Quantum teleportation is one of the most pioneering features of the quantum world. Typically, the quality of a teleportation protocol is solely judged by its average fidelity. In this work, we analyze the performance of teleportation in terms of both fidelity and the deviation in fidelity. Specifically, we define a quantity called teleportability score, which incorporates contributions from both the fidelity and its deviation. It also takes into account the sensitivity one requires for a protocol in which the teleportation of a quantum state is required in one or many intermediate steps. We compute the teleportability score in the noiseless scenario and find that it increases monotonically with the entanglement content of the resource state. The result remains same even if we consider an n-chain repeater-like configuration. However, in the presence of noise, the teleportability score, can sometime display a nonmonotonic behaviour with respect to the entanglement content of the initially shared resource state. Specifically, under local bit-flip and bit-phase-flip noise, lesser entangled states can have higher teleportability score for certain choice of system parameters. In the presence of global depolarizing noise, for low entangled resource states and high sensitivity requirements, the noisy states can have better a teleportability score in comparison to the noiseless scenario.

I. INTRODUCTION

Quantum information science is continually revolutionizing the fields of communication and computation. Pioneering quantum protocols include quantum cryptography [1, 2], quantum dense coding [3], quantum error correction [4], quantum teleportation [5] etc. The field of communication is one of the major beneficiaries due to the advent of these protocols, which have also been implemented experimentally in a variety of physical systems [6]. For example, quantum communication protocols like dense coding [3] out perform the classical ones (in terms of capacity) by a factor of 2. Furthermore, for secure communication, the security of quantum protocols [1, 2] is guaranteed by the laws of physics, unlike the classical case where security is derived from exponential complexity of some mathematical problem. These examples establish superiority of the protocols in the quantum domain.

Among the quantum communication protocols, quantum teleportation [5, 7] has been one of the central interests of study. It has been studied quite extensively both theoretically [8] and experimentally [9–15]. However, almost all of these works, characterize the performance of a given teleportation protocol solely by the average fidelity it yields. But such characterization is very limited since it does not incorporate the effect of fluctuations. Different input states can have widely varying fidelities keeping the average value of fidelity fixed. Such high fluctuations can be detrimental during implementation of some quantum information processing protocols, like that of quantum gates in quantum computation [16, 17]. Therefore, not only the average fidelity but also the distribution of the fidelity for various input states is what that determines the performance of teleportation. Analysis of fluctuation and distribution of fidelities have been studied [18] in the context of the performance of single qubit gates. Similar investigations for quantum gates have also been carried out using higher order moments [19]. In the avenue of quantum teleportation, studies of fluctuations using deviation in fidelities was first introduced in [20], and was later formalized in [21], where they analyzed quality of teleportation in the plane of fidelity and its deviation (see also [22, 23]).

In this work, we seek a quantitative answer while comparing the performance of two resource states (noiseless or noisy) for a given teleportation protocol. For this purpose, we have introduced a new performance indicator of quantum teleportation, ‘teleportability score’, which, apart from the average fidelity, also incorporates its fluctuations while rating. Specifically, we find that in the noiseless scenario, the fidelity deviation decreases with increasing values of fidelity, thereby the teleportability score is always higher for more entangled states. However, in the presence of local noise, such ordering of states with respect to entanglement is not always present. For certain parameter ranges, we find that lesser entangled states can yield better/equally good teleportability scores in comparison to states having higher entanglement content. This is due to the fact that, unlike in the noiseless scenario, in presence of noise, both fidelity as well as its fluctuation might increase when higher entangled states are used for teleportation. In this paper, we investigate the teleportability score in presence of both locally and globally noisy channels. The local noise models [24, 25] considered in this manuscript are namely, bit-flip, phase-flip, bit-phase-flip, amplitude damping and depolarizing noise. We consider these noises to act locally on both the qubits of the shared entangled state. We also compute the teleportability score in presence of global depolarizing (white) noise and when the resource state suffers from both global and local noises.

The paper is organized as follows. We formally define the teleportability score in Sec. II The analysis for the noiseless scenario is done in Sec. III. The fidelity and its deviation for a n-chain repeater-like setting is computed in Sec. III A. The investigation of the teleportability score in the presence of noise is presented in Sec. IV. The issue with local noise is dealt in Sec. IV A. The case studies with bit-flip, phase-flip, and bit phase-flip noises are given in Sec. IV A 1 - Sec. IV A 3. The issue of global noise and combination of local and global noises are addressed in Sec. IV B. Finally we conclude in Sec. V.
II. TELEPORTABILITY SCORE

The average fidelity of a given teleportation scheme is given by

$$F = \int \phi(\phi|\rho_{\phi}|\phi),$$  (1)

where $|\phi\rangle$ is the arbitrary state to be teleported and $\rho_{\phi}$ is the teleported state with $|\phi\rangle$ at the input. The averaging is performed over all possible input states. Whenever we mention fidelity, $F$ in this manuscript, we refer to the average fidelity. The fidelity obtained for a particular input state $|\phi\rangle$ is simply given by $F^{\phi} = \langle \phi|\rho_{\phi}|\phi\rangle$. The corresponding deviation in fidelity (standard deviation) reads

$$D = \sqrt{\int \phi(\phi|\rho_{\phi}|\phi)^2 - \left(\int \phi(\phi|\rho_{\phi}|\phi)\right)^2} = \sqrt{\int \phi(\phi|\rho_{\phi}|\phi)^2 - F^2}. \tag{2}$$

In the usual fidelity based rating of a teleportation scheme, for a given protocol, two resource states with same $F$ but different $D$’s are deemed to be equivalent. But one can easily argue that the state having a lower value of $D$ is clearly a better choice. Therefore one might expect a payoff in the rating scheme of teleportation protocols due to $D$. With the motivation of quantifying the quality of teleportation in a more general context, we define a new quantity, teleportability score as

$$\tau_k = F - k. D, \tag{3}$$

where $F$ and $D$ are the fidelity and its deviation and $k$ has to be chosen according to the sensitivity requirements to fluctuations in fidelity of a particular protocol in which the teleportation is used as an intermediate step. Greater sensitivity requirements would simply imply a higher pay off in the score due to $D$ which is ensured by choosing a higher value of $k$. However, practically, we cannot choose an arbitrarily large $k$. This simply indicates that there is a physical cutoff to the maximum amount of sensitivity we can demand out of the teleportation process.

Apart from its interpretation as the sensitivity parameter, there can be situations where $k$ assumes a mathematical significance. Suppose, we want to design a protocol which maximizes the fidelity for a fixed value of standard deviation. In such a constrained optimization problem, $k$ takes the role of a Lagrange’s multiplier. Naturally, $k$ is constrained to take non-negative values.

The best fidelity obtained in a classical (entanglement [26] free) scheme is $F^c = \frac{2}{3}$ [27, 28]. Following the same fidelity maximizing protocol, the corresponding fidelity deviation is also easily calculated to be $D^c = \frac{1}{3\sqrt{5}}$. So, we define the classical value of the teleportability score for a given $k$, following Eq. (3), as

$$\tau_k^c = F^c - k.D^c = \frac{1}{3}(2 - k/\sqrt{5}) \tag{4}$$

A state therefore would be considered to possess any quantum advantage only when for the given protocol, it’s teleportability score is higher than the classical limit, i.e., $\tau_k > \tau_k^c$. Once again, the classical teleportability score is computed by considering the fidelity maximizing protocol and not via the overall maximization of the teleportability score with respect to all classical protocols. Similar strategy (with respect to the choice of the protocol) is adopted in the quantum case as well which we discuss in the subsequent sections.

III. THE NOISELESS SCENARIO

The initial shared state to be utilized for teleportation, in the Schmidt form [29], is

$$|\psi^a\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}|11\rangle. \tag{5}$$

The maximal singlet fraction, $f_{max}^a$, for this shared state is given by $1/2 + \sqrt{\alpha(1-\alpha)}$, and following the prescription given in [30], the maximal teleportation fidelity reads

$$F = \frac{1}{3}(2f_{max}^a + 1) = \frac{2}{3} + \frac{2}{3}\sqrt{\alpha(1-\alpha)}. \tag{6}$$

The above fidelity can be achieved by performing Bell measurements [31] at the Alice’s end and appropriate Pauli unitaries at Bob’s end after 2-bits of classical communication, i.e., the usual teleportation protocol [5]. Specifically, when an arbitrary state, $|\psi\rangle = \cos \theta|0\rangle + e^{i\phi}\sin \theta|1\rangle$ is teleported using the above protocol, the fidelity reads

$$F^{\psi} = 1 - \frac{1}{2}(2 - 2\sqrt{\alpha(1-\alpha)})\sin^2 \theta, \tag{7}$$

which when uniformly averaged over the Bloch sphere parameters yields the same fidelity as in Eq. (6), thereby establishing the usual teleportation protocol as an optimal one (in terms of fidelity) for states of the structure as given in Eq. (5). The fidelity deviation, $D$, corresponding to the above protocol is given by

$$D = \frac{1}{3\sqrt{5}}\sqrt{1 + 4(1-\alpha)\alpha - 4\sqrt{(1-\alpha)\alpha}} = \frac{1}{\sqrt{5}}(1 - F). \tag{8}$$

Therefore, in the noiseless scenario, the deviation in fidelity can be completely specified in terms of the average fidelity, $F$. We notice a sort of win-win situation since in the noiseless case, greater the fidelity, lesser is the fluctuation (deviation). For the maximally entangled resource state ($\alpha = 1/2$), we get a unit teleportation fidelity, $F = 1$, and the corresponding deviation falls to zero ($D = 0$). In Fig. 1, we plot the teleportability scores for various $k$-values in the noiseless scenario and also depict the classical values of teleportability scores ($\tau_k^c$) for those $k$-values.

Therefore, the teleportability score in the noiseless scenario reads

$$\tau_k = F - k.D = (1 + \frac{k}{\sqrt{5}})F - \frac{k}{\sqrt{5}} \tag{9}$$
Note that, in the noiseless scenario, $\tau_k$ is an increasing function of $\alpha$ for all values of $k$. Therefore we can conclude that more entangled resource states yield higher teleportability score in the noiseless scenario. This result holds true when instead of one shared state, we have a repeater-like configuration consisting of $n$ entangled states. We shall investigate how this situation changes in presence of noise in the succeeding sections. But before that, we would investigate the fidelity and deviation in a repeater like scenario.

The corresponding average fidelity and deviation reads

$$F_n = \frac{2}{3} + \frac{2^n}{3} \{\alpha(1-\alpha)\}^{\frac{2}{n}}$$

$$D_n = \frac{1}{3\sqrt{5}} \sqrt{1 + 2^{2n}\{\alpha(1-\alpha)\}^{n} + 2^{n+1}\{\alpha(1-\alpha)\}^{\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{5}} (1 - F_n).$$

Note that the expressions in Eq. (11) reduces to that given in Eqs. (6) and (8) on substituting $n = 1$. Thus in the noiseless scenario, even for the $n$-chain setting, we have qualitatively the same results as in the case of a single shared entangled state. In the rest of the manuscript where we deal with noisy channels, we would restrict our investigation of the teleportability score for teleportation with a single noisy channel.

### IV. THE NOISY SCENARIO

The presence of noise is ubiquitous in nature and inevitably affects the performance of any protocol quantum or classical. The noiseless situation is rather ideal and one must go beyond the noiseless assumption to make predictions in a more realistic setting. In the context of this manuscript, the omnipresence of noise demands the re-examination of the performance of teleportation in the presence of imperfections. In this attempt, we consider noisy resource states and compute their teleportability score for various ranges of noise parameters. We consider the situation of both local and global noises and investigate their impact on the quality of teleportation. Situations where both global and local noises act together are also analyzed in subsequent sections.

#### A. Local noise

In this section, we consider the situation where the resource state for teleportation suffers from local noise in both its qubits with different rates. The local noise models we have considered are namely bit flip, phase flip, bit-phase flip, amplitude damping channel, and phase damping channel [24]. The a comparative study of the response of teleportability score to these different kinds of local noise is presented. Our analysis reveal a counter intuitive feature in the presence of local bit flip noise where for certain range of noise parameter and
sensitivity, we get more teleportability score with a lesser entangled resource state. In the noisy scenario, the win-win situation of higher fidelity with lesser deviation is lost and the teleportability score becomes a much more realistic quantifier of the performance of teleportation. Therefore, teleportability score, which assumed a passive role as a quantifier of the ‘goodness’ of teleportation in the noiseless scenario, attains an active role and becomes much more physically relevant in presence of noise.

![Graph](image)

**FIG. 3.** Teleportability score for different values of \(k\) for local bit-flip and biphase-flip noises with \(p = 0.7\) and \(q = 1.0\). Although the average fidelity increases with \(\alpha, \tau_k\), depending on how high \(k\) is might decrease with increasing \(\alpha\). All the axes are dimensionless.

1. **Bit-flip noise**

The bit-flip noise can be modeled by the action of the Pauli operator \(\sigma^x\). Unsurprisingly, it flips the state \(|0\rangle\) to \(|1\rangle\) and vice-versa. Given an arbitrary state, \(\rho\), it keeps it unaltered with a probability \(p\), and flips it’s bits with a probability of \(1 - p\). Naturally, \(0 \leq p \leq 1\). Therefore in the presence of bit flip noise in both the qubits with probabilities \(p\) and \(q\), an arbitrary resource state \(\rho_{12}\) becomes

\[
\rho \xrightarrow{\text{bit flip}} pq\rho_{12} + (1 - p)q\sigma_1^z\rho_{12}\sigma_1^z + p(1 - q)\sigma_2^z\rho_{12}\sigma_2^z + (1 - p)(1 - q)\sigma_1^z\sigma_2^z\rho_{12}\sigma_1^z\sigma_2^z. \tag{12}
\]

When an arbitrary state, \(|\eta\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle\), is to be teleported through such a channel, the fidelity of the output state with Bob with \(|\eta\rangle\) reads

\[
F^\eta = (pq + (1 - p)(1 - q)) \left[ 1 - \frac{1}{2} (1 - 2\sqrt{\alpha(1 - \alpha)}) \sin^2 \theta \right] + \frac{1}{2} (p + q - pq) \left[ 1 + 2 \cos 2\phi \sqrt{\alpha(1 - \alpha)} \right] \sin^2 \theta. \tag{13}
\]

Although the average fidelity and fidelity deviation can be computed easily from the above equation, but the corresponding expressions, especially for the deviation, becomes cumbersome. So, to simplify matters we assume \(q = 1.0\). Note that we would have obtained the same results if we have instead assumed \(p = 1.0\). The computed fidelity reads

\[
F^\text{bit-flip} = \frac{1}{3}(1 - p + 2p(1 + \sqrt{(1 - \alpha)/\alpha})) \tag{14}
\]

and the corresponding deviation is given by

\[
F^\text{bit-flip} = \frac{1}{3} \sqrt{1 + 4(1 - \alpha)\alpha - 4\sqrt{(1 - \alpha)\alpha} + 4(1 - p)^2(1 + 4(1 - \alpha)\alpha - 2\sqrt{(1 - \alpha)\alpha}) - 4(1 - p)(1 + 2(1 - \alpha)\alpha - 3\sqrt{(1 - \alpha)\alpha})} \tag{15}
\]

Clearly, the fidelity, \(F^\text{bit-flip}\), remains an increasing function of \(\alpha\) \((0 \leq \alpha \leq 1/2)\). However, unlike the noiseless case, \(D^\text{bit-flip}\), after some initial non-monotonics, also increases with increasing \(\alpha\). Therefore for high sensitivity requirements, the weight of the payoff term outweighs the gain in fidelity with increasing \(\alpha\). So, when the resource states suffer from local bit-flip noise, there are situations in which the hierarchy of the teleportability score in terms of entanglement is lost (See Fig. 3). Thus, in the presence of local bit flip noise, for certain choice system parameters, we get more teleportability score with less entanglement.

2. **Phase-flip noise**

The phase-flip noise is modeled by the action of the Pauli operator \(\sigma^z\). It keeps the state \(|0\rangle\) as is and adds a phase of \(e^{i\pi} = -1\) to \(|1\rangle\). Given an arbitrary state, \(\rho\), it keeps it unaltered with a probability \(p\), and phase-flips it with a probability of \(1 - p\). Again, \(0 \leq p \leq 1\). Therefore in the presence of phase-flip noise in both the qubits with probabilities \(p\) and \(q\), an arbitrary resource state \(\rho_{12}\) becomes

\[
\rho \xrightarrow{\text{phase flip}} pq\rho_{12} + (1 - p)q\sigma_1^z\rho_{12}\sigma_1^z + p(1 - q)\sigma_2^z\rho_{12}\sigma_2^z + (1 - p)(1 - q)\sigma_1^z\sigma_2^z\rho_{12}\sigma_1^z\sigma_2^z. \tag{16}
\]
When an arbitrary state, $|\eta\rangle = \cos \frac{\theta}{2}|0\rangle + e^{i\phi} \sin \frac{\theta}{2}|1\rangle$, is to be teleported through such a channel, the fidelity of the output state with Bob with $|\eta\rangle$ reads

$$F^\eta = \left( pq + (1-p)(1-q) \right) \left[ 1 - \frac{1}{2} \left( 1 - 2\sqrt{\alpha(1-\alpha)} \right) \sin^2 \theta \right]$$

$$+ \frac{1}{2} \left( p + q - 2pq \right) \left[ 1 - \frac{1}{2} \left( 1 + 2\sqrt{\alpha(1-\alpha)} \right) \sin^2 \theta \right].$$

Again for simplicity, we assume $q = 1$. The expressions of fidelity and fidelity deviation reads

$$F^\eta = \frac{2}{3} \left( 1 + (2p - 1)\sqrt{1 - \alpha} \right)$$

$$D^\text{phase-flip} = \frac{1 + 4(2p - 1)^2\alpha - 4(2p - 1)^2\alpha^2 - 4\sqrt{1 - \alpha}\alpha + 8(1 - p)\sqrt{1 - \alpha}\alpha}{3\sqrt{5}}$$

$$= \frac{1 - 2(2p - 1)\sqrt{1 - \alpha}\alpha}{3\sqrt{5}} = \frac{1 - F^\text{phase-flip}}{\sqrt{5}}$$

It is clear from the above expressions that $D^\text{phase-flip}$ decreases with increasing $F^\text{phase-flip}$. Therefore, like the noiseless setting, the usual ordering of resource states is retained in the presence of phase-flip noise.

3. Bitphase-flip noise

The bitphase-flip noise is modeled by the action of the Pauli operator $\sigma^y$. When an arbitrary resource state $\rho_{12}$ suffers bitphase-flip noise in both the qubits with probabilities $p$ and $q$, it reads as

$$\rho^\text{bitphase-flip} \rightarrow pq\rho_{12} + (1-p)q\sigma_1^y\rho_{12}\sigma_1^y + p(1-q)\sigma_2^y\rho_{12}\sigma_2^y + (1-p)(1-q)\sigma_1^y\sigma_2^y\rho_{12}\sigma_2^y\sigma_1^y.$$  

Note that the expressions of fidelity for an arbitrary input state to be teleported is same as that in for the bit-flip noise (see Eq. 13). Consequently the average fidelity and its deviation are also exactly same. Therefore, the physics of teleportability score, remains exactly the same as in case of bit-flip noise (see Fig. 3).

B. Global noise

Apart from the local noise models considered in the previous section, the resource state might also suffer from global noise. In such a situation, the environment interacts with the whole composite system. In this section, we analyze the response of teleportability score to global depolarizing noise where the whole bipartite state becomes a convex mixture of itself with white noise. We also consider the combined effect of both local and global depolarizing noise on the teleportability score of the resource state.

![Graph showing teleportability score when resource states suffer from global depolarizing noise as compared to the noiseless value for $k = 4$. All axes are dimensionless.](image)
The average teleportation fidelity $F^{\text{dep}}(p)$ can estimated as

$$F^{\text{dep}}(p) = \frac{2p}{3} \left( 1 + \sqrt{\alpha(1-\alpha)} \right) + \frac{1-p}{2} = pF + \frac{1-p}{2},$$

where $F = \frac{2}{3} + 2\sqrt{\alpha(1-\alpha)}$, the average fidelity in the noiseless case, see Eq. 6. Similarly, the fidelity deviation reads

$$D^{\text{dep}}(p) = \frac{p(1-2\sqrt{\alpha(1-\alpha)})}{3\sqrt{5}}.$$  \hfill (22)

Note that the deviation in fidelity is lower compared to the noiseless case. This allows for the possibility that for high sensitivity requirements, the teleportability score in the presence of global depolarizing noise becomes better compared to the noiseless case (see Fig. 4 for a typical example).

$$D^{\text{dep-local-global}}(p, p_1, p_2) = \frac{1}{24\sqrt{5}} \left[ (64 - 96p_2 + 51p_2^2) + 4(64 + 3p_2(7p_2 - 32)) \alpha(1-\alpha) - (256 - 384p_2 + 144p_2^2)\sqrt{\alpha(1-\alpha)} \right.$$

$$\left. + (64p_2^2 + 3{p_2}(3p_1 + 3p_2 - 4) - 96p_1) \left\{ 1 - 4\sqrt{\alpha(1-\alpha)} + 4\alpha(1-\alpha) \right\} \right.$$  \hfill (23)

$$+ p_1^2 \left\{ 51 - 12\sqrt{\alpha(-7\sqrt{\alpha} + 7\alpha + 12\sqrt{1-\alpha})} \right\} + p_2 \left\{ -17 + 4\sqrt{\alpha(-7\sqrt{\alpha} + 7\alpha + 12\sqrt{1-\alpha})} \right\} \right].$$

V. CONCLUSION

Traditionally, the performance of teleportation is calibrated by the average fidelity it yields. However, not all states are teleported equally, and the fidelity of each state can be widely dispersed for a fixed value of average fidelity. These fluctuations turn out to be detrimental during various quantum information processing tasks like implementation of quantum gates etc. Therefore, the characterization of the performance of teleportation via the average fidelity alone is very restrictive. In this work, we characterize teleportation via both average fidelity ($F$) and fidelity deviation ($D$), and reanalyze the usual teleportation protocol in presence of local and global noises.

In this work, we define teleportability score, $\tau$, as the difference between average fidelity and the pay off term due to fluctuations, $F - kD$, where, $k$ denotes the sensitivity of the score to fluctuations in fidelity. In the noiseless scenario, we find that for any value of $k$, $\tau_k$ is a monotonically increasing function of fidelity. In the presence of local bit-flip and bitphase-flip noise, if high sensitivity to fluctuations is imposed, although the fidelity alone increases with increasing entanglement of the shared resource state, the teleportability score on the other hand might decrease. It therefore reverses the known hierarchy of resource states in terms of their capability of teleportation. When the resource state is affected by global depolarizing noise, for low values of entanglement and high sensitivity demands, the noisy states can sometime outperform the noiseless ones in terms of the teleportability score.

Although, we have analyzed the performance of teleportation by considering both fidelity and its deviation, similar analysis can also be carried out for other protocols whose performance is quantified by the average fidelity.

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