THE CONTINUITY METHOD ON MINIMAL ELLIPTIC KÄHLER SURFACES

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Abstract. We prove that, on a minimal elliptic Kähler surface of Kodaira dimension one, the continuity method introduced by La Nave and Tian in [19] starting from any initial Kähler metric converges in Gromov-Hausdorff topology to the metric completion of the generalized Kähler-Einstein metric on its canonical model constructed by Song and Tian in [26].

1. Introduction

In [19], La Nave and Tian introduced a new approach to the Analytic Minimal Model Program. It is a continuity method of complex Monge-Ampère equations. In this note, we will study the geometric convergence of this continuity equation on minimal elliptic surfaces.

Let \( X \) be a minimal elliptic Kähler surface of Kodaira dimension \( \text{kod}(X) = 1 \). By definition (see, e.g., [22, Section I.3] or [26, Section 2.2]), there exists a holomorphic map \( f : X \to \Sigma \), determined by the pluricanonical system \( |mK_X| \) for sufficiently large integer \( m \), from \( X \) onto a smooth projective curve \( \Sigma \) (i.e., the canonical model of \( X \)), such that the general fiber is a smooth elliptic curve and all fibers are free of \((-1)\)-curves. Set \( \Sigma_{\text{reg}} := \{ s \in \Sigma | X_s := f^{-1}(s) \text{ is a nonsingular fiber} \} \) and let \( m_i F_i = X_{s_i} \) be the corresponding singular fiber of multiplicity \( m_i \), \( i = 1, \ldots, k \). We refer readers to [22, Section I.5] for several interesting examples of minimal elliptic surfaces.

In [26], Song and Tian proved that there exists a unique generalized Kähler-Einstein current \( \chi_\infty \) on \( \Sigma \), i.e., \( \chi_\infty \) is a closed positive \((1,1)\)-current on \( \Sigma \) such that \( \chi_\infty \) is smooth on \( \Sigma_{\text{reg}} \) and \( \text{Ric}(\chi_\infty) = -\sqrt{-1} \partial \bar{\partial} \log \chi_\infty \) is a well-defined \((1,1)\)-current on \( \Sigma \) satisfying

\[
\text{Ric}(\chi_\infty) = -\chi_\infty + \omega_{WP} + 2\pi \sum_{i=1}^{k} \frac{m_i - 1}{m_i} [s_i],
\]

where \( \omega_{WP} \) is the induced Weil-Petersson form and \([s_i]\) is the current of integration associated to the divisor \( s_i \) on \( \Sigma \).

In this paper we consider the following continuity method introduced by La Nave and Tian in [19] and Rubinstein in [24] starting from any initial Kähler metric \( \omega_0 \) on \( X \),

\[
\begin{aligned}
(1 + t)\omega(t) &= \omega_0 - t \text{Ric}(\omega(t)) \\
\omega(0) &= \omega_0.
\end{aligned}
\]

According to [19], a solution \( \omega = \omega(t) \) to (1.2) exists uniquely for all \( t \geq 0 \) and the Ricci curvature of \( \omega(t) \) satisfies

\[
\text{Ric}(\omega(t)) \geq -2\omega(t)
\]

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for all $t \geq 1$.

Our main result is the following

**Theorem 1.1.** Assume as above, we have

1. As $t \to \infty$, $\omega(t) \to f^*\chi_{\infty}$ as currents on $X$ and, for any given compact subset $V$ of $X_{\text{reg}}$, there exists a constant $\alpha_V \in (0,1)$ such that $\omega(t) \to f^*\chi_{\infty}$ in $C^{1,\alpha_V}(V,\omega_0)$-topology;
2. For any $s \in \Sigma_{\text{reg}}$, $(1+t)\omega(t)|_{X_s}$ converges in $C^\infty(X_s,\omega_0|_{X_s})$-topology to the unique flat metric in class $[\omega_0|_{X_s}]$ as $t \to \infty$.

Moreover, if we let $(X_\infty,d_\infty)$ be the metric completion of $(\Sigma_{\text{reg}},\chi_{\infty})$, then

3. $(X_\infty,d_\infty)$ is a compact length metric space and $X_\infty$ is homeomorphic to $\Sigma$ as a projective variety;
4. As $t \to \infty$, $(X,\omega(t)) \to (X_\infty,d_\infty)$ in Gromov-Hausdorff topology.

We would like to explain how our result fits the existing literatures. According to the Analytic Minimal Model Program proposed in [26, 27, 28, 19], the Kähler-Ricci flow and the continuity method should deform any Kähler metric on a smooth minimal model (i.e., a Kähler manifold with nef canonical line bundle), say $M$, to a canonical metric or its metric completion on its canonical model $M_{\text{can}}$ in Gromov-Hausdorff topology. If in addition we assume that the canonical line bundle $K_M$ of $M$ is semi-ample, then by semi-ample fibration theorem (see [21, 37]) there exists a fiber space map determined by the pluricanonical system of $K_M$:

$$\pi : M \to M_{\text{can}}.$$ (1.4)

If $\dim(M_{\text{can}}) = 0$, then $M$ is a Calabi-Yau manifold. By a classical result of Cao [3], Kähler-Ricci flow will deform any Kähler metric to the unique Ricci-flat Kähler metric in the same Kähler class smoothly. It can be easily checked that the same result holds for the continuity method.

If $\dim(M_{\text{can}}) = \dim(M)$, i.e., $M$ is a smooth minimal model of general type, the expected geometric convergence is obtained for Kähler-Ricci flow when $\dim(M) \leq 3$ (see [13, 29]). For the continuity method, the convergence is obtained for any dimension in [20].

The remaining case is $0 < \dim(M_{\text{can}}) < \dim(M)$. In this case, the geometric convergence of Kähler-Ricci flow is obtained in [33] (see [8, 10] for certain special cases) when $M_{\text{can}}$ is smooth and the semi-ample fibration (1.4) does not admit any singular fiber. In general, this problem is largely open. Our Theorem 1.1 confirms this expected picture for the continuity method when $\dim(M) = 2$. In fact, our argument can also apply to Kähler-Ricci flow on minimal elliptic Kähler surface of Kodaira dimension one if all its singular fibers are of type $mI_0$ (see Remark 3.12).

The rest of this paper is organized as follows. We will prove parts (1) and (2) of Theorem 1.1 in Section 2 and prove parts (3) and (4) in Section 3. The key observation in the proof is that the limit space $(X_\infty,d_\infty)$ is compact.

## 2. Estimates and local convergence

In this section we will derive necessary estimates and prove parts (1) and (2) of Theorem 1.1. As the first step, we will reduce (1.2) to a scalar equation of Monge-Ampère type as in [19, 20]. Let $\chi$ be the restriction on $\Sigma$ of a multiple of the Fubini-Study metric of a
projective space and Ω a smooth positive volume form on X with \( \sqrt{-1} \partial \bar{\partial} \log \Omega = f^* \chi \). Set \( \omega_t := \frac{1}{1+t} \omega_0 + \frac{t}{1+t} f^* \chi \). Then (1.2) can be reduced to the following equation of \( \varphi = \varphi(t) \)

\[
\begin{cases}
(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^2 = (1 + t)^{-1} e^{\frac{1+t}{1+t} \varphi} \Omega \\
\varphi(0) = 0.
\end{cases}
\]  

(2.1)

Namely, \( \omega_t + \sqrt{-1} \partial \bar{\partial} \varphi(t) \) solves (1.2) if \( \varphi(t) \) solves (2.1). We remark that the factor \((1 + t)^{-1}\) in the right hand side of (2.1) comes from the cohomology class and formal scale of volume, which in particular is crucial in obtaining a uniform bound of Kähler potential \( \varphi(t) \), see Lemma 2.1.

Next, following [31], we fix a smooth nonnegative function \( \sigma \) on \( X \), which vanishes exactly on singular fibers and satisfies \( \sigma \leq 1 \), \( \sqrt{-1} \partial \sigma \wedge \bar{\partial} \sigma \leq C f^* \chi \), \(-C f^* \chi \leq \sqrt{-1} \partial \bar{\partial} \sigma \leq C f^* \chi \) (2.2)

on \( X \) for some constant \( C \).

**Lemma 2.1.** There exists a constant \( C > 0 \) such that for any \( t \geq 1 \),

\[ \| \varphi(t) \|_{C^0(X)} \leq C, \]

or equivalently,

\[ \frac{C}{1+t} \Omega \leq \omega(t)^2 \leq \frac{C}{1+t} \Omega. \]  

(2.4)

**Proof.** It suffices to prove (2.3). Firstly note that \( \omega_t^2 \leq C \frac{\Omega}{1+t} \). By applying the maximum principle in (2.1), the upper bound of \( \varphi \) follows easily. Then (2.3) can be proved by the Moser iteration as in [26, 39]. We remark that this lemma also follows from general theory of degenerate complex Monge-Ampère equations, see [31]. □

**Lemma 2.2.** There exists a positive constant \( C \) such that for all \( t \geq 1 \),

\[ \text{tr} \omega f^* \chi \leq C. \]  

(2.5)

**Proof.** Along (1.2), the Schwarz Lemma argument (see, e.g., [38, 26]) gives

\[ \Delta_\omega (\log \text{tr} \omega f^* \chi - 2(A + 1) \varphi) \geq \text{tr} \omega f^* \chi - 4(A + 1) - 2 \]

for some fixed large constant \( A \). Using the maximum principle and the \( C^0 \)-estimate (2.3), we have proved (2.5). □

Using (2.5) and the same argument in [26], we can find a positive constant \( \lambda_1 \) such that for all \( t \geq 1 \),

\[ \sup_{X_s} \varphi - \inf_{X_s} \varphi \leq \frac{C(1 + t)^{-1}}{\sigma^{\lambda_1}}. \]  

(2.6)

Next, as in [26], we define a function \( \bar{\varphi} \) on \( \Sigma \) by

\[ \bar{\varphi}(s) = \frac{\int_{X_s} \varphi(\omega_0|_{X_s})}{\int_{X_s} \omega_0|_{X_s}}. \]

Then (2.6) implies that

\[ (1 + t)|\varphi - \bar{\varphi}| \leq \frac{C}{\sigma^{\lambda_1}}. \]  

(2.7)

**Lemma 2.3.**

\[ \Delta_\omega ((1 + t)(\varphi - \bar{\varphi})) \leq -\text{tr} \omega \omega_0 + \frac{1}{Vol(X_s)} \text{tr}_\omega \left( \int_{X_s} \omega_0^2 \right) + 2(1 + t). \]  

(2.8)
Proof. The proof is the same as [26, Lemma 5.9]. □

Next lemma can be easily checked by a direct computation.

Lemma 2.4. There exists a positive constant $C$ such that for all $t \geq 1$,
\[
\Delta_\omega \log tr_\omega((1 + t)^{-1}\omega_0) \geq -C tr_\omega \omega_0 - C.
\] (2.9)

Lemma 2.5. There exist positive constants $\lambda_2$ and $C$ such that for all $t \geq 1$,
\[
tr_\omega((1 + t)^{-1}\omega_0) \leq Ce^{C\sigma^{-\lambda_2}}.
\] (2.10)

Proof. Set $H = \sigma^{\lambda_2}(\log tr_\omega((1 + t)^{-1}\omega_0) - A(1 + t)(\varphi - \bar{\varphi}))$. By the Schwarz Lemma argument (see, e.g., [38, 26]), we have
\[
\Delta_\omega H \geq A/3 \sigma^{\lambda_2} tr_\omega \omega_0 + 2Re \left( \nabla H \nabla \sigma^{\lambda_2} \right) - 3A(1 + t)
\] if we choose $\lambda_2$ and $A$ large enough. Now by the maximum principle and (2.7), (2.10) follows. □

Corollary 2.6. There exist positive constants $C$ and $\lambda_2$ such that for all $t \geq 1$,
\[
Ric(\omega) \leq Ce^{C\sigma^{-\lambda_2}} \omega.
\] (2.11)

Proof. Along the continuity equation (1.2) we have, for $t \geq 1$,
\[
Ric(\omega) = t^{-1}\omega_0 - \frac{(1 + t)}{t} \omega.
\]
Thus (2.11) follows from Lemma 2.5 immediately. □

Combining (2.5) and (2.10) we have
\[
tr_\omega \omega_t \leq Ce^{C\sigma^{-\lambda_2}}.
\]
Then
\[
tr_\omega \omega \leq (tr_\omega \omega_t) \frac{\omega^2}{\omega^2_t}
\leq C(tr_\omega \omega_t) \frac{(1 + t)^{-1}\Omega}{(1 + t)^{-1}\omega_0 \wedge f^*\chi}
\leq Ce^{C\sigma^{-\lambda_2}} \sigma^{-\lambda_2}
\leq Ce^{C\sigma^{-\lambda_3}}.
\]
In conclusion,
\[
C^{-1}e^{-C\sigma^{-\lambda_3}} \omega_t \leq \omega \leq Ce^{C\sigma^{-\lambda_3}} \omega_t.
\] (2.12)

Let $t_j$ be any time sequence converging to $\infty$. Since the cohomology class of $\omega(t)$ is bounded and $\varphi(t)$ is uniformly bounded for all $t \geq 1$, using the weak compactness of currents, we may assume that $\omega(t_j)$ converges to a limit closed positive $(1,1)$-current $\chi_\infty$ (see, e.g., [4]), which a priori depends on the given sequence. Note that $\chi_\infty \in [f^*\chi]$, $\chi_\infty = f^*\chi + \sqrt{-1} \partial \bar{\partial} \varphi_\infty$ and $\varphi(t_j) \to \varphi_\infty$ in $L^1(X, \omega_0^2)$, which in particular implies that $\varphi_\infty \in L^\infty(X)$. Indeed, after passing to a subsequence, we may assume $\varphi(t_j) \to \varphi_\infty$ a.e. on $X$ and hence, by Lemma 2.1 $|\varphi_\infty| \leq C$ a.e. on $X$. Then, for any $x \in X$ and a fixed
local potential $u$ of $f^*\chi$ (i.e. $f^*\chi = \sqrt{-1}\partial\bar{\partial}u$) on $B_{\omega_0}(x, \rho_0)$, applying e.g. [12] Theorem K.15 gives
\[
|(u + \varphi_\infty)(x)| = \left| \lim_{\epsilon \to 0} \frac{\int_{B_{\omega_0}(x,\epsilon)} (u + \varphi_\infty) \omega_0^2}{\int_{B_{\omega_0}(x,\epsilon)} \omega_0^2} \right| \leq C
\]
and so
\[
|\varphi_\infty(x)| \leq C
\]
for some uniform constant $C$.

Moreover, using the estimates we have obtained above, we can assume that $\varphi(t_j) \to \varphi_\infty$ in $C^{1,\alpha}_{loc}(X_{\text{reg}}, \omega_0)$ for any $\alpha \in (0, 1)$.

**Lemma 2.7.** There exists a function $\hat{\varphi}_\infty \in PSH(\Sigma, \chi) \cap L^\infty(\Sigma)$ such that $f^*\hat{\varphi}_\infty = \varphi_\infty$.

**Proof.** We present a proof similar to [7, Theorem 6.3]. It suffices to show that $\varphi_\infty$ is constant on every fiber $X_s = f^{-1}(s)$. For $s \in \Sigma_{\text{reg}}$, i.e. $X_s$ is a nonsingular fiber, when restricting $\chi_\infty$ to such $X_s$, we see that $\sqrt{-1}\partial\bar{\partial}\varphi_\infty|_{X_s} \geq 0$. Hence $\varphi_\infty$ is constant on every nonsingular fiber $X_s$. For a singular fiber, $X_0$, by Hironaka’s theorems (see [16]) we fix an embedded resolution $\hat{f} : \hat{X} \to X$ of singularities of $X_0$, i.e., $\hat{X}$ is a compact complex manifold, $\hat{f} : \hat{X} \to X$ is a holomorphic surjective map and is biholomorphic over $X \setminus X_0$, and the proper transform of $X_0$, denoted by $\hat{X}_0$, is a smooth connected submanifold of $\hat{X}$. Now we pullback $\chi_\infty$ to $\hat{X}$ to obtain $\pi^*\chi_\infty = \pi^*f^*\chi + \sqrt{-1}\partial\bar{\partial}\pi^*\varphi_\infty$, which is a closed positive $(1,1)$-form on $\hat{X}$. Now, as before, we can restrict $\pi^*\varphi_\infty$ to $X_0$ to see that $\pi^*\varphi_\infty$ is plurisubharmonic on $\hat{X}_0$ and hence is constant on $\hat{X}_0$, which implies that $\varphi_\infty$ is constant on $X_0$.

Lemma 2.7 is proved. $\square$

In the following, we identify $\hat{\varphi}_\infty$ and $\varphi_\infty$.

On the other hand, let $\omega_{SF} := \omega_0 + \sqrt{-1}\partial\bar{\partial}\rho_{SF}$, where $\rho_{SF}$ is a smooth function on $X_{\text{reg}}$, be the semi-flat $(1,1)$-form defined by [26, Lemma 3.1]. Define $F := \frac{\Omega}{\omega_{SF}}$, which can be seen as a function $\in L^{1+r}(\Sigma, \chi)$ (see [26, 27, 14]) and is smooth on $\Sigma_{\text{reg}}$. It was proved in [26] (see also [17] for more general theory) that there exists a unique solution $\hat{\phi} \in PSH(\Sigma, \chi) \cap C^0(\Sigma) \cap C^\infty(\Sigma_{\text{reg}})$ to the following equation on $\Sigma$:
\[
\chi + \sqrt{-1}\partial\bar{\partial}\hat{\phi} = F e^{\hat{\phi}} \chi. \tag{2.13}
\]

Here we will use the above estimates to prove the existence of a bounded solution of (2.13). Precisely, we have

**Lemma 2.8.** $\varphi_\infty$ is a bounded solution of equation (2.13).

**Proof.** Since $\varphi_\infty$ is bounded, $\chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty$ takes no mass on pluripolar sets, e.g. $\Sigma \setminus \Sigma_{\text{reg}}$ (see e.g. [18]). Moreover, using $F \in L^{1+r}(\Sigma)$ and Hölder inequality, one easily sees that $e^{\varphi_\infty} F \chi$ also takes no mass on $\Sigma \setminus \Sigma_{\text{reg}}$. Therefore, it suffices to show that for any given $K \subset \subset \Sigma_{\text{reg}}$ and any given $\phi \in C^\infty_0(K)$,
\[
\int_K \phi(\chi + \sqrt{-1}\partial\bar{\partial}\varphi_\infty) = \int_K \phi F e^{\varphi_\infty} \chi. \tag{2.14}
\]

To this end, we use an argument similar to [31, Theorem 4.1]. Firstly, using the equation (2.1), we have
\[
\int_X (f^*\phi) e^{\frac{|\chi|}{\sqrt{2}}} \Omega = \int_X (f^*\phi)(1 + t)(\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi)^2. \tag{2.15}
\]
As \( t_j \to \infty \), the left hand side of (2.15) will go to
\[
\int_X (f^*\phi) e^{\varphi_0} \Omega = 2 \int_X (f^*\phi) e^{\varphi_0} F \omega_{SF} \wedge f^*\chi = 2 \int_\Sigma \phi F e^{\varphi_0} \chi \int_{X_s} \omega_{SF}|_{X_s}.
\] (2.16)

For the right hand side of (2.15), we have
\[
\int_X (f^*\phi)(1 + t)(\omega_t + \sqrt{-1} \partial \bar{\partial} \varphi)^2
= \int_X (f^*\phi)(1 + t) \left( \frac{1}{1 + t} \omega_0 + \sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi}) + \frac{t}{1 + t} f^*\chi + \sqrt{-1} \partial \bar{\partial} \bar{\varphi} \right)^2
= \int_X (f^*\phi) \frac{1}{1 + t} \omega_0^2
+ 2 \int_X (f^*\phi) \omega_0 \wedge \sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi})
+ \int_X (f^*\phi)(1 + t)(\sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi}))^2
+ 2 \int_X (f^*\phi)(1 + t) \sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi}) \wedge (\frac{t}{1 + t} f^*\chi + \sqrt{-1} \partial \bar{\partial} \bar{\varphi})
+ 2 \int_X (f^*\phi) \omega_0 \wedge (\frac{t}{1 + t} f^*\chi + \sqrt{-1} \partial \bar{\partial} \bar{\varphi})
= A_1 + A_2 + A_3 + A_4 + A_5.
\]

(1) The first term \( A_1 \) will go to zero as \( t \to \infty \).
(2) The second term
\[
A_2 = 2 \int_X (f^*\phi) \omega_0 \wedge \sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi})
= 2 \int_X (\varphi - \bar{\varphi}) \omega_0 \wedge \sqrt{-1} \partial \bar{\partial} (f^*\phi),
\]
which will go to zero as \( t \to \infty \) by (2.7).
(3) The third term
\[
A_3 = \int_X (f^*\phi)(1 + t)(\sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi}))^2
= \int_X (1 + t)(\varphi - \bar{\varphi}) \sqrt{-1} \partial \bar{\partial} (f^*\phi) \wedge \sqrt{-1} \partial \bar{\partial} (\varphi - \bar{\varphi})
= \int_X (1 + t)(\varphi - \bar{\varphi}) \sqrt{-1} \partial \bar{\partial} (f^*\phi) \wedge \sqrt{-1} \partial \bar{\partial} \varphi|_{X_s}.
\]
Note that \( (1 + t)(\varphi - \bar{\varphi}) \) is uniformly bounded on \( K \) by (2.7) and
\[
- (1 + t)^{-1} \omega_0|_{X_s} \leq \sqrt{-1} \partial \bar{\partial} \varphi|_{X_s} \leq C_K (1 + t)^{-1} \omega_0|_{X_s},
\]
by \((2.12)\). Thus \(A_3 \to 0\) as \(t \to \infty\).

(4) The fourth term

\[
A_4 = 2 \int_X (f^* \phi)(1 + t) \sqrt{-1} \partial \bar{\partial} (\phi - \bar{\phi}) \wedge \left( \frac{t}{1 + t} f^* \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi} \right)
\]

\[
= 2 \int_X (1 + t)(\phi - \bar{\phi}) \sqrt{-1} \partial \bar{\partial} (f^* \phi) \wedge \left( \frac{t}{1 + t} f^* \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi} \right)
\]

\[
= 0,
\]

since the term \(\sqrt{-1} \partial \bar{\partial} (f^* \phi) \wedge \left( \frac{t}{1 + t} f^* \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi} \right) = f^* (\sqrt{-1} \partial \bar{\partial} \phi \wedge \left( \frac{t}{1 + t} \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi} \right))\) and, obviously, \(\sqrt{-1} \partial \bar{\partial} \phi \wedge \left( \frac{t}{1 + t} \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi} \right)\) vanishes on \(\Sigma\) as \(\dim(\Sigma) = 1\).

(5) For the last term \(A_5\), first note that \((2.7)\) implies that \(\bar{\phi}(t_j) \to \varphi^\infty\) in \(L^\infty(K)\) as \(t_j \to \infty\). So we have

\[
A_5 = 2 \int_X (f^* \phi) \omega_0 \wedge \left( \frac{t_j}{1 + t_j} f^* \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi}(t_j) \right)
\]

\[
\to 2 \int_X (f^* \phi) \omega_0 \wedge (f^* \chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi})
\]

\[
= 2 \int \phi(\chi + \sqrt{-1} \partial \bar{\partial} \bar{\phi}) \int_{X_s} \omega_0|_{X_s}.
\] (2.17)

Combining \((2.15), (2.16), (2.17)\) and the fact that \(\int_{X_s} \omega_{SF}|_{X_s} = \int_{X_s} \omega_0|_{X_s} \equiv \text{constant}\), we obtain \((2.14)\).

Let \(\chi^\infty = \chi + \sqrt{-1} \partial \bar{\partial} \varphi^\infty\). Since the bounded solution of \((2.13)\) is unique (see e.g. [26]), we conclude that the above convergence holds without passing to a subsequence, i.e.,

**Lemma 2.9.** \(\varphi(t) \to \varphi^\infty\) in \(L^1(X, \omega_0^2)\) and \(C^{1,\alpha}_{loc}(X_{reg}, \omega_0)\) for any \(\alpha \in (0, 1)\) as \(t \to \infty\).

In particular \(\omega(t) \to f^* \chi^\infty\) in the sense of currents as \(t \to \infty\).

**Remark 2.10.** Alternatively, we can easily apply arguments in [26, Section 6] to conclude the \(L^1\)-convergence in Lemma [2.9].

Next we shall prove the interior \(C^{1,\alpha}\) estimate for \(\omega(t)\) on \(X_{reg}\), where we will apply an idea due to [11, 15] (see also [8]). Let \(B \subset \Sigma_{reg}\) small enough and \(U = f^{-1}(B)\). There exists a holomorphic function \(z(s)\) on \(B\) such that \(Im(z(s)) > 0\) and \(U\) is biholomorphic to \(B \times \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z(s)\), which is compatible with the projection to \(B\). Composing the quotient map \(B \times \mathbb{C} \to B \times \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}z(s)\) with this biholomorphism, we obtain a local biholomorphism \(p : B \times \mathbb{C} \to U\) such that \(f \circ p(s, w) = s\) for all \((s, w)\). Moreover, by [15] there exists a closed semi-positive real \((1, 1)\)-form \(\tilde{\omega}_{SF}\) on \(U\) such that \(\tilde{\omega}_{SF} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho\) for some \(\rho \in C^\infty(U, \mathbb{R})\), \(\tilde{\omega}_{SF}|_{X_s}\) is a flat metric on \(X_s\) for all \(s \in B\) and \(p^* \tilde{\omega}_{SF} = \sqrt{-1} \partial \bar{\partial} \eta\), where \(\eta(s, w)\) is a smooth real function on \(B \times \mathbb{C}\) satisfying

\[
\eta(s, \lambda w) = \lambda^2 \eta(s, w)
\] (2.18)

for any \(\lambda \in \mathbb{R}\) (note that, a priori, \(\tilde{\omega}_{SF}\) may be different from \(\omega_{SF}\) given by Lemma 3.1 of [26]).
Define $\lambda_t : B \times \mathbb{C} \to B \times \mathbb{C}$ by $\lambda_t(s, w) = (s, \sqrt{1 + t} \cdot w)$. Then using (2.18) we have

$$(1 + t)^{-1} \lambda_t^* p^* \tilde{\omega}_{SF} = (1 + t)^{-1} \lambda_t^* \sqrt{-1} \partial \bar{\partial} \eta$$

where we have used the fact that $\Omega$ only depends on $\chi$. Hence we arrive at

$$C^{-1} p^* (\tilde{\omega}_{SF} + f^* \chi) \leq \lambda_t^* p^* \omega(t) \leq C p^* (\tilde{\omega}_{SF} + f^* \chi)$$

on $B \times \mathbb{C}$. Notice that $p^* (\tilde{\omega}_{SF} + f^* \chi)$ is $C^\infty$ equivalent to $\omega_E$, where $\omega_E$ is the Euclidean metric on $B \times \mathbb{C}$. So for each given $K \subset B \times \mathbb{C}$ there exists a positive constant $C_K$ such that

$$C_K^{-1} \omega_E \leq \lambda_t^* p^* \omega(t) \leq C_K \omega_E$$

on $K$. Combining with (2.12) we have

$$- C_K \omega_E \leq \sqrt{-1} \partial \bar{\partial}(\varphi \circ p \circ \lambda_t) \leq C_K \omega_E.$$  

On the other hand, if we fix a $\beta \in C^\infty(B, \mathbb{R})$ such that $\chi = \sqrt{-1} \partial \bar{\partial} \beta$ on $B$, then on $B \times \mathbb{C}$ we have

$$\lambda_t^* p^* \omega = \lambda_t^* p^* \left( \frac{1}{1 + t} \omega_0 + \frac{t}{1 + t} f^* \chi + \sqrt{-1} \partial \bar{\partial} \varphi \right)$$

$$= \frac{1}{1 + t} \lambda_t^* p^* \omega_0 + \frac{t}{1 + t} p^* f^* \chi + \sqrt{-1} \partial \bar{\partial}(\varphi \circ p \circ \lambda_t)$$

$$= \frac{1}{1 + t} \lambda_t^* p^* (\tilde{\omega}_{SF} - \sqrt{-1} \partial \bar{\partial} \rho) + p^* f^* \chi + \sqrt{-1} \partial \bar{\partial}(\varphi \circ p \circ \lambda_t - \frac{1}{1 + t} \beta \circ p \circ f)$$

$$= p^* (\tilde{\omega}_{SF} + f^* \chi) + \sqrt{-1} \partial \bar{\partial}(\varphi \circ p \circ \lambda_t - \frac{1}{1 + t} \beta \circ p \circ f - \frac{1}{1 + t} \rho \circ p \circ \lambda_t).$$

Set $v = \varphi \circ p \circ \lambda_t - \frac{1}{1 + t} \beta \circ f \circ p - \frac{1}{1 + t} \rho \circ p \circ \lambda_t$ on $B \times \mathbb{C}$, which is uniformly bounded on $B \times \mathbb{C}$. Now we translate (2.21) to $B \times \mathbb{C}$ as

$$(\lambda_t^* p^* \omega)^2 = e^{\frac{t}{1 + t} \varphi \circ p \circ \lambda_t} \frac{1}{1 + t} \lambda_t^* p^* \Omega = e^{\frac{t}{1 + t} \varphi \circ p \circ \lambda_t} p^* \Omega,$$

where we have used the fact that $\Omega$ only depends on $s \in B$ since $\sqrt{-1} \partial \bar{\partial} \log \Omega = f^* \chi$. Hence we arrive at

$$\log(p^* (\tilde{\omega}_{SF} + f^* \chi) + \sqrt{-1} \partial \bar{\partial} v)^2 = \frac{t}{1 + t} \varphi \circ p \circ \lambda_t + \log p^* \Omega.$$  

Lemma 2.11. Given any small $K \subset B \times \mathbb{C}$, there exist two constants $C_K > 0$ and $\alpha_k \in (0, 1)$ such that for all $t \geq 1$,

$$\|\lambda_t^* p^* \omega\|_{C^{1, \alpha_k}(K, \omega_E)} \leq C_K.$$  

Proof. Firstly, we may assume that there exists a compact $L \subset B \times \mathbb{C}$ containing $K$ in its interior and some $K_1 \subset X_{\text{reg}}$ such that $p \circ \lambda_t(L) \subset K_1$. Then by (2.22) we can find a positive constant $C_1 = C_1(K_1)$ such that

$$|\Delta_{\omega_E} \varphi \circ p \circ \lambda_t| \leq C_1$$  

(2.25)
on $L$. Combining the fact that $\varphi \circ p \circ \lambda_t$ is uniformly bounded, we obtain by elliptic estimates (see [9]) that, for any fixed $\alpha \in (0, 1)$, we can find a slightly smaller subset $L_1$ of $L$ (still contains $K$ in its interior) and a constant $C_2$ depends on $K_1$ and the uniform bound of $\varphi$ such that
\[
\|\varphi \circ p \circ \lambda_t\|_{C^{1,\alpha}(L_1, \omega_E)} \leq C_2,
\] (2.26)
Similarly, as we have
\[
\Delta_{\omega_E} v = t r_{\omega_E} \lambda_t^* p^* \omega (t) - t r_{\omega_E} (p^* (\omega SF + f^* \chi))
\] (2.27)
and hence $\Delta_{\omega_E} v$ is bounded in $L$, there exists a slightly smaller subset $L_2$ (still contains $K$ in its interior) of $L_1$ and a positive constant $C_3$ such that
\[
\|v\|_{C^{1,\alpha}(L_2, \omega_E)} \leq C_3.
\] (2.28)
Now we can apply a complex version of Evans-Krylov theory (see e.g. [1, Theorem 3.1]) to conclude that for some $\alpha_0 \in (0, 1)$ and a slightly smaller subset $L_3$ (still contains $K$ in its interior) of $L_2$,
\[
\|v\|_{C^{2,\alpha_0}(L_3, \omega_E)} \leq C_4.
\] (2.29)
Equivalently,
\[
\|\lambda_t^* p^* \omega\|_{C^{\alpha_0}(L_3, \omega_E)} \leq C_4,
\] (2.30)
and hence
\[
\| (\lambda_t^* p^* \omega)^{-1}\|_{C^{\alpha_0}(L_3, \omega_E)} \leq C_5.
\] (2.31)
Furthermore, for $i \in \{1, 2\}$, differentiating (2.23) by $\partial_i$ gives
\[
\Delta_{\lambda_t^* p^* \omega} (\partial_i v) = \frac{t}{1 + t} \partial_i (\varphi \circ p \circ \lambda_t) + A,
\] (2.32)
where $A$ is a term whose $C^{\alpha_0}$-norm with respect to $\omega_E$ is uniformly bounded. Then combining (2.26), (2.30) and (2.31), we know that the coefficients and right hand side of (2.32) are in $C^{\alpha_0}(L_2, \omega_E)$ and we can apply Schauder estimates (see [9]) to conclude that for some $L_4 \subset \subset L_3$ (still contains $K$ in its interior),
\[
\|\partial_i v\|_{C^{2,\alpha_0}(L_4, \omega_E)} \leq C_6,
\] which implies
\[
\|v\|_{C^{3,\alpha_0}(L_4, \omega_E)} \leq C_7
\] and
\[
\|\lambda_t^* p^* \omega\|_{C^{2,\alpha_0}(L_4, \omega_E)} \leq C_8.
\] This lemma is proved.

Remark 2.12. (1) Note that, as mentioned during the above proof, the Hölder exponent in Lemma 2.11 is obtained by applying Evans-Krylov theory and hence depends on the chosen compact subset $K$.

(2) One may like to apply bootstrapping to derive $C^k$ estimates for all $k$. However, after having $C^{3,\alpha_0}$ bound on $v$, one may not obtain the $C^{3,\alpha}$ bound on $\varphi \circ p \circ \lambda_t$ and hence can't apply bootstrapping directly.

Consequently, as in [11] (see Lemma 4.5), we have

**Proposition 2.13.** Given any $K' \subset \subset U$, there exist two positive constants $C_{K'}$ and $\alpha_{K'} \in (0, 1)$ such that for all $t \geq 1$,
\[
\|\omega\|_{C^{1,\alpha_{K'}}(K', \omega_0)} \leq C_{K'}.
\] (2.33)
Combining with Lemma 2.9, we have
Theorem 2.14. For any given compact subset $V$ of $X_{reg}$, there exists a constant $\alpha_V \in (0,1)$ such that $\omega(t) \to f^*\chi_\infty$ in $C^{1,\alpha_V}(V,\omega_0)$-topology as $t \to \infty$.

Note that $\chi_\infty$ is exactly the unique generalized Kähler-Einstein metric on $\Sigma$ (see [26]). Hence we have proved part (1) of Theorem 1.1.

In the remaining part of this section, we shall give a proof of part (2) of Theorem 1.1 using the method developed in [33]. To this end, we firstly apply a translation to the continuity equation (1.2) in the following manner. Let $\eta$ and if we set $h$ where

$$\omega(t) \to \infty$$

we have

Lemma 2.15. There exist uniform constants $C > 1$ and $\lambda > 0$ such that for all $u \geq 2$ we have

1. $\|\psi(u)\|_{C^0(\Sigma)} \leq C$;
2. $tr_{\eta(u)} f^*\chi \leq C$;
3. $\lambda e^{-C\sigma^{-\lambda}} \eta_u \leq \eta(u) \leq C e^{C\sigma^{-\lambda}} \eta_u$;
4. $\psi(u) \to \varphi_\infty$ in $L^1(X,\omega_0)$- and $C^{1,\alpha}_{loc}(X_{reg},\omega_0)$-topology, for any $\alpha \in (0,1)$, as $u \to \infty$;
5. $tr_{\eta(u)} f^*\chi_\infty \leq C\sigma^{-\lambda}$.

Proof. Part (5) is concluded from part (2) and Lemma 2.16 in the next section. \qed

Next we prove an analogue of [33, Lemma 3.1] for [33, Lemma 4.6].

Lemma 2.16. There exists a positive function $H(u)$ with $H(u) \to 0$ as $u \to \infty$ such that

$$\sup_{\Sigma} e^{-C\sigma^{-\lambda}} |\psi(u) + \partial_u \psi(u) - \varphi_\infty| \leq H(u).$$

Proof. We begin with the following inequality:

$$\sup_{\Sigma} e^{-C\sigma^{-\lambda}} |\psi(u) - \varphi_\infty| \leq h(u),$$

where $h(u)$ is a positive function and will go to zero as $u \to \infty$. As we have Lemma 2.15, this can be checked by the same argument in the proof of [33, Lemma 4.3]. Thus it suffices to show

$$\sup_{\Sigma} e^{-C\sigma^{-\lambda}} |\partial_u \psi(u)| \leq H(u).$$

To this end, we collect some useful equalities as follows. By taking $u$-derivative of (2.35) and using the easy facts that $\Delta_\psi \psi = 2 - tr_\eta \eta_u$ and $\partial_u \eta_u = \chi - \eta_u$, we have

$$(1 - e^{-u})\Delta_\eta (\psi + \partial_u \psi) = -(1 - e^{-u})(tr_\eta f^*\chi - 1) + \partial_u \psi - e^{-u} \log \frac{e^u \eta(u)^2}{\Omega}$$

$$\chi.$$
and
\[
\Delta_\eta \left( (1 - e^{-u}) \partial_u \partial_u \psi + 2e^{-u} \partial_u \psi - (1 - 3e^{-u}) \psi \right) = \partial_u \partial_u \psi + e^{-u} \log \frac{e^u \eta(u)^2}{\Omega} + (1 - 3e^{-u}) tr_\eta f^* \chi + 4e^{-u} - 2 + (1 - e^{-u})|\partial_u \eta|^2. \tag{2.41}
\]

By using parts (1) and (2) of Lemma \ref{Lemma2.15} and the maximum principle, one can conclude that there exists a positive constant \( C \) such that for all \( u \geq 2 \),
\[
|\partial_u \psi|_{C^0(X)} \leq C \tag{2.42}
\]
from \eqref{2.39} and
\[
\partial_u \partial_u \psi \leq C \tag{2.43}
\]
from \eqref{2.40} and \eqref{2.42}. With \eqref{2.37}, \eqref{2.42} and \eqref{2.43}, we can apply the arguments in Lemma 4.6 of \cite{33} to obtain the desired conclusion \eqref{2.38}. The proof is now completed. \( \square \)

With Lemma \ref{Lemma2.16} and equation \eqref{2.39}, one can apply a maximum principle argument (see \cite{33} Lemma 4.7 for details) to conclude that there exist two positive constants \( C \) and \( \lambda \) such that for all \( u \geq 2 \),
\[
\sup_X e^{-C \sigma - \lambda} (tr_\eta f^* \chi_{\infty} - 1) \leq C \sqrt{H(u)}, \tag{2.44}
\]
where \( H(u) \) is the function satisfying Lemma \ref{Lemma2.16}.

Now we shall give a proof of part (2) of Theorem \ref{Theorem1.1}, which is equivalent to the following

**Proposition 2.17.** For any \( s \in \Sigma_{reg} \), \( e^u \eta(u)|_{X_s} \) converges in \( C^\infty(X_s, \omega_0|_{X_s}) \)-topology to the unique flat metric in class \( [\omega_0]|_{X_s} \) as \( u \to \infty \).

**Proof.** By part (3) of Lemma \ref{Lemma2.15} there exists a constant \( C > 1 \) such that for all \( u \geq 2 \),
\[
C^{-1} \omega_0|_{X_s} \leq e^u \eta(u)|_{X_s} \leq C \omega_0|_{X_s}. \tag{2.45}
\]
Adapting the arguments in the proof of \cite{35} Theorem 1.1, we can find constants \( C_k \) for all \( k \in \mathbb{N} \) such that for all \( u \geq 2 \),
\[
\|e^u \eta(u)|_{X_s}\|_{C^k(X_s, \omega_0|_{X_s})} \leq C_k. \tag{2.46}
\]
For the sake of convenience, we sketch a proof here by following \cite{35}. For any given \( x \in X_s \), we fix a small chart \( (U, (s, w)) \) in \( X \) centered at \( x \) such that \( f \) in this coordinate is given by \( f(s, w) = s \). Without loss of any generality, we assume \( U = \{(s, w) \in \mathbb{C}^2||s| < 1, |w| < 1\} \). Let \( B_r(0) \) be the standard disc in \( \mathbb{C} \) centered at \( 0 \in \mathbb{C} \) with radius \( r > 0 \). Define the maps \( F_u : B_r \times B_1 \to U \) for \( u \geq 0 \) as follows:
\[
F_u(s, w) = (se^{-\frac{u}{2}}, w).
\]
Then \( e^u F_u^* \eta \) satisfies
\[
Ric(e^u F_u^* \eta) = \frac{1}{e^u - 1} F_u^* \omega_0 - \frac{1}{e^u - 1} (e^u F_u^* \eta) \tag{2.47}
\]
on \( B_r \times B_1 \) and for any given \( M \subset \subset B_{\frac{r}{2}} \times B_1 \) one can find a constant \( C_M > 1 \) depending only on \( M \) such that
\[
C_M^{-1} \omega_E \leq e^u F_u^* \eta \leq C_M \omega_E \tag{2.48}
\]
on \( M \), where \( \omega_E \) is the Euclidean metric on \( \mathbb{C}^2 \). Note that \eqref{2.47} implies a uniform lower bound for Ricci curvature of \( e^u F_u^* \eta \) and both \( e^u F_u^* \omega_0 \) and its covariant derivative with respect to \( \omega_E \) are uniformly bounded for all \( u \geq 2 \). Thus one can modify Calabi’s \( C^3 \) estimate (see e.g. \cite{23, 25, 15}) to obtain the uniform \( C^1 \) estimate of \( e^u F_u^* \eta \) on a slightly
smaller $M_1 \subset M$. Moreover, by equation (2.47), the components of $e^u F_u \eta$ (resp. $F_u \omega_0$), say $g_{ij}$ (resp. $(g_0)_{ij}$), satisfy

$$
\Delta_g(g_{ij}) = g^{ap} g^{bp} \partial_i g_{ab} \partial_j g_{pq} - \frac{1}{e^u - 1} g_{ij} + \frac{1}{e^u - 1} (g_0)_{ij}.
$$

where $g$ is the metric associated to $e^u F_u \eta$. Thus, a standard bootstrapping argument (see e.g. [15]) will give the higher order estimates for $e^u F_u \eta$ and then, by restricting to $\{0\} \times B_1$ and using a finite cover by local charts of $X_s$, the desired estimate (2.46) follows.

On the other hand, for any fixed $\bar{s} \in V' \subset \Sigma_{reg}$ if we define a function $g$ on $f^{-1}(V') \times [2, \infty)$ by

$$
g = \frac{e^u \eta|_{X_s}}{\omega_{SF}|_{X_s}},
$$

then

$$
e^u \eta(u)|_{X_s} = g \cdot \omega_{SF}|_{X_s},
$$

and

$$
g = \frac{e^u \eta|_{X_s}}{\omega_{SF}|_{X_s}} = \frac{e^u \eta \wedge f^* \chi_\infty}{\omega_{SF} \wedge f^* \chi_\infty} = \frac{(tr \eta f^* \chi_\infty) e^{u \eta}}{2 \omega_{SF} \wedge f^* \chi_\infty} = \frac{(tr \eta f^* \chi_\infty) e^{u \eta - \psi}}{e^{\psi}}.
$$

Since (2.46) holds uniformly when $s$ varies in any compact subset of $\Sigma_{reg}$, we can make use of the estimates obtained above and [33, Lemma 2.4] to conclude that

$$
\sup_{f^{-1}(V')} |g - 1| \to 0
$$

and hence

$$
\|e^u \eta|_{X_s} - \omega_{SF}|_{X_s}\|_{C^0(X_s, \omega_0)|_{X_s}} \to 0
$$

as $u \to \infty$. Combining (2.46), we have proved Proposition 2.17, i.e., part (2) of Theorem 1.1. □

**Remark 2.18.** Note that the convergence (2.52) holds uniformly as $s$ varies in any given compact subset of $\Sigma_{reg}$. If we set $\tilde{\eta}_u = e^{-u} \omega_{SF} + (1 - e^{-u}) f^* \chi_\infty$, then as in [33] one easily obtains that for any given $V \subset X_{reg}$,

$$
\|\eta(u) - \tilde{\eta}_u\|_{C^0(V, \omega_0)} \to 0
$$

and hence

$$
\|\eta(u) - f^* \chi_\infty\|_{C^0(V, \omega_0)} \to 0
$$

as $u \to \infty$. 

3. Diameter bounds and Gromov-Hausdorff convergence

In this section we first recall the following lemma, which is proved in [26, Lemma 3.4] and [14, Section 3.3]. Recall that $\omega_{SF}$ is the semi-flat $(1,1)$-form on $X_{reg}$ defined by [26, Lemma 3.1] and $F = \frac{\Omega}{\omega_{SF} \wedge \chi}$.

**Lemma 3.1 ([26, 14]).** Let $\Delta_r$ be a disk centered at 0 of small diameter $r$ with respect to $\chi$ such that all fibers $X_s, s \neq 0$, are nonsingular. Let $\Delta_r^* = \Delta_r \setminus \{0\}$. Then there exist two constants $C > 1$ and $0 < \beta < 1$ such that for small $r$,

$$F(s) \leq \frac{C}{|s|^{2\beta}}$$

(3.1)

for all $s \in \Delta_r^*$.

**Remark 3.2.** Combining the results in [26] and [14], $\beta = \max\{\frac{5}{6}, 1 - \frac{1}{2m_1}, \ldots, 1 - \frac{1}{2m_k}\}$ will satisfy Lemma 3.1. Here $m_i$'s are the multiplicities of singular fibers as in Section 1.

Consequently, we have

**Proposition 3.3.** There exists a constant $C > 1$ such that for small $r$

$$\text{diam}(\Delta_r^*, \chi_{\infty}) \leq Cr^{1-\beta}.$$  

(3.2)

**Proof.** Without loss of any generality, we assume that $\Delta_r$ is the standard disc in $\mathbb{C}$ and $\chi \leq \omega_E$ on $\Delta_r$, where $\omega_E$ is the standard flat metric on $\Delta_r$. For any fixed two points $p_1, p_2$ in $\Delta_r^*$, we express them in polar coordinates by $p_1 = \rho_1 e^{\sqrt{-1} \theta_1}$, $p_2 = \rho_2 e^{\sqrt{-1} \theta_2}$, where $\rho_1, \rho_2 \in (0, r)$ and $\theta_1, \theta_2 \in [0, 2\pi)$. Without loss of any generality, we assume $\rho_1 < \rho_2$, $\theta_1 < \theta_2$. Set $p_3 = \rho_2 e^{\sqrt{-1} \theta_1}$. Then

$$d_{\chi_{\infty}}(p_1, p_2) \leq d_{\chi_{\infty}}(p_1, p_3) + d_{\chi_{\infty}}(p_3, p_2).$$

We connect $p_1$ and $p_3$ by $\gamma_1(t) = te^{\sqrt{-1} \theta_1}$, $t \in [\rho_1, \rho_2]$ and connect $p_3$ and $p_2$ by $\gamma_2(s) = \rho_2 e^{\sqrt{-1} \theta_1}$, $s \in [\theta_1, \theta_2]$. Then we have

$$L_{\chi_{\infty}}(\gamma_1) \leq C \int_{\rho_1}^{\rho_2} \sqrt{F(\gamma_1(t))} \, dt$$

and

$$L_{\chi_{\infty}}(\gamma_2) \leq C \rho_2 \int_{\theta_1}^{\theta_2} \sqrt{F(\gamma_2(s))} \, ds.$$  

Now by Lemma 3.1, we have

$$F(\gamma_1(t)) \leq C|\gamma_1(t)|^{-2\beta} = C|t|^{-2\beta}.$$  

Therefore

$$L_{\chi_{\infty}}(\gamma_1) \leq C \int_{\rho_1}^{\rho_2} |t|^{-\beta} \, dt \leq Cr^{1-\beta}.$$  

On the other hand,

$$L_{\chi_{\infty}}(\gamma_2) \leq C \rho_2^{1-\beta} \int_{\theta_1}^{\theta_2} ds \leq Cr^{1-\beta}.$$  

Thus the (3.3) is proved. \qed
Now it can be seen that the metric completion \((X_\infty,d_\infty)\) of \((\Sigma_{reg},\chi_\infty)\) is compact and \(X_\infty\) is homeomorphic to \(\Sigma\). In fact, for any \(s \in \Sigma_{reg}\) and \(s_i\), define \(d_\infty(s,s_i) = \lim_{t \to \gamma} d_\infty(s,r'_j)\), where \(\{r'_j\}\) is a sequence contained in \(\Sigma_{reg}\) and converges to \(s_i\). Note that Proposition 3.3 implies that this is well-defined. One can define \(d_\infty(s_i,s_j)\) similarly.

In particular, we have proved part (3) of Theorem 1.1 i.e.,

**Proposition 3.4.** \((X_\infty,d_\infty)\) is a compact length metric space and \(X_\infty\) is homeomorphic to \(\Sigma\) as a projective variety.

We will denote the metric completion of \((\Sigma_{reg},\chi_\infty)\) by \((\Sigma,d_\infty)\) and its diameter by \(D_\infty\).

In the following, without loss of any generality we assume \(\Sigma\{s_0\}\). For small \(\delta > 0\), let \(B_\infty(s_0,\delta)\) be the ball centered at \(s_0\) of radius \(\delta\) with respect to \(d_\infty\), \(K'_\delta := \Sigma\backslash B_\infty(s_0,\delta)\) and \(K_\delta := f^{-1}(K'_\delta)\). Similarly let \(H'_\delta = \Sigma\backslash B_\chi(s_0,\delta)\) and \(H_\delta := f^{-1}(H'_\delta)\). We remark that, if \(\delta\) is sufficiently small, one can assume that \(B_\chi(s_0,\delta)\) is a standard disc in \(\mathbb{C}\). Moreover Proposition 3.3 implies that there exists a uniform constant \(N > 1\) such that for small \(\delta\),

\[
B_\chi(s_0,\delta) \subset B_\infty(s_0,N\delta^{\frac{1}{N}}). \tag{3.3}
\]

To begin with, we have

**Lemma 3.5.**

\[
d_{GH}((K'_\delta,d_\infty),(\Sigma,d_\infty)) \leq \delta. \tag{3.4}
\]

**Lemma 3.6.** For any small \(\delta > 0\), there exists a constant \(T_\delta\) such that for all \(t \geq T_\delta\),

\[
diam(K_\delta,d_\omega(t)) \leq D_\infty + 1. \tag{3.5}
\]

**Proof.** For any fixed \(p,q \in K_\delta\), one can choose a piecewise smooth curve \(\gamma(z) \subset H'_{(\frac{\delta}{10N})^N}, z \in [0,1]\), connecting \(f(p)\) and \(f(q)\), such that

\[
L_{\chi_\infty}(\gamma) \leq d_\infty(f(p),f(q)) + \frac{\delta}{4}. \tag{3.6}
\]

This can be chosen as follows: first choose a piecewise smooth curve \(\tilde{\gamma} \subset \Sigma_{reg}\) connecting \(f(p),f(q)\) with \(L_{\chi_\infty}(\tilde{\gamma}) \leq d_\infty(f(p),f(q)) + \frac{\delta}{10}\). If \(\tilde{\gamma}\)∩\(B_\chi(s_0,(\frac{\delta}{10N})^N) = \emptyset\), we are done; otherwise we replace the part of \(\tilde{\gamma}\) contained in \(B_\chi(s_0,(\frac{\delta}{10N})^N)\) by a curve lies in \(\partial B_\chi(s_0,(\frac{\delta}{10N})^N)\) with length with respect to \(\chi_\infty\) no more than \(\frac{\delta}{10}\) (here we have identified \(B_\chi(s_0,(\frac{\delta}{10N})^N)\) with some small standard disc in \(\mathbb{C}\)). We obtain a curve \(\gamma\) as desired.

We will lift \(\gamma\) to a curve in \(H_{(\frac{\delta}{10N})^N}\) connecting \(p, q\).

First, without loss of any generality, we assume that \(\gamma(z), z \in [0,1]\), is smooth and covered by two open subsets \(U, V\) of \(\Sigma_{reg}\) such that \(f^{-1}(U) = U \times E, f^{-1}(V) = V \times E\), where \(E\) is a smooth fiber and both equalities mean diffeomorphisms. Fix a point \(r' := \gamma(z_1) \in U \cup V\). Define \(\gamma_1(z) = (\gamma(z), e_1)\) for some \(e_1 \in E\) with \(p = (\gamma(0), e_1), z \in [0, z_1]\) and \(\gamma_2(z) = (\gamma(z - 1), e_2)\) for some \(e_2 \in E\) with \(q = (\gamma(1), e_2), z \in [z_1 + 1, 2]\). Also connect \(\gamma_1(z_1)\) and \(\gamma_2(z_1 + 1)\) by a curve \(\gamma_3(z) \subset E, z \in [z_1, z_1 + 1]\). Now by connecting \(\gamma_1, \gamma_3\) and \(\gamma_2\) we obtain a curve \(\sigma(z) \subset H'_{(\frac{\delta}{10N})^N}, z \in [0, 2]\), connecting \(p\) and \(q\). Note that the diameter of smooth fibers over \(H'_{(\frac{\delta}{10N})^N}\) will go to zero uniformly as \(t \to \infty\), so we can choose \(\gamma_3\) with arbitrarily small length (with respect to \(\omega(t)\)) as long as \(t\) is large enough. 
Then we can find a $T_\delta$ such that for $t \geq T_\delta$,

$$d_{\omega(t)}(p, q) \leq L_{\omega(t)}(\sigma)$$

$$\leq L_{\omega}(\gamma_1) + L_{\omega}(\gamma_2) + \frac{\delta}{4}$$

$$\leq L_{f^*\chi^\infty}(\gamma_1) + L_{f^*\chi^\infty}(\gamma_2) + \frac{\delta}{2}$$

$$= L_{\chi^\infty}(\gamma) + \frac{\delta}{2}$$

$$\leq d_\infty(f(p), f(q)) + \frac{3\delta}{4}, \quad (3.7)$$

where in the third inequality we have used Theorem 2.14. Thus we obtain

$$d_{\omega(t)}(p, q) \leq L_{\omega(t)}(\sigma)$$

$$\leq L_{\omega(t)}(\gamma_1) + L_{\omega(t)}(\gamma_2)$$

$$\leq L_{f^*\chi^\infty}(\gamma_1) + L_{f^*\chi^\infty}(\gamma_2) + \frac{\delta}{2}$$

$$= L_{\chi^\infty}(\gamma) + \frac{\delta}{2}$$

$$\leq d_\infty(f(p), f(q)) + \frac{3\delta}{4}, \quad (3.7)$$

where in the third inequality we have used Theorem 2.14. Thus we obtain

$$diam(K_\delta, d_{\omega(t)}) \leq D_\infty + \delta.$$

We complete the proof by choosing small $\delta < 1$. \hfill \Box

**Lemma 3.7.** There exist positive constants $D_1$ and $T_1$ such that for all $t \geq T_1$,

$$diam(X, \omega(t)) \leq D_1. \quad (3.8)$$

**Proof.** Let $\epsilon > 0$ be arbitrary. By Lemma 3.6, we can choose a $K_\delta$ and $T_1$ such that for all $t \geq T_1$,

$$diam(K_\delta, d_{\omega(t)}) \leq D_\infty + 1. \quad (3.9)$$

Using the volume estimate (2.4) along the continuity method and the fact that $X \setminus X_{reg}$ has real codimension 2 (it is a proper analytic subvariety of $X$), by decreasing $\delta$ if necessary, we have

$$Vol_{\omega(t)}(X \setminus K_\delta) \leq (1 + t)^{-1}\epsilon \quad (3.10)$$

for all $t \geq T_1$.

Let $x_t \in X \setminus K_\delta$ be a point which achieves the maximal distance $R_t$ to $K_\delta$ in $(X, \omega(t))$. Then $B_{\omega(t)}(x_t, R_t) \subset X \setminus K_\delta$. On the one hand we have

$$\frac{Vol(B_{\omega(t)}(x_t, R_t + D_\infty + 1))}{Vol(B_{\omega(t)}(x_t, R_t))} \geq \frac{Vol_{\omega(t)}(X) - Vol_{\omega(t)}(X \setminus K_\delta)}{Vol_{\omega(t)}(B_{\omega(t)}(x_t, R_t))}$$

$$\geq \frac{(1 + t)^{-1}(C_1 - \epsilon)}{(1 + t)^{-1}\epsilon}$$

$$\geq C_2\epsilon^{-1}. \quad (3.11)$$

On the other hand, by the uniform lower bound of Ricci curvature (1.3), we have

$$\frac{Vol(B_{\omega(t)}(x_t, R_t + D_\infty + 1))}{Vol(B_{\omega(t)}(x_t, R_t))} \leq \frac{\int_{0}^{R_t + D_\infty + 1} sinh^3 v dv}{\int_{0}^{R_t} sinh^3 v dv}. \quad (3.12)$$

Thus if we choose $\epsilon$ small enough, $R_t$ will be uniformly bounded and the Lemma 3.7 is proved. \hfill \Box

**Lemma 3.8.** For any small $\epsilon > 0$, there exists a $K_\delta$ and a positive constant $T_2$ such that for all $t \geq T_2$

$$d_{GH}((K_\delta, d_{\omega(t)}), (X, d_{\omega(t)})) \leq \epsilon. \quad (3.13)$$
Proof. The argument in the proof of Lemma 3.7 implies that, for any \( \epsilon > 0 \), there exist a \( K_\delta \), two positive constants \( C_2 \) and \( T_2 \) such that for all \( t \geq T_2 \),
\[
C_2\epsilon^{-5} \leq \frac{\int_{0}^{R_t} \sinh^3 v \, dv}{\int_{0}^{R_t} \sinh^3 v \, dv}.
\]
(3.14)

Since we have known that \( R_t \) is uniformly bounded, (3.14) gives
\[
\int_{0}^{R_t} \sinh^3 v \, dv \leq C_3\epsilon^5,
\]
which implies that, if \( \epsilon \) small enough, for all \( t \geq T_2 \) we have
\[
R_t \leq \epsilon.
\]
Thus we conclude (3.13).

□

Lemma 3.9. For any fixed \( \delta > 0 \), there exists a \( \delta_0 \) and a \( T_3 > 0 \) such that for any \( p, q \in K_\delta \) and \( t \geq T_3 \), one can find a curve \( \gamma_t \subset K_{\delta_0} \) connecting \( p \) and \( q \) such that
\[
L_{\omega(t)}(\gamma_t) \leq d_{\omega(t)}(p, q) + \frac{\delta}{4}.
\]
(3.15)

Proof. This can be proved by an argument similar to that in the proof of Lemma 3.6. In fact, if the minimal geodesic \( \gamma_t \) in \((X, \omega(t))\) connecting \( p \) and \( q \) intersects \( f^{-1}(B_\chi(s_0, (\frac{\delta}{10N}))^N) \), then we may replace the part of \( \gamma_t \) contained in \( f^{-1}(B_\chi(s_0, (\frac{\delta}{10N}))^N) \) by lifting suitably a curve \( \gamma' \subset \partial B_\chi(s_0, (\frac{\delta}{10N})) \) and obtain a new curve, still denote it by \( \gamma_t \), satisfying (3.15). Now we complete the proof by choosing a sufficiently large \( T_3 \) and a \( \delta_0 < \delta \) with \( H_{\chi(\frac{\delta}{10N})^N} \subset K_{\delta_0} \).

□

Lemma 3.10. There exists a \( T_\delta > 0 \) such that for all \( t \geq T_\delta \),
\[
d_{GH}((K_\delta, d_{\omega(t)}), (K'_{\delta}, d_{\infty})) \leq \delta.
\]
(3.16)

Proof. Define a map \( g : K'_{\delta} \rightarrow K_{\delta} \) by mapping every point \( s \in K'_{\delta} \) to some chosen point \( g(s) \) in \( X \). Given arbitrary \( p, q \in K_\delta \) and \( s, r \in K'_{\delta} \), By Lemma 3.9, for \( t \) large enough, we can find a curve \( \gamma_t \subset K_{\delta_0} \) connecting \( p \) and \( q \) such that
\[
L_{\omega(t)}(\gamma_t) \leq d_{\omega(t)}(p, q) + \frac{\delta}{4}.
\]
(3.17)

By Theorem 2.14 we can find a \( T_\delta \) such that for all \( t \geq T_\delta \),
\[
L_{\omega(t)}(\gamma_t) \geq L_{f \circ \chi}(\gamma_t) - \frac{\delta}{4}
\[
\geq L_{\chi}(f \circ \gamma_t) - \frac{\delta}{4}
\[
\geq d_{\infty}(f(p), f(q)) - \frac{\delta}{4}.
\]
Therefore
\[
d_{\infty}(f(p), f(q)) \leq d_{\omega(t)}(p, q) + \delta.
\]
(3.18)
A similar argument gives
\[
d_{\infty}(s, r) \leq d_{\omega(t)}(g(s), g(r)) + \delta.
\]
(3.19)
On the other hand, using the argument in Lemma 3.6, we can show that, for all \( t \geq T_\delta \) (increase \( T_\delta \) if necessary),

\[
d_{\omega(t)}(g(s), g(r)) \leq d_\infty(s, r) + \delta,
\]

(3.20)

and

\[
d_{\omega(t)}(p, q) \leq d_\infty(f(p), f(q)) + \delta.
\]

(3.21)

Also of course we have, for any \( s \in K'_\delta \),

\[
d_\infty(s, f \circ g(s)) \equiv 0
\]

(3.22)

and for any \( p \in K_\delta \),

\[
d_{\omega(t)}(p, g \circ f(p)) \leq \delta
\]

(3.23)

for all \( t \geq T_\delta \). Combining (3.18), (3.19), (3.20), (3.21), (3.22) and (3.23), we conclude Lemma 3.10.

□

Now we are ready to prove part (4) of Theorem 1.1, i.e.,

**Theorem 3.11.** \((X, \omega(t)) \to (\Sigma, d_\infty)\) in Gromov-Hausdorff topology as \( t \to \infty \).

**Proof.** For any small \( \epsilon > 0 \), we fix a \( \delta > 0 \) satisfying

1. \( \delta < \frac{\epsilon}{4} \);
2. \( d_{GH}((K'_\delta, d_\infty), (\Sigma, d_\infty)) < \frac{\epsilon}{4} \);
3. \( d_{GH}((K_\delta, d_{\omega(t)}), (X, d_{\omega(t)})) < \frac{\epsilon}{4} \) for any \( t \geq T_\epsilon \);
4. \( d_{GH}((K_\delta, d_{\omega(t)}), (K'_\delta, d_\infty)) < \frac{\epsilon}{4} \) for any \( t \geq T_\epsilon \).

Note that (2), (3) and (4) are guaranteed by Lemma 3.5, Lemma 3.8 and Lemma 3.10 respectively. In fact the constant \( T_\epsilon \) may also depend on \( \delta \). However \( \delta \) is a fixed constant determined by \( \epsilon \). Therefore \( T_\epsilon \) only depends on \( \epsilon \) and Theorem 3.11 is proved. □

We end this paper by a remark on the Kähler-Ricci flow.

**Remark 3.12.** Consider the Kähler-Ricci flow \( \bar{\omega}(t) \) on \( X \) starting from \( \omega_0 \),

\[
\begin{align*}
\partial_t \bar{\omega}(t) &= -\text{Ric}(\bar{\omega}(t)) - \bar{\omega}(t), \\
\bar{\omega}(0) &= \omega_0,
\end{align*}
\]

which exists for all \( t \geq 0 \) [3, 30, 36] and converges to \( \text{f}^*\chi_\infty \) in \( C^\infty_{\text{loc}}(X_{\text{reg}}, \omega_0) \)-topology [8] (see also [32]). If \( f : X \to \Sigma \) has only singular fibers of type \( mI_0 \), i.e., every singular fiber \( X_s \) is a smooth elliptic curve of some non-trivial multiplicity \( m_i \) (see, e.g., [22] for more discussions on all possible singular fibers), it is shown in [35] that \( |\text{Rm}(\bar{\omega}(t))|_{\bar{\omega}(t)} \) is uniformly bounded on \( X \times [0, \infty) \). In particular, \( \text{Ric}(\bar{\omega}(t)) \) is uniformly bounded from below on \( X \times [0, \infty) \). Thus in this case we can apply the same arguments to show that, as \( t \to \infty \), \((X, \bar{\omega}(t)) \to (\Sigma, d_\infty)\) in Gromov-Hausdorff topology [4]. Note that if \( X \) has no singular fiber, the Gromov-Hausdorff convergence is known, see [8, 33].

\(^1\) V. Tosatti points out to us that if the only singular fibers are of type \( mI_0 \), then \( X \) has a global finite covering space which is an elliptic bundle and hence one can also use the similar arguments in [34] to conclude this result.
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