Computational Complexity of Three Central Problems in Itemset Mining *

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Abstract

Itemset mining is one of the most studied tasks in knowledge discovery. In this paper we analyze the computational complexity of three central itemset mining problems. We prove that mining confident rules with a given item in the head is NP-hard. We prove that mining high utility itemsets is NP-hard. We finally prove that mining maximal or closed itemsets is coNP-hard as soon as the users can specify constraints on the kind of itemsets they are interested in.

1 Introduction

Many techniques have been developed for itemset mining problems. Famous examples are mining frequent itemsets (Agrawal et al., 1993; Han et al., 2000), mining association rules (Agrawal et al., 1993; Szathmary et al., 2007), mining closed itemsets (Pasquier et al., 1999), mining high utility itemsets (Chan et al., 2003), etc. Most of these works have focused on improving the practical performance of the mining process, but few have conducted a theoretical analysis of the computational complexity of itemset mining problems.

Wijsen and Meersman (1998) have proved that it is NP-complete to decide whether there exists a valid quantitative rule. Angiulli et al. (2001) have proved that the problem of deciding whether there exists a non-redundant association rule of size at least $k$ that is frequent and the problem of deciding whether there exists a non-redundant association rule of size at least $k$ that is confident are both NP-complete. There also exist results on the computational complexity of

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mining maximal frequent itemsets. Yang (2004), and Zaki and Ogihara (1998) have proved that deciding whether there exists a maximal frequent itemset is polynomial. Boros et al. (2002) have proved that deciding whether there exist other maximal frequent itemsets than those in a given set is NP-complete.

In this paper we analyze the computational complexity of three well-known itemset mining problems. We prove that deciding whether there exists a confident rule that contains a given item in the head is NP-complete. This directly leads to the result that mining confident rules with a given item in the head is NP-hard. We then prove that deciding whether there exists an itemset with high utility is NP-complete. This directly leads to the result that mining high utility itemsets is NP-hard. We finally prove that deciding whether there exists an itemset that is maximal or closed w.r.t. those satisfying a set of user’s constraints is coNP-complete. This directly leads to the result that mining maximal or closed constrained itemsets is coNP-hard.

The paper is organized as follows. We start with some preliminary definitions and notations in Section 2. In Section 3 we study the problem of mining association rules that are confident. Section 4 reports our result on the problem of mining high utility itemsets. In Section 5 we study the problem of mining maximal or closed itemsets among itemsets subject to a set of user’s constraints. Section 6 concludes this work.

2 Preliminary Definitions and Notations

Let \( I = p_1, \ldots, p_n \) be a set of \( n \) distinct objects, called items. An itemset \( P \) is a non-empty subset of \( I \). A transactional dataset \( D \) is a bag of \( m \) itemsets \( t_1, \ldots, t_m \), called transactions.

The cover of an itemset \( P \) in \( D \), denoted by \( \text{cover}(P) \), is the bag of transactions from \( D \) containing \( P \). The frequency of an itemset \( P \) in \( D \), denoted by \( \text{freq}(P) \), is the cardinality of its cover, i.e. \( \text{freq}(P) = |\text{cover}(P)| \). Given a frequency threshold \( s \), an itemset \( P \) is frequent in \( D \) if \( \text{freq}(P) \geq s \). This condition is called the minimum frequency constraint.

Example 1 The dataset in Table 1 has 5 items and 5 transactions. The cover of \( CE \) is \( \text{cover}(CE) = \{t_3, t_4, t_5\} \). Its frequency is the cardinality of its cover, i.e. \( \text{freq}(CE) = |\text{cover}(CE)| = 3 \). If the frequency threshold \( s \) is equal to 2, \( CE \) is frequent.

| trans. | Items |
|--------|-------|
| \( t_1 \) | \( A \ B \ D \ E \) |
| \( t_2 \) | \( A \ C \) |
| \( t_3 \) | \( A \ B \ C \ E \) |
| \( t_4 \) | \( B \ C \ E \) |
| \( t_5 \) | \( A \ B \ C \ E \) |

Table 1: Dataset with five items and five transactions.
3 On Mining Confident Rules

3.1 Background on association rules

An association rule (Agrawal et al., 1993) is an implication of the form $X \rightarrow Y$, where $X$ and $Y$ are itemsets such that $X \cap Y = \emptyset$ and $Y \neq \emptyset$. $X$ represents the body of the rule and $Y$ represents its head. The confidence of a rule captures how often $Y$ occurs in transactions containing $X$, that is, $\text{conf}(X \rightarrow Y) = \frac{\text{freq}(X \cup Y)}{\text{freq}(X)}$. Given a confidence threshold $c$, a rule $X \rightarrow Y$ is confident if $\text{conf}(X \rightarrow Y) \geq c$.

Example 2 Consider the dataset presented in Table 1 and the confidence threshold $c = 60\%$. $B \rightarrow C$ is a confident association rule because $\text{conf}(B \rightarrow C) = \frac{\text{freq}\{B, C\}}{\text{freq}\{B\}} = \frac{3}{4} \geq c$.

3.2 Our result

In this subsection we analyze the computational complexity of mining confident rules. We prove that deciding whether there exists a confident rule with a given item in the head is NP-complete, which implies that mining confident rules that contain a given item in the head is NP-hard.

Theorem 1 Given a dataset $D$ on a set of items $I$, deciding whether there exists a rule that contains a given item in the head and has a confidence higher than a given threshold $c$ is NP-complete.

Proof. 

Membership. Given an association rule $X \rightarrow Y$, we check if the given item belongs to $Y$. This is linear in the size of the rule. We then traverse the dataset $D$ and compute the size of the covers of the itemsets $X$ and $X \cup Y$. This is linear in $|D|$. We then compute the ratio $\frac{\text{cover}(X \cup Y)}{\text{cover}(X)}$ and compare it to $c$ to decide if the rule is confident. This is linear in $\log|D|$.

Completeness. We reduce 3-SAT to the problem of deciding whether there exists a confident rule $X \rightarrow Y$ with a given item in the head. Let us call $z$ that item. Given a 3-SAT formula $F$ with $m$ 3-clauses on the set $V = \{v_1, \ldots, v_n\}$ of Boolean variables, we build the following instance. For clarity purpose, we denote the items in the dataset $D$ by $\text{pos}_1, \text{neg}_1, \ldots, \text{pos}_n, \text{neg}_n, \text{z}$. We denote by $\text{All}$ the set of all items. The confidence ratio $c$ is set to 0.5.

The dataset $D$ is:

1. $\text{All} \setminus \{z\}$ (n times)
2. $\text{All} \setminus \{\text{pos}_i\}, \forall v_i \in V$
3. $\text{All} \setminus \{\text{neg}_i\}, \forall v_i \in V$
4. $\text{All} \setminus \{\text{pos}_i, \text{neg}_i, z\}, \forall v_i \in V$ (2 times)
5. $\text{All} \setminus \{it_1, it_2, it_3, z\}$, for each clause $cl$ in $F$, where $it_i = \text{pos}_j$ if the $i$th literal in $cl$ is $v_j$, $it_i = \text{neg}_j$ if the $i$th literal in $cl$ is $\neg v_j$. 

3
The dataset contains $n + n + n + 2n + m$ transactions. The reduction is thus polynomial in size.

For instance, if $F = \{v_1 \lor \neg v_2 \lor v_3\}$ with $n = 3$, $D$ is:

|   | pos1 | neg1 | pos2 | neg2 | pos3 | neg3 |   |
|---|------|------|------|------|------|------|---|
| t1 |      |      |      |      |      |      |   |
| t2 |      |      |      |      |      |      |   |
| t3 |      |      |      |      |      |      |   |
| t4 |      |      |      |      |      |      |   |
| t5 |      |      |      |      |      |      |   |
| t6 |      |      |      |      |      |      |   |
| t7 |      |      |      |      |      |      |   |
| t8 |      |      |      |      |      |      |   |
| t9 |      |      |      |      |      |      |   |
| t10|      |      |      |      |      |      |   |
| t11|      |      |      |      |      |      |   |
| t12|      |      |      |      |      |      |   |
| t13|      |      |      |      |      |      |   |
| t14|      |      |      |      |      |      |   |
| t15|      |      |      |      |      |      |   |
| t16|      |      |      |      |      |      |   |

Suppose a formula $F$ is satisfiable. Let us denote by $S$ a solution of $F$. We construct the rule $X \rightarrow \{z\}$ such that $\text{pos}_i \in X$ and $\text{neg}_i \not\in X$ for each $i$ such that $S[v_i] = 1$, and $\text{pos}_i \not\in X$ and $\text{neg}_i \in X$ for each $i$ such that $S[v_i] = 0$. By construction of $D$, $X \rightarrow \{z\}$ appears in $n$ transactions (2) and (3). By construction again, $X$ appears in the $n$ transactions where $X \rightarrow \{z\}$ appears plus the $n$ transactions (1). $X$ does not appear in any transaction (4) because they all miss $\text{pos}_i$ and $\text{neg}_i$ for some $i$, whereas $X$ contains $\text{pos}_i$ or $\text{neg}_i$ for all $i$. Finally, as $S$ satisfies $F$, $X$ does not appear in any transaction (5) because these transactions all miss at least the item of $X$ corresponding to the literal satisfying the clause. As a result, the rule $X \rightarrow \{z\}$ has confidence $\frac{n}{2n} = 0.5 \geq c$.

Suppose now that $X \rightarrow Y$ is a confident rule with $\{z\} \in Y$. $Y$ contains $z$, so $X$ does not. Hence, $X$ appears at least in the $n$ transactions (1) where $Y$ does not appear. Now, $Y$ only appears in transactions (2) and (3) because it contains $z$. Thus, $X$ must appear in at least $n$ transactions (2) and (3) to reach the confidence of 50\%. For a given $i$, $X$ must contain at least one among $\text{pos}_i$ and $\text{neg}_i$, otherwise the two corresponding transactions (4) would cover $X$ and not $Y$, making confidence impossible to reach. Thus, $X$ (and must) appear in exactly $n$ transactions (2) and (3), which means that for each $i$, exactly one among $\text{pos}_i$ and $\text{neg}_i$ is in $X$. We then can build the mapping from the rule to the instantiation $S$ on $v_1, \ldots, v_n$ such that $S[v_i] = 1$ if $\text{pos}_i \in X$, and $S[v_i] = 0$ if $\text{neg}_i \in X$. We have $n$ transactions (1-3) covering $X \cup Y$ and $2n$ covering $X$. As transactions (5) do not contain $z$, they must not cover $X$, otherwise confidence cannot be reached. As a result, for every transaction (5), $X$ necessarily contains at least one item (other than $z$) which is not in the transaction. By construction of transactions (5) and thanks to the mapping from the rule to $S$, this item corresponds to the truth value of a Boolean variable that satisfies the clause of $F$ associated with the transaction. Therefore, $F$ is
Corollary 1 Given a dataset \( D \) on a set of items \( I \), finding a rule containing a given item in the head and having confidence higher than a given threshold \( c \) is NP-Hard.

4 On Mining High Utility Itemsets

4.1 Background on high utility itemset mining

In high utility itemset mining (Chan et al., 2003), each transaction \( t_j \) is associated with a vector \( v_j \) of \( n \) positive integers, where \( v_j(i) \) is the cardinality of item \( p_i \) in transaction \( t_j \). A utility function \( u \) is a vector of \( n \) positive integers, where \( u(i) \) is the utility of item \( p_i \). The utility can be seen as the profit obtained when someone buys item \( p_i \). The utility \( u(P, t_j) \) of an itemset \( P \) in a transaction \( t_j \) is \( 0 \) if \( P \nsubseteq t_j \), \( \sum p_i \in P(v_j(i) \cdot u(i)) \) otherwise. The utility \( u(P) \) is \( \sum_{t_j \in \text{cover}(P)} u(P, t_j) \). Given a utility threshold \( ut \), the itemset \( P \) is of high utility if and only if \( u(P) \geq ut \).

Example 3 In Table 2 we present the vectors \( v_j \) for every transaction \( t_j \) and the utility function \( u(i) \) for every item. The utility \( u(AC, t_1) \) of the itemset \( AC \) in \( t_1 \) is \( 0 \) because \( AC \nsubseteq t_1 \). The utility \( u(AC, t_2) \) of the itemset \( AC \) in \( t_2 \) is \( u(AC, t_2) = 4 \times 25 + 8 \times 12 = 196 \). The utility \( u(AC) \) of the itemset \( AC \) is \( u(AC) = u(AC, t_2) + u(AC, t_3) + u(AC, t_5) = 196 + 218 + 282 = 696 \). If the utility threshold \( ut \) is set to \( 660 \), \( AC \) is of high utility \((u(AC) \geq ut)\), but \( ACE \) is not \((u(ACE) = 656 < ut)\).

Table 2: Dataset (left) with cardinality vectors (middle) and utility function (right).

| trans. | Items   | \( v_j \) | A | B | C | D | E |
|--------|---------|-----------|---|---|---|---|---|
| t_1    | A B     | \( v_1 \) | 5 | 7 | 0 | 3 | 1 |
| t_2    | A       | \( v_2 \) | 4 | 0 | 8 | 0 | 0 |
| t_3    | A B C   | \( v_3 \) | 2 | 11| 14| 0| 3 |
| t_4    | B C E   | \( v_4 \) | 0 | 9 | 24| 0| 1 |
| t_5    | A B C   | \( v_5 \) | 6 | 5 | 11| 0| 2 |

| u      | A | B | C | D | E |
|--------|---|---|---|---|---|
|        | 25| 14| 36| 34|

4.2 Our result

In this subsection we analyze the computational complexity of mining itemsets of high utility. Several algorithms have been proposed for mining high utility itemsets (Liu et al., 2005; Fournier-Viger et al., 2014; Peng et al., 2017).
None of these algorithms has been proved to be polynomial in time. We prove that deciding whether there exists a high utility itemset is NP-complete, which implies that mining high utility itemsets is NP-hard. It is thus not possible that an algorithm for mining high utility itemsets is polynomial, unless \( P = NP \).

**Theorem 2** Given a dataset \( D \) on a set of items \( I \), given a utility function \( u \), deciding whether there exists an itemset with utility higher than a given threshold \( ut \) is NP-complete.

**Proof. Membership.** Checking that an itemset \( P \) is a witness to the existence of itemsets with utility higher than the threshold is done by computing the utility \( u(P, t_j) \) of \( P \) in each transaction \( t_j \) such that \( j \) is in the cover of \( P \) and to sum these utilities. All this is polynomial in \( |D| \).

**Completeness.** We reduce 1in3-Positive-3SAT, which is NP-complete, to the problem of deciding whether there exists an itemset with utility higher than the threshold \( ut \). Given a formula \( F \) with \( m \) positive 3-clauses on \( n \) Boolean variables \( v_1, \ldots, v_n \), we want to know whether there exists an assignment of the variables such that exactly one variable is true in each 3-clause. We build the dataset \( D \) on the set \( I = (p_1, \ldots, p_n) \) of items, where the item \( p_i \) represents the Boolean variable \( v_i \). An itemset \( P \) corresponds to the assignment of the variables of formula \( F \) such that \( p_i \in P \) if and only if \( v_i = 1 \). The utility function \( u \) returns 1 for every item. \( D \) contains \( 3m \) transactions and the utility threshold \( ut \) is set to \( 3nm^2 \). Each clause \( v_i \lor v_j \lor v_k \) in \( F \) is encoded by adding three transactions to the dataset \( D \). These three transactions have utility 1 for all items except \( p_i, p_j, p_k \). The first of the three transactions has utility \( 3nm \) for \( p_i \), the second transaction has utility \( 3nm \) for \( p_j \), and the third transaction has utility \( 3nm \) for \( p_k \). The remaining two unset items in each transaction have utility 0. The dataset contains \( 3m \) transactions. The reduction is thus polynomial in size.

For instance, if \( F = (v_1 \lor v_2 \lor v_3) \land (v_2 \lor v_4 \lor v_5) \) with \( n = 5 \) and \( m = 2 \), then \( ut = 3 \times 5 \times 2^2 = 60 \). The dataset \( D \) and the corresponding utilities and cardinalities (the vectors \( v_i \)) are presented in Table 3.

We show that there is a one-to-one mapping between itemsets with utility higher than the threshold and solutions to the 1in3-Positive-3SAT problem. Suppose that an itemset \( P \) has utility higher than the threshold \( 3nm^2 \). To reach this threshold, at least \( m \) of the occurrences of \( 3nm \) in \( D \) must participate to the sum because the sum of all other occurrences in \( D \) does not reach \( 3nm \) (it is equal to \( 3(n - 3)m \)). By construction of \( D \), if an itemset \( P \) contains more than one item corresponding to variables in a given clause in \( F \), none of the occurrences of \( 3nm \) in the three transactions encoding that clause can participate to the sum. As a result, to reach the threshold, for each triplet of transactions encoding a clause, an itemset \( P \) must contain exactly one of the three items corresponding to the three variables of the clause encoded by this triplet of transactions. By construction of these triplets of clauses, this means that assigning the set of Boolean variables corresponding to \( P \) to value 1 and the others to 0, we obtain a solution to \( F \).
Table 3: Example of an instance with $F = (v_1 \lor v_2 \lor v_3) \land (v_2 \lor v_4 \lor v_5)$, $m = 2$ and $n = 5$.

| trans. | Items  | $v_i$  | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|--------|--------|--------|-------|-------|-------|-------|-------|
| $t_1$  | $p_1$  | $p_4$  | $p_5$ |       |       |       |       |
| $t_2$  | $p_2$  | $p_4$  | $p_5$ |       |       |       |       |
| $t_3$  | $p_3$  | $p_4$  | $p_5$ |       |       |       |       |
| $t_4$  | $p_1$  | $p_2$  | $p_3$ |       |       |       |       |
| $t_5$  | $p_1$  | $p_3$  | $p_4$ |       |       |       |       |
| $t_6$  | $p_1$  | $p_3$  | $p_5$ |       |       |       |       |

Suppose now that an assignment $A$ of the $v_i$’s is solution to $F$. By definition, for each clause $v_i \lor v_j \lor v_k$, the itemset $P$ corresponding to $A$ contains exactly one item among $p_i, p_j, p_k$. By construction of the triplets of transactions representing the clauses, the sum of utilities contains an occurrence of $3nm$ per triplet, and thus reaches the threshold.

Consequently, deciding whether there exists an itemset with utility higher than a given threshold $ut$ is NP-complete. □

Corollary 2 Given a dataset $D$ on a set of items $I$, given a utility function $u$, finding an itemset with utility higher than a given threshold $ut$ is NP-hard.

5 On Mining Maximal or Closed Itemsets

5.1 Background on constrained theories and concise representations

Following Bonchi and Lucchese (2004), a constraint on itemsets is a function $c : 2^I \rightarrow \{true, false\}$. We say that an itemset $P$ satisfies a constraint $c$ if and only if $c(P) = true$. Given a set $C$ of constraints and a dataset $D$, the theory of $C$ is the set of itemsets satisfying the constraints in $C$:

$$Th_D(C) = \{P \in 2^I \mid \forall c \in C : c(P)\}$$

Constraints are used to specify the kind of properties the user wants the mined itemsets to satisfy. For instance, given a frequency threshold $s$, the minimum frequency constraint defined in Section 2 is denoted by $c_{freq}$:

$$c_{freq}(P) \iff freq(P) \geq s$$

The theory $Th_D(\{c_{freq}\})$ corresponds to the set of frequent itemsets. Users may define any kind of constraints so that $Th_D(C)$ corresponds to the itemsets they are interested in.
We now define two types of concise representations for a theory. The first one is defined w.r.t. inclusion. An itemset $P$ is maximal for a theory if and only if $P$ is in the theory and none of its supersets are in the theory, that is,

$$P \text{ is maximal for } Th_D(C) \iff P \in Th_D(C) \land \nexists Q \in Th_D(C) \mid Q \supset P$$

We can observe that when restricting our attention to the frequency constraint $c_{freq}$, itemsets that are maximal for $Th_D(\{c_{freq}\})$ correspond to maximal frequent itemsets (MFIs) (Mannila and Toivonen, 1997).

There exists a more restrictive type of concise representations that, in addition to inclusion, take into account the exact frequency of itemsets. An itemset $P$ is closed for a theory if and only if $P$ is in the theory and $P$ does not have any superset in the theory with the same frequency, that is,

$$P \text{ is closed for } Th_D(C) \iff P \in Th_D(C) \land \nexists Q \in Th_D(C) \mid Q \supset P \land freq(Q) = freq(P)$$

Example 4: In Example 1, we saw that the itemset $CE$ is frequent in the dataset described in Table 1. That is, $CE \in Th_D(\{c_{freq}\})$. $CE$ is not closed for $Th_D(\{c_{freq}\})$ because $BCE$ belongs to $Th_D(\{c_{freq}\})$ and $freq(BCE) = freq(CE)$. $BCE$ is closed for $Th_D(\{c_{freq}\})$ because it belongs to $Th_D(\{c_{freq}\})$ and none of its supersets have the same frequency. $BCE$ is not maximal for $Th_D(\{c_{freq}\})$ because $ABCE$ is a superset of $BCE$ and it also belongs to $Th_D(\{c_{freq}\})$.

Users often want to mine concise representations of a theory $Th_D(C)$, where $C$ is a set of constraints, usually containing the frequency constraint $c_{freq}$, but also other constraints specifying properties the returned itemsets should satisfy. There exist extremely efficient algorithms for mining concise representations of the theory of frequent itemsets $Th_D(\{c_{freq}\})$. We can cite the algorithm CHARM for mining maximal frequent itemsets (Zaki and Hsiao, 2002), or LCM for mining closed frequent itemsets (Uno et al., 2004). However, as noticed by Bonchi and Lucchese (2004), given a set of constraints $C$ containing $c_{freq}$, mining itemsets that are maximal or closed for the theory $Th_D(C)$ does not simply consist in generating the maximal/closed frequent itemsets and then remove those that do not satisfy the other constraints in $C$. It consists in mining those itemsets that are maximal/closed for $Th_D(C)$.

Existing approaches for mining itemsets that are maximal/closed for a theory, such as the one presented in (Négrevergne et al., 2013), are "multi-shot", in the sense that they perform several calls to a SAT or CSP solver. The result in the next subsection shows that it is not possible to do differently (that is, "one-shot") unless $coNP \subseteq NP$.

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1The Constraint Satisfaction Problem (CSP) is a powerful paradigm to solve combinatorial problems (Rossi et al., 2006).
5.2 Our result

In this subsection we prove that it is coNP-hard to find itemsets that are maximal or closed for a theory \( Th_D(C) \).

**Theorem 3** Given a dataset \( D \) on a set of items \( \mathcal{I} \) and a set \( C \) of constraints, deciding whether an itemset is maximal/closed for \( Th_D(C) \) is coNP-complete, and it remains so even if \( c_{freq} \in C \).

**Proof.**

*Membership.* Given an itemset \( P \), a witness to its non maximality/closeness is an itemset \( Q \supset P \) that satisfies \( C \), and in the case of closeness has the same frequency as \( P \). Checking that \( Q \) has the same frequency as \( P \) is linear in \(|D|\). Checking that \( Q \) satisfies \( C \) requires checking the \( C \) constraints, which is polynomial in \(|C| \cdot |D|\). Hence, the “no” answer admits a polynomial certificate, and deciding maximality or closeness for \( Th_D(C) \) is in coNP.

*Completeness.* We reduce 3-UNSAT, which is coNP-complete, to the problem of deciding whether an itemset \( \{z\} \) is maximal/closed. Given a 3-CNF formula \( F \) with \( m \) clauses on the set \( V = \{v_1, \ldots, v_n\} \) of Boolean variables, we want to decide whether \( F \) is unsatisfiable. A clause \( cl_j \) in \( F \) is a disjunction \( (l_{j_1} \lor l_{j_2} \lor l_{j_3}) \), where a literal \( l_i \) denotes either the variable \( v_i \) or its negation \( \neg v_i \).

We build the dataset \( D \) on the set \( \mathcal{I} = \{\text{posi}, \ldots, \text{posn}, \text{neg1}, \ldots, \text{negn}, cl_1, \ldots, cl_m, z\} \) of items. The intuition is that the pair of items \((\text{posi}, \text{negi})\) represents the Boolean variable \( v_i \) in \( V \) and the item \( cl_j \) represents the clause \( cl_j \) in \( F \). We define the function \( \text{item} \) such that \( \text{item}(l_i) = \text{posi} \) if \( l_i = v_i \), and \( \text{item}(l_i) = \text{negi} \) if \( l_i = \neg v_i \). We denote by \( \text{All} \) the set \( \{\text{posi}, \text{neg1}, \ldots, \text{posn}, \text{negn}, cl_1, \ldots, cl_m, z\} \) of all items.

The dataset \( D \) contains \( m \) transactions. For each clause \( cl_j \) in \( F \), \( D \) contains the transaction \( \text{All} \setminus \{cl_j\} \).

The set \( C \) is composed of the following constraints. For each variable \( v_i \in V \), \( C \) contains a constraint \( c_i^{\text{var}} \) defined by

\[
\forall P \in 2^F, c_i^{\text{var}}(P) \equiv |P \cap \{\text{posi}, \text{negi}\}| \neq 2
\]

For each clause \( cl_j = (l_{j_1} \lor l_{j_2} \lor l_{j_3}) \in F \), \( C \) contains a constraint \( c_j^{\text{clause}} \) defined by

\[
\forall P \in 2^F, c_j^{\text{clause}}(P) \equiv P \cap \bigcup_{i \in 1 \ldots n} \{\text{posi}, \text{negi}\} = \emptyset \lor P \cap \{cl_j, \text{item}(l_{j_1}), \text{item}(l_{j_2}), \text{item}(l_{j_3})\} \neq \emptyset
\]

Finally, \( C \) contains the frequency constraint \( c_{freq} \), with the frequency threshold \( s \) set to \( m \).

The dataset contains \( m \) transactions and the set \( C \) contains \( n + m + 1 \) constraints. The reduction is thus polynomial in size. We now show that deciding maximality/closeness of item \( z \) is equivalent to deciding unsatisfiability of formula \( F \).

Suppose the formula \( F \) is satisfiable. Let us denote by \( A \) an assignment satisfying \( F \). We construct the itemset \( P \) containing \( z \) and such that for each
such that $A[v_i] = 1$, $\text{posi} \in P$ and $\text{negi} \notin P$, and for each $i$ such that $A[v_i] = 0$, $\text{posi} \notin P$ and $\text{negi} \in P$. By construction of $P$, all constraints $c^{\text{var}}$ are satisfied. By construction of $P$, the fact that $A$ satisfies $F$ implies that all constraints $c^{\text{clause}}$ are satisfied too because for each clause $c_l = (l_{j_1} \lor l_{j_2} \lor l_{j_3}) \in F$, at least one literal is true in $A$, thus at least one of the items $\text{item}(l_{j_1}), \text{item}(l_{j_2}), \text{item}(l_{j_3})$ is in $P$. Finally, by construction of $P$ and $D$, $c^{\text{freq}}$ is satisfied because $P$ does not contain any item $c_l$ and thus all the $m$ transactions contain $P$. As a result, $z$ is neither maximal nor closed for $Th_D(C)$ because $P$ is a superset of $z$ satisfying $C$ and with same frequency $m$ as $z$.

Suppose now that $z$ is not maximal or not closed for $Th_D(C)$. This means that there exists a superset $P$ of $z$ satisfying $C$. Thanks to constraint $c^{\text{freq}}$ in $C$, we know that all $m$ transactions contain $P$. Hence, $P$ cannot contain any item $c_l$. Thus, thanks to constraints $c^{\text{clause}}$, we are guaranteed that for each clause $c_l = (l_{j_1} \lor l_{j_2} \lor l_{j_3}) \in F$, $P$ contains at least one of the items $\text{item}(l_{j_1}), \text{item}(l_{j_2}), \text{item}(l_{j_3})$. In addition, thanks to constraints $c^{\text{var}}$, we are guaranteed that $P$ does not contain both $\text{posi}$ and $\text{negi}$ for any variable $v_i \in V$. As a result, the assignment $A$ built by setting $A[v_i] = 1$ when $\text{posi} \in P$, and $A[v_i] = 0$ when $\text{posi} \notin P$, is a satisfying assignment for $F$.

Consequently, deciding whether an itemset $z$ is maximal/closed is equivalent to deciding whether a CNF formula $F$ is unsatisfiable, which is coNP-complete.

\[ \square \]

**Corollary 3** Given a dataset $D$ on a set of items $I$ and a set $C$ of user’s constraints, finding an itemset that is maximal/closed for $Th_D(C)$ is coNP-hard, and it remains so even if $c^{\text{freq}} \in C$.

6 Conclusion

In this paper we have analyzed the computational complexity of some well known itemset mining problems. We have proved that mining a confident rule that has a given item in the head is NP-hard, mining a high utility itemset is NP-hard, and mining a maximal or closed constrained itemsets is coNP-hard. We hope that these results will give directions on which algorithmic technique to choose for these problems.

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\[ \text{Observe that for some } v_i \in V, \text{ it is possible that neither } \text{posi} \text{ nor } \text{negi} \text{ are in } P. \text{ This can happen when all clauses are satisfied whatever value } v_i \text{ takes. In such a case, we chose to set } v_i \text{ to } 0 \text{ in } A. \]
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