Heat conduction in rotating relativistic stars

S. K. Lander\textsuperscript{1,*}, N. Andersson\textsuperscript{2}

\textsuperscript{1}Nicolaus Copernicus Astronomical Centre, Polish Academy of Sciences, Bartycka 18, 00-716 Warsaw, Poland, 
\textsuperscript{2}Mathematical Sciences and STAG Research Centre, University of Southampton, Southampton, SO17 1BJ, U.K.

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ABSTRACT
In the standard form of the relativistic heat equation used in astrophysics, information is propagated instantaneously, rather than being limited by the speed of light as demanded by relativity. We show how this equation nonetheless follows from a more general, causal theory of heat propagation in which the entropy plays the role of a fluid. In deriving this result, however, we see that it is necessary to make some assumptions which are not universally valid: the dynamical timescales of the process must be long compared with the explicitly causal physics of the theory, the heat flow must be sufficiently steady, and the spacetime static. Generalising the heat equation (e.g. restoring causality) would thus entail retaining some of the terms we neglected.

As a first extension, we derive the heat equation for the spacetime associated with a slowly-rotating star or black hole, in which the thermal conductivity becomes tensorial. In this case the heat equation features two new, non-dissipative, terms related to frame-dragging and the spacetime’s curvature. We show that the latter term is liable to be very small, and as a consequence demonstrate that a hotspot on a neutron star will be seen modulated at the rotation frequency by a distant observer.

Key words: accretion, accretion discs; conduction; gravitation; stars – rotation; stars – neutron

1 INTRODUCTION
Treating heat propagation in general relativity (GR) is a necessary ingredient for the quantitative modelling of compact objects. It becomes important during the collapse of a massive star into either a neutron star or black hole (Misner & Sharp 1964; Govender, Maharaj & Maartens 1998; Woosley, Heger & Weaver 2002; Sekiguchi 2010). It is needed to understand neutron-star cooling from birth, through the rapid proto-neutron star phase (Burrows & Lattimer 1986) to the slower, secular evolution of mature neutron stars (Van Riper 1991; Aguilara, Pons, Miralles 2008). It is also important in modelling short-timescale cooling following outbursts from accreting neutron stars (Cumming et al. 2017), for accretion physics around black holes (Yuan & Narayan 2014; Ressler et al. 2015), and in the very late stages of a binary neutron-star inspiral (Shibata, Taniguchi & Uryu 2005).

The standard relativistic heat equation is acausal, predicting that information propagates instantaneously, and thus violating a basic tenet of relativity that nothing can travel faster than light. It is now well established that causality can be restored in a natural way, by treating entropy as a fluid whose dynamics couple to those of the medium (e.g. the various fluid species of a neutron star). Relativistic thermal dynamics is naturally expressed in this multifluid framework, and can be applied in problems on short timescales where the finite propagation of entropy is important, or in situations where the spacetime is highly dynamic – like the merger of compact objects. In its full form it is, however, likely to be too computationally complex for many practical purposes.

Here we show how the usual form of the relativistic heat equation, for a spherical spacetime, may be recovered from the multifluid framework in a natural way. Along the way we show that it is necessary to make a few assumptions – probably safe ones for secular processes in a neutron-star or black-hole environment, but not necessarily in every astrophysical situation.

* skl@camk.edu.pl
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This therefore gives a diagnostic of when the familiar relativistic heat equation is not applicable: if any of these terms is not negligible. Having established the non-rotating result, we generalise our approach to find the heat equation governing a slowly-rotating star, keeping the terms of linear order in the rotation rate (which include frame-dragging). We drop second-order rotational terms (which cause the spacetime to deviate from sphericity), but note that these have been explored in the context of neutron-star cooling by Miralles, Van Riper & Lattimer (1993) and Negreiros, Schramm & Weber (2017).

This paper is aimed at an astrophysics audience, and so is intended to assume no specialist knowledge of the reader. For this reason, we begin with a brief review of the theory of relativistic thermal dynamics and the foliation of spacetime for numerical simulations, in order to make our discussion self-contained. We then derive, in turn, Fourier’s law and the heat equation; in each case beginning with the non-rotating result and then exploring the effect of slow rotation. We conclude with a specific example which demonstrates that the first-order rotational correction is negligible: the rotational modulation of a neutron-star hotspot as seen by a distant observer.

2 RELATIVISTIC THERMAL DYNAMICS

2.1 The problem of causality in heat conduction

The heat equation is so familiar that it is easy to forget one conceptually unsatisfactory feature of it. Although this paper is concerned with heat conduction in general relativity, let us begin in Newtonian gravity. The heat equation states that a temperature distribution $T$ evolves as:

$$\frac{\partial T}{\partial t} = -\frac{1}{C_V} \nabla \cdot Q,$$

where $C_V$ is the volumetric heat capacity. The heat flux $Q$ is, in turn, given by Fourier’s law:

$$Q = -\kappa \nabla T,$$

where $\kappa$ is the thermal conductivity. Substituting (2) into (1), however, we see that the resulting heat equation is parabolic – and so the characteristic propagation speed is infinite. Even in Newtonian physics, it is unreasonable to expect some physical quantity to be transmitted instantaneously, and efforts to rectify this deficiency in the heat equation go back decades. A natural reference point is the Cattaneo equation, in which causality is introduced – albeit in a phenomenological manner – through a time-dependent term in Fourier’s law:

$$t \frac{\partial Q}{\partial t} + Q = -\kappa \nabla T,$$

where $t$ is some small positive number, which can be thought of as a relaxation timescale for the medium (Herrera & Santos 1997). Now plugging this back into equation (1), we have a hyperbolic equation, known as the telegrapher’s equation; the problem of instantaneous propagation has been removed, but at the expense of introducing a term not clearly linked to any underlying microphysics.

The relativistic heat equation used for neutron stars and black-hole spacetimes emerges from the same derivation as in Newtonian gravity, but with all quantities redshifted, as they are seen by a distant observer (Misner & Sharp 1965; Thorne 1967; Van Riper 1991). The causality problem is therefore still present, but is now even more serious; it is fundamentally unacceptable in GR for any information to propagate beyond the speed of light, let alone at infinite speed. The first successful (and not ad-hoc) resolution to the problem was the Israel-Stewart approach (Israel & Stewart 1979), which posits an expansion of the entropy flux through a set of terms which encode deviations from thermal equilibrium. The various independent coefficients of this expansion need, however, to be fixed with additional constraints (either theoretical or experimental). The theory is thus rather complex, but nonetheless pragmatically motivated: it allows one to recover a causal heat equation, a relativistic analogue of the Cattaneo equation.

An alternative starting point is the multifluid (variational) approach of Carter – see e.g. Carter (1989), or Lopez-Monsalvo (2011) for a fuller account of the problem and relevant references. In this, entropy is regarded mathematically as a massless fluid, which satisfies continuity and Euler equations like any other fluid. Furthermore, the theory naturally allows for three kinds of interaction between different fluid species. One can have chemical reactions describing the creation or destruction of particles – although, in the case of entropy, the second law of thermodynamics dictates that the reaction rate must be non-negative. Secondly, dissipation is introduced through a series of scattering terms, which include scattering of the usual fluid particles with the entropy ‘particles’ (e.g. phonons). The third, and least familiar, interaction is entrainment. This is a non-dissipative coupling between two fluid species, which depends on the relative velocity between the two. In normal fluid dynamics, the momentum and velocity of a fluid are parallel; in the presence of entrainment the two can be misaligned, a phenomenon sometimes known as the Andreev-Bashkin effect (Andreev & Bashkin 1975; Alpar, Langer & Sauls 1984).

Within the multifluid framework, it is certainly permitted to allow for entrainment between the entropy fluid and other species, but the physical reason for doing so is not initially obvious. Indeed, in the original incarnation of his multifluid model, Carter dropped these terms for simplicity. It was quickly seen, however, that the resulting theory suffered from serious
instabilities (Olson & Hiscock 1990), casting doubt on the usefulness of the framework until Priou (1991) showed that the theory was indeed stable once the entrainment terms were restored. In fact, the resulting model can be shown to be equivalent to the Israel-Stewart theory up to first order in deviations from thermal equilibrium (Lopez-Monsalvo & Andersson 2011).

Despite its superficially obscure nature, entrainment between the entropy fluid and the matter fluid(s) in a system is also the key ingredient which allows one to derive a Cattaneo-type form of Fourier’s law, and thus restore causality to heat conduction within the multifluid approach – both in Newtonian gravity and general relativity (Lopez-Monsalvo 2011). Because it acts over timescales much shorter than those of many relativistic astrophysics problems, it might appear to be irrelevant for these. Let us sound two notes of caution, though. Firstly, the standard heat equation will not be adequate for modelling every hot relativistic system – especially not those involving dynamic spacetimes or processes with short characteristic timescales. Secondly, even when it should be valid, there is still a risk that the neglect of entrainment could lead to instabilities, as for Carter’s original approach. For these reasons, whilst we will indeed drop entropy entrainment during the course of deriving the heat equation, we will keep it for the first steps, to show where the explicitly causal terms lie.

2.2 The equations of entropy dynamics

Our starting point will be Andersson et al. (2017), hereafter AHDC, who described thermal dynamics in general relativity in a multifluid formalism, where entropy becomes, mathematically, just another fluid. We begin, as they do, in geometrised units where \( G = c = 1 \) (so that all quantities have dimensions with powers of length alone), then restore these factors afterwards. AHDC utilise the standard ‘3+1 split’, in which 4-dimensional spacetime is foliated into a nested set of 3-dimensional spacelike hypersurfaces threaded by a set of timelike worldlines which cut through each hypersurface perpendicularly; see e.g. Thorne & MacDonald (1982). In what follows, we will denote spacetime quantities using indices \( a, b \) (taking values from 0 to 3); and will use the indices \( i, j, k \) (taking values from 1 to 3) to denote quantities restricted to the hypersurfaces. We will give only a minimal description of the 3+1 split, and refer the reader to the notes of Gourgoulhon (2007) for a detailed, pedagogical description.

An observer travelling along a timelike worldline (often called an Eulerian observer) experiences proper time \( \tau \) and has a 4-velocity \( N \) (in index notation, \( N^a \)) given by \( d/d\tau \). The relationship between an observer’s proper time, and the ‘global time’ \( t \) measured by an observer at infinity, is encoded in the lapse \( \alpha \), defined as

\[
\alpha \equiv \frac{dt}{d\tau}
\]

along a worldline.

The notion of time variation may be expressed in terms of the 3+1 split, as the Lie derivative along the 4-vector

\[
t = \alpha N + \beta \quad \text{or} \quad t^a = \alpha N^a + \beta^a,
\]

where \( \beta \) is the ‘shift vector’: a 3-vector living in a spacelike hypersurface, so that

\[
\beta^a N_a = 0.
\]

We may therefore denote the shift vector \( \beta^a \) rather than \( \beta^a \), when it does not result in mismatched indices within the same expression. The shift vector may be arbitrarily specified, aside from the restriction of being spatial. Now, from the lapse and shift we can split the spacetime, writing \( x^a = (t, x^i) \), so that the line element reads

\[
d\ell^2 = g_{ab} dx^a dx^b = -(\alpha^2 - \beta_i \beta^i) dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j,
\]

where \( g_{ab} \) is the spacetime metric and \( \gamma_{ij} \) the 3-metric of the spatial hypersurfaces.

Returning to derivatives, we may use the linearity of the Lie derivative to show that

\[
\partial_i = \mathcal{L}_4 = \mathcal{L}_{\alpha N + \beta} = \mathcal{L}_\alpha N + \mathcal{L}_\beta.
\]

Since the Lie derivative of a covariant 4-vector \( v^a \) along some other 4-vector \( u^a \) is given by

\[
\mathcal{L}_u v^a = u^b v_{,a}^b + v_b^b u_{,a}^b,
\]

we see that for a scalar \( h \):

\[
\mathcal{L}_N h = \frac{1}{\alpha}(\partial_i - \mathcal{L}_\beta) h.
\]

The same is also true for a covariant 3-vector \( p_i \). To see this, first note that since \( N_a \) is timelike, \( N_a q^a = 0 \) for a 3-vector \( q^i \), and therefore \( 0 = N_a q^a = g_{ab} N_b^a q^a = N^a q_a \). Using this result and equation (9), we then find that:

\[
\mathcal{L}_N p_i = \alpha N^b_{,i} p_b + p_j (\alpha N^j,i)_i = \alpha N^b_{\alpha} p_{\alpha,i} + \alpha p_j N^j,i = \alpha \mathcal{L}_{\alpha N} p_i,
\]

and so

\[
\mathcal{L}_N p_i = \frac{1}{\alpha}(\partial_i - \mathcal{L}_\beta) p_i.
\]

In addition to the Lie derivative, we will briefly use the covariant derivative for spacetime quantities, denoting it with a
subscript semicolon; and will extensively use the covariant derivative projected into the hypersurfaces, $D_i$ (see AHDC for more details). We recall its form, in components, when acting on co- and contra-variant 3-vectors, $p_i$ and $q^i$ respectively:

\begin{align}
D_j p_i &= p_{i,j} - \Gamma^k_{ij} p_k, \\
D_j q^i &= q^i_{,j} + q^k \Gamma^i_{jk},
\end{align}

where the Christoffel symbols $\Gamma^i_{jk}$ (associated with the projected covariant derivatives) encode the difference between covariant and partial derivatives of a 3-vector which arises from the curvature of spacetime. In our context we only have derivatives within a hypersurface, for which these symbols involve derivatives of the 3-metric $g_{ij}$ only:

\begin{equation}
\Gamma^i_{jk} = \frac{1}{2} \gamma^{il} (\gamma_{jl,k} + \gamma_{jk,l} - \gamma_{jk,l}).
\end{equation}

Note that the 3-metric is diagonal in the cases we consider, i.e. $\gamma^i_i = 0$ for $i \neq l$.

After this general summary section, we will specify to stationary and axisymmetric spacetimes, i.e. spacetimes with azimuthal and timelike Killing vectors. The arbitrariness of the shift vector discussed above is then very useful, because we may define it so that $\alpha N + \beta$ is equal to the timelike Killing vector.

AHDC showed that the thermal dynamics of a star may be described by one scalar equation, the entropy equation; and one vector equation, the entropy momentum equation. The equations allow for a bulk flow $(\mathbf{v})$ of any fluid species $x$ is split as follows:

\begin{equation}
u^a_0 = N^a + v_0^a, \tag{16}
\end{equation}

where $v_0^a$ is the spacelike 3-velocity of the fluid within a hypersurface. As mentioned earlier, thermal dynamics in the multifluid formalism is described quite naturally by treating entropy as a massless ‘fluid’ (Prix 2004). Then, the entropy flux $n_s^a$ under the 3+1 split is given by

\begin{equation}n_s^a = n_s u^a_0 = s(N^a + v_0^a), \tag{17}
\end{equation}

where $n_s$ is the number density of the entropy fluid is the entropy density $s$. Like all fluids, the entropy has a continuity equation associated with it:

\begin{equation}(n_s^a)_a = \Gamma_a s \geq 0, \tag{18}
\end{equation}

where $\Gamma_a$ is the entropy production rate. The only thing which distinguishes this continuity equation from the usual form is that the production rate must be non-negative in this case, by the second law of thermodynamics. We shall call equation (18) the entropy equation.

The other fundamental quantity is the entropy 4-momentum:

\begin{equation}\mu_s^a = TN_a + S^a, \tag{19}
\end{equation}

where we have identified the time component of the 4-momentum as the temperature $T$, and denoted the entropy 3-momentum by $S^a$ (which is spacelike; it satisfies $N^a S_a = 0$). Again, in analogy with ordinary fluids, the non-conservation of the entropy 4-momentum is due to chemical reactions and frictional processes, as expressed in the momentum equation:

\begin{equation}(N^a + v_0^a) (\mu^a_0 - \mu^a_{\infty}) + \Gamma_a \mu^a_s = \Gamma_a T(N_a + v^a_0) + \sum_{m \neq s} R^{a,m} (\delta^a_m + v^m_0 u_a) (v_0^m - v^m_k), \tag{20}
\end{equation}

which may be obtained by combining equations (64), (65) and (72) from AHDC with equation (16) of this paper. Note that here, and later, we will generally refer to all fluid species other than the entropy as matter fluids (assumed to move together), with index m. Equation (20), being an intermediate algebraic step, is not the usual guise in which the momentum equation appears. Nonetheless, we can identify the familiar features by thinking of the spacetime coordinate with index 0 as time and coordinates 1, 2, 3 as spatial. Then, we see that equation (20) features a time-derivative of the momentum as well as its divergence – representing the notion from Newton’s second law that the imbalance of forces on a fluid source its net acceleration – together with various terms which describe the transfer of momentum into or out of the entropy-fluid component by dissipation or reactions.

As discussed in section 2.1, entropy entrainment is crucial for constructing a causal theory of thermal dynamics. Although we will not retain this effect later on, it is instructive to see where it features in these fundamental equations. In principle the entrainment term could then be propagated through the rest of our derivation of Fourier’s law to yield an explicitly causal final heat equation. It appears in the definition of the entropy 3-momentum:

\begin{equation}S^a_i = Tv^i_0 + \sum_{m \neq s} A^{a,m} n_m (v^i_s - v^i_0), \tag{21}
\end{equation}

from which it is easy to see that entrainment allows for the entropy momentum and velocity to be misaligned. The timescale on which a coefficient $A^{a,m}$ couples the matter and entropy dynamics should, therefore, be some analogue of the phenomenological relaxation timescale $t$ from the Cattaneo equation (3).

We have introduced the entropy (18) and entropy momentum (20) equations as they emerge from the multifluid framework.
A major part of the work of AHDC was to rewrite and simplify these to 3+1 forms closer to those used in numerical relativity. We refer the reader to their paper for derivations, and here simply present the required results. Firstly, the entropy equation may be rewritten as [AHDC equation (136)]:
\[
\partial_t (\gamma^{1/2} s) + D_i \left\{ \alpha \gamma^{1/2} \left[ \frac{Q^i}{T} - \frac{\beta^i}{\alpha s} \right] \right\} = \alpha \gamma^{1/2} \Gamma_s, \tag{22}
\]
where \( \gamma = g^{ab} \gamma_{ab} \) is the determinant of the spatial 3-metric, and where we have defined a more physically-motivated quantity – the heat flux – as
\[
Q^i \equiv s T v^i_s = s S^i_s - s \sum_{m \neq s} A^m m_m (v^i_m - v^i_s). \tag{23}
\]
We now turn to the expression for the entropy momentum equation given in AHDC equation (139), which features both \( Q^i \) and \( S^i_s \) explicitly. Using equation (23) then allows us to eliminate the entropy 3-momentum in favour of the heat flux and entropy entrainment terms. If we now neglect entrainment, the result is:
\[
\partial_t (\gamma^{1/2} Q_i) + D_j \left[ \gamma^{1/2} \left( \frac{\alpha Q^j}{sT} - \beta^j \right) Q_i \right] + s D_i (\alpha \gamma^{1/2} T) - \frac{Q^j}{s} D_i \left( \alpha \gamma^{1/2} \frac{Q_j}{s} \right) = \gamma^{1/2} \left( \alpha F^i_s - \alpha Q^i Q_j + Q_j D_i \beta^j \right), \tag{24}
\]
where the entropy entrainment terms – were they to be restored – would feature together with every instance of the heat flux, except for the two contravariant \( Q^j \) terms on the left-hand side. In equation (24) we have introduced two new quantities. The first is the extrinsic curvature
\[
K_{ij} = \frac{1}{2 \alpha} \left[ -\gamma_{ij,\ell} + \beta^k \gamma_{ijk} + \gamma_{ij,\ell} \beta^k + \gamma_{jk,\ell} \beta^i \right], \tag{25}
\]
which describes how a hypersurface curves within the spacetime in which it is embedded; and the second is a resistive term \( F^i_s \) which encodes the transfer of momentum away from the entropy component due to collisions with other particle species. The general form of this term is given by equation (136) of AHDC, but is simpler in our case, since the bulk flow \( \hat{v}^i = 0 \) and so the redshift factor \( \gamma = (1 - \hat{v}^i \hat{v}_i)^{-1/2} = 1 \). For clarity, let us consider only entropy-matter particle scattering, and neglect any additional resistive mechanisms (e.g. Joule heating from magnetic-field decay). Under these assumptions, the collision term from AHDC equation (136) reduces to:
\[
F^i_s = -\sum_{m \neq s} \frac{R^m_s}{s T} Q_i \equiv -\frac{R}{s T} Q_i. \tag{26}
\]
The system of equations for relativistic thermal dynamics summarised above, from AHDC, is appropriate for nonlinear evolutions in dynamic spacetimes using a 3+1 foliation of spacetime (even though, by equation (24), the system has already ceased to be valid for studying processes on timescales short compared with the medium’s thermal relaxation). However, there are many problems for which this is unduly general. In what follows, we will explore how these equations simplify in one typical astrophysical setting.

2.3 Rotating systems and observers

Let us consider heat conduction in the spacetime associated with a central object – either a star or a black hole – rotating at rate \( \Omega \). We will assume that the object is isolated, so that the spacetime remains stationary, and that the rotation is sufficiently slow that we may drop the second-order rotational terms which lead to deviations from spherical symmetry (e.g. the oblateness induced by rotation, in Newtonian or relativistic stars).

From here onwards, we will begin to restore the suppressed \( G \) and \( c \) factors (i.e. the dimensions) to the expressions from AHDC. By doing so, we will immediately be able to identify combinations of unfamiliar quantities from the multifluid framework with familiar transport properties, using dimensional analysis. Converting from geometrised to physical units is not especially simple, and so we establish the requisite conversions systematically, by beginning with the most fundamental quantities. To start with, it is natural to identify the spatial coordinates for our system with globally-defined spherical polar coordinates, \((ct, r, \theta, \varphi)\). The line element, with geometrising factors unsuppressed, is then
\[
dl^2 = -e^{2\Phi/r^2} c^2 dw^2 - 2 \frac{\omega}{c} r^2 \sin^2 \theta d\varphi dw + e^{2\Lambda/r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \tag{27}
\]
where \( \Phi = \Phi(r) \) and \( \Lambda = \Lambda(r) \) are the two metric functions, and \( \omega \) is the angular velocity at which frames near the central object are dragged with respect to an observer at infinity.

It is very useful to generalise the Newtonian notion of a corotating observer to the relativistic one of a zero-angular-momentum observer (ZAMO) \( (\text{Bardeen 1970; Bardeen, Press & Teukolsky 1972}) \). A ZAMO has a local rotational velocity of zero, and the mathematical description of physical processes is at its least complex with respect to such an observer. More specifically, one has a set of equations governing the physics of the system (e.g. a neutron star), in terms of a globally-defined system of coordinates. The local coordinate system of a ZAMO is encoded in a tetrad of orthonormal basis vectors; projecting
the globally-defined equations onto a ZAMO’s orthonormal tetrad results in a greatly simplified description of the physics. From this perspective we may regard $\omega$ as the variation, with respect to global time $t$, of the ZAMO’s azimuthal coordinate:

$$\omega = \frac{dx^3}{dt} = \frac{d\tilde{\omega}}{dt}.$$  

(28)

This quantity can be shown to be a function of the radial coordinate alone (Hartle 1967). It is given by the equation

$$\frac{1}{r^3} \frac{d}{dr} \left( r^3 e^{-\left(\Phi+\Lambda\right)/c^2} \frac{d\omega}{dr} \right) + \frac{\omega}{c} \frac{d}{dr} \left( e^{-\left(\Phi+\Lambda\right)/c^2} \right) \tilde{\omega} = 0,$$

(29)

where

$$\tilde{\omega} = \Omega - \omega.$$  

(30)

Comparing the two line elements – which means identifying the ZAMO 4-velocity $u^a$ with the normal $N^a$ – we see that the lapse and shift are given by:

$$\alpha = e^{\Phi/c^2}, \quad \beta^i = -\frac{\omega}{c} \delta^i_\theta.$$  

(31)

It is now clear that the $\beta_i\beta^i = \beta^2$ term in the 3+1 line element, equation (7), corresponds to second-order rotational corrections in our problem, and therefore may be neglected. A brief calculation using equation (25) shows that the extrinsic curvature $K_{ij}$ vanishes for a non-rotating star, but in a rotating system has one independent non-zero component, which in physical units is (using primes to denote derivatives with respect to $r$):

$$K_{r\varphi} = -\frac{1}{2} e^{-\Phi/c^2} r^2 \sin^2 \theta \omega'(r) = K_{\varphi r}.$$  

(32)

since $K_{ij}$ is symmetric (Gourgoulhon 2007).

### 2.4 Physical quantities and their dimensions

Let us pause to discuss the quantities involved in the equations for thermal dynamics and also their dimensions, which will help in the interpretation of physical quantities later on in our derivations. We denote by $M, L, T$ and $\Theta$ the dimensions of mass, length, time and temperature. The physical dimensions of the two basic thermal quantities, $T$ and $s$, are

$$[T_{phys}] = \Theta, \quad [s_{phys}] = [S]|L|^{-3} = ML^{-2}T^{-1}\Theta^{-1}$$

(33)

where $S$ is the true entropy (i.e. not per unit volume). In standard geometrised units for temperature-independent problems, one sets $G = c = 1$, and all quantities have dimensions which are powers of $L$. It is possible to extend this to relativistic thermal dynamics by setting $G = c = k_B = 1$, where $k_B$ is the Boltzmann constant; e.g. one divides entropy by $k_B$, so that its geometrised form is dimensionless. Since no algebra from AHDC involved factors of $k_B$ anyway though, their equations are the same in either system of geometrised units. Accordingly, we will proceed with the simpler $G = c = 1$ system, allowing the dimensions of each quantity to contain powers of both $L$ and $\Theta$.

In a $G = c = 1$ unit system, $T$ remains the same, but we must multiply $s$ by a prefactor combination of $G$ and $c$ to eliminate $T$ and $M$:

$$s_{geom} = \frac{G}{c^3} s_{phys} \implies [s_{geom}] = L^{-2} \Theta^{-1}.$$  

(34)

Next, all velocities in geometrised units are of the form

$$v^i_{geom} = \frac{v^i_{phys}}{c}$$  

(35)

and so are dimensionless. The heat flux $Q^i$ in geometrised units is therefore

$$Q^i_{geom} = s_{geom} T(v^i_s)_{geom} = \frac{G}{c^3} s_{phys} T(v^i_s)_{phys}$$

(36)

and has dimensions

$$[Q^i_{geom}] = L^{-2}.$$  

(37)

If we divide through by the geometrising prefactor we get

$$[Q^i_{phys}] = [c^5 |G|^{-1}] [Q^i_{geom}] = MT^{-3},$$  

(38)

which are indeed the expected physical units for heat flux. Next, because our physical coordinates are $(ct, r, \theta, \varphi)$, time derivatives must contain a $1/c$ factor, which is suppressed in geometrised units:

$$\left(\partial_t\right)_{geom} = \frac{1}{c} (\partial_t)_{phys}.$$  

(39)

This means a time derivative in geometrised units has dimensions $L^{-1}$. From (22) we then see that

$$[F^s_{geom}] = [(\partial_s s)_{geom}]$$  

(40)
and so
\[ \Gamma^\text{geom}_x = \frac{G}{c^2} \Gamma^\text{phys}_x. \] (41)

Using equation (24) we can determine that in geometrised units
\[ [\mathcal{F}^a] = [sD_i\alpha T] = L^{-2}\Theta^{-1} \times L^{-1} \times \Theta = L^{-3}, \] (42)
and so
\[ (\mathcal{F}^a)^\text{geom} = \frac{G}{c^2} (\mathcal{F}^a)^\text{phys}. \] (43)

Finally, by equation (26) we have
\[ [\mathcal{R}] = [\mathcal{F}^a][s][T][Q_i]^{-1} = L^{-3}L^{-2}\Theta^{-1}\Theta(L^{-2})^{-1} = L^{-3}. \] (44)
Again restoring the suppressed geometric prefactors, we have
\[ \mathcal{R}_\text{geom} = \frac{G}{c^2} \mathcal{R}_\text{phys} \] (45)
i.e.
\[ [\mathcal{R}_\text{phys}] = ML^{-3}T^{-1}. \] (46)

3 FOURIER’S LAW

The standard form of Fourier’s law, equation (2), relates the heat flux to the temperature gradient, with the thermal conductivity as the constant of proportionality. If heat conduction is different in different directions – e.g. due to the effect of a magnetic field – one needs to replace the scalar conductivity with a tensorial one \( \kappa \), so that \( \mathbf{Q} = -\kappa \cdot \nabla T \) (Urpin & Yakovlev 1980). As discussed in section 2.1, relativistic forms of Fourier’s law and the heat equation are by no means novel (e.g. Van Riper (1991); Aguilera, Pons, Miralles (2008)), and date back to at least the 1960s (Misner & Sharp 1965; Thorne 1967). However, these have been generalised from their Newtonian counterparts in a simple way, essentially by replacing flat-space quantities in the derivations of these equations by redshifted ones (i.e. the locally-measured value of some quantity in a spherical spacetime needs to be multiplied by a factor of \( e^{\Phi/c^2} \) to yield the value seen by a distant observer). This makes it difficult to see how causality could be restored to the heat equation, or what new terms would appear in a more complex relativistic system. Here, by contrast, we aim to derive relativistic forms of Fourier’s law and the heat equation from the manifestly causal multifluid formalism, in which the route to generalising the model is also clear. We will first recover the expected equation for Fourier’s law in a non-rotating star, then will consistently account for terms which appear at first order in the rotation.

Firstly, let us recall from AHDC the results:
\[ [(\gamma^{1/2})^a]_a = (\alpha \gamma^{1/2})_a = 0, \quad D_i(\alpha \gamma^{1/2}) = \partial_i(\alpha \gamma^{1/2}) - \Gamma^a_{ji} \gamma^{1/2} = 0. \] (47)
In addition
\[ (\gamma^{1/2})_t = (\alpha \gamma^{1/2})_t = 0 \] (48)
by the stationarity of the spacetime we consider; recall that we are not allowing for the spacetime itself to evolve here. Given these, we may pull these quantities out of the covariant and time derivatives, and cancel them. The entropy momentum equation (24) in geometrised form then reduces to:
\[ \partial_t Q_i - \beta^j D_j Q_i + \alpha s D_i T = \alpha \mathcal{F}_i - \alpha Q_i K_{ij} + Q_j D_i \beta^j, \] (49)
where we have used the fact that our shift vector is divergence-free, \( D_j \beta^j = 0 \). Comparing equations (12) and (13), we see that the first two terms of the above equation (49) may be rewritten as follows:
\[ \partial_t Q_i - \beta^j D_j Q_i = \partial_t Q_i - L_\beta Q_i + Q_j \beta^j + \beta^j \Gamma^k_{ji} Q_k = \alpha \mathcal{L}_N Q_i + Q_j \beta^j + \beta^j \Gamma^k_{ji} Q_k. \] (50)
Next, we expand and rearrange the covariant derivative from the right-hand side of equation (49):
\[ Q_j D_i \beta^j = Q_i \beta^j + Q_j \beta^j \Gamma^k_{ij} = Q_i \beta^j + Q_k \beta^j \Gamma^k_{ij}, \] (51)
where we have relabelled the indices and used the fact that the Christoffel symbols are symmetric in the lower two indices.

Now inserting the results (50) and (51) into the entropy momentum equation (still in geometrised form), a number of terms cancel, leaving us with:
\[ \frac{1}{8} \mathcal{L}_N Q_i + D_i T = \frac{1}{8} \mathcal{F}_i - \frac{1}{8} Q_i K_{ij}. \] (52)
Next we restore the suppressed \( G \) and \( c \) factors to return to physical units, and plug in the form of the scattering term (26), finding that
\[ \frac{1}{c^2 s} \mathcal{L}_N Q_i + D_i T = -\frac{\mathcal{R}}{8\sqrt{s}} Q_i - \frac{1}{c^2 s} Q_i K_{ij}. \] (53)
3.1 Non-rotating limit

We can understand equation (53) by looking at the limit in which $\omega \to 0$. This corresponds to no frame dragging, and therefore no rotation in relativistic gravity. We find that:

$$\frac{1}{c^2 \alpha s} \partial_t Q_i + D_i T = -\frac{R}{s^2 T} Q_i. \quad (54)$$

Comparing with the usual form of Fourier’s law (2), let us identify the following quantity as the heat conductivity $\kappa_0$:

$$\kappa_0 \equiv \frac{s^2 T}{R}. \quad (55)$$

We can use dimensional analysis as a consistency check of this definition, using results from section 2.4: we find that $[\kappa_0] = \text{MLT}^{-3} \Theta^{-1}$, which are indeed the expected physical units. Now, equation (54) becomes:

$$\frac{1}{c^2 \alpha s} \partial_t Q_i + D_i T = -\frac{1}{\kappa_0} Q_i. \quad (56)$$

This is not quite what we want though: the usual form of Fourier’s law has no time dependence, whereas here we find a time derivative of the heat flux. Equally though, this factor does not render equation (56) the kind of stable, causal expression resulting from the inclusion of the entropy entrainment – despite its superficial resemblance to equation (3) (see, e.g., Lopez-Monsalvo (2011)). This is because the prefactor of $\partial_t Q_i$ in equation (56) is not ‘tunable’ and has no connection with the medium through which heat propagates. Only in the limit of Newtonian dynamics, where $Q/c \propto v_i/c \to 0$, does the time-derivative term vanish automatically.

At this point we see that in GR we can recover the standard Fourier’s law only if the heat flow is approximately steady on a thermal timescale. More precisely, we want the timescale $\tau_Q$ for variations in the heat flux to satisfy:

$$\tau_Q \gg \frac{\kappa_0}{c^2 \alpha s}, \quad (57)$$

but the $c^2$ factor means that this assumption ought to be quite safe – at least for processes on secular timescales. Finally then, we can reach the desired result:

$$D_i T = -\frac{1}{\kappa_0} Q_i. \quad (58)$$

3.2 Slow rotation

Having thus identified the heat conductivity from the Newtonian limit of the entropy momentum equation, let us return to the general case, equation (53). For the same reasons as in the non-rotating case, we again want to assume the heat flow is steady over dynamical timescales. The natural notion of time variation in the foliation framework, however, is not with respect to global time, $\partial_t$, but rather a local expression given by the Lie derivative $\mathcal{L}_N$ along the normal to the hypersurfaces:

$$\mathcal{L}_N = \frac{1}{\alpha}(\partial_t - \mathcal{L}_\beta). \quad (59)$$

We therefore neglect this term from the left-hand side of equation (53), substituting in equation (55) for $\kappa_0$, to find that

$$D_i T = -\frac{1}{\kappa_0} Q_i - \frac{1}{c^2 \alpha s} Q^2 K_{ij}. \quad (60)$$

We next need to evaluate the term involving the extrinsic curvature, using equation (32). With $i$ set to $r, \theta, \varphi$ we find:

$$Q^r K_{rj} = Q^\theta K_{\theta j} = 0, \quad (61)$$

$$Q^\theta K_{\theta j} = 0, \quad (62)$$

$$Q^\varphi K_{\varphi j} = Q^r K_{r\varphi} = -\frac{1}{c^2 \alpha s} \kappa_0 \omega^2 r^2 \sin^2 \theta Q^r. \quad (63)$$

We now substitute results (61)-(63) back into equation (60); some simple algebra then yields:

$$Q^r = -\frac{\kappa_0}{r^2 \sin \theta} \partial_r T + \frac{e^{-2\Lambda - \Phi}/c^2 \kappa_0}{2c^2 s} \omega^2 r^2 \sin^2 \theta Q^\varphi, \quad (64)$$

$$Q^\theta = -\frac{\kappa_0}{r^2 \sin \theta} \partial_\theta T, \quad (65)$$

$$Q^\varphi = -\frac{\kappa_0}{r^2 \sin \theta} \partial_\varphi T + \frac{e^{-2\Lambda - \Phi}/c^2 \kappa_0}{2c^2 s} \omega^2 Q^r. \quad (66)$$

Unlike the non-rotating case, these expressions cannot be trivially rewritten in the form of Fourier’s law with a scalar heat conductivity – although the $\theta$-component is in fact already in the right form. We will see that the three equations can, however, be expressed in a Fourier-law form with a suitably defined tensor conductivity. To do so, the natural choice of coordinate system is not a global one, but rather those carried by a ZAMO. To transform equations so that they are with respect to such an observer, one typically needs to rewrite the individual vector components with respect to a ZAMO basis, and then projecting the resulting equations onto the ZAMO tetrad. Here, however, the equation is simple: there are no vector
We begin by defining the components of the gradient operator $\nabla$ acting on a scalar:

$$\nabla \psi = \frac{\partial \psi}{\partial x_i} \nabla_i, \quad \nabla^\theta = \frac{1}{r \sin \theta} \partial_r, \quad \nabla^\varphi = \frac{1}{r \sin \theta} \partial_\varphi,$$

which are valid in both global and ZAMO coordinate bases for our particular problem. We wish to cast our equations (64), (65) and (66) into a Fourier-law form, $Q^i = \kappa^i_j \nabla^j T$. To do so, we will rearrange them into expressions from which the components of the thermal-conductivity tensor $\kappa^i_j$ can be simply read off.

Let us start by using equation (66) to eliminate $Q^r$ from (64):

$$Q^r = -\kappa_0 e^{-2\Lambda/c^2} \frac{\partial_r T}{r} + \frac{e^{-(2\Lambda+\Phi)/c^2} \kappa_0 \omega'}{2c^2} \partial_\varphi T,$$

where, as always, we have neglected terms of order $\omega^2$. Similarly, we eliminate $Q^\varphi$ from equation (66) using equation (64), finding that to first order in $\omega$:

$$Q^\varphi = -\frac{\kappa_0}{r^2 \sin^2 \theta} \partial_\theta T + \frac{e^{-(2\Lambda+\Phi)/c^2} \kappa_0 \omega'}{2c^2} \partial_r T.$$

Now we project these equations into the ZAMO tetrad; the conversions we need are:

$$\hat{Q}^i = e^{\Lambda/c^2} Q^r, \quad \hat{Q}^\theta = r Q^\theta, \quad \hat{Q}^\varphi = r \sin \theta Q^\varphi,$$

where quantities with hats are referred to a ZAMO (Bardeen, Press & Teukolsky 1972). Equations (64), (65) and (66) may then be written:

$$\hat{Q}^i = -\kappa_0 \nabla^i T - \frac{\kappa_0 \omega'}{2c^2} e^{-(\Lambda+\Phi)/c^2} r \sin \theta \nabla^j \omega^i,$$

$$\hat{Q}^\theta = -\kappa_0 \nabla^\theta T,$$

$$\hat{Q}^\varphi = -\kappa_0 \nabla^\varphi T - \frac{\kappa_0 \omega'}{2c^2} e^{-(\Lambda+\Phi)/c^2} r \sin \theta \nabla^j \omega^i T.$$

At this point it is useful to define a second heat conductivity $\kappa_\omega$, whose magnitude is related to the frame dragging:

$$\kappa_\omega \equiv \frac{\kappa_0 \omega}{c^2}$$

Using the dimensions of its component parts from earlier, we can confirm that $\kappa_\omega$ does indeed have the correct dimensions, of MLT$^{-3}\Theta^{-1}$. By comparing the above equations for the components of the heat flux as measured by a ZAMO with the general expansion of $Q^i = \kappa^i_j \nabla^j T$ in components, we may read off the components of the heat conductivity tensor for a rotating star.

Fourier’s law for a rotating star may therefore be written, either with index or vector notation, as:

$$\hat{Q}^i = -\kappa^i_j \nabla^j T,$$

$$\hat{Q} = -\kappa \cdot \nabla T,$$

where

$$\kappa^i_j = \kappa_0 \delta^i_j + \kappa_\omega \frac{e^{-(\Lambda+\Phi)/c^2} \omega' \sin \theta}{2\omega} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This shows explicitly that rotation, like a magnetic field, introduces a directionality to heat conduction. Putting the equations into a Fourier-law form with a heat-conductivity tensor does not have much practical use, since one would just write out the individual components again when actually performing a calculation. But it is reassuring to see that the equations can be cast into the ‘right’ schematic form.

## 4 THE HEAT EQUATION

We used the entropy momentum equation (24) to derive Fourier’s law, above. Now we use the corresponding scalar entropy equation (22) to derive the heat equation. As for the derivation of Fourier’s law, we begin by taking the $\gamma^{1/2}$ and $\alpha \gamma^{1/2}$ factors out of the covariant and time derivatives and cancelling the former, leaving us with:

$$\partial_\beta s + \alpha D_t \left( \frac{Q^i}{T} \right) - D_t (\hat{s}^\beta) = \alpha \Gamma_s.$$

Note that this equation is the same in both physical and geometrised units; in the geometrised case each of the terms has a factor of $G/c^5$ suppressed, and so these may be cancelled. At this point we need to specify the form of the entropy creation
rate $\Gamma_s$. From AHDC equation (143), we see that in the absence of magnetic fields it is given by

$$\Gamma_s = \frac{\mathcal{R}}{s^2 T^3} Q^2 = \frac{Q^2}{\kappa_0 T^2},$$

(79)

which, together with the second law of thermodynamics, implies that $\mathcal{R} > 0$. The next part of the strategy will be to rewrite terms involving derivatives of the entropy, using the first law of thermodynamics:

$$dU = T dS - P dV + \sum_{x \neq s} \mu_x dN_x,$$

(80)

where $S$ and $U$ are the true entropy and internal energy (as opposed to the quantities per unit volume we use elsewhere), $P$ is pressure, $V$ volume and $N_x$ the number of $x$-particles. Let us assume the total number $N$ of matter particles is conserved, so that $dN = 0$; then, for a system with a single species of matter particle, the third term becomes $\mu dN = 0$. In the case of multiple particle species, we get a similar result if we make the additional assumption of chemical equilibrium; then $dN_x = 0$ for each species, and so the first law becomes:

$$dU = T dS - P dV.$$

(81)

From equations (78) and (81) we will derive the heat equation – first for a non-rotating star, to show that we recover the expected result, and then for the case of slow rotation.

### 4.1 Non-rotating limit

Taking the time derivative of the first law (81) per unit volume, we find that

$$\frac{\partial U}{\partial t} \bigg|_V = T \frac{\partial s}{\partial t} \bigg|_V,$$

(82)

where $U$ is internal energy per unit volume, and we recall that $s$ is entropy per unit volume. Now expand the left-hand side of this expression using the chain rule:

$$\frac{\partial U}{\partial t} \bigg|_V = \frac{\partial U}{\partial T} \frac{\partial T}{\partial t} \bigg|_V + C_V \frac{\partial T}{\partial t},$$

(83)

using the definition of $C_V$. Note that we do not have an additional $\partial U/\partial s$ term from applying the chain rule in equation (83), because $s$ and $T$ are thermodynamic conjugate pairs, and we may regard $U$ as being a function of either variable. Now comparing the above two equations, we can eliminate $\partial t/s$ in favour of $\partial T$ in the entropy equation, which for constant volume processes becomes:

$$C_V \partial_t T + TD_t \left( \frac{\alpha Q^2}{\kappa_0 T^2} \right) = \frac{\alpha Q^2}{\kappa_0 T^2}.$$

(84)

Use of the product rule on this expression then gives:

$$C_V \partial_t T + C_V \alpha D_t Q^2 - \alpha \frac{Q^2}{T} D_t T = \frac{\alpha Q^2}{\kappa_0 T^2}.$$

(85)

Rewriting the third term with Fourier’s law (58), we have

$$\alpha \frac{Q^2}{T} D_t T = \frac{\alpha Q^2}{\kappa_0 T},$$

(86)

which cancels with the right-hand-side term of (85). Then, by additionally using $D_t(\alpha \gamma^{1/2}) = D_t(\gamma^{1/2}) = 0$ on the resulting expression, we can manipulate it as follows:

$$C_V \alpha \gamma \partial_t T = \alpha^2 \gamma D_t(\kappa_0 D^T T) = D_t[\alpha(\gamma^{1/2} \kappa_0 D^T T) = \gamma D_t[\alpha \kappa_0 D^T (\alpha T)],$$

(87)

so giving us

$$C_V \partial_t (e^T) = D_t[e^T \kappa_0 D^T (e^T)] \text{ or } C_V \partial_t (e^T) = \nabla \cdot [e^T \kappa_0 \nabla (e^T)],$$

(88)

which is the usual way in which the relativistic heat equation is presented, in terms of the redshifted temperature $e^T$.

### 4.2 Slow rotation

For rotating stars the logic used in deriving the heat equation is the same, but because the spacetime is no longer static we again need to generalise the notion of time derivative to be the Lie derivative along the normal vector (see equation (59)). With this concept, equation (82) from the static case generalises to:

$$\mathcal{L}_N U \bigg|_V = T \mathcal{L}_N s \bigg|_V,$$

(89)

and the chain rule gives

$$\mathcal{L}_N U \bigg|_V = \frac{\partial U}{\partial t} \bigg|_V \mathcal{L}_N T \bigg|_V = C_V \mathcal{L}_N T \bigg|_V.$$

(90)
Combining these last two equations, as for the non-rotating case, gives
\[ \mathcal{L}_N s = \frac{C_V}{T} \mathcal{L}_N T. \] (91)

We now return to the entropy equation (78), expanding the covariant derivatives with the product rule and using the results \( D_i \beta^i = 0 \) and \((\partial_t s) \beta^i = \mathcal{L}_\beta s\) to show that:
\[ \partial_t s + \alpha \frac{D_i Q^i}{T} - \alpha \frac{Q^i D_i T}{T^2} - \mathcal{L}_\beta s = \frac{\alpha Q^2}{\kappa_0 T}. \] (92)

Although Fourier’s law now features a tensorial heat conductivity, equation (77), we know that for our assumption of slow rotation the spherical piece dominates, so that \( Q_i \approx -\kappa_0 \delta_i^j D_j T = -\kappa_0 D_i T \). Plugging this result and the definition of \( \mathcal{L}_N \) into equation (93), we find that:
\[ T \mathcal{L}_N s + D_i Q^i = \frac{Q^2}{\kappa_0 T} + \frac{1}{T} Q^i D_i T = 0, \] (93)

and replacing \( s \) with \( T \) then yields the heat equation for a rotating star,
\[ C_V \mathcal{L}_N T = -D_i Q^i, \] (94)
or, by expanding \( \mathcal{L}_N \) and using Fourier’s law,
\[ C_V \left( \frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial \varphi} \right) = D_i (\kappa_0^j D_j T) \] or \[ C_V \left( \frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial \varphi} \right) = \nabla \cdot (\kappa \nabla T). \] (95)

This result is in global coordinates, but Fourier’s law is expressed referred to a ZAMO. We therefore need to express the right-hand side of equation (94) in terms of the heat flux seen by a ZAMO. The piece involving \( \kappa_0 \) is straightforward and takes the same form as in the non-rotating case:
\[ D_i (\kappa_0 \delta_j^i D_j T) = D_i (\kappa_0 D_j T), \] (96)

Let us define the additional pieces of the heat flux due to frame-dragging (i.e. those with \( \kappa_\omega \) factors) as \( \tilde{Q}^i \). We need to take the divergence of these pieces and express them in ZAMO coordinates:
\[ D_i \tilde{Q}^i = D_t \tilde{Q}^t + D_o \tilde{Q}^\varphi = \left( \partial_t + \frac{2}{r} \frac{\Lambda'}{c^2} \right) \tilde{Q}^t + \partial_\varphi \tilde{Q}^\varphi = \left( \partial_t + \frac{2}{r} \frac{\Lambda'}{c^2} \right) \left( e^{-\Lambda/c^2} \tilde{Q}^t \right) + \frac{1}{r \sin \theta} \partial_\varphi \tilde{Q}^\varphi = e^{-\Lambda/c^2} \left( \partial_t + \frac{2}{r} \right) \tilde{Q}^t + \frac{1}{r \sin \theta} \partial_\varphi \tilde{Q}^\varphi, \] (97)

where we have used equation (70) and the following result from AHDC:
\[ D_i \tilde{v}^i = \gamma^{-1/2} \partial_t (\gamma^{1/2} \tilde{v}^i). \] (98)

Now plugging in \( \tilde{Q}^t \) and \( \tilde{Q}^\varphi \) from equations (71) and (73), a little algebra shows that
\[ D_i \tilde{Q}^i = -e^{-\Phi/(2 \Lambda)/c^2} \omega \left[ \partial_t (\kappa_\omega \partial_\varphi T) + \partial_\varphi (\kappa_\omega \partial_t T) \right] + \kappa_\omega \left[ \omega'' + \frac{2 \omega'}{r} - \frac{(\Phi' + \Lambda') \omega'}{c^2} \right] \partial_\varphi T. \] (99)

In order to replace the \( \omega'' \) term, let us return to Hartle’s equation, which may be rewritten in terms of unbarred \( \omega \) from equation (30). Using the fact that the rotation rate \( \Omega \) is constant, we find:
\[ \omega'' = \left( \frac{\Phi' + \Lambda'}{c^2} - \frac{4}{r} \right) \omega' - \frac{4}{c^2 r^2} (\Phi' + \Lambda') (\Omega - \omega). \] (100)

Plugging this result and equations (96) and (99) back into equation (94), we arrive at the general form of the heat equation for a slowly-rotating spacetime, as referred to a ZAMO:
\[ \frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial \varphi} = \frac{1}{C_V} D_t (\kappa_0 D_j T) + e^{-\Phi/(2 \Lambda)/c^2} \left\{ \omega' \left[ \partial_t (\kappa_\omega \partial_\varphi T) + \partial_\varphi (\kappa_\omega \partial_t T) \right] + \kappa_\omega \left[ \frac{2 \omega'}{r} + \omega'' + \omega' \right] \frac{c^2}{r^2} (\Phi' + \Lambda') (\Omega - \omega) \right\} \partial_\varphi T \} \] (101)

Finally, if we make the – probably mild – simplification that \( \kappa_\omega \) is axisymmetric, we may pull out the \( \varphi \) derivative from all the frame-dragging terms:
\[ \frac{\partial T}{\partial t} + \omega \frac{\partial T}{\partial \varphi} = \frac{1}{C_V} D_t (\kappa_0 D_j T) + e^{-\Phi/(2 \Lambda)/c^2} \kappa_\omega \partial_\varphi \left\{ 2 \omega' \partial_t T + \left[ \frac{\omega' \partial_t \kappa_\omega}{\kappa_\omega} + \frac{2 \omega'}{r} + \frac{(\omega')^2}{c^2} + \frac{4}{c^2 r^2} (\Phi' + \Lambda') (\Omega - \omega) \right] T \right\}. \] (102)

### 4.3 Physical nature of different terms in the heat equation

Equation (102) shows that a temperature distribution in the spacetime of a slowly-rotating star (as seen by a ZAMO) evolves under three influences: the right-hand-side terms depending on the standard thermal conductivity \( \kappa_0 \); the \( \varphi \)-derivative on the left-hand side, which encodes the frame-dragging seen by a ZAMO; and the right-hand-side terms depending on \( \kappa_\omega \), which originate from the term \( Q^j K_{ij} \) and so describe the variation of the heat flux within a hypersurface, due to the curvature of
the hypersurface itself within spacetime. The first thing to note is that these latter two effects only affect the evolution of a non-axisymmetric temperature distribution.

The usual heat equation (in both its Newtonian and relativistic forms) is a parabolic equation describing the spreading and decay of an initial temperature profile. To get a feeling for the new heat equation of equation (102), let us make an ansatz of a temperature distribution with harmonic azimuthal and time dependence, i.e.:

$$ T \sim \exp[(\tau + im\varphi) : \tau, \sigma \in \mathbb{R}]. $$

(103)

Clearly $\tau$ is associated with decay of $T$ and $\sigma$ with oscillations of $T$; this general ansatz therefore allows for both propagation and decay of a temperature distribution. Plugging in our ansatz, the left-hand side of equation (102) becomes

$$ [\tau + i(\sigma + \omega m)]T. $$

(104)

On the right-hand side, note that the first term (with $\kappa_0$) involves pieces with no $\varphi$-derivative, and a piece with a double $\varphi$-derivative. By contrast, the second term (premultiplied by $\kappa_\omega$) involves one $\varphi$ derivative. But since

$$ \frac{\partial T}{\partial \varphi} = imT, \quad \frac{\partial^2 T}{\partial \varphi^2} = (im)^2T = -m^2T, $$

we see that all $\kappa_0$ terms are real and all $\kappa_\omega$ terms are imaginary. Now comparing with the left-hand side, we recover the expected result that the standard term involving $\kappa_0$ is dissipative. The first-order rotational corrections to the heat equation, however, are conservative and describe the propagation of $T$. This is a purely relativistic effect. In fact, the $\varphi$ derivative on the left-hand side is the standard form for a frame-dragging term, which should indeed be conservative; the less obvious result is that the $\kappa_\omega$ terms are conservative too.

Dimensional analysis allows us to associate each of the three effects in equation (102) with a characteristic timescale. The familiar dissipation associated with the $\kappa_0$ term takes place on a timescale (typically long in astrophysical contexts) proportional to $C_V/\kappa_0$. The frame-dragging term from the left-hand side has a timescale of $1/\omega$, and the new thermodynamic terms on the right-hand side have a characteristic timescale of $C_V/\kappa_\omega$. Unless $\Omega$ is almost zero, it seems inevitable that the timescales will always be ordered as follows:

$$ \frac{C_V}{\kappa_\omega} \gg \frac{C_V}{\kappa_0} \gg \frac{1}{\omega}. $$

(106)

due to the $1/c^2$ factor in the definition of $\kappa_\omega$. We show this explicitly in the next section, for one particular example which has direct relevance for neutron-star observations.

## 5 ROTATIONAL MODULATION OF A NEUTRON-STAR HOTSPOT

There are a number of observations of hotspots on neutron stars which show modulation in time. In some cases the frequency at which they are modulated is the only way to determine their rotation frequency ([Strohmayer, Zhang & Swank 1997]). Some neutron stars accreting in low-mass X-ray binary systems produce X-ray bursts due to thermonuclear burning in the neutron-star ocean, and in some cases the bursts display almost coherent oscillations – typically in the range $\sim 300 - 600$ Hz ([Watts 2012]). These oscillations are believed to be related either to modes of the neutron-star ocean ([Hoy 2004]), or to an essentially confined hotspot ([Cumming & Bildsten 2000]) – though there are challenges with either model in explaining the small frequency drifts of $\sim 1$ Hz seen in the burst oscillations. In either case, the characteristic frequency is believed to be due to rotational modulation, which implicitly assumes that an observer at infinity really sees a hotspot on the stellar surface moving at the rotation rate. However, we now know from equation (102) that this is not quite true; there are additional rotational corrections – which might be important for rapidly-rotating systems, like most of those in which burst oscillations are observed – so let us check the size of these terms.

Ignoring temporarily the $\kappa_\omega$ terms, the frame-dragging term on the left-hand side of equation (102) tells us that a hotspot fixed on the surface of a neutron star will be rotationally-modulated for a ZAMO, but at a frequency of $\Omega - \omega$; in Newtonian gravity, the equivalent corotating observer would, of course, see the hotspot as stationary. However, the ZAMO itself moves at $+\omega$ with respect to an observer at infinity – so, for this latter observer the hotspot is indeed rotationally modulated at the expected frequency of $\Omega$.

Now we need to check the size of the $\kappa_\omega$ terms. We already know that these describe propagation, rather than dissipation, of a temperature distribution – so, if they are numerically significant they could affect how reliably one can associate modulation of a hotspot with stellar rotation.

We first need an expression for the frame-dragging frequency $\omega(r)$. To this end we solve the TOV equations for the neutron-star structure, closing the system with the SLy equation of state ([Douchin & Haensel 2001]), in order to find the metric functions $\Lambda$ and $\Phi$. These are then used as input to solve Hartle’s equation (29) for $\omega$. For a stellar model with a mass of $1.4M_\odot$ and radius 11.7 km, we find that the frame-dragging frequency at the stellar surface is 12.6% times that of the spin frequency.
Next, we take the following as representative values for the neutron-star ocean:

\[ \kappa_0 \approx 10^{17} \text{ g cm}^{-3}\text{K}^{-1}, \]
\[ C_V \approx 3k_Bn_i \approx 10^{17} \text{ g cm}^{-1}\text{s}^{-2}\text{K}^{-1}, \]
\[ \rho \approx 10^9 \text{ g cm}^{-3}, \]

where the density is that of the base of the ocean, \( \kappa_0 \) is the value for this density and \( T = 10^8 \text{ K} \), assuming a composition dominated by \(^{56}\text{Fe} \) (Potekhin et al. 1999), and \( C_V \) the heat capacity for a strongly-coupled Coulomb liquid (Mestel & Ruderman 1967). Combining the value of \( \rho \) with the condition of charge neutrality and the assumption that ions have equal numbers of protons and neutrons, we have that the number density of electrons \( n_e \), should be:

\[ n_e = n_p \approx \frac{\rho}{2m_b} \approx 3 \times 10^{32} \text{ cm}^{-3}, \]

where \( m_b \) is the baryon mass (to our level of approximation, the difference between neutron and proton masses is negligible). Next we need an expression for the entropy density \( s \). For this we turn to Cox & Giuli (1968), who calculate closed-form approximate expressions for the degeneracy parameter \( \eta \), and for \( s \) as a function of \( \eta \), in various limiting cases. The electrons at the base of the ocean are ultra-relativistic and highly degenerate. Using the relevant expression from Cox & Giuli (1968), we may calculate the degeneracy parameter \( \eta \) for the electrons, taking a temperature of \( 10^8 \text{ K} \):

\[ \eta \approx \frac{\hbar c}{k_B T} (3\pi^2 n_e)^{2/3} \approx 470. \]

Note that \( \eta \) is dimensionless. The large numerical value here confirms that we are in the high-degeneracy limit, \( \eta \gg 1 \). The corresponding entropy density \( s \) is given by:

\[ s \approx \frac{\pi^2 k_B n_e}{\eta} \approx 9 \times 10^{14} \text{ g cm}^{-1}\text{s}^{-2}\text{K}^{-1}. \]

Now we can estimate \( \kappa_\omega \):

\[ \kappa_\omega = \frac{\kappa_0 \omega}{c^2 s} \approx 0.7 \text{ g cm}^{-3}\text{K}^{-1} \]

for a star rotating at 400 Hz. The corresponding characteristic timescale is

\[ \frac{C_V}{\kappa_\omega} \approx 5 \times 10^9 \text{ yr}. \]

We see that \( \kappa_\omega \) is about \( 10^{17} \) times smaller than \( \kappa_0 \), implying that the propagation of a non-axisymmetric temperature perturbation is vastly slower than its decay, and therefore irrelevant in this case. It is therefore safe to associate the modulation of a hotspot, fixed on the neutron-star surface, with the stellar rotation rate.

6 SUMMARY

We have shown how the standard form of the relativistic heat equation follows from a causal theory of heat propagation, in which the entropy is treated as a fluid. At first order in rotation, two corrections emerge: one due to the frame-dragging seen by a local observer, and the other related to the curvature of the spacelike hypersurface in which the heat propagation occurs. The standard non-rotating result is, nonetheless, easily accurate enough for most purposes – and as a result we are able to show quantitatively that it is safe to associate the modulation of a neutron-star surface hotspot with the stellar rotation rate.

In deriving the heat equation, however, we find that it relies implicitly on various assumptions which will not be safe in all astrophysical situations. The dynamical timescale for the problem at hand must be long compared to the thermal relaxation time of the medium, or equivalently the timescale on which entrainment couples the entropy and matter fluids. Equally, the heat flux must be approximately steady over short timescales. Even in situations where these conditions are met, we should be mindful that a thermal evolution relying on the standard heat equation could still suffer genuine instabilities connected with the neglect of entropy entrainment.

It is not so surprising that the usual form of the heat equation may be recovered in the limiting case described above, where thermal information propagates almost ‘instantaneously’ – compared with any fluid dynamics – and where the spacetime is stationary. For more dynamical situations, however, the equations presented here are not applicable – and a future goal for the numerical simulations of hot relativistic systems should be to evolve the entropy dynamics directly.

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