Riemannian foliations with parallel or harmonic basic forms

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Abstract

In this paper, we consider a Riemannian foliation whose normal bundle carries a parallel or harmonic basic form. We estimate the norm of the O’Neill tensor in terms of the curvature data of the whole manifold. Some examples are then given.

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1 Introduction

In [2], J. F. Grosjean obtained some non-existence results on minimal submanifolds carrying parallel or harmonic forms. Indeed, given a Riemannian manifold \((M^m, g)\) admitting a parallel \(p\)-form and let \((N^n, h)\) be a Riemannian manifold satisfying a certain curvature pinching condition depending on \(m\) and \(p\), he proved that there is no minimal immersion from \(M\) into \(N\). His proof is based on computing the curvature term (which is zero in this case) in the Bochner Weitzenböck formula and using the Gauss formula relating the curvatures of \(M\) and \(N\). As a consequence, he deduced various rigidity results when \(N\) is the hyperbolic space \(\mathbb{H}^n\), the Riemannian product \(\mathbb{H}^r \times S^s\) or the complex hyperbolic space \(\mathbb{C}H^n\).

In the same spirit, he proved that for any compact manifold \((M^m, g)\) carrying a harmonic \(p\)-form (or a non-zero \(p\)th betti number \(b_p(M)\)) and isometrically immersed into a Riemannian manifold \((N^n, h)\), there exists at least a point \(x\) of \(M\) so that (see also [1])

\[
\frac{m}{\sqrt{p}} \left( \frac{p-1}{p} \right) |B(x)||H(x)| \geq k(x) - \left( \frac{p-1}{p} \right) ((m-1)\vec{K}_1 + \vec{\rho}_1)(x),
\]

and

\[
m \left( \frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}} \right) |B(x)||H(x)| \geq \text{Scal}^M(x) - (m-2)((m-1)\vec{K}_1 + \vec{\rho}_1)(x),
\]

where \(|B(x)|\), \(H(x)\), \(k(x)\) and \(\vec{\rho}_1(x)\) denote respectively the norm of the second fundamental form \(B\), the mean curvature of the immersion, the smallest eigenvalue of the Ricci curvature of \(M\), the largest eigenvalue of the curvature operator of \((N^n, h)\) and \(\vec{K}_1(x)\) is the largest sectional curvature of \(N\). These inequalities come from a lower bound of the curvature term (which is non-positive at the point \(x\)) in the Bochner Weitzenböck formula. Thus, if the manifold \((M^m, g)\) is minimally immersed into \((N^n, h)\) and satisfying the pinching condition

\[
\min_M (\text{Scal}^M) > (m-2)((m-1)\max_N(\vec{K}_1) + \max_N(\vec{\rho}_1)),
\]

then \((M^m, g)\) is a sphere of homology (see also [7]).
In this paper, we investigate the study of foliated manifolds whose normal bundle carries a particular form. In fact, we consider a Riemannian manifold \((M, g)\) equipped with a Riemannian foliation \(\mathcal{F}\), which roughly speaking, is the decomposition of \(M\) into submanifolds (called leaves) given by local Riemannian submersions to a base manifold. We assume that the normal bundle of the foliation carries a parallel \(p\)-form (resp. harmonic), with respect to the connection defined in Section \(2\). This corresponds locally to the existence of such a form on the base manifold of the submersions. When shifting the study from immersions to submersions, many objects are replaced by their dual. In particular, the O'Neill tensor \([6]\) plays the role of the second fundamental form and thus, we aim to estimate the norm of the O'Neill tensor in terms of different curvatures data of the manifold \(M\). The main tool is to use the transverse Bochner Weitzenböck formula for foliations \([4]\). Recall that this tensor completely determines the geometry of the foliation. Indeed, it vanishes if and only if the normal bundle of the foliation is integrable.

The paper is organized as follows. In Section \(2\) we recall some well-known facts on differential forms and review some preliminaries on Riemannian foliations. In Section \(3\) we treat the case where the normal bundle carries a parallel form. We compute the curvature term in the transverse Bochner Weitzenböck formula and relate it to the curvature of the manifold \(M\) using the O'Neill formulas. We then deduce a lower bound estimate for the O'Neill tensor (see Thm. \(3.3\) for \(p > 1\) and Cor. \(3.2\) for a rigidity result when \(p = 1\)). In the last section, we study the case where there exist a harmonic form. As before, we deduce a new estimate of the O'Neill tensor (see Thm. \(4.2\)).

### 2 Preliminaries

Let \((M, g)\) be a Riemannian manifold of dimension \(n\) and \(\nabla^M\) be the Levi-Civita connection associated with the metric \(g\). In all the paper, we make the following notations for the curvatures, \(R^M(X, Y) = \nabla^M [X, Y] - [\nabla^M X, Y] + [X, \nabla^M Y]\) and \(R^M_{XYZW} = g(R^M(X, Y)Z, W)\) for any \(X, Y, Z, W \in \Gamma(TM)\). We will denote respectively by \(K^M_0(x)\) and \(K^M_1(x)\) the smallest and the largest sectional curvature and by \(\rho^M_0(x)\) and \(\rho^M_1(x)\) the smallest and largest eigenvalue of the curvature operator \(p^M(X \wedge Y, Z \wedge W) = g(R^M(X, Y)Z, W)\) at a point \(x \in M\). Thus, we have the following inequalities

\[
\rho^M_0(x) \leq K^M_0(x) \leq K^M_1(x) \leq \rho^M_1(x).
\]

(2.1)

Now, let us recall some definitions on forms. The inner product of any two \(p\)-forms \(\alpha\) and \(\beta\) is defined as

\[
\langle \alpha, \beta \rangle = \frac{1}{p!} \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq n} \alpha(e_{i_1}, e_{i_2}, \cdots, e_{i_p})\beta(e_{i_1}, e_{i_2}, \cdots, e_{i_p}),
\]

where \(\{e_1, \cdots, e_n\}\) is an orthonormal frame of \(TM\). The interior product of a \(p\)-form \(\alpha\) with a vector field \(X\) is a \((p-1)\)-form defined by

\[
(X.\alpha)(X_1, \cdots, X_{p-1}) = \alpha(X, X_1, \cdots, X_{p-1}).
\]

More generally, the interior product of \(\alpha\) with \(s\) vector fields \(X_1, X_2, \cdots, X_s\) is a \((p-s)\)-form which is defined as the following

\[
((X_1 \wedge \cdots \wedge X_s).\alpha)(Y_1, \cdots, Y_{p-s}) = \alpha(X_s, \cdots, X_1, Y_1, \cdots, Y_{p-s}).
\]

As a consequence from the definition, we get the rule \(X.\omega(\wedge \theta) = (X.\omega) \wedge \theta + (-1)^p \omega \wedge (X.\theta)\), where \(p\) is the degree of \(\omega\). If the manifold is orientable, the Hodge operator \(*\) defined on a \(p\)-form \(\alpha\) satisfies the following property:

\[
X.(\ast\alpha) = (-1)^p X^* \ast (X^* \wedge \alpha).
\]

(2.2)

Assume now that \((M^n, g)\) is endowed with a Riemannian foliation \(\mathcal{F}\) of codimension \(q\). That means \(\mathcal{F}\) is given by an integrable subbundle \(L\) of \(TM\) of rank \(n - q\) such that the metric \(g\) satisfies the holonomy-invariance condition on the normal vector bundle \(Q = TM/L\); that is \(\mathcal{L}_X g|_Q = 0\) for all \(X \in \Gamma(L)\), where \(\mathcal{L}\) denotes the Lie derivative \([9]\). We call \(g\) a bundle-like metric. This latter condition gives rise to a transverse Levi-Civita connection on \(Q\) defined by \([10]\)

\[
\nabla_X Y = \begin{cases} 
\pi[X, Y], & \text{if } X \in \Gamma(L), \\
\pi(\nabla^M_X Y), & \text{if } X \in \Gamma(Q),
\end{cases}
\]
where \( \pi : TM \to Q \) is the projection. A fundamental property of the connection \( \nabla \) is that it is flat along the leaves, that is \( X_{\mathfrak{d}} R^\nabla = 0 \) for any \( X \in \Gamma(L) \). Thus, we can associate to \( \nabla \) all the curvatures data such as the transverse Ricci curvature \( \text{Ric}^\nabla \) and transverse scalar curvature \( \text{Scal}^\nabla \). A basic form \( \alpha \) on a Riemannian foliation is a differential form which depends locally on the transverse variables; that is satisfying the rules \( X_{\mathfrak{d}} \alpha = 0 \) and \( X_{\mathfrak{d}} d\alpha = 0 \) for any \( X \in \Gamma(L) \). It is easy to see that the exterior derivative \( d \) preserves the set of basic forms and its restriction to this set will be denoted by \( d_b \). We let \( \delta_b \) the formal adjoint of \( d_b \) with respect to the \( L^2 \)-product. Then we have

\[
d_b = \sum_{i=1}^q e_i \wedge \nabla e_i, \quad \delta_b = - \sum_{i=1}^q e_i \wedge \nabla e_i + \kappa, \]

where \( \{e_i\}_{i=1, \ldots, q} \) is an orthonormal frame of \( \Gamma(Q) \) and \( \kappa = \sum_{s=1}^{n-q} \pi(\nabla^M s \nabla s) \) is the mean curvature field of the foliation, which is assumed to be a basic 1-form. Here \( \{V_s\}_{s=1, \ldots, n-q} \) is an orthonormal frame of \( \Gamma(L) \). The basic Laplacian is defined as \( \Delta_b = d_b \delta_b + \delta_b d_b \). Recall that when the foliation is transversally orientable, the basic Hodge operator \( *_b \) is defined on the set of basic \( p \)-forms as being

\[
*_b \alpha = (-1)^{(n-q)(q-p)} * (\alpha \wedge \chi_F),
\]

where \( \chi_F \) is the volume form of the leaves. It is also a basic \( (q-p) \)-form which satisfies the same property as \( (2.2) \). In \([4]\), the authors define a new twisted exterior derivative \( \tilde{d}_b := d_b - \frac{1}{2} \kappa \wedge \) and prove that the associated twisted Laplacian \( \tilde{\Delta}_b := d_b \delta_b + \delta_b d_b \) commutes with the basic Hodge operator. In particular, this shows that the twisted cohomology group \((\text{i.e. the one associated with } d_b)\) satisfies the Poincaré duality. Here \( \delta_b := \delta_b - \frac{1}{2} \kappa \) denotes the \( L^2 \)-adjoint of \( d_b \). Moreover, they state the transverse Bochner-Weitzenböck formula for \( \tilde{\Delta}_b \)

\[
\tilde{\Delta}_b \alpha = \nabla^* \nabla \alpha + \frac{1}{4} |\kappa|^2 \alpha + R(\alpha),
\]

where \( R(\alpha) = - \sum_{j=1}^q e_j \wedge (e_j \wedge R^\nabla(e_i, e_j) \alpha) \). As for ordinary manifolds \( [2] \), the scalar product of \( R(\alpha) \) by \( \alpha \) gives after the use of the first Bianchi identity that

\[
\langle R(\alpha), \alpha \rangle = \sum_{1 \leq i, j \leq q} \text{Ric}^\nabla(e_i, \alpha, e_j, \alpha) - \frac{1}{2} \sum_{1 \leq i, j, k, l \leq q} R^\nabla_{ijkl} (e_j \wedge e_i \wedge e_k \wedge e_l). \tag{2.3}
\]

On the other hand, the geometry of a Riemannian foliation can be interpreted in terms of the so-called the O’Neill tensor \([6]\). It is a 2-tensor field given for all \( X, Y \in \Gamma(TM) \) by

\[
A_X Y = \pi^\perp(\nabla^M_{\pi^\perp(X)} \pi(Y)) + \pi(\nabla^M_{\pi^\perp(X)} \pi^\perp(Y)),
\]

where \( \pi^\perp \) denotes the projection of \( TM \) onto \( L \). By the bundle-like condition, the O’Neill tensor is a skew-symmetric tensor with respect to the vector fields \( Y, Z \in \Gamma(Q) \) and it is equal to \( A_Y Z = \frac{1}{2} \pi^\perp ([Y, Z]) \) and for any \( V \in \Gamma(L) \) we have \( g(A_Y V, Z) = - g(V, A_Y Z) \). Thus we deduce that the normal bundle is integrable if and only if the O’Neill tensor vanishes. If moreover the bundle \( L \) is totally geodesic, the foliation is isometric to a local product.

We point out that the curvature of \( M \) can be related to the one on the normal bundle \( Q \) via the O’Neill tensor by the formula \([9]\)

\[
R^M_{XYZW} = R^Q_{XYZW} - 2 g(A_X Y, A_Z W) + g(A_Y Z, A_X W) + g(A_Z X, A_Y W), \tag{2.4}
\]

where \( X, Y, Z, W \) are vector fields in \( \Gamma(Q) \). One can easily see by \([2, 4]\) that the norm of the O’Neill tensor \( |A|^2 = \sum_{1 \leq i, j \leq q} |A_{e_i} V_j|^2 \) can be bounded at any point by

\[
\text{Scal}^\nabla - q(q-1)K^M_1 \leq 3 |A|^2 \leq \text{Scal}^\nabla - q(q-1)K^M_0.
\]

In particular, if the transversal scalar curvature does not belong to the interval \([q(q-1)K^M_0, q(q-1)K^M_1]\), the normal bundle cannot be integrable.
3 Foliations with parallel basic forms

In this section, we discuss the case where the normal bundle of a Riemannian foliation carries a parallel basic form. That is a $p$-form $\alpha$ satisfying $\nabla \alpha = 0$.

**Proposition 3.1** Let $(M, g, \mathcal{F})$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension $q$. Assume that the normal bundle carries a parallel $p$-form $\alpha$. Then we have

$$0 \leq - \sum_{1 \leq i,j,l \leq q} R^M_{ijkl}(e_i \wedge \alpha, e_j \wedge \alpha) + \frac{1}{2} \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl}(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha)$$

$$+ \sum_{s=1}^{n-q} \left\{ \left( \sum_{i=1}^{q} A_{e_i} V_s \wedge e_i \wedge \alpha \right)^2 - 2 \sum_{i=1}^{q} |A_{e_i} V_s \wedge \alpha|^2 \right\},$$

(3.1)

where $\{e_i\}_{i=1,...,q}$ and $\{V_s\}_{s=1,...,n-q}$ are respectively orthonormal frames of $\Gamma(Q)$ and $\Gamma(L)$.

**Proof.** From Equation (2.4), we have the following formulas

$$R^\mathcal{F}_{ijkl} = R^M_{ijkl} + 2g(A_{e_i} e_j, A_{e_k} e_l) - g(A_{e_i} e_k, A_{e_j} e_l) - g(A_{e_k} e_i, A_{e_j} e_l),$$

(3.2)

and that,

$$\text{Ric}^\mathcal{F}_{ij} = \sum_{l=1}^{q} \left\{ R^M_{ijll} + 2g(A_{e_i} e_j, A_{e_l} e_l) - g(A_{e_i} e_l, A_{e_j} e_l) - g(A_{e_l} e_i, A_{e_j} e_l) \right\}.$$  

(3.3)

The existence of a parallel form $\alpha$ implies that $\langle R(\alpha), \alpha \rangle = 0$. Thus plugging these last two equations into (3.3), we get that

$$0 = \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl}(e_i \wedge \alpha, e_j \wedge \alpha) + 3g(A_{e_i} e_i, A_{e_j} e_j)(e_i \wedge \alpha, e_j \wedge \alpha)$$

$$+ \sum_{1 \leq i,j,k,l \leq q} \left\{ \frac{1}{2} R^M_{ijkl}(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) - g(A_{e_i} e_j, A_{e_k} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) \right\}$$

$$+ \frac{1}{2} g(A_{e_k} e_j, A_{e_i} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) + \frac{1}{2} g(A_{e_k} e_i, A_{e_j} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha).$$

(3.4)

The last two summations in the above equality are in fact equal. Indeed, using that the O’Neill tensor is antisymmetric, we find

$$\sum_{1 \leq i,j,k,l \leq q} g(A_{e_k} e_i, A_{e_j} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) = - \sum_{1 \leq i,j,k,l \leq q} g(A_{e_i} e_k, A_{e_j} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha)$$

$$= - \sum_{1 \leq i,j,k,l \leq q} g(A_{e_i} e_l, A_{e_j} e_k)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha)$$

$$= \sum_{1 \leq i,j,k,l \leq q} g(A_{e_i} e_l, A_{e_j} e_k)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha).$$

On the other hand, we have

$$\sum_{1 \leq i,j,k,l \leq q} g(A_{e_k} e_i, A_{e_j} e_l)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) = \sum_{1 \leq i,j,k,l \leq q} g(A_{e_k} e_j, V_s)(A_{e_k} e_i, V_s)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha)$$

$$= \sum_{1 \leq i,j,k,l \leq q} g(A_{e_k} V_s, e_j)(A_{e_k} V_s, e_i)(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha)$$

$$= \sum_{s=1}^{n-q} \left( \left( \sum_{i=1}^{q} A_{e_i} V_s \wedge e_i \wedge \alpha \right) \left( \sum_{k=1}^{q} A_{e_k} V_s \wedge e_k \wedge \alpha \right) \right)$$

$$= \sum_{i=1}^{q} \left( \sum_{k=1}^{q} A_{e_k} V_s \wedge e_i \wedge \alpha \right)^2.$$  

(3.5)
Also we have that
\[
\sum_{1 \leq i,j,l \leq q} g(A_{e_i} e_i, A_{e_j} e_j) (e_{i,j} \alpha, e_{j,i} \alpha) = \sum_{1 \leq i,j,l \leq q} g(A_{e_i} e_i, V_s) g(A_{e_j} e_j, V_s) \langle e_{i,j} \alpha, e_{j,i} \alpha \rangle
\]
\[
= \sum_{1 \leq i,j,l \leq q} g(A_{e_i} V_s, e_i) g(A_{e_j} V_s, e_j) \langle e_{i,j} \alpha, e_{j,i} \alpha \rangle
\]
\[
= \sum_{1 \leq i,j,l \leq q} (A_{e_i} V_s \alpha, A_{e_j} V_s \alpha) = \sum_{1 \leq i,j \leq n} |A_{e_i} V_s \alpha|^2.
\]

(3.6)

In order to estimate the last term in (3.4), we introduce the p-tensor
\[
B^+(\alpha)(X_1, \cdots, X_p) = \sum_{i=1}^q (e_{i,i} \alpha \wedge A_{e_i})(X_1, \cdots, X_p),
\]
for any \(X_1, \cdots, X_p \in \Gamma(Q)\). We now proceed the computation as in [2]. The norm of the tensor \(B^+(\alpha)\) is equal to
\[
|B^+(\alpha)|^2 = \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p, j \leq q} \langle (e_{i,i} \alpha \wedge A_{e_i})_{i_1, \cdots, i_p} (e_{j,j} \alpha \wedge A_{e_j})_{i_1, \cdots, i_p} \rangle
\]
\[
= \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p, j \leq q} (-1)^{r+t} g(A_{e_i} e_i, A_{e_j} e_j) a_{i_1, \cdots, i_p, j_1, \cdots, j_p}
\]
\[
+ \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p, j \leq q} (-1)^{r+t} g(A_{e_i} e_i, A_{e_j} e_i) a_{i_1, \cdots, i_p, j_1, \cdots, j_p}
\]
\[
+ \frac{1}{p!} \sum_{1 \leq i_1, \cdots, i_p, j \leq q} (-1)^{r+t} g(A_{e_i} e_i, A_{e_j} e_i) a_{i_1, \cdots, i_p, j_1, \cdots, j_p}
\]
\[
= \frac{1}{(p-1)!} \sum_{1 \leq i_1, \cdots, i_{p-1}, j, k \leq q} g(A_{e_i} e_k, A_{e_j} e_k) a_{i_1, \cdots, i_{p-1}, j_1 \cdots j_p}
\]
\[
- \frac{2}{p!} \sum_{1 \leq i_1, \cdots, i_p, j \leq q} g(A_{e_i} e_i, A_{e_j} e_i) a_{i_1, i_2, \cdots, i_p, j_1, \cdots, j_p}
\]

Since we can choose \(\frac{p(p-1)}{2}\) numbers \(r, t\) with \(r < t\) from the set \(\{1, \cdots, p\}\), the last equality can be reduced to
\[
|B^+(\alpha)|^2 = \frac{1}{(p-1)!} \sum_{1 \leq i_1, \cdots, i_{p-1}, j, k \leq q} g(A_{e_i} e_k, A_{e_j} e_k) a_{i_1, \cdots, i_{p-1}, j_1, \cdots, j_p}
\]
\[
- \frac{1}{(p-2)!} \sum_{1 \leq i_1, \cdots, i_{p-2}, j, k, l \leq q} g(A_{e_i} e_k, A_{e_j} e_l) a_{i_1, \cdots, i_{p-2}, j_1, \cdots, j_p}
\]
\[
= \sum_{1 \leq i, j, k \leq q} g(A_{e_i} e_k, A_{e_j} e_k) \langle e_{i,j} \alpha, e_{j,i} \alpha \rangle - \sum_{1 \leq i, j, k \leq q} g(A_{e_i} e_k, A_{e_j} e_l) \langle e_l \wedge e_{i,j} \alpha, e_k \wedge e_{j,i} \alpha \rangle
\]
\[
+ \sum_{1 \leq i, j, k \leq q} g(A_{e_i} e_i, A_{e_j} e_j) \langle e_{i,j} \alpha, e_{j,i} \alpha \rangle + \sum_{1 \leq i, j, k, l \leq q} g(A_{e_i} e_{i,l} A_{e_j} e_{j,k}) \langle e_{i,j} \alpha, e_{j,i} \alpha \rangle.
\]

(3.7)
Returning back to the Equation (3.4) and after plugging Equations (3.5), (3.6) and (3.7), we get the following

\[
0 = \sum_{1 \leq i,j \leq q} R^M_{ijkl}(e_i,\alpha, e_j, \omega) - \frac{1}{2} \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl}(e_j \wedge e_i, \alpha, e_l \wedge e_k, \omega) + |B^+(\alpha)|^2 + 2 \sum_{1 \leq \ell \leq q} |A_{e_{\ell}} V_s \wedge \omega|^2 - \sum_{s=1}^{n-q} |(\sum_{i=1}^{q} A_{e_{\ell}} V_s \wedge e_i)\omega|^2.
\]

Finally using the fact that $|B^+(\alpha)|^2 \geq 0$, we deduce the desired inequality. \qed

For $p = 1$, we find by (3.1) that the lowest sectional curvature $K^M_0$ should be non-positive. Hence we have

**Corollary 3.2** Let $(M, g, F)$ be a Riemannian manifold with positive sectional curvature and endowed with a Riemannian foliation $\mathcal{F}$. Then $M$ cannot carry a parallel basic 1-form.

In the following, we will treat the case $p \geq 2$. For that, we aim to estimate each term in Inequality (3.1). As in [2], we define the basic 2-form $\theta^{i_1,\ldots,i_p-2} = \frac{1}{2} \sum_{1 \leq i,j \leq q} \alpha_{ij}\omega e_i \wedge e_j$ Thus the second term of Inequality (3.1) can be bounded from above by

\[
\frac{1}{2} \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl}(e_j \wedge e_i, \alpha, e_l \wedge e_k, \omega) = \frac{2}{(p-2)!} \sum_{1 \leq i_1,\ldots,i_p-2 \leq q} \rho^M(\theta^{i_1,\ldots,i_p-2}, \theta^{i_1,\ldots,i_p-2}) \leq \frac{2}{(p-2)!} \rho^M_1 \sum_{1 \leq i_1,\ldots,i_p-2 \leq q} |\theta^{i_1,\ldots,i_p-2}|^2 = p(p-1)\rho^M_1 |\alpha|^2.
\]

Using the Cauchy-Schwarz inequality and the fact that $|v \wedge w, \omega| \leq |v||w, \omega|$ for any vectors $v, w$, the last term in (3.1) is bounded by

\[
|\sum_{i=1}^{q} A_{e_{\ell}} V_s \wedge e_i, \omega|^2 \leq q \sum_{i=1}^{q} |A_{e_{\ell}} V_s \wedge e_i, \omega|^2 \leq q \sum_{i=1}^{q} |A_{e_{\ell}} V_s, \omega|^2.
\]  

(3.9)

Now we state our main result:

**Theorem 3.3** Let $(M, g, F)$ be a Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension $q \geq 4$. Assume that the normal bundle carries a parallel $p$-form $\alpha$ with $2 \leq p \leq q-2$. Then we have

\[
(q-2)|\alpha|^2 \geq K^M_0(q(q-1) - (p(p-1) + (q-p)(q-p-1))\rho^M_1.
\]

**Proof.** Plugging the estimates in (3.8) and (3.9) into Inequality (3.1), we get that

\[
0 \leq - \sum_{1 \leq i,j \leq q} R^M_{ijkl}(e_i,\alpha, e_j, \omega) + p(p-1)\rho^M_1 |\alpha|^2 + (q-2) \sum_{1 \leq i \leq q} |A_{e_{\ell}} V_s, \omega|^2.
\]

(3.10)

Since $\alpha$ is a parallel $p$-form, the $(q-p)$-form $*_{\mathcal{F}} \alpha$ is also parallel. Thus replacing $\alpha$ by $*_{\mathcal{F}} \alpha$ in (3.10), we find the inequality

\[
0 \leq - \sum_{1 \leq i,j \leq q} R^M_{ijkl}(e_i,\alpha, e_j, \omega) + (q-p)(q-p-1)\rho^M_1 |\alpha|^2 + (q-2) \sum_{1 \leq i \leq q} |A_{e_{\ell}} V_s, \omega|^2.
\]

(3.11)
In the last term of (3.11), we use the equality (2.2) for the basic Hodge operator. Now the sum of Inequalities (3.10) and (3.11) gives the desired inequality after the use of

$$\sum_{1 \leq i, j, l \leq q} R^M_{ijl}((e_i \circ \alpha, e_j \circ \alpha) + \langle e_i \circ \alpha, e_j \circ \alpha \rangle) = \sum_{1 \leq i, j, l \leq q} R^M_{ijl}((e_i \circ \alpha, e_j \circ \alpha) + \langle e_i \circ \alpha, e_j \circ \alpha \rangle)$$

$$= \sum_{1 \leq i, j, l \leq q} R^M_{ijl}((e_i \circ \alpha, e_j \circ \alpha) + \langle e_i \circ \alpha, e_j \circ \alpha \rangle)$$

$$= \sum_{1 \leq i, j, l \leq q} R^M_{ijl}((e_i \circ \alpha, e_j \circ \alpha) + \delta_{ij} |\alpha|^2 - (e_i \circ \alpha, e_j \circ \alpha))$$

$$= \sum_{1 \leq i, j, l \leq q} R^M_{ijl} |\alpha|^2,$$

(3.12)

which is greater than $K^M_0 q(q-1) |\alpha|^2$.

We point out that the theorem is of interest only if

$$K^M_0 q(q-1) - (p(p-1) + (q-p)(q-p-1)|\rho|^2 > 0,$$

which means by (3.10) that the manifold $M$ is of positive sectional curvature.

**Example:** Let us consider the round sphere $S^{2m-1}$ equipped with the standard metric of constant curvature 1. We denote by $F$ the 1-dimensional Riemannian fibers given by the action

$$e^{2it\theta_1 z_1} \cdots e^{2it\theta_m z_m},$$

with $0 < \theta_1 < \theta_2 \leq \cdots \leq \theta_m \leq 1$. These foliations are Seifert fibrations (i.e. the fibers are compact) if and only if all $\theta'_i$'s are rational and the Hopf fibration corresponds to the case where $\theta_1 = \theta_2 = \cdots = \theta_m = 1$.

In the following, we will compute the O'Neill tensor of the foliation $F$ and study the optimality of the estimate in Theorem 3.3 Without loss of generality, we can assume that $\theta_1 = 1$. The vector $X$ that generates $F$ is given by

$$X = (iz_1, iz_2, \cdots, iz_m).$$

For an integer $l \in \{1, \cdots, m-1\}$ and $p \in \{1, \cdots, m-2\}$, we define the vector fields $Y_l$ and $W_p$ on the tangent space of $S^{2m-1}$ by the following

$$Y_l = (0, \cdots, 0, -\sum_{k=1}^m |z_k|^2)z_l, |z_l|^2 z_{l+1}, \cdots, |z_l|^2 z_m),$$

and,

$$W_p = (0, \cdots, 0, -\sum_{k=p+1}^m |\theta_k|^2 |z_k|^2)iz_p, \theta_p |z_p|^2 |iz_{p+1}|, \cdots, \theta_m |z_m|^2 |z_m).$$

We also denote by $W_{m-1}$ the vector field on $T(S^{2m-1})$ by

$$W_{m-1} = (0, \cdots, 0, -\theta_m |z_m|^2 |iz_{m-1}|, \theta_{m-1} |z_{m-1}|^2 |iz_m).$$

It is easy to see that the set $\{X, Y_l, W_p, W_{m-1} \}$ is an orthogonal frame of the tangent space of the sphere for any $l$ and $p$. Recall now that given an orthonormal frame $\{X/|X|, e_i = Z_i/|Z_i|\}$ of the tangent space of the round sphere for $i = 1, \cdots, 2m-2$, the norm of O'Neill tensor can be computed as follows

$$|A|^2 = \sum_{i, j} |A_{e_i e_j}|^2 = 4 \sum_{i, j} |\pi^{-1}(e_i, e_j)|^2 = \frac{1}{2|X|^2} \sum_{l, j} \frac{1}{|Z_l^2| |Z_j^2|} ([Z_l, Z_j], X)^2,$$

where $\pi^{-1}: TM \rightarrow \mathbb{R}X$ is the projection. On the one hand, a straightforward computation of the norms yields to

$$|X|^2 = |z_1|^2 + \sum_{k=2}^m \theta_k^2 |z_k|^2.$$
Moreover, for any \( l \) and \( p \), we have

\[
|Y_l|^2 = |z_l|^2 \left( \sum_{k=l+1}^{m} |z_k|^2 \right) \left( \sum_{k=l}^{m} |z_k|^2 \right),
\]

\[
|W_p|^2 = |z_p|^2 \left( \sum_{t=p+1}^{m} \theta_t^2 |z_t|^2 \right) \left( \sum_{s=p}^{m} \theta_s^2 |z_s|^2 \right).
\]

Also we find that,

\[
|W_{m-1}|^2 = (\theta_{m-1}^2 |z_{m-1}|^2 + \theta_m^2 |z_m|^2)|z_{m-1}|^2 |z_m|^2.
\]

On the other hand, the computation of the Lie brackets yields for any \( l \) to

\[
([Y_l, W_l], X) = -2\theta_l |z_l|^2 \left( \sum_{s=l+1}^{m} \theta_s^2 |z_s|^2 \right) \left( \sum_{k=l}^{m} |z_k|^2 \right),
\]

and for \( l > p \),

\[
([Y_l, W_p], X) = 2|z_l|^2 \theta_p |z_p|^2 \sum_{k=l+1}^{m} (\theta_t^2 - \theta_k^2) |z_k|^2.
\]

Also, we have that

\[
([Y_{m-1}, W_{m-1}], X) = -2|z_{m-1}|^2 |z_m|^2 \theta_{m-1} \theta_m (|z_{m-1}|^2 + |z_m|^2).
\]

The other Lie brackets are all equal to zero. Thus the O'Neill tensor is equal to

\[
[A]^2 = \frac{2}{|X|^2} \left( \frac{\theta_{m-1}^2 \theta_m^2 (|z_{m-1}|^2 + |z_m|^2)}{|X|^2} + \sum_{j=1}^{m-2} \theta_j^2 \sum_{s=j+1}^{m} \theta_s^2 |z_s|^2 (\sum_{k=j+1}^{m} |z_k|^2) \right) + \sum_{j=1}^{m-2} \sum_{i=j+1}^{m} \frac{|z_i|^2 |z_j|^2 (\sum_{k=i+1}^{m} \theta_k^2 |z_k|^2)}{|X|^2} (\sum_{k=j+1}^{m} |z_k|^2) \}
\]

We will now prove that the norm is constant if only if all \( \theta_i \)'s are equal to 1. Indeed, if we evaluate this norm when it corresponds to the cases where \( |z_{m-1}| \to 1, |z_i| \to 0, i \neq m \) and \( |z_{m-1}| \to 1, |z_i| \to 0, i \neq m-1 \), we find after identifying that \( \theta_{m-1} = \theta_m = \theta \). The value of the O'Neill tensor corresponding to the case \( |z_i|^2 = |z_m|^2 \to \frac{1}{2}, |z_i| \to 0, 2 \leq i \leq m-1 \) gives that \( \theta = 1 \). The same computation can be done successively to prove that \( \theta_i \)'s are equal to one for \( i \neq m, m-1 \) when considering the case \( |z_i|^2 = |z_m|^2 \to \frac{1}{2}, |z_i| \to 0 \). Comparing the lower bound of the inequality in Theorem 3.3 with the norm of the O'Neill tensor which is equal to 2(m − 1), we find that the optimality is realized for \( S^5 \).

Next, we will get another pinching condition which doesn’t require the positivity of the sectional curvature. We have

**Theorem 3.4** Under the same condition as in Theorem 3.3, we have

\[
(q - 2)|A|^2 \geq \text{Scal}^M - K_1^M(n - q)(n + q - 1) - (p(p - 1) + (q - p)(q - p - 1))\rho_1^M.
\]

**Proof.** The proof is a direct consequence from the fact that

\[
\sum_{1 \leq i, j \leq q} R_{\mu i \mu j}^M \geq \text{Scal}_M - K_1^M(n - q)(n + q - 1).
\]

The inequality in Theorem 3.4 is of interest if

\[
K_1^M(n - q)(n + q - 1) + (p(p - 1) + (q - p)(q - p - 1))\rho_1^M \leq \text{Scal}^M
\]

which with the use of \( \text{Scal}^M \leq K_1^M(n - 1) \) gives that \( K_1^M > 0 \).
4 Foliations with basic harmonic forms

In this section, we study the case of Riemannian foliation carrying a basic harmonic form. That is a $p$-form $\alpha$ such that $\Delta_b \alpha = 0$.

**Proposition 4.1** Let $(M, g, \mathcal{F})$ be a Riemannian manifold carrying a Riemannian foliation. Then, we have

$$2\langle R(\alpha), \alpha \rangle \geq -\frac{p-7}{3} \sum_{1 \leq i,j \leq q} \text{Ric}^q_{ij}(\alpha, \alpha) + \frac{p-1}{3} \sum_{1 \leq i,j,l \leq q} R^M_{ijkl}(\alpha, \alpha)$$

$$- \sum_{1 \leq i,j,k,l \leq q} R^M_{ijkl}(\alpha) - \sum_{s=1}^{n-q} \left\{ \sum_{i=1}^{q} |A_{e_i} V_s \omega|^2 + 2 \sum_{i=1}^{q} A_{e_i} V_s \wedge e_i \omega |^2 \right\},$$

for any basic $p$-form $\alpha$.

**Proof.** For $p = 1$, the inequality is clearly satisfied by (2.3). In order to prove the inequality for $p \geq 2$, we introduce as in [2] the operator

$$\mathcal{B}^{-\alpha} = \frac{1}{(p-2)!} \sum_{i_1, \ldots, i_{p-2}} (e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}}) \wedge A_e \otimes e_i^* \wedge \cdots \wedge e_i^*.$$ 

The norm of the tensor $\mathcal{B}^{-\alpha}$ is being defined as the sum

$$|\mathcal{B}^{-\alpha}|^2 = \frac{1}{(p-2)!} \sum_{k,l,i_1, \ldots, i_{p-2}} |(\mathcal{B}^{-\alpha})_{kl, i_1, \ldots, i_{p-2}}|^2.$$ 

Therefore, we compute

$$\frac{(p-2)!}{2} |\mathcal{B}^{-\alpha}|^2 = \frac{1}{2} \sum_{k,l,i_1, \ldots, i_{p-2}} |(\mathcal{B}^{-\alpha})_{kl, i_1, \ldots, i_{p-2}}|^2$$

$$= \frac{1}{2} \sum_{i_1, \ldots, i_{p-2}} ((e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_{p-2}}) \wedge A_e) \otimes e_i^* \wedge \cdots \wedge e_i^*$$

$$= \frac{1}{2} \sum_{i_1, \ldots, i_{p-2}} \alpha_{ii_1 \cdots i_{p-2}} \alpha_{ji_1 \cdots j_{p-2}} g(A_{e_i} e_i, A_{e_j} e_j) - \alpha_{i_1i_2 \cdots i_{p-2}} \alpha_{j_1j_2 \cdots j_{p-2}} g(A_{e_i} e_i, A_{e_j} e_j)$$

$$= (p-1)! \sum_{1 \leq i,j \leq q} \langle e_i \wedge e_i \alpha, e_j \wedge e_j \alpha \rangle g(A_{e_i} e_i, A_{e_j} e_j)$$

$$- (p-2)! \sum_{1 \leq i,j,k \leq q} \langle e_i \wedge e_j \wedge e_k \wedge e_k \alpha, e_i \wedge e_j \wedge e_k \alpha \rangle g(A_{e_i} e_i, A_{e_j} e_j).$$

Thus we deduce that

$$\frac{1}{2} |\mathcal{B}^{-\alpha}|^2 = (p-1) \sum_{1 \leq i,j \leq q} \langle e_i \wedge e_j \alpha, e_i \wedge e_j \alpha \rangle g(A_{e_i} e_i, A_{e_j} e_j)$$

$$- \sum_{1 \leq i,j,k \leq q} \langle e_i \wedge e_j \wedge e_k \wedge e_k \alpha, e_i \wedge e_j \wedge e_k \alpha \rangle g(A_{e_i} e_i, A_{e_j} e_j).$$

Plugging now Equations (3.2) and (3.3) into the above one, we find that

$$\frac{1}{2} |\mathcal{B}^{-\alpha}|^2 = \frac{p-1}{3} \left\{ \sum_{1 \leq i,j \leq q} \text{Ric}^q_{ij}(e_i \wedge e_j \alpha, e_i \wedge e_j \alpha) - \sum_{1 \leq i,j,l \leq q} R^M_{ijkl}(e_i \wedge e_j \alpha, e_i \wedge e_j \alpha) \right\}$$

$$+ \sum_{1 \leq i,j,k,l \leq q} \left\{ -R^q_{ijkl} + R^M_{ijkl} + 2g(A_{e_i} e_j, A_{e_k} e_l) - g(A_{e_k} e_i, A_{e_l} e_j) \right\} \langle e_i \wedge e_j \wedge e_k \wedge e_k \alpha, e_i \wedge e_j \wedge e_k \wedge e_k \alpha \rangle.$$

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Using Equations (2.9) and (3.7), we get
\[
\frac{1}{2} |B^- \alpha|^2 = \frac{p-1}{3} \left\{ \sum_{1 \leq i, j \leq q} \text{Ric}_g^{ij}(e_i \wedge \alpha, e_j \wedge \alpha) - \sum_{1 \leq i, j \leq q} R^M_{ijkl}(e_i \wedge \alpha, e_j \wedge \alpha) \right\} + 2(R(\alpha), \alpha) - 2 \sum_{1 \leq i, j \leq q} \text{Ric}_g^{ij}(e_i \wedge \alpha, e_j \wedge \alpha) + \sum_{1 \leq i, j, k \leq q} R^M_{ijkl}(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha).
\]
\[
+ 2 \sum_{s=1}^{n-q} \left\{ \sum_{i=1}^q |A e_i V_s \wedge \alpha|^2 - \sum_{1 \leq i, j, k, l \leq q} (e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) g(A e_i, e_l, A e_j, e_k) \right\}.
\]

Finally, after the use of Equation (3.7), we find with the help of (3.6) that
\[
\frac{1}{2} |B^- \alpha|^2 + |B^+ \alpha|^2 = \frac{p-1}{3} \left\{ \sum_{1 \leq i, j \leq q} \text{Ric}_g^{ij}(e_i \wedge \alpha, e_j \wedge \alpha) - \sum_{1 \leq i, j \leq q} R^M_{ijkl}(e_i \wedge \alpha, e_j \wedge \alpha) \right\} + 2(R(\alpha), \alpha) + \sum_{1 \leq i, j, k, l \leq q} R^M_{ijkl}(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) + \sum_{s=1}^{n-q} \left\{ \sum_{i=1}^q |A e_i V_s \wedge \alpha|^2 + 2 \sum_{i=1}^q |A e_i V_s \wedge \alpha|^2 \right\}.
\]

Since the l.h.s. of the equality above is non-negative, we finish the proof of the proposition.

We investigate now the case where the form \(\alpha\) is a basic harmonic form. We have

**Theorem 4.2** Let \((M, g, \mathcal{F})\) be a compact Riemannian manifold endowed with a Riemannian foliation of codimension \(q\). Assume that the normal bundle carries a basic harmonic \(p\)-form, there exists at least a point \(x \in M\) such that

\[
(2q + 1) |A|^2(x) \geq -\frac{p-7}{3} \text{Scal}^g(x) + \left( \frac{p-1}{3}q(q-1)K^M_0(x) - 2(p-p-1) + (q-p)q(p-1) \right)|\rho^M_1(x)|^2
\]

where \(2 \leq p \leq q - 2\).

**Proof.** As in the proof of Theorem 3.3, we use Inequality (3.9) in order to deduce that

\[
2\langle R(\alpha), \alpha \rangle \geq -\frac{p-7}{3} \sum_{1 \leq i, j \leq q} \text{Ric}_g^{ij}(e_i \wedge \alpha, e_j \wedge \alpha) + \frac{p-1}{3} \sum_{1 \leq i, j \leq q} R^M_{ijkl}(e_i \wedge \alpha, e_j \wedge \alpha)
\]

\[
- \sum_{1 \leq i, j, k, l \leq q} R^M_{ijkl}(e_j \wedge e_i \wedge \alpha, e_l \wedge e_k \wedge \alpha) - (2q + 1) \sum_{1 \leq i, \ldots, \alpha} |A e_i V_s \wedge \alpha|^2
\]

for any basic \(p\)-form \(\alpha\). Applying the above inequality for the \((q-p)\)-form \(*_b \alpha\) and then summing the two equations, we get by using (3.12) and (3.8)

\[
2(\langle R(\alpha), \alpha \rangle + \langle R(*_b \alpha), *_b \alpha \rangle) \geq -\frac{p-7}{3} \text{Scal}^g|\alpha|^2 + \frac{p-1}{3} q(q-1)K^M_0|\alpha|^2
\]

\[
-2(p-p-1) + (q-p)(q-p-1)|\rho^M_1| |\alpha|^2
\]

\[
-(2q + 1)|A|^2|\alpha|^2.
\]

If the form \(\alpha\) is now harmonic, i.e. \(d_b \alpha = \delta_b \alpha = 0\), then the twisted derivative is equal to \(d_b \alpha = -\frac{1}{2} \kappa \wedge \alpha\) and its adjoint is \(\delta_b \alpha = -\frac{1}{2} \kappa \wedge \alpha\).

Thus

\[
\int_M \langle \Delta_b \alpha, \alpha \rangle v_g = \int_M (|d_b \alpha|^2 + |\delta_b \alpha|^2) v_g = \frac{1}{4} \int_M |\alpha|^2|\alpha|^2 v_g.
\]

This implies by the transverse Bochner-Weitzenböck formula, that \(\int_M \langle R(\alpha), \alpha \rangle v_g \leq 0\). Since the basic Hodge operator commutes with the twisted Laplacian [2], the same inequality holds for \(*_b \alpha\). Thus, we get the required inequality. \(\square\)
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