REPRESENTATIONS OF NILPOTENT LIE GROUPS VIA MEASURABLE DYNAMICAL SYSTEMS

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Abstract. We study unitary representations associated to cocycles of measurable dynamical systems. Our main result establishes conditions on a cocycle, ensuring that ergodicity of the dynamical system under consideration is equivalent to irreducibility of its corresponding unitary representation. This general result is applied to some representations of finite-dimensional nilpotent Lie groups and to some representations of infinite-dimensional Heisenberg groups.

1. Introduction

A measurable dynamical system is a measure space \((X, \mu)\) endowed with a group action on the right \(X \times S \to X, (x, s) \mapsto x.s\), for which the measure \(\mu\) is quasi-invariant, hence \(d\mu(x.s) = j(x,s) d\mu(x)\) for a suitable a.e. defined positive measurable function \(j(\cdot, s)\) on \(X\). A scalar cocycle of this measurable dynamical system is a family \(\{a(\cdot, s)\}_{s \in S}\) of a.e. defined measurable functions on \(X\) with values in the unit circle \(T\), for which the map \(\pi_a : S \to \mathcal{B}(L^2(X, \mu))\) is a unitary representation, where

\[
\pi_a(s) : L^2(X, \mu) \to L^2(X, \mu), \quad (\pi_a(s) \varphi)(x) = j(x, s)^{1/2} a(x, s) \varphi(x.s).
\]

Our main abstract result is Theorem 2.4 which establishes conditions on the cocycle \(a\), ensuring that ergodicity of the above dynamical system is equivalent to irreducibility of the unitary representation \(\pi_a\). The unifying force of this result is then illustrated by a variety of applications, including unitary irreducible representations of finite-dimensional nilpotent Lie groups and some representations of infinite-dimensional Heisenberg groups.

Some preliminaries on measure theory.

Lemma 1.1. Let \((X, \mu)\) be any measure space and \(\mathcal{H} := L^2(X, \mu)\). For any \(\psi \in L^\infty(X, \mu)\) let \(M_\psi \in \mathcal{B}(\mathcal{H})\) be the multiplication-by-\(\psi\) operator, and define \(A := \{M_\psi : \psi \in L^\infty(X, \mu)\}\). If at least one of the following conditions is satisfied:

1. \(X\) is a locally compact space and \(\mu\) is a Radon measure;
2. one has \(\mu(X) < \infty\) and \(\mathcal{H}\) is separable;

then \(A\) is a maximal abelian self-adjoint subalgebra of \(\mathcal{B}(\mathcal{H})\).
Proof. If the first condition is satisfied then the assertion follows by [Di69, Ch. I, §7, no. 3, Th. 2]. If the second condition is satisfied, then the constant function 
1 ∈ L∞(X, µ) ⊆ L2(X, µ) is a cyclic vector for A, hence the conclusion follows by 
[SS08, Th. 2.3.4]. □

Definition 1.2. Let α: G × X → X be any group action by measurable transformations of a measure space (X, µ). The action α is called ergodic if for every measurable set A ⊆ X with µ(A) = 0 or µ(X \ A) = 0. (Here, for two sets X and Y, X Δ Y denotes the symmetric difference X Δ Y = (X \ Y) ∪ (Y \ X).)

Remark 1.3. In the framework of Definition 1.2 it is straightforward to check that the group action α is ergodic if and only if the equivalence classes of a.e. constant functions in L∞(X, µ) are the only elements φ ∈ L∞(X, µ) with φ ◦ αg = φ for all g ∈ G. This implies that if α is a transitive action (or more generally, if for every x ∈ X with its orbit G.x := {αg(x) | g ∈ G} one has µ(X \ G.x) = 0), then α is ergodic.

We refer to Ta03 for the role of ergodic actions in the theory of operator algebras.

2. General results

To begin with, we recall some ideas from [Is96, Ch. I, Subsect. 1.4].

Definition 2.1. A measurable dynamical system consists of a measure space (X, µ) endowed with a group action on the right

β: X × S → X, \ (x, s) → βs(x) =: x.s,

for which the measure µ is quasi-invariant. Then for every s ∈ S there is an a.e. defined positive function j(·, s) on X for which (βs)*(µ) = j(·, s)µ, where (βs)*(µ) denotes the pushforward of the measure µ through the map βs. Hence for every measurable set E ⊆ X one has

μ(βs(E)) = \int_E j(x, s)dµ(x).

A scalar cocycle of this measurable dynamical system is a family \{a(·, s)\}s∈S consisting of a.e. defined measurable functions on X with values in the unit circle \mathbb{T}, satisfying the conditions

a(x, s₁s₂) = a(x, s₁)a(x, s₁, s₂) and a(x, 1) = x

for a.e. x ∈ X and all s₁, s₂ ∈ S.

In the above setting we also define \mathcal{H} := L^2(X, µ) and for every s ∈ S,

πa(s): \mathcal{H} → \mathcal{H}, \quad π_a(s)φ = j(·, s)^{1/2}a(·, s)(φ ◦ β_s)(·).

Remark 2.2. In Definition 2.1 since \((β_{s₁s₂})_*(µ) = (β_{s₁})_*((β_{s₂})_*(µ))\) for all s₁, s₂ ∈ S, it is easily checked that the family \{a(·, s)\}s∈S satisfies the conditions of a scalar cocycle, except that the functions from this family take values in the multiplicative group \((0, ∞)\) instead of the unit circle \mathbb{T}.

The following result is a special case of [Is96, Ch. I, Props. 1.1–1.2] whose proof was not included therein, so we give the sketch of a proof here, for the sake of completeness.
Proposition 2.3. Assume the setting of Definition 2.4. Then the following assertions hold:

(i) The map $\pi_a: S \to \mathcal{B}(\mathcal{H})$ is a unitary representation.
(ii) If the representation $\pi_a$ is irreducible, then the action of $S$ on $X$ is ergodic.
(iii) If $S$ is a topological group and one has

$$\lim_{s \to 1} \mu(E \triangle (E.s)) = \lim_{s \to 1} \int_{E \triangle (E.s)} |j(\cdot, s)^{1/2} - 1|^2 d\mu = \lim_{s \to 1} \int_E (a(s, \cdot) - 1) d\mu = 0$$

for every measurable set $E \subseteq X$ with $\mu(E) < \infty$, then the representation $\pi_a$ is continuous.

Proof. Assertion (i) is based on a straightforward computation.

For Assertion (ii) note that for every measurable set $E \subseteq X$ which is $G$-invariant, the multiplication operator $M_{\chi_E} \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection whose image is invariant under $\pi_a(s)$ for all $s \in S$.

For Assertion (iii) we use that the values of $\pi_a$ are unitary operators on $\mathcal{H}$, hence an $(\varepsilon/3)$-argument shows that it suffices to check that $\lim_{s \to 1} \|\pi_a(s) \varphi - \varphi\| = 0$ for $\varphi$ in some subset of $\mathcal{H}$ that spans a dense linear subspace. Using the assumptions, one can check that $\lim_{s \to 1} \|\pi_a(s) \chi_E - \chi_E\| = 0$ for every measurable set $E \subseteq X$ with $\mu(E) < \infty$, and this completes the proof. \hfill \square

For the following theorem we recall that a multiplicity-free representation is a unitary representation whose commutant is commutative.

Theorem 2.4. Assume the setting of Definition 2.4, where $(X, \mu)$ satisfies either of the conditions in Lemma 1.1, and let

$$S_0 := \{s \in S \mid x.s = x \text{ for a.e. } x \in X\}.$$

If the set $\{a(\cdot, s) \mid s \in S_0\}$ spans a $w^*$-dense linear subspace of $L^\infty(X, \mu)$, then $\pi_a: S \to \mathcal{B}(\mathcal{H})$ is a multiplicity-free representation and moreover the following assertions are equivalent:

(i) The action of $S$ on $(X, \mu)$ is ergodic.
(ii) The representation $\pi_a$ is irreducible.

Proof. Recall that $\mathcal{H} = L^2(X, \mu)$ and for any $\psi \in L^\infty(X, \mu)$ we denote by $M_\psi \in \mathcal{B}(\mathcal{H})$ be the operator of multiplication by $\psi$. By Lemma 1.1, the operator algebra

$$\mathcal{A} := \{M_\psi \mid \psi \in L^\infty(X, \mu)\} \subseteq \mathcal{B}(\mathcal{H})$$

is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. To prove that $\pi_a$ is a multiplicity-free representation, we will show that $\pi_a(S)' \subseteq \mathcal{A}$. Hence we must prove that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S$, then $T \in \mathcal{A}$. In fact we will prove a stronger fact, namely if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}$.

If $s \in S_0$, then it is clear that $j(x, s) = 1$ and $\varphi(x.s) = \varphi(x)$ for a.e. $x \in X$, where $\varphi \in L^2(X, \mu)$ is arbitrary, and it then follows by the definition of $\pi_a$ that $\pi_a(s)$ is the operator of multiplication by $a(\cdot, s) \in L^\infty(X, \mu)$. Since the set $\{a(\cdot, s) \mid s \in S_0\}$ spans a $w^*$-dense linear subspace of $L^\infty(X, \mu)$ by hypothesis, it then follows that if $T \in \mathcal{B}(\mathcal{H})$ and $T\pi(s) = \pi(s)T$ for all $s \in S_0$, then $T \in \mathcal{A}$. We have seen above that $\mathcal{A}$ is a maximal self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence $\mathcal{A}' = \mathcal{A}$, and then $T \in \mathcal{A}$, as claimed above. This completes the proof of the fact that $\pi_a$ is a multiplicity-free representation.
Moreover, if the representation $\pi_a$ is irreducible, then the action of $S$ on $(X, \mu)$ is ergodic by Proposition 2.3(ii). Conversely, let us assume that the action of $S$ on $(X, \mu)$ is ergodic. In order to prove that the representation $\pi_a$ is irreducible, we must show that if $T \in B(\mathcal{H})$ satisfies $T \pi(s) = \pi(s)T$ for all $s \in S$, then $T$ is a scalar multiple of the identity operator on $\mathcal{H}$. In fact, using the condition $T \pi(s) = \pi(s)T$ for all $s \in S_0$, we obtain by the above reasoning that $T = M_\psi$ for some $\psi \in L^\infty(X, \mu)$. Then for all $s \in S$ and $\varphi \in L^\infty(X, \mu)$ one has

$$
\psi(x) j(x, s)^{1/2} \varphi(x, s) = (M_\psi \pi_a(s) \varphi)(x) = (T \pi_a(s) \varphi)(x) = (\pi_a(s) T \varphi)(x) = (\pi_a(s) M_\psi \varphi)(x) = j(x, s)^{1/2} \psi(x, s) \varphi(x, s)
$$

for a.e. $x \in X$. This implies that for all $s \in S$ one has $\psi(x) = \psi(x, s)$ for a.e. $x \in X$. Since $\psi \in L^\infty(X, \mu)$ and the action of $S$ on $(X, \mu)$ is ergodic, it then follows that $\psi$ is constant a.e. on $X$, hence the multiplication operator $T = M_\psi$ is a scalar multiplication of the identity operator on $\mathcal{H}$, and this completes the proof.

**Remark 2.5.** As we will see in the examples presented in the following sections of this paper, the group $S_0$ from Theorem 2.4 is an abstract version of the Lie subgroup that corresponds to a polarization of a nilpotent Lie algebra. More precisely, one can interpret the representation $\pi_a : S \to B(L^2(X, \mu))$ as the representation induced from the character $\chi_0 : S_0 \to \mathbb{T}$, $\chi_0(s) := a(x_0, s)$, for some fixed $x_0 \in X$ (if any) with $x_0.s = x_0$ for all $s \in S_0$. It is worth noting that if there exists such a point $x_0 \in X$, then the above $\chi_0$ is a group homomorphism because of the cocycle properties of $a$.

### 3. Applications to group actions on locally compact spaces

The following proposition establishes irreducibility of some unitary representations that play a very significant role in [BB09] and [BB10a]. See Examples 3.4–3.5 below for more specific information in this connection.

**Proposition 3.1.** Let $G$ be a group and $(X, \mu)$ be any locally compact space endowed with a Radon measure. Assume that $\alpha : G \times X \to X$, $(g, x) \mapsto \alpha_g(x)$, is an action of $G$ on $X$ by measure-preserving transformations. Let $\mathcal{F}$ be any $G$-invariant vector space of real measurable functions on $X$, with the corresponding representation $\lambda : G \to \operatorname{End}(\mathcal{F})$, $\lambda_g(f) := f \circ \alpha_g^{-1}$. Assume in addition that the linear span of the set $\{ \exp(if) \mid f \in \mathcal{F} \}$ is $w^*$-dense in $L^\infty(X, \mu)$. Then the following conditions are equivalent:

1. The action $\alpha$ is ergodic.
2. The unitary representation

$$
\pi : \mathcal{F} \rtimes_\lambda G \to B(L^2(X, \mu)), \quad (\pi(f, g) \varphi)(x) = e^{if(x)} \varphi(\alpha_g^{-1}(x))
$$

is irreducible.

**Proof.** This is just a special case of Theorem 2.4 with $S = \mathcal{F} \rtimes_\lambda G$. Indeed in this case $S_0 = (\mathcal{F}, +)$, and it acts trivially on $X$. The fact that group action of $\mathcal{F} \rtimes_\lambda G$ on $(X, \mu)$ is ergodic is equivalent to ergodicity of the action of $G$ on $(X, \mu)$, since the action of $\mathcal{F}$ on $(X, \mu)$ is trivial. $\square$
Remark 3.2. In Proposition 3.1 the ergodicity hypothesis is necessary for the representation $\pi$ to be irreducible, without imposing any condition on the linear span of the set $\{\exp(if) \mid f \in F\}$. This follows by Proposition 2.3[iii].

For the transitive group action of a connected simply connected nilpotent Lie group on itself by left translations, the following corollary implies that the unitary representations constructed in [BB09], Subsect. 2.4] are irreducible. Using suitable global coordinates on coadjoint orbits of nilpotent Lie groups and the transitivity of coadjoint action on its orbits, this corollary also recovers the result of [BB10a, Prop. 5.1(2)]. See Examples 3.4–3.5 below for more details in this connection.

Corollary 3.3. Let $G$ be a group and $X$ be a finite-dimensional real vector space with a Lebesgue measure $\mu$. Assume that $\alpha : G \times X \to X$, $(g,x) \mapsto \alpha_g(x)$, is an action of $G$ on $X$ by measure-preserving transformations. Let $F$ be any $G$-invariant vector space of real measurable functions on $X$, with the corresponding representation $\lambda : G \to \text{End}(F)$, $\lambda_g(f) := f \circ \alpha_{g^{-1}}$, and define the unitary representation $\pi : F \rtimes_G G \to B(L^2(X,\mu))$, $(\pi(f,g)\varphi)(x) = e^{if(x)}\varphi(\alpha_{g^{-1}}(x))$.

If the linear dual space of $X$ satisfies $X^* \subseteq F$, then the following conditions are equivalent:

(i) The action $\alpha$ is ergodic.

(ii) The representation $\pi$ is irreducible.

Proof. If the representation $\pi$ is irreducible, then $\alpha$ is ergodic by Remark 3.2.

Conversely, the result will follow by Proposition 3.1 as soon as we will have proved that the linear span of the set $\{\exp(it\xi) \mid \xi \in X^*\}$ is $w^*$-dense in $L^\infty(X,\mu)$. To check this, recall that the predual of the von Neumann algebra $L^\infty(X,\mu)$ is $L^1(X,\mu)$ and the corresponding duality pairing is

$$L^\infty(X,\mu) \times L^1(X,\mu) \to \mathbb{C}, \quad (\varphi,\psi) \mapsto \langle \varphi,\psi \rangle := \int_X \varphi \psi d\mu.$$ 

On the other hand, if $\psi \in L^1(X,\mu)$ and $0 = \langle \exp(it\xi),\psi \rangle = \int_X \exp(it\xi)\psi d\mu$ for all $\xi \in X^*$, then $\psi = 0$ by the injectivity property of the Fourier transform. It then follows by the Hahn-Banach theorem that indeed the linear span of the set $\{\exp(it\xi) \mid \xi \in X^*\}$ is $w^*$-dense in $L^\infty(X,\mu)$, and this completes the proof. \square

Example 3.4 ([BB09, Subsect. 2.4]). Let $G$ be any connected, simply connected, nilpotent Lie group with some fixed left invariant Haar measure, and $F \subseteq C^\infty(G)$ be a linear subspace of the space of smooth functions on $G$. satisfying the following conditions:

(1) The linear space $F$ is invariant under the representation of $G$ by left translations, $\lambda : G \to \text{End}(C^\infty(G))$, $\lambda_g \varphi(x) = \varphi(g^{-1}x)$. We denote again by $\lambda : G \to \text{End}(F)$ the restriction to $F$ of the above representation $\lambda$ of $G$.

(2) The mapping $G \times F \to F$, $(g,\varphi) \mapsto \lambda_g \varphi$ is continuous.

(3) We have $g^* \subseteq \{\phi \circ \exp_G \mid \phi \in F\}$.

We define $\pi : F \rtimes G \to B(L^2(G))$ by

$$(\pi(\phi,g)f)(x) = e^{i\phi(x)}f(g^{-1}x)$$

for all $\phi \in F$, $g \in G$, and $f \in L^2(G)$, and almost all $x \in G$. 

Hence $\pi$ is as in Proposition [3.1]. In order to apply that proposition we must check that the linear span of the set $\{\exp(i\phi) \mid \phi \in F\}$ is $w^*$-dense in $L^\infty(G)$, hence that if $\psi \in L^1(G)$ and $\int G \psi \exp(i\phi) = 0$ for all $\phi \in F$, then necessarily $\psi = 0$. To this end, using the above condition for $\phi = \xi \circ \log_G$ with arbitrary $\xi \in \mathfrak{g}^*$ (note that $\phi \in F$ by the hypothesis [3]), we obtain that the Fourier transform of $\psi$ is zero, hence $\psi = 0$. Finally, the right action of $F \rtimes G$ on $G$ given by

$$G \times (F \rtimes G) \to G, \quad (x, (g, \phi)) \mapsto g^{-1}x,$$

is transitive, hence ergodic (see Remark [1.3]), and then by Proposition [3.1] the representation $\pi$ is irreducible.

Let us also note that the above hypotheses on $F$ ensure that $F$ is an admissible function space in the sense of [BB10a, Def. 2.8].

Example 3.5 ([BB10a Prop. 5.1(2)]). Let $G$ be any connected, simply connected, nilpotent Lie group, with its center $Z$ and the corresponding Lie algebras $\mathfrak{z} \subseteq \mathfrak{g}$. Endow the coadjoint orbit $O$ with its Liouville measure and define

$$\tilde{\pi} : G \ltimes \mathfrak{g} \to B(L^2(O)), \quad (\tilde{\pi}(g, Y)f)(\xi) = e^{i\langle \xi, Y \rangle}f(\operatorname{Ad}_G^*(g^{-1})\xi).$$

Then the following assertions hold:

(i) The group $\tilde{G} := G \ltimes \mathfrak{g}$ is nilpotent and its center is $Z \times \mathfrak{z}$.

(ii) $\tilde{\pi}$ is a unitary irreducible representation of $\tilde{G}$.

We recall that the multiplication in the semi-direct product group $\tilde{G}$ is given by

$$(g_1, Y_1) \cdot (g_2, Y_2) = (g_1g_2, Y_1 + \operatorname{Ad}_G(g_1)Y_2) \quad (3.1)$$

and the bracket in the corresponding Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \ltimes \mathfrak{g}$ is defined by

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [X_1, Y_2] - [X_2, Y_1]).$$

An inspection of these equations shows that $\tilde{\mathfrak{g}}$ is a nilpotent Lie algebra with its center $\mathfrak{z} \times \mathfrak{z}$.

To see that $\tilde{\pi}$ is a representation we need to check that the function

$$a : O \times \tilde{G} \to \mathbb{T}, \quad a(\xi, (g, Y)) := e^{i\langle \xi, Y \rangle}$$

is a cocycle in the sense of Definition [2.1]. In fact, using the right action of $\tilde{G}$ on $O$ given by

$$O \times \tilde{G} \to O, \quad (\xi, (g, Y)) \mapsto \xi \circ \operatorname{Ad}_G(g) = \operatorname{Ad}_G^*(g^{-1})\xi, \quad (3.2)$$

it follows by (3.1) and the above definition of $a$ that

$$a(\xi, (g_1, Y_1), (g_2, Y_2)) = a(\xi, (g_1g_2, Y_1 + \operatorname{Ad}_G(g_1)Y_2))$$

$$= e^{i\langle \xi, Y_1 + \operatorname{Ad}_G(g_1)Y_2 \rangle}$$

$$= e^{i\langle \xi, Y_1 \rangle}e^{i\langle \xi, \operatorname{Ad}_G(g_1)Y_2 \rangle}$$

$$= e^{i\langle \xi, \mathfrak{z} \times \mathfrak{z} \rangle}a(\xi, (g_1, Y_1), (g_2, Y_2)).$$

The property $a(\xi, 1) = \xi$ for all $\xi \in O$ is clear from the definition of $a$. Also note that the Liouville measure on $O$ is invariant under the group action (3.2). It then follows by Proposition [2.3] that $\tilde{\pi}$ is a continuous unitary representation.

Moreover, to see that $\tilde{\pi}$ is irreducible we will use Corollary [3.3]. To this end, first note that the group action (3.2) is transitive, hence ergodic (see Remark [1.3]). Furthermore, recall that the mapping

$$O \to \mathfrak{g}_c^*, \quad \xi \mapsto \xi|_{\mathfrak{g}_c}$$
is a global chart which takes the Liouville measure of $\mathcal{O}$ to a Lebesgue measure on $\mathfrak{g}^*$, where $\mathfrak{e}$ is the jump index set of $\mathcal{O}$ with respect to some Jordan-Hölder basis in $\mathfrak{g}$ (see for instance [BB10a]). Then we can use the Fourier transform to see that the linear subspace generated by $\{e^{i(Y\cdot)} \mid Y \in \mathfrak{g}\}$ is weak*-dense in $L^\infty(\mathcal{O})$ ($\simeq L^1(\mathcal{O})^*$). Therefore we can use Corollary 3.3 to obtain that $\mathfrak{p}$ is irreducible.

In addition to the above properties of $\mathfrak{p}$ we also recall some additional information on the irreducible representation $\mathfrak{p}$ that was obtained in [BB10a Prop. 5.1(2)]. Firstly, the space of smooth vectors for the representation $\mathfrak{p}$ is $\mathcal{S}(\mathcal{O})$. Moreover, select any Jordan-Hölder basis $X_1, \ldots, X_n$ in $\mathfrak{g}$ and define

$$\tilde{X}_j = \begin{cases} (0, X_j) & \text{for } j = 1, \ldots, n, \\ (X_{j-n}, 0) & \text{for } j = n + 1, \ldots, 2n. \end{cases}$$

Then $\tilde{X}_1, \ldots, \tilde{X}_{2n}$ is a Jordan-Hölder basis in $\mathfrak{g}$ and the corresponding predual for the coadjoint orbit $\tilde{\mathcal{O}} \subseteq \mathfrak{g}^*$ associated with the representation $\mathfrak{p}$ is

$$\mathfrak{g}_\mathfrak{e} = \mathfrak{g}_\mathfrak{e} \times \mathfrak{g}_\mathfrak{e} \subseteq \mathfrak{g},$$

where $\mathfrak{e}$ is the set of jump indices for $\tilde{\mathcal{O}}$.

4. APPLICATION TO GAUSSIAN MEASURES ON HILBERT SPACES

We first recall here a few facts from [BB10b] and [BBM15].

**Definition 4.1.** If $\mathcal{V}$ is a real Hilbert space, $A \in \mathcal{B}(\mathcal{V})$ with $(Ax \mid y) = (x \mid Ay)$ for all $x, y \in \mathcal{V}$, and moreover $\ker A = \{0\}$, then the Heisenberg algebra associated with the pair $(\mathcal{V}, A)$ is the real Hilbert space $\mathfrak{h}(\mathcal{V}, A) = \mathcal{V} + \mathcal{V} + \mathbb{R}$ endowed with the Lie bracket defined by $[(x_1, y_1, t_1), (x_2, y_2, t_2)] = (0, 0, (Ax_1 \mid y_2) - (Ax_2 \mid y_1))$. The corresponding Heisenberg group $\mathbb{H}(\mathcal{V}, A) = (\mathfrak{h}(\mathcal{V}, A), *)$ is the Lie group whose underlying manifold is $\mathbb{H}(\mathcal{V}, A)$ and whose multiplication is defined by

$$(x_1, y_1, t_1) * (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1 + t_2 + ((Ax_1 \mid y_2) - (Ax_2 \mid y_1))/2)$$

for $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{H}(\mathcal{V}, A)$.

Let $\mathcal{V}_- = \{x \in \mathcal{V} \mid x \geq 0\}$ and every symmetric, nonnegative, injective, trace-class operator $K$ on $\mathcal{V}_-$ there is a unique probability Borel measure $\gamma$ on $\mathcal{V}_-$ with

$$(\forall x \in \mathcal{V}_-) \int_{\mathcal{V}_-} e^{i(x \mid y) - \frac{i}{2} (Kx \mid x)} - d\gamma(y) = e^{i(a \mid x) - \frac{i}{2} (Kx \mid x)}$$

and $\gamma$ is called the Gaussian measure with the mean $a$ and the variance $K$.

Now assume that $a = 0$ and let $\mathcal{V}_+ := \text{Ran } K$ and $\mathcal{V}_0 := \text{Ran } K^{1/2}$ be endowed with the scalar products given by $(Kx \mid Ky)_+ := (x \mid y)_-$ and $(K^{1/2}x \mid K^{1/2}y)_0 := (x \mid y)_-$, respectively, for all $x, y \in \mathcal{V}_-$, which turn the linear bijections $K: \mathcal{V}_- \to \mathcal{V}_+$ and $K^{1/2}: \mathcal{V}_- \to \mathcal{V}_0$ into isometries. We thus obtain the real Hilbert spaces

$$\mathcal{V}_+ \hookrightarrow \mathcal{V}_0 \hookrightarrow \mathcal{V}_-$$

where the inclusion maps are Hilbert-Schmidt operators, since $K^{1/2} \in \mathcal{B}(\mathcal{V}_-)$ is a Hilbert-Schmidt operator. Also, the scalar product of $\mathcal{V}_0$ extends to a duality pairing $(\cdot \mid \cdot)_0 : \mathcal{V}_- \times \mathcal{V}_+ \to \mathbb{R}$. 


We also recall that for every \( x \in \mathcal{V}_+ \) the translated measure \( d\gamma(-x + \cdot) \) is absolutely continuous with respect to \( d\gamma(\cdot) \) and we have the Cameron-Martin formula
\[
d\gamma(-x + \cdot) = \rho_x(\cdot) d\gamma(\cdot) \quad \text{with} \quad \rho_x(\cdot) = e^{i|\cdot|_0 - \frac{1}{2}(x|\cdot)_0}.
\]

**Definition 4.2.** Let \( \mathcal{V}_+ \) be a real Hilbert space with the scalar product denoted by \((x, y) \mapsto (x | y)_+\). Also let \( A: \mathcal{V}_+ \to \mathcal{V}_+ \) be a nonnegative, symmetric, injective, trace-class operator. Let \( \mathcal{V}_0 \) and \( \mathcal{V}_- \) be the completions of \( \mathcal{V}_+ \) with respect to the scalar products
\[
(x, y) \mapsto (x | y) := (A^{1/2} x | A^{1/2} y)_+\]
and
\[
(x, y) \mapsto (x | y)_- := (Ax | Ay)_+,
\]
respectively. Then the operator \( A \) uniquely extends to a nonnegative, symmetric, injective, trace-class operator \( K \in \mathcal{B}(\mathcal{V}_-), \) hence by the above observations one obtains the Gaussian measure \( \gamma \) on \( \mathcal{V}_- \) with variance \( K \) and mean 0.

One can also construct the Heisenberg group \( \mathbb{H}(\mathcal{V}_+, A) \). The *Schrödinger representation* \( \pi: \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma)) \) is defined by
\[
\pi(x, y, t) \phi = \rho_x(\cdot)^{1/2} e^{i(t+(\cdot)|y)_0 + \frac{1}{2}(x|y)_0} \phi(-x + \cdot)
\]
for \((x, y, t) \in \mathbb{H}(\mathcal{V}_+, A) \) and \( \phi \in L^2(\mathcal{V}_-, \gamma) \).

**Proposition 4.3.** The representation \( \pi: \mathbb{H}(\mathcal{V}_+, A) \to \mathcal{B}(L^2(\mathcal{V}_-, \gamma)) \) from Definition 4.2 is irreducible.

**Proof.** See for instance from [BB10a Rem. 3.6] or [BBM15]. \( \square \)

**Corollary 4.4.** In the above setting, the action by translations of \( \mathcal{V}_+ \) on \( (\mathcal{V}_-, \gamma) \) is ergodic.

**Proof.** In the present framework, the representation \( \pi \) is the unitary representation associated to the measure space \((\mathcal{V}_-, \gamma)\) acted on by the additive group \((\mathcal{V}_+, +)\) by translations. The cocycle of that measurable dynamical system which gives rise to the representation \( \pi \) is given by
\[
a(\cdot, (x, y, t)) = e^{i(t+(\cdot)|y)_0 + \frac{1}{2}(x|y)_0}
\]
for all \((x, y, t) \in \mathbb{H}(\mathcal{V}_+, A) \). The conclusion follows by Propositions 1.3 and 2.3 [BB].

and we are done. \( \square \)

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