Lewenstein-Sanpera decomposition for Bell decomposable states

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October 31, 2018

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Abstract

We propose a simple geometrical approach for finding the Lewenstein-Sanpera decomposition of Bell decomposable states of $2 \otimes 2$ quantum systems. We show that in these systems, the weight of the pure entangled part in the decomposition is equal to the concurrence of the state. It is also shown that the optimized separable part of L-S decomposition minimizes the von Neumann relative entropy. We also obtain the decomposition for a class of mixed states by using some LQCC actions. It is also shown that for these states the average concurrence of L-S decomposition is equal to their concurrence.

**Keywords:** Quantum entanglement, Bell decomposable states, Lewenstein-Sanpera decomposition, Concurrence

**PACs Index:** 03.65.U
1 Introduction

Entanglement is one of the most striking features of quantum mechanics [1, 2]. The non local character of an entangled system is usually manifested in quantum correlations between non interacting subsystems provided that they had only interaction in the past. A bipartite mixed state is said to be separable (non entangled) if it can be written as a convex combination of pure states

\[ \rho = \sum_i p_i \left| \phi_i^A \right\rangle \left\langle \phi_i^A \right| \otimes \left| \psi_i^B \right\rangle \left\langle \psi_i^B \right|, \]  

(1-1)

where \( \left| \phi_i^A \right\rangle \) and \( \left| \psi_i^B \right\rangle \) are pure states of subsystems A and B, respectively. In the case of pure states it is easy to check whether a given state is, or is not entangled. Entangled pure states do always violate Bell inequalities [3]. For mixed states, however, the statistical properties of the mixture can hide the quantum correlations embodied in the system, making thus the distinction between separable and entangled states enormously difficult.

In the pioneering paper [4], a very interesting description of entanglement was achieved by defining the best separable approximation (BSA) of a mixed state. In the case of 2-qubit system, it consists of a decomposition of the state into a linear combination of mixed separable part and a pure entangled one. In this way, the whole non-separability properties are concentrated in the pure part. It also provides a natural measure of entanglement given by the entanglement of the pure part (well defined for pure states) multiplied by the weight of the pure part in the composition.

In the Ref. [4], the numerical method for finding the BSA has been reported. Also in 2 \( \otimes \) 2 systems some analytical results for special states were found in [5]. An attempt to generalize the results of Ref. [4] is made in [6].

In [7] an algebraic approach to find BSA of a 2-qubit state is attempted. They have also showed that the weight of the entangled part in the decomposition is equal to the concurrence of the state.
In this paper we consider Bell decomposable (BD) states. We provide a simple geometrical approach and give an analytical expression for L-S decomposition, where our results are in agreement with those reported in \cite{4, 5}. Our method to find L-S decomposition is geometrically intuitive. We also see that the weight of the entangled part in the decomposition is equal to the concurrence of the state. It is also shown that separable state optimizing L-S decomposition, minimizes the von Neumann relative entropy introduced in \cite{8, 9} as a measure of entanglement. Starting from BD states, we perform local quantum operations and classical communications (LQCC) and find L-S decomposition for a generic two qubit system. We prove that for some special LQCC the obtained decomposition is optimal. It is also shown that for these cases average concurrence of the decomposition is equal to concurrence.

The paper is organized as follows. In section 2 we review BD states and present a perspective of their geometry. L-S decomposition of these states is obtained in section 3 via a geometric approach. We prove that this decomposition is optimal. Relation between L-S decomposition and relative entropy is discussed in section 4. It is shown that BSA also minimize von Neumann relative entropy. Effect of LQCC on L-S decomposition is studied in section 5. The paper is ended with a brief conclusion.

2 Bell Decomposable States

In this section we briefly review Bell decomposable (BD) states and some of their properties. A BD state is defined by

\[ \rho = \sum_{i=1}^{4} p_i \, |\psi_i\rangle \langle \psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_{i=1}^{4} p_i = 1, \quad (2.2) \]
where \( |\psi_i\rangle \) is Bell state given by

\[
|\psi_1\rangle = |\phi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle),
\]

\[
|\psi_2\rangle = |\phi^-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),
\]

\[
|\psi_3\rangle = |\psi^+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle),
\]

\[
|\psi_4\rangle = |\psi^-\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle).
\]

In terms of Pauli’s matrices, \( \rho \) can be written as

\[
\rho = \frac{1}{4} (I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i),
\]

where

\[
t_1 = p_1 - p_2 + p_3 - p_4,
\]

\[
t_2 = -p_1 + p_2 + p_3 - p_4,
\]

\[
t_3 = p_1 + p_2 - p_3 - p_4.
\]

From positivity of \( \rho \) we get

\[
1 + t_1 - t_2 + t_3 \geq 0,
\]

\[
1 - t_1 + t_2 + t_3 \geq 0,
\]

\[
1 + t_1 + t_2 - t_3 \geq 0,
\]

\[
1 - t_1 - t_2 - t_3 \geq 0.
\]

These equations form a tetrahedral with its vertices located at \((1, -1, 1), (-1, 1, 1), (1, 1, -1),\)

\((-1, -1, -1)\) \[10\]. In fact these vertices are Bell states given in Eqs. \((2-3)\) to \((2-6)\), respectively.

According to the Peres and Horodecki’s condition for separability \[11, 12\], a 2-qubit state is separable if and only if its partial transpose is positive. This implies that \( \rho \) given in Eq. \((2-7)\) is
separable if and only if \( t_i \)s satisfy Eqs. (2-9) and
\[
\begin{align*}
1 + t_1 + t_2 + t_3 & \geq 0, \\
1 - t_1 - t_2 + t_3 & \geq 0, \\
1 + t_1 - t_2 - t_3 & \geq 0, \\
1 - t_1 + t_2 - t_3 & \geq 0.
\end{align*}
\] (2-10)

Inequalities (2-9) and (2-10) form an octahedral with its vertices located at \( O_1^\pm = (\pm 1, 0, 0) \), \( O_2^\pm = (0, \pm 1, 0) \) and \( O_3^\pm = (0, 0, \pm 1) \). Hence, tetrahedral of Eqs. (2-9) is divided into five regions. Central regions, defined by octahedral, are separable states. There are also four smaller equivalent tetrahedral corresponding to entangled states. Each tetrahedral takes one Bell state as one of its vertices. Three other vertices of each tetrahedral form a triangle which is its common face with the octahedral (See Fig. 1).

3 Lewenstein-Sanpera decomposition

According to Lewenstein-Sanpera decomposition [4], any 2-qubit density matrix \( \rho \) can be written as
\[
\rho = \lambda \rho_{\text{sep}} + (1 - \lambda) |\psi\rangle \langle \psi|, \quad \lambda \in [0,1],
\] (3-11)
where \( \rho_{\text{sep}} \) is a separable density matrix and \( |\psi\rangle \) is a pure entangled state. The L-S decomposition of a given density matrix \( \rho \) is not unique and, in general, there is a continuum set of L-S decomposition to choose from. The optimal decomposition is unique in which \( \lambda \) is maximal, and
\[
\rho = \lambda^{(\text{opt})} \rho^{(\text{opt})}_{\text{sep}} + (1 - \lambda^{(\text{opt})}) |\psi^{(\text{opt})}\rangle \langle \psi^{(\text{opt})}|, \quad \lambda^{(\text{opt})} \in [0,1].
\] (3-12)
All other decomposition of the form \( \rho = \tilde{\lambda} \tilde{\rho}_{\text{sep}} + (1 - \tilde{\lambda}) |\tilde{\psi}\rangle \langle \tilde{\psi}| \), with \( \tilde{\lambda} \in [0,1] \) such that \( \tilde{\rho} \neq \rho^{(\text{opt})} \) necessarily implies that \( \tilde{\lambda} < \lambda^{(\text{opt})} \) [4].
In the following, we will refer to Eq. (3-11) as the optimal decomposition of $\rho$. The separable part $\rho_{sep}$ is called the best separable approximation (BSA) of $\rho$, and $\lambda$ is its separability.

Here in this section we obtain L-S decomposition for Bell decomposable states via a geometrical approach. Our results are in agreement with those reported by [4, 5]. In addition we present an explicit form for $\rho_{sep}$ and show that, pure entangled state $|\psi\rangle$ is Bell state which $\rho$ belongs to its entangled tetrahedral. For simplicity, we show in Fig. 2 entangled tetrahedral corresponding to singlet state [2-6].

Suppose $\rho$ is an entangled state parameterized as $\vec{t} = (t_1, t_2, t_3)$. We connect vertex $p$, which denotes singlet state, to point $\vec{t}$ and extend it to cut separable surface $O_1^-O_2^-O_3^-$ at $\vec{t}'$ corresponding to separable state $\rho_s$, where this line can defined by Eqs. $(1 + t_2)(x_1 - t_1) - (1 + t_1)(x_2 - t_2) = 0$ and $(1 + t_3)(x_2 - t_2) - (1 + t_2)(x_3 - t_3) = 0$. It can be easily seen that this line cuts plane $O_1^-O_2^-O_3^-$, defined by $x_1 + x_2 + x_3 + 1 = 0$, at point $\vec{t}' = (t_1', t_2', t_3')$

\begin{align*}
t_1' &= \frac{-1 + t_1 - t_2 - t_3}{3 + t_1 + t_2 + t_3}, \\
t_2' &= \frac{-1 - t_1 + t_2 - t_3}{3 + t_1 + t_2 + t_3}, \\
t_3' &= \frac{-1 - t_1 - t_2 + t_3}{3 + t_1 + t_2 + t_3}. \\
\end{align*}

Using Eq. (2-8) it is straightforward to obtain coordinates of $\rho_s$ in terms of parameters $p_i$ as

\begin{align*}
p_i' &= \frac{p_i}{2(1 - p_4)} \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad p_4' = \frac{1}{2}. \\
\end{align*}
Now, using the Eqs. (2-7), (3-13) we can write explicit form for separable state $\rho_s$:

$$\rho_s = \frac{1}{2(3 + t_1 + t_2 + t_3)} \begin{pmatrix} 1 + t_3 & 0 & 0 & t_1 - t_2 \\ 0 & 2 + t_1 + t_2 & -1 - t_3 & 0 \\ 0 & -1 - t_3 & 2 + t_1 + t_2 & 0 \\ t_1 - t_2 & 0 & 0 & 1 + t_3 \end{pmatrix}. \quad (3-15)$$

By convexity we can write $\rho$ as convex sum of $\rho_s$ and projector $|\psi^-\rangle \langle \psi^-|$:

$$\rho = \lambda \rho_s + (1 - \lambda) |\psi^-\rangle \langle \psi^-|. \quad (3-16)$$

Using Eqs. (3-13) and (3-16) we obtain

$$\lambda = \frac{|pt|}{|pt'|} = \frac{3 + t_1 + t_2 + t_3}{2} = 1 - C, \quad (3-17)$$

where $|pt|$ and $|pt'|$ are distances between points $p$, $t$ and also $p'$, $t'$ respectively, and $C$ is concurrence of $\rho$. [13]

Obviously Eq. (3-17) implies that the entanglement contribution of singlet state in L-S decomposition of the BD states is the same as its concurrence. The concurrence of a mixed state is defined as the minimum of the average concurrence over all decompositions of the state in terms of pure states [13]. This means that for the L-S decomposition given in (3-11) we have $C(\rho) \leq (1 - \lambda)C(\psi)$. Eq. (3-17) shows that optimal decomposition of BD states saturate this inequality.

Now, in order to show that $\lambda$ of Eq. (3-17) is maximal and thus the decomposition (3-16) with $\rho_s$ given in Eq. (3-13) is optimal, first we show that $\rho_s$ can be written in terms of product states. In fact $\rho_s$ of Eq. (3-17) can be written as a convex sum of three states corresponding to three vertices $O_1^-, O_2^-, O_3^-$ of octahedral,

$$\rho_s = \lambda_1^+ \rho_1^- + \lambda_2^+ \rho_2^- + \lambda_3^+ \rho_3^-, \quad (3-18)$$
where

\[
\rho_{1}^{-} = \frac{1}{2}(|x_{+}\rangle \langle x_{+}| \otimes |x_{-}\rangle \langle x_{-}| + |x_{-}\rangle \langle x_{-}| \otimes |x_{+}\rangle \langle x_{+}|),
\]
\[
\rho_{2}^{-} = \frac{1}{2}(|y_{+}\rangle \langle y_{+}| \otimes |y_{-}\rangle \langle y_{-}| + |y_{-}\rangle \langle y_{-}| \otimes |y_{+}\rangle \langle y_{+}|),
\]
\[
\rho_{3}^{-} = \frac{1}{2}(|z_{+}\rangle \langle z_{+}| \otimes |z_{-}\rangle \langle z_{-}| + |z_{-}\rangle \langle z_{-}| \otimes |z_{+}\rangle \langle z_{+}|),
\]

and \(|x_{\pm}\rangle\), \(|y_{\pm}\rangle\) and \(|z_{\pm}\rangle\) are eigenstates corresponding to eigenvalues \(\pm 1\) of \(\sigma_{x}\), \(\sigma_{y}\) and \(\sigma_{z}\), respectively. Considering the fact that for any point interior to equilateral triangles, the sum of the orthogonal distances of the point to three edges is equal to the height of triangle, for triangle \(O_{1}^{-}O_{2}^{-}O_{3}^{-}\) we have \(h_{1} + h_{2} + h_{3} = \sqrt{3}/2\), where \(h_{1}\), \(h_{2}\) and \(h_{3}\) are orthogonal distances from corresponding edges (See Fig. 3). After straightforward calculation we get

\[
h_{i} = -\sqrt{3} t'_{i},
\]

where \(t'_{i}, (i = 1, 2, 3)\) are coordinates of \(\rho_{s}\) given by (3-13).

Taking into account the fact that \(\lambda_{i}^{-}\) is proportional to \(h_{i}\) and \(\lambda = \lambda_{1}^{-} + \lambda_{2}^{-} + \lambda_{3}^{-}\), we get

\[
\lambda_{1}^{-} = \frac{1}{2}(1 - t_{1} + t_{2} + t_{3}),
\]
\[
\lambda_{2}^{-} = \frac{1}{2}(1 + t_{1} - t_{2} + t_{3}),
\]
\[
\lambda_{3}^{-} = \frac{1}{2}(1 + t_{1} + t_{2} - t_{3}).
\]

The same is true for other Bell decomposable states belonging to other maximally entangled tetrahedral.

Using above results we rewrite \(\rho\) given in Eq. (3-16) in terms of product states and pure entangled state

\[
\rho = \sum_{\alpha=1}^{6} \Lambda_{\alpha} |e_{\alpha}, f_{\alpha}\rangle \langle e_{\alpha}, f_{\alpha}| + |\psi^{-}\rangle \langle \psi^{-}|,
\]

(3-21)
where $|e_\alpha, f_\alpha\rangle$, $\alpha = 1, 2, ..., 6$ are product states defined by

$$
|e_1, f_1\rangle = |x_+\rangle \otimes |x_-\rangle, \quad |e_2, f_2\rangle = |x_-\rangle \otimes |x_+\rangle, \\
|e_3, f_3\rangle = |y_+\rangle \otimes |y_-\rangle, \quad |e_4, f_4\rangle = |y_-\rangle \otimes |y_+\rangle, \\
|e_5, f_5\rangle = |z_+\rangle \otimes |z_-\rangle, \quad |e_6, f_6\rangle = |z_+\rangle \otimes |z_-\rangle,
$$

(3-22)

and $\Lambda_\alpha$, $\alpha = 1, 2, ... 6$ are given by

$$
\Lambda_1 = \Lambda_2 = \frac{\lambda_1^-}{2}, \\
\Lambda_3 = \Lambda_4 = \frac{\lambda_2^-}{2}, \\
\Lambda_5 = \Lambda_6 = \frac{\lambda_3^-}{2}.
$$

(3-23)

(3-24)

(3-25)

Now with this notation, we are in position to prove that decomposition (3-16) with $\lambda$ given in (3-17) is optimal. To do this we show that all coefficients of product states appeared in (3-21) are maximal. According to [4] maximizing all the pairs $(\Lambda_\alpha, \Lambda_\beta)$ with respect to $\rho_{\alpha\beta} = \rho - \sum_{\alpha' \neq \alpha, \beta} \Lambda_{\alpha'} P_{\alpha'}$ and $(P_\alpha, P_\beta)$ is a necessary and sufficient condition to subtract the maximal separable matrix $\rho_s^\alpha$ from $\rho$, where for the sake of self-containment we quote the theorem 2 and the related lemmas of reference [4] below.

**Theorem 1** [4]

*Given the set $\Lambda_V$ of product vectors $|e, f\rangle \in \mathcal{R}(\rho)$, the matrix $\rho_s^\alpha = \sum_{\alpha} \Lambda_\alpha P_\alpha$ is the best separable approximation to $\rho$ iff a) all $\Lambda_\alpha$ are maximal with respect to $\rho_{\alpha\beta} = \rho - \sum_{\alpha' \neq \alpha, \beta} \Lambda_{\alpha'} P_{\alpha'}$ and the projector $P_\alpha$; b) all pairs $(\Lambda_\alpha, \Lambda_\beta)$ are maximal with respect to $\rho_{\alpha\beta} = \rho - \sum_{\alpha' \neq \alpha, \beta} \Lambda_{\alpha'} P_{\alpha'}$, and the projectors $(P_\alpha, P_\beta)$.***

**Lemma 2** [4] *\Lambda is maximal with respect to $\rho$ and $P = |\psi\rangle \langle \psi|$ iff a) if $|\psi\rangle \notin \mathcal{R}(\rho)$ then $\Lambda = 0$, and b) if $|\psi\rangle \in \mathcal{R}(\rho)$ then $\Lambda = (\langle \psi | \rho^{-1} |\psi\rangle)^{-1} > 0$.***
Lemma 3 A pair \((\Lambda_1, \Lambda_2)\) is maximal with respect to \(\rho\) and a pair of projectors \((P_1, P_2)\) iff: a) if \(|\psi_1\rangle, |\psi_2\rangle\) do not belong to \(\mathcal{R}(\rho)\) then \(\Lambda_1 = \Lambda_2 = 0\); b) if \(|\psi_1\rangle\) does not belong, while \(|\psi_2\rangle\) \(\in \mathcal{R}(\rho)\) then \(\Lambda_1 = 0, \Lambda_2 = \langle \psi_2 | \rho^{-1} | \psi_2 \rangle^{-1}\); c) if \(|\psi_1\rangle, |\psi_2\rangle\) \(\in \mathcal{R}(\rho)\) and \(\langle \psi_1 | \rho^{-1} | \psi_2 \rangle = 0\) then \(\Lambda_i = \langle \psi_i | \rho^{-1} | \psi_i \rangle^{-1}, i = 1, 2\); d) finally, if \(|\psi_1\rangle, |\psi_2\rangle\) \(\in \mathcal{R}(\rho)\) and \(\langle \psi_1 | \rho^{-1} | \psi_2 \rangle \neq 0\) then

\[
\Lambda_1 = \frac{\langle \psi_2 | \rho^{-1} | \psi_2 \rangle - \langle \psi_1 | \rho^{-1} | \psi_2 \rangle}{D},
\]

\[
\Lambda_2 = \frac{\langle \psi_1 | \rho^{-1} | \psi_1 \rangle - \langle \psi_1 | \rho^{-1} | \psi_2 \rangle}{D},
\]

where \(D = \langle \psi_1 | \rho^{-1} | \psi_1 \rangle \langle \psi_2 | \rho^{-1} | \psi_2 \rangle - \langle \psi_1 | \rho^{-1} | \psi_2 \rangle^2\).

First, we show that \(\Lambda_\alpha\)'s are maximal with respect to \(\rho_\alpha\) and \(P_\alpha\).

Matrices \(\rho_\alpha = \rho - \sum_{\alpha' \neq \alpha}^6 \Lambda_{\alpha'} P_{\alpha'} + (1 - \lambda) | \psi^- \rangle \langle \psi^- |\) with \(P_\alpha = | e_\alpha, f_\alpha \rangle \langle e_\alpha, f_\alpha |\) for \((i = 1, 2, ..., 6)\) have two zero eigenvalues and two non zero eigenvalues. In Bell basis its kernel and range are separated. After restriction to its range, it is straightforward to evaluate \(\rho_i^{-1}\) and we find that

\[
\langle e_i, f_i | \rho_i^{-1} | e_i, f_i \rangle = 1/\Lambda_i.
\]

In order to prove that the pair \((\Lambda_\alpha, \Lambda_\beta)\) are maximal with respect to \(\rho_\alpha\beta\) and the pair of projectors \((P_\alpha, P_\beta)\), we proceed as follows:

a) Matrices \(\rho_{i,i+1} = \Lambda_i P_i + \Lambda_{i+1} P_{i+1} + (1 - \lambda) | \psi^- \rangle \langle \psi^- |\) for \((i = 1, 3, 5)\) have a two dimensional range. In Bell basis its range and kernel are separated and one can obtain \(\langle e_i, f_i | \rho_{i,i+1}^{-1} | e_i, f_i \rangle = (\Lambda_{i+1} + (1 - \lambda))/\Gamma_i, \langle e_{i+1}, f_{i+1} | \rho_{i,i+1}^{-1} | e_{i+1}, f_{i+1} \rangle = (\Lambda_i + (1 - \lambda))/\Gamma_i\) and \(\langle e_i, f_i | \rho_{i,i+1}^{-1} | e_{i+1}, f_{i+1} \rangle = (1 - \lambda)/(2\Gamma_i)\), where \(\Gamma_i = \Lambda_i \Lambda_{i+1} + \frac{1}{2}(1 - \lambda)\). Using the above results together with Eqs. (3-26) we obtain the maximality of pair \((\Lambda_i, \Lambda_{i+1})\) with respect to \(\rho_{i,i+1}\) and the pair of projectors \((P_i, P_{i+1})\) for \(i = 1, 3\) and 5.

b) For other possibility of \(\alpha\) and \(\beta\), matrices \(\rho_{\alpha\beta} = \Lambda_\alpha P_\alpha + \Lambda_\beta P_\beta + (1 - \lambda) | \psi^- \rangle \langle \psi^- |\) have rank 3. Using the Bell basis we can evaluate \(\rho^{-1}\) and we find that \(\langle e_\alpha, f_\alpha | \rho^{-1} | e_\beta, f_\beta \rangle = 0\) for \(\alpha \neq \beta\),
\[ \langle e_\alpha, f_\alpha | \rho^{-1} | e_\alpha, f_\alpha \rangle = 1/\Lambda_\alpha. \] This completes the proof that \( \Lambda_\alpha \) of Eq. (3-20) are maximal and decomposition (3-16) is optimal.

Also it is worth to note that the decomposition (3-16) satisfies conditions for BSA of Ref. [7]. According to theorem 1 of Ref. [7], decomposition given in (3-16) is the optimal decomposition if and only if: rank(\( \rho_{sT}^B \)) = 3, i.e. \( \exists |\phi\rangle \rho_{sT}^B |\phi\rangle = 0 \), and either

\begin{enumerate}
\item[(i)] \( \exists_{\alpha > 0} (|\phi\rangle \langle \phi|)^T_B |\psi\rangle = - \alpha |\psi\rangle \), or
\item[(ii)] rank(\( \rho_s \)) = 3, i.e. \( \exists |\tilde{\phi}\rangle \rho_s |\tilde{\phi}\rangle = 0 \), and \( \exists_{\alpha, \nu \geq 0} (\nu |\tilde{\phi}\rangle \langle \tilde{\phi}| + (|\phi\rangle \langle \phi|)^T_B ) |\psi\rangle = - \alpha |\psi\rangle \).
\end{enumerate}

(3-27)

It is now straightforward to see that \( \rho_{sT}^B \) has three non vanishing eigenvalues, that is, its rank is 3. Its one dimensional kernel is along the Bell state \( |\psi_1\rangle \) given in Eq. (2-3). Actually the density matrices corresponding to the interior of tetrahedral satisfy condition (i) while those at its boundary satisfy condition (ii), respectively.

### 4 Relative entropy of entanglement and L-S decomposition

Vedral et al. in [8, 9] introduced a class of distance measures suitable for entanglement measures. According to their methods, entanglement measure for a given state \( \rho \) is defined as

\[ E(\rho) = \min_{\sigma \in D} D(\rho \parallel \sigma), \]

(4-28)

where \( D \) is any measure of distance (not necessarily a metric) between two density matrix \( \rho \) and \( \sigma \), and \( D \) is the set of all separable states. They have also shown that von Neumann relative entropy defined by

\[ S(\rho \parallel \sigma) = tr(\rho \ln \frac{\rho}{\sigma}), \]

(4-29)

satisfies three conditions that a good measure of entanglement must satisfy [8]. Here, we would like to emphasis that \( \rho_s \) given in Eq. (3-15) minimizes von Neumann relative entropy given in (4-29).
Authors in [8] have shown that for BD states given in Eq. (2-2), separable state $\sigma$ that minimize relative entropy is

$$p'_i = \frac{p_i}{2(1-p_4)} \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad p'_4 = \frac{1}{2}. \quad (4-30)$$

It is worth to note that the above equation is the same as Eq. (4-30), that is, separable state optimizing L-S decomposition minimizes von Neumann relative entropy, too.

## 5 L-S decomposition under LQCC

In this section we study the behavior of L-S decomposition under local quantum operations and classical communications (LQCC). A general LQCC is defined by [14, 15]

$$\rho' = \frac{(A \otimes B)\rho(A \otimes B)^\dagger}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)}, \quad (5-31)$$

where operators $A$ and $B$ can be written as

$$A \otimes B = U_A f^{\mu,a,m} \otimes U_B f^{\nu,b,n}, \quad (5-32)$$

where $U_A$ and $U_B$ are unitary operators acting on subsystems $A$ and $B$, respectively and the filtration $f$ defined by

$$f^{\mu,a,m} = \mu(I_2 + a m.\sigma), \quad (5-33)$$

$$f^{\nu,b,n} = \nu(I_2 + b n.\sigma).$$

As it is shown in Refs. [14, 15], the concurrence of the state $\rho$ transforms under LQCC of the form given in Eq. (5-31) as

$$C(\rho') = \frac{\mu^2 \nu^2 (1-a^2)(1-b^2)}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} C(\rho). \quad (5-34)$$

Performing LQCC on L-S decomposition of BD states we get
L-S decomposition for BD states

\[
\rho' = \frac{(A \otimes B)\rho(A \otimes B)^\dagger}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} = \lambda' \rho'_s + (1 - \lambda') |\psi'\rangle \langle \psi'|, \tag{5-35}
\]

with \( \rho'_s \) and \( |\psi'\rangle \) defined as

\[
\rho'_s = \frac{(A \otimes B)\rho_s(A \otimes B)^\dagger}{\text{tr}((A \otimes B)\rho_s(A \otimes B)^\dagger)} \tag{5-36}
\]

\[
|\psi'\rangle = \frac{(A \otimes B)|\psi^-\rangle}{\sqrt{\langle \psi^-| (AA^\dagger \otimes BB^\dagger) |\psi^-\rangle}}, \tag{5-37}
\]

respectively, and \( \lambda' \) is

\[
\lambda' = \frac{\text{tr}((A \otimes B)\rho_s(A \otimes B)^\dagger)}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} \lambda. \tag{5-38}
\]

Using Eq. (5-38), we get for the weight of entangled part in the decomposition (5-35)

\[
(1 - \lambda') = \frac{\langle \psi^-| (AA^\dagger \otimes BB^\dagger) |\psi^-\rangle}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} (1 - \lambda). \tag{5-39}
\]

Now we can easily evaluate the average concurrence of \( \rho' \) in the L-S decomposition given in (5-35)

\[
(1 - \lambda')C(|\psi'\rangle) = \frac{\mu^2 \nu^2 (1 - \alpha^2)(1 - \beta^2)}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} (1 - \lambda)C(|\psi\rangle), \tag{5-40}
\]

where, by Comparing the above equation with Eq. (5-34) we see that \((1 - \lambda)C(|\psi\rangle) \) (the average concurrence in the L-S decomposition) transforms like concurrence under LQCC.

In order to prove that the decomposition (5-35) is the optimal one, we rewrite \( \rho_s \) in terms of the product states given in Eq. (3-22)

\[
\rho_s = \sum_{\alpha=1}^{6} \Lambda_{\alpha} \begin{vmatrix} e_{\alpha}, f_{\alpha} \end{vmatrix} \langle e_{\alpha}, f_{\alpha} |. \tag{5-41}
\]

Now, performing LQCC action we get

\[
\rho'_s = \sum_{\alpha=1}^{6} \Lambda'_{\alpha} \begin{vmatrix} e'_{\alpha}, f'_{\alpha} \end{vmatrix} \langle e'_{\alpha}, f'_{\alpha} |, \tag{5-42}
\]

where

\[
\begin{vmatrix} e'_{\alpha}, f'_{\alpha} \end{vmatrix} = \frac{(A \otimes B)\begin{vmatrix} e_{\alpha}, f_{\alpha} \end{vmatrix}}{\sqrt{t(P_{\alpha})}}, \quad t(P_{\alpha}) = \langle e_{\alpha}, f_{\alpha} | (AA^\dagger \otimes BB^\dagger) |e_{\alpha}, f_{\alpha}\rangle, \tag{5-43}
\]
with $P_\alpha = |e_\alpha, f_\alpha\rangle \langle e_\alpha, f_\alpha|$ and

$$\Lambda'_\alpha = \frac{\langle e_\alpha, f_\alpha | (AA^\dagger \otimes BB^\dagger) | e_\alpha, f_\alpha \rangle}{\text{tr}((A \otimes B)\rho(A \otimes B)^\dagger)} \Lambda_\alpha. \quad (5-44)$$

First we show that $\Lambda'_\alpha$s are maximal with respect to $\rho'_\alpha$ and the projector $P'_\alpha$.

As we see the matrices $\rho_\alpha = \Lambda_\alpha P_\alpha + (1 - \lambda) |\psi^-\rangle \langle \psi^-|$ for ($\alpha = 1, 2, ..., 6$) transform as

$$\rho'_\alpha = \frac{(A \otimes B)\rho_\alpha (A \otimes B)^\dagger}{t(\rho_\alpha)} \quad \text{and} \quad t(\rho_\alpha) = \text{tr}((A \otimes B)\rho_\alpha (A \otimes B)^\dagger), \quad (5-45)$$

under LQCC. Using the fact that LQCC transformations are invertible [15, 16, 17], we can evaluate $\rho'^{-1}_\alpha$ as

$$\rho'^{-1}_\alpha = t(\rho_\alpha) (A^\dagger \otimes B^\dagger)^{-1} \rho^{-1}_\alpha (A \otimes B)^{-1}. \quad (5-46)$$

Using the above equation and Eq. (5-43) we get

$$\langle e'_\alpha, f'_\alpha | \rho'^{-1}_\alpha | e'_\alpha, f'_\alpha \rangle = \frac{t(\rho)}{t(P_\alpha)} \langle e_\alpha, f_\alpha | \rho^{-1}_\alpha | e_\alpha, f_\alpha \rangle = \Lambda'_\alpha. \quad (5-47)$$

The Eq. (5-47) shows that $\Lambda'_\alpha$s are maximal with respect to $\rho'_\alpha$ and the projector $P'_\alpha$.

In order to prove that the pair $(\Lambda'_\alpha, \Lambda'_\beta)$ are maximal with respect to $(\rho'_\alpha, \rho'_\beta)$ and $(P'_\alpha, P'_\beta)$, we proceed as follows:

a) Matrices $\rho_{i,i+1} = \Lambda_i P_i + \Lambda_{i+1} P_{i+1} + (1 - \lambda) |\psi^-\rangle \langle \psi^-|$ transform under LQCC as

$$\rho'_{i,i+1} = \frac{(A \otimes B)\rho_{i,i+1} (A \otimes B)^\dagger}{t(\rho_{i,i+1})}, \quad t(\rho_{i,i+1}) = \text{tr}((A \otimes B)\rho_{i,i+1} (A \otimes B)^\dagger), \quad (5-48)$$

Using the above equation and invertibility of LQCC we arrive at the following results

$$\langle e'_{i+1}, f'_{i+1} | \rho'^{-1}_{i,i+1} | e'_{i+1}, f'_{i+1} \rangle = \frac{t(\rho)}{t(P_i)} \left( \Lambda_i + \frac{1}{2} (1 - \lambda) \right) \frac{1}{\Gamma_i}, \quad (5-49)$$
\[ \langle e'_i, f'_i | \rho_{i,i+1}^{-1} | e'_i, f'_i \rangle = \frac{t(\rho)}{\sqrt{t(P_i) t(P_{i+1})}} \frac{(1 - \lambda)}{2\Gamma_i}, \]

where \( \Gamma_i = \Lambda_i \Lambda_{i+1} + \frac{1}{2}(1 - \lambda)(\Lambda_i \Lambda_{i+1}) \). Now using Eqs. (3-26) we get

\[ \frac{\langle e'_i, f'_i | \rho_{i,i+1}^{-1} | e'_i, f'_i \rangle - \langle e'_i, f'_i | \rho_{i,i+1}^{-1} | e'_{i+1}, f'_{i+1} \rangle}{D} = \Lambda_{i+1}' \]

\[ + (1 - \lambda) \frac{(t(P_{i+1}) - \sqrt{t(P_i) t(P_{i+1})})}{t(\rho)}, \quad (5-50) \]

\[ \frac{\langle e'_{i+1}, f'_{i+1} | \rho_{i,i+1}^{-1} | e'_{i+1}, f'_{i+1} \rangle - \langle e'_{i+1}, f'_{i+1} | \rho_{i,i+1}^{-1} | e'_{i+1}, f'_{i+1} \rangle}{D} = \Lambda_i' \]

\[ + (1 - \lambda) \frac{(t(P_{i+1}) - \sqrt{t(P_i) t(P_{i+1})})}{t(\rho)}. \quad (5-51) \]

b) For other values of \( \alpha \) and \( \beta \), matrices \( \rho_{\alpha,\beta} = \Lambda_{\alpha} P_{\alpha} + \Lambda_{\beta} P_{\beta} + (1 - \lambda) |\psi^{-}\rangle \langle\psi^{-}| \) transform under LQCC as

\[ \rho'_{\alpha,\beta} = \frac{(A \otimes B) \rho_{\alpha,\beta} (A \otimes B)^\dagger}{t(\rho_{\alpha,\beta})}, \quad t(\rho_{\alpha,\beta}) = tr((A \otimes B) \rho_{\alpha,\beta} (A \otimes B)^\dagger). \quad (5-52) \]

So with same procedure we can evaluate the following expressions

\[ \langle e'_\alpha, f'_\alpha | \rho'_{\alpha,\beta}^{-1} | e'_\alpha, f'_\alpha \rangle = \frac{1}{\Lambda'_\alpha}, \]

\[ \langle e'_\beta, f'_\beta | \rho'_{\alpha,\beta}^{-1} | e'_\beta, f'_\beta \rangle = \frac{1}{\Lambda'_\beta}, \quad (5-53) \]

\[ \langle e'_\alpha, f'_\alpha | \rho'_{\alpha,\beta}^{-1} | e'_\beta, f'_\beta \rangle = 0, \quad \text{for} \quad \alpha \neq \beta. \]

Eqs. (5-50) and (5-51) show that the pair \( (\Lambda'_\alpha, \Lambda'_\beta) \) are maximal with respect to \( \rho_{\alpha,\beta} \) and \( (P'_\alpha, P'_\beta) \) provided that \( t(P_i) = t(P_{i+1}) \). This restricts LQCC to special case of \( A = B \). Under these conditions the decomposition given in Eq. (5-35) is optimal. Actually considering the fact that the inequality \( C(\rho') \leq (1 - \lambda') C(\psi') \) is saturated by decomposition given in (5-35) we conjecture that this decomposition is optimal for a general LQCC, that is, \( \rho'_s \) is the BSA for \( \rho' \). This implies that, in general, the product states given in Eq. (5-43) is not the good product ensemble for the best
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separable part. To prove that the decomposition is the optimal one for the whole class of LQCC is still an open problem which is under investigation.

6 Conclusion

We have derived Lewenstein-Sanpera decomposition for Bell decomposable states from an entirely different approach. We show that for these systems, the weights of the pure entangled part in the decomposition is equal to the concurrence of the states. Optimality of the presented decomposition have been proved by using the theorems given in [4]. It is also shown that the optimized separable part of L-S decomposition minimizes the von Neumann relative entropy. We have also obtained Lewenstein-Sanpera decomposition for a large class of states obtained from BD states via some LQCC action. It is shown that for these states the average concurrence of the decomposition is equal to their concurrence.

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Figures Captions

Figure 1: All BD states are defined as points interior to tetrahedral. Vertices $P_1$, $P_2$, $P_3$ and $P_4$ denote projectors corresponding to Bell states Eqs. (2-3) to (2-6), respectively. Octahedral corresponds to separable states.

Figure 2: Entangled tetrahedral corresponding to singlet state. Line $pc$ denotes entangled Werner states. Points $t$ and $t'$ correspond to a generic BD state $\rho$ and associated BSA $\rho_s$. Vertex $p$ denote singlet state and other vertices are defined in Eq. (3-19).

Figure 3: $\rho_s$ can be written as a convex combination of separable states $\rho_1^-$, $\rho_2^-$ and $\rho_3^-$ with weights proportional to $h_1$, $h_2$ and $h_3$, respectively.
Figure 1:
Figure 2:
Figure 3: