ASYMPTOTIC SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS WITH PARTIAL DERIVATIVES AND WITH RAPIDLY VARYING KERNEL

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Abstract. In the paper, ideas of the Lomov regularization method are generalized to the Cauchy problem for a singularly perturbed partial integro-differential equation in the case when the integral term contains a rapidly varying kernel. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

Keywords: singularly perturbed, partial integro differential equation, regularization of an integral, solvability of iterative problems.

1. Introduction

In the paper, we consider the Cauchy problem for the partial integro-differential equation

$$\varepsilon \frac{\partial y(x,t,\varepsilon)}{\partial x} = a(x)y(x,t,\varepsilon) + \int_{0}^{x} e^{\frac{x-s}{\varepsilon}} \int_{s}^{t} \mu(\theta) y(s,t,\varepsilon) ds \, ds + h(x,t),$$

$$y(0,t,\varepsilon) = y_{0}(t) \quad ((x,t) \in [0,1] \times [0,1]) \quad (1)$$

with the rapidly varying kernel. The problem of constructing a regularized asymptotic solution of the problem (1) [1] is posed. Earlier, in [3-7], systems for ordinary integro-differential equations were mainly considered. In contrast to these works, in this paper we consider an partial integro-differential equation. Construction of asymptotical solutions for partial integro-differential equations with rapidly varying kernels from the position of the regularization method are considered in [8-9]. However, in [9], the case of a purely imaginary spectral value $\mu(x)$ of the kernel

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of an integral operator was considered. This article allows purely imaginary $\mu(x) = \pm i\omega(x)$ ($\omega(x) > 0$).

2. Regularization of the problem. We suppose that the following conditions are satisfied:

1. functions $a(x) \in C^\infty([0, T], C^1)$, $\mu(x) \in C^\infty([0, T], C^1)$,
   
2. $h(x, t) \in C^\infty([0, 1] \times [0, 1], C^1)$, the kernel $K(x, t, s)$ belongs to the space $K(x, t, s) \in C^\infty\left\{ [0 < s < x < 1, 0 \leq t \leq 1], C^1 \right\}$;
3. $a(x), \mu(x) \neq 0$, $a(x) \neq \mu(x)$ ($\forall x \in [0, 1]$);

2. Regularization of the problem.

To justify this fact, we need to show that the image $Jy$ on the regularizing functions (2) will be obviously the exact solution of the original problem (1). However, the problem (3) can not be considered as completely regularized since the integral operator

$$Jy = \int_0^x e^{\frac{1}{\varepsilon} \int_s^x \mu(\theta)d\theta} K(x, t, s)y(s, t, \frac{\varphi(s)}{\varepsilon}, \varepsilon)ds$$

is not regularized in it.

To regularize the integral term $Jy$ of the problem (3), we need to introduce the space $M_\varepsilon$ asymptotically invariant with respect to the operator $J$. This is done in this way. It is introduced the class $U$ of solutions of iterative problems (see below):

$$U = \left\{ y(x, t, \tau) : \begin{array}{c} y = y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} \\
y_0(x, t), \quad y_1(x, t), \quad y_2(x, t) \in C^\infty([0, 1] \times [0, 1]) \end{array} \right\},$$

and then the restriction of this class is taken at $\tau = \varphi(x)/\varepsilon$. This will be the space $M_\varepsilon$. To justify this fact, we need to show that the image $Jy(x, t, \tau)$ of the integral operator $J$ on the element of the space $U$ can be represented in the form of a power series

$$\sum_{k=0}^\infty \varepsilon^k \left( y_1^{(k)}(x, t)e^{\tau_1} + y_2^{(k)}e^{\tau_2} + y_0^{(k)}(x, t) \right)$$

convergent asymptotically for $\varepsilon \to +0$ (uniformly with respect to $(x, t) \in [0, 1 \times [0, 1]$). Deal with this procedure.

The image of the integral operator $J$ on an arbitrary element $y(x, t, \tau)$ of the space $U$ has the form

$$Jy(x, t, \varepsilon) = \int_0^x \varepsilon^{\frac{1}{\varepsilon} \int_s^x \mu(\theta)d\theta} K(x, t, s)y_1(s, t)e^{\frac{x(s)}{\varepsilon}}ds +$$

$$+ \int_0^x \varepsilon^{\frac{1}{\varepsilon} \int_s^x \mu(\theta)d\theta} K(x, t, s)y_2(s, t)e^{\frac{x(s)}{\varepsilon}}ds +$$

$$+ \int_0^x \varepsilon^{\frac{1}{\varepsilon} \int_s^x \mu(\theta)d\theta} K(x, t, s)y_0(s, t)ds.$$
Apply the operation of integration by parts to the each term.

\[
\int_0^z e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} K(x, t, s) y_1(s, t) e^{\sum_{\varphi_1(x)}} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z K(x, t, s) y_1(s, t) e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z K(x, t, s) y_1(s, t) e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z K(x, t, s) y_1(s, t) \frac{1}{a(s) - \mu(s)} \left. \left. e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} \right|_{s=0}^{s=x} \right. -
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z \left( \frac{\partial}{\partial s} K(x, t, s) y_1(s, t) \right) e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds.
\]

Introduce the notation \( I_0^t (K(x, t, s) y_1(s, t)) \equiv \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)}. \) Then the previous result of transformations can be written in the form

\[
\int_0^z e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} K(x, t, s) y_1(s, t) e^{\sum_{\varphi_1(x)}} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds =
\]

\[
= e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} \int_0^z e^{\frac{1}{z} \int_0^1 [\theta(0) - \mu(0)] \, d\theta} ds,
\]

where \( \tau_1 = \varphi_1(x, \varepsilon). \) Continuing this process, we obtain the series

\[
\int_0^z e^{\frac{1}{z} \int_0^1 \mu(0) \, d\theta} K(x, t, s) y_1(s, t) e^{\sum_{\varphi_1(x)}} ds = \sum_{k=0}^{\infty} (-1)^k \varepsilon^{k+1} \times \left( I_0^t (K(x, t, s) y_1(s, t)))_{s=x} e^{\tau_1} - (I_0^t (K(x, t, s) y_1(s, t)))_{s=0} e^{\tau_2} \right)
\]

(5)

where \( \tau_1 = \varphi_1(x, \varepsilon), \) and the operators \( I_0^t \) have the form:

\[
I_0^t (K(x, t, s) y_1(s, t)) \equiv \frac{K(x, t, s) y_1(s, t)}{a(s) - \mu(s)},
\]

\[
I_0^t (K(x, t, s) y_1(s, t)) \equiv \frac{1}{a(s) - \mu(s)} \frac{\partial}{\partial s} I_0^t (K(x, t, s) y_1(s, t)), \ldots,
\]

\[
I_0^t \ldots (K(x, t, s) y_1(s, t)) \equiv \frac{1}{a(s) - \mu(s)} \frac{\partial}{\partial s} I_0^t \ldots (K(x, t, s) y_1(s, t)), m \geq 1.
\]

(6)

The second summand in (4) is transformed as follows:

\[
\int_0^x e^{\frac{1}{x} \int_0^1 \mu(0) \, d\theta} K(x, t, s) y_2(s, t) e^{\sum_{\varphi_2(t)}} ds \equiv
\]

\[
\equiv \int_0^x e^{\frac{1}{x} \int_0^1 \mu(0) \, d\theta} K(x, t, s) y_2(s, t) e^{\sum_{\varphi_2(t)}} ds =
\]

\[
= e^{\frac{1}{x} \int_0^1 \mu(0) \, d\theta} \int_0^x K(x, t, s) y_2(s, t) ds = (\int_0^x K(x, t, s) y_2(s, t) ds) e^{\tau_2},
\]

\( \tau_2 = \frac{\varphi_2(t)}{x}. \)
And, finally, we have for the last summand in (4)
\[
\int_0^x e^{\frac{\varepsilon}{\tau}} J^* \mu(\theta) d\theta K(x, t, s)y_0(s, t) ds = \varepsilon \int_0^x \frac{K(x, t, s)y_0(s, t)}{-\mu(s)} ds \frac{d}{d\theta} J^* \mu(\theta) d\theta =
\]
\[
= \varepsilon \int_0^x \left( \frac{K(x, s, y_0(s, t))}{-\mu(x)} - \frac{K(x, s, y_0(0, t))}{-\mu(s)} e^{\tau_2^*} \right) e^{\frac{\varepsilon}{\tau}} J^* \mu(\theta) d\theta d\theta =
\]
\[
= \varepsilon \int_0^x \left( \frac{\partial}{\partial s} K(x, s, y_0(s, t)) \right) e^{\frac{\varepsilon}{\tau}} J^* \mu(\theta) d\theta d\theta =
\]
\[
= \sum_{k=0}^{\infty} \varepsilon^{k+1} \left[ I^k_0(K(x, s, y_0(s, t))_{s=0}) e^{\tau_2^*} - I^k_0(K(x, s, y_0(s, t))_{s=x}) \right]
\]
where the following operators are introduced:
\[
I^0_0(K(x, s, y_0(s, t))) \equiv \frac{K(x, s, y_0(s, t))}{-\mu(s)} y_0(s, t),
\]
\[
I^1_0(K(x, s, y_0(s, t))) \equiv -\frac{\partial}{\partial s} I^0_0(K(x, s, y_0(s, t))),
\]
\[
I^m_0(K(x, s, y_0(s, t))) \equiv -\frac{\partial}{\partial s} I^{m-1}_0(K(x, s, y_0(s, t))), m \geq 1.
\]

The asymptotic convergence of the series (5) and (8) is proved in the same way as the analogous statement in [1]. Let now \( \tilde{y}(x, t, \tau, \varepsilon) \) be an arbitrary function continuous with respect to \((x, t, \tau) \in [0, 1] \times [0, 1] \times \{ \text{Re} a(x) \} \) and having the asymptotic expansion
\[
\tilde{y}(x, t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k y_k(x, t, \tau), \quad y_k(x, t, \tau) \in U,
\]
convergent at \( \varepsilon \to +0 \) (uniformly with respect to \((x, t, \tau) \in [0, 1] \times [0, 1] \times \{ \text{Re} a(x) \})\). Introduce the operators \( R_m : U \to U \) acting on the each element \( y(x, t, \tau) \) of the space \( U \) according the law:
\[
Ry(x, t, \tau) \equiv R_0(y_1(x, t)) e^{\tau_1^*} + y_2(x, t) e^{\tau_2^*} + y_0(x, t)) = e^{\tau_2^*} \int_0^x K(x, t, s)y_2(s, t) ds,
\]
\[
R_{k+1}y(x, t, \tau) =
\]
\[
= (-1)^k \left[ I^k_1(K(x, t, s)y_1(s, t))_{s=x} e^{\tau_1^*} - I^k_1(K(x, t, s)y_1(s, t))_{s=0} e^{\tau_2^*} \right] +
\]
\[
+ \left[ I^k_0(K(x, t, s)y_0(s, t))_{s=0} e^{\tau_2^*} - I^k_0(K(x, t, s)y_0(s, t))_{s=x} \right]
\]
where the operators \( I^k_1 \) have the form (6), and operators \( I^k_0 \) have the form (9), \( k \geq 0 \).

Operators \( R_m \) are called the operators of order (with respect to \( \varepsilon \), since, applying to the function \( y(x, t, \tau) \), these operators separate terms of the order \( e^m \)). It is natural to define the extended operator for the integral operator \( I \) in the following way.

**Definition 1.** The operator \( I \) acting on each function \( \tilde{y}(x, t, \tau, \varepsilon) \) of the form (10) by the law
\[
J \tilde{y} \equiv \tilde{J} \left( \sum_{k=0}^{\infty} \varepsilon^k y_k(x, t, \tau) \right) = \sum_{r=0}^{\infty} \varepsilon^r \left( \sum_{k=0}^{r} R_{r-k} y_k(x, t, \tau) \right)
\]
is said to be a formal extension of the operator \( I \).

Now one can write the completely regularized problem (with respect to the original problem (1)):
\[
L \tilde{y}(x, t, \tau, \varepsilon) \equiv \varepsilon \frac{\partial^2 \tilde{y}}{\partial \tau^2} + a(x) \frac{\partial \tilde{y}}{\partial \tau} + \mu(x) \frac{\partial \tilde{y}}{\partial x} -
\]
\[
- a(x) \tilde{y} - J \tilde{y} = h(x, t), \quad \tilde{y}(0, t, 0, \varepsilon) = y^0(t).
\]
where \( \tilde{y}(x, t, \tau, \varepsilon) \) is the series (10).
3. Solvability of iterative problems. Substituting the series (10) into (13) and equating the coefficients of the same powers of $\varepsilon$, we obtain the following iterative problems:

$$
Ly_0(x, t, \tau) \equiv a(x) \frac{\partial y}{\partial \tau_1} + \mu(x) \frac{\partial y}{\partial \tau_2} - a(x)y_0 - R_0y_0 = h(x, t), \quad y_0(0, t, 0) = y^0(t); \quad (14_0)
$$

$$
Ly_1(x, t, \tau) = - \frac{\partial y_0}{\partial x} + R_1y_0, \quad y_1(0, t, 0) = 0; \quad (14_1)
$$

$$
Ly_k(x, t, \tau) = - \frac{\partial y_{k-1}}{\partial x} + R_1y_{k-1} + \ldots + R_ky_0, \quad y_k(0, t, 0) = 0, \ k \geq 1. \quad (14_k)
$$

Each iterative problem (14_k) has the form

$$
Ly(x, t, \tau) \equiv a(x) \frac{\partial y}{\partial \tau_1} + \mu(x) \frac{\partial y}{\partial \tau_2} - a(x)y - R_0y = H(x, t, \tau), \quad y(0, t, 0) = y_*(t), \quad (15)
$$

where $H(x, t, \tau) = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U$, $y_*(x) \in C^\infty[0, 1]$ are known functions, and $R_0y$ is the operator:

$$
R_0y(x, t, \tau) \equiv R_0(y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} + y_0(x, t)) = e^{\tau_2} \int_0^x K(x, t, s)y_2(s, t)ds.
$$

Let’s try to solve the problem (15). Substituting the element

$$
y(x, t, \tau) = y_1(x, t)e^{\tau_1} + y_2(x, t)e^{\tau_2} + y_0(x, t)
$$

of the space $U$ into (15), we obtain

$$
a(x)y_1(x, t)e^{\tau_1} + \mu(x)y_2(x, t)e^{\tau_2} - a(x)y_1(x, t)e^{\tau_1} - \mu(x)y_2(x, t)e^{\tau_2} + a(x)y_0(x, t)) - e^{\tau_2} \int_0^x K(x, t, s)y_2(s, t)ds = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t)).
$$

Equating separately the free terms and coefficients for identical exponentials, we obtain equations:

$$
-a(x)y_0(x, t) = h_0(x, t),
$$

$$
[a(x) - a(x)]y_1(x, t) = h_1(x, t), \quad (16)
$$

$$
[\mu(x) - a(x)]y_2(x, t) - \int_0^x K(x, t, s)y_2(s, t)ds = h_2(x, t).
$$

The first equation of (16) has the unique solution $y_0(x, t) = -\frac{h_0(x, t)}{a(x)}$. For solvability of the second system in (16), it is necessary and sufficient the fulfillment of the condition

$$
h_1(x, t) = 0, \quad (\forall x, t \in [0, 1] \times [0, 1]).
$$

The third equation of (16) is a Volterra equation of the second kind with the smooth kernel $G(x, t, s) = [\mu(x) - a(x)]^{-1}K(x, t, s)$, that’s why it has the unique solution in the space $C^\infty([0, 1] \times [0, 1])$. Thus, we prove the following

**Theorem 1.** Let the right-hand side in (15)

$$
H(x, t, \tau) = h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U,
$$

and conditions (i)–(ii) be satisfied. Then the equation (15) is solvable in the space $U$ if and only if

$$
h_1(x, t) = 0, \quad (\forall x, t \in [0, 1] \times [0, 1]). \quad (17)
$$
Remark. If (17) is satisfied, then (15) has the following solution in $U$:

$$y(x, t, \tau) = \alpha(x, t)e^{\tau} - \frac{h_0(x, t)}{a(x)} + \left[ \int_0^x R(x, t, s)[\mu(s) - a(s)]^{-1}h_2(s, t)ds + [\mu(x) - a(x)]^{-1}h_2(x, t) \right] e^{\tau},$$

(18)

where $R(x, t, s)$ is the resolvent of the kernel $G(x, t, s) = [\mu(x) - a(x)]^{-1}K(x, t, s)$, $\alpha(x, t) \in C^\infty([0, 1] \times [0, 1])$ is an arbitrary function.

Subject the solution (18) to the initial condition $y(0, t, 0) = y_*(t)$. Then we get

$$\alpha(0, t) - \frac{h_0(0, t)}{a(0)} = y_*(t) \quad \Leftrightarrow \quad \alpha(0, t) = y_*(t) + \frac{h_0(0, t)}{a(0)}.$$ (19)

However, the functions $\alpha_j(x, t)$ were not found completely. An additional requirement is required to solve problem (15). Such a requirement is dictated by iterative problems (14), from which it can be seen that the natural additional constraint is the condition

$$-\frac{\partial y}{\partial x} + R_1y + P(x, t, \tau) \equiv 0, \quad (\forall(x, t) \in [0, 1] \times [0, 1]),$$

(20)

where $P(x, t, \tau) = P_1(x, t)e^{\tau} + P_2(x, t)e^{2\tau} + P_0(x, t) \in U$ is the known vector-function. We show that the problem (15) has the unique solution in the space $U$ if (20) is satisfied.

**Theorem 2.** Let the conditions (i)–(ii) take place and the right-hand side $H(x, t) \equiv h_1(x, t)e^{\tau_1} + h_2(x, t)e^{\tau_2} + h_0(x, t) \in U$ satisfy the condition (17). Then the problem (15) is uniquely solvable in the space $U$ under the additional condition (20).

**Proof.** To use the condition (20), we calculate the expression $-\frac{\partial y}{\partial x} - R_1y$. Since

$$R_1y(x, t, \tau) = -\left[ \left( I_1^0(K(x, t, s)y_1(s, t)) \right)_{s=x}e^{\tau_1} - \left( I_1^0(K(x, t, s)y_1(s, t)) \right)_{s=0}e^{\tau_2} \right] +$$

$$+ \left[ \left( I_1^0(K(x, t, s)y_0(s, t)) \right)_{s=0}e^{\tau_2} - I_0^0(K(x, t, s)y_0(s, t)) \right]_{s=x},$$

$$y_1(s, t) = \alpha(s, t), \quad y_0(s, t) = -\frac{h_0(x, t)}{a(x)},$$

$$I_1^0(K(x, t, s)y_1(s, t)) = \frac{K(x, t, s)y_1(s, t)}{a(s) - \mu(s)},$$

we have

$$-\frac{\partial y}{\partial x} + R_1y + P(x, t, \tau) = -\frac{\partial(\alpha(x, t))}{\partial x} e^{-\tau} -$$

$$-\frac{\partial}{\partial x} \left[ \int_0^x R(x, t, s)[\mu(s) - a(s)]^{-1}h_2(s, t)ds + [\mu(x) - a(x)]^{-1}h_2(x, t) \right] e^{\tau} +$$

$$+ \left[ \int_0^x R(x, t, s)[\mu(s) - a(s)]^{-1}h_2(s, t)ds + [\mu(x) - a(x)]^{-1}h_2(x, t) \right] e^{\tau} +$$

$$+ \left[ \left( I_1^0(K(x, t, s)y_1(s, t)) \right)_{s=x}e^{\tau_1} - \left( I_1^0(K(x, t, s)y_1(s, t)) \right)_{s=0}e^{\tau_2} \right] +$$

$$+ \left[ \left( I_1^0(K(x, t, s)y_0(s, t)) \right)_{s=0}e^{\tau_2} - I_0^0(K(x, t, s)y_0(s, t)) \right]_{s=x} +$$

$$+ P_1(x, t)e^{\tau_1} + P_2(x, t)e^{\tau_2} + P_0(x, t),$$

therefore (20) takes the form

$$\left\{ \begin{array}{c}
\frac{\partial(\alpha(x, t))}{\partial x} + K(x, t, x)y_1(x, t) + \frac{P_1(x, t)e^{\tau_1} + P_2(x, t)e^{\tau_2} + P_0(x, t)}{a(x) - \mu(x)} = 0,
\end{array} \right.$$ (20)

Taking into account the initial condition (19), this equation has the unique solution

$$\alpha(x, t) = e^{q(x, t)} \left[ \alpha(0, t) + \int_0^x P_1(s, t)e^{-q(s, t)}ds \right],$$
where \( q(x,t) = \int_0^x K(s,t,s) \, ds \). Hence, under the conditions of Theorem 2, the solution (18) in the space \( U \) is uniquely determined. Theorem 2 is proved.

4. Justification of the asymptotic convergence of approximate solutions to the exact. Applying Theorems 1 and 2 to the iteration problems (14), we construct a series (10) with coefficients from the class \( U \). Let be the restriction of the \( y_N(t,x) = \sum_{k=0}^N \varepsilon^k y_k \left( x, t, \frac{x(t)}{\varepsilon} \right) \) \( N \)-th partial sum of this series at \( \tau = \frac{x(t)}{\varepsilon} \).

Just as in [2], it is easy to prove the following statement.

**Lemma.** Let conditions (i) – (iii) be fulfilled. Then the partial sum \( y_N(x,t) \) satisfies the problem (1) up to terms containing \( \varepsilon^{N+1} \), i.e.

\[
\varepsilon \frac{\partial y_N(x,t)}{\partial x} \equiv a(x)y_N(x,t) + h(x,t) + \int_0^x \varepsilon^k \int_s^x \mu(\theta)d\theta K(x,t,s) y(x,t,\varepsilon) \, ds + \\
\varepsilon \int_{\varepsilon^{N+1}}^x R_N(x,t,\varepsilon),
\]

\( y_N(0,t) = y_0(t), \forall (x,t) \in [0,1] \times [0,1] \),

where \( ||R_N(x,t,\varepsilon)||_{C([0,1] \times [0,1])} \leq R_N \) for all \( (x,t) \in [0,1] \times [0,1] \) and all \( \varepsilon \in (0,\varepsilon_N] \) (here \( R_N > 0 \) — some constant that does not depend on \( \varepsilon \in (0,\varepsilon_N] \), \( \varepsilon_N > 0 \) is sufficiently small).

Now consider the following problem:

\[
\varepsilon \frac{\partial y(x,t,\varepsilon)}{\partial x} \equiv a(x)y(x,t,\varepsilon) + \int_0^x \varepsilon^k \int_s^x \mu(\theta)d\theta K(x,t,s) y(x,t,\varepsilon) \, ds + \\
+ \Phi(x,t,\varepsilon), \quad y(0,t,\varepsilon) = 0, \quad (x,t) \in [0,1] \times [0,1].
\]

Let us prove its correct solvability in space \( C^1([0,1] \times [0,1]) \).

**Theorem 3.** Let conditions (i) – (iii) be fulfilled. Then, for sufficiently small \( \varepsilon \in (0,\varepsilon_0] \), problem (22) for any right-hand side \( \Phi(x,t,\varepsilon) \in C([0,1] \times [0,1]) \) has a unique solution \( y(x,t,\varepsilon) \) in space \( C^1([0,1] \times [0,1]) \), and the estimate

\[
||y(x,t,\varepsilon)||_{C([0,1] \times [0,1])} \leq \frac{K_0}{\varepsilon} ||\Phi(x,t,\varepsilon)||_{C([0,1] \times [0,1])},
\]

is holds, where \( \varepsilon_0 > 0 \) is a constant independent of \( \varepsilon \in (0,\varepsilon_0] \).

**Proof.** We introduce an additional unknown function

\[
z(x,t,\varepsilon) = \int_0^x \varepsilon^k \int_s^x \mu(\theta)d\theta K(x,t,s) y(x,t,\varepsilon) \, ds.
\]

Differentiating it by \( x \), we will have

\[
\varepsilon \frac{\partial z(x,t,\varepsilon)}{\partial x} = \mu(x) z(x,t,\varepsilon) + \varepsilon K(x,t,x) y(x,t,\varepsilon) + \\
+ \varepsilon \int_0^x \varepsilon^k \int_s^x \mu(\theta)d\theta \frac{\partial K(x,t,s)}{\partial x} y(x,t,\varepsilon) \, ds.
\]

For the vector function \( w = [y,z] \), we get the system

\[
\varepsilon \frac{\partial w(x,t,\varepsilon)}{\partial x} = \begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix} w(x,t,\varepsilon) + \\
\begin{pmatrix} K(x,t,x)z(x,t,\varepsilon) + \int_0^x \varepsilon^k \int_s^x \mu(\theta)d\theta \frac{\partial K(x,t,s)}{\partial x} y(x,t,\varepsilon) \, ds \\ \Phi(x,t,\varepsilon) \end{pmatrix}, w(0,t,\varepsilon) = 0.
\]

Let \( G(x,\eta,\varepsilon) \) be the fundamental Cauchy matrix of the differential system

\[
\varepsilon \frac{dG(x,\eta,\varepsilon)}{dx} = \begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix} G(x,\eta,\varepsilon), \quad G(x,x,\varepsilon) = I, \quad 0 \leq \eta \leq x \leq 1.
\]
Since the matrix \( \begin{pmatrix} a(x) & 1 \\ 0 & \mu(x) \end{pmatrix} \) is a matrix of simple structure and its eigenvalues \( a(x) \) and \( \mu(x) \) lie in the half-plane \( \Re \lambda \leq 0 \), the Cauchy matrix \( G(x, \eta, \varepsilon) \) is uniformly bounded, i.e.

\[
\|G(x, \eta, \varepsilon)\| \leq c_0 \forall (x, \eta, \varepsilon) : 0 \leq \eta \leq x \leq 1, \varepsilon > 0
\]

(see, for example, [10], pp. 119-120). We now write an integral system equivalent to system (24):

\[
w(x, t, \varepsilon) = x(t) G(x, \eta, \varepsilon) \begin{pmatrix} 0 \\ c \end{pmatrix} + \int_0^x G(x, \eta, \varepsilon) \begin{pmatrix} 0 \\ 0 \end{pmatrix} d\eta + \int_0^x G(x, \eta, \varepsilon) \begin{pmatrix} \Phi(\eta, t, \varepsilon) \\ 0 \end{pmatrix} d\eta.
\]

This Volterra integral system has for each \( \varepsilon > 0 \) a unique solution \( w(x, t, \varepsilon) \) in the class \( C([0, 1] \times [0, 1]) \). Substituting it into the indicated system, and passing to the norms in the resulting identity, we will have

\[
\|w(x, t, \varepsilon)\| \leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} + (c_1 + c_2) \int_0^x \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} d\eta.
\]

Applying to this inequality the Gronwall-Bellman lemma, we arrive at the inequality

\[
\|w(x, t, \varepsilon)\| \leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} + (c_1 + c_2) \int_0^x \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} d\eta.
\]

Applying this inequality the Gronwall-Bellman lemma, we arrive at the inequality

\[
\|w(x, t, \varepsilon)\| \leq \frac{c_0}{\varepsilon} \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} + (c_1 + c_2) \int_0^x \|\Phi(x, t, \varepsilon)\|_{C([0, 1] \times [0, 1])} d\eta.
\]

Theorem 4. Let the conditions (i)–(iii) be satisfied. Then for \( \varepsilon \in (0, \varepsilon_N) \), where \( \varepsilon_N > 0 \) is sufficiently small, the problem (1) has the unique solution \( y(x, t, \varepsilon) \in C([0, 1] \times [0, 1]) \), and the estimate

\[
\|y(x, t, \varepsilon) - y_{\varepsilon N}(x)\|_{C([0, 1] \times [0, 1])} \leq C_N \varepsilon^{N+1} \quad (N = 0, 1, 2, \ldots)
\]

takes place, where the constant \( C_N > 0 \) does not depend on \( \varepsilon \in (0, \varepsilon_N) \).

Proof. By the lemma, the partial sum \( y_{\varepsilon N}(x, t) \) satisfies the problem (21), therefore the remainder term \( r_N(x, t, \varepsilon) = y(x, t, \varepsilon) - y_{\varepsilon N}(x, t) \) satisfies the following problem:

\[
\varepsilon \frac{r_N(x, t, \varepsilon)}{dx} = a(x) r_N(x, t, \varepsilon) + \int_0^x e^{-\frac{1}{\varepsilon} \int_0^y a(\theta)d\theta} K(x, t, s) r_N(s, t, \varepsilon) ds - \varepsilon^{N+1} R_N(x, t, \varepsilon), \quad r_N(0, t, \varepsilon) = 0,
\]
Here the role of the function $\Phi(x, t, \varepsilon)$ (see (22) and (23)) plays function $-\varepsilon^{N+1} R_N(x, t, \varepsilon)$. From here we get the estimate $\|r_N(x, t, \varepsilon)\|_{C([0,1] \times [0,1])} \leq \varepsilon^N R_N$, fair for any $N = 0, 1, 2, \ldots$ and all $\varepsilon \in (0, \varepsilon_N]$, which means that for a partial sum $y_{\varepsilon,N+1}(x, t) \equiv y_{\varepsilon,N}(x, t) + \varepsilon^{N+1} y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon})$ the estimate

$$\|y(x, t, \varepsilon) - y_{\varepsilon,N+1}(x, t)\|_{C([0,1] \times [0,1])} \equiv \|y(x, t, \varepsilon) - y_{\varepsilon,N}(x, t)\|_{C([0,1] \times [0,1])} - \varepsilon^{N+1} y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon})\|_{C([0,1] \times [0,1])} \leq C_N \varepsilon^{N+1}.$$

is true. Using inequality $||a - b|| \geq ||a|| - ||b||$, we will have

$$\|y(t, \varepsilon) - y_{\varepsilon}(t)\|_{C([0,1] \times [0,1])} \leq C_N \varepsilon^{N+1} + \varepsilon^{N+1} \|y_{N+1}(x, t, \frac{\varphi(x)}{\varepsilon})\|_{C([0,1] \times [0,1])},$$

where we derive the usual estimate

$$\|y(t, \varepsilon) - y_{\varepsilon,N}(t)\|_{C([0,1] \times [0,1])} \leq C_N \varepsilon^{N+1},$$

where the constant $C_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_N]$. From this inequality it follows that the restriction of the series (10) for is an asymptotic series for the exact solution of problem (1) for the Theorem is proved.

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