COMPOSITE AND INVERSE OF MULTIVARIATE FUNCTIONS AND ALGEBRAIC SYSTEM OF EQUATIONS

Wu Zi qian

Xi’an Shiyou University. Email: woodschain@sohu.com. Address: Xi’an city, China.

Abstract

In this paper we extended composite and inverse to functions of many variables. As an application we gave formula solutions expressed by multivariate functions to general transcendental equations. We built algebraic system of equations. The equations contained in it will be algebraic equations if the elements of its basic set are numbers and the equations will be operator equations if the elements of its basic set are functions. Actually we will have a united concept for algebraic equations and operator equations. We discussed the probability to express these solutions in binary functions or unary functions.

MSC (2010): Primary 65H99; Secondary 33F10, 46S99, 46T99, 47J99, 47S99, 26B40.

Keywords: transcendental equations, symbolic computation, operator theory, Hilbert 13th problem, functional analysis, representation and superposition of functions.

1. Introduction

We will not give any reference about the composite and inverse of multivariate functions as the concepts for unary functions are very familiar for all of us and the concepts for multivariate functions are never stated before. One think the problem about general transcendental equations is closed after Abel’s having given the conclusion that there is no solution by radicals for a general polynomial algebraic equation if \( n \geq 5 \). This is a big mistake. Please reference 'Discrete Algebraic Equations and Discrete Operator
Equations’[6] or ‘Representation and Superposition of Discrete Functions and Equations with Parameterized Operations’ [7] for the history about solution by radicals for a general polynomial algebraic equation. It has been a short communication in ICM2010, Hyderabad[8]. This paper involves the Hilbert 13th problem. We supply references [1][2][3][4][5] about it.

2. Composite and Inverse of multivariate functions

It is import to know how many variables in a function for us because we will use these functions so frequently in our paper. Superscript with wavy line will be used to indicate the number of variables for a function and it will be omitted for a binary function.

For clearer expression we give new symbolics for several known operations, \(\varphi_a\) for addition, \(\varphi_m\) for multiplication, \(\varphi_d\) for division, \(\varphi_p\) for power, \(\varphi_r\) for root and \(\varphi_l\) for logarithm.

2.1 composite for multivariate functions

We will define composite between functions with same number of variables, two functions of n variables, \(\varphi^n_1\) and \(\varphi^n_2\). We should take a function of m \((m < n)\) variables \(\varphi^m\) as a special one of n variables \(\varphi^n\). Each variable of \(\varphi^m\) must be indicated to be which variable of \(\varphi^n\) because \(m < n\). Such as different special functions of 4 variables \(\varphi^4_1 = \varphi_p(x_1, x_4) = x_1^{x_4}\) and \(\varphi^4_2 = \varphi_p(x_4, x_2) = x_4^{x_2}\) are obtained by taking the same variable of \(\varphi_p\) as the different variable of \(\varphi^4\) so we introduce the concept of variables identify.

**Definition 2.1 Variables identify** Let \(\varphi^m(v_1, \ldots, v_j, \ldots, v_m)\) is a function of m variables, \(\varphi^n\) will be changed to a function of n \((m < n)\) variables \(\varphi^n\) by taking j-th variable of \(\varphi^m\) as the \(i_j\)-th variable of \(\varphi^n\) like:

\[
w^n(u_1, u_2, \cdots, u_n) = [A^n_{i_1, \ldots, i_j, \ldots, i_m}(w^m)](u_1, u_2, \cdots, u_n) = w^n(i_1, \cdots, i_j, \ldots, i_m)
\]

There are \(n \times (n - 1) \times \cdots \times (n - m + 1)\) combinations for a function of m variables \(\varphi^m\) that will be changed to a function of n variables \(\varphi^n\).

For example:
\[
\begin{align*}
w^4_1(x_1, x_2, x_3, x_4) &= x_1 + x_2 \quad \leftrightarrow \quad w^n_1 = A^n_{1, 2}(\varphi_a) \\
w^4_2(x_1, x_2, x_3, x_4) &= x_3 + x_4 \quad \leftrightarrow \quad w^n_2 = A^n_{3, 4}(\varphi_a) \\
w^4_3(x_1, x_2, x_3, x_4) &= x_1 + x_4 \quad \leftrightarrow \quad w^n_3 = A^n_{1, 4}(\varphi_a)
\end{align*}
\]

The same binary function \(\varphi_a\) is changed to different functions of 4 variables, \(w_1^4, w_2^4, w_3^4\) and \(w_3^4\) by different \(A^n_{1, 2}, A^n_{3, 4}\) and \(A^n_{1, 4}\) respectively.
Definition 2.2 composite for multivariate functions composite between two functions of n variables $\varphi_1^n$ and $\varphi_2^n$ is defined like:

$$\varphi_3^n = \varphi_1^n(x_1, \ldots, x_{i-1}, \varphi_2^n(x_1, \ldots, x_n), x_{i+1}, \ldots, x_n) \quad (2)$$

We introduce operator $C^n_i$ to express the relation between $\varphi_3^n$ and $\varphi_1^n, \varphi_2^n$. The subscript $i$ of $C^n_i$ means replacing $i$-th variable $x_i$ of $\varphi_1^n$ by $\varphi_2^n$.

$$\varphi_3^n = \varphi_1^n C^n_i \varphi_2^n \quad (3)$$

Note:
1 $\varphi_1^n$ or $\varphi_2^n$ may be functions with symbolic of variables like $x_1 + x_4$ or functions with no symbolic of variables like $A_1^{1,2}(\varphi_0)$.
2 $\varphi_2^n$ may be special function of n variables lacking several variables. For example, $(x_0^2 + x_3^2)C_0^4(x_1^2 + x_3^2) = (x_2^2 + x_3^2)^2 + x_2^3$, there is no $x_1, x_3$ in $\varphi_1^n$ and there is no $x_0, x_2$ in $\varphi_2^n$.
3 Replacement will not happen if $\varphi_1^n$ lacks the i-th variable which is defined by the subscript of $C^n_i$. In this situation the result of composite will be $\varphi_1^n$, such as $aC_1^n(x_1^2 + x_4^2) = a$.
4 The result of composite will be a function of n-1 variables if $\varphi_2^n$ is a constant like $(x_0 + x_1^2 + x_2^3)C_2^3a = x_0 + a^2 + x_2^3$.

Special situation:
if $n \geq 2, \varphi_2^n = x_j, i \neq j$, then each $x_i$ in $\varphi_1^n$ will be replaced by $x_j$ then we obtain a function of n-1 variables.

$$\psi^{n-1} = \varphi_1^n C^n_i x_j = \varphi_1^n C^n_i [A^n_j(e)]$$

Here $e$ is the identity function. $\psi^{n-1}$ is also called oblique projection of $\varphi_1^n$ about i-th variable and j-th variable. The oblique projection of $\varphi_1^n$ is depend on only $\varphi_1^n, x_i$ and $x_j$ so we give it another expression:

$$\psi^{n-1} = C^n_{i,j}(\varphi_1^n) \quad (4)$$

For example: $x^x = f^1(x) = [C^2_{1,2}(\varphi_p)](x)$

By the oblique projection $C^n_{1,2}$ we change the binary function $\varphi_p$ to a unary function $C^n_{1,2}(\varphi_p)$ and we reduce the number of ‘x’ from two to one and this is the thing we want to do when we solve equations.

The double branches expression, $w^4(u_1, u_2, u_3, u_4) = \varphi_3[\varphi_1(u_1, u_2), \varphi_2(u_3, u_4)] = (u_1 \varphi_1 u_2) \varphi_3(u_3 \varphi_2 u_4)$ is obtained by replacing two variables of $\varphi_3$ by $\varphi_1$ and $\varphi_2$ respectively. Thus $w^4$ can be written in $([A^4_{1,3}(\varphi_3)]C^4_{1,2}(\varphi_1))C^4_{3,4}(\varphi_2)$. 

3
It is also called the structural expression of \( w^4 \) because this expression contains all structural elements of \( w^4 \). Note, subscripts of \( A_{1,3}^4 \) is not 1,2 but 1,3. The expression will mean \( \varphi_3\{\varphi_1[x,\varphi_2(x,b)],\varphi_2(x,b)\} \) if we replace \( A_{1,3}^4 \) by \( A_{1,2}^4 \).

2.1 The inverse of function of many variables

**Definition 2.3** The inverse of function of many variables

Let \( \varphi^i(x_1, x_2, \cdots, x_i, \cdots, x_n) \) is a function of \( n \) variables. \( \xi^i_1: x_i \mapsto \varphi^i(x_1, x_2, \cdots, x_i, \cdots, x_n) \). \( \varphi^i \) is a invertible function of many variables about \( x_i \) if \( \xi^i_1 \) is bijection for any \( x_j (j = 1,2,\cdots,n, j \neq i) \).

For example, \( f(x_1, x_2) = x_1^3 + x_2^2 \) is invertible about variable \( x_1 \) and is not invertible about variable \( x_2 \).

we define \((\varphi^i)^{-i}\), the i-th inverse of \( \varphi^i \) as:

\[
x_i = (\varphi^i)^{-i}(x_1, \cdots, x_{i-1}, x_0, x_{i+1}, \cdots, x_m)
\]  

(5)

To express the relation between \( \varphi^i \) and \((\varphi^i)^{-i}\) we introduce inverse operator \( I_i \):

\[
(\varphi^i)^{-i} = I_i(\varphi^i)
\]  

(6)

**Definition 2.4** The piecewise invertible function of many variables

\[ x_0 = \varphi^i(x_1, \cdots, x_i, \cdots, x_n), \]

A non-reversible function of \( n \) variables \( \varphi^i \) defined on domain \( D \) is called a piecewise invertible function of many variables if we can divide \( D \) into several sub-domains properly, \( d_1 \cup d_2, \cdots, d_j, \cdots, \bigcup \cup d_m = D, d_1 \bigcap d_2, \cdots, d_j, \cdots, \bigcap \bigcap d_m = \emptyset \) and \( \varphi^i \) is invertible in each \( d_j, j = 1,\cdots,m \).

We consider each piece of \( \varphi^i \) being invertible on \( d_j \) as a function of \( n \) variables, \( \varphi^i_j \).

3. Application of the composite and Inverse for multivariate functions

Let us solve an equation which contains parameterized functions:

Replace the parameters \( u_1, u_2, u_3, u_4 \) in the expression \( \left([A^4_{1,3}(\varphi_3)]C^4_1[A^4_{1,2}(\varphi_1)]\right)(u_1, u_2, u_3, u_4) \) mentioned above by \( x,a,x,b \) respectively then we obtain equation:

\[
\left([A^4_{1,3}(\varphi_3)]C^4_1[A^4_{1,2}(\varphi_1)]\right)(x,a,x,b) = c
\]  

(6)
By the oblique projection of \( \{ [A_{1,3}^4(\varphi_3)]C_{1}^4[A_{1,2}^4(\varphi_1)] \}C_{3}^4[A_{3,4}^4(\varphi_2)] \) we have:
\[
[C_{1,3}^4 \left( \{ [A_{1,3}^4(\varphi_3)]C_{1}^4[A_{1,2}^4(\varphi_1)] \}C_{3}^4[A_{3,4}^4(\varphi_2)] \} \right) (x, a, b) = c
\]
Here we amuse \( C_{1,3}^4 \left( \{ [A_{1,3}^4(\varphi_3)]C_{1}^4[A_{1,2}^4(\varphi_1)] \}C_{3}^4[A_{3,4}^4(\varphi_2)] \) is invertible or piecewise invertible. By the inverse of \( C_{1,3}^4 \left( \{ [A_{1,3}^4(\varphi_3)]C_{1}^4[A_{1,2}^4(\varphi_1)] \}C_{3}^4[A_{3,4}^4(\varphi_2)] \) we obtain the solution finally:
\[
x = \left\{ I_{1} \left[ C_{1,3}^4 \left( \{ [A_{1,3}^4(\varphi_3)]C_{1}^4[A_{1,2}^4(\varphi_1)] \}C_{3}^4[A_{3,4}^4(\varphi_2)] \} \right] \right\} (c, a, b)
\]

4. Algebraic System of Equations

An algebraic system of equations (B,E) is constructed by a non-empty set B and a set E of equations constructed by the binary functions defined on B.

If elements of B are numbers then equations in (B,E) are algebraic equations. If elements of B are functions then equations in (B,E) are operator equations. B can be set of complex numbers or of real numbers or of integers. Algebraic systems of equations on set of complex numbers and on set of real numbers will be our main target. If B is a finite set then binary functions defined on it will be discrete ones.

5. Superposition Theorem

**Theorem 3.4 Kolmogorov’s Superposition Theorem** Let \( f^\hat{n}: [0, 1]^n \rightarrow \mathbb{R} \) be an arbitrary multivariate continuous function. Then it has the representation.
\[
f^\hat{n}(x_1, x_2, \cdots, x_n) = \sum_{q=0}^{2^n} \Phi^\hat{1}_q \left[ \sum_{p=1}^{n} \psi^\hat{1}_{q,p}(x_p) \right]
\]  

Here a set of inner functions \( \psi^\hat{1}_{q,p} \) is not unique but any selected set will be independent to \( f^\hat{n} \) then they are called remaining function.

Most of the results about Hilbert 13th problem are existence but there are some constructive ones. We selected one set of \( \psi^\hat{1}_{q,p} \) from Jürgen Braun and Michael Griebe[3] and then the relation between \( \varphi^\hat{1}_q \) and \( f^\hat{n} \) will be given like below:

**Definition 5.1 Decomposite operator** For expressing the relation between \( \varphi^\hat{1}_q \) and \( f^\hat{n} \), we introduce decomposite operator \( D_q \)
\[ \varphi_q^1 = D_q(f^n), \quad (0 \leq q \leq 2n) \quad (8) \]

It can be expressed too:

\[ f^n(x_1, x_2, \cdots, x_n) = \sum_{q=0}^{2n} D_q(f^n) \left[ \sum_{p=1}^{n} \psi_{q,p}^1(x_p) \right] \quad (9) \]

First A.N.Kolmogorov give a conclusion that any multivariate functions can be expressed by functions of three variables[1]. Further more V.I.Arnold proved that any functions of three variables can be expressed by binary functions[4],[5]. By this we can transfer the solutions expressed by multivariate functions to ones expressed by binary functions. By formula 8 we can obtain solutions expressed in unary function. Note, if we replaced x by \((y-a)/(b-a)\) then \([a,b]\) will become \([0,1]\) so formula 6 is suited to general domain.

Unary functions in Kolmogorov’s superposition theorem are continues. Hilberts 13th problem is open for smooth situation. We are looking forward the positive result for it. Superposition theorem can be extended to multivariate functions defined on finite set. Some results about it will be shown in another paper.

6. Discussion and expectation

Structural mathematics,topology and algebra have been the main role of mathematics for more than one hundred years since Hilbert’s solving Gordan problem. Meanwhile the method of classical mathematics is nearly forgotten by mathematicians. But resources in topology and algebra will be dried up in the near future and mathematics must go back to the path of classical mathematics as there are so mineral in it.

References

[1] A.N.Kolmogorov, On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, Dokl.Akad.Nauk SSSR 114 (1957), 953-956; English transl., Amer.Math. Soc.Transl. (2) 28 (1963), 55-59.

[2] D.Hilbert, Mathematical Problems, Bull.Amer.Math. Soc.8(1902),461-462.

[3] H.Umemura. Solution of algebraic equations in terms of theta constants. In D.Mumford, Tata.Lectures on Theta II, Progress in Mathematics 43, Birkh user, Boston, 1984.
[4] V.I. Arnold, On functions of three variables, Dokl. Akad. Nauk SSSR 114 (1957), 679-681; English transl., Amer. Math. Soc. Transl.(2) 28 (1963), 51C54.

[5] V.I. Arnold, On the representation of continuous functions of three variables by superpositions of continuous functions of two variables, Mat. Sb. 48 (1959), 3C74; English transl., Amer. Math. Soc. Transl.(2) 28 (1963), 61C147.

[6] Wu Zi Qian, Discrete Algebraic Equations and Discrete Operator Equations, IJMSEA. 1(2011), 309-326.

[7] Wu Zi Qian, Representation and Superposition of Discrete Functions and Equations with Parameterized Operations, arXiv:0909.3747.

[8] Wu Zi Qian, Discrete Algebraic Equations and Discrete Operator Equations, ICM 2010, ABSTRACTS, short communication, posters, 285-285, http://www.icm2010.in/wp-content/icmfiles/abstracts/Contributed-Abstracts-5July2010.pdf.