GENERALIZED HUYGENS TYPES INEQUALITIES FOR BESSEL
AND MODIFIED BESSEL FUNCTIONS

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Abstract. In this paper, we present a generalization of the Huygens types inequalities involving Bessel and modified Bessel functions of the first kind.

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1. Introduction

This inequality

\[ 2\frac{\sin x}{x} + \frac{\tan x}{x} > 3 \]

which holds for all \( x \in (0, \pi/2) \) is known in literature as Huygens’s inequality \(^5\). The hyperbolic counterpart of (1) was established in \(^8\) as follows:

\[ 2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3, \quad x > 0. \]

The inequalities (1) and (2) were respectively refined in \(^5\) as

\[ 2\frac{\sin x}{x} + \frac{\tan x}{x} > \frac{2x}{\sin x} + \frac{x}{\tan x} > 3 \]

for \( 0 < x < \frac{\pi}{2} \) and

\[ 2\frac{\sinh x}{x} + \frac{\tanh x}{x} > \frac{2x}{\sinh x} + \frac{x}{\tanh x} > 3, \quad x \neq 0. \]

In \(^12\), Zhu give some new inequalities of the Huygens type for circular functions, hyperbolic functions, and the reciprocals of circular and hyperbolic functions, as follows:

**Theorem A** The following inequalities

\[ (1 - p)\frac{\sin x}{x} + p\frac{\tan x}{x} > 1 > (1 - q)\frac{\sin x}{x} + q\frac{\tan x}{x}. \]
holds for all \( x \in (0, \pi/2) \) if and only if \( p \geq 1/3 \) and \( q \leq 0 \).

**Theorem B** The following inequalities

\[
(1-p) \frac{\sinh x}{x} + p \frac{\tanh x}{x} > 1 > (1-q) \frac{\sinh x}{x} + q \frac{\tanh x}{x}
\]

holds for all \( x \in (0, \infty) \) if and only if \( p \leq 1/3 \) and \( q \geq 0 \).

Recently, the author of this paper extend and sharpen inequalities (5) and (6) for the Bessel and modified Bessel functions to the following results in [7].

**Theorem C** Let \(-1 < \nu \leq 0\) and let \(j_{\nu,1}\) the first positive zero of the Bessel function \(J_\nu\) of the first kind. Then the Huygens type inequalities

\[
(1-p) J_{\nu+1}(x) + p \frac{J_{\nu+1}(x)}{J_\nu(x)} > 1 > (1-q) J_{\nu+1}(x) + q \frac{J_{\nu+1}(x)}{J_\nu(x)}
\]

holds for all \((x \in (0, j_{\nu,1})\), if and only if, \( p \geq \nu+1/\nu+2 \) and \( q \leq 0 \).

**Theorem D** Let \(\nu > -1\), the following inequalities

\[
(1-p) I_{\nu+1}(x) + p \frac{I_{\nu+1}(x)}{I_\nu(x)} > 1 > (1-q) I_{\nu+1}(x) + q \frac{I_{\nu+1}(x)}{I_\nu(x)}
\]

holds for all \( x \in (0, \infty) \) if and only if \( p \leq \nu+1/\nu+2 \) and \( q \geq 1 \).

For \( \nu > -1 \) and consider the function \( J_\nu : \mathbb{R} \rightarrow (-\infty, 1] \), defined by

\[
J_\nu(x) = 2^\nu \Gamma(\nu+1)x^{-\nu} J_\nu(x) = \sum_{n \geq 0} \left( \frac{-1}{4} \right)^n \frac{1}{(\nu+1)n!} x^{2n},
\]

where \( \Gamma \) is the gamma function, \((\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)\) for each \( n \geq 0 \), is the well-known Pochhammer (or Appell) symbol, and \( J_\nu \) defined by

\[
J_\nu(x) = \sum_{n \geq 0} \frac{(-1)^n(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)},
\]

stands for the Bessel function of the first kind of order \(\nu\). It is worth mentioning that in particular the function \( J_\nu \) reduces to some elementary functions, like sine and cosine. More precisely, in particular we have:

\[
J_{-1/2}(x) = \sqrt{\pi/2}x^{1/2} J_{-1/2}(x) = \cos x,
\]

\[
J_{1/2}(x) = \sqrt{\pi/2}x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x},
\]

For \( \nu > -1 \), let us consider the function \( I_\nu : \mathbb{R} \rightarrow [1, \infty) \), defined by

\[
I_\nu(x) = 2^\nu \Gamma(\nu+1)x^{-\nu} I_\nu(x) = \sum_{n \geq 0} \left( \frac{1}{4} \right)^n \frac{1}{(\nu+1)n!} x^{2n},
\]
where \( I_\nu \) is the modified Bessel function of the first kind defined by
\[
I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}, \text{ for all } x \in \mathbb{R}.
\]

It is worth mentioning that in particular we have
\[
\begin{align*}
I_{-1/2}(x) &= \sqrt{\pi/2}x^{1/2}, \\
I_{1/2}(x) &= \sqrt{\pi/2}x^{1/2}I_{-1/2}(x) = \frac{\sinh x}{x}, \\
I_{3/2}(x) &= 3\sqrt{\pi/2}x^{-3/2}I_{3/2}(x) = -3 \left( \frac{\sinh x}{x^3} - \frac{\cosh x}{x^2} \right).
\end{align*}
\]

In this note, we present a generalization of the Huygens type inequalities (1) and (2) for Bessel and modified Bessel functions.

2. Lemmas

In order to establish our main results, we need several lemmas, which we present in this section.

**Lemma 1.** \[10\] Let \( a_n \) and \( b_n \) \((n = 0, 1, 2, \ldots)\) be real numbers, and let the power series \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) and \( B(x) = \sum_{n=0}^{\infty} b_n x^n \) be convergent for \(|x| < R\). If \( b_n > 0 \) for \( n = 0, 1, \ldots \), and if \( \frac{a_n}{b_n} \) is strictly increasing (or decreasing) for \( n = 0, 1, 2, \ldots \), then the function \( \frac{A(x)}{B(x)} \) is strictly increasing (or decreasing) on \((0, R)\).

**Lemma 2.** \([6, 2, 9]\) Let \( f, g : [a, b] \to \mathbb{R} \) be two continuous functions which are differentiable on \((a, b)\). Further, let \( g' \neq 0 \) on \((a, b)\). If \( \frac{f''}{g''} \) is increasing (or decreasing) on \((a, b)\), then the functions \( \frac{f(x)-f(a)}{g(x)-g(a)} \) and \( \frac{f(x)-f(b)}{g(x)-g(b)} \) are also increasing (or decreasing) on \((a, b)\).

**Lemma 3.** (Turán type inequality for modified Bessel function) The following Turán type inequality
\[
I_\nu(x)I_{\nu+2}(x) < \frac{\nu+2}{\nu+1} I_{\nu+1}(x),
\]
holds for all \( \nu > -1 \) and \( x \in \mathbb{R} \). In particular, the following Turán type inequality
\[
cosh(x)(\cosh(x) - x\sinh(x)) < \sinh^2(x)
\]
is valid for all \( x \in \mathbb{R} \).

**Proof.** By using the Cauchy product
\[
I_\mu(x)I_\nu(x) = \sum_{n\geq 0} \frac{\Gamma(\nu+1)\Gamma(\mu+1)\Gamma(\nu+\mu+2n+1)x^{2n}}{2^n\Gamma(n+1)\Gamma(\nu+\mu+n+1)\Gamma(\mu+n+1)\Gamma(\nu+n+1)}
\]
we have
\[
\frac{\nu + 2}{\nu + 1} I_{\nu+1}(x) - I_{\nu+2}(x) = \sum_{n \geq 0} \frac{\Gamma(\nu + 1)\Gamma(\nu + 3)\Gamma(2\nu + 2n + 3)}{2^{2n}\Gamma(n + 1)\Gamma(2\nu + n + 3)\Gamma(\nu + n + 2)\Gamma(\nu + n + 3)} x^{2n} \geq 0
\]
for all \(x \in \mathbb{R}\) and \(\nu > -1\). On the other hand, observe that using (9), (10), and (14) in particular \(\nu = -1/2\), the Turán type inequality (14) becomes (15). \(\blacksquare\)

3. Main results

We first obtain the further result concerning the generalized Huygens inequality to the Bessel functions described as Theorem 1.

**Theorem 1.** Let \(-1 < \nu \leq 0\) and let \(j_{\nu,1}\) the first positive zero of the Bessel function \(J_\nu\) of the first kind. Then the Huygens types inequalities

\[
1 > \left(1 - \frac{\nu}{\nu + 1}\right) J_\nu(x) + \frac{\nu + 2}{\nu + 1} J_\nu(x) J_{\nu+1}(x)
\]

holds for all \(x \in (0, j_{\nu,1})\)

**Proof.** Let \(\nu > -1\), we consider the function

\[
F_\nu(x) = \frac{1 - J_\nu(x)}{J_\nu(x) - J_{\nu+1}(x)}, \quad 0 < x < j_{\nu,1}.
\]

For \(0 < x < j_{\nu,1}\), let

\[
f_{\nu,1}(x) = 1 - J_\nu(x) \quad \text{and} \quad f_{\nu,2}(x) = \frac{J_\nu(x)}{J_{\nu+1}(x) - J_\nu(x)}.
\]

Now, by again using the differentiation formula

\[
J'_\nu(x) = -\frac{x}{2(\nu + 1)} J_{\nu+1}(x)
\]

we get

\[
f'_{\nu,1}(x) = \frac{x J_{\nu+1}(x)}{2(\nu + 1)},
\]

and

\[
f'_{\nu,2}(x) = \frac{x J_{\nu+1}(x)}{2(\nu + 1)} + \frac{x}{2(\nu + 2)} J_\nu(x) J_{\nu+2}(x) - \frac{x}{2(\nu + 1)} J_{\nu+1}^2(x).
\]

Thus

\[
\frac{f'_{\nu,1}(x)}{f'_{\nu,2}(x)} = \frac{1}{1 + \frac{1}{J_{\nu+1}(x)} \left(\frac{(\nu+1)J_\nu(x)J_{\nu+2}(x)}{(\nu+2)J_{\nu+1}(x)} - 1\right)}.
\]

Let

\[
h_\nu(x) = \frac{(\nu + 1)J_\nu(x)J_{\nu+2}(x)}{(\nu + 2)J_{\nu+1}(x)} - 1.
\]
From the Turán type inequality \[ (20) \]
\[ J_{\nu+1}^2(x) - J_{\nu}(x)J_{\nu+2}(x) > 0. \]
where \( \nu > -1 \) and \( x \in (-j_{\nu-1}, j_{\nu+1}) \), we conclude that \( h_{\nu}(x) \leq 0 \) for all \( x \in (0, j_{\nu+1}) \). On the other hand, differentiation again and simplifying give

\[ (21) \]
\[ h_{\nu}'(x) = \frac{(\nu + 1)x}{(\nu + 2)J_{\nu+1}^2(x)} \left[ J_{\nu+1}(x)J_{\nu+2}(x) \left( \frac{J_{\nu}(x)J_{\nu+2}(x)}{\nu + 2} - \frac{J_{\nu+1}^2(x)}{2(\nu + 1)} \right) - \frac{J_{\nu}(x)J_{\nu+3}(x)}{2(\nu + 3)} \right]. \]

By \( (20) \) and \( (21) \) we easily get

\[ h_{\nu}'(x) \leq \frac{x \nu J_{\nu+1}(x)J_{\nu+2}(x)}{2(\nu + 2)^2} - \frac{x J_{\nu}(x)J_{\nu+2}(x)J_{\nu+3}(x)}{2(\nu + 2)(\nu + 3)J_{\nu+1}(x)}. \]

In fact, since the function \( \nu \mapsto J_{\nu}(x) \) is increasing \([3, \text{Theorem 3}]\) on \((-1, \infty)\) for all fixed \( x \in (-j_{\nu-1}, j_{\nu+1}) \), and \( J_{\nu}(x) \in (0, 1] \), we conclude that, by \( (21) \), the function \( h_{\nu}'(x) \) is decreasing on \((0, j_{\nu+1})\), for all \( \nu \in (-1, 0] \). Indeed, the function \( x \mapsto J_{\nu}(x) \) is decreasing on \([0, j_{\nu+1}] \) \([3, \text{Theorem 3}]\) and nonnegative, which implies that the function \( x \mapsto \frac{1}{J_{\nu+1}(x)} \left( \frac{(\nu + 1)J_{\nu}(x)J_{\nu+2}(x)}{(\nu + 2)J_{\nu+1}(x)} - 1 \right) \) is decreasing on \((0, j_{\nu+1})\), for all \( \nu \in (-1, 0] \), as a product of two functions one is increasing and nonnegative and other is decreasing and negative. So, the function \( x \mapsto \frac{J_{\nu}(x)}{J_{\nu+2}(x)} \) is increasing \((0, j_{\nu+1})\), for all \( \nu \in (-1, 0] \), and consequently the function \( x \mapsto F_{\nu}(x) \) is increasing \((0, j_{\nu+1})\), for all \( \nu \in (-1, 0] \), by Lemma \([2]\). Using L’Hospital rule and \([19]\) yields

\[ \lim_{x \to 0} F_{\nu}(x) = \frac{\nu + 2}{\nu + 1}. \]

Finally, for each \( \nu > -1 \) and \( x \in (0, j_{\nu+1}) \) one has \( 0 \leq J_{\nu}(x) \leq 1 \), and with this the proof of inequality \( (17) \) is complete.

**Remark 1.** Using the relation \( (9) \), \( (17) \) and \( j_{-1/2} = \pi/2 \) and from the extended Huygens type inequality \( (17) \) for \( \nu = -\frac{1}{2} \) we obtain the inequality \( (17) \).

In the next Theorem, we establish the analogue of inequality \( (17) \) involving the modified Bessel functions.

**Theorem 2.** Let \( \nu > -1 \). Then the Huygens types inequality

\[ (22) \]
\[ 1 > \left( 1 - \frac{\nu + 2}{\nu + 1} \right) I_{\nu}(x) + \frac{\nu + 2}{\nu + 1} I_{\nu+1}(x) \]

holds for all \( x \in (0, \infty) \).

**Proof.** Let \( \nu > -1 \), we define the function \( G_{\nu} \) on \((0, \infty)\) by

\[ G_{\nu}(x) = \frac{1 - I_{\nu}(x)}{I_{\nu+1}(x) - I_{\nu}(x)} = \frac{g_{\nu,1}(x)}{g_{\nu,2}(x)}, \]

where

\[ g_{\nu,1}(x) = \frac{1}{j_{\nu}(x)} \left( \frac{j_{\nu}(x)}{j_{\nu+1}(x)} - 1 \right), \]

and

\[ g_{\nu,2}(x) = \frac{1}{j_{\nu}(x)} \left( J_{\nu}(x) - J_{\nu+1}(x) \right). \]
where \( g_{\nu,1}(x) = 1 - I_\nu(x) \) and \( g_{\nu,2}(x) = \frac{I_\nu(x)}{I_{\nu+1}(x)} - I_\nu(x) \). By using the differentiation formula [11, p. 79]

\[
I_\nu'(x) = \frac{x}{2(\nu + 1)} I_{\nu+1}(x)
\]

can easily show that

\[
f_{\nu,1}'(x) = \frac{1}{1 + \frac{1}{I_{\nu+1}(x)} \left( \frac{(\nu+1)I_\nu(x)I_{\nu+2}(x)}{(\nu+2)I_{\nu+1}^2(x)} - 1 \right)}.
\]

Now, for \( \nu > -1 \), we define the function \( k_\nu \) by:

\[
k_\nu(x) = \frac{(\nu + 1)I_\nu(x)I_{\nu+2}(x)}{(\nu + 2)I_{\nu+1}^2(x)} - 1.
\]

From the Turán type inequality [14] (see Lemma 3), we conclude that \( k_\nu(x) \leq 0 \) for all \( x \in \mathbb{R} \). On the other hand, using the Cauchy product [16] we get

\[
\frac{(\nu + 1)I_\nu(x)I_{\nu+2}(x)}{(\nu + 2)I_{\nu+1}^2(x)} = \sum_{n=0}^{\infty} a_n x^{2n} \sum_{n=0}^{\infty} b_n x^{2n},
\]

where \( a_n(\nu) = \frac{\Gamma^2(\nu+2)\Gamma(\nu+2n+3)}{2^n\Gamma(n+1)\Gamma(\nu+n+1)\Gamma(n+\nu+3)} \) and \( b_n(\nu) = \frac{\Gamma^2(\nu+2)\Gamma(\nu+2n+3)}{2^n\Gamma(n+1)\Gamma(\nu+n+2)} \) for all \( n = 0, 1, \ldots \). So, for all \( n = 0, 1, \ldots \), we have

\[
c_n(\nu) = \frac{a_n(\nu)}{b_n(\nu)} = \frac{\Gamma^2(\nu + n + 2)}{\Gamma(\nu + n + 1)\Gamma(n + \nu + 3)} = \frac{\nu + n + 1}{\nu + n + 2},
\]

we conclude that \( c_n(\nu) \) is increasing for \( n = 0, 1, \ldots \), and the function \( x \mapsto k_\nu(x) \) is increasing on \((0, \infty)\), by Lemma 1. Since the function \( x \mapsto \frac{1}{I_{\nu+1}(x)} \) is decreasing and nonnegative on \((0, \infty)\) and the function \( x \mapsto k_\nu(x) \) is increasing and negative on \((0, \infty)\), we conclude that \( x \mapsto \frac{g_{\nu,1}(x)}{g_{\nu,2}(x)} \) is decreasing on \((0, \infty)\), and consequently the function \( x \mapsto G_\nu(x) \) is decreasing on \((0, \infty)\), by Lemma 1. Therefore, from the L'Hospital rule and (24) yields

\[
\lim_{x \to 0} G_\nu(x) = \frac{\nu + 2}{\nu + 1}.
\]

Moreover, using the fact \( I(x) \geq 1 \), we get the Huygens type inequality (22). So, the proof of Theorem 2 is complete.

\[\square\]

**Remark 2.**

1. From the relations (11) and (12) we find that the inequality (22) is the generalization of inequality (2).
2. Since the function \( x \mapsto G_\nu(x) \) is decreasing on \((0, \infty)\), and using the asymptotic formula [11, p. 377]

\[
I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{4\nu^2 - 1}{1!(8x)} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8x)^2} - \ldots \right].
\]
which holds for large values of $x$ and for fixed $\nu > -1$, we obtain

$$\lim_{x \to \infty} G_\nu(x) = 1.$$ 

Then, the following inequality [3]

$$I_{\nu+1}(x) \leq I_\nu(x),$$

holds for all $x \in \mathbb{R}$ and $\nu > -1$.

3. Using the relation (12) and (13) from the extended Huygens type inequality (22) for $\nu = 1/2$, we obtain the following inequality

$$9 > \frac{\sinh x}{x} \left( -6 + \frac{5x^3}{x \cosh x - \sinh x} \right),$$

which holds for all $x \in \mathbb{R}$.

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