Extended-valued topical and anti-topical functions on semimodules

Ivan Singer
Institute of Mathematics, P. O. Box 1-764, Bucharest, Romania
E-mail address: ivan.singer@imar.ro

Viorel Nitica
Department of Mathematics, West Chester University, West Chester, PA 19383 and Institute of Mathematics, P. O. Box 1-764, Bucharest, Romania
E-mail address: vnitica@wcupa.edu

Abstract

In the papers [I. Singer, Lin. Alg. Appl. 2010] and [I. Singer and V. Nitica, Lin. Alg. Appl. 2012] we have studied functions defined on a \( b \)-complete idempotent semimodule \( X \) over a \( b \)-complete idempotent semifield \( \mathcal{K} = (\mathcal{K}, \oplus, \otimes) \), with values in \( \mathcal{K} \), where \( \mathcal{K} \) may (or may not) contain a greatest element \( \sup \mathcal{K} \), and the residuation \( x/y \) is not defined for \( x \in X \) and \( y = \inf X \). In the present paper we assume that \( \mathcal{K} \) has no greatest element, then adjoin to \( \mathcal{K} \) an outside “greatest element” \( \top = \sup \mathcal{K} \) and extend the operations \( \oplus \) and \( \otimes \) from \( \mathcal{K} \) to \( \mathcal{K}_+ := \mathcal{K} \cup \{ \top \} \), so as to obtain a meaning also for \( x/\inf X \), for any \( x \in X \), and study functions with values in \( \mathcal{K}_+ \). In fact we consider two different extensions of the product \( \otimes \) from \( \mathcal{K} \) to \( \mathcal{K}_+ \), denoted by \( \otimes \) and \( \hat{\otimes} \) respectively, and use them to give characterizations of topical (i.e. increasing homogeneous, defined with the aid of \( \otimes \)) and anti-topical (i.e. decreasing anti-homogeneous, defined with the aid of \( \hat{\otimes} \)) functions in terms of some inequalities. Next we introduce and study for functions \( f : X \rightarrow \mathcal{K}_+ \) their conjugates and biconjugates of Fenchel-Moreau type with respect to the coupling functions \( \varphi(x,y) = x/y, \forall x, y \in X \), and \( \psi(x, (y, d)) := \inf \{ x/y, d \}, \forall x, y \in X, \forall d \in \mathcal{K}_+ \), and use them to obtain characterizations of topical and anti-topical functions. In the subsequent sections we consider for the coupling functions \( \varphi \) and \( \psi \) some concepts that have been studied in [A. M. Rubinov and I. Singer, Optimization, 2001] and [I. Singer, Optimization, 2004] for the so-called “additive min-type coupling functions” \( \pi_\mu : \mathbb{R}_+^n \times \mathbb{R}_+^m \rightarrow \mathbb{R}_+^m \) and \( \pi_\mu : A^n \times A^m \rightarrow A \) respectively, where \( A \) is a conditionally complete lattice ordered group and \( \pi_\mu(x, y) := \inf_{1 \leq i \leq n} (x_i + y_i), \forall x, y \in \mathbb{R}_+^m \) (or \( A^m \)). Thus, we study the polars of a set \( G \subseteq X \) for the coupling functions \( \varphi \) and \( \psi \), and we consider for a function \( f : X \rightarrow \mathcal{K}_+ \) the notion of support set of \( f \) with respect to the set \( \mathcal{T} \) of all “elementary topical functions”
\[\tilde{\iota}_y(x) := \frac{x}{y}, \forall x \in X, \forall y \in X \setminus \{\inf X\}\] and two concepts of support set of \(f\) at a point \(x_0 \in X\). The main differences between the properties of the conjugations with respect to the coupling functions \(\varphi, \psi\) and \(\pi_\mu\) and between the properties of the polars of a set \(G\) with respect to the coupling functions \(\varphi, \psi\) and \(\pi_\mu\) are caused by the fact that while \(\pi_\mu\) is symmetric, with values only in \(R_{\max}\) \((\text{resp. } A)\), \(\varphi\) and \(\psi\) are not symmetric and take values also outside \(R_{\max}\) \((\text{resp. } A)\).

**Key Words:** Semifield, semimodule, \(b\)-complete, extended product, extended-valued function, topical function, anti-topical function, elementary topical function, Fenchel-Moreau conjugate, biconjugate, support function, polar, bipolar, support set, downward set, subdifferential

**AMS classification:** Primary 06F07, 26B25; Secondary 52A01, 06F20

### 1 Introduction

In the previous papers \[16\] and \[17\], attempting to contribute to the construction of a theory of functional analysis and convex analysis in semimodules over semifields, we have studied topical functions over semifields, we have studied topical functions over a \(b\)-complete idempotent semifield \(K\). We recall that \(f : X \to K\) is called \textit{topical} if it is \textit{increasing} (i.e., the relations \(x', x'' \in X, x' \leq x''\) imply \(f(x') \leq f(x'')\), where \(\leq\) denotes the canonical order on \(K\), respectively on \(X\), defined by \(\lambda \leq \mu \iff \lambda \oplus \mu = \mu, \forall \lambda \in K, \forall \mu \in K\), respectively by \(x \leq y \iff x \oplus y = y, \forall x \in X, \forall y \in X\), and \textit{homogeneous} \((\text{i.e., } f(\lambda x) = \lambda f(x)\) for all \(x \in X, \lambda \in K\), where \(\lambda x \equiv \lambda \otimes x, \lambda f(x) = \lambda \otimes f(x)\); the fact that we use the same notations for addition \(\oplus\) both in \(K\) and in \(X\) and for multiplication \(\otimes\) both in \(K\) and in \(K \times X\) will lead to no confusion). These definitions will be used also when \(K\) is replaced by \(R = ((-\infty, +\infty), \oplus = \max, \otimes = +)\) although it is not a semifring, and \(X\) is replaced by \(R^n\). Let us also recall that an idempotent semiring \(K\), that is, a semifring with idempotent addition \(\oplus\) (i.e. such that \(\lambda \oplus \lambda = \lambda\) for all \(\lambda \in K\)) or an idempotent semimodule \(X\) (over an idempotent semiring \(K\)) is called \textit{\(b\)-complete}, if it is closed under the sum \(\oplus\) of any subset \((\text{order-})\) bounded from above and the multiplication \(\otimes\) distributes over such sums.

As in \[16\] and \[17\], we shall make the following \textit{basic assumptions}:

\[A(0') K = (K, \oplus, \otimes)\] is a \(b\)-\textit{complete} idempotent semifield \((\text{i.e., a } b\)-\textit{complete idempotent semiring in which every } \mu \in K \setminus \{\varepsilon\} \text{ is invertible for the multiplication } \otimes, \text{ where } \varepsilon \text{ denotes the neutral element of } (K, \oplus))\), and the supremum of each \((\text{order-})\) bounded from above subset of \(K\) belongs to \(K\); also, \(X\) is a \(b\)-\textit{complete idempotent semimodule over }\(K\). In the sequel we shall omit the word “idempotent”; this will lead to no confusion.

\[A(1)\] For all elements \(x \in X\) and \(y \in X \setminus \{\inf X\}\) the set \(\{\lambda \in K | \lambda y \leq x\}\) is \((\text{order-})\) bounded from above, where \(\leq\) denotes the canonical order on \(K\), respectively on \(X\).
Remark 1  
a) It is easy to see that an idempotent semifield $K$ has no greatest element $\sup K$, unless $K = \{\varepsilon\}$ or $K = \{\varepsilon, e\}$, where $\varepsilon$ and $e$ denote the neutral elements of $(K, \oplus)$ and $(K, \otimes)$ respectively [2] p. 27, Remark 2.1.2.5]. Indeed, let $K \neq \{\varepsilon\}$ and $K \neq \{\varepsilon, e\}$ and assume, a contrario, that $\sup K \in K$. Then

$$\sup K = (\sup K) \otimes e \leq (\sup K) \otimes (\sup K) \leq \sup K,$$

whence $(\sup K) \otimes (\sup K) = \sup K$; since $(\sup K)^{-1} \neq e$, multiplying both sides of this equality with $(\sup K)^{-1}$ we obtain $\sup K = e$, so $\lambda \leq e$ for all $\lambda \in K$. Since $K$ is a semifield, replacing here each $\lambda \in K \setminus \{\varepsilon\}$ by $\lambda^{-1}$ it follows that $\lambda \geq e$ and hence $\lambda = e$, for all $\lambda \in K \setminus \{\varepsilon\}$. Consequently, $K = \{\varepsilon, e\}$, in contradiction with our assumption. The converse statement is obvious: if $K = \{\varepsilon, e\}$, then $\sup K = e \in K$.

In the sequel, without any special mention, we shall assume that $K$ has no greatest element, since for $K = \{\varepsilon\}$ the statements are trivial and for $K = \{\varepsilon, e\}$ the subsequent results remain valid with similar but simpler proofs.

b) $K$ is commutative, by $(A0')$ and Iwasawa’s theorem (see e.g. [3]).

An important example of a pair $(X, K)$ satisfying $(A0')$ and $(A1)$ is obtained by taking

$$X = R^n_{\max} := ((R \cup \{-\infty\})^n, \oplus := \max, \otimes := +) \quad (1)$$

(with max and $+$ understood componentwise), and $K := R^1_{\max}$. The results of [11] on $R^n := (R, \oplus := \max, \otimes := +)^n$ can and will be expressed in the sequel as results on $R^n_{\max}$ replacing $R$ by $R \cup \{-\infty\}$ endowed with the usual operations max and $+$. Also, as has been observed in [11], many results on $R^n$ remain valid, essentially with the same proofs, for $X = R^n_{\max}$, the set of all bounded vectors $x = (x_i)_{i \in I}$ where $I$ is an arbitrary index set and $x_i \in R, \forall i \in I$, $\sup_{i \in I} |x_i| < +\infty$, endowed with the componentwise semimodule operations $x' \oplus x'' := (\max(x'_i, x''_i))_{i \in I}, \lambda x = (\lambda x_i)_{i \in I}$ and the componentwise order relation $x' \leq x'' \iff x'_i \leq x''_i, \forall i \in I$.

One of the main tools in [16] and [17] has been residuation. We recall that by $(A0')$ and $(A1)$, for each $y \in X \setminus \{\inf X\}$ (hence such that the set $\{\lambda \in K | \lambda y \leq x\}$ is (order-) bounded from above, by $(A1)$) there exists the residuation operation / defined by

$$x/y := \max\{\lambda \in K | \lambda y \leq x\}, \quad \forall x \in X, \forall y \in X \setminus \{\inf X\}, \quad (2)$$

where max denotes a supremum that is attained, and it has, among others, the following properties (see e.g. [4]):

$$(x/y)y \leq x, \quad \forall x \in X, \forall y \in X \setminus \{\inf X\}, \quad (3)$$

$$y/y = e, \quad \forall y \in X \setminus \{\inf X\}, \quad (4)$$

$$x/(\mu y) = \mu^{-1}(x/y), \quad \forall x \in X, \forall \mu \in K \setminus \{\varepsilon\}, \forall y \in X \setminus \{\inf X\}. \quad (5)$$

In [16] and [17] we have considered only functions $f : X \to K$, where $(X, K)$ is a pair satisfying $(A0')$ and $(A1)$. Therefore under the assumption of Remark
above \(x/\inf X\), i.e. the residuation \(x/y\) of \(2\) for \(x \in X\) and \(y = \inf X\), is not defined. In the present paper we shall adjoin to \(K\) an outside “greatest element” \(\sup K\) which we shall denote by \(\top\), and extending in a suitable way the operations \(+\) and \(\otimes\) from \(K\) to \(\overline{K} := K \cup \{\top\}\), we shall then study functions \(f : X \to \overline{K} := K \cup \{\top\}\), that one may call “extended-valued functions”; for example, we shall study topical (i.e. increasing homogeneous) and anti-topical (i.e. decreasing anti-homogeneous) functions defined on a semimodule \(X\) over \(K\) with values in \(\overline{K} := K \cup \{\top\}\). Naturally, the extension of the sum operation to sets \(M \subseteq \overline{K}\), which we shall denote again by \(+\), must be \(\oplus M := \sup M\) (in \(K\)) if \(M\) is bounded from above in \(K\) and \(\oplus M := \top = \sup K\) if \(M\) is not bounded from above in \(K\). Also, generalizing the lower addition \(+\) and upper

addition \(+\) on \(R\), of Moreau (see e.g. \([11]\)) we shall give two different extensions of the product \(\otimes\) from \(K\) to \(\overline{K}\), denoted respectively by \(\otimes\) (which will cause no confusion) and \(\otimes\), that are dual to each other in a certain sense, so as to obtain a meaning also for \(x/\inf X\), with any \(x \in X\), and in particular for \(\inf X/\inf X\).

Let us mention that in the particular case of the pair \((X,K) = (R^a,R^b)\) such extended products and a table of the values of the residuations \(x/y\) for all \(x,y \in R^\max \cup \{+\infty\}\) have been given in \([1,\text{Table 1}]\).

First, in Section 2 we shall introduce and study the extended addition \(+\) and the extended products \(\otimes\) and \(\odot\) in \(\overline{K} := K \cup \{\top\}\) and in Section 3 we shall use them to give characterizations of topical and anti-topical functions \(f : X \to \overline{K} := K \cup \{\top\}\) in terms of some inequalities. For the case of topical functions these characterizations extend some results of \([14,\text{Lemma 2.2}]\) each one of the products \(\otimes\) and \(\odot\) determines uniquely the other, the simple observation that a function \(f : X \to \overline{K}\) is anti-topical if and only if the function \(h(x) := f(x)^{-1}, \forall x \in X\), is topical, will permit us to deduce results on anti-topical functions from those of topical functions.

Another domain where topical and anti-topical functions \(f : X \to \overline{K} := K \cup \{\top\}\) play an important role is that of conjugate functions of Fenchel-Moreau type with respect to coupling functions \(\pi : X \times X \to K\), defined by \(f^\pi(y) := \sup_{x \in X} f(x)^{-1}\pi(x,y), \forall y \in X\). In Section 4 the extended products \(\otimes\) and \(\odot\) will permit us to introduce for functions \(f : X \to \overline{K} := K \cup \{\top\}\) their conjugates and biconjugates with respect to the coupling functions \(\varphi(x,y) = x/y, \forall x \in X, \forall y \in X, \text{and } \psi(x,(y,d)) := \inf\{x/y,d\}, \forall x \in X, \forall y \in X, \forall d \in \overline{K}\), and to use them for the study of topical and anti-topical functions. We shall also consider the “lower conjugates” of \(f\) with respect to these coupling functions, defined with the aid of the product \(\odot\), that are useful for the study of biconjugates. The main differences between the properties of the conjugations with respect to the coupling functions \(\varphi, \psi\) and the so-called “additive min-type coupling functions” \(\pi_{\mu} : R^\max \times R^\max \to R^\max\) resp. \(\pi_{\mu} : A^a \times A^a \to A\), where \(A\) is a conditionally complete lattice ordered group, studied previously e.g. in \([11]\), respectively \([15]\), defined by \(\pi_{\mu}(x,y) := \inf_{1 \leq i \leq n}(x_i \otimes y_i), \forall x = (x_i) \in R^\max\) (resp. \(A^n\), \(\forall y = (y_i) \in R^\max\) (resp. \(A^n\), are caused by the fact that while \(\pi_{\mu}\) is “symmetric” (i.e., \(\pi_{\mu}(x,y) = \pi_{\mu}(y,x), \forall x \in X, \forall y \in X\) and takes values only in \(R^\max\) (resp. \(A\)), \(\varphi\) and \(\psi\) are not symmetric and take also the value
$+\infty$ (resp. $\top$); for example, since $\pi_{\mu}(x, y)$ is topical both in $x$ and in $y$, while $\varphi(x, y)$ is topical as a function of $x$ and anti-topical as a function of $y$, it follows that while $f^{c(\pi_{\mu})} : R_{\max}^n \to \mathbb{R}$ is always a topical function, the conjugate function $f^{c(\varphi)} : X \to \mathcal{K}$ is always anti-topical. Note that the notions of "conditionally complete lattice ordered group" and "b-complete idempotent semifield" are equivalent (the only difference is the zero which is not difficult to add to the first notion).

In the subsequent sections we shall consider for the coupling functions $\varphi$ and $\psi$ some concepts that have been studied previously for the additive min-type coupling functions $\pi_{\mu} : R_{\max}^n \times R_{\max}^m \to R_{\max}$ and $\pi_{\mu} : A^n \times A^m \to A$, in [11] and [15] respectively. Thus, in Section 5 we shall study the polar set of a set $G \subseteq X$ with respect to the coupling functions $\varphi$ and $\psi$, and in Section 6 we shall consider the support set of a function $f : X \to \mathcal{K}$ with respect to the set $\tilde{T}$ of all "elementary topical functions" $\tilde{t}_y(x) := x/y$, $\forall x \in X$, $\forall y \in X \setminus \{\inf X\}$ and two concepts of support set of $f : X \to \mathcal{K}$ at a point $x_0 \in X$. While for functions $f : X \to \mathcal{K}$ the theory of conjugations $f \to f^{c(\varphi)}$ is of interest, we shall show that for subsets $G$ of $X$ the theory of polarities $G \to G_{\text{pol}(\varphi)}$ permits to obtain some relevant results. Similarly to the case of conjugates of functions, the main differences between the properties of the polar set of a set $G$ with respect to the coupling functions $\varphi, \psi$ and the additive min-type coupling function $\pi_{\mu} : R_{\max}^n \times R_{\max}^m \to R_{\max}$, are caused by the fact that while $\pi_{\mu}$ is symmetric and takes only values in $R_{\max}$, $\varphi$ and $\psi$ are not symmetric and take also the value $+\infty$ (resp. $\top$).

2 Extension of $\mathcal{K}$ to $\mathcal{K} = \mathcal{K} \cup \{\top\}$

**Definition 2** Let $\mathcal{K} = (\mathcal{K}, \oplus, \otimes)$ be a $b$-complete semifield that has no greatest element. We shall adjoin to $\mathcal{K}$ an outside element, which we shall denote by $\top$, and we shall extend the canonical order $\leq$ and the addition $\oplus$ from $\mathcal{K}$ to an (canonical) order $\leq$ and an addition $\oplus$ on $\mathcal{K} = \mathcal{K} \cup \{\top\}$ by

\[
\begin{align*}
\varepsilon \leq \alpha & \leq \top, \quad \forall \alpha \in \mathcal{K}, \\
\alpha \oplus \top &= \top \oplus \alpha = \top, \quad \forall \alpha \in \mathcal{K};
\end{align*}
\]

hence the equivalence $\alpha \leq \beta \iff \alpha \oplus \beta = \top$ remains valid for all $\alpha, \beta \in \mathcal{K}$. Furthermore, we shall extend the multiplication $\otimes$ from $\mathcal{K}$ to $\mathcal{K} := \mathcal{K} \cup \{\top\}$ to two multiplications $\otimes$ and $\hat{\otimes}$ by the following rules:

\[
\begin{align*}
\alpha \otimes \beta &= \alpha \otimes \beta, \quad \forall \alpha \in \mathcal{K}, \forall \beta \in \mathcal{K}, \\
\alpha \hat{\otimes} \top &= \top \hat{\otimes} \alpha = \top, \quad \forall \alpha \in \mathcal{K}, \\
\alpha \otimes \top &= \top \otimes \alpha = \top, \quad \forall \alpha \in \mathcal{K} \setminus \{\varepsilon\}, \\
\alpha \otimes \varepsilon &= \varepsilon \otimes \alpha = \varepsilon, \quad \forall \alpha \in \mathcal{K}.
\end{align*}
\]

We shall often denote the extended product $\otimes$ also by concatenation, which will cause no confusion.
For the inverses in $K$ with respect to $\otimes$ we shall make the convention
\[ \varepsilon^{-1} := T, \quad T^{-1} := \varepsilon, \]  
whence, by the above,
\[ \varepsilon^{-1}\varepsilon = T\varepsilon = \varepsilon \neq e, \quad \varepsilon^{-1}\varepsilon = T\varepsilon = T \neq e, \]  
\[ T^{-1}T = \varepsilon T = \varepsilon \neq e, \quad T^{-1}T = \varepsilon T = T \neq e. \]  
We shall call the set $K = K \cup \{T\}$ endowed with the operations $\oplus, \otimes$ and $\dot{\otimes}$ the minimal enlargement of $K$.

Remark 3 a) The product $\dot{\otimes}$ on $K$ is associative, i.e. we have
\[ (\alpha \dot{\otimes} \beta) \dot{\otimes} \gamma = \alpha \dot{\otimes} (\beta \dot{\otimes} \gamma), \quad \forall \alpha, \beta, \gamma \in K. \]  
Indeed, if $T$ occurs as a term in one side of (15), then that side must be equal to $T$ (by (9)) and hence one of the terms of the other side of (15), too, must be equal to $T$ (since if $\lambda, \mu \in K$ then $\lambda \otimes \mu = \lambda \otimes \mu \in K$ by (8), so $\lambda \otimes \mu \neq T$). On the other hand, if $T$ does not occur in any one of the terms of (15), then (15) holds by the usual associativity of $\otimes$ on $K$.

b) By the definition of $\otimes$ (for $\alpha \in K$) and by (10) (for $\alpha = e$), we have
\[ \alpha \otimes e = e \otimes \alpha = \alpha, \quad \forall \alpha \in K; \]  
Furthermore, by (8) and (9) we have
\[ \alpha \dot{\otimes} e = e \dot{\otimes} \alpha = \alpha, \quad \forall \alpha \in K, \]  
i.e., $e$ is the unit element of $K$ for both products $\otimes$ and $\dot{\otimes}$.

c) $\varepsilon^{-1}$ and $T^{-1}$ are called “inverses” only by abuse of language, as shown by (13) and (14).

d) We shall see that with the above definition, the notions and results of [16, 17] on functions $f : X \to K$, where $X$ is a semimodule over $K$, admit extensions in the above sense to functions $f : X \to K$, for the extended product $\otimes$ of (10), (11). Therefore in the sequel whenever we shall refer to a result of [17] or [16], we shall understand, without any special mention, its extension (using the above conventions) to functions $f : X \to K$, for the extended product $\otimes$ on $K$.

e) Note that a priori the extended products (actions) $\lambda \otimes x$ and $\lambda \dot{\otimes} x$, where $\lambda \in K$ and $x \in X$, need not be defined, except that by the definition of a semimodule $X$ over $K$, we have
\[ \lambda \inf X = \lambda \inf X := \inf X, \quad \forall \lambda \in K. \]  

Definition 4 We extend formula (18) to $\lambda = T$ by defining
\[ T \inf X = T \otimes \inf X := \inf X, \]  
and we define
\[ \lambda \dot{\otimes} \inf X := \lambda \otimes \inf X = \inf X, \quad \forall \lambda \in K. \]
Remark 5 If one would define in any way the products $\top x$, where $x \in X$, then for each “homogeneous” function $f : X \to K$, in the “extended” sense $f(\lambda x) = \lambda f(x), \forall x \in X, \forall \lambda \in K$, we would necessarily have

$$f(\top x) = \begin{cases} \top f(x) = \top & \text{if } f(x) \neq \varepsilon \\ \varepsilon f(x) = \varepsilon & \text{if } f(x) = \varepsilon, \end{cases}$$

so $f(\top x)$ could have only two values, namely either $\top$ or $\varepsilon$.

Definition 6 We define the extended residuation in any semimodule $X$ over $K$ by taking sup instead of max in (2), that is

$$x/y := \sup\{\lambda \in K | \lambda y \leq x\}, \forall x \in X, \forall y \in X. \quad (21)$$

Remark 7 a) In Definition 6 conditions $(A0')$ and $(A1)$ need not be assumed and using (21) and (18) we have

$$x/\inf X := \sup\{\lambda \in K | \lambda \inf X \leq x\} = \sup_{\lambda \in K} \lambda = \top, \forall x \in X; \quad (22)$$

note also that, by (21),

$$\inf X/y := \sup\{\lambda \in K | \lambda x \leq \inf X\} = \begin{cases} \varepsilon & \text{if } x \neq \inf X \\ \sup_{\lambda \in K} \lambda = \top & \text{if } x = \inf X. \end{cases} \quad (23)$$

On the other hand, if $(A0')$ and $(A1)$ hold, then the sup of (21) is attained in $K$ and thus $x/y$ of (21) for $y \neq \inf X$ coincides with $x/y$ of (2), so (2), (22) and (23) could be used as an alternative definition of the extended residuation $x/y$. Note that under $(A0')$ and $(A1)$ we need to take sup instead of max in (22), since the set $\{\lambda \in K | \lambda \inf X \leq x\}$ is not bounded in $K$ for any $x \in X$ and by Remark 1a) $\top \not\in K$.

b) For $x = \inf X$, from (22) and/or (23) it follows that

$$\inf X/\inf X = \top. \quad (24)$$

We shall use the following extension of formula (5):

Lemma 8 We have

$$x/(\mu y) = \mu^{-1} \hat{\otimes}(x/y) \quad \forall x, y \in X, \mu \in K. \quad (25)$$

Proof. For $y \in X \setminus \{\inf X\}$ and $\mu \in K \setminus \{\varepsilon\}$, (25) reduces to (5).

For $y \in X \setminus \{\inf X\}$ and $\mu = \varepsilon$ we have $x/(\varepsilon y) = x/\inf X = \top$ and

$$\varepsilon^{-1} \hat{\otimes}(x/y) = \top \hat{\otimes}(x/y) = \top.$$

For $y = \inf X$ and each $\mu \in K$ we have $x/(\mu \inf X) = x/\inf X = \top, \forall x \in X$, and

$$\mu^{-1} \hat{\otimes}(x/\inf X) = \mu^{-1} \hat{\otimes} \top = \top, \forall x \in X. \quad \blacksquare$$

The products $\otimes$ and $\hat{\otimes}$ on $\overline{K}$ are closely related, as shown by the following result of [14, Lemma 2.2], for which we give here a more transparent proof:
Theorem 9 The product $\hat{\otimes}$ is uniquely determined by $\otimes$, namely we have
\[ \lambda \hat{\otimes} \mu = (\lambda^{-1} \otimes \mu^{-1})^{-1}, \quad \forall \lambda \in \overline{K}, \forall \mu \in \overline{K}. \tag{26} \]

Equivalently, the product $\otimes$ is uniquely determined by $\hat{\otimes}$, namely we have
\[ \lambda \otimes \mu = (\lambda^{-1} \hat{\otimes} \mu^{-1})^{-1}, \quad \forall \lambda \in \overline{K}, \forall \mu \in \overline{K}. \tag{27} \]

Proof. Let us prove (26).

Case 1°. Both $\lambda \in K \setminus \{\varepsilon\}$ and $\mu \in K \setminus \{\varepsilon\}$. Then $\lambda \hat{\otimes} \mu = \lambda \otimes \mu$ and hence, since $K \setminus \{\varepsilon\}$ is a commutative semigroup, we have
\[ (\lambda \hat{\otimes} \mu)^{-1} = (\lambda \otimes \mu)^{-1} = \mu^{-1} \otimes \lambda^{-1} = \lambda^{-1} \otimes \mu^{-1}. \]

Case 2°. $\lambda = \varepsilon$ and $\mu \in \overline{K}$. Then $\varepsilon \hat{\otimes} \mu = \varepsilon \otimes \mu = \varepsilon$ if $\mu \neq \top$, $\varepsilon \hat{\otimes} \top = \varepsilon \otimes \top = \top$ if $\mu = \top$, $$(\varepsilon^{-1} \otimes \mu^{-1})^{-1} = (\top \otimes \mu^{-1})^{-1} = \left\{ \begin{array}{ll} \top^{-1} = \varepsilon & \text{if } \mu^{-1} \neq \varepsilon \\ \varepsilon^{-1} = \top & \text{if } \mu^{-1} = \varepsilon, \end{array} \right.$$ whence (26) holds also for $\lambda = \varepsilon, \mu \in \overline{K}$.

Case 3°. $\lambda = \top$ and $\mu \in \overline{K}$. Then $\top \hat{\otimes} \mu = \top$ and

\[ (\top^{-1} \otimes \mu^{-1})^{-1} = (\varepsilon \otimes \mu^{-1})^{-1} = \varepsilon^{-1} = \top, \]
so (26) holds also for $\lambda = \top, \mu \in \overline{K}$.

Case 4°. $\lambda \in K \setminus \{\varepsilon\}$ and $\mu \in \overline{K}$. If $\mu = \varepsilon$, then we have (26) by Case 2° with $\lambda$ and $\mu$ interchanged. If $\mu = \top$, then we have (26) by Case 3° with $\lambda$ and $\mu$ interchanged. Finally, if $\mu \in K \setminus \{\varepsilon\}$, then we are in case 1°.

This proves (26). Finally, interchanging in (26) $\lambda, \mu$ with $\lambda^{-1}$ and $\mu^{-1}$ respectively, we obtain (27). □

Remark 10 In the particular case when $K = R_{\max}, \overline{K} = \overline{R} = R_{\max} \cup \{+\infty\}$, endowed with the lower addition $\otimes = +$ and upper addition $\hat{\otimes} = +$, the rules (9)–(11) are satisfied (by the definitions of $+$ and $+$) and formula (26) becomes the following well-known observation of Moreau [10, formula (2.1)]:
\[ \lambda + \mu = -(-\lambda + -\mu), \quad \forall \lambda \in \overline{R}, \forall \mu \in \overline{R}. \tag{28} \]

In the sequel the following properties of equivalence of some inequalities involving the extended products $\otimes, \hat{\otimes}$ on $\overline{K}$ will be useful:

Lemma 11 For all $\lambda, \mu, \beta \in \overline{K}$:
A) The inequality
\[ \lambda \mu \leq \beta \tag{29} \]
is equivalent to
\[ \beta^{-1} \mu \leq \lambda^{-1}. \] (30)\\

B) The inequality
\[ \lambda \otimes \mu \geq \beta \] (31)

is equivalent to
\[ \beta^{-1} \otimes \mu \geq \lambda^{-1}. \] (32)

**Proof.** A) The inequalities (29) and (30) are equivalent if \( \lambda, \mu, \beta \in K \setminus \{\varepsilon\} \) (indeed, this follows immediately from the fact that \( \alpha\alpha^{-1} = \varepsilon \) for any \( \alpha \in K \setminus \{\varepsilon\} \)). Thus it remains to consider the cases when one of \( \lambda, \mu \) or \( \beta \) is \( \varepsilon \) or \( T \).

Case (I): \( \lambda = \varepsilon \). Then (29) means that \( \varepsilon = \varepsilon \mu \leq \beta \), which is true for all \( \mu, \beta \), and (30) means that \( \beta^{-1} \mu \leq \lambda^{-1} = T \), which is also true for all \( \mu, \beta \). Hence (29) \( \Leftrightarrow \) (30).

Case (IIa): \( \lambda = T \) and \( \mu = \varepsilon \). Then (29) means that \( \varepsilon = T \varepsilon \leq \beta \), which is true for all \( \beta \), and (30) means that \( \varepsilon = \beta^{-1} \mu \leq \lambda^{-1} \), which is also true for all \( \beta \). Hence (29) \( \Leftrightarrow \) (30).

Case (IIb): \( \lambda = T \) and \( \mu \neq \varepsilon \). Then (29) means that \( T = T \mu \leq \beta \), which implies that \( \beta = T \), whence \( \beta^{-1} \mu = \varepsilon \mu \leq \lambda^{-1} \), so (29) implies (30). In the reverse direction, (30) means that \( \beta^{-1} \mu \leq T^{-1} = \varepsilon \), whence either \( \beta^{-1} = \varepsilon \) or \( \mu = \varepsilon \). But, if \( \beta^{-1} = \varepsilon \), then \( \beta = T \), whence \( \lambda \mu \leq \beta \) and, on the other hand, if \( \mu = \varepsilon \), then \( \lambda \mu = \lambda \varepsilon \leq \beta \). Thus in either case (30) implies (29).

Case (III): \( \beta = \varepsilon \) or \( \beta = T \) : the proof of the equivalence (29) \( \Leftrightarrow \) (30) reduces to the above proofs of the cases \( \lambda = \varepsilon \) respectively \( \lambda = T \), since (29) and (30) are symmetric (by interchanging \( \lambda \) and \( \beta \) with \( \beta^{-1} \) and \( \lambda^{-1} \) respectively).

Case (IV): \( \mu = \varepsilon \). Then (29) means that \( \varepsilon = \lambda \varepsilon \leq \beta \), which is true for all \( \lambda, \beta \), and (30) means that \( \varepsilon = \beta^{-1} \mu \leq \lambda^{-1} \), which is also true for all \( \lambda, \beta \). Hence (29) \( \Leftrightarrow \) (30).

Case (Va): \( \mu = T \) and \( \lambda = \varepsilon \). Then (29) means that \( \varepsilon = \varepsilon T \leq \beta \), which is true for all \( \beta \), and (30) means that \( \beta^{-1} \mu \leq T = \lambda^{-1} \), which is also true for all \( \beta \). Hence (29) \( \Leftrightarrow \) (30).

Case (Vb): \( \mu = T \) and \( \lambda \neq \varepsilon \). Then (29) means that \( T = \lambda T \leq \beta \), which implies that \( \beta = T \), whence \( \beta^{-1} \mu = \varepsilon \mu \leq \lambda^{-1} \), so (29) implies (30). In the reverse direction, (30) means that \( \beta^{-1} \mu \leq T^{-1} = \varepsilon \), whence \( \beta^{-1} = \varepsilon \), so \( \beta = T \). Therefore \( \lambda \mu \leq T = \beta \), and thus (30) implies (29).

B) Using Theorem 4 passing to inverses of both sides and applying part A), and then again taking inverses in both sides and applying Theorem 4 we obtain

\[
\lambda \otimes \mu = (\lambda^{-1} \otimes \mu^{-1})^{-1} \geq \beta \Leftrightarrow \lambda^{-1} \otimes \mu^{-1} \leq \beta^{-1} \\
\Leftrightarrow \beta \otimes \mu^{-1} \leq \lambda \Leftrightarrow (\beta \otimes \mu^{-1})^{-1} \geq \lambda^{-1} \Leftrightarrow \beta^{-1} \otimes \mu \geq \lambda^{-1}. \] \[ \square \]

**Remark 12** a) In general, for \( \lambda, \mu, \beta \in K \), the inequality (29) is not equivalent to the inequality
\[ \mu \leq \lambda^{-1} \beta. \]
A counterexample is obtained by taking $\lambda = \beta = \varepsilon$ and $\mu \in \mathcal{K} \setminus \{\varepsilon\}$. Indeed, then (29) becomes $\varepsilon \leq \varepsilon$, thus it is true, while the second inequality becomes $\mu \leq T \varepsilon = \varepsilon$, which is false.

b) In general, for $\lambda, \mu, \beta \in \mathcal{K}$, the inequality (31) is not equivalent to the inequality $\mu \geq \lambda^{-1} \hat{\otimes} \beta$.

A counterexample is obtained by taking $\lambda = \beta = T$ and $\mu \in \mathcal{K} \setminus \{\varepsilon\}$. Indeed, then (31) becomes $T \geq T$, which is true, while the second inequality becomes $\mu \geq T$, which is false.

c) One cannot replace in Lemma 11 the inequalities by the opposite ones. For example $\varepsilon \varepsilon \geq \varepsilon$ does not imply that $\varepsilon^{-1} \varepsilon \geq \varepsilon^{-1}$, since $\varepsilon^{-1} \varepsilon = T \varepsilon = \varepsilon \not\geq T = \varepsilon^{-1}$.

Corollary 13 For any $\lambda, \mu, \nu \in \mathcal{K}$ we have the equivalence

$$\mu \otimes \nu \leq \lambda \Leftrightarrow \nu \leq \lambda \hat{\otimes} (\mu^{-1}).$$  \(\text{(33)}\)

Proof. By Theorem 9, given any $\lambda, \mu, \nu \in \mathcal{K}$, for the first inequality of (33) by Lemma 11A we have the equivalence $\mu \otimes \nu \leq \lambda \Leftrightarrow \lambda^{-1} \otimes \mu \leq \nu^{-1}$ and for the second inequality of (33) we have the equivalence $\nu \leq \lambda \hat{\otimes} (\mu^{-1}) = (\lambda^{-1} \otimes \mu)^{-1} \Leftrightarrow \nu^{-1} \geq \lambda^{-1} \otimes \mu$. \(\square\)

Remark 14 In the particular case when $\mathcal{K} = R_{\max}, \mathcal{K} = \mathcal{K} = R_{\max} \cup \{+\infty\}$, endowed with the lower addition $\otimes = +$ and upper addition $\hat{\otimes} = +$, Corollary 13 reduces to [10, p.119, Proposition 3c] (where the proof is different).

Finally, concerning “scalar residuation”, note that one can consider the semimodule $\mathcal{K} = \mathcal{K}$ over the semiring $\mathcal{K}$ (which is almost a semifield, since all elements of $\mathcal{K} \setminus \{\varepsilon, T\}$ are invertible), and by Remark 7a) one can define residuation in this semimodule (briefly, “in $\mathcal{K}$”) by (21). Let us give the following application of Corollary 13 which shows that residuation in $\mathcal{K}$ can be expressed with aid of the upper product and vice versa, the upper product $\lambda \hat{\otimes} \mu$ can be expressed with aid of residuation in $\mathcal{K}$:

Corollary 15 We have

$$\lambda/\mu = \lambda \hat{\otimes} (\mu^{-1}), \quad \forall \lambda \in \mathcal{K}, \forall \mu \in \mathcal{K},$$  \(\text{(34)}\)

$$\lambda \hat{\otimes} \mu = \lambda/(\mu^{-1}), \quad \forall \lambda \in \mathcal{K}, \forall \mu \in \mathcal{K}.$$  \(\text{(35)}\)

Proof. By the definition of residuation in $\mathcal{K}$ and by Corollary 13 we obtain

$$\lambda/\mu = \sup \{\nu \in \mathcal{K} | \mu \otimes \nu \leq \lambda\} = \sup \{\nu \in \mathcal{K} | \nu \leq \lambda \hat{\otimes} (\mu^{-1})\}$$

$$= \lambda \hat{\otimes} (\mu^{-1}), \quad \forall \lambda \in \mathcal{K}, \forall \mu \in \mathcal{K},$$

which proves (34). Finally, interchanging in (34) $\mu$ and $\mu^{-1}$ we obtain (35). \(\square\)
Remark 16 a) Alternatively, by (5) applied in $\mathcal{K}$ we have
\[ \frac{\lambda}{\mu\nu} = \mu^{-1} \otimes \frac{\lambda}{\nu}, \quad \forall \lambda \in \mathcal{K}, \forall \mu \in \mathcal{K}, \forall \nu \in \mathcal{K}, \] (36)
and taking here $\nu = e$, we obtain (34).

b) In the particular case when $\mathcal{K} = R_{\text{max}}$, $\mathcal{K} = \mathcal{K} = R_{\text{max}} \cup \{ +\infty \}$, endowed with the lower addition $\otimes = +$ and upper addition $\hat{\otimes} = +$, formula (35) has been proved in [1, Proposition 2.1].

3 Characterizations of topical and anti-topical functions $f : X \to \mathcal{K}$ in terms of some inequalities

Definition 17 Let $(X, \mathcal{K})$ be a pair satisising $(A0')$, $(A1)$, and let $\mathcal{K} = \mathcal{K} \cup \{ \top \}$ be the minimal enlargement of $\mathcal{K}$. A function $f : X \to \mathcal{K}$ is said to be
a) increasing (resp. decreasing), if $x', x'' \in X, x' \leq x''$ imply $f(x') \leq f(x'')$ (resp. $f(x') \geq f(x'')$);
b) homogeneous (resp. anti-homogeneous), if
\[ f(\lambda x) = \lambda f(x), \quad \forall x \in X, \forall \lambda \in \mathcal{K} \] (37)
(resp. if
\[ f(\lambda x) = \lambda^{-1} \hat{\otimes} f(x), \quad \forall x \in X, \forall \lambda \in \mathcal{K}; \] (38)
c) topical (resp. anti-topical), if it is increasing and homogeneous (resp. decreasing and anti-homogeneous).

Similarly to the fact that the two products $\otimes$ and $\hat{\otimes}$ on $\mathcal{K}$ are closely related, the topical and anti-topical functions with values in $\mathcal{K}$ are closely related, too, as shown by the following simple lemma:

Lemma 18 $f : X \to \mathcal{K}$ is anti-topical if and only if the function $h : X \to \mathcal{K}$ defined by
\[ h(x) := f(x)^{-1}, \quad \forall x \in X, \] (39)
is topical.

Proof. Clearly $f$ is decreasing if and only if $h$ is increasing.
Furthermore, let $x \in X, \lambda \in \mathcal{K}$. If $f$ is anti-homogeneous and $x \in X, \lambda \in \mathcal{K}$, then by Theorem [9] we have
\[ h(\lambda \otimes x) = f(\lambda \otimes x)^{-1} = (\lambda^{-1} \otimes f(x))^{-1} = \lambda \otimes f(x)^{-1} = \lambda \otimes h(x), \]
so $h$ is homogeneous. Conversely, if $h$ is homogeneous, then by Theorem [9] we have
\[ f(\lambda \otimes x) = h(\lambda \otimes x)^{-1} = (\lambda \otimes h(x))^{-1} = \lambda^{-1} \hat{\otimes} h(x)^{-1} = \lambda^{-1} \hat{\otimes} f(x), \]
so $f$ is anti-homogeneous. □

In the sequel we shall often use Lemma 18 and Theorem 9 above.
Lemma 19  a) Let $f : X \to \overline{K}$ be a topical function and for each $y \in X$ let $t_y : X \to \overline{K}$ be the function defined by

$$t_y(x) = f(y)x/y, \quad \forall x \in X.$$  \hfill (40)

Then we have

$$t_y \leq f, \quad t_y(y) = f(y).$$  \hfill (41)

Conversely, if $t = t_y : X \to \overline{K}$ is a function of the form

$$t(x) = t_y(x) = \alpha x/y, \quad \forall x \in X,$$  \hfill (42)

where $y \in X, \alpha \in \overline{K}$, satisfying (41), then we have

$$\alpha = f(y),$$

so $t_y$ is equal to (40).

b) Let $f : X \to \overline{K}$ be an anti-topical function. For any $y \in X$ define $q_y : X \to \overline{K}$ by

$$q_y(x) := (x/y)^{-1} \otimes f(y), \quad \forall x \in X.$$  \hfill (43)

Then

$$q_y \geq f, \quad q_y(y) = f(y).$$  \hfill (44)

Conversely, if $q = q_y : X \to \overline{K}$ is a function of the form

$$q(x) = q_y(x) = (x/y)^{-1} \otimes \alpha, \quad \forall x \in X,$$  \hfill (45)

where $y \in X, \alpha \in \overline{K}$, satisfying (44), then we have

$$\alpha = f(y),$$

so $q_y$ is equal to (43).

Proof. a) Observe first that for every homogeneous (and hence for every topical) function $f : X \to \overline{K}$ we have

$$f(\inf X) = \varepsilon.$$  \hfill (46)

Indeed, by (18), the homogeneity of $f$ and (11) we have

$$f(\inf X) = f(\varepsilon \inf X) = \varepsilon f(\inf X) = \varepsilon.$$

Now let $f$ be topical. If $y \in X \setminus \{\inf X\}$, then since $f$ is topical, by (10), (11) and (4) we have

$$t_y(x) = f(y)x/y = f((x/y)y) \leq f(x), \quad \forall x \in X,$$  \hfill (47)

$$t_y(y) = f(y)(y/y) = f(y)e = f(y).$$  \hfill (48)

On the other hand, if $y = \inf X$, then by (40), (46), (22), (11) and (10),

$$t_{\inf X}(x) = f(\inf X)(x/\inf X) = \varepsilon \top = \varepsilon \leq f(x), \quad \forall x \in X,$$  \hfill (49)

$$t_{\inf X}(\inf X) = f(\inf X)(\inf X/\inf X) = \varepsilon \top = \varepsilon = f(\inf X).$$  \hfill (50)

Conversely, if $t_y$ is a function of the form (42) satisfying (11) and $y \in X \setminus \{\inf X\}$, so $y/y = e$, then we have

$$\alpha = \alpha y/y = t_y(y) = f(y).$$

On the other hand, for $y = \inf X$, formulae (42), (22), (11) and (46) mean that

$$t_{\inf X}(x) = \alpha x/\inf X = \alpha \top, \quad \forall x \in X,$$  \hfill (51)
which, by (10), (11) and (46), implies that
\[ \alpha \top = t_{\inf X} = \epsilon, \]
Hence by (51) for
\[ x = \inf X \]
and the second part of (52), we obtain
\[ H \]
which, by (10), (11) and (46), implies that \( \alpha = \epsilon = f(\inf X) \).

b) Observe first that for every anti-homogeneous (and hence for every anti-topical) function \( f : X \to K \) we have
\[ f(\inf X) = \top. \]
Indeed, by Lemma 18 and part a) above, for every anti-homogeneous function \( f : X \to K \) and for \( h : X \to K \) of (39) we have
\[ f(\inf X)^{-1} = h(\inf X) = \epsilon, \]
whence (53).

Now let \( f \) be anti-topical. Then by Lemma 18 \( h : X \to K \) of (39) is topical and hence by Theorem 9 and part a) above, for each \( x \in X \) and each \( y \in X \) we have
\[ q_y(x) = (x/y)^{-1} \otimes h(y)^{-1} = ((x/y) \otimes h(y))^{-1} \geq h(x)^{-1} = f(x), \]
\[ q_y(y) = (y/y)^{-1} \otimes h(y)^{-1} = ((y/y) \otimes h(y))^{-1} = h(y)^{-1} = f(y). \]

Conversely, if \( q_y \) is a function of the form (43) satisfying (44), that is, by Theorem 9 if
\[ ((x/y) \otimes \alpha)^{-1} \]
then passing to inverses, we obtain
\[ (x/y) \otimes \alpha^{-1} \leq f(x)^{-1} = h(x) \quad \forall x \in X, \]
\[ (y/y) \otimes \alpha^{-1} = f(y)^{-1} = h(y). \]
Therefore, since by Lemma 18 \( h \) is topial, from part a) (with \( \alpha \) replaced by \( \alpha^{-1} \)) we obtain \( h(y) = \alpha^{-1} \), so \( f(y) = \alpha \). \( \Box \)

In [17] Theorem 5 we have shown that if \( (X, \mathcal{K}) \) is a pair satisfying (A0'), (A1), then a function \( f : X \to \mathcal{K} \) is topial if and only if \( f(\inf X) = \epsilon \) and
\[ f(y)x/y \leq f(x), \quad \forall x \in X, \forall y \in X \setminus \{ \inf X \} \]
(54)
(where the condition on \( y \) was needed in order to be able to define \( x/y \)). A similar characterization of topical functions \( f : X \to \mathcal{K} \) in which the inequality \( f(y)x/y \leq f(x) \) is replaced by \( f(y)s_{y,d}(x) \leq f(x), \forall x \in X, \forall y \in X \setminus \{ \inf X \} \), \( \forall d \in \mathcal{K} \), where
\[ s_{y,d}(x) := \inf\{ x/y, d \} = \inf\{ \max\{ \lambda \in \mathcal{K} | \lambda y \leq x \}, d \}, \]
\[ \forall x \in X, \forall y \in X \setminus \{ \inf X \} \]
\[ \forall d \in \mathcal{K}, \]
\[ \forall x \in X, \forall y \in X \setminus \{ \inf X \} \]
\[ \forall d \in \mathcal{K}, \]
\[ \forall x \in X, \forall y \in X \setminus \{ \inf X \} \]
has been given in [17, Theorem 16]. Now, with the aid of the multiplication \( \otimes \) on \( \overline{\mathcal{K}} \) defined above, we shall extend these results to functions \( f : X \to \overline{\mathcal{K}} \), replacing the conditions \( y \in X \setminus \{\inf X\} \) and \( d \in \mathcal{K} \) by \( y \in X \) and \( d \in \mathcal{K} \) respectively. To this end, for the case of topical functions we shall extend \( s_{y,d}(x) \) to all \( y \in X \) and \( d \in \mathcal{K} \) by (55) and

\[
s_{\inf X,d}(x) \overset{\text{def}}{=} \inf\{x/\inf X,d\} = \inf\{\top, d\} = d, \quad \forall x \in X, \forall d \in \mathcal{K}, \quad (56)
\]

\[
s_{y,\top}(x) \overset{\text{def}}{=} \inf\{x/y, \top\} = x/y, \quad \forall x \in X, \forall y \in X. \quad (57)
\]

Moreover, we shall also give corresponding characterizations of anti-topical functions, using the multiplication \( \hat{\otimes} \) defined above and the functions

\[
\overline{s}_{y,d}(x) := \sup\{(x/y)^{-1}, d\}, \quad \forall x \in X, \forall y \in X, \forall d \in \mathcal{K}; \quad (58)
\]

note that in particular for the extreme values \( d = \varepsilon \) and \( d = \top \) in (58) we have, respectively,

\[
\overline{s}_{y,\varepsilon}(x) = \sup\{(x/y)^{-1}, \varepsilon\} = (x/y)^{-1}, \quad \forall x \in X, \forall y \in X, \quad (59)
\]

\[
\overline{s}_{y,\top}(x) = \sup\{(x/y)^{-1}, \top\} = \top, \quad \forall x \in X, \forall y \in X. \quad (60)
\]

**Theorem 20** Let \((X, \mathcal{K})\) be a pair that satisfies (A0'), (A1).

a) For a function \( f : X \to \overline{\mathcal{K}} \) the following statements are equivalent:

1°. \( f \) is topical.

2°. We have (40) and

\[
f(y)x/y \leq f(x), \quad \forall x \in X, \forall y \in X. \quad (61)
\]

3°. We have (40) and

\[
f(y)s_{y,d}(x) \leq f(x), \quad \forall x \in X, \forall y \in X, \forall d \in \mathcal{K}. \quad (62)
\]

b) For a function \( f : X \to \overline{\mathcal{K}} \) the following statements are equivalent:

1°. \( f \) is anti-topical.

2°. We have (53) and

\[
f(y)\hat{\otimes}(x/y)^{-1} \geq f(x), \quad \forall x \in X, \forall y \in X. \quad (63)
\]

3°. We have (53) and

\[
f(y)\hat{\otimes}\overline{s}_{y,d}(x) \geq f(x), \quad \forall x \in X, \forall y \in X, \forall d \in \mathcal{K}. \quad (64)
\]

**Proof.** a) The implication 1° \( \Rightarrow \) 2° follows from Lemma (14a) and its proof.

2° \( \Rightarrow \) 3°. Assume 2°. If \( x \in X, y \in X \setminus \{\inf X\} \), then for any \( d \in \mathcal{K} \) we have, by (55) and 2°,

\[
f(y)s_{y,d}(x) = f(y)\inf\{x/y, d\} \leq f(y)x/y \leq f(x).
\]

If \( y = \inf X \), then by (40) we have

\[
f(\inf X)s_{\inf X,d}(x) = \varepsilon s_{\inf X,d}(x) = \varepsilon \leq f(x), \quad \forall x \in X, \forall d \in \mathcal{K}.
\]
3° ⇒ 2°. Assume 3°. If \( x \in X, y \in X \setminus \{\inf X\} \), then by 3° with any \( d \geq x/y \) we obtain

\[
f(y)x/y = f(y)\inf\{x/y, d\} = f(y)s_{y,a}(x) \leq f(x).
\]

Finally, if \( x \in X, y = \inf X \), then \( f(\inf X)/(\inf X) = \varepsilon \top = \varepsilon \leq f(x) \).

2° ⇒ 1°. Assume 2°. We need to show that \( f \) is increasing and homogeneous.

Assume that \( x, y \in X, y \leq x \). Then \( e \leq x/y \) and due to (16) and (61) one has

\[
f(y)/x = \inf\{x/y, d\} \leq f(x),
\]

so \( f \) is increasing.

Assume now that \( x \in X \setminus \{\inf X\} \) and \( \lambda \in \mathcal{K}\setminus\{\varepsilon\} \), so \( \lambda x \in X \setminus \{\inf X\}, x/x = e \) (by (4)) and \( \lambda x^{-1} = e \). Then by (61) with \( y = \lambda x \) we have \( f(\lambda x)x/\lambda x \leq f(x) \), whence using also (5),

\[
f(\lambda x) = \lambda f(\lambda x)(\lambda^{-1})x/x = \lambda f(\lambda x)x/\lambda x \leq \lambda f(x) \quad (65)
\]

On the other hand, for \( x = \inf X, \lambda \in \mathcal{K} \), we have, by (15), (40) and (11),

\[
f(\lambda \inf X) = f(\inf X) = \varepsilon = \lambda \varepsilon = \lambda f(\inf X) \quad (66)
\]

Moreover, if \( x \in X, \lambda = \varepsilon \), then by \( \varepsilon x = \inf X, \forall x \in X \), and (60) we have

\[
f(\varepsilon x) = f(\varepsilon) = \varepsilon = \varepsilon f(x) \quad (67)
\]

Furthermore, if \( y \in X \setminus \{\inf X\}, \lambda \in \mathcal{K} \), then by (4) and (61) with \( x = \lambda y \) we have

\[
\lambda f(y) = f(\lambda y)/(\lambda y) \leq f(\lambda y) \quad (68)
\]

On the other hand, for \( y = \inf X, \lambda \in \mathcal{K} \), we have

\[
\lambda f(\inf X) = \lambda \varepsilon = \varepsilon = f(\lambda \inf X) \quad (69)
\]

From (65)–(69) it follows that \( f \) is homogeneous.

b) 1° ⇔ 2°. By Lemma (18) \( f \) is anti-topical if and only if the function \( h \) of (39) is topical, which, by part a), is equivalent to \( h(\inf X) = \varepsilon \) and

\[
h(y) \otimes x/y \leq h(x), \quad \forall x \in X, y \in X,
\]

that is, to \( f(\inf X) = \top \) and

\[
f(y)^{-1} \otimes x/y \leq f(x)^{-1}, \quad \forall x \in X, \forall y \in X.
\]

Then, taking inverses in both sides, we get equivalence of the latter inequality with

\[
(f(y)^{-1} \otimes (x/y))^{-1} \geq f(x), \quad \forall x \in X, \forall y \in X.
\]

which, by Theorem (9) with \( \lambda = f(y), \mu = (x/y)^{-1} \), is equivalent to (63).
1° $\iff$ 3°. by Lemma 18 $f$ is anti-topical if and only if the function $h$ of (39) is topical, which, by part a), is equivalent to $f(\inf X) = \top$ and

$$f(y)^{-1} \otimes s_{y,d}(x) \leq f(x)^{-1}, \quad \forall x \in X, \forall y \in X, \forall d \in K,$$

that is, taking inverses in both sides, to $f(\inf X) = \top$ and

$$(f(y)^{-1} \otimes s_{y,d}(x))^{-1} \geq f(x), \quad \forall x \in X, \forall y \in X, \forall d \in K.$$

But, by Theorem 9 and the definitions of $s$ and $\mathfrak{s}$, for any $x \in X, y \in X$ and $d \in K$ we have

$$(f(y)^{-1} \otimes s_{y,d}(x))^{-1} = f(y) \otimes s_{y,d}(x)^{-1} = f(y) \otimes (\inf \{x/y, d\})^{-1} = f(y) \otimes \sup \{(x/y)^{-1}, d^{-1}\} = f(y) \otimes \mathfrak{s}_{y,d^{-1}}(x), (70)$$

so the above inequalities are equivalent to (64). □

**Remark 21** In the statements of Theorem 20 one can replace, equivalently, the inequalities by equalities. For example in Theorem 20 a) Statement 2° can be replaced, equivalently, by:

2'. We have (46) and

$$\sup_{y \in X} f(y)x/y = f(x), \quad \forall x \in X. \quad (71)$$

Indeed, for each $x \in X \setminus \{\inf X\}$ the sup in (71) is attained at $y = x$. On the other hand, for $x = \inf X$ we have $\sup_{y \in X} f(y)(\inf X/y) = \varepsilon = f(\inf X)$ (by (46)). Thus 2° $\Rightarrow$ 2'. The reverse implication is obvious.

Similarly, in Theorem 20 a) Statement 3° can be replaced, equivalently, by:

3'. We have (46) and

$$\sup_{(y,d) \in X \times K} f(y)s_{y,d}(x) = f(x), \quad \forall x \in X. \quad (72)$$

Indeed, for each $x \in X \setminus \{\inf X\}$ the sup in (72) is attained at $y = x, d \geq e$ (since then $f(y)s_{x,d}(x) = f(x) \inf \{x/x, d\} = f(x)e = f(x)$); furthermore, for $x = \inf X, y \in X \setminus \{\inf X\}$, so $\inf X/y = \varepsilon$, and any $d \in K$, we have, by (20) and (46),

$$\sup_{y \in X} f(y)s_{y,d}(\inf X) = \sup_{y \in X} f(y)\inf \{\inf X/y, d\} = \sup_{y \in X} f(y)\varepsilon = \varepsilon = f(\inf X);$$

finally, if $x = y = \inf X$, then for any $d \in K$ we have, by (46),

$$f(\inf X)s_{\inf X,d}(\inf X) = \varepsilon s_{\inf X,d}(\inf X) = \varepsilon = f(\inf X).$$

Thus 3° $\Rightarrow$ 3'. The reverse implication is obvious. The cases of Theorem 20b) are similar.
Corollary 22 a) If $f$ is topical and for some $y \in X$ we have $f(y) = \top$, then for each $x \in X$ either $f(x) = \top$ or $x/y = \varepsilon$.

b) If $f$ is anti-topical and for some $y \in X$ we have $f(y) = \varepsilon$, then for each $x \in X$ either $f(x) = \varepsilon$ or $(x/y)^{-1} = \top$.

c) A function $f : X \to \overline{\mathbb{K}}$ cannot be simultaneously topical and anti-topical.

Proof. a) If $f$ is topical, $x, y \in X$ and $f(y) = \top$, then by Theorem 20a), implication $1^o \Rightarrow 2^o$, we have $\top x/y \leq f(x)$. Hence if $f(x) \neq \top$, then by (110) we obtain $x/y = \varepsilon$.

The proof of part b) is similar, mutatis mutandis.

c) This follows from the fact that the values of $f$ at inf $X$ are $f(\inf X) = \varepsilon$ and $f(\inf X) = \top$ for topical and anti-topical functions respectively, and $\top \neq \varepsilon$. \hfill \Box

Now we can prove the following further characterizations of anti-topical functions:

Theorem 23 For a function $f : X \to \overline{\mathbb{K}}$ the following statements are equivalent:

1$. $f$ is anti-topical.

2$. We have $f(\inf X) = \top$ and

$$f(x) \otimes x/y \leq f(y), \quad \forall x \in X, \forall y \in X. \tag{73}$$

3$. We have $f(\inf X) = \top$ and

$$f(x) \otimes s_{y,a}(x) \leq f(y), \quad \forall x \in X, \forall y \in X, d \in \overline{\mathbb{K}}. \tag{74}$$

4$. We have $f(\inf X) = \top$ and

$$f(y)/(x/y) \geq f(x), \quad \forall x \in X, \forall y \in X. \tag{75}$$

Proof. $1^o \iff 2^o$. By Lemma 18 we have $1^o$ if and only if the function $h$ of (39) is topical, which, by Theorem 20a) is equivalent to $h(\inf X) = \varepsilon$ and $h(y) \otimes x/y \leq h(x)$, that is, to $f(\inf X) = \top$ and $f(y)^{-1} \otimes x/y \leq f(x)^{-1}, \forall x \in X, \forall y \in X$. But, by Lemma 11a), the latter is equivalent to $f(\inf X) = \top$ and (73).

$1^o \iff 3^o$. By Lemma 18 we have $1^o$ if and only if we have $h(\inf X) = \varepsilon$ and $h(y) \otimes s_{y,a}(x) \leq h(x), \forall x \in X, \forall y \in X$, that is, to $f(\inf X) = \top$ and $f(y)^{-1} \otimes s_{y,a}(x) \leq f(x)^{-1}, \forall x \in X, \forall y \in X$. But, by Lemma 11a), the latter is equivalent to $f(\inf X) = \top$ and (73).

$1^o \iff 4^o$. By Theorem 20a) we have $1^o$ if and only if we have $f(\inf X) = \top$ and (115). But, by Corollary 15 we have $f(y) \otimes (x/y)^{-1} = f(y)/(x/y), \forall x \in X, \forall y \in X$, which shows that $1^o \iff 4^o$. \hfill \Box

The following Corollary of Theorem 20 gives characterizations of the functions $f$ that satisfy the inequalities (61)-(64):

Corollary 24 Let $(X, K)$ be a pair that satisfies $(A0')$, $(A1)$.

a) For a function $f : X \to \overline{\mathbb{K}}$ the following statements are equivalent:
1°. We have (61).
2°. We have (62).
3°. Either \( f \) is topical or \( f \equiv \top \).

b) For a function \( f : X \to \mathcal{K} \) the following statements are equivalent:
1°. We have (63).
2°. We have (64).
3°. Either \( f \) is anti-topical or \( f \equiv \varepsilon \).

Proof. a) 1° \( \Rightarrow \) 3°. Assume that we have 1° and \( f \) is not topical, so \( f(\inf X) \neq \varepsilon \) (by Theorem 20a)). Then by (22) and (61) (applied to \( y = \inf X \)) we have
\[ f(\inf X) \top = f(\inf X)x/\inf X \leq f(x), \quad \forall x \in X, \]
whence, since \( f(\inf X) \top = \top \) (by \( f(\inf X) \neq \varepsilon \)), it follows that \( \top \leq f(x), \forall x \in X \), and hence \( f \equiv \top \).

3° \( \Rightarrow \) 2°. Assume 3°. If \( f \) is topical, then we have (62) by Theorem 20a). On the other hand, if \( f \equiv \top \), then (62) holds since \( \top \) is the greatest element of \( \mathcal{K} \).

Finally, the implication 2° \( \Rightarrow \) 1° holds since (62) for \( d = \top \) reduces to (61) (by (57)).

b) 1° \( \Rightarrow \) 3°. Assume that we have 1° and \( f \) is not anti-topical, so \( f(\inf X) \neq \top \) (by Theorem 20b)). Then by (22), \( \top - 1 \equiv \varepsilon \) and (63) applied to \( y = \inf X \), we have
\[ f(\inf X) \varepsilon = f(\inf X)\varepsilon(x/\inf X)^{-1} \geq f(x), \quad \forall x \in X, \]
whence, since \( f(\inf X) \varepsilon = \varepsilon \) (by \( f(\inf X) \neq \top \)), it follows that \( \varepsilon \geq f(x), \forall x \in X \), and hence \( f \equiv \varepsilon \).

3° \( \Rightarrow \) 2°. Assume 3°. If \( f \) is anti-topical, then we have (63) by Theorem 20b). On the other hand, if \( f \equiv \varepsilon \), then (63) holds since \( \varepsilon \) is the smallest element of \( \mathcal{K} \).

Finally, the implication 2° \( \Rightarrow \) 1° holds since (64) for \( d = \varepsilon \) reduces to (63) (by (59)). \( \square \)

Remark 25 The constant function \( f \equiv \top \) is not homogeneous, and hence not topical. Indeed, we have \( \top(\varepsilon x) = \top \), but \( \varepsilon \top(x) = \varepsilon, \forall x \in X \). However, \( f \equiv \top \) is anti-topical, since \( \top(\lambda x) = \top, \lambda^{-1}\top(x) = \top, \forall x \in X, \forall \lambda \in \mathcal{K} \). Let us also mention that the constant function \( f \equiv \varepsilon \) is topical and hence not anti-topical.

4 Characterizations of topical and anti-topical functions \( f : X \to \mathcal{K} \) using conjugates of Fenchel-Moreau type

We recall that for two sets \( X \) and \( Y \) and a “finite coupling function” \( \pi : X \times Y \to R \), respectively \( \pi : X \times Y \to A \), where \( A = (A, \oplus, \otimes) \) is a conditionally complete
Indeed, by (12) and (11) we have

Remark 27

a) For the constant function \( f \equiv \top \) we have

\[ \top^{c(\pi)}(y) = \varepsilon, \quad \forall y \in Y. \]  

(77)

Indeed, by (12) and (11) we have

\[ \top^{c(\pi)}(y) = \sup_{x \in X} (\top(x^{-1})\pi(x, y)) = \sup_{x \in X} \varepsilon\pi(x, y) = \varepsilon, \quad \forall y \in Y. \]

b) For the constant function \( f \equiv \varepsilon \) we have

\[ \varepsilon^{c(\pi)}(y) = \sup_{x \in X} \varepsilon(x^{-1})\pi(x, y) = \sup_{x \in X} \top\pi(x, y) = \begin{cases} \top & \text{if } \exists x_0 \in X, \pi(x_0, y) \notin \varepsilon \\ \varepsilon & \text{if } \pi(x, y) = \varepsilon, \forall x \in X. \end{cases} \]

(78)

c) If \( f : X \to \overline{K} \) and \( x_0 \in X \) are such that \( f(x_0) = \varepsilon \) (e.g., if \( f \) is homogeneous, or in particular, topical, and \( x_0 = \inf X \)) and if \( y_0 \in Y, \pi(x_0, y_0) \notin \varepsilon \), then

\[ f^{c(\pi)}(y_0) = \top. \]

(79)

Indeed, by (76) and \( \varepsilon^{-1} = \top \) we have \( f^{c(\pi)}(y_0) \geq f(x_0)^{-1}\pi(x_0, y_0) = \top\pi(x_0, y_0) \), whence by \( \pi(x_0, y_0) \notin \varepsilon \), (11) and (6) we obtain (79).

d) The abstract (axiomatic) theory of Fenchel-Moreau conjugations and the theory of kernel representations of lower semi-continuous linear mappings for idempotent semifield-valued functions (see e.g. [8] and the references therein) have developed in parallel and independently. The close connections between them have been shown in [14].

e) In general, one does not always have the equalities

\[ \sup_{x \in X} f(x)^{-1}\pi(x, y) = \sup_{x \in X} \pi(x, y)/f(x), \quad y \in X, \]

(80)

and the second term of (80) is not suitable to define a conjugate function by \( f^{c(\pi)}(y) := \sup_{x \in X} \pi(x, y)/f(x) \forall y \in X. \) Indeed, we shall prove these statements in Remark (29b) below. Let us only note here that by Corollary (15) we have

\[ \pi(x, y)/f(x) = f(x)^{-1}\hat{\pi}(x, y), \quad \forall x \in X, \forall y \in X, \]

(81)

while in Definition (26) we have \( f(x)^{-1}\hat{\pi}(x, y) \) instead of \( f(x)^{-1}\hat{\pi}(x, y) \).
Returning to Definition 26 here we shall be interested in Fenchel-Moreau conjugates first in the case where \((X, \mathcal{K})\) is a pair satisfying \((A0'), (A1)\) and \(Y := X\), with the coupling function \(\pi = \varphi : X \times X \to \mathcal{K}\) defined by

\[
\varphi(x, y) := \frac{x}{y} = y^\rho(x), \quad \forall x \in X, \forall y \in X. \tag{82}
\]

In a different context, related to the study of increasing positively homogeneous functions \(f : C \to \mathbb{R}_+\) defined on a cone \(C\) of a locally convex space \(X\) endowed with the order induced by the closure \(\overline{C}\) of \(C\), with values in \(\mathbb{R}_+ = \mathbb{R} \cup \{+\infty\}\), the coupling function \((82)\) has been considered in [6].

Second, we shall be interested in Fenchel-Moreau conjugates for the coupling function \(\pi = \psi : X \times (X \times K) \to \mathcal{K}\) defined by

\[
\psi(x, (y, d)) := \inf\{x/y, d\} = s_{y,d}(x), \quad \forall x \in X, \forall y \in X, \forall d \in \mathcal{K}. \tag{83}
\]

For \(\pi = \varphi\) and \(\pi = \psi\) Definition 26 leads to:

**Definition 28** If \((X, \mathcal{K})\) is a pair satisfying \((A0'), (A1)\), the Fenchel-Moreau conjugate of a function \(f : X \to \mathcal{K}\)

a) associated to the coupling function \(\varphi\) of \((82)\), or briefly, the \(\varphi\)-conjugate of \(f\), is the function \(f^{\varphi} : X \to \mathcal{K}\) defined by

\[
f^{\varphi}(y) := \sup_{x \in X} f(x)^{-1} x/y, \quad \forall y \in X; \tag{84}
\]

b) associated to the coupling function \(\psi\) of \((83)\), or briefly, the \(\psi\)-conjugate of \(f\), is the function \(f^{\psi} : X \times \mathcal{K} \to \mathcal{K}\) defined by

\[
f^{\psi}(y, d) = \sup_{x \in X} f(x)^{-1} s_{y,d}(x), \quad \forall y \in X, \forall d \in \mathcal{K}. \tag{85}
\]

**Remark 29** a) For any function \(f : X \to \mathcal{K}\), the conjugate function \(f^{\varphi} : X \to \mathcal{K}\) is decreasing (i.e. \(y_1 \leq y_2 \Rightarrow f^{\varphi}(y_1) \geq f^{\varphi}(y_2)\)) and antihomogeneous (i.e. \(f^{\varphi}(\lambda y) = \lambda^{-1} \otimes f^{\varphi}(y)\) for all \(y \in X\) and \(\lambda \in \mathcal{K}\), since by Lemma 8 we have

\[
f^{\varphi}(\lambda y) = \sup_{x \in X} f(x)^{-1} x/\lambda y = \sup_{x \in X} f(x)^{-1} x/y = \lambda^{-1} \otimes f^{\varphi}(y),
\]

so \(f^{\varphi}\) is anti-topical. This should be compared with the situation for the conjugates of \(f\) with respect to the so-called “additive min-type coupling functions” \(\pi_\mu : R^n_{\text{max}} \times R^n_{\text{max}} \to R_{\text{max}}\) defined [11][13] by

\[
\pi_\mu(x, y) = \min_{1 \leq i \leq n} (x_i + y_i), \quad \forall x = (x_i) \in R^n_{\text{max}}, \forall y = (y_i) \in R^n_{\text{max}}, \tag{86}
\]

with + denoting the usual addition on \(R_{\text{max}}\), and respectively \(\pi_\mu : A^n \times A^n \to A\), where \(A\) is a conditionally complete lattice ordered group, defined [17] by

\[
\pi_\mu(x, y) := \inf_{1 \leq i \leq n} (x_i \otimes y_i), \quad \forall x = (x_i) \in A^n, \forall y = (y_i) \in A^n. \tag{87}
\]
For example, in the latter case the conjugate $f^{c(\pi_\mu)} : A^n \rightarrow \overline{A}$ of a function $f : A^n \rightarrow \overline{A}$ (where $\overline{A}$ is the canonical enlargement of $A$) is a $\otimes$-topical function (see [15, Proposition 6.1]). Note that for $X = A^n$ and $\varphi, \pi_\mu$ of (82), (87) we have

$$\varphi(x, y) = x/y = \inf_{1 \leq i \leq n} (x_i \otimes (y_i^{-1})) = \pi_\mu(x, y^{-1}), \quad \forall x \in A^n, \forall y \in A^n,$$  \tag{88}

where $y^{-1} := (y_i^{-1})_{1 \leq i \leq n}$ (see e.g. [1, Remark 2.4(a)] for $R_{\max}^n$).

b) For the coupling function $\varphi : X \times X \rightarrow \overline{X}$ of (82) and any $f : X \rightarrow \overline{X}$ one can define a function $f^{c(\varphi)_2} : X \rightarrow \overline{X}$ by

$$f^{c(\varphi)_2}(y) := (x/y)/f(x) = f(x)^{-1} \odot x/y, \quad \forall y \in X.$$ \tag{89}

Then for $f \equiv \top$ and $y = \inf X$, by (54), (46), (24) and (89) we have

$$\top^{c(\varphi)}(\inf X) = \sup_{x \in X} \top(x)^{-1}x/\inf X = \sup_{x \in X} \varepsilon x/\inf X = \varepsilon,$$  \tag{90}

$$\top^{c(\varphi)_2}(\inf X) = \sup_{x \in X} \top(x)^{-1} \odot (x/\inf X) = \varepsilon \odot \top = \top.$$ \tag{91}

Thus $\top^{c(\varphi)}(\inf X) \neq \top^{c(\varphi)_2}(\inf X)$ and $\top^{c(\varphi)_2}(\inf X) = \top$, which proves the statements made in Remark (27), for the particular coupling function $\pi = \varphi$ (since usually, extending [12], a conjugation $f \rightarrow f^{c(\pi)}$ should satisfy $\top^{c(\pi)} = \varepsilon$).

c) By (55) (extended to all $y \in X$) and (57), for any $f : X \rightarrow \overline{X}$ and the extreme values $d = \varepsilon$ and $d = \top$ in (55) we have, respectively,

$$f^{c(\varphi)}(y, \varepsilon) = \sup_{x \in X} f(x)^{-1}s_y, \varepsilon(x) = \sup_{x \in X} f(x)^{-1} \varepsilon = \varepsilon, \quad \forall y \in X,$$  \tag{92}

$$f^{c(\varphi)}(y, \top) = \sup_{x \in X} f(x)^{-1}s_y, \top(x) = \sup_{x \in X} f(x)^{-1} \top x/y = f^{c(\varphi)}(y), \quad \forall y \in X.$$ \tag{93}

**Theorem 30** Let $(X, \mathcal{K})$ be a pair satisfying assumptions (A0') and (A1). For a function $f : X \rightarrow \overline{X}$ the following statements are equivalent:

1°. $f$ is topical.

2°. We have (46) and

$$f^{c(\varphi)}(y) = f(y)^{-1}, \quad \forall y \in X.$$ \tag{94}

3°. We have (46) and

$$f^{c(\varphi)}(y) \leq f(y)^{-1}, \quad \forall y \in X.$$ \tag{95}

4°. We have (46) and

$$f^{c(\varphi)}(y, d) = f(y)^{-1}, \quad \forall y \in X, \forall d \in \overline{X}\backslash\{\varepsilon\}.$$ \tag{96}

5°. We have (46) and

$$f^{c(\varphi)}(y, d) \leq f(y)^{-1}, \quad \forall y \in X, \forall d \in \overline{X}\backslash\{\varepsilon\}.$$ \tag{97}

21
Proof. 1° $\Rightarrow$ 2°. If 1° holds, then by Theorem (20a) and Lemma (11A) we have (96) and $f^{-1}(x/y) \leq f(y)^{-1}, \forall x \in X, \forall y \in X$. Consequently by (84) we obtain $f^c(y) \leq f(y)^{-1}$.

In the reverse direction, by (85) we have

$$f^c(y) = \sup_{x \in X} f(x)^{-1}x/y \geq f(y)^{-1}y/y, \quad y \in Y.$$ 

If $y \neq \inf X$, then $y/y = e$ and $f(y)^{-1}y/y = f(y)^{-1}$, so we obtain (94). If $y = \inf X$, then by (83) and (24) we have

$$f^c(y) = \sup_{x \in X} f(x)^{-1}x/\inf X \geq f(\inf X)^{-1} \inf X/\inf X = \top = \top,$$

whence $f^c(\inf X) = \top = e^{-1} = f(\inf X)^{-1}$.

The implication 2° $\Rightarrow$ 3° is obvious.

3° $\Rightarrow$ 1°. If 3° holds, then for all $x, y \in X$ we have, by (84) and (95), $f(x)^{-1}x/y \leq f^c(y) \leq f(y)^{-1}$, whence $f(y)x/y \leq f(x)$ (by Lemma (11)). Hence, by Theorem (20a), $f$ is topical.

1° $\Rightarrow$ 4°. If 1° holds, then by Theorem (20a) we have (49). Assume now 1° and let $x \in X, y \in X \setminus \{\inf X\}, d \in K \setminus \{\varepsilon\}$. Then by (35) (extended to all $y \in X$), 1° and Theorem (20a) we have

$$f(y)s_{y,d}(x) = f(y)\inf\{x/y, d\} \leq f(y)x/y \leq f(x),$$

whence $f(x)^{-1}s_{y,d}(x) \leq f(y)^{-1}$ (by Lemma (11)). Hence by (35) and since $x \in X$ has been arbitrary, we get

$$f^c(y, d) = \sup_{x \in X} \{f(x)^{-1}s_{y,d}(x)\} \leq f(y)^{-1}. \quad (99)$$

On the other hand, by (35), $y \in X \setminus \{\inf X\}$ and (41) we have

$$s_{y,d}(dy) = \inf\{dy/y, d\} = d.$$

Hence, by (35) and (100), for any $f : X \rightarrow K$ (not necessarily topical) and any $y \in X \setminus \{\inf X\}, d \in K \setminus \{\varepsilon\}$ we have

$$f^c(y, d) \geq f(dy)^{-1}s_{y,d}(dy) = f(dy)^{-1}d,$$

whence, since by 1° and $dd^{-1} = e$ there holds

$$f(dy)^{-1}d = (df(y))^{-1}d = dd^{-1}f(y)^{-1} = f(y)^{-1},$$

we obtain the opposite inequality to (99), and hence (96) for $y \in X \setminus \{\inf X\}, d \in K \setminus \{\varepsilon\}$.

Assume now that $y = \inf X$ and $d \in K \setminus \{\varepsilon\}$. Then by (22), (100) and (11),

$$f^c(\inf X, d) = \sup_{x \in X} f(x)^{-1}\inf\{x/\inf X, d\}$$

$$= \sup_{x \in X} f(x)^{-1}\inf\{\top, d\} = \sup_{x \in X} f(x)^{-1}d$$

$$\geq f(\inf X)^{-1}d = e^{-1}d = \top d = \top,$$

22
so we have \( f^{c}(\psi)(\inf X, d) = \top = \varepsilon^{-1} = f(\inf X)^{-1} \), and hence \( (96) \) for all \( d \in K\setminus\{\varepsilon\} \).

Finally, for \( d = \top \) we have, by \( (93) \), \( f^{c}(\psi)(y, \top) = f^{\psi}(\psi), \forall y \in X \), so the equalities of \( 3^{o} \) for \( d = \top \) and \( 2^{o} \) coincide. Hence by \( 1^{o} \) and the implication \( 1^{o} \Rightarrow 2^{o} \) proved above, we have \( (96) \) also for \( d = \top \).

The implication \( 4^{o} \Rightarrow 5^{o} \) is obvious.

If \( 5^{o} \) holds, then for all \( x, y \in X, d \in K\setminus\{\varepsilon\} \), we have, by \( (85) \) and \( (27) \), \( f(x)^{-1}s_{y,d}(x) \leq f^{c}(\psi)(y) \leq f(y)^{-1} \), whence \( f(y)s_{y,d}(x) \leq f(x) \) (by Lemma \( 11 \)). Furthermore, for \( d = \varepsilon \) we have \( f(x)^{-1}s_{y,\varepsilon}(x) = \varepsilon \leq f(y)^{-1} \), whence \( f(y)s_{y,\varepsilon}(x) \leq f(x), \forall x \in X, \forall y \in X \) (by Lemma \( 11 \)). Consequently, by Theorem \( 20a \), \( f \) is topical. \( \square \)

**Remark 31**

a) One cannot add \( d = \varepsilon \) to statement \( 4^{o} \), since by \( (12) \) we have \( f^{c}(\psi)(y, \varepsilon) = \varepsilon, \forall y \in X \), which shows that for a topical function \( f : X \to K \) (hence \( f \not\equiv \top \) by \( (10) \)) one cannot have \( f^{c}(\psi)(y, \varepsilon) = f(y)^{-1}, \forall y \in X \).

b) Alternatively, one can also prove the implication \( 3^{o} \Rightarrow 2^{o} \) of Theorem \( 30 \) as follows: Assume that \( 3^{o} \) holds, that is, \( f^{c}(\psi)(y) = \sup_{x \in X} f(x)^{-1}x/y \leq f(y)^{-1}, \forall y \in X. \) (101)

If \( y \neq \inf X \), then the sup in \( (101) \) is attained at \( x = y \), since \( f(y)^{-1}y/y = f(y)^{-1} \).

Furthermore, if \( y = \inf X \) and there exists \( x_{0} \in X \) such that \( f(x_{0}) \neq \top \), or equivalently, \( f(x_{0})^{-1} \neq \varepsilon \), then

\[
\sup_{x \in X} f(x)^{-1}(x/\inf X) \geq f(x_{0})^{-1}(x_{0}/\inf X) = \top,
\]

whence by \( (40) \) we obtain \( \sup_{x \in X} f(x)^{-1}(x/\inf X) = \top = \varepsilon^{-1} = f(\inf X)^{-1} \).

Finally, if \( f \equiv \top \), then

\[
\sup_{x \in X} x/y = \varepsilon = \top(y)^{-1}, \forall y \in X.
\]

Thus in all cases we have equality in \( (101) \).

c) Theorem \( 30 \) above should be compared with \( 11 \) Theorem 5.3 and \( 15 \) Theorem 6.2; according to the latter a function \( f : A^{n} \to \overline{A} \), where \( A \) and \( \overline{A} \) are as in Remark \( 29a \) above, is “\( \psi \)-topical” (i.e., increasing and “\( \psi \)-homogeneous”) if and only if for the coupling function \( \pi_{\mu} \) of \( (87) \) we have

\[
f^{c}(\pi_{\mu})(y) = [f(y)^{-1}]^{-1}, \forall y \in A^{n}. \) (102)

The reason for this discrepancy between \( (102) \) and Theorem \( 30 \) is shown by formula \( (88) \) above.

d) From Corollary \( 24a \), one obtains characterizations of the functions \( f \) that satisfy the equalities \( (94) - (97) \), namely, for a function \( f : X \to \overline{K} \) the following statements are equivalent:

\( 1^{o} \). We have \( (94) \).
2°. We have \([95]\).

3°. We have \([96]\).

4°. We have \([97]\).

5°. Either \(f\) is topical or \(f \equiv \top\).

Indeed, the implication \(1° \Rightarrow 2°\) is obvious. Furthermore, if we have \([95]\),
so \(\inf_{x \in X} f(x)^{-1} x/y \leq f(y)^{-1}, \forall y \in X\), then by Lemma \([11A]\) we obtain the inequalities \([91]\), whence by Corollary \([24k]\), either \(f\) is topical or \(f \equiv \top\). Conversely, if \(f\) is topical, then \([94]\) holds by Theorem \([30]\) while if \(f \equiv \top\), then \([94]\) holds since

\[
\begin{align*}
\sup_{x \in X} x/y = \sup_{x \in X} x/y = \varepsilon = \top(y)^{-1}, & \quad \forall y \in X.
\end{align*}
\]

Thus \(1° \Rightarrow 2° \Rightarrow 5° \Rightarrow 1°\).

Furthermore, the implication \(3° \Rightarrow 4°\) is obvious, the implication \(4° \Rightarrow 2°\)
holds by \(4°\) applied to \(d = \top\) (see \([93]\)), and the implication \(2° \Rightarrow 5°\) was proved above. Finally, if \(5°\) holds and \(f\) is topical, then \([96]\) holds by Theorem \([30]\) while if \(f \equiv \top\), then \([94]\) holds since

\[
\sup_{x \in X} x/y = \sup_{x \in X} x/y = \varepsilon = \top(y)^{-1}, \quad \forall y \in X, \forall d \in \mathcal{K} \setminus \{\varepsilon\}.
\]

Thus \(3° \Rightarrow 4° \Rightarrow 2° \Rightarrow 5° \Rightarrow 3°\), which completes the proof of the equivalences
\(1° \Leftrightarrow \ldots \Leftrightarrow 5°\).

Lemma 32 Let \((X, \mathcal{K})\) be a pair satisfying assumptions \((A0')\) and \((A1)\) and let
\(\varphi : X \times X \to \mathcal{K}\) be the coupling function \([82]\). Then

a) For each \(y \in X\) the partial function \(\varphi(., y)\) is topical.

b) For each \(x \in X\) the partial function \(\varphi(x, .)\) is anti-topical.

Proof. a) Let \(y \in X\). Then by the properties of extended residuation, for each \(x', x'', x, y \in X\) and \(\lambda \in \mathcal{K}\) we have

\[
\begin{align*}
x' & \leq x'' \Rightarrow \varphi(x', y) = x'/y \leq x''/y = \varphi(x'', y), \\
x & \in X, \lambda \in \mathcal{K} \Rightarrow \varphi(\lambda x, y) = (\lambda x)/y = \lambda(x/y) = \lambda\varphi(x, y).
\end{align*}
\]

b) Let \(x \in X\). Then by the properties of extended residuation, for each \(y', y'', y, x \in X\) and \(\lambda \in \mathcal{K}\) we have

\[
\begin{align*}
y' & \leq y'' \Rightarrow \varphi(x', y') = x'/y' \geq x/y'' = \varphi(x, y''), \\
y & \in X, \lambda \in \mathcal{K} \Rightarrow \varphi(x, \lambda y) = x/\lambda y = \lambda^{-1}\varepsilon x/y = \lambda^{-1}\varepsilon\varphi(x, y). \quad \Box
\end{align*}
\]

Remark 33 a) We know already part a), since in other words it says that the function \(y^0 : X \to \mathcal{K}\) defined by \([82]\) is topical (by \([16]\) when \(y \in X\setminus\{\inf X\}\) and since \((\inf X)^0 = \cdot/\inf X \equiv \top\) by \([22]\)).

b) Lemma \([32]\) should be compared with the fact that for the coupling function \(\pi_\mu : A^n \times A^n \to A\) of \([87]\) the partial functions \(\pi_\mu(., y)\) and \(\pi_\mu(x, .)\) are topical.
Definition 34 If \((X, \mathcal{K})\) is a pair satisfying \((A0'), (A1)\), the Fenchel-Moreau lower conjugate of a function \(f : X \to \mathcal{K}\),

a) associated to the coupling function \(\varphi\) of \((82)\), or briefly, the \(\varphi\)-lower conjugate of \(f\), is the function \(f^{\varphi} : X \to \mathcal{K}\) defined by

\[
f^{\varphi}(y) := \inf_{x \in X} f(x)^{-1} \hat{\odot}(x/y)^{-1} = \inf_{x \in X} (f(x) \odot (x/y)^{-1})^{-1}, \quad \forall y \in X; \quad (103)
\]

b) associated to the coupling function \(\psi\) of \((83)\), or briefly, the \(\psi\)-lower conjugate of \(f\), is the function \(f^{\psi} : X \to \mathcal{K}\) defined by

\[
f^{\psi}(y, d) := \inf_{x \in X} f(x)^{-1} \hat{\odot} \pi_{y,d}(x) = \inf_{x \in X} (f(x) \odot s_{y,d^{-1}}(x))^{-1}, \quad \forall y \in X, \forall d \in \mathcal{K}, \quad (104)
\]

with \(\pi_{y,d}(x)\) of \((70)\).

Here the last equalities in \((103)\) and \((104)\) follow from Theorem 3 and formula \((71)\).

Remark 35 a) For any function \(f : X \to \mathcal{K}\) such that \(f \not\equiv \varepsilon\) the lower conjugate function \(f^{\varphi} : X \to \mathcal{K}\) is topical. Indeed, since \(y_1 \leq y_2\) implies that \((x/y_1)^{-1} \leq (x/y_2)^{-1}\), \(f^{\varphi}\) of \((103)\) is increasing. Furthermore, if \(y \in X \setminus \{\inf X\}, \lambda \in \mathcal{K}\), then by \((6)\) we have

\[
f^{\varphi}(\lambda y) = \inf_{x \in X} f(x)^{-1} \hat{\odot}(x/\lambda y)^{-1} = \inf_{x \in X} f(x)^{-1} \hat{\odot}(\lambda^{-1} x/y)^{-1} = \lambda \inf_{x \in X} f(x)^{-1} \hat{\odot}(x/y)^{-1} = \lambda f^{\varphi}(y);
\]
on the other hand, if \(y = \inf X\) then since by \(f \not\equiv \varepsilon\) there exists \(x_0 \in X\) such that \(f(x_0)^{-1} \not\equiv \top\), we have

\[
f^{\varphi}(\lambda \inf X) = f^{\varphi}(\inf X) = \inf_{x \in X} f(x)^{-1} \hat{\odot}(x/\inf X)^{-1} \leq f(x_0)^{-1} \hat{\odot}(x_0/\inf X)^{-1} = f(x_0)^{-1} \hat{\odot} \varepsilon = \varepsilon,
\]

whence \(f^{\varphi}(\lambda \inf X) = \varepsilon = \lambda f^{\varphi}(\inf X)\), so \(f^{\varphi}\) of \((103)\) is homogeneous, and hence topical.

However, note that for the constant function \(\varepsilon\), that is, \(\varepsilon(x) \equiv \varepsilon, \forall x \in X\), we have

\[
\varepsilon^{\varphi}(y) = \varepsilon^{\varphi}(y, d) = \top, \quad \forall y \in X, \forall d \in \mathcal{K}; \quad (105)
\]

indeed,

\[
\varepsilon^{\varphi}(y) = \inf_{x \in X} \{\varepsilon^{-1} \hat{\odot}(x/y)^{-1}\} = \inf_{x \in X} \{\top \hat{\odot}(x/y)^{-1}\} = \top, \quad \forall y \in X,
\]

and the last equality of \((105)\) follows similarly. Consequently, by Remark 25, the lower conjugate function \(\varepsilon^{\varphi}\) is anti-topical.
b) By \(58\), \(59\) and \(60\), for any \(f : X \to \overline{\mathcal{K}}\) and the extreme values \(d = \varepsilon\) and \(d = \top\) in \(104\) we have, respectively,

\[
\begin{align*}
\theta^\varepsilon(y, \varepsilon) &= \inf_{x \in X} f(x)^{-1} \otimes \Psi_{y, \varepsilon}(x) = \inf_{x \in X} \{ f(x)^{-1} \otimes (x/y)^{-1} \} \quad (106) \\
&= f^\varepsilon(y), \quad \forall y \in X, \\
\theta^\varepsilon(y, \top) &= \inf_{x \in X} f(x)^{-1} \otimes \Psi_{y, \top}(x) = \inf_{x \in X} f(x)^{-1} \otimes \top \quad (107) \\
&= \top, \quad \forall y \in X.
\end{align*}
\]

**Theorem 36** Let \((X, \mathcal{K})\) be a pair satisfying assumptions \((A0')\) and \((A1)\). For a function \(f : X \to \overline{\mathcal{K}}\) the following statements are equivalent:

1°. \(f\) is anti-topical.
2°. We have \(53\) and

\[
\theta^\varepsilon(y) \geq f(y)^{-1}, \quad \forall y \in X. \quad (108)
\]

3°. We have \(53\) and

\[
\theta^\varepsilon(y) = f(y)^{-1}, \quad \forall y \in X. \quad (109)
\]

4°. We have \(53\) and

\[
f^\varepsilon(y, d) \geq f(y)^{-1}, \quad \forall y \in X, \forall d \in \mathcal{K}. \quad (110)
\]

5°. We have \(53\) and

\[
f^\varepsilon(y, d) = f(y)^{-1}, \quad \forall y \in X, \forall d \in \mathcal{K}. \quad (111)
\]

**Proof.** 1° \(\iff\) 3°. If 1° holds, then by Theorem \(20\) b) we have \(53\). Furthermore, by \(103\), Theorem \(20\) b) and Lemma \(11\) B) we have \(108\).

Let us prove now the opposite inequalities

\[
f^\varepsilon(y) \leq f(y)^{-1}, \quad \forall y \in X. \quad (112)
\]

If \(y \in X \setminus \{ \inf X \}\), then by \(103\) and \(11\), we have

\[
f^\varepsilon(y) = \inf_{x \in X} \{ f(x)^{-1} \otimes (x/y)^{-1} \} \leq f(y)^{-1} \otimes (y/y)^{-1} = f(y)^{-1}.
\]

Furthermore, for \(y = \inf X\) we have, by \(103\) and \(53\),

\[
f^\varepsilon(\inf X) = \inf_{x \in X} f(x)^{-1} \otimes \inf X \leq f(\inf X)^{-1} \otimes \inf X = \inf \infty = \top = f(\inf X)^{-1}.
\]

The implication 3° \(\Rightarrow\) 2° is obvious.

2° \(\Rightarrow\) 1°. If 2° holds, then for all \(x, y \in X\) we have, by \(103\) and \(108\),

\[
f(x)^{-1} \otimes (x/y)^{-1} \geq f^\varepsilon(y) \geq f(y)^{-1}. \quad \text{Hence } f(y)^{-1} \otimes (x/y)^{-1} \geq f(x) \text{ (by Lemma } 11\text{) and therefore, by Theorem } 20\text{ b), } f\text{ is anti-topical.} \]

26
1° ⇒ 5°. If 1° holds, then by Theorem [20] b) we have [53]. Assume now 1° and let \( x, y \in X, d \in K \). Then by [58], 1° and Theorem [20] b) we have

\[
f(y) \circ \sup \{ (x/y)^{-1}, d \} \geq f(y) \circ (x/y)^{-1} \geq f(x),
\]

and thus \( f(x)^{-1} \circ \sup \{ (x/y)^{-1}, d \} \geq f(y)^{-1} \) (by Lemma [11]). Hence by [104], and since \( x \in X \) was arbitrary, we get

\[
f^{\theta(y)}(y, d) = \inf_{x \in X} \{ f(x)^{-1} \circ \sup \{ (x/y)^{-1}, d \} \} \geq f(y)^{-1}.
\] (113)

In the opposite direction, by [104] and [58], for any \( f : X \to \tilde{K}, y \in X \setminus \{ \inf X \} \) and \( d \in K \setminus \{ \varepsilon \} \) we have

\[
f^{\theta(y)}(y, d) \leq f(d^{-1}y)^{-1} \circ \sup \{ (d^{-1}y/y)^{-1}, d \} = f(d^{-1}y)^{-1} \circ d.
\]

Hence, since by 1° and \( d \in K \setminus \{ \varepsilon \} \) we have

\[
f(d^{-1}y)^{-1} \circ d = (d \circ f(y))^{-1} \circ d = (d^{-1} \circ f(y)^{-1}) \circ d = f(y)^{-1},
\]

we obtain the opposite inequality to (113), and hence the equality (111) for \( y \in X \setminus \{ \inf X \} \).

Assume now that \( y = \inf X \) and \( d \in K \setminus \{ \varepsilon \} \). Then by (22), (53), (8) and (11) we have

\[
f^{\theta(y)}(\inf X, d) = \inf_{x \in X} f(x)^{-1} \circ \sup \{ (x/\inf X)^{-1}, d \}
= \inf_{x \in X} f(x)^{-1} \circ \sup \{ \varepsilon, d \} = \inf_{x \in X} f(x)^{-1} \circ d
\leq f(\inf X)^{-1} \circ d = 1^{-1} \circ d = \varepsilon \circ d = \varepsilon \circ f(\inf X)^{-1},
\]

so we obtain the opposite inequality to (113), and hence the equality (111) for \( y = \inf X, d \in K \setminus \{ \varepsilon \} \). Thus, the equality (111) holds for all \( y \in X, d \in K \setminus \{ \varepsilon \} \).

Finally, for \( y \in X, d = \varepsilon \) we have, by (106), 1° and the implication 1° ⇒ 2° proved above,

\[
f^{\theta(y)}(y, \varepsilon) = f^{\theta(y)}(y) = f(y)^{-1}.
\]

The implication 5° ⇒ 4° is obvious.

4° ⇒ 1°. If 4° holds, then by (104) and Lemma [11] b), we have [53] and [64]. Consequently, by Theorem [20] b), \( f \) is anti-topical. \(\square\)

The following Corollary of Theorem [35] gives characterizations of the functions \( f \) that satisfy (108)-(111):

**Corollary 37** Let \( (X, K) \) be a pair that satisfies (A0'), (A1). For a function \( f : X \to \tilde{K} \) the following statements are equivalent:

1°. We have (108).
2°. We have (109).
3°. We have (110).
4°. We have (111).
5°. Either \( f \) is anti-topical or \( f \equiv \varepsilon \).
Proof. $1^\circ \Rightarrow 5^\circ$. Assume that we have $1^\circ$ and $f$ is not anti-topical, so $f(\inf X) \neq \top$ (by Theorem 36). Then by (103) and (108) applied to $y = \inf X$ we have

$$\inf_{x \in X} \{ f(x)^{-1} \hat{\otimes} (x / \inf X)^{-1} \} = f^\theta(\inf X) \geq f(\inf X)^{-1},$$

whence, by (22) and since $f(\inf X)^{-1} > \varepsilon$, it follows that

$$f(x)^{-1} \hat{\otimes} \varepsilon = f(x)^{-1} \hat{\otimes} (x / \inf X)^{-1} > \varepsilon, \quad \forall x \in X.$$ 

Therefore we must have $f(x)^{-1} = \top, \forall x \in X$, so $f \equiv \varepsilon$.

$5^\circ \Rightarrow 4^\circ$. Assume $5^\circ$. If $f$ is anti-topical, then we have (111) by Theorem 36.

On the other hand, if $f \equiv \varepsilon$, then (111) holds since by (104) we have

$$\varepsilon^\theta(y, d) = \inf_{x \in X} \{ \varepsilon(x)^{-1} \hat{\otimes} \psi_{y, d}(x) \} = \top = \varepsilon(y)^{-1}, \quad \forall y \in X, \forall d \in K.$$

The implication $4^\circ \Rightarrow 2^\circ$ holds by (106) applied to $d = \top$. Finally, the equivalences $2^\circ \Leftrightarrow 1^\circ$ and $4^\circ \Leftrightarrow 3^\circ$ hold by (112).

Now we shall attempt to apply the second conjugates (i.e., conjugates of conjugates), or briefly, biconjugates, of a function $f : X \to K$, for the study of topical and anti-topical functions. In the particular case of the coupling function $\pi : R_n^{\max} \times R_n^{\max} \to R^{\max}$ of (86), in [11] it has been shown (see [11, Theorem 5.4 and Lemma 5.1]) that $f$ is topical if and only if $f_c(\pi) = f$ (see also [15, Theorem 6.3 and formula (6.32)] for an extension from $R_n^{\max}$ to $A_n$). This approach has used the so-called dual mappings of Moreau (see e.g. [10]), which we shall now try to adapt.

We recall that $K_X$ denotes the set of all functions $f : X \to K$.

Definition 38 Let $(X, K), (Y, K)$ be two pairs satisfying $(A0'), (A1)$.

a) For any coupling function $\pi : X \times Y \to K$ the coupling function $\overline{\pi} : Y \times X \to \overline{K}$ defined by

$$\overline{\pi}(y, x) := \pi(x, y), \quad \forall x \in X, \forall y \in Y \quad (114)$$

will be called the reflexion of $\pi$.

b) The dual of any mapping $u : K_X \to K_Y$ is the mapping $u' : K_Y \to K_X$ defined by

$$h^{u'} := \inf_{g \in K_X} g, \quad \forall h \in K_Y, \quad (115)$$

where we write $g^u$ and $h^{u'}$ instead of $u(g)$ and $u'(h)$ respectively.

c) The bidual of any mapping $u : K_X \to K_Y$ is the mapping $f \to (f^u)^{u'}$ of $K_X$ into $K_X$.

For the Fenchel-Moreau conjugation $u = c(\pi)$ (see Definition 26) we have
Lemma 39 If \((X, \mathcal{K}), (Y, \mathcal{K})\) are two pairs satisfying \((A0'), (A1)\), and \(\pi : X \times Y \to \mathcal{K}\) is a coupling function, then

\[
c(\pi)' = c(\pi).
\] (116)

Proof. By (76), Lemma (11A) and (114), for any \(g \in \mathcal{K}^X\) and \(h \in \mathcal{K}^Y\) we have the equivalences

\[
\begin{align*}
g^{c(\pi)}(y) & \leq h(y), \forall y \in Y \\
\iff & \ g(x)^{-1}\pi(x, y) \leq h(y), \forall x \in X, \forall y \in Y \\
\iff & \ h(y)^{-1}\pi(x, y) \leq g(x), \forall x \in X, \forall y \in Y \\
\iff & \ h(y)^{-1}\varphi(y, x) \leq g(x), \forall x \in X, \forall y \in Y \\
\iff & \ h^{c(\varphi)}(x) \leq g(x), \forall x \in X,
\end{align*}
\] (117)

whence by (116),

\[
h^{c(\pi)'}(x) = \inf_{g \in \mathcal{K}^X} g(x) = \inf_{g \in \mathcal{K}^X} \inf_{g^{c(\pi)}(x) \leq h} g(x) = h^{c(\varphi)}(x), \quad \forall x \in X. \quad \square
\]

Remark 40 a) In the particular case where \(X = Y\) and \(\pi = \varphi : X \times X \to \mathcal{K}\) is the coupling function (82), the \(\varphi\)-conjugate function \(f^{c(\varphi)}\) of any function \(f : X \to \mathcal{K}\) such that \(f(\inf X) \neq \top\) (e.g., of any topical function \(f\)) satisfies

\[
f^{c(\varphi)}(y) = \sup_{x \in X} f(x)^{-1}\varphi(x, y) = \sup_{x \in X} f(x)^{-1}\varphi(y, x) = \sup_{x \in X} f(x)^{-1}(y/x)
\]

\[
\geq f(\inf X)^{-1}(y/\inf X) = \top, \quad \forall y \in X,
\]

whence \(f^{c(\varphi)}(y) = \top, \forall y \in X\), so \(f^{c(\varphi)}\) is anti-topical, while for \(f \equiv \top\) we have

\[
\top^{c(\varphi)}(y) = \sup_{x \in X} \top(x)^{-1}(y/x) = \sup_{x \in X} \varepsilon y/x = \varepsilon, \quad \forall y \in X,
\]

so \(\top^{c(\varphi)}\) is topological. This should be compared with the facts that for any function \(f : X \to \mathcal{K}\), \(f^{c(\varphi)}\) is anti-topical (see Remark 29 a)) and for any function \(f : X \to \mathcal{K}\) such that \(f \neq \varepsilon\), \(f^{\theta(\phi)}\) is topological, while for \(f \equiv \varepsilon\), \(\varepsilon^{\theta(\phi)}\) is anti-topical (see Remark 15b).

b) The inequalities occurring in (117) are equivalent to each of

\[
\pi(x, y) \leq h(y)g(x), \quad \varphi(y, x) \leq g(x)h(y), \quad \forall x \in X, \forall y \in Y, \quad (118)
\]

which might be called “generalized Fenchel-Young inequalities”, because of the particular case of the so-called “natural coupling function” \(\pi : R \times R \to R\) defined by \(\pi(x, y) := xy, \forall x \in R, \forall y \in R\).

In the particular case where \(X = Y\) and \(\pi : X \times X \to \mathcal{K}\) is a symmetric coupling function, that is,

\[
\pi(x, y) = \varphi(y, x), \quad \forall x \in X, \forall y \in X, \quad (119)
\]

Lemma 39 reduces to the following:
**Corollary 41** If \((X, \mathcal{K})\) is a pair satisfying \((A0'), (A1)\), and \(\pi : X \times X \to \overline{\mathcal{K}}\) is a symmetric coupling function, then \(c(\pi)\) is "self-dual", that is,
\[
c(\pi)' = c(\pi).
\]

**Remark 42** In particular, for \(\mathcal{K} = R_{\text{max}}\) and \(X\) an arbitrary set, Corollary 41 has been obtained in [11, Lemma 5.1].

For the Fenchel-Moreau biconjugates \(f^{c(\varphi)c(\varphi)'} := (f^{c(\varphi)})^{c(\varphi)'}\) with respect to the coupling function \(\varphi\) of [82], we obtain

**Theorem 43** If \((X, \mathcal{K})\) is a pair satisfying \((A0'), (A1)\), then for every function \(f : X \to \overline{\mathcal{K}}\) we have
\[
f^{c(\varphi)c(\varphi)'} \leq f.
\]

If \(f : X \to \overline{\mathcal{K}}\) is topical, then
\[
f^{c(\varphi)c(\varphi)'} = f;
\]
the converse is not true, since \((122)\) is also satisfied e.g. for the anti-topical function \(f \equiv \top\).

**Proof.** For any function \(f : X \to \overline{\mathcal{K}}\), applying formula \((115)\) to \(h = f^{c(\varphi)}\) and \(u = c(\varphi)\) we obtain
\[
f^{c(\varphi)c(\varphi)'}(x) := \inf_{g \in X} \{g(x) \leq f(x), \quad \forall x \in X\},
\]
which proves \((121)\).

Furthermore, observe that for any function \(f : X \to \overline{\mathcal{K}}\) we have, using \((116)\) with \(\pi = \varphi\) of [82],
\[
f^{c(\varphi)c(\varphi)'}(x) = (f^{c(\varphi)})^{c(\varphi)}(x), \quad \forall x \in X.
\]

Now let \(f : X \to \overline{\mathcal{K}}\) be a topical function and let \(h = f^{c(\varphi)}\) and \(x \in X\). Then by \((114)\) for \(\pi = \varphi\) of [82] and by \(f^{c(\varphi)}(y) = f(y)^{-1}, \forall y \in X\) (see Theorem 30), we have
\[
f^{c(\varphi)c(\varphi)}(x) = h^{c(\varphi)}(x) = \sup_{y \in X} h(y)^{-1}(\varphi(y, x)) = \sup_{y \in X} h(y)^{-1}(y/x)
\]
\[= \sup_{y \in X} (f^{c(\varphi)}(y)^{-1}(y/x)) = \sup_{y \in X} f(y)(y/x) \geq f(x)(x/x). \quad (124)
\]

If \(x \neq \inf X\), then \(f(x)x/x = f(x)\), and hence by \((121)\),
\[
f^{c(\varphi)c(\varphi)}(x) \geq f(x)x/x = f(x).
\]
On the other hand, if \(x = \inf X\), then \(f(\inf X) = \varepsilon\) (since \(f\) is topical), and hence by \((124)\),
\[
f^{c(\varphi)c(\varphi)}(\inf X) \geq f(\inf X)(\inf X/\inf X) = f(\inf X)\inf X = \varepsilon = f(\inf X).
\]

30
Thus for any topical function \( f : X \to \mathcal{K} \) we have \( f^{c(\varphi)c(\varphi)}(x) \geq f(x), \forall x \in X \), which together with \( 121 \) yields the equality \( 122 \).

Finally, let us show that the converse implication is not true. Indeed, by Remarks \( 27a \) and \( 40a \) we have

\[
\top^{c(\varphi)c(\varphi)} = (\top^{c(\varphi)})^{c(\varphi)} = \varepsilon^{c(\varphi)} = \top,
\]

whence \( 122 \) for \( f \equiv \top \), which, since \( f \equiv \top \) is anti-topical (see Remark \( 25 \)), completes the proof. \( \square \)

**Remark 44** In \( [11] \) Theorem 5.4\] it has been shown that for any topical function \( f : R_{\text{max}}^n \to \mathbb{R} \) the equality \( 122 \) holds for \( \varphi \) replaced by the additive min-type coupling function \( \pi_\mu \) of \( 80 \) and conversely, every function \( f : R_{\text{max}}^n \to \mathbb{R} \) such that \( f^{c(\pi_\mu)c(\pi_\mu)'} = f \) is topical. The proof of \( [11] \) has used the fact that \( \pi_\mu \) is symmetric and hence self-dual, but this is not the case for \( \varphi \); furthermore, \( f^{c(\pi_\mu)} \) is topical for every function \( f : R_{\text{max}}^n \to \mathbb{R} \), but \( f^{c(\varphi)} \) is anti-topical for every function \( f : X \to \mathcal{K} \) (see Remark \( 29a \)), and therefore the arguments of the proof of \( [11] \) Theorem 5.4\] do not work for the case of \( f^{c(\varphi)c(\varphi)'} \).

Another way of characterizing topical and anti-topical functions with the aid of biconjugates is to combine conjugates with lower conjugates as follows:

**Theorem 45** If \( (X, \mathcal{K}) \) is a pair satisfying \( (A0)', (A1) \), then:

a) A function \( f : X \to \mathcal{K} \) is topical if and only if \( f \not\equiv \top \) and

\[
f^{c(\varphi)\theta(\varphi)} = f.
\]

(125)

b) A function \( f : X \to \mathcal{K} \) is anti-topical if and only if

\[
f^{\theta(\varphi)c(\varphi)} = f.
\]

(126)

**Proof.** a) If \( f : X \to \mathcal{K} \) is topical, then \( f \not\equiv \top \) (by \( 40 \)). Furthermore, \( f^{c(\varphi)} \) is anti-topical (by Remark \( 29a \)), and hence by Theorem \( 30 \) applied to the function \( f^{c(\varphi)} \), and Theorem \( 30 \) we obtain

\[
(f^{c(\varphi)})^{\theta(\varphi)}(y) = f^{c(\varphi)}(y)^{-1} = (f(y)^{-1})^{-1} = f(y), \quad \forall y \in X,
\]

so \( f \) satisfies \( 125 \).

Conversely, assume that \( f \) satisfies \( f \not\equiv \top \) and \( 127 \). Then \( f^{c(\varphi)} \not\equiv \varepsilon \), since otherwise by \( 127 \) and \( 105 \) we would obtain \( f(y) = (f^{c(\varphi)})^{\theta(\varphi)}(y) = \varepsilon^{\theta(\varphi)}(y) = \top(y), \forall y \in X \), in contradiction with the assumption \( f \not\equiv \top \). Consequently, by Remark \( 25a \), \( f = (f^{c(\varphi)})^{\theta(\varphi)} \) is topical.

b) If \( f : X \to \mathcal{K} \) is anti-topical, then \( f \not\equiv \varepsilon \) (by \( 53 \)). Consequently, by Remark \( 25a \), \( f^{\theta(\varphi)} \) is topical and hence, by Theorem \( 30 \) applied to \( f^{\theta(\varphi)} \), and Theorem \( 30 \) applied to \( f \), we obtain

\[
f^{\theta(\varphi)c(\varphi)}(y) = f^{\theta(\varphi)}(y)^{-1} = (f(y)^{-1})^{-1} = f(y), \quad \forall y \in X.
\]

Conversely, if \( f \) satisfies \( 120 \), then by Remark \( 29a \), \( f \) is anti-topical. \( \square \)
5 Polars of a set

We shall study the following concept of “polar set” in our framework:

**Definition 46** Let \((X, \mathcal{K})\) be a pair satisfying \((A_0)', (A_1)\), let \(\mathcal{K}\) be enlarged to \(\overline{\mathcal{K}} := \mathcal{K} \cup \{\top\}\) as above, and let \(\pi : X \times Y \to \overline{\mathcal{K}}\) be a coupling function. The \(\pi\)-polar of a subset \(G\) of \(X\) is the subset \(G^o(\pi)\) of \(Y\) defined by

\[
G^o(\pi) := \{y \in Y| \pi(g, y) \leq e, \forall g \in G\} = \{y \in Y| \sup_{g \in G} \pi(g, y) \leq e\}. \tag{127}
\]

In particular, for the coupling function \(\pi = \varphi : X \times X \to \overline{\mathcal{K}}\) of (82) we have the \(\varphi\)-polar of \(G\) defined by

\[
G^o := G^o(\varphi) := \{y \in Y| g/y \leq e, \forall g \in G\} = \{y \in Y| \sup_{g \in G} (g/y) \leq e\}, \tag{128}
\]

which we shall call simply the polar of \(G\).

**Remark 47**

a) In particular, for \(Y = X, \pi : X \times Y \to \overline{\mathcal{K}}\) of (82) and any (non-empty) set \(G \subseteq X\), by (128) and (22) we have \(\inf_X \not\in G^o\).

b) Generalizing a concept of [11, 15], given two pairs \((X, \mathcal{K}), (Y, \mathcal{K})\) satisfying assumptions \((A_0')\) and \((A_1)\), a coupling function \(\pi : X \times Y \to \overline{\mathcal{K}}\), where \(\overline{\mathcal{K}}\) is the minimal enlargement of \(\mathcal{K}\), and a subset \(G\) of \(X\), we shall call \(\pi\)-support function of \(G\) the function \(\sigma_{G,\pi} : Y \to \overline{\mathcal{K}}\) defined by

\[
\sigma_{G,\pi}(y) := \sup_{g \in G} \pi(g, y), \quad \forall y \in Y. \tag{129}
\]

In the sequel we shall consider the particular case when \(Y = X\) and \(\pi = \varphi : X \times X \to \overline{\mathcal{K}}\) of (82). In this case we shall call

\[
\sigma_G(y) := \sigma_{G,\varphi}(y) = \sup_{g \in G} (g/y), \quad \forall y \in X, \tag{130}
\]

the support function of \(G\). Note that for any set \(G \subseteq X, \sigma_G\) is an anti-topical function on \(X\); indeed, for each \(g \in G\), the function \(y \to g/y = \varphi(g, y), \forall y \in X\), is anti-topical (by Lemma 32b)) and the supremum of any family of anti-topical functions is anti-topical. Note also that for \(y = \inf X\) we have

\[
\sigma_G(\inf X) = \sup_{g \in G} (g/\inf X) = \top, \tag{131}
\]

and that the polar of \(G\) is the “e-level set” of \(\sigma_G\), that is,

\[
G^o = \{y \in X| \sigma_G(y) \leq e\}. \tag{132}
\]

c) We recall that a subset \(G\) of \(X\) is said to be downward, if

\[
g \in G, x \in X, x \leq g \Rightarrow x \in G, \tag{133}
\]
and upward if
\[ g \in G, x \in X, x \geq g \Rightarrow x \in G. \quad (134) \]

Let us observe that for any set \( G \) the polar \( G^o = G^{o(\varphi)} \) is an upward set; indeed, if \( y \in G^o, y' \in X, y \leq y' \), then \( y' \in X \setminus \{ \inf X \} \) and \( g/y' \leq g/y \leq e \), so \( y' \in G^o \). Thus \( G^o \) is upward. On the other hand, for the reflexion \( \overline{\varphi} \) of \( \varphi \) (see (114)), the \( \overline{\varphi} \)-polar set
\[ G^{o(\overline{\varphi})} := \{ y \in X|y/g \leq e, \forall g \in G \} \quad (135) \]
is downward; indeed, if \( y \in G^{o(\overline{\varphi})}, y' \leq y \), then \( y'/g \leq y/g \leq e, \forall g \in G \).

We recall (see e.g. \[13\] and the references therein) that for two sets \( X \) and \( Y \) the dual of any mapping \( \Delta : 2^X \to 2^Y \) (where \( 2^X \) denotes the family of all subsets of \( X \)) is the mapping \( \Delta' : 2^Y \to 2^X \) defined by
\[ \Delta'(P) := \{ x \in X|P \subseteq \Delta(\{ x \}) \}, \quad \forall P \subseteq Y; \quad (136) \]
Furthermore, we recall that a mapping \( \Delta : 2^X \to 2^Y \) is called a polarity (in \[13\], \[12\] and earlier references we have used the term duality) if
\[ \Delta(G) = \cap_{g \in G} \Delta(\{ g \}), \quad \forall G \subseteq X. \quad (137) \]
It is well-known and immediate that the dual \( \Delta' \) of any mapping \( \Delta : 2^X \to 2^Y \) is a polarity.

**Definition 48** If \( X, Y \) are two sets and \( \pi : X \times Y \to K \) is a coupling function, the polarity associated to \( \pi \), or briefly, the \( \pi \)-polarity, is the mapping \( \Delta_\pi : 2^X \to 2^Y \) defined by
\[ \Delta_\pi(G) := G^{o(\pi)}, \quad \forall G \subseteq X; \quad (138) \]
with \( G^{o(\pi)} \) of (127); the mapping \( \Delta = \Delta_\pi \) satisfies (137), so it is indeed a polarity.

We have the following extension of \[13\] Lemma 2.1:

**Lemma 49** For the dual \( \Delta'_\pi \) of the mapping \( \Delta = \Delta_\pi \) we have
\[ \Delta'_\pi = \Delta_{\overline{\pi}}, \quad (139) \]
where \( \overline{\pi} \) is the reflexion \[114\] of \( \pi \).

**Proof.** By (136), (138), (127), (114) and (138) (applied to \( \overline{\pi} \) and \( P \)), for any subset \( P \) of \( Y \) we have
\[
\Delta'_\pi(P) = \{ x \in X|P \subseteq \Delta_\pi(\{ x \}) \} = \{ x \in X|\pi(x, y) \leq e, \forall y \in P \} \\
= \{ x \in X|\pi(y, x) \leq e, \forall y \in P \} = \Delta_{\overline{\pi}}(P). \quad \square
\]

In the particular case when \( \pi \) is a symmetric coupling function, that is, satisfies (119), Lemma 49 reduces to the following:

33
Corollary 50 If \((X, K)\) is a pair satisfying (A0'), (A1), and \(\pi : X \times X \to \overline{K}\) is a symmetric coupling function, then \(\Delta_x : 2^X \to 2^X\) is self-dual, that is,
\[
\Delta'_x = \Delta_\pi.
\] (140)

Remark 51 In particular, for \(K = R_{\max}\) and \(X\) an arbitrary set, Corollary 50 has been obtained in [13, Lemma 2.1].

If \(\Delta : 2^X \to 2^Y\) is a polarity, with dual \(\Delta'\), then by definition
\[
\Delta'\Delta(G) := \Delta'((\Delta(G)), \quad \forall G \subseteq X,
\] (141) and a subset \(G\) of \(X\) is called \(\Delta'\Delta\)-convex if \(G = \Delta'\Delta(G)\), or equivalently (see [12]), if for each \(x \notin G\) there exists \(y \in Y\) such that
\[
G \subseteq \Delta'(\{y\}), \quad x \in X \setminus \Delta'(\{y\}).
\] (142)

In other words, a subset \(G\) of \(X\) is \(\Delta'\Delta\)-convex if and only if it is \(M\)-convex, i.e., “convex with respect to the family of sets” \(M := \{\Delta'(\{y\})| y \in Y\} \subseteq X\) (see e.g. [12] and the references therein).

For the \(\pi\)-polarity \(\Delta_\pi : 2^X \to 2^Y\) of (138), \(\Delta_\pi'\Delta_\pi\)-convexity of a set \(G \subseteq X\) means that for each \(x \notin G\) there should exist \(y \in Y\) such that \(y \in \Delta_\pi(G) \setminus \Delta_\pi(\{x\})\), or in other words such that
\[
\sup_{g \in G} \pi(g, y) \leq e, \quad \pi(x, y) \leq e.
\] (143)

For \(Y = X\) and the coupling function \(\pi = \varphi\) of (62), \(\Delta_\varphi'\Delta_\varphi\)-convexity of a set \(G \subseteq X\) means that for each \(x \notin G\) there should exist \(y \in X\) where \(y \in \Delta_\varphi(G) \setminus \Delta_\varphi(\{x\})\), or is equivalent (by Remark 47a)), \(y \in X \setminus \{\inf X\}\), such that
\[
\sup_{g \in G} (g/y) \leq e, \quad x/y \leq e.
\] (144)

In the particular case when \(\leq\) is a total order on \(K\), (143) and (144) become, respectively,
\[
\sup_{g \in G} \pi(g, y) \leq e < \pi(x, y),
\] (145)
and
\[
\sup_{g \in G} (g/y) \leq e < x/y.
\] (146)

Let us consider now the counterpart for “bipolars” \(\Delta_\varphi'\Delta_\varphi(G)\) of the problem mentioned in Remark 14 (namely, to study the functions \(f : X \to \overline{K}\) such that \(f^{(\varphi)(e)} = f\)). Some results on \(\Delta_\varphi'\Delta_\varphi\)-convex sets \(G \subseteq X\), without using this terminology, have been given in [16]. For example, as in [16], let us recall the following conditions on the relations between the topologies of \(K\) and \(X\) (for various other possible conditions see [2] and the references therein):

(A2) For each \(x \in X\) the function \(u_x : \lambda \in K \to \lambda x \in X\) is continuous, that is, for any \(x \in X, \lambda \in K\) and any net \(\{\lambda_n\} \subseteq K\) such that \(\lambda_n \to \lambda\) we have

34
\lambda_n x \to \lambda x \) (here we denote convergence both in \( \mathcal{K} \) and in \( X \) by \( \to \), which will lead to no confusion).

(A3) For each \( y \in X \setminus \{ \inf X \} \) the function \( y^{\circ} (.) = /y : X \to \mathcal{K} \) is continuous.

If (A2) holds, then a set \( G \subseteq X \) is said to be closed along rays if for each \( x \in X \) the set

\[ H_x := \{ \lambda \in \mathcal{K} | \lambda x \in G \} \quad (147) \]

is closed in \( \mathcal{K} \) (that is, for any \( x \in X \) and any net \( \{ \lambda_n \} \subset \mathcal{K} \) with \( \{ \lambda_n x \} \subset G \) such that \( \lambda_n \to \lambda \in \mathcal{K} \) we have \( \lambda x \in G \)).

We can recall now the following result of [16, Theorem 10]:

**Theorem 52** Let \((X, \mathcal{K})\) be a pair satisfying (A0'), (A1), (A2) and (A3). For a subset \( G \) of \( X \) let us consider the following statements:

1°. \( G \) is a closed downward set.

2°. \( G \) is closed along rays and downward.

3°. We have \( \Delta_\varphi \Delta_\varphi (G) = G \), that is, \( G \) is \( \Delta_\varphi \Delta_\varphi \)-convex, where \( \varphi \) is the coupling function \( \langle 82 \rangle \).

Then we have the implications \( 1^\circ \Rightarrow 2^\circ \Rightarrow 3^\circ \).

If the canonical order \( \leq \) on \( \mathcal{K} \) is total and if (A3) holds, then the statements \( 1^\circ \), \( 2^\circ \), and \( 3^\circ \) are equivalent to each other and to the following statements:

4°. For each \( x \in (X \setminus \{ \inf X \}) \) there exists \( \lambda \in X \setminus \{ \inf X \} \) satisfying \( \langle 146 \rangle \).

5°. For each \( x \in (X \setminus \{ \inf X \}) \) there exists \( \lambda \in X \setminus \{ \inf X \} \) such that

\[ \sup_{y \in G} \frac{g(y)}{y} < \frac{x}{y} . \quad (148) \]

In particular, if \( X = \mathcal{K}^n \), where \( \mathcal{K} = R_{\max} \), then the canonical order \( \leq \) and the lattice operations in \( \mathcal{K} \) and \( X \) coincide with the usual order and lattice operations on \( \mathcal{K}^n \) and conditions (A2) and (A3) are satisfied for the order topology (by [5], Corollary 2.11), which is known to coincide with the usual topologies on \( \mathcal{K} \) and \( X \) (see [5], the last observation before Proposition 2.9).

**6 Support set. Support set at a point**

For a pair \((X, \mathcal{K})\) satisfying (A0'), (A1), we shall denote by \( \tilde{T} \) the set of all elementary topical functions \( \tilde{t} = \tilde{t}_y : X \to \mathcal{K} \), that is, all functions of the form

\[ \tilde{t}(x) = \tilde{t}_y (x) := x/y, \quad \forall x \in X, \forall y \in X \setminus \{ \inf X \}; \quad (149) \]

we use here the notation \( \tilde{t}_y \) in order to avoid confusion with the functions \( t_y : X \to \mathcal{K} \) of \( \langle 40 \rangle \), \( \langle 42 \rangle \).

We recall that for a set \( X \), a coupling function \( \pi : X \times X \to R_{\max} \), respectively \( \pi : X \times X \to A \), where \( A = (A, \odot, \odot) \) is a conditionally complete lattice ordered group, and a function \( f : X \to \mathcal{K} \), respectively \( f : X \to A \), the \( \pi \)-support set of \( f \) is the subset of \( X \) defined \( \langle 11 \rangle \langle 15 \rangle \) by

\[ \text{Supp}(f, \pi) := \{ y \in X | \pi(x,y) \leq f(x), \forall x \in X \}. \quad (150) \]
More generally, for a pair \((X, \mathcal{K})\) satisfying (A0\'), (A1) and the coupling function \(\pi = \varphi : X \times X \to \mathcal{K}\) of (32) this leads us to define the following concept of “support set” that will be suitable in our framework:

**Definition 53** Let \((X, \mathcal{K})\) be a pair satisfying (A0\'), (A1). For a function \(f : X \to \mathcal{K}\) we shall call support set of \(f\) (with respect to \(\mathcal{T}\) of (149)) the subset of \(X \setminus \{\inf X\}\) defined by

\[
\text{Supp}(f, \mathcal{T}) := \{y \in X \setminus \{\inf X\} | \tilde{t}_y \leq f\} = \{y \in X \setminus \{\inf X\} | x/y \leq f(x), \forall x \in X\}. \tag{151}
\]

**Remark 54**

a) For any function \(f : X \to \mathcal{K}\), the support set \(\text{Supp}(f, \mathcal{T})\) is upward. Indeed, if \(y, y' \in \text{Supp}(f, \mathcal{T})\), then \(y' \in X \setminus \{\inf X\}\) and

\[
\tilde{t}_y(y') = x/y' \leq x/y = \tilde{t}_y(x), \quad \forall x \in X,
\]

so \(y' \in \text{Supp}(f, \mathcal{T})\).

b) If a function \(f : X \to \mathcal{K}\) is topical then

\[
f(x) = \max_{y \in \text{Supp}(f, \mathcal{T})} \tilde{t}_y(x), \quad \forall x \in X. \tag{152}
\]

Indeed, by (151) we have the inequality \(\geq\) in (152); furthermore, for any \(x \in X\), the equality in (152) is attained for \(y = f(x)^{-1}x \in X \setminus f^{-1}(\varepsilon)\) when \(f(x) \in \mathcal{K}\setminus \{\varepsilon\}\), because then \(x \in X \setminus \{\inf X\}\) (by the topicality of \(f\) and (10)) and

\[
\tilde{t}_y(x) = \tilde{t}_{f(x)^{-1}x}(x) = x/f(x)^{-1}x = f(x)x/x = f(x),
\]

and for any \(y \in \text{Supp}(f, \mathcal{T})\) when \(f(x) = \varepsilon\), because then \(\tilde{t}_y(x) = x/y \leq f(x) = \varepsilon\) implies \(\tilde{t}_y(x) = \varepsilon = f(x)\).

From (152), it follows that for two topical functions \(f_1, f_2 : X \to \mathcal{K}\) we have

\[
f_1 = f_2 \Leftrightarrow \text{Supp}(f_1, \mathcal{T}) = \text{Supp}(f_2, \mathcal{T}), \tag{153}
\]

which shows that for a topical function \(f : X \to \mathcal{K}\) the support set \(\text{Supp}(f, \mathcal{T})\) determines uniquely \(f\). This should be compared with [13] Proposition 7.1.

Generalizing another concept studied in [11] [15], we introduce:

**Definition 55** Let \((X, \mathcal{K})\) be a pair satisfying (A0\'), (A1), \(\pi : X \times X \to \mathcal{K}\) a coupling function and \(x_0 \in X\). For a function \(f : X \to \mathcal{K}\) we shall call support set of \(f\) at \(x_0\) (with respect to \(\mathcal{T}\) of (149)) the subset of \(X \setminus \{\inf X\}\) defined by

\[
\text{Supp}(f, \mathcal{T}; x_0) := \{y \in \text{Supp}(f, \mathcal{T}) | \pi(x_0, y) = f(x_0)\}, \tag{154}
\]

that is,

\[
\text{Supp}(f, \mathcal{T}; x_0) = \{y \in X \setminus \{\inf X\} | \pi(x, y) \leq f(x), \forall x \in X, \pi(x_0, y) = f(x_0)\}. \tag{155}
\]
For a pair \((X, \mathcal{K})\) satisfying \((A0')\), \((A1)\) and the coupling functions \(\pi = \varphi : X \times X \to \mathcal{K}\) and \(\psi : X \times (X \times \mathcal{K}) \to \mathcal{K}\) of \([82]\) and \([84]\) respectively, this leads us to define two concepts of “support set at a point” that will be suitable in our framework: \( \text{Supp}_X(f, \bar{T}; x_0) \) and \( \text{Supp}_{(X, \mathcal{K})}(f, \bar{T}; x_0) \).

**Definition 56** Let \( f : X \to \mathcal{K} \) be a function and let \( x_0 \in X \). We shall call \( X \)-support set of \( f \) at \( x_0 \) the set

\[
\text{Supp}_X(f, \bar{T}; x_0) := \{ y \in X \setminus \{ \inf X \} \mid \bar{t}_y \leq f, \bar{t}_y(x_0) = f(x_0) \},
\]  

(156)

where \( \bar{t}_y : X \to \mathcal{K} \) are the elementary topical functions \([149]\). In other words:

\[
\text{Supp}_X(f, \bar{T}; x_0) = \{ y \in X \setminus \{ \inf X \} \mid x/y \leq f(x), \forall x \in X, x_0/y = f(x_0) \}.
\]  

(157)

**Proposition 57** Let \( f : X \to \mathcal{K} \) be a topical function and let \( x_0 \in X \) be such that \( f(x_0) \in \mathcal{K}\{\varepsilon\} \). Then

\[
x_0 f(x_0)^{-1} \in \text{Supp}_X(f, \bar{T}; x_0) \neq \emptyset.
\]  

(158)

**Proof.** Since \( f(x_0) \in \mathcal{K}\{\varepsilon\} \), by \([3]\), the topicality of \( f \) and Lemma \([19a]\)

we have

\[
x/f(x_0)^{-1} x_0 = f(x_0)x/x_0 \leq f(x), \quad \forall x \in X.
\]  

(159)

Hence defining

\[
y_0 := f(x_0)^{-1} x_0,
\]  

(160)

we obtain \( f(y_0) = f(f(x_0)^{-1} x_0)) = e \neq \varepsilon \), so \( y_0 \in X \setminus f^{-1}(\varepsilon) \subseteq X \setminus \{ \inf X \} \) (by \([40]\)) and

\[
x/y_0 \leq f(x), \forall x \in X, \quad x_0/y_0 = x_0/f(x_0)^{-1} x_0 = f(x_0),
\]  

that is, \( y_0 \in \text{Supp}_X(f, \bar{T}; x_0) \neq \emptyset \). \( \square \)

**Theorem 58** Let \( f : X \to \mathcal{K} \) be a topical function and let \( x_0 \in X \) be such that \( f(x_0) \in \mathcal{K}\{\varepsilon\} \). For an element \( y \in X \setminus \{ \inf X \} \) the following statements are equivalent:

1°. \( y \in \text{Supp}_X(f, \bar{T}; x_0) \).

2°. We have

\[
f(y) = e, \bar{t}_y(x_0) = f(x_0).
\]  

(161)

3°. We have

\[
f(y) = e, \bar{t}_y(x_0) \geq f(x_0).
\]  

(162)

**Proof.** 1° ⇒ 2°. Assume that \( y \in \text{Supp}_X(f, \bar{T}; x_0) \). Then by \([156]\) and \([3]\) we have

\[
yf(x_0) = y\bar{t}_y(x_0) = y(x_0/y) \leq x_0.
\]  

(163)

Hence, since \( f \) is topical, we obtain

\[
f(y)f(x_0) = f(yf(x_0)) \leq f(x_0).
\]  

(164)
But, by our assumption we have \( f(x_0) \in \mathcal{K}\setminus\{\varepsilon\} \), whence multiplying (164) with \( f(x_0)^{-1} \), we obtain
\[
    f(y) \leq e. \tag{165}
\]

On the other hand, using the fact that by (4) we have
\[
    \tilde{t}_y(y) = y/y = e, \quad \forall y \in X\setminus\{\inf X\},
\]
and again that \( y \in \text{Supp}_X(f, \tilde{T}; x_0) \), one has:
\[
    e = \tilde{t}_y(y) \leq f(y). \tag{166}
\]

Combining (165) and (166) yields \( f(y) = e \).

Finally, from \( y \in \text{Supp}_X(f, \tilde{T}; x_0) \) and \( f(y) = e \) it follows that we have 2°.

The implication 2° \( \Rightarrow \) 3° is obvious.

3° \( \Rightarrow \) 1°. Assume that \( y \in X\setminus\{\inf X\} \) satisfies 3°. Then using \( f(y) = e \), the topicality of \( f \) and Lemma 19a), one has
\[
    \tilde{t}_y(x) = e(x/y) = f(y)(x/y) \leq f(x), \quad x \in X. \tag{167}
\]

On the other hand, by 3° we have
\[
    \tilde{t}_y(x_0) = x_0/y \geq f(x_0), \tag{168}
\]
and hence, by (167) (for \( x = x_0 \)) and (168), we obtain
\[
    \tilde{t}_y(x_0) = x_0/y = f(x_0). \tag{169}
\]

From (169) and (167) it follows that \( y \in \text{Supp}_X(f, \tilde{T}; x_0) \). \( \square \)

**Definition 59** Let \( f : X \to \overline{\mathcal{K}} \) be a function and let \( x_0 \in X \). We shall call \((X, \mathcal{K})\)-support set of \( f \) at \( x_0 \) the set
\[
    \text{Supp}_{(X, \mathcal{K})}(f, \tilde{T}; x_0) := \{(y, d) \in (X\setminus\{\inf X\}) \times \mathcal{K} | s_{y,d} \leq f, s_{y,d}(x_0) = f(x_0)\}, \tag{170}
\]
where \( s_{y,d} : X \to \overline{\mathcal{K}} \) are the functions (55), (56), (57).

**Proposition 60** Let \( f : X \to \overline{\mathcal{K}} \) be a topical function and let \( x_0 \in X \) be such that \( f(x_0) \in \mathcal{K}\setminus\{\varepsilon\} \). Then
\[
    (f(x_0)^{-1}x_0, f(x_0)) \in \text{Supp}_{(X, \mathcal{K})}(f, \tilde{T}; x_0) \neq \emptyset. \tag{171}
\]

**Proof.** If \( f \) is topical and \( f(x_0) \in \mathcal{K}\setminus\{\varepsilon\} \), then \( (f(x_0)^{-1}x_0, f(x_0)) \in (X\setminus\{\inf X\}) \times (\mathcal{K}\setminus\{\varepsilon\}) \) and by (169), the topicality of \( f \) and Theorem (20a) we have
\[
    \begin{align*}
        \inf \{x/f(x_0)^{-1}x_0, d\} &= \inf \{f(x_0)x/x_0, d\} \leq \inf \{f(x), d\} \\
        &\leq f(x), \quad \forall x \in X, \forall d \in \mathcal{K}. \tag{172}
    \end{align*}
\]
Consequently,

\[
\begin{align*}
{s_{f(x_0)}^{-1}x_0, f(x_0)}(x) &= \inf \{x/f(x_0)^{-1}x_0, f(x_0)\} \leq f(x), \quad \forall x \in X, \\
{s_{f(x_0)}^{-1}x_0, f(x_0)}(x) &= \inf \{x_0/f(x_0)^{-1}x_0, f(x_0)\} = f(x_0),
\end{align*}
\]

that is, \((f(x_0)^{-1}x_0, f(x_0)) \in \text{Supp}(X, K)(f, \tilde{T}; x_0) \neq \emptyset\). □

**Theorem 61** Let \(f : X \to \bar{K}\) be a topical function and let \(x_0 \in X\) be such that \(f(x_0) \in K \setminus \{\varepsilon\}\). For a pair \((y, d) \in (X \setminus \{\inf X\}) \times K\), the following statements are equivalent:

1°. \((y, d) \in \text{Supp}(X, K)(f, \tilde{T}; x_0)\).
2°. We have \(f(y) = e, s_{y,d}(x_0) = f(x_0)\). \hspace{1cm} (173)
3°. We have \(f(y) = e, s_{y,d}(x_0) \geq f(x_0)\). \hspace{1cm} (174)

**Proof.** 1° \(\Rightarrow\) 2°. Assume that \((y, d) \in \text{Supp}(X, K)(f, \tilde{T}; x_0)\). Then by (170) we have

\[
f(x_0) = s_{y,d}(x_0) = \inf \{x_0/y, d\} \leq x_0/y,
\]
whence by (3),

\[
yf(x_0) \leq y(x_0/y) \leq x_0.
\]

Hence, since \(f\) is topical, we obtain

\[
f(y)f(x_0) = f(yf(x_0)) \leq f(x_0).
\]

But, by our assumption we have \(f(x_0) \in K \setminus \{\varepsilon\}\), whence multiplying (176) with \(f(x_0)^{-1}\) we obtain

\[
f(y) \leq e. \hspace{1cm} (177)
\]

On the other hand, using again that \((y, d) \in \text{Supp}(X, K)(f, \tilde{T}; x_0)\), we have

\[
\inf \{x/y, d\} = s_{y,d}(x) \leq f(x), \quad \forall x \in X,
\]

which for \(x = dy\) gives, using (4) and that \(f\) is homogeneous,

\[
d = \inf \{dy/y, d\} \leq f(dy) = df(y).
\]

Hence, multiplying with \(d^{-1}\) (which exists by \(\varepsilon < f(x_0) = \inf \{x_0/y, d\} \leq d\)), we get

\[
e \leq f(y). \hspace{1cm} (178)
\]

Combining (177) and (178) yields \(f(y) = e\).

Finally, from \((y, d) \in \text{Supp}(X, K)(f, \tilde{T}; x_0)\) and \(f(y) = e\) it follows that we have 2°.

The implication 2° \(\Rightarrow\) 3° is obvious.

39
3° ⇒ 1°. Assume that the pair \((y,d) ∈ (X\{\inf X}) × K\) satisfies 3°. Then using \(f(y) = e\), the topicality of \(f\) and Theorem 20, we obtain

\[s_{y,d}(x) = es_{y,d}(x) = f(y)s_{y,d}(x) ≤ f(x), \quad ∀x ∈ X.\]  \hspace{1cm} (179)

On the other hand, by 3° we have \(s_{y,d}(x_0) ≥ f(x_0)\), and hence the equality \(s_{y,d}(x_0) = f(x_0)\), which, by (179), yields \((y,d) ∈ \text{Supp}(X,K)(f,\bar{T};x_0)\). □

Remark 62 We recall that for a set \(X\), a coupling function \(π: X × X → A\), where \(A = (A, ⊕, ⊗)\) is a conditionally complete lattice ordered group, a function \(f: X → A\) (where \(A\) is the minimal enlargement of \(A\)) and a point \(x_0 ∈ X\) with \(f(x_0) ∈ A\), the \(π\)-subdifferential of \(f\) at \(x_0\) is the subset of \(X\) defined \([15, 9]\]

\[∂_π f(x_0) := \{y_0 ∈ X | π(x,y_0) ⊗ π(x_0,y_0)^{-1} ⊗ f(x_0) ≤ f(x), ∀x ∈ X\}.\]  \hspace{1cm} (180)

For \(π = π_µ\) of (87) some properties of \(∂_π f(x_0)\), e.g. that it contains the \(π\)-support set of \(f\) at \(x_0\) and some other connections between these sets have been proved in \([15\text{ Section 8}]\). This suggests to attempt to introduce, in our framework of a pair \((X,K)\) satisfying \((A0′),(A1)\) and the coupling function \(ϕ: X × X → K\) of \([32]\), the following concept: We shall call \(ϕ\)-subdifferential of a function \(f: X → K\) at a point \(x_0\) with \(f(x_0) ∈ K\), the subset of \(X\) defined by

\[∂_ϕ f(x_0) := \{y_0 ∈ X | (x/y_0)(x_0/y_0)^{-1} f(x_0) ≤ f(x), ∀x ∈ X\}.\]  \hspace{1cm} (181)

The proofs of the properties of \(∂_π f(x_0)\) given in \([15\text{ Section 8}]\) lean heavily on the fact that for each \(x ∈ X\) the partial function \(π_µ(x,.)\) is topical, but in the case of the coupling function \(ϕ: X × X → K\) of \([32]\) for each \(x ∈ X\) the partial function \(ϕ(x,.)\) is anti-topical (see Lemma \([32]\)). Therefore those proofs of \([15]\) cannot be adapted to our general framework.

Acknowledgement

Viorel Nitica was partially supported by a grant from Simons Foundation 208729.

We would like to thank the anonymous referee for careful reading of this paper and for making useful suggestions.

References

[1] M. Akian, S. Gaubert, V. Nitica and I. Singer, Best approximation in max-plus semimodules. Lin. Alg. Appl. 435 (2011), 3261-3296.

[2] M. Akian and I. Singer, Topologies on lattice ordered groups, separation from closed downward sets and conjugations of type Lau. Optimization 52 (2003), 629-672.
[3] G. Birkhoff, Lattice Theory. Colloquium Publications, Vol. 25, American Mathematical Society, 1967.

[4] G. Cohen, S. Gaubert and J.-P. Quadrat, Duality and separation theorems in idempotent semimodules. Lin. Alg. Appl. 379 (2004), 395-422.

[5] G. Cohen, S. Gaubert, J.-P. Quadrat and I. Singer, Max-plus convex sets and functions. In: Idempotent Mathematics and Mathematical Physics (G. L. Litvinov and V. P. Maslov, eds.), Contemporary Math. 377 (2005), 105-129.

[6] J. Dutta, J.-E. Martínez-Legaz and A. M. Rubinov, Monotonic analysis over cones. Optimization 53 (2004), 129-146.

[7] S. Gaubert, Introduction aux systèmes dynamiques à événements discrets. Notes de cours, INRIA, 1999.

[8] G. L. Litvinov, V. P. Maslov and G. B, Shpiz, Idempotent functional analysis: An algebraic approach. Math. Notes 69(5) (2001), 696-729.

[9] J.-E. Martínez-Legaz and I. Singer, On conjugations for functions with values in extensions of ordered groups. Positivity 1 (1997), 193-218.

[10] J.-J. Moreau, Inf-convolution, sous-additivité, convexité des fonctions numériques. J. Math. Pures Appl. (9), 49 (1970), 109–154.

[11] A. M. Rubinov and I. Singer, Topical and sub-topical functions, downward sets and abstract convexity. Optimization 50 (2001), 307-351.

[12] I. Singer, Abstract Convex Analysis. Wiley-Interscience, New York, 1997.

[13] I. Singer, Further applications of the additive min-type coupling function. Optimization 51 (2002), 471-485.

[14] I. Singer, Some relations between linear mappings and conjugations in idempotent analysis. J. Math. Sciences 115 (2003), 2610-2630

[15] I. Singer, On radiant sets, downward sets, topical functions and sub-topical functions in lattice ordered groups. Optimization 53 (2004), 393-428.

[16] I. Singer, Elementary topical functions on b-complete semimodules over b-complete idempotent semifields. Lin. Alg. Appl. 433 (2010), 2139-2146.

[17] I. Singer and V. Nitica, Topical functions on semimodules and generalizations. Lin. Alg. Appl. 437 (2012), 2471-2488.