Dissipation-Range Fluid Turbulence and Thermal Noise

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We revisit the issue of whether thermal fluctuations are relevant for incompressible fluid turbulence, and estimate the scale at which they become important. As anticipated by Betchov in a prescient series of works more than six decades ago, this scale is about equal to the Kolmogorov length, even though that is several orders of magnitude above the mean free path. This result implies that the deterministic version of the incompressible Navier-Stokes equation is inadequate to describe the dissipation range of turbulence in molecular fluids. Within this range, the fluctuating hydrodynamics equation of Landau and Lifschitz is more appropriate. In particular, our analysis implies that both the exponentially decaying energy spectrum and the far-dissipation range intermittency predicted by Kraichnan for deterministic Navier-Stokes will be generally replaced by Gaussian thermal equipartition at scales just below the Kolmogorov length. Stochastic shell model simulations at high Reynolds numbers verify our theoretical predictions and reveal furthermore that inertial-range intermittency can propagate deep into the dissipation range, leading to large fluctuations in the equipartition length scale. We explain the failure of previous scaling arguments for the validity of deterministic Navier-Stokes equations at any Reynolds number and we provide a mathematical interpretation and physical justification of the fluctuating Navier-Stokes equation as an “effective field-theory” valid below some high-wavenumber cutoff Λ, rather than as a continuum stochastic partial differential equation. At Reynolds number around a million, comparable to that in Earth’s atmospheric boundary layer, the strongest turbulent excitations observed in our simulation penetrate down to a length-scale of about eight microns, still two orders of magnitude greater than the mean-free-path of air. However, for longer observation times or for higher Reynolds numbers, more extreme turbulent events could lead to a local breakdown of fluctuating hydrodynamics.

I. INTRODUCTION

The incompressible Navier-Stokes equation for the fluid velocity field $u(x,t)$ as a function of space $x$ and time $t$:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0. \quad (I.1)$$

since their original introduction \cite{I}, have been accepted for more than 100 years as the mathematical model of turbulence in molecular fluids at low Mach numbers and arbitrarily high Reynolds-numbers. These equations have been used as the starting point for statistical theories of turbulence, such as that of Reynolds \cite{II}. In particular, these equations were invoked by Kolmogorov in his celebrated 1941 theory of turbulence (K41), which postulated a universal scaling behavior in the dissipation range of turbulent flow \cite{III} and yielded the exact “4/5th-law” \cite{IV}. There are rigorous derivations of the Navier-Stokes equations at any fixed Reynolds number, no matter how large, in the limit of small Mach number and small Knudsen number, starting from the Boltzmann equation for low-density gases \cite{V,VI} and from stochastic lattice-gas models with no restriction on density \cite{VII}. In fact, by a well-known argument of Corrsin using the K41 turbulence theory \cite{VIII} (see also \cite{IX}, section 7.5), the hydrodynamic approximation which underlies the incompressible Navier-Stokes equation becomes increasingly better the higher the Reynolds number, because the Knudsen number decreases as an inverse power of Reynolds number. Mathematically, Leray \cite{IX,X} has shown that dissipative weak solutions of incompressible Navier-Stokes equation exist globally in time for any initial data of locally finite energy and also that these solutions remain smooth and unique locally in time for smooth initial data. The global smoothness remains an open question \cite{XII} and singularities might possibly appear in finite time for certain initial data at sufficiently high Reynolds number, as conjectured by Leray himself \cite{IX,X}. However, the existing rigorous derivations \cite{V,VI} show that, even if singularities occur so that Leray solutions are non-unique, nevertheless the empirical velocity field will satisfy some Leray solution of incompressible Navier-Stokes equations. The physical and mathematical foundations for basing a theory of turbulence on these equations seems thus quite secure.

There are effects intrinsic to molecular fluids that
are omitted, however, by the incompressible Navier-Stokes equations, most importantly thermal fluctuations\cite{13, 14}. These effects are described instead by an extension of the usual deterministic fluid equations known as fluctuating hydrodynamics, originally due to Landau and Lifshitz\cite{15}, Ch. XVII. For an incompressible fluid satisfying $\nabla \cdot \mathbf{u} = 0$ these equations have the form\cite{16, 20, 21}

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \nabla \cdot \bar{\mathbf{t}}$$ (I.2)

with $\bar{t}_{ij}(x, t)$ a fluctuating stress prescribed as a Gaussian random field with mean zero and covariance

$$\langle \bar{t}_{ij}(x, t) \bar{t}_{kl}(x', t') \rangle = \frac{2
u k_B T}{p} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \times \delta^3(x - x') \delta(t - t')$$ (I.3)

whose realizations are symmetric and traceless, and with Boltzmann’s constant $k_B \equiv 1.38 \times 10^{-23} \text{ m}^2 \text{ kg/sec}^2 \text{ K}$. These equations can be phenomenologically derived by requiring that the equations with the added stochastic terms have the Gibbs measure

$$P_G[\mathbf{u}] = \frac{1}{Z} \exp \left( -\frac{\rho}{2k_B T} \int_\Omega d^3 x |\mathbf{u}(x)|^2 \right)$$ (I.4)

as an invariant measure, in which case (I.3) is known as the fluctuation-dissipation relation. It is very difficult to give the equation (I.2), (I.3), as written, precise mathematical meaning as a stochastic partial differential equation and to show that it has (I.4) as an invariant measure (e.g. see\cite{22}). One approach to make mathematical sense of this equation is as an equivalent Onsager-Machlup action or large-deviations rate function\cite{23, 24}, and in this form it has been rigorously derived for a stochastic lattice gas\cite{7}. However, the common approach of statistical physicists is to regard (I.2), (I.3) as an effective, low-wavenumber field theory that should be truncated at some wave-number cutoff $\Lambda$ which is larger than an inverse gradient-length $\ell^{-1}_v$ of the fluid but smaller than an inverse microscopic length $\lambda^{-1}_{micr}$ (in a gas, the inverse mean-free-path $\lambda^{-1}_{mfp}$). In fact, there are formal physical derivations of fluctuating hydrodynamics in this sense for compressible fluids starting from molecular dynamics\cite{25, 27} and taking the low Mach-number limit yields precisely (I.2), (I.3). Incorporating such a cut-off $\Lambda$, the fluctuating hydrodynamic equations become well-defined stochastic ODE’s for a finite number of Fourier modes, and the corresponding Fourier truncated measure (I.4) is easily checked to be time-invariant. See Appendix A.

There have been, however, only a relatively few studies of the effects of thermal fluctuations on turbulent flows and most of these have focused on the role of weak noise in selecting a unique invariant measure for deterministic Navier-Stokes\cite{28, 30}. The presence of a new dimensional parameter in the fluctuating hydrodynamics equations (I.2), (I.3), the thermal energy $k_B T$, vitiated the similarity analysis of Kolmogorov\cite{3, 4}, who postulated that the only relevant dimensional parameters in the dissipation range of a turbulent flow are the kinematic viscosity $\nu$ and the mean energy dissipation-rate per mass $\varepsilon$. This violation of Kolmogorov’s 1941 analysis is in addition to the defect that he later noted himself\cite{31}, which is that space-time intermittency introduces dependence upon the outer length-scale $L$ of the flow. See\cite{9} for an extensive review. The interplay between these two additional dimensional parameters, $L$ and $k_B T$, is one of the major issues addressed in this work. For high Reynolds-number turbulent flows this interplay raises new questions regarding the precise formulation of fluctuating hydrodynamics, even within the “effective field-theory” point of view. Arguing from K41 theory, the gradient length $\ell_v$, or largest length below which the velocity field is smooth, should be the Kolmogorov scale $\eta = \nu^{3/4} \varepsilon^{-1/4}$. However, space-time intermittency makes the dissipative cut-off fluctuate\cite{32} (or\cite{9}, section 8.5.5) so that at some points $\ell_v < \eta$. In that case, does a cut-off length $\Lambda^{-1}$ exist which satisfies the necessary conditions $\ell_v \gg \Lambda^{-1} \gg \lambda^{-1}_{micr}$? If so, how should $\Lambda$ be chosen in practice? And, most importantly, what significant effects, if any, does thermal noise have on the dynamics and statistics of incompressible turbulence?

We develop answers to all of these questions in this work, and, in particular, argue that thermal noise has profound observable consequences for turbulence. In a following work we shall discuss the inertial range of scales, but here we deal with the dissipation range at scales smaller than the Kolmogorov length $\eta$. This range has been the subject of much theoretical and rigorous mathematical work within the framework of the incompressible Navier-Stokes equation for decades, for example, on the rate of decay of the spectrum\cite{33, 35} and on intermittency in the dissipation range\cite{36, 39, 40}. There have also been intensive recent efforts to study the dissipation range energy spectrum by direct numerical simulations (DNS) of the incompressible Navier-Stokes equations\cite{41, 43}. This is part of a larger program to determine the most extreme events and most singular, smallest-scale structures in a turbulent flow, at lengths far below the Kolmogorov scale\cite{44, 50}. The underlying science question which drives this work is whether the hydrodynamic equations can remain valid during such extreme turbulent events or whether strong singularities can lead to breakdown of the macroscopic, hydrodynamic description. We shall argue that much of this prior theory and simulation work is called into question for turbulence in real molecular fluids and may require substantial modifications, because effects of thermal noise become significant already at length-scales scales right around the Kolmogorov scale. Laboratory experiments are now attempting to probe these small length-scales\cite{42, 51, 52} but, as we shall discuss at length, all current experimental methods lack both the space-time resolution and the sensitivity to measure turbulent velocity fields accurately at sub-Kolmogorov scales. We regard this state of affairs as a crisis in turbulence research, which calls for the development of completely novel experimental techniques.
After the work in this paper was completed, we became aware of a series of remarkable papers published by Robert Betchov starting in the late 1950’s [53–55], which anticipated several of our key conclusions. Betchov not only recognized the significant effects that thermal noise could have on dissipation-range statistics and other phenomena in fluid turbulence, such as transition and predictability, but he also developed the framework of fluctuating hydrodynamics for incompressible fluids [52], independent of Landau and Lifschitz [15]. Betchov carried out a novel experimental investigation with a multi-jet flow created by a perforated box [53], designed to lower as much as possible the space resolution scale of extant hot-wire methods, in order to test his ideas. Unfortunately, despite improving upon the resolution and especially the accuracy of prior experiments by a couple of orders of magnitude, Betchov’s experiments nevertheless lacked the sensitivity required to verify his predicted results. Because he employed linearized equations for his theoretical analysis, Betchov’s predictions mainly regarded 2nd-order statistics, such as energy spectra and one-dimensional dissipation, but he studied also experimentally the velocity-derivative skewness and kurtosis. Our analysis goes well beyond that of Betchov, taking full account of the nonlinearity of the fluid equations of motion and associated phenomena such as inertial-range intermittency, which were unappreciated in his day. However, Betchov’s pioneering work should be more widely known and many of his ideas are still highly relevant today. We shall therefore compare his conclusions with our own results, which serve to confirm and extend his early insights. In the conclusion section V of our paper we shall briefly review Betchov’s experiments and place them in the context of current efforts.

We shall proceed in this paper by developing simple theoretical arguments which are then tested numerically in a reduced dynamical model of turbulence, the Sabra shell model [56, 57]. Our numerical results for this model provide, to our knowledge, the first empirical confirmation of Betchov’s essential predictions anywhere in the literature. Furthermore, based upon these simulations, we shall then formulate more refined theoretical predictions for the dissipation-range of real fluid turbulence. A short report of our most essential physical predictions has been submitted elsewhere [58], but we provide here full details of our numerical study and, furthermore, address the quite subtle and complex issues surrounding the hydrodynamic description of turbulent flows.

The detailed contents of our paper are as follows: In Section II we discuss the incompressible fluctuating hydrodynamics model (II.1) and its physical and mathematical foundations for describing turbulent fluid flow. This includes the formulation of the basic equations (II.1) and an extended dimensional analysis of turbulence taking into account thermal effects (II.2). In particular, we discuss how thermal noise breaks the scaling symmetry of deterministic incompressible Navier-Stokes (I.1) and why standard arguments on its validity for molecular fluids thus fail in the dissipation range of turbulent flows (II.3). We make also a preliminary evaluation of the effects of inertial-range intermittency by a phenomenological multifractal approach, in order to assess possible limitations to a hydrodynamic description of high-Reynolds turbulence (II.4). To test these theoretical ideas we develop in Section II a stochastic Sabra shell model of fluctuating hydrodynamics. We justify carefully the use of this “zero-dimensional” model despite some significant differences from fluctuating hydrodynamics in three space dimensions (III.1), and we discuss its meaning and numerical solution as an effective theory for low wavenumbers (III.2). The main Section IV of our paper presents numerical results on thermal noise effects in turbulence, obtained by simulating the shell model, which confirm our theoretical predictions and motivate additional conjectures for fluid turbulence. After describing the set-up of our turbulence simulations (IV.1), we present results on thermal effects in the statistics of modal energies (IV.2), a comparison of dissipation-range intermittency for the deterministic and stochastic model (IV.3), and a similar comparison of shell-model structure function scaling in the inertial-range as well as the dissipation-range (IV.4). Finally, Section V summarizes our conclusions and outlines directions for future work.

II. FLUCTUATING HYDRODYNAMICS OF TURBULENT FLOW

A. Formulation of the Equations

In order to keep things simple, we take as flow domain a periodic box or 3-torus, $\Omega = L^3$, with volume $V = |\Omega| = L^3$. Then, to discuss steady-state turbulent flow, we shall modify the equation (I.2) in two ways. First, we shall employ the standard theoretical artifice of adding to the local momentum balance an external body force $\rho f$ to drive the turbulent flow, assuming that this force is supported in Fourier space at very low wavenumbers $\sim 1/L$. Second, and very essentially, we shall assume that the velocity field is comprised of Fourier modes with wavenumbers $|k| < \Lambda$ only, a condition which may be expressed in terms of a projection operator $P_\Lambda$:

$$u(x) = P_\Lambda u(x) := \frac{1}{V} \sum_{|k| \leq \Lambda} e^{ik \cdot x} \hat{u}_k,$$

and then the fluctuating hydrodynamic equation (I.2) is modified to

$$\partial_t u + P_\Lambda (u \cdot \nabla) u = -\nabla p + \nu \Delta u + \left( \frac{2\nu k_B T}{\rho} \right)^{1/2} \nabla \cdot \eta_\Lambda + f,$nabla \cdot u = 0.$$

Separating out the covariance of the thermal noise facilitates our scaling analysis below. Thus $\eta_\Lambda(x, t)$ is a tensor.
spacetime Gaussian field with mean zero and covariance

\[ \langle \eta_{ij}(x, t) \eta_{kl}(x', t') \rangle = \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ij} \delta_{kl} \right) \times \delta^3(\mathbf{x} - \mathbf{x}') \delta(t - t'). \]

for

\[ \delta^3(\mathbf{x} - \mathbf{x}') = P_\Lambda \delta^3(\mathbf{x} - \mathbf{x}') = \frac{1}{V} \sum_{|k| \leq \Lambda} e^{ik \cdot \mathbf{x}}. \]

The kinematic pressure \( p \) in (II.1) is determined by the requirement that \( u \) be solenoidal and obviously satisfies the condition \( p = \rho \Lambda^2 \). In contrast to the original equation (I.2) with no UV cut-off \( \Lambda \), which is mathematically ill-defined \emph{a priori}, the equation (II.1) is equivalent to a system of Itô stochastic differential equations for the Fourier modes \( \mathbf{u}(k) \) of the velocity field and its solutions are stochastically well-posed. For example, see the lectures of Flandoli [59]. As usual in discussions of fluctuating hydrodynamics [16–19, 25–27], it is assumed that \( \Lambda \) can be chosen to satisfy 1/\( \ell_R \ll \Lambda \ll 1/\ell_{\text{micr}} \).

For a liquid, \( \ell_{\text{micr}} = \frac{\ell}{\ell_{\text{mfp}}} \) is the mean interparticle distance \( \ell_{\text{mfp}} \equiv n^{-1/3} \) defined in terms of particle number density \( n \). The condition \( \ell_{\text{mfp}} \ll 1 \) is a minimal requirement that coarse-graining cells of size \( 1/\Lambda \) should contain many molecules. For a low-density gas, the mean-free-path length \( \ell_{\text{mfp}} \gg \ell_{\text{entp}} \) and the condition \( \Lambda \ell_{\text{mfp}} \ll 1 \) guarantees that terms higher than 2nd-order in gradients can be neglected. We thus take \( \ell_{\text{micr}} = \max\{\ell_{\text{entp}}, \ell_{\text{mfp}}\} \).

For a high Reynolds-number turbulent flow where intermittency effects may cause the gradient length \( \ell_\nu \) to be much smaller than the traditional Kolmogorov length \( \ell_\eta \), it is not trivial that the crucial condition \( \ell_\nu \gg \ell_{\text{micr}} \) should be satisfied. Even if so, one must determine how to choose \( \Lambda \) in this range so that predictions are independent of the choice.

The latter problem is highly non-trivial and currently lacks an analytical solution, even at the physical level of rigor [60]. It is expected that the “bare viscosity” must be chosen to have a cut-off-dependent form \( \Lambda \equiv \Lambda(t, \mathbf{x}) \). This is appropriate to non-dimensionalize the equation (II.1) with Kolmogorov dissipation length and velocity scales

\[ \eta = \nu^{3/4} \varepsilon^{-1/4}, \quad u_\eta = (\varepsilon \nu)^{1/4}, \]

by setting

\[ \hat{u} = u/u_\eta, \quad \hat{x} = x/\eta, \quad \hat{t} = (u_\eta/\eta)t, \quad \hat{p} = p/u_\eta^2, \quad \hat{f} = f/F, \quad \hat{\Lambda} = \Lambda \eta. \]

where \( F \) is the typical magnitude of the force per mass (e.g. an r.m.s. value). The equations of motion in dimensionless form become

\[ \partial_t \hat{u} + P_\Lambda (\hat{u} \cdot \hat{\nabla}) \hat{u} = -\nabla \hat{p} + \hat{\Delta} \hat{u} + (2\theta_\eta)^{1/2} \nabla \cdot \hat{\eta}_\Lambda + F_\eta \hat{f}. \]

B. Dimensional Analysis

For our study of the turbulent dissipation range, it is appropriate to non-dimensionalize the equation (II.1) with Kolmogorov dissipation length and velocity scales
with dimensionless temperature and force magnitude:
\[ \theta_\eta = \frac{k_B T}{\rho u_\eta^2 \eta^3}, \quad F_\eta = \frac{F}{u_\eta^2} \]

(II.8)

However, it is more natural to non-dimensionalize the large-scale force with inertial-range units of length \( L \) and velocity \( U = (\varepsilon L)^{1/3} \), so that \( F_\eta = F/L = \frac{U L}{\nu} = \frac{\varepsilon^{1/3} L^{4/3}}{\nu} \)

(II.9)

the dimensionless force magnitude at the integral scale and the Reynolds number, respectively. Scaled in this manner and with hats omitted, the fluctuating hydrodynamic equation (II.1) becomes

\[ \partial_t \mathbf{u} + P_\eta (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{\rho}{\varepsilon} \nabla^2 \mathbf{u} + (2\theta_\eta)^{1/2} \nabla \cdot \mathbf{u} + \frac{F}{Re^{1/4}}. \]

(II.10)

The Reynolds-dependence of the final term just reflects that the direct effect of large-scale forcing will be negligible in the dissipation range for \( Re \gg 1 \).

The thermal noise term is also expected to be small at the Kolmogorov scale. The crucial parameter \( \theta_\eta \) is the ratio of the thermal energy to the energy of the Kolmogorov-scale velocity fluctuations \( u_\eta \) in a spatial region of diameter \( \sim \eta \). It is interesting to consider concrete numbers corresponding to typical values of physical constants for the turbulent atmospheric boundary layer (ABL), taken from the monograph [67]

\[ \varepsilon = 400 \text{ cm}^2/\text{sec}^3, \quad \nu = 0.15 \text{ cm}^2/\text{sec}, \quad \rho = 1.2 \times 10^{-3} \text{ g/cm}^3, \quad T = 300^\circ \text{K}, \]

(II.11)

which gives

\[ \eta = 0.54 \text{ mm}, \quad u_\eta = 2.78 \text{ cm/sec}, \quad \theta_\eta = 2.83 \times 10^{-8} \]

(II.12)

The very small value of \( \theta_\eta \) arises from the small value of Boltzmann’s constant in cgs units, \( k_B = 1.38 \times 10^{-16} \text{ erg/K} \).

Although this number is very small, it however rises rapidly at length-scales \( \ell < \eta \). It can be estimated very crudely by assuming an exponential decay for \( \ell < \eta \), so that the fluid velocity fluctuation level becomes

\[ u_\ell \sim u_\eta \exp(-\eta/\ell). \]

(II.13)

Thus,

\[ \theta_\ell = \frac{k_B T}{\rho u_\eta^2 \ell^3} \sim \theta_\eta \left( \frac{\eta}{\ell} \right)^3 \exp(\eta/\ell) \]

(II.14)

Because \( \theta_\ell \) increases so rapidly for decreasing \( \ell \), one can see that \( \theta_\ell \sim 1 \) already for \( \ell_{eq} \sim \eta/11 \). In the ABL, this length is of the order of 49 \( \mu \text{m} \). For comparison, the mean-free path length of air at room temperature and standard atmospheric pressure is \( \lambda_{mfp} = 68 \text{ mm} \).

Thus, thermal noise becomes of the same order as non-linear terms already at sufficiently large scales where a hydrodynamic description remains valid. Similar estimates hold for other natural turbulent flows, such as the upper ocean mixing layer, and in laboratory experiments performed with a variety of fluids. For example, in the water experiment of Debue et al. [53] at \( Re = 3 \times 10^5 \) and 20\(^\circ\)C, the Kolmogorov scale is \( \eta = 0.016 \text{ mm} \) and \( \theta_\eta = 2.5 \times 10^{-7} \), so that thermal fluctuations become relevant around length-scale \( \ell_{eq} \sim \eta/11 = 1.5 \text{ \mu m} \) which is still much larger than the mean-free-path length of water \( \lambda_{mfp} = 0.25 \text{ nm} \). These cases are typical. We therefore argue that theories of the “far dissipation range” of turbulence which omit thermal noise are of questionable relevance to molecular fluids in Nature.

The example of the ABL and others are reassuring that a cut-off \( \Lambda \) should exist satisfying the fundamental requirement \( \ell_\nu \gg \Lambda^{-1} \gg \ell_{micr} \) for validity of fluctuating hydrodynamics. Since the nonlinearity at length scales \( \ell < \eta \) is weak compared with the thermal noise and the viscous damping, one expects that the velocity fluctuations at those scales should reach thermal equilibrium with a Maxwell-Boltzmann distribution and an equipartition energy spectrum

\[ E(k) \sim \frac{k_B T}{\rho} \frac{4\pi k^2}{(2\pi)^3}. \]

(II.15)

See Appendix A. Since this spectrum is growing in \( k \), it must always exceed the spectrum of the turbulent velocity fluctuations at sufficiently high wavenumber. The physical origin of these large velocity fluctuations is the high speeds of the constituent molecules of the fluid, which are of the order of magnitude of the sound speed \( c_{th} \), or about 343 \text{ m/sec} in air and 1481 \text{ m/sec} in water. While these large velocities almost completely cancel in macroscopic spatial averages by the law of large numbers, the central limit theorem fluctuations grow \( \sim \ell^{-d/2} \) in space dimension \( d \) as the resolved length-scale \( \ell \) decreases toward molecular scales. The crossover wavenumber to see thermal effects can again be crudely estimated by equating the dissipation-range turbulence spectrum with the thermal spectrum

\[ u_\eta^2 \eta \exp(-k\eta) \sim \frac{k_B T}{\rho} k^2 \]

(II.16)

which implies \( \theta_\eta(k\eta)^2 \exp(k\eta) \sim 1 \) and yields an estimate of the crossover length \( \ell_{eq} \sim 1/k_{eq} \sim \eta/12 \) consistent with that found earlier.

At much larger wavenumbers than this, the velocity fluctuations should be close to Gaussian, with statistical independence of modes instantaneously in non-overlapping wavenumber bands. We therefore propose that the UV truncation wavenumber \( \Lambda \) of the fluctuating hydrodynamic equation (II.1) may be chosen anywhere in the range of \( k \) where the equipartition spectrum is achieved and where the high-pass filtered velocity \( \mathbf{u}^k(x, t) \) is a Gaussian random field. Assuming that such a range of wavenumbers exists, we expect that the cutoff \( \Lambda \) may be selected arbitrarily in this range and the
predictions of the model will be insensitive to the particular choice, as long as the bare viscosity $\nu_\Lambda$ is chosen appropriately. No analytical prescription currently exists for this choice but the prescription will be the same as that for the fluid in thermal equilibrium with the given temperature $T$, mass density $\rho$ and cutoff wavenumber $\Lambda$. Thus, it should suffice to choose $\nu_\Lambda$ by matching with equilibrium velocity-velocity correlations from molecular dynamics simulations, as in current practice [19]. Note that the equipartition range which we predict slightly beyond the Kolmogorov wavenumber will be in the “low-wavenumber” weak-coupling range of the thermal equilibrium fluid. One can verify this by substituting the definition (II.8) of $\theta_\eta$ into (II.4) for $Re_\theta^{th}$, giving

$$Re_\theta^{th} = \theta_\eta^{1/2}. \quad (\text{II.17})$$

We see that for $\theta_\eta$ with realistic values, $Re_\theta^{th} \ll 1$ and thus the nonlinear coupling in the thermal equipartition range should remain negligible for several decades of wavenumber above $1/\eta$.

The predicted energy spectrum which emerges from our arguments is illustrated in Figure 1 with a model spectrum proposed by von Kármán [68]

$$E(k) = C_K (\varepsilon L)^{2/3} \frac{L^5 k^4}{(1 + (kL)^2)^{11/6}} \exp(-bk\eta) + Ak^2 \quad (\text{II.18})$$

which has been supplemented with an exponential factor $\exp(-bk\eta)$ to represent decay in the traditional “far-dissipation range” and with also an additive contribution from the thermal equipartition spectrum (II.15) for $A = k_0 \rho T \frac{4\pi}{(2\pi)^3}$. In the plot we have used the parameters (II.11) and also a typical Reynolds number $Re = 10^7$ for the ABL taken from [67]. The exponential-decay factor is consistent with asymptotic predictions [35, 38] and rigorous upper bounds [30, 37] for deterministic Navier-Stokes dynamics, with the coefficient $b = 7$ chosen consistent with numerical observations [11] and with a conventional value $C_K = 1.6$ of the Kolmogorov constant [69].

There observed at this point that our picture of the turbulent energy spectrum was almost entirely anticipated by Betchov in his first paper on the subject [53]. Assuming that the hydrodynamic modes at small scales would reach energy equipartition, he arrived at an expression for the thermal spectrum $E_{\text{noise}}(k)$ identical to our Eq. (II.15), except for an extra overall factor of $3/2$. As previously noted by Hosokawa [28], Betchov did not take into account incompressibility and thus counted 3 degrees of freedom for each independent wavenumber mode rather than 2. By an argument essentially identical to ours, Betchov then determined the wavenumber $k_{eq}$ by matching $E_{\text{noise}}(k)$ with the turbulent energy spectrum $E(k)$ in the dissipation range. For the latter, he assumed a power-law form $E(k) \sim k^{-n}$, with $n = 7$ predicted by the theory of Heisenberg [70] and the same with $n = 6$ more consistent with Betchov’s own experimental results. Although he did not state a quantitative estimate of $k_{eq}$, Betchov plotted his theoretical result for $E_{\text{noise}}(k)$ together with his experimental data for $E(k)$ and, extrapolating the latter, they can be observed to cross at a wavenumber around 10 $\mu$m. In the final sentence of his paper, Betchov concluded that “the gap between turbulence and molecular agitation may not be as wide as is generally appreciated.” In his later work [55], he arrived at the more quantitative estimate that $k_{eq} \eta \approx 1$ by assuming that the turbulent spectrum drops essentially to zero for $k \eta > 1$. His schematic plot of the atmospheric spectrum, entirely analogous to our Fig. 1, indicated an equilibration scale there of order 1 mm. Betchov ended the paper [55] by noting that, because $\eta \gg \lambda_{mfp}$, his conclusions differ from those of von Neumann [71], who had argued that thermal noise becomes relevant only for length scales of order the mean free path and that separation of scales is thus “unambiguous” in turbulent flows.

It should be emphasized that these conclusions are very robust and do not depend upon any particular model of the energy spectrum in the turbulent dissipation-range, as long as that spectrum decays rapidly in wave-number.
We made our estimate of the equilibration scale using the model spectrum (II.18) with exponential decay, but Betchov \[52\] reached the same conclusion assuming a rapid power-law decay. In the Supplemental Materials we show similarly that the “intermediate dissipation range” predicted by Frisch & Vergassola \[40\] using multifractal phenomenological models \[31, 72, 73\] implies also that the equipartition scale is only about an order of magnitude below the Kolmogorov scale. The stretched exponential decay predicted for the dissipation-range spectrum by functional renormalization group arguments \[74\] will likewise lead to the same result. Of course, all of these arguments are phenomenological and the intuitive superposition of spectra for turbulent and thermal fluctuations depends upon the hypothesis that small scales will achieve the same equilibrium distribution in a turbulent flow as in a laminar one. This hypothesis is supported by the standard presumption of weak turbulent fluctuations in the dissipation range, but it must be tested empirically and could even be false due to rare, intermittent bursts of turbulence that penetrate to very small scales \[32\].

The physical arguments of Betchov and ourselves which lead to our proposed picture of the turbulent energy spectrum will be corroborated in section \[11\] by shell-model simulations. Furthermore, the direct effects of thermal noise will be found to exist in those simulations at length-scales much larger than \(1/\kappa_{eq}\) in more refined statistical measures, such as negative-order structure functions or “inverse structure functions” \[75\], which are more sensitive than the energy spectrum to rare low-amplitude events. Equipartition spectra similar to those predicted here have been observed in superfluid turbulence via numerical simulations of the Gross-Pitaevskii equation \[76\], where they correspond to a thermal bath of phonons at high wave-numbers created by the forward energy cascade. Since superfluids have strictly zero viscosity, there is no exact analogue in these simulations of a dissipation range and equipartition spectra there join directly with the Kolmogorov cascade spectra. The same is true for equipartition spectra observed in numerical simulations of decaying turbulence for the truncated Euler system \[77, 78\] where the \(k^3\) spectrum corresponds to thermalized wave-number modes near the spectral cutoff \(\Lambda\) of the model. In contrast to the fluctuating hydrodynamic equations (I.2), which model molecular fluids at mesoscopic scales, the truncated Euler system does not correspond directly to any physical system in nature or in the laboratory. For a more detailed comparison of truncated Euler and related systems with our results in this paper, see section \[V\].

C. Violation of Scale-Symmetry of Navier-Stokes

A possible objection can be raised to our claim that the “far dissipation range” of deterministic Navier-Stokes turbulence is physically unachievable, based upon the well-known space-time scaling symmetry \[19\], section 2.2; \[79\], section 1.2):

\[
u \to u' = \lambda u, \quad x \to x' = \lambda^{-1} x, \quad t \to t' = \lambda^{-2} t\]

which maps an incompressible Navier-Stokes solution \(u(x, t)\) to another solution \(u'(x', t')\) with the same Reynolds number \(Re' = Re\) and with molecular viscosity \(\nu\) also unchanged. This scale symmetry is equivalent to the familiar principle of hydrodynamic similarity. In the presence of an external body force \(f\), one must also take

\[
f \to f' = \lambda^3 f.
\]

This symmetry of incompressible Navier-Stokes equation is the basis of its rigorous derivation from the Boltzmann equation \[5, 6\] or from lattice gas models \[7\] through a scaling limit with \(\lambda \to 0\). Based on such mathematical treatments, one might conclude that the deterministic Navier-Stokes equation should hold in a turbulent flow to any desired accuracy by simply taking the outer length of the flow large enough and the r.m.s. velocity small enough, at whatever Reynolds number desired.

If, however, a solution of the fluctuating Navier-Stokes equation (I.2) in 3D is subjected to the same rescaling (II.19), then

\[
\frac{k_B T}{\rho} \to \frac{k_B T'}{\rho'} = \lambda^3 \frac{k_B T}{\rho} = \frac{\lambda^3 k_B T}{\rho}. \tag{II.21}
\]

Thus, thermal noise breaks the scaling symmetry of the deterministic Navier-Stokes equation, unless temperature can be decreased as \(T \to T' = \lambda^2 T\) and density increased as \(\rho \to \rho' = \lambda^{-5} \rho\), so that thermal noise satisfies (II.21) with kinematic viscosity held fixed \(\nu(\rho', T') = \nu(\rho, T)\). (For example, for an ideal gas \(\nu \propto T^{1/2}/\rho\) and one must take \(a = 2, b = 1\).) The presumed relation (II.21) is the reason that derivation of the incompressible Navier-Stokes equation through a hydrodynamic scaling limit corresponds to weak noise and leads to a large-deviations framework \[7\]. However, even this extended scaling symmetry of the fluctuating equation (I.2) is physically limited, since at low enough temperature and/or high enough density the fluid will undergo a phase transition to a binary gas-liquid mixture or to a solid state. If one instead keeps the temperature and density fixed, then the thermal noise does not become weaker and remains given by the fluctuation-dissipation relation (I.3) for both the solutions \(u\) and \(u'\), which are no longer statistically similar to each other. This violation of scale-symmetry is generally not observed in fluids because Boltzmann’s constant \(k_B\) is so small in macroscopic units (c.g.s. or m.k.s.). However, the breaking of hydrodynamic similarity due to thermal noise can be observed in microfluidics, for example in the mixing at fluid interfaces \[80\].

Specializing these general considerations for the fluctuating hydrodynamic equation (I.2) to the case of forced turbulent flow with time-average power input per mass \(\varepsilon = \langle u \cdot f \rangle\), the scalings (II.19), (II.20) imply that

\[
\varepsilon \to \varepsilon' = \lambda^3 \varepsilon. \tag{II.22}
\]
In decaying turbulence, the same result holds by Taylor’s relation $\epsilon \sim U^3/L$ [81]. The result (II.22) then implies that the Kolmogorov velocity and length transform as

$$ u_\eta = (\nu \epsilon)^{1/4} \to u'_\eta = \lambda u_\eta, \quad \eta = \nu^{3/4} \epsilon^{-1/4} \to \eta' = \lambda^{-1} \eta $$

(II.23)

consistent with the scaling (II.19). It is easy to check that along with the Reynolds number $Re$, also the dimensionless force amplitudes $f_\eta$ and $f_L$ defined in (II.8),(II.9) are invariant under the rescaling (II.19), but that

$$ \theta_\eta \to \theta'_\eta = \lambda \theta_\eta $$

(II.24)

when keeping $\rho$ and $k_B T$ fixed. This result holds because the kinetic energy of Kolmogorov-scale turbulent fluid eddies increases as $\lambda^{-3}$. It is thus possible, in principle, to observe deterministic Navier-Stokes predictions in the dissipation range by taking $u \to u' = \lambda u$ and $x \to x' = \lambda^{-1} x$ with $\lambda \ll 1$. However, in practice, $\lambda$ must be chosen exponentially small, since the relations (II.14),(II.16) show that deterministic Navier-Stokes predictions for the far-dissipation range will hold only up to a wavenumber $k \eta \sim \ln(1/\theta_\eta)$ growing as $\ln(1/\lambda)$. For example, in the case of the ABL we argued below (II.14) that noise would be relevant already at a length-scale $\ell \sim \eta/11$. To make the deterministic predictions valid down to $\ell \sim \eta/22$ would require that the integral length be made $4 \epsilon^{11} = 240,000$ times larger and r.m.s. velocities 240,000 times weaker [82].

Based on these considerations, we argue that an extended far-dissipation range spectrum described by deterministic Navier-Stokes will be practically unobservable both in natural flows and in laboratory experiments. The best hope of achieving a sizable exponential-decay range is probably with a highly viscous fluid at relatively low Reynolds numbers, so that both $\eta$ and $u_\eta$ are made as large as possible. In such a moderate Reynolds number turbulent flow, the dimensionless noise parameter $\theta_\eta$ will be as small as possible. This strategy should work best in liquids where viscosity can be increased by lowering temperature, whereas in gases large kinematic viscosity requires either high temperatures or low densities. In any case, there will be a fundamental difficulty in then, say, doubling the wavenumber extent of such a far-dissipation range spectrum, because this would require a further exponential decrease in $\theta_\eta$, which will be unachievable.

D. Intermittency and Validity of Fluctuating Hydrodynamics

The previous considerations have ignored the phenomenon of small-scale turbulence intermittency, which leads to enhanced fluctuations in the inertial-range that may be described phenomenologically by the Parisi-Frisch multifractal model [93]. This model postulates a dimension spectrum $D(h)$ of velocity Hölder exponents $h$ and suggests the existence of viscous cutoff lengths

$$ \eta_h \sim L Re^{-1/(1+h)} $$

(II.25)

depending upon $h$ which can be much smaller than the classical Kolmogorov length $\eta$, which corresponds to $h = 1/3$ [22]. This model has been invoked to predict an “intermediate dissipation range” (IDR) of scales [40], as an intermediate asymptotics at high-Re between the inertial range where viscosity effects are negligible and the far-dissipation range where viscosity dominates to produce exponential decay of velocity fluctuations. The far-dissipation range is itself predicted to suffer extremely large fluctuations, because the intermittency at lower wavenumbers is intensely magnified by the exponential drop-off in the spectrum [39]. We should therefore discuss how such intermittency might modify the conclusions in the previous section about the choice of $\Lambda$ and indeed about the validity of a fluctuating hydrodynamic description of turbulent flow.

We shall exploit here the Parisi-Frisch multifractal model to address these questions. We note that the multifractal concept of an IDR predicts a collapse of spectra at different $Re$ with a certain scaling [40], and this “multiscaling” has been verified in the GOY shell model [84] and to some degree in direct numerical simulations (DNS) of the Navier-Stokes equations; see [41], Fig.4. On the other hand, there are alternative theoretical ideas about the “near-dissipation range” of turbulent flows, for example, from functional renormalization group (FRG) analyses [74, 85]. Some recent experimental and DNS studies have given stronger support to the FRG predictions than to the multifractal IDR prediction [42, 43, 51]. We therefore invoke the multifractal model only as a heuristic to draw qualitative conclusions. We note, however, that several of the results we obtain below can be confirmed independently by rigorous mathematical arguments.

The considerations of Corrsin [8] on the validity of a hydrodynamic description within the K41 theory of turbulence can be easily extended to the multifractal model; see [86], section III(c). The essential point of Corrsin’s analysis is that viscosity $\nu$ and microscopic length $\lambda_{\text{micro}}$ are not independent quantities, but are instead linked by the standard estimate from kinetic theory

$$ \nu \sim \lambda_{\text{micro}} c_{\text{th}} $$

(II.26)

where $c_{\text{th}}$ is the thermal velocity/sound speed. Combining (II.26) with (II.25) then easily yields

$$ Kn_h := \lambda_{\text{micro}} / \eta_h \sim Ma Re^{-h/(1+h)} $$

(II.27)

as an estimate of the “local Knudsen number” at a point with Hölder exponent $h$, where $Ma = U/c_{\text{th}}$ is the Mach number based on the large-scale flow velocity $U$. We see that the scale separation required for validity of a hydrodynamic description, as quantified by the fundamental condition $Kn_h \ll 1$, will become progressively better with increasing $Re$ for $h > 0$ and $Kn_h \ll 1$ will hold marginally even for $h = 0$ if $Ma \ll 1$.

There is some evidence that the smallest Hölder exponent in incompressible fluid turbulence is $h_{\text{min}} = 0$ [87] (although this conclusion requires the assumption that
negative $h$ cannot occur, e.g. [9], section 8.3, which can be called into question; see section [VC]. If so, then there is a range of possible wavenumber cutoffs $\Lambda$ satisfying

$$1/\eta_h \ll \Lambda \ll 1/\lambda_{micr}. \quad \text{for all } h,$$

(II.28)
as long as $Ma \ll 1$. Taking into account the thermal noise, we can expect for each $h$ that its effects will be manifested in the length scales just slightly below $\eta_h$, where the local energy spectrum (defined e.g. by a wavelet transform) experiences exponential drop-off. In that case, we may further expect that Gaussian, thermal-equilibrium statistics will hold locally at any length-scale $\ell \ll \eta_h$. More precisely, we may consider a locally coarse-grained velocity

$$\bar{\mathbf{u}}(x) = \sum_n \mathbf{v}_n G_{\ell}(x-x_n)/\sum_n G_{\ell}(x_n) \quad \text{(II.29)}$$

where $G_{\ell}(r) = \ell^{-3}G(r/\ell)$ is a smooth, well-localized kernel function and where the sum $\sum_n$ is over molecules of the fluid with positions $x_n$ and velocities $\mathbf{v}_n$. Because the microscopic velocity distributions are close to Maxwellian and because of the central limit theorem [29], we expect to observe nearly Gaussian statistics [38]

$$P(\bar{\mathbf{u}}_x) \propto \exp\left(-C\rho \nu \ell^3 |\bar{\mathbf{u}}_x - \mathbf{v}|^2 / k_BT\right), \quad \ell_{intp} \ll \ell \ll \eta_h \quad \text{(II.30)}$$

with $\mathbf{v} = \bar{\mathbf{u}}_{\eta_h}$ locally at each point in space and time, with Hölder exponent $h = h(x, t)$. Thus, the primary effect of turbulent intermittency should be a strong fluctuation in space and time of the length scale $\eta_h$ below which Gaussian thermal equipartition sets in. Since the ratio $\eta_h/\lambda_{micr}$ grows with increasing $Re$, as we have argued earlier, it should be possible to choose a cutoff satisfying [II.28] without any restriction on $Re$ as long as $Ma \ll 1$.

Our tentative conclusion is that fluctuating Navier-Stokes equation in the form (II.1) should be valid with a suitable choice of wavenumber cutoff $1/\eta_h \ll \Lambda \ll 1/\lambda_{micr}$ for incompressible turbulent flows in the limit $Kn \ll 1$ and $Ma \ll 1$. We shall return to this important issue after presenting and discussing our numerical results for the shell model in the following section (see section [VC]). Most importantly, we have argued that this model predicts a strikingly different behavior than does the deterministic Navier-Stokes equation in the far-dissipation range of turbulent flow, at length scales somewhat smaller than the Kolmogorov scale $\eta$. Whereas the deterministic equations predict an exponential decay of the energy spectrum [35,38] and consequent enhanced intermittency [39,40], the fluctuating equations predict that the energy spectrum about only an order of magnitude above $1/\eta$ should rapidly bottom out and then rise again with increasing wavenumber $k$ in a thermal equipartition $k^3$ spectral range with Gaussian statistics.

### III. SABRA SHELL MODEL OF FLUCTUATING NAVIER-STOKES EQUATION

In the preceding section we have made two fundamental claims: first, that turbulence in low Mach-number molecular fluids is described by the incompressible FNS equations [12] with a suitable high-wavenumber cut-off $\Lambda$ and, second, that those equations predict a thermal equipartition range a decade or so below the Kolmogorov scale rather than a “far-dissipation range” with exponentially decaying energy spectrum. To check the first claim will require novel experimental investigations, which will be discussed later. The second claim is based on physically plausible reasoning, but can be verified by numerical simulations of the FNS equations. Such simulations are possible using existing numerical schemes such as the low Mach number FNS codes in [18,20], which employ finite-volume spatial discretizations and explicit or semi-implicit stochastic Runge-Kutta integrators in time. Motivated by the present study, such computations have been carried out and the results will be discussed briefly later. An alternative numerical approach would be based on the lattice Boltzmann method with thermal fluctuations incorporated at the kinetic level [39,40]. Neither of these numerical schemes even with the most powerful current computers can, however, reach Reynolds numbers comparable to those in the atmospheric boundary layer or even in many engineering flows. Such high Reynolds numbers are particularly relevant to the issue whether a hydrodynamic description is valid for the most extreme turbulent events [14,50], since inertial-range intermittency increases with $Re$. We shall therefore in this paper validate our physical arguments by numerical simulations of a simplified “shell model” of turbulence which can be solved at much higher Reynolds numbers. These models long been used as surrogates of incompressible Navier-Stokes equations, both for physical investigation of high-$Re$ turbulence [91,93] and for comparative mathematical study [94]. Below we discuss the model, introduce a suitable numerical scheme, and then present numerical simulation results on the dissipation-range of the model in a statistical steady state of high-Reynolds turbulence.

#### A. Introduction of the Model

The stochastic model that we consider is based upon the deterministic Sabra shell model [56,57,95,96] which describes the evolution of complex shell variables $\mathbf{u}_n$ for discrete wavenumbers $k_n = k_0 2^n$, $n = 0, 1, 2, ..., N$ via the coupled set of ODE’s

$$d\mathbf{u}_n/dt = B_n[\mathbf{u}] - \nu k_n^2 \mathbf{u}_n + f_n \quad \text{(III.1)}$$

with

$$B_n[\mathbf{u}] = i[k_{n+1} \mathbf{u}_{n+1} u_{n+1}^{*} + (1/2)k_n u_n^{*} u_{n-1}^{*} \mathbf{u}_{n-2}^{*}] \quad \text{(III.2)}$$

$$B_n[\mathbf{u}] = i[k_{n+1} \mathbf{u}_{n+1} u_{n+1}^{*} + (1/2)k_n u_n^{*} u_{n-1}^{*} \mathbf{u}_{n-2}^{*}] \quad \text{(III.2)}$$
Here \( u_n \) represents the “velocity” of an eddy of size \( 1/k_n \), the parameter \( \nu \) is “kinematic viscosity”, and \( f_n \) is an external body-force to stir the system. Shell models have long been studied as convenient surrogates for the Navier-Stokes equation in the numerical study of high-Reynolds-number turbulence [33, 97] but the Sabra model has been especially popular because it enjoys symmetries most similar to those of the incompressible fluid equations, roughly analogous to translation-invariance and scale-invariance. There is no “position space” in the shell model on which a translation group can act, but one can shift phases by defining

\[
\hat{u}^\phi(k, t) = e^{i\phi_n u_n(t)} \tag{III.3}
\]

and if the constraint \( \phi_{n+2} + \phi_{n+1} = \phi_n \) is satisfied, then \( \hat{u}^\phi \) is a solution of the Sabra model whenever \( u \) is a solution. This result holds as well with a deterministic force, if the latter is also transformed as \( f_n^\phi(t) = e^{i\phi_n f_n(t)} \). This symmetry is analogous to the action of space-translations by length displacement \( a \), acting on Fourier modes of the velocity field as

\[
\hat{u}^a(k, t) = e^{i k a} \hat{u}(k, t) \tag{III.4}
\]

Other basic symmetries of the inviscid Sabra model are invariance under continuous time scaling indexed by a real parameter \( \tau > 0 \)

\[
u_n^{(\tau)}(t) = \tau u_n(\tau t) \tag{III.5}
\]

which maps solutions over time interval \([0, T]\) to solutions over the interval \([0, \tau^{-1} T]\) and also under discrete space scaling indexed by natural number \( L \)

\[
u_n^{(L)}(t) = 2L u_{n+L}(t) \tag{III.6}
\]

where \( N^{(L)} = N - L \) and likewise the lowest shell index is shifted from \( M = 0 \) to \( M^{(L)} = M - L \). These are analogous to the scale symmetries of incompressible Euler, according to which for any \( \lambda, \tau > 0 \) the transformation

\[
\hat{u}^{(\lambda, \tau)}(\lambda x, \tau t) = \frac{\tau}{\lambda} \hat{u}(\lambda x, \tau t) \tag{III.7}
\]

maps every Euler solution \( u \) in the space-time domain \( \Omega \times [0, T] \) to another solution \( \hat{u}^{(\lambda, \tau)} \) in the space-time domain \( \lambda^{-1} \Omega \times [0, \tau^{-1} T] \). See [79], section 1.2. Addition of viscous damping in both shell models and real fluids leaves only the restricted symmetry group with the constraint \( \tau = \lambda^2 \) and this remaining scaling symmetry is that which leaves the Reynolds number invariant.

Here we add to the deterministic model \( \text{(II.1)} \) also thermal noise modeled by random Langevin forces

\[
du_n/dt = B_n[u] - \nu k_n^2 u_n + \left( \frac{2 \nu k_B T}{\varrho} \right)^{1/2} k_n \eta_n(t) + f_n, \tag{III.8}
\]

where the complex white-noises \( \eta_n(t) \) have covariance

\[
\langle \eta_n(t) \eta_n^*(t') \rangle = 2 \delta_{nn'} \delta(t - t'), \tag{III.9}
\]

for \( n = 0, 1, ..., N \). Note that the “translation-invariance” symmetry \( \text{(III.3)} \) still holds in the presence of thermal noise, but the remaining viscous scaling symmetry \( \text{(III.7)} \) with \( \tau = \lambda^2 \) is broken. This noisy Sabra model is motivated as the shell-model caricature of the fluctuating Navier-Stokes equation \( \text{(II.1)} \) at absolute temperature \( T \) and mass density \( \varrho \). Note that the “density” \( \varrho \) has units of mass, because the shell model is zero-dimensional, describing a scale hierarchy of turbulent eddies at a single point. As usual in statistical physics, we do not attempt to define this stochastic model for infinitely many shells \( N = \infty \), but instead truncate at some finite \( N \) whose choice is discussed further below. The resulting system of stochastic ODE’s is then well-posed globally in time (see Example 3.3 in [59]). The specific form of the noise term can be justified as that required to make the equilibrium Gibbs distribution

\[
P_G[u] = \frac{1}{Z} \exp(-\beta E[u]) \tag{III.10}
\]

the unique invariant measure for zero-forcing \( f = 0 \), where \( \beta = 1/k_B T \) and

\[
E[u] = \sum_{n=0}^{N} \frac{1}{2} \varrho |u_n|^2 \tag{III.11}
\]

is the shell-model analogue of fluid kinetic energy. In fact, this measure is in detailed balance for the dynamics or time-reversible; see Appendix A for the proof. It is not hard to verify by a modification of this argument that the Langevin forces in \( \text{(III.8)} \) are the only possible white-noise terms that make \( P_G \) invariant, a result often called the “Einstein relation” or “fluctuation-dissipation relation” in statistical physics. These are well-known results in the literature on the fluctuating Navier-Stokes equations, here simply extended to the Sabra shell model.

We should note that the infinite-\( N \) limit of our model \( \text{(III.8, III.9)} \) in the unforced case with \( f_n = 0 \) has been previously studied in [33]. There it was shown that the stochastic dynamical equations with \( N \leq +\infty \) define a time evolution which is strong in the probabilistic sense (i.e. for individual realizations of the noise) and that the corresponding probability measure \( \text{(III.10)} \) with \( N = +\infty \) is time-invariant. We do not have any need to consider such an infinite-\( N \) limit in our study, although mathematical results of this type do bear upon the \( N \)-independence of the statistical predictions of our model, which will be discussed more below. The paper [89] showed also that the unforced, deterministic, inviscid model with \( \nu = 0 \) makes sense in the limit \( N = +\infty \) with initial data chosen from the Gibbs measure \( \text{(III.10)} \), which is again time-invariant. This result is likewise not of direct physical interest. The turbulent flows studied here will be described by (weak) solutions of the inviscid shell model dynamics in the limit \( Re \to \infty, N - M \to \infty \), as mathematically studied in [90, 98], but these weak solutions will dissipate kinetic energy and will not have \( P_G \) as an invariant measure. There are additional in-
variant measures of the deterministic inviscid dynamics, associated for example to the “helicity” invariant $H = \sum_n (-1)^n k_n |u_n|^2$ \cite{93, 97}. However, these measures are not invariant in the presence of viscosity and thermal noise, and are also not of any direct physical interest.

To study our model in the dissipation range, we non-dimensionalize variables in analogy to (II.6), which brings the equation (III.8) to the form

\[ du_n/dt = B_n u_n - k_n^2 u_n + (2\theta_n)^{1/2} k_n \eta_n(t) + f_n/Re^{1/4}, \]

where

\[ Re = \frac{U}{\nu k_0}, \quad \theta_n = \frac{k_B T}{\rho_0 n^2} \]  

(III.13)

and the second parameter is the ratio of thermal energies to kinetic energies of Kolmogorov-scale fluid fluctuations. In these units, the shell index now ranges over values $n = M, ..., R$, where $M = -\lfloor \frac{3}{4} \log_2(Re) \rfloor$, $R = N - M$, with $\lfloor x \rfloor$ denoting the integer part of $x$, and now $n = 0$ corresponds to the Kolmogorov wavenumber. In a physically reasonable correspondence to real fluid turbulence, the parameter $\theta_n$ should be taken extremely small, but fixed independent of $Re$. In our numerical studies in this work, we shall adopt the precise value in (III.12) appropriate to the ABL and which has magnitude $\theta_n \sim 10^{-8}$, but our results do not depend qualitatively upon the particular choice of this parameter.

The existence of the $R \to \infty$ limit of the model (III.12), demonstrated in \cite{98} for $f_n = 0$ and conjectured here with $f_n \neq 0$, is consistent with its much more benign UV-behavior than that of the 3D fluctuating Navier-Stokes model (II.1). These differences in the two models arise both from the strictly local-in-wavenumber couplings of the shell model and also from its lower dimensionality, corresponding to a fluid in space dimension $d = 0$. If the RG analysis of \cite{16, 17} is carried out for the unstirred shell model (III.5) in thermal equilibrium, then it is found that that the dynamics is UV asymptotically-free for $k_n \gg u_{h}/\nu$, with $u_n := (2k_B T/\rho)^{1/2}$. The general result (II.17) implies that $R_n^{th} = \theta_n^{1/2} \ll 1$ and thus the wavenumbers $k_n \eta \gtrsim 1$ are deep in the UV asymptotic-free regime of the thermal Gibbs state. In other words, the modes of the model (III.12) with shell numbers $n \gtrsim 0$ have dynamics given nearly by uncoupled linear Langevin equations and thus variables $u_n$, $u_n'$ in that range with $n \neq n'$ are statistically independent not only instantaneously but also very nearly independent at unequal times. This statistical independence will be verified to hold in our numerical solutions to very good accuracy. It is therefore possible to increase $R \to R + 1$ while keeping the bare-viscosity as a fixed constant $\nu_L = \nu$ and any resultant change in the stochastic dynamics is unobservable (see section (III.B.3). As an aside, we remark that, conversely, the IR dynamics of the noisy shell model (III.8) in the unstirred, thermal equilibrium state at wavenumbers $k_n \ll u_{h}/\nu$ will be strongly coupled and the time-correlations should exhibit nontrivial scaling analogous to that in the $d = 1$ KPZ model \cite{17, 99}.

It must be emphasized that, as a consequence, our shell model (III.12) will underestimate the effects of thermal noise at high wavenumbers. This is due not only to the decrease of thermal noise effects for Sabra compared with increase for Navier-Stokes, but also due to the faster decay of the turbulent energy spectrum in Navier-Stokes without noise. As to the latter, the decay of the energy spectrum in the far-dissipation range of deterministic Navier-Stokes turbulence is expected to be exponential (up to a power-law prefactor), based upon physical theory \cite{35, 38}, rigorous mathematical arguments \cite{36, 37}, and numerical simulations \cite{41}. In the shell model, on the contrary, physical arguments \cite{100, 101}, rigorous bounds \cite{94} and numerical simulations \cite{102} support instead a stretched exponential decay:

\[ \langle |u_n|^2 \rangle \sim \exp(-c(k_n/\gamma)^\gamma), \quad \gamma = \log_2 \left( \frac{1 + \sqrt{5}}{2} \right) \]  

(III.14)

This slower decay in the shell-model means that deterministic nonlinear effects will persist to higher wavenumbers. In addition, the Gibbs measure (III.10) for the shell model corresponds to an energy spectrum

\[ E_n := \frac{\langle |u_n|^2 \rangle}{2k_n} = A/k_n \]  

(III.15)

with $A = k_B T/\rho$. In contrast to the spectrum (III.15) for FNS in $d = 3$, which is growing $\sim k^2$ at high-$k$, the corresponding equipartition spectrum (III.15) for the noisy shell model is decaying $\sim 1/k_n$. Because of these important differences of our model from reality, any thermal effects that we observe in our shell-model simulations should have appreciably greater analogues in real fluids.

**B. Numerical Integration Method**

We now discuss the method for numerical solution of the noisy Sabra model (III.12) which we shall use for our study of the steady-state statistics in this paper.

1. **Slaved Itô-Taylor scheme**

We employ here a slaved 3/2-strong-order Itô-Taylor scheme which was proposed by Lord and Rougemont for parabolic stochastic PDE’s (see \cite{103}, section 6). Because of the extreme stiffness of the shell model dynamics, it is desirable to use a scheme which explicitly solves the fast viscous dynamics by an integrating factor. The method we employ is a close analogue for stochastic equations of the slaved 2nd-order Adams-Bashforth method widely used for numerical simulation of deterministic shell models \cite{104}. The method in \cite{103} is based on stochastic Itô-Taylor expansions and is a slaved version of the 3/2-strong-order method of Platen & Wagner.
see [106], section 10.4 for a detailed discussion. The method of Lord-Rougemont [103] used a Fourier-Galerkin method for spatial discretization that kept Fourier modes $|n| \leq N$, which brings it very close to the formulation of the shell models. It must be emphasized once again however that our view of the noisy Sabra model as an “effective low-wavenumber theory”, although standard in the statistical physics literature, is quite different from the framework of stochastic PDE’s, which requires a limit $N \to \infty$. This difference will inform our discussion of convergence issues further below.

To state the numerical scheme explicitly, we write the noisy Sabra model (III.8) in the form

$$du_n = a_n dt + b_n dW_n, \quad n = 0, 1, ..., N$$

(III.16)

with

$$a_n := B_n[u] - \nu k_n^2 u_n + f_n, \quad b_n := \left(\frac{2k_n T}{\theta}\right)^{1/2} k_n.$$  

(III.17)

The method of [103] solves this system approximately for a discrete set of times $t_k$, $k = 0, 1, 2, ...$ by the iteration

$$u_n(t_{k+1}) = e^{-\nu k_n^2 \Delta t} \left\{ u_n(t_k) + \Delta t [B_n(t_k, u(t_k)) + f_n(t_k)] \right\}$$

$$+ \frac{\Delta t}{2} \frac{\nu k_n^2}{\Delta t} \left[ B_n(t_k, u(t_k)) + f_n(t_k) \right]$$

$$+ \frac{\Delta t}{2} \left[ k_{n+1} (a_{n+2} u_{n+1} + u_{n+2} a_{n+1}) - (1/2) k_n (a_{n+1} u_{n+1}^* + u_{n+1} a_{n+1}^*) + (1/2) k_n (a_{n-1} u_{n-1}^* + u_{n-1} a_{n-1}) \right]$$

$$+ b_n \left[ (1 + \nu k_n^2 \Delta t) D W_n(t_k) - \Delta Z_n(t_k) \right]$$

$$+ i \left[ k_{n+1} (b_{n+2} Z_{n+2}(t_k) u_{n+1} + u_{n+2} b_{n+1} \Delta Z_{n+1}(t_k)) - (1/2) k_n (b_{n+1} \Delta Z_{n+1}(t_k) u_{n-1} + u_{n+1} b_{n-1} \Delta Z_{n-2}(t_k)) \right]$$

$$+ (1/2) k_{n-1} (b_{n-1} \Delta Z_{n-1}(t_k) u_{n-2} + u_{n-1} b_{n-2} \Delta Z_{n-2}(t_k)) \right\},$$

(III.18)

for $n = 0, 1, ..., N$, where

$$\Delta W_n(t_k) := \int_{t_k}^{t_{k+1}} dW_n(t) = W_n(t_{k+1}) - W_n(t_k),$$

(III.19)

$$\Delta Z_n(t_k) := \int_{t_k}^{t_{k+1}} dt \left[ W_n(t) - W_n(t_k) \right].$$

(III.20)

The derivation of this scheme is sketched briefly in Appendix B for completeness. In a practical implementation the pairs of complex random variables $\Delta W_n(t_k)$, $\Delta Z_n(t_k)$ have real and imaginary parts which can be generated from independent $N(0,1)$ random variables by the method in [106], section 10.4, eq.(4.3). For all of our tests of convergence of this scheme we used the same model parameters employed in the long-time steady-state simulation, discussed in detail in the next section IV A.

The convergence proofs in the mathematical literature for this method and related ones [103, 107, 108] employ a joint limit $\Delta t \to 0$ and $N \to \infty$ and require a noise spectrum rapidly converging $b_n \to 0$ as $n \to \infty$. By contrast, the spectrum defined in (III.17) that corresponds to thermal noise has in fact diverging $b_n$, with $b_n \to \infty$ as $n \to \infty$. However, we do not take the limit $N \to \infty$ but consider instead a fixed large $N$, which makes our model a system of stochastic ODE’s, for which standard convergence proofs in the limit $\Delta t \to 0$ apply [109]. Our selection criterion for $N$ is that it must lie in the range of shell-numbers $n$ where the statistics are Gibbsian thermal equilibrium at temperature $T$, with energy equipartition and independent Gaussian distributions of the shell variables $u_n$. We shall show in section IV C for fixed individual realizations of the shell model state vector $u$ which are chosen from the long-time, turbulent steady-state of such an equipartition range in fact appears for all $n \geq N_e$, with some $N_e$, when these data $u$ are evolved under the stochastic dynamics (III.8) for a very short time $\tau_e \sim 1/k_{n_e} u_{n_e}$. Here $N_e = N_e(u)$ depends upon the state vector $u$ which is selected. We shall show furthermore that, for the range of integration times considered (300 large-eddy turnover times), there is a maximum value $N^*_e = \max \{N_e(u(t))\}$ over all times $t$. In our convergence tests below we shall select as initial data $u$ a particular realization $u^*$ such that $N^*_e = N_e(u^*)$, which corresponds to one of these most intense “bursts” which we encountered in our long numerical run. See section IV C where this state $u^*$ is more completely described and, also, Supplemental Materials. We only note here that this state $u^*$ was found to have $N^*_e = N_e(u^*) = n_\eta + 6$, with $n_\eta$ the Kolmogorov shell-number, in a simulation with shell-number cut-off $N = n_\eta + 7 = 22$. Thus, only the two highest wavenumber shells remained in thermal equilibrium for this intense bursting event $u^*$. This is presumably the most stringent test case for convergence of our numerical scheme, which should require the smallest time-step $\Delta t$ and largest truncation shell-number $N$. However, we have tested convergence of our numerical scheme as well for other initial data $u(0)$ selected from the turbulent steady-state and found very similar results.

2. Strong Convergence in $\Delta t$

To check strong (pathwise) convergence in the time-step $\Delta t$ in the model with $N = 22$ shells, we must compare numerical solutions with different time-steps $\Delta t$ for the same realization of the complex Brownian motions $W_n(t)$. This requires a statistically consistent choice of the random variables $\Delta W_n(t_k)$, $\Delta Z_n(t_k)$ for the different step-sizes $\Delta t$. We create such a consistent set by first constructing these pairs by the method of [106], section 10.4, for the smallest time-step $\Delta t = \delta t$. We then construct consistent values for $\Delta t = 2\delta t$ by the combinations
\[ \Delta W^{(1)}_n(t_k^{(1)}) = \Delta W(t_{2k+1}) + \Delta W(t_{2k}) \quad (\text{III.21}) \]

\[ \Delta Z^{(1)}_n(t_k^{(1)}) = \Delta Z(t_{2k+1}) + \Delta Z(t_{2k}) + \Delta t \Delta W(t_{2k}) \quad (\text{III.22}) \]

where \( t_k = k(\delta t) \) and \( t^{(1)}_k = k(2\delta t) \), for \( k = 0, 1, 2, \ldots \).

The formulas (III.21), (III.22) follow easily from the definitions (III.19), (III.20). This procedure may be iterated \( p \) times to produce consistent sets of random increments \( \Delta W^{(p)}_n(t^{(p)}_k), \Delta Z^{(p)}_n(t^{(p)}_k) \) for any time-step of the form \( \Delta t = 2^p \delta t \), with \( t^{(p)}_k = k(2^p \delta t) \) for \( k = 0, 1, 2, \ldots \).

As discussed in the previous section, we use as initial condition for our pathwise convergence study the state \( u^* \) from the strong “burst” which propagated to the highest shell. We define error as the expectation over noise realizations \( E \) of the norm \( \| \cdot \| \) of the difference \( u^\Delta(t) - u^{\delta t}(t) \), where \( u^\Delta(t) \) is the numerical solution starting from \( u^* \) using time-step \( \Delta t \) and \( u^{\delta t}(t) \) is the reference state obtained by integration with the finest time-step \( \delta t \), which we take as the “exact solution”. To make our convergence criterion most sensitive to the largest wavenumbers, we used the enstrophy norm associated to the space \( h_2^1 \), or

\[ \| u \|_{\text{ens}} := \sqrt{\sum_{n=0}^{N} k_n^2 |u_n|^2}. \quad (\text{III.23}) \]

However, we obtained similar results with norms for other spaces such as energy norm for the space \( l_2 \) and sup-norm for the space \( l_{\infty} \) and also for individual shells \( n \). We took \( t = 10^{-3} t_\eta \), where \( t_\eta = \eta/u_n \) is the Kolmogorov time, since this choice of time \( t \) was large enough to obtain appreciable evolution at the highest shells. The results which are plotted in Figure 2 show convincingly that the strong order of convergence of the method is 2.

This is consistent with the mathematical theory (103), which establishes at least \( \frac{3}{2} \)-order (see also Appendix B), but better than we had initially expected. We furthermore found the method to be stronger order 2 for all other initial data that we considered. To illuminate this unexpectedly large convergence rate, we estimated the local truncation error \( T(\Delta t) \) by calculating the error in the method with a single step \( E[\| u^\Delta(\Delta t) - u^{\delta t}(\Delta t) \|_{h_2^1}] \) for each stepsize \( \Delta t \). The results, plotted in Figure 2, show the scaling \( T(\Delta t) \propto (\Delta t)^{5/2} \) that was expected. This would produce a global error scaling as \( (\Delta t)^{3/2} \) in a number of steps \( N_{\text{steps}} \propto \frac{1}{\Delta t} \), if these errors accumulated without cancellation. The observed scaling of global error is explained if the errors at each step are in fact uncorrelated and of different signs, so that, by the central limit theorem, the total error then scales as \( (\Delta t)^{3/2} \sqrt{N_{\text{steps}}} \propto (\Delta t)^2 \).

In conclusion, our numerical scheme (III.18) converges pathwise with order at least \( 3/2 \) (and effectively 2) as \( \Delta t \to 0 \) for a choice of the cutoff \( N > N^* \),

\[ P_N(u, t|u(0), 0) = E[\delta(u - U_N(t); u(0), W)] \quad (\text{III.24}) \]

where \( U_N(t; u(0), W) \) is the (strong) solution of our stochastic shell model (III.8) for number of shells \( N \geq N_c(\delta t) \) with initial data \( u(0) \) and with the particular realization \( W = (W_n : n = 0, 1, \ldots, N) \) of the random Brownian motions over the time-interval \([0, t]\), and \( E[\cdot] \)
again denotes expectation over those random Brownian motions. We present evidence that this transition probability for the set of modes \( u_n(0) = u_n^\star : n = 0, 1, ..., N^\star_n \) is independent of the choice of cut-off \( N \geq N^\star_n \) and also independent of the initial data \( u_n(0) : n = N^\star_n + 1, ..., N \) of the modes with \( n > N^\star_n \) when those are selected at random from the thermal Gibbs distribution for those shells. If this result holds, then we have a well-defined stochastic Markov evolution for the modes \( u_n(0) : n = 0, 1, ..., N^\star_n \), which is independent of the cut-off \( N \). This gives a precise mathematical meaning to the stochastic shell-model [III.8] as a “low-wavenumber effective theory.”

The transition probability density [III.24] of the entire state vector \( u_n : n = 0, 1, ..., N^\star \) is obviously too unwieldy to investigate in its entirety, so we consider instead reduced or marginal PDF’s of \( u_n \) for specific shell-numbers \( n \leq N^\star_n \). We focus our attention on the shell modes with \( n \) near to \( N^\star_n \), since those must be most affected by the change in truncation \( N \). Here we present results for \( n = N^\star_n \) itself, but comparable results are found also for \( n < N^\star_n \). For the time lapse \( t \) in the transition PDF [III.24] we chose \( t = t_n \), one Kolmogorov time. This time should be sufficient for the influence of truncation \( N \) to propagate through the entire dissipation range. Once the transition PDF [III.24] has been verified to be independent of \( N \) for times \( t \leq t_n \), then this independence of course extends to the transition PDF’s for all times \( t > 0 \) by the Chapman-Kolmogorov equation. Here we presume that \( N \)-independence of the transition probability for the most singular event, \( u(0) = u^\star \), implies independence for any other choices of \( u(0) \).

For the study of effects of truncation, we used the same model parameters and time step \( \Delta t \) that were employed in the long-time steady-state simulation, discussed in detail in the next section [IVA]. Increasing \( N \) from its original value \( N = N^\star_n + 1 \) would require a smaller time-step and this would have been numerically expensive. We therefore chose to make the much more demanding test of reducing the cut-off \( N \) to the value \( N = N^\star_n \), taking \( u_{N^\star_n + 1} = 0 \) as boundary condition, and then comparing the transition PDF for the reduced value \( N = N^\star_n \) with that for the original value \( N = N^\star_n + 1 \). We emphasize that for the intense burst event \( u^\star \), only the highest two modes with \( n = N^\star_n, N^\star_n + 1 \) remained in thermal equilibrium, so that our reduction to \( N = N^\star_n \) leaves only a single mode in equipartition. Plotted in Fig. 4 are the reduced transition PDF’s for the real part \( x_{N^\star_n} = \Re(u_{N^\star_n}) \) with both \( N = N^\star_n \) and \( N = N^\star_n + 1 \), calculated by averaging over \( 8 \times 10^4 \) independent samples of the Brownian motions \( W = (W_n : n = 0, 1, ..., N^\star_n + 1) \). The two PDF’s are identical within Monte Carlo error, a strong evidence for statistical decoupling of mode \( u_{N^\star_n + 1} \) from the dynamics of the modes \( u_n \) with \( n < N^\star_n \). We obtained analogous results (not shown here) for the marginal transition PDF of the imaginary part \( y_{N^\star_n} = \Im(u_{N^\star_n}) \) and for the similar variables \( x_n, y_n \) with \( n < N^\star_n \).

These independence results support the claim that we have a well-defined stochastic dynamics whenever the cutoff \( N \) is selected larger than \( N^\star_n \), in agreement with the idea that the shell modes in thermal equilibrium are UV asymptotically free. These results in fact suggest that a convergence result should hold for the idealized mathematical limit \( N \to \infty \). This is hinted also by the rigorous results in [IIB] for the unforced, thermal equilibrium dynamics. Of course, even if such a limit result could be proved for the forced model, it would still not suffice to justify it physically, unless it could be shown even further that convergence is practically attained for a value of \( N^\star_n = \log_2(\Lambda/k_0) \) corresponding to a length-scale \( \Lambda^{-1} \) still much greater than \( \lambda_{mfp} \). Our results for the Sabra model are thus encouraging, because \( N^\star_n = n_\eta + 6 \) corresponds to a length-scale only 64 times smaller than \( \eta \). In the case of the ABL with \( \eta = 0.54 \) mm these events of most extreme intermittency would correspond to a length about 8.4 \( \mu m \), which is still 124 times greater than the mean-free path length of air, \( \lambda_{mfp} = 68 \) nm.

IV. NUMERICAL SIMULATION RESULTS

A. Set-up of the Simulations

We undertook to perform our simulations of the noisy Sabra model in dimensionless form with the dissipation scaling [III.12], so that \( \varepsilon = \nu = 1 \) and the Kolmogorov shell-number is set to \( n = 0 \). We wanted dimensionless parameters in correspondence with the atmospheric boundary layer (ABL), and thus the dimensionless temperature was taken to have the value \( \theta_t = 2.83 \times 10^{-8} \) in [III.12]. The range of shell numbers in our simulation was chosen as \( n = M, M + 1, ..., R \) with \( M = -15 \) and \( R = 7 \) and with constant stirring forces applied at the first two shells, \( M \) and \( M + 1 \). We aimed to achieve a Reynolds number \( Re = u_{rms}^2 M \) comparable to the value \( Re \sim 10^7 \) cited as typical in the ABL [67]. However, with our choice
of forcing, neither of the statistical quantities

\[ u_{rms}^2 = \sum_{n=M}^{R} \langle |u_n|^2 \rangle, \quad \varepsilon = \sum_{n=M}^{R} k_n^2 \langle |u_n|^2 \rangle \]  

(IV.1)

was under our precise control. We therefore adjusted the forcing until we obtained \( u_{rms} \sim O(10^2) \) and \( \varepsilon \sim O(1) \), which was satisfied with the stirring forces

\[ f_M = -0.008900918232183095 \]

\[ -0.0305497603210104 i, \]

\[ f_{M+1} = 0.005116337459331228 \]

\[ -0.018175040700335127 i, \]  

(IV.2)

which gave the precise value \( \varepsilon \approx 1.478 \), close to our target value of unity. We then rescaled all quantities to correct dissipation-scale units by taking

\[ u_n \rightarrow u_n/\varepsilon^{1/4}, \quad k_n \rightarrow k_n/\varepsilon^{1/4}, \quad f_n \rightarrow f_n/\varepsilon^{3/4} \]  

(IV.3)

which yielded \( u_{rms} = 56.48, \quad Re = 2.04 \times 10^6 \), and \( \theta_0 = 2.328 \times 10^{-8} \). All results presented below are in these Kolmogorov units.

The time-step of our simulation was chosen (in original units) to be equal to \( \Delta t = 10^{-5} \) which was about an order of magnitude smaller than the viscous time at the highest wavenumber, \( t_{vis} = 1/k_n^2 \approx 6.1 \times 10^{-5} \). Since one large-eddy turnover-time of the simulation was \( T = 1/k_M u_{rms} \approx 640 \) dimensionless time units, it was too time-consuming to calculate time-averages over many such times \( T \) in a single run of the model on one computer. We therefore took advantage of parallel computing by using a strategy rst computing a long run of the model for \( N_{samp} = 300 \) large-eddy turnover times with increased time-step \( \Delta t = 10^{-3} \) and then creating \( N_{samp} \) independent initial-data by sampling that under-resolved solution in intervals of one turnover time. These 300 independent initial data were then distributed over the nodes of a computer cluster and each integrated again for time \( T \) with the time-step \( \Delta t = 10^{-5} \). To avoid possible early-time artefacts from under-resolution, we discarded the first 100 steps of these well-resolved simulations in calculating all long-time averages (and, in fact, the statistics were checked to change negligibly also by including those initial time-steps).

Both the noisy model (III.8) and the deterministic model (III.1) were solved with the same numerical Taylor-Itô scheme (III.18), the latter simply by setting \( \theta_n = 0 \). The calculations were performed in double-precision arithmetic and only for the deterministic model at the highest wavenumbers did we approach the underflow level of double precision. In fact, it is one of the numerical advantages of stochastic models that the requirements on arithmetic precision are considerably reduced, a fact which has been previously stressed for climate modelling [109].

B. Energy Spectra with Thermal Noise

We first present numerical results on the simplest statistical quantity of interest, the energy spectrum. This result was already reported in [58], but it is included here also for completeness. In order to make thermal energy equipartition as evident as possible, we shall show the average \( \bar{\epsilon}_n \) of the kinetic energy per mass in the mode with shell-number \( n \)

\[ \epsilon_n = \frac{1}{2} |u_n|^2, \]  

(IV.4)

plotted versus wavenumber \( k_n \). This differs slightly from the standard energy spectrum for the shell model which is conventionally defined by \( E_n = \epsilon_n/k_n^2 \), as in (III.15). The quantity (IV.4) is more convenient because thermal equipartition gives a constant value independent of \( n \)

\[ \epsilon_n^{eq} = \theta_n, \]  

(IV.5)

in Kolmogorov dissipation units. In Fig. 5 we plot the energy in mode \( n \) given by (IV.4) for both the deterministic and noisy Sabra models.

The two spectra are indistinguishable in the inertial range, where both exhibit a power-law decay \( \propto k_n^{-\alpha} \) with exponent a bit larger than the K41 value \( \alpha = 5/3 \). The increase of \( \alpha \) above the K41 value is due to standard inertial-range intermittency effects, which do not
differ for the deterministic and noisy models. The behavior of the spectrum in the dissipation-range is, however, drastically different for the two models, with the spectrum of the deterministic model exhibiting a very rapid exponential-type decay and the spectrum of the noisy model saturating at the equilibrium exponential PDF to within numerical errors that arise mostly from the finite number of samples.

C. Dissipation-Range Intermittency

The dissipation range of 3D incompressible fluid turbulence is widely expected to exhibit extreme spatiotemporal intermittency because of the super-algebraic decay of the energy spectrum at high wavenumbers \( k \).

With thermal noise present, the modes at the highest wavenumber not only have energy spectrum in thermal equilibrium, but in fact have all statistics quite accurately described by the gaussian Gibbs measure (III.10). Plotted in Fig. 6 are the PDF’s of the modal energies \( \epsilon_n = \frac{1}{2} |u_n|^2 \) for the four highest shells \( n = 4, 5, 6, 7 \) compared with the exponential PDF \( p(\epsilon) = \beta e^{-\beta \epsilon} \) that follows from (III.10). The energy for the lowest of these modes, \( n = 4 \), agrees well with the exponential PDF out to about 7 standard deviations, but has a distinctly broader tail past this point. The three highest modes with \( n = 5, 6, 7 \) have energy PDF’s that are indistinguishable from the thermal-equilibrium exponential PDF.
inertial-range intermittency propagating into the dissipation range. This “near-dissipation range intermittency” is present also in the deterministic Sabra model and, in a spatiotemporal form, in turbulence modelled by the deterministic Navier-Stokes equation. This is the type of near-singularity due to extreme events intensively studied in recent works \[44–50\]. In fluid turbulence such intermittency is known to lead to strong fluctuations in the “local viscous/cutoff length” \(\eta\) of the velocity field is expected to be smooth, with a local H"{o}lder exponent of the velocity in the Parisi-Frisch multifractal model \[9, 83\].

Below the length-scale \(\eta/u\) the velocity field is expected to be smooth, with a local energy spectrum defined by a suitable band-pass filter that is exponentially decaying. These are the considerations that led Frisch and Vergassola to predict for velocity structure functions an “intermediate dissipation range” \[10\], bridging the inertial-range and the far-dissipation range. A definition of a “local viscous shellnumber” analogous to (IV.7) may be made also in shell models, both deterministic and noisy, by setting

\[
N_{vis}(u) := \min \left\{ n : \frac{|u_n|}{\nu k_n} \leq 1 \right\}.
\]  

and the predictions of \[40\] concerning the intermediate dissipation range were previously verified in a numerical simulation of the GOY shell model \[84\].

We expect that a similar physics lies behind the intermittency displayed in Fig. (b) for our noisy shell model, but with the rapid exponential decay of amplitudes below the viscous cutoff replaced in the noisy model by a thermal equipartition, like that exhibited by the average energy spectrum for shellnumbers \(n \geq 5\) in Fig. 3. As a consequence of the inertial-range intermittency, however, the shell-number at which this equipartition first sets in must fluctuate greatly from realization to realization. It is actually a somewhat subtle issue how to define precisely “energy equipartition” for an individual realization, because equipartition is a statistical concept. Even realizations selected from the thermal Gibbs state \[\text{III.10}\] show considerable fluctuations in energy from the equipartition value and some averaging in time is typically required to bring the modal energy close to the ensemble mean value even for such an equilibrium realization. As one possible measure of the “equilibration shellnumber” \(N_e(u)\) for an individual realization \(u\) from our turbulent simulation, we can average the modal energy \(\epsilon_n = (1/2)|u_n|^2\) over one Kolmogorov time \(t_\eta = \eta/u_\eta\) and we then identify \(N_e(u)\) as the smallest integer such that this local time-average \((1/2)|u_n|^2\) is below \(2\eta\) for all \(n \geq N_e(u)\). For examples of such locally-time-averaged realizations, see Figure 3 in \[58\]. We have plotted in Fig. 8 the PDF of the local equipartition shellnumber \(N_{vis}(u)\) obtained from our DNS with the above definition, together with the PDF of the viscous cutoff shell-number \(N_{vis}(u)\) defined in (IV.8).

Before examining the simulation results, we must first acknowledge that our definition of the “equilibration shell-number” \(N_e(u)\) suffers from a good bit of arbitrariness. We have therefore explored as well alternative definitions. For example, if averaging over time \(t_{avg}\) produces an equipartition spectrum down to shell \(N(t_{avg})\), then averaging over a longer time might extend that range. Since the natural viscous time-scale of the stochastic dynamics at one lower shell increases by \(4\), we have considered another possible definition by successively increasing the averaging time \(t_{avg}\) from \(\eta\) by factors of \(4\) and by redefining \(N_e(u) = N(4t_{avg})\) as long as \(N(4t_{avg}) < N(\eta)\) \[113\]. This alternative definition yielded slightly lower estimates of \(N_e(u)\) for some realizations \(u\), but with the same qualitative features. A completely different approach to defining \(N_e(u)\) would be to apply standard
distribution tests from mathematical statistics, such as 
p-values [114], to the hypothesis that the shell variables
$u_n, u_{K+1}, ..., u_N$ are drawn from the multivariate distri-
bution (III.11) and then define $N_e(u)$ to be the smallest
$K \leq N$ for which that hypothesis is accepted. However,
since any definition of the “equilibration shell-number”
seems to involve various subjective choices and since all
definitions that we have considered exhibit qualitatively
similar intermittency, we shall only discuss here the quan-
tity $N_e(u)$ defined in the previous paragraph.

We first observe in Fig. 8 that the equilibration shell-
number $N_e(u)$ does fluctuate substantially from realiza-
tion to realization, with non-vanishing probability to take
values from -2 to 6 in our ensemble of events gathered
from a simulation of 300 large-eddy turnover times. We
have no theoretical prediction for the PDF of $N_{eq}(u)$,
but we observe that it is fit fairly well in this core range by a standard discrete Gaussian PDF [115] of the form
$p(n) = e^{-(n-2)^2/2}/\Theta$, plotted as well in Fig. 8.

Most interestingly, the PDF’s of $N_{vis}(u)$ and $N_e(u)$ are
roughly similar in form, but that for $N_e(u)$ is shifted
to the right by $2 - 3$ shells. This observation suggests
that the same dynamics is responsible for the fluctua-
tions in both $N_{vis}(u)$ and $N_e(u)$. This supports the pic-
ture we have proposed that inertial-range intermittency
leads to a fluctuating viscous cutoff shellnumber, above
which the amplitude of $u_n$ drops drastically in $n$ with the
stretched-exponential decay (III.14) which is the same for
all realizations. In that case, within a small number of
shells which is nearly independent of $u$ the amplitude of the shell velocity drops to the level $u_{th}$ where thermal
equipartition is achieved. As a further test of this explana-
tion, we have also calculated with our DNS the PDF of
the shift $\Delta N(u) = N_e(u) - N_{eq}(u)$ in each realization,
with the result plotted in the inset of Fig. 8. This data
shows a probability of about $2/3$ for $\Delta N = 3$, and con-
siderably smaller probabilities $\approx 2/9$ for $\Delta N = 2$ and
$\approx 1/9$ for $\Delta N = 4$. We conclude that the shell veloc-
ity amplitude indeed drops to the equipartition level $u_{th}$
about 3 shells above the viscous cutoff shellnumber $N_{eq}(u)$
in every realization. The intermittency that appears in
the dissipation range of the noisy model thus appears to
be imprinted by small- and large-amplitude events that
propagate down from the inertial range. The few events
observed with $N_e(u) = -2$ may be described as very low-
intensity “lulls” and the handful of events with $N_e(u) = 6$
are extreme amplitude “bursts”.

The burst event $u$ that we considered in our conver-
gence studies in sections III.B.2-III.B.3 is the only real-
ization with $N_e(u) = 6$ that we encountered in our long
numerical simulation over 300 large-eddy turnover times.
As discussed there, even this largest value $N_e(u) = 6$
would correspond in the ABL to an “equilibration length”
about 124 times larger than the mean-free path of air.
Such a significant separation in scales suffices to
justify the validity of a hydrodynamic description even
for such extreme events. Of course, we cannot rule out
that running our shell model dynamics for much longer
times would produce much more intense events still, with
$N_e(u) > 6$. To properly identify the most extreme event
at our given Reynolds number $Re$ and dimensionless tem-
perature $\theta_v$ which achieves the largest value $N_e^*$ (possibly
$=\infty$) would require specialized tools of rare-event sam-
pling beyond the scope of the current work. However,
more systematic investigation of such extreme behaviors
is important work for the future.

The results of the present section demonstrate that siz-
able intermittency remains in the dissipation range of our
stochastic shell model with thermal noise. Our numeri-
cal results are consistent with the hypothesis that these
strong fluctuations are due to intermittent events, rang-
ing from “lulls” to “bursts”, that propagate in from the
inertial-range. However, a quantitative relationship re-
 mains to be established with inertial-range scaling. Fur-
thermore, it remains to be understood how such inter-
mittency manifests in standard statistical averages and
at what wavenumber scale. All of these issues are ad-
dressed in the following section.

D. Structure Functions

In order to study intermittency effects across both
inertial- and dissipation-ranges in our simulations, we use
$p$-th-order structure functions, which we define here as statistical “$p$-norms”
$$\|u_n\|_p = \langle |u_n|^p \rangle^{1/p} \quad (IV.9)$$
of the shell node $u_n$. The additional $p$th-root in [IV.9]
compared with the standard definition makes compar-
results for different choices of $p$ more transparent. Al-
though these are norms literally only for $p \geq 1$, we
FIG. 9 Structure functions (IV.9) for the deterministic model (heavy blue lines, — — — —) and for the noisy model (heavy red lines, — — — —). The predicted equipartition levels $[\Gamma(1 + p/2)]^{1/p}$ are indicated by the horizontal dashed red lines (---) and power-law fits in the inertial range are plotted as black dashed lines (---). The left panels show negative values (a) $p = -0.3$, (c) $p = -0.6$, (e) $p = -0.9$, and the right panels show positive values (b) $p = 1$, (d) $p = 3$, (f) $p = 6$. Standard errors of the mean for structure functions are smaller than the marker size. The insets show with heavy blue line (— — — —) the local stretching exponent (IV.10) and with horizontal dashed blue line (---) theoretical prediction $\gamma = \log_2 \left( \frac{1 + \sqrt{5}}{2} \right)$. 
consider all real values of \( p > -1 \), because negative \( p \) values give information about rare events which are smoother or more regular than typical. Similar information could be obtained also from so-called “inverse structure functions” \[116\] but we confine ourselves here to the more traditional direct structure functions. Note that the shell-model quantities \( \langle IV.9 \rangle \) correspond only in the inertial-range to the standard structure-functions defined by moments of velocity-increments in Navier-Stokes turbulence, but more generally they correspond to structure-functions of a suitably band-passed velocity field \( \mathbf{u}_n(x, t) \) at wavenumber magnitudes \( k_n \sim 2^n k_0 \). This distinction is crucial in the turbulent dissipation range, where first-order velocity-increments are completely dominated by the linear term in their Taylor series and are an inadequate tool to probe the energy spectrum and intermittency effects. An empirical study of incompressible fluid turbulence which aimed to investigate the same physics issues that we do here for shell models would need to employ a band-pass filter kernel \( f_n(k) \) with very rapid decay for \( |k| < k_n \), optimally vanishing identically for \( |k| < c k_n \) with some constant \( c > 0 \).

In Fig. \[9\] we plot the structure functions \( \|u_n\|_p/\eta_n \) versus \( n \) for the deterministic and noisy models, both normalized by the thermal velocity \( \eta_n = (2 \theta_n)^{3/2} \) in Kolmogorov units. With this normalization, the thermal equipartition value of the \( p \)-th-order function is \( \left[ \Gamma(1 + \frac{2}{p}) \right]^{1/p} \) in terms of the Euler \( \Gamma \)-function. We plot the structure functions for six representative values of \( p \), both positive and negative, over the entire range of \( n \). The first important observation is that the structure functions for the deterministic and noisy models are identical in the inertial-range, within numerical accuracy. This is not unexpected, because the thermal noise is extremely weak in the inertial-range and the direct effect on the dynamics should be negligible. The \( p \)-th-order functions here exhibit power-law scaling \( \propto k_n^{-\sigma_p} \), which is indicated by the straight-line fits in the log-log plots in Fig. \[9\]. These exponents are related by \( \sigma_p = \zeta_p/p \) to the standard structure-function scaling exponents \( \zeta_p \), and the values obtained from our linear fits agree for \( p > 0 \) with those previously appearing in the literature. See Fig. \[10\] where we plot our values together with those reported in \[56\], showing agreement within our error bars \[117\]. The decrease of \( \sigma_p \) with the order \( p \) is a reflection of temporal intermittency in the shell model dynamics, which has been long understood to be associated with strong “bursts” that propagate from low to high wavenumbers \[101\] \[118\] \[119\]. This inertial-range dynamics is essentially unaltered by the presence of weak thermal noise.

In the dissipation range, however, the \( p \)-th-order structure functions of the deterministic and noisy models are entirely different. Analogous to what is observed in the pair of energy spectra \( (p = 2) \) plotted in Fig. \[5\] the structure functions for the noisy model approach thermal equipartition at high wavenumbers whereas the same functions for the deterministic model exhibit a super-algebraic decay. In fact, the \( p \)-th-order structure functions of the deterministic model for all \( p \)-values exhibit the same stretched-exponential decay \[114\] as does the energy spectrum. This is verified in the insets of Fig. \[9\] which plot the local stretching exponents:

\[
\gamma_n^{(p)} = \log_2 \left( \frac{\|u_{n+1}\|_p}{\|u_n\|_p} \right) - \log_2 \left( \frac{\|u_{n+1}\|_p}{\|u_n\|_p} \right)
\]  

and the theoretical prediction \( \gamma = \log_2 \left( \frac{1+\sqrt{2}}{2} \right) \). Especially the negative \( p \)-values and \( p = 1 \) exhibit the predicted stretched-exponential, whereas \( p = 3 \) and \( p = 6 \) only approach this behaviour for large \( n \). The plausible explanation is that the intense “bursts” which dominate the structure functions for \( p = 3 \) and \( p = 6 \) penetrate to very high wavenumbers, so that for our deterministic simulation the cutoff \( R = 7 \) is too small to capture well the stretched-exponential. For \(-0.9 < p < 1\), on the other hand, the “lulls” or low-intensity events which dominate those \( p \)-values end at relatively low wavenumbers, so that the stretched-exponential above that wavenumber is well-resolved. The wavenumbers where the asymptotic behaviors first appear, equipartition or stretched-exponential, likewise strongly depend on the order \( p \) and, from Fig. \[9\] clearly increase with \( p \). This is directly associated with the strong fluctuations in the equipartition shellnum-
ber $N_e(u)$, observed in the statistics in Fig. 8. In consequence, the negative-order structure functions which get dominant contributions from ‘lulls’ with $N_e(u) \leq 0$ show direct effects of thermal noise at the Kolmogorov wavenumber and even lower wavenumbers. This should be true also for three-dimensional hydrodynamic turbulence, but obtaining those structure functions from a laboratory experiment would require extremely accurate velocity measurements, since the thermal velocity $u_{th}$ which sets the floor will be $3 \sim 4$ orders of magnitude smaller than the Kolmogorov velocity.

To establish a quantitative connection with inertial-range scaling, we have made also a very simple multifractal model of the PDF of the viscous cutoff shellnumber $N_{vis}(h)$, with the ansatz (in Kolmogorov units)

$$N_{vis}(h) = \text{Round} \left( \log_2(Re) \left( \frac{1}{1 + h} - \frac{3}{4} \right) \right), \quad (\text{IV.11})$$

where “Round” denotes rounding to the nearest integer, and where the PDF of the Hölder exponent $h$ is taken to be $P(h) \propto Re^{D(h)/(1+h)}$ for multifractal dimension spectrum $D(h)$. Using the Legendre transform relation $D(h) = \inf_{\phi} \{ \phi - \zeta_p \}$, we have determined the dimension spectrum from our numerical results for $\zeta_p$, with the resulting model PDF of $N_{vis}$ also plotted in Fig. 8.

Although our model is less sophisticated than the corresponding multifractal model developed for the PDF of $\eta(x, t)$ in 3D fluid turbulence [20], it matches reasonably well the PDF of $N_{vis}(u)$ from our DNS. This is consistent with an earlier study [84] verifying in the GOY shell model the predictions of the multifractal model for an “intermediate dissipation range” [10], since those predictions are based on the same assumption (IV.11). We conclude that the fluctuating viscous cutoff $N_{vis}(u)$ in our noisy Sabra model has statistics plausibly associated to inertial-range intermittency of “lulls” and “bursts”, and the equipartition shellnumber $N_{eq}(u)$ follows suit, occurring generally about just three shells higher.

V. DISCUSSION AND CONCLUSIONS

The main claim of this paper is that thermal noise fundamentally modifies the far-dissipation range of turbulent flow, leading to a thermal equipartition range in the turbulent energy spectrum at length-scales about $\eta/10$, one-tenth of the Kolmogorov length $\eta$, or even larger. If so, the correct equation to describe low-Mach-number fluid turbulence down to sub-Kolmogorov scales is not incompressible Navier-Stokes, but is instead the fluctuating hydrodynamics of Landau-Lifschitz [13] in its incompressible limiting form [16,20]. This conclusion was already anticipated in the pioneering papers by Betchov [53,54]. We have further explained why standard scaling arguments [9,7] for validity of the deterministic Navier-Stokes equation at arbitrarily high Reynolds numbers do not contradict our conclusions. We have also discussed interactions of turbulent intermittency with thermal noise effects that should lead to large spatiotemporal fluctuations in the length-scale at which thermal equipartition occurs in individual realizations. Finally, we have verified our various theoretical conclusions by simulations with a Sabra shell model of fluctuating hydrodynamics.

More questions are certainly raised by our results in this paper than can be currently answered definitively, and there is an urgent need for new computations, laboratory experiments, physical theory, and rigorous mathematical analysis to address them. Many of these questions were raised already by Betchov in his early works [53,54], as we discuss further below, and remain completely open to the present day.

A. Relations to Prior Theory

First, however, it is important to discuss relations of our work to earlier studies. There is a large body of work attempting to define fluctuating hydrodynamics equations of the form (1.2) as continuum stochastic partial-differential equations (SPDE’s), with the view that this is necessary for the understanding of turbulence. For example, the excellent book on SPDE’s in hydrodynamics [22] states in its preface that

“In a sentence, one of the purposes of the course was to understand the link between the Euler and Navier-Stokes equations or their stochastic versions and the phenomenological laws of turbulence.”

and the chapters of this book attempt to make mathematical sense of equations like (1.2) as continuum SPDE’s. This was also the point of view of the earlier paper on the stochastic shell model [98] which showed how to make sense of equations (11.3) in the limit $N \to \infty$ (for thermal equilibrium). An exact mathematical solution of this problem for fluctuating hydrodynamics would allow the limit $\Lambda \to \infty$ formally to be taken so that the equation (1.2) would make sense as an SPDE with the invariant measure (1.4). This is a problem of the same nature as the non-perturbative construction of a renormalized quantum field-theory and it has only been solved for simpler models such as the KPZ equation [99], where it is already extremely difficult [21]. We have argued that this point of view is, in fact, physically incorrect for fluctuating hydrodynamics and that at any finite Reynolds number, however large, the equations (1.2) must be regarded as low-wavenumber “effective field theories” with an explicit UV cut-off $\Lambda$, somewhat arbitrary, but chosen to satisfy the constraints $1/\mathcal{V} \ll \Lambda \ll 1/\lambda_{micro}$. This point of view is not novel, of course, but is standard in the field of fluctuating hydrodynamics [10,19,25,27] and there have also been renormalization group analyses of stochastically forced Navier-Stokes fluids to study systematically the effect of changing $\Lambda$, notably by Forster et al. [16,17], who included the case with stochastic force representing thermal noise as their “Model A”. This
paper carried out a Wilson Fourier-slicing RG analysis of the thermal fluid at equilibrium obtaining scale-dependent viscosity \( \nu \) and temperature \( T_i \) with effective dimensionless nonlinear coupling constant of the velocity fluctuations at scale \( \ell \) given in space dimension \( d \) by

\[
g_{\ell} = \left( \frac{k_B T_i}{\rho \nu^2 \ell^{d-2}} \right)^{1/2} \quad (V.1)
\]

These authors reached the conclusion that the thermal fluid is UV asymptotically free for \( d > 2 \), with coupling \( g_\ell \to 0 \) for \( \ell \to \infty \), corresponding to a linear Langevin model for relaxation of long-wavelength fluctuations (Onsager regression hypothesis). Conversely, they concluded that the thermal fluid is UV strongly coupled, with the constant \( g_\ell \) becoming large for \( \ell \to \infty \), and they equated this strong-coupling regime with turbulence:

“we will not attempt to treat the formidable and probably more interesting problem of the ultraviolet (short-distance, short-time) correlations described by (2.1), i.e., of fully developed turbulence...” \( \text{[17]} \)

Exactly the opposite situation was shown to hold for \( d < 2 \) by \( \text{[16, 17]} \), with now thermal velocity fluctuations UV asymptotically free and IR strongly coupled. As discussed in section \( \text{[14]} \), the shell model that we study numerically in this work is effectively a model in 0 space dimension and, according to the RG analysis of Forster et al., it is UV asymptotically free, so that the shell variables \( u_n \) are described by independent linear Langevin models for very large \( n \). Because 3D FNS is instead UV strongly coupled, it might be argued that thermal fluctuations in our shell model are qualitatively different at high wavenumbers than those for 3D fluids and that our shell model is thus unsuitable to test the effects of thermal noise in the turbulent dissipation range.

In fact, it is not hard to see that thermal velocity fluctuations both in 3D fluids and in our stochastic shell model are weakly coupled at the Kolmogorov dissipation length and down to much smaller scales. Note that the coupling constant \( g_\ell \) of Forster et al. in \( \text{(V.1)} \) coincides for \( \ell = \eta \) with the thermal Reynolds number \( \nu \eta^2 \), which we defined in \( \text{(14)} \) and thus \( \eta_n = \theta_n^{1/2} \) by \( \text{(17)} \). It follows that the coupling constant \( g_\ell \) at the Kolmogorov scale is tiny for realistic magnitudes of \( \theta_n \). The naive belief expressed in \( \text{[17]} \) that turbulent flows must correspond to large coupling constant \( g_\ell \) is not necessarily true.

It is true that this coupling will increase as \( \ell \) is further decreased, both in turbulent flows and even in a thermal equilibrium fluid at rest. One can estimate the wavenumber \( k_{\text{coup}} \) where coupling becomes strong by setting \( g_\ell \sim 1 \) in \( \text{(V.1)} \) and solving for \( k \sim 1/\ell \), yielding \( \text{(22)} \)

\[
k_{\text{coup}} = \left( \frac{k_B T_i}{\rho \nu^2} \right)^{1/2}. \quad (V.2)
\]

Simple estimates then imply that \( k_{\text{coup}} \) in \( \text{(V.2)} \) is so large that it is strictly outside the regime of validity of a hydrodynamic description! To see this, we can substitute the standard kinetic theory estimate for kinematic viscosity \( \nu \sim \lambda_{\text{micr}} c_{\text{th}} \), with \( c_{\text{th}} \) the thermal velocity/sound speed, into \( \sqrt{2} \), which yields

\[
k_{\text{coup}} \sim (n \lambda_{\text{micr}} \sqrt{\ell})^{1/2} \sim \left( \frac{\lambda_{\text{micr}}}{\ell^{1/2}} \right)^{1/2} \quad (V.3)
\]

where \( n \) is the particle number density of the fluid and \( \ell_{\text{intp}} := n^{-1/d} \) is the mean interparticle distance. In a liquid, \( \lambda_{\text{micr}} \sim \ell_{\text{intp}} \sim \ell_i \), where \( \ell_i \) is the radius of the molecule, and we see that \( k_{\text{coup}} \sim 1. \) In a gas described by the Boltzmann kinetic equation, \( k_{\text{coup}} \) is even larger.

The low-density limit for validity of the Boltzmann equation first identified by Bogolyubov \( \text{[123]} \) and Grad \( \text{[124]} \) is in which \( \lambda_{\text{micr}} / \ell_{\text{intp}} \gg 1 \) and \( \ell_{\text{intp}} / \ell_i \gg 1 \) with \( \lambda_{\text{micr}} \sim 1/n \ell_i^{d-1} \) held fixed. It is then easy to see that for such a Boltzmann gas \( \nu_{\text{coup}} \sim (\ell_{\text{intp}} / \ell_i)^{1/2} \gg 1 \). These considerations suggest that the strong-coupling problem encountered in the limit \( \Lambda \to \infty \) is of only academic mathematical interest and is not relevant to the physical description of molecular fluids in thermal equilibrium.

It is worth pointing out that the limit \( \Lambda \to \infty \) reappears in a different guise when taking the infinite Reynolds-number limit, \( Re = U L / \nu \to \infty \) with mean dissipation \( \varepsilon = U^3 / L \) and fluid parameters \( \rho, T \) all held fixed. In that case, \( \Lambda := L \lambda / \eta \to \infty \) so that the UV cutoff diverges to infinity when the fluctuating hydrodynamic equations are non-dimensionalized with the integral-scale quantities \( L \) and \( U \). This limit will be the subject of our following work \( \text{[125]} \). Here we just note that, assuming the validity of the Landau-Lifschitz equations \( \text{(1.2)} \) for arbitrarily large values of \( Re \), we find in \( \text{[125]} \) that the limiting velocity fields are singular (weak) solutions of the deterministic Euler equation. Thus, both the molecular noise and the molecular viscosity vanish in this limit. It is not entirely clear, however, that the fluctuating hydrodynamic equations \( \text{(1.2)} \) do remain valid for very large \( Re \), because increasing intermittency could allow extreme turbulent singularities to reach down to the microscopic length-scale \( \lambda_{\text{micr}} \). This possibility will be discussed more in section \( \text{(V.C)} \) below.

Finally, there has been much work on thermalization and equipartition spectra in various mathematical model fluid problems, especially truncated Euler \( \text{[77, 78, 126–128]} \), but also truncated Burgers \( \text{[129–132]} \), hypervisous Navier-Stokes & Burgers \( \text{[133–136]} \), and shell models \( \text{[137–140]} \). For a good overview of this large literature, see \( \text{[141]} \). In these various deterministic models, equipartition energy spectra \( \sim k^{d-1} \) for dimension \( d \) and Gaussian thermal statistics have been observed over certain ranges of wavenumber, rather similar to our observations. None of these works, however, have included stochastic terms to model the effects of thermal noise. There have been a few prior works on stochastic shell models, such as \( \text{[88, 132]} \), but with different noise than ours and with very different goals. Our aim in this work has been to use our stochastic shell model as a surrogate for 3D fluctuating Navier-Stokes equation, to assess
the effects of thermal noise in the turbulent dissipation range of molecular fluids and to understand the interactions between thermal noise and turbulent intermittency. As we have argued in depth, the shell model is suitable for this purpose. It is theoretically interesting that truncated 3D Euler can mimic many of the features of 3D FNS. Not only does the bath of thermalized hydrodynamic modes at high wavenumbers create an “effective viscosity” but also, following ideas of Kraichnan, it should create an “effective noise” satisfying a fluctuation-dissipation relation. Nevertheless, the predictions of 3D truncated Euler differ in several ways from those of 3D FNS, in particular lacking a viscous dissipation range at intermediate scales. Most significantly, 3D truncated Euler has never been proposed as a realistic model of the dissipation range of a molecular fluid, whereas 3D FNS is expected to be an accurate mesoscopic model down to almost microscopic length scales.

B. Prospects for Empirical Verification

The most important question raised by our work is the existence of the predicted thermal equipartition range in the sub-Kolmogorov scales, which we argued supplants the traditional “far-dissipation range” of deterministic Navier-Stokes. We thus find that deterministic Navier-Stokes and fluctuating Navier-Stokes make two radically different sets of predictions for the turbulent dissipation range, and it must now be determined which is correct. Current numerical codes for solving the incompressible fluctuating hydrodynamic equations are adequate to investigate turbulent flows at Taylor-scale Reynolds numbers up to 100 or so and such simulations should provide additional confirmation of our predictions. In fact, since the original submission of this paper, a preprint has appeared which reports on such a simulation, directly motivated by our work. Although those simulations reached only $Re_\lambda = 143$, that suffices to test our prediction of an equipartition range appearing at the Kolmogorov scale and Gaussian velocity statistics rather than strong intermittency in the far-dissipation range. Both of these predictions were fully verified; see for details. This numerical confirmation gives strong a posteriori validation to our methodology of using the stochastic shell model as a surrogate for 3D FNS. Our predictions for effects of inertial-range intermittency cannot yet be corroborated, because 3D FNS cannot currently be solved numerically at the high Reynolds numbers required. Nevertheless, our work and that of set the stage for a clash of two competing physical theories.

Ultimately, of course, the matter must be resolved by experiment. While the predictions of fluctuating hydrodynamics have been verified in many globally far-from-equilibrium situations and there is little doubt at all that thermal noise effects must be present at sub-Kolmogorov scales, the detailed predictions of fluctuating hydrodynamics can be legitimately questioned in turbulent flows where they have not yet been measured. The possibility exists that the local equilibrium assumption underlying the fluctuation-dissipation relation could break down, as already noted by Betchov (see section II.D). This is especially true since extreme turbulent intermittency could threaten the validity of any hydrodynamic description at all, at least locally.

More than 60 years after the early experimental attempt of Betchov, it remains a grand challenge to develop techniques which can measure the coarse-grained fluid velocities in Eq. at the relevant length-scales $\ell < \eta$. All traditional fluid-velocity measurement techniques have well-known limitations in achieving such fine spatial resolution. Betchov himself in his study used a standard technique of hot-wire anemometry to measure turbulent velocity fields. He made special efforts to minimize the high-frequency noise in the wires in order to increase their resolution and sensitivity. Furthermore, he investigated a novel multi-jet configuration in a “porcupine” box designed to create a nearly isotropic flow of high turbulence intensity, avoiding the weak electrical signal due to low turbulence intensity in grid-turbulence. Despite these efforts, the thermal noise spectrum of the fluid velocity predicted by Betchov remained about four orders of magnitude below the sensitivity of his measurements. See his Fig.6, where the highest wavenumbers of his measured spectrum are also clearly contaminated by electrical noise in the wire. These limitations of hot-wire technology remain to the present day. Here we may note that a recent study of grid turbulence in the Modane wind tunnel by hot-wires has remarked concerning the measured energy spectra that “all of them appear to increase as functions of $k$ beyond a wave number $k_M$” and that “the value of $k_M$ depends on the spectra, but is found to be typically $k_M \sim 3$.” This behavior, of course, naively accords with our predictions. The authors explain these observations however “as a contamination by the small-scale response of the hot wires” and we have no reason to doubt this conclusion, but it underlines the essential limitations of the hot-wire technology.

Another popular set of methods to determine fluid velocity vectors are those under the rubric of “particle-image velocimetry” or PIV, which do so by measuring the displacements of small, neutrally-bouyant solid beads that seed the flow. A preprint has appeared which reports on such a measurement, directly motivated by our work. Although those simulations reached only $Re_\lambda = 143$, that suffices to test our prediction of an equipartition range appearing at the Kolmogorov scale and Gaussian velocity statistics rather than strong intermittency in the far-dissipation range. Both of these predictions were fully verified; see for details. This numerical confirmation gives strong a posteriori validation to our methodology of using the stochastic shell model as a surrogate for 3D FNS. Our predictions for effects of inertial-range intermittency cannot yet be corroborated, because 3D FNS cannot currently be solved numerically at the high Reynolds numbers required. Nevertheless, our work and that of set the stage for a clash of two competing physical theories.

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over individual molecules. The motion of such submerged beads is known to be sensitive to the local thermal fluctuations of the velocity, but the effects appear only in the long-time tails of the particle velocity auto-correlation [138]. Furthermore, while the instantaneous velocities of micron-scale Brownian particles have been successfully measured in quiescent flows when confined by optical traps [149–150], it will be very difficult to measure the velocity of even one such freely-advected particle, let alone many. The large number of particles that must seed the flow in PIV introduce “ghost particles” that must be disambiguated by sophisticated post-processing techniques that introduce additional numerical noise in the inferred velocities [151]. Another velocity measurement technique exploiting tracer particles is “laser doppler velocimetry” or LDV, which has achieved micron-scale spatial resolution in turbulent boundary layers [152], but similar difficulties of interpretation and sensitivity appear. The sub-Kolmogorov scales of turbulent fluid flows remain a vast terra incognita of experimental science.

It may be more reasonable to hope first for indirect evidence of the effects of thermal noise. There are, in fact, many physical processes in turbulent flows which are recognized to involve sub-Kolmogorov scales in a fundamental way but which in a stationary, laminar fluid are known to be strongly influenced by thermal noise. These include high Schmidt/Prandtl-number scalar mixing [153], droplet and bubble formation [154], chemical reactions (combustion) [156–157] and locomotion of micro-organisms [158], among others. Current theory and numerical modelling of all these processes in turbulent flows omit thermal noise completely, e.g. in the case of high Schmidt/Prandtl-number scalar mixing [159,161], dynamics of droplets and bubbles [162–164], chemical combustion [165,167], and locomotion in turbulent flows [158,159]. The possibility exists in all of these cases of interesting interplay between turbulence and thermal effects, which might yield clear experimental signatures. These problems are all ripe also for numerical investigation by fluctuating hydrodynamics, which should spur development of novel schemes which are more efficient at the high Reynolds numbers required.

C. Validity of a Hydrodynamic Description

A key question underlying the current intense interest in extreme events and smallest scales in a turbulent flow [116–119] is whether such near-singular events may lead to a breakdown in the hydrodynamic approximation. In our shell model simulation with parameters appropriate to the ABL, the most singular event that we observed in 300 large-eddy turnover times penetrated down to a length scale 8.4 μm, which is still 124 times greater than the mean-free path length of air. The 3D FNS model should remain valid for such a singular event in the atmosphere. However, we cannot rule out that even more extreme events will occur if much longer times are con-
tion. It is worth observing that the fluctuating hydrodynamic equations can be derived microscopically with non-uniform spatial grids \[27\] so that a global wavenumber cutoff \(\Lambda\) need not be imposed and instead local cutoff lengths can be adapted to the particular solution.

We may note that the strong subsonic flow condition, \(Ma \ll 1\), does set some upper limit on the Reynolds numbers that are achievable within an incompressible fluid approximation. In both nature and in the laboratory, \(Re = UL/\nu\) is generally made larger by increasing \(L\) and/or \(U = (\varepsilon L)^{1/3}\), so that achieving very high \(Re\) at fixed \(\varepsilon\) requires \(Ma \simeq 1\) or even \(\gg 1\), as in astrophysical turbulence environments such as the molecular gas of the interstellar medium. In such compressible fluid turbulence strong shock discontinuities develop with \(h = 0\) and a shock-width of order \(\ll \lambda_{mfp}\), so that the fluid approximation breaks down locally and a Boltzmann kinetic equation is required to describe the internal structure of the shock \[171, 173\]. For compressible fluid turbulence, the thermal effects will be described by the fluctuating hydrodynamics of the compressible Navier-Stokes equations \[13, 14, 24–26\] or for large-\(Ma\) flows with strong shocks, by the nonlinear fluctuating Boltzmann equation \[174\]. It should be noted that even in the latter case, a hydrodynamic description is still valid at length scales \(\ell \gg \lambda_{mfp}\), because at scales much larger than the shock width the dynamics is accurately described by a weak Euler solution with an idealized discontinuity \[173\]. As we shall discuss in our following paper, the general description of turbulent inertial ranges by suitable (weak) solutions of the Euler equations \[176, 178\] is unchanged by thermal noise effects in the dissipation range.

New physical theory and new mathematical analysis are however demanded by our results. Existing derivations of the nonlinear fluctuating hydrodynamic equations \[25, 27\] are based upon the projection-operator methods of Zwanzig-Mori \[179, 180\]. Although these methods are formally exact and have seen recent mathematical and computational development \[181, 182\], they have not been fully justified from the point of view of rigorous statistical mechanics. In fact, important questions exist regarding the ultimate limits of validity of the fluctuating hydrodynamic equations, since those equations often work quite well for micro- and nano-scale fluid systems without a clear separation of scales. This suggests that the existing rigorous framework of hydrodynamic scaling limits \[2, 183\] may be too restrictive. The resulting stochastic hydrodynamic equations present also some very challenging questions regarding their mathematical formulation. As the existing formal microscopic derivations \[25, 27\] make clear, the fluctuating hydrodynamics equations are not stochastic PDE’s because they contain an explicit UV cutoff \(\Lambda\). Current mathematical theory (e.g. \[22\]) suffices to show that such cut-off models specify a well-posed dynamics for each finite value of \(\Lambda\), but the fundamental issue remains to be addressed that physical predictions should be \(\Lambda\)-independent. The reflexive response might be to attempt to show that a well-defined SPDE exists in the limit \(\Lambda \to \infty\), with a suitable choice of “bare” parameters, so that finite-\(\Lambda\) models can be regarded as “approximations” to this idealized continuum limit. Based on our arguments in section \[II A\] this point of view seems rather unphysical. It seems to us that a more natural goal is to establish some exact “renormalization group invariance” of the finite-\(\Lambda\) effective field-theories which expresses invariance of their predictions to changes of UV cut-off \(\Lambda\) and of other arbitrary features of the models, such as the numerical discretization.

### D. Future Directions

In this paper we have focused on the dissipation range of fully-developed, homogeneous turbulence, but there should be influences of thermal noise also on other turbulent flows and processes. Several of these novel effects and new directions of research were suggested already by Betchov. For example, he argued that thermal noise could play an important role in triggering transition to turbulence \[54\], sections II.E-G, a possibility currently being actively explored \[184, 185\], and that thermal noise could generate unpredictability in fully developed turbulence \[54\], section II.H, anticipating modern ideas on spontaneous stochasticity \[101, 186\]. In \[55\], Betchov suggested information-theoretic approaches as a possible means to distinguish Gaussian thermal fluctuations from turbulent fluctuations, in line with recent research \[187, 188\]. Betchov recognized as well that thermal noise could play an important role in triggering transition to turbulence \[54\], sections II.E-G, a possibility currently being actively explored \[184, 185\], and that thermal noise could generate unpredictability in fully developed turbulence \[54\], section II.H, anticipating modern ideas on spontaneous stochasticity \[101, 186\]. Betchov suggested information-theoretic approaches as a possible means to distinguish Gaussian thermal fluctuations from turbulent fluctuations, in line with recent research \[187, 188\]. Betchov recognized as well that thermal noise could play an important role in triggering transition to turbulence \[54\], sections II.E-G, a possibility currently being actively explored \[184, 185\], and that thermal noise could generate unpredictability in fully developed turbulence \[54\], section II.H, anticipating modern ideas on spontaneous stochasticity \[101, 186\]. In \[55\], Betchov suggested information-theoretic approaches as a possible means to distinguish Gaussian thermal fluctuations from turbulent fluctuations, in line with recent research \[187, 188\]. Betchov recognized as well that thermal noise could play an important role in triggering transition to turbulence \[54\], sections II.E-G, a possibility currently being actively explored \[184, 185\], and that thermal noise could generate unpredictability in fully developed turbulence \[54\], section II.H, anticipating modern ideas on spontaneous stochasticity \[101, 186\].

In consequence of these many important directions of research, this paper which focuses on the dissipation range is just the first in a series to study the influence of thermal noise on turbulent flows. In \[125\] we shall discuss its more subtle effects on the turbulent inertial-range in the limit \(Re \gg 1\), in particular the role of thermal noise in triggering Eulerian spontaneous stochasticity \[101, 186\]. We plan to make also a parametric study of the Reynolds-number dependence of the dissipation-
range intermittency discussed in this work, in order to explore possible limitations to the hydrodynamic description of small-scale fluid turbulence. In a work in preparation [193], we study high Schmidt-number turbulent mixing and we show that the exponentially decaying scalar spectrum theoretically predicted for the viscous-diffusive range [195] and verified numerically by deterministic Navier-Stokes simulations [196, 197] is erased by thermal noise and replaced by a $k^{-2}$ power-law spectrum associated to giant concentration fluctuations [198, 199]. Similar effects should be present also in the high magnetic Prandtl-number kinematic dynamo. Because of the universality of the fluctuation-dissipation relation, thermal noise is inextricably linked to dissipation and the two effects must always appear together.

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Appendices

A. Proof of the Fluctuation-Dissipation Relations

We here derive the fluctuation-dissipation relation (1.3) for the compressible Navier-Stokes equation and the corresponding result for the noisy Sabra model (III.8). The former result is well-known [13, 14, 16, 17, 25, 26] but we give the proof here for completeness and, also, to stress the close parallels to the same result for the Sabra model. A good general reference is [200].

1. FDR for Navier-Stokes

We begin with the truncated fluctuating Navier-Stokes system (I.1) which can be written as a system of SDE’s for the Fourier modes

$$ \hat{u}_k = \int_\Omega d^3x e^{-ik\cdot x} u(x) \quad (A.1) $$

of the velocity field satisfying $|k| < \Lambda$. Because of the reality condition

$$ \hat{u}_k = \hat{u}_{-k} \quad (A.2) $$

not all of these modes are independent. We take the modes whose wavevector lies in the half-set

$$ K^+ = \left\{ k : \begin{array}{ll} k_x > 0, & \text{or} \\ k_y > 0, & \text{if } k_x = 0, \text{ or} \\ k_z > 0, & \text{if } k_x = k_y = 0 \end{array} \right\} \quad (A.3) $$

as the independent complex modes. The fluctuating Navier-Stokes equation can then be written as

$$ \partial_t \hat{u}_{k,m} + ik_n \left( \delta_{mp} - \frac{k_n p_{mn}}{k^2} \right) \sum_{p+q=k} \hat{u}_{p,n} \hat{u}_{q,p} + \nu k^2 \hat{u}_{k,m} = \hat{q}_{k,m} \quad (A.4) $$

for $k \in K^+$ with $|k| < \Lambda$, and where $\hat{q}_{k}$ is a suitable random force, further specified below, that represents the thermal noise. Here we note that the wavevectors $p, q$ which are summed over in the expression

$$ B_{k,m}(\hat{u}, \hat{u}^*) = -ik_n \left( \delta_{mp} - \frac{k_n p_{mn}}{k^2} \right) \sum_{p+q=k} \hat{u}_{p,n} \hat{u}_{q,p} \quad (A.5) $$

may lie in the complementary set $K^- = -K^+$ and in the case that $p \in K^-$, then $\hat{u}_p$ should be interpreted instead as $\hat{u}_{-p}^*$. There is a corresponding equation of motion for the complex-conjugate variables

$$ \partial_t \hat{q}_{k,m} = B^*_{k,m}(\hat{u}, \hat{u}^*) + \hat{q}_{k,m} \quad (A.6) $$

with $B^*_{k,m}(\hat{u}, \hat{u}^*) := B_{k,m}(\hat{u}, \hat{u}^*)^*$ when $k \in K^+$ and $|k| < \Lambda$. Finally, we note that the random force $\hat{q}_{k}(t)$ is Gaussian with mean zero and covariance

$$ \langle \hat{q}_{k,m}(t) \hat{q}^{*}_{k',n}(t') \rangle = \frac{2\nu \kappa B T}{\rho} \delta_{kk'} \delta(t-t') \left( k^2 \delta_{mn} - k_m k_n \right) \quad (A.7) $$

with $\hat{q}_{k} = \hat{q}_{-k}$ and $k \hat{q}_{k} = 0$. To see the equivalence with (1.3), we note $\nabla \cdot \hat{\pi}$ in (1.2) can be written as

$$ \nabla \cdot \hat{\pi} = \hat{\pi} + \nabla \hat{\pi} \quad (A.8) $$

with the random scalar field $\hat{\pi}$ chosen so that $\nabla \cdot \hat{\pi} = 0$, and in that case $\hat{\pi}$ can be absorbed into the pressure and $\hat{q}$ is Gaussian with covariance

$$ \langle \hat{q}_{n}(x,t) \hat{q}_{n}(x',t') \rangle = \frac{2\nu \kappa B T}{\rho} (-\Delta_x \delta_{mn} + \nabla_x.m \nabla_x,n) \delta^3(\mathbf{x} - \mathbf{x'}) \delta(t-t') \quad (A.9) $$

The Fourier coefficient $\hat{q}_k(t)$ of $\hat{q}(x,t)$ then satisfies (A.7).

We now wish to show that the noise covariance (A.7) is correctly chosen so that the Gibbs measure

$$ P_G[\hat{u}, \hat{u}^*] = \frac{1}{Z} \exp \left[ -\mathcal{E}/k_B T \right] \quad (A.10) $$

for kinetic energy

$$ \mathcal{E} = \int_\Omega d^3x \frac{\rho}{2} |u(x)|^2 = \frac{\rho}{2V} \sum_{|k|<\Lambda} |\hat{u}_k|^2 = \frac{\rho}{V} \sum_{k \in K^+,|k|<\Lambda} |\hat{u}_k|^2 \quad (A.11) $$

is invariant under the stochastic dynamics (A.4) with no large-scale forcing. The proof is based on the Fokker-Planck equation for the probability distribution $P[\hat{u}, \hat{u}^*]$.
of the Fourier modes. We employ the standard device of treating \( \dot{u}_k \) and its complex conjugate \( \dot{u}_k^\ast \) as independent variables in complex differential calculus (Wirtinger derivatives). The Fokker-Planck equation equivalent to the coupled Langevin equations (A.4) is easily checked to be

\[
\partial_t P = \sum_{k \in K^+} \left[ -\frac{\partial}{\partial \dot{u}_k} \cdot \left( V_k[\dot{u}, \dot{u}^\ast] P \right) - \frac{\partial}{\partial \dot{u}_k^\ast} \cdot \left( V_k[\dot{u}^\ast, \dot{u}] P \right) + \frac{2\nu k_B T}{\rho} V(k^2 \mathbf{I} - kk) : \frac{\partial^2}{\partial \dot{u}_k \partial \dot{u}_k^\ast} P \right] := \mathcal{L}^\ast P, \tag{A.12}
\]

where

\[
V_k[\dot{u}, \dot{u}^\ast] = B_k[\dot{u}, \dot{u}^\ast] - \nu k^2 \dot{u}_k
\tag{A.13}
\]

and \( V_k^\ast[\dot{u}, \dot{u}^\ast] := V_k[\dot{u}^\ast, \dot{u}]^\ast \) are the components of the drift velocity in the phase-space.

The proof further uses two fundamental properties of the truncated Euler dynamics: the Liouville theorem on conservation of phase-volume:

\[
\sum_{k \in K^+} \left( \frac{\partial}{\partial \dot{u}_k} B_k[\dot{u}, \dot{u}^\ast] + \frac{\partial}{\partial \dot{u}_k^\ast} B_k^\ast[\dot{u}, \dot{u}^\ast] \right) = 0 \tag{A.14}
\]

and the conservation of kinetic energy:

\[
\sum_{k \in K^+} \left( B_k[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial \mathcal{E}}{\partial \dot{u}_k} + B_k^\ast[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial \mathcal{E}}{\partial \dot{u}_k^\ast} \right) = 0. \tag{A.15}
\]

The Liouville theorem for truncated Euler has been well-known since the work of Lee \cite{Lee1950} and, in fact, follows easily from definition (A.5) by the observation that

\[
\frac{\partial}{\partial \dot{u}_k} B_k[\dot{u}, \dot{u}^\ast] = -2i k \cdot \mathbf{u}(0), \quad \frac{\partial}{\partial \dot{u}_k^\ast} B_k^\ast[\dot{u}, \dot{u}^\ast] = 2i k \cdot \mathbf{u}(0)
\tag{A.16}
\]

The conservation of kinetic energy (A.15) follows from the detailed conservation for wavevector triads first noted by Onsager \cite{Onsager1931}. The consequence of (A.14), (A.15) is that

\[
\sum_{k \in K^+} \left[ -\frac{\partial}{\partial \dot{u}_k} \cdot (B_k[\dot{u}, \dot{u}^\ast] P_G) - \frac{\partial}{\partial \dot{u}_k^\ast} \cdot (B_k^\ast[\dot{u}, \dot{u}^\ast] P_G) \right] = 0,
\tag{A.17}
\]

so that \( P_G[\dot{u}, \dot{u}^\ast] \) is an invariant measure for the truncated Euler system.

The full stationarity condition \( \partial_t P_G = 0 \) therefore simplifies to

\[
\sum_{k \in K^+} \nu k^2 \left[ \frac{\partial}{\partial \dot{u}_k^\ast} \cdot \left( \dot{u}_k P_G \right) + \frac{\partial}{\partial \dot{u}_k} \cdot \left( \dot{u}_k^\ast P_G \right) \right] = -\frac{2\nu k_B T}{\rho} V \sum_{k \in K^+} (k^2 \mathbf{I} - kk) : \frac{\partial^2}{\partial \dot{u}_k \partial \dot{u}_k^\ast} P_G
\tag{A.18}
\]

It is worth observing that this is exactly the stationarity condition for the Gibbs measure under the linear Ornstein-Uhlenbeck dynamics

\[
\partial_t \dot{u}_k = -\nu k^2 \dot{u}_k + \xi_k, \quad \partial_t \dot{u}_k^\ast = -\nu k^2 \dot{u}_k^\ast + \xi_k^\ast \tag{A.19}
\]

corresponding to the fluctuating Stokes equation. On the other hand, the elementary derivatives

\[
\frac{\partial P_G}{\partial \dot{u}_k} = -\frac{\rho \dot{u}_k}{V k_B T} P_G, \quad \frac{\partial P_G}{\partial \dot{u}_k^\ast} = -\frac{\rho \dot{u}_k^\ast}{V k_B T} P_G, \quad \frac{\partial^2 P_G}{\partial \dot{u}_k \partial \dot{u}_k^\ast} = \left[ \frac{\rho}{V k_B T} (I - kk) - \frac{\rho^2 \dot{u}_k \dot{u}_k^\ast}{(V k_B T)^2} \right] P_G
\tag{A.20}
\]

imply that both sides of (A.18) are indeed equal to the same quantity

\[
\sum_{k \in K^+} \nu k^2 \left( 4 - \frac{2\rho |\dot{u}_k|^2}{V k_B T} \right) P_G.
\tag{A.21}
\]

Thus, \( P_G \) is an invariant measure, as claimed. Because of the non-degeneracy of the noise and the boundedness of the drift, this is in fact the unique invariant measure.

It is furthermore easy to show by the same arguments that the backward Fokker-Planck operator

\[
\mathcal{L} F = \sum_{k \in K^+} \left[ V_k[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial F}{\partial \dot{u}_k} + V_k^\ast[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial F}{\partial \dot{u}_k^\ast} + \frac{2\nu k_B T}{\rho} V(k^2 \mathbf{I} - kk) : \frac{\partial^2 F}{\partial \dot{u}_k \partial \dot{u}_k^\ast} \right]
\tag{A.22}
\]

for any real functions \( F[\dot{u}, \dot{u}^\ast] \), \( G[\dot{u}, \dot{u}^\ast] \) satisfies the following adjoint property with respect to the equilibrium Gibbs measure \( P_G \):

\[
\int \mathcal{L} F \cdot P_G d\dot{u} \cdot d\dot{u}^\ast = \int (\tilde{\mathcal{L}} G) F \cdot P_G d\dot{u} \cdot d\dot{u}^\ast \tag{A.23}
\]

where \( \dot{u} \cdot d\dot{u}^\ast = \prod_{k \in K^+} d\dot{u}_k \cdot d\dot{u}_k^\ast \) and where

\[
\tilde{\mathcal{L}} F = \sum_{k \in K^+} \left[ \tilde{V}_k[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial F}{\partial \dot{u}_k} + \tilde{V}_k^\ast[\dot{u}, \dot{u}^\ast] \cdot \frac{\partial F}{\partial \dot{u}_k^\ast} + \frac{2\nu k_B T}{\rho} V(k^2 \mathbf{I} - kk) : \frac{\partial^2 F}{\partial \dot{u}_k \partial \dot{u}_k^\ast} \right]
\tag{A.24}
\]

with

\[
\tilde{V}_k[\dot{u}, \dot{u}^\ast] = -B_k[\dot{u}, \dot{u}^\ast] - \nu k^2 \dot{u}_k
\tag{A.25}
\]

is the time-reversal \( \tilde{\mathcal{L}} \) of the operator \( \mathcal{L} \) under \( \mathbf{u} \to -\mathbf{u} \). The adjoint property (A.23) is equivalent to the detailed balance condition

\[
P[\mathbf{u}, t|\mathbf{u}_0, 0] P_G[\mathbf{u}_0] = P[-\mathbf{u}_0, t| -\mathbf{u}, 0] P_G[-\mathbf{u}]
\tag{A.26}
\]
for the transition probability densities of the nonlinear diffusion (A.12). Thus, the fluctuating Navier-Stokes dynamics is time-reversible in thermal equilibrium.

Finally, we derive for the Gibbs distribution \( P_G \) the spectrum of the mean energy per unit mass

\[
E = \frac{\langle \mathcal{E} \rangle}{\rho V} = \frac{1}{2 V^2} \sum_{|k| < \Lambda} |\langle \hat{u}_k \rangle|^2 \quad (A.27)
\]

which is conventionally considered for turbulence of an incompressible fluid. The variance of the Gaussian measure is given by

\[
|\langle \hat{u}_k \rangle|^2 = \frac{2 V k_B T}{\rho}, \quad (A.28)
\]

which expresses energy equipartition, taking into account the two “spin” degrees of freedom for the solenoidal Fourier modes. Then taking the infinite-volume limit

\[
\frac{1}{V} \sum_k \to \frac{1}{(2\pi)^3} \int d^3k \quad \text{for } V \to \infty \quad (A.29)
\]

and we see that \( E = \int_0^\Lambda dk \, E(k) \) with

\[
E(k) = \frac{k_B T}{\rho} \frac{4\pi k^2}{(2\pi)^3}. \quad (A.30)
\]

This is the \( k^2 \) equipartition spectrum first derived by Lee [62] and Hopf [64] for truncated Euler dynamics.

2. FDR for Sabra Model

Corresponding results for the Sabra shell model are obtained by identical arguments. The Fokker-Planck equation for the probability distribution \( P[u, u^*] \) of the vector \( u = (u_0, u_1, ..., u_N) \) of shell-variables is easily derived from the coupled Langevin equations (III.8) as:

\[
\partial_t P = \sum_{n=0}^N \left[ -\frac{\partial}{\partial u_n} \left( V_n[u, u^*] P \right) - \frac{\partial}{\partial u_n^*} \left( V_n^*[u, u^*] P \right) + \frac{4\nu k_B T}{\varrho} k_n^2 \frac{\partial^2}{\partial u_n \partial u_n^*} P \right] = \mathcal{L}^s P, \quad (A.31)
\]

where (now writing \( B_n[u] \) as \( B_n[u, u^*] \))

\[
V_n[u, u^*] = B_n[u, u^*] - \nu k_n^2 u_n \quad (A.32)
\]

and \( V_n^*[u, u^*] = V_n[u, u^*]^* \). The inviscid Sabra model dynamics also satisfies a Liouville theorem

\[
\sum_{n=0}^N \left( \frac{\partial}{\partial u_n} B_n[u, u^*] + \frac{\partial}{\partial u_n^*} B_n^*[u, u^*] \right) = 0, \quad (A.33)
\]

which is direct by inspection of eq. (III.2), and conserves kinetic energy:

\[
\sum_{n=0}^N \left( B_n[u, u^*] \frac{\partial \mathcal{E}}{\partial u_n} + B_n^*[u, u^*] \frac{\partial \mathcal{E}}{\partial u_n^*} \right) = 0 \quad (A.34)
\]

Consequently, the stationarity condition \( \partial_t P_G = 0 \) for the measure \( P_G[u, u^*] \) defined in (III.10) simplifies to

\[
\sum_{n=0}^N \nu k_n^2 \left[ \frac{\partial}{\partial u_n} \left( u_n P_G \right) + \frac{\partial}{\partial u_n^*} \left( u_n^* P_G \right) \right] = - \left( \frac{4\nu k_B T}{\varrho} \right) \sum_{n=0}^N k_n^2 \frac{\partial^2}{\partial u_n \partial u_n^*} P_G \quad (A.35)
\]

On the other hand, the elementary derivatives

\[
\frac{\partial P_G}{\partial u_n} = - \frac{\varrho u_n}{2 k_B T} P_G, \quad \frac{\partial P_G}{\partial u_n^*} = - \frac{\varrho u_n^*}{2 k_B T} P_G, \quad \frac{\partial^2 P_G}{\partial u_n \partial u_n^*} = - \left( \frac{\varrho^2 |u_n|^2}{(2 k_B T)^2} \right) P_G \quad (A.36)
\]

imply that both sides of (A.35) are equal to the same quantity

\[
\sum_{n=0}^N \nu k_n^2 \left( 2 - \frac{|u_n|^2}{k_B T} \right) P_G. \quad (A.37)
\]

Thus, the Gibbs distribution \( P_G \) is the unique invariant measure of the noisy Sabra model dynamics (III.8) when the external driving force is set to zero, \( f_n = 0 \). Furthermore, the analogue of the adjoint property (A.23) for fluctuating Navier-Stokes holds also for the noisy shell model, so that the dynamics is time-reversible under the transformation \( u_n \to -u_n \) in thermal equilibrium.

B. DERIVATION OF THE SLAVED TAYLOR-ITO SCHEME

For completeness we sketch here the derivation of the Taylor-Itô scheme from [103] for our noisy Sabra model (III.8), written as

\[
du_n = a_n dt + b_n dW_n \quad (B.1)
\]

with

\[
a_n = B_n[u] - \nu k_n^2 u_n + f_n, \quad b_n = \left( \frac{2 k_B T}{\varrho} \right)^{1/2} k_n \quad (B.2)
\]

Explicit integration of the linear viscous term gives

\[
u_n(t_{k+1}) = e^{-\nu k_n^2 \Delta t} [u_n(t_k) + \int_{t_k}^{t_{k+1}} c_n(t, u(t)) dt] + \int_{t_k}^{t_{k+1}} d_n(t) dW_n(t) \quad (B.3)
\]

with

\[
c_n(t, u(t)) = e^{\nu k_n^2 (t - t_k)} \langle B_n[u(t)] + f_n(t) \rangle \quad (B.4)
\]

and

\[
d_n(t) = e^{\nu k_n^2 (t - t_k)} b_n. \quad (B.5)\]
Taylor-expanding $d_n(t)$ as

\[ d_n(t) = e^{\nu k_n^2 (t-t_k)} b_n = [1 + \nu k_n^2 (t-t_k) + O((t-t_k)^2)] b_n \]  

(B.6)

and then substituting into the time-integral yields

\[
\int_{t_k}^{t_{k+1}} d_n(t) \, dW_n(t) \\
= b_n \Delta W_n(t_k) + \nu k_n^2 b_n \int_{t_k}^{t_{k+1}} (t-t_k) \, dW(t) + O((\Delta t)^{5/2}) \\
= b_n \Delta W_n(t_k) + \nu k_n^2 b_n (\Delta t \Delta W_n(t_k) - \Delta Z_n(t_k)) \\
\quad + O((\Delta t)^{5/2}) 
\]  

(B.7)

by an integration by parts.

To similarly expand $c_n(t, u(t))$ we must use the Itô formula

\[
c_n(t, u(t)) = c_n(t_k, u(t_k)) + \int_{t_k}^{t} ds \left( \partial_s + L \right) c_n(s, u(s)) \\
+ \sum_m \int_{t_k}^{t} ds \left[ b_m dW_m(s) \partial c_n \partial u_m + b_m dW_n(s) \partial c_n \partial u_n \right] \\
+ b_n dW_n^*(s) \partial c_n \partial u_n(s, u(t_k)) + R
\]  

(B.8)

where $L$ is the forward Kolmogorov operator

\[
L = \sum_m \left( a_m \frac{\partial}{\partial u_m} + a_m^* \frac{\partial}{\partial u_m^*} + 2b_n \frac{\partial^2}{\partial u_m \partial u_n} \right).
\]  

(B.9)

This implies that

\[
\int_{t_k}^{t_{k+1}} dt \ c_n(t, u(t)) \, dt = c_n(t_k, u(t_k)) \Delta t \\
+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} ds \ (\partial_s + L) c_n(s, u(s)) \\
+ \sum_m \int_{t_k}^{t} dt \int_{t_k}^{t} ds \ b_m \left[ dW_m(s) \partial c_n \partial u_m + dW_n(s) \partial c_n \partial u_n \right] \\
+ b_n \left[ (1 + \nu k_n^2 \Delta t) \Delta W_n(t_k) - \Delta Z_n(t_k) \right] + R
\]  

(B.10)

Substitution of the Itô formulæ for $(\partial_s + L)c_n(s, u(s))$ and $\partial c_n \partial u_m(s, u(s))$ then gives the Itô-Taylor series approximation to the desired order

\[
\int_{t_k}^{t_{k+1}} dt \ c_n(t, u(t)) \, dt = c_n(t_k, u(t_k)) \Delta t \\
+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} ds \ (\partial_s + L) c_n(t_k, u(t_k)) \\
+ \sum_m \int_{t_k}^{t} dt \int_{t_k}^{t} ds \ b_m \left[ dW_m(s) \partial c_n \partial u_m(t_k, u(t_k)) + dW_n(s) \partial c_n \partial u_n(t_k, u(t_k)) \right] \\
+ b_n \left[ (1 + \nu k_n^2 \Delta t) \Delta W_n(t_k) - \Delta Z_n(t_k) \right] + R
\]  

(B.11)

where $R$ is a stochastic remainder term. Straightforward calculations using (B.4) then give

\[
\int_{t_k}^{t_{k+1}} dt \ c_n(t, u(t)) = \Delta t [B_n(t_k, u(t_k)) + f_n(t_k)] \\
+ \frac{1}{2} (\Delta t)^2 \left( \nu k_n^2 \left[ B_n(t_k, u(t_k)) + f_n(t_k) \right] + \dot{f}_n(t_k) \right) \\
+ \frac{1}{2} (\Delta t)^2 \sum_m \left[ a_m \partial B_n \partial u_m(t_k, u(t_k)) + a_m^* \partial B_n \partial u_m(t_k, u(t_k)) \right] \\
+ 2b_m^2 \partial^2 B_n \partial u_m \partial u_m(t_k, u(t_k)) \\
+ b_n \left[ (1 + \nu k_n^2 \Delta t) \Delta W_n(t_k) - \Delta Z_n(t_k) \right] + R
\]  

(B.12)

Putting it all together gives the integration scheme

\[
u u_n(t_{k+1}) = e^{-\nu k_n^2 \Delta t} \left\{ U_n(t_k) + \Delta t [B_n(t_k, u(t_k)) + f_n(t_k)] \\
+ \frac{1}{2} (\Delta t)^2 \left( \nu k_n^2 \left[ B_n(t_k, u(t_k)) + f_n(t_k) \right] + \dot{f}_n(t_k) \right) \\
+ \frac{1}{2} (\Delta t)^2 \sum_m \left[ a_m \partial B_n \partial u_m(t_k, u(t_k)) + a_m^* \partial B_n \partial u_m(t_k, u(t_k)) \right] \\
+ 2b_m^2 \partial^2 B_n \partial u_m \partial u_m(t_k, u(t_k)) \\
+ b_n \left[ (1 + \nu k_n^2 \Delta t) \Delta W_n(t_k) - \Delta Z_n(t_k) \right] \right\} + R
\]  

(B.13)

This result may be compared with equation (6.1) of [103]. For the Sabra shell model, the only non-vanishing first-derivatives are

\[
\frac{\partial B_n}{\partial u_{n-2}} = \frac{1}{2} i k_{n-1} u_{n-1}, \quad \frac{\partial B_n}{\partial u_{n+2}} = i k_{n+1} u_{n+1}^* \\
\frac{\partial B_n}{\partial u_{n-1}} = \frac{1}{2} i k_{n-1} u_{n-2}, \quad \frac{\partial B_n}{\partial u_{n+1}} = -i k_{n+1} u_{n-1}^* \\
\frac{\partial B_n^*}{\partial u_{n-1}} = -\frac{1}{2} i k_{n} u_{n+1}, \quad \frac{\partial B_n^*}{\partial u_{n+1}} = i k_{n+1} u_{n+2}
\]  

(B.14)

while for all $m$

\[
\frac{\partial^2 B_n}{\partial u_m \partial u_m^*} = 0.
\]  

(B.15)

Substituting these results into (B.13) yields our numerical scheme (III.18) for the noisy Sabra model.
C. CONVERGENCE OF STEADY-STATE AVERAGES

Convergence of steady-state averages for our numerical study required a sufficiently large averaging time $T$ and proper resolution of the dynamics required a sufficiently large truncation wavenumber $N$ and sufficiently small time-step $\Delta t$. Here we describe the tests we have made that our choices of those parameters were sufficient.

In the section III.3 we already addressed at length the convergence of transition probabilities with respect to the high-wavenumber shell truncation $N$. Such convergence of transition probabilities implies convergence of the averages with respect to truncation wavenumber. The time-step was chosen as $\Delta t = 10^{-5}$ so as to be smaller than the viscous time $t_{\text{visc}} = 1/\nu k_{\text{m}}^2$ for the highest shell number $N$. Setting the external forcing to zero, $f_n = 0$ for all shells $n$, guarantees the gaussian Gibbs distribution for the external forcing and we checked that the time-step $\Delta t = 10^{-5}$ sufficed to reproduce that distribution to excellent accuracy for all shells, whereas reducing the step size to $\Delta t = 1 \times 10^{-4} \sim 3 \times 10^{-4}$ introduced errors for $n$ near $N$. In the forced simulation with $f_n$ chosen as in III.10, it was likewise found that the same same choice $\Delta t = 10^{-5}$ sufficed to produce Gaussian thermal equilibrium very accurately at the highest two shells (see Fig. 3) and further produced accurately the known stretched-exponential decay in the deterministic model run (see insets in Fig. 3). These consistency checks confirmed that our time-step $\Delta t = 10^{-5}$ was sufficiently small.

The total averaging time $T$ was taken to be 300 large eddy turnover times, based on convergence tests of the statistical averages presented in section IV. Dividing the total time into ten subintervals of times $T/10$ and calculating averages separately over each led to negligible changes, which suggested that we had acceptable convergence for $-0.9 < p < 6$. This was confirmed by the good agreement of our scaling exponents with the very accurate values obtained in [56], as shown in Fig. 10. Our work was not focused on high-precision of inertial-range scaling exponents, so that this was acceptable accuracy for our purposes. Crucially, the profound differences that we observed in the dissipation range between the statistics of the deterministic Sabra model III.1 and of the noisy Sabra model III.8 lie very far outside all error bars on the numerical calculations.

[1] C. L. Navier, “Mémoire sur les lois du mouvement des fluides,” Mem. Acad. Roy. Sci. 6, 389–440 (1823).
[2] O. Reynolds, “On the dynamical theory of turbulent incompressible viscous fluids and the determination of the criterion,” Phil. Trans. R. Soc. London A 186, 123–161 (1894).
[3] A.N. Kolmogorov, “The local structure of turbulence in incompressible viscous fluid for very large reynolds number,” Dokl. Akad. Nauk SSSR 30, 9–13 (1941).
[4] A. N. Kolmogorov, “Dissipation of energy in locally isotropic turbulence,” Dokl. Akad. Nauk. SSR 32, 16–18 (1941).
[5] Claude Bardos, François Golse, and David Levermore, “Fluid dynamic limits of kinetic equations. I. Formal derivations,” Journal of statistical physics 63, 323–344 (1991).
[6] Claude Bardos, François Golse, and C David Levermore, “Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation,” Communications on pure and applied mathematics 46, 667–753 (1993).
[7] Jeremy Quastel and H-T Yau, “Lattice gases, large deviations, and the incompressible Navier-Stokes equations,” Annals of mathematics, 51–108 (1998).
[8] Stanley Corssin, “Outline of some topics in homogeneous turbulent flow,” Journal of Geophysical Research 64, 2134–2150 (1959).
[9] U. Frisch, Turbulence: The Legacy of A. N. Kolmogorov (Cambridge University Press, 1995).
[10] Jean Leray, “Sur le mouvement d’un liquide visqueux emplissant l’espace,” Acta mathematica 63, 193–248 (1934).
[11] Robert Terrell, “Translation of “On the motion of a viscous liquid filling space”, by J. Leray,” arXiv preprint arXiv:1604.02484 (2016).
[12] Charles L Fefferman, “Existence and smoothness of the Navier-Stokes equation,” The Millennium Prize Problems 57, 67 (2006).
[13] J.O. de Zarate and J. Sengers, Hydrodynamic Fluctuations in Fluids and Fluid Mixtures (Elsevier Science, 2006).
[14] Rudolf Schmitz, “Fluctuations in nonequilibrium fluids,” Physics Reports 171, 1–58 (1988).
[15] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, Course of Theoretical Physics, Vol. 6 (Addision-Wesley, Reading, MA, 1959).
[16] Dieter Forster, David R Nelson, and Michael J Stephen, “Long-time tails and the large-eddy behavior of a randomly stirred fluid,” Physical Review Letters 36, 867 (1976).
[17] Dieter Forster, David R Nelson, and Michael J Stephen, “Large-distance and long-time properties of a randomly stirred fluid,” Physical Review A 16, 732 (1977).
[18] Florencio Balboa Usabiaga, John B Bell, Rafael Delgado-Buscalioni, Aleksandar Donev, Thomas G Fai, Boyce E Griffith, and Charles S Peskin, “Staggered schemes for fluctuating hydrodynamics,” Multiscale Modeling & Simulation 10, 1369–1408 (2012).
[19] Aleksandar Donev, Andy Nonaka, Yifei Sun, Thomas Fai, Alejandro Garcia, and John Bell, “Low mach number fluctuating hydrodynamics of diffusively mixing fluids,” Communications in Applied Mathematics and Computational Science 9, 47–105 (2014).
[20] Andrew Nonaka, Yifei Sun, John Bell, and Aleksandar Donev, “Low mach number fluctuating hydrodynamics of binary liquid mixtures,” Communications in Applied Mathematics and Computational Science 10, 163–204 (2015).
To prevent confusion, we note that the fluctuating hydrodynamics equations are essentially different, both physically and mathematically, from the Navier-Stokes equation with a spatially-smooth white-in-time force added to represent large-scale stirring [10, 11, 12]. When the forcing spectrum decays rapidly at high wavenumbers, then the random stirring in the latter case inputs energy which cascades inertially to small scales. Mathematically, one can then safely take any UV cutoff Λ to infinity with fixed viscosity ν and obtain a well-defined limiting spectrum defined by a stochastic partial differential equation (SPDE) [13]. On the contrary, the fluctuating hydrodynamic equations have stochastic forcing prescribed by the fluctuation-dissipation relation, so that the statistical steady-state is thermal equilibrium with time-reversible dynamics. See Appendix A. Because the forcing spectrum is now growing with wavenumber, it is not at all obvious how to define a limiting SPDE as Λ → ∞ [14]. In fact, the bare viscosity is now renormalized by the thermal fluctuations and becomes a scale-dependent quantity ν = 1/2νt. These important differences are discussed at greater length in the text.

G. Da Prato and M. Röckner, eds., *SPDE in Hydrodynamics: Recent Progress and Prospects: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, August 29 – September 5, 2005* Lecture Notes in Mathematics No. 1942 (Springer Berlin Heidelberg, 2008).

Robert Graham, “Onset of cooperative behavior in nonequilibrium steady states,” in *Order and fluctuation-dissipation relations in equilibrium and nonequilibrium statistical mechanics. Proceedings of the XVIIIth International Solvay Conference on Physics* (Wiley, New York, 1981) pp. 235–288.

Gregory L Eyink, “Dissipation and large thermodynamic fluctuations,” Journal of statistical physics 61, 533–572 (1990).

DN Zubarev and VG Morozov, “Statistical mechanics of nonlinear hydrodynamic fluctuations,” Physica A: Statistical Mechanics and its Applications 120, 411–467 (1983).

VG Morozov, “On the Langevin formalism for nonlinear and nonequilibrium hydrodynamic fluctuations,” Physica A: Statistical Mechanics and its Applications 126, 443–460 (1984).

Pep Español, Jesús G Anero, and Ignacio Zúñiga, “Microscopic derivation of discrete hydrodynamics,” The Journal of chemical physics 131, 244117 (2009).

Iwao Hosokawa, “Ensemble mechanics for the random-forced navier-stokes flow,” Journal of Statistical Physics 15, 87–104 (1976).

David Ruelle, “Microscopic fluctuations and turbulence,” Physics Letters A 72, 81–82 (1979).

Martin Macháček, “The role of thermal noise in stationary hydrodynamical turbulence,” Physics Letters A 128, 76–79 (1988).

Andrey Nikolaevich Kolmogorov, “A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number,” Journal of Fluid Mechanics 13, 82–85 (1962).

Giovanni Paladin and Angelo Vulpiani, “Degrees of freedom of turbulence,” Physical Review A 35, 1971 (1987).

Werner Heisenberg, “Zur statistischen Theorie der Turbulenz,” Zeit. f. Phys., 628–657 (1948).

Subrahmanyan Chandrasekhar, “Theory of turbulence,” Physical Review 102, 941 (1956).

Robert H Kraichnan, “The structure of isotropic turbulence at very high Reynolds numbers,” Journal of Fluid Mechanics 5, 497–543 (1959).

Uriel Frisch and Rudolf Morf, “Intermittency in nonlinear dynamics and singularities at complex times,” Physical review A 23, 2673 (1981).

C Foias, O Manley, and L Sirovich, “Empirical and Stokes eigenfunctions and the far-dissipative turbulent spectrum,” Physics of Fluids A: Fluid Dynamics 2, 464–467 (1990).

Lawrence Sirovich, Leslie Smith, and Victor Yakhot, “Energy spectrum of homogeneous and isotropic turbulence in far dissipation range,” Physical review letters 72, 344 (1994).

Robert H Kraichnan, “Intermittency in the very small scales of turbulence,” The Physics of Fluids 10, 2080–2082 (1967).

U Frisch and M Vergassola, “A prediction of the multifractal model: the intermediate dissipation range,” EPL (Europhysics Letters) 14, 439 (1991).

Sualeh Klarushid, Diego A Donzis, and KR Sreenivasan, “Energy spectrum in the dissipation range,” Physical Review Fluids 3, 082601 (2018).

Anastasia Gorbunova, Guillaume Balarac, Mickael Bourgoin, Léonie Canet, Nicolas Mordant, and Vincent Rossetto, “Analysis of the dissipative range of the energy spectrum in grid turbulence and in direct numerical simulations,” Physical Review Fluids 5, 044604 (2020).

Dhawal Buaria and Katepalli R Sreenivasan, “Dissipation range of the energy spectrum in high Reynolds number turbulence,” arXiv preprint arXiv:2004.06274 (2020).

PK Yeung, XM Zhai, and Katepalli R Sreenivasan, “Extreme events in computational turbulence,” Proceedings of the National Academy of Sciences 112, 12633–12638 (2015).

PK Yeung and K Ravikumar, “Advancing understanding of turbulence through extreme-scale computation: Intermittency and simulations at large problem sizes,” Physical Review Fluids 5, 110517 (2020).

Mohammad Farazmand and Themistoklis P Sapsis, “A variational approach to probing extreme events in turbulent dynamical systems,” Science advances 3, e1701533 (2017).

Dhawal Buaria, Alain Pumir, Eberhard Bodenschatz, and Pui-Keun Yeung, “Extreme velocity gradients in turbulent flows,” New Journal of Physics 21, 043004 (2019).

Dhawal Buaria, Alain Pumir, and Eberhard Bodenschatz, “Self-attenuation of extreme events in Navier–Stokes turbulence,” Nature communications 11, 1–7 (2020).

Dhawal Buaria, Eberhard Bodenschatz, and Alain Pumir, “Vortex stretching and enstrophy production in high Reynolds number turbulence,” Physical Review Fluids 5, 104602 (2020).

Florian Nguyen, J-P Laval, and Bérengère Dubrulle, “Characterizing most irregular small-scale structures in turbulence using local Hölder exponents,” Physical Review E 102, 063105 (2020).

Paul Debue, Denis Kuzay, Ewe-Wei Saw, François...
Daviaud, Bérengère Dubrulle, Léonie Canet, Vincent Rossetto, and Nicolás Wschebor, “Experimental test of the crossover between the inertial and the dissipative range in a turbulent swirling flow,” Physical Review Fluids 3, 024602 (2018).

[52] P Debue, V Valori, C Cuvier, F Daviaud, J-M Foucaut, J-P Laval, C Wiertel, V Padilla, and B Dubrulle, “Three-dimensional analysis of precursors to non-viscous dissipation in an experimental turbulent flow,” Journal of Fluid Mechanics 914 (2021).

[53] R Betchov, “On the fine structure of turbulent flows,” Journal of Fluid Mechanics 3, 205–216 (1957).

[54] R. Betchov, “Thermal agitation and turbulence,” in Rarefied Gas Dynamics, edited by L. Talbot (Academic Press, New York, 1961) p. 307–321, proceedings of the Second International Symposium on Rarefied Gas Dynamics, held at the University of California, Berkeley, CA, 1960.

[55] R Betchov, “Measure of the intricacy of turbulence,” The Physics of Fluids 7, 1160–1162 (1964).

[56] Victor S L’vov, Evgenii Podivilov, Anna Pomyalov, Itamar Procaccia, and Damien Vandembroucq, “Improved shell model of turbulence,” Physical Review E 58, 1811 (1998).

[57] Victor S L’vov, Evgenii Podivilov, and Itamar Procaccia, “Hamiltonian structure of the sabra shell model of turbulence: exact calculation of an anomalous scaling exponent,” EPL (Europhysics Letters) 46, 609 (1999).

[58] D. Bandak, G. Eyink, A. Mailybaev, and N. Goldenfeld, “Thermal noise competes with turbulent fluctuations at millimeter scales,” Phys. Rev. Lett., submitted (2021).

[59] Franco Flandoli, “An introduction to 3D stochastic fluid dynamics,” in SPDE in Hydrodynamics: Recent Progress and Prospects: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, August 29 - September 3, 2005, Lecture Notes in Mathematics, Vol. 1942, edited by G. Da Prato and M. Röckner (Springer Berlin Heidelberg, 2008).

[60] This problem is solved in principle by the Green-Kubo formulas for the transport coefficients obtained in the microscopic derivations of the fluctuating hydrodynamic equations, such as Eq.(2.56) of 26 or Eq.(48) of 27. However, evaluation of those formulas itself requires microscopic molecular dynamics simulations.

[61] J. M. Burgers, “On the application of statistical mechanics to the theory of turbulent fluid motion. VII.” Koninklijke Nederlandse Akademie van Wetenschappen 36, 620 (1933).

[62] T. D. Lee, “On some statistical properties of hydrodynamical and magneto-hydrodynamical fields,” Quarterly of Applied Mathematics 10, 69–74 (1952).

[63] Lars Onsager, “Statistical hydrodynamics,” Nuovo Cimento Suppl. 6, 279–287 (1949).

[64] Eberhard Hopf, “Statistical hydromechanics and functional calculus,” Journal of Rational Mechanics and Analysis 1, 87–123 (1952).

[65] Robert H Kraichnan, “Helical turbulence and absolute equilibrium,” Journal of Fluid Mechanics 59, 745–752 (1973).

[66] There is no assumption made here that the r.m.s. velocity \( u_{rms} \sim U \) and their ratio could be Re-dependent.

[67] J.R. Garratt, The Atmospheric Boundary Layer, Cambridge Atmospheric and Space Science Series (Cambridge University Press, 1994).

[68] Theodore Von Kármán, “Progress in the statistical theory of turbulence,” Proceedings of the National Academy of Sciences of the United States of America 34, 530 (1948).

[69] DA Donzis and KR Sreenivasan, “The bottleneck effect and the kolmogorov constant in isotropic turbulence,” Journal of fluid mechanics 657, 171 (2010).

[70] Werner Heisenberg, “Zur statistischen Theorie der Turbulenz,” Z.Physik 124, 628–657 (1948).

[71] J. Von Neumann, “Recent theories of turbulence (unpublished report to the Office of Naval Research, 1949),” in Collected Works of John Von Neumann, Vol. 6: Theory of Games, Astrophysics, Hydrodynamics and Meteorology, edited by A.H. Taub (Pergamon Press, Oxford, 1963) pp. 437–472.

[72] Zhen-Su She and Emmanuel Laveque, “Universal scaling laws in fully developed turbulence,” Physical review letters 72, 336 (1994).

[73] Victor Yakhot, “Mean-field approximation and a small parameter in turbulence theory,” Physical Review E 63, 026307 (2001).

[74] Malo Tarpin, Léonie Canet, and Nicolás Wschebor, “Breaking of scale invariance in the time dependence of correlation functions in isotropic and homogeneous turbulence,” Physics of Fluids 30, 055102 (2018).

[75] Mogens H Jensen, “Multiscaling and structure functions in turbulence: an alternative approach,” Physical review letters 83, 76 (1999).

[76] Vishwanath Shukla, Pablo D Mininni, Giorgio Krstulovic, Patricio Clark Di Leon, and Marc E Brachet, “Quantitative estimation of effective viscosity in quantum turbulence,” Physical Review A 99, 043605 (2019).

[77] Robert H Kraichnan, “Remarks on turbulence theory,” Advances in Mathematics 16, 305–331 (1975).

[78] Cyril Cichowlas, Pauline Bona¨ıti, Fabrice Debbasch, and Marc Brachet, “Effective dissipation and turbulence in spectrally truncated euler flows,” Physical review letters 95, 264502 (2005).

[79] A.J. Majda and A.L. Bertozzi, Vorticity and Incompressible Flow, Cambridge Texts in Applied Mathematics (Cambridge University Press, 2002).

[80] Kai Kadau, Charles Rosenblatt, John L Barber, Timothy C Germann, Zhibin Huang, Pierre Carliès, and Berni J Alder, “The importance of fluctuations in fluid mixing,” Proceedings of the National Academy of Sciences 104, 7741–7745 (2007).

[81] G. I. Taylor, “Statistical theory of turbulence, i.” Proc. Roy. Acad. Lond. A 151, 421–444 (1935).

[82] If relation \( x^2e^x = 1/\theta_n \) is solved for \( x = k\eta, \theta_n \), then satisfying it for \( x' = 2x, \theta'_n \) requires \( 1/\lambda = \theta_n/\theta'_n = 4e^x \).

[83] U. Frisch and G. Parisi, “On the singularity structure of correlation functions in isotropic and homogeneous turbulence,” Proceedings of the National Academy of Sciences 99, 530 (1998).

[84] John C Bowman, Charles R Doering, Bruno Eckhardt, Jahanshah Davoudi, Malcolm Roberts, and Jörg Schumacher, “Links between dissipation, intermittency, and helicity in the GOY model revisited,” Physica D 218, 1–10 (2006).

[85] Léonie Canet, Vincent Rossetto, Nicolás Wschebor, and Guillaume Balarac, “Spatiotemporal velocity-velocity correlation function in fully developed turbu-
lence,” Physical Review E 95, 023107 (2017).

[86] G. L. Eyink, “Turbulence Theory, Course Notes, Johns Hopkins University,” https://urldefense.com/v3/__https://edu/*eyink/Turbulence/notes.html__;fg!!DZ3fjg!qWAmkc11jp8D0pTvryMOLn0vbUbUYmJA1ZPPlJR3jaopKea25uzxxx3V44GS0sVui9 (2007-2020).

[87] Kartik P Iyer, Katepalli R Sreenivasan, and PK Yeong, “Scaling exponents saturate in three-dimensional isotropic turbulence,” Physical Review Fluids 5, 054605 (2020).

[88] The meaning of statistics here is in the sense of “local thermodynamic equilibrium.” Thus, the region of diameter $\eta$ at time $t$ may be partitioned into $(\eta_1/t)$ subregions of diameter $\ell$ and the distribution of values $\bar{u}$ over the subregions should be the given Gaussian distribution $(1.30)$ for $\ell_{av}/t \ll \eta$.

[89] M Gross, ME Cates, F Varnik, and R Adhikari, “Langevin theory of fluctuations in the discrete boltzmann equation,” Journal of Statistical Mechanics: Theory and Experiment 2011, P03030 (2011).

[90] Xiao Xue, Luca Biferale, Mauro Sbragaglia, and Federico Toschi, “A lattice boltzmann study on brownian motion of the component fluid,” Journal of Computational Science, 2011.

[91] Evgenii Borisovich Gledzer, “System of hydrodynamic type admitting two quadratic integrals of motion,” Sov. Phys. Dokl. 18, 216–217 (1973).

[92] Koji Ohkitani and Michio Yamada, “Temporal intermittency in the energy cascade process and local Lyapunov analysis in fully-developed model turbulence,” Progress of theoretical physics 81, 329–341 (1989).

[93] Luca Biferale, “Shell models of energy cascade in turbulence,” Annual review of fluid mechanics 35, 441–468 (2003).

[94] Dario Vincenzi and John D Gibbon, “How close are shell models to the 3d navier–stokes equations?” Nonlinearity 34, 5821 (2021).

[95] Peter Constantin, Boris Levant, and Edriss S Titi, “Analytic study of shell models of turbulence,” Physica D: Nonlinear Phenomena 219, 120–141 (2006).

[96] Peter Constantin, Boris Levant, and Edriss S Titi, “Regularity of inviscid shell models of turbulence,” Physical Review E 75, 016304 (2007).

[97] P.D. Ditlevsen, Turbulence and Shell Models (Cambridge University Press, 2010).

[98] Hakima Bessaih and Benedetta Ferrario, “Invariant gibbs measures of the energy for shell models of turbulence: the inviscid and viscous cases,” Nonlinearity 25, 1075 (2012).

[99] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang, “Dynamic scaling of growing interfaces,” Physical Review Letters 56, 889 (1986).

[100] Norbert Schörghofer, Leo Kadanoff, and Detlef Lohse, “How the viscous subrange determines inertial range properties in turbulence shell models,” Physica D: Nonlinear Phenomena 88, 40–54 (1995).

[101] Alexei A Mailybaev, “Spontaneous stochasticity of velocity in turbulence models,” Multiscale Modeling & Simulation 14, 96–112 (2016).

[102] Victor S L’vov, Itamar Procaccia, and Damien Vandembroucq, “Universal scaling exponents in shell models of turbulence: viscous effects are finite-size corrected to scaling,” Physical review letters 81, 802 (1998).

[103] Peter E Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations Stochastic Modelling and Applied Probability (Springer Berlin Heidelberg, 2013).

[104] Sebastian Becker, Arnulf Jentzen, and Peter E Kloeden, “An exponential wagner–platen type scheme for spdes,” SIAM Journal on Numerical Analysis 54, 2389–2426 (2016).

[105] Peter E Kloeden, Gabriel J Lord, Andreas Neuenkirch, and Tony Shardlow, “The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds,” Journal of Computational and Applied Mathematics 235, 1245–1260 (2011).

[106] TN Palmer, “Stochastic weather and climate models,” Nature Reviews Physics 1, 463–471 (2019).

[107] Shiyi Chen, Gary Doolen, Jackson R Herring, Robert H Kraichnan, Steven A Orszag, and Zhen Su She, “Far-dissipation range of turbulence,” Physical review letters 70, 3051 (1993).

[108] Jörg Schuchauer, Katepalli R Sreenivasan, and Victor Yakhot, “Asymptotic exponents from low-Reynolds-number flows,” New Journal of Physics 9, 89 (2007).

[109] Jörg Schuchauer, “Sub-kolmogorov-scale fluctuations in fluid turbulence,” EPL (Europhysics Letters) 80, 54001 (2007).

[110] Note that it is possible that $N(4t_{av}) > N(t_{av})$ because a new strong “burst” may enter the dissipation range between the times $t_{avg}$ and $4t_{avg}$.

[111] John I Marden, “Hypothesis testing: from p values to bayes factors,” Journal of the American Statistical Association 95, 1316–1320 (2000).

[112] Daniele Agostini and Carlos Amendola, “Discrete gaussian distributions via theta functions,” SIAM Journal on Applied Algebra and Geometry 3, 1–30 (2019).

[113] Stéphane Roux and Mogens H Jensen, “Dual multifractal spectra,” Physical Review E 80, 053011 (2013).

[114] Shiyi Chen, Gary Doolen, Jackson R Herring, Robert H Kraichnan, Steven A Orszag, and Zhen Su She, “Far-dissipation range of turbulence,” Physical review letters 70, 3051 (1993).

[115] Jörg Schuchauer, Katepalli R Sreenivasan, and Victor Yakhot, “Asymptotic exponents from low-Reynolds-number flows,” New Journal of Physics 9, 89 (2007).

[116] Jörg Schuchauer, “Sub-kolmogorov-scale fluctuations in fluid turbulence,” EPL (Europhysics Letters) 80, 54001 (2007).

[117] John I Marden, “Hypothesis testing: from p values to bayes factors,” Journal of the American Statistical Association 95, 1316–1320 (2000).

[118] Daniele Agostini and Carlos Amendola, “Discrete gaussian distributions via theta functions,” SIAM Journal on Applied Algebra and Geometry 3, 1–30 (2019).

[119] Stéphane Roux and Mogens H Jensen, “Dual multifractal spectra,” Physical Review E 80, 053011 (2013).

[120] Shiyi Chen, Gary Doolen, Jackson R Herring, Robert H Kraichnan, Steven A Orszag, and Zhen Su She, “Far-dissipation range of turbulence,” Physical review letters 70, 3051 (1993).

[121] Jörg Schuchauer, Katepalli R Sreenivasan, and Victor Yakhot, “Asymptotic exponents from low-Reynolds-number flows,” New Journal of Physics 9, 89 (2007).

[122] Jörg Schuchauer, “Sub-kolmogorov-scale fluctuations in fluid turbulence,” EPL (Europhysics Letters) 80, 54001 (2007).

[123] Note that it is possible that $N(4t_{av}) > N(t_{av})$ because a new strong “burst” may enter the dissipation range between the times $t_{avg}$ and $4t_{avg}$.

[124] John I Marden, “Hypothesis testing: from p values to bayes factors,” Journal of the American Statistical Association 95, 1316–1320 (2000).

[125] Daniele Agostini and Carlos Amendola, “Discrete gaussian distributions via theta functions,” SIAM Journal on Applied Algebra and Geometry 3, 1–30 (2019).

[126] Stéphane Roux and Mogens H Jensen, “Dual multifractal spectra,” Physical Review E 80, 053011 (2013).

[127] Shiyi Chen, Gary Doolen, Jackson R Herring, Robert H Kraichnan, Steven A Orszag, and Zhen Su She, “Far-dissipation range of turbulence,” Physical review letters 70, 3051 (1993).

[128] Jörg Schuchauer, Katepalli R Sreenivasan, and Victor Yakhot, “Asymptotic exponents from low-Reynolds-number flows,” New Journal of Physics 9, 89 (2007).

[129] Jörg Schuchauer, “Sub-kolmogorov-scale fluctuations in fluid turbulence,” EPL (Europhysics Letters) 80, 54001 (2007).

[130] Note that it is possible that $N(4t_{av}) > N(t_{av})$ because a new strong “burst” may enter the dissipation range between the times $t_{avg}$ and $4t_{avg}$.

[131] John I Marden, “Hypothesis testing: from p values to bayes factors,” Journal of the American Statistical Association 95, 1316–1320 (2000).

[132] Daniele Agostini and Carlos Amendola, “Discrete gaussian distributions via theta functions,” SIAM Journal on Applied Algebra and Geometry 3, 1–30 (2019).

[133] Stéphane Roux and Mogens H Jensen, “Dual multifractal spectra,” Physical Review E 80, 053011 (2013).

[134] Shiyi Chen, Gary Doolen, Jackson R Herring, Robert H Kraichnan, Steven A Orszag, and Zhen Su She, “Far-dissipation range of turbulence,” Physical review letters 70, 3051 (1993).
as \( k \) grows, then \( \kappa_{\text{coup}} \) is correspondingly lowered. A 
kinematic viscosity on order of magnitude \( \nu \sim \lambda_{\text{mirc}} c_s \) 
follows, as is well-known, from the kinetic theories of 
Boltzmann and Enskog. This is a reasonable estimate 
of the “bare viscosity” at length scales of order the 
mean-free-path, before it is dressed or renormalized by 
thermal fluctuations. Indeed, it is well-known that the 
Boltzmann-type kinetic equations neglect the effects of 
thermal fluctuation, which must be incorporated by ad-
ditional Langevin terms. See [?] [?] [?] and, for more 
recent work, [173]. Furthermore, this estimate of the 
“bare viscosity” is in agreement with the study of [19]. 
See their Appendix C, Figure 8.

[123] N.N. Bogoliubov, “Problems of a Dynamical Theory in Statis-
tical Physics,” in: Studies in Statistical Mechanics, vol. 1, 
ed. J. de Boer and G. E. Uhlenbeck (North-Holland, 
Amsterdam, 1962), p.1.

[124] Harold Grad, “On the kinetic theory of rarefied 
gasses,” Communications on pure and applied mathematics 
2, 331–407 (1949).

[125] D. Bandak, G. L. Eyink, N. Goldenfeld, and A. Mai-
lybaev, “Spontaneous stochasticity in fluid turbulence 
from thermal noise: How the ‘sverse’ of molecules in-
fluence weather and climate,” (2020), in preparation, 
2020.

[126] Giorgio Krstulovic and Marc-Étienne Brachet, “Two-
fluid model of the truncated euler equations,” Physica 
D: Nonlinear Phenomena 237, 2015–2019 (2008).

[127] Samriddhi Sankar Ray, Uriel Frisch, Sergei Nazarenko, 
and Takeshi Matsumoto, “Resonance phenomenon for 
the galerkin-truncated euler equations,” Physical Review 
E 84, 016301 (2011).

[128] Sugan D Murugan, Dheeraj Kumar, Subhro Bhat-
tacharjee, and Samriddhi Sankar Ray, “Many-body 
chaos in thermalized fluids,” Physical Review Letters 
127, 124501 (2021).

[129] Andrew J Majda and Ilya Timofeyev, “Remarkable stat-
istical behavior for truncated burgers–hopf dynamics,” 
Proceedings of the National Academy of Sciences 97, 
12413–12417 (2000).

[130] Divya Venkataraman and Samriddhi Sankar Ray, “The 
onset of thermalization in finite-dimensional equations 
of hydrodynamics: insights from the burgers equa-
tion,” Proceedings of the Royal Society A: Mathematical, 
Physical and Engineering Sciences 473, 20160585 
(2017).

[131] Patricio Clark Di Leoni, Pablo D Mininni, and Marc E 
Brachet, “Dynamics of partially thermalized solutions of 
the burgers equation,” Physical Review Fluids 3, 014603 
(2018).

[132] Sugan D Murugan, Uriel Frisch, Sergey Nazarenko, 
Nicolas Besse, and Samriddhi Sankar Ray, “Suppress-
ing thermalization and constructing weak solutions in 
truncated inviscid equations of hydrodynamics: Lessons 
from the burgers equation,” Physical Review Research 
2, 033202 (2020).

[133] Uriel Frisch, Susan Kurien, Rahul Pandit, Walter Pauls, 
Samriddhi Sankar Ray, Achim Wirth, and Jian-Zhou 
Zhu, “Hyperviscosity, galerkin truncation, and bot-
tlenecks in turbulence,” Physical review letters 101, 
144501 (2008).

[134] Uriel Frisch, Samriddhi Sankar Ray, Ganapati Sahoo, 
Debarghya Banerjee, and Rahul Pandit, “Real-space 
manifestations of bottlenecks in turbulence spectra,” 
Physical review letters 110, 064501 (2013).

[135] Debarghya Banerjee and Samriddhi Sankar Ray, “Tra-
nsition from dissipative to conservative dynamics in 
equations of hydrodynamics,” Physical Review E 90, 
041001 (2014).

[136] Rahul Agrawal, Alexandros Alexakis, Marc E Brachet, 
and Laurette S Tuckerman, “Turbulent cascade, bott-
leneck, and thermalized spectrum in hyperviscous flows,” 
Physical Review Fluids 5, 024601 (2020).

[137] Peter D Ditlivesen and IA Mogensen, “CASCADEs 
and statistical equilibrium in shell models of turbulence,” 
Physical Review E 53, 4785 (1996).

[138] Thomas Gilbert, Victor S L’vov, Anna Pomyalov, and 
Itamar Procaccia, “Inverse cascade regime in shell mod-
els of two-dimensional turbulence,” Physical review 
letters 89, 074501 (2002).

[139] Boris Levant, Fabio Ramos, and Edrriss S Titi, “On the 
statistical properties of the 3d incompressible navier-
stoikes-voigt model,” Communications in Mathematical 
Sciences 8, 277–293 (2010).

[140] Rithwik Tom and Samriddhi Sankar Ray, “Revisiting 
the sabra model: Statics and dynamics,” EPL (Euro-
physics Letters) 120, 34002 (2018).

[141] Samriddhi Sankar Ray, “Thermalized solutions, statis-
tical mechanics and turbulence: An overview of some 
recent results,” Pramana 84, 395–407 (2015).

[142] Takeshi Matsumoto, Michio Otsuki, Ooshida Takeshi, 
Susumu Goto, and Akio Nakahara, “Response function of 
turbulence computed via fluctuation-response relation 
of a langevin system with vanishing noise,” Physical 
Review E 89, 061002 (2014).

[143] Robert H Kraichnan, “Convergents to turbulence func-
tions,” Journal of Fluid Mechanics 41, 189–217 (1970).

[144] John B Bell, Andrew Nonaka, Alejandro L Garcia, 
and Gregory Eyink, “Thermal fluctuations in the dissipa-
tion rate of homogeneous isotropic turbulence,” arXiv 
preprint arXiv:2109.08761 (2021).

[145] Hamed Sadeghi, Philippe Lavoie, and Andrew Pollard, 
“Effects of finite hot-wire spatial resolution on turbu-
ience statistics and velocity spectra in a round turbulent 
free jet,” Experiments in Fluids 59, 40 (2018).

[146] M. Raffel, C.E. Willert, F. Scarano, C.J. Kähler, S.T. 
Werenly, and J. Kompenhans, Particle Image Velocime-
try: A Practical Guide (Springer International Publishing, 
2018).

[147] Bérengère Dubrulle, “Giant von Kármán exper-
iment,” https://urldefense.com/v3/__https://www-llb.cea.fr/Pisp/berengere.dubrulle/
/projects/#EXPLOIT/Gallery.html?_LiW!DZ3fjg!qWAmKc11JpqC0D0pTwzfMoLm0vBuBjAMJaiZPPIK3R83jaoPkeaz5uzz3yV

[148] Aleksandar Donev, John B Bell, Alejandro L Garcia, 
and Berni J Alder, “A hybrid particle-continuum method 
for hydrodynamics of complex fluids,” Multi-
scale Modeling & Simulation 8, 871–911 (2010).

[149] Tongcang Li, Simon Kheifets, David Medellin, and 
Mark G Raizen, “Observation of brownian motion 
in liquids at short times: instantaneous velocity
and memory loss,” science 343, 1493–1496 (2014).

[151] Shiyoung Tan, Ashwanth Salibindla, Ashik Ullah Mohammad Masuk, and Rui Ni, “Introducing openpl: new method of removing ghost particles and high-concentration particle shadow tracking,” Experiments in Fluids 61, 1–16 (2020).

[152] J. Czarske and L. Böttner, “Micro laser doppler velocimetry (µ-LDV),” in Encyclopedia of Microfluidics and Nanofluidics, edited by D. Li (Springer Science, New York, 2013).

[153] Aleksandar Donev, Thomas G Fai, and Eric Vanden-Eijnden, “A reversible mesoscopic model of diffusion in liquids: from giant fluctuations to fick’s law,” Journal of Statistical Mechanics: Theory and Experiment 2014, P04004 (2014).

[154] Anuj Chaudhri, John B Bell, Alejandro L Garcia, and Aleksandar Donev, “Modeling multiphase flows using fluctuating hydrodynamics,” Physical Review E 90, 033014 (2014).

[155] Mirko Gallo, Francesco Magaletti, Davide Cocco, and Carlo Massimo Cisciola, “Nucleation and growth dynamics of vapour bubbles,” Journal of Fluid Mechanics 883 (2020).

[156] A Lemarchand and B Nowakowski, “Fluctuation-induced and nonequilibrium-induced bifurcations in a thermalchemical system,” Molecular Simulation 30, 773–780 (2004).

[157] Amit Kumar Bhattacharjee, Kaushik Balakrishnan, Alejandro L Garcia, John B Bell, and Aleksandar Donev, “Fluctuating hydrodynamics of multi-species reactive mixtures,” The Journal of chemical physics 142, 224107 (2015).

[158] Ingo O Götzé and Gerhard Gompper, “Mesoscale simulations of hydrodynamic squirmer interactions,” Physical Review E 82, 041921 (2010).

[159] Diego A Donzis, KR Sreenivasan, and PK Yeung, “The batchelor spectrum for mixing of passive scalars in isotropic turbulence,” Flow, turbulence and combustion 85, 549–566 (2010).

[160] MP Clay, Dhawal Buaria, PK Yeung, and T Gotoh, “Gpu acceleration of a petascale application for turbulent mixing at high Schmidt number using openmp 4.5,” Computer Physics Communications 228, 100–114 (2018).

[161] Dhawal Buaria, Matthew P Clay, Katepalli R Sreenivasan, and PK Yeung, “Turbulence is an ineffective mixer when Schmidt numbers are large,” arXiv preprint arXiv:2004.06202 (2020).

[162] Izumi Saito and Toshiyuki Gotoh, “Turbulence and cloud droplets in cumulus clouds,” New Journal of Physics 20, 023001 (2018).

[163] Said Elghobashi, “Direct numerical simulation of turbulent flows laden with droplets or bubbles,” Annual Review of Fluid Mechanics 51, 217–244 (2019).

[164] Felix Milan, Luca Biferale, Mauro Strargaglia, and Federico Toschi, “Sub-kolmogorov droplet dynamics in isotropic turbulence using a multiscale lattice boltzmann scheme,” Journal of Computational Science 35, 101178 (2020).

[165] KR Sreenivasan, “Possible effects of small-scale intermittency in turbulent reacting flows,” Flow, turbulence and combustion 72, 115–131 (2004).

[166] James F Driscoll, “Turbulent premixed combustion: Flamelet structure and its effect on turbulent burning velocities,” Progress in Energy and Combustion Science 34, 91–134 (2008).

[167] T. Echekki and E. Mastorakos, Turbulent Combustion Modeling: Advances, New Trends and Perspectives, Fluid Mechanics and Its Applications (Springer Netherlands, 2010).

[168] William M Durham, Eric Climent, Michael Barry, Filippo De Lillo, Guido Boffetta, Massimo Cencini, and Roman Stocker, “Turbulence drives microscale patches of motile phytoplankton,” Nature communications 4, 1–7 (2013).

[169] Jeanette D Wheeler, Eleonora Secchi, Roberto Rusconi, and Roman Stocker, “Not just going with the flow: the effects of fluid flow on bacteria and plankton,” Annual review of cell and developmental biology 35, 213–237 (2019).

[170] Luis Caffarelli, Robert Kohn, and Louis Nirenberg, “Partial regularity of suitable weak solutions of the Navier-Stokes equations,” Communications on pure and applied mathematics 35, 771–831 (1982).

[171] Harold Meade Mott-Smith, “The solution of the Boltzmann equation for a shock wave,” Physical Review 82, 885 (1951).

[172] Hans Wolfgang Liepmann, R Narasimha, and Moustafa T Chahine, “Structure of a plane shock layer,” The Physics of Fluids 5, 1313–1324 (1962).

[173] E. Salomon and Michel Mareschal, “Usefulness of the Burnett description of strong shock waves,” Physical review letters 69, 269 (1992).

[174] Freddy Bouachet, “Is the Boltzmann equation reversible? a large deviation perspective on the irreversibility paradox,” Journal of Statistical Physics 181, 515–550 (2020).

[175] Shi-Hsien Yu, “Hydrodynamic limits with shock waves of the Boltzmann equation,” Communications on Pure and Applied Mathematics 58, 409–443 (2005).

[176] Gregory L Eyink, “Review of the Onsager ‘ideal turbulence’ theory,” arXiv preprint arXiv:1803.02223 (2018).

[177] Gregory L Eyink and Theodore D Drivas, “Cascades and dissipative anomalies in compressible fluid turbulence,” Physical Review X 8, 011022 (2018).

[178] Theodore D Drivas and Gregory L Eyink, “An Onsager singularity theorem for turbulent solutions of compressible Euler equations,” Communications in Mathematical Physics 359, 733–763 (2018).

[179] Robert Zwanzig, “Memory effects in irreversible thermodynamics,” Physical Review 124, 983 (1961).

[180] H. Grabert, Projection Operator Techniques in Nonequilibrium Statistical Mechanics, Springer Tracts in Modern Physics (Springer Berlin Heidelberg, 2006).

[181] Alexandra J Chorin, Ole H Hald, and Raz Kupferman, “Optimal prediction and the mori–zwanzig representation of irreversible processes,” Proceedings of the National Academy of Sciences 97, 2968–2973 (2000).

[182] Carmen Hijón, Pep Espa˜nol, Eric Vanden-Eijnden, and Rafael Delgado-Buscalioni, “Mori–zwanzig formalism as a practical computational tool,” Faraday discussions 144, 301–322 (2010).

[183] H. Spohn, Large Scale Dynamics of Interacting Particles, Theoretical and Mathematical Physics (Springer Berlin Heidelberg, 2012).

[184] Paolo Luchini, “Receptivity to thermal noise of the boundary layer over a swept wing,” AIAA Journal 55, 121–130 (2017).
[185] Alexander Fedorov and Anatoli Tumin, “Receptivity of high-speed boundary layers to kinetic fluctuations,” AIAA Journal 55, 2335–2348 (2017).

[186] Alexei A Mailybaev, “Spontaneously stochastic solutions in one-dimensional inviscid systems,” Nonlinearity 29, 2238 (2016).

[187] N Vladimirova, M Shavit, S Belan, and G Falkovich, “Second harmonic generation as a minimal model of turbulence,” arXiv preprint arXiv:2103.15468 (2021).

[188] Michal Shavit and Gregory Falkovich, “Singular measures and information capacity of turbulent cascades,” Physical Review Letters 125, 104501 (2020).

[189] O. Feliachi and F. Bouchet, “Dynamical large deviations for homogeneous systems with long range interactions and the Balescu-Guernsey-Lenard equation,” J. Stat. Phys., submitted (2021).

[190] John A Krommes, “Projection-operator methods for classical transport in magnetized plasmas, part 1. linear response, the braginskii equations and fluctuating hydrodynamics,” Journal of Plasma Physics 84 (2018).

[191] John A Krommes, “Projection-operator methods for classical transport in magnetized plasmas, part 2. nonlinear response and the burnett equations,” Journal of Plasma Physics 84 (2018).

[192] Siyao Xu and Alex Lazarian, “Magnetohydrodynamic turbulence and turbulent dynamo in partially ionized plasma,” New Journal of Physics 19, 065005 (2017).

[193] Glen Satten and David Ronis, “Fluctuations in finite systems: Time reversal symmetry, surface onsager reciprocal relations and fluctuating hydrodynamics,” Physica A: Statistical Mechanics and its Applications 125, 281–301 (1984).

[194] G. L. Eyink and A. Jafari, “High Schmidt-number turbulent advection giant concentration fluctuations,” to be submitted (2021).

[195] Robert H Kraichnan, “Convection of a passive scalar by a quasi-uniform random straining field,” Journal of Fluid Mechanics 64, 737–762 (1974).

[196] PK Yeung, S Xu, DA Donzis, and KR Sreenivasan, “Simulations of three-dimensional turbulent mixing for Schmidt numbers of the order 1000,” Flow, turbulence and combustion 72, 333–347 (2004).

[197] Matthew Paul Clay, Strained turbulence and low-diffusivity turbulent mixing using high performance computing, Ph.D. thesis, Georgia Institute of Technology (2017).

[198] Alberto Vailati and Marzio Giglio, “Giant fluctuations in a free diffusion process,” Nature 390, 262–265 (1997).

[199] Alberto Vailati, Roberto Cembino, Stefano Mazzoni, Christopher J Takacs, David S Cannell, and Marzio Giglio, “Fractal fronts of diffusion in microgravity,” Nature communications 2, 1–5 (2011).

[200] John D Ramshaw and Katja Lindenberg, “Augmented Langevin description of multiplicative noise and nonlinear dissipation in Hamiltonian systems,” Journal of statistical physics 45, 295–307 (1986).