Local regularity of axisymmetric solutions to the Navier–Stokes equations

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Abstract
In the note, a local regularity condition for axisymmetric solutions to the non-stationary 3D Navier–Stokes equations is proven. It reads that axially symmetric energy solutions to the Navier–Stokes equations have no Type I blowups.

Keywords Navier–Stokes equations · Axisymmetric solutions · Local regularity

1 Introduction
The aim of the note is to discuss potential singularities of axisymmetric solutions to the non-stationary 3D Navier–Stokes equations. Roughly speaking, we would like to show that if scale-invariant energy quantities of an axially symmetric solution are bounded then such a solution is smooth. By definition, potential singularities with bounded scale-invariant energy quantities are called Type I blowups. It is important to notice that our result does not follow from the so-called $\varepsilon$-regularity theory, where regularity is coming from smallness of those scale-invariant energy quantities.

Before stating and proving the main result of the note, see Theorem 2.1, we are going to remind basic notions from the mathematical theory of the Navier–Stokes equations.

For simplicity, let us consider the following Cauchy problem for the Navier–Stokes equations:

$$
\partial_t v + v \cdot \nabla v - \Delta v = -\nabla q, \quad \text{div } v = 0
$$

in $Q_+ = \mathbb{R}^3 \times ]0, \infty[ \text{ and }$

$$
v|_{t=0} = u_0
$$
in $\mathbb{R}^3$, where $u_0 \in C_{0,0}^\infty(\mathbb{R}^3) := \{ v \in C_{0}^\infty(\mathbb{R}^3) : \text{div } v = 0 \}$. One of the main problems of the mathematical theory of viscous incompressible fluids is the global well-posedness of the Cauchy problem (1.1) and (1.2). A plausible approach (which, of course, is not unique) is to prove the global existence of a solution and then to prove its uniqueness. It was done by J. Leray many years ago in [10], who introduced the notion which is known now as a weak Leray–Hopf solution.

**Definition 1.1** A divergence free velocity field $v$ is a weak Leray–Hopf solution to the Cauchy problem (1.1) and (1.2) if it has the following properties:

1. $v \in L_\infty(0, \infty; J) \cap L_2(0, \infty; J_2)$, where $J$ is the closure of $C_{0,0}^\infty(\mathbb{R}^3)$ in $L_2(\mathbb{R}^3)$ and $J_2$ the closure of $C_{0,0}^\infty(\mathbb{R}^3)$ with respect to the semi-norm $\left( \int_{\mathbb{R}^3} |\nabla w|^2 dx \right)^{1/2}$;  
2. the function $t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) dx$ is continuous on $[0, \infty[$ for any $w \in L_2(\mathbb{R}^3)$;  
3. the Navier–Stokes equations is satisfied as the variational identity  
$$ \int_Q \left\{ v \cdot \partial_t w + v \otimes v : \nabla w - \nabla v \cdot \nabla w \right\} dxdt = 0 $$  
for any test vector-valued function $w \in C_0^\infty(Q_+)$ with div $w = 0$;  
4. $\| v(\cdot, t) - u_0(\cdot) \|_{L_2(\mathbb{R}^3)} \to 0$ as $t \downarrow 0$;  
5. global energy inequality  
$$ \frac{1}{2} \int_{\mathbb{R}^3} |v(x, t)|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 dxdt' \leq \frac{1}{2} \int_{\mathbb{R}^3} |u_0|^2 dx $$  
holds for all $t \geq 0$.

There is no information about the pressure in Definition 1.1. But one can easily recover the pressure by means of the linear theory and the following estimate for the pressure takes place

$$ \| q \|_{L_2^2(Q_+)} \leq c \| v \|_{L_2(Q_+)}^2. $$

There is an important class of weak solutions, which is closely related to the uniqueness of solutions to the initial boundary value problems for the Navier–Stokes equations.

**Definition 1.2** Let $v$ be a weak Leray–Hopf solution to (1.1) and (1.2). It is a strong solution to the Cauchy problem on the set $Q_T = \mathbb{R}^3 \times ]0, T[ \text{ if } \nabla v \in L_{2,\infty}(Q_T)$.  

The proposition, proved by Leray in [10], essentially reads the following. Assume that $v$ a weak Leray–Hopf solution to (1.1) and (1.2). Then there exists a number
\[ T \geq c \| \nabla u_0 \|_{L^4_2(\mathbb{R}^3)}^{-4}, \] where \( c \) is an universal constant, such that \( v \) is strong solution in \( Q_T \).

Another important result proved by Leray is the so-called weak-strong uniqueness. Assume that \( v^1 \) is another weak Leray–Hopf solution with the same initial data \( u_0 \), then \( v^1 = v \) on \( Q_T \).

So, as it follows from the above statement, in order to prove uniqueness of weak Leray–Hopf solution on the interval \([0, T]\), it is enough to show that \( \nabla v \in L^\infty_2(Q_T) \). In other words, the problem of unique solvability of the Cauchy problem in the energy class can be reduced to the problem of regularity of weak Leray–Hopf solutions.

As usual in the theory of non-linear equations, we do not need to prove that \( \nabla v \in L^\infty_2(Q_T) \) for some \( T > 0 \). In fact, it is enough to prove a weaker regularity result and then the remaining part of the proof of regularity follows from the linear theory.

For example, one of the convenient spaces for such intermediate regularity is the space \( L^\infty_\infty(Q_T) \).

So, the first time when a singularity occurs can be defined as follows:

\[
\limsup_{t \uparrow T} \| v(\cdot, t) \|_{L^\infty_\infty(\mathbb{R}^3)} = \infty.
\]

To study regularity of weak Leray–Hopf solutions by classical PDE’s methods, we should mimic energy solutions on the local (in space-time) level. To this end, the pressure should be involved into considerations. The corresponding setting has been already discussed by Caffarelli–Kohn–Nirenberg, who have introduced the notion of suitable weak solutions to the Navier–Stokes equations, see [1,11,14,15].

**Definition 1.3** Let \( \omega \subset \mathbb{R}^3 \) and \( T_2 > T_1 \). \( w \) and \( r \) is a suitable weak solution to the Navier–Stokes in \( Q_\ast = \omega \times [T_1, T_2] \) if:

1. \( w \in L^\infty_2(Q_\ast), \nabla w \in L^2_2(Q_\ast), r \in L^\frac{3}{2}(Q_\ast); \)
2. \( w \) and \( r \) satisfy the Navier–Stokes equations in the sense of distributions;
3. for a.a. \( t \in [T_1, T_2] \), the local energy inequality

\[
\int_\omega \varphi(x, t)|w(x, t)|^2 \, dx + 2 \int_1^t \int_\omega \varphi |\nabla w|^2 \, dx \, dt' \leq \int_1^t \int_\omega [\| w \|^2(\partial_t \varphi + \Delta \varphi) + w \cdot \nabla \varphi (|w|^2 + 2r)] \, dx \, dt'
\]

holds for all non-negative \( \varphi \in C^1_0(\omega \times [T_1, T_2 + (T_2 - T_1)/2]) \).

In order to state a typical result of the regularity theory for suitable weak solutions, introduce the notation for parabolic cylinders (balls): \( Q(z_0, R) = B(x_0, R) \times [t_0 - R^2, t_0], B(x_0, R) = \{ x \in \mathbb{R}^3 : |x - x_0| < R \} \), and \( z_0 = (x_0, t_0) \).

**Proposition 1.4** 1. There are universal constants \( \varepsilon \) and \( c_0 \) such that for any suitable weak solution \( v \) and \( q \) in \( Q(z_0, R) \) satisfying the assumption

\[
C(z_0, R) + D(z_0, R) < \varepsilon,
\]

where
\[ C(z_0, R) = \frac{1}{R^2} \int_{Q(z_0, R)} |v|^3 \, dz, \quad D(z_0, R) = \frac{1}{R^2} \int_{Q(z_0, R)} |q|^3 \, dz, \]
the velocity field \( v \) is Hölder continuous in \( Q(z_0, R/2) \) and
\[ \sup_{z \in Q(z_0, R/2)} |v(z)| \leq C_0. \]

2. There is a universal constant \( \varepsilon > 0 \) such that for any suitable weak solution \( v \) and \( q \) in \( Q(z_0, R) \) satisfying the assumption
\[ g(z_0) := \min \{ \limsup_{r \to 0} E(z_0, r), \limsup_{r \to 0} A(z_0, r), \limsup_{r \to 0} C(z_0, r) \} < \varepsilon, \]
where
\[ A(z_0, R) = \sup_{t_0 - R^2 < t < t_0} \frac{1}{R} \int_{B(x_0, R)} |v(x, t)|^2 \, dx, \quad E(z_0, R) = \frac{1}{R} \int_{Q(z_0, R)} |
\nabla v|^2 \, dz, \]
the point \( z_0 \) is a regular point of \( v \), i.e., there exists \( 0 < r \leq R \) such that \( r \in L_\infty(Q(z_0, r)) \).

All the involved quantities are invariant with respect to the Navier–Stokes scaling
\[ v(x, t) \to \lambda v(\lambda x, \lambda^2 t), \quad q(x, t) \to \lambda^2 q(\lambda x, \lambda^2 t). \]
We often call them energy scale invariant quantities.

Proposition 1.4 describes the so-called \( \varepsilon \)-regularity theory of suitable weak solutions. In this context, it is interesting to verify whether or not a given weak Leray–Hopf solution has the property to be a suitable weak one in subdomains.

**Theorem 1.5** Among of all weak Leray–Hopf solutions to the Cauchy problem with the same initial data \( u_0 \), there exists at least one solution which is a suitable weak solution in any \( Q(z_0, R) \subset Q_+ \).

**Corollary 1.6** A weak Leray–Hopf solution, having the properties indicated in Theorem 1.5, is, in fact, a turbulent solution, i.e., there exists a set \( S \) of full measure, containing zero, with the following property: for any \( s \in S \), the inequality
\[ \int_{\mathbb{R}^3} |v(x, t)|^2 \, dx + 2 \int_t^s \int_{\mathbb{R}^3} |\nabla v|^2 \, dx \, dt' \leq \int_{\mathbb{R}^3} |v(x, s)|^2 \, dx \]
holds for any \( t \leq s \).
Notion of turbulent solutions has been introduced in [10].

Finally, we would like to define different types of blowups (singularities).

Let $v$ and $q$ be a suitable weak solution to the Navier–Stokes equations in $Q(z_0, R)$. Obviously, if

$$
g(z_0) = \min \{\limsup_{r \to 0} E(z_0, r), \limsup_{r \to 0} A(z_0, r), \limsup_{r \to 0} C(z_0, r)\} > 0,$$

then $z_0$ is a singular point.

**Definition 1.7** Let $z_0$ be a singular point. It is a Type I blowup if $g(z_0) < \infty$. The point $z_0$ is of Type II if $g(z_0) = \infty$.

It is useful to notice that if $g(z_0) < \infty$, see [16], then

$$
G(z_0) = \max \{\limsup_{r \to 0} E(z_0, r), \limsup_{r \to 0} A(z_0, r), \limsup_{r \to 0} C(z_0, r), \limsup_{r \to 0} D_0(z_0, r)\} < \infty.
$$

Here,

$$
D_0(z_0, r) = \frac{1}{r^2} \int_{Q(z_0, r)} |q - [q]_{B(x_0, r)}|^2 \, dz.
$$

In fact, as it follows from [21], see Lemma 2.1 there, $D_0(z_0, r)$ can be replaced with

$$
D(z_0, r) = \frac{1}{r^2} \int_{Q(z_0, r)} |q|^2 \, dz.
$$

2 **Axial symmetry and blowups**

There are many papers on regularity of axially symmetric solutions. We cannot pretend to cite all good works in this direction. For example, let us mention papers: [2–4, 7–9, 13, 25, 26].

In this section, we assume that $\Omega$ is the unit cylinder centred at the origin, i.e.,

$$
\Omega = C = \{x = (x', x_3), x' = (x_1, x_2) : |x'| < 1, |x_3| < 1\}.
$$

In what follows, it will be convenient to replace balls $B(x_0, r)$ with cylinders $\mathcal{C}(x_0, r) = \{x = (x', x_3) : |x' - x_0'| < r, |x_3 - x_{03}| < r\}$. We let $\mathcal{C}(r) = \mathcal{C}(0, r)$.

In our standing assumption, it is supposed that a suitable weak solution $v$ and $q$ to the Navier–Stokes equations in $Q = C \times ]-1, 0[$ is axially symmetric with respect to the axis $x_3$. The latter means the following: if we introduce the corresponding cylindrical
coordinates \((\varrho, \varphi, x_3)\) and use the corresponding representation \(v = v_\varrho e_\varrho + v_\varphi e_\varphi + v_3 e_3\), then \(v_\varrho, v_\varphi = v_{\varphi, \varphi} = v_3, \varphi = q, \varphi = 0\).

The main statement of the note reads that axisymmetric solutions have no Type I singularities.

**Theorem 2.1** Assume that a pair \(v, q\) is axially symmetric suitable weak solution to the Navier–Stokes equations in \(Q\) and the origin \(z = 0\) is a singular point of \(v\). Then it is a Type II blowup.

**Proof** Our first observation is related to Lemma 3.3 in [20]. It is stated there that:

\[
\sup_{z \in Q(1/2)} \sigma(z) \leq C(M) \left( \int_{Q(3/4)} |\sigma(z)|^{10/3} \, dz \right)^{3/10}, \tag{2.1}
\]

where \(\sigma := \varrho v_\varphi = v_2 x_1 - v_1 x_2\) and

\[
M = \left( \int_{Q(3/4)} |\overline{v}(z)|^{10/3} \, dz \right)^{3/10}, \quad \overline{v} = v_\varrho e_\varrho + v_3 e_3.
\]

However, estimate (2.1) has been proven in [20] under the additional assumption that there are no singular points if \(t < 0\). Let us try to modify the proof of Lemma 3.3 in [20] to a general case of axially symmetric suitable weak solutions. Indeed, the basic part of the proof will remain to be Moser iterations.

So, let us show that, under assumptions of Theorem 2.1, the left hand side of (2.1) is finite. As in the proof of Lemma 3.3 of [20], we hope to estimate \(\omega^{10/3}\) in a smaller domain by \(\omega^5\) in larger domain, where \(\omega = \sigma^m\) with a suitable number \(m \geq 1\). To this end, let us try to understand how smooth axially symmetric suitable weak solutions are. Denote by \(S\) the set of singular points of \(v\). It is well known that \(S\) has 1D Hausdorff measure zero, \(x' = 0\) for any \(z = (x, t) \in S\), and any spatial derivative of \(v\) is Hölder continuous in \(C \times (-1, 0]\) \(\setminus S\).

It is easy to check that \(\sigma\) satisfies the equation

\[
\partial_t \sigma + \left( v + \frac{2(x', 0)}{|x'|^2} \right) \cdot \nabla \sigma - \Delta \sigma = 0 \tag{2.2}
\]

in \(Q \setminus \{(x' = 0) \times ]-1, 0]\}. Given \(N > 0\), set \(\sigma_N(x, t) = \sigma(x, t)\) if \(|\sigma(x, t)| \leq N\), \(\sigma_N(x, t) = N\) if \(\sigma > N\), and \(\sigma_N(x, t) = -N\) if \(\sigma < -N\). Take two cut-off functions: the first of them is \(\psi = \psi(x, t)\), vanishing in a neighbourhood of the parabolic boundary of \(Q\) and the second one is \(\phi = \phi(x')\), vanishing in a neighbourhood of the axis of symmetry, multiple the left hand side of equation (2.2) by \(\sigma^{2m-1}_N \psi^4 \phi^2\) and integrate by parts. As a result, three different terms appear and they will be treated
separately. For the first one, we have

\[
\int_{-1}^{t_*} \int_C \partial_t \sigma \sigma_N^{2m-1} \psi^4 \phi^2 \, dz = \frac{1}{2m} \int_C \sigma_N^{2m} \psi^4 \phi^2 \big|_{(x,t_*)} \, dx
\]

\[
+ \int_C (\sigma - \sigma_N) \sigma_N^{2m-1} \psi^4 \phi^2 \big|_{(x,t_*)} \, dx
\]

\[
- \frac{1}{2m} \int_{-1}^{t_*} \int_C \sigma_N^{2m} \partial_t \psi^4 \phi^2 \, dz - \int_{-1}^{t_*} \int_C (\sigma - \sigma_N) \sigma_N^{2m-1} \partial_t \psi^4 \phi^2 \, dz.
\]

Since the second term on the right hand side of the above identity is non-negative, it can be dropped out:

\[
\int_{-1}^{t_*} \int_C \partial_t \sigma \sigma_N^{2m-1} \psi^4 \phi^2 \, dz \geq \frac{1}{2m} \int_C \sigma_N^{2m} \psi^4 \phi^2 \big|_{(x,t_*)} \, dx
\]

\[
- \frac{1}{2m} \int_{-1}^{t_*} \int_C \sigma_N^{2m} \partial_t \psi^4 \phi^2 \, dz - \int_{-1}^{t_*} \int_C (\sigma - \sigma_N) \sigma_N^{2m-1} \partial_t \psi^4 \phi^2 \, dz.
\]

Denoting \(b = 2(x', 0)|x'|^{-2}\), transform the second term as follows:

\[
\int_{-1}^{t_*} \int_C (v + b) \cdot \nabla \sigma \sigma_N^{2m-1} \psi^4 \phi^2 \, dz
\]

\[
= - \frac{1}{2m} \int_{-1}^{t_*} \int_C (v + b) \cdot \nabla (\psi^4 \phi^2) \sigma_N^{2m} \, dz
\]

\[
- \int_{-1}^{t_*} \int_C (v + b) \cdot \nabla (\psi^4 \phi^2)(\sigma - \sigma_N) \sigma_N^{2m-1} \, dz
\]

Finally, for the third term, we have

\[
- \int_{-1}^{t_*} \int_C \Delta \sigma \sigma_N^{2m-1} \psi^4 \phi^2 \, dz
\]

\[
= \frac{2m - 1}{m^2} \int_{-1}^{t_*} \int_C |\nabla \sigma_N|^2 \psi^4 \phi^2 \, dz - \frac{1}{2m} \int_{-1}^{t_*} \int_C \sigma_N^{2m} \Delta(\psi^4 \phi^2) \, dz
\]
\[ - \int_{-1}^{t_*} \int_C (\sigma - \sigma_N) \cdot \sigma_N^{2m-1} \Delta (\psi^4 \phi^2) \, dz. \]

Combining previous relationships, we find the following energy inequality

\[
\frac{1}{2m} \int_C \sigma_N^{2m} \psi^4 \phi^2 \big|_{(x, t_*)} \, dx + \frac{2m - 1}{m^2} \int_{-1}^{t_*} \int_C |\nabla \sigma_N^m| \psi^4 \phi^2 \, dz \\
\leq \frac{1}{2m} \int_{-1}^{t_*} \int_C \sigma_N^{2m} (\partial_t (\psi^4 \phi^2) + \Delta (\psi^4 \phi^2)) \, dz \\
+ \int_{-1}^{t_*} \int (\sigma - \sigma_N) \sigma_N^{2m-1} (\partial_t (\psi^4 \phi^2) + \Delta (\psi^4 \phi^2)) \, dz \\
+ \frac{1}{2m} \int_{-1}^{t_*} \int (\psi^4 \phi^2) \sigma_N^{2m} \, dz \\
+ \int_{-1}^{t_*} \int (\psi^4 \phi^2) (\sigma - \sigma_N) \sigma_N^{2m-1} \, dz.
\]

Now, selecting a special non-negative cut-off function \( \phi \) so that \( \psi(x') = 0 \) if \( 0 < |x'| < \varepsilon/2 \), \( \psi(x') = 1 \) if \( |x'| > \varepsilon \), and \( |\nabla^k \phi| \leq c \varepsilon^{-k} \), \( k = 0, 1, 2 \), let us see what happens if \( \varepsilon \to 0 \). We start with the two most important terms:

\[
I_1 = \int_{-1}^{t_*} \int_C |\psi^4 \phi| |\nabla \phi| (|\sigma| + |\sigma_N|) |\sigma_N|^{2m-1} \, dz
\]

and

\[
I_2 = \int_{-1}^{t_*} \int_C |b| \psi^4 \phi |\nabla \phi| (|\sigma| + |\sigma_N|) |\sigma_N|^{2m-1} \, dz.
\]

As to \( I_1 \), it is easy to see

\[
I_1 \leq c \int_{-1}^{t_*} 2\pi \int_{-1}^{1} \int_{\varepsilon/2 < \rho < \varepsilon} |\psi|^2 N^{2m-1} \rho \, d\rho \, dx \, dt \to 0
\]
as $\varepsilon \to 0$. The second term is the most difficult one. Indeed,

$$I_2 \leq c \int_{-1}^{t_*} 2\pi \int_{-1}^{1/2} \int_{-1}^{1/2} \int_{-1/2}^{1/2} \frac{1}{\varepsilon} N^{2m-1} |v_\varphi| d\varphi dx_3 dt \leq$$

$$\leq c N^{2m-1} \frac{1}{\varepsilon} \left( \int_{-1}^{t_*} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |v_\varphi|^2 d\varphi \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} dz \right)^{1/2} \to 0$$

as $\varepsilon \to 0$. It remains to estimate the first two terms in the energy inequality. In the worst case scenario, we proceed as follows:

$$I_3 = c N^{2m-1} \frac{1}{\varepsilon^2} \int_{-1}^{t_*} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |\sigma| dz = c N^{2m-1} \frac{1}{\varepsilon} \int_{-1}^{t_*} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} |v_\varphi| dz \to 0$$

as $\varepsilon \to 0$, see bounds for $I_2$. So, passing to the limit in the energy inequality, we find

$$\frac{1}{2m} \int_{C} \sigma_N^{2m} \psi^4(x,t_*) dx + \frac{2m-1}{m^2} \int_{-1}^{t_*} \int_{C} |\nabla \sigma_N^m|^2 \psi^4 dz \leq$$

$$\leq \frac{1}{2m} \int_{-1}^{t_*} \int_{C} \sigma_N^{2m} (\partial_1 \psi^4 + \Delta \psi^4) dz +$$

$$+ \int_{-1}^{t_*} \int_{C} (\sigma - \sigma_N) \sigma_N^{2m-1} (\partial_1 \psi^4 + \Delta \psi^4) dz +$$

$$+ \frac{1}{2m} \int_{-1}^{t_*} \int_{C} (v + b) \cdot \nabla \psi^4 \sigma_N^2 m dz +$$

$$+ \int_{-1}^{t_*} \int_{C} (v + b) \cdot \nabla \psi^4 (\sigma - \sigma_N) \sigma_N^{2m-1} dz.$$

Now, pick up a cut-off function exactly as in [20], i.e., $\psi(x,t) = \Phi(x) \chi(t)$ so that $0 \leq \Phi \leq 1$, $\Phi = 0$ outside $C(r_1)$, $\Phi = 1$ in $C(r)$, $0 \leq \chi \leq 1$, $\chi(t) = 0$ if $t \leq -r_1^2$, $\chi(t) = 1$ if $t > -r_2^2$, and

$$\nabla^k \Phi \leq \frac{c}{|r_1 - r|^k}, \quad k = 1, 2, \quad |\partial_1 \chi| \leq \frac{c}{(r_1 - r)^2},$$
where $1/2 \leq r < r_1 \leq 3/4$. Letting $\omega = \sigma^m$ and $\omega_N = \sigma^m_N$, we derive from the energy inequality the basic estimate:

$$|\psi^2\omega_N|_{2, Q}^2 := \sup_{-1 < t < 0} \|\psi^2\omega_N\|^2_{L^2(Q)} + \|\nabla(\psi^2\omega_N)\|^2_{L^2(Q)} \leq \frac{c}{(r_1 - r)^2} \int_{Q(r_1)} \psi^2\omega_N dz + cJ_1 + cJ_2,$$

where

$$J_1 = \frac{1}{r_1 - r} \int_{Q(r_1)} |v||\omega||\psi^2\omega_N| dz$$

and

$$J_2 = \frac{1}{r_1 - r} \int_{Q(r_1)} \psi^3|b||\omega|^{\frac{1}{m}} |\omega_N|^{2 - \frac{1}{m}} dz.$$

Now, we split the proof into three steps.

**Step I** We assume that $m = 4/3$, $r_1 = 3/4$, and $r = 5/8$. Then we have

$$\frac{c}{(r_1 - r)^2} \int_{Q(r_1)} \psi^2\omega_N dz \leq c \frac{r_1}{(r_1 - r)^2} \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^\frac{4}{5}.$$

For the second term, we are going to exploit the Hölder inequality. Indeed,

$$J_1 \leq \frac{1}{r_1 - r} \left( \int_{Q_1(r_1)} |v|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} \left( \int_{Q_1(r_1)} |\omega|^5 dz \right)^{\frac{2}{5}} \left( \int_{Q_1(r_1)} |\psi^2\omega_N|^{\frac{10}{7}} dz \right)^{\frac{3}{10}}.$$

Finally, taking into account that $\omega = (|x'||v|')^{\frac{1}{3}}$, we estimate the third term:

$$J_2 \leq c \int_{Q_1(r_1)} \frac{1}{|x'|} \omega^2 dz \leq c \int_{Q_1(r_1)} |v|^{\frac{5}{3}} dz \leq c \left( \int_{Q(r_1)} |v|^{\frac{10}{7}} dz \right)^{\frac{4}{5}}.$$

Using the known multiplicative inequality

$$\|\psi^2\omega_N\|_{L^{10}(Q(r_1))} \leq c|\psi^2\omega_N|_{2, Q(r_1)}$$

(2.5)
and then Young inequality, we can pass to the limit as $N \to \infty$ and conclude that

$$\int_{Q(5/8)} |\sigma|^{(\frac{4}{3})^2 \frac{5}{2}} dz = \int_{Q(5/8)} |\sigma|^{\frac{4}{3}} dz < \infty. \quad (2.6)$$

**Step II** Now, assume that $m \geq (4/3)^2$, $r_1 \leq 5/8$. The first two terms in energy inequality (2.3) can be evaluated in the same way as in Step I. So, let us focus on the third term:

$$J_2 \leq \frac{1}{r_1 - r} \int_{Q(r_1)} \psi^3 |b| |\omega| \frac{9}{16} |\omega_N|^{\frac{23}{16}} dz$$

$$\leq \frac{1}{r_1 - r} \int_{Q(r_1)} |b| |\omega| \frac{9}{16} |\psi^2 \omega_N|^{\frac{23}{16}} dz$$

For $s = 40/21$, it follows from the Hölder inequality that

$$J_2 \leq \frac{1}{r_1 - r} \left( \int_{Q(r_1)} |b|^s dx \right)^{\frac{1}{s}} \left( \int_{-r_1^2}^{0} \left( \int_{Q(r_1)} |\omega|^\frac{s'}{16} |\psi^2 \omega_N|^{\frac{s'}{16}} dx \right)^{\frac{1}{s'}} dx \right)^{\frac{1}{s'}}$$

where $s' = s/(s - 1) = 40/19$ and

$$J_3 = \left( \int_{Q(r_1)} |\omega|^\frac{s'}{16} |\psi^2 \omega_N|^{\frac{s'}{16}} dx \right)^{\frac{1}{s'}}$$

$$\leq \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dx \right)^{\frac{9}{40}} \left( \int_{Q(r_1)} |\psi^2 \omega_N|^{\frac{s'}{16}} dx \right)^{\frac{40 - s'}{40 s'}}.$$

Applying Hölder inequality one more time, we find

$$\int_{-r_1^2}^{0} \left( \int_{Q(r_1)} |\omega|^\frac{s'}{16} |\psi^2 \omega_N|^{\frac{s'}{16}} dx \right)^{\frac{1}{s'}} dx \leq \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{9}{40}}$$

$$\times \left( \int_{-r_1^2}^{0} \left( \int_{Q(r_1)} |\psi^2 \omega_N|^{\frac{s'}{16}} dx \right)^{\frac{40 - s'}{40 s'}} dt \right)^{\frac{31}{40}}.$$
Let us introduce numbers

\[ p = \frac{s'115}{2(40 - s'9)} = \frac{23}{4}, \quad q = \frac{40 - s'9}{31s'} = \frac{115}{62}. \]

It is easy to check that \( 3/p + 2/q - 3/2 = 1/10 \) and thus the known multiplicative inequality implies the bound

\[
\|\psi^2 \omega_N\|_{p,q, Q(r_1)}^{\frac{23}{36}} := \left( \int_{-r_1^2}^{0} \left( \int_{Q(r_1)} |\psi^2 \omega_N|^{\frac{s'115}{2(40 - s'9)}} dx \right)^{\frac{40 - s'9}{31s'}} dt \right)^{\frac{31}{40}} \leq c(r_1^{\frac{10}{11}} \|\psi^2 \omega_N\|_{2, Q(r_1)}^{\frac{23}{36}}).
\]

So, the final estimate for the \( J_3 \) has the form

\[
J_3 \leq \frac{c}{r_1 - r} \left( \int_{Q(r_1)} |b|^s dx \right)^{\frac{1}{s}} \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{9}{35}} \left( r_1^{\frac{10}{11}} \|\psi^2 \omega_N\|_{2, Q(r_1)}^{\frac{23}{36}} \right)
\]

\[
\leq \frac{c}{r_1 - r} r_1^{\frac{23}{35}} \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{9}{35}} \left( r_1^{\frac{10}{11}} \|\psi^2 \omega_N\|_{2, Q(r_1)}^{\frac{23}{36}} \right).
\]

Now, our energy estimate can be re-written in the following way:

\[
|\psi^2 \omega_N|_{2, Q}^2 \leq c \frac{r_1}{(r_1 - r)^{\frac{3}{10}}} \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{4}{5}}
\]

\[
+ \frac{c}{r_1 - r} \left( \int_{Q_1(r_1)} |v|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} \left( \int_{Q_1(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{3}{5}} |\psi^2 \omega_N|_{2, Q}
\]

\[
+ \frac{c}{r_1 - r} r_1^{\frac{23}{35}} \left( \int_{Q(r_1)} |\omega|^\frac{5}{2} dz \right)^{\frac{9}{35}} \left( r_1^{\frac{10}{11}} \|\psi^2 \omega_N\|_{2, Q(r_1)}^{\frac{23}{36}} \right).\]
After application of the Young inequality, we arrive at the estimate

\[ |\psi^2\omega_N|^2 \leq \frac{c}{r_1 - r} \left( \left( \frac{r_1}{r_1 - r} \right)^{\frac{23}{27}} + \left( \int_{Q(r_1)} |v|^{\frac{10}{7}} \, dz \right)^{\frac{3}{7}} \right) \int_{Q(r_1)} |\omega|^\frac{5}{3} \, dz \],

which, by the multiplicative inequality and by passing to the limit as \( N \to \infty \), leads us to the final estimate of Step II:

\[ \left( \int_{Q(r)} |\omega|^{\frac{10}{7}} \right)^{\frac{3}{10}} \leq \frac{c \sqrt{r_1 - r}}{r_1 - r} \left( \left( \frac{r_1}{r_1 - r} \right)^2 + \left( \int_{Q(r_1)} |v|^{\frac{10}{7}} \, dz \right)^{\frac{3}{10}} \right) \int_{Q(r_1)} |\omega|^\frac{5}{3} \, dz \],

where

\[ M = 1 + \left( \int_{Q(r_1)} |v|^{\frac{10}{7}} \, dz \right)^{\frac{3}{10}}. \]

**Step III.** For \( k = 2, 3, \ldots \), set

\[ m = m_k = \left( \frac{4}{3} \right)^k, \quad r_1 = r^{(k)} = \frac{1}{2} + \frac{1}{2^{k+1}}, \quad r = r^{(k+1)}, \quad Q_k = Q(r^k). \]

Then, the following sequence of inequalities follows from the inequality (2.7):

\[ \left( \int_{Q_{k+1}} |\sigma|^{\frac{10m_k}{3}} \right)^{\frac{3}{10m_k}} \leq \left( \frac{cM}{\sqrt{r^{(k+1)} - r^{(k)}}} \left( \frac{r^{(k+1)}}{r^{(k+1)} - r^{(k)}} \right)^2 \right)^{\frac{1}{m_k}} \left( \int_{Q_k} |\sigma|^{\frac{5m_k}{2}} \, dz \right)^{\frac{2}{5m_k}}. \]
Observing that $10m_k/3 = 5m_{k+1}/2$ and applying Moser’s arguments, we find

$$\sup_{z \in Q(1/2)} |\sigma(z)| \leq c(M) \left( \int_{Q(5/8)} |\sigma|^{5m_2/2} \right)^{2/m^2}.$$ 

The right hand side of the above inequality is finite and can be estimated by a constant depending only on $M$ as it has been shown in Step I.

So, finally, we conclude that

$$\sup_{z \in Q(1/2)} |\sigma(z)| = \sup_{z \in Q(1/2)} |x'| v_\varphi(z) | \leq c(M). \quad (2.9)$$

Now, returning to the proof Theorem 2.1, assume, in contrary, that $z = 0$ is a Type I blowup, i.e.,

$$0 < g = g(0) < \infty. \quad (2.10)$$

As it has been already noticed, (2.10) implies that

$$G = G(0) < \infty. \quad (2.11)$$

So, without lose of generality, we may assume that

$$0 < L_0 = \sup_{0 < r < 1} A(r) + \sup_{0 < r < 1} C(r) + \sup_{0 < r < 1} E(r) + \sup_{0 < r < 1} D(r) < \infty. \quad (2.12)$$

Here, for example,

$$A(r) = A(0, r) = \sup_{-r^2 < t < 0} \frac{1}{r} \int_{C(r)} \left| v(x, t) \right|^2 dx$$

and so on.

Now, we can rescale our function $v$ and $q$ around the origin as it has been described, for example, in [19], see Theorem 3.5 there. Let $\lambda_k \to 0$ be a sequence and let

$$u^k(y, s) = \lambda_k v(x, t), \quad p^k(y, s) = \lambda_k^2 q(x, t),$$

where $x = \lambda_k y$ and $t = \lambda_k^2 s$. Passing $k \to \infty$, we can find limit functions $u$ and $p$ of sequences $u^k$ and $p^k$ that have the properties $(A)$:

(i) $u$ is a local energy ancient solution in $Q_- = \mathbb{R}^3 \times ]-\infty, 0[$, i.e., the pair $u$ and $p$ is a suitable weak solution in $Q(R)$ for any $R > 0$;

(ii) $u$ and $p$ is axially symmetric solution to the Navier–Stokes equations in $Q_-$;

(iii) for any $R > 0$,
\[ A(u, R) + E(u, R) + C(u, R) + D(p, R) \leq L_0 < \infty. \]

In addition, as it follows from (2.9),
\[ \Gamma = \varrho u \varphi \in L_\infty(Q_-) \tag{2.13} \]
and the velocity \( u \) is not trivial in the sense
\[ \frac{1}{a^2} \int_{Q(a)} |u|^3 d\tau \geq \varepsilon(L_0) > 0 \tag{2.14} \]
for all \( 0 < a \leq 1 \), see also [18].

Since \( u \) and \( p \) is an axially symmetric suitable weak solution, there exists a closed set \( S^\Gamma \) in \( Q_- \), whose 1D-parabolic measure \( \mathbb{R}^3 \times \mathbb{R} \) is equal to zero and \( x' = 0 \) for any \( z = (x', x_3, t) \in S^\Gamma \), such that any spatial derivative of \( u \) (and thus of \( \Gamma \)) is Hölder continuous in \( Q_- \setminus S^\Gamma \). In particular, for all \( t < 0 \), the function \( \Gamma(0, x_3, t) = 0 \) for all \( x_3 \in \mathbb{R} \setminus S^\Gamma_t \), where \( S^\Gamma_t \) is a closed set of measure zero in \( \mathbb{R} \).

Moreover, by the known multiplicative inequalities and by \( (A)_{iii} \), we have
\[ \sup_{R > 0} \frac{1}{R^2} \left( \int_{-R^2}^0 \left( \int_{C(R)} |u|^3 \, dx \right)^{\frac{4}{3}} \, dt \right)^{\frac{3}{4}} \leq c(L_0). \tag{2.15} \]

And the function \( \Gamma \) satisfies the following heat equation with the drift
\[ \partial_t \Gamma + \left( u + \frac{(x', 0)}{|x'|^2} \right) \cdot \nabla \Gamma - \Delta \Gamma = 0 \tag{2.16} \]
in \( \mathbb{R}^3 \setminus \{x' = 0\} \times (-\infty, 0] \).

It has been shown in [12] that if \( \Gamma \) is a Lipschitz continuous function in \( Q_- \) then, under the above conditions, it must be identically equal to zero. This could reduce our problem to the case with no drift, i.e., \( u_\varphi = 0 \). Now, the task is to show that the above Liouville type theorem remains to be true in our situation as well.

First, we need to understand differentiability properties of the function \( \Gamma \) in the domain \( |x'| > 0 \). To this end, we observe that
\[ |\partial_{i} \Gamma(z) - \Delta \Gamma(z)| \leq \sup_{z= \{x, t\} \in P(\delta, R; R) \times \mathbb{R}^2, 0[} |u(z)| + 2/\delta |\nabla \Gamma(z)| \]
for any \( 0 < \delta < R \), where \( P(a, b; h) = \{x : a < |x'| < b, |x_3| < h\} \). Since \( u \) is axially symmetric, the first factor on the right hand side is finite. This, by iteration, see, for instance, [23,24] yields
\[ \Gamma \in W_p^{2,1}(P(\delta, R; R) \times \mathbb{R}^2, 0[]) \]
for any $0 < \delta < R < \infty$ and for any finite $p \geq 2$.

We also know that (it follows from the partial regularity theory) that, for any $t < 0$,

$$\Gamma(x', x_3, t) \to 0 \quad \text{as} \quad |x'| \to 0$$

(2.17)

for all $x_3 \in \mathbb{R}^3 \setminus S^\Gamma$.

Now, define the class $\mathcal{V}$ of functions $\pi : Q_- \to \mathbb{R}$ possessing the properties:

(i) there exists a closed set $S^\pi$ in $Q_-$, whose 1D-parabolic measure $\mathbb{R}^3 \times \mathbb{R}$ is equal to zero and $x' = 0$ for any $z = (x', x_3, t) \in S^\pi$, such that any spatial derivative is Hölder continuous in $Q_- \setminus S^\pi$;

(ii)

$$\pi \in W^{2,1}_2 \left( [P(\delta, R; R) \times ] - R^2, 0] \right) \cap L_\infty(Q_-)$$

for any $0 < \delta < R < \infty$.

Now, we can state an analog of Lemma 4.2 of [12] for the class $\mathcal{V}$. □

**Lemma 2.2** Let $u$ and $p$ have all the properties (A). Let $\pi \in \mathcal{V}$ be a non-negative function satisfying the following conditions:

$$\partial_t \pi + \left( u + 2 \frac{(x', 0)}{|x'|^2} \right) \cdot \nabla \pi - \Delta \pi \leq 0$$

(2.18)

in $\mathbb{R}^3 \setminus \{x' = 0\} \times ] - \infty, 0[$;

$$\pi(0, x_3, t) \geq k_R$$

(2.19)

for all $t \in ] - (2R)^2, 0[, \ x_3 \in ] - 2R, 2R[ \setminus S^\pi_t$, where $S^\pi_t = \{x_3 : (0, x_3, t) \in S^\pi\}$, and for some $k > 0$;

$$\pi \leq M_0 k_R$$

(2.20)

in $Q(2R)$ and for some $M_0 \geq 1$. Then

$$\pi \geq \beta k_R$$

(2.21)

in $Q(R/2)$, where $\beta$ depends on $M_0$ and $L_0$ only.

We can easily deduce from the above lemma that $\Gamma = 0$. Indeed, let $k_R := \frac{1}{2} \text{osc}_{z \in Q(2R)} \Gamma(z)$. Then either $\pi = \Gamma - \inf_{z \in Q(2R)} \Gamma(z)$ or $\pi = \sup_{z \in Q(2R)} \Gamma(z) - \Gamma$ satisfies condition (2.19) and (2.20) with $M_0 = 2$. So, (2.21) implies $\text{osc}_{z \in Q(R/2)} \Gamma \leq (1 - \beta/2) \text{osc}_{z \in Q(2R)} \Gamma$ and iterations give the result.

Now, let us outline a proof of Lemma 2.2, following the paper [12]. It is based on the inequality:
\[
\int_{Q_-} \left( -\pi \partial_t \eta - \pi \Delta \eta - (u + 2x'/|x'|^2) \cdot \nabla \eta \right) dx \, dt \\
\geq 4\pi_0 \int_{\infty}^{0} \int_{\infty}^{\infty} \pi(0, x_3, t) \eta(0, x_3, t) dx_3 dt
\]

for any non-negative test function \( \eta \) with compact support in \( Q_- \). Here, \( \pi_0 = 3.14 \ldots \). Although the above inequality has been proven in [12] under the assumption that \( \pi \) is Lipschitz, it is true for functions \( \pi \) from the class \( \mathcal{V} \) as well. Of course, it should be verified with the help of a cut-off around \( x_3 \)-axis similar to the derivation of estimate (2.9).

Selecting the test function \( \eta \) in a special way, see the proof of Lemma 4.2 of [12], and repeating the arguments there, it is possible to show that there exists \( \tilde{t} \in [-R^2, -3R^2/4] \) such that

\[
|\{x \in C(R) : \pi(x, \tilde{t}) > \kappa k_R\}| \geq \delta |C(R)|, \quad (2.22)
\]

where positive numbers depends on \( M_0 \) and \( L_0 \) only. The latter inequality should imply (2.21) and in fact it is the main result of a certain theory developed in [12] for the class of Lipschitz functions \( \pi \). This theory with minor changes can be also adapted to the class \( \mathcal{V} \). In order to see that, let us state two important inequalities related to particular equation (2.16). To this end, assume that a function \( \pi \in \mathcal{V} \) satisfies condition (2.19) and the equation

\[
\partial_t \pi + \left( u + 2 \frac{(x', 0)}{|x'|^2} \right) \cdot \nabla \pi - \Delta \pi = 0 \quad (2.23)
\]

in \( \mathbb{R}^3 \setminus \{x' = 0\} \times ] - \infty, 0[. \) Then, the first inequality is as follows:

\[
\sup_{z \in C(\tau R) \times ] - (\mu R)^2, 0[} (k - \pi(z))^+ \leq c(\tau, \mu, L_0) \left( \frac{1}{|Q(2R)|} \int_{Q(2R)} (k - \pi(z))^q dz \right)^\frac{1}{q}, \quad q = \frac{40}{9} \quad (2.24)
\]

for any \( 0 < \tau < 2 \), for any \( 0 < \mu < 2 \), and for any \( k \leq k_R \). It is the so-called Moser inequality and can be proved in the same way as the corresponding inequality for \( v \) (even easier since \( \pi \) is bounded).

The second inequality is simply the energy one:

\[
\int_{C(2R)} ((\pi(x, t) - k)_-)^2 \eta(x, t) dx + 2 \int_{t_0 - \theta R^2}^t \int_{C(2R)} |\nabla (\pi - k)_-|^2 dx d\tau
\]
for any $0 < k \leq k_R$, for any cut-off function $\eta$ vanishing near $\partial C(2R)$, and for any $t_0 - \theta R^2 \leq t \leq t_0$. Here, $t_0 \leq 0$ and $\theta > 0$ obey the restriction $-4R^2 < t_0 - \theta R^2$.

All other arguments to deduce (2.21) from (2.22) in [12] are either algebraical or based on the general function inequalities and can be repeated without changes.

Now, since $\Gamma = 0$, $u_\varphi = 0$ as well and thus $u = u_\varphi e_\varphi + u_3 e_3$. Let us notice that any axially symmetric suitable weak solution with no swirl, i.e., $u_\varphi = 0$, is smooth in the sense that any spatial derivative of $u$ is Hölder continuous in space-time (it can be done by considering a problem for $\eta = \omega_\varphi / \varrho$, where $\omega_\varphi = u_3, \varphi - u_\varphi, 3$, and reduction of it to spatial dimension 5, see, for example, [6]). In particular, the function $u$ is a continuous function in $Q(R)$ for any $R > 0$. The latter contradicts restriction (2.14) for sufficiently small $a$. Hence, the origin cannot be a Type I singularity. So, as, by assumptions, $z = 0$ is a singular point, it should be Type II blowup.

### 3 Axially symmetric solutions to the Navier–Stokes equations that are locally in critical spaces

Theorem 2.1 reads: axially symmetric solutions to the Navier–Stokes equations have no Type I blowups. Therefore, any additional assumptions that exclude singularities of Type II, could be, in fact, sufficient conditions of regularity for axially symmetric solutions. For example, boundedness of norms in certain critical spaces is exactly such a case. It is known that solutions, belonging to critical space like $H^{1/2}$ or $L^3$, are smooth, see [5]. As to weaker critical spaces like $L^{3,\infty}$, $BMO^{-1}$, or $\dot{B}^{-1,\infty}$, it is still unknown whether solutions, belonging to these spaces, have no blowups. However, we can show that the assumption of boundedness of suitable weak solutions in the above critical spaces excludes Type II singularities, see papers [8,17,22]. Such an assumption and Theorem 2.1 would imply regularity of axially symmetric solutions.

Let us consider first the case of the space $L_\infty(-1, 0; BMO^{-1}(C))$. Here, in 3D case, $v = \text{rot } \omega$ with divergence free vector field $\omega \in L_\infty(-1, 0; BMO(Q; \mathbb{R}^3))$ for which

$$
\|v(\cdot, t)\|_{BMO^{-1}(C)} := \|\omega(\cdot, t)\|_{BMO(C)}
$$

$$
= \sup \left\{ \frac{1}{|C(r)|} \int_{C(x, r)} |\omega(y, t) - [\omega]_{C(x, r)}(t)|dy : C(x, r) \subset C \right\}
$$

$$
\leq M < \infty
$$
for a.a. \( t \in ] -1, 0 [ \). Then, according Proposition 1.1 of [17],

\[
\sup_{z_0 \in \overline{Q}(1/2), 0 < r < 1/4} A(z_0, r) + D(z_0, r) + E(z_0, r) + C(z_0, r) < \infty. \tag{3.1}
\]

and, hence, \( z = 0 \) is a regular point, see Theorem 2.1.

In the second case, \( v \in L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)) \). Here, there is a problem with localisation in the space \( \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3) \). To avoid unessential issues, we shall consider the problem of regularity in the whole space, i.e., in \( Q_T = \mathbb{R}^3 \times ]0, T[ \). Let \( v \in L_\infty(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)) \) be a suitable weak solution to the Navier–Stokes equations in \( Q_T \). By Theorem 1.2 of [22], for \( z_0 \in \mathbb{R}^3 \times ]0, T[ \), we have the estimate

\[
\sup_{0 < r < r_0} G(z_0, R) \leq c_0(r_0^{1/2} + \|v\|^2_{L_\infty(0,T;\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3))} + \|v\|^6_{L_\infty(0,T;\dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3))}),
\]

where \( r_0 \leq \frac{1}{2} \min\{1, t_0\} \) and \( c_0 \) depends \( C(z_0, 1) \) and \( D(z_0, 1) \). And again by Theorem 2.1, such solution should be regular, for example, Hölder continuous.

The last case, \( v \in L_\infty(-1, 0; L^{3, \infty}(\mathcal{C})) \), is the easiest one as directly from Hölder inequality it follows that

\[
\frac{1}{r} \int_{\mathcal{C}(x,r)} |v(y, t)|^2 dy \leq c \|v(\cdot, t)\|_{L^{3, \infty}(\mathcal{C}(x,r))}^2
\]

and thus

\[
\sup_{z_0 \in \overline{Q}(0,1/4)} \sup_{0 < r < 1/4} A(z_0, r) \leq c \|v(\cdot, t)\|_{L^{3, \infty}(\mathcal{C})}^2.
\]

Then, an estimate of type (3.1) can be deduced from [16], see Lemma 1.8, with the same conclusion, by Theorem 2.1, that \( z = 0 \) is a regular point.

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References

1. Caffarelli, L., Kohn, R.-V., Nirenberg, L.: Partial regularity of suitable weak solutions of the Navier–Stokes equations. Comm. Pure Appl. Math. XXXV, 771–831 (1982)
2. Chae, D., Lee, J.: On the regularity of the axisymmetric solutions of the Navier–Stokes equations. Math. Z. 239, 645–671 (2002)
3. Chen, C., Fang, D., Zhang, T.: Regularity of 3D axisymmetric Navier–Stokes equations. Discret. Contin. Dyn. Syst. - A 37(4), 1923–1939 (2017)
4. Chen, C., Strain, R.M., Yau, H., Tsai, T.: Lower bounds on the blow-up rate of the axisymmetric Navier–Stokes equations II. Commun. Part. Diff. Equa. 34, 203–232 (2009)
5. Escauriaza, L., Seregin, G., Sverak, V.: $L^{3,\infty}$-solutions to the Navier–Stokes equations and backward uniqueness, Uspekhi Matematicheskii Nauk, v. 58, 2(350), pp. 3–44. English translation in Russian Mathematical Surveys, 58 2, pp. 211–250 (2003)
6. Kang, K.: Regularity of axially symmetric flows in half-space in three dimensions. SIAM J. Math. Anal. 35(6), 1636–1643 (2004)
7. Ladyzhenskaya, O.A.: On unique solvability of the three-dimensional Cauchy problem for the Navier–Stokes equations under the axial symmetry. Zap. Nauchn. Sem. LOMI 7, 155–177 (1968)
8. Lei, Z., Zhang, Q.: A Liouville theorem for the axially symmetric Navier–Stokes equations. J. Funct. Anal. 261, 2323–2345 (2011)
9. Leonardi, S., Malek, J., Necas, J., Pokorny, M.: On axially symmetric flows in $\mathbb{R}^3$. ZAA 18, 639–649 (1999)
10. Leray, J.: Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63, 193–248 (1934)
11. Lin, F.-H.: A new proof of the Caffarelli–Kohn–Nirenberg theorem. Commun. Pure Appl. Math. 51, 241–257 (1998)
12. Nazarov, A.I., Ural'tseva, N.N.: The Harnack inequality and related properties for solutions to elliptic and parabolic equations with divergence-free lower-order coefficients. St. Petersb. Math. J. 23(1), 93–115 (2012)
13. Neustupa, J., Pokorny, M.: Axisymmetric flow of Navier–Stokes fluid in the whole space with non-zero angular velocity components. Math. Bohem. 126, 469–481 (2001)
14. Scheffer, V.: Partial regularity of solutions to the Navier–Stokes equations. Pac. J. Math. 66, 535–552 (1976)
15. Scheffer, V.: Hausdorff measure and the Navier–Stokes equations. Commun. Math. Phys. 55, 97–112 (1977)
16. Seregin, G.: Estimates of suitable weak solutions to the Navier–Stokes equations in critical Morrey spaces. Zapiski Nauchn. Seminar, POMI 336, 199–210 (2006)
17. Seregin, G.: Note on bounded scale-invariant quantities for the Navier–Stokes equations. Zapiski POMI 397, 150–156 (2011)
18. Seregin, G.: Remark on Wolf’s condition for boundary regularity of Navier–Stokes equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 444, Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 45, 124–132 (2016)
19. Seregin, G.A., Shilkin, T.N.: Liouville-type theorems for the Navier–Stokes equations. Russ. Math. Surv. 73(4), 661–724 (2018)
20. Seregin, G., Sverak, V.: On type I singularities of the local axi-symmetric solutions of the Navier–Stokes equations. Commun. PDE’s 34, 171–201 (2009)
21. Seregin, G., Zajaczkowski, W.: A sufficient condition of regularity for axially symmetric solutions to the Navier–Stokes equations. SIAM J. Math. Anal. 39, 669–685 (2007)
22. Seregin, G., Zhou, D.: Regularity of solutions to the Navier–Stokes equations in $B^{-1}_{\infty,\infty}$. Zaptiki POMI 407, 119–128 (2018)
23. Solonnikov, V.A.: Estimates of solutions to the non-stationary Navier–Stokes system. Zapiski Nauchn. Seminar LOMI 28, 153–231 (1973)
24. Solonnikov, V.A.: Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 123–156; translation in Russian Math. Surveys 58, no. 2, 331–365 (2003)
25. Ukovskii, M.R., Yudovich, V.L.: Axially symmetric motions of ideal and viscous fluids filling all space. Prikl. Mat. Meh. 32, 59–69 (1968)
26. Zhang, P., Zhang, T.: Global axisymmetric solutions to the three-dimensional Navier–Stokes equations system. Int. Math. Res. Not. 2014, 610–642 (2014)

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