ASYMPTOTIC NONVANISHING OF SYZYGIES
OF ALGEBRAIC VARIETIES

JINHYUNG PARK

Abstract. We establish precise nonvanishing results for asymptotic syzygies of smooth projective varieties. This refines Ein–Lazarsfeld’s asymptotic nonvanishing theorem. Combining with the author’s previous asymptotic vanishing result, we completely determine the asymptotic shapes of the minimal free resolutions of the graded section modules of a line bundle on a smooth projective variety as the positivity of the embedding line bundle grows.

1. Introduction

The purpose of this paper is to address the main theme of [8] and [19]: the asymptotic behavior of syzygies of algebraic varieties is surprisingly uniform. After the pioneering work of Green [16], there has been a considerable amount of research to understand syzygies of algebraic varieties. It is an interesting problem to describe the overall asymptotic behavior of syzygies of graded section modules of a line bundle on a smooth projective variety as the positivity of the embedding line bundle grows (see [16, Problem 5.13] and [7, Problem 4.4]). The influential paper [8] of Ein–Lazarsfeld opens the door to asymptotic syzygies of algebraic varieties, and the asymptotic nonvanishing theorem was proved there. In the present paper, we provide a new approach to nonvanishing of asymptotic syzygies, and together with the author’s asymptotic vanishing theorem [19], we exhibit the uniform behavior of all asymptotic syzygies.

Throughout the paper, we work over an algebraically closed field $k$ of arbitrary characteristic. Let $X$ be a smooth projective variety of dimension $n$, and $B$ be a line bundle on $X$. For an integer $d \geq 1$, set

$$L_d := \mathcal{O}_X(dA + P) \quad \text{and} \quad r_d := h^0(X, L_d) - 1,$$

where $A$ is an ample divisor and $P$ is an arbitrary divisor on $X$. We assume that $d$ is sufficiently large so that $L_d$ is very ample and $r_d = \Theta(d^n)$. Here, for a nonnegative function $f(d)$ defined for positive integers $d$, we define:

- $f(d) \geq \Theta(d^k) \iff$ there is a constant $C_1 > 0$ such that $f(d) \geq C_1 d^k$ for any sufficiently large positive integer $d$;
- $f(d) \leq \Theta(d^k) \iff$ there is a constant $C_2 > 0$ such that $f(d) \leq C_2 d^k$ for any sufficiently large positive integer $d$;
- $f(d) = \Theta(d^k) \iff$ there are constants $C_1, C_2 > 0$ such that $C_1 d^k \leq f(d) \leq C_2 d^k$ for any sufficiently large positive integer $d$. 

Date: May 29, 2023.

2020 Mathematics Subject Classification. 14C20, 14J60, 13D02.

Key words and phrases. asymptotic syzygies, Koszul cohomology, algebraic varieties, line bundles.

J. Park was partially supported by the National Research Foundation (NRF) funded by the Korea government (MSIT) (NRF-2019R1A6A1A10073887 and NRF-2022M3C1C8094326).
For simplicity, we write $f(d) = \Theta(1)$ if $f(d)$ is a constant including 0 for any sufficiently large positive integer $d$. Let $S_d := \bigoplus_{m \geq 0} S^m H^0(X, L_d)$. By Hilbert syzygy theorem, the finitely generated graded section $S_d$-module

$$R_d = R(X, B; L_d) := \bigoplus_{m \in \mathbb{Z}} H^0(X, B \otimes L_d^m)$$

admits a minimal free resolution

$$0 \rightarrow R_d \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_t \rightarrow 0,$$

where

$$E_p = \bigoplus_q K_{p,q}(X, B; L_d) \otimes \mathbb{K} S_d(-p - q).$$

Here the Koszul cohomology group $K_{p,q}(X, B; L_d)$ can be regarded as the space of $p$-th syzygies of weight $q$. When $X \subseteq \mathbb{P} H^0(X, L_d) = \mathbb{P}^{r_d}$ is projectively normal, the minimal free resolution of $R_d$ with $B = \mathcal{O}_X$ contains all information about the defining equations of $X$ in $\mathbb{P}^{r_d}$ and their syzygies. Based on the experience of the case of curves, it was widely believed that the minimal free resolutions of $R_d$ become simpler as $d$ increases. However, Ein–Lazarsfeld [8] showed that this had been misleading, and they instead proposed that there would be a uniform asymptotic vanishing and nonvanishing behavior of $K_{p,q}(X, B; L_d)$ when $d$ is sufficiently large.

It is elementary to see that

$$K_{p,q}(X, B; L_d) = 0 \text{ for } q \geq n + 2.$$  

The cases $q = 0$ and $q = n + 1$ are well understood due to Green, Schreyer, Ottaviani–Paoletti, and Ein–Lazarsfeld: [8, Proposition 5.1 and Corollary 5.2] state that

$$K_{p,0}(X, B; L_d) \neq 0 \iff 0 \leq p \leq h^0(B) - 1;$$

$$K_{p,n+1}(X, B; L_d) \neq 0 \iff r_d - n - h^0(X, \omega_X \otimes B^{-1}) + 1 \leq p \leq r_d - n.$$  

For $1 \leq q \leq n$, let $c_q(d)$ be the number such that

$$K_{c_q(d),q}(X, B; L_d) \neq 0 \text{ and } K_{p,q}(X, B; L_d) = 0 \text{ for } 0 \leq p \leq c_q(d) - 1,$$

and $c'_q(d)$ be the number such that

$$K_{r_d-c'_q(d),q}(X, B; L_d) \neq 0 \text{ and } K_{p,q}(X, B; L_d) = 0 \text{ for } r_d - c'_q(d) + 1 \leq p \leq r_d.$$  

After interesting nonvanishing results of Ottaviani–Paoletti [18] and Eisenbud–Green–Hulek–Popescu [12], Ein–Lazarsfeld proved the asymptotic nonvanishing theorem ([8, Theorem 4.1]): For each $1 \leq q \leq n$, if $d$ is sufficiently large, then

$$K_{p,q}(X, B; L_d) \neq 0 \text{ for } \Theta(d^{q-1}) \leq p \leq r_d - \Theta(d^{n-1}).$$  

In particular, $c_q(d) \geq \Theta(d^{q-1})$ and $c'_q(d) \leq \Theta(d^{n-1})$. In [8, Conjecture 7.1], Ein–Lazarsfeld conjectured that

$$K_{p,q}(X, B; L_d) = 0 \text{ for } 0 \leq p \leq \Theta(d^{q-1}),$$

and this was confirmed by the present author [19, Theorem 1.1] using Raicu’s result in the appendix of [20]. In particular, $c_q(d) = \Theta(d^{q-1})$.

Despite the aforementioned results on asymptotic syzygies of algebraic varieties, at least two problems remain. First, it is unclear whether vanishing and nonvanishing of $K_{p,q}(X, B; L_d)$ can alternate in a few steps after $c_q(d)$ or before $r_d - c'_q(d)$. Second, the previous results do not say anything about $K_{p,q}(X, B; L_d)$ for $r_d - \Theta(d^{n-1}) \leq p \leq r_d$. In this paper, we completely resolve these two issues: We show that vanishing and nonvanishing of $K_{p,q}(X, B; L_d)$ can alternate only at $c_q(d)$ and $r_d - c'_q(d)$, and we give estimations for $c_q(d)$ and $c'_q(d)$. Consequently, we could
determine the precise vanishing and nonvanishing range of \( p \) for \( K_{p,q}(X, B; L_d) \). It is worth noting that \( c'_q(d) \) heavily depends on \( H^{q-1}(X, B) \) while \( c_q(d) \) depends only on \( d \) asymptotically.

**Theorem 1.1.** Let \( X \) be a smooth projective variety of dimension \( n \geq 1 \), and \( B \) be an arbitrary divisor on \( X \). Fix an index \( 1 \leq q \leq n \). Then there exist functions \( c_q(d) \) and \( c'_q(d) \) with

\[
(1.1) \quad c_q(d) = \Theta(d^{n-1}) \quad \text{and} \quad c'_q(d) = \begin{cases} \Theta(d^{n-q}) & \text{if } H^{q-1}(X, B) = 0 \text{ or } q = 1 \\ q - 1 & \text{if } H^{q-1}(X, B) \neq 0 \text{ and } q \geq 2 \end{cases}
\]

such that if \( d \) is sufficiently large, then

\[
(1.2) \quad K_{p,q}(X, B; L_d) \neq 0 \iff c_q(d) \leq p \leq r_d - c'_q(d).
\]

To prove Theorem 1.1, we do not use Ein–Lazarsfeld’s asymptotic nonvanishing theorem \([8, \text{Theorem 4.1}]\), but we adopt their strategy in \([8]\). Let \( H \) be a suitably positive very ample line bundle on \( X \), and choose a general member \( X \in |H| \). Put

\[
\overline{L_d} := L_d|_{\overline{X}}, \quad \overline{B} := B|_{\overline{X}}, \quad \overline{H} := H|_{\overline{X}}, \quad V'_d := H^0(X, L_d \otimes H^{-1}).
\]

We have noncanonical splitting

\[
K_{p,q}(X, B|_{\overline{X}}; L_d) = \bigoplus_{j=0}^p \wedge^j V'_d \otimes K_{p-j,q}(\overline{X}, \overline{B}; L_d) \quad \text{and} \quad K_{p,q}(X, B \otimes \overline{H}; L_d) = \bigoplus_{j=0}^p \wedge^j V'_d \otimes K_{p-j,q}(\overline{X}, \overline{B} \otimes \overline{H}; L_d).
\]

There are natural maps

\[
\theta_{p,q} : K_{p+1, q-1}(X, B \otimes \overline{H}; L_d) \longrightarrow K_{p,q}(X, B; L) \quad \text{and} \quad \theta'_{p,q} : K_{p,q}(X, B; L_d) \longrightarrow K_{p,q}(X, B; L_d).
\]

When \( q = 1 \) or \( 2 \), the map \( \theta_{p,q} \) should be modified (see Section 3). In \([8, \text{Sections 3 and 4}]\), the secant constructions were introduced to carry nonzero syzygies by highly secant planes. This essentially shows that the map \( \theta_{p,q} \) is nonzero for \( \Theta(d^{n-1}) \leq p \leq r_d - \Theta(d^{n-1}) \). Instead of utilizing the secant constructions, in this paper, we apply the asymptotic vanishing theorem \([19, \text{Theorem 1.1}]\) to get the estimations of \( c_q(d) \) and \( c'_q(d) \) (Proposition 4.4) and to see that the maps \( \theta_{c_q(d), q} \) and \( \theta'_{r_d-c'_q(d), q} \) are nonzero (Proposition 4.6). The latter statement means that there are syzygies \( \alpha \) and \( \beta \) in

\[
(*) \quad K_{c_q(d)+1-j_0, q-1}(\overline{X}, \overline{B} \otimes \overline{H}; L_d) \quad \text{and} \quad K_{r_d-c'_q(d)-j'_0, q}(\overline{X}, \overline{B}; L_d)
\]

for some \( 0 \leq j_0 \leq c_q(d) + 1 \) and \( \dim V'_d - c'_q(d) \leq j'_0 \leq \dim V'_d \) that are lifted to syzygies in \( K_{c_q(d), q}(X, B; L_d) \) and \( K_{r_d-c'_q(d), q}(X, B; L_d) \) via the maps \( \theta_{c_q(d), q} \) and \( \theta'_{r_d-c'_q(d), q} \), respectively. Since (*) survives in

\[
K_{p,q}(X, B \otimes \overline{H}; L_d) \quad \text{for } c_q(d) \leq p \leq r_d - \Theta(d^{n-1}) \quad \text{and} \quad K_{p,q}(X, B; L_d) \quad \text{for } \Theta(d^{n-1}) \leq p \leq r_d - c'_q(d),
\]

we can argue that the syzygies \( \alpha \) and \( \beta \) in (*) are also lifted to syzygies in \( K_{p,q}(X, B; L_d) \) for \( c_q(d) \leq p \leq r_d - c'_q(d) \) via the maps \( \theta_{p,q} \) and \( \theta'_{p,q} \), respectively (Theorem 3.1).

The paper is organized as follows. After recalling preliminary results on syzygies of algebraic varieties in Section 2, we show how to lift syzygies from hypersurfaces (Theorem 3.1) in Section 3. Section 4 is devoted to the proof of Theorem 1.1. Finally, in Section 5, we present complementary results and open problems on asymptotic syzygies of algebraic varieties.

**Acknowledgements.** The author is very grateful to Lawrence Ein, Yeongrak Kim, and Wenbo Niu for inspiring discussions.
2. Preliminaries

In this section, we collect relevant basic facts on Koszul cohomology and Castelnuovo–Mumford regularity.

2.1. Koszul Cohomology. Let \( V \) be an \( r \)-dimensional vector space over an algebraically closed field \( k \), and \( S := \bigoplus_{m \geq 0} S^m V \). Consider a finitely generated graded \( S \)-module \( M \). The Koszul cohomology group \( K_{p,q}(M,V) \) is the cohomology of the Koszul-type complex

\[
\wedge^{p+1} V \otimes M_{q-1} \xrightarrow{\delta} \wedge^p V \otimes M_q \xrightarrow{\delta} \wedge^p \otimes M_{q+1},
\]

where the Koszul differential \( \delta \) is given by

\[
\delta(s_1 \wedge \cdots \wedge s_p \otimes t) \mapsto \sum_{i=1}^{p} (-1)^i s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_p \otimes s_i t.
\]

Then \( M \) has a minimal free resolution

\[
0 \leftarrow M \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_r \leftarrow 0,
\]

where \( E_p = \bigoplus_q K_{p,q}(M,V) \otimes_k S(-p-q) \).

We may regard \( K_{p,q}(M,V) \) as the vector space of \( p \)-th syzygies of weight \( q \). Let

\[
0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0
\]

be a short exact sequence of finitely generated graded \( S \)-modules. By [16, Corollary (1.d.4)] (see also [2, Lemma 1.24]), this induces a long exact sequence

\[
\cdots \rightarrow K_{p+1,q-1}(M,V) \rightarrow K_{p+1,q-1}(M'',V) \rightarrow K_{p,q}(M',V) \rightarrow K_{p,q}(M,V) \rightarrow \cdots.
\]

Consider an injective map

\[
\iota' : \wedge^{p+1} V \rightarrow V \otimes \wedge^p V, \quad s_1 \wedge \cdots \wedge s_{p+1} \mapsto \sum_{i=1}^{p+1} (-1)^i s_i \otimes s_1 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_{p+1}.
\]

It is straightforward to check that the following diagram commutes:

\[
\begin{array}{ccc}
\wedge^{p+1} V \otimes M_q & \xrightarrow{\delta} & \wedge^p V \otimes M_{q+1} \\
\iota' \otimes \text{id}_{M_q} & \downarrow & \iota' \otimes \text{id}_{M_{q+1}} \\
V \otimes \wedge^p V \otimes M_q & \xrightarrow{-\text{id}_V \otimes \delta} & V \otimes \wedge^p \otimes M_{q+1}.
\end{array}
\]

Then the map \( \iota' \) induces a map

\[
\iota : K_{p+1,q}(M,V) \rightarrow V \otimes K_{p,q}(M,V).
\]

This map is the glueing of the evaluation maps \( \text{ev}_x : K_{p+1,q}(M,V) \rightarrow K_{p,q}(M,V) \) for \( x \in V^\vee \) in [2, Subsection 2.2.1].

We now turn to the geometric setting. Let \( X \) be a projective variety, \( B \) be a coherent sheaf on \( X \), and \( L \) be a very ample line bundle on \( X \). Put \( V := H^0(X,L) \) and \( S := \bigoplus_{m \geq 0} S^m V \). Then the section module

\[
R(X,B;L) := \bigoplus_{m \in \mathbb{Z}} H^0(X,B \otimes L^m),
\]
is finitely generated graded $S$-module. We define the Koszul cohomology group as

$$K_{p,q}(X, B; L) := K_{p,q}(R(X, B; L), V).$$

In this paper, $L$ is always assumed to be sufficiently positive, so we have

$$H^0(X, B \otimes L^{-m}) = 0 \text{ for } m > 0.$$ 

Then $K_{p,q}(X, B; L) = 0$ for $q < 0$. It is clear that if $K_{p_0,q}(X, B; L) = 0$ for $0 \leq q \leq q_0$, then $K_{p,q}(X, B; L) = 0$ for $p \geq p_0$ and $0 \leq q \leq q_0$. Let $M_L$ be the kernel bundle of the evaluation map ev: $H^0(X, L) \otimes \mathcal{O}_X \to L$. We have a short exact sequence

$$0 \to M_L \to H^0(X, L) \otimes \mathcal{O}_X \to L \to 0.$$ 

We frequently use the following well-known facts.

**Proposition 2.1** (cf. [8, Proposition 3.2, Corollary 3.3, Remark 3.4], [19, Proposition 2.1]).

Assume that

$$H^i(X, B \otimes L^m) = 0 \text{ for } i > 0 \text{ and } m > 0.$$ 

For any $p \geq 0$, the following hold:

1. If $q \geq 2$, then
   $$K_{p,q}(X, B; L) = H^1(X, \wedge^{p+1} M_L \otimes B \otimes L^{q-1}) = H^2(X, \wedge^{p+2} M_L \otimes B \otimes L^{q-2}) = \cdots = H^{q-1}(X, \wedge^{p+q-1} M_L \otimes B \otimes L).$$ 

Consequently, $K_{p,q}(X, B; L_d) = 0$ for $p \geq r_d - q$, and $K_{p,q}(X, B; L_d) = 0$ for $q \geq \dim X + 2$.

2. If $q \geq 2$ and $H^{q-1}(X, B) = H^q(X, B) = 0$, then $K_{p,q}(X, B; L) = H^q(X, \wedge^{p+q} M_L \otimes B)$.

3. If $q = 1$, then
   $$K_{p,1}(X, B; L) = \text{coker} \left( \wedge^{p+1} H^0(X, L) \otimes H^0(X, B) \to H^0(X, \wedge^p M_L \otimes B \otimes L) \right).$$

If $H^1(X, B) = 0$, then $K_{p,1}(X, B; L) = H^1(X, \wedge^{p+1} M_L \otimes B)$.

4. If $q = 0$ and $H^0(X, B \otimes L^{-m}) = 0$ for $m > 0$, then $K_{p,0}(X, B; L) = H^0(X, \wedge^p M_L \otimes B)$.

**Proof.** We have a short exact sequence

$$0 \to \wedge^{p+1} M_L \to \wedge^{p+1} H^0(X, L) \otimes \mathcal{O}_X \to \wedge^p M_L \otimes L \to 0.$$ 

Using the Koszul-type complex and chasing through the diagram, we see that

$$K_{p,q}(X, B; L) = \text{coker} \left( \wedge^{p+1} H^0(X, L) \otimes H^0(X, B \otimes L^{q-1}) \to H^0(X, \wedge^p M_L \otimes B \otimes L^q) \right).$$ 

Then the proposition easily follows. See [2, Section 2.1], [7, Section 1], [8, Section 3].

**Proposition 2.2** (cf. [1, Proposition 2.4], [8, Proposition 3.5]). Put $n := \dim X$. Assume that $X$ is smooth, $B$ is a line bundle, and

$$H^i(X, B \otimes L^m) = 0 \text{ for } i > 0 \text{ and } m > 0 \text{ or } i < n \text{ and } m < 0.$$ 

For any $p \geq 0$, the following hold:

1. If $q = n + 1$, then $K_{p,n+1}(X, B; L) = K_{r-p-n,0}(X, \omega_X \otimes B^{-1}; L)^\vee$.

2. If $q = n$, then there is an exact sequence
   $$\wedge^{p+n} H^0(X, L) \otimes H^{n-1}(X, B) \to K_{p,n}(X, B; L) \to K_{r-p-n,1}(X, \omega_X \otimes B^{-1}; L_d)^\vee \to 0.$$ 

If $H^{n-1}(X, B) = 0$, then $K_{p,n}(X, B; L) = K_{r-p-n,1}(X, \omega_X \otimes B^{-1}; L)^\vee$.
(3) If $2 \leq q \leq n - 1$, then there is an exact sequence
\[ \wedge^{p+q}H^0(X, L) \otimes H^{q-1}(X, B) \rightarrow K_{p,q}(X, B; L) \rightarrow K_{r-p-n,n+1-q}(X, \omega_X \otimes B^{-1}; L_d) \rightarrow \wedge^{p+q}H^0(X, L) \otimes H^q(X, B). \]
If $H^{q-1}(X, B) = H^q(X, B) = 0$, then $K_{p,q}(X, B; L) = K_{r-p-n,n+1-q}(X, \omega_X \otimes B^{-1}; L_d)$.

(4) If $q = 1$, then there is an exact sequence
\[ 0 \rightarrow K_{p,1}(X, B; L) \rightarrow K_{r-p-n,n}(X, \omega_X \otimes B^{-1}; L_d) \rightarrow \wedge^{p+1}H^0(X, L) \otimes H^1(X, B) \]
If $H^1(X, B) = 0$, then $K_{p,1}(X, B; L) = K_{r-p-n,n}(X, \omega_X \otimes B^{-1}; L_d)$.

(5) If $q = 0$, then $K_{p,0}(X, B; L) = K_{r-p-n,n+1}(X, \omega_X \otimes B^{-1}; L)$.

Proof. Let $V := H^0(X, L)$. From (2.1), we get an exact sequence
\[ \wedge^{p+q}V \otimes H^{q-1}(X, B) \rightarrow H^{q-1}(X, \wedge^{p+q}M_L \otimes B \otimes L) \rightarrow H^q(X, \wedge^{p+q}M_L \otimes B) \rightarrow \wedge^{p+q}V \otimes H^q(X, B). \]
By Proposition 2.1, $H^{q-1}(X, \wedge^{p+q}M_L \otimes B \otimes L) = K_{p,q}(X, B; L)$ when $q \geq 2$. By Serre duality,
\[ H^q(X, \wedge^{p+q}M_L \otimes B) = H^{n-q}(X, \wedge^{p+q}M_L^\vee \otimes \omega_X \otimes B^{-1}) \wedge. \]
Since $M_L = r$ and $\det M_L = L^{-1}$, it follows that $\wedge^{p+q}M_L^\vee = \wedge^{r-p-q}M_L \otimes L$. Thus
\[ H^q(X, \wedge^{p+q}M_L \otimes B) = H^{n-q}(X, \wedge^{r-p-q}M_L \otimes \omega_X \otimes B^{-1} \otimes L). \]
Using Proposition 2.1, the assertions easily follow. See [2, Section 2.3], [8, Section 3].

In the situation of Proposition 2.2, if we further assume $H^i(X, B) = 0$ for $1 \leq i \leq n - 1$, i.e., $R(X, B; L)$ and $R(X, \omega_X \otimes B^{-1}; L)$ are Cohen–Macaulay, then
\[ K_{p,q}(X, B; L) = K_{r-p-n,n+1-q}(X, \omega_X \otimes B^{-1}; L). \]

Lemma 2.3. If $H^q(X, M_L \otimes \wedge^p M_L \otimes B) = 0$ and $H^q(X, B) = 0$, then $H^q(X, \wedge^{p+1}M_L \otimes B) = 0$.

Proof. Let $V := H^0(X, L)$. Consider the commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \rightarrow & M_L \otimes \wedge^p M_L & \rightarrow & V \otimes \wedge^p M_L & \rightarrow & \wedge^p M_L \otimes L & \rightarrow & 0 \\
0 & \rightarrow & \wedge^{p+1} M_L & \rightarrow & \wedge^{p+1} V \otimes \mathcal{O}_X & \rightarrow & \wedge^p M_L \otimes L & \rightarrow & 0,
\end{array}
\]
which gives rise to the following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
H^{q-1}(X, \wedge^p M_L \otimes B \otimes L) & \rightarrow & H^q(X, M_L \otimes \wedge^p M_L \otimes B) & \rightarrow & V \otimes H^q(X, \wedge^p M_L \otimes B) \\
H^{q-1}(X, \wedge^p M_L \otimes B \otimes L) & \rightarrow & H^q(X, \wedge^{p+1} M_L \otimes B) & \rightarrow & \wedge^{p+1} V \otimes H^q(X, B).
\end{array}
\]
Since $H^q(X, B) = 0$, the middle vertical map is surjective. Thus the lemma follows.

2.2. Castelnuovo–Mumford Regularity. Let $X$ be a projective variety, and $L$ be a very ample line bundle on $X$. A coherent sheaf $\mathcal{F}$ on $X$ is said to be $m$-regular with respect to $L$ if
\[ H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m - i)) = 0 \] for $i > 0$.

By Mumford’s theorem ([17, Theorem 1.8.5]), if $\mathcal{F}$ is $m$-regular with respect to $L$, then $\mathcal{F} \otimes L^{m+\ell}$ is globally generated, the multiplication map
\[ H^0(X, \mathcal{F} \otimes L^m) \otimes H^0(X, L^\ell) \rightarrow H^0(X, \mathcal{F} \otimes L^{m+\ell}) \]
is surjective, and $\mathcal{F}$ is $(m + \ell)$-regular with respect to $L$ for every $\ell \geq 0$. 
Lemma 2.4 (cf. [3, Corollary 3.2]). If $\mathcal{O}_X$ is $k$-regular with $k \geq 1$ and $\mathcal{F}$ is $m$-regular with respect to $L$, then there are finite dimensional vector spaces $W_N, \ldots, W_1, W_0$ over $k$ and a resolution of $\mathcal{F}$ of the form

$$W_N \otimes L^{-m-Nk} \rightarrow \cdots \rightarrow W_1 \otimes L^{-m-k} \rightarrow W_0 \otimes L^{-m} \rightarrow \mathcal{F} \rightarrow 0.$$  

Proof. By [17, Theorem 1.8.5], $\mathcal{F} \otimes L^m$ is globally generated. Letting $W_0 := H^0(X, \mathcal{F} \otimes L^m)$, we have a short exact sequence

$$0 \rightarrow M_0 \rightarrow W_0 \otimes L^{-m} \rightarrow \mathcal{F} \rightarrow 0.$$  

By [17, Theorem 1.8.5], the map

$$W_0 \otimes H^0(X, L^{(m+k)-m-1}) \rightarrow H^0(X, \mathcal{F} \otimes L^{(m+k)-1})$$

is surjective. Note that

$$H^i(X, L^{(m+k)-m-i}) = 0 \text{ for } i \geq 1 \text{ and } H^{i-1}(X, \mathcal{F} \otimes L^{(m+k)-i}) = 0 \text{ for } i \geq 2.$$  

Thus $M_0$ is $(m+k)$-regular with respect to $L$. Replacing $\mathcal{F}$ by $M_0$ and continuing the arguments, we obtain the lemma. \hfill \Box

3. Lifting Syzygies from Hypersurfaces

The aim of this section is showing how to lift syzygies from hypersurfaces (see Theorem 3.1). This is the main ingredient of the proof of Theorem 1.1. We start by setting notations. Let $X$ be a smooth projective variety, $B$ be a line bundle on $X$, and $L$ be a very ample line bundle on $X$. Assume that $n := \dim X \geq 2$. Take a very ample line bundle $H$ on $X$, and suppose that

\begin{align*}
H^i(X, B \otimes L^m) &= 0 \text{ for } i > 0 \text{ and } m > 0 \text{ or } i < n \text{ and } m < 0; \\
H^i(X, B \otimes H \otimes L^m) &= H^i(X, B \otimes H^{-1} \otimes L^m) = 0 \text{ for } 1 \leq i \leq n-1 \text{ and } m \in \mathbb{Z}; \\
H^0(X, B \otimes H \otimes L^m) &= H^0(X, B \otimes L^m) = 0 \text{ for } m < 0; \\
H^n(X, B \otimes H \otimes L^m) &= H^n(X, B \otimes H^{-1} \otimes L^m) = 0 \text{ for } m > 0.
\end{align*}

In particular, $R(X, B \otimes H; L)$ and $R(X, B \otimes H^{-1}; L)$ are Cohen–Macaulay. Choose a general member $\overline{X} \in |H|$, and put

- $\underline{L} := L|_{\overline{X}}$, $\underline{B} := B|_{\overline{X}}$, $\underline{H} := H|_{\overline{X}}$;
- $V := H^0(X, L)$, $V' := H^0(X, L \otimes H^{-1})$, $\overrightarrow{V} := H^0(X, \underline{L})$;
- $r := \dim V - 1$, $v' := \dim V'$, $\overrightarrow{r} := \dim \overrightarrow{V} - 1$

so that $r = v' + \overrightarrow{r}$. Fix a splitting $V = V' \oplus \overrightarrow{V}$. As in [8, Lemma 3.12], we get

$$\wedge^{p+1} M_{\underline{L}}|_{\overline{X}} = \bigoplus_{j=0}^{p+1} \wedge^j V' \otimes \wedge^{p+1-j} M_{\overrightarrow{\underline{L}}} \text{ for } p \geq 0.$$  

By Proposition 2.1, we obtain

- $K_{p,q}(X, \underline{L}; L) = \bigoplus_{j=0}^{p} \wedge^j V' \otimes K_{p-j,q}(\overline{X}, \underline{B}; \underline{L})$ for $p, q \geq 0$;
- $K_{p,q}(X, \overrightarrow{H} \otimes \overrightarrow{L}; \overrightarrow{L}) = \bigoplus_{j=0}^{p} \wedge^j V' \otimes K_{p-j,q}(\overline{X}, \overrightarrow{B} \otimes \overrightarrow{H}; \overrightarrow{L})$ for $p, q \geq 0$. 

\begin{align*}
K_{p,q}(X, \underline{L}; L) &= \bigoplus_{j=0}^{p} \wedge^j V' \otimes K_{p-j,q}(\overline{X}, \underline{B}; \underline{L}) \\
K_{p,q}(X, \overrightarrow{H} \otimes \overrightarrow{L}; \overrightarrow{L}) &= \bigoplus_{j=0}^{p} \wedge^j V' \otimes K_{p-j,q}(\overline{X}, \overrightarrow{B} \otimes \overrightarrow{H}; \overrightarrow{L})
\end{align*}
Now, put $S := \bigoplus_{m \geq 0} S^m V$ and $\overline{S} := \bigoplus_{m \geq 0} \overline{S^m V}$. Consider the following short exact sequence
\begin{equation}
0 \to B \to B \otimes H \to \overline{B} \otimes \overline{H} \to 0.
\end{equation}
By (3.1), we have an exact sequence of finitely generated graded $S$-modules
\begin{equation}
0 \to R(X, B; L) \to R(X, B \otimes H; L) \to R(X, \overline{B} \otimes \overline{H}; L) \to H^1(X, B) \to 0.
\end{equation}
Let $\overline{R}(X, B \otimes \overline{H}; L)$ be the kernel of the map $R(X, \overline{B} \otimes \overline{H}; L) \to H^1(X, B)$, which is a finitely generated graded $\overline{S}$-module. Then we get a short exact sequence of finitely generated graded $S$-modules
\begin{equation}
0 \to \overline{R}(X, B; L) \to R(X, B \otimes H; L) \to \overline{R}(X, B \otimes \overline{H}; L) \to 0.
\end{equation}
By [16, Corollary (1.d.4)], this induces a connecting map
\begin{equation}
\theta_{p,q} : \overline{K}_{p+1,q-1}(X, B \otimes \overline{H}; L) \to K_{p,q}(X, B; L).
\end{equation}
Notice that
\begin{equation}
\overline{K}_{p+1,q-1}(X, B \otimes \overline{H}; L) = \bigoplus_{j=0}^{p+1} \wedge^j V' \otimes K_{p+1-j,q-1}(X, B; L).
\end{equation}
On the other hand, we also have a short exact sequence of finitely generated graded $S$-modules
\begin{equation}
0 \to \overline{R}(X, B; L) \to R(X, B \otimes H; L) \to H^1(X, B) \to 0.
\end{equation}
Since
\begin{equation*}
K_{p,q}(H^1(X, B), V) = \begin{cases}
\wedge^p V \otimes H^1(X, B) & \text{if } q = 0 \\
0 & \text{if } q \geq 1,
\end{cases}
\end{equation*}
we get an exact sequence
\begin{equation}
0 \to \overline{K}_{p+1,0}(X, B \otimes \overline{H}; L) \xrightarrow{\psi_0} K_{p+1,0}(X, B \otimes \overline{H}; L) \xrightarrow{\phi} \wedge^{p+1} V \otimes H^1(X, B) \xrightarrow{\overline{\psi}_1} K_{p,1}(X, B \otimes \overline{H}; L) \xrightarrow{\overline{\psi}_1} K_{p,1}(X, B \otimes \overline{H}; L) \to 0
\end{equation}
and an isomorphism
\begin{equation}
\overline{\psi}_{q-1} : \overline{K}_{p+1,q-1}(X, B \otimes \overline{H}; L) \to K_{p+1,q-1}(X, B \otimes \overline{H}; L) \text{ for } q \geq 3.
\end{equation}
In view of (3.6), there is a map
\begin{equation}
\overline{\psi}_{p+1-j,q-1} : \overline{K}_{p+1-j,q-1}(X, B \otimes \overline{H}; L) \to K_{p+1-j,q-1}(X, B \otimes \overline{H}; L)
\end{equation}
such that
\begin{equation*}
\psi_{q-1} = \bigoplus_{j=0}^{p+1} \text{id} \wedge^j V' \otimes \overline{\psi}_{p+1-j,q-1}.
\end{equation*}
Next, consider the following short exact sequence
\begin{equation}
0 \to B \otimes H^{-1} \to B \to \overline{B} \to 0.
\end{equation}
By (3.1), we have a short exact sequence of finitely generated graded $S$-modules
\begin{equation}
0 \to R(X, B \otimes H^{-1}; L) \to R(X, B; L) \to R(X, \overline{B}; L) \to 0.
\end{equation}
By [16, Corollary (1.d.4)], this induces a restriction map
\begin{equation}
\theta'_{p,q} : K_{p,q}(X, B; L) \to K_{p,q}(X, \overline{B}; L).
\end{equation}
**Theorem 3.1.** Fix an index \( q \geq 1 \). Then we have the following:

1. Suppose that the map \( \theta_{p,q} \) in (3.5) is a nonzero map for \( p = c \) with \( 0 \leq c \leq r - \tau - 1 \). Then the map \( \theta_{p,q} \) is a nonzero map for \( c \leq p \leq r - \tau - 1 \), and consequently,

\[
K_{p,q}(X, B; L) \neq 0 \quad \text{for} \quad c \leq p \leq r - \tau - 1.
\]

2. Suppose that the map \( \theta'_{p,q} \) in (3.10) is a nonzero map for \( p = r - c' \) with \( 0 \leq c' \leq \tau \). Then \( \theta'_{p,q} \) is a nonzero map for \( \tau \leq p \leq r - c' \), and consequently,

\[
K_{p,q}(X, B; L) \neq 0 \quad \text{for} \quad \tau \leq p \leq r - c'.
\]

**Proof.** (1) There is \( \alpha_{c+1} \in K_{c+1,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L) \) such that \( \theta_{c,q}(\alpha_{c+1}) \neq 0 \). We may assume that

\[
\alpha_{c+1} = s'_1 \wedge \cdots \wedge s'_{j_0} \otimes \alpha' \in \wedge^{j_0} V' \otimes K_{c+1-j_0,q-1}(X, \mathcal{B}; L) \subseteq K_{c+1,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L)
\]

for some \( 0 \leq j_0 \leq c + 1 \). We proceed by induction on \( p \). For \( c \leq p - 1 \leq r - \tau - 2 \), we may assume that there is

\[
\alpha_p = s'_1 \wedge \cdots \wedge s'_j \otimes \alpha' \in \wedge^j V' \otimes K_{c+1-j,q-1}(X, \mathcal{B}; L) \subseteq K_{p,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L),
\]

where \( j = p - (c + 1 - j_0) \), such that \( \theta_{p-1,q}(\alpha_p) \neq 0 \). Consider the commutative diagram

\[
\begin{array}{ccc}
K_{p+1,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L) & \xrightarrow{\theta_{p,q}} & K_{p,q}(X, B; L) \\
\downarrow{i} & & \downarrow{i} \\
V \otimes K_{p,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L) & \xrightarrow{\text{id}_V \otimes \theta_{p-1,q}} & V \otimes K_{p-1,q}(X, B; L).
\end{array}
\]

Take any \( s'_{j+1} \in V' \) with \( s'_1 \wedge \cdots \wedge s'_j \wedge s'_{j+1} \neq 0 \), and let

\[
\alpha_{p+1} := s'_1 \wedge \cdots \wedge s'_{j+1} \otimes \alpha' \in \wedge^{j+1} V' \otimes K_{c+1-j_0,q-1}(X, \mathcal{B}; L) \subseteq K_{p+1,q-1}(X, \mathcal{B} \otimes \mathcal{H}; L).
\]

Then

\[
\iota(\alpha_{p+1}) = \sum_{i=1}^{j+1} ((-1)^i s'_i \otimes \alpha'_j \wedge \cdots \wedge \hat{\alpha}'_i \wedge \cdots \wedge s'_{j+1} \otimes \alpha') + \sum_{i=1}^{j+1} s'_1 \wedge \cdots \wedge s'_{j+1} \otimes \iota(\alpha').
\]

Notice that

\[
\text{id}_V \otimes \theta_{p-1,q}(\alpha_p) = (-1)^{j+1} s'_{j+1} \otimes \theta_{p-1,q}(\alpha_p) \neq 0
\]

in \( \langle s'_{j+1} \rangle \otimes K_{p-1,q}(X, B; L) \). Observe then that all other terms in \( \iota(\alpha_{p+1}) \) go into complements of \( \langle s'_{j+1} \rangle \otimes K_{p-1,q}(X, B; L) \) via the map \( \text{id}_V \otimes \theta_{p-1,q} \). Thus

\[
(\iota \circ \theta_{p,q})(\alpha_{p+1}) = ((\text{id}_V \otimes \theta_{p-1,q}) \circ \iota)(\alpha_{p+1}) \neq 0,
\]

and hence, \( \theta_{p,q}(\alpha_{p+1}) \neq 0 \).

(2) There is \( \beta_{r-c'} \in K_{r-c',q}(X, B; L) \) such that \( \theta'_{r-c',q}(\beta_{r-c'}) \) has a nonzero term

\[
s'_1 \wedge \cdots \wedge s'_{j_0} \otimes \beta' \in \wedge^{j_0} V' \otimes K_{r-c-j_0,q}(X, \mathcal{B}; L) \subseteq K_{r-c,q}(X, \mathcal{B}; L)
\]

for some \( r - \tau - c' \leq j_0 \leq r - \tau \). We proceed by reverse induction on \( p \). For \( \tau + 1 \leq p + 1 \leq r - c' \), we may assume that there is \( \beta_{p+1} \in K_{p+1,q}(X, B; L) \) such that \( \theta'_{p+1,q}(\beta_{p+1}) \) has a nonzero term

\[
s'_1 \wedge \cdots \wedge s'_{j+1} \otimes \beta' \in \wedge^{j+1} V' \otimes K_{r-c-j,q}(X, \mathcal{B}; L) \subseteq K_{p+1,q}(X, \mathcal{B}; L),
\]
where \( j + 1 = p + 1 - (r - c' - j_0) \). Consider the commutative diagram

\[
\begin{array}{ccc}
K_{p+1,q}(X, B; L) & \xrightarrow{\theta'_{p+1,q}} & K_{p+1,q}(X, \overline{B}; L) \\
\downarrow{\iota} & & \downarrow{\iota} \\
V \otimes K_{p,q}(X, B; L) & \xrightarrow{\text{id}_V \otimes \theta'_{p,q}} & V \otimes K_{p,q}(X, \overline{B}; L).
\end{array}
\]

We have

\[
\iota(s_1' \land \cdots \land s_{j+1}' \otimes \beta') = \sum_{j=1}^{j+1} (-1)^j s_1' \land \cdots \land s_j' \land s_{j+1}' \otimes \beta' + s_1' \land \cdots \land s_{j+1}' \otimes \iota(\beta').
\]

Note that all terms of \( \theta'_{p+1,q}(\beta_{p+1}) \) not in \( \land^{j+1}V' \otimes (\beta') \) go into complements of \( V' \otimes \land^{j}V' \otimes (\beta') \) via the map \( \iota \). Observe then that the term

\[
(-1)^{j+1} s_{j+1}' \land s_1' \land \cdots \land s_j' \land \beta' \in V' \otimes \land^{j}V' \otimes K_{r-c'-j_0,q}(X, \overline{B}; L) \subseteq V \otimes K_{p,q}(X, \overline{B}; L)
\]

of \( (\iota \circ \theta'_{p+1,q})(\beta_{p+1}) \) cannot be cancelled in \( V' \otimes \land^{j}V' \otimes (\beta') \). Thus there is \( \beta_p \in K_{p,q}(X, B; L) \) such that \( (\text{id}_V \otimes \theta'_{p,q})((-1)^{j+1} s_{j+1}' \land \beta_p) \) has the above term, so \( \theta'_{p,q}(\beta_p) \) has a nonzero term

\[
s_1' \land \cdots \land s_j' \land \beta' \in \land^{j}V' \otimes K_{r-c'-j_0,q}(X, \overline{B}; L) \subseteq K_{p,q}(X, \overline{B}; L).
\]

We complete the proof. \( \square \)

4. Precise Asymptotic Nonvanishing Theorem

After establishing key steps as propositions, we finish the proof of Theorem 1.1 at the end of this section. We start by setting notations. Let \( X \) be a smooth projective variety of dimension \( n \geq 1 \), and \( B \) be a line bundle on \( X \). For an integer \( d \geq 1 \), set

\[
L_d := \mathcal{O}_X(dA + P) \quad \text{and} \quad r_d := h^0(X, L_d) - 1,
\]

where \( A \) is an ample divisor and \( P \) is an arbitrary divisor on \( X \). We assume throughout that \( d \) is sufficiently large so that \( L_d \) is a sufficiently positive very ample line bundle and \( r_d = \Theta(d^m) \). Furthermore, we have

\[
H^i(X, B \otimes L_d^m) = 0 \quad \text{for} \quad i > 0 \quad \text{and} \quad m > 0 \quad \text{or} \quad i < n \quad \text{and} \quad m < 0.
\]

For \( 1 \leq q \leq n \), let \( c_q(d) \) be the number such that

\[
K_{c_q(d),q}(X, B; L_d) \neq 0 \quad \text{and} \quad K_{p,q}(X, B; L_d) = 0 \quad \text{for} \quad 0 \leq p \leq c_q(d) - 1,
\]

and \( c'_q(d) \) be the number such that

\[
K_{r_d-c'_q(d),q}(X, B; L_d) \neq 0 \quad \text{and} \quad K_{p,q}(X, B; L_d) = 0 \quad \text{for} \quad r_d - c'_q(d) + 1 \leq p \leq r_d.
\]

If \( K_{p,q}(X, B; L_d) = 0 \) for all \( p \), then we set \( c_q(d) := r_d + 1 \) and \( c'_q(d) := r_d + 1 \). We will see in Proposition 4.4 that this cannot happen. Recall from [19, Theorem 1.1] that \( c_q(x) \geq \Theta(d^{q-1}) \).

**Lemma 4.1.** Assume that \( H^i(X, B) = 0 \) for \( 1 \leq i \leq n - 1 \), i.e., \( R(X, B; L_d) \) is Cohen–Macaulay. Fix an index \( 1 \leq q \leq n \). Then we have the following:

1. If \( K_{p,q}(X, B; L_d) = 0 \) for some \( p_0 \leq \Theta(d^{q-1}) \), then \( K_{p,q}(X, B; L_d) = 0 \) for \( p \leq p_0 \).
2. If \( K_{p,q}(X, B; L_d) = 0 \) for some \( p_0 \geq r_d - \Theta(d^{n-q}) \), then \( K_{p,q}(X, B; L_d) = 0 \) for \( p \geq p_0 \).
Proof. By Proposition 2.2, [8, Proposition 5.1], and [19, Theorem 1.1], if \(0 \leq q' \leq q - 1\), then
\[
K_{p,q'}(X, B; L_d) = K_{r_d-p-n,n+1-q'}(X, \omega_X \otimes B^{-1}; L_d)^{\vee} = 0 \quad \text{for } p \geq r_d - \Theta(d^{m-q'}). 
\]
Thus \(K_{p,q}(X, B; L_d) = 0\) for \(0 \leq q' \leq q\), so the assertion (2) follows. Now, by Proposition 2.2, the assertion (2) implies the assertion (1).

Remark 4.2. If \(R(X, B; L_d)\) is Cohen–Macaulay, then [8, Theorem 4.1], [19, Theorem 1.1], and Lemma 4.1 imply Theorem 1.1. In general, we know from [8, Theorem 4.1] and [19, Theorem 1.1] that \(c_q(d) = \Theta(d^{q-1})\). Using Boij–Söderberg theory [4, 13], one can show that vanishing and nonvanishing of \(K_{p,q}(X, B; L_d)\) do not alternate for a while after \(c_q(d)\). This means that
\[
K_{p,q}(X, B; L_d) \neq 0 \quad \text{for } c_q(d) \leq p \leq r_d - \Theta(d^{n-1}).
\]
However, we will not use this remark in our proof of Theorem 1.1.

Lemma 4.3. Theorem 1.1 holds when \(n = 1\).

Proof. When \(n = 1\), [8, Proposition 5.1 and Corollary 5.2] imply
\[
K_{p,1}(X, B; L_d) \neq 0 \quad \text{for } h^0(B) \leq p \leq r_d - h^0(X, \omega_X \otimes B^{-1}) - 1.
\]
This shows that \(c_1(d) = \Theta(1)\) and \(c'_1(d) = \Theta(1)\). Then Lemma 4.1 implies the lemma.

Assume henceforth that \(n = \dim X \geq 2\). Let \(H\) be a very ample line bundle (independent of \(d\)) on \(X\) such that
\[
H^i(X, B \otimes H) = H^i(X, B \otimes H^{-1}) = 0 \quad \text{for } 1 \leq i \leq n - 1.
\]
As \(d\) is sufficiently large, we have
\[
H^i(X, B \otimes H \otimes L^m_d) = H^i(X, B \otimes H^{-1} \otimes L^m_d) = 0 \quad \text{for } 1 \leq i \leq n - 1 \text{ and } m \in \mathbb{Z}.
\]
This means that \(R(X, B \otimes H; L_d)\) and \(R(X, B \otimes H^{-1}; L_d)\) are Cohen–Macaulay. Clearly,
\[
H^0(X, B \otimes H \otimes L^m_d) = H^0(X, B \otimes H^{-1} \otimes L^m_d) = 0 \quad \text{for } m < 0;
\]
\[
H^n(X, B \otimes H \otimes L^m_d) = H^n(X, B \otimes H^{-1} \otimes L^m_d) = 0 \quad \text{for } m > 0;
\]
Choose a general member \(\overline{X} \in |H|\), and put
\[
\overline{L}_d := L_d|_{\overline{X}}, \quad \overline{B} := B|_{\overline{X}}, \quad \overline{H} := H|_{\overline{X}};
\]
\[
V_d := H^0(X, L_d), \quad V'_d := H^0(X, L_d \otimes H^{-1}), \quad V_d := H^0(X, L_d);
\]
\[
r_d := \dim V_d - 1, \quad V'_d := \dim V'_d, \quad \tau_d := \dim V_d - 1
\]
such that \(r_d = v'_d + \tau_d\) and \(\tau_d = \Theta(d^{m-1})\). As in the previous section, fix a splitting \(V_d = V'_d \oplus V_d\). Then (3.2) and (3.3) hold. Furthermore, the short exact sequences (3.4) and (3.9) induce the map \(\theta_{p,q}\) in (3.5) and \(\theta'_{p,q}\) in (3.10), respectively. In view of [16, Corollary (1.d.4)], they fit into the following exact sequences
\[
(3.3a) \quad K_{p+1,q-1}(X, B \otimes H; L_d) \rightarrow \overline{K}_{p+1,q-1}(X, \overline{B} \otimes \overline{H}; L_d) \xrightarrow{\theta_{p,q}} K_{p,q}(X, B; L_d) \rightarrow K_{p,q}(X, B \otimes H; L_d);
\]
\[
(3.3b) \quad K_{p,q}(X, B \otimes H^{-1}; L_d) \rightarrow K_{p,q}(X, B; L_d) \xrightarrow{\theta'_{p,q}} K_{p,q}(X, \overline{B}; L_d) \rightarrow K_{p,q+1}(X, B \otimes H^{-1}; L_d).
\]

Proposition 4.4. For each \(1 \leq q \leq n\), we have
\[
c_q(d) = \Theta(d^{q-1}) \quad \text{and} \quad c'_q(d) = \begin{cases} 
\Theta(d^{n-q}) & \text{if } H^{q-1}(X, B) = 0 \text{ or } q = 1 \\
q - 1 & \text{if } H^{q-1}(X, B) \neq 0 \text{ and } q \geq 2.
\end{cases}
\]
By induction, $H^q(X, B) = 0$ for $q \geq 2$. Then

$$K_{r_d-q+1}(X, B; L_d) = H^q(X, \wedge d M_L \otimes B \otimes L_d) = H^q(X, B) \neq 0.$$ 

Since $K_{p,q}(X, B; L_d) = 0$ for $p \geq r_d - q$, it follows that $c'(d) = q - 1$. Suppose that $H^q(X, B) = 0$ when $q \geq 2$ or $q = 1$. By Proposition 2.2 and [19, Theorem 1.1], $K_{p,q}(X, B; L) \subseteq K_{r_d-p-n,n+1-q}(X, \omega X \otimes B^{-1}; L_d) \cap 0$ for $0 \leq r_d - p - n \leq \Theta(d^{n-q})$, so $c'(d) \geq \Theta(d^{n-q})$. For $2 \leq q \leq n$, let $\tau_{q-1}(d)$ be the number such that

$$K_{r_d-\tau_{q-1}(d),q-1}(X, B \otimes H; L_d) = K_{r_d-\tau_{q-1}(d),q-1}(X, B \otimes H; L_d) \neq 0$$

for $2 \leq q \leq n$. Possibly replacing $H$ by more positive $H$ (still independent of $d$), we may assume that

(4.4) $\theta_0(X, B) > \theta_0(X, B)$. 

Then $K_{r_d-\tau_{q-1}(d),q-1}(X, B \otimes H; L_d) \neq 0$. Putting $\tau_{q-1}(d) = \theta_0 = \Theta(d^{n-1})$, we get from (3.6) that

$$K_{r_d-\tau_{q-1}(d),q-1}(X, B \otimes H; L_d) \neq 0.$$ 

For $1 \leq q \leq n$, thanks to (4.1), Proposition 2.2 and [19, Theorem 1.1] yield that

$$K_{r_d-\tau_{q-1}(d),q-1}(X, B \otimes H; L_d) = K_{r_d-\tau_{q-1}(d)-n,n+2-q}(X, \omega X \otimes B^{-1} \otimes H^{-1}; L_d) \cap 0 = 0,$$

since $\tau_{q-1}(d) - n = \Theta(d^{n-q}) < \Theta(d^{n+1-q})$. Then the map

$$\theta_{r_d-\tau_{q-1}(d)-1} : K_{r_d-\tau_{q-1}(d)-1}(X, B \otimes H; L_d) \rightarrow K_{r_d-\tau_{q-1}(d)-1, q}(X, B; L_d)$$

in (3.3a) is a nonzero injective map. Thus $c'(d) = \tau_{q-1}(d) + 1 = \Theta(d^{n-q})$, so $c'(d) = \Theta(d^{n-q})$.

Next, we consider $c_q(d)$. We know from [19, Theorem 1.1] that $c_q(d) \geq \Theta(d^{q-1})$. When $q = n$, Proposition 2.2 says that there is a surjective map

$$K_{p,n}(X, B; L_d) \rightarrow K_{r_d-p-n,1}(X, \omega X \otimes B^{-1}; L_d) \cap 0.$$ 

As we have seen in the previous paragraph that $K_{r_d-p-n,1}(X, \omega X \otimes B^{-1}; L_d) \cap 0 \neq 0$ for some $p = \Theta(d^{n-1})$, we have $c_n(d) \leq \Theta(d^{n-1})$. Hence $c_n(d) = \Theta(d^{n-1})$. Assume that $1 \leq q \leq n - 1$. Let $\tau_q(d)$ be the number such that

$$K_{\tau_q(d),q}(X, B; L_d) \neq 0$$

by induction, $\tau_q(d) = \Theta(d^{q-1})$. Note that

$$K_{\tau_q(d),q}(X, B; L_d) \neq 0$$

thanks to (3.3). By [19, Theorem 1.1],

$$K_{\tau_q(d)-1,q+1}(X, B \otimes H^{-1}; L_d) = 0$$

since $\tau_q(d) - 1 = \Theta(d^{q-1}) < \Theta(d^q)$. Then the map

$$\theta_{\tau_q(d),q} : K_{\tau_q(d),q}(X, B; L_d) \rightarrow K_{\tau_q(d),q}(X, B; L_d)$$

in (3.3b) is a nonzero surjective map. Thus $c_q(d) \leq \tau_q(d) = \Theta(d^{q-1})$, so $c_q(d) = \Theta(d^{q-1})$. 

\[\Box\]
Next, we prove the following technical lemma.

**Lemma 4.5.** Let $B'$ be a line bundle on $X$ (independent of $d$), and $L$ be a very ample line bundle on $X$ (independent of $d$) such that

$$
H^i(X, L^m) = 0 \quad \text{for } i > 0 \text{ and } m > 0 \text{ or } i < n \text{ and } m < 0;
$$

$$
H^i(X, B' \otimes L^m) = 0 \quad \text{for } i > 0 \text{ and } m > 0 \text{ or } i < n \text{ and } m < 0;
$$

$$
H^i(X, M_{L_d} \otimes L^m) = 0 \quad \text{for } i > 0 \text{ and } m > 0.
$$

Put $H := L^{n+1}$. For $1 \leq p \leq \Theta(d^{q-1})$ and $1 \leq q \leq n+1$, we have the following:

(1) If $q \geq 2$ and $H^{q-1}(X, \wedge^p M_{L_d} \otimes B' \otimes L_d) = 0$, then $H^{q-1}(X, \wedge^{p+1} M_{L_d} \otimes B' \otimes H \otimes L_d) = 0$.

(2) If $q \leq n$ and $H^q(X, \wedge^p M_{L_d} \otimes B') = 0$, then $H^q(X, \wedge^{p+1} M_{L_d} \otimes B' \otimes H) = 0$.

**Proof.** Notice that $\mathcal{O}_X$ and $M_{L_d}$ are $(n+1)$-regular with respect to $L$. By Lemma 2.4, there are finitely dimensional vector spaces $\ldots, W_1, W_0$ over $k$ and an exact sequence

$$
\cdots \rightarrow W_1 \otimes H^{-1} \rightarrow W_0 \otimes H^{-1} \rightarrow M_{L_d} \rightarrow 0.
$$

(1) By Lemma 2.3, it is sufficient to prove that

$$
H^{q-1}(X, M_{L_d} \otimes \wedge^p M_{L_d} \otimes B' \otimes H \otimes L_d) = 0.
$$

For this purpose, consider the exact sequence from (4.5):

$$
\cdots \rightarrow W_1 \otimes \wedge^p M_{L_d} \otimes B' \otimes H^{-1} \otimes L_d \rightarrow W_0 \otimes \wedge^p M_{L_d} \otimes B' \otimes L_d
$$

$$
\quad \quad \quad \quad \quad \quad \rightarrow M_{L_d} \otimes \wedge^p M_{L_d} \otimes B' \otimes H \otimes L_d \rightarrow 0.
$$

In view of [17, Proposition B.1.2], it suffices to show that

$$
H^{q-1+i}(X, \wedge^p M_{L_d} \otimes B' \otimes H^{-i} \otimes L_d) = K_{p-q-i+1,q+i}(X, B' \otimes H^{-i}; L_d). = 0 \quad \text{for } i \geq 0.
$$

When $i = 0$, this is the given condition. When $i \geq 1$, this follows from [19, Theorem 1.1] since $p - q - i + 1 \leq \Theta(d^{q-1}) < \Theta(d^{q+i-1})$.

(2) By Lemma 2.3, it is sufficient to prove that

$$
H^q(X, M_{L_d} \otimes \wedge^p M_{L_d} \otimes B' \otimes H) = 0.
$$

Let $M_0$ be the kernel of the map $W_0 \otimes H^{-1} \rightarrow M_{L_d}$ in (4.5). We have a short exact sequence

$$
0 \rightarrow M_0 \otimes B' \otimes H \rightarrow W_0 \otimes B' \rightarrow M_{L_d} \otimes B' \otimes H \rightarrow 0.
$$

As $H^q(X, \wedge^p M_{L_d} \otimes B') = 0$, the claim (4.6) is implied by the injectivity of the map

$$
\rho: H^{q+1}(X, \wedge^p M_{L_d} \otimes B' \otimes H) \rightarrow W_0 \otimes H^{q+1}(X, \wedge^p M_{L_d} \otimes B').
$$

This map fits into the following commutative diagram

$$
\begin{array}{c}
H^q(X, \wedge^p M_{L_d} \otimes M_0 \otimes B' \otimes H \otimes L_d) \rightarrow W_0 \otimes H^q(X, \wedge^p M_{L_d} \otimes B' \otimes L_d) \\
\downarrow \\
H^{q+1}(X, \wedge^p M_{L_d} \otimes M_0 \otimes B' \otimes H) \rightarrow W_0 \otimes H^{q+1}(X, \wedge^p M_{L_d} \otimes B') \\
\downarrow \psi \\
\wedge^p V_d \otimes H^{q+1}(X, M_0 \otimes B' \otimes H) \rightarrow W_0 \otimes \wedge^p V_d \otimes H^{q+1}(X, B').
\end{array}
$$

For the injectivity of the map $\rho$, it is enough to check that $\psi$ and $\varphi$ are injective. From (4.5), we have an exact sequence

$$
\cdots \rightarrow W_2 \otimes H^{-3} \rightarrow W_1 \otimes H^{-2} \rightarrow M_0 \rightarrow 0.
$$
By \[19, \text{Theorem 1.1},\]
\[H^q(X, \wedge^p M_{L_d} \otimes B' \otimes H^{-i-1} \otimes L_d) = K_{p-q-i, q+i+1}(X, B' \otimes H^{-i-1}; L_d) = 0 \quad \text{for } i \geq 0 \]
since \(p - q - i \leq \Theta(d^{q-1}) < \Theta(d^{q+i})\). By \[17, \text{Proposition B.1.2},\]
\[H^q(X, \wedge^p M_{L_d} \otimes M_0 \otimes B' \otimes H \otimes L_d) = 0,\]
so \(\psi\) is injective. On the other hand, we get from (4.7) that
\[H^{n+2-q}(X, M_0 \otimes B' \otimes H) = W_0 \otimes H^{n+2-q}(X, B'),\]
so \(\varphi\) is an isomorphism.

Now, take a very ample line bundle \(L\) on \(X\) (independent of \(d\)) such that

(4.8a) \[H^i(X, L^m) = 0 \quad \text{for } i > 0 \quad \text{and} \quad m > 0 \quad \text{or} \quad i < n \quad \text{and} \quad m < 0;\]
(4.8b) \[H^i(X, B \otimes L^m) = 0 \quad \text{for } i > 0 \quad \text{and} \quad m > 0 \quad \text{or} \quad i < n \quad \text{and} \quad m < 0;\]
(4.8c) \[H^i(X, M_{L_d} \otimes L^m) = 0 \quad \text{for } i > 0 \quad \text{and} \quad m > 0;\]
(4.8d) \[H^i(X, M_{L_d} \otimes \omega_X \otimes B^{-1} \otimes L^m) = 0 \quad \text{for } i > 0 \quad \text{and} \quad m > 0;\]

By Proposition 4.4, we can take an integer \(c \geq c_1(d) + 1, c'_n(d) - n + 1\) independent of \(d\). Successively applying Lemma 4.5 and possibly replacing \(L\) by a higher power of \(L\) (still independent of \(d\)), we may assume that

\[K_{c-1, 1}(X, B \otimes L^{n+1}; L_d) = H^1(X, \wedge^c M_{L_d} \otimes B \otimes L^{n+1}) = 0;\]
\[K_{c-1, 1}(X, \omega_X \otimes B^{-1} \otimes L^{n+1}; L_d) = H^1(X, \wedge^c M_{L_d} \otimes \omega_X \otimes B^{-1} \otimes L^{n+1}) = 0.\]

By Lemma 4.1, we have

(4.9a) \[K_{c_1(d), 1}(X, B \otimes L^{n+1}; L_d) = H^1(X, \wedge^{c_1(d)+1} M_{L_d} \otimes B \otimes L^{n+1}) = 0;\]
(4.9b) \[K_{c'_n(d)-n, 1}(X, \omega_X \otimes B^{-1} \otimes L^{n+1}; L_d) = H^1(X, \wedge^{c'_n(d)-n+1} M_{L_d} \otimes \omega_X \otimes B^{-1} \otimes L^{n+1}) = 0.\]

From now on, replace \(H\) by \(H := L^{n+1}\). Then (4.1) holds thanks to (4.8b), so \(R(X, B \otimes H; L_d)\) and \(R(X, B \otimes H^{-1}; L_d)\) are Cohen–Macaulay. Clearly, (4.2) is satisfied.

**Proposition 4.6.** Assume that \(n \geq 2\). For \(1 \leq q \leq n\), we have the following:

1. If \(K_{p,q}(X, B; L_d) = 0\) for \(p < c_q(d)\), then \(K_{p+1,q}(X, B \otimes H; L_d) = 0\). Consequently, the map

\[\theta_{c_q(d), q} : \overline{K}_{c_q(d)+1,q-1}(X, \overline{B} \otimes \overline{P}; L_d) \rightarrow K_{c_q(d), q}(X, B; L_d)\]

in (4.3a) is a nonzero surjective map.

2. If \(K_{p,q}(X, B; L_d) = 0\) for \(p > r_d - c'_q(d)\), then \(K_{p-1,q}(X, B \otimes H^{-1}; L_d) = 0\). Consequently, the map

\[\theta'_{r_d-c'_q(d), q} : K_{r_d-c'_q(d), q}(X, B; L_d) \rightarrow K_{r_d-c'_q(d), q}(X, \overline{B}; L_d)\]

in (4.3b) is a nonzero injective map.

**Proof.** (1) Recall that \(R(X, B \otimes H; L_d)\) is Cohen–Macaulay. Then Lemma 4.1 says that

\[K_{c_q(d), q}(X, B \otimes H; L_d) = 0 \implies K_{p+1,q}(X, B \otimes H; L_d) = 0 \quad \text{for } p \leq c_q(d) - 1.\]

Thus it suffices to show that \(K_{c_q(d), q}(X, B \otimes H; L_d) = 0\). The case \(q = 1\) is nothing but (4.9a). Assume that \(2 \leq q \leq n\). Note that the given condition is

\[H^{q-1}(X, \wedge^{c_q(d)+q-2} M_{L_d} \otimes B \otimes L_d) = K_{c_q(d)-1,q}(X, B; L_d) = 0.\]
Then Lemma 4.5 (1) yields
\[ K_{c_0(d),q}(X, B \otimes H; L_d) = H^{q-1}(X, \wedge^{c_0(d)}M_{L_d} \otimes B \otimes H \otimes L_d) = 0. \]

(2) Recall that \( R(X, B \otimes H^{-1}; L_d) \) is Cohen–Macaulay. Then Lemma 4.1 says that
\[ K_{r_d-c'_0(d),q}(X, B \otimes H^{-1}; L_d) = 0 \implies K_{p-1,q}(X, B \otimes H^{-1}; L_d) = 0 \]
for \( p \geq r_d - c'_0(d) + 1 \).

Thus it suffices to show that \( K_{r_d-c'_0(d),q}(X, B \otimes H^{-1}; L_d) = 0 \). By Proposition 2.2,
\[ K_{r_d-c'_0(d),q}(X, B \otimes H^{-1}; L_d) = K_{c'_0(d)-n,n+1-q}(X, \omega_X \otimes B^{-1} \otimes H; L_d)'. \]

We need to show that
\[ H^{n+1-q}(X, \wedge^{c'_0(d)-q}M_{L_d} \otimes \omega_X \otimes B^{-1} \otimes H) = 0. \]

When \( q = n \), (4.10) is the same to (4.9b). Assume that \( 1 \leq q \leq n - 1 \). If \( q \geq 2 \) and \( H^{q-1}(X, B) \neq 0 \), then \( c'_0(d) = q - 1 \) so that (4.10) holds by (4.9b). Assume \( H^{q-1}(X, B) = 0 \) when \( q \geq 2 \). Then \( c'_0(d) = \Theta(d^{n-q}) \). The given condition and Serre duality yield
\[ H^{n+1-q}(X, \wedge^{c'_0(d)-q}M_{L_d} \otimes \omega_X \otimes B^{-1} - 1) = H^{n+1-q}(X, \wedge^{c'_0(d)-q}M_{L_d} \otimes \omega_X \otimes B^{-1} \otimes L_d') = H^{q-1}(X, \wedge^{c'_0(d)-q}M_{L_d} \otimes B \otimes L_d)'. \]

Then the claim (4.10) follows from Lemma 4.5 (2). \( \square \)

Theorem 1.1 now follows at once from the previous propositions and Theorem 3.1.

Proof of Theorem 1.1. By Lemma 4.3, we may assume that \( n \geq 2 \). Note that (1.1) is proved in Proposition 4.4. For \( 1 \leq q \leq n \), by Theorem 3.1 (1) and Proposition 4.6 (1),
\[ K_{p,q}(X, B; L_d) \neq 0 \text{ for } c_0(d) \leq p \leq r_d - \tau_d - 1. \]

On the other hand, [8, Proposition 5.1] says \( K_{p,0}(X, B; L_d) = 0 \) for \( p > \Theta(1) \). Thus we obtain (1.2) for \( q = 1 \). For \( 2 \leq q \leq n \), by Theorem 3.1 (2) and Proposition 4.6 (2),
\[ K_{p,q}(X, B; L_d) \neq 0 \text{ for } \tau_d \leq p \leq r_d - c'_0(d). \]

As \( r_d - \tau_d - 1 = \Theta(d^n) > \Theta(d^{n-1}) = \tau_d \), we obtain (1.2) for \( 2 \leq q \leq n \). \( \square \)

5. Complements and Problems

In this section, we show some additional results, and discuss some open problems. Recall that the asymptotic vanishing theorem ([19, Theorem 1.1]) holds for singular varieties with coherent sheaves. Precisely, let \( X \) be a projective variety of dimension \( n \), and \( B \) be a coherent sheaf on \( X \). For an integer \( d \geq 1 \), let \( L_d := \Theta_{X}(dA + P) \), where \( A \) is an ample divisor and \( P \) is an arbitrary divisor on \( X \). For each \( 1 \leq q \leq n + 1 \), if \( d \) is sufficiently large, then
\[ K_{p,q}(X, B; L_d) = 0 \text{ for } 0 \leq p \leq \Theta(d^{n-1}). \]

We expect that Theorem 1.1 also holds in this setting.

Conjecture 5.1. Theorem 1.1 still holds when \( X \) is a projective variety and \( B \) is a coherent sheaf on \( X \) with \( \text{Supp } B = X \).

Note that the expected nonvanishing of \( K_{p,q}(X, B; L_d) \) for \( q > \dim \text{Supp } B + 1 \) may not hold.

Remark 5.2. In the proof of Theorem 1.1, we use the assumption that \( X \) is smooth and \( B \) is a line bundle only when we apply Serre duality. Thus Theorem 1.1 holds when \( X \) is Cohen–Macaulay and \( B \) is a vector bundle.
From now on, we assume that $X$ is smooth and $B$ is a line bundle as in Theorem 1.1. In the remaining, fix an index $1 \leq q \leq n$. It is very natural to study the asymptotic growth of $c_q(d)$ and $c'_q(d)$ as $d \to \infty$. In the spirit of [23], we give an effective upper bound for each of $c_q(d)$ and $c'_q(d)$. For this purpose, we introduce some notations. Choose suitably positive very ample divisors $H_1, \ldots, H_{n-1}$ on $X$ such that
\[ \overline{X}_i := H_1 \cap \cdots \cap H_i \]
is a smooth projective variety for every $0 \leq i \leq n - 1$. Note that $\overline{X}_0 = X$. For each $0 \leq i \leq n - 1$, put
\[ \overline{\mathcal{I}}_i := \mathcal{O}_X(H_{i+1})|_{\overline{X}_i}, \quad \overline{\mathcal{I}}_i := B|_{\overline{X}_i}, \quad \overline{\mathcal{I}}_i := B(H_1 + \cdots + H_i)|_{\overline{X}_i}; \]
\[ \overline{L}_d := L_d|_{\overline{X}_i}, \quad \overline{\pi}(d) := h^0(\overline{X}_i, \overline{L}_d) - 1 = \Theta(d^{n-i}), \quad \overline{\pi}_n(d) := 0, \]
and assume that (4.1), (4.2), (4.4) hold for
\[ X = \overline{X}_i, \quad B = \overline{\mathcal{I}}_i, \quad \overline{\pi}(d) := h^0(\overline{X}_i, \overline{L}_d) - 1 = \Theta(d^{n-i}), \quad \overline{\pi}_n(d) := 0. \]

**Proposition 5.3.** $c_q(d) \leq \overline{\pi}_{n+1-q}(d) - q + 1$ and $c'_q(d) \leq \overline{\pi}_q(d) + q$.

**Proof.** We proceed by induction on $n = \dim X$. When $n = 1$, the assertion is trivial. Assume that $n \geq 2$. For $c'_q(d)$, we may assume that $H^{q-1}(X, B) = 0$ or $q = 1$. In the proof of Proposition 4.4, we proved that
\[ \theta_{r_d-q}(d-q): \overline{K}_{r_d-q}(d-q-1)(X, \overline{B} \otimes \overline{\mathcal{I}}_d; L_d) \to K_{r_d-q}(d-q)(X, B; L_d) \]
is nonzero, so $c'_q(d) \leq \overline{\pi}_q(d) + q$. We also have $K_{r_d-q}(d-1, 1)(X, \omega_X \otimes B^{-1}; L_d) \neq 0$. By Proposition 2.2, $K_{r_1(d)-n+1, 1}(X, B; L_d) \neq 0$. Thus we obtain $c_n(d) \leq \overline{\pi}_1(d) - n + 1$. For $1 \leq q \leq n - 1$, in the proof of Proposition 4.4, we proved that
\[ \theta_{r_{n+1-q}(d)-q+1}(d-q+1)(X, B; L_d) \to K_{r_{n+1-q}(d)-q+1}(X, B; L_d) \]
is nonzero, so $c_q(d) \leq \overline{\pi}_{n+1-q}(d) - q + 1$. \hfill \Box

**Remark 5.4.** In Proposition 5.3, we do not assume that $R(X, B; L_d)$ is Cohen–Macaulay. However, when $R(X, B; L_d)$ is Cohen–Macaulay, by a more careful analysis, one can improve bounds for $c_q(d)$ and $c'_q(d)$ as in [23]. In particular, one can recover [8, Theorem 6.1]: If $X = \mathbb{P}^n, \ B = \mathcal{O}_{\mathbb{P}^n}(b), L_d = \mathcal{O}_{\mathbb{P}^n}(d)$ and $b \geq 0, d \gg 0$, then
\[ c_q(d) \leq \frac{d + q}{q} - \frac{d - b - 1}{q} - q \quad \text{and} \quad c'_q(d) \leq \frac{d + n - q}{n - q} - \frac{n + b}{q + b} + q. \]
We leave the details to interested readers.

In characteristic zero, David Yang [22, Theorem 1] confirmed that $c_1(d)$ is a constant. This gives an answer to [8, Problem 7.2]. On the other hand, for Veronese syzygies, Ein–Lazarsfeld [10, Conjecture 2.3] conjectured that equalities hold in (5.1) whenever $d \geq b + q + 1$. In particular, $c_q(d)$ and $c'_q(d)$ are polynomials. One may hope that the same is true in general.

**Question 5.5 (cf. [6, Remark 3.2]).** (1) Does the limit
\[ \lim_{d \to \infty} \frac{c_q(d)}{d^{q-1}} \]
exist? If so, is the function $c_q(d)$ a polynomial of degree $q - 1$ for sufficiently large $d$? What can one say about the leading coefficient $a_{q-1} := \lim_{d \to \infty} c_q(d)/d^{q-1}$ of $c_q(d)$?
(2) Suppose that $H^{q-1}(X, B) = 0$ if $q \geq 2$. Does the limit
\[
\lim_{d \to \infty} \frac{c'_q(d)}{d^{n-q}}
\]
exist? If so, is the function $c'_q(d)$ a polynomial of degree $n - q$ for sufficiently large $d$? What can one say about the leading coefficient $a'_{n-q} := \lim_{d \to \infty} c'_q(d)/d^{n-q}$ of $c'_q(d)$?

When $\text{char}(k) = 0$, a geometric meaning of the constant $c_1(d)$ was explored as follows. Ein–Lazarsfeld–Yang [11, Theorem A] proved that if $B$ is $p$-jet very ample, then $c_1(d) \geq p + 1$. Agostini [1, Theorem A] proved that if $c_1(d) \geq p + 1$, then $B$ is $p$-very ample. These results are higher dimensional generalizations of the gonality conjecture on syzygies of algebraic curves, which was established by Ein–Lazarsfeld [9] and Rathmann [21]. On the other hand, Eisenbud–Green–Hulek–Popescu [12] related the nonvanishing of $K_{p,q}(X, L_d)$ to the existence of special secant planes. In particular, if the property $N_k$ holds for $L_d$, then $L_d$ is $(k+1)$-very ample. It would be exceedingly interesting to know whether the nonexistence of special secant planes implies the vanishing of certain $K_{p,q}(X, L_d)$.

When $X$ is a smooth projective curve and $\text{char}(k) = 0$, Rathmann [21, Theorem 1.2] showed that if $H^1(X, L_d) = 0$ and $H^1(X, B^{-1} \otimes L_d) = 0$, then $K_{p,1}(X, B; L_d) = 0$. It is natural to extend this effective result to higher dimensions. When $B = \mathcal{O}_X$ and $L_d = \mathcal{O}_X(K_X + dA)$, the following problem is closely related to Mukai’s conjecture [7, Conjecture 4.2].

**Problem 5.6.** (1) Suppose that $c_q(d)$ is a polynomial of degree $q - 1$ for sufficiently large $d$. Find an effective bound for $d_0$ such that $c_q(d)$ becomes a polynomial for $d \geq d_0$.

(2) Suppose that $H^{q-1}(X, B) = 0$ if $q \geq 2$ and $c'_q(d)$ is a polynomial of degree $n-q$ for sufficiently large $d$. Find an effective bound for $d'_0$ such that $c'_q(d)$ becomes a polynomial for $d \geq d'_0$.

Now, we turn to the asymptotic behaviors of the Betti numbers
\[
\kappa_{p,q}(X, B; L_d) := \dim K_{p,q}(X, B; L_d).
\]
Ein–Erman–Lazarsfeld conjectured that the Betti numbers $\kappa_{p,q}(X, B; L_d)$ are normally distributed [5, Conjecture B], and they verified the conjecture for curves [5, Proposition A]. The normal distribution conjecture suggests the following unimodality conjecture.

**Conjecture 5.7.** The Betti numbers $\kappa_{p,q}(X, B; L_d)$ form a unimodal sequence.

As the cases of very small $p$ and very large $p$ for $\kappa_{p,q}(X, B; L_d)$ are negligible in the normal distribution conjecture, the unimodality conjecture is not a consequence of the normal distribution conjecture. Finally, we verify the unimodality conjecture for curves following the strategy of Erman [15] based on Boij–Söderberg theory. Eisenbud–Schreyer [13] and Boij–Söderberg [4] showed that the Betti table of a graded module over a polynomial ring is a positive rational sum of pure diagrams (see [14, Theorem 2.2] for the precise statement). Let $C$ be a smooth projective curve, $B$ be a line bundle, and $L$ be a very ample line bundle of sufficiently large degree $d$. Put $c := h^0(X, B)$ and $c' := h^1(X, B)$. By [8, Proposition 5.1 and Corollary 5.2], $\kappa_{p,0}(C, B; L) = 0$ for $p \geq c$ and $\kappa_{p,2}(C, B; L) = 0$ for $p \leq r - 1 - c'$, where $r := h^0(X, L) - 1$. By Riemann–Roch theorem, $r = d - g \approx d$ since $d$ is sufficiently large. Let $\pi$ be the Betti table of $R(C; B; L)$. Then [14, Theorem 2.2] says that
\[
\pi = \sum_{i=0}^c \sum_{j=0}^{c'} a_{i,j} \pi_{i,j}
\]
for some rational numbers $a_{i,j} \geq 0$ with $\sum_{i=0}^c \sum_{j=0}^{c'} a_{i,j} = d$.

where each $\pi_{i,j}$ is the pure diagram of the form:
Then (1) implies that
\[ \kappa_{p,0}(\pi_{i,j}) = \frac{(r-1)!(i-p)(r-j+1-p)}{(r+1-p)!p!} \] for 0 \leq p \leq i - 1;
\[ \kappa_{p,1}(\pi_{i,j}) = \frac{(r-1)!(p+1-i)(r-j-p)}{(r-p)!(p+1)!} \] for i \leq p \leq r - j - 1;
\[ \kappa_{p,2}(\pi_{i,j}) = \frac{(r-1)!(p+2-i)(p-r+j+1)}{(r-p-1)!(p+2)!} \] for r - j \leq p \leq r - 1.

**Proposition 5.8.** (1) The Betti table of \( R(C,B;L) \) is asymptotically pure:

\[ \frac{a_{i,j}}{d} \to \begin{cases} 
1 & \text{if } i = c \text{ and } j = c' \\
0 & \text{otherwise} 
\end{cases} \quad \text{as } d \to \infty. \]

(2) \( \kappa_{p,0}(C,B;L), \ldots, \kappa_{c-1,0}(C,B;L) \) is increasing, and \( \kappa_{r-c',2}(C,B;L), \ldots, \kappa_{r-1,2}(C,B;L) \) is decreasing.

(3) The Betti numbers \( \kappa_{p,1}(C,B;L) \) form a unimodal sequence.

**Proof.** (1) Let \( \overline{a}_{i,j} := a_{i,j}/d \). Then (5.2) says

\[ \sum_{i=0}^{c} \sum_{j=0}^{c'} \overline{a}_{i,j} = 1. \]

Notice that \( \kappa_{0,0}(\pi_{i,j}) \to i/d \) and \( \kappa_{r-1,2}(\pi_{i,j}) \to j/d \) as \( d \to \infty \). Since \( \kappa_{0,0}(C,B;L) = c \) and \( \kappa_{r-1,2}(C,B;L) = c' \), it follows from (5.2) that

\[ \sum_{i=0}^{c} \sum_{j=0}^{c'} \overline{a}_{i,j} \to c \quad \text{and} \quad \sum_{i=0}^{c} \sum_{j=0}^{c'} j \overline{a}_{i,j} \to c' \quad \text{as } d \to \infty. \]

Then we have

\[ \sum_{i=0}^{c} \sum_{j=0}^{c'} (c-i) \overline{a}_{i,j} \to 0 \quad \text{and} \quad \sum_{i=0}^{c} \sum_{j=0}^{c'} (c'-j) \overline{a}_{i,j} \to 0 \quad \text{as } d \to \infty. \]

Thus \( \overline{a}_{i,j} \to 0 \) as \( d \to \infty \) unless \( i = c \) and \( j = c' \), and hence, \( \overline{a}_{c,c'} \to 1 \) as \( d \to \infty \).

(2) As \( d \gg 0 \), we have \( r \approx d \) and

\[ \kappa_{p,0}(\pi_{i,j}) \approx \frac{i-p}{p!}d^{p-1} \quad \text{for } 0 \leq p \leq i - 1. \]

Then (1) implies that

\[ \kappa_{p,0}(C,B;L) \approx \frac{c-p}{p!}d^{p} \quad \text{for } 0 \leq p \leq c - 1. \]

Thus the first assertion holds, and the second assertion follows from Proposition 2.2.

(3) For \( (r-2+c)/2 \leq p \leq r-j-1 \), we find

\[ \frac{\kappa_{p+1,1}(\pi_{i,j})}{\kappa_{p,1}(\pi_{i,j})} = \frac{(r-p)(p+2-i)(r-j-p-1)}{(p+2)(p+1-i)(r-j-p)} \leq 1 \]
since $r - p \leq p + 2$ and
\[(p + 2 - i)(r - j - p - 1) - (p + 1 - i)(r - j - p) = (r - j - p - 1) - (p + 1 - i) \leq 0.
\]

For $(r - 2 + c)/2 \leq p \leq r - 2$, we get
\[
\kappa_{p+1,1}(C, B; L) = \sum_{i=0}^{c} \sum_{j=0}^{\min(c', r - p - 2)} a_{i,j} \kappa_{p+1,1}(\pi_{i,j}) \leq \sum_{i=0}^{c} \sum_{j=0}^{\min(c', r - p - 1)} a_{i,j} \kappa_{p,1}(\pi_{i,j}) = \kappa_{p,1}(C, B; L),
\]
so the Betti numbers $\kappa_{p,q}(C, B; L)$ with $(r - 2 + c)/2 \leq p \leq r - 1$ form a decreasing sequence. Now, as in [5, Proof of Proposition A], we compute
\[
(5.3) \quad \kappa_{p,1}(C, B; L) = \chi(C, \Lambda^p M_L \otimes B \otimes L) - \binom{r + 1}{p + 1} c = \binom{r}{p} \left( b - \frac{pd}{r} - \frac{(r + 1)c}{p + 1} \right)
\]
for $c - 1 \leq p \leq r - c'$, where $b := r + \deg B + 1$. For $(r - 1)/2 \leq p \leq (r - 2 + c)/2$, we have
\[
\frac{\kappa_{p+1,1}(C, B; L)}{\kappa_{p,1}(C, B; L)} = \frac{(r - p)(b - (p + 1)d/r - (r + 1)c/(p + 2))}{(p + 1)(b - 2d/r - (r + 1)c/(p + 1))} \leq 1
\]
since $r - p \leq p + 1$ and
\[
\left( b - \frac{(p + 1)d}{r} - \frac{(r + 1)c}{p + 2} \right) - \left( b - \frac{pd}{r} - \frac{(r + 1)c}{p + 1} \right) = -\frac{d}{r} + \frac{(r + 1)c}{(p + 1)(p + 2)} \approx -1 + \frac{dc}{2d \cdot d/2} < 0.
\]

Thus $\kappa_{p+1,1}(C, B; L) \leq \kappa_{p,1}(C, B; L)$. We have shown that the Betti numbers $\kappa_{p,q}(C, B; L)$ with $(r - 1)/2 \leq p \leq r - 1$ form a decreasing sequence. By Proposition 2.2, the Betti numbers $\kappa_{p,q}(C, B; L)$ with $0 \leq p \leq (r - 1)/2$ form an increasing sequence. \hfill \Box

**Remark 5.9.** When $B = \mathcal{O}_C$, Proposition 5.8 (1) is the main theorem of [15]. In view of Proposition 5.8 (2), one may expect that nonzero entries of the top row $(q = 0)$ of the Betti table form an increasing sequence and nonzero entries of the bottom row $(q = n + 1)$ of the Betti table form a decreasing sequence in higher dimensions.

**Example 5.10.** Recall that a log-concave sequence of positive terms is unimodal. It is tempting to expect that $\kappa_{p,1}(C, B; L)$ form a log-concave sequence. Unfortunately, this may fail when $p$ is small. For instance, let $C$ be a general smooth projective complex curve of genus 3, and $B := \omega_C(x)$ for a point $x \in C$. Note that $B$ is not base point free, $\deg B = 5$, and $h^0(C, B) = 3$. If $L$ is a very ample line bundle on $C$ of degree $d \gg 0$, then [9, Theorem C] says that $\kappa_{1,1}(C, B; L)$ is a polynomial in $d$ of degree
\[
\gamma_1(B) := \dim \{ \xi \in C_2 \mid H^0(C, B) \to H^0(\xi, B|_{\xi}) \text{ is not surjective} \} = 1
\]
(as $h^1(C, B(-\xi)) = h^0(C, \mathcal{O}_C(\xi - x)) = 1 \iff x \in \xi$)

On the other hand, from (5.3), we find
\[
\kappa_{2,1}(C, B; L) = \left( \binom{d-3}{2} \right) \left( d + 3 - \frac{2d}{d-3} - \frac{(d-2)d}{4} \right) \approx \frac{3}{2} d^2;
\]
\[
\kappa_{3,1}(C, B; L) = \left( \binom{d-3}{3} \right) \left( d + 3 - \frac{3d}{d-3} - \frac{(d-2)d}{4} \right) \approx \frac{1}{24} d^4.
\]
Thus $\kappa_{2,1}(C, B; L)^2 < \kappa_{1,1}(C, B; L) \cdot \kappa_{3,1}(C, B; L)$. 
References

[1] Daniele Agostini, *Asymptotic syzygies and higher order embeddings*, Int. Math. Res. Not. **2022** no.4, 2934–2967.

[2] Marian Aprodu and Jan Nagel, *Koszul cohomology and algebraic geometry*, University Lecture Series, **52** (2010), Amer. Math. Soc., Providence, RI.

[3] Donu Arapura, *Frobenius amplitude and strong vanishing theorems for vector bundles* (with an appendix by Dennis S. Keeler), Duke Math. J. **121** (2004), 231–267.

[4] Mats Boij and Jonas Söderberg, *Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case*, Algebra Number Theory **6** (2012), 437–454.

[5] Lawrence Ein, Daniel Erman, and Robert Lazarsfeld, *Asymptotics of random Betti tables*, J. Reine Angew. Math. **702** (2015), 35–75.

[6] Lawrence Ein, Daniel Erman, and Robert Lazarsfeld, *A quick proof of nonvanishing for asymptotic syzygies*, Algebra Geom. **3** (2016), 211-222.

[7] Lawrence Ein and Robert Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), 51–67.

[8] Lawrence Ein and Robert Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Invent. Math. **190** (2012), 603–646.

[9] Lawrence Ein and Robert Lazarsfeld, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 301–313.

[10] Lawrence Ein and Robert Lazarsfeld, *Syzygies of projective varieties of large degree: recent progress and open problems*, Algebraic Geometry: Salt Lake City 2015, Proc. Sympos. Pure Math. **97** (2018), Amer. Math. Soc., Providence, RI, 223–242.

[11] Lawrence Ein, Robert Lazarsfeld, and David Yang, *A vanishing theorem for weight-one syzygies*, Algebra Number Theory **10** (2016), 1965–1981.

[12] David Eisenbud, Mark Green, Klaus Hulek, and Sorin Popescu, *Restricting linear syzygies: algebra and geometry*, Compos. Math. **141** (2005), 1460–1478.

[13] David Eisenbud and Frank-Olaf Schreyer, *Betti numbers of graded modules and cohomology of vector bundles*, J. Amer. Math. Soc. **22** (2009), 859–888.

[14] David Eisenbud and Frank-Olaf Schreyer, *Betti numbers of syzygies and cohomology of coherent sheaves*, Preceedings of the International Congress of Mathematicians. Volume II (2010), Hindustan Book Agency, New Delhi, 586–602.

[15] Daniel Erman, *The Betti table of a high-degree curve is asymptotically pure*, Recent advances in algebraic geometry, London Math. Soc. Lecture Note Ser. **417** (2015), Cambridge Univ. Press, Cambridge, 200–206.

[16] Mark Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), 125–171.

[17] Robert Lazarsfeld, *Positivity in algebraic geometry I. Classical Setting: line bundles and linear series*, A Series of Modern Surveys in Math. **48** (2004), Springer-Verlag, Berlin.

[18] Giorgio Ottaviani and Raffaella Paoletti, *Syzygies of Veronese embeddings*, Compos. Math. **125** (2001), 31–37.

[19] Jinhyung Park, *Asymptotic vanishing of syzygies of algebraic varieties*, Comm. Amer. Math. Soc. **2** (2022), 133–148.

[20] Claudiu Raicu, *Representation stability for syzygies of line bundles on Segre–Veronese varieties*, J. Eur. Math. Soc. (JEMS) **18** (2016), 1201–1231.

[21] Jürgen Rathmann, *An effective bound for the gonality conjecture*, preprint, arXiv:1604.06072.

[22] David Yang, *$S_n$-equivariant sheaves and Koszul cohomology*, Res. Math. Sci. **1** (2014), Art. 10, 6 pp.

[23] Xin Zhou, *Effective non-vanishing of asymptotic adjoint syzygies*, Proc. Amer. Math. Soc. **142** (2014), 2255–2264.

Department of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea

Email address: parkjh13@kaist.ac.kr