Holonomies of gauge fields in twistor space 3: gravity as a square of $\mathcal{N} = 4$ theory

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Abstract

In a recent paper, we show that an S-matrix functional for graviton amplitudes can be described by an $\mathcal{N} = 8$ supersymmetric gravitational holonomy operator in twistor space. In this paper, we obtain an alternative expression for the gravitational holonomy operator such that it can be interpreted as a square of an $\mathcal{N} = 4$ holonomy operator for frame fields, by taking a sum of certain shuffles over ordered indices. The new expression leads to amplitudes of not only spin-2 gravitons but also spin-0 massless particles. We discuss that the squared model is favored as a theory of quantum gravity.
1 Introduction

In recent years there has been much attention to the relation between $\mathcal{N} = 4$ super Yang-Mills theory and $\mathcal{N} = 8$ supergravity. An explicit relation between the two theories at the level of classical scattering amplitudes is first obtained in [1] by taking the field-theory limit of the so-called Kawai-Lewellen-Tye (KLT) relation between tree-level amplitudes of open and closed string theories [2]. Roughly speaking, this relation gives an expression of graviton amplitudes in terms of a square of gluon counterparts, with certain multiplicity factors. Field theoretic construction of a gravitational theory by use of this relation has been studied in earlier works of Bern and others [3, 4]. What is suggested in Bern’s approach is that one may reduce the ultraviolet behavior of $\mathcal{N} = 8$ supergravity to that of $\mathcal{N} = 4$ super Yang-Mills theory if one utilizes structural similarities between the two theories.

It is not until the work of Witten [5], which generalizes Nair’s observation on the so-called maximally helicity violating (MHV) amplitudes of gluons in a twistor-space framework [6], that many researchers start realizing that Bern’s approach is in fact very promising in showing the ultraviolet finiteness of $\mathcal{N} = 8$ supergravity as a theory of quantum gravity. This is partly because recent developments in the helicity-based calculation of gluon amplitudes show that the amplitudes can significantly be simplified, even at loop levels, by use of the MHV amplitudes (or vertices). It is therefore natural to apply these developments to gravitational theories using the above-mentioned relation between gauge theory and gravity. In fact, there are a plentiful number of papers on this specific subject. For some earlier works, see for example [7]-[15]. For very recent papers, see also [16]-[21].

In the present paper, following these lines of developments, we investigate the “squared” relation between gauge theory and gravity in a recently proposed holonomy formalism [22, 23]. In [23] we construct a gravitational holonomy operator in twistor space, interpreting gravity as a gauge theory with nontrivial Chan-Paton factors. We then show that an S-matrix functional for graviton amplitudes can be expressed in terms of a supersymmetric version of the holonomy operator. Motivated by Bern’s approach, in this paper we shall change our interpretation of gravity to obtain an alternative expression for the gravitational holonomy operator such that we can easily understand it as a square of a gauge-theory holonomy operator with $\mathcal{N} = 4$ extended supersymmetry. We shall also check that the alternative expression does reproduce the correct graviton amplitudes.

This paper is organized as follows. In the next section, we review the construction of a gravitational holonomy operator, following [23], and present its explicit definition. In section 3, we treat the summations that appear in the gravitational holonomy operator in a different manner so that it can easily be regarded as a square of a gauge-theory holonomy operator. In section 4, we consider supersymmetrization of the holonomy operators and confirm that the new expression also correctly leads to graviton amplitudes. Lastly, we shall present some concluding remarks.
2 Review of a gravitational holonomy operator

Definition

In this section we review the construction of a gravitational holonomy operator which is proposed in a recent paper [23]. The gravitational holonomy operator is defined by

\[ \Theta^{(H)}_{R,\gamma}(u, \bar{u}) = \text{Tr}_{R,\gamma} \mathcal{P} \exp \left[ \sum_{m \geq 3} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H \right] \]

where \( H \) is called a comprehensive graviton field and is defined by the following set of equations.

\[ H = \sqrt{8 \pi G_N} \sum_{1 \leq i < j \leq m} \sum_{\sigma \in S_{r-1}} \sum_{\tau \in S_{m-r-2}} \left( \sum_{h_{\mu_i}} g_i^{(h_{\mu_i})} \otimes g_j^{(00)} \right) \omega_{ij} \omega_{\lambda_i \lambda_j} \]

Coupling constant, numbering indices and braid diagrams

First of all, \( G_N \) denotes the Newton constant. In the natural unit (\( c = \hbar = 1 \)), this is equivalent to the inverse square of the Planck mass \( M_{Pl} \):

\[ G_N = \frac{1}{M_{Pl}^2} = 6.7088 \times 10^{-39} \left[ \frac{1}{\text{GeV}} \right]^2. \]

The numbering indices \( i, j \) take values of \( 1, 2, \cdots, m \). We split these into \( \{2, 3, \cdots, r\} \) and \( \{r+1, r+2, \cdots, m-1\} \) (\( 2 \leq r \leq m-2 \)) and consider transpositions (or permutations) of the two distinct sets of indices. The transpositions are labeled by

\[ \sigma = \left( \begin{array}{c} 2 \cdots r \\ \sigma_2 \cdots \sigma_r \end{array} \right), \quad \tau = \left( \begin{array}{c} r+1 \cdots m-2 \\ \tau_{r+1} \cdots \tau_{m-2} \end{array} \right). \]

In the rest of this section, we shall explain the notations of the above expressions one by one. The reader may find the following description lengthy but what we shall do is nothing but to present the definition of quantum gravity in the holonomy formalism. Thus we find it important to review the definition in a consistent manner. We try to make the discussion as much concise as possible; for details of the definition, the reader may refer to [23].
The sum of the transpositions $\sigma$ can be denoted as a sum over $\sigma \in S_{r-1}$ where $S_{r-1}$ represents the rank-$(r-1)$ symmetric group. Similarly the sum of the transpositions $\tau$ can be denoted by a sum over $\tau \in S_{m-r-2}$. We fix the rest of the numbering indices, 1, $m-1$ and $m$, out of the permutations. For convenience, we denote this fact by

$$\sigma_1 = \sigma_{r+1} = 1, \quad \tau_{m-1} = \tau_r = m - 1.$$

The above permutations of the indices are schematically shown in Figure 1 where we draw braid diagrams for $\sigma$’s and $\tau$’s separately. In the figure, the elements of $\sigma$’s and $\tau$’s are chosen at random, while the symbol P denotes an ascending ordering of the elements. Structure of each braid diagram depends on a specific choice of the permutation; the structure is shown by a thick down-arrow in Figure 1.

Figure 1: Braid diagrams corresponding to the permutations of $\sigma$’s and $\tau$’s as well as the identity permutation of the index $m$ — the symbol P denotes an ascending ordering of the arguments. When two lines are crossing each other, we consider that a line with an arrow is closer to us, crossing over the other line without an arrow.

The index $\mu_i$ in (2.3) is a composite numbering index in a sense that it covers both $\sigma$’s and $\tau$’s. Similarly the index $\lambda_i$ in (2.4) is a composite numbering index; the difference from $\mu_i$ is that the indices of $\sigma$’s are positively shifted by one while those of $\tau$’s are negatively shifted by one. For the full definition of $H$, we need to define

$$\lambda_r = 1, \quad \lambda_{r+1} = m - 1, \quad \lambda_{m+1} = \sigma_2.$$

The first two relations are in accord with (2.9). Information of $\lambda_{m+1}$ is necessary in defining the gravitational holonomy operator (2.1).

Chan-Paton factors of gravitons and frame fields

Now we explain the meaning of the graviton operator $g_i^{(h_{i\mu_i})}$ in (2.5). A graviton labeled by a particular numbering index corresponds to a particular strand in the braid diagrams in Figure 1. $h_{i\mu_i}$ represents the helicity of the $i$-th graviton, taking a value of $h_{i\mu_i} \equiv h_i h_{\mu_i} =$
Permutations, where \( \sigma_p \) and \( T \) are defined as

\[
T^i = \begin{cases} 
T_{\sigma_i} & \text{for } i = 1, 2, \ldots, r \\
T_{\tau_i} & \text{for } i = r + 1, r + 2, \ldots, m - 1 \\
T^m & \text{for } i = m 
\end{cases} \tag{2.11}
\]

where \( p_{\sigma_i} \) and \( p_{\tau_i} \) represent a product of four-momenta for the \( i \)-th and the \( j \)-th gravitons. The meaning of the bracket will be clarified in a moment (see (2.24) for the definition). We fix \( T^1, T^{m-1} \) and \( T^m \) to the identity. This is related to the fact that the spinor momenta \( u_i \) preserves the \( SL(2, C) \) symmetry which we discuss later. In the above expressions, \( p_{\sigma_{i<j}} \) and \( p_{\tau_{i<j}} \) are defined as follows.

\[
p_{\sigma_{i<j}} = \begin{cases} 
p_{\sigma_i} & \text{for } \sigma_i < \sigma_j \\
0 & \text{otherwise} \end{cases} \quad p_{\tau_{i<j}} = \begin{cases} 
p_{\tau_i} & \text{for } \tau_i < \tau_j \\
0 & \text{otherwise} \end{cases} \tag{2.15}
\]

The Chan-Paton factors of gravitons are expressed in terms of the products of four-momenta, with certain combinatoric structures. This is natural if we notice that a graviton is composed of two frame fields and that their Chan-Paton factors are given by translational operators on the tangent spaces. An explicit form of the frame-field operator \( e^{(\mp)}_i \) can be defined as

\[
e^{(\pm)}_i = e_i^{(\pm)a}(\sqrt{2}p_i)^a = e_i^{(\pm)A\bar{A}} p_i^{A\bar{A}} \tag{2.16}
\]

where we split the tangent-space index \( a (= 0, 1, 2, 3) \) into the two-component indices \( A \) and \( \bar{A} \) both of which take values of \( (1, 2) \). As is seen in a moment, the factor of \( \sqrt{2} \) arises from the use of spinor momenta. \( p^{A\bar{A}}_i \) is a translational operator in the tangent space. Since the tangent space is generally given by a copy of the coordinate space, we can interpret \( p^{A\bar{A}}_i \) as the four-momentum of the \( i \)-th graviton.

**Spinor momenta, twistor space and products of four-momenta**

Since \( p^{A\bar{A}}_i \) satisfies the on-shell condition \( p^2_i = 0 \), it can be written in terms of two-
component spinor momenta \( u^A_i \) and \( \bar{u}^\dot{A}_i \). Explicitly, this can be written as

\[
p^A_i \dot{A} = (\sigma^a)^{A\dot{A}} p_{ia} = u^A_i \bar{u}^\dot{A}_i
\]

where \( \sigma^a = (1, \sigma) \), with \( \sigma \) and the ordinary \((2 \times 2)\) Pauli matrices and the \((2 \times 2)\) identity matrix, respectively. Explicit forms of the spinor momenta are then given by

\[
u^A_i = \frac{1}{\sqrt{p_0 - p_3}} \left( \begin{array}{c} p_1 - ip_2 \\ p_0 - p_3 \end{array} \right), \quad \bar{u}^\dot{A}_i = \frac{1}{\sqrt{p_0 - p_3}} \left( \begin{array}{c} p_1 + ip_2 \\ p_0 - p_3 \end{array} \right)
\]

where we omit the numbering index for simplicity. Notice that we can take \( \bar{u}^\dot{A}_i \) as a conjugate of \( u^A_i \), i.e., \( \bar{u}^\dot{A}_i = (u^A_i)^* \) by requiring that the four-momenta are real.

Lorentz transformations of \( u^A_i \) are given by

\[
u^A_i \rightarrow (gu)^A_i
\]

where \( g \in SL(2, \mathbb{C}) \) is a \((2 \times 2)\)-matrix representation of \( SL(2, \mathbb{C}) \); the complex conjugate of this relation leads to Lorentz transformations of \( \bar{u}^\dot{A}_i \). Four-dimensional Lorentz transformations are realized by a combination of these, that is, the four-dimensional Lorentz symmetry is given by \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \). Scalar products of \( u^A_i \)'s or \( \bar{u}^\dot{A}_i \)'s, which are invariant under the corresponding \( SL(2, \mathbb{C}) \), are expressed as

\[
u_i \cdot \nu_j \equiv (u_i u_j) = \epsilon_{AB} u^A_i u^B_j, \quad \bar{u}_i \cdot \bar{u}_j \equiv [\bar{u}_i \bar{u}_j] = \epsilon^{AB} \bar{u}_i^\dot{A} \bar{u}_j^\dot{B}
\]

where \( \epsilon_{AB} \) is the rank-2 Levi-Civita tensor. This can be used to raise or lower the indices, e.g., \( u_B = \epsilon_{AB} u^A \). Notice that these products are zero when \( i \) and \( j \) are identical.

For a theory with conformal invariance, such as a theory of electromagnetism or \( \mathcal{N} = 4 \) super Yang-Mills theory, we can impose scale invariance on the spinor momentum, i.e.,

\[
u^A_i \sim \lambda u^A_i, \quad \lambda \in \mathbb{C} - \{0\}
\]

where \( \lambda \) is non-zero complex number. With this identification, we can regard the spinor momentum \( u^A_i \) as a homogeneous coordinate of the complex projective space \( \mathbb{CP}^1 \). In the spinor-momenta formalism, we identify this \( \mathbb{CP}^1 \) as an \( S^2 \) fiber of the twistor space \( \mathbb{CP}^3 \). In this formulation, four-dimensional spacetime coordinates \( x_{AA} \) emerges form the twistor-space condition

\[
v^A_A = x_{AA} u^A
\]

where \( v^A_A \) is another two-component complex spinor and the twistor space is defined by a four-component spinor \( Z_I = (u^A, v^A) \) \((I = 1, 2, 3, 4)\) that satisfies the scale invariance

\[
Z_I \sim \lambda Z_I, \quad \lambda \in \mathbb{C} - \{0\}.
\]

In terms of the spinor momenta, products of four-momenta in the form of \((2.17)\) can be defined as

\[
p_i^{A\dot{A}} p_{j\dot{A}A} = (u_i u_j) [\bar{u}_i \bar{u}_j] = 2p_i^a p_{ja} \equiv \langle p_i \cdot p_j \rangle
\]
where we use the expressions in (2.20). This shows an explicit meaning of the brackets that appear in (2.12) and (2.13). It also explains the factor $\sqrt{2}$ in (2.16).

Frame-field holonomy: bialgebraic operator and logarithmic one-form

As mentioned earlier, the sign $\pm$ in (2.16) represents an analog of helicity for the frame field. In fact, in our construction we consider the frame field as a massless gluon, with its Chan-Paton factor specified by (2.16). This means that we can define a holonomy operator for the frame field:

$$\Theta^{(E)}_{R,\gamma}(u) = \text{Tr}_{R,\gamma} P \exp \left[ \sum_{m \geq 2} \oint_{\gamma} E \wedge E \wedge \cdots \wedge E \right]$$  \hspace{1cm} (2.25)

$$E = \sum_{1 \leq i < j \leq n} E_{ij}$$ \hspace{1cm} (2.26)

$$E_{ij} = e_i^{(+)} \otimes e_j^{(0)} + e_i^{(-)} \otimes e_j^{(0)}$$ \hspace{1cm} (2.27)

$$\omega_{ij} = d \log(u_i u_j) = d(u_i u_j)/(u_i u_j)$$ \hspace{1cm} (2.28)

where the operators $e_i^{(\pm)}$ and $e_i^{(0)}$ obey the $SL(2, \mathbb{C})$ algebra. Explicitly this can be expressed as

$$[e_i^{(+)} , e_j^{(-)}] = 2e_i^{(0)} \delta_{ij}, \quad [e_i^{(0)} , e_j^{(+)}] = e_i^{(+)} \delta_{ij}, \quad [e_i^{(0)} , e_j^{(-)}] = -e_i^{(-)} \delta_{ij}$$ \hspace{1cm} (2.29)

where Kronecker’s deltas show that the non-zero commutators are obtained only for $i = j$. The remaining of commutators, those expressed otherwise, all vanish. As shown in (2.28), $\omega_{ij}$ is a logarithmic one-form in terms of the Lorentz invariant product of the spinor momenta $u_i$ and $u_j$. $\omega_{ij}$ in (2.2) is also defined by this logarithmic one-form. The bialgebraic operator $E$ in (2.26) is what we may call the comprehensive frame field.

Configuration space, “path” ordering and braid trace

Physical variables of the comprehensive frame field $E$ is given by $n$ spinor momenta. Since these are symmetric to each other, a physical configuration space of $E$ and hence that of $\Theta^{(E)}_{R,\gamma}(u)$ can be defined by $C = \mathbb{C}^n/S_n$ where $S_n$ denotes the rank-$n$ symmetric group. The symbol $\gamma$ in (2.25) represents a closed path on $C$ along which the integral is evaluated. On the other hand, the symbol $R$ in (2.25) denotes the representation of the algebra of the Chan-Paton factor.

The symbol $P$ in (2.25) denotes an ordering of the numbering indices. The meaning of the action of $P$ on the exponent of (2.25) can explicitly be written as

$$P \sum_{m \geq 2} \oint_{\gamma} E \wedge E \wedge \cdots \wedge E_m = \sum_{m \geq 2} \oint_{\gamma} E_{12} E_{23} \cdots E_{m1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m1}$$

$$= \sum_{m \geq 2} \frac{1}{2^{m+1}} \sum_{(h_1, h_2, \cdots, h_m)} (-1)^{h_1 + h_2 + \cdots + h_m} \times e_1^{(h_1)} \otimes e_2^{(h_2)} \otimes \cdots \otimes e_m^{(h_m)} \oint_{\gamma} \omega_{12} \wedge \cdots \wedge \omega_{m1}$$ \hspace{1cm} (2.30)

where $h_i = \pm \pm 1 \ (i = 1, 2, \cdots, m)$ denotes the “helicity” of the $i$-th frame field. In obtaining the above expression, we use an ordinary definition of commutators for bialgebraic
operators. For example, using the commutation relations (2.29), we can calculate \([E_{12}, E_{23}]\) as
\[
[E_{12}, E_{23}] = e_1^{(+)} \otimes e_2^{(+)} \otimes e_3^{(0)} - e_1^{(+)} \otimes e_2^{(-)} \otimes e_3^{(0)} + e_1^{(-)} \otimes e_2^{(+)} \otimes e_3^{(0)} - e_1^{(-)} \otimes e_2^{(-)} \otimes e_3^{(0)}.
\] (2.31)

In (2.30), we also define \(e_1^{(\pm)} \otimes e_2^{(h_2)} \otimes \cdots \otimes e_m^{(h_m)} \otimes e_1^{(0)}\) as
\[
e_1^{(\pm)} \otimes e_2^{(h_2)} \otimes \cdots \otimes e_m^{(h_m)} \otimes e_1^{(0)} = \frac{1}{2} \{ e_1^{(0)}, e_1^{(\pm)} \} \otimes e_2^{(h_2)} \otimes \cdots \otimes e_m^{(h_m)} = \pm \frac{1}{2} e_1^{(\pm)} \otimes e_2^{(h_2)} \otimes \cdots \otimes e_m^{(h_m)}
\] (2.32)

where we implicitly use an antisymmetric property for the indices due to the wedge products.

The trace \(\text{Tr}_{R,\gamma}\) in the definition (2.25) means a trace over the Chan-Paton factors of the frame fields. As discussed in [23], this trace includes not only a trace over the translational operators but also that of braid generators. The latter, a so-called braid trace, is realized by a sum over permutations of the numbering indices. Thus the trace \(\text{Tr}_{R,\gamma}\) over the exponent of (2.25) can be expressed as
\[
\text{Tr}_{R,\gamma} P \sum_{m \geq 2} \oint \frac{E \wedge \cdots \wedge E}{m} = \sum_{m \geq 2} \sum_{\sigma' \in S_{m-1}} \oint_{\gamma} E_{1\sigma'_1} E_{\sigma'_2 \sigma'_3} \cdots E_{\sigma'_m} \omega_{1\sigma_2} \wedge \omega_{\sigma'_2 \sigma'_3} \wedge \cdots \wedge \omega_{\sigma'_m} (2.33)
\]

where the sum of \(\sigma' \in S_{m-1}\) is now taken over the permutations \(\sigma' = \begin{pmatrix} 2 & 3 & \cdots & m \\ \sigma'_2 & \sigma'_3 & \cdots & \sigma'_m \end{pmatrix}\).

The exponent of \(\Theta^H_{R,\gamma}(u, \bar{u})\)

We now return to the gravitational case. The above meanings of \(\text{Tr}_{R,\gamma}\) and \(P\) can also be applied to (2.1). (Regarding what \(\gamma\) and an integral around it mean in a gravitational theory, we shall consider in the next section.) We can then define a gravitational analog of (2.33). Its explicit form is given by
\[
\text{Tr}_{R,\gamma} P \oint_{\gamma} \frac{H \wedge H \wedge \cdots \wedge H}{m} = (8\pi G_N)^{\frac{m}{2}} \text{Tr}_{R,\gamma} \oint_{\gamma} H_{12} H_{23} \cdots H_{m-1} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m-1}
\]
\[
= (8\pi G_N)^{\frac{m}{2}} \left( \frac{1}{2m+1} \right)^2 \sum_{\sigma \in S_{m-1}} \sum_{r \in S_{m-r-2}} \prod_{i=2}^r T^{i \sigma_i} \prod_{i=r+1}^{m-2} T^{r_i} \sum_{(h_{11}, h_{22}, \cdots, h_{mm})} g_{11}^{(h_{11})} \otimes g_{22}^{(h_{22})} \otimes g_{33}^{(h_{33})} \cdots \otimes g_{rr}^{(h_{rr})} \otimes g_{r+1}^{(h_{r+1})} \otimes g_{r+2}^{(h_{r+2})} \cdots \otimes g_{m-2}^{(h_{m-2})} \otimes g_{m-1}^{(h_{m-1})} \otimes g_{mm}^{(h_{mm})}
\]
\[
\times \oint_{\gamma} \omega_{12} \wedge \omega_{23} \wedge \cdots \wedge \omega_{m-1} \wedge \omega_{m-1}
\]
\[
\times \oint_\gamma \omega_{\sigma_2 \sigma_3} \land \omega_{\sigma_3 \sigma_4} \land \cdots \land \omega_{\sigma_{r-1} \sigma_r} \land \omega_{\sigma_1} \\
\land \omega_{1 m - 1} \land \omega_{m - 1 \tau_{r+1}} \land \omega_{\tau_{r+1} \tau_{r+2}} \land \cdots \land \omega_{\tau_{m-2} m} \land \omega_{m 2} \\
+ \mathcal{P}(23 \cdots m - 2) 
\]  

(2.34)

where the sum of \((h_{11}, h_{2\sigma_2}, \cdots, h_{mm})\) is taken over any combinations of \(h_{i\mu_i} = (++, +-, --, ---)\). Explicit forms of \(T^{\sigma_i}\)’s and \(T^{\tau_i}\)’s are defined in (2.12) and (2.13), respectively.

In (2.34), a sum over possible metrics is realized by two distinct sums over the permutations of \(\sigma = \left( \begin{array}{c} 2 \cdots r \\ \sigma_2 \cdots \sigma_r \end{array} \right)\) and \(\tau = \left( \begin{array}{c} r + 1 \cdots m - 2 \\ \tau_{r+1} \cdots \tau_{m-2} \end{array} \right)\). We shall call this set of sums a split sum. A braid trace, on the other hand, is realized by \(\mathcal{P}(23 \cdots m - 2)\) which indicates the terms obtained by permutations of the overall elements \(\{2, 3, \cdots, m - 2\}\). These realizations reflect the fact that we split the numbering indices as shown in (2.3). In fact, this feature is pertinent to KLT-inspired graviton amplitudes in general. In the next section, we treat the numbering indices in a more democratic manner and consider an alternative definition of the gravitational holonomy operator (2.1).

3 Gravitational holonomy in a squared form

In this section, we present a main result of this paper. Namely, we shall obtain an alternative expression for the exponent of the gravitational holonomy operator, which is different from (2.34), such that we can interpret \(\Theta^{(H)}_{R, \gamma}(u, \bar{u})\) as a square of \(\Theta^{(E)}_{R, \gamma}(u)\).

A split sum and a homogeneous sum

In the previous section, we have two distinct permutations (2.8). Accordingly, the numbering elements are split into two parts. Under the ordering conditions, \(\sigma_2 < \sigma_3 < \cdots < \sigma_r\) and \(\tau_{r+1} < \tau_{r+2} < \cdots < \tau_{m-2}\), these elements can uniquely be determined. The braid trace is then realized by a sum over permutations of the overall elements \(\{2, 3, \cdots, m - 2\}\). This sum (or trace) should be taken on top of the split sum, \(i.e.,\), the two distinct sums over \(\sigma\)’s and \(\tau\)’s. There is another way of calculating the Chan-Paton factor in (2.34). This can be carried out by assigning \(\sigma\)’s and \(\tau\)’s to the overall elements \(\{2, 3, \cdots, m - 2\}\) homogeneously. Namely, the elements of both \(\sigma\)’s and \(\tau\)’s can take any values in the overall elements, given that they satisfy the ordering conditions. A primordial form of this alternative expression is first introduced in the study of graviton amplitudes [8]. We briefly review its results in the appendix of the present paper. In the following, we shall use these results in relation to the homogeneous sum and give an interpretation of \(\Theta^{(H)}_{R, \gamma}\) as a square of \(\Theta^{(E)}_{R, \gamma}\).

Symmetry of holonomy operator and characterization of braid trace

To begin with, we first remind ourselves that the Chan-Paton factor of (2.34) has an \(SL(2, \mathbb{C})\) symmetry. In the spinor-momenta formalism, this symmetry is relevant to the Lorentz invariance of the spinor momenta as shown in (2.19). In the Yang-Mills case, the corresponding Chan-Paton factor has a \(U(1)\) symmetry. (Notice that it is a Chan-Paton factor of a Yang-Mills holonomy operator \(per se\), not that of a gauge field.) This corresponds
to the fact that there is a single type of permutation, labeled by $\sigma'$, in the expression (2.33). In terms of the braid trace $\text{Tr}_\gamma$, this means that there is a single loop, say $\gamma_1$, that labels the braid trace. On the other hand, in the gravitational case, we have the $SL(2, \mathbb{C})$ symmetry. Thus, as discussed in [23], the gravitational braid trace can be characterized by three distinct loops. These may be chosen by $(\gamma_1, \gamma_{m-1}, \gamma_m)$ where the indices $(1, m-1, m)$ correspond to the fixed numbering indices in Figure 1.

There must be a correspondence between the loops $(\gamma_1, \gamma_{m-1}, \gamma_m)$ and the elements of $SL(2, \mathbb{C})$ algebra, say, a set of generators $(t^{(+)}, t^{(-)}, t^{(0)})$. As discussed in [23], a natural way to realize this correspondence is to assign orderings to the numbering indices for each of the loops. We then make the numbering indices in a descending order for the loop $\gamma_1$ and in an ascending order for the loop $\gamma_{m-1}$, along with certain orientations of the loops. The loop $\gamma_m$ which corresponds to $t^{(0)}$ does not have a notion of ordering. Hence, it is natural to think that the loop $\gamma_m$ involves only one numbering element, otherwise we may have redundant $U(1)$ symmetries. The gravitational braid trace is therefore essentially characterized by the ladder generators $t^{(\pm)}$ of $SL(2, \mathbb{C})$.

**Use of the homogeneous sum**

We denote the elements of $\gamma_1$ by $\{\sigma_2, \sigma_3, \ldots, \sigma_r\}$ and those of $\gamma_{m-1}$ by $\{\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{m-2}\}$ $(2 \leq r \leq m-3)$. Then the three disconnected loops can be created by the three braid diagrams in Figure 1. In the case of a split sum, we have split the numbering elements into $\{2, 3, \ldots, r\}$ and $\{r+1, r+2, \ldots, m-2\}$. Thus, for a specific choice of $r$, this fixes the choice of the elements for $\sigma$’s and $\tau$’s. In the case of a homogeneous sum, however, we assume that we can choose the numbering elements for $\sigma$’s (and $\tau$’s) arbitrarily so that there are $m-3C_{r-1}$ such choices for a fixed $r$. Using the notations in (2.3)-(2.5) and the results (A.9)-(A.11) in the appendix, we can then write down a homogeneous version of the expression (2.34) as

$$
\text{Tr}_{R, \gamma} P \oint_{\gamma} H \wedge H \wedge \cdots \wedge H
= \left(8\pi G_N\right)^{\frac{m}{2}} \left(\frac{1}{2m+1}\right)^2 \sum_{\{\sigma, \tau\} = \{2, 3, \ldots, m-2\}} \mathcal{C}(\mu_1 \mu_2 \cdots \mu_m)
\times \left(\prod_{i=1}^{m} T^{\lambda_i} \sum_{\{h_{\mu_1 \lambda_1}, h_{\mu_2 \lambda_2}, \ldots, h_{\mu_m \lambda_m}\}} g_{\mu_1 \lambda_1}^{(h_{\mu_1 \lambda_1})} \otimes g_{\mu_2 \lambda_2}^{(h_{\mu_2 \lambda_2})} \otimes \cdots \otimes g_{\mu_m \lambda_m}^{(h_{\mu_m \lambda_m})} \mathcal{C}(\lambda_1 \lambda_2 \cdots \lambda_m) + \mathcal{P}(\sigma|\tau)\right)
+ \mathcal{P}(\sigma|\tau)
\left(3.1\right)
$$

where

$$
\mathcal{C}(\mu_1 \mu_2 \cdots \mu_m) = \mathcal{C}(1 \sigma_2 \cdots \sigma_r \tau_{r+1} \cdots \tau_{m-2} m - 1 m)
= \oint_{\gamma} \omega_{1 \sigma_2} \wedge \omega_{\sigma_2 \sigma_3} \wedge \cdots \wedge \omega_{m-1 m} \wedge \omega_{m1}\left(3.2\right)
$$

and the same for $\mathcal{C}(\lambda_1 \lambda_2 \cdots \lambda_m) = \mathcal{C}(\sigma \cdots \sigma_r 1 m - 1 \tau_{r+1} \cdots \tau_{m-2} m)$. The sum of $\{\sigma, \tau\}$ in (3.1) is taken over all possible combinations for the elements $\{\sigma_2, \ldots, \sigma_r, \tau_{r+1}, \ldots, \tau_{m-2}\}$ such that the ordering conditions $\sigma_2 < \cdots < \sigma_r$ and $\tau_{r+1} < \cdots < \tau_{m-2}$ are preserved. As
in the case of (2.34), the sum of \((h_{\mu_1\lambda_1}, h_{\mu_2\lambda_2}, \ldots, h_{\mu_m\lambda_m})\) is taken over any combinations of \(h_{\mu_i\lambda_i} = (++, +-, -+, --)\) with \(i = 1, 2, \ldots, m\). Notice that the factor \(\prod_{i=1}^{m} T^{\lambda_i}\) is determined only by the permutations \(\sigma\) and \(\tau\). Thus it is also equal to \(\prod_{i=1}^{m} T^{\mu_i}\). In other words, this product sum is in one-to-one correspondence with the braid diagrams in Figure 1, and is therefore uniquely determined once we choose the permutations \(\sigma\) and \(\tau\). The symbol \(\mathcal{P}(\sigma|\tau)\) in (3.1) denotes the terms obtained by the permutations of \(\sigma\)'s and \(\tau\)'s, i.e.,
\[
\mathcal{P}(\sigma|\tau) = \mathcal{P}(\sigma_2\sigma_3 \cdots \sigma_r) \times \mathcal{P}(\tau_{r+1}\tau_{r+2} \cdots \tau_{m-2})
\] (3.3)
where, as in (2.34), \(\mathcal{P}(\sigma_2\sigma_3 \cdots \sigma_r)\) denotes terms obtained by permutations of \(\sigma\)'s, and the same for \(\mathcal{P}(\tau_{r+1}\tau_{r+2} \cdots \tau_{m-2})\).

A double braid-trace and the squared form

As in the previous case, \(\mathcal{P}(\sigma|\tau)\) can be regarded as a realization of a braid trace. The double appearance of \(\mathcal{P}(\sigma|\tau)\) then suggests that the braid trace over gravitons can be replaced by a double braid-trace over frame fields. This interpretation is in accord with the idea that the graviton is describable in terms of a product of frame fields even at the level of comprehensive field operators. As we shall see in the next section, the double appearance of \(\mathcal{P}(\sigma|\tau)\) also supports the use of functional derivatives with respect to the frame-field operators (or source functions, to be precise) in obtaining graviton amplitudes generated by the gravitational holonomy operator. Notice that \(\mathcal{P}(\sigma|\tau)\)'s appear before taking the homogeneous sum of \(\{\sigma, \tau\}\). Thus, the eventual expression of (3.1) is independent of the choices of \(\sigma\)'s and \(\tau\)'s, but the squared structure appears inside the homogeneous sum and, in this respect, we need to label the indices of frame fields by \((\sigma, \tau)\) or equivalently by \((\mu, \lambda)\).

Motivated by these considerations, we now introduce a new notation:
\[
\text{Tr}_{R_{\mu,\gamma}\sigma|\tau} \mathcal{P} \oint_{\gamma[\sigma|\tau]} E \wedge E \wedge \cdots \wedge E
\] = \text{Tr}_{R_{\mu,\gamma}\sigma|\tau} \oint_{\gamma[\sigma|\tau]} E_{\mu_1\mu_2} E_{\mu_2\mu_3} \cdots E_{\mu_{m}\mu_1} \omega_{\mu_1\mu_2} \wedge \omega_{\mu_2\mu_3} \wedge \cdots \wedge \omega_{\mu_{m}\mu_1}
\]
= \(E_{\mu_1\mu_2} E_{\mu_2\mu_3} \cdots E_{\mu_{m}\mu_1} C(\mu_1\mu_2 \cdots \mu_m) + \mathcal{P}(\sigma|\tau)\) (3.4)

where we denote the closed path by \(\gamma_{\sigma|\tau}\) to indicate that the permutations over the numbering indices are separately taken for \(\sigma\)'s and \(\tau\)'s. We also label the representation of the algebra of a braid group by \(R_{\mu}\), which reflects that the comprehensive frame fields are labeled by \(\mu_i\) \((i = 1, 2, \ldots, m)\) in the above expression.

Using the notation (3.4), we can rewrite (3.1) as
\[
\text{Tr}_{R_{\mu,\gamma}} \mathcal{P} \oint_{\gamma} H \wedge H \wedge \cdots \wedge H
\]
= \((8\pi G_N)^{\frac{m}{2}} \sum_{\{\sigma, \tau\} = \{2, 3, \ldots, m-2\}} \langle \text{Tr}_{R_{\mu,\gamma}} \mathcal{P} \oint_{\gamma} E \wedge E \wedge \cdots \wedge E, \text{Tr}_{R_{\lambda,\gamma}} \mathcal{P} \oint_{\gamma} E \wedge E \wedge \cdots \wedge E \rangle \rangle_{\gamma = \gamma_{\sigma|\tau}}
\]

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\[ (8\pi G_N)^{\frac{m}{2}} \left( \frac{1}{2m+1} \right)^2 \sum_{\{\sigma, \tau\} = \{2,3,\ldots,m-2\}} \text{Tr}_{R_{\mu, \gamma}} \left[ C(\mu_1 \mu_2 \cdots \mu_m) \right] \times \text{Tr}_{R_{\lambda, \gamma}} \left[ \prod_{i=1}^{m} T^{\lambda_i} \sum_{(h_{\mu_1, \ldots, h_{\mu_m, \lambda_m})} g^{(h_{\mu_1, \lambda_1})} \otimes \cdots \otimes g^{(h_{\mu_m, \lambda_m})} C(\lambda_1 \lambda_2 \cdots \lambda_m) \right]_{\gamma = \gamma_{\sigma|\tau}} \] (3.5)

where the bracket in the middle denotes a set of products between Chan-Paton factors of the frame fields, as defined in (2.24). The set of products take a form of \( \prod_{i=1}^{m} T^{\lambda_i} \) in the last line. As mentioned below (3.2), this factor is dependent only on the choice of \( (\sigma, \tau) \), i.e.,

\[ \prod_{i=1}^{m} T^{\lambda_i} = \prod_{i=2}^{r} T^{\sigma_i} \prod_{i=r+1}^{m-2} T^{\tau_i} = \prod_{i=1}^{m} T^{\mu_i}. \] (3.6)

This reflects the fact that the holonomy operator inherently describes a multi-particle system so that its Chan-Paton factor depends on comprehensive information about all the particles. The information is given by a specific permutation of particles in a form of an irreducible representation of the braid diagrams. As analyzed in [23], the factor (3.6) is indeed in one-to-one correspondence with the braid diagrams in Figure 1, once we impose irreducibility up to certain Reidemeister moves of the strands. The specific choice of \( \lambda_i \), in relation to \( \mu_i \), has been made such that we have non-vanishing Chan-Paton factors. In this sense, the indices \( \mu_i \) and \( \lambda_i \) labels the (irreducible) representation of the diagrams. This means that, before carrying out the homogeneous sum, the gravitational holonomy operator can and should be labeled by the representation \( R_{\mu, \lambda} \). Notice that in the Yang-Mills case, we do not have to consider the product of Chan-Paton factors such as (3.6). Thus, in taking the homogeneous sum, a representation of the algebra of a braid group, or a representation of the Iwahori-Hecke algebra, for the frame-field holonomy operator can be labeled by a single index. This explains the notations \( R_{\mu} \) and \( R_{\lambda} \) in (3.5).

Using the expression (3.5), we can then express the gravitational holonomy operator as a homogeneous sum over squares of the frame-field holonomy operator:

\[ \Theta_{R_{\gamma}, \gamma}^{(H)}(u, \bar{u}) = \sum_{\{\sigma, \tau\}} \Theta_{R_{\mu, \gamma}, \gamma_{\sigma|\tau}}^{(H)}(u, \bar{u}) = \sum_{\{\sigma, \tau\}} \left( \Theta_{R_{\mu, \gamma}, \gamma_{\sigma|\tau}}^{(E)}(u) \cdot \Theta_{R_{\lambda, \gamma}, \gamma_{\sigma|\tau}}^{(E)}(u) \right)_{\gamma = \gamma_{\sigma|\tau}} \] (3.7)

\[ \Theta_{R_{\mu, \gamma}, \gamma_{\sigma|\tau}}^{(E)}(u) = \text{Tr}_{R_{\mu, \gamma_{\sigma|\tau}}} \text{P exp} \left[ \sum_{m \geq 2} \oint \gamma_{\sigma|\tau} E \wedge E \wedge \cdots \wedge E \right] \] (3.8)

\[ \Theta_{R_{\lambda, \gamma}, \gamma_{\sigma|\tau}}^{(E)}(u) = \text{Tr}_{R_{\lambda, \gamma_{\sigma|\tau}}} \text{P exp} \left[ \sum_{m \geq 2} \oint \gamma_{\sigma|\tau} E \wedge E \wedge \cdots \wedge E \right] \] (3.9)

where we make the coupling constant absorbed into each of the frame-field operators. We specify the representation of the frame-field holonomy operators by \( R_{\mu} \) and \( R_{\lambda} \). This corresponds to the fact that the exponent of \( \Theta_{R_{\gamma}, \gamma}^{(H)}(u, \bar{u}) \) is given by the expression (3.5). The homogeneous sum of \( \{\sigma, \tau\} \) is taken over all possible combinations for the elements \( \{\sigma_2, \ldots, \sigma_r, \tau_{r+1}, \ldots, \tau_{m-2}\} = \{2,3,\ldots,m-2\} \) such that the ordering conditions \( \sigma_2 < \sigma_3 < \cdots < \sigma_r \) and \( \tau_{r+1} < \tau_{r+2} < \cdots < \tau_{m-2} \) are preserved.

The homogeneous sum: a sum over \((k,l)\)-shuffles
We now briefly discuss how the homogeneous sum appears naturally in the framework of holonomy formalism. The factor of \( C(\mu_1 \mu_2 \cdots \mu_m) \) in (3.2) gives an iterated (loop) integral over a series of the logarithmic one-forms. Generally, a product of iterated integrals can be defined as

\[
\int_{\tilde{\gamma}_k} \omega_1 \omega_2 \cdots \omega_k \int_{\tilde{\gamma}_l} \omega_{k+1} \omega_{k+2} \cdots \omega_{k+l} = \sum_{\tilde{\sigma} \in S_{k,l}} \int_{\tilde{\gamma}_{k+l}} \omega_{\tilde{\sigma}_1} \omega_{\tilde{\sigma}_2} \cdots \omega_{\tilde{\sigma}_{k+l}}
\]

(3.10)

where \( \omega_1, \omega_2, \cdots, \omega_{k+l} \) are arbitrary differential one-forms. The symbol \( \tilde{\gamma}_n \) denotes an open path in \( \mathbb{C}^n \). The sum of the permutations \( \tilde{\sigma} \in S_{k,l} \) is taken over the so-called \((k, l)\)-shuffles that satisfy the ordering conditions:

\[
\tilde{\sigma}_1 < \tilde{\sigma}_2 < \cdots < \tilde{\sigma}_k, \\
\tilde{\sigma}_{k+1} < \tilde{\sigma}_{k+2} < \cdots < \tilde{\sigma}_{k+l}.
\]

(3.11)

The sum of \( \tilde{\sigma} \in S_{k,l} \) is therefore essentially the same as the homogeneous sum. Applying the relation (3.10) to loop integrals along \((\gamma_1, \gamma_{m-1}, \gamma_m)\) that we have defined in the beginning of this section, we can then obtain an expression

\[
\sum_{\{\sigma, \tau\}} \oint_{\gamma_{|\sigma| \tau}} \omega_{\mu_1} \omega_{\mu_2} \cdots \omega_{\mu_m} = \oint_{\gamma_1} \omega_1 \omega_2 \cdots \omega_r \oint_{\gamma_{m-1}} \omega_{r+1} \omega_{r+2} \cdots \omega_{m-1} \oint_{\gamma_m} \omega_m = \oint_{\gamma_1} \omega_2 \omega_3 \cdots \omega_{r+1} \omega_{r+2} \cdots \omega_{m-2} \oint_{\gamma_m} \omega_m = \sum_{\{\sigma, \tau\}} \oint_{\gamma_{|\sigma| \tau}} \omega_{\lambda_1} \omega_{\lambda_2} \cdots \omega_{\lambda_m}
\]

(3.12)

where we use the cyclic property of the loop integrals along \( \gamma_1 \) and \( \gamma_{m-1} \). The sum of \( \{\sigma, \tau\} \) denotes the homogeneous sum, being the same as the one defined in (3.1). As mentioned earlier, the overall path \( \gamma = \gamma_{|\sigma| \tau} \) is decomposed into three closed paths \((\gamma_1, \gamma_{m-1}, \gamma_m)\). By identifying \( \omega_i \) as the logarithmic one-form \( \omega_{i+1} \) in (3.12), we can then obtain the relation

\[
\sum_{\{\sigma, \tau\}} C(\mu_1 \mu_2 \cdots \mu_m) = \sum_{\{\sigma, \tau\}} C(\lambda_1 \lambda_2 \cdots \lambda_m).
\]

(3.13)

This equation means that the factors of \( C(\mu_1 \mu_2 \cdots \mu_m) \) and \( C(\lambda_1 \lambda_2 \cdots \lambda_m) \) are equivalent under the homogeneous sum or the sum over the \((r - 1, m - 2 - r)\)-shuffles. Thus, in this sense, the subset of the gravitational holonomy operator, denoted by \( \Theta_{H\mu\lambda, \gamma_{|\sigma| \tau}}^{(H)}(u, \bar{u}) \) in (3.7), can be interpreted as a square of the same theory.

**General covariance and diffeomorphism**

Lastly, as a summary of this section, we now consider some physical aspects of the squared expression (3.1) or (3.5). There are essentially two sums to be taken in the holonomy formalism of gravity. As emphasized in [23], these are given by the following two sums:

1. a sum over all possible metrics that guarantees general covariance of the theory; and
2. a sum over permutations of the numbering elements, or a braid trace, that is necessary for diffeomorphism invariance.

For the original expression (2.34), as discussed in the previous section, the former sum is realized by the split sum \[ \sum_{\sigma \in S_{r-1}} \sum_{\tau \in S_{m-r-1}} \] and the latter is represented by the terms of \( \mathcal{P}(23 \cdots m - 2) \). In this section, we have shown that the double appearance of \( \mathcal{P}(\sigma|\tau) \) in (3.1) can be interpreted as a double braid-trace in (3.5). Thus, in the squared expression, the braid trace is realized by the double-permutation terms, which we denote here as \( \mathcal{P}(\sigma|\tau) + \mathcal{P}(\sigma|\tau) \), while the sum over metrics is realized by the homogeneous sum \[ \sum_{\{\sigma,\tau\} = \{2, 3, \cdots, m-2\}} \].

Therefore, for either case, we can make physically clear interpretations to the summations that appear in the expressions of the gravitational holonomy operator. These interpretations are summarized in Table 1.

| Sum over metrics (general covariance) | split sum | homogeneous sum |
|--------------------------------------|-----------|-----------------|
| Braid trace (diffeomorphism)         | single-permutation terms | double-permutation terms |
| Quantities of interest               | gravitons | frame fields     |
| Relevant expression                  | (2.34)    | (3.1), (3.5)    |
| Gravitational theory                 | as a gauge theory | as a square of gauge theory |

Table 1: Interpretation of a sum over metrics and a braid trace in expressions of the gravitational holonomy operator

4 S-matrix functionals for graviton amplitudes

So far, we have discussed how the gravitational holonomy operator can be expressed as a square of the frame-field holonomy operator. In this section, we utilize the new expression to obtain an S-matrix functional for graviton amplitudes. For this purpose, we first review how we obtain an S-matrix functional for the maximally helicity violating (MHV) graviton amplitudes in the split-sum case. We then consider the homogeneous-sum case and show that the MHV S-matrix functional can also be described in terms of a supersymmetric version of the operator (3.7). For the completion of the analysis, we shall also consider S-matrix functionals for non-MHV amplitudes in general.

The split-sum case

In the holonomy formalism, physical information is embedded in the operator \( g_i^{(h_{\mu\nu})} \) in (2.5). This operator is in a momentum-space representation. Let \( x \) be the four-dimensional spacetime coordinate. In an \( x \)-space representation, the operator is then expressed as

\[ g_i^{(h_{\mu\nu})}(x) = \int d\mu(p_i) \frac{g_i^{(h_{\mu\nu})}}{e^{ix \cdot p_i}} \]  

(4.1)
where \(d\mu(p_i)\) denotes a four-dimensional Lorentz invariant measure, known as the Nair measure.

It is known that the most convenient prescription to an S-matrix functional for the MHV amplitudes is to supersymmetrize the operator (4.1). In the present case, we consider an \(\mathcal{N} = 8\) extended supersymmetry. The relevant Grassmann variables are expressed as \(\theta^\alpha_A\), with \(A = 1, 2\) and \(\alpha = 1, 2, \cdots, 8\). In the spinor-momenta formalism, it is convenient to introduce the “projected” Grassmann variables:

\[
\xi^\alpha = \theta^\alpha_A u^A \quad (\alpha = 1, 2, \cdots, 8). \tag{4.2}
\]

For the later convenience, we further split the index \(\alpha\) into two parts:

\[
\alpha = (\alpha_1, \alpha_2), \quad \alpha_1 = 1, 2, 3, 4, \quad \alpha_2 = 5, 6, 7, 8. \tag{4.3}
\]

We can then write down a supersymmetrization of (4.1) as

\[
\begin{align*}
g_i^{(h_{\mu_1})}(x, \theta) &= \int d\mu(p_i) \ g_i^{(h_{\mu_1})}(\xi_i) \ e^{ix \cdot p_i} \bigg|_{\xi_i^\alpha = \theta^\alpha_A u^A} \tag{4.4} \\
g_i^{(h_{\mu_1})}(\xi_i) &= T^{\mu_i} g_i^{(h_{\mu_1})}(\xi_i) = T^{\mu_i} e_i^{(h_1)a}(\xi_i) e_i^{(h_2)a}(\xi_i) \tag{4.5}
\end{align*}
\]

where \(g_i^{(h_{\mu_1})}(\xi_i)\) in the second equation can be considered as a supersymmetrization of the graviton operator \(g_i^{(h_{\mu_1})}\) defined in (2.5). Accordingly, \(e_i^{(h_1)a}(\xi_i)\) \((a = 0, 1, 2, 3)\) correspond to a supersymmetric version of the frame fields \(e_i^{(h_2)}\) in (2.16) and are defined as

\[
\begin{align*}
e_i^{(+a)}(\xi_i) &= e_i^{(+a)} \\
e_i^{(+\frac{1}{2}a)}(\xi_i) &= \xi_i^{\alpha_1} e_i^{(+\frac{1}{2}a)}_{\alpha_1} \\
e_i^{(0a)}(\xi_i) &= \frac{1}{2} \xi_i^{\alpha_1} \xi_i^{\beta_1} e_i^{(0a)}_{\alpha_1 \beta_1} \\
e_i^{(-\frac{1}{2}a)}(\xi_i) &= \frac{1}{3!} \xi_i^{\alpha_1} \xi_i^{\beta_1} \xi_i^{\gamma_1} e_{\alpha_1 \beta_1 \gamma_1} e_i^{(-\frac{1}{2}a)} \\
e_i^{(-a)}(\xi_i) &= \xi_i^{\alpha_1} \xi_i^{\beta_1} \xi_i^{\gamma_1} e_i^{(-a)}
\end{align*} \tag{4.6}
\]

where each of \(\alpha_1, \beta_1, \cdots\) takes a value of 1, 2, 3 or 4. Similarly, \(e_i^{(h_{\mu_1})}(\xi_i)\)'s are defined as

\[
\begin{align*}
e_i^{(+a)}_{\mu_1}(\xi_i) &= e_i^{(+a)}_{\mu_1} \\
e_i^{(+\frac{1}{2}a)}_{\mu_1}(\xi_i) &= \xi_i^{\alpha_2} e_i^{(+\frac{1}{2}a)}_{\mu_1 \alpha_2} \\
e_i^{(0a)}_{\mu_1}(\xi_i) &= \frac{1}{2} \xi_i^{\alpha_2} \xi_i^{\beta_2} e_i^{(0a)}_{\mu_1 \alpha_2 \beta_2} \\
e_i^{(-\frac{1}{2}a)}_{\mu_1}(\xi_i) &= \frac{1}{3!} \xi_i^{\alpha_2} \xi_i^{\beta_2} \xi_i^{\gamma_2} e_{\alpha_2 \beta_2 \gamma_2} e_{\mu_1}^{(-\frac{1}{2}a)} \\
e_i^{(-a)}_{\mu_1}(\xi_i) &= \xi_i^{\delta_2} \xi_i^{\epsilon_2} \xi_i^{\zeta_2} e_{\mu_1}^{(-a)}
\end{align*} \tag{4.7}
\]

where each of \(\alpha_2, \beta_2, \cdots\) takes a value of 5, 6, 7 or 8. Notice that either \(\hat{h}_i\) or \(\hat{h}_{\mu_1}\) represents a helicity of an \(\mathcal{N} = 4\) supersymmetric frame field. The symbol \(\hat{h}_{\mu_1}\) then denotes a
supersymmetrization of $h_{\mu i} \equiv h_i h_{\mu i}$, i.e., $\hat{h}_{\mu i} \equiv \hat{h}_i \hat{h}_{\mu i}$. We use $\xi_\alpha^\alpha$'s, rather than $\xi_\alpha^\alpha$'s, in (4.7). This comes from the fact that we interpret the graviton (4.4) as a point-like operator in $\mathcal{N} = 8$ chiral superspace. Alternatively, we can interpret $\xi_i$ as chiral superpartners of the tangent-space coordinate $x_a \ (a = 0, 1, 2, 3)$, with spacetime not being supersymmetrized.

A supersymmetric gravitational holonomy operator $\Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta)$ is then defined by substitution of $g_i^{(h_{\mu i})}(x, \theta)$ into $g_i^{(h_{\mu i})}$ in (2.2).

Using the supersymmetric holonomy operator $\Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta)$, we can define an S-matrix functional for the MHV graviton amplitudes as

$$F_{\text{MHV}} \left[ g_i^{(h_{\mu i})} \right] = \exp \left[ \frac{i}{8\pi G_N} \int d^4x \, d^6\theta \, \Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta) \right]$$

(4.8)

where $g_i^{(h_{\mu i})} \ (i = 1, 2, \cdots)$ denotes an operator or a source function associated with the expression $g_i^{(h_{\mu i})} = T^\mu g_{\mu i}^{(h_{\mu i})}$.

Now, from the general formula for graviton amplitudes (A.9) and (A.10) in the appendix, we find that the MHV graviton amplitudes can be expressed as follows.

$$\mathcal{M}_{\text{MHV}}^{(s_- t_-)}(u, \bar{u}) = i(8\pi G_N)^{\frac{n}{2} - 1}(-1)^{n+1}(2\pi)^4 \delta^{(4)}(\sum p_i) \tilde{M}_{\text{MHV}}^{(s_- t_-)}(u, \bar{u})$$

(4.9)

$$\tilde{M}_{\text{MHV}}^{(s_- t_-)}(u, \bar{u}) = \sum_{\sigma \in S_{r-1}} \sum_{\tau \in S_{n-r-2}} \prod_{i=2}^r T_{\sigma i} \prod_{i=r+1}^{n-1} T_{\tau i} \tilde{C}_{\text{MHV}}^{(s_- t_-)}(12 \cdots n)$$

$$\times \tilde{C}_{\text{MHV}}^{(s_- t_-)}(\sigma_2 \sigma_3 \cdots \sigma_{r-1} n - 1 \tau_{r+1} \tau_{r+2} \cdots \tau_{n-2} n) + \mathcal{P}(23 \cdots n - 2)$$

(4.10)

$$\tilde{C}_{\text{MHV}}^{(s_- t_-)}(12 \cdots n) = \frac{(u_s u_t)^4}{(u_1 u_2)(u_2 u_3) \cdots (u_n u_1)}$$

(4.11)

where we label the two negative-helicity gravitons by $(s_- t_-)$, with the rest of gravitons having helicity $++ (= +2)$. From (4.8) and (4.10), we find that the MHV graviton amplitudes $\tilde{M}_{\text{MHV}}^{(s_- t_-)}(u, \bar{u})$ are indeed generated by $F_{\text{MHV}} \left[ g_i^{(h_{\mu i})} \right]$ as

$$\frac{\delta}{\delta g_{i_{\mu i}}^{(++)}(x)} \otimes \cdots \frac{\delta}{\delta g_{i_{\mu i}}^{(-\sigma)}(x)} \otimes \cdots$$

$$\cdots \otimes \frac{\delta}{\delta g_{i_{\mu i}}^{(++)}(x)} \otimes \cdots \frac{\delta}{\delta g_{i_{\mu i}}^{(+\sigma)}(x)} \mathcal{F}_{\text{MHV}} \left[ g_i^{(h_{\mu i})} \right] \bigg|_{g_i^{(h_{\mu i})}(x) = 0}$$

$$= i(8\pi G_N)^{\frac{n}{2} - 1} \tilde{M}_{\text{MHV}}^{(s_- t_-)}(u, \bar{u})$$

(4.12)

where we use the result (2.34) and the Grassman integral

$$\int d^6\theta \, \prod_{\alpha=1}^8 \xi^\alpha \prod_{\beta=1}^8 \xi^\beta \bigg|_{\xi^\alpha = \theta^\alpha u^A_i} = (u_s u_t)^8$$

(4.13)
In obtaining (4.12), we also use the normalization relation
\[ \oint_\gamma d(u_1 u_2) \wedge d(u_2 u_3) \wedge \cdots \wedge d(u_n u_1) = 2^{n+1} \]  
(4.14)
for the spinor momenta. Under a permutation of the numbering indices, a sign factor arises in the above expression. We disregard this sign factor since physical quantities are given by the square of the amplitudes. As discussed below (3.4), we can incorporate the information on permutations into the closed path \( \gamma \) on \( C = \mathbb{C}^n / S_n \). Thus, we may make this sign factor absorbed into the above normalization. Notice that only the MHV-type helicity configurations are survived in the above calculation (4.12). The rest of the helicity configurations are prohibited due to the Grassmann integral (4.13).

The MHV amplitude (4.9) is expressed in a momentum-space representation. In an \( x \)-space representation, this can be written as
\[ \mathcal{M}_{MHV}^{(s_-, t_-, \ldots)}(x) = \prod_{i=1}^{n} \int d\mu_i \mathcal{M}^{(s_-, t_-, \ldots)}_{MHV}(u, \bar{u}) . \]  
(4.15)
In terms of the S-matrix functional (4.8), this MHV amplitude can also be generated as
\[ (-1)^{n+1} \mathcal{M}^{(s_-, t_-, \ldots)}_{MHV}(x) \]
where, again, the sign factor \((-1)^{n+1}\) may be irrelevant to physical observables. Notice that in the above calculation the momentum-conservation delta function in (4.9) naturally arises.

The homogeneous-sum case

We now consider an alternative expression for the MHV S-matrix functional by use of the expression (3.7) where the homogeneous sum appears. As summarized in Table 1, the physical quantities of interest in this case are the frame fields rather than the gravitons. Consequently, the gravitational theory is now given by a “square” of \( N = 4 \) theory for the frame fields. We then have two types of Grassmann variables:
\[ \xi^\alpha_{\mu_i} = \theta^A_{\alpha} u^A_{\mu_i} \quad (\alpha = 1, 2, 3, 4; A = 1, 2), \]  
(4.17)
\[ \eta^\beta_{\lambda_i} = \theta^B_{\beta} u^B_{\lambda_i} \quad (\beta = 1, 2, 3, 4; B = 1, 2). \]  
(4.18)
One may find that the use of indices \( \mu_i \) and \( \lambda_i \) is redundant. As emphasized in the previous section, however, the squared structure appears before taking the homogeneous sum of \( \{ \sigma, \tau \} \). Thus labeling the numbering indices by \( \mu_i \) and \( \lambda_i \) is appropriate for our purpose. Of course, eventually the homogeneous sum is taken so that the final form is independent of the choice of \( (\mu, \lambda) \) or that of \( (\sigma, \tau) \).

Using the above Grassmann variables, we can define supersymmetric operators for the frame fields:
\[ e^{(h_{\mu_i})}_{\mu_i}(x, \theta) = \int d\mu(p_{\mu_i}) e^{(h_{\mu_i})}_{\mu_i}(\xi_{\mu_i}) e^{ix_{\mu_i}} \bigg|_{\xi_{\mu_i} = \theta^A_{\alpha} u^A_{\mu_i}} \]  
(4.19)
\[ e^{(\hat{h}_{\lambda_i})}(x', \theta') = \int d\mu(p_{\mu_i}) e^{(\hat{h}_{\lambda_i})}(\eta_{\lambda_i}) e^{ix'p_{\mu_i}} \bigg|_{\eta_{\lambda_i}^a = \theta^a_{\mu_i} u^B_{\lambda_i}} \]  

(4.20)

where \( d\mu(p_{\mu_i}) \) and \( d\mu(p_{\lambda_i}) \) denote the Nair measures for \( p^{AA}_{\mu_i} = u^{A}_{\mu_i} \hat{u}^{A}_{\mu_i} \) and \( p^{BB}_{\lambda_i} = u^{B}_{\lambda_i} \hat{u}^{B}_{\lambda_i} \), respectively. As shown in (2.22), the spacetime coordinates \( x_{AA} \), \( x_{BB} \) are defined in terms of twistor-space variables:

\[ v_{\mu_i A} = x_{AA} u^{A}_{\mu_i}, \]  

(4.21)

\[ v'_{\lambda_i B} = x_{BB} u^{B}_{\lambda_i} \]  

(4.22)

where \( x_{AA} \) and \( x'_{BB} \) are two distinct coordinates but \( u^{A}_{\mu_i} \) and \( u^{B}_{\lambda_i} \) are those spinor momenta that are defined on the same physical configuration space \( \mathcal{C} = \mathbb{C}^n / \mathcal{S}_a \). In the holonomy formalism, physical variables are given by the spinor momenta. Thus the emergence of two distinct spacetimes \( x, x' \) is possible but it does seem unnatural in modeling a physical theory. In the following, we shall consider a gravitational theory such that \( x' \)-dependence becomes immaterial. Our strategy is to define a supersymmetric gravitational holonomy operator \( \Theta^{(H)}(u, \bar{u}; x, \theta, x', \theta') \), which is analogous to the above \( \Theta^{(H)}(u, \bar{u}; x, \theta) \), and obtain an MHV S-matrix functional from it by integrating out the \( x' \)-dependence. As we shall see later, it turns out that this construction is also suitable for the generation of the non-MHV amplitudes.

The frame-field operators (4.19) and (4.20) are analogs of the supersymmetric graviton operator \( g^{(h_{\mu_i})}(x, \theta) \) defined in (4.4). In terms of (4.19) and (4.20), the new graviton operator \( g^{(\hat{h}_{\mu_i}, \lambda_i)}(x, \theta, x', \theta') \) can be expressed as

\[ g^{(\hat{h}_{\mu_i}, \lambda_i)}(x, \theta, x', \theta') = \int d\mu(p_{\mu_i}) d\mu(p_{\lambda_i}) g^{(\hat{h}_{\mu_i}, \lambda_i)}(\xi_{\mu_i}, \eta_{\lambda_i}) e^{ixp_{\mu_i}} e^{ix'p_{\lambda_i}} \bigg|_{\xi_{\mu_i}^a = \theta^a_{\mu_i} u^{A}_{\mu_i}, \eta_{\lambda_i}^a = \theta^a_{\lambda_i} u^{B}_{\lambda_i}} \]  

(4.23)

\[ g^{(\hat{h}_{\mu_i}, \lambda_i)}(\xi_{\mu_i}, \eta_{\lambda_i}) = T_{\lambda_i} g^{(\hat{h}_{\mu_i}, \lambda_i)}(\xi_{\mu_i}, \eta_{\lambda_i}) = T_{\lambda_i} e_{\mu_i}^{(h_{\mu_i})a}(\xi_{\mu_i}) e^{(h_{\lambda_i})a}(\eta_{\lambda_i}) \]  

(4.24)

where \( g^{(\hat{h}_{\mu_i}, \lambda_i)}(\xi_{\mu_i}, \eta_{\lambda_i}) \) in the second equation is an analog of \( g^{(h_{\mu_i})}(\xi_{\mu_i}) \) in (4.5) with two types of Grassmann variables (4.17) and (4.18). The supersymmetric frame fields \( e^{(h_{\mu_i})a}(\xi_{\mu_i}) \) are now define by

\[ e_{\mu_i}^{(+)}(\xi_{\mu_i}) = e_{\mu_i}^{(+)} \]  

\[ e_{\mu_i}^{(\mp)}(\xi_{\mu_i}) = e_{\mu_i}^{(\mp)} \]  

\[ e_{\mu_i}^{(0)}(\xi_{\mu_i}) = \frac{1}{2} e_{\mu_i}^{\epsilon_{\mu_i}^{(0)}} \]  

\[ e_{\mu_i}^{(-)}(\xi_{\mu_i}) = e_{\mu_i}^{(-)} \]  

\[ e_{\mu_i}^{(1)}(\xi_{\mu_i}) = e_{\mu_i}^{(1)} \]  

\[ e_{\mu_i}^{(2)}(\xi_{\mu_i}) = e_{\mu_i}^{(2)} \]  

\[ e_{\mu_i}^{(3)}(\xi_{\mu_i}) = e_{\mu_i}^{(3)} \]  

\[ e_{\mu_i}^{(4)}(\xi_{\mu_i}) = e_{\mu_i}^{(4)} \]  

(4.25)

where each of \( \alpha_1, \alpha_2, \alpha_3, \cdots \) takes a value of 1, 2, 3 or 4. Similarly, the other set of the supersymmetric frame fields are defined by

\[ e_{\lambda_i}^{(+)}(\eta_{\lambda_i}) = e_{\lambda_i}^{(+)} \]
\[(+\frac{1}{2})^a (\eta_{\lambda_i}) = \eta^{\beta_1}_{\lambda_i} \eta^{(\frac{1}{2})^a}_{\beta_i} \]
\[e_{\lambda_i}^{(0)a}(\lambda_i) = \frac{1}{2} \eta^{\beta_1}_{\lambda_i} \eta^{\beta_2}_{\lambda_i} e_{\lambda_i}^{(0)a} \]
\[e_{\lambda_i}^{-\frac{1}{2}a}(\lambda_i) = \frac{1}{3!} \eta^{\beta_1}_{\lambda_i} \eta^{\beta_2}_{\lambda_i} \eta^{\beta_3}_{\lambda_i} e_{\lambda_i}^{-\frac{1}{2}a} \]
\[e_{\lambda_i}^{(-)a}(\lambda_i) = \eta^{\beta_1}_{\lambda_i} \eta^{\beta_2}_{\lambda_i} \eta^{\beta_3}_{\lambda_i} e_{\lambda_i}^{(-)a} \]

(4.26)

where each of $\beta_1, \beta_2, \beta_3, \cdots$ takes a value of 1, 2, 3 or 4.

A supersymmetric gravitational holonomy operator $\Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta')$ is then defined by substitution of $g^{(\tilde{h}_{\mu_i})}(x, \theta, x', \theta')$ into $g^{(h_{\mu_i})}$ in (2.2).

The operator $\Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta')$ can also be written as a supersymmetrization of the expression in (3.7), i.e.,

\[
\Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta') = \sum_{\{\sigma, \tau\}} \Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta')
\]
\[
= \sum_{\{\sigma, \tau\}} \left\langle \Theta^{(E)}_{R,\gamma}(u; x, \theta) \cdot \Theta^{(E)}_{R,\gamma}(u; x', \theta') \right\rangle_{\gamma=\gamma_{\sigma|\tau}}
\]

(4.27)

where $\Theta^{(E)}_{R,\gamma}(u; x, \theta)$ and $\Theta^{(E)}_{R,\gamma}(u; x', \theta')$ are defined as follows.

1. $\Theta^{(E)}_{R,\gamma}(u; x, \theta)$ is obtained by substitution of $e^{(h_{\mu_i})}(x, \theta)$ into $e^{(h_{\mu_i})}$ in the definition of $\Theta^{(E)}_{R,\gamma}(u; x, \theta)$ given by (3.8). An explicit expansion form of (3.8) can be obtained from (3.4) and (2.30). Notice that the frame-field operator $e^{(h_{\mu_i})}$ enters in the comprehensive frame field $E$ as defined in (2.26) and (2.27).

2. $\Theta^{(E)}_{R,\gamma}(u; x', \theta')$ is obtained by substitution of $e^{(h_{\lambda_i})}(x', \theta')$ into $e^{(h_{\mu_i})}$ in the definition of $\Theta^{(E)}_{R,\gamma}(u; x', \theta')$ given by (3.9).

Using the squared holonomy operator $\Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta')$ in (4.27), we can define an S-matrix functional for a subset of the MHV graviton amplitudes as

\[
\mathcal{F}_{MHV} \left[ e^{(h_{\mu_i})} \cdot e^{(h_{\lambda_i})} \right] = \exp \left[ \frac{i}{8\pi G_N} \int d^4x \, d^8\theta \, d^8\theta' \, \Theta^{(H)}_{R,\gamma}(u, \tilde{u}; x, \theta, x', \theta') \right]
\]

(4.28)

where $e^{(h_{\mu_i})} \cdot e^{(h_{\lambda_i})}$ denotes a source function that is associated with the composite operator $g^{(h_{\mu_i},\lambda_i)} = e^{(h_{\mu_i})} \cdot e^{(h_{\lambda_i})}$. For simplicity, we here express a product of the frame fields on the tangent space by a dot product rather than using the tangent-space index $a \ (= 0, 1, 2, 3)$.

As reviewed in the appendix, the graviton amplitudes of arbitrary helicity configuration can generally be expressed in the form of (A.11). Applying this expression to the MHV graviton amplitudes, we can easily check that the amplitudes $\tilde{M}_{MHV}^{(s, \tau, t)}(u, \tilde{u})$ can be generated...
by \( \mathcal{F}_{MHV} \left[ e_{(h_{\mu_i})} \cdot e_{(h_{\lambda_i})} \right] \) as follows.

\[
\sum_{\{\sigma, \tau\}} \left[ \bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\mu_i})}(x)} \right] \cdot \left[ \bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\lambda_i})}(x')} \right] \mathcal{F}_{MHV} \left[ e_{(h_{\mu_i})} \cdot e_{(h_{\lambda_i})} \right] \bigg|_{e_{(h_{\mu_i})}(x) = e_{(h_{\lambda_i})}(x') = 0} = i(8\pi G_N)^{\frac{d-4}{d-2}} \tilde{M}_{MHV}^{(s-t-)}(u, \bar{u})
\]

where the (direct) product sums of the functional derivatives are defined by

\[
\bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\mu_i})}(x)} \equiv \frac{\delta}{\delta e^{(+)}(x)} \cdots \frac{\delta}{\delta e^{(-)}(x)} \cdots \frac{\delta}{\delta e^{(+)}(x)} \equiv \bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\lambda_i})}(x')} \equiv \frac{\delta}{\delta e^{(+)}(x')} \cdots \frac{\delta}{\delta e^{(-)}(x')} \cdots \frac{\delta}{\delta e^{(+)}(x')}.
\]

The expression (4.29) is a homogeneous-sum version of the expression (4.12). Notice that the Grassmann integrals over \( \theta \) and \( \theta' \) pick up only the MHV-type helicity configuration since the integrals vanish unless we have the following factors:

\[
\int d^\theta \xi^{\mu_1} \xi^{\mu_2} \xi^{\mu_3} \xi^{\mu_4} \bigg|_{\xi^{\mu_i} = \theta^\alpha_i u^\alpha_i} = (u_{\mu_1} u_{\mu_2})^4,
\]

\[
\int d^\theta' \eta^{\lambda_1} \eta^{\lambda_2} \eta^{\lambda_3} \eta^{\lambda_4} \bigg|_{\eta^{\lambda_i} = \theta'^\beta_i u'^\beta_i} = (u_{\lambda_1} u_{\lambda_2})^4.
\]

In the homogeneous-sum case, as shown in (3.1), the gravitational operator is denoted by \( T_{\lambda_i} g_{(h_{\mu_i})}^{(h_{\mu_i})} = g_{\mu_i}^{(h_{\mu_i})} \). Thus the helicity of the \( \mu_i \)-th graviton is labeled by \( h_{\mu_i} \). In a practical calculation of (4.29), we first set \( (\sigma_2, \sigma_3, \ldots, \sigma_r) = (2, 3, \ldots, r) \) and \( (\tau_{r+1}, \tau_{r+2}, \ldots, \tau_{n-2}) = (r+1, r+2, \ldots, n-2) \), or equivalently \( (\mu_1, \mu_2, \ldots, \mu_n) = (1, 2, \ldots, n) \) and \( (\lambda_1, \lambda_2, \ldots, \lambda_n) = (2, 3, \ldots, r, 1, n-1, r+1, r+2, \ldots, n-2, n) \), and then take the double permutation of \( P(\sigma, \tau) \) before carrying out the homogeneous sum where the helicity information is synchronized with the numbering indices. Thus the choices of the functional derivatives (4.30), (4.31) correctly lead to the MHV configurations of the amplitudes \( \tilde{M}_{MHV}^{(s-t-)}(u, \bar{u}) \).

In the \( x \)-space representation, the MHV amplitudes can be generated as

\[
\sum_{\{\sigma, \tau\}} \left[ \bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\mu_i})}} \right] \cdot \left[ \bigotimes_{(s-t-)} \frac{\delta}{\delta e_{(h_{\lambda_i})}} \right] \mathcal{F}_{MHV} \left[ e_{(h_{\mu_i})} \cdot e_{(h_{\lambda_i})} \right] \bigg|_{e_{(h_{\mu_i})} = e_{(h_{\lambda_i})}} = (-1)^{n+1} \mathcal{M}_{MHV}^{(s-t-)}(x).
\]

As in the case of (4.16), the energy-conservation delta function naturally arises from the functional derivatives with respect to \( e_{(h_{\mu_i})} \)'s. Notice that in this representation the momenta of gravitons are equivalent to those of the frame fields; we do not have to make the latter momenta be half of the former as usually prescribed for the momenta of closed and open strings in superstring theory.
Obtaining the expression (4.29) or (4.34) is the main objective of the present paper. It shows that an S-matrix functional for the MHV graviton amplitudes can also be described in terms of the supersymmetric gravitational holonomy operator, $\Theta_{\hat{\gamma}_{\mu}, (\nu)(\alpha)}^{(H)}(u, \bar{u}; x, \theta, x', \theta')$ defined in (4.27), by use of the homogeneous sum. This gives a concrete realization of the rough idea that gravity can be considered as a square of gauge theory at the level of construction of the holonomy operators in twistor space. For the completion of our analysis, we shall consider a generalization to the non-MHV amplitudes in what follows.

The non-MHV amplitudes

Generalization of the above analysis to the non-MHV amplitudes can be carried out straightforwardly by use of the so-called Cachazo-Svrcek-Witten (CSW) rules [25]. The rules are summarized by the expressions (A.4) and (A.5) in the appendix. In the language of functional integrals, these rules can be realized succinctly by use of the following S-matrix functional [23]:

$$\mathcal{F} \left[ g_{(h_{i\mu})} \right] = \widetilde{W}^{(H)} \mathcal{F}_{\text{MHV}} \left[ g_{(h_{i\mu})} \right]$$

$$\widetilde{W}^{(H)} = \exp \left[ \int d^4x d^4y \frac{\delta}{\delta g_{(++)}^{(h_{1\mu})}(x)} \otimes \frac{\delta}{\delta g_{(--)}^{(h_{1\mu})}(y)} \right]$$

(4.35)

(4.36)

where $q$ in (4.36) is a momentum transferred between the vertices at $x$ and $y$. This momentum transfer plays the same role as $q_{ij}$ in (A.5) for the next-to-MHV amplitudes. (The contraction operator (4.36) that realizes the CSW rules is first introduced in [26] for gluon amplitudes.) The general S-matrix functional (4.35) is defined in terms of the MHV S-matrix functional (4.8) for the split-sum case.

Using (4.35), we can generate tree-level graviton amplitudes in general as

$$\delta \frac{\delta}{\delta g_{(h_{1\mu})}^{(h_{1\mu})}}(x_1) \otimes \delta \frac{\delta}{\delta g_{(h_{2\mu})}^{(h_{2\mu})}}(x_2) \otimes \cdots \otimes \frac{\delta}{\delta g_{(h_{n\mu})}^{(h_{n\mu})}}(x_n) \mathcal{F} \left[ g_{(h_{i\mu})} \right] \bigg|_{g_{(h_{i\mu})}^{(h_{1\mu})}(x)=0} = i(8\pi G_N)^{\frac{3}{2}} \tilde{M}^{(h_{1\mu}, h_{2\mu}, \ldots, h_{n\mu})}(u, \bar{u})$$

(4.37)

where the helicity $h_{i\mu}$ $(i = 1, 2, \ldots, n)$ takes a value of $(++, --)$. Other configurations, such as $(--, -+)$, are ruled out due to the Grassmann integral in (4.13).

Notice that the particular assignment for the index $\alpha$ in (4.3) is crucial to extract the helicities of $(++, --)$. Without such an assignment, particles with $(+-, --)$ helicities would emerge. The operators $g_{i\mu}^{(+-)}$ and $g_{i\mu}^{(--)}$ are to represent stable and electrically neutral particles without mass or spin which we may regard as candidates for the origin of dark matter. Although this is nothing but an intuitive speculation, observational evidence of dark matter and dark energy suggests that there might be operators like $g_{i\mu}^{(+-)}$ and $g_{i\mu}^{(--)}$ to be incorporated in a full gravitational theory. In the present formalism, this can be carried out by relaxing the assignment (4.3) and using instead the assignments of (4.17), (4.18); this lead to a gravitational theory as a square of an $\mathcal{N} = 4$ supersymmetric gauge theory for frame fields such that the gravitational theory includes operators involving $g_{i\mu}^{(+-)}$ and $g_{i\mu}^{(--)}$. 

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In the present paper, we further consider a gravitational holonomy operator $\Theta$ in (4.28) rather than $F_{\mu\nu}$ in (4.8). Using the former, in comparison to the forms in (4.35)-(4.37), we can define an alternative expression for the non-MHV S-matrix functional:

$$
F[e^{(h_{\mu_i})} \cdot e^{(h_{\lambda_i})}] = \tilde{W}^{(EE)} F_{\text{MHV}} [e^{(h_{\mu_i})} \cdot e^{(h_{\lambda_i})}] 
$$

$$
\tilde{W}^{(EE)} = \exp \left[ \int d^4x d^4y \frac{\delta}{q^2} \delta e^{(+)}_\mu(x) \otimes \delta e^{(-)}_\nu(y) \right] \cdot \exp \left[ \int d^4x' d^4y' \frac{\delta}{q^2} \delta e^{(+)}_\mu(x') \otimes \delta e^{(-)}_\nu(y') \right]
$$

where $q$ ($q'$) denotes a momentum transfer between the vertices at $x$ ($x'$) and $y$ ($y'$). Notice that, in the homogeneous-sum case, graviton amplitudes are factorized by the MHV vertices for frame fields while, in the split-sum case, they are factorized by the graviton MHV vertices. This explains why we have two $q$'s in (4.39) while there is a single $q$ in (4.36).

Using the new S-matrix functional (4.38), we can also generate the non-MHV graviton amplitudes as

$$
\sum_{(\sigma,\tau)} \left[ \bigotimes \frac{\delta}{\delta e^{(h_{\mu_i})}_\mu(x)} \right] \cdot \left[ \bigotimes \frac{\delta}{\delta e^{(h_{\lambda_i})}_\lambda(x')} \right] \left[ \bigotimes \frac{\delta}{\delta e^{(h_{\mu_i})}_\mu(x)} \right] \left[ \bigotimes \frac{\delta}{\delta e^{(h_{\lambda_i})}_\lambda(x')} \right] e^{(h_{\mu_i})}_\mu(x) = e^{(h_{\lambda_i})}_\lambda(x') = 0
$$

where the sets of functional derivatives are now defined by

$$
\bigotimes \frac{\delta}{\delta e^{(h_{\mu_i})}_\mu(x)} \equiv \frac{\delta}{\delta e^{(h_{\mu_1})}_\mu(x_1)} \otimes \frac{\delta}{\delta e^{(h_{\mu_2})}_\mu(x_2)} \otimes \cdots \otimes \frac{\delta}{\delta e^{(h_{\mu_n})}_\mu(x_n)},
$$

$$
\bigotimes \frac{\delta}{\delta e^{(h_{\lambda_i})}_\lambda(x')} \equiv \frac{\delta}{\delta e^{(h_{\lambda_1})}_\lambda(x'_1)} \otimes \frac{\delta}{\delta e^{(h_{\lambda_2})}_\lambda(x'_2)} \otimes \cdots \otimes \frac{\delta}{\delta e^{(h_{\lambda_n})}_\lambda(x'_n)}.
$$

The expression (4.40) confirms that the non-MHV S-matrix functional can indeed be obtained in terms of the holonomy operator $\Theta^{(H)}_{R_{\mu\lambda} \gamma\sigma|x,y}(u, \bar{u}; x, \theta, x', \theta')$ defined in (4.27). Notice that the helicities $h_{\mu_i}\lambda_i = h_{\mu_i} h_{\lambda_i}$ ($i = 1, 2, \cdots, n$) in (4.40) can take any combinations including $(-, +)$. Since graviton operators are defined by $g^{(h_{\mu_i}\lambda_i)}_{\mu_1\lambda_1} = e^{(h_{\mu_i})}_\mu \cdot e^{(h_{\lambda_i})}_\lambda$, this means that the above formulation suggests the existence of particles labeled by $g^{(+,-)}_{\mu_i\lambda_i}$ and $g^{(-,+)}_{\mu_i\lambda_i}$.

### 5 Conclusion remarks

In the present paper, we further consider a gravitational holonomy operator $\Theta^{(H)}_{R_{\mu\lambda} \gamma\sigma|x}(u, \bar{u})$ that has been developed in [23]. The holonomy operator is defined in twistor space, with $u, \bar{u}$
denoting spinor momenta of gravitons defined in a $\mathbb{CP}^1$-fiber of the twistor space. In section 2, we first review the construction of $\Theta^{(H)}_{R,\gamma}(u, \bar{u})$ and how it can be interpreted as a holonomy operator of gauge fields with a certain combinatoric Chan-Paton factor. The structure of the Chan-Paton factor, explicitly given in (2.11)-(2.14), is the same as the structure of a Chan-Paton factor in graviton amplitudes that has been obtained by Bern et al. in [3]. As shown in [23], this relation is utilized to obtain an S-matrix functional for graviton amplitudes in terms of a supersymmetric version of the gravitational holonomy operator, $\Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta)$, with $\mathcal{N} = 8$ extended supersymmetry. Here $x$ denotes a spacetime coordinate that emerges from the twistor space and $\theta^\alpha (\alpha = 1, 2, \cdots, 8)$ denotes Grassmann variables that compose $\mathcal{N} = 8$ chiral superspace. The construction of such an S-matrix functional is also reviewed in section 4.

We present the main results of this paper in section 3. There we give an alternative expression for a gravitational holonomy operator such that it can be interpreted as a square of an $\mathcal{N} = 4$ holonomy operator for frame fields. The expression is motivated by the previous work [8] and is obtained by use of what we call the homogeneous sum. This sum is taken by certain shuffles over ordered numbering indices. An explicit form of $\Theta^{(H)}_{R,\gamma}(u, \bar{u})$ with such a sum is shown in (3.7). Supersymmetrization of this expression, $\Theta^{(H)}_{R,\gamma}(u, \bar{u}; x, \theta, x', \theta')$, is considered in section 4 and is explicitly shown in (4.27). In section 4, we also show that this squared expression can also be used to define an S-matrix functional for the general non-MHV graviton amplitudes.

The homogeneous sum that appears in the new expression is equivalent to a sum over what is called $(k, l)$-shuffles ($k, l \in \mathbb{N}$) in mathematics. Such a sum appears, for example, in (a) the definition of Laplace expansion formula for determinants in terms of the so-called Plücker coordinates, (b) the definition of a wedge product of a differential $k$-form and a differential $l$-form, and (c) the definition of a product of iterated integrals defined by a set of differential one-forms, say, $\omega_1, \omega_2, \cdots, \omega_{k+l}$. We have seen an explicit definition for the case of (c) in (3.10). In (3.10)-(3.13), we then argue that the gravitational holonomy operator can be interpreted as a square of a same theory which is represented by a frame-field holonomy operator. Regarding the case of (a), it suggests that $\Theta^{(H)}_{R,\gamma}(u, \bar{u})$ may be interpreted as some determinant. The holonomy operator is related to a Wess-Zumino-Witten (WZW) action, or more precisely to the current correlator of a WZW model. The WZW action, on the other hand, is closely related to a chiral Dirac determinant. (For the relation between the WZW action and the gluon amplitudes in this context, see [26].) It is then natural to interpret $\Theta^{(H)}_{R,\gamma}(u, \bar{u})$ as a chiral Dirac determinant suitably defined in twistor space. Details of this relation are currently under study.

Lastly, we would like to discuss that the squared expression we obtain in this paper is theoretically more natural than the previously known expression. In the holonomy formalism of gravity, we need to take essentially two sums. One is a sum over all possible metrics that guarantees general covariance of the theory, and the other is a sum over permutations of the numbering indices, or a braid trace, that guarantees diffeomorphism invariance of the theory. In the squared expression, the former sum is realized by the homogeneous sum and the latter is realized by double-permutation terms, denoted as $\mathcal{P}(\sigma|\tau) + \mathcal{P}(\sigma|\tau)$, in (3.1). The double permutation can also be written as a double braid-trace in (3.5). On the other
hand, in the original expression (2.34), the sum over metrics are realized by the split sum
\[ \sum_{\sigma \in S_r} \sum_{\tau \in S_{m-r-1}} r - 1 \]
and the braid trace is realized by \( \mathcal{P}(23 \cdots m - 2) \). Thus, for either case, we can make physically clear interpretations to the sums that appear in the expressions of \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}) \). These interpretations are summarized in Table 1.

Our preference for the squared expression arises upon supersymmetrization of \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}) \). In the split-sum case, the supersymmetric operator is defined by \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}; x, \theta) \), with Grassmann variables \( \theta^\alpha (\alpha = 1, 2, \cdots 8) \) split into two parts, \( \alpha_1 = 1, 2, 3, 4 \) and \( \alpha_2 = 5, 6, 7, 8 \), as shown in (4.3). Although this setting leads to the correct graviton amplitudes, there are no a priori reasons to choose this particular splitting. In this sense, it is an artificial setting. This problem does not occur in the homogeneous-sum case where the supersymmetric operator is defined by \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}; x, \theta, x', \theta') \), with Grassmann variables, \( \theta^\alpha (\alpha = 1, 2, 3, 4) \) and \( \theta^\beta (\beta = 1, 2, 3, 4) \), as shown in (4.17) and (4.18). In this case, there are no restrictions on the indices \( \alpha \) and \( \beta \). As a consequence, this theory contains particles of helicity configuration \((+-, --)\) in addition to the pure-gravity helicity configuration \((++ , --)\). The extra particles are massless spin-zero particles with no electric charges. These particles are also expected to be stable as the ordinary gravitons. Thus we can naturally interpret these as candidates for the origin of dark matter. Research on this speculative idea will be reported in a future paper.

There is another theoretical reason to prefer the squared theory to the original one. In the holonomy formalism, the physical variables are given by a set of spinor momenta defined on a \( \mathbb{CP}^1 \) fiber of twistor space \( \mathbb{CP}^3 \). An underlining space of interest is thus the twistor space without which we would not construct physical operators in four-dimensional spacetime. In other words, in analogy with the language of a WZW model, we can define and identify a target space of the holonomy operator by the twistor space. In terms of this terminology, the target space of the supersymmetric holonomy operator \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}; x, \theta, x', \theta') \) is given by \( \mathbb{CP}^{3|8} \), while that of \( \Theta_{R,\gamma}^{(H)}(u, \bar{u}; x, \theta, x', \theta') \) is given by \( \mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4} \). Notice that \( \mathbb{CP}^{3|4} \) is a super Calabi-Yau manifold but \( \mathbb{CP}^{3|8} \) is not. This means that one can construct a superstring theory, which of course contains quantum gravity, on \( \mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4} \) but not on \( \mathbb{CP}^{3|8} \). Thus, from this perspective as well, it is natural to favor the squared theory.

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A Graviton amplitudes

In this appendix, we review some expressions of graviton amplitudes in relation to those of gluon counterparts. Most of the following results are obtained in [22, 23]. We here simply give those expressions that are of direct relevance to the present paper.
Gluon amplitudes, MHV amplitudes and the CSW rules

We first consider the gluon amplitudes. In the spinor-momenta formalism, the simplest way of describing the gluon amplitudes is to factorize the amplitudes in terms of the maximally helicity violating (MHV) amplitudes. The MHV amplitudes are the scattering amplitudes of \((n - 2)\) positive-helicity gluons and 2 negative-helicity gluons or the other way around. In a momentum-space representation, the MHV tree amplitudes of gluons are expressed as

\[
A^{(r_{-}s_{-})}(u, \bar{u}) = ig^n (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \hat{A}^{(r_{-}s_{-})}_{\text{MHV}}(u) \tag{A.1}
\]

where the elements \(r\) and \(s\) denote the numbering indices of the negative-helicity gluons, \(g\) represents the Yang-Mills coupling constant, and \(t_c\)'s are the Chan-Paton factors of gluons. \(u_i\) denotes the two-component spinor momentum of the \(i\)-th gluon \((i = 1, 2, \cdots, n)\). In terms of \(u_i^A\) \((A = 1, 2)\) and its complex conjugate \(\bar{u}_i^\dot{A}\) \((\dot{A} = 1, 2)\), the four-dimensional gluon momentum \(p_i^{A\dot{A}}\) is parametrized by

\[
p_i^{A\dot{A}} = u_i^A \bar{u}_i^\dot{A}. \tag{A.3}\]

This parametrization is explicitly shown in (2.17) and (2.18). Of particular interest in the spinor-momenta formalism is that the MHV gluon amplitudes \(\hat{A}^{(r_{-}s_{-})}_{\text{MHV}}(u)\) is purely holomorphic in terms of the spinor momentum \(u_i\). For the MHV graviton amplitudes, however, it no longer holds since the Chan-Paton factors of gravitons in the spinor-momenta formalism are composed of a set of four-dimensional graviton momenta analogous to (A.3). An explicit form of the MHV graviton amplitudes is given in (4.10).

The non-MHV gluon amplitudes, or the general gluon amplitudes, can be expressed in terms of the MHV amplitudes \(\hat{A}^{(r_{-}s_{-})}_{\text{MHV}}(u)\). Prescription for such expressions is called the Cachazo-Svrcek-Witten (CSW) rules \cite{25}. For the next-to-MHV (NMHV) amplitudes, which contain 3 negative-helicity gluons and \((n - 3)\) positive-helicity gluons, the CSW rules can be expressed as

\[
\hat{A}^{(r_{-}s_{-}t_{-})}_{\text{NMHV}}(u) = \sum_{(i,j)} \hat{A}^{(i_{+}r_{-}s_{-}t_{-}j_{+})}_{\text{MHV}}(u) \frac{\delta_{kl}}{q_{ij}^2} \hat{A}^{(l_{-}(j+1)_{+} \cdots t_{-}(i-1)_{+})}_{\text{MHV}}(u) \tag{A.4}
\]

where the sum is taken over all possible choices for \((i, j)\) that satisfy the ordering \(i < r < s < j < t \pmod{n}\). The numbering indices for the negative-helicity gluons are now given by \(r, s\) and \(t\). The momentum transfer \(q_{ij}\) between the two MHV vertices can be expressed by a set of the gluon four-momenta:

\[
q_{ij} = p_i + p_{i+1} + \cdots + p_r + \cdots + p_s + \cdots + p_j. \tag{A.5}\]
The non-MHV amplitudes are then obtained by iterative use of the relation (A.4). Thus, in principle, we can express the general gluon amplitudes as

$$\mathcal{A}^{(l_1 h_2 \cdots h_n)}(u, \bar{u}) = i g^{n-2} (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right) \tilde{\mathcal{A}}^{(l_1 h_2 \cdots h_n)}(u)$$

(A.6)

$$\tilde{\mathcal{A}}^{(l_1 h_2 \cdots h_n)}(u) = \sum_{\sigma \in \mathcal{S}_n} \text{Tr}(t^{e_1} t^{e_2} t^{e_3} \cdots t^{e_n}) \, \hat{C}(1\sigma_2 \sigma_3 \cdots \sigma_n)$$

(A.7)

where $h_i = \pm$ denotes the helicity of the $i$-th gluon and $\hat{C}(1\sigma_2 \sigma_3 \cdots \sigma_n)$ denotes a function of the Lorentz-invariant scalar products $(u_i u_j)$. The simplest form of this function is given in the case of the MHV amplitudes:

$$\hat{C}_{\text{MHV}}^{(r-s-)}(1\sigma_2 \sigma_3 \cdots \sigma_n) = \frac{(u_{r} u_{s})^4}{(u_{1} u_{\sigma_2})(u_{\sigma_2} u_{\sigma_3}) \cdots (u_{\sigma_n} u_{1})}.$$  

(A.8)

By use of the CSW rules, we can then obtain $\hat{C}$'s of any helicity configurations. Notice that $\hat{C}_{\text{MHV}}$'s are holomorphic in terms of the scalar products $(u_i u_j)$ but $\hat{C}$'s are not holomorphic in general due to the factor of $q_{ij}$'s in (A.5).

**Graviton amplitudes: uses of the split sum and the homogeneous sum**

In terms of such $\hat{C}$'s, one can express tree-level graviton amplitudes. According to [3], an explicit form of the graviton amplitudes is given by

$$\tilde{\mathcal{M}}^{(l_1 \mu_1, 2h_2 \cdots h_{n\mu n})}(u, \bar{u}) = i (8\pi G_N)^{3/2} \frac{1}{2} (-1)^{n+1} (2\pi)^4 \delta^{(4)} \left( \sum_{i=1}^{n} p_i \right)$$

$$\times \, \tilde{\mathcal{M}}^{(l_1 \mu_1, 2h_2 \cdots h_{n\mu n})}(u, \bar{u})$$

(A.9)

$$\tilde{\mathcal{M}}^{(l_1 \mu_1, 2h_2 \cdots h_{n\mu n})}(u, \bar{u}) = \sum_{\sigma \in \mathcal{S}_{n-1}} \sum_{r \in \mathcal{S}_{n-r-2}} \prod_{i=2}^{r} T^{\sigma_i} \prod_{i=r+1}^{n-1} T^{\tau_i} \hat{C}(12 \cdots n)$$

$$+ \hat{C}(\sigma_2 \sigma_3 \cdots \sigma_n 1 n - 1 \tau_{r+1} \tau_{r+2} \cdots \tau_{n-2} n) + \mathcal{P}(23 \cdots n - 2)$$

(A.10)

where the indices $\mu_i$ follow the definition (2.3). Also $T^{\sigma_i}$ and $T^{\tau_i}$ are given by (2.12) and (2.13), respectively, with $m$ replaced by $n$.

The expression (A.10) uses the split sum that we discuss in section 3. In terms of the homogeneous sum, this can be rewritten as

$$\tilde{\mathcal{M}}^{(l_1 \mu_1, 2h_2 \cdots h_{n\mu n})}(u, \bar{u})$$

= \sum_{\{\sigma, \tau\} = \{2, 3, \cdots, m-2\}} \hat{C}(\mu_1 \mu_2 \cdots \mu_m) \left( \prod_{i=1}^{m} T^{\lambda_i} \hat{C}(\lambda_1 \lambda_2 \cdots \lambda_m) + \mathcal{P}(\sigma | \tau) + \mathcal{P}(\sigma | \tau) \right)$$

(A.11)

where the indices $\lambda_i$ follow the definition (2.4) and $\mathcal{P}(\sigma | \tau)$ is defined by (3.3). The expression (A.11) is first obtained in [8]. Notice that the factor of $\prod_{i=1}^{m} T^{\lambda_i}$ can be replaced by $\prod_{i=1}^{m} T^{\lambda_i}$ since this product sum is determined only by the permutations $\sigma$ and $\tau$. The homogeneous sum in (A.11) is taken over the all possible combinations for the elements $\{\sigma_2, \sigma_3, \cdots, \sigma_r, \tau_{r+1}, \tau_{r+2}, \cdots, \tau_{m-2}\}$ such that the ordering conditions $\sigma_2 < \sigma_3 < \cdots < \sigma_r$ and $\tau_{r+1} < \tau_{r+2} < \cdots < \tau_{m-2}$ are preserved. In mathematical literature, this sum is sometimes called a sum over the $(r - 1, m - 2 - r)$-shuffles.
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