SOME EXAMPLES OF LIFTING PROBLEMS FROM QUOTIENT ALGEBRAS

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Abstract. We consider three lifting questions: Given a $C^*$-algebra $I$, if there is a unital $C^*$-algebra $A$ contains $I$ as an ideal, is every unitary from $A/I$ lifted to a unitary in $A$? is every unitary from $A/I$ lifted to an extremal partial isometry? is every extremal partial isometry from $A/I$ lifted to an extremal partial isometry? We show several constructions of $I$ which serve as working examples or counter-examples for above questions.

1. Introduction

Let $\mathcal{E}(A)$, or just $\mathcal{E}$, denote the set of extreme points of the unit ball of a unital $C^*$-algebra $A$. Recall that elements in $\mathcal{E}$ are characterized as the partial isometries $u$ in $A$ satisfying $(1 - uu^*)A(1 - u^*u) = 0$ by R. V. Kadison [10]. We call them extremal partial isometries and call the projections $1 - uu^*$, $1 - u^*u$ defect projections. In [4], Brown and Pedersen defined the notion of extremal richness for $C^*$-algebra $A$ which means quasi-invertible elements are dense in $A$ as an analogue of stable rank one for possibly infinite $C^*$-algebras. (We say $T$ in $A$ is quasi-invertible if $T$ has closed range and the kernel projections of $T^*$ and $T$ are centrally orthogonal in $A$, or if $T$ is in $A^{-1}\mathcal{E}A^{-1}$. For more equivalent definitions, see Theorem 1.1 in [4].) We denote by $A_q^{-1}$ the set of quasi-invertible elements. As a result, stable rank one $C^*$-algebras are characterized within the class of extremally rich $C^*$-algebras by the property that all extreme points of the unit ball are unitaries or $A_q^{-1} = A^{-1}$ where $A^{-1}$ is the set of invertible elements of $A$.

Suppose we have an extension of $C^*$-algebras:

$$
0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0
$$

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It is well known that if we have an extension of an extremally rich $C^*$-algebra $I$ by an extremally rich $C^*$-algebra $B$, we cannot deduce $A$ is extremally rich even in the finite case. The obstacle, as in the analogous problem with stable rank one, can be expressed as a lifting problem but with special properties. In fact, Brown and Pedersen proved the following theorem [3, Theorem 6.1].

**Theorem 1.1.** If $A$ is an extension of $I$ by $B$ and both $I$ and $B$ are extremally rich, then $A$ is extremally rich if and only if extremal partial isometries in $B$ are liftable to extremal partial isometries in $A$ and $PAQ$ is extremally rich whenever $P$ is projection of the form $1 - uu^*$ where $u \in \mathcal{E}(A)$ and $Q$ projection of the from $1 - v^*v$ where $v \in \mathcal{E}(I)$.

However, there are some special ideals $I$ for which the hypotheses in Theorem 1.1 can be simplified:

**Corollary 1.2.** Let $I$ be a $C^*$-algebra of stable rank one. Then $A$ is extremally rich if and only if $B$ is extremally rich and extremal partial isometries in $B$ lift.

This corollary implies the following: Whenever $I$ embedded as an ideal in a unital extremally rich $C^*$-algebra and $u$ is an extremal partial isometry in a extremally rich $C^*$-algebra $A/I$, there is an extremal partial isometry in $A$ which lifts $u$.

Motivated by above example, but not necessarily related to extension, we can ask whether we can find examples of $I$ such that whenever there exists $A$ containing $I$ as a closed ideal, certain lifting questions below from any quotient $C^*$-algebra $B = A/I$ to $A$ are affirmative:

1. Is every unitary in $B$ lifted to a unitary in $A$?
2. Is every unitary in $B$ lifted to an extremal partial isometry in $A$?
3. Is every extremal partial isometry in $B$ lifted to an extremal partial isometry in $A$?

We cannot expect a positive answer in great generality due to the nature of the problem; In fact, it is not very difficult to find counter examples for these questions. Interesting direction, however, is to find an affirmative answer. For example, it is well-known that if $I$ has an approximate identity consisting of projections, then every unitary in $B$ lifts to a partial isometry in $A$.

**Remark 1.3.** (i) When $I$ and $B$ are of stable rank one, an affirmative answer to (1) is equivalent to the stable rank one property for $A$. 
(ii) When $I$ and $B$ are of stable rank one, an affirmative answer to (2) is equivalent to extremal richness for $A$.

(iii) When $I$ has stable rank one and $B$ is extremally rich, an affirmative answer to (3) is equivalent to extremal richness for $A$.

2. Examples

Throughout this article $H$ will denote an infinite dimensional Hilbert space and $B(H)$ the set of bounded operators on $H$ and $K(H)$ (shortly $K$) the set of compact operators on $H$. Given a $C^*$-algebra $I$, $M(I)$ will denote the multiplier algebra of $I$ and $C(I)$ will denote the corona algebra of $I$, that is, $M(I)/I$. In view of Remark 1.3-(i), it is still interesting if we restrict $I$ to be the class of stable rank one $C^*$-algebras. But, even in this case, question (1) is not true for every stable rank one $C^*$-algebras. In fact, there is a well-known counter example: the Toeplitz extension

$$0 \longrightarrow K \longrightarrow T \longrightarrow C(S^1) \longrightarrow 0$$

where $T$ is the $C^*$-algebra generated by the unilateral shift $S$ on $H$ is such an example by the remark 1.3-(i) and the fact $\text{tsr}(T) \neq 1$. (See [14, 4.13].) However, there is a $C^*$-algebra of stable rank one which serve as an answer for question (1).

Definition 2.1. ([5, p.2]) We say a (non-unital) $C^*$-algebra $I$ has good index theory if whenever $I$ is embedded as an ideal in a unital $C^*$-algebra $A$ and $u$ is a unitary in $A/I$ such that $\partial_1([u]) = 0$ where $\partial_1 : K_1(A/I) \rightarrow K_0(I)$, there is a unitary in $A$ which lifts $u$.

It was noted by Brown and Pedersen that G. Nagy proved that any stable rank one $C^*$-algebra has good index theory [5]. Thus if we can show the existence of $I$ with $\text{tsr}(I) = 1$ such that $\partial_1 : K_1(B) \rightarrow K_0(I)$ is trivial, question (1) holds for $I$.

Example 2.2. Let

$$I = \{(f_n) \mid f_n \in C(\mathbb{T}, M_n(\mathbb{C})) \text{ and } f_n \rightarrow f \text{ in } C(\mathbb{T}, K)\}$$

Here $f_n \rightarrow f$ in $C(\mathbb{T}, K)$ means $\sup_{t \in \mathbb{T}} \|f_n(t) - f(t)\|$ goes to 0 as $n$ goes to infinity. Note that $I$ has stable rank one. Let $C(\mathbb{T}, B(H)_{*,S})$ be the set of functions $m : \mathbb{T} \rightarrow B(H)$ which are continuous with respect to the $*$-strong operator topology on $B(H)$, and which satisfy $\|m\|_{\infty} := \sup_{t \in \mathbb{T}} \|m(t)\| < \infty$.

It is not hard to show that

$$M(I) = \{(F_n) \mid F_n \in C(\mathbb{T}, M_n(\mathbb{C})) \text{ and } F_n \rightarrow_s F \text{ in } C(\mathbb{T}, B(H)_{*,S})\}$$
Here \( F_n \rightarrow_s F \) means \( F_n f_n \rightarrow F f \) and \( F_n^* f_n \rightarrow F^* f \) in \( I \) for each \((f_n) \in I\). Next we want to show \( K_0(\iota) : K_0(I) \rightarrow K_0(M(I)) \) is injective where \( \iota : I \rightarrow M(I); \) since the sequence \( \alpha = (z_n) \) in \( K_0(I) \) is eventually constant, we may assume \( I \) has only finite dimensional irreducible representations. Let \( \pi \) be such an irreducible representation and suppose \( K_0(\iota)(\alpha) = 0 \). Then \( K_0(\bar{\pi})(K_0(\iota)(\alpha)) = 0 \) where \( \bar{\pi} : M(I) \rightarrow B(H) \) is a unique map such that \( \bar{\pi} \circ \iota = \pi \). Note that we have \( \pi(I) = \bar{\pi}(I) \) since \( \pi \) is finite dimensional representation. Consequently, we have \( K_0(\pi)(\alpha) = 0 \) in \( K_0(\pi(I)) \). By the Remark 5.12 in [3], \( \alpha = 0 \).

From the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I & \xrightarrow{j} & A & \longrightarrow & B & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \xrightarrow{\iota} & M(I) & \longrightarrow & C(I) & \longrightarrow & 0
\end{array}
\]

we have

\[
\begin{array}{ccc}
K_0(I) & \xrightarrow{K_0(\iota)} & K_0(M(I)) \\
\downarrow & & \downarrow \\
K_0(\iota) & \xrightarrow{K_0(j)} & K_0(A)
\end{array}
\]

Thus it follows that \( K_0(j) \) is also injective. Thus from the exactness of six term exact sequence of K-theory we know the map \( \partial_1 : K_1(B) \rightarrow K_0(I) \) is trivial.

Next we show there are counter-examples to question (2).

**Example 2.3.** Let

\[
I = \{ (a_n) \in \prod M_{2n}(\mathbb{C}) \mid a_n \rightarrow (\begin{smallmatrix} D & 0 \\ 0 & E \end{smallmatrix}) \text{ in } M_2(K) \}
\]

Then

\[
M(I) = \{ (T_n) \in \prod M_{2n}(\mathbb{C}) \mid T_n \rightarrow (\begin{smallmatrix} M & 0 \\ 0 & N \end{smallmatrix}) \text{ in } M_2(B(H)_{s-s}) \}
\]

Now let’s fix the basis \( \mathfrak{B} \) of Hilbert space \( H \) as follows;

\[
\{ \cdots, w_n, \cdots, w_2, w_1, v_1, v_2, \cdots v_n, \cdots \}
\]

Let \( e_{xy} \) be a rank one projection such that \( e_{xy}(z) := \langle y, z \rangle x \), and take \( T_n = \sum_{i=1}^{n-1} e_{w_i w_{i+1}} + \sum_{j=1}^{n-1} e_{v_{j+1} v_j} + e_{w_n v_n} \). In fact, \( T_n \) is of the
following form with respect to the basis $\mathfrak{B}$:

$$
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \ddots & 0 & 0 & \ddots & 0 & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}
$$

Then $1 - T_n^* T_n = e_{w_1 w_1}$, $1 - T_n T_n^* = e_{v_1 v_1}$. Therefore $\pi((T_n))$ is an unitary element of $C(I)$ where $\pi$ is the natural quotient map. But it cannot be liftable to an extremal partial isometry: Assume there is an element $(a_n) \in I$ such that $T_n + a_n$ is extremal partial isometry in $M(I)$. Since $M_{2n}(\mathbb{C})$ has stable rank one, $T_n + a_n$ must be a unitary, and it happens only when $a_n = e_{v_1 v_1}$. Then $T_n + a_n$ cannot converge to the operator of the form $(* \ 0 \ 0 \ 0 \ \cdots \ 0 \ 0 \ 0 \ 0 \ 0)$. It is a contradiction.

More sophisticated example can be found as follows.

**Example 2.4.** Let $D$ be the stabilization of cone or suspension. Assume there exist a projection $p$ in the corona algebra of $D$ such that which does not lift but its $K_0$-class does lift. (For the construction of such a projection, see the example 5.13 in [11].) If we let $a$ be the self adjoint element which lifts $p$ in $M(D)$, we take $I$ to be the $C^*$-algebra generated by $a$ so that the quotient $I/D$ is isomorphic to $\mathbb{C}$. Then the Busby invariant is determined by sending 1 to $p$, and we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & D & \xrightarrow{j} & I & \rightarrow & \mathbb{C} & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow p & & \downarrow & & \| \\
0 & \rightarrow & D & \rightarrow & M(D) & \rightarrow & C(D) & \rightarrow & 0
\end{array}
\]

By the long exact sequence, we have

\[
\begin{array}{cccc}
0 & \rightarrow & K_0(I) & \rightarrow & K_0(\mathbb{C}) & \rightarrow & K_1(D) & \rightarrow & 0 \\
\downarrow & & \downarrow p & & \| & & \| & & \| \\
0 & \rightarrow & K_0(M(D)) & \rightarrow & K_0(C(D)) & \rightarrow & K_1(D) & \rightarrow & 0
\end{array}
\]

Since $\partial_0([p]_0) = 0$, $\partial_0 : K_0(\mathbb{C}) \rightarrow K_1(D)$ becomes trivial. Thus $K_0(I) \cong K_0(\mathbb{C})$. In particular, $K_0(I)$ is non-trivial. Consequently, we found a (stably) projectionless stable rank one $C^*$-algebra such that its
$K_0$-group is non-trivial. Now stabilize this algebra and call it $I$ again. We consider an extension of $I$ by $C(T)$ corresponding to a unitary $u$ in the coronal algebra with non-trivial class $K_1$-class. $u$ can’t be lifted to a unitary (If so, $[u]_1 = 0$ which is a contradiction), and it can’t be lifted to a partial isometry either because there are no non-zero projections available to defect projections of the partial isometry.

On the other hand, we have a certain class of $C^*$-algebras (e.g. elementary $C^*$-algebras) such that question (2) is true.

**Proposition 2.5.** Let $I$ be the $C^*$-algebra such that $M(I)$ is extremally rich. Suppose a unital $C^*$-algebra $A$ contains $I$ as an ideal. Then any unitary $u$ in $A/I$ can be liftable to an extremal partial isometry.

**Proof.** From the diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & I & \longrightarrow & A & \longrightarrow & A/I & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \tau & \\
0 & \longrightarrow & I & \longrightarrow & M(I) & \longrightarrow & C(I) & \longrightarrow & 0
\end{array}
$$

we see $\tau(u)$ is also a unitary in $C(I)$. Since $M(I)$ is an extremally rich $C^*$-algebra, we can find an extremal partial isometry $w$ which is an inverse image of $\tau(u)$ under the map $\pi$ so that we have $\pi(w) = \tau(u)$. Hence $(w, u)$ is the extremal partial isometry in $A$ which lifts $u$ since we can view $A$ as the pullback construction of $A/I$ and $M(I)$. \qed

**Remark 2.6.** We do not know the result of Proposition 2.5 is true under the weaker hypothesis $C(I)$ is extremally rich. In addition, $I$ in Example 2.3 and Example 2.4 are (stable rank one) $C^*$-algebras such that $M(I)$ is not extremally rich.

**Corollary 2.7.** Let $I$ be the $C^*$-algebra such that $M(I)$ is extremally rich and any Busby invariant into $C(I)$ is extreme point preserving map. Suppose a unital $C^*$-algebra $A$ contains $I$ as an ideal. Then any extremal partial isometry $u$ in $A/I$ can be lifted to an extremal partial isometry.

**Proof.** If a map is extreme point preserving, it sends an extremal partial isometry to an extremal partial isometry. Thus the result follows from above diagram. \qed

We have seen an example of $I$ such that any extremal partial isometry in $A/I$ can’t be lifted to a partial isometry in $A$ for some $C^*$-algebra $A$ which contains $I$ as an ideal in Example 2.4. If this does not happen, we can show the lifting question (3) might be true for a certain class of idealizers of $I$. 
Proposition 2.8. Let I be the $C^*$-algebra such that any extremal partial isometry in $A/I$ is lifted to a partial isometry whenever a $C^*$-algebra $A$ contains $I$ as an ideal. If $A$ is an extremally rich $C^*$-algebra, then any extremal partial isometry in $A/I$ can be lifted to an extremal partial isometry in $A$.

Proof. Let $q$ be the quotient map from $A$ to $A/I$. If $u$ is an extremal partial isometry in $B = A/I$, it can be lifted to a partial isometry $v$ in $A$. Since $A$ is an extremally rich $C^*$-algebra, $v$ has an extremal extension $w$ in $\mathcal{E}(A)$ such that $v = wv^*w = vv^*w$ [I Proposition 2.6].

Now let $I^+$ and $I^-$ be the defect ideals of $u$ and consider the map $\pi^+ \oplus \pi^- : B \to B/I^+ \oplus B/I^-$ which is injective. Since $\pi^+ \oplus \pi^- (q(v)) = \pi^+ \oplus \pi^- (q(w))$, $q(v) = q(w) = u$. Hence an extremal partial isometry $w$ is an inverse image of the extremal partial isometry $u$. \qed

However, the following examples show us that affirmative examples for question (3) might be hard to get without putting a condition on $I$.

Example 2.9. Let $A = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(B(H)) \mid B, C \in K \right\}$

Then $I = M_2(K)$ is an ideal of $A$. And $B = A/M_2(K)$ is isomorphic to $Q(H) \oplus Q(H)$. Now consider $\begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix}$ where $S$ and $T$ are isometries such that $1 - SS^*$ and $1 - TT^*$ are not compacts. Then $\pi \left( \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} \right)$ becomes an extremal partial isometry in $Q(H) \oplus Q(H)$ where $\pi : I \to B$. We can check, however, that there is no extremal partial isometry which lifts $\pi \left( \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} \right)$: Suppose there is $R \in M_2(K)$ such that $V = \begin{pmatrix} S & 0 \\ 0 & T^* \end{pmatrix} + R$ is an extremal partial isometry. Since $A$ contains $M_2(K)$ as an ideal, either one of defect projections of $V$ must be zero. But note that $1 - V^*V = \begin{pmatrix} 0 & 0 \\ 0 & 1 - TT^* \end{pmatrix}$ is compact is in $M_2(K)$. Therefore $1 - VV^* = 0$ implies $1 - TT^*$ is compact which is a contradiction. The other case is also similar.

The following clever example is due to Larry Brown although it was slightly generalized by the author.

Example 2.10. Let $A$ be a unital $C^*$-algebra such that $RR(A) = 0$, $\text{tsr}(A) = 1$ and $K_1(A) = 0$. Then it is known that $\text{tsr}(C[0, 1] \otimes A) = 1$...
Now let’s consider $I$ as $C[0,1] \otimes (A \otimes K)$ which has stable rank one too. Note that $M(A \otimes K)$ is equipped with the strict topology. Then $M(I) = C([0,1], M(A \otimes K))$ which has stable rank one too. Note that $M(A \otimes K)$ is equipped with the strict topology. Then $M(I) = C([0,1], M(A \otimes K))$ which has stable rank one too. Note that $M(A \otimes K)$ is equipped with the strict topology.

If we denote $[\mathcal{F}_A]$ by the set of homotype classes of Freehold elements in $M(A \otimes K)$, there is Mingo’s index map $\partial : [\mathcal{F}_A] \to K_0(A)$ which is actually an isomorphism [13]. Suppose $K_0(A)$ is an ordered group and let $e$ be an order unit. Then let $f_0 \in M(I)$ such that $f_0(1) = 1$ and $f_0(t)$ is an isometry with $\partial(f_0(t)) = -2e$ for $t < 1$. Let $u$ be a co-isometry in $M(A \otimes K)$ with the index $e$ and $f$ be $f_0u$. Then we can see that $\pi(f)$ is an isometry in $C(I)$ where $\pi : M(I) \to C(I)$. Assume there is a $k \in I$ such that $f + k$ is an extremal partial isometry in $M(I)$ which is a continuous path of extremal partial isometries. Since $M(A \otimes K)$ is a prime $C^*$-algebra, using index theory, $f(1) + k(1)$ must be co-isometry of index $e$. It follows that there is a unitary $v$ such that $u + k(1) = uv$ [6, Theorem 2.1]. Since $uv - u$ is in $A \otimes K$, $u$ is a Fredholm element in $M(A \otimes K)$, and $v \in 1 + A \otimes K$. Finally, let $f_1 = fu$ and $k_1$ be $f + k - f_1 = f(1 - v) + k \in I$. Note that $k_1(1) = 0$ and $f + k = f_1 + k_1$. Thus $f_1(t) + k_1(t)$ must be an isometry for $t < 1$ since $f$ has negative index for $t < 1$ and $f(t) + k(t)$ is an extremal partial isometry (isometry or co-isometry in this case) for each $t$. Since $f_1$ has a non-trivial kernel which is the range of $v^*(1 - u^*u)$, we know that $|k(t)| \geq 1$ for $t < 1$ which is a contradiction.

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