Where the Tsallis Statistic is valid?

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In the present paper are analyzed the conditions for the validity of the Tsallis Statistics. The same have been done following the analogy with the traditional case: starting from the microcanonical description of the systems and analyzing the scaling properties of the fundamental macroscopic observables in the Thermodynamic Limit. It is shown that the Generalized Legendre Formalism in the Tsallis Statistic only could be applied for one special class of the bordering systems, those with non exponential growth of the accessible states density in the thermodynamic limit and zero-order divergence behavior for the fundamental macroscopic observables, systems located in the chaos threshold.

I. INTRODUCTION

In the last years many researchers have been working in the justification of the Tsallis Formalism, but many of them have pretend to do it in the context of the Information Theory[1, 2, 3] without appeal to the microscopic properties of the systems. Our opinion is that all of these attempts lead to the conclusion of the non uniqueness of the entropy concept by mean of probabilistic interpretation. That is why we consider that the statistical description of nonextensive systems has to start from the microscopic characteristics of them.

Boltzmann[4], Gibbs[5], Einstein[6, 7] and Ehrenfests[8] recognized the hierarchical primacy of the microcanonical ensemble with regard to the others statistical ensembles. These last ones can be derived starting from the first by the consideration of certain particular conditions: the extensivity postulates. This is the essential ideas to generalize the traditional results for the justification of the Tsallis Statistics, to put it on the level of the microscopic description, in the ground of the Mechanics.

With these ideas A. Plastino and A. R. Plastino[9] have proposed one way to justify the q-generalized canonical ensemble with similar arguments employed by Gibbs himself in deriving his canonical ensemble. It is based in the consideration of a closed system composed by a subsystem with a weakly interaction with a finite thermal bath. They showed that the macroscopic characteristics of the subsystem are described by a Tsallis potential distribution, relating the q entropy index with the finiteness of the last one.

Another attempt was made by S. Abe and A. K. Rajagopal[10, 11], with the same idea: a closed system composed by a subsystem with a weakly interaction with a very large thermal bath, but this time analyzing the behavior of the systems around the equilibrium, considering the last one as the most probable configurations. He shown that the Tsallis generalization of the Boltzmann-Gibbs Distribution can be obtained if the counting rule is modified, introducing the Tsallis generalization of the logarithmic function for arbitrary entropic index.

However we disagree with these interpretations. So far it has been said that the statistic of Tsallis allows to extend the study for systems that are anomalous for the traditional thermodynamic, systems with of long-range correlations, due to the presence of long-range interactions with a dynamics of non markovianity stochastic processes, where the entropic index gives a measure of the non extensivity of a system, an intrinsic characteristic of the same[12]. That is the reason why we find the identification of this parameter with the finiteness of a thermal bath an artificial, outside of the supposed application context that has the Statistic of Tsallis.

For example, for the Plastino and Plastino analysis, What is that we consider by a thermal bath in astrophysical systems or a atomic nucleus? What is that we consider by a finite thermal bath in the turbulent fluids or a non screened plasma?

By other hand, for the Abe and Rajagopal analysis, they insisted that there is an arbitrariness in selection of the counting rule that which determinate the form of the distribution. Although the selection of the counting rule do not have so much influence for configurations of the system near from the equilibrium, for configurations far from the equilibrium this selection is very important since this fact mark the difference among all the generalized canonical ensembles. In their works they do not establish a criterium that allows to define in univocal manner the selection of the counting rule.

However, this is not the case of the Boltzmann-Gibbs Statistics, because its counting rule is supported by the scaling
behavior of the states density in the Thermodynamic Limit (ThL), by mean of the scaling behavior of the fundamental macroscopic observables with the increasing of the degree of freedom of the system, and its Thermodynamic Formalism, based on the Legendre Transformation between the Thermodynamic potentials, by equivalence between the microcanonical and the canonical ensembles in the ThL too.

At the present work we pretend to analyze the conditions to satisfy by the system in order to justify the application of the Tsallis Statistics in them. This time, we will pay special attention to equivalency of the ensembles in the ThL

II. MICROCANONICAL ENSEMBLE: ITS GEOMETRICAL ASPECTS AND SCALING PROPERTIES.

It is very well know that the microcanonical ensemble are equivalent to the traditional canonical and Gran-canonical ensembles in the ThL of the infinitely many particles interacting by short range interactions if the system is homogeneous.

However, in the nature we find examples of systems where the thermodynamic limit does not take place, since they are not composed by a huge number of particles. We find examples of these in the molecular and atomic clusters. A very interesting problem is the extension of the macroscopic description for this kind of systems as well as the question about when could be considered that these systems are found in the thermodynamic limit.

An important step toward the resolution of these problems was accomplished by D.H.E. Gross with the development of the Microcanonical Thermostatistics \[13, 14\], formalism based on the consideration of the microcanonical ensemble. In this approach is possible to accomplish the description of some finite systems, extending thus the study of phase transitions in them. On the other hand, it is attractive the possibility that has the same of exploring the behavior of the macroscopic observables with the increasing of the number of particles, until the convergency in some cases in an ordinary extensive system, permitting thus to give a valuation about when the thermodynamic limit is reached in the system.

In our opinion, this study revealed two very useful conclusions:

- Any closed system could be exhaustively described by the microcanonical ensemble without matter how many particles it has. In fact, this is the only way to generalize the equilibrium thermodynamics for arbitrary closed system without appealing to anything outside of the mechanics.

- It is possible that in the thermodynamical limit the microcanonical ensemble could be equivalent to some generalized canonical ensemble.

Following the analogy with the traditional case, it would be possible to associate to this generalized canonical ensemble a probabilistic interpretation of the entropy concept with the same style of Shannon -Boltzmann-Gibbs entropy and the non extensive entropy of Tsallis, being this entropy equivalent with some generalized definition of the Boltzmann entropy in the thermodynamic limit. In the last case, it is really interesting point because open the door to a possibility to obtain the entropic index \(q\) of the Tsallis formalism without the necessity of invoke to additional postulates.

A common denominator of all Probabilistic Thermodynamics Formalisms (PThF), those derived from a probabilistic interpretation of the entropy concept, is the description of the equilibrium macroscopic state of the systems by means of the intensive parameters of the generalized canonical distributions, \(\beta\), that is to say, they are supported by the validity of the zero principle of the Thermodynamics.

In our previous works \[15, 16\] we centered in the geometrical aspects that posses the Probabilistic Distribution Functions (PDF) of the different ensembles. In the case of the microcanonical ensemble we showed that its PDF is invariant under the transformation group of local reparametrizations or Diffeomorfism of the space of the integrals of movement, \(Diff(\mathbb{S}_N)\), being this the maximal symmetry group that a geometrical theory could have, being this associated with the local properties of a general space. By other hand, all the PDF derived from the PThF must depend on the integrals of movement in a lineal combination with the canonical intensive parameters:

\[
p(X; \beta, N) = F(\beta, N; \beta \cdot I_N(X))
\]  

therefore, the most general symmetry group that preserve this functional form is the general lineal group, \(GL(\mathbb{R}^n)\)(\(n\) is the dimension of \(\mathbb{S}_N\)), which is related with the euclidean vectorial spaces. It is supposed that a spontaneous symmetry breaking happens during the thermodynamic limit, from \(Diff(\mathbb{S}_N)\) to \(GL(\mathbb{B}^n)\), where \(\mathbb{B}^n\) is the space of the generalized canonical parameters. The way in that this hypothetical symmetry breaking happens will determine the specific form of the statistics in the ThL.
What we understand by a spontaneous symmetry breaking? Ordinarily, the microcanonical ensemble could be described equivalently through any representation of the abstract space of the integrals of movement of the distribution. However, with the increasing of the degree of freedom of the system, some specific representation will be more adequate to describe the macroscopic state because they reflect in better manner the general properties of the system: i.e. in the case of the traditional systems, the extensivity.

In complete analogy with the traditional analysis, we identify the cause of this spontaneous symmetry breaking with the scaling properties of the fundamental macroscopic observables, the behavior of the integrals of movement and the accessible states density with the increasing of the degree of freedom of the system.

In general, the scaling properties of the systems in the ThL could be diverse. In order to be more specific, we will consider here the following scaling behavior of the state density:

\[ a) \quad W(I, N) \simeq \exp(\gamma(I, N) N^\tau) \]
\[ b) \quad W(I, N) \simeq \gamma(I, N) N^\tau \]
\[ c) \quad W(I, N) \simeq \gamma(I, N) \ln N \]
\[ d) \quad W(I, N) \simeq N^{\gamma(I, N)} \]

where \( \gamma(I, N) \) is an scaling invariant function of the integrals of movement, and \( \tau \) is an scaling invariant parameter. Extensivity is associated with the dependency a) with \( \tau = 1 \), when the interacting forces have a short-range and there is independency between different parts of the system. Any deviation of this behavior is a signal of non-extensivity, the presence of long-range correlation in the system. The systems with scaling behavior given by (2.a) with \( 0 < \tau < 1 \) we will refer as pseudoextensive systems and the others by bordering systems. The reasons for the last denominations will be understand later.

In order to aim this discussion in the analysis of the conditions for the validity of the Tsallis formalism, we will fit all above behaviors by:

\[ W(I, N) \simeq e_q[\gamma(I, N) \ln_p(N)] \]

where \( e_q(x) \) and \( \ln_q(x) \) are the generalized q-exponential and q-logarithmic function of the Tsallis formalism:

\[ e_q(x) = [1 + (1 - q) x]^{\frac{1}{1-q}} \quad \ln_q(x) = \frac{x^{1-q} - 1}{1-q} \equiv e_q^{-1}(x) \]

In this case, it is convenient define the entropy by:

\[ S_{B-T} = \ln_q[W] \]

in order to access to the invariant scaling function \( \gamma(I, N) \), because it contains all the useful information that we can extract from the macroscopic description of the system. We recognize immediately the Tsallis generalization of the Boltzmann entropy.

III. THE LEGENDRE FORMALISM

In this section we will pay attention to the relationship between the microcanonical and the canonical ensemble. To do this we will reproduce first the general analysis realized by D.H.E. Gross in deriving his microcanonical thermostatistics. After, in analogy with the previous exposition, we will consider the q-generalization of the canonical ensemble of Tsallis.

A. The case of pseudoextensive systems.

For this kind of system we will consider the usual definition of Boltzmann entropy with \( q = 1 \):

\[ S_B = \ln[W] \]

In this case, the canonical PDF will be given by:

\[ \omega(X; \beta, N, a) = \frac{1}{Z(\beta, N, a)} \exp[-\beta \cdot I_N(X; a)] \]

where \( Z(\beta, a, N) \) is the partition function.
\[ Z(\beta, N, a) = \int \exp[-\beta \cdot I_N(X;a)]dX \] (8)

Introducing the delta function in the above integral, it can be rewritten as:

\[ Z(\beta, N, a) = \int \exp[-\beta \cdot I]dI \int \delta[I - I_N(X;a)]dX \] (9)

The integration of the delta function yields the density of states:

\[ W(I, N, a) = \Omega(I, N, a) \delta I_o = \left( \int \delta[I - I_N(X;a)]dX \right) \delta I_o \] (10)

where \( \delta I \) is a suitable constant volume element to make \( W \) dimensionless. We arrive finally to the Laplace Transformation of the states density:

\[ Z(\beta, N, a) = \int \exp[-\beta \cdot I]W(I, N, a) \frac{dI}{\delta I_o} \] (11)

The Laplace Transformation establishes the connection between the Boltzmann entropy with the fundamental thermodynamics potential of the canonical distribution, the Planck Potential:

\[ P(\beta, N, a) = -\ln Z(\beta, N, a) \] (12)

We adopted the Planck Potential because it does not introduce any preference with a specific integral of movement of the microcanonical distribution, in order to be consistent with the \( GL(R^n) \) invariance. The last integral can be rewritten as:

\[ \exp[-P(\beta, N, a)] = \int \exp[-\beta \cdot I + S_B(I, N, a)] \frac{dI}{\delta I_o} \] (13)

Immediately, we recognized in the exponential argument the Legendre Transformation between the thermodynamic potentials. If the integrals of movements and the Boltzmann entropy have the same scaling behavior in the ThL, this integral will have a very sharp peak around the maximum value if this maximum exists and therefore, the main contribution to this integral will come from this maximum.

If we define \( \tilde{P}(\beta, N, a) \) by:

\[ \tilde{P}(\beta, N, a) = \max_{I=I_M} [\beta \cdot I - S_B(I, N, a)] \] (14)

The maximization yields:

\[ \beta_\mu = \frac{\partial}{\partial I_\mu} S_B \bigg|_{I=I_M} \] (15)

being this the usual relation between the canonical intensive parameters and the entropy. Introducing the curvature tensor:

\[ (K_B)_{\mu\nu} = \frac{\partial}{\partial I_\mu} \frac{\partial}{\partial I_\nu} S_B \bigg|_{I=I_M} \] (16)

the Laplace transformation will be estimated by:

\[ \exp[-P(\beta, N, a)] \approx \int \exp[-\tilde{P}(\beta, N, a)] \exp \left[ -\frac{1}{2} (I - I_M)^\mu \cdot (-K_B)_{\mu\nu} \cdot (I - I_M)^\nu \right] \frac{dI}{\delta I_o} \] (17)

\[ \approx \exp[-\tilde{P}(\beta, N, a)] \frac{1}{\delta I_o \det^\frac{1}{2} \left( -\frac{1}{2\pi} (K_B)_{\mu\nu} \right)} \] (18)

The above integral will be well defined if all the eigenvalues of the curvature tensor are negatives, if the Boltzmann entropy if locally concave around the maximum. In this case, in the canonical ensemble there will be small fluctuations of the integrals of movement around its mean values with a standard deviation:
\[
\sqrt{(I^\mu - I_M^\mu)(I^\nu - I_M^\nu)} \sim \frac{1}{N^2}
\]

Thus, the Planck potential could be obtained by mean of the Legendre formalism:

\[
P(\beta, N, a) \simeq \beta \cdot I - S_B(I, N, a)
\]  

and the Boltzmann entropy will be equivalent with the Shannon-Boltzmann-Gibbs entropy:

\[
S_{S-B-G} = -\sum_k p_k \ln p_k \simeq S_B
\]

However, if the concavity condition for the entropy is not hold, there will be a catastrophe in the Laplace transformation and the canonical ensemble will not be able to describe the system. Ordinarily, this situation is a signal of the occurrence of phase transitions in the system.

As it has been shown, although the pseudoextensive systems are nonextensive in the usual sense, their study could be carry out with the same formalism employing for the ordinary extensive systems with appropriate selection of the representation of the space of integrals of movement. The Eq. (15) leads to condition of the homogeneous scaling of the integrals of movement and the entropy in order to satisfy the requirement of the scaling invariance of the canonical parameters, the validity of the zero principle of the thermodynamics. Here we have the cause of the spontaneous symmetry breaking. Although in the microcanonical ensemble all the representation of the integrals of movement are equivalent, in the generalized canonical ensemble only are possible those representations with an homogeneous nondegenerated scaling. To understand the term nondegenerate let us show the following example. Let be \( A \) and \( B \) two integrals of movement with different scaling behavior:

\[
A \sim N^a; \ B \sim N^b \text{ with } a > b
\]

In another representation, these integrals could be represented equivalently in the microcanonical ensemble by:

\[
I^\pm = A \pm B
\]

but in this case the scaling behavior of them are:

\[
I^\pm \approx A \sim N^a
\]

In the ThL \( I^\pm \) have an homogeneous scaling but they are not independent. The canonical parameters derived of \( I^\pm, \beta^\pm = \frac{\partial S_B}{\partial I^\pm} \), will be identically in the ThL. This leads to the trivial vanishing of the curvature tensor determinant in ThL:

\[
\lim_{N \to \infty} N^{2a} \det (K_{\mu\nu}) = \lim_{N \to \infty} N^{2a} \det \left( \partial_+ \partial_+ S_B \partial_- \partial_+ S_B \right) = 0
\]

This is an example of an homogeneous degenerate scaling representation. These representations are inadmissible for the generalized canonical ensemble. The above considerations give a criterion of an homogeneous nondegenerated scaling representation, the non trivial vanishing of the curvature tensor determinant:

\[
\lim_{N \to \infty} \det (N^+ \partial_+ \partial_+ S_q) \neq 0
\]

B. The bordering systems.

The previous analysis showed that the possible application of the formalism of Tsallis can be found for the bordering systems. An exponential growth of the density of accessible states during the ThL indicates that this system will be strongly chaotic due the spectacular quantity of states that it will be able to access during its evolution in the time. However, using the previous argument, the bordering systems will have a weaker chaotic behavior.

The \( q \)-generalization of the Boltzmann-Gibbs PDF from the Tsallis Formalism[17, 18] is given by:
\[ \omega_q(X; \beta, N, a) = \frac{1}{Z_q(\beta, N, a)} e_q[-\beta \cdot I_N(X; a)] \] (27)

where \( Z_q(\beta, a, N) \) is the q-generalized partition function:

\[ Z_q(\beta, N, a) = \int e_q[-\beta \cdot I_N(X; a)] dX \] (28)

For this ensemble the q-generalized Laplace Transformation will be give by:

\[ Z_q(\beta, N, a) = \int e_q[-\beta \cdot I] W(I, N, a) \frac{dI}{\delta I_o} \] (29)

The Laplace Transformation establishes the connection between the fundamental potentials of both ensembles, the q-generalized Planck potential:

\[ P_q(\beta, N, a) = -\ln_q[Z_q(\beta, N, a)] \] (30)

and generalized Boltzmann entropy given in Eq.(5), by the expression:

\[ e_q[-P_q(\beta, N, a)] = \int e_q[-\beta \cdot I] e_q[S_{B-T}(I, N, a)] \frac{dI}{\delta I_o} \] (31)

The q-logarithmic function satisfy the subadditivity relation:

\[ \ln_q[xy] = \ln_q[x] + \ln_q[y] + (1-q) \ln_q[x] \ln_q[y] \] (32)

From it is derived the identity:

\[ e_q[x] e_q[y] = e_q[x + y + (1-q) xy] \] (33)

The last identity allows to rewrite Eq.(31) as:

\[ e_q[-P_q(\beta, N, a)] = \int e_q[-\beta \cdot I + S_{B-T}(I, N, a) - (1-q) (\beta \cdot I) S_{B-T}(I, N, a)] \frac{dI}{\delta I_o} \] (34)

In the Tsallis case, the lineal form of the Legendre Transformation is violated and therefore, in this case the ordinary Legendre Formalism do not establish the correspondence between the two ensembles.

The nonlineal term in the q-exponential argument violate too the homogeneous scaling of all macroscopic observables for arbitrary scaling behavior. The only possibility to preserve the homogeneous scaling in the q-exponential argument is that all fundamental macroscopic observables have the same zero-order divergency behavior. The function \( f(x) \) will have a zero-order divergency behavior in the infinite limit if it satisfy the conditions:

\[ \left( \lim_{x \to \infty} f(x) = \infty \right) \land \left( \lim_{x \to \infty} \frac{f(x)}{x^\alpha} = 0; \forall \alpha > 0 \right) \] (35)

The above conditions are only satisfied by systems with scaling behavior given in (2.6). This kind of scaling behavior must correspond to systems with so weak chaotic regimen, systems belonging to the chaos threshold. This result have been supported by an entire series of works that indicate that the formalism of Tsallis should be associated to those systems with a dynamic behavior located in the edge of the chaos (see in [19, 20]).

For this case, we have to assume the nonlineal Tsallis generalization of the Legendre Formalism[21, 22] given by:

\[ \tilde{P}_q(\beta, N, a) = \text{Max} [c_q \beta \cdot I - S_{B-T}(I, N, a)] \] (36)

where \( c_q = 1 + (1-q) S_q \). We recognized the generalization of the Legendre Transformation for the normalized q-expectations values. The maximization leads to the relation:

\[ \beta = \frac{\nabla S_{B-T}}{1 + (1-q) S_{B-T}} (1 - (1-q) \beta \cdot I) \] (37)

using the identity:
\[ \nabla S_B = \frac{\nabla S_{B-T}}{1 + (1 - q) S_{B-T}} \]  

where \( S_B \) is the usual Boltzmann entropy, we can rewrite Eq.(37) as:

\[ \beta = \nabla S_B \left( 1 - (1 - q) \beta \cdot I \right) \]  

solving this equation by successive interactions we find:

\[ \beta = \frac{\nabla S_B}{(1 + (1 - q) I \cdot \nabla S_B)} \]  

valid only if the following restriction is hold:

\[ |(1 - q) I \cdot \nabla S_B| < 1 \]  

The last condition is a very interesting result because it allows to limit the values of the entropy index. If we apply arbitrarily this formalism to a pseudoextensive system then \( I \cdot \nabla S_B \) will not bound in the ThL and the condition only will be satisfy if \( q \equiv 1 \), found again that the Tsallis statistics only will be valid for the bordering systems. There are many examples in the literature in which the same have been applied indiscriminately without matter if the systems are extensive or not: i.e. ideal gas, black body equilibrium emission, and others. In some cases, the authors of such as works have introduced some artificial modifications to the original Tsallis formalism in order to obtain the same results of the classical thermodynamics, i.e. the \( q \)-dependent Boltzmann constant (see for example in [23]). The above results indicate the non applicability of the Tsallis Statistic for these kind of systems.

As we see, the Tsallis formalism introduces a correlation to the canonical intensive parameters of the Boltzmann-Gibbs PDF. However, an important second condition have to be satisfy for the validity of the same, the stability of the maximum.

This condition leads to the \( q \)-generalization of the Microcanonical Thermostatistics of D.H.E.Gross . In this approach, the stability of the Legendre formalism rest in the concavity of the entropy, the negative definition of the quadratic forms of the curvature tensor. In the Tsallis case, the curvature tensor have to be modified by:

\[ (K_q)_{\mu\nu} = \frac{1}{1 - (1 - q) P_q} \left( (2 - q) \frac{\partial}{\partial I^n} \frac{\partial}{\partial I^\nu} S_{B-T} + (1 - q) \left( \beta_{\mu} \frac{\partial}{\partial I^\nu} S_{B-T} + \beta_{\nu} \frac{\partial}{\partial I^\mu} S_{B-T} \right) \right) \]  

Using the definition, we can rewrite Eq.(34) as:

\[ e_q \left[ -P_q (\beta, N, a) \right] \simeq \int e_q \left[ -\tilde{P}_q (\beta, N, a) \right] e_q \left[ \frac{1}{2} \left( I - I_M \right)^\mu \cdot \left( -K_q \right)_{\mu\nu} \cdot \left( I - I_M \right)^\nu \right] \frac{dI}{\delta I_0} \]  

The maximum will be stable if the eigenvalues of the q-curvature tensor are negative and very large. In this case, in the \( q \)-generalized canonical ensemble there will be small fluctuations of the integrals of movement around the its \( q \)-expectation values. The integration of Eq.(43) yields:

\[ e_q \left[ -P_q (\beta, N, a) \right] \simeq e_q \left[ -\tilde{P}_q (\beta, N, a) \right] \frac{1}{\delta I_0 \det \left( -\frac{1 - q}{2\pi} (K_q)_{\mu\nu} \right) \Gamma \left( \frac{2 - q}{1 - q} \right)} \]  

Let \( K_q \) be the quantity:

\[ K_q^{-1} = \frac{1}{\delta I_0 \det \left( -\frac{1 - q}{2\pi} (K_q)_{\mu\nu} \right) \Gamma \left( \frac{2 - q}{1 - q} + \frac{1}{2} n \right)} \]  

rewriting Eq.(44) again:

\[ e_q \left[ -P_q (\beta, N, a) \right] \simeq e_q \left[ -\tilde{P}_q (\beta, N, a) + \ln_q \left[ K_q^{-1} \right] - (1 - q) \ln_q \left[ K_q^{-1} \right] \tilde{P}_q (\beta, N, a) \right] \]  

arriving finally to the condition:
\[ R(q; I, N, a) = \left| \frac{\ln_q \left[ K^{-1} \right]}{P_q(\beta, N, a)} \right| \ll 1 \quad (47) \]

The last condition could be considered as a *optimization problem* since the entropic index is the only one independent variable in the functional dependency of the physical quantities. The specific value of \( q \) could be chosen in order to minimize the function \( R(q; I, N, a) \) for all the possible values of the integrals of movement. This way, the problem of the determination of the entropic index could be solved in the frame of the microcanonical theory without appeal to the computational modeling or the experiment.

Thus, the \( q \)-generalized Planck potential could be obtained by mean of the generalized Legendre transformation:

\[ P_q(\beta, N, a) \simeq c_q \beta \cdot I - S_{B-T}(I, N, a) \quad (48) \]

and the \( q \)-generalization of the Boltzmann entropy will be equivalent with the Tsallis entropy in the ThL:

\[ S_q = - \sum p^q_k \ln p_k \simeq S_{B-T} \quad (49) \]

If the condition (41) or (47) are not satisfied, the Tsallis canonical ensemble will not be able to describe the system in the ThL, there will be a catastrophe of the generalized Laplace Transformation.

**IV. FINAL REMARKS**

We have analyzed the conditions to satisfy by the systems in order to in the thermodynamic limit its study through the microcanonical ensemble would be equivalent to some generalized canonical ensemble, for the special cases of the Boltzmann-Gibbs Distributions and its generalization in the Tsallis statistics. This has been carried out in analogy with the traditional methodology, starting from the properties of scaling of the fundamental observables of the system during the thermodynamic limit.

We have proven that the traditional Thermoestadistics is valid for the case of the pseudo-extensive systems, those with an exponential growth of the accessible states density in the thermodynamic limit, that which allows to extend to their application to some non extensive systems with this kind of behavior. In this analysis we have checked the importance that acquires the local reparametrization invariance of the space of the integrals of movement for the microcanonical ensemble during the passage from this description to the canonical description, with the occurrence of a spontaneous symmetry breaking by mean of the properties of scaling of the system.

On the other hand, when is carried out this analysis for the case of the Tsallis generalization of the Boltzmann-Gibbs Distributions, we have shown that the same only can be valid for one special class of the bordering systems, those with non exponential growth of the density of accessible states in the thermodynamic limit and zero-order divergency behavior of the fundamental macroscopic observables, systems located in the chaos threshold. In this context we have shown an entire series of results of the Tsallis formalism that in this approach they appear in a natural way: the \( q \)-expectation values, the generalized Legendre transformations of between the thermodynamic potentials, as well as the conditions for the validity of the same one, having *a priori* the possibility to estimate the value of the entropic index, without the necessity of appeal to the computational modelation or the experiment.

Of this study it is suggested the possible existence of an entire spectrum of the bordering systems that they are not covered by the Tsallis statistic. We refer to those with a scaling given by the conditions (2.b) and (2.d).

We summarized the results of this work in the following table:

| Generic System Name | Scaling behavior of the states density | Chaotic Dynamical Regimen | Generalized Canonical Statistic |
|---------------------|---------------------------------------|---------------------------|--------------------------------|
| Pseudo-Extensive    | Exponential                           | strongly chaotic          | Boltzmann-Gibbs                |
| Bordering Systems   | Potential, Logarithmic                | weakly chaotic            | Unknown                        |

Table#1. Equivalence of the microcanonical ensemble in the thermodynamic limit with a generalized canonical ensemble, equilibrium properties and dynamical connections.
REFERENCES

[1] Evaldo M. F. Curado, General Aspects of the Thermodynamical Formalism, Brazilian J. of Phys. Vol. 29, no. 1 (1999).
[2] Sumiyoshi Abe, Axioms and uniqueness theorem for the Tsallis entropy, http://arXiv.org/abs/cond-mat/0005538.
[3] R. Salazar; A.R. Plastino; R. Toral, Weakly Nonextensive Thermostatistic and Ising Model with long-range interactions, http://arXiv.org/abs/cond-mat/0005379.
[4] L. Boltzmann, Über die Beziehung eines allgemeinen mechanischen Satzes Zum Hauptsatz der Wärmelehre. Sitzungsbericht der akademie der Wissenschaften, Wien, II:67-73, 1877.
[5] J.W. Gibbs, Elementary Principles in Statistical Physics, Volume II of The Collected works of J. Williard Gibbs, Yale University Press, 1902.
[6] A. Einstein, Eine Theorie der Grundlagen der Thermodynamik, Annal. Phys 11:170-187, 1903.
[7] A. Einstein, Zur allgemeinen molekularen. Theorie der Wärme, Annal. Phys 14:354-362, 1904.
[8] P. Ehrenfest and T. Ehrenfest. The Conceptual Foundation of the Statistical Approach in Mechanics, Cornell University Press, Ithaca NY, 1959.
[9] A. Plastino; A. R. Plastino, Tsallis Entropy and Jaynes’ Information Theory Formalism, Brazilian J. of Phys. Vol. 29, no. 1 (1999).
[10] S. Abe and A. K. Rajagopal, Nonuniqueness of Canonical Ensemble Theory arising from Microcanonical Basis, Phys. Lett. A272, 341 (2000).
[11] S. Abe and A. K. Rajagopal, Macroscopic thermodynamics of equilibrium characterized by power-law canonical distributions, http://arXiv.org/abs/cond-mat/0009400.
[12] C. Tsallis, Nonextensive Statistic: A possible measure of complexity, Brazilian J. of Phys., Vol. 29, no. 1 (1999).
[13] D. H. E. Gross, Microcanonical Statistical Mechanics of some non extensive systems, http://arXiv.org/abs/cond-mat/0004268.
[14] D. H. E. Gross, Microcanonical thermodynamics: Phase transitions in Small systems, 66 Lectures Notes in Physics, World scientific, Singapore (2001) and refs. therein.
[15] L. Velazquez, F. Guzman, Some geometrical aspects of microcanonical distribution, http://arXiv.org/abs/cond-mat/0102459.
[16] L. Velazquez, F. Guzman, Geometrical aspects in Equilibrium Thermodynamics, http://arXiv.org/abs/cond-mat/0105364.
[17] C. Tsallis, J. Stat. Phys. 52, 479 (1988).
[18] A set of mini-reviews is available in Nonextensive Statistical Mechanics and Thermodynamics, eds. S.R.A. Salinas and C. Tsallis, Braz. J. Phys. 29 (1999).
[19] U. Tirnakli, G. F. J. Garin, C. Tsallis, Generalization of the Komolgorov-Sinai entropy: Logistic- and Periodic-like maps at a Chaos Thresholds, http://arXiv.org/abs/cond-mat/0005210.
[20] V. Latora, M. Baranger, A. Rapisarda, C. Tsallis, The rate of entropy at the edge of chaos, preprint [cond-mat/9907412], (1999).
[21] S. Abe, S. Martinez, F. Penini and A. Plastino; Phys. Lett. A (2001), in press [cond-mat/0011012], (2001).
[22] Franck Jedrzejewski, Probabilistic properties of nonextensive thermodynamic, http://arXiv.org/abs/cond-mat/0105386.
[23] S. Abe, S. Martinez, F. Penini and A. Plastino; in press [cond-mat/006109], (2000).