On an optimal interpolation formula in $K_2(P_2)$ space
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Abstract. The paper is devoted to construction of an optimal interpolation formula in $K_2(P_2)$ Hilbert space. Here the interpolation formula consists of a linear combination $\sum_{\beta=0}^{N} C_\beta(z) \varphi(x_\beta)$ of given values of a function $\varphi$ from the space $K_2(P_2)$. The difference between functions and the interpolation formula is considered as a linear functional called the error functional. The error of the interpolation formula is estimated by the norm of the error functional. We obtain the optimal interpolation formula by minimizing the norm of the error functional by coefficients $C_\beta(z)$ of the interpolation formula. The obtained optimal interpolation formula is exact for trigonometric functions $\sin \omega x$ and $\cos \omega x$. At the end of the paper we give some numerical results which confirm our theoretical results.

Keywords: extremal function, error functional, Hilbert space, optimal interpolation formula, optimal coefficients.

Mathematics Subject Classification (2010): 41A15.

1 Introduction and statement of the Problem

There are algebraic and variational approaches in the spline theory. In the algebraic approach splines are considered as some smooth piecewise polynomial functions. In the variational approach splines are elements of Hilbert or Banach spaces minimizing certain functionals. The first spline functions were constructed from pieces of cubic polynomials. After that, this construction was modified, the degree of polynomials increased. The theory of splines based on variational methods studied and developed, for example, by J.H.Ahlberg et al. [1], C. de Boor [3], A.R.Hayotov, G.V.Milovanivi´ c and Kh.M.Shadimetov [5], I.J.Schoenberg [9], L.L.Schumaker [7], S.L.Sobolev [8], V.A.Vasilenko [12] and others.

The present work is also devoted to the variational method of construction of optimal interpolation formulas.

Assume, we are given the table of the values $\varphi(x_\beta)$, $\beta = 0,1,...,N$ of a function $\varphi$ at the points $x_\beta \in [0,1]$. It is required to approximate the
function $\varphi$ by another more simple function $P_\varphi$, i.e.

$$\varphi(x) \equiv P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x) \cdot \varphi(x_\beta)$$

(1.1)

which satisfies the following interpolation conditions

$$\varphi(x_\beta) = P_\varphi(x_\beta), \beta = 0, 1, ..., N.$$  

Here $C_\beta(x)$ and $x_\beta (\in [0,1])$ are the coefficients and the nodes of the interpolation formula (1.1), respectively. By $K_2(P_2)$ we denote the class of all functions $\varphi$ defined on $[0,1]$ which posses an absolutely continuous first derivative on $[0,1]$ and whose second derivative is in $L_2(0,1)$. The class $K_2(P_2)$ under the pseudo-inner product

$$\langle \varphi, \psi \rangle = \int_{0}^{1} \left( \varphi''(x) + \omega^2 \varphi(x) \right) \left( \psi''(x) + \omega^2 \psi(x) \right) dx.$$  

is a Hilbert space if we identify functions that differ by $\sin(\omega x)$ and $\cos(\omega x)$, where $\omega \neq 0$. Here we consider the norm

$$\| \varphi | K_2(P_2) \| = \left\{ \int_{0}^{1} \left( \varphi''(x) + \omega^2 \varphi(x) \right)^2 dx \right\}^{1/2}.$$  

For the fixed $z \in [0,1]$ the error

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx = \varphi(z) - P_\varphi(z) = \varphi(z) - \sum_{\beta=0}^{N} C_\beta(z) \cdot \varphi(x_\beta)$$

(1.2)

of the interpolation formula (1.1) is a linear functional.

Here, in fixed $z \in [0,1]$,

$$\ell(x, z) = \delta(x - z) - \sum_{\beta=0}^{N} C_\beta(z) \delta(x - x_\beta),$$

and is the error functional of the interpolation formula (1.1) belongs to the space $K_2^*(P_2)$. Here $K_2^*(P_2)$ is the conjugate space to the space $K_2(P_2)$, $\delta$ is Dirac’s delta-function.
By the Cauchy-Schwarz inequality the absolute value of the error (1.2) is estimated as follows
\[ |(\ell, \varphi)| \leq \|\varphi|_{K_2^*(P_2)}\| \cdot \|\ell|_{K_2^*(P_2)}\|, \]
where
\[ \|\ell|_{K_2^*(P_2)}\| = \sup_{\varphi, \|\varphi\| \neq 0} \frac{|(\ell, \varphi)|}{\|\varphi|_{K_2^*(P_2)}\|.} \]

Therefore, in order to estimate the error of the interpolation formula (1.1) on functions of the space \(K_2^*(P_2)\) it is required to find the norm of the error functional \(\ell\) in the conjugate space \(K_2^*(P_2)\). That is we get the following problem.

**Problem 1.1.** Find the norm of the error functional \(\ell\) of the interpolation formula (1.1) in the space \(K_2^*(P_2)\).

It is clear that the norm of the error functional \(\ell\) depends on the coefficients \(C_\beta(z)\) and the nodes \(x_\beta\). The problem of minimization of the quantity \(\|\ell\|\) by coefficients \(C_\beta(z)\) is the linear problem and by nodes \(x_\beta\) is, in general, nonlinear and complicated problem. We consider the problem of minimization of the quantity \(\|\ell\|\) by coefficients \(C_\beta(z)\) when nodes \(x_\beta\) are fixed.

The coefficients \(\mathring{C}_\beta(z)\) (if there exist) satisfying the equality
\[ \|\mathring{\ell}|_{K_2^*(P_2)}\| = \inf_{C_\beta(z)} \|\ell|_{K_2^*(P_2)}\| \tag{1.3} \]
are called the optimal coefficients and corresponding interpolation formula \(\mathring{P}_\varphi(z) = \sum_{\beta=0}^N \mathring{C}_\beta(z)\varphi(x_\beta)\) is called the optimal interpolation formula in the space \(K_2^*(P_2)\). Therefore, for construction of the interpolation formula we should solve the next problem.

**Problem 1.2.** Find the coefficients \(\mathring{C}_\beta(z)\) which satisfy equality (1.3) when the nodes \(x_\beta\) are fixed.

It should be noted that Problems 1.1 and 1.2 were solved in [2] in the Hilbert space \(W_2^{(m,m-1)}\). There the optimal interpolation formulas, which are exact for any polynomial of degree \(m-2\) and for the function \(\exp(-x)\), were obtained.

The rest of the paper is organized as follows. In Section 2, using the extremal function, the norm of the error functional is found. Existence and
uniqueness of the optimal interpolation formula of the form (1.1) is discussed in Section 3. Section 4 is devoted to construction of the optimal interpolation formula. Finally, in Section 5 some numerical results are presented.

2 The extremal function and the norm of the error functional $\ell$

Here we find explicit form of the norm of the error functional $\ell$.

For finding the explicit form of the norm of the error functional $\ell$ in the space $K_2(P_2)$ we use its extremal function which was introduced by Sobolev [8, 9]. The function $\psi_\ell$ from $K_2(P_2)$ space is called the extremal function for the error functional $\ell$ if the following equality is fulfilled

$$ (\ell, \psi_\ell) = \|\ell|K_2^*(P_2)\| \cdot \|\psi_\ell|K_2(P_2)\|. $$

According to the Riesz theorem any linear continuous functional $\ell$ in a Hilbert space is represented in the form of an inner product. So, in our case, for any function $\varphi$ from $K_2(P_2)$ space we have

$$ (\ell, \varphi) = \langle \psi_\ell, \varphi \rangle. \quad (2.1) $$

Here $\psi_\ell$ is the function from $K_2(P_2)$ is defined uniquely by functional $\ell$ and is the extremal function.

It is easy to see from (2.1) that the error functional $\ell$, defined on the space $K_2(P_2)$, satisfies the following equalities

$$ (\ell, \sin(\omega x)) = 0, \quad (2.2) $$

$$ (\ell, \cos(\omega x)) = 0. \quad (2.3) $$

The equalities (2.2) and (2.3) mean that our interpolation formula is exact for the functions $\sin(\omega x)$ and $\cos(\omega x)$.

The equation (2.1) was solved in [5] and for the extremal function $\psi_\ell$ was obtained the following expression

$$ \psi_\ell(x) = (\ell * G_2)(x) + d_1 \sin(\omega x) + d_2 \cos(\omega x), \quad (2.4) $$

where

$$ G_2(x) = \frac{\text{sgn} x}{4\omega^3} [\sin(\omega x) - \omega x \cos(\omega x)], \quad (2.5) $$
Optimal interpolation formulas in $K_2(P_2)$ space

* is the operation of convolution which for the functions $f$ and $g$ is defined as follows

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dy = \int_{-\infty}^{\infty} f(y)g(x - y) \, dy.$$ (2.6)

Now we obtain the norm of the error functional $\ell$. Since the space $K_2(P_2)$ is the Hilbert space then by the Riesz theorem we have

$$(\ell, \psi_\ell) = \|\ell\| \cdot \|\psi_\ell\| = \|\ell\|^2.$$ (2.7)

Hence, using (2.4) and (2.5), taking into account (2.6) and (2.7), we get

$$\|\ell\|^2 = (\ell, \psi_\ell) = \int_{-\infty}^{\infty} \ell(x, z)\psi_\ell(x) \, dx$$

$$= \int_{-\infty}^{\infty} \left( \delta(x - z) - \sum_{\beta=0}^{N} C_\beta(z)\delta(x - x_\beta) \right)$$

$$\times \left( G_2(x - z) - \sum_{\beta=0}^{N} C_\beta(z)G_2(x - x_\beta) \right) \, dx.$$

Hence, keeping in mind that $G_2(x)$, defined by (2.5), is the even function, we have

$$\|\ell\|^2 = (-1)^m \left( \sum_{\beta=0}^{N} \sum_{\gamma=0}^{N} C_\beta(z)C_\gamma(z)G_2(x_\beta - x_\gamma) - 2 \sum_{\beta=0}^{N} C_\beta(z)G_2(z - x_\beta) \right).$$ (2.8)

Thus, Problem 1.1 is solved.

Further, we solve Problem 1.2.

3 Existence and uniqueness of the optimal interpolation formula

Assume that the nodes $x_\beta$ of the interpolation formula (1.1) are fixed. The error functional (1.2) satisfies the conditions (2.2) and (2.3). The norm of the error functional $\ell$ is a multivariable function with respect to the coefficients
$C_\beta(z)$ ($\beta = 0, N$). For finding the point of the conditional minimum of the expression (2.8) under the conditions (2.2) and (2.3) we apply the Lagrange method.

Consider the function

$$\Psi(C_0(z), C_1(z), ..., C_N(z), d_1, d_2)$$

$$= \| \ell \|^2 - 2 \left( d_1(z)(\ell, \sin(\omega x)) + d_2(z)(\ell, \cos(\omega x)) \right).$$

Equating to 0 the partial derivatives of the function $\Psi$ by $C_\beta(z)$ ($\beta = 0, N$), $d_1(z)$ and $d_2(z)$, we get the following system of $N + 3$ linear equations of $N + 3$ unknowns

$$\sum_{\gamma=0}^{N} C_\gamma(z) G_2(x_\beta - x_\gamma) + d_1(z) \sin(\omega x_\beta) + d_2(z) \cos(\omega x_\beta) = G_2(z - x_\beta),$$

$$\beta = 0, 1, ..., N,$$  \hspace{1cm} (3.1)

$$\sum_{\gamma=0}^{N} C_\gamma(z) \sin(\omega x_\gamma) = \sin(\omega z),$$  \hspace{1cm} (3.2)

$$\sum_{\gamma=0}^{N} C_\gamma(z) \cos(\omega x_\gamma) = \cos(\omega z),$$  \hspace{1cm} (3.3)

where $G_2(x)$ is defined by equality (2.5).

The system (3.1)-(3.3) has a unique solution and this solution gives the minimum to $\| \ell \|^2$ under the conditions (3.2) and (3.3).

The uniqueness of the solution of the system (3.1)-(3.3) is proved as the uniqueness of the solution of the system (24)-(26) of the work [11].

Therefore, in fixed values of the nodes $x_\beta$ the square of the norm of the error functional $\ell$, being quadratic function of the coefficients $C_\beta(z)$, has a unique minimum in some concrete value $C_\beta(z) = \hat{C}_\beta(z)$.

Remark 3.1. It should be noted that by integrating both sides of the system (3.1)-(3.3) by $z$ from 0 to 1 we get the system (3.1)-(3.1) of the work [5]. This means that by integrating the optimal interpolation formula (1.1) in the space $K_2(P_2)$ we get the optimal quadrature formula of the form (1.1) in the same space (see [5]).
Remark 3.2. It is clear from the system (3.1)-(3.3) that for the optimal coefficients the following are true

\[ \hat{C}_\beta(h\gamma) = \begin{cases} 1, & \gamma = \beta, \quad \gamma = 0, 1, ..., N, \quad \beta = 0, 1, ..., N. \\ 0, & \gamma \neq \beta. \end{cases} \]

Below for convenience the optimal coefficients \( \hat{C}_\beta(z) \) we remain as \( C_\beta(z) \).

4 The algorithm for computation of coefficients of the optimal interpolation formula

In the present section we give the algorithm for solution of the system (3.1)-(3.3). Below mainly is used the concept of discrete argument functions and operations on them. The theory of discrete argument functions is given, for instance, in [9, 11]. For completeness we give some definitions about functions of discrete argument.

Assume that the nodes \( x_\beta \) are equal spaced, i.e. \( x_\beta = h\beta, \quad h = \frac{1}{N}, \quad N = 1, 2, ..., \)

Definition 4.1. The function \( \varphi(h\beta) \) is a discrete argument function (or discrete function) if it is given on some set of integer values of \( \beta \).

Definition 4.2. The inner product of two discrete functions \( \varphi(h\beta) \) and \( \psi(h\beta) \) is given by

\[ [\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta), \]

if the series on the right hand side converges absolutely.

Definition 4.3. The convolution of two functions \( \varphi(h\beta) \) and \( \psi(h\beta) \) is the inner product

\[ \varphi(h\beta) \ast \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma). \]

Now we turn on to our problem.
Suppose that $C_{\beta}(z) = 0$ when $\beta < 0$ and $\beta > N$. Thus we have the following problem.

**Problem 4.1** Find the discrete functions $C_{\beta}(z)$, $\beta = 0, 1, ..., N$, $d_1(z)$ and $d_2(z)$ which satisfy the system (3.2)-(3.3).

Further we investigate Problem 4.1 which is equivalent to Problem 1.2. Instead of $C_{\beta}(z)$ we introduce the following functions

$$v_2(h\beta) = G_2(h\beta) * C_\beta(z),$$
$$w_2(h\beta) = v_2(h\beta) + d_1(z)\sin(\omega h\beta) + d_2\cos(\omega h\beta).$$

(4.1)

Now we should express the coefficients $C_{\beta}(z)$ by the function $u(h\beta)$. For this we use the operator $D_2(h\beta)$ which satisfies the equality

$$D_2(h\beta) * G_2(h\beta) = \delta(h\beta),$$

(4.2)

where $\delta(h\beta)$ is equal to 0 when $\beta \neq 0$ and is equal to 1 when $\beta = 0$, i.e. $\delta(h\beta)$ is the discrete delta-function.

In [4] the operator $D_2(h\beta)$ which satisfies equation (4.2) is constructed and its some properties are studied.

The following theorems are proved in [4].

**Theorem 4.4.** The discrete analogue of the differential operator $\frac{d^4}{dx^4} + 2\omega^2 \frac{d^2}{dx^2} + \omega^4$ satisfying the equation (4.2) has the form

$$D_2(h\beta) = p \begin{cases} A\lambda^{\beta-1}, & |\beta| \geq 2, \\ 1 + A, & |\beta| = 1, \\ C + \frac{A}{\lambda}, & \beta = 0. \end{cases}$$

(4.3)

where

$$p = \frac{2\omega^3}{\sin(\omega h) - \omega h \cos(\omega h)},$$
$$C = \frac{2\omega \cos(2\omega h) - \sin(2\omega h)}{\sin(\omega h) - h \omega \cos(\omega h)},$$
$$A = \frac{(\lambda^2 - 1)(\sin(h\omega) - h \omega \cos(h\omega))^2}{(2\omega h)^2 \sin^4(h\omega) - \lambda^2},$$
$$\lambda = \frac{2\omega - \sin(2\omega) - 2\sin(\omega)\sqrt{(h\omega)^2 - \sin^2(h\omega)}}{2(h\omega \cos(h\omega) - \sin(h\omega))}, \quad |\lambda| < 1.$$

**Theorem 4.5.** The discrete analogue $D_2(h\beta)$ of the differential operator $\frac{d^4}{dx^4} + 2\omega^2 \frac{d^2}{dx^2} + \omega^4$ satisfies the following equalities

1) $D_2(h\beta) * \sin(h\omega \beta) = 0,$
2) $D_2(h\beta) * \cos(h\omega \beta) = 0,$
3) $D_2(h\beta) \ast (h\omega\beta) \cos(h\omega\beta) = 0,$
4) $D_2(h\beta) \ast (h\omega\beta) \sin(h\omega\beta) = 0.$

Then taking into account (4.2), (4.3), using Theorems 4.4 and 4.5, for optimal coefficients we have

$$C_\beta(z) = D_2(h\beta) \ast u_2(h\beta).$$ (4.4)

Thus if we find the function $u_2(h\beta)$ then the optimal coefficients $C_\beta(z)$ will be found from equality (4.4).

In order to calculate the convolution (4.4) it is required to find the representation of the function $u_2(h\beta)$ at all integer values of $\beta$. From equality (4.1) we get that $u_2(h\beta) = G_2(z - h\beta)$ when $h\beta \in [0, 1]$. Now we find the representation of the function $u_2(h\beta)$ when $\beta < 0$ and $\beta > N$.

Since $C_\beta(z) = 0$ when $h\beta \notin [0, 1]$ then

$$C_\beta(z) = D_2(h\beta) \ast u_2(h\beta) = 0, \quad h\beta \notin [0, 1].$$

Now we calculate the convolution $v_2(h\beta) = G_2(h\beta) \ast C_\beta(z)$ when $h\beta \notin [0, 1]$.

Suppose $\beta \leq 0$ and $\beta \geq N$ then taking into account equalities (2.5), (3.2) and (3.3), we have

$$u_2(h\beta) = \begin{cases} 
    d^-_1(z) \sin(\omega h\beta) + d^-_2(z) \cos(\omega h\beta) + \frac{h\beta}{4\omega^2} \cos(\omega(h\beta - z)), & \beta \leq 0, \\
    G_2(z - h\beta), & 0 \leq \beta \leq N, \\
    d^+_1(z) \sin(\omega h\beta) + d^+_2(z) \cos(\omega h\beta) + \frac{h\beta}{4\omega^2} \cos(\omega(h\beta - z)), & \beta \geq N, 
\end{cases}$$ (4.5)

where $d^-_1(z)$, $d^+_1(z)$, $d^-_2(z)$ and $d^+_2(z)$ are unknowns.

From (4.5) when $\beta = 0$ and $\beta = N$ we get

$$d^-_2(z) = G_2(z),$$ (4.6)

$$d^+_2(z) = \frac{1}{\cos(\omega)} \left[ G_2(z - 1) + \frac{\cos(\omega(1 - z))}{4\omega^2} - d^+_1(z) \sin(\omega) \right].$$ (4.7)

Thus, putting (4.6) and (4.7) to (4.5) we have the following explicit form of
the function $u_2(h\beta)$:

$$u_2(h\beta) = \begin{cases}
  d_1^-(z) \sin(\omega h\beta) + G_2(z) \cos(\omega h\beta) + \frac{h\beta}{4\omega^2} \cos(\omega(h\beta - z)), & \beta \leq 0, \\
  G_2(z - h\beta), & 0 \leq \beta \leq N, \\
  d_1^+(z) \frac{\sin(\omega h\beta - 1)}{\cos(\omega)} + \frac{\cos(\omega h\beta)}{\cos(\omega)} \left[ G_2(z - 1) + \frac{\cos(\omega(1-z))}{4\omega^2} \right], & \beta \geq N.
\end{cases}$$

(4.8)

In the last expression of the function $u_2(h\beta)$ we have only two unknowns $d_1^-(z)$ and $d_1^+(z)$.

Hence using (4.3) and (4.8) we get the following problem.

**Problem 4.2.** Find the solution of the equation

$$D_2(h\beta) * u_2(h\beta) = 0, \ h\beta \notin [0, 1],$$

having the form (4.8). Here $d_1^-(z)$ and $d_1^+(z)$ are unknowns.

Unknowns $d_1^-(z)$ and $d_1^+(z)$ we find from the equation

$$D_2(h\beta) * u_2(h\beta) = 0 \quad (4.9)$$

when $\beta = -1$ and $\beta = N + 1$. From the last equation

From the system (4.9) in the case $\beta = -1$ and after some simplifications we have the following system of equations

$$A_{11} d_1^-(z) + A_{12} d_1^+(z) = S_1 \quad (4.10)$$

where

$$A_{11} = -\sin(2\omega h) - C \sin(\omega h) - \frac{A \sin(\omega h)}{\lambda(\lambda^2 - 2\lambda \cos(\omega h) + 1)},$$

$$A_{12} = \frac{A\lambda \lambda^N \sin(\omega h)}{\cos(\omega)(\lambda^2 - 2\lambda \cos(\omega h) + 1)},$$

$$S_1 = \frac{1}{4\omega^2} \left( Ch \cos(\omega(h + z)) + 2h \cos(\omega(2h + z)) - \frac{A\lambda \lambda^N \cos(\omega(1-z))}{\cos(\omega)} \right) K_1$$

$$+ \frac{Ah}{\lambda^2} K_2 + Ah\lambda^N K_3 - AK_4 - \frac{A\lambda \lambda^N G_2(z-1)}{\cos(\omega)} K_1$$

$$- G_2(z) \left( C \cos(\omega h) + 1 + \cos(2\omega h) + \frac{A}{\lambda^2} K_5 \right),$$

$$K_1 = \frac{\lambda(\cos(\omega(1+h)) - \lambda \cos(\omega h) - 1)}{\lambda^2 - 2\lambda \cos(\omega h) + 1},$$

$$K_2 = \frac{\lambda(\lambda^2 \cos(\omega(h-z)) - 2\lambda \cos(\omega z) + \cos(\omega(h+z)))}{(\lambda^2 - 2\lambda \cos(\omega h) + 1)^2},$$
Optimal interpolation formulas in $K_2(P_2)$ space

\[ K_3 = \sum_{\gamma=1}^{\infty} \lambda^\gamma (N + \gamma) \cos(\omega(h(N + \gamma) - z)), \]
\[ K_4 = \sum_{\gamma=0}^{N} \lambda^\gamma G_2(z - h\gamma), \]
\[ K_5 = \frac{\lambda(\cos(\omega h) - \lambda)}{\lambda^2 - 2\lambda \cos(\omega h) + 1}. \]

Now, from (4.9) in the case $\beta = N + 1$ doing some calculations we get the next equation

\[ A_{21} d_1^-(z) + A_{22} d_1^+(z) = S_2, \quad (4.11) \]

where

\[ A_{21} = -\frac{A\lambda^{N+1} \sin(\omega h)}{\lambda^2 - 2\lambda \cos(\omega h) + 1}, \]
\[ A_{22} = \frac{1}{\cos(\omega)} \left[ C \sin(\omega h) + \sin(2\omega h) + \frac{A \sin(\omega h)}{\lambda(\lambda^2 - 2\lambda \cos(\omega h) + 1)} \right], \]
\[ S_2 = \frac{1}{4\omega^2} \left( A\lambda^{N} hK_2 + \cos(\omega(1 - z)) + Ch(N + 1) \cos(\omega(h(N + 1) - z)) \right. \]
\[ + h(N + 2) \cos(\omega(h(N + 2) - z)) + \frac{AhK_3}{\lambda^2} - \frac{\cos(\omega(1-z))K_6}{\cos(\omega)} \left. \right) \]
\[ - A\lambda^{N} K_7 - G_2(z)A\lambda^{N} K_5 - \frac{G_2(z-1)}{\cos(\omega)} K_6, \]
\[ K_6 = \cos(\omega) + C \cos(\omega h(N + 1)) + \cos(\omega h(N + 2)) + \frac{AK_1}{\lambda^2}, \]
\[ K_7 = \sum_{\gamma=0}^{N} \lambda^{-\gamma} G_2(z - h\gamma), \]
\[ K_8 = \frac{\lambda(\sin(\omega h + 1)) - \lambda \sin(\omega))}{\lambda^2 - 2\lambda \cos(\omega h) + 1}. \]

Then solving the system (4.10), (4.11) of equations we get $d_1^-(z)$ and $d_1^+(z)$. Finally, from (4.1) for $\beta = 0, 1, ..., N$ we get the explicit formulas for optimal coefficients as claimed in the following theorem.

**Theorem 4.6.** Coefficients of the optimal interpolation formula (1.1) with
equal spaced nodes in the space $K_2(P_2)$ have the following form

$$C_0(z) = p \left[ -d^-_1(z) \sin(\omega h) + d^-_2(z) \cos(\omega h) - \frac{1}{4\omega^2} h \cos(\omega(h + z)) + CG_2(z) \right. + G_2(z - h) + \frac{A}{\lambda} \left[ \sum_{\gamma=0}^{N} \lambda^\gamma G_2(z - h\gamma) + M_1 + \lambda^N N_1 \right],$$

$$C_{\beta}(z) = p \left[ G_2(z - h(\beta - 1)) + CG_2(z - h\beta) + G_2(z - h(\beta + 1)) \right. + \frac{A}{\lambda} \left[ \sum_{\gamma=0}^{N} \lambda^{\beta - \gamma} G_2(z - h\gamma) + \lambda^\beta M_1 + \lambda^{N - \beta} N_1 \right], \beta = 1, 2, \ldots, N - 1.$$

$$C_N(z) = p \left[ d^+_1(z) \sin(\omega h(N + 1)) + d^+_2(z) \cos(\omega h(N + 1)) \right. - \frac{1}{4\omega^2} h(N + 1) \cos(\omega(h(N + 1) - z)) + CG_2(z - 1) \left. + G_2(z - h(N - 1)) + \frac{A}{\lambda} \left[ \sum_{\gamma=0}^{N} \lambda^{N - \gamma} G_2(z - h\gamma) + \lambda^N M_1 + N_1 \right] \right],$$

Here

$$M_1 = -d^-_1(z) \frac{\lambda \sin(\omega h)}{\lambda^2 - 2\lambda \cos(\omega h) + 1} + d^-_2(z) K_5 - \frac{h}{4\omega^2} K_2,$$

$$N_1 = d^+_1(z) K_8 + d^+_2(z) K_1 - \frac{h}{4\omega^2} K_3,$$

$$d^-_1(z) = \frac{S_1 \cdot A_{22} - S_2 \cdot A_{12}}{A_{11} \cdot A_{22} - A_{21} \cdot A_{12}},$$

$$d^+_1(z) = \frac{S_2 \cdot A_{11} - S_1 \cdot A_{21}}{A_{11} \cdot A_{22} - A_{21} \cdot A_{12}},$$

$$d^-_2(z) = G_2(z),$$

$$d^+_2(z) = \frac{1}{\cos(\omega)} \left[ G_2(z - 1) + \frac{\cos(\omega(1 - z))}{4\omega^2} - d^+_1(z) \sin(\omega) \right].$$

5 Numerical results

In this section we give some numerical results.

First, when $N = 5$ using Theorem 4.6, we get the graphs of the coefficients of the optimal interpolation formulas

$$\varphi(z) \equiv \hat{P}_\varphi(z) = \sum_{\beta=0}^{5} \hat{C}_\beta(z) \varphi(h\beta), \ z \in [0, 1].$$
They are presented in Fig 1. These graphical results confirm Remark 3.2 for the case $N = 5$, i.e. for the optimal coefficients the following hold

$$\tilde{C}_\beta(h \gamma) = \delta_{\beta \gamma}, \quad \beta, \gamma = 0, 1, \ldots, 5,$$

where $\delta_{\beta \gamma}$ is the Kronecker symbol.

Now, in numerical examples, we interpolate the functions

$$\varphi_1(z) = z^2, \quad \varphi_2(z) = e^z \quad \text{and} \quad \varphi_3(z) = \sin z$$

by optimal interpolation formulas of the form (1.1) in the cases $N = 5, 10$, using Theorem 4.6. For the functions $\varphi_i, i = 1, 2, 3$, the graphs of absolute errors $|\varphi_i(z) - \tilde{P}_\varphi_i(z)|, i = 1, 2, 3$, are given in Fig 2, Fig 3, Fig 4. In these Figures one can see that by increasing value of $N$ absolute errors between optimal interpolation formulas and given functions are decreasing.
Fig. 3. Graphs of absolute errors for $N = 5$ and $N = 10$: $|\exp(z) - P_{\exp}(z)|$.

Fig. 4. Graphs of absolute errors for $N = 5$ and $N = 10$: $|\sin(z) - P_{\sin}(z)|$.

The Figure 4 shows the exactness of our optimal interpolation formula for the function $\sin(x)$. 

\[z\]
\[\begin{array}{c}
0.2 & 0.4 & 0.6 & 0.8 & 1 \\
\end{array}\]
\[\begin{array}{c}
1.4 \times 10^{-4} \\
1.0 \times 10^{-4} \\
6.0 \times 10^{-5} \\
\end{array}\]
\[\begin{array}{c}
1.5 \times 10^{-4} \\
1.0 \times 10^{-4} \\
5.0 \times 10^{-5} \\
\end{array}\]
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