Rare and radiative kaon decays

Giancarlo D’Ambrosio
INFN-Sezione di Napoli, Via Cintia, 80126 Napoli, Italia
E-mail: gdambros@na.infn.it

Abstract. We discuss theoretical issues in radiative rare kaon decays. The interest is twofold: to extract useful short-distance information and understand the underlying dynamics. We emphasize channels where either we can understand non-perturbative aspects of QCD or there is a chance to test the Standard Model. An interesting channel, $K^+ \to \pi^+\pi^0e^+e^-$, is studied also in connection with the recent experimental NA48 results. Motivated by LHCB results on $K_S \to \mu^+\mu^-$ we discuss other channels like $K_{S,L} \to l^+l^-l^+l^-$ and $K_S \to \pi^+\pi^-l^+l^-$. Motivated by recent theoretical work by Buras and collaborators we study also the $K^± \to \pi^±l^+l^-$ form factor.

1. Introduction

Kaon decays are an important place to study non-perturbative aspects of QCD and test the Standard Model. Indeed some channels are completely dominated by long-distance dynamics, such as the CP-conserving amplitude for $K \to \pi\pi$ and others, like $K \to \pi\nu\bar{\nu}$ [1], which are described in terms of pure short-distance physics. In this review we will be mostly concerned with kaon decays involving electromagnetic interactions and thus long-distance phenomena are not negligible. However, as we shall see, it is still possible in these channels to extract the short-distance component with an accurate analysis. Indeed there are plenty of motivations to look for new physics (NP) in these kaon decays [2].

The channels which will be considered here are $K \to \pi\pi\gamma$, $K \to \pi\pi\nu\bar{\nu}$, $K_{S,L} \to l^+l^-$, $K_{S,L} \to l^+l^-l^+l^-$ and $K_S \to \pi^+\pi^-l^+l^-$. Recently, Buras and collaborators, motivated by their previous work where weak hadronic matrix elements are evaluated in their model (BBG model) and also by recent lattice results, are investigating New Physics to explain the experimental value of $\epsilon'$ [2, 3, 4, 5], Motivated by this interest we study also the $K^± \to \pi^±l^+l^-$ form factor in the BBG model [6].

2. $K \to \pi\pi\gamma$ and $K \to \pi\pi\nu\bar{\nu}$-decays

2.1. $K \to \pi\pi\gamma$

Let’s discuss $K(p) \to \pi(p_1)\pi(p_2)\gamma(q)$ decays: according to gauge and Lorentz invariance we decompose $K(p) \to \pi(p_1)\pi(p_2)\gamma(q)$ decays, in electric ($E$) and magnetic ($M$) amplitudes [7]. Particularly interesting are the recent interesting NA48/2 data regarding $K^+ \to \pi^+\pi^0\gamma$ decays [8].

Due to the $\Delta I = 3/2$ suppression of the bremsstrahlung, interference between $E_B$ and the electric dipole ($E_{DE}$) and magnetic transitions ($M_{DE}$) can be measured. Defining $z_i = p_i \cdot q/m_K^2 \quad z_3 = p_K \cdot q/m_K^2$ and $z_3z_+ = m^2 - W^2$, ($W^2$ is defined in this equation) we can study
also the interesting CP bound was obtained [8]:

$$\frac{\Gamma(K^+ \to \pi^+\pi^0\gamma) - \Gamma(K^- \to \pi^-\pi^0\gamma)}{\Gamma(K^+ \to \pi^+\pi^0\gamma) + \Gamma(K^- \to \pi^-\pi^0\gamma)} < 1.5 \times 10^{-3} \text{ at 90\% CL.} \quad (2)$$

2.2. $K \to \pi\pi\nu\nu$-decays

Historically kaon four body semileptonic decays, $K_{\pi\pi}$ have been studied as a tool to tackle final state rescattering effects in $K \to \pi\pi$-decays: crucial to this goal has been finding an appropriate set of kinematical variables which would allow i) to treat the system as two body decay in dipion mass $M_{\pi\pi}$ and dilepton mass $M_{\ell\ell}$ and ii) to identify appropriate kinematical asymmetries to extract observables crucially dependent on final state interaction [9].

In Fig. 1 we show the traditional kinematical variables for the four body kaon semileptonic decay which allow to write the four body phase space $\Phi$ in terms of the two two-body phase spaces: $\Phi_{\pi}$ and $\Phi_{\ell}$ from [9, 10]. We extend the $K_{\pi\pi}$ kinematical and dynamical description of the amplitude to describe the $K \to \pi\pi\nu\nu$-decays; in particular we write the long distance contribution in Fig. 2 as

$$M_{LD} = \frac{e}{q^2} [\pi(k^-)\gamma^\mu v(k_+)] H_\mu(p_1, p_2, q), \quad (3)$$
where $H_\mu$ is the electroweak hadronic vector, which can be written in terms of three form factors $F_{1,2,3}$:

$$H_\mu(p_1,p_2,q) = F_1p_1^\mu + F_2p_2^\mu + F_3\varepsilon^{\mu\nu\rho\sigma}p_1\nu p_2\rho q_\sigma.$$

Then

$$\frac{d^3\Gamma}{dT_1^\gamma dT_2^\gamma dq^2d\cos\theta_1d\phi} = A_1 + A_2\sin^2\theta_1 + A_3\sin^2\theta_1\cos^2\phi$$

$$+ A_4\sin 2\theta_1 \cos\phi + A_5\sin \theta_1 \cos \phi + A_6 \cos \theta_1$$

$$+ A_7 \sin \theta_1 \sin \phi + A_8 \sin 2\theta_1 \sin \phi + A_9 \sin^2 \theta_1 \sin 2\phi,$$

where $\theta_1$ and $\phi$ are two variables for $K_{L\ell\gamma}$ decays and $A_i$ are dynamical functions that can be parameterized in terms of the three form factors. The amplitudes $A_{8,9}$, odd in $\theta_1$, are also linearly dependent on the final state, establishing a clear way to determine them; while $A_{5,6,7}$ are generated by interference with the axial leptonic current.

One can easily show that the Bremsstrahlung, direct emission and electric interference terms contribute to $A_{1,4}$. In contrast, $A_{8,9}$ receive contributions from the electric-magnetic interference terms (BM and EM) and therefore capture long-distance induced P-violating terms. $A_{5,6,7}$ are also P-violating terms but generated through the interference of $Q_\pi A_{\gamma}$ with long distances.

Short distance physics can be studied in $A_{5,6,7}$ [11] and in the diplane angular asymmetry proportional to $A_{8,9}$. This last observable is large and has been measured by KTeV and NA48 [12, 13]. However this observable is proportional to electric (bremsstrahlung) and magnetic interference, both contributions known already from $K_{L} \to \pi^+\pi^-\gamma$. In fact it was known that these contributions were large and they may obscure smaller but more interesting short distance physics effects.

We have performed a similar analysis for the decay $K^+ \to \pi^+\pi^0\ell^+\ell^-$ trying to focus on i) short distance physics and ii) all possible Dalitz plot analyses to disentangle all possible interesting long and short distance effects [10]. This decay has not been observed yet, and the interesting physics is hidden by bremsstrahlung [10, 14, 15]

$$B(K^+ \to \pi^+\pi^0\ell^+\ell^-)_B \sim (330 \pm 15) \cdot 10^{-8}$$

$$B(K^+ \to \pi^+\pi^0\ell^+\ell^-)_M \sim (6.14 \pm 1.30) \cdot 10^{-8},$$

and so Dalitz plot analysis is necessary in order to capture the more interesting direct emission contributions. The $K^+ \to \pi^+\pi^0\ell^+\ell^-$-amplitude is then written as in Eq. 3.

We have studied the decay $K^{\pm} \to \pi^\pm\pi^0\ell^+\ell^-$, dominated by long distance through one photon exchange in Fig. 2. We computed, the large long distance contributions and the relatively small short distance ones in Ref. [10, 15]. We have succeeded in describing $K^{\pm} \to \pi^\pm\pi^0\ell^+\ell^-$-optimizing the theoretical and experimental knowledge of $K(p) \to \pi(p_1)\pi(p_2)\gamma(q)$ decays.

Relatively to the $K^{\pm} \to \pi^\pm\pi^0\gamma$ the possibility to go kinematically at large $q^2$ opens the possibility to beat the bremsstrahlung: at large dilepton invariant mass the bremsstrahlung can be even 100 times smaller than the magnetic contribution, at the price, however, of decreasing the statistics. Indeed compared to $K^{\pm} \to \pi^\pm\pi^0\gamma$ decays the possibility to measure also the $q^2$-differential rates we have the possibility to measure also the $q^2$-differential rates

$$\frac{\partial^2\Gamma_{IB}}{\partial T_1^\gamma \partial W^2 \partial q^2}$$

$$d\Phi = \frac{1}{4m_K^5} (2\pi)^5 \int ds_\pi \int ds_\ell \lambda^{1/2}(m_K^2, p_\pi^2, q^2) \Phi_\pi \Phi_\ell.$$

$$\Phi_\ell = \frac{1}{4m_K^5} (2\pi)^5 \int ds_\ell \lambda^{1/2}(m_K^2, p_\ell^2, q^2) \Phi_\ell.$$
Then defining \( q^2 = M_{ee}^2 \) and the \( \pi\pi \) invariant mass \( p_{\pi\pi}^2 = M_{\pi\pi}^2 \), we can write

\[
d^5\Phi = \frac{1}{2^{14}\pi^6m_K^3} \frac{1}{8\pi} \sqrt{1 - \frac{4m^2_{\pi\pi}}{q^2}} \lambda^{1/2}(m^2_{K\pi\pi}, p^2_{\pi}, q^2) \lambda^{1/2}(p^2_{\pi}, m^2_{\pi\pi}, m^2_{\pi\pi}) dp_{\pi}^2 dq_{\pi}^2 d\cos \theta_{\pi} d\cos \theta_{\pi} d\phi, \quad (9)
\]

**Figure 3.** \( q \)-dependence of Bremsstrahlung (solid line), the dashed lines (from bigger to smaller) are 100M, 100BE and 300E. Error bars are omitted. The band corresponds to changing counterterms .

The further advantage is that at each \( q^2 \) we can measure the different Dalitz plot, as we can see in (4), obtained at \( \sqrt{q^2} = 50 \text{ MeV} \). Also several short distance observables can be measured by appropriate kinematical analyses [10, 15]. In Table 1 we show the ratios of Bremsstrahlung (B) to magnetic (M) and to various interferences. Our effort has been to write the \( K^\pm \rightarrow \pi^+\pi^0e^+e^- \) amplitude in terms of experimental known Bremsstrahlung (B) and DE, electric (E) and magnetic (M), transitions [10]

**Table 1.** Branching ratios for the Bremsstrahlung and the relative weight of the rest of the contributions for different cuts in \( q \), starting at \( q_{min} \) (first row) and ending at 180 MeV. In the last column we have also included the parity-odd magnetic-electric interference term.

| \( q_c \) (MeV) | B \( [10^{-6}] \) | B/M | B/E | B/BE | B/BM |
|-----------------|----------------|-----|-----|------|------|
| \( 2m_l \)      | 418.27         | 71  | 4405| 128  | 208  |
| 55              | 5.62           | 12  | 118 | 38   | 44   |
| 100             | 0.67           | 8   | 30  | 71   | 36   |
| 180             | 0.003          | 12  | 5   | -19  | 44   |

The big news is that NA48/2 has reported a final measurement of this branching: the experiment selects 3 reconstructed tracks coming from one decay vertex, then Particle ID for \( e^+/e^- \) separation; then two reconstructed \( \gamma \) clusters are compatible with \( \pi^0 \) mass [16]. The number of \( K^\pm \) decays (kaon flux) is measured by using the reference channel \( K^\pm \rightarrow \pi^\pm\pi^0(\gamma) \). 5076 events have been selected with 289 background; the error is dominated by external error on \( \mathcal{B}(\pi^0 \rightarrow e^+e^-\gamma) \)

\[
(4.06 \pm 0.06_{\text{stat.}} \pm 0.04_{\text{syst.}} \pm 0.13_{\text{ext.}}) \times 10^{-6}
\]
to be compared to our prediction dominated from Bremsstrahlung \cite{10,15}

\[ B(IB) = (4.19) \times 10^{-6} \quad \text{no Isospin breaking} \quad B(TOT) = (4.29) \times 10^{-6} \]

\[ B(IB) = (4.10) \times 10^{-6} \quad \text{with Isospin breaking} \quad B(TOT) = (4.19) \times 10^{-6} \]

The amplitudes \( A_{8,9} \), odd in \( \theta_f \), are also linearly dependent on the final state, establishing a clear way to determine them; while \( A_{5,6,7} \) are generated by interference with the axial leptonic current.

CP violation has been also studied in the \( K \to \pi\pi\gamma \) and \( K \to \pi\pi e e \) decays.

3. Bardeen Buras Gerard approach and \( K^{\pm} \to \pi^{\pm}l^+l^- \) form factor

In Refs. \cite{3,4}, the authors compute loop calculations employing a cut-off regulator instead of the dimensional regularization method. Consequently, their results exhibit a quadratic dependence on the physical cut-off \( M \) which is lost in the usual chiral perturbative calculations. This quadratic dependence is a crucial ingredient in the matching of the meson and quark pictures. They argue that one can obtain a parametrization of non-perturbative QCD effects by matching a low-energy Lagrangian valid up to the scale \( M \), to the logarithmic behavior of relevant Wilson coefficients at high-energy. In this work we refer to this computational method as the Bardeen-Buras-Gérard framework (BBG). Recently Buras has advocated this determination to claim New Physics to explain the experimental value of \( \epsilon' \) \cite{5}. BBG evaluate \( K \to \pi\pi \) chiral loop with a dimensional cut-off (M) and match the quadratic divergences of the \( Q_i(M) \) weak matrix elements with the log of the short distance Wilson coefficient as described by the equation

\[ H_{\text{eff}} = \sum_i c_i(\mu) \ Q_i(\mu) \quad (10) \]

We have studied in the BBG contest the \( K^{\pm} \to \pi^{\pm}l^+l^- \) form factor described in CHPT by a loop function, \( W_{\pi\pi}^{\pm}(z) \) and a polynomial in \( z \) with experimentally determined coefficients \( a_+ \) and \( b_+ \) \cite{17}. These parameterize the intermediate region between low and high energy regimes. Our goal is to predict the values of these two coefficients using BBG framework \cite{6}. Indeed in Fig. 3 we show: i) (left plot) the \( a_+ \) as a function of \( M \) without the vectors contributions and ii) (right plot) adding the vectors contributions particularly important are the novel weak vector couplings.

4. \( K_{S,L} \to l^+l^- \), \( K_{S,L} \to l^+l^-l^+l^- \) and \( K_S \to \pi^+\pi^-l^+l^- \)

The recent LHCB limit on \( K_S \to \mu\mu \) \cite{13} in Table 4 is close to test interesting New Physics (NP) models \cite{18}. A high precision measurement can test the short distance (SD) SM but it requires to improve the long distance (LD) prediction \cite{18,19} with auxiliary channels \cite{20}. \( K_L \to \mu \mu \): the small ratio SD/LD \( \sim \frac{1}{30} \) may obscure an experimental improvement on the rate \cite{18}. The situation would be a bit ameliorated if the still unknown sign of \( A(K_L \to \gamma\gamma) \) would be either theoretically or experimentally determined. Help to this ambiguity could come from the experimental study of \( K_{S,L} \to l^+l^-l^+l^- \) \cite{20} As shown in Table 4 these channels are at reach in a high intensity machine and they may also give LD distance info needed for a better control of \( K_L \to \mu\mu \). These four body decays have also a peculiar feature, similarly to \( K_{S,L} \to \pi^+\pi^-e^+e^- \), the two different helicity amplitudes interfere; then one can measure the sign \( K_L \to \gamma^+\gamma^\star \to l^+l^-l^+l^- \) by studying the time interference \( K_S \ K_L \) which it has a decay length \( 2\Gamma_S \) \cite{20}.

\[ K_S \to \pi^+\pi^-l^+l^- \]

Actually only \( K^+ \to \pi^+\pi^0e^+e^- \) has been studied so far \cite{10} while \( K_S \to \pi^+\pi^-\mu^+\mu^- \), more feasible experimentally, is in progress; however generic features can be already extracted from
Figure 5. Left: In blue, the variation of $a_+$ as a function of $M$ in GeV. The dotted green curve represents the contribution proportional to $C_-(M^2)$ and the dashed orange curve the one proportional to $C_7(M^2)$. The vertical dashed line stands for the matching scale Right: $a_+$ as a function of $M$ in the three different frameworks: ‘BBG no vect.’ where vectors are not included, ‘BBG(vect)($a$)’ represents the contribution coming only from diagrams with vectors coupled strongly and ‘BBG(vect) ($a$) + ($b$) ’ is the case where vectors with weak couplings were included. The vertical line indicates the value $M = 0.7$ GeV.

Table 2. Interesting channels: PDG values vs theoretical estimates

| Channel                      | PDG                  | Prospects               |
|------------------------------|----------------------|-------------------------|
| $K_S \rightarrow \mu\mu$    | $< 9 \times 10^{-9}$ at 90% CL | (LD)(5.0 ± 1.5) $\cdot 10^{-12}$ NP $< 10^{-11}$ |
| $K_L \rightarrow \mu\mu$    | $(6.84 \pm 0.11) \times 10^{-9}$ | difficult: SD $<<$ LD |
| $K_S \rightarrow \mu\mu\mu\mu$ | –                     | SM LD $\sim 2 \times 10^{-14}$ |
| $K_S \rightarrow e\bar{e}\mu\mu$ | –                     | $\sim 10^{-11}$ |
| $K_S \rightarrow e\bar{e}e\bar{e}$ | –                     | $\sim 10^{-10}$ |
| $K_S \rightarrow \pi^+\pi^-\mu^+\mu^-$ | –                     | SM LD $\sim 10^{-14}$ |

Table I in Ref. [10]: we have less than 10 MeV phase space which can be extracted from the last lines of the Table I in Ref. [10] telling us that i) the Branching is expected $O(10^{-14})$ and the novel purely electric and magnetic contribution are relatively enhanced with respect to the less interesting bremsstrahlung.
5. Conclusions
In these proceedings we have studied theoretical issues in radiative rare kaon decays. An interesting channel, \( K^+ \to \pi^+ \pi^0 e^+ e^- \), is studied also in connection with the recent experimental NA48 results. Motivated by LHCb results on \( K_S \to \mu^+ \mu^- \) we discuss other channels like \( K_{S,L} \to l^+ l^- l^+ l^- \). Motivated by recent theoretical work by Buras and collaborators we study also the \( K^\pm \to \pi^\pm l^+ l^- \) form factor.

Acknowledgments
I wish to thank the organizers of KAON 2016, in particular Cristina Lazzeroni.

6. References
[1] V. Cirigliano, G. Ecker, H. Neufeld, A. Pich and J. Portoles, Rev. Mod. Phys. 84, 399 (2012) doi:10.1103/RevModPhys.84.399 [arXiv:1107.6001 [hep-ph]].
[2] A. J. Buras, arXiv:1609.05711 [hep-ph].
[3] W. A. Bardeen, A. J. Buras and J. M. Gerard, Nucl. Phys. B 293 (1987) 787. doi:10.1016/0550-3213(87)90091-5; A. J. Buras, J. M. Gérard and W. A. Bardeen, "Large \( N_c \) Approach to Kaon Decays and Mixing 28 Years Later: \( \Delta l = 1/2 \) Rule,\( \hat{B}_K \) and \( \Delta M_K \)”, Eur. Phys. J. C 74, 2871 (2014) [arXiv:1609.05711 [hep-ph]].
[4] J. M. Gerard, Acta Phys. Polon. B 21 (1990) 257.
[5] A. J. Buras, M. Gorbahn, S. Jager and M. Jamin, JHEP 1511, 202 (2015) doi:10.1007/JHEP11(2015)202 [arXiv:1507.06345 [hep-ph]].
[6] E. Coluccio Leskow, G. D’Ambrosio, D. Greynat and A. Nath, Phys. Rev. D 93, no. 9, 094031 (2016) doi:10.1103/PhysRevD.93.094031 [arXiv:1603.09721 [hep-ph]].
[7] G. D’Ambrosio and D. N. Gao, JHEP 0010, 043 (2000) doi:10.1088/1126-6708/2000/10/043 [hep-ph/0010122]; L. Cappiello and G. D’Ambrosio, Phys. Rev. D 75, 094014 (2007) [arXiv:hep-ph/0702292].
[8] J. R. Batley et al. [NA48/2 Collaboration], Eur. Phys. J. C 68, 75 (2010) [arXiv:1004.0494 [hep-ex]].
[9] J. Bijnens, G. Colangelo, G. Ecker and J. Gasser, arXiv:hep-ph/9411311 and references there
[10] L. Cappiello, O. Cata, G. D’Ambrosio and D. -N. Gao, Eur. Phys. J. C 72, 1872 (2012) [arXiv:1112.5184 [hep-ph]].
[11] L. M. Sehgal and M. Wanninger, Phys. Rev. D 46, 1035 (1992) [Erratum-ibid. D 46, 5209 (1992)]; P. Heiliger and L. M. Sehgal, Phys. Rev. D 48, 4146 (1993) [Erratum-ibid. D 60, 079902 (1999)]; J. K. Elwood, M. B. Wise, M. J. Savage, Phys. Rev. D52, 5095 (1995). [hep-ph/9504288]; J. K. Elwood, M. B. Wise, M. J. Savage, J. W. Walden, Phys. Rev. D53, 4078-4081 (1996). [hep-ph/9506287].
[12] E. Abouzaid et al. [KTeV Collaboration], Phys. Rev. Lett. 96, 101801 (2006) [hep-ex/0508010]; A. Lai et al. [NA48 Collaboration], Eur. Phys. J. C 30, 33 (2003).
[13] The Review of Particle Physics (2016) C. Patrignani et al. (Particle Data Group), Chin. Phys. C, 40, 100001 (2016).
[14] H. Pichl, Eur. Phys. J. C 20, 371 (2001) [hep-ph/0010284].
[15] S. R. Gevorkyan and M. H. Misheva, Eur. Phys. J. C 74, no. 5, 2860 (2014) [arXiv:1403.1053 [hep-ph]].
[16] Brigitte Bloch-Devaux On behalf of the NA48/2 collaboration “Study of the K to ? ?0 e+ e- decay with NA48/2 @ CERN” : these proceedings.
[17] G. D’Ambrosio, G. Ecker, G. Isidori and J. Portoles, JHEP 9808, 004 (1998) doi:10.1088/1126-6708/1998/08/004 [hep-ph/9808289].
[18] G. Isidori and R. Unterdröfer, JHEP 0401, 009 (2004) [hep-ph/0311084].
[19] G. Ecker and A. Pich, Nucl. Phys. B 366, 189 (1991); G. D’Ambrosio, D. Greynat and G. Vulvert, Eur. Phys. J. C 73, 2678 (2013) [arXiv:1309.5736 [hep-ph]].