Reduction for $SL(3)$ pre-buildings

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Abstract. Given an $SL(3)$ spectral curve over a simply connected Riemann surface, we describe in detail the reduction steps necessary to construct the core of a pre-building with versal harmonic map whose differential is given by the spectral curve.

1. Introduction

Let $X$ be a Riemann surface with a spectral covering $Σ \subset T^*X$ for the group $SL(3)$. In [15] we proposed in general terms a reduction process that would construct a versal $Σ$-harmonic map to an $SL(3)$ pre-building. It was conjectured that if the $Σ$-spectral network [6, 7, 8] has no BPS states then the reduction process should be well-defined.

This conjecture would lead to a precise calculation of the WKB exponents for singular perturbations whose spectral curve has no BPS states, generalizing the known picture [5] for quadratic differentials and $SL(2)$.

The purpose of the present paper is to provide more details on the reduction process, particularly about the combinatorial structure of the singularities that occur and how they are arranged at each reduction step. We will show (although the proofs are sometimes only sketches) that the reduction steps are well-defined if there are no BPS states. It is left for later to show that the process finishes in finitely many steps.

Assume $X$ is complete and simply connected. For the present one should think of it as being the complex plane with $Σ$ a spectral covering similar to the one considered in our original example [14]. The case of the universal covering of a compact Riemann surface would pose additional problems of non-finiteness of the set of singularities, and the right notion of convergence seems less clear.

The process and results will be summed up in §8 and the reader is referred there.

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In the remainder of the introduction, we review the motivation for the reduction process considered here. A building $B$ for the group $SL(3)$ is a piecewise linear cell complex that is covered by copies of the standard apartment $A = \mathbb{R}^2$, and indeed any two points of $B$ are contained in a common apartment. One of the main properties characterizing a building is that it is negatively curved.

A harmonic map $h : X \rightarrow B$ is a continuous map such that any point $x \in X$, except for a discrete set of singularities, admits a neighborhood $x \in U \subset X$ such that there is an apartment $A \subset B$ with $h : U \rightarrow A \cong \mathbb{R}^2$ being a harmonic map.

The differential $dh$ is naturally a triple of real 1-forms $(\eta_1, \eta_2, \eta_3)$ with $\eta_1 + \eta_2 + \eta_3 = 0$. These are well-defined up to permutation. Now, these real harmonic forms are real parts of holomorphic 1-forms $\eta_i = \Re \phi_i$ and the collection $\{\phi_1, \phi_2, \phi_3\}$ defines the spectral curve $\Sigma \subset T^*X$. We say that $h$ is a $\Sigma$-harmonic map.

For a given spectral curve we would like to understand the $\Sigma$-harmonic maps to buildings. We conjectured in [14, 15] that, under a certain genericity hypothesis, there should be an essentially uniquely defined map $h_\phi : X \rightarrow B^\text{pre}_\phi$ depending only on the spectral curve $\Sigma = \{\phi_1, \phi_2, \phi_3\}$ with the following properties:

1) the pre-building $B^\text{pre}_\phi$ is a negatively curved complex built out of enclosures in $A$ [15], and
2) any $\Sigma$-harmonic map to a building $h : X \rightarrow B$ factors through an embedding $B^\text{pre}_\phi \rightarrow B$ isometric for the Finsler and vector distances.

The conjectured genericity hypothesis for existence of $h_\phi$ is that the spectral network associated to $\Sigma$ should not have any BPS states [6, 7, 8].

Before getting to the proposed method for constructing $B^\text{pre}_\phi$, let us consider the implications for exponents of WKB problems. There are several different ways of getting harmonic mappings to buildings, such as Gromov-Schoen’s theory [9]. Parreau interpreted boundary points of the character variety as actions on buildings [22]. In [14] we extended Parreau’s theory slightly for the situation of WKB problems, getting a control on the differential. Suppose $\nabla_t$ is a singular perturbation of flat connections, for $t$ a large parameter. There are two typical ways of getting $\nabla_t$, the Riemann-Hilbert situation

$$\nabla_t = \nabla_0 + t\varphi$$

or by solution of Hitchin’s equations for the Higgs bundle $(E, t\varphi)$. In either case, there is an associated limiting Higgs bundle $(E, \varphi)$ and we let $\Sigma$ be its spectral curve.

For $P, Q \in X$ let $T_{PQ}(t) : E_P \rightarrow E_Q$ denote the transport for $\nabla_t$. For an ultrafilter $\omega$ on $t \rightarrow \infty$ define the exponent

$$\nu^\omega_{PQ} := \lim_{\omega} \frac{1}{t} \log \|T_{PQ}(t)\|.$$ 

There is a similar vector exponent [14] that is a point in the positive Weyl chamber of $A$.

The groupoid version of Parreau’s theory [22, 14] gives a map to a building $h_\omega : X \rightarrow B_\omega$ such that the Finsler distance (resp. vector distance) between $h_\omega(P), h_\omega(Q)$ is the exponent $\nu^\omega_{PQ}$ (resp. the vector exponent).

We showed in [14] for the Riemann-Hilbert situation that $h_\omega$ is a $\Sigma$-harmonic map. Mochizuki [20] showed this for the Hitchin WKB problem.
If there exists a $\Sigma$-harmonic map $h_\phi$ satisfying the properties (1), (2) above, then it follows that $\nu_{\phi Q}$ is calculated as the Finsler distance between $h_\phi(P), h_\phi(Q)$. In particular, it depends only on the spectral curve $\Sigma$. Independence of the choice of ultrafilter means that the ultrafilter limit used to define the exponent is actually a limit, and we obtain a calculation of the WKB exponents for our singular perturbation.

Turn now to the reduction process for constructing $B_{\phi}^{\text{pre}}$. A conjectural yet detailed picture of this construction was set out in [15] and readers are referred there for a full explanation. The first step was to make an initial construction. That is essentially what we shall be calling $Z^{\text{init}}$ below, although the initial construction for the pre-building has to include additional small parallelogram-shaped regions corresponding to the folded pieces $\tilde{Q}_i$ that we’ll meet in §3 below. Our $Z^{\text{init}}$ has these trimmed off.

The main problem of the initial construction is that it contains points of positive curvature, referred to as 4-fold points or 4$_2$ points below. These have to collapse in some way under any harmonic map to a building $B$ since $B$ is negatively curved. The construction of the pre-building consists of successively doing such a collapsing operation.

Unfortunately, the direction of collapsing at a 4-fold point is not well-defined, rather there are two possibilities. For that reason, the notion of scaffolding was introduced in [15]. The initial scaffolding formalizes the existence of small neighborhoods $U$ that map, in an unfolded way, into appartments of the building. It follows that $U$ should not be folded in the map $h_\phi$ to $B_{\phi}^{\text{pre}}$. The edges gotten by gluing together sectors are, on the other hand, to be folded further, and this collection of data is complete: every edge is either marked as folded or unfolded (open) in the scaffolding.

The scaffolding tells us which direction to fold at a 4$_2$ point. The remaining difficulty is to propagate the information of the scaffolding into the new constructions obtained by collapsing. It was conjectured in [15] that this should be possible, under the hypothesis of absence of BPS states in the original spectral network.

The purpose of the present paper is to prove this conjecture on propagation of the scaffolding. We show how to make a series of reduction steps and how to propagate the scaffolding and other required information so that the series of reduction steps is well-defined.

The present work does not yet result in a full construction of $B_{\phi}^{\text{pre}}$. Notably missing is a convergence statement saying that the process stops in a (locally) finite number of steps. Some parts of our arguments are also sketches rather than full proofs, and we don’t provide here a justification for the choice of initial construction (Principle 5.2).

It was observed in [15] that each step of the reduction process towards the pre-building, could be accompanied by a trimming of the just glued-together parallelograms. If one does that, then the sequence of constructions is a sequence of 2-dimensional surfaces. This point of view will be useful for the program of generalizing Bridgeland-Smith’s work on stability conditions, discussed briefly in §9 at the end. Also, the combinatorics of the reduction steps are all contained in this sequence of surfaces. Therefore, in the present paper we shall include the trimming operation as part of our reduction steps. In order to construct the pre-building
one should put back in the pieces that were trimmed off—the procedure for doing that was explained in \([15]\).

2. Spectral networks in \(X\)

Start by considering spectral networks in \(X\). Away from the branch points, the spectral curve \(\Sigma\) consists of three holomorphic 1-forms \(\phi_1, \phi_2, \phi_3\) and setting \(\phi_{ij} := \phi_i - \phi_j\) these define foliation lines \(f_{ij}\) where \(\Re \phi_{ij} = 0\). Assume the ramification of \(\Sigma\) consists of simple branch points. At a branch point, two indices are interchanged, picking out one of the foliations \(\Re \phi_{ij} = 0\) whose singular leaves starting from the branch point are the initial edges of the spectral network.

Away from the caustics where the three foliation lines are tangent, the coordinates \(\Re \phi_i\) define a flat structure on \(X\) by local identification with the standard apartment \(A := \{(x_1, x_2, x_3) \in \mathbb{R}^3, \ x_1 + x_2 + x_3 = 0\} \cong \mathbb{R}^2\) for buildings of the group \(SL(3)\). We use this flat structure when speaking of angles.

We are going to add some extra singularities, so suppose given a finite subset \(S \subset X\) and for each \(p \in S\) one of the three foliations \(f_p\). A spectral network graph is a map from a trivalent graph \(G\), that can have endpoints as well as at most one infinite end, to \(X\) that sends endpoints of \(G\) to elements of \(S\), that sends edges to foliation lines, and that sends trivalent vertices to collisions \([7]\), points where foliation lines \(f_{ij}, f_{jk}, f_{ik}\) meet at \(120^\circ\) angles. We require that an endpoint of the graph going to \(p \in S\) has adjoining edge going to a foliation line for the given foliation \(f_p\).

A foliation line in \(X\) is an SN-line if there exists a spectral network graph such that the given foliation line is in the image of an edge of the graph that is either adjacent to an endpoint, or is an infinite end.

A BPS state \([6, 7, 8]\) is a compact spectral network graph. Our main hypothesis will be that these don’t exist.

We now note how to add singularities while conserving this hypothesis.

The original set \(S_0\) of singular points is equal to the set of ramification points of the spectral curve, assumed to be simple ramifications. The foliation line \(f_p\) at a ramification point is the one determined by the two sheets of the spectral curve that come together at that point.

The following proposition will allow us successively to add points to the set of singularities.

**Proposition 2.1.** Suppose given a set of singularities \(S_{i-1}\) such that the resulting spectral network doesn’t have BPS states. Choose a point \(p_i\) in general position along the interior of a caustic curve, let \(f_{p_i}\) be one of the foliation lines at \(p_i\) and let \(S_i := S_{i-1} \cup \{p_i\}\). Then the spectral network associated to \(S_i\) also doesn’t have any BPS states.

**Proof.** Suppose we are given a spectral network graph \(\beta : G \to X\) that is a BPS state for \(S_i\). Consider a nearby point \(p_i(\epsilon)\) obtained by moving it a small distance \(\epsilon\) in one direction along the caustic curve (the caustic is generically transverse to the foliation lines otherwise it would constitute a BPS state itself). We may assume that the BPS state follows to a nearby one \(\beta(\epsilon) : G \to X\). The foliation lines are defined by differential forms \(\Re \phi_{ij}\) so this gives us a way of measuring transverse distances; in these terms it makes sense to talk about the distance between one of
the foliation lines and an adjacent one for the same foliation. We should choose for each edge of the graph a sign for the form in question. Now, label edges of the graph \(e \in \text{edge}(G)\) by integers \(k(e)\) such that the edge \(e\) moves by \(k(e)\epsilon\). An edge \(e'\) adjoining an endpoint of the graph that goes to a previous singularity \(q \in S_{i-1}\) has to be labeled with \(k(e') = 0\), since \(q\) doesn’t move with \(\epsilon\). There is a balancing condition on the labels at the collision points. The edges that end in \(p_i\) from one direction have to be labeled by \(k = 1\) whereas the edges that end in \(p_i\) from the other direction have to be labeled by \(k = -1\). However, the balancing condition at the trivalent vertices plus the condition that all the other endpoint labels are zero, means that the number of \(k = 1\) labels and the number of \(k = -1\) labels at \(p_i\) have to be equal. Therefore, we can pair up edges coming in from one direction with edges going out in the other, gluing these edges together pairwise. This results in a new graph \(G' \to X\) that is a BPS state for the previous set of singularities \(S_{i-1}\).

**Boundedness**—Let us mention a boundedness hypothesis that will be useful. Suppose that there exists a compact subset \(K \subset X\) whose boundary \(\partial K\) consists of foliation lines, such that the corners of \(\partial K\) are convex in the sense that each corner consists of foliation lines separated by two sectors inside \(K\). And, we suppose that all ramification points of the spectral curve, and all caustics, are contained in the interior of \(K\). Since the additional singularities were chosen on caustics, it follows that \(S \subset K\).

With this hypothesis, a spectral network graph has no collisions outside of \(K\), and any edge that leaves \(K\) continues as an infinite end. Indeed, all caustics are contained in the interior of \(K\), so \(X - K\) has a flat structure modeled on the standard appartment \(A\). Suppose \(x \in X - K\) were a collision point, say the first one outside of \(K\). Then the two incoming edges must exit from \(K\), but two distinct foliation lines that exit from \(K\) and meet, must exit from the same edge because of the convexity hypothesis on the corners. If they exit from the same edge and meet, then they must meet at a 60° angle, contradicting the hypothesis that they form incoming edges of a collision. Thus, such a collision cannot occur.

Referring to the reduction steps that will be discussed later, any regions that are to be folded together have to stay inside \(K\) by the same kind of considerations. Therefore, we may follow our compact subset along into the sequence of constructions \(Z\) that will occur; the compact subset will contain all modifications, and \(X - K\) remains untouched as a flat space contained in each of our series of constructions.

Although we don’t discuss the question of convergence in the present paper, the existence of a series of compact subsets that contain all the modifications means that one could envision an argument using decrease of the area to conclude termination of the reduction process. That would require bounding from below the size of the reduction steps.

### 3. The initial construction

The *initial construction* is a construction \(Z^{\text{init}}\) obtained by a first process of folding together certain regions in \(X\) and trimming away the resulting pieces. We consider a collection of regions \(Q_i \subset X\) covering the caustics, such that \(Q_i\) are bounded by foliation lines meeting in two points \(q_i, q'_i\) symmetrical with respect to the caustic. In any \(\phi\)-harmonic map to a building \(h : X \to B\), \(q_i\) and \(q'_i\) have to
map to the same point, and indeed $h|_{Q_i}$ has to factor through a map $\tilde{Q}_i \to B$ where $\tilde{Q}_i$ is $Q_i$ folded in two along the caustic $[14, 15]$.

Here is a picture of the regions $Q_i$ in $X$

![Regions Q_i in X](image)

and here is what their images look like in any harmonic map to a building:

![Harmonic images](image)

We assume that the boundaries of the regions $Q_i$ are formed by new spectral network lines gotten after adding singularities along the caustics according to Proposition 2.1. That will be used for the refracting property in §5.

Let $\bar{X}$ denote the quotient of $X$ by the equivalence relation induced by the quotients $Q_i \to \tilde{Q}_i$. Now let $X^{\text{init}}$ be the result of trimming off the folded-together pieces, in other words it is the closure in $\bar{X}$ of $X - \bigcup Q_i$, or equivalently the image of $X - \bigcup Q_i^o$.

The space $X^{\text{init}}$ is a topological surface, and we have cut out the caustics. Thus, $X^{\text{init}}$ is provided with a geometric structure locally modeled on the standard apartment $A$; the three foliation lines at any point correspond to the three standard directions in $A$. In particular $X^{\text{init}}$ has a flat metric. The conformal structure for
this flat metric is different from the original structure of Riemann surface on $X$. It has singular points of positive and negative curvature, namely 8-fold points of negative curvature whose total angle is $480^\circ$ and 4-fold points of positive curvature whose total angle is $240^\circ$.

View $X^{\text{init}}$ as corresponding to a construction, i.e. a presheaf on the site of enclosures as discussed in [15]. More precisely we have a construction $Z^{\text{init}}$ whose usual set of points is $Z^{\text{init}}(p) = X^{\text{init}}$. It is provided with a map

$$h^{\text{init}} : X - \bigcup Q_i^o \to Z^{\text{init}}(p)$$

that sends foliation lines in $X$ to segments in $Z^{\text{init}}$.

This construction will be the starting point of our reduction process. It is provided with a scaffolding [15]: certain edges (the image of $\partial \bigcup Q_i^o$) are designated as “fold edges” while all the other edge germs are designated as “open” or “unfolded”. If $h : X \to B$ is any $\phi$-harmonic map to a building, then its restriction to $X - \bigcup Q_i^o$ factors through a map $Z^{\text{init}} \to B$ that respects the scaffolding, in the sense that fold edges are folded and open edges are unfolded.

4. Structures

Let $Z$ be a construction. We say that $Z$ is complete if the associated metric space is complete. It is normal if the link at any point is connected. We say that $Z$ is ecarinate if, at any edge there are two half-planes. Notice that $Z^{\text{init}}$ satisfies these conditions, and our reduction process will conserve them, so let us consider only complete normal ecarinate and simply connected constructions $Z$. In particular the set of usual points $Z(p)$ is a complete 2-manifold.

If $z \in Z(p)$ is a point then the link $Z_z$ is a connected graph such that each vertex (corresponding to a germ of edge at $z$) is contained in two edges (corresponding to germs of sectors at $z$). Therefore, the link is a polygon. By the parity property the polygon has an even number of edges. We assume that the number of edges in the link at any point is 4, 6 or 8. Most points are flat, meaning that their links are hexagons. The other points are called 4-fold, or 8-fold respectively. The 4-fold points are points of positive curvature, and the 8-fold points are points of negative curvature.

A scaffolding consists of the following structures:

1. A marking of each edge in $Z$ as either open (o) or folded (f), such that the fold edges are those from a discrete collection $F$ of straight edges in $Z$.
2. An orientation assigned to each fold edge.
3. A marking of some subset of the fold edges said to be refracting.

Our initial construction $Z^{\text{init}}$ contains a scaffolding in which all the edges are already marked as refracting, see [15] below, and our reduction process will preserve the refracting condition so we henceforth consider only fully refracting scaffoldings.

We assume that $Z^{\text{init}}$ and its scaffolding have the property that there exists a harmonic mapping to a building such that the fold edges are folded and the open edges are unfolded. This constrains the local type of singularities. However, the number of possibilities is still rather large.

We describe here a list standard examples that will be sufficient for our reduction process—the statement that we remain within this standard list is indeed one of the main conclusions of our treatment in the present paper.
The notation will consist of a boldface number saying how many sectors there are, and a subscript saying how many folded lines in the scaffolding there are. Arrows in the pictures indicate the orientations of the scaffolding edges.

For example, points of type $6_0$ and $8_0$ are respectively 6-fold and 8-fold points with no adjoining fold edges. A point of type $6_2$ is a 6-fold point with a single straight fold edge (it is nonsingular); a $6_3$ point has three folded edges separated by $120^\circ$, and a $6_4$ point is the same with an additional folded edge. These may be pictured as follows:

\[ \begin{array}{ccc}
6_2 & \rightarrow & 6_3 \\
\end{array} \]

Next we picture the 8-fold and 4-fold points. Note that the pictures cannot be conformally correct for the angles; all drawn sectors represent sectors of 60° in $\mathbb{Z}$.

The 8-fold points have at most 2 fold edges; and if there are 2 of them, they are separated by either 1 or 4 sectors. The fold edge orientations go outward, so after the $8_0$ picture there are three possibilities:

\[ \begin{array}{ccc}
8_1 & \rightarrow & 8_2 \\
\end{array} \]

Recall [15] that at a 4-fold point, at least two of the four edges are folded, and if an edge is folded then so is the opposite one. Our standard type is when only two edges are folded, and the edge orientations are inwards towards the singularity, so it has the following picture:

\[ \begin{array}{ccc}
4_2 & \rightarrow \\
\end{array} \]

**Definition 4.1.** We say that the singularities of the scaffolding are initial if at any point of $\mathbb{Z}$ the picture is either $6_0$ (a smooth point not on the scaffolding), $6_2$ (an interior point of an edge of the scaffolding), an 8-fold point of the form $8_1, 8_2, 8_2'$, or a 4-fold point $4_2$. We say that the singularities of the scaffolding are standard if at any point the local picture is one of the ones we have described, namely:

$6_0, 6_2, 6_3, 6_4, 8_0, 8_1, 8_2, 8_2', 4_2$
shown above.

Included in the above definitions are compatibility of the orientations of the fold lines with the singularity types as drawn in the pictures. Recall that edges are oriented outward at 8-fold points and inward at 4-fold points. For initial scaffoldings we obtain the following characterization:

**Remark 4.2.** Given a scaffolding with initial singularities, the collection of fold lines decomposes into a disjoint union of piecewise linear curves called the post-caustics, whose endpoints are of type $8_i$, and along which the 4-fold points alternate with 8-fold points. In particular, the number of 8-fold points on a connected post-caustic is 1 more than the number of 4-fold points.

5. The refraction property

Using Proposition 2.1 to add spectral network lines along the boundaries of the regions $Q_i$ used to define the initial construction, will give that the scaffolding of the initial construction $Z^{\text{init}}$ is a fully refracting scaffolding. The “refracting properties” with respect to the spectral network are:

(R1) At singular points of the scaffolding, all unfolded edges are initial edges of the spectral network; and

(R2) When a spectral network line crosses over a fold edge, it can continue on the other side in either one of the two available directions.

Suppose we are given a construction $Z$ and a fully refracting scaffolding. A spectral network graph for the refracting spectral network is a map from a graph $G \to Z$ satisfying the following properties:

—edges of $G$ go to segments in $Z$ that are straight except when they cross the fold edges;

—edges do not go along fold edges of the scaffolding;

—when edges cross over the fold edges they can “refract”, that is to say they go out of the fold edge on the other side in either one of the two directions;

—trivalent vertices of the graph go to collisions at $6_0$ points i.e. points not on fold edges of the scaffolding; and

—endpoints of the graph go to singular points of the scaffolding or $8_0$ points of $Z$, with the adjoining edges going outward in any non-fold directions.

Before choosing the regions $Q_i \subset X$ that will be folded and trimmed to get $X^{\text{init}}$, add points to the set of singularities of our spectral network, using Proposition 2.1. Add these points along the caustic curves at the places where the boundaries of the $Q_i$ meet the caustics, so that the boundary curves of the $Q_i$ become spectral network curves. Notice furthermore that at the corners $q_i, q'_i$, all three directions outward to the rest of $X$ are spectral network lines, two from continuing the boundary curves and the third by collision of the two boundary curves.

Our above somewhat heuristic discussion leads to the following principle.

**Principle 5.1.** If the original spectral network of $X$ had no BPS states, then the refracting spectral network of $Z^{\text{init}}$ has no BPS states.

We also need to know something about the arrangement of singularities of the scaffolding for $Z^{\text{init}}$. We state this as another principle:

**Principle 5.2.** The initial construction $Z^{\text{init}}$ may be chosen to have only initial singularities (Definition 4.1). The fold edges of the scaffolding are thus arranged into post-caustics as described in Remark 4.2.
We don’t give here a formal justification for the possibility of choosing the initial construction in this way, but note that it is what happens in the pictures we have considered. For example, see [15] for a picture leading to an $8_2$ point.

6. Reduction

This section begins the discussion of a step in the reduction process. The first question is to show the existence of a $4_2$ point about which some collapsing can be done.

**Proposition 6.1.** Suppose given a construction with a refracting scaffolding, such that there are no BPS states in the resulting spectral network. Suppose that the fold edges of the scaffolding are oriented and the singularities are all from our standard list. Make a directed graph using the singularities as vertices, except that we separate a $6_4$ singularity into two vertices. The edges of the graph are the fold edges with their orientations; at a $6_4$ point the “spine” (consisting of the two fold edges that are opposite) goes to one of the vertices and the other two edges go to the other one. This directed graph has no directed loops.

**Proof.** (Sketch)—Consider a path parallel to a directed loop, just to one side of it. This will satisfy the collision and refracting conditions, so taken together with the appropriate initial SN lines coming from singularities, it constitutes a BPS state. □

**Corollary 6.2.** If there are no BPS states and if the set of fold lines in the scaffolding is nonempty, then there must be a $4_2$ point.

**Proof.** In the directed graph described in Proposition 6.1 our hypothesis that there are no BPS states implies that there are no directed loops. Therefore, the orientations of edges make the graph into a poset; since it is finite it has a minimal vertex. The only type of point on the scaffolding that has all fold edges pointed inward is a $4_2$ point, therefore there exists a $4_2$ point. □

We now consider an extended collapsing operation at a $4_2$ point $a$. For this, consider two pieces $R_1$ and $R_2$ that match up, and share a common vertex $a$ and common edges $f$ and $f'$. Let $b$ (resp. $b'$) denote the endpoint of $f$ (resp. $f'$) different from $a$. The $R_i$ are assumed to be constructions that can be considered...
as isomorphic to a subset $R$ in an abstract parallelogram $P$. Let us also label the corresponding vertex $a$ and the corresponding edges $f, f'$ in $P$. The embeddings $R_i \cong R \subset P$ preserve $a, f, f'$. We can now describe the configuration of $R$: the vertex $a$ is an obtuse vertex of $P$, and $f, f'$ are the full edges of $P$ meeting $a$. Hence the points also labelled $b, b' \in P$ are the two acute vertices. We assume that $R$ is a union of finitely many sub-parallelograms $R(2j - 1) \subset P$ (see the picture below for the numbering) such that $R(2j - 1)$ contains $a$ as obtuse vertex, that is to say the edges of $R(2j - 1)$ will be segments in $f, f'$ starting at $a$. Let $t(2j - 1)$ denote the vertices of $R(2j - 1)$ opposite to $a$. We assume that the order $R(1), R(3), \ldots, R(2k + 1)$ is such that $t(1)$ is on the same edge of $P$ as $b$, and they go in a consecutive sequence until $t(2k + 1)$ is on the same edge of $P$ as $b'$. The $t(1), t(3), \ldots, t(2k + 1)$ are the other convex corners of $R$ after $a, b, b'$. Let $s(2), s(4), \ldots, s(2k)$ denote the concave corners of $R$ in between them, so that $s(2j)$ lies between $t(2j - 1)$ and $t(2j + 1)$. Let $g(1)$ be the edge from $b$ to $t(1)$ and let $g(2k + 2)$ be the edge from $t(2k + 1)$ to $b'$. Let $g(2j)$ be the edge from $t(2j - 1)$ to $s(2j)$ and $g(2j + 1)$ be the edge from $s(2j)$ to $t(2j + 1)$. Thus, $g(2j)$ and $f$ are parallel, and $g(2j - 1)$ and $f'$ are parallel.

Now consider the same points in the regions $R_i$, indicated as $s_i(2j)$ and $t_i(2j - 1)$, with edges $g_i(j)$. The configuration of this maximal collapsing pair of regions may be pictured as in Figure [1].

We assume that $R$ is a maximal such region with corresponding regions $R_i \subset Z$, such that the following conditions are satisfied:

—there are no singularities in the interior of $R_i$;

—the only singularities on the interior of the edges $f, f'$ are $6_4$ points where $f$ or $f'$ is the straight fold edge (spine).

**Proposition 6.3.** Under the above maximality conditions, we have the following properties:

1. The vertices $b, b'$ are singularities, and furthermore these singularities are not $6_4$ points with edge $f$ or $f'$ on the spine;

2. For each $j$ there is exactly one of the $s_1(2j), s_2(2j)$ that is an 8-fold singularity, and the other is nonsingular;

3. There are no 8-fold singularities in the interiors of the edges $g_i(j)$;

4. If $q_1$ is a 4-fold or singular 6-fold point on an edge $g_1(j)$ of $R_1$ then the corresponding point $q_2$ on $R_2$ is not a singularity, and vice-versa, and this also applies for corners $t_i(2j - 1)$.

**Proof.** Notice in general that there can’t be a fold edge going from a point on one of the edges $g_i(j)$ into $R_i$ in the middle direction between the directions of $f$ and $f'$ (vertical in Figure [1]); that would have to meet the 4-fold point making it of type $4_4$, or meet an edge in a $6_4$ point but oriented in the wrong direction.

It follows that two corresponding points $q_1 \in R_1$ and $q_2 \in R_2$ can’t both be singularities, for they are joined by a common foliation line that refracts at an edge $f, f'$; we have seen that it can’t be a fold line so it would be an initial SN-line from both singularities resulting in a BPS state. We get 4 and part of 2.

For 1, if $b$ (resp. $b'$) were nonsingular, or a $6_4$ point with edge $f$ (resp. $f'$) along the spine, then one could continue the regions $R_i$ further along the corresponding edge.

For 2, if both points are nonsingular then the regions are not maximal, so one must be singular, say on $R_1$. However, if it were a 6-fold point then both lines going
into $R_1$ parallel to $f$ and $f'$ would be fold lines; these would have to meet $f$ and $f'$ in $6_3$ or $6_4$ points, resulting in fold lines going back in the opposite direction in $R_2$. These two would meet in the corresponding point of $R_2$ but that would have to be singular, contradicting the statement above.

For 3, notice that if some segments of one of these edges are folded, then the orientations of the fold segments are all the same (it follows from the consideration of the first paragraph). But if we had an 8-fold point on the interior of one of these edges, it would send out either a fold line or an SN-line, and that would contradict the existence of the 8-fold singularity given in part 2.

We now consider how fold lines, SN lines and singularities can be arranged along a pair of edges $g_1(j), g_2(j)$. Each edge $g_i(j)$ has two endpoints, $q_i^{(0)}$ that is either $s_i(j)^{0}$ or $b, b'$, and $q_i^{(m_{ij}+1)} = t_i(j)^{0}$.

**Lemma 6.4.** One of the two edges $g_1(j)$ or $g_2(j)$ is an SN line, and has no singularities in its interior (so $m_{ij} = 0$), or at the endpoint $q_i^{(1)}$. It points towards the endpoint $q_i^{(1)} = t_i(j)^{0}$.

**Proof.** One of the two $q_i^{(0)}$ is an 8-fold point. At our admissible possibilities $8_0, 8_1, 8_2, 8_2'$ there are never two fold edges separated by three sectors. Therefore, at this point either the outgoing edge along $g_i(j)$, or the edge that goes into $R_i$ in the opposite direction (that is to say separated by three sectors interior to $R_i$), are SN-lines. If $g_i(j)$ is an SN-line then we obtain the desired conclusion. Notably, there are no singular points along this edge otherwise that would create a BPS state. In the other case, the SN line reflects at $f$ or $f'$ and comes back in the other region $R_i'$, eventually joining the edge $g_i(j)$ so that edge is an SN line. Again in that case it has no singularities. The SN line points away from $q_i^{(0)}$ in both cases, so towards $q_i^{(1)}$.

On the edge $g_i(j)$ not concerned by the above lemma, let $q_i^{(1)}, \ldots, q_i^{(m_{ij})}$ denote the singularities in order along the interior of $g_i(j)$, starting from the nearest to $q_i^{(0)}$ (recall that was $s_i(j)^{0}$ or $b, b'$). The internal singularities are 4 or 6-fold points.

**Lemma 6.5.** All the segments $q_i^{(u-1)}, q_i^{(u)}$ are fold lines for $1 \leq u \leq m_{ij}$. Furthermore, for any $1 \leq u < m_{ij}$ the singularity $q_i^{(u)}$ is a $6_4$ point with the connecting segments of the edge $g_i$ being its spine. These are oriented in the direction away from $q_i^{(0)}$.

**Proof.** If any of these segments were SN lines that would create a BPS state. Note that if $q_i^{(0)}$ is not the singular one of the two points $s_1(j)^{0}, s_2(j)^{0}$, then an SN line from $q_i^{(1)}$ to $q_i^{(0)}$ would continue to one of the edges $f, f'$ and reflect and hit the corresponding point $s_i(j)^{0}$ on the other piece, that is then a singularity by part 2 of Proposition 6.3.

If $1 \leq u < m_{ij}$ then both segments $q_i^{(u-1)}, q_i^{(u)}$ and $q_i^{(u)}, q_i^{(u+1)}$ are fold edges. The only type of singularity with two fold edges separated by three sectors is a $6_4$ point and these edges are the spine. For the orientation, note that all the segments from $q_i^{(0)}$ up to $q_i^{(u+1)}$ are fold edges, and $q_i^{(0)}, q_i^{(1)}$ is oriented outwards from $q_i^{(0)}$. By the rule for orientations at $6_4$ it follows inductively that all the segments are oriented the same way.
When we do our glue and trim operation, the combination of any edge with a folded edge has the same marking as the first edge. So, for the problem of deciding which edges are in the scaffolding, the only indeterminacy is whether the edge \( q_i^{(m_j)} q_i^{(m_j+1)} \) is a fold edge or an SN edge. This could mean the entire edge \( g_i(j) \) if \( m_j = 0 \).

The following lemma says that the spectral network or fold lines created in the glued and trimmed construction will satisfy the refracting property.

**Lemma 6.6.** If the edge \( q_i^{(m_j)} q_i^{(m_j+1)} \) is an SN line, then it points in the direction towards the endpoint \( q_i^{(m_j+1)} = t_i(j') \). As a corollary, if a spectral network line crosses into \( R_1 \) from anywhere across the edge \( g_1 \), it is reflected from \( f \) or \( f' \) and creates two SN lines going out from \( R_2 \) at the corresponding point in both outward directions. Same with 1, 2 interchanged.

**Proof.** The point \( q_i^{(m_j)} \) is a singularity so all outgoing non-folded edges are initial for the spectral network.

\[ \square \]

7. The new construction

Let us now proceed with the collapsing operation of glueing together \( R_1 \) and \( R_2 \), and trimming off the resulting piece. After doing this we obtain the new construction \( Z^\text{new} \). It is again earinate, complete, normal and simply connected.

Let \( g(j) \) denote the edges in \( Z^\text{new} \) obtained by identifying \( g_1(j) \subset R_1 \) with \( g_2(j) \subset R_2 \). Similarly let \( s(j) \) (resp. \( t(j) \)) denote the points resulting from the identification of \( s_1(j) \) and \( s_2(j) \) (resp. \( s_1(j) \) and \( s_2(j) \)). The images of \( b, b' \) are again denoted \( b, b' \). We identify \( Z = R_1 \cup R_2 \) with \( Z^\text{new} - \bigcup g(j) \).

A point \( s(j) \) is obtained by gluing together an 8-fold point and a 6-fold point, having removed 4 sectors from each one. It follows that \( s(j) \) is a 6-fold point.

If say \( t_1(j) \) is a \( 4_2 \) then \( t_2(j) \) is a \( 6_0 \) point, indeed the edges \( g_1(j) \) and \( g_1(j + 1) \) must be folded so by Lemma 6.4 both edges \( g_2(j) \) and \( g_2(j + 1) \) are SN lines. It follows in this case that \( t(j) \) is a \( 6_0 \) point.

If \( t_i(j) \) are both 6-fold points then \( t(j) \) is an 8-fold point.

Suppose that an edge say \( g_1(j) \) contains some singularities \( q_1^{(1)}, \ldots, q_1^{(m)} \). As we have seen, \( q_1^{(1)} \), \ldots, \( q_1^{(m-1)} \) are \( 6_4 \) points with spine along \( q_1(j) \). One can see that these become \( 6_2 \) points in \( Z^\text{new} \). The segments of \( g_1(j) \) that are folded, become unfolded segments in \( Z^\text{new} \) since the opposite edge \( g_2(j) \) is an SN line and a folded segment glued to an unfolded one yields an unfolded segment in the new construction.

In order to determine the scaffolding of \( Z^\text{new} \) it remains to see what becomes of the last segment \( q_1^{(m)} q_1^{(m+1)} \), in case \( g_1(j) \) contains some singularities, or of the whole of \( g(j) \) in case neither side contains singularities in the interior. This depends on the singularities at \( t_i(j') \) and on the last singular point \( q_1^{(m)} \) if it is there, or else on \( s_i(j'') \).

One must do an analysis of cases. The conclusion is that the labeling of the resulting segment of \( g(j) \) as folded or unfolded, is determined by these singularities. In some cases the answer is indeterminate at \( t(j') \) but determined by the other endpoint. The determinations of fold/unfold coming from the two ends of the segment must be the same, because we are supposing the existence of some harmonic map to a building compatible with the scaffolding.
If a segment becomes folded, then it is oriented outwards from the 8-fold point \( t(j') \) (if it is a folded segment then we were not in the case where one \( t_i(j') \) is a 4-fold point, treated above).

We have seen in Lemma 6.6 that such a folded segment will satisfy the refracting condition for reflection of SN lines. We have determined the scaffolding of \( Z^{\text{new}} \), satisfying the refracting condition, and with orientations of edges.

Notice that the two edges meeting in \( t(j) \) cannot both have singular points in the interior, as that would have meant that there was an intersection of fold lines in the interior of \( R_1 \) or \( R_2 \).

What remains to be verified is that the new points \( s(j), t(j) \), and the images of the \( q_i(m) \) if they are there, fall into the standard list of possible singularities; that the required spectral network lines emanating from these points exist; and that the orientations of fold lines are compatible with the allowable orientations at these new singularities.

Discuss first the compatibilities of orientations of fold lines. An 8-fold point \( s_i(j) \) becomes a 6-fold point \( s(j) \). Any fold lines not on the edges \( g(j), g(j+1) \) will stay oriented in the outgoing direction. There may be one or two new fold lines on the edges \( g(j), g(j+1) \), but these will now orient inwards towards \( s(j) \). Indeed, this only happens if \( g(j) \) (resp. \( g(j+1) \)) has no interior singularities and \( t(j') \) is an 8-fold point. See Table 1 below for compatibility.

Consider singularities interior to the edges. Suppose say \( g_1(j) \) contains a sequence of singular points with last one \( q_1(m) \). Let \( q^{(m)} \) denote its image in \( Z^{\text{new}} \).

The segment \( t(j')q^{(m)} \) might or might not be folded. If it is folded, it is oriented outward from \( t(j') \) hence inward towards \( q^{(m)} \). We should check that this is compatible with the singularity type of \( q^{(m)} \). If \( q_1^{(m)} \) is a 4-fold point then so is \( q^{(m)} \) and this compatibility holds. If our new segment is folded it means that the previous segment \( q_1^{(m)}t_1(j') \) had to be unfolded. Recall that \( q_i^{(m-1)}q_1^{(m)} \) is folded and oriented towards \( q_1^{(m)} \). The direction into \( R_1 \) that is parallel to \( f \) or \( f' \) is folded, and this fold line hits one of the edges \( f \) or \( f' \) at a 6_4 point getting reflected back into \( R_2 \) and eventually forming a fold line that will participate in the local picture at the point \( q^{(m)} \).

Using these facts there are two possibilities. First, \( q_1^{(m)} \) could be a 6_3 point whose fold edge along \( g_1(j) \), the segment \( q_1^{(m-1)}q_1^{(m)} \), is oriented inwards. The new segment \( t(j')q^{(m)} \) is not folded, and the resulting point \( q^{(m)} \) is a nonsingular 6_2 point with compatible orientations of fold edges (see the middle two lines of Table 2).

The other possibility is that \( q_1^{(m)} \) be a 6_4 point. In this case the new segment \( t(j')q^{(m)} \) must be folded, and oriented inwards towards \( q^{(m)} \) since \( t(j') \) is an 8-fold point, whereas \( q^{(m-1)}q_1^{(m)} \) is unfolded. The resulting point \( q^{(m)} \) is a 6_4 point. The inward oriented edge of its spine comes from the inward oriented edge of the spine of \( q_1^{(m)} \) reflected into \( R_2 \). See the next-to-last line of Table 2 below for the other fold edges and their orientations.

This completes the discussion of compatibility of the orientations of new scaffolding edges.

We need to verify that the new singular types are all contained in the list that we are using. This will be done by writing down tables of the possibilities. The cases of the points \( b, b' \) are similar and are left to the reader.
Let us assume that we have a singular point on \( R_1 \). Also, for singularities inside edges, we consider edges \( g_1(j) \) that are parallel to \( f \). The other cases are the same by symmetry.

It will be convenient to establish a convention for speaking of directions in \( R_1, R_2 \) at a singularity \( s, t, q \). Number them as follows: \( \langle 0 \rangle \) corresponds to the direction from our singularity, towards the interior of \( R_1 \), not parallel to \( f \) or \( f' \) (“up” in the picture). Then \( \langle 1 \rangle, \langle 2 \rangle, \ldots \) are the directions obtained by turning clockwise 1, 2, \ldots sectors. On the other side, \( \langle -1 \rangle, \langle -2 \rangle, \ldots \) are the directions obtained by turning counterclockwise that many sectors. These join up: at a 6-fold point \( \langle 3 \rangle = \langle -3 \rangle \) while at an 8-fold point \( \langle 4 \rangle = \langle -4 \rangle \).

The edge \( f \) is oriented \( \langle 1 \rangle \), and \( f' \) is oriented \( \langle -1 \rangle \).

In our tables below we will be listing the folded directions at singular points. In this case, a notation \( <i> \) means the edge germ emanating from the singularity in the specified direction. With this notation, in the orientation of our scaffolding, the fold edge is said to be oriented outwards. The same edge oriented inwards towards the singularity will be denoted \( (i) \).

We now consider a singularity of the form \( s_1(j) \). It is an 8-fold point, and the corresponding \( s_2(j) \) is a nonsingular 6-fold point (either \( 6_0 \) or \( 6_2 \)). These glue together to form a point \( s(j) \in Z^{new} \). Table 1 gives the structure of the scaffolding at \( s(j) \) as a function of the structure at \( s_1(j) \).

In order to fill in the table, recall that four sectors are removed from the neighborhood of \( s_1(j) \), as well as from the neighborhood of \( s_2(j) \); then the remaining two sectors from \( s_2(j) \) are glued back in to give the neighborhood of \( s \). We make the convention that edge germs at \( s(j) \) are numbered starting with the middle edge of the two sectors from \( s_2(j) \) being \( \langle 0 \rangle \); the two indeterminate lines are \( \langle 1 \rangle, \langle -1 \rangle \), then remaining \( \langle 2 \rangle, \langle 3 \rangle, \langle -2 \rangle \). These latter correspond to the directions \( \langle 3 \rangle, \langle 4 \rangle, \langle -3 \rangle \) respectively at \( s_1(j) \).

We include a column in the table to say what is happening at the point \( s_2(j) \). Note that it is a nonsingular 6-fold point, hence either \( 6_0 \) or \( 6_2 \). If it is \( 6_2 \) then the direction of the fold line comes from the direction of the fold line at \( s_1(j) \) that goes in direction either \( \langle 1 \rangle \) or \( \langle -1 \rangle \). This extra column will be most useful in our third table below.

We don’t include configurations that are obtained by symmetry (changing \( <i> \) to \( <\bar{i}> \)) from ones that were already included, and we don’t include configurations (such as \( 8_1, \langle 0 \rangle \)) that are ruled out.

Along an edge \( g_1(j) \) parallel to \( f \) suppose given singularities \( q_1^{(1)}, \ldots, q_1^{(m)} \). We have seen that for \( 1 \leq u < m \) the \( q_1^{(u)} \) have to be \( 6_4 \) points with spine along \( g_1(j) \) resulting in a nonsingular \( 6_2 \) point in \( Z^{new} \) (as shows up in the first lines of the next table).

Let us consider now the configurations for \( q_1^{(m)} \) and resulting configurations for \( q^{(m)} \) in \( Z^{new} \), shown in Table 2. Recall that in this case, the three sectors of \( R_1 \) and the three sectors of \( R_2 \) are cut out, and the remaining ones are glued together. We number the edges at the new point \( q^{(m)} \) as follows: the edges \( \langle 2 \rangle, \langle 3 \rangle \) exterior to \( R_1 \) keep the same numbers, whereas \( \langle 2 \rangle, \langle 3 \rangle \) exterior to \( R_2 \) become respectively \( \langle 0 \rangle, \langle -1 \rangle \) (in practice an edge \( \langle -1 \rangle \) at \( q_1^{(m)} \) reflects becoming \( \langle 2 \rangle \) at the nonsingular point \( q_2^{(m)} \in R_2 \) opposite \( q_1^{(m)} \) hence \( \langle 0 \rangle \) at \( q^{(m)} \)); the directions \( \langle 1 \rangle \) and \( \langle -2 \rangle \) correspond to the edge \( g(j) \). Recall that the direction \( \langle 1 \rangle \) is by hypothesis folded...
on $g_1(j)$ so the new edge (1) is unfolded. If $\langle -2 \rangle$ is folded at $q_1^{(m)}$ then it becomes unfolded at $q^{(m)}$, whereas if it is unfolded then it can become either.

### Table 1. Structure at $s(j)$

| $s_1(j)$ | fold edges | $s_2(j)$ | $s(j)$ | new fold edges |
|----------|------------|---------|-------|----------------|
| 8_0      | none       | 6_0     | 6_0   | none           |
| 8_1      | (1)_1      | 6_2     | (2)_2 | 6_0           |
| 8_1      | (2)_1      | 6_0     | 6_0   | none           |
| 8_1      | (3)_1      | 6_0     | 6_2   | $\langle -2 \rangle$ |
| 8_1      | (4)_1      | 6_0     | 6_1   | $\langle -1 \rangle$ |
| 8_2      | (1)_1, (2)_1 | 6_2     | (2)_2 | 6_2   |
| 8_2      | (1)_1, (2)_1 | 6_2     | (2)_2 | 6_2   |
| 8_2      | (2)_1, (3)_1 | 6_0     | 6_2   | $\langle -2 \rangle$ |
| 8_2      | (3)_1, (4)_1 | 6_0     | 6_1   | $\langle -1 \rangle$ |

We now turn to the case of the singular points $t_1(j)$ glueing to the nonsingular $t_2(j)$ to yield $t(j)$. In this case, two sectors are removed from the neighborhood of $t_1(j)$, two sectors removed from the neighborhood of $t_2(j)$, and the remaining sectors are put back together. There are four remaining sectors from $t_2(j)$. We make the following labeling conventions, with subscripts indicating sectors coming from neighborhoods of $t_1(j)$ or $t_2(j)$:

- $(2)_1 \mapsto (3)_1$, $(3)_1 \mapsto (4)_1$, $(2)_2 \mapsto (3)_2$, $(3)_2 \mapsto (4)_2$, $(2)_3 \mapsto (3)_3$, $(3)_3 \mapsto (4)_3$, $\langle -3 \rangle \mapsto \langle -2 \rangle$, $\langle -2 \rangle \mapsto \langle -1 \rangle$, $\langle -1 \rangle \mapsto \langle -2 \rangle$.

In this case the structure of $t_2(j)$ is not determined by that of $t_1(j)$ since it could depend on the singularities along the adjacent edges, so it is included in the table. The possibilities are unfolded, $6_2$ folded in direction $\langle 1 \rangle$, $\langle -2 \rangle$ or $6_2$ folded in direction $\langle -1 \rangle$, $\langle 2 \rangle$. If for example there is a fold in direction $\langle 1 \rangle$, $\langle -2 \rangle$ then it came from a fold line in direction $\langle 1 \rangle$ at the singularity $s_1(j)$ that was reflected on the
edge \( f' \), and in this case the edge \( g_1(j) \) is unfolded, in particular the direction \( \langle 1 \rangle \) at \( t_1(j) \) is unfolded. Similarly in the other direction. Again we omit cases that can be obtained by symmetry.

**Table 3. Structure at \( t(j) \)**

| \( t_1(j) \) | fold edges | \( t_2(j) \) | \( t(j) \) | new fold edges |
|-------------|------------|-------------|-------------|---------------|
| \( 6_0 \)   | none       | \( 6_0 \)   | \( 8_0, 8_1, 8_2 \) | \( (2), (2) \)  |
| \( 6_1 \)   | none       | \( 6_2 \)   | \( -1 \rangle_2, \langle 2 \rangle_2 \) | \( 8_1, 8'_2 \) |
| \( 6_2 \)   | \( \langle 1 \rangle_1, \langle -2 \rangle_1 \) | \( 6_0 \) | \( 8_1, 8'_2 \) | \( (3), (2) \) |
| \( 6_3 \)   | \( \langle 1 \rangle_1, \langle -1 \rangle_1, \langle 3 \rangle_1 \) | \( 6_0 \) | \( 8_1 \) | \( (4) \) |
| \( 6_4 \)   | \( \langle 1 \rangle_1, \langle -1 \rangle_1, \langle 2 \rangle_1, \langle 3 \rangle_1 \) | \( 6_0 \) | \( 8'_2 \) | \( (3), (4) \) |
| \( 4_2 \)   | \( \langle 1 \rangle_1, \langle -1 \rangle_1 \) | \( 6_0 \) | \( 6_0 \) | none |

In some rows of the table, the answer is not determined by the information local to \( t_1(j) \). In those cases we have included the various possibilities. Notice that the marking of edges \( (2), \langle -2 \rangle \) will be determined from what happens in the two previous tables at the adjacent singularities on these segments.

**Corollary 7.1.** The scaffolding of \( Z^{\text{new}} \) is well-defined, with orientations of the fold edges. From the tables, the types of local pictures of the scaffolding for \( Z^{\text{new}} \) are in our standard list \( 4_2, 6_3, 6_4, 6_0, 8_0, 8_1, 8_2, 8'_2 \). The orientations of the fold edges at these singularities are compatible with the allowable configurations.

**Proposition 7.2.** Define the refracting spectral network of \( Z^{\text{new}} \) to be generated by an initial SN line going in every non-fold direction from each of the singularities, and closed under collisions as well as refraction upon crossing fold lines of the new scaffolding. Then, any SN-line of this new spectral network, outside of the edges \( g(j) \), is contained in the previous spectral network of \( Z \). SN lines along non-folded segments of the \( g(j) \) are with reversed orientation with respect to those of \( Z \). Under the assumption that there were no BPS states in the refracting spectral network of \( Z \), then there are no BPS states in the refracting spectral network of \( Z^{\text{new}} \).

**Proof.** (Sketch)—We verify in each of the cases contained in the tables, that there are SN lines in \( Z - (R_1 \cup R_2) \) corresponding to all non-folded outward directions from singular points. In the case of non-folded edges that are segments of the \( g(j) \), there were SN lines in the non-folded segments of \( g_i(j) \) going in the opposite direction. Switch the directions of these, and when these new SN lines meet a \( 6_2 \) point, notice that it came from a singular point and the refracted directions are among the directions containing SN lines of \( Z \) (or we continue along the next segment of \( g(j) \) to the next \( 6_2 \) point).

At an 8-fold singularity obtained from the third table, the SN lines in all directions are generated by the SN lines along the segments of \( g_i(j) \) and \( g'_i(j) \), sometimes by using the collision process in \( Z \).
We should check that there are no BPS states in the new spectral network. Concerning lines not on the edges \( g(j) \) this comes from the inclusion into the previous spectral network in \( Z - (R_1 \cup R_2) \), and the existence of SN lines going outwards from any new singularities as noted above.

We therefore need to consider new SN lines along segments of \( g(j) \). For this, let us notice that in Table 1, whenever a singular \( 6_3 \) or \( 6_4 \) is created, the edges going along \( g(j) \) and \( g(j + 1) \), there denoted \( \langle 1 \rangle \) and \( \langle -1 \rangle \), are folded. Furthermore, whenever a \( 6_2 \) is created, one of those two edges is folded so a BPS state between the two adjacent \( s(j), s(j + 1) \) is not created. Similarly, in 2 when a \( 6_4 \) or point is created, the segment after it on \( g(j) \), denoted there \( \langle -2 \rangle \), is folded.

This has only been a sketch of proof, a more detailed discussion is needed in order to follow through all possible SN lines that might start with directions along the edges \( g(j) \).

\[\Box\]

8. Scholium

We now review what has been done (or sketched) above. From \( X \) we created an initial complete normal ecarinate construction \( Z^{\text{init}} \) and we are assuming that this is done following Principles 5.1 and 5.2.

Thus \( Z^{\text{init}} \) is provided with a fully refracting scaffolding, whose associated spectral network has no BPS states. Initially the scaffolding has only \( 8_1, 8_2, 8'_2, 4_2 \) singularities, so the endpoints of the post-caustics are \( 8_1 \) points, and the 8-fold and 4-fold points alternate.

The reduction process will consist of a sequence of reduction steps starting with \( Z^{\text{init}} \). Let us denote by \( Z \) the construction obtained after a certain number of steps. Our goal is to describe the next reduction step.

Our construction \( Z \) is again complete, ecarinate, normal, and it is provided with a refracting scaffolding with oriented fold edges. Our assumptions are as follows:

— that the refracting spectral network generated by the scaffolding has no BPS states;
— and that the types of points in the scaffolding are in the standard list of Definition 4.1 taking account orientations of edges.

Suppose that there are some remaining fold edges. By Corollary 6.2 there exists a \( 4_2 \) point. The reduction step will be to collapse a neighborhood of this \( 4_2 \) point in a good way.

Choose a maximal collapsing pair \( R_1, R_2 \) at this vertex. These regions are arranged with singularities and edges satisfying the properties of Proposition 6.3 and the subsequent discussion.

We then glue together \( R_1 \) and \( R_2 \), and trim away the resulting piece (except for the union of edges \( g(j) \)), to get a new construction \( Z^{\text{new}} \). This is the result of a single step of our reduction process.

The main point is to verify that \( Z^{\text{new}} \) is provided with a well-defined refracting scaffolding that still satisfies the required properties. We have seen that the configurations at singularities in \( Z \) determine the fold edges at the new points in \( Z^{\text{new}} \). Indeed, this is the case at points of the form \( s(j) \) and \( q^{(m)} \), and the only indeterminacy at points \( t(j) \) is along segments that will be connected either to points \( q^{(m)} \) or \( s(j) \) so the fold edges are determined. We have also assigned orientations to the fold edges.
Specific analysis of each case allows to fill in Tables 1, 2, 3. These show that the new singularities are only of the types in our standard list. Furthermore, we see here that the types of singularities are compatible with the assigned orientations of the fold edges in the new scaffolding.

We noted along the way that the fold edges of the new scaffolding have the required refracting effect on spectral network lines. The sketch of proof of Proposition 7.2 shows how the new spectral network is a subset of the previous one, apart from the spectral network lines that might have switched directions along the unfolded segments of the edges \(g(j)\), and this spectral network doesn’t have any BPS states.

This completes the verification that our new construction \(Z^{\text{new}}\) has the required structures and satisfies the required properties so it can be used as the starting point in a next step of the reduction process.

9. Further questions

We have described a single step of the reduction process. The main question will now be to obtain a convergence statement saying that the process ends in finitely many steps with a construction \(Z^{\text{core}}\) whose scaffolding has an empty set of edges. Suppose it does end. The only singularities of \(Z^{\text{core}}\) are 8\(0\) points of negative curvature

This reduced construction will be the core of the pre-building that we are conjecturing to exist in [15]. In order to get the pre-building, the steps of putting back in the pieces that have been trimmed off, need to be done as described in [15].

The construction of the core \(Z^{\text{core}}\) may be seen as a 2-dimensional analogue of the Stallings core graph [23, 24]. It should be interesting to compare these combinatorics to the ones of [1, 2, 11].

In current work with Fabian Haiden, we hope to apply this operation

\[
X \mapsto Z^{\text{init}} \mapsto Z^{\text{core}}
\]

in order to generalize the work of Bridgeland and Smith constructing stability conditions [3] from \(SL(2)\) to \(SL(3)\).

The core \(Z^{\text{core}}\) has a natural flat structure with geometry modeled on the standard apartment \(A\) for \(SL(3)\). This geometric structure carries with it a natural cyclic 3-fold spectral covering with ramification at the 8-fold singular points. The 8-fold singular points are negatively curved conical points for the flat metric, with angles of \(480^\circ\). The flat metric determines a conformal and hence complex structure; this is a modification of the complex structure of the original Riemann surface \(X\), and the cyclic covering corresponds to a cubic differential. This modification looks to be a discrete or possibly “tropical” analogue of Labourie’s result [17, Conjectures 1.6, 1.7] [18], replacing a general spectral curve by a cubic differential.

The first conjecture is that a minimization process provides a stability condition corresponding to the cubic differential, on the category of sections of the perverse Schober with fiber \(A^{CY}_2\) corresponding to the cyclic spectral covering.

If we can do that, then a procedure for defining the stability condition for a general \(SL(3)\)-spectral curve \(\Sigma\) will be to define the categories \(D_{\leq \theta}\) of objects of phase \(\leq \theta\), for any \(\theta\) by considering the core \(Z^{\text{core}}(e^{i\theta}\Sigma)\) and using the stability condition given by the conjectured minimization process of the previous paragraph.
These $t$-structures should each satisfy the required axioms for taking part in a stability condition. Therefore, taken together for all phases they should define a stability condition. We are just beginning to work on this program.

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