DEFINING RELATIONS OF MINIMAL DEGREE OF THE TRACE ALGEBRA OF $3 \times 3$ MATRICES

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Abstract. The trace algebra $C_{nd}$ over a field of characteristic $0$ is generated by all traces of products of $d$ generic $n \times n$ matrices, $n, d \geq 2$. Minimal sets of generators of $C_{nd}$ are known for $n = 2$ and $n = 3$ for any $d$ as well as for $n = 4$ and $n = 5$ and $d = 2$. The defining relations between the generators are found for $n = 2$ and any $d$ and for $n = 3, d = 2$ only. Starting with the generating set of $C_{3d}$ given by Abeasis and Pittaluga in 1989, we have shown that the minimal degree of the set of defining relations of $C_{3d}$ is equal to $7$ for any $d \geq 3$. We have determined all relations of minimal degree. For $d = 3$ we have also found the defining relations of degree $8$. The proofs are based on methods of representation theory of the general linear group and easy computer calculations with standard functions of Maple.

Introduction

Let $K$ be any field of characteristic $0$. All vector spaces, tensor products, algebras considered in this paper are over $K$. Let $X_i = \left( x_{pq}^{(i)} \right), p, q = 1, \ldots, n, i = 1, \ldots, d,$ be $d$ generic $n \times n$ matrices. We consider the pure (or commutative) trace algebra $C_{nd}$ generated by all traces of products $\text{tr}(X_{i_1} \cdots X_{i_k})$. The algebra $C_{nd}$ coincides with the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on $d$ matrices of size $n \times n$. General results of invariant theory of classical groups imply that the algebra $C_{nd}$ is finitely generated. Theory of PI-algebras provides upper bounds for the generating sets of the algebras $C_{nd}$. The Nagata-Higman theorem states that the polynomial identity $x^n = 0$ implies the identity $x_1 \cdots x_N = 0$ for some $N = N(n)$. If $N$ is minimal with this property, then $C_{nd}$ is generated by traces of products $\text{tr}(X_{i_1} \cdots X_{i_k})$ of degree $k \leq N$. This estimate is sharp if $d$ is sufficiently large. A description of the defining relations of $C_{nd}$ is given by the Razmyslov-Procesi theory $[17, 16]$ in the language of ideals of the group algebras of symmetric groups. For a background on the algebras of matrix invariants see, e.g. $[12, 10]$ and for computation aspects of the theory see $[9]$.

Explicit minimal sets of generators of $C_{nd}$ and the defining relations between them are found in few cases only. It is well known that, in the Nagata-Higman theorem, $N(2) = 3$, $N(3) = 6$, and $N(4) = 10$, which gives bounds for the degrees of the generators of the algebras $C_{2d}, C_{3d},$ and $C_{4d}$, respectively. Nevertheless, the defining relations of $C_{nd}$ are explicitly given for $n = 2$ and any $d$, see e.g. $[10]$ for

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details, and for \( n = 3, d = 2 \), see the comments below. For \( n = 3, d \geq 3 \) and \( n \geq 4 \) and \( d \geq 2 \), nothing is known about the concrete form of the defining relations with respect to fixed minimal systems of generators.

Teranishi [18] found a system of 11 generators of \( C_{32} \). It follows from his description that, with respect to these generators, \( C_{32} \) can be defined by a single relation of degree 12. The explicit (but very complicated) form of the relation was found by Nakamoto [15], over \( \mathbb{Z} \), with respect to a slightly different system of generators. Abeasis and Pittaluga [1] found a system of generators of \( C_{3d} \), for any \( d \geq 2 \), in terms of representation theory of the symmetric and general linear groups, in the spirit of its usage in theory of PI-algebras. Aslaksen, Drensky and Sadikova [3] gave the defining relation of \( C_{32} \) with respect to the set found in [1]. Their relation is much simpler than that in [15]. For \( C_{42} \), a set of generators was found by Teranishi [18, 19] and a minimal set by Drensky and Sadikova [11], in terms of the approach in [1]. Djoković [7] gave another minimal set of 32 generators of \( C_{42} \) consisting of traces of products only. He found also a minimal set of 173 generators of \( C_{52} \).

As usually in invariant theory, the determination of generators and defining relations is simpler, if one has some additional information about the algebras of invariants. In particular, it is very useful to know the Hilbert (or Poincaré) series of the algebra. Again, the picture is completely clear for \( n = 2 \). The only other cases, when the Hilbert series are explicitly given, are \( n = 3, d = 2 \) (Teranishi [18]) and \( d = 3 \) (Berele and Stembridge [5]), \( n = 4, d = 2 \) (Teranishi [19] (with some typos) and corrected by Berele and Stembridge [5]). Recently Djoković [7] has calculated also the Hilbert series of \( C_{52} \) and \( C_{62} \).

The minimal generating set of \( C_{3d} \) given in [1] consists of

\[
g = g(d) = \frac{1}{240}d(5d^6 + 19d^4 - 5d^3 + 65d^2 + 636)
\]

homogeneous trace polynomials \( u_1, \ldots, u_g \) of degree \( \leq 6 \). In more detail, the number of polynomials \( u_i \) of degree \( k \) is \( g_k \), and

\[
g_1 = d, g_2 = \frac{1}{2}(d+1)d, g_3 = \frac{1}{3}d(d^2 + 2), g_4 = \frac{1}{24}(d+1)d(d-1)(5d-6),
\]

\[
g_5 = \frac{1}{30}d(d-1)(d-2)(3d^2 + 4d + 6), g_6 = \frac{1}{48}(d+2)(d+1)d(d-1)(d^2 - 3d + 4).
\]

Hence \( C_{3d} \) is isomorphic to the factor algebra \( K[y_1, \ldots, y_g]/I \). Defining \( \deg(y_i) = \deg(u_i) \), the ideal \( I \) is homogeneous. For \( d = 3 \), surprisingly, the comparison of the Hilbert series of \( C_{33} \cong K[y_1, \ldots, y_9]/I \) given in [5], with the Hilbert series of \( K[y_1, \ldots, y_9] \), gives that any homogeneous minimal system of generators of the ideal \( I \) contains no elements of degree \( \leq 6 \), three elements of degree 7 and 30 elements of degree 8. The purpose of the present paper is to find the defining relations of minimal degree for \( C_{3d} \) and any \( d \geq 3 \), with respect to the generating set in [1]. It has turned out that the minimal degree of the relations is equal to 7 for all \( d \geq 3 \), and there are a lot of relations of degree 7. (Compare with the single relation of degree 12 in the case \( d = 2 \).) The dimension of the vector space of relations of degree 7 is equal to

\[
r_7 = r_7(d) = \frac{2}{7!}(d+1)d(d-1)(d-2)(41d^3 - 86d^2 + 114d - 360).
\]

For \( d = 3 \) we have computed also the homogeneous relations of degree 8. The defining relations are given in the language of representation theory of \( GL_d \). There
is a simple algorithm which gives the explicit form of all relations of degree 7, and of degree 8 for \( d = 3 \). The proofs involve basic representation theory of \( GL_d \) and develop further ideas of [3, 11] and our recent paper [4] combined with computer calculations with Maple. In the case \( d = 3 \) we have used essentially the Hilbert series of \( C_{33} \) from [3] which has allowed to reduce the number of computations. Our methods are quite general and we believe that they can be successfully used for further investigation of generic trace algebras and other algebras close to them.

1. Preliminaries

In what follows, we fix \( n = 3 \) and \( d \geq 3 \) and denote by \( X_1, \ldots, X_d \) the \( d \) generic \( 3 \times 3 \) matrices. Often, when the value of \( d \) is clear from the context, we shall denote \( C_{3d} \) by \( C \). It is a standard trick to replace the generic matrices with generic traceless matrices. We express \( X_i \) in the form

\[
X_i = \frac{1}{3} \text{tr}(X_i) e + x_i, \quad i = 1, \ldots, d,
\]

where \( e \) is the identity \( 3 \times 3 \) matrix and \( x_i \) is a generic traceless matrix. Then

\[
C_{3d} \cong K[\text{tr}(X_1), \ldots, \text{tr}(X_d)] \otimes C_0,
\]

where the algebra \( C_0 \) is generated by the traces of products \( \text{tr}(x_{i_1} \cdots x_{i_k}) \), \( k \leq 6 \). Hence the problem for the defining relations of \( C \) can be replaced by a similar problem for \( C_0 \).

As in the case of “ordinary” generic matrices, without loss of generality we may replace \( x_1 \) by a generic traceless diagonal matrix. Changing the variables \( x_{pp}^{(i)} \), we may assume that

\[
x_1 = \begin{pmatrix}
 x_{11}^{(1)} & 0 & 0 \\
 0 & x_{22}^{(1)} & 0 \\
 0 & 0 & -(x_{11}^{(1)} + x_{22}^{(1)})
\end{pmatrix}, \quad (1)
\]

\[
x_i = \begin{pmatrix}
 x_{i1}^{(i)} & x_{i2}^{(i)} & x_{i3}^{(i)} \\
 x_{i2}^{(i)} & x_{i2}^{(i)} & x_{i3}^{(i)} \\
 x_{i3}^{(i)} & x_{i3}^{(i)} & -(x_{i1}^{(i)} + x_{i2}^{(i)})
\end{pmatrix}, \quad i = 2, \ldots, d.
\]

Till the end of the paper we fix these \( d \) generic traceless matrices. Let \( C_0^+ = \omega(C_0) \) be the augmentation ideal of \( C_0 \). It consists of all trace polynomials \( f(x_1, \ldots, x_d) \in C_0 \) without constant terms, i.e., satisfying the condition \( f(0, \ldots, 0) = 0 \). Any minimal system of generators of \( C_0 \) lying in \( C_0^{+} \) forms a basis of the vector space \( C_0^{+} \). Conversely, if a system of polynomials \( f_1, \ldots, f_g \) forms a basis of \( C_0^{+} \) modulo \( (C_0^{+})^2 \), and each \( f_i \) is a linear combination of traces of products \( \text{tr}(x_{i_1} \cdots x_{i_k}) \) then it is a minimal generating system of \( C_0 \). The algebra \( C = C_{3d} \) is \( \mathbb{Z} \)-graded assuming that the trace \( \text{tr}(X_{i_1} \cdots X_{i_k}) \) is of degree \( k \), and this grading is inherited by \( C_0 \). Similarly, \( C \) (and also \( C_0 \)) has a more precise \( \mathbb{Z}^d \)-multigrading induced by the condition that \( X_1, \ldots, X_d \) are, respectively, of multi-degree \( (1, 0, \ldots, 0, 0), \ldots, (0, 0, \ldots, 0, 1) \). The considerations below, stated for the \( \mathbb{Z} \)-grading hold also for the \( \mathbb{Z}^d \)-multigrading. The numbers \( g_1, g_2, \ldots, g_6 \) of elements of degree \( 1, 2, \ldots, 6 \), respectively, in any homogeneous minimal system of generators
is an invariant of $C$. Any homogeneous minimal system $\{f_1, \ldots, f_h\}$ of generators of $C_0$ consists of $g_2, \ldots, g_6$ elements of degree $2, \ldots, 6$, and $h = g_2 + \cdots + g_6$. Hence

$$C_0 \cong K[z_1, \ldots, z_h]/J,$$

with isomorphism defined by $z_j + J \to f_j$, $j = 1, \ldots, h$. If $u_j(z_1, \ldots, z_h)$, $j = 1, \ldots, r$, is a system of generators of the ideal $J$, then $u_j(f_1, \ldots, f_h) = 0$, $j = 1, \ldots, r$, is a system of defining relations of $C_0$ with respect to the system of generators $\{f_1, \ldots, f_h\}$. Any homogeneous system of polynomials in $J$ (where, by definition $\deg(z_j) = \deg(f_j)$) which forms a basis of the vector space $J$ modulo the subspace $JK[z_1, \ldots, z_h]^\perp$, is a minimal system of generators of the ideal $J$. We denote by $r_k$ the number of elements of degree $k$ in such a system. Clearly, $r_k$ is the dimension of the homogeneous component of degree $k$ of the vector space $J/JK[z_1, \ldots, z_h]^\perp$.

Now we summarize the necessary background on representation theory of $GL_d$. We refer e.g. to [14] for general facts and to [8] for applications in the spirit of the problems considered here. All $GL_d$-modules which appear in this paper are completely reducible and are direct sums of irreducible polynomial modules. The irreducible polynomial representations of $GL_d$ are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_1 \geq \cdots \geq \lambda_d \geq 0$. We denote by $W(\lambda) = W_\lambda(\lambda)$ the corresponding irreducible $GL_d$-module, assuming that $W_\lambda(\lambda) = 0$ if $\lambda_{d+1} \neq 0$. The group $GL_d$ acts in the natural way on the $d$-dimensional vector space $K \cdot x_1 + \cdots + K \cdot x_d$ and this action is extended diagonally on the free associative algebra $K \langle x_1, \ldots, x_d \rangle$.

The module $W(\lambda) \subset K \langle x_1, \ldots, x_d \rangle$ is generated by a unique, up to a multiplicative constant, homogeneous element $w_\lambda$ of degree $\lambda_j$ with respect to $x_j$, called the highest weight vector of $W(\lambda)$. It is characterized by the following property.

**Lemma 1.1.** Let $1 \leq i < j \leq d$ and let $\Delta_{ij}$ be the derivation of $K \langle x_1, \ldots, x_d \rangle$ defined by $\Delta_{ij}(x_j) = x_i$, $\Delta_{ij}(x_k) = 0$, $k \neq j$. If $w(x_1, \ldots, x_d) \in K \langle x_1, \ldots, x_d \rangle$ is multihomogeneous of degree $\lambda = (\lambda_1, \ldots, \lambda_d)$, then $w(x_1, \ldots, x_d)$ is a highest weight vector for some $W(\lambda)$ if and only if $\Delta_{ij}(w(x_1, \ldots, x_d)) = 0$ for all $i < j$. Equivalently, $w(x_1, \ldots, x_d)$ is a highest weight vector for $W(\lambda)$ if and only if

$$g_{ij}(w(x_1, \ldots, x_d)) = w(x_1, \ldots, x_d), \quad 1 \leq i < j \leq d,$$

where $g_{ij}$ is the linear operator of the $d$-dimensional vector space which sends $x_j$ to $x_i + x_j$ and fixes the other $x_k$.

**Proof.** The lemma is a partial case of a result by De Concini, Eisenbud, and Procesi [6], see also Almkvist, Dicks, and Formanek [2]. In the version which we need, the first part of the lemma was established by Koshlukov [13]. The equivalence follows from the fact that the kernel of any locally nilpotent derivation $\Delta$ coincides with the fixed points of the related exponential automorphism $\exp(\Delta) = 1 + \Delta/1! + \Delta^2/2! + \cdots$, and $g_{ij} = \exp(\Delta_{ij})$. \hfill $\Box$

If $W_i$, $i = 1, \ldots, m$, are $m$ isomorphic copies of the $GL_d$-module $W(\lambda)$ and $w_i \in W_i$ are highest weight vectors, then the highest weight vector of any submodule $W(\lambda)$ of the direct sum $W_1 \oplus \cdots \oplus W_m$ has the form $\xi_1 w_1 + \cdots + \xi_m w_m$ for some $\xi_i \in K$. Any $m$ linearly independent highest weight vectors can serve as a set of generators of the $GL_d$-module $W_1 \oplus \cdots \oplus W_m$. 
It is convenient to work with an explicit copy of $W(\lambda)$ in $K\langle x_1, \ldots, x_d \rangle$ obtained in the following way. Let

$$s_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma)x_{\sigma(1)} \cdots x_{\sigma(k)}$$

be the standard polynomial of degree $k$. (Clearly,

$$s_2(x_1, x_2) = x_1x_2 - x_2x_1 = [x_1, x_2]$$
is the commutator of $x_1$ and $x_2$.) If the lengths of the columns of the diagram of $\lambda$ are, respectively, $k_1, \ldots, k_p$, $p = \lambda_1$, then

$$(4) \quad w_{\lambda} = w_{\lambda}(x_1, \ldots, x_{k_1}) = s_{k_1}(x_1, \ldots, x_{k_1}) \cdots s_{k_p}(x_1, \ldots, x_{k_p})$$
is the highest weight vector of a submodule $W(\lambda) \subset K\langle x_1, \ldots, x_d \rangle$. Sometimes we shall write $w_{\lambda} = w_{\lambda}(x_1, \ldots, x_d)$, even when $k_1 < d$.

Recall that the $\lambda$-tableau

$$T = (a_{ij}), \quad a_{ij} \in \{1, \ldots, d\}, \quad i = 1, \ldots, d, \quad j = 1, \ldots, \lambda_i,$$
is semistandard if its entries do not decrease from left to right in rows and increase from top to bottom in columns. The following lemma gives a basis of the vector subspace $W(\lambda) \subset K\langle x_1, \ldots, x_d \rangle$. It also provides an algorithm to construct this basis.

**Lemma 1.2.** Let $T = (a_{ij})$ be a semistandard $\lambda$-tableau such that its $i$-th row contains $b_{i,j}$ times $i$, $b_{i,j+1}$ times $i + 1$, $\ldots$, $b_{i,d}$ times $d$. Let $w(x_1, \ldots, x_d)$ be the highest weight vector of $W(\lambda) \subset K\langle x_1, \ldots, x_d \rangle$ and let

$$u_T(x_{11}, x_{12}, \ldots, x_{1d}, x_{21}, \ldots, x_{dd})$$
be the multihomogeneous component of degree $b_{iq}$ in $x_{iq}$, $q = i, i + 1, \ldots, d$, of the polynomial

$$w(x_{11} + x_{12} + \cdots + x_{1d}, x_{21} + x_{22} + \cdots + x_{2d}, \ldots, x_{dd}).$$

When $T$ runs on the set of semistandard $\lambda$-tableaux, the polynomials

$$v_T = v_T(x_1, \ldots, x_d) = u_T(x_1, x_2, \ldots, x_d, x_2, \ldots, x_d, \ldots, x_d)$$
form a basis of the vector space $W(\lambda)$.

**Proof.** By standard Vandermonde arguments, the polynomial $u_T(x_{11}, x_{12}, \ldots, x_{dd})$ is a linear combination of some $w(\sum \alpha_{1j}x_{1j}, \ldots, \sum \alpha_{dj}x_{dj})$, $\alpha_{ij} \in K$. Hence $v_T(x_1, \ldots, x_d)$ is a linear combination of $w(\sum \alpha_{1j}x_j, \ldots, \sum \alpha_{dj}x_j)$ and belongs to the $GL_d$-module $W(\lambda)$ generated by $w(x_1, \ldots, x_d)$. Without loss of generality, it is sufficient to consider the case when $W(\lambda)$ is generated by the element $[\lambda]$. The polynomial $w(x_{11} + x_{12} + \cdots + x_{1d}, x_{22} + \cdots + x_{2d}, \ldots, x_{dd})$ is a product of evaluations

$$s_{k_1}(x_{11} + x_{12} + \cdots + x_{1d}, x_{22} + \cdots + x_{2d}, \ldots, x_{kk}, x_{k+1} + \cdots + x_{k,k})$$
of standard polynomials. Hence $u_T(x_{11}, x_{12}, \ldots, x_{dd})$ is a linear combination of monomials starting with $x_{\sigma(1)}q_1 \cdots x_{\sigma(k_1)}q_{k_1}$. We order the variables by $x_1 > \cdots > x_d$ and consider the lexicographic order of $K\langle x_1, \ldots, x_d \rangle$. The first column of the semistandard tableau $T$ contains $a_{11} < \cdots < a_{k1}$ and in each row $a_{ij} \leq a_{ij}$ for all $j = 2, \ldots, \lambda_i$. This easily implies that the leading monomial of $v_T$ starts with $x_{a_{11}}, x_{a_{21}}, \ldots, x_{a_{k1}}$. Hence the first $k_1$ variables in the leading monomial are indexed by the entries of the first column of the tableau. Continuing in the same way, we obtain that the next $k_2$ variables in the leading monomial are indexed with
the entries of the second row, etc. Hence, for fixed $\lambda$, the leading monomial of $v_T$ determines completely the tableau $T$, and the polynomials $v_T$ are linearly independent. Since $W(\lambda)$ has a basis which is in 1-1 correspondence with the semistandard $\lambda$-tableaux, and the number of $v_T$ is equal to the number of semistandard tableaux, we obtain that the polynomials $v_T$ form a basis of $W(\lambda)$. \hfill $\Box$

If $W$ is a $GL_d$-submodule or a factor module of $K(x_1, \ldots, x_d)$, then $W$ inherits the $\mathbb{Z}^d$-grading of $K(x_1, \ldots, x_d)$. Recall that the Hilbert series of $W$ with respect to its $\mathbb{Z}^d$-multigrading is defined as the formal power series

$$H(W, t_1, \ldots, t_d) = \sum_{k_1, \ldots, k_d \geq 0} \dim(W^{(k_1, \ldots, k_d)}) t_1^{k_1} \cdots t_d^{k_d},$$

with coefficients equal to the dimensions of the homogeneous components $W^{(k_1, \ldots, k_d)}$ of degree $(k_1, \ldots, k_d)$. It plays the role of the $GL_d$-character of $W$: If

$$W \cong \sum_{\lambda} m(\lambda)W(\lambda),$$

then

$$H(W; t_1, \ldots, t_d) = \sum_{\lambda} m(\lambda)S(\lambda(t_1, \ldots, t_d),$$

where $S(\lambda(t_1, \ldots, t_d))$ is the Schur function associated with $\lambda$, and the multiplicities $m(\lambda)$ are determined by $H(W, t_1, \ldots, t_d)$. One of the possible ways to introduce Schur functions is via Vandermonde-like determinants. For a partition $\mu = (\mu_1, \ldots, \mu_d)$, define the determinant

$$V(\mu_1, \ldots, \mu_d) = \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_d^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \cdots & t_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{\mu_d} & t_2^{\mu_d} & \cdots & t_d^{\mu_d} \end{vmatrix}.$$

Then the Schur function is

$$S(\lambda(t_1, \ldots, t_d)) = \frac{V(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_{d-1} + 1, \lambda_d)}{V(d-1, d-2, \ldots, 1, 0)}.$$

The dimension of $W(\lambda)$ is given by the formula

$$\dim(W_d(\lambda)) = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

The decomposition of the tensor product $W_d(\lambda) \otimes W_d(\mu)$ of two irreducible $GL_d$-modules $W_d(\lambda)$ and $W_d(\mu)$ is given by the Littlewood-Richardson rule. We shall need it in one case only:

$$W_3(2^2) \otimes W_3(2^2) \cong W_3(4^2) \oplus W_3(4,3,1) \oplus W_3(4,2^2).$$

But even in this case we can check the equality verifying directly the equality of symmetric functions

$$H(W_3(2^2) \otimes W_3(2^2), t_1, t_2, t_3) = S(2^2)(t_1, t_2, t_3)$$

$$= S_{(2^2)}(t_1, t_2, t_3) + S_{(4,3,1)}(t_1, t_2, t_3) + S_{(4,2^2)}(t_1, t_2, t_3).$$

In all other cases it will be sufficient to use the Young rule which is a partial case of the Littlewood-Richardson one:

$$W_d(\lambda_1, \ldots, \lambda_d) \otimes W_d(p) \cong \sum_{i} W_d(\lambda_1 + p_1, \ldots, \lambda_d + p_d),$$
where the sum runs on all nonnegative integers $p_1, \ldots, p_d$ such that $p_1 + \cdots + p_d = p$ and $\lambda_i \geq \lambda_{i+1} + p_{i+1}$, $i = 1, \ldots, d - 1$, and its dual version

$\sum W_d(\lambda, \ldots, \lambda_d) \otimes W_d(1^p) \cong \sum W_d(\lambda_1 + \varepsilon_1, \ldots, \lambda_d + \varepsilon_d),$

where the sum is on all partitions $(\lambda_1 + \varepsilon_1, \ldots, \lambda_d + \varepsilon_d)$ such that $\varepsilon_i = 0, 1, \varepsilon_1 + \cdots + \varepsilon_d = p$.

In the $q$-th symmetric tensor power

$W^\otimes q = W \otimes_s \cdots \otimes_s W$

of the $GL_d$-module $W$, we identify the tensors $w^{(1)} \otimes \cdots \otimes w(q)$ and $w \otimes \cdots \otimes w$, $\sigma \in S_q$. If $W = W_1 \oplus \cdots \oplus W_k$, then

$W^\otimes q = \bigoplus W_i^\otimes q_1 \otimes \cdots \otimes W_k^\otimes q_k,$

$q_1 + \cdots + q_k = q.$

There is no general combinatorial rule for the decomposition of the symmetric tensor powers of $W_d(\lambda)$. We shall need the following partial results due to Thrall [20], see also [21]:

$K[W_d(2)] = \sum_{q \geq 0} W_d(2)^\otimes q = \sum W_d(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_d),$

where the sum is over all partitions $\lambda$.

$W_d(p) \otimes_s W_d(p) = \sum_{0 \leq k \leq p/2} W_d(p + 2k, p - 2k),$

$W_d(1^p) \otimes_s W_d(1^p) = \sum_{0 \leq k \leq p/2} W_d(2p-2k, 1^4k).$

Besides, we shall need the decomposition

$W_3(2^2) \otimes_s W(2^2) \subset W_3(4^2) \oplus W_3(4, 4^2).$

The easiest way to check [23] is to use [20], hence

$W_3(2^2) \otimes_s W(2^2) \subset W_3(4^2) \oplus W_3(4, 3, 1) \oplus W_3(4, 2^2).$

Therefore

$W_3(2^2) \otimes_s W(2^2) = \varepsilon_1 W_3(4^2) \oplus \varepsilon_2 W_3(4, 3, 1) \oplus \varepsilon_3 W_3(4, 2^2),$

$\varepsilon_i = 0, 1.$

The Hilbert series of $W_3(2^2) \otimes_s W_3(2^2)$ contains the summand $t_1^4 t_2^4$ and $S_1(4^2)$ is the only Schur function among $S_1(4^2)$, $S_1(4,3,1)$, $S_1(4,2^2)$ which contains $t_1^4 t_2^4$. This implies that $W_3(4^2)$ participates in the decomposition of $W_3(2^2) \otimes_s W_3(2^2)$ and $\varepsilon_1 = 1$. Finally, we apply dimension arguments:

$\dim(W_3(2^2) \otimes_s W(2^2)) = \dim(W_3(4^2)) + \varepsilon_2 \dim(W_3(4, 3, 1)) + \varepsilon_3 \dim(W_3(4, 2^2)).$

Since

$\dim(W_3(2^2)) = 6, \quad \dim(W_3(2^2) \otimes_s W(2^2)) = \binom{6 + 1}{2} = 21,$

$\dim(W_3(4^2)) = \dim(W_3(4, 3, 1)) = 15, \quad \dim(W_3(4, 2^2)) = 6,$

we obtain the only possibility $\varepsilon_2 = 0, \varepsilon_3 = 1.$

The action of $GL_d$ on $K[x_1, \ldots, x_d]$ is inherited by the algebras $C_{3d}$ and $C_6$. Now we discuss the approach of Abeasis and Pittaluga [1] for the special case $n = 3$. (Pay attention that the partitions in [1] are given in “Francophone” way,
Corollary 1.4. The algebra $C_{3d}$ has a system of generators of degree $\leq 6$. Without loss of generality we may assume that this system consists of traces of products $\text{tr}(X_{i_1} \cdots X_{i_r})$. Let $U_k$ be the subalgebra of $C_{3d}$ generated by all traces $\text{tr}(X_{i_1} \cdots X_{i_l})$ of degree $l \leq k$. Clearly, $U_k$ is also a $GL_d$-submodule of $C_{3d}$. Let $C_{3d}(k+1)$ be the homogeneous component of degree $k+1$ of $C_{3d}$. Then the intersection $U_k \cap C_{3d}(k+1)$ is a $GL_d$-module and has a complement $G_{k+1}$ in $C_{3d}(k+1)$, which is the $GL_d$-module of the “new” generators of degree $k + 1$. We may assume that $G_{k+1}$ is a submodule of the $GL_d$-module spanned by traces of products $\text{tr}(X_{i_1} \cdots X_{i_{k+1}})$ of degree $k + 1$. The $GL_d$-module of the generators of $C_{3d}$ is

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_6.$$ 

**Proposition 1.3.** (Abeasis and Pittaluga [II]) The $GL_d$-module $G$ of the generators of $C_{3d}$ decomposes as

$$G = W(1) \oplus W(2) \oplus W(3) \oplus W(1^3) \oplus W(2^2) \oplus W(2, 1^2)$$

$$\oplus W(3, 1^2) \oplus W(2^2, 1) \oplus W(1^5) \oplus W(3^2) \oplus W(3, 1^3).$$

Each module $W(\lambda) \subset G$ is generated by the “canonical” highest weight vector $\text{tr}(w_\lambda(X_1, \ldots, X_d))$, where $w_\lambda$ is given in [II].

**Corollary 1.4.** The numbers $g_k$ of generators of degree $k \leq 6$ in any homogeneous minimal system of generators of $C_{3d}$ are

$$g_1 = d, g_2 = \frac{1}{2}(d + 1)d, g_3 = \frac{1}{3}d(d^2 + 2), g_4 = \frac{1}{24}(d + 1)d(d - 1)(5d - 6),$$

$$g_5 = \frac{1}{30}(d + 1)(d - 1)(3d^2 + 4d + 6), g_6 = \frac{1}{48}(d + 2)(d + 1)d(d - 1)(d^2 - 3d + 4).$$

The total number of generators is

$$g = g(d) = \frac{1}{240}d(5d^5 + 19d^4 - 5d^3 + 65d^2 + 636)$$

**Proof:** The number $g_k$ is equal to the dimension of the $GL_d$-submodule $G_k$ of the $GL_d$-module $G$ of generators of $C_{3d}$. Applying the formula (5) we obtain that the dimensions of the $GL_d$-modules

$$W(1), W(2), W(3), W(1^3), W(2^2), W(2, 1^2),$$

$$W(3, 1^2), W(2^2, 1), W(1^5), W(3^2), W(3, 1^3).$$

are, respectively,

$$d, \binom{d + 1}{2}, \binom{d + 2}{3}, \binom{d}{3}, \frac{d}{2} \binom{d + 1}{3}, 3 \binom{d + 1}{4},$$

$$6 \binom{d + 2}{5}, d \binom{d + 1}{4}, \binom{d}{5}, 3 \binom{d + 2}{4}, \binom{d + 1}{2}, 10 \binom{d + 2}{6}.$$ 

This easily implies the results, because

$$\dim(G_3) = \dim(W(3)) + \dim(W(1^3)), \dim(G_4) = \dim(W(2^2)) + \dim(W(2, 1^2)),$$

$$\dim(G_5) = \dim(W(3, 1^2)) + \dim(W(2^2, 1)) + \dim(W(1^5)),$$

$$\dim(G_6) = \dim(W(3^2)) + \dim(W(3, 1^3)),$$

and $g = g_1 + g_2 + \cdots + g_6$. 

□
In the sequel we shall need the Hilbert series of $C_{33}$ calculated by Berele and Stembridge \[5\]:

$$H(C_{33}, t_1, t_2, t_3) = \frac{p(t_1, t_2, t_3)}{q(t_1, t_2, t_3)},$$

where

$$p = 1 - e_2 + e_3 + e_1 e_4 + e_2^2 + e_1^2 e_3 - e_2 e_3 - 2 e_1 e_2 e_3 + e_3^2 + e_2^2 e_3$$

$$- e_1^2 e_2 e_3 + 2 e_1 e_3^2 + e_1^3 e_3 + e_2^3 e_2 - e_1^2 e_2 e_2^2 - e_1 e_3^3 - 2 e_1 e_3 e_2^2$$

$$+ 2 e_2 e_3^3 - e_2^3 e_3 + e_1^3 e_3 + 2 e_1^2 e_2 e_3 - 2 e_1 e_3^4 - e_1^2 e_3^2 + e_1 e_2^2 e_3$$

$$+ e_2 e_3^4 - e_2^3 e_3 - 2 e_2^2 e_3^2 + e_1 e_2 e_3^4 + 2 e_1 e_2 e_3^5 - e_3^6 - e_2^2 e_3^5$$

$$+ e_1 e_3^6 - e_2 e_3^6 - e_1^3 e_3 - e_2^3 e_3 - e_3^6,$$

$$q = \left( \prod_{i=1}^{3} (1 - t_i)(1 - t_i^2)(1 - t_i^3) \right) \left( \prod_{1 \leq i < j \leq 3} (1 - t_i t_j)^2 (1 - t_i^2 t_j)(1 - t_i^3 t_j) \right) \left( 1 - t_1 t_2 t_3 \right),$$

and

$$e_1 = t_1 + t_2 + t_3, \quad e_2 = t_1 t_2 + t_1 t_3 + t_2 t_3, \quad e_3 = t_1 t_2 t_3$$

are the elementary symmetric polynomials in three variables. Since the Hilbert series of the tensor product is equal to the product of the Hilbert series of the factors, and

$$H(K[\text{tr}(X_1), \ldots, \text{tr}(X_d)], t_1, \ldots, t_d) = \frac{1}{(1 - t_1) \cdots (1 - t_d)},$$

(1) implies that

$$H(C_{33}, t_1, t_2) = \frac{H(C_0, t_1, t_2, t_3)}{(1 - t_1)(1 - t_2)(1 - t_3)}.$$ 

In this way, for $d = 3$,

$$H(C_0, t_1, t_2, t_3) = (1 - t_1)(1 - t_2)(1 - t_3)H(C_{33}, t_1, t_2).$$

2. The symmetric algebra of the generators

We consider the symmetric algebra

$$S = K[G_2 \oplus \cdots \oplus G_6]$$

of the $GL_d$-module of the generators of the algebra $C_0$. Clearly, the grading and the $GL_d$-module structure of $G_2 \oplus \cdots \oplus G_6$ induce a grading and the structure of a $GL_d$-module also on $S$. The defining relations of the algebra $C_0$ are in the square of the augmentation ideal $\omega(S)$ of $S$. Since we are interested in the defining relations of degree 7 for $C_0$ for any $d \geq 3$ and of degree 8 for $d = 3$, we shall decompose the homogeneous components of degree 7, respectively, 8 of the ideal $\omega^2(S)$ into a sum of irreducible $GL_d$-modules, respectively, $GL_3$-modules. Then we shall find explicit generators of those irreducible components which may give rise to relations.

**Lemma 2.1.** The following $GL_d$-module isomorphisms hold:

(15) \[ W(2) \otimes_s W(2) \cong W(4) \oplus W(2^2), \]

(16) \[ W(3) \otimes W(2) \cong W(5) \oplus W(4, 1) \oplus W(3, 2), \]

(17) \[ W(1^3) \otimes W(2) \cong W(3, 1^2) \oplus W(2, 1^3), \]

(18) \[ W(2^2) \otimes W(2) \cong W(4, 2) \oplus W(3, 2, 1) \oplus W(2^3), \]
The homogeneous components (29) and (30) are obtained from (7); (28) and (30) follow from (8). Finally, (31) and (23) follow from (12).

Equations (20), (22) are partial cases of (10), (11), respectively; (15) and (20) follow from (10), (11), respectively.

Proof. Equations (20), (22) are partial cases of (10), (11), respectively; (15) and (20) follow from (10), (11), respectively. Equations (10), (11), (15), (19), (21), (24), (26), (27), and (29) are obtained from (7); (28) and (30) follow from (8). Finally, (31) and (23) are calculated applying, respectively, (17) and (5) to (17).

Proposition 2.2. The homogeneous components \( (\omega^2(S))^{(k)} \) of degree \( k \leq 7 \) of the square \( \omega^2(S) \) of the augmentation ideal of the symmetric algebra of \( G_2 \oplus \cdots \oplus G_6 \) decomposes as

\[
(\omega^2(S))^{(4)} = W(4) \oplus W(2^2),
\]

\[
(\omega^2(S))^{(5)} = W(5) \oplus W(4, 1) \oplus W(3, 2) \oplus W(3, 1^2) \oplus W(2, 1^3),
\]

\[
(\omega^2(S))^{(6)} = 2W(6) \oplus 3W(4, 2) \oplus 2W(4, 1^2)
\]

\[ \oplus 2W(3, 2, 1) \oplus 2W(3, 1^3) \oplus 3W(2^2) \oplus W(2^2, 1^2) \oplus W(2, 1^4), \]

\[
(\omega^2(S))^{(7)} = W(7) \oplus W(6, 1) \oplus 3W(5, 2) \oplus 3W(5, 1^2)
\]

\[ \oplus W(4, 3) \oplus 5W(4, 2, 1) \oplus 3W(4, 1^3) \oplus 3W(3^2, 1) \oplus 4W(3, 2^2)
\]

\[ \oplus 6W(3, 2, 1^2) \oplus 2W(3, 1^4) \oplus 2W(2^3, 1) \oplus 3W(2^3, 1^3) \oplus 2W(2, 1^5). \]
Proof. By (9), the homogeneous component of degree \( k \) of \( \omega^2(S) \) has the form
\[
(\omega^2(S))^{(k)} = \bigoplus G_2^{\otimes q_2} \otimes \cdots \otimes G_6^{\otimes q_6}, \quad 2q_2 + \cdots + 6q_6 = k, q_2 + \cdots + q_6 \geq 2.
\]
The decomposition of \( G_2, \ldots, G_6 \) is given in Proposition 1.3. Again, the equality (9) implies that \( (\omega^2(S))^{(k)} = 0 \) for \( k < 4 \). For \( k = 4 \) we use (15) and obtain
\[
(\omega^2(S))^{(4)} = G_2 \otimes_s G_2 = W(2) \otimes_s W(2) = W(4) \oplus W(2^2).
\]
For \( k = 5 \), we derive
\[
(\omega^2(S))^{(5)} = G_3 \otimes G_2 = (W(3) \oplus W(1^3)) \otimes W(2)
\]
and the decomposition follows from (16) and (17). For \( k = 6 \) we have
\[
(\omega^2(S))^{(6)} = G_4 \otimes G_2 \oplus G_3 \otimes G_2 \oplus G_2 \otimes G_2
\]
and we use the decompositions given in (18) – (32). The decomposition of \( (\omega^2(S))^{(7)} \) is obtained in a similar way and makes use of (24) – (32). \( \square \)

**Proposition 2.3.** The following elements of \( S = K[G_2 \oplus \cdots \oplus G_6] \) are highest weight vectors:

- For \( \lambda = (2, 1^3) \):
  \[
  w = \sum_{\sigma \in S_4} \text{sign}(\sigma) \text{tr}(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \text{tr}(x_{\sigma(4)}, x_1);
  \]

- For \( \lambda = (3, 1^3) \):
  \[
  w_1 = \sum_{\sigma \in S_4} \text{sign}(\sigma) \text{tr}(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) x_1 + s_3(x_1, x_{\sigma(1)}, x_{\sigma(2)}),
  w_2 = \sum_{\sigma \in S_4} \text{sign}(\sigma) \text{tr}(x_{\sigma(4)}^2) \text{tr}(s_3(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})),
  \]

- For \( \lambda = (2^2, 1^2) \):
  \[
  w = \sum_{\sigma \in S_4, \tau \in S_2} \text{sign}(\sigma \tau) \text{tr}(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) x_{\tau(1)} + s_3(x_{\tau(1)}, x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)})) \text{tr}(x_{\sigma(4)} x_{\tau(2)}),
  \]

- For \( \lambda = (2, 1^4) \):
  \[
  w = \sum_{\sigma \in S_4} \text{sign}(\sigma) \text{tr}(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \text{tr}(s_3(x_{\sigma(4)}, x_{\sigma(5)}), x_1)).
  \]

For \( \lambda = (2, 1^3), (2^2, 1^2), (2, 1^4), \) every highest weight vector \( w \in W(\lambda) \subset \omega^2(S) \) is equal, up to a multiplicative constant, to the corresponding \( w \). For \( \lambda = (3, 1^3) \) the highest weight vectors are linear combinations of \( w_1 \) and \( w_2 \).

**Proof.** By Proposition 1.3, the submodules \( W(2), W(3), W(1^3), W(2^2) \) of \( G_2 \oplus \cdots \oplus G_6 \) are generated, respectively, by
\[
  v_1 = \text{tr}(x_1^2), v_2 = \text{tr}(x_3^2), v_3 = \text{tr}(s_3(x_1, x_2, x_3)), v_4 = \text{tr}(s_3(x_1, x_2, x_3) x_1).
\]
The trace polynomials \( \text{tr}(x_1 x_2) \) and \( \text{tr}(s_3(x_1, x_2, x_3) x_4) \) are linearizations of \( v_1 \) and \( v_4 \), respectively. Hence the trace polynomials \( w, w_1, w_2 \) defined in the proposition belong to the ideal \( \omega^2(S) \). They have the necessary skew-symmetries, as the “canonical” highest weight vectors \( w_\lambda \) from (11). Hence all \( w \) and \( w_1, w_2 \) are highest weight vectors. Clearly, they are nonzero in \( S \) and their number coincides with the multiplicities of \( W(\lambda) \) in the decomposition of \( \omega^2(S) \).
Hence, every highest weight vector $w \in W(\lambda) \subset \omega^2(S)$ can be expressed as their linear combination. \hfill \Box

**Proposition 2.4.** The following elements of $S = K[G_2 \oplus \cdots \oplus G_6]$ are highest weight vectors:

For $\lambda = (4, 1^3)$:

$$w_1 = (\text{tr}(s_3(x_1, x_2, x_3)(x_1 x_4 + x_4 x_1)) - \text{tr}(s_3(x_1, x_2, x_4)(x_1 x_3 + x_3 x_1))$$

$$+ \text{tr}(s_3(x_1, x_3, x_4)(x_1 x_2 + x_2 x_1)) + 3\text{tr}(s_3(x_2, x_3, x_4)x_1^2)\text{tr}(x_1^2)$$

$$+ 5(-\text{tr}(s_3(x_1, x_2, x_3)x_1^2)\text{tr}(x_1x_4))$$

$$+ \text{tr}(s_3(x_1, x_2, x_4)x_1^2)\text{tr}(x_1x_3) - \text{tr}(s_3(x_1, x_3, x_4)x_1^2)\text{tr}(x_1x_2)),$$

$$w_2 = (\text{tr}(s_3(x_1, x_2, x_3)x_4) - \text{tr}(s_3(x_1, x_2, x_4)x_3) + \text{tr}(s_3(x_1, x_3, x_4)x_2)$$

$$+ 3\text{tr}(s_3(x_2, x_3, x_4)x_1)\text{tr}(x_1^2) + 4(-\text{tr}(s_3(x_1, x_2, x_3)\text{tr}(x_1^2))$$

$$+ \text{tr}(s_3(x_1, x_2, x_4)x_1)\text{tr}(x_1^2) - \text{tr}(s_3(x_1, x_3, x_4)x_1)\text{tr}(x_1^2)),$$

$$w_3 = (\text{tr}(s_3(x_2, x_3, x_4))\text{tr}(x_1^2) - \text{tr}(s_3(x_1, x_3, x_4))\text{tr}(x_1^2)$$

$$+ \text{tr}(s_3(x_1, x_2, x_4))\text{tr}(x_1^2) - \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(x_1^2)).$$

For $\lambda = (3, 2^2)$:

$$w_1 = \sum_{\sigma \in S_3} \text{sgn}(\sigma)\text{tr}(s_3(x_1, x_2, x_3)x_{\sigma(1)}x_{\sigma(2)})\text{tr}(x_1x_{\sigma(3)}),$$

$$w_2 = \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(s_3(x_1, x_2, x_3)),$$

$$w_3 = \text{tr}([x_1, x_2]^2)\text{tr}(x_1x_3) + \text{tr}([x_1, x_3]^2)\text{tr}(x_1^2) + \text{tr}([x_2, x_3]^2)\text{tr}(x_1^2)$$

$$- \text{tr}([x_1, x_2][x_1, x_3])\text{tr}(x_1x_3) + 2\text{tr}([x_1, x_2][x_1, x_3])\text{tr}(x_1x_3)$$

$$+ 2\text{tr}(x_1x_2)(-\text{tr}(x_1x_2)\text{tr}(x_1^2) + \text{tr}(x_1x_3)\text{tr}(x_2x_3))$$

$$+ 2\text{tr}(x_1x_2)(-\text{tr}(x_1x_2)\text{tr}(x_1^2) + \text{tr}(x_1x_3)\text{tr}(x_2x_3))$$

$$+ \text{tr}(x_1(x_2x_3 + x_3x_2))(-\text{tr}(x_1^2)\text{tr}(x_2x_3) + \text{tr}(x_1x_2)\text{tr}(x_1x_3)).$$

For $\lambda = (3, 2, 1^2)$:

$$w_1 = (\text{tr}(s_3(x_1, x_2, x_3)(x_2x_4 + x_4x_2)) - \text{tr}(s_3(x_1, x_2, x_4)(x_2x_3 + x_3x_2))$$

$$+ 4\text{tr}(s_3(x_2, x_3, x_4)(x_1x_2 + x_2x_1)) + 2\text{tr}(s_3(x_1, x_3, x_4)x_1^2)\text{tr}(x_1^2)$$

$$+ (-\text{tr}(s_3(x_1, x_2, x_4)(x_1x_4 + x_4x_1)) + \text{tr}(s_3(x_1, x_2, x_4)(x_1x_3 + x_3x_1))$$

$$- 6\text{tr}(s_3(x_1, x_2, x_3)(x_1x_2 + x_2x_1)) - 8\text{tr}(s_3(x_2, x_3, x_4)x_1^2)\text{tr}(x_1x_2)$$

$$+ 5(-\text{tr}(s_3(x_1, x_2, x_3)(x_1x_2 + x_2x_1))\text{tr}(x_1x_4)$$

$$+ \text{tr}(s_3(x_1, x_2, x_1)(x_1x_2 + x_2x_1))\text{tr}(x_1x_3))$$

$$+ 10\text{tr}(s_3(x_1, x_2, x_3)x_1^2)\text{tr}(x_2x_4) - \text{tr}(s_3(x_1, x_2, x_4)x_1^2)\text{tr}(x_2x_3)$$

$$+ \text{tr}(s_3(x_1, x_3, x_4)x_1^2)\text{tr}(x_2x_3),$$

$$w_2 = \text{tr}(x_1^2)(-\text{tr}(s_3(x_1, x_2, x_3)|x_2x_4) - \text{tr}(s_3(x_2, x_3, x_4)|x_1x_2)$$

$$+ \text{tr}(s_3(x_1, x_2, x_4)|x_2x_3) + 3\text{tr}(s_3(x_2, x_3, x_4)|x_1x_2)$$

$$+ \text{tr}(x_1x_2)(\text{tr}(s_3(x_1, x_2, x_3)|x_1x_4) - \text{tr}(s_3(x_1, x_3, x_4)|x_1x_2)$$

$$- \text{tr}(s_3(x_1, x_2, x_4)|x_1x_3) - \text{tr}(s_3(x_1, x_3, x_4)|x_1x_2))$$

$$+ 3\text{tr}(x_1x_3)\text{tr}(s_3(x_1, x_2, x_4)|x_1x_2) - 3\text{tr}(x_1x_4)\text{tr}(s_3(x_1, x_2, x_3)|x_1x_2).$$
\[ w_3 = \text{tr}([x_1, x_2]^2)\text{tr}(s_3(x_1, x_3, x_4)) - \text{tr}(x_1, x_2)[x_1, x_3]\text{tr}(s_3(x_1, x_2, x_4)) + \text{tr}(x_1, x_2)[x_1, x_3]\text{tr}(s_3(x_1, x_2, x_4)), \]
\[ w_4 = -2\text{tr}(s_3(x_2, x_3, x_4)x_2)\text{tr}(x_1^2) + 2\text{tr}(s_3(x_1, x_3, x_4)x_2) + \text{tr}(s_3(x_2, x_1, x_4)x_1)\text{tr}(x_1^2)x_2 - 2\text{tr}(s_3(x_1, x_2, x_4)x_2)\text{tr}(x_1^2)x_3 + 2\text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(x_1^2)x_4 - 2\text{tr}(s_3(x_1, x_3, x_4)x_1)\text{tr}(x_1^2)x_2 + \text{tr}(s_3(x_1, x_2, x_4)x_1)\text{tr}(x_1^2)/x_2 - \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(x_1^2)/x_2/4 + x_4, x_2)), \]
\[ w_5 = \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(s_3(x_1, x_2, x_4)) - \text{tr}(s_3(x_1, x_2, x_4)x_1)\text{tr}(s_3(x_1, x_2, x_3)), \]
\[ w_6 = (\text{tr}(x_1^2)\text{tr}(x_2^2) - \text{tr}(x_1x_2)^2)\text{tr}(s_3(x_1, x_3, x_4)) + (-\text{tr}(x_1^2)\text{tr}(x_2x_4) + \text{tr}(x_1x_2)^2 + \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(x_1x_2)) + \text{tr}(x_1^2)\text{tr}(x_2x_4) - \text{tr}(x_1x_2)^2 + \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(x_1x_2). \]

For \( \lambda = (3, 1^4) \):
\[ w_1 = \text{tr}(s_3(x_1, x_2, x_3, x_4)x_1\text{tr}(s_3(x_1, x_2, x_4)), \quad \sigma(1) = 1. \]
\[ w_2 = \sum_{\sigma \in S_6} \text{sign}(\sigma)\text{tr}(s_3(x_1, x_{\sigma(1)}, x_{\sigma(2)}))\text{tr}(s_3(x_1, x_{\sigma(3)})), \]
\[ w_3 = \text{tr}(s_3(x_1, x_2, x_3, x_4)) - \text{tr}(x_1, x_2)[x_1, x_3]\text{tr}(s_3(x_1, x_2, x_4)) + \text{tr}(x_1, x_2)[x_1, x_3]\text{tr}(s_3(x_1, x_2, x_4)), \]
\[ w_4 = -2\text{tr}(s_3(x_2, x_3, x_4)x_2)\text{tr}(x_1^2) + 2\text{tr}(s_3(x_1, x_3, x_4)x_2) + \text{tr}(s_3(x_2, x_1, x_4)x_1)\text{tr}(x_1^2)x_2 - 2\text{tr}(s_3(x_1, x_2, x_4)x_2)\text{tr}(x_1^2)x_3 + 2\text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(x_1^2)x_4 - 2\text{tr}(s_3(x_1, x_3, x_4)x_1)\text{tr}(x_1^2)x_2 + \text{tr}(s_3(x_1, x_2, x_4)x_1)\text{tr}(x_1^2)/x_2/4 + x_4, x_2)), \]
\[ w_5 = \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(s_3(x_1, x_2, x_4)) - \text{tr}(s_3(x_1, x_2, x_4)x_1)\text{tr}(s_3(x_1, x_2, x_3)), \]
\[ w_6 = (\text{tr}(x_1^2)\text{tr}(x_2^2) - \text{tr}(x_1x_2)^2)\text{tr}(s_3(x_1, x_3, x_4)) + (-\text{tr}(x_1^2)\text{tr}(x_2x_4) + \text{tr}(x_1x_2)^2 + \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(x_1x_2)) + \text{tr}(x_1^2)\text{tr}(x_2x_4) - \text{tr}(x_1x_2)^2 + \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(x_1x_2). \]

For \( \lambda = (2^3, 1^1) \):
\[ w_1 = \text{tr}(s_3(x_1, x_2, x_3, x_4)x_1)\text{tr}(s_3(x_1, x_2, x_4)), \]
\[ w_2 = (3\text{tr}(s_3(x_1, x_2, x_3)x_4) + \text{tr}(s_3(x_2, x_3, x_4)x_1)) + \text{tr}(s_3(x_1, x_3, x_4)x_2) + \text{tr}(s_3(x_2, x_3, x_4)x_1) - \text{tr}(s_3(x_1, x_2, x_4)x_3) + \text{tr}(s_3(x_2, x_3, x_4)x_1)\text{tr}(s_3(x_1, x_2, x_3)) + 4\text{tr}(s_3(x_1, x_2, x_4)x_3) - \text{tr}(s_3(x_1, x_2, x_3)x_2) + 4\text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(s_3(x_1, x_2, x_3)). \]

For \( \lambda = (2^2, 1^3) \):
\[ w_1 = \text{tr}(x_1, x_2)\text{tr}(x_1, x_3) + \text{tr}(x_1, x_3), \]
\[ w_2 = \sum_{\sigma \in S_5} \sum_{\tau \in S_2} \text{sign}(\sigma\tau)\text{tr}(s_3(x_{\tau(1)}, x_{\sigma(1)}, x_{\sigma(2)}))\text{tr}(s_3(x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\tau(2)}).} \]
For $\lambda = (2, 1^5)$:

\[
\begin{align*}
  w_1 &= \sum_{\sigma \in S_6} \text{sign}(\sigma) \tr(s_5(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})) \tr(x_{\sigma(6)}), \\
  w_2 &= \sum_{\sigma \in S_6} \text{sign}(\sigma) \tr(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)} x_1)) \tr(s_3(x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)})) \\
  &\quad + \sum_{\sigma \in S_6} \text{sign}(\sigma) \tr(s_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \tr(s_3(x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)})).
\end{align*}
\]

For $\lambda = (4, 1^3), (3, 2^2), (3, 2, 1^2), (3, 1^4), (2^3, 1^3), (2, 1^5)$, every highest weight vector $w \in W(\lambda) \subset \omega^2(S)$ is equal to a linear combination of $w_1$.

**Proof.** The computations are similar to those in the proof of Proposition 2.2. For a fixed $\lambda \vdash 7$, the number of $w_i$ can be calculated from the equations (24) – (32) and Proposition 2.2. The concrete form of $w_1$ is found using Lemmas 1.1 and 1.2. We shall demonstrate the process in one case only.

Let $\lambda = (3, 2^2)$. Consider the tensor product $W(3) \otimes (W(2) \otimes_s W(2)) \subset \omega^2(S)$. The equation (31) gives that the module $W(3, 2^2)$ participates with multiplicity 1. More precisely, applying (7) to (15), we see that $W(3, 2^2)$ appears as a submodule of the component $W(3) \otimes W(2^2)$ of $W(3) \otimes (W(2) \otimes_s W(2))$. Since $W(3) \subset G_3$ and $W(2) = G_2$ are generated by $\tr(x_1^2)$ and $\tr(x_1)$, respectively, Lemma 1.2 gives that they have bases

\[
\begin{align*}
  \{\tr(x_i^3), \tr(x_i^j), i \neq j, & \quad \tr(x_i(x_j x_k + x_k x_j)), i < j < k\}, \\
  \{\tr(x_i^2), \tr(x_i x_j), & \quad i \neq j\}.
\end{align*}
\]

The submodule $W(2^2)$ of $W(2) \otimes_s W(2)$ is generated by

\[
u(x_1, x_2) = \tr(x_1^2) \tr(x_1 x_2) - \tr(x_1 x_2) x_2.
\]

(Direct verification shows that $u(x_1, x_2 + x_1) = u(x_1, x_2)$ and we apply Lemma 1.1.) Its partial linearization in $x_2$ is

\[
v(x_1, x_2, x_3) = \tr(x_1^2) \tr(x_2 x_3) - \tr(x_1 x_2) \tr(x_1 x_3).
\]
We are looking for an element \( w = w(x_1, x_2, x_3) \) in \( W(3) \otimes W(2^2) \subset W(3) \otimes (W(2) \otimes W(2)) \) which is homogeneous of multidegree \((3, 2^2)\) and satisfies the conditions \( \Delta_{21}(w) = \Delta_{22}(w) = w \). All such elements are of the form

\[
\begin{align*}
w &= \zeta_1 \text{tr}(x_1^3) u(x_2, x_3) + \zeta_2 \text{tr}(x_1^2 x_2) v(x_3, x_1, x_2) + \zeta_3 \text{tr}(x_1 x_2^2) v(x_2, x_1, x_3) \\
&\quad + \zeta_4 \text{tr}(x_1 x_2^2) u(x_1, x_3) + \zeta_5 \text{tr}(x_1^2 x_3 + x_3 x_2) v(x_1, x_2, x_3) + \zeta_6 \text{tr}(x_1 x_3^2) u(x_1, x_2).
\end{align*}
\]

Direct verifications show that

\[
\begin{align*}
\Delta_{21}(w) &= (2\zeta_1 + \zeta_2) \text{tr}(x_1^3) u(x_3, x_1, x_2) \\
&\quad + (\zeta_2 + 2\zeta_4) \text{tr}(x_1^2 x_2) u(x_1, x_3) + (\zeta_3 + 2\zeta_5) \text{tr}(x_1 x_2^2) v(x_1, x_2, x_3), \\
\Delta_{31}(w) &= (2\zeta_1 + \zeta_3) \text{tr}(x_1^3) v(x_2, x_1, x_3) \\
&\quad + (-\zeta_2 + 2\zeta_5) \text{tr}(x_1^2 x_2) v(x_1, x_2, x_3) + (\zeta_3 + 2\zeta_6) \text{tr}(x_1 x_2^2) u(x_1, x_2), \\
\Delta_{32}(w) &= (-\zeta_2 + \zeta_3) \text{tr}(x_1^2 x_2) v(x_2, x_1, x_3) \\
&\quad + 2(\zeta_4 + \zeta_5) \text{tr}(x_1 x_2^2) v(x_1, x_2, x_3) + (\zeta_5 + \zeta_6) \text{tr}(x_1 (x_2 x_3 + x_3 x_2)) u(x_1, x_2).
\end{align*}
\]

Hence we obtain the homogeneous linear system

\[
\begin{align*}
2\zeta_1 + \zeta_2 &= 0, \\
2\zeta_1 + \zeta_3 &= 0, \\
-\zeta_2 + 2\zeta_5 &= 0, \\
-\zeta_2 + \zeta_3 + 2(\zeta_4 + \zeta_5) &= 0, \\
\zeta_5 &= 0.
\end{align*}
\]

Up to a multiplicative constant, the only solution of the system is

\[
\begin{align*}
\zeta_1 = 1, \quad \zeta_2 = -2, \quad \zeta_3 = -1,
\end{align*}
\]

which is equal to \( w_4 \). In practice, in most of the cases we have used a slightly different algorithm to determine \( w_i \). We have considered \( w_i \) with unknown coefficients. Then we have evaluated it in the trace algebra \( C_{3d} \) instead of in the symmetric algebra \( S \), in order to use the programs which we already had. Requiring that

\[
g_{kl}(\omega^2(S)) = \omega^2(S), \quad 1 \leq k < l \leq d,
\]

we have obtained the possible candidates for \( w_i \). Since the number of candidates has coincided with the number predicted in Proposition 2.5, we have concluded that the \( w_i \)'s really are the needed highest weight vectors.

\[ \square \]

**Proposition 2.5.** For \( d = 3 \), the homogeneous component \((\omega^2(S))^{(8)}\) of degree 8 of the square \( \omega^2(S) \) of the augmentation ideal of the symmetric algebra of \( G_2 \oplus \cdots \oplus G_6 \) decomposes as

\[
(\omega^2(S))^{(8)} = 2W_3(8) \oplus W_3(7, 1) \oplus 4W_3(6, 2) \oplus 3W_3(6, 1^2)
\]

\[ \oplus 2W_3(5, 3) \oplus 6W_3(5, 2, 1) \oplus 4W_3(4^2) \oplus 7W_3(4, 3, 1) \oplus 9W_3(4, 2^2) \oplus 4W_3(2^3, 2). \]

**Proof.** The considerations are similar to those in the proof of Proposition 2.2 and involves the equations

\[
\begin{align*}
W_3(3^2) \otimes W_3(2) &\cong W_3(5, 3) \oplus W_3(4, 3, 1) \oplus W_3(3^2, 2), \\
W_3(3, 1^2) \otimes W_3(3) &\cong W_3(6, 1^2) \oplus W_3(5, 2, 1) \oplus W_3(4, 3, 1), \\
W_3(3, 1^2) \otimes W_3(1^3) &\cong W_3(4, 2^2), \\
W_3(2^2, 1) \otimes W_3(3) &\cong W_3(5, 2, 1) \oplus W_3(4, 2^2), \\
W_3(2^2, 1) \otimes W_3(1^3) &\cong W_3(3^2, 2), \\
W_3(2^2) \otimes W_3(2^2) &\cong W_3(4^2) \oplus W_3(4, 2^2),
\end{align*}
\]
\[ W_3(2^2) \otimes W_3(2, 1^2) \cong W_3(4, 3, 1) \oplus W_3(3^2, 2), \]
\[ W_3(2, 1^2) \otimes_s W_3(2, 1^2) \cong W_3(4, 2^2), \]
\[ W_3(2^2) \otimes (W_3(2) \otimes W_3(2)) \cong W_3(6, 2), \]
\[ \oplus W_3(5, 2, 1) \oplus W_3(4^2) \oplus W_3(4, 3, 1) \oplus 2W_3(4, 2^2), \]
\[ W_3(2, 1^2) \otimes (W_3(2) \otimes W_3(2)) \cong W_3(6, 1^2) \oplus W_3(5, 2, 1) \oplus W_3(4, 3, 1) \oplus W_3(3^2, 2), \]
\[ (W_3(3) \otimes W_3(3)) \otimes W_3(2) \cong W_3(8) \]
\[ \oplus W_3(7, 1) \oplus 2W_3(6, 2) \oplus W_3(5, 3) \oplus W_3(5, 2, 1) \]
\[ \oplus W_3(4^2) \oplus W_3(4, 3, 1) \oplus W_3(4, 2^2), \]
\[ (W_3(3) \otimes W_3(1^3)) \otimes W_3(2) \cong W_3(6, 1^2) \oplus W_3(5, 2, 1) \oplus W_3(4, 3, 1), \]
\[ (W_3(1^3) \otimes W_3(1^3)) \otimes W_3(2) \cong W_3(4, 2^2), \]
\[ W_3(2) \otimes W_3(2) \otimes W_3(2) \cong W_3(8) \oplus W_3(6, 2) \oplus W_3(4^2) \oplus W_3(4, 2^2). \]

The proof of these equations uses the Young rule \( \mathcal{Y} \) and \( \mathcal{S} \), the formulas (10) – (12), (6), and (13). For example, by (10), \( W_3(2) \otimes W_3(2) \cong W_3(4) \oplus W_3(2^2). \) Hence
\[ W_3(2^2) \otimes (W_3(2) \otimes W_3(2)) \cong W_3(2^2) \otimes (W_3(4) \oplus W_3(2^2)), \]
by (6)
\[ W_3(2^2) \otimes W_3(4) \cong W_3(6, 2) \oplus W_3(5, 2, 1) \oplus W_3(4, 2^2), \]
by (6)
\[ W_3(2^2) \otimes W_3(4^2) \cong W_3(4^2) \oplus W_3(4, 3, 1) \oplus W_3(4, 2^2), \]
and we obtain the decomposition for \( W_3(2^2) \otimes (W_3(2) \otimes W_3(2)). \) In few cases we use also that
\[ W_d(\lambda_1 + 1, \ldots, \lambda_d + 1) \cong W_d(1^d) \otimes W_d(\lambda_1, \ldots, \lambda_d). \]
For example,
\[ W_3(2, 1^2) \otimes W_3(2, 1^2) \cong (W_3(1^3) \otimes W_3(1)) \otimes_s (W_3(1^3) \otimes W_3(1)) \]
\[ = (W_3(1^3) \otimes W_3(1^3)) \otimes (W_3(1) \otimes_s W_3(1)) \cong W_3(2^3) \oplus W_3(2) \cong W_3(4, 2^2). \]

\[ \square \]

**Proposition 2.6.** For \( d = 3 \), the following elements of \( S = K[G_2 \oplus \cdots \oplus G_6] \) are highest weight vectors:

For \( \lambda = (4, 3, 1) : \)
\[ w_1 = \sum_{\sigma \in S_3} \text{sign}(\sigma) \text{tr}([x_1, x_2]^2 x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}), \]
\[ w_2 = -\text{tr}(s_3(x_1, x_2, x_3)x_2^2) \text{tr}(x_1 x_2^2) \]
\[ + \text{tr}(s_3(x_1, x_2, x_3)(x_1 x_2 + x_2 x_1)\text{tr}(x_1^2 x_2) - \text{tr}(s_3(x_1, x_2, x_3) x_2^2 \text{tr}(x_1^2)), \]
\[ w_3 = \text{tr}([x_1, x_2]^2) \text{tr}(s_3(x_1, x_2, x_3)) x_3, \]
\[ w_4 = \text{tr}([x_1, x_2] [x_1, x_3]) (\text{tr}(x_1^2) \text{tr}(x_2^2)) \]
\[ - \text{tr}^2(x_1 x_2) - \text{tr}(x_1) \text{tr}(x_2) x_2^2 x_3 x_2 - \text{tr}(x_1 x_2) \text{tr}(x_1 x_3)), \]
\[ w_5 = \text{tr}(s_3(x_1, x_2, x_3) x_3) (\text{tr}(x_1^2)^2 x_2^2 - \text{tr}^2(x_1 x_2)), \]
\[ w_6 = (\text{tr}(x_1 x_2)^2) (\text{tr}(x_2 x_3) - \text{tr}(x_1 x_3)^2 - \text{tr}(x_1 x_2) \text{tr}(x_1 x_3)) + (\text{tr}(x_1^2) x_2 x_3) \text{tr}(x_1^2) + (\text{tr}(x_1^2) x_2 x_3) \text{tr}(x_1^2) \text{tr}(x_1 x_2) \]
\[ - \text{tr}(x_1 x_2 x_3) \text{tr}(x_1 x_2) + 3 \text{tr}(x_1 x_3) \text{tr}(x_1 x_2) \text{tr}(x_1 x_3) + \text{tr}(x_1^2 x_2) \text{tr}(x_1 x_2) \text{tr}(x_1 x_3) + \text{tr}(x_1^3 x_2) \text{tr}(x_1 x_2) \text{tr}(x_1 x_3) \text{tr}(x_1 x_3) \text{tr}(x_1 x_2) \text{tr}(x_1 x_3) + \text{tr}(x_1^3) \text{tr}(x_1 x_2) \text{tr}(x_1 x_3) - 2 \text{tr}(x_1^2 x_2) \text{tr}(x_1^2 x_3) \text{tr}(x_1 x_2) \]
For \( \lambda = (4, 2^2) \):

\[
\begin{align*}
    w_1 &= \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(s_3(x_1, x_2, x_3)), \\
    w_2 &= \sum_{\sigma \in S_3} \text{sign}(\sigma)\text{tr}(s_3(x_1, x_2, x_3)x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)})\text{tr}(x_1^2x_2^2), \\
    w_3 &= \text{tr}([x_1, x_2]^2)\text{tr}([x_1, x_3]^2) - \text{tr}^2([x_1, x_2][x_1, x_3]), \\
    w_4 &= \text{tr}^2(s_3(x_1, x_2, x_3)x_1), \\
    w_5 &= \text{tr}([x_2, x_3]^2)\text{tr}^2(x_1^2) - 2\text{tr}([x_1, x_3][x_2, x_3])\text{tr}(x_1^2)\text{tr}(x_1x_2) \\
        &\quad + 2\text{tr}([x_1, x_2][x_2, x_3])\text{tr}(x_1^2)\text{tr}(x_1x_3) + \text{tr}([x_1, x_3]^2)\text{tr}(x_1x_2) \\
        &\quad - 2\text{tr}([x_1, x_2][x_1, x_3])\text{tr}(x_1x_2)\text{tr}(x_1x_3) + \text{tr}([x_1, x_2]^2)\text{tr}(x_1x_3), \\
    w_6 &= \text{tr}([x_1, x_3]^2)(\text{tr}(x_1^2)\text{tr}(x_2^2) - \text{tr}(x_1x_2)) \\
        &\quad - 2\text{tr}([x_1, x_2][x_1, x_3])\text{tr}(x_1x_2)\text{tr}(x_1x_3) - \text{tr}(x_1x_2)\text{tr}(x_1x_3)](\text{tr}(x_1^2)\text{tr}(x_2^2) - \text{tr}(x_1x_2)) \\
    w_7 &= (-4\text{tr}(x_1x_2)\text{tr}(x_1x_3) + \text{tr}^2(x_1(x_2x_3 + x_3x_2)))\text{tr}(x_1^2) \\
        &\quad + 4(2\text{tr}(x_1^2x_2)\text{tr}(x_1x_3) - \text{tr}(x_1x_2)\text{tr}(x_1(x_2x_3 + x_3x_2)))\text{tr}(x_1x_2) \\
        &\quad + 4(2\text{tr}(x_1x_2)\text{tr}(x_1x_3) - \text{tr}(x_1x_2)\text{tr}(x_1(x_2x_3 + x_3x_2)))\text{tr}(x_1x_3) \\
        &\quad + 4(-\text{tr}(x_1^2)\text{tr}(x_1x_2) + \text{tr}(x_1x_2))\text{tr}(x_1x_3) \\
        &\quad + 4(\text{tr}(x_1^2)\text{tr}(x_1x_2) + \text{tr}(x_1x_2))\text{tr}(x_1x_3), \\
    w_8 &= \text{tr}^2(s_3(x_1, x_2, x_3))\text{tr}(x_1^2), \\
    w_9 &= (\text{tr}(x_1^2)\text{tr}(x_2^2)\text{tr}(x_3^2) - \text{tr}(x_1x_2)\text{tr}(x_1x_3)) \\
        &\quad - \text{tr}(x_1x_2)\text{tr}(x_2x_3) + 2\text{tr}(x_1x_2)\text{tr}(x_1x_3)\text{tr}(x_2x_3) - \text{tr}(x_1x_2)\text{tr}(x_1x_3)\text{tr}(x_2x_3)\text{tr}(x_1^2) \text{tr}(x_2^2).
\end{align*}
\]

For \( \lambda = (3^2, 2) \):

\[
\begin{align*}
    w_1 &= -\text{tr}([x_1, x_3]^2[x_1, x_2])\text{tr}(x_1^2) \\
        &\quad + \text{tr}([x_1, x_2][x_1, x_3][x_2, x_3] + [x_2, x_3][x_1, x_3])\text{tr}(x_1x_2) \\
        &\quad - 2\text{tr}([x_1, x_2]^2[x_2, x_3])\text{tr}(x_1x_3) - \text{tr}([x_1, x_3]^2[x_1, x_2])\text{tr}(x_1^2) \\
        &\quad + 2\text{tr}([x_1, x_2]^2[x_1, x_3])\text{tr}(x_2x_3) - \text{tr}([x_1, x_2]^3)\text{tr}(x_1^2), \\
    w_2 &= \text{tr}(s_3(x_1, x_2, x_3))\text{tr}(s_3(x_1, x_2, x_3)x_1) \\
        &\quad + \text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(s_3(x_1, x_2, x_3)x_1)) \\
        &\quad - \text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(s_3(x_1, x_2, x_3)x_1) \\
        &\quad - \text{tr}(s_3(x_1, x_2, x_3)x_3)\text{tr}(s_3(x_1, x_2, x_3)x_1)), \\
    w_3 &= \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(x_1x_2)\text{tr}(x_1x_3) - \text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(x_1x_2)\text{tr}(x_1x_3) \\
        &\quad + \text{tr}(s_3(x_1, x_2, x_3)x_3)\text{tr}(x_1x_2)\text{tr}(x_1x_3), \\
    w_4 &= \text{tr}(s_3(x_1, x_2, x_3)x_1)\text{tr}(x_1x_2)\text{tr}(x_1x_3) - \text{tr}(x_1x_3)\text{tr}(x_2x_3)) \\
        &\quad + \text{tr}(s_3(x_1, x_2, x_3)x_2)\text{tr}(x_1x_2)\text{tr}(x_1x_3) \\
        &\quad + \text{tr}(s_3(x_1, x_2, x_3)x_3)\text{tr}(x_1x_2)\text{tr}(x_1x_3).
\end{align*}
\]

In each of the cases, every highest weight vector \(w \in W_3(\lambda) \subset \omega^2(S)\) is equal to a linear combination of \(w_i\).
The Hilbert series of the symmetric algebra is

Clearly, the Hilbert series of the kernel

Proof. The considerations use the decomposition of $(\omega^2(S))^{(8)}$ given in Proposition 2.5. The highest weight vectors have been found as those in Propositions 2.3 and 2.4. Of course, if we already know the explicit form of the (candidates for) highest weight vectors $w_i$, we can check that they are linearly independent in $\omega^2(S)$ and satisfy the requirements of Lemma 1.1. Since for each $\lambda$ the number of the highest weight vectors $w_i$ coincides with the multiplicity of $W_3(\lambda) \subset \omega^2(S)$ from Proposition 2.6, we conclude that every highest weight vector $w \in W_3(\lambda) \subset \omega^2(S)$ is equal to a linear combination of $w_i$. □

Finally, we shall calculate the Hilbert series of the kernel of the natural homomorphism $S \to C_0$ for $d = 3$.

Lemma 2.7. Let $d = 3$ and let $J$ be the kernel of the natural homomorphism $S \to C_0$. Let the Hilbert series of $J$ be

where $h_k$ is the homogeneous component of degree $k$ of $H(J,t_1,t_2,t_3)$. Then $h_k = 0$ for $k \leq 6$,

$$h_7 = S_{(3,2^2)}(t_1,t_2,t_3),$$

$$h_8 = S_{(4,3,1)}(t_1,t_2,t_3) + 2S_{(4,2^2)}(t_1,t_2,t_3) + S_{(3^2,2)}(t_1,t_2,t_3).$$

Proof. Clearly, the Hilbert series of the kernel $J$ is equal to the difference of the Hilbert series of $S$ and $C_0$. For $d = 3$ we have that

$$G_2 \oplus \cdots \oplus G_6 = W(2) \oplus W(3) \oplus W(1^3)$$

$$\oplus W(2^2) \oplus W(2,1^2) \oplus W(3,1^2) \oplus W(2^2,1) \oplus W(3^2)$$

and its Hilbert series is

$$H(G_2 \oplus \cdots \oplus G_6,t_1,t_2,t_3) = \sum_{a_{k_1,k_2,k_3}} a_{k_1,k_2,k_3} t_1^{k_1} t_2^{k_2} t_3^{k_3}$$

$$= S_{(2)} + S_{(3)} + S_{(1^3)} + S_{(2^2)} + S_{(2,1^2)} + S_{(3,1^2)} + S_{(2^2,1)} + S_{(3^2)}.$$ 

The Hilbert series of the symmetric algebra is

$$H(S,t_1,t_2,t_3) = \prod \frac{1}{(1 - t_1^{k_1} t_2^{k_2} t_3^{k_3})^{a_{k_1,k_2,k_3}}}.$$ 

Now the result follows by evaluation of the coefficients $a_{k_1,k_2,k_3}$ and expanding the first several homogeneous components of the difference of the Hilbert series of $S$ and the Hilbert series of $C_0$, which is given in (14). □

Corollary 2.8. For $d = 3$, the algebra $C_0$ has a minimal system of defining relations with the property that the relations of degree 7 and 8 form $GL_3$-modules isomorphic, respectively, to $W_3(3,2^2)$ and $W_3(4,3,1) + 2W_3(4,2^2) + W_3(3^2,2)$.

Proof. If $J$ is the kernel of the natural homomorphism $S \to C_0$, then a minimal homogeneous system of generators of $J$ is obtained as a factor space of $J$ modulo $J_{\omega}(S)$. Since $\omega(S)$ contains no homogeneous elements of degree 1, we obtain that the multihomogeneous components of total degree 7 and 8 of $J$ and $J/J_{\omega}(S)$ are of the same dimension. Hence $J^{(7)}$ and $J^{(8)}$ are isomorphic as $GL_3$-modules to $(J/J_{\omega}(S))^{(7)}$ and $(J/J_{\omega}(S))^{(8)}$, respectively, and the conclusion follows from the expressions of $h_7$ and $h_8$ given in Lemma 2.7. □
3. Main results

Now we present the explicit defining relations of degree 7 of the algebra $C_{3d}$ for any $d \geq 3$ and of degree 8 for the algebra $C_{33}$, with respect to the generators of Abeasis and Pittaluga [1]. As we already mentioned, by (11) it is sufficient to give the defining relations of the algebra $C_0$ generated by traces $\text{tr}(x_{i_1} \cdots x_{i_d})$ of products of the traceless matrices $x_i$. As in the previous sections, we denote by $S$ the symmetric algebra of the $GL_d$-module $G_2 \oplus \cdots \oplus G_6$ of generators of $C_0$ and call defining relations of $C_0$ the expressions $f = 0$, where $f$ is an element of the kernel $J$ of the natural homomorphisms $S \to C_0$.

**Theorem 3.1.** Let $d \geq 3$. The algebra $C_0$ does not have any defining relations of degree $\leq 6$. The $GL_d$-module structure of the homogeneous defining relations of degree 7 of $C_0$, i.e., of the component $J^{(7)}$ in $S$ is

$$J^{(7)} = W_d(4,1^3) \oplus W_d(3,2^2) \oplus W_d(3,2,1^2) \oplus W_d(2^2,1^5) \oplus W_d(2,1^5).$$

In the notation of Proposition 2.4, the defining relations of $C_0$ which are highest weight vectors are:

- For $\lambda = (4,1^3)$:
  - $(33) \quad 12w_1 - 15w_2 - 20w_3 = 0$;

- For $\lambda = (3,2^2)$:
  - $2w_1 - w_2 + 2w_3 = 0$.

- For $\lambda = (3,2,1^2)$:
  - $-6w_1 + 10w_3 - 15w_4 + 40w_6 = 0$.

- For $\lambda = (2^3,1)$:
  - $12w_1 + w_2 = 0$.

- For $\lambda = (2^2,1^3)$:
  - $w_2 = 0$.

- For $\lambda = (2,1^5)$:
  - $2w_1 - 5w_2 = 0$.

**Proof.** In all cases the idea is the same. We already know that there are no relations of degree $\leq 11$ for $d = 2$ and the only $GL_3$-module of relations is isomorphic to $W_3(3,2^2)$. Hence, we have to consider the cases in Proposition 2.4 only.

We shall consider in detail the case $\lambda = (4,1^3)$. The possible relations $w = 0$ are linear combinations of $w_1, w_2, w_3$. We assume that

$$w = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 = 0$$

and evaluate $w$ on the traceless matrices (2) and (3). The coefficients of the monomials $(x_{i_1}^{(1)})^4 x_{i_2}^{(2)} x_{i_3}^{(3)} x_{i_4}^{(4)}$ and $(x_{i_1}^{(1)})^4 x_{i_2}^{(2)} x_{i_3}^{(3)} x_{i_4}^{(4)}$ are, respectively, $20\xi_1 - 8\xi_2 + 18\xi_3$ and $20\xi_1 + 12\xi_3$. Hence the equation (34) implies that

$$20\xi_1 - 8\xi_2 + 18\xi_3 = 20\xi_1 + 12\xi_3 = 0.$$

Up to a multiplicative constant, the only solution of this system is

$$\xi_1 = 12, \quad \xi_2 = -15, \quad \xi_3 = -20.$$

Hence, there is only one possible candidate for a defining relation which is a highest weight vector of some $W_d(4,1^3)$. We evaluate once again (33) on (2) and (3) for these values of $\xi_1, \xi_2, \xi_3$ and obtain that $w(x_1, x_2, x_3, x_4) = 0$. Hence the multiplicity of $W_d(4,1^3)$ in $J$ is equal to 1 and the corresponding relation is (33). We want to
mention that the case $\lambda = (3,1^4)$ does not participate in the statement of the theorem, because the multiplicity of $W_d(3,1^4)$ in $J$ is 0. \hfill \square

**Corollary 3.2.** The dimension of the defining relations of degree 7 of the algebra $C_{3d}$ is equal to

$$r_7 = r_7(d) = \frac{2}{71}(d + 1)d(d - 1)(d - 2)(41d^3 - 86d^2 + 114d - 360).$$

**Proof.** Since $C_{3d}$ does not satisfy relations of degree $\leq 6$, and the constants are the only elements of degree 0 in $K[\text{tr}(X_1), \ldots, \text{tr}(X_d)]$, the dimension of the relations of degree 7 of $C_{3d}$ coincides with this dimension in $C_0$. Now the proof is complete using Theorem 3.1 and the dimension formula (5) for $W_d(\lambda)$. \hfill \square

**Theorem 3.3.** Let $d = 3$. The $\text{GL}_d$-module structure of the homogeneous component $J^{(8)}$ of degree 8 in $S$ is

$$J^{(8)} = W_3(4,3,1) \oplus 2W_3(4,2^2) \oplus W_3(3,2^2,2).$$

In the notation of Proposition 2.6, the defining relations which are highest weight vectors are:

For $\lambda = (4,3,1)$:

\begin{equation}
-6w_1 - 18w_2 + 3w_3 + 3w_5 - 8w_7 = 0;
\end{equation}

For $\lambda = (4,2^2)$: All nontrivial linear combinations of

$$w_1 - 15w_2 + 3w_3 + \frac{21}{4}w_4 - \frac{5}{2}w_5 + \frac{5}{2}w_6 - 3w_7 + 2w_9 = 0,$$

$$-36w_2 + 6w_3 + \frac{27}{2}w_4 - 6w_5 + 6w_6 - 9w_7 + w_8 + 6w_9 = 0.$$

For $\lambda = (3^2,2)$:

$$6w_1 + 2w_2 - 3w_3 - 3w_4 = 0.$$

The number $r_8$ of the defining relations of degree 8 of any homogeneous minimal system of defining relations of the algebra $C_{33}$ is equal to 30.

**Proof.** The decomposition of $J^{(8)}$ is given in Corollary 2.8. The explicit form of the highest weight vectors is obtained as in the proof of Theorem 3.1. The number of defining relations of degree 8 in any homogeneous minimal system of defining relations for $C_{33}$ is equal to the dimension of the relations $J^{(8)}$ of $C_0$. For the proof that $r_8 = 30$ it is sufficient to use the dimension formula (5) for $W_d(\lambda)$. \hfill \square

**Remark 3.4.** Using Lemma 1.2 we can find an explicit basis of the set of defining relations of degree 7 for $C_0$, $d \geq 3$, and of degree 8 for $C_0$, $d = 3$.

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