In particular,

- an isomorphism
  
  $K_0 \rightarrow F_p$

  can be extended in time polynomial in the input length $l(K)$ to a probabilistic polynomial in $l(K)$ time isomorphism

  $K \rightarrow F_p^n$;

- there exists an isomorphism $F_p^n \rightarrow K$ computable in polynomial in $l(K)$ time.

If char $K = p$ and $p$ is known, we can easily find an isomorphism $F_p \rightarrow K_0$. Indeed, the standard representation of $F_p$ is

$$F_p = \mathbb{Z}/p\mathbb{Z}$$

and elements of $F_p$ are represented by integers

$$0, 1, 2, \ldots, p - 1,$$

coset representatives of $p\mathbb{Z}$ in the additive group $\mathbb{Z}$. Let $1$ be the unit in $K_0$, then the map

$$n \mapsto 1 + 1 + \cdots + 1 \ (n \text{ times})$$

is a desired isomorphism $F_p \rightarrow K_0$; it is computable in linear in $\log p$ time by the classical double-and-add method.

The existence of the reverse isomorphism $F_p \leftarrow K_0$ would follow from a solution of the discrete logarithm problem in $K_0$. In particular, this means that

for small primes $p$, for every black box field $K$ of order $p^n$

there is a probabilistic polynomial in $n \log p$ time isomorphism to an explicitly given field $F_p^n$. }
This means that, for small primes $p$, there are no secure homomorphic encryption schemes based on finite fields of characteristic $p$.

4. Black box groups

In various classes of black box problems for groups the isomorphism type of the encrypted group $A$ could be known in advance or unknown.

Some of the problems arising in relation to black box groups could be formulated in a wider context of arbitrary algebraic structures.

4.1. Black box problems for algebraic structures. Given a black box algebraic structure $X$, we usually deal with one of the following problems and wish to find solution in probabilistic polynomial in $l(X)$ time.

A. Verification problem. Is $X$ isomorphic to the structure $A$ given is some explicit form?

B. Identification problem. Determine, with the given degree of certainty, the isomorphism type of $X$, that is, find an algebraic structure $A$ given in some explicit form and such that $X$ encrypts a structure $B$ isomorphic to $A$.

C. Constructive recognition. Find isomorphisms

$$X \leftrightarrow A$$

between $X$ and an explicitly given algebraic structure $A$ computable in probabilistic time polynomial in $l(X)$.

4.2. Structure recovery. The following problem is specific for groups, it could be seen as a weaker form of constructive recognition for a wide class of simple groups.

D. Structure recovery. Suppose that a black box group $X$ encrypts a concrete and explicitly given group $G$ of Lie type (for example, a classical matrix group) over an explicitly given finite field $\mathbb{F}_q$. To achieve a structure recovery in $X$ means to construct, in probabilistic polynomial in $l(X)$ time,

- a black box field $K$ encrypting $\mathbb{F}_q$, and
- probabilistic polynomial time isomorphisms

$$\Psi : G(K) \rightarrow X$$

and

$$\Psi^{-1} : X \rightarrow G(K).$$

Structure recovery plays crucial role in algorithms developed in the proof of Theorems [5.1 and 5.2].

4.3. Monte–Carlo and Las Vegas. By the nature of our axioms, all algorithms for black box algebraic structures as randomized objects (and, in particular, all expected solutions for problems A – D above) are probabilistic Monte-Carlo algorithms.

Recall that a Monte-Carlo algorithm is a randomized algorithm which gives a correct output to a decision problem with probability strictly bigger than $\frac{1}{2}$. The probability of having incorrect output can be made arbitrarily small by running the algorithm sufficiently many times. A special case of Monte–Carlo algorithms is a Las Vegas algorithm which either outputs a correct answer or reports failure. A detailed comparison of Monte–Carlo and Las Vegas algorithms, both from practical and theoretical point, can be found in [6].
In the case of black box groups, many algorithms can be made Las Vegas under some additional information about the encrypted group.

4.4. **Examples of black box groups.** As we have already explained in Section 2.4, theory of black box groups, as better developed, could serve as a paradigm for study of security of homomorphic encryption. There are two older areas where black box groups appear:

4.4.1. *The Miller-Rabin primality test.* As explained in [14], the celebrated Miller-Rabin primality test [68] amounts to treating the multiplicative group \((\mathbb{Z}/n\mathbb{Z})^*\) as a black box \(X\) (with random elements produced by a random numbers generator) and testing whether \(X\) encrypts the cyclic group \(\mathbb{Z}_{n-1}\); if the answer is \(NO\), \(n\) is not a prime number.

The Miller-Rabin primality test is an archetypal black box verification problem.

4.4.2. *Matrix groups.* An important example of a black box group is provided by a group \(G\) generated in an ambient matrix group \(\text{GL}_n(\mathbb{F}_{p^k})\) by several matrices \(g_1, \ldots, g_l\) [49]. The Babai’s algorithm [5] or a much more efficient product replacement algorithm [30] produces a sample of (almost) independent elements from a distribution on \(G\) which is close to the uniform distribution – see a discussion and further development in [8, 38, 63, 64, 65]. We can, of course, multiply, invert, compare matrices. Therefore the computer routines for these operations together with the sampling of the product replacement algorithm run on the tuple of generators \((g_1, \ldots, g_l)\) can be viewed as a black box \(X\) encrypting the group \(G\). The group \(G\) could be unknown—in which case we are interested in its isomorphism type—or its isomorphism type could be known, as it happens in a variety of other black box problems.

So matrix groups provide a variety of problems of every class – from verification to constructive recognition.

Translated into the homomorphic encryption context, recognition of matrix groups becomes a problem of recovery of Alice’s secret homomorphism

\[ E : A \rightarrow \text{GL}_n(\mathbb{F}_{p^k}) \]

(a representation of \(A\), in terminology of representation theory of groups). From the cryptographic point of view, this problem is mostly of theoretical interest because the cost of computing the homomorphisms \(E\) and \(E^{-1}\) is likely to be incomparably higher than the cost of matrix operation in \(\text{GL}_n(\mathbb{F}_{p^k})\).

There is considerable literature on black box matrix groups, here is a list: [3, 7, 10, 11, 13, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 41, 42, 43, 44, 45, 46, 47, 50, 51, 52, 53, 54, 55, 57, 58, 59, 60, 61, 62, 66, 73, 74]. Most of these works are conditional on the assumption that the Discrete Logarithm Problem can be solved in the field \(\text{GL}_n(\mathbb{F}_{p^k})\); many of them also assume the availability of the so-called \(\text{SL}_2\)-Oracle which, given a subgroup \(L\) in \(\text{GL}_n(\mathbb{F}_{p^k})\) isomorphic to \(\text{SL}_2(\mathbb{F}_{p^l})\), constructs an isomorphism between \(L\) and \(\text{SL}_2(\mathbb{F}_{p^l})\).

Our approach to black box groups was developed for solving some harder problems in matrix groups recognition without use of the Discrete Logarithm and \(\text{SL}_2\)-Oracles [15, 21]. Moreover, our paper [15] eliminates the need for \(\text{SL}_2\)-Oracle in the study of black box matrix groups.
When working on our paper [15], we discovered that the homomorphic encryption context sheds new light on, and allows to clarify, the nature of the key concepts in the black box group theory—this is what we discuss in this paper.

5. TWO BLACK BOX GROUPS

Two procedures described in Theorems 5.1 and 5.2 below are reformulations of principal results of papers [21] and [15] in a homomorphic encryption setup. In our discussion, they demonstrate the depth of structural analysis involved and suggest that a similarly deep but revealing structural theory can be developed for other algebraic structures if they are sufficiently rich for use in homomorphic encryption. Also, it is worth noting that the procedures do not use any assumptions about the encryption homomorphism $E$, the analysis is purely algebraic.

5.1. $\text{SL}_3(\mathbb{F}_q)$.

**Theorem 5.1.** Assume that Alice and Bob run a homomorphic encryption protocol based on the special linear group $A = \text{SL}_3(\mathbb{F}_q)$ (that is, the group of $3 \times 3$ matrices over $\mathbb{F}_q$ with determinant 1), $q$ odd, with Bob doing computations with cryptoelements using a black box $X$. Assume that Bob knows $A$, including the representation of the field $\mathbb{F}_q$ used by Alice.

Then:

(a) As shown in [21], Bob can solve the structure recovery problem for the black box $X$ and construct, in probabilistic polynomial in $\log q$ time,

- a black box field $K$ encrypting $\mathbb{F}_q$, and
- probabilistic polynomial in $\log q$ time isomorphisms

$$\Psi : \text{SL}_3(K) \rightarrow X$$

and

$$\Psi^{-1} : X \rightarrow \text{SL}_3(K).$$

(b) If, in addition, Bob has an algorithm for probabilistic polynomial in $\log q$ time isomorphisms between a black box field $K$ and the explicitly given field $\mathbb{F}_q$ (for example, if the characteristic of $\mathbb{F}_q$ is small, see Theorem 3.1), he gets probabilistic polynomial in $\log q$ isomorphisms $X \leftrightarrow \text{SL}_3(\mathbb{F}_q)$.

(c) Since Alice’s group $A$ is also standard $\text{SL}_3(\mathbb{F}_q)$, then, under assumptions of (b), Bob gets an image of Alice’s data transformed by an automorphism

$$\delta : \text{SL}_3(\mathbb{F}_q) \rightarrow \text{SL}_3(\mathbb{F}_q).$$

(d) Automorphisms of the group $\text{SL}_3(\mathbb{F}_q)$ are well known: every automorphism of $\text{SL}_3(\mathbb{F}_q)$ is a product of a conjugation by a matrix from $\text{GL}_3(\mathbb{F}_q)$, the inverse-transpose automorphism and a field automorphism induced by an automorphism of the field $\mathbb{F}_q$. Therefore if Bob can run a few instances of known plaintexts attacks against Alice, he can compute the automorphism $\delta$ and after that read plaintexts of all Alice’s cryptoelements.

(e) Moreover, under assumptions of (b) and (d), Bob can compute the inverse of $\delta$ and pass to Alice, as answers to Alice’s requests, values of his choice.
5.2. A more sophisticated example: $\text{PGL}_2(\mathbb{F}_q)$.

**Theorem 5.2.** Assume that Alice and Bob run a homomorphic encryption protocol over the group $A = \text{PGL}_2(\mathbb{F}_q)$, $q$ odd, with Bob doing computations with cryptoelements using a black box $X$. Assume that Bob knows $A$, including the representation of the field $\mathbb{F}_q$ used by Alice.

Then:

(a) As shown in [15], Bob can solve the structure recovery problem for the black box $X$ and construct, in probabilistic polynomial in $\log q$ time,
- a black box field $K$ encrypting $\mathbb{F}_q$, and
- probabilistic polynomial time isomorphisms
  \[
  \Psi : \text{SO}_3(K) \longrightarrow X
  \]
  and
  \[
  \Psi^{-1} : X \longrightarrow \text{SO}_3(K).
  \]

(b) If, in addition, Bob has an algorithm for probabilistic polynomial in $\log q$ time, isomorphisms between a black box field $K$ and the explicitly given field $\mathbb{F}_q$ (for example, if the characteristic of $\mathbb{F}_q$ is small, see Theorem 3.1), he gets two ways probabilistic polynomial in $\log q$ isomorphisms $X \leftrightarrow \text{SO}_3(\mathbb{F}_q)$.

(b') The Alice’s group $A = \text{PGL}_2(\mathbb{F}_q)$ and the group $\text{SO}_3(\mathbb{F}_q)$ are isomorphic, but have different representations: $\text{PGL}_2(\mathbb{F}_q)$ by $2 \times 2$ projective matrices, $\text{SO}_3(\mathbb{F}_q)$ by $3 \times 3$ matrices over $\mathbb{F}_q$. But computing efficient deterministic isomorphisms $\text{SO}_3(\mathbb{F}_q) \leftrightarrow \text{PGL}_2(\mathbb{F}_q)$ is a simple problem of linear algebra.

(c) Since Alice’s group $A$ is an explicitly given $\text{PGL}_2(\mathbb{F}_q)$, then, under assumptions of (b), Bob gets an image of Alice’s data transformed by an automorphism
  \[
  \delta : \text{PGL}_2(\mathbb{F}_q) \longrightarrow \text{PGL}_2(\mathbb{F}_q).
  \]

(d) Automorphisms of the group $\text{PGL}_2(\mathbb{F}_q)$ are well known: every automorphism of $\text{PGL}_2(\mathbb{F}_q)$ is a product of a conjugation by a matrix from $\text{GL}_2(\mathbb{F}_q)$ and a field automorphism induced by an automorphism of the field $\mathbb{F}_q$. Therefore if Bob can run a few instances of known plaintexts attacks against Alice, he can compute the automorphism $\delta$ and after that read plaintexts of all Alice’s cryptoelements.

(e) Moreover, under assumptions of (b) and (d), Bob can compute the inverse of $\delta$ and pass to Alice, as answers to Alice’s requests, values of his choice.

5.3. **Black box projective planes.** An interesting stage in proofs of Theorems 5.1 and 5.2 is construction of a black box projective plane $P$, a structure with two sorts of elements, points and lines, and with two partial operations performed by a black box: computing the line $p \lor q$ incident to two distinct points $p \neq q$, and computing the point $m \land n$ incident to two distinct lines $m \neq n$. The black box field $K$ is obtained by coordinatization of the projective plane $P$, and the action of $X$ on $P$ by collineations (again, performed by a black box) produces, in appropriately chosen projective coordinates on $P$, the homomorphisms $X \longrightarrow \text{PGL}_3(K)$ which have images $\text{PSL}_3(K)$ and $\text{SO}_3(K)$, when $X$ encrypts $\text{SL}_3(\mathbb{F})$ and $\text{PGL}_2(\mathbb{F})$, respectively. The map $X \longrightarrow \text{PSL}_3(K)$ is then lifted to $X \longrightarrow \text{SL}_3(K)$.

The ways how the projective plane $P$ is constructed are very different in the two theorems; in particular, the projective plane of Theorem 5.2 has an additional
operation (performed by a black box): polarity, an involution which sends points to lines and lines to points while preserving the incidence relation. This polarity is invariant under the action by \(X\), which forces \(X\) to be an orthogonal group in dimension 3.

5.4. **Discussion.** Points (d) and (e) in Theorems 5.1 and 5.2 above look as serious vulnerabilities of homomorphic encryptions of \(\text{PGL}_2(\mathbb{F}_q)\) and \(\text{SL}_3(\mathbb{F}_q)\).

We come to conclusion that homomorphic encryption of groups \(\text{PGL}_2(\mathbb{F}_q)\) and \(\text{SL}_3(\mathbb{F}_q)\) is no more secure than homomorphic encryption of the field \(\mathbb{F}_q\). As a consequence of Theorem 5.1 homomorphic encryption of \(\text{PGL}_2(\mathbb{F}_q)\) and \(\text{SL}_3(\mathbb{F}_q)\) for \(q = p^k\) does not survive a known plaintext attack when the prime \(p > 2\) is small.

One of the reasons why the groups \(\text{PGL}_2(\mathbb{F}_q)\) and \(\text{SL}_3(\mathbb{F}_q)\) happened to be vulnerable is that they have rich internal structure which allows us to build new black boxes and carry our sophisticated algebraic constructions. We believe that the same applies to all classical matrix groups:

**Conjecture 5.3.** Theorems 5.1 and 5.2 remains true if we replace groups \(\text{SL}_3(\mathbb{F}_q)\) and \(\text{PGL}_2(\mathbb{F}_q)\) by any classical matrix group of dimension \(n\) over \(\mathbb{F}_q\) – with the only difference that algorithms which reduce an attack on encryption of a group to a similar attack on encryption of a field have complexity polynomial in \(n \log q\).

We have already achieved significant progress towards a proof of this conjecture. Theorems 5.1 and 5.2 serve as a basis of induction on \(n\).

This allows us to formulate a more general conclusion:

**Conjecture 5.4.** Homomorphic encryption of classical matrix groups over finite fields is not safer than homomorphic encryption of their underlying fields.

6. **Morphisms and protomorphisms**

In this section we explain some crucial ideas behind the proof of Theorems 5.1 and 5.2 in particular, we give examples where graphs of homomorphisms are treated as (black box) groups – we emphasised this point in Section 1.

6.1. **Morphisms.** For a black box group \(X\) encrypting a group \(G\), the *canonical projection* \(\pi_X : X \rightarrow G\) maps every cryptoelement in \(X\) to the corresponding element in \(G\). We are not making any assumptions about the feasibility of computing this map in practice.

Given two black boxes \(X\) and \(Y\) encrypting finite groups \(G\) and \(H\), respectively, we say that a map \(\zeta\) which assigns cryptoelements from \(X\) to cryptoelements from \(Y\) is a *morphism* of black box groups, if

- the map \(\zeta\) is computable in probabilistic time polynomial in \(l(X)\) and \(l(Y)\), and
- there is a homomorphism \(\phi : G \rightarrow H\) such that the following diagram is commutative:

\[
\begin{array}{ccc}
X & \xrightarrow{\zeta} & Y \\
\downarrow{\pi_X} & & \downarrow{\pi_Y} \\
G & \xrightarrow{\phi} & H
\end{array}
\]
where $\pi_X, \pi_Y$ are the canonical projections of $X$ and $Y$ onto $G$ and $H$, respectively.

We shall say in this situation that a morphism $\zeta$ encrypts the homomorphism $\phi$.

To give one example, morphisms arise naturally when a black box group $X$ is given by a generating set of cryptoelements and we replace a generating set for the black box group $X$ by a more convenient one and start sampling the product replacement algorithm for the new generating set; in fact, we replace a black box for $X$ and deal with a morphism $Y \to X$ from the new black box $Y$ into $X$.

We say that a morphism $\zeta$ is an embedding, or an epimorphism, etc., if $\phi$ has these properties. Dotted arrows are reserved for abstract (that is, unknown, or of unknown complexity) homomorphisms, including natural projections

$$X \xrightarrow{\pi_X} \pi(X);$$

the latter are not necessarily morphisms, since, by the very nature of black box problems, we are not given an efficient procedure for constructing the projection of a black box onto the (explicitly given) group it encrypts.

6.2. Black box subgroups. If we have an embedding of black box groups $Y \to X$, we shall say that $Y$ is a subgroup of $X$.

In our papers [15, 21], black box subgroups are constructed in one of the following ways:

- We generate $Y$ by some cryptoelements $y_1, \ldots, y_m \in X$ and use some version of the product replacement algorithm [30] for random sampling.
- Given black box subgroups $Y_1, \ldots, Y_k$ in $X$, we generate a subgroup
  $$Y = \langle Y_1, \ldots, Y_k \rangle$$
  by taking generating sets in $Y_i$ and combining them into a generating set in $Y$.
- $Y$ is the centralizer in $X$ of an involution or a proto-involution in the sense of Section 5.5 when we apply the procedure described in [14] to “populate” $Y$ and eventually find a generating set for $Y$.

6.3. Morphisms as black box groups. We recall again that a map

$$G \xrightarrow{\phi} H$$

from a group to a group is a homomorphism of groups if and only if its graph

$$F = \{(g, \phi(g)) : g \in G\}$$

is a subgroup of $G \times H$.

Recall construction of direct products of black boxes (Section 2.2): if $X$ encrypts $G$ and $Y$ encrypts $H$ then the black box $X \times Y$ produces pairs of cryptoelements $(x, y)$ by sampling $X$ and $Y$ independently, with operations carried out componentwise in $X$ and $Y$; of course, $X \times Y$ encrypts $G \times H$.

This allows us to treat a morphism

$$X \xrightarrow{\zeta} Y$$

of black box groups as a black box subgroup $Z \to X \times Y$ encrypting $F$:

$$Z = \{(x, \zeta(x)) : x \in X\}$$
with the natural projection
\[
\pi_Z : Z \to F,
\]
\[
(x, \zeta(x)) \mapsto (\pi_X(x), \phi(\pi_X(x))).
\]

In practice it means that we can find cryptoelements \(x_1, \ldots, x_k\) generating \(X\) with known images \(y_1 = \zeta(x_1), \ldots, y_k = \zeta(x_k)\) in \(Y\) and then use the product replacement algorithm to run a black box for the subgroup
\[
Z = \langle (x_1, y_1), \ldots, (x_k, y_k) \rangle \subseteq X \times Y
\]
which is of course exactly the graph \(\{(x, \zeta(x))\}\) of the homomorphism \(\zeta\). Random sampling of the black box \(Z\) returns cryptoelements \(x \in X\) with their images \(\zeta(x) \in Y\) already attached.

6.4. **Protomorphisms.** Let \(X\) and \(Y\) be two black box groups encrypting \(G\) and \(H\), respectively, and \(\pi\) the canonical projection of \(X \times Y\) onto \(G \times H\). A **protomorphism** \(Z\) between black box groups \(X\) and \(Y\) is a black box subgroup \(Z < X \times Y\) such that \(\pi(Z)\) is the graph of a homomorphism from \(G\) to \(H\) or from \(H\) to \(G\)—the direction of homomorphism is not set here. We say that \(Z\) **encrypts** this homomorphism.

We frequently construct new black boxes from the given ones, and in these constructions cryptoelements in \(X\) act as pointers to other black boxes. Therefore it is convenient to think of elements of black boxes as other black boxes—the same way as in the ZF set theory all objects are sets, with some sets being elements of others. A projective plane constructed in \[15\] provides a good example: it could be seen as consisting of points and lines, where a “line” is a black box that produces random “points” on this line and a “point” is a black box that produces random “lines” passing through this point.

In a black box group \(X\), it is frequently useful to associate with an element encrypted by a cryptoelement \(x \in X\) a black box for the graph of a specific homomorphism, namely, the conjugation by \(x\), viewed as a subgroup of the direct product \(X \times X\), the latter provided with group operations and equality relation in the obvious way:
\[
C_x = \{(y, y^x) : y \in X\}.
\]
Treating a homomorphism \(X \to Y\) of black box groups \(X\) and \(Y\) as a black box subgroup in their direct product \(X \times Y\) has happened to be an effective way of working with automorphisms of black box groups, as can be seen, for example, in “reification of involutions”, see Section \[6.5\] more detail can be found in \[15\].

6.5. **Amalgamation of local proto-involutions.** Let \(X\) be a black box group encrypting a group \(G\). Expanding the terminology from the previous section, a proto-involution \(F\) on \(X\) is a black box subgroup \(F < X \times X\) for the graph of an involutive automorphism of \(X\).

Assume that black box subgroups \(Y_1, \ldots, Y_k\) in \(X\) are encrypting, respectively, subgroups \(H_1, \ldots, H_k\) in \(G\), and assume that \(\langle H_1, \ldots, H_k \rangle = G\). Assume that \(\phi_1, \ldots, \phi_k\) are involutive automorphisms of subgroups \(H_1, \ldots, H_k\), respectively, and \(F_i\) are proto-involutions on \(Y_i\) encrypting \(\phi_i\), \(i = 1, \ldots, k\). We say that the system of proto-involutions \(F_1, \ldots, F_k\) is **consistent** if there exists an automorphism \(\phi\) of \(G\) such that \(\phi_i = \phi |_{H_i}\) for all \(i = 1, \ldots, k\).
Theorem 6.1 (Amalgamation of local proto-involutions). If $F_1, \ldots, F_k$ is a consistent system of proto-involutions on black box subgroups $Y_1, \ldots, Y_k$ of $X$, then

$$ F = \langle F_1, \ldots, F_k \rangle $$

is a proto-involution on $X$.

Proof. The proof is self-evident. \hfill \Box

We shall call $F$ the amalgam of proto-involutions $F_1, \ldots, F_k$.

6.6. Reification. Theorem 6.1 was systematically used in proofs of Theorems 5.1 and 5.2. It was very useful because proto-involutions can be converted into actual involutions, that is, cryptoelements in $X$:

Theorem 6.2 (Reification Theorem, [15]). Let $X$ be a black box groups encrypting a simple group $G$ of Lie type and odd characteristic. Assume that $F$ is a proto-involution on $X$ encrypting an involutive internal automorphism of $G$ induced by some (unknown to us) involution $t \in G$. Then we can construct, in probabilistic time polynomial in $l(X)$, an involution $t \in X$ encrypting $t$.

This is an example of the technique borrowed from the computer science and known as reification. According to a widely accepted description,

Reification is the process by which an abstract idea about a computer program is turned into an explicit data model or other object created in a programming language. A computable/addressable object – a resource – is created in a system as a proxy for a non computable/addressable object.

In Theorem 6.2 a proto-involution is a proxy object used for a construction of a specific object, a string in $X$ encrypting an involution with specified properties. Black box projective planes and fields mentioned in Section 5 are also proxies. The power of abstract algebra is in its ability to define, when necessary, new algebraic structures; the black box algebra efficiently constructs, when necessary, proxy structures represented by new black boxes which use given black boxes for computational engines.

6.7. Augmentation of a black box group by a proto-involution. Semidirect products of black box groups arise in a situation when we have two black box group $X$ and $Y$ and a polynomial time in $l(X)$ and $l(Y)$ procedure for the action of $Y$ on $X$ by automorphisms,

$$ X \times Y \rightarrow X $$

$$(x, y) \mapsto x^y. $$

Then the semidirect product $X \rtimes Y$ is defined, in the usual way, as the set of pairs $X \times Y$ with multiplication

$$(x_1, y_1) \circ (x_2, y_2) := \left( x_1 x_2^{-1} y_1 y_2 \right).$$

The following theorem is very simple and very useful; it provides an “external” version of reification of involutions, compare with Theorem 6.2.
Theorem 6.3 (Augmentation of a black box group by a proto-involution). If $F < X \times X$ is a proto-involution on $X$ representing an involutive automorphism $\phi$ on $G$, we can construct an involutive automorphism $\phi$ of $F$ by setting 

$$\phi: (x, x') \mapsto (x', x)$$

for $(x, x') \in F$.

Then the semidirect product $F \rtimes \{1, \phi\}$ is a black box encrypting $G \rtimes \langle \phi \rangle$.

Theorems 6.1 and 6.3 can be generalized, with appropriate modification of definitions, to automorphisms of arbitrary order. They demonstrate the power of the simple principle: in homomorphic encryption, the graph of an encrypted homomorphism might have a sophisticated internal structure which provides a window of opportunity for its cryptoanalysis.

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