Ellipsoids of maximal volume in convex bodies

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Abstract. The largest discs contained in a regular tetrahedron lie in its faces. The proof is closely related to the theorem of Fritz John characterising ellipsoids of maximal volume contained in convex bodies.

§0. Introduction.

In 1948, Fritz John proved that each convex body in $\mathbb{R}^n$ contains an unique ellipsoid of maximal volume. Thus, each convex body has an affine image whose ellipsoid of maximal volume is the Euclidean unit ball, $B_2^n$. John characterised these affine images with the following theorem.

Theorem [J]. The Euclidean ball is the ellipsoid of maximal volume contained in the convex body $C \subset \mathbb{R}^n$ if and only if $B_2^n \subset C$ and, for some $m \geq n$, there are Euclidean unit vectors $(u_i)^m$, on the boundary of $C$, and positive numbers $(c_i)_1^m$ for which

\begin{align*}
a) \quad & \sum_i c_i u_i = 0 \\
b) \quad & \sum_i c_i u_i \otimes u_i = I_n, \text{ the identity on } \mathbb{R}^n.
\end{align*}

The $u_i$’s of the theorem, are points of contact of the unit sphere $S^{n-1}$ with the boundary of $C$. The theorem says that weights may be distributed on the collection of such points so that, a) the centre of mass of the distribution is at the origin and b) the inertia tensor of the distribution is the identity. The first condition shows that the contact points do not all lie “on one side” of the sphere, and the second, that they do not all lie “close to proper subspace”.

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Condition b) shows that the $u_i$'s behave like an orthonormal basis in that the inner product can be represented

$$\langle x, y \rangle = \sum_i c_i \langle u_i, x \rangle \langle u_i, y \rangle, \quad x, y \in \mathbb{R}^n.$$ 

It follows immediately from the equality of the traces of the operators in b) that

$$\sum_i c_i = n. \quad (1)$$

At each $u_i$, the supporting hyperplane to $C$ (is unique and) is perpendicular to $u_i$ (since this is true for the Euclidean ball). Hence, the set $K = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1, \ 1 \leq i \leq m\}$ contains $C : C$ is sandwiched between $B_2^n$ and $K$. From this it follows (and this was one of the motivations for John's theorem) that $C$ is contained in $nB_2^n$, the Euclidean ball of radius $n$. To see this, suppose $x \in K$ and $\|x\| = r$. Since

$$-r \leq \langle x, u_i \rangle \leq 1 \text{ for every } i,$$

then

$$0 \leq \sum_i c_i (1 - \langle x, u_i \rangle) (r + \langle x, u_i \rangle)$$

$$= r \sum_i c_i + (1 - r) \sum_i c_i \langle x, u_i \rangle - \sum_i c_i \langle x, u_i \rangle^2.$$ 

Properties a) and b) and (1) show that the latter is

$$rn - \|x\|^2 = rn - r^2.$$ 

Hence $r \leq n$.

John’s theorem has been used many times in the theory of finite-dimensional normed spaces. For symmetric convex bodies (the unit balls of normed spaces) condition a) is redundant and a stronger estimate $\|x\| \leq \sqrt{n}$ can be obtained for $x \in K$. As a consequence, every $n$-dimensional normed space is isomorphic, with isomorphism constant at most $\sqrt{n}$, to $n$-dimensional Euclidean space.

It was pointed out to me a few months ago by Prof. A. Pełczyński that the literature does not contain any very explicit proof of the easy (if) part of John’s theorem. The first section of this article contains a quick proof of this assertion. At about the same time, several people (not quite independently) asked me for a proof of the intuitively obvious
fact that the largest discs contained in a regular tetrahedron, lie in its faces. The second section of this article consists of a proof of (the analogue of) this fact for \(k\)-dimensional Euclidean balls inside regular \(n\)-dimensional simplices. For each \(n\) and \(k < n\), the regular \(n\)-dimensional simplex which circumscribes \(B_2^n\), contains a \(k\)-dimensional Euclidean ball of radius

\[
\sqrt{\frac{n(n+1)}{k(k+1)}} = r(n, k) \text{ (say),}
\]

in each of its \(k\)-dimensional faces. The relationship between Sections 1 and 2 of this article is elucidated in Section 3 where it is shown that if \(C\) is a convex body in \(\mathbb{R}^n\) whose ellipsoid of maximal volume is \(B_2^n\), then \(C\) does not contain \(k\)-dimensional ellipsoids whose volume is larger than that of a \(k\)-dimensional ball of radius \(r(n, k)\).

The result of Section 3, and, a fortiori, that of Section 2, could certainly be "checked by hand". It is enough to show that the convex hull of \(B_2^n\) and a "large" \(k\)-dimensional ellipsoid, contains an \(n\)-dimensional ellipsoid of volume larger than \(B_2^n\). But the calculations involved are messy. The argument presented in Section 3 is a compelling, if simple, illustration of the usefulness of John’s characterisation.

§1. The 'if' part of John’s theorem.

**Proposition 1.** Let \((u_i)^m_1\) be a sequence of unit vectors in \(\mathbb{R}^n\) and \((c_i)^n_1\) a sequence of positive numbers satisfying

a) \(\sum_i c_i u_i = 0\) and  
b) \(\sum_i c_i u_i \otimes u_i = I_n\).

Then the set \(K = \{x \in \mathbb{R}^n: \langle x, u_i \rangle \leq 1, 1 \leq i \leq m\}\) contains an unique ellipsoid of maximal volume, the Euclidean unit ball.

**Proof.** Let \(E\) be the ellipsoid,

\[
\left\{ x \in \mathbb{R}^n: \sum_1^n \alpha_j^{-2}(x - y, v_j)^2 \leq 1 \right\}
\]

for some \(y \in \mathbb{R}^n\), orthonormal basis \((v_j)^n_1\) and positive numbers \((\alpha_j)^n_1\). The problem is to show that if \(E \subset K\), then \(\Pi \alpha_j \leq 1\) with equality only if \(\alpha_j = 1\) for all \(j\), and \(y = 0\).
Now, for each $i$, $1 \leq i \leq m$, the point

$$x_i = y + \left( \sum_{j=1}^{n} \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{-\frac{1}{2}} \sum_{j=1}^{n} \alpha_j^2 \langle u_i, v_j \rangle v_j$$

belongs to $E$ and so $\langle u_i, x_i \rangle \leq 1$ for each $i$. Hence

$$\langle u_i, y \rangle + \left( \sum_{j=1}^{n} \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq 1$$

for each $i$. Multiply by $c_i$, sum over $i$ and use the fact that $\sum_i c_i u_i = 0$ to get

$$\sum_i c_i \left( \sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq \sum_i c_i = n.$$ 

Since $\sum_i c_i \langle u_i, x \rangle^2 = \|x\|^2$ for all $x$, and the $v_j$'s form an orthonormal basis,

$$\sum_j \alpha_j = \sum_j \sum_i \alpha_j c_i \langle u_i, v_j \rangle^2$$

$$= \sum_i c_i \left( \sum_j \alpha_j \langle u_i, v_j \rangle^2 \right)$$

$$\leq \sum_i c_i \left( \sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \left( \sum_j \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}}$$

$$= \sum_i c_i \left( \sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq n.$$ 

By the $AM\backslash GM$ inequality $\Pi \alpha_j \leq 1$. There is equality only if $\alpha_j = 1$ for all $j$ in which case (2) says that

$$\langle u_i, y \rangle + \|u_i\| \leq 1 \quad \text{for all } i,$$

i.e. $\langle u_i, y \rangle \leq 0$ for all $i$. Since $\sum c_i \langle u_i, y \rangle = 0$, this implies that $\langle u_i, y \rangle = 0$ for all $i$ and so $y = 0$. \qed
§2. The simplex.

**Proposition 2.** Let $T$ be a regular solid simplex in $\mathbb{R}^n$ of internal radius 1. For $1 \leq k \leq n - 1$, the largest $k$-dimensional Euclidean balls in $T$ are those of radius

$$\sqrt{\frac{n(n+1)}{k(k+1)}}$$

which lie in $k$-dimensional faces of $T$.

The proof of Proposition 2 makes use of the following well-known application of Caratheodory’s theorem.

**Lemma 3.** If $(x_i)_{i=1}^m$ is a sequence in $\mathbb{R}^k$ of diameter at most $d$, there is a point $x \in \mathbb{R}^k$ with

$$\|x_i - x\| \leq r = d \sqrt{\frac{k}{2(k+1)}}$$

for all $i$. The bound is sharp only if $(x_i)_{i=1}^m$ includes some $k+1$ points, all at a distance $r$ from their average.

**Proof.** Let $x$ be the point of $\mathbb{R}^k$ which minimises $\max_i \|x_i - x\|$ and suppose that this maximum is $s$. Than $x$ is in the convex hull of those $x_i$’s from which it has distance $s$ since otherwise there would be a small perturbation, $y$, of $x$ with $\max_i \|x_i - y\| < s$. By Caratheodory’s theorem, $x$ is a convex combination of some $k+1$ of the $x_i$’s at distance $s$ from $x$: say

$$x = \sum_{i=1}^{k+1} \lambda_i x_i.$$ 

Then
\[ s^2 = \sum_{i=1}^{k+1} \lambda_i \|x_i - x\|^2 \]
\[ = \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j \|x_i - x_j\|^2 \]
\[ = \frac{1}{2} \sum_{i \neq j} \lambda_i \lambda_j \|x_i - x_j\|^2 \]
\[ \leq \frac{1}{2} d^2 \sum_{i \neq j} \lambda_i \lambda_j = \frac{1}{2} d^2 \left( (\sum \lambda_i)^2 - \sum \lambda_i^2 \right) \]
\[ \leq \frac{d^2 k}{2(k+1)} \]

since \( \sum \lambda_i^2 \geq \frac{1}{k+1} \left( \sum \lambda_i \right)^2 = \frac{1}{k+1} \) by the Cauchy-Schwartz inequality.

For the estimate to be sharp, one needs that \( \lambda_i = \frac{1}{k+1}, \ 1 \leq i \leq k + 1 \), implying the second assertion of the lemma.

\[ \square \]

**Proof of Proposition 2.** Assume that \( T \) is given by

\[ T = \{ x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1, \ 1 \leq i \leq n + 1 \} \]

for an appropriate sequence \( (u_i)_{1}^{n+1} \) of unit vectors and note that

\[ \sum_i u_i = 0 \quad (3) \]
\[ \sum_i u_i \otimes u_i = \frac{n+1}{n} I_n. \quad (4) \]

Let \( \mathcal{E} = \{ x \in \mathbb{R}^n : \sum_{j=1}^{k} \langle x - y, v_j \rangle^2 \leq r^2, \langle x - y, v_j \rangle = 0, k + 1 \leq j \leq n \} \) be a \( k \)-dimensional ball contained in \( T \), for some \( y \in \mathbb{R}^n, r > 0 \) and orthonormal basis \( (v_j)_{1}^{n} \). As in the proof of Proposition 1,

\[ \langle u_i, y \rangle + r \left( \sum_{j=1}^{k} \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq 1 \text{ for all } i. \]
Let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto $\text{span}(v_j)_1^k$. Then for all $i$

$$\langle u_i, y \rangle + r\|P u_i\| \leq 1. \quad (5)$$

Summing over $i$ and using (3),

$$r \sum_i \|P u_i\| \leq n + 1$$

so it is enough to show that

$$\sum_i \|P u_i\| \geq \sqrt{\frac{k(k+1)(n+1)}{n}} \quad (6)$$

for every orthogonal projection of rank $k$.

Now, the set $\{u_i\}_{1}^{n+1}$ has diameter

$$\sqrt{\frac{2(n+1)}{n}}$$

and hence the set $\{P u_i\}_{1}^{n+1}$ has diameter at most this. Since $\{P u_i\}_{1}^{n+1}$ sits in an Euclidean space of dimension $k$, Lemma 3 shows that there is a point $x \in P(\mathbb{R}^n)$ with

$$\|P u_i - x\| \leq \sqrt{\frac{k(n+1)}{(k+1)n}} \quad \text{for all } i. \quad (7)$$

From identity (4),

$$\sum_i P u_i \otimes P u_i = \frac{n+1}{n} P$$

and equating traces,

$$\sum_i \|P u_i\|^2 = \frac{n+1}{n} k.$$

Since $\sum_i P u_i = 0$, 

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\[
\frac{n+1}{n} k = \sum_i \langle Pu_i, Pu_i \rangle \\
= \sum_i \langle Pu_i, Pu_i - x \rangle \\
\leq \sum_i \| Pu_i \| \cdot \| Pu_i - x \| \\
\leq \sqrt{\frac{k(n+1)}{(k+1)n} \sum_i \| Pu_i \|},
\]
giving the desired inequality (6).

Now, suppose the maximum radius is attained. Then there is equality in (7) so, by Lemma 3 again, the set \{Pu_i\} includes \(k + 1\) points all at distance \(\sqrt{\frac{k(n+1)}{(k+1)n}}\) from their average. But every \(k + 1\) of the \(u_i\)'s are at this distance from their average. So \(P\) is an isometry on the affine hull of some \(k + 1\), \(u_i\)'s: i.e. the range of \(P\) is parallel to this affine hull. This implies that \(E\) lies in a \(k\)-dimensional subspace parallel to some \(k\)-dimensional face of \(T\). This fact determines all the numbers \(\| Pu_i \|\) and hence the numbers \(\langle u_i, y \rangle\) since there is equality in (5) for all \(i\). These numbers determine \(y\). \(\square\)

§3. The general case.

**Proposition 4.** Let \((u_i)_1^m\), \((c_i)_1^m\) and \(K\) be as in Proposition 1. If \(E\) is a \(k\)-dimensional ellipsoid in \(K\) then the \((k\)-dimensional) volume of \(E\) is no larger than that of a \(k\)-dimensional ball of radius \(\sqrt{\frac{n(n+1)}{k(k+1)}}\).

This proposition cannot be proved using the argument of Proposition 2 as it stands, since for a general sequence \((u_i)_1^m\) satisfying the hypotheses and orthogonal projection \(P\) of rank \(k\), \(\sum_i c_i \| Pu_i \|\) may be as small as \(k\). This complicates the argument somewhat: Proposition 2 is isolated because it has a simpler proof. For the proof of Proposition 4, Lemma 3 is replaced by an easier observation.

**Lemma 5.** If \((x_i)_1^m\) is a sequence of vectors with \(\sum x_i = 0\) and \((u_i)_1^m\), a sequence of unit vectors in some Euclidean space then
\[
\left( \sum_i \langle x_i, u_i \rangle \right)^2 \leq \sum_{i,j} \| x_i \| \cdot \| x_j \| (1 - \langle u_i, u_j \rangle).
\]
Proof. By homogeneity, it may be assumed that $\sum ||x_i|| = 1$. Set $\lambda_i = ||x_i||$, $1 \leq i \leq m$ and $u = \sum \lambda_i u_i$. Then,

$$\left( \sum_i \langle x_i, u_i \rangle \right)^2 = \left( \sum \langle x_i, u_i - u \rangle \right)^2 \leq \left( \sum ||x_i|| \cdot ||u_i - u|| \right)^2 = \left( \sum \lambda_i ||u_i - u|| \right)^2 \leq \sum \lambda_i ||u_i - u||^2 = 1 - ||u||^2 \leq \sum_{i,j} \lambda_i \lambda_j (1 - \langle u_i, u_j \rangle). \square$$

Proof of Proposition 4. Let $\mathcal{E}$ be the ellipsoid

$$\left\{ x \in \mathbb{R}^n : \sum_{k}^{k} \alpha_j^{-2} (x - y, v_j)^2 \leq 1, \quad \langle x - y, v_j \rangle = 0, \quad k + 1 \leq j \leq n \right\}$$

for some $y \in \mathbb{R}^n$, orthonormal basis $(v_j)_j^{n}$ and positive numbers $(\alpha_j)_j^{k}$. The problem is to show that

$$\left( \prod_{1}^{k} \alpha_j \right)^{\frac{1}{k}} \leq \sqrt{\frac{n(n + 1)}{k(k + 1)}}.$$

It certainly suffices to show that

$$\sum_{1}^{k} \alpha_j \leq \sqrt{k n(n + 1)} \frac{1}{k + 1}.$$

As in the proof of Proposition 1,

$$\langle u_i, y \rangle + \left( \sum_{j=1}^{k} \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq 1$$

for every $i$. Define $T: \mathbb{R}^n \to \mathbb{R}^n$ by
\[ Tx = \sum_{j=1}^{k} \alpha_j \langle x, v_j \rangle v_j. \]

Then

\[ \langle u_i, y \rangle + \|Tu_i\| \leq 1 \text{ for every } i. \] (8)

Also

\[ \sum_{1}^{k} \alpha_j^2 = \sum_{i,j} \alpha_j^2 c_i \langle u_i, v_j \rangle^2 = \sum_{i} c_i \|Tu_i\|^2 \] (9)

and

\[ \sum_{1}^{k} \alpha_j = \sum_{i,j} \alpha_j c_i \langle u_i, v_j \rangle^2 = \sum_{i} c_i \langle Tu_i, u_i \rangle. \] (10)

The proof divides into two parts, the first of which effectively handles the \( u_i \)'s which are far apart (as if the body were symmetric) while the second, and more complicated part, handles the \( u_i \)'s which are close together (as if the body were a simplex).

Since \( \|Tu_i\| \leq 1 - \langle u_i, y \rangle \) for each \( i \) (by (8))

\[ \sum_{i} c_i \|Tu_i\|^2 \leq \sum_{i} c_i (1 - \langle u_i, y \rangle)^2 \]

\[ = \sum_{i} c_i - \left( \sum_{i} c_i u_i, y \right) + \sum_{i} c_i \langle u_i, y \rangle^2 \]

\[ = n + \|y\|^2. \]

So by (9)

\[ \frac{1}{k} \left( \sum_{i} c_i \alpha_j \right)^2 \leq \sum_{j} \alpha_j^2 \leq n + \|y\|^2. \] (11)

On the other hand, set \( x_i = c_i Tu_i, 1 \leq i \leq m \) and observe that \( \sum x_i = T \left( \sum c_i u_i \right) = 0. \)

Then Lemma 5 and (10) show that
\[
\left( \sum_{j=1}^{k} \alpha_j \right)^2 = \left( \sum_i \langle x_i, u_i \rangle \right)^2 \\
\leq \sum_{i,j} \|x_i\| \cdot \|x_j\| (1 - \langle u_i, u_j \rangle) \\
= \sum_{i,j} c_i c_j (1 - \langle u_i, u_j \rangle) \|Tu_i\| \cdot \|Tu_j\|.
\]

Since \(c_i c_j (1 - \langle u_i, u_j \rangle) \geq 0\) for all \(i\) and \(j\), (8) can be applied again to give

\[
\left( \sum \alpha_j \right)^2 \leq \sum_{i,j} c_i c_j (1 - \langle u_i, u_j \rangle)(1 - \langle u_i, y \rangle)(1 - \langle u_j, y \rangle).
\]

Expanding this product and using the fact that \(\sum_i c_i u_i = 0\) one obtains

\[
\left( \sum \alpha_j \right)^2 \leq \left( \sum c_i \right)^2 - \sum_{i,j} c_i c_j \langle u_i, u_j \rangle \langle u_i, y \rangle \langle u_j, y \rangle.
\]

Two applications of the identity \(\sum_i c_i \langle u_i, x \rangle \langle u_i, y \rangle = \langle x, y \rangle\) show that

\[
\left( \sum \alpha_j \right)^2 \leq n^2 - \|y\|^2. \tag{12}
\]

Finally, this inequality may be added to (11) to give

\[
\left( 1 + \frac{1}{k} \right) \left( \sum \alpha_j \right)^2 \leq n^2 + n,
\]

and hence

\[
\left( \sum \alpha_j \right)^2 \leq \frac{kn(n+1)}{k+1}
\]
as required. \(\square\)

**Remarks.** For \(k = 1\), Proposition 4 states that if \(C\) is a convex body whose ellipsoid of maximal volume is \(B_n^2\) then \(\text{diam}(C) \leq \sqrt{2n(n+1)}\). This fact can be proved more simply: if \(x, y \in C\) then \(\|x\|, \|y\| \leq n\), as explained in the introduction, and (with the usual notation)
\[ 0 \leq \sum_i c_i (1 - \langle u_i, x \rangle)(1 - \langle u_i, y \rangle) \]
\[ = n + \langle x, y \rangle \]
so that

\[ ||x - y||^2 = ||x||^2 + ||y||^2 - 2\langle x, y \rangle \]
\[ \leq 2n^2 + 2n = 2n(n + 1). \]

The fact that \( ||x|| \leq n \) for \( x \in C \) could be deduced from inequality (12) of the above proof, with “\( k = 0 \)”. 

References

[J] F. John, Extremum problems with inequalities as subsidiary conditions, Courant Anniversary Volume, Interscience, New York, 1948, 187-204.