QUOTIENTS OF HYPERSURFACES IN WEIGHTED
PROJECTIVE SPACE

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Abstract. In [1] some quotients of one-parameter families of Calabi-Yau
varieties are related to the family of Mirror Quintics by using a construction due
to Shioda. In this paper, we generalize this construction to a wider class of
varieties. More specifically, let $A$ be an invertible matrix with non-negative
integer entries. We introduce varieties $X_A$ and $M_A$ in weighted projective
space and in $P^n$, respectively. The variety $M_A$ turns out to be a quotient of
a Fermat variety by a finite group. As a by-product, $X_A$ is a quotient of a
Fermat variety and $M_A$ is a quotient of $X_A$ by a finite group. We apply this
construction to some families of Calabi-Yau manifolds in order to show their
birationality.

1. Introduction

Hypersurfaces in weighted projective space have been investigated by many au-
thors in connection with Physics, in particular Mirror Symmetry: see, for instance,
[3], [4] and [21]. The usual quintic threefold is an example of Calabi-Yau manifold
in ordinary projective space. Its mirror can be described in terms of quotients of
a one-parameter family of quintics, the Dwork pencil. In [1], we investigated other
one-parameter families of Calabi-Yau manifolds and related them to the family
of Mirror quintics. Our main tool was a construction due to Shioda [19], which
clarified previous work in [3].

In the present paper we generalize this construction to hypersurfaces in weighted
projective space. Let $A$ be an invertible matrix of size $n$ with non-negative integer
entries. For such a matrix we define a weighted homogeneous polynomial $F_A$, a sum
of $n$ monomials. Its zero locus gives a projective scheme $X_A$ in weighted projective
space $W_{P^{n-1}}(q_1, \ldots, q_n)$. The weights are determined by the relation $Aq = de$,
where $q = (q_1, \ldots, q_n)$, $e = (1, 1, \ldots, 1)$ and $d$ is the smallest positive integer
such that $dA^{-1}$ has integer entries. If the total degree of $F_A$ equals the degree of
the anticanonical bundle and the singularities of $X_A$ are canonical, then $X_A$ is a
Calabi-Yau manifold as explained in Section 2. It is therefore interesting to study
the quotients of $X_A$ in relation with the mirror varieties of $X_A$. For this purpose,
we introduce a manifold $M_A \subset P^n$, which we refer to as the Shioda quotient.
It is the image of a suitable map (first introduced by Shioda in [19] and applied to
a similar context in [1]) of a Fermat variety. Actually, we prove that the Shioda
quotient is the quotient of a Fermat variety by a finite group. As a by-product, it
is also the quotient of $X_A$ by a finite group. We describe these groups in detail.

It is natural to investigate deformation families of $X_A$ defined by $F_{A,t} = F_A -
tx_1 \cdots x_n$, where $x_i$ are variables of degree $q_i$. They are particularly meaningful
when the Hodge number $h^{2,1}$ is one, since then $F_{A,t}$ gives a versal deformation.

Some examples of deformation families of $X_A$ when $X_A$ is a Calabi-Yau manifold in weighted projective space are given in [20]. Their mirror all have $h^{2,1} = 1$. We investigate the birational classes associated to these families and prove they are birational to the mirror families of the first four hypergeometric Calabi-Yau families studied by Rodriguez Villegas [18] and listed in [15, p. 134].

2. Preliminaries

2.1. Weighted projective spaces. We briefly recall the definition of weighted projective space. Let $(q_1, \ldots, q_n)$ be a sequence of positive integers. As customary, set

$$\mathbb{P}^{n-1}(q_1, \ldots, q_n) := (\mathbb{C}^n \setminus \{0\}) / \sim,$$

where the equivalence relation is

$$(x_1, \ldots, x_n) \sim (\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n)$$

whenever $\lambda \in \mathbb{C} \setminus \{0\}$. Recall that $\mathbb{P}^{n-1}(q_1, \ldots, q_n) \cong \mathbb{P}^{n-1}(q_1/a_1, \ldots, q_n/a_n)$, where $a_i = \text{l.c.m.}(d_1, \ldots, \hat{d}_i, \ldots, d_n)$ and $d_i = \text{g.c.d.}(q_1, \ldots, \hat{q}_i, \ldots, q_n)$. Weighted projective space are almost always singular. As proved in [6], the singular locus of $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ can be described in the following way. Let $p$ be a prime. Let $I(x) = \{j; x_j \neq 0\}$. Then define

$$\text{Sing}_p(\mathbb{P}^{n-1}) := \{x \in \mathbb{P}^{n-1}(q_1, \ldots, q_n) : p | q_i \text{ for any } i \in I(x)\}.$$

The singular locus of the weighted projective space is given by the union over all primes of $\text{Sing}_p(\mathbb{P}^{n-1})$.

As explained, for instance, in [5], weighted projective space is a toric variety. Set $Q := \sum_i q_i$. We recall that the canonical sheaf of $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ is given by $\mathcal{O}(-Q)$, which is not always a line bundle. As proved in [5], Lemma 3.5.6., the canonical sheaf is a line bundle if and only if $q_i | Q$ for all $i = 1, \ldots, n$. Under this assumption, weighted projective space is a Fano toric variety.

Some of our hypersurfaces are Calabi-Yau varieties. Following [5], a (possibly singular) Calabi-Yau variety is an $m$-dimensional normal compact variety $X$ which satisfies the following conditions:

(i) $X$ has at most Gorenstein canonical singularities;
(ii) the dualizing sheaf of $X$ is trivial;
(iii) $H^1(X, \mathcal{O}_X) = \ldots = H^{m-1}(X, \mathcal{O}_X) = 0$.

Let $f$ be a weighted homogeneous polynomial such that the zero locus $\{f = 0\}$ is quasi-smooth (according to Definition 3.1.5 in [7]). Since a quasi-smooth scheme has finite quotient singularities ([7, Thm. 3.1.6]), the locus $\{f = 0\}$ is normal and Cohen-Macaulay; furthermore, it is Gorenstein with dualizing sheaf $\mathcal{O}(d)$, where $d$ is the weighted degree of $f$. Assume that $\{f = 0\}$ has at most canonical singularities. Following the arguments in Proposition 4.1.3 in [5], it follows that $\{f = 0\}$ is a Calabi-Yau variety when $d = Q$. Notice that if there exists a crepant resolution of $\{f = 0\}$, the singularities are canonical by definition.
2.2. The Shioda maps. Let $A$ be an invertible matrix with non-negative integer entries:

$$A = (a_{ij}) \ (\in M_n(\mathbb{Z})), \ a_{ij} \in \mathbb{Z}_{\geq 0}, \ \det(A) \neq 0.$$ 

For such a matrix we define a polynomial in $n$ variables, a sum of $n$ monomials:

$$F_A := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}} = x_1^{a_{11}} x_2^{a_{12}} \ldots x_n^{a_{1n}} + x_1^{a_{21}} x_2^{a_{22}} \ldots x_n^{a_{2n}} + \ldots$$

Let $d$ be the smallest positive integer such that $B := dA^{-1}$ is in $M_n(\mathbb{Z})$. Then set

$$q := Be,$$

where $e = (1, \ldots, 1)$ and $q = (q_1, \ldots, q_n)$. Clearly, this implies that

$$Aq = de.$$

The zero locus $X_A = Z(F_A)$ is a (not necessarily smooth or irreducible) projective variety $X_A$, which is contained in $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$. Let $m$ be the greatest common divisor of the $q_i$’s. Define $a_i = q_i/m$. Thus we have

$$X_A \subset \mathbb{P}^{n-1}(q_1, \ldots, q_n) \cong \mathbb{P}^{n-1}(a_1, \ldots, a_n).$$

We have a rational map $\phi_A$ from $\mathbb{P}^{n-1}$ of degree $d$ to $X_A$ defined by:

$$\phi_A : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}(q_1, \ldots, q_n),$$

$$(y_1 : \ldots : y_n) \mapsto (x_1 : \ldots : x_n), \quad x_j = \prod_{k=1}^{n} y_k^{b_{jk}}.$$

Notice that each $y_j$ has degree one, so $\deg(x_j) = \sum_k b_{jk} = q_j$. Hence $\phi_A$ is indeed a rational map from $\mathbb{P}^{n-1}$ to $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$.

Assume further that $d = Q = \sum_j q_j$, so that the variety $X_A$ gives a Calabi-Yau provided the singularities are canonical. We can read this condition on the coefficients of the matrix $A^{-1}$. In fact, we have:

$$d = Q = \sum_j q_j = t^e q = d^e eA^{-1} e,$$

which gives

$$t^e A^{-1} e = 1;$$

in other words, $\sum_j a'_{ij} = 1$, where $a'_{ij}$ are the entries of $A^{-1}$.

We define a rational map

$$q_A : \mathbb{P}^{n-1}(q_1, \ldots, q_n) \to \overline{\text{im}(q_A)} := \overline{M_A} \subset \mathbb{P}^n$$

in the following way:

$$(x_1 : \ldots : x_n) \mapsto (u_0 : u_1 : \ldots : u_n) := \left( \prod_{j=1}^{n} x_j : \prod_{j=1}^{n} x_j^{a_{1j}} : \ldots : \prod_{j=1}^{n} x_j^{a_{nj}} \right).$$

Since $\deg(x_i) = q_j$ and $Aq = de$, we have

$$\deg(u_0) = \sum_j q_j = d, \quad \deg(u_k) = \sum_j a_{kj} q_j = d,$$

hence $q_A$ is well-defined.
Finally, we describe the composition \( q_A \circ \phi_A \), which will be used in the next section. First, as \( BA = AB = dI_n \) (where \( I_n \) is the identity matrix) we have \( \sum_j a_{ij}b_{jk} = d\delta_{ik} \), where \( \delta_{ik} \) is the Kronecker delta. Second, we set
\[
q' := d^tA^{-1}e = d^tA^{-1} = i^tB;
\]
so \( q'_k = \sum_j b_{jk} \). This said, it is easy to check that the composition \( q_A \circ \phi_A : X_d \subset \mathbb{P}^{n-1} \to \overline{M_A} \subset \mathbb{P}^n \) is given by
\[
(u_0 : u_1 : \ldots : u_n) = \left( \prod_{k=1}^n y_k^{q'_k} : y_1^d : \ldots : y_n^d \right),
\]
where \( X_d \) is the Fermat variety \( \{ \sum_{i=1}^n y_i^d = 0 \} \) and \( \overline{M_A} \) is the closure of the image of \( X_d \).

Let us consider the projection
\[
\pi : \overline{M_A} \subset \mathbb{P}^n \to V \cong \mathbb{P}^n \quad \text{and} \quad (u_0 : u_1 : \ldots : u_n) \to (u_1 : \ldots : u_n),
\]
where \( V \) is the closure of \( \pi(\overline{M_A}) \). It is easy to check that \( V \) is isomorphic to the \( \mathbb{P}^{n-1} \) given by \( \sum_{i=1}^n u_i = 0 \) as the Fermat equation has to be satisfied.

From now on, we will assume that \( q'_k \) is strictly positive for any \( k \). By direct inspection, we obtain the following equations for the image of the Fermat variety \( X_d \) under \( q_A \circ \phi_A \):
\[
u_1 + \ldots + u_n = 0, \quad u_0^d = u_1^{q'_1} \ldots u_n^{q'_n}.
\]

Set \( m' := g.c.d(d, q'_1, \ldots, q'_n) \). Hence \( d = m'a' \) and \( q'_k = d_k^m \), so the composition \( q_A \circ \phi_A \) is given by
\[
(u_0 : \ldots : u_n) = \left( \prod_{k=1}^n y_k^{q'_k} : y_1^d : \ldots : y_n^d \right) = \left( \prod_{k=1}^n (y_k^{m'})^{a_k} : (y_1^{m'})^{a'} : \ldots : (y_n^{m'})^{a'} \right).
\]

By composing (2.7) with the map \( t_k = y_k^{m'} \) for \( k = 1, \ldots, n \), we get
\[
(u_0 : \ldots : u_n) = \left( \prod_{k=1}^n t_k^{a_k} : t_1^{a'} : \ldots : t_n^{a'} \right),
\]
so the equations defining \( \overline{M_A} \) are the following:
\[
(u_0 : \ldots : u_n) = \left( \prod_{k=1}^n t_k^{a_k} : t_1^{a'} : \ldots : t_n^{a'} \right).
\]

In the next section, we will show that under our assumptions on the \( q_j \)'s, the equation (2.8) define a very singular model in \((n - 1)\)-dimensional projective space for a manifold with \( h^{n-2,0} = 1 \) of degree \( a' (> n \text{ in general}) \).

3. The Shioda Quotient

In this section we assume that \( A \in M_n(\mathbb{Z}_{\geq 0}) \) is an invertible matrix such that \( [2.3] \) holds. We will introduce "natural" automorphism groups and study the quotients by these groups.
3.1. The automorphism groups. Let $\zeta = \zeta_d$ be a generator of the cyclic group of $d$-th roots of unity, where $d$ is the smallest positive integer such that $dA^{-1}$ has integer entries. For $k = (k_1, \ldots, k_n) \in (\mathbb{Z}/d\mathbb{Z})^n$ we define an automorphism $g_k$ of $\mathbb{P}^{n-1}$ by

$$g_k(y_1 : \ldots : y_n) := (\zeta^{k_1}y_1 : \ldots : \zeta^{k_n}y_n).$$

Note that $a, b \in \mu_d^*$ define the same automorphism iff $a - b = (k, \ldots, k)$ for some $k \in \mu_d$. Define $\Gamma_d$ to be the quotient group

$$\Gamma_d := \mu_d^*/\langle (g_{(1,1,\ldots,1)} \rangle \subset Aut(\mathbb{P}^{n-1})$$

Notice that $\Gamma_d \cong \mu_d^{n-1}$; hence $\# \Gamma_d = d^{n-1}$. The group $\Gamma_d$ is a subgroup of the automorphism group of the Fermat variety $X_d$.

The map $q_A \circ \phi_A$ is invariant under the subgroup of $\Gamma_d$ given by

$$\Gamma(q') := \left\{ g_k : k = (k_1, \ldots, k_n); \sum_j k_jq'_j \equiv 0 \mod d \right\} / \langle (g_{(1,\ldots,1)}) \rangle.$$ 

In other words, we have

$$(q_A \circ \phi_A)(g_k(y_1 : \ldots : y_n)) = (q_A \circ \phi_A)(y_1 : \ldots : y_n)$$

for all $g_k$ in $\Gamma(q')$ and all $(y_1, \ldots, y_n) \in X_d$. Notice that $g_{(1,\ldots,1)}$ is an element of $\Gamma(q')$. In fact, by (2.3):

$$\sum_j q'_j = \text{eq}' = d^i e \text{eq}' = d.$$

The coordinate functions of the Shioda map $\phi_A$ are products of the $y_i$. If $\phi_A(y_1 : \ldots : y_n) = (x_1 : \ldots : x_n)$, then

$$\phi_A(g_k(y_1 : \ldots : y_n)) = \left(\zeta^{k_1}x_1 : \ldots : \zeta^{k_n}x_n\right).$$

As $x_j = \prod y_j^{b_j}$, the column vector $k'$ is obtained from the column vector $k$ as $k' = Bk$. Thus we get a homomorphism

$$\Gamma(q') \rightarrow Aut(X_A), \quad g_k \rightarrow g_{Bk},$$

which is well defined since $Be \equiv 0 \mod d$.

The kernel (image resp.) of this homomorphism will be denoted by $\Gamma_A$ ($H_A$ resp.). Notice that $\Gamma_A$ is the subgroup of $\Gamma(q')$, which is generated by the images of the $g_k$ such that $Bk \equiv 0 \mod d$.

3.2. The birational model. We recall that two rational maps between algebraic varieties $f_i : X \rightarrow Y_i$ for $i = 1, 2$ are said to be birationally equivalent if there is a Zariski open subset $U$ of $X$ and there are Zariski open subsets $U_i \subset Y_i$ with an isomorphism $\phi : U_1 \rightarrow U_2$ such that $\phi \circ f_1 = f_2$ on $U$.

**Theorem 3.1.** Let $A$ be an invertible $n \times n$ matrix with integer entries such that $X_A$ is irreducible and (2.3) holds. Then the composition $q_A \circ \phi_A$ is birational to the quotient map $X_d \rightarrow X_d/\Gamma(q')$; hence $X_d/\Gamma(q')$ is birational to $\overline{M}_A$.

**Proof.** The composition $q_A \circ \phi_A$ is given by (2.4). Also, recall the map $\pi$ defined in (2.3). The composition of $q_A \circ \phi_A$ and $\pi$ yields a map from the Fermat variety $X_d$ to $V \cong \mathbb{P}^{n-1}$, which corresponds to an abelian extension with group $\Gamma_d$ of function fields - recall that $X_d$ is the Fermat variety and $y_i = y_i^d$. This means that $X_d \rightarrow V$ is the quotient for the group $\Gamma_d$, namely $X_d/\Gamma_d = V$. Therefore, by abelian Galois
theory, each subfield is obtained as an invariant field under a finite subgroup of \( \Gamma_d \).

Thus, the map \( X_d \to \overline{M}_A \) corresponds to a quotient by a finite subgroup of \( \Gamma_d \).

Now, we show that this subgroup is isomorphic to \( \Gamma(q') \). The map \( \pi \) corresponds to an abelian extension of function fields with group \( \mathbb{Z}/a'\mathbb{Z} \), where \( d = m'a' \) and \( m' = \gcd(d, q_1, \ldots, q_n) \). The group \( \Gamma_d \) acts on \( \overline{M}_A \) only through the variable \( u_0 \), and \( g_k : u_0 \to \zeta_m^k u_0 \), where \( \zeta_m \) is a primitive \( m' \)-th root of unity. The kernel of this action is exactly \( \Gamma(q') \), hence the map \( X_d \to \overline{M}_A \) corresponds to an extension of function fields with group \( \Gamma(q') \). Hence, the claim follows. \( \square \)

By the Theorem above, the order of the group \( \Gamma(q') \) is \( q^{n-2}m' \).

**Corollary 3.2.** The maps \( \phi_A : X_d \to X_A \) and \( q_A : X_A \to \overline{M}_A \) are birational to quotient maps. In particular, \( \overline{M}_A (\overline{M}_A \text{ resp.}) \) is birational to \( X_d/\Gamma_A (X_A/H_A \text{ resp.}) \).

**Proof.** By Theorem 3.1, the composition of \( \phi_A \) and \( q_A \) is a quotient map, namely:

\[
X_d \to X_A \to \overline{M}_A \approx X_d/\Gamma(q').
\]

The proof follows easily from arguments similar to those in Theorem 2.6 in \( \square \).

Now, we prove that the equations (2.6) and (2.8) give a very singular model for a manifold with \( h^{n-2,0} = 1 \). Assume, \( q_i' > 0 \). We recall that the vector space of holomorphic \((n-2)\)-forms on a smooth hypersurface \( X = Z(F) \) of degree \( d \geq n \) in \( \mathbb{P}^{n-1} \) has a basis of the form

\[
\omega_{b,F} = \text{Res}_X \left( \sum_{i=1}^n (-1)^i y_1^i \cdots y_n^i \frac{\sum_{i=1}^n (-1)^i y_1^i \cdots y_n^i \wedge dy_1 \wedge \cdots \wedge dy_n}{F} \right),
\]

where \( b = (b_1, \ldots, b_n) \) and \( b_i \in \mathbb{Z}_{\geq 0}, \sum_i b_i = d - n \).

**Proposition 3.3.** There exists a unique holomorphic \((n-2)\)-form on any resolution of \( \overline{M}_A \).

**Proof.** The action of an element \( g_k \) in \( \Gamma(q') \) on \( \omega_{b,F} \) is given by \( \zeta^{\sum(b_i+1)k_i} \), where \( b_i \in \mathbb{Z}_{\geq 0} \). A form is invariant with respect to \( \Gamma(q') \) if and only if \( b_i + 1 = q_i' \) for all \( i \). The unique invariant form \( \omega_{b,F} \) with vector \( b = q' - e \) descends to a form on the quotient \( X_d/\Gamma(q') \approx \overline{M}_A \). \( \square \)

### 3.3. Some examples

**Example A.** Let us consider \( F_A := x_1^5 + x_2^{10} + x_3^{10} + x_4^{10} + x_5^3 = 0 \) in weighted projective space \( W\mathbb{P}^4(2,1,1,1,5) \). It is easy to check that the corresponding hypersurface is smooth and does not intersect the singularities of \( W\mathbb{P}^4(2,1,1,1,5) \), which are two isolated points.

The matrix \( A \) is given by \( \text{diag}(5,10,10,10,2) \). The matrix \( B \) is \( \text{diag}(2,1,1,1,5) \) and \( d = 10 \). The condition \( 'eA^{-1}e = 1 \) is satisfied. Moreover,

\[
q = (2,1,1,1,5), \quad q' = (2,1,1,1,5)
\]

so \( q_i' > 0 \) for any \( i = 1, \ldots, 5 \). The equations cutting out \( \overline{M}_A \) in \( \mathbb{P}^5 \) are given by

\[
u_1 + u_2 + u_3 + u_4 + u_5 = 0,
\]

\[
u_0^1 = u_1^2u_2u_3u_4u_5^5.
\]
The integer $d = 10$. Generators for the groups $\Gamma(q')$, $\Gamma_A$ and $H_A$ are as follows. Consider the elements $v_1, \ldots, v_4 \in (\mathbb{Z}/10\mathbb{Z})^5$ given by:

$$v_1 = (0, 0, 0, 5, 1), \quad v_2 = (0, 0, 1, 4, 1),$$
$$v_3 = (1, 0, 0, 3, 1), \quad v_4 = (0, 1, 0, 4, 1),$$

then $q' \cdot v_i \equiv 0 \mod 10$ for $i = 1, \ldots, 4$. Moreover, we have $e = 8v_1 + v_2 + v_3 + v_4$; hence

$$\Gamma(q') = \langle v_1, v_2, v_3 \rangle \cong \mu_{10}^3.$$  

The group $\Gamma_A$ is isomorphic to $\mu_{10}$ and is generated by $(5, 0, 0, 0, 6)$. Finally, set

$$w_1 = (0, 0, 1, 4, 5), \quad w_2 = (2, 0, 0, 3, 5), \quad w_3 = (0, 1, 0, 4, 5).$$

As $w_1 + w_2 + w_3 = q'$, we have

$$H_A = \langle w_1, w_2 \rangle \cong \mu_2 \subset Aut(X_A).$$

The isomorphism $H_A \cong \mu_2$ was first suggested in [11]. Finally, as mentioned in [16], notice that $h^{2,1}(X_A) = 1$.

**Example B.** Let us consider the equation $F_A := x_1^5x_5 + x_2^3 + x_3x_4^2 + x_2x_3^3 = 0$ in weighted projective space $\mathbb{P}^4(1, 5, 5, 4, 10)$. In this case, we do not have a Fano toric variety since $4$ does not divide $25 = 1 + 5 + 5 + 4 + 10$. The zero locus does not intersect the singularities of $\mathbb{P}^4(1, 5, 5, 4, 10)$ and $F_A$ is a smooth variety. The matrices $A$ and $B$ are given by

$$A := \begin{pmatrix}
15 & 0 & 0 & 0 & 1 \\
0 & 5 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 \\
0 & 0 & 1 & 5 & 0 \\
0 & 1 & 0 & 0 & 2
\end{pmatrix}, \quad B := \begin{pmatrix}
10 & 1 & 0 & 0 & -5 \\
0 & 30 & 0 & 0 & 0 \\
0 & 0 & 30 & 0 & 0 \\
0 & 0 & -6 & 30 & 0 \\
0 & -15 & 0 & 0 & 75
\end{pmatrix}.$$

The integer $d$ equals 150. Moreover, we have

$$q = 6(1, 5, 5, 4, 10), \quad q' = (10, 16, 24, 30, 70)$$

and $\sigma' = \gcd(d, q_1', \ldots, q_5') = 2$. Notice that $q_i' > 0$ for any $i = 1, \ldots, 5$.

A birational model of $\overline{M}_A$ is cut out by the equations

$$(3.1) \quad u_0 = u_1^5u_2u_3^{12}u_4^{15}u_5, \quad u_1 + \ldots + u_5 = 0.$$  

Consider the vectors $r_1 \in \mathbb{Z}/d\mathbb{Z}$ given by:

$$r_1 = (0, 0, 75, 0, 0), \quad r_2 = (0, 1, 1, 0, 8),$$
$$r_3 = (1, 0, 0, 0, 2), \quad r_4 = (0, 0, 1, 6), \quad r_5 = (0, 0, 5, 0, 9);$$

then $9e = r_1 + 9r_2 + 9r_3 + 9r_4 - 15r_5$. The vectors $r_i$ generate $\Gamma(q')$; in fact, the following holds:

$$\Gamma(q') \cong \langle r_1, r_2, r_3, r_4, r_5 \rangle/\langle r_1 + 9r_2 + 9r_3 + 9r_4 - 15r_5 \rangle \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/150\mathbb{Z})^3$$

By using Magma, it is possible to check that

$$\Gamma_A \cong (\mathbb{Z}/150\mathbb{Z})^3,$$

$$H_A \cong \mathbb{Z}/2\mathbb{Z}.$$  

The group $H_A$ is generated by $g(0.75, 0.75, 0.75)$, which maps $(x_1 : \ldots : x_5)$ to $(x_1 : -x_2 : -x_3 : x_4 : -x_5)$. Recall that $X_A \to \overline{M}_A \approx X_A / H_A$ is a double cover.

**Example C.** Consider the weighted homogeneous polynomial

$$F_A := x_1^2 + x_2^3 + x_3^{18} + x_4^{18} + x_5^{18}$$
It gives a quasi-smooth locus in weighted projective space $\mathbb{P}^4(9,6,1,1,1)$. The matrices $A$ and $B$ are given by $\text{diag}(2,3,18,18,18)$ and $\text{diag}(9,6,1,1,1)$, respectively.

The integer $d$ is equal to 18 and we have
\[ q = q' = (9,6,1,1,1). \]

Notice that $q'_i > 0$ for any $i$. Moreover, using Magma we found that
\[ \Gamma(q) \cong (\mathbb{Z}/18\mathbb{Z})^3; \]
\[ H_A \cong \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}; \]
\[ \Gamma_A \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z}. \]

The Calabi-Yau $X_A$ has a singularity at $[-1,1,0,0,0]$, which is a singularity of $\mathbb{P}^4(9,6,1,1,1)$. As explained in [13], this singularity can be blown-up so as to get a Calabi-Yau in the (toric) blow-up of $\mathbb{P}^4(9,6,1,1,1)$.

**Example D.** When the group $H_A$ is trivial, it is possible to write down an explicit birational inverse between from $\overline{M}_A$ to $X_A$. We show it in one specific example. Let us consider the polynomial
\[ F_A := x_1^5 + x_2^9 x_3 + x_3^9 x_4 + x_4^{10} + x_5^2 \]
where $A$ is the matrix
\[
A := \begin{pmatrix}
5 & 0 & 0 & 0 & 0 \\
0 & 9 & 1 & 0 & 0 \\
0 & 0 & 9 & 1 & 0 \\
0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]

The variety $X_A := Z(F_A)$ is contained in $\mathbb{P}^4(2,1,1,1,5)$. The matrix $B$ is given by
\[
B := \begin{pmatrix}
162 & 0 & 0 & 0 & 0 \\
0 & 90 & -10 & 1 & 0 \\
0 & 0 & 90 & -9 & 0 \\
0 & 0 & 0 & 81 & 0 \\
0 & 0 & 0 & 0 & 405
\end{pmatrix},
\]
where $AB = BA = 810I$. The map $q_A : \mathbb{P}^4(2,1,1,1,5) \rightarrow \mathbb{P}^5$
\[
(x_1 : \ldots : x_5) \mapsto (u_0 : \ldots : u_5) := (x_1 x_2 \ldots x_5 : x_1^5 : x_2^9 x_3 : x_3^9 x_4 : x_4^{10} x_2 : x_5^2)
\]
maps $X_A$ to the variety
\[
\overline{M}_A := Z(u_1 + \ldots + u_5, -u_1^{810} + u_1^{162} u_2^{90} u_3^{73} u_4 u_5^{405}) \subset \mathbb{P}^5.
\]

An explicit birational inverse for $q_A$ is given by the following map:
\[
\begin{align*}
M^{162} x_1 &= u_1^{65} u_2^{54} u_3^{12} u_4^{15} u_5^{162}, \\
M^{81} x_2 &= u_1^{-2} u_2^{-8} u_3^{-11} u_4^{-8} u_5^{-5}, \\
M^{81} x_3 &= u_1^{18} u_2^{19} u_3^{-1} u_4 u_5^{45}, \\
M^{81} x_4 &= u_2^9 u_3^{-10} u_4^{-7}, \\
M^{405} x_5 &= u_1^{81} u_2^{90} u_3^{-10} u_4 u_5^{203}.
\end{align*}
\]
where $M = x_1^2 x_2^3 x_3 x_4 x_5^2$.

4. A ONE-DIMENSIONAL FAMILY

Let us consider the one-parameter family $X \to \mathbb{P}^1$ of degree $d$ hypersurfaces in $\mathbb{P}^{n-1}$ with $X_t = X_{d,t} = Z(F_{d,t})$, where

$$F_{d,t} = \sum_{i=1}^{n} y_i^d - t y_1 q'_1 \cdots y_n q'_n.$$

Clearly, this is a one-dimensional deformation of the Fermat variety $X_d$. If we apply the map $\phi_A: X_{d,t} \subset \mathbb{P}^{n-1} \to X_{A,t} \subset W \mathbb{P}(q_1, \ldots, q_n)$, the image of $X_{d,t}$ is given by $X_{A,t} = Z(F_{A,t})$, where

$$F_{A,t} = F_A - t x_1 x_2 \cdots x_n.$$

Under the composition $q_A \circ \phi_A$ the image of (4.1) is given by the equations

$$\sum_{i=1}^{n} u_i - tu_0 = 0, \quad u_0 = u_1^{q'_1} \cdots u_n^{q'_n}.$$

If $t \neq 0$, we solve for $u_0$ and get the equation

$$\left( \sum_{i}^{n} u_i \right)^d = t^d u_1^{q'_1} \cdots u_n^{q'_n}.$$

The group $\Gamma(q')$ acts on each $X_{A,t}$ since $\sum q'_i$ equals $d$. Denote by $\overline{\mathcal{M}} \to \mathbb{P}^1$ the family given by (4.3). By the universal property of the quotient there exists a map $\Psi$ between $\mathcal{X}/\Gamma(q')$ and $\overline{\mathcal{M}}$, which commutes with the projection map on $\mathbb{P}^1$.

**Proposition 4.1.** The map $\Psi$ yields a birational morphism from $\mathcal{X}/\Gamma(q')$ to $\overline{\mathcal{M}}$.

**Proof.** It suffices to compare the degree of the quotient map $\mathcal{X} \to \mathcal{X}/\Gamma(q')$, which is $\#\Gamma(q')$, with that of the map $\mathcal{X} \to \overline{\mathcal{M}}$. Let

$$(l_0 : l_1 : \cdots : l_n) = \left( \prod_{k}^{n} y_k^{q'_k} : y_1^d : \cdots : y_n^d \right)$$

be a generic point in $\overline{\text{Im}(q_A \circ \phi_A)}$. Thus, we have $y_j = \zeta^{k_j} \sqrt[d]{l_j}$, where $\zeta$ is a primitive $d$-th root of unity. Hence, we get

$$\zeta^{\sum q'_j k_j} \sqrt[d]{l_1^{q'_1} \cdots l_n^{q'_n}} = l_0.$$

On the other hand, by the equation of $F_{A,t}$, we must have

$$\sum_{j} q'_j k_j \equiv 0 \mod d.$$

Recall that $\sum q'_j \equiv 0 \mod d$, so we can take the quotient of the set (4.3) of solutions by $(1,1,\ldots,1)$. The degree of $\mathcal{X} \to \overline{\mathcal{M}}$ is thus equal to $\#\Gamma(q')$. 

$\Box$
4.1. Some Birational Families. Let us examine some Calabi-Yau varieties $X_A$ which have a one-dimensional versal deformation family, so it may be described by the family in the section above. In Schimmrigk’s list (see [20]), we found twelve entries with $h^{2,1} = 1$. Since the deformation space is one dimensional, we take into account one-dimensional families corresponding to these entries of the list. The generic members of the families, which are denoted by the same letter, have the same Euler characteristic. We have

\[ A_1(t) = x_1^4 x_3 + x_2 x_5^3 + x_7^2 x_4 + x_7^2 + x_2^3 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(75, 84, 86, 98, 343), \]

\[ A_2(t) = x_1^3 x_2 + x_2^2 x_3 + x_7^2 + x_4 + x_2^5 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(43, 48, 56, 98, 147), \]

\[ A_3(t) = x_2^2 + x_1^3 x_3 + x_7^2 + x_4 + x_4 x_5^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(48, 49, 56, 86, 153) \]

\[ A_4(t) = x_1^2 x_2 + x_1^3 x_3 + x_2^2 + x_4 + x_4 x_5^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(42, 43, 49, 75, 134), \]

\[ B_1(t) = x_1^{10} x_2 + x_2^{10} x_3 + x_3^{10} + x_2 + x_2^3 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(73, 80, 90, 162, 405), \]

\[ B_2(t) = x_1^9 x_2 + x_2^9 x_3 + x_3^9 + x_2 + x_2^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(64, 72, 73, 115, 324), \]

\[ B_3(t) = x_1^9 x_2 + x_2^9 x_3 + x_3^9 + x_4 + x_3 + x_2^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(40, 45, 73, 81, 166), \]

\[ B_4(t) = x_1^9 + x_2^9 + x_3^9 + x_4 + x_3 + x_2^5 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(40, 45, 63, 64, 148), \]

\[ C_1(t) = x_1^6 + x_2^5 + x_2 + x_2 + x_2^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(52, 60, 63, 75, 125), \]

\[ C_2(t) = x_1^5 + x_2^5 + x_2 + x_4 + x_4 + x_2^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(48, 50, 60, 63, 79), \]

\[ D_1(t) = x_1^5 + x_2^4 + x_2 + x_2^4 + x_2 + x_4 + x_4 + x_2^2 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(41, 48, 51, 52, 64), \]

\[ D_2(t) = x_1^5 + x_2^4 + x_2 + x_2^4 + x_2 + x_4 + x_4 + x_2^5 - t x_1 x_2 x_3 x_4 x_5 \subset A^1 \times W_0^4(51, 60, 64, 65, 80). \]

Let $V_5(t), V_6(t), V_8(t), V_{10}(t)$ be the four hypergeometric families on page 134 in [15]. For each of these families $V_j(t), j = 5, 6, 8, 10$, there is a group acting on the family such that the mirror $W_j(t)$ of $V_j(t)$ can be described as a resolution of the quotient $V_j(t)/G$. The singular members have one orbit of ordinary nodes under the action of $G$ and the resolution of the quotient is a rigid Calabi-Yau threefold, i.e., $h^{2,1} = 0$.

Theorem 4.2. The following birational equivalences hold:

\[ A_1(t) \cong A_2(t) \cong A_3(t) \cong A_4(t) \cong W_8(t), \]

\[ B_1(t) \cong B_2(t) \cong B_3(t) \cong B_4(t) \cong W_{10}(t), \]

\[ C_1(t) \cong C_2(t) \cong W_6(t), \]

\[ D_1(t) \cong D_2(t) \cong W_5(t). \]

Proof. It is easy to check that the general member of the families above is a singular Calabi-Yau in four weighted projective space. The singular locus is a rational curve. For some values of $t$ there are extra singularities that are ordinary nodes, namely:

| $A_i(t)$ | $t^8 = 2^{16}$ | $B_i(t)$ | $t^{10} = 800000$ |
|----------|----------------|----------|-----------------|
| $C_i(t)$ | $t^6 = 3^{2} 2^{4}$ | $D_i(t)$ | $t^5 = 5^3$ |
Let us focus on the two families $D_1(t)$ and $D_2(t)$. The proof for the other cases can be dealt with analogously. Define the two families

$$X_{d_1,t} = \left\{ \sum_{i=1}^{5} y_i^{320} - \prod_{i} y_i^{64} = 0 \right\},$$

$$X_{d_2,t} = \left\{ \sum_{i=1}^{5} y_i^{1280} - \prod_{i} y_i^{256} = 0 \right\}.$$

Let $A_1$ and $A_2$ be the following matrices:

$$A_1 := \begin{pmatrix} 5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \quad A_2 := \begin{pmatrix} 0 & 0 & 5 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Using Magma we found that the groups $H_{A_1}$ and $H_{A_2}$ are trivial - this happens for all 12 families.

We thus have

$$q_{A_1} \circ \phi_{A_1} : X_{d_1,t} \to \overline{M}_{A_1,t}$$

$$(y_1 : y_2 : \ldots : y_5) \mapsto \left( \prod_{i} y_i^{64} : y_1^{320} : \ldots : y_5^{320} \right)$$

and, similarly,

$$q_{A_2} \circ \phi_{A_2} : X_{d_2,t} \to \overline{M}_{A_2,t}$$

$$(y_1 : y_2 : \ldots : y_5) \mapsto \left( \prod_{i} y_i^{256} : y_1^{1280} : \ldots : y_5^{1280} \right)$$

It is easy to check that $\overline{M}_{A_1,t} \cong \overline{M}_{A_2,t} \cong \{ \sum_i u_i - tu_0, u_0^5 = u_1 \ldots u_5 \} \approx W_5(t)$.

Since $H_{A_1}$ and $H_{A_2}$ are trivial, $D_1(t)$ and $D_2(t)$ are birational since they are both birational to $W_5(t)$.

\[\square\]

4.2. Picard-Fuchs equations. When $X_A$ is a Calabi-Yau hypersurface, the Hodge number $h^{2,1}(X_A)$ gives the number of independent parameters of deformations of $X_A$. There exists a system of partial differential equations, the so called GKZ-hypergeometric system (see [9]), which yield Picard-Fuchs equations for the variation of periods along families with central fiber $X_A$. When $h^{2,1}(X_A) = 1$, the Picard-Fuchs equation can also be found via a generalization of the Griffiths-Dwork method for hypersurfaces in weighted projective space: see, for instance, [16].

5. $\overline{M}_A$ and the mirror family of Calabi-Yau hypersurfaces in weighted projective space.

Assume $X_A$ is a Calabi-Yau manifold (as defined in Section 2) in weighted projective space $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$, where $q_i | Q$ and $Q = \sum_j q_j$. Batyrev’s mirror construction (see, for instance, [12]) depends only on the polytope $\Delta$ associated to the toric variety $\mathbb{P}^{n-1}(q_1, \ldots, q_n)$ and not on the matrix $A$. As explained, for instance in [12], the Calabi-Yau varieties in the mirror family $W \to \mathbb{P}^1$ of a general section of the anticanonical bundle $\mathcal{O}(Q)$, with $Q = \sum_j q_j$, can be represented
as compactifications of complete intersections of the affine hypersurfaces in \((\mathbb{C}^*)^n\) given by
\begin{equation}
    t_1 + \ldots + t_n = 1, \quad t_1^{q_1} \ldots t_n^{q_n} = x.
\end{equation}

Let \(W_x\) be the fiber over the point \(x \in \mathbb{P}^1\). By comparing (4.2) and (5.1), the following holds.

**Proposition 5.1.** The compactification of \(W_1\) is given by the equations (4.2) that define the Shioda quotient \(\tilde{M}_{t^A,1}\) for any matrix \(A\).

**Proof.** Let \(A\) be a matrix as in Section 1. If we start from the family \(F_{t^A,t}\), the equations (4.2) become:
\begin{equation}
    \sum_i u_i - tu_0 = 0, \quad u_0^d = u_1^{q_1} \ldots u_n^{q_n}.
\end{equation}

Since \(\sum_j q_j = d\), the claim follows. \(\square\)

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