Associated quantum vector bundles and symplectic structure on a quantum plane

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Abstract

We define a quantum generalization of the algebra of functions over an associated vector bundle of a principal bundle. Here the role of a quantum principal bundle is played by a Hopf-Galois extension. Smash products of an algebra times a Hopf algebra $H$ are particular instances of these extensions, and in these cases we are able to define a differential calculus over their associated vector bundles without requiring the use of a (bicovariant) differential structure over $H$. Moreover, if $H$ is coquasitriangular, it coacts naturally on the associated bundle, and the differential structure is covariant.

We apply this construction to the case of the finite quotient of the $SL_2$ function Hopf algebra at a root of unity ($q^3 = 1$) as the structure group, and a reduced 2-dimensional quantum plane as both the “base manifold” and fibre, getting an algebra which generalizes the notion of classical phase space for this quantum space. We also build explicitly a differential complex for this phase space algebra, and find that levels 0 and 2 support a (co)representation of the quantum symplectic group. On this phase space we define vector fields, and with the help of the $Sp_q$ structure we introduce a symplectic form relating 1-forms to vector fields. This leads naturally to the introduction of Poisson brackets, a necessary step to do “classical” mechanics on a quantum space, the quantum plane.

Keywords: non commutative geometry, fibre bundles, quantum groups, symplectic structures.

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1 Introduction

The Gel'fand-Naimark theorem [1] tells us that we can equivalently study a topological manifold through its algebra of (continuous) functions. Non commutative (NC) geometry builds upon this fact, considering arbitrary non commutative algebras as algebras of would be functions on hypothetical non commutative (or "quantum") spaces. In this context we have a standard description of vector fibre bundles as (finitely generated, projective) $A$-modules, where $A$ is the algebra encoding the "base manifold". Essentially, elements of this module should be thought as sections of the fibre bundle. In the commutative (classical) case, this correspondence encodes all the structure of the bundle, as asserted by the Serre-Swan theorem.

However, any bundle is, in particular, a manifold (the "total space"). Therefore, in the quantum case we should expect its noncommutative version to be encoded by an algebra. Of course, in the same way that the "total" manifold is obviously not enough to define a classical bundle, in the quantum case we will need an algebra with additional structure. One of the physical motivations for trying to characterize a quantum vector bundle using an algebra is the following. One could try to generalize the standard way of doing classical mechanics to some quantum space. But this needs first the construction of a phase space (that means, the cotangent bundle) over such NC space, and then of a differential algebra on this phase space. Hence, having a module as a phase space is not enough to reach this aim, and we again see the need of an algebra to encode the structure of the total space for such a vector bundle. It should be noted here that an alternative—and more direct—approach is the one taken in [2], where the starting quantum space is considered as a phase space.

As principal fibre bundles are characterized, in NC geometry, by Hopf-Galois extensions (algebras), we first construct (section 2) a quantum generalization of a vector bundle as associated vector bundle to a principal bundle. We illustrate this construction with the example of a quantum plane. Although developed independently, this definition of quantum vector bundles turned out to be essentially the same as the one introduced in [3] and [4]. In section 3 we write down the definition of a differential complex over such bundles. The second part of this work (section 4) is devoted to the definition of a symplectic structure on a quantum bundle over a reduced quantum plane, namely the phase space for the $2D$ reduced quantum plane at $q^8 = 1$. We first define vector fields, their pairing with 1-forms, and a symplectic 2-form. Finally, we define Poisson brackets and we mention some of its properties.

2 Quantum bundles

Let $H$ be a Hopf algebra, and $B \subset P$ two algebras. One says that $P$ is a (right) extension of $B$ by $H$ if $P$ is a (right) $H$-comodule algebra such that

$$B = P^{C_{Ho}} = \{ b \in P / \delta_R b = b \otimes 1_H \}.$$ 

In particular, this means that we can always say that an $H$-comodule algebra $P$ is an $H$-extension of its coinvariant subalgebra $P^{C_{Ho}}$. Moreover, if the map

$$\beta : P \otimes_B P \longrightarrow P \otimes H$$

1
\[ \beta = (m_P \otimes \text{id}) \circ (\text{id} \otimes \delta_R) \]
\[ \beta(p \otimes_B p') \equiv pp_0' \otimes p_1' \quad \text{with} \quad \delta_R p' = p_0' \otimes p_1' \in P \otimes H \]

is bijective, one says that \( P \) is a (right) Hopf-Galois extension of \( B \) by \( H \).

Classically, a Hopf-Galois extension is the dual object to a principal fibre bundle. In fact, in such a case the algebra \( B \) is simply the algebra of functions on the base manifold, \( P \) the algebra of functions on the total space, and \( H \) the algebra of functions on the structure group. The condition involving the \( \beta \) function encodes the fact that all the fibres are isomorphic to the structure group that acts freely on the bundle.

For \( A \) a left \( H \)-module algebra, the smash product algebra \( A\#H \) is defined to be the tensor product \( A \otimes H \) as a vector space, with the modified multiplication
\[ (a\#h)(b\#g) \equiv a(h_1 \triangleright b) \# h_2 g \quad , \quad a, b \in A \quad , \quad h, g \in H \quad , \quad (1) \]

where \( h_1 \otimes h_2 = \Delta h \) is the coproduct. Such a smash product is automatically a right \( H \)-comodule algebra, with
\[ \delta_R(a\#h) \equiv (a\#h_1) \otimes h_2 \quad . \quad (2) \]

Moreover, this is a particular case of a (right) Hopf-Galois extension of \( A \) by \( H \), as it can be easily seen. In fact, it corresponds to a particular case of a globally trivial (quantum) principal bundle, as it has a (dual) analogue of a global section. General trivial bundles are given by cross product algebras \([5]\), in which smash product data is replaced by a weakened form through the introduction of a cocycle (be aware that some authors just call cross product algebras what we have here called smash ones). In the commutative "limit", the action of \( H = C(G) \) on \( A = C(M) \) reduces to the trivial action and we get back the standard tensor product algebra of functions on a globally trivial \( M \times G \) manifold.

Suppose that we now want to build an associated vector bundle, given some principal bundle \( E \). Classically, being \( G \) the structure group of \( E \), one first chooses a vector space \( V \) with a left \( G \)-action. Then the associated vector bundle \( F \) with fibre \( V \) is defined to be the quotient of \( E \times V \) by the equivalence relation \((e, gv) \sim (eg, v) \), \( e \in E, v \in V, g \in G \).

Dually, we need a left \( H \)-comodule algebra \( W \) playing the role of the algebra \( C(V) \). Now, the associated vector bundle will be encoded by
\[ Q \equiv P \square_H W \quad (3) \]

which corresponds to the algebra of functions on \( F \). Here the cotensor product \( P \square_H W \) is defined to be
\[ P \square_H W \equiv \{ p_i \otimes w_i \in P \otimes W \quad / \quad \delta_R p_i \otimes w_i = p_i \otimes \delta_L w_i \} \quad . \]

As it can be checked easily, this subspace of \( P \otimes W \) is invariant under the product of the tensor product algebra. Therefore, \( P \square_H W \) is in fact an algebra, with
\[ (p \otimes w)(p' \otimes w') = (pp' \otimes ww') \quad \text{whenever} \quad p \otimes w, p' \otimes w' \in P \square_H W \quad . \]

If we take \( P \) to be a smash product, \( P = A\#H \), then we will have \( Q = P \square_H W = A\#H \square_H W \), a vector subspace of \( A \otimes H \otimes W \). But we see that the left coaction \( \delta_L \) on
$W$ is a vector space isomorphism between $H \boxtimes W$ and $W$, and hence $id \otimes \delta_L$ is a vector space isomorphism between $A \otimes H \boxtimes W$ and $A \otimes W$. Using this isomorphism we can now map the general product found in $Q$ to this isomorphic space, finding that $A \otimes W$ is an algebra —that we call $A \boxtimes W$— with

$$(a \boxtimes v)(b \boxtimes w) = a (v_{-1} \triangleright b) \boxtimes v_0 w.$$ 

So, in this particular case we will simply take $Q = A \boxtimes W$ as the algebra encoding the associated vector bundle. We also mention that, if $H$ is coquasitriangular with universal $r$-form $r$, there is a left coaction $\Delta_L$ of $H$ on $Q$ given by

$$\Delta_L (a \boxtimes w) = a_{-1} w_{-1} \otimes (a_0 \boxtimes w_0).$$

Here, the left $H$-comodule structure on $A$ ($\delta_L a = a_{-1} \otimes a_0$) should be related to the left $H$-action on $A$ by coquasitriangularity,

$$h \triangleright a \equiv r(a_{-1} \otimes h) a_0.$$ 

### 2.1 Phase space for a quantum plane

As an example of the previous construction, and to use it afterwards, we define here a bundle over a quantum plane. Let $q$ be a cubic root of unity, $q^3 = 1$, and call $M$ the reduced quantum plane defined by

$$xy = q yx, \quad x^3 = y^3 = 1.$$ 

(4)

In order to obtain a “phase space” over $M$ we obviously need to take $A = M$. By analogy with the classical case we also take $W = M$. Therefore the natural candidate for $H$ is the quantum group $F = Fun(SL_q(2, \mathbb{C})) / a^3 = d^3 = 1, \quad b^3 = c^3 = 0$ —and not its dual, see the discussion about classical counterparts above. We refer to $[6, 7]$ for a much more detailed analysis of such objects and further references on the subject, and in particular for explicit formulas of the Hopf structure on $F$ according to our conventions. See also $[8]$, where tangent and cotangent bimodules associated to a Hopf-Galois extension of the $2d$ quantum plane by the biparametric quantum group $Fun(GL_q(p, 2))$ are built and analysed. Obviously, this bigger Hopf algebra can be reduced to $F$ suitably choosing the parameter $p$ and taking the corresponding quotient when $q$ is a root of unit; therefore this construction provides a pair of tangent/cotangent bimodules for $M$. Actually, the algebra $Fun(SL_q(2, \mathbb{C}))$ itself is a Hopf-Galois extension by $F$ of its classical counterpart $Fun(SL(2, \mathbb{C}))$; this was mentioned by $[9]$ and explicitly shown in $[10]$. However, we do not use this interesting property in the present paper.

Now we need both a left coaction and a left action of $F$ on $M$. The coaction we take is the standard one, given by

$$\delta_L \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \otimes \left( \begin{array}{c} x \\ y \end{array} \right).$$

Of course, for the action we could take the trivial one, but that would reduce the smash product $M \# F$ to a tensor product... the results would be uninteresting. However, using the fact that $F$ is coquasitriangular (because $Fun(SL_q(2, \mathbb{C}))$ is so $[11]$) with universal
r-form $r(T^j_i \otimes T^k_i) \equiv q R_{jk}^i$, we can map left comodules to left modules. Doing so with the comodule $(\mathcal{M}, \delta_L)$ we get the following non-trivial left action of $\mathcal{F}$ on $\mathcal{M}$:

| $\triangleright$ | $x$ | $y$ |
|-----------------|-----|-----|
| $a$             | $q^2 x$ | $q y$ |
| $b$             | $0$ | $(q^2 - 1) x$ |
| $c$             | $0$ | $0$ |
| $d$             | $q x$ | $q^2 y$ |

Just to simplify the formulas, we introduce a more compact notation for the generators of $\mathcal{M} \Box \mathcal{M}$, defining

$$x \equiv x \Box 1 \quad p_x \equiv 1 \Box x$$

$$y \equiv y \Box 1 \quad p_y \equiv 1 \Box y$$

(5)

Now the product (“commutation”) relations on $\mathcal{M} \Box \mathcal{M}$ may be written as

$$p_x x = q^2 x p_x \quad p_x p_y = q p_y p_x$$

$$p_x y = q y p_x + (q^2 - 1) x p_y \quad p_x^3 = 1$$

$$p_y x = q x p_y \quad p_y^3 = 1$$

(6)

These relations are reminiscent of those of the Wess-Zumino (reduced) complex [12, 6], but they are not the same. Of course, due to the notation (5), the relations (6) are still valid in this bigger algebra. We also note that, disregarding the cubic relations, this algebra has the same Poincaré series as the commutative one (functions over a 2 + 2-dimensional phase space).

### 3 Differential calculus on a vector bundle

Now we will show that it is possible to build a differential calculus over a quantum vector bundle of the type $Q = A \Box W$, making use only of covariant differential calculi $\Omega(A), \Omega(W)$ —that is, no forms “on the quantum group $H$” are ever needed, as would be the case for a more general $Q$.

As we said, we need

- $\Omega(A)$ a left $H$-module differential calculus, $h \triangleright d a = d(h \triangleright a)$.
- $\Omega(W)$ a left $H$-comodule differential calculus, $\delta_L(dw) = (id \otimes d)\delta_L w$.

Since $A \Box W$ coincides with $A \otimes W$ as a vector space, we also take as a vector space the following equality:

$$\Omega^1(Q) = \Omega^1(A \otimes W) = \Omega^1(A) \otimes W \oplus A \otimes \Omega^1(W)$$.
The differential operator $D : Q \mapsto \Omega^1(Q)$ should be taken as

$$D(a \square w) = da \square w + a \square dw.$$  

Only the bimodule structure should be made non-trivial, to reflect the fact that $Q = A \square W$ has a product which is different from the one in $A \otimes W$ (but similar to the product needed in $\Omega(A \# H)$, see [13]). The good choice is (the linear extension of)

$$(a \square v) (db \square w + c \square dt) \equiv a (v_{-1} \triangleright db) \square v_0 w + a (v_{-1} \triangleright c) \square v_0 dt$$

$$(db \square w + c \square dt) (a \square v) \equiv db (w_{-1} \triangleright a) \square w_0 v + c (t_{-1} \triangleright a) \square dt_0 v$$

The generalization to a higher order differential calculus is pretty straightforward. As vector spaces one should take

$$\Omega(A \square W) \equiv \bigoplus_{n \geq 0} \Omega^n(A \square W)$$

$$\Omega^n(A \square W) \equiv \bigoplus_{k=0}^{n} \left[ \Omega^{n-k}(A) \otimes \Omega^k(W) \right]$$

The differential operator is graded, so we must set

$$D : \Omega^n(A \square W) \longrightarrow \Omega^{n+1}(A \square W)$$

$$D(\alpha \square \omega) = d\alpha \square \omega + (-1)^k \alpha \square d\omega, \quad \alpha \in \Omega^k(A), \ \omega \in \Omega^{n-k}(W).$$

We also need a product for this differential calculus, extending the bimodule structure already written:

$$(\alpha \square \nu)(\beta \square \omega) = (-1)^{(n-k)j} \alpha(\nu_{-1} \triangleright \beta) \square \nu_0 \omega$$  

$$\alpha \in \Omega^k(A), \ \nu \in \Omega^{n-k}(W), \ \beta \in \Omega^j(A), \ \omega \in \Omega^{m-j}(W).$$

### 3.1 Differential algebra on $\mathcal{M} \square \mathcal{M}$

Let’s now show explicit formulas for our example, the differential algebra $\Omega(\mathcal{M} \square \mathcal{M})$ on the phase space of the quantum plane $\mathcal{M}$. To this end we obviously need the differential algebra $\Omega(\mathcal{M})$ [13, 14]. It is given by

$$xdx = q^2 dx x \quad dx^2 = 0$$
$$xdy = q dy x + (q^2 - 1) dx y \quad dy^2 = 0$$
$$ydx = q dx y \quad dx dy = -q^2 dy dx$$
$$ydy = q^2 dy y$$

Here we use again the shortened notation, calling $dz \square 1 = dz$, $1 \square dz = dp_z$, with $z = x, y$. In this way, the product relations happen to be those found at level zero, already shown
in (8), plus the following level 1 equalities,

\[
\begin{align*}
x \, dx &= q^2 \, dx \, x \\
y \, dx &= q \, dx \, y \\
x \, dy &= q \, dy \, x + (q^2 - 1) \, dx \, y \\
y \, dy &= q^2 \, dy \, y \\
dp_x \, x &= q^2 \, dp_x \\
dp_x \, y &= q \, dp_x + (q^2 - 1) \, dp_x \, y \\
dp_y \, x &= q \, dp_y \\
dp_y \, y &= q^2 \, dp_y \\
\end{align*}
\]

(10)

and finally the level 2 ones,

\[
\begin{align*}
dx^2 &= dy^2 = 0 \\
dp_x \, dx &= -q^2 \, dx \, dp_x \\
dp^2_x &= dp^2_y = 0 \\
dp_x \, dy &= -q \, dy \, dp_x + (1 - q^2) \, dx \, dp_y \\
dx \, dy &= -q^2 \, dy \, dx \\
dp_x \, dp_y &= -q^2 \, dp_y \, dp_x \\
dp_y \, dx &= -q \, dx \, dp_y \\
dp_y \, dy &= -q^2 \, dy \, dp_y \\
\end{align*}
\]

(11)

Hence, any monomial can be reordered in a predetermined way, and \( \Omega(\mathcal{M} \Box \mathcal{M}) \) has the same Poincaré series as its classical counterpart.

Notice that the algebra spanned by \( \{dx, dy\} \) is the Manin dual of the algebra spanned by \( \{x, y\} \), and the same is true for \( \{dp_x, dp_y\} \) and \( \{p_x, p_y\} \). However, we should stress the fact that this fails to be true for \( \{dx, dy, dp_x, dp_y\} \) and \( \{x, y, p_x, p_y\} \).

4 Symplectic structure on \( \mathcal{M} \)

4.1 Symplectic q-group on phase space

In can be checked that levels 0 and 2 of the differential complex \( \Omega(Q) \) on the phase space \( Q = \mathcal{M} \Box \mathcal{M} \) (relations (3) are corepresentation spaces for the \( \check{Sp}_q(2) \) quantum group
(in addition to being also $SL_q(2)$ covariant!). In fact, using the following identifications,

\[
\begin{align*}
    x &\leftrightarrow x_1 & dp_x &\leftrightarrow \xi_1 \\
    y &\leftrightarrow x_2 & dp_y &\leftrightarrow \xi_2 \\
    p_x &\leftrightarrow x_3 & dx &\leftrightarrow \xi_3 \\
    -p_y &\leftrightarrow x_4 & -dy &\leftrightarrow \xi_4
\end{align*}
\]  

(12)

the product relations at levels 0 and 2 of the differential algebra $\Omega(Q)$ can be rewritten

\[
\begin{align*}
    (\hat{R}_{Sp} - q)(x \otimes x) &= 0 \\
    \left(\hat{R}_{Sp}^2 - \hat{R}_{Sp} + 1\right)(\xi \otimes \xi) &= 0 .
\end{align*}
\]

Here $\left(\hat{R}_{Sp}\right)_{ij}^{kl} = (R_{Sp})_{ji}^{kl}$, and $R_{Sp}$ is the $R$-matrix of the $Sp_q(2)$ quantum group to be found in [14]. Remark that here $\xi_i \neq dx_i$ . . . this is why this symmetry is not found at level 1.

4.2 Vector fields

In differential geometry, vector fields are defined as the $C(M)$-module of derivations $\text{Der}(C(M))$ on $C(M)$. However, this definition is not longer a good one for the quantum case, as the space $\text{Der}(A)$ would only be a module over the center of the corresponding algebra $A$. So this gives rise to different possibilities for defining vector fields. But having already a differential algebra $\Omega(A)$ for a quantum space, the natural thing to do is to build vector fields as objects dual to 1-forms. In general, they happen to be twisted derivations.

In the same way as in the case of the 2d quantum plane [12], here we define derivative operators on the phase space $Q = M \boxtimes M$ as the ones such that

\[
d = dx \partial_x + dy \partial_y + dp_x \partial_{p_x} + dp_y \partial_{p_y} .
\]

(13)

The operators $\partial$ exist and are well defined because any 1-form can be expanded in a unique way using $dx, dy, dp_x, dp_y$ as generators of $\Omega^1(Q)$ as a right $Q$-module. More generally, one can define in a similar way derivations for an associative algebra $Q$ such that the uniqueness condition of the right expansion applies (replacing $dx, dy, dp_x, dp_y$ by a set of generators $\{dz_i\}$ of $\Omega^1(Q)$ as a $Q$-bimodule). The derivatives $\partial$ happen to be twisted derivative operators, as they will generally not satisfy Leibniz’s rule. Operators $\partial$ naturally generate a $Q$-(bi)module, remembering that elements of $Q$ may also be thought as (multiplicative) operators $Q \mapsto Q$. This space $D^1$ is the bimodule of first order differential operators. We also introduce a bilinear mapping $\langle , \rangle : \Omega^1 \otimes D^1 \mapsto Q$, pairing one-forms and vector fields as follows:

\[
\langle df, g \rangle \equiv X(f)g, \quad f, g \in Q, \quad X \in D^1 .
\]

One can show that this pairing is well defined, given the choice in [13].
4.3 Symplectic 2-form

Classically, a symplectic form is a closed non-degenerate 2-form. The first condition may be automatically fulfilled in the NC context by taking \( \omega = w_{ij} \xi_i \xi_j \), with \( w_{ij} \in \mathbb{C} \). Moreover, higher order forms are classically defined to be wedge (antisymmetric) products of 1-forms, and a pairing of such forms with vector fields is defined componentwise. On the contrary, here we are using forms which do not correspond to the antisymmetrization of a product of 1-forms. Therefore this fact should be taken care of in the pairing with vector fields.

So, we define the evaluation of a symplectic form \( \omega \) on a pair of two vector fields using \( A_{Sp} = \left( \hat{R}_{Sp}^2 - q^2 \right) / \left( q - q^{-1} \right) \), the \( Sp_q(2) \) antisymmetrizer (classically, it enters in the wedge product):

\[
\omega(X, Y) \equiv w_{ij} A_{ij,kl} \langle \xi_k, X \rangle \langle \xi_l, Y \rangle .
\]

In this way equivalent expressions of \( \omega \) obtained by reordering \( \xi_i, \xi_j \) using (11) result in the same value of the above pairing with \( X \) and \( Y \), since the symmetrizer \( S_{Sp} = - \left( q^2 / 2 \right) \left( \hat{R}_{Sp}^2 - \hat{R}_{Sp} + 1 \right) \) is involved in such a reordering. Now, any closed \( \omega \) provides a map between vector fields and 1-forms, through the relation

\[
\omega(X df, \cdot) = \langle df, \cdot \rangle .
\]

The non-degeneracy of the symplectic form is implemented by requiring this mapping to be one-to-one [13].

Notice that, for an arbitrary associative algebra \( Q \), symplectic forms associated to the differential calculus \( \Omega_{Der}(Q) \) (see [13]) lead to Poisson brackets that are distinct from those described here.

For the case \( Q = M \square M \) we can simply chose the non-degenerate 2-form

\[
\omega = q (dx dp_x + dy dp_y) ,
\]

and we get

\[
X_{dx} = q^2 \partial_{p_x} \quad X_{dp_x} = -q \partial_x \\
X_{dy} = q^2 \partial_{p_y} \quad X_{dp_y} = -q \partial_y .
\]

4.4 Poisson brackets

Having the 2-form \( \omega \), and a way to evaluate it on a pair of vector fields, we can now introduce Poisson brackets on \( Q = A \square W \) by

\[
\{ f, g \} \equiv \omega(X_{df}, X_{dg}) \quad f, g \in M \square M \\
= \langle df, X_{dg} \rangle = X_{dg}(f) .
\]

The last expression makes it evident that these Poisson brackets are twisted derivations, considered as operators on their first variable, since \( X_{dg} \) itself is so.

In the case \( Q = M \square M \), the brackets amongst generators are easy to get, and one finds that the only non-zero ones are

\[
\{ x, p_x \} = -q \quad \{ p_x, x \} = q^2 \\
\{ y, p_y \} = -q \quad \{ p_y, y \} = q^2 .
\]

It can also be checked that the brackets \( \{ 1, f(x, y, p_x, p_y) \} \), \( \{ f(x, y, p_x, p_y), 1 \} \), and \( \{ f(x, y), g(x, y) \} \) are all vanishing.
4.5 Final comments

The Poisson brackets introduced above may be used to define an analogue of canonical equations of motion, after choosing a Hamiltonian function $h \in \mathcal{M} \square \mathcal{M}$, by $\partial_t f = \{f, h\}$. However, a real structure should be first defined on the phase space and its differential algebra, in order to select a real symplectic form $\omega$. Moreover, we must remark that the “time evolution” determined by these equations is twisted ($\partial_t$ is not a derivation).

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