Compressive buckling analysis of orthotropic composite plates restrained by stringers

Jianwen Feng
Aero Engine Academy of China, 21 Shunxing Road, Beijing 101304
E-mail: jianwenfeng@hotmail.com

Abstract. Closed section stiffeners have much larger rotational stiffness than open section stiffeners and contribute significantly to the panels’ buckling strength. The buckling analysis formulae developed for open-section stringer-reinforced panels underestimate the buckling strength of closed-section stringer-stiffened panels, such as composite fuselage panels stiffened by hat-shaped stringers. This study proposes a unified approach for compressive buckling analysis of stiffened composite plates, which takes into account the contribution of stringers’ rotational stiffness, and achieves a closed-form solution.

1. Introduction
Stiffened panels are primary structural elements of airframes, where stringers add out-of-plane bending stiffness to skins, prevent the panel from global buckling, and reduce the width of skin bays. Metal wings and fuselages are often stiffened by open section stringers, e.g. T-shaped and Z-shaped stringers. Metal structural members are susceptible to corrosion, and open section profiles have advantages of preventing humidity accumulation and facilitating inspections. Composites are however more resistant to corrosion, and therefore composite fuselages, e.g. Boeing 787 and Airbus 350XWB, often adopt hat-shaped (Omega) stringers.

For compressive buckling analysis of open-section stringer-stiffened panels, stringers' rotational stiffness is considered according to the ratio between the skin bay width $b$ and the skin thickness $t$ [1, 2]:

- When $b/t<40$, the stringer's rotational stiffness is negligible compared with the skin's bending stiffness, and therefore the skin pocket's lateral edges are considered as simply supported.
- When $b/t>110$, the stringer's stiffness is relatively high, and therefore the lateral edges are approximated as clamp.
- When $b/t$ lies between 40 and 110, the support coefficient is determined by interpolation.

However, such strategies cannot apply to stringers with closed sections because torsional stiffness of closed-section members is typically orders of magnitude larger than that of open-section members. Figure 1 shows two distinct buckling modes of panels with close-section stringers (left) and open-section stringers (right). The two panels both use hat-shaped stringers, but the stringer on the right panel is inversely mounted resulting in an open-section profile. For the left panel, the stringers keep strait in the buckling mode, but for the right panel the stringers rotate around the attachment line. This trivial example confirms that it is inadequate to use the formulae developed for open-section stringers to predict buckling strength of panels stiffened by closed-section stringers, otherwise the result would be too conservative.
Figure 1. Buckling modes of stiffened panels with hat-shaped stringers (left) and inverse hat-shaped stringers (right).

2. The proposed approach
A number of methods are developed to study the stability of composite structure [3-14]. Finite element methods are versatile and powerful, but for fuselage zones with regular geometry, from the standpoint of aeronautical engineers of stress department, a closed-form formula is more efficient. Thus, by taking into account stringers’ rotational restraint, a new method is proposed for the analysis of compressive buckling of stiffened panels. This method is developed from the Lévy solution of orthogonal plate buckling. The stringers supply rotational restraint to the skin pockets, which depends on the rotational stiffness of the stringer.

2.1. The Lévy solution
A graphical illustration of a plate buckling model is shown in figure 2. The governing equation for compressive buckling of a symmetric and orthotropic \( D_{00}, D_{20} = 0 \) plate is obtained from the theory of plates and the classical lamination theory [15, 16] as:

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} = N_{\sigma} \frac{\partial^2 w}{\partial x^2},
\]

in which \( w \) is the out-of-plane deflection with the unit of \( \text{mm} \); \( D_{11}, D_{22}, D_{12}, D_{66} \) are entries of the bending stiffness matrix with the unit of \( \text{N mm} \), which are obtained following the classical lamination theory [15, 16]; and \( N_{\sigma} \) is negative valued, indicating a compressive load.

Figure 2. Geometry and loading definition of plate buckling.

If the two loading edges are simply supported, the Lévy solution can be employed to obtain a closed-form solution for the critical buckling load [17, 18]. By the Lévy solution, the deflection \( w \) takes the form

\[
w(x, y) = F(y) \sin \frac{m\pi x}{a},
\]

(2)
in which \( m \) is an integer. Substituting Equation (2) into Equation (1) yields an ordinary differential equation about \( F(y) \):

\[
D_2 a^4 \frac{d^2 F}{dy^2} - 2(D_{12} + 2D_{00}) a^2 m^2 \pi^2 F(y) + \left( N_\alpha a^2 m^2 \pi^2 + D_2 m^4 \pi^4 \right) F(y) = 0 .
\]  (3)

It deserves highlight that the simple support boundary on the loading edges is a prerequisite for the Lévy solution, but this is not a fatal limitation to this method. Most panels used in airframes are "long" plates, with typical a/b ratio ranging from 3-5 and many identical buckling waves are developed along the longitudinal direction, leading to the buckling strength independent of end edge support condition \[19\]. Therefore, in engineering design, the end boundary is usually assumed as simple support.

Due to the geometric constraints on the two lateral edges, the critical buckling load of a plate is larger than that of a simply supported column, therefore

\[
N_\alpha < -\frac{D_2 m^2 \pi^2}{a^2} ,
\]  (4)
in which the right hand side is the critical buckling load of a simply supported column. Let

\[
A = D_2 a^4 > 0
\]

\[
B = 2(D_{12} + 2D_{00}) a^2 m^2 \pi^2 > 0 ,
\]  (5)

\[
C = N_\alpha a^2 m^2 \pi^2 + D_2 m^4 \pi^4 < 0
\]

the general solution to Equation (3) is:

\[
F(y) = k_1 \cosh(\alpha y) + k_2 \sinh(\alpha y) + k_3 \cos(\beta y) + k_4 \sin(\beta y) ,
\]  (6)
in which, the two coefficients \( \alpha \) and \( \beta \) in Equation (6) are given by:

\[
\alpha = \sqrt{\frac{B^2 - 4AC}{2A^2} + \frac{B}{2A}} ,
\]  (7)

\[
\beta = \sqrt{\frac{B^2 - 4AC}{2A^2} - \frac{B}{2A}} ,
\]  (8)

and \( k_1 - k_4 \) are four undetermined coefficients. When the solution takes the Lévy form, the simple support on the two loaded edges are satisfied automatically. On the two unloaded edges, the out-of-plane displacement is vanished, therefore

\[
w(x,0) = 0 \Rightarrow F(0) = 0 ,
\]  (9)

\[
w(x,b) = 0 \Rightarrow F(b) = 0 .
\]  (10)

If the stiffness of the stringers is very weak, the stringers will buckle together with the panel, then Equations (9) and (10) will no longer hold, leading to the so-called “global buckling”. The global buckling mode can be readily checked \[16, 20\], while this study focuses only on the situation when skin buckling occurs before the global buckling mode. For most part of fuselage, global bucking is not permitted prior to the local skin bucking by setting a minimum stiffened ratio. Applying the two conditions in Equations (9) and (10) yields:

\[
k_1 = -k_i
\]

\[
k_2 = \text{csch}(ab) \left[ k_i \cos(b\beta) - k_i \cosh(ab) - k_i \sin(b\beta) \right]
\]  (11)

The two unknown coefficients \( k_i \) and \( k_4 \) can be determined by rotational boundary conditions.
2.2. The rotational boundary condition
The plate's lateral sides are restrained against free rotation by stringers, resulting a similar boundary condition as the Windenburg's problem [6], which is utilized to studying a web's stability by taking into account the flanges' restraint. The equilibrium of the stringer-plate interaction and the stringer's rotational stiffness are two factors to establish the rotational boundary condition.

![Figure 3. Beam torsion model.](image)

Firstly, the stringer is isolated to conduct the stress analysis. Consider a beam subjected to a torque moment $T$ as shown in figure 3, the linear elastic constitutive law reads:

$$T = GJ \frac{d\phi}{dx}, \quad (12)$$

in which $GJ$ is the rotational stiffness of the cross section. For open sections,

$$GJ = 4 \sum_{i=1}^{n} (b_i D_{ii}), \quad (13)$$

where $n$ is the number of cross sectional segments and $b_i$ is the width of the $i$-th segment. For closed sections,

$$GJ = \frac{4A^2}{\sum_{i=1}^{n} b_i A_{66,i}}, \quad (14)$$

where $A$ is the enclosed area of the cross section. If $T$ is generated by distributed torques $t$ acting along the longitudinal direction, the following equilibrium condition can be obtained by differentiating both sides of Equation (12):

$$\frac{dT}{dx} = -t = GJ \frac{d^2\phi}{dx^2}. \quad (15)$$

If the distributed torque of the stringer is supplied by the plate's lateral edge, then the equilibrium condition is given by:

$$M_{yy}(x, b) = -\frac{dT(x)}{dx},$$

$$M_{yy}(x, 0) = \frac{dT(x)}{dx}, \quad (16)$$

in which $M_{yy}$ is the plate's bending moment in the $y$ direction. Recalling

$$M_{yy}(x, y) = -D_{12} \frac{\partial^2 w}{\partial y^2} - D_{12} \frac{\partial^2 w}{\partial x^2}, \quad (17)$$

and $\phi = dw/dy$, then the rotational boundary conditions of the two lateral edges are given by substituting Equation(16) and Equation (17) into Equation (15):
Substituting Equations (6) and (11) into Equation (18), two homogeneous linear equations about $k_i, k_4$ are obtained:

\[
\begin{align*}
D_{22}a^2(\alpha^2 + \beta^2)\sinh(ab) + GJa m^2\pi^2\left[\cosh(ab) - \cos(\beta b)\right]k_1 &+ \\
Gjm^2\pi^2\left[\alpha \sin(\beta b) - \beta \sinh(ab)\right]k_4 &= 0
\end{align*}
\]

\[
\begin{align*}
D_{22}a^2(\alpha^2 + \beta^2)\sinh(ab)\cos(\beta b) + GJa m^2\pi^2\left[-\alpha + \alpha \cosh(ab) \cos(\beta b) + \beta \sinh(ab) \sin(\beta b)\right]k_1 &+ \\
-D_{22}a^2(\alpha^2 + \beta^2)\sinh(ab)\sin(\beta b) + GJa m^2\pi^2\left[-\alpha \cosh(ab) \sin(\beta b) + \beta \sinh(ab) \cos(\beta b)\right]k_4 &= 0
\end{align*}
\]  

The critical value of the compressive load is determined by setting the determinant of Equation (19) to zero, which leads to

\[t_0 + t_1GJ + t_2GJ^2 = 0,\]

in which,

\[t_0 = D_{22}a^4(\alpha^2 + \beta^2)^2\sinh(ab)\sin(\beta b),\]

\[t_1 = 2D_{22}a^2(\alpha^2 + \beta^2)m^2\pi^2\left[\alpha \cosh(ab) \sin(\beta b) - \beta \sinh(ab) \cos(\beta b)\right],\]

\[t_2 = m^4\pi^4\left[2\alpha\beta[1 - \cosh(ab) \cos(\beta b)] + (\alpha^2 - \beta^2)\sinh(ab) \sin(\beta b)\right].\]

For each value of $m$, there exist a solution of $N_n$, and the first buckling load is the solution with the smallest absolute value. If the rotational stiffness of the stringer is 0, then the solution is $t_0 = 0$, so $\beta b = n\pi, n = 1,2L$. This is solution corresponds to a buckling plate with four edges simply supported. If the rotational stiffness of the stringer is infinite, then the solution is $t_2 = 0$, which corresponds to the buckling plate with two lateral edges clamped [18].

2.3. Numerical scheme
Note that the solution to a buckling plate with four edges simply supported takes the form:

\[N_n^{ss}(n) = -\frac{\pi^2\left[D_1m^4 + 2(D_{12} + 2D_m)m^2n(a/b)^2 + D_{22}n^2(a/b)^4\right]}{a^2m^2}.\]

For uniaxial buckling, the critical buckling load always takes place when $n = 1$. It is highlighted that $\beta b = \pi$ with $N_n^{ss} = N_n^{ss}(1)$ and $\beta b = 2\pi$ with $N_n^{ss} = N_n^{ss}(2)$, etc. If $\beta b = \pi$,

\[t_0 = 0,\]

\[t_1 = 2D_{22}a^2(\alpha^2 + \beta^2)m^2\pi^2\beta \sinh(ab) > 0,\]

\[t_2 = 2m^4\pi^4a\beta[1 + \cosh(ab)] > 0.\]

hence the left hand side of Equation (20) is positive when $N_n^{ss} = N_n^{ss}(1)$. If $\beta b = 2\pi$
\[ t_0 = 0, \]
\[ t_1 = -2D_{22}a^3 \left( \alpha^2 + \beta^2 \right) m^2 \pi^2 \beta \sinh(ab) < 0, \]  \hspace{1cm} (26)
\[ t_2 = 2m^4 \pi^4 a \beta \left[ 1 - \cosh(ab) \right] < 0. \]

hence the left hand side of Equation (20) is negative when \( N_{ss} = N_{ss}^{es}(2) \). The solution must lie between \( N_{ss}^{es}(1) \) and \( N_{ss}^{es}(2) \). Therefore a bisection algorithm can be used to find the root to \( N_{ss} \) for each \( m \), and the final solution is the root with the smallest absolute value.

3. Finite element verification

The proposed theoretical method is compared with the commercial finite element package Abaqus for verification. As shown in figure 4, the compressive buckling of a laminate with two lateral edges restraint by beams is considered here. The mechanical properties of the unidirectional tape are: \( E_1 = 150 \) GPa, \( E_2 = 10 \) GPa, \( G_{12} = 5 \) GPa and \( v_{12} = 0.3 \). Two unidirectional tape thicknesses \( t \) are used: \( t = 0.1 \) mm or \( t = 0.2 \) mm. The stack of the laminate is \([45/90/-45/0/-45/45]\). The shear modules of the beam section is \( G = 30 \) GPa, while the torsion constant \( J \) varies from 0 to infinity to simulate the rotational stiffness from simple support to clamp.

In the Abaqus model, the degree of freedom (DOF) of the \( z \)-directional translation of all the four sides are restrained, while the DOF of all rotations are released. The DOF of the \( y \)-directional translation of the two lateral sides are freed to guarantee unidirectional compression. The two beams are attributed with "beam general section" and tied to the lateral sides of the plate.

![Figure 4. The finite element model.](image)

The absolute values of the critical buckling strains are listed in table 1, in which the results predicted by Abaqus and the proposed method are indicated by "FEM" and "Lévy", respectively. The two methods show a satisfied correlation.

| \( J \)          | \( t=0.2 \) | \( t=0.1 \) | \( t=0.2 \) | \( t=0.1 \) |
|------------------|------------|------------|------------|------------|
|                  | FEM        | Lévy       | Error      | FEM        | Lévy       | Error      |
| 0, Simple support| 1.19E-03   | 1.22E-03   | 2.22%      | 2.99E-04   | 3.20E-04   | 7.26%      |
| 10               | 1.33E-03   | 1.43E-03   | 7.95%      | 3.48E-04   | 3.56E-04   | 2.23%      |
| 100              | 1.53E-03   | 1.66E-03   | 8.30%      | 4.50E-04   | 4.70E-04   | 4.29%      |
| 1000             | 1.84E-03   | 1.83E-03   | -0.68%     | 4.79E-04   | 5.00E-04   | 4.50%      |
| 10000            | 1.92E-03   | 2.00E-03   | 4.18%      | 4.82E-04   | 5.03E-04   | 4.36%      |
| 100000           | 1.93E-03   | 2.01E-03   | 4.09%      | 4.83E-04   | 5.03E-04   | 4.27%      |
| 1000000          | 1.93E-03   | 2.01E-03   | 4.25%      | 4.83E-04   | 5.03E-04   | 4.27%      |
| Inf, Clamp       | 2.00E-03   | 2.01E-03   | 0.60%      | 5.01E-04   | 5.03E-04   | 0.51%      |
4. Examples
Two stiffened panel examples are employed to demonstrate the application of the proposed method. One panel is stiffened by T-shaped stringers while the other is stiffened by hat-shaped stringers. The stacks of the skins are [45/90/-45/0/-45/45], and the stack of the stringers are [45/0/-45/90/0]. The panels' lengths are 600mm and the cross sectional profiles are shown in figure 5. The two stingers are deliberately designed to have the same cross sectional area and axial stiffness but distinct torsional stiffnesses.

Figure 5. Cross sectional dimensions of the two panels.

The width $b$ of the skin pocket is set as the axis-to-axis distance of adjacent stringers. The thickness of the unidirectional tape of the stringers is 0.2mm, while the thickness of the unidirectional tape of the skins changes from 0.1 to 0.24 mm, and as a result the $b/t$ ratio varies from 60 to 170. The loading edges are simply supported and three lateral boundary conditions are used: the first is simple support condition, the second is clamp support condition, and the third is the proposed method termed as “rotational support”, taking into account the stringers' rotational restraint. The critical buckling strains (in absolute value) are plotted in figure 6.

Figure 6. Critical buckling strains of stiffened panels.

As shown in figure 6, there is a huge gap between the results obtained by assuming simple support and clamp boundary conditions. For open-section stringers, the result considering stringers’ rotational restraint is close to the result of simple support condition for lower $b/t$ values but as the $b/t$ ratio increases, it approaches the result obtained by the clamp support condition. This phenomenon is identical to that of stiffened metal panels. But for closed-section stringers, the result considering stringers’ rotational restraint is much closer to the result of clamp support condition.
Unfortunately, due to the diversity of stack sequences and the stiffness entries as a result, it is not able to define a few design coefficients or curves for engineers to refer to. Instead, spreadsheets or applets are used for composite analysis. A sample of the hat-shaped panel example is illustrated by a step-by-step explanation in the appendix to demonstrate the application and programming implement of the proposed method.

5. Conclusions
Closed-section stringers’ rotational stiffness contributes significantly to panels’ compressive stability and the restraint is close to clamp condition. A closed-form solution is developed to take into account stringers’ rotational stiffness. The new formula can facilitate stress analysis of composite fuselage panels.

In this study, the plate width is set as the axis-to-axis distance of adjacent stringers, while in some literature, the plate width is set as the foot-to-foot distance of adjacent stringers. The compressive buckling strength is often sensitive to plate width. Such a dispute is still an open question and deserves further verification with experiments.

References
[1] Niu M C-Y 1997 Airframe stress analysis and sizing (Hong Kong: Conmilit Press)
[2] Gerard G 1946 Effective Width of Elastically Supported Flat Plates Journal of the Aeronautical Sciences 13 518-24
[3] Iuspa L and Ruocco E 2008 Optimum topological design of simply supported composite stiffened panels via genetic algorithms Computers & Structures 86 1718-37
[4] Ruocco E and Fraldi M 2012 Critical behavior of flat and stiffened shell structures through different kinematical models: A comparative investigation Thin-Walled Structures 60 205-15
[5] Ruocco E 2015 Elastic/plastic buckling of moderately thick plates and members Computers & Structures 158 148-66
[6] Mittelstedt C and Schagerl M 2010 A composite view on Windenburg's problem: Buckling and minimum stiffness requirements of compressively loaded orthotropic plates with edge reinforcements International Journal of Mechanical Sciences 52 471-84
[7] Chen Q and Qiao P 2015 Post-buckling behavior of imperfect laminated composite plates with rotationally-restrained edges Composite Structures 125 117-26
[8] Bouazza M, Tounsi A, Adda-Bedia E A and Megueni A 2010 Thermoelastic stability analysis of functionally graded plates: An analytical approach Computational Materials Science 49 865-70
[9] Bouazza M, Lairedj A, Benseddiq N and Khalki S 2016 A refined hyperbolic shear deformation theory for thermal buckling analysis of cross-ply laminated plates Mechanics Research Communications 73 117-26
[10] Becheri T, Amara K, Bouazza M and Benseddiq N 2016 Buckling of symmetrically laminated plates using nth-order shear deformation theory with curvature effects Steel and Composite Structures 21 1347-68
[11] Vosoughi A R, Darabi A, Anjabin N and Topal U 2017 A mixed finite element and improved genetic algorithm method for maximizing buckling load of stiffened laminated composite plates Aerospace Science and Technology 70 378-87
[12] Bouazza M, Zenkour A M and Benseddiq N 2018 Closed-from solutions for thermal buckling analyses of advanced nanoplates according to a hyperbolic four-variable refined theory with small-scale effects Acta Mechanica
[13] Wu Z, Raju G and Weaver P M 2018 Optimization of postbuckling behaviour of variable thickness composite panels with variable angle tows: Towards “Buckle-Free” design concept International Journal of Solids and Structures 132-133 66-79
[14] Oliveri V and Milazzo A 2018 A Rayleigh-Ritz approach for postbuckling analysis of variable angle tow composite stiffened panels Computers & Structures 196 263-76
[15] Reddy J N and Miravete A 1995 *Practical analysis of composite laminates* (Boca Raton: CRC)
[16] Kassapoglou C 2013 *Design and analysis of composite structures: with applications to aerospace structures* (Chichester: John Wiley & Sons)
[17] Timoshenko S P 1936 *Theory of Elastic Stability* (New York: McGraw-Hill Book Company)
[18] Reddy J N 2006 *Theory and analysis of elastic plates and shells* (Boca Raton: CRC)
[19] ESDU03001 2003 Elastic buckling of long, flat, symmetrically-laminated (AsBoDf), composite stiffened panels and struts in compression
[20] Seide P and Stein M 1949 Compressive buckling of simply supported plates with longitudinal stiffeners (Washington: NACA)

**Appendix**

The scenario in which the unidirectional tape thickness of both the hat-shaped stringer and skin is 0.2mm in Section 4 is selected for this step-by-step demonstration. The dimensions of the plate are $a = 600 \text{mm}$, $b = 200 \text{mm}$.

**Step 1: Computing the laminate stiffness**

The stacking of the skin and stringer are $[45/90/-45/45]_s$ and $[45/0/-45/90/0]_s$, respectively and material properties are given in Section 3. By the classical lamination theory, the laminate stiffness are given in Table 2:

|                  | Skin   | Stringer | Skin   | Stringer |
|------------------|--------|----------|--------|----------|
| $A_{11}$         | 1.392E5| 1.621E5  | $D_{11}$| 5.247E4  |
| $A_{12}$         | 6.122E4| 3.302E4  | $D_{12}$| 2.794E4  |
| $A_{22}$         | 1.392E5| 1.058E5  | $D_{22}$| 8.402E4  |
| $A_{16}$         | 0      | 0        | $D_{16}$| 9.014E3  |
| $A_{26}$         | 0      | 0        | $D_{26}$| 9.014E3  |
| $A_{66}$         | 6.597E4| 3.699E4  | $D_{66}$| 3.023E4  |

**Step 2: Computing $GJ$**

According to the dimensions illustrated in figure 5, the area enclosed by the stringer is

$$A = \frac{5 + 32}{2} \times 18 = 333 \text{mm}^2.$$  

(27)

The torsional stiffness $GJ$ is calculated by Equation (14) as:

$$GJ = \frac{4 \times 333^3 \times 333}{3.699 \times 10^7 + 22.5 \times 3.699 \times 10^7 + 22.5 \times 3.699 \times 10^7 + 32 \times 6.597 \times 10^7} = 2.415 \times 10^8 \text{N/mm}^2$$

(28)

A stringer supplies rotational restraint to its two neighbour skin pockets, therefore its available stiffness for each plate is half of its torsional stiffness. Thus, the rotational stiffness adopted in the following computation will be half of its nominal value.

**Step 3: Solving Equation (20) for each $m$**

For simplicity, only the situation of $m=1$ is described in details. The upper bound and lower bound of the critical buckling load are determined by Equation (24) for $n=1$ and $n=2$, respectively.

The upper bound is:

$$N_{cr}^{u}(1) = - \frac{\pi^4 \left[ 52470 \times 1 + 2 \times (27940 + 2 \times 30230) \times 1 \times 1 \times (600/200)^2 + 84020 \times 1 \times (600/200)^4 \right]}{600 \times 1}$$

(29)

$$= -231.7 \text{N/mm}$$

The lower bound is:
When the upper and lower bounds are determined, the bisection algorithm is employed to find the solution to Equation (20). Variables during the iteration are listed in Table 3.

Table 3. Process of the bisection process.

| n   | \(N_{cr}^{n}\) | \(\alpha\) by Equation(7) | \(\beta\) by Equation(8) | LHS. of Equation (20) |
|-----|----------------|---------------------------|---------------------------|------------------------|
| 1st | \(-231.7\)     | 0.01744                   | 0.01571                   | 2.38E16                |
| 2nd | \(-3161\)      | 0.03232                   | 0.03142                   | -2.40E18               |
| 3rd | \((-231.7 - 3161)/2 = -1696\) | 0.02780 | 0.02675 | -9.16E17 |
| 4th | \((-231.7 - 1696)/2 = -964.1\) | 0.02429 | 0.02307 | -1.81E17 |
| ... | ...            | ...                       | ...                       | ...                    |
| Last| -572.2         | 0.02148                   | 0.02009                   | 0                      |

The critical buckling load for \(m = 1\) is \(-572.2 \, \text{N/mm}\).

A same procedure as the situation \(m = 1\) is adopted to compute the critical buckling loads corresponding to different values of \(m\). The critical buckling loads are listed in Table 4.

Table 4. Critical buckling loads v.s. \(m\).

| \(m\) | \(N_{cr}\) |
|-------|-----------|
| 1     | -572.2    |
| 2     | -248.5    |
| 3     | -160.2    |
| 4     | -131.2    |
| 5     | -124.7    |
| 6     | -129.4    |
| 7     | -141.1    |
| 8     | -157.9    |

The critical buckling load corresponding to \(m = 5\) is the first buckling load, \(-124.7 \, \text{N/mm}\).

Step 4: computing critical compressive strains

Strains are preferred to stresses in structural analysis of composites. Giving the plate load, the critical strain is computed by the classical lamination theory. The normal strains of the orthotropic plate are given by:

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}^{-1} \begin{bmatrix}
N_{ss} \\
N_{pp}
\end{bmatrix} =
\begin{bmatrix}
1.392E5 & 6.122E4 \\
6.122E4 & 1.392E5
\end{bmatrix}^{-1} \begin{bmatrix}
-124.7 \\
0
\end{bmatrix} =
\begin{bmatrix}
-1.111 \times 10^{-3} \\
4.886 \times 10^{-4}
\end{bmatrix}
\] (31)