RÉNYI DIVERGENCE AND THE CENTRAL LIMIT THEOREM

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ABSTRACT. We explore properties of the $\chi^2$ and more general Rényi (Tsallis) distances to the normal law. In particular we provide necessary and sufficient conditions for the convergence to the normal law in the central limit theorem using these distances. Moreover, we derive exact rates of convergence in these distances with respect to an increasing number of summands.

1. Introduction

Given random elements $X$ and $Z$ in a measurable space $(\Omega, \mu)$ with densities $p$ and $q$ (with respect to $\mu$), the $\chi^2$-distance of Pearson

$$\chi^2(X, Z) = \int \frac{(p-q)^2}{q} d\mu$$

represents an important measure of deviation of the distribution $P$ of $X$ from the distribution $Q$ of $Z$, which has been frequently used especially in Statistics and Information Theory (cf. e.g. [Le], [L-V], [V]). It is a rather strong distance-like quantity, which may be related to and included in the hierarchy of Rényi divergences (relative $\alpha$-entropies)

$$D_\alpha(X||Z) = \frac{1}{\alpha - 1} \log \int \left(\frac{p}{q}\right)^\alpha q d\mu \quad (\alpha > 0)$$

or equivalently, the Rényi divergence powers or the relative Tsallis entropies $T_\alpha(X||Z) = \frac{1}{\alpha - 1} \left[e^{(\alpha-1)D_\alpha} - 1\right]$ (which do not depend on the choice of the dominating measure $\mu$). The most important indexes are $\alpha = 0$, $\alpha = \frac{1}{2}$ (Hellinger distance), $\alpha = 1$ (Kullback-Leibler distance) and $\alpha = 2$ (quadratic Rényi/Tsallis divergence), in which case $T_2 = \chi^2$ and $T_2 = \log(1 + \chi^2)$.

The functionals $D_\alpha$ and $T_\alpha$ are non-decreasing in $\alpha$, so, for growing indexes the distances are strengthening. In the range $0 < \alpha < 1$, all $D_\alpha$ are comparable to each other and are metrically equivalent to the total variation $\|P - Q\|_{TV}$. However, the informational divergence $D = D_1 = T_1$ (called also entropic distance or relative entropy),

$$D(X||Z) = \int p \log \frac{p}{q} d\mu,$$

is much stronger, and this applies even more so to $D_\alpha$ with $\alpha > 1$. The difference between the different $D_\alpha$’s appears in applications like the central limit theorem (CLT for short), which is studied in this paper. Here we consider the $\chi^2$-divergence in the simplest situation of independent, identically distributed (i.i.d.) summands.

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For i.i.d. random variables $X, X_1, X_2, \ldots$ such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, introduce the normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \quad (n = 1, 2, \ldots)$$

together with their distributions $F_n$, which hence approach the standard normal law $\Phi$ in the weak sense. For convergence in the CLT using strong distances, recall that convergence in total variation was addressed in the 1950’s by Prokhorov [Pr]. He showed that $\|F_n - \Phi\|_{TV}$ tends to zero as $n \to \infty$, if and only if $F_n$ has a non-trivial absolutely continuous component for some $n = n_0$, i.e., $\|F_{n_0} - \Phi\|_{TV} < 2$ (in particular, this is true, if $X$ has density). A similar description is due to Barron [B] in the 1980’s for the Kullback-Leibler distance: $D(Z_n || Z)$ tends to zero for $Z \sim N(0, 1)$, if and only if $D(Z_n || Z) < \infty$ for some $n = n_0$. The latter condition is fulfilled for a large family of underlying distributions, in particular, when $X$ has density $p$ such that

$$\int_{-\infty}^{\infty} p(x) \log p(x) \, dx < \infty.$$ 

Different aspects of such strong CLT’s, including the non-i.i.d. situation and the problem of rates or Berry-Esseen bounds, were studied by many authors, and we refer an interested reader to [Li], [S-M], [A-B-B-N], [B-J], [J], [B-C-G2-4], [B-C-K], [B-C].

As for convergence in $D_\alpha$ with $\alpha > 1$, not much is known so far. This case seems to be quite different in nature, and here the distance restricts the range of applicability of the CLT quite substantially. When focusing on the particular value $\alpha = 2$, we are concerned with the behavior of the quantity

$$\chi^2(Z_n, Z) = \int_{-\infty}^{\infty} \frac{(p_n(x) - \varphi(x))^2}{\varphi(x)} \, dx,$$

where $p_n$ denotes the density of $Z_n$ and $\varphi$ is the standard normal density. The finiteness of this integral already requires the existence of all moments of $X$ (and actually the existence of a “Gaussian moment”). This condition is to be expected, but the convergence to zero, and even the verification of the boundedness of $\chi^2(Z_n, Z)$ in $n$ is rather delicate. This problem has been studied in the early 1980’s by Fomin [F] in terms of the exponential series (using Cramer’s terminology) for the density of $X$,

$$p(x) = \varphi(x) \sum_{k=1}^{\infty} \frac{\sigma_k}{2^k k!} H_{2k}(x),$$

where $H_r$ denotes the $r$-th Chebyshev-Hermite polynomial. As a main result, he proved that $\chi^2(Z_n, Z) = O(\frac{1}{n})$ as $n \to \infty$, assuming that $p$ is compactly supported, symmetric, piecewise differentiable, such that the series coefficients satisfy $\sup_{k \geq 2} \sigma_k < 1$. This sufficient condition was verified for the uniform distribution on the interval $(-\sqrt{3}, \sqrt{3})$ (this specific length is caused by the assumption $\mathbb{E}X^2 = 1$). However, for many other examples, Fomin’s result does not seem to provide an applicable and satisfactory answers.

Fortunately, more or less simple necessary and sufficient conditions can be stated for the convergence in $\chi^2$ by using the Laplace transform of the distribution of $X$. One of the purposes of this paper is to provide the following characterization of a class which may be called the “domain of $\chi^2$-attraction to the normal law”.

**Theorem 1.1.** We have $\chi^2(Z_n, Z) \to 0$ as $n \to \infty$, if and only if $\chi^2(Z_n, Z)$ is finite for some $n = n_0$, and

$$\mathbb{E} e^{tX} < e^{t^2} \quad \text{for all real } t \neq 0.$$  \hspace{1cm} (1.1)
In this case the $\chi^2$-divergence admits an Edgeworth-type expansion

$$\chi^2(Z_n, Z) = \sum_{j=1}^{s-2} \frac{c_j}{n^j} + O\left(\frac{1}{n^{s-1}}\right) \quad \text{as } n \to \infty,$$

which is valid for every $s = 3, 4, \ldots$ with coefficients $c_j$ representing certain polynomials in the moments $\alpha_k = \mathbb{E} X^k$, $k = 3, \ldots, j + 2$.

For $s = 3$ this expansion simplifies to

$$\chi^2(Z_n, Z) = \frac{\alpha^2}{6n} + O\left(\frac{1}{n^2}\right),$$

and if $\alpha_3 = 0$ (as in the case of symmetric distributions), one may turn to the next moment of order $s = 4$, for which (1.2) yields

$$\chi^2(Z_n, Z) = \frac{(\alpha_4 - 3)^2}{24n^2} + O\left(\frac{1}{n^3}\right).$$

Let us note that the property $\chi^2(Z_n, Z) < \infty$ is rather close to the subgaussian condition (1.1). In particular, it implies that (1.1) is fulfilled for all $t$ large enough, as well as near zero, due to the variance assumption. It may happen, however, that (1.1) is fulfilled for all $t \neq 0$ except just one value $t = t_0$ (and then there will be no CLT for the $\chi^2$-distance). Various examples illustrating these conditions together with the convergence in $\chi^2$ will be given in the end of the paper.

A similar characterization continues to hold in the multidimensional case for mean zero i.i.d. random vectors $X, X_1, X_2, \ldots$ in $\mathbb{R}^d$ normalized to have identity covariance. Here we endow the Euclidean space with the canonical norm and scalar product. Moreover, one may extend these results to the range of indexes $\alpha > 1$, arriving at the following statement, where by $\alpha^* = \frac{\alpha}{\alpha - 1}$ we denote the conjugate index.

**Theorem 1.2.** Let $Z$ denote a random vector in $\mathbb{R}^d$ having a standard normal distribution. Then $D_\alpha(Z_n || Z) \to 0$ as $n \to \infty$, if and only if $D_\alpha(Z_n || Z)$ is finite for some $n = n_0$, and

$$\mathbb{E} e^{t(X)} < e^{\alpha^* |t|^2/2} \quad \text{for all } t \in \mathbb{R}^d, \ t \neq 0. \quad (1.4)$$

In this case, we necessarily have $D_\alpha(Z_n || Z) = O(1/n)$, and even $D_\alpha(Z_n || Z) = O(1/n^2)$, provided that the distribution of $X$ is symmetric about the origin.

Thanks to the existence of all moments of $X$, an Edgeworth-type expansion for $D_\alpha$ and $T_\alpha$ also holds similarly to (1.2), involving the mixed cumulants of the components of $X$. Such expansion shows in particular an equivalence

$$D_\alpha(Z_n || Z) \sim T_\alpha(Z_n || Z) \sim \frac{\alpha}{2} \chi^2(Z_n, Z),$$

provided that these distances tend to zero. Let us also note that the restriction imposed by (1.4) is asymptotically vanishing as $\alpha$ approaches 1. This means that we may expect to arrive at Barron’s theorem in the limit, though this is not rigorously shown here.

As a closely related issue, the Renyi divergence appears naturally in the study of normal approximation for densities $p_n$ of $Z_n$ in the form of non-uniform local limit theorems. Like in dimension one, denote by $\varphi$ the standard normal density in $\mathbb{R}^d$. 
Theorem 1.3. Suppose that $D_\alpha(Z_n||Z)$ is finite for some $n = n_0$, and let the property (1.4) be fulfilled. Then, for all $n$ large enough and for all $x \in \mathbb{R}^d$,

$$|p_n(x) - \varphi(x)| \leq \frac{c}{\sqrt{n}} e^{-|x|^2/(2\alpha^*)}$$

(1.5)

with some constant $c$ which does not depend on $n$. Moreover, the rate $1/\sqrt{n}$ on the right may be improved to $1/n$, provided that the distribution of $X$ is symmetric about the origin.

Thus, (1.5) is implied by the convergence $D_\alpha(Z_n||Z) \to 0$. Non-uniform bounds in the normal approximation have been intensively studied in the literature, cf. [Pe1-2], [I-L], [A1-2]. However, existing results start with weaker hypotheses (e.g. moment assumptions) and either provide a polynomial error of approximation with respect to $x$ (such as $1/(1+|x|^3)$), or deal with narrow zones contained in regions $|x| = o(\sqrt{n})$.

The paper consists of two parts. In the first part results about the functional $D_\alpha$ are collected, including moment (exponential) inequalities and special properties of characteristic functions. Moreover, a number of remarkable algebraic properties of the $\chi^2$-distance will be derived. They are related to the associated exponential series, the behavior under convolutions and heat semi-group transformations, and in higher dimensions – to the super additivity of $\chi^2$ with respect to its marginals. As a by-product, we establish the existence of densities in terms of the so-called normal moments. The second part is entirely devoted to the proof of Theorems 1.1-1.3. Employing an Edgeworth expansion for densities (together with the results from the first part), this proof heavily relies on the tools of Complex Analysis. To simplify the presentation, almost all proofs will be stated for the one dimensional case, deferring the modifications needed to extend Theorems 1.1-1.3 to higher dimensions to separate sections.

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Part I: The $D_\alpha$ and $\chi^2$-divergence from the normal law

2. Background on Rényi Divergence

First let us briefly review some general properties of the Rényi divergences. More details can be found in the recent paper by van Erven and Harremoës [E-H]; cf. also [Le], [S], [G-S].

Let $(\Omega, \mu)$ be a measure space (with a $\sigma$-finite measure), and let $X$ and $Z$ be random elements with values in $\Omega$, having distributions $P$ and $Q$ with densities $p = \frac{dP}{d\mu}$, $q = \frac{dQ}{d\mu}$, respectively. The following basic definitions go back to the work of Rényi [R].

**Definition 2.1.** Let $0 < \alpha < \infty$, $\alpha \neq 1$. The Rényi divergence of $P$ from $Q$ and the corresponding divergence power or relative Tsallis entropy of index $\alpha$ are the quantities

$$D_\alpha(X||Z) = D_\alpha(P||Q) = \frac{1}{\alpha - 1} \log \int \left(\frac{p}{q}\right)^\alpha q \, d\mu,$$

$$T_\alpha(X||Z) = T_\alpha(P||Q) = \frac{1}{\alpha - 1} \left[ \int \left(\frac{p}{q}\right)^\alpha q \, d\mu - 1 \right].$$

The divergence $D_\alpha$ admits an axiomatic characterization via certain postulates. As a natural generalization of the Kullback-Leibler distance, the definition of $T_\alpha$ was introduced by Tsallis in [T] (within the so-called “nonextensive thermostatistical formalism”), cf. also [B-T-P]. Both quantities are related by monotone transformations, namely

$$D_\alpha = \frac{1}{\alpha - 1} \log \left(1 + (\alpha - 1)T_\alpha\right), \quad T_\alpha = \frac{1}{\alpha - 1} \left[e^{(\alpha-1)D_\alpha} - 1\right].$$

Thus, when they are small, these quantities are equivalent. Both represent directional distances. In particular, $D_\alpha(P||Q) \geq 0$ and $D_\alpha(P||Q) = 0$, if and only if $P = Q$.

The Rényi divergence with $0 < \alpha < 1$ possesses some unique features, like for example an obvious skew symmetry

$$D_\alpha(P||Q) = \frac{\alpha}{1-\alpha} D_\alpha(Q||P),$$

where the coefficient on the right is equal to 1 when $\alpha = \frac{1}{2}$. In this case, $D_\alpha$ represents a function of the square of the Hellinger metric:

$$D_{1/2}(P||Q) = -2 \log \left(1 - \frac{1}{2} \text{Hel}^2(P, Q)\right).$$

Another remarkable property is the equivalence of all $D_\alpha$ in this range: If $0 < \alpha < \beta < 1$, then

$$\frac{\alpha}{1-\alpha} \frac{1-\beta}{1-\beta} D_\beta(P||Q) \leq D_\alpha(P||Q) \leq D_\beta(P||Q).$$

When $\alpha \in (0, 1)$ is fixed, $D_\alpha(P||Q)$ is a continuous function of the tuple $(P, Q)$ with respect to the total variation distance in both coordinates. Conversely, it majorizes the total variation distance between $P$ and $Q$. Gilardoni [G] has shown that

$$D_\alpha(P||Q) \geq \frac{\alpha}{2} \|P - Q\|^2_{TV}.$$

This extends the classical Pinsker inequality for the Kullback-Leibler distance (when $\alpha = 1$), with best constant due to Csiszár, cf. [Pi], [Cs].

The following general property is important for comparing the Rényi divergence with different values of $\alpha$.

**Proposition 2.2.** For all probability measures $P$ and $Q$ on $\Omega$, the functions $\alpha \to D_\alpha(P||Q)$ and $\alpha \to T_\alpha(P||Q)$ are non-decreasing.
The monotonicity of \( D_\alpha \) is discussed in [E-H], Theorem 3. As for \( T_\alpha \), let \( 0 < \alpha < \beta \), \( \alpha, \beta \neq 1 \). The functions \( c \rightarrow e^{ct_0} - 1 \) with fixed \( t_0 \geq 0 \) and \( t \rightarrow \frac{e^{ct} - 1}{t} \) are non-decreasing in \( c \geq 0 \) and \( t > 1 \), respectively. Hence, in case \( \alpha > 1 \), we get, using monotonicity of \( D_\alpha \),

\[
T_\alpha(P||Q) = \frac{1}{\alpha - 1} \left[ e^{(\alpha - 1)D_\alpha(P||Q)} - 1 \right] \leq \frac{1}{\alpha - 1} \left[ e^{(\alpha - 1)D_\beta(P||Q)} - 1 \right] \\
\leq \frac{1}{\beta - 1} \left[ e^{(\beta - 1)D_\beta(P||Q)} - 1 \right] = T_\beta(P||Q).
\]

In case \( \alpha < 1 \), we use the property that the function \( c \rightarrow 1 - e^{-ct_0} \) is non-decreasing in \( c \geq 0 \), while \( t \rightarrow \frac{1 - e^{-ct}}{t} \) is non-increasing on the half-axis \(-\infty < t < 1\). This yields

\[
T_\alpha(P||Q) = \frac{1}{1 - \alpha} \left[ 1 - e^{-(1-\alpha)D_\alpha(P||Q)} \right] \leq \frac{1}{1 - \alpha} \left[ 1 - e^{-(1-\alpha)D_\beta(P||Q)} \right] \\
\leq \frac{1}{1 - \beta} \left[ 1 - e^{-(1-\beta)D_\beta(P||Q)} \right] = T_\beta(P||Q).
\]

The values \( 0 < \alpha < 1 \) and \( 1 < \alpha < \infty \), for which the Rényi divergence was defined explicitly, are called simple. The monotonicity of \( D_\alpha(P||Q) \) with respect \( \alpha \) allows to extend this function to the missing values \( \alpha = 0, \alpha = 1 \) and \( \alpha = \infty \), which are called extended values:

\[
D_0(P||Q) = \lim_{\alpha \downarrow 0} D_\alpha(P||Q), \quad D_\infty(P||Q) = \lim_{\alpha \rightarrow \infty} D_\alpha(P||Q), \\
D_1(P||Q) = \lim_{\alpha \uparrow 1} D_\alpha(P||Q).
\]

It is easy to check that \( D_0(P||Q) = -\log Q\{p(x) > 0\} \) and \( D_\infty(P||Q) = \log \text{ess sup}_P \frac{p(x)}{q(x)} \)
with the convention that \( 0/0 = 0 \).

The extended index \( \alpha = 0 \) may be used to characterize an absolute continuity or singularity of two given probability distributions: \( D_0(P||Q) = 0 \), if and only if \( Q \) is absolutely continuous with respect to \( P \), and \( D_0(P||Q) = \infty \), if and only if \( P \) and \( Q \) are orthogonal to each other. This can be illustrated by the Gaussian dichotomy – the property saying that any two Gaussian measures are either absolutely continuous to each other or orthogonal, cf. [S], p. 366.

The extended index \( \alpha = 1 \) leads to the Kullback-Leibler distance

\[
D(X||Z) = D(P||Q) = \int p \log \frac{p}{q} d\mu,
\]
also known as the relative entropy or the informational divergence. Motivated by works of Shannon and Wiener on communication engineering, this quantity was introduced by Kullback and Leibler [K-L] under the name “the information of \( P \) relative to \( Q \)” (though using a different notation). Note that in this case \( D_1 = T_1 = D \). It follows immediately that \( D(P||Q) = \infty \), if \( P \) is not absolutely continuous with respect to \( Q \).

As was already mentioned, in the particular case \( \alpha = 2 \), we arrive at the definition of the quadratic Renyi divergence and the quadratic Renyi divergence power also known as the \( \chi^2 \)-distance:

\[
D_2(X||Z) = \log \int \frac{p^2}{q} d\mu, \quad \chi^2(X, Z) = T_2(X||Z) = \int \frac{p^2}{q} d\mu - 1.
\]

In all cases, by the Csiszár-Pinsker inequality for \( \alpha = 1 \), we have the relations

\[
\frac{1}{2} \|P - Q\|^2_{TV} \leq D(X||Z) \leq D_2(X||Z) \leq \chi^2(X, Z).
\]

Another important property of these distances is the contractivity under mappings.
Proposition 2.3. For any measurable map $S$ from $\Omega$ to any measurable space $\Omega'$,
\[ D_\alpha(S(X)||S(Z)) \leq D_\alpha(X||Z) \quad (\alpha \geq 1). \quad (2.1) \]

Proof. Suppose that $D_\alpha(X||Z)$ is finite, so that the distribution $P$ is absolutely continuous with respect $Q$. Introducing $\xi = p/q$, $\beta = \alpha/(\alpha - 1)$ with $\alpha > 1$, one may write
\[ (1 + (\alpha - 1) T_\alpha(X||Z))^{1/\alpha} = (\mathbb{E}_Q \xi^\alpha)^{1/\alpha} = \sup_{\mathbb{E}_Q \eta^\beta \leq 1} \mathbb{E}_Q \xi \eta \]
\[ = \sup_{\mathbb{E}_Q \eta^\beta \leq 1} \mathbb{E}_P \eta = \sup_{\mathbb{E}_\eta(Z)^\beta \leq 1} \mathbb{E}_\eta(X), \]
that is,
\[ 1 + (\alpha - 1) T_\alpha(X||Z) = \sup_{\mathbb{E}_\eta(Z)^\beta \leq 1} (\mathbb{E} \eta(X))^\alpha, \quad (2.2) \]
where the supremum is taken over all measurable functions $\eta : \Omega \to \mathbb{R}_+$ such that $\mathbb{E} \eta(Z)^\beta \leq 1$.

Similarly
\[ 1 + (\alpha - 1) T_\alpha(S(X),S(Z)) = \sup_{\mathbb{E}_\eta(S(Z))^{\beta'} \leq 1} (\mathbb{E} \eta(S(X)))^\alpha = \sup_{\mathbb{E} \eta'(X)^\alpha \leq 1} \mathbb{E}_\eta'(X), \]
where the second supremum on the right has been restricted to the class of functions of the form $\eta' = \eta(S)$. Hence, this supremum does not exceed the right-hand side of (2.2), thus proving (2.1) for $T_\alpha$. \hfill \Box

The property (2.1) is closely related to the so called data processing inequality in Information Theory, namely
\[ D_\alpha(P_\mathfrak{A}||Q_\mathfrak{A}) \leq D_\alpha(P||Q), \]
where $P_\mathfrak{A}$ and $Q_\mathfrak{A}$ denote restrictions of the measures $P$ and $Q$ to an arbitrary $\sigma$-subalgebra $\mathfrak{A}$ in $\Omega$ (cf. [E-H], Theorem 1).

3. Pearson-Vajda Distances

Writing $\chi^2(X, Z) = \int \frac{(p-q)^2}{q} \, d\mu$, the $\chi^2$-distance may be regarded as a particular member in the family of Pearson-Vajda distances [N], described below.

Definition 3.1. For $\alpha \geq 1$, the $\chi_\alpha$-distance of $P$ from $Q$ is defined by
\[ \chi_\alpha(X, Z) = \chi_\alpha(P, Q) = \int \left| \frac{p - q}{q} \right|^\alpha q \, d\mu = \| p - q \|^\alpha_{L^\alpha(\frac{1}{\alpha^2} \, d\mu)}. \]

As in the previous section, here $X$ and $Z$ denote random elements in $(\Omega, \mu)$, having distributions $P$ and $Q$ with densities $p = \frac{dP}{d\mu}$, $q = \frac{dQ}{d\mu}$. The quantity $\chi_\alpha(P, Q)$ (which is often denoted $\chi^\alpha$) does not depend on the choice of the dominating measure $\mu$.

Clearly, the function $\chi_\alpha^{1/\alpha}$ is non-decreasing in $\alpha$, and when $\alpha = 1$, we arrive at the total variation distance between $P$ and $Q$.

For our further purpose, it will be useful to relate the Rényi divergence power $T_\alpha$ to $\chi_\alpha$. Both quantities are metrically equivalent, as seen by the following elementary observation.
Proposition 3.2. For all \( \alpha > 1 \),
\[
T_{\alpha} \leq \frac{1}{\alpha - 1} \left[ (1 + \chi_{\alpha}^{1/\alpha})^\alpha - 1 \right],
\]
where \( T_{\alpha} = T_{\alpha}(P||Q) \) and \( \chi_{\alpha} = \chi_{\alpha}(P||Q) \). Conversely,
\[
T_{\alpha} \geq \frac{3}{16} \min\{\chi_{\alpha}, \chi_{\alpha}^{2/\alpha}\} \quad (1 < \alpha \leq 2), \quad T_{\alpha} \geq 3^{-\alpha} \chi_{\alpha} \quad (\alpha \geq 2).
\]

Proof. By the triangle inequality in \( L^\alpha(\frac{1}{\eta^{\alpha-1}} \, d\mu) \),
\[
\chi_{\alpha}^{1/\alpha} = \|p - q\|_{L^\alpha(\frac{1}{\eta^{\alpha-1}} \, d\mu)} \geq \left\| p \right\|_{L^\alpha(\frac{1}{\eta^{\alpha-1}} \, d\mu)} - \|q\|_{L^\alpha(\frac{1}{\eta^{\alpha-1}} \, d\mu)} = \int \left( \left( \frac{E}{q} \right)^\alpha \, q \, d\mu \right)^{1/\alpha} - 1 = (1 + (\alpha - 1)T_{\alpha})^{1/\alpha} - 1,
\]
which proves (3.1).

To argue in the opposite direction, put \( \xi = p/q \). Since \( dQ = q \, d\mu \), we may write
\[
T_{\alpha} = \frac{1}{\alpha - 1} \left[ E \xi - 1 \right], \quad \chi_{\alpha} = E |\xi - 1|^{\alpha},
\]
where the expectations are taken on the probability space \((\Omega, Q)\). We have \( \xi \geq 0 \) and \( E \xi = 1 \).

Consider the random variable \( \eta = \xi - 1 \geq -1 \) and the function
\[
\psi(t) = E (1 + t\eta)^{\alpha} - 1, \quad t \geq 0,
\]
so that \( \psi(1) = E \xi - 1 = 1 \). This function is differentiable in \( t > 0 \), with continuous derivatives
\[
\psi'(t) = \alpha \eta (1 + t\eta)^{\alpha - 1}, \quad \psi''(t) = \alpha (\alpha - 1) \eta^2 (1 + t\eta)^{\alpha - 2}.
\]

Since \( \psi(0) = \psi'(0) = 0 \), by the Taylor integral formula,
\[
\psi(1) = \int_0^1 (1 - t) \psi''(t) \, dt = \alpha (\alpha - 1) \eta^2 \int_0^1 (1 - t)(1 + t\eta)^{\alpha - 2} \, dt.
\]

Case 1 \( \alpha \leq 2 \). Since the function \( t \to (1 + t\eta)^{\alpha - 2} \) is convex on \((0, \infty)\), Jensen’s inequality with respect to the probability measure \( d\nu(t) = 2(1 - t) \, dt \) on \((0, 1)\) yields
\[
\int_0^1 (1 - t)(1 + t\eta)^{\alpha - 2} \, dt \geq \frac{1}{2} \left( 1 + \eta \int t \, d\nu(t) \right)^{\alpha - 2} = \frac{1}{2} \left( 1 + \frac{1}{3} \eta \right)^{\alpha - 2}.
\]

Therefore,
\[
\psi(1) \geq \frac{1}{2} \alpha (\alpha - 1) \eta^2 \left( 1 + \frac{1}{3} \eta \right)^{\alpha - 2}.
\]

On the set \( A = \{ |\eta| \leq 1 \} \), the expression \( \eta^2 (1 + \frac{1}{3} \eta)^{\alpha - 2} \) is bounded from below by \( \left( \frac{3}{4} \right)^{2-\alpha} \eta^2 \), and on the set \( B = \{ |\eta| > 1 \} \), by \( \eta^2 \cdot \left( \frac{3}{4} \eta \right)^{\alpha - 2} \). Hence
\[
\psi(1) \geq \frac{1}{2} \alpha (\alpha - 1) \left( \frac{3}{4} \right)^{2-\alpha} \eta^2 \left( 1 + \eta \right)^{1_B}.
\]

For our range of \( \alpha \)’s we may simply use \( \alpha \left( \frac{3}{4} \right)^{2-\alpha} \geq \frac{3}{4} \), so that
\[
\psi(1) \geq \frac{3}{8} (\alpha - 1) \eta^2 \left( 1 + \eta \right)^{1_B}.
\]
By Markov’s inequality, 
\[ \frac{1}{\mathbb{P}(A)} \mathbb{E} \eta^2 1_A \geq \left( \frac{1}{\mathbb{P}(A)} \mathbb{E} |\eta|^\alpha 1_A \right)^{2/\alpha}, \]
so \( \mathbb{E} \eta^2 1_A \geq (\mathbb{E} |\eta|^\alpha 1_A)^{2/\alpha} \) and thus 
\[ \mathbb{E} (\eta^2 1_A + |\eta|^\alpha 1_B) \geq U = u_0^{2/\alpha} + u_1, \quad \text{where} \quad u_0 = \mathbb{E} |\eta|^\alpha 1_A, \quad u_1 = \mathbb{E} |\eta|^\alpha 1_B. \]
Fixing the value \( u = u_0 + u_1 \), in case \( u_1 \geq \frac{1}{2} u \) we have \( U \geq u_1 \geq \frac{1}{2} u \), while in case \( u_0 \geq \frac{1}{2} u \) we have \( U \geq u_0 \geq (\frac{1}{2} u)^{2/\alpha} \geq \frac{1}{2} u^{2/\alpha} \). In both cases, \( U \geq \frac{1}{2} \min(u, u^{2/\alpha}) \), that is, 
\[ \mathbb{E} (\eta^2 1_A + |\eta|^\alpha 1_B) \geq \frac{1}{2} \min\{\mathbb{E} |\eta|^\alpha, (\mathbb{E} |\eta|^\alpha)^{2/\alpha}\}. \]
As a result, 
\[ T_\alpha = \frac{1}{\alpha - 1} \psi(1) \geq \frac{3}{16} \min\{\mathbb{E} |\eta|^\alpha, (\mathbb{E} |\eta|^\alpha)^{2/\alpha}\} = \frac{3}{16} \min\{\chi_\alpha, \chi_\alpha^{2/\alpha}\}, \]
which yields the first inequality in (3.2).

**Case \( \alpha > 2 \).** Let us return to the Taylor integral formula 
\[ \psi(1) = \alpha(\alpha - 1) \mathbb{E} \eta^2 \int_0^1 (1 - t)(1 + t\eta)^{\alpha - 2} dt, \]
where we now restrict integration to the interval \( (\frac{1}{3}, \frac{2}{3}) \) to get 
\[ \psi(1) \geq \frac{\alpha(\alpha - 1)}{3} \mathbb{E} \eta^2 \int_{1/3}^{2/3} (1 + t\eta)^{\alpha - 2} dt. \]
Since \( \eta \geq -1 \), in case \( \eta \leq 0 \), we have \( 1 + t\eta \geq 1 + \frac{2}{3} \eta \geq \frac{1}{3} \eta \). In case \( \eta \geq 0 \), we similarly have \( 1 + t\eta \geq t\eta \geq \frac{1}{3} \eta \). In both cases, \( 1 + t\eta \geq \frac{1}{3} |\eta| \), hence 
\[ \psi(1) \geq \frac{\alpha(\alpha - 1)}{3} \mathbb{E} \eta^2 \int_{1/3}^{2/3} \left( \frac{1}{3} |\eta| \right)^{\alpha - 2} dt = \frac{\alpha(\alpha - 1)}{3^\alpha} \mathbb{E} |\eta|^\alpha, \]
and therefore 
\[ T_\alpha = \frac{1}{\alpha - 1} \psi(1) \geq \frac{\alpha - 1}{3^\alpha} \mathbb{E} |\eta|^\alpha = \frac{\alpha - 1}{3^\alpha} \chi_\alpha. \]

\[ \square \]

4. Basic Exponential Inequalities

We now focus on the particular case, where \( \Omega = \mathbb{R} \) is the real line with Lebesgue measure \( \mu \), and where \( Z \sim N(0, 1) \) is a standard normal random variable, i.e., with density 
\[ \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}. \]
Given a random variable \( X \), the Rényi divergence and the Tsallis distance of index \( \alpha > 1 \) with respect to \( Z \) are then given by the formulas 
\[ (\alpha - 1) D_\alpha(X||Z) = \log \int_{-\infty}^{\infty} p(x)^\alpha \varphi(x) dx, \quad (\alpha - 1) T_\alpha(X||Z) = \int_{-\infty}^{\infty} \frac{p(x)^\alpha}{\varphi(x)^{\alpha-1}} dx - 1, \]
where \( p \) is density of \( X \). If the distribution of \( X \) is not absolutely continuous with respect to \( \mu \), then we automatically have \( D_\alpha(X||Z) = T_\alpha(X||Z) = \infty \). These quantities are finite, if, for
example, $p$ is bounded and $\mathbb{E} e^{(\alpha-1)X^2/2} < \infty$. In fact, the finiteness of $D_\alpha(X||Z)$ or $T_\alpha(X||Z)$ implies a similar property. In the sequel, we put

$$\beta = \alpha^* = \frac{\alpha}{\alpha - 1}.$$  

**Proposition 4.1.** If $T_\alpha = T_\alpha(X||Z) < \infty$, then $X$ must have an absolutely continuous distribution with

$$\mathbb{E} e^{cX^2} \leq \frac{C}{(1 - 2\beta c)^{2\beta}} \text{ for all } c < \frac{1}{2\beta},$$

where $C = (1 + (\alpha - 1)T_\alpha)^{1/\alpha}$. It is possible that $T_\alpha < \infty$, while $\mathbb{E} e^{\frac{1}{2\beta}X^2} = \infty$.

In particular, if $T_\alpha$ is finite, $X$ must have finite moments of any order.

**Proof.** Let $X$ have density $p$ such that the integral $C = (\int_{-\infty}^{\infty} \frac{p(x)^\alpha}{\varphi(x)^{1/\beta}} \, dx)^{1/\alpha}$ is finite. By the Hölder inequality with dual exponents $(\beta, \alpha)$,

$$\mathbb{E} e^{cX^2} = \int_{-\infty}^{\infty} p(x) e^{cx^2} \varphi(x) \frac{1}{\varphi(x)^{1/\beta}} \, dx \leq C \left( \int_{-\infty}^{\infty} e^{\beta cx^2} \varphi(x) \, dx \right)^{1/\beta} \frac{C}{(1 - 2\beta c)^{2\beta}}.$$

This proves the first assertion. For the second assertion, one may consider a density of the form $p(x) = \frac{a}{1 + |x|^{1/2}} e^{-\frac{1}{2\beta}x^2}$, where $a$ is a normalizing constant. Then $T_\alpha < \infty$ and $\mathbb{E} e^{\frac{1}{2\beta}X^2} = \infty$. □

As an alternative (although almost equivalent) variant of Proposition 4.1, we also have:

**Proposition 4.2.** If $T_\alpha = T_\alpha(X||Z) < \infty$, then for all $t \in \mathbb{R}$,

$$\mathbb{E} e^{tX} \leq Ce^{\beta t^2/2},$$

(4.1)

where $C = (1 + (\alpha - 1)T_\alpha)^{1/\alpha}$. In particular, for any $r \geq 0$,

$$\mathbb{P}\{X \geq r\} \leq Ce^{-\frac{1}{2\beta}r^2}.$$

Indeed, arguing as before, if $p$ is density of $X$,

$$\mathbb{E} e^{tX} = \int_{-\infty}^{\infty} p(x) e^{tx} \, dx = \int_{-\infty}^{\infty} p(x) \cdot e^{tx} \varphi(x)^{1/\beta} \, dx \leq C \left( \int_{-\infty}^{\infty} e^{\beta tx} \varphi(x) \, dx \right)^{1/\beta} = Ce^{\beta t^2/2}.$$

This bound cannot be deduced from the bound of Proposition 4.1. In fact, the coefficient $C$ in (4.1) may be chosen to be smaller than 1 for large values of $|t|$. The next assertion will be one of the steps needed in the proof of the sufficiency part of Theorems 1.1-1.2.
Proposition 4.3. If $T_\alpha(X||Z) < \infty$, then
\[
\lim_{|t| \to \infty} \mathbb{E} e^{t X} e^{-\frac{\beta t^2}{2}} = 0.
\]

Proof. Let $p$ be the density of $X$ and write $\mathbb{E} e^{t X} = \int_{-\infty}^{\infty} e^{tx} p(x) \, dx$. Here integration over the positive half-axis may be split into the two intervals. First, given $t > 0$, by the Hölder inequality,
\[
\int_0^{\beta t/2} e^{tx} p(x) \, dx = \int_0^{\beta t/2} p(x) e^{\frac{\beta}{2}x} \cdot e^{tx - \frac{\beta}{2} x} \, dx 
\leq \left( \int_{-\infty}^{\infty} p(x) e^{\frac{\alpha}{2}x} \, dx \right)^{1/\alpha} \left( \int_0^{\beta t/2} e^{tx - \frac{\beta}{2} x} \, dx \right)^{1/\beta} 
\leq \frac{1}{(2\pi)^{1/(2\beta)}} \left( 1 + (\alpha - 1)T_\alpha(X||Z) \right)^{1/\alpha} \left( \frac{\beta t}{2} \right)^{1/\beta} e^{3\beta t^2/8},
\]
where we used the monotonicity of $\beta t x - \frac{1}{2} x^2$ in the interval $0 \leq x \leq \beta t$ (in order to estimate the last integral). Similarly,
\[
\int_{\beta t/2}^{\infty} e^{tx} p(x) e^{\frac{\beta}{2}x} \cdot e^{tx - \frac{\beta}{2} x} \, dx \leq \left( \int_{-\infty}^{\beta t/2} p(x) e^{\frac{\alpha}{2}x} \, dx \right)^{1/\alpha} \left( \int_{-\infty}^{\infty} e^{tx - \frac{\beta}{2} x} \, dx \right)^{1/\beta} \leq \delta(t) e^{\beta t^2/2}
\]
with $\delta(t) \to 0$ as $t \to \infty$. Collecting these bounds, we get
\[
\mathbb{E} e^{t X} 1_{\{X > 0\}} e^{-\frac{\beta t^2}{2}} \leq (2\pi)^{-1/(2\beta)} \left( 1 + (\alpha - 1)T_\alpha(X||Z) \right)^{1/\alpha} \left( \frac{\beta t}{2} \right)^{1/\beta} e^{-\beta t^2/8} + \delta(t) \to 0.
\]
Since also $\mathbb{E} e^{t X} 1_{\{X < 0\}} \to 0$ as $t \to \infty$, the conclusion follows. \hfill \Box

5. Laplace and Weierstrass Transforms

Although in general the critical constant in the exponent $c = 1/(2\beta)$ cannot be included in the statement of Proposition 4.1, this turns out possible for sufficiently many normalized convolutions of the distribution of $X$ with itself. Given independent copies $X_1, \ldots, X_n$ of $X$, here we consider “Gaussian” moments for the normalized sums
\[
Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}.
\]
The following statement is crucial both in the necessity and sufficiency parts of the proof of Theorems 1.1-1.2. We always assume that $Z \sim N(0, 1)$.

Proposition 5.1. If $T_\alpha = T_\alpha(X||Z) < \infty$, then $\mathbb{E} e^{\frac{1}{2\beta} Z_n^2} < \infty$ for all $n \geq \alpha$, and
\[
\mathbb{E} e^{\frac{1}{2\beta} Z_n^2} \leq 3^n \left( 1 + (\alpha - 1)T_\alpha \right)^{\frac{n}{2}}. \tag{5.1}
\]
Moreover, putting $\chi_\alpha = \chi_\alpha(X, Z)$, we have
\[
\|\mathbb{E} e^{\frac{1}{2\beta} Z_n^2} - \mathbb{E} e^{\frac{1}{2\beta} Z^2}\| \leq 3^n \left( 1 + \chi_\alpha^{1/\alpha} \right)^n - 1. \tag{5.2}
\]

Thus, when $X$ is close to $Z$ in the sense of the Pearson-Vajda distance, we also obtain closeness of the corresponding Gaussian moments of $Z_n$ and $Z$ with fixed $n \geq \alpha$. Recall that
\( \chi_\alpha \) in (5.2) can be estimated from above in terms of \( T_\alpha \) according to Proposition 3.2 (while these distances coincide in case \( \alpha = 2 \)).

As for the inequality (5.1), one may equivalently rephrase it in terms of the Laplace transform of the distribution of \( Z_n \). Let us state one immediate corollary.

**Corollary 5.2.** Let \( T_\alpha = T_\alpha(X\|Z) \) be finite. Then the function \( \psi(t) = \mathbb{E} e^{tX} e^{-\beta t^2/2} \) is integrable with any power \( n \geq \alpha \), and up to some \( n \)-dependent constant \( c_n \),

\[
\int_{-\infty}^{\infty} \psi(t)^n \, dt \leq c_n (1 + (\alpha - 1)T_\alpha)^{\frac{n}{\alpha}}. \tag{5.3}
\]

The argument uses the contractivity properties of the Weierstrass transform, which is defined by the equality

\[
W_t u(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{2t}} u(y) \, dy, \quad x \in \mathbb{R}, \ t > 0.
\]

For short in the sequel we denote by \( L^\alpha \) the Lebesgue space \( L^\alpha(\mathbb{R}, dx) \) of all measurable functions on the real line with finite norm

\[
\|u\|_\alpha = \left( \int_{-\infty}^{\infty} |u(x)|^\alpha \, dx \right)^{1/\alpha}, \quad \alpha \geq 1,
\]

with usual convention \( \|u\|_\infty = \text{ess sup}_x |u(x)| \).

We refer an interested reader to [H-W] for a detail account on the Weierstrass transform, and here only mention one property. Since \( W_t u \) represents the convolution of \( u \), namely, with the Gaussian density \( \varphi_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)} \), we have, by Jensen’s inequality, \( \|W_t u\|_\alpha \leq \|u\|_\alpha \) for all \( \alpha \geq 1 \) and \( t > 0 \). That is, \( W_t \) acts as a contraction from \( L^\alpha \) to \( L^\alpha \).

This implies that \( W_t \) is a bounded operator from \( L^\alpha \) to \( L^\gamma \) with any \( \gamma > \alpha \). Indeed, by Hölder’s inequality, \( \|W_t u(x)\| \leq \|\varphi_t\|_\beta \|u\|_\alpha (\beta = \alpha^*) \), and since \( \|\varphi_t\|_\beta = (2\pi t)^{-1/(2\alpha)} \beta^{-1/(2\beta)} \), we get

\[
\|W_t u\|_\infty \leq (2\pi t)^{-1/(2\alpha)} \beta^{-1/(2\beta)} \|u\|_\alpha.
\]

More generally, given \( \alpha < \gamma < \infty \), we have

\[
\int_{-\infty}^{\infty} |W_t u(x)|^\gamma \, dx = \int_{-\infty}^{\infty} |W_t u(x)|^{\gamma - \alpha} |W_t u(x)|^\alpha \, dx \leq (2\pi t)^{\frac{\alpha - \gamma}{2\alpha}} \beta^{\frac{\alpha - \gamma}{2\beta}} \|u\|_\alpha^\alpha.
\]

Hence

\[
\|W_t u\|_\gamma \leq (2\pi t)^{\frac{\alpha - \gamma}{2\alpha}} \beta^{\frac{\alpha - \gamma}{2\beta}} \|u\|_\alpha, \quad \alpha \leq \gamma \leq \infty. \tag{5.4}
\]

In fact, since \( \frac{\alpha - \gamma}{\gamma} = \frac{1}{\gamma} - \frac{1}{\alpha} \) may vary from zero to \( -\frac{1}{\alpha} \), the latter bound can be made independent of \( \gamma \), namely, in the indicated range

\[
\|W_t u\|_\gamma \leq \max \{1, (2\pi t)^{-\frac{1}{\alpha}}\} \|u\|_\alpha.
\]

The inequality (5.4) is what we need for the proof of Proposition 5.1.

**Proof of Proposition 5.1.** Let \( p \) be the density of \( X \). The Weierstrass transform can be applied to the function

\[
u(x) = \varphi(x)^{-1/\beta} \, p(x),
\]
which has finite norm $\|u\|_\alpha = (1 + (\alpha - 1)T_n)^{1/\alpha}$. Putting $\bar{x} = \frac{1}{n}(x_1 + \cdots + x_n)$, the expectation we have to estimate is

$$E e^{\frac{1}{n}Z^2_n} = \int_{\mathbb{R}^n} e^{\frac{1}{n} \bar{x}^2} p(x_1) \cdots p(x_n) \, dx_1 \cdots dx_n$$

$$= (2\pi)^{-\frac{n-1}{2\beta}} \int_{\mathbb{R}^n} \exp \left\{ \frac{n}{2\beta} \bar{x}^2 - \frac{1}{2\beta} (x_1^2 + \cdots + x_n^2) \right\} u(x_1) \cdots u(x_n) \, dx_1 \cdots dx_n$$

$$= (2\pi)^{-\frac{n-1}{2\beta}} \int_{\mathbb{R}^n} \exp \left\{ - \frac{1}{4\beta n} \sum_{i=1}^n Q_i \right\} u(x_1) \cdots u(x_n) \, dx_1 \cdots dx_n,$$

where $Q_i = \sum_{j=1}^n (x_i - x_j)^2$. First, we apply Hölder’s inequality and put $t = 2\beta$, to get

$$E e^{\frac{1}{n}Z^2_n} \leq (2\pi)^{-\frac{n-1}{2\beta}} \prod_{i=1}^n \left( \int_{\mathbb{R}^n} \exp \left\{ - \frac{1}{4\beta} \sum_{i=1}^n Q_i \right\} u(x_1) \cdots u(x_n) \, dx_1 \cdots dx_n \right)^{1/n}$$

$$= (2\pi)^{-\frac{n-1}{2\beta}} (2\pi t)^{\frac{n-1}{2t}} \int_{-\infty}^\infty (W_t u(x_1))^{n-1} u(x_1) \, dx_1,$$

where on the second step, inside the $i$-th integral in the product we performed the integration over the variables $x_j$, $j \neq i$, which yielded the value $(2\pi t)^{-\frac{n-1}{2t}} (W_t u(x_i))^{n-1}$. By Hölder’s inequality once more, and applying (5.4) with $\gamma = \beta(n - 1)$, which satisfies $\gamma \geq \alpha$ due to the assumption $n \geq \alpha$, we see that the last one dimensional integral does not exceed

$$\left( \int_{-\infty}^\infty (W_t u(x_1))^{\gamma} \, dx_1 \right)^{\frac{1}{\gamma}} \|u\|_\alpha = \|W_t u\|_{\gamma}^{-1} \|u\|_\alpha \leq \left( (2\pi t)^{\frac{\gamma}{\gamma - \beta}} \beta^{\frac{\gamma}{\gamma - \beta}} \|u\|_\alpha \right)^{n-1} \|u\|_\alpha.$$ 

Hence $E e^{\frac{1}{n}Z^2_n} \leq c_{n,\alpha} \|u\|_\alpha^n$ with constant

$$c_{n,\alpha} = (2\pi)^{-\frac{n-1}{2\beta}} (2\pi t)^{\frac{n-1}{2t}} (2\pi t)^{\frac{n-1}{2t}} \beta^{\frac{n-1}{2}} \frac{\alpha - \gamma}{\alpha} \beta^{\frac{n-1}{2}} \frac{\alpha - \gamma}{\alpha} = (2\pi)^{-\frac{1}{2t}} (2\pi t)^{\frac{n}{2t}} \beta^{\frac{n}{2t}} \beta^{\frac{1}{2t}} < 3^n.$$

This proves (5.1). It is also interesting to note that $c_{n,\alpha} \to 1$ as $\alpha \to 1$.

Obviously, this argument can easily be extended to not necessarily equal positive functions. Namely, for the integral

$$I = I(p_1, \ldots, p_n) = \int_{\mathbb{R}^n} e^{\frac{1}{n} \bar{x}^2} p_1(x_1) \cdots p_n(x_n) \, dx_1 \cdots dx_n$$

we similarly obtain

$$|I| \leq (2\pi)^{-\frac{n-1}{2\beta}} \prod_{i=1}^n \left( \int_{\mathbb{R}^n} \exp \left\{ - \frac{1}{4\beta} \sum_{i=1}^n Q_i \right\} |u(x_1)| \cdots |u(x_n)| \, dx_1 \cdots dx_n \right)^{1/n}$$

$$= (2\pi)^{-\frac{n-1}{2\beta}} (2\pi t)^{\frac{n-1}{2t}} \prod_{i=1}^n \left( \int_{-\infty}^\infty |u_i(x_i)| \prod_{j \neq i} (W_t |u_j|)(x_i) \, dx_i \right)^{1/n},$$
where \( u_j = \varphi^{-1/\beta} p_j \). An application of Hölder’s inequality together with (5.4) allows one to estimate the last integral by

\[
\|u_i\|_\alpha \prod_{j \neq i} \|W_t u_j\|_\gamma \leq \|u_i\|_\alpha \prod_{j \neq i} (2\pi t)^{\frac{\alpha - \gamma}{2\gamma}} \beta^{\frac{\alpha - \gamma}{2\gamma}} \|u_j\|_\alpha = (2\pi t)^{\frac{n-1}{n} \frac{\alpha - \gamma}{\beta \gamma}} \beta^{\frac{n-1}{\beta \gamma}} \|u_1\|_\alpha \ldots \|u_n\|_\alpha.
\]

This leads to

\[
|I(p_1, \ldots, p_n)| \leq c_{n,\alpha} \|u_1\|_\alpha \ldots \|u_n\|_\alpha \tag{5.5}
\]

with the same constant as before (so that \( c_{n,\alpha} < 3^n \)).

We use the latter bound to derive the second inequality (5.2). Let us split the density of \( X \) as \( p = \varphi + \varphi^{1/\beta} v \), such that \( \|v\|_\alpha^2 = \chi_c(X, Z) \). Hence we get a decomposition

\[
\mathbb{E} e^{\frac{1}{2\alpha} Z_n^2} = \int_{\mathbb{R}^n} e^{\frac{1}{2\alpha} \beta^2} p(x_1) \ldots p(x_n) \, dx_1 \ldots dx_n
\]

\[
= \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} \int_{\mathbb{R}^n} e^{\frac{1}{2\alpha} \beta^2} \varphi(x_1) \ldots \varphi(x_k) \varphi^{1/\beta} (x_{k+1}) v(x_{k+1}) \ldots v^{1/\beta} (x_n) v(x_n) \, dx_1 \ldots dx_n.
\]

We apply (5.5) with \( p_1 \) replaced by \( \varphi \), and with \( p_{k+1} \) replaced with \( \varphi^{1/\beta} v \) (that is, \( u_j = \varphi^{1/\alpha} \) for \( j \leq k \) and \( u_j = v \) for \( j > k \)). Moving the first term with \( k = 0 \) of this decomposition to the left, we then get the bound

\[
\left| \mathbb{E} e^{\frac{1}{2\alpha} Z_n^2} - \mathbb{E} e^{\frac{1}{2\alpha} Z^2} \right| \leq c_{n,\alpha} \sum_{k=1}^{n} \frac{n!}{k! (n-k)!} \|\varphi^{1/\alpha}\|_{\alpha}^k \|v\|_{\alpha}^{n-k} = c_{n,\alpha} (1 + \|v\|_{\alpha}^n - 1).
\]

\[\square\]

6. Connections with Fourier Transform

In the next sections, we restrict ourselves to the particular interesting index \( \alpha = 2 \), that is, to the \( \chi^2 \)-distance from the standard normal law,

\[
\chi^2(X, Z) = \int_{-\infty}^{\infty} \frac{p(x)^2}{\varphi(x)} \, dx - 1.
\]

In this case, necessary and sufficient conditions for finiteness of this divergence may be given in terms of the characteristic function

\[
f(t) = \mathbb{E} e^{itX}, \quad t \in \mathbb{R}.
\]

**Proposition 6.1.** The condition \( \chi^2(X, Z) < \infty \) insures that \( f(t) \) has square integrable derivatives of any order. Moreover, in that case

\[
1 + \chi^2(X, Z) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} |f^{(n)}(t)|^2 \, dt.
\]

**Proof.** By the very definition,

\[
1 + \chi^2(X, Z) = \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} x^{2n} p(x)^2 \, dx.
\]
We know that $f$ has finite derivatives of any order given by
\[ f^{(n)}(t) = \mathbb{E}(iX)^n e^{itX} = \int_{-\infty}^{\infty} (ix)^n e^{itx} p(x) \, dx. \]

It remains to apply Plancherel’s theorem. \qed

In view of Proposition 4.1, existence of $\chi^2(X, Z)$ does not guarantee existence of the “Gaussian” moment $\mathbb{E} e^{X^2/4}$. Nevertheless, it is true for the normalized convolution of the distribution of $X$ with itself, as indicated in Proposition 5.1. In fact, in this case inequality (5.1) can be stated more precisely as
\[ \mathbb{E} e^{\frac{1}{4}(\overline{X} + \tilde{X})^2} \leq 2 (1 + \chi^2(X, Z)), \]
where $\tilde{X}$ is an independent copy of $X$. Equivalently, there is a corresponding refinement of inequality (5.3) in Corollary 5.2 (without any convolution).

**Proposition 6.2.** In any case
\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(iy)^2 e^{-2y^2} \, dy \leq 1 + \chi^2(X, Z). \]

The argument is based on the following general observation which may be of independent interest.

**Lemma 6.3.** Given a function $p$ on the real line, suppose that the function $g(x) = p(x) e^{x^2/4}$ belongs to $L^2$. Then the Fourier transforms
\[ f(t) = \int_{-\infty}^{\infty} e^{itx} p(x) \, dx, \quad \rho(t) = \int_{-\infty}^{\infty} e^{itx} g(x) \, dx \]
are connected by the identity
\[ f(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2} \rho(u) \, du \quad (t \in \mathbb{R}), \] (6.1)
which may analytically be extended to the complex plane. Moreover,
\[ \int_{-\infty}^{\infty} |f(iy)|^2 e^{-2y^2} \, dy = \int_{-\infty}^{\infty} |\rho(t)|^2 e^{-2t^2} \, dt. \] (6.2)

Thus, the characteristic function $f$ appears as the Weierstrass transform of the function $g$. While Proposition 5.1 and its Corollary 5.2 are key ingredients of the proof of Theorem 1.2, Lemma 6.2 can be used as an alternative approach to Theorem 1.1 for the particular case $\alpha = 2$. Lemma 6.3 and Proposition 6.2 can be adapted to cover the range $1 < \alpha \leq 2$ by considering the Fourier transform on the Lebesgue space $L^\alpha$. However, these results do not extend to indexes $\alpha > 2$.

Returning to the $L^2$-case, note that $g$ does not need to be integrable, so, one should understand $\rho$ as a $L^2$-limit $\rho(t) = \lim_{N \to \infty} \int_{-N}^{N} e^{itx} g(x) \, dx$ in the norm of the space $L^2$.

Note also that the second integral in (6.2) can be bounded by the squared $L^2$-norm of $\rho$, which is, by the Plancherel theorem, equal to
\[ 2\pi \|g\|_2^2 = 2\pi \int_{-\infty}^{\infty} |p(x)|^2 e^{x^2/2} \, dx. \]
When $p$ is density of $X$, the last expression is exactly $\sqrt{2\pi} (1 + \chi^2(X,Z))$, thus proving Proposition 6.2.

Proof of Lemma 6.3. First assume that $p$ is a compactly supported; in particular, both $p$ and $g$ are integrable and have analytic Fourier transforms. By Fubini’s theorem,

$$f(t) = \int_{-\infty}^{\infty} e^{itz} g(x) \left[ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ixu - u^2} du \right] dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} \left[ \int_{-\infty}^{\infty} e^{i(t-u)x} g(x) dx \right] du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(u-t)^2} \rho(u) du = \frac{1}{\sqrt{\pi}} e^{-t^2} \int_{-\infty}^{\infty} e^{2u^2 - u^2} \rho(u) du,$$

and we obtain (6.1). Moreover, a change of variable, we have

$$\sqrt{\pi} f\left(\frac{iz}{2}\right) e^{-z^2/4} = \int_{-\infty}^{\infty} e^{izu - u^2} \rho(u) du \quad (z \in \mathbb{R}),$$

which means that the left-hand side represents the Fourier transform of the function $e^{-u^2} \rho(u)$. Hence, by Plancherel’s theorem,

$$\|e^{-u^2} \rho(u)\|_2^2 = \frac{1}{2} \int_{-\infty}^{\infty} |f\left(\frac{iz}{2}\right)|^2 e^{-z^2/4} dz = \int_{-\infty}^{\infty} |f(iz)|^2 e^{-2y^2} dy,$$

thus proving (6.2).

In the general case, we have $p \in L^1 \cap L^2$, and arguing as in the proof of Proposition 4.1 (for the case $\alpha = 2$), we also get

$$\int_{-\infty}^{\infty} e^{cx^2} |p(x)| dx \leq \frac{C}{(1-4c)^{1/4}} < \infty \quad \text{for all} \quad c < \frac{1}{4},$$

where $C^2 = \int_{-\infty}^{\infty} \frac{p(x)^2}{\varphi(x)} dx$. In particular, $f$ is an entire function. Let $p_N$ be the restriction of $p$ to $[-N, N]$, $g_N(x) = p_N(x) e^{x^2/4}$, and put

$$f_N(t) = \int_{-\infty}^{\infty} e^{itz} p_N(x) dx, \quad \rho_N(t) = \int_{-\infty}^{\infty} e^{itz} g_N(x) dx.$$

According to the previous step, for all $t \in \mathbb{R}$,

$$f_N(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2} \rho_N(u) du. \quad (6.4)$$

By the Lebesgue dominated convergence theorem, we have $f_N(t) \to f(t)$ for all real $t$ and $
\|g_N - g\|_2 \to 0 \quad \text{as} \quad N \to \infty$. By the continuity of the Fourier transform on $L^2$, we obtain

$$\|\rho_N - \rho\|_2 \to 0,$$

which in turn implies

$$\int_{-\infty}^{\infty} e^{-(t-u)^2} \rho_N(u) du \to \int_{-\infty}^{\infty} e^{-(t-u)^2} \rho(u) du.$$

Hence, in the limit (6.4) yields the desired identity (6.1). Its right-hand side is well-defined and finite for all complex $t$, and clearly represents an entire function. Moreover, as before, one may apply Plancherel’s theorem, leading to (6.3) and therefore to (6.2).
7. Exponential Series

The $\chi^2$-distance from the standard normal law on the real line admits a nice description in terms of a so-called exponential series (following Cramér’s terminology) as well. Let us some introduce basic notations and recall several well-known facts. By $H_k$ we denote the $k$-th Chebyshev-Hermite polynomial

$$H_k(x) = (-1)^k \left( e^{-x^2/2} \right)^{(k)} e^{x^2/2}, \quad k = 0, 1, 2, \ldots \quad (x \in \mathbb{R}),$$

so that $\varphi^{(k)}(x) = (-1)^k H_k(x) \varphi(x)$ in terms of the standard normal density. In particular,

$$H_0(x) = 1, \quad H_2(x) = x^2 - 1,$$
$$H_1(x) = x, \quad H_3(x) = x^3 - 3x.$$

Each $H_k$ is a polynomial of degree $k$ with integer coefficients, with leading coefficient 1. Depending on $k$ being even or odd, $H_k$ contains even resp. odd powers only. These polynomials

may be defined explicitly via

$$H_k(x) = \mathbb{E} (x + iZ)^k, \quad Z \sim N(0, 1).$$

All $H_k$ are orthogonal to each other on the real line with weight function $\varphi(x)$, and moreover – they form a complete orthogonal system in the Hilbert space $L^2(\mathbb{R}, \varphi(x) dx)$. Their $L^2$-norms

are given by

$$\mathbb{E} H_k(Z)^2 = \int_{-\infty}^{\infty} H_k(x)^2 \varphi(x) dx = k!$$

Equivalently, the Hermite functions $\varphi_k = H_k \varphi$ form a complete orthogonal system in $L^2(\mathbb{R}, \frac{dx}{\varphi(x)})$, and their $L^2$-norms in this space are given by $\int_{-\infty}^{\infty} \varphi_k(x)^2 dx = k!$ Summarizing we have:

**Proposition 7.1.** Any complex valued function $u = u(x)$ with $\int_{-\infty}^{\infty} |u(x)|^2 e^{x^2/2} dx < \infty$ admits a unique representation in the form of the orthogonal series

$$u(x) = \varphi(x) \sum_{k=0}^{\infty} \frac{c_k}{k!} H_k(x), \quad (7.1)$$

which converges in $L^2(\mathbb{R}, \frac{dx}{\varphi(x)})$. The coefficients are given by $c_k = \int_{-\infty}^{\infty} u(x) H_k(x) dx$, and we have Parseval’s identity

$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{k!} = \int_{-\infty}^{\infty} \frac{|u(x)|^2}{\varphi(x)} dx.$$

The functional series (7.1) representing $u$ is called an exponential series. The question of its pointwise convergence is rather delicate similar to the pointwise convergence of ordinary Fourier series based on trigonometric functions. In Cramér’s paper [Cr], the following two propositions are stated, together with an explanation of the basic ingredients of the proof.

**Proposition 7.2.** If $u(x)$ is vanishing at infinity and has a continuous derivative such that

$$\int_{-\infty}^{\infty} |u'(x)|^2 e^{x^2/2} dx < \infty,$$

it may be developed in an exponential series, which is absolutely and uniformly convergent for $-\infty < x < \infty$. 
Proposition 7.3. If \( u(x) \) has bounded variation in every finite interval, and if
\[
\int_{-\infty}^{\infty} |u(x)| e^{x^2/4} \, dx < \infty,
\]
then the exponential series for \( u(x) \) converges to \( \frac{1}{2} (u(x+) + u(x-)) \). The convergence is uniform in every finite interval of continuity.

The integral condition of Proposition 7.3 is illustrated in [Cr] on the example of the Gaussian functions \( u(x) = e^{-\lambda x^2} \) (\( \lambda > 0 \)). In this case, the corresponding exponential series can be explicitly computed, and at \( x = 0 \) it is given by the series
\[
\frac{1}{\sqrt{2\lambda}} \sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2 4^k} \left( 1 - \frac{1}{2\lambda} \right)^k.
\]
This series is absolutely convergent for \( \lambda > \frac{1}{4} \), simply convergent for \( \lambda = \frac{1}{4} \) and divergent for \( \lambda < \frac{1}{4} \).

8. Normal Moments

Let \( X \) be a random variable with density \( p \), and let \( Z \) be a standard normal random variable (which is assumed to be independent of \( X \)). Applying Proposition 7.1 to \( p \), we obtain the following: If
\[
\int_{-\infty}^{\infty} p(x)^2 e^{x^2/2} \, dx < \infty,
\]
then \( p \) admits a unique representation in the form of the exponential series
\[
p(x) = \varphi(x) \sum_{k=0}^{\infty} \frac{c_k}{k!} H_k(x),
\]
which converges in \( L^2(\mathbb{R}, \frac{dx}{\varphi(x)}) \). Here, the coefficients are given by
\[
c_k = \int_{-\infty}^{\infty} H_k(x) p(x) \, dx = \mathbb{E} H_k(X) = \mathbb{E} (X + iZ)^k,
\]
which we call the normal moments of \( X \). In particular, \( c_0 = 1 \), \( c_1 = \mathbb{E}X \), \( c_2 = \mathbb{E}X^2 - 1 \).

In general, these moments exist, as long as the \( k \)-th absolute moments of \( X \) are finite. These moments are needed to develop the characteristic function of \( X \) in a Taylor series around zero as follows:
\[
f(t) = \mathbb{E} e^{itX} = e^{-t^2/2} \sum_{k=0}^{N} \frac{c_k}{k!} (it)^k + o(|t|^N), \quad t \to 0.
\]
In particular, \( c_k = 0 \) for \( k \geq 1 \) in case \( X \) is standard normal, similarly to the property of the cumulants
\[
\gamma_k(X) = \frac{d^k}{i^k dt^k} \log f(t)|_{t=0}
\]
with \( k \geq 3 \) (where we use the branch of the logarithm determined by \( \log f(0) = 0 \)).

Let us emphasize one simple algebraic property of normal moments.

Proposition 8.1. Let \( X \) be a random variable such that \( \mathbb{E}X = 0 \), \( \mathbb{E}X^2 = 1 \) and \( \mathbb{E}|X|^k < \infty \) for some integer \( k \geq 3 \), and let \( Z \sim N(0,1) \). The following three properties are equivalent:
a) \( \gamma_j(X) = 0 \) for all \( j = 3, \ldots, k - 1; \)
b) \( \mathbb{E}H_j(X) = 0 \) for all \( j = 3, \ldots, k - 1; \)
c) \( \mathbb{E}X^j = \mathbb{E}Z^j \) for all \( j = 3, \ldots, k - 1. \)

In this case
\[
\gamma_k(X) = \mathbb{E}H_k(X) = \mathbb{E}X^k - \mathbb{E}Z^k. \tag{8.4}
\]

**Proof.** Let us first describe the structure of the coefficients in (8.3) used for \( N = k. \) Repeated differentiation of the identity \( f(t) e^{t^2/2} = \mathbb{E} e^{it(x + iZ)} \) yields \( \frac{df}{dt} \bigg|_{t=0} = \mathbb{E}(X + iZ)^j. \) Hence, we get indeed \( c_j = \mathbb{E}H_j(X) \) for all \( j \leq k. \)

Now, assuming that b) holds, the expansion (8.3) simplifies to
\[
f(t) = e^{-t^2/2} \left( 1 + \frac{c_k}{k!} (it)^k \right) + o(|t|^k), \tag{8.5}
\]
so that \( \log f(t) = -\frac{1}{2} t^2 + \frac{c_k}{k!} (it)^k + o(|t|^k). \) The latter expansion immediately yields a). The argument may easily be reversed in order to show that a) \( \Rightarrow \) b) as well. Next, differentiating (8.5) \( j \) times at zero, \( j \leq k - 1, \) we get that \( \mathbb{E}X^j = H_j(0). \) But, we obtain a similar equality \( \mathbb{E}Z^j = H_j(0) \) when writing (8.5) for \( g(t) = e^{-t^2/2}. \) Hence, c) follows from b). Moreover, differentiating (8.5) \( k \) times at zero, we arrive at \( \mathbb{E}X^k = \mathbb{E}Z^k + c_k, \) which is the second equality in (8.4). Again, the argument may be reversed in the sense that, starting from c), we obtain (8.5) and therefore b). Thus, all the three properties are equivalent.

Finally, the first equality in (8.4) is obtained when differentiating the expression \( \log f(t) = -\frac{1}{2} t^2 + \frac{c_k}{k!} (it)^k + o(|t|^k) \) \( k \) times. \( \square \)

In general (without the above conditions a) – c)), the moments of \( X \) may be expressed easily in terms of the normal moments. Indeed, the Chebyshev-Hermite polynomials have generating function
\[
\sum_{k=0}^{\infty} H_k(x) \frac{z^k}{k!} = e^{xz - z^2/2},
\]
which follows, for example, from the identity \( H_k(x) = \mathbb{E}(x + iZ)^k. \) Here \( x \) may \( z \) may be any complex numbers. Equivalently,
\[
e^{xz} = e^{z^2/2} \sum_{i=0}^{\infty} H_i(x) \frac{z^i}{i!} = \sum_{i,j=0}^{\infty} H_i(x) \frac{z^{i+2j}}{i! j! 2^j}.
\]
Expanding \( e^{xz} \) into the power series and comparing the coefficients in front of \( z^k, \) we get
\[
x^k = k! \sum_{j=0}^{([k/2])} \frac{1}{(k-2j)! j! 2^j} H_{k-2j}(x).
\]
Hence, if \( \mathbb{E}|X|^k < \infty, \) then
\[
\mathbb{E}X^k = k! \sum_{j=0}^{([k/2])} \frac{1}{(k-2j)! j! 2^j} \mathbb{E}H_{k-2j}(X). \tag{8.6}
\]

Now, let us describe the connection between normal moments and the \( \chi^2 \)-distance. The series in (8.3) is absolutely convergent as \( N \to \infty, \) when \( f \) is analytic in the complex plane.
Hence we have the expansion
\[ f(t) = e^{-t^2/2} \sum_{k=0}^{\infty} \frac{c_k}{k!} (it)^k, \quad t \in \mathbb{C}, \]  
which holds, in particular, assuming condition (8.1). Moreover, using the Parseval identity as in Proposition 7.1, we have
\[ \sum_{k=0}^{\infty} \frac{|c_k|^2}{k!} = \int_{-\infty}^{\infty} \frac{p(x)^2}{\varphi(x)} \, dx. \]  
Since right-hand side is related to $\chi^2$-distance from the standard normal law, we arrive at the following relation:

**Proposition 8.2.** If $\chi^2(X, Z) < \infty$, then
\[ \chi^2(X, Z) = \sum_{k=1}^{\infty} \frac{1}{k!} (\mathbb{E}H_k(X))^2. \]  

For the quadratic Renyi divergence, we thus have
\[ D_2(X||Z) = \log(1 + \chi^2(X, Z)) = \log \sum_{k=0}^{\infty} \frac{1}{k!} (\mathbb{E}H_k(X))^2. \]  

Recall that, if $\chi^2(X, Z) < \infty$, then $X$ has finite moments of any order, and moreover, $\mathbb{E} e^{cX^2} < \infty$ for any $c < \frac{1}{4}$. Hence, the normal moments $\mathbb{E}H_k(X)$ are well defined and finite, so that the representation for $\chi^2(X, Z)$ makes sense.

We now show a converse to Proposition 8.2.

**Proposition 8.3.** Let $X$ be a random variable with finite moments of any order. If the series in (8.9) is convergent, then $X$ has an absolutely continuous distribution with finite distance $\chi^2(X, Z)$.

It looks surprising that a simple sufficient condition for the existence of a density $p$ of $X$ can be formulated in terms of moments of $X$ only. Note that if $X$ is bounded, then it has finite moments of any order, and the property $\chi^2(X, Z) < \infty$ just means that $p$ is in $L^2$. Thus, we have:

**Corollary 8.4.** A bounded random variable $X$ has an absolutely continuous distribution with a square integrable density, if and only if the series in (8.9) is convergent.

**Proof of Proposition 8.3.** Let $C^2 = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathbb{E}H_k(X))^2$ be finite ($C \geq 1$). Then $|\mathbb{E}H_k(X)| \leq C \sqrt{k!}$ and from the formula (8.6) we get
\[ |\mathbb{E}X^k| \leq k! \sum_{j=0}^{[k/2]} \frac{1}{(k-2j)! j! 2^j} |\mathbb{E}H_{k-2j}(X)| \leq C k! \sum_{j=0}^{[k/2]} \frac{1}{(k-2j)! j! 2^j}. \]

In particular,
\[ \mathbb{E}X^{2k} \leq C(2k)! \sum_{j=0}^{k} \frac{1}{\sqrt{(2k-2j)! j! 2^j}}. \]
Using \( \frac{(2k)!}{(2k-2j)!} \leq (2k)^{2j} \), we obtain that

\[
\mathbb{E} X^{2k} \leq C \sqrt{(2k)!} \sum_{j=0}^{k} \frac{\sqrt{(2k)!}}{j! 2^j} \leq C \sqrt{(2k)!} \sum_{j=0}^{k} \frac{k^j}{j!} < C \sqrt{(2k)!} \sum_{j=0}^{\infty} \frac{j!}{k!} = C e^k \sqrt{(2k)!}
\]

Thus, \( \mathbb{E} X^{2k} < C e^k \sqrt{(2k)!} \) for all \( k \). This estimate implies that \( \mathbb{E} e^{cX^2} < \infty \) for some \( c > 0 \). In particular, \( X \) has an entire characteristic function \( f(t) = \mathbb{E} e^{itX} \) which thus admits a power series representation (8.7), where necessarily \( c_k = \mathbb{E} H_k(X) \). Consider the \( N \)-th partial sum of that series,

\[
f_N(t) = e^{-t^2/2} \sum_{k=0}^{N} c_k \frac{(it)^k}{k!}.
\]

It represents the Fourier transform of the function \( p_N(x) = \varphi(x) \sum_{k=0}^{N} c_k \frac{H_k(x)}{k!} \) which is the \( N \)-th partial sum of the exponential series in (8.2). Since, by the assumption,

\[
\sum_{k=0}^{\infty} \frac{c_k^2}{k!} < \infty,
\]

\( p_N \) converge to some \( p \) in \( L^2(\mathbb{R}, dx/\varphi(x)) \), by Proposition 7.1. In particular, \( p_N \) converge in \( L^2(\mathbb{R}, dx) \), and by Plancherel’s theorem, \( f_N \) also converge in \( L^2(\mathbb{R}, dx) \) to the Fourier transform \( \hat{p} \) of \( p \). But \( f_N(t) \to f(t) \) for all \( t \), so \( f(t) = \hat{p}(t) \) almost everywhere. Thus we conclude that \( f \) belongs to \( L^2(\mathbb{R}, dx) \) and is equal to the Fourier transform of \( p \). Hence, \( X \) has an absolutely continuous distribution, and \( p \) is density of \( X \).

It remains to use once more the orthogonal series (8.2). By Proposition 7.1, we have Parseval’s equality (8.8), which means that \( \chi^2(X,Z) = \sum_{k=0}^{\infty} \frac{c_k^2}{k!} < \infty. \) □

There is a natural generalization of the identity (8.9) in terms of the \( \chi^2 \)-distance for the random variables

\[
X_t = \sqrt{t} X + \sqrt{1-t} Z, \quad 0 \leq t \leq 1,
\]

where \( Z \sim N(0,1) \) is independent of \( X \).

**Proposition 8.5.** If \( \chi^2(X,Z) < \infty \), then, for all \( t \in [0,1] \),

\[
\chi^2(X_t, Z) = \sum_{k=1}^{\infty} \frac{t^k}{k!} (\mathbb{E} H_k(X))^2.
\]

This in turn yields another description of the normal moments via the derivatives of the \( \chi^2 \)-distance:

\[
(\mathbb{E} H_k(X))^2 = \frac{d^k}{dt^k} \chi^2(X_t, Z)|_{t=0}, \quad k = 1, 2, \ldots
\]
Proof. It is known, e.g., as a direct consequence of the identity \( H_k(x) = \mathbb{E} (x + iz)^k \), that the Hermite polynomials satisfy the binomial formula

\[
H_k(\alpha x + \beta y) = \sum_{i=0}^{k} C_k^i \alpha^i \beta^{k-i} H_i(x) H_{k-i}(y), \quad x, y \in \mathbb{R},
\]

whenever \( \alpha^2 + \beta^2 = 1 \). In particular, \( \mathbb{E} H_k(aX + bZ) = a^k \mathbb{E} H_k(X) \), which may be used in the formula (8.9) with \( a = \sqrt{t} \) and \( b = \sqrt{1-t} \).

9. Behavior of Rényi divergence under Convolutions

The obvious question, when describing convergence in the central limit theorem in the \( D_\alpha \)-distance is, it remain finite for sums of independent summands with finite \( D_\alpha \)-distances? The answer is affirmative and is made precise in the following:

**Proposition 9.1.** Let \( X \) and \( Y \) be independent random variables. Given \( \alpha > 1 \), for all \( a, b \in \mathbb{R} \) such that \( a^2 + b^2 = 1 \), we have

\[
D_\alpha(aX + bY || Z) \leq D_\alpha(X || Z) + D_\alpha(Y || Z),
\]

where \( Z \sim N(0,1) \). Equivalently,

\[
1 + (\alpha - 1) T_\alpha(aX + bY || Z) \leq (1 + (\alpha - 1) T_\alpha(X || Z)) (1 + (\alpha - 1) T_\alpha(Y || Z)).
\]

The statement may be extended by induction to finitely many independent summands \( X_1, \ldots, X_n \) by the relation

\[
D_\alpha(a_1 X_1 + \cdots + a_n X_n || Z) \leq D_\alpha(X_1 || Z) + \cdots + D_\alpha(X_n || Z),
\]

where \( a_1^2 + \cdots + a_n^2 = 1 \). Note that for the relative entropy \( \alpha = 1 \), there is a much stronger property, namely

\[
D(a_1 X_1 + \cdots + a_n X_n || Z) \leq \max\{D(X_1 || Z), \ldots, D(X_n || Z)\},
\]

which follows from the entropy power inequality (cf. [D-C-T]). However, this is no longer true for \( D_\alpha \). Nevertheless, for the normalized sums \( Z_n = (X_1 + \cdots + X_n)/\sqrt{n} \) with i.i.d. summands, Proposition 9.1 guarantees a sublinear growth of the Rényi divergence with respect to \( n \). More precisely, we have

\[
D_\alpha(Z_n || Z) \leq n D_\alpha(X_1 || Z).
\]

**Proof of Proposition 9.1.** Let \( Z \) be an independent copy of \( Z \), so that the random vector \( \tilde{Z} = (Z, Z') \) is standard normal in \( \mathbb{R}^2 \). From Definition 2.1 it follows that the Rényi distance of the random vector \( \tilde{X} = (X, Y) \) to \( \tilde{Z} \) is given by

\[
D_\alpha(\tilde{X} || \tilde{Z}) = D_\alpha(X || Z) + D_\alpha(Y || Z').
\]

Hence, by the contractivity property (2.1), cf. Proposition 2.3, we get

\[
D_\alpha(S(\tilde{X}) || S(\tilde{Z})) \leq D_\alpha(X || Z) + D_\alpha(Y || Z'),
\]

for any Borel measurable function \( S : \mathbb{R}^2 \to \mathbb{R} \). It remains to apply this inequality to the linear function \( S(x, y) = ax + by \). \( \square \)
Let us describe a simple alternative argument in the case \( \alpha = 2 \), which relies upon normal moments only. One may assume that both \( D_2(X|Z) \) and \( D_2(Y|Z) \) are finite, so that \( X \) and \( Y \) have finite moments of any order. In addition, without loss of generality, let \( a, b > 0 \).

From the binomial formula (8.10) it follows that

\[
\mathbb{E}H_k(aX + bY) = \sum_{i=0}^{k} C_k^i a^{i} b^{k-i} \mathbb{E}H_i(X) \mathbb{E}H_{k-i}(Y).
\]

By Cauchy’s inequality,

\[
(\mathbb{E}H_k(aX + bY))^2 \leq \sum_{i=0}^{k} C_k^i (a^i b^{k-i})^2 \sum_{j=0}^{k} C_k^j (\mathbb{E}H_i(X))^2 (\mathbb{E}H_{k-j}(Y))^2
\]

\[
= \sum_{i=0}^{k} C_k^i (\mathbb{E}H_i(X))^2 (\mathbb{E}H_{k-i}(Y))^2.
\]

This gives

\[
\frac{(\mathbb{E}H_k(aX + bY))^2}{k!} \leq \sum_{i=0}^{k} \frac{(\mathbb{E}H_i(X))^2}{i!} \frac{(\mathbb{E}H_{k-i}(Y))^2}{(k-i)!},
\]

and summation over all integers \( k \geq 0 \) leads to

\[
\sum_{k=0}^{\infty} \frac{(\mathbb{E}H_k(aX + bY))^2}{k!} \leq \sum_{i=0}^{\infty} \frac{(\mathbb{E}H_i(X))^2}{i!} \sum_{j=0}^{\infty} \frac{(\mathbb{E}H_j(Y))^2}{j!}.
\]

But, by Proposition 8.2, this inequality is the same as

\[
1 + \chi^2(aX + bY, Z) \leq (1 + \chi^2(X, Z)) (1 + \chi^2(Y, Z)),
\]

which is exactly (9.1) for \( \alpha = 2 \).

One may also ask whether or not \( \chi^2(aX + bY, Z) \) remains finite, when \( \chi^2(X, Z) \) is finite, and \( Y \) is “small” enough. If \( p \) is density of \( X \), the density of \( aX + bY \) is given by

\[
q(x) = \frac{1}{|a|} \mathbb{E} p\left(\frac{x - bY}{a}\right), \quad x \in \mathbb{R},
\]

which is a convex mixture of densities on the line. Applying Cauchy’s inequality, we have

\[
\frac{q(x)^2}{\varphi(x)} \leq \frac{1}{a^2} \mathbb{E} \frac{p(x - bY)^2}{\varphi(x)},
\]

and using \( (ax + by)^2 \leq x^2 + y^2 \), we get an elementary bound

\[
\int_{-\infty}^{\infty} q(x)^2 \varphi(x) dx \leq \frac{1}{|a|} \mathbb{E} \int_{-\infty}^{\infty} \frac{p(x)^2}{\varphi(ax + bY)} dx
\]

\[
\leq \frac{1}{|a|} \mathbb{E} \int_{-\infty}^{\infty} \sqrt{2\pi} p(x)^2 e^{\frac{1}{2}(x^2 + y^2)} dx = \frac{1}{|a|} (1 + \chi^2(X, Z)) \mathbb{E} e^{Y^2/2}.
\]

That is, we arrive at:

**Proposition 9.2.** Let \( X \) and \( Y \) be independent random variables. For all \( a, b \in \mathbb{R} \) such that \( a^2 + b^2 = 1 \), we have

\[
1 + \chi^2(aX + bY, Z) \leq \frac{1}{|a|} (1 + \chi^2(X, Z)) \mathbb{E} e^{Y^2/2}, \quad Z \sim N(0, 1).
\]
Let us now describe two examples of i.i.d. random variables $X, X_1, \ldots, X_n$ such that for the normalized sums $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$, we have

$$\chi^2(Z_1, Z) = \cdots = \chi^2(Z_{n_0-1}, Z) = \infty, \quad \chi^2(Z_{n_0}, Z) < \infty, \quad (9.3)$$

where $n_0 > 1$ is a given prescribed integer.

**Example 9.3.** Suppose that $X$ has density of the form

$$p(x) = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/2\sigma^2} d\pi(\sigma^2), \quad x \in \mathbb{R}, \quad (9.4)$$

where $\pi$ is a probability measure on the positive half-axis. The existence of $\chi^2(X, Z)$ implies that $\sigma^2 < 2$ for $\pi$-almost all $\sigma^2$, i.e., $\pi$ should be supported on the interval $(0, 2)$. Squaring the equality (9.4) and integrating over $x$, we find that

$$1 + \chi^2(X, Z) = \int_{-\infty}^{\infty} p(x)^2 \, dx = \int_0^2 \int_0^2 \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1^2 \sigma_2^2}} d\pi(\sigma_1^2) d\pi(\sigma_2^2).$$

It is easy to see that the last double integral is convergent, if and only if

$$\int_0^1 \int_0^1 \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2}} d\pi(\sigma_1^2) d\pi(\sigma_2^2) < \infty \quad \text{and} \quad \int_1^2 \int_1^2 \frac{1}{\sqrt{4 - (\sigma_1^2 + \sigma_2^2)}} d\pi(\sigma_1^2) d\pi(\sigma_2^2) < \infty.$$

These conditions may be simplified in terms of the distribution function $F(\varepsilon) = \pi\{\sigma^2 \leq \varepsilon\}, 0 \leq \varepsilon \leq 2$, by noting that

$$F(\varepsilon/2)^2 \leq (\pi \otimes \pi)\{(\varepsilon_1^2 + \varepsilon_2^2 \leq \varepsilon\} \leq F(\varepsilon)^2.$$

Hence, the first integral is convergent, if and only if

$$\int_0^1 \frac{1}{\sqrt{\varepsilon}} dF(\varepsilon)^2 = F(1)^2 + \frac{1}{2} \int_0^1 \frac{F(\varepsilon)^2}{\varepsilon^{3/2}} \, d\varepsilon$$

is finite. A similar description applies to the second double integral.

Let us summarize: We have $\chi^2(X, Z) < \infty$ for the random variable $X$ with density (9.4), if and only if the mixing probability measure $\pi$ is supported on the interval $(0, 2)$, and its distribution function $F$ satisfies

$$\int_0^1 \frac{F(\varepsilon)^2}{\varepsilon^{3/2}} \, d\varepsilon < \infty, \quad \int_1^2 \frac{(1 - F(\varepsilon))^2}{(2 - \varepsilon)^{3/2}} \, d\varepsilon < \infty. \quad (9.5)$$

Based on this description, we now investigate convolutions. Note that $Z_n$ has density of a similar type as before

$$p_n(x) = \int_0^\infty \frac{1}{\sigma \sqrt{2\pi}} e^{-x^2/2\sigma^2} d\pi_n(\sigma^2).$$

More precisely, if $\xi_1, \ldots, \xi_n$ are independent copies of a random variable $\xi$ distributed according to $\pi$, then the mixing measure $\pi_n$ can be recognized as the distribution of the normalized sum $S_n = 1/n (\xi_1 + \cdots + \xi_n)$. Therefore, by (9.5), $\chi^2(Z_n, Z) < \infty$, if and only if $P\{S_n < 2\} = 1$ (which is equivalent to the property that $\pi$ is supported on $(2, \infty)$), and

$$\int_0^1 \frac{F_n(\varepsilon)^2}{\varepsilon^{3/2}} \, d\varepsilon < \infty, \quad \int_1^2 \frac{(1 - F_n(\varepsilon))^2}{(2 - \varepsilon)^{3/2}} \, d\varepsilon < \infty,$$
where \( F_n \) is the distribution function of \( S_n \). Since \( F(\varepsilon/n)^n \leq F_n(\varepsilon) \leq F(\varepsilon)^n \), which is needed near zero, and using similar relations near the point 2, these conditions may be simplified to
\[
\int_0^1 \frac{F(\varepsilon)^{2n}}{\varepsilon^{3/2}} \, d\varepsilon < \infty, \quad \int_1^2 \frac{(1 - F(\varepsilon))^{2n}}{(2 - \varepsilon)^{3/2}} \, d\varepsilon < \infty. \tag{9.6}
\]

Now, for simplicity, suppose that \( \pi \) is supported on \((0, 1)\), so that the second integral in (9.6) is vanishing, and let \( F(\varepsilon) \sim \varepsilon^n \) for \( \varepsilon \to 0 \) with parameter \( \kappa > 0 \) (where the equivalence is understood up to a positive factor). Then, the first integral in (9.6) will be finite, if and only if \( n > 1/(4\kappa) \). Choosing \( \kappa = \frac{1}{4(n_0 - 1)} \), we obtain the required property (9.3).

**Example 9.4.** Consider a density of the form
\[
p(x) = \frac{a_k}{1 + |x|^{2k}} e^{-x^2/4}, \quad x \in \mathbb{R},
\]
where \( a_k \) is a normalizing constant, \( k = n_0 - 1 \), and let \( f_1 \) denote its Fourier transform (i.e., the characteristic function). Define the distribution of \( X \) via its characteristic function
\[
f(t) = \alpha f_1(t) + (1 - \alpha) \frac{\sin(\gamma t)}{\gamma t}
\]
with a sufficiently small parameter \( \alpha > 0 \) and \( \gamma = \sqrt{3 \frac{1 + \alpha f_1''(0)}{1 - \alpha}} \). It is easy to check that \( f''(0) = -1 \), which guarantees that \( \mathbb{E}X = 0, \mathbb{E}X^2 = 1 \). Furthermore, it is not difficult to show that the densities \( p_n \) of \( Z_n \) admit the two-sided bounds
\[
\frac{b_n'}{1 + |x|^{n/2k}} e^{-x^2/4} \leq p_n(x) \leq \frac{b_n''}{1 + |x|^{n/2k}} e^{-x^2/4} \quad (x \in \mathbb{R}),
\]
up to some positive \( n \)-dependent factors. Hence, again we arrive at the property (9.3).

10. **Superadditivity of \( \chi^2 \) with Respect to Marginals**

A multidimensional version of Theorem 1.1 requires to involve some other properties of the \( \chi^2 \)-distance in higher dimensions. The contractivity under mappings,
\[
\chi^2(S(X), S(Z)) \leq \chi^2(X, Z),
\]
has already been shown in Proposition 2.3 in a general setting. This inequality may be considerably sharpened, when distance is measured to the standard normal law in \( \Omega = \mathbb{R}^d \). In order to compare the behavior of \( \chi^2 \)-divergence with often used information-theoretic quantities, recall the definition of Shannon entropy and Fisher information,
\[
h(X) = -\int_{\mathbb{R}^d} p(x) \log p(x) \, dx, \quad I(X) = \int_{\mathbb{R}^d} \frac{|
abla p(x)|^2}{p(x)} \, dx,
\]
where \( X \) is a random vector in \( \mathbb{R}^d \) with density \( p \) (assuming that the above integrals are well-defined). These functionals are known to be subadditive and superadditive with respect to the components: If we write \( X = (X', X'') \) with \( X' \in \mathbb{R}^{d_1}, X'' \in \mathbb{R}^{d_2} \) \((d_1 + d_2 = d)\), then one always has
\[
h(X) \leq h(X') + h(X''), \quad I(X) \geq I(X') + I(X'') \tag{10.1}
\]
cf. \([L], [C]\). Both \( h(X) \) and \( I(X) \) themselves are not yet distances, so one also considers the relative entropy and the relative Fisher information with respect to other distributions. In
particular, in case of the standard normal random vector $Z \sim N(0, I_d)$ and random vectors $X$ with mean zero and identity covariance matrix $I_d$, they are given by
\[
D(X||Z) = h(Z) - h(X), \quad I(X||Z) = I(X) - I(Z).
\]
Hence, it immediately follows from (10.1) that these information-theoretic distances are both superadditive, that is,
\[
D(X||Z) \geq D(X'||Z') + D(X''||Z''), \quad I(X||Z) \geq I(X'||Z') + I(X''||Z''),
\]
where $Z'$ and $Z''$ are standard normal in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively (both inequalities become equalities, when $X'$ and $X''$ are independent).

We now establish a similar property for the $\chi^2$-distance, which can be more convenient stated in the setting of a Euclidean space $H$, say of dimension $d$, with norm $| \cdot |$ and inner product $\langle \cdot, \cdot \rangle$. If $X$ is a random vector in $H$ with density $p$, and $Z \sim N(0, I_d)$ is a normal random vector with mean zero and an identity covariance operator $I_d$, then (according to the abstract definition),
\[
\chi^2(X, Z) = \int_H \frac{p(x)^2}{\varphi(x)} \, dx - 1 = \int_H \frac{(p(x) - \varphi_d(x))^2}{\varphi(x)} \, dx,
\]
where $\varphi(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2} \ (x \in H)$ is the density of $Z$.

**Proposition 10.1.** Given a random vector $X$ in $H$ and an orthogonal decomposition $H = H' \oplus H''$ into two linear subspaces $H', H'' \subset H$ of dimensions $d_1, d_2 \geq 1$, for orthogonal projections $X' = \text{Proj}_{H'}(X)$, $X'' = \text{Proj}_{H''}(X)$, we have
\[
\chi^2(X, Z) \geq \chi^2(X', Z') + \chi^2(X'', Z''),
\]
where $Z, Z', Z''$ are standard normal random vectors in $H, H', H''$, respectively.

Note, however, that (10.2) won’t become an equality for independent components $X', X''$.

**Proof.** Let $H = \mathbb{R}^d$ and $X = (\xi_1, \ldots, \xi_d)$. Note that $\chi^2(X, Z)$ is invariant under orthogonal transformations $U$ of the space, i.e., $\chi^2(U(X), Z) = \chi^2(X, Z)$. Hence, without loss of generality, one may assume that $X' = (\xi_1, \ldots, \xi_{d_1})$ and $X'' = (\xi_{d_1+1}, \ldots, \xi_d)$. Moreover, to simplify the argument (notationally), let $d_1 = d_2 = 1$.

The finiteness of the distance $\chi^2(X, Z)$ means that the random vector $X = (\xi_1, \xi_2)$ has density $p = p(x_1, x_2) \ (x_i \in \mathbb{R})$ such that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, x_2)^2 e^{(x_1^2 + x_2^2)/2} \, dx_1 \, dx_2 < \infty.
\]
Since the Hermite functions $\varphi_{k_1, k_2}(x_1, x_2) = \varphi(x_1)\varphi(x_2) H_{k_1}(x_1) H_{k_2}(x_2)$ form a complete orthogonal system in $L^2(\mathbb{R}^2)$ (where now $\varphi$ denotes the one dimensional standard normal density), the density $p$ admits a unique representation in the form of the exponential series
\[
p(x_1, x_2) = \varphi(x_1)\varphi(x_2) \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \frac{c_{k_1, k_2}}{k_1! k_2!} H_{k_1}(x_1) H_{k_2}(x_2), \quad \text{(10.3)}
\]
which converges in $L^2(\mathbb{R}, \frac{dx_1 dx_2}{\varphi(x_1)\varphi(x_2)})$. Here the coefficients (mutual normal moments) are given by
\[
c_{k_1, k_2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_{k_1}(x_1) H_{k_2}(x_2) p(x_1, x_2) \, dx_1 \, dx_2 = \mathbb{E} H_{k_1}(\xi_1) H_{k_2}(\xi_2),
\]
and we have Parseval’s equality
\[ 1 + \chi^2(X, Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{p(x_1, x_2)^2}{\varphi(x_1)\varphi(x_2)} \, dx_1 \, dx_2 = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} c_{k_1, k_2}^2 \frac{k_1! \, k_2!}{k_1! \, k_2!}, \]  
(10.4)

Now, integrating (10.3) over \( x_2 \) and separately over \( x_1 \), we obtain similar representations for the marginal densities
\[ p_1(x_1) = \varphi(x_1) \sum_{k_1=0}^{\infty} \frac{c_{k_1, 0} \, H_{k_1}(x_1)}{k_1!}, \quad p_2(x_2) = \varphi(x_2) \sum_{k_2=0}^{\infty} \frac{c_{0, k_2} \, H_{k_2}(x_2)}{k_2!}, \]
hence, by Proposition 8.1,
\[ \chi^2(\xi_1, \xi) = \sum_{k_1=1}^{\infty} \frac{c_{k_1, 0}^2}{k_1!} \quad \text{and} \quad \chi^2(\xi_2, \xi) = \sum_{k_2=1}^{\infty} \frac{c_{0, k_2}^2}{k_2!} \quad (\xi \sim N(0, 1)). \]
Obviously, the quantities \( \chi^2(\xi_1, \xi) \) and \( \chi^2(\xi_2, \xi) \) appear as summands in (10.4), thus showing the
inequality. \( \square \)
11. Asymptotic Expansions and Lower Bounds

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, with characteristic function $f(t) = \mathbb{E}e^{itX}$. Then the normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

weakly converge in distribution to the standard normal law: $Z_n \Rightarrow Z$ for $Z \sim N(0,1)$. In this connection the following question arises: When is it true that $D_\alpha(Z_n||Z) \to 0$ or equivalently $T_\alpha(Z_n||Z) \to 0$ as $n \to \infty$? And if so, what is the rate of convergence?

We shall give a complete solution of this problem in the next sections. First we shall describe here asymptotic expansions for “truncated” $T_\alpha$-distances, which yield reasonable lower bounds for $T_\alpha(Z_n||Z)$. More precisely, given $M > 0$, we have an obvious estimate

$$T_\alpha(Z_n||Z) \geq \frac{1}{\alpha - 1} \left( I(M) - 1 \right)$$

with

$$I(M) = \int_{|x| \leq M} \left( \frac{p_n(x)}{\phi(x)} \right)^\alpha \phi(x) \, dx,$$

where $p_n$ denotes the density of $Z_n$. We will see that, under suitable conditions (like the ones in Theorems 1.1-1.2), while choosing $M = M_n(s) = \sqrt{2(s-1) \log n}$ with a fixed integer $s \geq 2$, inequality (11.1) can be reversed up to an error term of order $o(n^{-(s-1)})$. This reduces our task to the study of the asymptotic behavior of the integrals $I(M_n(s))$, using the following result due to Petrov (cf. [Pe1-2], [B-C-G1]).

**Proposition 11.1.** Suppose that $X$ has a finite absolute moment of order $k \geq 3$, and assume that $Z_n$ admits a density in $L^2$ for some $n$. Then, for all $n$ large enough, $Z_n$ have continuous bounded densities $p_n$ satisfying uniformly in $-\infty < x < \infty$

$$p_n(x) = \phi(x) + \phi(x) \sum_{\nu=1}^{k-2} \frac{q_\nu(x)}{n^{\nu/2}} + o\left( \frac{1}{n^{(k-2)/2}} \right) \frac{1}{1 + |x|^k}.$$  

(11.3)

In this formula

$$q_\nu(x) = \sum_{m=1}^{\nu} H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left( \frac{\gamma_{m+2}}{(m+2)!} \right)^{k_m},$$

(11.4)

where $\gamma_r$ denotes the $r$-th cumulant of $X$. The summation extends over all non-negative integer solutions $(k_1, k_2, \ldots, k_\nu)$ to the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$, and where we put $l = k_1 + k_2 + \cdots + k_\nu$. The sum in (11.3) defines a polynomial in $x$ of degree at most $3(k-2)$.

For example, for $k = 3$ (11.3) yields

$$p_n(x) = \phi(x) + \frac{\gamma_3}{3! \sqrt{n}} H_3(x) \phi(x) + o\left( \frac{1}{n} \right) \frac{1}{1 + |x|^3},$$

where $\gamma_3$ is the third cumulant of $X$. This reduces our task to the study of the asymptotic behavior of the integrals $I(M_n(s))$. We will see that, under suitable conditions (like the ones in Theorems 1.1-1.2), while choosing $M = M_n(s) = \sqrt{2(s-1) \log n}$ with a fixed integer $s \geq 2$, inequality (11.1) can be reversed up to an error term of order $o(n^{-(s-1)})$. This reduces our task to the study of the asymptotic behavior of the integrals $I(M_n(s))$, using the following result due to Petrov (cf. [Pe1-2], [B-C-G1]).
where \( \gamma_3 = \alpha_3 = \mathbb{E}X^3 \) and \( H_3(x) = x^3 - 3x \). More generally, if the first cumulants of \( X \) up to order \( k - 1 \) are vanishing (i.e., the fist \( k - 1 \) moments of \( X \) are the same as for a standard normal law), then (11.3) simplifies to the expression

\[
p_n(x) = \varphi(x) + \frac{\gamma_k}{k!} H_k(x) \varphi(x) n^{-\frac{k-2}{2}} + o(n^{-\frac{k-2}{2}}) \frac{1}{1 + |x|^k} \quad (\gamma_3 = \cdots = \gamma_{k-1} = 0).
\]

This local limit theorem may be used to derive:

**Lemma 11.2.** Under the assumptions of Proposition 11.1 with \( k = 2s \) \( (s \geq 2) \), the following expansion holds

\[
I(M_n(s)) = 1 + \sum_{j=1}^{s-1} \frac{b_j}{n^j} + o(n^{-(s-1)}) \quad (11.5)
\]

with

\[
b_j = \sum \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m_1!\cdots m_{2j-1}!} \int_{-\infty}^{\infty} q_1(x)^{m_1} \cdots q_{2j-1}(x)^{m_{2j-1}} \varphi(x) \, dx. \quad (11.6)
\]

Here the sum extends over all non-negative integer numbers \( m_1, \ldots, m_{2j-1} \) such that \( m_1 + 2m_2 + \cdots + (2j-1)m_{2j-1} = 2j \), and where \( m = m_1 + \cdots + m_{2j-1} \). In particular, if \( \gamma_j = 0 \) for \( j = 3, \ldots, s-1, \ s \geq 3 \), then

\[
I(M_n(s)) = 1 + \alpha(\alpha-1) \frac{\gamma_s^2}{2s!} \frac{1}{n^{s-2}} + O(n^{-(s-1)}). \quad (11.7)
\]

Using (11.4), one can evaluate the integrals in (11.5) and rewrite them as polynomials in the cumulants \( \gamma_3, \ldots, \gamma_{2j+1} \), which in turn may be expressed polynomially in terms of the moments \( \alpha_r = \mathbb{E}X^r, \ r \leq 2j+1 \).

**Proof.** The representation (11.3) with \( k = 2s \) may be written as

\[
\frac{p_n(x)}{\varphi(x)} = 1 + R_n(x) + \frac{\varepsilon_n(x)}{n^{s-1}} \frac{1}{\varphi(x)(1 + |x|^{2s})}, \quad R_n(x) = \sum_{\nu=1}^{s-2} \frac{q_{\nu}(x)}{n^{\nu/2}},
\]

where \( \sup_x |\varepsilon_n(x)| = o(1) \) as \( n \to \infty \). Since every polynomial \( q_{\nu} \) has degree at most \( \nu = 3(2s-2) \), we necessarily have \( |R_n(x)| \leq C_\nu \sqrt{n} (1 + |x|^\nu) \) up to some constant \( C \). It follows that

\[
\left| \frac{p_n(x)}{\varphi(x)} - 1 \right| \leq C \sqrt{n} (1 + |x|^\nu) + \frac{o(n^{-(s-1)})}{\varphi(x)(1 + |x|^{2s})} \leq \delta_n \to 0
\]

as \( n \to \infty \) uniformly in \( |x| \leq M_n(s) \). Using the Lipschitz property of the power function near the point 1, we thus obtain that

\[
\left( \frac{p_n(x)}{\varphi(x)} \right)^\alpha = (1 + R_n(x))^\alpha + \frac{o(n^{-(s-1)})}{\varphi(x)(1 + |x|^{2s})},
\]

so that

\[
I(M_n(s)) = \int_{|x| \leq M_n(s)} (1 + R_n(x))^\alpha \varphi(x) \, dx + o(n^{-(s-1)}). \quad (11.8)
\]

Using a Taylor expansion of \( x^\alpha \)

\[
(1 + R_n(x))^\alpha = 1 + \sum_{m=1}^{2s-2} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} R_n(x)^m + \frac{C_n(x)}{n^{s-1/2}} (1 + |x|^\nu(2s-1))
\]
with \( \sup_x |C_n(x)| \leq C \) (where \( C \) is a constant). Thus, integration in (11.8) leads to

\[
I(M_n(s)) = 1 + \sum_{m=1}^{2s-2} \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m!} \int_{|x| \leq M_n(s)} R_n(x)^m \varphi(x) \, dx + o(n^{-(s-1)}).
\]

Here the integrals may be extended to the whole real line at the expense of an error at most \( o\left(\frac{1}{n^{s-1}}\right) \). Indeed, with some constant \( C_l \) depending on \( l \geq 1 \), we have

\[
\int_{|x| > M_n(s)} |x|^l \varphi(x) \, dx \leq C_l M_n(s)^{l-1} e^{-M_n(s)^2/2} = O\left(\frac{\log \frac{1}{x} n}{n^{s-1}}\right),
\]

which may be used in the polynomial bound on \( R_n \) (together with the factor \( 1/\sqrt{n} \)). Thus,

\[
I(M_n(s)) = 1 + \sum_{m=1}^{2s-2} \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m!} \int_{-\infty}^{\infty} R_n(x)^m \varphi(x) \, dx + o(n^{-(s-1)}).
\]

Using a multinomial expansion, we get

\[
R_n(x)^m = \sum_{m_1+\ldots+m_{2s-2}=m} \frac{m!}{m_1!\ldots m_{2s-2}!} n^{-N/2} q_1(x)^{m_1} \ldots q_{2s-2}(x)^{m_{2s-2}} \varphi(x) \, dx,
\]

where \( N = m_1 + 2m_2 + \cdots + (2s-2) m_{2s-2} \). That is, up to a \( o(n^{-(s-1)}) \)-term, one can describe \( I(M_n(s)) - 1 \) as the sum

\[
\sum \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m_1!\ldots m_{2s-2}!} n^{-N/2} \int_{-\infty}^{\infty} q_1(x)^{m_1} \ldots q_{2s-2}(x)^{m_{2s-2}} \varphi(x) \, dx,
\]

where the summation extends over all integers \( m_1, \ldots, m_{2s-2} \geq 0 \), not all zero, such that \( m = m_1 + m_2 + \cdots + m_{2s-2} = 2s - 2 \). This representation simplifies thanks to the following property of Hermite polynomials:

\[
\int_{-\infty}^{\infty} H_{\nu_1}(x) \ldots H_{\nu_k}(x) \varphi(x) \, dx = 0 \quad (\nu_1 + \cdots + \nu_k \text{ is odd}).
\]

Hence, it follows from (11.4) that a similar property holds for \( q_j \)'s as well, so that the integral in (11.9) is vanishing, as long as \( N \) is odd. Restricting ourselves to the values \( N = 2j \), we necessarily have \( m_l = 0 \) for \( l > 2j \), and (11.8) becomes

\[
\sum \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{m_1!\ldots m_{2j}!} n^{-j} \int_{-\infty}^{\infty} q_1(x)^{m_1} \ldots q_{2j}(x)^{m_{2j}} \varphi(x) \, dx,
\]

where the summation extends over all \( m_1, \ldots, m_{2j} \geq 0 \) such that \( m_1 + 2m_2 + \cdots + 2j m_{2j} = 2j \) and with \( m = m_1 + \cdots + m_{2j} \). Finally, we may exclude the case \( m_{2j} = 1 \), \( m_l = 0 \) for \( l < 2j \), where again the above integral is vanishing. As a result, we arrive at the required expansion (11.5) with coefficients (11.6). Finally, in the second assertion, we necessarily have \( b_j = 0 \) for \( j = 1, \ldots, s - 3 \) and \( b_{s-2} = \gamma_s^2/2s! \) and we obtain (11.7). \( \square \)

Note that the integral in (11.10) is zero as well, provided that \( m = 1 \) (i.e., only one \( m_l = 1 \)). For the index \( \alpha = 2 \), the factor in front of the integral in (11.6) is vanishing unless \( m \leq 2 \). Hence, we are reduced to tuples \( m_1, \ldots, m_{2j-1} \) such that \( m_l = 1 \) holds for two different indexes, say, \( l = \nu_1 \) and \( l = \nu_2 \), and also for tuples where \( m_l = 2 \) holds for one \( l \) only. Hence, the description of the coefficients may be simplified to

\[
b_j = \sum_{\nu_1, \nu_2 \geq 0 \atop \nu_1 + \nu_2 = 2j} \int_{-\infty}^{\infty} q_{\nu_1}(x) q_{\nu_2}(x) \varphi(x) \, dx \quad (\alpha = 2).
\]
Recall that if $T_\alpha(Z_n||Z)$ is finite, then $\mathbb{E} e^{cZ_n^2} < \infty$, and hence $\mathbb{E} e^{cX^2} < \infty$ for some $c > 0$ (so that $X$ has finite moments of all orders). In addition, $Z_n$ must have a density in $L^2$. Therefore, all conditions of Lemma 11.1 are fulfilled, and in view of the lower bound (11.1), Lemma 11.2 yields:

**Proposition 11.3.** For every fixed $s = 3, 4, \ldots$, we have, as $n \to \infty$,

$$T_\alpha(Z_n||Z) \geq \frac{1}{\alpha - 1} \sum_{j=1}^{s-2} b_j \frac{1}{n^j} + O\left(\frac{1}{n^{s-1}}\right)$$

with coefficients given in (11.6). In particular, if $\gamma_j = 0$ for $j = 3, \ldots, s - 1$ and $\gamma_s \neq 0$, then

$$T_\alpha(Z_n||Z) \geq \frac{\alpha \gamma_s^2}{2s!} \frac{1}{n^{s-2}} + O\left(\frac{1}{n^{s-1}}\right).$$

(11.11)

The last lower bound extends to $D_\alpha$ as well (which is equivalent to $T_\alpha$ when these two distances are small). Hence we get:

**Corollary 11.4.** If, for some integer $K > 1$,

$$\liminf_{n \to \infty} \frac{\log D_\alpha(Z_n||Z)}{\log n} < -K,$$

then $\gamma_j = 0$ for all $j = 3, \ldots, K$. In particular, the random variable $X$ is standard normal, if and only if

$$\liminf_{n \to \infty} \frac{\log D_\alpha(Z_n||Z)}{\log n} = -\infty.$$

Combining the lower bound (11.2) with the upper bound (9.2) yields:

**Corollary 11.5.** Let $D_\alpha(X||Z) < \infty$, with $\gamma_j = 0$, $j = 3, \ldots, s - 1$, and $\gamma_s \neq 0$ for some $s \geq 3$. Then as $n \to \infty$

$$\left(1 + O\left(\frac{1}{n}\right)\right) \frac{\gamma_s^2}{2s!} \frac{1}{n^{s-2}} \leq D_\alpha(Z_n||Z) \leq nD_\alpha(X||Z).$$

12. Necessity Part in Theorem 1.2 ($d = 1$)

Again, let $X, X_1, X_2, \ldots$ denote i.i.d. random variables with characteristic function $f(t) = \mathbb{E} e^{itX}$, and let $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. The necessity part in Theorem 1.2 does not require any moment assumptions on the mean and variance. As a preliminary step, the next proposition provides a subgaussian bound on the Laplace transform $f(iy) = \mathbb{E} e^{-yX}$ subject to the sublinear growth of $D_\alpha(Z_n||Z)$. Recall that $\alpha > 1$ is fixed, and we denote its conjugate value by $\beta = \alpha/(\alpha - 1)$.

**Lemma 12.1.** If $\liminf_{n \to \infty} \left[ \frac{1}{n} D_\alpha(Z_n||Z) \right] = 0$, then

$$f(iy) \leq e^{\beta y^2/2}, \quad y \in \mathbb{R}. \quad (12.1)$$
Proof. Indeed, by Proposition 4.2, applied to $Z_n$ in place of $X$, for all $y \in \mathbb{R}$,

$$f(iy/\sqrt{n})^n \leq (1 + (\alpha - 1)T_\alpha(Z_n||Z))^{1/\alpha} e^{\beta y^2/2},$$

and after a change of the variable we get

$$f(iy) \leq \exp \left\{ \frac{1}{\alpha n} \log \left( 1 + (\alpha - 1)T_\alpha(Z_n||Z) \right) \right\} e^{\beta y^2/2}.$$

But $\lim \inf_{n \to \infty} \left[ \frac{1}{n} D_\alpha(Z_n||Z) \right] = 0$, if and only if $\lim \inf_{n \to \infty} \left[ \frac{1}{\alpha} \log(1+(\alpha-1)T_\alpha(Z_n||Z)) \right] = 0$. Hence, we arrive at the required conclusion by letting $n \to \infty$ along a suitable subsequence. \(\square\)

In other words, if $f(iy_0) > e^{\beta y_0^2/2}$ holds for some $y_0 \in \mathbb{R}$, then $D_\alpha(Z_n||Z) \geq c n$ holds for some positive constant $c$. Thus, in this case $D_\alpha(Z_n||Z)$ has a maximal growth rate, in view of the sublinear upper bound (9.2).

The assumption of Lemma 12.1 is fulfilled, when $D_\alpha(Z_n||Z) \to 0$, which provides a slightly weakened variant of the necessary condition (1.4) in Theorem 1.2 for dimension $d = 1$ (replacing the strict inequality with a non-strict inequality). To arrive at a more precise condition, we have to add another preliminary step.

**Lemma 12.2.** If $\lim_{n \to \infty} D_\alpha(Z_n||Z) = 0$, then, for any integer $k \geq \alpha/2$,

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f(iy/\sqrt{kn})^{2kn} e^{-\beta y^2} dy = \sqrt{\pi} (\alpha - 1).$$

(12.2)

**Proof.** The characteristic function of $Z_n$ is given by $f_n(t) = f(t/\sqrt{n})^n$. Hence, the integral in (12.2) is just

$$\int_{-\infty}^{\infty} \left( \mathbb{E} e^{-\gamma Z_{nk}} \right)^2 e^{-\beta y^2} dy = \int_{-\infty}^{\infty} \mathbb{E} e^{-\gamma (Z_{nk} + Z'_{nk})} e^{-\beta y^2} dy$$

$$= \int_{-\infty}^{\infty} \mathbb{E} e^{-\gamma Z_{2nk}} e^{-\beta y^2} dy = \sqrt{\beta/\pi} \mathbb{E} e^{1/2} Z_{2nk}^2,$$

where by $Z_{nk}'$ we denoted an independent copy of $Z_{nk}$. On the other hand, since $Z_{nk}$ is a normalized sum of $k$ independent copies of $Z_n$, we may apply Proposition 5.1 with $X$ replaced by $Z_n$ and with $n$ replaced by $2k$. In this case, inequality (5.2) tells us that, whenever $2k \geq \alpha$, we have

$$|\mathbb{E} e^{1/2} Z_{2nk}^2 - \mathbb{E} e^{1/2} Z^2| \leq c_{2k} \left( (1 + \chi_\alpha(Z_n||Z)\alpha^{-1}) k - 1 \right), \quad Z \sim N(0,1).$$

Since, by the assumption, $\chi_\alpha(Z_n, Z) \to 0$ as $n \to \infty$, the limit in (12.2) is equal to $\sqrt{\beta/\pi} \mathbb{E} e^{1/2} Z^2$. \(\square\)

**Proof of the necessity part in Theorem 1.2 for $d = 1$.** Let $D_\alpha(Z_n||Z) \to 0$ as $n \to \infty$. Given a fixed number $\delta > 0$, let us decompose

$$\int_{-\infty}^{\infty} f(iy/\sqrt{nk})^{2nk} e^{-\beta y^2} dy = I_1 + I_2$$

$$= \left( \int_{|y| \leq \delta \sqrt{nk}} + \int_{|y| > \delta \sqrt{nk}} \right) f(iy/\sqrt{nk})^{2nk} e^{-\beta y^2} dy. \quad (12.3)$$
The characteristic function $f$ is entire, and $f(0) = 1$, hence it is non-vanishing in some disc $|t| < R$ on the complex plane. Define $g(t) = \log f(t)$ for $|t| < R$, choosing the branch of the logarithm according to the condition $\log f(0) = 0$. The function $g$ is analytic in the same disc and admits a power series representation

$$g(t) = -\frac{1}{2}t^2 + \sum_{k=3}^{\infty} b_k t^k.$$ 

Clearly, for a suitable value $r \in (0, R)$ and some constant $C$, we have $\sum_{k=3}^{\infty} |b_k| t^k \leq C|t|^3$ in the disc $|t| \leq r$, so that

$$f(y/\sqrt{nk})^{2nk} = \exp\{y^2 + \theta y^3/\sqrt{n}\} \quad \text{for real } y \in [-r/\sqrt{n}, r\sqrt{n}],$$

where $\theta$ is a quantity such that $|\theta| \leq C$ and $k \geq \alpha/2$ is a fixed integer. Assuming that $\delta \leq \min\{r, 1/(2C)\}$, this relation allows us to rewrite the integral $I_1$ as

$$I_1 = \int_{|y| \leq \delta \sqrt{n}} e^{-(\beta-1)y^2 + \theta y^3/\sqrt{n}} dy.$$ 

Here the term $\theta y^3/\sqrt{n}$ in the above exponent may be removed at the expense of an error of order $O(\frac{1}{\sqrt{n}})$. This is justified by the bounds

$$\int_{|y| \leq \delta \sqrt{n}} |e^{-(\beta-1)y^2 + \theta y^3/\sqrt{n}} - e^{-(\beta-1)y^2}| dy \leq \int_{|y| \leq \delta \sqrt{n}} C\frac{y^3}{\sqrt{n}} e^{-(\beta-1)y^2 + \theta y^3/\sqrt{n}} dy \leq C \int_{|y| \leq \delta \sqrt{n}} y^3 e^{-(\beta-1)y^2/2} dy = O\left(\frac{1}{\sqrt{n}}\right).$$

Hence

$$I_1 = \int_{|y| \leq \delta \sqrt{n}} e^{-(\beta-1)y^2} dy + O\left(\frac{1}{\sqrt{n}}\right) = \sqrt{\pi}(\alpha - 1) + O\left(\frac{1}{\sqrt{n}}\right), \quad n \to \infty.$$ 

Applying this result in (12.3), the equality (12.2) implies that $I_2 \to 0$, or equivalently

$$\int_{|u| > \delta} f(iu) e^{-\beta u^2/2} 2nk \, du = o\left(\frac{1}{\sqrt{n}}\right) \quad \text{as } n \to \infty, \quad (12.4)$$

which holds for any sufficiently small $\delta > 0$, and since the integrand is non-negative, for any smaller fixed $\delta > 0$ as well.

Now, the function $\psi(u) = f(iu) e^{-\beta u^2/2}$ is analytic and satisfies $0 < \psi(u) \leq 1$ on the real line, cf. (12.1). In order to show that $\psi(u) < 1$ for all $u \neq 0$, suppose for a moment that $\psi(u_0) = 1$ for some $u_0 > 0$. Obviously $u_0$ has to be local maximum point, which implies $\psi'(u_0) = 0$. Hence the power series representation at this point, that is

$$\psi(u) - 1 = c_l (u - u_0)^l + \sum_{j=l+1}^{\infty} c_j (u - u_0)^j$$

starts with a non-zero term $c_l \neq 0$ for some $l \geq 2$. Since $\psi(u) - 1 \leq 0$ for all $u \in \mathbb{R}$, all coefficients are real numbers, and moreover, $l = 2m$ is even ($m \geq 1$) and $c_l < 0$. Hence, in some neighborhood $|u - u_0| \leq r_0 < u_0$ and for some constants $c_1, c_0 > 0$, we have $\psi(u) \geq 1 - c_1(u - u_0)^{2m} \geq e^{-c_0(u - u_0)^2m}$. Now choosing $\delta = u_0 - r_0$, this neighborhood is contained in
(δ, ∞), and with some constant c > 0 we get
\[
\int_{|u| > \delta} (f(iu) e^{-\beta u^2/2})^{2nk} du \geq \int_{|u - u_0| < \delta} \psi(u)^{2nk} du \\
\geq \int_{|u - u_0| < \delta} \exp \left\{ -2nk \cdot c_0 (u - u_0)^{2m} \right\} du \\
= 2 \int_{0}^{\delta} \exp \left\{ -2nk \cdot c_0 x^{2m} \right\} dx \geq \frac{c}{n^{1/(2m)}},
\]
which contradicts to the asymptotic relation (12.4). The case \( u_0 < 0 \) is similar, and thus we necessarily arrive at \( \psi(u) < 1 \) for all real \( u \neq 0 \). □

13. Pointwise Upper Bounds for Convolutions of Densities

Before turning to the sufficiency part in Theorem 1.2, we shall derive several upper bounds for the densities \( p_n \) of the normalized sums \( Z_n \). In general, bounds for the density \( p(x) \) of \( X \) at individual points \( x \) cannot be deduced from \( D_{\alpha}(X|\|Z) < \infty \). However, this is possible after several convolutions of \( p \) with itself. The following observation holds without assuming that \( X \) has mean zero and variance one. Put
\[
\psi(u) = f(iu)e^{-\beta u^2/2} = \mathbb{E} e^{-uX} e^{-\beta u^2/2} = e^{-\beta u^2/2} \int_{-\infty}^{\infty} e^{-ux} p(x) dx, \quad u \in \mathbb{R},
\]
where \( f \) is the characteristic function of \( X \) and \( \beta = \frac{\alpha}{\alpha - 1} \). As usual, \( Z \) denotes a standard normal random variable.

**Proposition 13.1.** Given a random variable \( X \) such that \( T_\alpha = T_\alpha(X|\|Z) < \infty \), we have, for all \( x \in \mathbb{R} \) and \( n \geq n_\beta = \max(\beta, 2) \),
\[
p_n(x) \leq \frac{A_\alpha \sqrt{n}}{(2\pi)^{1/2}} e^{-x^2/(2\beta)} \left\{ -\frac{x}{\beta \sqrt{n}} \right\} \psi\left( -\frac{x}{\beta \sqrt{n}} \right)^{n-n_\beta}, \tag{13.1}
\]
where \( A_\alpha = (1 + (\alpha - 1)T_\alpha)^{\frac{\alpha}{2}} \) in case \( 1 < \alpha \leq 2 \) and \( A_\alpha = (1 + (\alpha - 1)T_\alpha)^{\frac{\alpha}{2}} \) in case \( \alpha > 2 \).

In particular, under the condition (1.1), that is, when \( \psi \leq 1 \), we arrive at the following subgaussian pointwise bound
\[
p_n(x) \leq \frac{A_\alpha \sqrt{n}}{(2\pi)^{1/2}} e^{-x^2/(2\beta)},
\]
which may be effective in the region \( |x| \gg \sqrt{\log n} \). It can be sharpened further for larger values of \( |x| \) by virtue of Proposition 4.3. Combined with (13.1), it immediately provides an exponential pointwise bound (with respect to \( n \)).

**Corollary 13.2.** If \( T_\alpha(X|\|Z) < \infty \), there exist constants \( x_0 > 0 \) and \( \delta \in (0, 1) \) depending on the density \( p \) only, such that, for all \( n \) large enough,
\[
p_n(x) \leq \delta^n e^{-x^2/(2\beta)} \psi\left( -\frac{x}{\beta \sqrt{n}} \right)^{n/2}, \quad \text{whenever} \ |x| \geq x_0 \sqrt{n}. \tag{13.2}
\]
Here the last $\psi$-term is (13.2) will become crucial for bounding $T_\alpha(Z_n||Z)$.

**Proof of Proposition 13.1.** Since $\mathbb{E} e^{yX} < \infty$ for all $y \in \mathbb{R}$, the characteristic function $f_n(t) = \mathbb{E} e^{itZ_n} = f(t/\sqrt{n})^n$ is extended as an entire function to the complex plane. Since $p$ belongs to $L^\alpha(\mathbb{R}, dx)$, an application of the Hausdorff-Young inequality implies that $f_n$ is integrable whenever $n \geq \max(\beta, 2)$. In this case $Z_n$ has a continuous density given by the Fourier inversion formula

$$p_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t/\sqrt{n})^n \, dt = \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} e^{-itx} f(t/\sqrt{n})^n \, dt.$$  

Moreover, since the family $\{e^{hx}p_n(x)\}_{0 \leq h \leq y}$ is compact in $L^1(\mathbb{R})$, $f_n(t)$ tends to zero at infinity uniformly in every strip $|\text{Im} t| \leq y < \infty$ (by the Riemann-Lebesgue lemma). Applying Cauchy’s theorem to rectangle contour $[-T, T] \cup [T+i\eta] \cup [T+i\eta, -T+i\eta] \cup [-T+i\eta, -T]$, the inversion formula may therefore be written as

$$p_n(x) = e^{yx} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f((t+i\eta)/\sqrt{n})^n \, dt$$  

(13.3) for any fixed $y > 0$. Without loss of generality, let $x < 0$.

**Case $\alpha > 2, n \geq 2$.** Using $|f(t+i\eta)| \leq f(i\eta) \, (t, y \in \mathbb{R})$ and changing variable in (13.3), we get

$$p_n(x) \leq e^{yx} f(i\eta/\sqrt{n})^{n-2} \sqrt{n} \int_{-\infty}^{\infty} |f(t+i\eta/\sqrt{n})|^2 \, dt.$$  

(13.4)

The function $t \to f(t+i\eta/\sqrt{n}) = \mathbb{E} e^{itX-yX/\sqrt{n}}$ is the Fourier transform of $g(u) = e^{-yu/\sqrt{n}} p(u)$. Hence, by Parseval’s identity,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t+i\eta/\sqrt{n})|^2 \, dt = \int_{-\infty}^{\infty} e^{-2yu/\sqrt{n}} p(u)^2 \, du.$$  

To estimate the latter integral, factorize the integrand as $\left( e^{-2yu/\sqrt{n}} \phi(u)^{2/\beta} \right) \frac{\phi(u)^2}{\phi(u)^{2/\beta}}$ and apply Hölder’s inequality with exponents $r = \frac{\alpha}{\alpha - 1}$, $r^* = \frac{\alpha}{2}$. Thus, up to the factor $(1 + (\alpha - 1)T_\alpha)^{2/\alpha}$, this integral can be estimated from above by

$$\left( \int_{-\infty}^{\infty} e^{-2yu/\sqrt{n}} \phi(u)^{2/\beta} \, du \right)^{1/r} = \frac{1}{\sqrt{2\pi}} \left( \frac{\alpha - 2}{2\alpha - 2} \right)^{1/2r} e^{\beta y^2/n} \leq \frac{1}{\sqrt{2\pi}} e^{\beta y^2/n}.$$  

This gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t+i\eta/\sqrt{n})|^2 \, dt \leq \frac{1}{\sqrt{2\pi}} (1 + (\alpha - 1)T_\alpha)^{2/\alpha} e^{\beta y^2/n},$$  

and (13.4) results in the upper bound

$$p_n(x) \leq \sqrt{\frac{n}{2\pi}} (1 + (\alpha - 1)T_\alpha)^{2/\alpha} e^{\beta y^2/n} f(i\eta/\sqrt{n})^{n-2}$$  

$$= \sqrt{\frac{n}{2\pi}} (1 + (\alpha - 1)T_\alpha)^{2/\alpha} e^{\beta y^2/2} \psi(y/\sqrt{n})^{n-2}.$$  

Choosing here $y = -x/\beta$, we arrive at (13.1).

**Case $1 < \alpha \leq 2, n \geq \beta$.** Again using $|f(t+i\eta)| \leq f(i\eta) \, (t, y \in \mathbb{R})$ and changing variable, we obtain from (13.3) that

$$p_n(x) \leq e^{yx} f(i\eta/\sqrt{n})^{n-\beta} \sqrt{n} \int_{-\infty}^{\infty} |f(t+i\eta/\sqrt{n})|^\beta \, dt.$$  

(13.5)
Now, since $\beta \geq 2$, we are allowed to apply the classical Hausdorff-Young inequality
\[
\left( \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t + iy/\sqrt{n})|^\beta dt \right)^{1/\beta} \leq \|g\|_{\alpha} = \left( \int_{-\infty}^{\infty} e^{-\alpha yu/\sqrt{n}} p(u)^\alpha du \right)^{1/\alpha}.
\]
To estimate the latter integral, factorize its integrand as \( e^{-\alpha yu/\sqrt{n}} \varphi(u) \) and use the inequality \( e^{-\alpha yu/\sqrt{n}} \varphi(u) \leq (2\pi)^{-\alpha/2} e^{\frac{\alpha^2 y^2}{2n}} \). This gives
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(t + iy/\sqrt{n})|^\beta dt \leq \left( (2\pi)^{-\alpha/2} e^{\frac{\alpha^2 y^2}{2n}} (1 + (\alpha - 1)T_\alpha) \right)^{\beta/\alpha} = \frac{1}{\sqrt{2\pi}} e^{\frac{\beta}{2n} - \frac{\alpha}{\alpha - 1}}.
\]
Hence, (13.5) results in the upper bound
\[
p_n(x) \leq \sqrt{\frac{n}{2\pi}} \left( 1 + (\alpha - 1)T_\alpha \right)^{\frac{1}{\alpha-1}} e^{\frac{\beta}{2n} - \frac{\alpha}{\alpha - 1}} f(iy/\sqrt{n})^{\alpha - \beta}
\]
\[
= \sqrt{\frac{n}{2\pi}} \left( 1 + (\alpha - 1)T_\alpha \right)^{\frac{1}{\alpha-1}} e^{\frac{\beta}{2n} - \frac{\alpha}{\alpha - 1}} \psi(y/\sqrt{n})^{\alpha - \beta}.
\]
Again choosing \( y = -x/\beta \), we arrive at (13.1). \( \square \)

14. Sufficiency Part in Theorem 1.2 \((d = 1)\).

Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables such that \( \mathbb{E}X = 0 \), \( \mathbb{E}X^2 = 1 \), with characteristic function \( f(t) = \mathbb{E}e^{itX} \). As before, put \( \psi(u) = f(iu) e^{-\beta u^2/2} \), \( \beta = \frac{\alpha}{\alpha - 1} \), and let \( Z \sim N(0,1) \). Assuming that the condition (1.4) is fulfilled, i.e., \( \psi(u) < 1 \) for all real \( u \neq 0 \), here it will be shown that the normalized sums
\[
Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}
\]
do satisfy \( T_\alpha(Z_n||Z) \to 0 \) as \( n \to \infty \), as long as \( T_\alpha(Z_{n_0}||Z) < \infty \) for some \( n_0 \). We also derive an asymptotic expansion for this distance which is rather similar to (1.2) in case \( \alpha = 2 \). For simplicity, let us assume that \( n_0 = 1 \), so that \( X \) has density \( p \) with \( T_\alpha(X||Z) < \infty \) (the general case \( n_0 \geq 1 \) is rather similar and needs only minor modifications). In particular, all \( Z_n \) have densities \( p_n \) which are continuous and bounded for all \( n \) large enough.

In Section 11, we considered integrals of the form
\[
I_0 = \int_{|x| \leq M_n} \frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} dx \quad \text{with} \quad M_n = \sqrt{2(l - 1) \log n} \quad (l = 3, 4, \ldots)
\]
According to Proposition 11.1 with \( k = 2l \) and Lemma 11.2, these integrals admit an asymptotic expansion
\[
I_0 = 1 + \sum_{j=1}^{l-1} \frac{b_j}{n^j} + o(n^{-(l-1)})
\]
(14.1)
which may be simplified in terms of the cumulants of \( X \) as
\[
I_0 = 1 + \alpha(\alpha - 1) \frac{\gamma_2^2}{2s!} \frac{1}{n^{s-2}} + O(n^{-(s-1)})
\]
(14.2)
when \( \gamma_j = 0 \) for \( j = 3, \ldots, l - 1 \). Hence, for the proof of Theorem 1.2 (in dimension one), it remains to bound the integral of \( p_n^\alpha/\varphi^{\alpha-1} \) over the complementary region \( |x| > M_n \) by a
polynomially small quantity (with respect to \( n \)). More precisely, it will be sufficient to show that, for any large enough \( l \geq 3 \) and some constant \( \kappa > 0 \),
\[
\int_{|x| > M_n} \frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} \, dx = O\left( \frac{1}{n^{\kappa}} \right), \quad n \to \infty.
\tag{14.3}
\]

To this aim, we need to properly estimate \( p_n(x) \) for \( |x| > M_n \), which can be done based on the pointwise bounds of the previous section. For definiteness, let us consider the half-axis \( x < -M_n \), which we split into three intervals reflecting the possible different behavior of these densities. Namely, define
\[
I_1 = \int_{-\infty}^{-x_0\sqrt{n}} \frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} \, dx, \quad I_2 = \int_{-x_0\sqrt{n}}^{-x_1\sqrt{n}} \frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} \, dx, \quad I_3 = \int_{-x_1\sqrt{n}}^{-M_n} \frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} \, dx
\]
with parameters \( 0 < x_1 < x_0 \) and assuming that \( M_n < x_1\sqrt{n} \) (otherwise, \( I_3 = 0 \)).

Applying inequality (13.2), we obtain that, for all \( n \) large enough and with some \( \delta \in (0, 1) \) and \( x_0 > 0 \),
\[
I_1 \leq (2\pi)^{\frac{\alpha+1}{2}} \delta^{\alpha n} \int_{-\infty}^{-x_0\sqrt{n}} \psi\left( -\frac{x}{\beta\sqrt{n}} \right) \alpha n/2 \, dx
\]
\[
\leq (2\pi)^{\frac{2\alpha+1}{2}} \delta^{\alpha n} \beta \sqrt{n} \int_{-\infty}^{-x_0\sqrt{n}} \psi(u) u^\delta \, du, \quad m \leq \frac{\alpha n}{2},
\]
where on the last step we used \( \psi \leq 1 \). By Corollary 5.2, cf. (5.3), the last integral is convergent whenever \( m \geq \alpha \). One may take, for example, \( m = [\alpha]+1 \), which ensures the condition \( m \leq \frac{\alpha n}{2} \) for all sufficiently large \( n \). Hence
\[
I_1 \leq C \delta^{\alpha n} \quad (n \geq n_1)
\]
with some constants \( C > 0 \), \( x_0 > 0 \) and \( \delta < \delta_1 < 1 \), depending on the density \( p \) only.

To estimate the integral \( I_2 \) (with any fixed number \( 0 < x_1 < x_0 \)), we employ Proposition 13.1. By the condition (1.4), the function \( \psi \) is bounded away from 1 on any compact interval in \((-\infty, 0)\), so, \( \delta_2 = \max_{-x_0 \leq u \leq -x_1} \psi(u) < 1 \). Hence, by inequality (13.1),
\[
I_2 \leq A_\alpha n^{\alpha/2} \int_{-x_0\sqrt{n}}^{-x_1\sqrt{n}} \psi\left( -\frac{x}{\beta\sqrt{n}} \right)^{n-n_\beta} \, dx
\]
\[
= A_\alpha n^{(\alpha+1)/2} \int_{-x_0/2}^{-x_1/2} \psi(u)^{n-n_\beta} \, du \leq A_\alpha n^{(\alpha+1)/2} (x_0 - x_1) \delta_2^{-n-n_\beta}
\]
which again decays exponentially fast like \( I_1 \).

It remains to properly estimate the integral \( I_3 \) with some (prescribed) \( x_1 > 0 \). In order to estimate \( p_n(x) \) in \([-x_1\sqrt{n}, -M_n]\), we use the bound (13.1) once more. As discussed in Section 12, the function \( h(u) = \log f(iu) \) is analytic in some disc \( |u| \leq r \), and since \( h(0) = 0 \), \( h'(0) = 1/2 \), we have \( h(u) \sim \frac{1}{2} u^2 \) near zero. Hence \( |h(u)| \leq \frac{1+\beta}{4} |u|^2 \) throughout this disc, when \( r \) is sufficiently small, implying \( |f(iu)| \leq e^{(1+\beta)|u|^2/4} \). Hence for \( u \) real, \( |u| \leq r \), we have \( \psi(u) \leq e^{-(\beta-1)|u|^2/4} \), which implies
\[
\psi\left( -\frac{x}{\beta\sqrt{n}} \right)^{n-n_\beta} \leq \psi\left( -\frac{x}{\beta\sqrt{n}} \right)^{n/2} \leq \exp\left\{ -\frac{\beta - 1}{4} \frac{x^2}{2\beta^2} \right\} = e^{-x^2/(8\alpha\beta)}
\]
for all \( n \geq 2 \max(\beta, 2) \) and \( -\beta r \sqrt{n} < x < 0 \). Therefore, by (13.1), in this interval
\[
\frac{p_n(x)^\alpha}{\varphi(x)^{\alpha-1}} \leq A_\alpha n^{\alpha/2} e^{-x^2/(8\beta)},
\]
which results with \( x_1 = \beta r \) in
\[
I_3 \leq A_0^n n^{\alpha/2} \int_{-x_1 \sqrt{n}}^{-M_n} e^{-x^2/(8\beta)} \, dx
\]
\[
\leq \sqrt{2\pi\beta} A_0^n n^{\alpha/2} e^{-M_n^2/(8\beta)} = \sqrt{2\pi\beta} A_0^n n^{-(4\beta \pi \alpha - 2)},
\]
where we used a well-known inequality \( \int_M^\infty \varphi(x) \, dx \leq \frac{1}{2} e^{-M^2/2} (M > 0) \).

Collecting these bounds, we obtain that \( I_1 + I_2 + I_3 = o(n^{-1/8\beta}) \) for a sufficiently large \( l \).
A similar relation holds true for integrals over the half-axis \( x > M_n \), which proves (14.3).

Since \( T_\alpha(Z_n||Z) = \frac{1}{\alpha - 1} (I_0 + I_1 + I_2 + I_3 - 1) \), and using the expansions (14.1)-(14.2), we conclude that, for any \( s = 3, 4, \ldots \),
\[
T_\alpha(Z_n||Z) = \frac{1}{\alpha - 1} \sum_{j=1}^{s-2} \frac{b_j}{n^j} + O(n^{-(s-1)})
\]
with coefficients \( b_j \) described in (11.6). Moreover, in terms of the cumulants of \( X \), (14.4) simplifies to
\[
T_\alpha(Z_n||Z) = \alpha \frac{\gamma_j^2}{2s!} \frac{1}{n^{s-2}} + O(n^{-(s-1)}) \quad \text{in case } \gamma_j = 0 \text{ for } j = 3, \ldots, s - 1.
\]
Since \( D_\alpha \) and \( T_\alpha \) are equivalent (when these quantities are small), the last relation holds true for the Rényi distance \( D_\alpha(Z_n||Z) \) as well. Thus, Theorem 1.2 is proved in dimension one.  \( \square \)

15. Non-uniform Local Limit Theorem

Here we prove Theorem 1.3 in dimension one, still keeping the basic assumptions \( EX = 0 \), \( EX^2 = 1 \). We shall state it in a more precise form, by using the cumulants \( \gamma_k \) of \( X \). We remind that \( \beta = \frac{\alpha}{\alpha - 1} \) (\( \alpha > 1 \)).

**Theorem 15.1.** Suppose that \( D_\alpha(Z_n||Z) \) is finite for some \( n = n_0 \), and assume that condition (1.4) holds. If \( \gamma_3 = \cdots = \gamma_{s-1} = 0 \) for some \( s \geq 3 \), then
\[
\sup_{x \in \mathbb{R}} \frac{|p_n(x) - \varphi(x)|}{\varphi(x)^{1/\beta}} = \frac{a_s |\gamma_s|}{s!} n^{-\frac{s-2}{2}} + O(n^{-\frac{s-2}{4}}),
\]
(15.1)
where
\[
a_s = \sup_{x \in \mathbb{R}} \left[ \varphi(x)^{1/\alpha} |H_s(x)| \right].
\]

In case \( s = 3 \) we thus obtain the inequality (1.5), and if \( EX^3 = 0 \) (and hence \( \gamma_3 = 0 \), one may turn to the next moment of order \( s = 4 \), which yields the rate \( 1/n \) in (15.1). As for the cumulant coefficient, let us recall that \( \gamma_s = E H_s(X) = EX^s - EZ^s \) (cf. Proposition 8.1).

To compare these results with Proposition 11.1, note that, assuming the existence of moments of order \( s \), and that \( Z_n \) has a bounded continuous density \( p_n \) for large \( n \), the Edgeworth expansion (11.3) allows to derive a weaker statement, such as
\[
\sup_{x \in \mathbb{R}} (1 + |x|^s) \frac{|p_n(x) - \varphi(x)|}{\varphi(x)} = \frac{a'_s |\gamma_s|}{s!} n^{-\frac{s-2}{2}} + o(n^{-\frac{s-2}{4}}),
\]
where \( a'_s = \sup_{x \in \mathbb{R}} (1 + |x|^s) |H_s(x)| \varphi(x) \) (still assuming that the moments of \( X \) of orders less than \( s \) are the same as for the standard normal law).
Note in addition that the condition (1.4) is almost necessary for the conclusion such as (15.1) and even for a weaker one. Indeed, suppose that
\[
\liminf_{n \to \infty} \sup_{x \in \mathbb{R}} \frac{p_n(x) - \varphi(x)}{\varphi(x)^{1/\beta}} < \infty, \tag{15.2}
\]
so that
\[
p_n(x) \leq \varphi(x) + C_n \varphi(x)^{1/\beta}, \quad \liminf_{n \to \infty} C_n < \infty.
\]
Multiplying this inequality by \( e^{tx} \) and integrating, we get
\[
(\mathbb{E} e^{tX/\sqrt{n}})^n = \mathbb{E} e^{tZ_n} \leq e^{t^2/2} + BC_n e^{\beta t^2/2}, \quad B = (2\pi)^{(1-1/\beta)/2} \sqrt{\beta}.
\]
Now substitute \( t \) with \( t\sqrt{n} \) and raise the above inequality to the power \( 1/n \). Letting \( n \to \infty \) along a suitable subsequence, we arrive in the limit at
\[
\mathbb{E} e^{tX} \leq e^{3t^2/2}, \quad t \in \mathbb{R}.
\]
Thus, this subgaussian property is indeed implied by the local limit theorem in the form (15.2).

**Proof of Theorem 15.1.** Here in contrast with the proof of Theorem 1.2, we need to consider a decomposition into a smaller number of zones. For simplicity, let \( n_0 = 1 \), and as before, define
\[
M_n = \sqrt{2(l - 1) \log n}
\]
with parameter \( l \geq s - 1 \), assuming that is sufficiently large. Then (11.3) yields the desired equality (15.1), provided that the supremum on the left is taken over the interval \( |x| \leq M_n \). Hence, it will be sufficient to bound the two suprema
\[
J_1 = \sup_{|x| \geq x_1 \sqrt{n}} \frac{p_n(x)}{\varphi(x)^{1/\beta}}, \quad J_2 = \sup_{M_n \leq |x| \leq x_1 \sqrt{n}} \frac{p_n(x)}{\varphi(x)^{1/\beta}}
\]
by polynomially small quantities (with respect to \( n \)) with some \( x_1 > 0 \) and assuming that \( M_n < x_1 \sqrt{n} \) (otherwise, \( J_2 = 0 \)).

To this aim, we again invoke the bounds of Proposition 13.1 and Corollary 13.2. The assumption (1.4) means that the function \( \psi(u) = \mathbb{E} e^{-uX} e^{-\beta hu^2/2} \) satisfies \( \psi(u) < 1 \) for all \( u \neq 0 \). Hence, the bound (13.2) yields, for all \( n \) large enough,
\[
\frac{p_n(x)}{\varphi(x)^{1/\beta}} \leq \delta^n, \quad |x| \geq x_0 \sqrt{n},
\]
which is valid with some \( \delta \in (0, 1) \) and \( x_0 > 0 \). Moreover, since \( \delta_2 = \max_{x_1 \leq |u| \leq x_0} \psi(u) < 1 \) for any \( x_1 \in (0, x_0) \), the bound (13.1) yields
\[
\frac{p_n(x)}{\varphi(x)^{1/\beta}} \leq A_\alpha \sqrt{n} \delta_2^{n - n_\beta} \leq \delta_1^n \quad (n \geq n_\beta = \max(\beta, 2))
\]
with some \( \delta < \delta_2 < 1 \). Both estimates imply \( J_1 = O(\delta_1^n) \) as \( n \to \infty \) for any \( x_1 > 0 \).

Moreover, as shown in the proof of the sufficiency part of Theorem 1.2, we have for some \( x_1 > 0 \),
\[
\frac{p_n(x)}{\varphi(x)^{1/\beta}} \leq A_\alpha \sqrt{n} e^{-2x^2/(8\alpha^2\beta)}, \quad |x| \leq x_1 \sqrt{n}.
\]
This gives
\[
J_2 \leq A_\alpha \sqrt{n} e^{-M_n^2/(8\alpha^2\beta)} = A_\alpha n^{-\left(\frac{1}{2(\beta - 1)}\right)} \leq A_\alpha n^{-\kappa},
\]
where the last inequality holds for any prescribed value of \( \kappa > 0 \) by a suitable choice of \( l \). \( \square \)
16. The Multidimensional Case

Let us now turn to the multidimensional variant of Theorems 1.1-1.3. We will denote by $Z$ a standard normal random vector in $\mathbb{R}^d$, i.e., having mean zero and an identity covariance matrix. Given i.i.d. random vectors $X, X_1, X_2, \ldots$ in $\mathbb{R}^d$ with mean zero and identity covariance, consider the normalized sums

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \quad (n = 1, 2, \ldots)$$

We need to show that $D_\alpha(Z_n\|Z) \to 0$ as $n \to \infty$, if and only if $D_\alpha(Z_n\|Z)$ is finite for some $n = n_0$, and

$$\mathbb{E} e^{\langle X, t \rangle} < e^{\beta|t|^2/2} \quad \text{for all } t \in \mathbb{R}^d, \ t \neq 0. \quad (16.1)$$

Moreover, in this case $D_\alpha(Z_n\|Z) = O(1/n)$, and $D_\alpha(Z_n\|Z) = O(1/n^2)$ when the distribution of $X$ is symmetric about the origin. In fact, a more precise Edgeworth-type expansion holds for $T_\alpha(Z_n\|Z)$ in powers of $1/n$ similarly to (14.4)-(14.5), with the coefficients being polynomials of mixed cumulants of the components of $X$.

As for the proof of the theorems, much of the analysis developed before about the convergence in $T_\alpha$ (or $D_\alpha$), as well pointwise upper bounds on the densities $p_n$ of $Z_n$, may easily be extended from dimension one to an arbitrary dimension $d$. Actually, the contractivity property of the functional $D_\alpha$ (Proposition 2.3) allows one to reduce the necessity part in Theorem 1.2 to the one dimensional case using a standard Wold type device. Indeed, consider the i.i.d. sequence $\langle X_i, \theta \rangle$ with unit vectors $\theta$. Then, assuming that $D_\alpha(Z_n\|Z) \to 0$ as $n \to \infty$, we get

$$D_\alpha(\langle Z_n, \theta \| \langle Z, \theta \rangle \rangle \leq D_\alpha(Z_n\|Z) \to 0.$$ 

Since $\mathbb{E} \langle X_i, \theta \rangle = 0$, $\mathbb{E} \langle X_i, \theta \rangle^2 = 1$, and $\langle Z, \theta \rangle \sim N(0, 1)$, we are ready to apply the one dimensional variant of this theorem which gives

$$\mathbb{E} e^{r\langle X, \theta \rangle} < e^{\beta r^2/2} \quad \text{for all } r \neq 0.$$ 

This is exactly the condition (16.1), thus proving the necessity part in Theorem 1.2.

Note that, as in dimension one (cf. Proposition 4.1), the finiteness of $D_\alpha(X\|Z)$ guarantees that $\mathbb{E} e^{c|X|^2} < \infty$ for all $c < 1/(2\beta)$. In particular, the characteristic function $f(t) = \mathbb{E} e^{\langle X, t \rangle}$ now extends as an entire function to the $d$-dimensional complex space $\mathbb{C}^d$. Most important properties of the densities $p_n$ of $Z_n$ rely upon the function

$$\psi(u) = f(iu) e^{-\beta|u|^2/2} = \mathbb{E} e^{-\langle X, u \rangle} e^{-\beta|u|^2/2} \quad (u \in \mathbb{R}^d).$$

**Lemma 16.1.** If $T_\alpha = T_\alpha(X\|Z) < \infty$, then $\psi(u)$ tends to zero as $|u| \to \infty$ and belongs to $L^k(\mathbb{R}^d)$ for any integer $k \geq \alpha$. Moreover, up to some $(k, d)$-dependent constants $c_{k, d}$, we have

$$\int_{\mathbb{R}^d} \psi(u)^k \, du \leq c_{k, d} \left(1 + (\alpha - 1)T_\alpha \right)^{\frac{k}{2}}. \quad (16.2)$$

The first assertion is a multidimensional analog of Proposition 4.3; it can be proved with very similar arguments as in dimension one. The second assertion generalizing Corollary 5.2 can be proved by using the contractivity properties of the $d$-dimensional Weierstrass transform

$$W_t u(x) = \frac{1}{(2\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} u(y) \, dy, \quad x \in \mathbb{R}^d, \ t > 0.$$
In particular, in $\mathbb{R}^d$ the inequality (5.1) takes the form $\mathbb{E} e^{\frac{x}{\beta/n}} |Z_k|^2 \leq c_{k,d} (1 + (\alpha - 1)T_\alpha)^{\frac{k}{\beta}}$, from which (16.2) easily follows. In case $\alpha = 2$, one may adapt Lemma 6.3 as well to the multidimensional situation with its Parseval identity in $\mathbb{R}^d$. Furthermore, Proposition 6.2 is extended as

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \psi(u)^2 du \leq 1 + \chi^2(X, Z),$$

thus refining (16.2) for $k = 2$.

Repeating the arguments as in Section 13, one may also extend the corresponding upper pointwise bounds on the densities.

**Lemma 16.2.** If $T_\alpha(X||Z) < \infty$, then for all $x \in \mathbb{R}^d$ and $n \geq n_\beta = \max(\beta, 2)$,

$$p_n(x) \leq A_{\alpha,d} n^{d/2} e^{-|x|^2/(2\beta)} \psi\left( -\frac{x}{\beta\sqrt{n}} \right)^{n-n_\beta},$$

where $A_{\alpha,d}$ depends on $(\alpha, d)$ only. In particular, there exist constants $x_0 > 0$ and $\delta \in (0,1)$ depending on the density $p$ such that for all $n$ large enough

$$p_n(x) \leq \delta^n e^{-|x|^2/(2\beta)} \psi\left( -\frac{x}{\beta\sqrt{n}} \right)^{n/2}, \quad \text{whenever } |x| \geq x_0\sqrt{n}. \quad (16.4)$$

**Proof of Theorem 1.2** (Sufficiency part) and **Theorem 1.3.** Assume that $n_0 = 1$. Hence $Z_n$ admits density $p_n$ for any $n \geq 1$. We need to derive the asymptotic behavior of

$$(\alpha - 1) T_\alpha(Z_n||Z) = \int_{\mathbb{R}^d} w_n^\alpha(x) \, dx - 1, \quad w_n(x) = \frac{p_n(x)}{\varphi(x)^{1/\beta}},$$

where $\varphi$ is the standard normal density on $\mathbb{R}^d$. To this aim, it is natural to split the integration into the four shell-type regions. The asymptotic behavior of the integrals

$$I_0 = \int_{|x| < M_n} w_n^\alpha(x) \, dx, \quad M_n = \sqrt{2(l-1) \log n},$$

may be studied as in dimension one (cf. Lemma 11.2) by virtue of the Edgeworth expansion for $p_n(x)$ on the balls $|x| < M_n$ with a non-uniform error term. To this aim, a multidimensional variant of Proposition 11.1 is used as stated in the monograph [BR-R], Theorem 19.2: Uniformly in $\mathbb{R}^d$

$$p_n(x) = \varphi_s(x) + o\left(n^{-(s-2)/2}\right) \frac{1}{1 + |x|^s}, \quad \varphi_s(x) = \varphi(x) + \varphi(x) \sum_{k=1}^{s-2} \frac{q_k(x)}{n^{k/2}}, \quad (16.5)$$

where each $q_k$ represents a polynomial whose coefficients involve mixed cumulant of the components of $X$ of order up to $k + 2$. In particular, if the distribution of $X$ is symmetric about the origin, then $q_1(x) = 0$ and thus there is no $1/\sqrt{n}$ term in the sum (16.5).

In this way, we will arrive at the Edgeworth-type expansion for $I_0$ similarly to dimension one, which readily implies that $I_0 - 1 = O(1/n)$ in general, and $I_0 - 1 = O(1/n^2)$ when the distribution of $X$ is symmetric. As a result, it remains to establish a polynomial smallness of the integrals

\begin{align*}
I_1 &= \int_{|x| > x_0\sqrt{n}} w_n^\alpha(x) \, dx, \quad I_2 = \int_{x_1\sqrt{n} < |x| < x_0\sqrt{n}} w_n^\alpha(x) \, dx, \quad I_3 = \int_{M_0 < |x| < x_1\sqrt{n}} w_n^\alpha(x) \, dx
\end{align*}
with \( x_1 > 0 \) being any fixed small number, and \( x_0 > x_1 \) depending on the density \( p \). The bounds (16.2)-(16.4) allow us to properly estimate these integrals as functions of \( n \), by modifying the arguments from the previous section. Using (16.4) and (16.2) with \( k = [\alpha] + 1 \) and assuming that \( \psi \leq 1 \), we get for all \( n \) large enough

\[
I_1 \leq C_1 \int_{|x| > x_0 \sqrt{n}} \psi \left( -\frac{x}{\beta \sqrt{n}} \right)^{n/2} dx \leq C_2 \delta^{\alpha n} n^{d/2} \int_{\mathbb{R}^d} \psi(u)^k du \leq C_3 \delta_1^n
\]

with some constants \( C_j, x_0 > 0 \) and \( 0 < \delta < \delta_1 < 1 \) which do not dependent on \( n \).

For the region of \( I_2 \), thanks to condition (1.4), we have \( \delta_2 = \max_{x_0 \leq |u| \leq x_1} \psi(u) < 1 \). Hence, by (16.3), putting \( n_1 = n - \max(\beta, 2) \), we obtain that with some constants \( C_j > 0 \)

\[
I_2 \leq C_1 n^{d\alpha/2} \int_{x_1 \sqrt{n} < |x| < x_0 \sqrt{n}} \psi \left( -\frac{x}{\beta \sqrt{n}} \right)^{n_1} dx
\]

\[
= C_2 n^{d(\alpha + 1)/2} \int_{x_1 \sqrt{n} < |x| < x_0 \sqrt{n}} \psi(u)^{n_1} du \leq C_3 n^{d(\alpha + 2)/2} x_0^d \delta_1^{n_1}
\]

which is decaying exponentially fast like \( I_1 \).

Finally, using the analyticity of \( f \), we have \( \psi(u) \leq e^{-(\beta - 1)|u|^2/4} \) in a sufficiently small ball \( |u| < r \), so that

\[
\psi \left( -\frac{x}{\beta \sqrt{n}} \right)^{n_1} \leq \psi \left( -\frac{x}{\beta \sqrt{n}} \right)^{n/2} \leq e^{-x^2/(8\alpha \beta)}, \quad |x| < \beta r \sqrt{n},
\]

for all \( n \geq 2 \max(\beta, 2) \). Therefore, by (16.2), in this ball \( w_n(x) \leq A_{\alpha, d}^n n^{d\alpha/2} e^{-|x|^2/(8\alpha \beta)} \), which gives with \( x_1 = \beta r \)

\[
I_3 \leq C_1 n^{d\alpha/2} \int_{M_n < |x| < x_1 \sqrt{n}} e^{-|x|^2/(8\alpha \beta)} dx
\]

\[
< C_2 n^{d\alpha/2} \mathbb{P}\{|Z|^2 > M_n^2/(8d\alpha \beta)\} \leq C_3 n^{d\alpha/2} e^{-M_n^2/(8d\alpha \beta)} = C_3 n^{-(\delta_1^{n_1} - 2)}.
\]

Collecting these bounds, we get that \( I_1 + I_2 + I_3 = o(n^{-1/8d\alpha \beta}) \) for all sufficiently large \( l \), thus proving Theorem 1.2.

For the proof of Theorem 1.3 in \( \mathbb{R}^d \), we need to investigate the suprema

\[
J_0 = \sup_{|x| \leq M_n} \frac{|p_n(x) - \varphi(x)|}{\varphi(x)^{1/\beta}}, \quad J_1 = \sup_{|x| \geq x_1 \sqrt{n}} \frac{p_n(x)}{\varphi(x)^{1/\beta}}, \quad J_2 = \sup_{M_n \leq |x| \leq x_1 \sqrt{n}} \frac{p_n(x)}{\varphi(x)^{1/\beta}}
\]

with some \( x_1 > 0 \) and assuming that \( M_n < x_1 \sqrt{n} \). An application of the expansion (16.5) implies that \( J_0 = O(1/\sqrt{n}) \) in general and \( I_0 = O(1/n) \) when the distribution of \( X \) is symmetric.

The polynomial smallness of \( J_1 \) and \( J_2 \) (for sufficiently large values of \( l \) in the definition of \( M_n \)) follows from Lemma 16.2, by repeating the arguments of the proof of Theorem 15.1. □

### 17. Some Examples and Counter-Examples

Given a random variable \( X \) such that \( \mathbb{E}X = 0, \mathbb{E}X^2 = 1 \), consider the function \( \psi(t) = e^{-t^2} \mathbb{E}e^{tX} \ (t \in \mathbb{R}) \). As before, put

\[
Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}},
\]

where \( X_j \)'s are independent copies of \( X \). One immediate consequence of Theorem 1.1 (with \( n_0 = 1 \)) is the following characterization.
Theorem 17.1. Assume that the random variable $X$ has a density $p$ such that
\[ \int_{-\infty}^{\infty} p(x)^2 e^{x^2/2} \, dx < \infty. \]  
(17.1)
Then $\chi^2(Z_n, Z) \to 0$ as $n \to \infty$ for $Z \sim N(0, 1)$, if and only if
\[ \psi(t) < 1 \quad \text{for all } t \neq 0. \]  
(17.2)

The assumption (17.1) is fulfilled, for example, when $X$ is bounded and has a square integrable density. We now illustrate Theorem 17.1 and the more general Theorem 1.2 with a few examples (mostly in dimension one).

Uniform distribution. Let $X$ be uniformly distributed on the segment $[-\sqrt{3}, \sqrt{3}]$. The characteristic function of $X$ is given by $f(t) = \sin(t\sqrt{3})/(t\sqrt{3})$, and for imaginary values $t = iy$, we have the simple estimate
\[ f(iy) = \sinh(y\sqrt{3})/y\sqrt{3} < e^{y^2/2}, \quad y \in \mathbb{R} \ (y \neq 0), \]  
so that (17.2) does hold. In this case the first moments are given by $\alpha_2 = 1$, $\alpha_3 = 0$, $\alpha_4 = \frac{5}{6}$. Therefore, by Theorem 17.1, $\chi^2(Z_n, Z) \to 0$ as $n \to \infty$. Moreover, Theorem 1.1 provides an asymptotic expansion (1.3) which becomes
\[ \chi^2(Z_n, Z) = \frac{3}{50n^2} + O\left(\frac{1}{n^3}\right). \]

In fact, the property (17.3) means that the condition (1.4) of a more general Theorem 1.2 is fulfilled in the whole range of indexes $\alpha > 1$. Using the formula (14.5), we therefore obtain a stronger assertion $T_\alpha(Z_n||Z) = \frac{\alpha}{2} \chi^2(Z_n, Z) + O\left(\frac{1}{n^3}\right)$, and a similar one for $D_\alpha$.

Convex mixtures of centered Gaussian measures. Consider the densities of the form
\[ p(x) = \int_{0}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \, d\pi(\sigma^2), \quad x \in \mathbb{R}, \]
where $\pi$ is a (mixing) probability measure on the positive half-axis with $\int_{0}^{\infty} \sigma^2 d\pi(\sigma^2) = 1$. The random variable $X$ with this density has mean zero and variance one, and its distribution is equal to that of $\sqrt{\xi} Z$, where $\xi$ is independent of $Z \sim N(0, 1)$ and is distributed according to $\pi$. As in Example 9.3, $\chi^2(Z_n, Z) < \infty$ for some $n = n_0$, if and only if $\pi$ is supported on the interval $(0, 2)$, and its distribution function $F(\varepsilon) = \pi((0, \varepsilon])$ satisfies
\[ \inf_n \int_{0}^{1} \frac{F(\varepsilon)2n}{\varepsilon^{3/2}} \, d\varepsilon < \infty, \quad \inf_n \int_{1}^{2} \frac{(1 - F(\varepsilon))2n}{(2 - \varepsilon)^{3/2}} \, d\varepsilon < \infty. \]  
(17.4)

On the other hand, the distribution of $X$ has the Laplace transform
\[ \mathbb{E} e^{tX} = \int_{0}^{\infty} e^{\sigma^2 t^2/2} \, d\pi(\sigma^2) = \mathbb{E} e^{\xi t^2/2}, \quad t \in \mathbb{R}. \]
Hence, the condition $\chi^2(Z_n, Z) < \infty$ guarantees that (17.2) is fulfilled. Without that condition, $\mathbb{E} e^{tX} < e^{t^2}$ for all $t \neq 0$, if and only if $\mathbb{P}\{\xi \leq 2\} = 1$ and $\mathbb{P}\{\xi = 2\} < 1$. Here, $\mathbb{P}\{\xi = 2\} = 1$ is not possible in view of the second moment assumption $\mathbb{E}X^2 = \mathbb{E} \xi = 1$.

Hence, one concludes that $\chi^2(Z_n, Z) \to 0$ as $n \to \infty$, if and only if the measure $\pi$ is supported on the interval $(0, 2)$ and satisfies the condition (17.4). In this case, we obtain the

Rényi divergence from the normal law and the CLT
expansion (1.3) which reads
\[
\chi^2(Z_n, Z) = \frac{3(m - 1)^2}{8n^2} + O\left(\frac{1}{n^4}\right), \quad m = \int_0^\infty \sigma^4 \, d\pi(\sigma^2).
\]

**Distributions with Gaussian component.** Consider random variables of the form
\[
X = a\xi + bZ \quad (a^2 + b^2 = 1, \ a, b > 0)
\]
assuming that \(\mathbb{E}\xi = 0, \mathbb{E}\xi^2 = 1\), and where \(Z \sim N(0, 1)\) is independent of \(\xi\). The distribution of \(X\) is a convex mixture of shifted Gaussian measures on the line with variance \(b^2\). It admits a density
\[
p(x) = \frac{1}{b} \mathbb{E} \varphi\left(\frac{x - a\xi}{b}\right), \quad x \in \mathbb{R}.
\]
To ensure finiteness of \(\chi^2(X, Z)\) (and even finiteness of \(\chi^2(Z_n, Z)\) with some \(n\)), the random variable \(\xi\) should have a finite Gaussian moment, or equivalently, the Laplace transform of the distribution of \(\xi\) should admit a subgaussian bound
\[
\mathbb{E} e^{t\xi} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R},
\]
with some finite \(\sigma > 0\). Let \(\sigma\) be an optimal value in this inequality (necessarily \(\sigma \geq 1\)). It then follows that \(\mathbb{E} e^{t\xi} < \infty\) whenever \(c < 1/(2\sigma^2)\).

Squaring the formula for \(p(x)\), we easily find an expression for the \(\chi^2\)-distance, namely,
\[
1 + \chi^2(X, Z) = \frac{1}{\sqrt{1 - a^4}} \mathbb{E} \exp\left\{\frac{a^2}{2(1 - a^2)} \left(\frac{2}{1 + a^2} (\xi + \eta)^2 - (\xi^2 + \eta^2)\right)\right\},
\]
where \(\eta\) is an independent copy of \(\xi\). Using \((\xi + \eta)^2 \leq 2\xi^2 + 2\eta^2\), we are lead to a simple upper bound
\[
1 + \chi^2(X, Z) \leq \frac{1}{\sqrt{1 - a^4}} \left(\mathbb{E} e^{2(1 + \sigma^2)\xi^2}\right)^2.
\]
Hence, \(\chi^2(X, Z) < \infty\) whenever \(a < a_\sigma = \frac{1}{\sqrt{\sigma^2 - 1}}\), which is automatically fulfilled in case \(\sigma^2 \leq 2\). Moreover, for all \(t \neq 0\),
\[
\mathbb{E} e^{tX} = \mathbb{E} e^{a\xi} e^{b^2 t^2/2} \leq e^{(\sigma^2 a^2 + b^2) t^2/2} = e^{((\sigma^2 - 1)a^2 - 1) t^2/2} < e^{t^2}
\]
under the same constraint \(a < a_\sigma\). Thus we conclude, by applying Theorem 17.1, that \(\chi^2(Z_n, Z) \to 0\) as \(n \to \infty\), if \(a < \frac{1}{\sqrt{\sigma^2 - 1}}\). In case \(\sigma^2 \leq 2\), this convergence holds for all admissible parameters \((a, b)\).

**Distributions with finite Gaussian moment.** Suppose that a random variable \(X\) with mean zero and variance one has finite Gaussian moment \(M = \mathbb{E} e^{cX^2} \ (c > 0)\). It is well-known that the property (17.5) is fulfilled for some \(\sigma \geq 1\); moreover, one can show that an optimal value satisfies \(\sigma^2 \leq \frac{4\log M}{c\log 2}\). This means that condition (1.4) is fulfilled for any \(\alpha > 1\) such that \(\beta < \sigma^2\). We conclude that, if \(D_\alpha(X\|Z) < \infty\), then \(D_\alpha(Z_n\|Z) \to 0\) with any \(\alpha < \frac{\sigma^2}{\sigma^2 - 1}\).

**Conditions in terms of exponential series.** Consider a symmetric density of the form
\[
p(x) = \varphi(x) \sum_{k=0}^\infty \frac{\sigma_k}{2^kk!} H_{2k}(x), \quad x \in \mathbb{R},
\]
where
with $\sigma_0 = 1$ and $\sigma_1 = 0$ (which means that $\mathbb{E}X^2 = 1$ for the random variable with density $p$). In view of Section 6, condition (17.1) is fulfilled, if and only if the series

$$\chi^2(X, Z) = \sum_{k=2}^{\infty} \frac{(2k)!}{4^k k!^2} \sigma_k^2 \sim \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}} \sigma_k^2$$

is convergent (which is fulfilled automatically, when $p$ is compactly supported and bounded). Assuming additionally that $\sup_{k \geq 2} \sigma_k \leq 1$, we also have

$$\mathbb{E} e^{tX} = e^{t^2/2} \left[ 1 + \sum_{k=2}^{\infty} \frac{\sigma_k}{k!} \left( \frac{t^2}{2} \right)^k \right] \leq e^{t^2/2} \left( e^{t^2/2} - \frac{t^2}{2} \right) < e^{t^2}, \quad t \neq 0.$$

Hence, in this case, by Theorem 17.1, $\chi^2(Z_{n}, Z) \to 0$ as $n \to \infty$. Moreover, according to the expansion (1.3), we have $\chi^2(Z_{n}, Z) = O(1/n^2)$. This assertion strengthens the result of [F] (under weaker assumptions).

**Log-concave probability distributions.** More examples including those in higher dimensions illustrate the multidimensional Theorem 1.2 within the class of densities $p(x) = e^{-V(x)}$ supported on some open convex region $\Omega \subset \mathbb{R}^d$, where $V$ is a $C^2$-convex function with Hessian satisfying $V''(x) \geq c I_d$ in the sense of positive definite matrices ($c > 0$). The probability measures with such densities are known to admit logarithmic Sobolev inequalities (via the Bakry-Emery criterion). In particular, they satisfy transport-entropy inequalities which in turn can be used to get a subgaussian bound on the Laplace transform such as

$$\mathbb{E} e^{ru(X)} \leq e^{r^2/(2c)}, \quad r \in \mathbb{R}.$$ 

Here, $u$ may be an arbitrary function on $\mathbb{R}^d$ with Lipschitz semi-norm $\|u\|_{\text{Lip}} \leq 1$, such that $\mathbb{E} u(X) = 0$ (cf. [B-G], [O-V]). In particular, if $\mathbb{E}X = 0$, one may choose an arbitrary linear function $u(x) = \langle x, \theta \rangle$ with $|\theta| = 1$. Hence, the condition (1.4) will be fulfilled, as long as $c > \frac{1}{\beta}$. Moreover, the property $D_{\alpha}(X||Z) < \infty$ will also hold in this case, since necessarily

$$V(x) \geq V(x_0) + \langle V'(x_0), x - x_0 \rangle + \frac{c}{2} |x - x_0|^2$$

for all $x, x_0 \in \Omega$. Applying Theorem 1.2, we get:

**Corollary 17.2.** If a random vector $X$ in $\mathbb{R}^d$ with mean zero and identity covariance matrix has density $p = e^{-V}$ such that $V'' \geq c I_d$ ($0 < c \leq 1$) on the supporting open convex region, then $D_{\alpha}(Z_n||Z) \to 0$ as $n \to \infty$, whenever $\alpha < \frac{1}{1-c}$.

18. Convolution of Bernoulli with Gaussian

One might wonder whether or not it is possible to replace the condition (1.1) in Theorem 1.1 with a slightly weaker requirement like $\mathbb{E} e^{tX} \leq e^{t^2}$ (hoping e.g. that the strict inequality would automatically hold, in view of the assumption $\mathbb{E}X^2 = 1$). The answer is negative, as the following statement shows:

**Proposition 18.1.** There exists a random variable $X$ with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\chi^2(X, Z) < \infty$ for $Z \sim N(0, 1)$, and such that the inequality

$$\mathbb{E} e^{tX} < e^{t^2} \quad (18.1)$$
is fulfilled for all $t \neq 0$ except for exactly one point $t_0 \neq 0$.

Since (18.1) is violated (although at one point only), Theorem 1.1 implies that convergence $\chi^2(Z_n, Z) \to 0$ does not hold any more.

Let us describe explicitly one family of distributions satisfying the assertion of this proposition. Returning to one of the previous examples, consider random variables of the form

$$X_p = a\xi + bZ \quad (a, b > 0),$$

assuming that $\xi$ takes two values $q$ and $-p$ with probabilities $p$ and $q$, respectively ($p, q > 0, p + q = 1$), and where $Z \sim N(0, 1)$ is independent of $\xi$. Clearly, $EX_p = 0$, and we have the constraint

$$EX_p^2 = pq a^2 + b^2 = 1.$$  \hspace{1cm} (18.2)

The density $w$ of $X_p$ represents a convex mixture of two shifted Gaussian densities,

$$w(x) = p \varphi \left( \frac{x - aq}{b} \right) + q \varphi \left( \frac{x + ap}{b} \right),$$

and the condition $\chi^2(X, Z) < \infty$ obviously holds (since necessarily $b < 1$).

Now, let $\sigma^2 = \sigma^2(p, q)$ denote the smallest positive constant such that the following inequality holds

$$\mathbb{E} e^{t\xi} = pe^{qt} + qe^{-qt} \leq e^{\sigma^2 t^2/2}, \quad t \in \mathbb{R}. \quad \hspace{1cm} (18.3)$$

This is the so-called subgaussian constant for the Bernoulli distribution. Since $\mathbb{E} e^{tX_p} = \mathbb{E} e^{t\xi} e^{b^2 t^2/2}$, (18.3) yields

$$\mathbb{E} e^{tX_p} \leq e^{(\sigma^2 a^2 + b^2) t^2/2}, \quad t \in \mathbb{R},$$

with an optimal constant $\sigma^2 a^2 + b^2$ in the exponent on the right-hand side. Thus, according to the requirement (18.1), we get another constraint $\sigma^2 a^2 + b^2 = 2$. Combining it with (18.2), we find that necessarily

$$a^2 = \frac{1}{\sigma^2 - pq}, \quad b^2 = \frac{\sigma^2 - 2pq}{\sigma^2 - pq},$$

which makes sense provided that $\sigma^2 > 2pq$. The subgaussian constant for the Bernoulli distribution is known to be (cf. [B-H-T], Proposition 2.3)

$$\sigma^2 = \frac{p - q}{2 (\log p - \log q)}.$$  \hspace{1cm} (18.4)

It is easy to see that (18.3) becomes equality for $t_0 = -2 (\log p - \log q)$, which is a unique non-zero point with such property, as long as $p \neq q$.

Hence we conclude that the random variable $X = X_p$ satisfies the assertion of Proposition 18.1, if and only if

$$\frac{p - q}{2 (\log p - \log q)} > 2pq.$$  \hspace{1cm} (18.4)

This inequality holds, provided that $p$ is sufficiently close to 0 or 1 (although it is not true for a neighborhood of $1/2$). For example, one may choose $p = 1/6$. More precisely, for some constant $p_0 \in (0, \frac{1}{2})$, (18.4) holds for $p$ from the set $(0, p_0) \cup (1 - p_0, 1)$, while for $p$ from $(p_0, 1 - p_0)$ it holds with an opposite inequality sign.
REFERENCES

[A1] Amosova, N. N. Narrow zones of local normal attraction. (Russian) Teor. Veroyatnost. i Primen. 35 (1990), no. 1, 138–143. Translation in: Theory Probab. Appl. 35 (1990), no. 1, 140–145 (1991).

[A2] Amosova, N. N. A remark on a local limit theorem for large deviations. (Russian) Teor. Veroyatnost. i Primen. 35 (1990), no. 4, 754–756. Translation in: Theory Probab. Appl. 35 (1990), no. 4, 758–760 (1991).

[A-B-B-N] Artstein, S.; Ball, K. M.; Barthe, F.; Naor, A. On the rate of convergence in the entropic central limit theorem. Probab. Theory Related Fields 129 (2004), no. 3, 381–390.

[B-C] Bally, V.; Caramellino, L. Asymptotic development in the CLT in total variation distance. Bernoulli 22 (2016), 2442–2485.

[B] Barron, A. R. Entropy and the central limit theorem. Ann. Probab. 14 (1986), no. 1, 336–342.

[B-J] Barron, A. R.; Johnson, O. Fisher information inequalities and the central limit theorem. Probab. Theory Related Fields 129 (2004), no. 3, 391–409.

[B-RR] Bhattacharya, R. N.; Ranga Rao, R. Normal approximation and asymptotic expansions. John Wiley & Sons, Inc. 1976. Also: Soc. for Industrial and Appl. Math., Philadelphia, 2010.

[B-C-G1] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Non-uniform bounds in local limit theorem in case of fractional moments. I. Math. Methods of Statistics, 20 (2011), no. 3, 171–191; II. Math. Methods of Statistics, 20 (2011), no. 4, 269–287.

[B-C-G2] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Rate of convergence and Edgeworth-type expansion in the entropic central limit theorem. Ann. Probab. 41 (2013), no. 4, 2479–2512.

[B-C-G3] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Berry-Esseen bounds in the entropic central limit theorem. Probab. Theory Related Fields 2014 (159), 435–478.

[B-C-G4] Bobkov, S. G.; Chistyakov, G. P.; Götze, F. Fisher information and the central limit theorem. Probab. Theory Related Fields 159 (2014), issue 1-2, 1–59.

[B-C-K] Bobkov, S. G.; Chistyakov, G. P.; Kösters, H. The entropic Erdös-Kac limit theorem. J. Theor. Probab. 28 (2015), no. 4, 1520–1555.

[B-G] Bobkov, S. G.; Götze, F. Exponential integrability and transportation cost related to logarithmic Sobolev inequalities. J. Funct. Anal. 163 (1999), no. 1, 1–28.

[B-H-T] Bobkov, S. G.; Houdr´e, C.; Tetali, P. The subgaussian constant and concentration inequalities. Israel J. Math. 156 (2006), 255–283.

[B-P-T] Borland, L.; Plastino, A. R.; Tsallis, C. Information gain within nonextensive thermostatistics. J. Math. Phys. 39 (1998), no. 12, 6490–6501.

[C] Carlen, E. A. Superadditivity of Fisher’s information and logarithmic Sobolev inequalities. J. Funct. Anal. 101 (1991), no. 1, 194–211.

[Cr] Cramér, H. On some classes of series used in mathematical statistics. Proc. 6th Scand. Math. Congr. Copenhagen, 1925, 399–425. See also: Harald Cramér. Collected works. Volume I, Ed. A. Martin-Löf, Springer-Verlag, 438–464.

[Cs] Csiszár, I. Information-type measures of difference of probability distributions and indirect observations. Studia Sci. Math. Hungar. 2 (1967), 299–318.

[D-C-T] Dembo, A., Cover, T. M., Thomas, J. A. Information-theoretic inequalities. IEEE Trans. Inform. Theory, 37 (1991), no. 6, 1501–1518.
[E-H] van Erven, T., Harremoës, P. Rényi divergence and Kullback-Leibler divergence. IEEE Trans. Inform. Theory 60 (2014), no. 7, 3797–3820.

[F] Fomin, S. V. The central limit theorem: convergence in the norm $\|u\| = (\int_{-\infty}^{\infty} u^2(x) e^{-x^2/2} dx)^{1/2}$. (Russian) Problems of the theory of probability distributions, VII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 119 (1982), 218–229.

[G-S] Gibbs, A. L.; Su, F. E. On choosing and bounding probability metrics. Intern. Statist. Review 70 (2002), issue 3, 419–435.

[G] Gilardoni, G. L. On Pinsker’s and Vajda’s type inequalities for Csiszar’s $f$-divergences. IEEE Trans. Inform. Theory 56 (2010), no. 11, 5377–5386.

[J] Johnson, O. Information theory and the central limit theorem. Imperial College Press, London, 2004, xiv+209 pp.

[H-W] Hirschman, I. I.; Widder, D. V. The convolution transform. Princeton University Press, Princeton, N. J., 1955. x+268 pp.

[I-L] Ibragimov, I. A., Linnik, Yu. V. Independent and stationary sequences of random variables. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov. Translation from the Russian edited by J. F. C. Kingman. Wolters-Noordhoff Publishing, Groningen, 1971, 443 pp.

[K] Kullback, S. A lower bound for discrimination in terms of variation. IEEE Trans. Inform. Theory, T-13, 1967, 126–127.

[K-L] Kullback, S., Leibler, R. A. On information and sufficiency. Ann. Math. Statistics 22 (1951), 79–86.

[Le] LeCam, L. M. Asymptotic Methods in Statistical Decision Theory. Springer Series in Statistics. Springer-Verlag, New York, 1986. xxvi+742 pp.

[L] Lieb, E. H. Some convexity and subadditivity properties of entropy. Bull. Amer. Math. Soc. 81 (1975), 1–13.

[L-V] Liese, F.; Vajda, I. Convex statistical distances. With German, French and Russian summaries. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 95. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1987. 224 pp.

[Li] Linnik, Ju. V. An information-theoretic proof of the central limit theorem with the Lindeberg condition. Theory Probab. Appl. 4 (1959), 288–299.

[N] Nielsen, F. On the Chi square and higher-order Chi distances for approximating $f$-divergences. arXiv: 1309.3029v2 [cs.IT] 18 Sep 2013.

[O-V] Otto, F.; Villani, C. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. J. Funct. Anal. 173 (2000), no. 2, 361–400.

[Pe1] Petrov, V. V. Local limit theorems for sums of independent random variables. (Russian) Teor. Veroyatnost. i Primenen. 9 (1964), 343–352.

[Pe2] Petrov, V. V. Sums of Independent Random Variables. Berlin Heidelberg New York (1975).

[Pi] Pinsker, M. S. Information and information stability of random variables and processes. Translated and edited by Amiel Feinstein Holden-Day, Inc., San Francisco, Calif.-London-Amsterdam, 1964, xii+243 pp.

[Pr] Prohorov, Yu. V. A local theorem for densities. (Russian) Doklady Akad. Nauk SSSR (N.S.) 83, (1952), 797–800.

[R] Rényi, A. On measures of entropy and information. Proc. 4th Berkeley Symp. Math. Statist. and Prob. Vol. I, pp. 547–561, 1961, Univ. California Press, Berkeley, Calif.
[S] Shiryaev, A. N. Probability. Springer-Verlag, 1996.

[S-M] Sirazhdinov, S. H.; Mamatov, M. On mean convergence for densities. Theory Probab. Appl., 7 (1962), No. 4, 424–428.

[Sz] Szegő, G. Orthogonal polynomials. 3rd ed., American Math. Soc. Publications 23. Amer. Math. Soc., Providence, RI.

[T] Tsallis, C. Generalized entropy-biased criterion for consistent testing. Phys. Rev. E 58 (1998), 1442–1445.

[V] Vajda, I. Theory of Statistical Inference and Information. Kluwer Academic Publishers, Dordrecht-Boston-London, 1989, 432 pp.

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