Effects of motion in cavity QED

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We consider effects of motion in cavity quantum electrodynamics experiments where single cold atoms can now be observed inside the cavity for many Rabi cycles. We discuss the timescales involved in the problem and the need for good control of the atomic motion, particularly the heating due to exchange of excitation between the atom and the cavity, in order to realize nearly unitary dynamics of the internal atomic states and the cavity mode which is required for several schemes of current interest such as quantum computing. Using a simple model we establish ultimate effects of the external atomic degrees of freedom on the action of quantum gates. The performance of the gate is characterized by a measure based on the entanglement fidelity and the motional excitation caused by the action of the gate is calculated. We find that schemes which rely on adiabatic passage, and are not therefore critically dependent on laser pulse areas, are very much more robust against interaction with the external degrees of freedom of atoms in the quantum gate.

The realization of unitary dynamics of single quantum systems is a field of great current interest. Firstly, emerging experimental possibilities have led to the possibility of testing in the laboratory many of the thought experiments regarding aspects of quantum mechanics such as superposition, non-locality and entanglement which have puzzled physicists since its inception. Secondly a new theoretical understanding of the possibilities of quantum entanglement has led to interest in the efficacy of unitary quantum evolution in problems of computation and communication. In order to enforce unitary evolution for a given system it is necessary to overcome any and all couplings to other degrees of freedom. This is the problem of decoherence which typically leads to increasingly classical seeming evolution as couplings to the environment are increased. It is now the case however that several experimental systems approach the idealized situation where a few degrees of freedom are isolated almost entirely from their surroundings with sufficient experimental control that almost arbitrary unitary evolutions can be effected. These systems include the motional and internal electronic states of ions in Paul traps and the system in which we are interested here, single atoms in high finesse Fabry-Perot microcavities.

Although experiments in cavity quantum electrodynamics (CQED) are far from realizing full-blown quantum computers, this system is particularly interesting since the environmental noises and couplings to extraneous systems are particularly well understood. Current experiments are able to observe the interaction of single atoms with the mode of the cavity over many Rabi cycles due to the use of magneto-optical trapping and cooling of the atoms. Current experimental efforts are aimed at trapping the atom inside the cavity for essentially arbitrary lengths of time. Obtaining complete control over the motion of the atom while inside the cavity has proven to be a challenging experimental task particularly in the face of very large heating rates in current experiments. These are due to the repeated exchange of excitation between the atom and the field, with the associated momentum kicks to the atom. In this paper therefore we briefly discuss effects of the motion of the atom in current and future experiments in CQED with single cold atoms. In particular we will be interested in the level of control of the motion which will be necessary in order to have a system which corresponds to a quantum gate with a high level of accuracy.

This paper is structured as follows. In section we discuss the interaction of a single atom with a cavity mode and the effects of motion on current experiments in cavity QED. We discuss the reasons for seeking to confine the atom in a potential that is independent of the cavity mode. In section we discuss two schemes for quantum computation in cavity QED, set up a model to include motional effects and a measure of how closely such a gate system approaches the ideal unitary evolution. In section we discuss the effects of motion on a Raman scheme for quantum computing and in section we give a parallel treatment for a scheme based on adiabatic passage. In section we conclude.

I. SINGLE ATOM IN A CAVITY

In the experiments of very cold Cesium atoms are dropped into tiny single-mode Fabry-Perot cavities. The master equation for the system is well known. The Hamiltonian for a two level atom interacting with a single driven mode of the electromagnetic field in an optical cavity using the electric dipole and rotating wave approximations (in the interaction picture with respect to the driving laser frequency) is
which depends on the cavity mode volume and the dipole matrix elements for the relevant atomic transition. The final term describes the driving of the cavity by a coherent (laser) driving field of amplitude $E$, chosen here to be real. The atomic transition frequency is $\omega_0$, the cavity has a resonance at the frequency $\omega_c$ and the driving frequency is $\omega_L$. The cavity mode function is $\psi(r) = \cos(k_L x)\exp(-(y^2 + z^2)/w_0^2)$, describing the Gaussian standing wave structure of the field in the Fabry-Perot cavity, the optical wavelength $\lambda_L = 852.359$ nm for the Cesium transition employed ($k_L = 2\pi/\lambda_L$).

Dissipation in the system is due to cavity losses and spontaneous emission. By treating the modes external to the cavity as heat reservoirs at zero temperature it is possible to derive the standard master equation for the density operator of the system \([1,\rho]\),

$$\dot{\rho} = \frac{i}{\hbar}[H,\rho] + 2\kappa D[a]\rho + 2\gamma D[\hat{\sigma}_-]\rho. \quad (1.2)$$

The superoperator $D[c]$ acting on a density matrix $\rho$ is $D[c]\rho = c\rho c^\dagger - \frac{1}{2}c^\dagger c\rho - \frac{1}{2}\rho c^\dagger c$. The dipole decay rate is $\gamma$, while the cavity field decay constant is $\kappa$. The third term describes the effect of spontaneous emission and $\hat{\sigma}_-$ is an operator describing both the change of internal state and the momentum kick on the atom due to a single spontaneous emission.

These experiments represent an important improvement on previous work in that each atom remains in the cavity for many Rabi cycles. The cooling of the atoms prior to entering the cavity effects a separation of timescales of the dynamics of the external degrees of freedom of the atom and the other degrees of freedom in the problem. The variation of the coupling due to the atomic motion, frequency with which the atom passes through wavelengths of the standing wave in the cavity, is, at least initially, much less than the other frequencies involved,

$$|p \cdot \nabla \psi(r)| \ll g_0, \kappa, \gamma. \quad (1.3)$$

However the driving of the cavity field and the interaction of this field with the atom leads to disturbance of the motion which can be understood in terms of a semi-classical theory of the mechanical effects of light in the cavity. In particular the effect of the dipole force in trapping atoms near the antinodes of the cavity field was observed in \([3]\). However the rapid exchange of excitation between the atom and the cavity mode leads to an increased momentum diffusion or heating in this system over a free-space standing-wave. The effect of heating was probably very significant in \([6,8]\). These semi-classical parameters can be calculated numerically, as a function of atomic velocity, through a matrix continued fraction calculation just as in the free-space theory. Simulations of the classical trajectories of the atoms inside the cavity can be performed in three dimensions using a Langevin equation approach with the semiclassical force, friction and momentum diffusion acting on a classical point particle \([3,8]\). It was found that the atom is in the cavity long enough to be significantly heated and that only a few atoms will be sufficiently slow that their motion along the standing wave can be tracked. The heating of the motion means that the atoms will eventually boil out of even the very deep potential wells that can be set up by the dipole force due to the very large field gradients inside the cavity.

One way to reduce the noise on the atom is to move into a highly detuned regime, the dispersive limit of CQED, in which the atom induces a phase shift on the field and the the dipole force provides a nearly conservative potential for the atom. This corresponds to the far off resonance trapping of atoms in optical lattices. In this limit both the cavity field and the atomic internal state can be adiabatically eliminated and a master equation written for the quantum mechanical motional state alone \([1]\)

$$\dot{\rho} = \frac{1}{i\hbar}[H',\rho] + \frac{2g_d^4 E^2}{\kappa^2 \Delta^2} D[\cos^2(k_L x)]\rho \quad (1.4a)$$

$$H' = \frac{p^2}{2m} - \hbar g_0^2 E^2 \kappa^2 \Delta \cos^2(k_L x). \quad (1.4b)$$

The Lindblad term describes heating due to light scattering caused by cavity assisted spontaneous emission. Essentially this is an extra contribution to the light scattering heating present in far off resonant optical lattices and takes place
even though the atom is in principle never excited. Such a regime does provide the hope of long trapping times but there are technical difficulties associated with attaining sufficiently high detunings to fully realize the model, although effects such as light scattering due to free space spontaneous emission could easily be included in this treatment.

There are several reasons to have some other means of trapping the atom in the cavity. Firstly even in the far detuned regime driving the cavity field does not give a particularly slow heating environment for the atom due to the increased light scattering out through the cavity mirrors. Secondly, quantum computing and other interesting schemes for this atom cavity system tend to require that the cavity field is initially in the vacuum state and is not driven and that the motion of the atom is undisturbed during the action of the gate. So it is enticing to consider loading the cavity and trapping the atom with an optical lattice, perhaps another — lower finesse — mode of the Fabry-Perot, or with an ion trap. We anticipate that in any practical realization of these models there will be a means of confining the atom other than the cavity field alone.

It is important that the heating rate $\gamma_{\text{heat}}$ of the atom in this potential be slow compared to the other dynamics of the system and we anticipate that as in the dispersive regime considered above the system will preserve the situation present initially in current experiments, that the frequency $\omega$ associated with the atomic motion is small compared to $\kappa, \gamma$ which are in turn small compared to the coherent coupling $g$. Conditions which will ultimately realize unitary evolution of the atomic internal states and the cavity field will therefore be,

\[
g \gg \kappa, \gamma; \quad g \gg \omega \gg \gamma_{\text{heat}}.
\]

As we will see below there is also a regime in which the mechanical motion of the atom is much faster than the internal state evolution and therefore effectively decouples from it. However the heating of the atom would still have to be negligible for at least a few Rabi cycles so that a gate could be performed before the motional state had to be reset. This requires

\[
\omega \gg g \gg \gamma_{\text{heat}}
\]

which implies very much larger quality factors than current or near future optical lattice or ion trap technology could provide. Finally the requirement that the action of the gate not affect the motion of the atom too greatly will mean that transitions between harmonic oscillator states require more energy than is provided by the atom emitting or absorbing a photon. Thus we also require that the recoil frequency $\omega_r = \hbar k^2 L / 2m$ for the atom and transition under consideration is small compared to the motional frequency

\[
\omega \gg \omega_r.
\]

II. QUANTUM COMPUTING IN CQED

There are several proposals for realizing quantum gates in CQED with point dipoles. We wish to consider here the unavoidable effects of the motion and thus understand the level of control of the motion which will be necessary to realize these schemes with a high fidelity. In particular we wish to compare a system based on controlling the times for which a given interaction is turned on and off with one which relies on an adiabatic passage through eigenstates of a time-dependent Hamiltonian and for which the interaction time is not critical.

A. Raman Scheme

A model of the first type is given by van Enk et al [12] which employs a Raman transition in a cavity to effect a two bit quantum gate. The procedure obtains conditional dynamics for the two atomic internal states through the exchange of a cavity photon between the two atoms, which are imagined to be confined to the antinode of the cavity field. It is also assumed the atoms can each be driven through the side of the cavity by a separate laser. Each atom has two states $|0\rangle_i, |1\rangle_i$ which form the qubit, an auxiliary level $|r\rangle_i$ which is coupled to the cavity and an excited state $|e\rangle_i$ from which the laser driving is detuned. A Raman transition is employed since this reduces the effect of spontaneous emission. The interaction Hamiltonian for the atom cavity interaction with the excited state adiabatically eliminated is

\[
H = \sum_{j=1,2} \frac{gf_j(t)}{2} |1\rangle_j\langle e| a + \text{H.c.} \tag{2.1}
\]
where the atoms have the level structure shown in figure [1]. The function \( f_i(t) < 1 \) describes some laser driving pulse shape and the constant \( g \) describes the effective Raman coupling. Since both atoms interact with the cavity mode this Hamiltonian can be used to build a quantum gate. This gate can be designed so that population in \(|0\rangle\) rarely has to interact with the cavity thereby reducing the errors. In [2] a sequence of pulses is described which realizes the universal two-bit gate

\[
|0\rangle_1|0\rangle_2 \rightarrow |0\rangle_1|0\rangle_2; \quad |1\rangle_1|0\rangle_2 \rightarrow -|1\rangle_1|0\rangle_2
\]

\[
|0\rangle_1|1\rangle_2 \rightarrow |0\rangle_1|1\rangle_2; \quad |1\rangle_1|1\rangle_2 \rightarrow |1\rangle_1|1\rangle_2.
\]

B. Adiabatic Passage via Dark State

A model of the second type is given by Pellizzari et al [3] who show how to perform a controlled-NOT and various other quantum gates by encoding two qubits onto four levels of a single atom and employing laser driving to achieve the conditional dynamics. Information about one of the qubits is transferred back and forth between the two atoms in the gate by transferring coherences between the ground states of one atom to the other through an adiabatic passage involving excitation of the cavity field. This adiabatic passage is through a dark state based on the single atom dark states discussed in [4] and thus suppresses spontaneous emission without employing a Raman transition since the excited states of the atoms are in principle never occupied. The interaction Hamiltonian is very similar to the one for the previous system

\[
H = \sum_{j=1,2} \frac{g}{2} |e\rangle_{jj}\langle r| A + \frac{\Omega_j(t)}{2} |e\rangle_{jj}\langle 1| A + \text{H.c.} \quad (2.2)
\]

and this Hamiltonian has the dark states

\[
|D_0\rangle = |r, r, 0\rangle \equiv |r\rangle_1|r\rangle_2|0\rangle_c,
\]

\[
|D_1\rangle \propto \Omega_1 g |r, 1, 0\rangle + \Omega_2 g |1, r, 0\rangle - \Omega_1 \Omega_2 |r, r, 1\rangle.
\]

Here we have labeled the states in the same way as in the previous model although \(|r\rangle\) could in fact be used as the logical zero of the qubit. With the second atom initially prepared in \(|r\rangle\), switching on the laser driving the second atom, so that \(\Omega_2\) is initially large, and then slowly increasing the driving on the first atom while decreasing the driving on the second, transfers the state of the first atom on \(|r\rangle, |1\rangle\] to the second atom. With appropriate driving on the transition \(|r\rangle \leftrightarrow |0\rangle\] the logical state of the first atom is transferred to the logical state of the second atom. In order to perform a gate it is necessary to consider more than these three levels on atoms and the two qubits can be transferred to coherences between four ground states of the second atom on which the gate can be performed through Raman transitions between the ground states. Since this laser driving can be achieved in such a way that all of the field gradients in the vicinity of the atom will be small we can disregard motional effects for this part of the evolution. What is of interest is the motional state dependence of the actual adiabatic transfer of coherence through the cavity field described here so we will restrict ourselves to the simpler three level system and investigate the behavior of the dark state when motional states and the position dependence of the coupling to the cavity are included.

C. Errors in Quantum Computers

Several strategies for overcoming the effects of unwanted couplings to the environment — including the motion of the atom — are possible. One approach, that of quantum error correcting codes, first described by Shor and Steane [5,6], and encoded gate operations, follows from the realization that errors during the storage of a quantum state or its manipulation can be corrected by coding the qubits in larger Hilbert spaces made up of several identical quantum systems. This would correspond in our case to several atoms in a single cavity or perhaps several cavities each with their own atoms in order to achieve the redundancy necessary to overcome the effects of environmental noise. In this case it can be shown that given sufficient resources any quantum computation can be performed as long as the fundamental error rate for each of the systems is below some threshold value [17]. If this approach is taken the critical thing to know for a particular candidate system is the fundamental error rate due to a given coupling to the environment and what can be done to modify this rate. Other strategies are system specific and revolve around characterizing the errors that occur in a given physical realization of a quantum computer and correcting for
those errors alone, perhaps to all orders. Work along these lines is very far advanced in the case of CQED quantum computing, and references therein.

In the work of van Enk et al. it is necessary that a stationary property holds for the coupling to the environment. This essentially requires that the evolution operators which entangle the basis vectors of the computational subspace with the environment depend only on the length of time it takes to operate the gate and that they commute with each other. This allows schemes which symmetrize the effect of the noise on all of the basis states of the qubits by allowing them to interact with the environment at different times. Studying the effects of motion on the CQED quantum computer is important because, unlike photon absorption, spontaneous emission and systematic errors in the driving times, errors induced by the motion do not obey this stationary property. This is because the free evolution term and the position dependent coupling term do not commute. In any situation where the motion of the atom is comparable to the coupling $g$, so that the evolution operators associated with these terms in the Hamiltonian will be significantly non-commuting, there will be a departure from the conditions required for the correction schemes of to work ideally. Therefore the rate of errors due to the motion will be one contribution to the noise processes which limit the performance of the CQED quantum computer with currently available simplified approaches to error correction.

**D. Model for Motion of Atoms**

We need a model to describe the fundamental effects of motion in CQED models such as those discussed above. The model we will use is motivated by the differences of timescales discussed in the previous section and the knowledge of correction schemes for photon losses and spontaneous emission to all orders. The important rates are therefore the coherent coupling rate achieved for the gate, the frequency associated with the motion of atom and the recoil frequency which describes the effect of the emission or absorption of a single photon on the motion of the atom. Position dependence of the coupling combined with the initial spread of the motional state and motion of the atom in the trapping potential will be sources of noise in the computation. We will assume that the particle is trapped in a harmonic potential with a low heating rate. If the atom is cold and confined in a far off resonance dipole trap then this will be the most significant contribution to the effects of the motion although the oscillation frequency may depend on the internal state of the atom. If the oscillation rate in the potential is small compared to the effective coupling frequency $g$ then in effect we are just adjusting the length scale of the initial state of motion and the precise shape of the potential and any dependence of the potential on the internal state will not significantly change the results. We will leave the consideration of heating of the atom during the gate action for future work although in the limit in which the motion is rather slower than the coupling and the heating rate is slower again than this then the only significant effects of heating will be to lead to initially thermal states of the motion which we will consider in the following.

Assuming then that each atom is trapped in a standing wave cavity in a harmonic potential that is the same regardless of the internal state of the atom we have the Hamiltonian

$$H_i = \frac{\eta}{2} \left( \langle \tilde{A}_i | \tilde{A}_i ^\dagger \rangle | \tilde{A}_i + \tilde{A}_i ^\dagger \rangle \right) + \omega \tilde{A}_i ^\dagger \tilde{A}_i + \tilde{\sigma}_i + \tilde{\sigma}_i ^\dagger, \tag{2.3}$$

where we have defined the operator $\tilde{\sigma}_i = |1\rangle_i \langle r| a$ and $A_i$ is the lowering operator for the motional state of the atom. The recoil frequency $\omega_r = \hbar k_r^2 / 2m$ and the oscillation frequency $\omega$ define a Lamb-Dicke parameter $\eta = \sqrt{\omega_r/\omega}$. In the ideal situation the atom is tightly confined compared to the wavelength of the light and this Lamb-Dicke parameter is very small. We assume that the couplings to the lasers driving the atoms are position independent, this makes sense because these lasers reach the atom through the sides of the cavity and the beam width will in general be much larger than a wavelength. This then is the model depicted schematically in figure (2). There will in fact be some interaction between errors caused by the motion and by cavity decay, which is the most important feature left out of our model, however the purpose of this work is to identify the ultimate sources of error due to the motion in the situation in which the effects of cavity decay can in principle be reversed to all orders as in [12].

**E. Characterising Imperfect Gates**

We also need a means of characterizing the success and failure of a gate. In essence we need a measure of the distance between the actual evolution and the ideal one. Several such measures of the distance between superoperators have been used or proposed in related work and references therein. In this case we are just interested in a simple measure which is physically motivated for quantum gates. We will employ a simple modification the entanglement fidelity introduced...
in [21]. This is related to the overlap or fidelity of a state \( \rho \) to some desired pure state \( |\psi\rangle \), \( F = \langle \psi | \rho | \psi \rangle \). \( F \) is one if and only if \( \rho = |\psi\rangle \langle \psi| \). The entanglement fidelity for a noisy evolution \( \mathcal{E} \) on some state \( \rho \) of a system \( Q \) is

\[
F_e(\rho, \mathcal{E}) = \langle \psi^{\text{RQ}} | (\mathcal{T} R \otimes \mathcal{E}) (|\psi^{\text{RQ}}\rangle \langle \psi^{\text{RQ}}|) |\psi^{\text{RQ}}\rangle
\]

(2.4)

where \( |\psi^{\text{RQ}}\rangle \) is a pure state of \( Q \) and a fictional auxiliary system \( R \) such that \( \text{Tr}_R(|\psi^{\text{RQ}}\rangle \langle \psi^{\text{RQ}}|) = \rho \) and \( \mathcal{T} R \) is the identity superoperator on \( R \). It is shown in [23] that \( F_e \) is independent of the particular purification \( |\psi^{\text{RQ}}\rangle \) chosen. The entanglement fidelity can be thought of as characterizing how well the state and its entanglement are preserved by \( \mathcal{E} \). It is shown in [22] that

\[
F_e(\rho, \mathcal{E}) = \min_{\rho^{\text{RQ}}, \mathcal{E}'} F((\mathcal{E}' \otimes \mathcal{I}^Q)(\rho^{\text{RQ}}), (\mathcal{E}' \otimes \mathcal{E})(\rho^{\text{RQ}}))
\]

where \( F \) is the fidelity of mixed states defined in [23] and describes how close two density matrices are to each other. \( \rho^{\text{RQ}} \) is an extension of the state \( \rho \) to the combined system such that \( \text{Tr}_R(\rho^{\text{RQ}}) = \rho \) and \( \mathcal{E}' \) is an arbitrary evolution on the auxiliary space \( R \). Thus the entanglement fidelity corresponds to the worst possible fidelity of the system state after the evolution \( \mathcal{E} \) to its initial state regardless of how the system is entangled with the environment and of what dynamics \( \mathcal{E}' \) the environment is undergoing. The entanglement fidelity provides a good measure of the preservation of a state in the memory of a quantum computer which could be entangled with many other qubits in the computer and where these qubits could be undergoing arbitrary evolutions as part of the computation. Moreover if \( \rho = \sum p_i |\psi_i\rangle \langle \psi_i| \) then the entanglement fidelity is less than or equal to the average fidelity under \( \mathcal{E} \) of the ensemble making up \( \rho \), \( F_e \leq \sum p_i F_e(\mathcal{E}(|\psi_i\rangle \langle \psi_i|), |\psi_i\rangle \langle \psi_i|) \). But on the other hand if the fidelity of all of the pure states \( |\psi_i\rangle \) with support on \( \rho \) is close to one then the entanglement fidelity is close to one also [24].

Motivated by these considerations we will use a gate entanglement fidelity which measures how close \( \mathcal{E} \) is to the ideal unitary evolution \( U \) over the whole computational subspace of \( \{|0\rangle_1 |1\rangle_2 |0\rangle_2 |1\rangle_2 \} \) by

\[
F_{eg}(\mathcal{E}, U) = \langle \psi^{\text{RQ}} | U^\dagger (\mathcal{T} R \otimes \mathcal{E}) (|\psi^{\text{RQ}}\rangle \langle \psi^{\text{RQ}}|) U |\psi^{\text{RQ}}\rangle
\]

(2.5)

where the \( \rho \) is the completely mixed state on the computational subspace \( \text{Tr}_R(|\psi^{\text{RQ}}\rangle \langle \psi^{\text{RQ}}|) = \mathcal{T} C / 4 \). Thus if \( F_{eg} \) is close to one then the gate is close to ideal for all initial states of the two qubits regardless of how they are entangled with the other qubits in computer and of how these other qubits are being manipulated during the gate operation. This measure has the property of measuring not just how close the evolution is to the ideal evolution for any pure state on the computational subspace but also how well the evolution preserves entanglement between the state of the system and the state of other systems which may be part of the quantum computer.

### III. GATE FIDELITY FOR RAMAN SCHEME

Position dependence of the coupling and motion of the atom in the trapping potential will be a source of noise. In order that the atom in fact be localized near the antinode of the cavity it should occupy a motional state of low position dependence of the coupling and motion of the atom in the trapping potential will be a source of noise. If we restrict our interest just to one atom in the cavity for the moment then we can calculate the Schrödinger picture ket where the internal and cavity states are initially \( |1\rangle_1 |0\rangle_c \) and then leave the laser on such that in the idealized (point-dipole) case we end in the state \( |r\rangle_1 |1\rangle_c \). We wish to leave open the possibility of tailoring the length of the laser pulse such that the fidelity of the final state is optimized by using a pi-pulse appropriate to the mean-squared position of the atom. Thus we choose the interaction time \( t = \pi (1 + \delta) / g \) where \( \delta \) will be of order \( \omega_r / \omega \) and will be chosen to maximize the fidelity of the final state. We will consider initially just a number state of the atom and perform the thermal average at the end of the calculation.

\[
H = H_0 + V;
\]

\[
H_0 = \frac{g}{2} (\hat{\sigma} + \hat{\sigma}^\dagger) + \omega A^\dagger A;
\]

\[
V = -\frac{\gamma^2 g}{4} (A + A^\dagger)^2 (\hat{\sigma} + \hat{\sigma}^\dagger)
\]

(3.1a)

(3.1b)

(3.1c)
The overall Schrödinger picture state after the evolution is

\[
\begin{align*}
\left(1 - \frac{\pi^2 \delta^2}{4} + \frac{\pi^2 \eta^2}{4} (2\bar{n} + 1) - \left(\frac{\pi \eta^2}{4}\right)^2 (8\bar{n}^2 + 8\bar{n} + 1) \\
- \left(\frac{\eta^2 g}{2\omega}\right)^2 \sin^2(\pi\omega/g) \left(\bar{n}^2 + \bar{n} + \frac{1}{2}\right)
\right) |r\rangle_{1c} |n\rangle_{m1}
+ i \left(\frac{\pi \eta^2}{4} (2n + 1) - \frac{\pi \delta}{2}\right) |1\rangle_{1c} |n\rangle_{m1}
+ \left(\frac{\eta^2 g}{8\omega} \sqrt{(n + 1)(n + 2)} \left(1 - e^{-2\pi\omega/g}\right)\right) |1\rangle_{1c} |n + 2\rangle_{m1}
- \left(\frac{\eta^2 g}{8\omega} \sqrt{n(n - 1)} \left(1 - e^{2\pi\omega/g}\right)\right) |1\rangle_{1c} |n - 2\rangle_{m1}
\right)
\end{align*}
\]

where we have only retained those terms which turn out to affect the fidelity and entropy up to fourth order in the Lamb-Dicke parameter and have disregarded an overall phase. Clearly the majority of the population is in the desired final state, there is also population left in the original internal state and a superposition of motional states.

We assume that the initial motional state is in fact a thermal state of average excitation \( \bar{n} \). The thermal averaging can be performed by summing the series for the terms in the reduced density matrix of the internal and cavity states resulting from each of the individual initial number states since these are just geometric series or their derivatives.

The fidelity for this interaction with the cavity is

\[
F = 1 - \frac{\pi^2 \delta^2}{4} + \frac{\pi^2 \eta^2}{4} (2\bar{n} + 1) - \left(\frac{\pi \eta^2}{4}\right)^2 (8\bar{n}^2 + 8\bar{n} + 1) \\
- \left(\frac{\eta^2 g}{2\omega}\right)^2 \sin^2(\pi\omega/g) \left(\bar{n}^2 + \bar{n} + \frac{1}{2}\right).
\]

We may have sufficient control over the length of the laser pulse that we can choose \( \delta \) so as to maximize this quantity, thus giving us the best possible fidelity of the evolution. Setting \( \delta = \eta^2 (2\bar{n} + 1)/2 \) gives us

\[
F_{\text{opt}} = 1 - \left(\frac{\pi \eta^2}{2}\right)^2 (\bar{n}^2 + \bar{n}) \\
- \left(\frac{\eta^2 g}{2\omega}\right)^2 \sin^2(\pi\omega/g) \left(\bar{n}^2 + \bar{n} + \frac{1}{2}\right).
\]

These expressions show the basic behavior of the system in a number of regimes. In the most relevant limit that the harmonic oscillation is much slower than the internal dynamics \( \omega \ll g \), we get \( F_{\text{opt}} \approx 1 - 2 \left(\pi \eta^2/2\right)^2 \left(\bar{n}^2 + \bar{n} + \frac{1}{2}\right) \). In this case the motion of the atom is irrelevant during the time for a pi-pulse and so in this case the parameters \( \eta \) and \( \bar{n} \) essentially just define the initial position spread and coherence length of the atomic motional state. This is the fidelity that would be achieved for any such initial state regardless of the details of the atomic motion on longer timescales.

The opposite limit of very fast motion \( g \ll \omega \) amounts to a rotating wave approximation for the mechanical motion in which the atom oscillates in the potential many times during a single operation, \( F_{\text{opt}} \approx 1 - (\pi \eta^2/2)^2 \left(\bar{n}^2 + \bar{n}\right) \). This limit is attractive since it suggests that if the oscillator is sufficiently cold then the effects of motion could be overcome simply by modifying the naive length for a pi-pulse of the system. However this still requires the assumption that the heating rate of the motion \( \omega_{\text{heat}} \ll g \), which implies an enormous quality factor for the mechanical motion. The laser power required to achieve \( g \ll \omega \) in a far off resonant optical trap would probably be prohibitive in any case. Noise in current ion trap experiments would result in heating rates that were at least comparable with the couplings \( g \) so such a regime would appear to be unfeasible with near future technology.

Neither will the computing operations leave the motional state unmodified. The state will be heated until eventually it will become necessary to cool the motion of the atom. An indication of this can be found by calculating the excitation of the motional state after one exchange on excitation between the atom and the cavity. This is entirely due to contributions resulting from transitions between motional states at some stage during the evolution and as such depends on trigonometric functions of the ratio between coupling and mechanical oscillation frequencies and is independent of small changes in the length of the laser driving,

\[
\langle A^1 A \rangle - \bar{n} = \left(\frac{\eta^2 g}{2\omega}\right)^2 \sin^2(\pi\omega/g) (2\bar{n} + 1).
\]

7
So that in the rotating wave regime $g \ll \omega$ the effective decoupling of the internal state and cavity dynamics from the motion means that the motional state is unaffected by the action of the gate. On the other hand in the more realistic situation $\omega \ll g$, $(A^\dagger A) - \bar{n} \simeq (\pi \eta^2/2)^2 (2\bar{n} + 1)$.

It is straightforward to extend this calculation to the full evolution of the quantum gate with two atoms in the cavity described above and to evaluate the entanglement fidelity for the gate operation

$$F_{eg} = 1 - \pi^2 \eta^4 \left( \bar{n}^2 + \bar{n} + \frac{1}{8} \right) - \frac{\eta^2 g^2}{4\omega} \sin^2(\pi \omega/g) \left( 1 + \cos^2(\pi \omega/g) \right) \left( \bar{n}^2 + \bar{n} + \frac{1}{2} \right)$$

which we give here for the situation in which the driving is not optimized. In this limit of fidelity close to one the entanglement fidelity essentially reduces to the average of the fidelities of the gate operation on each of the four basis states of the computational subspace, although this is not true in general.

The motion of the atom will be excited by the action of the gate depending on the actual initial state of the gate. However in the operation of the gate this initial state could be any superposition of these and could be entangled with the state of other qubits in the computer. As a measure of the overall heating of the motion we will calculate $n_i = \text{Tr} \left( A_i^\dagger A_i E (|\psi^{\text{RC}}\rangle \langle \psi^{\text{RC}}|) \right)$ for a purification on the computational subspace of $\rho^C = \mathcal{I}^C/4$ as discussed above.

This basically assumes no knowledge of the internal state and therefore averages the effect of the motion for each of the four basis states of the computational subspace. These excitation parameters can be calculated

$$n_1 - \bar{n} = \left( \frac{\eta^2 g^2}{2\omega} \right)^2 \sin^2(\pi \omega/g) (2\bar{n} + 1),$$

$$n_2 - \bar{n} = \left( \frac{\eta^2 g^2}{4\omega} \right)^2 \sin^2(2\pi \omega/g) (2\bar{n} + 1).$$

The different dependence on $\omega/g$ is due to the motional states of the atoms being excited at different times during the action of the gate.

We have also performed numerical simulations of the Hamiltonian (2.3) with laser pulses as described above to all orders in the Lamb-Dicke parameter, employing a number state expansion of the motional operators. Here we will plot results for initial states where both the cavity and the atomic motion are initially in the ground states $|\psi^{\text{RC}}\rangle = |\psi^{\text{RC}}\rangle |0_c0_r\rangle |0_{1m0}\rangle |0_{2m0}\rangle$ where $\text{Tr}_C (|\psi^{\text{RC}}\rangle \langle \psi^{\text{RC}}|) = \mathcal{I}^C/4$. In figure (3) the gate entanglement fidelity for this procedure is plotted as a function of the Lamb-Dicke parameter along with the analytic approximation resulting from equation (3.5). This approximation is seen to hold up to reasonably large values of $\eta$. We considered recoil frequencies sufficiently small that the entanglement fidelity and motional excitation were effectively independent of $\omega$ except through $\eta$, the actual values used in these simulations were $g = 1, \omega_r = 0.0005$. Once $\eta > 0.4$, errors are in the region of 10%. The motional excitation of the first atom is plotted in figure (4) along with the approximation of equation (3.6a).

IV. ADIABATIC PASSAGE AND MOTION

A comparison of the previous scheme with one involving adiabatic passage is motivated by the fact that adiabatic passage schemes do not depend on the pulse area of the laser pulses. The position variation of the coupling means that different parts of the wave function see different pulse areas and so may be under or over-rotated by the driving laser. All that is required for the adiabatic theorem to hold is that the Hamiltonian is varied sufficiently slowly that non-adiabatic transitions, the rate of which depend on the energy separation between the eigenstates, do not occur, see for example [23]. The energy spacing of the eigenstates from neighboring eigenstates is determined by the size of the coupling $g$ so it might be hoped that it would be practical to perform the transfer sufficiently slowly that all the population in a wide area around the antinode of the field was transferred through the dark state with high fidelity.

Another way of seeing this is to consider the eigenstates of the Hamiltonian with motion included. Dark states exist which can effect the transfer as long as $\omega \ll g$ — a kind of Raman-Nath regime for the gate. The values $i, j, k, l, m$ in the ket $|i, j, k, l, m\rangle \equiv |i_11j_22k_3|l_{1m1}|m_{m2}\rangle$ refer to the internal state of the first atom, of the second atom, the cavity state, the motional state of the first atom and of the second atom, respectively. The states

$$|D_1\rangle \propto \Omega_1 g|r, 1, 0, n_1, n_2\rangle + \Omega_2 g|1, r, 0, n_1, n_1\rangle$$
\[-\Omega_1 \Omega_2 |r, r, 1, n_1, n_2\rangle
\]
\[-\eta^2 \Omega_1 \Omega_2 \left(A_1^† + A_1\right)^2 |r, r, 1, n_1, n_2\rangle
\]
\[-\eta^2 \Omega_1 \Omega_2 \left(A_2^† + A_2\right)^2 |r, r, 1, n_1, n_2\rangle\]

are eigenstates of the Hamiltonian including atomic motion up to $O(\eta^4)$. Thus terms in the Hamiltonian of $O(\eta^2)$ do not cause errors in the adiabatic passage as they do in the Raman scheme. As a result we could expect fidelities for the process differing from one by numbers of $O(\eta^8)$.

We simulated the adiabatic transfer of coherence discussed above for $g = 1, \omega_r = 0.0001$. We used Gaussian pulse profiles $f_i(t) = \exp \left(-\frac{(g(t - t_i)/40)^2}{2}\right)$ where the two pulses were separated by a time $\Delta t = t_2 - t_1 = 80/g$. This choice resulted in transfer with high fidelity and little population of the atomic excited states for the point dipole atom. Figure (5) plots the fidelity of the transfer for different values of $\eta$. Any gate built on this principle will be limited by the fidelity of this procedure. Laser pulses into the side of the cavity will have very much less affect on the motion and will be achievable with high fidelity presuming the atoms can be addressed separately by the lasers. Thus we do not show a full gate entanglement fidelity for the gate described in [13] here but it will be of the same order as the fidelity for the transfer if laser operations are performed accurately. The striking feature of figure (5) is that for a much larger range of $\eta$ the fidelity is essentially undisturbed by the motion. As $\eta$ is increased the transfer takes place with increased cavity excitation and motional excitation, but without any increase in the excited state population, as suggested by the approximate dark state above. Thus for $\eta \simeq 0.4$ errors are still only $0.2\%$ although the fidelity has begun a sharp decline. Simulation of the internal and external degrees of freedom for two atoms as well as the cavity mode requires a very large Hilbert space which is computationally intensive and so we have not explored values of $\eta$ beyond those plotted here. So that the point at which the fidelity becomes usefully small is yet to be established.

V. CONCLUSIONS

We have investigated the effect of motion on experiments in cavity QED. In particular we discussed the necessity of good control over the motional state in order to realize the dynamics expected for models involving point dipole atoms. The ultimate limitations that the motional state places on quantum computing in CQED systems was discussed for both a Raman scheme and one involving adiabatic passage via a dark state to transfer information between the atoms. The scheme involving adiabatic passage was found to be extremely robust to the precise nature of the atomic motion which may be an important consideration in future experimental implementations of similar schemes.

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FIG. 1. Level structure of the atoms in cavity QED quantum computing models. Information is typically encoded on the states \{|0\rangle, |1\rangle\}.

FIG. 2. Schematic of imagined CQED quantum gate. Two laser beams drive atoms which are harmonically trapped at antinodes of a high finesse microcavity

FIG. 3. Entanglement fidelity for quantum gate with motion as a function of the Lamb-Dicke parameter $\eta$. Both atoms are initially in the ground states of their motion. The solid line is from numerical calculations to all orders of $\eta$ while the dotted line represents an analytical approximation up to $O(\eta^4)$.

FIG. 4. Graph of average excitation $\langle a^†a \rangle$ of one atom after the action of the quantum gate as a function of the Lamb-Dicke parameter $\eta$. The atoms are initially in the ground state of their motion. The solid line represents the results of numerical computations to all orders of $\eta$ while the dotted line represents an analytical approximation to $O(\eta^4)$.
FIG. 5. Fidelity of the adiabatic transfer of from one atom to another as a function of the Lamb-Dicke parameter with atoms initially in the ground state of their motion. Note the improved performance compared to figure (3).

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$|e\rangle$  

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$|1\rangle$  

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$|0\rangle$  

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laser field

cavity field
Driving Lasers
Lamb-Dicke Parameter

Motional State Excitation

Lamb-Dicke Parameter

Motional State Excitation
Adiabatic Transfer Fidelity vs. Lamb-Dicke Parameter