Load impedance perturbation formulas for class-E power amplifiers

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Abstract: This paper considers how a class-E amplifier behaves when the load deviates from its nominal impedance. The Kirchhoff’s law and Fourier series formulate the circuit into three simultaneous equations. These equations are decomposed into zeroth- and first-order terms to apply the perturbation technique. The zeroth-order solutions meet the zero-voltage and zero-current switching conditions. The first-order formulas express the DC current consumption, RF output amplitude and phase, equivalent output impedance, and periodic turn-on power dissipation in terms of the load resistance and reactance deviations.

Keywords: zero-voltage switching, zero-current switching, turn-on capacitor voltage, load deviation, equivalent output impedance, Poincaré distance

Classification: Transmission Systems and Transmission Equipment for Communications

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1 Introduction

In radio communication, broadcasting, and wireless power transfer systems, an RF power amplifier is indispensable. Sokal presented a switch-mode but sinusoidal-
wave output amplification scheme named class E. This scheme effectively works to nullify the periodic turn-on power loss and to achieve a high conversion efficiency [1, 2]. However, it often suffers from load-impedance deviations because the class-E operation is carried out only when it is precisely loaded with its nominal impedance. To address this problem, Suetsugu et alia performed deliberate considerations and experiments on how a circuit behaves in case the load is offset from the nominal impedance [3, 4, 5]. Because the class-E operation involves periodic discontinuity and reactive resonance, the formulation becomes quite tough, particularly in off-nominal situations. To find elegant explicit expressions on the load deviation effects, this paper applies a perturbation technique to class-E circuit equations.

2 Circuit equations

We consider the class-E power amplifier shown in Fig. 1, where DC power is input at the left-hand port, goes through a choke coil, is converted into an RF power by a transistor, is filtered by an LC resonator, and finally outputs a sinusoidal wave at the right-hand port. The transistor is assumed to function as an ideal on-off switch exclusively controlled by the base-input signal regardless of the instantaneous collector-to-emitter voltage or even its polarity. The switch turns off at \( t = 0 \), turns on at \( t = T/2 \), and repeats this sequence for time period \( T \). The series LC resonator forces the output current to assume a sinusoidal waveform.

![Class-E power amplifier circuit scheme.](image)

The sinusoidal current assumption and Kirchhoff’s current law lead to

\[
i_{RF}(t) = I_{RF} \sin(\omega t + \theta), \quad i(t) = I_{DC} - I_{RF} \sin(\omega t + \theta)
\]

where \( I_{DC}, I_{RF} > 0 \) and \( \omega = 2\pi/T \). Immediately after the switch turns off at \( t = 0 \), the current \( i(t) \) starts to charge the capacitor \( C \), resulting in capacitor voltage

\[
v(t) = \frac{1}{C} \int_0^t i(t) \, dt = \frac{I_{DC}}{C} t + \frac{I_{RF}}{\omega C} \{\cos(\omega t + \theta) - \cos \theta\}
\]

in the OFF state until the switch turns on at \( t = T/2 \). The voltage is kept at zero during the ON state because the capacitor is short-circuited by the transistor.

Because this on-off sequence is periodically repeated, the voltage waveform is a periodic function of time, which can be expanded into a Fourier series

\[
v(t) = V_{DC} + RI_{RF} \sin(\omega t + \theta) + XI_{RF} \cos(\omega t + \theta) + \cdots
\]
where . . . implies higher-order harmonic terms. Coefficient \( X \) denotes the series LC resonator’s reactance: \( X = \omega L_0 - 1/\omega C_0 \). According to the Fourier series theory, the DC term in (3) calculates

\[
V_{DC} = \frac{1}{T} \int_0^{T/2} v(t) \, dt
\]

(4)

The fundamental-harmonic coefficients in (3) calculate

\[
RI_{RF} = \frac{2}{T} \int_0^{T/2} v(t) \sin(\omega t + \theta) \, dt,
X_{RF} = \frac{2}{T} \int_0^{T/2} v(t) \cos(\omega t + \theta) \, dt
\]

(5)

Note that the integrals are all truncated halfway because \( v(t) \) vanishes right after the switch turns on at \( t = T/2 \). Substituting (3) into (4) and (5) yields

\[
\pi^2 I_{DC} - 2(2\sin \theta + \pi \cos \theta)I_{RF} = 4\pi \omega CV_{DC}
\]

(6)

\[
(\pi \cos \theta - 2\sin \theta)I_{DC} - (2\cos^2 \theta + \pi \omega C)I_{RF} = 0
\]

(7)

\[
2(2\cos \theta + \pi \sin \theta)I_{DC} - (\pi + 2\sin 2\theta - 2\pi \omega CX)I_{RF} = 0
\]

(8)

These equations dominate the class-E amplifier’s behavior. They hold true even if the circuit does not satisfy the soft-switching condition. The problem is that they involve manifold trigonometry, and are thus too difficult to be solved at once for three unknowns \( I_{DC} \), \( I_{RF} \), and \( \theta \).

### 3 Decomposition for perturbation

To solve the above three circuit equations in an analytical manner, we employ a perturbation technique. All the functions and variables are decomposed into their zeroth- and first-order terms as

\[
v(t) = v_0(t) + v_1(t), \quad i(t) = i_0(t) + i_1(t)
\]

(9)

\[
I_{DC} = I_{DC0} + I_{DC1}, \quad I_{RF} = I_{RF0} + I_{RF1}, \quad \theta = \theta_0 + \theta_1
\]

(10)

\[
Z = Z_0 + Z_1 = R + jX = (R_0 + R_1) + j(X_0 + X_1)
\]

(11)

where subscript “0” implies a nominal class-E operation, and subscript “1” indicates a possible deviation from the nominal operation. Each deviation may be upward or downward, but it is assumed to be much smaller than the original magnitude. Considering practical system applications, we should be able to keep \( \omega \), \( C \), and \( V_{DC} \) constant.

By applying the above-mentioned decompositions to (1), (2), (6), (7), and (8), we extract their zeroth- and first-order terms as follows:

\[
i_0(t) = I_{DC0} - I_{RF0} \sin(\omega t + \theta_0)
\]

(12)

\[
i_1(t) = I_{DC1} - I_{RF1} \sin(\omega t + \theta_0) - I_{RF0}\theta_1 \cos(\omega t + \theta_0)
\]

(13)

\[
v_0(t) = \frac{I_{DC0}}{C} t + \frac{I_{RF0}}{\omega C} \{\cos(\omega t + \theta_0) - \cos \theta_0\}
\]

(14)

\[
v_1(t) = \frac{I_{DC1}}{C} t + \frac{I_{RF1}}{\omega C} \{\cos(\omega t + \theta_0) - \cos \theta_0\}
- \frac{I_{RF0}}{\omega C} \theta_1 \{\sin(\omega t + \theta_0) - \sin \theta_0\}
\]

(15)
\[
\begin{align*}
\pi^2 I_{DC0} - 2(2 \sin \theta_0 + \pi \cos \theta_0)I_{RF0} &= 4\pi \omega CV_{DC} \quad (16) \\
\pi^2 I_{DC1} - 2(2 \sin \theta_0 + \pi \cos \theta_0)I_{RF1} &= 2\theta_1 (2 \cos \theta_0 - \pi \sin \theta_0)I_{RF0} \quad (17) \\
(\pi \cos \theta_0 - 2 \sin \theta_0)I_{DC0} - (2 \cos^2 \theta_0 + \pi \omega CR_0)I_{RF0} &= 0 \quad (18) \\
(\pi \cos \theta_0 - 2 \sin \theta_0)I_{DC1} - (2 \cos^2 \theta_0 + \pi \omega CR_0)I_{RF1} &= \theta_1 (\pi \sin \theta_0 + 2 \cos \theta_0)I_{DC0} - (2\theta_1 \sin 2\theta_0 - \pi \omega CR_1)I_{RF0} \quad (19) \\
2(2 \cos \theta_0 + \pi \sin \theta_0)I_{DC0} - (\pi + 2 \sin 2\theta_0 - 2\pi \omega CX_0)I_{RF0} &= 0 \quad (20) \\
2(2 \cos \theta_0 + \pi \sin \theta_0)I_{DC1} - (\pi + 2 \sin 2\theta_0 - 2\pi \omega CX_0)I_{RF1} &= 2\theta_1 (2 \sin \theta_0 - \pi \cos \theta_0)I_{DC0} + 2(\theta_1 \cos 2\theta_0 - \pi \omega CX_1)I_{RF0} \quad (21)
\end{align*}
\]

4 Zeroth-order solutions

Imposing the zero-current switching (ZCS) and zero-voltage switching (ZVS) conditions upon (12) and (14) at \( t = T/2 \), we obtain

\[ I_{DC0} + I_{RF0} \sin \theta_0 = 0, \quad \pi I_{DC0} = 2I_{RF0} \cos \theta_0 \quad (22) \]

Feeding these relationships back into (18) and (20) and then eliminating \( \theta \) from the resulting equations yields

\[
(\omega CR_0)^2 + \left( \omega CX_0 + \frac{2}{\pi^2} - \frac{1}{2} \right)^2 = \left( \frac{2}{\pi^2} \right)^2 \quad (23)
\]

\[
(\omega CR_0)^2 + (\omega CX_0 - 1)^2 = \frac{4}{\pi^2} + \frac{1}{4} \quad (24)
\]

Impedance \( Z_0 = R_0 + jX_0 \) that satisfies these equations plots circular arcs marked “ZCS” and “ZVS” as shown in Fig. 2. These arcs are called geodesic lines in hyperbolic geometry. They intersect with each other at

\[
\omega CR_0 = \frac{1}{\pi} \cdot \frac{8}{\pi^2 + 4} = 0.184, \quad \omega CX_0 = \frac{1}{2} \cdot \frac{\pi^2 - 4}{\pi^2 + 4} = 0.212 \quad (25)
\]

This point specifies the nominal load for class-E operation. Although \( R_0 \) and \( X_0 \) are normalized with \( \omega C \), their ratio becomes a dimension-free constant.

Fig. 2. Load impedance loci for class-E operation: \( 1/\omega C = 50 \Omega \).
\[ Q = \frac{X_0}{R_0} = \frac{\pi(\pi^2 - 4)}{16} = 1.15 \] (26)

This is called the series-resonator loaded \( Q \). The constant-\( Q \) contour is plotted and shown as a broken arc in Fig. 2, along which the ZVS-ZCS intersection moves according to \( \omega C \). From (12), (14), (16), and (22), the zeroth-order variables and waveforms are found as

\[ I_{DC0} = \pi \omega CV_{DC}, \quad I_{RF0} = \frac{1}{2} \sqrt{\pi^2 + 4I_{DC0}}, \quad \tan \theta_0 = -\frac{2}{\pi} \] (27)

\[ i_0(t) = \left( 1 + \cos \omega t - \frac{\pi}{2} \sin \omega t \right) I_{DC0} \] (28)

\[ v_0(t) = \pi \left( \omega t + \sin \omega t + \frac{\pi}{2} \cos \omega t - \frac{\pi}{2} \right) V_{DC} \] (29)

5 First-order solutions

Once the above zeroth-order solutions are known, we can regard (17), (19), and (21) as equations only for first-order unknowns \( I_{DC1}, I_{RF1}, \) and \( \theta_1 \). Because the equations are all linear with respect to the three unknowns, we analytically solve them as

\[ \frac{I_{DC1}}{I_{DC0}} = -\frac{\pi^2 - 4}{\pi^2 + 4} \left( \frac{R_1}{R_0} + \frac{\pi^2 X_1}{4 X_0} \right) = -0.423 \frac{R_1}{R_0} - 1.04 \frac{X_1}{X_0} \] (30)

\[ \frac{I_{RF1}}{I_{RF0}} = -\frac{\pi^2}{\pi^2 + 4} \left( \frac{R_1}{R_0} + \frac{\pi^2 - 4 X_1}{8 X_0} \right) = -0.712 \frac{R_1}{R_0} - 0.522 \frac{X_1}{X_0} \] (31)

\[ \theta_1 = -\frac{\pi^2 - 4 X_1}{32 X_0} = -0.576 \frac{X_1}{X_0} \text{[rad]} = -33.0 \frac{X_1}{X_0} \text{[deg]} \] (32)

These results appear simple, but are so informative that we can predict the amplifier’s basic behavior by focusing on its DC input and RF output currents with respect to the load impedance deviation. The above three formulas tell us some significant points.

The DC current consumption (30) increases if the load resistance or reactance decreases. Note that the load-reactance deviation is identical to the resonator-reactance deviation because they are connected in series as shown in Fig. 1. This information is useful for consideration of the DC power supply as well as the heat generation due to the transistor’s on-state resistance. The RF output current magnitude (31) behaves in a manner similar to that in (30). In contrast, the RF output phase (32) is not affected by the resistance deviation, but is merely delayed by the reactance increment. In other words, the amplifier’s equivalent output impedance \( Z_{\text{out}} \) must be real. Therefore, we can simply formulate \( Z_{\text{out}} \) as follows.

The Ohm’s law on the load enables us to decompose the RF output voltage \( V_{RF} \) into

\[ V_{RF0} = R_0 I_{RF0}, \quad V_{RF1} = I_{RF1} R_0 + I_{RF0} R_1 \] (33)

Applying (31) to this decomposition, the output impedance is expressed as

\[ Z_{\text{out}} = \frac{V_{RF1}}{I_{RF1}} = -R_0 - \frac{I_{RF0}}{I_{RF1}} R_1 = \frac{4}{\pi^2} R_0 = 0.405 R_0 \] (34)

where the first \( I_{RF1} \) has a negative sign because the output impedance should be observed backward from the load.
Above $Z_{out}$ is particularly crucial when we consider the load deviation as a standing-wave problem. Imagine, for any reason, part of the transmitted power is coming back as a wave from the load to the amplifier. This returning wave, in turn, detects above $Z_{out}$ at the amplifier’s output port as shown in Fig. 3. Because $Z_{out}$ differs from $R_0$, the wave reflects back to the load again.

By considering this phenomenon from a wave-engineering viewpoint, we deduce that amplifier output reflectance $\Gamma$ (or $S_{22}$), standing-wave ratio $\rho$, and Poincaré distance $D$ between $Z_{out}$ and $R_0$ as

$$
\Gamma = \frac{Z_{out} - R_0}{Z_{out} + R_0} = \frac{4 - \pi^2}{4 + \pi^2},
\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{\pi^2}{4} = 2.47,
D = \ln \rho = 2(\ln \pi - \ln 2) = 0.903
$$

This result is apparently different from the typical behavior of linear RF circuits, from which we expect that $Z_{out}$ matches the nominal load $R_0$, and hence $D$ becomes zero. This is exactly a special feature of the class-E operation.

One final note we can find out from the aforementioned first-order waveform is the shunt capacitor’s residual voltage at the moment of turn-on. Applying (30) and (31) to (15), we reach

$$
v_1(T/2) = \frac{\pi}{2} \left( \pi R_1 - \frac{\pi^2 - 4}{4} X_1 \right) I_{DC0} = (4.93 R_1 - 2.30 X_1) I_{DC0} \tag{36}
$$

This undesired voltage remains due to the combination of $R_1$ and $X_1$, namely the complex load deviation, which would soft-land onto zero volt if the amplifier were loaded with its nominal impedance given by the zeroth-order solution in (25) or at the intersection shown in Fig. 2. Given $I_{DC0}$ in (27), it is so simple for (36) to estimate the turn-on excess power dissipation in the transistor using the well-known formula

$$P_{sw} = \omega C v_1^2(T/2)/4\pi.$$

6 Conclusion

A perturbation approach has successfully derived easy-to-use formulas for class-E power amplifiers. The zeroth-order solution locates the nominal load impedance at the intersection of the ZVS and ZCS geodesic arcs projected onto a Smith chart. The amplifier’s equivalent output impedance is found out to be 40% of the nominal load resistance, which is clearly distinct from that of linear RF circuits used with a matched load. The shunt capacitor’s residual turn-on voltage formula enables us to quickly predict the transistor’s excess heat generation from both resistance and reactance deviations. The formulas presented in this paper provide an intuitive and
lucid view on the class-E operation, which will offer a constructive bridge between power-electronics engineers and radio-wave engineers, through which we can gather momentum toward future works on high-power high-frequency transmission system development.

Acknowledgments

The author thanks Minoru Mizutani for technical assistance. This work is funded by Cross-Ministerial Strategic Innovation Promotion Program (SIP), MLIT CART Program, and Aichi Prefecture Knowledge Hub Priority Research Project.