On the entropic derivation of the $r^{-2}$
Newtonian gravity force

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Abstract

Following Verlinde’s conjecture, we show that Tsallis’ classical free
particle distribution at temperature $T$ can generate Newton’s grav-
tational force’s $r^{-2}$ distance’s dependence. If we want to repeat the
concomitant argument by appealing to either Boltzmann-Gibbs’ or
Renyi’s distributions, the attempt fails and one needs to modify the
conjecture.

Keywords: Tsallis’, Boltzmann-Gibbs’, and Renyi’s distributions, clas-
sical partition function, entropic force.
1 Introduction

Eight years ago, Verlinde [1] advanced a conjecture that links gravity to an entropic force, so that gravity would result from information regarding the positions of material bodies. His model joins a thermal gravity-treatment to 't Hooft’s holographic principle. This would entail that gravitation should be viewed as an emergent phenomenon. Verlinde’s notion received much attention, of course (just as an example, see [2]). For an excellent overview on the statistical mechanics of gravitation, the reader is directed to Padmanabhan’s article [4], and references therein.

Verlinde’s work attracted efforts on cosmology, the dark energy hypothesis, cosmological acceleration, cosmological inflation, and loop quantum gravity. The literature is immense [3]. In particular, an important contribution to information theory is that of Guseo [5], who has proved that the local entropy function, related to a logistic distribution, is a catenary and vice versa. This special invariance may be explained, at a deeper level, through the Verlinde conjecture on the origin of gravity, as an effect of the entropic force. Guseo advances a novel interpretation of the local entropy in a system, as quantifying a hypothetical attraction force that the system would exert [5].

This paper deals with none of these issues, though. We just show that extremely simple classical reasoning based on the Tsallis, probability distributions straightforwardly proves the conjecture. In Boltzmann-Gibbs and Renyi’s instance, one needs to modify the conjecture to achieve a similar result.

2 Tsallis’ q-entropy of the free particle

Tsallis’ q-partition function for a free particle of mass \( m \) in \( \nu \) dimensions reads [6]

\[
Z(\nu) = V(\nu) \int \left[ 1 + (1 - q) \frac{\beta p^2}{2m} \right]^{\frac{1}{q-1}} d^\nu p,
\]

(2.1)

with the particle probability distribution \( \xi(p) \) being

\[
\xi = \frac{1}{Z(\nu)} \left[ 1 + (1 - q) \frac{\beta p^2}{2m} \right]^{\frac{q}{q-1}},
\]

(2.2)
where $V_\nu$ is the volume of an hypersphere in $\nu$ dimensions and we assume $q > 1$. (2.1) can be recast as

$$Z_\nu = \frac{2 \pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} V_\nu \int_0^\infty \left[ 1 + (1 - q) \beta \frac{p^2}{2m} \right]^{\frac{1}{q-1}} p^{\nu-1} dp.$$ (2.3)

With the change of variables $x^2 = \frac{p^2}{2m}$ one has

$$Z_\nu = \frac{(2m\pi)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} V_\nu \int_0^1 \left[ 1 + (1 - q) \beta x \right]^{\frac{1}{q-1}} x^{\frac{\nu}{2}-1} dx,$$ (2.4)

that after integration becomes

$$Z_\nu = V_\nu \frac{(2m\pi)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{(q-1) \beta} \frac{\Gamma\left(\frac{q}{q-1}\right)}{\Gamma\left(\frac{q}{q-1} + \frac{\nu}{2}\right)}.$$ (2.5)

The mean energy is

$$<U_\nu> = \frac{V_\nu}{Z_\nu} \int \left[ 1 + (1 - q) \beta \frac{p^2}{2m} \right]^{\frac{1}{q-1}} p^{2m} d^\nu p,$$ (2.6)

or

$$<U_\nu> = \frac{V_\nu (2m\pi)^{\frac{\nu}{2}}}{Z_\nu \Gamma\left(\frac{\nu}{2}\right)} \int_0^1 \left[ 1 + (1 - q) \beta x \right]^{\frac{1}{q-1}} x^{\frac{\nu}{2}-1} dx,$$ (2.7)

so that after integration we find

$$<U> = \frac{\nu}{2(q-1) \beta} \frac{\Gamma\left(\frac{1}{q-1} + \frac{\nu}{2} + 1\right)}{\Gamma\left(\frac{1}{q-1} + \frac{\nu}{2} + 2\right)},$$ (2.8)

and finally

$$<U> = \frac{\nu}{2q + \nu(q-1) \beta}.$$ (2.9)

For the entropy one has [6]

$$S_\nu = \ln_q Z_\nu + Z_\nu^{1-q} \beta <U>.$$ (2.10)
3 The Tsallis entropic force

We specialize things now to $\nu = 3$ and $q = \frac{4}{3}$. Why do we select this special value $q = \frac{4}{3}$? There is a solid reason. This is because

$$S_\nu = \ln_q Z_\nu + Z_\nu^{1-q} \beta < U >_\nu.$$  

Since the entropic force is to be defined as proportional to the gradient of $S$, there is a unique $q$-value for which the dependence on $r$ of the entropic force is $\sim r^{-2}$ when $\nu = 3$. Thus we obtain, for $q = \frac{4}{3}$,

$$Z = \left(\frac{6m\pi}{\beta}\right)^{\frac{4}{3}} \frac{8\pi}{\Gamma\left(\frac{11}{2}\right)} r^3, \quad (3.1)$$

$$< U > = \frac{9}{11\beta}. \quad (3.2)$$

Following Verlinde [1] we define the entropic force as

$$\vec{F}_e = -\frac{\lambda(m, M)}{\beta} \vec{\nabla} S, \quad (3.3)$$

where $\lambda$ is a numerical parameter depending on the masses involved, $m$ and a new one $M$ that we place at the center of the sphere. Thus,

$$\vec{F}_e = -\frac{24}{11} \left[\frac{\Gamma\left(\frac{11}{2}\right)}{8\pi}\right]^{\frac{1}{3}} \left(\frac{k_B T}{6m\pi}\right)^{\frac{1}{2}} \frac{\lambda(m, M)}{r^2} \vec{e}_r, \quad (3.4)$$

where $\vec{e}_r$ is the radial unit vector. We see that $F_e$ acquires an appearance quite similar to that of Newton’s gravitation, as conjectured by Verlinde en [1]. Note that entropic force vanishes at zero temperature, in agreement with Thermodynamics’ third law [7].

4 An illustrative example

Assume that we deal with a large mass $M$ and a very small one $m$. One has

$$\vec{F}_e = -\frac{24}{11} \left[\frac{\Gamma\left(\frac{11}{2}\right)}{8\pi}\right]^{\frac{1}{3}} \left(\frac{k_B T}{6m\pi}\right)^{\frac{1}{2}} \frac{\lambda(m, M)}{r^2} \vec{e}_r = -\frac{GmM}{r^2} \vec{e}_r. \quad (4.1)$$
We obtain for \( \lambda(m, M) \)

\[
\lambda^2(m, M) = \frac{121 \pi^2 G^2 m^3 M^2}{k_B T 24 2^\frac{3}{4} \left[ \Gamma \left( \frac{11}{2} \right) \right]^\frac{3}{2}}.
\]  

(4.2)

If we select \( M = \text{Sun mass} \) \( m = \text{Jupiter mass} \) \( T = 3^\circ K \) then \( \lambda(m, M) = 2.63 \times 10^{72} \text{Kg meters}^2 \). When \( m = \text{Earth mass} \), then \( \lambda(m, M) = 3.22 \times 10^{68} \text{Kg meters}^2 \).

4.1 Energies involved

In [8], different \( q \)-values have been associated to energies of CERN experiments [9, 10]. \( q \)-Statistics is seen to be meaningful at very high energies (TeVs) for \( q = 1.15 \), high ones (GeVs) for \( q = 1.001 \), and at low energies (MeVs) for \( q = 1.000001 \). Then we see that \( q = \frac{4}{3} \) should be associated with an energy of (TeVs), an energy that can be expected to arise shortly after the Big Bang, where quantum gravity effects should be apparent.

5 The Boltzmann-Gibbs entropy of the free particle

Now the classical partition function \( Z_\nu \) is

\[
Z_\nu = V_\nu \int e^{-\frac{\pi^2 \nu^2}{2m} d^\nu p},
\]

(5.1)

with \( V_\nu \)

\[
V_\nu = \frac{2\pi^\frac{\nu}{2}}{\Gamma \left( \frac{\nu}{2} \right)} r^\nu.
\]

(5.2)

Since

\[
\int e^{-\frac{\pi^2 \nu^2}{2m} d^\nu p} = \left( \frac{2\pi m}{\beta} \right)^\frac{\nu}{2} \frac{\pi^\frac{\nu}{2}}{\Gamma \left( \frac{\nu}{2} + 1 \right)} r^\nu,
\]

(5.3)

we have

\[
Z_\nu = \left( \frac{2\pi m}{\beta} \right)^\frac{\nu}{2} \frac{\pi^\frac{\nu}{2}}{\Gamma \left( \frac{\nu}{2} + 1 \right)} r^\nu.
\]

(5.4)

so that the mean energy \( <U>_\nu \) is

\[
<U>_\nu = \frac{V_\nu}{Z_\nu} \int \frac{p^2}{2m} e^{-\frac{\pi^2 \nu^2}{2m} d^\nu p}.
\]

(5.5)
We appeal now to the well known relation:

\[
\int \frac{p^2}{2m} e^{-\beta \frac{p^2}{2m}} d^\nu p = \left( \frac{2\pi m}{\beta} \right)^{\frac{\nu}{2}} \frac{\nu}{2\beta},
\]

so that

\[
< U >_\nu = \frac{\nu}{2\beta},
\]

which leads to an entropy:

\[
S_\nu = \ln Z_\nu + \frac{\nu}{2}.
\]

### 6 The Boltzmann-Gibbs entropic force

Our hyper-sphere’s area \( A_\nu \) is

\[
A_\nu = \frac{2\pi^\frac{\nu}{2}}{\Gamma \left( \frac{\nu}{2} \right)} r^{\nu-1}.
\]

The hyper-sphere’s volume, as a function of its area reads

\[
V_\nu = \left[ \frac{\Gamma \left( \frac{\nu}{2} \right)}{2^{\frac{\nu}{2}} \pi^{\frac{\nu}{2} - 1}} \right] A_\nu^{\frac{\nu}{2} - 1} \frac{\nu}{2^{\frac{\nu}{2}} \pi^{\frac{\nu}{2} - 1}}.
\]

The derivative of \( S_\nu \) with respect to \( A_\nu \) is

\[
\frac{\partial S_\nu}{\partial A_\nu} = \frac{\nu}{\nu - 1} \frac{1}{A_\nu}.
\]

Specialize things now to \( \nu = 3 \). Following Verlinde [1], with a slight modification, we define the entropic force that arises out of forcing the particle of mass \( m \) to remain enclosed in a given volume as

\[
F_e = -\frac{\lambda(m, M) k_B T}{\beta} \frac{\partial S_3}{\partial A_3} = -\frac{\lambda}{\beta} \frac{3}{2} \frac{1}{A_3},
\]

Replacing \( A_3 \)’s value in (6.4) we find

\[
F_e = -\lambda(m, M) k_B T \frac{3 \Gamma \left( \frac{3}{2} \right)}{2^{\frac{3}{2}} \pi^{\frac{3}{2}}} \frac{1}{r^2},
\]

or

\[
F_e = -\frac{3\lambda(m, M) k_B T}{8\pi} \frac{1}{r^2}.
\]

We see again that \( F_e \) acquires an appearance quite similar to that of Newton’s gravitation, as conjectured by Verline in [1].
A second illustrative example

Let us replace the enclosing effect of a spherical cavity by the gravitational one of a large mass $M$ on a very small one $m$, that is,

$$F_e = -\frac{3\lambda(m, M)k_b T}{8\pi} \frac{1}{r^2} = -\frac{GmM}{r^2},$$

(7.1)

and deduce $\lambda(m, M)$ as

$$\lambda(m, M) = \frac{8\pi GmM}{3Tk_b}.$$  

(7.2)

If we select $M=$Sun mass $m=$Jupiter mass, $T=3^\circ K$ then $\lambda(m, M) = 4, 6 \times 10^{71}$ meters. When $m=$Earth mass, then $\lambda(m, M) = 1, 5 \times 10^{69}$ meters.

The Renyi entropic force

In Renyi’s approach to our problem the entropy is [11]-[22]

$$Z_\nu = V_\nu \left[ \left( \frac{2m\pi}{\alpha - 1}\right)^{\frac{\nu}{2}} \frac{\Gamma\left(\frac{\alpha}{\alpha-1}\right)}{\Gamma\left(\frac{\alpha}{\alpha-1} + \frac{\nu}{2}\right)} \right] \quad \alpha > 1,$$

$$Z_\nu = V_\nu \left[ \left( \frac{2m\pi}{1 - \alpha}\right)^{\frac{\nu}{2}} \frac{\Gamma\left(\frac{1}{1-\alpha}\right)}{\Gamma\left(\frac{1}{1-\alpha} + \frac{\nu}{2}\right)} \right] \quad \alpha < 1,$$

(8.1)

(8.2)

that for $\nu = 3$ becomes

$$Z_3 = \gamma(\alpha, m, \beta) A_3^{\frac{3}{2}} \quad A_3 = 4\pi r^2,$$

(8.3)

while for the mean energy one has

$$<U>_\nu = \frac{\nu}{2\alpha + \nu(\alpha - 1)} \beta \quad \alpha > 1,$$

$$<U>_\nu = \frac{\nu}{2 - (\nu + 1)(1 - \alpha)} \beta \quad \alpha < 1,$$

(8.4)

(8.5)

and for the entropy

$$S = \ln Z + \ln[1 + (1 - \alpha)\beta <U>]\frac{1}{1 - \alpha}.$$  

(8.6)
The second term on the right hand of (8.6) is independent of \( r \). Additionally,

\[
\ln \mathcal{Z}_3 = \frac{3}{2} \ln A_3 + \ln[\gamma(m, \beta)] + \ln(3\sqrt{4\pi}).
\]  

(8.7)

Slightly modifying, as in the BG case, Verlinde’s entropic form we have

\[
F_e = -\lambda(m, M) \frac{\partial S_3}{\beta \partial A_3} = -\frac{\lambda}{\beta} \frac{3}{8\pi r^2}.
\]  

(8.8)

We see that (8.8) coincides with (6.6). Renyi’s entropic force is just Boltzmann-Gibbs’ one.

9 Conclusions

We have presented three very simple classical realizations of Verlinde’s conjecture. The Tsallis one, for \( q = 4/3 \) seems to be ”cleaner”, as the entropic force is directly associated to the gradient of Tsallis’ entropy \( S_q \), which acts as a ”potential”, as Verlinde prescribes. This is not so in the classical BG and Renyi instances, in which one has to modify Verlinde’s \( F_e \) definition. The Tsallis case also gives interesting indications regarding the energies involved. Remarkably enough, Boltzmann-Gibbs’ and Renyi’s entropic forces coincide.

Strictly speaking, Verlinde’s conjecture can be unambiguously proved for the Tsallis entropy with \( q = 4/3 \). The Boltzmann-Gibbs and Renyi demonstrations correspond to a modified version of Verlinde’s conjecture. Of course, ours is a very preliminary, if significant, effort. A much more elaborate model would be desired.
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