THE BÄCKLUND TRANSFORMS OF PETERSON’S DEFORMATIONS OF QUADRICS

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Abstract. In trying to provide explicit deformations of quadrics the starting point of our investigation is to use Bianchi’s link between real deformations of totally real regions of real paraboloids and various totally real forms of the sine-Gordon equation coupled with Bianchi’s simple observation that the vacuum soliton of these totally real forms of the sine-Gordon equation provides precisely Peterson’s deformations of such quadrics in order to derive explicit Bäcklund transforms of Peterson’s deformations of quadrics. Based also on Bianchi’s approach of the Bäcklund transformation for quadrics via common conjugate systems and in analogy to the solitons of the sine-Gordon equation corresponding at the level of the geometric picture to the solitons of the pseudo-sphere we propose a model for the solitons of quadrics.

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1. Introduction

During the years 1899-1906 the theory of deformation (through bending) of general quadrics got the attention of geometers (mainly Bianchi, Calapso, Darboux, Guichard, Peterson and Titeica); as a consequence of their results the classical differential geometry of surfaces underwent a fundamental change. This theory culminated with Bianchi’s discovery in 1906 of the Bäcklund (B) transformation for general quadrics and the applicability correspondence provided by the Ivory affinity (ACPIA). However no explicit examples of deformations built on Bianchi’s approach exist in literature except mainly for the solitons of the (pseudo-)sphere. As any other integrable system one method to produce explicit solutions is to begin with the vacuum soliton as seed and build its B transforms. However in our case another seeds (namely Peterson’s deformations of quadrics) will be amenable to explicit computations of their B transforms. The starting point of our investigation is to use Bianchi’s link from (II, ch VI) between real deformations of totally real regions of real paraboloids...
and various totally real forms of the sine-Gordon equation coupled with Bianchi’s simple observation that the vacuum soliton of these totally real forms of the sine-Gordon equation provides precisely Peterson’s deformations of such quadrics to derive explicit deformations of quadrics.

The condition that a conjugate system (that is the second fundamental form is missing mixed terms) on a quadric is common to a Peterson’s 1-dimensional family of deformations of the quadric is projective invariant; also the condition that the lines of coordinates are planar is a projective invariant. On the complex unit sphere such conjugate systems with planar lines of coordinates are given by orthogonal systems of circles, that is the axes of the two pencils of planes containing the circles are polar reciprocal with respect to the sphere; according to Bianchi this condition is projective invariant, so it is valid for all quadrics. For general quadrics when one of the axes is a principal axis for the quadric one can derive explicit formulae for Peterson’s 1-dimensional family of deformations of quadrics.

The starting point of Bianchi’s investigation was results of Calapso, Darboux and Servant according to which for any real deformation of a totally real region of a quadric the conjugate system common to the deformation and quadric (any two surfaces in a point-wise correspondence admit a common conjugate system) is isothermal-conjugate system of coordinates on the deformation (that is the second fundamental form is a multiple of the identity, modulo some signs as required by curvature and signature of the ambient space considerations): he introduced as an auxiliary variable a (hyperbolic) angle. According to Bianchi the (hyperbolic) angle between one of Peterson’s conjugate system lines and one of the lines of the isothermal-conjugate system is a solution of some totally real form of the sine-Gordon equation for general real paraboloids: this is the geometric link between the sine-Gordon equation and totally real deformations of totally real regions of real paraboloids.

Note also that Calapso in [3] has completed Bianchi’s approach of the B transformation of deformations of 2-dimensional quadrics via common conjugate systems from paraboloids to quadrics with center (QC), but his approach for QC is different from Bianchi’s outline. The condition that the conjugate system on a quadric is a conjugate system on one of its deformations was known to Calapso for a decade, but the Bäcklund transformation for general quadrics via the Ivory affinity eluded Calapso since the common conjugate system is a-priori best suited for the B transformation only at the analytic level (which makes it also the best suited tool to provide explicit examples).

In what concerns totally real deformations of totally real regions with positive linear element of real paraboloids our main result is to put Bianchi’s machinery to work to churn out explicit formulae for the B transforms of Peterson’s deformations of such quadrics up to including the third iteration of the B transformation.

In what concerns totally real deformations of totally real regions of other quadrics our main result is to complete Bianchi’s elegant approach of the B transformation via common conjugate systems and then use this to churn out explicit formulae for the B transforms of Peterson’s deformations of such quadrics up to including the third iteration of the B transformation. Calapso’s approach from [3] (another completion of Bianchi’s approach of the B transformation via common conjugate systems to general quadrics) is similar in the main ideas to Bianchi’s approach but different in the fact that he uses only the common conjugate system, without paying attention to the change from the initial conjugate system common to a Peterson’s 1-dimensional family of deformations of quadrics. The totally real forms of the sine-Gordon equation are replaced for quadrics with center by another equation

Once a case of a general QC (the general case) being solved, all other complex types of quadrics should be amenable to explicit computations of the B transforms of Peterson’s deformations of such quadrics by similar computations. Since multiplication by $i$ exchanges both the signature of the totally real surface and of the ambient Lorentz space, from a totally real point of view one needs only discuss deformations in $\mathbb{R}^2 \times \sqrt{\epsilon} \mathbb{R}$, $\epsilon = \pm 1$ of quadrics with positive linear element (there are for example isotropic quadrics without center that cannot be realized as real quadrics, but admit real deformations) and deformations in $\mathbb{R}^2 \times i \mathbb{R}$ of quadrics with linear element of signature $(1, 1)$. 

\[2\]
The deformation problem for positive definite linear element is elliptic for real deformations of surfaces of positive Gauß curvature and for totally real deformations in Lorentz spaces of signature (2, 1) of surfaces of negative Gauß curvature and hyperbolic otherwise.

Thus for the hyperbolic sine-Gordon and sinh-Gordon equation the seed and the leaf will admit asymptotic lines and will be applicable to the same totally region of the real quadric; for the elliptic sine-Gordon and sinh-Gordon equation the seed and the leaf will not admit asymptotic lines and will be applicable to different totally regions of the real quadric (the applicability becomes ideal in Petzer’s denotation); one needs composition of B transformations to get back surfaces applicable to the initial totally real region.

In what concerns the solitons of quadrics we take as model the fact that the solitons of the sine-Gordon equation (with the (vacuum) 0-soliton \( \omega = 0 \)) correspond at the level of the geometric picture to the 0-soliton being the axis of the tractrix (thus it is a degenerate surface), the 1-solitons (B transforms of the 0-soliton) being the Dini helicoids (which include the real pseudo-sphere) and thus one can find the \( n \)-solitons, \( n \geq 2 \) by explicit formula via the Bianchi Permutability Theorem (BPT).

Based on this model the 0-soliton should be a degenerated surface (curve or point) and one must be able to explicitly compute the 1-solitons (B transforms of the 0-soliton); after that the \( n \)-solitons, \( n \geq 2 \) will be amenable to explicit computations via the same BPT.

2. Totally real forms of the sine-Gordon equation and their solitons

Consider the classical hyperbolic sine-Gordon equation

\[
\omega_{uv} - \omega_{uu} = \cos \omega \sin \omega
\]

in conjunction with real deformations \( x \subset \mathbb{R}^3 \) of the pseudo-sphere (it represents the Gauß equation in Chebyshev coordinates \( (u + v, u - v) \) which are further asymptotes) and the classical symmetric Bäcklund (B) transformation \( \omega_1 = B_{\sigma_1}(\omega_0), \omega_0 = B_{\sigma_0}(\omega_1), \sigma_0 = -\sigma_1 \in \mathbb{R}^* \)

\[
\omega_{1v} - \omega_{0u} = \frac{\sigma_1 \sin(\omega_1 + \omega_0) + \sigma_1^{-1} \sin(\omega_1 - \omega_0)}{2},
\]

\[
\omega_{1u} - \omega_{0v} = \frac{\sigma_1 \sin(\omega_1 + \omega_0) - \sigma_1^{-1} \sin(\omega_1 - \omega_0)}{2}, \quad 0 \leftrightarrow 1
\]

of its solutions together with its 1-solitons \( \omega_0 = 0, \omega_1 = \pm 2 \tan^{-1} e^{c_1 \frac{\tau_1}{\tau_2} + \tau_2 + c_1} \), \( c_1 \in \mathbb{R} \) and Bianchi Permutability Theorem (BPT)

\[
\tan \omega_3 - \omega_0 = \frac{\sigma_2 + \sigma_1}{\sigma_2 - \sigma_1} \tan \frac{\omega_2 - \omega_1}{2} \quad \text{for} \quad \omega_1 = B_{\sigma_1}(\omega_0), \omega_2 = B_{\sigma_2}(\omega_0),
\]

\[
B_{\sigma_2} \circ B_{\sigma_1}(\omega_0) = B_{\sigma_2}(\omega_1) = \omega_3 = B_{\sigma_1}(\omega_2) = B_{\sigma_1} \circ B_{\sigma_2}(\omega_0).
\]

Note that \( B \) admits the complex conjugate \( \sigma_2 = \sigma_1 \in \mathbb{C} \setminus \mathbb{R}, \omega_2 = \bar{\omega}_1 \subset \mathbb{C}, \omega_3, \omega_0 \subset \mathbb{R} \) version; with certain rationality conditions one obtains at the level of the geometric picture breathers.

Note also that as it was pointed out by Bianchi when he originally introduced his BPT in 1890 the BPT does not exclude the case \( \sigma_2 = \sigma_1 \) as being the trivial \( \omega_3 = \omega_0 \), but allows it as a limiting case \( \sigma_2 \to \sigma_1 \) and an application of L’Hospital; for example for 2 solitons if we let \( c_2 = c_2(\sigma_2), c_2(\sigma_1) = c_1, c'_2(\sigma_1) = c'_1, \) then \( \tan \frac{\omega_3}{2} = \frac{\sigma_1 + 1}{\sigma_3} - \frac{\sigma_1 - 1}{\sigma_3} + c_1 \) depends on two constants \( c, c_1 \) besides the spectral parameter \( \sigma_1 \), as expected (the B transformation should introduce one constant besides the spectral parameter).

In order to assure that further \( n \)-th iterates \( \mathcal{M}_n \) of the B transformation (moving Möbius configurations) in Bianchi’s denotation give the same result independently of the chosen path of composition of the B transformation we need to check only for the third iteration. This is to be expected, since by discretization the B transformation corresponds to the first derivative, the BPT \( (\mathcal{M}_2) \) corresponds to the commuting of second order derivatives (roughly the Gauß-Weingarten equations or equivalently the flat connection form corresponds) and the third Möbius configuration \( \mathcal{M}_3 \).
corresponds to the commuting of the third order derivatives (roughly the \textit{Gau\ss-Codazzi-Mainardi-Peterson} (GCMP) equations; as we know there are no conditions beyond the GCMP equations for a surface). Moreover the BPT with a leg of the Bianchi quadrilateral infinitesimal precisely describes the B transformation and thus the BPT encodes all necessary algebraic information needed to prove the existence of the B transformation and similarly the third M"{o}bius configuration encodes all necessary algebraic information needed to prove the validity of the BPT.

Since the odd \(M_{2n+1}\), \(n > 0\) M"{o}bius configuration does not depend on \(\omega_0\) we can use \(\omega_0 = 0\) to get \(M_3\)

\[
(e^{i\omega_2+i\omega_4} - e^{i\omega_1+i\omega_7})(\frac{\sigma_2}{\sigma_3} - \frac{\sigma_3}{\sigma_2}) + (e^{i\omega_1+i\omega_4} - e^{i\omega_2+i\omega_7})(\frac{\sigma_3}{\sigma_1} - \frac{\sigma_1}{\sigma_3}) + (e^{i\omega_1+i\omega_2} - e^{i\omega_4+i\omega_7})(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}) = 0.
\]

Note again that we can have \(\sigma_2 = \bar{\sigma}_1 \in \mathbb{C} \setminus \mathbb{R}\), \(\sigma_3 \in \mathbb{R}\), \(\omega_2 = \bar{\omega}_1 \subset \mathbb{C}\), \(\omega_4, \omega_7 \subset \mathbb{R}\).

Note also that unlike the BPT the \(M_3\) configuration is symmetric in all variables (it has the symmetries of a regular tetrahedron).

Similarly by considering purely imaginary \(\omega = i\theta\) in the sine-Gordon equation we have the hyperbolic sinh-Gordon equation

\[
(\theta_{\nu\nu} - \theta_{\nuu}) = \cosh \theta \sinh \theta
\]

with the symmetric B transformation \(\theta_1 = B_{\sigma_1}(\theta_0), \theta_0 = B_{\sigma_0}(\theta_1), \sigma_0 = -\sigma_1 \in \mathbb{R}^*\)

\[
\theta_{1\nu} - \theta_{0\nu} = \frac{\sigma_1 \sinh(\theta_1 + \theta_0) + \sigma_1^{-1} \sinh(\theta_1 - \theta_0)}{2},
\]

\[
\theta_{1u} - \theta_{0u} = \frac{\sigma_1 \sinh(\theta_1 + \theta_0) - \sigma_1^{-1} \sinh(\theta_1 - \theta_0)}{2}, \quad 0 \to 1
\]

of its solution together with its 1-solitons \(\theta_0 = 0\), \(\theta_1 = \pm 2 \tanh^{-1} e^\frac{\sigma_1 - \sigma_1^{-1}}{2} u + c_1, c_1 \in \mathbb{R}, -1 < \tanh(\theta_1) < 1\), BPT

\[
\tanh \frac{\theta_3 - \theta_0}{2} = \frac{\sigma_2 + \sigma_1}{\sigma_2 - \sigma_1} \tanh \frac{\theta_2 - \theta_1}{2} \text{ for } \theta_1 = B_{\sigma_1}(\theta_0), \quad \theta_2 = B_{\sigma_2}(\theta_0),
\]

\[
B_{\sigma_2} \circ B_{\sigma_1}(\theta_0) = B_{\sigma_2}(\theta_2) = \theta_3 = B_{\sigma_1}(\theta_2) = B_{\sigma_1} \circ B_{\sigma_2}(\theta_0)
\]

and third M"{o}bius configuration \(M_3\)

\[
(e^{\theta_2 + \theta_4} - e^{\theta_1 + \theta_7})(\frac{\sigma_2}{\sigma_3} - \frac{\sigma_3}{\sigma_2}) + (e^{\theta_1 + \theta_4} - e^{\theta_2 + \theta_7})(\frac{\sigma_3}{\sigma_1} - \frac{\sigma_1}{\sigma_3}) + (e^{\theta_1 + \theta_2} - e^{\theta_4 + \theta_7})(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}) = 0.
\]

Note that (6) admits the complex conjugate \(\sigma_2 = \bar{\sigma}_1 \in \mathbb{C} \setminus \mathbb{R}\), \(\theta_2 = \bar{\theta}_1 \subset \mathbb{C}\), \(\theta_3, \theta_0 \subset \mathbb{R}\) version and we may also have \(\sigma_2 = \bar{\sigma}_1 \in \mathbb{C} \setminus \mathbb{R}\), \(\sigma_3 \in \mathbb{R}\), \(\theta_2 = \bar{\theta}_1 \subset \mathbb{C}\), \(\theta_4, \theta_7 \subset \mathbb{R}\).
Similarly by considering purely imaginary coordinate \( u \) in the sine-Gordon equation we have the elliptic sine-Gordon equation
\[
\omega_{vv} + \omega_{uu} = \cos \omega \sin \omega
\]
and by considering purely imaginary coordinate \( u \) and purely imaginary \( \omega = i\theta \) in the sine-Gordon equation we have the elliptic sinh-Gordon equation
\[
\theta_{vv} + \theta_{uu} = \cosh \theta \sinh \theta
\]
with the symmetric B transformation \( \theta_1 = B_{\sigma_1}(\omega_0) \), \( \omega_0 = B_{\sigma_0}(\theta_1) \), \( \sigma_0 = -\sigma_1 \in S^1 \)
\[
i\theta_1 - i\partial_u \omega_0 = \frac{\sigma_1 \sin(i\theta_1 + \omega_0) + \sigma_1^{-1} \sin(i\theta_1 - \omega_0)}{2}, \quad 0 \leftrightarrow 1
\]
of their solution together with its \( \pm \)solitons \( \omega_0 = 0 \), \( \theta_1 = \pm 2 \tanh^{-1} e^{-\frac{\sigma_1^{-1} - \sigma_1}{2} u + \frac{\sigma_1^{-1} + \sigma_1}{2} v + c_1} \), \( c_1 \in \mathbb{R} \),
\[-1 < \tanh(\theta_1) < 1; \quad \theta_0 = 0, \quad \omega_1 = \pm 2 \tanh^{-1} e^{-\frac{\sigma_1^{-1} - \sigma_1}{2} u - \frac{\sigma_1^{-1} + \sigma_1}{2} v + c_1}, \quad c_1 \in \mathbb{R} \),
\[
\tan \frac{\omega_3 - \omega_0}{2} = i \frac{\sigma_2 + \sigma_1}{\sigma_2 - \sigma_1} \tanh \frac{\theta_2 - \theta_1}{2} \quad \text{for} \quad \theta_1 = \sigma_1(\omega_0), \quad \theta_2 = \sigma_2(\omega_0),
\]
\[
\tan \frac{\omega_3 - \omega_1}{2} = -i \frac{\sigma_2 + \sigma_1}{\sigma_2 - \sigma_1} \tan \frac{\omega_2 - \omega_1}{2} \quad \text{for} \quad \omega_1 = \sigma_1(\theta_1), \quad \omega_2 = \sigma_2(\theta_1),
\]
and third Möbius configuration \( \mathcal{M}_3 \)
\[
(e^{i\theta_2 + i\theta_1} - e^{i\theta_1 + i\theta_2})(\frac{\sigma_2}{\sigma_3} - \frac{\sigma_3}{\sigma_2}) + (e^{i\theta_2 + i\theta_1} - e^{i\theta_2 + i\theta_1})(\frac{\sigma_3}{\sigma_1} - \frac{\sigma_1}{\sigma_3}) + (e^{i\theta_1 + i\theta_2} - e^{i\theta_1 + i\theta_2})(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}) = 0,
\]
\[
(e^{i\omega_2 + i\omega_1} - e^{i\omega_1 + i\omega_2})(\frac{\sigma_2}{\sigma_3} - \frac{\sigma_3}{\sigma_2}) + (e^{i\omega_2 + i\omega_1} - e^{i\omega_2 + i\omega_1})(\frac{\sigma_3}{\sigma_1} - \frac{\sigma_1}{\sigma_3}) + (e^{i\omega_1 + i\omega_2} - e^{i\omega_1 + i\omega_2})(\frac{\sigma_1}{\sigma_2} - \frac{\sigma_2}{\sigma_1}) = 0.
\]
Note that \( (10) \) admits the complex conjugate \( \sigma_2 = \bar{\sigma}_1 \in \mathbb{C} \setminus \mathbb{R} \), \( \theta_2 = -\bar{\theta}_1 \subset \mathbb{C} \), \( \omega_3, \omega_0 \subset \mathbb{R} \); \( \omega_2 = -\bar{\omega}_1 \subset \mathbb{C} \), \( \theta_3, \theta_0 \subset \mathbb{R} \) version and we may also have \( \sigma_2 = \bar{\sigma}_1 \in \mathbb{C} \setminus \mathbb{R} \), \( \sigma_3 \in \mathbb{R} \), \( \theta_2 = -\bar{\theta}_1 \subset \mathbb{C} \), \( \theta_4, \theta_7 \subset \mathbb{R} \); \( \omega_2 = -\bar{\omega}_1 \subset \mathbb{C} \), \( \omega_4, \omega_7 \subset \mathbb{R} \).

3. Bianchi’s Bäcklund transformation for real quadrics via common conjugate systems

3.1. Real deformations of (the imaginary region of) the real hyperbolic paraboloid.

Consider the general confocal real hyperbolic paraboloids in an isometric-conjugate parametrization invariant under the Ivory affinity between confocal quadrics
\[
x_z = x_z(\alpha, \beta) := \sqrt{a_1^2 - z\alpha} \quad \sqrt{-a_2^2 + z\sqrt{\epsilon} \beta}, \quad a_1 > z > 0 > a_2,
\]
\[
a_1^{-1} - a_2^{-1} = 1, \quad \alpha \in \mathbb{R}, \quad \beta \in \mathbb{R}, \quad \epsilon = \pm 1, \quad \alpha^2 a_1^{-1} - \beta^2 a_2^{-1} + \epsilon > 0
\]
(the case \( 0 > z > a_2 \) is realized by a rigid motion \((c_1, c_3) \leftrightarrow (c_2, -c_3)\) and we have imaginary region for \( \epsilon = -1 \)) with positive definite linear element, second fundamental form and Christoffel symbols of \( x_0 \):
\[
|dx_0|^2 = a_1 d\alpha^2 - a_2 d\beta^2 + (\alpha d\alpha - \beta d\beta)^2, \quad N_0^2 d^2 x_0 = \sqrt{-\epsilon d\alpha^2 + d\beta^2} \frac{\sqrt{H}}{\sqrt{H}}, \quad H := \frac{\alpha^2}{a_1} - \frac{\beta^2}{a_2} + \epsilon;
\]
\[
-\epsilon \Gamma_2 = \Gamma_1^1 = (\log \sqrt{H})_{\alpha}, \quad -\epsilon \Gamma_1 = \Gamma_2^2 = (\log \sqrt{H})_{\beta}, \quad \Gamma_1^2 = \Gamma_2^1 = 0.
\]
We have the GCPM equations
\[
g_{2p}[(\Gamma_1^1)_{2} - (\Gamma_1^2)_{1} + \Gamma_1^q \Gamma_1^p - \Gamma_1^q \Gamma_1^p] = R_1^2 = h_{11} h_{22} - h_{12}^2,
\]
\[
R_{22} = -h_{12} h_{12} + h_{11} h_{22} - h_{12}^2;
\]
\[
R_{22} = h_{12} h_{12} + h_{11} h_{22} - h_{12}^2.
\]
\((hh_{12})1 - (hh_{11})2 + \Gamma_{12}^m h_{m_1} - \Gamma_{11}^m h_{m_2} = 0, (hh_{12})2 - (hh_{22})1 + \Gamma_{12}^m h_{m_2} - \Gamma_{22}^m h_{m_1} = 0.\)

We have a distinguished tangent vector field \(\mathcal{V}_0 := \epsilon(\log \sqrt{H})_x x_0 \alpha - (\log \sqrt{H})_\beta x_0 \beta;\) it has the properties \(|\mathcal{V}_0|^2 = 1 - \frac{\epsilon}{H}, \mathcal{V}_0^T x_0 \alpha = \alpha, \mathcal{V}_0^T x_0 \beta = -\beta.\)

Note also that the condition

\[
(\Gamma_{11}^2 h_{22})_\alpha = (\Gamma_{22}^1 h_{11})_\beta = -2\Gamma_{11}^2 \Gamma_{22}^1
\]

that \((\alpha, \beta)\) is common to an 1-dimensional Peterson’s family of deformations \(x\) of \(x_0\) is satisfied.

Given a real deformation \(x \in \mathbb{R}^3\) of a real region \(x_0\) (that is, \(\epsilon = 1\)) there exists a conjugate system \((u, v)\) common to both \(x_0\) and \(x\) (this is true for any two surfaces in a point-wise correspondence). Denote with \(\mathcal{V}_0\) the quantities of interest in the GCMP equations (namely the Christoffel symbols and the second fundamental form) of \(x_0\) referred to the \((u, v)\) coordinates and similarly with \(\bar{\mathcal{V}}_0\) those of \(x\). We have \(\alpha_u \alpha_v - \beta_u \beta_v = 0\) and from the Gauß equation \((\alpha_u^2 - \beta_u^2)(\alpha_v^2 - \beta_v^2) < 0;\) assume (by changing \(u\) and \(v\) if necessary) \(\alpha_u^2 - \beta_u^2 > 0\). With \(\lambda := \text{sgn}(\alpha_u) \sqrt{\alpha_u^2 - \beta_u^2}, \mu := \text{sgn}(\beta_v) \sqrt{\beta_v^2 - \alpha_v^2}\) we have \(\bar{h}_{11} = \frac{\lambda^2}{\mu^{2}}, \bar{h}_{12} = 0, \bar{h}_{22} = -\frac{1}{\mu^2}.\) From the general formula for the change of Christoffel symbols \(\tilde{\alpha}_{i\ell} = \alpha_{i\ell} + \alpha_{i\alpha} \alpha_{\alpha\ell} + \alpha_{i\beta} \beta_{\beta\ell} = \beta_{i\ell} + \beta_{i\alpha} \alpha_{\alpha\ell} + \beta_{i\beta} \beta_{\beta\ell}\) we get \(\bar{\mathcal{V}}_{12} \alpha_u = \bar{\mathcal{V}}_{12} \alpha_v = \alpha_{uv}, \bar{\mathcal{V}}_{12} \beta_u = \bar{\mathcal{V}}_{12} \beta_v = \beta_{uv}, \bar{\mathcal{V}}_{11} \alpha_u + \bar{\mathcal{V}}_{11} \alpha_v = \alpha_{uu} + \lambda^2 (\log \sqrt{H})_u, \bar{\mathcal{V}}_{11} \beta_u + \bar{\mathcal{V}}_{11} \beta_v = \beta_{uu} - \lambda^2 (\log \sqrt{H})_\beta; \bar{\mathcal{V}}_{22} \alpha_u + \bar{\mathcal{V}}_{22} \alpha_v = \alpha_{vv} - \mu^2 (\log \sqrt{H})_u, \bar{\mathcal{V}}_{22} \beta_u + \bar{\mathcal{V}}_{22} \beta_v = \beta_{vv} - \mu^2 (\log \sqrt{H})_\beta, \) so \(\bar{\mathcal{V}}_{12} = (\log \lambda)_u, \bar{\mathcal{V}}_{11} = (\log(\lambda^2))_u, \bar{\mathcal{V}}_{22} = (\log(\mu^2))_v, \bar{\mathcal{V}}_{12} = \frac{\mu^2}{\lambda^2} (\log \sqrt{H})_u.\) From the CMP equations of \(x_0, x\) we have

\[
(\bar{h}_{11})_v = \bar{\mathcal{V}}_{12} \bar{h}_{22} - \bar{\mathcal{V}}_{22} \bar{h}_{12}, (\bar{h}_{11})_u = \bar{\mathcal{V}}_{12} \bar{h}_{22} - \bar{\mathcal{V}}_{22} \bar{h}_{12},
\]

Keeping account of the Gauß equation \(\bar{h}_{11} \bar{h}_{22} = \bar{h}_{12} \bar{h}_{22}\) one can multiply the first equations respectively with \(\bar{h}_{11}, \bar{h}_{22}\) (and the second equations respectively with \(\bar{h}_{22}, \bar{h}_{12}\)), subtract them and get rid respectively of the \(\bar{\mathcal{V}}_{12}, \bar{\mathcal{V}}_{22}\) terms: \((\log(\bar{h}_{12}^2 - \bar{h}_{21}^2))_v = 2(\log \lambda)_v, (\log(\bar{h}_{22}^2 - \bar{h}_{12}^2))_u = 2(\log \mu)_u.\) Thus \(\bar{h}_{11}^2 - \bar{h}_{21}^2 = \phi(u) \lambda^2, \bar{h}_{22}^2 - \bar{h}_{12}^2 = \varphi(v) \mu^2;\) after a change of the \(u\) and \(v\) variables one can absorb \(\phi(u), \varphi(v)\) up to opposite signs \(\epsilon_1 := \pm 1, \epsilon_2 = -\epsilon_1\) (here we have again from the Gauß equation \(\bar{h}_{11}^2 > \bar{h}_{21}^2 \Rightarrow \bar{h}_{22}^2 < \bar{h}_{12}^2\)). We have \(\bar{h}_{11}^2 > \bar{h}_{21}^2 - \epsilon_1 \lambda^2 = \lambda^2((\frac{\mu}{\lambda} - \epsilon_1), \bar{h}_{22}^2 = \bar{h}_{22}^2 + \epsilon_1 \mu^2 = \mu^2(\frac{\lambda}{\mu} + \epsilon_1);\) from the Gauß equation \(\bar{h}_{11} \bar{h}_{22} = \bar{h}_{11} \bar{h}_{22}\) we get \(H = -\epsilon_1 \mu^2 + \epsilon_2 \lambda^2;\) by performing, if necessary, the change \((\alpha, \mu) \mapsto (\beta, \mu)\) we can choose \(\epsilon_1 := -1.\) Thus the second fundamental form of \(x\) is \(\frac{\mu^2 - \lambda^2}{H};\) we are led to consider the hyperbolic angle \(\theta\) between the conjugate systems \((\alpha, \beta)\) and \((u, v), that is \([\begin{array}{c} \alpha_u \\ \beta_u \\ \alpha_v \\ \beta_v \end{array}] = \frac{[\begin{array}{c} \lambda C \\ \lambda S \\ \mu S \\ \mu C \end{array}]}{C := \cosh \theta, S := \sinh \theta (\text{note that by doing this the sign of } \theta \text{ is decided by that of } \beta_u). \text{ Imposing the compatibility conditions } (\lambda C)_v = (\mu S)_u, (\lambda S)_v = (\mu C)_u \text{ we get } \mu_u = \lambda \theta_v, \lambda_v = \mu \theta_u; \text{ differentiating } H = \mu^2 - \lambda^2 \text{ with respect to } u, \text{ respectively } v \text{ we are led to consider the hyperbolic sinh-Gordon equation } \) as the compatibility condition of the completely integrable linear system in \(\alpha, \beta, \lambda, \mu: \)

\[
\left(\begin{array}{c} \alpha \\ \beta \\ \lambda \\ \mu \end{array}\right) = \left(\begin{array}{c} \lambda C du + \mu S dv \\ \lambda S du + \mu C dv \\ -\lambda \theta du + (S \frac{\alpha_1}{\alpha_2} - C \frac{\alpha_2}{\alpha_1} + \lambda \theta du) dv \\ \lambda \theta du + (S \frac{\alpha_1}{\alpha_2} - C \frac{\alpha_2}{\alpha_1} + \lambda \theta du) dv \\ \end{array}\right), \mu^2 - \lambda^2 = H.
\]

Note that a solution \(\theta\) of (11) will produce an 1-dimensional family of deformations \(x\) of \(x_0\) (from the original 4-dimensional space of solutions of the differential part of (12) the prime integral property \(\mu^2 - \lambda^2 = H\) removes a constant and translation in \(u, v\) another two). The condition that an 1-dimensional family of deformations \(x\) of \(x_0\) with common conjugate system is of Peterson’s type (that is (11) is satisfied in the \((u, v)\) coordinates is invariant under changes of variables \((u, v)\) into
themselves; in our case we need \((\log \frac{1}{\mu})_{uv} = 0\), but this adjoined to \((12)\) will be over-determined; as we shall see later the condition \((\log \frac{1}{\mu})_{uv} = 0\) will be preserved by the B transformation.

Note that if we assume that the common conjugate system on \(x_0\), \(x\) is isothermic-conjugate on \(x\) (Darboux), then from the Gauß equations we obtain immediately that the second fundamental form of \(x\) is \(\frac{\lambda u(du^2 - dv^2)}{\sqrt{H}}\); everything else except \(\mu_u\), \(\lambda_u\), \(H = \mu - \lambda^2\) follows immediately as previously. The remaining needed information follows immediately from the CMP equations of \(x\):

\[
d(\log \frac{u}{\sqrt{H}}) = (\Gamma_{12}^2 + \Gamma_{11}^2)du + (\Gamma_{12}^1 + \Gamma_{11}^1)dv \text{ become } d(\log \frac{\mu^2 - \lambda^2}{H}) = 0; \text{ by a same homothety in the}
\]

\[(u, v) \text{ variables and a choice of sign we can assume } \mu^2 - \lambda^2 = 1.
\]

By similar computations if \(x \subset \mathbb{R}^3\) is a real deformation of an imaginary region \(x_0\) (that is \(\epsilon = -1\)), then with \(\lambda := \pm \sqrt{\alpha_1^2 + \beta_1^2}\), \(\alpha := \pm \sqrt{\alpha_2^2 + \beta_2^2}\) (the signs may vary when we shall consider the B transformation) the second fundamental form of \(x\) is \(\frac{\lambda(x u^2 - dv^2)}{\sqrt{H}}\), \(H = \mu^2 + \lambda^2\) and we are led to consider the hyperbolic sine-Gordon equation \((1)\) as the compatibility condition of the completely integrable linear system in \(\alpha, \beta, \lambda, \mu\):

\[
\begin{pmatrix}
\alpha \\
\beta \\
\lambda \\
\mu
\end{pmatrix}
\begin{pmatrix}
\lambda C du - \mu S dv \\
\lambda S du + \mu C dv \\
-\lambda \omega_u du + (S_{\alpha_1} - C_{\alpha_2} + \lambda \omega_u) dv \\
\lambda \omega_v du + (S_{\alpha_1} - C_{\alpha_2} + \lambda \omega_v) dv
\end{pmatrix} = 0, \quad \mu^2 + \lambda^2 = H.
\]

One can put everything in a matrix notation: with \(R := \begin{bmatrix} C & \omega S \\
S & C \end{bmatrix}, \delta := \text{diag}[du \ dv], V := [\alpha \beta]^T, \Lambda := [\lambda \mu]^T, A' := \text{diag}[a_1^{-1} \ a_2^{-1}], \Omega := R^{-1} R_u dv + R^{-1} R_v du, \mathcal{E} := \text{diag}[1 - \epsilon]\) we have the sine(sinh)-Gordon equation

\[
d \wedge \Omega - \delta R^{-1} A' \wedge R \delta, \quad R^{-1} dR \wedge \delta - \delta \wedge \Omega = 0 \Leftrightarrow
\]

\[
e_{1}^{T} [(R^{-1} R_u)_{u} - (R^{-1} R_v)_{v} + R^{-1} A'R]_{2} = 0
\]

as the compatibility condition for the completely integrable linear system

\[
d \begin{bmatrix} V \\
A
\end{bmatrix} = \begin{bmatrix} 0 \\
-\delta R^{-1} A' \\
\Omega
\end{bmatrix}, \quad A^T \mathcal{E} A = -V^T \mathcal{E} A' V - 1.
\]

With \(R' := I_{2} - z A'\) consider two points \(x_{0}^{0}, x_{1}^{0} \in x_{0}\) in the same totally real region of \(x_{0}\) (corresponding to \(\epsilon = \pm 1\)) such that \(x_{0}^{0}, x_{1}^{0}\) are in the symmetric tangency configuration

\[
x_{1}^{0} \in T_{x_{0}^{0}} x_{0} \Leftrightarrow x_{1}^{0} = x_{0}^{0} + [x_{0}^{0} x_{1}^{0}]_{0} (\sqrt{R_{x} V_{1}} - V_{0}) \Leftrightarrow
\]

\[
(\sqrt{R_{x} V_{1}} - V_{0})^{T} \mathcal{E} (\sqrt{R_{x} V_{1}} - V_{0}) = -\epsilon z H_{1}, \quad 0 \Leftrightarrow 1.
\]

Thus a quadratic functional relationship is established between \(\alpha_0, \beta_0, \alpha_1, \beta_1\) and only three among them remain functionally independent:

\[
d V_{0}^{2} T \mathcal{E} (\sqrt{R_{x} V_{1}} - V_{0}) = -d V_{1}^{2} T \mathcal{E} (\sqrt{R_{x} V_{0}} - V_{1}).
\]

Bianchi’s main theorem on the theory of deformations of quadrics states in (our case) that given the seed deformation \(x^{0} \subset \mathbb{R}^3\) of the totally real region \(x_{0}^{0} \subset x_{0}\) (that is \(|dx^{0}|^2 = |dx^{0}_{0}|^2\) the differential system

\[
x^{1} = x^{0} + [x_{0}^{0} x_{0}^{0}]_{0} (\sqrt{R_{x} V_{1}} - V_{0}), \quad |dx^{1}|^2 = |dx^{0}|^2
\]

obtained by imposing the ACPIA \(|dx^{1}|^2 = |dx^{0}_{1}|^2\) is completely integrable (thus it admits two 1-dimensional family of solutions (leaves) \(x^{1} = B_{x, \epsilon_{1}}(x^{0}) \subset \mathbb{R}^3\), \(\epsilon_{1} = \pm 1\) whose determination requires the integration of a Ricatti equation), that \(x^{0}, x^{1}\) are the focal surfaces of a Weingarten
congruence (congruence of lines on whose two focal surfaces the asymptotic directions correspond; since conjugate directions are harmonically conjugate to the asymptotic ones all conjugate systems correspond in this case), that $x^1$ is applicable to $x_0^0$ (in our case of the same totally real region of $x_0$ as $x_0^0$) and that we have the symmetry $0 \leftrightarrow 1$. Its simplest proof uses parametrization by rulings, since they behave well with respect to the metric properties of the Ivory affinity and for the particular configuration $x^0 = x_0^0$ the leaves $x^1$ become rulings on $x_z$; it appears elsewhere so we shall not insist on it here.

Using (15) we get by differentiating (16)

$$dx^1 = -cdV_1^T \mathcal{E}(\sqrt{R_z^2}V_0 - V_1)\mathcal{V}^0 + [x_0^0 \ x_0^0] \sqrt{R_z^2}dV_1$$

$$+N^0(N^0)^T [dx_{\alpha_0}^0 \ dx_{\beta_0}^0](\sqrt{R_z^2}V_1 - V_0), \quad N^0 := \frac{x_0^0 \times x_{\beta_0}^0}{\sqrt{-a_1a_2H_0}},$$

so the ACPIA becomes

$$(N^0)^T [dx_{\alpha_0}^0 \ dx_{\beta_0}^0](\sqrt{R_z^2}V_1 - V_0) = \frac{\epsilon_1}{\sqrt{H_0}}dV_1^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\sqrt{R_z^2}V_1 - V_0),$$

(17)

$$\epsilon_0 = \epsilon_1 = \pm 1, \ 0 \leftrightarrow 1.$$

Using \[ \begin{pmatrix} \frac{\partial}{\partial \alpha_0} \\ \frac{\partial}{\partial \beta_0} \end{pmatrix} = \begin{pmatrix} C_0 & -S_0 \\ S_0 & C_0 \end{pmatrix} \frac{\partial}{\partial \phi}, \ \begin{pmatrix} \frac{\partial}{\partial \alpha_0} \\ \frac{\partial}{\partial \beta_0} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \alpha_0} \\ \frac{\partial}{\partial \beta_0} \end{pmatrix}, \] (15) and (17) we get with $r_j := \sqrt{1 - za_j^2}$, $j = 1, 2$:

$$\mu_0[(r_1 \alpha_1 - \alpha_0)C_0 - \epsilon(r_2 \beta_1 - \beta_0)S_0] = \epsilon_1[(r_2 \beta_0 - \beta_1)\alpha_{1u} - (r_1 \alpha_0 - \alpha_1)\beta_{1u}],$$

$$\lambda_0[(r_2 \beta_1 - \beta_0)C_0 - (r_1 \alpha_1 - \alpha_0)S_0] = \epsilon_1[(r_1 \alpha_0 - \alpha_1)\beta_{1u} - (r_2 \beta_0 - \beta_1)\alpha_{1u}],$$

$$\lambda_0[(r_1 \alpha_1 - \alpha_0)C_0 - (r_1 \alpha_1 - \alpha_0)S_0] = -(r_2 \beta_0 - \beta_1)\beta_{1u} - (r_2 \beta_0 - \beta_1)\alpha_{1u}].$$

If these equations are $I - IV$ and using $(r_2 \beta_1 - \beta_0)^2 - \epsilon(r_1 \alpha_1 - \alpha_0)^2 = zH_1 > 0$, then by considering $I^2 - \epsilon III^2$, $IV^2 - \epsilon II^2$, $I \cdot II - III \cdot IV$ we obtain $\alpha_{1u}^2 - \epsilon \beta_{1u}^2 > 0$, $\beta_{1u}^2 - \epsilon \alpha_{1u}^2 > 0$, $\alpha_{1u}^2 - \epsilon \beta_{1u}^2 = 0$, so

$$\begin{pmatrix} \alpha_{1u} & \alpha_{1v} \\ \beta_{1u} & \beta_{1v} \end{pmatrix} = \begin{pmatrix} \lambda_1C_1 & \epsilon_1 \mu_1 S_1 \\ \lambda_1 S_1 & \mu_1 C_1 \end{pmatrix}.$$  Because of the symmetry $0 \leftrightarrow 1$ and using $I$, $II$ we obtain that the second fundamental form of $x^1$ is $\frac{\lambda_1 \mu_1 (da^2 - dc^2)}{\sqrt{H_1}}$, so $(u, v)$ is also isothermic-conjugate on $x^1$ and the B transformation preserves the orientation of $(u, v)$; from $II^2 - \epsilon III^2$ we obtain $\mu_1^2 - \epsilon \lambda_1^2 = H_1$ and we have complete symmetry $0 \leftrightarrow 1$ also at the level of the isothermic-conjugate system $(u, v)$. Thus

$$\alpha_1 = r_1 \alpha_0 - \epsilon_1 \sqrt{\epsilon}(\epsilon C_1 \lambda_0 - \epsilon_1 S_1 \mu_0), \quad \beta_1 = r_2 \beta_0 - \epsilon_1 \sqrt{\epsilon}(S_1 \lambda_0 - \epsilon_1 C_1 \mu_0), \quad \epsilon_0 = -\epsilon_1, \ 0 \leftrightarrow 1,$$

(18)

$$\begin{pmatrix} V_1 \\ \Lambda_1 \end{pmatrix} = \epsilon_1 \sqrt{\epsilon} \begin{pmatrix} I_2 & 0 \\ 0 & \epsilon_1 R_0^{-1} \end{pmatrix} \begin{pmatrix} D & -I_2 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & R_1 \end{pmatrix} \begin{pmatrix} V_0 \\ \Lambda_0 \end{pmatrix}, \quad D := \sqrt{\frac{R_z^2}{\epsilon_1^2 \epsilon}}, \ 0 \leftrightarrow 1.$$

Finally differentiating $V_1$ we obtain the B transformation at the analytic level

$$dR_1 \Lambda_1 = -R_1 \epsilon_1 \Omega_0 - R_1 \epsilon_1 \delta R_0^{-1} DR_1 \epsilon_1 + DR_0 \delta;$$

with diag[\(\frac{\sigma_1 - \sigma_2}{2} \quad \frac{\sigma_1 + \sigma_2}{2}\)] := $-D \mathcal{E}$, this becomes the B transformations [2] respectively [3] between solutions $\omega_0$, $\epsilon_1 \omega_1$ of (11) and respectively $\theta_0$, $\epsilon_1 \theta_1$ of (11).

The BPT states that if $x^1 = B_{z,\epsilon_1}(x^0)$, $j = 1, 2$, then one can find only by algebraic computations and derivatives a surface $x^3$ such that $B_{z_2,\epsilon_2}(x^1) = x^3 = B_{z_1,\epsilon_1}(x^2)$, that is $B_{z_1,\epsilon_1} \circ B_{z_2,\epsilon_2} = B_{z_2,\epsilon_2} \circ B_{z_1,\epsilon_1}$. Again one can derive the analytic BPT for the B transformation of the hyperbolic sine(sinh)-Gordon equation from the geometric picture, just as we did for the B transformation itself, but we take advantage of the already completed algebraic transformation of solutions to derive the BPT at the analytic level, following that we shall then use these analytic computations to get the geometric realization of solutions in space.
We have
\[
\begin{pmatrix}
I_2 & 0 & 0 & I_2 \\
0 & E_2 R_1 & 0 & 0 \\
D_2 & -I_2 & I_2 & D_2 \\
I_2 & 0 & 0 & I_2
\end{pmatrix}
= \begin{pmatrix}
I_2 & 0 & 0 & I_2 \\
D_1 & -I_2 & I_2 & D_1 \\
0 & R_1 E_2 R_0^{-1} & 0 & 0 \\
D_2 & -I_2 & I_2 & D_2
\end{pmatrix}
\]
\[
= \begin{pmatrix}
I_2 & 0 & 0 & I_2 \\
0 & E_1 R_2^{-1} & 0 & 0 \\
D_1 & -I_2 & I_2 & D_1 \\
0 & R_2 E_1 R_0^{-1} & 0 & 0
\end{pmatrix}
\]
\[
R_3 E_2 E_0 R_0^{-1} = (D_1 R_2 E_1 E_2 R_1 - D_2)(D_1 - D_2 R_2 E_2 R_1 R_1^{-1})^{-1};
\]

again this is just (3) and (5) and for \(\epsilon_1 \epsilon_2 \omega_3\) (resp \(\epsilon_1 \epsilon_2 \theta_3\)) we have only two cases \(\epsilon_1 = \pm \epsilon_2\) in what concerns the dependence on \(\epsilon_1\), \(\epsilon_2\) because the BPT requires a \(\mathbb{Z}_2\) co-cycle condition on these signs.

We have the space realization
\[
x_3 = x^0 - \epsilon(\sqrt{R_2} V_3 - V_1) T \mathcal{E}(\sqrt{R_2} V_0 - V_1) V_0 + \sqrt{R_1 \epsilon}_0 (\sqrt{R_2} V_3 - V_1) V_0 \]

For the \(M_3\) configuration consider \((D_1 D_3)^{-1} [(D_2 D_3)] D_1 R_1 E_1 (D_1 D_3) - D_2 R_2 E_2 - D_3 R_3 E_3^{-1} - D_3 E_3^2 = R_3 E_3 E_2 R_0^{-1} - R_1 E_1 E_2 E_0^{-1} - D_1 D_3^2 \]

Thus we have \(D_1 R_2 E_2 E_0 R_0^{-1} - D_3 R_3 E_3 E_0^{-1} - D_3 R_3 E_3^{-1} - D_3 E_3^2 = (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1}
\]

Similarly \((D_3 R_3 E_3 R_3^{-1} - D_1 E_1 E_2 E_0^{-1} - D_1 E_1 E_2 E_0^{-1} - D_1 E_1 E_2 E_0^{-1} = (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1} - (D_3 - D_3^2) (D_1 R_1 E_1 - D_2 R_2 E_2) E_0^{-1}
\]

Finally to get the first three iterations of the B transformation for Peterson’s real deformations of totally real regions of the real hyperbolic paraboloid it is enough to give only \(\alpha_0, \beta_0, \mu_0, \mu_0\) and the space realization of \(x_0\); everything else will follow according to the established algebraic formulae.

For \(\epsilon = 1\) and \(\theta_0 = 0\) we have \(\alpha_0 = \alpha_0(u), \lambda_0 = \lambda_0(u) = \alpha_0'(u), \beta_0 = \beta_0(u), \mu_0 = \mu_0(u) = \beta_0'(u), \mu_0^2 + \frac{\alpha_0^2}{\alpha_0} = \lambda_0^2 + \frac{\alpha_0^2}{\alpha_0} + 1, \) so \(\alpha_0 = \sqrt{\alpha_0} \sinh s \sin \frac{u}{\sqrt{\alpha_0}}, \beta_0 = -\alpha_0 \cosh s \sin \frac{u}{\sqrt{\alpha_0}}\) and have

Peterson’s 1-dimensional family
\[
x_3(x_0, \alpha_0, \beta_0) := \int_0^{\mathfrak{a}_0} \sqrt{a_1 - \frac{\epsilon^2}{\sinh^2 s}} \, dt
\]

For \(\epsilon = -1\) and \(\omega_0 = 0\) we have \(\alpha_0 = \alpha_0(u), \lambda_0 = \lambda_0(u) = \alpha_0'(u), \beta_0 = \beta_0(u), \mu_0 = \mu_0(u) = \beta_0'(u), \) so \(\alpha_0 = \sqrt{\alpha_0} \sinh s \sin \frac{u}{\sqrt{\alpha_0}}, \beta_0 = -\alpha_0 \cosh s \sin \frac{u}{\sqrt{\alpha_0}}\) and have

Note that one can begin with different Peterson’s deformations as seed: we take the pencil of planes passing through the \(e_3\) axis; the other pencil of planes will be planes perpendicular on the \(e_3\) axis.

Thus for \(\epsilon = 1\) we have
\[x_3 = x_0(\alpha, \beta) := \left[\sqrt{\alpha_0} \epsilon \cosh \beta \sqrt{-a_2 \epsilon^2 \sinh \beta \frac{\alpha_0}{\alpha_0}} \right] \text{ with linear element, second fundamental form and Christoffel symbols}\]
\(|dx_0|^2 = e^{2\alpha}[(a_1 - a_2)(\cosh \beta d\alpha + \sinh \beta d\beta)^2 + a_2(d\alpha^2 - d\beta^2) + e^{2\alpha} d\alpha^2],\quad N_0 d^2 x_0 = \frac{-d\alpha^2 + d\beta^2}{e^{-\beta} \sqrt{H}}, \quad \vec{H} := \sinh^2 \beta + a_1^{-1} + e^{-2\alpha}, \quad \vec{G}_1 := 0, \quad \vec{G}_2 := 1, \quad 2 - \vec{G}_1 = \vec{G}_2 = -(\log \sqrt{H})_{\vec{a}}, \quad -\vec{G}_1 = \vec{G}_2 = (\log \sqrt{H})_{\vec{b}}.

Again with \(\bar{\lambda} := \text{sgn}(\bar{\alpha})\sqrt{\bar{a}_u^2 - \bar{b}_u^2}, \quad \bar{\mu} := \text{sgn}(\bar{\beta})\sqrt{\bar{b}_u^2 - \bar{a}_u^2}, \quad \begin{bmatrix} \bar{\alpha}_u & \bar{\alpha}_v \\ \bar{\beta}_u & \bar{\beta}_v \end{bmatrix} = \begin{bmatrix} \lambda C & \bar{\mu} \bar{S} \\ \lambda S & \bar{\mu} C \end{bmatrix}, \quad \bar{C} := \cosh \theta, \quad \bar{S} := \sinh \theta \quad \text{we have} (\mu^2, \lambda^2, H) = e^{2\bar{\alpha}}(\bar{\mu}^2, \lambda^2, \bar{H}), \quad \text{so} \quad \bar{\mu}^2 - \bar{\lambda}^2 = H, \quad \bar{h}_{11} = \frac{\bar{\mu}^2}{e^{-2\beta}} \quad \bar{h}_{12} = 0, \quad \bar{h}_{22} = -\frac{\bar{\mu}^2}{e^{-2\beta}} \quad \text{and we have} \quad \bar{G}_1 = (\log e^\bar{\lambda})_{\bar{v}}, \quad \bar{G}_2 = (\log e^\bar{\mu})_u, \quad \bar{G}_1 = \frac{\bar{\mu}^2}{\bar{\mu}^2 - \bar{\lambda}^2} - (\log \frac{\bar{\lambda}}{\bar{\mu}})_v, \quad \bar{G}_2 = \frac{\bar{\mu}^2}{\bar{\mu}^2 - \bar{\lambda}^2} - (\log \frac{\bar{\lambda}}{\bar{\mu}})_u.

Thus we get the differential system
\[
(20) \quad d \begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \\ \bar{\mu} \end{bmatrix} = \begin{bmatrix} \lambda C du + \bar{\mu} \bar{S} dv \\ \lambda S du + \bar{\mu} C dv \\ \bar{C} \bar{\lambda} du + (\bar{S} e^{-2\beta} + \bar{C} \sinh \beta \cosh \beta + \bar{\lambda} \bar{\theta}_u) dv \end{bmatrix}, \quad \bar{\mu}^2 - \bar{\lambda}^2 = H
\]

with compatibility condition
\[
(21) \quad \bar{\theta}_{vv} - \bar{\theta}_{uu} = (\cosh 2\beta + 2e^{-2\alpha})\bar{S} \bar{C}.
\]

Note that we need further manipulation, since the dependence of \(\bar{\alpha}, \bar{\beta} \) on \((u, v)\) in \((21)\) is a-priori undetermined, but that it is not our interest right now, since we are interested only in the solution \(\theta = 0\). In this case we have \(\bar{\alpha} = \bar{\alpha}(u), \quad \bar{\lambda} = \bar{\lambda}(u) = \bar{\alpha}'(u), \quad \bar{\beta} = \bar{\beta}(v), \quad \bar{\mu} = \bar{\mu}(v) = \bar{\beta}'(v), \quad \bar{\beta}^2(v) - \sinh^2 \beta(v) = \bar{\alpha}^2(u) + e^{-2\beta}(u) + a_1^{-1} = c.

3.2. Imaginary deformations of (the imaginary region of) the real hyperbolic paraboloid.

In this case \(x^1\) will be applicable to \(x_0^1\) of a different totally real region of \(x_0\) as \(x_0\), so the confocal \(x_1^2\) will change type from a real metric point of view. Thus we are led to consider the elliptic paraboloids
\[
x_z = x_z(\alpha, \beta) := \sqrt{a_1 - z\alpha} \quad \sqrt{a_2 - z\sqrt{\epsilon} \beta} \quad \frac{\alpha^2 + \epsilon \beta^2 + z}{2}, \quad a_2 > z
\]

(the case \(z > a_1\) is realized by a rigid motion \((e_1, e_3) \leftrightarrow (e_2, -e_3)\) and we have imaginary region for \(\epsilon = -1\) confocal to the given hyperbolic one \(x_0\). The Ivory affinity
\[
x_0 \mapsto \text{diag}(\sqrt{1 - z_{a_1}^{-1}}, \sqrt{1 - z_{a_2}^{-1}}) x_0 + \tilde{z} = \sqrt{a_1 - z\alpha} \quad \sqrt{a_2 - z\sqrt{\epsilon} \beta} \quad \frac{\alpha^2 - \epsilon \beta^2 + z^2}{2} \quad \text{takes in this case the real (imaginary) region of } x_0 \text{ to the imaginary (real) one of } x_z.
\]

If \(x \subset \mathbb{R}^2 \times i \mathbb{R}\) is an imaginary deformation of a totally real region \(\subset x_0\), then by similar computations its second fundamental form is \(i\frac{\lambda \eta(d\alpha^2 + d\beta^2)}{\sqrt{H}}\), \(H = \mu^2 + \epsilon \lambda^2\) and we are led to consider the elliptic sinh-Gordon equation \((\bar{S})\) as the compatibility condition of the completely integrable linear system in \(\alpha, \beta, \lambda, \mu, \epsilon:\

\[
(22) \quad d \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda C du + \mu S dv \\ \lambda S du + \mu C dv \\ (C \lambda a_1 - S \lambda a_2 - \mu \lambda u) du + \mu \lambda u dv \\ \lambda \mu u dv + (S \lambda a_1 - C \lambda a_2 - \mu \lambda u) dv \end{bmatrix}, \quad \mu^2 + \lambda^2 = H
\]

for \(\epsilon = 1\) and the elliptic sine-Gordon equation \((\bar{S})\) as the compatibility condition of the completely integrable linear system in \(\alpha, \beta, \lambda, \mu; \epsilon:\

\[
(23) \quad d \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \lambda C du - \mu S dv \\ \lambda S du + \mu C dv \\ (C \lambda a_1 + S \lambda a_2 + \mu \omega u) du - \mu \omega u dv \\ \lambda \omega u dv + (S \lambda a_1 - C \lambda a_2 + \mu \omega u) dv \end{bmatrix}, \quad \mu^2 - \lambda^2 = H
\]

for \(\epsilon = -1\).
With $r_1 := \sqrt{1 - za^2}$, $r_2 := \sqrt{1 + za^2}$ we have

$$x^1 = x^0_1 + (r_1 \alpha_1 - \alpha_0)x^0_{\alpha_0} + (- \epsilon_2 \beta_1 - \beta_0)x^0_{\beta_0}, \quad x^1 = x^0_1 + (r_1 \alpha_1 - \alpha_0)x^0_{\alpha_0} + (- \epsilon_2 \beta_1 - \beta_0)x^0_{\beta_0},$$

$$x^1 = - \epsilon (r_1 \alpha_0 - \alpha_1)(\frac{\alpha_0}{a_1 H_0} x^0_{\alpha_0} + \frac{\beta_0}{a_2 H_0} x^0_{\beta_0}) + r_1 x^0_{\alpha_0} + i \frac{1}{\sqrt{H_0}} N^0,$$

$$x^1 = - \epsilon (r_2 \beta_0 - \beta_1)(\frac{\alpha_0}{a_1 H_0} x^0_{\alpha_0} + \frac{\beta_0}{a_2 H_0} x^0_{\beta_0}) - \epsilon_2 x^0_{\beta_0} - i \frac{1}{\sqrt{H_0}} N^0,$$

$$N^0 := \frac{1}{\sqrt{-a_1 a_2 H_0}} \subset (i \mathbb{R})^2 \times \mathbb{R}, \quad \epsilon_0 = \epsilon_1 = \pm 1, \quad (0, \epsilon) \leftrightarrow (1, -\epsilon).$$

We get $(r_1 \alpha_1 - \alpha_0)(N^0)^T dx^0_{\alpha_0} + (- \epsilon_2 \beta_1 - \beta_0)(N^0)^T dx^0_{\beta_0} = \frac{i}{\sqrt{H_0}} \epsilon_1 (r_2 \beta_0 - \beta_1) d\beta_1 + (r_1 \alpha_0 - \alpha_1) d\alpha_1$, or using $\partial_0 = \partial_0^0 \partial_0^1 - \partial_0^1 \partial_0^0$, $\partial_0 = \epsilon \partial_0^0 \partial_0^1 + \partial_0^1 \partial_0^0$, $- \epsilon (r_1 \alpha_0 - \alpha_1) d\alpha_0 + (- \epsilon_2 \beta_1 - \beta_0) d\beta_0 = (r_1 \alpha_0 - \alpha_1) d\alpha_1 + (r_2 \beta_0 - \beta_1) d\beta_1$.

If these equations are $I - IV$ and using $(\epsilon_2 \beta_0 - \beta_1)^2 + (r_1 \alpha_0 - \alpha_1)^2 = -z H_0 > 0$, then by considering $I^2 + c I^3$ and $IV^2 + c IV^3$, we obtain $\alpha^2_{1uv} + \beta^2_{1uv} > 0$, $\alpha^2_{1uv} + \beta^2_{1uv} > 0$, $\alpha^2_{1uv} + \beta^2_{1uv} = 0$.

Finally differentiating these with respect to $u, v$ we obtain that the B transformation [2] between solutions $\omega_0, \epsilon_1 \theta_1$ and respectively $\theta_0, \epsilon_1 \omega_1$ of [1] and [3] has an influence the algebraic transformations of solutions of [2] respectively [2] as follows:

3.3. Real deformations of (the imaginary region of) the real elliptic paraboloid.

3.4. Imaginary deformations of (the imaginary region of) the real elliptic paraboloid.

3.5. Real deformations of (the imaginary region of) the real hyperboloid with one sheet.

Consider the general confocal real hyperboloids with one sheet in an isothermic-conjugate parametrization invariant under the Ivory affinity between confocal quadrics

$$x_z = x_z(\alpha, \beta) := [\sqrt{a_1 - z \cos \beta \sec \alpha} \sqrt{a_2 - z \sin \beta \sec \alpha} \sqrt{-a_3 + z \tan \alpha}]^T, \quad a_1 > a_2 > z > 0 > a_3,$$

$$\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \beta \in \mathbb{R}$$

with linear element, second fundamental form and Christoffel symbols of $x_0$

$$|dx_0|^2 = a_1 (d(\cos \beta \sec \alpha))^2 + a_2 (d(\sin \beta \sec \alpha))^2 - a_3 (d(\tan \alpha))^2;$$

$$N_0^T d^2 x_0 = \frac{\partial^2 - \beta^2}{\cos \alpha \sqrt{H}}, \quad H := a_1^{-1} \cos^2 \beta + a_2^{-1} \sin^2 \beta - a_3^{-1} \sin^2 \alpha;$$

$$-\Gamma_{12}^1 = \Gamma_{11}^1 - 2 \tan \alpha = (\log \sqrt{H})_\alpha, \quad -\Gamma_{22}^2 = \Gamma_{22}^2 = (\log \sqrt{H})_\beta, \quad \Gamma_{12}^1 = \tan \alpha, \quad \Gamma_{12}^1 = 0.$$
We also have a distinguished tangent vector field $\mathbf{v}_0 := ((\log \sqrt{H})_\alpha + \tan \alpha) x_{0\alpha} - (\log \sqrt{H})_\beta x_{0\beta}$; it has the properties

$$dx_{0\alpha} = \mathbf{v}_0 \, d\alpha - N_0 dN_0^T x_{0\alpha} - \tan \alpha \, dx_0,$$

$$dx_{0\beta} = -\mathbf{v}_0 d\beta - N_0 dN_0^T x_{0\beta} + \tan \alpha (x_{0\beta} d\alpha + x_{0\alpha} d\beta).$$

(25)

Given a real deformation $x \subset \mathbb{R}^3$ of a real region $x_0$ there exists a conjugate system $(u, v)$ common to both $x_0$ and $x$ (this is true for any two surfaces in a point-wise correspondence). Denote with $\bar{z}$ the quantities of interest in the GCMP equations (namely the Christoffel symbols and the second fundamental form) of $x_0$ referred to the $(u, v)$ coordinates and similarly with $\tilde{z}$ those of $x$. We have $\alpha_u \alpha_v - \beta_u \beta_v = 0$ and from the Gauß equation $(\alpha^2_u - \beta^2_u)(\alpha^2_v - \beta^2_v) < 0$; after $\alpha_u^2 - \beta_u^2 > 0$. With $\lambda := \text{sgn}(\alpha_u) \sqrt{\alpha^2_u - \beta^2_u}$, $\mu := \text{sgn}(\alpha_v) \sqrt{\beta^2_v - \alpha^2_v}$ we have $\bar{h}_{11} = \frac{x^2}{\cos \alpha \sqrt{H}}$, $\bar{h}_{12} = 0$, $\bar{h}_{22} = -\frac{\mu^2}{\cos \alpha \sqrt{H}}$

From the general formula for the change of Christoffel symbols $\frac{\partial^2 \tilde{\Gamma}_c}{\partial u \partial v} = \frac{\partial^2 \bar{u}_c}{\partial \bar{u}_a \partial \bar{u}_b} + \frac{\partial \bar{u}_a}{\partial u} \frac{\partial \bar{u}_b}{\partial v} \bar{\Gamma}_{jk}$ we are interested only in $\bar{\Gamma}_1^{12}$, $\bar{\Gamma}_1^{22}$: $\alpha_{uv} + 2 \tan \alpha_{u\alpha} \alpha_{v\beta} = \bar{\Gamma}_1^{12} \alpha_u + \bar{\Gamma}_1^{22} \alpha_v$, $\beta_{uv} + \tan \alpha (\alpha_v \beta_u + \alpha_u \beta_v) = \bar{\Gamma}_1^{12} \beta_u + \bar{\Gamma}_2^{12} \beta_v$, so $\bar{\Gamma}_1^{12} = (\log \frac{\alpha_v}{\cos \alpha})_u$, $\bar{\Gamma}_2^{12} = (\log \frac{\beta_u}{\cos \alpha})_u$. From the CMP equations of $x_0$, $x$ we have

\[
(\bar{h}_{11})_v = \bar{\Gamma}_1^{12} \bar{h}_{11} - \bar{\Gamma}_1^{22} \bar{h}_{22},

(\bar{h}_{12})_u = \bar{\Gamma}_2^{12} \bar{h}_{11} - \bar{\Gamma}_2^{22} \bar{h}_{22},

(\bar{h}_{22})_u = \bar{\Gamma}_2^{22} \bar{h}_{11},
\]

Keeping account of the Gauß equation $\bar{h}_{11} \bar{h}_{22} = \bar{h}_{11} \bar{h}_{22}$ one can multiply the first equations respectively with $\bar{h}_{11}$, $\bar{h}_{22}$ (and the second equations respectively with $\bar{h}_{22}$, $\bar{h}_{11}$), subtract them and get rid respectively of the $\bar{\Gamma}_1^{12}$, $\bar{\Gamma}_1^{22}$ terms: $(\log(\bar{h}_{11}^{-1} - \bar{h}_{11}))_v = 2(\log \frac{x^2}{\cos \alpha \sqrt{H}})_v$, $(\log(\bar{h}_{22}^{-1} - \bar{h}_{22}))_u = 2(\log \frac{\mu^2}{\cos \alpha \sqrt{H}})_u$.

Thus $\bar{h}_{11} = \log \frac{x^2}{\cos \alpha \sqrt{H}}$, $\bar{h}_{22} = \log \frac{\mu^2}{\cos \alpha \sqrt{H}}$; after a change of the $u$ and $v$ variables one can absorb $\phi(u), \varphi(v)$ up to opposite signs $\epsilon_1 := \pm 1$: $\bar{h}_{11} = \epsilon_1 \log \frac{x^2}{\cos \alpha}, \bar{h}_{22} = \epsilon_1 \log \frac{\mu^2}{\cos \alpha}$ (again from the Gauß equation we have $\bar{h}_{11} = \frac{\mu^2}{\cos \alpha}$). Now from the Gauß equation $\bar{h}_{11} \bar{h}_{22} = \bar{h}_{11} \bar{h}_{22}$ we get $H = -\epsilon_1 \mu^2 + \epsilon_1 \lambda^2$; we can choose $\epsilon_1 := -1$. Thus the second fundamental form of $x$ is $\frac{\lambda \mu (d^2 - d\lambda^2)}{\cos \alpha \sqrt{H}}$ and we are led to consider the hyperbolic angle $\theta$ between the conjugate systems $(\alpha, \beta)$ and $(u, v)$, that is $\begin{bmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{bmatrix} = \begin{bmatrix} \lambda C & \mu S \\ \lambda S & \mu C \end{bmatrix}$, $C := \cosh \theta$, $S := \sinh \theta$. Note (by doing this the sign of $\theta$ will be decided by that of $\beta_u$). Imposing the compatibility conditions $(\lambda C)_v = (\mu S)_u$, $(\lambda S)_v = (\mu C)_u$ we get $\mu_u = \lambda \theta_u$, $\lambda_u = \mu \theta_u$; differentiating $H = \mu^2 - \lambda^2$ with respect to $u$, respectively $v$ we are led to consider the differential system in $\alpha, \beta, \lambda, \mu$:

(26)

$$d \begin{bmatrix} \alpha \\ \beta \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} C \lambda \mu u + S \mu dv \\ S \lambda \mu u + C \mu dv \\ -\frac{1}{2} H_\alpha C - \frac{1}{2} H_\beta S + \mu \theta_u [du + \mu \theta_u dv] \\ \lambda \theta_u du + [\frac{1}{2} H_\alpha S + \frac{1}{2} H_\beta C + \lambda \theta_u] dv \end{bmatrix}, \mu^2 - \lambda^2 = H$$

with the modified hyperbolic sinh-Gordon equation

(27)

$$\theta_{uv} - \theta_{uu} = \frac{1}{2} (H_{\alpha\alpha} + H_{\beta\beta}) CS$$

as the compatibility condition. As opposed to the case of paraboloids, where $H$ was quadratic, has the inconvenience of depending on $\alpha, \beta$ and the dependence of $\alpha, \beta$ on $(u, v)$ being undetermined, so further manipulation is required to obtain an equation depending on $\theta$ only (it will be a third or fourth order differential equation), but it is not our interest to do that now; note also that its B transformation will reveal itself later.

Note that if we assume the common conjugate system on $x_0$, $x$ is isothermic-conjugate on $x$ (Darboux), then from the Gauß equations we obtain immediately that the second fundamental form of $x$ is $\frac{\lambda \mu (d^2 - d\lambda^2)}{\cos \alpha \sqrt{H}}$, everything else except $\mu_v$, $\lambda_u$, $H = \mu^2 - \lambda^2$ follows immediately as previously. The remaining needed information follows immediately from the CMP equations of $x$: we have $\bar{\Gamma}_1^{12} = (\log \frac{\lambda}{\cos \alpha})_v$, $\bar{\Gamma}_2^{12} = \frac{\mu}{\lambda} \theta_v - \frac{\lambda}{\mu} (\log \sqrt{H})_u$, $\bar{\Gamma}_1^{22} = \frac{\lambda}{\mu} \theta_u - \frac{\lambda}{\mu} (\log \sqrt{H})_v$, $\bar{\Gamma}_2^{22} = (\log \frac{\mu}{\cos \alpha})_u$ and
\[
(\log \frac{\mu}{\cos \sqrt{\nu}}) \nu = \Gamma_{12}^1 + \Gamma_{11}^1, \quad (\log \frac{\mu}{\cos \sqrt{\nu}}) \mu = \Gamma_{12}^1 + \Gamma_{12}^2 \text{ respectively become } \mu_{\nu} = \frac{a^2 - \lambda^2}{\lambda} (\log \sqrt{H}) \nu + \lambda \theta \mu, \quad \lambda \mu = - \frac{a^2 - \lambda^2}{\lambda} (\log \sqrt{H}) \mu + \mu \theta \nu, \text{ so } d \log \frac{e^{2 - \lambda^2}}{H} = 0; \text{ by a same homothety in the } (u, v) \text{ variables we and a choice of sign we can assume } a^2 - \lambda^2 = 1.
\]

Next we derive the algebraic computations of the \textit{tangency configuration} (TC) and of the B transformation.

Note
\[
(28) \quad - x_0 x_\alpha x_\beta T + x_\beta x_\alpha T + \sec^2 \alpha x_\beta x_\alpha T = \sec^2 \alpha \text{ diag }[a_1 - z \quad a_2 - z \quad a_3 - z].
\]

Consider two points \(x_0, x_0 \in x_0; \) with \(V_0^3 := x_1 - x_0, \text{ } N_0^3 := -2 \partial_2 z = 0 \) the relation
\[
(29) \quad (-x_{\alpha 0}^3 (x_{\alpha 0}^3 T + x_{\beta 0}^3 T N_0^3 = - \sec^2 \alpha [x_1 (V_0^3 T N_0^3 + (V_0^3 + z N_0^3)])
\]
follows immediately from (28).

Consider now the symmetric TC \(V_0^3 T N_0^3 = (V_0^3)^T T N_0^1 \) \(0 .\) Multiplying (29) on the left respectively with \((N_0^3)^T, (V_0^3 \times N_0^3)^T\) we get two algebraic consequences of the TC:
\[
- [(x_{\alpha 0}^3 T N_0^3)^2 + [(x_{\beta 0}^3 T N_0^3)^2 = - \zeta \sec^2 \alpha_1 = \frac{dx_1^2 - |dx_0^2|}{dx_1^2 + dx_3^2}
\]
\[
(x_{\alpha 1}^1 - x_{\beta 1}^1 T (I - 2 N_0^1 (N_0^3)^T)(x_{\alpha 0}^1 + x_{\beta 1}^1 T V_0^3)] = 0.
\]

Note that \(x_{\alpha 0} \pm x_{\beta 0} \) are rung on \(x_0 \) and thus their length is preserved under the Ivory affinity.

With \((R_0, t_0) \) being the \textit{rigid motion} provided by the \textit{Ivory affinity} (RMPIA) such that
\[(R_0, t_0)^3 \mu = x_0^3 x_0^1 x_0^3 x_0^2 x_0^3 x_0^1 x_1^0 x_0^0 - x_0^3 x_1^1 x_0^0 x_0^1 x_1^1 x_0^0 - x_0^3 x_1^1 x_0^0 x_0^1 x_1^1 x_0^0 - x_0^3 x_1^1 x_0^0 x_0^1 x_1^1 x_0^0 \text{ we have } x_0^3 x_0^3 x_0^1 x_0^0 - x_0^3 x_1^1 x_0^0 x_0^1 x_1^1 x_0^0 \text{ since by changing the ruling family on } x_1^1 \text{ the action of the RMPIA on } T_0 x_0 \text{ does not change.}
\]

Multiplying (29) on the left with \(R_0^3 \) and using (30) we obtain
\[
(31) \quad x_1^2 = x_0^2 + \cos^2 \alpha_0 (x_{\alpha 0}^3 T x_{\beta 0}^3 T + x_{\beta 0}^3 T x_{\beta 0}^3 T N_0^1, \text{ } 0) = 1.
\]

Differentiating the relation \((V_0^3)^T T N_0^3 = 0 \) we obtain
\[
(32) \quad (dx_0^3)^T T dN_0^3 = - (x_0^3)^T dN_0^3 = - (N_0^3)^T dN_0^1.
\]

Now Bianchi’s main theorem on the deformation of quadrics (the existence and inversion of the Bäcklund transformation and the ACPIA) states (in our case) that the TC coupled with
\[
(33) \quad x_1^1 = x_0^1 + \cos^2 \alpha_0 (x_{\alpha 0}^3 T x_{\beta 0}^3 T + x_{\beta 0}^3 T x_{\beta 0}^3 T N_0^1, |dx_1^2|^2 = |dx_0^2|^2
\]
is a differential system in involution (completely integrable) given the \textit{seed deformation} \(x_0^3 \subset \mathbb{R}^3 \) of \(x_0^3 \) (that is \(|dx_0^3|^2 = |dx_0^2|^2 \), that the 1-dimensional family of solutions (leaves) \(x_1^1 \) is given by the integration of a Riccati equation, that \(x_0^1 \) and \(x_1^1 \) are the focal surfaces of a Weingarten congruence (congruence of lines on whose two focal surfaces the asymptotic directions correspond; since conjugate directions are harmonically conjugate to the asymptotic ones all conjugate systems correspond in this case) and that we have the symmetry 0 \(\in 1 \).

If \((R_0, t_0) \subset O_3(\mathbb{R}) \times 3 \) is the rolling of \(x_0^3 \) on \(x_0^3 \) (that is \((R_0, t_0)^3 (x_0^3, dx_0^3) := (R_0 x_0^0 + t_0, R_0 dx_0) = (x_0^0, dx_0^3), N_0^3 := R_0 N_0^3, \) then \(R_0^T dR_0 N_0^3 = R_0^T dN_0^0 - dN_0^0, R_0^T dR_0 dR_0 V_0^1 = dR_0^3 + (V_0^3)^T R_0^T dR_0 N_0^3 = (I - N_0^3 (N_0^3)^T) dR_0^3 - N_0^3 (dN_0^0)^T (x_0^1 - x_0^3) \) and \(|dx_0^3|^2 = |dx_0^2|^2 \) becomes
\[
|(dN_0^0)^T (x_0^1 - x_0^0)|^2 = |dx_1^2|^2 - |dx_0^2|^2 + [(N_0^3)^T dR_0 x_0^1 T (x_0^1, dR_0 x_0^1), \text{ so}
\]
\[
(34) \quad (dN_0^0)^T (x_0^1 - x_0^0) = e_1 (N_0^3)^T (x_{\beta 0}^3 dR_0 1 + x_{\alpha 0}^3 dR_0 1), \quad e_1 := \pm 1
\]
(each choice of \(e_1 \) corresponds to a ruling family). Using
\[
\begin{bmatrix}
\frac{\partial}{\partial \mu_0} \\
\frac{\partial}{\partial \nu_0}
\end{bmatrix} = 
\begin{bmatrix}
\frac{\partial}{\partial \nu_0} - \frac{\partial}{\partial \mu_0} \\
\frac{\partial}{\partial \mu_0} - \frac{\partial}{\partial \nu_0}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial \mu_0} \\
\frac{\partial}{\partial \nu_0}
\end{bmatrix}, \quad (32), \quad (33)
\]
and (34) we get:
\[
\mu_0 \left[ \text{C}_0 (x_{\alpha 0}^3 T + S_0 (x_{\alpha 0}^3 T) \right] = e_1 [1 + \alpha_1 (x_{\alpha 0}^3 T + \beta_1 (x_{\alpha 0}^3 T) ] N_0^3,
\]
\[
\lambda_0 \left[ \text{C}_0 (x_{\beta 0}^3 T + S_0 (x_{\beta 0}^3 T) \right] = e_1 [1 + \beta_1 (x_{\alpha 0}^3 T + \alpha_1 (x_{\alpha 0}^3 T) ] N_0^0,
\]
\[
\lambda_0 \left[ \text{C}_0 (x_{\beta 0}^3 T + S_0 (x_{\beta 0}^3 T) \right] = e_1 [1 + \beta_1 (x_{\alpha 0}^3 T + \alpha_1 (x_{\alpha 0}^3 T) ] N_0^0,
\]
\[ \lambda_0 [C_0 (x_{z_0}^0)^T + S_0 (x_{z_0}^0)^T ] N_0^0 = -[\alpha_1 u (x_{z_0}^1)^T + \beta_1 u (x_{z_0}^1)^T ] N_0^0. \]

\[ \rho_0 [C_0 (x_{z_0}^0)^T + S_0 (x_{z_0}^0)^T ] N_0^1 = -[\beta_1 u (x_{z_0}^1)^T + \alpha_1 u (x_{z_0}^1)^T ] N_0^0. \]

Consider the case \( z < 0 \). If these equations are \( I - IV \) and using (30) (here we use \( z < 0 \), then by considering \( I^2 - III^2 \), \( IV^2 - I^2 \), \( I - II - III \cdot IV \) we obtain \( \alpha_1^2 - \beta_1^2 > 0 \), \( \beta_1^2 - \alpha_1^2 > 0 \), \( \alpha_1 \alpha_1 v - \beta_1 \beta_1 v = 0 \), so \[ \begin{bmatrix} \alpha_{1u} & \alpha_{1v} \\ \beta_{1u} & \beta_{1v} \end{bmatrix} = \begin{bmatrix} \lambda_1 C_1 & \mu_1 S_1 \\ \lambda_1 S_1 & \mu_1 C_1 \end{bmatrix}. \] Because of the symmetry \( 0 \leftrightarrow 1 \) and using \( I, II \) we obtain that the second fundamental form of \( x^1 \) is \( \frac{\lambda_1 \mu_1 (d^2 a^2 - d^2 b^2)}{\cos \alpha_1 \sqrt{T_1}} \), so \( (u, v) \) is also isothermic-conjugate on \( x^1 \) and the B transformation preserves the orientation of \( (u, v) \) (for \( z > 0 \) it changes it); from \( I^2 - III^2 \) we obtain \( \mu_1^2 - \lambda_1^2 = H_1 \) and we have complete symmetry \( 0 \leftrightarrow 1 \) also at the level of the isothermic-conjugate system \( (u, v) \). Thus

\[ (x_{z_0}^0)^T N_0^0 = \epsilon_1 \sqrt{-z} \sec \alpha_1 \sec \alpha_1 (C_0 \lambda_1 + \epsilon_1 S_0 \mu_1), \]

\[ (x_{z_0}^0)^T N_0^1 = -\epsilon_1 \sqrt{-z} \sec \alpha_1 \sec \alpha_1 (S_0 \lambda_1 + \epsilon_1 C_0 \mu_1), \]

\( \epsilon_0 := -\epsilon_1, 0 \leftrightarrow 1 \).

With \( r_j := \sqrt{1 - z a_j^2} \) we have the TC \( r_1 \cos \beta_0 \cos \beta_1 + r_2 \sin \beta_0 \sin \beta_1 = \cos \alpha_0 \cos \alpha_1 + r_3 \sin \alpha_0 \sin \alpha_1 \) and the above become

\[ \sin \alpha_0 \cos \alpha_1 - r_3 \cos \alpha_0 \sin \alpha_1 = \epsilon_1 \sqrt{-z} (C_0 \lambda_1 + \epsilon_1 S_0 \mu_1), \]

\[ -r_1 \sin \beta_0 \cos \beta_1 + r_2 \cos \beta_0 \sin \beta_1 = -\epsilon_1 \sqrt{-z} (S_0 \lambda_1 + \epsilon_1 C_0 \mu_1), \]

\( \epsilon_0 := -\epsilon_1, 0 \leftrightarrow 1 \), or

\[ \begin{bmatrix} \sin \alpha_0 \cos \alpha_1 \\ \cos \alpha_0 \sin \alpha_1 \\ \sin \beta_0 \cos \beta_1 \\ \cos \beta_0 \sin \beta_1 \end{bmatrix} = \begin{bmatrix} \epsilon_1 \sqrt{-z} & 1 & r_3 & 1 \\ 1 & 1 & -1 & (C_0 \lambda_1 + \epsilon_1 S_0 \mu_1) \\ 1 & 1 & -1 & (S_0 \lambda_1 + \epsilon_1 C_0 \mu_1) \end{bmatrix}. \]

We have

\[ \begin{align*}
(C_0 \lambda_1 + \epsilon_1 S_0 \mu_1) u & = (S_0 \lambda_1 + \epsilon_1 C_0 \mu_1) (\theta_{0u} + \epsilon_1 \theta_{1u}) - C_0 (\frac{1}{2} H_{1 \alpha_1} C_1 + \frac{1}{2} H_{1 \beta_1} S_1), \\
(C_0 \lambda_1 + \epsilon_1 S_0 \mu_1) v & = (S_0 \lambda_1 + \epsilon_1 C_0 \mu_1) (\theta_{0v} + \epsilon_1 \theta_{1v}) + \epsilon_1 S_0 (\frac{1}{2} H_{1 \alpha_1} S_1 + \frac{1}{2} H_{1 \beta_1} C_1), \\
(S_0 \lambda_1 + \epsilon_1 C_0 \mu_1) u & = (C_0 \lambda_1 + \epsilon_1 S_0 \mu_1) (\theta_{0u} + \epsilon_1 \theta_{1u}) - S_0 (\frac{1}{2} H_{1 \alpha_1} C_1 + \frac{1}{2} H_{1 \beta_1} S_1), \\
(S_0 \lambda_1 + \epsilon_1 C_0 \mu_1) v & = (C_0 \lambda_1 + \epsilon_1 S_0 \mu_1) (\theta_{0v} + \epsilon_1 \theta_{1v}) + \epsilon_1 C_0 (\frac{1}{2} H_{1 \alpha_1} S_1 + \frac{1}{2} H_{1 \beta_1} C_1) \end{align*} \]

and the B transformation of (27) in conjunction with solutions of (26) reveals itself:

\[ \begin{align*}
\theta_{0u} + \epsilon_1 \theta_{1u} & = -\frac{\epsilon_1}{2} \sqrt{-z} \left[ (\sin \alpha_0 \sin \alpha_1 + r_3 \cos \alpha_0 \cos \alpha_1) S_0 C_1 - (r_1 \sin \beta_0 \sin \beta_1 + r_2 \cos \beta_0 \cos \beta_1) C_0 S_1 \right], \\
\theta_{0v} + \epsilon_1 \theta_{1v} & = -\frac{\epsilon_1}{2} \sqrt{-z} \left[ (r_1 \sin \beta_0 \sin \beta_1 + r_2 \cos \beta_0 \cos \beta_1) S_0 C_1 - (r_1 \sin \beta_0 \sin \beta_1 + r_2 \cos \beta_0 \cos \beta_1) S_0 C_1 \right]
\end{align*} \]

3.6. Calapso’s Bäcklund transformation for real quadrics of revolution.

3.7. Darboux’s integral formula for deformations of the real paraboloid of revolution.

4. THE SOLITONS OF QUADRICS

In analogy to the situation for the link between the solitons of the sine-Gordon equation and the solitons of the pseudo-sphere (when the 0-soliton is the axis of the tractrix) we are interested in finding degenerate deformations of quadrics (that is the seed collapses to a curve or point) as 0-solitons and then in finding explicit formulae of their B transforms.

For real deformations of the real hyperbolic paraboloid from the differential system (12) we have

\[ \alpha = \alpha(v), \beta = \beta(v), \mu = \mu(v), \theta = \theta(v), \alpha' = \mu S, \beta' = \mu C, -C_{\alpha_1} + S_{\alpha_2} + \mu \theta' = 0, \mu' = S_{\alpha_1} - C_{\alpha_2}, \mu^2 = \frac{a_1^2}{a_2^2} - \frac{a_2^2}{a_1^2} + 1 \] and \( \theta \) will satisfy the (hyperbolic) pendulum equation \( \theta'' = \text{SC} \).

We have \( \theta^2 = \frac{C^2 + S^2 + \epsilon c}{2} \), so the solution \( \theta \) is given in terms of elliptic functions

\[ \epsilon = \pm 1. \]

Then we take \( \alpha \) solution of the second order ODE \( \alpha'' = 2(\log S)' \alpha' + \frac{\alpha}{\alpha_1} = 0 \) and \( \beta := -\alpha' (C_{\alpha_1} - S_{\alpha_2}), \mu := \frac{1}{2} (C_{\alpha_1} - S_{\alpha_2}) \) and we have \( \alpha' = \mu S, \beta' = \mu C, \mu' = S_{\alpha_1} - C_{\alpha_2}, \mu^2 - \frac{a_1^2}{a_2^2} + \frac{a_2^2}{a_1^2} = \epsilon c; \) this last constant can be normalized to \( 1 \) by a choice of constant in the initial value of \( \alpha \). Note the ODE of \( \alpha \) can be brought to the form \( (C^2 + S^2 + c) \frac{d^2 \alpha}{dt^2} - 2 \frac{C^2}{S^2} (C^2 + c) \frac{d\alpha}{dt} + 2 \frac{a_1}{a_2} = 0 \) and the prime integral above mentioned to the form \( \frac{C^2 + S^2 + c}{2S^2 + c} (\frac{d\alpha}{dt})^2 + \frac{a_1}{1 - a_1} (\frac{C^2 + S^2 + c}{2S^2 + c} \frac{d\alpha}{dt})^2 - \frac{a_2^2}{a_1^2} = 1. \)
The homogeneous part can be integrated by quadrature ($\frac{d\log \alpha}{d\theta}$ will depend algebraically on $C, S$) and then by the standard variation of parameters argument one can solve this prime integral for $\alpha = \alpha(\theta)$.

Thus all quantities of interest can be found by explicit formulae; the only remaining question is if one can find explicit formulae for the B transforms of the solutions of the hyperbolic pendulum equation.

Again a change from the $v$ variable to the $\theta_0$ variable is in order ($\theta_1 = \theta_1(u, \theta_0)$):

$$\theta_{1\theta_0} = \epsilon \frac{\sqrt{2}}{\sqrt{C_0^2 + S_0^2 + c}} (\frac{\sigma_1 + \sigma_1^{-1}}{2} S_1 C_0 + \frac{\sigma_1 - \sigma_1^{-1}}{2} C_1 S_0),$$

$$\theta_{1u} = \epsilon \frac{\sqrt{C_0^2 + S_0^2 + c}}{\sqrt{2}} + \frac{\sigma_1 - \sigma_1^{-1}}{2} S_1 C_0 + \frac{\sigma_1 + \sigma_1^{-1}}{2} C_1 S_0.$$  
The last equation is separable (we consider $\theta_0 = ct$); by quadrature one can find the solution depending on a constant of $u$ (function of $\theta_0$); in turn by replacing the result in the first equation one can find the function of $\theta_0$ up to a constant.

Thus the 0-solitons will depend on two constants and each iteration of the B transformation will introduce two constants.

For $c = \pm 1$ the elliptic function will degenerate to hyperbolic trigonometric ones and $\theta_0$ will turn out to be the 1-solitons of the hyperbolic sinh-Gordon equation with $\sigma = 1$; then one can apply directly the BPT to find $\theta_1$.

Also since we already know the 1-solitons of the hyperbolic sinh-Gordon equation from Peterson’s deformations of quadrics, a space realization of solitons is possible in this particular case.

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