Improved Design and Numerical Calculation of Chaotic Circuit in Jerk System

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Abstract. Chaotic secure communication has always been a hot topic in nonlinear circuit research. In this paper, an improved jerk chaotic circuit is designed by using cell circuits and nonlinear function circuits. Simulation results show that the circuit can generate multi-scroll chaotic attractors. The position and number of chaotic attractors can be controlled by adding or deleting the number of nonlinear function circuit or by adjusting its parameters. Theoretical analysis and numerical calculation show that the circuit can describe jerk system, and the calculation results are consistent with the circuit simulation results. The circuit is not only simple in structure, easy to operate and high in experimental accuracy, but also easy to control chaos, which is widely used in secure communication.

Keywords: Jerk system, chaotic circuit, symbol function, scroll attractor, secure communication.

1. Introduction
In the 1960s, the "Butterfly Effect" caused a chaotic research storm in the world. Since then, chaos has been widely studied in physics, mathematics, communication, ecology, economy and other fields [1-3]. Chaos circuit widely used in secure communication, image encryption and other aspects is a hot research field [4-5]. Since J.C. Sprott proposed the jerk system chaotic system and its circuit [6-7], jerk system has become a chaotic system of interest for many researchers because of its simple equation form and easy circuit implementation [8-11]. In this paper, firstly, according to the concept of modular circuit, a simple chaotic circuit of jerk system is designed. Secondly, the simulation experiment is carried out on the circuit simulation software. The symbolic function is used as the nonlinear function to control the position and number of the equilibrium points by changing it to achieve the purpose of chaos control. Thirdly, the equation of the circuit is deduced theoretically, and the numerical calculation is carried out to see whether the calculation results are consistent with the circuit simulation results. Finally, the application of this method is discussed.
2. Circuit design of jerk system

2.1. Modular unit

J.C. Sprott, an American physicist, proposed the following equations for the jerk system

\[ \dddot{x} = -\alpha \ddot{x} - \dot{x} + f(x) \]  

(1)

On the basis of Ref [12] and [13], a cell circuit is obtained, as shown in Fig. 1.

![Cell Circuit](image1)

According to the circuit theory, the evolution equation of voltage with time of Fig.1 can be listed as

\[ \frac{dV_0}{dt} = -\frac{1}{C_1} \left( \frac{V_{i1}}{R_1} + \frac{V_{i2}}{R_2} + \frac{V_{i3}}{R_3} + i(x) \right) \]  

(2)

In order to make formula (1) produce chaos, \( i(x) \) in Fig.1 must be a non-linear function. In this paper, we take the symbolic function as follows

\[ i(x) = -I_0 \text{sgn}(x \pm x_0) \]  

(3)

Where \( I_0 \) is a constant and \( \text{sgn}(x) \) is a symbolic function. To realize formula (3), the comparator is composed of operational amplifiers, as shown in Fig. 2. Depending on the principle of comparator, let \( V_0 = 0 \), when \( x > 0 \), \( x_{out} = -|V_{sat}| \); when \( x < 0 \), \( x_{out} = +|V_{sat}| \). Namely

\[ x_{out} = -\text{sgn}(x)|V_{sat}| \]  

(4)

Where \( V_{sat} \) is the saturation voltage. When \( V_0 \neq 0 \),

\[ x_{out} = -\text{sgn}(x + V_0)|V_{sat}| \]  

(5)

According to equation (5), the nonlinear current is

\[ i(x) = -\frac{|V_{sat}|\text{sgn}(x+V_0)}{R} = -I_0 \text{sgn}(x + V_0) \]  

(6)

This is formula (3), where \( V_0 \) is \( x_0 \).

2.2. Jerk circuit

Using the above cell circuit, an improved circuit for implementing the jerk system can be combined, as shown in Fig.3.
Tl083 is selected as the integrated operational amplifier in the circuit, and its operating voltage is ±15V, and the actual saturation voltage is ±|V_{sat}| = ±13.5V. Fig.3 shows the parameters of other circuit elements.

2.3. Circuit simulation of jerk system
The circuit is built on Multisim according to Fig.3. The working voltage of the integrated operational amplifier is ±15V, and the output of U3 and U4 are connected to the X and Y interfaces of the oscilloscope. The positive input of the operational amplifier of U5 is grounded and R3 is adjusted to 12.5kΩ. We ran the circuit and got the phase diagram of a double-scroll attractor from oscilloscope, as shown in Fig.4.

Referring to Ref [14], on the basis of double scroll attractor circuit, if even number of nonlinear function circuits are paralleled with U5, even scroll attractors such as four-scroll, six-scroll and eight-scroll can be obtained; similarly, odd number circuits can also generate odd scroll attractor. In this experiment, two U5 units are connected in parallel, and the voltage of their operational amplifier positive input is set to ±2V and R3 is adjusted to 20kΩ. The circuit is operated to obtain a four-scroll attractor, as shown in Fig.5.
Fig. 5 Phase diagram of jerk chaotic circuit when two symbol function circuits are connected in parallel

We continuously paralleled the nonlinear circuits, respectively set the positive input voltage of the operational amplifier to $\pm 4V$ and $\pm 6V$, and no longer modified the resistance of $R_3$ to obtain a six-scroll and an eight-scroll attractor, as shown in Fig. 6 and Fig. 7.

Fig. 6 Phase diagram of jerk chaotic circuit when four symbol function circuits are connected in parallel

Fig. 7 Phase diagram of jerk chaotic circuit when six symbol function circuits are connected in parallel

In order to obtain odd attractors, two $U_5$ units are paralleled in the circuit, and $\pm 1V$ voltage is applied to the positive input terminal of their operational amplifier respectively to obtain a three-scroll attractor, as shown in Fig. 8. Then, four $U_5$ units are paralleled, and $\pm 3V$ voltage is applied to obtain a five-scroll attractor, as shown in Fig. 9.
3. Theoretical analysis and numerical calculation

3.1. Circuit state equation
Let $U_1$, $U_3$ and $U_4$ output corresponding voltages in Fig.3 as $V_{01}$, $V_{03}$, and $V_{04}$ respectively, and $U_5$ outputs $i(x)$. According to Fig.3 and equation (2), the state equation of jerk system is

$$\begin{align*}
\frac{dV_{03}}{dt} &= \frac{1}{R_8C_3} V_{04} \\
\frac{dV_{04}}{dt} &= \frac{1}{R_5C_2} V_{01} \\
\frac{dV_{01}}{dt} &= -\frac{1}{C_1} \left( \frac{V_{03}}{R_4} + \frac{V_{04}}{R_2} + \frac{V_{01}}{R_3} + i(x) \right)
\end{align*}$$

Let $x = \frac{V_{03}}{|V_{sat}|}$, $y = \frac{V_{04}}{|V_{sat}|}$, $z = \frac{V_{01}}{|V_{sat}|}$, $\tau = \frac{t}{t_0}$, where $|V_{sat}|$ is voltage and $t_0 = R_0C_0$ is time scale transformation factor. Then the dimensionless form of equation (7) is

$$\begin{align*}
\frac{dx}{d\tau} &= ay \\
\frac{dy}{d\tau} &= bz \\
\frac{dz}{d\tau} &= -c[(x + \beta y + \alpha z) + g(x)]
\end{align*}$$
Here \( a = \frac{R_0 C_0}{R_4 C_3}, b = \frac{R_0 C_0}{R_5 C_2}, c = \frac{R_0 C_0}{R_4 C_1}, \beta = \frac{R_4}{R_2}, \alpha = \frac{R_4}{R_1}, g(x) = \frac{C_0}{C_1} \frac{i(x)}{V/R_0} - \frac{C_0}{C_1} i(x), \) let \( l(x) = \frac{i(x)}{|V_{sat}|/R_0} = \text{sgn}(x), \) then \( g(x) = \frac{C_0}{C_1} \frac{i(x)}{V/R_0} = \frac{C_0}{C_1} \text{sgn}(x). \) According to Fig.3, \( R_0 = R_2 = R_4 = R_5 = R_9, C_0 = C_1 = C_2 = C_3, \) then \( a = b = c = \beta = 1, f(x) = -i(x) - x = -\text{sgn}(x) - x. \) Therefore, equation (8) is simplified as

\[
\begin{align*}
\frac{dx}{d\tau} &= y \\
\frac{dy}{d\tau} &= z \\
\frac{dz}{d\tau} &= -y - az + f(x)
\end{align*}
\]  

(9)

This is exactly the Jerk system given in equation (1). Where \( \alpha \) is a parameter, \( f(x) \) is composed of symbolic functions, which are nonlinear.

3.2. Theoretical analysis

The equilibrium point of equation (9) is \((-i(x), 0, 0)\). According to the definition of symbolic function, there are \( x_+ = -1, x_- = 1. \) The Jacobian matrix of system (9) is

\[
J(x_\pm, \alpha) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -1 & -\alpha
\end{pmatrix}
\]  

(10)

It can be seen that the two equilibrium points \((-1, 0, 0)\) and \((1, 0, 0)\) have the same characteristic equation

\[
\lambda^3 - \alpha \lambda^2 - \lambda - 1 = 0
\]  

(11)

By solving equation (11), it can be seen that when the parameter \( \alpha \) is appropriate, these equilibrium points satisfy the condition of saddle-focus. If the characteristic equation satisfies the condition of singular saddle ring in a certain range of parameters, chaotic oscillation may occur.

If \( f(x) = -\text{sgn}(x) - x, \) that is, \( i(x) = -I_0 \text{sgn}(x), \) the double attractor phase diagram can be obtained in a certain parameter range. According to the method of constructing multi-scroll chaotic attractors in Ref [11], when

\[
i(x) = -I_0 \text{sgn}(x) + I_0 \sum_{j=0}^{N-1} (\text{sgn}(x + 2jl_0) + \text{sgn}(x - 2jl_0))
\]  

(12)

An attractor which has 2N scrolls can be generated by jerk system. When

\[
i(x) = I_0 \sum_{j=0}^{N-1} (\text{sgn}(x + (2j + 1)l_0) + \text{sgn}(x - (2j + 1)l_0))
\]  

(13)

An attractor which has 2N+1 scroll can be generated.

3.3. Numerical calculation

The fourth-order Runge Kutta method is used to solve equation (9). Set the parameter \( \alpha = 0.8, \) the initial value is 0, the final value is 2000, and the step size is 0.01. Let the nonlinear function \( I(x) = \text{sgn}(x). \) The phase diagram of double scroll attractor is obtained by solving equation (9), as shown in Fig.10.

If the initial value and step size are constant, take the parameter \( \alpha = 0.5 \) and let \( I_0=1 \) and \( N=2 \) in equation (12), then

\[
i(x) = \text{sgn}(x) + \text{sgn}(x + 2) + \text{sgn}(x - 2)
\]
The above equation is put into equation (9) to calculate the four-scroll attractor, as shown in Fig.11.

![Double-scroll attractor](image1.png)  ![Four-scroll attractor](image2.png)

**Fig.10** Double-scroll attractor  **Fig.11** Four-scroll attractor

Similarly, if the initial value and step size remain unchanged, take the parameter $\alpha = 0.5$, and let $I_0=1$, $N=3$ and $4$ in equation (12), respectively. Input the program to obtain six-scroll and eight-scroll attractors, as shown in Fig.12 and Fig.13.

![Six-scroll attractor](image3.png)  ![Eight-scroll attractor](image4.png)

**Fig.12** Six-scroll attractor  **Fig.13** Eight-scroll attractor

If the initial value and step length are constant, take the parameter $\alpha = 0.5$, and let $I_0=-1$, $N = 1$ and $2$ in equation (13), and put them into equation (9), three-scroll and five-scroll attractors can be obtained, as shown in Fig.14 and Fig.15.

![Three-scroll attractor](image5.png)  ![Five-scroll attractor](image6.png)

**Fig.14** Three-scroll attractor  **Fig.15** Five-scroll attractor
It can be seen that the numerical calculation is in good agreement with the circuit simulation results.

4. Conclusions
Through the above research, the following conclusions can be drawn

(1) It is proved by experiment and theory that a jerk chaotic system can be realized by the combination of several operational amplifier cell circuits and nonlinear function circuits.

(2) It is easy to control the position and number of equilibrium points by using step function (such as symbolic function), so that the position and number of chaotic attractors can be predicted in advance to achieve the purpose of chaos control. Moreover, this kind of control only needs to increase or decrease the number of nonlinear circuits, which is very easy to realize and has high application value in engineering.

(3) The experimental results can be visualized by computer or oscilloscope.

(4) The design idea of the chaotic circuit can be extended to other chaotic systems.

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