Relevant Perturbations Of The $SU(1,1)/U(1)$ Coset

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Abstract

It is shown that the space of cohomology classes of the $SU(1,1)/U(1)$ coset at negative level $k$ contains states of relevant conformal dimensions. These states correspond to the energy density operator of the associated nonlinear sigma model. We exhibit that there exists a subclass of relevant operators forming a closed fusion algebra. We make use of these operators to perform renormalizable perturbations of the $SU(1,1)/U(1)$ coset. In the infra-red limit, the perturbed theory flows to another conformal model. We identify one of the perturbative conformal points with the $SU(2)/U(1)$ coset at positive level. From the point of view of the string target space geometry, the given renormalization group flow maps the euclidean black hole geometry described by the $SU(1,1)/U(1)$ coset into the sphere described by the $SU(2)/U(1)$ coset.

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1 Introduction

It has been realized that nonunitary Wess-Zumino-Novikov-Witten (WZNW) models play a significant role in string theory \cite{1}-\cite{9}. In some sense these models appear to be more fundamental than ordinary unitary WZNW models. We have recently exhibited that the latter can be obtained from nonunitary WZNW models through renormalization group flows \cite{10},\cite{11}. The curious fact about these flows is that in the presence of two dimensional quantum gravity cosmological minkowskian string solutions can smoothly flow to euclidean string solutions, whenever the corresponding CFT’s admit proper space-time interpretations \cite{12}.

However, nonunitary WZNW models, which are based on nonunitary affine groups, turn out to be very complicated systems because their spectra contain states of negative norm. Therefore, in order for these models to make sense, one has to find a certain way to get rid of negative normed states. Witten has proposed that WZNW models on compact groups at negative level can be understood via analytic continuation to noncompact groups \cite{7},\cite{8}. It has been argued that nonunitary states of the latter can be eliminated by a coset construction \( G/H \), where \( G \) is a noncompact group and \( H \) is its maximal compact subgroup \cite{2}.

As yet the problem of nonunitary WZNW models has not been solved completely. Perhaps, the \( SU(1,1)/U(1) \) coset is the only case which has been studied in detail \cite{4}. It turns out that unitary \( N = 2 \) superconformal models as well as physical states of two dimensional black holes can be extracted from unitary representations of \( SU(1,1) \) \cite{1},\cite{2},\cite{7},\cite{9}. These unitary representations are described as the \( SU(1,1)/U(1) \) coset with \( U(1) \) being the compact subgroup of \( SU(1,1) \).

The aim of the present paper is to explore relevant perturbations on the \( SU(1,1)/U(1) \) gauged WZNW model. We will exhibit that the physical subspace of cohomological classes of the \( SU(1,1)/U(1) \) gauged WZNW model has relevant conformal operators corresponding to highest weight vectors of unitary Virasoro representations. At the same time, these unitary states are descendants of highest weight states of nonunitary (finite dimensional) representations of the affine group. These unitary Virasoro representations have been
missed in the previous analysis of the $SU(1,1)/U(1)$ coset [2], [3]. A nongauged version of these representations has been discussed in [13]. We will show that these new relevant operators obey a fusion algebra which allows us to use them to perform renormalizable perturbations on the $SU(1,1)/U(1)$ coset. These perturbations are different from canonical perturbations of gauged WZNW models considered in [11] (see also [15]). The new type of relevant perturbations to be discussed in this paper provides renormalization group flows between noncompact target space geometries (like euclidean black holes) and compact geometries. This is, in fact, a new sort of topology change generated by relevant quasimarginal conformal operators but not (truly) marginal as in the case of Calabi-Yau manifolds [14].

The paper is organized as follows. In section 2 we will construct new relevant conformal operators which belong to $\text{Ker}Q/\text{Im}Q$ of the $SU(1,1)/U(1)$ gauged WZNW model at negative level $k$. Here $Q$ is the corresponding BRST operator. In section 3 we will study the fusion rules of these BRST invariant relevant operators. In particular we will exhibit a subclass of operators which form closed fusion algebras. In section 4 we will apply the relevant operators to perform renormalizable perturbations of the $SU(1,1)/U(1)$ coset in the limit $k \to -\infty$. We will discuss the renormalization group flow from the $SU(1,1)/U(1)$ coset to the infra-red conformal point. Finally, in the last section we will summarize our results and comment on them.

2 $SU(1,1)/U(1)$ coset

Let us consider the level $k$ WZNW model defined on the group manifold $G$ corresponding to the Lie group $G$. The action of the theory is given as follows [16], [17].

$$S_{WZNW}(g,k) = -\frac{k}{4\pi} \int \left[ \text{Tr}|g^{-1}\text{d}g|^2 + \frac{i}{3} \text{d}^{-1}\text{Tr}(g^{-1}\text{d}g)^3 \right],$$

(2.1)

where $g$ is the matrix field taking its values on the Lie group $G$. For compact groups the Wess-Zumino term [18] is well defined only modulo $2\pi$ [16], therefore, the parameter $k$ must be an integer in order for the quantum theory to be single valued with the multivalued classical action. For noncompact groups there are no topological restrictions for
$k$. The theory possesses the affine symmetry $\hat{G} \times \hat{G}$ which entails an infinite number of conserved currents $[15],[17]$. The latter can be derived from the basic currents $J$ and $\bar{J}$,

$$J \equiv J^a t^a = -\frac{k}{2} g^{-1} \partial g,$$

$$(2.2)$$

$$\bar{J} \equiv \bar{J}^a t^a = -\frac{k}{2} \bar{\partial} g g^{-1},$$

satisfying the equations of motion

$$\bar{\partial} J = 0, \quad \partial \bar{J} = 0. \quad (2.3)$$

In eqs. (2.2) $t^a$ are the generators of the Lie algebra $\mathcal{G}$ associated with the Lie group $G$,

$$[t^a, t^b] = f^{abc} t^c, \quad (2.4)$$

with $f^{abc}$ the structure constants.

WZNW models based on compact groups are well understood when the level $k$ is a positive integer. For nonnegative integer $k$ a positive definite Hilbert space, encompassing all the states of the conformal field theory, is defined by representations of the unitary affine algebra $[16],[17],[19]$. At the same time, the situation with negative integer $k$ is far from being well understood. It is clear that the theory is no longer unitary because there arise negative normed states in the spectrum. This is also true for WZNW models on noncompact groups. These theories are nonunitary due to indefinite Killing metric. However, it has been noticed by Dixon et al. $[2]$ that the coset $SU(1,1)/U(1)$, which involves a noncompact group, gives rise to a unitary CFT with a positive definite Hilbert space $[2],[3]$. Although the Hilbert space of the affine $SU(1,1)$ algebra may not be positive definite, one may still construct a positive definite Hilbert space after the projection provided by the compact $U(1)$ subgroup. It has been argued that the same procedure of gauging out the maximal compact subgroup of a given noncompact group should lead to unitary CFT’s in general case $[4]$.

According to Witten’s conjecture, WZNW models on compact group manifolds have to be understood as WZNW models on noncompact group manifolds after appropriate analytic continuation (Wick rotation) of the compact group to a noncompact group. The
point to be made is that analytic continuation does not spoil the hermicity condition of the Virasoro generators corresponding to the affine-Sugawara stress-energy tensor of the WZNW model. Indeed, one can check that $L_n^\dagger = L_{-n}$ holds before and after analytic continuation, where $L_n$ are generators of the Virasoro algebra (see definition of $L_n$ below). For generators of the affine Lie algebra this is not true. This observation allows us to guess that unitary Virasoro representations of the analytically continued theory have to be unitary representations of the original model before analytic continuation.

In particular, the noncompact group $SU(1, 1)$ can be thought of as being analytically continued from the compact $SU(2)$. Apparently, an analytically continued affine algebra will inherit the same level of the affine algebra of the compact affine Lie group. The WZNW model on the latter requires the level to be integer. Therefore, the distinctive feature of WZNW models on noncompact groups obtained by analytic continuation is that their levels are integer. Because of this fact, our interest in WZNW models on noncompact groups will be restricted to those having integer level.

Moreover, we will mainly discuss the WZNW model on $SU(1, 1)$ at negative level $k$. This model will be thought of as being obtained by analytic continuation of the WZNW model on $SU(2)$ at negative level $k$. Let us discuss the spectrum of the WZNW model on the noncompact group $SU(1, 1)$. The ground states in our model are the states which are annihilated by the modes $J_{n>0}^a$, where

$$J_n^a = \oint \frac{dw}{2\pi i} w^n J^a(w). \quad (2.5)$$

These states will fall into representations of the global $SU(1, 1)$ algebra and since we are requiring unitarity, these will first of all be unitary representations. All such unitary irreducible representations have been classified \cite{20} and since the algebra is noncompact, they will be infinite dimensional. Denoting the ground state by $|0; j, m\rangle$, where

$$g_{ab} J_0^a J_0^b |0; j, m\rangle = j(j+1) |0; j, m\rangle, \quad J_0^3 |0; j, m\rangle = m |0; j, m\rangle, \quad (2.6)$$

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*Our convention is

$$f_{31}^3 = f_{13}^3 = f_{31}^1 = 1, \quad g^{ab} = \frac{1}{2} f_c^{bd} f_d^{ca} = \text{diag}(-1, -1, 1), \quad g_{ac} g^{cb} = \delta_a^b.$$
we have the following nontrivial possibilities

1. \[ C_0^j : j = -\frac{1}{2} + i\kappa \text{ and } m = 0, \pm 1, \pm 2, \ldots, \]
2. \[ C_{1/2}^j : = -\frac{1}{2} + i\kappa \text{ and } m = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, \]
3. \[ E_0^j : -\frac{1}{2} \leq j < 0 \text{ and } m = 0, \pm 1, \pm 2, \ldots, \]
4. \[ D_+^j : j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \text{ and } m = -j, -j + 1, \ldots, \]
5. \[ D_-^j : j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots \text{ and } m = j, j - 1, \ldots, \]

where \( \kappa \) is real and non-zero. The first three classes, \( C_0^j, C_{1/2}^j \) and \( E_0^j \) are termed continuous, since there the eigenvalue of the Casimir operator is a continuous parameter. The last two representations \( D_\pm^j \) are called discrete.

We also will be interested in a nonunitary finite dimensional representation of \( SU(1,1) \) with \( j = \) half-integer and \( -j \leq m \leq j \). This representation contains both positive and negative norm states. We will denote this representation \( \Phi_{jm}^j \).

It has been proven \[ \cite{2},\cite{4} \] that the projection of the classes \( C_0^j, C_{1/2}^j, E_0^j, D_\pm^j \) onto the \( SU(1,1)/U(1) \) coset leads to the unitary representations of the Virasoro algebra. We are going to exhibit that there are more unitary representations of the Virasoro algebra associated with the \( SU(1,1)/U(1) \) coset. These representations will be shown to originate from the nonunitary finite dimensional representation \( \Phi_{jm}^j \) at \( j = 1 \).

For our purposes, it will be convenient to make use of the Lagrangian formulation of coset constructions \[ \cite{21}-\cite{23} \]. A generic \( G/H \) coset can be described as a combination of ordinary conformal WZNW models and the ghost-like action,

\[
S_{G/H} = S_{WZNW}(g,k) + S_{WZNW}(h,-k-2c_V(H)) + S_{Gh}(b,c,\bar{b},\bar{c}), \quad (2.7)
\]

where \( h \) takes values on the subgroup \( H \) of \( G \), \( c_V(H) \) is defined according to

\[
f_k^l f_j^k = -c_V(H)g^{ij}, \quad i,j,k,l = 1,2, \ldots \text{ dim } H. \quad (2.8)
\]

The last term in eq. (2.7) is the contribution from the ghost-like fields,

\[
S_{Gh} = \text{Tr} \int d^2z \ (b\partial c + \bar{b}\partial \bar{c}). \quad (2.9)
\]
The physical states are defined as cohomology classes of the nilpotent operator $Q$,

$$Q = \oint \frac{dz}{2\pi i} \left[ : c_a (\tilde{J}^a + J\_H^a) : (z) - \frac{1}{2} f^{ab}_c : c_a c_b c^c : (z) \right] ,$$ (2.10)

where we have used the following notations

$$J\_H^a = -\frac{k}{2} g^{-1} \partial g|_H,$$ (2.11)

$$\tilde{J} = \frac{(k + 2 c_V (H))}{2} h^{-1} \partial h .$$

Here the current $J\_H$ ia a projection of the $G$-valued current $J$ on the subalgebra $H$ of $G$.

We have already mentioned that unitary Virasoro representations corresponding to the unitary classes of $SU(1, 1)$ can be extracted by projecting out the $U(1)$ compact subgroup of $SU(1, 1)$. From the point of view of the Lagrangian approach all these representations belong to Ker$Q$/Im$Q$. Let us turn to the finite dimensional nonunitary representation $\Phi^j_m$. Let us take the adjoint representation $\Phi^1_m$ which we will denote $\phi^a$, $a = 1, 2, ..., \dim G$. The ground state corresponding to this operator has the conformal dimension

$$\Delta_\phi = \frac{c_V (G)}{k + c_V (G)} .$$ (2.12)

The unitarity conditions, which come out from the infinite dimensional representations, require $[4]$

$$k < -c_V (G) .$$ (2.13)

If condition (2.13) is satisfied, $\Delta_\phi < 0$. Hence, a Virasoro representation on this highest weight state will be nonunitary. This amounts to the statement that at level zero the finite dimensional representation $\phi^a$ does not lead to unitary Virasoro representations.

Let us consider the level one descendant state of the nonunitary representation of the affine Lie algebra. Namely $[13]$

$$|O^L\rangle = O^L(0)|0\rangle , \quad O^L(z) = L_{ab} : J^a (z) \phi^b (z) : .$$ (2.14)

Here the normal ordered product is defined according to $[10]$

$$O^L(z) = L_{ab} \oint \frac{dw}{2\pi i} \frac{J^a (w) \phi^b (z)}{w - z} ,$$ (2.15)
where the product in the numerator of the integrand is understood as an operator product expansion (OPE). The contour in eq. (2.15) goes anticlockwise around point $z$. It is easy to see that the given product does not contain singular terms provided the matrix $L_{ab}$ is symmetrical\footnote{Indeed, the field $\phi$ is a WZNW primary vector. Therefore, its OPE with the affine current $J$ is as follows

$$J^a(w)\phi^b(z) = \frac{f_{ab}^{\alpha}}{w-z} \phi^\alpha(z) + \text{reg. terms.}$$

After substitution of this formula into eq. (2.15), one can see that only regular terms will contribute provided $L_{ab}$ is a symmetrical matrix.}. According to definition (2.5) of the affine generators, one can present the state $|O^L\rangle$ as follows

$$|O^L\rangle = L_{ab} J^a_{-1} |\phi^b\rangle. \quad (2.16)$$

This state is no longer a highest weight vector of the affine Lie algebra. At the same time, $|O^L\rangle$ is a highest weight vector of the Virasoro algebra. Indeed, one can check that

$$L_0 |O^L\rangle = \Delta_0 |O^L\rangle, \quad (2.17)$$

$$L_{m>0} |O^L\rangle = 0.$$ 

Here the Virasoro generators $L_n$ are given by the contour integrals

$$L_n = \oint \frac{dw}{2\pi i} w^{n+1} T(w), \quad (2.18)$$

where $T(w)$ is the holomorphic component of the affine-Sugawara stress-energy tensor of the conformal WZNW model,

$$T(z) = \frac{g_{ab} : J^a J^b : (z)}{k + c_V(G)}. \quad (2.19)$$

In eqs. (2.17), $\Delta_O$ is the conformal dimension of the operator $O^L$. It is not difficult to find that

$$\Delta_O = 1 + \frac{c_V(G)}{k + c_V(G)}. \quad (2.20)$$

From this formula it is clear that when condition $k \leq -2c_V(G)$ (which is consistent with (2.13)) is fulfilled the conformal dimension of $O^L$ is in the range between 0 and 1, i.e. it
is positive. The Virasoro central charge is also positive. Indeed,
\[ c_{WZNW}(k) = \frac{k \dim G}{k + c_V(G)} = \dim G - \frac{c_V(G) \dim G}{k + c_V(G)} > \dim G > 1. \quad (2.21) \]
Thus, the operator $O^L$ lies in the unitary range of the Kac-Kazhdan determinant \[24\] and, hence, it provides a unitary representation of the Virasoro algebra. However, it is easy to verify that the given operator is not annihilated by $Q$. Therefore, $O^L$ does not belong to the physical subspace of the gauged WZNW models. Fortunately, there is a way to modify the operator $O^L$ without spoiling its properties so that it will belong to $\text{Ker}Q/\text{Im}Q$. We will do this modification for the case of the $SU(1,1)/U(1)$ gauged WZNW model.

First of all, let us restrict ourselves to $O^L$ with a diagonal matrix $L_{ab} = \lambda_a g_{ab}$ (no summation over indices). Note that for $SU(2)$ the diagonal form of $L_{ab}$ is the normal one, but in the case of $SU(1,1)$ there are additional nondiagonalizable forms. In this context, the equation
\[ L_{ab} = \lambda_a g_{ab} \quad (2.22) \]
is a certain restriction which reminds us that the group $SU(1,1)$ originates from $SU(2)$ via analytic continuation.

Now let us define a new operator
\[ \hat{O}^L = O^L + N \tilde{J}^3 \phi^3, \quad (2.23) \]
where $\tilde{J}^3$ is the current associated with the compact subgroup $H = U(1)$,
\[ \tilde{J}^3(w)\tilde{J}^3(z) = \frac{-k/2}{(w-z)^2} + \text{reg. terms.} \quad (2.24) \]
Here the minus comes from the fact that the level of the affine algebra $H$ acquires the opposite sign to the level of $\hat{G}$. The constant $N$ is to be determined from the condition
\[ Q|\hat{O}^L\rangle = 0, \quad (2.25) \]
where the nilpotent charge $Q$ is given by
\[ Q = \oint \frac{dw}{2\pi i} : c (\tilde{J}^3 + J^3) : (z). \quad (2.26) \]
The second term in eq. (2.10) vanishes in eq. (2.26) because $U(1)$ is Abelian.
It is more convenient to present $Q$ in terms of modes

$$Q = \sum_{n=-\infty}^{+\infty} :c_{-n}(\bar{J}^3_n + J^3_n) :.$$

(2.27)

A canonical choice for the vacuum state of the ghost Fock space is an $SL(2, C)$ invariant state $|0\rangle_{Gh}$, which is annihilated by $L_{Gh,n}$ for $n = 0, \pm 1$. This requires that

$$c_n|0\rangle_{Gh} = 0, \quad n \geq 1,$$

(2.28)

$$b_n|0\rangle_{Gh} = 0, \quad n \geq 0.$$

Acting with $Q$ on $|\hat{O}^L\rangle$ one finds

$$c_0(\bar{J}^3_0 + J^3_0)|\hat{O}^L\rangle + c_{-1}(\bar{J}^3_1 + J^3_1)|\hat{O}^L\rangle = 0.$$

(2.29)

This condition yields two equations

$$\lambda_1 = \lambda_2,$$

(2.30)

$$N = \lambda_3 - \frac{2}{k}(\lambda_1 + \lambda_2),$$

where $k$ obeys condition (2.13).

Thus, we have found an operator which is annihilated by $Q$. In other words, we have proved that the operator $\hat{O}^L$ with $\lambda_i$ and $N$ given by eqs. (2.30) belongs to the physical subspace $\text{Ker}Q/\text{Im}Q$. Therefore, this operator $\hat{O}^L$ has to be considered on the same footing with the unitary infinite dimensional representations of $SU(1,1)$. However, we have not proven that the fusion algebra between $\hat{O}^L$ and other operators, commuting with $Q$, will be free from negative normed operators. In fact, in the large $|k|$ limit it is not difficult to prove that the $\text{Ker}Q/\text{Im}Q$ space is evidently unitary [4]. More discussion of the unitarity of $\text{Ker}Q/\text{Im}Q$ will be given in the next section.

3 Fusion rules of $\hat{O}^L$

It is obvious that the BRST invariant operator $\hat{O}^L$ and the operator $O^L$ share one and the same conformal dimension given by eq. (2.20). Of course, the conformal dimension
of $\hat{O}^L$ is defined with respect to the Virasoro operator $\hat{L}_0$ of the gauged WZNW model. The point to be made is that the unitarity condition in eq. (2.13) indicates that $\hat{O}^L$ is a relevant operator. That is,

$$\Delta_{\hat{O}} = 1 + \frac{c \nu(G)}{k + c \nu(G)} < 1.$$  (3.31)

Moreover, in the large $|k|$ limit, $\hat{O}^L$ becomes a relevant quasimarginal operator. This fact makes the operator $\hat{O}^L$ of a great interest because it can be used as a perturbing operator on the given CFT. In the case under consideration the CFT is the $SU(1,1)/U(1)$ coset which has the target space interpretation of the two dimensional euclidean black hole\footnote{The black hole is euclidean because the $U(1)$ subgroup is chosen to be compact. For the noncompact $U(1)$ the target space geometry of the $SU(1,1)/U(1)$ coset corresponds to the lorentzian black hole \cite{7,9}.} \cite{7,9}. Therefore, perturbations of the $SU(1,1)/U(1)$ coset would amount to perturbations of the euclidean two dimensional black hole. We will study given perturbations by the operator $\hat{O}^L$ in the next section.

Clearly, operators $\hat{O}^L$ with different arbitrary diagonal matrices $L_{ab}$ obeying conditions (2.30) will give rise to Virasoro primary vectors with the same conformal dimension given by eq. (3.31). However, their fusion algebras may be different. From now on we would like to focus on a particular subclass of operators $\hat{O}^L$ which satisfy the following fusion

$$\hat{O}^L \cdot \hat{O}^L = [\hat{O}^L] + [I] + ..., \quad (3.32)$$

where the square brackets denote the contributions of $\hat{O}^L$ and identity operator $I$ and the corresponding descendants of $\hat{O}^L$ and $I$, whereas dots stand for all other admitted operators with different conformal dimensions. In fact, we will show that only BRST-exact operators and operators of irrelevant conformal dimensions are admitted to replace dots in eq. (3.32).

We are going to translate the fusion algebra (3.32) in an equation for the matrix $L_{ab}$. To this end, it is convenient to introduce the following operator

$$\Psi^{ab}(z) \equiv J^{(a)}( -1 \phi^{(b)})(z) = \oint \frac{dy}{2\pi i} \frac{J^{(a)}(y)\phi^{(b)}(z)}{y - z}, \quad (3.33)$$

where indices $a, b$ are symmetrized. Obviously,

$$O^L = \frac{1}{2} L_{ab} \Psi^{ab}. \quad (3.34)$$
To start with, let us compute the following OPE
\[
\phi^c(w)\Psi^{ab}(z) = \frac{C^{ab,c}}{(w-z)^\Delta} \phi^d(z) + \ldots, \tag{3.35}
\]
where dots stand for all other operators with different conformal dimensions. To proceed we also have to calculate
\[
\phi^a(w)\phi^b(z) = \sum_I (w-z)^{\Delta_I - 2\Delta} C^{ab}_I [\Phi^I(z)], \tag{3.36}
\]
where \([\Phi^I]\) are conformal classes of all Virasoro primaries \(\Phi^I\) arising in the fusion of two \(\phi\)'s. It is convenient to set \(z\) to zero in eq. (3.36). Then after acting on the \(SL(2,C)\) vacuum, eq. (3.36) yields
\[
\phi^a(w)|\phi^b\rangle = w^{-\Delta} C^{ab}_c |\phi^c\rangle + \ldots \tag{3.37}
\]
Acting with \(J^k_0\) on both sides of eq. (3.37), we find
\[
[J^k_0, \phi^a(w)]|\phi^b\rangle + f^{kbc}_a |\phi^c\rangle = C^{ab}_c f^{kd}_c |\phi^d\rangle + \ldots, \tag{3.38}
\]
where we took into account the following quantization conditions
\[
J^a_0 |\phi^b\rangle = f^{abc}_c |\phi^c\rangle, \quad J^a_{m \geq 1} |\phi^b\rangle = 0. \tag{3.39}
\]
The first commutator on the left hand side of eq. (3.38) can be calculated according to
\[
[J^k_n, \phi^a(w)] = \oint \frac{dz}{2\pi i} \frac{z^n J^k(z)\phi^a(w)}{z-w} = \frac{w^n f^{ka} \phi^d(w)}{z-w} = w^n f^{ka} \phi^d(w), \tag{3.40}
\]
where one has to use the definition of \(J^a_n\) in eq. (2.5).

Finally, we arrive at the following consistency condition
\[
f^{kca}_c C^{eb}_d + f^{kcb}_c C^{ec}_d = C^{ab}_c f^{kc}_d. \tag{3.41}
\]
Obviously, the solution to this equation is
\[
C^{ab}_c = A f^{ab}_c, \tag{3.42}
\]
where \(A\) is arbitrary constant whose value is to be fixed by appropriate normalization.
By applying the operators $J^a_n$ to both sides of eq. (3.37) one can fix order by order all the higher power terms on the right hand side of eq. (3.37). One more term on the right hand side of eq. (3.37) will be found later on.

Now we turn to the OPE given by eq. (3.35). Again acting with $J^k_0$ on both sides of eq. (3.35) we obtain

$$[J^k_0, \phi^c(w)|\Psi^{ab}] + \phi^c(w)J^k_0|\Psi^{ab}) = w^{-\Delta_c}C^{ab,c}_{d}f^k|\phi^c) + ...$$  \hspace{1cm} (3.43)

The commutator on the left hand side of eq. (3.43) can be computed according to formula (3.40). Whereas

$$J^k_0|\Psi^{ab}) = J^k_0J^{(a}_{-1}|\phi^b) = f^k_d|\Psi^{db}).$$  \hspace{1cm} (3.44)

All in all, we find the consistency condition

$$f^k_dC^{db,c}_{n} + f^k_cC^{ab,d}_{n} = C^{ab,c}_{d}f^k_n,$$  \hspace{1cm} (3.45)

which yields the following solution

$$C^{ab,c}_{d} = \frac{A}{2}(f^b_d f^{ea} + f^a_e f^{eb}),$$  \hspace{1cm} (3.46)

where one can show that $A$ is the same normalization constant as in eq. (3.42).

Now let us compute

$$\Psi^{ab}(z)|\Psi^{cd}) = z^{-2\Delta_O+\Delta_c}C^{ab,cd}_n|\phi^n) + z^{-\Delta_O}C^{ab,cd}_{mn}|\Psi^{mn} + ...$$  \hspace{1cm} (3.47)

The coefficient $C^{ab,cd}_n$ can be found from the equation

$$[J^k_0, \Psi^{ab}(z)|\Psi^{cd}) + \Psi^{ab}(z)J^k_0|\Psi^{cd}) = \Psi^{a}(z)f^k_n|\phi^c) + ...$$  \hspace{1cm} (3.48)

The latter gives rise to the consistency condition

$$f^k_dC^{(ab,cd}_n = C^{ab,(cd}_e f^k_e = C^{ab,cd}_e f^k_e.$$  \hspace{1cm} (3.49)

This yields

$$C^{ab,cd}_e = \frac{A}{4}[f^a_e f^b_d f^c_n + f^a_e f^b_c f^d_n + f^c_d f^a_e f^b_n + f^d_n f^a_e f^b_n] + (a \leftrightarrow b).$$  \hspace{1cm} (3.50)
Further, the coefficient $C_{mn}^{ab,cd}$ is fixed from

$$
[J_1^k, \Psi^{ab}(z)]|\Psi^{cd}\rangle + \Psi^{ab}(z) J_1^k |\Psi^{cd}\rangle = z^{-\Delta_O} C_{mn}^{ab,cd} J_1^k |\Psi^{mn}\rangle + \ldots \quad (3.51)
$$

Here one has to use the following relations

$$
J_1^k |\Psi^{ab}\rangle = J_1^k J_1^k \phi^a(z) |= f^{k(a}_b \phi^b(\phi^e) + \frac{k}{2} g^{k(a}_b |\phi^b\rangle. \quad (3.52)
$$

Whereas the commutator on the left hand side of eq. (3.51) is given by

$$
[J_1^k, \Psi^{ab}(z)] = - f^{k(a}_b \phi^b(z) + \frac{k}{2} g^{k(a}_b |\phi^b\rangle + z f^{k(a}_b \Psi^{db}(z). \quad (3.53)
$$

Taking into account eqs. (3.52), (3.53) and (3.35), we come to the following consistency condition

$$
- f^{k(a}_b \phi^b \phi^e C^{cd,e} + \frac{k}{2} g^{k(a}_b |\phi^b\rangle + f^{k(c}_d \phi^e C^{ab,e} + \frac{k}{2} g^{k(c}_d |\phi^e\rangle + f^{k(a}_b \phi^{de}}{C^{ab,cd}} \quad (3.54)
$$

Bearing in mind the expressions for the coefficients $C^{ab,cd}$, one can compute the coefficient $C_{mn}^{ab,cd}$ from eq. (3.54).

Now it becomes clear that in order to have the operator $O^L$ on the right hand side of the OPE given by eq. (3.32), we have to impose the following condition

$$
L_{ab} L_{cd} C_{mn}^{ab,cd} \sim L_{mn}. \quad (3.55)
$$

Taking into account eq. (3.54) we obtain

$$
\left( \frac{k}{2} g^{k(m}_p \phi^e - f^{k(m}_p \phi^e} \right) L_{mn}
$$

$$
= L_{ab} L_{cd} \left[ \frac{k}{2} (g^{k(a}_b \phi^b \phi^e C^{cd,e} + \frac{k}{2} g^{k(a}_b |\phi^b\rangle + \frac{k}{2} g^{k(c}_d |\phi^e\rangle + f^{k(a}_b \phi^{de}}{C^{ab,cd}} \right] . \quad (3.56)
$$

This is the equation which yields matrices $L_{ab}$ giving rise to the following OPE

$$
O^L O^L = [O^L] + [I] + \ldots , \quad (3.57)
$$

where dots stand for conformal classes of Virasoro primaries of irrelevant conformal dimensions. Indeed, it is transparent that the equation (3.56) is invariant under adjoint
transformations of the matrix $L_{ab}$. These transformations are generated by $J^a_0$. In virtue of this fact, the right hand side of eq. (3.57) must belong to the equivalence class (or the orbit) generated by the adjoint action of the global group $G$. Within this class the OPE in eq. (3.57) is a scalar under global $G$ transformations. There are no other relevant Virasoro primary operators but $O^L$ and $I$ which possess the given property. Hence, the OPE for operators $O^L$ whose matrices $L$ obey the equation (3.56) is closed on $O^L$ and $I$, and other admitted operators with irrelevant conformal dimensions (as : $(O^L)^2 :$).

The curious fact is that in the limit $k = 0$, the obtained equation (3.56) goes to the master Virasoro equation [20]. However, this limit is beyond validity of the unitarity condition in eq. (2.13).

We cannot go on with the operator $O^L$ because it does not belong to the physical subspace of the $SU(1,1)/U(1)$ gauged WZNW model. The BRST invariant operator is $\hat{O}^L$ given by eq. (2.22). The additional term in $\hat{O}^L$ changes the conditions on the matrix $L_{ab}$. In order to take into account these modifications, we have to compute the following symmetrized OPE

$$\phi^{(a}(z)|\phi^{b)} = A \ C^{ab}_{mn} |\Psi^{mn}⟩ + ... ,$$

where we have already extracted the normalization constant $A$ from the coefficient. Acting with $J^1_k$ on both sides of eq. (3.58), we obtain

$$[J^1_k, \phi^{(a}(z)]|\phi^{b)}⟩ = A \ C^{ab}_{mn} J^1_k |\Psi^{mn}⟩ + ...$$

Now taking into account formula (3.40), we arrive at the following consistency condition

$$- f^e_k (a f^b_c) = C^{ab}_{mn} \left[ \frac{k}{2} g^{k(m} \delta^n_c) - f^e_k (m f^n_c) \right],$$

from which one can find the coefficient $C^{ab}_{mn}$. In fact, we are interested only in the coefficient $C^{33}_{mn}$ which is given by

$$C^{33}_{mn} = \frac{1}{(k-1)^2 - 9} \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
which is the condition of invertibility of the matrix \( \frac{k}{2} g^{k(m\delta n)} - f^{k(mf n)c} \) for the case \( SU(1, 1) \).

Now we can write down the modified equation for the matrix \( \hat{L}_{ab} \)

\[
\left( \frac{k}{2} g^{k(m\delta n)} - f^{k(mf n)f} \right) \left( \hat{L}_{mn} + \frac{k}{2} AN^2 C_{mn}^{33} \right)
\]

\[
= \hat{L}_{ab} \hat{L}_{cd} \left[ \frac{k}{2} \left( g^{k(aC_{p}^{cd,b})} + g^{k(cC_{p}^{ab,d})} \right) - f^{k(a) f e c d} C_{p}^{e} - f^{k(c) f e c d} C_{p}^{a b} + f^{k(a) C_{p}^{b} e c d} \right],
\]

where the constant \( N \) is given by eqs. (2.29).

The point to be made is that in the derivation of the equation (3.63) we did not use the mode expansions for conformal operators. These expansions, which in principle can be defined, are not much of use because we are dealing with operators of rational conformal dimensions. At the same time, the method of consistency conditions we have employed here allows us to handle OPE’s both for positive and negative values of \( k \).

In what follows our interest will be in the limit \( k \to -\infty \). In particular, we will need to know only the large \( |k| \) solution of the equation (3.63). First of all note that in this limit, \( C_{mn}^{33} \to 0 \). Therefore, when \( k \to -\infty \) there is no difference between eqs. (3.56) and (3.63). Moreover, in the large \( |k| \) limit the equation (3.63) reduces to the following

\[
g^{k(m\delta n)} \hat{L}_{mn} = \hat{L}_{ab} \hat{L}_{cd} \left( g^{k(aC_{p}^{cd,b})} + g^{k(cC_{p}^{ab,d})} \right) + O(1/k).
\]

A solution to this equation which fits the conditions in eqs. (2.30) is

\[
\hat{L}_{ab} = \frac{g_{ab}}{2A} + O(1/k).
\]

It is convenient to normalize \( \hat{L} \) to

\[
\hat{L}_{ab} = \sqrt{\frac{1}{2} g_{ab} + O(1/k)}.
\]

At the given value of \( \hat{L} \) the operator \( \hat{O}^L \) will satisfy the OPE in eq. (3.32). Since eq. (3.32) is a gauge invariant version of eq. (3.57), there will be only gauge invariant extensions of the operators emerging in eq. (3.57) as well as \( Q \)-exact operators. Thus, modulo \( Q \)-exact terms the OPE given by eq. (3.32) must be closed on operators \( \hat{O}^L \) and \( I \), and other operators of irrelevant conformal dimensions. In the case of fusion between \( \hat{O}^L \) and
operators from the unitary representations of $SU(1, 1)$, the fusion algebra will be closed on the unitary representations. Indeed, we first can consider fusion between $O^L$ and the unitary representations. Because $O^L$ does not change the tensor structure with respect to the global group $SU(1, 1)$, all fusion of $O^L$ have to be closed on global representations of $SU(1, 1)$. In the case of the BRST invariant operator $\hat{O}^L$, a part of the fusion algebra can be deduced from the fusion algebra of $O^L$ by the BRST procedure. In addition to the BRST symmetrized terms there may appear $Q$-exact terms in the fusion of $\hat{O}^L$. All potential operators of irrelevant conformal dimensions in eq. (3.32) will belong to the unitary range of the Kac-Kazhdan determinant. Thus, modulo $Q$-exact terms the operator $\hat{O}^L$ along with the unitary representations form the closed unitary fusion algebra. However, the whole space $\ker Q / \text{im} Q$ is still not unitary. Indeed, the condition of annihilation by the BRST operator $Q$ is not sufficient to get rid of all nonunitary states. For example, the state $|\phi^3\rangle$ obeys the BRST symmetry, i.e. $Q|\phi^3\rangle = 0$. Thus, $|\phi^3\rangle$ belongs to $\ker Q / \text{im} Q$. However, the conformal dimension of $\phi^3$ is nonpositive. Therefore, the descendant state $L_{-1}|\phi^3\rangle$ is negative normed. This amounts to the nonunitarity of the “physical” subspace of the $SU(1, 1)/U(1)$ gauged WZNW model. Hence, the procedure of gauging out the maximal compact subgroup of the noncompact group does not automatically lead to the unitarity of the gauged WZNW model.

4 Relevant perturbations

In the previous sections we have exhibited that there are relevant operators in the space of cohomological classes of the BRST operator of the gauged $SU(1, 1)/U(1)$ model. We have shown that these operators are highest weight vectors of unitary Virasoro representations and that their OPE’s are closed without introducing new relevant conformal operators. These properties allow us to make use of the operators $\hat{O}^L$ to perform renormalizable perturbations on the $SU(1, 1)/U(1)$ coset.

To this end, we first consider relevant perturbations of the nonunitary $SU(1, 1)$ WZNW model. The perturbative theory is described by the following action

$$S(\epsilon) = S_{WZWN}^* - \epsilon \int d^2z \ O^{\bar{L}}(z, \bar{z}),$$

(4.67)
where the first term on the right hand side of eq. (4.67) is the conformal action of the WZNW model on $SU(1,1)$, whereas the second term is the perturbation by the operator
\[ O^{L,L}(z,\bar{z}) = O^L(z) \, \bar{O}^L(\bar{z}). \] (4.68)

Note that within perturbation theory one can apply the theorem of holomorphic factorization to understand the factorization in eq. (4.68).

From now on we will be interested in the limit $k \to -\infty$. This is the classical limit for the WZNW model on $SU(1,1)$. Correspondingly the operator $O^{L,L}$ can be presented in the form
\[ O^{L,L}(z,\bar{z}) \xrightarrow{k \to -\infty} G_{\mu\nu} \partial x^\mu \partial x^\nu + B_{\mu\nu} \partial x^\mu \partial x^\nu, \] (4.69)

where $x^\mu$ are coordinates on the group manifold $SU(1,1)$, whereas the metric and the antisymmetric tensor are given by
\[ G_{\mu\nu} = -\frac{k}{8} \hat{L}_{ab} \hat{\phi}^a \hat{\phi}^b \epsilon^{\mu}_a \epsilon^{\nu}_b, \] \[ B_{\mu\nu} = -\frac{k}{8} \hat{L}_{ab} \hat{\phi}^a \hat{\phi}^b \epsilon^{\mu}_a \epsilon^{\nu}_b. \] (4.70)

Here $\epsilon^{\mu}_a$ and $\epsilon^{\nu}_b$ define left- and right-invariant Killing vectors respectively. Note that when $\hat{L} = \hat{\phi}$, $B_{\mu\nu} = 0$.

Thus, the operator $O^{L,L}$ corresponds to the energy density of the nonlinear sigma model with metric and antisymmetric field given by eqs. (4.70). The renormalizability of the gauged sigma model together with eq. (3.56) will ensure the following OPE
\[ O^{L,L} \, O^{L,L} = [O^{L,L}] + [I] + ... , \] (4.71)

where dots stand for operators with irrelevant conformal dimensions. This OPE agrees with eq. (3.57).

We proceed to calculate the renormalization beta function associated with the coupling $\epsilon$. Away of criticality, when $\epsilon \neq 0$, the renormalization group equation is given by (see e.g. [27])
\[ \frac{d\epsilon}{dt} \equiv \beta = (2 - 2\Delta_O)\epsilon - \pi C \epsilon^2 + O(\epsilon^3), \] (4.72)
where the conformal dimension $\Delta_O$ is given by eq. (2.20), whereas the coefficient $C$ is to be computed from the three point function

$$\langle O^{L,L}(z_1, \bar{z}_1)O^{L,L}(z_2, \bar{z}_2)O^{L,L}(z_3, \bar{z}_3)\rangle = C\|O\|^2 \prod_{i<j}\frac{1}{|z_{ij}|^{2\Delta_O}}$$

with $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. Here

$$\|O\|^2 = \langle O^{L,L}(1)O^{L,L}(0)\rangle.$$

In the large $|k|$ limit one can take into account eq. (3.64) to obtain

$$\|O\|^2 = \frac{k^2}{4\dim G}L^{ab}L_{ab}\bar{L}\bar{L} = k^2 \left(\frac{3}{16} + \mathcal{O}(1/k)\right),$$

where the factor $(1/\dim G)$ stems from the normalization of $\phi^{ab}$.

$$\langle \phi^{ab}(1)\phi^{\bar{a}\bar{b}}(0)\rangle = \frac{g^{ab}g^{\bar{a}\bar{b}}}{\dim G}.$$

Whereas from eq. (3.56) it follows that

$$C = 1.$$ 

With the given $C$ we can find a nontrivial fixed point of the beta function in eq. (4.72):

$$\epsilon^* = -\frac{2\gamma(G)}{\pi k} = -\frac{4}{\pi k}.$$ 

At this value of $\epsilon$ the theory in eq. (4.67) becomes a new CFT whose Virasoro central charge can be estimated by the Cardy-Ludwig formula

$$c(\epsilon^*) = c_{WZNW} - \frac{(2 - 2\Delta_O)^3\|O\|^2}{C^2} + \text{higher in } 1/k \text{ terms.}$$

We find

$$c(\epsilon^*) = \frac{3k}{k + 2} + \frac{12}{k} + \mathcal{O}(1/k^2).$$

It turns out that the CFT with the given Virasoro central charge can be identified with an exact CFT due to the following relation

$$\frac{(-k)^3}{(-k) + 2} = \frac{3k}{k + 2} + \frac{12}{k} + \mathcal{O}(1/k^2).$$
Note that $k$ is thought of as being negative integer, so that $-k$ is positive integer. Thus, the expression on the left hand side of eq. (4.80) coincides with the Virasoro central charge of the WZNW model on the compact group $SU(2)$ at positive level. In other words, the perturbation by the operator $O^{L,L}(z, \bar{z})$ gives rise to the following renormalization group flow:

$$SU(1, 1)_{-|k|<4} \rightarrow SU(2)_{|k|-4}. \tag{4.82}$$

This is a curious result. Indeed, from the point of view of the target space-time geometry the left hand side of the flow in eq. (4.82) corresponds to the noncompact group manifold. Whereas the right hand side of the flow describes the compact group manifold. Apparently these two target spaces have different topologies. Thus, the renormalization group flow at hands provides a certain mechanism of topology change in the target space. It is different from the topology change mechanism of Calabi-Yau manifolds. There topology changes under truly marginal deformations. Now we have exhibited that relevant perturbations also lead to topology change of the target space geometry. This may, perhaps, result in a new type of mirror symmetry.

Now let us turn to the case of the BRST invariant perturbation of the $SU(1, 1)/U(1)$ coset. There are two ways to approach this problem. The first one is to work out the gauge Ward identities for correlation functions dressed with the quantum gauge fields. This approach mimics the method of studying CFT’s coupled to 2d gravity. Gauge dressed correlators are defined as follows

$$\langle\langle \cdots \rangle\rangle = \int DA D\bar{A} \langle \cdots e^{-\int \text{Noether terms}} \rangle, \tag{4.83}$$

\footnote{Formally, the formula for the Virasoro central charge of the $SU(1, 1)$ WZNW model at positive level also fits eq. (4.81). However, for the given CFT there are no unitary representations in the spectrum due to the unitarity condition (2.13). At the same time, the perturbation by the operator $O^{L,L}$ must preserve the unitarity of the positive normed representations of $SU(1,1)$ in the course of perturbation. Since we know that such unitary representations exist, we make our choice between the two options in favour of the WZNW model on the compact group manifold.}

\footnote{The finite shift in the level of the WZNW model on the compact group can be seen within the fermi-bose equivalence of the non-Abelian Thirring model. Due to this shift the level of the unitary WZNW model may take value 1, 2 and so on.}
where the functional integrals over the gauge fields are computed according to the Faddeev-Popov method.

The second approach is based on the BRST formalism according to which all gauge dressed correlation functions are written as correlators of BRST invariant operators. In particular for $O^{L,\bar{L}}$ one will have

$$\langle\langle O^{L,\bar{L}}(z_1, \bar{z}_1) O^{L,\bar{L}}(z_2, \bar{z}_2) \cdots O^{L,\bar{L}}(z_n, \bar{z}_n) \rangle\rangle \sim \langle \hat{O}^{L,\bar{L}}(z_1, \bar{z}_1) \hat{O}^{L,\bar{L}}(z_2, \bar{z}_2) \cdots \hat{O}^{L,\bar{L}}(z_n, \bar{z}_n) \rangle, \quad (4.84)$$

where the operator $O^{L,\bar{L}}$ corresponds to the following gauge invariant expression

$$O^{L,\bar{L}} = -\frac{k^2}{4} \left\{ \text{Tr}(\partial g \partial g^{-1}) + 2\text{Tr}[Ag^{-1}\partial g - \bar{A}\bar{g}g^{-1} + \bar{A}g^{-1}Ag - AA] \right\}, \quad (4.85)$$

with $A = A_z \cdot t^3$, $\bar{A} = A_{\bar{z}} \cdot t^3$ and $L = \bar{L} = 1$. Whereas the operator $\hat{O}^{L,\bar{L}}$ is defined as follows

$$\hat{O}^{L,\bar{L}}(z, \bar{z}) = \hat{O}^L(z) \hat{O}^{\bar{L}}(\bar{z}), \quad (4.86)$$

with $\hat{O}^L$ given by eq. (2.23).

Correspondingly, one can study the renormalization group flows in the presence of the quantum gauge fields following the two ways just described above. The important point to be made is that Polyakov has proved that there is no renormalization of the gauge coupling constant [28]. Therefore, all renormalization will amount to the renormalization of the perturbation parameter $\epsilon$ in the perturbative action

$$S_{gauged}(\epsilon) = S_{\text{coset}}^* - \epsilon \int d^2z \hat{O}^{L,\bar{L}}(z, \bar{z}), \quad (4.87)$$

where $S_{\text{coset}}^*$ is the conformal action of the gauged $SU(1,1)/U(1)$ model.

Due to eq. (3.63), the coefficient $C$ in the three point function of the BRST invariant operator $\hat{O}$ is equal to one. Therefore, to leading orders in $\epsilon$, the renormalization group equation for the coupling $\epsilon$ coincides with eq. (4.72). Hence, all fixed points of the perturbed WZNW model remain critical points of the perturbed gauged WZNW model. Thus, the point $\epsilon^*$ given by eq. (4.78) has to be a conformal point of the theory in eq. (4.87). However, correlation functions undergo the gauge dressing which affects the Virasoro central charge at the infra-red conformal point. Indeed, the dressed two-point
function is given by
\[
\langle\langle O^L(1)O^L(0)\rangle\rangle = \langle\hat{O}^L(1)\hat{O}^L(0)\rangle = k^2 \left(\frac{1}{8} + \mathcal{O}(1/k)\right). \quad (4.88)
\]
This expression differs from the one in eq. (4.75). Correspondingly the Cardy-Ludwig formula changes. The alterations obscure the interpretation of the CFT at the IR critical point. For example, the formula for the Virasoro central charge now is given by
\[
c(\epsilon^*) = \frac{3k}{k+2} - 1 + \frac{8}{k} + \mathcal{O}(1/k^2).
\quad (4.89)
\]
It is not clear to which exact CFT this perturbative central charge may correspond.

Instead of the SU(1,1)/U(1) coset, one can start with the slightly modified CFT described by the direct sum of the SU(1,1)/U(1) coset and a free compact scalar field. The action is
\[
S_{SU(1,1)/U(1)} = S_{SU(1,1)/U(1)} + \frac{1}{4\pi} \int d^2z \partial X \bar{\partial} X. \quad (4.90)
\]
Now the space Ker\(Q/\text{Im}Q\) contains a new relevant operator
\[
O_X = \partial X \phi^3. \quad (4.91)
\]
Note that the scalar field \(X\) is inert under the gauge transformations. There exists a combination of the operators \(\hat{O}^L\) and \(O_X\) which obeys the fusion algebra in eq. (3.32):
\[
O^L_X = \hat{O}^L + E O_X, \quad (4.92)
\]
where \(E\) is arbitrary constant whose physical meaning will be clarified presently. The matrix \(L\) depends on \(E\) through the consistency condition which can be derived in a similar way to how we derived the equation (3.63).

If we choose
\[
E = \pm i \sqrt{|k|/2}, \quad (4.93)
\]
then the norm of \(O^L_X\) will coincide with eq. (4.75). Therefore, after perturbation of the CFT in eq. (4.90) by the operator \(O^L_X\), the system arrives at the IR critical point with the following Virasoro central charge
\[
c(\epsilon^*) = \frac{2k}{k+2} + \frac{12}{k} + \mathcal{O}(1/k^2). \quad (4.94)
\]
The last formula allows us to identify the given CFT with the \((SU(2) \times U(1))/U(1)\) coset construction. All in all, we come to conclusion that at the critical point \(\epsilon^*\) the perturbed \((SU(1, 1)/U(1)) \times U(1)\) coset coincides with the action of the gauged \((SU(2) \times U(1))/U(1)\) model at positive level. This will amount to the following renormalization group flow

\[
\frac{SU(1, 1)_{k<4}}{U(1)} \times U(1) \longrightarrow \frac{SU(2)_{|k|-4} \times U(1)}{U(1)}.
\]

(4.95)

From the point of view of the target space geometry, the left hand side of eq. (4.95) describes a two-dimensional electrically charged euclidean black hole [30], whereas the right hand side of the flow is the product of the two-dimensional sphere and a circle. One might expect to have the last geometry for the equilibrium of two extreme black holes. In this connection, the constant \(E\) defines electric charge of the black hole, whereas the equation (4.93) coincides with the condition for extreme black holes. However, this conjecture has to be more carefully investigated. The aforementioned effect once again illustrates target space topology change triggered by relevant perturbations.

5 Conclusion

We have started with the analysis of the spectrum of the \(SU(1, 1)/U(1)\) coset at negative level and proceeded to define relevant operators corresponding to highest weight vectors of unitary Virasoro representations. We found that these representations come into being as level one descendants of the highest weight vector of the nonunitary (finite dimensional) representation of the noncompact affine Lie algebra at negative level. We have established that these relevant operators can be arranged to form the closed fusion algebra.

We have performed the large \(|k|\) renormalizable perturbation of the \((SU(1, 1)/U(1)) \times U(1)\) coset and found that the perturbed model has a nontrivial conformal point. It has been displayed that this perturbative conformal point corresponds to the \((SU(2) \times U(1))/U(1)\) coset at positive level. Thus, we have exhibited the new mechanism of topology change in the target space along the renormalization group flow. Therefore, it might be interesting to understand whether or not this topology change is related to a new kind of duality symmetry or mirror symmetry of string theory.
There is another quite interesting issue which we left for further investigation. Namely, one can consider the $N = 2$ supersymmetric generalization of the $SU(1,1)/U(1)$ coset. This theory describes the unitary $N = 2$ superconformal discrete series [1]. Our conjecture is that the perturbation we discussed in the present paper can provide flows between $c > 3$ and $c < 3 \ N = 2$ series [31].

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