A polynomial algorithm for the k-cluster problem on interval graphs

George B. Mertzios
Department of Computer Science,
RWTH Aachen University
mertzios@cs.rwth-aachen.de
1st February 2008

Abstract

This paper deals with the problem of finding, for a given graph and a given natural number k, a subgraph of k nodes with a maximum number of edges. This problem is known as the k-cluster problem and it is NP-hard on general graphs as well as on chordal graphs. In this paper, it is shown that the k-cluster problem is solvable in polynomial time on interval graphs. In particular, we present two polynomial time algorithms for the class of proper interval graphs and the class of general interval graphs, respectively. Both algorithms are based on a matrix representation for interval graphs. In contrast to representations used in most of the previous work, this matrix representation does not make use of the maximal cliques in the investigated graph.

Keywords: Interval graph, proper interval graph, polynomial algorithm, dynamic programming.

AMS classification: 05C85, 05C75, 68R10, 05C62.

1 Introduction

A graph $G$ is called an interval graph if its nodes can be assigned to intervals on the real line so that two nodes are adjacent in $G$ if and only if their assigned intervals intersect. The set of intervals assigned to the nodes of $G$ is called a realization of $G$. A proper interval graph is an interval graph that has an intersection model, in which no interval contains another one strictly. Interval and proper interval graphs have been studied extensively in the literature and several linear-time algorithms are known for their recognition [1, 2, 3]. They are important for their applications to scheduling problems, biology, VLSI circuit design, as well as to psychology and social sciences in general [4, 5].

The class of interval graphs is of major importance, while studying the complexity of several difficult optimization problems, which are solvable in polynomial time on them, but NP-hard in the general case. Some of these problems are the maximum clique [6], the maximum independent set [6, 7], the Hamiltonian cycle and the Hamiltonian path [8].

This paper deals with the problem of finding, for a given graph and a given natural number k, a subgraph on k nodes and of maximum number of edges. This problem is called the k-cluster problem. Until now it is known that the k-cluster problem is NP-hard as a generalization of the maximum clique problem. It remains NP-hard, even when restricted to comparability graphs, as well as on bipartite graphs and chordal graphs [9]. On the other side, it has been proved that
there are polynomial algorithms for the k-cluster problem on cographs, as well as on k-trees and split graphs [9]. Furthermore, it has been proved that the decision version of the k-cluster problem is solvable in polynomial time, when searching for fixed-density k-subgraphs, while it remains NP-hard, when searching for a k-subgraph with density at least \( f(k) = \Omega(k^\varepsilon) \) edges, for some \( \varepsilon > 0 \) [10]. Finally, there are also some other polynomial time algorithms designed for the k-cluster problem on some special classes of the proper interval graphs, e.g., of the graphs, whose clique graph is a simple path [11].

In the present work, it is proved that the k-cluster problem on proper interval graphs, as well as on the general class of interval graphs, is solvable in polynomial time and thus the corresponding open problem stated in [9] is answered. To this end, a matrix representation, which characterizes these classes of graphs, is used here. This representation does not use their maximal cliques, as the vast variety of the existing characterizations do.

2 The interval graphs in the general case

Without loss of generality, we may suppose that all intervals in a realization of an interval graph are closed, i.e. of the form \([a, b]\). However, this representation is too general. To this end, a more suitable interval representation form is presented in Definition 1 [12]. Recall that an interval graph can be recognized in linear time [1, 2]. In the following, suppose we are given a realization of an interval graph \(G\) on \(n\) nodes.

**Definition 1.** A representation of \(n\) intervals, having the following properties, is called a **Normal Interval Representation (NIR) form**:

1. all intervals are of the form \(\lbrack i, j\rbrack\), where \(0 \leq i < j \leq n\),
2. exactly one interval begins at \(i\), for every \(i \in \{0, 1, \ldots, n-1\}\).

Suppose we are given a realization of the interval graph \(G\). It can be converted to another realization of the same graph, in which all \(2n\) endpoints are distinct in the real line. This can be done simply by disturbing them sufficiently, so that the structure of the graph remains unchanged, under the condition that the relative order of the left endpoints of any two intervals is not being reversed. After that, the arbitrary closed interval \([a, b]\) may be replaced by \([a, b)\), since the intersection of any two intervals, if such occurs, is a non-trivial interval. In the sequel, any interval’s right endpoint may be moved to the next greater interval’s left endpoint in the current realization, resulting thus in exactly \(n + 1\) distinct endpoints altogether. Finally, all these endpoints may be moved bijectively to the points \(0, 1, \ldots, n\), obtaining thus an NIR form of \(G\) in linear time \(O(n)\).

**Lemma 1.** An arbitrary graph is an interval graph iff it can be represented by the NIR form.

**Proof.** An NIR form is clearly a set of intervals and thus it corresponds to an interval graph. Conversely, since any interval graph can be represented by an NIR form, this representation holds as a characterization of interval graphs.

Since no two intervals in the NIR form share a common left endpoint, it is possible to define a perfect order over them. Let the \(i\)th interval be \([i-1, b]\). Now recall the Heaviside function:

\[
H(x) := \begin{cases} 
1, & \text{if } x \geq 0 \\
0, & \text{otherwise}
\end{cases}
\]
Definition 2. Consider the \(i^{th}\) interval \([i-1, b)\) of the NIR form of the interval graph \(G\), for which we define the quantity \(x_{i} := b - i\). Then, the square matrix

\[
H_{G}(i, j) := \begin{cases} 
H(x_{j} + j - i), & \text{if } i > j \\
0, & \text{otherwise}
\end{cases}
\]

is called the Normal Interval Representation (NIR) matrix of \(G\).

In the above definition the quantity \(x_{i}\) equals the number of intervals among the \((i + 1)^{th}, \ldots, n^{th}\) ones that intersect with the \(i^{th}\) one. \(H_{G}\) is a lower triangular matrix with zero diagonal, having a chain of \(x_{i}\) consecutive 1’s under the \(i^{th}\) diagonal element and all the remaining matrix entries being zero. It can be seen also as the lower triangular portion of the adjacency matrix of \(G\), where however rows and columns are ordered in a particular way. Specifically, the \(i^{th}\) interval of \(G\) is represented schematically by the \(i^{th}\) column of \(H_{G}\). Figure I[a] shows an example of the form of \(H_{G}\).

Denote further the desired \(k\)-subgraph of \(G\) with the maximum number of edges as \(C_{k}\). Join the variable \(z_{i} \in \{0, 1\}\) to the \(i^{th}\) interval. The case \(z_{i} = 1\) indicates that the \(i^{th}\) node of \(G\), i.e. the \(i^{th}\) interval of its NIR form, is included in \(C_{k}\). Let now \(1 \leq j < i \leq n\). The \(j^{th}\) and the \(i^{th}\) intervals intersect in \(C_{k}\) if and only if the quantity \(z_{j} \cdot z_{i} \cdot H(x_{j} + j - i) \in \{0, 1\}\) equals one. Indeed, in this case both intervals have been chosen in \(C_{k}\), i.e. \(z_{i} = z_{j} = 1\) and, simultaneously, the \(j^{th}\) interval ends strictly further than \(i - 1\), where the \(i^{th}\) one begins, i.e. \(H(x_{j} + j - i) = 1\). Thus, the number of intersections among the \(k\) intervals of the realization of \(C_{k}\) equals

\[
\sum_{i=2}^{n} \sum_{j=1}^{i-1} z_{j} \cdot z_{i} \cdot H(x_{j} + j - i) = z^{T} \cdot H_{G} \cdot z
\]

where \(z = [z_{1} \ z_{2} \ \cdots \ z_{n}]^{T}\) and \(H_{G}\) is the NIR matrix of \(G\).

Since \(C_{k}\) has exactly \(k\) nodes, exactly \(k\) entries of the vector \(z\) are one. Thus, the k-cluster problem on \(G\) is equivalent to finding the appropriate subset \(I \subseteq \{1, 2, \ldots, n\}\) of the satisfied entries of \(z\), with \(|I| = k\), so that the following quantity is maximized:

\[
\sum_{i,j \in I; i < j} H_{G}(i, j) = \sum_{i,j \in I} H(x_{j} + j - i)
\]

\(\text{Lemma 2.}\) Any maximal clique of \(G\) corresponds bijectively to a row of its NIR matrix \(H_{G}\), in which at least one of its unit elements or its zero diagonal element does not have any chain of 1’s below it.

\(\text{Proof.}\) Consider an arbitrary row of \(H_{G}\), let it be the \(i^{th}\) one, in which exactly the \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{r}^{th}\) elements equal one. Clearly, the \(i^{th}\) and the \(j^{th}\) intervals intersect for every \(j \in \{i_{1}, i_{2}, \ldots, i_{r}\}\), since \(H_{G}(i, j) = 1\). The \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{r}^{th}\) intervals of \(G\) intersect each other also, due to the NIR form of \(H_{G}\). Thus, the \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{r}^{th}\) intervals build a clique \(Q\) in \(G\). Consider now the case that in this row at least one of its \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{r}^{th}\) elements, say the \(j^{th}\) one, does not have any chain of 1’s below it. Suppose also that there exists another clique \(Q’\) in \(G\), which strictly includes \(Q\). Since \(H_{G}(\ell_{1}, j) = H_{G}(i, \ell_{2}) = 0\) for every \(\ell_{1} > i\) and \(\ell_{2} \in \{1, 2, \ldots, i\} \setminus \{i_{1}, i_{2}, \ldots, i_{r}\}\), the \(\ell_{1}^{th}\) and the \(j^{th}\), as well as the \(i^{th}\) and the \(\ell_{2}^{th}\) intervals, do not intersect. Therefore, \(Q’\) can not be a clique, which is a contradiction. Thus, \(Q\) is a maximal clique.

Conversely, let \(Q\) be a maximal clique in \(G\), which contains the \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{|Q|}^{th}\) intervals of its NIR form, where \(i_{1} < i_{2} < \ldots < i_{|Q|}\). Consider now the \(i_{|Q|}^{th}\) row of \(H_{G}\). Since \(Q\) is a clique, the \(i_{1}^{th}, i_{2}^{th}, \ldots, i_{|Q|-1}^{th}\) intervals intersect with the \(i_{|Q|}^{th}\) one and therefore \(H_{G}(i_{|Q|}, j) = 1\) for every \(j \in \{i_{1}, i_{2}, \ldots, i_{|Q|-1}\}\). Suppose \(i_{|Q|} < n\). Then, if \(H_{G}(i_{|Q|} + 1, j) = 1\) for every
consider now the case that $G$ is a proper interval graph. Since $G$ is also an interval graph, it can be represented by the NIR form, which however has an additional property, as described in Definition 3.

**Definition 3.** An NIR form of $n$ intervals is called a Stair Normal Interval Representation (SNIR) form, iff it has the following additional property:

If for the intervals $[a, b)$ and $[c, d)$, $a < c$ holds, then $b \leq d$ also holds.

**Lemma 3.** An arbitrary proper interval graph $G$ can be converted to the SNIR form.

**Proof.** Suppose we are given an arbitrary realization of $G$, in which no interval contains another strictly. Consider the case that in this realization the left endpoint of the interval $v_1 = [a, b)$ is strictly less than the left endpoint of the interval $v_2 = [c, d)$, i.e., $a < c$. Then the same also do their right endpoints respectively. i.e., $b < d$, since otherwise $v_2$ would strictly include $v_1$, which is a contradiction. Since $G$ is also an interval graph, it can be converted to the NIR form, as described above. Suppose that $v_1$ and $v_2$ are converted to the intervals $v'_1 = [a', b')$ and $v'_2 = [c', d')$ in the resulting NIR form respectively. Then, $a' < c'$ holds, since the relative order of the interval left points $a$ and $c$ is not being reversed during the conversion of $G$ to the NIR form; also $b' \leq d'$ holds, since the right endpoints $b$ and $d$ may be “aligned” by the left interval endpoints of the graph. Thus, the obtained NIR form satisfies the condition of Definition 3, i.e., it is an SNIR form. Note that in the special case of two initially identical intervals, i.e., $a = c$ and $b = d$, we obtain the same right endpoints $b' = d'$ for them in the resulting NIR form, while their left endpoints are ordered by increasing order, i.e., in this case the obtained NIR form is also an SNIR form.

**Definition 4.** The NIR matrix $H_G$ that corresponds to the SNIR form of a proper interval graph $G$ is called the Stair Normal Interval Representation (SNIR) matrix of $G$.

**Definition 5.** Consider the SNIR matrix $H_G$ of the proper interval graph $G$. The matrix element $H_G(i, j)$ is called a pick of $H_G$, iff:

1. $i \geq j$,
2. if $i > j$ then $H_G(i, j) = 1$,
3. $H_G(i, k) = 0$, for every $k \in \{1, 2, ..., j - 1\}$ and
4. $H_G(\ell, j) = 0$, for every $\ell \in \{i + 1, i + 2, ..., n\}$.

Given the pick $H_G(i, j)$ of $H_G$, the set

$$S := \{H_G(k, \ell) : i \geq k \geq \ell \geq j\}$$

of matrix entries is called the stair of $H_G$, which corresponds to this particular pick.
Recall that the left and the right endpoints of the \(i\)th interval in the SNIR form of \(G\) correspond to the \(i\)th and the \((x_i + i)\)th elements of the \(i\)th column of \(H_G\) respectively. Therefore, due to Definition 3 it holds that \(x_i + i \geq x_j + j\) for \(i > j\). Consequently, any stair of \(H_G\) consists of unit matrix elements, except of the diagonal elements of \(H_G\), while the corresponding pick is the lower most left matrix entry of this stair. As it is seen in Figure 1(b) the SNIR matrix \(H_G\) has a stair-shape and equals the union of all its stairs. A stair of \(H_G\) can be also recognized in this figure, where the corresponding pick is marked with a circle.

Lemma 4. An arbitrary graph is a proper interval graph iff it can be represented by the SNIR form.

Proof. Due to Lemma 3 any proper interval graph can be represented by the SNIR form. Conversely, the SNIR form is clearly a set of intervals, where no one of which includes strictly another one, i.e., it is a realization of a proper interval graph.

Lemma 5. Any stair of the SNIR matrix \(H_G\) corresponds bijectively to a maximal clique in \(G\).

Proof. Due to Lemma 2 every maximal clique of \(G\) corresponds bijectively to a row of \(H_G\), in which at least one of its unit elements or its zero diagonal element does not have any chain of 1’s below it. However, since \(G\) is a proper interval graph and due to Definition 5, such a row corresponds bijectively to a pick of \(H_G\) and therefore to a stair of it, as it is shown in Figure 1(b).

4 The k-cluster problem on proper interval graphs

Due to Lemma 4 a proper interval graph \(G\) is equivalent to an SNIR matrix \(H_G\). Denote by \(S_1, S_2, ..., S_m, m \leq n - 1\), the stairs of \(H_G\), numbered from the top to the bottom. Due to Lemma 5 these stairs correspond bijectively to the maximal cliques \(Q_1, Q_2, ..., Q_m\) of \(G\). Denote for simplicity \(S_0 := \emptyset\) and \(Q_0 := \emptyset\). Every stair \(S_i\) constitutes together with its previous stairs \(S_1, S_2, ..., S_{i-1}\) a submatrix \(H_i := H_{G_i}\) of \(H_G\) that is equivalent to the subgraph \(G_i := \bigcup_{\ell=1}^i Q_\ell\) of \(G\), which remains also a proper interval graph. In particular, \(H_m = H_G\) is equivalent to \(G_m = G\). We develop further a dynamic programming algorithm for the j-cluster problem on
G_i, which makes use of the optimal solutions of the q-cluster problems on G_{i-1}, for q = 1, 2, ..., j.

The critical observation here is that the arbitrary i-th stair S_i of H_G contains at least one row that does not belong to the previous stair S_{i-1}, i.e. S_i \setminus S_{i-1} \neq \emptyset and therefore Q_i \setminus Q_{i-1} \neq \emptyset. Suppose that the pick of S_i is the matrix element H_G(a_i, b_i). Then, the maximal clique Q_i has |Q_i| = a_i - b_i + 1 nodes, namely the b_i, b_i + 1, ..., a_i ones.

Denote now by f_i(j, x, x') the value of an optimal solution of the j-cluster problem on G_i, including exactly x nodes of the clique Q_i \setminus Q_{i-1} and exactly x' nodes of the clique Q_i \cap Q_{i-1}. Clearly, 0 \leq x \leq |Q_i \setminus Q_{i-1}|, 0 \leq x' \leq |Q_i \cap Q_{i-1}| and x + x' \leq j. Then, the value of an optimal solution of the j-cluster problem on G_i is f_i(j) = \max_{x,x'} \{f_i(j, x, x')\}. Note that obviously for the j-cluster problem on a single stair H_1 = S_1 we should require that x' = 0 and x = j, as also that Q_1 has at least j nodes, since otherwise we should include also j - x > 0 nodes of Q_0 = \emptyset, which is a contradiction. Therefore, the following initial conditions hold for i = 1 and j = 1, 2, ..., k:

\[
f_i(j, x, 0) = \begin{cases} 
\left( \frac{j}{2} \right), & \text{if } x = j \leq |Q_i| \\
-\infty, & \text{otherwise}
\end{cases}
\] (3)

If j \leq |Q_i|, then any subclique of Q_i on j nodes is clearly an optimal solution. Otherwise, consider the case j > |Q_i|. The recursive computation of f_i(j, x, x'), which is presented below, makes use of the values f_{i-1}(q, r, r') for q = 1, 2, ..., j, where x = |Q_i \setminus Q_{i-1}|, x' = |Q_i \cap Q_{i-1}|, r = |Q_{i-1} \setminus Q_{i-2}| and r' = |Q_{i-1} \cap Q_{i-2}|. We distinguish the cases Q_i \cap Q_{i-2} \neq \emptyset and Q_i \cap Q_{i-2} = \emptyset, or equivalently S_i \cap S_{i-2} \neq \emptyset and S_i \cap S_{i-2} = \emptyset. In the case Q_i \cap Q_{i-2} \neq \emptyset an optimal solution may include y nodes of Q_{i-1} \setminus Q_{i-2}, z nodes of Q_i \cap Q_{i-2}, w nodes of Q_{i-1} \setminus Q_i and u nodes of the remaining part of G_i. In the opposite case Q_i \cap Q_{i-2} = \emptyset, an optimal solution may include y nodes of Q_i \cap Q_{i-1}, z nodes of Q_{i-1} \setminus (Q_i \cup Q_{i-2}), w nodes of Q_{i-1} \cap Q_{i-2} and u nodes of the remaining part of G_i. Both situations are illustrated in Figure[2]. As it can be easily verified, for all these sets the following hold:

Case Q_i \cap Q_{i-2} \neq \emptyset : Case Q_i \cap Q_{i-2} = \emptyset :

\[
\begin{align*}
0 \leq x & \leq x_0 := |Q_i \setminus Q_{i-1}|, & 0 \leq x & \leq x_0 := |Q_i \setminus Q_{i-1}| \\
= a_i - a_{i-1} & = a_i - a_{i-1} \\
0 \leq y & \leq y_1 := |Q_{i-1} \setminus Q_{i-2}|, & 0 \leq y & \leq y_2 := |Q_i \cap Q_{i-1}| \\
= a_{i-1} - a_{i-2} & = a_{i-1} - b_i + 1 \\
0 \leq z & \leq z_1 := |Q_i \cap Q_{i-2}|, & 0 \leq z & \leq z_2 := |Q_i \setminus (Q_i \cup Q_{i-2})| \\
= a_{i-2} - b_i + 1 & = b_i - a_{i-2} - 1 \\
0 \leq w & \leq w_1 := |Q_{i-1} \setminus Q_i|, & 0 \leq w & \leq w_2 := |Q_{i-1} \cap Q_{i-2}| \\
= b_i - b_{i-1} & = a_{i-2} - b_{i-1} + 1 \\
0 \leq u & \leq u_1 := b_{i-1} - 1 & 0 \leq u & \leq u_2 := b_{i-1} - 1
\end{align*}
\] (4)

The case Q_i \cap Q_{i-2} \neq \emptyset occurs exactly when b_i \leq a_{i-2}, i.e. H(a_{i-2} - b_i) = 1, while the opposite case Q_i \cap Q_{i-2} = \emptyset occurs exactly when H(b_i - a_{i-2} - 1) = 1. Thus, since x, y, z, w and u add up to j, we can summarize the relations in (4) to the following, for the general case:

\[
\begin{align*}
0 \leq x & \leq x_0 \\
0 \leq y & \leq y_1 \cdot H(a_{i-2} - b_i) + y_2 \cdot H(b_i - a_{i-2} - 1) \\
0 \leq z & \leq z_1 \cdot H(a_{i-2} - b_i) + z_2 \cdot H(b_i - a_{i-2} - 1) \\
0 \leq w & \leq w_1 \cdot H(a_{i-2} - b_i) + w_2 \cdot H(b_i - a_{i-2} - 1) \\
0 \leq u & \leq u_1 \cdot H(a_{i-2} - b_i) + u_2 \cdot H(b_i - a_{i-2} - 1) \\
x + y + z + w + u & = j
\end{align*}
\] (5)
Figure 2: The split of the SNIR matrix $H_G$ for the recursion of the $k$-cluster problem on a proper interval graph $G$, in the cases (a) $S_i \cap S_{i-2} \neq \emptyset$ and (b) $S_i \cap S_{i-2} = \emptyset$.

For simplicity, let $\zeta_1 = z \cdot H(a_{i-2} - b_i)$ and $\zeta_2 = z \cdot H(b_i - a_{i-2} - 1)$. Now, the value $f_i(j, x, x')$ can be computed by using the top-down approach of the following equation, for both cases $Q_i \cap Q_{i-2} \neq \emptyset$ and $Q_i \cap Q_{i-2} = \emptyset$:

$$f_i(j, x, x + \zeta_1) = \begin{cases} \binom{j}{2}, & \text{if } x + y + \zeta_1 = j \leq |Q_i| \\ \max_{y, z, w, u \in S_i} \left\{ f_{i-1}(j - x, y + \zeta_2, \zeta_1 + w) + \binom{x}{2} + x (y + \zeta_1) \right\}, & \text{otherwise} \end{cases} \tag{6}$$

Finally, the dynamic programming Algorithm 1 returns the value of an optimal solution of the $k$-cluster problem on $G$. After applying some necessary modifications, it will return the optimal solution, instead of its value.

**Algorithm** Proper-Interval-$k$-cluster problem($G$):

**Input:** An arbitrary realization of a proper interval graph $G$  

**Output:** The value of an optimal solution of the $k$-cluster problem on $G$

1. Construct the SNIR matrix $H_G$. Let that $H_G$ has the $m$ stairs $S_1, S_2, ..., S_m$ that correspond to the maximal cliques $Q_1, Q_2, ..., Q_m$ of $G$  

2. If $m = 1$ Then Return $f_1(k) = f_1(k, k, 0)$, computed from (3);  

   Else Return $f_m(k) = \max\{f_m(k, x, x') : 0 \leq x \leq |Q_i \setminus Q_{i-1}|, 0 \leq x' \leq |Q_i \cap Q_{i-1}|, x + x' \leq k\}$, computed from (6)

Algorithm 1: The value of an optimal solution of the $k$-cluster problem on the proper interval graph $G$.

**Theorem 1.** The $k$-cluster problem is solvable in $O(nk^5)$ time on proper interval graphs.
Proof. The computation of a single \( f_i(j) \) in the Algorithm 1 takes at most \( O(j^4) = O(k^4) \) time due to the combinations of the \( x, y, z, w, u \), such that they sum up to \( j \), since \( x, y, z \) and \( w \) may vary and \( u = j - x - y - z - w \) is then uniquely determined by them. Every \( f_i(j) \) is computed for all \( i \in \{1, 2, ..., m\} \) and \( j \in \{1, 2, ..., k\} \), i.e., altogether at most \( m \cdot k = O(nk) \) quantities are computed. Thus, since any proper interval graph can be recognized and converted to the SNIR form in linear time, the k-cluster problem can be solved in \( O(nk^5) \) time on any proper interval graph. 

Note that in the presented analysis the subgraph that corresponds to the obtained optimal solution is not necessarily connected. Lemma 6 proposes a modification to the Algorithm 1, in order to find an optimal solution, under the additional constraint of connectivity.

**Lemma 6.** The Algorithm 1 returns the value of an optimal solution of the k-cluster problem on proper interval graphs, under the additional constraint of connectivity, if the following additional condition to \( (7) \) is required:

\[
y + \zeta_1 \geq 1, \text{ if } x > 0. \tag{7}
\]

After this modification, the runtime of the proposed algorithm remains \( O(nk^5) \).

Proof. The proof is done by induction. If \( i = 1 \), then the obtained solution is always connected, as an induced subgraph of a clique. Suppose now that \( i > 1 \) and \( x > 0 \). It follows that we use \( x \geq 1 \) nodes of \( Q_i \), which are not included in \( Q_j \), for any \( j < i \). Therefore, in order to construct a connected subgraph, it is equivalent to require that at least one node of \( Q_i \cap G_{i-1} = Q_i \cap Q_{i-1} \) is included, i.e., a node which is simultaneously connected to the \( x \) nodes of \( Q_i \setminus Q_{i-1} \) and to at least one node of the remaining graph \( G_{i-1} \). However, as described above, we include in the constructed subgraph exactly \( y + z \) nodes of \( Q_i \cap Q_{i-1} \) if \( Q_i \cap Q_{i-2} \neq \emptyset \) and exactly \( y \) nodes of \( Q_i \cap Q_{i-1} \) if \( Q_i \cap Q_{i-2} = \emptyset \). Namely, we include exactly \( y + \zeta_1 \) nodes of \( Q_i \cap Q_{i-1} \) in the general case. Therefore, in order to construct a connected subgraph, it is equivalent to require that \( y + \zeta_1 \geq 1 \). Finally, the asymptotic complexity of the proposed algorithm remains obviously unchanged, when requiring the additional condition \( (7) \) to the conditions \( (5) \). □

## 5 The k-cluster problem on interval graphs

In this section we propose a polynomial dynamic programming algorithm for the k-cluster problem on interval graphs, whose complexity status was an open question \(^9\). The proposed algorithm constitutes a generalization of Algorithm 1 for proper interval graphs. Due to Lemma 1, an interval graph \( G \) is equivalent to a NIR matrix \( H_G \). In the following consider an interval graph \( G \) on \( n \) nodes, as well as its NIR matrix \( H_G \).

Due to Lemma 2 any maximal clique of \( G \) corresponds bijectively to a row of the NIR matrix \( H_G \), in which at least one of its unit elements or its zero diagonal element does not have any chain of 1’s below it. The maximal clique, which refers to such a row, contains all intervals, i.e., nodes, which correspond to the unit elements and the zero diagonal element of this row. Denote these maximal cliques of \( G \) by \( Q_1, Q_2, ..., Q_m \), \( m \leq n - 1 \), numbered from the top to the bottom, as well as \( Q_0 := \emptyset \). Suppose also that the maximal clique \( Q_\ell \) occurs at the \( a_\ell^\ell \)th row of \( H_G \) and denote by \( |Q_\ell| \) the number of nodes of \( Q_\ell \). It holds clearly that \( Q_i \setminus Q_{i-1} \neq \emptyset \) for all \( i = 1, 2, ..., m \). Every maximal clique \( Q_i \) constitutes together with its previous maximal cliques \( Q_1, Q_2, ..., Q_{i-1} \) a subgraph \( G_i \) of \( G \), which remains also an interval graph. Similarly to Section 4 for the proper interval graphs, we develop further a dynamic programming algorithm for the j-cluster problem on \( G_i \), which makes use of the optimal solutions of the q-cluster problems on \( G_{i-1} \), for \( q = 1, 2, ..., j \).
An optimal solution may include $y$ nodes of $(Q_i \cap Q_{i-1}) \setminus Q_{i-2}$, $z$ nodes of $Q_{i-1} \setminus (Q_i \cup Q_{i-2})$, $w$ nodes of $Q_i \cap Q_{i-2}$, $u$ nodes of $(Q_{i-1} \cap Q_{i-2}) \setminus Q_i$ and $v$ nodes of the remaining part of $G_i$, as it is illustrated in Figure 3. We compute in Appendix A the split of the NIR matrix $H_G$ and we obtain the following relations for the variables $x, y, z, w, u$ and $v$:

\[
0 \leq x \leq |Q_i \setminus Q_{i-1}| = a_i - a_{i-1}
\]

\[
0 \leq y \leq |(Q_i \cap Q_{i-1}) \setminus Q_{i-2}| = \sum_{\ell=a_{i-2}+1}^{a_{i-1}-1} H(\ell + x_{\ell} - a_i)
\]

\[
0 \leq z \leq |Q_{i-1} \setminus (Q_i \cup Q_{i-2})| = a_{i-1} - a_{i-2} - \sum_{\ell=a_{i-2}+1}^{a_{i-1}-1} H(\ell + x_{\ell} - a_i)
\]

\[
0 \leq w \leq |Q_i \cap Q_{i-2}| = \sum_{\ell=1}^{a_{i-2}} H(\ell + x_{\ell} - a_i)
\]

\[
0 \leq u \leq |(Q_{i-1} \cap Q_{i-2}) \setminus Q_i| = \sum_{\ell=1}^{a_{i-2}} H(\ell + x_{\ell} - a_{i-1}) - \sum_{\ell=1}^{a_{i-2}} H(\ell + x_{\ell} - a_i) - \sum_{\ell=1}^{a_{i-2}} H(\ell + x_{\ell} - a_{i-1}) \cdot H(a_i - \ell - x_{\ell} - 1) - x + y + z + w + u + v = j
\]

Now, the value $f_i(j, x, x')$ can be computed by using the top-down approach of the following equation:

\[
f_i(j, x, y+w) = \begin{cases} 
\binom{j}{2}, & \text{if } x + y + w = j \leq |Q_i| \\
\max_{y, z, w, u, v \in \mathbb{S}} \left\{ f_{i-1}(j - x, y+z, w + u) + \binom{x}{2} + x(y+w) \right\}, & \text{otherwise}
\end{cases}
\]

Finally, the dynamic programming Algorithm 2, similarly to Algorithm 1, returns the value of an optimal solution of the k-cluster problem on $G$. After applying some necessary modifications, it will return the optimal solution, instead of its value.

**Algorithm** Interval-k-cluster problem($G$):

**Input:** An arbitrary realization of an interval graph $G$

**Output:** The value of an optimal solution of the k-cluster problem on $G$

1. Construct the NIR matrix $H_G$. Let that $G$ has the $m$ maximal cliques $Q_1, Q_2, ..., Q_m$
2. If $m = 1$ Then Return $f_1(k) = f_1(k, k, 0)$, computed from (3);
   Else Return $f_m(k) = \max\{f_m(k, x, x') : 0 \leq x \leq |Q_i \setminus Q_{i-1}|, 0 \leq x' \leq |Q_i \cap Q_{i-1}|, x + x' \leq k\}$, computed from (9)

Algorithm 2: The value of an optimal solution of the k-cluster problem on the interval graph $G$.

**Theorem 2.** The k-cluster problem is solvable in $O(nk^6)$ time on interval graphs.

**Proof.** The computation of a single $f_i(j)$ in the Algorithm 2 takes at most $O(j^5) = O(k^5)$ time due to the combinations of the $x, y, z, w, u, v$, such that they sum up to $j$, since $x, y, z, w$ and $v$ may vary and $v = j - x - y - z - w - u$ is then uniquely determined by them. Every $f_i(j)$ is computed for all $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., k\}$, i.e., altogether at most $m \cdot k = O(nk)$ quantities are computed. Thus, since any interval graph can be recognized and converted to the NIR form in linear time, the k-cluster problem can be solved in $O(nk^6)$ time on any interval graph. 

\[\square\]
Figure 3: The split of the NIR matrix $H_G$ for the recursion of the k-cluster problem on an interval graph $G$.

**Lemma 7.** The proposed algorithm returns the value of an optimal solution of the k-cluster problem on interval graphs, under the additional constraint of connectivity, if the following additional condition is required to the conditions (9):

$$y + w \geq 1, \text{ if } x > 0.$$  \hspace{1cm} (10)

After this modification, the runtime of the proposed algorithm remains $O(nk^6)$.

**Proof.** The proof is done by induction. If $i = 1$, then the obtained solution is always connected, as an induced subgraph of a clique. Suppose now that $i > 1$ and $x > 0$. It follows that we use $x \geq 1$ nodes of $Q_i$, which are not included in $Q_j$, for any $j < i$. Therefore, in order to construct a connected subgraph, it is equivalent to require that at least one node of $Q_i \cap Q_{i-1} = Q_i \cap Q_{i-1}$ is included, i.e., a node which is simultaneously connected to the $x$ nodes of $Q_i \setminus Q_{i-1}$ and to at least one node of the remaining graph $G_{i-1}$. However, as described above, we include in the constructed subgraph exactly $y + w$ nodes of $Q_i \cap Q_{i-1}$. Therefore, in order to construct a connected subgraph, it is equivalent to require that $y + w \geq 1$. Finally, the asymptotic complexity of the proposed algorithm remains obviously unchanged, when requiring the additional condition (10) to the conditions (9).

\hfill $\square$

### 6 Conclusions

In this paper an efficient matrix representation that characterizes the interval graphs, as well as its restriction on the proper interval graphs is used, which leads to a simple polynomial time algorithm for the k-cluster problem on these classes of graphs. This problem is known to be NP-hard on an arbitrary graph, as a generalization of the maximum clique problem, as well as on the chordal graphs. In contrary, its complexity on interval and proper interval graphs was an open question.

**Acknowledgment**

I wish to thank Professor Philippe Baptiste and Dr. Maxim Sviridenko for reading the manuscript and improving the presentation.
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A The split of the NIR matrix $H_G$

We remind at first that it is assumed that the maximal clique $Q_\ell$ occurs at the $\ell^{th}$ row of $H_G$, for $\ell = 1, 2, \ldots, m$. Suppose that $a_{i-1} < \ell \leq a_i$. If the chain of 1’s under the $\ell^{th}$ diagonal element of $H_G$ stops at a row, which is higher than the $a_i^{th}$ one, then a maximal clique would occur between $Q_{i-1}$ and $Q_i$, which is a contradiction. Thus, the chain under the $\ell^{th}$ diagonal element stops either at the $a_i^{th}$ row, or even lower. Suppose now that $\ell \leq a_{i-1}$. If $\ell \in Q_i$, then also $\ell \in Q_{i-1}$, since the chain under the $\ell^{th}$ diagonal element stops either at the $a_i^{th}$ row, or even lower, i.e. strictly lower than the $a_{i-1}^{th}$ row. Therefore, the elements of $Q_i \setminus Q_{i-1}$ are exactly the $(a_{i-1} + 1)^{th}, \ldots, a_i^{th}$ diagonal elements. Thus,

$$|Q_i \setminus Q_{i-1}| = a_i - a_{i-1}$$  (11)

In order to compute the value $|(Q_i \cap Q_{i-1}) \setminus Q_{i-2}|$, we have to compute how many of the $1^{st}$, $2^{nd}$, ..., $a_{i-1}^{th}$ diagonal elements belong to $Q_i$ and to $Q_{i-1}$, but not to $Q_{i-2}$. For $1 \leq \ell \leq a_{i-1}$, the arbitrary $\ell^{th}$ diagonal element belongs to $Q_i$ exactly when its chain of 1’s reaches the $a_i^{th}$ row, i.e. exactly when $\ell + x_\ell \geq a_i$, or equivalently $H(\ell + x_\ell - a_i) = 1$. In this case, it belongs also to $Q_{i-1}$, since $a_{i-1} < a_i$. Further, for $1 \leq \ell \leq a_{i-2}$, if $H(\ell + x_\ell - a_i) = 1$ then the $\ell^{th}$ diagonal element belongs also to $Q_{i-2}$ and therefore not to $(Q_i \cap Q_{i-1}) \setminus Q_{i-2}$. It follows that

$$|(Q_i \cap Q_{i-1}) \setminus Q_{i-2}| = \sum_{\ell=a_{i-2}+1}^{a_{i-1}} H(\ell + x_\ell - a_i)$$  (12)

Now, the sets $(Q_i \cap Q_{i-1}) \setminus Q_{i-2}$ and $Q_{i-1} \setminus (Q_i \cup Q_{i-2})$ partition the set $Q_{i-1} \setminus Q_{i-2}$, which has $a_{i-1} - a_{i-2}$ nodes, due to (11). Thus, it follows from (12) that

$$|Q_{i-1} \setminus (Q_i \cup Q_{i-2})| = a_{i-1} - a_{i-2} - \sum_{\ell=a_{i-2}+1}^{a_{i-1}} H(\ell + x_\ell - a_i)$$  (13)

In order to compute the value $|Q_i \cap Q_{i-2}|$, we have to compute how many of the $1^{st}$, $2^{nd}$, ..., $a_{i-2}^{th}$ diagonal elements belong simultaneously to $Q_{i-2}$ and to $Q_i$. For $1 \leq \ell \leq a_{i-2}$, the $\ell^{th}$ one belongs to $Q_i$ exactly when its chain of 1’s reaches the $a_i^{th}$ row, i.e. exactly when $\ell + x_\ell \geq a_i$, or equivalently $H(\ell + x_\ell - a_i) = 1$. In this case, if $\ell \neq a_{i-2}$, then its chain reaches also the $a_{i-2}^{th}$ row, which means that it belongs also to $Q_{i-2}$, while the $a_{i-2}^{th}$ one belongs always to $Q_{i-2}$. It follows that

$$|Q_i \cap Q_{i-2}| = \sum_{\ell=1}^{a_{i-2}} H(\ell + x_\ell - a_i)$$  (14)

Similarly, in order to compute the value $|(Q_{i-1} \cap Q_{i-2}) \setminus Q_i|$, we have to compute how many of the $1^{st}$, $2^{nd}$, ..., $a_{i-1}^{th}$ diagonal elements belong simultaneously to $Q_{i-1}$ and to $Q_{i-2}$ but not to $Q_i$. For $1 \leq \ell \leq a_{i-2}$, the $\ell^{th}$ one belongs to $Q_{i-1}$ exactly when $H(\ell + x_\ell - a_{i-1}) = 1$. In this case it belongs also to $Q_{i-2}$, since $a_{i-2} < a_{i-1}$. Further, it does not belong to $Q_i$ exactly when $\ell + x_\ell < a_i$, or equivalently $H(a_i - \ell - x_\ell - 1) = 1$. It follows that

$$|(Q_{i-1} \cap Q_{i-2}) \setminus Q_i| = \sum_{\ell=1}^{a_{i-2}} H(\ell + x_\ell - a_{i-1}) \cdot H(a_i - \ell - x_\ell - 1)$$  (15)

Finally, the complementary part in $G_i$ of the sets in (11)-(15) has

$$a_{i-2} - \sum_{\ell=1}^{a_{i-2}} H(\ell + x_\ell - a_i) - \sum_{\ell=1}^{a_{i-2}} H(\ell + x_\ell - a_{i-1}) \cdot H(a_i - \ell - x_\ell - 1)$$  (16)

nodes, since $G_i$ has overall $a_i$ nodes.