Gravitational energy in small regions for the quasilocal expressions in orthonormal frames

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Abstract
The Möller tetrad gravitational energy-momentum expression was recently evaluated for a small vacuum region using orthonormal frames adapted to Riemann normal coordinates. However the result was not proportional to the Bel-Robinson tensor $B_{\alpha\beta\mu\nu}$. Treating a modified quasilocal expressions in a similar way, we found one unique combination that gives a multiple of $B_{\alpha\beta\mu\nu}$ which provides a non-negative gravitational energy-momentum in the small sphere approximation. Moreover, in addition to $B_{\alpha\beta\mu\nu}$, we found a certain tensor $S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu}$ which gives the same “energy-momentum” density in vacuum. Using this tensor combination, we obtained an infinite set of solutions that provides a positive gravitational energy within the same limit.

1 Introduction
Finding an appropriate quasilocal expression for the gravitational energy is an open problem in general relativity. How do we even know that gravitational energy exists? Tidal friction from the tidal force of the Moon on the Earth leads to the slowing of the rate of Earth rotation. The lengthing of the day is the indirect evidence for the gravitational energy.

All matter and all other interaction fields, through their energy-momentum density, act as the source of gravity. These sources exchange energy-momentum with the gravitational field; all attempts to identify an energy-momentum density for gravity itself led to reference frame dependent quantities (i.e. pseudotensors), a reflection of the fact that energy-momentum of an isolated gravitating system is inherently non-local. This feature can be understood in terms of the equivalence principle: gravity cannot be detected at a point. However, one can get around this difficulty using the idea of quasilocal energy-momentum, i.e., associated with a closed 2-surface surrounding a region [1].

The Bel-Robinson tensor has a desirable property as it provides a non-negative value, although it cannot be interpreted as a “stress energy” for gravity directly because it has the wrong dimension. The energy density has dimension $\text{cm}^{-2}$, while $B_{\alpha\beta\mu\nu}t^\alpha t^\beta t^\mu t^\nu$ has dimension $\text{cm}^{-4}$, where $t^\alpha$ is the timelike unit normal. However, the small sphere region limit can resolve this mismatch. In the small sphere limit, the quasilocal expression for the energy-momentum density should be a multiple of $r^5B_{\mu000} \sim \frac{4}{3}\pi r^3(r^2B_{\mu000})$ [1], where $r$ is the radius of the Euclidean volume 3-ball. Indeed $r^2B_{0000}$ matches the energy density dimension.

A positivity of gravitational energy proof was obtained in orthonormal frames [2]. Success on a large scale automatically implies the positivity on the small region.
limit. Recently Møller’s tetrad gravitational energy-momentum expression \[3\] was evaluated for a small vacuum region using orthonormal frames adapted to Riemann normal coordinates. This result for the gravitational energy in the small sphere approximation is not positive definite. Treating a modified quasilocal expression \[4\] in a similar way, we found one unique combination that gives a multiple of the Bel-Robinson tensor which means that the gravitational energy is definitely non-negative.

The Bel-Robinson tensor component \(B_{0000}\) or \(B_{00}\) gives the non-negative gravitational “energy” density. We discovered that the sum of the tensor components \(S_{0000} + K_{0000}\) or \(S_{00} + K_{00}\) offers the same “energy” density value. Based on this criteria, we also obtained an infinite set of solutions that provide a positive gravitational energy within the same small sphere limit using certain modified quasilocal expressions.

2 Orthonormal frames and quadratic curvature

The quasilocal quantities for small regions can be studied by Taylor expanding the Hamiltonian, including the divergence of its boundary term in a small spatial region surrounding a point. The reference is the flat space geometry at this origin. The orthonormal frame satisfies

\[
e^{\alpha}_{\ a}(0) = \delta^{\alpha}_{\ a}, \quad \partial_i e^{\alpha}_{\ a}(0) = 0, \quad \Gamma^{\alpha}_{\ \beta i}(0) = 0, \quad (1)
\]

\[
\partial^2_{ij} e^{\alpha}_{\ a}(0) = -\frac{1}{6}(R^{\alpha}_{\ iaj} + R^{\alpha}_{\ jai}), \quad \partial_j \Gamma^{\alpha}_{\ \beta i}(0) = \frac{1}{2} R^{\alpha}_{\ \beta ji}, \quad (2)
\]

where the Latin letters refer to coordinate frames (holonomic frames) and Greek letter means the orthonormal frames (non-holonomic frames).

The Bel-Robinson tensor \[5\] is defined as

\[
B_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu} - R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu} + \frac{1}{2} g_{\alpha\beta} R_{\lambda\rho\mu\nu} R^{\lambda\rho}, \quad (3)
\]

where the dual curvature is \(*R_{\alpha\beta\mu\nu} = \frac{1}{2} \epsilon_{\alpha\beta\lambda\sigma} R^{\lambda\mu\sigma\nu}\). Furthermore, the tensors \(S_{\alpha\beta\mu\nu}\) and \(K_{\alpha\beta\mu\nu}\) in vacuum are defined as

\[
S_{\alpha\beta\mu\nu} := R_{\alpha\mu\lambda\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\nu\lambda\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{1}{4} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\rho\sigma\tau} R^{\lambda\rho\sigma\tau}, \quad (4)
\]

\[
K_{\alpha\beta\mu\nu} := R_{\alpha\lambda\beta\sigma} R_{\mu\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\beta} R_{\mu\sigma}^{\lambda\sigma} - \frac{3}{8} g_{\alpha\beta} g_{\mu\nu} R_{\lambda\rho\sigma\tau} R^{\lambda\rho\sigma\tau}. \quad (5)
\]

The identity in vacuum \(R_{\lambda\rho\sigma\mu} R^{\lambda\rho\sigma\nu} = \frac{1}{4} g_{\mu\nu} R_{\lambda\rho\sigma\tau} R^{\lambda\rho\sigma\tau}\) is useful.

It is known that the Bel-Robinson tensor is completely symmetric, we have found the identity

\[
3B_{\alpha\beta\mu\nu} \equiv B_{\alpha\beta\mu\nu} + B_{\alpha\mu\beta\nu} + B_{\alpha\nu\beta\mu} = S_{\alpha\beta\mu\nu} + S_{\alpha\mu\beta\nu} + S_{\alpha\nu\beta\mu} + K_{\alpha\beta\mu\nu} + K_{\alpha\mu\beta\nu} + K_{\alpha\nu\beta\mu}. \quad (6)
\]
Technically, the alternative form can be written as
\[ B_{\alpha\beta\mu\nu} \equiv B_{\alpha(\beta\mu\nu)} \equiv S_{\alpha(\beta\mu\nu)} + K_{\alpha(\beta\mu\nu)}. \] (7)

The tensors \( S_{\alpha\beta\mu\nu} \) and \( K_{\alpha\beta\mu\nu} \) are both symmetric at the first pair and last pair of indices.

It turns out that in vacuum the small region energy density has a RNC Taylor series expansion of the form
\[ t^\mu_\nu = t^\mu_{ij}x^ix^j, \] (8)
the corresponding energy-momentum is
\[ P_\mu = \int_{t=0} t^\nu_{\mu ij}x^ix^jd\Sigma_\nu \]
\[ = t^0_{\mu ab} \int_{t=0} x^a x^bd^3x \]
\[ = t^0_{\mu ab} \frac{\delta^{ab}}{3} \int r^2d^3x \]
\[ = t^0_{\mu a} \frac{4\pi r^5}{15}, \] (9)
where \( a, b = 1, 2, 3 \). Note that \( t^0_{\mu a} = t^0_{\mu c} - t^0_{\mu 0} \). In particular \( B^0_{\mu a} = B^0_{\mu 00} \) as \( B_{\alpha\beta\mu\nu} \) is completely traceless. More covariantly
\[ t_{0\mu 00} := t_{\alpha\mu\beta\gamma}t^{\alpha\beta\gamma}. \] (10)

The “energy-momentum” associated with the Bel-Robinson tensor is
\[ B_{\alpha\beta\mu\nu}t^\beta_\mu t^\nu_\nu = (E_{ab}E^{ab} + H_{ab}H^{ab}, 2\epsilon_{c}^{ab}E_{ad}H^{db}), \] (11)
where the electric part \( E_{ab} \) and magnetic part \( H_{ab} \) are defined in terms of the Weyl tensor as follows:
\[ E_{ab} := C_{ambn}t^mt^n, \quad H_{ab} := *C_{ambn}t^mt^n. \] (12)

Moreover for \( S_{\alpha\beta\mu\nu} + K_{\alpha\beta\mu\nu} \), we have the following identity related with the Bel-Robinson tensor components
\[ S_{\mu 000} + K_{\mu 000} \equiv B_{\mu 000} \equiv B_{\mu 0l}^l \equiv S_{\mu 0l}^l + K_{\mu 0l}^l. \] (13)
It means that \( S_{\mu 000} + K_{\mu 000} \) or \( S_{\mu 0l}^l + K_{\mu 0l}^l \) have the same physical quantities as \( B_{\mu 000} \) or \( B_{\mu 0l}^l \).

### 3 Modified quasilocal boundary expressions

For a first order Lagrangian density:
\[ \mathcal{L} = dq \wedge p - \Lambda(q, p) \] (14)
where $q$ is an f-form, $p$ is a 3-form and $\Lambda$ is the potential. The modified quasilocal expressions [4, 6] can be briefly summarized as follows

$$B_{c_1, c_2}(N) = B_p(N) + c_1 i_N \Delta q \wedge \Delta p + \epsilon c_2 \Delta q \wedge i_N \Delta p,$$

(15)

where $B_p(N) = i_N \eta \wedge \Delta p - \epsilon \Delta q \wedge i_N p$, $\epsilon = (-1)^f$ with f-form, $c_1$ and $c_2$ are real numbers, $\Delta q = q - \eta$, $\Delta p = p - \eta$, $\eta$ and $\eta$ are the background reference values. For GR

$$\mathcal{L} = R^\alpha_\beta \wedge \eta_\alpha \beta,$$

(16)

so let

$$q \rightarrow \Gamma^\alpha_\beta, \quad p \rightarrow \frac{1}{2k} \eta_\alpha \beta.$$

(17)

Allowing for the background connection $\Gamma^\alpha_\beta = 0$, then (15) becomes

$$2k B_{c_1, c_2}(N) = \Gamma^\alpha_\beta \wedge i_N \eta_\alpha \beta + c_1 i_N \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha \beta - c_2 \Gamma^\alpha_\beta \wedge i_N \Delta \eta_\alpha \beta.$$

(18)

When $(c_1, c_2) = (0, 0), (0, 1), (1, 0)$ and $(1, 1)$, the four quasilocal boundary expressions with the simplest boundary conditions are

$$B_p(0, 0) = \Gamma^\alpha_\beta \wedge i_N \eta_\alpha \beta,$$

(19)

$$B_d(0, 1) = \Gamma^\alpha_\beta \wedge i_N \eta \beta,$$

(20)

$$B_c(1, 0) = \Gamma^\alpha_\beta \wedge i_N \eta_\alpha \beta + i_N \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha \beta,$$

(21)

$$B_q(1, 1) = \Gamma^\alpha_\beta \wedge i_N \eta \beta + i_N \Gamma^\alpha_\beta \wedge \Delta \eta_\alpha \beta.$$

(22)

In terms of the superpotential, rewrite the modified quasilocal expressions (18) as

$$2k B_{c_1, c_2}(N) = -\frac{1}{2} N^\mu \left\{ U_\mu^{ij} + c_1 U_\mu^{[ij]} - c_2 U_\mu^{[ij]} \right\} \epsilon_{ij}. $$

(23)

The tetrad teleparallel gauge current expression in orthonormal frames is

$$U_\mu^{[ij]} = -\epsilon \gamma^\beta \sigma \delta^\alpha_{\beta m} \delta^\rho^\tau \gamma e^i e_j e^\rho,$$

(24)

and in RNC

$$c_1 U_\mu^{[ij]} = -\frac{c_1}{12} \epsilon \gamma^\beta \sigma R^\alpha_{\beta \gamma \mu} R^\epsilon_{\xi \kappa \lambda} e^i e^j_\pi x^\gamma x^\xi x^\kappa \delta^\rho^\sigma_\tau \gamma + O(x^4),$$

(25)

$$c_2 U_\mu^{[ij]} = \frac{c_2}{12} \epsilon \gamma^\beta \sigma R^\alpha_{\beta \gamma \lambda} R^\epsilon_{\xi \kappa \tau} e^i e^j_\pi x^\gamma x^\xi x^\kappa \delta^\rho^\sigma_\tau \gamma + O(x^4).$$

(26)

It should be noted that both the tetrad teleparallel gauge current $U_\mu^{[ij]}$ and the associated energy-momentum density $\partial_j (U_\mu^{[ij]})$ is a tensor. In contrast, the Møller 1961 expression $M U_\mu^{[ij]}$ is a tensor but the corresponding energy-momentum density $\partial_j (M U_\mu^{[ij]})$ is not a tensor. Precisely it is a pseudotensor, which means it depends on the coordinates in a non-covariant way. As $M U_\mu^{[ij]} = e^\mu_h U_\mu^{[ij]}$, modify the superpotential of (23) to

$$2k U_\mu^{[ij]} = M U_\mu^{[ij]} + e^\mu_h \left( c_1 U_\mu^{[ij]} - c_2 U_\mu^{[ij]} \right).$$

(27)
Then the corresponding pseudotensor becomes locally in a small region in RNC
\[ 2\kappa t_h^i = 2\kappa \partial_j U_h^{[ij]} \]
\[ = e 2G_{h^i}^j + \frac{e}{24} \left\{ 2(2 - 3c_2)B_h^i \xi_\kappa - (1 - 3c_1 + 3c_2)S_h^i \xi_\kappa + 2(c_1 - 2c_2)K_h^i \xi_\kappa \right\} x^\xi x^\kappa \]
\[ + O(\text{Ricci}, x) + O(x^3). \]  
(28)

Regarding whether the above expression is good for the gravitational energy, there are three limits we can consider. They are inside matter (interior mass density), at spatial infinity (ADM mass energy) and in vacuum (positive gravitational energy). The first two tests are relatively mild in general, but the last one is not, because it is very sensitive if we insist to obtain the Bel-Robinson tensor. For example, the Møller 1961 expression fulfills the first two tests while it fails the third examination [3].

Test (i): Inside matter. The energy density inside matter at the origin is
\[ \mathcal{E} = -t_0^0(0) = -\frac{G_0^0(0)}{\kappa} = -T_0^0(0) = \rho. \]  
(29)

Test (ii): At spatial infinity. The total energy of the pseudotensor agrees with the ADM mass formula [7]
\[ E = \frac{1}{2\kappa} \oint N^0 U_0^{[\mu\nu]} \epsilon_{\mu\nu} = \frac{1}{2\kappa} \lim_{r \to \infty} \sum_{i,j=1}^3 \oint \left( h_{i,j,i} - h^{i,j} \right) N^j dA, \]  
(30)

where the integrals are taken over a sphere of constant \( r \) and \( N^j = x^j/r \) is the outward normal to this sphere.

Test (iii): In vacuum. Consider (28) by eliminating the tensors \( S_{\alpha\beta\mu\nu} \) and \( K_{\alpha\beta\mu\nu} \) when \( (c_1, c_2) = \left( \frac{2}{3}, \frac{1}{3} \right) \), the gravitational energy-momentum density in the small sphere region limit is
\[ t_\alpha^\beta = \frac{1}{12\kappa} B_\alpha^\beta \xi_\kappa x^\xi x^\kappa. \]  
(31)

This is the first desired result we found in vacuum. It is proportional to the Bel-Robinson tensor which is an invariant strength measurement of the non-negative gravitational energy density within a very small region. Explicitly
\[ B_{0000} = E_{ab} E^{ab} + H_{ab} H^{ab} \geq 0. \]  
(32)

The second desired result for the non-negative gravitational energy-momentum in the small sphere limit is
\[ P_\mu = (-E, \vec{P}) \]
\[ = -\frac{1}{48\kappa} \int \left\{ 2(2 - 3c_2)B_{\mu0ij} - (1 - 3c_1 + 3c_2)S_{\mu0ij} + 2(c_1 - 2c_2)K_{\mu0ij} \right\} x^i x^j d^3x \]
\[ = -\frac{2c_1 - 1}{240G} \int B_{\mu000}, \]  
(33)

provided \( c_1 \geq 1/2 \) and the unique combination \( c_1 + c_2 = 1 \), which is the constraint such that the coefficients of \( S_{\mu0i}^j \) and \( K_{\mu0i}^j \) are the same. There is an infinite set of solutions because of the constant \( c_1 \). When \( (c_1, c_2) = (1, 0) \), the quasilocal expression for this set is \( B_{c}(1, 0) \) as mentioned in [21] which has a simple boundary condition.
4 Conclusion

Once the positivity energy proof is achieved for some particular expression, the small sphere limit is guaranteed for the positive gravitational energy calculation. Recently, the Møller tetrad gravitational energy-momentum expression was evaluated for a small vacuum region using orthonormal frames adapted to Riemann normal coordinates \[3\]. Treating a modified quasilocal expressions in a similar way, we found one unique combination that gives a multiple of the Bel-Robinson tensor.

The components of the Bel-Robinson tensor \( B_{0000} = B_{00l} \) gives the non-negative “energy” density which has a nice property for the gravitational field. However, besides this tensor, we found that \( S_{0000} + K_{0000} = S_{00l} + K_{00l} \) gives the same physical value. Based on this property, we also obtained an infinite set of solutions that provide the positive gravitational energy within the same region limit using certain modified quasilocal expressions.

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