ABSTRACT

We examine the consistency of the thermodynamics of the most general class of conformally flat solution with an irrotational perfect fluid source (the Stephani Universes). For the case when the isometry group has dimension \( r \geq 2 \), the Gibbs-Duhem relation is always integrable, but if \( r < 2 \) it is only integrable for the particular subclass (containing FRW cosmologies) characterized by \( r = 1 \) and by admitting a conformal motion parallel to the 4-velocity. We provide explicit forms of the state variables and equations of state linking them. These formal thermodynamic relations are determined up to an arbitrary function of time which reduces to the FRW scale factor in the FRW limit of the solutions. We show that a formal identification of this free parameter with a FRW scale factor determined by FRW dynamics leads to an unphysical temperature evolution law. If this parameter is not identified with a FRW scale factor, it is possible to find examples of solutions and formal equations of state complying with suitable energy conditions and reasonable asymptotic behavior and temperature laws.

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I. Introduction

The “Stephani Universes” is the generic name for a class of metrics comprising the most general conformally flat solution with an irrotational perfect fluid source\(^1\)\(^\text{–}\)\(^6\). These solutions, admitting (in general) no isometries, generalize FRW spacetimes (their particular case with vanishing 4-acceleration), and could have a physical interest as simple inhomogeneous and anisotropic cosmological models. However, it is well known that Stephani Universes (except for the FRW subcase) do not admit a barotropic equation of state \(p = p(\rho)\), where \(p\) and \(\rho\) are the pressure and matter-energy density, perhaps explaining why so few references are found in the literature\(^3\),\(^7\),\(^8\) studying the physics of (non-FRW) Stephani Universes. However, barotropic equations of state might be too restrictive\(^9\), and so we aim in this paper to verify whether one can find more arguments to gauge the physical viability of these solutions besides simply dismissing them for not admitting a barotropic equation of state.

Coll and Ferrando\(^10\) addressed the question of the consistency of the thermodynamical equations with Einstein field equations, deriving rigorously the criterion to verify if a single component perfect fluid source of a given exact solution admits what these authors denote a “thermodynamic scheme”. However, Coll and Ferrando did not go beyond the admissibility of their consistency criterion, that is, into the physics of the fluids: note that it is perfectly possible to have unphysical fluids whose thermodynamics is formally correct. In a recent paper\(^11\), we have expanded and complemented the work of Coll and Ferrando by applying their criterion to irrotational, non-isentropic and geodesic fluids (the perfect fluid Szekeres solutions). Regarding the Stephani Universes, Bona and Coll\(^7\) did apply the work of Coll and Ferrando to these solutions, claiming that non-barotropic cases only admit a thermodynamic scheme if \(r \geq 2\) and presenting a very brief discussion of the thermodynamics of non-barotropic Stephani Universes with \(r = 2\). By providing a specific counterexample, we prove in this paper that the result of Bona and Coll is incorrect (see section V). We also provide a deeper discussion of the thermodynamics of non-barotropic Stephani Universes with \(r < 2\) admitting a thermodynamic scheme. It is important to specify that the study of this thermodynamics assumes the matter source to be a single
component perfect fluid. If the source were a mixture of perfect fluids, the study of its thermodynamics would involve looking at chemical potentials, and so would be an altogether different problem, a problem which will not be addressed in this paper. We assume henceforth that all mention of the term fluid indicates a single component fluid. The contents of this paper are described below.

We present in section II a summary of the equations of the thermodynamics of a general relativistic perfect fluid, together with the conditions for admissibility of a thermodynamic scheme. We rephrase the conditions derived by Coll and Ferrando in terms of differential forms expanded in a coordinate basis adapted to the comoving frame in which the Stephani Universes are usually described. These conditions are applied in section III to the Stephani Universes, yielding an interesting result: these solutions do not admit a thermodynamic scheme in general, that is, with unrestricted values of their free parameters. However, under suitable restrictions of these parameters, we find a specific subclass of non-barotropic Stephani Universes which does admit a thermodynamic scheme when $r < 2$. This subclass is the counter example to the work of Bona and Coll mentioned above.

In section IV, we derive for the non-barotropic Stephani Universes complying with a thermodynamic scheme explicit expressions of all state variables: $\rho$ and $p$, particle number density $n$, specific entropy $S$ and temperature $T$, as well as two-parameter equations of state linking them. The latter turn out to be difficult to interpret as there is no clue on how to fix the only time dependent free parameter of the solutions and its relation with the matter energy density. We explore in section V the strategy which consists in formally identifying these quantities with the scale factor and matter-energy density of a FRW limiting spacetime assumed to comply with a “gamma law”equation of state. The resulting temperature evolution law is unphysical and does not reduce in the FRW limit to that expected for a FRW cosmology with such an equation of state. On the other hand, if this identification with FRW parameters is abandoned, we show in section VI that it is possible to obtain temperature evolution laws and equations of state which, being more formal than physical, are not altogether unphysical and do comply with suitable energy conditions. Conclusions are presented and summarized in section VII.
We prove in the Appendix that Stephani Universes admitting a thermodynamic scheme are characterized invariantly by admitting: (a) a spacelike Killing vector \((r = 1)\) and (b) a conformal Killing vector field parallel to the 4-velocity. In fact, we show that these Stephani Universes comprise the most general class of solutions with an irrotational perfect fluid source admitting this type of conformal symmetry.

II. Thermodynamics of a non-isentropic irrotational perfect fluid.

Consider the energy–momentum tensor for a perfect fluid

\[
T^{ab} = (\rho + p)u^a u^b + p g^{ab} \tag{1}
\]

where \(\rho\), \(p\) and \(u^a\) are the matter-energy density, pressure and 4-velocity, respectively. This tensor satisfies the conservation law \(T^a_{;b} = 0\) which implies the contracted Bianchi identities

\[
\dot{\rho} + (\rho + p)\Theta = 0 \tag{2a}
\]

\[
h^b_a p_{;b} + (\rho + p)\dot{u}_a = 0 \tag{2b}
\]

where \(\Theta = u^a_{;a}\), \(\dot{u}_a = u_{a;b} u^b\) and \(h^b_a = \delta^b_a + u_a u^b\) are respectively the expansion, 4-acceleration and projection tensor and \(\dot{\rho} = u^a \rho_{;a}\). The thermodynamics of a perfect fluid is essentially contained in the matter conservation law, the condition of vanishing entropy production and the Gibbs-Duhem relation. The first two are given by

\[
(nu^a)_{;a} = 0 \tag{3a}
\]

\[
(nS u^a)_{;a} = 0 \tag{3b}
\]

where \(n\) is the particle number density and \(S\) is the specific entropy. Condition (3a) inserted in (3b) leads to \(u^a S_{;a} = \dot{S} = 0\), so that \(S\) is conserved along the fluid lines but is not a universal constant. In the latter case we have: \(dS = 0\), and the fluid is isentropic,
admitting a barotropic equation of state. The Gibbs-Duhem relation can be given as the
1-form

$$\omega = dS = \frac{1}{T} \left[ d \left( \frac{\rho}{n} \right) + p d \left( \frac{1}{n} \right) \right]$$

(4)

where $T$ is the temperature. The necessary and sufficient condition for the integrability of (4)

$$\omega \wedge d\omega = 0 \quad \text{necessary and sufficient}$$

(5)

subjected to fulfilment of the conservation laws (2) and (3), are the conditions which Coll
and Ferrando denote admissibility of a “thermodynamic scheme”. These conditions were
given by these authors as

$$(\dot{\rho}d\dot{\rho} - \dot{\rho}d\dot{\rho}) \wedge d\rho \wedge d\rho = 0$$

(6)

Another integrability condition, not examined by Coll and Ferrando, is

$$d\omega = 0 \quad \text{sufficient}$$

(7)

The perfect fluid source of the Stephani Universes is characterized by an irrotational
(hence, hypersurface orthogonal) and shear-free 4-velocity. For such a fluid source there
exist\textsuperscript{2,4,5} local comoving coordinates $(t, x^i)$, such that the metric, 4-velocity, 4-acceleration,
expansion and projection tensor are given by

$$ds^2 = -N^2 dt^2 + L^2 \delta_{ij} dx^i dx^j$$

(8a)

$$N = \frac{L_{,i}/L}{\Theta/3}$$

$$u^a = N^{-1} \delta_t^a \quad \dot{u}_a = (\log N)_{,a} \delta_t^a$$

(8b)

$$h_{ab} = g_{ij} \delta^i_a \delta^j_b$$

(8c)
where $\Theta = \Theta(t)$ and the metric function $L$ is (in general) a function of all the coordinates $(t, x^i)$. In this representation $\dot{X} = (1/N)X_t$ for all scalar functions and the Bianchi identities and conservation laws (2) and (3) become

\begin{align*}
\rho,_{t} + (\rho + p)(\log n),_{t} &= 0 \quad (9a) \\
p,_{i} + (\rho + p)(\log N),_{i} &= 0 \quad (9b)
\end{align*}

\begin{equation}
n = \frac{n_0(x^i)}{L^3} \quad (9c)
\end{equation}

\begin{equation}
S = S(x^i) \quad (9d)
\end{equation}

where $n_0(x^i)$ appearing in (9c) is an arbitrary function denoting the conserved particle number distribution. In the the coordinate basis of 1-forms $(dt, dx^i)$ associated with the comoving frame (8), the Gibbs-Duhem relation reads

\begin{equation}
\omega = S,_{i}dx^i = \frac{1}{T}\left[\left(\frac{\rho}{n}\right),_{i} + p\left(\frac{1}{n}\right),_{i}\right]dx^i \quad (10)
\end{equation}

where the $t$ component of $\omega$ in this coordinate basis vanishes due to (9d). A sufficient integrability condition of (10) is given by

\begin{equation}
d\omega = W_{ti}\frac{dt \wedge dx^i}{nT} + W_{ij}\frac{dx^i \wedge dx^j}{nT} = 0 \quad (11)
\end{equation}

\begin{align*}
W_{ti} &= \frac{p_{[i}n_{,t]} - n^2T_{[i}T_{,t}S_{,i]}}{n} = (\rho + p)\left(\left(\frac{n,_{i}T,_{t}}{n} - \dot{u}_t\frac{n,_{t}}{n}\right) - \left(\rho,_{i}\frac{T,_{t}}{T} + p,_{i}\frac{n,_{i}}{n}\right)\right) \\
W_{ij} &= \frac{p_{[i}n_{,j]} + n^2T_{[i}S_{,j]}}{n} = \frac{T_{[i}p_{,j]} - (\rho + p)\left(\frac{T_{[i} + T\dot{u}_{[i}}}{T}\right) \frac{n,_{j]}{n}}{n}
\end{align*}

where square brackets denote antisymmetrization on the corresponding indices. The necessary and sufficient condition (5) is given by
\[ d\omega \wedge \omega = X_{ijk} \frac{dx^i \wedge dx^j \wedge dx^k}{n^3 T^2} + X_{tij} \frac{dt \wedge dx^i \wedge dx^j}{n^3 T^2} = 0 \] (12)

\[ X_{ijk} = -\rho_{[i}p_{,j}n_{,k]} \]

\[ X_{tij} = \rho_{[t}p_{,i}n_{,j]} \]

Conditions (12) are entirely equivalent to (6) provided by Coll and Ferrando. One can obtain the latter form of the former simply by using (2) and (3) (in their forms (9)). However, (11) and (12) are more intuitive than (5) and (6), as they directly incorporate state variables such as \( n, S \) and \( T \), and their relations with \( \rho \) and \( p \). Condition (12) is also more practical than (6), as it is easier to use \( n \) and \( S \) from (9c) and (9d) than to compute the set \( (\rho, \dot{\rho}, p, \dot{p}) \) in exact solutions in which these quantities can be quite cumbersome. The sufficient condition (11), not examined by Coll and Ferrando, is also helpful, since if its fulfilment guarantees that (6) (or (12)) holds.

As shown in the following section, if a solution of Einstein equations is available (thus providing \( \rho \) and \( p \) in terms of the metric functions) it is straightforward to verify the admissibility of the thermodynamic scheme. This we will do for the Stephani Universes which are particular cases of (8), and to do so we suggest the following procedure: (a) solve the conditions (12) and substitute the solution into (10), thus identifying possible (non-unique) forms for \( S \) and \( T \); insert the obtained forms of \( T \) and \( n \) into (11) in order to verify if further restrictions follow from the sufficient conditions. If these conditions hold, the equations of state linking the state variables \( (\rho, p, n, S, T) \) (together with their functional relation with respect to the metric functions) follow directly from integrating them.

III. The Stephani Universes.

The Stephani Universes are described by the particular case of (8a) given by:

\[ L(t, x^i) = \frac{R}{1 + 2A_i x^i + (A^2 + (k/4)R^2)\delta_{ij}x^i x^j} \] (13)
with $A_i(t) = (A_x(t), A_y(t), A_z(t))$, $A^2 \equiv \delta^{ij} A_i A_j$, $k(t)$ and $R(t)$ are arbitrary functions. Notice that the Stephani Universes contain FRW spacetimes as the particular case $A_i = 0$, $k = k_0 = \text{const.}$ in (8) and (13). As it is well known that the latter are the subclass of barotropic Stephani Universes, we assume hereafter (and unless stated otherwise) that all mention of Stephani Universes excludes their FRW subclass. The field equations associated with (8) and (13) are

$$\rho = \rho(t) = \frac{\Theta^2}{3} + 3k$$

(14a)

$$p = -\rho - \frac{\rho, t}{3L, t / L}$$

(14b)

where the contracted Bianchi identity (9a) has been used as a definition of $p$. The remaining Bianchi identities and conservation laws are given by equations (9b-d), while the Gibbs-Duhem 1-form (10) and the integrability conditions (11) and (12) follow as

$$S_{ti} = \frac{(\rho + p)}{T} \left( \frac{1}{n} \right)_{i}$$

(15)

$$W_{ti} = \left[ (\rho + p) \frac{T, t}{T} - p, t \right] n, i - (\rho + p) u_i n, t = 0$$

(16a)

$$W_{ij} = 0 \Rightarrow \Omega_{[i} (\log n)_{, j]} = 0$$

(16b)

where

$$\Omega_i \equiv (\log T), i + \dot{u}_i = (\log TN), i$$

(16c)

$$X_{tij} = 0 \Rightarrow p_{[i} n_{, j]} = 0 \Rightarrow (\log N)_{[i, i} (\log n)_{, j]} = 0$$

(17)

Inserting the forms of $N$ and $n$ given by (8a) (9c) and (13) into the necessary and sufficient condition (17) yields a general solution of the latter given by

$$a(t) \log \left( \frac{L, t}{L} \right) + b(t) \log \left( \frac{f}{L^3} \right) = \log (c(t))$$

(18)
where \((a, b, c)\) are arbitrary functions. This integrability condition is not satisfied in general, that is for arbitrary forms of the free functions \((A_i, k, R)\) appearing in the metric function (13). Bona and Coll\(^7\) have claimed that this condition is only satisfied by Stephani Universes with isometry groups of dimension \(r \geq 2\). However, it is possible to provide a particular case of this metric (characterized by \(r = 1\), see Appendix) which satisfies (16) and (17) and so, leads to well defined forms for \(T\) and \(S\). This case is characterized by the existence of a conformal Killing vector field parallel to the 4-velocity (see Appendix), the corresponding forms of \(A_i\) and \(k\) are

\[
A_i = a_i R, \quad k = \frac{k_0}{R^2} + \frac{b_0}{R} - 4\delta^{ij}a_ia_j \tag{19a}
\]

where \((a_i, k_0, b_0)\) are arbitrary constants and \(R\) remains arbitrary. With these parameter values, (18) holds with

\[
a(t) = 1, \quad b(t) = 1/3, \quad c(t) = -(1/R)_t \tag{19b}
\]

\[
n_0 = f^{-3} \quad \text{where:} \quad f \equiv 1 + \frac{1}{4}k_0\delta_{ij}x^ix^j \tag{19c}
\]

Inserting (19a) and (19c) into (15) yields

\[
TS_i = -3(\rho + p)(Lf)^4(F/f)_i \tag{20a}
\]

where

\[
F = 2a_ix^i + \frac{1}{4}b_0\delta_{ij}x^ix^j \tag{20b}
\]

This equation shows how non unique forms of \(S\) and \(T\) emerge if we demand only the fulfilment of condition (17) (or (18)). A general expression for these quantities, in agreement with (15), is given by

\[
S = S(\sigma) \quad \sigma \equiv -\frac{F}{f} \tag{21a}
\]

9
\[ T = \frac{3(\rho + p)}{S' n^{4/3}} \]  \hspace{1cm} (21b)

where a prime denotes derivative with respect to \( \sigma \) and (9c) (13) have been used to eliminate \( L \) in terms of \( n \). Regarding the sufficient conditions (16), \( T \) given by (21b) satisfies (16a). This is easily verified by eliminating \( \dot{u}_i \) from (16c) and inserting this result together with \( T_{t}/T \) computed from (21b) into (16a). On the other hand, inserting (21b) into (16b) and using (17) leads to the condition \( S'' = 0 \), so that \( S \) is a linear function of \( \sigma \). This yields the following forms for \( T \) and \( S \):

\[ S = S_0 + \sigma \]  \hspace{1cm} (23a)

\[ T = \frac{3(\rho + p)}{n^{4/3}} \]  \hspace{1cm} (23b)

which are compatible with both sets of conditions (16) and (17). A discussion on the interpretation of the expressions derived above is provided in the following sections.

**IV. Formal equations of state.**

The particular subclass of Stephani Universes complying with the thermodynamic scheme, as characterized by the parameter restrictions (19a), is described by the conformally FRW metric

\[ ds^2 = \Phi^2 \left[ -dt^2 + \frac{R^2 \delta_{ij} dx^i dx^j}{(1 + \frac{1}{4} k_0 \delta_{ij} x^i x^j)^2} \right] \]  \hspace{1cm} (24a)

\[ \Phi = \frac{1 + \frac{1}{2} k_0 \delta_{ij} x^i x^j}{1 + \frac{1}{4} k_0 \delta_{ij} x^i x^j + R (2 a_i x^i + \frac{1}{4} b_0 \delta_{ij} x^i x^j)} = \frac{f}{f + RF} \]  \hspace{1cm} (24b)

Its corresponding field equations are

\[ \frac{1}{3} \rho = \left( \frac{R_t}{R} \right)^2 + \frac{k_0}{R^2} + \frac{b_0}{R} - 4 \delta^{ij} a_i a_j \]  \hspace{1cm} (25a)

\[ p = -\rho - \frac{1}{3} R \rho_{,R} (1 - R \sigma) \]  \hspace{1cm} (25b)
where we have chosen $\Theta/3 = R_t/R$ and $\sigma$ is given by (21a). These Stephani Universes can be characterized invariantly by the existence of a conformal Killing vector field parallel to the 4-velocity (see Appendix). Assuming the forms of $T$ and $S$ from (23), the term $\sigma$ appearing in (25b) (and in other expressions) is a linear function of $S$. Also, since $\rho$ and $R$ are both functions of $t$, all terms involving $R$ and $\rho, R$ can be expressed as functions of $\rho$. This results in the following forms for generic equations of state, expressing $p$, $n$ and $T$ in terms of $\rho$ and $S$

\[ p(\rho, S) = -\rho - \frac{1}{3} R \rho, R \left[ 1 + (S - S_0) R \right] \]  

(26a)

\[ n(\rho, S) = \left[ \frac{1}{R} + S - S_0 \right]^3 \]  

(26b)

\[ T(\rho, S) = \frac{-3 R^5 \rho, R}{\left[ 1 + (S - S_0) R \right]^3} \]  

(26c)

These are formal equations of state, in the sense that they are not dictated by physical considerations and imposed before solving the constraints of the field equations (physical equations of state), but arise as a consequence of imposing the fulfilment of the thermodynamic scheme on metric functions whose spacial dependence has been fixed by imposing conformal flatness (a geometric constraint: vanishing of the Weyl tensor). The best one can do in this case is to verify if these formal thermodynamic relations could be manipulated in such a way that solutions (24) could describe physically reasonable cosmologies. However, equations (25) are still undetermined: a choice of $\rho(R)$ must be made in order to determine these equations and to be able to integrate the Friedmann-like equation (25a) to yield the time evolution of the metric. Unfortunately, there is no clear cut way guiding one on how to select $\rho(R)$. Various possibilities are explored below.

V. FRW limit.

The metric (24) bears a close resemblance to a FRW metric. As equations (24)-(25) reveal, the term $\sigma$, related to the spacially dependent entropy density, is the term that makes these solutions inhomogeneous and anisotropic (i.e. non-FRW). In fact, their
FRW limit follows if \( b_0 \to 0 \) and \( a_i \to 0 \), so that \( \sigma \to 0 \). Under this limit, the metric and field equations (25) become the metric and field equations of a perfect fluid FRW spacetime with scale factor \( R \) and \( k_0 = 0, \pm 1 \) marking the curvature of the spacial sections. This correspondence and resemblance to FRW cosmologies motivates us to verify if these solutions could be considered as some sort of “near-FRW” cosmologies and if they could be examined within the framework of a FRW limit. Hence we pose the question of whether the state variables and their equations of state become those one would expect of a “near-FRW” cosmology if we assume that \( R \) can be fixed as it were a FRW scale factor. Along these lines, we notice that the state variables \( \rho, p \) and \( n \), given by (25a), (25b), (26a) and (26b), also tend to their FRW values, but the limiting form of \( T \) in (26c) takes the strange form \( T \to -R^{5}\rho_{,R} \). In order to see what sort of temperature law and equations of state correspond to this FRW limit, we assume a “gamma law” equation of state \( p_0 = (\gamma - 1)\rho \), where \( p_0 \) is given by setting \( S = S_0 \) in (26a), the form of \( \rho(R) \) becomes

\[
\rho(R) = \left( \frac{R}{R_0} \right)^{-3\gamma}
\]

and so the various forms (26) of the equation of state become

\[
p(\rho, S) = (\gamma - 1)\rho + \frac{1}{3}\gamma R_0 \rho^{1-1/3\gamma}(S - S_0) \tag{27a}
\]

\[
n(\rho, S) = \left[ \rho^{1/3\gamma} + S - S_0 \right]^3 \tag{27b}
\]

\[
T(\rho, S) = \frac{\gamma R_0^{4}\rho^{1-1/3\gamma}}{\left[ \rho^{1/3\gamma} + S - S_0 \right]^3} \tag{27c}
\]

Irrespective of the interpretation of these strange formal thermodynamic relations, notice that \( p \) and \( n \) do reduce to their FRW values in the FRW limit, though \( T \to \rho^{1-4/3\gamma} \propto R^{4-3\gamma} \), a temperature evolution law which has no relation to that expected at the FRW limit: for \( \gamma = 1 \) (dust), \( T \propto R \) instead of \( T = \text{const.} \) and for \( \gamma = 4/3 \) (radiation), \( T \propto \text{const.} \) instead of \( T \propto R^{-1} \). Therefore, this formal identification of \( R \) with the FRW scale factor leads to contradictory results. Of course, one could also consider \( R \) as a FRW
scale factor associated with other equations of state, however, the presence of the strange term \(-R^5 \rho_R\) in (26c) makes it highly improbable for this temperature evolution law to have any meaningful correspondence with that of a limiting FRW cosmology.

VI. Examples complying with minimal physical requirements.

Looking at the equations of state (26) as formal thermodynamic relations, which hopefully might provide at least a gross approximation to physical relations, and if one is prepared to abandon the identification of \(R\) with a specific FRW scale factor, it is possible at least to verify if the free parameters \((a_i, k_0, b_0)\) and \(R(t)\) can be selected to assure that solutions (24) and equations (26) comply with minimal physical conditions, such as energy conditions and an acceptable asymptotic behavior. In order to examine equations (24)-(26) within this framework, consider the coordinate transformation

\[
\begin{align*}
  x &= 2 \Gamma(\chi/2) \sin \theta \cos \varphi \\
  y &= 2 \Gamma(\chi/2) \sin \theta \sin \varphi \\
  z &= 2 \Gamma(\chi/2) \cos \theta
\end{align*}
\]

which brings the metric (24) into the simple form

\[
ds^2 = \frac{-dt^2 + R^2 [d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2 \theta d\varphi^2)]}{[1 + R (2 \Sigma(\chi) V(\theta, \varphi) + b_0 \Sigma^2(\chi/2))^2]}
\]

\[
\Sigma(\chi) = \begin{cases} 
  \sin \chi & k_0 = 1 \\
  \sinh \chi & k_0 = -1
\end{cases}
\]

\[
V(\theta, \varphi) = a_1 \sin \theta \cos \varphi + a_2 \sin \theta \sin \varphi + a_3 \cos \theta
\]

The state variables in (26) are unaffected by this coordinate transformation, with \(\sigma\) and the function \(n_0\) in (19c) which provides the conserved particle number at an initial hypersurface \(t = \text{const.}\) now given by

\[
\sigma = 2 \Sigma(\chi) V(\theta, \varphi) + b_0 \Sigma^2(\chi/2)
\]
\[ n_0 = \begin{cases} \cos^6\left(\frac{1}{2}\chi\right) & k_0 = 1 \\ \cosh^6\left(\frac{1}{2}\chi\right) & k_0 = -1 \end{cases} \] (29b)

From (28c), the term \( V \) containing the dependence on \((\theta, \varphi)\) is bounded, hence \( \sigma = -F/f \) might only diverge along \( \chi \to \pm\infty \) (irrespective of the value of \( \rho = \rho(t) \)) for the case \( k_0 = -1 \). This fact can be examined from another angle: the function \( \sigma \) (and so, the entropy density \( S \)) can be expressed in terms of \( n_0 \) as

\[ S = S_0 + \sigma = S_0 + 4n_0^{1/6}(1 - k_0n_0^{1/3})^{1/2} + b_0(1 - k_0n_0^{1/3}) \] (29c)

and so, for \( k_0 = -1 \), the initial particle number \( n_0 \) becomes infinite as \( \chi \to \pm\infty \), this infinite concentration of particles causes the entropy density \( S \) and the pressure to diverge along these limits. This means that solutions with \( k_0 = -1 \) have an undesirable asymptotic behavior. On the other hand, the locus \( R = 0 \) marks another singularity, analogous to a FRW big bang, while the vanishing of the denominator in (28a) indicates an asymptotically deSitter evolution \( (p \to -\rho, n \to \infty) \) characterized by the unphysical behavior \( T \to \infty \).

Therefore, for the formal equations of state (26) to have any physical meaning, we must choose \( k_0 = 1 \) and demand the condition \( 1 - R\sigma \neq 0 \) to hold, together with the dominant and weak energy conditions which can be combined into the restriction: \( 0 \leq p/\rho \leq 1 \).

Also, all state variables must diverge at the big-bang singularity \( R = 0 \). From equations (26), the conditions \( 0 \leq p/\rho \leq 1 \) and \( T \to \infty \) hold at the limit \( R \to 0 \) if \( \rho \approx R^{-(4+m)} \) for \( m > 0 \) at this limit, leading to the following asymptotic values:

\[ \frac{p}{\rho} \approx \frac{1}{3}(m + 1) \] (30a)

\[ n \approx \frac{1}{R^3} \approx \rho^{3/(m+4)} \] (30b)

\[ T \approx \frac{m + 4}{R^m} \approx (m + 4)\rho^{m/(m+4)} \] (30c)

as \( R \to 0 \). Since \( R \) is no longer constrained to be interpreted as a sort of FRW scale factor in a FRW limit, we can devise a simple example complying with the conditions mentioned.
above and the asymptotic limits (30) by choosing a simple power law $\rho = (R_0/R)^6$, where $R_0$ is a constant. This choice leads to the following state variables

$$p(\rho, S) = \rho + 2R_0(S - S_0)^{5/6} \quad (31a)$$

$$n(\rho, S) = \left[\rho^{1/6} + S - S_0\right]^3 \quad (31b)$$

$$T(\rho, S) = \frac{6R_0^4\rho^{5/6}}{[\rho^{1/6} + S - S_0]^3} \quad (31c)$$

which yield a stiff fluid equation of state $p/\rho \to 1$ and $T \to \infty$ near the big-bang singularity ($R = 0$ and/or $\rho \to \infty$). However, as $R \to \infty$ (or $\rho \to 0$), the dominant energy condition could be violated: $p/\rho \to \infty$. This can be avoided by selecting the remaining arbitrary constants $(a_i, b_0)$ in such a way that $S_0 > S$ and $1 - R\sigma = 1 + R(S - S_0) > 0$ holds everywhere and the Friedmann-like equation

$$(\frac{R_{,t}}{R})^2 = \frac{R_0^6}{3R^6} - \frac{1}{R^2} - \frac{b_0}{R} + 4\delta^{ij}a_ia_j \quad (32a)$$

obtained by substituting $k_0 = 1$ and $\rho = (R_0/R)^6$ into (25a), has no solutions $R(t)$ allowing for $R \to \infty$, or equivalently, that $\rho$ does not vanish along the time evolution of the fluid. These conditions require (32a) to have a real positive root and

$$\frac{R_{,,t}}{R} = -\frac{2}{3} \left(\frac{R_0}{R}\right)^6 - \frac{b_0}{2R} + 4\delta^{ij}a_ia_j < 0 \quad (32b)$$

along this root, hence $R(t)$ is convex. If we assume $R = R_0 > 0$ to be the value along which $R_{,t}/R = \Theta/3$ vanishes, equation (32a) fixes $R_0$ in terms of the parameters $(a_i, b_0)$ as the positive root of:

$$\left(\frac{1}{3} + 4\delta^{ij}a_ia_j\right)R_0^2 - b_0R_0 - 1 = 0 \quad (32c)$$

which substituted into (32b) yields the condition of convexity. It is not difficult to find combinations of parameters $(a_i, b_0)$ so that the fluid has the desired type of kinematic evolution (qualitatively analogous to a standard “closed” FRW cosmology) and physically
correct behavior of the state variables: that is, to have the ratio \( p/\rho = 1 \) and \( T \to \infty \) at the big bang evolving to values in the range \( 0 \leq p/\rho < 1 \) and cooling to \( T \) finite as the fluid reaches its maximum expansion, bounces and then recollapses with \( p/\rho = 1 \) and \( T \to \infty \) at the big crunch. Such an example is illustrated in figure 1.

VII. Conclusions.

We have investigated the consistency of the thermodynamic equations following from the condition of existence of a thermodynamic scheme for the Stephani Universes, whose source is a non-isentropic perfect fluid (thus, not admitting a barotropic equation of state). This work has aimed at improving the study of this type of solutions, as classical fluid models generalizing FRW cosmologies, in contrast to a widespread attitude of simply disregarding them for not admitting a barotropic equation of state.

For the particular subclass of Stephani Universes admitting a thermodynamic scheme, the resulting equations of state have an ellusive interpretation, as there is no blue print on how to select the time dependent free parameter of the solutions, an arbitrary function \( R \) reducing to the FRW scale factor in the FRW limit. We have shown that by formally identifying this parameter with the FRW scale factor of a FRW cosmology satisfying a “gamma law” equation of state leads to an unphysical temperature evolution law, totally unrelated to that of their limiting FRW cosmology. The question of how to select these parameters in a convenient way remains unsolved, though the adequate theoretical framework to carry this task has been presented in section VI. We have shown that combinations of free parameters exist so that the formal equations of state comply with minimal physical requirements.

We have shown that Stephani Universes (other than the FRW subclass or the subclass presented in previous sections) are not compatible, in general, with a thermodynamic scheme. This fact seems to disqualify these solutions as classical fluids of physical interest. However, the latter can still be useful if they are examined under a less restrictive framework than that of the simple perfect fluid. In this context, the Stephani Universes (like other perfect fluid solutions with a shear-free 4-velocity) can be recast as exact solutions.
for a fluid with a bulk viscous stress\textsuperscript{12}. From this point of view, the thermodynamics is not only totally different as that of the perfect fluid case but much less restrictive and more amenable to satisfy the criteria for constructing inhomogeneous and anisotropic cosmological models of physical interest.

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Appendix. Isometries and conformal symmetries.

(a) The metric (24) admits a one-parameter group of isometries.

Being conformally flat, (24) admits a $G_{15}$ of conformal symmetries. Various isometry subgroups are readily identified. If $k_0 = 0$, irrespective of the values of the remaining constants $a_i$, (24) becomes spherically symmetric (this can be verified through re-scalings of the form $x^i = \bar{x}^i + c^i$). Consider now (24) with $k_0 = \pm 1$ and $a_i$ arbitrary nonzero constants. Since we are assuming this spacetime to be non-static ($\Theta = 3R_{\ell t}/R \neq 0$), its Killing vectors (if they exist) must all be tangent to the hypersurfaces orthogonal to the 4-velocity\textsuperscript{15}. Spherical symmetry is easily identified by setting $a_i = 0$ and $b_0 \neq 0$, while FRW subcases follow from $a_i = 0 = b_0$. However, for general values of these constant parameters, it is not easy to identify at first glance the existence of isometries groups $G_1$ or $G_3$ acting along the hypersurface orthogonal to the 4-velocity.

In order to find out if (24) admits isometries, and if so, the dimension of their orbits,
we follow the result of theorem 3 in Bona and Coll\textsuperscript{6}. Identifying carefully the parameters used by these authors: $a$, $b$ and $\phi$ (see their equations (2)-(4)) with the corresponding parameters in (8) and (13), we find that their function $a$ is our function $R$, their vector $b$ is the vector formed by the functions $A_i$ in (13), while $\phi = A^2 + (k/4) \ast R^2$, that is, the coefficient of the quadratic term in the denominator of (13). The case complying with the thermodynamic scheme (the metric (24)), is defined by (19a), thus we have for this metric $b = (a_1, a_2, a_3)R$ and $\phi = b_0R$. Therefore, metric (24) with arbitrary values of its free parameters $R$, $a_i$, $b_0$, corresponds to the case: $\dot{b} \neq 0$, $\ddot{b} \neq 0$ with $\dot{b} \wedge \ddot{b} = 0$ and with $\phi = b_0 \dot{R} \neq 0$. According to the classification of page 616 of Bona and Coll\textsuperscript{6}, the metrics (24) admit a one parameter isometry group with $r = 1$, a case these authors identify as axially symmetric.

(b) All Stephani Universes associated with the metric (24) admit a conformal Killing vector parallel to the 4-velocity.

We have proven that the metric (24) describes a class of Stephani Universes admitting a thermodynamic scheme and having an isometry group of dimension $r < 2$. On the other hand, it is well known\textsuperscript{16} that the 4-velocity in FRW spacetimes ($u^a = \delta^a_t$ in the comoving coordinates of (24)) is a conformal Killing vector, satisfying $\xi_{(a;b)} = \psi_{(o)} g_{ab}$ with scale factor $\psi_{(o)} = R_{,t}$. Since (24) is conformally related to a FRW metric, the vector field $\xi^a = \delta^a_t$ in (24) (parallel to the 4-velocity $u^a = (-\Phi)^{-1}\delta^a_t$) is a conformal Killing vector $\xi^a$ in (24), satisfying $\xi_{(a;b)} = \psi g_{ab}$, with conformal factor $\psi$ given by\textsuperscript{16}

$$
\psi = \psi_{(o)} + \xi^a (\log \Phi),_a = R_{,t} \left(1 - \frac{F}{f + RF} \right) \quad (A1)
$$

(c) The metric (24) describes the most general perfect fluid spacetime admitting a conformal Killing vector parallel to the 4-velocity.

It is known\textsuperscript{17} that the existence of a conformal symmetry of this type ($\xi^a = \Omega u^a$, satisfying $\xi_{(a;b)} = \psi g_{ab}$) requires the fluid 4-velocity to be shear-free, with the remaining kinematic parameters given by
\[ \dot{u}_a = u_{a;b}u^b = h^b_a(\log \Omega)_b \quad (A2) \]

\[ \omega_{ab} = u_{[a;b]} + \dot{u}_{[a}u_{b]} = \Omega(\Omega^{-1}\xi_{[a};b]) \quad (A3) \]

\[ \frac{1}{3}\Theta = \frac{1}{3}u^a_{;a} = u^a(\log \Omega)_a = \frac{\psi}{\Omega} \quad (A4) \]

Consider the irrotational case \( \omega_{ab} = 0 \), as described by the comoving coordinates of (8) but with \( L \) not necessarily equal to (13). The specific form of the 4-acceleration in (A2) implies the constraint

\[ \left(1/L\right)_t = \frac{1}{3}\Theta R J(x^i) \]

where \( J \) is an arbitrary function. Inserting this constraint into (8), and demanding the fulfilment of the Einstein field equations \( G_{ij} = 0 \) and \( G_{ii} - G_{jj} = 0 \) \((i \neq j)\), the metric (8) becomes (24a) with \( J = f \).

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Figure Caption

Figure 1. Thermodynamic quantities associated with the example presented in section VI.

We chose the following values of the constant free parameter $s$: $a_1 = 10^{-4}$, $a_2 = 9 \times 10^{-5}$, $a_3 = 1.1 \times 10^{-4}$, $b_0 = -9/20$, so that the positive solution of (32c) is $R_0 \approx 1.18$. Figure (1a) displays $\Theta/3 = R,\theta/R$ vs. $R$, showing that the latter is bounded between $R = 0$ and $R = R_0$, with the latter constant being the value of $R$ at which $\Theta$ vanishes. This means that $\rho$ evolves between $\rho = 1$ at $R = R_0$ and infinity at $R = 0$ (big bang and big crunch singularities). Figure (1b) exhibits the ratio $p/\rho$ in terms of $R$ and the “radial” coordinate $\chi$, for $\theta = \pi/2$ and $\phi = 0$. Since the constants $a_i$ are very small, the plot looks qualitatively analogous for all values of these “angular” coordinates. Notice how the constraint $0 < p/\rho \leq 1$ holds throughout the evolution of the fluid, from $p/\rho = 1$ at $R = 0$, keeping the same value along $\chi = 0$ and evolving to $p/\rho \approx 0$ at $R = R_0$ and $\chi \approx \pm \pi$. Figures (1c) and (1d) display equations of state (31a) and (31c) The ratio $p/\rho$ and $T$ are plotted in terms of arctan($\rho$) in the range (arctan(1), $\pi/2$) for various values of $S = S_0 - \sigma$ in the range between $S = 0$ and $S = S_0 = 9/20$. Notice that $p/\rho \rightarrow 1$ and $T \rightarrow \infty$ as arctan($\rho$) $\rightarrow \pi/2$ (that is: $\rho \rightarrow \infty$) for all values of $S$. As $\rho \rightarrow 1$, $T$ and $p/\rho$ decrease at various rates depending on the value of $S$. The parameter values in these plots do not follow from any physical consideration, we simply aim to illustrate that free parameters exist so that the solutions satisfy the minimal physical conditions discussed in section VI. These plots were obtained with the symbolic computing program MAPLE.