Universal Khovanov homology for singular tangles and a categorified Vassiliev skein relation

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In this paper, we construct an invariant for singular tangles as an extension of Bar-Natan’s complex of cobordisms for ordinary tangles, aka the universal Khovanov homology, in view of Vassiliev theory. More precisely, we introduce a morphism, called the genus-one morphism, realizing crossing change on the universal Khovanov complex and prove that its mapping cone yields an invariant of singular tangles satisfying a categorified analogue of Vassiliev skein relation. As a result, we obtain extensions of variants of Khovanov homology to singular links; examples include Lee homology and Bar-Natan homology as well as Khovanov homology with arbitrary coefficients. From the viewpoint of Vassiliev invariants, we also prove that our extension satisfies the FI relation.

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1 Introduction

In view of seminal works of Birman [4] and Birman-Lin [5], quantum invariants of knots are related to Vassiliev invariants. This aspect is, however, unclear in case of link homologies. In an attempt to establish it, the first problem we have to solve is the following.

Problem 1.1. Extend a link homology to singular links so that a categorified version of Vassiliev skein relation holds.

Since Vassiliev invariants had been treated in some general settings, we prefer to “tangle” homology; for example, Vassiliev invariant classifies pure braids (cf. Kohno [12]), and Milnor invariants of string links are known as Vassiliev types (cf. Bar-Natan [1] and Lin [17]). Hence, in addition to Problem 1.1, we also interested in the problem below.

Problem 1.2. Define singular tangle homology based on Vassiliev theory.

The goal of this paper is to give answers to the problems above in the case of the universal Khovanov homology. More precisely, we construct a chain complex whose homotopy type is a singular tangle invariant (Main Theorem A) and show that compositions of singular tangles are encoded in terms of composition of cobordisms (Main Theorem B). Furthermore, we obtain a categorified version of the FI relation (Main Theorem C), which is one of the fundamental relations appearing in Kontsevich’s universal construction of Vassiliev invariants [13].

We obtained an answer to Problem 1.1 in our previous paper [8] in the case of Khovanov homology with coefficients in the field $\mathbb{F}_2$ of two elements. Indeed, we constructed a chain map $\hat{\Phi}$, called the genus-one morphism realizing crossing-changes on Khovanov complexes with coefficients in $\mathbb{F}_2$ and showed that Khovanov homology extends to a singular link invariant with an isomorphism

$$C^{*,*} \left( \begin{array}{c}
X \\
\mathbb{F}_2
\end{array} \right) \cong \text{Cone} \left( C^{*,*} \left( \begin{array}{c}
X \\
\mathbb{F}_2
\end{array} \right) \xrightarrow{\hat{\Phi}} C^{*,*} \left( \begin{array}{c}
\bigotimes \\
\mathbb{F}_2
\end{array} \right) \right), \quad (1.1)$$

here the right hand side is the mapping cone. Note that (1.1) categorifies Vassiliev skein relation in terms of graded Euler characteristics; specifically, it recovers Vassiliev skein
relation of the Jones polynomial. This property distinguishes our crossing-change from others, e.g. the one considered by Hedden and Watson [7], which yields Jones skein relation instead.

On the other hand, there are variants of Khovanov homology such as Lee homology [16] or Bar-Natan homology [2] as well as Khovanov homology with integral coefficients, and the construction above does not work for them unfortunately. Aiming at extending the results in [8] to them, we use Bar-Natan’s category of “picture” [2] with a little bit different notations to emphasize the relation to topological field theories (cf. [14]). Namely, let $k$ be a fixed coefficient ring. For compact oriented 0-dimensional manifolds $Y_0$ and $Y_1$, let us denote by $k\text{Cob}_2(Y_0, Y_1)$ the $k$-linear category with cobordisms $Y_0 \to Y_1$ as objects and formal sums of 2-bordisms (aka. cobordisms with corners [15]) between them with coefficients in $k$ as morphisms. Bar-Natan introduced three relations on the category called $(S)$, $(T)$, and $(4Tu)$; the quotient category will be denoted by $\text{Cob}_2^\ell(Y_0, Y_1)$ in this paper. He then constructed a chain complex $[D]$ in $\text{Cob}_2^\ell(\partial^- D, \partial^+ D)$ for each tangle diagram $D$ and proved the following theorem.

**Theorem 1.3** (Bar-Natan [2, Theorem 1]). The homotopy type of $[D]$ is invariant under Reidemeister moves so that it defines an invariant of tangles.

In case of link diagrams, one can recover major variants of Khovanov homology from the complex $[D]$: if $A$ is the Frobenius algebra over $k$ giving either the original Khovanov homology, Lee homology, or Bar-Natan homology, then it turns out the associated 2-dimensional TQFT $\text{Cob}_2(\emptyset, \emptyset) \to \text{Mod}_k$ induces a functor $Z_A : \text{Cob}_2^\ell(\emptyset, \emptyset) \to \text{Mod}_k$. The link homology induced by $A$ is obtained as the homology of the complex $Z_A([D])$. In other words, $[-]$ is “universal” among such variants, and for this reason, we call it the universal Khovanov complex (though it may sometimes refer to the complex appearing in [19]).

We extend the universal Khovanov homology to singular tangle diagrams as follows. We lift the genus-one morphism $\hat{\Phi}$ in (1.1) to the universal Khovanov homology based on the morphism

$$
\begin{array}{cccc}
\text{ cubic diagram } & \to & \text{ cube of chain complexes } \\
\end{array}
$$

In fact, in view of Viro’s exact sequence [23], it turns out that this induces a morphism of chain complexes

$$
\hat{\Phi} : \begin{bmatrix} X \end{bmatrix} \to \begin{bmatrix} X \end{bmatrix}.
$$

(1.2)

For a singular tangle diagram $D$, the genus-one morphisms $\hat{\Phi}$ on the double points give rise to a cube of chain complexes. We then define $[D]$ by “folding” the cube as with Khovanov’s original construction of Khovanov homology.
Main Theorem A. The homotopy type of the chain complex \([D]\) defines an invariant of singular tangles such that there is an isomorphism
\[
\begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix} \cong \text{Cone} \left( \begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix} \overset{\hat{f}}{\to} \begin{bmatrix}
\begin{array}{c}
\end{array}
\end{bmatrix} \right).
\]

Furthermore, if \(D\) is a singular link diagram, then with respect to the Frobenius algebra \(\mathbb{F}_2[x]/(x^2)\), the induced chain complex over \(\mathbb{F}_2\) agrees with the extended Khovanov complex obtained in [3].

Bar-Natan [2, Theorem 2] proved that the universal Khovanov homology on tangles admits operations of a planar algebra [11]. In view of the topological field theory, we rather formulate them in terms of compositions of cobordisms. Namely, the gluing and the disjoint union of cobordisms yield the following \(k\)-bilinear functors:
\[
(-) \ast (-) : \text{Cob}_2(Y_1, Y_2) \times \text{Cob}_2(Y_0, Y_1) \to \text{Cob}_2(Y_0, Y_2),
\]
\[
(-) \otimes (-) : \text{Cob}_2(Y_0, Y_1) \times \text{Cob}_2(Y_0', Y_1') \to \text{Cob}_2(Y_0 \amalg Y_0', Y_1 \amalg Y_1').
\]

Main Theorem B. (1) Let \(D\) be the composition of singular tangle diagrams \(D'\) and \(D''\). Then, there is an isomorphism
\[
[D] \cong [D'] \ast [D''].
\]

(2) Let \(D\) be the tensor product of tangle diagrams \(D'\) and \(D''\). Then, there is an isomorphism
\[
[D] \cong \begin{cases} 
[D'] \otimes [D''] & \text{if } D' \text{ is an even tangle,} \\
[D'] \otimes \rho_0([D'']) & \text{if } D' \text{ is an odd tangle,}
\end{cases}
\]

here \(\rho_0\) is the functor realizing the orientation reversing of cobordisms.

According to Vassiliev’s study [21], the homology classes of the space of knots suggest varieties of relations between singular knots. The basic framework is as follows: let us denote by \(\mathcal{M}\) the space of generic smooth maps \(S^1 \to \mathbb{R}^3\) equipped with Whitehead \(C^\infty\)-topology. Then, Thom-Boardman theory (see [6, Chapter VI]) gives rise to a stratification \(\mathcal{M} = \bigcup_i \mathcal{M}_i\): for example,

- \(\mathcal{M}_0\) consists of smooth embeddings;
- \(\mathcal{M}_1\) consists of smooth immersions with exactly one double point;
- \(\mathcal{M}_2\) consists of
  - (a) smooth injections with exactly one critical point and
  - (b) smooth immersion with exactly two double points.
Although $\mathcal{M}$ is not finite dimensional, it turns out that each stratum $\mathcal{M}_i \subset \mathcal{M}$ has codimension exactly $i$. Since knot invariants can be seen as cohomology classes on $\mathcal{M}$, “Poincaré duality” hence implies that the homology class of $\mathcal{M}_i$ yields degree $i$ relations. For instance, Vassiliev skein relation comes from the stratum $\mathcal{M}_1$.

The $FI$ relation is one of such relations: for a singular knot invariant $v$ with values in an abelian group, it is represented as

$$v \begin{bmatrix} \chi \end{bmatrix} = 0 \quad .$$

(1.3)

In view of Vassiliev theory, the equation is derived as follows: for a point in $\mathcal{M}_2$ of type (a) above, its neighborhood in $\mathcal{M}$ is depicted as in Fig. 1.1 and let us consider the two paths (1) and (2). If a knot $K$ moves along the path (1), the value of $v(K)$ does not change. On the other hand, if it goes along (2), $v(K)$ is subject to Vassiliev skein relation since it crosses the “wall” $\mathcal{M}_1$. Comparing the effects of the two paths, we obtain the equation (1.3).

The FI relation is fundamental in Vassiliev theory; in fact, it naturally appears in Kontsevich’s construction of the universal Vassiliev invariant [13]. Actually, our extension of the universal Khovanov complex satisfies a categorified analogue.

Main Theorem C. The complex below is contractible, i.e. the identity is null-homotopic

$$\begin{bmatrix} \chi \end{bmatrix}$$

To the best of our knowledge, there is no singular tangle homology except ours known to satisfy the categorified FI relation in the sense of Main Theorem C. In other words, Main Theorem C distinguishes our invariant from the others.

The plan of this paper is as follows. We review the universal Khovanov complex for ordinary tangles in Section 2. In particular, the category $\text{Cob}^\ell_2(Y_0, Y_1)$ is defined in terms of cobordisms with corners; we mainly follow [20] for this material. We also define the
universal bracket complex as the “unshifted” version of the universal Khovanov complex. The checkerboard colorings are discussed to determine the orientations on cobordisms.

We then define the genus-one morphism $\hat{\Phi}$ in Section 3. In Section 4, we show the invariance of $\hat{\Phi}$ under the moves of singular tangles. Using the result, in Section 5, we then extend the universal Khovanov complex to singular tangles. The compositions and the FI relations are discussed.

2 The universal Khovanov complex

2.1 Cobordisms of manifolds with corners

In order to develop tangle homology, we need the notion of cobordism of manifolds with corners. We here give a brief sketch, and for details, we refer the reader to [9, Definition 1], [15], and [20].

Let $Y_0$ and $Y_1$ be closed oriented 0-manifolds (i.e. finite sets with a label \{-, +\} on each element), and let $W_0$ and $W_1$ be two cobordisms from $Y_0$ to $Y_1$. Then, a 2-bordism from $W_0$ to $W_1$ is a compact oriented 2-manifold $S$ with corners such that

- $\partial S$ is a union of submanifolds: $\partial S = \partial_0 S \cup \partial_1 S$;
- it is equipped with orientation-preserving diffeomorphisms

$$s_0 : \overline{W_0} \amalg W_1 \xrightarrow{\cong} \partial_0 S, \quad s_1 : (\overline{Y_0} \amalg Y_1) \times [0, 1] \xrightarrow{\cong} \partial_1 S,$$

here $\overline{W_0}$ and $\overline{Y_0}$ are respectively the manifolds $W_0$ and $Y_0$ with the reversed orientation.

In this case, we write $S : W_0 \rightarrow W_1 : Y_0 \rightarrow Y_1$ or $S : W_0 \rightarrow W_1$ simply.

Definition 2.1. For closed oriented 0-manifolds $Y_0$ and $Y_1$, we define a category $\text{Cob}_2(Y_0, Y_1)$ as follows:

- the objects are cobordisms $W : Y_0 \rightarrow Y_1$;
- the morphisms are diffeomorphism classes of 2-bordisms $S : W_0 \rightarrow W_1 : Y_0 \rightarrow Y_1$, where only diffeomorphisms that preserve orientations and structure maps are considered;
- the composition is given by gluing.

Remark 2.2. By Collar Neighborhood Theorem [15, Lemma 2.1.6], every chain of bordisms actually admits gluing. It also turns out that gluing is unique up to diffeomorphisms of bordisms. In general, though a choice of such diffeomorphisms is not canonical, it can be done within a canonical choice of an isotopy class [20].

In view of Remark 2.2, the composition is associative. For a cobordism $W : Y_0 \rightarrow Y_1$, the identity on $W$ is represented by the trivial 2-bordism $W \times [0, 1]$. Hence, $\text{Cob}_2(Y_0, Y_1)$ is a category.
Convention. In this paper, we always use the “bottom-to-top” convention for cobordisms and the “left-to-right” one for 2-bordisms as in Fig. 2.1

The gluing also gives rise to a functor. Indeed, if \( Y_0, Y_1 \) and \( Y_2 \) are closed oriented 0-manifolds, then we define a functor

\[
(-) \ast (-): \mathbf{Cob}_2(Y_1, Y_2) \times \mathbf{Cob}_2(Y_0, Y_1) \to \mathbf{Cob}_2(Y_0, Y_2)
\]

as follows (see [20] for details):

- for objects, if \( W: Y_0 \to Y_1 \) and \( W': Y_1 \to Y_2 \), then choose a gluing \( \tilde{W} \) of \( W \) and \( W' \) along \( Y_1 \), and set \( W \ast W' := \tilde{W} \) (which is often denoted by \( W' \circ W \));
- for morphisms, if \( S: W_0 \to W_1 : Y_0 \to Y_1 \) and \( S': W_0' \to W_1' : Y_1 \to Y_2 \), then \( S \ast S' \) is the diffeomorphism class of a gluing of \( S \) and \( S' \) along \( Y_1 \) with respect to \( W_0' \circ W_0 \) and \( W_1' \circ W_1 \).

Lemma 2.3 ([20]). The construction above actually defines a unique functor (2.1) up to a canonical isomorphism.

The disjoint union of manifolds gives rise to another functor

\[
(-) \otimes (-): \mathbf{Cob}_2(Y_0, Y_1) \times \mathbf{Cob}_2(Y_0', Y_1') \to \mathbf{Cob}_2(Y_0 \amalg Y_0', Y_1 \amalg Y_1')
\]

This functor is associative in the sense that it defines an essentially unique functor

\[
\mathbf{Cob}_2(Y_0^{(1)}, Y_1^{(1)}) \times \cdots \times \mathbf{Cob}_2(Y_0^{(r)}, Y_1^{(r)})
\to \mathbf{Cob}_2(Y_0^{(1)} \amalg \cdots \amalg Y_0^{(r)}, Y_1^{(1)} \amalg \cdots \amalg Y_1^{(r)})
\]

In particular, in the case \( Y_0 = Y_1 = \emptyset \), we obtain a symmetric monoidal structure on the category \( \mathbf{Cob}_2(\emptyset, \emptyset) \). We further introduce two functors, both of which are given by the orientation reversion:

\[
\rho_0: \mathbf{Cob}_2(Y_0, Y_1) \to \mathbf{Cob}_2(\overline{Y}_0, \overline{Y}_1)
\]

on cobordisms \( W \mapsto \overline{W} \)

on 2-bordisms \( S \mapsto \overline{S} \)

\[
\rho_2: \mathbf{Cob}_2(Y_0, Y_1)^{op} \to \mathbf{Cob}_2(Y_0, Y_1)
\]

on cobordisms \( W \mapsto W \)

on 2-bordisms \( S \mapsto S \)

These functors respect gluing and disjoint union.
2.2 The category $\text{Cob}^\ell_2(Y_0, Y_1)$

Let $k$ be a commutative ring and $\mathcal{C}$ a $k$-linear category. We define a category $\text{Mat}(\mathcal{C})$ as follows [18, VIII.2, Exercise 6]:

- An object is a tuple $(A_1, A_2, \ldots, A_n)$, which is denoted by $\bigoplus_{i=1}^n A_i$ of $n \in \mathbb{N}$ and $A_i \in \mathcal{C}$.

- For objects $\bigoplus_{i=1}^n A_i$ and $\bigoplus_{j=1}^m B_j$, a morphism is defined by the set $\{f_{ij} : A_i \to B_j\}_{i=1, j=1}^n$ here $f_{ij}$ is a morphism of $\mathcal{C}$.

- Compositions of morphisms are defined in the same way as the matrix multiplication.

**Notation 2.4.** For $\{f_{ij}\} \in \text{Mat}(\mathcal{C})$, we often denote it by $\sum_{i,j} f_{ij}$ if there is no danger of confusion.

Let $Y_0$ and $Y_1$ be compact oriented $0$-manifolds. For a fixed commutative ring $k$, we extend the category $\text{Cob}_2(Y_0, Y_1)$ to a $k$-linear category $k\text{Cob}_2(Y_0, Y_1)$ with $\text{Ob}(k\text{Cob}_2(Y_0, Y_1)) := \text{Ob}\text{Cob}_2(Y_0, Y_1)$ and $k\text{Cob}_2(Y_0, Y_1)(W_0, W_1)$ being the free $k$-module generated by $\text{Cob}_2(Y_0, Y_1)(W_0, W_1)$. We introduce the following relations on $k\text{Cob}_2(Y_0, Y_1)(W_0, W_1)$ for each cobordisms $W_0$ and $W_1$.

- $(S)$ $S \amalg S^2 = 0$ for each $2$-bordism $S$, here $S^2$ is the $2$-dimensional sphere;

- $(T)$ $S\amalg T^2 = 2 \cdot S$ for each $2$-bordism $S$, here $T^2$ is the $2$-dimensional torus $T^2 = S^1 \times S^2$;

- $(4Tu)$ $S_1 + S_2 - S_3 - S_4 = 0$ for each quadruple of $2$-bordisms $S_1$, $S_2$, $S_3$, and $S_4$ which are identical outside disks and tubes that are depicted as follows:

![Diagram](image)

We denote by $k\text{Cob}_2(Y_0, Y_1)/\mathcal{L}$ the quotient category and set

$$\text{Cob}^\ell_2(Y_0, Y_1) := \text{Mat}(k\text{Cob}_2(Y_0, Y_1)/\mathcal{L}) \ .$$

**Lemma 2.5.** The two functors (2.1) and (2.2) induce $k$-bilinear functors

$$(-) * (-) : \text{Cob}^\ell_2(Y_1, Y_2) \times \text{Cob}^\ell_2(Y_0, Y_1) \to \text{Cob}^\ell_2(Y_0, Y_2) \ ,$$

$$(-) \otimes (-) : \text{Cob}^\ell_2(Y_0, Y_1) \times \text{Cob}^\ell_2(Y_0', Y_1') \to \text{Cob}^\ell_2(Y_0 \amalg Y_0', Y_1 \amalg Y_1') \ .$$

In particular, $\text{Cob}^\ell_2(\emptyset, \emptyset)$ is a symmetric monoidal category with $k$-bilinear monoidal product.
Similarly, since the relations \([S]\) and \([T]\) and \([4Tu]\) are stable under orientation reversion, we also have functors below induced by (2.3):

\[
\rho_0 : \text{Cob}_2^\ell(Y_0, Y_1) \rightarrow \text{Cob}_2^\ell(Y_0, Y_1), \\
\rho_2 : \text{Cob}_2^{\ell}(Y_0, Y_1)^{\text{op}} \rightarrow \text{Cob}_2^{\ell}(Y_0, Y_1). 
\]  

We further extend these functors to complexes in the following way: let \(\mathcal{A}, \mathcal{B}, \) and \(\mathcal{C}\) be \(k\)-linear categories with \(\mathcal{C}\) being additive. If \(F : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}\) is a \(k\)-bilinear functor, then, for bounded chain complexes \(X\) in \(\mathcal{A}\) and \(Y\) in \(\mathcal{B}\), we define a chain complex \(F(X, Y)\) in \(\mathcal{C}\) by setting

\[
F(X, Y)^n := \bigoplus_{p+q=n} F(X^p, Y^q), \\
d^n_{F(X,Y)} := \sum_{p+q=n} (F(d^p_X, \text{id}_Y) + (-1)^p F(\text{id}_X, d^q_Y)).
\]

We denote by \(\text{Ch}^b(\mathcal{A})\) the category of bounded chain complexes and chain maps in \(\mathcal{A}\). Then, the assignment above yields a \(k\)-bilinear functor

\(F : \text{Ch}^b(\mathcal{A}) \times \text{Ch}^b(\mathcal{B}) \rightarrow \text{Ch}^b(\mathcal{C})\)

which extends the original \(F\). Applying the construction to the functors in Lemma 2.5, we in particular obtain functors

\[
(-) * (-) : \text{Ch}^b(\text{Cob}_2^\ell(Y_1, Y_2)) \times \text{Ch}^b(\text{Cob}_2^\ell(Y_0, Y_1)) \rightarrow \text{Ch}^b(\text{Cob}_2^\ell(Y_0, Y_2)), \\
(-) \otimes (-) : \text{Ch}^b(\text{Cob}_2^\ell(Y_0, Y_1)) \times \text{Ch}^b(\text{Cob}_2^\ell(Y_0', Y_1')) \rightarrow \text{Ch}^b(\text{Cob}_2^\ell(Y_0 \amalg Y_0', Y_1 \amalg Y_1')).
\]

We also extend the functors \(\rho_0\) and \(\rho_2\) in (2.4) by

\[
\rho_0 : \text{Ch}^b(\text{Cob}_2^\ell(Y_0, Y_1)) \rightarrow \text{Ch}^b(\text{Cob}_2^\ell(Y_0, Y_1)), \\
\rho_2 : \text{Ch}^b(\text{Cob}_2^{\ell}(Y_0, Y_1))^{\text{op}} \rightarrow \text{Ch}^b(\text{Cob}_2^{\ell}(Y_0, Y_1)).
\]

2.3 The universal bracket complex

In this section, we construct the universal bracket complex of tangle diagrams. To begin with, we define the modules of signs.

Let \(\mathcal{S}\) be the totally ordered set. For each subset \(A \subset \mathcal{S}\), we set \(E_A := \emptyset \in \text{Cob}_2^\ell(\emptyset, \emptyset)\), which is the unit in the monoidal structure. For each \(a \in \mathcal{S}\), we define the morphisms \(\tilde{a}\), \((\wedge a)\), and \((a \wedge)\) as follows. Let \(\mu_a = \# \{a' \in A \mid a' < a\}\) and \(\nu_a = \# \{a' \in A \mid a' > a\}\).
Then, we set
\[
\tilde{a}: E_A \to E_{A\setminus\{a\}} := \begin{cases} (-1)^{\mu_a} & \text{if } a \in A, \\ 0 & \text{if } a \notin A, \end{cases} \tag{2.9}
\]
\[
(\land a): E_A \to E_{A\cup\{a\}} := \begin{cases} (-1)^{\mu_a} & \text{if } a \notin A, \\ 0 & \text{if } a \in A, \end{cases} \tag{2.10}
\]
\[
(a\land): E_A \to E_{A\cup\{a\}} := \begin{cases} (-1)^{\nu_a} & \text{if } a \notin A, \\ 0 & \text{if } a \in A. \end{cases} \tag{2.11}
\]

**Notation 2.6.** We often denote \((a\land)\) by \(a\). Let \(D\) be a tangle diagram, which we regard as a planar graph with boundary neatly embedded in \(\mathbb{R} \times [0,1]\). We denote by \(c(D)\) the set of crossings in \(D\) and call each subset \(s \subset c(D)\) a state on \(D\); we write \(|s|\) the cardinality. For each state \(s\) on \(D\), we write \(D_s\) the compact 1-dimensional neat submanifold of \(\mathbb{R} \times [0,1]\) obtained by smoothing each crossing of \(D\) according to \(s\):

\[
\begin{array}{ccc}
\text{smoothing} & c \notin s & c \\
\text{1-smoothing} & c & \text{1-smoothing} \\
\end{array}
\]

Hence, each \(D_s\) is a neat submanifold of \(\mathbb{R} \times [0,1]\). We endow \(D_s\) with an orientation as follows: recall that a checkerboard coloring \(\chi\) on the complement \((\mathbb{R} \times [0,1]) \setminus D\) is a mapping

\[
\chi: \pi_0((\mathbb{R} \times [0,1]) \setminus D) \to \{\text{white, black}\}
\]

which distinguishes adjacent components. If a checkerboard coloring \(\chi\) on the complement of \(D\) is fixed, it induces a checkerboard coloring on \((\mathbb{R} \times [0,1]) \setminus D_s\) for each state \(s\) which we also write \(\chi\) by abuse of notation. Then, we denote by \(D^\chi_s\) the manifold \(D_s\) equipped with the canonical orientation on the boundary of the black component with respect to \(\chi\); i.e. \(D^\chi_s = \partial(\chi^{-1}\{\text{black}\})\) as oriented manifolds. Note that there are exactly two checkerboard colorings. Namely, if \(\chi\) is a checkerboard coloring on \((\mathbb{R} \times [0,1]) \setminus D\), then the other is obtained by swapping all the values of \(\chi\), which we denote by \(-\chi\). In this case, the 1-manifold \(D^{-\chi}_s\) is identified with \(D^\chi_s\) with the reversed orientation.

Since the induced orientation on the boundary \(\partial D^\chi_s\) does not depend on the state \(s\), we in particular write \(\partial D^\chi := \partial D^\chi_s\) and

\[
\partial^- D^\chi := \partial D^\chi \cap (\mathbb{R} \times \{0\}), \quad \partial^+ D^\chi := \partial D^\chi \cap (\mathbb{R} \times \{1\})
\]

In fact, the orientations on them are determined locally by the rules as in Fig. 2.2. Thus, for each state \(s\) on \(D\), we may regard \(D^\chi_s\) as an object of the category \(\text{Cob}_2(\partial^- D^\chi, \partial^+ D^\chi)\) and hence of \(\text{Cob}_2(\partial^- D^\chi, \partial^+ D^\chi)\).

For a tangle diagram \(D\) with a checkerboard coloring \(\chi\) on \((\mathbb{R} \times [0,1]) \setminus D\), we define a graded \(k\)-module \(\{D^\chi\}\) by

\[
\{D^\chi\}^i := \bigoplus_{s \subset c(D), |s| = i} D^\chi_s \otimes E_s \in \text{Cob}_2(\partial^- D^\chi, \partial^+ D^\chi)
\]

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for each integer \( i \in \mathbb{Z} \). We in addition endow \( \{D^\chi\} \) with a differential as follows: for each pair \((s, c)\) of a state \( s \subset c(D) \) and a crossing \( c \in c(D) \) with \( c \notin s \), notice that, \( D_s \) and \( D_{s \cup \{c\}} \) is identical except on a neighborhood of the crossing \( c \) where they are of the following forms regardless of the orientation:

\[
\begin{array}{c|c|c}
D & D_s & D_{s \cup \{c\}} \\
\hline
\includegraphics[width=1cm]{D}\quad & \includegraphics[width=1cm]{Ds} & \includegraphics[width=1cm]{DsUc} \\
\end{array}
\]

We define a cobordism \( S_{s;c} : D_s \to D_{s \cup \{c\}} \) by

\[
\begin{array}{c}
\includegraphics[width=1cm]{D}
\end{array} \quad \to \quad \begin{array}{c}
\includegraphics[width=1cm]{DsUc}
\end{array}
\]

(2.12)
on the neighborhood and the identity elsewhere. Thanks to the stability of checkerboard colorings under smoothing, \( S_{s;c} \) has an obvious orientation which make \( S_{s;c} \) as an oriented cobordism \( D^\chi_s \to D^\chi_{s \cup \{c\}} \). Hence, we obtain a morphism

\[
S_{s;c} \otimes (\wedge c) : D^\chi_s \otimes E_s \to D^\chi_{s \cup \{c\}} \otimes E_{s \cup \{c\}}
\]

We then define the differential by

\[
d^i := \sum_{s \subset c(D), |s|=i, c \in c(D) \setminus \{s\}} S_{s;c} \otimes (\wedge c) : \{D^\chi\}^i \to \{D^\chi\}^{i+1}
\]

(2.13)

We call \( \{D^\chi\} \) the universal bracket complex of \( D \).

The following results show that the universal bracket complex respects the operation on tangles in terms of the functors (2.5) and (2.6).
Proposition 2.7 ([2, Theorem 2]). Let $D$ be the composition of two tangle diagrams $D'$ and $D''$ as in Fig. 2.3. For a checkerboard coloring $\chi$ on $(\mathbb{R} \times [0,1]) \setminus D$, let us write $\chi'$ and $\chi''$ respectively the induced coloring on the complements of $D'$ and $D''$. Then, there is an isomorphism

$$\langle [D^\chi] \rangle \cong \langle [D']^{\chi'} \rangle * \langle [D'']^{\chi''} \rangle$$

(2.14)

in the category $\text{Ch}^b(\text{Cob}_2(\partial^+ D^\chi, \partial^- D^\chi))$.

Proposition 2.8 ([2, Theorem 2]). Let $D$ be the tensor product of tangle diagrams $D'$ and $D''$ as in Fig. 2.3. For a checkerboard coloring $\chi$ on $(\mathbb{R} \times [0,1]) \setminus D$, we write $\chi'$ and $\chi''$ respectively the induced coloring on the complements of $D'$ and $D''$. Then, there is an isomorphism

$$\langle [D^\chi] \rangle \cong \langle [D'^\chi] \rangle \otimes \langle [D''^\chi] \rangle$$

(2.15)

in the category $\text{Ch}^b(\text{Cob}_2(\partial^+ D^\chi, \partial^- D^\chi))$.

2.4 The universal Khovanov complex

We now introduce the complex $[D]$ that is an invariant of tangles. We always assume a tangle $T$ to be “generic” so that the image of $T$ under the projection $\mathbb{R}^2 \times [0,1] \to \mathbb{R} \times [0,1]$ defines a tangle diagram $D$; in this case, we call $D$ the diagram of $T$. We say that two tangles are isotopic if they are connected by an ambient smooth isotopy which is the identity on the boundary. A connected component of $(\mathbb{R} \times [0,1]) \setminus D$ is said to be negatively unbounded if it contains the point $(-x, \frac{1}{2})$ for arbitrarily large $x > 0$.

Notation 2.9. If $W = \{W^i, d^i\}_i$ is a chain complex, then we define a chain complex $W[k]$ by

$$W[k]^i := W^{i-k}, \quad d_{W[k]} := (-1)^k d^i.$$

Definition 2.10. Let $D$ be a tangle diagram with $n_-$ negative crossings. Let $\chi_w$ be the checkerboard coloring with negatively unbounded white component. Then, we set

$$[D] := \langle [D^\chi] \rangle [-n_-] \in \text{Cob}_2^b(\partial^- D^\chi, \partial^+ D^\chi)$$

(2.15)

and call it the universal Khovanov complex of $D$.

If $D$ and $D'$ are the diagrams of two isotopic tangles, then the restriction on the isotopies guarantees that $\partial D = D \cap (\mathbb{R} \times \{0,1\})$ and $\partial D' = D' \cap (\mathbb{R} \times \{0,1\})$ are mutually identical. This in particular implies that we have

$$\partial D^\chi = \partial D'^\chi$$

as oriented 0-manifolds provided $\chi$ and $\chi'$ have the same color at the negatively unbounded components. It follows that $[D]$ and $[D']$ lie in the same category.

Theorem 2.11 ([2, Theorem 1]). The homotopy type of $[D]^1$ is an isotopy invariant of tangles.
2.5 The universal bracket complex as a mapping cone

Definition 2.12 (mapping cone). Let $\mathcal{A}$ be an additive category. If $f : X \to Y$ is a chain map between chain complexes in $\mathcal{A}$, then the mapping cone $\text{Cone}(f)$ is a chain complex defined as follows:

- as an object of $\mathcal{A}$, we have 
  $$\text{Cone}(f)^i = Y^i \oplus X^{i+1};$$
- the differential $d^i = d^i_{\text{Cone}(f)} : \text{Cone}(f)^i \to \text{Cone}(f)^{i+1}$ is presented by the matrix
  $$d^i_{\text{Cone}(f)} := \begin{pmatrix} d_Y^i & f \\ 0 & -d_X^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \to Y^{i+1} \oplus X^{i+2}.$$

Since $\text{Cone}(f)$ is actually a chain complex, we call it the mapping cone of $f$.

For a tangle diagram $D$, fix a crossing $c \in c(D)$, and set $D^{(0)}$ and $D^{(1)}$ the diagrams obtained from $D$ by applying $0$- and $1$-smoothing to $c$ respectively. We hence have a canonical identification $c(D^{(0)}) = c(D^{(1)}) = c(D) \setminus \{c\}$. Then, the saddle cobordism induces the morphism

$$\delta_c := \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} : \langle \langle D^{(0)} \rangle \rangle \to \langle \langle D^{(1)} \rangle \rangle.$$

Proposition 2.13. In the situation above, there is an isomorphism

$$\langle \langle D \rangle \rangle \cong \text{Cone}(-\delta_c)[1].$$

Proof. Note that, for each $s \subset c(D) \setminus \{c\}$, there are identifications

$$(D_s^{(0)})^X = D_s^X, \quad (D_s^{(1)})^X = D_{s\text{II}(c)}^X.$$

We hence define a morphism $\langle \langle (D^{(1)})^X \rangle \rangle \oplus \langle \langle (D^{(0)})^X \rangle \rangle \to \langle \langle D^X \rangle \rangle$ consisting of

$$\text{id} \otimes (\land c) : (D_s^{(1)})^X \otimes E_s \to D_{s\text{II}(c)}^X \otimes E_{s\text{II}(c)}, \quad \text{id} \otimes \text{id} : (D_s^{(0)})^X \otimes E_s \to D_s^X \otimes E_s.$$

By comparing the differentials, one can easily verify that this is actually an isomorphism of chain complexes.

2.6 Duality with respect to mirroring

To conclude the section, we see the dualities of the universal Khovanov complex in terms of functors (2.7) and (2.8). In order to establish them, we need some technical materials on the modules of signs. Let $\mathcal{S}$ be a finite totally ordered set, say $n = |\mathcal{S}|$. For a subset $A \subset \mathcal{S}$, we write $\varepsilon_A$ the sign of the $(|A|, n - |A|)$-shuffle and think of it as a morphism

$$\varepsilon_A : E_A \to E_{\mathcal{S} \setminus A} \in \text{Cob}^d_2(\emptyset, \emptyset).$$
It then turns out that the diagram below commutes:

\[
\begin{array}{ccc}
E_{\text{Hall}(c)} & \xrightarrow{\partial} & E_{\text{A}} \\
\varepsilon_{\text{Hall}(c)} & & \varepsilon_{\text{A}} \\
E_{(S\setminus A)\setminus(c)} & \xrightarrow{(-1)^{n-1}(\wedge c)} & E_{S\setminus A}
\end{array}
\]  \tag{2.16}

In terms of the universal bracket complex, the dualities are stated as follows.

**Proposition 2.14.** Let \( D_{\text{mir}} \) be the mirror image of a tangle diagram \( D \) with \( n \) crossings. Then, for every checkerboard coloring \( \chi \), there are isomorphisms

\[
\rho_0(\langle D^\chi \rangle) \cong \langle D^{-\chi} \rangle, \quad \rho_2(\langle D^\chi \rangle) \cong \langle (D_{\text{mir}})^\chi \rangle[-n].
\]

**Proof.** Since the first isomorphism is obvious, we prove the second. We identify the set \( c(D_{\text{mir}}) \) of crossings in \( D_{\text{mir}} \) with \( c(D) \). Hence, for each state \( s \subset c(D) \), there is a canonical identification \( D^\chi_s = (D_{\text{mir}}^\chi)_s \) with \( s = c(D) \setminus s \). We set

\[
\iota_s := (-1)^{|s|}\text{id} \otimes \varepsilon_s : D^\chi_s \otimes E_s \rightarrow (D_{\text{mir}}^\chi)_s \otimes E_s \tag{2.17}
\]

and write \( \iota^i : \langle D^\chi \rangle^{-i} \rightarrow \langle (D_{\text{mir}})^\chi \rangle^{n+i} \) the induced morphism. We assert that the family \( \iota = \{\iota^i\}_i \) defines a morphism of chain complexes \( \rho_2(\langle D^\chi \rangle) \rightarrow \langle (D_{\text{mir}})^\chi \rangle[-n] \). Indeed, for each \( s \subset c(D) \) and \( c \in c(D) \setminus s \), since we have \( \rho_2((c\wedge)) = \partial : E_{\text{Hall}(c)} \rightarrow E_s \), the square [2.16] yields a commutative square

\[
\begin{array}{ccc}
D^\chi_s \otimes E_{\text{Hall}(c)} & \xrightarrow{(-1)^{|s|}\rho_2(S^D_{s,c}\otimes(\wedge c))} & D^\chi_s \otimes E_s \\
\iota_s \otimes \varepsilon_{\text{Hall}(c)} \downarrow & & \iota_s \downarrow \\
(D^\chi_s \setminus(c)) \otimes E_{\text{Hall}(c)} & \xrightarrow{(-1)^{n}S_{s,c}^{D_{\text{mir}}}_{s,c} \otimes(\wedge c)} & (D^\chi_{s_{\text{mir}}}) \otimes E_{s_{\text{mir}}}
\end{array}
\]  \tag{2.18}

here \( S^D_s \) and \( S^{D_{\text{mir}}}_s \) are the saddle cobordisms [2.12] which appear in the differentials. This implies that \( \iota \) is a morphism of chain complexes. Since it is obviously an isomorphism, this completes the proof.

**Corollary 2.15.** Let \( D_{\text{mir}} \) be the mirror image of a tangle diagram \( D \) with \( n \) crossings. Then, there is an isomorphism

\[
\rho_2([D]) \cong [D_{\text{mir}}].
\]

### 3 Genus-one morphism

We now define a morphism of chain complexes

\[
\hat{\Phi} : \langle \chi \rangle \rightarrow \langle \chi \rangle[1].
\]  \tag{3.1}
Lemma 3.1 ([22 Proposition 3.1.3]). Suppose we have a sequence

\[ X \xrightarrow{f} Y \xrightarrow{g} Z \]

of chain morphisms in an additive category \( \mathcal{A} \). If there is a chain homotopy \( H : gf \Rightarrow 0 \), that is, \( dH + Hd = -gf \), then the morphism \( g \) factors through a morphism \( \hat{g} : \text{Cone}(f) \to Z \) given by

\[ \hat{g}^i = (g - H) : \text{Cone}(f)^i = Y^i \oplus X^{i+1} \to Z^i \]  

(3.2)

following the canonical morphism \( Y \to \text{Cone}(f) \).

We define the morphism \( \Phi \) on the universal bracket complex \( \langle \langle \cdot \rangle \rangle \) induced by the following cobordism:

\[ \quad \begin{array}{c}
\begin{array}{c}
\xrightarrow{\delta^-}
\end{array}
\end{array} \begin{array}{c}
\xrightarrow{\Phi}
\end{array} \begin{array}{c}
\xrightarrow{\delta^+}
\end{array} \quad .
\]  

(3.3)

We also have the following single saddle operations:

\[ \quad \begin{array}{c}
\begin{array}{c}
\xrightarrow{\delta^-}
\end{array}
\end{array} \begin{array}{c}
\xrightarrow{\Phi}
\end{array} \begin{array}{c}
\xrightarrow{\delta^+}
\end{array} \quad .
\]  

(3.4)

Proposition 3.2. In the situation above, the compositions \( \Phi \delta^- \) and \( \delta^+ \Phi \) are zero. Consequently, the sequence (3.4) induces a morphism of chain complexes

\[ \hat{\Phi} = \otimes \hat{c} : \langle \langle \chi \rangle \rangle \to \langle \langle \chi \rangle \rangle \]  

[1].

Proof. The first statement follows from the equations:

\[ \quad \begin{array}{c}
\begin{array}{c}
\delta^-
\end{array}
\end{array} \begin{array}{c}
\Phi
\end{array} \begin{array}{c}
\delta^+
\end{array} \quad .
\]  

We show the latter. By Proposition 2.13, we have identifications

\[ \text{Cone}(-\delta-) \cong \langle \langle \chi \rangle \rangle [-1], \quad \text{Cone}(-\delta+) \cong \langle \langle \chi \rangle \rangle [-1]. \]

Hence, in view of Lemma 3.1 Proposition 3.2 yields a morphism of chain complexes \( \hat{\Phi} \) as required.
In what follows, the morphism $\hat{\Phi}$ is referred to as the \textit{genus-one morphism}. The morphism $\hat{\Phi}$ induces a morphism

$\left[ \begin{array}{c} \chi \\ \end{array} \right] \rightarrow \left[ \begin{array}{c} \chi \\ \end{array} \right]$.

Moreover, it is of degree 0 with respect to Euler graded TQFT \[2, 14\].

\textbf{Remark 3.3.} In the definition of the morphism $\Phi$, for the position of the 1-handle attaching in the second term, if we switch “left” to “right” and define $\Phi'$, we have $\Phi' = -\Phi$ thanks to the relation (4Tu) in Section 2.2.

\textbf{Proposition 3.4.} Let $D_-$ and $D_+$ be the same tangle diagram except for a crossing $c$ whose sign is negative and positive, respectively. Let $D_+^{\text{mir}}$ and $D_-^{\text{mir}}$ be the mirror images of $D_-$ and $D_+$, respectively. Let $\chi$ be a checkerboard coloring. The crossing in $D_+^{\text{mir}}$ corresponding to $c$ is denoted by $c^{\text{mir}}$. Let $\hat{\Phi}_c : \langle \langle D_-^{\chi} \rangle \rangle \rightarrow \langle \langle D_+^{\chi} \rangle \rangle$ be the genus-one morphism that is applied to $c$. Then the following diagram commutes:

\[
\rho_2 (\langle \langle D_+^{\chi} \rangle \rangle) \xrightarrow{\rho_2 (\hat{\Phi}_c)} \rho_2 (\langle \langle D_-^{\chi} \rangle \rangle) \\
\langle \langle D_+^{\text{mir} \chi} \rangle \rangle([-n]) \xrightarrow{\hat{\Phi}_c^{\text{mir}}} \langle \langle D_-^{\text{mir} \chi} \rangle \rangle([-n])
\]

\textit{here the vertical isomorphisms are the ones in Proposition 2.14.}

Proposition 3.4 is verified by the direct computation, so we omit the proof.

\textbf{Remark 3.5.} We note that, though there are other choices on morphisms of chain complexes of the form (3.1), some popular ones fail to have of bidegree $(0, 0)$ in the case of Khovanov homology. For example, a canonical way of the crossing change is realized as the composition:

\[
\langle \langle X^\chi \rangle \rangle \xrightarrow{\beta} \langle \langle \rangle \rangle \xrightarrow{\alpha} \langle \langle X^\chi \rangle \rangle [-1].
\]

M. Hedden and L. Watson \[7, Section 3.1\] used this morphism to derive a categorified version of the Jones skein relation.

\section{4 Invariance}

In this section, we see that the genus-one morphism $\hat{\Phi}$ defined in Proposition 3.2 is invariant under moves involved with singular links. Namely, according to \[13\], two singular link diagrams represent isotopic singular links if and only if they are connected by the following moves in addition to Reidemeister moves:

\[
\begin{align*}
\begin{array}{c}
\leftrightarrow \\
\end{array} \\
\begin{array}{c}
\leftrightarrow \\
\end{array} \\
\begin{array}{c}
\leftrightarrow \\
\end{array} \\
\end{align*}
\]

(4.1)
Motivated by this fact, we aim at proving the invariance of $\hat{\Phi}$ under these moves in the sense of the following propositions, where the checkerboard colorings are omitted from the notation for simplicity.

**Proposition 4.1.** There is a chain-homotopy commutative diagram

$$
\begin{array}{c}
\langle \langle c_- \rangle \rangle \xrightarrow{\hat{\Phi}_c} \langle \langle c_+ \rangle \rangle[1] \\
\cong \downarrow \quad \quad \quad \quad \quad \cong \downarrow \\
\langle \langle c'_- \rangle \rangle \xrightarrow{\hat{\Phi}_c'} \langle \langle c'_+ \rangle \rangle[1]
\end{array}
$$

(4.2)

with vertical edges being chain homotopy equivalences.

**Proposition 4.2.** There is a chain-homotopy commutative diagram

$$
\begin{array}{c}
\langle \langle a_- \rangle \rangle \xrightarrow{\hat{\Phi}_a} \langle \langle a_+ \rangle \rangle[1] \\
\cong \downarrow \quad \quad \quad \quad \quad \cong \downarrow \\
\langle \langle a_+ \rangle \rangle \xrightarrow{\hat{\Phi}_b} \langle \langle b_+ \rangle \rangle[1]
\end{array}
$$

(4.3)

with vertical edges being chain homotopy equivalences.

**Remark 4.3.** We note that a move in (4.1) suggests that we should have another chain-homotopy commutative diagram; namely the following:

$$
\begin{array}{c}
\langle \langle c_- \rangle \rangle \xrightarrow{\hat{\Phi}_c} \langle \langle c_+ \rangle \rangle[1] \\
\cong \downarrow \quad \quad \quad \quad \quad \cong \downarrow \\
\langle \langle c'_- \rangle \rangle \xrightarrow{\hat{\Phi}_c'} \langle \langle c'_+ \rangle \rangle[1]
\end{array}
$$

(4.4)

Actually, it is given rise to by the square (4.2) thanks to Proposition 2.14 and Proposition 3.4.
4.1 Homotopy coherence of mapping cones

As the genus-one morphism \( \tilde{\Phi} \) is obtained from the sequence (3.4), the invariance stated in Proposition 4.1 and Proposition 4.2 will be inherited from that of (3.4). We then begin with a discussion on this kind of inheritance of invariance.

We first see that mapping cones have functoriality with respect not only to commutative squares but also to chain-homotopy commutative ones. Let \( \mathcal{A} \) be an additive category, and suppose we are given a chain-homotopy commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{u} & & \downarrow{v} \\
X & \xrightarrow{F} & Y
\end{array}
\]

in other words, \( F \) is a chain homotopy with \( d_Y F + F d_{X'} = f u - v f' \). We define a morphism

\[
F^i := \begin{pmatrix} v^i & -F^i \\ 0 & u^i+1 \end{pmatrix} : Y^i \oplus X^{i+1} \to Y^i \oplus X^{i+1}
\]

for each integer \( i \in \mathbb{Z} \). It turns out that the family \( F_* = \{ F^i \} \) forms a morphism of complexes \( F_* : \text{Cone}(f') \to \text{Cone}(f) \) which makes the following diagram commute (strictly):

\[
\begin{array}{ccc}
Y' & \xrightarrow{\text{Cone}(f')} & X'[-1] \\
\downarrow{v} & & \downarrow{u} \\
Y & \xrightarrow{\text{Cone}(f)} & X[-1]
\end{array}
\]

Actually, the construction is invariant under chain homotopies in the following sense.

**Lemma 4.4.** Suppose we are given a chain-homotopy commutative diagram as below:

\[
\begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{u'} & & \downarrow{v'} \\
X & \xrightarrow{F} & Y
\end{array}
\]

We define a chain homotopy \( F' : v' f' \Rightarrow f u' \) by \( F' := f U + F + V f' \) and write \( F_*' \) : \( \text{Cone}(f') \to \text{Cone}(f) \) the morphisms of complexes induced by \( F \) and \( F' \) respectively. Then, there is a chain homotopy \( \Psi : F_* \Rightarrow F_*' \) given by

\[
\Psi^i = \begin{pmatrix} V^i & 0 \\ 0 & U^{i+1} \end{pmatrix} : Y^i \oplus X^{i+1} \to Y^{i-1} \oplus X^i
\]

**Corollary 4.5.** In the chain-homotopy commutative square (4.5), suppose in addition that \( u \) and \( v \) are both chain homotopy equivalences. Then the induced morphism \( F_* \) is also a chain homotopy equivalence.
The mapping cones have further chain-homotopy coherence.

**Proposition 4.6.** Suppose we have a chain-homotopy commutative diagram

\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & Y' \\
\downarrow{u} & \searrow{F} & \uparrow{v} \\
X & \xrightarrow{f} & Y \\
\end{array} \quad \begin{array}{ccc}
Y' & \xrightarrow{g'} & Z' \\
\downarrow{w} & \nearrow{G} & \uparrow{\Psi} \\
Y & \xrightarrow{g} & Z \\
\end{array} \] (4.7)

of chain complexes such that \( gf = 0 \) and \( g'f' = 0 \) together with a family of morphisms \( \Psi = \{ \Psi^i : X^n \to Z^{i-2} \} \) satisfying the equation

\[ d\Psi - \Psi d = g \circ F + G \circ f' . \]

Write \( \hat{g} : \text{Cone}(f) \to Z \) and \( \hat{g}' : \text{Cone}(f') \to Z' \) the morphisms of complexes induced by \( g \) and \( g' \) respectively. Then, there is a chain homotopy depicted as below:

\[ \begin{array}{ccc}
\text{Cone}(f') & \xrightarrow{\hat{g}'} & Z' \\
\downarrow{F_*} & \swarrow{\hat{G}} & \downarrow{w} \\
\text{Cone}(f) & \xrightarrow{\hat{g}} & Z \\
\end{array} \] (4.8)

More precisely, \( \hat{G}^i : \text{Cone}(f')^i \to Z^{i-1} \) is presented by the matrix

\[ \hat{G} = \begin{pmatrix} G & -\Psi \end{pmatrix} : Y^i \oplus X^{i+1} \to Z^{i-1} . \]

**Proof.** Using the explicit formulas (3.2) and (4.6), we have

\[
d\hat{G} + \hat{G}d - \hat{g}F_* + w\hat{g}' \\
= d_Z (G' - \Psi) + (G' - \Psi) \begin{pmatrix} d_{Y'} & f' \\ 0 & -d_{X'} \end{pmatrix} - (g' \begin{pmatrix} v & -F' \\ 0 & u \end{pmatrix} + (w g' \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}) \\
= (d_Z G + Gd_{Y'} - g v + w g' - d_Z \Psi + \Psi d_{X'} + g F + G f') .
\]

The last term vanishes by virtue of the assumption, so we obtain the result. \( \square \)

**Remark 4.7.** The dual of Proposition 4.6 also holds. Namely, if the diagram (4.7) and the family \( \Psi \) are given as in Proposition 4.6, they induce a chain-homotopy commutative square

\[ \begin{array}{ccc}
X' & \xrightarrow{\hat{f}'} & \text{Cone}(g')[1] \\
\downarrow{u} & \searrow{\hat{F}} & \uparrow{G_*} \\
X & \xrightarrow{\hat{f}} & \text{Cone}(g)[1] \\
\end{array} \]

where \( \hat{F} \) is given by the matrix

\[ \hat{F}^i := \begin{pmatrix} -\Psi \\ F \end{pmatrix} : X^n \to Z^{i-2} \oplus Y^{i-1} . \]
Corollary 4.8. Suppose we are given a chain-homotopy commutative diagram

\[
\begin{array}{c}
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & W' \\
\downarrow F & & \downarrow G & & \downarrow H & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & W
\end{array}
\]

such that

\[gf = 0, \quad hg = 0, \quad g'f' = 0, \quad h'g' = 0\]

together with the following data:

(i) families of morphisms \(\Psi = \{\Psi^i : X^i \rightarrow Z^{i-2}\}_i\) and \(\Xi = \{\Xi^i : Y^i \rightarrow W^{i-2}\}_i\) satisfying

\[d\Psi - \Psi d = gF + Gf', \quad d\Xi - \Xi d = hG + Hg';\]

(ii) a family of morphisms \(\Gamma : \{\Gamma^i : X^i \rightarrow W^{i-3}\}\) satisfying

\[d\Gamma + \Gamma d = h\Psi - \Xi f'.\]

Then, the diagram gives rise to a chain-homotopy commutative square

\[
\begin{array}{c}
\text{Cone}(f') & \xrightarrow{\Gamma_*} & \text{Cone}(h')[1] \\
\downarrow F_* & & \downarrow H_* \\
\text{Cone}(f) & \xrightarrow{\Gamma_*} & \text{Cone}(h)[1]
\end{array}
\]

where the chain homotopy \(\Gamma_*\) is given by the following matrix:

\[
\Gamma_*^i = \left(\begin{array}{cc}
-\Xi & \Gamma \\
g & -\Psi
\end{array}\right) : Y^i \oplus X^{i+1} \rightarrow W^{i-2} \oplus Z^{i-1}.
\]

Proof. Applying Proposition 4.6 with \(H = 0\) and \(H' = 0\), we obtain the following chain-homotopy commutative diagram

\[
\begin{array}{c}
\text{Cone}(f') & \xrightarrow{\bar{g}'} & Z' & \xrightarrow{k'} & W' \\
\downarrow F_* & & \downarrow H & & \downarrow \\
\text{Cone}(f) & \xrightarrow{\bar{g}} & Z & \xrightarrow{h} & W
\end{array}
\]

We note that both of the horizontal compositions vanish while the assumption on \(\Xi\) and \(\Gamma\) implies

\[d_W (\Xi - \Gamma) + (\Xi - \Gamma) \begin{pmatrix} dY' & f' \\ 0 & -dX' \end{pmatrix} = h\bar{G} + H\bar{g}'.\]

Therefore, applying Proposition 4.6 again (or its dual more precisely; see Remark 4.7), one obtains the result.
4.2 Proof of Proposition 4.1

We now begin the proof of Proposition 4.1 making use of Corollary 4.8. For this, we first construct a chain-homotopy commutative square in the following form:

\[
\begin{array}{cccc}
\langle \langle a b \rangle \rangle & \xrightarrow{-\delta^R} & \langle \langle a^\prime b^\prime \rangle \rangle & \xleftarrow{\Phi^R} & \langle \langle a b \rangle \rangle \\
\gamma & \xleftarrow{F} & -\omega & \xrightarrow{\Phi^L} & \gamma \\
\langle \langle a^\prime b^\prime \rangle \rangle & \xrightarrow{-\delta^L} & \langle \langle a^\prime b^\prime \rangle \rangle & \xleftarrow{\omega} & \langle \langle a^\prime b^\prime \rangle \rangle
\end{array}
\]

(4.9)

here \(\delta^R\) and \(\delta^L\) are the saddle operations representing the appropriate smoothing changes on the crossings, say, in the right and the left of the vertical strands; while \(\Phi^R\) and \(\Phi^L\) are the morphisms given in (3.3) on those crossings. On the other hand, \(\gamma\) and \(\omega\) are morphisms given as follows (see Section 2.3 for sign symbols):

\[
\begin{align*}
\gamma & := \otimes \text{id} + \otimes b^\prime \tilde{b} + \otimes a^\prime \tilde{a} + \otimes (a'b')_{\text{r}}(ab) \\
\omega & := \otimes a^\prime \tilde{b} + \otimes b^\prime \tilde{b} - \otimes a^\prime \tilde{a} - \otimes b^\prime \tilde{a}
\end{align*}
\]

(4.10)

Lemma 4.9. The morphisms \(\gamma\) and \(\omega\) given in (4.10) are chain homotopy equivalences.

Proof. It is obvious that \(\gamma\) is even an isomorphism. On the other hand, recall that Bar-Natan defined in [2, pp.1458] chain homotopy equivalences

\[
R_{\Pi} : \langle \langle \rangle \rangle \mapsto \langle \langle \rangle \rangle [1] : \bar{R}_{\Pi}
\]

(4.11)

that are given as follows:

\[
\begin{align*}
R^i_{\Pi} & := \otimes \tilde{b} - \otimes \tilde{a} : \langle \langle a b \rangle \rangle^i \to \langle \langle \rangle \rangle^i, \\
\bar{R}^i_{\Pi} & := \otimes b^i + \otimes a^i : \langle \langle \rangle \rangle^i \to \langle \langle a b \rangle \rangle^i
\end{align*}
\]

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It is easily seen that $\omega$ is realized as a composition of $R_{II}$ and $\overline{R}_{II}$ with respect to different pairs of strands. Hence it is also a chain homotopy equivalence.

To complete the diagram (4.9), we define the following families of morphisms:

$$F^i := (-1)^i \left( \begin{array}{c}
\text{Diagram}
\end{array} \right) : \begin{array}{c}
\langle \langle a \ b \rangle \rangle^i \to \langle \langle a' \ b' \rangle \rangle^{i-1}
\end{array},$$

$$G^i := (-1)^i \left( \begin{array}{c}
\text{Diagram}
\end{array} \right) : \begin{array}{c}
\langle \langle a \ b \rangle \rangle^i \to \langle \langle a' \ b' \rangle \rangle^{i-1}
\end{array},$$

$$H^i := (-1)^{i+1} \left( \begin{array}{c}
\text{Diagram}
\end{array} \right) : \begin{array}{c}
\langle \langle a \ b \rangle \rangle^i \to \langle \langle a' \ b' \rangle \rangle^{i-1}
\end{array}.\quad(4.12)$$

By direct computations, one can prove that the families $F = \{F^i\}_i$, $G = \{G^i\}_i$, and $H = \{H^i\}_i$, given in (4.12) define chain homotopies

$$F : \omega \delta R \Rightarrow -\delta L \gamma, \quad G : \omega \Phi R \Rightarrow -\Phi L \omega, \quad H : -\gamma \delta R \Rightarrow -\delta L \omega.$$

We next construct families $\Psi_- = \{\Psi_-^i\}_i$ and $\Psi_+ = \{\Psi_+^i\}_i$, of morphisms

$$\Psi_+^i : \begin{array}{c}
\langle \langle a \ b \rangle \rangle^i \to \langle \langle a' \ b' \rangle \rangle^{i-2}
\end{array}, \quad \Psi_-^i : \begin{array}{c}
\langle \langle a' \ b' \rangle \rangle^i \to \langle \langle a \ b \rangle \rangle^{i-2}
\end{array}$$

which are coherences of the left two squares and the right ones in (4.9) in the sense of Proposition 4.6, that is, they satisfy

$$d\Psi_- - \Psi_- d = \Phi^L F + \Phi R \delta^L, \quad (4.13)$$

$$d\Psi_+ - \Psi_+ d = \delta^L G + H \Phi R. \quad(4.14)$$

**Lemma 4.10.** We define

$$\Psi_-^i := \langle \langle a \ b \rangle \rangle \otimes (ab) : \begin{array}{c}
\langle \langle a \ b \rangle \rangle^i \to \langle \langle a' \ b' \rangle \rangle^{i-2}
\end{array}.$$

$$\Psi_+^i := \langle \langle a' \ b' \rangle \rangle \otimes (ab) : \begin{array}{c}
\langle \langle a' \ b' \rangle \rangle^i \to \langle \langle a \ b \rangle \rangle^{i-2}
\end{array}.\quad(4.15)$$
Then, they satisfy the equations (4.13) and (4.14) respectively.

Proof. We only show the equation (4.13); actually (4.14) is obtained by rotating (4.13) in 180 degrees around the z-axis with a little care about the change of the sign of the morphism $\Phi$ (see Remark 3.3).

By direct computations, we obtain

$$d\Psi_i = (-1)^i \left( \otimes a'_i(ab) + (-1)^i \otimes b'_i(ab) \right),$$

$$-\Psi_{i+1}d = (-1)^i \left( \otimes \tilde{b} - (-1)^i \otimes \tilde{a} \right).$$

On the other hand, as for the right hand side of (4.13), we have

$$(\Phi^L)^i F^i = (-1)^i \left( \left( \begin{array}{c} \vdots \quad \vdots \quad \vdots \quad \vdots \\ -1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ -1 \end{array} \right) \otimes b'_i(ab) \right) + (-1)^i \left( \left( \begin{array}{c} \vdots \quad \vdots \quad \vdots \quad \vdots \\ -1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ -1 \end{array} \right) \otimes \tilde{b} \right)$$

and

$$G^i(\delta^R)^i = (-1)^i \left( \otimes a'_i(ab) + (-1)^i \otimes b'_i(ab) \right) + (-1)^i \left( \otimes \tilde{b} - (-1)^i \otimes \tilde{a} \right).$$

Comparing the terms, we obtain

$$(\text{the first term of } d\Psi_i) = (\text{the first term of } G^i(\delta^R)^i) \text{,}$$

$$(\text{the second term of } -\Psi_{i+1}d) = (\text{the fourth term of } G^i(\delta^R)^i). \quad (4.16)$$

In addition, due to the relation (4T$u$) with respect to tubes attached to the disks in the cobordisms

$$\begin{array}{c}
\end{array}$$
we also obtain the equations

\[
\begin{align*}
\text{(the second term of } d\Psi_-) &= \text{(the first term of } (\Phi^L)^i F^i) \\
&\quad + \text{(the second term of } G^i(\delta^R)^i) \\
\text{(the second term of } -\Psi^{i+1} d) &= \text{(the first term of } (\Phi^L)^i F^i) \\
&\quad + \text{(the third term of } G^i(\delta^R)^i).
\end{align*}
\]

Adding (4.16) and (4.17), we obtain the result. \qed

Finally, we apply Corollary 4.8 to the diagram 4.9. In fact, we have

\[
\delta^R \Psi_- = \left( \begin{array}{c}
\otimes \langle \begin{array}{c}
a \\
b
\end{array} \rangle
\end{array} \right) + \left( \begin{array}{c}
\otimes \langle \begin{array}{c}
a \\
b
\end{array} \rangle
\end{array} \right) = \Psi_- + \delta^L.
\]

Hence, all the assumptions in Corollary 4.8 are satisfied with respect to \(\Psi_-\), \(\Psi_+\), and \(\Gamma = 0\). Therefore, a chain homotopy as in (4.2) is induced. Moreover, by Lemma 4.9, all the vertical arrows in (4.9) are chain homotopy equivalences. It then follows from Corollary 4.5 that the vertical arrows in (4.2) are chain homotopy equivalences. This completes the proof of Proposition 4.1.

4.3 Proof of Proposition 4.2

In contrast to the arguments in the previous section, the proof of Proposition 4.2 is relatively easy. In fact, it is a consequence of the following lemma, whose proof is left to the reader as it is mostly straightforward.

**Lemma 4.11.** The following diagram commutes:

\[
\begin{array}{c}
\left\langle \begin{array}{c}
a_-
\end{array} \right\rangle \quad \left\langle \begin{array}{c}
b_-
\end{array} \right\rangle \\
\downarrow \quad \downarrow \Phi_a
\end{array}
\xrightarrow{\text{R}}
\begin{array}{c}
\left\langle \begin{array}{c}
a_+
\end{array} \right\rangle \\
\downarrow \Phi_b
\end{array}
\]

\[
\left\langle \begin{array}{c}
a_\circ
\end{array} \right\rangle \quad \left\langle \begin{array}{c}
b_\circ
\end{array} \right\rangle
\]

\[
\left[1\right]
\]

\[
\left\langle \begin{array}{c}
a_\circ
\end{array} \right\rangle \quad \left\langle \begin{array}{c}
b_\circ
\end{array} \right\rangle
\]

\[
\left[1\right]
\]

Here the morphisms \(\bar{R}_\Pi\) and \(\bar{R}_\Pi^o\) are the ones given in (4.11) while the latter is applied to 180-degree-rotated diagrams.

We write \(R_\Pi^o\) the chain homotopy inverse to \(R_\Pi^o\) in the diagram (4.18) described in (4.11). Using the chain homotopy \(\text{id} \Rightarrow R_\Pi^o \bar{R}_\Pi\), one can define a chain homotopy \(H\) as
in the diagram below:

\[
\begin{array}{c}
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\end{array}
\xrightarrow{\phi^+_a}
\begin{array}{c}
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\end{array}
\xrightarrow{\phi^+_b}
\begin{array}{c}
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\end{array}
\xrightarrow{H}
\begin{array}{c}
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\end{array}
\xrightarrow{R_{II}}
\begin{array}{c}
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\langle \langle a \rangle \rangle \\
\langle \langle b \rangle \rangle \\
\end{array}
\]

Since the morphism \( R_{II} \) is also a chain homotopy equivalence, the left vertical arrow above is a chain homotopy equivalence. Therefore, we finally obtain Proposition 4.2.

### 5 Universal Khovanov complex for singular tangles

Using the invariance of the genus-one morphism \( \hat{\Phi} \) proved in Section 4, we can now extend the universal Khovanov complex to singular tangles so that an analogue of Vassiliev skein relation holds.

#### 5.1 Definition

We first extend the universal bracket complex \( \langle \langle - \rangle \rangle \) to singular tangle diagrams. For a singular tangle diagram \( D \), we denote by \( c^2(D) \) the set of double points of \( D \). We call each subset \( r \subset c^2(D) \) a resolution scheme of \( D \) and denote by \( |r| \) the cardinality of \( r \).

For each resolution scheme \( r \), we obtain an ordinary tangle diagram (i.e. without double points) \( D_r \) which is identical to \( D \) except near the double points where we have

\[
D \quad D_r (b \notin r) \quad D_r (b \in r)
\]

\[
\begin{array}{c}
\chi \quad \chi^- \quad \chi^+
\end{array}
\]

for each \( b \in c^2(D) \). In particular, a checkerboard coloring \( \chi \) on \((\mathbb{R} \times [0, 1]) \setminus D\) induces that on \( D_r \) for each resolution scheme \( r \). Hence, if \( b \notin r \), the genus-one morphism yields a morphism of chain complexes

\[
\hat{\Phi}_{r,b} : \langle \langle D_r^\chi \rangle \rangle \to \langle \langle D_r^\chi \rangle \rangle [1].
\]

We now define \( \langle \langle D^\chi \rangle \rangle \) as a graded object in \( \text{Cob}_2^f(\partial^- D^\chi, \partial^+ D^\chi) \) by

\[
\langle \langle D^\chi \rangle \rangle := \bigoplus_{r \subset c^2(D)} \langle \langle D_r^\chi \rangle \rangle [2|r|] \otimes E_r.
\]
The differential $d_D = \{d^i_D\}$ consists of $d^i_D : \langle\langle D \chi\rangle\rangle^i \to \langle\langle D \chi\rangle\rangle^{i+1}$ which is componentwisely given by

$$d^{i+2|\tau|}_D \otimes id_{E_{\tau}} + \sum_{b \in c^\tau(D) \setminus \tau} \hat{\Phi}_{\tau,b} \otimes (\wedge b) :$$

$$\langle\langle D \chi\rangle\rangle^i \otimes E_{\tau} \to \langle\langle D \chi\rangle\rangle^i \oplus \bigoplus_{b \in c^\tau(D) \setminus \tau} \langle\langle D \chi\rangle\rangle^i \otimes E_{\tau \cup \{b\}}$$

for each resolution scheme $\tau$.

We see that the complex $\langle\langle D \chi\rangle\rangle$ is also realized as an iterated mapping cone. For this, fix a double point $b \in c^\tau(D)$ and let $D(-)$ and $D(+)$ be diagrams obtained from $D$ by resolving $b$ into positive and negative crossings respectively. The set $c^\tau(D \pm)$ is then identified with $c^\tau(D) \setminus \{b\}$ so that, for each resolution scheme $\tau \subset c^\tau(D) \setminus \{b\}$, we have $D(-) = D_{\tau}$ and $D(+) = D_{\tau \cup \{b\}}$. Under these identifications, one can see that the inclusion and the projection yield the following exact sequence of morphisms of chain complexes in $\text{Cob}_2(\partial^- D \chi, \partial^+ D \chi)$:

$$\langle\langle D(-) \chi\rangle\rangle^i \otimes E_{\tau} \to \langle\langle D(+) \chi\rangle\rangle^i \otimes E_{\tau \cup \{b\}}$$

(5.1)

On the other hand, for each $\tau \subset c^\tau(D) \setminus \{b\}$, we have genus-one morphism $\langle\langle D(-) \chi\rangle\rangle \to \langle\langle D(+) \chi\rangle\rangle$. As $\tau$ varies in resolution schemes on $D(-)$, it turns out that it defines a morphism

$$\hat{\Phi} : \langle\langle D(-) \chi\rangle\rangle \to \langle\langle D(+) \chi\rangle\rangle$$

(5.2)

which we again call genus-one morphism.

**Proposition 5.1.** In the situation above, there is an isomorphism

$$\langle\langle D \chi\rangle\rangle \cong \text{Cone} \left( \langle\langle D(-) \chi\rangle\rangle \xrightarrow{\hat{\Phi}} \langle\langle D(+) \chi\rangle\rangle[1] \right)[1]$$

(5.3)

in the category $\text{Ch}^b(\text{Cob}_2(\partial^- D \chi, \partial^+ D \chi))$ so that the associated exact sequence is exactly (5.1).

The proof is almost identical to Proposition 2.13 and so omitted.

We now extend the universal Khovanov complex to singular tangles diagrams by normalizing the degree of the universal bracket complex. The argument is almost parallel to the case of ordinary tangles.

**Definition 5.2.** Let $D$ be a singular tangle diagram with $n_-$ negative crossings and $n_\times$ double points. We write $\chi_w$ the checkerboard coloring with negatively unbounded white component. Then, we set

$$[D] := \langle\langle D \chi_w\rangle\rangle[-n_- - 2n_\times] \in \text{Cob}_2(\partial^- D \chi_w, \partial^+ D \chi_w)$$

(5.4)

and call it the universal Khovanov complex of $D$. 

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It is obvious that, if $D$ has no double point, the definition agrees with the ordinary universal Khovanov complex. In Section 5.3, we see that $[-]$ defines an invariant of singular tangles up to chain homotopies.

**Corollary 5.3.** In the same situation as Proposition 5.1, there is an isomorphism

$$[D] \cong \text{Cone} \left( [D^{-}] \xrightarrow{\hat{\Phi}} [D^{+}] \right).$$

We may think of Corollary 5.3 as a categorified analogue of Vassiliev skein relation. Namely, it gives rise to a distinguished triangle

$$\cdots \rightarrow \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] \xrightarrow{\hat{\Phi}} \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] \rightarrow \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] \rightarrow \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] [-1] \xrightarrow{\hat{\Phi}} \cdots$$  \hspace{1cm} (5.5)

in the homotopy category (with the standard triangulated structure). In fact, if we write $[-]$ the image of the universal Khovanov complex $[-]$ in the $K$-group, (5.5) yields the equation

$$\left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] = \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right] - \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right],$$

which is exactly the Vassiliev skein relation.

**Example 5.4.** Evaluating the sequence (5.5) with the Euler-graded TQFT associated with the Frobenius algebra $k[x]/(x^2)$, we obtain the following long exact sequence of Khovanov homologies with coefficients in $k$:

$$\cdots \rightarrow \text{Kh}^{i,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \xrightarrow{\hat{\Phi}^*} \text{Kh}^{i,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \longrightarrow \text{Kh}^{i,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right)$$

\hspace{2cm} \overbrace{\phantom{\text{Kh}^{i+1,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \xrightarrow{\hat{\Phi}} \text{Kh}^{i+1,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \rightarrow \cdots}}^{	ext{Kh}^{i+1,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \xrightarrow{\hat{\Phi}} \text{Kh}^{i+1,j} \left( \left[ \begin{array}{c} \chi \\ \vline \\
\end{array} \right]; k \right) \rightarrow \cdots}

In case $k$ is a field, one can recover the Vassiliev skein relation for the Jones polynomial by taking the graded Euler characteristics.

**5.2 Composition formulas**

As seen in Proposition 2.8 and Proposition 2.7, the universal bracket complexes behave well for compositions and tensor products of tangle diagrams. Actually, there are analogous isomorphisms for singular tangle diagrams. We define compositions and tensor products of singular tangle diagrams in the same manner as the non-singular case.
Theorem 5.5. Let $D$ be the composition of two singular tangle diagrams, say $D'$ and $D''$. For a checkerboard coloring $\chi$ on $(\mathbb{R} \times [0, 1]) \setminus D$, let us write $\chi'$ and $\chi''$ respectively the induced coloring on the complements of $D'$ and $D''$. Then, there is an isomorphism

$$\langle \langle D^x \rangle \rangle \cong \langle \langle D^{x'} \rangle \rangle \otimes \langle \langle D^{x''} \rangle \rangle$$

in the category $\text{Ch}^b(\text{Cob}_2^\ell(\partial^- D^x, \partial^+ D^x)) = \text{Ch}^b(\text{Cob}_2^\ell(\partial^- D^{x'}, \partial^+ D^{x''}))$.

Proof. We may identify the double points in $D'$ and $D''$ with those in $D$; in other words, $c^2(D) = c^2(D') \amalg c^2(D'')$. For a resolution scheme $r \subset c^2(D)$, we write $r' := r \cap c^2(D')$ and $r'' := r \cap c^2(D'')$. Then, by Proposition 2.7 implies that there is an isomorphism

$$\langle \langle D^x_r \rangle \rangle \cong \langle \langle D^{x'}_{r'} \rangle \rangle \otimes \langle \langle D^{x''}_{r''} \rangle \rangle .$$

In addition, for any pair of integers $(p, q)$, the morphism

$$(-1)^q \ast \text{id} : \langle \langle D^{x''}_{r''} \rangle \rangle^{i - p} \ast \langle \langle D^{x'}_{r'} \rangle \rangle^{j - q} \to \langle \langle D^{x''}_{r''} \rangle \rangle^{i - p} \ast \langle \langle D^{x'}_{r'} \rangle \rangle^{j - q}$$

defines an isomorphism

$$\langle \langle D^{x''}_{r''} \rangle \rangle[p] \ast \langle \langle D^{x'}_{r'} \rangle \rangle[q] \cong (\langle \langle D^{x''}_{r''} \rangle \rangle \ast \langle \langle D^{x'}_{r'} \rangle \rangle)[p + q] .$$

Thanks to the identifications (5.7) and (5.8), we obtain (5.6) as an isomorphism of graded objects in $\text{Cob}_2^\ell(\partial^- D^x, \partial^+ D^x)$. Furthermore, for each $b \in c^2(D) \setminus r$, the genus-one morphism $\hat{\Phi}_{r, b} : \langle \langle D^x_r \rangle \rangle \to \langle \langle D^x_{r \cup \{b\}} \rangle \rangle[1]$ with respect to $b$ is given by

$$\hat{\Phi}_{r, b} = \begin{cases} \text{id} \ast \hat{\Phi}_{r', b} : \langle \langle D^{x''}_{r''} \rangle \rangle \ast \langle \langle D^{x'}_{r'} \rangle \rangle \to \langle \langle D^{x''}_{r'' \cup \{b\}} \rangle \rangle \ast \langle \langle D^{x'}_{r' \cup \{b\}} \rangle \rangle[1] \quad b \in c^2(D') , \\ \hat{\Phi}_{r'', b} \ast \text{id} : \langle \langle D^{x''}_{r''} \rangle \rangle \ast \langle \langle D^{x'}_{r'} \rangle \rangle \to \langle \langle D^{x''}_{r'' \cup \{b\}} \rangle \rangle \ast \langle \langle D^{x'}_{r' \cup \{b\}} \rangle \rangle[1] \ast \langle \langle D^{x'}_{r' \cup \{b\}} \rangle \rangle \quad b \in c^2(D'') . \end{cases}$$

Using this formula, one can easily verify that the differentials on the complexes in (5.6) agree with each other. Hence, the result follows.

The same argument also shows the following.

Theorem 5.6. Let $D$ be the tensor product of two singular tangle diagrams, say $D'$ and $D''$. For a checkerboard coloring $\chi$ on $(\mathbb{R} \times [0, 1]) \setminus D$, we write $\chi'$ and $\chi''$ respectively the induced coloring on the complements of $D'$ and $D''$. Then, there is an isomorphism

$$\langle \langle D^x \rangle \rangle \cong \langle \langle D^{x'} \rangle \rangle \otimes \langle \langle D^{x''} \rangle \rangle$$

in the category $\text{Ch}^b(\text{Cob}_2^\ell(\partial^- D^x, \partial^+ D^x)) = \text{Ch}^b(\text{Cob}_2^\ell(\partial^- D^{x'} \otimes D^{x''}, \partial^+ D^{x'} \otimes D^{x''}))$.

Note that the isomorphism (5.8) also exists for the universal bracket complexes for singular tangles. Consequently, our Main Theorem B now follows from Theorem 5.5 and Theorem 5.6.
5.3 Invariance

Using the results obtained in Section 5.2, we now prove our Main Theorem A; namely, we show that the complex \([D]\) is invariant under the local moves (4.1) up to chain homotopies so that it defines an invariant of singular tangles. To be more precise, by an (oriented) singular tangle, we mean a compact oriented immersed submanifold \(T \subset \mathbb{R}^2 \times [0, 1]\) of dimension 1 such that

(a) it has only finitely many transverse double points as singularities; and

(b) it is a neat submanifold near the boundary \(\mathbb{R}^2 \times \{0, 1\}\).

We always assume a singular tangle \(T\) to be “generic” so that the image of \(T\) under the projection \(\mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R} \times [0, 1]\) defines a singular tangle diagram \(D\); in this case, we call \(D\) the diagram of \(T\). We also consider isotopies between them in the same way as the non-singular case (see Section 2.4).

Proof of Main Theorem A. We first show that \([–]\) defines an isotopy invariant of tangles; in other words, we show that, if \(D\) and \(D'\) are the diagrams of isotopic singular tangles, then there is a chain homotopy equivalence \([D] \cong [D']\). Since chain homotopy equivalences compose, we may assume that \(D\) and \(D'\) are connected by a single elementary move; that is, one of the moves (4.1) and Reidemeister moves. Furthermore, by virtue of Theorem 5.5 and Theorem 5.6, we are reduced to the case where \(D\) and \(D'\) are exactly the local tangles involved with the move. For Reidemeister moves, the result is nothing but the invariance of the universal Khovanov complexes for ordinary tangles. On the other hand, for moves (4.1), the result is exactly Proposition 4.1, Remark 4.3, and Proposition 4.2 respectively.

Now, the isomorphism in the statement is exactly Corollary 5.3. To verify the last statement, set \(k = \mathbb{F}_2\) and let \(Z: \text{Cob}^k_2(\emptyset, \emptyset) \rightarrow \text{Mod}_{\mathbb{F}_2}\) be the 2-dimensional TQFT associated with the Frobenius algebra \(\mathbb{F}_2[x]/(x^2)\). One can easily verify that the image \(Z(\hat{\Phi})\) of the genus-one morphism coincides with the genus-one map introduced in [8, Section 3.2]. Combining this observation with Corollary 5.3, we can conclude that, for every singular link diagram \(D\), the image \(Z([D])\) is isomorphic to the complex constructed in [8, Section 4.2]. This completes the proof.

5.4 The FI relation

We now prove that our extension of the universal Khovanov complex satisfies a categorized version of the FI relation. For a technical reason, we instead consider the universal bracket complex.

Theorem 5.7. The complex below is contractible, i.e. the identity is null-homotopic, for any checkerboard coloring \(\chi\):

\[
\left\langle \chi \right\rangle.
\]
Proof. The assertion is equivalent to saying that the genus-one morphism of the form
\[
\hat{\Phi} : \langle \langle c^x \chi \rangle \rangle \to \langle \langle c^+ \chi \rangle \rangle [1]
\]
is a chain homotopy equivalence. Using the relation \((4Tu)\) in Section 2.2, we have
\[
\hat{\Phi} = \otimes \circ \tilde{c}^- - \otimes \tilde{c}^- : \langle \langle c^x \chi \rangle \rangle \to \langle \langle c^+ \chi \rangle \rangle [1].
\]
On the other hand, recall that Bar-Natan [2] constructed the following chain homotopy equivalences associated to Reidemeister moves of type I:
\[
\begin{align*}
\mathcal{R}^-_I &= \otimes c^\sim \otimes c^- : \langle \langle \chi \rangle \rangle \to \langle \langle c^x \chi \rangle \rangle [-1], \\
\mathcal{R}^+_I &= \otimes \text{id} - \otimes \text{id} : \langle \langle \chi \rangle \rangle \to \langle \langle c^+ \chi \rangle \rangle.
\end{align*}
\]
Hence, it suffices to show that the following diagram commutes:
\[
\begin{array}{ccc}
\langle \langle \chi \rangle \rangle & \xrightarrow{\mathcal{R}^-_I} & \langle \langle c^x \chi \rangle \rangle [-1] \\
\downarrow{\mathcal{R}^+_I} & & \downarrow{\hat{\Phi}} \\
\langle \langle c^+ \chi \rangle \rangle & \xrightarrow{\hat{\Phi}} & \langle \langle c^+ \chi \rangle \rangle
\end{array}
\]
This is verified by the direct computation, and the proof is now completed. 

Our Main Theorem C clearly follows from Theorem 5.7. Furthermore, in view of Theorem 5.5 and Theorem 5.6, Theorem 5.7 implies that \( \langle\langle D^x \rangle\rangle \simeq 0 \) as soon as \(D\) contains a double point of the form of (5.7).

5.5 Examples
We conclude the paper with some examples of our invariant for singular links. Recall that Khovanov [11] classified Frobenius algebras which give rise to link invariants. Namely,
for a fixed coefficient ring $k$, and for two elements $h, t \in k$, we define a $k$-algebra $C_{h,t} := k[x]/(x^2 - hx - t)$ with a Frobenius algebra structure given by

\[
\Delta(1) := 1 \otimes x + x \otimes 1 - hx \otimes x , \quad \Delta(x) := x \otimes x + t1 \otimes 1 , \\
\varepsilon(1) := 0 , \quad \varepsilon(x) := 1 .
\]

If we denote by $Z_{h,t} : \text{Cob}_2(\emptyset, \emptyset) \to \text{Mod}_k$ the associated TQFT, then it turns out that it induces a symmetric monoidal $k$-linear functor $\text{Cob}_2^\ell(\emptyset, \emptyset) \to \text{Mod}_k$, which is again written $Z_{h,t}$ by abuse of notation. Then, for a singular link diagram $D$, we define

\[
[D]_{h,t} := Z_{h,t}(\llbracket D \rrbracket) .
\]

As a consequence of Main Theorem [A], the complex $[\llbracket D \rrbracket]_{h,t}$ is an invariant of the singular link defined by $D$.

We compute the complex $[\llbracket - \rrbracket]_{h,t}$ for the following three diagrams:

\[
D^{(1)} = \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array}, \\
D^{(2)} = \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array}, \\
D^{(3)} = \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array}.
\]

Unwinding the definition, one sees that the chain complex $\langle \langle D^{(1)} \rangle \rangle$ is isomorphic to the cochain complex associated to the following skew-commutative diagram in $\text{Cob}_2^\ell(\emptyset, \emptyset)$:

\[
\begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array} \xrightarrow{\delta} \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array} \xrightarrow{\Phi} \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array} \xrightarrow{\delta} \begin{array}{c}
\begin{tikzcd}
\bullet & \bullet \\
\bullet & \bullet
\end{tikzcd}
\end{array} ,
\end{array}
\]

where the morphisms with label $\delta$ are saddles while the ones with $\Phi$ are the morphisms given in (3.3). Applying the functor $Z_{h,t}$, we obtain a skew-commutative diagram below:

\[
\begin{array}{c}
\begin{tikzcd}
C_{h,t} \otimes C_{h,t} & C_{h,t} \\
C_{h,t}
\end{tikzcd}
\end{array} \xrightarrow{\mu} \begin{array}{c}
\begin{tikzcd}
C_{h,t} \otimes C_{h,t} & C_{h,t} \\
C_{h,t}
\end{tikzcd}
\end{array} \xrightarrow{0} \begin{array}{c}
\begin{tikzcd}
C_{h,t} \otimes C_{h,t} & C_{h,t} \\
C_{h,t}
\end{tikzcd}
\end{array} \xrightarrow{\Delta} \begin{array}{c}
\begin{tikzcd}
C_{h,t} \otimes C_{h,t} & C_{h,t} \\
C_{h,t}
\end{tikzcd}
\end{array} \xrightarrow{\mu} ,
\end{array}
\]

where $\mu$ and $\Delta$ are the multiplication and the comultiplication of the Frobenius algebra $C_{h,t}$ respectively and $\varphi : C_{h,t} \otimes C_{h,t} \to C_{h,t} \otimes C_{h,t}$ is given by

\[
\varphi(a \otimes b) := a \otimes bx - ax \otimes b .
\]

It turns out that the bottom row of (5.11) is exact so that we obtain an isomorphism

\[
H^i\left( \llbracket D^{(1)} \rrbracket \right)_{h,t} \cong \begin{cases} 
\ker \mu & i = -3 , \\
\coker \Delta & i = 0 , \\
0 & \text{otherwise}.
\end{cases}
\]

\[
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\]
A similar computation also shows that

\[ H^i \left( \left[ D^{(2)} \right]_{h,t} \right) \cong \begin{cases} \ker \mu & i = -1, \\ \coker \Delta & i = 2, \\ 0 & \text{otherwise}. \end{cases} \]

Finally, by virtue of Corollary 5.3 we have a long exact sequence

\[ \cdots \rightarrow H^{i-1} \left( \left[ D^{(3)} \right]_{h,t} \right) \rightarrow H^i \left( \left[ D^{(1)} \right]_{h,t} \right) \xrightarrow{\hat{\Phi}} H^i \left( \left[ D^{(2)} \right]_{h,t} \right) \rightarrow H^i \left( \left[ D^{(3)} \right]_{h,t} \right) \rightarrow \cdots \]

of \( k \)-modules. Note that, as seen in (5.12) and (5.13), the cohomology groups of \( \left[ D^{(1)} \right]_{h,t} \) and \( \left[ D^{(2)} \right]_{h,t} \) are direct summands of the free \( k \)-module \( C_{h,t} \otimes C_{h,t} \) and hence all projective. Therefore, we obtain an isomorphism

\[ H^i \left( \left[ D^{(3)} \right]_{h,t} \right) \cong \begin{cases} \ker \mu & i = -4, \\ \ker \mu \oplus \coker \Delta & i = -1, \\ \coker \Delta & i = 2, \\ 0 & \text{otherwise}. \end{cases} \]

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