Using the Hopf algebra structure of QFT in calculations

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Abstract

We employ the recently discovered Hopf algebra structure underlying perturbative Quantum Field Theory to derive iterated integral representations for Feynman diagrams. We give two applications: to massless Yukawa theory and quantum electrodynamics in four dimensions.

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I. INTRODUCTION

Quantum field theories (QFT) of interest to particle physicists are plagued by UV divergences. Typically, we are confronted with theories which favour local interactions, which means that we can remove these divergences order by order in the loop expansion. This iterative removal of divergences, all the while maintaining the locality of the theory, has to fulfill certain combinatorial properties, succinctly summarized by Zimmermann’s forest formula. It was a great achievement when the self-consistency of the whole renormalization procedure was proven [1]. While there is little doubt that renormalization theory allows one to obtain sensible answers from an a priori ill-defined theory, only recently has it become clear that its underlying mathematical structure is by no means accidental, for it relates QFT to basic mathematical notions, well-known in low dimensional topology, number theory [2–6] and even in numerical analysis [7].

In this paper, we want to show to what extent the renormalization of Feynman graphs can be incorporated in the language of generalized iterated integrals, and how the intricacies of full-fledged QFT, including spin and Lorentz structure, still can be described in terms of such generalized iterated integrals in the spirit of [6].

We will begin by describing how the Hopf algebra structure, associated with renormalization, can be related to rooted trees, through the nature of the Feynman graphs. In that opening section we also show how two kinds of renormalization schemes can be connected by a convolution product; this convolution is the group product naturally related to the Hopf algebra, and dutifully delivers the renormalization group. The next section explains how rooted trees acquire decorations, according to the topology of the Feynman diagram, and how their evaluation is completely determined by vertex weights and generalized tree factorials. The following section concerns the representation of Feynman diagrams as iterated one-dimensional integrals and how this can be tied to the renormalization group, through appropriate choices of end-points. Finally we treat the case of massless Yukawa theory and quantum electrodynamics in four dimensions; the former sets the scene for tackling scalar theories, while the latter contains the necessary matrix generalization to spin, where the occurrence of form-factors can complicate the argument. The paper ends with a brief section embodying our conclusions and highlighting topics for further development.

II. FORMULARY FOR RENORMALIZATION

In this section we will briefly describe the Hopf algebra structure of renormalization. We start with a basic formulation directly on graphs and then introduce the correspondence to rooted trees.

A. Basics

Let us consider a Feynman graph $\Gamma$ as a set, consisting of vertices and edges of several sorts. The edges correspond to various types of propagators, representing inverse differential operators of free relativistic wave equations with boundary conditions in accord with causality. The vertices correspond to local interactions associated with a Lagrangian which
is a polynomial in fields and derivatives. In this way, at the same time \( \Gamma \) denotes a graph which we can draw on paper as well as a unique analytical expression.

Regarding \( \Gamma \) as a set of edges and vertices, we can formally consider the power set of this set, and distinguish those elements of this power set which themselves correspond to superficially divergent Feynman graphs.

Such a Feynman graph, together with all its superficially divergent subgraphs, constitutes a Hopf algebra structure which is isomorphic to a Hopf algebra of decorated rooted trees. This isomorphism is described in Fig. 1 and is detailed in [3,4]. (For the reader’s convenience, we have added an Appendix which summarizes the crucial points.) By this isomorphism, the standard Hopf algebra operations on rooted trees correspond to established notions of physicists.

Admissible cuts on the sum of rooted trees representing \( \Gamma \) are in one-to-one correspondence with divergent subgraphs. Those are determined by power-counting, thereby allowing detection of all sectors in the analytic expression \( \Gamma \) which correspond to divergent subintegrations in their own right, while the forests of renormalization theory then correspond to arbitrary cuts [2–4].

This viewpoint leads to a Hopf algebra structure [2–6], which we can succinctly formulate on overall divergent Feynman graphs \( \Gamma \) as

\[
\begin{align*}
\Delta[1] &= 1 \otimes 1 \\
\Delta[\Gamma] &= \sum_{\gamma \subseteq X \Gamma} \gamma \otimes \Gamma/\gamma \\
S[1] &= 1 \\
S[\Gamma] &= -\Gamma - \sum_{\gamma \subseteq X \Gamma} \gamma \otimes \Gamma/\gamma \\
\bar{e}[1] &= 1 \\
\bar{e}[\Gamma] &= 0.
\end{align*}
\]

In this notation, the sum \( \sum_{\gamma \subseteq X \Gamma} \) is a sum over all superficially divergent subgraphs, including the case \( \gamma = \Gamma \) and \( \gamma = \emptyset \). \( \Gamma/\gamma \) is obtained by the contraction of \( \gamma \subset \Gamma \) to a point in \( \Gamma \). The algebra element corresponding to the empty set is, as usual, the unit 1 of the algebra. Hence, the coproduct determines all sectors in the graph which are ill-defined and require renormalization. The sum \( \sum_{\gamma \subseteq X \Gamma} \) is a similar sum, with the two cases \( \gamma = \Gamma \) and \( \gamma = \emptyset \) excluded.

\( \mathcal{F} \) is the set of all forests of a Feynman graph, and this set is in one-to-one correspondence with all cuts at the rooted tree \( t_\Gamma \) representing the graph \( \Gamma \), while divergent subgraphs are in one-to-one correspondence with all admissible cuts, as already mentioned. \( n_F \) is the number of boxes of the forest \( F \), or the number of elementary cuts under the above correspondence [3]. Since formulae (1)-(8) are rather abstract for the uninitiated reader, we give an illuminating example in Fig. 1; a proper derivation of these correspondences can be found in [2–4] while the Appendix summarizes the crucial notions.

Hopf algebras, as embodied in (1)-(8) are Hopf algebras of rooted trees for an appropriately chosen set of decorations [4]. With this Hopf algebra structure, the naive renormalized
function $\Gamma[\Gamma]$ associated with $\Gamma$ vanishes:

$$\Gamma[\Gamma] := m[(S \otimes id)\Delta[\Gamma]] = \bar{\epsilon}[\Gamma] = 0.$$  \hfill (8)

Spelling the formula out, one indeed finds that each term appears twice with opposite signs. This must be so as the above formula employs the graph itself as the counterterm of a divergent graph, and hence eliminates all divergent graphs. Equation (8) represents a well-defined renormalization scheme, but is not of much use in practice. It has a few merits in the large $N_C$ expansion however \cite{8}.

Usually we want to subtract in accordance with certain renormalization conditions, and hence our counterterms should surely remove divergences of integrals but may not completely nullify all superficially divergent bare Green functions. We have to choose non-trivial renormalization schemes to obtain non-vanishing but finite results. The essential modification is to allow for a map $R$ which modifies the bare unrenormalized Feynman graph without changing its divergence. In \cite{6} it was shown that all renormalization schemes can be described on the same footing, allowing for tree-dependent variations of dimensionful parameters in bare Green functions.

\textbf{B. Schemes and scale transformations}

We can use the results of \cite{3} to recast the behaviour under a change of parameters and the dependence on renormalization schemes in a nice manner. We first modify the antipode via a renormalization map $R$, and in consequence get a finite renormalized Green function $\Gamma_R[\Gamma]$ \cite{2,6}. The $R$ dependent antipode, the $Z$-factor $S_R[\Gamma]$ of a bare expression $\Gamma$ and the renormalized Green function $\Gamma_R[\Gamma]$ explicitly read

$$S_R[\Gamma] = R \left[ -\Gamma - \sum_{\gamma \subset X} S_R[\gamma] \Gamma/\gamma \right]$$ \hfill (9)

$$= R \left[ \sum_{F \in \mathcal{F}} (-1)^{n_F+1} F_R(\gamma_F) \Gamma/\gamma_F \right]$$ \hfill (10)

$$\Gamma_R[\Gamma] = m[(S_R \otimes id)\Delta[\Gamma]]$$ \hfill (11)

$$= [id - R] \left( \sum_{F \in \mathcal{F}} (-1)^{n_F} F_R(\gamma_F) \Gamma/\gamma_F \right).$$ \hfill (12)

Here, $\gamma_F$ is the subgraph corresponding to the forest $F$, and $F_R$ is the evaluation of this subgraph using $R$ to evaluate boxes of forests corresponding to cuts, cf. Fig. 2.

Introducing the convolution product $(\phi \ast \psi)(\Gamma) = m[(\phi \otimes \psi)\Delta(\Gamma)]$ we find

$$\Gamma_R[\Gamma] = (S_R \ast id)[\Gamma],$$ \hfill (13)

which makes the renormalized Green function transparent as a ratio of a bare Green function $\Gamma \equiv id(\Gamma)$ to a counterterm $S_R(\Gamma)$. Hence our notation emphasizes the fact that the renormalized Green function is a function of the bare Green function as well as the chosen renormalization scheme. A detailed discussion of all its properties can be found in \cite{6}. Here
we only remark that the group structure embodied in the renormalization group is a direct consequence of the structure of a renormalized Green function as a convolution product \[\Gamma\].

Of course, bare functions \(\Gamma\) are parametrized by masses, external momenta and regularization (cut-off, dimensional-regularization scale \(\mu\), etc.). We can vary such parameters. Essentially, the choice of a renormalization scheme amounts to a choice of conditions for external parameters. Hence, a renormalized Feynman graph is determined by the choice of two sets of such parameters, one for the bare diagrams, one for the counterterm diagrams. If we want to highlight the dependence of the renormalized Green function on such parameter sets \(i, k\) say, we explicitly write \(\Gamma_{R,i,k}\); this assigns to a Feynman graph a renormalized expression with a specified set of dimensionful parameters \(k\) used in the bare Green functions and a set of parameters \(i\) used for counterterms. For example, an on-shell renormalized off-shell Green function will be calculated by using bare Feynman diagrams with off-shell momenta and imposing on-shell conditions to fix them in the counterterms.

Now, from \[\Gamma\], we know that the most general renormalization scheme, including for example minimal subtraction (MS), can be captured by suitable modifications of these scales. There one discovers that the change of schemes and scales can be succinctly summarized by the following formula

\[
\Gamma_{R,i,k}(\Gamma) = \left[\Gamma_{R,i,j} \ast \Gamma_{R,j,k}\right](\Gamma). \tag{14}
\]

In \(\Gamma_{R,i,j}\), the first subscript at the renormalization map \(R\) indicates the choice of scales for the counterterms, while the second one indicates the set of external scales used in the bare diagrams. Equation (14) shows how an intermediate set of scales \(j\) allows for group-like transformation laws. From this, one obtains renormalization group equations, operator product expansions and cohomological properties of renormalization \[\Gamma\] with ease. Depending on the choice of parameters to be varied, one arrives at Wilson’s viewpoint by varying a cut-off \[\mu\] or at the Callan-Symanzik equation from a variation of the renormalization point. Details will be given in future work.

C. Example

The following example, of a three-loop fermion self-energy contribution in QED in four dimensions, will make the above ideas more transparent and their implementation more concrete. We first calculate the coproduct for the diagram:

\[
\Delta\left[\begin{array}{c}
\end{array}\right] = \begin{array}{c}
\end{array} \otimes 1 + 1 \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes \begin{array}{c}
\end{array}.
\]

The antipode \(S_R[\begin{array}{c}
\end{array}]\) is then found to be

\[
S_R[\begin{array}{c}
\end{array}] = -R[\begin{array}{c}
\end{array}] + R\left[R\left[\begin{array}{c}
\end{array}\right] \begin{array}{c}
\end{array}\right]\right] + R\left[R\left[\begin{array}{c}
\end{array}\right] \begin{array}{c}
\end{array}\right]\right] + R\left[R\left[\begin{array}{c}
\end{array}\right] \begin{array}{c}
\end{array}\right]\right] - R\left[R\left[\begin{array}{c}
\end{array}\right] \begin{array}{c}
\end{array}\right]\right].
\]

Whenever we see \(R\), this means we have to evaluate the analytic expression corresponding to the Feynman graph on which \(R\) acts according to some chosen rules. For example, in minimal
subtraction we would introduce a regularization, say the dimensional method, evaluate the bare Feynman graph as some Laurent series in the corresponding regularization parameter, and then read $R$ as the projection onto the pole term. On the other hand, in on-shell BPHZ renormalization, we would introduce no regularization, but let $R$ modify the integrand in a way such that it fulfills certain constraints when external momenta are on-shell. From the results in [8] we know that all such schemes are fully equivalent. In particular the result of any scheme can be recast into BPHZ form with appropriately chosen constraints for external parameters.

Whatever scheme is chosen, the renormalized Green function for the example finally becomes

$$\Gamma_R\left[\begin{array}{c} \text{amplitude} \\ \text{amplitude} \end{array}\right] = (id - R) \left[\begin{array}{c} \text{amplitude} \\ \text{amplitude} \end{array}\right] - (id - R) \left[\begin{array}{c} R \left[\begin{array}{c} \text{amplitude} \\ \text{amplitude} \end{array}\right] \right]$$

Here, $id$ is the identity map which leaves the bare diagram unchanged. The same result is obtained when one works with rooted trees.

III. FEYNMAN DIAGRAMS AS REPRESENTATIONS OF ROOTED TREES

Particularly simple classes of Feynman graphs are given by graphs which are represented by rooted trees which have the same decoration at each vertex, and which give a multiplicative representation defined below. We will soon see that the general case can be reduced to similar representations, possibly allowing for appropriate matrix structures to incorporate necessary modifications due to spin. Hence it is legitimate to start the study of the renormalization problem on graphs as simple as the ones shown in Fig. 3. These simplest graphs are completely governed by the combinatorics of undecorated rooted trees. Indeed, tree-factorials and vertex weights to be defined below are sufficient to determine such graphs completely [4,5].

A. Undecorated trees

Let us first recapitulate some basic notions of rooted trees, following [3]. For further details we refer the reader to the Appendix. Let $T^{[0]}$ be the set of vertices of the rooted tree $T$ with $n$ vertices. For any vertex $v$ of $T$ which is not the root, let $t_v = P^c(T)$, where $c$ is the single cut which only removes the one edge incoming to $v$ (all edges are oriented away from the root). If $v$ is the root $r$, set $t_r = T$. Also, $w(v) = \#(t_v)$ is the number of vertices of the tree $t_v$. Then, we define the tree factorial by

$$T! := \prod_{v \in T^{[0]}} w(v). \quad (15)$$

Hence, for the root, $w(r) = n = \#(T)$. Fig. 3 gives instructive examples.
In the following, a variable $z$ plays the role of a $q^2/\mu^2$ in field theory, and $x$ plays the role of the regularization parameter, and can be identified with $(4-D)/2$ in dimensional regularization.

We now consider representations of $\mathcal{H}$ defined as follows. Assume that we are given a set of functions $B_k(x)$ which are Laurent series in the regularization parameter $x$, each having a first-order pole. Using vertex weights as defined above we can construct a function

$$\phi_z(t) := \prod_{v \in T[0]} B_{w(v)}(x)z^{-nx} = B_t(x) z^{-nx}, \quad (16)$$

where $n$ is the number of vertices of $t$ and $z$ is to be regarded as the scale parametrizing the representation. We have also abbreviated $\prod_{v \in T[0]} B_{w(v)}(x) = B_t(x)$. Quite a number of interesting applications can be brought into this form \[5,6\]. Typically, one finds such representations whenever one iterates a (scalar) Feynman diagram in terms of itself, as described by a rooted tree \[5,6\]. Even if one only iterates one-loop integrals, such representations offer a variety of interesting number-theoretic properties realized in the same manner by different quantum field theories, ranging from the absence of transcendentals in counterterms derived from ladder graphs \[13–15\] (which represent trees of the form $B^k_t(e)$, \[6\]) to the absence of non-rational coefficients in quenched QED \[16\] to similar phenomena when summing up all diagrams with Connes Moscovici weights \[5\].

Thus bare Green functions are given by $\phi_z(t)$. Then we define the MS renormalization scheme by setting

$$R_{MS}[\phi_z] = \langle \phi_1 \rangle, \quad (17)$$

where angle brackets denote projection onto the pole part of the Laurent series in $x$ inside the brackets, and we set $z = 1$ inside these brackets. Next, let the minimal subtracted counterterm be defined by

$$S_{R_{MS}}(\phi_z)(t) = - \langle \phi_z(t) + \sum S_{R_{MS}}(\phi_z(t(1))\phi_z(t(2))) \rangle, \quad (18)$$

which is the direct translation of (9) using the map from Feynman graphs to decorated rooted trees, and let the renormalized Green function be defined by $\Gamma_{MS}(\phi_z)(t) = [S_{R_{MS}} \circ \phi_z \times \phi_z](t)$, in accordance with (11), as expected. The reader should have no difficulties confirming that for $t_2$, the rooted tree with two vertices,

$$S_{R_{MS}}(\phi_z)(t_2) = - <B_2 B_1> + <<B_1 B_1> := Z_{t_2}^{MS}, \quad (19)$$

and

$$\Gamma_{R_{MS} \phi_z}(t_2) = B_2 B_1 z^{-2x} - <B_1 B_1 z^{-x} - <B_2 B_1> + <<B_1 B_1 >. \quad (20)$$

Therefore the antipode $Z_i^{MS} := S_{R_{MS}}(\phi_z)(t)$ in MS reads, for the first few trees,

$$Z_{t_1}^{MS} = - <B_1 >, \quad (21)$$

$$Z_{t_2}^{MS} = - <B_2 B_1> + <<B_1 B_1 >, \quad (22)$$

$$Z_{t_3}^{MS} = - <B_3 B_2 B_1> + <<B_1 B_2 B_1> + <<B_2 B_1 B_1 >$$

$$- <<B_1 B_1 B_1 >, \quad (23)$$

$$Z_{t_4}^{MS} = - <B_3 B_2^2> + 2 <<B_1 B_2 B_2 >$$

$$- <<B_1 B_1 B_1 >, \quad \text{etc.} \quad (24)$$
which can be summarized by the formula \[6\]

\[
Z_{t}^{MS} = \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_{C}} \left[ \prod_{i} < B_{t_{i}} > \right] B_{t_{R}} > .
\]  

(25)

This being the result in the MS scheme, it was shown in \[6\] how to express it in terms of a BPHZ type scheme, where the use of a regularization scheme can be completely avoided. Translated to the above representations, what we want to achieve is the following: the above functions representing rooted trees depend on a function \(B_{t}(x)\), determined by a chosen tree, a regularization parameter and the chosen one-loop graph (the decoration) which is iterated according to the tree structure. Further, there is the scale \(z\), delivered in simple bare Feynman diagrams typically by a variable, such as the square of a single momentum, \(q^{2}\).

An on-shell scheme would simply subtract at a chosen point \(q^{2}/\mu^{2} =: z = 1\), for example. An MS scheme would also modify the Laurent series \(B_{t}\), by projecting to \(< B_{t} >\). In the former case, one typically obtains the difference of integrands evaluated at scales \(z\) versus \(1\), and hence one may subtract integrands before having to regularize. In the latter case, one modifies Laurent series after regularization and integration. We want to show how appropriate subtraction points \(\mu_{t}\) can be defined so that the MS renormalized functions equal on-shell renormalized functions subtracted at \(\mu_{t}\).

The idea is to use tree indexed parameters \(\mu_{t}\). The antipode \(Z_{t}^{\mu}\) in a subtraction scheme using tree-indexed parameter sets \(\mu_{t}\) reads

\[
Z_{t}^{\mu} = \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_{C}} \left[ \prod_{i} B_{t_{i}}^{C} \mu_{t_{i}}^{-\#(t_{i}^{C})x} \right] B_{t_{R}}^{\mu} \mu_{t_{R}}^{-\#(t_{R})x},
\]  

(26)

where \(P^{C}(t) = \prod_{i} t_{i}\), which one immediately obtains by standard formulas \[6\]. The first two trees \(t_{1}, t_{2}\) deliver immediately

\[
Z_{t_{1}}^{\mu} = -B_{t_{1}} \mu_{t_{1}}^{-x}
\]

(27)

\[
Z_{t_{2}}^{\mu} = -B_{t_{2}} \mu_{t_{2}}^{-2x} + B_{t_{1}} B_{t_{1}} \mu_{t_{1}}^{-2x},
\]

(28)

as the reader should confirm using the equation (26). For the definition of full cuts, appearing in the summation above, we refer the reader to the Appendix.

Equating \(Z_{t}^{MS}\) to \(Z_{t}^{\mu}\) determines \(\mu_{t}\) recursively \[6\]:

\[
\mu_{t} = \exp \left[ \left( \frac{1}{x} \right) \log(B_{t}^{'}/B_{t}) \right],
\]

(29)

where

\[
B_{t}^{'} := \sum_{\text{full cuts } C \text{ of } t} (-1)^{n_{C}} \left[ \prod_{i} S_{RMS}(\phi_{z})(t_{i}^{C}) \right] S_{RMS}(\phi_{z})(t_{R}).
\]

(30)

One also confirms that the MS-renormalized Green function can now be expressed through the on-shell subtracted one, where the on-shell condition for \(\phi_{z}(t)\) signifies subtraction at \(z = \mu_{t}\) in this case. Thus

\[
\Gamma_{RMS}(\phi_{z})(t) = \Gamma_{R_{\mu}}(\phi_{z})(t)
\]

(31)

holds. Also, one immediately derives that \(B_{t}^{'}/B_{t} = 1 + \mathcal{O}(x)\), so that we properly arrive at well-defined conditions on the external parameters \(\mu_{t}\) to express the renormalized Green function in MS through a BPHZ subtracted Green function with tree-indexed boundaries.
B. Decorated trees

What is the situation that confronts us in general? In the previous section, we dressed every vertex of a rooted tree with a decoration $B_{w(v)}$, and then assigned the function $\phi_z(t) = z^{-\#(t)x} \prod B_{w(v)}$ to a rooted tree, the product being taken over all vertices of the tree. (Note that $\#(t) = n = w(r)$, where $w(r)$ is the vertex-weight of the root.) Clearly, the vertex weight $w(r)$ of a root of an undecorated tree is simply the number $n$ of vertices of the tree. This equality does not survive in the general case.

But if we only very slightly modify this equation and write

$$\phi_z(t) = z^{-w(r)x} \prod B_{w(v)}.$$  \hfill (32)

we obtain the form which permits all necessary generalizations. Such generalizations include the following cases:

- In general, decorations can be provided by analytic expressions of any loop order. We will have to attach appropriate weights to the vertices themselves, and hence will have to generalize the concept of vertex weights.

The crucial step is to devise some sensible notation. An arbitrary rooted tree has vertices decorated by graphs corresponding to analytic expressions without subdivergences which themselves can have any loop order. We still assume that these analytic expressions can be described by scalar functions. Loop integrations of decorations at vertices below the vertex considered will alter the measure by an amount $(k^2)^{w(v_i)}$ where $v_i$ is the generalized vertex weight defined in Fig. 4 and $k$ is an appropriate external momentum for the decoration at $v_i$. At each vertex we indicate the loop number of its decoration. The generalized vertex weight is then the sum of all these loop numbers, assigned to the vertex in question and all vertices below it. From this, one obtains the generalized tree factorial, which is to be used in (32).

- We might consider graphs which do not come from scalar theories. In order to cope with form factors, one needs to devise an appropriate matrix calculus; this will be worked out in a detailed example in a section below. Essentially, the master equation (32) remains valid, but becomes now a matrix equation. This is the most non-trivial generalization, and we will discuss it in detail presently.

In the same manner, one can use matrices to keep track of internal structure of a graph: at some stage one has to bookkeep the location of subdivergences in a larger graph, and this leads to a matrix calculus which is a straightforward generalization of the example which we consider below. Indeed, one can enumerate all vertices of a graph and can construct suitable matrices which store the information about the nature of these vertices and the propagators connecting them. In this manner, the previously discussed changes of measures due to subdivergences operate on entries of such matrices, and it is a straightforward notational exercise to set-up a convenient calculus. We will not consider the most general case here, but rather present below an example for QED which should make the idea sufficiently clear.
• In general, divergent subgraphs depend on masses and more than one external momentum; hence on multiple scales. For example, a vertex subdivergence will depend on more than one internal momentum usually and this can spoil the multiplicative form of (32). This difficulty can be solved by a proper decomposition into sums of expressions, each of the form (32). This was described already in [2]. One proceeds by decomposing any function of multiple scales into one of a single scale and another finite multiple scale function. This is always possible by adding zero in appropriate ways.

The final result is that from a local QFT we essentially get analytic expressions which can be expressed in the form (32). We now utilize this fact to show that they essentially behave as generalized iterated integrals.

IV. FEYNMAN DIAGRAMS FROM GENERALIZED ITERATED INTEGRALS

Now that we have established that (32) is an acceptable prototype expression into which Feynman diagrams can be resolved, we want to utilize the fact that the simplest representation of such form can be obtained from generalized iterated integrals. Such generalized iterated integrals were prominent in [3] and give a convenient means to investigate renormalization. Generalized iterated integrals are distinguished from ordinary iterated integrals by the fact that boundaries of the domains of integration are determined as a function of the rooted tree which the integral represents. In this respect, ordinary iterated integrals use a function which is constant on rooted trees. For such a generalization, the shuffle symmetry of iterated integrals is lost.

The extra freedom gained by using this generalization will allow us to rewrite Feynman diagrams as such generalized iterated integrals. We will see that the generalization only affects non-leading UV divergences, and hence we will obtain later the result that the leading UV divergences of Feynman integrals still obey a shuffle identity.

At this point, it might be worth stressing that if we were able to reduce Feynman diagrams completely to ordinary iterated integrals we would obtain considerable progress in understanding the analytical and transcendental structure of Green functions, as the theory of ordinary iterated integrals is quite rich and highly developed. We hope that a similar theory will be emerging in the future for generalized iterated integrals.

A. Iterated Integrals

Essentially, we want to go one step further as in [3] and express contributions to Green functions in QFT as iterated integrals. We will start our considerations by reminding ourselves of some basic properties of iterated integrals [10,11], specializing to the case of a single function \( f(x) \) with associated one-form \( f(y)dy \) on the real line. We assume that \( f(y) \) behaves \( \sim 1/y \) for large \( y \).

Iterated integrals built with the help of \( f \) are parametrized by an integer \( n \) and two real numbers \( a, b \) say. They are defined by
\[ F_{a,b}^{[0]} = 1, \quad \forall a, b \in \mathbb{R}, \quad (33) \]
\[ F_{a,b}^{[n]} = \int_a^b f(x) F_{a,x}^{[n-1]} \, dx, \quad \forall n > 0. \quad (34) \]

Hence we can write them as an integral over the simplex,
\[ F_{a,b}^{[n]} = \int_{a \le x_1 \le \cdots \le x_n \le b} f(x_1) \cdots f(x_n) dx_1 \cdots dx_n. \quad (35) \]

We will also consider
\[ F_{a,b}^{[T]} = \int_a^b dx f(x) \Pi_i F_{a,x}^{[t_i]} \]
where \( T \) is a rooted tree with \( B_-(T) = \Pi_i t_i \).

A generalization which turns out to be quite useful in practice is to let even the boundaries \( a, b \) be indexed by decorated rooted trees. This is similar to what we did before when we expressed the results of an MS calculation by an on-shell scheme with suitably adjusted tree-dependent parameters. Here, parameters are provided by boundaries in the iterated integral. Hence, we define an iterated integral which serves as a bare Green function
\[ G_{b,\infty}(t) = \int_b^\infty f(x) \prod_i G_{x,\infty}(t_i) \, dx. \quad (36) \]

The product is again over all branches \( t_i \) of \( B_-(t) \), and we set \( G_{b,\infty}(e) = 1 \). This is the same as before, except that we now label the lower outer boundary by a decorated rooted tree \( t \) and set the upper boundary to infinity. We assume that a renormalization map \( R_a \) shifts the outer lower boundary \( b_t \) to another value \( a_t \), and that the coproduct action extends to this label. We define \( \phi_b \) to be the map which sends \( t \to G_{b,\infty}(t) \).

It is worthwhile citing here an example taken from [6] which exhibits the intricate nature of a change of scales in full generality. For the rooted tree with two vertices, \( t_2 \), having coproduct
\[ \Delta(t_2) = t_2 \otimes e + e \otimes t_2 + t_1 \otimes t_1 \quad (37) \]
we formally obtain a counterterm
\[ S_{Ra}(\phi_b)(t_2) = -R_a[\phi_b(t_2) + m[(S_{Ra} \circ \phi_b \otimes \phi_b)\Delta'(t_2)]] \]
(where \( \Delta'(T) = \Delta(T) - [1 \otimes T] - [T \otimes 1] \)) as
\[ S_{Ra}(\phi_b)(t_2) = \left[ \int_{a_1}^\infty \int_{a_1}^\infty - \int_{a_1}^\infty \right] f(x)f(y)dydx. \quad (38) \]
and a renormalized iterated integral \( \Gamma_{a,b}(t_2) = m[(S_{Ra} \circ \phi_b \otimes \phi_b)\Delta(t_2)] \) as
\[ \Gamma_{a,b}(t_2) = \left[ \int_{b_1}^\infty \int_{a_1}^\infty - \int_{b_1}^\infty \right] f(x)f(y)dydx. \quad (39) \]

In this notation, \( a, b \) are to be regarded as representing actually a whole set of constants \( a_t, b_t \), parametrizing the relevant scales for the decorated tree under consideration.
We now want to change the set of scales and reexpress how $\Gamma_{a,b}$ can be obtained from renormalized Green functions $\Gamma_{a,s}$ and $\Gamma_{s,b}$. Obviously, the convolution

$$\Gamma_{a,b}(t) = [\Gamma_{a,s} \ast \Gamma_{s,b}](t)$$

holds. This form of Chen’s Lemma describes what happens if we change the renormalization point. It works for the simple case of iterated integrals in the same manner as for full Green functions of QFT. $\Gamma_{a,b} = \Gamma_{a,s} \ast \Gamma_{s,b}$ now becomes

$$\left[ \int_{b_2}^\infty \int_x^\infty - \int_{b_1}^\infty \int_{a_1}^\infty + \int_{a_1}^\infty \int_{a_1}^\infty - \int_{a_2}^\infty \int_x^\infty \right] f(x)f(y)dydx$$

This is evidently true, as the reader may readily check.

Finiteness of $\Gamma_{a,b}$ now imposes conditions on the tree-indexed parameters, a fact which we will utilize in the next section. Actually, we will determine these parameters in a way which reproduces the results of a renormalized QFT.

V. COUNTERTERMS AND RENORMALIZED GREEN FUNCTIONS AS ITERATED INTEGRALS

We are now in the fortunate position of being able to transform Feynman diagrams to generalized iterated integrals. To that end, let us consider the simplest case of an iterated integral based on the single one-form $f(y)dy = y^{-1} - x dy$, which diverges logarithmically when integrated to infinity and $x \to 0$.

Then, we obtain the bare iterated integral,

$$G_{b,\infty}(t) = \frac{1}{t^{1-x-\#(t)}x} b_t^{-\#(t)x}$$

and associated counterterms $S_{R_\mu}(\phi_b)(t)$, which in this case are simply

$$S_{R_\mu}(t) = -R_\mu[\phi_b(t) + m][(S_{R_\mu} \otimes id)(\phi_b \otimes \phi_b)]\Delta'] = \phi_\mu(S(t)), $$

with $R_\mu(\phi_b) = \phi_\mu$. Thus we arrive at the renormalized integral

$$\Gamma_{\mu,b}(t) = \phi_\mu(P^C(t))\phi_b(R^C(t)).$$

Explicitly,

$$S_{R_\mu}(t_2) = -\frac{1}{2\mu_2^2}\mu_2^{-2x} + \left[ \frac{1}{x\mu_1^{-x}} \right]^2$$
\[ \Gamma_{\mu,b}(t_2) = \frac{1}{2x^2}b_{t_2}^{-2x} - \frac{1}{x^2}b_{t_1}^{-x} \mu_{t_1}^{-x} + \frac{1}{x^2}\mu_{t_1}^{-x} \mu_{t_1}^{-x} - \frac{1}{2x^2}t_{t_2}^{-2x}, \]

which is nonsingular when \( x \to 0 \).

Next determine the set of parameters \( \mu_t \) by iteratively setting

\[ S_{R_{\mu}}(t) = S_R(\phi_z(t)), \]

wherein we identify the parameter \( x \) in the iterated integrals with the regularization parameter for bare Green functions in QFT and use the representation (32) for those Green functions. A similar equation is used to determine the set of parameters \( b_t \) in the iterated integrals such that

\[ \Gamma_{\mu,b}(t) = \Gamma_R(\phi_z(t)). \]

If we use a on-shell subtraction at the point \( z = 1 \) for example, one simply finds

\[ \mu_t = \exp \left( \frac{1}{\#(t)x} \log[t^x \#(t)B_t(x)] \right). \]

Note that this quantity has no essential singularity at \( x = 0 \) due to the fact that \( t^x \#(t)B_t \) has the form \( 1 + \mathcal{O}(x) \) in our adopted normalization. In this simple on-shell scheme, \( b_t \) turns out to be

\[ b_t = \mu_t z^{-\#(t)x} \]

and the resulting generalized iterated integral is a well-defined finite quantity which reproduces the renormalized Green function.

Changing renormalization conditions or schemes is now evidently a convolution of iterated integrals. For example, transforming to a \( \text{MS} \) scheme amounts to a convolution similar to the one described before, and we would eventually have

\[ \Gamma_{\mu,\text{MS},b}(t) = [\Gamma_{\mu,\text{MS},\mu} \ast \Gamma_{\mu,b}](t). \]

The second Green function on the rhs uses the same \( \mu_t, b_t \) as defined above, the first on the rhs uses the same \( \mu_t \) but determines \( \mu_t^{\text{MS}} \) using minimal subtracted QFT Green functions. Such convolutions were already applied in [4] and systematically derived in [3]. They essentially reduce the calculation of renormalized Green functions to a determination of appropriate functions \( B_t \) from which all other results follow via the algebraic structure of the Hopf algebra. Note that the convolution also permits us to describe the change of scales in bare Green functions; for example \( \phi_z(t) \to \phi_{\rho z}(t) \) is described by

\[ \phi_{\rho z}(t) = [\phi_x \ast \Gamma_{R_z}(\phi_{\rho z})](t), \]

which immediately translates to iterated integrals as the convolution \( \Gamma_{\mu,b'} = \Gamma_{\mu,b} \ast \Gamma_{b,b'} \) where \( b'_t = b_t \rho^{-\#(t)x} \).

In the following we determine some of the \( B_t \) explicitly, to illustrate the methods.
A. Massless Yukawa theory

Let us apply the previous ideas to massless Yukawa theory, with interaction
\[ g_0 \bar{\psi}_0 \gamma_5 \psi_0 \phi_0 = Z_1 g \bar{\psi} \gamma_5 \psi \phi. \]

For illustrative purposes we consider a three-loop contribution to the fermion self-energy and an analogous one for the vertex part, as sketched in Fig. 5.

In order to describe the results, in the context of dimensional regularization, the parameter \( x \) is identified with \( (4 - D)/2 \equiv \epsilon \) and \( z \) with the scale factor \( \mu^2/p^2 \), where \( p \) is the external momentum and \( \mu \) is the renormalization mass scale which arises via the dimensionality of the bare coupling: \( g_0^2 \equiv (-4\pi)^{-x} \mu^{2x} 4\pi \alpha/(1 - x) \). The calculations bring in supplementary integrals which are readily calculated in any dimension \( D = 4 - 2x \), namely

\[ g_0^2 I(p, nx) = -ig_0^2 \int \frac{d^D k/(2\pi)^D}{k^2((k + p)^2(1 + nx))} = \frac{\alpha z^x}{4\pi} i_{n+1}/(p^2)^{nx}, \]

where

\[ i_{n+1} = \frac{\Gamma(1 - (n + 1)x) \Gamma((n + 1)x)}{\Gamma(1 + nx) \Gamma(2 - (n + 2)x)}, \]

and

\[ g_0^2 J(p, nx) = -ig_0^2 \int \frac{[1 + p.k/p^2] d^D k/(2\pi)^D}{k^2((k + p)^2(1 + nx))} = \frac{\alpha z^x}{4\pi} j_{n+1}/(p^2)^{nx}, \]

where

\[ j_{n+1} = \frac{\Gamma(2 - (n + 1)x) \Gamma((n + 1)x)}{\Gamma(1 + nx) \Gamma(3 - (n + 2)x)}. \]

Those functions \( i_{n+1}, j_{n+1} \) play the role of the functions \( B_w(v) \) of the previous considerations. Applying the on-shell subtraction which sets \( z \rightarrow 1 \), \( R_1 \), to this graph, yields the following result for the self-energy integral:

\[ \left( \frac{\alpha}{4\pi} \right)^3 \left[ (z^{3x} - 1) j_1^2 j_3 - 2(z^{2x} - 1) j_1^2 j_2 + (z^x - 1) j_1^3 \right]; \quad z \equiv \mu^2/p^2, \]

which is obtained as \( m[(S_{R_1} \otimes id) \Delta(j_t(z))] \) where \( j_t(z) = \prod_v j_w(v) [\alpha z^{-x}]^\#(t) \) plays the role of \( \phi(z(t)); \Delta \) acts on the tree parametrizing \( j_t \) such that \( \Delta(j_t(z)) = \sum j_{t(w)}(z) \otimes j_{t(v)}(z) \) and \( R_1[j_t(z)] = j_t(1) \).

Similarly, the following result for the vertex integral at zero meson momentum is obtained:

\[ -\left( \frac{\alpha}{4\pi} \right)^3 \left[ (z^{3x} - 1) j_1^2 i_3 - 2(z^{2x} - 1) j_1^2 i_2 + (z^x - 1) j_1^2 i_1 \right]; \quad z \equiv \mu^2/p^2, \]

in which we evidently see, as before, the \( (id - R) \) structure of renormalized Green functions. The limit \( x \rightarrow 0 \) may readily be taken in the renormalized answers, producing

\[ \left( \frac{\alpha}{4\pi} \right)^3 \left[ \frac{U^3}{24} + \frac{U^2}{16} + \frac{U}{16} \right]; \quad U \equiv \ln(\mu^2/p^2). \]
for the self-energy and

\[-3 \left( \frac{\alpha}{4\pi} \right)^3 \left[ \frac{U^3}{12} + \frac{U^2}{4} + \frac{U}{2} \right]; \quad U \equiv \ln(\mu^2/p^2)\]

for the vertex part. The latter carries an extra weight factor 3 because of the way that the interior self-energies can be distributed.

### B. Quantum electrodynamics

In this case the renormalization scale \( \mu \) enters through \( e_0^2 \equiv (-4\pi)^{-x}\mu^{2x}4\pi\alpha/\Gamma(1-x) \), where \( \alpha \) now has the significance of the fine structure constant. Because we are dealing with the massless version, the self-energy assumes the form, \( \Sigma(p) = \not{p}A(p^2) \), while the vertex at zero-momentum transfer contains two terms (form-factors):

\[
\Gamma_\mu(p,p) = \gamma_\mu F(p^2) - \not{p}\gamma_\mu \not{p}G(p^2)/p^2,
\]

of which only the first requires (infinite) renormalization. The form factors can be found in the usual manner by tracing appropriately:

\[
\text{Tr}[\gamma \cdot \Gamma] = 2^{2-x}[(4-2x)F - xG],
\]

\[
\text{Tr}[\not{p}p \cdot \Gamma] = p^22^{2-x}[F - G],
\]

and it is helpful to define the auxiliary integral

\[
e_0^2H(p,nx) = \frac{\alpha(1-x)z^n h_{n+1}}{4\pi(p^2)^{nx}},
\]

where

\[
h_{n+1} = -\frac{(1-x)\Gamma((n+1)x)\Gamma(1-(n+1)x)}{2\Gamma(2+nx)\Gamma(3-(n+2)x)}.
\]

We should also exhibit a combination which features prominently in actual computations, viz.

\[
i_{n+1} + Dh_{n+1} = \Gamma(2-(n+1)x)\Gamma(1+(n+1)x)/\Gamma(3-(n+2)x)\Gamma(2+nx); \quad D = 4-2x.
\]

Lowest order in \( \alpha \) calculations give (\( z \equiv \mu^2/p^2 \))

\[
A = \left( \frac{\alpha}{2\pi} \right) j_1 z^x, \quad (42a)
\]

\[
F = (1-x)^2 \left( \frac{\alpha}{2\pi} \right) h_1 z^x,
\]

\[
G = \left( \frac{\alpha}{2\pi} \right) [i_1 + Dh_1] z^x, \quad (42b)
\]

leading simply to the on-shell subtracted expressions,
\[ A_{os} = \left( \frac{\alpha}{2\pi} \right) j_1[z^x - 1], \quad (43a) \]

\[ \Gamma_{\mu,os} = \left( \frac{\alpha}{2\pi} \right) j_1 \left[ \gamma_{\mu}(z^x - 1) - 2x\frac{p_{\mu}}{\not p} z^x \right]. \quad (43b) \]

It is very easy to continue this process to higher orders for the self-energy function \( A(p^2) \) in the Feynman gauge say; all we need do is to substitute \( g^2 \) in massless Yukawa theory by \( 2e^2 \) in QED, the factor of 2 arising from the gamma-matrix trace. However the vertex computations are subtler, because they involve the two form factors \( F \) and \( G \). Here we require knowledge of the generic integral,

\[ i\epsilon_0^2 \int \frac{d^Dk}{(4\pi)^{D/2}k^2} \gamma_{\nu} \frac{1}{\not p + \not k} \left[ f_n \gamma_{\mu} - g_n \frac{(\not p + \not k)\gamma_{\mu}(\not p + \not k)}{(p+k)^2} \right] \left( \frac{\mu^2}{(p+k)^2} \right)^{nx} \frac{1}{\not p + \not k} \gamma_{\nu} \]

\[ \equiv \left( \frac{\alpha}{2\pi} \right) \left[ f_{n+1} \gamma_{\nu} - g_{n+1} \frac{(\not p + \not k)\gamma_{\mu}(\not p + \not k)}{(p+k)^2} \right] \left( \frac{\mu^2}{p^2} \right)^{(n+1)x}, \quad (44) \]

starting with \( f_0 = 1, g_0 = 0 \). In this way we straightforwardly arrive at the recurrence property,

\[ \mathbf{v}_{n+1} = \mathbf{M}_{n+1} \mathbf{v}_n, \]

where \( \mathbf{v}_i \) is a column vector \((f_i, g_i)^T\) with \( \mathbf{v}_0 = (1, 0)^T \) and \( \mathbf{M}_n \) is a 2 \times 2 matrix with entries \((M_n)_{rs}\) given by

\[ (M_n)_{11} \equiv \frac{(1-x)^2\Gamma(nx)\Gamma(1+nx)}{\Gamma(2+(n-1)x)\Gamma(3-(n+1)x)}, \quad (M_n)_{12} \equiv \frac{\Gamma(1-nx)\Gamma(nx)}{\Gamma(1+(n-1)x)\Gamma(2-(n+1)x)} \]

\[ (M_n)_{21} \equiv \frac{\Gamma(2-nx)\Gamma(1+nx)}{\Gamma(2+(n-1)x)\Gamma(3-(n+1)x)}, \quad (M_n)_{22} = 0. \quad (45) \]

We now introduce the matrix-function \( \mathbf{R} \) with entries \( R_{11} = R_1 \) and 0 elsewhere, where \( R_1 \) is again the map which sends \( z = \frac{\mu^2}{p^2} \to 1 \). Further, we use the 2 \times 2 unit matrix \( \mathbf{I} \) such that an insertion of a fermion self-energy amounts to an insertion of matrices of the type \( \mathbf{A}_k := \mathbf{I} \frac{\alpha}{2\pi} j_k \) into a string of matrices. Then we can express the result for any diagram \( \Gamma(t) \), which iterates one-loop vertices at zero-momentum transfer and also has various combinations of one-loop fermion self-energies as subdivergences, by the formula

\[ \Gamma(t) = \left[ \prod_{v \in \ell[0]} B_{w(v)}^{i(v)} z^{-w(v)x} \right], \]

where \( t \) is a decorated rooted tree allowing for one of two possible decorations \( i(v) \) at each vertex: either \( i(v) \) evaluates to a one-loop vertex function or to a one-loop fermion self-energy. As both decorations are of one-loop order, \( w(v) \) is the usual vertex weight discussed before. We then set \( B_{j}^{i(v)} := \mathbf{M}_j \) if the vertex \( v \) of \( t \) is decorated by the one-loop vertex-function, and \( B_{j}^{i(v)} = \mathbf{A}_j \) if the vertex \( v \) is decorated by a one-loop fermion self-energy. Various examples are drawn in Fig. 6.
Renormalization now amounts to inserting the matrix $R$ into the string of matrices at places which correspond to cuts at the decorated tree. Hence the formulae for the counterterms and renormalized Green functions remain essentially unchanged. For the Feynman graph with the tree $t$ shown in Fig. 6, we find the bare Green function
\[ \Gamma(t) = A_1 A_1 M_3 v_0 z^{-3x}. \]

The counterterm being $Z_R = - R[\Gamma(t) + m((Z_R \otimes id)(\Gamma \otimes \Gamma)\Delta(t))]$, we get
\[ -R[A_1 A_1 M_3 v_0] + 2R[A_1]R[A_1 M_2 v_0] - R[A_1]R[A_1]R[M_1 v_0]. \]

Finally the renormalized Green function $m((Z_R \otimes id)(\Gamma \otimes \Gamma)\Delta(t))$ leads to the on-shell subtracted result (aside from a weight factor of 3)
\[ \left( \frac{\alpha}{2\pi} \right)^3 j_1^2 \left[ \gamma_\mu \left( (M_3)_{11}(z^{3x} - 1) - 2(M_2)_{11}(z^{2x} - 1) + (M_1)_{11}(z^x - 1) \right) \right. \]
\[ \left. -(M_3)_{12} + 2(M_2)_{12} - (M_1)_{12} \right] \Gamma_{\gamma_\mu} \hat{p}/p^2. \]

The renormalized contribution may be found by proceeding to the delicate limit $x \to 0$, when the above answer collapses into
\[ -\left( \frac{\alpha}{2\pi} \right)^3 \left[ \gamma_\mu \frac{2U^3 + 3U^2 + 15U - 9}{48} \right. \right. \]
\[ \left. \left. - \frac{\Gamma_{\gamma_\mu}}{p^2} \frac{2U^2 - 2U + 3}{16} \right] ; \quad U = \ln(\mu^2/p^2). \]

Other diagrammatic contributions may be extracted in much the same way; we have evaluated many more results for the renormalized self-energy $A_{os}$ and the vertex at zero momentum transfer $\Gamma_{\mu,os}$. A useful check on the calculations, that we have carried out, is to verify the Ward identity, $\Gamma(p,p) = \partial \Sigma(p)/\partial p_\mu$, since a particular contribution to $A$ will produce a combination of vertex graphs with appropriate weights.

C. A relaxed shuffle product

Proper iterated integrals satisfy a shuffle identity $[[11,11]]$. Due to this shuffle identity, one can express all tree-iterated integrals representing rooted trees as proper iterated integrals. The latter are based on trees without sidebranching and form a closed Hopf subalgebra of $H$. The generalized iterated integrals which we used to summarize some restricted results of QFT do not obey a shuffle algebra, and hence we get non-trivial representations of the full Hopf algebra $[[11,11]]$.

However one can easily establish that the leading coefficient of bare Feynman diagrams still obeys such an identity, by making use of the expression in terms of generalized iterated integrals. For that it suffices to realize that if the $\mu_t$ become $t$-independent, we get a proper iterated integral. But to leading order, we have
\[ B_t = \frac{1}{t^x #(t)} F(t), \]
with $F(t) = 1 + O(x)$, by definition; so to leading order, $F(t) = 1 \forall t$, and then $B_t = 1/t^x #(t)$, which can be easily obtained as the iterated integral $G_{1,\infty}(t)$ defined previously. This explains relations known to practitioners of QFT that hold between leading coefficients of the overall divergence of graphs representing different trees.
VI. COMMENTS AND CONCLUSIONS

In this and previous papers we have developed a number of mathematical tools which clarify and simplify the renormalization procedure for any renormalizable quantum field theory. These are encapsulated in formulae such as (11), (13), (15), (26) and (30); we have provided two nontrivial examples where these ideas can be fruitfully applied without too much effort. Therefore, it is our firm belief and hope that all these tools will enable the practitioner of QFT to obtain the amplitudes associated with renormalized Feynman diagrams by purely combinatorial means. We also anticipate that the connection between these amplitudes and iterated one-dimensional integrals, whose endpoints define the renormalization scheme, will also lead to substantial progress in automating the method, since they provide a significant step along this direction. The mathematical tools emphasize the fundamental connection between geometry, topology, number theory and Feynman diagrams that is emerging [17,18,13,19–21].

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APPENDIX

A. The Hopf Algebra

We follow [3,4] closely. A rooted tree \( t \) is a connected and simply-connected set of oriented edges and vertices such that there is precisely one distinguished vertex which has no incoming edge. This vertex is called the root of \( t \). Further, every edge connects two vertices and the fertility \( f(v) \) of a vertex \( v \) is the number of edges outgoing from \( v \). The trees being simply-connected, each vertex apart from the root has a single incoming edge.

As in [3], we consider the (commutative) algebra of polynomials over \( \mathbb{Q} \) in rooted trees; hence the multiplication \( m(t,t') \) of two rooted trees means drawing them next to each other in arbitrary order. Observe that any rooted tree \( t \) with root \( r \) yields \( f(r) \) trees \( t_1, \ldots, t_{f(r)} \) which are the trees attached to \( r \). The unit element of this algebra is 1, corresponding, as a rooted tree, to the empty set.

Let \( B_- \) be the operator which removes the root \( r \) from a tree \( t \):

\[
B_- : t \rightarrow B_-(t) = t_1 t_2 \cdots t_{f(r)}. \tag{48}
\]

Fig. 7 depicts an example. Also let \( B_+ \) be the operation which maps a monomial of \( n \) rooted trees to a new rooted tree \( t \) which has a root \( r \) with fertility \( f(r) = n \) which connects to the \( n \) roots of \( t_1, \ldots, t_n \).

\[
B_+ : t_1 \cdots t_n \rightarrow B_+(t_1 \cdots t_n) = t. \tag{49}
\]
This is clearly the inverse to the action of $B_-$. For any rooted tree $t$ one has

$$B_+(B_-(t)) = B_-(B_+(t)) = t,$$

and Fig. 8 provides one such example. We further set $B_-(t_1) = 1, B_+(1) = t_1$.

We will introduce a Hopf algebra on such rooted trees by taking the opportunity to cut such trees in pieces. We start with the most elementary possibility. An elementary cut is a cut of a rooted tree at a single chosen edge, as indicated in Fig. 8.

Before introducing the coproduct we finally introduce the notion of an admissible cut, also called a simple cut. It is any assignment of elementary cuts to a rooted tree $t$ such that any path from any vertex of the tree to the root has at most one elementary cut. Fig. 10 depicts such a situation. An admissible cut $C$ maps a tree to a monomial in trees. If the cut $C$ contains $n$ elementary cuts, it induces a map

$$C : t \to C(t) = \prod_{i=1}^{n+1} t_{j_i}.$$  

(51)

Note that precisely one of these trees $t_{j_i}$ will contain the root of $t$. Let us denote this distinguished tree by $R^C(t)$. The monomial which is delivered by the $n-1$ other factors is denoted by $P^C(t)$. The definitions of $C, P, R$ can be extended to monomials of trees in the obvious manner, by choosing a cut $C_i$ for every tree $t_{j_i}$ in the monomial:

$$C(t_{j_1} \ldots t_{j_n}) := C^1(t_{j_1}) \ldots C^n(t_{j_n}),$$

$$P^C(t_{j_1} \ldots t_{j_n}) := P^C_1(t_{j_1}) \ldots P^C_n(t_{j_n}),$$

$$R^C(t_{j_1} \ldots t_{j_n}) := R^C_1(t_{j_1}) \ldots R^C_n(t_{j_n}).$$

Let us now establish the Hopf algebra structure. Following [2,3] we first define the counit and the coproduct. The counit \( \bar{e} : A \to \mathbb{Q} \) is simple:

$$\bar{e}(X) = 0$$

for any $X \neq 1$,

$$\bar{e}(1) = 1.$$

The coproduct $\Delta$ is defined by the equations

$$\Delta(1) = 1 \otimes 1$$

(52)

$$\Delta(t_1 \ldots t_n) = \Delta(t_1) \ldots \Delta(t_n)$$

(53)

$$\Delta(t) = t \otimes 1 + (id \otimes B_+)[\Delta(B_-(t))],$$

(54)

which defines the coproduct on trees with $n$ vertices iteratively through the coproduct on trees with a lesser number of vertices. See Fig. 11 for an example. Actually, the coproduct can be written as

$$\Delta(t) = 1 \otimes t + t \otimes 1 + \sum_{\text{adm. cuts } C \text{ of } t} P^C(t) \otimes R^C(t) =: 1 \otimes t + t \otimes 1 + \Delta'(t),$$

(55)

which defines $\Delta'$. 

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Up to this point we have established a bialgebra structure, but it is actually a Hopf algebra. Following [2,3] we find the antipode $S$ as

$$S(1) = 1$$
$$S(t) = -t - \sum_{\text{adm. cuts } C \text{ of } t} S[P^C(t)]R^C(t) = -t - m[(S \otimes \text{id})\Delta(t)].$$  \hspace{1cm} (56)

An alternative formula for the antipode, which one may easily derive by induction on the number of vertices [2,3] is

$$S(t) = - \sum_{\text{all cuts } C \text{ of } t} (-1)^{n_C} P^C(t)R^C(t),$$

where $n_C$ is the number of single cuts in $C$. This time, we have a non-recursive expression, summing over all cuts $C$ and relaxing the restriction to admissible cuts. The overall minus sign can be incorporated in the sum if we attach an incoming edge to the root. All cuts which remove this edge are then full cuts, all the other ones are normal cuts. A Feynman graph corresponding to a tree with $n$ vertices allows then for $2^n$ cuts. The $2^{n-1}$ full cuts deliver the counterterm, the $2^n - 1$ normal cuts eliminate the subdivergence to deliver the result of the $R$-bar operation [2,3].

So far we have established a Hopf algebra on rooted trees, using the set of rooted trees, the commutative multiplication $m$ for elements of this set, the unit 1 and counit $\bar{e}$, the coproduct $\Delta$ and antipode $S$. We call this Hopf algebra $\mathcal{H}_R$. Continuing in the manner of [2–4], we may label the vertices of rooted trees by Feynman graphs without subdivergences, in the sense described in the paper and in detail in [3,4].

Let us also mention that

$$m[(S \otimes \text{id})\Delta(t)] = \bar{e}(t) = 0 = \sum S(t_{(1)})t_{(2)},$$  \hspace{1cm} (58)

where we introduced Sweedler’s notation $\Delta(t) =: \sum t_{(1)} \otimes t_{(2)}$, and $\text{id}$ is the identity map $\mathcal{H}_R \to \mathcal{H}_R$. We conclude by defining $\#(t)$ to be the number of vertices of a rooted tree $t$. This extends to a monomial of rooted trees in the obvious manner: $\#(\prod_i T_i) = \sum_i \#(T_i)$.

**B. The Hopf Algebra Structure of Graphs and Forest Formula**

The results of [4] show that for each Feynman graph $\Gamma$, we obtain a sum of associated rooted tree $T_\Gamma$ and a coproduct given by

$$\Delta(T_\Gamma) = 1 \otimes T_\Gamma + T_\Gamma \otimes 1 + \sum_{\gamma \subset \chi} T_\gamma \otimes T_{\Gamma/\gamma}. $$  \hspace{1cm} (59)

Here, $T_\Gamma$ is a sum of rooted trees with decorations, primitive elements in the Hopf algebra of rooted trees, which are obtained from Feynman graphs without subdivergences. As the map $T_\Gamma \leftrightarrow \Gamma$ is one-to-one, we can directly formulate the Hopf algebra on graphs $\Gamma$, as in section II.

To the coproduct (59) belongs an antipode given by

$$S(T_\Gamma) = -T_\Gamma - \sum_{\gamma \subset \Gamma} S[T_\gamma]T_{\Gamma/\gamma}, $$  \hspace{1cm} (60)
as one straightforwardly checks. Because it is an antipode in a Hopf algebra of rooted trees, it can be written as a sum over all cuts. Set $T_{\Gamma} = \sum_i T_i$ for some decorated rooted trees $T_i$. Then,

$$S(T_{\Gamma}) = \sum_i \sum_{\text{all cuts } C_i \text{ of } T_i} (-1)^{n_{C_i}} P^{C_i}(T_i) R^{C_i}(T_i).$$

(61)

Each such cut corresponds to a renormalization forest, obtained by boxing the corresponding subgraphs in $\Gamma$, and vice versa [3].

Now, let $\phi$ be a $\mathbb{Q}$-linear map which assigns to $T_{\Gamma}$ the corresponding Feynman integral. Further, let $\phi_R = \tau_R \circ \phi$ be a map which assigns to $T_{\Gamma}$ the corresponding Feynman integral, evaluated under some renormalization condition $R$. Hence, from $T_{\Gamma}$ we obtain via $\phi$ a Feynman integral $\phi(T_{\Gamma})$ in need of renormalization. $\tau_R$ modifies this Feynman integral, in such a way that the result contains the divergent part of this integral. Essentially, $\tau_R$ extracts the divergences of $\phi(T_{\Gamma})$ in a meaningful way [12]. Hence, as $\tau_R$ isolates divergences faithfully, differences $(id - \tau_R)(\phi(T_{\Gamma}))$ eliminate infinities in Feynman integrals. Depending on the chosen renormalization scheme $R$, one can adjust finite parts to fulfill renormalization conditions. A detailed study of this freedom from the Hopf algebra viewpoint can be found in [3].

We remind the reader of Sweedler’s notation: $\Delta(T_{\Gamma}) = \sum T_{\Gamma(1)} \otimes T_{\Gamma(2)}$. Let us consider the antipode $\bar{e}(T_{\Gamma})$ in the same notation:

$$0 = \bar{e}(T_{\Gamma}) = \sum S(T_{\Gamma(1)}) T_{\Gamma(2)}. $$

The above map vanishes identically, and it can also be written as

$$m[(S \otimes id)\Delta(T_{\Gamma})] \equiv \bar{e}(T_{\Gamma}) = 0. $$

But this map gives rise to a much more interesting map, by composition with $\phi$,

$$T_{\Gamma} \to \Gamma_R := m[(S_R \otimes id)(\phi \otimes \phi)\Delta(T_{\Gamma})], $$

for it associates the renormalized Feynman integral $\Gamma_R$ to the Feynman graph $\Gamma$ represented by a unique sum of rooted trees.

Its usual definition,

$$\Gamma_R = (id - \tau_R) \left[ \Gamma + \sum_{\gamma \subset \Gamma} Z_{\gamma} \Gamma / \gamma \right], $$

(62)

is recovered if we define

$$S_R[\phi(T_{\gamma})] \equiv Z_{\gamma} = -\tau_R(\gamma) - \tau_R \left[ \sum_{\gamma' \subset \gamma} Z_{\gamma' \gamma / \gamma'} \right].$$

(63)

This map is derived from the antipode

$$S[T_{\gamma}] = -T_{\gamma} - \sum_{\gamma' \subset \gamma} S[T_{\gamma'}] T_{\gamma / \gamma'}. $$

(64)
Using $\phi$ to lift this to Feynman graphs, and using the freedom to alter corresponding analytic expressions according to renormalization schemes $R$ one ends up with (63).

Note that if one defines

$$\phi_R = S_R \circ \phi \circ S,$$

one has $S_R \circ \phi = \phi_R \circ S$ and hence

$$S_R[\phi(T_\gamma)] = \tau_R \left[ -\phi(T_\gamma) - \sum_{\gamma' \subset \gamma} \phi_R(S[T_{\gamma'}])\phi(T_{\gamma/\gamma'}). \right] \tag{65}$$

Hence, in accordance with [2–4] we determine the $Z$-factor of a graph $\gamma$ as derived from the antipode in the Hopf algebra of rooted trees. In (62), we recovered the original forest formula in its recursive form. The non-recursive form is recovered with the same ease, using (61) instead of (60) [2–4].
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FIGURES

FIG. 1. A Feynman graph, its divergent subgraphs, its forests, and the corresponding tree with appropriate full and normal cuts. From this one calculates the antipode and the renormalized Green function.

FIG. 2. Forests and counterterms as cuts and antipodes. Each forests (thick grey lines) contains an analytic expression which has to be evaluated using $R$.

FIG. 3. Vertex weights and tree factorials. For the tree $t$ given at the left in the upper row we indicate the vertex weights attached to each vertex. We also give the Feynman diagram corresponding to a chosen common one-loop self-energy decoration at each vertex.

FIG. 4. Generalized vertex weights and tree factorials which can appear if there are decorations of arbitrary loop order. The factorial of this decorated tree is $2 \times 5 = 10$.

FIG. 5. A Yukawa self-energy at three loops.

FIG. 6. Examples of Feynman graphs, their decorated rooted trees and the corresponding matrix calculus. From top to bottom, the bare function $\Gamma(t)$, evaluates for the indicated decorated rooted trees to $M_1 v_0 z^{-x}$, $A_1 M_2 v_0 z^{-2x}$, $A_1 M_2 M_3 v_0 z^{-3x}$.

FIG. 7. The action of $B_-$ on a rooted tree.

FIG. 8. The action of $B_+$ on a monomial of trees.

FIG. 9. An elementary cut $c$ splits a rooted tree $t$ into two components, the fall-down $P^c(t)$ and the piece which is still connected to the root, $R^c(t)$.

FIG. 10. An admissible cut $C$ acting on a tree $t$, leading to a monomial of trees. One of the factors, $R^C(t)$, contains the root of $t$.

FIG. 11. The coproduct, worked out for the trees $t_1, t_2, t_3, t_2$, from top to bottom. In the last line we give one full and all normal cuts of the tree.
FIG. 11.