Branched Hamiltonians for a Class of Velocity Dependent Potentials

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Abstract. Hamiltonians that are multivalued functions of momenta are of topical interest since they correspond to the Lagrangians containing higher-degree time derivatives. Incidentally, such classes of branched Hamiltonians lead to certain not too well understood ambiguities in the procedure of the quantization. Within this framework we pick up a model which samples the latter ambiguities and which, simultaneously, turns out to be amenable to a transparent analytic and perturbative treatment.

1. Introduction

Models of classical systems with branched structures [1], in either coordinate (x) space or in its momentum (p) counterpart, have of late been a subject of active theoretical enquiry [2–9]. The key idea is that classical Lagrangians possessing time derivatives in excess of quadratic powers inevitably lead to p becoming a multi-valued function of velocity (v) thereby yielding a multivalued class of Hamiltonian systems.

Branched Hamiltonians in the classical context, and their quantized forms, have been recently discussed by Shapere and Wilczek [2]. Following it, Curtright and Zachos [3] analyzed certain representative models for a classical Lagrangian described by a pair of convex, smoothly tied functions of v. The underlying v turns out to be a double-valued function of p. Proceeding to the quantum domain shows that the double-valued Hamiltonians thus obtained have the inherent feature of being expressible in a supersymmetric form in the p space. Subsequently, a class of nonlinear systems whose Hamiltonians exhibit branching was explored by Bagchi et al [4] who also considered the possibility of quantization for some specific cases of the underlying coupling parameter.

In this paper we present a class of velocity-dependent Lagrangians which define a canonical momentum that yields exactly a pair of velocity variables in fractional terms. As a consequence, the corresponding Hamiltonians develop a branching character. An interesting aspect of our scheme is that it is well-suited for a perturbative treatment.
2. Branched Hamiltonians: A brief review

Let us briefly review the example of a branched system that was put forward in [3]. It was noted that a typical classical model of branched Hamiltonians results from a non-conventional form of the Lagrangian say, for the one given by

\[ L = (v - 1)^{2k-1} - V(x) \]  

where the traditional kinetic-energy term features the replacement of a typical quadratic form by a fractional function of “velocity” \( v \) while the function \( V(x) \) stands for a convenient local interaction potential. The fractional powers of the difference \( v - 1 \) was invoked to make plausible connections to known phenomenology such as the supersymmetric pairing. In detail, the \( (2k + 1) \)st root was required to be real and positive or negative for \( v > 1 \) or \( v < 1 \), respectively. Correspondingly, the quantity \( v \) turned out to be a double-valued function of \( p \).

Working out the standard steps leads to the following two branches:

\[ H_{\pm} = p \pm \frac{1}{4k-2} \left( \frac{1}{\sqrt{p}} \right)^{2k-1} + V(x) . \]  

Note that the \( k = 1 \) case speaks of the canonical supersymmetric structure [10] for the difference \( H_{\pm} - V(x) \) namely, \( p \pm \frac{1}{2\sqrt{p}} \), but in the momentum space if viewed as a quantum mechanical system. The spectral and boundary condition linkages of these Hamiltonians are not difficult to set up.

3. A velocity dependent potential

Against the above background we consider setting up of an extended Lagrangian model having a velocity dependent potential \( U(x,v) \) that gives rise to a branched Hamiltonian under Legendre transformation:

\[ L(x,v) = C(v - 1)^{2k+1} - U(x,v) \text{ where } C = \frac{2k - 1}{2k + 1} \left( \frac{1}{4} \right)^{2k+1} \]  

where we assume \( U(x,v) \) to be given in a separable form \( U(x,v) = f(v) + V(x) \), \( f(v) \) and \( V(x) \) are certain functions of \( v \) and \( x \) respectively.

Using the standard definition of the canonical momentum, we find that it is given by

\[ p = \left( \frac{1}{4} \right)^{2k+1} (v - 1)^{2k+1} - f'(v) . \]  

The above equation is too complicated to put down the multivalued nature of velocity in a tractable closed form.

If we try to determine the associated branches of the Hamiltonian corresponding to this Lagrangian (3), \( H_{\pm} \) emerge in a mixed form involving the momentum \( p \), the function \( f(v) \) and its derivative.

\[ H_{\pm} = p \pm \frac{1}{4} \left[ p + f'(v) \right]^{-\frac{2k+1}{2}} \left( \frac{2k + 1}{2k - 1} \right) \left( p + f'(v) \right)^{-1} + U(x,v) . \]  

Since a Hamiltonian has to be a function of the coordinate and the corresponding canonical momentum, \( H_{\pm} \) as derived above is of little use.
We note that the case $k = 1$ is particularly interesting to understand the spectral properties of $L(x, v)$. Explicitly, the Lagrangian assumes the simple but a general form

$$L = 3 \left( \frac{1}{4} \right)^{\frac{2}{3}} (v - 1)^{\frac{1}{3}} - f(v) - V(x).$$  

(6)

A sample choice for $f(v)$ could be

$$f(v) = \lambda v + 3\delta (v - 1)^{\frac{1}{3}}$$

(7)

with $\lambda (\geq 0)$ and $\delta \left( < 4^{-\frac{2}{3}} \right)$ being suitable real constants. Observe that the presence of $\delta$ rescales the kinetic energy coefficient which now enjoys a parametric representation.

The above form of $f$ facilitates determination of the canonical momentum $p$ in a closed form as given by

$$p = \mu (v - 1)^{-\frac{2}{3}} - \lambda$$

(8)

where $\mu = 4^{-\frac{2}{3}} - \delta > 0$. On inversion, we find a pair of relations for the velocity depending on $p$:

$$v_{\pm}(p) = 1 \mp \mu^{\frac{3}{2}} (p + \lambda)^{-\frac{3}{2}}.$$  

(9)

As a consequence, we run into two branches of the Hamiltonian which we put down in the form

$$H_{\pm} - V(x) = (p + \lambda) \pm \frac{2\gamma}{\sqrt{p + \lambda}}.$$  

(10)

For the ease of notation, note that we have replaced $\mu^{3/2}$ with $\gamma$. In the special case where $\lambda = 0$

![Figure 1](image_url)

**Figure 1.** When $\lambda = 1$ and $\gamma = \frac{1}{2}$, $H_{\pm} - V(x)$ branches correspond to the upper and lower curves respectively.

and $\gamma = \frac{1}{2}$, we recover the Hamiltonian derived in [3]. However, the presence of the parameter $\gamma$ in (10) is nontrivial as our following treatment of perturbative analysis will show. In Figure 1, we have given a graphical illustration (for $\lambda = 1$ and $\gamma = \frac{1}{2}$) of the behavior of the two branches of the Hamiltonian against some typical values of the momentum variable. As in the $\lambda = 0$ case of [3] here also we encounter a cusp asymptotically with regard to $p$ for a fixed $\gamma$. 


4. Lowest excitations and the Fourier transform

After one decides to consider just small excitations of our quantum system over a local or global minimum \( (x_0) \) of a generic analytic potential \( V(x) \), one may put the origin of the coordinate axis to this minimum, \( x \to y = x - x_0 \), and write down the Taylor series

\[
V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2}(x - x_0)^2V''(x_0) + \ldots
\]  

(11)

Recall that \( V'(x_0) = 0 \) and the zero of the energy scale can be shifted in such a manner that \( V(x_0) = 0 \). Finally, the series is truncated after the first non-trivial term yielding, in \emph{ad hoc} units,

\[
V(x_0 + y) = y^2.
\]  

(12)

After a Fourier transform to the momentum space, we get a transformed quantum form of the Hamiltonian guided by the second-order differential operator,

\[
H = -\frac{d^2}{dp^2} + W(p)
\]  

(13)

containing a one-parametric family of pseudo-potentials

\[
W(p) = p + \frac{2\gamma}{\sqrt{p}}.
\]  

(14)

Here, the original subscript \( \pm \) entering Eq. (10) may be perceived as equivalent to an optional switch between positive coupling-type parameter \( \gamma > 0 \) and its negative alternative \( \gamma < 0 \). Besides such a freedom of the sign of the dynamical characteristic, the consequent quantum-theory interpretation of the model requires also a few nontrivial mathematical addenda. The form Eq. (14) matches with Eq. (10) for \( \lambda = 0 \) which will now be our point of inquiry.

First of all, the most natural tentative candidate

\[
H\phi_n(p) = E_n\phi_n(p), \quad p \in (-\infty, \infty)
\]  

(15)

for the quantum Schrödinger equation living on the whole real line of momenta (i.e., with \( \phi_n(p) \in L^2(\mathbb{R}) \)) is characterized by the asymptotically linear decrease of the pseudo-potential (14) along the left half-line. Hence, the negative half-axis of momenta \( p \) must be excluded, \emph{a priori}, as unphysical. In other words, the acceptable wave functions \( \phi_n(p) \) should vanish, identically, whenever \( p \in (-\infty, 0) \). The consistent quantization of our model must be based on the modified, half-line version of Eq. (15), viz., on Schrödinger equation

\[
H\phi_n(p) = E_n\phi_n(p), \quad p \in (0, \infty)
\]  

(16)

such that (cf. also [2] and [3])

\[
\phi_n(p) \in L^2(\mathbb{R}^+).
\]  

(17)

Still, the discussion is not yet complete. Due care must be also paid to the fact that the inverse-square-root singularity of \( W(p) \) in the origin is “weak” (see, e.g., Ref. [11] for a detailed explanation of the rigorous, “extension theory” mathematical contents of this concept). In the language of physics, such a comment means that the information about possible bound states and physics represented by Eq. (16) with constraint (17) is incomplete.

In the rest of this paper (i.e., in sections 5 and 6) we shall, therefore, describe the two alternative versions of the completion of the missing, phenomenology-representing information.
5. Eligible “missing” boundary conditions at small $\gamma$ and $p = 0$

As we emphasized above, the existence of the usual discrete spectrum of bound states can only be guaranteed via an additional physical boundary condition at $p = 0$. Although, from the point of view of pure mathematics, the choice of such a condition is flexible and more or less arbitrary, the necessary suppression of this unwanted freedom can rely upon several forms of the physics-based intuition.

Let us split the problem into two subcategories. In a simpler scenario we shall assume that the central core is repulsive and strong (i.e., that our parameter is positive and large, $\gamma \gg 1$). This possibility will be discussed in the next section 6. For the present, let us admit that the (real) value of $\gamma$ is arbitrary and that the regular nature of our ordinary differential Schrödinger equation near $p = 0$ implies that the integrability condition (17) itself still does not impose any constraint upon the energy $E$ [11]. A fully explicit and constructive demonstration of such an observation may be based on the routine reduction of (16) to its simplified, leading-order form

$$- \sqrt{p} \frac{d^2}{dp^2} \psi(p) + 2 \gamma \psi(p) = 0. \quad (18)$$

Being valid at the very small (though still positive) values of $p \ll 1$ this equation is exactly solvable in terms of Bessel functions [12]. Thus, one may choose either $\gamma > 0$ or $\gamma < 0$.

After some algebra we obtain the respective two-parametric families of the general solutions which depend on two parameters $C_{1,2}$ or $D_{1,2}$ and which remain energy-dependent. At small $p$ they behave, respectively, as follows,

$$\psi(p) = C_1 \sqrt{p} I_{2/3} \left( \frac{4\sqrt{2}}{3} \sqrt{\gamma p^3} \right) + C_2 \sqrt{p} K_{2/3} \left( \frac{4\sqrt{2}}{3} \sqrt{\gamma p^3} \right) \quad (19)$$

and

$$\psi(p) = D_1 \sqrt{p} J_{2/3} \left( \frac{4\sqrt{2}}{3} \sqrt{-\gamma p^3} \right) + D_2 \sqrt{p} Y_{2/3} \left( \frac{4\sqrt{2}}{3} \sqrt{-\gamma p^3} \right) \quad (20)$$

On this purely analytic background, one of the most natural resolutions of the paradox of the ambiguity of the physical boundary conditions at $p = 0$ may be based on the brute-force choice of the parameters $C_{1,2}$ or $D_{1,2}$ in these formulae.

Finally, let us emphasize that intuitively by far the most plausible requirement of the absence of the jump in the wave functions at $p = 0$, i.e., the Dirichlet boundary condition

$$\lim_{p \to 0} \psi(p) = 0 \quad (21)$$

would remove the latter ambiguity of quantization in the most natural manner. The resulting pair of the requirements

$$C_2 = 0, \quad D_2 = 0 \quad (22)$$

may be then recommended as easily derived from the well known approximate formulae for the Bessel functions near the origin [12].

6. Perturbation-theory analysis at large $\gamma \gg 1$

In a purely formal spirit one could complement the above-recommended Dirichlet boundary condition (21) by its Neumann vanishing-derivative analogue

$$\lim_{p \to 0} \psi'(p) = 0 \quad (23)$$
or, more generally, by a suitable Robin boundary condition. In this context it is worth adding that with a systematic strengthening of the repulsive version of the barrier (i.e., with the growth of the positive coupling constant $\gamma$) the specification of the additional boundary conditions at $p = 0$ becomes less and less relevant because the two alternative energy levels will degenerate in the limit $\gamma \to \infty$.

The most immediate explanation of this phenomenon may be provided by perturbation theory. In the dynamical regime, when the parameter is large, a perturbative approach seems to be particularly well suited. With $\gamma \gg 1$, we look at the absolute minimum of the potential $W(p)$ which occurs at $p_0$, say. This value is, incidentally, unique

$$p_0 = \gamma^{\frac{2}{3}} \gg 1$$

With the construction of a Taylor series in its vicinity,

$$W(p) = W(p_0) + (p - p_0)W'(x_0) + \frac{1}{2}(p - p_0)^2W''(p_0) + \ldots$$

we observe that the first term, which is given by

$$W(p_0) = 3\gamma^{\frac{2}{3}}$$

in very large in this scenario. In contrast, all of the further Taylor coefficients remain very small and asymptotically negligible,

$$W''(p_0) = \frac{3}{2}\gamma^{-\frac{2}{3}}, \quad W'''(p_0) = -\frac{15}{4}\gamma^{-\frac{4}{3}} \ldots$$

Clearly then, with $\gamma \gg 1$, $H$ can be expressed as

$$H = -\frac{d^2}{dp^2} + 3\gamma^{\frac{2}{3}} + \frac{3}{4}\gamma^{-\frac{2}{3}}(p - p_0)^2 + \mathcal{O}(\gamma^{-\frac{4}{3}}(p - p_0)^3).$$

After one re-scales the axis $p = \rho q$, equation (16) acquires the modified form

$$\tilde{H}\tilde{\phi}_n(q) = E_n\rho^2\tilde{\phi}_n(q)$$

where,

$$\tilde{H} = -\frac{d^2}{dq^2} + 3\rho^2\gamma^{\frac{2}{3}} + \frac{3}{4}\rho^4\gamma^{-\frac{2}{3}}(q - q_0)^2 + \mathcal{O}(\rho^5\gamma^{-\frac{4}{3}}(q - q_0)^3).$$

One may now set

$$\rho = \left(\frac{4}{3}\right)^{\frac{1}{4}}\gamma^{\frac{1}{6}}$$

yielding the very weakly perturbed harmonic-oscillator Hamiltonian

$$\tilde{H} = -\frac{d^2}{dq^2} + (q - q_0)^2 + \gamma\sqrt{12} + \mathcal{O}(\rho^5\gamma^{-\frac{4}{3}}(q - q_0)^3).$$

In full analogy to many models with similar structure (cf., the study [13] containing further references), the exact solvability of the model in the leading-order harmonic-oscillator approximation proves sufficient because in the domain of large $\gamma \gg 1$ the contribution of the anharmonic corrections becomes negligible.
7. Summary
To summarize, we looked at the particular example of a non-conventional form of a velocity-dependent Lagrangian that leads to a double-valued structure of the associated Hamiltonian for some specific choice of the underlying coupling parameter. We showed that our scheme allows for a perturbative analysis by constructing a Taylor series near the vicinity of the absolute minimum of the potential.

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