Chaotic Friedmann - Robertson - Walker Cosmology

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ABSTRACT: We show that the dynamics of a spatially closed Friedmann - Robertson - Walker Universe conformally coupled to a real, free, massive scalar field, is chaotic, for large enough field amplitudes. We do so by proving that this system is integrable under the adiabatic approximation, but that the corresponding KAM tori break up when non adiabatic terms are considered. This finding is confirmed by numerical evaluation of the Lyapunov exponents associated with the system, among other criteria. Chaos sets strong limitations to our ability to predict the value of the field at the Big Crunch, from its given value at the Big Bang.

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I - Introduction

In this paper, we shall present analytical and numerical evidence of chaotic behavior in a cosmological model, consisting of a spatially closed, Friedmann - Robertson - Walker (FRW) Universe, filled with a conformally coupled but massive real scalar field. Although this model is far too simplified to be considered realistic, its simplicity itself makes it an interesting testing ground for the implications of chaos for cosmology, either classical, semi-classical or quantum. Moreover, the occurrence of chaotic motion in this example supports the conjecture of a farther reaching connection between chaos at the classical level, particle creation at the semiclassical one, and decoherence in the full quantum cosmological treatment (see below).

The research on chaotic cosmological models began with the work by the Russian school\cite{1} and by C. Misner and coworkers \cite{2} on chaos in Bianchi type IX vacuum cosmologies. Although the hopes initially placed on these models (such as a resolution of the horizon problem from the mixing effect associated to chaos) could not be sustained, Bianchi IX is still by far the best studied chaotic cosmology\cite{3}, both because of its intrinsic interest, and as a test bench for more general questions, such as whether chaos is an observer dependent phenomenon\cite{4}. In the broader context of General Relativity, chaos has been considered in connection to geodesic motion, both in cosmological\cite{5} and Black Hole space times\cite{6}.

The relative paucity of examples of relativistic chaos\cite{7} makes it hard to disentangle the general features (if any) of these phenomena from the “miracles” proper to each peculiar manifestation, such as the occurrence of the Gauss’ map hidden in Bianchi IX dynamics, which led to the discovery of cosmological chaos in the first place\cite{1}. It is therefore of the utmost importance to develop a systematic search for instances of relativistic chaos, to enhance our battery of examples, and therefore better to aim future research.

As pointed out in an earlier communication\cite{8}, the class of near integrable systems (NIS) is an interesting field for such a search. A NIS is an integrable system which becomes non integrable under the effect of a perturbation. The non integrable perturbed dynamics
may or may not be, in turn, chaotic. A typical situation occurs when the unperturbed dynamics displays an unstable fixed point, asymptotically joined to itself by a nontrivial orbit (the so called “homoclinic loop”). If the loop is destroyed by the perturbation, then a “stochastic layer” forms in its neighborhood. There are subsets in this layer where the dynamics is equivalent to a Bernoulli shift. This is the same degree of chaotic behavior characteristic of the Bianchi IX example. A detailed discussion of this scenario, geared to its application in relativistic problems, has been given in Ref.8.

The case at hand in this paper belongs to a wider class where there are no homoclinic points in the unperturbed dynamics. However, in a certain sense, the perturbation both creates them and destroys the corresponding homoclinic loops, allowing the formation of stochastic layers. This road to chaos results from the overlapping of several resonances between the external perturbation and the unperturbed motion. In the neighborhood of a resonant region in phase space, one of these resonances dominates over the rest; because of this dominant resonance, KAM tori are destroyed in this region. In particular, the torus for which the resonance is exact dissapears, leaving behind a discrete set of both stable and unstable fixed points. The unstable fixed points are connected to each other through doubly asymptotic orbits or separatrices; chaos occurs whenever the separatrices are further destroyed by the effect of the secondary resonant terms. In the neighborhood of a destroyed separatrix, it is possible to find (measure zero) invariant sets where the restricted dynamics is equivalent to a Bernoulli flow. More generally, chaotic behavior, under the form of extreme sensitivity regarding initial conditions, is displayed in a set of full measure around the separatrix, the so called stochastic layer.

The particular model we shall present here has been chosen both because of its simplicity and because of its relevance to the discussion of the ties between classical and quantum cosmological models. Indeed, similar models have been used by Hawking and Page to discuss the relationship between the cosmological and thermodynamic arrow of time, in the framework of Quantum Cosmology. Since in this model the system behaves as a Bernoulli flow in appropriate invariant sets, time reversal invariance may be spontaneously
broken, and thus a “chaotic” arrow of time appears\textsuperscript{[14]}. The behavior of this arrow supports Page’s picture of the relationship between the cosmological and the thermodynamic ones.

In the cosmological literature more generally, fundamental scalar fields are usually considered within the framework of Inflationary models. In these models, however, minimal coupling is often preferred to conformal one\textsuperscript{[15]}. Thorough analysis of the dynamics of a minimally coupled field in a FRW background have been developed by several authors\textsuperscript{[16]}; they have found that Inflationary cosmologies act as an attractor in phase space, but little or no evidence of chaos. Further studies of non minimally coupled fields in open universes have yielded similar results\textsuperscript{[17]}. In the conformally coupled model we shall present here, on the other hand, “inflationary” periods, where the radius $a$ of the Universe increases by several orders of magnitude, are a common feature of the solutions to the full dynamics. In this model, “inflation” is not powered by an effective cosmological constant, but rather by the effective Newton’s constant becoming negative for large values of the field \textsuperscript{[18]}.

Another source of interest in this model is that it is one of the simplest cosmologies where particle creation by the gravitational field occurs \textsuperscript{[19]}. Indeed, particle creation in models of this kind has been analyzed by Hartle and others \textsuperscript{[20]}. In a perturbative analysis, the spectrum of created particles is closely related to the Fourier decomposition of $a^2(\eta)$, where $\eta$ stands for conformal time, and $a$ is the “Radius of the Universe” (see below). This fact, and the connection between particle creation and the breakdown of the WKB approximation through Stokes’ phenomenon \textsuperscript{[21]}, suggest that particle creation is indeed the effect of resonances between the evolution of $a$ and that of the scalar field. If this were true, then a strong correlation between classical chaos and semiclassical particle creation should exist. The relationship of semiclassical particle creation to quantum cosmological decoherence has been discussed elsewhere\textsuperscript{[22]}.

In this paper we shall investigate our cosmological model both through analytical (perturbative) and numerical (non perturbative) methods. After showing how the full Hamiltonian can be split into an integrable part plus a perturbation, we shall show how the KAM tori of the integrable part are destroyed and replaced by stochastic layers under the effect of the perturbation. We shall analyze the onset of chaos through three methods
of increasing sophistication. We shall begin by employing Chirikov’s resonance overlap criterium\cite{11} to show that the stochastic layer of a given resonance is wide enough to intrude into the layers of its neighbors. Then we shall employ the Melnikov Integral\cite{9} to show that the separatrix created by each resonance is split by the effect of the secondary perturbations. Finally, we shall investigate the dynamics close to a separatrix by constructing the corresponding separatrix or standard map \cite{11}. We shall use this map to discuss the important issue of whether the sensitive dependence on initial conditions proper to chaos is strong enough to lead to observable effects within the lifespan of the Universe. It should be remembered that Bianchi IX models have been found lacking in this regard\cite{2}.

Numerical analysis will allow us to go beyond perturbation theory. We shall present results from numerical integration of the full equations of motion, both in the chaotic and non chaotic regimes; plots of the field against $a$, where the change in the topology of the orbits, subsequent to chaos, can be observed; numerical estimates of the Lyapunov coefficients\cite{23} in the stochastic layers; and finally, a plot of the values of the field at the Big Crunch vs. those at the Big Bang, for fixed initial geometry, to demonstrate the loss of predictibility subsequent to chaos.

The paper is organized as follows. We introduce the model in the following section, where we also develop its perturbative analysis. Our numerical results are presented in Section III. Finally, we briefly state our conclusions in Section IV.

**II - The model and its perturbative treatment**

**II.1 - The model**

Let us begin by introducing the model and how we shall split its Hamiltonian into an unperturbed part plus a perturbation.

As already discussed, our cosmological model assumes a FRW spatially closed geometry, that is, a line element
\[
    ds^2 = a^2(\eta)[ -d\eta^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) ]
\] (1)

Where \( 0 \leq \varphi \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq \chi \leq \pi, \) and \( \eta \) stands for “conformal” time. For concreteness, we shall consider only models starting from a cosmic singularity, that is, we restrict \( \eta \) to be positive, with \( a(0) = 0 \). Also, as indicated by the dynamics, we shall assume that after the Big Crunch (that is, when \( a \) returns to 0), a new cosmological cycle begins, now with \( a \leq 0 \). Therefore, a complete periodic orbit describes the birth and death of two Universes.

The gravitational dynamics is described by the Einstein-Hilbert action

\[
    S_g = \int d^4 x \sqrt{-g} m_p^2 R
\] (2)

Where the determinant of the metric \(-g = a^4 \sin^2 \chi \sin \theta\), and the scalar curvature

\[
    R = 6\left(\frac{\ddot{a}}{a^3} + \frac{1}{a^2}\right)
\] (3)

(We shall use MTW conventions throughout \textsuperscript{[24]}.) A dot represents a \( \eta \) derivative, and \( m_p \) is Planck’s mass. For simplicity, we shall assume \( m_p = \sqrt{1/12v} \), where \( v = 2\pi^2 \) is the conformal volume of an spatial surface.

The action for a conformally coupled, massive, real scalar field is given by

\[
    S_f = \int d^4 x \sqrt{-g} \left( \frac{-1}{2} \right) [\partial_{\mu} \Phi \partial^{\mu} \Phi + (m^2 + (1/6) R) \Phi^2]
\] (4)

Where \( m^2 \) is the mass. Consistency with the symmetries of the background geometry demands the field be homogeneous. Parametrizing the field as \( \Phi = \phi/v^{(1/2)} \), performing the spatial integrals, and discarding total derivatives, we are led to a dynamical system with two degrees of freedom, \( a \) and \( \phi \), and Hamiltonian
\[ H = \left( \frac{1}{2} \right) \left[ -(\pi^2 + a^2) + (p^2 + \phi^2) + m^2 a^2 \phi^2 \right] \] (5)

Where \( \pi \) and \( p \) are the momenta conjugated to \( a \) and \( \phi \) respectively. At this point we must recall that, because we have relinquished our gauge freedom in writing the line element as in Eq.(1), we are missing one of Einstein’s equations, namely, the Hamiltonian constraint\(^{[24]} \). We reintroduce this constraint as a restriction on allowable initial conditions

\[ H = 0 \] (6)

Which is clearly respected by the dynamics.

When \( m^2 = 0 \), the Hamiltonian (5) is obviously integrable; for nonvanishing \( m^2 \), this is no longer so obvious, and we must resort to perturbative methods ( or solve the dynamics numerically, see next section ). It is tempting to consider the massless Hamiltonian as the unperturbed one, with the last term in Eq. (5) as perturbation. This is questionable, however, on the grounds that for “macroscopic” Universes, very easily we obtain \( m^2 a^2 \gg 1 \).

For example, for our own Universe \( a \sim 10^{60} \), and for a mass of 1 eV, we get \( m \sim 10^{-28} \), so \( ma \sim 10^{32} \). In this regime, the last term in (5) is in no way small compared with the other ones.

We shall therefore proceed in a different way. Let us observe that, if the evolution of \( a \) is slow compared to the oscillations in \( \phi \), then its main effect is to produce an adiabatic change in the frequency of the latter. We therefore introduce the “adiabatic” amplitude and phase \( j \) and \( \phi \)

\[ \phi = \sqrt{\frac{2j}{\omega}} \sin \varphi \] (7)

\[ p = \sqrt{2\omega j} \cos \varphi \] (8)
Where $\omega^2 = 1 + m^2 a^2$ is the instantaneous frequency of the field. This transformation is canonical; it can be accomplished by means of the generating functional

$$S_1 = Pa + \left(\frac{\omega \phi^2}{2 \tan \varphi}\right)$$

But, because of the $a$ dependence in $\omega$, we are forced to change the geometrical momentum from $\pi$ to $P$, according to

$$\pi = P + \frac{m^2 a j}{2(1 + m^2 a^2)} \sin 2\varphi$$

We now substitute the new variables in the Hamiltonian, and rewrite this as

$$H = -(H_0 + \delta H)$$

where the unperturbed Hamiltonian

$$H_0 = \left(\frac{1}{2}\right)[P^2 + a^2] - j \sqrt{1 + m^2 a^2}$$

is obviously integrable ( $H_0$ and $j$ are constants of motion in involution ), and the perturbation

$$\delta H = \frac{m^2 a P j}{2(1 + m^2 a^2)} \sin 2\varphi + \left[\frac{m^2 a j}{4(1 + m^2 a^2)}\right]^2 (1 - \cos 4\varphi)$$

remains small both for small and large Universes.

II.2 - Solving the unperturbed dynamics

The dynamics of $a$, as generated by $H_0$, is obviously bounded. The point $a = 0$ is a fixed point; it is stable if $m^2 j \leq 1$, and unstable otherwise. In this second case, there is an
homoclinic loop associated with it. However, this orbit does not satisfy the Hamiltonian
constraint, Eq. (6); rather, we have $H_0 = -j$ on the homoclinic loop.

The equations of motion are simpler if written in terms of a new variable $X = \sqrt{1 + m^2 a^2}$, rather than $a$ itself. The transformation is canonical if we associate to $X$ the
momentum

$$P_X = \frac{XP}{m\sqrt{X^2 - 1}} \quad (14)$$

In terms of the new variables $(X, P_X)$, the unperturbed Hamiltonian reads

$$H_0 = \left[ \frac{m^2(X^2 - 1)}{2X^2} \right] P_X^2 + \left( \frac{1}{2m^2} \right) [(X - m^2 j)^2 - (m^2 j)^2 - 1] \quad (15)$$

For fixed $H_0 = h$ and $j$, provided $h \geq -j$, $X$ ranges from 1 to the classical turning
point $X_T = m^2 j + m\sqrt{K}$, where $K = (1/m^2) + (mj)^2 + 2h$ (The reasons for the somewhat
unusual notation shall be clear below).

Although the Hamiltonian Eq. (15) can be explicitly integrated (the solution involves
the use of elliptic integrals $^{[25]}$), for our purposes it will be better to concentrate on a
particular case, where a number of simplifications will be available. Concretely, we shall
consider the large $j$ limit, with $H_0 \sim 0$ and $m$ fixed. In this limit, we find $X \gg 1$
for most of the orbit. Indeed, from Hamilton’s equations, we get $\dot{X} \sim 2m\sqrt{h + j}\sqrt{X - 1}$ when
$X \sim 1$ and $H_0 = h$. Therefore, even if $X$ starts at 1, $X - 1$ becomes of order unity after a
lapse $\delta t \sim (m\sqrt{h + j})^{-1}$, which is small in the case under consideration. Now, for $X \gg 1$,
the Hamiltonian Eq. (15) simplifies to

$$H_0 = \left( \frac{m^2}{2} \right) P_X^2 + \left( \frac{1}{2m^2} \right) [(X - m^2 j)^2 - (m^2 j)^2 - 1] \quad (16)$$

If we parametrize

$$P_X = \frac{\sqrt{K}}{m} \cos 2\alpha \quad (17)$$

9
\[ X = m^2 j + m\sqrt{K} \sin 2\alpha \]  

(18)

Eq. (16) reduces to

\[ H_0 = \left( \frac{1}{2} \right) [K - (mj)^2 - \frac{1}{m^2}] \]  

(19)

As \( X \) goes from 1 to \( X_T \), that is, over a quarter of an orbit, \( \sin 2\alpha \) goes from \(-\frac{(m^2 j - 1)}{m\sqrt{K}}\) to 1. For large \( j \) and \( H_0 \sim 0 \), however, \( K \) is very close to \((mj)^2\). So the end points can be taken as \( \alpha \sim \pm \pi/4 \), in which case \( K \) is precisely the associated action variable. We have succeeded in integrating the unperturbed motion, only that, because the reparametrization eqs. (17) and (18) involves \( j \), the angle canonically conjugated to it is no longer \( \varphi \), but

\[ \theta = \varphi - m\sqrt{K} \cos 2\alpha \]  

(20)

The transformation eqs. (17), (18) and (20), from variables \((X, P_X, \varphi, j)\) to new variables \((\alpha, K, \theta, j' = j)\) is canonical; indeed, it is generated by

\[ S_2 = j' \varphi + \frac{(X - m^2 j')^2}{2m^2 \tan 2\alpha} \]  

(21)

In the following, we shall omit the prime on \( j' \).

II.3 - Analysis of the perturbation

Having reduced the unperturbed Hamiltonian Eq. (12) to action - angle form (19), let us return to the perturbation, Eq. (13). We notice, first of all, that the first term in the perturbation dominates the second (except close to the turning points). For our purposes, it shall be sufficient to retain only the dominant perturbation.

We also notice that, \( m\sqrt{K} \) being large, \( \sin 2\varphi \equiv \sin 2(\theta + m\sqrt{K} \cos 2\alpha) \) (cfr. Eq. (20)) is a strongly oscillating function of \( \alpha \). Thus, in computing the Fourier expansion of
the perturbation, we may substitute the non oscillatory factor $m^2 a P_j / 2 (1 + m^2 a^2)$ by its mean value over a quarter orbit, which is easily found to be $(j / \pi) \ln X_T - (X_T^2 - 1) / 2 X_T^2 \sim (j \ln X_T) / \pi$.

The Fourier expansion of the oscillatory factor itself is

$$\sin 2(\theta + m \sqrt{K} \cos 2\alpha) = J_0(2m \sqrt{K}) \sin 2\theta + \sum_{n=1}^{\infty} J_n(2m \sqrt{K}) \sin \delta^n_+ + \sin \delta^n_- \quad (22)$$

Where $J_n$ is the usual Bessel function, and $\delta^n_\pm = (2\theta \pm 2n(\alpha + (\pi/4)))$. Therefore, resonances occur whenever

$$\omega_j \pm n \omega_K = 0 \quad (23)$$

$\omega_j, K$ being the frequencies associated to $\theta$ and $\alpha$, respectively. From Eq.(19), we have $\omega_j \sim m^2 j$ (we have reinstated the proper sign, cfr Eq.(11)), and $\omega_K \sim -1/2$. We thus find a tower of resonances, corresponding to all positive values of $n$; the $n$-th resonance occurs at $j_n \sim n/2m^2$, independently of $K$. If we further impose the Hamiltonian constraint, then $K$ must take the value $K_n \sim (mj_n)^2 = (n/2m)^2$.

The $n$-th resonant term in the Hamiltonian, with its proper sign, reads

$$\left( -\frac{j}{\pi} \right) (\ln X_T) J_n(2m \sqrt{K}) \sin(2\theta + 2n(\alpha + (\pi/4))) \quad (24)$$

To analyze the perturbed motion it is sufficient to approximate the prefactor by its value at resonance. So doing, and using the proper asymptotic form for the Bessel function$^{[26]}$, Eq.(24) reduces to

$$\left( -\frac{\epsilon}{m^2} \right) n^{2/3} (\ln n) \sin(2\theta + 2n(\alpha + (\pi/4))) \quad (25)$$

where $\epsilon$ is a numerical coefficient, $\epsilon \sim 0.111827...
II.4 - Solving the perturbed motion

To analyze the motion in the presence of the perturbation, let us focus first on the neighborhood of the \( n \)-th resonance (the calculations below are simplest if \( n = 4k + 1 \) for some \( k \), which we shall assume). Let us keep only the dominant resonant term Eq. (25) in the Hamiltonian, and introduce the new action variables \( \xi = (j - j_n)/2 \) and \( \kappa = -(K - K_n - n(j - j_n))/2 \), which vanish at the resonant point. These variables are canonically conjugated to the angles \( \psi = 2(\theta + n\alpha) \) and \( \tau = -2\alpha \), respectively (the transformation is generated by \( S_3 = [K_n - 2(\kappa - n\xi)]\alpha + [j_n + 2\xi]\theta \)). In terms of the new variables, the resonant Hamiltonian becomes (cfr. Eqs. (19) and (25))

\[ H_n = \kappa + 2m^2\xi^2 - \left( \frac{\omega_0^2}{4m^2} \right) \cos \psi \]  

(26)

Where \( \omega_0^2 = 4\epsilon n^{2/3} \ln n \).

Because the Hamiltonian (26) is linear in \( \kappa \), it can be considered as resulting from the “parametrization” of a one degree of freedom system[27]. To analyze the resulting motion, it is convenient to “deparametrize” it, that is, to promote \( \tau \) to the role of “time”. The evolution of the “true” degree of freedom \( \psi \) as \( \tau \) unfolds is described by the Hamiltonian \( E = -\kappa \).

From Eq. (26) and the Hamiltonian constraint (6), it is obvious that \( E \) is simply the Hamiltonian of a non linear pendulum, \( \omega_0 \) being the frequency of small oscillations around the stable equilibrium point \( \psi = 0 \). The pendulum also has an unstable equilibrium point \( \psi = \pm \pi \), joined to itself by a separatrix.

Following Chirikov[11], we define the “width” of the resonance as the maximum value of \( \xi \) along the separatrix, \( \Delta \xi \sim \omega_0/2m^2 \). Since \( \omega_0 \sim n^{1/3} \), it is clear that for large \( n \) each resonance is much wider that the separation \( 1/2m^2 \) between resonances. Thus, according to Chirikov’s criterium, the behavior of the perturbed system must be chaotic.

To obtain a more detailed picture, let us add to the resonant Hamiltonian Eq. (26) the first secondary resonance \( (n = 4k + 2) \)
\[ \delta H_n \sim (\omega_0^2/4m^2) \sin(\psi - \tau) \] (27)

(The same term is added to the “deparametrized” Hamiltonian \( E \)). We can now apply the criterium that chaos will occur if the secondary resonance is able to destroy the separatrix created by the primary resonance. This can be determined by computing the total change in the unperturbed pendulum Hamiltonian, induced by the perturbation, as the pendulum swings along the separatrix, which is given by the Melnikov Integral\([9]\)

\[ I = -\omega_0^2 \int_{-\infty}^{+\infty} d\tau \xi \cos(\psi - \tau) \] (28)

In our case\([11]\), \( I = -\Delta E \cos \psi_0 \), where \( \psi_0 \) is the value of \( \psi \) at \( \tau = 0 \), and

\[ \Delta E = \frac{\pi e^{\frac{\pi}{\omega_0}}}{m^2 \sinh \frac{\pi}{\omega_0}} \] (29)

The fact that \( I \) displays isolated zeroes as a function of \( \psi_0 \) is again proof of the presence of chaos when the perturbation is turned on\([9]\).

II.5 - The Separatrix Map

The oscillatory behavior of the Melnikov Integral proves implicitly that the dynamics of our system, restricted to suitable invariant sets, is equivalent to a Bernoulli flow. By constructing the so-called “separatrix map”\([11]\), we shall be able to show explicitly some of those sets, and therefore acquire a much more immediate insight on the evolution of the Universe near a resonance.

To build the separatrix map we shall adopt the picture, emerging from the previous subsection, of the dynamics of the conformal field as that of a pendulum, with angle \( \psi \) and conjugated momentum \( \xi \), subject to the perturbation Eq.(27).

Let us suppose that, at a certain instant \( \tau_r \), the “pendulum” has energy \( E_r \) and coordinate \( \psi_r \). If we follow the unperturbed motion over a closed orbit, the energy will
keep constant, while the phase \( \varphi = \tau - \psi \) of the perturbation shall increase to \( \varphi_{r+1} = \varphi_r + 2\pi/\omega(E_r) \). Observe that the phase shift is independent of the phase.

If we consider the effect of the perturbation, over the same lapse, we observe that \( E \) does not remain constant. For motion close to the separatrix, the change in \( E \) is approximately the same than at the separatrix itself, and so we find

\[
E_{r+1} = E_r - \Delta E \cos \varphi_r 
\]

The energy shift induces changes in the characteristic frequencies \( \omega(E) \) of the unperturbed orbits, and thus the total phase shift is also affected. As a first approximation, we may write

\[
\varphi_{r+1} = \varphi_r + \frac{2\pi}{\omega(E_{r+1})} 
\]

Eqs. (30) and (31) define the separatrix map. Observe that the phase shift is no longer independent of the phase.

The separatrix map preserves areas. It also has a double sequence of fixed points \( x_k^\pm = (E_k, \pm \pi/2) \), where \( k \) is an integer, and \( \omega(E_k) = 1/k \). For large \( k \), the fixed points approach the separatrix, where\[^{[11]}\] \( \omega(E) \sim \pi \omega_0/\ln(32E_0/|E_0 - E|) \), \( E_0 \equiv \omega_0^2/4m^2 \). Restricting ourselves to motion below the separatrix, for simplicity, we find \( E_0 - E_k \sim 32E_0 \exp(-k\pi\omega_0) \).

The behavior of the separatrix map close to a fixed point is determined by its linear part. It can be shown that all \( x_k^+ \) fixed points are hyperbolic, while the \( x_k^- \) points are hyperbolic only for large enough \( k \). For our purposes, it is sufficient to concentrate on the properties of the map near the \( x^+ \) fixed points.

For large enough \( k \), the eigenvalues of the linearized map around \( x_k^+ \) are \( \lambda_k \) and \( 1/\lambda_k \), where
\[ \lambda_k \equiv \Delta E \left. \frac{d}{dE} \left( \frac{2\pi}{\omega(E)} \right) \right|_{E_k} \] (32)

For large \( k \), \( \lambda_k \sim (\Delta E/16\omega_0 E_0) \exp(k\pi \omega_0) \) diverges exponentially. The map is stretching in the direction \( v^+_k = \cos \beta_+ \partial \varphi + \sin \beta_+ \partial E \), where \( \tan \beta_+ \sim \Delta E/(\lambda_k + 1) \), and contracting in the direction \( v^-_k = -\sin \beta_- \partial \varphi + \cos \beta_- \partial E \), where \( \tan \beta_- \sim (1/\Delta E)(1 - (1/\lambda_k)) \).

We can now see chaos arising. A small parallelogram, with a vertex on \( x^+_k \) and sides along the directions \( v^+ \) and \( v^- \), is stretched along the former, and contracted along the latter. Since \( v^+ \) is essentially the \( \varphi \) direction, the angular span of the original parallelogram is dilated to many times \( 2\pi \). After modulo \( 2\pi \) identification, therefore, the image is superimposed to the original parallelogram, in a way essentially equivalent to Arnold’s “Cat Map”\[10\]. By a standard procedure, then, it is possible to identify, in our parallelogram, an invariant (null) set where the separatrix map is Bernoulli\[9\].

Clearly, the procedure works only if the map is strongly dilating. As a ballpark estimate, we can place the “border of chaos” at the value of \( E_k \) for which \( \lambda_k \sim 1 \). Thus, we are led to estimate the width of the stochastic layer around the departing separatrix at \( \delta E \sim 2\Delta E/\omega_0 \). Deep in the resonance region, that is, for the \( n \)th resonance, \( n \) large, Eq.(29) yields \( \Delta E \sim \omega_0/m^2 \) (recall that \( \omega_0 \sim n^{1/3} \) is large itself), and so the width of the stochastic layer approaches a constant value \( \delta E \sim 2/m^2 \). This value is about four times the distance between the resonances themselves, so the stochastic layers merge and a stochastic sea is formed: any initial condition in this region falls within the stochastic layer of some resonance \[11\].

Almost more important to us, these results show that the loss of information associated to chaotic behavior is observable within the span of a single Universe. Indeed, we have identified chaos in the oscillations of the scalar field, parametrized against the angle variable associated to the radius of the Universe. Now, over most of the stochastic layer, the field performs many oscillations in the lapse between the Big Bang and the Big Crunch. This results from the fact that the frequency associated with the expansion and contraction
of the Universe is nearly constant for large $m^2j$, while the frequencies associated to the field oscillations scale as $\omega_0 \sim n^{1/3}$. For example, at the outer rim of the stochastic layer, the frequency of field oscillations is $\sim (\pi/2)(\omega_0/\ln 2\omega_0)$, much larger than the cosmological frequency. To reach field frequencies comparable to the cosmic one, we must approach the separatrix, up to energies $E_0 - E \sim 32E_0 \exp(-2\pi\omega_0)$. Since the width of the stochastic layer does not shrink with $n$, we see that the fraction of it where the frequency of field oscillations is comparable to the rate of cosmic expansion and contraction, is negligible for deep resonances.

To summarize the content of this section, we have shown that for our simple cosmological model, the Hamiltonian can be split into an integrable and a non integrable part, the integrable one being the main determinant of the dynamics both for small and large Universes. Using perturbative methods, we have shown that the perturbation destroys the invariant tori of the unperturbed motion. For $m^2j \gg 1$, $j$ being the adiabatic amplitude of field oscillations, a stochastic sea is formed, through the merger of the stochastic layers of the individual resonances. Chaos manifest itself through seemingly random phase shifts in the field oscillations as the Universe expands and recollapses. Thus severe limitations to our ability to predict the future behavior of the field should arise, even within the span of a single Universe.

III - Numerical treatment of the model

although the analysis in the previous section amounts to rather impressive evidence of chaotic behavior in our model, it suffers from the limitations of the perturbative approach we have chosen. By solving the model numerically, we shall be able to go beyond perturbation theory, and thus find an independent confirmation of the results above.

In our numerical work, we have used the “physical” variables $(a, \pi, \phi, p)$, whose equations of evolution follow from the Hamiltonian Eq. (5); the simplicity of these equations makes this approach more appealing than other, more sophisticated, alternatives.
To solve the model, we have used a Runge Kutta\textsuperscript{[28]} 5\textsuperscript{th} order routine, implemented on an IBM compatible PC 486. As a check on the numerical code, in all runs we have surveyed the value of the Hamiltonian, finding that it never exceeded a threshold of $10^{-13}$. For simplicity, we shall set $m = 1$ throughout.

Figures (1) and (2) show the evolution of a “typical Universe” in the region of “weak chaos” ($m^2 j \sim 0.5$, in the notation of Section II.1). Figure (1) shows the evolution of the scale factor, and Figure (2) that of the field. The nonlinearity is clearly visible, together with a sharp rise of the amplitude associated to the radius of the Universe in the last oscillations. Observe the simultaneous change in the frequency of field oscillations.

In what follows, we shall subject our model to a series of numerical experiments, searching for unambiguous signals of chaotic, rather than complicated, behavior. To this end, we shall study its Lyapunov exponents, the projection of an orbit on configuration space, and finally the relationship between the values of the field at two consecutive zero crossings of the radius of the Universe.

III.1 - Lyapunov exponents. May chaos be quantified?

It is possible to obtain a quantitative measure of chaos, through the determination of Lyapunov exponents. These are defined by considering the deformation of a $\epsilon$-ball of initial conditions in phase space, in an ellipsoid of principal axis $\lambda_i$ at time $t$. If we order the axis by size, then the $i$ -th Lyapunov exponent is given by the double limit $L_i = \lim_{\epsilon \to 0} \lim_{t \to \infty} (\ln \lambda_i / t)$. In a Hamiltonian system, the Lyapunov exponents add to zero, as a consequence of Liouville theorem; if the system, moreover, is conservative, then two of the Lyapunov exponents are zero, reflecting the invariance of the energy shells, and the homogeneity of time. Thus, in a conservative system with two degrees of freedom, such as ours, only the largest Lyapunov exponent carries meaningful information.

While a rigorous determination of Lyapunov exponents is usually numerically prohibitive, it is possible to estimate them, following a proposal by Wolf et al.\textsuperscript{[23]}. The idea is to construct “local” Lyapunov exponents by taking the average over time of the logarithms
of the eigenvalues of the linearized evolution operator. While these “local” Lyapunov exponents show a marked transient behavior over short times, they approach the true Lyapunov exponent as time goes to infinity.

Figures (3), (4) and (5) show plots of the largest Lyapunov exponent for three different “cosmologies”, corresponding to \( m^2 j \sim .5, 50 \) and 5,000, respectively. In the latter cases, we have cut off the early transient, to allow for a larger vertical scale. Beyond the transient regime, it can be seen that the greatest Lyapunov exponents are clearly positive, taking values in the intervals from \(.4\) to \(.8\), \(2\) to \(3\), and \(3.5\) to \(4\), respectively. This is consistent with the results of Section II.

III.2 - Orbits in configuration space. May chaos be proven?

In spite of their usefulness to the determination of chaos, Lyapunov exponents are sometimes schewed in studies of relativistic problems, since they are generally gauge dependent [4]. More reliable information can be obtained from the analysis of Poincare sections. A Poincare section is the set of points in which a given orbit of the system intersects, in a given direction, a given plane in phase space. If the motion is regular, the Poincare sections shall fit in smooth curves, while this will not occur for chaotic motion.

Similarly, we can study the projection of a trajectory of our system on the \((a, \phi)\) plane. If a second constant of motion \( C \), in involution with the Hamiltonian \( H \), existed, making the system integrable\([29]\), then at every point of the trajectory it would be possible to solve the equations \( C = \text{constant} \) and \( H = \text{constant} \) for \( \pi \) and \( p \), and thereby reduce Hamilton’s equations to parametric equations for a curve \( \phi = \phi(a) \). For example, if \( m = 0 \), then we may take \( C = (p^2 + \phi^2)/2 \). In these circumstances, the projection of the orbit on the \((a, \phi)\) plane fits into a smooth curve. Faillure to do so, on the other hand, evidences the non existence of a second constant of motion ( disregarding pathological cases ).

Figures (6), (7) and (8) display plots of \( \phi \) vs \( a \), again for \( m^2 j \) equal to \(.5, 50 \) and \(5,000\), respectively. It can be seen that, while in the first case it seems possible to fit the
resulting graph into a smooth Lissajous curve, this is no longer so in the latter cases. This
again confirms the results from Section II.

III.3 - Field values at consecutive cosmic singularities. May chaos be observed?

In the above numerical tests, we have allowed the radius of the Universe to cross zero
several times. However, in the “real” world our observations cannot be projected beyond
the cosmological singularity. Thus it is crucial to determine whether the loss of accuracy
in the prediction of the future behavior of orbits, associated to chaos, is strong enough to
lead to observable results within a single Universe.

In order to do that, we analyzed what happens with Universes starting from the same
geometrical condition ( that is, same \( a \) and \( \pi \) ) but different values of \( \phi \) ( and \( p \) given by
the Hamiltonian constraint ). We begin our computation a little while after the Big Bang,
ending the numerical calculation when \( a \) becomes negative for the first time ( that is, on
the Big Crunch ). Repeating this procedure several times, a plot of the final value \( \phi_f \) of
the field against the initial one \( \phi_i \) can be obtained. For regular behavior, this plot should
allow the prediction of \( \phi_f \) given \( \phi_i \) with the proper accuracy.

Figures (9), (10) and (11) show the results of numerical experiments such as described,
once more for \( m^2 j \sim .5, 50 \) and \( 5,000 \), respectively. In the plot Fig. (9), all simulations
started at \( a = .001 \) and \( \pi = -1 \), while \( \phi_i \) ranged from .001 to 1; to obtain Fig (10), we set
the initial values of \( a \) and \( \pi \) at .001 and \(-10 \), respectively, allowing \( \phi_i \) to range from .01
to 10; finally, Fig. (11) corresponds to the values \( a_i = .1, \pi_i = -100, .1 \leq \phi_i \leq 100 \).

The outcome of the simulations indicates that, while \( \phi_f \) is easily predicted from \( \phi_i \)
in the first case, prediction becomes impossible in the latter ones. Indeed, the correlation
coefficient between \( \phi_f \) and \( \phi_i \), which gives .933 for the data in Fig. (9), falls to .049 and
.103 for Figs. (10) and (11), respectively. Moreover, regular “islands” which can still be
seen in Fig (10) ( e. g., for \( \phi_i \) between .3 and .45 ) have quite dissapeared from Fig (11).

In conclusion, numerical simulation of the model strongly confirms the results of the
analysis in Section II. Moreover, chaos places severe limitations in our ability to predict
the future behavior of the Universe, even before the Cosmic Singularity is reached for the first time.

**IV - Conclusions**

The analytical and numerical analysis above shows that the dynamics of a spatially closed Friedmann - Robertson - Walker Universe coupled to a conformal but massive scalar field displays chaotic behavior. Indeed, in appropriate invariant sets, the dynamics of this cosmological model is equivalent to a Bernoulli shift, and thus essentially indistinguishable from a purely random process\[30\]. The analysis above tells us how to identify those sets.

The random element we have alluded to appears in the solutions of the model through the nonlinear oscillations of the field as the Universe evolves from the initial cosmic singularity towards recollapse. As a result, our ability to predict the future behavior of the field is limited. This phenomenon demonstrably occurs within a single cosmic episode, and thus it is, in principle, observable.

The interest of this result probably lies less in its direct impact on present theories of the Universe, than in the avenues it opens for further research. As a description of the Universe, the model we have discussed here is obviously oversimplified. More seriously, the chaotic behavior we have analyzed requires high values of the field energy density, a regime where quantum effects should not be disregarded.

In our view, the interest of this model lies in its sheer simplicity. In this sense, we believe this model to be an almost ideal ground where to investigate the general issues associated with chaos and cosmology. For one thing, this model is so simple that its analysis at the quantum and semiclassical levels is not essentially harder than at the classical one. This affords an excellent opportunity to discuss the semiclassical limit of Quantum Cosmology in a highly non trivial framework, an issue which we will discuss in a separate communication.
On the other hand, the model we have presented is far from displaying the full richness of behavior than chaos may bring to cosmology. We continue our research on this most varied and rewarding field.

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FIGURE CAPTIONS

Figure 1 Evolution of the scale factor for a Robertson-Walker cosmology in a weak chaotic situation \((\phi_i = 1 \text{ and } \pi_i = -1)\). After several oscillations there is an unusual growth of the scale factor, this corresponds to an “inflationary-like” stage.

Figure 2 Evolution of the field \(\phi\) corresponding to the initial conditions of Fig.1. It can be appreciated the oscillatory behavior with a sudden change, in frequency and amplitude, during the “inflationary” stages.

Figure 3 Maximal Lyapunov exponent \((\phi_i = 1, \pi_i = -1 \text{ and discarding a transient stage near } \eta = 0)\). Its value is clearly greater than zero.

Figure 4 Maximal Lyapunov exponent \((\phi_i = 10, \pi_i = -10 \text{ and discarding a transient stage near } \eta = 0)\).

Figure 5 Maximal Lyapunov exponent \((\phi_i = 1, \pi_i = -100 \text{ and discarding a transient stage near } \eta = 0)\). It can be appreciated that its value increase with the increase of initial condition, as predicted in section II.

Figure 6 Projection of the phase space trajectory onto the \(\phi, a\) plane. In this situation \((\phi_i = 1 \text{ and } \pi_i = -1)\), it behaves as a smooth Lissajous curve.

Figure 7 Same as Fig.6, \((\phi_i = 1 \text{ and } \pi_i = -100)\). The trajectory is very irregular, as indicated by its high Lyapunov exponent.

Figure 8 Same as Fig.6. This corresponds to the Lyapunov shown in Fig.4 \((\phi_i = 10 \text{ and } \pi_i = -10)\). See the increase of amplitude in the scale factor in the last two cases.

Figure 9 Field at the Big Crunch against field at the Big Bang. The initial data are: \(a_i = 0.001 \text{ (for computational conveniences)}, \pi_i = -1 \text{ and } \phi_i \text{ running from zero to one. There is a simple relation between } \phi_f \text{ and } \phi_i\).

Figure 10 Same as Fig.9, with \(a_i = 0.001, \pi_i = -10 \text{ and } \phi_i \text{ running from zero to ten. “Islands” of stability may still be seen, for example between (1.5 and 3), (3 and 4.5) and (5 and 6).} \)
Figure 11 Same as Fig.9, with $a_i = 0.1$, $\pi_i = -100$ and $\phi_i$ running from zero to one hundred. In this situation there is no clear correlation between initial and final data.