The spectral radius of graphs with no $K_{2,t}$ minor

V. Nikiforov*

Abstract

Let $t \geq 3$ and $G$ be a graph of order $n$, with no $K_{2,t}$ minor. If $n > 400t^6$, then the spectral radius $\mu(G)$ satisfies

$$\mu(G) \leq \frac{t-1}{2} + \sqrt{n + \frac{t^2 - 2t - 3}{4}},$$

with equality if and only if $n \equiv 1 \pmod{t}$ and $G = K_1 \lor \lfloor n/t \rfloor K_t$.

For $t = 3$ the maximum $\mu(G)$ is found exactly for any $n > 40000$.

AMS classification: 15A42; 05C35.

Keywords: spectral radius; forbidden minor; spectral extremal problem.

1 Introduction and main results

A graph $H$ is called a minor of a graph $G$ if $H$ can be obtained by contracting edges of a subgraph of $G$. Write $H \not\subset G$ if $H$ is not a minor of $G$. The spectral radius $\mu(G)$ of a graph $G$ is the largest eigenvalue of its adjacency matrix. In this note we study the following question:

Question 1 How large can $\mu(G)$ be if $G$ a graph of order $n$ and $K_{2,t} \not\subset G$?

Particular cases of this question have been studied before: for example, Yu, Shu and Hong [7] showed that if $G$ is a graph of order $n$ and $K_{2,3} \not\subset G$, then

$$\mu(G) = 3/2 + \sqrt{n - 7/4}. \tag{1}$$

Unfortunately, bound (1) it is not attained for any $G$, although it is tight up to an additive term approaching $1/2$. Likewise, Benediktovich [1] studied 2-connected graphs with no $K_{2,4}$ minors and gave a few bounds similar to (1), but gave no summary result.

To outline the case $t = 2$, let $n$ be odd and $F_2(n)$ be the friendship graph, that is, a set of $\lfloor n/2 \rfloor$ triangles sharing a single common vertex. If $n$ is even, let $F_2(n)$ be obtained by hanging an extra edge to the common vertex of $F_2(n-1)$.

In [5] and [8], it was shown that if $G$ is a graph of order $n$, with no $K_{2,2}$, then $\mu(G) < \mu(F_2(n))$, unless $G = F_2(n)$.

Incidentally, $K_{2,2} \not\subset F_2(n)$; thus, Question 1 is settled for $t = 2$. We shall show that the situation is similar for any $t \geq 3$ and $n$ large.

*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA; email: vnikifrv@memphis.edu
First, we extend the family \( \{F_t (n)\} \) for \( t > 2 \). Given graphs \( F \) and \( H \), write \( F \lor H \) for their join and \( F + H \) for their disjoint union. Suppose that \( t \geq 3 \) and \( n \geq t + 1 \); set \( p = \lfloor (n - 1) / t \rfloor \) and let \( n - 1 = pt + s \).

Now, let \( F_t (n) := K_1 \lor (pK_t + K_s) \); in particular, if \( s = 0 \), let \( F_t (n) := K_1 \lor pK_t \). Clearly, the graph \( F_t (n) \) is of order \( n \) and \( K_{2,t} \not\subseteq F_t (n) \).

It is not hard to find that \( \mu (F_t (n)) \) is the largest root of the cubic equation

\[
(x - s + 1) (x^2 - (t - 1) x - n + 1) + s (t - s) = 0,
\]

and satisfies the inequality

\[
\mu (F_t (n)) \leq \frac{t - 1}{2} + \sqrt{n + \frac{t^2 - 2t - 3}{4}},
\]

with equality if and only if \( n \equiv 1 \pmod{t} \), i.e., if \( s = 0 \).

Our first result answers Question 1 for \( t = 3 \) and \( n \) large:

**Theorem 2** If \( G \) is a graph of order \( n > 40000 \) and \( K_{2,3} \not\subseteq G \), then \( \mu (G) < \mu (F_3 (n)) \), unless \( G = F_3 (n) \).

A similar theorem may hold also for \( t > 3 \), but our general result is somewhat weaker:

**Theorem 3** Let \( t \geq 4 \) and \( n \geq 400t^6 \). If \( G \) is a graph of order \( n \) and \( K_{2,t} \not\subseteq G \), then

\[
\mu (G) \leq \frac{t - 1}{2} + \sqrt{n + \frac{t^2 - 2t - 3}{4}}.
\]

Equality holds if and only if \( n \equiv 1 \pmod{t} \) and \( G = F_t (n) \).

Before proving these theorems, let us note that if \( t \geq 4 \) and \( n \geq 400t^6 \), then

\[
\mu (F_t (n)) > \frac{t - 1}{2} + \sqrt{n + \frac{t^2 - 2t - 3}{4}} - \frac{t (t + 1)}{8n},
\]

so bound (2) is quite tight.

The proofs of Theorems 2 and 3 are based on a structural lemma inspired by [6]. Write \( M_t (n) \) for the set of graphs of order \( n \), with no \( K_{2,t} \) minors, and with maximum spectral radius.

**Lemma 4** Let \( t \geq 3 \), \( n > 16 (t - 1)^4 (5t - 3)^2 \), and \( G \in M_t (n) \). If \( x \) is an eigenvector to \( \mu (G) \), then the maximum entry of \( x \) corresponds to a vertex of degree \( n - 1 \).

**Proof** Let \( t, n \) and \( G \) be as required. Hereafter, let \( V := \{v_1, \ldots, v_n\} \) be the vertex set of \( G \); let \( \Gamma_G (v) \) be the set of the neighbors of \( v \in V \), and set \( d_G (v) := |\Gamma_G (v)| \); the subscript \( G \) is omitted if \( G \) is understood. Also, \( G - v \) stands for the graph obtained by omitting the vertex \( v \).

Clearly \( G \) is connected, as otherwise \( G \) there is a graph \( H \) with no \( K_{2,t} \) minor such that \( \mu (H) > \mu (G) \), contradicting \( G \in M_t (n) \).

Set for short, \( \mu := \mu (G) \), and let \( x := (x_1, \ldots, x_n) \) be a unit eigenvector to \( \mu \) such that \( x_1 \geq \cdots \geq x_n \). We have to show that \( d (v_1) = n - 1 \).

Let \( A \) be the adjacency matrix of \( G \) and set \( B = [b_{ij}] := A^2 \). Note that \( b_{ij} \) is equal to the number of 2-walks starting at \( v_i \) and ending at \( v_j \); hence, if \( i \neq j \), then \( b_{ij} \leq t - 1 \), as \( K_{2,t} \not\subseteq G \). Since \( Bx = \mu^2 x \), for any vertex \( u \), we see that

\[
\mu^2 x_u = d (u) x_u + \sum_{i \in V \setminus \{u\}} b_{iu} x_i \leq d (u) x_u + (t - 1) \sum_{i \in V \setminus \{u\}} x_i \leq d (u) x_u + (t - 1) (\sqrt{n} - x_u).
\]
The last inequality follows from \((x_1 + \cdots + x_n)^2 \leq n (x_1^2 + \cdots + x_n^2) = n\). We find that
\[
d(u) \geq \mu^2 + t - 1 - \frac{(t-1)\sqrt{n}}{x_u}.
\] (3)

On the other hand, if \(u \neq v_1\), then \(d(u) + d(v_1) \leq n + t - 1\), as \(K_{2,t} \not\subseteq G\). Using (3), we get
\[
n + t - 1 \geq 2\mu^2 + 2(t-1) - \frac{(t-1)\sqrt{n}}{x_u} - \frac{(t-1)\sqrt{n}}{x_1} \geq 2\mu^2 + 2(t-1) - \frac{2(t-1)\sqrt{n}}{x_u}.
\] In view of \(\mu^2 > n - 1\), we obtain
\[
x_u \leq \frac{2(t-1)\sqrt{n}}{2\mu^2 - n + t - 1} < \frac{2(t-1)\sqrt{n}}{\sqrt{n}} = \frac{2(t-1)}{\sqrt{n}}.
\] (4)

Assume for a contradiction that \(d(v_1) \leq n - 2\); let \(H\) be the graph induced in \(G\) by the set \(V \setminus (\Gamma(v_1) \cup \{v_1\})\) and suppose that \(v\) is a vertex with minimum degree in \(H\). Since \(K_{2,t} \not\subseteq G\), Theorem 1.1 of [2] implies that \(d_H(v) \leq t\), and since \(v\) and \(v_1\) have at most \(t - 1\) common neighbors, we see that \(d_G(v) \leq 2t - 1\).

Next, remove all edges incident to \(v\) and join \(v\) to \(v_1\). Write \(G'\) for the resulting graph, which is of order \(n\) and \(K_{2,t} \not\subseteq G'\). As \(G \in M_t(n)\), we see that
\[
0 \leq \mu - \mu(G') \leq 2x_u \sum_{i \in \Gamma(v)} x_i - 2x_1 x_u.
\]

Thus, bound (4) implies an upper bound on \(x_1\)
\[
x_1 \leq \sum_{i \in \Gamma(v)} x_i \leq d_G(v) \frac{2(t-1)}{\sqrt{n}} \leq \frac{2(t-1)(2t-1)}{\sqrt{n}}.
\] (5)

Finally, we apply (4) and (5) to show that \(\mu\) is bounded in \(n\)
\[
\mu = 2 \sum_{\{i,j\} \in E(G)} x_i x_j \leq 2x_1 \sum_{i \in \Gamma(v_1)} x_i + 2 \sum_{\{i,j\} \in E(G-v_1)} x_i x_j \\
\leq 8(t-1)^2 (2t-1) d(v_1) + 8(t-1)^2 (|E(G)| - d(v_1)) \\
= \frac{16(t-1)^3 d(v_1)}{n} + \frac{8|E(G)|(t-1)^2}{n}.
\]

Since \(d(v_1) < n\), and Theorem 1.1 of [2] gives \(2|E(G)| \leq (t+1)(n-1)\), we find that
\[
n - 1 < \mu^2 < \frac{16(t-1)^4 (5t-3)^2}{n},
\]
contradicting the premises. Hence, \(d(v_1) = n - 1\).

\[\square\]

**Proof of Theorem 3** Let \(G \in M_t(n)\), \(\mu := \mu(G)\), and \(x := (x_1, \ldots, x_n)\) be a unit eigenvector to \(\mu\) such that \(x_1 \geq \cdots \geq x_n\). Lemma 4 implies that \(d(v_1) = n - 1\). Clearly \(\mu x_1 \leq (n - 1) x_2\) and since \(d(v_2) \leq t\), we see that \(\mu x_2 \leq x_1 + (t-1) x_2\). Therefore,
\[
(\mu - t + 1) x_1 \leq n - 1,
\]
\[
(\mu - t + 1) x_1 \leq n - 1,
\]
\[\[\text{We use Theorem 1.1 of [2] solely to lower the bound on } n; \text{ otherwise just as good is an older result of Mader [4] implying that } |E(G)| \geq (4r+8)n \text{ forces a } K_{2,r} \text{ minor.}\]

\[\]
implying (2). If equality holds in (2), then \( x_2 = x_3 = \cdots = x_n \) and \( \mu x_2 = x_1 + d(u) x_2 \) for \( u = 2, \ldots, n \). Hence, \( G - v_1 \) is \((t - 1)\)-regular. To complete the proof, we show that \( G - v_1 \) is a union of disjoint \( K_t \)'s.

Assume for a contradiction that \( G - v_1 \) has a component \( H \) that is non-isomorphic to \( K_t \), and let \( h \) be the order of \( H \). Clearly \( h \geq t + 2 \), for if \( h = t + 1 \), any two nonadjacent vertices in \( H \) have \( t - 1 \) common neighbors, which together with \( v_1 \) form a \( K_{t+1} \).

Further, since \( K_{2,t} \not\subseteq G \), we see that \( K_{1,t} \not\subseteq H \). As shown in [3], these conditions on \( H \) imply that \( |E(H)| \leq h + t(t - 3)/2 \), contradicting the identity \( |E(H)| = (t - 1)h/2 \). Hence, \( G - v_1 \) is a union of disjoint \( K_t \)'s, completing the proof of Theorem 3.

**Proof of Theorem 2** Let \( G \in \mathcal{M}_t(n), \mu := \mu(G) \), and \( x := (x_1, \ldots, x_n) \) be a unit eigenvector to \( \mu \) such that \( x_1 \geq \cdots \geq x_n \). Lemma 4 implies that \( d(v_1) = n - 1 \). Since \( G - v_1 \) has no vertex of degree more than 2, its components are paths, triangles, or isolated vertices, as otherwise \( G \) contains a \( K_{2,3} \) minor.

Since \( G - v_1 \) is edge maximal, it may have at most one component that is not a triangle, say the component \( H \). If \( H \) is an isolated vertex or an edge, we are done, so suppose that \( H \) is a path of order \( h \), and let \( v_{k+1}, \ldots, v_{k+h} \) be the vertices along the path. Clearly \( h \geq 4 \).

Suppose first that \( h \) is odd, say \( h = 2s + 1 \) and \( s \geq 2 \). By symmetry, \( x_{k+i} = x_{k+i-s+1} \) for any \( i \in [s] \). Remove the edges \( \{v_{k+s-1}, v_{k+s}\}, \{v_{k+s+2}, v_{k+s+3}\} \); add the edges \( \{v_{k+s}, v_{k+s+2}\}, \{v_{k+s-1}, v_{k+s+3}\} \); and write \( G' \) for the resulting graph. Clearly \( K_{2,3} \not\subseteq G' \) has no \( K_{2,3} \) minor, as \( H \) is replaced by a shorter path and a disjoint triangle. On the other hand,

\[
\sum_{\{i,j\} \in E(G')} x_i x_j = \sum_{\{i,j\} \in E(G)} x_i x_j - x_{k+s-1} x_{k+s} - x_{k+s+2} x_{k+s+3} + x_{k+s} x_{k+s+2} + x_{k+s+1} x_{k+s+3}
\]

Since \( G \in \mathcal{M}_t(n) \), we get \( \mu(G') = \mu \); hence \( x \) is an eigenvector to \( \mu \) of \( G' \). But \( v_{k+s}, v_{k+s+1}, \) and \( v_{k+s+2} \) are symmetric in \( G' \), implying that \( x_{k+s} = x_{k+s+1} = x_{k+s+2} \). Now, using the eigenequations of \( G \), we find that \( x_{k+1} = \cdots = x_{k+h} \), which is a contradiction, in view of

\[
\mu x_{k+1} = x_{k+2} + x_1 \quad \text{and} \quad \mu x_{k+2} = x_{k+3} + x_{k+1} + x_1.
\]

Next, suppose that \( h \) is even, say \( h = 2s \), and let \( s \geq 3 \). By symmetry, \( x_{k+i} = x_{k+i-s+1} \) for any \( i \in [s] \). Remove the edges \( \{v_{k+s-1}, v_{k+s}\}, \{v_{k+s+2}, v_{k+s+3}\} \); add the edges \( \{v_{k+s}, v_{k+s+2}\}, \{v_{k+s-1}, v_{k+s+3}\} \); and write \( G' \) for the resulting graph. Clearly \( K_{2,3} \not\subseteq G' \), as \( H \) is replaced by a shorter path and a disjoint triangle. On the other hand,

\[
\sum_{\{i,j\} \in E(G')} x_i x_j = \sum_{\{i,j\} \in E(G)} x_i x_j - x_{k+s-1} x_{k+s} - x_{k+s+2} x_{k+s+3} + x_{k+s} x_{k+s+2} + x_{k+s+1} x_{k+s+3}
\]

Since \( G \in \mathcal{M}_t(n) \), we get \( \mu(G') = \mu \); hence \( x \) is an eigenvector to \( \mu \) of \( G' \). But \( v_{k+s}, v_{k+s+1}, \) and \( v_{k+s+2} \) are symmetric in \( G' \), implying that \( x_{k+s} = x_{k+s+1} = x_{k+s+2} \). This fact leads to a contradiction precisely as above.

\[\text{See also Section 1.2 of [2] where the result is stated more fittingly for our use.}\]
It remains the case $h = 4$. By symmetry, $x_{k+1} = x_{k+4}$ and $x_{k+2} = x_{k+3}$. Remove the edge $\{v_{k+1}, v_{k+2}\}$, add the edge $\{v_{k+2}, v_{k+4}\}$, and write $G'$ for the resulting graph. Clearly $K_{2,t} \not\subset G'$ and

$$\sum_{\{i,j\} \in E(G')} x_i x_j = \sum_{\{i,j\} \in E(G)} x_i x_j - x_{k+1} x_{k+2} + x_{k+2} x_{k+4} = \sum_{\{i,j\} \in E(G)} x_i x_j.$$

Since $G \in \mathcal{M}_t(n)$, we get $\mu(G') = \mu$; hence $x$ is an eigenvector to $\mu(G')$, implying the contradicting eigenequations

$$\mu(G') x_{k+1} = x_1 \quad \text{and} \quad \mu(G') x_{k+4} = x_{k+2} + x_{k+3} + x_1.$$

The proof of Theorem 2 is completed. \hfill \Box

References

[1] V.I. Benediktovich, Spectral radius of $K_{2,4}$-minor free graph (in Russian), Dokl. Nats. Akad. Nauk Belarusi 59 (2015), 5–12.

[2] M. Chudnovsky, B. Reed, and P. Seymour, The edge-density for $K_{2,t}$ minors, J. Combin. Theory Ser. B, 101 (2011) 18–46.

[3] G. Ding, T. Johnson, and P. Seymour, Spanning trees with many leaves, J. Graph Theory 37 (2001) 189–197.

[4] W. Mader, Existenz $n$-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, Abh. Math. Sem. Univ. Hamburg 37 (1972) 86–97.

[5] V. Nikiforov, Bounds on graph eigenvalues II, Linear Algebra Appl. 427 (2007) 183–189.

[6] M. Tait and J. Tobin, Three conjectures in extremal spectral graph theory, preprint available at arXiv:1606.01916.

[7] G. Yu, J. Shu, and Y. Hong, Bounds of spectral radius of $K_{2,3}$-minor free graphs, Electronic J. Linear Algebra, 23 (2012) 171–179.

[8] M. Zhai and B. Wang, Proof of a conjecture on the spectral radius of $C_4$-free graphs, Linear Algebra Appl. 437 (2012) 1641–1647.