Polytope duality for families of $K3$ surfaces associated to singularities $Q_{16}$ and $S_{16}$

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Abstract

There are strange dual pairs of bimodal singularities that are not assigned an invertible projectivisation in [4]. We study families of $K3$ surfaces associated to such pairs.

Key words: families of $K3$ surfaces, strange duality, polytope duality, bimodal singularities

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1 Introduction

Ebeling and Takahashi [5] introduced a notion of strange duality for invertible polynomials by Berglund-Hübsch mirror construction [2]. Mase and Ueda [6] studied an extension of strange duality to polytope duality when bimodal singularities admit an invertible projectivisation, which are given in the study of Ebeling and Ploog [4] of distinguished basis. In order to complete our list, we concern two pairs of strange duality: $Q_{16}$ and $S_{16}$ that are defined by

$Q_{16}: f_{Q_{16}} = x^4 z + y^3 + x z^2, \quad S_{16}: f_{S_{16}} = x^4 y + x z^2 + y^2 z.$

The matrices of exponents of the defining polynomials are respectively given by

$A_{f_{Q_{16}}} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad A_{f_{S_{16}}} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix},$

which are both symmetric. Thus, the singularities are self-dual.

In [4] a projectivisations of these singularities are determined as follows:

$Q_{16}: F_{Q_{16}} = X^4 Z + Y^3 + X Z^2 + W^6 Z + W^7 Y, \quad S_{16}: F_{S_{16}} = X^4 Y + X Z^2 + Y^3 Z + W^5 Z + W^6 Y.$

Note that these polynomials are not invertible. Recall that a polynomial $F$ is invertible if the matrix of exponents of $F$ is an invertible matrix.

The polynomial $F_{Q_{16}}$ is an anticanonical section of the weighted projective space $\mathbb{P}(2, 3, 7, 9)$, and $F_{S_{16}}$ of $\mathbb{P}(2, 3, 5, 7)$. It is known that both spaces $\mathbb{P}(2, 3, 7, 9)$, and $\mathbb{P}(2, 3, 5, 7)$ are Fano 3-folds as is classified by Yonemura [7], thus general anticanonical sections are $K3$ surfaces with at most Gorenstein singularities as is shown by Batyrev [1]. It was concluded in [6] that all strange-dual pairs for bimodal singularities admitting an invertible projectivisation extend to polytope duality. Even though the singularities
Let \( w = (w_0, w_1, w_2, w_3) \) be a weight system. If \( w_0 \leq w_1 \leq w_2 \leq w_3 \), and any three \( w_0, w_1, w_3 \) out of four are prime, then \( w \) is well-posed. Let \( \Delta^+ := \Delta(w_0 + w_1 + w_2 + w_3) \) be the 3-dimensional polytope associated to \( \mathbb{P}(w) \). For a weight system \( w \), we define the weighted projective space with weight system \( \mathbb{P}(w) = \mathbb{P}(w_0, w_1, w_2, w_3) := \text{Proj} \mathbb{C}[W, X, Y, Z] \).

It is known that weighted projective spaces are toric 3-folds: we denote by \( \Delta^+ = \Delta(w_0, w_1, w_2, w_3) \) the 3-dimensional polytope associated to \( \mathbb{P}(w) \).

The anticanonical divisor of \( \mathbb{P}(w) \) is isomorphic to \( \mathcal{O}(-d) \) (see [3]). Thus the global sections of it are polynomials of weighted degree \( d \), which we simply call anticanonical sections. Indeed, let \( M_w \) be a lattice of rank 3 defined by

\[
M_w := \{(i, j, k, l) \in \mathbb{Z}^4 \mid w_0i + w_1j + w_2k + w_3l = 0\}
\]
with a basis, $\Delta_w$ is embedded into $\mathbb{R}^3$, and a monomial $W^{i+1}X^{j+1}Y^{k+1}Z^{l+1}$ of weighted degree $d$ is corresponding to an element $(i, j, k, l)$ of $M_w$.

The weighted projective spaces that are Fano, namely, the anticanonical divisor is ample, are classified by Yonemura into 95 classes \[2\].

Let $M$ be a lattice of rank 3, and $N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be its dual with a natural pairing $\langle , \rangle : N \times M \to \mathbb{Z}$ and $\langle , \rangle_\mathbb{R}$ is the extension to $\mathbb{R}$-coefficients, and $\Delta$ be a 3-dimensional convex hull of finite number of points in $M \otimes \mathbb{R}$, which we simply call a polytope. Define the polar dual polytope $\Delta^*$ of $\Delta$ by

$$\Delta^* := \{ y \in N \otimes \mathbb{R} | \langle y, x \rangle \geq -1 \text{ for all } x \in \Delta \}.$$ 

A polytope $\Delta$ with all vertices being integral points is reflexive if $\Delta$ contains the only integral points in its interior and the polar dual $\Delta^*$ has also all vertices integral. In general, if a polytope $\Delta$ is reflexive, the associated projective space $\mathbb{P}_\Delta$ is a Fano 3-fold, and its general anticanonical sections are $K3$ surfaces with at most Gorenstein singularities (see \[3\]).

For non-degenerate isolated singularities ($f = 0$) and ($f' = 0$) in $C^3$ with projectivisations $F$ in $\mathbb{P}(w) = \mathbb{P}(w_0, w_1, w_2, w_3)$ and $F'$ in $\mathbb{P}(w') = \mathbb{P}(w'_0, w'_1, w'_2, w'_3)$ as anticanonical sections of Fano weighted projective spaces, families $\mathcal{F}_\Delta$ and $\mathcal{F}_{\Delta'}$ of $K3$ surfaces with at most Gorenstein singularities associated to reflexive polytopes $\Delta$ and $\Delta'$ (c.f. \[4\]) are said polytope dual if the following relations hold:

$$\Delta_F \subset \Delta \subset \Delta_{(w)}, \quad \Delta_{F'} \subset \Delta' \subset \Delta_{(w')}, \quad \text{and} \quad \Delta^* \simeq \Delta'.$$

### 3 Main Result

Recall our main theorem that is proved in this section.

**Theorem 3.1** Let $w$ be a weight system $(2, 3, 7, 9)$ (resp. $(2, 3, 5, 7)$) and $\Delta$ be a reflexive polytope such that $\Delta_F \subset \Delta \subset \Delta_w$, where $F$ is $F_{Q_{16}}$ (resp. $F_{S_{16}}$). The polar dual polytope $\Delta'$ of $\Delta$ is not a subpolytope of the polytope $\Delta_w$. In particular, the strange duality for the singularity $Q_{16}$ (resp. $S_{16}$) does not extend to the polytope duality between subfamilies of $K3$ surfaces in $\mathbb{P}(w)$.

**Proof.**

**Singularity of type $Q_{16}$**. The singularity is defined by a polynomial $f = x^4z + y^2 + x^2z$, and take a projectivisation $F = X^4Z + Y^3 + XZ^2 + W^6Z + W^7Y$ in accordance of \[3\]. Let $M$ be a lattice of rank 3 defined by

$$M := \{(i, j, k, l) \in \mathbb{Z}^4 | 2i + 3j + 7k + 9l = 0 \}.$$ 

By taking a basis $\{e_1, e_2, e_3\}$ of $M$ by

$$e_1 = (8, 0, -1, -1), \quad e_2 = (6, -1, 0, -1), \quad e_3 = (5, -1, -1, 0),$$

the polytope $\Delta_{(2,3,7,9)}$ is given by a convex hull of vertices

$$(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -1, 1), (-1, 2, -1), (4, -3, -3),$$

respectively corresponding to monomials

$$W^6X, W^7Y, W^6Z, XZ^2, Y^3, X^7.$$
The Newton polytope $\Delta_F$ is a convex hull of vertices

\[(0, 1, 0), (0, 0, 1), (0, -1, 1), (-1, 2, -1), (2, -2, -1),\]

where $(2, -2, -1)$ is corresponding to the monomial $X^4Z$. Since the polar dual of the face

$$\text{Conv}\{(0, 1, 0), (-1, 2, -1), (2, -2, -1)\}$$

is a rational vertex $(-4/3, -1, 1/3)$, the Newton polytope is not reflexive.

Any reflexive polytope $\Delta$ satisfying $\Delta_F \subset \Delta \subset \Delta_{(2,3,7,9)}$ is thus a convex hull of vertices

\[(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -1, 1), (-1, 2, -1), (2, -2, -1), (n+1, -n, -n)\]

with $n = 1$.

As long as there is an edge

$$\Gamma = \text{Conv}\{(0, -1, 1), (-1, 2, -1)\}$$

in $\Delta$, since the polar dual of $\Gamma$ is

$$\Gamma^* = \text{Conv}\{(8, 6, 5), (2, 0, -1)\},$$

and thus there are 5 lattice points in $\Gamma^*$, the polar dual polytope $\Delta' := \Delta^*$ of $\Delta$ should contain an edge with 5 lattice points. However, by a direct check, there does not exist such an edge in or inside of the polytope $\Delta_{(2,3,5,7)}$. Thus, the polytope $\Delta'$ is not a subpolytope of $\Delta_{(2,3,5,7)}$. Therefore the assertion is verified.

**Singularity of type S_{16}.** The singularity is defined by a polynomial $f = x^4y + xz^2 + y^2z$, and take a projectivisation $F = X^4Y + XZ^2 + Y^2Z + W^5 + W^6 Y$ in accordance of \([4]\). Let $M$ be a lattice of rank 3 defined by

$$M := \{(i, j, k, l) \in \mathbb{Z}^4 : 2i + 3j + 5k + 7l = 0\}.$$

By taking a basis $\{e_1, e_2, e_3\}$ of $M$ by

$$e_1 = (6, 0, -1, -1), e_2 = (5, -1, 0, -1), e_3 = (4, -1, -1, 0),$$

the polytope $\Delta_{(2,3,5,7)}$ is given by a convex hull of vertices

\[(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, -1, 1), (-1, 1, 0), (-1, 2, -1), (2, -1, -2), (3, -2, -2),\]

respectively corresponding to monomials

$$W^7X, W^6Y, W^5Z, XZ^2,$$

$$Y^2Z, WY^3, X^4Y, WX^5.$$

The Newton polytope $\Delta_F$ is a convex hull of vertices

\[(0, 1, 0), (0, 0, 1), (0, -1, 1), (-1, 1, 0), (2, -1, -2).\]

Since the polar dual of the face

$$\text{Conv}\{(0, 0, 1), (0, -1, 1), (2, -1, -2)\}$$

is a rational vertex $(-3/2, 0, -1)$, the Newton polytope is not reflexive.
Any reflexive polytope $\Delta$ satisfying $\Delta_F \subset \Delta \subset \Delta_{(2,3,7,9)}$ is thus a convex hull of vertices 

$$(1,0,0), (0,1,0), (0,0,1), (0,-1,1), (-1,1,0), (2,-1,-2), (n+1,-n,-n)$$

with $n = 1, 2$, or with $(-1,2,-1)$.

As long as there is an edge 

$$\Gamma = \text{Conv}\{(0,-1,1), (-1,1,0)\}$$

in $\Delta$, since the polar dual of $\Gamma$ is 

$$\Gamma^* = \text{Conv}\{(6,5,4), (1,0,-1)\},$$

and thus there are 4 lattice points in $\Gamma^*$, the polar dual polytope $\Delta' := \Delta^*$ of $\Delta$ should contain an edge with 4 lattice points. However, by a direct check, there does not exist such an edge in or inside of the polytope $\Delta_{(2,3,7,9)}$. Thus, the polytope $\Delta'$ is not a subpolytope of $\Delta_{(2,3,7,9)}$. Therefore the assertion is verified. \(\square\)

4 Conclusion

Combining our result with \cite{6}, all but singularities that are not assigning an invertible projectivisation, the strange duality for bimodal singularities extends to the polytope duality.

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