INDEFINITE LQ OPTIMAL CONTROL WITH PROCESS STATE INEQUALITY CONSTRAINTS FOR DISCRETE-TIME UNCERTAIN SYSTEMS

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Abstract. Uncertainty theory is a branch of axiomatic mathematics that deals with human uncertainty. Based on uncertainty theory, this paper discusses linear quadratic (LQ) optimal control with process state inequality constraints for discrete-time uncertain systems, where the weighting matrices in the cost function are assumed to be indefinite. By means of the maximum principle with mixed inequality constraints, we present a necessary condition for the existence of optimal state feedback control that involves a constrained difference equation. Moreover, the existence of a solution to the constrained difference equation is equivalent to the solvability of the indefinite LQ problem. Furthermore, the well-posedness of the indefinite LQ problem is proved. Finally, an example is provided to demonstrate the effectiveness of our theoretical results.

1. Introduction. The stochastic linear quadratic (LQ) optimal control problem was pioneered by Wonham [22], which has been extensively studied [25, 26, 4, 24, 2, 20] in the past decades. It is known that the dynamic programming [4, 24] and the stochastic maximum principle [2, 20] are two main techniques to solve stochastic optimal control problem. For the stochastic LQ optimal control problem, the early reference assumed that the control weight is positive definite and the state weight is positive semidefinite. It has been shown recently that for stochastic systems, LQ problem with indefinite control weights could make sense [5, 11]. Owing to many applications, LQ problem with indefinite control weights have drawn increasing attention ranging from pollution control [5] to portfolio selection [28]. To some extent, the influence of stochastic noises compensate the negative control weights to make the problem well-posed.

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As we know, probability theory has been used to deal with stochastic phenomena for a long time. Before applying it in practice, we should first obtain the probability distribution via statistics, or test the probability distribution to make sure it is close enough to the real frequency, either of which is based on a lot of observed data. However, due to the technological or economical difficulties, we sometimes have no samples. In this case, we have to invite some domain experts to evaluate their belief degree about the chances that the possible events happen. According to Kahneman and Tversky [10], humans tend to overweight unlikely events, so the belief degree generally has a much larger range than the real frequency. As a result, the probability theory is not applicable in this case, otherwise some counterintuitive results may be derived. In order to rationally deal with personal belief degrees, uncertainty theory was founded by Liu [12] in 2007, and refined by Liu [13] in 2010 based on an uncertain measure which satisfies normality, duality, subadditivity and product axioms. So far, the content of uncertainty theory has been developed to a fairly complete system for modeling human uncertainty and has been applied to uncertain programming [21], facility location problem [7], stock model [6], product control problem [16], and so on.

Based on the uncertainty theory, Zhu [29] proposed an uncertain optimal control in 2010, and gave an optimality equation as a counterpart of Hamilton-Jacobi-Bellman equation. After that, there are a lot of uncertain optimal control problems having been solved. For example, Sheng and Zhu [18] studied an optimistic value model of uncertain optimal control problem and proposed an equation of optimality to solve the model, Yan and Zhu [23] established a linear-quadratic control problem for discrete-time switched systems with subsystems perturbed by uncertainty and the analytical expressions are derived for both the optimal objective function and the optimal switching strategy, Shu and Zhu [19] considered an optimal control problem for an uncertain continuous-time singular system and obtained the equation of optimality for the uncertain singular system.

It is worth being mentioned that some constraints are of considerable importance in many practical problems [3, 9], so the constrained LQ issue has a concrete application background. Thus, some researchers discussed stochastic LQ optimal problems with indefinite control weights and constraints [27, 15]. Inspired by the stochastic indefinite LQ control control and uncertain optimal control, we propose an indefinite LQ optimal control with process state inequality constraints for discrete-time uncertain systems, in which the state and control in dynamics depend on linear uncertain noises. This control model comes from the practical problems where the noise disturbances are lack of observation data to get probability distribution. We have to invite some domain experts to give the belief degree that the noises happen. In this situation, it is more suitable to choose uncertain variable as the noises in the control model. To the best of our knowledge, this problem has not been investigated in the literature and remains open.

The organization of this paper is as follows. In section 2, we give some definitions about uncertainty theory. Section 3 presents an indefinite LQ optimal control with inequality constraints and gives a necessary condition for the existence of optimal controllers. Section 4 shows that the solvability of the constrained difference equation is sufficient for the well-posedness of the indefinite LQ problem. Section 5 applies the result to a numerical example. Section 6 presents the main conclusions.

For convenience, we adopt the following notations. $\mathbb{R}^n$ is the real $n$-dimensional Euclidean space; $\mathbb{R}^{m \times n}$ the set of all $m \times n$ matrices; $M^T$ the transpose of a matrix.
2. Preliminaries. In this section, we will review some basic concepts and results in uncertainty theory and Moore-Penrose inverse of a matrix.

2.1. Some concepts about uncertainty theory. Uncertain measure \( \mathcal{M} \) is a real-valued set-function on a \( \sigma \)-algebra \( \mathcal{L} \) over a nonempty set \( \Gamma \) satisfying the following axioms:

**Axiom 1.** (Normality Axiom) \( \mathcal{M}(\Gamma) = 1 \) for the universal set \( \Gamma \).

**Axiom 2.** (Duality Axiom) \( \mathcal{M}(\Lambda) + \mathcal{M}(\Lambda^c) = 1 \) for any event \( \Lambda \).

**Axiom 3.** (Subadditivity Axiom) For every countable sequence of events \( \Lambda_1, \Lambda_2, \cdots \), we have

\[
\mathcal{M}\left( \bigcup_{i=1}^{\infty} \Lambda_i \right) \leq \sum_{i=1}^{\infty} \mathcal{M}(\Lambda_i).
\]

The triplet \((\Gamma, \mathcal{L}, \mathcal{M})\) is called an uncertainty space. In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu [14] as follows.

**Axiom 4.** (Product Axiom) Let \((\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)\) be uncertainty spaces for \( k = 1, 2, \cdots \). Then, the product uncertain measure \( \mathcal{M} \) on the product \( \sigma \)-algebra satisfies

\[
\mathcal{M}\left( \prod_{k=1}^{\infty} \Lambda_k \right) = \bigwedge_{k=1}^{\infty} \mathcal{M}_k(\Lambda_k),
\]

where \( \Lambda_k \) are arbitrarily chosen events from \( \mathcal{L}_k \) for \( k = 1, 2, \cdots \), respectively.

**Definition 2.1.** (Liu [12]) An uncertain variable is a function \( \xi \) from an uncertainty space \((\Gamma, \mathcal{L}, \mathcal{M})\) to the set \( \mathbb{R} \) of real numbers such that for any Borel set \( B \) of real numbers, the set

\[
\{ \xi \in B \} = \{ \gamma \in \Gamma | \xi(\gamma) \in B \},
\]

is an event in \( \mathcal{L} \).

**Definition 2.2.** (Liu [12]) The uncertainty distribution \( \Phi \) of an uncertain variable \( \xi \) is defined by

\[
\Phi(x) = \mathcal{M}\{ \gamma \in \Gamma | \xi(\gamma) \leq x \},
\]

for any real number \( x \).

**Definition 2.3.** (Liu [13]) An uncertainty distribution \( \Phi(x) \) is said to be regular if it is a continuous and strictly increasing function with respect to \( x \) at which \( 0 < \Phi(x) < 1 \), and

\[
\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.
\]

**Example 1.** (Liu [12]) An uncertain variable \( \xi \) is called linear if it has a linear uncertainty distribution

\[
\Phi(x) = \begin{cases} 
0, & \text{if } x \leq a \\
(x-a)/(b-a), & \text{if } a \leq x \leq b \\
1, & \text{if } x \geq b
\end{cases}
\]

denoted by \( \mathcal{L}(a, b) \) where \( a \) and \( b \) are real numbers with \( a < b \).
Definition 2.4. (Liu [12]) Let \( \xi \) be an uncertain variable. Then the expected value of \( \xi \) is defined by

\[
E[\xi] = \int_0^{+\infty} M\{\xi \geq r\} \, dr - \int_{-\infty}^0 M\{\xi \leq r\} \, dr
\]

provided that at least one of the two integrals is finite.

Lemma 2.5. (Liu [13]) Let \( \xi \) be an uncertain variable with regular uncertainty distribution \( \Phi \). If the expected value exists, then

\[
E[\xi] = \int_0^1 \Phi^{-1}(\alpha) \, d\alpha.
\]

Example 2. Let \( \xi \sim \mathcal{L}(a, b) \) be a linear uncertain variable. Then its inverse uncertainty distribution is \( \Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b \), and its expected value is

\[
E[\xi] = \int_0^1 ((1 - \alpha)a + \alpha b) \, d\alpha = \frac{a + b}{2}.
\]

Independence is an important concept in uncertainty theory.

Definition 2.6. (Liu [14]) The uncertain variables \( \xi_1, \xi_2, \ldots, \xi_m \) are said to be independent if

\[
M \left\{ \bigcap_{i=1}^m (\xi_i \in B_i) \right\} = \min_{1 \leq i \leq m} M\{\xi_i \in B_i\},
\]

for any Borel sets \( B_1, B_2, \ldots, B_n \) of real numbers.

Theorem 2.7. (Liu [13]) Let \( \xi \) and \( \eta \) be independent uncertain variables with finite expected values. Then for any real numbers \( a \) and \( b \), we have

\[
E[a\xi + b\eta] = aE[\xi] + bE[\eta].
\]

Remark 1. Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed.

Remark 2. Let

\[
\xi = \begin{pmatrix}
\xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\
\xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\
\cdots & \cdots & \cdots & \cdots \\
\xi_{p1} & \xi_{p2} & \cdots & \xi_{pq}
\end{pmatrix}
\]

where \( \xi_{ij} \) are uncertain variables for \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, q \). The expected value of \( \xi \) is provided by

\[
E[\xi] = \begin{pmatrix}
E[\xi_{11}] & E[\xi_{12}] & \cdots & E[\xi_{1q}] \\
E[\xi_{21}] & E[\xi_{22}] & \cdots & E[\xi_{2q}] \\
\cdots & \cdots & \cdots & \cdots \\
E[\xi_{p1}] & E[\xi_{p2}] & \cdots & E[\xi_{pq}]
\end{pmatrix}.
\]

2.2. Moore-Penrose inverse of a matrix. For later use, we give the following lemmas about Moore-Penrose inverse of a matrix.

Lemma 2.8. (Penrose [17]) Let a matrix \( M \in \mathbb{R}^{m \times n} \) be given. Then there exists a unique matrix \( M^+ \in \mathbb{R}^{n \times m} \) such that

\[
\begin{align*}
MM^+M & = M, M^+MM^+ = M^+ \\
(MM^+)^\top & = MM^+, (M^+M)^\top = M^+M.
\end{align*}
\]

The matrix \( M^+ \) is called the Moore-Penrose pseudoinverse of \( M \).
Lemma 2.9. (Penrose [17]) Let matrices $L$, $M$ and $N$ be given with appropriate sizes. Then the matrix equation $LXM = N$ has a solution $X$ if and only if $LL^*NN^+ = N$. Moreover, any solution to $LXM = N$ is represented by $X = L^+NP + Y - L^*YM^+M^*$, where $Y$ is a matrix with an appropriate size.

3. Indefinite LQ optimal control with inequality constraints.

3.1. Problem setting. Consider the indefinite LQ optimal control with process state inequality constraints for discrete-time uncertain systems as follows:

$$\begin{aligned} &\inf_{u_k, u_{k+1}, \ldots, u_{N-1}} J(x_0, u) = \sum_{k=0}^{N-1} E\left[x_k^TQ_kx_k + u_k^TR_ku_k\right] + E\left[x_N^TQ_Nx_N\right] \\
&\text{subject to} \\
&x_{k+1} = A_kx_k + B_ku_k + \lambda_k(A_kx_k + B_ku_k)\xi_k, \quad k = 0, 1, \ldots, N-1, \\
&E\left[x_k^T\right] \leq c_k, \quad k = 1, 2, \ldots, N, \quad x(0) = x_0, 
\end{aligned}$$

where $\lambda_k \in \mathbb{R}$ and $|\lambda_k| \leq 1$. The vector $x_k \in X(\Gamma, \mathbb{R}^n)$ is a state vector with the initial state $x_0 \in \mathbb{R}^n$ and $u_k \in U(\mathbb{R}^n, U_k)$ is a control vector subject to a constraint set $U_k \subset \mathbb{R}^m$, where $X(\Gamma, \mathbb{R}^n)$ denotes the function space of all uncertain vectors (measurable functions from $\Gamma$ to $\mathbb{R}^n$), and $U(\mathbb{R}^n, U_k)$ denotes the function space of all functions from $\mathbb{R}^n$ to $U_k$. In this paper we assume $U_k = \mathbb{R}^m$. Denote $u = (u_0, u_1, \ldots, u_{N-1})$. Moreover, $Q_0, Q_1, \ldots, Q_N$ and $R_0, R_1, \ldots, R_{N-1}$ are real symmetric matrices with appropriate dimensions. In addition, $c_k \geq 0 (k = 1, 2, \ldots, N)$ is constant, the coefficients $A_0, A_1, \ldots, A_{N-1}$ and $B_0, B_1, \ldots, B_{N-1}$ are matrices with appropriate dimensions determined from the context. Besides, the noises $\xi_0, \xi_1, \ldots, \xi_{N-1}$ are independent linear uncertain variables $\mathcal{L}(-1,1)$.

Notice that the difference equation $x_{k+1} = A_kx_k + B_ku_k + \lambda_k(A_kx_k + B_ku_k)\xi_k$ in the problem (1) means that the state $x_{k+1}$ in the $(k+1)$th stage is linear with the state $x_k$ in the $k$th stage and the control $u_k$ with the disturbance $\lambda_k(A_kx_k + B_ku_k)\xi_k$. Generally speaking, the disturbance is not larger than the main part of the system. For the convenience of computation in the sequel, the disturbance is described by an uncertain variable $\xi_k$ (noise) associated with the amplitude $\lambda_k(A_kx_k + B_ku_k)$ which guarantees the disturbance is not larger than the main part $A_kx_k + B_ku_k$.

It is observed that the state and control weighting matrices in the objective functional are not required to be definite. Therefore problem (1) is called indefinite LQ optimal control problem. Next we give the following definitions.

Definition 3.1. The indefinite LQ problem (1) is called well-posed if

$$V(x_0) = \inf_{u_k, u_{k+1}, \ldots, u_{N-1}} J(x_0, u) > -\infty, \forall x_0 \in \mathbb{R}^n.$$ 

Definition 3.2. A well-posed problem is called solvable, if for $x_0 \in \mathbb{R}^n$, there is a control sequence $(u_0, u_1^*, \ldots, u_{N-1}^*)$ that achieves $V(x_0)$. In this case, the control $(u_0^*, u_1^*, \ldots, u_{N-1}^*)$ is called an optimal control sequence.

3.2. Equivalent deterministic problem. In this subsection, we will transform the indefinite LQ optimal control problem (1) into an equivalent deterministic optimal control problem with constraints.

Let $X_k = E[x_k^T]$. Since state $x_k \in \mathbb{R}^n$, $x_k^T$ is a $n \times n$ matrix which elements are uncertain variables, and $X_k$ is a symmetric deterministic matrix ($k =$
Theorem 3.3. If the indefinite LQ problem (1) is solvable by a feedback control sequence
\[ u_k = C_kx_k, \quad \text{for } k = 0, 1, \ldots, N - 1, \]
where \( C_0, C_1, \ldots, C_{N-1} \) are constant matrices, then it is equivalent to the following deterministic optimal control problem
\[
\begin{aligned}
&\min_{C_k \in \mathbb{R}^{N \times N}} J(X_0, C) = \sum_{k=0}^{N-1} \text{tr}[Q_k + C_k^TR_kC_k]X_k + \text{tr}[Q_NX_N] \\
\text{subject to} &
\begin{aligned}
X_{k+1} &= (1 + \frac{1}{3}\lambda_k^2)(A_kX_kA_k^T + A_kX_kC_k^TB_k^T + B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T, \\
&+ B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T), \quad k = 0, 1, \ldots, N - 1, \quad X_0 = x_0x_0^T, \\
\text{tr}(X_k) &\leq 0, \quad k = 1, 2, \ldots, N.
\end{aligned}
\end{aligned}
\]

Proof. Assume the indefinite LQ problem (1) is solvable by a feedback control
\[ u_k = C_kx_k, \quad \text{for } k = 0, 1, \ldots, N - 1, \]
where \( C_0, C_1, \ldots, C_{N-1} \) are constant matrices. Let \( X_k = E[x_kx_k^T] \) for \( k = 0, 1, \ldots, N - 1 \). Then we have
\[
\begin{aligned}
X_{k+1} &= E[x_{k+1}x_{k+1}^T] \\
&= E\{[A_k + B_kC_k + \lambda_k(A_k + B_kC_k)]\xi_k|x_kx_k^T[A_k^T + C_k^TB_k^T + \lambda_k(A_k^T + C_k^TB_k^T)]\xi_k\} \\
&= A_kX_kA_k^T + A_kX_kC_k^TB_k^T + B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T \\
&+ E[U_k\xi_k + V_k\xi_k^2],
\end{aligned}
\]
where
\[
\begin{aligned}
U_k &= 2\lambda_k(A_kX_kA_k^T + A_kX_kC_k^TB_k^T + B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T) \\
V_k &= \lambda_k^2(A_kX_kA_k^T + A_kX_kC_k^TB_k^T + B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T).
\end{aligned}
\]

It is obvious that \( \lambda_kU_k = 2V_k \). Note that since \( \xi_k \) and \( \xi_k^2 \) are not independent, we will calculate \( E[U_k\xi_k + V_k\xi_k^2] \) as follows.
(i) If \( V_k = 0 \), we obtain
\[ E[U_k\xi_k + V_k\xi_k^2] = E[U_k\xi_k] = U_kE[\xi_k] = 0. \]
(ii) If \( V_k \neq 0 \), we know that \( \lambda_k \neq 0 \) and \( |\frac{2}{\lambda_k}| \geq 2 \). According to Example 2 in [30], we have
\[
E[U_k\xi_k + V_k\xi_k^2] = E\left[\frac{2}{\lambda_k}V_k\xi_k + V_k\xi_k^2\right] = V_kE\left[\frac{2}{\lambda_k}\xi_k + \xi_k^2\right] = \frac{1}{3}V_k.
\]
Combined with case (i) and case (ii), it is concluded that
\[ E[U_k\xi_k + V_k\xi_k^2] = \frac{1}{3}V_k. \quad (4) \]
Substituting (4) into (3), we have
\[
X_{k+1} = (1 + \frac{1}{3}\lambda_k^2)(A_kX_kA_k^T + A_kX_kC_k^TB_k^T + B_kC_kX_kA_k^T + B_kC_kX_kC_k^TB_k^T).
\]
The associated cost function is expressed equivalently as

\[
J(X_0, C) = \min \sum_{0 \leq k \leq N-1} tr[(Q_k + C_k^T R_k C_k)X_k] + tr[Q_N X_N], \quad (6)
\]

and the constraints \(E[x_k^T x_k] \leq c_k (k = 1, \cdots, N)\) becomes

\[
tr(X_k) \leq c_k, \quad \text{for } k = 1, \cdots, N. \quad (7)
\]

By (5), (6) and (7), we obtain the desired deterministic optimal control problem (2).

**Remark 3.** Obviously, if problem (1) has a linear feedback optimal control solution \(u^*_k = C_k^* x_k\) \((k = 0, 1, \cdots, N-1)\), then \(C_k^* (k = 0, 1, \cdots, N-1)\) is the optimal solution of problem (2).

### 3.3. A necessary condition for state feedback control.

The next theorem shows that a necessary condition for the optimal linear state feedback control with deterministic gains of the indefinite LQ problem (1) can be obtained by applying the maximum principle with mixed inequality constraints [8].

**Theorem 3.4.** If the indefinite LQ problem (1) is solvable by a feedback control sequence

\[
u_k = C_k x_k, \quad \text{for } k = 0, 1, \cdots, N-1, \quad (8)
\]

where \(C_0, C_1, \cdots, C_{N-1}\) are constant matrices, then there exist symmetric matrices \(H_k (k = 0, 1, \cdots, N-1)\) and \(\gamma_k \geq 0 \in \mathbb{R} (k = 1, 2, \cdots, N)\) such that

\[
\begin{align*}
H_k &= Q_k + \left(1 + \frac{1}{3} \lambda_k^2\right)A_k^T H_{k+1} A_k + \gamma_k I - M_k L_k^T M_k \\
L_k L_k^T M_k - M_k &= 0 \\
L_k &= R_k + \left(1 + \frac{1}{3} \lambda_k^2\right)B_k^T H_{k+1} B_k \geq 0 \\
M_k = (1 + \frac{1}{3} \lambda_k^2)B_k^T H_{k+1} A_k, \quad k = 0, 1, \cdots, N-1 \\
H_N &= Q_N + \gamma_N I \\
\gamma_k [tr(X_k) - c_k] &= 0, \quad k = 1, 2, \cdots, N,
\end{align*}
\]

where (9) is a constrained difference equation. Moreover

\[
C_k = -L_k^T M_k + Y_k - L_k^T L_k Y_k, \quad (10)
\]

with \(Y_k \in \mathbb{R}^{m \times n}, k = 0, 1, \cdots, N-1\), being any given matrices.

**Proof.** Assume the uncertain LQ problem (1) is solvable by

\[
u_k = C_k x_k, \quad \text{for } k = 0, 1, \cdots, N-1,
\]

where the matrices \(C_0, C_1, \cdots, C_{N-1}\) are viewed as the control to be determined. We know that \(C_0, C_1, \cdots, C_{N-1}\) are also the optimal solution of problem (2). Thus, we consider the deterministic optimal control problem (2) as follows. Denote

\[
\begin{aligned}
J(X_0, C) &= \sum_{k=0}^{N-1} tr[(Q_k + C_k^T R_k C_k)X_k] + tr[Q_N X_N] \\
g_k(X_k) &= tr(X_k) - c_k, \quad k = 0, 1, \cdots, N, \\
h_{k+1}(X_k, C_k) &= (1 + \frac{1}{3} \lambda_k^2)(A_k x_k A_k^T + A_k x_k C_k^T B_k^T + B_k C_k x_k A_k^T + B_k C_k x_k C_k^T B_k^T) - X_{k+1}, \quad k = 0, 1, \cdots, N-1.
\end{aligned}
\]
The Lagrangian function is defined as follows
\[
\mathcal{L} = J(X_0, C) + \sum_{k=1}^{N} \gamma_k g_k(X_k) + \sum_{k=0}^{N-1} \text{tr}[H_{k+1} h_{k+1}(X_k, C_k)],
\]
where \(\gamma_k \geq 0 \in \mathbb{R}\) \((k = 1, 2, \cdots, N)\) are the Lagrangian multiplier of the inequality constraints \(g_k(X_k) \leq 0\) \((k = 1, 2, \cdots, N)\), and the matrices \(H_{k+1} (k = 0, 1, \cdots, N - 1)\) are the Lagrangian multipliers of the equality constraints \(h_{k+1}(X_k, C_k) = 0\) \((k = 0, 1, \cdots, N - 1)\).

By means of maximum principle with mixed inequality constraints [8], we have
\[
\frac{\partial \mathcal{L}}{\partial C_k} = 0 \quad (k = 0, 1, \cdots, N - 1),
\]
(11)
and the complementary slackness condition
\[
\gamma_k [\text{tr}(X_k) - c_k] = 0 \quad (k = 1, 2, \cdots, N).
\]
(13)
Based on the partial rule of gradient matrices [1], (11) can be transformed into
\[
[R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k] C_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} A_k = 0.
\]
(14)
Let
\[
\begin{cases}
L_k = R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k \\
M_k = (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} A_k.
\end{cases}
\]
(15)
Then we have \(L_k L_k^+ M_k = M_k\) and \(L_k C_k + M_k = 0\). Applying Lemma 2.8, the general solution of (14) is given by
\[
C_k = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k, \quad Y_k \in \mathbb{R}^{m \times n}
\]
(16)
for \(k = 0, 1, \cdots, N - 1\).

By (12), firstly we have
\[
H_N = \frac{\partial \mathcal{L}}{\partial X_N}
\]
(17)
for \(k = N\), which leads to
\[
H_N = Q_N + \gamma N I.
\]
(18)
Secondly, we obtain
\[
H_k = \frac{\partial \mathcal{L}}{\partial X_k}
\]
for \(k = 0, 1, \cdots, N - 1\). That is
\[
H_k = Q_k + (1 + \frac{1}{3} \lambda_k^2) A_k^T H_{k+1} A_k + C_k^T [R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k] C_k
\]
\[
+ (1 + \frac{1}{3} \lambda_k^2) A_k^T H_{k+1} B_k C_k + (1 + \frac{1}{3} \lambda_k^2) C_k^T B_k^T H_{k+1} A_k + \gamma_k I.
\]
(19)
Substituting (16) into (19), it follows
\[
H_k = Q_k + (1 + \frac{1}{3} \lambda_k^2) A_k^T H_{k+1} A_k + \gamma_k I - M_k^+ L_k^+ M_k.
\]
(20)
The associated objective functional

\[ J(x_0, u) = \sum_{k=0}^{N-1} E \left[ x_k^T Q_k x_k + u_k^T R_k u_k \right] + E \left[ x_N^T Q_N x_N \right] \]

by applying (18) and (20), we obtain

\[ \lambda \]

Denote the unitary eigenvector with respect to \( \lambda \).

A completion of square implies

\[ \text{tr} \left[ (Q_k + C_k^T R_k C_k) x_k \right] + \text{tr} \left[ H_{k+1} x_{k+1} \right] - tr \left[ H_k x_k \right] \]

\[ + tr \left[ (Q_N - H_N) x_N \right] + x_0^T H_0 x_0. \]

Since \( x_{k+1} = (1 + \frac{1}{3} \lambda_k^2) (A_k x_k B_k^T + B_k C_k x_k A_k^T + B_k C_k x_k C_k^T B_k^T), \)

by applying (18) and (20), we obtain

\[ J(X_0, C) \]

\[ = \sum_{k=0}^{N-1} \left\{ \text{tr} \left[ (Q_k + C_k^T R_k C_k) + (1 + \frac{1}{3} \lambda_k^2) (A_k^+ H_{k+1} A_k + C_k^T B_k^+ H_{k+1} A_k \right. \]

\[ + A_k^+ H_{k+1} B_k C_k + C_k^T B_k^+ H_{k+1} B_k C_k) - H_k \right\} X_k \right\} + \text{tr} \left[ (Q_N - H_N) X_N \right] \]

\[ + x_0^T H_0 x_0 \]

\[ = \sum_{k=0}^{N-1} \text{tr} \left\{ \left[ (1 + \frac{1}{3} \lambda_k^2) A_k^+ H_{k+1} A_k - H_k \right] + (1 + \frac{1}{3} \lambda_k^2) C_k^T B_k^+ H_{k+1} A_k \right. \]

\[ + (1 + \frac{1}{3} \lambda_k^2) A_k^+ H_{k+1} B_k C_k + C_k^T \left[ R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^+ H_{k+1} B_k \right] C_k \} X_k \right\} \]

\[ - tr(\gamma_N X_N) + x_0^T H_0 x_0 \]

\[ = \sum_{k=0}^{N-1} \text{tr} \left[ (M_k^T L_k^T + M_k + M_k^T C_k + C_k^T L_k C_k) x_k \right] - \sum_{k=1}^{N} tr(\gamma_k x_k) + x_0^T H_0 x_0. \]

A completion of square implies

\[ J(X_0, C) \]

\[ = \sum_{k=0}^{N-1} \text{tr} \left[ (C_k + L_k^T M_k)^T L_k (C_k + L_k^T M_k) x_k \right] - \sum_{k=1}^{N} tr(\gamma_k x_k) + x_0^T H_0 x_0. \]  

(21)

Substituting (13) into (21) yields that

\[ J(X_0, C) \]

\[ = \sum_{k=0}^{N-1} \text{tr} \left[ (C_k + L_k^T M_k)^T L_k (C_k + L_k^T M_k) x_k \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0. \]

Next, we prove that \( L_k \) must satisfy

\[ L_k = R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^+ H_{k+1} B_k \geq 0, \text{ for } k = 0, 1, \cdots, N - 1. \]  

(22)

If it is not so, there is a \( L_p \) for \( p \in \{0, 1, \cdots, N - 1\} \) with a negative eigenvalue \( \lambda \). Denote the unitary eigenvector with respect to \( \lambda \) as \( v_\lambda \) (i.e., \( v_\lambda^T v_\lambda = 1 \) and
$L_p v_\lambda = \lambda v_\lambda$). Let $\delta \neq 0$ be an arbitrary scalar and construct a control sequence $\hat{u} = (u_1, \hat{u}_2, \cdots, \hat{u}_{N-1})$ as follows
\[
\hat{u}_k = \begin{cases} 
-L_k^+ M_k x_k, & k \neq p \\
\delta |\lambda|^{-\frac{1}{2}} v_\lambda - L_k^+ M_k x_k, & k = p.
\end{cases}
\] (23)

The associated cost functional becomes
\[
J(x_0, \hat{u})
= \sum_{k=0}^{N-1} tr \left[ (\hat{C}_k + L_k^+ M_k)^T L_k (\hat{C}_k + L_k^+ M_k) X_k \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0
= \sum_{k=0}^{N-1} E \left[ (\hat{u}_k + L_k^+ M_k x_k)^T L_k (\hat{u}_k + L_k^+ M_k x_k) \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0
= \left[ \frac{\delta}{|\lambda|^{\frac{1}{2}}} v_\lambda \right]^T \left[ \frac{\delta}{|\lambda|^{\frac{1}{2}}} v_\lambda \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0
= -\delta^2 - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0.
\]

Letting $\delta \to \infty$, it yields $J(x_0, \hat{u}) \to -\infty$, which contradicts the well-posedness of problem (1).

3.4. Special cases. We have obtained that $L_k \geq 0$ in the constrained difference equation (9) of Theorem 3.4. The following corollaries are special cases of the above result if we let $L_k > 0$ and $L_k = 0$, respectively.

**Corollary 1.** The indefinite LQ problem (1) is uniquely solvable if and only if $L_k > 0$ for $k = 0, 1, \cdots, N - 1$. Moreover, the unique optimal control is given by
\[
u_k = -L_k^{-1} M_k x_k, \text{ for } k = 0, 1, \cdots, N - 1.
\]

**Proof.** By using Theorem 3.4, we immediately obtain the corollary.

**Corollary 2.** If $L_k = 0$ for $k = 0, 1, \cdots, N - 1$, then any admissible control of the indefinite LQ problem (1) is optimal and the constrained difference equation (9) reduces to the following linear system
\[
\begin{align*}
H_k &= Q_k + (1 + \frac{1}{3} \lambda_k^2) A_k^T H_{k+1} A_k + \gamma_k I \\
R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k &= 0 \\
B_k^T H_{k+1} A_k &= 0, \ k = 0, 1, \cdots, N - 1 \\
H_N &= Q_N + \gamma_N I \\
\gamma_k [tr(X_k) - c_k] &= 0, \ k = 1, 2, \cdots, N.
\end{align*}
\] (24)

**Proof.** Firstly, letting $L_k = 0$ in (9), then the constrained difference equation (9) reduces to the linear system (24). Secondly, letting $L_k = 0$ in (21), it is shown that
\[
J(x_0, u) = x_0^T H_0 x_0 - \sum_{k=1}^{N} c_k \gamma_k,
\]
which implies that $V(x_0) = x_0^T H_0 x_0 - \sum_{k=1}^{N} c_k \gamma_k$ for any admissible control. Then any admissible control of the indefinite LQ problem (1) is optimal.
Next we consider the following indefinite LQ control problem with equality constraint on the terminal state as follows:

\[
\begin{aligned}
\inf_{u_k} \quad & J(x_0, u) = \sum_{k=0}^{N-1} \left[ x_k^T Q_k x_k + u_k^T R_k u_k \right] + E \left[ x_N^T Q_N x_N \right] \\
\text{subject to} & \\
& x_{k+1} = A_k x_k + B_k u_k + \lambda_k (A_k x_k + B_k u_k) \xi_k, \quad k = 0, 1, \cdots, N-1, \\
& E[x_N^T x_N] = c, \quad x(0) = x_0.
\end{aligned}
\]  

(25)

Corollary 3. If the indefinite LQ problem (25) is solvable by a feedback control sequence

\[ u_k = C_k x_k, \quad \text{for} \ k = 0, 1, \cdots, N-1, \]  

(26)

where \( C_0, C_1, \cdots, C_{N-1} \) are constant matrices, then there exist symmetric matrices \( H_k \) and a \( \mu \in \mathbb{R} \), such that

\[
\begin{aligned}
H_k &= Q_k + \left(1 + \frac{1}{3} \lambda_k^2\right) A_k^T H_{k+1} A_k - M_k^T L_{k}^+ M_k \\
L_k L_k^+ M_k - M_k &= 0 \\
L_k &= R_k + \left(1 + \frac{1}{3} \lambda_k^2\right) B_k^T H_{k+1} B_k \geq 0 \\
M_k &= (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} A_k \\
H_N &= Q_N + \mu I
\end{aligned}
\]

for \( k = 0, 1, \cdots, N-1 \). Moreover

\[ C_k = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k, \]  

(28)

with \( Y_k \in \mathbb{R}^{m \times n}, \ k = 0, 1, \cdots, N-1 \), being any given matrices. Furthermore, the optimal cost of the indefinite LQ problem (25) is

\[ V(x_0) = x_0^T H_0 x_0 - c \mu. \]

Proof. Let \( \text{tr} [X_N] = c \) in the problem (1) and \( \mu \) be the lagrange multiplier of this equality constraint in Theorem 3.4. According to Theorem 3.4, we known that \( g(X_N) = \text{tr} [X_N] - c = 0 \). By the similar process as in Theorem 3.3 and Theorem 3.4, we obtain the corollary directly.

\[ \square \]

4. Solvability and well-posedness of the indefinite LQ problem.

4.1. Sufficiency of the the constrained difference equation. In this section, it is shown that the solvability of the constrained difference equation (9) is sufficient for the solvability of the indefinite LQ problem (1). Moreover, any optimal control can be obtained via the solution of the constrained difference equation (9).

Theorem 4.1. The indefinite LQ problem (1) is solvable if there exist symmetric matrices \( H_k \) (\( k = 0, 1, \cdots, N-1 \)) and \( \gamma_k \geq 0 \in \mathbb{R} \) (\( k = 1, 2, \cdots, N \)) satisfying the constrained difference equation (9). Moreover, the indefinite LQ problem (1) is solvable by

\[
\begin{aligned}
u_k &= -\left[R_k + \left(1 + \frac{1}{3} \lambda_k^2\right) B_k^T H_{k+1} B_k\right]^+ \left[(1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} A_k\right] x_k, \quad k = 0, 1, \cdots, N-1.
\end{aligned}
\]  

(29)
Proof. Let \( H_k \) and \( \gamma_k \geq 0 \in \mathbb{R} \) solve the constrained difference equation (9). Then
\[
J(x_0, u)
= \sum_{k=0}^{N-1} E \left[ x_k^T Q_k x_k + u_k^T R_k u_k \right] + E \left[ x_N^T Q_N x_N \right]
= \sum_{k=0}^{N-1} \left\{ E \left[ x_k^T Q_k x_k + u_k^T R_k u_k \right] + E \left[ x_{k+1}^T H_{k+1} x_{k+1} \right] - E \left[ x_k^T H_k x_k \right] \right\}
+ E \left[ x_N^T Q_N x_N \right] - E \left[ x_N^T H_N x_N \right] + x_0^T H_0 x_0
= \sum_{k=0}^{N-1} \left\{ tr \left[ (Q_k + C_k^T R_k C_k) x_k \right] + tr \left[ H_{k+1} x_{k+1} \right] - tr \left[ H_k x_k \right] \right\}
+ tr \left[ (Q_N - H_N) x_N \right] + x_0^T H_0 x_0
= \sum_{k=0}^{N-1} \left[ \left( 1 + \frac{1}{3} \lambda_k^2 \right) A_k^T H_{k+1} A_k - H_k \right] + \left( 1 + \frac{1}{3} \lambda_k^2 \right) C_k^T B_k^T H_{k+1} A_k
+ \left( 1 + \frac{1}{3} \lambda_k^2 \right) A_k^T H_{k+1} B_k C_k + C_k^T \left[ R_k + \left( 1 + \frac{1}{3} \lambda_k^2 \right) B_k^T H_{k+1} B_k \right] C_k \right] X_k
- tr(\gamma_N X_N) + x_0^T H_0 x_0
= \sum_{k=0}^{N-1} \left[ M_k^T L_k^T M_k + C_k^T M_k + M_k^T C_k + C_k^T L_k C_k \right] X_k - \sum_{k=1}^{N} tr(\gamma_k X_k) + x_0^T H_0 x_0.

By using (13), a completion of square implies
\[
J(X_0, C)
= \sum_{k=0}^{N-1} tr \left[ (C_k + L_k^T M_k)^T L_k (C_k + L_k^T M_k) X_k \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0. \tag{30}
\]

The (30) shows that the indefinite LQ problem (1) is solvable by the feedback control
\[
u_k = -C_k x_k = -L_k^T M_k x_k, \quad k = 0, 1, \cdots, N - 1.
\]
\]

Remark 4. The Theorem 4.1 can be viewed as the converse of Theorem 3.4, which means that the solvability of the indefinite LQ problem (1) is equivalent to the solvability of the constrained difference equation (9).

4.2. Well-posedness of the indefinite LQ problem. Next we will discuss the well-posedness of the indefinite LQ problem (1).

Theorem 4.2. The indefinite LQ problem (1) is well-posed if there exist symmetric matrices \( H_k \) (\( k = 0, 1, \cdots, N - 1 \)) and \( \gamma_k \geq 0 \in \mathbb{R} \) (\( k = 1, 2, \cdots, N \)) satisfying the constrained difference equation (9). Moreover, the optimal value of the indefinite LQ problem (1) is
\[
V(x_0) = x_0^T H_0 x_0 - \sum_{k=1}^{N} c_k \gamma_k.
\]
Proof. Let $H_k$ ($k = 0, 1, \cdots, N - 1$) and $\gamma_k \geq 0 \in \mathbb{R}$ ($k = 1, 2, \cdots, N$) solve the constrained difference equation (9). The same process as in Theorem 4.1, we obtain

$$J(X_0, C) = \sum_{k=0}^{N-1} \text{tr} \left[ (C_k + L^+_k M_k)^T L_k (C_k + L^+_k M_k) X_k \right] - \sum_{k=1}^{N} c_k \gamma_k + x_0^T H_0 x_0.$$  \hfill (31)

Since $L_k \geq 0$ ($k = 0, 1, \cdots, N - 1$) in (31), we know that the cost function of the indefinite LQ problem (1) is bounded from below by

$$x_0^T H_0 x_0 - \sum_{k=1}^{N} c_k \gamma_k,$$

which means that the indefinite LQ problem (1) is well-posed. By (31), the optimal value of the indefinite LQ problem (1) is

$$V(x_0) = \inf_{C_k, 0 \leq k \leq N-1} J(X_0, C) = x_0^T H_0 x_0 - \sum_{k=1}^{N} c_k \gamma_k, \forall x_0 \in \mathbb{R}^n.$$ 



4.3. General expression for the optimal control set. In the following, we present a general expression for the optimal control set based on the solution of the constrained difference equation (9).

**Theorem 4.3.** Assume that $H_k$ ($k = 0, 1, \cdots, N - 1$) and $\gamma_k \geq 0 \in \mathbb{R}$ ($k = 1, 2, \cdots, N$) solve the constrained difference equation (9). A sufficient and necessary condition that $u_k$ is in the set of all optimal feedback controls for indefinite LQ problem (1) is that

$$u_k = -(L^+_k M_k + Y_k - L^+_k L_k Y_k) x_k + Z_k = L^+_k M_k Z_k, k = 0, 1, \cdots, N - 1,$$ \hfill (32)

where $Y_k \in \mathbb{R}^{m \times n}$ and $Z_k \in \mathbb{R}^m$, $k = 0, 1, \cdots, N - 1$, are arbitrary variables with appropriate size.

**Proof.** Sufficiency. Let $H_k$ ($k = 0, 1, \cdots, N - 1$) and $\gamma_k \geq 0 \in \mathbb{R}$ ($k = 1, 2, \cdots, N$) solve the constrained difference equation (9). The same process as to the Theorem
3.4, we obtain
\[
J(x_0, u) = \sum_{k=0}^{N-1} E[x_k^TQ_kx_k + u_k^TR_ku_k] + E[x_N^TQ_Nx_N] \\
= \sum_{k=0}^{N-1} tr\left\{ Q_k + (1 + \frac{1}{3}\lambda_k^2)A_k^TH_{k+1}A_k - H_k \right\} + (1 + \frac{1}{3}\lambda_k^2)C_k^TB_k^TH_{k+1}A_k \\
+ (1 + \frac{1}{3}\lambda_k^2)A_k^TH_{k+1}B_kC_k + C_k^TR_k + (1 + \frac{1}{3}\lambda_k^2)B_k^TH_{k+1}B_k \right\} C_k \} X_k \\
- tr(\gamma_NX_N) + x_0^TH_0x_0 \\
= \sum_{k=0}^{N-1} E[x_k^TM_k^TL_k^TM_kx_k + 2x_k^TM_k^Tu_k + u_k^TL_ku_k] - \sum_{k=1}^{N} c_k\gamma_k + x_0^TH_0x_0. \quad (33)
\]

By denoting \( T_k^1 = -(Y_k - L_k^+L_kY_k) \) and \( T_k^2 = -(Z_k - L_k^+L_kZ_k) \), we have
\[
L_kT_k^1 = 0, \quad L_kT_k^2 = 0. \quad (34)
\]

According to the last formula of (33) and (34), we obtain
\[
J(x_0, u) = \sum_{k=0}^{N-1} E[u_k + (L_k^+M_k + T_k^1)x_k + T_k^2]^TL_k[u_k + (L_k^+M_k + T_k^1)x_k + T_k^2] \\
- \sum_{k=1}^{N} c_k\gamma_k + x_0^TH_0x_0.
\]

Since \( L_k \geq 0 \) for \( k = 0, 1, \ldots, N-1 \), we know that the control \( u_k = -[(L_k^+M_k + T_k^1)x_k + T_k^2] \) minimizes \( J(x_0, u) \) with the optimal value \(-\sum_{k=1}^{N} c_k\gamma_k + x_0^TH_0x_0 \) for \( k = 0, 1, \ldots, N-1 \).

Necessity. If any control sequence \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_{N-1}) \) which minimizes the cost function \( J(x_0, u) \). Then we have
\[
J(x_0, \tilde{u}) = \sum_{k=0}^{N-1} E[(\tilde{u}_k + L_k^+M_kx_k)^TL_k(\tilde{u}_k + L_k^+M_kx_k)] - \sum_{k=1}^{N} c_k\gamma_k + x_0^TH_0x_0,
\]
for \( k = 0, 1, \cdots, N-1 \). The above equality implies that
\[
\sum_{k=0}^{N-1} E[(\tilde{u}_k + L_k^+M_kx_k)^TL_k(\tilde{u}_k + L_k^+M_kx_k)] = 0, \quad k = 0, 1, \cdots, N-1.
\]

Since \( L_k \geq 0 \) for \( k = 0, 1, \cdots, N-1 \), we get the following equivalent condition
\[
L_k(\tilde{u}_k + L_k^+M_kx_k) = 0, \quad k = 0, 1, \cdots, N-1.
\]
We see that $\tilde{u}_k$ solves the following equation

$$L_k \tilde{u}_k + L_k L_k^T M_k x_k = 0, \ k = 0, 1, \ldots, N - 1. \tag{35}$$

By using Lemma 2.8 with $L = L_k, M = I, N = -L_k L_k^T M_k x_k$, it is easy to verify that

$$LL^+ M MM^+ = N.$$ 

We obtain the solution of (35) with

$$\tilde{u}_k = -L_k^+ M_k x_k + Z_k - L_k^+ L_k Z_k, \ Z_k \in \mathbb{R}^m, \ k = 0, 1, \ldots, N - 1.$$ 

As in (16), the optimal control can be represented by

$$u_k = -(L_k^+ M_k + Y_k - L_k^+ L_k Y_k) x_k + Z_k - L_k^+ M_k Z_k, \ k = 0, 1, \ldots, N - 1.$$ 

\[\Box\]

5. **Numerical example.** In this section, we report our numerical experiments based on the approach developed in the previous sections. In the constrained indefinite LQ control problem for discrete-time uncertain systems, we give out a set of specific parameters of the coefficients as follows:

$$x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ E[x_1 x_1^T] \leq c_1 = 2, \ E[x_2 x_2^T] \leq c_2 = 2.25, \ N = 2,$$

and

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ B_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \lambda_0 = -\frac{\sqrt{6}}{2}, \ \lambda_1 = \frac{\sqrt{6}}{2}.$$ 

The state weights and the control weights are as follows

$$Q_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ Q_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \ Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ R_0 = 4, \ R_1 = -1.$$ 

According to the necessary condition of Theorem 3.4, we obtain

$$\begin{cases} H_k = Q_k + (1 + \frac{1}{3} \lambda_k^2) A_k^T H_{k+1} A_k + \gamma_k I - M_k^T L_k^T M_k \\ L_k L_k^T M_k - M_k = 0 \\ L_k = R_k + (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k \geq 0 \\ M_k = (1 + \frac{1}{3} \lambda_k^2) B_k^T H_{k+1} B_k, \ k = 0, 1. \tag{36} \\ H_2 = Q_2 + \gamma_2 I \\ \gamma_k [tr(X_k) - c_k] = 0, \ \gamma_k \geq 0 \text{ and } tr(X_k) \leq 0, \ k = 1, 2, \\ X_{k+1} = (1 + \frac{1}{3} \lambda_k^2)(A_k X_k A_k^T + A_k X_k C_k^T B_k^T + B_k C_k X_k A_k^T) + B_k C_k X_k C_k^T B_k^T, \ k = 0, 1, X_0 = x_0 x_0^T. \end{cases}$$

Firstly, we have

$$X_0 = x_0 x_0^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Secondly, we solve the equations (36) as the following four cases: (i) $\gamma_1 = 0, \ \gamma_2 = 0$; (ii) $\gamma_1 = 0, \ \gamma_2 > 0$; (iii) $\gamma_1 > 0, \ \gamma_2 = 0$; (iv) $\gamma_1 > 0, \ \gamma_2 > 0$. The analysis of above cases show that there are no feasible solution about $X_1$ and $X_2$ in cases
We can obtain a group of feasible solutions about \( X_1 \) and \( X_2 \) by solving out \( \gamma_1 = 0 \) and \( \gamma_2 = 2 \) in case (ii), which are given below by

\[
X_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1.5 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 2.25 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then we obtain the terminal condition

\[
H_2 = Q_2 + \gamma_2 I = \gamma_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.
\]

In order to find the optimal controls and optimal cost value in this example, we will construct the optimal feedback control law by \( u_k = C_k x_k \) (\( k = 1, 0 \)) stage by stage in reverse order.

Stage 2: For \( k = 1 \), we obtain

\[
L_1 = R_1 + (1 + \frac{1}{3} \lambda_1^2) B_1^T H_2 B_1 = 2 > 0, \quad M_1 = (1 + \frac{1}{3} \lambda_1^2) B_1^T H_2 A_1 = (3, 0),
\]

\[
H_1 = Q_1 + (1 + \frac{1}{3} \lambda_1^2) A_1^T H_2 A_1 + \gamma_1 I - M_1^T L_1^{-1} M_1 = \begin{pmatrix} -2.5 & 0 \\ 0 & 3 \end{pmatrix}.
\]

The optimal feedback control is \( u_1 = C_1 x_1 \) where

\[
C_1 = -L_1^{-1} M_1 = (-1.5, 0).
\]

Stage 1: For \( k = 0 \), we have

\[
L_0 = R_0 + (1 + \frac{1}{3} \lambda_0^2) B_0^T H_1 B_0 = 0.25 > 0, \quad M_0 = (1 + \frac{1}{3} \lambda_0^2) B_0^T H_1 A_0 = (-3.75, 0),
\]

\[
H_0 = Q_0 + (1 + \frac{1}{3} \lambda_0^2) A_0^T H_1 A_0 - M_0^T L_0^{-1} M_0 = \begin{pmatrix} -61 & 0 \\ 0 & 3.5 \end{pmatrix}.
\]

The optimal feedback control is \( u_0 = C_0 x_0 \) where

\[
C_0 = -L_0^{-1} M_0 = (15, 0).
\]

Finally, the optimal cost value is

\[
V(x_0) = x_0^T H_0 x_0 - \sum_{k=1}^{2} c_k \gamma_k = -1.
\]

**Remark 5.** Note that in this example, the state weight \( Q_0 \) is negative definite, \( Q_1 \) is negative semidefinite, and \( Q_2 \) is positive semidefinite, the control weight \( R_1 \) is negative definite.

6. **Conclusion.** This paper has investigated the constrained indefinite LQ control for discrete-time systems with state and control dependent on uncertain noise. We first transform the indefinite LQ control problem into an equivalent deterministic optimal control problem. Then, we present a necessary condition for the existence of optimal linear state feedback control by means of the maximum principle with mixed inequality constraints. Moreover, the existence of a solution to the constrained difference equation is equivalent to the solvability of the indefinite LQ problem. Furthermore, the well-posedness of the indefinite LQ problem is discussed. For future work, we will consider indefinite LQ control in infinite time horizon for discrete-time uncertain systems.
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