Coulomb corrections to electron scattering on the extended source and the proton charge radius.

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It is shown that the account for the proton charge form factor in the Coulomb corrections to the electron-proton scattering cross section noticeably diminishes the difference between the value of the proton charge radius \( r_E \), extracted from the \( e\bar{p} \) scattering data, and that following from the muonic hydrogen data. For the electron energy much higher than the electron mass but much smaller than \( r_E^{-1} \approx 230 \) MeV, the relative correction has the universal form \( \delta r_E / r_E = -\pi \alpha / 2 \), where \( \alpha \) is the fine structure constant.

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INTRODUCTION

One of the unsolved puzzles in the elementary particle physics widely discussed now is the huge difference, five standard deviations, between the value of the proton charge radius \( r_E \), extracted from the \( e\bar{p} \) scattering data, Refs. [1,2], and that following from the muonic hydrogen data, Refs. [3,4]. Though careful revisions of the theoretical results for the muonic hydrogen atom have been performed, no contribution which could noticeably shift the value of the proton charge radius was found, see review in Ref. [5]. Concerning the result following from the \( e\bar{p} \) scattering data, the most problematic point is the account for the radiative corrections, and consensus was not achieved whether this account was performed correctly, see discussion in Refs. [6,7]. Though there are many papers devoted to the calculation of the radiative corrections to \( e\bar{p} \) scattering, see also Refs. [8–13], the authors of the experimental paper [1] and the author of Ref. [6] came to the opposite conclusions on the importance of account for the proton form factor in the calculations of the Coulomb corrections to the \( e\bar{p} \) scattering cross section, see also Refs. [2,7].

In the present paper we derive explicitly the shift \( \delta r_E \) of the proton charge radius which appears due to the account for the proton charge form factor in the leading Coulomb corrections. These corrections are the contribution of the two-photon exchange amplitude calculated in the external field approximation. The derived shift is essentially larger than the model uncertainties declared by the authors of Ref. [2]. Moreover, it diminishes noticeably the difference between the value of the proton charge radius following from the muonic hydrogen data and that following from the \( e\bar{p} \) scattering data.

COULOMB CORRECTIONS

In the \( e\bar{p} \) scattering experiment, the most accurate value of the proton charge radius are obtained at the smallest value of the momentum transfer \( Q = p_2 - p_1 \), where \( p_1 \) and \( p_2 \) are the initial and final electron momenta, respectively, we set \( \hbar = c = 1 \) throughout the paper. In the experiment [1], the minimal value of \( Q \) was 60 MeV which is much smaller than the proton mass. Besides, the lowest electron energy \( E = 180 \) MeV in this experiment is also essentially smaller than the proton mass, so that the external field approximation is appropriate to study the influence of the charge form factor on the Coulomb corrections to the \( e\bar{p} \) scattering amplitude and the cross section.

Let us consider the electron scattering cross section in the external potential \( V(r) \), having the Fourier transform \( V_F(Q^2) = -4 \pi \alpha F(Q^2)/Q^2 \), where \( F(Q^2) \) is the charge form factor of the proton, and \( \alpha \approx 1/137 \) is the fine structure constant. The cross section reads (see, e.g., Ref. [14])

\[
\frac{d\sigma}{d\Omega} = \sum_{\lambda_2} |M_{\lambda_2}\lambda_1|^2, \quad M_{\lambda_2}\lambda_1 = -\frac{E}{2\pi} u_{\lambda_2 p_2}^\dagger \int dr \exp(-ip_2 \cdot r) V(r) \psi_{\lambda_1 p_1}^{(\text{in})}(r),
\]

where \( E = \sqrt{p_{1,2}^2 + m^2} \) is the electron energy, \( m \) is the electron mass, \( \psi_{\lambda_1 p_1}^{(\text{in})}(r) \) is a positive-energy solution of the Dirac equation in the external field, the asymptotic form of \( \psi_{\lambda_1 p_1}^{(\text{in})}(r) \) at large \( r \) contains the plane wave and the spherical divergent wave, \( u_{\lambda_2 p_2} \) is the free Dirac spinor normalized to \( u_{\lambda_2 p_2}^\dagger u_{\lambda_2 p_2} = 1 \), \( \lambda_1 = \pm 1 \) and \( \lambda_2 = \pm 1 \).
enumerate the independent solutions of the Dirac equation. It is shown in Ref. [15] that the wave function \( \psi_{\lambda_1 p_1}^{(in)} (r) \) has the form
\[
\psi_{\lambda_1 p_1}^{(in)} (r) = [g_0(p_1, r) + \alpha \cdot g_1(p_1, r) + \Sigma \cdot g_2(p_1, r)] u_{\lambda_1 p_1},
\]
where \( \alpha = \gamma^0 \gamma_i, \Sigma = -\gamma^0 \gamma_i, \) and \( \gamma^\mu \) are the Dirac matrices. Note that the perturbation expansion of the functions \( g_0(p_1, r), g_1(p_1, r), \) and \( g_2(p_1, r) \) starts from the terms \( V^0, V^1 \) and \( V^2 \), respectively. We are going to calculate the matrix element \( M_{\lambda_2 \lambda_1} \) taking into account the contributions \( V^1 \) and \( V^2 \) only (the one-photon exchange and the two-photon exchange). Therefore, we can neglect the term with \( g_2(p_1, r) \). Let us introduce the quantities
\[
(G_0, G_1) = -\frac{E}{2\pi} \int dr \exp(-ip_2 \cdot r)V(r)(g_0(p_1, r), g_1(p_1, r)).
\]
Due to the parity conservation, the function \( G_1 \) can be represented as \( f_1 p_1 + f_2 p_2 \), where \( f_{1,2} \) depend on \( p_1 \cdot p_2 \). Therefore, using the Dirac equation for \( u_{\lambda_1 p_1}, u_{\lambda p_2}^\dagger \) and neglecting all terms of order \( m/E \) we obtain
\[
M_{\lambda_2 \lambda_1} = Tu_{\lambda_2 p_2}^\dagger u_{\lambda_1 p_1}, \quad T = G_0 + \frac{\nu \cdot G_1}{\nu^2},
\]
where \( \nu = (p_1 + p_2)/2E \).

Remarkably, in Eq. (4) the Coulomb corrections enter the scattering amplitude via the overall scalar factor \( T \). Using the conventional perturbation theory we find for the linear and quadratic in \( V \) terms of \( T = T_1 + T_2 \),
\[
T_1 = -\frac{EV_F(Q^2)}{2\pi},
\]
\[
T_2 = \frac{E}{(2\pi)^2} \int ds \frac{V_F(\chi_+)V_F(\chi_-)[2E + (\nu \cdot s)/\nu^2]}{s^2 + 2E(\nu \cdot s) - Q^2/4 - i0},
\]
\[
\chi_\pm = (s \pm Q/2)^2.
\]
Up to \( V^3 \) terms, the cross section reads
\[
d\sigma = (1 + \delta)d\sigma_B, \quad \delta = 2\text{Re}(T_1^*T_2)/|T_1|^2,
\]
where \( d\sigma_B \) is the Born cross section. Since \( T_1 \) is the real quantity, one should calculate the real part of \( T_2 \). Eq. (6) is in agreement with the corresponding result of Ref. [8].

For a pointlike particle, when \( F(Q^2) = 1 \), the Coulomb correction \( \delta \) is reduced to the Feshbach correction \( \delta_F \), Ref. [17]
\[
\delta_F = \frac{\pi \alpha}{1 + \sin^2 \theta} \sin \theta = \frac{Q}{2\mu}.
\]
It was explicitly stated in Ref. [2] that the Coulomb corrections to the cross section were taken into account in the data analysis in the experiment [1] solely via the correction-factor \( (1 + \delta_F) \). In the next Section we show that this is not sufficient for the accuracy declared in Refs. [1, 2].

**CHARGE RADIUS**

Let us discuss the effect of finite proton charge radius on the Coulomb corrections. We consider small-angle scattering when \( Q \ll E, r_E^{-1} \), where \( r_E^{-1} \approx 230 \text{ MeV} \). In this case \( F(Q^2) \approx 1 - r_E^2 Q^2/6 \), so that
\[
T_1 = \frac{2E\alpha}{Q^2} F(Q^2) \approx 2E\alpha \left( \frac{1}{Q^2} - \frac{r_E^2}{6} \right).
\]
Therefore, the \( Q \)-independent term in \( T_2 \) can imitate the effect of \( r_E \) in the Born term. Thus, the correction \( \delta r_E^2 \), coming from the account of the finite proton charge radius has the form
\[
\delta r_E^2 = \frac{3\alpha}{\pi^2} \text{Re} \int d\sigma \frac{[1 - F^2(s^2)]|2E + (\nu \cdot s)|}{s^4[s^2 + 2E(\nu \cdot s) - i0]}.
\]
If \( E \ll r_E^{-1} \), then the main contribution to the integral in Eq. (9) is given by the region \( s \sim E \) and we can replace 
\[ 1 - F^2(s^2) \] by \( s^2 r_E^2 / 3 \). Then we obtain
\[
\delta r_E^2 = -\pi \alpha r_E^2, \quad \delta r_E / r_E = -\frac{\pi \alpha}{2} \approx -0.0115 .
\] (10)

Thus, at \( E \ll r_E^{-1} \) the relative correction \( \delta r_E / r_E \) is independent of the shape of the form factor. It is two times larger than the statistical error and five times larger than the model error presented in Ref. [1].

For \( E \gg r_E^{-1} \), the main contribution to the integral in Eq. (9) is given by the region \( s \sim r_E^{-1} \), and we find in this limit,
\[
\delta r_E^2 = -\frac{12 \alpha}{\pi E} \int \frac{ds}{s^2} [1 - F^2(s^2)] .
\] (11)

In contrast to the asymptotics (10), the asymptotics (11) is model-dependent. The widely used fit of the form factor has the dipole form
\[
F(Q^2) = \left[ 1 + \frac{Q^2}{\Lambda^2} \right]^{-2}, \quad \Lambda = 840 \text{ MeV} .
\] (12)

For this form factor, the correction Eq. (11) reads
\[
\frac{\delta r_E}{r_E} = \frac{35 \alpha \Lambda}{64 E} .
\] (13)

In Fig. 1 we show the dependence of the relative correction \( \delta r_E / r_E \) on \( E \) for the dipole form factor (solid line). In fact, this dependence is not very sensitive to the shape of the form factor at a given \( r_E \). This statement is demonstrated in Fig. 1 by the dashed line obtained for the form factor
\[
F(Q^2) = \left[ 1 + \frac{2 Q^2}{\Lambda^2} \right]^{-1} ,
\] (14)

having the same expansion at small \( Q^2 \) as the expansion of the dipole form factor (12). One can see that \( \delta r_E / r_E \) is very sensitive to the energy \( E \), and becomes \( \delta r_E / r_E = -0.82 \cdot 10^{-2} \) for \( E = 180 \text{ MeV} \), corresponding to the minimal energy in the experiment Ref.[1]. This correction is still essentially (more than three times) larger than the model error presented in Ref.[2].

**CONCLUSION**

We have demonstrated that, for the model uncertainty declared in Refs.[1, 2], the account for the Coulomb corrections via the factor \((1 + \delta_F)\), with \( \delta_F \) being the Feshbach correction, Eq. (7), is not sufficient. The account for the proton charge form factor in the Coulomb corrections is essentially larger than the model uncertainty of Refs.[1, 2] and diminishes noticeably the difference between the value of the proton charge radius \( r_E \), extracted from the \( ep \) scattering data, and that following from the muonic hydrogen data. For the electron energy much higher than the electron mass but much smaller than \( r_E^{-1} \approx 230 \text{ MeV} \), the relative correction has the universal form \( \delta r_E / r_E = -\pi \alpha / 2 \).

Certainly, for the energies, corresponding to the experimental conditions in [1], there are some corrections to Eq. (9) coming from the recoil and virtuality effects. However, they can not change the conclusion on the importance of the account for the proton form factor at the calculation of the Coulomb corrections to provide the accuracy declared in Refs.[1, 2].

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FIG. 1: The dependence of $\delta r_E/r_E$ on $E$. Solid line corresponds to the dipole form factor (12), dashed line corresponds to the form factor in Eq. (14).

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