LOCAL $\varepsilon$-REGULARITY CRITERIA FOR THE FIVE DIMENSIONAL STATIONARY NAVIER-STOKES EQUATIONS

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ABSTRACT. We establish interior and boundary $\varepsilon$-regularity criteria at one scale for suitable weak solutions to the five dimensional stationary incompressible Navier-Stokes equations which improve previous results in [18] and [35]. Our proof is based on an iteration argument, Campanato’s method, and interpolation techniques.

1. Introduction

In this paper, we consider the five dimensional stationary incompressible Navier-Stokes equations with unit viscosity:

$$\begin{cases}
    u \cdot \nabla u - \Delta u + \nabla p = f & \text{in } \Omega, \\
    \nabla \cdot u = 0 & \text{in } \Omega,
\end{cases}$$

where $u = (u_1(x), \ldots, u_5(x)) \in \mathbb{R}^5$ is the velocity field, $p = p(x) \in \mathbb{R}$ is the pressure, and $f = (f_1(x), \ldots, f_5(x)) \in \mathbb{R}^5$ is the given external force. We prove that any suitable weak solution $u$ of (1.1) is locally H"older continuous in $\Omega$ when certain scale invariant quantities are small. Here $\Omega$ is either the unit ball $B_1$ or the unit upper half ball $B_1^+$ in $\mathbb{R}^5$. In the upper half ball case, we impose the zero Dirichlet boundary condition on a flat portion of the boundary

$$u = 0 \quad \text{on } \partial \Omega \cap \{x_5 = 0\}.$$  

(1.2)

The regularity problem of the Navier-Stokes equations is an important question in fluid dynamics, and much effort has been made in the literature. In [16, 21], Leray and Hopf proved the existence of weak solutions to the time-dependent three dimensional incompressible Navier-Stokes equations for any given $L^2$ initial data. However, the regularity and uniqueness of such weak solutions are still open. Many criteria which ensure the regularity of weak solutions have been developed with various conditions on the velocity, the vorticity, or the pressure, and the most famous one is the Ladyženskaja-Prodi-Serrin criteria, see [1, 2, 21, 27, 34] and the references therein. On the other hand, there is an extensive literature devoting to estimating the size of possible singularity set. In a series of papers [28, 30], Scheffer studied the Hausdorff measure of possible singular set for weak solutions of three dimensional Navier-Stokes equations. In [3], Caffarelli, Kohn, and Nirenberg introduced the notion of suitable weak solutions and proved that the one dimensional Hausdorff measure of singular set is zero for any suitable weak solution of three dimensional Navier-Stokes equations. A simplified proof was given later by Lin in [23] when the external force $f$ is zero and the pressure $p \in L^{3/2}(Q_1)$. In [37], Tian and Xin proved a local uniform $L^\infty$ bound of $\nabla u$ for smooth solutions when either the scaled local $L^2$ norm of the vorticity or the scaled local total energy is small. In [40], Vasseur
gave a new proof of the result in [3] by using the De Giorgi iteration. In particular, he proved that if the quantity
\[
\sup_{t \in [-1,0]} \int_{B_1} |u(t,x)|^2 \, dx + \int_{Q_1} |\nabla u|^2 \, dx \, dt + \int_{-1}^{0} \|p\|^{	ilde{q}}_{L^1_t(L^1_x(B_1))} \, dt
\]
is sufficiently small, then \(u\) is regular in \(B_{1/2}\). The above result was recently extended to the boundary case in [6], where it was also proved that any suitable weak solution \(u\) of the three dimensional incompressible Navier-Stokes equations is regular in \(Q_{1/2}^+\) provided that
\[
\|u\|_{L^q_t(L^1_x(Q_{1}^+))} + \|p\|_{L^4_t(L^1_x(Q_{1}^+))}
\]
is sufficiently small, where \(q > 5/2\) and \(\tilde{q} > 1\).

Considerable attention has also been paid to the stationary Navier-Stokes equations. In [12], the author proved the existence of regular solutions to the \(n\) dimensional stationary incompressible Navier-Stokes equations, where the integer \(n \in \{2,3,4\}\). See also [39]. Since five is the smallest dimension in which the stationary Navier-Stokes equations are super-critical, there is a great number of papers devoted to this case. See, for instance, [18,35,36] and the references therein. In [35], for the five dimensional stationary incompressible Navier-Stokes equations (1.1), Struwe proved that any suitable weak solution \(u\) is Hölder continuous near \(x \in \Omega\) provided that
\[
\limsup_{r \to 0} \frac{1}{r} \int_{B_r(x)} |\nabla u|^2 \, dx = 0
\]
is small and the external force \(f \in L^q(\Omega)\), where \(q > 5/2\) and \(\Omega \subset \mathbb{R}^5\) is an open set. The above result was extended to the boundary case by Kang in [18] under the condition that either
\[
\limsup_{r \to 0} \frac{1}{r} \int_{B(x,r) \cap \Omega} |\nabla u|^2 \, dx = 0
\]
or
\[
\liminf_{r \to 0} \frac{1}{r^2} \int_{B(x,r) \cap \Omega} |u|^3 \, dx = 0,
\]
where \(x \in \partial \Omega\) and \(\Omega\) is any smooth domain in \(\mathbb{R}^5\). For more relevant research about the higher dimensional stationary Navier-Stokes equations, we refer readers to [3,5,7,10,17,19,22] and the references therein.

In this paper, we use an iteration argument and interpolation technique to establish local \(\varepsilon\)-regularity criteria at one scale for suitable weak solutions to the five dimensional stationary incompressible Navier-Stokes equations in either the ball \(B_1\) or the upper half ball \(B_1^+\). Our main result, Theorems 1.1, reads that a suitable weak solution \(u\) in \(B_1\) is regular in \(B_{1/2}\) provided that
\[
\|u\|_{L^q(B_1)} + \|f\|_{L^2(B_1)}
\]
is sufficiently small, where \(q\) is any exponent great than \(5/2\). In Theorem 1.2, we obtain a similar result near the boundary. Compared to the conditions (1.3), (1.4), and (1.5) in the previous papers [18] and [35], our smallness condition on the \(L^q\) norm of \(u\) is substantially weaker.

Let us give an outline of the proof. Following the ideas in [6], we use Campanato’s characterization of Hölder continuity and an iteration argument to prove the regularity of \(u\). We decompose \(u\) as \(u = w + v\), where \(v\) is a harmonic function.
We estimate \( w \) by the \( L^p \) estimates of elliptic equations and the uniform decay rates of certain scale invariant quantities. For \( v \), we use the properties of harmonic functions. To be more precise, we first show that the values of the scale invariant quantities \( A + E \), \( A^+ + E^+ \), \( G \), and \( G^+ \) in a small ball can be controlled by their values in a larger ball. We refer the reader to Section 2 for the definitions of \( A + E \), and other scale invariant quantities. This part of the argument is standard. See, for instance, [14, 15, 23]. Then we derive the smallness of \( E \) and \( E^+ \) in Lemmas 3.3 and 4.3 under the conditions of Theorems 1.1 and 1.2 respectively. This is a key estimate in the proofs. Based on the above results, we show certain uniform decay rates of the scale invariant quantities by induction in Lemmas 3.4 and 4.4. Combining the uniform decay rates and the \( L^p \) estimates for elliptic equations, we obtain the Hölder continuity of \( u \) in either \( B_{1/2} \) or \( B_{1/2}^+ \) by using Campanato’s characterization of Hölder continuity. Compared to the interior case, the main obstacle in showing the \( \varepsilon \)-regularity criteria in the upper half ball \( B_{1/2}^+ \) is to estimate the pressure term up to the boundary. More precisely, when we estimate the quantity \( G \) in \( B_1 \), the pressure is decomposed as the sum of a harmonic function and a term which can be controlled by the Calderón-Zygmund estimate. However, this method does not seem to work for the boundary case. In the upper half ball \( B_{1/2}^+ \), we adopt the pressure decomposition originally due to Seregin [31], and then apply the known solvability and boundary regularity results for the linear Stokes systems (see [11, 13, 26, 33]). Compared to [6], we only need the smallness assumption of \( u \) and \( f \) thanks to the local \( W^{k,p} \) estimate for linear Stokes systems. Besides, due to the presence of the external force \( f \) and the different Sobolev embedding inequality in five dimensions, it is quite delicate to choose suitable exponents for the scale invariant quantities.

The main results of this paper are stated as follows. The notation in Theorems 1.1 and 1.2 is introduced in Section 2.

**Theorem 1.1.** Let \( \Omega = B_1 \) and the pair \( (u, p) \) be a suitable weak solution to the five dimensional stationary incompressible Navier-Stokes equations (1.1). Let \( p \in L^1(\Omega) \) and \( f \in L^q(\Omega) \) for some \( q \in (5/2, 10/3) \). There exists a positive constant \( \varepsilon \) such that if

\[
\int_{B_1} |u|^q \, dx + \int_{B_1} |f|^2 \, dx < \varepsilon,
\]

then \( u \) is regular in \( B_{1/2} \).

**Theorem 1.2.** Let \( \Omega = B_{1/2}^+ \) and the pair \( (u, p) \) be a suitable weak solution to the five dimensional stationary incompressible Navier-Stokes equations (1.1) with the boundary condition (1.2). Let \( p \in L^1(\Omega) \) and \( f \in L^q(\Omega) \) for some \( q \in (5/2, 10/3) \). There exists a positive constant \( \varepsilon \) such that if

\[
\int_{B_{1/2}^+} |u|^q \, dx + \int_{B_{1/2}^+} |f|^2 \, dx < \varepsilon,
\]

then \( u \) is regular in \( B_{1/2}^+ \).

In this paper, we only consider the flat boundary for simplicity. The boundary regularity result in Theorem 1.2 still holds true for general \( C^2 \) boundary by following the argument, for example, in [32]. We emphasize that the strict inequality \( q > 5/2 \) is necessary when we derive the smallness of \( E \) and \( E^+ \) in Lemmas 3.3 and 4.3 and
the uniform decay rates for the scale invariant quantities in Lemmas 3.4 and 3.3. At the time of this writing, it is not clear to us whether the result still holds when \( q = 5/2 \). Besides, even though our results refine the known regularity criteria, they probably do not improve the estimate of the Hausdorff dimension of the singular set.

The remainder of this paper is organized as follows. In Section 2, we introduce some notation and state a lemma which will be frequently used in the proof of the main results. The interior \( \varepsilon \)-regularity for suitable weak solutions is proved in Section 3. We prove the corresponding boundary \( \varepsilon \)-regularity in Section 4. Throughout this paper, we use \( N \) to denote various constants which may change from line to line. We also use the expression \( N = N(\cdots) \) for the constant which depends on the contents between the parentheses.

2. Preliminaries

In this section, we introduce some notation and the definition of suitable weak solutions which will be used throughout the paper. We also present a key lemma which will be used to prove our main results.

For any point \( x_0 = (x_0^1, x_0^2, \ldots, x_0^5) \in \mathbb{R}^5 \), we use the notation

\[
B(x_0, \rho) = \{ x \in \mathbb{R}^5 : |x - x_0| < \rho \} \quad \text{and} \quad B^+(x_0, \rho) = B(x_0, \rho) \cap \mathbb{R}^5_+,
\]

to denote the balls in \( \mathbb{R}^5 \) and half balls in \( \mathbb{R}^5_+ \) with center at \( x_0 \) and radius \( \rho > 0 \). For the convenience of notation, we denote

\[
B_\rho = B(0, \rho), \quad B^+_\rho = B^+(0, \rho).
\]

For any nonempty open set \( \Omega \in \mathbb{R}^5 \), we use the abbreviation

\[
(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx
\]

to denote the average of \( u \) in \( \Omega \), where \( |\Omega| \) as usual denotes the Lebesgue measure of \( \Omega \).

We introduce the following scale invariant quantities:

\[
A(x_0, r) = \frac{1}{r^7} \int_{B(x_0, r)} |u|^2 \, dx,
\]

\[
C(x_0, r) = \frac{1}{r^{5-q}} \int_{B(x_0, r)} |u|^q \, dx,
\]

\[
E(x_0, r) = \frac{1}{r} \int_{B(x_0, r)} |\nabla u|^2 \, dx,
\]

\[
G(x_0, r) = \frac{1}{r^{\frac{n-4}{4-n}}} \int_{B(x_0, r)} |p - (p)_{B(x_0, r)}|^\frac{4(n+1)}{4-n-4} \, dx,
\]

\[
P(x_0, r) = \frac{1}{r^7} \int_{B(x_0, r)} |p - (p)_{B(x_0, r)}| \, dx,
\]

\[
F(x_0, r) = r \int_{B(x_0, r)} |f|^2 \, dx,
\]
where \( c \in \left( \frac{5-q}{6q-5}, \frac{1}{4} \right) \) is any fixed positive constant and \( r \in (0, \rho] \). Similarly, we also define the following scale invariant quantities:

\[
A^+(x_0, r) = \frac{1}{r^d} \int_{B^+(x_0, r)} |u|^2 \, dx,
\]

\[
C^+(x_0, r) = \frac{1}{r^q} \int_{B^+(x_0, r)} |u|^q \, dx,
\]

\[
E^+(x_0, r) = \frac{1}{r} \int_{B^+(x_0, r)} |\nabla u|^2 \, dx,
\]

\[
G^+(x_0, r) = \frac{1}{r^\varepsilon} \int_{B^+(x_0, r)} |p - (p)_{B^+(x_0, r)}|^\frac{5(1+\varepsilon)}{4} \, dx,
\]

\[
P^+(x_0, r) = \frac{1}{r^\beta} \int_{B^+(x_0, r)} |p - (p)_{B^+(x_0, r)}| \, dx,
\]

\[
F^+(x_0, r) = r \int_{B^+(x_0, r)} |f|^2 \, dx,
\]

where \( c \in \left( \frac{5-q}{6q-5}, \frac{1}{4} \right) \) is any fixed positive constant and \( r \in (0, \rho] \).

For simplicity, we often denote \( A(r) = A(0, r) \) and similarly for other scale invariant quantities. We see that all these quantities are invariant under the natural scaling:

\[
u_\lambda := \lambda u(\lambda x), \quad p_\lambda := \lambda^2 p(\lambda x), \quad f_\lambda := \lambda^3 f(\lambda x), \quad (2.1)
\]

where \( \lambda > 0 \) is any constant.

Let \( \Omega \) be a domain in \( \mathbb{R}^5 \) and \( \Gamma \in \partial \Omega \). We denote \( \mathcal{C}_0^\infty(\Omega, \Gamma) \) to be the space of divergence-free infinitely differentiable vector fields which vanishes near \( \Gamma \). Let \( J(\Omega, \Gamma) \) be the closure of \( \mathcal{C}_0^\infty(\Omega, \Gamma) \) in \( H^1(\Omega) \).

In this paper, we focus on an important type of solutions called suitable weak solutions. The definition of suitable weak solutions was first introduced in a celebrated paper \(^3\) by Caffarelli, Kohn, and Nirenberg.

**Definition 2.1.** We say that a pair \((u, p)\) is a suitable weak solution to the stationary Navier-Stokes equations \((1.1)\) in \( \Omega \) vanishing on \( \Gamma \) if \((u, p) \in J(\Omega, \Gamma) \times L^1(\Omega) \) satisfies \((1.1)\) in the sense of distribution and for any non-negative function \( \psi \in C^\infty(\overline{\Omega}) \) vanishing in a neighborhood of the boundary \( \partial \Omega \setminus \Gamma \), we have the local energy inequality

\[
2 \int_{\Omega} |\nabla u|^2 \psi \, dx \leq \int_{\Omega} \left[ |u|^2 \Delta \psi + (|u|^2 + 2p) u \cdot \nabla \psi + 2f \cdot u \psi \right] \, dx. \quad (2.2)
\]

The following key lemma will be frequently used in the proofs of our main results.

**Lemma 2.2.** Suppose \( f(\rho_0) < N_0 \). If there exist \( \alpha > \beta > 0 \) and \( N_1, N_2 > 0 \) such that for any \( 0 < r < \rho \leq \rho_0 \), it holds that

\[
f(r) \leq N_1 (r/\rho)^\alpha f(\rho) + N_2 \rho^\beta,
\]

then there exist positive constants \( N_3 \) and \( N_4 \) depending on \( N_0, N_1, N_2, \alpha \) and \( \beta \) such that

\[
f(r) \leq N_3 (r/\rho_0)^\beta f(\rho_0) + N_4 \rho^\beta,
\]

where \( r \in (0, \rho_0] \) is any constant.
Proof. See, for example, Lemma 2.1 in Chapter 3 of [13]. □

3. Interior $\varepsilon$-regularity

This section is devoted to the proof of Theorem 1.1. According to Campanato’s characterization of Hölder continuity, we only need to prove
\[
\int_{B(x,\rho)} |u - (u)_{B(x,\rho)}|^2 \, dx \leq N \rho^{5+\alpha}
\]
for some index $\alpha \in (0,1)$. To this end, we decompose $u$ as $u = w + v$, where $v$ is a harmonic function. We estimate the mean oscillation of $w$ by the $L^p$ estimates for elliptic equations and the uniform decay estimates of the scale invariant quantities, which in turn is proved by an induction argument. To obtain the above decay estimates, we need all these scale invariant quantities to be small. However, from the assumptions in Theorem 1.1, we only have the smallness of $C$ and $F$. For this, we apply the local $W^{k,p}$ estimate for linear Stokes systems to get the smallness of $P$, and then show the smallness of $E$ by an iteration argument based on the fact that the values of $A + E$ and $G$ in a smaller ball can be controlled by their values in a larger ball. For the mean oscillation of $v$, we use the properties of harmonic functions. More precisely, we first obtain the smallness of $E(3/4)$ in Lemma 3.3 by using an iteration argument as well as the estimates in Lemmas 3.1 and 3.2. Then, based on the above results and the condition (1.9), we derive the uniform decay estimates of $A + E$, $G$, and $P$ in Lemma 3.4. Finally, by the decay estimates of these scale invariant quantities and the $L^p$ estimate for elliptic equations, we prove the Hölder continuity of $u$ in the ball $B_{1/2}$ by using Campanato’s characterization of Hölder continuity.

Throughout this section, we use the pair $(u,p)$ to represent a suitable weak solution to the incompressible Navier-Stokes equations (1.1).

First of all, we rewrite the equation into
\[-\Delta u + \nabla p = f - \text{div}(u \otimes u).\]
Applying the local $W^{k,p}$ estimate for linear Stokes systems (see, for instance, in [18 Theorem 3.8]), we have
\[
\|p - (p)_{B_{1/2}}\|_{L^{5/2}(B_{1/2})} \leq N\|u\|_{L^4(B_1)}^2 + N\|f\|_{L^{5/2}(B_1)} + N\|u\|_{L^5(B_1)}.
\]
Therefore, by using (1.6) and after a scaling, we may assume that
\[
\int_{B_1} |u|^q \, dx + \int_{B_1} |p - (p)_{B_1}| \, dx + \int_{B_1} |f|^2 \, dx < \varepsilon,
\]
which in turn is proved by an induction argument. To obtain the above decay estimates, we need all these scale invariant quantities to be small. However, from the assumptions in Theorem 1.1, we only have the smallness of $C$ and $F$. For this, we apply the local $W^{k,p}$ estimate for linear Stokes systems to get the smallness of $P$, and then show the smallness of $E$ by an iteration argument based on the fact that the values of $A + E$ and $G$ in a smaller ball can be controlled by their values in a larger ball. For the mean oscillation of $v$, we use the properties of harmonic functions. More precisely, we first obtain the smallness of $E(3/4)$ in Lemma 3.3 by using an iteration argument as well as the estimates in Lemmas 3.1 and 3.2. Then, based on the above results and the condition (1.9), we derive the uniform decay estimates of $A + E$, $G$, and $P$ in Lemma 3.4. Finally, by the decay estimates of these scale invariant quantities and the $L^p$ estimate for elliptic equations, we prove the Hölder continuity of $u$ in the ball $B_{1/2}$ by using Campanato’s characterization of Hölder continuity.

In the next lemma, we show that the values of $A + E$ and $G$ in smaller ball can be controlled by their values in a larger ball by the following two lemmas.

Lemma 3.1. For any $\gamma \in (0,1/2]$ and $B(x_0,\rho) \subset B_1$, we have
\[
A(x_0,\gamma\rho) + E(x_0,\gamma\rho)
\]
\[
\leq N \left[ \gamma^2 A(x_0,\rho) + \gamma^{-2} A(x_0,\rho) \frac{\delta}{4} E(x_0,\rho)^{\frac{\delta}{4}} + \gamma^{-2} A(x_0,\rho)^{\frac{\delta}{4}} + \gamma^{-2} A(x_0,\rho)^{\frac{\delta}{4}} G(x_0,\rho)^{\frac{4-c}{(4+c)}} + \gamma^{-2} A(x_0,\rho)^{\frac{\delta}{4}} G(x_0,\rho)^{\frac{4-c}{(4+c)}} + \gamma^{-4} F(x_0,\rho) \right],
\]
where $N = N(c)$ is a positive constant independent of $\gamma$ and $\rho$. 
**Proof.** The proof is more or less standard. For the sake of completeness, we give the detail of proof. By the scale invariant property, we may assume $\rho = 1$.

For the test function
\[ \Gamma(x) = \left( \gamma^2 + |x - x_0|^2 \right)^{-\frac{3}{2}}, \]
one has
\[ \Delta \Gamma(x) < 0 \quad \text{in } \mathbb{R}^5, \quad \Delta \Gamma(x) \leq -N\gamma^{-5} \quad \text{in } B(x_0, \gamma). \]

We choose a suitable smooth cutoff function $\phi(x) \in C_c^\infty(B(x_0, 1))$ which satisfies
\[ 0 \leq \phi(x) \leq 1 \quad \text{in } B(x_0, 1), \quad \phi(x) = 1 \quad \text{in } B(x_0, 1/2), \]
\[ |\nabla \phi(x)| \leq N, \quad |\nabla^2 \phi(x)| \leq N \quad \text{in } B(x_0, 1). \]

A simple calculation yields
\[ \Gamma(x)\phi(x) \geq N\gamma^{-3}, \quad \forall \ x \in B(x_0, \gamma), \quad (3.3) \]
\[ |\Gamma(x)\phi(x)| \leq N\gamma^{-3}, \quad \forall \ x \in B(x_0, 1), \quad (3.4) \]
\[ |\nabla \Gamma(x)\phi(x) + \Gamma(x)\nabla \phi(x)| \leq N\gamma^{-4}, \quad \forall \ x \in B(x_0, 1) \quad (3.5) \]
and
\[ |\Gamma(x)\Delta \phi(x) + 2\nabla \Gamma(x) \cdot \nabla \phi(x)| \leq N, \quad \forall \ x \in B(x_0, 1). \quad (3.6) \]

Taking $\psi = \Gamma(x)\phi(x)$ in the energy inequality, we have
\[
- \int_{B(x_0, \gamma)} |u|^2 \Delta \Gamma(x)\phi(x) \, dx + 2 \int_{B(x_0, \gamma)} |\nabla u|^2 \Gamma(x)\phi(x) \, dx \\
\leq \int_{B(x_0, 1)} |u|^2 \Gamma(x) \Delta \phi(x) + 2 \nabla \Gamma(x) \cdot \nabla \phi(x) | \, dx \\
+ \int_{B(x_0, 1)} \left( |u|^2 + |p - (p)_{B(x_0, 1)}| \right) |u| \cdot |\nabla \Gamma(x)\phi(x) + \Gamma(x)\nabla \phi(x)| \, dx \\
+ 2 \int_{B(x_0, 1)} |f \cdot u| |\Gamma(x)\phi(x)| \, dx.
\]

By the properties and Young’s inequality, one obtains
\[
\gamma^{-5} \int_{B(x_0, \gamma)} |u|^2 \, dx + \gamma^{-3} \int_{B(x_0, \gamma)} |\nabla u|^2 \, dx \\
\leq N \left[ \int_{B(x_0, 1)} |u|^2 \, dx + \gamma^{-4} \int_{B(x_0, 1)} \left( |u|^2 + |p - (p)_{B(x_0, 1)}| \right) |u| \, dx \\
+ \gamma^{-6} \int_{B(x_0, 1)} f^2 \, dx \right],
\]
which yields
\[
A(x_0, \gamma) + E(x_0, \gamma) \leq N \left[ \gamma^2 A(x_0, 1) \\
+ \gamma^{-2} \int_{B(x_0, 1)} \left( |u|^2 + |p - (p)_{B(x_0, 1)}| \right) |u| \, dx + \gamma^{-4} F(x_0, 1) \right]. \quad (3.7)
\]
For the second term on the right-hand side of (3.7), by Hölder’s inequality and the Sobolev embedding inequality, we have
\[
\int_{B(x_0,1)} |u|^3 \, dx \leq \left( \int_{B(x_0,1)} |u|^\frac{4}{3} \, dx \right)^{\frac{3}{4}} \left( \int_{B(x_0,1)} |u|^2 \, dx \right)^{\frac{1}{4}} \\
\leq N \left( \int_{B(x_0,1)} |\nabla u|^2 \, dx + \int_{B(x_0,1)} |u|^2 \, dx \right)^{\frac{1}{4}} \left( \int_{B(x_0,1)} |u|^2 \, dx \right)^{\frac{3}{4}} \\
\leq N \left( A(x_0,1)^{\frac{1}{4}} E(x_0,1)^{\frac{3}{4}} + A(x_0,1)^{\frac{2}{3}} \right)
\]
and
\[
\int_{B(x_0,1)} |p - (p)_{B(x_0,1)}| |u| \, dx \\
\leq \left( \int_{B(x_0,1)} |p - (p)_{B(x_0,1)}|^{\frac{5(1+\gamma)}{4+2\gamma}} \, dx \right)^{\frac{4+2\gamma}{5(1+\gamma)}} \left( \int_{B(x_0,1)} |u|^{\frac{5(1+\gamma)}{4+2\gamma}} \, dx \right)^{\frac{4+2\gamma}{5(1+\gamma)}} \\
\leq N \left( \int_{B(x_0,1)} |p - (p)_{B(x_0,1)}|^{\frac{5(1+\gamma)}{4+2\gamma}} \, dx \right)^{\frac{4+2\gamma}{5(1+\gamma)}} \left( \int_{B(x_0,1)} |u|^2 \, dx \right)^{\frac{5(1+\gamma)}{4+2\gamma}} \\
\quad \cdot \left( \int_{B(x_0,1)} |\nabla u|^2 \, dx + \int_{B(x_0,1)} |u|^2 \, dx \right)^{\frac{5(1+\gamma)}{4+2\gamma}} \\
\leq N \left( A(x_0,1)^{\frac{9}{1+4\gamma}} E(x_0,1)^{\frac{4+6\gamma}{1+4\gamma}} G(x_0,1)^{\frac{4+6\gamma}{1+4\gamma}} + A(x_0,1)^{\frac{2}{3}} G(x_0,1)^{\frac{4+6\gamma}{1+4\gamma}} \right),
\]
where \(N = N(c)\) is some positive constant. Plugging above two inequalities into (3.7), the lemma is proved. \(\square\)

**Lemma 3.2.** For any \(\gamma \in (0, 3/5]\) and \(B(x_0, \rho) \subset B_1\), we have
\[
G(x_0, \gamma \rho) \leq N \gamma^{-\frac{10+15\gamma}{4+2\gamma}} \left( \gamma \frac{25}{1+\gamma} P(x_0, \rho)^{\frac{5(1+\gamma)}{4+2\gamma}} + A(x_0, \rho)^{\frac{5(1+\gamma)}{4+2\gamma}} E(x_0, \rho)^{\frac{5(1+\gamma)}{4+2\gamma}} \right) \\
+ F(x_0, \rho)^{\frac{5(1+\gamma)}{4+2\gamma}} ,
\]
where \(N = N(c)\) is a positive constant independent of \(\gamma\) and \(\rho\).

**Proof.** By the scale invariant property, we assume \(\rho = 1\). Taking divergence on both sides of the first equation in (1.1), since \(u\) is divergence free, we have
\[
\Delta p = -D_{ij} (u_i u_j) + \nabla \cdot f \\
= -D_{ij} \left[ (u_i - (u_i)_{B(x_0,1)})(u_j - (u_j)_{B(x_0,1)}) \right] + \nabla \cdot f.
\]

Let \(\eta(x)\) be a suitable smooth cutoff function in \(B(x_0,1)\) satisfying \(0 \leq \eta \leq 1\) and \(\eta = 1\) in \(B(x_0, 2/3)\). We decompose the pressure term as follows
\[
p = \tilde{p} + \tilde{h},
\]
where \(\tilde{p}\) is the Newtonian potential of
\[
-D_{ij} \left[ (u^i - (u^i)_{B(x_0,1)})(u^j - (u^j)_{B(x_0,1)}) \eta(x-x_0) \right] + \nabla \cdot [f \eta(x-x_0)].
\]
By applying the Calderón-Zygmund estimate, the Sobolev embedding inequality, and Hölder’s inequality, one has
\[
\int_{B(x_0,1)} |\tilde{p}|^{\frac{5(1+\gamma)}{4+2\gamma}} \, dx
\]
\[ \leq N \left[ \int_{B(x_0,1)} |u - (u)_{B(x_0,1)}|^{\frac{10(1+c)}{4}} dx + \int_{B(x_0,1)} \left| \Delta^{-1} \nabla \cdot [f \eta(x - x_0)] \right|^{\frac{5(1+c)}{4}} dx \right] \]

\[ \leq N \left[ \left( \int_{B(x_0,1)} |u|^2 dx \right)^{\frac{5(1-4c)}{4(1+c)}} \left( \int_{B(x_0,1)} |\nabla u|^2 dx \right)^{\frac{5(1+6c)}{4(1+c)}} \right] \]

\[ + \left( \int_{B(x_0,1)} |f|^{1+c} dx \right)^{\frac{5}{4+c}} \]

\[ \leq N \left[ \left( \int_{B(x_0,1)} |u|^2 dx \right)^{\frac{5(1-4c)}{4(1+c)}} \left( \int_{B(x_0,1)} |\nabla u|^2 dx \right)^{\frac{5(1+6c)}{4(1+c)}} \right] \]

\[ + \left( \int_{B(x_0,1)} |f|^2 dx \right)^{\frac{5(1+c)}{4+c}} \right], \quad (3.10) \]

where \( N = N(c) \).

Since \( \tilde{h} \) is a harmonic function in \( B(x_0,2/3) \), any Sobolev norm of \( \tilde{h} \) in \( B(x_0,\gamma) \) can be controlled by the \( L^p \) norm of it in \( B(x_0,2/3) \), where \( p \in [1, +\infty] \). Thus, we have

\[ \int_{B(x_0,\gamma)} |\tilde{h} - (\tilde{h})_{B(x_0,\gamma)}|^{\frac{5(1+c)}{4+c}} dx \]

\[ \leq N \gamma^{\frac{25}{4+c}} \int_{B(x_0,\gamma)} |\nabla \tilde{h}|^{\frac{5(1+c)}{4+c}} dx \leq N \gamma^{\frac{25}{4+c}} \sup_{B(x_0,\gamma)} |\nabla \tilde{h}|^{\frac{5(1+c)}{4+c}} \]

\[ \leq N \gamma^{\frac{25}{4+c}} \left( \int_{B(x_0,2/3)} |\tilde{h} - (p)_{B(x_0,1)}| dx \right)^{\frac{5(1+c)}{4+c}} \]

\[ \leq N \gamma^{\frac{25}{4+c}} \left[ \left( \int_{B(x_0,1)} |p - (p)_{B(x_0,1)}| dx \right)^{\frac{5(1+c)}{4+c}} + \int_{B(x_0,1)} |\tilde{h}|^{\frac{5(1+c)}{4+c}} dx \right] \quad (3.11) \]

The combination of (3.10) and (3.11) yields

\[ \int_{B(x_0,\gamma)} |p - (p)_{B(x_0,1)}|^{\frac{5(1+c)}{4+c}} dx \]

\[ \leq N \left( \int_{B(x_0,\gamma)} |\tilde{h} - (\tilde{h})_{B(x_0,\gamma)}|^{\frac{5(1+c)}{4+c}} dx + \int_{B(x_0,\gamma)} |\tilde{h}|^{\frac{5(1+c)}{4+c}} dx \right) \]

\[ \leq N \gamma^{\frac{25}{4+c}} \left( \int_{B(x_0,1)} |p - (p)_{B(x_0,1)}| dx \right)^{\frac{5(1+c)}{4+c}} + \left( \int_{B(x_0,1)} |u|^2 dx \right)^{\frac{5(1+6c)}{4(1+c)}} \]

\[ \cdot \left( \int_{B(x_0,1)} |\nabla u|^2 dx \right)^{\frac{5(1-4c)}{4(1+c)}} + \left( \int_{B(x_0,1)} |f|^2 dx \right)^{\frac{5(1+c)}{4+c}} \right], \]

where \( N = N(c) \). The conclusion of Lemma 3.2 follows immediately. \( \square \)

The following lemma shows that \( E(3/4) \) can be sufficiently small under the conditions of Theorem 1.1. This lemma is needed for us to derive the decay estimates of scale invariant quantities with respect to the radius.

**Lemma 3.3.** Let \( \Omega = B_1 \) and \((u, p)\) be a suitable weak solution to (1.1) satisfying the condition of Theorem 1.1. Then we have

\[ E(3/4) \leq N^{\beta}, \]
where $N$ and $\beta$ are positive constants which depend only on $c$ and $q$.

Proof. Let $\rho_k = 1 - 2^{-(k+1)}$ and $B_k = B_{\rho_k}$, where $k$ is any positive integer. We choose cutoff functions $\psi_k$ which satisfy

$$\text{supp } \psi_k \subset B_{k+1}, \quad \psi_k = 1 \text{ in } B_k,$$

$$|D\psi_k| \leq N2^k, \quad |D^2\psi_k| \leq N2^{2k} \text{ in } B_{k+1}.$$  

By the energy inequality (2.2), we have

$$2 \int_{B_k} |\nabla u|^2 \, dx \leq N \int_{B_{k+1}} \left( 2^k |u|^3 + 2^{k+1} |p - (p)_{B_{k+1}}| |u| + 2^{2k} |u|^2 + 2 |f||u| \right) \, dx. \quad (3.12)$$

Due to the condition (3.1) and Hölder’s inequality, we have

$$A(\rho_{k+1}) \leq NC(\rho_{k+1})^{\frac{2}{q}} \leq N \varepsilon^{\frac{2}{q}}. \quad (3.13)$$

Next we use (3.1) and (3.13) to estimate each term on the right-hand side of (3.12).

For the first term on the right-hand side of (3.12), by Hölder’s inequality, the Sobolev embedding inequality and (3.13), one has

$$\int_{B_{k+1}} |u|^3 \, dx = \int_{B_{k+1}} |u|^a |u|^{3-a} \, dx \leq \left( \int_{B_{k+1}} |u|^a \, dx \right)^{\frac{q}{q-a}} \left( \int_{B_{k+1}} |u|^{\frac{(3-a)q}{q-a}} \, dx \right)^{\frac{q-a}{q}}$$

$$\leq N \int_{B_{k+1}} |u|^q \, dx \leq N \left( \int_{B_{k+1}} |u|^2 \, dx \right)^{\frac{q-5}{2(q-5)}} \left( \int_{B_{k+1}} |\nabla u|^2 \, dx \right)^{\frac{q-2}{2(q-5)}} + \rho_k^{-2} \int_{B_{k+1}} |u|^2 \, dx \leq N \rho_k^{-2} \int_{B_{k+1}} |u|^2 \, dx$$

$$\leq N \rho_k^{-2} \left( A(\rho_{k+1}) \frac{\rho_{k+1}^{5(q-5)+2q}}{4q(q-5)} C(\rho_{k+1}) \varepsilon \frac{\rho_k^{-2}}{q} E(\rho_{k+1}) \frac{\rho_{k+1}^{5(q-5)+2q}}{4q(q-5)} \right)$$

$$\leq N \left( \varepsilon \frac{\rho_{k+1}^{5(q-5)+2q}}{4q(q-5)} E(\rho_{k+1}) \frac{\rho_{k+1}^{5(q-5)+2q}}{4q(q-5)} + \varepsilon \right), \quad (3.14)$$

where $N = N(a, q)$ is some positive constant. Here, we take the constant $a \in (0, 1)$ to make sure that all the exponents in the above inequality are positive.

Similarly, for the second term on the right-hand side of (3.12), it holds that

$$\int_{B_{k+1}} |p - (p)_{B_{k+1}}| |u| \, dx \leq \left( \int_{B_{k+1}} |p - (p)_{B_{k+1}}|^2 \frac{2(1+c)}{4-c} \, dx \right)^{\frac{4-c}{2(1+c)}} \left( \int_{B_{k+1}} |u|^{\frac{5(1+c)}{4-c}} \, dx \right)^{\frac{4-2c}{5(1+c)}}$$

$$\leq \left( \int_{B_{k+1}} |\nabla p|^{1+c} \, dx \right)^{\frac{1}{1+c}} \left( \int_{B_{k+1}} |u|^2 \, dx \right)^{\frac{2(1+c)-5c}{4(1+c)(q-2)}} \left( \int_{B_{k+1}} |u|^q \, dx \right)^{\frac{3-7c}{(1+c)(q-2)}}, \quad (3.15)$$
where the exponents \( \frac{q+6c-5-5c}{5(1+c)(q-2)} \) and \( \frac{3-7c}{5(1+c)(q-2)} \) are positive due to the fact \( c \in \left( \frac{5-q}{6q-5}, \frac{1}{4} \right) \).

For the first term on the right-hand side of (3.13), by the \( W^{1,p} \) estimate for elliptic equation (3.9), we have
\[
\left( \int_{B_{k+2}} |\nabla p|^{1+c} \, dx \right)^{\frac{1}{1+c}} \leq N \left[ \left( \int_{B_{k+2}} |f|^{1+c} \, dx \right)^{\frac{1}{1+c}} + \left( \int_{B_{k+2}} |u \cdot \nabla u|^{1+c} \, dx \right)^{\frac{1}{1+c}} \right. \\
+ \left( \rho_{k+2} - \rho_{k+1} \right)^{-\frac{1+c}{1+c}} \int_{B_{k+2}} |p - (p)_{B_{k+2}}| \, dx \right], \tag{3.16}
\]
where \( N = N(c) > 0 \) is some constant.

For the second term on the right-hand side of the above inequality, by Hölder’s inequality and the Sobolev embedding inequality, one obtains
\[
\|u \cdot \nabla u\|_{L^{1+c}(B_{k+2})} \leq \|u\|_{L^{\frac{5(1+c)}{k+c}}(B_{k+2})} \|\nabla u\|_{L^{2}(B_{k+2})} \\
\leq N \left( \|\nabla u\|_{L^{2}(B_{k+2})}^{\frac{5}{k+c}} + \rho_{k+2}^{\frac{5}{k+c}} \|u\|_{L^{2}(B_{k+2})} \|\nabla u\|_{L^{2}(B_{k+2})} \right) \\
\leq N \left( \|u\|_{L^{2}(B_{k+2})} \|\nabla u\|_{L^{2}(B_{k+2})} + \rho_{k+2}^{\frac{5}{k+c}} \|u\|_{L^{2}(B_{k+2})} \|\nabla u\|_{L^{2}(B_{k+2})} \right), \tag{3.17}
\]
where \( N = N(c) \) is some positive constant.

The combination of (3.16) and (3.17) yields
\[
\left( \int_{B_{k+1}} |\nabla p|^{1+c} \, dx \right)^{\frac{1}{1+c}} \leq N \left[ \rho_{k+2}^{\frac{5}{k+c}} \left( \int_{B_{k+2}} |f|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{B_{k+2}} |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B_{k+2}} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \right. \\
+ \rho_{k+2}^{\frac{5}{k+c}} \left( \int_{B_{k+2}} |u|^2 \, dx \right) \left( \int_{B_{k+2}} |\nabla u|^2 \, dx \right) \right. \\
+ 2 \left( \frac{k+1+6c}{k+3} \right)^{\frac{1}{2}} \int_{B_{k+2}} |p - (p)_{B_{k+2}}| \, dx \right] \\
\leq N \left[ \rho_{k+2}^{\frac{5}{k+c}} F(\rho_{k+2}) \right. \\
+ \rho_{k+2}^{\frac{5}{k+c}} A(\rho_{k+2}) \frac{2^{\frac{k+5}{1+c}} E(\rho_{k+2})^{\frac{1+k}{1+c}}}{\frac{2^{k+1} A(\rho_{k+2})^{\frac{k+5}{1+c}}}{\rho_{k+2}^{\frac{2}{1+c}}}} \\
+ \frac{2^{\frac{k+5}{1+c}} A(\rho_{k+2})^{\frac{1+k}{1+c}}}{\rho_{k+2}^{\frac{2}{1+c}}} E(\rho_{k+2})^{\frac{1}{k+c}} \right],
\]
where \( N = N(c) \) > 0. Inserting the above inequality into (3.13), by (3.1) and (3.13), we have
\[
\int_{B_{k+1}} |p - (p)_{B_{k+1}}| \, dx \\
\leq N \left( \rho_{k+2}^{2} A(\rho_{k+2})^{\frac{2^{k+5+c} - 5 - 5c}{5(1+c)(q-2)}} C(\rho_{k+2})^{\frac{3-7c}{5(1+c)(q-2)}} \right. \\
+ \rho_{k+2}^{2} A(\rho_{k+2})^{\frac{2^{k+5+c} - 20 - 9q}{5(1+c)(q-2)}} C(\rho_{k+2})^{\frac{3-7c}{5(1+c)(q-2)}} \right. \\
+ \rho_{k+2}^{2} A(\rho_{k+2})^{\frac{2^{k+5+c} - 20 - 9q}{5(1+c)(q-2)}} C(\rho_{k+2})^{\frac{3-7c}{5(1+c)(q-2)}} \right. \\
+ 2 \left( \frac{k+3(1+6c)}{k+1+6c} \right)^{\frac{3-8c}{k+1+6c}} C(\rho_{k+2})^{\frac{3-7c}{5(1+c)(q-2)}} P(\rho_{k+2}) \right).
\[
\leq N \left( \varepsilon^{\frac{a}{3q}} + \varepsilon^{\frac{2-3a}{2q}} E(\rho_{k+2})^{\frac{1+6c}{1+6c}} + \varepsilon^\frac{\delta}{2} E(\rho_{k+2})^{\frac{1}{2}} + 2^{\frac{(k+3)(1+6c)}{4+q} - \frac{a+1}{2q}} \varepsilon^{\frac{\delta}{4}} \right),
\]

where all the exponents of scale invariant quantities are positive due to the facts $c \in \left( \frac{5-a}{6q-5}, \frac{1}{2} \right)$ and $q \in \left( \frac{2}{3}, \frac{10}{3} \right)$.

For the last two terms on the right-hand side of (3.12), by (3.1), (3.14), and Hölder’s inequality, we obtain

\[
\int_{B_{k+1}} |u|^2 \, dx \leq N^2 \rho_{k+1}^3 \left( \int_{B_{k+1}} |u|^2 \, dx \right)^\frac{3}{2}
\]

\[
\leq N^2 \rho_{k+1}^3 \left( A(\rho_{k+1}) \frac{q+5a-10\delta}{6q} C(\rho_{k+1}) + A(\rho_{k+1}) + C(\rho_{k+1}) \right)
\]

\[
\leq N \left( \varepsilon^{\frac{a}{3q}} E(\rho_{k+1})^{\frac{5(q-aq+2\delta)}{6q}} + \varepsilon^{\frac{\delta}{4}} \right)
\]

and

\[
\int_{B_{k+1}} |f||u| \, dx \leq \left( \int_{B_{k+1}} |f|^2 \, dx \right)^\frac{1}{2} \left( \int_{B_{k+1}} |u|^2 \, dx \right)^\frac{1}{2}
\]

\[
\leq \rho_{k+1} A(\rho_{k+1})^{\frac{1}{2}} F(\rho_{k+1})^{\frac{1}{2}} \leq N \varepsilon^{\frac{2q}{7q}},
\]

where $N = N(a, q) > 0$.

The combination of (3.12), (3.14), and (3.18)-(3.20) yields

\[
E(\rho_k) \leq 2^{2k} N \left( \varepsilon^{\frac{a}{3q}} + \varepsilon^{\frac{2-3a}{2q}} E(\rho_{k+2})^{\frac{5(q-aq+2\delta)}{6q}} + \varepsilon^\frac{\delta}{4} + \varepsilon^{\frac{2q}{7q}} + \varepsilon^{\frac{a}{3q}} E(\rho_{k+2}) \right)^{\frac{1+6c}{1+6c}} + \varepsilon^\frac{\delta}{4} + \varepsilon^{\frac{2q}{7q}} + \varepsilon^{\frac{a}{3q}} E(\rho_{k+2})^{\frac{1}{2}}
\]

\[
\leq 2^{2k} N \left( \varepsilon^{\frac{a}{3q}} + \varepsilon^{\frac{2-3a}{2q}} E(\rho_{k+2})^{\frac{5(q-aq+2\delta)}{6q}} + \varepsilon^\frac{\delta}{4} + \varepsilon^{\frac{2q}{7q}} + \varepsilon^{\frac{a}{3q}} E(\rho_{k+2}) \right)^{\frac{1+6c}{1+6c}} + \varepsilon^\frac{\delta}{4} + \varepsilon^{\frac{2q}{7q}} + \varepsilon^{\frac{a}{3q}} E(\rho_{k+2})^{\frac{1}{2}}
\]

where $N = N(a, c, q)$. Since $c \in \left( \frac{5-a}{6q-5}, \frac{1}{2} \right)$, if we take $a \in \left( \frac{q}{3q-10}, 1 \right)$, then all the exponents of $E(\rho_{k+2})$ in the above inequality are positive and smaller than one. Thus, by Young’s inequality, for any $\delta > 0$, we have

\[
E(\rho_k) \leq \delta^2 E(\rho_{k+2}) + 2^{4k} N \varepsilon^\beta,
\]

where $N = N(c, q, \delta) > 0$, $\beta = \beta(c, q) > 0$ are some constants.

Let $\delta = 3^{-4}$. Multiplying both sides of the above inequality by $\delta^k$ and summing over $k$ from 1 to infinity, one obtains

\[
\sum_{k=1}^{\infty} \delta^k E(\rho_k) \leq \sum_{k=3}^{\infty} \delta^k E(\rho_k) + N \varepsilon^\beta \sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^{4k},
\]

which implies

\[
E(\frac{3}{4}) \leq N \varepsilon^\beta.
\]

The lemma is proved.
Based on the above three lemmas, we will verify the decay estimates of $A(x, \rho)$, $E(x, \rho)$, $G(x, \rho)$, and $P(x, \rho)$ with respect to radius when $A(\rho_0)$, $E(\rho_0)$, $F(\rho_0)$, and $P(\rho_0)$ are sufficiently small.

**Lemma 3.4.** There exists a constant $\bar{\varepsilon} > 0$ satisfying the following property. If we have

$$A(\rho_0) + E(\rho_0) + F(\rho_0) \frac{2-\alpha}{1-\alpha} + P(\rho_0) \frac{2-\gamma}{\gamma+1} \leq \bar{\varepsilon},$$  \hspace{1cm} (3.21)

then there exists sufficiently small constant $l \in \left(0, \frac{4q-10}{5q-6}\right)$ such that for any $x \in B_{\rho_0/6}$ and $\rho \in (0, \rho_0/2)$, the following inequalities hold

$$A(x, \rho) + E(x, \rho) \leq N \hat{\varepsilon} \left(\frac{\rho}{\rho_0}\right)^{2-l},$$  \hspace{1cm} (3.22)

$$G(x, \rho) \leq N \hat{\varepsilon} \left(\frac{\rho}{\rho_0}\right)^{\frac{5(1+\alpha)(1+\delta)}{4\alpha^+}},$$  \hspace{1cm} (3.23)

and

$$P(x, \rho) \leq N \hat{\varepsilon} \left(\frac{\rho}{\rho_0}\right)^{1+l},$$  \hspace{1cm} (3.24)

where $N$ is a positive constant depending on $c, l$, but independent of $\varepsilon, \rho$ and $x$.

**Proof.** We use an induction argument. Let $\rho_k = \tilde{\rho}^{(1+\beta)^k}$, where $\tilde{\rho} = \rho_0/2$ and $\beta > 0$ is some small constant which will be specified later. We fix an auxiliary parameter $\alpha \in (2 - l, 2)$ by a scaling argument, we first consider the special case: $\tilde{\rho}^\alpha = N \hat{\varepsilon}$. For any $x \in B_{\rho_0/6}$, it is sufficient for us to prove the following decay estimates:

$$A(x, \rho_k) + E(x, \rho_k) \leq \rho_k^\alpha, \quad G(x, \rho_k) \leq \rho_k^{\frac{2-\alpha}{1-\alpha}}, \quad P(x, \rho_k) \leq \rho_k^{\frac{2-\gamma}{\gamma+1}}. \hspace{1cm} (3.25)$$

Let us first verify the above estimates when $k = 0$. For any $x \in B_{\rho_0/6}$, since $\tilde{\rho} = \rho_0/2$, by the condition (3.21) and the fact $B(x, \tilde{\rho}) \subset B_{\rho_0}$, we get

$$A(x, \tilde{\rho}) + E(x, \tilde{\rho}) + F(x, \tilde{\rho}) \frac{2-\alpha}{1-\alpha} + P(x, \tilde{\rho}) \frac{2-\gamma}{\gamma+1} \leq N \hat{\varepsilon}. \hspace{1cm} (3.26)$$

For $G(x, \tilde{\rho})$, since $B(x, 5\tilde{\rho}/3) \subset B_{\rho_0}$, by (3.21) and Lemma 3.2, we have

$$G(x, \tilde{\rho}) \leq N \left(P(x, 5\tilde{\rho}/3)^{\frac{5(1+\alpha)-1}{4\alpha^-}} + A(x, 5\tilde{\rho}/3)^{\frac{5(1-4\epsilon)}{4\epsilon}} E(x, 5\tilde{\rho}/3)^{\frac{5(1+6\epsilon)}{4\epsilon}} + F(x, 5\tilde{\rho}/3)^{\frac{5(1+\alpha)}{4\alpha^-}}\right) \hspace{1cm} (3.27)$$

where $N = N(c, l)$. Due to (3.21) and (3.24), we can choose $N > 0$ such that $\tilde{\rho}^\alpha = N \hat{\varepsilon}$. Hence, we obtain

$$A(x, \tilde{\rho}) \leq \tilde{\rho}^\alpha, \quad E(x, \tilde{\rho}) \leq \tilde{\rho}^\alpha, \quad G(x, \tilde{\rho}) \leq \tilde{\rho}^{\frac{2-\alpha}{1-\alpha}}, \quad P(x, \tilde{\rho}) \leq \tilde{\rho}^{\frac{2-\gamma}{\gamma+1}}. \hspace{1cm} (3.28)$$

Thus, we proved (3.25) for $k = 0$.

Suppose (3.25) holds for $0$ to $k$, we want to prove that it is also true for $k + 1$.

We first estimate $A(x, \rho_{k+1}) + E(x, \rho_{k+1})$. Taking $\gamma = \rho_k^\beta$ and $\rho = \rho_k$ in (3.2), we have

$$A(x, \rho_{k+1}) + E(x, \rho_{k+1})$$
\[
\leq \left( \rho_k^{2\beta+\alpha} + \rho_k^{-2\beta+\frac{3\alpha}{4}} + \rho_k^{-2\beta+\frac{3\alpha}{4} + \frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)}} + \rho_k^{-2\beta+\frac{5(\alpha+1)(1+l)}{4(4-c)(2-l)}} + \rho_k^{-2\beta+\frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)}} \right) \\
+ \rho_k^{-2\beta+\frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)}} + 6 \cdot \frac{\rho_k}{q} \| f \|_{L^q}^2 \\
\leq N \left( \rho_k^{2\beta+\alpha} + \rho_k^{-2\beta+\frac{3\alpha}{4}} + \rho_k^{-2\beta+\frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)}} + \rho_k^{-2\beta+\frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)}} \right),
\]
where \( N = N(c) \). If we take \( \beta < \min \left\{ \frac{\alpha}{2(\alpha+2)}, \frac{3\alpha}{4(4-c)(4-\alpha+2)}, \frac{6q - \alpha q - 10}{(4+c)q} \right\} \), then we have
\[
\min \left\{ 2\beta + \alpha, -2\beta + \frac{3\alpha}{2}, -2\beta + \frac{\alpha(4+l)}{4-2l}, 6 - \frac{10}{q} - 4\beta \right\} > \alpha(1+\beta).
\]
Hence, there exists a small constant \( \xi > 0 \) such that
\[
A(x, \rho_{k+1}) + E(x, \rho_{k+1}) \leq N \rho_k^{(1+\beta) + \xi(1+\beta)} \leq N \rho_k^{\alpha + \xi}. \tag{3.29}
\]
We choose a sufficiently small \( \tilde{\xi} \) which satisfies
\[
N \rho_k^{\xi} \leq N \tilde{\rho}^{\tilde{\xi}} < N(\tilde{\xi}) \frac{1}{2} \leq 1.
\]
Inserting the above inequality into (3.29), we have
\[
A(x, \rho_{k+1}) + E(x, \rho_{k+1}) \leq \rho_k^{\xi}. \tag{3.30}
\]
Next we bound \( G(x, \rho_{k+1}) \). Taking \( \gamma = \rho_k^{\beta} \) and \( \rho = \rho_k \) in (3.8), by (3.23) and H"older's inequality, one has
\[
G(x, \rho_{k+1}) \leq N \left( \rho_k^{\frac{5(1-c)\beta}{4-c}} + \frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)} \right) \| f \|_{L^q}^{\frac{5(1+c)}{4-c}} \\
+ \rho_k^{\frac{5(1-c)\beta}{4-c}} + \frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)} \| f \|_{L^q}^{\frac{5(1+c)}{4-c}} \leq N \left( \rho_k^{\frac{5(1-c)\beta}{4-c}} + \frac{5\alpha(1+c)(1+l)}{4(4-c)(2-l)} \right), \tag{3.31}
\]
where \( N = N(c) \). If we take
\[
0 < \beta < \frac{(1+c)(6q - \alpha q - 10 - (3q + \alpha q - 5)l)}{q(1+c)(1+l)\alpha + (2-3c)(2-l)}
\]
and
\[
0 < \frac{2q - 10}{5q - 5} < \frac{6q - \alpha q - 10}{3q + \alpha q - 5},
\]
then all the exponents of \( \rho_k \) on the right-hand side of (3.31) are greater than \( \frac{5\alpha(1+c)(1+l)(1+\beta)}{(4-c)(2-l)} \). Hence, there exists a small constant \( \xi > 0 \) such that
\[
G(x, \rho_{k+1}) \leq N \rho_k^{\frac{5\alpha(1+c)(1+l)(1+\beta)}{(4-c)(2-l)}} \leq N \rho_k^{\alpha(1+\beta) + \xi}. \tag{3.32}
\]
By choosing \( \tilde{\xi} \) sufficiently small, one has
\[
N \rho_k^{\xi} < N \tilde{\rho}^{\tilde{\xi}} < N(\tilde{\xi}) \frac{1}{2} < 1.
\]
Inserting the above inequality into (3.32), we have
\[
G(x, \rho_{k+1}) \leq \rho_k^{\frac{5\alpha(1+c)(1+l)}{(4-c)(2-l)}}. \tag{3.33}
\]
For $P(x, \rho_{k+1})$, by Hölder’s inequality, we have
\[
\int_{B(x, \rho_{k+1})} |p - (p)_{B(x, \rho_{k+1})}| \, dx \leq N\rho_{k+1}^{\frac{4(c-1)}{5(1+c)}} \left( \int_{B(x, \rho_{k+1})} |p - (p)_{B(x, \rho_{k+1})}|^{\frac{5(1+c)}{4}} \, dx \right)^{\frac{4}{5(1+c)}}.
\] (3.34)

The combination of (3.32) and (3.34) implies
\[
P(x, \rho_{k+1}) \leq NG(x, \rho_{k+1}) \leq N\rho_{k+1}^{\frac{4(c-1)}{5(1+c)}},
\] (3.35)
where $N = N(c)$. We choose $\tilde{\varepsilon}$ sufficiently small such that
\[
N\rho_{k+1}^{\frac{4(c-1)}{5(1+c)}} < N\tilde{\varepsilon}^{\frac{4(c-1)}{5(1+c)}} < 1.
\]

Inserting the above inequality into (3.35), we have
\[
P(x, \rho_{k+1}) \leq \rho_{k+1}^{\frac{4(c-1)}{5(1+c)}}.
\] (3.36)

Using (3.25), (3.30), (3.33), and (3.36), by induction we obtain (3.25) for any integer $k \geq 0$.

For any $\rho \in (0, \rho_0/2)$, there exists a positive integer $k$ such that $\rho_{k+1} \leq \rho < \rho_k$. By (3.25), we obtain
\[
A(x, \rho) \leq \left( \frac{\rho_k}{\rho_{k+1}} \right)^{\frac{3}{2}} A(x, \rho_k) \leq \rho_k^{\frac{\alpha-3\beta}{2} - \frac{\alpha-3\beta}{4}} \leq \rho^{2-l},
\] (3.37)

\[
E(x, \rho) \leq \frac{\rho_k}{\rho_{k+1}} E(x, \rho_k) \leq \rho_k^{\frac{\alpha-\beta}{2} - \frac{\alpha-\beta}{4}} \leq \rho^{2-l},
\]

\[
G(x, \rho) \leq \left( \frac{\rho_k}{\rho_{k+1}} \right)^{\frac{10-15\alpha}{40-15\alpha}} G(x, \rho_k) \leq \rho_k^{\frac{5(1+c)(1+\beta)}{2(1+c)(1+3\beta)}} \leq \rho \leq \rho_{k+1}^{\frac{5(1+c)(1+\beta)}{4(1+c)(1+3\beta)}},
\] (3.38)

and
\[
P(x, \rho) \leq \left( \frac{\rho_k}{\rho_{k+1}} \right)^{\frac{3}{2}} P(x, \rho_k) \leq \rho_k^{\frac{\alpha+3\beta}{2} - \frac{\alpha+3\beta}{4}} \leq \rho^{1+l}.
\] (3.39)

where we take $\beta$ sufficiently small such that $\beta \in \left( 0, \frac{(1+l)(\alpha-2l)}{2-l(4+1l)} \right)$.

Hence, the lemma is proved when $\rho^\alpha = N\tilde{\varepsilon}$.

Next we consider the general case. Recall (2.1). By taking $\lambda = \rho_0/(N\tilde{\varepsilon})^{\frac{1}{2}}$, we know that $(u_\lambda, p_\lambda)$ is also a suitable weak solution to (1.1) in $B(x, (N\tilde{\varepsilon})^{\frac{1}{2}})$.

Moreover,
\[
A(x, \rho) = \frac{1}{\rho} \int_{B(x, \rho)} |u(y)|^2 \, dy = \left( \frac{\lambda}{\rho} \right)^3 \int_{B(x, \rho/\lambda)} |u_\lambda(y)|^2 \, dy,
\]
\[
E(x, \rho) = \frac{1}{\rho} \int_{B(x, \rho)} |\nabla u(y)|^2 \, dy = \frac{\lambda}{\rho} \int_{B(x, \rho/\lambda)} |\nabla u_\lambda(y)|^2 \, dy,
\]
\[
G(x, \rho) = \frac{1}{\rho^{10-15\alpha}} \int_{B(x, \rho)} |p(y) - (p)_{B(x, \rho)}| \frac{5(1+c)}{4-c} \, dy
\]
have with the zero boundary condition. Hence, by the Calderón-Zygmund estimate, we
\[ u = \left( \frac{\lambda}{\rho} \right)^{10} \int_{B(x, \rho/\lambda)} |p\lambda(y) - (p\lambda)_{B(x, \rho/\lambda)}|^{\frac{5}{2(1+c)}} \, dy \]
as well as
\[ P(x, \rho) = \frac{1}{\rho^3} \int_{B(x, \rho)} |p(y) - (p)_{B(x, \rho)}| \, dy = \left( \frac{\lambda}{\rho} \right)^3 \int_{B(x, \rho/\lambda)} |p\lambda(y) - (p\lambda)_{B(x, \rho/\lambda)}|^2 \, dy. \]

Applying \( (3.37)-(3.39) \) to \( (u_\lambda, p_\lambda) \), we get
\[ A(x, \rho) + E(x, \rho) \leq \left( \frac{\rho}{\lambda} \right)^{2-l} \leq N \varepsilon \frac{4}{\rho_0} \left( \frac{\rho}{\rho_0} \right)^{2-l} \leq N \varepsilon \frac{4}{\rho_0} \left( \frac{\rho}{\rho_0} \right)^{2-l}, \]
\[ G(x, \rho) \leq \left( \frac{\rho}{\lambda} \right)^{\frac{5(1+c)(1+l)}{2(4-c)}} \leq N \varepsilon \frac{4}{\rho_0} \left( \frac{\rho}{\rho_0} \right)^{\frac{5(1+c)(1+l)}{2(4-c)}}, \]
as well as
\[ P(x, \rho) \leq \left( \frac{\rho}{\lambda} \right)^{1+l} \leq N \varepsilon \frac{4}{\rho_0} \left( \frac{\rho}{\rho_0} \right)^{1+l}, \]
where \( N = N(c, l) \). Thus, the lemma is proved.

In the rest of this section, we will utilize Lemma \( 3.4 \) to prove Theorem \( 1.1 \). Let \( \rho_0 = 3/4 \). By the condition \( 3.1 \), Hölder’s inequality which gives
\[ A(1) \leq NC(1)^{\frac{\varepsilon}{\gamma}} \leq N \varepsilon \]
and Lemma \( 3.3 \) we can choose \( \varepsilon > 0 \) sufficiently small such that the condition \( 3.21 \) holds. Hence, we obtain the decay estimates \( 3.22-3.24 \) in Lemma \( 3.4 \).

Let \( x \in B_{1/8} \) and \( \rho \in (0, 3/8) \). We decompose the suitable weak solution \( u \) of \( 1.1 \) as \( u = w + v \), where \( w \) satisfies the equation
\[ \Delta w_i = \partial_i (p - (p)_{B(x, \rho)}) + \partial_i(u_n) + f_i \]
with the zero boundary condition. Hence, by the Calderón-Zygmund estimate, we have
\[ \| \nabla w_i \|_{L^{\frac{\nu}{\rho}}(B(x, \rho))} \leq N \left( \| p - (p)_{B(x, \rho)} \|_{L^{\frac{\nu}{\rho}}(B(x, \rho))} + \| u_i \|_{L^{\frac{\nu}{\rho}}(B(x, \rho))} + \| f_i \|_{L^{\frac{\nu}{\rho}}(B(x, \rho))} \right). \] 
(3.40)

We first estimate the pressure term on the right-hand side of \( 3.40 \). By \( 3.22 \) and \( 3.24 \) in Lemma \( 3.4 \) we have
\[ G(x, \rho) \leq N \rho^{\frac{5(1+c)(1+l)}{4(4-c)}}, \quad P(x, \rho) \leq N \rho^{1+l}, \]
where \( N = N(c, l) > 0 \) is some constant. Thus, by Hölder’s inequality and the above estimates, one derives
\[ \left( \int_{B(x, \rho)} |p - (p)_{B(x, \rho)}|^{\frac{\nu}{\rho}} \, dx \right)^{\frac{\nu}{\rho}} \leq \left( \int_{B(x, \rho)} |p - (p)_{B(x, \rho)}| \, dx \right)^{\frac{n-1}{n+1}} \left( \int_{B(x, \rho)} |p - (p)_{B(x, \rho)}|^{\frac{5(1+c)(1+l)}{4(4-c)\nu}} \, dx \right)^{\frac{4(4-c)\nu}{5(1+c)(1+l)}} \leq N \rho^{n-1} G(x, \rho)^{\frac{3(4-c)}{5(1+c)}} \leq N \rho^{n+2l}. \] 
(3.41)
For the second term on the right-hand side of (3.40), due to (3.22) in Lemma 3.4, we have
\[ A(x, \rho) + E(x, \rho) \leq N\rho^{-2l}, \]
where \( N = N(c, l) \). Thus, by the Sobolev embedding inequality and the above decay rate, one obtains
\[
\left( \int_{B(x, \rho)} |u|^{20} \, dx \right)^{\frac{7}{5}} \leq N \left( \int_{B(x, \rho)} |\nabla u|^2 \, dx + \rho^{-2} \int_{B(x, \rho)} |u|^2 \, dx \right)^{\frac{7}{5}}
\]
\[
\leq N\rho A(x, \rho) \left( E(x, \rho) + A(x, \rho) \right)^{\frac{7}{5}} \leq N\rho^{7-2l}, \tag{3.42}
\]
where \( N = N(c, l) \).

For the last term on the right-hand side of (3.40), by Hölder’s inequality, one derives
\[
\rho^{\frac{2}{7}} \left( \int_{B(x, \rho)} |f|^{10} \, dx \right)^{\frac{7}{5}} \leq N\rho^{9-\frac{10}{7}} \left( \int_{B(x, \rho)} |f|^q \, dx \right)^{\frac{7}{5}}
\]
\[
= N\rho^{9-\frac{10}{7}} \|f\|^2_{L^q(B_1)}. \tag{3.43}
\]

The combination of the Sobolev-Poincaré inequality and (3.40)–(3.43) yields
\[
\int_{B(x, \rho)} |w - (w)_{B(x, \rho)}|^2 \, dx \leq N \left( \int_{B(x, \rho)} |\nabla w|^{10} \, dx \right)^{\frac{7}{5}}
\]
\[
\leq N \left[ \left( \int_{B(x, \rho)} |p - (p)_{B(x, \rho)}|^{10} \, dx \right)^{\frac{7}{5}} + \left( \int_{B(x, \rho)} |u|^{20} \, dx \right)^{\frac{7}{5}} \right]
\]
\[
+ \rho^2 \left( \int_{B(x, \rho)} |f|^{10} \, dx \right)^{\frac{7}{5}} \right] \leq N \left( \rho^{5+2l} + \rho^{7-2l} + \rho^{9-\frac{10}{7}} \|f\|^2_{L^q(B_1)} \right), \tag{3.44}
\]
where \( N = N(c, l) \).

Since any Sobolev norm of harmonic function \( v - (v)_{B(x, \gamma \rho)} \) in \( B(x, \gamma \rho) \) can be controlled by the \( L^p \) norm of it in \( B(x, \rho) \) for any \( p \in [1, +\infty] \), we have
\[
\int_{B(x, \gamma \rho)} |v - (v)_{B(x, \gamma \rho)}|^2 \, dx \leq N(\gamma \rho)^2 \int_{B(x, \gamma \rho)} |\nabla u|^2 \, dx
\]
\[
\leq N(\gamma \rho)^7 \sup_{B(x, \gamma \rho)} |\nabla u|^2
\]
\[
\leq N\gamma^7 \int_{B(x, \rho)} |v - (v)_{B(x, \rho)}|^2 \, dx, \tag{3.45}
\]
where \( \gamma \in (0, 1) \) is any constant.

By (3.44) and (3.45), one derives
\[
\int_{B(x, \gamma \rho)} |u - (u)_{B(x, \gamma \rho)}|^2 \, dx
\]
\[
\leq \int_{B(x, \gamma \rho)} |v - (v)_{B(x, \gamma \rho)}|^2 \, dx + \int_{B(x, \gamma \rho)} |w - (w)_{B(x, \gamma \rho)}|^2 \, dx
Lemma 3.1, we have
\[ N \leq \gamma^7 \int_{B(x,\rho)} |v - (v)_{B(x,\rho)}|^2 \, dx + \rho^{5+2l} + \rho^{7-2l} + \rho^{9-\frac{30}{q}} \| f \|_{L^q(B_1)}^2 \]
\[ \leq N \gamma^7 \int_{B(x,\rho)} |u - (u)_{B(x,\rho)}|^2 \, dx + \rho^{5+2l} + \rho^{7-2l} + \rho^{9-\frac{30}{q}} \| f \|_{L^q(B_1)}^2 , \quad (3.46) \]
where \( N = N(c, l) \).

Due to the condition of \( l \) in Lemma 3.4, by taking a sufficiently small constant \( l \in \left( 0, \frac{4g-10}{3g-6} \right) \), we have
\[ \min \{ 5 + 2l, 7 - 2l, 9 - 10/q \} > 5 + l. \]
The combination of (3.46) and Lemma 2.2 implies that
\[ \int_{B(x,\rho)} |u - (u)_{B(x,\rho)}|^2 \, dx \leq N \rho^{5+l} \]
for any \( x \in B_{1/8} \) and \( \rho \in (0, 3/8) \). Hence, by Campanato’s characterization of Hölder continuity, \( u \) is Hölder continuous in \( B_{1/2} \).

4. Boundary \( \varepsilon \)-regularity

In this section, we give the proof Theorem 1.2. We first prove the smallness of \( E^+(15/16) \) in Lemma 4.4 by using Lemmas 4.1 and 4.2 and an iteration argument. Then, based on the above results and the condition (1.7), we establish the uniform decay estimates of \( A^+, E^+, G^+, \) and \( P^+ \) in Lemma 4.4. Finally, by the decay estimates of scale invariant quantities and the \( L^p \) estimate for elliptic equations, we obtain the Hölder continuity of \( u \) by Campanato’s characterization of Hölder continuity. Throughout this section, we use the pair \( (u, p) \) to represent a suitable weak solution to the incompressible Navier-Stokes equations (1.1) with the boundary condition (1.2).

First, as in the interior case, by using (1.7) and [18, Theorem 3.8], without loss of generality, we may assume that
\[ \int_{B_1^+} |u|^q \, dx + \int_{B_1^+} |p - (p)_{B_1^+}| \, dx + \int_{B_1^+} |f|^2 \, dx < \varepsilon, \quad (4.1) \]

In the following lemmas, we prove that the values of \( A^+ + E^+ \) and \( G^+ \) in a smaller half ball can be controlled by their values in a larger half ball.

**Lemma 4.1.** Let \( \Omega = B_1^+ \). For any \( \gamma \in (0, 1/2) \), \( x_0 \in \partial \Omega \cap \{ x_5 = 0 \} \) and \( B^+(x_0, \rho) \subset B_1^+ \), we have
\[ A^+(x_0, \gamma \rho) + E^+(x_0, \gamma \rho) \]
\[ \leq N \left[ \gamma^2 E^+(x_0, \rho) + \gamma^{-2} E^+(x_0, \rho)^{\frac{3}{2}} + \gamma^{-2} E^+(x_0, \rho)^{\frac{3}{4}} G^+(x_0, \rho)^{\frac{3}{4}} + \gamma^{-4} F^+(x_0, \rho) \right], \]
where \( N = N(c) \) is a positive constant independent of \( \gamma \) and \( \rho \).

**Proof.** By the scale invariant property, we assume \( \rho = 1 \). Similar to the proof of Lemma 5.1, we have
\[ A^+(x_0, \gamma) + E^+(x_0, \gamma) \]
\[ \leq N \left[ \gamma^2 A^+(x_0, 1) + \gamma^{-2} A^+(x_0, 1)^{\frac{3}{2}} E^+(x_0, 1)^{\frac{3}{4}} + \gamma^{-2} A^+(x_0, 1)^{\frac{3}{4}} \right] \]
Let \( x_0 \) into (4.2), we obtain the lemma. \( \square \)

Inserting the boundary Poincaré inequality and \( B_r \) where \( N \)

\[
N > \]

Proof. Moreover, we have the following estimate

\[
\]
and the boundary condition
\[ w = 0 \quad \text{on } \{ x : x = (x^1, \ldots, x^4, 0) \} \cap \partial \tilde{B}. \]
By (4.5) and the stationary case of the estimate of pressure in Theorem 1.2 of [33] (see also [18]), one has
\[
\| \nabla p_2 \|_{L^{1+\epsilon}(B^+_{r}))} \leq N_2 \gamma^\frac{2}{3} \int \| \nabla p_2 \|_{L^{1+\epsilon}(B^+_{r}))} \\
\leq N_2 \gamma^\frac{2}{3} \int \left( \| u \|_{L^{1+\epsilon}(B^+_{r}))} + \| \nabla u \|_{L^{1+\epsilon}(B^+_{r}))} + \| p_2 \|_{L^{1}(B^+_{r}))} \right) \\
\leq N_2 \gamma^\frac{2}{3} \int \left( \| u \|_{L^{1+\epsilon}(B^+_{r}))} + \| \nabla v \|_{L^{1+\epsilon}(B^+_{r}))} + \| p \|_{L^{1}(B^+_{r}))} + \| (p) B^+_{r} \|_{L^{1}(B^+_{r}))} \right) \\
\leq N_2 \gamma^\frac{2}{3} \int \left( \| u \|_{L^{2}(B^+_{r}))} + \| \nabla u \|_{L^{2}(B^+_{r}))} + \| p \|_{L^{1}(B^+_{r}))} \right),
\]
where \( r' \in (1 + c, +\infty) \) and \( N = N(c, r') \) are some positive constants.
Combining (4.5) and (4.6), by the Sobolev-Poincaré inequality, we have
\[
\left\| p - (p) B^+_{r} \right\|_{L^{\frac{5(1+\epsilon)}{4}}(B^+_{r}))} \leq N \left[ \left\| \nabla p_1 \right\|_{L^{1+\epsilon}(B^+_{r}))} + \left\| \nabla p_2 \right\|_{L^{1+\epsilon}(B^+_{r}))} \right] \\
\leq N \left[ \left\| u \right\|_{L^{2}(B^+_{r}))} \left\| \nabla u \right\|_{L^{2}(B^+_{r}))} + \left\| p \right\|_{L^{2}(B^+_{r}))} \right) \\
+ \gamma \left( \left\| u \right\|_{L^{2}(B^+_{r}))} + \left\| \nabla u \right\|_{L^{2}(B^+_{r}))} + \left\| p \right\|_{L^{2}(B^+_{r}))} \right),
\]
where \( N = N(c, r') \). Hence, we get
\[
G^+(x_0, \gamma) \leq N \gamma \frac{2(-\epsilon)}{4} \left[ A^+(x_0, 1) \frac{5(1+\epsilon)}{4} \epsilon \right] + A^+(x_0, 1) \frac{5(1+\epsilon)}{4} + F^+(x_0, 1) \frac{5(1+\epsilon)}{4} \\
+ \gamma \frac{2}{4} \left[ A^+(x_0, 1) \frac{5(1+\epsilon)}{4} \epsilon \right] + E^+(x_0, 1) \frac{5(1+\epsilon)}{4} \epsilon \right] + P^+(x_0, 1) \frac{5(1+\epsilon)}{4} \epsilon \right].
\]
Inserting the boundary Poincaré inequality (4.3) into the above inequality, the conclusion of Lemma 4.2 follows immediately.

By Lemmas 4.1 and 4.2, we show below that \( E^+(15/16) \) can be sufficiently small under the condition (4.1). This lemma is needed for us to prove the decay estimates of scale invariant quantities with respect to the radius.

**Lemma 4.3.** Let \( \Omega = B^+_{1} \) and \((u, p)\) be a suitable weak solution to (1.1) satisfying the condition of Theorem 1.2. Then we have
\[
E^+(15/16) \leq N \epsilon^\beta,
\]
where \( N, \beta \) are some positive constants which depend only on \( c \) and \( q \).

**Proof.** Let \( \rho_k = 1 - 2^{-4k} \) and \( B^+_{k} = B^+_{\rho_k} \), where \( k \) is any positive integer. We choose the domains \( \tilde{B}_k \) satisfying
\[
B^+_{k+1} \subset \tilde{B}_k \subset B^+_{k+2}.
\]
For each $k$, we choose cut-off functions $\psi_k$ such that
\[
\text{supp } \psi_k \subset B_{k+1}, \quad \psi_k = 1 \quad \text{in } B_k,
\]
\[
|D\psi_k| \leq N 2^k, \quad |D^2\psi_k| \leq N 2^{2k} \quad \text{in } B_{k+1}^+.
\]

By the energy inequality (2.2), we have
\[
2 \int_{B_{k+1}^+} |\nabla u|^2 \, dx
\leq N \int_{B_{k+1}^+} \left( 2^k |u|^3 + 2^{k+1} |p - (p)_{B_{k+1}^+}| |u| + 2^{2k} |u|^2 + 2 |f| |u| \right) \, dx. \tag{4.7}
\]

Next we estimate each term on the right-hand side of the above inequality.

By (4.1) and Hölder’s inequality, we have
\[
A^+(\rho_{k+1}) \leq NC^+(\rho_{k+1})^{\frac{4}{3}} \leq NC^+(1)^{\frac{4}{3}} \leq N^{\frac{4}{3}}. \tag{4.8}
\]

For the first term on the right-hand side of (4.7), similar to (3.14), by Hölder’s inequality, the Sobolev embedding inequality and (4.8), we derive that
\[
\int_{B_{k+1}^+} |u|^3 \, dx \leq N \left( A^+(\rho_{k+1})^{\frac{2(q - q_1)}{q}} C^+(\rho_{k+1})^{\frac{2q}{q}} E^+(\rho_{k+1})^{\frac{N(2 - q) + 2q}{q}} \right)
\]
\[
+ A^+(\rho_{k+1})^{\frac{3-n}{2}} \leq N \left( \varepsilon \frac{2(q - q_1)}{q} + \frac{2q}{q} \right), \tag{4.9}
\]

where $N = N(a, q)$. Here, we take the constant $a \in (0, 1)$ to make sure all the exponents in the above inequality are positive.

For the third term on the right-hand side of (4.7), similar to (3.19) in Lemma 3.3, we have
\[
\int_{B_{k+1}^+} |u|^2 \, dx \leq N \left( A^+(\rho_{k+1})^{\frac{2(q - q_1)}{q}} C^+(\rho_{k+1})^{\frac{2q}{q}} E^+(\rho_{k+1})^{\frac{N(2 - q) + 2q}{q}} \right)
\]
\[
+ A^+(\rho_{k+1})^{\frac{3-n}{2}} \leq N \left( \varepsilon \frac{2(q - q_1)}{q} + \frac{2q}{q} \right), \tag{4.10}
\]

where $N = N(a, q)$.

For the last term on the right-hand side of (4.7), by Hölder’s inequality, we have
\[
\int_{B_{k+1}^+} |f| |u| \, dx \leq NA^+(\rho_{k+1})^{\frac{q}{2}} F^+(\rho_{k+1})^{\frac{4}{2}} \leq N^{\frac{4}{2} + \frac{q}{2}}. \tag{4.11}
\]

Next we bound the second term on the right-hand side of (4.7). Analogous to (3.15), we have
\[
\int_{B_{k+1}^+} |p - (p)_{B_{k+1}^+}| |u| \, dx \leq \left( \int_{B_{k+1}^+} |\nabla p|^{1+c} \, dx \right)^{\frac{1}{1+c}} \left( \int_{B_{k+1}^+} |u|^2 \, dx \right)^{\frac{q+6c-5c}{3(1+c)(q-2)}}
\]
\[
\cdot \left( \int_{B_{k+1}^+} |u|^q \, dx \right)^{\frac{3-7c}{3(1+c)(q-2)}}, \tag{4.12}
\]

where the exponents $\frac{q+6c-5c}{3(1+c)(q-2)}$ and $\frac{3-7c}{3(1+c)(q-2)}$ are positive due to the fact $c \in \left( \frac{5-q}{6q-5}, \frac{1}{4} \right)$.
To deal with the first term on the right-hand side of (4.12), we decompose the velocity $u$ and the pressure $p$ as follows

$$u = v_k + w_k, \quad p = p_k + h_k,$$

where $(v_k, p_k)$ satisfy the boundary value problem

$$\begin{cases}
-\Delta v_k + \nabla p_k = -u \cdot \nabla u + f & \text{in } \tilde{B}_k, \\
\nabla \cdot v_k = 0 & \text{in } \tilde{B}_k, \\
(p_k)_{\partial \tilde{B}_k} = 0, \\
v_k = 0 & \text{on } \partial \tilde{B}_k.
\end{cases}$$

By the assumptions in Theorem 1.2 and the stationary case of the estimate in Lemma 4.4 of [1], one has

$$\|u_k\| + |p_k| + |\nabla p_k|_{L^{1+\varepsilon}(\tilde{B}_k)} \leq 2^{b_k} N \left[ \|u \cdot \nabla u\|_{L^{1+\varepsilon}(\tilde{B}_k)} + \|f\|_{L^{1+\varepsilon}(\tilde{B}_k)} \right]$$

$$\leq 2^{b_k} N \left[ \|u\|_{L^\frac{6}{5}(\tilde{B}_k)}^{1+\varepsilon} \|\nabla u\|_{L^2(\tilde{B}_k)}^{1+\varepsilon} + \|u\|_{L^2(\tilde{B}_k)} \|\nabla u\|_{L^2(\tilde{B}_k)} + \|f\|_{L^2(\tilde{B}_k)} \right]$$

$$\leq 2^{b_k} N \left[ A^+(\rho_{k+2}) \frac{1+\varepsilon}{n+1} E^+(\rho_{k+2})^{\frac{n+\varepsilon}{n+1}} + A^+(\rho_{k+2})^{\frac{n}{n+1}} E^+(\rho_{k+2})^{\frac{2}{n+1}} + F^+(\rho_{k+2})^{\frac{n}{n+1}} \right],$$

(4.13)

where the positive constants $b$ and $N$ depend only on $c$.

Since $(w_k, h_k)$ is a suitable weak solutions to the following equations

$$\begin{cases}
-\Delta w_k + \nabla h_k = 0 & \text{in } \tilde{B}_k, \\
\nabla \cdot w_k = 0 & \text{in } \tilde{B}_k, \\
w_k = 0 & \text{on } \partial B_k \cap \{x^5 = 0\},
\end{cases}$$

by (4.13) and the stationary case of the estimate in Lemma 4.5 of [6], we have

$$\|h_k - (h_k)_{B_{k+1}^+}\|_{L^{1+\varepsilon}(B_{k+1}^+)} + \|\nabla h_k\|_{L^{1+\varepsilon}(B_{k+1}^+)}$$

$$\leq 2^{b_k} N \left[ \|u\|_{L^1(B_{k+2}^+)} + \|h_k - (h_k)_{B_{k+2}^+}\|_{L^1(B_{k+2}^+)} \right]$$

$$\leq 2^{b_k} N \left[ \|u\|_{L^1(B_{k+2}^+)} + \|v\|_{L^1(B_{k+2}^+)} + \|p - (p)_{B_{k+2}^+}\|_{L^1(B_{k+2}^+)} + \|p_k\|_{L^1(B_{k+2}^+)} \right]$$

$$\leq 2^{b_k} N \left[ C^+(\rho_{k-3})^{\frac{1}{2}} + A^+(\rho_{k-3})^{\frac{n}{n+1}} E^+(\rho_{k-3})^{\frac{2}{n+1}} + A^+(\rho_{k-3})^{\frac{2}{n+1}} E^+(\rho_{k-3})^{\frac{1+\varepsilon}{n+1}} + F^+(\rho_{k-3})^{\frac{1+\varepsilon}{n+1}} + P^+(\rho_{k-3}) \right],$$

(4.14)

where $b$ and $N$ depend only on $c$.

The combination of (4.12), (4.13) and (4.14) yields

$$\int_{B_{k+1}^+} |p - (p)_{B_{k+1}^+}| |u| \, dx$$

$$\leq 2^{b_k} N \left[ C^+(\rho_{k+3})^{\frac{1}{2}} + A^+(\rho_{k+3})^{\frac{n}{n+1}} E^+(\rho_{k+3})^{\frac{2}{n+1}} + A^+(\rho_{k+3})^{\frac{2}{n+1}} E^+(\rho_{k+3})^{\frac{1+\varepsilon}{n+1}} + F^+(\rho_{k+3})^{\frac{1+\varepsilon}{n+1}} + P^+(\rho_{k+3}) \right] \cdot A^+(\rho_{k+3})^{\frac{2}{n+1}} C^+(\rho_{k+3})^{\frac{1+\varepsilon}{n+1}}$$

$$\leq 2^{b_k} N \left( \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{\frac{2}{n+1} - \frac{1+\varepsilon}{n+1}}{\frac{1+\varepsilon}{n+1}}} E^+(\rho_{k+3})^{\frac{1+\varepsilon}{n+1}} + \varepsilon^{\frac{1+\varepsilon}{n+1}} E^+(\rho_{k+3})^{\frac{1+\varepsilon}{n+1}} + \varepsilon^{\frac{1+\varepsilon}{n+1}} + \varepsilon^{1+\varepsilon} \right)$$
where \(b\) and \(N\) depend only on \(c\).

Inserting (4.9)-(4.11) and the above inequality into (4.7), one derives

\[
E^+ (\rho_k) \leq 2^{(2+b)k} N \left( \varepsilon \frac{a^5 + 10a}{q^2} E^+ (\rho_{k+3}) \frac{5(q^2 + 2a)}{4a} + \varepsilon^\frac{2}{3} E^+ (\rho_{k+3}) \frac{5(q^2 + 2a)}{4a} + \varepsilon^\frac{2}{3} + \varepsilon^\frac{2}{3} \right) + \varepsilon^\frac{2}{3} E^+ (\rho_{k+3}) \frac{5(q^2 + 2a)}{4a} + \varepsilon^\frac{2}{3} + \varepsilon^\frac{2}{3} + \varepsilon^\frac{2}{3},
\]

where \(b = b(c)\) and \(N = N(a, c, q)\) are some positive constants. Since \(c \in \left( \frac{5(q^2 - 10a)}{4a} \right)\), if we take \(a \in \left( \frac{q^2 - 10a}{4a}, 1 \right)\), then all the exponents of \(E^+ (\rho_{k+3})\) are positive and smaller than one. Hence, by (4.15) and Young’s inequality, for any \(\delta > 0\), we have

\[
E^+ (\rho_k) \leq \delta^3 E^+ (\rho_{k+3}) + 2^{(2+b)k} N \varepsilon^\beta,
\]

where \(N = N(c, q, \delta)\) and \(\beta = \beta(c, q)\) are some positive constants.

Let \(\delta = 3^{-(2+b)}\). Multiplying both sides of the above inequality by \(\delta^k\) and summing over \(k\) from 1 to infinity, we have

\[
\sum_{k=1}^\infty \delta^k E^+ (\rho_k) \leq \sum_{k=1}^\infty \delta^k E^+ (\rho_k) + N \varepsilon^\beta \sum_{k=1}^\infty \left( \frac{2}{3} \right)^{(2+b)k},
\]

which implies

\[
E^+ (15/16) \leq N \varepsilon^\beta.
\]

The lemma is proved. \(\square\)

In the following lemma, we will verify the decay estimates of \(A^+ (x, \rho) + E^+ (x, \rho),\) \(G^+ (x, \rho),\) and \(P^+ (x, \rho)\) with respect to radius based on the above three lemmas and the condition (4.1). This is a key lemma for us to prove Theorem 1.2.

**Lemma 4.4.** There exists a constant \(\varepsilon > 0\) satisfying the following property. If

\[
A^+ (\rho_0) \frac{2-\varepsilon}{2(2+\varepsilon)} + E^+ (\rho_0) \frac{2-\varepsilon}{2(2+\varepsilon)} + F^+ (\rho_0) \frac{2-\varepsilon}{2(2+\varepsilon)} + P^+ (\rho_0) \leq \varepsilon,
\]

then there exists sufficiently small \(l \in \left( 0, \frac{4(q - 10q)}{10q - 20q} \right)\) such that for any \(x \in B^+_{\rho_0/5}\) and \(\rho \in (0, \rho_0/5)\), we have

\[
A^+ (x, \rho) + E^+ (x, \rho) \leq N \varepsilon \frac{(2-\varepsilon)(2q - 5 - 4q)}{6q - 10q} \left( \frac{\rho}{\rho_0} \right)^{(2-\varepsilon)(2q - 5 - 4q)} \left( \frac{\rho_0}{\rho} \right)^{(2-\varepsilon)(2q - 5 - 4q)},
\]

(4.17)

\[
G^+ (x, \rho) \leq N \varepsilon \frac{(1+q)(1+q)(2q - 5 - 4q)}{(4-\varepsilon)(1+q)} \left( \frac{\rho}{\rho_0} \right)^{(5+q)(1+q)(2q - 5 - 4q)} \left( \frac{\rho_0}{\rho} \right)^{(5+q)(1+q)(2q - 5 - 4q)},
\]

(4.18)

and

\[
P^+ (x, \rho) \leq N \varepsilon \frac{(1+q)(2q - 5 - 4q)}{6q - 10q} \left( \frac{\rho}{\rho_0} \right)^{(1+q)(2q - 5 - 4q)} \left( \frac{\rho_0}{\rho} \right)^{(1+q)(2q - 5 - 4q)},
\]

(4.19)

where \(N\) is a positive constant depending on \(c, l,\) and \(r' \in (1 + c, +\infty)\) in Lemma 4.2 but independent of \(\varepsilon, \rho_0,\) and \(\rho.\)
Proof. We prove this lemma by an iteration argument. We discuss two cases according to the position of $x$. We first consider the case when $x \in B_{\rho_0/5} \cap \{x^0 = 0\}$ and derive the decay estimates of scale invariant quantities. Then, by another iteration argument, we extend previous results to the general case $x \in B_{\rho_0}^+$.

Similar to the interior case, we denote $\rho_k = \tilde{\rho}^{(1+\beta)^k}$, where $\tilde{\rho} = 3\rho_0/5$ and $\beta > 0$ is some small constant which will be specified later.

Case 1: $x \in B_{\rho_0/5} \cap \{x^5 = 0\}$. We fix the parameter $\alpha \in (2-l, 2)$. By the scale invariant property, we first assume $\tilde{\rho}^\alpha = N\varepsilon$. We only need to prove the following decay estimates

$$A^+(x, \rho_k) + E^+(x, \rho_k) \leq \rho_k^\alpha, \quad G^+(x, \rho_k) \leq \rho_k \frac{5\alpha(1+c)(1+1)}{5\alpha(1+c)} \frac{E}{(1+c)}^{(1+c)} \quad P^+(x, \rho_k) \leq \rho_k \frac{\alpha(1+c)}{2}.$$ (4.20)

Let us prove the above estimates for $k = 0$. Since $B^+(x, \rho) \subset B_{\rho_0}^+$, by (4.11), we have

$$A^+(x, \rho) \leq N A^+(\rho_0) \leq N\varepsilon \frac{2(1+c)}{5(1+c)} \leq N\varepsilon = \tilde{\rho}^\alpha$$ (4.21)

and

$$E^+(x, \rho) \leq N E^+(\rho_0) \leq N\varepsilon \frac{2(1+c)}{5(1+c)} \leq N\varepsilon = \tilde{\rho}^\alpha.$$ (4.22)

For $G^+(x, \rho)$ and $P^+(x, \rho)$, since $B^+(x, 4\tilde{\rho}/3) \subset B_{\rho_0}^+$, by Lemma 4.1 and H"older’s inequality, we obtain

$$G^+(x, \rho) \leq N \left( E^+(x, 4\tilde{\rho}/3) \frac{5\alpha(1+c)}{5\alpha(1+c)} + F^+(x, 4\tilde{\rho}/3) \frac{5\alpha(1+c)}{5\alpha(1+c)} \right)$$

$$+ E^+(x, 4\tilde{\rho}/3) \frac{5\alpha(1+c)}{5\alpha(1+c)} + P^+(x, 4\tilde{\rho}/3) \frac{5\alpha(1+c)}{5\alpha(1+c)} \right)$$

$$\leq N \left( E^+(\rho_0) \frac{5\alpha(1+c)}{5\alpha(1+c)} + F^+(\rho_0) \frac{5\alpha(1+c)}{5\alpha(1+c)} + E^+(\rho_0) \frac{5\alpha(1+c)}{5\alpha(1+c)} + P^+(\rho_0) \frac{5\alpha(1+c)}{5\alpha(1+c)} \right)$$

$$\leq N \left( \varepsilon \frac{2(1+c)}{5(1+c)} + \varepsilon \frac{2(1+c)}{5(1+c)} + \varepsilon \frac{2(1+c)}{5(1+c)} + \varepsilon \frac{2(1+c)}{5(1+c)} \right)$$

$$\leq (N\varepsilon)^{\frac{5\alpha(1+c)}{5\alpha(1+c)}} = \rho \frac{5\alpha(1+c)}{5\alpha(1+c)}.$$ (4.23)

and

$$P^+(x, \rho) \leq N G^+(x, \rho) \frac{5\alpha(1+c)}{5\alpha(1+c)} \leq (N\varepsilon)^{\frac{1+c}{1+c}} = \rho \frac{1+c}{1+c},$$ (4.24)

where $N = N(c, l, r')$. Combining (4.21) - (4.24), we get (4.20) for $k = 0$.

To prove (4.20) for $k > 0$, since $A^+(x, \rho_k) + E^+(x, \rho_k)$ and $P^+(x, \rho_k)$ can be estimated by the same method as in Lemma 3.4, we only consider $G^+(x, \rho_k)$. We suppose that (4.20) holds for $k = 1, \ldots, k_0 \geq m$, where $m$ is an integer to be specified later. We will show that the estimate of $G^+$ also holds for $\rho_{k_0+1}$.

Taking $\beta = (1+\beta)^{m+1} - 1$, $\gamma = \rho \beta_{k_0-m}$, and $\rho_0 = \rho_{k_0-m}$, by Lemma 4.2, we have

$$G^+(x, \rho_{k_0+1}) \leq N \left[ \rho_{k_0-m} \frac{5(1+2\cdot 3\cdot \beta)}{4(1+c)} + \rho_{k_0-m} \frac{5\alpha(1+c)}{2(1+c)} + \rho_{k_0-m} \frac{5(2-3\cdot \beta)}{4(1+c)} + \rho_{k_0-m} \frac{5\alpha(1+c)}{2(1+c)} \right]$$

$$+ \rho_{k_0-m} \frac{5(1+2\cdot 3\cdot \beta)}{4(1+c)} \|f\|_{L^q}$$

$$+ \rho_{k_0-m} \frac{5(2-3\cdot \beta)}{4(1+c)} + 5\beta \frac{1}{4(1+c)} \left( \rho_{k_0-m} + \rho_{k_0-m} \right),$$ (4.25)
where \( N = N(c, l, r') \). If we choose
\[
\frac{3al}{(6 - 10/r')(2 - l) - 2a(1 + l)} < \ddot{\beta} < \frac{(1 + c)[6q - \alpha q - 10 - (3q + \alpha q - 5)]l}{q((1 + c)(1 + l)\alpha + (2 - 3c)(2 - l))},
\]
then all the exponents on the right-hand side of (4.25) are greater than
\[
\frac{5(1 + c)(1 + l)(1 + \ddot{\beta})}{(4 - c)(2 - l)}.
\]
Here, it is sufficient to take a small \( l \in \left(0, \frac{4q - 10}{16q - 25}\right) \). Indeed we can choose \( \beta = l^2 \) and take a sufficiently large integer \( m \) of order \( 1/l \) so that \( \ddot{\beta} \sim l \). A simple calculation shows that (4.26) hold.

Hence, there exists a small constant \( \xi > 0 \) such that
\[
G^+(x, \rho_{k_0 + 1}) \leq N\rho_{k_0 + m}^{\frac{5(1 + c)(1 + l)}{(4 - c)(2 - l)} + \xi} \leq N\rho_{k_0 + 1}^{\frac{5(1 + c)(1 + l)}{6 - 10/r' - 2a(1 + l)} + \xi}.
\]

By taking \( \varepsilon > 0 \) sufficiently small, one has
\[
N\rho_{k_0 + 1}^\xi < N\rho^{\xi} \leq N(N\varepsilon)^{\xi} < 1.
\]

Inserting the above inequality into (4.27), we obtain
\[
G^+(x, \rho_{k_0 + 1}) \leq \rho_{k_0 + 1}^{\frac{5(1 + c)(1 + l)}{(4 - c)(2 - l)}}.
\]
Thus, the estimate of \( G^+ \) in (4.20) holds for \( k = k_0 + 1 \) provided that it holds for \( k = 1, \ldots, k_0 \). Now analogous to the case \( k = 0 \), by choosing \( \varepsilon \) sufficiently small, we can obtain (4.20) for \( k = 1, \ldots, m \). Hence, by induction we have (4.20) for any integer \( k \geq 0 \).

For any \( \rho \in (0, 3\rho_0/5) \), there exists a positive integer \( k \) such that \( \rho_{k+1} \leq \rho < \rho_k \). Similar to the calculations in (3.37)-(3.39), we have
\[
A^+(x, \rho) \leq \rho^{2-l}, \quad E^+(x, \rho) \leq \rho^{2-l}, \quad G^+(x, \rho) \leq \rho^{\frac{5(1 + c)(1 + l)}{6 - 10/r' - 2a(1 + l)}}, \quad P^+(x, \rho) \leq \rho^{1+l}.
\]

Hence, (4.20) are proved for any \( x \in B_{\rho_0/5} \cap \{x_5 = 0\} \) when \( \tilde{\rho}^a = N\varepsilon \).

For the general case, due to the scale invariant property, we derive the decay estimates of all the scale invariant quantities by imposing additional scaling factors on the right-hand side of (4.20). To be precise, if \((u, p)\) is a suitable weak solution to (1.1) in \( B^+(x, \rho_0) \), by taking \( \lambda = \rho_0/(N\varepsilon)^{1/2} \), we know that \((u_\lambda, p_\lambda)\) is also a suitable weak solution to (1.1) in \( B^+(x, (N\varepsilon)^{1/2}) \). Hence, similar to the interior case, we have
\[
A^+(x, \rho) + E^+(x, \rho) \leq N\varepsilon^{2-l} \left( \frac{\rho}{\rho_0} \right)^{2-l} \leq N\varepsilon^{2-l} \left( \frac{\rho}{\rho_0} \right)^{2-l}, \quad (4.28)
\]
\[
G^+(x, \rho) \leq N\varepsilon^{\frac{5(1 + c)(1 + l)}{6 - 10/r' - 2a(1 + l)}} \left( \frac{\rho}{\rho_0} \right)^{\frac{5(1 + c)(1 + l)}{6 - 10/r' - 2a(1 + l)}}, \quad (4.29)
\]
and
\[
P^+(x, \rho) \leq N\varepsilon^{\frac{1+l}{2}} \left( \frac{\rho}{\rho_0} \right)^{1+l} \leq N\varepsilon^{\frac{1+l}{2}} \left( \frac{\rho}{\rho_0} \right)^{1+l}, \quad (4.30)
\]
where \( N \) is a positive constant depending on \( c, l \) and \( r' \).
Thus, by the conclusions of the boundary decay estimates (4.28)-(4.30), we have

\[ x \text{ projection of two cases to derive the decay estimates of scale invariant quantities. We denote the} \]

\[ \text{projection of } x \text{ on the boundary by } x^*. \]

**Case 2:** \( x \in \overline{B_{\rho_0/5}^+} \). By comparing \( d_x \), the distance of \( x \) to the boundary \( \{x_5 = 0\} \), with \( \rho \in (0, \rho_0/5) \), the radius of the ball around \( x \), we further consider two cases to derive the decay estimates of scale invariant quantities. We denote the projection of \( x \) on the boundary by \( x^* \).

**Case 2.1:** \( \rho \geq d_x/2 \). Since \( B^+(x, \rho) \subset B^+(x^*, 3\rho) \), we have

\[ A^+(x, \rho) \leq N A^+(x^*, 3\rho), \quad E^+(x, \rho) \leq N E^+(x^*, 3\rho), \]

\[ G^+(x, \rho) \leq N G^+(x^*, 3\rho), \quad P^+(x, \rho) \leq N P^+(x^*, 3\rho). \]

Thus, by the conclusions of the boundary decay estimates (4.28)-(4.30), we have

\[ A^+(x, \rho) + E^+(x, \rho) \leq N \varepsilon^2 \left( \frac{\rho}{\rho_0} \right)^{2-l}, \]

\[ G^+(x, \rho) \leq N \varepsilon^{\frac{2(2+\gamma+1+1)}{4(2-\gamma)}} \left( \frac{\rho}{\rho_0} \right)^{\frac{2(1+\gamma+1+1)}{4-l}}, \]

and

\[ P^+(x, \rho) \leq N \varepsilon^{\frac{1+l}{r}} \left( \frac{\rho}{\rho_0} \right)^{\frac{1+l}{r}}, \]

where \( N = N(c, l, r') \).

**Case 2.2:** \( \rho < d_x/2 \). Since

\[ B^+(x, \rho) = B(x, \rho) \subset B(x, d_x) \subset B^+(x^*, 2d_x), \]

applying the boundary results (4.28) and (4.30) to \( x^* \), we have

\[ A(x, d_x) + E(x, d_x) \leq N \left( A^+(x^*, 2d_x) + E^+(x^*, 2d_x) \right) \leq N \varepsilon^2 \left( \frac{2d_x}{\rho_0} \right)^{2-l} \] (4.31)

and

\[ P(x, d_x) \leq N P^+(x^*, 2d_x) \leq N \varepsilon^{\frac{1+l}{r}} \left( \frac{2d_x}{\rho_0} \right)^{1+l}. \] (4.32)

where \( N = N(c, l, r) \). Combining (4.31) and (4.32), we obtain

\[ A(x, d_x) + E(x, d_x) + P(x, d_x) \leq N \varepsilon^\frac{2}{r} \left( \frac{2d_x}{\rho_0} \right)^{2-l} \leq N \varepsilon^{\frac{2+2d_x}{4x-10} - \frac{2d_x}{3+10}} \left( \frac{2d_x}{\rho_0} \right)^{2-l}. \] (4.33)

For \( F(x, d_x) \), when \( \left( \frac{2d_x}{\rho_0} \right) < \varepsilon^{-\frac{2d_x}{2+2d_x-2d_x-10}} \), by Hölder’s inequality, one has

\[ F(x, d_x) = F^+(x, d_x) \leq N d_x^{6-\frac{2d_x}{q}} \| f \|_{L^q(B_{\rho_0}^+)}^2. \] (4.34)

Otherwise, by the condition (4.19), one has

\[ F(x, d_x) \leq NF^+(x^*, 2d_x) \leq N \left( \frac{2d_x}{\rho_0} \right)^{2-l} \left( \frac{2d_x}{\rho_0} \right)^{2-l}. \] (4.35)

Since \( \rho_0 \leq 1 \), by (4.34) and (4.35), we obtain

\[ F(x, d_x)^{\frac{2}{2-l(1+1)}} \leq N \varepsilon^{\frac{2+2d_x}{4x-10} - \frac{2d_x}{3+10}} \left( \frac{2d_x}{\rho_0} \right)^{2-l}. \] (4.36)

Thus, by (4.33) and (4.36), if we choose \( \varepsilon \) sufficiently small such that

\[ N \varepsilon^{\frac{2+2d_x}{4x-10} - \frac{2d_x}{3+10}} \left( \frac{2d_x}{\rho_0} \right)^{2-l} \leq \varepsilon, \]
where \( \varepsilon \) is from Lemma [3.4] then the condition (3.21) in Lemma [3.4] is satisfied. Hence, from (3.22) - (3.24), we have

\[
A^+(x, \rho) + E^+(x, \rho) = A(x, \rho) + E(x, \rho)
\]

\[
\leq N \left[ \varepsilon \left( \frac{2d_x}{\rho_0} \right)^{2-l} \right] \frac{1}{\varepsilon} \left( \frac{\rho}{2d_x} \right)^{2-l}
\]

\[
\leq N \varepsilon \left( \frac{2d_x}{\rho_0} \right)^{2-l} \left( \frac{\rho}{2d_x} \right)^{2-l}
\]

where \( \tilde{\rho} \) is from Lemma 3.4, then the condition (3.21) in Lemma 3.4 is satisfied. Let \( \varepsilon > 0 \) be sufficiently small such that the condition (4.13) holds with \( \rho_0 = \frac{15}{16} \). Hence, by Lemma 1.3, we obtain the decay estimates (4.17), (4.19). Let \( x \in B^+_{3/16} \) and \( \rho_0 = \frac{15}{16} \). According to the position of \( x \), we discuss two cases to prove the regularity of \( u \).

Let us first consider the case when \( x \in B^+_{3/16} \cap \{ x^5 = 0 \} \). We decompose the suitable weak solution \( u \) of (1.1) as \( u = w + v \), where \( w \) satisfies the equation

\[
\Delta w_i = \partial_i (p - (p)_{B^+(x, \rho)}) + \partial_j (u_i u_j) + f_i \quad \text{in } B^+(x, \rho)
\]

with the zero Dirichlet boundary condition, where \( p \in (0, 3/16] \). By the \( L^p \) estimate for elliptic equations, we have

\[
\| \nabla w \|_{L^\infty(B^+(x, \rho))} \leq N \left( \| p - (p)_{B^+(x, \rho)} \|_{L^\infty(B^+(x, \rho))} + \| u \|_{L^\infty(B^+(x, \rho))} + \rho \right) \]

\[
\leq N \left( \| p - (p)_{B^+(x, \rho)} \|_{L^\infty(B^+(x, \rho))} + \| u \|_{L^\infty(B^+(x, \rho))} + \rho \right),
\]

where \( N = N(c, l, r') \). The lemma is proved. \( \square \)

The rest of this section is devoted to the proof of Theorem 1.2. We will use Lemma 1.3 to prove that \( u \) is Hölder continuous in \( B^+_{1/2} \) by Campanato’s characterization of Hölder continuity and a covering argument. By the condition (4.11), Hölder’s inequality which gives

\[
A^+(1) \leq NC^+(1)^{\frac{3}{2}} \leq N \varepsilon^{\frac{3}{2}}
\]

and Lemma 1.3 we can choose \( \varepsilon > 0 \) sufficiently small such that the condition (4.16) holds with \( \rho_0 = \frac{15}{16} \). Hence, by Lemma 1.3, we obtain the decay estimates (4.17), (4.19). Let \( x \in B^+_{3/16} \) and \( \rho_0 = \frac{15}{16} \). According to the position of \( x \), we discuss two cases to prove the regularity of \( u \).

Let us first consider the case when \( x \in B^+_{3/16} \cap \{ x^5 = 0 \} \). We decompose the suitable weak solution \( u \) of (1.1) as \( u = w + v \), where \( w \) satisfies the equation

\[
\Delta w_i = \partial_i (p - (p)_{B^+(x, \rho)}) + \partial_j (u_i u_j) + f_i \quad \text{in } B^+(x, \rho)
\]

with the zero Dirichlet boundary condition, where \( p \in (0, 3/16] \). By the \( L^p \) estimate for elliptic equations, we have

\[
G^+(x, \rho) \leq N \rho \left( \frac{5(1+c)(1+l)(1-\rho)}{2(4-c)} \right) \]

\[
P^+(x, \rho) \leq N \rho \left( \frac{1+c}{2} \right) ,
\]
where $N = N(c, l, r') > 0$ is some constant. Hence, by Hölder’s inequality and the above decay estimates, we have
\[
\left( \int_{B^+(x, \rho)} |p - (p)_{B^+(x, \rho)}|^{\frac{3}{2}} \, dx \right)^{\frac{2}{3}} \\
\leq \left( \int_{B^+(x, \rho)} |p - (p)_{B^+(x, \rho)}| \, dx \right)^{\frac{6c-1}{1+16c}} \left( \int_{B^+(x, \rho)} |p - (p)_{B^+(x, \rho)}|^{\frac{3(4c-1)}{3(4c-1)+5\rho^4}} \, dx \right)^{\frac{3(4c-1)}{3(4c-1)+5\rho^4}} \\
\leq \rho^3 P^+(x, \rho)^{\frac{6c-1}{1+16c}} G^+(x, \rho)^{\frac{3(4c-1)}{3(4c-1)+5\rho^4}} \leq N \rho^{5+l-l^2},
\] (4.39)
where $N = N(c, l, r') > 0$ and $\rho \in (0, 3/16)$.

For the second term on the right-hand side of (4.38), due to (4.17), one has
\[
A^+(x, \rho) + E^+(x, \rho) \leq N \rho^{\frac{(2-l)^2}{2}},
\]
where $N = N(c, l, r') > 0$ is some constant. Thus, by the Sobolev embedding inequality and the above decay rate, we derive
\[
\left( \int_{B^+(x, \rho)} |u|^{\frac{2q}{q-1}} \, dx \right)^{\frac{q-1}{2q}} \\
\leq N \left( \int_{B^+(x, \rho)} |u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B^+(x, \rho)} |\nabla u|^2 \, dx + \rho^{-2} \int_{B^+(x, \rho)} |u|^2 \, dx \right)^{\frac{1}{2}} \\
\leq N \rho^3 A^+(x, \rho)^{\frac{1}{2}} \left( E^+(x, \rho) + A^+(x, \rho) \right)^{\frac{1}{2}} \leq N \rho^{7+l-4l},
\] (4.40)
where $N = N(c, l, r') > 0$ and $\rho \in (0, 3/16)$.

For the last term on the right-hand side of (4.38), by Hölder’s inequality, one has
\[
\rho^2 \left( \int_{B^+(x, \rho)} |f|^{\frac{16}{9}} \, dx \right)^{\frac{9}{16}} \leq N \left( \int_{B^+(x, \rho)} |f|^q \, dx \right)^{\frac{3}{2}} \rho^{-\frac{4q}{9}},
\] (4.41)
where $\rho \in (0, 3/16)$ is any constant.

Thus, by the Sobolev-Poincaré inequality and (4.39)-(4.41), we have
\[
\int_{B^+(x, \rho)} |w - (w)_{B^+(x, \rho)}|^2 \, dx \\
\leq N \left( \int_{B^+(x, \rho)} |\nabla w|^{\frac{16}{9}} \, dx \right)^{\frac{9}{16}} \\
\leq N \left[ \left( \int_{B^+(x, \rho)} |p - (p)_{B^+(x, \rho)}|^{\frac{3}{2}} \, dx \right)^{\frac{2}{3}} + \left( \int_{B^+(x, \rho)} |u|^{\frac{2q}{q-1}} \, dx \right)^{\frac{q-1}{2q}} \right]^{\frac{1}{2}} \\
+ \rho^2 \left( \int_{B^+(x, \rho)} |f|^{\frac{16}{9}} \, dx \right)^{\frac{9}{16}} \\
\leq N \left( \rho^{5+l-l^2} + \rho^{7+l-4l} + \rho^{-\frac{4q}{9}} \|f\|_{L^q(B^+(x, \rho))}^2 \right),
\] (4.42)
where $N = N(c, l, r') > 0$.

Due to the boundary Poincaré inequality and the fact that the $L^\infty$ norm of the gradient of harmonic function $v$ in $B^+(x, \gamma \rho)$ can be controlled by its $L^p$ norm in $B^+(x, \rho)$ for any $p \in \{1, +\infty\}$, we obtain
\[
\int_{B^+(x, \gamma \rho)} |v - (v)_{B^+(x, \gamma \rho)}|^2 \, dx \leq N(\gamma \rho)^2 \int_{B^+(x, \gamma \rho)} |\nabla u|^2 \, dx
\]
\[
\int_{B^+(x, \rho)} \left| u - (u)_{B^+(x, \rho)} \right|^2 \, dx \\
\leq N\gamma^7 \int_{B^+(x, \rho)} \left| v - (v)_{B^+(x, \rho)} \right|^2 \, dx, \quad (4.43)
\]

where \( \gamma \in (0, 1/2) \).

The combination of (4.42) and (4.43) implies

\[
\int_{B^+(x, \rho)} \left| u - (u)_{B^+(x, \rho)} \right|^2 \, dx \\
\leq \int_{B^+(x, \rho)} \left| v - (v)_{B^+(x, \rho)} \right|^2 \, dx + \int_{B^+(x, \rho)} \left| w - (w)_{B^+(x, \rho)} \right|^2 \, dx \\
\leq N\gamma^7 \int_{B^+(x, \rho)} \left| v - (v)_{B^+(x, \rho)} \right|^2 \, dx + \rho^{5+l-l^2} + \rho^{7+l^2-4l} + \rho^{9-\frac{36}{8q}} \| f \|^2_{L^q(B^+(x, \rho))} \\
\leq N\gamma^7 \int_{B^+(x, \rho)} \left| u - (u)_{B^+(x, \rho)} \right|^2 \, dx + \rho^{5+l-l^2} + \rho^{7+l^2-4l} + \rho^{9-\frac{16}{q}} \| f \|^2_{L^q(B^+(x, \rho))}, \quad (4.44)
\]

where \( N = N(c, l, r') > 0 \). By the condition of \( l \) in Lemma 2.2 and by taking \( l \in \left(0, \frac{4q-10}{16q-25}\right) \), we have

\[
\min \left\{ 5 + l - l^2, 7 + l^2 - 4l, 9 - 10/q \right\} > 5 + l/2.
\]

From (4.44) and Lemma 2.2 we obtain

\[
\int_{B^+(x, \rho)} \left| u - (u)_{B^+(x, \rho)} \right|^2 \, dx \leq N\rho^{5+l/2} \quad (4.45)
\]

for any \( x \in B_{3/16} \cap \{ x_5 = 0 \} \) and \( \rho \in (0, 3/16) \).

Next we discuss the case when \( x \in B_{3/16}^+ \). Let \( x^* \) be the projection of \( x \) on the boundary \( \{ x_5 = 0 \} \) and \( d_x \) be the distance between \( x \) and the flat boundary. Let \( \rho \in (0, 1/16) \). According to the values of \( d_x \) and \( \rho \), we divide the proof into two cases.

**Case 1:** \( \rho \geq d_x/2 \). In this case, we have \( B^+(x, \rho) \subset B^+(x^*, 3\rho) \). By (4.45), we derive

\[
\int_{B^+(x, \rho)} \left| u - (u)_{B^+(x, \rho)} \right|^2 \, dx \\
\leq N \int_{B^+(x^*, 3\rho)} \left| u - (u)_{B^+(x^*, 3\rho)} \right|^2 \, dx \leq N\rho^{5+l/2}, \quad (4.46)
\]

where \( N = N(c, l, r') \).

**Case 2:** \( \rho < d_x/2 \). In this case, by the decay estimates (4.17)-(4.19) in Lemma 4.4 we have

\[
A(x, \rho) + E(x, \rho) = A^+(x, \rho) + E^+(x, \rho) \leq N\rho^{\frac{(2-l)}{2}},
\]

\[
G(x, \rho) = G^+(x, \rho) \leq N\rho^{\frac{5(1+c)(1+l)(2-l)}{4(2-c)}},
\]

and

\[
P(x, \rho) = P^+(x, \rho) \leq N\rho^{\frac{(1+c)(2-l)}{2}},
\]

where \( N = N(c, l, r') \).
Similar to the calculations in (4.39)-(4.44), by the above decay estimates, we obtain
\[
\int_{B^+(x,\rho)} |u - (u)_{B^+(x,\rho)}|^2 \, dx = \int_{B(x,\rho)} |u - (u)_{B(x,d)}|^2 \, dx
\]
\[
\leq N\left(\frac{\rho}{d}\right)^7 \int_{B(x,d)} |u - (u)_{B(x,d)}|^2 \, dx
\]
\[
+ N\left(d^{5+l^2} + d^{7+l^2} - 4l + d^{9-10l_2} \|f\|_{L^2(B_1)}^2\right)
\]
for any \( \rho < d < d_x/2 \), where \( N = N(c,l,r') \). Thus, by Lemma 2.2 and the condition of \( l \), one has
\[
\int_{B^+(x,\rho)} |u - (u)_{B^+(x,\rho)}|^2 \, dx
\]
\[
\leq N\left(\frac{\rho}{d}\right)^5 + l/2 \int_{B(x,d_x)} |u - (u)_{B(x,d_x)}|^2 \, dx + N\rho^{5+l/2}, \tag{4.47}
\]
where \( N = N(c,l,r') \).

For the first term on the right-hand side of (4.47), by (4.46), we reach
\[
\int_{B(x,d_x)} |u - (u)_{B(x,d_x)}|^2 \, dx = \int_{B^+(x,d_x)} |u - (u)_{B^+(x,d_x)}|^2 \, dx \leq N d_x^{5+l/2}.
\]
Inserting the above inequality into (4.47), for any \( x \in B^+_3/16 \) and \( \rho \in (0,1/16) \), we have
\[
\int_{B^+(x,\rho)} |u - (u)_{B^+(x,\rho)}|^2 \, dx \leq N \rho^{5+l/2},
\]
where \( N = N(c,l,r') \). Hence, by Campanato’s characterization of Hölder continuity around the boundary, we see that \( u \) is Hölder continuous in \( B^+_1/4 \). The conclusion of Theorem 1.2 follows by a covering argument.

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