DYNAMICS OF REGULARLY RAMIFIED RATIONAL MAPS: I.
JULIA SETS OF MAPS IN ONE-PARAMETER FAMILIES

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Abstract. In [6], regularly ramified rational maps are constructed and Julia sets of these maps in some one-parameter families are explored through computer-generated pictures. It is observed that they have classifications similar to the Julia sets of maps in the families \( f^c_n(z) = z^n + c \), where \( n \geq 2 \) and \( c \) is a complex number. A new type of Julia set is also presented, which has not appeared in the literature. We call such a Julia set an exploded McMullen necklace. We prove in this paper: if a map \( f \) in the one-parameter families given in [6] has a superattracting fixed point of order greater than 2, then its Julia set \( J(f) \) is either connected, a Cantor set, or a McMullen necklace (either exploded or not); if such a map \( f \) has a superattracting fixed point of order equal to 2, then \( J(f) \) is either connected or a Cantor set.

1. Introduction. Let \( \hat{\mathbb{C}} \) be the Riemann sphere. A rational map \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is said to be regularly ramified if there exists a group \( G \) of conformal automorphisms of \( \hat{\mathbb{C}} \) such that two points \( z_1 \) and \( z_2 \) have the same image under \( f \) if and only if there is an element \( g \) of \( G \) with \( g(z_1) = z_2 \) ([8]). In fact, there are only five types of such maps, which correspond to the only five types of finite Kleinian groups acting on \( \hat{\mathbb{C}} \) ([5]). Let \( G \) be such a group. Then the quotient space \( \hat{\mathbb{C}}/G \) of \( \hat{\mathbb{C}} \) under \( G \) is conformally equivalent to \( \hat{\mathbb{C}} \) and hence the projection map from \( \hat{\mathbb{C}} \) to \( \hat{\mathbb{C}}/G = \hat{\mathbb{C}} \) is a regularly ramified rational map, denoted by \( R_G \).

Up to conjugacy by a Möbius transformation, a finite Kleinian group \( G \) is equal to one of the following five groups:

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1. A cyclic group $G_1$ of order $\nu$ generated by $z \mapsto e^{\frac{2\pi i}{\nu}}z$, where $\nu$ is any positive integer;
2. A dihedral group $G_2$ generated by the symmetries of a regular polygon of $\nu$ sides, where $\nu$ is a positive integer $\geq 2$;
3. The tetrahedral group $G_3$ generated by the symmetries of a regular tetrahedron;
4. The octahedral group $G_4$ generated by the symmetries of a regular octahedron or its dual, a cube;
5. The icosahedral group $G_5$ generated by the symmetries of a regular icosahedron or its dual, a regular dodecahedron.

Elements of each group $G_j$ are rotations of finite order. Fixed points of non-identity elements of $G_j$ are critical points of the corresponding projection map $R_{G_j}$. Points on the same orbit under $G_j$ are mapped by $R_{G_j}$ to the same point. Since each group $G_j$ has only two or three distinct orbits of fixed points, the projection map $R_{G_j}$ has two or three critical values. Up to conjugacy by a Möbius transformation, a regularly ramified rational map $f$ is written as $f = A \circ R_{G_j}$, for some Möbius transformation $A$ and some $1 \leq j \leq 5$. Clearly, $A \circ R_{G_j} \circ g = A \circ R_{G_j}$, for any $g \in G_j$, hence we call $f = A \circ R_{G_j}$ a regularly ramified rational map with $G_j$ invariance. It also follows that for every point $q \in \hat{\mathbb{C}}$, all pre-images of $q$ under $f$ have equal indices (meaning that $f$ has same local degrees at all pre-images of $q$). Furthermore, the modular space of $A \circ R_{G_j}$ in terms of dynamics is a two-dimensional space and the space of $A \circ R_{G_j}$, $2 \leq j \leq 5$, is a three-dimensional space. The former has been studied by Milnor in [9], but the latter has not been investigated much yet. Milnor showed in [9] that $A \circ G_1$ cannot
have Herman rings in its Fatou set and its Julia set is either connected or a Cantor set. Some one-parameter and two-parameter families of $A \circ G_2$ have been studied by Devaney and his collaborators in several papers, among which [3] is most related to this paper. Some two-parameter families of $A \circ G_2$ are investigated in [14] and [15]. We are interested in studying classifications of the Julia sets of regularly ramified rational maps $A \circ R_{G_j}$ with $2 \leq j \leq 5$, especially when $3 \leq j \leq 5$.

Julia sets of regularly ramified maps $A \circ R_{G_j}$ with $2 \leq j \leq 5$ are first explored in [6] through computer-generated pictures. Each of those maps $A \circ R_{G_j}$ satisfies the following two assumptions:

1. There exists a critical point fixed by the map, denoted by $p$;
2. There exists one critical value contained in $(A \circ R_{G_j})^{-1}(\{p\}) \setminus \{p\}$, denoted by $q$.

Without loss of generality, one can arrange $p$ at the origin and $q$ at infinity. Such a map $A \circ R_{G_j}$ is called a normalized projection map in [6], which is now called in this paper a normalized regularly ramified rational map and is briefly denoted by $A \circ R_{G_j}$.

The goal of this paper is to classify Julia sets of normalized regularly ramified rational maps $A \circ R_{G_j}$ in some one-parameter families, where $2 \leq j \leq 5$. We establish two classification results; that is, if the order of the fixed critical point $p$ is more than 2, then the classification is a tetrachotomy; if the order of the fixed point $p$ is 2, then the classification is a trichotomy.

The paper is organized as follows:

Section 2 is a warm-up section. One-parameter families of maps of form $A \circ R_{G_2}$ or $A \circ R_{G_2} \circ \phi$ are explored in this section, where $A$ and $\phi$ are two Möbius transformations. They are different from the family $F_\nu^c(z) = z^\nu + c$. But the strategy and techniques to draw classification results on the Julia sets of the maps in these one-parameter families are quite similar to those applied to the family $F_\nu^c$, and in fact one of these families can be embedded into a two-parameter family studied in [14]. So we mostly give statements of results without recapitulating details from the works of [3] and [14] to prove them.

Section 3 is the main work of this paper. We give proofs to classification results of the Julia sets of the maps in two one-parameter families of maps of form $A \circ R_{G_4}$ or $A \circ R_{G_4} \circ \phi$. In Subsection 3.1, we give statements of three main results: nonexistence of Herman rings, a tetrachotomy of the Julia sets of the maps in one family, and a trichotomy of the Julia sets of the maps in another family. In Subsection 3.2, we prove the result on nonexistence of Herman rings for the maps in either of the two families. In Subsection 3.3, we prove the tetrachotomy result. In Subsection 3.4, we prove the trichotomy result.

Section 4 is the last section. The first half considers two one-parameter families of maps of form $A \circ R_{G_5}$ or $A \circ R_{G_5} \circ \phi$ explored in [6] and states, without proofs, three main results: nonexistence of Herman rings, a tetrachotomy of the Julia sets of the maps in one family, and a trichotomy of the Julia sets of the maps in another family. The second half constructs two one-parameter families of maps of form $A \circ R_{G_3}$, or $A \circ R_{G_3} \circ \phi$ satisfying the two assumptions given in this section. Then we finish the section by stating the corresponding results for the maps in these two families.

2. The dihedral case. Let $\nu$ be a positive integer $\geq 2$. The symmetric group of a regular polygon of $\nu$ sides, called a dihedral group and denoted by $G_2$, can be
generated by rotations $z \mapsto e^{\frac{2\pi i}{\nu}} z$ and $z \mapsto \frac{1}{z}$. The orbit of fixed points of group elements of order $\nu$ is given by

$$\text{Orb}_{G_{2}}(0) = \{0, \infty\},$$

one orbit of fixed points of order 2 is given by

$$\text{Orb}_{G_{2}}(1) = \{e^{\frac{2\pi i}{\nu}} : k = 0, 1, \ldots, \nu - 1\},$$

and another orbit of fixed points of order 2 is given by

$$\text{Orb}_{G_{2}}(e^{\pi i}) = \{e^{\pi i} e^{\frac{2\pi ki}{\nu}} : k = 0, 1, \ldots, \nu - 1\}.$$ 

A normalized projection map $A \circ R_{G_{2}}$ is given in [6] by

$$f_{(2,\nu)}(z) = \frac{z\nu}{(z\nu - 1)^2}.$$ 

The critical points of $f_{(2,\nu)}(z)$ are comprised of the points in

$$\text{Orb}_{G_{2}}(0) \cup \text{Orb}_{G_{2}}(1) \cup \text{Orb}_{G_{2}}(e^{\pi i}).$$

Julia sets are explored in [6] for rational maps $A \circ R_{G_{2}}$ in the following one-parameter families

$$F_{c}^{\nu}(z) = z \nu + \frac{c}{z\nu}, \quad (2.1)$$

where $\nu$ is an integer greater than 1 and $\lambda$ is a nonzero complex number. From computer-generated pictures, the Julia sets of rational maps in this family have a similar classification as the rational maps in the following family of singularly perturbed monomials

$$f_{(2,\nu)}^{\lambda}(z) = \frac{\lambda z\nu}{(z\nu - 1)^2},$$

where $\nu$ is an integer $\geq 2$. The family $F_{c}^{\nu}$ has been extensively studied by Professor Robert L. Devaney and his collaborators. The Julia sets of the maps $F_{c}^{\nu}$ have a major difference in classification when $\nu > 2$ and $\nu = 2$ ([2]). That is, when $\nu > 2$, there is a so-called McMullen domain containing the origin in the parameter plane such that for each nonzero parameter $c$ in the domain, the Julia set of the map $F_{c}^{\nu}$ is a Cantor set of disjoint Jordan curves surrounding the origin, which we briefly call a McMullen necklace ([7]); when $\nu = 2$, such a McMullen domain doesn’t exist. Whether $\nu > 2$ or $\nu = 2$, it is proved that the Julia set of $F_{c}^{\nu}$ is connected if the nonzero critical values don’t escape to infinity ([4]). When the nonzero critical values do escape to infinity, the main result of [3] covers the statement that if $\nu > 2$, then the Julia set of $F_{c}^{\nu}$ is a Cantor set, a McMullen necklace or a Sierpinski curve; if $\nu = 2$, then the Julia set of $F_{c}^{\nu}$ is a Cantor set or a Sierpinski curve, where by a Sierpinski curve we mean a planar set homeomorphic to the well-known Sierpinski carpet fractal. Our computer-generated pictures of Julia sets indicate that the family $f_{(2,\nu)}^{\lambda}(z)$ behaves very similarly to $F_{c}^{\nu}(z)$ when $\nu > 2$ and when $\nu = 2$.

It is given in [6] that $f_{(2,\nu)}^{\lambda}(z)$ is conjugated to

$$\tilde{f}_{(2,\nu)}^{\lambda}(z) = z\nu - \frac{2}{\lambda z\nu} + \frac{1}{\lambda z^{\nu - 2}}$$

by the Möbius transformation $z \mapsto \frac{-\sqrt{\lambda}}{z}$. Clearly, $f_{(2,\nu)}^{\lambda}$ and $F_{c}^{\nu}(z)$ are two distinct one-parameter families of singularly perturbed monomials. Therefore, one cannot apply the results of the family $F_{c}^{\nu}(z)$ to draw same results for $f_{(2,\nu)}^{\lambda}$. On the other
hand, \( f^\lambda_{(2, \nu)} \) is a one-parameter family embedded in the following two-parameter family
\[
F_{a, c}^\nu(z) = z^\nu + \frac{c}{z^\nu} + a, 
\]
where \( c \neq 0 \) and \( \nu > 2 \). Classification of the Julia sets of the maps in this two-parameter family is given in [14]. Strategies, frameworks and techniques in [3] and [14] can be applied quite directly to classify the Julia sets of the maps in the family \( f^\lambda_{(2, \nu)} \) and hence in the family \( f^\lambda_{(2, \nu)} \). Without rewriting those frameworks and details, we give the statements of results for the family \( f^\lambda_{(2, \nu)} \). Using the method of [13] (or a simplified one given in [4]) to show the nonexistence of Herman rings for maps in the family \( F^\nu \), we first obtain the following theorem.

**Theorem 2.1.** For each \( \nu \geq 2 \) and each nonzero parameter \( \lambda \), \( f^\lambda_{(2, \nu)} \) has no Herman rings.

Let \( \mathcal{A}(0) \) be the global attracting basin of \( f^\lambda_{(2, \nu)} \) at the origin and let \( B(0) \) be the immediate attracting basin at the origin. Let \( T = (f^\lambda_{(2, \nu)})^{-1}(B(0)) \setminus B(0), \) which is called the trap door of \( f^\lambda_{(2, \nu)} \) (if it is not empty). Furthermore, we let \( v_\nu \) be the value of \( f^\lambda_{(2, \nu)} \) at a point on the orbit \( \text{Orb}_{\mu}(e^{\pi i}) \). In fact, \( v_\nu = -\frac{1}{2} \). We found that if \( \nu > 2 \), there is a tetrachotomy classification of Julia sets and four types of Julia sets are illustrated in Figure 2 for \( \nu = 4 \); if \( \nu = 2 \), there is a trichotomy classification and three types of Julia sets are illustrated in Figure 3.

**Theorem 2.2** (Tetrachotomy, Consequence of [14]). Assume that \( \nu > 2 \) and let \( J(f^\lambda_{(2, \nu)}) \) be the Julia set of \( f^\lambda_{(2, \nu)} \). Then the following tetrachotomy holds:

1. If \( v_\nu \in B(0) \), then \( J(f^\lambda_{(2, \nu)}) \) is a Cantor set.
2. If \( T \) is not empty and \( v_\nu \in T \), then \( J(f^\lambda_{(2, \nu)}) \) is a McMullen necklace.
3. If \( T \) is not empty and \( v_\nu \in (f^\lambda_{(2, \nu)})^{-j}(T) \) for some \( j > 0 \), then \( J(f^\lambda_{(2, \nu)}) \) is a Sierpinski curve.
4. If \( v_\nu \notin \mathcal{A}(0) \), \( J(f^\lambda_{(2, \nu)}) \) is connected.

**Theorem 2.3** (Trichotomy, Consequence of [3] and [14]). Assume that \( \nu = 2 \). Then the following trichotomy holds.

1. If \( v_\nu \in B(0) \), then \( J(f^\lambda_{(2, \nu)}) \) is a Cantor set.
2. If \( v_\nu \in \mathcal{A}(0) \setminus B(0) \), then \( J(f^\lambda_{(2, \nu)}) \) is a Sierpinski curve.
3. If \( v_\nu \notin \mathcal{A}(0) \), \( J(f^\lambda_{(2, \nu)}) \) is connected.

In the next part of this section, by proving the following proposition we show that for each \( \nu \geq 2 \), there exist actual parameter values of \( \lambda \) such that \( J(f^\lambda_{(2, \nu)}) \) is a Cantor set; for each \( \nu > 2 \), there exist actual values of \( \lambda \) such that \( J(f^\lambda_{(2, \nu)}) \) is a McMullen necklace; if \( \nu = 2 \), there is no value of \( \lambda \) such that \( J(f^\lambda_{(2, \nu)}) \) is a McMullen necklace.

**Proposition 2.4.**

1. For each \( \nu \geq 2 \), if \( |\lambda| \) is small enough, then \( v_\nu \in B(0) \) and hence the Julia set of \( f^\lambda_{(2, \nu)} \) is a Cantor set.
2. For each \( \nu > 2 \), if \( |\lambda| \) is large enough, then \( T \) is not empty and \( v_\nu \in T \) and hence the Julia set of \( f^\lambda_{(2, \nu)} \) is a McMullen necklace.
3. For \( \nu = 2 \), if \( T \) is not empty, then \( v_\nu \notin T \).
Figure 2. Four types of Julia sets for maps in the family $f^\lambda_{(2,4)}$.
In (a), a Cantor set with $\lambda = 2$; in (b), a non-escaping case of $v_\lambda$ with $\lambda = 3 + 3i$; in (c), a Sierpinski curve with $\lambda = 5$; and in (d), a McMullen necklace with $\lambda = 13$.

To prove (1) of Proposition 2.4, we conjugate $f^\lambda_{(2,\nu)}$ to $g^\lambda_{(2,\nu)}(z) = \frac{z^{-\nu}}{z^{\nu - 1} - 1}$ by using the map $z \mapsto \frac{1}{z}$. For brevity of notation, we prove a corresponding result for the family $f^\mu_n(z) = \mu(z^{2n-1})^2 z^n$ instead of $g^\lambda_{(2,\nu)}(z)$, where $\mu = \frac{1}{\lambda}$ and $n = \nu$.

Clearly, $f^\mu_n$ is a rational map invariant under pre-composition by the elements of the dihedral group; that is, $f^\mu_n(z) = f^\mu_n(\omega_n z)$ for any $n^{th}$ root $\omega_n$ of unity and $f^\mu_n(z) = f^\mu_n(1/z)$. It is easy to check that $f^\mu_n(z)$ has $2n + 2$ critical points $\{0, \infty, e^{\pi i k/n}, k = 1, \ldots, 2n\}$ and three critical values $0, \infty$ and $-4\mu$, it has a super-attracting fixed point at $\infty$ and $0$ is mapped to $\infty$. Thus, $v = -4\mu$ is the only critical value depending on $\mu$. In details, the critical points $\{e^{2\pi i k/n}, k = 1, \ldots, n\}$ of multiplicity 2 are mapped to 0; the critical point 0 of multiplicity $n$ is mapped to $\infty$; $\infty$ is a superattracting fixed point $f^\mu_n$ of order $n$; the critical points $\{e^{\pi i (2k+1)/2n}, k = 1, \ldots, n\}$ of multiplicity 2 are mapped to $v = -4\mu$.

Let $B(\infty)$ be the immediate attracting basin of $f^\mu_n$ at $\infty$. Now in order to prove (1) of Proposition 2.4, it is equivalent to show $v = -4\mu \in B(\infty)$ if $\mu$ is large enough. It suffices to show that for each $n \geq 2$, $v \in B(\infty)$ if $|\mu| > \frac{\sqrt{3} + 2\sqrt{2}}{4}$.

Lemma 2.5. If $|\mu| > \frac{\sqrt{3} + 2\sqrt{2}}{4}$ and $|z| \geq \sqrt{3 + 2\sqrt{2}}$, then $|f^\mu_n(z)| > \sqrt{3 + 2\sqrt{2}}$. 


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Figure 3. Three types of Julia sets for maps in the family $f^λ_{(2,2)}$.
In (a), a Cantor set with $λ = 1$; in (b), a non-escaping case of $ν_λ$ with $λ = 3 + 5i$; in (c) and (d), Sierpinski curves with $λ = -4$ and $λ = 10$ respectively.

Proof. Assume that $|μ| > \frac{\sqrt{3+2\sqrt{2}}}{4}$. Let $|z| ≥ \sqrt{3+2\sqrt{2}}$ and $z^n = re^{iθ}$. Then

$$z^n + \frac{1}{z^n} - 2 = re^{iθ} + \frac{1}{r}e^{-iθ} - 2 = \left( r \cos(θ) + \frac{1}{r} \cos(θ) - 2 \right) + i \left( r \sin(θ) - \frac{1}{r} \sin(θ) \right).$$

Thus,

$$\left| z^n - 2 + \frac{1}{z^n} \right|^2 = \left( r + \frac{1}{r} \right) \cos(θ) - 2 + \left( r - \frac{1}{r} \right) \sin^2(θ)$$
$$= \left( r + \frac{1}{r} \right)^2 \cos^2(θ) - 4 \left( r + \frac{1}{r} \right) \cos(θ) + 4 + \left( r - \frac{1}{r} \right)^2 \sin^2(θ)$$
$$= \left( r^2 + \frac{1}{r^2} \right) + 2(\sin^2(θ) - \cos^2(θ)) + 4 - 4 \left( r + \frac{1}{r} \right) \cos(θ)$$
$$= \left( r^2 + \frac{1}{r^2} \right) + 2(2\cos^2(θ) - 1) + 4 - 4 \left( r + \frac{1}{r} \right) \cos(θ)$$
$$≥ r^2 + \frac{1}{r^2} + 2 - 4 \left( r + \frac{1}{r} \right)$$
$$= r^2 - 4r - 4 + \frac{1}{r^2} + 2.$$

Let $h(r) = r^2 - 4r - 4 + \frac{1}{r^2} + 2$. Then $h'(r) = 2r - 4 + \frac{4}{r^2} - \frac{2}{r}$. When $r ≥ 3 + 2\sqrt{2}$, $h'(r) > 0$ and hence the minimum of $h(r)$ on $[3 + 2\sqrt{2}, \infty)$ is attained at $r = 3 + 2\sqrt{2}$. 
It follows
\[
|z^n - 2 + \frac{1}{z^n}|^2 \geq r^2 - 4r - \frac{1}{r} + \frac{1}{r^2} + 2
\geq (3 + 2\sqrt{2})^2 + 4(3 + 2\sqrt{2}) - 4(3 - 2\sqrt{2}) + (3 - 2\sqrt{2})^2 + 2 = 16.
\]
Hence,
\[
|f_n^\mu(z)| \geq |\mu|\sqrt{16} = 4|\mu| > \sqrt{3 + 2\sqrt{2}}.
\]

Corollary 2.6. Assume that \( n \geq 2 \). Then for any \( |\mu| > \frac{\sqrt{3 + 2\sqrt{2}}}{4} \),
\[
D = \{ z : |z| \geq \sqrt{3 + 2\sqrt{2}} \} \subset B(\infty) \text{ and hence } \nu = -4\mu \in B(\infty).
\]

Proof. Let \( |\mu| > \frac{\sqrt{3 + 2\sqrt{2}}}{4} \) and \( z \in D \). By the previous lemma, \( f_n^\mu \) maps \( D \) into the interior of \( D \). Using the Schwartz Lemma, we conclude that the iterates of \( z \) under \( f_n^\mu \) converges to the unique fixed point \( \infty \). Hence, \( z \in B(\infty) \). Then \( D \subset B(\infty) \) and \( \nu = -4\mu \in B(\infty) \).

It is clear that Corollary 2.6 implies (1) of Proposition 2.4.

To prove (2) of Proposition 2.4, we use the conjugated version \( \tilde{f}_\lambda^{(2,\nu)} \) of \( f_\lambda^{(2,\nu)} \) given in (2.2). For brevity of notation, we let \( a = \frac{1}{\lambda^2 - 1} \) and consider the following family
\[
\tilde{f}_\nu^{(2,\nu)}(z) = z^{\nu} - 2a + \frac{a^2}{z^{\nu}}.
\]

It is equivalent to show that if \(|a| \neq 0\) is sufficiently small, then \( \tilde{f}_\nu^{(2,\nu)} \) has a non-empty trap door \( T \) containing 0 and its critical value \(-4a \in T\) for each \( \nu > 2 \).

Main ideas to prove these claims are similar to those used to obtain same results for the family (2.1) ([3]). We summarize them as follows. Let \(|a| \neq 0\) be sufficiently small. Then \( \tilde{f}_\nu^{(2,\nu)} \) maps the circle \( \{ z : |z| = \frac{1}{\lambda} \} \) into the disk \( \{ z : |z| < \frac{1}{\lambda} \} \) and the circle \( \{ z : |z| = 2 \} \) outside of the disk \( \{ z : |z| \leq 2 \} \), and it has no critical point in the ring \( R = \{ z : \frac{1}{\lambda} \leq |z| \leq 2 \} \). Therefore, \( \tilde{f}_\nu^{(2,\nu)} \) has no branched point on \( R \) and \( (\tilde{f}_\nu^{(2,\nu)})^{-1}(R) \) is a ring contained in \( R \). It follows that \( \cap_{j=0}^{\infty} (\tilde{f}_\nu^{(2,\nu)})^{-j}(R) \) is a connected set separating \( B(\infty) \) from the origin and hence \( T \) is not empty. Clearly, if \(|a| \neq 0\) is sufficiently small, then \( \tilde{f}_\nu^{(2,\nu)} \) maps \( \{ z : |z| \geq 2 \} \) into itself and hence \( \{ z : |z| \geq 2 \} \subset B(\infty) \). It is also clear that if \( \nu > 2 \), then \( \tilde{f}_\nu^{(2,\nu)}(-4a) \to \infty \) as \( a \to 0 \). This implies that \(-4a \in T\) if \( \nu > 2 \) and \(|a| \neq 0\) is small enough.

Next, we apply Riemann-Hurwitz formula to prove (3) of Proposition 2.4; that is, we show if \( \nu = 2 \), then \(-4a \) cannot fall into the trap door \( T \) even if it exists. Our proof applies the same idea used in ([3]) to classify the Julia sets of the rational maps in the family (2.1). Let us first recall Riemann-Hurwitz formula.

Lemma 2.7 (Riemann-Hurwitz formula [1, §5.4, pp. 85-89]). Let \( f \) be a rational map defined from \( \hat{\mathbb{C}} \) to itself. Assume that
(1) \( V \) is a domain in \( \hat{\mathbb{C}} \) with finitely many boundary components;
(2) \( U \) is a component of \( f^{-1}(V) \); and
(3) there are no critical values of \( f \) on \( \partial V \).
Then there exists an integer $d \geq 1$ such that $f$ is a branched covering map from $U$ onto $V$ with degree $d$ and

$$\chi(U) + \delta_f(U) = d \cdot \chi(V),$$

where $\chi(\cdot)$ denotes the Euler characteristic and $\delta_f(U)$ denotes the total number of the critical points of $f$ in $U$ (counted with multiplicity).

Suppose that $T$ exists and $-4a \in T$. Applying Riemann-Hurwitz formula to the map $\tilde{f}_a^{(2,\nu)} : B(\infty) \to B(\infty)$, we can see $\chi(B(\infty)) = 1$. This means $B(\infty)$ is simply connected. Furthermore, we know $T$ is simply connected. The two critical values $0$ and $-4a$ contained in $T$ have their corresponding critical points on two different orbits of a corresponding dihedral group. Using the fact that the dihedral group is generated by two elements fixing two critical points on different orbits respectively, we can show that $(\tilde{f}_a^{(2,\nu)})^{-1}(T)$ is connected (this method is also used to prove the first half of Lemma 3.16). Furthermore, by applying Riemann-Hurwitz formula again, we know that $(\tilde{f}_a^{(2,\nu)})^{-1}(T)$ is a ring, which separates $B(\infty)$ from $T$. Let $D(0)$ and $D(\infty)$ denote the connected components of $\mathbb{C} \setminus (\tilde{f}_a^{(2,\nu)})^{-1}(T)$ containing $T$ and $B(\infty)$ respectively. Let $A(0) = D(0) \setminus T$ and $A(\infty) = D(\infty) \setminus B(\infty)$, and let $A = \mathbb{C} \setminus (T \cup B(\infty))$. Then $A(0)$ and $A(\infty)$ are two disjoint annuli contained in the annulus $A$ and they share one boundary with $A$ respectively. Therefore,

$$\text{Modulus}(A(0)) + \text{Modulus}(A(\infty)) < \text{Modulus}(A).$$

On the other hand, $\tilde{f}_a^{(2,\nu)} : A(0) \to A$ and $\tilde{f}_a^{(2,\nu)} : A(\infty) \to A$ are regularly covering maps of degree $2$. Hence,

$$\text{Modulus}(A(0)) = \frac{1}{2} \text{Modulus}(A) \quad \text{and} \quad \text{Modulus}(A(\infty)) = \frac{1}{2} \text{Modulus}(A).$$

This is a contradiction to the previous strict inequality. Therefore, if $T$ is not empty, then $-4a \notin T$.

Let $G_2$ be the symmetry group of a regular polygon with $\nu$ sides. Even if $\nu > 2$, one can construct normalized regularly ramified rational maps $A \circ R_{G_2}$ with a superattracting fixed point $p$ of order 2. For example, let $\nu$ be an even positive integer (denoted by $2\nu$ with $\nu > 1$) and take $p = 1$ and $q = -1$. After conjugation by the map $\phi(z) = \frac{z}{z+1}$, one of such maps can be expressed as

$$h_{(2,2\nu)}(z) = \frac{z^2 \prod_{k=1}^{\nu-1}(z - \phi(e^{\frac{2\pi i k}{2\nu}}))^2 \prod_{k=\nu+1}^{2\nu-1}(z - \phi(e^{\frac{2\pi i k}{2\nu}}))^2}{(z^2 - 1)^{2\nu}},$$

for which $0$ is a superattracting fixed point of order 2, $\infty$ is mapped to 0, and the two critical points of order 4 are mapped to $\infty$. Therefore, one can consider the following one-parameter family

$$h_{(2,2\nu)}^\lambda(z) = \lambda h_{(2,2\nu)}(z),$$

where $\lambda$ is a complex parameter. In Figure 4, three types of Julia sets are given for the family

$$h_{(2,4)}^\lambda(z) = \lambda h_{(2,4)}(z) = \lambda \frac{z^2(z^2 + 1)^2}{(z^2 - 1)^4}.$$
Figure 4. Three types of Julia sets for maps in the family \( h_{(2,4)}^\lambda \).
In (a), \( \lambda = 2 \), a Cantor set; in (b), \( \lambda = 3.467 \), an approximation of a non-escaping case of \( v_\lambda \) with \( v_\lambda \) in the Julia set; in (c), \( \lambda = 7 \), a Sierpinski curve; in (d), \( \lambda = -7 \), a non-escaping case of \( v_\lambda \) with \( v_\lambda \) in the Fatou set.

**Theorem 2.8.** For each \( \nu > 1 \) and \( \lambda \neq 0 \), \( h_{(2,2\nu)}^\lambda \) has no Herman rings.

**Theorem 2.9.** Let \( \nu > 1 \). The following trichotomy holds:
1. If \( v_\lambda \in B(0) \), then \( J(h_{(2,2\nu)}^\lambda) \) is a Cantor set.
2. If \( v_\lambda \in A(0) \setminus B(0) \), then \( J(h_{(2,2\nu)}^\lambda) \) is a Sierpinski curve.
3. If \( v_\lambda \notin A(0) \), \( J(h_{(2,2\nu)}^\lambda) \) is connected.

As pointed out in the introduction, this section is a warm-up section. So we claim the main results for the previous three one-parameter families of the maps of form \( A \circ R_{G_2} \) or \( A \circ R_{G_4} \circ \phi \) without providing proofs. Detailed proofs of similar results for the maps of form \( A \circ R_{G_4} \) or \( A \circ R_{G_4} \circ \phi \) are given in the next sections. The proofs of the main results in this section can be constructed in a straightforward way by using the strategies and techniques presented in the next section.

### 3. The octahedral case.

**3.1. Statements of results.** In this section, we prove the classification results on the Julia sets of normalized regularly ramified rational maps \( A \circ R_{G_4} \) in some one-parameter families, that are observed in [6] through computer-generated pictures. We continue to use the notation of \( p \) and \( q \) given in the introduction. By setting the fixed attracting point \( p \) at 0 and the point \( q \) at \( \infty \), these maps \( A \circ R_{G_4} \) form one-parameter families, which can be divided into two groups according to the order of \( p \). If the order of \( p \) is bigger than 2, then the classification of Julia sets is a tetrachotomy; if the order of \( p \) is equal to 2, then the classification is a trichotomy.
For example, if the order of 0 is 4 and the order of the critical points mapped to $\infty$ is 2, then this family is given by

$$f_\lambda(z) = f_{(2,3,4)}^\lambda(z) = \frac{\lambda z^4(z^4 - 1)^4}{(z^4 + 1)^2(z^4 - (1 + \sqrt{2})^4)^2(z^4 - (1 - \sqrt{2})^4)^2},$$

where $\lambda$ is a complex parameter.

Recall that $f_\lambda$ is invariant under pre-composition by any element of the symmetry group $G_4$ of a regular octahedron. Given a fixed point $x$ of some (non-identity) element of $G_4$, the stabilizer $G_4(x)$ is the collection of all elements of $G_4$ fixing $x$. In this case, all elements of $G_4(x)$ share the same rotation axis. Note that the Julia set $J(f_\lambda)$ is symmetric with respect to the axis of any element of $G_4(x)$; if $x$ belongs to a Fatou component $\Omega$ of $f_\lambda$ then $\Omega$ is symmetric with respect to $G_4(x)$.

Now we consider that the octahedron is embedded on the Riemann sphere $\hat{\mathbb{C}}$ with six vertices at $0, \infty, e^{i\pi k/2}$, $k = 0, 1, 2, 3$, and with edges and faces on $\mathbb{C}$. Then the 6 vertices are the critical points of $f_\lambda$ of order 4 which are mapped to 0; the 12 middle points of the edges $e^{i(2k+1)/4}(1 \pm \sqrt{2})e^{i\pi k/2}$, $k = 0, 1, 2, 3$, are the critical points of order 2 which are mapped to $\infty$; the 8 centers of the faces $(1 \pm \sqrt{3})(1 - i)e^{i\pi k/2}$, $k = 0, 1, 2, 3$, are the critical points of order 3 which are mapped to the free critical value $v_\lambda$.

If we denote by $\text{Crit}(f_\lambda)$ the set of critical points of $f_\lambda$, then

$$\text{Crit}(f_\lambda) = \mathbb{C}(2) \cup \mathbb{C}(3) \cup \mathbb{C}(4),$$

where

$$\mathbb{C}(2) = \{e^{i(2k+1)x_1}/4, (1 \pm \sqrt{2})e^{i\pi x_1}/4 : k = 0, 1, 2, 3\},$$

$$\mathbb{C}(3) = \{(1 \pm \sqrt{3})(1 - i)/2e^{i\pi x_1}/4 : k = 0, 1, 2, 3\},$$

$$\mathbb{C}(4) = \{0, \infty, e^{i\pi x_1}/4 : k = 0, 1, 2, 3\} = \{0, \infty, \pm 1, \pm i\}.$$

If $z \in \mathbb{C}(k)$, then the local degree of $f_\lambda$ is $k$, $k = 2, 3, 4$, and

$$f_\lambda(\mathbb{C}(2)) = \{\infty\}, f_\lambda(\mathbb{C}(3)) = \{v_\lambda\}, f_\lambda(\mathbb{C}(4)) = \{0\},$$

where $v_\lambda = -\lambda/108$. The critical values of $f_\lambda$ are 0, $\infty$ and $v_\lambda = -\lambda/108$, and 0 is a super-attracting fixed point. Define the attracting basin of $f_\lambda$ at 0 as

$$\mathcal{A}(0) = \{z \in \hat{\mathbb{C}} : f_\lambda^n(z) \to 0 \text{ as } n \to \infty\}.$$

Let $B(0)$ and $B(\infty)$ be the connected Fatou components of $f_\lambda$ containing 0 and $\infty$ respectively. In fact, $B(0)$ is called the immediate attracting basin of $f_\lambda$ at 0. It is possible that $B(0) = B(\infty)$, for example, if $v_\lambda \in B(0)$ then $B(0) = B(\infty) = \mathcal{A}(0)$.

But if $v_\lambda \notin B(0)$, then $B(\infty) \neq B(0)$ and both of them are simply connected, $\mathcal{A}(0)$ consists of infinitely many connected components, and $B(\infty)$ is the connected component of $f_\lambda^{-1}(B(0)) \setminus B(0)$ containing $\infty$ (which is called the trap door $T$ of $f_\lambda$). See later lemmas. For each point $w$ on the Riemann sphere, we denote by $B(w)$ the Fatou component of $f_\lambda$ containing $w$ if it exists.

In this section, we first prove the following two main theorems for $f_\lambda$.

**Theorem 3.1.** Each rational map $f_\lambda$ has no Herman rings.

**Theorem 3.2.** For the maps in the family $f_\lambda$ with $\lambda \neq 0$, the following tetrachotomy holds:

1. If $v_\lambda \in B(0)$ (in this case, $B(0) = B(\infty) = \mathcal{A}(0)$), then $J(f_\lambda)$ is a Cantor set.
Figure 5. Four types of Julia sets for maps in the family $f^\lambda_{(2,3,4)}$.

In (a), $\lambda = 20$, a Cantor set; in (b), $\lambda = 40 + 40i$, a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Fatou set; in (c), $\lambda = 500$, a Sierpinski curve; in (d) $\lambda = 1125$, an approximation of a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Julia set; in (e), $\lambda = 1500$, an exploded McMullen necklace; (f) is a zoom of (e) in the middle.

2. If $v_\lambda \in A(0) \setminus (B(0) \cup B(\infty))$, then the Julia set $J(f_\lambda)$ is a Sierpinski curve.
3. If $v_\lambda \in B(\infty) \setminus B(0)$, then (i) $J(f_\lambda)$ is not connected; (ii) each connected component of $f_\lambda^{-m}(B(i^k))$ is simply connected for each $m \geq 0$ and $k = 1, 2, 3, 4$; (iii) each connected component of $f_\lambda^{-m}(B(\infty))$ is multiple connected with connectivity number equal to $4^m + 2$ for each $m \geq 1$; (iv) every simply connected
Figure 6. Three types of Julia sets for maps in the family $h^\lambda_{(2,3,4)}$ (Note that $\infty$ is fixed). In (a), $\lambda = 1000$, a Cantor set; in (b), $\lambda = 890.5$, an approximation of a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Julia set; in (c), $\lambda = 380i$, a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Fatou set; in (d), $\lambda = 290$, a Sierpinski curve.

Fatou component $B$ of $f_\lambda$ is surrounded by a Cantor set of Jordan curve components of $J(f_\lambda)$. We call such a Julia set an exploded McMullen necklace.

4. If $v_\lambda \notin \mathcal{A}(0)$, then the Julia set $J(f_\lambda)$ is connected.

Remark 3.3. Patterns of connectivity numbers of Fatou components of rational maps are investigated in [1] and [11]. We have not found the pattern of connectivity numbers of the Fatou components of $f_\lambda = f^\lambda_{(2,3,4)}$ in the literature.

In the second half of this section, we consider the case when the order of the fixed critical point $p$ is 2. One example of such a one-parameter family was given in [6], for which $p$ is set at $\infty$. Precisely, it is defined there as follows. Let

$$\phi(z) = \frac{(1 + \sqrt{2})z - 1}{z + (1 + \sqrt{2})}.$$

Define

$$h^\lambda_{(2,3,4)}(z) = \lambda f^1_{(2,3,4)}(\phi(z)).$$

Thus $h^\lambda_{(2,3,4)}$ is a one-parameter family of normalized regularly ramified rational maps of form $A \circ R_{G_4} \circ \phi$, for which 0 and $\infty$ are critical points of order 2 and $\infty$ is fixed. We briefly denote by

$$h_\lambda = h^\lambda_{(2,3,4)},$$

and let $B(\infty)$ denote the immediate attracting basin of $h_\lambda$ at $\infty$ and $T$ the connected component of $h_\lambda^{-1}(B(\infty)) \setminus B(\infty)$ containing 0 (called the trap door $T$ of $h_\lambda$ if it is not empty). Furthermore,

$$\text{Crit}(h_\lambda) = \phi^{-1}(\mathcal{C}(2)) \cup \phi^{-1}(\mathcal{C}(3)) \cup \phi^{-1}(\mathcal{C}(4)).$$
and $h_{\lambda}$ has the same three critical values as $f_{\lambda}$. Now we state the classification theorem for the family $h_{\lambda}$. The main difference of this case is that the free critical value $v_3$ can never enter the trap door $T$ if it is not empty. Thus the Julia set $J(f_{\lambda})$ cannot be an exploded McMullen necklace. In the last part of this section, we prove the following two theorems for $h_{\lambda}$.

**Theorem 3.4.** Each rational map $h_{\lambda}$ has no Herman rings.

**Theorem 3.5.** For the maps in the family $h_{\lambda}$ with $\lambda \neq 0$, the following trichotomy holds:

1. If $v_\lambda \in B(\infty)$ (in this case, $B(0) = A(0)$), then $J(h_\lambda)$ is a Cantor set.
2. If $v_\lambda \in A(\infty) \setminus B(\infty)$, then the Julia set $J(h_\lambda)$ is a Sierpinski curve.
3. If $v_\lambda \notin A(\infty)$, then the Julia set $J(h_\lambda)$ is connected.

### 3.2. Proofs of Theorems 3.1 and 3.4.

A uniform proof is presented in this subsection to show nonexistence of Herman rings for $f_{\lambda}$ and $h_{\lambda}$. For this purpose, we use $f$ to stand for either $f_{\lambda}$ or $h_{\lambda}$. Let $J$ denote the Julia set of $f$, and $B(0)$ and $B(\infty)$ denote the Fatou components of $f$ containing $0$ and $\infty$ respectively. We like to point out that the proof is not affected by whether or not $B(0) = B(\infty)$.

Before we present the proof, let us first have an agreement on notation concerning the finite Kleinian group related to $f_{\lambda}$ and $h_{\lambda}$. Let $G_{(2,3,4)}$ be the group of the symmetries of the regular octahedron with vertices at the points in $C(4)$ and $G'_{(2,3,4)} = \phi^{-1} \circ G_{(2,3,4)} \circ \phi$. Through the rest of this section, we will use $G_{(2,3,4)}$ to denote either $G_{(2,3,4)}$ or $G'_{(2,3,4)}$; that is, if the involved map is $f_{\lambda}$, then $G_{(2,3,4)}$ is $G_{(2,3,4)}$; but if the involved map is $h_{\lambda}$, then $G_{(2,3,4)}$ is $G'_{(2,3,4)}$. Similarly, for $n = 2, 3$ or $4$, we use $C(n)$ to denote either $C(n)$ or $\phi^{-1}(C(n))$; that is, if the involved map is $f_{\lambda}$, then $C(n)$ is $C(n)$; but if the involved map is $h_{\lambda}$, then $C(n)$ is $\phi^{-1}(C(n))$.

**Proof.** Suppose that $f$ has a cycle of Herman rings $\{R_0, R_1, \ldots, R_{p-1}\}$. Then $f^p$ is conjugate to the irrational rotation $z \mapsto \lambda z$ on $R_0$, where $\lambda = \exp(2\pi i \alpha)$ and $\alpha$ is an irrational number. For each $0 \leq k \leq p - 1$, $f : R_k \to R_{k+1}$ is a conformal mapping. Take a Jordan curve $\gamma$ in $R_0$ invariant under $f^p$ which separates two connected components of $\partial R_0$. Let $U$ be the bounded connected component of $C \setminus \gamma$. Then $U \cap J \neq \emptyset$. Since $\gamma_m \equiv f^m(\gamma) \subset R_k$, where $k = m \mod p$, and $R_k$ is disjoint from $B(0)$ and $B(\infty)$ for all $k$, we have $\{f^m(\gamma)\}_{m=0}^{\infty}$ is uniformly bounded. Let $U_k$ denote the bounded component of $C \setminus \gamma_k$ for $k = 0, 1, \ldots, p - 1$ and $U_p = U_0$.

We first show that $U_k$ cannot contain any pole of $f$. Note that $C(2)$ is the set of all poles of $f$. So we show $U_k \cap C(2) = \emptyset$ for each $k = 0, 1, \ldots, p - 1$. On the contrary, if this is not true, then some $U_k$ contains at least an element $y \in C(2)$. Note that $f$ injectively maps the boundary curve $\gamma_k$ of $U_k$ onto the curve $\gamma_{k+1}$, and the winding number of $\gamma_{k+1}$ with respect to the origin is 0, 1 or $-1$. Let $N(f, U_k)$ denote the number of zeros of $f$ in $U_k$ and $P(f, U_k)$ denote the number of poles of $f$ in $U_k$. By the argument principle, we obtain

$$N(f, U_k) - P(f, U_k) = 1, 0 \text{ or } -1.$$ 

Clearly, $C(4)$ is the set of all zeros of $f$. Set $k_j = \sharp(U_k \cap C(j))$ for $j = 2, 4$. We obtain that $N(f, U_k) = 4k_4, P(f, U_k) = 2k_2$ and

$$4k_4 - 2k_2 = 1, 0 \text{ or } -1.$$  \hspace{1cm} (3.1)

The equation (3.1) has an integer solution if and only if $4k_4 - 2k_2 = 0$. Thus, $k_2 = 2k_4$. Let $V_k = U_k \setminus R_k$. Since $f : R_k \to R_{k+1}$ is conformal, all zeros and poles
in $U_k$ belong to $V_k$. By the assumption $k_2 \neq 0$, we know $k_4 \neq 0$ and $U_k$ contains at least one point $x \in \mathbb{C}(4)$ not equal to $\infty$, that is, $k_4 \geq 1$. Next, we show $k_4$ has to be equal to 6 by eliminating other choices of $k_4$, for which we use the following observation.

**Observation.** Let $g \in G_{(2,3,4)}$. If $\gamma_k$ separates the fixed points of $g$, then $g(\gamma_k) = \gamma_k$ and hence $g(U_k) = U_k$.

Now let $g$ be an element of order 4 with one fixed point at $x$. If $k_4 = 1$, then the other fixed point of $g$ is not in $U_k$, and hence $g(U_k) = U_k$ by the above observation. Then the orbit of $y$ under $g$ belongs to $U_k$. It follows $k_2 \geq 4$ and $k_4 \geq 2$. Let $x_1$ and $x_2$ denote two points in $\mathbb{C}(4) \cap U_k$. There are two cases to consider depending on whether or not $x_1$ and $x_2$ are fixed points of the same element $g$ of $G_{(2,3,4)}$.

**Case 1.** Assume $x_1$ and $x_2$ are fixed points of two different elements of $G_{(2,3,4)}$. Let $g$ be an element of $G_{(2,3,4)}$ of order 4 with a fixed point at $x_1$. Using the observation, we conclude the orbit of $x_2$ under $g$ is contained in $U_k$. Hence, $k_4 \geq 5$. If $k_4 = 5$, then $k_2 = 10$ and there exits $g \in G_{(2,3,4)}$ such that only one fixed point of $g$ belongs to $U_k$. By the observation, the orbits of points of $U_k \cap \mathbb{C}(2)$ are contained in $U_k$ and hence $k_2$ is divisible by 4. Thus, $k_2 = 12$ and hence $k_4 = 6$.

**Case 2.** Assume $x_1$ and $x_2$ are fixed points of the same element $g$ of $G_{(2,3,4)}$ of order 4. Note that $U_k$ is a simply connected domain and both $g^2(U_k)$ and $U_k$ contain $x_1$ and $x_2$. Hence, $g^2(U_k) \cup U_k$ is a connected domain and it is also invariant under $g^2$. Then this union is either a connected domain with connectivity number at least 2 or it is the whole Riemann sphere. In the former situation, the boundaries of $g^2(U_k)$ and $U_k$ intersect and the intersecting points are critical points of $f$. So this situation cannot happen since $\gamma_k \subset R_k$ and $f$ is univalent on $R_k$. Therefore, the union $g^2(U_k) \cup U_k$ is the Riemann sphere. Since $U_k$ and $g^2(U_k)$ contain the same number of the critical points in $\mathbb{C}(4)$, $U_k$ has to contain at least two more points of $\mathbb{C}(4)$ besides $x_1$ and $x_2$, which are denoted by $x_3$ and $x_4$. Furthermore, we show it is impossible to have the situation in which $k_4 = 4$ and $x_3$ and $x_4$ are the fixed points of the same element $g'$ of $G_{(2,3,4)}$ of order 4. Otherwise, the complement $W_k$ of $U_k$ contains exactly two points of $U_k \cap \mathbb{C}(4)$ that are the fixed points of the same element $g''$ of $G_{(2,3,4)}$ of order 4, denoted by $x_5$ and $x_6$. Using the above argument, we conclude that $W_k$ contains two more elements of $U_k \cap \mathbb{C}(4)$ besides $x_5$ and $x_6$. This is impossible. Hence, either $k_4 \geq 5$ or $k_4 = 4$ with $x_3$ and $x_4$ being fixed points of two different elements of $G_{(2,3,4)}$ of order 4. If $k_4 \geq 5$, then we can show $k_4 = 6$ by using the same argument at the end of the previous paragraph. If the latter situation happens, we can conclude $k_4 = 6$ by applying the observation to the elements of order 4 with one fixed point at $x_3$ and $x_4$ respectively.

In summary, we have shown $k_4 = 6$ and then $k_2 = 12$, which is impossible since $\infty \notin U_k$. This means $U_k$ contains no pole of $f$ for all $k = 0, 1, \ldots, p - 1$. Then $f : U_k \to U_{k+1}$ is holomorphic for $k = 0, 1, \ldots, p - 1$. Since $f$ maps $\gamma_k$ onto $\gamma_{k+1}$ injectively, $f : U_k \to U_{k+1}$ is in fact conformal by using the argument principle. Thus, $\{f^n\}_{n=1}^{\infty}$ is a normal family on $U_0$, which contradicts $U_0 \cap J \neq \emptyset$. Therefore, we conclude that $f$ has no Herman rings in its Fatou set. \hfill \Box

### 3.3. Proof of Theorem 3.2

This is a long subsection. We first prepare some background and lemmas before giving a proof to Theorem 3.2.

Let $B(0)$ be the immediate attracting basin of $f_\lambda$ at the origin. We use $M$ to denote the connected component of $\hat{\mathbb{C}} \setminus B(0)$ containing $\infty$ if exists. In the following,
we first mention a symmetric property of $M$. This set is studied in Proposition 3.18, which contains some main ingredients to prove Theorem 3.2.

Let $U$ be a subset of $\hat{\mathbb{C}}$ and $\lambda \in \mathbb{C}$. We denote by $\lambda U := \{\lambda z : z \in U\}$ and by $\partial U$ the boundary of $U$, and denote by $|\lambda(U)|$ the cardinality of $U$. Recall that $f_\lambda$ is invariant under precomposition by any element $g$ of $G_{(2, 3, 4)}$. In particular, for any $\omega \in \{i^k : k = 1, 2, 3, 4\}$, the map $z \mapsto \omega z$ belongs to $G_{(2, 3, 4)}$; the map $h : z \mapsto \frac{1}{z}$ also belongs to $G_{(2, 3, 4)}$. Therefore, the following lemma holds.

Lemma 3.6. (1) For any $z \in \hat{\mathbb{C}}$ and $g \in G_{(2, 3, 4)}$, $f_\lambda(g(z)) = f_\lambda(z)$. In particular, $f_\lambda(\omega z) = f_\lambda(z)$ and $f_\lambda(h(z)) = f_\lambda(z)$ for any $\omega \in \{i^k : k = 1, 2, 3, 4\}$ and $h(z) = \frac{1}{z}$.

(2) For any Fatou component $U$, both $\omega U$ and $h(U)$ are Fatou components. In particular, $\omega B(0) = B(0)$, $\omega B(\infty) = B(\infty)$, $\omega M = M$, $h(B(0)) = B(\infty)$, and $h(B(\infty)) = B(0)$.

(3) Let $U$ be a Fatou component of $f_\lambda$. Then either $iU = U$ (for which we say that $U$ has a 4-order rotation symmetry) or $U$, $iU$, $i^2U$ and $i^3U$ are pairwise disjoint.

(4) For any $z \in \hat{\mathbb{C}}$ and $p_1, p_2 \in f_\lambda^{-1}(z)$, there is an element $g \in G_{(2, 3, 4)}$ such that $g(p_1) = p_2$.

The following lemma is given in [15].

Lemma 3.7 ([15, Lemma 2.9]). If a rational function $f$ has no Herman rings and each Fatou component contains at most one critical value, then the Julia set of $f$ is connected.

Obviously, if a simply connected domain $U \subset \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ does not contain any critical value, then $f_\lambda^{-1}(U)$ consists of exactly 24 connected components and each of them is simply connected. For the case that $U$ contains the critical value $v_\lambda$, we have the following lemma.

Lemma 3.8. If $U \subset \hat{\mathbb{C}} \setminus \{0, \infty\}$ is a simply connected domain and $v_\lambda \in U$, then $f_\lambda^{-1}(U)$ consists of 8 simply connected components.

Proof. Assume that $U \subset \hat{\mathbb{C}} \setminus \{0, \infty\}$ is a simply connected domain containing $v_\lambda$. Let $V$ be a connected component of $f_\lambda^{-1}(U)$. Since $f_\lambda^{-1}(v_\lambda) = \mathbb{C}(3)$ and the local degree of $f_\lambda$ at every point $c \in \mathbb{C}(3)$ is 3, we obtain that $f_\lambda$ is a branched covering map from $V$ onto $U$ with degree $\deg(f_\lambda|V) = 3k_3$, where $k_3 = \sharp(\mathbb{C}(3) \cap V)$. By Lemma 2.7,

$$\chi(V) + 2k_3 = 3k_3\chi(U).$$

Then $k_3 = \chi(V) \leq 2$. Note that $0 \notin U$ implies $\infty \notin V$. Hence $V$ is a proper sub-domain of $\hat{\mathbb{C}}$, which and $\chi(V) < 2$. It follows that $\chi(V) = 1$ and $k_3 = 1$. This means $V$ is simply connected and contains only one point of $\mathbb{C}(3)$. Therefore, $f_\lambda^{-1}(U)$ consists of 8 simply connected components.

Lemma 3.9. If $v_\lambda \in B(0)$, then $B(0)$ is completely invariant under $f_\lambda$ and $J(f_\lambda) = \partial B(0)$.

Proof. We first see that $f_\lambda$ maps $B(0)$ onto itself. Set $k_3 = \sharp(B(0) \cap \mathbb{C}(3))$ and $k_4 = \sharp(B(0) \cap \mathbb{C}(4))$. Since $v_\lambda \in B(0)$, it follows that $3k_3 = 4k_4$. Then $k_3 = 4$ and $k_4 = 3$, or $k_3 = 8$ and $k_4 = 6$. By the previous lemma 3.6, $B(0)$ has a 4-order rotation symmetry. Hence $0 \in B(0)$ and $k_4 \geq 3$ imply $k_4 \geq 4$. So $k_4 = 6$ and $k_3 = 8$. Thus, $\mathbb{C}(4) \cup \mathbb{C}(3) \subset B(0)$. Since $f_\lambda^{-1}(\{0\}) = \mathbb{C}(3) \subset B(0)$, it follows that $B(0)$ is completely invariant under $f_\lambda$. Therefore, $J(f_\lambda) = \partial B(0)$. 

\[\square\]
Lemma 3.10. If \( v_\lambda \notin B(0) \), then \( B(0) \neq B(\infty) \), and \( B(0) \) and \( B(\infty) \) are simply connected.

Proof. Suppose \( B(0) = B(\infty) \); that is, \( \infty \in B(0) \). Since 0 is a super-attracting fixed point, we can choose a small simply connected neighborhood \( \Omega_0 \) of 0 such that \( \infty \notin \overline{\Omega_0} \), \( f_\lambda(\overline{\Omega_0}) \subset \Omega_0 \) and \( \partial \Omega_0 \) is a Jordan curve. For \( m \geq 0 \), let \( \Omega_m \) be the connected component of \( f_\lambda^{-m}(\Omega_0) \) containing \( \Omega_0 \). Then we have \( \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m \subset \cdots \) and \( B(0) = \bigcup_{m \geq 0} \Omega_m \). Since \( \infty \in B(0) \), there must exist \( m_0 \geq 1 \) such that \( \infty \in \Omega_{m_0} \setminus \overline{\Omega}_{m_0-1} \). By Lemma 2.7, it follows that \( \Omega_{m_0-1} \) is simply connected. Now, we consider the branching covering map \( f_\lambda : \Omega_{m_0} \to \Omega_{m_0-1} \). Set \( k_4 = \#(\Omega_{m_0} \cap C(4)) \geq 2 \). Then by Lemma 2.7, we obtain

\[
\chi(\Omega_{m_0}) + 3k_4 = 4k_4.
\]

Then \( \chi(\Omega_{m_0}) = k_4 \geq 2 \), which is impossible since \( \chi(\Omega_{m_0}) \) is at most equal to 1. Therefore, \( \infty \notin B(0) \). It follows that each \( \Omega_m \) is simply connected and hence \( B(0) = \bigcup_{m \geq 0} \Omega_m \) is simply connected. Using Lemma 2.7 again, we obtain that \( B(\infty) \) is simply connected too.

Now we start to prove Theorem 3.2. The proof is quite long since there are four cases to consider. Before doing that, let us recall one more known result.

Lemma 3.11 ([1, Theorem 9.8.1]). Let \( f \) be a rational map with degree at least two. If all of the critical points of \( f \) lie in the immediate attracting basin of a (super)attracting fixed point of \( f \), then the Julia set of \( f \) is a Cantor set.

By Lemma 3.9, if \( v_\lambda \in B(0) \), then \( B(0) \) is completely invariant under \( f_\lambda \). Hence all critical points of \( f_\lambda \) are contained in \( B(0) \). It follows from Lemma 3.11 that \( J(f_\lambda) \) is a Cantor set.

Proposition 3.12. If \( v_\lambda \in B(0) \), then the Julia set \( J(f_\lambda) \) is a Cantor set.

In the second part of this subsection, we prove that if \( v_\lambda \in A(0) \setminus (B(0) \cup B(\infty)) \), then the Julia set is a Sierpinski curve. Recall that by a Sierpinski curve we mean a planar set homeomorphic to the well-known Sierpinski carpet fractal. In [12], Whyburn given a topological characterization of the set.

Theorem 3.13 ([12, Theorem 3]). Any non-empty planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpinski carpet.

The previous characterization of a Sierpinski curve is used in [3] to prove Julia sets to be such curves for maps in the family \( f_\lambda(z) = z^n + \lambda/z^m \). We use it in this paper too. In the following, we first recall a result on local connectivity of Julia sets. By a hyperbolic rational map we mean a rational map with every critical point approaching an attracting periodic cycle under iteration.

Lemma 3.14 ([8]). (1) If the Julia set of a hyperbolic rational map is connected, then it is locally connected.

(2) If \( U \) is a simply connected Fatou component of a hyperbolic rational map, then the boundary \( \partial U \) is locally connected.

Lemma 3.15. If \( f_\lambda \) is hyperbolic and \( v_\lambda \notin B(0) \), then \( \partial B(\infty) \cap \partial B(0) = \emptyset \). Moreover, the boundaries of components of \( A(0) = \bigcup_{m=0}^\infty f_\lambda^{-m}(B(0)) \) are pairwise disjoint. In particular, if \( v_\lambda \in A(0) \setminus B(0) \), then the same conclusions hold.
Proof. From Lemma 3.10, \( B(0) \neq B(\infty) \) and both of them are simply connected. By the Böttcher theorem, there exists a unique conformal map 
\[
\phi : B(0) \to \Delta
\]
such that 
\[
\phi \circ f_\lambda(z) = (\phi(z))^4,
\]
where \( \Delta = \{z : |z| < 1\} \). For a given angle \( \theta \in [0, 1) \), by definition, the internal ray in \( B(0) \) with angle \( \theta \) is \( \gamma(\theta) = \{z \in B(0) : \phi(z) = r e^{\theta} : r \in (0, 1)\} \). Obviously, \( f_\lambda(\gamma(\theta)) = \gamma(4\theta) \). If \( \gamma(\theta) \to z \in \partial B(0) \) as \( r \to 1 \), we say that the internal ray \( \gamma(\theta) \) lands at \( z \), and \( z \) is a landing point of \( \gamma(\theta) \). Since \( f_\lambda \) is hyperbolic, \( \partial B(0) \) is locally connected by Lemma 3.10 and Lemma 3.14. (2). From the Carathéodory theorem, all internal rays are landing and every point on \( \partial B(0) \) is a landing point of an internal ray.

Let \( D \) be a component of \( f_\lambda^{-m}(B(0)) \) and \( \mu \) be a component of \( f_\lambda^{-m}(\gamma(\theta)) \) in \( D \). If \( \gamma(\theta) \) lands at \( z \in \partial B(0) \), then \( \mu \) lands on \( \partial D \), which may land at a point or several points in \( f_\lambda^{-m}(z) \).

Suppose that \( \partial B(0) \cap \partial B(\infty) \neq \emptyset \), say \( z \in \partial B(0) \cap \partial B(\infty) \). Then there is an internal ray \( \gamma(\theta) \) in \( B(0) \) and a component \( \mu \) of \( f_\lambda^{-1}(\gamma(\theta)) \) in \( B(\infty) \) which both land at \( z \) and mapped by \( f_\lambda \) to the external ray \( \gamma(4\theta) \) landing at \( f_\lambda(z) \in \partial B(0) \). This implies that \( z \) is a critical point, which contradicts the assumption that \( f_\lambda \) is hyperbolic. So \( \partial B(0) \) and \( \partial B(\infty) \) are disjoint. By the same argument, we conclude that the boundaries of all components of \( f_\lambda^{-1}(B(0)) \) are pairwise disjoint.

Now suppose that there are two components \( D_1 \) and \( D_2 \) of \( A(0) \) such that \( \partial D_1 \cap \partial D_2 \neq \emptyset \). Then there are smallest integers \( m, k \geq 0 \) such that \( f_\lambda^{-m}(D_1) = B(0) \) and \( f_\lambda^{-k}(D_2) = B(0) \) respectively. If \( k = m \), then there are internal rays \( \mu_1 \) and \( \mu_2 \) in \( D_1 \) and \( D_2 \) respectively such that both of them land at a point \( z \in \partial D_1 \cap \partial D_2 \) and \( \gamma = f_\lambda^{-m}(\mu_1) = f_\lambda^{-m}(\mu_2) \) is an internal ray in \( B(0) \) landing at \( f_\lambda^{-m}(z) \). Hence, \( z \) is a critical point of \( f_\lambda^{-m} \). This is a contradiction. Now we assume \( m \neq k \), say \( m < k \). Let \( f_\lambda^{-k}(D_2) = B^* \). Then \( f_\lambda^{-k}(D_1) = B(0) \) and \( B^* \) is a connected component of \( f_\lambda^{-1}(B(0)) \setminus B(0) \). It also follows that \( \partial B(0) \cap \partial B^* \neq \emptyset \). This is a contradiction. So we have proved \( \partial D_1 \cap \partial D_2 = \emptyset \). Therefore, the boundaries of the components of \( A(0) = \bigcup_{m=0}^{\infty} f_\lambda^{-m}(B(0)) \) are pairwise disjoint.

We have proved in Lemma 3.10 that if \( v_\lambda \notin B(0) \), then both \( B(0) \) and \( B(\infty) \) are simply connected. It follows that if \( v_\lambda \in A(0) \setminus (B(0) \cup B(\infty)) \), then each component of \( f_\lambda^{-m}(B(0)) \) is simply connected with locally connected boundary for each \( m \geq 0 \). When \( v_\lambda \in B(\infty) \setminus B(0) \), \( f_\lambda^{-1}(B(\infty)) \) is connected and multiply connected with locally connected boundary components. There are two different ways to see why \( f_\lambda^{-1}(B(\infty)) \) is connected. We describe one and present another in the proof. Connect the two critical values \( \infty \) and \( v_\lambda \) by a curve \( \Gamma \) contained in \( B(\infty) \). Up to pre-composition and post-composition by Möbius transformations, our rational map is the projection map from \( \hat{\mathbb{C}} \) to \( \hat{\mathbb{C}}/G_1 = \hat{\mathbb{C}} \). Thus, the preimage \( f_\lambda^{-1}(\Gamma) \) is a connected skeleton through all critical points in \( \mathbb{C}(2) \cup \mathbb{C}(3) \). Let \( U \) be the connected component \( U \) of \( f_\lambda^{-1}(B(\infty)) \) containing \( f_\lambda^{-1}(\Gamma) \). Then \( f_\lambda : U \to B(\infty) \) is a holomorphic map of degree 24. This implies \( f_\lambda^{-1}(B(\infty)) = U \), which means \( f_\lambda^{-1}(B(\infty)) \) is connected. See an illustration of this domain in Figure 7.

Lemma 3.16. If \( v_\lambda \in B(\infty) \setminus B(0) \), then \( f_\lambda^{-1}(B(\infty)) \) is connected and \( \partial f_\lambda^{-1}(B(\infty)) \) has 6 connected components. Furthermore for each \( n \geq 1 \), every connected component of \( \partial f_\lambda^{-n}(B(\infty)) \) is locally connected.
Proof. Let $U$ be a connected component of $f_\lambda^{-1}(B(\infty))$. Since $v_\lambda \in B(\infty)$, there is a point $z_1$ of $C(3)$ and a point $z_2$ of $C(2)$ contained in $U$. Let $g_1$ and $g_2$ be non-identity elements of $G_4$ with fixed point at $z_1$ and $z_2$ respectively. Then $G_4$ is generated by $g_1$ and $g_2$. For any other point $z'_1$ on the orbit of $z_1$ under the group $G_4$, there is a finite product $g_{k_1}g_{k_2}\cdots g_{k_j}$ of $g_1$ and $g_2$ such that $g_{k_1}g_{k_2}\cdots g_{k_j}(z_1) = z'_1$. Clearly, $U$ is invariant under $g_1$ and $g_2$; that is, $g_1(U) = U$ and $g_2(U) = U$. This implies that $g_{k_1}g_{k_2}\cdots g_{k_j}(U) = U$ and hence $z'_1 \in U$. Thus, $C(3) \subset U$. Similarly, we can show $C(2) \subset U$. Now it is clear that $f_\lambda : U \to B(\infty)$ is a holomorphic map of degree 24. Therefore, $f_\lambda^{-1}(B(\infty))$ has only one component, which means it is connected. Applying Riemann-Hurwitz formula, we obtain that $\partial f_\lambda^{-1}(B(\infty))$ has 6 connected components, which means its connectivity number is 6.

A rational map $f$ is said to be subhyperbolic if every critical point in the Julia set is preperiodic and every critical point in the Fatou set is attracted to an attracting cycle. Morosawa established in [10] a sufficient condition for boundary of a Fatou component to be a Jordan curve.

**Lemma 3.17** (Lemma 6, [10]). Let $R$ be a subhyperbolic rational map and $U$ a forward invariant (attracting) Fatou component of $R$. If there exists a complementary component $W$ of $U$ and a Fatou component $D$ such that $D \cup R^{-1}(D) \subset W$, then the boundary of $U$ is a Jordan curve.

We recall that $M$ is the connected component of $\hat{\mathbb{C}} \setminus \overline{B(0)}$ containing $\infty$ and $B(z)$ is the Fatou component containing a point $z$. 

**Figure 7.** Here $B = B(0)$ and $T = B(\infty)$; the shadowed domain is an illustration for the domain $f_\lambda^{-1}(B(\infty))$ proved in Lemma 3.16; $A_{in}$ and $A_{out}$ stand for the two annuli used in the proof of Proposition 3.20.
Proposition 3.18. If $v_\lambda \in \mathcal{A}(0) \setminus B(0)$, then the following properties hold:

1. $v_\lambda \in M$;
2. $f_\lambda^{-1}(B(v_\lambda)) \cup B(v_\lambda) \subset M$;
3. $\partial B(0)$ is a Jordan curve and hence $M = \hat{C} \setminus \overline{B(0)}$;
4. the boundary of every connected component of $f_\lambda^{-n}(B(0))$ is a Jordan curve for each $n \geq 0$.

Proof. (1) Suppose that $v_\lambda \notin M$. Then there is a simply connected component $U$ of $\hat{C} \setminus \overline{B(0)}$ not equal to $M$ and containing $v_\lambda$. By Lemma 3.8, $f_\lambda^{-1}(U)$ has 8 simply connected components and each of them contains exactly one critical point in $C(3)$. Since there is no critical value on the boundary of $U$, the closures of these 8 components are pairwise disjoint. Let $V$ be such a component. Then $i^k V$ is a component of $f_\lambda^{-1}(U)$, where $k = 1, 2, 3$. Since $\partial U \subset \partial B(0)$ and $f_\lambda^{-1}(\partial B(0)) = \partial B(0) \cup \partial B(\infty) \cup \bigcup_{k=1}^{3} \partial B(i^k)$, we obtain $\partial V \subset \partial B(0) \cup \partial B(\infty) \cup \bigcup_{k=0}^{3} \partial B(i^k)$. Clearly, $\partial V$ is connected since $V$ is simply connected. Furthermore, boundaries of connected components of $f_\lambda^{-1}(B(0))$ are pairwise disjoint. Therefore, $\partial V$ is contained in the boundary of one component of $f_\lambda^{-1}(B(0))$. Without loss of generality, we may assume $\partial V \subset \partial B(0)$. Then by the order-$4$ rotation symmetry of $\partial B(0)$, we obtain $\partial (i^k V) \subset \partial B(0)$ for each $k = 0, 1, 2, 3$. Other components of $f_\lambda^{-1}(U)$ are expressed by $h(i^k V)$, $k = 0, 1, 2, 3$, where $h : z \mapsto \frac{1}{z}$. Note that $h(\partial B(0)) = \partial B(\infty)$ and hence $\partial h(i^k V) \subset \partial B(\infty)$ for each $k = 0, 1, 2, 3$. Now we take a point $z \in \partial U$ and consider the number of the preimages of $z$. Clearly, $f_\lambda : V \to U$ is a branched covering with degree $3$ and it defines analytically on a neighborhood of $V$. We also know $\partial U$ is locally connected since it is a connected subset of a connected and locally connected set $\partial B(0)$ without any interior point. Using the fact that there is no critical value on $\partial U$, we can conclude that $z$ also has exactly three preimages on $\partial V$. Similarly, $z$ has three preimages on the boundary of each component of $f_\lambda^{-1}(U)$. Therefore, $z$ has $24$ preimages on $\bigcup_{k=0}^{3} (\partial (i^k V) \cup \partial h(i^k V)) \subset \partial B(0) \cup \partial B(\infty)$. Now we consider $z$ as a point on $\partial B(0)$ and use $f_\lambda : B(i^k) \to B(0)$ for each $k = 0, 1, 2, 3$. Applying a similar argument, we know $z$ has $4$ preimages on $\partial B(i^k)$ for each $k = 0, 1, 2, 3$. Since the boundaries of the components of $f_\lambda^{-1}(B(0))$ are pairwise disjoint, it follows that $z$ has $16$ more distinct preimages on $\bigcup_{k=0}^{3} \partial B(i^k)$. All together, $z$ has at least $40$ distinct preimages. This is impossible since the degree of $f_\lambda$ is $24$. Thus, we conclude that $v_\lambda \in M$.

(2) We have proved in (1) that $v_\lambda \in M$. There are two cases to consider based on whether or not $v_\lambda \in B(\infty)$.

Case 1. $v_\lambda \in B(\infty)$. Then $B(v_\lambda) = B(\infty)$. It is relatively easy to show $f_\lambda^{-1}(B(v_\lambda)) \subset M$. From Lemma 3.16, we know that $f_\lambda^{-1}(B(v_\lambda))$ is connected and $\partial f_\lambda^{-1}(B(v_\lambda))$ has $6$ connected components. Clearly, $f_\lambda^{-1}(B(v_\lambda)) \subset \hat{C} \setminus \overline{B(0)}$. Suppose $f_\lambda^{-1}(B(v_\lambda))$ is not contained in $M$. Then it is contained in another component of $\hat{C} \setminus \overline{B(0)}$, denoted by $U$. It follows that $\partial U \subset \partial B(0)$ and $\partial h(U) \subset \partial B(\infty)$, where $h : z \mapsto \frac{1}{z}$. Using $\partial B(0) \cap \partial B(\infty) = \emptyset$ and $B(\infty) \subset M$, we can conclude that $h(U) \subset M$. Hence $h(f_\lambda^{-1}(B(v_\lambda))) \subset h(U) \subset M$. On the other hand, we also know $h(f_\lambda^{-1}(B(v_\lambda))) \subset f_\lambda^{-1}(B(v_\lambda))$ and $f_\lambda^{-1}(B(v_\lambda))$ is connected. Therefore, $h(f_\lambda^{-1}(B(v_\lambda))) = f_\lambda^{-1}(B(v_\lambda))$. Thus, $f_\lambda^{-1}(B(v_\lambda)) \subset M$.

Case 2. $v_\lambda \in \mathcal{A}(0) \setminus (B(0) \cup B(\infty))$. Under this condition, every Fatou component contains at most one critical value. Using the fact that $f_\lambda$ has no Herman rings
and Riemann-Hurwitz formula, we can see that all Fatou components are simply connected. Then the Julia set $J(f_\lambda)$ is connected from Lemma 3.7. Let $B(v_\lambda)$ be the Fatou component containing $v_\lambda$. Then $f_\lambda^{-1}(B(v_\lambda))$ consists of exactly 8 components and every one contains a critical point in $C(3)$. We claim that $f_\lambda^{-1}(B(v_\lambda))) \subset M$. Suppose that there is a component of $f_\lambda^{-1}(B(v_\lambda))$, say $D$, disjoint from $M$. Then by the symmetry of $M$, $v^jD$, $j = 0, 1, 2, 3$, are all disjoint from $M$. Let $V_j$ be the component of $\hat{C} \setminus B(0)$ containing $v^jD$, $j = 0, 1, 2, 3$. Then $\partial V_j \subset \partial B(0)$ and $V_j$ is also disjoint from $M$ for every $j = 0, 1, 2, 3$. On one hand, $B(\infty) \subset M$ implies $\partial B(\infty) \subset \overline{M}$; on the other hand, $\partial M \subset \partial B(0)$ and $\partial B(0) \cap \partial B(\infty) = \emptyset$ imply that $\partial B(\infty) \cap \partial M = \emptyset$. All together, we obtain $B(\infty) \subset M$ and $\partial B(\infty) \subset M$. Now using $\partial h(V_j) \subset \partial B(\infty)$, both $h(V_j)$ and $M$ are simply connected and $h(V_j) \cap B(0) = \emptyset$, we conclude $h(V_j) \subset M$. Hence, $h(v^jD) \subset h(V_j) \subset M$ for each $j = 0, 1, 2, 3$.

Now we take a point $z \in \partial B(v_\lambda)$ and a point $z^* \in \partial M$. By Lemma 3.10 and Lemma 3.14, $\partial B(0)$ is locally connected. Then $\partial M$ is also locally connected. Thus, there is a curve $\gamma$ in $M$ connecting $z$ and $z^*$ such that $\gamma \cap f_\lambda^{-1}(B(0)) = \{z^*\}$. For each $j = 0, 1, 2, 3$, $f_\lambda : v^jD \rightarrow B(v_\lambda)$ is a branched covering map of degree 3 and hence there are 3 preimages of $z$ on $\partial v^jD$. We consider the preimages of $\gamma$ intersecting $v^jD$. They are three curves emanating from the critical point in $v^jD$ and landing at three points on the boundary of $V_j$. The landing points are distinct (otherwise one of them becomes a critical point on $\partial V_j$). Thus, $z^*$ has 3 distinct preimages on $\partial V_j$ for each $j = 0, 1, 2, 3$. Hence, $z^*$ have 24 distinct preimages on the boundaries of $V_j$’s and $h(V_j)$’s, which are contained in $\partial B(0) \cup \partial B(\infty)$. On the other hand, by considering $z^*$ as a point on the boundary of $B(0)$, we know that it has 4 distinct preimages on $\partial B(v^k)$ for each $k = 1, 2, 3, 4$. All together, we have found 40 distinct preimages for $z^*$. This is impossible since the degree of $f_\lambda$ is 24. Therefore, the assumption that there is a component $D$ of $f_\lambda^{-1}(B(v_\lambda))$ disjoint from $M$ cannot hold. This means $D \cap M \neq \emptyset$ and hence $D \subset M$. By the order-4 rotation symmetry of $M$, we know $v^jD \subset M$ for each $j = 0, 1, 2, 3$. We have proved that $h(v^jD) \subset M$ for each $j = 0, 1, 2, 3$. Therefore, $f_\lambda(B(v_\lambda)) \cup B(v_\lambda) \subset M$.

(3) Using (2) and Lemma 3.17, we obtain $\partial B(0)$ is a Jordan curve and hence $\hat{C} \setminus B(0) = M$.

(4) By (3), $B(0)$ is simply connected and its boundary is a Jordan curve. Using Riemann-Hurwitz formula, we know each connected component of $f_\lambda^{-n}(B(0))$ is simply connected for each $n \geq 1$. Since $v_\lambda \in A(0) \setminus B(0)$, $\partial B(0)$ is disjoint from the post-critical set $P(f_\lambda)$ of $f_\lambda$, where $P(f_\lambda) = \{f_\lambda^n(c) : c \in \text{Crit}(f_\lambda) \text{ and } n \geq 0\}$. Thus, the boundary of each connected component of $f_\lambda^{-n}(\partial B(0))$ is a Jordan curve for each $n \geq 0$.

Now we show that if $v_\lambda \in A(0) \setminus (B(0) \cup B(\infty))$, then the Julia set $J(f_\lambda)$ is a Sierpinski curve.

**Proposition 3.19.** If $v_\lambda \in A(0) \setminus (B(0) \cup B(\infty))$, then the Julia set $J(f_\lambda)$ is a Sierpinski curve.

**Proof.** Since $v_\lambda \in A(0) \setminus (B(0) \cup B(\infty))$, every Fatou component contains at most one critical value. Because $f_\lambda$ has no Herman ring, every Fatou component is simply connected. It follows from Lemma 3.7 that the Julia set $J(f_\lambda)$ is connected. By Lemma 3.14, $J(f_\lambda)$ is also locally connected. It is well known that $J(f_\lambda)$ is compact and nowhere dense in $\hat{C}$. By Proposition 3.18, the boundary of every Fatou component is a Jordan curve and by Lemma 3.15, the boundaries of all
Fatou components are pairwise disjoint. Applying Theorem 3.13, we conclude that $J(f_\lambda)$ is a Sierpinski curve. 

In the next part of this subsection, we consider the third case of the tetrachotomy given in Theorem 3.2, that is when $B(\infty) \neq B(0)$ and $v_\lambda \in B(\infty)$. In this case, $B(\infty)$ is called the trap door for $f_\lambda$, which is often denoted by $T$ in the papers by Devaney and his collaborators.

Let $\gamma \subset \mathbb{C}$ be a Jordan curve and denote by $D(\gamma)$ the bounded component of $\mathbb{C} \setminus \gamma$. Given two Jordan curves $\gamma_1$ and $\gamma_2$ with $\gamma_1 \subset D(\gamma_2)$, denote by $A(\gamma_1, \gamma_2)$ the closed annulus bounded by $\gamma_1, \gamma_2$. We use $D(0)$ and $D(\infty)$ to denote the connected components of $\mathbb{C} \setminus f^{-1}_\lambda(B(\infty))$ containing $B(0)$ and $B(\infty)$ respectively, and use $D(i^k)$ to denote the connected component of $\mathbb{C} \setminus f^{-1}_\lambda(B(\infty))$ containing $B(i^k)$ for each $k = 1, 2, 3, 4$. Then $f_\lambda : \partial D(i^k) \to \partial B(\infty)$ is a degree-4 covering map.

**Proposition 3.20.** If $v_\lambda \in B(\infty) \setminus B(0)$, then the Julia and Fatou sets of $f_\lambda$ have the following properties.

1. The Julia set $J(f_\lambda)$ is not connected and it has uncountably infinitely many Jordan curve components.
2. All connected components of $f_\lambda^{-m}(B(i^k))$ are simply connected for all $m \geq 1$ and $k = 1, 2, 3, 4$.
3. Every simply connected Fatou component $B$ of $f_\lambda$ is surrounded by a Cantor set of Jordan curve components of $J(f_\lambda)$, denoted by $MC(B)$ the union of these Jordan curves.
4. The preimage $f^{-1}_\lambda(B(\infty))$ is multiply connected with connectivity number 6, and each connected component of $f^{-m}_\lambda(B(\infty))$ is multiply connected with connectivity number $4^m + 2$ for each $m \geq 1$.

**Proof.** Keep in mind that $v_\lambda \in B(\infty) \setminus B(0)$ means that $B(\infty) \neq \emptyset$, $B(\infty) \cap B(0) = \emptyset$ and $v_\lambda \in B(\infty)$.

(1) From Lemma 3.16, $f^{-1}_\lambda(B(\infty))$ is connected with connectivity number 6. Thus, the Julia set $J(f_\lambda)$ is not connected. By Proposition 3.18, we also know that the boundary of each connected component of $\partial f^{-1}_\lambda(B(\infty))$ is a Jordan curve. Denote by $D(\infty)$, $D(0)$ and $D(i^k)$, $k = 1, 2, 3, 4$, the (simply) connected components of the complement of $f^{-1}_\lambda(B(\infty))$ containing $\infty$, 0 and $i^k$, $k = 1, 2, 3, 4$, respectively. In fact, $B(\infty) \subset D(\infty)$, $B(0) \subset D(0)$ and $B(i^k) \subset D(i^k)$, $k = 1, 2, 3, 4$. 

Since both $f_\lambda : \partial B(\infty) \to \partial B(0)$ and $f_\lambda : \partial D(\infty) \to \partial B(\infty)$ are covering maps of degree 4, it follows that $f_\lambda : A(\partial D(\infty), \partial B(0)) \to A(\partial B(0), \partial B(\infty))$ is a covering map of degree 4, where $A(\partial D(\infty), \partial B(\infty))$ is the annulus bounded by $\partial D(\infty)$ and $\partial B(\infty)$ and $A(\partial B(0), \partial B(\infty))$ is the annulus bounded by $\partial B(0)$ and $\partial B(\infty)$. By the same argument, we obtain that $f_\lambda : A(\partial D(0), \partial B(0)) \to A(\partial B(\infty), \partial B(0))$ is a covering map of degree 4 as well.

Let $A_{out} = A(\partial D(0), \partial B(0))$ and $A_{in} = A(\partial D(\infty), \partial B(\infty))$. It is easy to see that the attractor of the iterated function system $\{(f_\lambda|_{A_{out}})^{-1}, (f_\lambda|_{A_{in}})^{-1}\}$ is a Cantor set of Jordan curves, denoted by $MC(B(0))$ the union of these Jordan curves. Obviously, $MC(B(0)) \subset J(f_\lambda)$ and all Jordan curves of $MC(B(0))$ separate the two Fatou components $B(0)$ and $B(\infty)$. 
(2) Under the assumption, it is clear that \( B(i^k) \) is simply connected and then each component of \( f_{\lambda}^{-1}(B(i^k)) \) is simply connected, where \( k = 1, 2, 3, 4 \) and \( j \) is a positive integer.

(3) By the same argument used in (1), we know
\[
f_{\lambda} : A(\partial B(i^k), \partial D(i^k)) \to A(\partial B(0), \partial B(\infty))
\]
is a covering map of degree 4, where \( k = 1, 2, 3, 4 \).

Since \( \mathcal{MC}(B(0)) \subset A(\partial B(0), \partial B(\infty)) \), there is a Cantor set of Jordan curves, denoted their union by \( \mathcal{MC}(B(i^k)) \), contained in the annulus \( A(\partial B(i^k), \partial D(i^k)) \) and surrounding the simply Fatou component \( B(i^k) \) such that \( f_{\lambda}(\mathcal{MC}(B(i^k))) = \mathcal{MC}(B(0)) \), where \( k = 1, 2, 3, 4 \).

Given any simply connected Fatou component \( B \), there is a smallest integer \( m \) such that \( f_{\lambda}^m(B) = B(i^k) \) for some \( k \in \{1, 2, 3, 4\} \). Since \( D(i^k) \cap P(f_{\lambda}) = \emptyset \), there is a simply connected component \( D \) of \( f_{\lambda}^{-m}(D(i^k)) \) such that \( B \subset D \) and \( f_{\lambda}^m : D \to D(i^k) \) is a conformal map. Note that there is a Cantor set of Jordan curve components of \( J(f_{\lambda}) \), namely \( \mathcal{MC}(B(i^k)) \), contained in the annulus \( A(\partial B(i^k), \partial D(i^k)) \). The pullback of \( \mathcal{MC}(B(i^k)) \) by \( f_{\lambda}^m \) is also a Cantor set of Jordan curve components of \( J_{\lambda} \), denoted by \( \mathcal{MC}(B) \), contained in the annulus \( A(\partial B, \partial D) \).

(4) Let \( D \) be a connected component of \( f_{\lambda}^{-m}(B(\infty)) \) for \( m \geq 2 \). Since the connectivity number of \( f_{\lambda}^{-1}(B(\infty)) \) is 6, we can show that the connectivity number of \( f_{\lambda}^{-m}(B(\infty)) \) is \( 4^m + 2 \) by induction and repeatedly applying Riemann-Hurwitz formula.

At last, we consider the case when the critical value \( v_{\lambda} \) is not attracted to the super-attracting fixed point 0 under iteration; that is, \( v_{\lambda} \notin A(0) \). We have proved that \( f_{\lambda} \) has no Herman rings in its Fatou set. Applying Lemma 3.7, we obtain the following.

**Proposition 3.21.** If \( v_{\lambda} \notin A(0) \), then the Julia set \( J(f_{\lambda}) \) is connected.

**Proof.** Since \( v_{\lambda} \notin A(0) \), \( B(0) \neq B(\infty) \) and they are simply connected. Clearly, each Fatou component contains at most one critical value. By Theorem 3.1, \( f_{\lambda} \) has no Herman ring. Using Riemann-Hurwitz formula, we can show that every Fatou component is simply connected. Therefore, the Julia set \( J(f_{\lambda}) \) is connected by Lemma 3.7.

Propositions 3.12, 3.19, 3.20 and 3.21 together imply Theorem 3.2.

### 3.4. Proof of Theorem 3.5

Recall that for each map \( h_{\lambda} \) with \( \lambda \neq 0 \), \( \infty \) is an attracting fixed point of order 2, \( B(\infty) \) is the immediate attracting basin of \( h_{\lambda} \) at \( \infty \) and \( T \) is the connected component of \( h_{\lambda}^{-1}(B(\infty)) \setminus B(\infty) \) containing \( 0 \). Similar to the previous section, we also denote \( T \) by \( B(0) \). Recall that \( v_{\lambda} \) is the critical value of \( h_{\lambda} \) at the critical points in \( \phi^{-1}(C(3)) \).

We apply Lemma 3.11 to derive (1) of Theorem 3.5. It suffices to show that all critical points are contained in \( B(\infty) \); that is to develop a version of Lemma 3.9 for \( h_{\lambda} \).

**Lemma 3.22.** If \( v_{\lambda} \in B(\infty) \), then \( B(\infty) \) is completely invariant under \( h_{\lambda} \) and \( J(h_{\lambda}) = \partial B(\infty) \).

**Proof.** Let \( k_3 = 3(B(\infty) \cap \phi^{-1}(C(3))) \) and \( k_2 = 3(B(\infty) \cap \phi^{-1}(C(2))) \). Since \( v_{\lambda} \in B(\infty) \), it follows that \( 3k_3 = 2k_2 \). Then \( k_2 \geq 1 \) and \( 3|k_2 \) imply that \( k_2 \geq 3 \). Using the invariance of \( B(\infty) \) under the elements of \( G_{(2,3,4)} \) of order 2 with at least
one fixed point contained in \( B(\infty) \cap \phi^{-1}(C(2)) \), we can conclude \( k_2 \geq 4 \). Then \( k_2 \geq 6 \) since \( 3|k_2 \). Hence \( k_3 \geq 4 \). Now we consider the orbits of the points of \( B(\infty) \cap \phi^{-1}(C(2)) \) under the elements of \( G_{1,2,3,4} \) of order 3 and with at least one fixed point in \( B(\infty) \cap \phi^{-1}(C(3)) \). Since there are at least two such elements with different rotation axes, we can see that the orbits of the points of \( B(\infty) \cap \phi^{-1}(C(2)) \) under these two elements contain at least 10 points. Thus, \( k_\alpha \geq 10 \). Then \( k_\alpha = 12 \) since \( 3|k_2 \). It follows \( \phi^{-1}(C(2)) \subset B(\infty) \) and the map \( h_\lambda : B(\infty) \to B(\infty) \) has degree equal to 24. Therefore, \( B(\infty) \) is completely invariant under \( h_\lambda \) and \( J(h_\lambda) = \partial B(\infty) \).

The classification of the Julia sets of the maps in the family \( h_\lambda \) is similar to the one of the maps in the family \( f_\lambda \) with one exception: absence of (exploded) McMullen necklace. The main reason is the following proposition.

**Proposition 3.23.** For each map \( h_\lambda \) with \( \lambda \neq 0 \), \( v_\lambda \notin T \) if \( T \) is not empty.

**Proof.** Suppose that \( T \) is not empty and \( v_\lambda \in T \). Using Riemann-Hurwitz formula, we can show that \( B(\infty) \) and \( T \) are simply connected. Using a similar argument to obtain Lemma 3.16, we can prove that \( h_\lambda^{-1}(T) \) is connected. Let \( D(0) \) and \( D(\infty) \) denote the connected components of \( \hat{\mathbb{C}} \setminus h_\lambda^{-1}(T) \) containing \( T \) and \( B(\infty) \) respectively. Let \( A(0) = D(0) \setminus T \) and \( A(\infty) = D(\infty) \setminus B(\infty) \), and let \( A = \mathbb{C} \setminus (T \cup B(\infty)) \). Then \( A(0) \) and \( A(\infty) \) are two disjoint annuli contained in the annulus \( A \) and they share one boundary with \( A \) respectively. Therefore,

\[
\text{Modulus}(A(0)) + \text{Modulus}(A(\infty)) < \text{Modulus}(A).
\]

On the other hand, \( h_\lambda : A(0) \to A \) and \( h_\lambda : A(\infty) \to A \) are regularly covering maps of degree 2. Hence,

\[
\text{Modulus}(A(0)) = \frac{1}{2} \text{Modulus}(A) \quad \text{and} \quad \text{Modulus}(A(\infty)) = \frac{1}{2} \text{Modulus}(A).
\]

This is a contradiction to the previous strictly inequality. Therefore, if \( T \) is not empty, then \( v_\lambda \notin T \). \( \square \)

After obtaining Proposition 3.23, there is no essentially new ideas required to prove (2) of Theorem 3.5 compared with the work to show (2) of Theorem 3.2. At first, proofs of Lemmas 3.10 and 3.15 can be modified quite straightforward to obtain similar results for \( h_\lambda \), which can be stated as follows.

**Lemma 3.24.** If \( v_\lambda \notin B(\infty) \), then \( B(0) \neq B(\infty) \), and \( B(0) \) and \( B(\infty) \) are simply connected.

**Lemma 3.25.** If \( h_\lambda \) is hyperbolic and \( v_\lambda \notin B(\infty) \), then \( \partial B(0) \cap \partial B(\infty) = \emptyset \). Moreover, the boundaries of components of \( A(\infty) = \bigcup_{m=0}^{\infty} h_\lambda^{-m}(B(\infty)) \) are pairwise disjoint. In particular, if \( v_\lambda \in A(\infty) \setminus B(\infty) \), then the same conclusions hold.

Next, we develop a version of Proposition 3.18 for \( h_\lambda \). Similarly, let \( M \) denote the connected component of \( \hat{\mathbb{C}} \setminus B(\infty) \) containing 0, and \( B(z) \) denote the Fatou component of \( h_\lambda \) containing a point \( z \).

**Proposition 3.26.** If \( v_\lambda \in A(\infty) \setminus B(\infty) \), then the following properties hold:

1. \( v_\lambda \in M \);
2. \( h_\lambda^{-1}(B(v_\lambda)) \cup B(v_\lambda) \subset M \);
3. \( \partial B(\infty) \) is a Jordan curve and hence \( M = \hat{\mathbb{C}} \setminus \overline{B(\infty)} \).
4. **the boundary of every connected component of \( h_{\lambda}^{-n}(B(\infty)) \)** is a Jordan curve for each \( n \geq 0 \).

The ideas to prove Proposition 3.18 can be used to show Proposition 3.26. In fact, we can organize a proof for Proposition 3.26 slightly simpler than the one for Proposition 3.18.

**Proof.** (1) Suppose that \( v_{\lambda} \notin M \) and let \( U \) be a simply connected component of \( \mathbb{C} \setminus B(0) \) not equal to \( M \) and containing \( v_{\lambda} \). It is easy to see that Lemma 3.8 holds for \( h_{\lambda} \). Then \( h_{\lambda}^{-1}(U) \) has 8 simply connected components \( V \) and each of them contains exactly one critical point in \( \phi^{-1}(C(3)) \). Since there is no critical value on the boundary of \( U \), the closures of these 8 components are pairwise disjoint. Using \( \partial U \subset \partial B(\infty) \subset h_{\lambda}^{-1}(\partial B(\infty)) \), we know there is a component \( V \) of \( h_{\lambda}^{-1}(U) \) such that \( \partial V \subset \partial B(\infty) \). Now we take a point \( z \in \partial U \) and consider the number of the preimages of \( z \). Clearly, \( h_{\lambda} : V \to U \) is a branched covering with degree 3 and it is analytic on a neighborhood of \( V \). We also know \( \partial U \) is locally connected since it is a connected subset of a connected and locally connected set \( \partial B(\infty) \) without any interior point. Using the fact that there is no critical value on \( \partial U \), we can conclude that \( z \) also has exactly three preimages on \( \partial V \), which is contained in \( \partial B(\infty) \). On the other hand, \( z \) is a point on \( \partial B(\infty) \), so we can count the preimages of \( z \) by using all inverse branches of \( h_{\lambda} \) on \( B(\infty) \). By the previous Lemma 3.25, the connected components of \( h_{\lambda}^{-1}(B(\infty)) \) have pairwise disjoint closures. Then \( z \) has 22 different preimages on the boundaries of 11 connected components of \( h_{\lambda}^{-1}(B(\infty)) \) different from \( B(\infty) \). Altogether, \( z \) has at least 25 distinct preimages. This is impossible. Thus, we conclude that \( v_{\lambda} \in M \).

(2) Using Proposition 3.23, we only need to handle the case when \( v_{\lambda} \in A(\infty) \setminus (B(\infty) \cup B(0)) \). Then each Fatou component of \( h_{\lambda} \) contains at most one critical value. Using the fact that \( h_{\lambda} \) has no Herman rings and the Riemann-Hurwitz formula, we can see that all Fatou components are simply connected. Then the Julia set \( J(h_{\lambda}) \) is connected from Lemma 3.7. Let \( B(v_{\lambda}) \) be the Fatou component containing \( v_{\lambda} \). Then \( h_{\lambda}^{-1}(B(v_{\lambda})) \) consists of exactly 8 components and each one contains a critical point in \( \phi^{-1}(C(3)) \). We show \( h_{\lambda}^{-1}(B(v_{\lambda})) \subset M \). Suppose not, then there exists a component \( D \) of \( h_{\lambda}^{-1}(B(v_{\lambda})) \) disjoint from \( M \). Let \( V \) be the component of \( \mathbb{C} \setminus B(\infty) \) containing \( D \). Then \( \partial V \subset \partial B(\infty) \) and \( V \) is also disjoint from \( M \).

Now we take a point \( z \in \partial B(v_{\lambda}) \) and a point \( z^* \in \partial M \). By Lemma 3.24 and Lemma 3.14, \( \partial B(\infty) \) is locally connected. Then \( \partial M \) is also locally connected. Using Lemma 3.25, we know that if a component of \( h_{\lambda}^{-1}(B(\infty)) \) intersects \( M \), then its closure is contained in \( M \). Thus, there is a curve \( \gamma \) in \( M \) connecting \( z \) and \( z^* \) such that \( \gamma \cap h_{\lambda}^{-1}(B(\infty)) = \{ z^* \} \). The map \( h_{\lambda} : D \to B(v_{\lambda}) \) is a branched covering map of degree 3 and hence there are 3 preimages of \( z \) on \( \partial D \). We consider the preimages of \( \gamma \) intersecting \( D \). They are three curves emanating from the critical point in \( D \), crossing three distinct points on \( \partial D \), and landing at three points on the boundary of \( V \). The landing points on \( \partial V \) are distinct (otherwise one of them becomes a critical point on \( \partial V \)). Thus, \( z^* \) has 3 distinct preimages on \( \partial V \), which belong to \( \partial B(\infty) \). Now using \( z^* \) as a point on \( \partial B(\infty) \) and counting the preimages of \( z^* \) under all inverse branches of \( h_{\lambda} \) on \( B(\infty) \), it has 22 distinct preimages on the boundaries of 11 connected components of \( h_{\lambda}^{-1}(B(\infty)) \) different from \( B(\infty) \). Therefore, \( z^* \) has at least 25 distinct preimages. This is a contradiction. Thus, we obtain \( h_{\lambda}^{-1}(B(v_{\lambda})) \subset M \).
(3) By (2) and Lemma 3.17, $\partial B(\infty)$ is a Jordan curve and $\hat{C} \setminus \overline{B(\infty)} = M$.

(4) By (3), $B(\infty)$ is simply connected and its boundary is a Jordan curve. Then the Riemann-Hurwitz formula implies that every connected component of $h_\infty^{-n}(B(\infty))$ is simply connected for each $n \geq 1$. Since $v_\lambda \in A(\infty) \setminus B(\infty)$, $\partial B(\infty)$ is disjoint from the post-critical set $P(h_\lambda) = \{ h_\lambda^n(c) : c \in \text{Crit}(h_\lambda) \text{ and } n \geq 0 \}$. Thus, the boundary of every connected component of $h_\infty^{-n}(\partial B(\infty))$ is a Jordan curve for each $n \geq 0$.

In summary, Lemma 3.22 and Lemma 3.11 imply (1) of Theorem 3.5; Propositions 3.23 and 3.26 and Theorem 3.13 imply (2) of Theorem 3.5; the proof of (3) of Theorem 3.5 goes as the same as the one of (4) of Theorem 3.2.

4. The tetrahedral and icosahedral cases. In this last section, we consider one-parameter families of normalized regularly ramified rational maps of form $A \circ R_{G_j}$ or $A \circ R_{G_j} \circ \phi$ for $j = 3$ or $j = 5$, where $A$ and $\phi$ are Möbius transformations. We first consider such maps related to $G_5$ since they have been explored in [6] through computer generated pictures. Some one-parameter families of rational maps related to $G_3$ are also explored in [6], but those maps don’t satisfy the two assumptions for the regularly ramified rational maps investigated in this paper. Therefore, we need to construct the ones of form $A \circ R_{G_3}$ or $A \circ R_{G_3} \circ \phi$ satisfying the two assumptions, which we carry out in the second half of this section.

4.1. The icosahedral case. A regular icosahedron can be embedded on the Riemann sphere with 12 vertices arranged at points in the set

$$C(5) = \left\{ 0, \infty, \frac{1 \pm \sqrt{5}}{2} e^{\frac{2k\pi i}{5}} : k = 0, 1, 2, 3, 4 \right\},$$

the centers of 20 faces arranged at points in the set

$$C(3) = \left\{ \frac{-3 - \sqrt{5} \pm \sqrt{10 + 2\sqrt{5}}}{4} e^{\frac{2k\pi i}{5}}, \frac{-3 + \sqrt{5} \pm \sqrt{10 - 2\sqrt{5}}}{4} e^{\frac{2k\pi i}{5}} : k = 0, \ldots, 4 \right\},$$

and the middle points of 30 edges arranged at points in the set

$$C(2) = \left\{ \pm \frac{ie^{\frac{2k\pi i}{5}}}{5} : k = 0, \ldots, 4 \right\}.$$

The following one-parameter family of normalized regularly ramified rational maps $A \circ R_{G_3}$ is given in [6]:

$$f_{(2,3,5)}^\lambda(z) = \frac{\lambda z^5}{(z^{10} + 1)^2} \prod_{j=0,1} \left( \frac{z^5}{2} \right) - \frac{1}{2} \left( \frac{1 + (-1)^j \sqrt{5}}{2} \right)^5 \frac{(z^5 - \frac{1 + (-1)^j \sqrt{5}}{2})^5}{2} (z^5 - \frac{1 + (-1)^j \sqrt{5}}{2} \sqrt{10 + 2\sqrt{5}})^2 (z^5 - \frac{1 + (-1)^j \sqrt{5}}{2} \sqrt{10 - 2\sqrt{5}})^2.$$

where $\lambda$ is a complex parameter. All points of $C(5)$ are critical points of $f_{(2,3,5)}^\lambda$ of order 5 and are mapped to 0; all points of $C(2)$ are critical points of order 2 and are mapped to $\infty$; all points of $C(3)$ are critical points of order 3 and are mapped to the same non-zero value, denoted by $v_\lambda$. Clearly, 0 is fixed by $f_{(2,3,5)}^\lambda$ and $\infty$ is mapped to 0. Similar to $f_{(2,3,4)}^\lambda$, we define $B(0)$ and $B(\infty)$ for $f_{(2,3,5)}^\lambda$. Julia sets of the rational maps in this family are explored through computer-generated pictures in [6]. Using the same strategies, but modified details, to prove Theorems 3.1 and 3.2, we obtain corresponding results for the maps in the family $f_{(2,3,5)}^\lambda$. Four types of Julia set for the maps in the family $f_{(2,3,5)}^\lambda$ are illustrated in Figure 8.
Theorem 4.1. Each rational map $f_{(2,3,5)}^λ$ has no Herman rings.

Theorem 4.2. For the maps in the family $f_{(2,3,5)}^λ$ with $\lambda \neq 0$, the following tetra-chootomy holds:

1. If $v_\lambda \in B(0)$ (in this case, $B(0) = B(\infty) = A(0)$), then $J(f_{(2,3,5)}^λ)$ is a Cantor set.
2. If $v_\lambda \in A(0) \setminus (B(0) \cup B(\infty))$, then the Julia set $J(f_{(2,3,5)}^λ)$ is a Sierpinski curve.
3. If $v_\lambda \in B(\infty) \setminus B(0)$, then (i) $J(f_{(2,3,5)}^λ)$ is not connected; (ii) $B(0)$ and $B(\infty)$ are simply connected, and for each $m \geq 0$ and each $w \in C(5) \setminus \{0, \infty\}$,
each connected component of \((f_{(2,3,5)}^λ)^{-m}(B(w))\) is simply connected; (iii) for each \(m \geq 1\), each connected component of \((f_{(2,3,5)}^λ)^{-m}(B(\infty))\) is multiply connected with connectivity number \(2(5^m) + 2\); (iv) every simply connected Fatou component \(B\) of \(f_{(2,3,5)}^λ\) is surrounded by a Cantor set of Jordan curves.

4. If \(v_\lambda \notin \mathcal{A}(0)\), then the Julia set \(J(f_{(2,3,5)}^λ)\) is connected.

Now we consider another type of one-parameter family of normalized regularly ramified rational maps of form \(A \circ R_{G_5} \circ \phi\) similar to the one considered for the octahedral case in the previous section. These rational maps fix one endpoint of a rotation axis of an element of \(G_5\) of order 2, map the antipodal point \(q\) of \(p\) to \(p\), and map all fixed points of rotation axes of elements of \(G_5\) of order 5 to \(q\). Through conjugation by a Möbius transformation, we may assume that \(p\) is arranged at \(\infty\) and \(q\) at 0. There is a short cut to construct such rational maps, which goes as follows. Let

\[
\phi(z) = \frac{iz + 1}{z + i},
\]

and define

\[
h_{(2,3,5)}^λ(z) = \lambda f_{(2,3,5)}^λ(\phi(z)).
\]

Then \(h_{(2,3,5)}^λ\) is a family of normalized regularly ramified rational maps of form \(A \circ R_{G_5} \circ \phi\), where \(A\) is a Möbius transformation. All points in \(\phi^{-1}(\mathbb{C}(2))\), including 0 and \(\infty\), are critical points of \(h_{(2,3,5)}^λ\) of order 2 and are mapped to \(\infty\); all points in \(\phi^{-1}(\mathbb{C}(5))\) are critical points of order 5 and are mapped to 0; all points in \(\phi^{-1}(\mathbb{C}(3))\) are critical points of order 3 and are mapped to the same value, denoted by \(v_λ\) again.

Julia sets of the maps in this family are explored in [6] and a trichotomy is observed there. Similar to \(h_{(2,3,4)}^λ\), we define \(\mathcal{A}(\infty)\) and \(B(\infty)\) for \(h_{(2,3,5)}^λ\). Using the same strategies, but modified details, to prove Theorems 3.4 and 3.5 for \(h_{(2,3,4)}^λ\), we draw similar conclusions for \(h_{(2,3,5)}^λ\). Three types of Julia set for the maps in \(h_{(2,3,5)}^λ\) are illustrated in Figure 9.

**Theorem 4.3.** Each rational map \(h_{(2,3,5)}^λ\) has no Herman rings.

**Theorem 4.4.** For the maps in the family \(h_{(2,3,5)}^λ\) with \(\lambda \neq 0\), the following trichotomy holds:

1. If \(v_λ \in B(\infty)\) (in this case, \(\mathcal{A}(\infty) = B(\infty)\)), then \(J(h_{(2,3,5)}^λ)\) is a Cantor set.
2. If \(v_λ \in \mathcal{A}(\infty) \setminus B(\infty)\), then the Julia set \(J(h_{(2,3,5)}^λ)\) is a Sierpinski curve.
3. If \(v_λ \notin \mathcal{A}(\infty)\), then the Julia set \(J(h_{(2,3,5)}^λ)\) is connected.

4.2. **The tetrahedral case.** As the last part of this paper, we consider one-parameter families of normalized regularly ramified rational maps of form \(A \circ R_{G_3} \circ \phi\) or \(A \circ R_{G_3} \circ \phi\). In [6], a regular tetrahedron is imbedded on a Riemann Sphere with 3 vertices at points in the set

\[
C_0(3) = \left\{ \infty, \frac{\sqrt{2}}{2} e^{\frac{2k\pi i}{3}} : k = 0, 1, 2 \right\},
\]

centers of 3 faces at points in the set

\[
C_{cf}(3) = \left\{ 0, -\sqrt{2} e^{\frac{2k\pi i}{3}} : k = 0, 1, 2 \right\}.
\]
Figure 9. Three types of Julia sets for maps in the family $h_{(2,3,5)}^\lambda$ (Note that $\infty$ is fixed). In (a), $\lambda = 15000 - 30000i$, a Cantor set; in (b), $\lambda = 12580 - 19760i$, an approximation of a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Julia set; in (c), $\lambda = 9000 + 5000i$, a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Fatou set; in (d), $\lambda = 3500 - 6000i$, a Sierpinski curve.

and middle points of 6 edges at points in the set $C(2) = \left\{ \frac{\sqrt{2}(1 \pm \sqrt{3})}{2} e^{\frac{2k\pi}{3}i} : k = 0, 1, 2 \right\}$.

To construct one example of the first type of one-parameter family of regularly ramified rational maps of form $A \circ R_{G_3}$, we choose two points from $C_v(3)$ for $p$ and $q$, for example, take $p = \frac{\sqrt{2}}{2}$ and $q = \infty$. Let $\phi(z) = z - \frac{\sqrt{2}}{2}$. Then

$$\phi(C_v(3)) = \left\{ \infty, 0, \frac{\sqrt{2}}{2} e^{\frac{2k\pi}{3}i}, -\frac{\sqrt{2}}{2} e^{\frac{2k\pi}{3}i} : k = 0, 1, 2 \right\},$$

$$\phi(C_c(3)) = \left\{ -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} e^{\frac{2k\pi}{3}i} : k = 0, 1, 2 \right\}$$

and

$$\phi(C(2)) = \left\{ \frac{\sqrt{2}(1 \pm \sqrt{3})}{2} e^{\frac{2k\pi}{3}i} - \frac{\sqrt{2}}{2} : k = 0, 1, 2 \right\}.$$
Now we can obtain an explicit example of the first type of one-parameter family as follows:

\[
 f_{(2,3,3)}^\lambda(z) = \lambda \frac{z^3 \left( z - \left( \frac{\sqrt{2}}{2} e^{\frac{2\pi i}{3}} - \frac{\sqrt{2}}{2} \right) \right)^3 \left( z - \left( \frac{\sqrt{2}}{2} e^{\frac{2\pi i}{3}} - \frac{\sqrt{2}}{2} \right) \right)^3}{\left( z + \frac{\sqrt{2}}{2} \right)^3 \prod_{k=0}^{2} \left( z + \frac{\sqrt{2}}{2} e^{\frac{2\pi i}{3}} + \frac{\sqrt{2}}{2} \right)^3}.
\]

Clearly, all points of \( \phi(C_v(3)) \cup \phi(C_{cf}(3)) \) are critical points of \( f_{(2,3,3)}^\lambda \) of order 3. It is also easy to see that 0 is an attracting fixed point, \( \infty \) is mapped to 0, and all points of \( \phi(C_{cf}(3)) \) are mapped to \( \infty \). Furthermore, all points of \( \phi(C(2)) \) are the only other critical points of \( f_{(2,3,3)}^\lambda \) (see [6]) at which \( f_{(2,3,3)}^\lambda \) attains the same value, denoted by \( v_\lambda \) again. Similar to \( f_{(2,3,4)}^\lambda \), we define \( B(0) \), \( A(0) \) and \( B(\infty) \) for \( f_{(2,3,3)}^\lambda \).

Using the same strategies, but modified details, to prove Theorems 3.1 and 3.2, we obtain corresponding results for the maps in the family \( f_{(2,3,3)}^\lambda \). Four types of Julia set for the maps in the family \( f_{(2,3,3)}^\lambda \) are illustrated in Figure.

With similar strategy and details, we obtain similar versions of Theorems 3.1 and 3.2 for the maps in the family \( f_{(2,3,3)}^\lambda \), for which their Julia sets are illustrated in Figure 10.

**Theorem 4.5.** Each rational map \( f_{(2,3,3)}^\lambda \) has no Herman rings.

**Theorem 4.6.** For the maps in the family \( f_{(2,3,3)}^\lambda \) with \( \lambda \neq 0 \), the following tetraochotomy holds:

1. If \( v_\lambda \in B(0) \) (in this case, \( B(0) = B(\infty) = A(0) \)), then \( J(f_{(2,3,3)}^\lambda) \) is a Cantor set.
2. If \( v_\lambda \in A(0) \setminus (B(0) \cup B(\infty)) \), then the Julia set \( J(f_{(2,3,3)}^\lambda) \) is a Sierpinski curve.
3. If \( v_\lambda \in B(\infty) \setminus B(0) \), then (i) \( J(f_{(2,3,3)}^\lambda) \) is not connected; (ii) \( B(0) \) and \( B(\infty) \) are simply connected, and for each \( m \geq 0 \) and each \( w \in \phi(C_v(3)) \setminus \{0, \infty\} \), each connected component of \( (f_{(2,3,3)}^\lambda)^{-m}(B(w)) \) is simply connected; (iii) for each \( m \geq 1 \), each connected component of \( (f_{(2,3,3)}^\lambda)^{-m}(B(\infty)) \) is multiply connected with connectivity number \( 2(3^m) + 2 \); (iv) every simply connected Fatou component \( B \) of \( f_{(2,3,3)}^\lambda \) is surrounded by a Cantor set of Jordan curve components of \( J(f_{(2,3,3)}^\lambda) \).
4. If \( v_\lambda \notin A(0) \), then the Julia set \( J(f_{(2,3,3)}^\lambda) \) is connected.

To provide an example of the second type of one-parameter family of regularly ramified rational maps of form \( A \circ R_{G_3} \), we choose two points from \( C(2) \) for \( p \) and \( q \), for example, take \( p = \frac{\sqrt{2(1+\sqrt{3})}}{2} \) and \( q = \frac{\sqrt{2(1-\sqrt{3})}}{2} \). Let

\[
 \varphi(z) = \frac{z - \frac{\sqrt{2(1+\sqrt{3})}}{2}}{z - \frac{\sqrt{2(1-\sqrt{3})}}{2}}.
\]

Then \( \varphi(C_v(3)) = \{ \pm i, \pm (2 + \sqrt{3}) \} \).
Figure 10. Four types of Julia sets for maps in the family $f_{(2,3,3)}^\lambda$.
In (a), $\lambda = 10$, a Cantor set; in (b), $\lambda = -200$, a Sierpinski curve;
in (c), $\lambda = 30$, a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Fatou set;
in (d), $\lambda = 290$, an approximation of a non-escaping case of $v_\lambda$ with $v_\lambda$ in the Julia set; in (e), $\lambda = 500$, an exploded McMullen necklace; (f) is a zoom of (e) in the middle. The black point in (a) or (f) stands for the origin, which is in the Fatou set.

and

$$\varphi(C(2)) = \left\{ 0, \infty, \frac{1 + \sqrt{3}}{2}(1 \pm i), -\frac{1 + \sqrt{3}}{2}(1 \pm i) \right\}.$$
Figure 11. Three types of Julia sets for maps in the family \( h^\lambda_{(2,3,3)} \).

In (a), \( \lambda = 20i \), a Cantor set; in (b), \( \lambda = 27.2899i \), an approximation of a non-escaping case of \( v_\lambda \) with \( v_\lambda \) in the Julia set; in (c), \( \lambda = 60 \), a non-escaping case of \( v_\lambda \) with \( v_\lambda \) in the Fatou set; in (d), \( \lambda = 120i \), a Sierpinski curve.

As an example, we obtain the following one-parameter family:

\[
h^\lambda_{(2,3,3)}(z) = \lambda \frac{z^2 \prod_{z_0 \in \varphi(C(2)) \setminus \{0, \infty\}} (z - z_0)^2}{\prod_{z_0 \in \varphi(C_c(3))} (z - z_0)^3} = \lambda \frac{z^2 [z^4 + (2 + \sqrt{3})^2]^2}{(z^2 + 1)^3 [z^2 - (2 + \sqrt{3})^2]^3}.
\]

Clearly, points of \( \varphi(C(2)) \) and \( \varphi(C_c(3)) \) are critical points of \( h^\lambda_{(2,3,3)} \) of order 2 and 3 respectively. It is obvious that 0 is an attracting fixed point, \( \infty \) is mapped to 0, and all points of \( \varphi(C_c(3)) \) are mapped to \( \infty \). Furthermore, all points of \( \varphi(C_c(3)) \) are the only other critical points of \( h^\lambda_{(2,3,3)} \) (see [6]) at which \( h^\lambda_{(2,3,3)} \) attains the same value, denoted by \( v_\lambda \). Similarly, we define \( B(0) \), \( A(0) \) and \( B(\infty) \) for \( h^\lambda_{(2,3,3)} \). With similar strategies and detail to prove Theorems 3.4 and 3.5, we obtain corresponding results for the maps in the family \( h^\lambda_{(2,3,3)} \), for which three types of Julia sets are illustrated in Figure 11.

**Theorem 4.7.** Each rational map \( h^\lambda_{(2,3,3)} \) has no Herman rings.

**Theorem 4.8.** For the maps in the family \( h^\lambda_{(2,3,3)} \) with \( \lambda \neq 0 \), the following trichotomy holds:

1. If \( v_\lambda \in B(0) \) (in this case, \( A(0) = B(0) \)), then \( J(h^\lambda_{(2,3,3)}) \) is a Cantor set.
2. If \( v_\lambda \in A(0) \setminus B(0) \), then the Julia set \( J(h^\lambda_{(2,3,3)}) \) is a Sierpinski curve.
3. If \( v_\lambda \notin A(0) \), then the Julia set \( J(h^\lambda_{(2,3,3)}) \) is connected.
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