Abstract

We solve the problem of a Bose or Fermi gas in $d$-dimensions trapped by $\delta \leq d$ mutually perpendicular harmonic oscillator potentials. From the grand potential we derive their thermodynamic functions (internal energy, specific heat, etc.) as well as a generalized density of states. The Bose gas exhibits Bose-Einstein condensation at a nonzero critical temperature $T_c$ if and only if $d + \delta > 2$, and a jump in the specific heat at $T_c$ if and only if $d + \delta > 4$. Specific heats for both gas types precisely coincide as functions of temperature when $d + \delta = 2$. The trapped system behaves like an ideal free quantum gas in $d + \delta$ dimensions. For $\delta = 0$ we recover all known thermodynamic properties of ideal quantum gases in $d$ dimensions, while in 3D for $\delta = 1, 2$ and 3 one simulates behavior reminiscent of quantum wells, wires and dots, respectively.

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1 Introduction

Ultra-cooled bosonic clouds trapped in a harmonic oscillator (HO) external potential mimic the behavior of bosons confined by realistic potentials as in opto-magnetic traps in the region of small oscillations. Bose-Einstein condensation (BEC) has been now observed with $^{87}$Rb, $^{27}$Na, $^{7}$Li, $^{3}$H, $^{37}$Rb, $^{3}$He and $^{41}$K neutral bosonic atoms, the upper and lower prefixes being the nuclear mass (number of nucleons in the nucleus) and proton numbers, respectively.
Trapped quantum gases have been discussed in general by several authors \[8\] - \[14\]. The first calculation of the properties of a Bose gas in an isotropic harmonic trap was reported by de Groot et al. \[8\]; Bagnato et al. \[9\] reported theoretical thermodynamic properties of a Bose gas confined by a generic power-law potential trap; Ketterle et al. \[10\] and Pathria \[13\] considered the BEC of a finite number of particles confined in a 3D HO and concluded that the thermodynamic-limit approximation is good. For a review of BEC in trapped dilute Bose gases see Ref. \[15\]. Trapped Fermi gases have also gained interest as possible precursors of a paired-fermion condensate at lower temperatures \[16\] - \[18\], and have been studied experimentally in ultracold fermionic clouds, e.g., with $^{40}$K neutral atoms in opto-magnetic traps \[19\] - \[22\].

On the other hand, the discovery of the quasi-2D superconductors such as cuprate \[23\] - \[25\] or the quasi-1D superconductors such as the organo-metallics (or Bechgaard salts) \[26\] - \[28\] have also motivated the study of confinement of quantum gases.

In this paper we describe boson or fermion harmonic-oscillator trapping in order to better understand these lower-dimensional structures. Since the system dimensionality modifies the nature of BEC, we seek an exact and complete solution in the thermodynamic limit to the non-interacting Bose or Fermi gas problem in $d$-dimensions constrained by a number $\delta$ of perpendicular HO external potentials. We show that it is possible to map this problem into that of a free gas but in a higher dimensionality of $d + \delta$. For example, confinement \[29\] - \[32\] in 3D by a 1D HO potential collapses the system to a quasi-2D “slab” reminiscent of a quantum well. Confinement by a 2D (or 3D) HO potential leads to a wire-like quasi-1D (or dot-like quasi-0D) system.

In Sec. 2 we calculate the thermodynamic (or grand) potential for the non-interacting Bose or Fermi gas in $d$-dimensions with $\delta$ mutually-perpendicular harmonic traps. In Sec. 3 we deduce the thermodynamic properties of these systems and extract a generalized density of states, find their thermodynamic limit, and exhibit their mapping to free gases in higher dimensions. In Sec. 4 we specialize to a trapped boson gas and obtain its critical BEC temperature, its condensate fraction and specific heat cusp or jump. We also summarize our findings for a 3D boson gas trapped by 1, 2 or 3 HO’s. In Sec. 5 we specialize to a trapped fermion gas. Sec. 6 contains our conclusions.

### 2 A $d$-dimensional quantum gas trapped by $\delta \leq d$ HO’s

We consider a $d$-dimensional noninteracting boson or fermion gas trapped by $\delta = 1, 2, ..., d$ mutually-perpendicular harmonic-oscillator potentials, the particles otherwise moving freely in the remaining $d - \delta$ directions. The Hamiltonian for a single boson or fermion of mass $m$ is $H = \sum_{i=1}^{d} p_i^2 / 2m + \frac{1}{2} m \omega^2 \sum_{j=d-\delta+1}^{d} r_j^2$ and its eigenvalues are

$$\varepsilon\{n_i, \nu_j\} = \frac{2\pi^2 \hbar^2}{m L^2} \sum_{i=1}^{d-\delta} n_i^2 + \hbar \omega \sum_{j=1}^{\delta} (\nu_j + 1/2)$$

(1)

where $L$ is the size of the “box” associated with the $d - \delta$ free dimensions and where $n_i = 0, \pm 1, \pm 2, ...$ while $\nu_j = 0, 1, 2, ...$. We may write the grand potential $\Omega(T, V, \mu)$ in generalized form (see p. 134 of \[33\]) as

$$\Omega(T, V, \mu) = U - TS - \mu N = \delta_{n,-1} \Omega_0 - \frac{k_B T}{a} \sum_{\{n_i, \nu_j\}} \ln[1 + a e^{-\beta (\varepsilon\{n_i, \nu_j\} - \mu)}].$$

(2)
Here \( V \equiv L^{d-\delta} x_0^{2\delta} \) is a confinement volume with \( x_0 \equiv \sqrt{h/m \omega} \) the oscillator length parameter, \( U \) the internal energy, \( T \) the absolute temperature, \( S \) the entropy, \( \mu \) the chemical potential, \( N \) the number of particles, \( a = -1 \) for bosons, \( a = 1 \) for fermions and \( a \rightarrow 0 \) in the classical case, \( \delta \) is the Kronecker delta function, and \( \beta \equiv 1/k_BT \).

In the case of a Bose gas it is convenient to separate out in the sum (2), the lowest energy state from the excited states. We thus defined \( \Omega \equiv -(k_BT/a) \ln [1 + ae^{-\beta(h\omega/2-\mu)}] \) corresponding to the ground state contribution to the grand potential. Using the logarithm expansion \( \ln(1+x) = -\sum_{l=1}^{\infty} (-x)^l/l \) valid for \( |x| < 1 \), (3) becomes

\[
\Omega(T, V, \mu) = \delta_{a=-1} \Omega_0 + \frac{k_BT}{a} \sum_{\{n_i, \nu_j\}} \sum_{l=1}^{\infty} \frac{(-ae^{-\beta(\epsilon_{n_i, \nu_j})})^l}{l}.
\]

Next, consider only the excited states as the ground state will be treated separately for the boson gas. In the continuous limit where \( h^2/mL^2 \ll k_BT \) and \( h\omega \ll k_BT \), the summations over \( n_i \) and \( \nu_j \) can be approximated by integrals, namely \( \sum_n \rightarrow (2s+1) \int d\nu_n \). Thus

\[
\Omega(T, V, \mu) = \delta_{a=-1} \Omega_0 - \frac{(2s+1)k_BT}{a} \sum_{\{n_i, \nu_j\}} \sum_{l=1}^{\infty} \frac{(-ae^{\beta \mu})^l}{l} \int_{-\infty}^{\infty} dn_1 e^{-\beta(h^2 2\pi^2/mL^2) n_1^2} \times \int_{-\infty}^{\infty} dn_2 e^{-\beta(h^2 2\pi^2/mL^2) n_2^2} \ldots \int_{-\infty}^{\infty} dn_{d-\delta} e^{-\beta(h^2 2\pi^2/mL^2) n_{d-\delta}^2} \times \int_0^{\infty} d\nu_1 e^{-\beta h\omega (\nu_1+1/2)} \ldots \int_0^{\infty} d\nu_{d} e^{-\beta h\omega (\nu_{d}+1/2)}.
\]

where \( s \) is the particle spin, with fermions having \( s = -1/2 \) and bosons \( s = 0 \). The integrals are elementary and give

\[
\Omega(T, V, \mu) = \delta_{a=-1} \Omega_0 - \frac{2s+1}{a} e^{-[(d+\delta)/2]+1} \left( \frac{mL^2}{2\pi \hbar^2} \right)^{(d-\delta)/2} (h\omega)^{-\delta} \sum_{l=1}^{\infty} \frac{(-ae^{\beta (\mu-\delta \hbar\omega/2)})^l}{l^{(d-\delta)/2+1}}.
\]

The infinite sum is expressible in terms of the polylogarithm function \( Li_{\sigma}(z) \), since

\[
-aLi_{\sigma}(-az) = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} dx \frac{x^{\sigma-1}}{z^{-1}e^x + a} = -\frac{1}{a} \sum_{l=1}^{\infty} \frac{(-az)^l}{l^\sigma}.
\]

For \( \sigma \geq 1 \) this reduces to Bose-Einstein (BE) integrals \( g_{\sigma}(z) \) when \( a = -1 \) and to Fermi-Dirac (FD) integrals \( f_{\sigma}(z) \) when \( a = 1 \), as defined in Appendix D and E of Ref. [33], and \( z \equiv e^{\beta \mu} \) is the fugacity. Using (1) in (3) leaves

\[
\Omega(T, V, \mu) = \delta_{a=-1} \Omega_0 - \frac{1}{a} \frac{A_{d+\delta}}{\beta^{(d+\delta)/2+1}} Li_{(d+\delta)/2+1}(-az_1)
\]

where

\[
A_{d+\delta} = \frac{2s+1}{(h\omega)^{\delta}} \left( mL^2/2\pi \hbar^2 \right)^{(d-\delta)/2}
\]

and

\[
z_1 = ze^{-\beta \hbar\omega/2} = e^{\beta (\mu-\delta \hbar\omega/2)}.
\]
3 Thermodynamic properties

From (7) it is possible to find the thermodynamic properties for a monatomic gas using the relation

\[ d\Omega = -SdT - PdV - Nd\mu. \]  

(10)

In this representation the grand potential \( \Omega(T, V, \mu) = -PV \) is the fundamental relation leading to all the thermodynamic properties of the system since

\[ N = -\left( \frac{\partial \Omega}{\partial \mu} \right)_{T,V}, \quad S = -\left( \frac{\partial \Omega}{\partial T} \right)_{V,\mu}, \quad P = -\left( \frac{\partial \Omega}{\partial V} \right)_{T,\mu} = -\frac{\Omega}{V}. \]  

(11)

Using (7) and (11) the particle number is given by

\[ N = -\frac{A_{d+\delta}}{a\beta(d+\delta)/2} L_i(d+\delta)/2(-az_1), \]  

(12)

where we used the relation

\[ \left( \frac{\partial L_i(d+\delta)/2+1}{\partial T} \right)_{T,V} = \beta L_i(d+\delta)/2(-az_1). \]  

(13)

The entropy follows on substituting (7) in the first equation of (11), giving

\[ S/k_B = -[(d + \delta)/2 + 1] \frac{A_{d+\delta}}{a\beta(d+\delta)/2} L_i(d+\delta)/2+1(-az_1) - N \ln z_1, \]  

(14)

where we used the number equation (12) and the relation

\[ \left( \frac{\partial L_i(d+\delta)/2+1}{\partial T} \right)_{V,\mu} = \frac{1}{z_1} \left( \frac{\partial z_1}{\partial T} \right)_{V,\mu} L_i(d+\delta)/2(-az_1). \]  

(15)

Thus (14) becomes

\[ S/Nk_B = \frac{[(d + \delta)/2 + 1] L_i(d+\delta)/2+1(-az_1)}{L_i(d+\delta)/2(-az_1)} - \ln z_1. \]  

(16)

The internal energy is obtained from (see p. 159 of Ref. [33])

\[ U(T, V) = -k_B T^2 \left[ \frac{\partial}{\partial T} \left( \frac{\Omega}{k_B T} \right) \right]_{V, z}. \]  

(17)

Substituting (7) here we find that

\[ U(T, V) = N \frac{\hbar \omega \delta}{2} - \frac{d + \delta}{2} \Omega, \]  

(18)

and since \( \Omega = -PV \) then

\[ PV = \frac{2}{d + \delta}(U - Nh\omega\delta/2). \]  

(19)

Using (8) and (12) the internal energy (18) can be rewritten as

\[ \frac{U(T, V)}{Nk_B T} = \left[ \frac{\beta \hbar \omega \delta}{2} + \frac{d + \delta}{2} \frac{L_i(d+\delta)/2+1(-az_1)}{L_i(d+\delta)/2(-az_1)} \right]. \]  

(20)

The specific heat at constant volume \( C_V \) then follows from

\[ C_V = \left[ \frac{\partial}{\partial T} U(T, V) \right]_{N,V}. \]  

(21)
and gives
\[
\frac{C_V}{Nk_B} = \frac{d + \delta}{2} \left[ \frac{d + \delta}{2} + 1 \right] \frac{Li_{(d+\delta)/2+1}(-az_1)}{Li_{(d+\delta)/2}(-az_1)} - \frac{d + \delta}{2} \frac{Li_{(d+\delta)/2}(-az_1)}{Li_{(d+\delta)/2-1}(-az_1)} \tag{22}
\]
where we have used the relation
\[
\frac{1}{z_1} \left( \frac{\partial z_1}{\partial T} \right)_{N,V} = -k_B \beta \frac{d + \delta}{2} \frac{Li_{(d+\delta)/2}(-az_1)}{Li_{(d+\delta)/2-1}(-az_1)} \tag{23}
\]
which can be extracted from the (vanishing) derivative with respect to \(T\) of the number equation \(\frac{dN}{dT} = 0\). Since \(z_1 \equiv e^{\beta(\mu - \hbar\omega/2)} \to 0, 1\) then implies that \(-aLi_\sigma(-az_1) \to z_1\), with \(\sigma = (d + \delta)/2, (d + \delta)/2\) or \((d + \delta)/2 + 1\). Then \(\frac{C_V}{Nk_B} \to \frac{d + \delta}{2} \left[ 1 + \alpha \frac{z_1}{2(\delta + \delta)/2+1} \left( 1 - \frac{d + \delta}{2} \right) \right] \tag{24}\),

which for \(\delta = 3\) gives the classical Dulong-Petit law for \(T \to \infty\) or \(z_1 = 0\), while for \(\delta = 0\) we obtain the classical limit for ideal gases of bosons or fermions. The first correction to unity in \(\ref{24}\) for \(d + \delta < 2\) is clearly negative for \(a = -1\) (bosons) and positive for \(a = +1\) (fermions), while for \(d + \delta > 2\) it is precisely the opposite. Thus we obtain known results obtained for ideal gases for \(\delta = 0\) [Refs. \[35\] (bosons), \[36\] (fermions)]

We now recover the results obtained in Refs. \[37\]-\[39\] dealing with the equivalence of the specific heat as a function of \(T\) of ideal Bose and Fermi gases in two dimensions. Here this equivalence is obtained more generally for \(d + \delta = 2\). If both gases are at the same temperature and have the same number density \(n_B = n_F\), where \(n_B \equiv N_B/V\) is the Bose and \(n_F \equiv N_F/V\) is the Fermi density, taking \(\delta + \delta = 2\) in \(\ref{12}\) gives
\[
n_B = \frac{A_2 Li_1(z_1B)}{\beta V} = -\frac{A_2 Li_1(-z_1F)}{\beta V} = n_F, \tag{25}
\]
where as before \(V\) was defined just below \(\ref{2}\), \(z_1B \equiv e^{\beta(\mu - \hbar\omega/2)}\) and \(z_1F \equiv e^{\beta(\mu - \hbar\omega/2)}\) are the fugacities with \(\mu_B\) and \(\mu_F\) the chemical potentials for bosons and fermions, respectively. Using Landen’s relations \[38\] the polylogarithm functions \(Li_\sigma(z)\) satisfy \(Li_1(x) = -Li_1(y)\) and \(Li_2(x) = -Li_2(y) - 1/2 [Li_1(y)]^2\), where \(x \to y\) satisfy the Euler transformation \(y \equiv -x/(1-x)\) with \(x\) real \(< 1\). Substituting these relations in \(\ref{29}\), we obtain
\[
z_1F = z_1B / (1 - z_1B). \tag{26}
\]
The energy of the Bose gas \(U(T, V)_B\) taking \(a = -1\) in \(\ref{20}\) with \(d + \delta = 2\), is
\[
\frac{U(T, V)_B}{Nk_B T} = \left[ \frac{\hbar\omega\delta}{2} + \frac{Li_2(-z_1F)}{Li_1(-z_1F)} \right]. \tag{27}
\]
Substituting \(\ref{20}\) in \(\ref{27}\) we obtain
\[
\frac{U(T, V)_B}{N} = \left[ \frac{\hbar\omega\delta}{2} + \beta^{-1} \frac{Li_2(-z_1F)}{Li_1(-z_1F)} + 1/2 \beta^{-1} Li_1(-z_1F) \right] = \left[ \frac{U(T, V)_F}{N} + 1/2 \beta^{-1} Li_1(-z_1F) \right], \tag{28}
\]
where \(U(T, V)_F\) is the Fermi gas energy. Substituting \(\ref{25}\) in \(\ref{28}\) the last term in \(\ref{28}\) is proportional to \(n_F\). Hence, the energies of the Bose and Fermi gases differ only by a \(T\)-independent term and so, from \(\ref{21}\), the specific heats for boson and fermion gases precisely coincide when \(d + \delta = 2\), or
\[
[C_V(N, T)]_B = [C_V(N, T)]_F. \tag{29}
\]
3.1 Mapping to higher-\(d\) and equivalent mass

Using \(\mathcal{A}\) and \(\mathcal{B}\), equation (12) can be rewritten as

\[
N = \frac{A_{d+\delta}}{\Gamma([d+\delta]/2)} \int_0^\infty dx \frac{e^{(d+\delta)/2}(d+\delta/2-1)}{\Gamma([d+\delta/2])} \int_{\hbar\omega\delta/2}^\infty dx \frac{e^{(\varepsilon - \hbar\omega\delta/2)/(d+\delta)/(d+\delta)}}{e^{\beta(\varepsilon/\mu) + a}}
\]

\[
\equiv \int_{\hbar\omega/2}^\infty d\varepsilon \mathcal{N}(\varepsilon) n(\varepsilon),
\]  

(30)

where \(n(\varepsilon) = \left[e^{\beta(\varepsilon/\mu) + a}\right]^{-1}\) is the BE \((a = -1)\) or FD \((a = +1)\) distribution, and \(\mathcal{N}(\varepsilon)\) is the density of states (DOS). Substituting \(A_{d+\delta}\) from \(\mathcal{A}\) into \(\mathcal{B}\) we identify this generalized DOS \(\mathcal{N}(\varepsilon)\) as

\[
\mathcal{N}(\varepsilon) = (2s + 1) \left(\frac{2\pi\hbar}{m\omega L^2}\right)^{\delta} \left(\frac{mL^2}{2\pi\hbar^2}\right)^{(d+\delta)/2} \frac{e^{(\varepsilon - \hbar\omega\delta/2)/(d+\delta)/(d+\delta)}}{\Gamma([d+\delta]/2)}.
\]

(31)

If \(\delta = 0\) we recover the DOS for a free gas confined in a “box” of sides \(L\)

\[
\mathcal{N}_0(\varepsilon) = (2s + 1) \left(\frac{mL^2}{2\pi\hbar^2}\right)^{d/2} \frac{e^{d/2-1}}{\Gamma(d/2)}.
\]

(32)

Comparing \(\mathcal{A}\) with \(\mathcal{B}\) in \((d + \delta)\)-dimensions

\[
\mathcal{N}_0(\varepsilon) = (2s + 1) \left(\frac{mL^2}{2\pi\hbar^2}\right)^{(d+\delta)/2} \frac{e^{(d+\delta)/2-1}}{\Gamma([d+\delta]/2)},
\]

(33)

we observe that except for the (negligible) zero-point energy of \(\hbar\omega\delta/2\), \(\mathcal{A}\) and \(\mathcal{B}\) are identical if in \(\mathcal{B}\) an equivalent particle mass \(m^*\) defined by

\[
m^* = \left(\frac{\hbar}{\omega L^2}\right)^{2\delta/(d+\delta)} m^{(d-\delta)/(d+\delta)}
\]

(34)

is introduced. Then

\[
\mathcal{N}(\varepsilon) = (2s + 1) \left(m^*/2\pi\hbar^2\right)^{(d+\delta)/2} \frac{e^{(d+\delta)/2-1}}{\Gamma([d+\delta]/2)}.
\]

(35)

In general, therefore, the effect of trapping a quantum gas renormalizes the particle mass \(m \to m^*\) in accordance with \(\mathcal{D}\) and increases the dimensionality \(d \rightarrow d + \delta\) by the number of oscillators.

3.2 Thermodynamic limit

Substituting the coefficient \(A_{d+\delta}\) from \(\mathcal{B}\) into \(\mathcal{C}\) gives

\[
N = (2s + 1) \left(\frac{m}{2\pi\hbar^2}\right)^{(d+\delta)/2} \frac{2\pi\hbar}{m\omega} \delta \frac{x_0^{-2\delta V}}{\Gamma([d+\delta]/2)} \int_{\hbar\omega\delta/2}^\infty d\varepsilon \frac{e^{(\varepsilon - \hbar\omega\delta/2)/(d+\delta)/(d+\delta)}}{e^{\beta(\varepsilon/\mu) + a}}
\]

(36)

the volume \(V\) being defined just below \(\mathcal{B}\). The proper thermodynamic limit then holds if \(N \to \infty, L \to \infty, \omega \to 0\) while keeping the ratio \(N/V = N/L^{d-\delta}a_0^{2\delta} \propto N\omega^{\delta}/L^{d-\delta}\) constant. This result was obtained for \(d = 3\) and \(\delta = 3\) in Ref. \(\mathcal{F}\). For a free gas, i.e. \(\delta = 0\), we recover the usual thermodynamic limit, i.e., \(N \to \infty, L \to \infty\), while \(N/L^d\) = constant.
4 Trapped bosons

In this section we study a system of \( N \) noninteracting bosons in \( d \) dimensions trapped by \( \delta \leq d \) mutually-perpendicular harmonic oscillators, and otherwise free in the remaining \( d - \delta \) directions. Let the boson number be

\[
N = N_0(T) + N_{k>0}(T)
\]  

(37)

where \( N_0(T) = - (\partial \Omega_0 / \partial \mu)_{T,V} \) is the number of bosons in the lowest energy state, with \( \Omega_0 \) defined just below (6), while \( N_{k>0}(T) \) is given by (12) with \( a = -1 \). Thus

\[
N = N_0(T) + \frac{A_d + \delta}{\beta^{(d+\delta)/2}} g_{(d+\delta)/2}(z_1),
\]  

(38)

where from (6) we introduce the Bose function \( g_\sigma(z) \) which for \( z = 1 \) and \( \sigma > 1 \) is identical to the Riemann Zeta function \( \zeta(\sigma) \).

Since for \( T > T_c \), \( N_0(T) \) is negligible compared with \( N \), while for \( T < T_c \), \( N_0(T) \) is a sizeable fraction of \( N \), at \( T = T_c \), \( N_0(T_c) \sim 0 \). The critical temperature \( T_c \) of BEC is found from the condition \( N_{k>0}(T_c, z_1 = 1) \sim N \), so that (38) leads to

\[
k_B T_c = \left( \frac{N}{A_d + \delta g_{(d+\delta)/2}(1)} \right)^{2/(d+\delta)}.
\]  

(39)

From (6) the infinite series \( g_\sigma(1) \) diverges for \( \sigma \leq 1 \) implying from (39) that BEC will occur with critical temperature \( T_c \neq 0 \) if and only if \( (d + \delta)/2 > 1 \). For \( \delta = 0 \) and \( d = 3 \) with \( n \equiv N/L^3 \) (39) reduces to the familiar formula \( T_c \sim 3.31 h^2 n^{2/3}/m k_B \) of “ordinary” BEC, since \( g_{3/2}(1) = \zeta(3/2) \sim 2.612 \). On the other hand, substituting \( \delta = 3 \) and \( d = 3 \) in (38) and (6), we recover the result obtained in [43].

\[
k_B T_c \approx 0.94 h \omega N^{1/3}.
\]  

(40)

Ensher et al., [40] compared the experimental \( T_c \) obtained for real traps with the theoretical value (40) and found good agreement.

From (38) and (6) we obtain the condensate fraction,

\[
N_0(T)/N = 1 - N_{k>0}(T)/N(T_c) = 1 - (T/T_c)^{(d+\delta)/2}.
\]  

(41)

The specific heat follows from (22) for \( a = -1 \) and from \( -a L \delta(-az) [\sigma, z_1] = g_\sigma(z_1) \). We obtain for \( T > T_c \)

\[
\frac{C_V}{N k_B} = \frac{d + \delta}{2} \left[ \frac{(d + \delta)}{2} + 1 \right] \frac{g_{(d+\delta)/2+1}(z_1)}{g_{(d+\delta)/2}(z_1)} - \frac{d + \delta}{2} \frac{g_{(d+\delta)/2}(z_1)}{g_{(d+\delta)/2-1}(z_1)},
\]  

(42)

while for \( T \leq T_c, z_1 = 1 \) it follows directly from (20) and (21) that

\[
\frac{C_V}{N k_B} = \frac{d + \delta}{2} \left( \frac{d + \delta}{2} + 1 \right) (T/T_c)^{(d+\delta)/2} \frac{g_{(d+\delta)/2+1}(1)}{g_{(d+\delta)/2}(1)}.
\]  

(43)

The specific heat jump at \( T_c \) is then

\[
\frac{\Delta C_V}{N k_B} = \frac{C_V(T_c^-) - C_V(T_c^+)}{N k_B} = \left( \frac{d + \delta}{2} \right)^2 \frac{g_{(d+\delta)/2}(1)}{g_{(d+\delta)/2-1}(1)}.
\]  

(44)

and, since \( g_\sigma(1) \) diverges for \( \sigma \leq 1 \), will be nonzero if and only if \( (d + \delta)/2 > 2 \).
The entropy for \( a = -1 \) follows from (14). Since \(-\alpha L_i\sigma(-\alpha z)\sigma, z_1 = g_\sigma(z_1)\) and using (12), in terms of the critical temperature \( T_c \) it becomes

\[
S/Nk_B = [(d+\delta)/2+1] (T/T_c)^{(d+\delta)/2} \frac{g_{(d+\delta)/2+1}(z_1)}{g_{(d+\delta)/2}(1)} - \ln z_1. \tag{45}
\]

For \( T \leq T_c, z_1 = 1 \), so that this becomes

\[
S/Nk_B = [(d+\delta)/2+1] (T/T_c)^{(d+\delta)/2} \frac{g_{(d+\delta)/2+1}(1)}{g_{(d+\delta)/2}(1)} \to 0
\]

which complies with the third law of thermodynamics.

For 3D bosons trapped by 1, 2 or 3 harmonic oscillators we summarize our results in Table 1. Since for \( z_1 = 1 \) the series \( g_\sigma(z_1) \) for \( \sigma > 1 \) coincides with \( \zeta(\sigma) \), we require the following values: \( \zeta(3/2) \approx 2.612, \zeta(2) \approx 1.645, \zeta(5/2) \approx 1.341, \zeta(3) \approx 1.202, \zeta(3/2) \approx 1.127, \) and \( \zeta(4) \approx 1.082 \). In Fig. 1 we show the condensate fraction for \( \delta = 1, 2 \) and 3. In Fig. 2 are shown their internal energy; specific heat at constant volume (having a jump discontinuity if and only if \( d + \delta > 4 \)); entropy and chemical potential.

| \( \delta \) | 3 | 2 | 1 |
| --- | --- | --- | --- |
| \( N(\varepsilon) \) | \( \frac{1}{2}(\hbar\omega)^{-3}(\varepsilon - \frac{3}{2}\hbar\omega)^2 \) | \( \frac{2^{3/2}L}{\pi x_0}(\hbar\omega)^{-5/2}(\varepsilon - \hbar\omega)^{3/2} \) | \( \frac{L^2}{2\pi x_0} (\hbar\omega)^{-2}(\varepsilon - \frac{3}{2}\hbar\omega) \) |
| \( N_0/N \) | \( 1 - \left( \frac{\hbar\omega}{4} \right)^3 \) | \( 1 - \left( \frac{\hbar\omega}{4} \right)^{5/2} \) | \( 1 - \left( \frac{\hbar\omega}{4} \right)^2 \) |
| \( T_c \) | \( \frac{\hbar\omega}{2k_B} + 3 \left( \frac{T_c}{\hbar\omega} \right)^3 \frac{g(\varepsilon)}{\zeta(3)} \) | \( \frac{\hbar\omega}{k_B} + \frac{5}{2} \left( \frac{T_c}{\hbar\omega} \right)^{5/2} \frac{g(\varepsilon)}{\zeta(5/2)} \) | \( \frac{\hbar\omega}{2k_B} + 2 \left( \frac{T_c}{\hbar\omega} \right)^2 \frac{g(\varepsilon)}{\zeta(2)} \) |
| \( U/Nk_BT \) | \( \frac{3}{2} \frac{\hbar\omega}{k_B} + 3 \left( \frac{T_c}{\hbar\omega} \right)^3 \frac{g(\varepsilon)}{\zeta(3)} \) | \( \frac{5}{2} \left( \frac{T_c}{\hbar\omega} \right)^{5/2} \frac{\zeta(\delta)/\zeta(\delta/2)}{\zeta(\delta/2)} \) | \( 6 \left( \frac{T_c}{\hbar\omega} \right)^2 \frac{\zeta(\delta)}{\zeta(2)} \) |
| \( C/V/Nk_B (T < T_c) \) | \( 12 \frac{g(\varepsilon)}{\zeta(3)} - 9 \frac{g(\varepsilon)}{\zeta(2)} \) | \( 6 \frac{g(\varepsilon)}{\zeta(2)} - 4 \frac{g(\varepsilon)}{\zeta(2)} \) | 0 |
| \( C/V/Nk_B (T > T_c) \) | \( \frac{9\zeta(3)}{\zeta(2)} \simeq 6.57 \) | \( \frac{25\zeta(5/2)}{\zeta(2)} \simeq 3.20 \) | 0 |
| \( \Delta C/V/Nk_B \) | \( \frac{1}{3}(U - \frac{2}{3}N\hbar\omega) \) | \( \frac{2}{3}(U - N\hbar\omega) \) | 0 |
| \( PV \) | \( \frac{1}{3}(U - \frac{2}{3}N\hbar\omega) \) | \( \frac{2}{3}(U - N\hbar\omega) \) | 0 |
| \( S/NkB \) | \( \frac{4}{5}(\frac{T_c}{\hbar\omega})^{3} \frac{g(\varepsilon)}{\zeta(3)} - \ln\varepsilon \) | \( \frac{7}{5} \left( \frac{T_c}{\hbar\omega} \right)^{3/2} \frac{g(\varepsilon)}{\zeta(3/2)} - \ln\varepsilon \) | \( 3 \left( \frac{T_c}{\hbar\omega} \right)^2 \frac{g(\varepsilon)}{\zeta(2)} - \ln\varepsilon \) |

Table 1. Thermodynamic quantities, as defined in text, for a 3D boson gas trapped by \( \delta = 3, 2, 1 \) harmonic oscillators, with the oscillator length parameter \( x_0 \equiv (\hbar/m\omega)^{1/2} \).

An ideal Bose gas in \( d \)-dimensional space trapped by \( \delta \leq d \) harmonic oscillators has its geometric dimensionality effectively reduced. The BEC temperature expression (39) for a trapped noninteracting Bose gas shows that BEC can occur if and only if \( (d + \delta)/2 > 1 \) as otherwise the term \( g_{(d+\delta)/2}(1) \) diverges forcing \( T_c \) to vanish. Thus BEC is possible in 2D provided \( \delta \geq 1 \).

Experiments with dilute boson gases confined in the realistic confining potentials of opto-magnetic traps in the region of small oscillations where (BEC) has been observed, can be viewed as a Bose gas in 3D with \( \delta = 3 \). Table 2 illustrates some parameters in bosonic vapor systems where BEC has thus far been observed.
Using (46) and the asymptotic expansion for \( e^{\beta \epsilon(\mu(T))} \), we obtain the BEC transition temperature; \( n \) the boson number density.

5 Trapped fermions

Finally, consider a system of \( N \) noninteracting fermions in \( d \) dimensions trapped by \( \delta \) \((\leq d)\) mutually perpendicular harmonic oscillators, and otherwise free in the remaining \( d-\delta \) directions. Since \( \left[ e^{\beta (\epsilon - \mu(T))} + 1 \right]^{-1} \rightarrow \theta(E_F - \epsilon) \), with \( \mu(0) \equiv E_F \equiv \hbar^2 k_F^2 / 2m \) the Fermi energy, \( k_F \) being the Fermi wavenumber, we see from (30) with \( a = +1 \) that

\[
N \rightarrow \frac{2 A_{d+\delta}}{(d+\delta)} \Gamma([d+\delta/2]) (E_F - \hbar \omega \delta / 2)^{(d+\delta)/2}
\]

(46)

where in the last step we neglected \( \hbar \omega \delta / 2 \) compared with \( E_F \). The fermion number density with \( s = 1 / 2 \), if \( \delta = 0 \) is obtained \( [36, 41] \), substituting (3) in (46), as

\[
n \equiv N \frac{k_F^d}{L^d} \frac{2}{2d-2\pi^{d/2}d! \Gamma(d/2)},
\]

(47)

which reduces to the familiar results \( n = 2k_F / \pi, k_F^2 / 2\pi \) and \( k_F^3 / 3\pi^2 \) for \( d = 1, 2 \) and 3, respectively.

Recalling that \(-Li_{s+1}(z) \equiv f_\sigma(z)\) which are the FD integrals, the internal energy from (22) can be expressed as

\[
\frac{U(T,V)}{Nk_B T} = \left[ \beta \hbar \omega \delta / 2 + \frac{d+\delta}{2} \right] \frac{f_{(d+\delta)/2}(z_1)}{f_{(d+\delta)/2}(z_2)}.
\]

(48)

Using (46) and the asymptotic expansion for \( f_{d/2}(z) \) for \( T \rightarrow 0 \) (Ref. [36], App. B), (48) becomes

\[
\frac{U(T) - \hbar \omega \delta / 2}{Nk_B T_F} \rightarrow \frac{(d+\delta)}{(d+\delta+2)} \left[ 1 + \frac{\beta \hbar \omega \delta}{2} \right] \frac{\pi^2}{12} \left( \frac{T}{T_F} \right)^2.
\]

(49)

From (22) the specific heat as \( T \rightarrow 0 \) is then

\[
\frac{C_V(T)}{Nk_B} \rightarrow (d+\delta) \frac{\pi^2}{6} \left( \frac{T}{T_F} \right),
\]

(50)

while the entropy \( S = \int_0^T dT' C_V(T') / T' \) becomes

\[
\frac{S}{Nk_B} \rightarrow (d+\delta) \frac{\pi^2}{6} \left( \frac{T}{T_F} \right).
\]

(51)

which again is in agreement with the third law of thermodynamics. These results are reflected in Fig. 3. Table 3 summarizes results for 3D fermions with \( \delta = 1, 2, 3 \) harmonic oscillators.
Table 3. Thermodynamic quantities, as defined in text, for a 3D fermion gas trapped by $\delta = 1, 2, 3$ harmonic oscillators.

| $\delta$ | 3 | 2 | 1 |
|----------|---|---|---|
| $N$ | $\frac{2}{3}(\hbar \omega)^{-3} E_F^3$ | $\frac{4}{5}(\frac{mL^2}{2\pi \hbar^2})^{1/2}(\hbar \omega)^{-2} E_F^{5/2}$ | $(\frac{mL^2}{2\pi \hbar^2})(\hbar \omega)^{-1} E_F^2$ |
| $\frac{U}{Nk_B T}$ | $\frac{3\hbar \omega}{2k_B T} + \frac{3f_3(z_1)}{f_3(z_1)}$ | $\frac{\hbar \omega}{k_B T} + \frac{5f_5/2(z_1)}{2f_5/2(z_1)}$ | $\frac{\hbar \omega}{2k_B T} + \frac{2f_5(z_1)}{f_3(z_1)}$ |
| $\frac{C_V}{Nk_B}$ | $12\frac{f_2(z_1)}{f_3(z_1)} - 9\frac{f_3(z_1)}{f_2(z_1)}$ | $\frac{f_3(z_1)}{f_2(z_1)} - \ln z_1$ | $\frac{f_3(z_1)}{f_2(z_1)} - \ln z_1$ |
| $\frac{P V}{Nk_B T}$ | | | |
| $\frac{S}{Nk_B}$ | $4\frac{f_4(z_1)}{f_3(z_1)} - \ln z_1$ | $7\frac{f_5/2(z_1)}{2f_5/2(z_1)} - \ln z_1$ | $7\frac{f_5/2(z_1)}{2f_5/2(z_1)} - \ln z_1$ |

6 Conclusions

After constructing the grand potential, thermodynamic properties were determined along with the densities of states of ideal boson and fermion gases in $d$ dimensions trapped by $\delta$ mutually perpendicular harmonic (HO) oscillators. Trapping maps the system into a free gas with a new dimensionality increased by the number of trapping oscillators, specifically, $d \rightarrow d + \delta$, and renormalizes the particle masses $m \rightarrow m^*$ according to (34).

In particular, we detailed how 3D boson and fermion gases trapped by 1, 2 or 3 mutually-perpendicular HO wells map into a free gas in 4, 5 and 6 dimensions, respectively. Also, we found that in a trapped boson gas Bose-Einstein condensation with critical temperature $T_c \neq 0$ occurs if and only if $d + \delta > 2$ so that for $\delta \geq 1$, $d$ need not be restricted to $d > 2$.

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Figure 1: Condensate fraction for a 3D boson gas trapped by $\delta = 1, 2$ or 3 harmonic oscillators.
Figure 2: Thermodynamic variables as functions of temperature $T$, as defined in text, for a 3D boson gas trapped by $\delta = 1, 2$ or 3 harmonic oscillators.
Figure 3: Same as Fig. 2 but for a fermion gas.