ON THE CR ANALOGUE OF FRANKEL CONJECTURE AND A SMOOTH REPRESENTATIVE OF THE FIRST KOHN-ROSSI COHOMOLOGY GROUP

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Abstract. In this note, we affirm the Frankel conjecture in a closed, spherical, strictly pseudoconvex CR manifold with positive constant Tanaka-Webster scalar curvature. More precisely, we first give a criterion of pseudo-Einstein contact forms and then affirm the CR analogue Frankel conjecture via a smooth representative of the first Kohn-Rossi cohomology group which is served as a generalization of the Frankel conjecture for Sasakian manifolds.

1. Introduction

The well-known Riemann mapping theorem states that every simply connected domain $\Omega$ properly contained in $\mathbb{C}$ is biholomorphically equivalent to the open unit disc. In the paper of [CJ], Chern and Ji proved a generalization of the Riemann mapping theorem.

Proposition 1.1. If $\Omega$ is a bounded, simply connected, strictly convex domain in $\mathbb{C}^{n+1}$ and its connected smooth boundary $\partial \Omega$ has a spherical CR structure, then it is biholomorphic to the unit ball and $M = \partial \Omega$ is the standard CR $(2n + 1)$-sphere.

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It is also known from Burns and Shnider ([BS Proposition 1.5.]) that if $M$ is the compact spherical boundary of a Stein manifold, then either $M$ is the standard CR sphere or $\pi_1(M)$ is infinite.

In Kaehler geometry, it was conjectured by Frankel ([F]) that a closed Kaehler manifold with positive bisectional curvature is biholomorphic to the complex projective space. The Frankel conjecture was proved in later 1970s independently by Mori ([M]) and Siu-Yau ([SY]). However Sasakian geometry (that is, its pseudohermitian torsion tensor vanishes) is an odd dimensional counterpart of Kaehler geometry, it is natural to ask for CR analogue of Frankel conjecture for Sasakian manifolds. In fact, this is proved by He and Sun ([HS]):

**Proposition 1.2.** The universal covering of any closed Sasakian $(2n+1)$-manifold of positive pseudohermitian bisectional curvature must be CR equivalent to the standard CR sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$.

Note that in view of Proposition [1.2] it involves the existence problem of transversely Kaehler-Einstein metrics (pseudo-Einstein contact structures) with positive pseudohermtian bisectional curvature and Sasakian-Einstein metrics in a closed Sasakian manifold. From this inspiration, by studying the existence theorem of pseudo-Einstein contact structures in a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of vanishing first Chern class for $n \geq 2$ (Theorem [1.1] Theorem [1.4] Theorem [1.2] and Theorem [1.3]), we affirm the Frankel conjecture (Theorem [1.5] Theorem [1.6] and Corollary [1.2]) which is served as a generalization of the Frankel conjecture for Sasakian manifolds.

More precisely, based on Theorem [1.1] and Theorem [1.4] we can show that any closed, spherical, strictly pseudoconvex CR $(2n + 1)$-manifold $(M, J, \theta)$ of pseudo-Einstein contact form $\theta$ with the positive constant Tanaka-Webster scalar curvature $R$ must be Sasakian space form and manifolds always admit Riemannian metrics with positive Ricci curvature, so they must have finite fundamental group ([CC]). Then manifolds must be a finite quotient of a standard CR sphere ([1]). Therefore the universal covering of $M$ must be globally CR
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equivalent to a standard CR sphere (Theorem 4.3). Then we are able to affirm the Frankel conjecture when \( M \) is spherical and its Tanaka-Webster scalar curvature is positive constant.

A strictly pseudoconvex CR \((2n + 1)\)-manifold is called pseudo-Einstein if the pseudohermitian Ricci curvature tensor is function-proportional to its Levi metric

\[
R_{\alpha\beta} = \frac{R}{n} h_{\alpha\beta}
\]

for \( n \geq 2 \). It is equivalent to saying the following quantity is vanishing ([Lee], [H], [CKL])

\[
W_\alpha \triangleq \left( R_{\alpha\alpha} - i A_{\alpha\beta} \right) = 0.
\]

Then the pseudo-Einstein condition (1.1) can be replaced by (1.2) when \( n \geq 1 \). From this, we define ([H], [PH], [CCC]) the CR analogue of \( Q \)-curvature by

\[
Q := - \text{Re} \left[ (R_{\alpha\alpha} - i A_{\alpha\beta}) \right] = -\frac{1}{2} (W_{\alpha\beta} + W_{\alpha\beta}).
\]

J. Lee ([Lee]) showed an obstruction to the existence of a pseudo-Einstein contact form \( \theta \), which is the vanishing of first Chern class \( c_1(T_{1,0}M) \) for a closed, strictly pseudoconvex \((2n + 1)\)-manifold \((M, J, \theta)\) with \( n \geq 2 \). Then Lee conjectured that any closed strictly pseudoconvex CR \((2n + 1)\)-manifold of the vanishing first Chern class \( c_1(T_{1,0}M) \) for \( n \geq 2 \) admits a global pseudo-Einstein structure.

To set up the method, we recall J. J. Kohn’s Hodge theory for the \( \overline{\partial}_b \) complex ([K]). Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold and \( \eta \in \Omega^{0,1}(M) \) be a smooth \((0,1)\)-form on \( M \) with

\[
\overline{\partial}_b \eta = 0.
\]

Then there exists a smooth complex-valued function \( \varphi = u + iv \in C^\infty_\mathbb{C}(M) \) and a smooth \((0,1)\)-form \( \gamma \in \Omega^{0,1}(M) \) for \( \gamma = \gamma_\varphi \theta^\varphi \) such that

\[
(\eta - \overline{\partial}_b \varphi) = \gamma \in \ker(\Box_b),
\]
where $\Box_b = 2\left(\bar{\partial}_b \partial^*_b + \partial^*_b \bar{\partial}_b\right)$ is the Kohn-Rossi Laplacian. In this paper, we assume $c_1(T_{1,0}M) = 0$. Then there is a pure imaginary 1-form

$$\sigma = \sigma_\alpha \theta^\alpha - \sigma_\alpha \theta^\alpha + i\sigma_0 \theta$$

with

$$d\omega^\alpha_\alpha = d\sigma$$

for the pure imaginary Webster connection form $\omega^\alpha_\alpha$. As in Lemma 3.3, we choose the $(0,1)$-form $\eta \in \Omega^{0,1}(M)$

$$\eta = \sigma_\alpha \theta^\alpha.$$ 

Then $\sigma_\alpha \theta^\alpha$ is $\bar{\partial}_b$-closed and the Kohn-Rossi solution is

$$\varphi_\alpha = \sigma_\alpha - \gamma_\alpha.$$ 

The first Bochner type formula is proved as in [Lee]

$$\int_M Ric(\gamma, \gamma) \, d\mu + \frac{1}{(n-1)} \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu = 0.$$ 

Here $Ric(\gamma, \gamma) = R_{\alpha\beta} \gamma^\alpha_\alpha \gamma^\beta_\beta$.

We observe that the CR $Q$-curvature is vanishing when it is pseudo-Einstein. On the other hand, it is unknown whether there is any obstruction to the existence of a contact form $\theta$ of vanishing CR $Q$-curvature ([CCC], [CKS]). Our first goal is to justify the case where a contact form $\theta$ is pseudo-Einstein whenever its CR $Q$-curvature is CR pluriharmonic consisting of infinite dimensional kernel of the CR Paneitz operator $P_0$ in a closed strictly pseudoconvex CR $(2n + 1)$-manifold $(M, J, \theta)$ for $n \geq 2$. The following proposition is due to (1.7) that
Proposition 1.3. (Lee) Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$. Suppose that

\[ \int_M \text{Ric} (\gamma, \gamma) \, d\mu \geq 0. \]  

Then

(i) $\tilde{\theta} = e^{\frac{2\pi}{n+1} \theta}$ is a pseudo-Einstein contact form.

(ii) $\theta$ is also a pseudo-Einstein contact form if the CR $Q$-curvature of $\theta$ is CR-pluriharmonic (i.e. $Q^\perp = 0$).

In general, we hope to replace the nonnegative assumption (1.8) by more natural pseudohermitian curvatures (1.9) which are a combination of pseudohermitian Ricci curvature and torsion. We reference this pseudohermitian curvature quantity with our previous results as in [CC]. More precisely, we have the second key Bochner type formula (3.18)

\[ \int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu + \frac{1}{2(n-1)} \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha}|^2 \, d\mu + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu = 0. \]

Here $\text{Tor} (\gamma, \gamma) := i (A_{\alpha \beta} \gamma_{\alpha} \gamma_{\beta} - A_{\alpha \beta} \gamma_{\alpha} \gamma_{\beta}) = 2 \text{Re}(i (A_{\alpha \beta} \gamma_{\alpha} \gamma_{\beta}))$ for $\gamma = \gamma_{\alpha} \theta^\alpha$. Now, based on Theorem 4.1, we have

Theorem 1.1. Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$. Suppose that

\[ \int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu \geq 0. \]  

Then

(i) $\tilde{\theta} = e^{\frac{2\pi}{n+1} \theta}$ is a pseudo-Einstein contact form.

(ii) $\theta$ is also a pseudo-Einstein contact form if the CR $Q$-curvature of $\theta$ is CR-pluriharmonic.

Next, we would like to give a criterion of existence of pseudo-Einstein contact forms with the nonpositive curvature assumption

\[ \int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu \leq 0. \]
We observe that it follows from (4.3) that if the CR $Q$-curvature is CR-pluriharmonic, then

\begin{equation}
\int_M \text{Tor}'(\gamma, \gamma) \, d\mu \leq 0.
\end{equation}

In fact, as consequences of Theorem 1.4, the Bochner formulae (3.20), and (3.19), we have

**Theorem 1.2.** Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$ and its CR $Q$-curvature be CR-pluriharmonic. Assume that

\begin{equation}
0 \geq \int_M (\text{Ric} - \frac{1}{2} \text{Tor})(\gamma, \gamma) \, d\mu \geq \frac{1}{2} \int_M \text{Tor}'(\gamma, \gamma) \, d\mu.
\end{equation}

Then both $\tilde{\theta} = e^{\frac{2\pi i}{n+2}} \theta$ and $\theta$ are pseudo-Einstein contact forms.

**Theorem 1.3.** Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$ with $d\omega_\alpha = d\sigma$ and its CR $Q$-curvature be CR-pluriharmonic. Then $\tilde{\theta}$ is pseudo-Einstein if and only if

$$
\eta \in \ker (\Box_b)
$$

and

$$
\int_M \text{Tor}'(\gamma, \gamma) \, d\mu = 0.
$$

In general, it follows from the Bochner formula (3.19) and Theorem 1.2, we have

**Theorem 1.4.** Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$, $n \geq 2$ with $d\omega_\alpha = d\sigma$ for $\sigma = \sigma_\alpha \theta^\alpha - \sigma_\alpha \theta^\alpha + i\sigma_0 \theta$. Assume that $\eta = \sigma_\alpha \theta^\alpha$ is a smooth representative of the first Kohn-Rossi cohomology group $H_{\overline{\partial}_b}^{0,1}(M)$ (i.e. $\eta \in \ker (\Box_b)$). Then $\tilde{\theta}$ is pseudo-Einstein if and only if

\begin{equation}
\int_M \text{Tor}'(\gamma, \gamma) \, d\mu = 0.
\end{equation}

Here $\text{Tor}'(\gamma, \gamma) := i(A_{\alpha}^\beta \gamma_\alpha - A_{\alpha}^\beta \gamma_\alpha)$ and $\eta = \gamma$. 
Remark 1.1. 1. We observe that if the first Kohn-Rossi cohomology group $H^0_{\partial b}(M)$ is vanishing (for instance the pseudohermitian Ricci curvature is positive), then
\[
\gamma = 0.
\]
It follows from Lemma 3.4 that $\tilde{\theta} = e^{\frac{-u}{n+2}}\theta$ is a pseudo-Einstein contact form.

2. If $\theta$ is a pseudo-Einstein contact form, it follows from J. Lee ([Lee]) that $d\omega^\alpha_{\alpha} = d\sigma$ holds with
\[
\sigma = -i\frac{R}{n}\theta.
\]
Then, by choosing $\eta = \sigma u^{\beta\alpha} = 0 = \gamma$, the assumption as in Theorem 1.4 holds trivially. Conversely, suppose that
\[
d\omega^\alpha_{\alpha} = id(f\theta)
\]
for some smooth, real-valued function $f$, it follows from Theorem 1.4 that both $\tilde{\theta}$ and $\theta$ are pseudo-Einstein contact forms.

Now if $J$ is spherical and $\theta$ is pseudo-Einstein (1.1) for $n \geq 2$, it follows from (1.9) that
\[
R_{\beta\pi\lambda\sigma} = \frac{R}{n(n+2)}[h_{\beta\pi}h_{\lambda\sigma} + h_{\lambda\sigma}h_{\beta\pi}] + \frac{R}{n(n+1)(n+2)}[\delta^\alpha_{\beta}h_{\alpha\pi\lambda\sigma} + \delta^\alpha_{\lambda}h_{\beta\pi\sigma}].
\]

**Theorem 1.5.** Let $(M, J, \theta)$ be a closed, spherical, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T^{1,0}M) = 0$, $n \geq 2$. Suppose that (1.9) holds. Then If $\theta$ has the positive constant Tanaka-Webster scalar curvature and the CR-pluri harmonic $Q$-curvature, then the universal covering of $M$ is CR equivalent to the standard CR sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$.

As a consequence of Theorem 1.5 and Proposition 1.3, we have

**Corollary 1.1.** Let $(M, J, \theta)$ be a closed, spherical, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T^{1,0}M) = 0$, $n \geq 2$ with nonnegative pseudohermitian Ricci curvature. Then if $\theta$ has the positive constant Tanaka-Webster scalar curvature and the CR-pluriharmonic $Q$-curvature, then the universal covering of $M$ is CR equivalent to the standard CR sphere $(S^{2n+1}, \hat{J}, \hat{\theta})$. 
Theorem 1.6. Let \((M, J, \theta)\) be a closed, spherical, strictly pseudoconvex CR \((2n+1)\)-manifold of \(c_1(T^{1,0}M) = 0, n \geq 2\) with \(d\omega_\alpha = d\sigma, \sigma = \sigma_\alpha^\alpha \bar{\tau} - \sigma_\alpha^\alpha + i\sigma_0 \theta\). Assume that \(\eta = \sigma_\alpha^\alpha \bar{\tau}\) is a smooth representative of the first Kohn-Rossi cohomology group \(H_{\bar{\theta}_\sigma}^{0,1}(M)\) with condition (1.12). Then if \(\theta\) has the positive constant Tanaka-Webster scalar curvature, then the universal covering of \(M\) is CR equivalent to the standard CR sphere \((S^{2n+1}, \widehat{J}, \widehat{\theta})\).

Corollary 1.2. Let \((M, J, \theta)\) be a closed, spherical, strictly pseudoconvex CR \((2n+1)\)-manifold of \(c_1(T^{1,0}M) = 0, n \geq 2\). Suppose that (1.11) holds. Then the universal covering of \(M\) is CR equivalent to the standard CR sphere \((S^{2n+1}, \widehat{J}, \widehat{\theta})\) if \(\theta\) has the positive constant Tanaka-Webster scalar curvature and the CR-pluriharmonic \(Q\)-curvature.

For \(n = 1\), we refer to the authors’ previous work where one needs the positivity condition of the CR Paneitz operator in a closed spherical strictly pseudoconvex CR 3-manifold as in \([CKL]\).

We briefly describe the methods used in our proofs. In section 2, we introduce some basic materials in a pseudohermitian \((2n+1)\)-manifold. In section 3, we will derive some crucial results such as the CR Bochner-type formula. In section 4, we give the proofs of Theorem 1.1, Theorem 1.3, Theorem 1.2, Theorem 1.3, Theorem 1.5, and Theorem 1.6 and then affirm a partial answer of the CR Frankel conjecture in a closed, spherical, strictly pseudoconvex CR \((2n+1)\)-manifold.

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2. Preliminaries

In this section, we recall some ingredients needed to prove the main results in this paper. We first introduce some basic materials in a pseudohermitian \((2n+1)\)-manifold (see \([Lee]\)).
Let \((M, \xi)\) be a \((2n + 1)\)-dimensional, orientable, contact manifold with contact structure \(\xi\). A CR structure compatible with \(\xi\) is an endomorphism \(J : \xi \to \xi\) such that \(J^2 = -1\). We also assume that \(J\) satisfies the following integrability condition: If \(X\) and \(Y\) are in \(\xi\), then so are \([JX, Y] + [X, JY]\) and \(J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]\).

Let \(\{T, Z_\alpha, Z_{\bar{\alpha}}\}\) be a frame of \(T^* M \otimes \mathbb{C}\), where \(Z_\alpha\) is any local frame of \(T^* T^1, 0\), \(Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T^* T^0, 1\), and \(T\) is the characteristic vector field. Then \(\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}\), which is the coframe dual to \(\{T, Z_\alpha, Z_{\bar{\alpha}}\}\), satisfies

\[
d\theta = i h_{\alpha \beta} \theta^\alpha \wedge \theta^{\bar{\beta}}
\]

for some positive definite hermitian matrix of functions \((h_{\alpha \bar{\beta}})\). We also call such \(M\) a strictly pseudoconvex CR \((2n + 1)\)-manifold. The Levi form \(\langle \cdot, \cdot \rangle_{L_\theta}\) is the Hermitian form on \(T^* T^1, 0\) defined by

\[
\langle Z, W \rangle_{L_\theta} = -i \langle d\theta, Z \wedge \overline{W} \rangle.
\]

We can extend \(\langle \cdot, \cdot \rangle_{L_\theta}\) to \(T^* T^0, 1\) by defining \(\langle Z, \overline{W} \rangle_{L_\theta} = \overline{\langle Z, W \rangle_{L_\theta}}\) for all \(Z, W \in T^* T^1, 0\). The Levi form naturally induces a Hermitian form on the dual bundle of \(T^* T^1, 0\), denoted by \(\langle \cdot, \cdot \rangle_{L_\theta^*}\), and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over \(M\) with respect to the volume form \(d\mu = \theta \wedge (d\theta)^n\), we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of \((J, \theta)\) is the connection \(\nabla\) on \(T^* M \otimes \mathbb{C}\) (and extended to tensors) given in terms of a local frame \(Z_\alpha \in T^* T^1, 0\) by

\[
\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,
\]

where \(\omega_\alpha^\beta\) are the 1-forms uniquely determined by the following equations:
\[ d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \]
\[ 0 = \tau_\alpha \wedge \theta^\alpha, \]
\[ 0 = \omega_\alpha^\beta + \omega_\beta^\alpha. \]

We can write (by the Cartan lemma) \( \tau_\alpha = A_{\alpha \gamma} \theta^\gamma \) with \( A_{\alpha \gamma} = A_{\gamma \alpha} \). The curvature of Tanaka-Webster connection, expressed in terms of the coframe \( \{ \theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}} \} \), is

\[ \Pi_\beta^\alpha = \Pi_{\bar{\beta}}^{\bar{\alpha}} = d\omega^\alpha_\beta - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma, \]
\[ \Pi_0^\alpha = \Pi_\alpha^0 = \Pi_0^\bar{\beta} = \Pi_{\bar{\beta}}^0 = \Pi_0^0 = 0. \]

Webster showed that \( \Pi_\beta^\alpha \) can be written

\[ \Pi_\beta^\alpha = R^\alpha_{\beta} \rho \theta^\rho \wedge \theta^\beta + W^\alpha_{\beta} \rho \theta^\rho \wedge \theta - W^\alpha_{\beta \rho} \theta^\beta \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha \]

where the coefficients satisfy

\[ R_{\beta \alpha \rho \sigma} = R^\alpha_{\beta \alpha \rho \sigma} = R^\alpha_{\beta \alpha \rho \sigma} = R^\alpha_{\beta \alpha \rho \sigma}, \quad W_{\beta \gamma} = W_{\gamma \beta}. \]

Here \( R^\gamma_{\delta \alpha \beta} \) is the pseudohermitian curvature tensor, \( R_{\alpha \bar{\beta}} = R^\gamma_{\alpha \gamma \bar{\beta}} \) is the pseudohermitian Ricci curvature tensor and \( A_{\alpha \beta} \) is the pseudohermitian torsion tensor. Furthermore, we denote

\[ Tor(X, Y) := h_{\alpha \beta} T_{\alpha \beta}(X, Y) = i(A_{\alpha \beta} X^\beta Y^\alpha - A_{\alpha \beta} X^\alpha Y^\beta) \]

for any \( X = X^\alpha Z_\alpha, Y = Y^\alpha Z_\alpha \) in \( T_{1,0} \). We will denote components of covariant derivatives with indices preceded by comma; thus write \( A_{\alpha \beta, \gamma} \). The indices \( \{0, \alpha, \bar{\alpha}\} \) indicate derivatives with respect to \( \{T, Z_\alpha, Z_{\bar{\alpha}}\} \). For derivatives of a scalar function, we will often omit the comma, for instance, \( u_\alpha = Z_\alpha u, \ u_{\alpha \bar{\beta}} = Z_{\bar{\beta}} Z_\alpha u - \omega_\alpha^\gamma (Z_{\bar{\beta}}) Z_\gamma u \). For a smooth real-valued function \( u \), the subgradient \( \nabla_b \) is defined by \( \nabla_b u \in \xi \) and \( \langle Z, \nabla_b u \rangle_{L_b} = du(Z) \) for all vector fields \( Z \) tangent to the contact plane. Locally, we denote \( \nabla_b u = \sum_\alpha u_{\bar{\alpha}} Z_\alpha + u_\alpha Z_{\bar{\alpha}} \). We also
denote $u_0 = Tu$. We can use the connection to define the subhessian as the complex linear map $(\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \to T_{1,0} \oplus T_{0,1}$ by

$$(\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u.$$ 

In particular,

$$|\nabla_b u|^2 = 2 \sum_\alpha u_\alpha u_\bar{\alpha}, \quad |\nabla_b^2 u|^2 = 2 \sum_{\alpha,\beta} (u_{\alpha\beta} u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta}).$$

Also

$$\Delta_b u = Tr ((\nabla^H)^2 u) = \sum_\alpha (u_{\alpha\bar{\alpha}} + u_{\alpha\alpha}).$$

**Definition 2.1.** ([Lee], [CJ]) Let $(M, \theta)$ be a closed strictly pseudoconvex CR $(2n + 1)$-manifold with $n \geq 2$.

(i) We define the first Chern class $c_1(T_{1,0}M) \in H^2(M, \mathbb{R})$ for the holomorphic tangent bundle $T^{1,0}M$ by

$$c_1(T^{1,0}M) = \frac{i}{2\pi} [d\omega^\alpha_\alpha] = \frac{i}{2\pi} [R_{\alpha\beta} \theta^\alpha \wedge \theta^\beta + A_{\alpha\mu} \theta^\mu \wedge \theta - A_{\alpha\bar{\mu}} \theta^{\bar{\mu}} \wedge \theta].$$

(ii) We call a CR structure $J$ spherical if the Chern curvature tensor

$$(2.2) \quad C_{\beta\alpha\lambda\sigma} = R_{\beta\alpha\lambda\sigma} - \frac{1}{n+2} [R_{\beta\alpha\lambda}h_{\lambda\sigma} + R_{\lambda\sigma}h_{\beta\bar{\sigma}} + \delta^\alpha_\beta R_{\lambda\sigma} + \delta^\alpha_\lambda R_{\beta\sigma}] + \frac{R}{(n+1)(n+2)} [\delta^\alpha_\beta h_{\lambda\sigma} + \delta^\alpha_\lambda h_{\beta\sigma}]$$

vanishes identically.

**Remark 2.1.**

1. Note that $C_{\alpha\sigma\lambda\sigma} = 0$. Hence $C_{\beta\alpha\lambda\sigma}$ is always vanishing for $n = 1$.

2. We observe that the spherical structure is CR invariant and a closed spherical CR $(2n + 1)$-manifold $(M, J)$ is locally CR equivalent to $(S^{2n+1}, \tilde{J})$.

3. ([KT]) In general, a spherical CR structure on a $(2n + 1)$-manifold is a system of coordinate charts into $S^{2n+1}$ such that the overlap functions are restrictions of elements of
PU(n + 1, 1). Here PU(n + 1, 1) is the group of complex projective automorphisms of the unit ball in \( C^{n+1} \) and the holomorphic isometry group of the complex hyperbolic space CH\(^n\).

**Definition 2.2.** (i) Let \((M, \xi, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold. Define

\[
P \varphi = \sum_{\alpha=1}^{n}(\varphi_{\bar{\alpha} \beta} + inA_{\beta \alpha} \varphi^\alpha)\theta^\beta = (P_\beta \varphi)\theta^\beta, \quad \beta = 1, 2, \ldots, n
\]

which is an operator that characterizes CR-pluriharmonic functions ([Lee] for \( n = 1 \) and [GL] for \( n \geq 2 \)). Here \( P_\beta \varphi = \sum_{\alpha=1}^{n}(\varphi_{\bar{\alpha} \beta} + inA_{\beta \alpha} \varphi^\alpha) \) and \( \overline{P} \varphi = (\overline{P_\beta \varphi})\theta^\beta \), the conjugate of \( P \). Moreover, we define

\[
(2.3) \quad P_0 \varphi = \delta_b(P \varphi) + \overline{\delta_b(\overline{P \varphi})}
\]

which is the so-called CR Paneitz operator \( P_0 \). Here \( \delta_b \) is the divergence operator that takes (1, 0)-forms to functions by \( \delta_b(\sigma_\alpha \theta^\alpha) = \sigma_\alpha \). Hence \( P_0 \) is a real and symmetric operator and

\[
\int_M (P \varphi + \overline{P} \varphi, \delta_b \varphi)_{\nu_\theta} d\mu = -\int_M (P_0 \varphi) \varphi d\mu.
\]

(ii) We call the Paneitz operator \( P_0 \) with respect to \((J, \theta)\) essentially positive if there exists a constant \( \Lambda > 0 \) such that

\[
(2.4) \quad \int_M P_0 \varphi \cdot \varphi d\mu \geq \Lambda \int_M \varphi^2 d\mu.
\]

for all real smooth functions \( \varphi \in (\ker P_0)^\perp \) (i.e. perpendicular to the kernel of \( P_0 \) in the \( L^2 \) norm with respect to the volume form \( d\mu = \theta \wedge d\theta \)). We say that \( P_0 \) is nonnegative if

\[
\int_M P_0 \varphi \cdot \varphi d\mu \geq 0
\]

for all real smooth functions \( \varphi \).

**Remark 2.2.** 1. The space of kernel of the CR Paneitz operator \( P_0 \) is infinite dimensional, containing all CR-pluriharmonic functions. However, for a closed pseudohermitian \((2n + 1)\)-manifold \((M, \xi, \theta)\) with \( n \geq 2 \), it was shown ([GL]) that

\[
(2.5) \quad \ker P_\beta = \ker P_0.
\]
2. ([GL], [CC]) The CR Paneitz $P_0$ is always nonnegative for a closed pseudohermitian $(2n + 1)$-manifold $(M, \xi, \theta)$ with $n \geq 2$.

3. ([Lee]) A real-valued smooth function $u$ is said to be CR-pluriharmonic if, for any point $x \in M$, there is a real-valued smooth function $v$ such that

$$\overline{\theta}_b(u + iv) = 0.$$  \hfill (2.6)

### 3. The Bochner-Type Formulae

In this section, we first derive some essential lemmas. Recall that the transformation law of the connection under a change of pseudohermitian structure was computed in [Lee2, Sec. 5]. Let $\hat{\theta} = e^{2f\theta}$ be another pseudohermitian structure. Then we can define an admissible coframe by $\hat{\theta}^\alpha = e^f(\theta^\alpha + 2f^\alpha \theta)$. With respect to this local coframe, the connection 1-form and the pseudohermitian torsion are given by

$$\hat{\omega}^\alpha_\beta = \omega^\alpha_\beta + 2(f^\alpha_\beta \theta^\gamma - f^\gamma \theta^\alpha_\beta) + \delta^\alpha_\beta (f^\gamma \theta^\gamma - f^\gamma \theta^\gamma)$$

$$+ i(f^\alpha_\beta + f^\beta_\alpha + 4 \delta^\alpha_\beta f^\gamma f^\gamma)\theta, \hfill (3.1)$$

and

$$\hat{A}_{\alpha\beta} = e^{-2f}(A_{\alpha\beta} + 2if_{\alpha\beta} - 4if_{\alpha}f_{\beta}). \hfill (3.2)$$

respectively. Thus the Webster curvature transforms as

$$\hat{R} = e^{-2f}(R - 2(n + 1)\Delta_b f - 4n(n + 1)f^\gamma f^\gamma). \hfill (3.3)$$

Here covariant derivatives on the right side are taken with respect to the pseudohermitian structure $\theta$ and an admissible coframe $\theta^\alpha$. Note also that the dual frame of $\{\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^\gamma\}$ is given by $\{\hat{T}, \hat{Z}_\alpha, \hat{Z}_{\hat{\gamma}}\}$, where

$$\hat{T} = e^{-2f}(T + 2if^\gamma Z_{\hat{\gamma}} - 2if^\gamma Z_{\gamma}), \quad \hat{Z}_\alpha = e^{-f}Z_\alpha.$$  

Now we derive the following transformation property for the CR-pluriharmonic operator and CR Paneitz operator.
**Lemma 3.1.** Let $\theta$ and $\hat{\theta}$ be contact forms in a $(2n + 1)$-dimensional pseudohermitian manifold $(M, \xi)$. If $\hat{\theta} = e^{2f}\theta$, then we have

$$
\hat{R}_\alpha - in\hat{A}_{\alpha\beta,\beta} = e^{-3f}[R_\alpha - inA_{\alpha\beta,\beta} - 2(n + 2)P_\alpha f]
+ 2nc^{-2f}(\hat{R}_{\alpha\beta} - \frac{\hat{R}^n}{n}h_{\alpha\beta})f^\beta.
$$

**Proof.** By the contracted Bianchi identity, we have

$$
\frac{n-1}{n}(R_\alpha - inA_{\alpha\beta,\beta}) = (R_{\alpha\beta} - \frac{R^n}{n}h_{\alpha\beta})^\beta.
$$

Also, by [Lee2, P 172]

$$
(R_{\alpha\beta} - \frac{R^n}{n}h_{\alpha\beta}) - 2(n + 2)(f_{\alpha\beta} - \frac{1}{n}f_\gamma h_{\alpha\beta}) = \hat{R}_{\alpha\beta} - \frac{\hat{R}^n}{n}h_{\alpha\beta}.
$$

Following the same computation as the proof of Lemma 5.4 in [H], by using (3.1), (3.2), and (3.3), we compute

$$
\hat{R}_\alpha = \hat{Z}_\alpha \hat{R} = e^{-f}Z_\alpha e^{-2f}(R - 2(n + 1)\Delta_b f - 2n(n + 1)|\nabla_b f|^2)
= e^{-3f}[R_\alpha - 2W f_\alpha + 4(n + 1)(\Delta_b f + n|\nabla_b f|^2)f_\alpha
- 2(n + 1)(f_\gamma \gamma + f_\tau f_\alpha) - 4n(n + 1)(f_\gamma f_\gamma + f_\tau f_\alpha)],
$$

$$
i\hat{A}_{\alpha\beta,\gamma} = i(\hat{Z}_\beta \hat{A}_{\alpha\beta} - \hat{\omega}_\beta^\gamma(\hat{Z}_\tau)\hat{A}_{\beta\tau} - \hat{\omega}_\beta^\gamma(\hat{Z}_\tau)\hat{A}_{\tau\beta})
= ie^{-f}[(Z_\tau + 2f_\tau)\hat{A}_{\alpha\beta} + 2(\delta_{\alpha\gamma} \hat{A}_{\beta\tau} + \delta_{\beta\gamma} \hat{A}_{\alpha\tau})f^\delta]
= ie^{-f}(Z_\tau + 2f_\tau)e^{-2f}(A_{\alpha\beta} + 2if_{\alpha\beta} - 4if_\alpha f_\beta)
+ 2e^{-3f}[\delta_{\beta\gamma}(iA_{\alpha\tau} - 2f_{\alpha\tau} + 4f_\alpha f_\tau) + \delta_{\alpha\gamma}(iA_{\beta\tau} - 2f_{\beta\tau} + 4f_\beta f_\tau)]f^\delta
= e^{-3f}[iA_{\alpha\beta,\tau} - 2f_{\alpha\beta,\tau} + 4(f_\alpha \beta f_\beta + f_\alpha f_\beta)\tau]
+ 2e^{-3f}[\delta_{\beta\gamma}(iA_{\alpha\tau} - 2f_{\alpha\tau} + 4f_\alpha f_\tau) + \delta_{\alpha\gamma}(iA_{\beta\tau} - 2f_{\beta\tau} + 4f_\beta f_\tau)]f^\delta.
$$

Contracting the second equation with respect to the Levi metric $\hat{h}_{\alpha\beta} = h_{\alpha\beta}$ yields

$$
i\hat{A}_{\alpha\beta,\beta} = e^{-3f}[iA_{\alpha\beta,\beta} - 2f_{\alpha\beta}^\beta + 4(f_\alpha \beta f_\beta + f_\alpha f_\beta^\beta)
+ 2(n + 1)(iA_{\alpha\beta} - 2f_{\alpha\beta} + 4f_\alpha f_\beta)f^\beta].$$
Thus
\[
\hat{R}_\alpha - in\hat{A}_{\alpha\beta} = e^{-3f[R_\alpha - inA_{\alpha\beta}] - 2(n + 1)(f^\beta_\beta + f^\beta_\alpha) + 2nf_{\alpha\beta}
- 2Rf_\alpha - 2n(n + 1)iA_{\alpha\beta}f^\beta + 4(n + 1)(f^\beta_\beta + f^\beta_\alpha)f_\alpha
- 4n(n + 1)f^\beta_\alpha f_\beta - 4n(f^\beta_\beta + f^\beta_\alpha f_\alpha)].
\]

By using the commutation relations ([Lee2, Lemma 2.3])
\[
-2(n + 1)f^\beta_\alpha + 2nf_{\alpha\beta} = -2f^\beta_\alpha - 2nR_{\alpha\beta}f^\beta - 2inA_{\alpha\beta}f^\beta,
\]
and
\[
f_{\alpha\beta} - f^\beta_\alpha = ih_{\alpha\beta}f_0.
\]


Then (3.4) follows easily.

\[\square\]

Lemma 3.2. ([Lee]) Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\). Then there is a pure imaginary 1-form
\[
\sigma = \sigma_\alpha \theta^\alpha - \sigma_\alpha \theta^\alpha + i\sigma_0 \theta
\]
with \(d\omega^\alpha = d\sigma\) such that
\[
(3.6) \quad \sigma_{\beta,\alpha} = \sigma_{\alpha,\beta} = \sigma_{\alpha,\beta}
\]
and
\[
(3.7) \quad \begin{cases}
R_{\alpha\beta} = \sigma_{\beta,\alpha} + \sigma_{\alpha,\beta} - \sigma_0 h_{\alpha\beta}, \\
A_{\alpha\beta} = \sigma_{\alpha,0} + i\sigma_{0,\alpha} - A_{\alpha\beta} \sigma^\beta.
\end{cases}
\]
Lemma 3.3. If \((M, J, \theta)\) is a closed, strictly pseudoconvex CR \((2n+1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\). Then there exist \(u \in C^\infty(M)\) and \(\gamma = \gamma_{\overline{\alpha}\theta} \in \Omega^{0,1}(M)\) such that

\[(3.8) \quad W_\alpha = 2P_\alpha u + i n (A_{\alpha\beta} \gamma_\beta - \gamma_{\alpha,0})\]

and

\[(3.9) \quad \gamma_{\overline{\alpha},\overline{\beta}} = \gamma_{\overline{\beta},\overline{\alpha}} \quad \text{and} \quad \gamma_{\overline{\alpha},\alpha} = 0.\]

Proof. By choosing

\[\eta = \sigma_{\overline{\alpha}\theta},\]

as in (1.4), where \(\sigma\) is chosen from Lemma 3.2, then from (3.6)

\[\nabla_{\overline{\theta}} \eta = 0\]

and there exists

\(\varphi = u + iv \in C^\infty_c(M)\)

and

\(\gamma = \gamma_{\overline{\alpha}\theta} \in \Omega^{0,1}(M) \cap \ker(\Box_b)\)

such that

\[(3.10) \quad \sigma_\pi = \varphi_\pi + \gamma_\pi.\]

Note that

\[\Box_b \gamma = 0 \implies \nabla_{\overline{\theta}} \gamma = 0 = \nabla_{\overline{\theta}}' \gamma \implies \gamma_{\overline{\alpha},\overline{\beta}} = \gamma_{\overline{\beta},\overline{\alpha}} \quad \text{and} \quad \gamma_{\overline{\pi},\alpha} = 0\]

and

\[(3.11) \quad \sigma_\alpha = (\overline{\varphi})_\alpha + \gamma_\alpha.\]

Here \(\gamma_\alpha = \overline{\gamma_\pi}\). From the first equality in (3.7),

\[(3.12) \quad R = \sigma_{\pi,\mu} + \sigma_{\mu,\pi} - n \sigma_0.\]
Therefore
\[ \sigma_{\mu, \overline{\nu}} = (\overline{\varphi})_{\mu \overline{\nu}} + \gamma_{\mu, \overline{\nu}} \text{ by (3.11)} \]
\[ = (\overline{\varphi})_{\mu \overline{\nu}} \text{ by (3.9)} \]
\[ = (\overline{\varphi})_{\mu \overline{\nu}} + i n(\overline{\varphi}),_{0\alpha} \]
\[ = (\overline{\varphi})_{\mu \overline{\nu}} + i n \left[ (\overline{\varphi})_{\alpha 0} + A_{\alpha \beta}(\overline{\varphi})_{\beta} \right] \]
and
\[ \sigma_{\overline{\mu}, \nu} = \varphi_{\overline{\mu} \nu} \text{ by (3.10) and (3.9)} . \]

It follows that
\[ W_{\alpha} = (R_{\alpha} - i nA_{\alpha \beta, \overline{\beta}}) \]
\[ = \sigma_{\overline{\mu}, \nu} + \sigma_{\mu, \overline{\nu}} - i n\sigma_{\alpha, 0} + i nA_{\alpha \beta} \sigma_{\overline{\gamma}} \text{ by (3.7) and (3.12)} \]
\[ = \varphi_{\overline{\mu}, \nu} + (\overline{\varphi})_{\mu \overline{\nu}} + i nA_{\alpha \beta}(\overline{\varphi})_{\overline{\gamma}} - i n\gamma_{\alpha, 0} + i nA_{\alpha \beta} \left( \varphi_{\overline{\gamma}} + \gamma_{\overline{\gamma}} \right) . \]
\[ = 2 \left( u_{\overline{\mu}, \nu} + i nA_{\alpha \beta} u_{\overline{\beta}} \right) + i n \left( A_{\alpha \beta} \gamma_{\overline{\beta}} - \gamma_{\alpha, 0} \right) \]
\[ = 2 P_{\alpha} u + i n \left( A_{\alpha \beta} \gamma_{\overline{\beta}} - \gamma_{\alpha, 0} \right) . \]

□

We also recall Lemma 6.2 in [Lee] that states

**Lemma 3.4.** If \((M, J, \theta)\) is a closed, strictly pseudoconvex CR \((2n+1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\), then \(\tilde{\theta} = e^{\frac{2\theta}{n+2}}\theta\) is a pseudo-Einstein contact form if and only if

\[ (3.13) \quad \gamma_{\overline{\mu}, \beta} + \gamma_{\beta, \overline{\mu}} = 0 \]

for all \(\alpha, \beta \in I_n\).

**Remark 3.1.** Note that the conformal factor \(e^{\frac{2\theta}{n+2}}\) is different from Lee’s paper by \(\frac{1}{n+2}\) due to the different setting between (3.10) and [Lee (6.4)].

In this paper, we have another criterion for \(\tilde{\theta} = e^{\frac{2\theta}{n+2}}\theta\) to be a pseudo-Einstein contact form.
Lemma 3.5. Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\). Then \(\tilde{\theta} = e^{\frac{2u}{n+2}}\theta\) is a pseudo-Einstein contact form if and only if

\[(A_{\alpha\beta} \gamma^\beta - \gamma_{\alpha,0}) = 0.\]  

Proof. If \(\tilde{\theta} = e^{\frac{2u}{n+2}}\theta\) is a pseudo-Einstein contact form, that is

\[(3.15) \quad \tilde{\nabla}_{\alpha\beta} = \frac{\tilde{R}_{\alpha\beta}}{n} h_{\alpha\beta},\]

it follows from (3.4) that

\[\tilde{W}_\alpha = e^{-\frac{2u}{n+2}} \left[ W_\alpha - 2(n+2) P_\alpha \left( \frac{u}{n+2} \right) \right] + 2ne^{-\frac{2u}{n+2}} \left( \tilde{R}_{\alpha\beta} - \frac{\tilde{R}_{\alpha\beta}}{n} h_{\alpha\beta} \right) \left( \frac{u}{n+2} \right)_{,\beta},\]

and by (3.15)

\[(3.16) \quad W_\alpha = 2(n+2) P_\alpha \left( \frac{u}{n+2} \right) = 2P_\alpha u.\]

Then, by Lemma (3.3), we obtain

\[(3.17) \quad (A_{\alpha\beta} \gamma^\beta - \gamma_{\alpha,0}) = 0.\]

Conversely, assume that \((A_{\alpha\beta} \gamma^\beta - \gamma_{\alpha,0}) = 0\), then

\[0 = ni \int_M (A_{\alpha\beta} \gamma^\beta - \gamma_{\alpha,0}) \gamma_{\alpha} d\mu \]

\[= ni \int_M A_{\alpha\beta} \gamma^\beta \gamma_{\alpha} d\mu - \int_M (\gamma_{\alpha,\beta} - \gamma_{\alpha,\beta} - R_{\alpha\beta} \gamma^\beta) \gamma_{\alpha} d\mu \]

\[= ni \int_M A_{\alpha\beta} \gamma^\beta \gamma_{\alpha} d\mu + \int_M Ric(\gamma, \gamma) d\mu - \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu.\]

Hence

\[\int_M Ric(\gamma, \gamma) d\mu - \frac{n}{2} \int_M Tor(\gamma, \gamma) d\mu - \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu = 0.\]

It is proved as in [Lee] that

\[(n - 1) \int_M Ric(\gamma, \gamma) d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + (n - 1) \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu = 0.\]
Then

\[(n - 1) \int_M \text{Tor} (\gamma, \gamma) \, d\mu + 2 \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu = 0.\]

On the other hand, it follows from (3.26) that

\[
\sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta} + \gamma_{\beta, \gamma}|^2 d\mu
= 2 \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 d\mu + (n - 1) \int_M \text{Tor}(\gamma, \gamma) d\mu.
\]

Hence

\[
\sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta} + \gamma_{\beta, \gamma}|^2 d\mu = 0.
\]

It follows from (3.13) that \(\tilde{\theta} = e^{\frac{2u}{n+2}}\theta\) is a pseudo-Einstein contact form. \(\square\)

In particular, if the pseudohermitian is vanishing, it is straightforward to obtain

\[\gamma_{\alpha, 0} = 0.\]

Therefore, we recapture that \(\tilde{\theta} = e^{\frac{2u}{n+2}}\theta\) is a pseudo-Einstein contact form as following:

**Corollary 3.1.** Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of \(c_1(T_{1,0}M) = 0\) and vanishing torsion \(A_{\alpha\beta} = 0\) for \(n \geq 2\). Then \(\tilde{\theta} = e^{\frac{2u}{n+2}}\theta\) is a pseudo-Einstein contact form.

**Proof.** Since \(\gamma_{\alpha, 0} = 0\) and \(A_{\alpha\beta} = 0\), by the commutation relations ([Lee]) and (3.8),

\[
0 \leq n \int_M |\gamma_{\alpha, 0}|^2 \, d\mu
= n \int_M \gamma_{\alpha, 0} \gamma_{\alpha, 0} d\mu
= i \int_M \gamma_{\alpha, 0} \left( R_{00} - 2 u_{\beta\beta\alpha} \right) d\mu
= -i \int_M \gamma_{\alpha} \left( R_{\alpha, 0} - 2 u_{\beta\beta\alpha} \right) d\mu
= -i \int_M \gamma_{\alpha} \left( R_{\alpha, 0} - 2 u_{\beta\beta\alpha} \right) d\mu
= i \int_M \gamma_{\alpha, 0} \left( R_{00} - 2 u_{\beta\beta\alpha} \right) d\mu
= 0.
\]
Then
\[ \gamma_{\alpha,0} = 0 \]
and since \( A_{\alpha\beta} = 0 \)
\[ (A_{\alpha\beta}\gamma_{\beta} - \gamma_{\alpha,0}) = 0. \]
It follows from (3.14) that \( \tilde{\theta} = e^{\frac{2u}{n+2}}\theta \) is a pseudo-Einstein contact form. \( \square \)

Next we come out with the following key Bochner-type formulae for \( \gamma = \gamma_\pi \theta^\pi \).

**Theorem 3.1.** Let \( (M, J, \theta) \) be a closed, strictly pseudoconvex CR \((2n+1)\)-manifold of \( c_1(T_{1,0}M) = 0 \) for \( n \geq 2 \). Then

(i)
\[
(3.18) \quad \int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha,\beta}|^2 \, d\mu + \frac{1}{2(n-1)} \sum_{\alpha, \beta} \int_M |\gamma_{\pi,\beta} + \gamma_{\beta,\pi}|^2 \, d\mu = 0.
\]

(ii)
\[
(3.19) \quad \frac{n}{2} \int_M \text{Tor}' (\gamma, \gamma) \, d\mu - \int_M (Q + P_0 u) \, d\mu + \frac{n}{2(n-1)} \sum_{\alpha, \beta} \int_M |\gamma_{\pi,\beta} + \gamma_{\beta,\pi}|^2 \, d\mu = 0.
\]

(iii)
\[
(3.20) \quad \int_M (\text{Ric} - \frac{1}{2} \text{Tor} - \frac{1}{2} \text{Tor}') (\gamma, \gamma) \, d\mu + \frac{1}{n} \int_M (Q + P_0 u) \, d\mu + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha,\beta}|^2 \, d\mu = 0.
\]

Here \( \text{Tor} (\gamma, \gamma) := i(A_{\alpha\pi\beta}\gamma_{\alpha\beta} - A_{\alpha\beta\pi}\gamma_{\pi}) \) and \( \text{Tor}' (\gamma, \gamma) := i(A_{\alpha\pi\beta}\gamma_{\alpha\beta} - A_{\alpha\beta\pi}\gamma_{\pi}) \).

**Proof.** From the equality (3.8)
\[ W_\alpha = 2P_\alpha u + i_n (A_{\alpha\beta}\gamma_{\beta} - \gamma_{\alpha,0}) , \]
we are able to get
\[
(R_{\alpha\beta} - i_n A_{\alpha\beta,\beta}) \gamma_{\pi} = W_\alpha \gamma_{\pi} \]
\[
= 2 (u_\beta\gamma_{\beta\alpha} + i_n A_{\alpha\beta} u_{\beta}) \gamma_{\pi} + i_n (A_{\alpha\beta}\gamma_{\beta} - \gamma_{\alpha,0}) \gamma_{\pi} \]
\[
= 2 (u_\beta\gamma_{\beta\alpha} + i_n A_{\alpha\beta} u_{\beta}) \gamma_{\pi} + i_n A_{\alpha\beta}\gamma_{\beta} \gamma_{\pi} - (\gamma_{\alpha,\beta\beta} - \gamma_{\alpha,\beta\beta} - R_{\alpha\beta}\gamma_{\beta}) \gamma_{\pi}.
\]
Taking the integration over $M$ of both sides and its conjugation, we have, by the fact that $\gamma_{\alpha, \bar{\alpha}} = 0$,

\[ \int_M (Ric - \frac{n}{2} Tor - \frac{n}{2} Tor^\prime) (\gamma, \gamma) \, d\mu - \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \bar{\beta}}|^2 \, d\mu \]

\[ + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu - n \int_M Tor (d\bar{u}, \gamma) \, d\mu \]

\[ = 0. \]  

Here $Tor (d\bar{u}, \gamma) = i(A_{\alpha \bar{\beta}} u_\alpha \gamma_{\beta} - A_{\alpha \bar{\beta}} u_\beta \gamma_{\alpha}).$

On the other hand, it follows from equality (3.8) that

\[ (R_{\alpha \bar{\beta}} - \text{in} A_{\alpha \bar{\beta}}) u_\alpha u_\beta = W_\alpha u_\alpha = [2P_\alpha u + \text{in} (A_{\alpha \beta} \gamma_{\bar{\beta}} - \gamma_{\alpha, 0})] u_\alpha. \]

By the fact that $\gamma_{\alpha, \bar{\alpha}} = 0$ again, we see that

\[ \int_M \gamma_{\alpha, 0} u_{\alpha \bar{\alpha}} d\mu = -\int_M \gamma_\alpha u_{\alpha \bar{\alpha}} d\mu \]

\[ = -\int_M \gamma_\alpha (u_{\alpha \bar{\alpha}} - A_{\alpha \bar{\beta}} u_\beta) \, d\mu \]

\[ = \int_M A_{\alpha \bar{\beta}} u_\beta \gamma_\alpha d\mu. \]

It follows from (3.22) and (3.23) that

\[ 2 \int_M Q \, u \, d\mu + 2 \int_M (P_0 u) \, u d\mu \]

\[ = \text{in} \int_M [(A_{\alpha \beta} u_{\bar{\beta}} \gamma_{\alpha} - A_{\alpha \bar{\beta}} u_\beta \gamma_{\alpha}) - \text{conj}] \, d\mu \]

\[ = -2n \int_M Tor (d\bar{u}, \gamma) \, d\mu. \]

That is

\[ \int_M Q \, u \, d\mu + \int_M (P_0 u) \, u \, d\mu = -n \int_M Tor (d\bar{u}, \gamma) \, d\mu. \]

Thus by (3.21),

\[ \int_M (Ric - \frac{n}{2} Tor - \frac{n}{2} Tor^\prime) (\gamma, \gamma) \, d\mu - \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \bar{\beta}}|^2 \, d\mu \]

\[ + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 \, d\mu + \int_M (Q + P_0 u) u \, d\mu \]

\[ = 0. \]
On the other hand, since
\[
\sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\pi}|^2 d\mu
\]

\[= 2 \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\pi}|^2 d\mu + (\int_M \gamma_{\alpha,\pi} \gamma_{\beta,\pi} d\mu + \text{conj})
\]

and by commutation relations,
\[
\int_M \gamma_{\alpha,\pi} \gamma_{\beta,\pi} d\mu
\]

\[= - \int_M \gamma_{\alpha,\pi} \gamma_{\beta,\pi} d\mu
\]

\[= i(n-1) \int_M A_{\alpha,\pi} \gamma_{\alpha,\pi} d\mu.
\]

Hence
\[
\sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\pi}|^2 d\mu
\]

\[= 2 \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + (n-1) \int_M \text{Tor}(\gamma, \gamma) d\mu.
\]

This and (3.25) implies
\[
\int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) d\mu - \frac{n}{2} \int_M \text{Tor}' (\gamma, \gamma) d\mu - \frac{1}{2} \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\pi}|^2 d\mu
\]

\[+ \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + \int_M (Q + P_0 u) u d\mu
\]

\[= 0.
\]

Furthermore (Lee),
\[
(n-1) \int_M \text{Ric} (\gamma, \gamma) d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu + (n-1) \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu = 0.
\]

This and (3.26) implies
\[
\int_M \left( \text{Ric} - \frac{1}{2} \text{Tor} \right) (\gamma, \gamma) d\mu + \frac{1}{2(n-1)} \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\pi}|^2 d\mu + \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta}|^2 d\mu = 0.
\]

By combining (3.27) and (3.18),
\[
-\frac{n}{2} \int_M \text{Tor}' (\gamma, \gamma) d\mu - \frac{n}{2(n-1)} \sum_{\alpha,\beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\pi}|^2 d\mu + \int_M (Q + P_0 u) u d\mu = 0.
\]
By combining (3.25) and (3.28),
\[
\int_M (Ric - \frac{1}{2} Tor - \frac{1}{2} Tor') (\gamma, \gamma) d\mu + \frac{1}{n} \int_M (Q + P_0 u) u d\mu + \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta}|^2 d\mu = 0.
\]

□

Finally, we come out with another key Bochner-type formula for \(\eta = \sigma_\pi \theta^\pi\).

**Theorem 3.2.** Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of \(c_1(T_{1,0}M) = 0\) with
\[
d\omega^\alpha_\alpha = d\sigma
\]
for \(\sigma = \sigma_\pi \theta^\pi - \sigma_\pi \theta^\pi + i\sigma_0 \theta\). Then
\[
\int_M \sum_{\alpha < \beta} |\sigma_{\alpha, \beta} + \sigma_{\beta, \alpha}|^2 d\mu + \frac{1}{2} \int_M \sum_{\alpha = 1} (\sigma_{\pi, \alpha} + \sigma_{\alpha, \pi})^2 d\mu - \frac{1}{2n} \int_M (\sigma_{\pi, \alpha} + \sigma_{\alpha, \pi})^2 d\mu
\]
\[
= \int_M \sum_{\alpha < \beta} |\sigma_{\pi, \alpha} + \sigma_{\alpha, \pi}|^2 d\mu + \frac{2}{n} \int_M \sum_{\alpha < \beta} [Re(\sigma_{\alpha, \pi}) - Re(\sigma_{\beta, \pi})]^2 d\mu
\]
\[
= (n - 1) \left\{ \frac{1}{2m} \int_M R(\sigma_{\pi, \alpha} + \sigma_{\alpha, \pi}) d\mu - \frac{1}{2} \int_M Tor' (\eta, \eta) d\mu \right\}
\]
\[
= -\frac{(n-1)}{n} \left( \int_M Qu d\mu + \frac{n}{2} \int_M Tor' (\gamma, \gamma) d\mu \right).
\]

Here we have the Einstein summation convention. In particular,
\[
\int_M Qu d\mu + \frac{n}{2} \int_M Tor' (\gamma, \gamma) d\mu \leq 0.
\]

**Proof.** We compute by commutation relation \([\text{Lee}]\) that
\[
\int_M (\sigma_{\alpha, \pi})(\sigma_{\beta, \pi}) d\mu = -\int_M (\sigma_{\alpha, \pi})(\sigma_{\beta, \pi}) d\mu
\]
\[
= \int_M (\sigma_{\alpha, \beta})(\sigma_{\beta, \pi}) d\mu + i \int_M (\sigma_{\alpha, 0})(\sigma_{\pi}) d\mu + \int_M (R_{\beta, \pi, \rho, \sigma})(\sigma_{\beta, \pi}) d\mu.
\]

Then, for \(\sigma_{\beta, \pi} = \sigma_{\pi, \beta}\)
\[
2 \int_M (\sigma_{\alpha, \pi})(\sigma_{\beta, \pi}) d\mu = 2 \int_M |\sigma_{\alpha, \beta}|^2 d\mu + 2 \int_M Ric(\sigma, \sigma) d\mu
\]
\[
i \int_M [(\sigma_{\alpha, 0})(\sigma_{\pi}) - (\sigma_{\pi, 0})(\sigma_{\alpha})] d\mu.
\]
Now by (3.7),

\[
\int_M (\sigma_{\alpha,\bar{\beta}}) (\sigma_{\beta,\bar{\beta}}) d\mu = \int_M |\sigma_{\alpha,\beta}|^2 d\mu + \int_M Ric(\eta,\eta) d\mu - \frac{1}{2} \int_M Tor'(\eta,\eta) d\mu \\
- \frac{1}{2} \int_M Tor(\eta,\eta) d\mu + \frac{1}{2} \int_M [\sigma_{0,\alpha}) (\sigma_{\alpha}) + (\sigma_{0,\bar{\alpha}})(\sigma_{\bar{\alpha}})] d\mu.
\]

Note that it follows from ([Lee], [DT, page 307]) that

\[
(n-1) \int_M |\sigma_{\alpha,\beta}|^2 d\mu = - \int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu - (n-1) \int_M Ric(\eta,\eta) d\mu + n \int_M (\sigma_{\alpha,\bar{\beta}})(\sigma_{\beta,\bar{\beta}}) d\mu.
\]

Thus by (3.7) again,

\[
\int_M (\sigma_{\alpha,\bar{\beta}}) (\sigma_{\beta,\bar{\beta}}) d\mu = - \frac{1}{n-1} \int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu + \frac{n}{n-1} \int_M (\sigma_{\alpha,\bar{\beta}})(\sigma_{\beta,\bar{\beta}}) d\mu \\
- \frac{1}{2} \int_M Tor'(\eta,\eta) d\mu - \frac{1}{2} \int_M Tor(\eta,\eta) d\mu \\
- \frac{1}{2n} \int_M (\sigma_{\alpha,\bar{\alpha}} + \sigma_{\bar{\alpha},\alpha}) (\sigma_{\beta,\bar{\beta}} + \sigma_{\bar{\beta},\beta} - R) d\mu.
\]

Hence

\[
\frac{1}{n-1} \int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu - \frac{1}{n(n-1)} \int_M (\sigma_{\alpha,\alpha})(\sigma_{\beta,\beta}) d\mu \\
+ \frac{1}{2n} \int_M (\sigma_{\alpha,\alpha})^2 + (\sigma_{\alpha,\bar{\alpha}})^2 \ d\mu \\
= \frac{1}{2} \int_M R(\sigma_{\alpha,\alpha} + \sigma_{\alpha,\bar{\alpha}}) d\mu - \frac{1}{2} \int_M Tor'(\eta,\eta) d\mu - \frac{1}{2} \int_M Tor(\eta,\eta) d\mu.
\]

By the commutation relation again,

\[
\frac{1}{2} \int_M [(\sigma_{\alpha,\alpha})^2 + (\sigma_{\alpha,\bar{\alpha}})^2] d\mu \\
= \frac{1}{2} \int_M (\sigma_{\alpha,\bar{\alpha}})(\sigma_{\beta,\bar{\beta}} + \sigma_{\bar{\alpha},\beta})(\sigma_{\beta,\bar{\alpha}}) d\mu - \frac{n-1}{2} \int_M Tor(\eta,\eta) d\mu.
\]

This implies

\[
\frac{1}{n-1} \int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu - \frac{1}{n(n-1)} \int_M (\sigma_{\alpha,\bar{\beta}})(\sigma_{\beta,\bar{\beta}}) + \frac{1}{2} ((\sigma_{\alpha,\alpha})^2 + (\sigma_{\alpha,\bar{\alpha}})^2) \ d\mu \\
+ \frac{1}{2n(n-1)} \int_M (\sigma_{\alpha,\bar{\alpha}})(\sigma_{\beta,\bar{\beta}} + \sigma_{\bar{\alpha},\beta})(\sigma_{\beta,\bar{\alpha}}) d\mu \\
= \frac{1}{2n} \int_M R(\sigma_{\alpha,\alpha} + \sigma_{\alpha,\bar{\alpha}}) d\mu - \frac{1}{2} \int_M Tor'(\eta,\eta) d\mu.
\]
Then
\[
\frac{1}{n-1} \left\{ \int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu - \frac{1}{n} \int_M [(\sigma_{\alpha,\pi}) (\sigma_{\pi,\beta}) + \frac{1}{2} ((\sigma_{\pi,\alpha})^2 + (\sigma_{\alpha,\pi})^2)] d\mu \right. \\
+ \frac{1}{2} \int_M (\sigma_{\alpha,\pi} \sigma_{\beta,\pi} + \sigma_{\pi,\alpha} \sigma_{\beta,\pi}) d\mu \left. \right\} \\
= \frac{1}{2n} \int_M R(\sigma_{\alpha,\pi} + \sigma_{\alpha,\pi}) d\mu - \frac{1}{2} \int_M Tor'(\eta, \eta) d\mu.
\]

We compute the left-hand sides again
\[
\int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu = \int_M \sum_{\alpha=1} |\sigma_{\alpha,\pi}|^2 d\mu + \int_M \sum_{\alpha=1} |\sigma_{\alpha,\pi}|^2 d\mu + \int_M \sum_{\alpha>\beta} |\sigma_{\alpha,\beta}|^2 d\mu
\]
and
\[
\frac{1}{2} \int_M (\sigma_{\alpha,\pi} \sigma_{\beta,\pi} + \sigma_{\pi,\alpha} \sigma_{\beta,\pi}) d\mu
= \frac{1}{2} \int_M \sum_{\alpha=1} (\sigma_{\alpha,\pi} \sigma_{\alpha,\pi} + \sigma_{\alpha,\pi} \sigma_{\alpha,\pi}) d\mu + \int_M \sum_{\alpha<\beta} (\sigma_{\alpha,\beta} \sigma_{\beta,\pi} + \sigma_{\pi,\alpha} \sigma_{\beta,\pi}) d\mu.
\]
Hence
\[
\int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu + \frac{1}{2} \int_M (\sigma_{\alpha,\pi} \sigma_{\beta,\pi} + \sigma_{\pi,\alpha} \sigma_{\beta,\pi}) d\mu
= \frac{1}{2} \int_M \sum_{\alpha=1} (\sigma_{\alpha,\pi} \sigma_{\alpha,\pi} + \sigma_{\alpha,\pi} \sigma_{\alpha,\pi})^2 d\mu + \int_M \sum_{\alpha<\beta} |\sigma_{\alpha,\beta} + \sigma_{\beta,\pi}|^2 d\mu.
\]
On the other hand,
\[
-\frac{1}{n} \int_M [(\sigma_{\alpha,\pi}) (\sigma_{\pi,\beta}) + \frac{1}{2} ((\sigma_{\pi,\alpha})^2 + (\sigma_{\alpha,\pi})^2)] d\mu = -\frac{1}{2n} \int_M (\sigma_{\alpha,\pi} + \sigma_{\alpha,\pi})^2 d\mu.
\]
This implies
\[
\int_M \sum_{\alpha,\beta} |\sigma_{\alpha,\beta}|^2 d\mu - \frac{1}{n} \int_M [(\sigma_{\alpha,\pi}) (\sigma_{\pi,\beta}) + \frac{1}{2} ((\sigma_{\pi,\alpha})^2 + (\sigma_{\alpha,\pi})^2)] d\mu
+ \frac{1}{2} \int_M (\sigma_{\alpha,\pi} \sigma_{\beta,\pi} + \sigma_{\pi,\alpha} \sigma_{\beta,\pi}) d\mu
= \int_M \sum_{\alpha<\beta} |\sigma_{\alpha,\beta} + \sigma_{\beta,\pi}|^2 d\mu + \frac{1}{2} \int_M \sum_{\alpha=1} (\sigma_{\alpha,\pi} + \sigma_{\alpha,\pi})^2 d\mu - \frac{1}{2n} \int_M (\sigma_{\alpha,\pi} + \sigma_{\alpha,\pi})^2 d\mu.
\]
We observe that
\[
\frac{1}{2} \int_M \sum_{\alpha=1} (\sigma_{\pi,\alpha} + \sigma_{\alpha,\pi})^2 d\mu - \frac{1}{2n} \int_M (\sigma_{\pi,\alpha} + \sigma_{\alpha,\pi})^2 d\mu
= \frac{2}{n} \int_M \sum_{\alpha<\beta} [Re(\sigma_{\alpha,\pi}) - Re(\sigma_{\beta,\pi})]^2 d\mu.
\]
Then
\[\int_M \sum_{\alpha<\beta} |\sigma_{\alpha,\beta} + \sigma_{\beta,\alpha}|^2 d\mu + \frac{1}{2} \int_M \sum_{\alpha=1} (\sigma_{\alpha,\alpha} + \sigma_{\alpha,\overline{\alpha}})^2 d\mu - \frac{1}{2n} \int_M (\sigma_{\alpha,\alpha} + \sigma_{\alpha,\overline{\alpha}})^2 d\mu
\]
\[= \int_M \sum_{\alpha<\beta} |\sigma_{\alpha,\beta} + \sigma_{\beta,\alpha}|^2 d\mu + \frac{2}{n} \int_M \sum_{\alpha<\beta} [\text{Re}(\sigma_{\alpha,\beta}) - \text{Re}(\sigma_{\beta,\alpha})]^2 d\mu
\]
\[= (n-1) \left\{ \frac{1}{2n} \int_M R(\sigma_{\alpha,\alpha} + \sigma_{\alpha,\overline{\alpha}}) d\mu - \frac{1}{2} \int_M \text{Tor}'(\eta, \eta) d\mu \right\}.
\]

Finally, by Bianchi identity \cite{Lee} (2.13) and \(\gamma_{\alpha,\alpha} = 0\)

\[\frac{1}{2n} \int_M R(\sigma_{\alpha,\alpha} + \sigma_{\alpha,\overline{\alpha}}) d\mu - \frac{1}{2} \int_M \text{Tor}'(\eta, \eta) d\mu
\]
\[= -\frac{1}{2n} \int_M \left[ R_{\alpha} + inA_{\alpha,\beta} \right] d\mu + \text{conj}]
\[= -\frac{1}{2n} \int_M \left[ W_{\alpha} \right] d\mu - \frac{1}{2} \int_M \text{Tor}'(\gamma, \gamma) d\mu.
\]

\[\square\]

4. THE CR FRANKEL CONJECTURE

Now, with the help of the lemmas in the last section, we are able to give the proof of Theorem 1.1 and Theorem 1.5 and then affirm a partial answer of the CR Frankel conjecture in a closed, spherical, strictly pseudoconvex CR \((2n+1)\)-manifold.

**Lemma 4.1.** Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n+1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\). Then

(i)

\[Q_{\ker} = 0.
\]

(ii)

\[Q^\perp + P_0 u^\perp = 0,
\]

if \(\tilde{\theta} = e^{\frac{2\pi}{n+2}} \theta\) is a pseudo-Einstein contact form.
(iii) \[
\int_M \text{Tor}'(\gamma, \gamma) \, d\mu \leq 0,
\]
if the CR $Q$-curvature is CR-pluriharmonic. i.e. $Q^\perp = 0$. Here $Q = Q_{\ker} + Q^\perp$. $Q^\perp$ is in $(\ker P_0)^\perp$ which is perpendicular to the kernel of self-adjoint Paneitz operator $P_0$ in the $L^2$ norm with respect to the volume form $d\mu = \theta \wedge d\theta$.

**Proof.** (i) We observe that the equality (3.8) still holds if we replace $u$ by $(u + CQ_{\ker})$. It follows from the Bochner-type formula (3.27) that

\[
\int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu - \frac{1}{2} \int_M \text{Tor}'(\gamma, \gamma) \, d\mu - \frac{1}{2} \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha}|^2 d\mu \\
+ \sum \int_M |\gamma_{\alpha, \beta}|^2 d\mu + \int_M (P_0 u) \, d\mu + \int_M Q u d\mu + C \int_M (Q_{\ker})^2 d\mu \\
= 0.
\]

However, if $\int_M (Q_{\ker})^2 d\mu$ is not zero, this will lead to a contradiction by choosing the constant $C << -1$ or $C >> 1$. Then we are done.

(ii) If $\tilde{\theta} = e^{\frac{2u}{n+2}} \theta$ is a pseudo-Einstein contact form, it follows from Lemma 3.3 that

\[
(A_{\alpha \beta} \gamma_{\beta} - \gamma_{\alpha, 0}) = 0.
\]

Then from Lemma 3.3

\[
W_\alpha = 2P_\alpha u.
\]

Hence

\[
(W_\alpha)_{\beta \gamma} = 2(P_\alpha u)_{\beta \gamma}.
\]

Taking its conjugacy in both sides

\[
-Q = P_0 u
\]

and then from (4.1)

\[
Q^\perp + P_0 u^\perp = 0.
\]
(iii) It follows from (4.1) and (3.30) that

$$\int_M \text{Tor}'(\gamma, \gamma) \, d\mu \leq 0,$$

if the CR $Q$-curvature is CR-pluriharmonic.

It follows from Lemma 4.1 and the CR Bochner-type formulae (3.27), (3.18), one can derive the following:

**Theorem 4.1.** Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$. Assume that

$$\int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu \geq 0.$$

Then

(i) $\tilde{\theta} = e^{\frac{2u}{n+2}} \theta$ is a pseudo-Einstein contact form.

(ii) $\theta$ is also a pseudo-Einstein contact form if the CR $Q$-curvature of $\theta$ is CR-pluriharmonic (i.e. $Q^\perp = 0$).

**Proof.** It follows from (3.18) that

$$\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha} = 0.$$

Hence, by Lemma 3.4, $\tilde{\theta} = e^{\frac{2u}{n+2}} \theta$ is a pseudo-Einstein contact form. On the other hand, if the CR $Q$-curvature is CR-pluriharmonic (i.e. $Q^\perp = 0$), then by (4.2) and (4.1),

$$u^\perp = 0$$

for $u = u_{\ker} + u^\perp$. Thus by (3.16),

$$W_\alpha = 0.$$

Then $\theta$ is also a pseudo-Einstein contact form. $\Box$
Corollary 4.1. Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of \(c_1(T_{1,0}M) = 0\) for \(n \geq 2\). Assume that
\[
\int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu \geq 0.
\]

Then
\[
(4.4) \quad \int_M \text{Tor}' (\gamma, \gamma) \, d\mu = 0.
\]

Here \(\text{Tor}' (\gamma, \gamma) := i(A_{\alpha\beta,\beta} \gamma_{\alpha} - A_{\alpha\beta,\alpha} \gamma_{\beta}) = 2 \text{Re}(i(A_{\alpha\beta,\beta} \gamma_{\alpha}))\).

Proof. It follows from (3.18) and the assumption that
\[
\int_M (\text{Ric} - \frac{1}{2} \text{Tor}) (\gamma, \gamma) \, d\mu = 0
\]
and
\[
0 = \sum_{\alpha, \beta} \int_M |\gamma_{\alpha,\beta} + \gamma_{\beta,\alpha}|^2 \, d\mu = \sum_{\alpha, \beta} \int_M |\gamma_{\alpha,\beta}|^2 \, d\mu.
\]

Hence by (3.27), we have
\[
\int_M Q u \, d\mu + \int_M (P_0 u) \, d\mu - \frac{n}{2} \int_M \text{Tor}' (\gamma, \gamma) \, d\mu = 0.
\]

Finally, it follows from (4.2) that
\[
\int_M Q u \, d\mu + \int_M (P_0 u) \, d\mu = 0
\]
and then
\[
\int_M \text{Tor}' (\gamma, \gamma) \, d\mu = 0.
\]

Now by combining the Bochner formulae (3.19), we have
Theorem 4.2. Let \((M, J, \theta)\) be a closed, strictly pseudoconvex CR \((2n + 1)\)-manifold of 
\(c_1(T_{1,0}M) = 0\) with 
\[d\omega_\alpha^\alpha = d\sigma\]
for \(\sigma = \sigma_\alpha \theta^\alpha - \sigma_\alpha \theta^\alpha + i\sigma_0 \theta\). Assume that 
\[(4.5) \quad \eta = \sigma_\alpha \theta^\alpha \in \ker(\Box_b)\].

Then \(\tilde{\theta}\) is pseudo-Einstein if and only if 
\[(4.6) \quad \int_M \text{Tor}'(\gamma, \gamma) d\mu = 0.\]

In fact, \(\theta\) is also pseudo-Einstein.

Remark 4.1. We observe that \(\eta = \sigma_\alpha \theta^\alpha\) is a smooth representative of the first Kohn-Rossi cohomology group \(H^{0,1}_{\Box_b}(M)\) if and only if 
\[\sigma_{\alpha,\beta} = 0 \quad \text{and} \quad \sigma_{\alpha,\beta} = \sigma_{\beta,\alpha}.\]

However, \(\sigma_{\alpha,\beta} = \sigma_{\beta,\alpha}\) holds if \(d\omega_\alpha^\alpha = d\sigma\). If \(\sigma_{\alpha,\beta} = 0\), then 
\[\int_M \text{Tor}'(\eta, \eta) d\mu = i \int_M (A_{\alpha\beta} \sigma_{\alpha,\beta} - A_{\alpha\beta} \sigma_{\alpha,\beta}) d\mu = 0.\]

This is the case as in the proof of the following Corollary 4.2, in which we have 
\[\gamma_{\alpha,\beta} = \sigma_{\alpha,\beta} = 0.\]

Proof. It follows from \((2.6), (1.6),\) and \((4.5)\) that
\[\sigma_\pi = \gamma_\pi\]
and
\[u^\perp = 0.\]
Here we use the fact that the Kohn-Rossi cohomology group $H^0_{\partial_b}(M)$ has a unique smooth representative $\gamma \in \ker(\Box_b)$. This implies
\[
\int_M (Q + P_0u)ud\mu = \int_M (Q^\perp + P_0u^\perp)u^\perp d\mu = 0.
\]
It follows from Bochner formula (3.19) that
\[
(4.7) \quad \frac{n}{2} \int_M Tor' (\gamma, \gamma) d\mu + \frac{n}{2(n-1)} \sum_{\alpha, \beta} \int_M |\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha}|^2 d\mu = 0.
\]
Then
\[
\int_M Tor' (\eta, \eta) d\mu = 0
\]
if and only if
\[
\int_M \sum_{\alpha, \beta} |\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha}|^2 d\mu = 0.
\]
That is
\[
\gamma_{\alpha, \beta} + \gamma_{\beta, \alpha} = 0.
\]
All these imply that $\tilde{\theta} = e^{2u_n} \theta$ is a pseudo-Einstein contact form as well as $\theta$ due to $u^\perp = 0$. \qed

**Corollary 4.2.** Let $(M, J, \theta)$ be a closed, strictly pseudoconvex CR $(2n + 1)$-manifold of $c_1(T_{1,0}M) = 0$ for $n \geq 2$ with
\[
\int_M (\text{Ric} - \frac{1}{2}Tor - \frac{1}{2}Tor') (\gamma, \gamma) d\mu \geq 0.
\]
Assume that
\[
Q^\perp = 0.
\]
Then both $\tilde{\theta} = e^{2u_n} \theta$ and $\theta$ are pseudo-Einstein contact forms.

**Proof.** It follows from the Bochner formula (3.20) and assumptions that
\[
u^\perp = 0
\]
and

\[ \sigma_{\alpha,\beta} = \gamma_{\alpha,\beta} = 0. \]

Here we use the fact that for \( n \geq 2 \)

\[ \int_M (P_0 u^\perp)u^\perp d\mu \geq 0. \]

Hence

\[ \eta = \sigma \theta^\perp \in \ker(\square_b) \]

and

\[ \int_M Tr'(\eta, \eta) d\mu = 0. \]

The results of the Corollary follow from Theorem 4.2 and (3.19) easily. \( \square \)

Similarly, we have

**Corollary 4.3.** Let \((M, J, \theta)\) be a closed, strictly pseudoconvex \( CR (2n + 1) \)-manifold of \( c_1(T_{1,0}M) = 0 \) for \( n \geq 2 \) with \( d\omega^\alpha = d\sigma \) and its \( CR Q \)-curvature be \( CR \)-pluriharmonic. Then \( \tilde{\theta} \) is pseudo-Einstein if and only if (4.5) and (4.6) hold.

Then Proposition 1.3, Theorem 1.1, Theorem 1.4, Theorem 1.2, and Theorem 1.3 follow from Theorem 4.1, Theorem 4.2, Corollary 4.2, and Corollary 4.3.

Before we affirm the partial result of the CR frankel conjecture, we derive the key Bochner type formula in a closed, spherical pseudo-Einstein, strictly pseudoconvex \( CR (2n + 1) \)-manifold.

**Lemma 4.2.** Let \((M, J, \theta)\) be a closed, spherical, strictly pseudoconvex \( CR (2n + 1) \)-manifold with the pseudo-Einstein contact form \( \theta \) for \( n \geq 2 \). Then

\[ 0 = \frac{n+2}{n+1} \int_M k \sum_{\alpha,\gamma} |A_{\alpha\gamma}|^2 d\mu + \int_M \sum_{\alpha,\gamma,\sigma} |A_{\alpha\gamma,\sigma}|^2 d\mu + \frac{1}{n-1} [\int_M \sum_{\alpha,\gamma,\beta} |A_{\alpha\gamma,\beta}|^2 d\mu - n \int_M \sum_{\alpha} A_{\alpha\overline{\beta},\overline{\beta}} A_{\alpha\gamma} d\mu]. \]

Here \( k := \frac{R}{n} \).
**Proof.** Since $\theta$ is pseudo-Einstein, it follows that

$$R_{\alpha\beta} = \frac{R}{n} h_{\alpha\beta} := k h_{\alpha\beta}$$

Here $k := \frac{R}{n}$. Since $J$ is spherical, it follows from (2.2) and (4.8) that

$$R_{\beta\alpha\lambda\sigma} = k_{n} h_{\beta\alpha} + \frac{nk}{(n+1)(n+2)} \left[ \delta_{\beta}^{\alpha} h_{\lambda\sigma} + \delta_{\lambda}^{\alpha} h_{\beta\sigma} \right]$$

Again by [Lee, (2.15)],

$$A_{\alpha\rho,\beta\gamma} = ih_{\rho\beta} A_{\alpha\rho,0} + R_{\alpha}^{\kappa} A_{\kappa\rho}^{\beta\gamma} + R_{\rho}^{\kappa} A_{\kappa\alpha}^{\beta\gamma} + A_{\alpha\rho,\gamma\beta}.$$
Hence
\[ i(n - 1)A_{\alpha \gamma,0} + R_{\alpha \sigma}A_{\gamma}^\sigma - R_{\alpha}^\rho \gamma A_{\rho}^\sigma = A_{\alpha \beta,\overline{\sigma} \gamma} - A_{\alpha \gamma,\overline{\beta}} \]
and thus
\[ i(n - 1)A_{\alpha \gamma,0} + kA_{\alpha \gamma} - R_{\alpha}^\rho \gamma A_{\rho}^\sigma = A_{\alpha \beta,\overline{\sigma} \gamma} - A_{\alpha \gamma,\overline{\beta}}. \]
On the other hand,
\[
R_{\alpha}^\rho \gamma A_{\rho}^\sigma = \frac{k}{n+2}[h_{\alpha \overline{\sigma}}h_{\gamma \sigma} + h_{\gamma \overline{\sigma}}h_{\alpha \sigma}]A_{\rho}^\sigma
+ \frac{k}{(n+1)(n+2)}[\delta_{\rho}^\alpha h_{\gamma \sigma} + \delta_{\sigma}^\rho h_{\alpha \sigma}]A_{\rho}^\sigma
= \frac{2k}{n+1}A_{\alpha \gamma}.
\]
All these imply
\[ i(n - 1)A_{\alpha \gamma,0} + \frac{n-1}{n+1}kA_{\alpha \gamma} = A_{\alpha \beta,\overline{\sigma} \gamma} - A_{\alpha \gamma,\overline{\beta}} \]
for \( n \geq 2 \). Thus (4.11) follows. Next, from (4.10) and (4.11), we obtain
\[
A_{\alpha \gamma,\sigma}^\sigma = iA_{\alpha \gamma,0} + 2kA_{\alpha \gamma} + A_{\alpha \gamma,\overline{\sigma}}^\sigma
= \frac{n+2}{n+1}kA_{\alpha \gamma} + \frac{n}{n-1}(A_{\alpha \beta,\overline{\sigma} \gamma} - A_{\alpha \gamma,\overline{\beta}}) + A_{\alpha \gamma,\overline{\sigma}}^\sigma.
\]
We integrate both sides with \( A_{\alpha \gamma} \) to get
\[ 0 = \frac{n+2}{n+1}\int_M k\sum_{\alpha \gamma}|A_{\alpha \gamma}|^2 d\mu + \int_M \sum_{\alpha \gamma,\sigma}|A_{\alpha \gamma,\sigma}|^2 d\mu
+ \frac{1}{n-1}[\int_M \sum_{\alpha \gamma,\beta}|A_{\alpha \gamma,\overline{\beta}}|^2 d\mu - n\int_M \sum_{\alpha} A_{\overline{\alpha} \sigma} A_{\alpha \gamma}^\sigma d\mu]. \]

\[ \square \]

**Theorem 4.3.** Let \((M, J, \theta)\) be a closed, spherical, strictly pseudoconvex CR \((2n + 1)\)-manifold with pseudo-Einstein contact form \( \theta \) of positive constant Tanaka-Webster scalar curvature. Then the universal covering of \( M \) must be globally CR equivalent to a standard CR sphere.

**Proof.** Since
\[ R_{\alpha \overline{\sigma} \beta \gamma} = R_{\alpha} - i(n - 1)A_{\alpha \beta,\overline{\gamma}}, \]
if $R_{\alpha \beta} = \frac{R}{n} h_{\alpha \beta}$ and $R$ is constant, then

$$A_{\alpha \gamma, \gamma} = 0.$$ 

It follows from Lemma 4.2 that if $k > 0$

$$\frac{n + 2}{n + 1} \int_M \sum_{\alpha, \gamma} |A_{\alpha \gamma}|^2 d\mu + \int_M \sum_{\alpha, \gamma, \sigma} |A_{\alpha \gamma, \sigma}|^2 d\mu + \frac{1}{n - 1} \int_M \sum_{\alpha, \gamma, \beta} |A_{\alpha \gamma, \beta}|^2 d\mu = 0$$

and

$$A_{\alpha \gamma} = 0.$$ 

Moreover, it follows from (4.9) that

$$R_{\beta \pi \lambda \sigma} = \frac{R}{n(n + 1)} [h_{\beta \pi} h_{\lambda \sigma} + h_{\lambda \pi} h_{\beta \sigma}].$$

Hence $(M, \theta)$ is a closed, Spherical, Sasakian CR $(2n + 1)$-manifold of positive constant pseudohermitian bisectional curvature. Hence manifolds always admit Riemannian metrics with positive Ricci curvature, so they must have finite fundamental group ([CC]). It follows from ([T]) that the universal covering of $M$ is CR equivalent to a CR standard Sphere $S^{2n+1}$ in $\mathbb{C}^{n+1}$.

Then the proof of Theorem 1.5 is completed.

**References**

[BS] D. Burns, Jr. and S. Shnider, Spherical Hypersurfaces in Complex Manifolds, Inventiones math. 33 (1976), 223-246.

[CaC] J. Cao and S.-C. Chang, Pseudo-Einstein and $Q$-Flat Metrics with Eigenvalue Estimates on CR-Hypersurfaces, Indiana Univ. Math. J., Vol. 56, No. 6 (2007), 2840-2857.

[CC] S.-C. Chang and H.-L. Chiu, Nonnegativity of CR Paneitz operator and its Application to the CR Obata’s Theorem in a Pseudohermitian $(2n + 1)$-Manifold, Journal of Geometric Analysis, Vol. 19 (2009), 261-287.

[CCC] S.-C. Chang, J.-H. Cheng and H.-L. Chiu, The Fourth-order $Q$-curvature flow on a CR 3-manifold, Indiana Univ. Math. J., Vol. 56, No. 4 (2007), 1793-1826.
S.-C. Chang, T.-J. Kuo, C. Lin, Pseudo-Einstein structure, eigenvalue estimate for the CR Paneitz operator and its applications to uniformization theorem, arXiv:1807.08898 [math.DG].

S.-C. Chang, T.-J. Kuo and T. Saotome, On existence of the vanishing CR $Q$-curvature in a closed CR 3-manifold, preprint.

S.-C. Chang and Chin-Tung Wu, Short-time existence theorem for the CR torsion flow, arXiv:1804.06585.

S.-S. Chern and S.-Y. Ji, On the Riemann mapping theorem, Annals of Math., Vol 144 (1996), 421-439.

S. Dragomir and G. Tomassini, Differential Geometry and Analysis on CR manifolds, Progress in Mathematics, Volume 246, Birkhauser 2006.

T. Frankel, Manifolds with positive curvature, Pacific J. Math. 11 (1961), 165-174.

C. Fefferman and K. Hirachi, Ambient Metric Construction of $Q$-Curvature in Conformal and CR Geometries, Math. Res. Lett., 10, No. 5-6 (2003), 819-831.

G. B. Folland, Subelliptic Estimates and Function Spaces on Nilpotent Lie Groups, Arkiv for Mat. 13 (1975), 161-207.

G. B. Folland and E. M. Stein, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group, Comm. Pure Appl. Math., 27 (1974), 429-522.

A. R. Gover and C. R. Graham, CR Invariant Powers of the Sub-Laplacian, J. Reine Angew. Math. 583 (2005), 1-27.

C. R. Graham and J. M. Lee, Smooth Solutions of Degenerate Laplacians on Strictly Pseudoconvex Domains, Duke Math. J., 57 (1988), 697-720.

A. Greenleaf: The first eigenvalue of a Sublaplacian on a Pseudohermitian manifold. Comm. Part. Diff. Equ. 10(2) (1985), no.3 191–217.

K. Hirachi, Scalar pseudohermitian invariants and the Szego kernel on three-dimensional CR manifolds, Complex Geometry, Lect. Notes in Pure and Appl. Math. 143, 67-76, Dekker (1993).

W. He and S. Sun, Frankel conjecture and Sasaki geometry, Advances in Mathematics, 291 (2016), 912–960.

J.M. Lee, Pseudo-Einstein Structures on CR manifolds, Amer. J. Math. 110 (1988), 157-178.

J. M. Lee, The Fefferman Metric and Pseudohermitian Invariants, Trans. A.M.S., 296 (1986), 411-429.

J.J. Kohn, Boundaries of Complex Manifolds, Proc. Conf. on Complex Analysis, Minneapolis,1964, Springer-Verlag, 81–94 (1965).
[KT] Y. Kamishima and T. Tsuboi, CR-structures on Seifert manifolds, Invent. math. 104 (1991), 149-163.

[M] S. Mori, Projective Manifolds with Ample Tangent Bundles. Ann. of Math. 110 (1979), 593-606.

[SY] Yum-Tong Siu and Shing-Tung Yau, Compact Kaehler Manifolds of Positive Bisectional Curvature, Inventiones math. 59 (1980), 189-204.

[T] S. Tanno, Sasakian manifolds with constant $\phi$-holomorphic sectional curvature, Tôhoku Math. Journ. 21 (1969), 501-507.

[Ta] N. Tanaka, A Differential Geometric Study on Strongly Pseudoconvex Manifolds, 1975, Kinokuniya Co. Ltd., Tokyo.

[We] S. M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geom. 13 (1978), 25-41.

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