Asymptotic dimension and Novikov-Shubin invariants for open manifolds

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Abstract

A trace on the C*-algebra $A$ of quasi-local operators on an open manifold is described, based on the results in [36]. It allows a description à la Novikov-Shubin [31] of the low frequency behavior of the Laplace-Beltrami operator. The 0-th Novikov-Shubin invariant defined in terms of such a trace is proved to coincide with a metric invariant, which we call asymptotic dimension, thus giving a large scale “Weyl asymptotics” relation. Moreover, in analogy with the Connes-Wodzicki result [7, 8, 45], the asymptotic dimension $d$ measures the singular traceability (at 0) of the Laplace-Beltrami operator, namely we may construct a (type II$_1$) singular trace which is finite on the $*$-bimodule over $A$ generated by $\Delta^{-d/2}$. 
0 Introduction.

The inspiration of this paper came from the idea of Connes’ of defining the dimension of a noncommutative compact manifold in terms of the Weyl asymptotics, namely as the inverse of the order of growth of the eigenvalues of differential operators of order one (the Dirac operator for example). Moreover Connes observed that a noncommutative measure (trace) may be attached to such noncommutative dimension via the Dixmier trace, setting \( \tau(a) = \text{tr}_\omega(a|D|^{-d}) \), where \( a \) is a “function” on the noncommutative manifold, \( D \) is the Dirac operator, \( d \) is the noncommutative dimension and \( \text{tr}_\omega \) is the (logarithmic) Dixmier trace. According to the identification of the Dixmier trace with the Wodzicki residue, such trace gives back the ordinary integration in the case of commutative Riemannian manifolds.

In this paper we present a large scale analogue of these results for the case of commutative noncompact manifolds.

We exhibit a large scale Weyl asymptotics, i.e. a correspondence between some asymptotic dimension and the low frequency behavior of the Laplace-Beltrami operator. Then we show that such asymptotic dimension carries a (noncommutative) integration in terms of a type II\(_1\) singular trace, which is the low frequency analogue of the Dixmier trace.

The definition of asymptotic dimension is given in the context of metric dimension theory, as a suitable large scale analogue of the metric dimension of Kolmogorov and Tihomirov. Concerning the low frequency behavior, in case of a noncompact manifold arising as universal covering of a compact one, there is a set of numbers, the so-called Novikov-Shubin invariants \( \alpha_p \), which are a measure of the low frequency behavior of the Laplacian (on \( p \)-forms) on the covering.

Since Atiyah, in his seminal paper on the \( \Gamma \)-trace, which replaced the ordinary trace in the statement of the index theorem for covering manifolds \( \Gamma \to M \to X \) and brought to a definition of the Betti numbers for coverings as \( \tau(\chi(0)(\Delta)) \), Novikov and Shubin conjectured in that the behavior of \( \tau(\chi([0,\lambda](\Delta))) \) when \( \lambda \to 0 \) should contain interesting topological information.

Indeed the efforts of Novikov-Shubin, Lott and Gromov-Shubin proved that Novikov-Shubin numbers are indeed invariant under homotopies of the base manifold. The relations between Novikov-Shubin invariants and the singular traceability of some \( \Gamma \)-invariant pseudodifferential operators is the object of a separate paper.

In the case of open manifolds, J. Roe proved an index theorem in which he had to replace the \( \Gamma \)-trace of Atiyah (which, at least for amenable coverings, may be seen as an average on the discrete group \( \Gamma \)) with an “average on the exhaustion” trace.

We show that for open manifolds with bounded geometry and regular polynomial growth the replacement of the Atiyah trace with the (suitably regularized) Roe trace allows us to define the 0-th Novikov-Shubin invariant for open manifolds.
Then, the large scale Weyl asymptotics takes the form of the coincidence of the asymptotic dimension with the 0-th Novikov-Shubin invariant, which we prove assuming the isoperimetric inequality of Grigor’yan [20]. Such a relation shows in particular that the 0-th Novikov-Shubin invariant, being a metric object, is independent of all the limiting procedures involved in its definition.

The construction of the asymptotic (noncommutative) measure instead, depends on the singular traceability (at 0) of $\Delta^{-\alpha/2}$, when the 0-th Novikov-Shubin invariant is a finite $\alpha \neq 0$.

For this we need more general singular traces, as singular traces on type I factors, as studied by Dixmier [10], Varga [12] and Albeverio-Guido-Ponosov-Scarlatti [1], only apply to compact operators, like the negative powers of $\Delta$ on a compact manifold, but, in the case of non-type-I algebras, it has been shown in [22] the existence of singular traces which are finite on suitable unbounded $\tau$-compact operators, like negative powers of the Laplacian on a noncompact manifold.

The theory of singular traces on C*-algebras developed in [23] may then be used to construct a type $II_1$ singular trace on the unbounded operators affiliated to a natural C*-algebra of operators on the manifold.

This paper is organized as follows.

In section 1, a natural extension of the notion of Kolmogorov-Tihomirov dimension [26] to nonnecessarily totally bounded metric spaces is used, as a kind of analogy, to introduce an asymptotic dimension for metric spaces. It is proved that this dimension is invariant under rough isometries.

In section 2, after some preliminaries on open manifolds of bounded geometry and using recent estimates for the heat kernel by Coulhon-Grigor’yan [11], we provide a relation between the asymptotic dimension and the long time behavior of the heat kernel of the manifold (Corollary 2.7), and establish a connection with N. Th. Varopoulos’ notion of asymptotic dimension for semigroups of operators [44], as applied to the heat semigroup (Corollary 2.14); finally we compare our definition with an analogous one given recently by E. B. Davies [15] for cylindrical ends.

In section 3 we introduce the C*-algebra of almost local operators on a manifold of bounded geometry, as the norm closure of the finite propagation speed operators, and show that $C_0$ functional calculi of the Laplace-Beltrami operator are almost local (Corollary 3.6); then, regularizing a previous construction by J. Roe [36], we exhibit a weight on $B(L^2(M))$, when $M$ is a manifold of bounded geometry and regular polynomial growth (Proposition 3.10), which becomes a semifinite, lower semicontinuous trace on the C*-algebra of almost local operators after two successive procedures of regularization have been performed (note that these two procedures are described abstractly) still retaining, though, the same value of the original weight on the heat semigroup (Corollary 3.22).

In the last section, after a brief exposition of the theory of singular traces on C*-algebras, which is the subject of a separate publication, we define the (0-th) Novikov-Shubin invariant $\alpha_0(M)$ for an open manifold $M$ of bounded geometry and regular polynomial growth (Definition 4.13) and show (Corollary 4.15) an asymptotic analogue of Wodzicki-Connes result, namely that $\Delta^{-\alpha_0(M)/2}$ is sin-
gularly traceable at 0, which is a statement on the asymptotic behavior of the “small eigenvalues” of the Laplacian. Here we observe that while in Wodzicki-Connes result only logarithmic divergences appear, because manifolds are locally regular, in our context different divergences appear, and we recover the logarithmic one in case of “asymptotic regularity”, for example if a discrete group acts on the manifold, [24]. Finally, under more restrictive hypotheses, we show that the Novikov-Shubin invariant coincides with the asymptotic dimension of the manifold (Theorem 4.18). This may be seen as a generalization of a result by Varopoulos that $\alpha_0(M) = \text{growth}(\Gamma)$, because of the rough-isometry invariance of the asymptotic dimension, and Proposition 2.4.

1 Asymptotic dimension.

The main purpose of this section is the introduction of an asymptotic dimension for metric spaces. To our knowledge, the notion of asymptotic dimension in the general setting of metric dimension theory has not been studied, even though Davies [15] proposed a definition in the case of cylindrical ends of a Riemannian manifold.

We shall give a definition of asymptotic dimension for a general metric space, based on the (local) Kolmogorov dimension [26] and state its main properties. We compare our definition with Davies’ and also with the notion of dimension at infinity for semigroups [14] in Section 2.

1.1 Kolmogorov-Tihomirov metric dimension

In this subsection we recall a definition of metric dimension due to Kolmogorov and Tihomirov [26]. Quoting from their paper, a dimension “corresponds to the possibility of characterizing the “massiveness” of sets in metric spaces by the help of the order of growth of the number of elements of their most economical $\varepsilon$-coverings, as $\varepsilon \to 0$’. Set functions retaining the general properties of a dimension (cf. Theorem 1.3) have been studied by several authors. Our choice of the Kolmogorov dimension is due to the fact that it is suitable for the kind of generalization we need in this paper, namely it quite naturally produces a definition of asymptotic dimension.

In the following, unless otherwise specified, $(X, \delta)$ will denote a metric space, $B_X(x, R)$ the open ball in $X$ with centre $x$ and radius $R$, $n_r(\Omega)$ the least number of open balls of radius $r$ which cover $\Omega \subset X$, and $\nu_r(\Omega)$ the largest number of disjoint open balls of radius $r$ centered in $\Omega$.

The following lemma is proved in [26]. Due to some notational difference, we include a proof.

**Lemma 1.1.** $n_r(X) \geq \nu_r(X) \geq n_{2r}(X)$.

**Proof.** We have only to prove the second inequality when $\nu_r$ is finite. Let us assume that $\{B(x_i, r)\}_{i=1}^{\nu_r(X)}$ are disjoint balls centered in $X$ and observe that, for any $y \in X$, $\delta(y, \bigcup_{i=1}^{\nu_r(X)} B(x_i, r)) < r$, otherwise $B(y, r)$ would be disjoint
from \( \bigcup_{i=1}^{\nu_r(X)} B(x_i, r) \), contradicting the maximality of \( \nu_r \). So for all \( y \in X \) there is \( j \) s.t. \( \delta(y, B(x_j, r)) < r \), that is \( X \subset \bigcup_{i=1}^{\nu_r(X)} B(x_i, 2r) \), which implies the thesis.

Kolmogorov and Tihomirov [26] defined a dimension for totally bounded metric spaces \( E \) as

\[
    d_0(E) := \limsup_{r \to 0} \frac{\log n_r(E)}{\log(1/r)}.
\]  

(1.1)

Then we may give the following definition.

**Definition 1.2.** Let \((X, \delta)\) be a metric space. Then, denoting by \( B(X) \) the family of bounded subsets of \( X \), the metric Kolmogorov dimension of \( X \) is

\[
    d_0(X) := \sup_{B \in B(X)} \limsup_{r \to 0} \frac{\log n_r(B)}{\log(1/r)}.
\]

Then the following proposition trivially holds.

**Proposition 1.3.** If \( \{B_n\} \) is an exhaustion of \( X \) by bounded subsets, namely \( B_n \) is increasing and for any bounded \( B \) there exists \( n \) such that \( B \subseteq B_n \), one has \( d_0(X) = \lim_n d_0(B_n) \). In particular,

\[
    d_0(X) = \lim_{R \to \infty} \limsup_{r \to 0} \frac{\log n_r(B_X(x, R))}{\log(1/r)} \tag{1.2}
\]

**Remark 1.4.** If bounded subsets of \( X \) are not totally bounded, we could define \( d_0(X) \) as the supremum over totally bounded subsets. These two definitions, which agree e.g. on proper spaces, may be different in general. For example an orthonormal basis in an infinite dimensional Hilbert space has infinite dimension according to Definition 1.2, but has zero dimension in the other case. A definition of metric dimension which coincides with \( d_0 \) on bounded subsets of \( \mathbb{R}^p \) has been given by Tricot [41] in terms of rarefaction indices.

Let us now show that this set function satisfies the basic properties of a dimension [33, 41].

**Theorem 1.5.** The set function \( d_0 \) is a dimension, namely it satisfies

(i) If \( X \subset Y \) then \( d_0(X) \leq d_0(Y) \).

(ii) If \( X_1, X_2 \subset X \) then \( d_0(X_1 \cup X_2) = \max\{d_0(X_1), d_0(X_2)\} \).

(iii) If \( X \) and \( Y \) are metric spaces, then \( d_0(X \times Y) \leq d_0(X) + d_0(Y) \).

**Proof.** Property (i) easily follows from formula (1.2).

Now we prove (ii). The inequality \( d_0(X_1 \cup X_2) \geq \max\{d_0(X_1), d_0(X_2)\} \) follows from (i). For the converse inequality, let \( x_i \in X_i \), and set \( \delta := \delta(x_1, x_2) \),
$d_1 = d_0(X_1), d_2 = d_0(X_2)$, with e.g. $d_1 \geq d_2$. If $d_1 = \infty$ the property is trivial, so we may suppose $d_1 \in \mathbb{R}$. Then

$$B_{X_1 \cup X_2}(x_1, R) \subset B_{X_1}(x_1, R) \cup B_{X_2}(x_2, R + \delta)$$

therefore

$$n_r(B_{X_1 \cup X_2}(x_1, R)) \leq n_r(B_{X_1}(x_1, R)) + n_r(B_{X_2}(x_2, R + \delta)).$$

(1.3)

Now, $\forall R > 0$,

$$\limsup_{r \to 0} \frac{\log n_r(B_{X_1}(x_1, R))}{\log(1/r)} \leq d_1$$

i.e. $\forall R, \varepsilon > 0$ there is $r_0 = r_0(\varepsilon, R)$ s.t. for all $0 < r < r_0$, $n_r(B_{X_1}(x_1, R)) \leq r^{-(d_1 + \varepsilon)}$, and $n_r(B_{X_2}(x_2, R + \delta)) \leq r^{-(d_2 + \varepsilon)}$ hence, by (1.3),

$$n_r(B_{X_1 \cup X_2}(x, R)) \leq r^{-(d_1 + \varepsilon)}(1 + \varepsilon^{d_1 - d_2}).$$

Finally,

$$\lim_{R \to \infty} \limsup_{r \to 0} \frac{\log n_r(B_{X_1 \cup X_2}(x, R))}{\log(1/r)} \leq d_1 + \varepsilon,$$

that is

$$d_0(X_1 \cup X_2) \leq \max\{d_0(X_1), d_0(X_2)\} + \varepsilon$$

and the thesis follows by the arbitrariness of $\varepsilon$.

The proof of part (iii) is postponed.

Kolmogorov dimension is indeed quasi-isometry invariant, as next proposition shows.

**Proposition 1.6.** Let $X,Y$ be metric spaces, and $f : X \to Y$ a surjective quasi-isometry, namely $f$ satisfies

$$c_1 \delta_X(x_1, x_2) \leq \delta_Y(f(x_1), f(x_2)) \leq c_2 \delta_X(x_1, x_2).$$

Then $d_0(X) = d_0(Y)$.

**Proof.** By hypothesis we have $f(B_X(x, \rho/c_2)) \subset B_Y(f(x), \rho) \subset f(B_X(x, \rho/c_1))$.

So that, with $y_j = f(x_j)$, $n := n_r(B_Y(f(x), R))$,

$$f(B_X(x, R/c_2)) \subset B_Y(f(x), R) \subset \bigcup_{j=1}^{n} B_Y(y_j, r)$$

$$\subset \bigcup_{j=1}^{n} f(B_X(x_j, r/c_1)) = f\left(\bigcup_{j=1}^{n} B_X(x_j, r/c_1)\right)$$

which implies $n_{r/c_1}(B_X(x, R/c_2)) \leq n_r(B_Y(f(x), R))$.

Since quasi-isometries are injective, we may repeat the same argument for $f^{-1}$,
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and we get $n_{c^2 \cdot r}(B_Y(f(x), c_1 R)) \leq n_r(B_X(x, R))$, so that $n_{r/c_1}(B_X(x, R/c_2)) \leq n_{r/c_2}(B_X(x, R/c_1))$. Finally

$$
\limsup_{r \to 0} \frac{\log n_{r/c_1}(B_X(x, R/c_2))}{\log(c_1/r) - \log c_1} \leq \limsup_{r \to 0} \frac{\log n_r(B_Y(f(x), R))}{\log(1/r)} \leq \limsup_{r \to 0} \frac{\log n_{r/c_2}(B_X(x, R/c_1))}{\log(c_2/r) - \log c_2}
$$
and the thesis follows.

**Proof of Theorem 1.5** (continued). By the preceding Proposition, we may endow $X \times Y$ with any metric quasi-isometric to the product metric, i.e.

$$
\delta_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max\{\delta_X(x_1, x_2), \delta_Y(y_1, y_2)\}.
$$
Then the thesis follows easily by $n_r(B_{X \times Y}((x, y), R)) \leq n_r(B_X(x, R)) n_r(B_Y(y, R))$.

**Remark 1.7.** Kolmogorov and Tihomirov assign a metric dimension to a totally bounded metric space $E$ when \( \exists \lim_{r \to \infty} \) in equation (1.1), and consider upper and lower metric dimensions in the general case. We observe that if the $\lim inf$ is considered, the classical dimensional inequality \([33]\) stated in Theorem 1.5 (iii) is replaced by $d_0(X \times Y) \geq d_0(X) + d_0(Y)$.

### 1.2 Asymptotic dimension

The function introduced in the previous subsection can be used to study local properties of metric spaces. In this paper we are mainly interested in the investigation of the large scale behavior of these spaces, so we need a different tool. Looking at equation (1.2), it is natural to set the following

**Definition 1.8.** Let $(X, \delta)$ be a metric space. We call

$$
d_\infty(X) := \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R},
$$
the asymptotic dimension of $X$.

Let us remark that, as $n_r(B_X(x, R))$ is nonincreasing in $r$, the function

$$
r \mapsto \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R}
$$
is nonincreasing too, so the $\lim_{r \to 0}$ exists.

**Proposition 1.9.** $d_\infty(X)$ does not depend on $x$. 

Proof. Let \( x, y \in X \), and set \( \delta := \delta(x, y) \), so that \( B(x, R) \subset B(y, R + \delta) \subset B(x, R + 2\delta) \). This implies,

\[
\frac{\log n_r(B(x, R))}{\log R} \leq \frac{\log n_r(B(y, R + \delta))}{\log(R + \delta)} \leq \frac{\log n_r(B(x, R + 2\delta))}{\log(R + 2\delta)}
\]

so that, taking \( \limsup_{R \to \infty} \) and then \( \lim_{r \to \infty} \) we get the thesis.

\[\square\]

Lemma 1.10.

\[
d_\infty(X) = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R}
\]

Proof. Follows easily from lemma 1.1.

\[\square\]

Theorem 1.11. The set function \( d_\infty \) is a dimension, namely it satisfies

(i) If \( X \subset Y \) then \( d_\infty(X) \leq d_\infty(Y) \).

(ii) If \( X_1, X_2 \subset X \) then \( d_\infty(X_1 \cup X_2) = \max\{d_\infty(X_1), d_\infty(X_2)\} \).

(iii) If \( X \) and \( Y \) are metric spaces, then \( d_\infty(X \times Y) \leq d_\infty(X) + d_\infty(Y) \).

Proof. (i) Let \( x \in X \), then \( B_X(x, R) \subset B_Y(x, R) \) and the claim follows easily.

(ii) By part (i), we get \( d_\infty(X_1 \cup X_2) \geq \max\{d_\infty(X_1), d_\infty(X_2)\} \). Let us prove the converse inequality.

Let \( x_i \in X_i \), \( i = 1, 2 \), and set \( \delta = \delta(x_1, x_2), a = d_\infty(X_1), b = d_\infty(X_2) \), with e.g. \( a \leq b \). Then, \( \forall \varepsilon, r > 0 \exists R_0 = R_0(\varepsilon, r) \) s.t. \( \forall R > R_0 \)

\[
n_r(B_{X_1}(x_1, R)) \leq R^{a+\varepsilon}
\]

\[
n_r(B_{X_2}(x_2, R + \delta)) \leq R^{b+\varepsilon},
\]

hence, by inequality (\[\text{[1.3]}\]),

\[
n_r(B_{X_1 \cup X_2}(x_1, R)) \leq R^{a+\varepsilon} + R^{b+\varepsilon}
\]

\[
= R^{b+\varepsilon}(1 + R^{a-b}).
\]

Finally,

\[
\frac{\log n_r(B_{X_1 \cup X_2}(x_1, R))}{\log R} \leq b + \varepsilon + \frac{\log(1 + R^{a-b})}{\log R}.
\]

Taking the \( \limsup_{R \to \infty} \) and then \( \lim_{r \to \infty} \) we get

\[
d_\infty(X_1 \cup X_2) \leq \max\{d_\infty(X_1), d_\infty(X_2)\} + \varepsilon
\]

and the thesis follows by the arbitrariness of \( \varepsilon \).

The proof of part (iii) is analogous to that of part (iii) in Theorem 1.5, where we may use Proposition 1.16 because quasi-isometries are rough isometries.

\[\square\]
Remark 1.12. In part (ii) of the previous theorem we considered $X_1$ and $X_2$ as metric subspaces of $X$. If $X$ is a Riemannian manifold and we endow the submanifolds $X_1$, $X_2$ with their geodesic metrics this property does not hold in general. A simple example is the following. Let $f(t) := (t \cos t, t \sin t)$, $g(t) := (-t \cos t, -t \sin t)$, $t \geq 0$ planar curves, and set $X$, $Y$ for the closure in $\mathbb{R}^2$ of the two connected components of $\mathbb{R}^2 \setminus (G_f \cup G_g)$, where $G_f$, $G_g$ are the graphs of $f$, $g$, and endow $X$, $Y$ with the geodesic metric. Then $X$ and $Y$ are roughly-isometric to $[0, \infty)$ (see below) so that $d_\infty(X) = d_\infty(Y) = 1$, while $d_\infty(X \cup Y) = 2$.

Remark 1.13. As for the local case, the choice of the lim sup in Definition 1.8 is the only one compatible with the classical dimensional inequality stated in Theorem 1.11 (iii). This will motivate our choice of the lim sup in formula (4.2) for the 0-th Novikov-Shubin invariant.

Definition 1.14. Let $X,Y$ be metric spaces, $f: X \to Y$ is said to be a rough isometry if there are $a \geq 1$, $b, \varepsilon \geq 0$ s.t.

(i) $a^{-1} \delta_X(x_1, x_2) - b \leq \delta_Y(f(x_1), f(x_2)) \leq a \delta_X(x_1, x_2) + b$, for all $x_1, x_2 \in X$,

(ii) $\bigcup_{x \in X} B_Y(f(x), \varepsilon) = Y$

It is clear that the notion of rough isometry is weaker then the notion of quasi isometry introduced in the preceding subsection and, since any compact set is roughly isometric to a point, $d_0$ is not rough-isometry invariant. We shall show that the asymptotic dimension is indeed invariant under rough isometries.

Lemma 1.15. (\cite{3}, Proposition 4.3) If $f: X \to Y$ is a rough isometry, there is a rough isometry $f^-: Y \to X$, with constants $a, b^-, \varepsilon^-$, s.t.

(i) $\delta_X(f^- \circ f(x), x) < c_X, x \in X$,

(ii) $\delta_Y(f \circ f^-(y), y) < c_Y, y \in Y$.

Proposition 1.16. Let $X,Y$ be metric spaces, and $f: X \to Y$ a rough isometry. Then $d_\infty(X) = d_\infty(Y)$.

Proof. Let $x_0 \in X$, then for all $x \in B_X(x_0, r)$ we have

$\delta_Y(f(x), f(x_0)) \leq a \delta_X(x, x_0) + b < ar + b$

so that

$f(B_X(x_0, r)) \subset B_Y(f(x_0), ar + b)$.

Then, with $n := n_r(B_Y(f(x_0), aR + b))$,

$f(B_X(x_0, R)) \subset \bigcup_{j=1}^n B_Y(y_j, r)$,
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which implies
\[ f^- \circ f(B_X(x_0, R)) \subset \bigcup_{j=1}^{n} f^-(B_Y(y_j, r)) \]
\[ \subset \bigcup_{j=1}^{n} B_X(f^-(y_j), ar + b^-). \]

Let \( x \in B_X(x_0, R) \), and \( j \) be s.t. \( f^- \circ f(x) \in B_X(f^-(y_j), ar + b^-) \), then
\[ \delta_X(x, f^-(y_j)) \leq \delta_X(x, f^- \circ f(x)) + \delta_X(f^- \circ f(x), f^-(y_j)) < c_X + ar + b^- \]
so that
\[ B_X(x_0, R) \subset \bigcup_{j=1}^{n} B_X(f^-(y_j), ar + b^- + c_X), \]
which implies \( n_{ar+b^-+c_X}(B_X(x_0, R)) \leq n_r(B_Y(f(x_0), aR + b)). \)

Finally
\[ d_\infty(X) = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x_0, R))}{\log R} \]
\[ = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_{ar+b^-+c_X}(B_X(x_0, R))}{\log R} \]
\[ \leq \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_Y(f(x_0), aR + b))}{\log R} \]
\[ = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_Y(f(x_0), R))}{\log R} \]
\[ = d_\infty(Y) \]
and exchanging the roles of \( X \) and \( Y \) we get the thesis.

In what follows we show that when \( X \) is equipped with a suitable measure, the asymptotic dimension may be recovered in terms of the volume asymptotics for balls of increasing radius, like the local dimension detects the volume asymptotics for balls of infinitesimal radius.

**Definition 1.17.** A Borel measure \( \mu \) on \((X, \delta)\) is said to be uniformly bounded if there are functions \( \beta_1, \beta_2 \), s.t. \( 0 < \beta_1(r) \leq \mu(B(x, r)) \leq \beta_2(r) \), for all \( x \in X, r > 0 \).

That is \( \beta_1(r) := \inf_{x \in X} \mu(B(x, r)) > 0 \), and \( \beta_2(r) := \sup_{x \in X} \mu(B(x, r)) < \infty \).

**Proposition 1.18.** If \((X, \delta)\) has a uniformly bounded measure, then every ball in \( X \) is totally bounded (so that if \( X \) is complete it is locally compact).

**Proof.** Indeed, if there is a ball \( B = B(x, R) \) which is not totally bounded, then there is \( r > 0 \) s.t. every \( r \)-net in \( B \) is infinite, so \( n_r(B) \) is infinite, and \( \nu_r(B) \) is infinite too. So that \( \beta_2(R) \geq \mu(B) \geq \sum_{i=1}^{r(B)} \mu(B(x_i, r)) \geq \beta_1(r) \nu_r(B) = \infty \), which is absurd. \( \square \)
Proposition 1.19. If $\mu$ is a uniformly bounded Borel measure on $X$ then

$$d_\infty(X) = \limsup_{R \to \infty} \frac{\log \mu(B(x, R))}{\log R}.$$  

Proof. As $\bigcup_{i=1}^{\nu_r(B(x,R))} B(x_i, r) \subset B(x, R + r) \subset \bigcup_{j=1}^{\nu_r(B(x,R+r))} B(y_j, r)$, we get $\beta_2(r) \nu_r(B(x, R + r)) \geq \mu(B(x, R + r)) \geq \beta_1(r) \nu_r(B(x, R)) \geq \beta_1(r) \nu_x(B(x, R))$, by Lemma [1.1]. So that

$$\beta_1(r/2) \leq \frac{\mu(B(x, R + r/2))}{\nu_r(B(x, R))}, \quad \frac{\mu(B(x, R))}{\nu_r(B(x, R))} \leq \beta_2(r),$$

and the thesis follows easily. \qed

Let us conclude this subsection with some examples.

Example 1.20.  

(i) $\mathbb{R}^n$ has asymptotic dimension $n$.  

(ii) Set $X := \bigcup_{n \in \mathbb{Z}} \{(x, y) \in \mathbb{R}^2 : \delta((x, y), (n, 0)) < \frac{1}{4}\}$, endowed with the Euclidean metric, then $d_0(X) = 2$, $d_\infty(X) = 1$.  

(iii) Set $X = \mathbb{Z}$ with the counting measure, then $d_0(X) = 0$, and $d_\infty(X) = 1$.  

(iv) Let $X$ be the unit ball in an infinite dimensional Banach space. Then $d_0(X) = +\infty$ while $d_\infty(X) = 0$.

Example 1.21. Set $X := \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x^\alpha\}$, endowed with the Euclidean metric, where $\alpha \in (0, 1]$. Then $d_\infty(X) = \alpha + 1$.

Proof. This metric space has a uniformly bounded Borel measure, the Lebesgue area, so we can use Proposition [1.19]. Set $x_0 := (0, 0)$, and $B_R := B_X(x_0, R)$. Then, if $R \geq \sqrt{4^{1+\alpha}r^{2/\alpha} + R^{4+\alpha}r^2}$, $B_R \subset Q_1 \cup Q_2$, where $Q_1 := \{(x, y) \in \mathbb{R}^2 : -2r \leq x \leq (2r)^{1/\alpha} + 2r, |y| \leq 4r\}$, and $Q_2 := \{(x, y) \in \mathbb{R}^2 : (2r)^{1/\alpha} \leq x \leq R, |y| \leq 2x^\alpha\}$. Now, if $x_R > 0$ is s.t. $x_R^2 + x_R^{2\alpha} = R^2$, we get

$$\text{area}(B_R) \geq \frac{2}{\alpha + 1} R^{\alpha + 1}$$

$$\text{area}(Q_1) = 4r(4r + (2r)^{1/\alpha})$$

$$\text{area}(Q_2) = \frac{4}{\alpha + 1} (R^{1+\alpha} - (2r)^{1+1/\alpha}),$$

so that

$$\lim_{R \to \infty} \frac{(\alpha + 1) \log x_R}{\log R} \leq \liminf_{R \to \infty} \frac{\log \text{area}(B_R)}{\log R} \leq \limsup_{R \to \infty} \frac{\log \text{area}(B_R)}{\log R} \leq \alpha + 1$$

and, as $\lim_{R \to \infty} \frac{\log x_R}{\log R} = \lim_{x \to \infty} \frac{\log x}{\log \sqrt{x^2 + x^2}} = 1$, we get the thesis. \qed
2 A semigroup formula for the asymptotic dimension of an open manifold

2.1 Open manifolds of bounded geometry

In this subsection, after some preliminary results on open manifolds of bounded geometry, we give a formula for the asymptotic dimension in terms of the asymptotics of the heat kernel. This opens the way for the abstract treatment of the following subsection.

Several definitions of bounded geometry for an open manifold (i.e. a Riemannian, complete, noncompact manifold) are usually considered. They all require some uniform bound (either from above or from below) on some geometric objects, such as: injectivity radius, sectional curvature, Ricci curvature, Riemann curvature tensor etc. (For all unexplained notions see e.g. Chavel’s book [3]).

In this paper the following form is used, but see [36] and references therein for a different approach.

Definition 2.1. Let $(M,g)$ be an $n$-dimensional complete Riemannian manifold. We say that $M$ has bounded geometry if it has positive injectivity radius, sectional curvature bounded from above by some constant $c_1$, and Ricci curvature bounded from below by $(n-1)c_2g$.

Theorem 2.2. ([3], p.119,123) Let $M$ be a complete Riemannian manifold of bounded geometry. Then there are positive real functions $\beta_1, \beta_2$ s.t.

(i) $0 < \beta_1(r) \leq \text{vol}(B(x,r)) \leq \beta_2(r)$, for all $x \in X$, $r > 0$, that is the volume form is a uniformly bounded measure (cf. Definition [1.17]),

(ii) $\lim_{r \to 0} \frac{\beta_2(r)}{\beta_1(r)} = 1$.

Proof. (i) We can assume $c_2 < 0 < c_1$ without loss of generality. Then, denoting with $V_\delta(r)$ the volume of a ball of radius $r$ in a manifold of constant sectional curvature equal to $\delta$, we can set $\beta_1(r) := V_{c_1}(r \wedge r_0)$, and $\beta_2 := V_{c_2}(r)$, where $r_0 := \min\{\text{inj}(M),\sqrt{c_1}\}$, and $\text{inj}(M)$ is the injectivity radius of $M$.

(ii)

$$\lim_{r \to 0} \frac{\beta_2(r)}{\beta_1(r)} = \lim_{r \to 0} \frac{V_{c_2}(r)}{V_{c_1}(r)} = \lim_{r \to 0} \frac{\int_0^r S_{c_2}(t)^{n-1}dt}{\int_0^r S_{c_1}(t)^{n-1}dt} = \left(\lim_{r \to 0} \frac{S_{c_2}(r)}{S_{c_1}(r)}\right)^{n-1} = 1$$

where (cfr. [3], formulas (2.48), (3.24), (3.25)) $V_\delta(r) = \frac{n\sqrt{\pi}}{\Gamma(n/2)} \int_0^r S_\delta(t)^{n-1}dt$, and

$$S_\delta(r) := \begin{cases} \frac{1}{\sqrt{\delta}} \sinh(r\sqrt{\delta}) & \delta < 0 \\ \frac{r}{\sqrt{\delta}} & \delta = 0 \\ \frac{1}{\sqrt{\delta}} \sinh(r\sqrt{\delta}) & \delta > 0. \end{cases}$$
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Conditions under which the inequality in Theorem 1.11 (iii) becomes an equality are often studied in the case of (local) dimension theory (cf. [33, 38]). The following proposition gives such a condition for the asymptotic dimension.

**Proposition 2.3.** Let $M, N$ be complete Riemannian manifolds of bounded geometry, which admit asymptotic dimension in a strong sense, that is

$$d_\infty(M) \equiv \lim_{r \to \infty} \lim_{R \to \infty} \frac{\log n_r(B_M(o, R))}{\log R},$$

and analogously for $N$. Then

$$d_\infty(M \times N) = d_\infty(M) + d_\infty(N).$$

**Proof.** As $\text{vol}(B_{M \times N}(x, y, R)) = \text{vol}(B_M(x, R))\text{vol}(B_N(y, R))$, we get

$$d_\infty(M \times N) = \lim_{R \to \infty} \frac{\log \text{vol}(B_{M \times N}((x, y), R))}{\log R} = \lim_{R \to \infty} \frac{\log \text{vol}(B_M(x, R))}{\log R} + \lim_{R \to \infty} \frac{\log \text{vol}(B_N(y, R))}{\log R} = d_\infty(M) + d_\infty(N).$$

As the asymptotic dimension is invariant under rough isometries, it is natural to substitute the continuous space with a coarse graining, which destroys the local structure, but preserves the large scale structure. To state it more precisely, recall ([3], p. 194) that a discretization of a metric space $M$ is a graph $G$ determined by an $\varepsilon$-separated subset $G$ of $M$ for which there is a $R > 0$ s.t. $M = \bigcup_{x \in G} B_M(x, R)$. The graph structure on $G$ is determined by one oriented edge from any $x \in G$ to any $y \in G$, $y \neq x$, denoted $<x, y>$, precisely when $\delta_M(x, y) < 2R$. Define the combinatorial metric on $G$ by $\delta_c(x, y) := \inf\{\sum_{i=0}^n \delta(x_i, x_{i+1}) : (x_0, \ldots, x_{n+1}) \in \text{Path}_n(x, y), \ n \in \mathbb{N}\}$, where $\text{Path}_n(x, y) := \{(x_0, \ldots, x_{n+1}) : x_i \in G, \ x_0 = x, \ x_{n+1} = y, <x_i, x_{i+1}> \in G\}$.

**Proposition 2.4.** ([3], Theorem 4.9) Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below. Then $M$ is roughly isometric to any of its discretizations, endowed with the combinatorial metric. Therefore $M$ has the same asymptotic dimension of any of its discretizations.

The previous result, together with the invariance of the asymptotic dimension under rough isometries, shows that, when $M$ has a discrete group of isometries $\Gamma$ with a compact quotient, the asymptotic dimension of the manifold coincides with the asymptotic dimension of the group, hence with its growth (cf. [24]), hence, by the result of Varopoulos [43], it coincides with the 0-th Novikov-Shubin invariant.
Let $M$ be a complete Riemannian manifold, and recall ([14], Chapter 5) that $\Delta$, the Laplace-Beltrami operator on $M$, is essentially self-adjoint and positive on $L^2(M)$, and the semigroup $e^{-t\Delta}$ has a strictly positive $C^\infty$ kernel, $p_t(x, y)$, on $(0, \infty) \times M \times M$, called the heat kernel. Recall the following results, which will be useful in the sequel, and where we use $V(x, r) := \text{vol}(B(x, r))$, for simplicity.

**Proposition 2.5.** Let $M$ be an $n$-dimensional complete Riemannian manifold with Ricci curvature bounded below, and let $E$ be the infimum of the spectrum of $-\Delta$, then for all $\varepsilon > 0$, there are $c, c' > 0$, s.t.

(i) \[ p_t(x, y) \leq \begin{cases} c V(x) x^{1/2} V(y, x) x^{-1/2} e^{-\delta(x, y)^2} & 0 < t \leq 1 \\ c V(x, 1)^{-1/2} V(y, 1)^{-1/2} e^{\varepsilon(x, y)^2} t \geq 1 \end{cases} \]

(ii) \[ |\nabla_x p_t(x, y)| \leq \begin{cases} c't^{-n/2} (t^{-1} + \delta(x, y))^{1/2} e^{-\delta(x, y)^2} & 0 < t \leq 1 \\ c' e^{\varepsilon(t)} (t^{-1} + \delta(x, y))^{1/2} e^{-\delta(x, y)^2} & t \geq 1 \end{cases} \]

As a consequence $p_t$ is uniformly continuous on a neighborhood of the diagonal of $M \times M$.

**Proof.** (i) is ([12], theorems 16, 17). Observe now that, with $r_1 := \min\{1, \text{inj}(M), \frac{\pi}{\sqrt{\varepsilon}}\}$, we have $V(x, t) \geq V(x, r_1) \geq V_{c_1}(r_1)$ for any $t \geq 1$. On the other hand, if $t \in [r_1, 1]$, we have $\frac{V(x, t)}{V(x, 1)} \geq V(x, r_1) \geq V_{c_1}(r_1)$, while, if $t \in (0, r_1)$, from $\frac{V(x, t)}{V(x, 1)} \geq \frac{V_{c_1}(t)}{V_{c_1}(1)}$ and $\lim_{t \to 0} \frac{V_{c_1}(t)}{V_{c_1}(1)} = \frac{\sqrt{\varepsilon}}{\Gamma(n/2 + 1)} > 0$ (use the formulas in the proof of Theorem 2.2), there follows $a > 0$ s.t. $\frac{V(x, t)}{t^n} \geq a$ for any $t \in (0, r_1)$.

Putting all things together we get a simplified version of the estimates (i)

\[ p_t(x, y) \leq \begin{cases} ce^{-n/2} e^{-\delta(x, y)^2} & 0 < t \leq 1 \\ ce^{\varepsilon(t)} t^{-n/2} e^{-\delta(x, y)^2} & t \geq 1 \end{cases} \]

(ii) follows from ([13], theorem 6), using the simplified estimates above.

Finally, for any $\delta_0 < r_1$, $x \in M$, $y \in B(x, \delta_0)$, we have $|p_t(x, y) - p_t(x, x)| \leq \sup |\nabla_y p_t(x, y)| \delta(x, y)$, and from (ii) we get the uniform continuity. \qed

The following proposition shows the deep connection between the heat kernel and the volume of balls.

**Theorem 2.6.** ([1], Corollary 7.3) ([2], Proposition 5.2)

Let $M$ be a complete Riemannian manifold, and set $\lambda_1(U)$ for the first Dirichlet eigenvalue of $-\Delta$ in $U$. Then the following are equivalent
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(i) there are $\alpha, \beta > 0$ s.t. for all $x \in M$, $r > 0$, and all regions $U \subset B(x, r)$,

$$\lambda_1(U) \geq \frac{\alpha r^2 \left(\frac{V(x, r)}{\text{vol}(U)}\right)^\beta}{\text{vol}(U)},$$

(ii) there are $A, C, C' > 0$ s.t. for all $x \in M$, $r > 0$,

$$V(x, 2r) \leq AV(x, r)$$

$$\frac{C}{V(x, \sqrt{r})} \leq p_r(x, x) \leq \frac{C'}{V(x, \sqrt{r})}.$$ 

Following [11] we call (2.1) the volume doubling property.

As a consequence of the above results we have the following

**Corollary 2.7.** Let $M$ be a complete Riemannian manifold of bounded geometry, and assume one of the equivalent properties of the previous Theorem. Then $d_\infty(M) = \limsup_{t \to \infty} -\frac{2 \log V(x_0, x_0)}{\log t}$.

**Proof.** Follows from Theorem 2.2.

For a different result under weaker hypotheses, see Corollary 2.14.

Before closing this subsection we observe that the volume doubling property is a key notion for our work, as it is a weak form of polynomial growth condition, and still guarantees the finiteness of the asymptotic dimension (for manifolds of bounded geometry).

**Proposition 2.8.** Let $M$ be a complete Riemannian manifold of bounded geometry, and suppose the volume doubling property (2.1) holds. Then $d_\infty(M) \leq \log_2 A$.

**Proof.** Let $R > 1$, and $n \in \mathbb{N}$ be s.t. $2^n - 1 < R \leq 2^n$. Then $V(x, R) \leq V(x, 2^n) \leq A^n V(x, 1)$, so that

$$1 \leq \frac{V(x, R)}{V(x, 1)} \leq A^n \leq AR^{\log_2 A}.$$ 

Therefore $d_\infty(M) = \limsup_{R \to \infty} \frac{\log V(x, R)}{\log R} \leq \log_2 A.$

### 2.2 Asymptotic dimension of some semigroups of bounded operators

Based on the notion of dimension at infinity due to Varopoulos, Saloff-Coste, Coulhon [14], see also [10], we define the asymptotic dimension of a semigroup of bounded operators on a measure space.
**Definition 2.9.** Let \((X, \Omega, \mu)\) be a measure space, and \(T_t : L^1(X, \Omega, \mu) \to L^\infty(X, \Omega, \mu)\) be a semigroup of bounded operators. Then we set
\[
d_\infty(T) := \liminf_{t \to \infty} \frac{-2 \log \|T_t\|_{1 \to \infty}}{\log t}.
\]

**Theorem 2.10.** ([L], Theorem II.4.3)
Let \((X, \Omega, \mu)\) be a measure space, and suppose given \(T_t \in \text{B}(L^1(X, \Omega, \mu) \cap L^\infty(X, \Omega, \mu))\), which, for any \(p \in [1, \infty]\), extends to a semigroup on \(L^p\), of class \(C^0\) if \(p < \infty\). Let \(A\) be the generator, and suppose that \(T_t\) is equicontinuous on \(L^1\) and \(L^\infty\), bounded analytic on \(L^2\), and \(\|T_t\|_{1 \to \infty} < \infty\). Let \(0 < \alpha < \frac{d}{2}\). Then the following are equivalent
(i) \(\|T_t\|_{2n/(n-2\alpha)} \leq C(\|A^{\alpha/2} \|_2 + \|A^{\alpha/2} \|_{2n/(n-2\alpha)})\), \(f \in \mathcal{D}\)
(ii) \(\|T_t f\|_{2n/(n-2\alpha)} \leq C\|A^{\alpha/2} \|_2\), \(f \in \mathcal{D}\)
(iii) \(\|T_t\|_{1 \to \infty} \leq C t^{-n/2},\) \(t \in (1, \infty),\)
where \(\mathcal{D} := \text{span} \{f^\infty \varphi(t)T_t f dt : \varphi \in C^\infty_0(0, \infty), f \in L^\infty(X, \Omega, \mu), \mu\{f \neq 0\} < \infty\}\).

**Proposition 2.11.** Let \(\{T_t\}\) be as in the previous Theorem. Then the following are equivalent
(i) \(\|T_t\|_{1 \to \infty} \leq C t^{-n/2},\) \(t \geq 1\)
(ii) \(\|T_t\|_{1 \to \infty} \leq C t^{-n/2},\) \(t > t_0 > 1\).

**Proof.** (ii) \(\Rightarrow\) (i)
Let \(t > 1\) and observe that \(\|T_t\|_{1 \to \infty} = \|T_1T_{t-1}\|_{1 \to \infty} \leq \|T_1\|_{1 \to \infty}\|T_{t-1}\|_{1 \to 1} \leq k\|T_t\|_{1 \to \infty} := M\), where \(k := \sup_{t > 0} \|T_t\|_{1 \to 1} < \infty\) because \(T_t\) is equicontinuous on \(L^1\). So that, with \(C_0 := \max\{C, M^{n/2}\}\), we get the thesis. \(\square\)

**Proposition 2.12.** \(d_\infty(T) = \sup\{n > 0 : \|T_t\|_{1 \to \infty} \leq C t^{-n/2},\) \(t \geq 1\}\).

**Proof.** Set \(d\) for the supremum. Then for all \(\varepsilon > 0\), there is \(t_0 > 1\) s.t. \(\|T_t\|_{1 \to \infty} \leq t^{-\left(d_\infty(T) - \varepsilon\right)/2}\), for all \(t \geq t_0\), and, by previous proposition, \(d_\infty(T) - \varepsilon \leq d\). Conversely \(\|T_t\|_{1 \to \infty} \leq t^{-\left(d - \varepsilon\right)/2}\), for all \(t \geq 1\) implies \(d - \varepsilon \leq d_\infty(T)\). \(\square\)

Using a recent result by Coulhon-Grigor'yan ([L]) we can show the relation between the asymptotic dimension of the heat kernel semigroup and the asymptotic dimension of the underlying manifold.

**Theorem 2.13.** ([L], Theorem 2.7)
Let \(M\) be a complete Riemannian manifold. If there are \(x_0 \in M, A > 0\) s.t. \(V(x_0, 2t) \leq AV(x_0, t)\), for all \(t > 0\), then there is \(c > 0\) s.t.
\[
\sup_{x \in M} p_t(x, x) \geq \frac{c}{V(x_0, \sqrt{t})}.
\]
Corollary 2.14. Let $M$ be a complete Riemannian manifold of bounded geometry, and assume there are $x_0 \in M$, $A > 0$ s.t. $V(x_0, 2t) \leq AV(x_0, t)$, for all $t > 0$. Then $d_\infty(e^{-t\Delta}) \leq d_\infty(M)$.

Proof. Follows immediately from the previous results if we recall that $\|e^{-t\Delta}\|_{1 \to \infty} = \sup_{x \in M} p_t(x, x)$.

2.3 Asymptotic dimension of some cylindrical ends

In this subsection we want to compare our work with a recent work of E.B. Davies'. In [15] he defines the asymptotic dimension of cylindrical ends of a Riemannian manifold $M$ as follows. Let $E \subset M$ be homeomorphic to $(1, \infty) \times A$, where $A$ is a compact Riemannian manifold. Set $\partial E := \{1\} \times A$, $E_r := \{x \in E : \delta(x, \partial E) < r\}$, where $\delta$ is the restriction of the metric in $M$. Then $E$ has asymptotic dimension $D$ if there is a positive constant $c$ s.t.

$$c^{-1}r^D \leq \text{vol}(E_r) \leq cr^D,$$  \hspace{1cm} (2.2)

for all $r \geq 1$. He does not assume bounded geometry for $E$. If one does, the two definitions coincide as in the following

Proposition 2.15. With the above notation, if the volume form on $E$ is a uniformly bounded measure (as in Definition 1.17), or in particular if $E$ has bounded geometry (as in Definition 2.1), and there is $D$ as in (2.2), then $d_\infty(E) = D$.

Proof. Choose $o \in E$, and set $\delta := \delta(o, \partial E)$, $\Delta := \text{diam}(\partial E)$. Then it is easy to prove that $E_{R-\delta-\Delta} \subset B_E(o, R) \subset E_{R+\delta}$.

Then $c^{-1}(R - \delta - \Delta)^D \leq \text{vol}(B_E(o, R)) \leq c(R + \delta)^D$, and from 1.19 the thesis follows.

Motivated by ([15], example 16), let us set the following

Definition 2.16. Let us say that $E$ is a standard end of local dimension $N$ if it is homeomorphic to $(1, \infty) \times A$, endowed with the metric $ds^2 = dx^2 + f(x)^2d\omega^2$, and with the volume form $d\text{vol} = f(x)^{N-1}dx d\omega$, where $(A, \omega)$ is an $(N - 1)$-dimensional compact Riemannian manifold, and $f$ is an increasing smooth function.

Proposition 2.17. The volume form on a standard end $E$ is a uniformly bounded measure. Therefore, if $E$ satisfies (2.2), we get $d_\infty(E) = D$.

Proof. It is easy to show that, for $(x_0, p_0) \in E$, $r < x_0 - 1$,

$$[x_0 - r/2, x_0 + r/2] \times B_A \left(p_0, \frac{r/2}{f(x_0 + r/2)} \right) \subset B_E((x_0, p_0), r) \subset [x_0 - r, x_0 + r] \times B_A \left(p_0, \frac{r}{f(x_0 - r)} \right)$$
So that
\[
\int_{x_0-r/2}^{x_0+r/2} f(x)^{N-1} dx V_A \left( p_0, \frac{r/2}{f(x_0 + r/2)} \right) \leq V_E((x_0, p_0), r)
\]
which implies
\[
rf(x_0 - r/2)^{N-1} V_A \left( p_0, \frac{r/2}{f(x_0 + r/2)} \right) \leq V_E((x_0, p_0), r)
\]
\[
\leq 2rf(x_0 + r)^{N-1} V_A \left( p_0, \frac{r}{f(x_0 - r)} \right)
\]
As for \( x_0 \to \infty \), \( V_A(p_0, \frac{r}{f(x_0 - r)}) \sim c \left( \frac{r}{f(x_0 - r)} \right)^{N-1} \), and the same holds for \( V_A(p_0, \frac{r/2}{f(x_0 + r/2)}) \), we get the thesis.

**Corollary 2.18.** Let \( E \) be the standard end of local dimension \( N \) and asymptotic dimension \( D \) in (16), example 16, which is homeomorphic to \((1, \infty) \times S^{N-1} \), endowed with the metric \( ds^2 = dr^2 + r^{2(D-1)/(N-1)}d\omega^2 \), and with the volume form \( dvol = r^{D-1}drd\omega \). Then \( d_\infty(E) = D \).

Observe that \( d_\infty(M) \) makes sense for any metric space, hence for any cylindrical end, while Davies’ asymptotic dimension does not. Indeed let \( E := (1, \infty) \times S^1 \), endowed with the metric \( ds^2 = dr^2 + f(r)^2 d\omega^2 \), and with the volume form \( dvol = f(r)drd\omega \), where \( f(r) := \frac{d}{dr}(r^2 \log r) \). Then \( d_\infty(E) = 2 \), but \( vol(E_r) \) does not satisfy one of the inequalities in (2.2).

Before closing this section we observe that the notion of standard end allows us to construct an example which shows that we could obtain quite different results if we used lim inf instead of lim sup in the definition of the asymptotic dimension. It makes use of the following function
\[
f(x) = \begin{cases} 
\sqrt{x} & x \in [1, a_1] \\
2 + b_{n-1} + c_{n-1} + (x - a_{2n-1}) & x \in [a_{2n-1}, a_{2n}] \\
2 + b_{n-1} + c_n + \sqrt{x - a_{2n} - 1} & x \in [a_{2n}, a_{2n+1}] 
\end{cases}
\]
where \( a_0 := 0, a_n - a_{n-1} := 2^n, b_n := \sum_{k=1}^{n} \sqrt{2^{2k+1}} + 1, c_n := \sum_{k=1}^{n} (2^{2k} - 1), n \geq 1 \).

**Proposition 2.19.** Let \( M \) be the Riemannian manifold obtained as a \( C^\infty \) regularization of \( C \cup_\varphi E \), where \( C := \{(x, y, z) \in \mathbb{R}^3 : (x - 1)^2 + y^2 + z^2 = 1, x \leq 1 \} \), with the Euclidean metric, \( E := [1, \infty) \times S^1 \), endowed with the metric \( ds^2 = dx^2 + f(x)^2 d\omega^2 \), and with the volume form \( dvol = f(x)dxd\omega \), where \( \varphi \) is the identification of \( \{y^2 + z^2 = 1, x = 1\} \) with \( \{1\} \times S^1 \). Then the volume form is a uniformly bounded measure, \( d_\infty(M) \geq 2 \) but \( d_\infty(M) \leq 3/2 \), where \( d_\infty(M) := \lim_{r \to \infty} \lim \inf \frac{\log n_r(B_d(x, R))}{\log R} \).
Proof. Set $o := (0, 0, 0) \in M$, then it is easy to see that, for $n \to \infty$, $a_n \sim 2^{2n}$, $b_n \sim c_n \sim 2^{2n}$, and
\[
\text{area}(B_M(o, a_{2n})) \sim \frac{1}{2} a_{2n}^2
\]
\[
\text{area}(B_M(o, a_{2n-1})) \sim \frac{5}{3} b_{2n-2}^{3/2}
\]
so that, calculating the limit of $\frac{\log \text{area}(B_M(o, R))}{\log R}$ on the sequence $R = a_{2n}$ we get 2, while on the sequence $R = a_{2n-1}$ we get 3/2. The thesis follows easily, using Proposition 1.19. \hfill \Box

3 A trace for open manifolds

This section is devoted to the construction of a trace on (a suitable subalgebra of) the bounded operators on $L^2(M)$, where $M$ is an open manifold of bounded geometry. The basic idea for this construction is due to J. Roe [36], who considers regularly exhaustible open manifolds. In our case we may (and will) restrict to exhaustions by spheres with linearly increasing radii. Moreover we shall regularize (three times) this trace, in order to get a semicontinuous semifinite trace on the C*-algebra of almost local operators. As observed by Roe, this trace is strictly related to the trace constructed by Atiyah [2] in the case of covering manifolds, and may therefore be used to define the (0-th) Novikov-Shubin invariant for open manifolds, as we do in Section 4.

3.1 The C*-algebra of almost local operators

Recall [37] that an operator $A \in B(L^2(M))$ has finite propagation speed if there is a constant $u(A) > 0$ s.t. for any compact subset $K$ of $M$, any $\varphi \in L^2(M)$, supp $\varphi \subset K$, we have supp $A\varphi \subset \text{Pen}(K, u(A)) := \{ x \in M : \delta(x, K) \leq u(A) \}$. Let us denote with $A_0$ the set of finite propagation speed operators. $A_0$ may be characterized as follows

Proposition 3.1.
(i) $A \in A_0$ iff, for any measurable set $\Omega$, $AE_{\Omega} = E_{\text{Pen}(\Omega, u(A))} AE_{\Omega}$, where $E_X$ is the multiplication operator by the characteristic function of the set $X$;
(ii) $A \in A_0$ iff, for any functions $\varphi, \psi \in L^2(M)$ with $\delta(\text{supp } \varphi, \text{supp } \psi) \geq u(A)$, one has $(\varphi, A\psi) = 0$.

Proof. (i) is obvious.
(ii) $(\Rightarrow)$ is easy.
(ii) $(\Leftarrow)$ The hypothesis implies that supp $A\psi \subset M \setminus \text{supp } \varphi$ for all $\varphi$ s.t. $M \setminus \text{supp } \varphi \subset \text{Pen}(\text{supp } \psi, u(A))$. The thesis follows. \hfill \Box
Proposition 3.2. The set $A_0$ of finite propagation speed operators is a $\ast$-algebra with identity.

Proof. Let $K$ be a compact subset of $M$, $\varphi \in L^2(M)$, supp $\varphi \subset K$, and $A, B \in A_0$. Then supp $(A + B)\varphi \subset$ supp $A\varphi \cup$ supp $B\varphi$, which implies $u(A + B) \leq u(A) \lor u(B)$. Moreover supp $(AB)\varphi \subset$ Pen(supp $B\varphi, u(A) \in Pen(K, u(A) + u(B))$, so that $u(AB) \leq u(A) + u(B)$.

As $(A^\ast \psi, \varphi) = (\psi, A\varphi) = 0$ for all $\psi, \varphi \in L^2(M)$, with $\delta($supp $\psi, $supp $\varphi) \geq u(A)$, that is supp $\varphi \cap$ Pen(supp $\psi, u(A)) = \emptyset$, we get supp $A^\ast \psi \subset$ Pen(supp $\psi, u(A)$), which implies $u(A^\ast) \leq u(A)$, and exchanging the roles of $A, A^\ast$, we get $u(A) = u(A^\ast)$.

The norm closure of $A_0$ will be denoted by $A$ and will be called the $C^\ast$-algebra of almost local operators. Now we show that Gaussian decay for the kernel of a positive operator $A$ is a sufficient condition for $A$ to belong to $A$.

Theorem 3.3. Let $M$ be an open $n$-manifold of bounded geometry. If $A$ is a self-adjoint bounded operator on $L^2(M)$, with kernel $a(x, y)$, and there are positive constants $c, \alpha, \delta_0$ s.t.

$$|a(x, y)| \leq c e^{-\alpha \delta(x,y)^2}, \quad \delta(x,y) \geq \delta_0$$

then $A \in A$.

In order to prove the theorem, we need some lemmas.

Lemma 3.4. Let $A$ be a bounded self-adjoint operator on $L^2(M)$, with measurable kernel. Then

$$\|A\| \leq \sup_{x \in M} \int_M |a(x,y)| dy$$

Proof. Since $A$ is self-adjoint, $a(x,y)$ is symmetric, hence

$$\|A\|_{1 \to 1} = \sup\{|(f, Ag)| : f \in L^\infty(M), \|f\|_\infty = 1, g \in L^1(M), \|g\|_1 = 1\}$$

$$\leq \sup_{x \in M} \int_M |a(y,x)| dy = \|A\|_{\infty \to \infty}$$

The thesis easily follows from Riesz-Thorin interpolation theorem.

Lemma 3.5. Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-increasing measurable function. Then

$$\sup_{x \in M} \int_M \varphi(\delta(x,y)) dy \leq C_n \int_0^\infty \varphi(r) S_{c_2}(r)^{n-1} dr$$

where $C_n := \frac{n \sqrt{\pi}}{\Gamma(n/2+1)}$, and $S_{c_2}(r) := \frac{1}{\sqrt{-c_2}} \sinh(r \sqrt{-c_2})$
Proof. From Theorem 2.2 we get 
\[ V(x,r) \leq C_n \int_0^r S_{c_2}(t)^{n-1} dt. \]
Then
\[ \int_M \varphi(\delta(x,y)) dy = \int_0^\infty \varphi(r) dV(x,r) \leq C_n \int_0^\infty \varphi(r) S_{c_2}(r)^{n-1} dr, \]
where the equality is in e.g. (25, Theorem 12.46), and the inequality holds because \( \varphi \) is non-increasing and positive, and \( V(x,0) = 0 \).

Proof of Theorem 3.3. Let \( \rho > \delta_0 \), and decompose 
\[ A = A_0 + A', \]
with \( a'_{\rho}(x,y) := a(x,y) \chi(\delta(x,y)) \). Then \( A_0 \in \mathcal{A}_0 \), and \( |a'_{\rho}(x,y)| \leq c' \varphi(\delta(x,y)) \), where
\[ \varphi(r) := \begin{cases} e^{-\alpha r^2} & 0 \leq r < \rho \\ e^{-\alpha r^2} & r \geq \rho. \end{cases} \]
By Lemmas 3.4, 3.5 we get
\[ \|A - A_0\| = \|A'\| \leq \sup_{x \in M} \int_M |a'_{\rho}(x,y)| dy \leq c' \sup_{x \in M} \int_M \varphi(\delta(x,y)) dy \leq c' \int_0^\infty \varphi(r) S_{c_2}(r)^{n-1} dr \leq c' e^{-\alpha \rho^2} \int_0^\rho S_{c_2}(r)^{n-1} dr + c'' \int_{\rho}^\infty e^{-\alpha r^2 + (n-1)r\sqrt{c_2}} dr \rightarrow 0, \quad \rho \rightarrow \infty \]
and the thesis follows.

Finally we conclude that \( C_0 \)-functional calculus of the Laplace operator belongs to \( \mathcal{A} \).

Corollary 3.6. Let \( M \) be an open manifold of bounded geometry. Then \( \varphi(\Delta) \in \mathcal{A} \), for any \( \varphi \in C_0([0, \infty)) \).

Proof. By Proposition 2.5 and Theorem 3.3 we obtain that \( e^{-t\Delta} \in \mathcal{A} \), for any \( t > 0 \). Since \( \{e^{-t\lambda}\}_{t \geq 0} \) generates a dense \( * \)-algebra of \( C_0([0, \infty)) \), the thesis follows by Stone-Weierstrass theorem.

3.2 A functional described by J. Roe

In the rest of this paper \( M \) is a complete Riemannian \( n \)-manifold of bounded geometry (as in Definition 2.1) and of regular polynomial growth, that is
\[ \lim_{r \rightarrow \infty} \frac{V(x, r + R) - V(x, r)}{V(x, r)} = 1 \]
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for all $x \in M, R > 0$.

Following Moore-Schochet [28], we recall that an operator $T$ on $L^2(M)$ is called locally trace class if, for any compact set $K \subset M$, $E_K T E_K$ is trace class, where $E_K$ denotes the projection given by the characteristic function of $K$. It is known that the functional $\mu_T(K) := Tr(E_K T E_K)$ extends to a Radon measure on $M$. To state next definition we need some preliminary notions.

**Lemma 3.7.** Let $M$ be as above, and $\mu$ be a measure on $M$. Then the following are equivalent

(i) there are $x_0 \in M, r_0 > 0, c > 0$ s.t. $\mu(B(x_0, r)) \leq c V(x_0, r), r \geq r_0$;

(ii) $\limsup_{r \to \infty} \frac{\mu(B(x_0, r))}{V(x_0, r)}$ is finite and independent of $x \in M$.

**Proof.** (ii) $\Rightarrow$ (i) is easy. 

(i) $\Rightarrow$ (ii) To prove the converse, we observe that, with $\delta := \delta(x_0, x)$,

$$\frac{\mu(B(x_0, r))}{V(x_0, r)} \leq \frac{\mu(B(x, r + \delta))}{V(x, r + \delta)} \frac{V(x_0, r + 2\delta)}{V(x_0, r)} \leq \frac{\mu(B(x, r + 2\delta))}{V(x, r + \delta)} \frac{V(x, r + 3\delta)}{V(x, r + \delta)} \frac{V(x_0, r + 2\delta)}{V(x_0, r)}$$

Taking the $\limsup_{r \to \infty}$, and making use of regular polynomial growth, the thesis follows.

**Definition 3.8.** A measure $\mu$ on $M$ is said to be dominated by the volume measure $\text{vol}$ in the large, denoted $\mu \prec_\infty \text{vol}$, if the equivalent conditions of the previous Lemma hold.

Define $J_{0+}$ as the set of positive locally trace class operators $T$, such that $\mu_T \prec_\infty \text{vol}$, and set $c_T := \limsup_{r \to \infty} \frac{\mu_T(B(x_0, r))}{V(x_0, r)}$.

**Lemma 3.9.** $J_{0+}$ is a hereditary (positive) cone in $\mathcal{B}(L^2(M))$.

**Proof.** If $T \in J_{0+}$, and $0 \leq A \leq T$, then $Tr(AB^*) \leq Tr(BT^*)$, for any $B \in \mathcal{B}(L^2(M))$, and the thesis follows.

Let $\omega$ be a translationally invariant state on $L^\infty([0, \infty))$, and consider the functional $\varphi$ on $\mathcal{B}(L^2(M))_+$ given by

$$\varphi(A) := \begin{cases} \omega \left( \frac{\mu_A(B(x_0, r))}{V(x, r)} \right) & A \in J_{0+} \\ +\infty & A \in \mathcal{B}(L^2(M))_+ \setminus J_{0+} \end{cases}$$

Observe that the functional $\varphi$ is very similar to the functional defined by J. Roe in [31]. Indeed regular polynomial growth implies that $\{B(x, kr)\}_{k \in \mathbb{N}}$ is a regular exhaustion according to [31]. The further hypothesis that $\omega$ is translationally invariant will play a crucial role in our construction.
Proposition 3.10.
(i) $\omega$ is a generalized limit on $[0, \infty)$
(ii) $\varphi$ is a weight on $B(L^2(M))$
(iii) $\varphi$ does not depend on $x \in M$.

Proof. (i) Since $\omega(1) = 1$, $\omega$ vanishes on compact support functions, therefore on $C_0([0, \infty))$, by continuity. Hence, when it exists, $\lim_{t \to \infty} f(t) = \omega(f)$, for $f \in C_0([0, \infty))$.

(ii) Positivity of $\varphi$ is obvious, while linearity follows from Lemma 3.9 and the observation that $\mu_A + \mu_B = \mu_A + \mu_B$.

(iii) Let $x, y \in M$, $\delta := \delta(x, y)$. Then, for any $A \in J_{0+}$

$$\mu_A(B(x, r)) \leq \mu_A(B(y, r + \delta)) \leq \mu_A(B(x, r + 2\delta)),$$

from which it follows

$$\varphi_x(A) := \omega \left( \frac{\mu_A(B(x, r))}{V(x, r)} \right) \leq \omega \left( \frac{\mu_A(B(y, r + \delta))}{V(y, r + \delta)} \right) \leq \omega \left( \frac{\mu_A(B(x, r + 2\delta))}{V(x, r + 2\delta)} \right) \leq \omega \left( \frac{\mu_A(B(y, r + \delta))}{V(y, r + \delta)} \right) \leq \omega \left( \frac{V(y, r + \delta)}{V(x, r)} - 1 \right) \leq \omega \left( \frac{V(y, r + \delta)}{V(x, r)} \right) \leq \omega \left( \frac{V(x, r + 2\delta)}{V(x, r)} \right).$$

Since $\omega$ is a generalized limit, we have

$$\left| \omega \left( \frac{\mu_A(B(y, r + \delta))}{V(y, r + \delta)} \right) \right| \leq \limsup_{r \to \infty} \frac{\mu_A(B(y, r + \delta))}{V(y, r + \delta)} \left( \frac{V(y, r + \delta)}{V(x, r)} - 1 \right).$$

Because of regular polynomial growth

$$1 \leq \lim_{r \to \infty} \frac{V(y, r + \delta)}{V(x, r)} \leq \lim_{r \to \infty} \frac{V(x, r + 2\delta)}{V(x, r)} = 1.$$

Therefore, by translation invariance,

$$\omega \left( \frac{\mu_A(B(y, r + \delta))}{V(x, r)} \right) = \omega \left( \frac{\mu_A(B(y, r + \delta))}{V(y, r + \delta)} \right) = \omega \left( \frac{\mu_A(B(y, r))}{V(y, r)} \right) = \varphi_y(A).$$

Analogically we show that

$$\omega \left( \frac{\mu_A(B(x, r + 2\delta))}{V(x, r)} \right) = \omega \left( \frac{\mu_A(B(x, r))}{V(x, r)} \right) = \varphi_x(A).$$

Then inequalities in (3.1) read $\varphi_x(A) \leq \varphi_y(A) \leq \varphi_x(A)$, and the thesis follows. \qed
The algebra $A$, being a $C^*$-algebra, contains many unitary operators, and is indeed generated by them. The algebra $A_0$ may not, but all unitaries in $A$ may be approximated by elements in $A_0$. Such approximants are $\delta$-unitaries, according to the following

**Definition 3.11.** An operator $U \in \mathcal{B}(L^2(M))$ is called $\delta$-unitary, $\delta > 0$, if $\|U^*U - 1\| < \delta$, and $\|UU^* - 1\| < \delta$.

Let us denote with $U_\delta$ the set of $\delta$-unitaries in $A_0$ and observe that, if $\delta < 1$, $U_\delta$ consists of invertible operators, and $U \in U_\delta$ implies $U^{-1} \in U_\delta/(1-\delta)$.

**Proposition 3.12.** The weight $\varphi$ is $\varepsilon$-invariant for $\delta$-unitaries in $A_0$, namely, for any $\varepsilon \in (0,1)$, there is $\delta > 0$ s.t., for any $U \in U_\delta$, and $A \in A_+$,

$$(1 - \varepsilon)\varphi(A) \leq \varphi(UAU^*) \leq (1 + \varepsilon)\varphi(A).$$

**Lemma 3.13.** If $T \in \mathcal{J}_{0+}$, then $ATA^* \in \mathcal{J}_{0+}$ for all $A \in A_0$.

**Proof.** For any $B := B_M(x,r)$ we have

$$\frac{\mu_{ATA^*}(B)}{V(x,r)} = \frac{\text{Tr}(E_BATA^*E_B)}{V(x,r)} = \frac{\text{Tr}(E_BAE_B(x,r+u(A))TE_B(x,r+u(A))A^*E_B)}{V(x,r)} \leq \|A^*E_BA\| \frac{\text{Tr}(E_B(x,r+u(A))TE_B(x,r+u(A)))}{V(x,r)} \frac{V(x,r+u(A))}{V(x,r)} \leq \|A\|^2 \frac{V(x,r+u(A))}{V(x,r)}$$

and, from regular polynomial growth, we get the thesis $\mu_{ATA^*} \prec \infty \text{vol}$.

**Proof of Proposition 3.12.** As in the proof of the previous Lemma we get

$$\varphi(UAU^*) \leq \|U\|^2 \omega \left( \frac{\mu_A(B_M(x,r+u(U)))}{V(x,r+u(U))} \right) \frac{V(x,r+u(U))}{V(x,r)}$$

which gives, as in the proof of Proposition 3.10(iii),

$$\varphi(UAU^*) \leq \|U\|^2 \varphi(A).$$

Choose now $\delta < \varepsilon/2$, and $U \in U_\delta$, so that $U^{-1} \in U_{2\delta}$, and $\varphi(UAU^*) \leq (1 + \delta)\varphi(A) < (1 + \varepsilon)\varphi(A)$. Replacing $A$ with $UAU^*$, and $U$ with $U^{-1}$, we obtain

$$\varphi(A) \leq \|U^{-1}\|^2 \varphi(UAU^*) \leq (1 + 2\delta)\varphi(UAU^*) < (1 + \varepsilon)\varphi(UAU^*)$$

and the thesis easily follows.

Finally we observe that, from the proof of Lemma 3.13 the following is immediately obtained
Lemma 3.14. If $A \in A_0$ and $\|A\| \leq 1$, then $\varphi(ATA^*) \leq \varphi(T)$, for any $T \in A_{0+}$.

Remark 3.15. The use of a translation invariant state $\omega$ is the first regularization w.r.t. the original Roe’s procedure. The request of some invariance on $\omega$ closely recalls Dixmier traces versus Varga traces (see [16, 42]), where the invariance requirement yields a larger domain for the singular trace. With this choice we get a much stronger property then the trace property in [36], namely bimodule-trace property. Indeed, our $\varepsilon$-invariance for $\delta$-unitaries obviously implies invariance under conjugation with unitaries in $A_0$. In order to get a trace on $A$, we need one more regularization, which makes $\varphi|_A$ a semicontinuous trace on $A$. This procedure will be discussed in the next subsection.

3.3 A construction of semicontinuous traces on C*-algebras

In this subsection we consider a unital C*-algebra $A$, with a dense *-subalgebra $A_0$. First we observe that, with each weight on $A$, namely a functional $\varphi_0 : A_+ \to [0, \infty]$, satisfying the property $\varphi_0(\lambda A + B) = \lambda \varphi_0(A) + \varphi(B)$, $\lambda > 0$, $A, B \in A_+$, we may associate a (lower-)semicontinuous weight $\varphi$ with the following procedure

$$\varphi(A) := \sup \{ \psi(A) : \psi \in A_+^*, \psi \leq \varphi_0 \} \quad (3.2)$$

Indeed, it is known that $\varphi(A) = \sup_{\psi \in \mathcal{F}(\varphi_0)} \psi(A)$

where $\mathcal{F}(\varphi_0) := \{ \psi \in A_+^* : \exists \varepsilon > 0, (1 + \varepsilon)\psi < \varphi_0 \}$. Moreover the following holds

Theorem 3.16. The set $\mathcal{F}(\varphi_0)$ is directed, namely, for any $\psi_1, \psi_2 \in \mathcal{F}(\varphi_0)$, there is $\psi \in \mathcal{F}(\varphi_0)$, s.t. $\psi_1, \psi_2 \leq \psi$.

From this theorem easily follows

Corollary 3.17. Let $\varphi_0$ be a weight on the C*-algebra $A$, and $\varphi$ be defined as in (3.3). Then

(i) $\varphi$ is a semicontinuous weight on $A$

(ii) $\varphi = \varphi_0$ iff $\varphi_0$ is semicontinuous.

The weight $\varphi$ will be called the semicontinuous regularization of $\varphi_0$.

Proof. (i) From Theorem 3.10, $\varphi(A) = \sup_{\psi \in \mathcal{F}(\varphi_0)} \psi(A) = \lim_{\psi \in \mathcal{F}(\varphi_0)} \psi(A)$, whence linearity and semicontinuity of $\varphi$ easily follow.

(ii) is a well known result by Combes [6].
Proposition 3.18. Let $\tau_0$ be a weight on $A$ which is $\varepsilon$-invariant by $\delta$-unitaries in $A_0$ (as in Proposition 3.13). Then the associated semicontinuous weight $\tau$ satisfies the same property.

Proof. Fix $\varepsilon < 1$ and choose $\delta \in (0, 1/2)$, s.t. $U \in \mathcal{U}_d$ implies $|\tau_0(UAU^*) - \tau_0(A)| < \varepsilon \tau_0(A)$, $A \in A_+$. Then, for any $U \in \mathcal{U}_{d/2}$, so that $U^{-1} \in \mathcal{U}_d$, and any $\psi \in A^*_+$, $\psi \leq \tau_0$, we get

$$\psi \circ \ad(U) \leq \tau_0(UAU^*) \leq (1 + \varepsilon)\tau_0(A),$$

for $A \in A_+$, i.e. $(1 + \varepsilon)^{-1}\psi \circ \ad U \leq \tau_0$. Then

$$\tau(UAU^*) = (1 + \varepsilon) \sup_{\psi \leq \tau_0} (1 + \varepsilon)^{-1}\psi \circ \ad(U) \leq (1 + \varepsilon)^{-1}\psi \circ \ad(U) \leq (1 + \varepsilon)^{-1}\psi \circ \ad(U) \leq (1 + \varepsilon)\tau(A).$$

Since $U^{-1} \in \mathcal{U}_d$, replacing $U$ with $U^{-1}$ and $A$ with $UAU^*$, we get $\tau(A) \leq (1 + \varepsilon)\tau(UAU^*)$. Combining the last two inequalities, we get the result.

Proposition 3.19. The weight $\tau$ is a semicontinuous trace on $A$, namely, setting $\mathcal{J}_+ := \{A \in A_+ : \tau(A) < \infty\}$, and extending $\tau$ to the linear span $\mathcal{J}$ of $\mathcal{J}_+$, we get

(i) $\mathcal{J}$ is an ideal in $A$

(ii) $\tau(AB) = \tau(BA)$, for all $A \in \mathcal{J}$, $B \in A$.

Proof. (i) Let us prove that $\mathcal{J}_+$ is a unitary invariant face in $A_+$, and it suffices to prove that $A \in \mathcal{J}_+$ implies $U^*AU \in \mathcal{J}_+$, for all $U \in \mathcal{U}(A)$, the set of unitaries in $A$. Suppose on the contrary that there is $U \in \mathcal{U}(A)$ s.t. $\tau(UAU^*) = \infty$. Then there is $\psi \in A^*_+$, $\psi \leq \tau_0$, s.t. $\psi(UAU^*) > 2\tau(A) + 2$. Then we choose $\delta < 3$ s.t. $V \in \mathcal{U}_d$ implies $\tau(VAV^*) < 2\tau(A)$, and an operator $U_0 \in A_0$ s.t.

$$||U - U_0|| < \frac{1}{3\||A||||\psi||}. The inequalities

$$||U_0U_0^* - 1|| = ||U^*U_0U^*_0 - U^*|| \leq ||U^*U_0 - 1||||U_0^*|| + ||U_0^* - U^*|| \leq \delta$$

and analogously for $||U_0^*U_0 - 1|| < \delta$, show that $U_0 \in \mathcal{U}_d$. Then, since $|\psi(U_0AU^*_0)| - \psi(UAU^*)| \leq 3||\psi||||A||||U - U_0|| < 1$, we get

$$2\tau(A) \geq \tau(U_0AU^*_0) \geq \tau(U_0AU^*_0) - 1 \geq 2\tau(A) + 1$$

which is absurd.

(ii) We only have to show that $\tau$ is unitary invariant. Take $A \in \mathcal{J}_+$, $U \in \mathcal{U}(A)$. For any $\varepsilon > 0$ we may find a $\psi \in A^*_+$, $\psi \leq \tau_0$, s.t. $\psi(UAU^*) > \tau(UAU^*) - \varepsilon$, as, by (i), $\tau(UAU^*)$ is finite. Then, arguing as in the proof of (i), we may find $U_0 \in A_0$, so close to $U$ that

$$|\psi(U_0AU^*_0) - \psi(UAU^*)| < \varepsilon$$

and

$$(1 - \varepsilon)\tau(A) \leq \tau(U_0AU^*_0) \leq (1 + \varepsilon)\tau(A).$$
Then
\[ \tau(A) \geq \frac{1}{1 + \varepsilon} \tau(U_0 A U_0^*) \geq \frac{1}{1 + \varepsilon} \psi(U_0 A U_0^*) \geq \frac{1}{1 + \varepsilon} (\psi(U A U^*) - \varepsilon) \geq \frac{1}{1 + \varepsilon} (\tau(U A U^*) - 2\varepsilon). \]

By the arbitrariness of \( \varepsilon \) we get \( \tau(A) \geq \tau(U A U^*). \) Replacing \( A \) with \( U A U^* \), we get the thesis.

The third regularization we need turns \( \tau \) into a (lower semicontinuous) semifinite trace, namely guarantees that
\[ \tau(A) = \sup \{ \tau(B) : 0 \leq B \leq A, B \in \mathcal{J}_+ \} \]
for all \( A \in \mathcal{A}_+ \). This regularization is well known (see e.g. [17], Section 6), and amounts to represent \( \mathcal{A} \) via the GNS representation \( \pi \) induced by \( \tau \), define a normal semifinite faithful trace \( \text{tr} \) on \( \pi'(\mathcal{A}) \), and finally pull it back on \( \mathcal{A} \), that is \( \text{tr} \circ \pi \). It turns out that \( \text{tr} \circ \pi \) is (lower semicontinuous and) semifinite on \( \mathcal{A} \), \( \text{tr} \circ \pi \leq \tau \), and \( \text{tr} \circ \pi(A) = \tau(A) \) for all \( A \in \mathcal{J}_+ \), that is \( \text{tr} \circ \pi \) is a semifinite extension of \( \tau \), and \( \text{tr} \circ \pi = \tau \) iff \( \tau \) is semifinite.

We still denote by \( \tau \) its semifinite extension. As follows from the construction, semicontinuous semifinite traces are exactly those of the form \( \text{tr} \circ \pi \), where \( \pi \) is a tracial representation, and \( \text{tr} \) is a n.s.f. trace on \( \pi'(\mathcal{A}) \).

### 3.4 The regularized trace on the C*-algebra of almost local operators

Now we apply the regularization procedure described in the previous subsection to Roe’s functional. First we observe that \( \tau_0 := \varphi|_\mathcal{A} \) is not semicontinuous.

**Proposition 3.20.** The set \( \mathcal{N}_0 := \{ T \in \mathcal{A}_+ : \tau_0(T) = 0 \} \) is not closed. In particular, there are operators \( T \in \mathcal{A}_+ \) s.t. \( \tau_0(T) = 1 \) but \( \tau(T) = 0 \) for any (lower-)semicontinuous trace \( \tau \) dominated by \( \tau_0 \).

**Proof.** Recall from Theorem 2.2(i) that there are positive real functions \( \beta_1, \beta_2 \) s.t. \( 0 < \beta_1(r) \leq V(x, r) \leq \beta_2(r) \), for all \( x \in M, r > 0 \), and \( \lim_{r \to 0} \beta_2(r) = 0 \). Therefore we can find a sequence \( r_n \searrow 0 \) s.t. \( \sum_{n=1}^{\infty} \beta_2(r_n) < \infty \). Fix \( o \in M \), and set \( X_n := \{(x_1, x_2) \in M \times M : n \leq \delta(x_1, o) \leq n + 1, \delta(x_1, x_2) \leq r_n \} \), \( Y_n := \bigcup_{k=1}^{n} X_k \), \( n \leq \infty \), and finally let \( T_n \) be the integral operator whose kernel, denoted \( k_n \), is the characteristic function of \( Y_n \). Since \( k_n \) has compact support, if \( n < \infty \), \( \tau_0(T_n) = 0 \). On the contrary, since \( Y_\infty \) contains the diagonal of
$M \times M$, clearly $\tau_0(T_\infty) = 1$. Finally

$$
\|T_\infty - T_n\| \leq \sup_{x \in M} \int_M \chi_{\cup_{k=n+1}^{\infty} X_k}(x, y) dy 
\leq \sup_{x \in M} \sum_{k=n+1}^{\infty} \int_M \chi_{X_k}(x, y) dy 
\leq \sup_{x \in M} \sum_{k=n+1}^{\infty} V(x, r_k) 
\leq \sum_{k=n+1}^{\infty} \beta_2(r_k) \to 0.
$$

This proves both the assertions. \qed

Finally we give a sufficient criterion for a positive operator $T$ to satisfy $\tau_0(T) = \tau(T)$, where $\tau$ is the semicontinuous semifinite regularization described in the previous subsection.

**Proposition 3.21.** Let $A \in J_{0+}$ be an integral operator, whose kernel $a(x, y)$ is a uniformly continuous function in a neighborhood of the diagonal in $M \times M$, namely

$$
\forall \varepsilon > 0, \exists \delta > 0 : \delta(x, y) < \delta \Rightarrow |a(x, y) - a(x, x)| < \varepsilon \quad (3.3)
$$

Then $\tau_0(A) = \tau(A)$.

**Proof.** Consider first a family of integral operators $B_\delta$, with kernels

$$
b_\delta(x, y) := \frac{\beta_1(\delta)}{\beta_2(\delta)} \chi_{\Delta_\delta}(x, y) V(x, \delta),
$$

where $\Delta_\delta := \{(x, y) \in M \times M : \delta(x, y) \leq \delta\}$. Then $\sup_{x \in M} \int_M b_\delta(x, y) dy = \frac{\beta_1(\delta)}{\beta_2(\delta)} \leq 1$, and $\sup_{y \in M} \int_M b_\delta(x, y) dx \leq \frac{\sup_{x \in M} V(y, \delta)}{\beta_2(\delta)} \leq 1$, which imply $\|B_\delta\| \leq 1$, by Riesz-Thorin theorem.

Fix $o \in M$, set $E_r$ for the multiplication operator by the characteristic function of $B(o, r)$, and observe that

$$
Tr(E_r B_\delta B_\delta^* E_r) = \int_{B(o, r)} dx \int_M b_\delta(x, y)^2 dy 
\leq \frac{\beta_1(\delta)}{\beta_2(\delta)} V(o, r) \leq \frac{V(o, r)}{\beta_2(\delta)}
$$

Therefore $\tau_0(B_\delta B_\delta^*) \leq \beta_2(\delta)^{-1}$. This implies that $\psi_\delta := \tau_0(B_\delta \cdot B_\delta^*)$ belongs to $A_{\tau}$, and $\psi_\delta \leq \tau_0$ by Lemma 3.14. By the results of the previous subsection, we
have $\psi_\delta(A) \leq \tau(A) \leq \tau_0(A)$, for any $A \in A_+$. Take now $A \in A_+$ satisfying (3.3), for a pair $\varepsilon > 0$, $\delta > 0$, and, setting $\beta(\delta) := (\frac{\psi_\delta(\delta)}{\tau_0(\delta)})^2$ to improve readability, compute

$$|\text{Tr}(E_r B_\delta AB_\delta^* E_r) - \text{Tr}(E_r AE_r)|$$

$$\leq |\text{Tr}(E_r B_\delta AB_\delta^* E_r) - \beta(\delta)\text{Tr}(E_r AE_r)| + (1 - \beta(\delta))\text{Tr}(E_r AE_r)$$

$$\leq \int_{B(o,r)} dx \int_{B(x,\delta) \times B(x,\delta)} b_\delta(x,y)a(y,z) - a(x,x)b_\delta(x,z) dydz$$

$$+ (1 - \beta(\delta))\text{Tr}(E_r AE_r)$$

$$\leq 3\varepsilon \int_{B(o,r)} dx \int_{B(x,\delta) \times B(x,\delta)} b_\delta(x,y)b_\delta(x,z) dydz$$

$$+ (1 - \beta(\delta))\text{Tr}(E_r AE_r)$$

$$\leq 3\varepsilon \beta(\delta)V(o,r) + (1 - \beta(\delta))\text{Tr}(E_r AE_r)$$

This implies $|\psi_\delta(A) - \tau_0(A)| \leq 3\varepsilon \beta(\delta) + (1 - \beta(\delta))\tau_0(A)$. By the arbitrariness of $\varepsilon$ and Theorem 2.2(ii), we get the thesis. 

**Corollary 3.22.** For any $t > 0$ $\tau_0(e^{-t\Delta}) = \tau(e^{-t\Delta})$, where $\Delta$ is Laplace-Beltrami operator.

**Proof.** Follows from Propositions 2.5 and 3.21.

---

4 Singular traces for open manifolds

4.1 Singular traces on C*-algebras

In this subsection we shall briefly recall how to construct type II$_1$ singular traces on a C*-algebra with a semicontinuous semifinite trace, as is treated in [23]. As it is known [17], if $\tau$ is a semicontinuous semifinite trace on a C*-algebra $A$ and $\pi_{\tau}$ denotes the GNS representation, there is a normal semifinite faithful trace on $M := \pi_{\tau}(A)'$ (which we still denote by $\tau$) such that $\tau = \tau \cdot \pi_{\tau}$. The main problem is that while type I$_\infty$ singular traces (like Dixmier traces, see [16, 1]) are defined on suitable ideals of a semifinite von Neumann algebra $M$, and therefore they can be “pulled back” via $\pi_{\tau}$ on $A$, type II$_1$ singular traces, which are needed here, are defined on bimodules of measurable operators affiliated to $M$. Then we need a notion of operator affiliated to $A$ that allows us to construct an *-bimodule over $A$ of such operators and a trace on it. Moreover we need to extend $\pi_{\tau}$ to such a bimodule, this extension taking values in the measurable operators affiliated to $M$, and then “pull back” the singular traces as before. Indeed we shall see that measurable operators affiliated to $A$ form what we may call a $\tau$ almost everywhere bimodule, in the sense that the usual
bimodule properties will hold only up to a zero trace projection, provided that operations are intended in a strong sense, as in [39]. From the technical point of view, we make use of the ideas of Segal [39], Nelson [29] and Christensen [5] on noncommutative integration, adapting them to the case of C*-algebras.

The main problem will be the possible absence of enough projections in $A$, to carry out Christensen construction, and therefore we shall construct a $\ast$-algebra containing $A$ on which the trace naturally extends and with enough projections in it. In the following $(A, \tau)$ will be a norm closed, unital $\ast$-algebra of operators acting on a Hilbert space $H$ together with a semicontinuous semifinite trace.

**Definition 4.1.** We say that a projection $e \in A''$ is essentially clopen (w.r.t. $(A, \tau)$) if for all $\varepsilon > 0$, exist $a_-, a_+ \in A$ s.t.

\[
0 \leq a_- \leq e \leq a_+ \leq 1, \quad \tau(a_+ - a_-) < \varepsilon.
\]

We shall denote by $E$ the class of $\tau$-finite essentially clopen projections.

**Proposition 4.2.** [23] The set $E$ with the operations $\lor$, $\land$ is a lattice.

**Theorem 4.3.** [23] There exists a $\ast$-subalgebra $C$ of $A''$ with the following properties

(i) $A \cup E \subset C$

(ii) If $x \in C$, for any $\varepsilon > 0$ there exist $a_-, a_+ \in A$ such that $a_- < x < a_+$ and $\tau(a_+ - a_-) < \varepsilon$.

(ii) The GNS representation $\pi_\tau$ extends to a $\ast$-homomorphism (still denoted by $\pi_\tau$) of $C$ to $\pi_\tau(A)''$.

According to the preceding theorem $C$ is equipped with a (positive) trace, still denoted by $\tau$, given by the pull back of the trace on $\pi_\tau(A)''$, which is the unique extension of the trace on $A$. The construction of $C$ is rather involved, indeed its elements are not explicitly characterized, while its definition resembles that of the enveloping Borel algebra in [32], therefore we shall not describe it here. Now we pass to the definition of affiliated operators.

**Definition 4.4.** A sequence $\{e_n\}$ of essentially clopen projections is called a **Strongly Dense Domain (SDD)** if $e_n$ is $\tau$-finite and $\tau(e_n) \to 0$. We shall denote by $e$ the projection sup$_n e_n$.

Let us remark that, since the trace $\tau$ is not faithful, $e$ is not necessarily 1. Nevertheless it is easy to show that $e^\perp \in E$ and $\tau(e) = 0$. Now let us consider a linear operator $T$ acting on $H$. If $T$ is neither densely defined nor closed then its adjoint is a closed operator from a proper subspace $K_1$ to another proper subspace $K_2$ of $H$. We shall denote by $T^+$ the closed, densely defined operator given by $T^+|_{K_1} = i_2 \cdot T^*$, where $i_2$ is the embedding of $K_2$ into $H$, and by $T^+|_{K_1^\perp} = 0$. Then we denote by $T^a$ the closed densely defined operator $(T^+)^*$. Let us recall that an operator $T$ on $H$ is said to be **affiliated** to a von Neumann algebra $M$ $(T \in M)$ if all elements of $x \in M'$ send its domain into itself and $Tx\eta = xT\eta$, for any $\eta$ in $D(T)$.
Definition 4.5. We call $\tilde{C}$ the family of closed, densely defined operators affiliated to $A''$ for which there exists a SDD $\{e_n\}$ such that

(i) $e_nH \subset \mathcal{D}(T)$
(ii) $e_nH \subset \mathcal{D}(T^*)$
(iii) $Te_n \in C$
(iv) $T^*e_n \in C$.

If $T, S \in \tilde{C}$, $a \in A$, we consider the following (strong sense) operations

$T \oplus S := (T + S)^\natural$, \quad $a \odot T := (a \cdot T)^\natural$, \quad $T \odot a := (T \cdot a)^\natural$.

We also introduce the relation of $\tau$-a.e. equivalence, which turns out to be an equivalence relation, among operators in $\tilde{C}$, namely $T$ is equivalent to $S \tau$-a.e. if there exists a common SDD $\{e_n\}$ for $T$ and $S$ such that, setting $H_0 := \cup_n e_n H$, we have $e_T|_{H_0} = e_S|_{H_0}$. We remark that, while this relation may appear too weak, it becomes an equality as soon as the trace is faithful on $\tilde{C}$. In fact strong sense operations too become the usual strong sense operations defined by Segal in the case of a faithful trace on a von Neumann algebra, therefore, as follows by next theorem, the class of operators in $\tilde{C}$ which are $0$ a.e. are in the kernel of the extension of the GNS representation $\pi_\tau$. In the following we shall denote by $\pi$ the GNS representation of $A$ associated with the trace $\tau$, by $M$ the von Neumann algebra $\pi(A)'$, and by $\tilde{M}$ the algebra of measurable operators affiliated to $M$.

Theorem 4.6. \[23\] The set $\tilde{C}$ is closed under strong sense operations, and the usual properties of a $^*$-bimodule over $A$ hold $\tau$-almost everywhere. Moreover the GNS representation extends to a map from $\tilde{C}$ to $\tilde{M}$ which preserves strong sense operations.

Let us recall that, if $A \in \tilde{M}$, its distribution function and non-decreasing rearrangement are defined as follows (cf. e.g. [19, 22])

$\lambda_A(t) := \tau(\chi_{(t, +\infty)}(|A|))$
$\mu_A(t) := \inf\{s \geq 0 : \lambda_A(s) \leq t\}$.

We may define the distribution function (and therefore the associated non-decreasing rearrangement) w.r.t. $\tau$ of an operator $A \in \tilde{C}$ as $\lambda_A(t) = \lambda_{\pi(A)}(t)$, and we get $\mu_A = \mu_{\pi(A)}$. Then the preimage $\tilde{C} \subset \tilde{C}$ under $\pi$ of the set $\tilde{M} := \{A \in \tilde{M} : \lambda_A(t_0) < \infty \text{ for some } t_0 > 0\}$ is an a.e. $^*$-bimodule over $A$. Let us observe that, if $A \in \tilde{C}$ is a positive (unbounded) continuous functional calculus of an element in $A$, then $\chi_{(t, +\infty)}(A)$ belongs to $\mathcal{E}$ a.e., therefore its distribution function may be defined without using the representation $\pi$

$\lambda_A(t) = \tau(\chi_{(t, +\infty)}(A))$.

We may carry out the construction of singular traces (with respect to $\tau$) as it has been done in [22]. However, since only type $\Pi_1$ traces will be used in the following, we shall restrict to this case, which corresponds to eccentricity at 0.
Definition 4.7. An operator $T \in \mathcal{C}$ is called eccentric (at 0) if either

$$\int_0^1 \mu_T(t) < \infty \quad \text{and} \quad \limsup_{t \to 0} \frac{\int_0^t \mu_T(s)ds}{\int_0^{2t} \mu_T(s)ds} = 1$$

or

$$\int_0^1 \mu_T(t) = \infty \quad \text{and} \quad \liminf_{t \to 0} \frac{\int_0^t \mu_T(s)ds}{\int_0^{2t} \mu_T(s)ds} = 1.$$ 

The following proposition trivially holds

Proposition 4.8. Let $(\mathcal{A}, \tau)$ be a $C^*$-algebra with a semicontinuous semifinite trace, $\pi$ the associated GNS representation, $T \in \mathcal{C}$, and let $X(T)$ denote the *-bimodule over $\mathcal{A}$ generated by $T$ in $\mathcal{C}$, while $X(\pi(T))$ denotes the *-bimodule over $\mathcal{M}$ generated by $\pi(T)$ in $\mathcal{M}$. Then

(i) $T$ is eccentric if and only if $\pi(T)$ is eccentric.

(ii) $\pi(X(T)) \subset X(\pi(T)).$

As in the case of von Neumann algebras, with any eccentric operator (at 0) in $\mathcal{C}$ we may associate a singular trace, where the word singular refers to the original trace $\tau$. Indeed such singular trace will vanish on bounded operators. Of course singular traces may be described as the pull-back of the singular traces on $\mathcal{M}$ via the (extended) GNS representation. On the other hand, explicit formulas may be written in terms of the non decreasing rearrangement. We write these formulas for the sake of completeness. First we observe that, by definition, if $T \in \mathcal{C}$ is eccentric (at 0) there exists a pure state $\omega$ on $\mathcal{C}_b(0, \infty)$ which is a generalized limit in 0, namely is an extension of the Dirac delta in 0 on $C[0, \infty)$ to $\mathcal{C}_b(0, \infty)$, such that

if $\int_0^1 \mu_T(t) < \infty$ then $\omega \left( \frac{\int_0^t \mu_A(s)ds}{\int_0^{2t} \mu_T(s)ds} \right) = 1$

if $\int_0^1 \mu_T(t) = \infty$ then $\omega \left( \frac{\int_0^t \mu_A(s)ds}{\int_0^{2t} \mu_T(s)ds} \right) = 1.$

Then the singular trace associated with $\tau$, $T$ and $\omega$ may be written as follows on the a.e. positive elements of $X(T)$, i.e. elements whose image under $\pi$ is positive

$$\int_0^1 \mu_T(t) < \infty \quad \Rightarrow \quad \tau_\omega(A) := \omega \left( \frac{\int_0^t \mu_A(s)ds}{\int_0^{2t} \mu_T(s)ds} \right), \quad A \in X(T)_+$$

(4.1)

$$\int_0^1 \mu_T(t) = \infty \quad \Rightarrow \quad \tau_\omega(A) := \omega \left( \frac{\int_0^t \mu_A(s)ds}{\int_0^{2t} \mu_T(s)ds} \right), \quad A \in X(T)_+$$

According to the previous analysis, some results in [22] may be rephrased as follows
**Theorem 4.9.** The functionals defined in formula \((4.1)\) extend to traces on the a.e. \(*\)-bimodule over \(\mathcal{A} X(T)\). They also naturally extend to traces on \(X(T) + \mathcal{A}\).

Now, for any \(T \in \mathcal{C}\), we set
\[
\alpha(T) := \left( \lim \inf_{t \to 0} \frac{\log \mu_T(t)}{\log \frac{1}{t}} \right)^{-1}
\]

As we shall see in the following, this number may be considered as a generalized Novikov-Shubin invariant of \(T\). A sufficient condition for being singularly traceable (at 0) is given in terms of this number.

**Theorem 4.10.** Let \(T \in \mathcal{C}\) with \(\alpha \equiv \alpha(T)\). If \(\alpha = 1\) then \(T\) is eccentric, hence singularly traceable. In general, if \(\alpha \in (0, \infty)\) then \(T^\alpha\) is eccentric at 0.

**Proof.** The first statement is proved in [24]. Then, by the properties of the non-increasing rearrangement, \(\mu_{T^\alpha}(t) = \mu_T(t)^\alpha\), therefore
\[
\alpha(T^\alpha) = \left( \lim \inf_{t \to 0} \frac{\log \mu_{T^\alpha}(t)}{\log \frac{1}{t}} \right)^{-1} = \left( \lim \inf_{t \to 0} \frac{\log(\mu_T(t))^\alpha}{\log \frac{1}{t}} \right)^{-1}
= \left( \alpha \lim \inf_{t \to 0} \frac{\log \mu_T(t)}{\log \frac{1}{t}} \right)^{-1} = 1
\]

\[\square\]

### 4.2 A singular trace associated with the Laplacian

In this subsection we consider an open manifold with bounded geometry and regular polynomial growth, i.e. the same hypotheses assumed in section 3.

**Theorem 4.11.** Let \(M\) be an open manifold with bounded geometry and regular polynomial growth. Then
\[
\alpha(\Delta^{-1}) = \lim \sup_{t \to 0} \frac{\log \tau(e_\Delta(t))}{\log t} = \lim \sup_{t \to \infty} \frac{\log \tau(e^{-t\Delta})}{\log \frac{1}{t}}
\]
where \(e_\Delta\) denotes the spectral family of \(\Delta\).

We need the following Lemma

**Lemma 4.12.** Let \(\lambda : \mathbb{R}_+ \to \mathbb{R}_+\) be a non-increasing, right continuous function, \(\mu(t) := \inf\{s \geq 0 : \lambda(s) \leq t\}\). Then
\[
\left( \lim \inf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}} \right)^{-1} = \lim \sup_{s \to \infty} \frac{\log \lambda(s)}{\log \frac{1}{s}}
\]
where the values 0 and \(\infty\) are allowed.
Proof. First recall that \( \mu \) is non increasing and right continuous and that \( \lambda(t) \equiv \inf\{s \geq 0 : \mu(s) \leq t\} \). Then let \( t_n \to 0 \) be a sequence such that \( \lim_{n \to \infty} \frac{\log \mu(t_n)}{\log t_n} = \lim_{t \to 0} \frac{\log \mu(t)}{\log t} \), and let \( t'_n := \inf\{s \geq 0 : \mu(s) = \mu(t_n)\} = \min\{s \geq 0 : \mu(s) = \mu(t_n)\} \) where the last equality holds because of right continuity. Then

\[
\lim_{t \to 0} \frac{\log \mu(t)}{\log t} \leq \lim_{n \to \infty} \frac{\log \mu(t'_n)}{\log t'_n} \leq \lim_{n \to \infty} \frac{\log \mu(t_n)}{\log t_n} = \lim_{t \to 0} \frac{\log \mu(t)}{\log t}
\]

namely we may replace \( t_n \) with \( t'_n \) to reach the lim inf. Also, \( \lambda(\mu(t'_n)) = \inf\{t \geq 0 : \mu(t) \leq \mu(t'_n)\} = t'_n \), therefore

\[
\lim_{t \to 0} \frac{\log \mu(t)}{\log t} = \lim_{n \to \infty} \frac{\log \mu(t'_n)}{\log t'_n} = \lim_{n \to \infty} \frac{\log \mu(t'_n)}{\log \lambda(\mu(t'_n))} \geq \lim_{s \to 0} \frac{\log s}{\log \lambda(s)} = \left( \limsup_{s \to 0} \frac{\log \lambda(s)}{\log s} \right)^{-1}
\]

For the converse inequality, let \( s_n \to \infty \) be a sequence for which \( \lim_{n \to \infty} \frac{\log \lambda(s_n)}{\log s_n} = \limsup_{s \to 0} \frac{\log \lambda(s)}{\log s} \). As before, \( s'_n := \inf\{s \geq 0 : \lambda(s) = \lambda(s_n)\} = \min\{s \geq 0 : \lambda(s) = \lambda(s_n)\} \) still brings to the lim sup and verifies \( \mu(\lambda(s'_n)) = s'_n \). \( \Box \)

Proof of Theorem 4.11. First we prove the first equality

\[
\alpha(\Delta^{-1}) = \left( \liminf_{t \to 0} \frac{\log \mu(\Delta^{-1}(t))}{\log t} \right)^{-1} = \limsup_{s \to \infty} \frac{\log \lambda(\Delta^{-1}(s))}{\log s} \leq \limsup_{s \to \infty} \frac{\log \tau(\chi_{(-\infty,0)}(\Delta^{-1}(s)))}{\log s}
\]

where the second equality follows by Lemma 4.12. For the last equality let us set, in analogy with [21], \( N(\lambda) := \tau(e_{\Delta}(\lambda)), \vartheta(t) = \tau(e^{-t\Delta}) \). Then it follows that \( \vartheta \) is the Laplace transform of the Stieltjes measure defined by \( N(\lambda) \)

\[
\vartheta(t) = \int e^{-t\lambda} dN(\lambda),
\]

and the last equality follows by the Tauberian theorem contained in the appendix of [21], provided that we show that \( \vartheta(t) = O(t^{-\delta}) \) for some \( \delta > 0 \).

On the other hand, under the assumptions of bounded geometry, Varopoulos [13] proved that the heat kernel on the diagonal has a uniform inverse-polynomial bound, more precisely, in the strongest form due to [4], we have

\[
\sup_{x,y \in M} p_t(x,y) \leq Ct^{-1/2}
\]
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for a suitable constant $C$. Then

$$\vartheta(t) = \tau(e^{-t\Delta}) = (\text{gen}) \lim_{r \to \infty} \frac{\int_{B(o,r)} p_t(x,x) \, dvol(x)}{V(o,r)} \leq Ct^{-1/2}$$

which concludes the proof. \qed

**Definition 4.13.** Let $M$ be an open manifold with bounded geometry and regular polynomial growth. Then the (0-th) Novikov-Shubin invariant of $M$ is defined as

$$\alpha_0(M) = 2\alpha(\Delta^{-1}) = 2 \limsup_{t \to 0} \frac{\log(N(t))}{\log t} = 2 \limsup_{t \to \infty} \frac{\log(\vartheta(t))}{\log \frac{1}{t}}. \quad (4.2)$$

**Remark 4.14.** We have chosen J. Lott’s normalization [27] for the Novikov-Shubin number $\alpha_0(M)$ because Laplace operator is a second order differential operator, and this normalization gives the equality between $\alpha_0(M)$ and the asymptotic dimension of $M$, cf. Theorem 4.18.

Our choice of the $\limsup$ in (4.2), in contrast with J. Lott’s choice [27], is motivated by our interpretation of $\alpha_0(M)$ as a dimension. On the one hand, it is compatible with the classical properties of a dimension as stated in Theorem 1.11, cf. also Remark 1.13, on the other hand, a noncommutative measure corresponds to such a dimension via a singular trace, according to Theorem 4.10, cf. [24].

**Corollary 4.15.** Let $M$ be an open manifold with bounded geometry and regular polynomial growth. Then there exists a singular trace on $\overline{\mathcal{C}}$ which is finite on the $^*$-bimodule over $\mathcal{A}$ generated by $\Delta^{-\alpha_0(M)/2}$.

**Remark 4.16.** This singular trace is the global (or asymptotic) counterpart of the Wodzicki residue, in the form of Connes, namely it is a singular trace which is finite exactly on the operators with a prescribed asymptotic behavior. Such an asymptotic behavior is that of a suitable power of the Laplace operator, i.e. that of a geometric pseudo-differential operator with a suitable order. The problem is that such an order seems to depend on the trace on $M$, which in turn depends on a dilation invariant limit procedure. Moreover, in the case of the local singular trace, such an order has a geometric meaning, is indeed the dimension of the manifold. These two questions will be completely solved in the next subsection, though under more stringent hypotheses on the manifold.

4.3 The asymptotic dimension and the 0-th Novikov Shubin invariant

In this subsection, besides bounded geometry and regular polynomial growth, we shall also assume the isoperimetric inequality which was the subject of Theorem...
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2.7, namely there are \(\alpha, \beta > 0\) s.t. for all \(x \in M, r > 0\), and all regions \(U \subset B(x, r)\), the first Dirichlet eigenvalue of \(-\Delta\) in \(U\), \(\lambda_1(U)\), satisfies

\[
\lambda_1(U) \geq \frac{\alpha}{r^2} \left( \frac{V(x, r)}{\text{vol}(U)} \right)^\beta.
\]

First we observe that in this case the volume of the balls of a given radius is uniformly bounded.

**Lemma 4.17.** If the previous hypotheses hold, then

\[
\gamma^{-1} \leq \frac{V(x, r)}{V(y, r)} \leq \gamma.
\]

**Proof.** The Lemma easily follows by a result of Grigor’yan ([20], Proposition 5.2), where it is shown that the isoperimetric inequality above implies the existence of a constant \(\gamma\) such that

\[
\gamma^{-1} \left( \frac{R}{r} \right)^{\alpha_1} \leq \frac{V(x, R)}{V(y, r)} \leq \gamma \left( \frac{R}{r} \right)^{\alpha_2}
\]

for some positive constants \(\alpha_1, \alpha_2\), for any \(R \geq r\), and \(B(x, R) \cap B(y, r) \neq \emptyset\). \(\square\)

**Theorem 4.18.** Let \(M\) be as above. Then the asymptotic dimension of \(M\) coincides with the 0-th Novikov-Shubin invariant, namely \(d_\infty(M) = \alpha_0(M)\).

**Proof.** First, from Theorem 2.7 and the previous Lemma, we get

\[
\frac{C\gamma^{-1}}{V(o, \sqrt{t})} \leq \frac{\int_{B(o, r)} C V(x, \sqrt{t}) \, d\text{vol}(x)}{V(o, r)} \leq \frac{\int_{B(o, r)} C V(x, \sqrt{t}) \, d\text{vol}(x)}{V(o, r)} \leq \frac{C'}{V(o, \sqrt{t})}
\]

therefore, by definition of the trace \(\tau\),

\[
\frac{C\gamma^{-1}}{V(o, \sqrt{t})} \leq \tau(e^{-t\Delta}) \leq \frac{C'}{V(o, \sqrt{t})}
\]

hence, finally,

\[
d_\infty(M) = 2 \limsup_{t \to \infty} \frac{\log V(o, t)}{2 \log t} = 2 \limsup_{t \to \infty} \frac{\log(C\gamma^{-1} V(o, \sqrt{t})^{-1})}{\log \frac{1}{t}} \leq \alpha_0(M) = 2 \limsup_{t \to \infty} \frac{\log \tau(e^{-t\Delta})}{\log \frac{1}{t}} \leq 2 \limsup_{t \to \infty} \frac{\log(C'\gamma V(o, \sqrt{t})^{-1})}{\log \frac{1}{t}} = 2 \limsup_{t \to \infty} \frac{\log V(o, t)}{2 \log t} = d_\infty(M)
\]

\(\square\)
Remark 4.19. On the one hand the previous result shows that the 0-th Novikov-Shubin invariant is intrinsically defined, since it coincides with a rough-isometry invariant. On the other hand, the singular trace described in the previous subsection is finite (and non trivial) on the geometric operator $\Delta^{-\alpha/2}$, namely on a pseudo-differential operator of degree $-d_\infty$. In this case too such a degree plays the role of a dimension, and more precisely coincides with the asymptotic dimension of the manifold.

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