Quaternion-based bilinear factor matrix norm minimization for color image inpainting

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Abstract—As a new color image representation tool, quaternion has achieved excellent results in the color image processing, because it treats the color image as a whole rather than as a separate color space component, thus it can make full use of the high correlation among RGB channels. Recently, low-rank quaternion matrix completion (LRQMC) methods have proven very useful for color image inpainting. In this paper, we propose three novel LRQMC methods based on three quaternion-based bilinear factor (QBF) matrix norm minimization models. Specifically, we define quaternion double Frobenius norm (Q-DFN), quaternion double nuclear norm (Q-DNN) and quaternion Frobenius/nuclear norm (Q-FNN), and then show their relationship with quaternion-based matrix Schatten-$p$ norm for certain $p$ values. The proposed methods can avoid computing quaternion singular value decompositions (QSVD) for large quaternion matrices, and thus can effectively reduce the calculation time compared with existing (LRQMC) methods. The experimental results demonstrate the superior performance of the proposed methods over some state-of-the-art low-rank (quaternion) matrix completion methods.

Index Terms—Color image inpainting, quaternion matrix completion, bilinear factor matrix norm, low-rank.

I. INTRODUCTION

Low-rank matrix completion (LRMC)-based techniques have made a great success in the application of image inpainting. Most of the existing LRMC methods generally can be divided into two categories [1]: matrix rank minimization and matrix factorization. Matrix rank minimization is generally achieved by sorts of rank approximation regularizers. For example, nuclear norm [2], [3] which is the tightest convex relaxation of the NP-hard rank minimization function [4]. Assigning different weights with different singular values, the authors in [5], [6] proposed the weighted nuclear norm minimization (WNNM) algorithm, which can better approximate the rank function. Besides, as a generalization of nuclear norm minimization (NNM) and WNNM, the authors in [7], [8] proposed the Schatten-$p$ norm minimization. Combining WNNM and the Schatten-$p$ norm, the authors in [9] proposed the weighted Schatten-$p$ norm minimization. However, the main solution strategy for these kinds of methods require computing singular value decompositions (SVD) which requires increasingly cost as matrix sizes and rank increase. On the other hand, matrix factorization generally factorizes the original larger matrix into at least two much smaller matrices [10]–[13]. These approaches benefit from fast numerical methods for optimization and easy kernelization [14]. The main issue of these kinds of methods is the lack of the rank values in many cases. To tackle this problem, the authors in [10] proposed a rank estimation strategy, the effectiveness of which has been proven by many applications and pieces of literature [13], [15].

Most of the LRMC algorithms mentioned above have obtained excellent performance for grayscale images. When handling color images, these algorithms usually processes each color channel independently using the monochromatic model or processes the concatenation of three color channels using the concatenation model [16], [17]. However, these two schemes may not make full use of the high correlation among RGB channels, thus they may cause unsatisfactory results.

Recently, quaternion, as an elegant color image representation tool, has attracted much attention in the field of color image processing. For instance, it has achieved excellent results in the following applications: color image filtering [18], color image edge detection [19], color image denoising [20], [21], color image watermarking [22], color face recognition [23], color image inpainting [24] and so on. By using quaternion algebra, a color image is encoded as a pure quaternion matrix, that means it processes a color image holistically as a vector field and handles the coupling between the color channels naturally [18], [25], [26], and thus color information of source image is fully used. More recently, the authors in [24] extended traditional LRMC methods to quaternion field, and proposed a general low-rank quaternion matrix completion (LRQMC) model based on several nonconvex rank functions including quaternion nuclear norm (QNN), Laplace function and German function. These kinds of methods show promising performance for color image inpainting. Nonetheless, these methods need to compute the quaternion singular
value decompositions (QSVD) for large quaternion matrices in each iteration, which are calculated by their equivalent complex matrices with twice sizes and thus suffer from high computational complexity and time-consuming. Therefore, it is necessary to design more efficient algorithms for LRQMC.

In this paper, we develop three novel LRQMC methods based on three quaternion-based bilinear factor (QBF) matrix norm minimization models for color image inpainting. The contributions of this paper can be summarized as

- We propose three novel LRQMC methods based on three QBF matrix norm minimization models including quaternion double Frobenius norm (Q-DFN)-based, quaternion double nuclear norm (Q-DNN)-based and quaternion Frobenius/nuclear norm (Q-FNN)-based models. Compared with traditional LRMC-based methods, the proposed models process three RGB channels information in a parallel way and thus can preserve the correlation among the color channels well. Compared with existing LRQMC methods, the proposed models only need to handle two much smaller factor quaternion matrices, thus it can effectively reduce the time consumption caused by the calculation of QSVD.
- We show the relationship among quaternion-based matrix Schatten-$p$ (Q-Schatten-$p$) norm for certain $p$ values with the defined three norms. Then, the three models are optimized by applying the alternating direction method of multipliers (ADMM) framework. An effective rank-estimation method is used to estimate the quaternion rank adaptively as the number of iterations increases.
- The experimental results on real color images show their empirical convergence and illustrate their competitive performance over several state-of-the-art methods.

The remainder of this paper is organized as follows. Section II introduces some notations and preliminaries for quaternion algebra. Section III revisits the matrix completion theory and related works, then gives our three quaternion-based matrix completion methods. Section IV provides some experiments to illustrate the performance of our algorithms, and compare it with several state-of-the-art methods. Finally, some conclusions are drawn in Section V.

II. NOTATIONS AND PRELIMINARIES

In this section, we first summarize some main notations and then introduce some basic knowledge of quaternion algebra.

A. Notations

In this paper, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ respectively denote the real space, complex space, and quaternion space. A scalar, a vector, and a matrix are written as $a$, $\mathbf{a}$, and $\mathbf{A}$, respectively. $\bar{a}$, $\mathbf{\bar{a}}$, and $\mathbf{A}$ respectively represent a quaternion scalar, a quaternion vector, and a quaternion matrix. $(\cdot)^*$, $(\cdot)^{-1}$, $(\cdot)^T$, and $(\cdot)^H$ denote the conjugation, inverse, transpose, and conjugate transpose, respectively. $|\cdot|$, $\|\cdot\|_F$, and $\|\cdot\|_*$ are respectively the absolute value or modulus, the Frobenius norm, and the nuclear norm. $(\cdot, \cdot)$ denotes the inner product operation. $\operatorname{tr}\{\cdot\}$ and $\operatorname{rank}(\cdot)$ denote the trace and rank operators respectively. $\Re(\cdot)$ denotes the real part of quaternion (scalar, vector, and matrix). $\mathbf{I}$ represents the identity matrix with appropriate size.

B. Basic knowledge of quaternion algebras

Quaternion algebras have attracted extensive attention in recent years, especially in the field of signal and image processing [24], [27], etc.. Some basic knowledge of quaternion algebras can be found in the Appendix A. In the following, we give two theorems about quaternion matrix.

**Theorem 1.** (Quaternion singular value decomposition (QSVD) [28]): For any quaternion matrix $\mathbf{Q} \in \mathbb{H}^{M \times N}$ with rank $r$, there exist unitary quaternion matrices $\mathbf{U} \in \mathbb{H}^{M \times M}$ and $\mathbf{V} \in \mathbb{H}^{N \times N}$ such that

$$\mathbf{Q} = \mathbf{U} \begin{pmatrix} \mathbf{D}_r & 0 \\ 0 & 0 \end{pmatrix} \mathbf{V}^H,$$

where $\mathbf{D}_r = \operatorname{diag}\{d_1, \ldots, d_r\} \in \mathbb{R}^{r \times r}$ and the $d$’s are the positive singular values of $\mathbf{Q}$.

The way that to obtain the QSVD of a quaternion matrix can be found in the Appendix A. Following the Theorem 1, we can see that the rank of a quaternion matrix $\mathbf{Q} \in \mathbb{H}^{M \times N}$ is equal to the number of its positive singular values [29].

**Theorem 2.** Let $\mathbf{X} \in \mathbb{H}^{M \times N}$ be an arbitrary quaternion matrix. If $\operatorname{rank}(\mathbf{X}) = r \leq d$, then $\mathbf{X}$ can be written into a quaternion matrix product form $\mathbf{X} = \mathbf{U} \mathbf{V}^H$, where $\mathbf{U} \in \mathbb{H}^{M \times d}$ and $\mathbf{V} \in \mathbb{H}^{N \times d}$ are two quaternion matrices of smaller sizes and they meet $\operatorname{rank}(\mathbf{U}) = \operatorname{rank}(\mathbf{V}) = r$.

The proof of Theorem 2 can be found in the Appendix B.

III. PROBLEM FORMULATION

This section firstly revisits the matrix completion theory and related works, then gives our three quaternion-based matrix completion methods.
A. Matrix completion

The matrix completion problem consists of recovering a data matrix from its partial entries by the well-known rank optimization model \[29]:

\[
\min_{X} \lambda \| \mathbf{X} \|_{S_p}, \text{s.t.}, \mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{T}) = \mathbf{0},
\]

where \( \lambda \) is a nonnegative parameter, \( \mathbf{X} \in \mathbb{R}^{M \times N} \) is a completed output matrix, \( \mathbf{T} \in \mathbb{R}^{M \times N} \) is an incomplete input matrix and \( \mathcal{P}_{\Omega} \) is the unitary projection onto the linear space of matrices supported on the entries set \( \Omega \), defined as

\[
(\mathcal{P}_{\Omega}(\mathbf{X}))_{mn} = \begin{cases} 
\mathbf{x}_{mn}, & (m, n) \in \Omega, \\
0, & (m, n) \notin \Omega.
\end{cases}
\]

Directly solve the problem (2) is difficult as the rank minimization problem is known as NP-hard [30]. Various heuristics approaches have been developed to solve this problem. These methods usually adopt the convex or non-convex surrogates to replace the rank function in (2), and formulate this problem into the following general form

\[
\min_{X} \lambda \| \mathbf{X} \|_{S_p}, \text{s.t.}, \mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{T}) = \mathbf{0},
\]

where \( \| \mathbf{X} \|_{S_p} \) is the Schatten-\( p \) norm \((0 < p < \infty)\) of matrix \( \mathbf{X} \), and defined as

\[
\| \mathbf{X} \|_{S_p} := \left( \sum_{k} \sigma_{k}^{p}(\mathbf{X}) \right)^{1/p},
\]

where \( \sigma_{k}(\mathbf{X}) \) denotes the \( k \)-th singular value of \( \mathbf{X} \). When \( p = 1 \) the Schatten-1 norm is the well-known nuclear norm \( \| \mathbf{X} \|_{*} \), which has been widely applied in low-rank matrix approximation problems. For example, there are many matrix completion approaches [9], [11], [31], [32] by using nuclear norm \( \| \mathbf{X} \|_{*} \), or weighted nuclear norm \( \| \mathbf{X} \|_{w*} \) to replace the first term in (3). In addition, as the non-convex surrogate for the rank function, the Schatten-\( p \) quasi-norm with \( 0 < p < 1 \) makes a closer approximation to the rank function than the nuclear norm [33]. Therefore, the Schatten-\( p \) quasi-norm has attracted a great deal of attention in low-rank matrix approximation problems [7], [8], [24]. Besides, the authors in [13] adopted the following two bilinear factor (BF) matrix norms (double nuclear norm \( \| \mathbf{X} \|_{D-N} \) and Frobenius/nuclear norm \( \| \mathbf{X} \|_{F-N} \)) to replace the Schatten-\( p \) norm and obtained an impressive performance

\[
\| \mathbf{X} \|_{D-N} = \min_{U,V} \frac{1}{4} (\|U\|_{*} + \|V\|_{*})^{2},
\]

\[
\| \mathbf{X} \|_{F-N} = \min_{U,V} \frac{1}{4} (\|U\|_{F}^{2} + 2\|V\|_{*} + \|V\|_{*})^{3/2}.
\]

B. Quaternion matrix completion

The traditional low-rank approximation based matrix completion methods are inherently designed in the real settings for grayscale image inpainting and thus may suffer from performance degradation of color image case. Quaternion matrix completion can be regarded as the generalization of the traditional matrix completion in the quaternion number field, which, for color image inpainting, allows different color channels to talk to each other rather than each channel being independently manipulated. We define the quaternion-based matrix Schatten-\( p \) norm (Q-Schatten-\( p \)) as follows.

**Definition 1.** (Q-Schatten-\( p \)): Given a quaternion matrix \( \mathbf{X} \in \mathbb{H}^{M \times N} \), the Q-Schatten-\( p \) norm \((0 < p < \infty)\) is defined as

\[
\| \mathbf{X} \|_{Q-S_p} := \left( \sum_{k} \sigma_{k}^{p}(\mathbf{X}) \right)^{1/p},
\]

where \( \sigma_{k}(\mathbf{X}) \) denotes the \( k \)-th singular value of quaternion matrix \( \mathbf{X} \). When \( p = 1 \) the above quaternion-based matrix Schatten-1 norm is quaternion nuclear norm (QNN) denoted as \( \| \mathbf{X} \|_{Q} \) [21], [24]. However, analogous to the traditional nuclear norm based approaches, the QNN-based methods involve computing the QSVD of a large quaternion matrix in each iteration, and thus suffer from high computational complexity and time-consuming. Hence, it is necessary to design more efficient algorithms.

C. Proposed quaternion-based matrix completion models

The Theorem 4 allows us to make bilinear factorization of larger quaternion matrices. Thus, in the following, we first define the quaternion double Frobenius norm (Q-DFN). Then, motivated by the definition of double nuclear norm and Frobenius/nuclear norm in [13], we also define the quaternion double nuclear norm (Q-DNN) and the quaternion Frobenius/nuclear norm (Q-FNN) in this section.

**Definition 2.** (Q-DFN): Given a quaternion matrix \( \mathbf{X} \in \mathbb{H}^{M \times N} \) with rank(\( \mathbf{X} \)) = \( r \leq d \), we decompose it into two much smaller quaternion factor matrices \( \mathbf{U} \in \mathbb{H}^{M \times d} \) and \( \mathbf{V} \in \mathbb{H}^{N \times d} \) such that \( \mathbf{X} = \mathbf{UV}^{H} \). Then the quaternion double Frobenius norm is defined as

\[
\| \mathbf{X} \|_{Q-DFN} := \min_{U,V} \frac{1}{2} \| \mathbf{U} \|_{F}^{2} + \frac{1}{2} \| \mathbf{V} \|_{F}^{2}.
\]

**Definition 3.** (Q-DNN): Given a quaternion matrix \( \mathbf{X} \in \mathbb{H}^{M \times N} \) with rank(\( \mathbf{X} \)) = \( r \leq d \), we decompose it into two much smaller quaternion factor matrices \( \mathbf{U} \in \mathbb{H}^{M \times d} \) and \( \mathbf{V} \in \mathbb{H}^{N \times d} \) such that \( \mathbf{X} = \mathbf{UV}^{H} \). Then the quaternion double nuclear norm is defined as

\[
\| \mathbf{X} \|_{Q-DNN} := \min_{U,V} \frac{1}{4} (\| \mathbf{U} \|_{*} + \| \mathbf{V} \|_{*})^{2}.
\]

**Definition 4.** (Q-FNN): Given a quaternion matrix \( \mathbf{X} \in \mathbb{H}^{M \times N} \) with rank(\( \mathbf{X} \)) = \( r \leq d \), we decompose it into two much smaller quaternion factor matrices \( \mathbf{U} \in \mathbb{H}^{M \times d} \) and \( \mathbf{V} \in \mathbb{H}^{N \times d} \) such that \( \mathbf{X} = \mathbf{UV}^{H} \). Then the quaternion Frobenius/nuclear norm is defined as

\[
\| \mathbf{X} \|_{Q-FNN} := \min_{U,V} \frac{1}{4} (\| \mathbf{U} \|_{F}^{2} + 2\| \mathbf{V} \|_{*} + \| \mathbf{V} \|_{*})^{3/2}.
\]

In the following three theorems, we give the relationships among the Q-Schatten-\( p \) norm, the quaternion double
Frobenius norm, the quaternion double nuclear norm, and the quaternion Frobenius/nuclear norm.

**Theorem 3.** The quaternion double Frobenius norm is in essence the Q-Schatten-1 norm (or QNN), i.e.,

\[
\|X\|_{Q-DFN} = \|X\|_{Q-s_{1/2}}.
\]

**Theorem 4.** The quaternion double nuclear norm is in essence the Q-Schatten-1/2 norm, i.e.,

\[
\|X\|_{Q-DNN} = \|X\|_{Q-s_{1/2}}.
\]

**Theorem 5.** The quaternion Frobenius/nuclear norm is in essence the Q-Schatten-1/2 norm, i.e.,

\[
\|X\|_{Q-FNN} = \|X\|_{Q-s_{2/3}}.
\]

The proofs of **Theorem 3**, **Theorem 4**, and **Theorem 5** can be found in the Appendix C.

Based on the **Definition 2**, **Definition 3**, and the **Definition 4**, we present the following three quaternion-based matrix completion models

\[
\begin{align*}
\min_{U,V,X} & \quad \frac{\lambda}{2} \left( \|U\|^2_F + \|V\|^2_F \right), \\
\text{s.t.,} & \quad P_2(X - T) = 0, \quad X = UV^H, \\
\end{align*}
\]

\[
\begin{align*}
\min_{U,V,X} & \quad \frac{\lambda}{2} \left( \|U\|^2_* + \|V\|^2_* \right), \\
\text{s.t.,} & \quad P_2(X - T) = 0, \quad X = UV^H, \\
\end{align*}
\]

\[
\begin{align*}
\min_{U,V,X} & \quad \frac{\lambda}{3} \left( \|U\|^2_F + 2\|V\|^2_* \right), \\
\text{s.t.,} & \quad P_2(X - T) = 0, \quad X = UV^H, \\
\end{align*}
\]

The above models have at least two obvious advantages. First of all, the above models are built in the quaternion field, which, compared with the traditional real matrix-based method, can directly deal with the problem of color image inpainting, and can make full use of the connection between color channels. In addition, compared with the existing quaternion matrix completion methods, the above models only need to handle two much smaller factor quaternion matrices, thus it can effectively reduce the time consumption caused by the calculation of QSVD.

### D. Optimization process

The problem (14) can be solved by minimizing the following augmented Lagrangian function

\[
\begin{align*}
\mathcal{L}_\mu(U,V,X,F) &= \mathcal{L}_\mu(U,V,X,F) + \frac{\mu}{2} \left( \|X - UV^H\|^2_F + \|F\|^2_F \right) \\
&= \frac{\lambda}{2} \left( \|U\|^2_F + \|V\|^2_F \right) + \left( F, X - UV^H \right) \\
&\quad + \mu \|X - UV^H\|^2_F + \frac{1}{2} \|P_2(X - T)\|^2_F, \\
\end{align*}
\]

where \( \mu > 0 \) is the penalty parameter, \( \mathcal{F} \) is Lagrange multiplier.

**Updating \( U \) and \( V \):** In the \( \tau + 1 \)-th iteration, fixing the other variables at their latest values, \( U^{\tau+1} \) and \( V^{\tau+1} \) are respectively the optimal solutions of the following problems

\[
\mathcal{U}^{\tau+1} = \arg \min_U \left( \frac{1}{2} \|X - U(V^\tau)^H + F^\tau/\mu\|^2_F + \frac{\lambda}{2\mu} \|\tilde{U}\|^2_F \right), \\
\mathcal{V}^{\tau+1} = \arg \min_V \left( \frac{1}{2} \|X - U(V^\tau)^H + F^\tau/\mu\|^2_F + \frac{\lambda}{2\mu} \|\tilde{V}\|^2_F \right). \\
\]

Theorem 5. Let \( \mathcal{Q}(U) := \frac{1}{2} \|X - U(V^\tau)^H + F^\tau/\mu\|^2_F + \frac{\lambda}{2\mu} \|\tilde{U}\|^2_F \) and \( \mathcal{G}(V) := \frac{1}{2} \|X - U(V^\tau)^H + F^\tau/\mu\|^2_F + \frac{\lambda}{2\mu} \|\tilde{V}\|^2_F \). We see that \( \mathcal{Q}(U) \) and \( \mathcal{G}(V) \) are real functions of quaternion variable, thus the left and right generalized HR (GHR) derivatives are identical [35]. Using the related theories of quaternion matrix derivatives in [35], the gradient of \( \mathcal{Q}(U) \) can be calculated as

\[
\frac{\partial \mathcal{Q}(U)}{\partial U^*} = \frac{1}{2} \left( \mathcal{C}_1 - \mathcal{U}(V^\tau)^H \mathcal{C}_1 - \mathcal{U}(V^\tau)^H \right) \\
+ \frac{\lambda}{\mu} \mathcal{C}_1 \mathcal{U}^* \mathcal{U}^H \mathcal{C}_1 \mathcal{U}^H \\
+ \frac{1}{2} \mathcal{C}_1 \mathcal{U}^H \mathcal{U}^H \mathcal{U}^H + \mathcal{C}_1 \mathcal{U}^H \mathcal{U}^H \mathcal{U}^H + \frac{\lambda}{\mu} \mathcal{U}^H \mathcal{U}^H \mathcal{U}^H. \\
\]

where \( \mathcal{C}_1 = \mathcal{X} + \mathcal{F}^\tau/\mu \). By the similar approach, we can obtain the gradient of \( \mathcal{G}(V) \)

\[
\frac{\partial \mathcal{G}(V)}{\partial V^*} = \frac{1}{4} \left( \mathcal{V}(U^{\tau+1})^H U^{\tau+1} - \mathcal{C}_1 \mathcal{U}^H \mathcal{U}^H \mathcal{U}^H + \lambda \mathcal{I} \right). \\
\]

Setting (20) and (21) to zero, we can obtain the unique solutions of \( U^{\tau+1} \) and \( V^{\tau+1} \) separately as

\[
U^{\tau+1} = \mathcal{C}_1 \mathcal{V}^\tau \left( (V^\tau)^H V^\tau + \frac{\lambda}{\mu^2} \mathcal{I} \right)^{-1} \\
V^{\tau+1} = \mathcal{C}_1^H \mathcal{U}^{\tau+1} \left( (U^{\tau+1})^H U^{\tau+1} + \frac{\lambda}{\mu^2} \mathcal{I} \right)^{-1}. \\
\]

**Updating \( \tilde{X} \):** In the \( \tau + 1 \)-th iteration, fixing the other variables at their latest values, \( X^{\tau+1} \) is the optimal solution of the following problem

\[
\tilde{X}^{\tau+1} = \arg \min_X \left( \frac{1}{2} \|P_2(X - T)\|^2_F + \frac{\mu}{2} \|X - (U^{\tau+1} (V^{\tau+1})^H - F^\tau/\mu)\|^2_F \right). \\
\]

Then, we can directly obtain the optimal \( X^{\tau+1} \) as

\[
X^{\tau+1} = P_2 \left( U^{\tau+1} (V^{\tau+1})^H - F^\tau/\mu + \frac{\mu}{\mu^2} \mathcal{I} \right). \\
\]

where \( \Omega^\complement \) is the complement of \( \Omega \), and we have used the fact that \( P_2(T) = 0 \) in (25).

**Updating \( \tilde{F} \) and \( \mu \):** The multiplier \( \tilde{F}^{\tau+1} \) is directly obtained by

\[
\tilde{F}^{\tau+1} = \tilde{F}^\tau + \mu^2 (X^{\tau+1} - U^{\tau+1} (V^{\tau+1})^H). \\
\]
And the penalty parameter $\mu^{r+1}$ is
\[
\mu^{r+1} = \min(\beta \mu^r, \mu_{\text{max}}),
\]
where $\mu_{\text{max}}$ is the default maximum value of the penalty parameter $\mu$, and $\beta \geq 1$ is a constant parameter.

Introducing the auxiliary variables $\dot{A}_U$ and $\dot{A}_V$, the problems (15) and (16) can be reformulated into the following equivalent formulations
\[
\min_{\mathbf{U}, \mathbf{V}, \dot{A}_U, \dot{A}_V, \mathbf{X}} \frac{\lambda}{2} (\|\dot{A}_U\|_2 + \|\dot{A}_V\|_2),
\]
subject to $\mathbf{U} = \dot{A}_U, \mathbf{V} = \dot{A}_V, \mathcal{P}_t(\mathbf{X} - \mathbf{T}) = 0, \mathbf{X} = \dot{U}\dot{V}^H$.
\[
\min_{\mathbf{U}, \mathbf{V}, \dot{A}_U, \dot{A}_V, \mathbf{X}} \frac{\lambda}{3} (\|\dot{U}\|_2^3 + 2\|\dot{A}_V\|_2^3),
\]
subject to $\mathbf{V} = \dot{A}_V, \mathcal{P}_t(\mathbf{X} - \mathbf{T}) = 0, \mathbf{X} = \dot{U}\dot{V}^H$.

Note that, (28) and (29) split the interdependent terms such that they can be solved independently. Then, the problems (28) and (29) can be solved by the ADMM framework.

We first solve the problem (28) by minimizing the following augmented Lagrangian function
\[
\mathcal{L}_\mu(\dot{U}, \dot{V}, \dot{A}_U, \dot{A}_V, \dot{F}_1, \dot{F}_2, \dot{F}_3) = \frac{\lambda}{2} (\|\dot{A}_U\|_2 + \|\dot{A}_V\|_2) + (\dot{F}_1, \dot{U} - \dot{A}_U) + \frac{\mu}{2} (\|\dot{U}\|_2^3 + \|\dot{V} - \dot{A}_V\|_2^3) + \frac{1}{2} \|\mathcal{P}_t(\mathbf{X} - \mathbf{T})\|_2^2,
\]
where $\dot{F}_1, \dot{F}_2$ and $\dot{F}_3$ are Lagrange multipliers.

**Updating $\dot{U}$ and $\dot{V}$**: In the $\tau + 1$-th iteration, fixing the other variables at their latest values, $\dot{U}^{\tau+1}$ and $\dot{V}^{\tau+1}$ are respectively the optimal solutions of the following problems
\[
\dot{U}^{\tau+1} = \arg \min_{\dot{U}} (\dot{F}_1^\tau, \dot{U} - \dot{A}_U^\tau) + (\dot{F}_1^\tau, \dot{X}^\tau - \dot{U}(\dot{V}^\tau)^H)
\]
\[
+ \frac{\mu^\tau}{2} \left(\|\dot{U} - \dot{A}_U^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}(\dot{V}^\tau)^H\|_2^3\right)
\]
\[
= \arg \min_{\dot{U}} (\dot{U} - \dot{A}_U^\tau) + \dot{F}_1^\tau/\mu^\tau)\|_2^3
\]
\[
+ \|\dot{X}^\tau - \dot{U}(\dot{V}^\tau)^H + \dot{F}_1^\tau/\mu^\tau\|_2^3,
\]
\[
\dot{V}^{\tau+1} = \arg \min_{\dot{V}} (\dot{F}_2^\tau, \dot{V} - \dot{A}_V^\tau) + (\dot{F}_2^\tau, \dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H)
\]
\[
+ \frac{\mu^\tau}{2} \left(\|\dot{V} - \dot{A}_V^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H\|_2^3\right)
\]
\[
= \arg \min_{\dot{V}} (\dot{V} - \dot{A}_V^\tau) + \dot{F}_2^\tau/\mu^\tau)\|_2^3
\]
\[
+ \|\dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H + \dot{F}_2^\tau/\mu^\tau\|_2^3.
\]
Let $A(\dot{U}) := \|\dot{U} - \dot{A}_U^\tau + \dot{F}_1^\tau/\mu^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}(\dot{V}^\tau)^H + \dot{F}_1^\tau/\mu^\tau\|_2^3$, and $B(\dot{V}) := \|\dot{V} - \dot{A}_V^\tau + \dot{F}_2^\tau/\mu^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H + \dot{F}_2^\tau/\mu^\tau\|_2^3$, and $E(\dot{V}) := \|\dot{V} - \dot{A}_V^\tau + \dot{F}_2^\tau/\mu^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H + \dot{F}_2^\tau/\mu^\tau\|_2^3$. The gradient of $A(\dot{U})$ can be calculated as
\[
\frac{\partial A(\dot{U})}{\partial \dot{U}} = \frac{\partial}{\partial \dot{U}} \left(\left(\dot{U} - \dot{A}_U^\tau + \dot{F}_1^\tau/\mu^\tau\right) \dot{U}^H + \dot{F}_1^\tau/\mu^\tau\right)
\]
\[
+ \frac{\partial}{\partial \dot{U}} \left(\left(\dot{U} - \dot{A}_U^\tau + \dot{F}_1^\tau/\mu^\tau\right) \dot{V}^H + \dot{F}_1^\tau/\mu^\tau\right)
\]
\[
+ \frac{\partial}{\partial \dot{U}} \left(\left(\dot{U} - \dot{A}_U^\tau + \dot{F}_1^\tau/\mu^\tau\right) \dot{X}^\tau - \dot{U}(\dot{V}^\tau)^H + \dot{F}_1^\tau/\mu^\tau\right)
\]
\[
= \left\{\begin{array}{ll}
\dot{F}_1^\tau/\mu^\tau & \text{if } \dot{U} = \dot{A}_U^\tau + \dot{F}_1^\tau/\mu^\tau \\
0 & \text{otherwise}
\end{array}\right.
\]
\[
\text{and } B(\dot{V}) := \|\dot{V} - \dot{A}_V^\tau + \dot{F}_2^\tau/\mu^\tau\|_2^3 + \|\dot{X}^\tau - \dot{U}^{\tau+1}\dot{V}^H + \dot{F}_2^\tau/\mu^\tau\|_2^3.
\]
\[
\text{The closed-form solutions of (36) and (37) can be obtained by the quaternion singular value thresholding (QSVT) [24], i.e.,}
\]
\[
\dot{A}_U^{\tau+1} = D\frac{\lambda}{2\mu^\tau} \left(\dot{U}^{\tau+1} + \dot{F}_1^\tau/\mu^\tau\right),
\]
\[
\dot{A}_V^{\tau+1} = D\frac{\lambda}{2\mu^\tau} \left(\dot{V}^{\tau+1} + \dot{F}_2^\tau/\mu^\tau\right).
\]
The QSVT operator \( \mathcal{D}_\theta(M) \) is defined as
\[
\mathcal{D}_\theta(M) = \mathcal{P} \text{diag} \{ \max(\sigma_i(M) - \delta, 0) \} Q^H,
\]
where \( M = \mathcal{P} \text{diag} \{ \sigma_i(M) \} Q^H \) is the QSVD of quaternion matrix \( M \).

### Updating \( \hat{X} \):
In the \( \tau + 1 \)-th iteration, fixing the other variables at their latest values, \( \hat{X}^{\tau+1} \) is the optimal solution of the following problem
\[
\hat{X}^{\tau+1} = \arg\min_{\hat{X}} \frac{1}{2} \| \mathcal{P}_\Omega(X - T) \|_F^2 + \frac{\mu^\tau}{2} \| X - (\hat{U}^{\tau+1} (\hat{V}^{\tau+1} H) - \hat{F}_4^\tau) \|_F^2.
\]
(40)

Then, we can directly obtain the optimal \( \hat{X}^{\tau+1} \) as
\[
\hat{X}^{\tau+1} = \mathcal{P}_\Omega \left( \hat{U}^{\tau+1} (\hat{V}^{\tau+1} H) - \hat{F}_4^\tau / \mu^\tau \right) + \frac{\mu^\tau}{1 + \mu^\tau} \mathcal{P}_\Omega \left( \hat{U}^{\tau+1} (\hat{V}^{\tau+1} H) - \hat{F}_4^\tau + T \right),
\]
(41)
where \( \Omega \) is the complement of \( \Omega \), and we have used the fact that \( \mathcal{P}_\Omega(T) = 0 \) in (41).

### Updating \( \hat{F}_1, \hat{F}_2, \hat{F}_3 \) and \( \mu \):
The multipliers \( \hat{F}_1^{\tau+1}, \hat{F}_2^{\tau+1}, \hat{F}_3^{\tau+1} \) are directly obtained by
\[
\begin{align*}
\hat{F}_1^{\tau+1} &= \hat{F}_1^\tau + \mu^\tau (\hat{U}^{\tau+1} - \hat{A}_1^{\tau+1}), \\
\hat{F}_2^{\tau+1} &= \hat{F}_2^\tau + \mu^\tau (\hat{V}^{\tau+1} - \hat{A}_2^{\tau+1}), \\
\hat{F}_3^{\tau+1} &= \hat{F}_3^\tau + \mu^\tau (\hat{X}^{\tau+1} - \hat{U}^{\tau+1} (\hat{V}^{\tau+1} H))
\end{align*}
\]
(42)
And the penalty parameter \( \mu^{\tau+1} \) is
\[
\mu^{\tau+1} = \min(\beta \mu^\tau, \mu_{\text{max}}).
\]
(43)

### E. Rank estimation
In most cases, we do not know the true rank, \( d \), of quaternion matrix data. However, it is essential for the success of the proposed methods. Thus, in this section, we develop a method for estimating the rank of quaternion matrix data. This method starts from an input overestimated rank \( d^* \) of quaternion matrix \( X \). Assume that the rank of \( X \) is \( d^* \). We compute the singular values of \( \hat{X}^{\tau+1} \) and ordered non-increasing, i.e., \( \sigma_1^\tau \geq \sigma_2^\tau \geq \ldots \geq \sigma_{d^*}^\tau \). Then, we compute the quotient sequence \( \delta_m^\tau = \sigma_m^\tau / \sigma_{m+1}^\tau \), \((m = 1, \ldots, d^* - 1)\). Assume that
\[
p^\tau = \arg\max_{1 \leq m \leq d^* - 1} \delta_m^\tau,
\]
and define
\[
\delta^\tau = (d^* - 1) \sigma_{d^*}^\tau / \sum_{m \neq d^*} \sigma_m^\tau.
\]
(44)
where the value of \( \delta^\tau \) represents how many times the maximum drop (occurring at the \( \sigma_{d^*}^\tau \)) is larger than the average of the rest of drops \( \{\delta_m^\tau\} \). Once \( \delta^\tau > 20 \), i.e., there is a large drop in the estimated rank, we reduce \( d^* \) to \( p^\tau \). This adjustment is done only one time and works sufficiently well in our test.

Finally, as described above, the Q-DFN-based quaternion matrix completion method (i.e., the solutions of the problem [46]) can be summarized in TABLE I. The Q-DDN-based quaternion matrix completion method (i.e., the solutions of the problem [13]) can be summarized in TABLE II. Similarly, we also present the Q-FNN-based quaternion matrix completion method (i.e., the solutions of the problem [15]) in TABLE III and provide the details in the Appendix D.
The stopping criterion of the three algorithms is defined as following relative error

\[
\text{RE} := \frac{\|\tilde{U}^T (\tilde{V}^T)^H - X^r\|_F}{\|T\|_F} \leq \text{tol},
\]

where, \(\text{tol} > 0\) (we set \(\text{tol} = 1 \times 10^{-4}\) in the experiments) is the stopping tolerance.

IV. EXPERIMENTAL RESULTS

In this section, several experiments on some natural color images are conducted to evaluate the effectiveness of the proposed Q-DFN-based, Q-DNN-based and Q-FNN-based quaternion matrix completion methods. All the experiments are run in MATLAB 2014b under Windows 10 on a personal computer with 1.60GHz CPU and 8GB memory.

**Compared methods:** In the simulations, we compare the proposed Q-DNN and Q-FNN with six state-of-the-art low-rank matrix completion methods including:

- **D-N** [13]: a BF-based real matrix completion method which minimize the double nuclear norm.
- **F-N** [13]: a BF-based real matrix completion method which minimize the Frobenius/nuclear norm.
- **WNMM** [6]: a real matrix completion method which uses the weighted nuclear norm regularization term.
- **MC-NC** [37]: a real matrix completion method based on Non-Convex relaxation.
- **LRQA-2** [24]: a quaternion matrix completion method based a nonconvex rank surrogate (laplace function).
- **RegL1-ALM** [12]: a robust BF-based real matrix completion method which utilizes the L1-norm rather than Frobenius norm to build loss function.

**Parameter and initialization setting:** For Q-DFN, Q-DNN and Q-FNN, we set \(\lambda = 0.05\sqrt{\max(M, N)}, \mu_{\text{max}} = 10^{20}\) and \(\beta = 1.03\). Let \(\mu_0 = 10^{-3}\) (\(\mu_0 = 10^{-2}\) for Q-DNN), \(A^0_U = I, A^0_V = I\), and the appropriate \(d^0\) is chosen from \{40, 60, 80, 100, 120\}. In addition, all compared methods are from the source codes and the parameter settings are based on the suggestions in the original papers.

**Quantitative assessment:** To evaluate the performance of proposed methods, except visual quality, we employ two widely used quantitative quality indexes, including the peak signal-to-noise ratio (PSNR) and the structure similarity (SSIM) [38].

The widely used 8 color images with different sizes are selected as the test samples shown in Fig.1. Similar to the settings in [13], [24], [39], all real matrix completion methods among the compared methods are performed on each channel of the test images individually.

Fig.2 shows the empirical convergence of the proposed three methods for color image inpainting on all images with MR = 0.50, where MR denotes the missing ratio of pixels (the larger the value of MR, the more pixels are lost).

**Compared methods:** For Q-DFN, Q-DNN and Q-FNN, we set \(\lambda = 0.05\sqrt{\max(M, N)}, \mu_{\text{max}} = 10^2\) and \(\beta = 1.03\). Let \(\mu_0 = 10^{-3}\) (\(\mu_0 = 10^{-2}\) for Q-DNN), \(A^0_U = I, A^0_V = I\), and the appropriate \(d^0\) is chosen from \{40, 60, 80, 100, 120\}. In addition, all compared methods are from the source codes and the parameter settings are based on the suggestions in the original papers.

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**Compared methods:** For Q-DFN, Q-DNN and Q-FNN, we set \(\lambda = 0.05\sqrt{\max(M, N)}, \mu_{\text{max}} = 10^2\) and \(\beta = 1.03\). Let \(\mu_0 = 10^{-3}\) (\(\mu_0 = 10^{-2}\) for Q-DNN), \(A^0_U = I, A^0_V = I\), and the appropriate \(d^0\) is chosen from \{40, 60, 80, 100, 120\}. In addition, all compared methods are from the source codes and the parameter settings are based on the suggestions in the original papers.
| Images | Indexes | D-N | F-N | WNNM | MC-NC | LRQA-2 | RegL1-ALM | Q-DFN | Q-DNN | Q-FNN |
|--------|---------|-----|-----|------|-------|--------|-----------|-------|-------|-------|
| Image(1) | PSNR | 20.956 | 21.313 | 23.549 | 26.002 | 25.767 | 20.467 | 25.332 | **24.085** | 26.170 |
| | SSIM | 0.404 | 0.413 | 0.430 | 0.407 | 0.375 | 0.508 | 0.364 | 0.737 | 0.569 |
| | time(s) | 5.54 | 5.06 | 41.20 | 28.32 | 92.44 | 154.19 | 60.10 | 66.05 | 73.79 |
| Image(2) | PSNR | 21.067 | 23.259 | 23.648 | 25.604 | 25.679 | 21.777 | 25.291 | **25.901** | 26.017 |
| | SSIM | 0.834 | 0.877 | 0.884 | 0.926 | 0.929 | 0.877 | 0.928 | **0.931** | 0.937 |
| | time(s) | 5.54 | 5.06 | 41.20 | 28.32 | 92.44 | 154.19 | 60.10 | 66.05 | 73.79 |
| Image(3) | PSNR | 19.722 | 22.804 | 22.847 | 25.899 | 27.453 | 27.795 | 27.721 | **27.969** | 28.040 |
| | SSIM | 0.860 | 0.854 | 0.857 | 0.913 | 0.928 | 0.928 | 0.928 | **0.930** | 0.932 |
| | time(s) | 5.54 | 5.06 | 41.20 | 28.32 | 92.44 | 154.19 | 60.10 | 66.05 | 73.79 |
| Image(4) | PSNR | 18.998 | 19.774 | 19.774 | 18.945 | 20.013 | 21.665 | 19.423 | **21.780** | 21.606 |
| | SSIM | 0.764 | 0.762 | 0.718 | 0.775 | 0.850 | 0.764 | **0.856** | 0.852 | 0.848 |
| | time(s) | 5.54 | 5.06 | 41.20 | 28.32 | 92.44 | 154.19 | 60.10 | 66.05 | 73.79 |

**Fig. 3.** (a) is the original image. (b) is the observed image \((MR = 0.70)\). (c)-(k) are the inpainting results of D-N, F-N, WNNM, MC-NC, LRQA-2, RegL1-ALM, Q-DFN, Q-DNN and Q-FNN, respectively. (l) summaries the PSNR values, SSIM values and the running time of all methods. The figure is viewed better in zoomed PDF.
### Table IV

Quantitative assessment indexes (PSNR/SSIM) of different methods on the eight color images (bold fonts denote the best performance; underline ones represent the second-best results).

| Images | MR = 0.50 | MR = 0.75 | MR = 0.85 |
|--------|---------|---------|---------|
| Image(1) | 21.4060/421 | 21.4060/422 | 28.3880/605 |
| Image(2) | 22.9900/858 | 22.9770/857 | 29.3500/957 |
| Image(3) | 21.8330/742 | 21.8330/742 | 27.6900/991 |
| Image(4) | 21.0480/780 | 21.0480/780 | 23.8880/000 |
| Image(5) | 23.6700/801 | 23.6700/802 | 29.8500/921 |
| Image(6) | 21.9120/875 | 21.9120/879 | 24.9600/929 |
| Image(7) | 19.3630/894 | 19.3630/894 | 26.8050/950 |
| Image(8) | 25.1440/924 | 25.1440/926 | 32.5080/962 |
| Aver. | 22.205 | 22.000 | 27.641 |

| Images | MR = 0.75 |
|--------|---------|
| Image(1) | 20.7610/398 | 20.7590/393 | 21.1890/379 |
| Image(2) | 20.8710/827 | 20.8760/827 | 21.8500/841 |
| Image(3) | 20.3260/706 | 20.3310/707 | 22.3670/741 |
| Image(4) | 19.3580/751 | 19.3550/751 | 18.1070/671 |
| Image(5) | 22.3880/769 | 22.3870/769 | 24.7460/827 |
| Image(6) | 19.9330/813 | 19.9060/815 | 20.0140/816 |
| Image(7) | 18.5280/824 | 18.5280/824 | 21.8500/870 |
| Image(8) | 24.5080/910 | 24.4900/911 | 27.1820/923 |
| Aver. | 20.817 | 20.829 | 22.163 |

Fig. 4. Average runtime (second) of all methods for color image inpainting on the eight testing color images with different MRs.

reduce the time consumption caused by the calculation of quaternion singular value decompositions (QSVD). The alternating direction method of multipliers (ADMM) framework is applied to solve the models, which indeed guarantees the convergence (as we empirically show in the experiments) of the proposed algorithms. Furthermore, experimental results on real color image inpainting also demonstrate the effectiveness of the developed methods.

One of our future work aims to extend the proposed models to some other low-rank quaternion matrix approximate problems such as color face recognition, color image superresolution, and color image denoising. Recently, some convolutional neural networks (CNN)-based approaches can complete images with very large missing blocks based on numerous training samples. We also would like to extend our ideas to the quaternion-based convolutional neural networks (QCNN) framework in the next works.

### Appendix A

**Basic Knowledge of Quaternion Algebras**

Quaternion space \( \mathbb{H} \) was first introduced by W. Hamilton in 1843, which is an extension of the complex space \( \mathbb{C} \).
A quaternion \( q \in \mathbb{H} \) with a real component and three imaginary components is defined as

\[
q = q_0 + q_1 i + q_2 j + q_3 k,
\]

where \( q_l \in \mathbb{R} \) \((l = 0, 1, 2, 3)\), and \( i, j, k \) are imaginary numbers and obey the quaternion rules that

\[
i^2 = j^2 = k^2 = ijk = -1, \\
i j = -ji = k, jk = -kj = i, ki = -ik = j.
\]

\( q \) can be decomposed into a real part \( \Re(q) := q_0 \) and an imaginary part \( \Im(q) := q_1 i + q_2 j + q_3 k \) such that \( q = \Re(q) + \Im(q) \). If the real part \( \Re(q) = 0 \), \( q \) is named a pure quaternion. Given two quaternions \( p \) and \( q \), the sum and multiplication of them are respectively

\[
p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k
\]

and

\[
pq = (pq_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)i + (p_0q_2 - p_1q_3 + p_2q_0 + p_3q_1)j + (p_0q_3 + p_1q_2 - p_2q_1 + p_3q_0)k.
\]

It is noticeable that the multiplication of two quaternions is not commutative so that in general \( pq \neq qp \). The conjugate and the modulus of a quaternion \( q \) are, respectively, defined as follows

\[
q^* = q_0 - q_1 i - q_2 j - q_3 k, \\
|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.
\]

Analogously, a quaternion matrix \( Q = \left(q_{mn}\right) \in \mathbb{H}^{M \times N} \) is written as \( Q = Q_0 + Q_1 i + Q_2 j + Q_3 k \), where \( Q_l \in \mathbb{H}^{M \times N} \) \((l = 0, 1, 2, 3)\). \( Q \) is named a pure quaternion matrix when \( \Re(Q) = Q_0 = 0 \). The quaternion matrix Frobenius norm is defined as \( \| Q \|_F = \sqrt{\sum_{m=1}^{M} \sum_{n=1}^{N} |q_{mn}|^2} = \sqrt{\text{tr}\{Q^H Q\}} \). Using Cayley-Dickson notation \([43]\), \( Q \) can be expressed as \( Q = Q_0 + Q_1 i + Q_2 j + Q_3 k \), \( Q_0 = Q_0 + Q_1 i \in \mathbb{C}^{M \times N} \), \( Q_0 = Q_2 + Q_3 k \in \mathbb{C}^{M \times N} \). Then the quaternion matrix \( Q \) can be denoted as an equivalent complex matrix

\[
Q_c = \left( \begin{array}{cc} Q_0 & Q_1 \\ -Q_1^* & Q_0^* \end{array} \right) \in \mathbb{C}^{2M \times 2N}.
\]

Based on \([17]\), the Q-SVD of quaternion matrix \( Q \) can be obtained by applying the classical complex SVD algorithm to \( Q_c \). The relation between the Q-SVD of quaternion matrix \( Q \in \mathbb{H}^{M \times N} \) and the SVD of its equivalent complex matrix \( Q_c \in \mathbb{C}^{2M \times 2N} \) (\( Q_c = UD\bar{V}^H \)) is defined as \([44]\)

\[
D = \text{row}_{\text{odd}}(\text{col}_{\text{odd}}(D)) \quad U = \text{col}_{\text{odd}}(U) + \text{col}_{\text{odd}}(-U_2^*)j, \\
V = \text{col}_{\text{odd}}(V) + \text{col}_{\text{odd}}(-V_2^*)j,
\]

such that \( Q = UD\bar{V}^H \), where

\[
U = \left( \begin{array}{c} U_1 \quad M \times 2M \\ U_2 \quad M \times 2M \end{array} \right), \\
V = \left( \begin{array}{c} V_1 \quad N \times 2N \\ V_2 \quad N \times 2N \end{array} \right),
\]

and \( \text{row}_{\text{odd}}(M), \text{col}_{\text{odd}}(M) \) respectively extract the odd rows and odd columns of matrix \( M \). Readers can find more details on quaternion algebra in \([28]\), \([45]\), \([46]\).

**APPENDIX B**

The proof of Theorem 2

Proof: According to the Q-SVD in Theorem 1, there exist unitary quaternion matrices \( \hat{A} \in \mathbb{H}^{M \times M} \) and \( \hat{B} \in \mathbb{H}^{N \times N} \) such that

\[
\hat{X} = \hat{A} \left( \begin{array}{cc} D_d & 0 \\ 0 & 0 \end{array} \right) \hat{B}^H,
\]

where \( D_d = \text{diag}\{d_1, 1, \ldots, d_r, d_{r+1}, \ldots, d_d\} \in \mathbb{H}^{d \times d} \) and the \( d_1, \ldots, d_r \) are the positive singular values of \( X \), \( d_{r+1}, \ldots, d_d \) are equal to 0. Then, \([49]\) can be rewritten as

\[
\hat{X} = \hat{A} \left( \begin{array}{c} D_d^\frac{1}{2} \\ 0 \end{array} \right)_{M \times d} \left( \begin{array}{c} D_d \frac{1}{2} \\ 0 \end{array} \right)_{d \times N} d_{d \times N} \hat{B}^H.
\]

Let \( \hat{U} = \hat{A} \left( \begin{array}{c} D_d^\frac{1}{2} \\ 0 \end{array} \right) \in \mathbb{H}^{M \times d} \) and \( \hat{V} = \hat{B} \left( \begin{array}{c} D_d \frac{1}{2} \\ 0 \end{array} \right) \in \mathbb{H}^{N \times d} \), they obviously meet \( \text{rank}(\hat{U}) = \text{rank}(\hat{V}) = r \), and \( \hat{X} = \hat{U}\hat{V}^H \).
APPENDIX C
THE PROOFS OF THEOREM 3, THEOREM 4 AND THEOREM 5

To prove the Theorem 3, Theorem 4 and Theorem 5, we first give the following lemma.

Lemma 1. Let two quaternion matrices \( U \in \mathbb{H}^{M \times d} \) and \( V \in \mathbb{H}^{N \times d} \) be given and let \( p > 0 \), denote \( K = \min(M, N, d) \), the following inequality holds for the decreasingly ordered singular values of \( UV^H \), \( U \) and \( V \):

\[
\sum_{k}^{K} k^p (UV^H) \leq \sum_{k}^{K} k^p (U)k^p (V). \tag{50}
\]

Proof:

\[
\sum_{k}^{K} k^p (UV^H) = \frac{1}{2} \sum_{k}^{K} k^p (P(UV^H)) = \frac{1}{2} \sum_{k}^{K} k^p (P(U)P(V^H)) \leq \frac{1}{2} \sum_{k}^{K} k^p (P(U))k^p (P(V^H)) = \sum_{k}^{K} k^p (U)k^p (V),
\]

where the inequality follows from \[47\] (Theorem 3.3.14).

Proof of Theorem 3 Since \( X = UV^H \), where \( U \in \mathbb{H}^{M \times d} \) and \( V \in \mathbb{H}^{N \times d} \), denote \( K = \min(M, N, d) \), we have

\[
\|X\|_* = \sum_{k}^{K} \sigma_k(X) = \sum_{k}^{K} \sigma_k(UV^H) \leq \sum_{k}^{K} \sigma_k(U)\sigma_k(V)
\]

\[
\leq \left( \sum_{k}^{K} \sigma_k^2(U) \right)^{1/2} \left( \sum_{k}^{K} \sigma_k^2(V) \right)^{1/2}
\]

\[
\leq \frac{1}{2} \left( \sum_{k}^{K} \sigma_k^2(U) \right) + \frac{1}{2} \left( \sum_{k}^{K} \sigma_k^2(V) \right)
\]

\[
\leq \frac{1}{2} \left( \sum_{k}^{\min(M,d)} \sigma_k^2(U) \right) + \frac{1}{2} \left( \sum_{k}^{\min(N,d)} \sigma_k^2(V) \right)
\]

\[
= \frac{1}{2} \left( \|U\|_F^2 + \|V\|_F^2 \right),
\]

where the first inequality follows from Lemma 1 as \( p = 1 \), the second inequality follows from the well-known Holder’s inequality \[48\]. the third inequality holds due to the Jensen’s inequality \[49\], and since we always have \( K = \min(M, N, d) \leq \min(M, d) \), \( K = \min(M, N, d) \leq \min(N, d) \), thus the last inequality holds. On the other hand, let \( \hat{U}_* = \hat{A}_X\hat{D}_X^2 \) and \( \hat{V}_* = \hat{B}_X\hat{D}_X^2 \), where \( \hat{X} = \hat{A}_X\hat{D}_X\hat{B}_X^H \) is the QSVD of \( X \). Then, we have \( X = \hat{U}_*\hat{V}_*^H \) and \( \|X\|_{Q-S_{1/2}} = \frac{1}{2} \left( \|\hat{U}_*\|_F^2 + \|\hat{V}_*\|_F^2 \right)^2 \).

Hence, from above, we have

\[
\min_{X=UV^H} \frac{1}{2} \|U\|_F^2 + \frac{1}{2} \|V\|_F^2 = \|X\|_{*}.
\]

Proof of Theorem 4 Since \( X = UV^H \), where \( U \in \mathbb{H}^{M \times d} \) and \( V \in \mathbb{H}^{N \times d} \), denote \( K = \min(M, N, d) \), we have

\[
\|X\|_{Q-S_{1/2}}^2 = \sum_{k}^{K} \sigma_k^{1/2}(X) = \sum_{k}^{K} \sigma_k^{1/2}(UV^H)
\]

\[
\leq \sum_{k}^{K} \sigma_k^{1/2}(U)\sigma_k^{1/2}(V)
\]

\[
\leq \left( \sum_{k}^{K} \sigma_k(U) \right)^{1/2} \left( \sum_{k}^{K} \sigma_k(V) \right)^{1/2}
\]

\[
\leq \frac{1}{2} \left( \sum_{k}^{K} \sigma_k(U) \right) + \frac{1}{2} \left( \sum_{k}^{K} \sigma_k(V) \right)
\]

\[
\leq \frac{1}{2} \left( \sum_{k}^{\min(M,d)} \sigma_k(U) \right) + \frac{1}{2} \left( \sum_{k}^{\min(N,d)} \sigma_k(V) \right)
\]

\[
= \frac{1}{2} \left( \|\hat{U}_*\|_F^2 + \|\hat{V}_*\|_F^2 \right),
\]

where the first inequality follows from Lemma 1 as \( p = \frac{1}{2} \), the second inequality follows from the well-known Holder’s inequality \[48\], the third inequality holds due to the Jensen’s inequality \[49\], and since we always have \( K = \min(M, N, d) \leq \min(M, d) \), \( K = \min(M, N, d) \leq \min(N, d) \), thus the last inequality holds. On the other hand, let \( \hat{U}_* = \hat{A}_X\hat{D}_X^2 \) and \( \hat{V}_* = \hat{B}_X\hat{D}_X^2 \), where \( \hat{X} = \hat{A}_X\hat{D}_X\hat{B}_X^H \) is the QSVD of \( X \). Then, we have \( X = \hat{U}_*\hat{V}_*^H \) and \( \|X\|_{Q-S_{1/2}} = \frac{1}{2} \left( \|\hat{U}_*\|_F^2 + \|\hat{V}_*\|_F^2 \right)^2 \).

Hence, from above, we have

\[
\min_{X=UV^H} \frac{1}{4} \|\hat{U}_*\|_F^2 + \|\hat{V}_*\|_F^2 = \|X\|_{Q-S_{1/2}}.
\]

Proof of Theorem 5 Since \( \hat{X} = \hat{U}_*\hat{V}_*^H \), where \( \hat{U} \in \mathbb{H}^{M \times d} \) and \( \hat{V} \in \mathbb{H}^{N \times d} \), denote \( K = \min(M, N, d) \), we have

\[
\|X\|_{Q-S_{2/3}}^2 = \sum_{k}^{K} \sigma_k^{2/3}(X) = \sum_{k}^{K} \sigma_k^{2/3}(UV^H)
\]

\[
\leq \sum_{k}^{K} \sigma_k^{2/3}(U)\sigma_k^{2/3}(V)
\]

\[
\leq \left( \sum_{k}^{K} \sigma_k^{2/3}(U) \right)^{2/3} \left( \sum_{k}^{K} \sigma_k^{2/3}(V) \right)
\]

\[
\leq \frac{1}{3} \left( \sum_{k}^{K} \sigma_k^{2/3}(U) \right) + \frac{2}{3} \left( \sum_{k}^{K} \sigma_k(V) \right)
\]

\[
\leq \frac{1}{3} \left( \sum_{k}^{\min(M,d)} \sigma_k(U) \right) + \frac{2}{3} \left( \sum_{k}^{\min(N,d)} \sigma_k(V) \right)
\]

\[
= \frac{1}{3} \left( \|\hat{U}_*\|_F^2 + 2\|\hat{V}_*\|_F^2 \right),
\]

where the first inequality follows from Lemma 1 as \( p = \frac{2}{3} \), the second inequality follows from the well-known Holder’s inequality \[48\], the third inequality holds due to the Jensen’s inequality \[49\], and since we always have \( K = \min(M, N, d) \leq \min(M, d) \), \( K = \min(M, N, d) \leq \min(N, d) \), thus the last inequality holds. On the other hand, let \( \tilde{U}_* = \tilde{A}_X\tilde{D}_X^2 \) and \( \tilde{V}_* = \tilde{B}_X\tilde{D}_X^2 \), where \( \tilde{X} = \tilde{A}_X\tilde{D}_X\tilde{B}_X^H \) is the QSVD of \( X \). Then, we have \( X = \tilde{U}_*\tilde{V}_*^H \) and \( \|X\|_{Q-S_{2/3}} = \frac{1}{2} \left( \|\tilde{U}_*\|_F^2 + \|\tilde{V}_*\|_F^2 \right)^2 \).

Hence, from above, we have

\[
\min_{X=UV^H} \frac{1}{4} \|\tilde{U}_*\|_F^2 + \|\tilde{V}_*\|_F^2 = \|X\|_{Q-S_{2/3}}.
\]
\[
\min(M, d), K = \min(M, N, d) \leq \min(N, d), \text{ thus the last inequality holds.} \text{ On the other hand, let } U = X^T \Sigma X \text{ and } V = X^T \Sigma X \text{, where } X = X^T \Sigma X \Sigma X^T \text{ is the QSVD of } X. \text{ Then, we have } X = U \Sigma V^H \text{ and } \|X\|_{Q-S_{2/3}} = \left(\|U\|_F^2 + 2\|V\|_F^2\right)^{3/2}. \]

Hence, from above, we have
\[
\min_{X=UV^H} \left(\|U\|_F^2 + 2\|V\|_F^2\right)^{3/2} = \|X\|_{Q-S_{2/3}}.
\]

**APPENDIX D**

**SOLVING THE PROBLEM**

Similar to the problem (28), the problem (29) is solved by minimizing the following augmented Lagrangian function
\[
\mathcal{L}_\mu(U, V, A, \Sigma, X, F_1, F_2) = \frac{\lambda}{3} \left(\|U\|_F^2 + 2\|A\|_F^2\right) + \langle F_1, V - \tilde{A} \rangle + \langle F_2, X - U V^H \rangle + \mu \left(\|\tilde{V} - \tilde{A} \Sigma \|_F^2\right),
\]
where \(\mu > 0\) is the penalty parameter, \(F_1\) and \(F_2\) are Lagrange multipliers.

**Updating \(U\) and \(V\):**
\[
\tilde{U}^{r+1} = \arg\min_U \frac{1}{2} \|X - \tilde{U}(\tilde{V}^r)^H + \tilde{F}_2^r / \mu^r\|_F^2 + \lambda \frac{\|\tilde{U}\|_F^2}{3} + \frac{\mu}{2} \left(\|\tilde{V} - \tilde{A} \|_F^2\right)
\]
\[
\tilde{V}^{r+1} = \arg\min_V \frac{1}{2} \|\tilde{X} - \tilde{U}^{r+1} \tilde{V}^H + \tilde{F}_1^r / \mu^r\|_F^2 + \frac{\|\tilde{V} - \tilde{A}\|_F^2}{\lambda} + \frac{\mu}{2} \left(\|\tilde{V} - \tilde{A}\|_F^2\right).
\]

By the similar way as [33], we can obtain the optimal solution of \(\tilde{U}^{r+1}\) and \(\tilde{V}^{r+1}\) as
\[
\tilde{U}^{r+1} = \left(\mu^r X + \tilde{F}_2^r / \mu^r\right)
\]
\[
\tilde{V}^{r+1} = \left(\tilde{A} \Sigma X + \tilde{F}_1^r / \mu^r\right)^H \left(1 + (\tilde{U}^{r+1})^H \tilde{U}^{r+1}\right)^{-1}.
\]

**Updating \(A\):**
\[
\tilde{A}^{r+1} = \arg\min_{\tilde{A}} \frac{1}{2} \|\tilde{X} - \tilde{U}^{r+1} \tilde{A} \Sigma\|_F^2 + \frac{\lambda}{\mu^r} \|\tilde{A}\|_F^2
\]
\[
\tilde{X}^{r+1} = \arg\min_{\tilde{X}} \frac{1}{2} \|\tilde{X} - \tilde{U}^{r+1}(\tilde{V}^{r+1})^H - \tilde{F}_1^r / \mu^r\|_F^2 + \frac{\lambda}{\mu^r} \|\tilde{X} - \tilde{U}^{r+1}(\tilde{V}^{r+1})^H\|_F^2.
\]

Then, we can directly obtain the optimal \(X^{r+1}\)
\[
\tilde{X}^{r+1} = \mathcal{P}_{\Omega c}(\tilde{U}^{r+1}(\tilde{V}^{r+1})^H - \tilde{F}_1^r / \mu^r)
\]
\[
+ \mathcal{P}_{\Omega t}\left(\mu^r \tilde{U}^{r+1}(\tilde{V}^{r+1})^H - \tilde{F}_2^r / \mu^r + \tilde{T}\right),
\]
where \(\Omega^c\) is the complement of \(\Omega\), and we have used the fact that \(\mathcal{P}_{\Omega t}(\mathcal{T}) = 0\) in [59].

**Updating \(F_1\) and \(F_2\):**
\[
\tilde{F}_1^{r+1} = \tilde{F}_1^r + \mu^r (\tilde{U}^{r+1} - \tilde{A}^{r+1})
\]
\[
\tilde{F}_2^{r+1} = \tilde{F}_2^r + \mu^r (\tilde{X}^{r+1} - \tilde{U}^{r+1}(\tilde{V}^{r+1})^H),
\]
\[
\mu^{r+1} = \min(\beta \mu^r, \mu_{\text{max}}).
\]

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