I. INTRODUCTION

Quantum technologies have shown their significant influence in communication and computing. On the one hand, many quantum cryptographic protocols have been proposed for protecting the security and privacy [1,2]. On the other hand, quantum computing which makes use of quantum mechanical principles, such as superposition and entanglement, shows tremendous potential that outperforms the conventional computing in time complexity in solving many problems, including Boolean function computing [3,4], matrix computing [5,6] and machine learning [7,8].

Matrices encountered in practical computations often have some special structures, and many problems can be transformed into solving linear systems with structures. Among various matrix structures, the ones of circulant, Toeplitz, Hankel types and their generalization called circulant/Toeplitz/Hankel-like are best-known and well-studied [9,10]. As of now, some efficient quantum algorithms have been proposed for solving linear systems with these displacement structures. The quantum algorithm for the Poisson equation is the earliest work (as far as we know) on solving Toeplitz linear systems involving a specific kind of banded Toeplitz matrices [11]. In 2016, a quantum algorithm for solving sparse circulant systems was proposed by Mahasinghe and Wang [12].

Whereafter, the quantum algorithm for solving circulant systems with the bounded spectral norm was presented in Ref. [21], where no assumption on the sparseness is demanded. By constructing associated circulant matrices, Wan et al [22] proposed a quantum algorithm for solving Toeplitz systems in Wiener class (defined in Sec.III.C), which is an asymptotic quantum algorithm of which the error is related to the dimension of the Toeplitz matrices.

The quantum algorithms introduced above can achieve excellent performance under certain circumstances and are powerful for solving large families of problems. However, these algorithms followed different ideas and employed different techniques that the quantum algorithm for linear systems with a specific displacement structure is not inspiring for other types. Moreover, it is intractable to generalize these algorithms to solve linear systems with the same type of generalized structures. Then, an interesting question is can we design quantum algorithms for linear systems with various types of displacement structures in a unified way. A unified treatment of such structured matrices can often provide conceptual and computational benefits and may also give insight into finding some new applicable instances. Combined with the method of block-encoding, we answer the question in the affirmative.

A block-encoding of a matrix $M$ is a unitary $U$ that encodes $M/\alpha$ as its top left block, where $\alpha \geq \|M\|$ is a scaling factor. Given a way to implement block-encodings of some matrices, many operations on the matrices can be done, including linear system solving [13,23,24]. Nevertheless, it is worth noting the complexity of the quantum algorithm for linear systems based...
on block-encodings has a linear dependence on scaling factors, and implementing block-encodings with preferred scale factors often requires ingenious design. Although there are some methods to implement the block-encodings for several specific matrices, such as sparse matrices, density operators, POVM operators, Gram matrices, matrices stored in a quantum-accessible data structure, directly applying the methods mentioned can not give rise to appealing quantum linear system solvers for the matrices with displacement structures. Exploiting the structures of such matrices to implement their block-encodings with favorable scaling factors and less cost of time, space or memory deserves specialized study.

In this paper, we devise an approach that decomposes \( n \times n \) dense matrices into linear combinations of unitaries (LCU), and implement block-encodings of matrices with displacement structures following the idea of LCU lemma [27]. The proposed block-encodings can give rise to efficient quantum algorithms for a set of linear systems with displacement structures, including the linear systems, such as Toeplitz systems in Wiener class and circulant systems with the bounded spectral norm, whose quantum algorithms can be improved by our method, and the linear systems without specialized quantum algorithm before, such as some Toeplitz/Hankel-like linear systems. More specifically, the main contributions of this paper are as follows:

1) We first deduce parameterized representations of \( n \times n \) dense matrices by decomposing them into linear combinations of unitaries, which provide a way that the structured matrices of interest can be represented and treated similarly. The proposed LCU decompositions possess several desirable features for implementing block-encodings. (i) The elementary components unitaries are displacement matrices that can be easily implemented; (ii) The decomposition coefficients are the elements of the displacement of the decomposed matrices that can be easily calculated; (iii) For the structured matrices of interest, the number of decomposed items is roughly \( O(n) \). Especially, this decomposition method provides a representation with \( 2n - 1 \) parameters for Toeplitz or Hankel matrices, which is optimal in terms of the number of parameters.

2) Based on the proposed LCU decompositions, we then construct efficient quantum circuits in two different data access models commonly used in various quantum algorithms, i.e., the black-box model and the QRAM data structured model, to implement the \( \epsilon \)-approximate block-encodings of matrices with displacement structures. If a matrix is given in the QRAM data structure model, it will often lead to a low-complexity construction. Otherwise, the construction scheme in the black-box model may be adopted since it requires less on matrix storage. In both models, we implement block-encodings of which the scaling factors are proportional to the \( l_1 \)-norm \( \chi \) of the displacement of the structured matrices. For the structured matrices with small \( \chi \), the linear system solvers based on the proposed block-encodings provide a quadratic speedup with respect to the dimension over classical algorithms in the black-box model and an exponential speedup in the QRAM data structure model.

3) We show that many matrices with displacement structures that are frequently encountered in various practical problems can harness the potential advantages of the proposed constructions. With the quantum linear system solver developed in the block-encoding framework, we obtain the following algorithms. (i) A quantum algorithm for Toeplitz linear systems in the Wiener class, which is an exact algorithm that the error is independent of the dimension of the Toeplitz matrices. It positively answers the open question raised in [22] and can provide computational benefits when rigorous precision is required. (ii) A quantum algorithm for circulant linear systems with the bounded spectral norm, providing a quadratic improvement in the dependence on the condition number and an exponential improvement in the dependence on the precision over the quantum algorithm proposed in [21]. (iii) An efficient quantum algorithm for linear systems with Toeplitz/Hankel-like structures. In particular, we also obtain a quantum algorithm for banded Toeplitz/Hankel linear systems without the use of any black box or QRAM, which may be more convenient when constructing practical quantum circuits. At last, we show that the proposed quantum algorithm for Toeplitz linear systems can be used in linear prediction of time series, which provides a concrete example to illustrate that the quantum speedup is practically achievable.

II. PRELIMINARIES

A. The Matrices With Displacement Structure

The matrices with displacement structures arise pervasively in many contexts. Below, we display three popular classes of such structured matrices and their generalizations, which are also our focus in this paper.

A Toeplitz matrix \( T_n \) is a matrix of size \( n \times n \) whose elements along each diagonal are constants. More clearly,

\[
T_n = \begin{pmatrix}
t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-1)} \\
t_1 & t_0 & t_{-1} & \cdots & \vdots \\
t_2 & t_1 & t_0 & \cdots & t_{-2} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
t_{(n-1)} & \cdots & t_2 & t_1 & t_0
\end{pmatrix}.
\]
where \( t_{i,k} = t_{i-k} \), \( T_n \) is determined by the sequence \( \{t_j\}_{j=-\infty}^{n-1} \).

There is a common special case of Toeplitz matrix called circulant matrix whose every row is a right cyclic shift of the row above it:

\[
C_n = \begin{pmatrix}
c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\
c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_{n-1} & \cdots & c_1 & c_0 & c_{n-1}
\end{pmatrix}.
\] (2)

Since the circulant matrix has some fantastic properties, it has been studied in specialty no matter in the classical or quantum setting.

Another representative class of matrices with displacement structures is the Hankel matrix. A matrix \( H_n \) is called Hankel matrix if it has the form

\[
H_n = \begin{pmatrix}
h_0 & h_1 & h_2 & \cdots & h_{n-1} \\
h_1 & h_2 & \cdots & \vdots & \vdots \\
h_2 & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & h_{n-1} & h_n \\
h_{n-1} & h_n & \cdots & h_{2n-3} & h_{2n-2}
\end{pmatrix}.
\] (3)

The entry \( h_{i,k}(i, k = 0, 1, \ldots, n-1) \) of \( H_n \) is equal to \( h_{i+k} \) for given sequence \( \{h_j\}_{j=0}^{2n-2} \). In other words, the skew-diagonals of a Hankel matrix are constants.

There are some natural generalizations of these structured matrices called Toeplitz/Hankel-like matrices (circulant-like matrices are regarded as a special case of Toeplitz-like matrices). A matrix is said to be Toeplitz/Hankel-like if there are a few elements on some diagonals/skew-diagonals of the matrix that are not equal to the others.

The computations with these structured matrices, both Toeplitz/Hankel matrices and Toeplitz/Hankel-like matrices, are widely applied in the various areas of sciences and engineering. For example, in time series analysis, the covariance matrices of weakly stationary processes are Toeplitz matrices, see [28]. The visual tracking framework of [29] requires a base sample image to generate multiple virtual samples which correspond to circulant matrices. The solvability of certain classical interpolation problems is connected with Hankel matrices, see [13]. The numerical solutions of some partial differential equations with mixed boundary conditions can be obtained by solving the Toeplitz-like linear systems generated by the discretization of the finite difference method, see [30]. Other applications involve polynomial computations [17], image restoration [31], machine learning [32], compressed sensing [33], and so on.

Compared to general matrices, the structures in these matrices can be exploited to perform algebraic operations, such as matrix-vector multiplication, inversion, and matrix exponential, with much less running time and memory space. There are a number of classical methods that have been presented to solve the linear systems with such structures [31]. However, the time complexity of these methods is \( \Omega(n) \), it is still a hard task to tackle the problems with very large \( n \) on a classical computer.

### B. The Framework of Block-encodings

In this section, we review the framework of block-encodings introduced in [24, 25].

**Definition 1** (Block-encoding). Suppose that \( M \) is an \( s \)-qubit operator, \( \alpha, \epsilon \in \mathbb{R}^+ \) and \( a \in \mathbb{N} \). Then we say that the \((s+a)\)-qubit unitary \( U \) is an \((\alpha; a; \epsilon)\)-block-encoding of \( M \), if

\[
\| M - \alpha((0)^{\otimes a} \otimes I)U((0)^{\otimes a} \otimes I) \| \leq \epsilon. \tag{4}
\]

Given a block-encoding \( U \) of a matrix \( M \), one can produce the state \( M|\psi\rangle\|M|\psi\rangle\| \) by applying \( U \) to a initial state \( |0\rangle|\psi\rangle \). Low and Chuang [24] presented Hamiltonian simulation algorithm under the framework of block-encodings by combining techniques qubitization and quantum signal processing, which can simulate sparse Hamiltonians with optimal complexity. Taking this Hamiltonian simulation algorithm as a subroutine, Chakraborty et al. [24] developed several useful tools within the block-encoding framework such as singular value estimation and quantum linear system solver. In fact, they also point out that one can implement any smooth function of a Hamiltonian when given a block-encoding of this Hamiltonian by using the techniques developed in [34]. Furthermore, the method of block-encoding has been applied to the study of machine learning, and many quantum algorithms have been presented such as quantum clustering algorithm [35], quantum classification algorithm [36] and quantum algorithms for semidefinite programming problems [25, 37].

Although the block-encoding can be applied to various algorithms for various computational problems, we will narrow our goal to the detail study of solving linear systems, and then analyze the improvements brought by the method proposed in this paper. Here we describe an informal version of the complexity result of the quantum algorithm for linear systems in the framework of block-encoding. Given a \((\alpha; a; \epsilon)\)-block-encoding \( U \) of \( M \), there is a quantum algorithm that produces a state that is \( \epsilon \)-close to \( M^{-1}|b\rangle\|M^{-1}|b\rangle\| \) in time \( O(\kappa_M(\alpha(a + T_U))log\frac{1}{\epsilon} + T_b)) \), where \( \kappa_M \) is the condition number of \( M \), \( T_U \) is the running time to implement \( U \), \( T_b \) is the running time to prepare the state \( |b\rangle \). This result suggests that one need to efficiently construct block-encodings with small scaling factors.

### C. The Data Access Model

We now specify the data access model involved in the proposed quantum algorithms for solving linear system \( Mx = b \).
For the right-hand-side vector $b$, a unitary that produces the quantum state $|b\rangle = \sum_i b_i |i\rangle/\|\sum_i b_i |i\rangle\|$ is required in the quantum linear system solving algorithm. It is shown that some specialized algorithms can be used to generate $|b\rangle$ efficiently under certain conditions \cite{32, 33}, or our quantum algorithm may be used as a subroutine while $|b\rangle$ can be prepared by another part of a larger quantum algorithm. Alternatively, if an efficiently-implementable state preparation procedure can not be provided, we assume $b$ can be accessed in the same way as the coefficient matrix, which will be introduced below.

For the coefficient matrix $M$, we are given two different data access models which are most commonly used in various quantum algorithms. In the first data access model, the elements of matrix $M \in \mathbb{C}^{n \times n}$ are accessed by a black box $O_M$ acting as

$$O_M|i\rangle|k\rangle|0\rangle = |i\rangle|k\rangle|m_{i,k}\rangle \quad i, k = 0, 1, \cdots, n - 1.$$  

This model is often referred to as the black-box model \cite{10, 11}. The access operation can be made efficiently when $m_{i,k}$ are efficiently computable or a QRAM is provided.

Kerendis and Prakash \cite{42} introduced a different data access model called the QRAM data structure model. This data access model stores data in QRAM with a binary tree structure and allows access in superposition. When the data is given as an $n \times n$ matrix, the matrix is stored in the binary trees by rows, and an additional binary tree is required to store the norms of the rows. Obviously, the memory requirement and the complexity of constructing this data structure must be $O(n^2)$. Although many quantum algorithms in this model do not take the complexity of constructing data structure, reducing memory requirement and thereby reducing the complexity of constructing data structure has many practical implications.

III. METHODS AND RESULTS

A. LCU Decomposition of Matrices

In this section, we deduce parameterized representations of $n \times n$ matrices by decomposing them into linear combinations of unitaries, where the unitaries used as elementary components are easy to implement and the decomposition coefficients are easy to calculate. Without loss of generality, we assume that $n$ is always a power of two.

For a better understanding, we first introduce some necessary background information about matrix displacement.

**Definition 2.** For a given pair of operator matrices $(A, B)$, and a matrix $M \in \mathbb{C}^{n \times n}$, the linear displacement operators $\mathcal{L}(M) : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ of Stein type is defined by:

$$\mathcal{L}(M) = \Delta_{A, B}[M] = M - AMB,$$  

and that of Sylvester type is defined by:

$$\mathcal{L}(M) = \nabla_{A, B}[M] = AM - MB.$$

The image $\mathcal{L}(M)$ of the operator $\mathcal{L}$ is called the displacement of the matrix $M$. According to the specific structure of the matrix $M$, one can instantiate the operator matrices $A$ and $B$ with desirable properties. Here, for our purposes, we introduce one of the customary choices of $A$ and $B$, the unit $f$-circulant matrix $Z_f$, which we will define next.

**Definition 3** (unit $f$-circulant Matrix). For a real-valued scalar $f$, an $n \times n$ unit $f$-circulant matrix is defined as follows,

$$Z_f = \begin{pmatrix} 0 & 0 & \cdots & f \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}. $$

It is easy to verify that $Z_0, Z_{-1}$ are unitary matrices, as well as $Z_1, Z_{-1}, i = 0, 1, \cdots, n - 1$. By inverting the displacement operators with operator matrices $(Z_1, Z_{-1})$, we then demonstrate how to decompose an $n \times n$ matrix as linear combinations of unitaries.

**Theorem 1.** Let $M \in \mathbb{C}^{n \times n}$, $m_{i,k}$ be the $k$-th element of the $i$-th row of $M$, $J = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$ is the reversal matrix, and

$$g(k) := \begin{cases} 0 & k = 0, 1, 2 \cdots n - 2, \\ 1 & k = n - 1, \end{cases}$$

then $M$ can be decomposed as:

i) $$M = \frac{1}{2} \sum_{i,k=0}^{n-1} \tilde{m}_{i,k} Z_1^i J Z_{-1}^{n-1-k},$$

where $\tilde{m}_{i,k} = m_{i,k} - (-1)^{g(k)} m_{(i-1) \mod n, (k+1) \mod n}$ is the $k$-th element of the $i$-th row of matrix $\Delta_{Z_1, Z_{-1}}[M]$.

ii) $$M = \frac{1}{2} \sum_{i,k=0}^{n-1} \tilde{m}_{i,k} Z_1^i Z_{-1}^{n-1-k},$$

where $\tilde{m}_{i,k} = m_{(i-1) \mod n, k} - (-1)^{g(k)} m_{i, (k+1) \mod n}$ is the $k$-th element of the $i$-th row of matrix $\nabla_{Z_1, Z_{-1}}[M]$.

**Proof.** See Appendix A.

We call these two decompositions Stein type and Sylvester type, respectively. From this theorem, using the displacement matrices \{J, Z_i, Z_{-i}, i = 0, 1, \cdots, n - 1\} as the elementary components, one can decompose an $n \times n$ matrix into linear combinations of these simple unitaries, and the decomposition coefficients are the
elements of the displacement of the matrix which can be easily calculated. The proposed LCU decompositions actually provide a way to parameterize the decomposed matrices, that is, we can use the elements of \( \mathcal{L}(M) \) as a parameterized representation of \( M \).

### B. Implement Block-encodings of the matrices with displacement structures

In this section, we will show that the Toeplitz/Hankel matrices and their generalizations can generate elegant parameterized representations with nearly \( O(n) \) parameters when associated with operator matrices \( (Z_1, Z_{-1}) \). Furthermore, we will illustrate how to implement the block-encodings of such matrices in detail.

Taking the Toeplitz matrices as an example, we compute their Sylvester displacement first:

\[
\nabla Z_{1, Z_{-1}}[T_n] = \begin{pmatrix} t_{n-1} - t_{-1} & t_{n-2} - t_{-2} & \cdots & t_0 \\
0 & 0 & \cdots & t_{-(n-1)} + t_1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & t_{-1} + t_{n-1} \end{pmatrix}.
\]

(12)

Then \( T_n \) can be decomposed into a linear combination of unitaries as follow,

\[
T_n = \frac{1}{2} [2t_0I + (t_1 + t_{-(n-1)})Z_1^1 + \ldots + (t_{n-1} + t_1)Z_{n-1}^{n-1} + (t_1 - t_{-(n-1)})Z_1^{-1} + \ldots + (t_{n-1} - t_1)Z_{n-1}^{-1}].
\]

(13)

We would like to emphasize that since the Toeplitz matrices are represented with \( 2n - 1 \) parameters, this decomposition should be optimal on the number of items.

To implement the block-encodings of Toeplitz matrices from Eq. (13), we define two state preparation operators as follows,

\[
V(\sqrt{\mathcal{V}[T_n]})|0\rangle = |V(\sqrt{\mathcal{V}[T_n]})\rangle
\]

\[
= \frac{1}{\sqrt{T_n}} \sum_{j=0}^{n-1} \sqrt{t_j} + t_{-[n-j \text{ mod } n]}|j\rangle
\]

\[
+ \frac{1}{\sqrt{T_n}} \sum_{j=n}^{2n-1} \sqrt{t_{j-n} - t_{-[2n-j \text{ mod } n]}|j\rangle
\]

\[
\equiv \frac{1}{\sqrt{T_n}} \sum_{j=0}^{2n-1} \sqrt{t_j}|j\rangle,
\]

(14)

\[
V(\sqrt{\mathcal{V}[T_n]^*})|0\rangle = |V(\sqrt{\mathcal{V}[T_n]^*})\rangle = \frac{1}{\sqrt{T_n}} \sum_{j=0}^{2n-1} \sqrt{t_j^*}|j\rangle,
\]

(15)

where \( \chi T_n = \sum_{j=0}^{2n-1} \sqrt{t_j} \), and the square root operation takes the main square root of \( t_j \) and \( t_j^* \). Then, we define a controlled unitary

\[
select U_{T_n} = \sum_{j=0}^{n-1} |j\rangle \langle j| \otimes Z_j^1 + \sum_{j=n}^{2n-1} |j\rangle \langle j| \otimes Z_{n-1}^{n-1}.
\]

(16)

Since \( \sqrt{t_j} (\sqrt{t_j^*})^* = \tilde{t}_j \), it is easy to verify that

\[
T_n = \frac{\chi T_n}{2} (|0\rangle \langle V(\sqrt{\mathcal{V}[T_n]^*})| \otimes I) select U_{T_n} (V(\sqrt{\mathcal{V}[T_n]}) \otimes I)|0\rangle),
\]

(17)

which means that \( V(\sqrt{\mathcal{V}[T_n]^*}) select U_{T_n} V(\sqrt{\mathcal{V}[T_n]}) \) is a block-encoding of \( T_n \).

Implementing the block-encoding of \( T_n \) is to implement \( V(\sqrt{\mathcal{V}[T_n]}) \), \( V(\sqrt{\mathcal{V}[T_n]^*}) \), and select \( U_{T_n} \). For quantum state preparation operators, we need to construct a quantum circuit that prepare \( |V(\sqrt{\mathcal{V}[T_n]})\rangle \) reversibly. In the black-box model, this can not be done by the traditional black-box quantum state preparation because its success probability can not be increased arbitrarily close to certainty, which will make the final error uncontrollable. We provide a reversible algorithm called steerable black-box quantum state preparation, based on fixed-point amplitude amplification, which can prepare a quantum state with the success probability increasing arbitrarily close to certainty. Since it will be called multiple times as a key subroutine of our algorithm and may be of independent interest to other quantum algorithms, we make a formal statement here.

**Lemma 1.** For a vector \( x \in \mathbb{C}^{n \times 1} \), \( |x|_{\max} = \max_i |x_i| \) is a small constant, and the elements are given by a black box \( O_x \) acting as

\[
O_x |i\rangle |0\rangle \rightarrow |i\rangle |x_i\rangle.
\]

(18)

Then, the steerable black-box quantum state preparation algorithm generate a state, up to a global phase, that is \( \epsilon_{\max} \)-approximation of

\[
|x\rangle = \sqrt{\frac{1}{|x|_1}} \sum_{i=0}^{n-1} \sqrt{x_i} |i\rangle
\]

(19)

with success probability at least \( 1 - \delta^2 \), using \( O\left(\sqrt{\frac{n \log(1/\delta)}{\epsilon_{\max}}}\right) \) queries of \( O_x \) and additional \( O\left(\sqrt{\frac{n \log(1/\delta)}{\epsilon_{\max}}\log(\frac{n}{\epsilon_{\max}})}\right) \) elementary gates.

**Proof.** See Appendix B.

The operators defined by Eqs. (15), (16) and (17) are actually the operators of LCU circuit [27] which has been used in many quantum algorithms [24, 43, 44]. Here, we implement this circuit with two different data access models introduced in Sec. [II.C]. The method in the QRAM data structure model is especially useful for the structured matrices whose displacements have been stored in the data structure. And the method in the black-box model will have a wider range of applications because of the flexibility of its implementation. We summarize the results as follows.
Theorem 2. Let \( T_n \in \mathbb{C}^{n \times n} \) be a Toeplitz matrix. (i) If the elements of \( T_n \) are provided by a black box \( O_{T_n} \), i.e.,
\[
O_{T_n}(|i\rangle|k\rangle) = |i\rangle|k\rangle |t_{i,k}\rangle
\]
one can implement a \( (\sqrt{\frac{n\log(n\chi_T/e)}{\epsilon}}, \frac{n\chi_T}{\epsilon}) \)-block-encoding of \( T_n \) with \( O\left(\frac{\sqrt{n\log(n\chi_T/e)}}{\sqrt{\epsilon}}\right) \) uses of \( O_{T_n} \) and additionally using \( O\left(\frac{\sqrt{n\log(n\chi_T/e)}}{\epsilon}\right) \) elementary gates. 
(ii) If the nonzero elements of Sylvester displacement \( V \) of \( T_n \), i.e., \( \{\tilde{t}_{i,k}\} \) are stored in the QRAM data structure as shown in lemma 6, one can implement a \( (\sqrt{\frac{n\log(n\chi_T/e)}{\epsilon}}, \frac{n\chi_T}{\epsilon}) \)-block-encoding of \( T_n \) with gate complexity \( O(\text{polylog}(n\chi_T/e)) \) and memory cost \( O(n) \).

Proof. See Appendix C.

Moreover, we show that how to implement the block-encoding of the Toeplitz-like matrices. Let \( T_L \in \mathbb{C}^{n \times n} \) be a Toeplitz-like matrix, \( (T_L)_{i,k} = \tilde{r}_{i,k}, (\nabla Z_L, I_z)_{i,k} = \tilde{r}_{i,k} \). We first define two state preparation operators as follows,
\[
V_{(\nabla[Z_L])} |0\rangle |0\rangle = V_{(\nabla[Z_L])} = \frac{1}{\sqrt{\chi_T}} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \sqrt{\tilde{r}_{i,k}} |i\rangle |k\rangle,
\]
(20)
\[
V_{(\nabla[Z_L])}^\dagger |0\rangle |0\rangle = V_{(\nabla[Z_L])}^\dagger = \frac{1}{\sqrt{\chi_T}} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \sqrt{\tilde{r}_{i,k}^*} |i\rangle |k\rangle,
\]
(21)
where \( \chi_T = \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} |\tilde{r}_{i,k}| \), and the square root operation takes the main square root of \( \tilde{r}_{i,k} \) and \( \tilde{r}_{i,k}^* \). Then, we define
\[
\text{select } U_{T_L} = \left( \sum_{i=0}^{n-1} |i\rangle \langle i| \otimes I \otimes Z^1 \right) \left( \sum_{k=0}^{n-1} I \otimes |k\rangle \langle k| \otimes Z^{n-1-k} \right).
\]
(22)

Since \( \sqrt{\tilde{r}_{i,k}}(\sqrt{\tilde{r}_{i,k}^*})^* = \tilde{r}_{i,k} \), it is easy to verify that
\[
T_L = \frac{\chi_T}{2} \langle 0| (V_{(\nabla[Z_L])} \otimes I) \text{select } U_{T_L} (V_{(\nabla[Z_L])} \otimes I) |0\rangle,
\]
(23)
which means that \( (V_{(\nabla[Z_L])} \otimes I) \text{select } U_{T_L} (V_{(\nabla[Z_L])} \otimes I) \) is a block-encoding of \( T_L \).

It seems difficult to implement the state preparation operators \( V_{(\nabla[Z_L])} \) and \( V_{(\nabla[Z_L])}^\dagger \) with complexity less than \( O(n) \) in the black-box model. However, as shown in Eq (12), the Sylvester displacement of a Toeplitz matrix has non-zero elements only along its first row and last column. For a Toeplitz-like matrix \( T_L \), there are a few elements on some diagonals of the matrix that are not equal to the others. Then, it can be directly verified that the sub-matrix left by deleting the first row and last column of the Sylvester displacement of the Toeplitz-like matrix is a sparse matrix. Based on this observation, we construct the block-encodings of the Toeplitz-like matrices, and the results are summarized as follows.

Corollary 1. Let \( T_L \in \mathbb{C}^{n \times n} \) be a Toeplitz-like matrix. Suppose that the sub-matrix left by deleting the first row and last column of \( \nabla Z_L, I_z \) \( T_L \) is \( (d-1) \)-row-sparse, i.e., there are at most \( (d-1) \) nonzero elements in each row. (i) If the elements of \( T_L \) are provided by a black box \( O_{T_L} \), i.e.,
\[
O_{T_L}(|i\rangle|k\rangle) = |i\rangle|k\rangle |\tau_{i,k}\rangle
\]
and a black box that computes the positions of the distinct elements on diagonals of the Toeplitz-like matrices is provided, one can implement a \( (\sqrt{\frac{n\log(n\chi_T/e)}{\epsilon}}, \frac{n\chi_T}{\epsilon}) \)-block-encoding of \( T_L \) with gate complexity \( O(\text{polylog}(n\chi_T/e)) \) and memory cost \( O(dn\log n) \).

Proof. See Appendix D.

Remark 1. One might be confused about the QRAM data structure used in this paper, which stores \( \tilde{m}_{i,k} \) for a matrix \( M \). In fact, in most of quantum algorithms using this data structure, such as [24, 26], the stored entries are \( m_{i,k}^p, p \in [0, 2] \). Since \( \tilde{m}_{i,k} \), as defined below Eq. (11), can be calculated as efficiently as \( m_{i,k}^p, p \in [0, 2] \), our assumption about such data structure is not stronger than the assumption in the previous algorithms.

Remark 2. The result (i) and result (ii) in theorem 2, as well as corollary 1, use different data access models, where the black-box model queries the elements of structured matrices, while the QRAM data structure model requires the elements of their displacements have been stored. Due to the different data access models, a direct comparison of the complexity of these results is inappropriate. And it is unwise to claim which result is more advantageous based on the complexity. In practical applications, one should choose the appropriate method according to the form of the data to be obtained.

For a circulant matrix \( C_n \), computing its Sylvester displacement, the LCU decomposition of \( C_n \) is \( C_n = \sum_{j=1}^{n-1} c_j Z_j^1 \). Similar to the implementation of the block-encodings of the Toeplitz matrices, we can implement block-encodings of \( C_n \). Since the number of decomposed items of the circulant matrices is less than that of the Toeplitz matrices, the resources required to implement these block-encodings are less than those stated in theorem 2. The same conclusion holds for circulant-like matrices.

For a Hankel matrix \( H_n \), it can be decomposed as follows by computing their Stein displacements,
\[
H_n = \frac{1}{2} [h_{n-1} J + (h_n + h_0) Z_1^1 J + \cdots + (h_{2n-2} + h_{n-1}) Z_{n-2}^1 J Z_{n-1}^1] \tag{24}
\]
\[
+ \cdots + (h_0 - h_n) J Z_{n-1}^n] .
\]
Note that $JZ_{-1} = -Z_{-1}^{-1}J$, and this decomposition is equivalent to

$$H_n = \frac{1}{2}(2h_{n-1}J + (h_n + h_0)Z_1^{-1}J + \cdots + (h_{2n-2} + h_{n-2})Z_1^{-1}J + (h_{2n-2} - h_{n-2})Z_1^{-1}J + \cdots + (h_n - h_0)Z_1^{-1}J).$$

(25)

Since $J = \sigma_x \otimes \sigma_x$ (Pauli-X operator), we can implement an $\epsilon$-approximate block-encoding of $H_n$ by constructing a quantum circuit similar to the block-encoding implementation of $T_n$, where the scaling factor is $\chi_{H_n, /2} = \sum_{n=0}^{n-1} \sum_{k=0}^{n-1} |\hat{b}_{i,j}|/2$. Also, the block-encodings of the Hankel-like matrices can be implemented similarly to that of the Toeplitz-like matrices.

In many cases, such as visual tracking [41], we need to extend the non-Hermitian matrices with displacement structures to Hermitian. For the extended matrices

$$\overline{M} = \begin{pmatrix} 0 & M \\ M^\dagger & 0 \end{pmatrix},$$

(26)

let $U$ be a $(\chi; a; \epsilon)$-block-encoding of $M$, then we can implement a $(\chi; a; \epsilon)$-block-encoding of $\overline{M}$ by using the method of implementing block-encoded matrices [24]. The cost of implementing this block-encoding is nearly twice the cost of implementing $U$. Therefore, without losing generality, we can assume that the matrices studied in the following sections are Hermitian.

C. Quantum Algorithm for Linear Systems with Displacement Structures

As mentioned, given a block-encoding $U$ of a matrix $M$, one can perform a number of useful operations on $M$. In particular, combining the variable-time amplitude amplification technique [41] and the idea of implementing smooth functions of block-Hamiltonians [24], Chakraborty et al [24] presented a quantum algorithm for linear systems within the block-encoding framework. We invoke the complexity of this algorithm as follows.

**Lemma 2.** (Variable-time quantum linear systems algorithm [24]) Let $H$ be an $n \times n$ Hermitian matrix, $\lambda_r$ are the nonzero eigenvalues of $H$ such that $\lambda_r \in [-1, -1/\kappa_H] \cup [1/\kappa_H, 1]$, where $\kappa_H > 2$ is the condition number of $H$. Suppose that there is a $(\alpha; a; \delta)$-block-encoding $U$ of $H$, where $\delta = o(\epsilon/(\kappa_H^2 \log^2(\frac{\epsilon}{\alpha})))$, and $U$ can be implemented in time $T_U$. Also suppose the state $|b\rangle$ can be prepared in time $T_b$. Then there exists a quantum algorithm that produces a state that is $\epsilon$-close to $H^{-1}|b\rangle||H^{-1}|b\rangle||$ in time

$$O(\kappa_H (\alpha(a + T_U)\log^2(\frac{\kappa_H}{\epsilon}) + T_b)\log(\kappa_H)).$$

As mentioned in Sec. [41], there are some methods that can prepare right-hand-side state $|b\rangle$ in time $O(\text{polylog } n)$ under certain conditions. Even if such an efficient state preparation procedure can not be provided, the complexity of preparing $|b\rangle$ will not exceed the complexity of implementing block-encodings of structured matrices in the same data access model. More specifically, in the black-box model, the query complexity of preparing an $n$-dimensional quantum state is $O(\sqrt{n})$ [41]. In the QRAM data structured model, one can prepare the quantum state $|b\rangle$ with complexity $O(\text{polylog } n)$ [42]. Here, following the assumption of previous quantum algorithms [8, 23], we neglect the error in producing $|b\rangle$ since this error is independent of the design of the quantum algorithm.

Therefore, according to lemma 2, the method proposed in theorem 2 can induce a quantum algorithm to solve the structured linear systems with complexity (i) $O(\kappa_H \sqrt{n} \text{polylog}(1/\epsilon))$ in the black-box model (We use the symbol $\hat{O}$ to hide redundant poly-logarithmic factors. And since the elementary gates requirement in the black-box model is larger than the query complexity by logarithmic factors, we will not describe them individually from now on.); (ii) $O(\kappa_H \text{polylog}(n / \epsilon))$ in the QRAM data structure model. Obviously, this algorithm is expected to be efficient for matrices of which the $\chi$ is small.

Many matrices with displacement structures encountered in a diverse range of applications satisfy this criteria. One of the typical examples should be Toeplitz matrices in the Wiener class [28, 31]. This kind of matrices are usually obtained by the discretization of some continuous problems. More specifically, let $C_{2\pi}$ be the set of all $2\pi$-periodic continuous real-valued functions defined on $[0, 2\pi]$. Let $T_n$ be the $n \times n$ Toeplitz matrices of which the elements of every diagonal are given by the Fourier coefficients of a function $f \in C_{2\pi}$, i.e.,

$$t_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{2\pi} f(\lambda) e^{-ij\lambda} d\lambda, \quad j = 0, \pm 1, \pm 2, \cdots.$$  

(27)

The function $f$ is called the generating function of the sequence of Toeplitz matrices $T_n(1 \leq n < \infty)$. The sequence of Toeplitz matrices $T_n(1 \leq n < \infty)$ of which the sequence $\{t_j\}$ is absolutely summable is said to be in the Wiener class. That is to say, for Toeplitz matrices in Wiener class, there must be a constant $\rho$, such that

$$\sum_{j=-\infty}^{\infty} |t_j| < \rho.$$  

(28)

Thus, for Toeplitz matrices in the Wiener class, we have

$$\lambda T_n = 2|t_0| + |t_1 + t_{-(n-1)}| + \cdots + |t_{n-1} + t_1| + |t_1 - t_{-(n-1)}| + \cdots + |t_{n-1} - t_{-1}| \leq 2 \sum_{j=-(n-1)}^{n-1} |t_j| < 2\rho.$$  

(29)

The complexity of the quantum algorithm for solving the Toeplitz systems in Wiener class is (i)
\(\tilde{O}(\kappa_{\mathcal{C}_n}, \sqrt{n}\text{polylog}(1/\epsilon))\) in the black-box model; (ii) \(\tilde{O}(\kappa_{\mathcal{C}_n}, \text{polylog}(n/\epsilon))\) in the QRAM data structure model. When the Toeplitz matrices are well-conditioned (We call a matrix \(M\) well-conditioned of which \(\kappa_M \in O(\text{polylog} n)\)) and \(1/\epsilon \in O(\text{poly} n)\), the quantum algorithm is (i) quadratically faster than the classical methods in the black-box model; (ii) exponentially faster than the classical methods in the QRAM data structure model.

As of now, some work regarding Toeplitz matrices have been studied in the quantum setting. In 2018, Wan et al. [22] adopted associated circulant matrices to approximate the Toeplitz matrices in Wiener class and solved the circulant linear systems by accessing the values of the generating function at specific points in parallel. It is an asymptotic quantum algorithm of which the error is related to the dimension of the Toeplitz matrices. Whether there is an exact quantum algorithm that the error is independent of the dimension is raised as an open question in [22]. The algorithm suggested in this section gives the answer, and it is more advantageous when rigorous precision is required or the Toeplitz matrices and their associated circulant matrices do not approach quickly as the dimension increases. Additionally, for the cases where no generating function is provided, our algorithm can improve the dependence on the condition number and precision since the complexity of the quantum algorithm proposed in [22] has a quadratic dependence on the condition number and a linear dependence on the precision.

Besides the Toeplitz matrices in the Wiener class, for the circulant matrices \(\mathcal{C}_n\), it is often the case in practical applications that \(c_j\) are nonnegative for all \(j\), and the spectral norm \(|\mathcal{C}_n| = \sum_{j=0}^{n-1} c_j\) of \(\mathcal{C}_n\) are constants. Thus, \(\chi_{\mathcal{C}_n}\) will be bounded by some constants, and the quantum algorithm based on proposed block-encodings can solve these circulant linear systems with complexity (i) \(\tilde{O}(\kappa_{\mathcal{C}_n}, \sqrt{n}\text{polylog}(1/\epsilon))\) in the black-box model; (ii) \(\tilde{O}(\kappa_{\mathcal{C}_n}, \text{polylog}(n/\epsilon))\) in the QRAM data structure model.

For the circulant matrices described above, based on the observation of LCU decomposition of \(\mathcal{C}_n\), Zhou et al. [21] used the method of simulating Hamiltonian with a truncated Taylor series [42] and HHL algorithm [8] to solve the associated linear systems. Under the assumption that there is an oracle that can prepare the state \(\frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \sqrt{c_j} |j\rangle\) in time \(O(\text{polylog} n)\), the complexity of the quantum algorithm proposed in [21] is \(\tilde{O}(n^{2/3}, \text{polylog}(n)/\epsilon)\). However, it is not always feasible to prepare this initial state with such a running time, especially in the black-box model. In this paper, we show in detail how to implement the preparation of this state in two different data access models. In particular, when using the same data access model, the algorithm proposed in this paper brings us complexity improvement on \(\kappa_{\mathcal{C}_n}\) and \(1/\epsilon\), which comes from the use of updated technique for Hamiltonian simulation and linear system solver.

There are some Hankel matrices of which \(\sum_{j=0}^{\infty} |h_j|\) are convergent, such as

\[
H_{i,k} = \frac{1}{(i+k+1)!} 
\]

which arise in determining the covariance structure of an iterated Kolmogorov diffusion [49]. In addition, there are also some Hankel matrices generated by the discretization of some functions [52], just as the Toeplitz matrices in the Wiener class. Then, we can implement block-encodings of these Hankel matrices with bounded scaling factors, which will derive a quantum algorithm that solves the Hankel linear systems with significant speedup (same as that for Toeplitz systems in Wiener class) in both data access models.

Immediately, for the Toeplitz/Hankel-like matrices, if they satisfy a constraint similar to one of the above forms, we can then solve the linear systems with such structures efficiently by using the block-encodings constructed in corollary 1. For example, finding a greatest common divisor of univariate polynomials involves solutions of Toeplitz-like systems and in some cases, as shown in [51], the elements of the displacement of the coefficient matrices are absolutely summable. We would like to emphasize the method proposed in this paper will result in efficient quantum linear systems solvers for the structured matrices of which the \(\chi\) is small, not only for the matrices introduced in this section.

IV. APPLICATION TO TIME SERIES ANALYSIS

Note that the quantum algorithm introduced in last subsection always outputs a state encoding the solution of the linear system in its amplitudes. Reading out all the classical information of the solution is time-consuming. To illustrate that the quantum speedup is practically achievable, we provide a concrete example where the coefficient matrix satisfies the specification and some useful information can be extracted from the output.

More specifically, we apply the quantum algorithm for the Toeplitz systems in Wiener class to solve the linear prediction problem of time series. Predicting the future value of a discrete time stochastic process with a set of past samples of the process is one of the most important problems in time series analysis. For linear prediction, we need to estimate the predicted value by a linear combination of the past samples.

To present the problem clearly, we first introduce some terminology used in signal processing, for details, see [52]. Let \(u(k)\) be a discrete-time stationary zero-mean complex-valued process. A finite impulse response (FIR) linear filter of order \(n\) is of the form

\[
\hat{u}(i) = \sum_{k=1}^{n} w_k u(i-k)
\]

where \(\hat{u}(i)\) is the filter output based on the data \(\{u(k)\}_{k=-n}^{0}\) and \(\{w_k\}_{k=1}^{n}\) are the impulse responses of
the filter. For the situation of linear prediction, the desired response is \( u(i) \), representing the actual sample of the input process at time \( i \). The difference between the desired response \( u(i) \) and the filter output \( \hat{u}(i) \) is called the estimation error. To estimate the desired response, we should choose the impulse responses \( \{w_k\}_{k=1}^{n} \) by making the estimation error as small as possible in some statistical sense.

According to Wiener filter theory, when the estimation error are optimized in the mean-square-error sense, the impulse responses \( \{w_k\}_{k=1}^{n} \) are given by the solution of the linear system

\[
Rw = r. \tag{32}
\]

Here,

\[
R = \begin{pmatrix}
    r(0) & r(1) & \cdots & r(n-1) \\
    r^*(1) & r(0) & \cdots & r(n-2) \\
    \vdots & \vdots & \ddots & \vdots \\
    r^*(n-1) & r^*(n-2) & \cdots & r(0)
\end{pmatrix}, \tag{33}
\]

\[
r = \begin{pmatrix}
    r^*(1) \\
    r^*(2) \\
    \vdots \\
    r^*(n)
\end{pmatrix}, \tag{34}
\]

where \( r(k) = E[u(j)u^*(j-k)] \) (\( E \) is the expectation operator) is the autocovariances of the input process for lag \( k \). This linear system is commonly called the Wiener-Hopf equations.

Note that the covariance matrix \( R \) is an \( n \times n \) Hermitian Toeplitz matrix and is almost always positive definite. For a discrete-time stationary process, if the autocovariances of the process are absolutely summable, i.e., \( \sum_{k=-\infty}^{\infty}|r(k)| < \infty \), then the function \( f(\lambda) \) that takes \( r(k) \) as its Fourier coefficients is called the power spectral density function of the process. The power spectral density functions ordinarily exist for the stochastic processes encountered in the physical sciences and engineering. Thus, \( R \) is a Toeplitz matrix generated by \( f(\lambda) \) and in the Wiener class. Moreover, the eigenvalues \( \lambda_k \) of a Hermitian Toeplitz matrix satisfy

\[
f_{\min} \leq \lambda_k \leq f_{\max}, \tag{35}\]

where \( f_{\min}, f_{\max} \) represent the smallest value and the largest value of the generating function respectively. When the spectral density function is bounded (that can be guaranteed by the continuity of \( f(\lambda) \) on \([0, 2\pi]\)), the condition number of \( R \) will also be bounded.

For the case of known statistics, i.e., the autocovariances of the stationary process are known, one can calculate the elements of the covariance matrix \( R \) by the “black box”. Alternatively, the covariance matrix \( R \) can be stored in the QRAM data structure as shown in lemma \([41]\) in advance. Similarly, the vector \( r \) can also be provided with two different data access models. Then, we can prepare quantum state \( |r \rangle \) with complexity (i) \( O(\sqrt{n}/|r|_2) \) in the black-box model \([41]\); (ii) \( O(\text{polylog}(n/e)) \) in the QRAM data structure model \([42]\). Note that \( |r|_2 \) will be a constant. By calling the quantum algorithm for solving the Toeplitz systems, we can get a quantum state \( |u \rangle \) in \( \text{polylog}(n) \) time.

\[
\text{DISCUSSION}
\]

There are some special cases of matrices with displacement structures that can be decomposed into linear combinations of displacement matrices with a few items. The simplest case is banded Toeplitz matrices, \( T_n \), of which the \( t_k = 0, |k| > q \) for a constant \( q \). The linear systems of banded Toeplitz matrices occur in many applications, involving the numerical solution of certain differential equations, the modeling of queueing problems, digital filtering, and so on.

Computing the Sylvester displacements of banded Toeplitz matrices, they can be decomposed as follows,

\[
T_n = \frac{1}{2}[t_0 I + t_1 Z_1^1 + \ldots + t_n Z_n^1 + t_{-1} Z_n^0 + \ldots + t_{-q} Z_n^{q-1}] + t_1 Z_{-1}^1 + \ldots + t_{q} Z_{-1}^{q-1} + (- t_{-q} Z_{-1}^q + \ldots + (- t_{-1}) Z_{-1}^{n-1}). \tag{36}
\]

Similarly to Eqs. \([13]\), \([15]\), and \([16]\), we can define two state preparation operators \( V_0, V_1 \) and a controlled unitary operator \( U \). Then, the unitaries \( V_0, V_1 \) and \( U \) can be implemented by using the generic state preparation algorithm described in \([53]\), which requires a gate cost of \( O(q) \). Also, the controlled unitary operator \( U \) can be implemented with \( O(\log n) \) primitive gates, as the quantum circuit of each \( Z_j \) only requires \( O(\log n) \) primitive gates. Thus, we can efficiently implement a \( (\chi_1/2; [\log(4q + 1)]; \epsilon) \)-block-encoding of \( T_n \), where \( \chi_0 = \sum_{j=-q}^{q} |t_j| \). It should be noted that the implementation scheme proposed here may provide more facilitation when constructing practical circuits of the block-encodings of such matrices since...
it does not require any oracle or QRAM. Combined with quantum algorithm for linear systems within the block-encoding framework, it can offer an exponential improvement in the dimension of the linear systems over classical methods.

In the black-box model, one can use the method of [23] to efficiently implement a block-encoding for a sparse matrix, where the scaling factor linearly depends on the sparsity (the maximum number of nonzero entries in any row or column) of the matrix. Since the structured matrices studied in this paper are not sparse, the scaling factor of this block-encoding will be \( O(n) \). Then, the quantum algorithm for structured linear systems based on such block-encoding can not provide speedup compared with the classical algorithm. We note that [10] provided a way to improve the scaling factor. However it require that the upper bound on the the \( p \)-norm of the rows of the matrix is known, which is not the circumstances considered in this paper.

In the QRAM data structure model, the method stated in [24, 26] also implements block-encodings based on the assumption that the powers of the elements of a matrix are stored in the quantum-accessible data structure beforehand. For a matrix with a displacement structure, the scaling factor produced by the method of [24, 26] can be in the same order of magnitude as that in this paper. However, it should be noted that constructing the data structure that stores some entries about the matrices may constitute the main restriction of this data access model. With respect to this, our method is more advantageous. More specifically, the method of [24, 26] would require a QRAM with data structure storing \( O(n^2) \) entries for the matrices with displacement structures. In our method, since we represent these structured matrices with \( O(n) \) entries of their displacements, a QRAM with data structure storing \( O(n) \) entries is required. Obviously, the data structure in our method can be constructed more rapidly and uses less memory space, so that our method will be more favorable when solving problems involving matrices with displacement structures.

As analyzed above, the origin of the advantages of our algorithm is the succinct parameterized representations of the matrices with displacement structures which is attributed to the Stein and Sylvester type LCU decompositions. There are also some intuitive methods to perform an LCU decomposition. Typically, we specify a set of unitaries that are easy to implement as the basis, and then calculate the decomposition coefficients by solving a linear system with \( n^2 \) unknown parameters. Alternatively, one can decompose the matrix into a sum of tensor products of Pauli operators, while the number of decomposition items will be considerably larger than ours. These decompositions are high-complexity to construct, and may not even fit to implement block-encodings. In general, it is not easy to find a desirable decomposition.

In particular, in this paper, by the proposed LCU decompositions, we make the implementation of block-encodings of matrices with displacement structure closely related to the task of preparing \( O(n) \)-dimensional quantum states. Since there are \( \Omega(n) \) parameters when defining an \( n \times n \) matrix with displacement structure, the query complexity in the black-box model and the memory cost in the QRAM data structure model are nearly optimal. It is an open question whether one can bypass the preparation of \( n \)-dimensional quantum states and implement block-encodings of the matrices with displacement structures with fewer resources.

It should be noted that although the proposed decompositions in this paper are also available for general dense matrices, the quantum algorithms based on these decompositions can not bring significant improvements over the known results due to the number of decomposition items of \( n^2 \). For the same reason, even if other unitaries are used as the basis, a universal LCU decomposition generally cannot give rise to quantum algorithms that surpass existing methods for all dense matrices. Nevertheless, it is still worth exploring specialized decomposition for some specific structured matrices to implement favorable block-encodings, which can result in a significant reduction in the complexity of the quantum linear system solver.

Following Tang’s breakthrough work [56], there is a large class of classical algorithms of which the running time is poly-logarithmic in the dimension. However, this type of speedup is only achievable for the matrices involved to be of low rank. It is not applicable to use the dequantization method to solve linear systems with displacement structures since these matrices are not low-rank in general. Besides, these dequantized algorithms have higher overhead than the quantum algorithms in practice due to their large polynomial dependence on the rank and the other parameters. For the computation of displacement structured matrices, it is an interesting open problem to explore classical algorithms with similar overheads to quantum algorithms.

VI. CONCLUSION

In this paper, we demonstrated that several important classes of matrices with displacement structures can be represented and treated similarly by decomposing them into linear combinations of displacement matrices. Based on the devised decompositions, we implemented block-encodings of these structured matrices in two different data access models, and introduced efficient quantum algorithms for solving the linear system with such structures. The obtained quantum linear systems solvers improved the known results and also motivated some new instances, see Table 1 for a brief summary. In particular, we provided a concrete example to illustrate these quantum algorithms can be used to solve the problems of practical interest with significant speedup.
Algorithm
The same order of magnitude
Asymptotic algorithm
Applicable to general
Non-asymptotic algorithm
Two different data access model
An oracle of preparing
Remark
Comparison of complexity
Finite difference discretization
pleteness, we restate here.
The proof of these results can be found in \[17\]. For completeness, we restate here.

| Coefficient Matrix | Algorithm | Remark | Comparison of complexity |
|--------------------|-----------|--------|--------------------------|
| Toeplitz matrices  | Wan et al. [22] | Asymptotic algorithm | Improve the dependence on the condition number and precision, when generating function is unknown |
| in the Wiener class | Theorem 2-based | Non-asymptotic algorithm | |
| Circulant matrices | Zhou and Wang [21] | An oracle of preparing | Improve the dependence on the condition number and precision, when using the same data access model |
| with bounded spectral norm | Theorem 2-based | Two different data access model | |
| Discretized Laplacian | Cao et al. [19] | Finite difference discretization of the Poisson equation | The same order of magnitude |
| (specific banded | Discussion-based | Applicable to general banded Toeplitz matrices | |
| Toeplitz matrices) | | | |
| Toeplitz/Hankel-like | No specialized quantum algorithm | | |
| matrices with small \( \chi \) | Corollary 1-based | | |

The presented methods can actually be extended to solve many important computational problems having ties to the structured matrices studied in this paper, such as structured least squares problems and computation of the structured matrices exponential. Also, we hope our work can inspire the study of matrices with displacement structures on near-term quantum devices, since we have shown that these matrices can be decomposed into some easy-to-implement unitaries. At last, there are many other structured matrices such as Cauchy, Bezout, Vandermonde, Loewner, and Pick matrices which are widely employed in various areas. Designing quantum algorithms for these structured matrices with a dramatic computational acceleration and a major memory-space decrease is worthy of further study.

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**Appendix A: Proof of the theorem 1**

In this appendix, we prove the conclusion in theorem 1. There are some well-known fundamental results, and the proof of these results can be found in \[17\]. For completeness, we restate here.

**Lemma 3.** \((17)\) For matrices \( A, B, M \in \mathbb{C}^{n \times n} \), and \( k \geq 1 \), we have

\[
M = A^k MB^k + \sum_{i=0}^{k-1} A^i \Delta_{A,B}(M) B^i. \quad (A1)
\]

**Proof.** It is trivial when \( k = 1 \). We assume the identity is true for \( k \). Then, multiplying the identity on the left by \( A \) and right by \( B \),

\[
AMB = A^{k+1} MB^{k+1} + \sum_{i=0}^{k-1} A^{i+1} \Delta_{A,B}(M) B^{i+1}
\]

\[
= A^{k+1} MB^{k+1} + \sum_{i=0}^{k} A^i \Delta_{A,B}(M) B^i - \Delta_{A,B}(M),
\]

\[
M = A^{k+1} MB^{k+1} + \sum_{i=0}^{k} A^i \Delta_{A,B}(M) B^i. \quad (A2)
\]

Thus, the identity is true for \( k + 1 \). According to mathematical induction, it is true for all natural numbers. \( \blacksquare \)

**Lemma 4.** \((17)\) If \( A \) is an \( a \)-potent matrix of order \( n \) and \( B \) is a \( b \)-potent matrix of order \( n \), i.e., \( A^n = aI \) and \( B^n = bI \), then

\[
M = \frac{1}{1 - ab} \sum_{i=0}^{n-1} A^i \Delta_{A,B}(M) B^i. \quad (A3)
\]

**Proof.** This conclusion is a direct inference of lemma 3 when \( k = n \) and \( A^n = aI \) and \( B^n = bI \). \( \blacksquare \)

**Lemma 5.** \((17)\) \( \nabla_{A,B} = A \Delta_{A^{-1},B} \), if the operator matrix \( A \) is nonsingular, and \( \nabla_{A,B} = -\Delta_{A,B^{-1}} B \), if the operator matrix \( B \) is nonsingular.

**Proof.** Note that if \( A \) is nonsingular, then \( AM - MB = A(M - A^{-1} MB) \); if \( B \) is nonsingular, then \( AM - MB = -(M - AM B^{-1}) B \). \( \blacksquare \)

**Definition 4** \((f\)-circulant Matrix\). The \( f\)-circulant matrix, \( Z_f(v) \), generated by a unit \( f\)-circulant matrix and...
a given vector $v = [v_0, \ldots, v_{n-1}]^T$ is defined as follows:

$$Z_f(v) = (v \ Z_f v \ Z_f^2 v \ \cdots \ Z_f^{n-1} v)$$

$$= \begin{pmatrix}
  v_0 & f v_1 & \cdots & f v_{n-1} \\
  v_1 & v_0 & \cdots & f v_2 \\
  \vdots & \vdots & \ddots & \vdots \\
  v_{n-1} & \cdots & v_1 & v_0
\end{pmatrix}. \quad (A4)$$

It turns out that a matrix $M$ can be expressed as the sums of the products of $f$-circular matrices and reversal matrix, by inverting the displacement operators.

**Theorem 3.** ([17]) If a matrix $M \in \mathbb{C}^{n \times n}$ satisfies $\mathcal{L}(M) = GHT^T$ where $G = [g_1 \ldots g_r], H = [h_1 \ldots h_r] \in \mathbb{C}^{n \times r}, e, f \text{ are constants},$ then $M$ can be expressed as:

i) $$M = \frac{1}{1 - ef} \sum_{j=1}^{r} Z_e(g_j) Z_f(J h_j)^T J,$$  \quad (A5)

where $\mathcal{L}(M) = \Delta z_e z_f(M)$, and $e f \neq 1$.

ii) $$M = \frac{1}{e - f} \sum_{j=1}^{r} Z_e(g_j) Z_f(J h_j),$$  \quad (A6)

where $\mathcal{L}(M) = \nabla z_e z_f(M)$, and $e \neq f$.

**Proof.** From lemma ([4]) let $A = Z_e, B = Z_f, e f \neq 1,$ then we have

$$M = \frac{1}{1 - ef} \sum_{i=0}^{n-1} Z_i^e \Delta z_e z_f(M) Z_i^f$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} \sum_{i=0}^{n-1} Z_i^e g_j h_i^T Z_i^f$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} (g_j h_j^T + Z e g_j h_j^T Z_f + Z^2 e g_j h_j^T Z_f^2 + \cdots + Z^{n-1} e g_j h_j^T Z_f^{n-1})$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} [g_j \ Z_e(g_j) \ Z^2 e g_j \ \cdots \ Z^{n-1} e g_j]$$

$$\cdot [h_j \ Z^f h_j \ Z^2 f h_j \ \cdots \ Z^f(h_j)^{n-1} h_j]^T$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} Z_e(g_j)$$

$$\cdot [J h_j \ J Z_f h_j \ J(Z_f)^2 h_j \ \cdots \ J(Z_f)^{n-1} h_j]^T$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} Z_e(g_j) [J \cdot Z_f(J h_j)]^T$$

$$= \frac{1}{1 - ef} \sum_{j=1}^{r} Z_e(g_j) Z_f(J h_j)^T J$$  \quad (A7)

by using the facts $J^2 = I$ and $Z_f = J Z_f^T J$.

Furthermore, according to the lemma ([4])

$$\Delta z^{T}_{1/n}, z_f[M] = Z^{T}_{1/n} \nabla z_e z_f[M],$$

where $Z_f^{-1} = Z_f^{T}_{1/n},$ then we can deduce the conclusion for Sylvester type. \quad \blacksquare

Now, we demonstrate how to decompose an $n \times n$ matrix into linear combinations of unitaries. For our purposes, we choose $(Z_1, Z_{-1})$ as the operator matrices. Note that

$$Z_1(g_j) = \begin{pmatrix}
  g_j^0 & g_j^{n-1} & \cdots & g_j^1 \\
  g_j^1 & g_j^0 & \cdots & g_j^{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  g_j^{n-1} & \cdots & g_j^1 & g_j^0
\end{pmatrix},$$

$$ Z_{-1}(h_j) = \begin{pmatrix}
  h_j^0 & -h_j^{n-1} & \cdots & -h_j^1 \\
  h_j^1 & h_j^0 & \cdots & -h_j^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  h_j^{n-1} & \cdots & h_j^1 & h_j^0
\end{pmatrix},$$

$$Z_{-1}(h_j) = h_j^0 Z_{-1}^0 + h_j^{n-1} Z_{-1}^{n-1} + \cdots + h_j^1 Z_{-1}^1,$$

where

$$g_j = (g_j^0, g_j^1, \ldots, g_j^{n-1})^T, \quad h_j = (h_j^0, h_j^1, \ldots, h_j^{n-1})^T.$$  \quad (A10)

Then, on the one hand
\[ M = \frac{1}{2} \sum_{j=1}^{r} Z_1(g_j) Z_{-1}(Jh_j) \]
\[ = \frac{1}{2} \sum_{j=1}^{r} \left( g_j^0 Z_0^0 + g_j Z_1^1 + \cdots + g_j^{n-1} Z_i^{n-1} \right) \left( h_j^{n-1} Z_{-1}^1 + h_j^{n-2} Z_{-1}^1 + \cdots + h_j Z_{-1}^1 \right) \]
\[ = \frac{1}{2} \sum_{j=1}^{r} \left( g_j^0 h_j^{n-1} Z_i^{0} Z_{-1}^1 + g_j^0 h_j^{n-2} Z_i^{1} Z_{-1}^1 + \cdots + g_j^0 h_j Z_i^{n-1} Z_{-1}^1 \right) \]
\[ + g_j^1 h_j^{n-1} Z_i^{0} Z_{-1}^1 + g_j^1 h_j^{n-2} Z_i^{1} Z_{-1}^1 + \cdots + g_j^1 h_j Z_i^{n-1} Z_{-1}^1 \]
\[ + \cdots + \]
\[ + g_j^{n-1} h_j^{n-1} Z_i^{0} Z_{-1}^1 + g_j^{n-1} h_j^{n-2} Z_i^{1} Z_{-1}^1 + \cdots + g_j^{n-1} h_j Z_i^{n-1} Z_{-1}^1 \]
\[ = \frac{1}{2} \sum_{j=1}^{r} \sum_{i,k=0}^{n-1} g_j^i h_j^k Z_i^{n-1-k} \]
\[ = \frac{1}{2} \sum_{i,k=0}^{n-1} \sum_{j=1}^{r} g_j^i h_j^k Z_i^{n-1-k}. \]

On the other hand, since
\[ \nabla_{z_i,z_{-1}}[M] = G H^T = \sum_{j=1}^{r} g_j h_j^T, \] (A12)

it is immediately verified that
\[ \hat{m}_{i,k} = \sum_{j=1}^{r} g_j^i h_j^k, \quad i, k = 0, 1, \ldots, n - 1, \] (A13)

where \( \hat{m}_{i,k} \) is the \( k \)-th element of the \( i \)-th row of matrix \( \nabla_{z_i,z_{-1}}[M] \).

Therefore,
\[ M = \frac{1}{2} \sum_{i,k=0}^{n-1} \hat{m}_{i,k} Z_i^{n-1-k}. \] (A14)

The decomposition for Stein type can be proved in the same way.

**Appendix B: Steerable black-box quantum state preparation**

The scenario for steerable black-box state preparation is as follows. For a vector \( \mathbf{x} = (x_0, x_1, \ldots, x_{n-1})^T \), we are provided a black box that returns target elements, i.e.,
\[ O_x|i\rangle|0\rangle \rightarrow |i\rangle|x_i\rangle. \]

The task is to prepare a quantum state that is \( \epsilon \)-close to
\[ |x\rangle = \frac{1}{\sqrt{||\mathbf{x}||_1}} \sum_{i=0}^{n-1} \sqrt{x_i}|i\rangle \]
with a adjustable bound on the success probability \( 1 - \delta^2 \).

The quantum algorithm for preparing this state contains two steps:

1. Prepare the initial state
   
   (i) Start with a uniform superposition state and perform the black box to have
   \[ \sum_{i=1}^{n-1} \frac{1}{\sqrt{n}} |i\rangle x_i \alpha. \] (B1)

   (ii) Add a qubit and perform controlled rotation to yield
   \[ \sum_{i=0}^{n-1} \frac{1}{\sqrt{n}} |i\rangle (\sqrt{x_i}|0\rangle + \sqrt{1 - x_i}|1\rangle), \] (B2)

   where \( |x_i|_{\text{max}} = \text{max}_i |\sqrt{x_i}| \) is a small constant.

   (iii) Uncompute the black box to obtain
   \[ \sum_{i=0}^{n-1} \frac{1}{\sqrt{n}} |i\rangle (\sqrt{x_i}|0\rangle + \sqrt{1 - x_i}|1\rangle). \] (B3)

Denote this state with \( |\psi\rangle_{1,2} \), and this step can be regarded as a unitary operator \( U_a \) that \( U_a|0\rangle|1\rangle = |\psi\rangle_{1,2} \). Note that \( |\psi\rangle_{1,2} \) can be rewritten as follow,
\[ |\psi\rangle_{1,2} = \sqrt{P_0} |\alpha\rangle_{1,2} + \sqrt{1 - P_0} |\beta\rangle_{1,2} \] (B4)

where
\[ P_0 = \frac{||\mathbf{x}||_1}{n |\mathbf{x}|_{\text{max}}^2}, \] (B5)

\[ |\alpha\rangle_{1,2} = \sum_{i=0}^{n-1} \sqrt{x_i} |i\rangle_1 |0\rangle, \] (B6)
2. Amplify the amplitude of getting $|\alpha\rangle$

In this step, we apply the fixed-point quantum search algorithm proposed in [44] to amplify the success probability with an adjustable bound. More specifically, define conditional phase shift operator

$$S_0^\phi = I_{1,2} + (e^{i\phi} - 1)I_1 \otimes |0\rangle\langle 0|_2,$$

and

$$S_1^\phi = I_{1,2} + (e^{i\phi} - 1)\exp(I_1 \otimes |0\rangle\langle 0|_2).$$

The algorithm performs the following sequence of generalized Grover operator

$$G(\phi_t, \varphi_l)G(\phi_{l-1}, \varphi_{l-1}) \cdots G(\phi_1, \varphi_1),$$

where $G(\phi_j, \varphi_j) = -U_aS_a^\phi U_a^\dagger S_a^\phi$. The condition on the phases $\{\varphi_j, \phi_j, 1 \leq j \leq l\}$ was indicated in [44]:

$$\varphi_j = -\arccot(\sqrt{1 - \gamma^2 tan(2\pi j / L)})$$

where $L = 2l + 1, \gamma = \frac{T_{1/l}(1/\delta)}{\sqrt{1 - P_0}}$, $\delta \in (0, 1)$ and $T_{1/l}(x)$ is the $l$-th Chebyshev polynomial of the first kind. After $l$ iterations, the final state, up to a global phase, will be

$$|\psi_1\rangle = \sqrt{P_L}|\alpha\rangle + \sqrt{1 - P_L}|\beta\rangle$$

where $P_L = 1 - \delta^2 T_{1/l}^2(1/\delta)/\sqrt{1 - P_0}$ is the success probability. It was shown that for a given $\delta$ and a known lower bound $P_{\min}$ of $P_0$, the following condition of $L$ holds:

$$L \geq \frac{\log(2/\delta)}{\sqrt{P_{\min}}}$$

can ensure $P_L \geq 1 - \delta^2$.

We now show that the $P_{\min}$ can be provided by using amplitude estimation. More specifically, for any $\epsilon_0 > 0$, amplitude estimation can approximate the probability $P_0$ up to additive error $P_0 + \epsilon_0$ with $O(1/(\epsilon_0 \sqrt{P_0}))$ uses of the standard Grover operator. Let the output of amplitude estimation to be $P_0'$, then $P_0 \geq \frac{P_0'}{1 + \epsilon_0}$. Thus, we can take $\frac{P_0'}{1 + \epsilon_0}$ as a low bound of $P_0$. Note that $\epsilon_0$ is the relative error of estimated $P_0$ and we only need the low bound of $P_0$, so we can set $\epsilon_0$ be a small constant like 1/2. Then the query complexity of this step is $O(\sqrt{n}/\sqrt{\|x\|_1})$.

Reviewing the cost of amplitude estimation and fixed-point amplitude amplification, we now analyze the complexity of this approach. Clearly, the query complexity is $O(\frac{\log(1/\delta)\sqrt{n}}{\sqrt{\|x\|_1}})$. The gate complexity of this approach is dominated by the gate complexity of fixed-point amplitude amplification which is given by the gate complexity of the generalized Grover operator multiplied by the number of iterations. The conditional phase shift operators $S_a^\phi$ and $S_a^\phi$ can be implemented by using $O(\log(n))$ elementary gates. The gate complexity of $U_a$ depend on the precision of controlled rotation. Note that performing the controlled rotation with error $\epsilon_r$ will generate a state that is $O(\sqrt{\epsilon_r})$-close to $|\psi_1\rangle$. Then, after performing the fixed-point amplitude amplification, we can get a state that is $O(\sqrt{\epsilon_r} + \sqrt{P_0})$-close to $|\psi_1\rangle$, since the error is also amplified by the fixed-point amplitude amplification. To obtain overall error $O(\epsilon_p)$, $\epsilon_r$ should be $O(\epsilon_p \sqrt{P_0}/\sqrt{n})$. Also it is shown that the controlled rotation can be performed with error $\epsilon_r$ using $O(\text{polylog}(\frac{1}{\epsilon_r}))$ elementary gates. Thus, the gate complexity of $U_a$ is $O(\text{polylog}(\frac{n}{\epsilon_p \sqrt{P_0}}))$. Putting these all together, we can obtain a state, up to a global phase, that is $\epsilon_p$-close to $|x\rangle$ with success probability at least $1 - \delta^2$, using $O(\sqrt{\log(1/\delta)}/\sqrt{\|x\|_1})$ queries of $O_x$ and $O(\sqrt{\log(1/\delta)}/\sqrt{\|x\|_1}) \text{polylog}(\frac{n}{\epsilon_p \sqrt{P_0}})$ elementary gates.

Appendix C: Proof of theorem 2

Now, we show how to implement the block-encoding of $T_n$, i.e., the quantum state preparation operators and the controlled unitary, in the two date access models introduced in Sec. II C.

1. Black-box model

In the black-box model, we are given the black box

$O_{T_n}(i)|k\rangle = |i\rangle|k\rangle|t_{i,k}\rangle, \quad i, k = 0, 1, \cdots, n - 1.$

Then, we can construct black box $O_1$ satisfying

$O_1|j\rangle = |j\rangle|t_j + t_{(-(n - j) \mod n)}\rangle, \quad j = 0, 1, \cdots, n - 1,$

by calling the black box $O_{T_n}$ twice to query the elements in the site $|j\rangle$, and $|0\rangle,(n - j) \mod n)$ and following an addition operation [57, 58]. More specifically, $O_1$ can be constructed as follows.

(1) Compute the index of site by using X-gates and quantum modular subtractor [57, 58]:

$$|j\rangle_{a_1} |0\rangle_{a_2} |0\rangle_{b_1} |0\rangle_{b_2} |0\rangle_{a_3} |0\rangle_{b_3} \rightarrow |j\rangle_{a_1} |0\rangle_{a_2} |0\rangle_{b_1} (n - j) \mod n |0\rangle_{b_2} |0\rangle_{a_3} |0\rangle_{b_3}. \quad \text{(C1)}$$

(2) Perform $O_{T_n}$ on registers $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ respectively:

$$|j\rangle_{a_1} |0\rangle_{a_2} |0\rangle_{b_1} (n - j) \mod n |0\rangle_{b_2} |0\rangle_{a_3} \rightarrow |j\rangle_{a_1} |0\rangle_{a_2} |0\rangle_{b_1} (n - j) \mod n |0\rangle_{b_2} |t_{j,a_3} |t_{-(n-j) \mod n}\rangle_{b_3}. \quad \text{(C2)}$$
(3) Perform quantum addition operation on registers \(\{a_3, b_3\}\) to yield
\[
|j\rangle_{a_1}|0\rangle_{a_2}|0\rangle_{b_1}[(n-j) \mod n]_{b_2}|t_j\rangle_{a_3}[t_j + (n-j) \mod n]_{b_3}.
\] (C3)

(4) Reverse the computation on registers \(\{a_3, b_2\}\):
\[
|j\rangle_{a_1}|0\rangle_{a_2}|0\rangle_{b_1}[(n-j) \mod n]_{b_2}|t_j + (n-j) \mod n]_{b_3}.
\] (C4)

The mapping on registers \(a_1\) and \(b_3\) is actually the \(O_1\).

Similarly, we can construct black box \(O_2\) satisfying
\[
O_2|j\rangle|0\rangle = |j\rangle|(t_j - (n-j) \mod n)\rangle, \quad j = 0, 1, \ldots, n-1.
\] (C5)

Since the gates required for constructing \(O_1\) and \(O_2\) are negligible compared to other subroutines of the quantum algorithm, we did not consider their complexity in this paper.

With these two black boxes \(O_1\) and \(O_2\), it is feasible to generate a controlled black box \(O_{1\&2}\) of the form
\[
|0\rangle\langle 0| \otimes O_1 + |1\rangle\langle 1| \otimes O_2.
\] (C6)

Note that the black box \(O_{1\&2}\) query in superposition acting as
\[
O_{1\&2} \frac{1}{\sqrt{2^n}(\sum_{j=0}^{n-1} |j\rangle\langle j| + \sum_{j=0}^{n-1} |1\rangle\langle j|)} = \frac{1}{\sqrt{2^n}(\sum_{j=0}^{2n-1} |j\rangle\langle t_j|)}.
\] (C7)

Thus, using this black box, we can approximately implement \(V_{(\mathcal{X}, \mathcal{Y})}\) by the steerable black-box quantum state preparation algorithm. Similarly, we can approximately implement \(V_{(\mathcal{Y}, \mathcal{X})}\) by constructing a black box \(O_{1\&2}^t\) that returns \(t_j^t\).

When implementing select \(U_{T_n}\), directly using the controlled circuit may take \(O(n \log n)\) elementary gates. To make the implementation more efficient, we take an idea similar to that in Ref. [21].

More specifically, note that
\[
Z_j = \sum_{a=0}^{n-1} |(a+j) \mod n\rangle\langle a|.
\] (C8)

\[
Z_j^t = \left( \sum_{b=0}^{n-1} |(b+j) \mod n\rangle\langle b| \right) \left( \sum_{b=0}^{n-1-j} |b\rangle\langle b| - \sum_{b=n-j}^{n-1} |b\rangle\langle b| \right).
\] (C9)

Thus, the action of select \(U_{T_n}\) on the basis states is
\[
0 \leq j \leq n-1, \quad 0 \leq e \leq n-1
\]
\[
n \leq j \leq 2n-1, \quad 0 \leq e \leq 2n-1 - j
\]
\[
n \leq j \leq 2n-1, \quad 2n-j \leq e \leq n-1.
\] (C10)

The mapping on registers \(a_1, a_2\) and \(c_3\) is
\[
U_{f_1} |j\rangle|e\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = (-1)^{|f_1(j,e)|} |j\rangle|e\rangle|0\rangle - |1\rangle|\sqrt{\frac{2}{2}}.
\] (C12)

On the other hand, using quantum modular adder [57, 58], which requires \(O(\log n)\) elementary gates, we can implement
\[
U_{add_1} |j\rangle|e\rangle = |j\rangle|(e + j) \mod n|\rangle.
\] (C13)

Therefore, by using \(U_{add_1}\) and \(U_{f_1}\), we can implement select \(U_{T_n}\) equivalently (Due to linearity, the implementation is available for any state). In summary, select \(U_{T_n}\) can be implemented in time \(O(polylog n)\).

Now, we analyse the error incurred due to imperfect state preparation. Let \(U_{\mathcal{V}}\) and \(U_{\mathcal{V}^*}\) be unitaries that perform the steerable black-box quantum state preparation with the black boxes \(O_{1\&2}\) and \(O^t_{1\&2}\). Define \(U_{\mathcal{V}}\) and \(U_{\mathcal{V}^*}\) as shown below,
\[
U_{\mathcal{V}} |0\rangle_{a_1}|0\rangle_{a_2} = \sqrt{1 - \delta^2} |V_{(\mathcal{X}, \mathcal{Y})}\rangle_{a_1}|0\rangle_{a_2} + \delta |\zeta\rangle_{a_1}|1\rangle_{a_2},
\] (C14)

\[
U_{\mathcal{V}^*} |0\rangle_{a_1}|0\rangle_{a_2} = \sqrt{1 - \delta^2} |V_{(\mathcal{Y}, \mathcal{X})}\rangle_{a_1}|0\rangle_{a_2} + \delta |\zeta\rangle_{a_1}|1\rangle_{a_2},
\] (C15)
where $|\zeta\rangle = \sum_{j=0}^{2n-1} \zeta_j |j\rangle$ is a quantum state similar in form to Eq. (17). Obviously, $U_Y$ and $U_{Y^\dagger}$ are $\epsilon_p$-approximation of $U_V$ and $U_{V^\dagger}$, respectively. To simplify the notation, we rewrite Toeplitz matrices as $T_n = \frac{1}{2} \sum_{j=0}^{2n-1} \tilde{t}_j U_j$ by using the symbol $U_j$ to represent $Z_{1j}^1$ and $Z_{1j}^2$ in the order of Eq. (13). Similarly, the operator defined by Eq. (16) can be rewritten as select $U = \sum_{j=0}^{2n-1} |j\rangle \langle j|_{a1} \otimes U_j$, where $U_j$ perform on the register $s$. Let $|0\rangle_a = |0\rangle_{a_1} |0\rangle_{a_2}$, we have

\[
\begin{align*}
(0|_a U^\dagger_V \otimes I_s)(&\text{select } U \otimes I_{a_2}) (U_V|_a \otimes I_s) \\
= (0|_{a_1} 0|_{a_2} U^\dagger_V \otimes I_s)(\sqrt{\epsilon^2 - \delta^2} \sum_{j=0}^{2n-1} \sqrt{\frac{\epsilon}{\chi}} \zeta_j |j\rangle_a_1 |0\rangle_{a_2} \otimes U_j + \delta \sum_{j=0}^{2n-1} \zeta_j |j\rangle_a_1 |1\rangle_{a_2} \otimes U_j) \\
= (\sqrt{1 - \delta^2} \sum_{j=0}^{2n-1} \sqrt{\frac{\epsilon}{\chi}} \zeta_j |j\rangle_a_1 |0\rangle_{a_2} + \delta \sum_{j=0}^{2n-1} \zeta_j |j\rangle_a_1 |1\rangle_{a_2}) (\sqrt{1 - \delta^2} \sum_{j=0}^{2n-1} \sqrt{\frac{\epsilon}{\chi}} \zeta_j |j\rangle_a_1 |0\rangle_{a_2} U_j + \delta \sum_{j=0}^{2n-1} \zeta_j |j\rangle_a_1 |1\rangle_{a_2} U_j) \\
= (1 - \delta^2) \sum_{j=0}^{2n-1} \frac{\epsilon}{\chi} U_j + \epsilon^2 \sum_{j=0}^{2n-1} \zeta_j^2 U_j 
\end{align*}
\]

Then, computing the first term on the right hand side of the inequality,

\[
||M - \frac{\chi}{2} (0|_a U^\dagger_V \otimes I_s)(&\text{select } U \otimes I_{a_2}) (U_V|_a \otimes I_s)\|| \\
= \frac{1}{2} \sum_{j=0}^{2n-1} \tilde{t}_j U_j - \frac{1}{2} (1 - \delta^2) \sum_{j=0}^{2n-1} \tilde{t}_j U_j - \frac{1}{2} \chi \delta^2 \sum_{j=0}^{2n-1} \zeta_j^2 U_j \\
= \frac{1}{2} \delta^2 \sum_{j=0}^{2n-1} \tilde{t}_j U_j - \frac{1}{2} \chi \delta^2 \sum_{j=0}^{2n-1} \zeta_j^2 U_j \\
\leq \chi \delta^2
\]

(C18)

In addition, since $U_Y$ and $U_{Y^\dagger}$ are $\epsilon_p$-approximation of $U_V$ and $U_{V^\dagger}$, respectively, the second term on the right hand side of the Eq. (C10) can be bounded by $\chi_p$. Let $\chi^2$ and $\chi_p$ not larger than $\epsilon/2$, we can get the result (i) of theorem 2.

2. QRAM data structure model

For QRAM data structure model, the quantum state preparation operators can be implemented using the method of [12]. More specifically,

Lemma 6. ([12]) Suppose that $x \in \mathbb{C}^{n \times 1}$ is stored in a QRAM data structure, i.e., the entry $x_i$ is stored in $i$-th leaf of a binary tree, the internal node of the tree stores the sum of the modulus of elements in the subtree rooted at it. Then, there is a quantum algorithm that can generate an $\epsilon_p$-approximation of $|x\rangle = \frac{1}{\sqrt{\lVert x \rVert_1}} \sum_{i=0}^{n-1} \sqrt{t_i}|i\rangle$ with gate complexity $O(\log(n/\epsilon_p))$.

Obviously, if $\{\tilde{t}_j\}_{j=0}^{n-1}$ and $\{\tilde{t}_j\}_{j=2n-1}$ are stored in such data structure respectively, there are two unitaries that generate the states:

\[
U_1|0\rangle = \frac{1}{\sqrt{\chi_1}} \sum_{j=0}^{n-1} \sqrt{\tilde{t}_j}|j\rangle, \quad U_2|0\rangle = \frac{1}{\sqrt{\chi_2}} \sum_{j=0}^{n-1} \sqrt{\tilde{t}_{j+n}}|j\rangle.
\]

(C19)

where $\chi_1 = \sum_{j=0}^{n-1} |\tilde{t}_j|$, $\chi_2 = \sum_{j=2n-1}^{n-1} |\tilde{t}_j|$.

Since $\chi_1$ and $\chi_2$ are known, which are stored in the root of the binary trees, we can prepare a state

\[
\sqrt{\chi_1} |0\rangle \otimes |0\rangle + \sqrt{\chi_2} |1\rangle \otimes |0\rangle.
\]

Then, performing a controlled unitary $|0\rangle|0\rangle \otimes U_1 + |1\rangle|1\rangle \otimes U_2$, we can get the state $\frac{1}{\sqrt{\chi_1}} \sum_{j=0}^{2n-1} \sqrt{\tilde{t}_j}|j\rangle$.

Thus, $V_{\lVert T_n \rVert}$ can be implemented in QARM data structure model with complexity $O(\log(n/\epsilon_p))$. Similarly, we can implement $V_{\lVert T_n \rVert^{-1}}$ with the same cost.

In addition, the select $U_{T_n}$ can be implemented in the same way as in the black-box model. Taking into account the amplification of the error, we can implement the block-encoding with complexity $O(\log(n/\epsilon_p))$ in QRAM data structure model. Besides, according to the constructed data structure, the memory cost in this data access model is $O(n)$. 


Appendix D: Proof of corollary \[D\]

1. Black-box model

For a Toeplitz-like matrix $T_L$, given a black box $O_{T_L}$ that query the $k$-th non-zero element of the $i$-th row of $T_L$,

$$O_{T_L} |i, k\rangle = |i, k\rangle |\tau_{i,k}\rangle,$$

the following map can be performed by querying the black box $O_{T_L}$ twice,

$$O_{T_L} |i, k\rangle = |i, k\rangle |\tau_{(i-1)\text{mod} n, k} - (-1)^{g(k)} \tau_{i, (k+1)\text{mod} n}\rangle.$$

$O_{T_L}$ actually returns the $k$-th non-zero element of the $i$-th row of the Sylvester displacements of Toeplitz-like matrices, i.e., $\tau_{i,k}$.

In addition, if a black box that computes the positions of the distinct elements (the element is different from the previous element on the same diagonal) on diagonals of the Toeplitz-like matrices is provided, we can construct a black box that computes the positions of non-zero elements of the Sylvester displacements of Toeplitz-like matrices, i.e.,

$$O_{T_L}^p |i, k\rangle = |i, f(i, k)\rangle,$$

where the function $f(i, k)$ gives the column index of the $k$-th nonzero element in row $i$ of $\nabla_z, z_{-1}[T_L]$.

When implementing the state preparation operators, if the steerable black-box quantum state preparation is directly called, the query complexity should be $O(n)$. To overcome this obstacle, we first prepare a uniform superposition state that only represents the position of the non-zero elements of $\nabla_z, z_{-1}[T_L]$. The specific process is as follows.

Prepare an initial state as:

$$\frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)|0\rangle_2|0\rangle_3.$$

(D1)

Apply Hadamard gates to Reg 2 and Reg 3 controlled by Reg 1:

$$\frac{1}{\sqrt{2}}|0\rangle_1 + |1\rangle_1||0\rangle_2|0\rangle_3$$

$$|0\rangle_1 \otimes I_2 \otimes H_3 + |1\rangle_1 \otimes H_2 \otimes I_3 \otimes H_2 \otimes H_3 \otimes H_3 \otimes H_3$$

$$\frac{1}{\sqrt{2}}|0\rangle_1 |0\rangle_2 |\sum_{k=0}^{n-1} |k\rangle + \sum_{i=0}^{2n} |i\rangle |\sum_{k=0}^{d-1} |k\rangle |\sum_{k=0}^{d-1} |k\rangle.$$

(D2)

Using a controlled-$O_{T_L}^p$, i.e., $|0\rangle_1 \otimes I_{2,3} + |1\rangle_1 |1\rangle_1 \otimes O_{T_L}^p$, we can prepare:

$$\frac{1}{\sqrt{2^n}}|0\rangle_1 |0\rangle_2 |\sum_{k=0}^{n-1} |k\rangle + \sum_{k=0}^{n-1} |i\rangle |\sum_{k=0}^{d-1} |f(i, k)\rangle.$$

(D3)

Add an ancillary qubit and perform a controlled rotation:

$$\frac{1}{\sqrt{2^n}}|0\rangle_1 |0\rangle_2 \sum_{k=0}^{n-1} |k\rangle |\sum_{k=0}^{d-1} |f(i, k)\rangle.$$

(D4)

Amplify the amplitude of $|0\rangle_4$. Since the amplitude is known, it can be amplified to exactly 1 by using Long’s amplitude amplification with zero theoretical failure rate \[59\]. The obtained state is denoted as $|\Phi_{\text{init}}\rangle$:

$$|\Phi_{\text{init}}\rangle = \frac{1}{\sqrt{n(d+1)}}|0\rangle_1 |0\rangle_2 \sum_{k=0}^{n-1} |k\rangle |\sum_{k=0}^{d-1} |f(i, k)\rangle,$$

(D5)

We run Long’s amplitude amplification again to get the quantum state $|\Phi_{\text{init}}\rangle$:

$$|\Phi_{\text{init}}\rangle = \frac{1}{\sqrt{n+(n-1)d}}|0\rangle_1 |0\rangle_2 \sum_{k=0}^{n-1} |k\rangle |\sum_{k=0}^{d-1} |f(i, k)\rangle.$$

(D6)

Note that the success probability of getting $|\Phi_{\text{init}}\rangle$ is $\frac{d+1}{2d} \geq \frac{17}{2}$, and the success probability of getting $|\Phi_{\text{init}}\rangle$ is $\frac{n}{n(d+1)} \geq \frac{17}{2}$, thus only a few iterations are required for the amplitude amplifications.

With the quantum state $|\Phi_{\text{init}}\rangle$ and the black box $O_{T_L}$, we can approximatively implement $V(\nabla[T_L])$ and $V(\nabla[T_L])$ by the steerable black-box quantum state preparation algorithm. The query complexity is $O\left(\sqrt{\frac{n \cdot \log(1/d)}{ \frac{d}{2d}}} \right)$ and $O\left(\sqrt{\frac{n \cdot \log(1/d)}{ \frac{d}{2d}}} \right)$ elementary gates are required.

To implement select $U_{T_L}$, we first observe its action on the basis states. Notice that

$$\text{select } U_{T_L} |i\rangle |k\rangle |\epsilon\rangle = \begin{cases} |i\rangle |k\rangle |(i + e - k - 1) \mod n\rangle \quad & \text{where } 0 \leq e \leq k, \\ -|i\rangle |k\rangle |(i + e - k - 1) \mod n\rangle \quad & \text{where } k < e \leq n - 1. \end{cases}$$

(D7)

Then, on the one hand, let

$$f_2(k, e) = \begin{cases} 0 & 0 \leq e \leq k, \\ 1 & k < e \leq n - 1. \end{cases}$$

(D8)

Similar to the calculation of $f_1$, we can construct a quantum circuit to implement

$$U_{f_2} |k\rangle |\epsilon\rangle |0\rangle - |1\rangle \sqrt{2},$$

(D9)
On the other hand, using quantum adders \[^{54}58\] which requires \(O(\log n)\) elementary gates, we can implement
\[
U_{\text{add2}}(i\langle k|e) = |i\rangle|k\rangle((i + e - k - 1) \mod n). \quad (D10)
\]
Therefore, select \(U_{T_L}\) can be implemented by \(U_{\text{add2}}\) and \(I \otimes U_{f_2}\) in time \(O(\text{polylog } n)\).

Based on the above conclusions and following the error analysis in \[^{11}\] we can infer the result (i) of corollary \[^{11}\]

2. QRAM data structure model

For QRAM data structure model, the quantum state preparation operators can be implemented as follows,

**Lemma 7.** Let \(T_L \in \mathbb{C}^{n \times n}, \|\tilde{\tau}_L\|_1\) be the 1-norm of the \(i\)-th row of \(\nabla Z_{i-1, z^-1}[T_L]\). Suppose that \(\nabla Z_{i-1, z^-1}[T_L]\) is stored in a QRAM data structure, more specifically, for the \(i\)-th row of \(\nabla Z_{i-1, z^-1}[T_L]\), the entry \(\tilde{\tau}_{i,k}\) is stored in \(k\)-th leaf of a binary tree, the internal node of the tree stores the sum of the modulus of elements in the subtree rooted at it, and an additional binary tree of which the \(i\)-th leaf stores \(\|\tilde{\tau}_L\|_1\). Then, there is a quantum algorithm that can perform the following maps with \(\epsilon_r\)-precision in time \(O(\text{polylog}(n/\epsilon_r))\):

\[
P : |i\rangle|0\rangle \mapsto \frac{\sum_{k=0}^{n-1} \sqrt{\tilde{\tau}_{i,k}}|i\rangle|k\rangle}{\sqrt{\|\tilde{\tau}_L\|_1}}, \quad (D11)
\]

\[
P' : |i\rangle|0\rangle \mapsto \frac{\sum_{k=0}^{n-1} \sqrt{\tilde{\tau}_{i,k}}|i\rangle|k\rangle}{\sqrt{\|\tilde{\tau}_L\|_1}}, \quad (D12)
\]

\[
Q : |0\rangle|k\rangle \mapsto \frac{\sum_{i=0}^{n-1} \sqrt{\tilde{\tau}_L}|i\rangle|k\rangle}{\sqrt{\|\tilde{\tau}_L\|}}. \quad (D13)
\]

This conclusion can be directly derived from the results in \[^{42}\]. Obviously,

\[
V_{(\nabla[T_L])}|0\rangle|0\rangle = PQ|0\rangle|0\rangle = \frac{1}{\sqrt{NT_L}} \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \sqrt{\tilde{\tau}_{i,k}}|i\rangle|k\rangle. \quad (D14)
\]

Similarly, we can efficiently implement \(V_{(\nabla[T_L])}\) with \(P'\) and \(Q\).

Note that the select \(U_{T_L}\) can be implemented in the same way as in \[^{11}\]. Taking into account the amplification of the error, we can implement the block-encoding with complexity \(O(\text{polylog}(n\sqrt{\|T_L\|}/\epsilon))\) in QRAM data structure model. Moreover, the memory cost in this data access model is easy to calculate as \(O(\text{dnlog } n)\).

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