CHARACTERIZATIONS OF GELFAND RINGS AND THEIR DUAL RINGS

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Abstract. In this paper, new criteria for the maximality of primes, pm-rings, clean rings and mp-rings are given. The equivalency of some of the classical criteria are also proved by new and simple methods. The dual notion of clean ring is defined, we call it purified ring. Then some non-trivial characterizations for purified rings are given. It is also proved that if the topology of a scheme is Hausdorff then the affine opens of that scheme is stable under taking finite unions (and nonempty finite intersections). In particular, every compact scheme is an affine scheme.

1. Introduction

This paper is devoted to study two very fascinating classes of rings which are so called pm-rings and mp-rings. A ring $A$ is said to be a pm-ring (or, Gelfand ring) if each prime ideal of $A$ is contained in a unique maximal ideal of $A$. Dually, a ring $A$ is called a mp-ring if each prime of $A$ contains a unique minimal prime of $A$. In this paper, we give new criteria for the maximality of primes, pm-rings, clean rings and mp-rings. These criteria have geometric nature and considerably simplify the proofs specially the equivalency of some of the classical criteria. In fact, this study bringing us new results such that “contributions to Theorems 3.3, 1.11 and 1.11 by providing new criteria, Corollary 3.7, Theorem 3.5, Corollary 4.6, Proposition 4.9, Theorem 5.3, Corollary 5.4, Theorem 5.5, Corollary 5.6, Proposition 5.7, Theorem 5.9, Remark 5.11 and Theorem 5.15” are amongst the most important ones.

The class of clean rings, as a subclass of pm-rings, is another amazing class of rings which is also investigated in this paper, see Theorem 4.11. Recall that a ring $A$ is called a clean ring if each element of it can be written as the sum of an idempotent and an invertible elements of that
ring. Theorem 4.11 in particular, tells us that if \( A \) is a clean ring then a system of equations \( f_i(x_1, ..., x_n) = 0 \) with \( i = 1, ..., d \) over \( A \) has a solution in \( A \) provided that this system has a solution in each local ring \( A_m \) with \( m \) a maximal ideal of \( A \). Clean rings have been extensively studied in the literature over the recent years, see e.g. [1], [6], [10], [11], [14], [18], [22] and [24]. Theorem 4.11 can be considered as the culmination and strengthen of all of these results. Then we introduce and study a new class of rings that is, purified rings. In fact, a ring \( A \) is said to be a purified ring if for every distinct minimal primes \( p \) and \( q \) of \( A \) then there exists an idempotent \( e \in A \) such that \( e \in p \) and \( 1 - e \in q \). Purified rings as we expected, like as clean rings, are so fascinating. In Theorem 5.15 we characterize purified rings. This result, in particular, tells us that if \( A \) is a reduced purified ring then a system of equations \( f_i(x_1, ..., x_n) = 0 \) over \( A \) has a solution in \( A \) provided that this system has a solution in each domain \( A/p \) with \( p \) a minimal prime of \( A \). This result also shows that purified rings are the dual of clean rings.

These two topics, specially pm-rings, have been the main subjects of many articles in the literature over the years and are still of current interest, see e.g. [1], [3], [4], [5], [7], [8], [9], [13], [21], [22], [23] and [26]. The beautiful article [13] can be viewed as a starting point of investigations for pm-rings in the commutative case. The article [23] is another interesting work that the category of pm-rings has been studied from a geometric point of view. In fact, in [23, Theorem 1] it is shown that the category of compact locally ringed spaces with the global section property as a full subcategory of the category of ringed spaces is anti-equivalent to the category of pm-rings.

It is a truism that the dual notions can behave very differently in algebra, for instance projective and injective modules. As we shall observe, the same is true for pm-rings and mp-rings. Indeed, every fact which holds on pm-rings can not be necessarily dualized on mp-rings and vice versa.

Mp-rings, as noted above, have been less studied in the literature than pm-rings. This may be because of that the pm-rings are tied up with the Zariski topology, see Theorem 4.3. By contrast, we show that the mp-rings are tied up with the flat topology, see Theorem 5.3. The flat topology is less known than the Zariski topology in the literature. It is worth mentioning that the flat topology behaves completely as the dual of the Zariski topology, for more details see §2 and also see [27].
Intuitively, the prime spectrum of a pm-ring can be analogized as the Alps whose the summits of the mountains are the maximal ideals, and the prime spectrum of a mp-ring can be analogized as the icicles whose the tips of the icicles are the minimal primes.

Most of the mathematicians which are involved in algebraic geometry are concerned primarily with the problem of when the underlying space of a scheme is separated (Hausdorff). Note that characterizing the separability of the Zariski topology of a scheme is not as easy to understand as one may think at first. This is because we are used to the topology of locally Hausdorff spaces, but the Zariski topology in general is not locally Hausdorff. Indeed, Corollary 3.7 and Theorem 3.5 give a complete answer to their question. In particular, it is proved that the underlying space of a separated scheme or more generally a quasi-separated scheme is Hausdorff if and only if every point of it is a closed point.

We have also found counterexamples for two claims in the literature, see Remarks 4.8 and 5.8. Consequently the results “Proposition 4.9, Corollary 4.10 and Theorem 5.9” are obtained. A mathematician must be very careful on the accuracy of the results in the literature when using them. The best way to be assured on the accuracy of the used results is to prove or read all of the arguments. In fact, in writing an article there are two major factors that the researchers should be concern on them. The first one is “the originality and correctness of own results” and the second major factor is “the accuracy of the used results”.

2. Preliminaries

Here we recall some material which is needed in the sequel.

In this paper all of the rings are commutative.

Local rings, zero dimensional rings and rings of continuous functions are typical examples of pm-rings. If $A$ is a ring then by [17, Theorem 6] there exists a ring $B$ such that the primes of $A$ have precisely the reverse order of the primes of $B$. Using this result, then $A$ is a pm-ring if and only if $B$ is a mp-ring. It is important to notice that the duality [17, Theorem 6] is just in the topological level. In general, it is not a
geometric duality (e.g. it does not say nothing on the algebraic properties of the ring \(B\)).

A morphism of rings \(\varphi : A \to B\) induces a morphism \(\theta = \text{Spec}(\varphi) : \text{Spec}(B) \to \text{Spec}(A)\) between the corresponding affine schemes where the function \(\theta\) between the underlying spaces maps each prime \(p\) of \(B\) into \(\varphi^{-1}(p)\). The map \(\theta\) sometimes is also denoted by \(\varphi^\star\).

A ring \(A\) is said to be absolutely flat (or, von-Neumann regular) if each \(A\)-module is \(A\)-flat. This is equivalent to the statement that each element \(f \in A\) can be written as \(f = f^2 g\) for some \(g \in A\). Every prime ideal of an absolutely flat ring is a maximal ideal.

Let \(A\) be a ring. Then there exists a (unique) topology over \(\text{Spec}(A)\) such that the collection of subsets \(V(f) = \{p \in \text{Spec}(A) : f \in p\}\) with \(f \in A\) forms a sub-basis for the opens of this topology. It is called the flat topology. Therefore, the collection of subsets \(V(I)\) where \(I\) runs through the set of finitely generated ideals of \(A\) forms a basis for the flat opens. In the literature, the flat topology is also called the inverse topology. Moreover there is a (unique) topology over \(\text{Spec}(A)\) such that the collection of subsets \(D(f) \cap V(g)\) with \(f, g \in A\) forms a sub-basis for the opens of this topology. It is called the patch (or, constructible) topology. It follows that the flat topology is quasi-compact. The flat topology behaves as the dual of the Zariski topology. For instance, if \(p\) is a prime ideal of \(A\) then its closure with respect to the flat topology originates from the canonical ring map \(A \to A_p\). In fact, \(\Lambda(p) = \{q \in \text{Spec}(A) : q \subseteq p\}\). Here \(\Lambda(p)\) denotes the closure of \(\{p\}\) in \(\text{Spec}(A)\) with respect to the flat topology. By contrast, the Zariski closure of this point comes from the canonical ring map \(A \to A/p\). It is proved that \(\text{Max}(A)\) is Zariski quasi-compact and flat Hausdorff. Dually, \(\text{Min}(A)\) is flat quasi-compact and Zariski Hausdorff. It is well known that the Zariski closed subsets of \(\text{Spec}(A)\) are precisely of the form \(\text{Im} \pi^\star\) where \(\pi : A \to A/I\) is the canonical ring map and \(I\) is an ideal of \(A\). One can show that the patch closed subsets of \(\text{Spec}(A)\) are precisely of the form \(\text{Im} \varphi^\star\) where \(\varphi : A \to B\) is a ring map. Moreover, the flat closed subsets of \(\text{Spec}(A)\) are precisely of the form \(\text{Im} \varphi^\star\) where \(\varphi : A \to B\) is a flat ring map. For more details see [27].
Theorem 2.1. Let \( I \) be an ideal of a ring \( A \). Then \( A/I \) is \( A \)-flat if and only if \( \text{Ann}(f) + I = A \) for all \( f \in I \).

Proposition 2.2. If each prime ideal of a ring \( A \) is a finitely generated ideal then \( A \) is a noetherian ring. In particular, every finite product of noetherian rings is a noetherian ring.

Surjective ring maps are special cases of epimorphisms of rings. As a specific example, the canonical ring map \( \mathbb{Z} \to \mathbb{Q} \) is an epimorphism of rings which is not surjective. A morphism of rings is called a flat epimorphism of rings if it is both a flat ring map and an epimorphism of rings. If \( S \) is a multiplicative subset of a ring \( A \) then the canonical morphism \( A \to S^{-1}A \) is a typical example of flat epimorphisms of rings. It is well known that if \( A \to B \) is an epimorphism of rings then the induced map \( \text{Spec}(B) \to \text{Spec}(A) \) is injective.

Proposition 2.3. If \( \varphi : A \to B \) is a flat epimorphism of rings then for each prime \( q \) of \( B \) the induced morphism \( \varphi_q : A_p \to B_q \) is an isomorphism of rings where \( p = \varphi^{-1}(q) \).

The following result is due to Grothendieck and has found interesting applications in this paper.

Theorem 2.4. The map \( f \mapsto D(f) \) is a bijection from the set of idempotents of a ring \( A \) onto the set of clopen (both open and closed) subsets of \( \text{Spec}(A) \).

Remark 2.5. If an ideal \( I \) of a ring \( A \) is generated by a set of idempotents of \( A \) then \( I \) is called a regular ideal of \( A \). Every maximal element of the set of proper regular ideals of \( A \) is called a max-regular ideal of \( A \). The set of max-regular ideals of \( A \) is called the pierce spectrum of \( A \) and denoted by \( \text{Sp}(A) \). It is a compact and totally disconnected topological space whose basis opens are of the form \( U_f = \{ M \in \text{Sp}(A) : f \notin M \} \) where \( f \) is an idempotent of \( A \), and the map \( \text{Spec}(A) \to \text{Sp}(A) \) given by \( p \mapsto (f : f \in p, f = f^2) \) is well-defined, continuous and surjective, see [27, Lemma 3.18]. It follows that \( C \) is a connected component of \( \text{Spec}(A) \) if and only if \( C = V(M) \) where \( M \) is a max-regular ideal of \( A \), see [27, Theorem 3.17]. Therefore \( \text{Sp}(A) \) is canonically homeomorphic to \( \text{Spec}(A) / \sim \), the space of connected components of \( \text{Spec}(A) \).
A quasi-compact and Hausdorff space is called a compact space. A topological space is called a normal space if every two disjoint closed subsets admit disjoint open neighborhoods. A subspace $Y$ of a topological space $X$ is called a retraction of $X$ if there exists a continuous map $\gamma : X \to Y$ such that $\gamma(y) = y$ for all $y \in Y$. Such a map $\gamma$ is called a retraction map.

**Theorem 2.6.** Let $X$ be a compact and totally disconnected topological space. Then the set of clopens of $X$ forms a basis for the topology of $X$. If moreover, $X$ has an open covering $C$ with the property that every open subset of each member of $C$ is a member of $C$ then there exist a finite number $W_1, \ldots, W_q \in C$ of pairwise disjoint clopens of $X$ such that $X = \bigcup_{k=1}^{q} W_k$.

**Lemma 2.7.** If $\varphi : X \to Y$ is a continuous map of topological spaces such that $X$ is quasi-compact and $Y$ is Hausdorff then it is a closed map.

By a closed immersion of schemes we mean a morphism of schemes $\varphi : X \to Y$ such that the map $\varphi$ between the underlying spaces is injective and closed map and the ring map $\varphi^\sharp_x : \mathcal{O}_{Y, \varphi(x)} \to \mathcal{O}_{X, x}$ is surjective for all $x \in X$.

**Theorem 2.8.** A morphism of rings $A \to B$ is surjective if and only if the induced morphism $\text{Spec}(B) \to \text{Spec}(A)$ is a closed immersion of schemes.

**Theorem 2.9.** If a scheme can be written as the disjoint union of a finite number of affine opens then it is an affine scheme.

### 3. Maximality of primes

**Lemma 3.1.** If $p$ and $q$ are distinct minimal primes of $A$ then $A_p \otimes_A A_q = 0$. 
Proof. If \( A_p \otimes_A A_q \neq 0 \) then it has a prime ideal \( P \). Thus in the following pushout diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_2} & A_q \\
\downarrow{\pi_1} & & \downarrow{\mu} \\
A_p & \xrightarrow{\lambda} & A_p \otimes_A A_q
\end{array}
\]

we have \( \lambda^{-1}(P) = pA_p \) and \( \mu^{-1}(P) = qA_q \) where \( \pi_1 \) and \( \pi_2 \) are the canonical morphisms. It follows that \( p = \pi_1^{-1}(pA_p) = \pi_2^{-1}(qA_q) = q \). But this is a contradiction. \( \square \)

Lemma 3.2. Let \( p \) and \( q \) be prime ideals of a ring \( A \). Then \( A_p \otimes_A A_q = 0 \) if and only if there exist \( f \in A \setminus p \) and \( g \in A \setminus q \) such that \( fg = 0 \).

Proof. To see the implication “\( \Rightarrow \)”, let \( M = A_p \). Then \( M_q \simeq A_p \otimes_A A_q = 0 \). Thus the image of the unit of \( A_p \) under the canonical map \( M \to M_q \) is zero. Hence there exists some \( g \in A \setminus q \) such that \( g/1 = 0 \) in \( A_p \). It follows that there is some \( f \in A \setminus p \) such that \( fg = 0 \).

The converse implication is also proved easily. \( \square \)

Let \( A \) be a ring. Consider the relation \( S = \{(p, q) \in X^2 : A_p \otimes_A A_q \neq 0 \} \) on \( X = \text{Spec}(A) \). This relation is reflexive and symmetric. Let \( \sim_S \) be the equivalence relation generated by \( S \). Thus \( p \sim_S q \) if and only if there exists a finite set \( \{p_1, ..., p_n\} \) of prime ideals of \( A \) with \( n \geq 2 \) such that \( p_1 = p, p_n = q \) and \( A_{p_i} \otimes_A A_{p_{i+1}} \neq 0 \) for all \( 1 \leq i \leq n - 1 \). Note that it may happen that \( p \sim_S q \) but \( A_p \otimes_A A_q = 0 \).

In the following result new criteria for the maximality of primes are given. In fact, the criteria (\textit{iii}), (\textit{iv}) and (\textit{viii}) are classical and the remaining are new. The equivalency of the classical criteria are also proved by new and simple methods. Zariski, flat and patch topologies on \( \text{Spec} A \) are denoted by \( \mathcal{Z}, \mathcal{F} \) and \( \mathcal{P} \), respectively.

Theorem 3.3. For a ring \( A \) the following conditions are equivalent.

(i) \( \text{Spec}(A) = \text{Max}(A) \).

(ii) If \( p \) and \( q \) are distinct primes of \( A \) then there exist \( f \in A \setminus p \) and
\[ g \in A \setminus q \text{ such that } fg = 0. \]

(iii) \( Z \) is Hausdorff.

(iv) \( Z = P \).

(v) \( F \) is Hausdorff.

(vi) \( Z = F \).

(vii) If \( p \) is a prime ideal of \( A \) then the canonical map \( \pi : A \to A_p \) is surjective.

(viii) \( A/\mathfrak{N} \) is absolutely flat where \( \mathfrak{N} \) is the nil-radical of \( A \).

(ix) Every flat epimorphism of rings with source \( A \) is surjective.

(x) If \( p \) is a prime of \( A \) then \([p] = \{p\}\).

**Proof.**

(i) \( \Rightarrow \) (ii) : It follows from Lemmas 3.1 and 3.2.

(ii) \( \Rightarrow \) (iii) : There is nothing to prove.

(iii) \( \Rightarrow \) (iv) : If \( X = \text{Spec}(A) \) then the map \( \varphi : (X, P) \to (X, Z) \) given by \( x \mapsto x \) is a homeomorphism, see Lemma 2.7. Thus \( Z = P \).

(iv) \( \Rightarrow \) (i) : If \( p \) is a prime of \( A \) then \( V(p) = \{p\} \) and so \( p \) is a maximal ideal.

(i) \( \Rightarrow \) (v) : If \( p \) and \( q \) are distinct primes of \( A \) then by the hypothesis, \( p + q = A \). Thus there are \( f \in p \) and \( g \in q \) such that \( f + g = 1 \). Therefore \( V(f) \cap V(g) = \emptyset \).

(v) \( \Rightarrow \) (i) : Let \( p \) be a prime of \( A \). There exist a maximal ideal \( m \) of \( A \) such that \( p \subseteq m \). Thus \( p \in \Lambda(m) \). By the hypothesis, \( \Lambda(m) = \{m\} \). Therefore \( p = m \).

(v) \( \Rightarrow \) (vi) : By a similar argument as applied in the implication (iii) \( \Rightarrow \) (iv), we get that \( F = P \). Then apply the equivalency (v) \( \Leftrightarrow \) (iv).

(vi) \( \Rightarrow \) (i) : If \( p \) is a prime of \( A \) then \( \Lambda(p) = V(p) \) and so \( p \) is a maximal ideal.

(ii) \( \Rightarrow \) (vii) : It suffices to show that the induced morphism \( \pi_q : A_q \to (A_p)_q \) is surjective for all \( q \in \text{Spec}(A) \). Clearly \( \pi_p \) is an isomorphism. If \( q \neq p \) then by Lemma 3.2, \( (A_p)_q \simeq A_p \otimes_A A_q = 0 \).

(vii) \( \Rightarrow \) (i) : For each \( f \in A \setminus p \) there exists some \( g \in A \) such that \( g/1 = 1/f \) in \( A_p \). Thus there exists an element \( h \in A \setminus p \) such that \( h(1 - fg) = 0 \). It follows that \( 1 - fg \in p \) and so \( A/p \) is a field.

(i) \( \Rightarrow \) (viii) : Let \( R := A/\mathfrak{N} \). If \( p \) is a prime of \( R \) then \( R_p \) is a field, because \( pR_p = \mathfrak{N}' = \mathfrak{N}A_p = 0 \) where \( \mathfrak{N}' \) is the nil-radical of \( R_p \). Therefore every \( R \)-module is \( R \)-flat.

(viii) \( \Rightarrow \) (i) : If \( p \) is a prime of \( A \) then \( p/\mathfrak{N} \) is a maximal ideal of \( A/\mathfrak{N} \) and so \( p \) is a maximal ideal of \( A \).

(ix) \( \Rightarrow \) (vii) : There is nothing to prove.

(iii) \( \Rightarrow \) (ix) : Let \( \varphi : A \to B \) be a flat epimorphism of rings and let \( \theta : \text{Spec}(B) \to \text{Spec}(A) \) be the induced morphism between the
corresponding affine schemes. By Theorem 2.8 it suffices to show that \( \theta \) is a closed immersion of schemes. The map \( \theta \) between the underlying spaces is injective since \( \varphi \) is an epimorphism of rings. The map \( \theta \) is also a closed map since Spec(\( B \)) is quasi-compact and Spec(\( A \)) is Hausdorff, see Lemma 2.7. It remains to show that if \( q \) is a prime ideal of \( B \) then \( \theta_q^\# : \mathcal{O}_{\text{Spec}(A), p} \to \mathcal{O}_{\text{Spec}(B), q} \) is surjective where \( p = \theta(q) = \varphi^{-1}(q) \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{O}_{\text{Spec}(A), p} & \xrightarrow{\theta_q^\#} & \mathcal{O}_{\text{Spec}(B), q} \\
\downarrow & & \downarrow \\
A_p & \xrightarrow{\varphi_q} & B_q
\end{array}
\]

where the vertical arrows are the canonical isomorphisms and \( \varphi_q \) is induced by \( \varphi \). By Proposition 2.3, \( \varphi_q \) is an isomorphism of rings. Therefore \( \theta_q^\# \) is an isomorphism.

\( (x) \Rightarrow (i) \): Let \( p \) be a prime of \( A \). There is a maximal ideal \( m \) of \( A \) such that \( p \subseteq m \). By Lemma 3.2, \( A_p \otimes_A A_m \neq 0 \). Thus \( m \in [p] \) and so \( p = m \).

\( (i) \Rightarrow (x) \): Let \( m \) be a maximal ideal of \( A \) and \( m' \in [m] \). Thus there exists a finite set \( \{m_1, ..., m_n\} \) of maximal ideals of \( A \) with \( n \geq 2 \) such that \( m_1 = m, m_n = m' \) and \( A_{m_i} \otimes_A A_{m_{i+1}} \neq 0 \) for all \( 1 \leq i \leq n - 1 \). Thus by Lemma 3.1, \( m = m_1 = ... = m_n = m' \). \( \square \)

**Remark 3.4.** In Theorem 3.3 we provided a geometric proof for the implication \( (iii) \Rightarrow (ix) \). In what follows a purely algebraic proof is given for this implication. It is well-known that if \( \varphi : A \to B \) is a flat epimorphism of rings then for each prime ideal \( p \) of \( A \) we have either \( pB = B \) or that the induced morphism \( A_p \to B_p \) is an isomorphism. If \( pB = B \) then \( B_p \simeq A_p \otimes_A B = 0 \) because if \( A_p \otimes_A B \neq 0 \) then it has a prime ideal \( P \) and so in the following pushout diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\pi} & & \downarrow{\mu} \\
A_p & \xrightarrow{\lambda} & A_p \otimes_A B
\end{array}
\]

we have \( \lambda^{-1}(P) = pA_p \) since by the hypothesis Spec(\( A \)) = Max(\( A \)). Thus \( p = \varphi^{-1}(q) \) where \( q := \mu^{-1}(P) \). It follows that \( pB \subseteq q \neq B \), a contradiction. Therefore \( \varphi \) is surjective.
Theorem 3.5. If the topology of a scheme \( X \) is Hausdorff then every finite union of affine opens of \( X \) is an affine open. In particular, every compact scheme is an affine scheme.

Proof. By induction it suffices to prove the assertion for two cases, hence let \( U \) and \( V \) be two affine opens of \( X \). Every affine open of \( X \) is closed, because in a Hausdorff space each quasi-compact subspace is closed. It follows that \( W = U \cap V \) is a clopen (both open and closed) of \( U \). Therefore \( W, U \setminus W \) and \( V \setminus W \) are affine opens, see Theorem 2.4. Thus by Theorem 2.9, \( U \cup V \) is an affine open. □

We use the above theorems to obtain more geometric results:

Corollary 3.6. The category of compact (affine) schemes is anti-equivalent to the category of zero dimensional rings. □

Corollary 3.7. Let \( X \) be a scheme which has an affine open covering such that the intersection of any two elements of this covering is quasi-compact. Then the underlying space of \( X \) is Hausdorff if and only if every point of \( X \) is a closed point.

Proof. The implication “⇒” is obvious since each point of a Hausdorff space is a closed point. Conversely, if \( X = \text{Spec}(A) \) is an affine scheme then every prime of \( A \) is a maximal ideal. Thus by Theorem 3.3 (iii), \( \text{Spec} A \) is Hausdorff. For the general case, let \( x \) and \( y \) be two distinct points of \( X \). By the hypothesis, there exist affine opens \( U \) and \( V \) of \( X \) such that \( x \in U \), \( y \in V \) and \( U \cap V \) is quasi-compact. If either \( x \in V \) or \( y \in U \) then the assertion holds. Because, by what we have proved above, every affine open of \( X \) is Hausdorff. Therefore we may assume that \( x \notin V \) and \( y \notin U \). But \( W := V \setminus (U \cap V) \) is an open subset of \( X \) because every quasi-compact (=compact) subset of a Hausdorff space is closed. Clearly \( y \in W \) and \( U \cap W = \emptyset \). □

Remark 3.8. The hypothesis of Corollary 3.7 is not limitative at all. Because a separated scheme or more generally a quasi-separated scheme has this property, see [20, Proposition 3.6] or [15, Ex. 4.3] for the separated case and [12, Tag 054D] for the quasi-separated case.
4. **PM-rings**

Let $A$ be a ring and consider the following relation $R = \{(p, q) \in X^2 : p + q \neq A\}$ on $X = \text{Spec}(A)$. Clearly it is reflexive and symmetric. Let $\sim_R$ be the equivalence relation generated by $R$. Then $p \sim_R q$ if and only if there exists a finite set $\{p_1, ..., p_n\}$ of primes of $A$ with $n \geq 2$ such that $p_1 = p$, $p_n = q$ and $p_i + p_{i+1} \neq A$ for all $1 \leq i \leq n - 1$. Note that it may happen that $X/ \sim_R = \{[m] : m \in \text{Max} A\} = \{[p] : p \in \text{Min} A\}$. If $m$ and $m'$ are maximal ideals of $A$ then $m \sim_R m'$ if and only if there exists a finite set $\{m_1, ..., m_n\}$ of maximal ideals of $A$ such that $m_1 = m$, $m_n = m'$ and each $m_i \cap m_{i+1}$ contains a prime ideal of $A$.

**Proposition 4.1.** Let $A$ be a pm-ring and $m$ a maximal ideal of $A$. Then $[m] = \{p \in \text{Spec} A : p \subseteq m\}$.

**Proof.** Let $p \in [m]$. There exists a maximal ideal $m'$ of $A$ such that $p \subseteq m'$. It follows that $m \sim_R m'$. Thus there exists a finite set $\{p_1, ..., p_n\}$ of primes of $A$ with $n \geq 2$ such that $p_1 = m$, $p_n = m'$ and $p_i + p_{i+1} \neq A$ for all $1 \leq i \leq n - 1$. By induction on $n$ we shall prove that $m = m'$. If $n = 2$ then $m + m' \neq A$ and so $m = m'$. Assume that $n > 2$. We have $p_{n-2} + p_{n-1} \neq A$ and $p_{n-1} \subseteq m'$. Thus by the hypothesis, $p_{n-2} \subseteq m'$ and so $p_{n-2} + m' \neq A$. Thus in the equivalency $m \sim_R m'$ the number of involved primes is reduced to $n - 1$. Therefore by the induction hypothesis, $m = m'$. □

**Lemma 4.2.** Let $S$ be a multiplicative subset of a ring $A$. Then the canonical morphism $\pi : A \to S^{-1}A$ is surjective if and only if $\text{Im} \pi^* = \{p \in \text{Spec}(A) : p \cap S = \emptyset\}$ is a Zariski closed subset of $\text{Spec}(A)$.

**Proof.** The map $\pi^* : \text{Spec}(S^{-1}A) \to \text{Spec}(A)$ is a homeomorphism onto its image. If $\text{Im} \pi^*$ is Zariski closed then $\pi^*$ is a closed map. If $q \in \text{Spec}(S^{-1}A)$ then the morphism $A_p \to (S^{-1}A)_q$ induced by $\pi$ is an isomorphism where $p = \pi^{-1}(q)$. Therefore the morphism $(\pi^*, \pi^*_q) : \text{Spec}(S^{-1}A) \to \text{Spec}(A)$ is a closed immersion of schemes. Thus by Theorem 2.8, $\pi$ is surjective. Conversely, if $\pi$ is surjective then $\text{Im} \pi^* = V(\text{Ker} \pi)$. □

If $p$ is a prime ideal of a ring $A$ then the image of each $f \in A$ under the canonical map $\pi_p : A \to A_p$ is also denoted by $f_p$. 

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In the following result, the criteria (ii), (iv), (viii), (ix) and (x) are new and the remaining are classical. But we also prove the equivalency of some of these classical criteria by new methods. See [13, Theorem 1.2] for the classical criteria.

**Theorem 4.3.** For a ring \( A \) the following conditions are equivalent.

(i) \( A \) is a pm-ring.

(ii) If \( m \) and \( n \) are distinct maximal ideals of \( A \) then \([m] \neq [n]\).

(iii) If \( m \) and \( n \) are distinct maximal ideals of \( A \) then there exist \( f \in A \setminus m \) and \( g \in A \setminus n \) such that \( fg = 0 \).

(iv) If \( m \) is a maximal ideal of \( A \) then the canonical map \( \pi : A \to A_m \) is surjective.

(v) \( \text{Max}(A) \) is a Zariski retraction of \( \text{Spec}(A) \).

(vi) \( \text{Spec}(A) \) is a normal space with respect to the Zariski topology.

(vii) For each \( f \in A \) there exist \( g, h \in A \) such that \((1 + fg)(1 + f'h) = 0 \) where \( f' = 1 - f \).

(viii) If \( m \) is a maximal ideal of \( A \) then \( \Lambda(m) = \{ p \in \text{Spec}(A) : p \subseteq m \} \) is a Zariski closed subset of \( \text{Spec}(A) \).

(ix) If \( m \) and \( n \) are distinct maximal ideals of \( A \) then \( \ker \pi_m + \ker \pi_n = A \).

(x) If \( m \) and \( n \) are distinct maximal ideals of \( A \) then there exists some \( f \in A \) such that \( f_m = 0 \) and \( f_n = 1 \).

**Proof.** (i) \(\Rightarrow\) (ii) : See Proposition 4.1.

(ii) \(\Rightarrow\) (i) : Let \( p \) be a prime of \( A \) such that \( p \subseteq m \) and \( p \subseteq n \) for some maximal ideals \( m \) and \( n \) of \( A \). It follows that \( m \sim_R n \) and so \([m] = [n]\).

Thus by the hypothesis, \( m = n \).

(i) \(\Rightarrow\) (iii) : If \( A_m \otimes_A A_n \neq 0 \) then it has a prime ideal \( P \). Thus in the following pushout diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\pi_2} & A_n \\
\downarrow{\pi_1} & & \downarrow{\mu} \\
A_m & \xrightarrow{\lambda} & A_m \otimes_A A_n
\end{array}
\]

we have \( p \subseteq m \) and \( p \subseteq n \) where \( p = (\lambda \circ \pi_1)^{-1}(P) \). This is a contradiction. Therefore \( A_m \otimes_A A_n = 0 \). Then apply Lemma 3.2.

(iii) \(\Rightarrow\) (i) : This is straightforward.

(iii) \(\Rightarrow\) (iv) : It suffices to show that the induced morphism \( \pi_n : A_n \to (A_m)_n \) is surjective for all \( n \in \text{Max}(A) \). Clearly \( \pi_m \) is an isomorphism.

If \( n \neq m \) then by Lemma 3.2 \( (A_m)_n \cong A_m \otimes_A A_n = 0 \).

(iv) \(\Rightarrow\) (iii) : Choose some \( h \in n \setminus m \) then there exists some \( a \in A \) such
that $1/h = a/1$ in $A_m$. Thus there exists some $f \in A \setminus m$ such that $f(ah - 1) = 0$. Clearly $g := ah - 1 \in A \setminus n$.

(i) ⇒ (v) : (This argument is a simplified version of the elegant proof of [13 Theorem 1.2]). Consider the function $\gamma : \text{Spec}(A) \to \text{Max}(A)$ where $\gamma(p)$ is the maximal ideal of $A$ containing $p$. It suffices to show that if $f \in A$ then $\gamma^{-1}(V(f) \cap \text{Max}(A)) = V(I)$ where $I = \bigcap_{q \in \mathfrak{m}} q$.

To see this, it suffices to show that if $F$ is a prime of $A$ such that $I \subseteq p$ then $f \in \gamma(p)$. We have $I \cap ST = \emptyset$ where $S = A \setminus p$ and $T = A \setminus \bigcup_{m \in V(f) \cap \text{Max}(A)} m$. Thus there exists a prime $q$ of $A$ such that $I \subseteq q$ and $q \cap ST = \emptyset$. It follows that $q \subseteq p$ and $q \subseteq \bigcup_{m \in V(f) \cap \text{Max}(A)} m$.

Hence $q + Af$ is a proper ideal of $A$. Thus there exists a maximal ideal $m$ of $A$ such that $q \subseteq m$ and $f \in m$. By the hypothesis, $\gamma(p) = \gamma(q) = m$.

(v) ⇒ (i) : Let $p$ be a prime of $A$ such that $p \subseteq m$ for some maximal ideal $m$ of $A$. By the hypothesis there exists a retraction map $\varphi : \text{Spec}(A) \to \text{Max}(A)$. Clearly $m \in \{p\}$ and so $m = \varphi(m) \in \{\varphi(p)\} = V(\varphi(p)) \cap \text{Max}(A) = \{\varphi(p)\}$. It follows that $\varphi(p) = m$.

(i) ⇒ (vi) : Let $E = V(I)$ and $F = V(J)$ be two disjoint closed subsets of $\text{Spec}(A)$ where $I$ and $J$ are ideals of $A$. It follows that $I + J = A$ and so $\gamma(E) \cap \gamma(F) = \emptyset$ where the function $\gamma : \text{Spec}(A) \to \text{Max}(A)$ maps each prime of $A$ into the maximal ideal of $A$ containing it. The map $\gamma$ is continuous, see the implication $(i) \Rightarrow (v)$. The space $\text{Max}(A)$ is also Hausdorff, see the implication $(i) \Rightarrow (iii)$. Thus by Lemma 2.7 $\gamma$ is a closed map. But $\text{Max}(A)$ is a normal space because it is well known that every compact space is a normal space. Thus there exist disjoint open neighborhoods $U$ and $V$ of $\gamma(E)$ and $\gamma(F)$ in $\text{Max}(A)$. It follows that $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are disjoint open neighborhoods of $E$ and $F$ in $\text{Spec}(A)$.

(vi) ⇒ (iii) : There exist $f \in A \setminus m$ and $g \in A \setminus n$ such that $D(fg) = \emptyset$. Thus $fg$ is nilpotent and so there exists a natural number $n \geq 1$ such that $f^n g^n = 0$.

(i) ⇔ (vii) : See [7 Theorem 4.1].

(iv) ⇔ (viii) : See Lemma 12.

(iv) ⇒ (ix) : We have $V(\text{Ker } \pi_m) = \{p \in \text{Spec}(A) : p \subseteq m\}$. It follows that Ker $\pi_m + \text{Ker } \pi_n = A$.

(ix) ⇒ (iii) : There are $f' \in \text{Ker } \pi_m$ and $g' \in \text{Ker } \pi_n$ such that $f' + g' = 1$. Thus there exist $f \in A \setminus m$ and $g \in A \setminus n$ such that $ff' = gg' = 0$. It follows that $fg = 0$.

(ix) ⇒ (x) : There exists some $f \in \text{Ker } \pi_m$ such that $1 - f \in \text{Ker } \pi_n$.

(x) ⇒ (iii) : There exist $g \in A \setminus m$ and $h \in A \setminus n$ such that $fg = 0$ and
\((1 - f)h = 0\). It follows that \(gh = 0\). □

**Remark 4.4.** Clearly pm-rings are stable under taking quotients, and mp-rings are stable under taking localizations. Consider the prime ideals \(p = (x/1)\) and \(q = (y/1)\) in \(A = (k[x, y])_P\) where \(k\) is a domain and \(P = (x, y)\). Then \(A\) is a pm-ring but \(S^{-1}A\) is not a pm-ring where \(S = A \setminus p \cup q\). Dually, \(k[x, y]\) is a mp-ring but the quotient \(A = k[x, y]/I\) is not a mp-ring because \(p = (x + I)\) and \(q = (y + I)\) are two distinct minimal primes of \(A\) which are contained in the prime ideal \((x+I, y+I)\) where \(I = (xy)\). If for each prime \(p\) of a ring \(A\) the set \(\text{Spec}(A/p)\) is totally ordered (with respect to the inclusion) then each localization of \(A\) is a pm-ring. Dually, if for each prime \(p\) of a ring \(A\) the set \(\text{Spec}(A_p)\) is totally ordered then each quotient of \(A\) is a mp-ring.

**Corollary 4.5.** [7, Theorem 3.3 and p. 103] The product of a family of rings \((A_i)\) is a pm-ring if and only if each \(A_i\) is a pm-ring. □

**Corollary 4.6.** If \(A\) is a pm-ring with a finitely many minimal primes then \(A\) is canonically isomorphic to \(\prod_{m \in \text{Max}(A)} A_m\).

**Proof.** If \(m\) is a maximal ideal of a ring \(A\) then \(\text{Ker } \pi_m \subseteq m\) and \(\bigcap_{m \in \text{Max}(A)} \text{Ker } \pi_m = 0\) where \(\pi_m : A \to A_m\) is the canonical map. Because take some \(f\) in the intersection, if \(f \neq 0\) then \(\text{Ann}(f) \neq A\), so there exists a maximal ideal \(m\) of \(A\) such that \(\text{Ann}(f) \subseteq m\), but there is some \(g \in A \setminus m\) such that \(fg = 0\) since \(f\) have been chosen from the intersection. But this is a contradiction and we win. If \(A\) is a pm-ring and \(m\) and \(n\) are distinct maximal ideals of \(A\) then by Theorem 4.3 (ix), \(\text{Ker } \pi_m + \text{Ker } \pi_n = A\). If moreover \(\text{Min}(A)\) is a finite set then \(\text{Max}(A)\) is a finite set. Thus by the Chinese remainder theorem, \(A\) is canonically isomorphic to \(\prod_{m \in \text{Max}(A)} A_m\). □

Noetherian pm-rings are completely characterized:

**Corollary 4.7.** [8, Theorem 1.4] A ring is a noetherian pm-ring if and only if it is isomorphic to a finite product of noetherian local rings. □

If \(A\) is a pm-ring then the polynomial ring \(A[x]\) is not a pm-ring. For example, if \(k\) is a field then \(k[x]\) is not a pm-ring.
Remark 4.8. Note that the main result of [25, Theorem 2.11 (iv)] by Harold Simmons is not true and the gap is not repairable. It claims that if $\text{Max}(A)$ is Zariski Hausdorff then $A$ is a pm-ring. As a counterexample for the claim, let $p$ and $q$ be two distinct prime numbers, $S = \mathbb{Z} \setminus (p\mathbb{Z} \cup q\mathbb{Z})$ and $A = S^{-1}\mathbb{Z}$. Then $\text{Max}(A) = \{S^{-1}(p\mathbb{Z}), S^{-1}(q\mathbb{Z})\}$ is Zariski Hausdorff because $D(p/1) \cap \text{Max}(A) = \{S^{-1}(q\mathbb{Z})\}$ and $D(q/1) \cap \text{Max}(A) = \{S^{-1}(p\mathbb{Z})\}$. But $A$ is not a pm-ring since it is a domain. See also Simmons’ erratum for [25]. In Proposition 4.9 we correct his mistake.

Proposition 4.9. If $\text{Max}(A)$ is Zariski Hausdorff and $\mathfrak{M} = \mathfrak{J}$ then $A$ is a pm-ring where $\mathfrak{J}$ is the Jacobson radical of $A$.

Proof. Let $\mathfrak{m}$ and $\mathfrak{m}'$ be distinct maximal ideals of $A$ both containing a prime $\mathfrak{p}$ of $A$. By the hypotheses, there are $f \in A \setminus \mathfrak{m}$ and $g \in A \setminus \mathfrak{m}'$ such that $(D(f) \cap \text{Max}(A)) \cap (D(g) \cap \text{Max}(A)) = \emptyset$. It follows that $fg \in \mathfrak{J}$. Thus there exists a natural number $n \geq 1$ such that $f^n g^n = 0$. Then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. This is a contradiction, hence $A$ is a pm-ring. □

Corollary 4.10. Let $A$ be a ring. Then $\text{Max}(A)$ is Zariski Hausdorff if and only if $A/\mathfrak{J}$ is a pm-ring. □

If $\mathfrak{p}$ is a prime ideal of a ring $A$ then clearly $\Lambda(\mathfrak{p}) = \text{Im} \pi_p^*$ is contained in $V(\text{Ker} \pi_p)$ where $\pi_p : A \to A_p$ is the canonical map. By Theorem 4.3 $A$ is a pm-ring if and only if $\Lambda(\mathfrak{m}) = V(\text{Ker} \pi_m)$ for all maximal ideals $\mathfrak{m}$ of $A$.

By a system of equations over a ring $A$ we mean a finite number of equations $f_i(x_1, ..., x_n) = 0$ with $i = 1, ..., d$ where each $f_i(x_1, ..., x_n) \in A[x_1, ..., x_n]$. We say that this system has a solution in $A$ if there exists an $n$-tuple $(c_1, ..., c_n) \in A^n$ such that $f_i(c_1, ..., c_n) = 0$ for all $i$.

In Theorem 4.11 we have improved the interesting result of [10, Theorem 1.1] by adding (i), (iii) and (vi) as new equivalents. The criteria (i) and (iii) are very powerful tools to investigate clean rings more deeply. For instance, the equivalency of the classical criteria (vii) and (viii) are proved by new and very simple methods (these classical criteria can be found in [24]). Theorem 4.11 also generalizes the technical result of [14, Proposition 2] from zero-dimensional rings to clean rings.
and from particular system of equations to arbitrary systems. Following the suggestion of [10], Theorem 1.1, we use the similar ideas of the proof of [14, Proposition 2] to deduce the implication (iv) ⇒ (i).

**Theorem 4.11.** For a ring $A$ the following conditions are equivalent.

(i) If a system of equations over an $A$-algebra $B$ has a solution in each ring $B_m$ with $m$ a maximal ideal of $A$, then that system has a solution in the ring $B$.

(ii) $A$ is a clean ring.

(iii) If $m$ and $m'$ are distinct maximal ideals of $A$ then there exists an idempotent $e \in A$ such that $e \in m$ and $1 - e \in m'$.

(iv) $A$ is a pm-ring and $\text{Max}(A)$ is totally disconnected with respect to the Zariski topology.

(v) $A$ is a pm-ring and for each maximal ideal $m$ of $A$ the ideal $\ker \pi_m$ is generated by a set of idempotents of $A$.

(vi) The connected components of $\text{Spec}(A)$ are precisely of the form $\Lambda(m)$ where $m$ is a maximal ideal of $A$.

(vii) For each $f \in A$ there exists an idempotent $e \in A$ such that $e \in A f$ and $1 - e \in A(1 - f)$.

(viii) The idempotents of $A$ can be lifted modulo each ideal of $A$ (i.e., if $I$ is an ideal of $A$ and $f - f^2 \in I$ for some $f \in A$, then there exists an idempotent $e \in A$ such that $f - e \in I$).

**Proof.** For the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) see the proof of [10, Theorem 1.1].

(v) ⇒ (iii) : If $m$ and $m'$ are distinct maximal ideals of $A$ then by Theorem 4.3, $\ker \pi_m + \ker \pi_{m'} = A$. Thus there exists an idempotent $e \in \ker \pi_m \subseteq m$ such that $e \notin m'$. It follows that $1 - e \in m'$.

(iv) ⇒ (vi) : If $m$ is a maximal ideal of $A$ then $A/\ker \pi_m$ has no nontrivial idempotents since by Theorem 4.3 (iv) it is canonically isomorphic to $A_m$. Moreover, $\ker \pi_m$ is a regular ideal of $A$, see the implication (iv) ⇒ (v). It follows that $\ker \pi_m$ is a max-regular ideal of $A$. Hence, by [27, Theorem 3.17], $V(\ker \pi_m)$ is a connected component of $\text{Spec}(A)$.

Conversely, if $C$ is a connected component of $\text{Spec}(A)$ then $\gamma(C)$ is a connected subset of $\text{Max}(A)$ where $\gamma : \text{Spec}(A) \to \text{Max}(A)$ is the retraction map, see Theorem 4.3. Therefore there exists a maximal ideal $m$ of $A$ such that $\gamma(C) = \{m\}$ because $\text{Max}(A)$ is totally disconnected. We have $C \subseteq \gamma^{-1}(\{m\}) = \Lambda(m) = V(\ker \pi_m)$. It follows that $C = V(\ker \pi_m)$.

(vi) ⇒ (iv) : Clearly $A$ is a pm-ring because distinct connected components are disjoint. The map $\varphi : \text{Max}(A) \to \text{Spec}(A)/\sim$ given by
\( \mathfrak{m} \rightsquigarrow \Lambda(\mathfrak{m}) \) is bijective. It is also continuous and closed map because 
\( \varphi = \pi \circ i \) and \( \text{Spec}(A)/\sim \) is Hausdorff where \( i : \text{Max}(A) \rightarrow \text{Spec}(A) \) is 
the canonical injection and \( \pi : \text{Spec}(A) \rightarrow \text{Spec}(A)/\sim \) is the canonical projection. Therefore by Remark 2.5, \( \text{Max}(A) \) is totally disconnected. 
(iV) \( \Rightarrow \) (i) : Consider the system of equations \( f_i(x_1, \ldots, x_n) = 0 \) with 
\( i = 1, \ldots, d \) where \( f_i(x_1, \ldots, x_n) \in B[x_1, \ldots, x_n] \). (If \( \varphi : A \rightarrow B \) is 
the structure morphism then as usual \( a.1_B = \varphi(a) \) is simply denoted by \( a \) for all \( a \in A \).) Using the calculus of fractions, then we may find a 
positive integer \( N \) and polynomials \( g_i(y_1, \ldots, y_n, z_1, \ldots, z_n) \) over \( B \) such that 
\[
f_i(b_1/s_1, \ldots, b_n/s_n) = g_i(b_1, \ldots, b_n, s_1, \ldots, s_n)/(s_1 \ldots s_n)^N
\]
for all \( i \) and for every \( b_1, \ldots, b_n \in B \) and \( s_1, \ldots, s_n \in S \) where \( S \) is a 
multiplicative subset of \( A \). If the above system has a solution in each 
ring \( B_a \) then there exist \( b_1, \ldots, b_n \in B \) and \( c, s_1, \ldots, s_n \in A \setminus \mathfrak{m} \) such 
that \( c g_i(b_1, \ldots, b_n, s_1, \ldots, s_n) = 0 \) for all \( i \). This leads us to consider 
\( \mathcal{C} \) the collection of those opens \( W \) of \( \text{Max}(A) \) such that there exists 
\( b_1, \ldots, b_n \in B \) and \( c, s_1, \ldots, s_n \in A \setminus \bigcup \mathfrak{m} \) so that 
\[c g_i(b_1, \ldots, b_n, s_1, \ldots, s_n) = 0.\]
Clearly \( \mathcal{C} \) covers \( \text{Max}(A) \) and if \( W \in \mathcal{C} \) then every open subset of \( W \) 
is also a member of \( \mathcal{C} \). Thus by Theorem 2.6 we may find a finite 
number \( W_1, \ldots, W_q \in \mathcal{C} \) of pairwise disjoint clopens of \( \text{Max}(A) \) such 
that \( \text{Max}(A) = \bigcup_{k=1}^q W_k \). Using the retraction map \( \gamma : \text{Spec}(A) \rightarrow \text{Max}(A) \) and Theorem 2.4 then the map \( f \rightsquigarrow D(f) \cap \text{Max}(A) \) is a 
bijection from the set of idempotents of \( A \) onto the set of clopens of 
\( \text{Max}(A) \). Therefore there exist orthogonal idempotents \( e_1, \ldots, e_q \in A \) 
such that \( W_k = D(e_k) \cap \text{Max}(A) \). Clearly \( \sum_{k=1}^q e_k \) is an idempotent and 
\[
D(\sum_{k=1}^q e_k) = \text{Spec}(A).\]
It follows that \( \sum_{k=1}^q e_k = 1 \). For each \( k = 1, \ldots, q \) 
there exist \( b_{1k}, \ldots, b_{nk} \in B \) and \( c_k, s_{1k}, \ldots, s_{nk} \in A \setminus ( \bigcup \mathfrak{m} ) \) such that 
\[
c_k g_i(b_{1k}, \ldots, b_{nk}, s_{1k}, \ldots, s_{nk}) = 0 \quad \text{for all } i.\]
For each \( j = 1, \ldots, n \) setting 
\[
b'_j = \sum_{k=1}^q e_k b_{jk} \quad \text{and} \quad s'_j = \sum_{k=1}^q e_k s_{jk}.
\]
Note that for each natural number \( p \geq 0 \) we have then \( (b'_j)^p = \sum_{k=1}^q e_k (b_{jk})^p \) and \( (s'_j)^p = \sum_{k=1}^q e_k (s_{jk})^p \). It follows that 
\[
c' g_i(b'_1, \ldots, b'_n, s'_1, \ldots, s'_n) = 0
\]
for all $i$ where $c' = \sum_{k=1}^{q} e_k c_k$. Because fix $i$ and let
\[ g_i(y_1, \ldots, y_n, z_1, \ldots, z_n) = \sum_{0 \leq i_1, \ldots, i_{2n} < \infty} r_{i_1, \ldots, i_{2n}} y_1^{i_1} \cdots y_n^{i_n} z_1^{i_{n+1}} \cdots z_n^{i_{2n}}. \]

Then
\[ c'g_i(b'_1, \ldots, b'_n, s'_1, \ldots, s'_n) = \]
\[ \sum_{0 \leq i_1, \ldots, i_{2n} < \infty} r_{i_1, \ldots, i_{2n}} \left( \sum_{k=1}^{q} e_k (b_{1k})^{i_1} \cdots (b_{nk})^{i_n} (s_{1k})^{i_{n+1}} \cdots (s_{nk})^{i_{2n}} \right) = \]
\[ \left( \sum_{t=1}^{q} e_t c_t \right) \left( \sum_{k=1}^{q} e_k g_i(b_{1k}, \ldots, b_{nk}, s_{1k}, \ldots, s_{nk}) \right) = \]
\[ \sum_{k=1}^{q} e_k c_k g_i(b_{1k}, \ldots, b_{nk}, s_{1k}, \ldots, s_{nk}) = 0. \]

But $c'$ is invertible in $A$ since $c' \notin m$ for all $m \in \text{Max}(A)$. Hence, $g_i(b'_1, \ldots, b'_n, s'_1, \ldots, s'_n) = 0$ for all $i$. Similarly, each $s'_j$ is invertible in $A$. Therefore $f_i(b'_1/s'_1, \ldots, b'_n/s'_n) = g_i(b'_1, \ldots, b'_n, s'_1, \ldots, s'_n)/(s'_1 \cdots s'_n)^{\mathbb{N}} = 0$ for all $i$. Hence, the n-tuple $(b'_1, \ldots, b'_n) \in B^n$ is a solution of the above system where $b'_j := b'_j \varphi(s'_j)^{-1}$.

(i) $\Rightarrow$ (vii): It suffices to show that the system of equations
\[
\begin{cases}
X = X^2 \\
X = fY \\
1 - X = (1 - f)Z
\end{cases}
\]
has a solution in $A$. If $A$ is a local ring with the maximal ideal $m$ then the system having the solution $(0, 0, 1/(1 - f))$ or $(1, 1/f, 0)$, according as $f \in m$ or $f \notin m$. Using this, then by the hypothesis the system has a solution for every ring $A$ (not necessarily local).

(vii) $\Rightarrow$ (iii): If $m$ and $m'$ are distinct maximal ideals of $A$ then there are $f \in m$ and $g \in m'$ such that $f + g = 1$. By the hypothesis, there exist an idempotent $e \in A$ and elements $a, b \in A$ such that $e = af$ and $1 - e = b(1 - f)$. It follows that $e \in m$ and $1 - e \in m'$.

(i) $\Rightarrow$ (viii): It suffices to show that the system of equations
\[
\begin{cases}
X = X^2 \\
f - X = (f^2 - f)Y
\end{cases}
\]
has a solution in $A$. If $A$ is a local ring with the maximal ideal $m$ then the system having the solution $(0, 1/(f - 1))$ or $(1, 1/f)$, according as $f \in m$ or $f \notin m$. Using this, then by the hypothesis the system has a solution for every ring $A$ (not necessarily local).

(viii) $\Rightarrow$ (iii): If $m$ and $m'$ are distinct maximal ideals of $A$ then
there exist \( f \in m \) and \( g \in m' \) such that \( f + g = 1 \). It follows that \( f - f^2 \in mm' \). So by the hypothesis, there exists an idempotent \( e \in A \) such that \( f - e \in mm' \). This implies that \( e \in m \) and \( 1 - e \in m' \). 

**Corollary 4.12.** [1, Theorem 9] *If \( A/\mathfrak{M} \) is a clean ring then \( A \) is a clean ring.*

**Proof.** It implies from Theorem 4.11 (vi). □

**Corollary 4.13.** [1, Corollary 11] *Every zero dimensional ring is a clean ring.*

**Proof.** If \( A \) is a zero dimensional ring then by Theorem 3.3 (viii), the Zariski and patch topologies over \( \text{Max}(A) \) are the same things and so it is totally disconnected. Then apply Theorem 4.11 (iv). □

**Corollary 4.14.** [24, Proposition 1.5] *Let \( I \) be an ideal of a ring \( A \) which is contained in the Jacobson radical. If \( A/I \) is a clean ring and the idempotents of \( A \) can be lifted modulo \( I \) then \( A \) is a clean ring.*

**Proof.** It implies from Theorem 4.11 (iii). □

5. **Mp-rings**

The following result is proved like Proposition 4.1 with a little difference.

**Proposition 5.1.** Let \( A \) be a mp-ring and \( p \) a minimal prime of \( A \). Then \( [p] = \{q \in \text{Spec}(A) : p \subseteq q\} \).

**Proof.** Let \( q \in [p] \). There exists a minimal prime \( p' \) of \( A \) such that \( p' \subseteq q \). It follows that \( p \sim_R p' \). Thus there exists a finite set \( \{q_1, \ldots, q_n\} \) of primes of \( A \) with \( n \geq 2 \) such that \( q_1 = p, q_n = p' \) and \( q_i + q_{i+1} \neq A \) for all \( 1 \leq i \leq n - 1 \). By induction on \( n \) we shall prove that \( p = p' \). If \( n = 2 \) then \( p + p' \neq A \) and so by the hypothesis, \( p = p' \). Assume that \( n > 2 \). There exists a minimal prime \( p'' \) of \( A \) such that \( p'' \subseteq q_{n-1} \). We have \( q_{n-1} + p' \neq A \). Thus by the hypothesis, \( p' = p'' \). It follows that \( q_{n-2} + p' \neq A \). Thus in the equivalency \( p \sim_R p' \) the number of involved
primes is reduced to \( n - 1 \). Therefore by the induction hypothesis, \( p = p' \). □

**Remark 5.2.** We observed that if \( A \) is a pm-ring then \( \text{Max}(A) \) is Zariski Hausdorff. Dually, if \( A \) is a mp-ring then \( \text{Min}(A) \) is flat Hausdorff. Because if \( p \) and \( q \) are distinct minimal primes of \( A \) then \( p + q = A \). Thus there are \( f \in p \) and \( g \in q \) such that \( f + g = 1 \). So \( V(f) \cap V(g) = \emptyset \).

**Theorem 5.3.** For a ring \( A \) the following conditions are equivalent.

(i) \( A \) is a mp-ring.

(ii) If \( p \) and \( q \) are distinct minimal primes of \( A \) then \([p] \neq [q]\).

(iii) \( \text{Min}(A) \) is a flat retraction of \( \text{Spec}(A) \).

(iv) \( \text{Spec}(A) \) is a normal space with respect to the flat topology.

(v) If \( p \) is a minimal prime of \( A \) then \( V(p) \) is a flat closed subset of \( \text{Spec}(A) \).

(vi) If \( p \) and \( q \) are distinct minimal primes of \( A \) then \( p + q = A \).

(vii) \( A/\mathfrak{N} \) is a mp-ring.

**Proof.**

(i) ⇒ (ii) : See Proposition 5.1.

(ii) ⇒ (i) : Easy.

(i) ⇒ (iii) : Consider the function \( \gamma_1 : \text{Spec}(A) \rightarrow \text{Min}(A) \) where for each prime \( p \) of \( A \) then \( \gamma_1(p) \) is the minimal prime of \( A \) contained in \( p \). It suffices to show that \( \gamma_1^{-1}(V(f) \cap \text{Min}(A)) \) is a flat open of \( \text{Spec}(A) \) for all \( f \in A \). By the Hochster’s theorem [17, Theorem 6], there exists a ring \( B \) and a homeomorphism \( \theta : (\text{Spec}(B), \mathcal{Z}) \rightarrow (\text{Spec}(A), \mathcal{F}) \) such that if \( p \subseteq p' \) are primes of \( B \) then \( \theta(p') \subseteq \theta(p) \) where \( \mathcal{Z} \) (resp. \( \mathcal{F} \)) denotes the Zariski (resp. flat) topology. By the hypothesis, \( B \) is a pm-ring. Thus by Theorem 4.3 (v), there exists a continuous function \( \gamma_2 : \text{Spec}(B) \rightarrow \text{Max}(B) \) such that for each prime \( p \) of \( B \) then \( \gamma_2(p) \) is the maximal ideal of \( B \) containing \( p \). Therefore if \( p \) is a prime of \( B \) then \( \gamma_1(\theta(p)) = \theta(\gamma_2(p)) \). It follows that

\[
\theta^{-1}\left(\gamma_1^{-1}(V(f) \cap \text{Min}(A))\right) = \gamma_2^{-1}\left(\theta^{-1}(V(f)) \cap \text{Max}(B)\right).
\]

Thus by Theorem 4.3 (v), \( \gamma_1 \) is continuous with respect to the flat topology.

(iii) ⇒ (i) : It is proved exactly like the proof of the implication (v) ⇒ (i) in Theorem 4.3.

(i) ⇔ (iv) ⇔ (v) : Apply the homeomorphism \( \theta \) and Theorem 4.3.
(i) ⇔ (vi) ⇔ (vii) : Easy. □

Finding a direct proof to Theorem 5.3 (without using Theorem 4.3 and the Hochster’s theorem) would be certainly a serious challenge to the readers.

**Corollary 5.4.** If there exists a retraction map from Spec($A$) onto Max($A$) (or onto Min($A$)) then it is unique. □

**Theorem 5.5.** If for each minimal prime $p$ of $A$, $A/p$ is $A$–flat then $A$ is a mp-ring. If moreover $A$ is a reduced ring the the converse holds.

**Proof.** For the implication “⇒” we prove a stronger assertion that if $I$ is a proper ideal of $A$ then it contains at most one minimal prime of $A$. This in particular shows that $A$ is a mp-ring. Let $p$ and $q$ be minimal primes of $A$ which are contained in $I$. If $f \in p$ then by Theorem 2.1, Ann($f$) + $p$ = $A$. Thus there exist $g \in$ Ann($f$) and $h \in p$ such that $g + h = 1$. It follows that $f(1 - h) = 0$. But $1 - h \notin q$. Therefore $f \in q$ and so $p = q$. Conversely, assume that $A$ is a reduced mp-ring. Let $p$ be a minimal prime of $A$ and $f \in p$. If Ann($f$) + $p$ ≠ $A$ then there exists a maximal ideal $m$ of $A$ such that Ann($f$) + $p$ ⊆ $m$. By the hypotheses, $pA_m = 0$. Hence there exists some $g \in A \setminus m$ such that $fg = 0$. But this is a contradiction. Thus by Theorem 2.1, $A/p$ is $A$–flat. □

Note that if $p$ is a prime of $A$ such that $A/p$ is $A$–flat then $p$ is a minimal prime of $A$.

**Corollary 5.6.** Let $A$ be a reduced mp-ring. Then Min(Ann($f$)) ⊆ Min($A$) for all $f \in A$. In particular, $A/$ Ann($f$) is a mp-ring.

**Proof.** Let $p \in$ Min(Ann($f$)). There exists a minimal prime $q$ of $A$ such that $q \subseteq p$. By Theorem 5.5, $f \notin q$. It follows that Ann($f$) ⊆ $q$ and so $q = p$. □

**Proposition 5.7.** A ring is a noetherian reduced mp-ring if and only if it is isomorphic to finite product of noetherian domains.
Proof. If \( A \) is a noetherian reduced mp-ring then \( \text{Min}(A) \) is a finite set and for distinct minimal primes \( p \) and \( q \) of \( A \) we have \( p + q = A \). Thus by the Chinese remainder theorem, \( A \) is canonically isomorphic to \( \prod_{p \in \text{Min}(A)} A/p \). The converse implication is also easily deduced, see Proposition 2.2. □

If \( A \) is a ring then clearly \( \text{Ann}(f) + \text{Ann}(g) \subseteq \text{Ann}(fg) \) for all \( f, g \in A \). In Theorem 5.9, it is shown that the equality holds if and only if \( A \) is a reduced mp-ring.

Remark 5.8. Note that [2, Lemma \( \alpha \)] is not true and consequently the proof of the key result [2, Lemma \( \beta \)] is not correct since it is profoundly based on Lemma \( \alpha \). In fact, Lemma \( \alpha \) claims that if \( p \) is a prime ideal of a ring \( A \) then \( \bigcap_{q \in A(p)} q = \{ f \in A : \text{Ann}(f) \not\subseteq p \} \). In what follows we give a counterexample for Lemma \( \alpha \). If \( p \) is a minimal prime of \( A \) then by Lemma \( \alpha \), \( p = \{ f \in A : \text{Ann}(f) \not\subseteq p \} \). This in particular implies that every ring with a unique prime ideal is a field. But this is not true. As a specific example, let \( p \) be a prime number and \( n \geq 2 \) then \( \mathbb{Z}/p^n\mathbb{Z} \) has a unique prime ideal which is not a field. In Theorem 5.9, we give a correct proof and more accurate expression of Lemma \( \beta \). Also in Proposition 5.17 we give a right expression of Lemma \( \alpha \) and a proof of it.

Theorem 5.9. For a ring \( A \) the following conditions are equivalent.
(i) \( A \) is a reduced mp-ring.
(ii) If \( fg = 0 \) then \( \text{Ann}(f) + \text{Ann}(g) = A \).
(iii) \( \text{Ann}(f) + \text{Ann}(g) = \text{Ann}(fg) \) for all \( f, g \in A \).
(iv) For each minimal prime \( p \) of \( A \) and for each \( f \in p \) there exists some \( g \in p \) such that \( fg = f \) and \( \text{Ann}(f^2) = \text{Ann}(f) \).

Proof. (i) \( \Rightarrow \) (ii) : Let \( fg = 0 \) for some \( f, g \in A \). If \( \text{Ann}(f) + \text{Ann}(g) \neq A \) then there is a maximal ideal \( m \) of \( A \) such that \( \text{Ann}(f) + \text{Ann}(g) \subseteq m \). Let \( p \) be the minimal prime of \( A \) such that \( p \subseteq m \). We may assume that \( f \in p \). By Theorem 5.5, \( A/p \) is \( A \)-flat. Thus by Theorem 2.1, \( \text{Ann}(f) + p = A \). But this is a contradiction since \( \text{Ann}(f) + p \subseteq m \). Therefore \( \text{Ann}(f) + \text{Ann}(g) = A \).

(ii) \( \Rightarrow \) (i) : Let \( p \) and \( q \) be two distinct minimal primes of \( A \). By Lemma 3.1, \( A_p \otimes_A A_q = 0 \). Thus by Lemma 3.2, there are elements \( f \in A \setminus p \) and \( g \in A \setminus q \) such that \( fg = 0 \). Thus by the hypothesis,
there are elements $a \in \text{Ann}(f)$ and $b \in \text{Ann}(g)$ such that $a + b = 1$. It follows that $a \in p$ and $b \in q$. Hence $p + q = A$ and so $A$ is a mp-ring. Let $f$ be a nilpotent element of $A$. Thus there exists the least positive natural number $n$ such that $f^n = 0$. We show that $n = 1$. If $n > 1$ then by the hypothesis, $\text{Ann}(f^{n-1}) = \text{Ann}(f) + \text{Ann}(f^{n-1}) = A$. It follows that $f^{n-1} = 0$. But this is in contradiction with the minimality of $n$.

$(ii) \Rightarrow (iii)$: If $a \in \text{Ann}(fg)$ then $(af)g = 0$. Thus by the hypothesis, $\text{Ann}(af) + \text{Ann}(g) = A$. Hence there are $b \in \text{Ann}(af)$ and $c \in \text{Ann}(g)$ such that $b + c = 1$. We have $a = ab + ac$, $ab \in \text{Ann}(f)$ and $ac \in \text{Ann}(g)$. Thus $a \in \text{Ann}(f) + \text{Ann}(g)$.

$(iii) \Rightarrow (ii)$: There is nothing to prove.

$(i) \iff (iv)$: It implies from Theorems 5.5 and 2.1. □

One direction of the following result is due to M. Contessa, see [9, Theorem 4.3].

**Corollary 5.10.** The product of a family of rings $(A_i)$ is a reduced mp-ring if and only if each $A_i$ is a reduced mp-ring.

**Proof.** It is an immediate consequence of Theorem 5.9 (ii). □

**Remark 5.11.** Here we give a second proof for the implication $(i) \Rightarrow (ii)$ of Theorem 5.9. Although the proof is a little long but some interesting ideas are introduced during the proof. For example, Corollary 5.6 was discovered during this proof. Now we present the proof. If $fg = 0$ then $D(f) \cap D(g) = \emptyset$. Thus there exists flat opens $U$ and $V$ of $\text{Spec}(A)$ such that $D(f) \subseteq U$, $D(g) \subseteq V$ and $U \cap V = \emptyset$, see Theorem 5.3 (iv). Note that the basis flat opens of $\text{Spec}(A)$ are precisely of the form $V(I)$ where $I$ is a finitely generated ideal of $A$. Hence we may write $U = \bigcup_{\alpha} V(I_{\alpha})$ where each $I_{\alpha}$ is a finitely generated ideal of $A$.

But $D(f)$ is flat quasi-compact since in a quasi-compact space every closed is quasi-compact. Thus there are a finitely many $I_1, ..., I_n$ from the ideals $I_{\alpha}$ such that $D(f) \subseteq \bigcup_{i=1}^n V(I_i) = V(I) \subseteq U$ where $I = I_1 ... I_n$.

Similarly, there exists a (finitely generated) ideal $J$ of $A$ such that $D(g) \subseteq V(J) \subseteq V$. Thus $V(I) \cap V(J) = \emptyset$. It follows that $I + J = A$. Hence there are elements $a \in I$ and $b \in J$ such that $a + b = 1$. We have $D(f) \subseteq V(a)$ and $D(g) \subseteq V(b)$. By Corollary 5.6, $a \in \sqrt{\text{Ann}(f)}$ and $b \in \sqrt{\text{Ann}(g)}$. Thus $\sqrt{\text{Ann}(f)} + \sqrt{\text{Ann}(g)} = A$. It follows that
Ann(f) + Ann(g) = A.

The following definition is the dual notion of clean ring.

**Definition 5.12.** A ring \( A \) is said to be a **purified ring** if for every distinct minimal primes \( p \) and \( q \) of \( A \) then there exists an idempotent \( e \in A \) such that \( e \in p \) and \( 1 - e \in q \).

Every integral domain or more generally every ring with a unique minimal prime is a purified ring. Purified rings are stable under taking localizations. A finite product of rings is a purified ring if and only if each factor is a purified ring.

**Proposition 5.13.** Every zero dimensional ring is a purified ring.

**Proof.** It implies from Theorem 4.11 (iii). \( \Box \)

**Proposition 5.14.** A ring \( A \) is a purified ring if and only if \( A/\mathfrak{N} \) is a purified ring.

**Proof.** Let \( A/\mathfrak{N} \) be a purified ring and \( p \) and \( q \) distinct minimal primes of \( A \). Then there exists an idempotent \( f + \mathfrak{N} \in A/\mathfrak{N} \) such that \( f \in p \) and \( 1 - f \in q \). Using Theorem 2.4 then it is not hard to see that the idempotents of a ring \( A \) can be lifted modulo its nil-radical. So there exists an idempotent \( e \in A \) such that \( f - e \in \mathfrak{N} \). It follows that \( e \in p \) and \( 1 - e = (1 - f) + (f - e) \in q \). \( \Box \)

The following result is the culmination of reduced purified rings.

**Theorem 5.15.** For a reduced ring \( A \) the following conditions are equivalent.

(i) \( A \) is a purified ring.
(ii) \( A \) is a mp-ring and \( \text{Min}(A) \) is totally disconnected with respect to the flat topology.
(iii) Every minimal prime of \( A \) is generated by a set of idempotents.
(iv) The connected components of \( \text{Spec}(A) \) are precisely of the form \( V(p) \) where \( p \) is a minimal prime of \( A \).
(v) If a system of equations over \( A \) has a solution in each ring \( A/p \)
with \( p \) a minimal prime of \( A \), then that system has a solution in \( A \).

(vi) The idempotents of \( A \) can be lifted by each localization \( S^{-1}A \) where \( S \) is a multiplicative subset of \( A \).

**Proof.** (i) \( \Rightarrow \) (ii) : If \( p \) and \( q \) are distinct minimal primes of \( A \) then there exists an idempotent \( e \in A \) such that \( p \in V(e) \) and \( q \in V(1 - e) \). We also have \( V(e) \cup V(1 - e) = \text{Spec}(A) \). Therefore \( \text{Min}(A) \) is totally disconnected with respect to the flat topology.

(ii) \( \Rightarrow \) (iii) : Let \( p \) be a minimal prime of \( A \) and \( f \in p \). By Remark 5.2 \( \text{Min}(A) \) is flat Hausdorff. It is also flat quasi-compact. Therefore by Theorem 2.6 there exists a clopen \( U \subseteq \text{Min}(A) \) such that \( p \in U \subseteq V(f) \cap \text{Min}(A) \). Then by Theorem 2.4 there exists an idempotent \( e \in A \) such that \( p \in V(e) = \gamma^{-1}(U) \) where \( \gamma : \text{Spec}(A) \to \text{Min}(A) \) is the retraction map, see Theorem 5.3. We have \( \gamma^{-1}(U) \subseteq V(f) \). Thus there exist a natural number \( n \geq 1 \) and an element \( a \in A \) such that \( f^n = ae \). It follows that \( 1 - e \in \text{Ann}(f^n) \). But by Theorem 5.9 \( \text{Ann}(f^n) = \text{Ann}(f) \). Therefore \( f = fe \).

(iii) \( \Rightarrow \) (i) : Let \( p \) and \( q \) be distinct minimal primes of \( A \). Then there exists an idempotent \( e \in p \) such that \( e \notin q \). It follows that \( 1 - e \in q \).

(ii) \( \Rightarrow \) (iv) : If \( p \) is a minimal prime of \( A \) then it is a max-regular ideal of \( A \), see the implication (ii) \( \Rightarrow \) (iii) . Thus by [27, Theorem 3.17], \( V(p) \) is a connected component of \( \text{Spec}(A) \). Conversely, if \( C \) is a connected component of \( \text{Spec}(A) \) then there exists a minimal prime \( p \) of \( A \) such that \( \gamma(C) = \{ p \} \). But we have \( C \subseteq \gamma^{-1}(\{ p \}) = V(p) \). It follows that \( C = V(p) \).

(iv) \( \Rightarrow \) (ii) : Clearly \( A \) is a mp-ring because distinct connected components are disjoint. The map \( \text{Min}(A) \to \text{Spec}(A)/\sim \) given by \( p \mapsto V(p) \) is a homeomorphism. Thus by Remark 2.5 \( \text{Min}(A) \) is flat totally disconnected.

(ii) \( \Rightarrow \) (v) : Assume that the system of equations \( f_i(x_1, ..., x_n) = 0 \) over \( A \) has a solution in each ring \( A/p \). Thus for each minimal prime \( p \) of \( A \) there exist \( b_1, ..., b_n \in A \) such that \( f_i(b_1, ..., b_n) \in p \) for all \( i \). This leads us to consider \( \mathcal{C} \), the collection of those opens \( W \) of \( \text{Min}(A) \) such that there exist \( b_1, ..., b_n \in A \) so that \( f_i(b_1, ..., b_n) \in \bigcap_{p \in W} p \) for all \( i \). Clearly \( \mathcal{C} \) covers \( \text{Min}(A) \) and if \( W \in \mathcal{C} \) then every open subset of \( W \) is also a member of \( \mathcal{C} \). Thus by Theorem 2.6 we may find a finite number \( W_1, ..., W_q \in \mathcal{C} \) of pairwise disjoint clopens of \( \text{Min}(A) \) such that \( \text{Min}(A) = \bigcup_{k=1}^q W_k \). Using Theorem 2.4 and the retraction map \( \gamma : \text{Spec}(A) \to \text{Min}(A) \) of Theorem 5.3 then the map...
Thus $f \sim V(f) \cap \text{Min}(A)$ is a bijection from the set of idempotents of $A$ onto the set of clopens of $\text{Min}(A)$. Therefore there are orthogonal idempotents $e_1, \ldots, e_q \in A$ such that $W_k = V(1 - e_k) \cap \text{Min}(A)$. Thus $\sum_{k=1}^{q} e_k$ is an idempotent and $D(\sum_{k=1}^{q} e_k) = \text{Spec}(A)$. It follows that $\sum_{k=1}^{q} e_k = 1$. For each $k = 1, \ldots, q$ there exist $b_{1k}, \ldots, b_{nk} \in A$ such that $f_i(b_{1k}, \ldots, b_{nk}) \in \bigcap_{p \in W_k} p$ for all $i$. For each $j = 1, \ldots, n$ setting $b_j' = \sum_{k=1}^{q} e_kb_{jk}$. Note that if $p \geq 0$ is a natural number then

$$(b_j')^p = \sum_{k=1}^{q} e_k(b_{jk})^p.$$  

It follows that $f_i(b_1', \ldots, b_n') = \sum_{k=1}^{q} e_kf_i(b_{1k}, \ldots, b_{nk})$ for all $i$. Now if $p$ is a minimal prime of $A$ then $p \in W_t$ for some $t$. We have $e_tf_i(b_{1t}', \ldots, b_{nt}') = e_tf_i(b_{1t}, \ldots, b_{nt}) \in p$. This implies that $f_i(b_1', \ldots, b_n') \in p$. Therefore $f_i(b_1', \ldots, b_n') \in \bigcap_{p \in \text{Min}(A)} p = 0$ for all $i$.

$(v) \Rightarrow (vi)$: If $a/s \in S^{-1}A$ is an idempotent then there exists some $t \in S$ such that $ast(a - s) = 0$. It suffices to show that the following system

$$\begin{cases}
X = X^2 \\
st(a - sX) = 0
\end{cases}$$

has a solution in $A$. If $A$ is an integral domain then the above system having the solution $0_A$ or $1_A$, according as $ast = 0$ or $a = s$. Using this, then by the hypothesis the above system has a solution for every ring $A$ (not necessarily domain).

$(vi) \Rightarrow (i)$: If $p$ and $q$ are distinct minimal primes of $A$ then by Theorem 2.4 there exists an idempotent $f \in S^{-1}A$ such that $D(f) = \{S^{-1}q\}$ and $D(1 - f) = \{S^{-1}p\}$ where $S = A \setminus (p \cup q)$. By the hypothesis, there exists an idempotent $e \in A$ such that $e/1 = f$. It follows that $e \in p$ and $1 - e \in q$. □

**Corollary 5.16.** If the product of a family of rings $(A_i)$ is a reduced purified ring then each $A_i$ is a reduced purified ring.

**Proof.** Let $f_k/s_k \in S_k^{-1}A_k$ be an idempotent. Let $S$ be the set of all $(t_i) \in A = \prod_i A_i$ such that $t_k \in S_k$ and $t_i = 1$ for all $i \neq k$. Then clearly $S$ is a multiplicative set and $f/s \in S^{-1}A$ is an idempotent where $f = (f_i)$ and $s = (s_i)$ such that $f_i = 0$ and $s_i = 1$ for all $i \neq k$. Thus by Theorem 5.15 (vi), there exists an idempotent $e = (e_i) \in A$ such
that \( c/1 = f/s \). It follows that \( e_k/1 = f_k/s_k \). Therefore by Theorem 5.15 (vi), \( A_k \) is a reduced purified ring. \( \square \)

The converse of Corollary 5.16 is unknown for the authors. Prove or disprove of it would be certainly a non-trivial result.

The following result was proved by our student M.R. Rezaee Huri.

**Proposition 5.17.** Let \( p \) be a prime ideal of a ring \( A \). Then \( f \in \bigcap_{q \in \Lambda(p)} q \) if and only if there exists some \( g \in A \setminus p \) such that \( fg \) is nilpotent.

**Proof.** If \( f \in \bigcap_{q \in \Lambda(p)} q \) then \( f/1 \in \bigcap_{q \in \Lambda(p)} qA_p = \mathfrak{N} \) where \( \mathfrak{N} \) is the nilradical of \( A_p \). Thus there exist some \( g \in A \setminus p \) and a natural number \( n \geq 1 \) such that \( f^ng = 0 \). It follows that \( fg \) is nilpotent. \( \square \)

The following result is an immediate consequence of Proposition 5.17.

**Corollary 5.18.** ([16, Lemma 1.1] and [19, Lemma 3.1]) A prime ideal \( p \) of \( A \) is a minimal prime of \( A \) if and only if for each \( f \in p \) there exists some \( g \in A \setminus p \) such that \( fg \) is nilpotent. \( \square \)

In a subsequent work, we will investigate the geometric aspects of pm-rings and reduced mp-rings.

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