Some properties of Graph Laplacians of cyclic groups

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Abstract

In this paper we investigate a spectra of the Laplacian matrix of cyclic groups using the properties of their characteristic polynomials. We have proved several assertions about the relationship between the spectra of different groups.

Keywords: Graph Laplacians, cyclic groups.

1 Introduction

Let us consider a graph $G$ with the vertex set $V = \{1, \ldots, n\}$ and the edge set $E$.

Definition 1.1. The Laplacian matrix of the Graph $G$ is a matrix $L(G) = (a_{i,j})_{i,j \in V}$, with

$$a_{i,j} = \begin{cases} 
-1 & \text{if } ij \in E \\
 d(i) & \text{if } i = j \\
 0 & \text{otherwise}
\end{cases}$$

where $d(i) = |\{e \in E | i \in e\}|$ is the degree of the vertex $i$.

Definition 1.2. The Cayley Graph of a discrete group $L$ with a system of generators $S$ is the graph whose vertices are the elements of the group $L$ and whose edges are determined by the following condition: if $g$ and $s$ belong to $L$ then there is an edge from $g$ to $f$ if and only if $f = g * s$ for some $s \in S \cup S^{-1}$.

Let us consider the Cayley graph of the group $Z_n$. Note that the Laplacian is a nonnegative operator so all eigenvalues are greater or equal to 0. If $n = 1$, then the Laplacian of the Cayley graph of this group is $\begin{pmatrix} 0 \end{pmatrix}$. This matrix has only one eigenvalue which is zero. If $n = 2$, then the Laplacian of the Cayley graph of $Z_2$ is the matrix

$$\begin{pmatrix} 1 & -1 \\
 -1 & 1 \end{pmatrix}$$

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The eigenvalues of the Laplacian are $\lambda = 0$ and $\lambda = 2$. The Laplacian of $Z_n, n > 3$ is the next matrix:

$$
\begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & -1 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
-1 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix}_n
$$

This matrix is circulant and its eigenvalues are known. In this paper we use another method instead of the well-known method of Gray (see [1]) by using spectrum to investigate the properties of the spectra and the characteristic polynomials of the Laplacians of cyclic groups. Let find the determinant of the following matrix. Set $a = 2 - \lambda$.

$$A_n := \begin{pmatrix}
a & -1 & 0 & \ldots & 0 & -1 \\
-1 & a & -1 & \ldots & 0 & 0 \\
0 & -1 & a & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a & -1 \\
-1 & 0 & 0 & \ldots & -1 & a
\end{pmatrix}_{n-1} + (-1)^n
$$

$$L_{n-1} := \begin{pmatrix}
a & -1 & 0 & \ldots & 0 & 0 \\
-1 & a & -1 & \ldots & 0 & 0 \\
0 & -1 & a & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a & -1 \\
0 & 0 & 0 & \ldots & -1 & a
\end{pmatrix}_{n-1} + \begin{pmatrix}
a & -1 & 0 & \ldots & 0 & 0 \\
-1 & a & -1 & \ldots & 0 & 0 \\
0 & -1 & a & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a & -1 \\
0 & 0 & 0 & \ldots & -1 & a
\end{pmatrix}_{n-2}$$

2
\[
A_n = aL_{n-1} - 2L_{n-2} - 2
\]
We can complete the table of coefficients of \( A_n \) from (2) and the table of coefficients of \( L_n \).

2 Main results

Lemma 2.1. \( \forall n \in N, \forall k \subseteq [1,..,n] : L_n = L_{n-k}L_k - L_{n-k-1}L_{k-1} \).

Proof. \( L_n = aL_{n-1} - L_{n-2} = a(aL_{n-2} - L_{n-3}) - L_{n-2} = (a^2 - 1)L_{n-2} - aL_{n-3} = L_2L_{n-2} - L_1L_{n-3} \).

Assume \( L_n = L_kL_{n-k} - L_{k-1}L_{n-k-1} \). Then \( L_n = L_kL_{n-k} - L_{k-1}L_{n-k-1} = L_k(aL_{n-k-1} - L_{n-k-2}) = L_{k-1}L_{n-k-1} = (aL_k - L_{k-1})L_{n-k-1} - L_kL_{n-k-2} = L_{k+1}L_{n-k-1} - L_kL_{n-k-2} \).

Lemma 2.2. \( \forall n \in N : L_{n-1}^2 = L_{n-2}L_n + 1 \).

Proof. \( L_1 = a, L_2 = a^2 - 1, L_3 = a^3 - 2a, L_2^2 = L_3L_1 + 1 \). Assume \( L_{k-1}^2 = L_{k-2}L_k + 1 \). Then \( L_k^2 = L_{k-1}L_{k+1} + 1 \);

\[
L_{k-1}^2 = L_k - 2L_{k-2}L_k + 1; \quad L_{k-1}^2 = L_k - 2aL_{k-1}L_{k-2} + 1; \quad L_{k-1}^2 + L_{k-2}^2 - 1 = aL_{k-1}L_{k-2}
\]

(1)

\[
L_k^2 = L_{k-1}L_{k+1} + 1; \quad (aL_{k-1} - L_{k-2})^2 = L_{k-1}(aL_k - L_{k-1}) + 1; \quad a^2L_{k-1}^2 - 2aL_{k-1}L_{k-2} + L_{k-2}^2 = aL_kL_{k-1} - L_{k-1}^2 + 1; \quad a^2L_{k-1}^2 - 2aL_{k-1}L_{k-2} = aL_kL_{k-1} - L_{k-1}^2 - L_{k-2}^2 + 1. \]

Then by Equation (1) \( aL_{k-1}^2 - aL_{k-1}L_{k-2} - aL_kL_{k-1} = 0; \ aL_{k-1}(aL_{k-1} - L_{k-2}) - aL_kL_{k-1} = 0; \ aL_kL_{k-1} - aL_kL_{k-1} = 0. \)
Lemma 2.3. If $\lambda$ is the eigenvalue of Laplacian of $Z_n$, then $\lambda$ is the eigenvalue of the Laplacian of $Z_{2^n}, \forall k \in N$.

Proof. $A_{2n} = aL_{2n-1} - 2L_{2n-2} - 2$. Then by Lemma 2.1 we have $L_{2n-1} = L_nL_{n-1} - L_{n-1}L_n$ and $A_{2n} = aL_{n}L_{n-1} - aL_{n-1}L_n - 2L_{2n-2} - 2 = a^2L_{2n-1} - 2aL_{n-1}L_n - 2L_{2n-2} - 2$. The following are routine calculations: $L_{2n-2} = L_{2n-1} - L_{2n-2} - 2$: $A_{2n} = a^2L_{2n-1} - 2aL_{n-1}L_n - 2L_{2n-2} - 2 = A_{2n} = (a^2L_{2n-1} - 4aL_{n-1}L_n + 4L_{2n-2} - 4) - 2L_{n-1} + 2aL_{n-1}L_n + 2$. Then by Lemma 2.2 $-2L_{n-1} - 2L_{n-2} + 2aL_{n-1}L_n + 2 = 0$. So $A_{2n} = a^2L_{n-1} - 4aL_{n-1}L_n - 2L_{2n-2} - 4 = (aL_{n-1} - 2L_{n-2})^2 - 4 = (aL_{n-1} - 2L_{n-2} - 2(aL_{n-1} - 2L_{n-2} + 2) = A_n(n + 4)$.

Note that $A_2 = a^2 - 4$ is not the determinant of the Laplacian of $Z_2$ and $A_1 = a - 2$ is not the determinant of the Laplacian of the trivial group $E$. But $\lambda = 0$ is the eigenvalue of all Laplacian because each Laplacian is a singular matrix. However, by the Table 2 we see that $\lambda = 2$ is the eigenvalue of Laplacian with the multiplicity 2 of $Z_{4k}, k \in N$.

Note that $\lambda = 2$ is not the eigenvalue of the Laplacian of $Z_{4k-2}, k \in N$. It is easy to see that $A_{4k-2}(0) = -2$.

Theorem 2.1. If $\lambda$ is the eigenvalue of the Laplacian of $Z_n, n \geq 3$, then $\lambda$ is the eigenvalue of the Laplacian of $Z_{kn}, \forall k \in N$.

Proof. Lemma 2.3 yields that $A_n$ is a divisor of $A_{2n}$. Now suppose $A_n$ is a divisor of $A_{mn}, \forall m \leq k$.

$A_n = aL_{n-1} - 2L_{n-2} - 2 = L_n - L_{n-2} - 2; A_{(k+1)n} = A_{(k+1)n} = L_{(k+1)n} - L_{(k+1)n} - 2 = L_{kn}L_n - L_{kn-1}L_{n-1} = L_{kn}L_{n-1} + L_{kn-1}L_{n-3} - 2 = L_{kn}(L_n - L_{n-2} - 2) + 2L_{kn} - 2 + L_{kn-1}L_{n-3} - L_{kn-1}L_{n-1} = A_nL_{kn} + 2L_{kn} - 2 + L_{kn-1}L_{n-3} + L_{kn-3}L_{n-1} - (L_{kn-2}L_n - L_{kn-2}L_{n-2} - 2L_{kn-2}) - 2L_{kn-2} - 2L_{kn-2} = 0$. So $A_{(k+1)n} = A_nL_{kn} - A_nL_{kn-2} + 2L_{kn-2} + A_{kn} + B$.

$B = 2 + L_{kn-1}L_{n-3} + L_{kn-3}L_{n-1} - 2L_{kn-2}L_{n-2} = 2 + L_{kn-2}L_{n-4} + L_{(k+1)n-4} + L_{kn-4}L_{n-2} + L_{(k+1)n-4} - 2L_{kn-3}L_{n-3} = 2L_{(k+1)n-4} = \ldots = 2 + L_{(k-1)n+3}L_1 + L_{(k-1)n+3}L_3 = 2$. $L_{(k-1)n+4} = (a^2 - 1)L_{(k-1)n} - aL_{(k-1)n-1} + (a^2 - 1)L_{(k-1)n} = 2aL_{(k-1)n+4} + 2aL_{(k-1)n-1} + 2 = 2aL_{(k-1)n} + L_{(k-1)n-1} - L_{(k-1)n-2} + L_{(k-1)n-2} + 2 = -L_{(k-1)n} + L_{(k-1)n-2} + 2 = -A_{(k-1)n}$.

So $A_{(k+1)n} = A_nL_{kn} - A_nL_{kn-2} + 2A_{kn} - A_{(k-1)n} = (A_n + 2)A_{kn} + 2A_{kn} - A_{(k-1)n}$ and $A_n$ is a divisor of $A_{(k+1)n}$.

For example we can prove that the Laplacian spectra of $Z_3 \times Z_3$ and $Z_6$ are different. The graph of $Z_3 \times Z_3$ is isomorphic to the complement of the graph of $Z_6$. It is well known that if $\lambda \neq 0$ is the eigenvalue of $L(G)$, then $n - \lambda$ is the eigenvalue of $L(G^C)$, see [2]. Since $\lambda = 4$ is the eigenvalue of $Z_6$ it follows that $\lambda = 2$ is the eigenvalue of $Z_2 \times Z_3$ and $\lambda = 2$ is not the eigenvalue of $Z_6$. Therefore, the spectra of isomorphic groups can be different. Note that $Z_3 \times Z_3 \neq Z_3$ but their graphs and Laplacian spectra coincide.

Lemma 2.4. $A_{kn+p} = (A_{p} + 2)A_{kn} + 2A_{p} - A_{kn-p}$.

Proof. $A_{kn+p} = L_{kn+p} - L_{kn+p-2} - 2 = L_{kn}L_p - L_{kn-1}L_{p-1} - L_{kn-1}L_{p-2} + L_{kn-1}L_{p-3} - 2 = L_{kn}(L_p - L_{p-2} - 2) + 2L_{kn} - 2 + L_{kn-1}L_{n-3} - L_{kn-1}L_{n-2} = A_{p}L_{kn} + 2L_{kn} + 2L_{kn} - 2 + L_{kn-1}L_{p-3} - aL_{kn-1}L_{p-1} + L_{kn-3}L_{p-1} + \ldots$
\[ L_{kn-2}L_{p-2} - L_{kn-2}L_{p-2} = A_pL_{kn} + 2L_{kn} - 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - (L_{kn-2}L_p - L_{kn-2}L_{p-2} - 2L_{kn-2}) - 2L_{kn-2} - 2L_{kn-2}L_{p-2} = A_pL_{kn} - A_pL_{kn-2} + 2(L_{kn} - L_{kn-2} - 2) + 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - 2L_{kn-2}L_{p-2} = A_pL_{kn} - A_pL_{kn-2} + 2A_p + B. \]

\[ B = 2 + L_{kn-1}L_{p-3} + L_{kn-3}L_{p-1} - 2L_{kn-2}L_{p-2} = 2 + L_{kn-2}L_{p-4} + L_{(k+1)n-4} + L_{kn-4}L_{p-2} + L_{(k+1)n-4} - 2L_{kn-3}L_{p-3} - 2L_{(k+1)n-4} = \ldots = 2 + L_{(k-1)n+3+(n-p)L} + L_{(k-1)n+1+(n-p)L} - 2L_{(k-1)n+2+(n-p)L} = \ldots = -A_{(k-1)n+(n-p)}. \]

So \( A_{kn+p} = A_pL_{kn} - A_pL_{kn-2} + 2A_{kn} = A_{(k-1)n+(n-p)} = (A_n + 2)A_{kn} + 2A_p = -A_{kn-p}. \)

By Theorem 2.1 we get

\[ A_{n+p} = A_n(A_p + 2) + 2A_p - A_{n-p}, \quad p < n \]

\[ \square \]

**Theorem 2.2.** If \( \lambda \neq 2 \) is the eigenvalue of Laplacian of \( Z_n \) and \( Z_m \), then \( \lambda \) is the eigenvalue of the Laplacian of \( Z_k \), where \( d \) is the greatest common divisor of \( m \) and \( n \). Moreover, if \( \lambda = 4 \) is the eigenvalue of the Laplacian of \( Z_n \), then \( \exists k \in N : n = 4k \) or \( n = 2. \)

**Proof.** Note that if \( \lambda = 4 \), then \( A_2(2 - \lambda) = 0 \). Assume that \( \lambda \neq 2 \) is the eigenvalue of the Laplacian of \( Z_n \) and \( Z_m \) when \( m > n, m = n + k \), and that the greatest common divisor of \( m \) and \( n \) is 1. Set \( a = 2 - \lambda \). Then \( A_{n+k}(a) = A_n(a) = 0 \). By the (4) we have: \( A_{2n+k}(a) = A_{n+k}(a)(A_n(a) + 2) + 2A_n(a) - A_k(a) = -A_k(a) \)

\[ A_{2n+2k}(a) = A_{n+k}(a)(A_n(a) + 2) + 2A_n(a) - A_k(a) = -A_k(a). \]

But \( A_{2n+2k}(a) = A_{2(n+k)}(a) = 0 \) by the (4).1. So \( A_k(a) = 0 \). If \( k \) and \( n \) have the common divisor \( 1 \) then \( m \) and \( n \) have the common divisor \( 1 \) too. So the greatest common divisor of \( k \) and \( \min(n, n+k) \) is 1. Continuing this procedure for the \( k \) and \( \min(n, n+k) \) we obtain the following:

\[ A_{\min(k, \min(n, n+k))}(a) = A_{[k-\min(n, n+k)]}(a) = 0 \]

In addition, the greatest common divisor of \( \min(k, \min(n, n+k)) \) and \( [k-\min(n, n+k)] \) is 1. Continuing this procedure further we prove for some \( p \) that \( A_p(a) = A_1(a) = 0 \). So if the greatest common divisor of \( m \) and \( n \) is 1, then \( A_{n+k}(a) = A_n(a) = A_1(a) = 0 \Rightarrow a = 2 \) and \( \lambda = 0 \).

Note then the multiplicity of the first eigenvalue \( \lambda = 0 \) is equal to the number of components of graph (see [3][2]). So for all cyclic groups the multiplicity of \( \lambda = 0 \) is 1.

**Lemma 2.5.** \( A_n(a) = aA_{n-1}(a) - A_{n-2}(a) + 2A_1(a), n \geq 3 \)

**Proof.** \( A_n = aL_{n-1} - 2L_{n-2} - 2 = a(aL_{n-2} - L_{n-3}) - 2L_{n-2} - 2 = a^2L_{n-2} - L_{n-3} - 2L_{n-2} = 2aL_{n-3} - 2a + aL_{n-3} + 2a - 2L_{n-2} = a(aL_{n-2} - 2L_{n-3} - 2) + aL_{n-3} + 2a - 2L_{n-2} = aA_{n-1} + aL_{n-3} + 2a - 2L_{n-3} - L_{n-4} = aA_{n-1} - aL_{n-3} + 2L_{n-4} + 2 + 2a - 4 = aA_{n-1} - A_{n-2} + 2A_1. \)

\[ \square \]

**Lemma 2.6.** \( A_{kn} = A_k \circ (A_n + 2) \)

**Proof.** \( A_{2n} = A_n(A_n + 4) = (A_n + 2 - 2)(A_n + 4) = (A_n + 2)^2 - 4 = A_2 \circ (A_n + 2). \) Now assume \( \forall m \leq k : A_{mn} = A_n \circ (A_n + 2). \)

\[ A_{(k+1)n}(a) = A_{kn}(a)(A_n(a) + 2) + 2A_n(a) - A_{(k-1)n}(a) = (A_n(a) + 2)A_k \circ (A_n(a) + 2) - A_{(k-1)n} \circ (A_n(a) + 2) = A_{(k+1)n}(a) \circ (A_n(a) + 2). \]

\[ \square \]
Proof. Assume that

$$Z$$

By Lemma 2.6 we get

$$\exists \text{ Laplacian of } Z$$

A

Laplacian of

Z

of

$$[3]$$ D. Cvetkovic, M. Doob, I. Gutman, and A. Torgasev. Recent results in the theory of graph spectra, Ann.

Discret. Math. 36, North Holland, 1988.

Proof. By Lemma 2.6 we see that

$$\exists$$

Laplacian of

Z

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Z

$$[2]$$ Turker Biyikoglu, Josef Leydold, Peter F. Stadler. Laplacian Eigenvectors of Graphs: Frobenius and Faber-Krahn Type Theorems. Springer, 2007.

Corollary 2.1. If $$\lambda$$ is the eigenvalue of the Laplacian of $$Z_n$$, then $$\forall m \in N : P_m(\lambda_0) = -A_m(2 - \lambda_0) = \lambda_1$$, where $$\lambda_1$$ is the eigenvalue of the Laplacian of $$Z_n$$.

Proof. By Lemma 2.6 we get

$$A_{mn}(2 - \lambda_0) = \prod_{j=1}^{m} (A_m(2 - \lambda_0) + 2j) = \prod_{j=1}^{m} (A_m(2 - \lambda_0) + j) = 0.$$  

Thus, $$\exists \lambda_1 : P_m(\lambda_0) = -A_m(2 - \lambda_0) = \lambda_1$$, where $$\lambda_1$$ is the eigenvalue of the Laplacian of $$Z_n$$.

Corollary 2.2. $$P_k(\lambda) = \lambda_i$$, where $$\lambda_i$$ is the eigenvalue of the Laplacian of $$Z_n$$, $$\iff$$ $$\lambda$$ is the eigenvalue of the Laplacian of $$Z_n$$.

Proof. By Lemma 2.6 we see that

$$P_k = (-1)^{n-1}(P_k - \lambda_j),$$

where $$\lambda_j$$ are the eigenvalues of the Laplacian of $$Z_n$$.

References

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[2] Turker Biyikoglu, Josef Leydold, Peter F. Stadler. Laplacian Eigenvectors of Graphs: Frobenius and Faber-Krahn Type Theorems. Springer, 2007.

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