New Congruences Modulo 2, 4, and 8 for the Number of Tagged Parts Over the Partitions with Designated Summands

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Abstract. Recently, Lin introduced two new partition functions $PD_t(n)$ and $PDO_t(n)$, which count the total number of tagged parts over all partitions of $n$ with designated summands and the total number of tagged parts over all partitions of $n$ with designated summands in which all parts are odd. Lin also proved some congruences modulo 3 and 9 for $PD_t(n)$ and $PDO_t(n)$, and conjectured some congruences modulo 8. In this paper, we prove the congruences modulo 8 conjectured by Lin and also find many new congruences and infinite families of congruences modulo some small powers of 2.

Key Words: Partitions with designated summands; Tagged Part; Dissection formula; Congruence.

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1. Introduction

In [2], Andrews, Lewis and Lovejoy introduced and studied a new class of partitions, partitions with designated summands. Partitions with designated summands are constructed by taking ordinary partitions and tagging exactly one of each part size. For example, there are 10 partitions of 4 with designated summands, namely,

$$4', \ 3' + 1', \ 2' + 2, \ 2 + 2', \ 2' + 1' + 1, \ 2' + 1 + 1', \ 1' + 1 + 1 + 1, \ 1 + 1' + 1 + 1, \ 1 + 1 + 1' + 1, \ 1 + 1 + 1 + 1'.$$

The total number of partitions of $n$ with designated summands is denoted by $PD(n)$. Hence, $PD(4) = 10$. Andrews, Lewis and Lovejoy [2] also studied $PDO(n)$, the number of partitions of $n$ with designated summands in which all parts are odd. From the above example, $PDO(4) = 5$. Further studies on $PD(n)$ and $PDO(n)$ were carried out by Chen, Ji, Jin, and Shen [5], Baruah and Ojah [3], and Xia [8].

Recently, Lin [7] introduced two new partition functions $PD_t(n)$ and $PDO_t(n)$, which count the total number of tagged parts over all partitions of $n$ with designated summands and the total number of tagged parts over all partitions of $n$ with designated summands in which all parts are odd, respectively. From the partitions of 4 with designated summands given above, we note that $PD_t(4) = 13$ and $PDO_t(4) = 6$. Lin [7] proved that the generating functions of $PD_t(n)$ and $PDO_t(n)$ are

$$\sum_{n=0}^{\infty} PD_t(n)q^n = \frac{1}{2} \left( \frac{f_3^5}{f_1^3f_6^2} - \frac{f_6}{f_1f_2f_3} \right) \quad (1.1)$$
and
\[
\sum_{n=0}^{\infty} \text{PDO}_t(n)q^n = \frac{q f_1^2 f_2^2 f_3^2 f_4^2}{f_1^2 f_6},
\]
where as usual, for any complex number \(a\) and \(|q| < 1\),
\[
(a; q)_\infty := \prod_{n=1}^{\infty} (1 - aq^n)
\]
and for any positive integer \(k\), \(f_k := (q^k; q^k)_\infty\).

Lin [7] also derived several congruences modulo small powers of 3 for \(\text{PD}_t(n)\) and \(\text{PDO}_t(n)\). For example, for any nonnegative integers \(n\) and \(k\),
\[
\begin{align*}
\text{PD}_t(3n) &\equiv 0 \pmod{3}, \\
\text{PD}_t(3n + 2) &\equiv 0 \pmod{3}, \\
\text{PD}_t(36n + 21) &\equiv 0 \pmod{9}, \\
\text{PD}_t(36n + 33) &\equiv 0 \pmod{9}, \\
\text{PD}_t(48n + 20) &\equiv 0 \pmod{9}, \\
\text{PD}_t(48n + 36) &\equiv 0 \pmod{9}, \\
\text{PD}_t(72n + 42) &\equiv 0 \pmod{9}, \\
\text{PD}_t(72n + 66) &\equiv 0 \pmod{9}, \\
\text{PDO}_t(8n) &\equiv 0 \pmod{9}, \\
\text{PDO}_t(24n) &\equiv 0 \pmod{27}, \\
\text{PDO}_t(36n) &\equiv 0 \pmod{27}, \\
\text{PDO}_t(36n + 24) &\equiv 0 \pmod{27}, \\
\text{PDO}_t(8 \cdot 5^{2k+1}(30n + 6a + 5)) &\equiv 0 \pmod{27},
\end{align*}
\]
where \(a = 1, 2, 3, 4\).

Very recently, Adansie, Chern and Xia [1] found the following two infinite families of congruences modulo 9.

For any nonnegative integers \(n\) and \(k\),
\[
\begin{align*}
\text{PD}_t(3^{2k+1}(9n + 2)) &\equiv 0 \pmod{9}, \\
\text{PD}_t(3^{2k+1}(9n + 7)) &\equiv 0 \pmod{9},
\end{align*}
\]
and
\[
\begin{align*}
\text{PDO}_t(3^{2k+1}(9n + 2)) &\equiv 0 \pmod{9}, \\
\text{PDO}_t(3^{2k+1}(9n + 7)) &\equiv 0 \pmod{9}.
\end{align*}
\]

By analyzing a large number of values of \(\text{PD}_t(n)\) and \(\text{PDO}_t(n)\) via MAPLE, Lin [7] further speculated the existence of congruences modulo small powers of 2. For example, he conjectured that, for any nonnegative integer \(n\),
\[
\begin{align*}
\text{PD}_t(48n + 28) &\equiv 0 \pmod{8}, \\
\text{PD}_t(48n + 46) &\equiv 0 \pmod{8}, \\
\text{PDO}_t(8n + 6) &\equiv 0 \pmod{8},
\end{align*}
\]
In this paper, we prove the above congruences and also find many new congruences and infinite families of congruences modulo 2 and 4.

The following theorem states the exact generating functions of \( \text{PDO}_t(8n+6) \) and \( \text{PDO}_t(8n+7) \) that immediately implies the congruences \( (1.7) \) and \( (1.8) \).

**Theorem 1.1.** For any nonnegative integer \( n \), we have

\[
\sum_{n=0}^{\infty} \text{PDO}_t(8n+6)q^n = 8 \left( 2 \frac{f_{16}^2 f_{10}^2}{f_1^4 f_3^4 f_5^4} - q \frac{f_{28}^2 f_{3}^4 f_{12}^4}{f_1^2 f_2^2 f_8^2} - 16q^2 \frac{f_2^2 f_4^4 f_8^4 f_{12}^4}{f_1^4 f_6^2} \right) \tag {1.9}
\]

and

\[
\sum_{n=0}^{\infty} \text{PDO}_t(8n+7)q^n = 8 \left( \frac{f_{14}^2 f_3^4 f_6^2 f_8^2}{f_1^4 f_3^4 f_5^4 f_6^2} + 2 \frac{f_{9}^2 f_5^4 f_4^5 f_6}{f_1^4 f_3^2 f_8^2} + 4q \frac{f_2^2 f_3^4 f_4^2 f_5^4 f_{12}^4}{f_1^2 f_6^2} \right). \tag {1.10}
\]

In the following theorem and corollary, we present our new congruences and infinite families of congruences modulo 2 and 4 for \( \text{PD}_t(n) \).

**Theorem 1.2.** For any nonnegative integers \( k, \ell \) and \( n \), we have

- \( \text{PD}_t(24n+12) \equiv 0 \pmod{2} \), \( (1.11) \)
- \( \text{PD}_t(24n+21) \equiv 0 \pmod{2} \), \( (1.12) \)
- \( \text{PD}_t(48n+30) \equiv 0 \pmod{2} \), \( (1.13) \)
- \( \text{PD}_t(144n+102) \equiv 0 \pmod{2} \), \( (1.14) \)
- \( \text{PD}_t(216n+153) \equiv 0 \pmod{2} \), \( (1.15) \)
- \( \text{PD}_t(36n+21) \equiv 0 \pmod{4} \), \( (1.16) \)
- \( \text{PD}_t(36n+33) \equiv 0 \pmod{4} \), \( (1.17) \)
- \( \text{PD}_t(2^{2k} \cdot 12n) \equiv \text{PD}_t(12n) \pmod{4} \), \( (1.18) \)
- \( \text{PD}_t(3^k \cdot 2^{2k}(24n+12)) \equiv \text{PD}_t(24n+12) \pmod{4} \), \( (1.19) \)
- \( \text{PD}_t(96n+60) \equiv 0 \pmod{4} \), \( (1.20) \)
- \( \text{PD}_t(96n+84) \equiv 0 \pmod{4} \), \( (1.21) \)
- \( \text{PD}_t(144n+84) \equiv 0 \pmod{4} \), \( (1.22) \)
- \( \text{PD}_t(144n+120) \equiv 0 \pmod{4} \), \( (1.23) \)
- \( \text{PD}_t(144n+132) \equiv 0 \pmod{4} \), \( (1.24) \)
- \( \text{PD}_t(3^k(288n+204)) \equiv \text{PD}_t(288n+204) \equiv 0 \pmod{4} \), \( (1.25) \)
- \( \text{PD}_t(864n+792) \equiv 0 \pmod{4} \), \( (1.26) \)
- \( \text{PD}_t(1728n+1224) \equiv 0 \pmod{4} \), \( (1.27) \)
- \( \text{PD}_t(2592n+1080) \equiv 0 \pmod{4} \), \( (1.28) \)
- \( \text{PD}_t(36n+30) \equiv 0 \pmod{4} \), \( (1.29) \)
Corollary 1.3. For any positive integers $k$, $\ell$ and any nonnegative integer $n$, we have

\[
\begin{align*}
\text{PD}_t(3^\ell \cdot 2^{2k}(8n + 5)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^\ell \cdot 2^{2k}(8n + 7)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^\ell \cdot 2^{2k}(12n + 7)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^\ell \cdot 2^{2k}(12n + 11)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3 \cdot 2^{2k+1}(6n + 5)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^{\ell+1} \cdot 2^{2k}(24n + 17)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^2 \cdot 2^{2k+1}(12n + 11)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^2 \cdot 2^{2k+1}(24n + 17)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(3^3 \cdot 2^{2k+1}(12n + 5)) & \equiv 0 \pmod{4}, \\
\text{PD}_t(2 \cdot 3^k(6n + 5)) & \equiv 0 \pmod{4}.
\end{align*}
\]

Proof. Congruences (1.20)–(1.22) and (1.24) may be rewritten as

\[
\begin{align*}
\text{PD}_t(24(4n + 2) + 12) & \equiv 0 \pmod{4}, \\
\text{PD}_t(24(4n + 3) + 12) & \equiv 0 \pmod{4}, \\
\text{PD}_t(24(6n + 3) + 12) & \equiv 0 \pmod{4},
\end{align*}
\]

and

\[
\begin{align*}
\text{PD}_t(24(6n + 5) + 12) & \equiv 0 \pmod{4},
\end{align*}
\]

respectively. From (1.19) and the above congruences, we easily arrive at the first four infinite families of congruences of the corollary. Since the other congruences can also be proved in a similar way, we omit the details. \qed

We organize the rest of the paper as follows. In Section 2 we present some 2- and 3-dissections that will be used in the subsequent sections. In Section 3 we prove Theorem 1.1 whereas Section 4 is devoted to proving the congruences (1.5) and (1.6). In Section 5 we present the proofs of our new congruences in Theorem 1.2.

2. Some 2- and 3-dissections

In this section, we present some useful 2- and 3-dissections.

Lemma 2.1. We have

\[
\begin{align*}
\frac{1}{f_1^2} &= \frac{f_5^5}{f_5^5 f_{16}^2} + 2 \frac{f_2^2 f_1^2}{f_2^2 f_8^2}, \\
f_1^2 &= \frac{f_2 f_5^5}{f_4^2 f_{16}^2} - 2 \frac{f_2^2 f_{16}^2}{f_8^2}.
\end{align*}
\]
We have
\[ \frac{1}{f_1} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_2^4 f_8^4}{f_2^{10}}, \]  
(2.3)
\[ f_1^4 = \frac{f_4^{10}}{f_2^4 f_8^4} - 4q \frac{f_2^4 f_8^4}{f_4^4}, \]  
(2.4)
\[ f_1 f_3 = \frac{f_2 f_4^2 f_8^4}{f_4 f_6 f_2^4} - q \frac{f_4 f_6 f_2^4}{f_2 f_8 f_8 f_2^4}, \]  
(2.5)
\[ \frac{1}{f_1 f_3} = \frac{f_8^6 f_2^4}{f_2^4 f_4 f_6 f_2^4} + q \frac{f_4^5 f_2^4}{f_4 f_6 f_8 f_8 f_2^4}, \]  
(2.6)
\[ \frac{f_3}{f_3} = \frac{f_4^3}{f_4} - 3q f_2^3 f_2^1, \]  
(2.7)
\[ \frac{f_3}{f_1} = \frac{f_4^6 f_6^4}{f_2 f_4 f_4} + 3q f_2^3 f_6 f_2^1, \]  
(2.8)
\[ \frac{f_3^2}{f_1} = \frac{f_4^6 f_6^4}{f_2 f_4 f_4} + q f_4, \]  
(2.9)
\[ \frac{f_3^2}{f_2} = \frac{f_2 f_4^2 f_8^4}{f_4 f_6 f_2^4} - 2q \frac{f_2^2 f_8 f_1^2 f_2^4}{f_4 f_6 f_6 f_2^4}, \]  
(2.10)
\[ \frac{f_3^2}{f_2} = \frac{f_4^6 f_6^4}{f_2 f_4 f_4} + 2q \frac{f_4 f_6^4 f_8 f_2^4}{f_2^4 f_2^1}, \]  
(2.11)

Proof. Identities (2.1) and (2.3) are the 2-dissections of \( \varphi(q) \) and \( \varphi(q^2) \) (see [6 Eqs. (1.9.4) and (1.10.1)]), where

\[ \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^5 f_4^2}. \]

Replacing \( q \) by \(-q\) in (2.1) and (2.3), and then using

\[ (-q; -q)_\infty = \frac{f_2^2}{f_1 f_4}, \]  
(2.12)
we readily arrive at (2.2) and (2.4), respectively. Identities (2.5), (2.6), (2.7), (2.9), (2.10), and (2.11) are Eqs. (30.12.1), (30.12.3), (22.1.13), (22.1.14), (30.10.2), and (30.10.4), respectively, in [6]. Finally, (2.8) follows from (2.7) by replacing \( q \) by \(-q\) and then using (2.12). \( \square \)

Lemma 2.2. We have

\[ \frac{f_1^3}{f_2} = \frac{f_6^2}{f_1^8} - 2q \frac{f_3 f_8^2}{f_6 f_9}, \]  
(2.13)
\[ \frac{f_2}{f_1^2} = \frac{f_6^4 f_6^6}{f_3 f_3^3} + 2q \frac{f_6^3 f_9}{f_3^3 f_8^2} + 4q^2 \frac{f_6^2 f_8^2}{f_3^2}, \]  
(2.14)
\[ f_3^3 = f_3 a(q^3) - 3q f_9^3, \]  
(2.15)
\[ \frac{1}{f_1^3} = a^2(q^3) \frac{f_3^3}{f_3^{10}} + 3qa(q^3) \frac{f_9^3}{f_3^{11}} + 9q^2 \frac{f_9^3}{f_3^{12}}, \]  
(2.16)
\[
\frac{1}{f_1 f_2} = a(q^6) \frac{f_3^3}{f_4^3 f_6^3} + q a(q^3) \frac{f_3^3}{f_4^3 f_6^3} + 3 q^2 \frac{f_3^3 f_3^3}{f_4^3 f_6^3},
\]

where
\[
a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6 \sum_{n=0}^{\infty} \left( \frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}} \right).
\]

Proof. The first identity is equivalent to the 3-dissection of \(\varphi(-q)\) (see [6, Eq. (14.3.2)]). The second can be obtained from the first by replacing \(q\) with \(\omega q\) and \(\omega^2 q\) and then multiplying the two results, where \(\omega\) is a primitive cube root of unity. Identities (2.15), (2.16) and (2.17) are in [6] Eqs. (21.3.1), (39.2.8) and (22.9.4). □

We also recall the following useful results from [6] Eqs. (22.1.12), (22.11.8) and (22.11.9), where the first is a 2-dissection of \(a(q)\):
\[
a(q) = a(q^4) + 6 q \frac{f_4 f_6}{f_2 f_6},
\]
\[
a(q) + 2 a(q^2) = 3 \frac{f_2 f_6^6}{f_1 f_6^6},
\]
\[
a(q) + a(q^2) = 2 \frac{f_2^3 f_3}{f_1^3 f_6^2}.
\]

We end this section by noting the following congruences which can be easily established:
\[
a(q) \equiv 1 \pmod{2},
\]
\[
a^2(q) \equiv 1 \pmod{4},
\]
\[
f_1^2 \equiv f_2 \pmod{2},
\]
\[
f_1^4 \equiv f_2^2 \pmod{4},
\]
\[
f_1^8 \equiv f_2^4 \pmod{8}.
\]

We will frequently use the identities and congruences of this section in the subsequent sections, sometimes without referring to these.

3. Proof of Theorem 1.1

We have
\[
\sum_{n=0}^{\infty} \text{PDO}_t(n) q^n = q \frac{f_2 f_{12}}{f_6} \cdot \frac{f_3^2}{f_1^2} = q \frac{f_2 f_{12}}{f_6} \left( f_2^4 f_6 f_{12} + 2 q \frac{f_4 f_6 f_{24}}{f_2 f_{12}} \right),
\]
from which we extract
\[
\sum_{n=0}^{\infty} \text{PDO}_t(2n) q^n = 2 q f_2 f_4 f_6 f_{12} \cdot \frac{f_3}{f_1^3},
\]
Number of tagged parts over the partitions of designated summands

\[
= 2q f_2 f_4 f_6 f_{12} \left( \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^2} \right).
\]

From the above, we extract

\[
\sum_{n=0}^\infty \text{PDO}_t(4n + 2)q^n
\]

\[
= 2 \frac{f_2^7}{f_6} \cdot \frac{1}{f_1^2} \cdot f_3^4
\]

\[
= 2 \frac{f_2^7}{f_6} \left( \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_2^2 f_4^4}{f_2^{10} f_8^2} \right)^2 \left( \frac{f_1^{10}}{f_2^6 f_4^2} - 4q^3 \frac{f_2^3 f_4^4}{f_2^2} \right)
\]

\[
= 2 \frac{f_2^{28} f_8^{10}}{f_2^{21} f_3^3 f_8^4 f_2^{24}} + 16q \frac{f_4^{16} f_6^{10}}{f_2^{17} f_3 f_8^4 f_2^{24}} + 32q^2 \frac{f_4^4 f_8^{10} f_1^2}{f_2^3 f_6^3 f_2^{24}}
\]

\[- 8q^3 \frac{f_4^2 f_6 f_8^4}{f_2^{21} f_3 f_8^4 f_2^{12}} - 64q^4 \frac{f_1^{16} f_6 f_8^4}{f_2^{17} f_8^2 f_2^{12}} - 128q^5 \frac{f_4^1 f_6 f_8^2 f_4^4}{f_2^{13} f_1^2 f_2^{12}},
\]

from which we extract

\[
\sum_{n=0}^\infty \text{PDO}_t(8n + 6)q^n = 16 \frac{f_2^{16} f_6^{10}}{f_1^{17} f_3 f_8^4 f_2^{24}} - 8q \frac{f_2^{28} f_3 f_1^{12}}{f_2^{21} f_8^4 f_6^2} - 128q^2 \frac{f_4^1 f_3 f_1^{12}}{f_1^{13} f_2^2},
\]

which is (1.9).

Next, from (3.1) we also extract

\[
\sum_{n=0}^\infty \text{PDO}_t(2n + 1)q^n = \frac{f_2^4 f_1^4}{f_4 f_2} \cdot \frac{1}{f_1^2} \cdot f_3^4
\]

\[
= \frac{f_2^4 f_1^4}{f_4 f_2} \left( \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_2^2 f_8^4}{f_2^{10} f_8^2} \right) \left( \frac{f_1^{14}}{f_2^6 f_8^2} + 4q \frac{f_2^2 f_8^4}{f_2^{10}} \right)
\]

from which we have

\[
\sum_{n=0}^\infty \text{PDO}_t(4n + 3)q^n
\]

\[
= 4 \frac{f_2^2 f_4^4}{f_6} \cdot \frac{1}{f_1^2} \cdot f_3^4
\]

\[
= 4 \frac{f_2^2 f_4^4}{f_6} \left( \frac{f_5^8}{f_2^{10} f_8^2} + 2q \frac{f_4^2 f_6 f_{12}^2}{f_2^{10} f_8^2} \right)^2 \left( \frac{f_4^1 f_6 f_{12}^2}{f_2^6 f_8^2} + 2q \frac{f_4^2 f_6 f_{12}^2 f_8 f_{24}}{f_2^6 f_8^2} \right)
\]

\[
= 4 \frac{f_4^{12} f_6 f_8^4 f_1^{12}}{f_2^{14} f_8^4 f_2^{16}} + 16q \frac{f_4^{12} f_6 f_8^4 f_1^{12}}{f_2^{13} f_8^4 f_2^{16}} + 8q \frac{f_4^{14} f_6 f_1^{12} f_8^2}{f_2^{14} f_8^4 f_2^{16}}
\]

\[
+ 16q^2 \frac{f_4^{12} f_6 f_8^4 f_1^{12} f_8^2 f_{24}}{f_2^{12} f_8^2 f_2^{16}} + 32q^2 \frac{f_4^1 f_6 f_{12}^2 f_8^2 f_{24}}{f_8^2 f_2^{16}} + 32q^3 \frac{f_4^{5} f_6 f_8^4 f_1^{12} f_8^2 f_{24}}{f_2^6 f_2^{16}},
\]

from which we extract

\[
\sum_{n=0}^\infty \text{PDO}_t(8n + 7)q^n = 8 \frac{f_2^{14} f_3 f_6^4 f_8^2}{f_1^{14} f_3 f_8^4 f_1^{12}} + 16 \frac{f_2^9 f_3 f_4^4 f_6}{f_1^{13} f_8^2} + 32q \frac{f_2^5 f_3 f_4^3 f_8^2 f_{12}^2}{f_8^2 f_1^{12} f_6},
\]
which is (1.10).

4. Proofs of (1.5) and (1.6)

We have

\[ 2 \sum_{n=0}^{\infty} \PD_t(n) q^n = \frac{f_3^5}{f_6^2} \cdot \frac{1}{f_3} - \frac{f_6}{f_3} \cdot \frac{1}{f_1 f_2} \]

\[ = \frac{f_3^5}{f_6^2} \left( a^2(q^3) \frac{f_9}{f_3^3} + 3qa(q^3) \frac{f_9}{f_3^2} + 9q^2 \frac{f_9}{f_3 f_6} \right) + \frac{f_6}{f_3} \left( a(q^6) \frac{f_9}{f_3 f_6} + qa(q^3) \frac{f_{18}}{f_3^3} + 3q^2 \frac{f_9 f_{18}}{f_3^3} \right), \quad (4.1) \]

from which we extract

\[ 2 \sum_{n=0}^{\infty} \PD_t(3n + 1) q^n = 3a(q) \frac{f_9}{f_1 f_2} - a(q) \frac{f_3^5}{f_1^2 f_2^3} \]

\[ = \left( 3 \frac{f_3^6}{f_1^2 f_2^3} - \frac{f_6^3}{f_1 f_2^3} \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( 3 \frac{f_2 f_6^3}{f_1^2 f_2^3} - 1 \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( a(q) + 2a(q^2) - 1 \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( a(q) + a(-q) + 2a(q^2) - 1 - a(-q) \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( 2a(q^4) + 2a(q^2) - 1 - a(-q) \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( 4 \frac{f_6^3 f_6}{f_1 f_2^3} - 1 \right) a(q) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( \left( 4 \frac{f_6^3 f_6}{f_1 f_2^3} - 1 \right) \left( a(q^4) + 6q \frac{f_2^2 f_2^3}{f_2^3 f_6} \right) \right) \]

\[ = \frac{f_6^3}{f_1 f_2^3} \left( \left( 4 \frac{f_6^3 f_6}{f_1 f_2^3} - 1 \right) \left( a(q^4) + 6q \frac{f_2^2 f_2^3}{f_2^3 f_6} \right) \right) \]

\[ - \left( a^2(q^4) - 36q^2 \frac{f_4^3 f_4^3}{f_2^3 f_6^3} \right) \]

\[ = \frac{f_6^3}{f_2^3} \left( \left( 4 \frac{f_6^3 f_6}{f_2^3 f_1^3} \right) a(q^4) - a(q^4) - a^2(q^4) + 36q^2 \frac{f_4^3 f_4^3}{f_2^3 f_6^3} \right) \]

\[ + q \left( 24 \frac{f_3}{f_2} - 6 \frac{f_2^2 f_2^2}{f_2 f_6} \right) \left( \frac{f_1^4}{f_2^3 f_8^3} + 4q \frac{f_2^4 f_8^3}{f_2 f_6^3} \right). \]
Extracting the terms involving $q^{2n+1}$ from both sides of the above and then dividing by 2, we find that

$$
\sum_{n=0}^{\infty} \text{PD}_4(6n + 4)q^n = \frac{f_3^3}{f_1^3} \left( 2 \frac{f_2^2 f_4^4}{f_1^{10}} \left( 4 \frac{f_2^6 f_3}{f_1^3 f_3^3} a(q^2) - a(q^2) - a^2(q^2) + 36q \frac{f_2^4 f_4^4}{f_1^7 f_3^2} \right) + \frac{f_4^2}{f_1^4 f_4^4} \left( 12 \frac{f_2^8}{f_1^4} - 3 \frac{f_2^2 f_4^2}{f_1 f_3} \right) \right).
$$

Taking congruences modulo 8, we have

$$
\sum_{n=0}^{\infty} \text{PD}_4(6n + 4)q^n
\equiv \frac{f_3^3}{f_1^3} \left( 6 \frac{f_2^2 f_4^4}{f_1^{10}} a(q^2) + 4 \frac{f_2^{22}}{f_1^{18} f_4^4} + 5 \frac{f_2^{16} f_6^2}{f_1^{15} f_3 f_4^4} \right)
\equiv \frac{f_3^3}{f_1^3} \left( 6 \frac{f_2^2 f_4^4}{f_1^{10}} a(q^2) + 6 \frac{f_2^2 f_4^4}{f_1^{10}} + 4 \frac{f_2^{22}}{f_1^{18} f_4^4} + 5 \frac{f_2^{16} f_6^2}{f_1^{15} f_3 f_4^4} \right)
\equiv (6f_4^4 f_6^2 a(q^2) + 10f_4^2 f_6^2) \cdot \frac{1}{f_1 f_3} + 5f_6^2 \cdot \frac{f_3^2}{f_1^2}
\equiv (6f_4^4 f_6^2 a(q^2) + 10f_4^2 f_6^2) \left( \frac{f_5^2 f_7 f_9^5}{f_2 f_4 f_6 f_8 f_24} + q \frac{f_4^2 f_2^2}{f_2^2 f_6 f_8 f_12} \right)
+ 5f_6^2 \left( \frac{f_4^4 f_6^2 f_12^2}{f_2^4 f_8 f_24} + 2q \frac{f_4^2 f_6 f_8 f_24}{f_4^2 f_12} \right).
\quad (4.2)
$$

We extract

$$
\sum_{n=0}^{\infty} \text{PD}_4(12n + 4)q^n
\equiv (6f_2^2 f_3^2 a(q) + 10f_2^2 f_3^2) \frac{f_2^2 f_6^4}{f_1^3 f_2 f_3 f_4 f_12} + 5 \frac{f_2^4 f_3^2 f_6^2}{f_1^3 f_4 f_12}
\equiv 6 \frac{f_2^2 f_4^2}{f_6} \cdot a(q) \cdot \frac{f_3^2}{f_1^2} + 10f_4^2 + 5 \frac{f_6^2}{f_4 f_12} \cdot f_1^3 f_3
\equiv 6 \frac{f_2^2 f_4^2}{f_6} \left( a(q^4) + 6q \frac{f_2^4 f_12^2}{f_2 f_6} \right) \left( \frac{f_4^4 f_6 f_7 f_9^2}{f_5^2 f_8 f_24} + 2q \frac{f_4^2 f_6 f_8 f_24}{f_2 f_12} \right)
+ 10f_4^2 + 5 \frac{f_6^2}{f_4 f_12} \left( \frac{f_2^2 f_4^4 f_12^4}{f_2^4 f_6 f_24^2} - q \frac{f_4^2 f_6^2 f_24^3}{f_2 f_8^2 f_12^2} \right)
\equiv 6 \left( a(q^4) \frac{f_6^4 f_12^2}{f_4^2 f_8 f_24^2} + 12q^2 \frac{f_4^5 f_8 f_12^2 f_24}{f_4^2} + q \left( 2a(q^4) \frac{f_2^2 f_6 f_8 f_24}{f_2^2 f_12} \right)
+ 6 \frac{f_8^4 f_12}{f_2^2 f_6 f_8 f_24} \right) \right) + 10f_4^2 + 5 \left( \frac{f_3^2 f_6^5 f_12^4}{f_1^3 f_6 f_24^2} - 3q \frac{f_2^2 f_6^2 f_8^2 f_12^2}{f_4 f_24} \right).
$$
\[
+ 3q^2 \frac{f_1^5 f_6^3 f_{24}^2}{f_2 f_8 f_{12}} - q^3 \frac{f_4^{11} f_6^5 f_{24}^6}{f_2 f_8 f_{12}} \pmod{8},
\]
from which we extract
\[
\sum_{n=0}^{\infty} \text{PD}_t(24n + 4)q^n \\
\equiv 6a(q^2) \frac{f_2^6 f_6^2}{f_1^4 f_4 f_{12}} + 10f_2^2 + 5 \left( \frac{f_1^3 f_4 f_6^{11}}{f_1^2 f_4^2 f_{12}} + 3q \frac{f_2^5 f_3^3 f_{12}^2}{f_2^4 f_4 f_6} \right) \\
\equiv 6a(q^2) \frac{f_4 f_6^2}{f_{12}} + 10f_2^2 + 5 \left( \frac{f_2^3 f_4 f_6^3}{f_1^2 f_{12}} \cdot \frac{f_1^3}{f_4} + 3q \frac{f_2^5 f_3^3}{f_4 f_6} \cdot \frac{f_4^3}{f_1} \right) \\
\equiv 6a(q^2) \frac{f_4 f_6^2}{f_{12}} + 10f_2^2 + 5 \left( \frac{f_2^3 f_4 f_6^3}{f_1^2 f_{12}} - 3q \frac{f_2^5 f_3^3}{f_4 f_6} \right) \\
+ 3q \frac{f_2^5 f_3^2}{f_4^2 f_6} \left( \frac{f_2^3 f_4^2}{f_2^2 f_{12}} + q \frac{f_2^3}{f_4} \right) \\
\equiv 6a(q^2) \frac{f_4 f_6^2}{f_{12}} + 10f_2^2 + 5 \left( \frac{f_2^3 f_4 f_6^3}{f_1^2 f_{12}} + 3q^2 \frac{f_2^5 f_3^2}{f_4 f_6} \right) \pmod{8}.
\]
Therefore,
\[
\text{PD}_t(48n + 28) \equiv 0 \pmod{8},
\]
which is (1.5).

Now, from (4.2), we also extract
\[
\sum_{n=0}^{\infty} \text{PD}_t(12n + 10)q^n \\
\equiv (6f_2^2 f_3^2 a(q) + 10f_2^2 f_3^2) \frac{f_2^6 f_6^2}{f_1^4 f_4^2 f_{12}} + 10f_2^3 f_4 f_6 f_{12} \\
\equiv 6f_2^2 f_6^2 \cdot a(q) + 10f_2^2 f_6^2 f_{12} + 10f_2 f_6 f_{12} \\
\equiv 6f_2^2 f_6^2 \left( a(q^4) + 6q f_4^2 f_{12}^2 \right) + 10f_2 f_6 f_{12} + 10f_2 f_6 f_{12} \pmod{8},
\]
from which we extract
\[
\sum_{n=0}^{\infty} \text{PD}_t(24n + 22)q^n \equiv 36 \frac{f_2^2 f_6^4}{f_3^3} \equiv 36f_2^2 f_6^3 \pmod{8},
\]
Thus,
\[
\text{PD}_t(48n + 46) \equiv 0 \pmod{8},
\]
which is (1.6).
5. Proof of Theorem 1.2

From (4.1) we extract
\[
2 \sum_{n=0}^{\infty} \text{PD}_t(3n)q^n = \frac{f_3^3}{f_1^3 f_2^2} a^2(q) - \frac{f_3^3}{f_1^3 f_2^2} a(q^2)
\]
\[
\equiv \frac{1}{f_1^3} \cdot \frac{f_3^3}{f_1} - \frac{a(q^2)}{f_1^3} \cdot \frac{f_3^3}{f_1}
\]
\[
\equiv \left( \frac{1}{f_1^4} - \frac{a(q^2)}{f_1^3} \right) \left( \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_3^3 f_2}{f_4} \right) \pmod{4},
\]
(5.1)
from which we extract
\[
2 \sum_{n=0}^{\infty} \text{PD}_t(6n)q^n
\]
\[
\equiv \left( \frac{1}{f_2^2} - \frac{a(q)}{f_2} \right) \frac{f_2^3 f_3^3}{f_1^3 f_6}
\]
\[
\equiv \frac{f_2}{f_6} \cdot \frac{f_2^3}{f_1} - \frac{f_2}{f_6} \cdot \frac{f_2^3}{f_1} \cdot a(q)
\]
\[
\equiv \frac{f_2}{f_6} \left( \frac{f_4^4 f_6 f_{12}^2}{f_2^2 f_{12}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^2 f_{12}} \right)
\]
\[
- \frac{f_2}{f_6} \left( \frac{f_4^4 f_6 f_{12}^2}{f_2^2 f_{12}} + 2q \frac{f_4 f_6^2 f_8 f_{24}}{f_2^2 f_{12}} \right) \left( a(q^4) + 6q \frac{f_4^3 f_{12}^2}{f_2 f_6} \right) \pmod{4},
\]
(5.2)
Therefore,
\[
2 \sum_{n=0}^{\infty} \text{PD}_t(12n)q^n
\]
\[
\equiv \frac{f_2 f_6^2}{f_4 f_{12}} - \frac{f_1 f_4^2 f_3 f_6^2}{f_3 f_6^2 f_4 f_{12}} \cdot a(q^2)
\]
\[
\equiv \frac{f_2 f_6^2}{f_4 f_{12}} - \frac{f_3^2 f_6^2}{f_4 f_{12}} \cdot a(q^2) \pmod{4},
\]
from which we readily arrive at
\[
\text{PD}_t(24n + 12) \equiv 0 \pmod{2},
\]
which is (1.11).

Next, extracting the terms involving \(q^{2n+1}\) from both sides of (5.2), and then dividing by 2, we have
\[
\sum_{n=0}^{\infty} \text{PD}_t(12n + 6)q^n
\]
\[
\equiv \frac{f_2 f_3 f_4 f_{12}}{f_3 f_6} - a(q^2) \cdot \frac{f_2 f_3 f_4 f_{12}^2}{f_3^3 f_6} - 3 \frac{f_2^6 f_4^4}{f_3^3 f_6 f_4 f_{12}}
\]
\[
\equiv \frac{f_2 f_4 f_{12}}{f_6} \cdot \frac{f_3}{f_1^2}
\]
\[
\equiv \frac{f_2 f_4 f_{12}}{f_6} \left( \frac{f_6^3 f_6^3}{f_2^2 f_{12}} + 3q \frac{f_2^3 f_6 f_{12}^2}{f_2^2} \right) \pmod{2},
\]
from which we extract
\[\sum_{n=0}^{\infty} \text{PD}_t(24n + 6)q^n \equiv \frac{f_1 f_2 f_6}{f_3} \cdot \frac{f_2^6 f_3^6}{f_1^9 f_6^2} \equiv f_2^3 \pmod{2},\]
from which we further extract
\[\text{PD}_t(48n + 30) \equiv 0 \pmod{2},\]
which is \([1.13]\), and
\[\sum_{n=0}^{\infty} \text{PD}_t(48n + 6)q^n \equiv f_1^3 \equiv f_3 a(q^3) - 3q f_9^3 \pmod{2},\]
which implies
\[\text{PD}_t(144n + 102) \equiv 0 \pmod{2},\]
which is \([1.14]\).

Now, from (5.1) we extract
\[2 \sum_{n=0}^{\infty} \text{PD}_t(6n + 3)q^n \equiv \left( \frac{1}{f_2^2} - \frac{a(q)}{f_2^2} \right) \frac{f_6^3}{f_2} \equiv f_6^3 \pmod{4},\]
from which we extract
\[\sum_{n=0}^{\infty} \text{PD}_t(12n + 9)q^n \equiv f_1^3 = f_3 a(q^3) - 3q f_9^3 \pmod{2}.\] \hspace{1cm} (5.3)
This implies
\[\text{PD}_t(24n + 21) \equiv 0 \pmod{2},\]
which is \([1.12]\). Furthermore,
\[\text{PD}_t(36n + 21) \equiv \text{PD}_t(36n + 33) \equiv 0 \pmod{2},\]
which are weaker versions of \([1.16]\) and \([1.17]\).

From (5.3) we also extract
\[\sum_{n=0}^{\infty} \text{PD}_t(72n + 9)q^n \equiv f_1^3 = f_3 a(q^3) - 3q f_9^3 \pmod{2},\] \hspace{1cm} (5.4)
from which we further extract
\[\text{PD}_t(216n + 153) \equiv 0 \pmod{2},\]
which is \([1.13]\).

From (4.1), we extract
\[2 \sum_{n=0}^{\infty} \text{PD}_t(3n)q^n = \frac{f_3^3}{f_1^3 f_2^3} a^2(q) - \frac{f_3^3}{f_1^3 f_2^3} a(q^2)\]
Taking congruences modulo 8, we have

\[
\sum_{n=0}^{\infty} \text{PD}_t(3n)q^n \equiv \frac{f_6^2}{f_2^4} \cdot \frac{1}{f_1 f_3} + 5 \frac{f_6}{f_2} \cdot f_3 + 2
\]

\[
\equiv \frac{f_6^2}{f_2^4} \left( \frac{f_8^2 f_1^5}{f_2^3 f_4 f_6^2 f_4^2 f_2^2} + q - \frac{f_4^2 f_2 f_8^2 f_1^4 f_2^4 f_8 f_1^2}{f_4^2 f_6^2 f_8^2 f_2^4} \right)
\]

\[
+ 5 \frac{f_6}{f_2} \left( \frac{f_2 f_8^2 f_1^4 f_2^4 f_1^2}{f_2^3 f_4 f_6 f_2^3} - q \frac{f_4 f_6 f_2^4 f_2^4}{f_2^3 f_8^2 f_2^4} \right) + 2, \tag{5.5}
\]

from which we extract

\[
\sum_{n=0}^{\infty} \text{PD}_t(6n + 3)q^n
\]

\[
\equiv \frac{f_2}{f_1 f_6} + 3 \frac{f_2}{f_1 f_6} \cdot f_2^2 f_3^2
\]

\[
\equiv \frac{f_2}{f_1 f_6} + 3 \frac{f_2}{f_1 f_6} \cdot \left( \frac{f_2 f_8 f_1^5}{f_2^3 f_4 f_6^2 f_4^2 f_2^2} - q \frac{f_4 f_6 f_2^4 f_2^4}{f_2^3 f_8^2 f_2^4} \right)
\]

\[
\equiv \frac{f_2}{f_1 f_6} + 3 \frac{f_2}{f_1 f_6} \cdot \left( \frac{f_2 f_8 f_1^5}{f_2^3 f_4 f_6^2 f_4^2 f_2^2} - q f_4 f_6 f_2^4 f_2^4 + q^2 f_4 f_6 f_2^4 f_2^4 \right) \pmod{8}.
\]

Extracting the terms involving \(q^{2n+1}\) from both sides, and then dividing by 2, we have

\[
\sum_{n=0}^{\infty} \text{PD}_t(12n + 9)q^n \equiv \frac{f_6^4}{f_2^3} \pmod{4},
\]

from which we extract

\[
\text{PD}_t(36n + 21) \equiv 0 \pmod{4}
\]

and

\[
\text{PD}_t(36n + 22) \equiv 0 \pmod{4},
\]

which are (1.16) and (1.17), respectively.

Now, from (5.5), we extract

\[
\sum_{n=0}^{\infty} \text{PD}_t(6n)q^n \equiv \frac{f_2^2 f_6^3}{f_1^3 f_2^3 f_3^3} + 5 \frac{f_2^2 f_6^4}{f_1^3 f_2^3 f_3^3} + 2
\]

\[
\equiv \frac{f_2^2 f_6^3}{f_1^3 f_2^3 f_3^3} + 5 \frac{f_2^2 f_6^4}{f_1^3 f_2^3 f_3^3} + 2
\]
From the above two, we have
\[ \frac{f_3^5 f_6^5}{f_2^5 f_{12}^5} \cdot \frac{f_1^2}{f_3^4} - \frac{f_2^2 f_4^4}{f_2^2 f_{12}^2} \cdot \frac{1}{f_1^4} + 2 \]
\[ \equiv \frac{f_3^5 f_6^5}{f_2^5 f_{12}^5} \left( \frac{f_2 f_4^2 f_{12}^5}{f_6 f_8 f_{24}} - 2q \frac{f_2 f_8 f_{12} f_{24}}{f_4 f_6^4} \right) \]
\[ - \frac{f_3^5 f_6^5}{f_2^5 f_{12}^5} \left( \frac{f_1^4}{f_{12}^4 f_8} + 4q \frac{f_2^2 f_4^4}{f_6^{10}} \right) + 2 \pmod{8}, \quad (5.6) \]
which yields
\[ 2 \sum_{n=0}^{\infty} PD_t(12n)q^n \equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \cdot \frac{1}{f_4^2} - \frac{f_6^2}{f_3^4} + 2 \]
\[ \equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \left( \frac{f_4^14 f_8^4}{f_{12}^4 f_6^4} + 4q \frac{f_2^4 f_8^4}{f_6^{10}} \right) \]
\[ - f_6^2 \left( \frac{f_{12}^4 f_6^4 f_8^4}{f_6^4 f_{24}^4} + 4q^3 f_{12}^4 f_{24}^4 \right) + 2 \pmod{8}, \quad (5.7) \]
from which we extract
\[ 2 \sum_{n=0}^{\infty} PD_t(24n)q^n \equiv \frac{f_2^4}{f_6} \cdot \frac{f_3^2}{f_1^2} - \frac{f_6^2}{f_3^4} + 2 \]
\[ \equiv \frac{f_2^4}{f_6} \left( \frac{f_4^14 f_6^4 f_{12}^5}{f_{12}^4 f_6^4 f_8 f_{24}} + 2q \frac{f_4^14 f_6^4 f_8 f_{12} f_{24}}{f_4 f_{12}^5} \right) \]
\[ - f_6^2 \left( \frac{f_{12}^4 f_6^4 f_8^4}{f_6^4 f_{24}^4} + 4q^3 f_{12}^4 f_{24}^4 \right) + 2 \pmod{8}. \quad (5.8) \]
We extract
\[ 2 \sum_{n=0}^{\infty} PD_t(48n)q^n \equiv \frac{f_2^4 f_6^2}{f_4 f_{12}} \cdot \frac{1}{f_4^2} - \frac{f_6^2}{f_3^4} + 2 \pmod{8}. \quad (5.9) \]
From (5.7) and (5.9), we arrive at
\[ PD_t(12n) \equiv PD_t(48n) \pmod{4}, \]
which, by iteration, gives (1.13).
We also extract from (5.7)
\[ \sum_{n=0}^{\infty} PD_t(24n + 12)q^n \equiv 2f_4^3 - 2q f_{12}^3 \]
\[ \equiv 2f_{12} a(q^{12}) - 6q^4 f_{36}^3 - 2q f_{12}^3 \pmod{4}, \quad (5.10) \]
from which we extract
\[ \sum_{n=0}^{\infty} PD_t(24(3n + 1) + 12)q^n \equiv 2f_4^3 - 2q f_{12}^3 \pmod{4}. \]
From the above two, we have
\[ PD_t(24(3n + 1) + 12) = PD_t(3(24n + 12)) \equiv PD_t(24n + 12) \pmod{4}. \]
Thus, for any nonnegative integer \( \ell \),

\[
\text{PD}_t(3^\ell (24n + 12)) \equiv \text{PD}_t(24n + 12) \pmod{4}.
\]

Combining the above with (1.18), we readily arrive at (1.19).

Next, from (5.8) we also have

\[
2 \sum_{n=0}^{\infty} \text{PD}_t(24n) q^n \equiv \frac{f_2^2}{f_6} \cdot \frac{f_2}{f_1^2} - \frac{f_6^2}{f_3^2} + 2
\]

\[
\equiv \frac{f_3^2}{f_6} \left( \frac{f_6 f_9^6}{f_3^8 f_3^3} + 2q \frac{f_6 f_9^3 f_3^3}{f_3^3} + 4q^2 \frac{f_6^2 f_3^3}{f_3^6} \right) - \frac{f_6^2}{f_4^4} + 2 \pmod{8},
\]

from which we extract

\[
\sum_{n=0}^{\infty} \text{PD}_t(72n + 48) q^n \equiv 2 \frac{f_2 f_6^3}{f_4^4}
\]

\[
\equiv 2 \frac{f_6^3}{f_2} \pmod{4},
\]

from which we further extract

\[
\text{PD}_t(144n + 120) \equiv 0 \pmod{4},
\]

which is (1.23).

From (5.10), we extract

\[
\sum_{n=0}^{\infty} \text{PD}_t(48n + 12) q^n \equiv 2 f_2^3 \pmod{4}
\]

(5.11)

and

\[
\sum_{n=0}^{\infty} \text{PD}_t(48n + 36) q^n \equiv 2 f_6^3 \pmod{4},
\]

(5.12)

which readily implies

\[
\text{PD}_t(96n + 60) \equiv 0 \pmod{4}
\]

and

\[
\text{PD}_t(96n + 84) \equiv 0 \pmod{4},
\]

which are (1.20) and (1.21), respectively. Furthermore, equating the coefficients of \( q^{3n+1} \) and \( q^{3n+2} \) from both sides of (5.12), we arrive at

\[
\text{PD}_t(144n + 84) \equiv 0 \pmod{4}
\]

and

\[
\text{PD}_t(144n + 132) \equiv 0 \pmod{4},
\]

which are (1.22) and (1.24), respectively.
From (5.11) and (5.12), we also have
\[ \sum_{n=0}^{\infty} \text{PD}_t(96n + 12)q^n \equiv \sum_{n=0}^{\infty} \text{PD}_t(288n + 36)q^n \equiv 2f_1^3 \equiv 2 (f_3a(q^3) - 3qf_3^3) \pmod{4}, \]
from which we extract
\[ \text{PD}_t(288n + 204) \equiv \text{PD}_t(3(288n + 204)) \equiv 0 \pmod{4}, \]
which, by iteration, yields (1.25).
From (5.8), we extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(48n + 24)q^n \equiv \frac{f_2f_3f_4f_{12}}{f_1^3 f_6} - 2q \frac{f_6^2 f_{12}^4}{f_3^3} \]
\[ \equiv \frac{f_2f_4f_{12}}{f_6} \cdot \frac{f_3}{f_1^3} - 2qf_{12}^3 \]
\[ \equiv \frac{f_2f_4f_{12}}{f_6} \left( \frac{f_6^3 f_9^3}{f_2^3 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^3} \right) - 2qf_{12}^3 \pmod{4}, \]
from which we further extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(96n + 24)q^n \equiv \frac{f_2^7 f_3^2}{f_1^8 f_6} \]
\[ \equiv \frac{f_3^3}{f_6} \cdot f_2^3 \]
\[ \equiv \frac{f_3^2}{f_6} \left( f_6a(q^6) - 3q^2 f_3 f_{18} \right) \pmod{4} \tag{5.13} \]
and
\[ \sum_{n=0}^{\infty} \text{PD}_t(96n + 72)q^n \equiv 3 \frac{f_2^3 f_6^3}{f_1^6} - 2f_6^3 \]
\[ \equiv 2f_6^3 + 3f_2 f_6 \cdot \frac{1}{f_1^2} \]
\[ \equiv 2f_6^3 + 3f_2 f_6 \left( \frac{f_8^5}{f_2^3 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^3 f_8} \right) \pmod{4}. \tag{5.14} \]
From (5.13) we extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(288n + 216)q^n \equiv f_6^3 \cdot \frac{f_2^2}{f_2} \]
\[ \equiv f_6^3 \left( \frac{f_5^2}{f_1^3} - 2q \frac{f_4^2 f_{18}^2}{f_6 f_9} \right) \pmod{4}, \]
from which we extract
\[ \text{PD}_t(864n + 792) \equiv 0 \pmod{4}, \]
which is (1.26). We further extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(864n + 216)q^n = \frac{f_3^2}{f_6} \cdot f_2^3 \]
\[ \equiv \frac{f_3^2}{f_6} \left( f_6 a(q^6) - 3q^2 f_9^3 \right) \quad (\text{mod } 4), \]
from which we have
\[ \text{PD}_t(2592n + 1080) \equiv 0 \quad (\text{mod } 4), \]
which is (1.28).

From (5.14) we extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(192n + 72)q^n \]
\[ \equiv 2f_3^3 + 3\frac{f_3^2 f_4}{f_1^4 f_2^4} \]
\[ \equiv 2f_3^3 + 3f_3 \cdot \frac{f_4}{f_2^3} \]
\[ \equiv 2f_3^3 + 3f_3^3 \left( \frac{f_1^4 f_6}{f_1^4 f_6} + 2q^2 \frac{f_1^4 f_6}{f_1^4 f_6} + 4q^4 \frac{f_1^4 f_6}{f_1^4 f_6} \right) \quad (\text{mod } 4), \]
from which we extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(576n + 72)q^n \equiv 2f_1^3 + 3\frac{f_3^4 f_4 f_6}{f_1^8 f_1^{12}} \]
\[ \equiv \left( 2 + 3\frac{f_6}{f_1^{12}} \right) \cdot f_1^3 \]
\[ \equiv \left( 2 + 3\frac{f_6}{f_1^{12}} \right) \left( f_3 a(q^3) - 3q f_9^3 \right) \quad (\text{mod } 4), \]
from which we further extract
\[ \text{PD}_t(1728n + 1224) \equiv 0 \quad (\text{mod } 4), \]
which is (1.27).

From (5.6), we extract
\[ \sum_{n=0}^{\infty} \text{PD}_t(12n + 6)q^n \equiv -\frac{f_2 f_3 f_4 f_6}{f_3^2 f_6} - 2\frac{f_4^4 f_1^4 f_4^{12}}{f_1^{12} f_6^2} \]
\[ \equiv 2f_4^3 + 3\frac{f_2 f_4 f_6}{f_6} \cdot \frac{f_3^2}{f_1^3} \]
\[ \equiv 2f_4^3 + 3\frac{f_2 f_4 f_6}{f_6} \left( \frac{f_4^6 f_6}{f_2^9 f_1^{12}} + 3q \frac{f_4^2 f_6 f_1^{12}}{f_2^9} \right) \]
\[ \equiv \left( 2 + 3\frac{f_6}{f_1^{12}} \right) \cdot f_4^3 + q f_1^{12} \cdot \frac{f_6^2}{f_4^2} \quad (5.15) \]
\[ \equiv \left( 2 + 3 \frac{f_6^2}{f_{12}} \right) (f_{12}a(q^{12}) - 3q^4 f_{36}^3) + q f_{12}^3 \left( \frac{f_{18}^2}{f_{36}} - 2q^2 \frac{f_6 f_{36}^2}{f_{12} f_{18}} \right) \pmod{4}, \]

from which we extract

\[ \text{PD}_t(36n + 30) \equiv 0 \pmod{4}, \]

which is (1.29), and

\[ \sum_{n=0}^{\infty} \text{PD}_t(36n + 18) q^n \equiv 2 q f_{12}^3 + 3q f_{12}^3 \cdot \frac{f_{18}^2}{f_{36}} + \frac{f_6 f_{12}^2}{f_{12}} \cdot f_{12}^3 \equiv 2 q f_{12}^3 + 3q f_{12}^3 \left( \frac{f_{18}^2}{f_{36}} - 2q^2 \frac{f_6 f_{36}^2}{f_{12} f_{18}} \right) + \frac{f_6^2}{f_{12}} (f_{12}a(q^{12}) - 3q^4 f_{36}^3) \pmod{4}. \]

From the above we extract

\[ \text{PD}_t(108n + 90) \equiv 0 \pmod{4}, \]

which is (1.30), and

\[ \sum_{n=0}^{\infty} \text{PD}_t(108n + 54) q^n \equiv \left( 2 + 3 \frac{f_6^2}{f_{12}} \right) \cdot f_{12}^3 + q f_{12}^3 \cdot \frac{f_6^2}{f_{12}} \pmod{4}. \quad (5.16) \]

From (5.15) and (5.16), we have

\[ \text{PD}_t(9(12n + 6)) \equiv \text{PD}_t(12n + 6) \pmod{4}, \]

which, upon iteration, yields (1.31). This completes the proof.

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