THE SINGULARITY CATEGORY OF AN ALGEBRA WITH RADICAL SQUARE ZERO

XIAO-WU CHEN

Abstract. To an artin algebra with radical square zero, a regular algebra in the sense of von Neumann and a family of invertible bimodules over the regular algebra are associated. These data describe completely, as a triangulated category, the singularity category of the artin algebra. A criterion on the Hom-finiteness of the singularity category is given in terms of the valued quiver of the artin algebra.

1. Introduction

Let $R$ be a commutative artinian ring. All algebras, categories and functors are $R$-linear. We recall that an $R$-linear category is Hom-finite provided that all the Hom sets are finitely generated $R$-modules.

Let $A$ be an artin $R$-algebra. Denote by $A$-mod the category of finitely generated left $A$-modules, and by $D^b(A$-mod) the bounded derived category. Following [14], the singularity category $D_{sg}(A)$ is the quotient triangulated category of $D^b(A$-mod) with respect to the full subcategory formed by perfect complexes; see also [3, 12, 10, 14, 2] and [13]. Here, we recall that a complex in $D^b(A$-mod) is perfect provided that it is isomorphic to a bounded complex consisting of finitely generated projective modules.

The singularity category measures the homological singularity of an algebra in the sense that an algebra $A$ has finite global dimension if and only if its singularity category $D_{sg}(A)$ vanishes. In the meantime, the singularity category captures the stable homological features of an algebra [3]. A fundamental result of Buchweitz and Happel states that for a Gorenstein algebra $A$, the singularity category $D_{sg}(A)$ is triangle equivalent to the stable category of (maximal) Cohen-Macaulay $A$-modules [3, 10]. This implies in particular that the singularity category of a Gorenstein algebra is Hom-finite and has Auslander-Reiten triangles. We point out that Buchweitz and Happel’s result specializes to Rickard’s result [15] on self-injective algebras. However, for non-Gorenstein algebras, not much is known about their singularity categories [4].

Our aim is to describe the singularity category of an algebra with radical square zero. We point out that such algebras are usually non-Gorenstein [5]. In what follows, we describe the results in this paper.

We denote by $r$ the Jacobson radical of $A$. The algebra $A$ is said to be with radical square zero provided that $r^2 = 0$. In this case, $r$ has a natural $A/r$-$A/r$-bimodule structure. Set $r^{⊗0} = A/r$ and $r^{⊗i+1} = r^{⊗i} A/r$ for $i ≥ 0$. Then there are obvious algebra homomorphisms $End_{A/r}(r^{⊗i}) → End_{A/r}(r^{⊗i+1})$ induced by $r^{⊗i} A/r$. We
denote by $\Gamma(A)$ the direct limit of this chain of algebra homomorphisms. It is a regular algebra \( (\mathbb{T}, S) \) in the sense of von Neumann. We call $\Gamma(A)$ the associated regular algebra of $A$. In most cases, the algebra $\Gamma(A)$ is not semisimple.

For $n \in \mathbb{Z}$ and $i \geq \max\{0, n\}$, $\text{Hom}_{A/r}(r^{\otimes i}, r^{\otimes i-n})$ has a natural $\text{End}_{A/r}(r^{\otimes i-n})$-$\text{End}_{A/r}(r^{\otimes i})$-bimodule structure. Set $K^n(A)$ to be the direct limit of the chain of maps $\text{Hom}_{A/r}(r^{\otimes i}, r^{\otimes i-n}) \to \text{Hom}_{A/r}(r^{\otimes i+1}, r^{\otimes i+1-n})$, which are induced by $r^{\otimes i-n}$. Then $K^n(A)$ is naturally a $\Gamma(A)$-$\Gamma(A)$-bimodule for each $n \in \mathbb{Z}$. Observe that $K^0(A) = \Gamma(A)\Gamma(A)$ as bimodules, and that composition of maps induces $\Gamma(A)$-$\Gamma(A)$-bimodule morphisms $\phi^{n,m} : K^n(A) \otimes_{\Gamma(A)} K^m(A) \to K^{n+m}(A)$ for all $n, m \in \mathbb{Z}$. These bimodules $K^n(A)$ are called the associated bimodules of $A$.

Recall that for an algebra $\Gamma$, a $\Gamma$-$\Gamma$-bimodule $K$ is invertible provided that the functor $K \otimes_{\Gamma} -$ induces an auto-equivalence on the category of left $\Gamma$-modules.

**Theorem A.** Let $A$ be an artin algebra with radical square zero. Use the notation as above. Then the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the maps $\phi^{n,m}$ are isomorphisms of bimodules.

Since the algebra $\Gamma(A)$ is regular, the category $\text{proj} \Gamma(A)$ of finitely generated right projective $\Gamma(A)$-module is a semisimple abelian category. The invertible bimodule $K^1(A)$ induces an auto-equivalence $\Sigma_A = - \otimes_{\Gamma(A)} K^1(A) : \text{proj} \Gamma(A) \to \text{proj} \Gamma(A)$. We observe that the category $\text{proj} \Gamma(A)$ has a unique triangulated structure with $\Sigma_A$ its shift functor; see Lemma 3.4. This unique triangulated category is denoted by $(\text{proj} \Gamma(A), \Sigma_A)$.

The following result describes the singularity category of an artin algebra with radical square zero, which is based on a result by Keller and Vossieck (\cite{12}).

**Theorem B.** Let $A$ be an artin algebra with radical square zero. Use the notation as above. Then we have a triangle equivalence

$$
\mathbf{D}_{\text{sg}}(A) \simeq (\text{proj} \Gamma(A), \Sigma_A).
$$

We are interested in the $\text{Hom}$-finiteness of singularity categories. For this, we recall the notion of valued quiver of an artin algebra $A$. Choose a complete set of representatives of pairwise non-isomorphic simple $A$-modules $\{S_1, S_2, \cdots, S_n\}$. Set $\Delta_i = \text{End}_{A}(S_i)$; they are division algebras. Observe that $\text{Ext}^1_A(S_i, S_j)$ has a natural $\Delta_j$-$\Delta_i$-bimodule structure. The valued quiver $Q_A$ of $A$ is defined as follows: its vertex set is $\{S_1, S_2, \cdots, S_n\}$, here we identify each simple module $S_i$ with its isoclass; there is an arrow from $S_i$ to $S_j$ whenever $\text{Ext}^1_A(S_i, S_j) \neq 0$, in which case the arrow is endowed with a valuation $(\dim_{\Delta_i} \text{Ext}^1_A(S_i, S_j), \dim_{\Delta_i, 0p} \text{Ext}^1_A(S_i, S_j))$; here $\Delta_i^{0p}$ denotes the opposite algebra of $\Delta_i$. We say that the valuation of $Q_A$ is trivial provided that all the valuations are $(1, 1)$. Recall that a vertex in a valued quiver is a source (resp. sink) provided that there is no arrow ending (resp. starting) at it. For a valued quiver, to adjoin a (new) source (resp. sink) is to add a vertex together with some valued arrows starting (resp. ending) at this vertex. For details, we refer to \cite{III.1}.

The following result characterizes the $\text{Hom}$-finiteness of the singularity category in terms of valued quivers.

**Theorem C.** Let $A$ be an artin algebra with radical square zero. Then the following statements are equivalent:

1. the singularity category $\mathbf{D}_{\text{sg}}(A)$ is $\text{Hom}$-finite;
2. the associated regular algebra $\Gamma(A)$ is semisimple;
3. the valued quiver $Q_A$ is obtained from a disjoint union of oriented cycles with the adjoining valuation by repeatedly adjoining sources or sinks.
The paper is structured as follows. In Section 2, we collect some basic facts on singularity categories and recall a basic result of Keller and Vossieck ([12]), which is applied to Ω-

![Image](https://example.com/image.png)

We recall a basic result due to Keller and Vossieck ([12]), which is applied to Ω-

![Image](https://example.com/image.png)

We introduce the notion of cyclicization of an algebra, which is used in the proof of Theorem C in Section 5.

For artin algebras, we refer to [11]. For triangulated categories, we refer to [11] and [9].

2. Preliminaries

In this section, we collect some facts on singularity categories of artin algebras. We recall a basic result due to Keller and Vossieck ([12]), which is applied to Ω-

![Image](https://example.com/image.png)

We introduce the notion of cyclicization of an algebra, which is used in the proof of Theorem C in Section 5.

For artin algebras, we refer to [11]. For triangulated categories, we refer to [11] and [9].

The following two results are known; compare [14, Lemma 1.11] and [3, Lemma 2.2.2].

**Lemma 2.1.** Let $X^\bullet$ be a complex in $D_{sg}(A)$ and $n_0 > 0$. Then for any $n$ large enough, there exists a module $M$ in $\Omega^{n_0}(A)$-mod) such that $X^\bullet \simeq q(M)[n]$.

**Proof.** Take a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet$ a bounded above complex of projective modules ([11] Lemma I.4.6]). Take $n \geq n_0$ such that $H^i(X^\bullet) = 0$ for all $i \leq n_0 - n$. Consider the good truncation $\sigma^{-n}P^\bullet = \cdots \to 0 \to M \to P^{1-n} \to \cdots$ of $P^\bullet$, which is quasi-isomorphic to $P^\bullet$. Then the cone of the obvious chain map $\sigma^{-n}P^\bullet \to M[n]$ is perfect, which becomes an isomorphism in $D_{sg}(A)$. This shows that $X^\bullet \simeq q(M)[n]$. We observe that $M$ lies in $\Omega^{n_0}(A)$-mod).

□
Lemma 2.2. Let $0 \to M \to P^{1-n} \to \cdots \to P^0 \to N \to 0$ be an exact sequence with each $P^i$ projective. Then we have an isomorphism $q(N) \simeq q(M)[n]$ in $\text{D}_{\text{sg}}(A)$. In particular, for an $A$-module $M$, we have a natural isomorphism $q(\Omega^n(M)) \simeq q(M)[-n]$.

Proof. The stalk complex $N$ is quasi-isomorphic to $\cdots \to 0 \to M \to P^{1-n} \to \cdots \to P^0 \to 0 \to \cdots$. This gives rise to a morphism $N \to M[n]$ in $\text{D}^b(A)$, whose cone is perfect. Then this morphism becomes an isomorphism in $\text{D}_{\text{sg}}(A)$. □

Consider the composite $q': \text{A-mod} \hookrightarrow \text{D}^b(\text{A-mod}) \xrightarrow{\mathbb{L}} \text{D}_{\text{sg}}(A)$; it vanishes on projective modules. Then it induces uniquely a functor $\text{A-mod} \to \text{D}_{\text{sg}}(A)$, which is still denoted by $q'$. Then Lemma 2.2 yields, for each $n \geq 0$, the following commutative diagram

$$
\begin{array}{ccc}
\text{A-mod} & \xrightarrow{\Omega^n} & \text{A-mod} \\
\downarrow q' & & \downarrow q' \\
\text{D}_{\text{sg}}(A) & \xrightarrow{[-n]} & \text{D}_{\text{sg}}(A).
\end{array}
$$

We refer to [3, Lemma 2.2.2] for a similar statement.

The functor $q'$ induces a natural map

$$\Phi^0: \text{Hom}_A(M,N) \to \text{Hom}_{\text{D}_{\text{sg}}(A)}(q(M),q(N))$$

for any modules $M,N$. Let $n \geq 1$. Lemma 2.2 yields a natural isomorphism $q(M) \xrightarrow{\sim} q(\Omega^n(M))[n]$. Then we have a map

$$\Phi^n: \text{Hom}_A(\Omega^n(M),\Omega^n(N)) \to \text{Hom}_{\text{D}_{\text{sg}}(A)}(q(M),q(N))$$

given by $\Phi^n(f) = (\theta_{\Omega^n(M)})^{-1} \circ (\Phi^0(f)[n]) \circ \theta_{\Omega^n(N)}$.

Consider the chain of maps $\text{Hom}_A(\Omega^n(M),\Omega^n(N)) \to \text{Hom}_A(\Omega^{n+1}(M),\Omega^{n+1}(N))$ induced by the syzygy functor. It is routine to verify that $\Phi^n$ are compatible with this chain of maps. Then we have an induced map

$$\Phi: \lim \text{Hom}_A(\Omega^n(M),\Omega^n(N)) \longrightarrow \text{Hom}_{\text{D}_{\text{sg}}(A)}(q(M),q(N)).$$

We recall the following basic result.

Proposition 2.3. (Keller-Vossieck) Let $M,N$ be $A$-modules as above. Then the map $\Phi$ is an isomorphism.

Proof. The statement follows from [12, Exemple 2.3]. We refer to [2, Corollary 3.9(1)] for a detailed proof. □

Recall that an additive category $A$ is idempotent split provided that each idempotent $e: X \to X$ splits, that is, it admits a factorization $X \xrightarrow{u} Y \xrightarrow{v} X$ with $u \circ v = \text{Id}_X$. For example, a Krull-Schmidt category is idempotent split ([6, Appendix A]). In particular, for an artin algebra $A$, the stable category $A\text{-mod}$ is idempotent split.

Corollary 2.4. The singularity category $\text{D}_{\text{sg}}(A)$ of an artin algebra $A$ is idempotent split.

Proof. By Lemma 2.1 it suffices to show that for each module $M$, an idempotent $e: q(M) \to q(M)$ splits. The above proposition implies that for a large $n$, there is an idempotent $e^n: \Omega^n(M) \to \Omega^n(M)$ in $A\text{-mod}$ which is mapped by $\Phi$ to $e$. The idempotent $e^n$ splits as $\Omega^n(M) \xrightarrow{\text{Id}_Y} \Omega^n(M)$ with $u \circ v = \text{Id}_Y$ in $A\text{-mod}$. Then the idempotent $e$ factors as $q(M) \xrightarrow{(q((u))[n]) \circ \theta_{\Omega^n(M)}} q(Y)[n] \xrightarrow{(\theta_{\Omega^n(M})^{-1} \circ (q((v))[n]) \circ q((u))[n])} q(M)$. □
Let $\mathcal{A}$ be an additive category. For a subcategory $\mathcal{C}$, denote by $\text{add} \mathcal{C}$ the full subcategory of $\mathcal{A}$ formed by direct summands of finite direct sums of objects in $\mathcal{C}$. For any algebra $\Gamma$, denote by $\text{proj} \Gamma$ the category of finitely generated right projective $\Gamma$-modules. We observe that $\text{proj} \Gamma = \text{add} \Gamma_1$.

An artin algebra $A$ is called $\Omega^\infty$-finite provided that there exists a module $E$ and $n \geq 0$ such that $\Omega^{n+1}(A\text{-mod}) \subseteq \text{add} (A \oplus E)$. In this case, we call $E$ an $\Omega^\infty$-generator of $A$.

**Proposition 2.5.** Let $A$ be an $\Omega^\infty$-finite algebra with an $\Omega^\infty$-generator $E$. Then we have $D_{sg}(A) = \text{add} q(E)$. Consequently, we have an equivalence of categories $D_{sg}(A) \cong \text{proj} \text{End}_{D_{sg}(A)}(q(E))$, which sends $q(E)$ to $\text{End}_{D_{sg}(A)}(q(E))$.

**Proof.** Observe that $\Omega^{n+1}(A\text{-mod}) \subseteq \Omega^n(A\text{-mod})$. Then we may assume that $\text{add} (A \oplus E) \supseteq \text{add} \Omega^n(A\text{-mod}) = \text{add} \Omega^{n_0+1}(A\text{-mod}) = \cdots$ for $n_0$ large enough.

For the first statement, it suffices to show that each object $X^\bullet$ in $D_{sg}(A)$ belongs to $\text{add} q(E)$. By Lemma 2.2, $X^\bullet \cong q(M[n_1])$ for a module $M \in \Omega^{n_0}(A\text{-mod})$ and $n_1 > 0$. Since $\text{add} \Omega^{n_0}(A\text{-mod}) = \text{add} \Omega^{n_0+n_1}(A\text{-mod})$, we may assume that $M \oplus N \in \Omega^{n_0+n_1}(A\text{-mod})$ for some module $N$. Take an exact sequence $0 \to M \oplus N \to P_{1-n_1} \to \cdots \to P_0 \to L \to 0$ with each $P^n$ projective and $L \in \Omega^n(A\text{-mod})$. By Lemma 2.2, $q(L) \cong q(M \oplus N)[n_1]$ and then $X^\bullet$ is a direct summand of $q(L)$. Observing that $L \in \text{add} (A \oplus E)$, we are done with the first statement.

The second statement follows from the projectivization; see [1, Proposition II.2.1]. The functor is given by $\text{Hom}_{D_{sg}(A)}(q(E), \cdot)$. We point out that Corollary 2.4 is needed here. □

### 3. Algebras with radical square zero

In this section, we study the singularity category of an algebra with radical square zero, and prove Theorem A and B. An explicit example is given at the end.

Let $A$ be an artin algebra. Denote by $r$ the Jacobson radical of $A$. The algebra $A$ is said to be with radical square zero provided that $r^2 = 0$. In this case, $r$ has an $A/r \times A/r$-bimodule structure, which is induced from the multiplication of $A$.

Denote by $A\_{ssmod}$ the full subcategory of $A\_{mod}$ formed by semisimple modules. We observe that $r \otimes_{A/r} S = 0$ for a simple projective module $S$. Then the functor $r \otimes_{A/r} \cdot : A\_{ssmod} \to A\_{ssmod}$ is well defined. We observe that the syzygy functor $\Omega$ sends semisimple modules to semisimple modules, and then we have the restricted functor $\Omega : A\_{ssmod} \to A\_{ssmod}$.

The following result is implicitly contained in the proof of [1, Lemma X.2.1].

**Lemma 3.1.** There is a natural isomorphism $\Omega \cong r \otimes_{A/r} -$ of functors on $A\_{ssmod}$.

**Proof.** Let $X$ be a semisimple module with a projective cover $P \to X$. Tensoring $P$ with the natural exact sequence of $A$-$A$-bimodules $0 \to r \to A \to A/r \to 0$ yields $\Omega(X) \cong r \otimes_A P$. Using isomorphisms $r \otimes_A P \cong r \otimes_{A/r} P \otimes rP$ and $P \otimes rP \cong X$, we get an isomorphism $\Omega(X) \cong r \otimes_{A/r} X$. It is routine to verify that this isomorphism is natural in $X$. □

Recall that an algebra $\Gamma$ is regular in the sense of von Neumann provided that for each element $a$ there exists $a'$ such that $a' a = a$. For example, a semisimple algebra is regular. Then a direct limit of semisimple algebras is regular. For details, we refer to [7, Theorem and Definition 11.24].

Recall that for an artin algebra $A$ with radical square zero, there is a chain of algebra homomorphisms $\text{End}_{A/r}(r^{\otimes i}) \to \text{End}_{A/r}(r^{\otimes i+1})$ induced by $r \otimes_{A/r} -$ . Here, $r^{\otimes 0} = A/r$ and $r^{\otimes i+1} = r \otimes_{A/r} (r^{\otimes i})$. We set $\Gamma(A)$ to be the direct limit of this
chain. Since each algebra \( \text{End}_{A/r}(r^{\otimes i}) \) is semisimple, the algebra \( \Gamma(A) \) is regular. It is called the \textit{associated regular algebra} of \( A \). We refer to [5] 19.26B, Example for a related construction.

We recall the \textit{associated \( \Gamma(A) \)-\( \Gamma(A) \)-bimodules} \( K^n(A) \) of \( A, n \in \mathbb{Z} \). For \( i \geq \max\{0, n\} \), \( \text{Hom}_{A/I}(r^{\otimes i}, r^{\otimes i-n}) \) has a natural \( \text{End}_{A/I}(r^{\otimes i-n})\text{-End}_{A/I}(r^{\otimes i}) \)-bimodule structure. Consider a chain of maps \( \text{Hom}_{A/I}(r^{\otimes i}, r^{\otimes i-n}) \rightarrow \text{Hom}_{A/I}(r^{\otimes i+1}, r^{\otimes i+1-n}), \) which are induced by \( r \otimes_{A/I} - \), and define \( K^n(A) \) to be its direct limit. Then \( K^n(A) \) is naturally a \( \Gamma(A) \)-\( \Gamma(A) \)-bimodule for each \( n \in \mathbb{Z} \). Observe that \( K^0(A) = \Gamma(A) \Gamma(A) \text{ as } \Gamma(A) \)-\( \Gamma(A) \)-bimodules.

**Proposition 3.2.** Let \( A \) be an artin algebra with radical square zero. Then there is a natural isomorphism

\[
K^n(A) \simeq \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r), q(A/r)[n])
\]

for each \( n \in \mathbb{Z} \).

**Proof.** Consider the case \( n \leq 0 \) first. In this case, by Lemmas 2.2 and 3.1 we have \( q(A/r)[n] \simeq q(\Omega^{-n}(A/r)) \simeq q(r^{\otimes -n}) \). Then Proposition 2.3 yields an isomorphism \( \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r), q(A/r)[n]) \simeq \lim \text{Hom}_{A/I}(\Omega(A/r), \Omega(r^{\otimes -n})). \) By Lemma 3.1 again we have \( \Omega(A/r) \simeq r^{\otimes i} \) and \( \Omega(r^{\otimes -n}) = r^{\otimes i-n} \). Then we have a surjective map \( \psi: K^n(A) \rightarrow \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r), q(A/r)[n]) \). On the other hand, every morphism \( f: r^{\otimes i} \rightarrow r^{\otimes i-n} \) that is zero in \( A\text{-mod} \) necessarily factors through a semisimple projective module. However, the functor \( r \otimes_{A/I} - \) vanishes on semisimple projective modules. Then \( r \otimes_{A/I} f \) is zero. This forces that \( \psi \) is injective. We are done in this case.

For the case \( n > 0 \), we observe that \( \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r), q(A/r)[n]) \) is isomorphic to \( \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r)[n], q(A/r)) \), and by the same argument as above, it is isomorphic to \( \lim \text{Hom}_{A/I}(r^{\otimes i+n}, r^{\otimes i}). \) Then we get a surjective map \( K^n(A) \rightarrow \text{Hom}_{D_{\mathfrak{sa}}(A)}(q(A/r), q(A/r)[n]) \). Similarly as above, we have that this map is injective.

**Remark 3.3.** In the case \( n = 0 \), the above isomorphism is an isomorphism \( \Gamma(A) \simeq \text{End}_{D_{\mathfrak{sa}}(A)}(q(A/r)) \) of algebras. Then for an arbitrary \( n \), the above isomorphism becomes an isomorphism of \( \Gamma(A) \)-\( \Gamma(A) \)-bimodules.

Recall that an abelian category \( \mathcal{A} \) is \textit{semisimple} provided that each short exact sequence splits. For example, for a regular algebra \( \Gamma \), the category \( \text{proj } \Gamma \) of finitely generated right projective \( \Gamma \)-modules is a semisimple abelian category. Here, we use the fact that all finitely presented \( \Gamma \)-modules are projective; see [7] Theorem and Definition 11.24(a).

The following observation is well known.

**Lemma 3.4.** Let \( \mathcal{A} \) be a semisimple abelian category, and let \( \Sigma \) be an auto-equivalence on \( \mathcal{A} \). Then there is a unique triangulated structure on \( \mathcal{A} \) with \( \Sigma \) the shift functor.

The obtained triangulated category in this lemma will be denoted by \( (\mathcal{A}, \Sigma) \).

**Proof.** We use the fact that each morphism in \( \mathcal{A} \) is isomorphic to a direct sum of morphisms of the forms \( K \rightarrow 0, I \overset{\text{Id}}{\rightarrow} I \) and \( 0 \rightarrow C \). Then all possible triangles are a direct sum of the following trivial triangles \( K \rightarrow 0 \rightarrow \Sigma(K) \overset{\text{Id}_{\Sigma(K)}}{\rightarrow} \Sigma(K), I \overset{\text{Id}}{\rightarrow} I \rightarrow 0 \rightarrow \Sigma(I) \) and \( 0 \rightarrow C \overset{\text{Id}_{C}}{\rightarrow} C \rightarrow \Sigma(0). \)

**Proposition 3.5.** Let \( A \) be an artin algebra with radical square zero and let \( \Gamma(A) \) be its associated regular algebra. Then there is a triangle equivalence

\[
\Psi: D_{\mathfrak{sa}}(A) \simeq (\text{proj } \Gamma(A), \Sigma)
\]
for some auto-equivalence $\Sigma$ on $\text{proj } \Gamma(A)$, which sends $q(A/r)$ to $\Gamma(A)$.

**Proof.** We observe that for any $A$-module $M$, its syzygy $\Omega(M)$ is semisimple. Hence we have $\Gamma^1(A\text{-mod}) \subseteq \text{add } (A \oplus A/r)$. We apply Proposition 2.5 to obtain an equivalence of categories $D_{\text{sg}}(A) \simeq \text{proj } \text{End}_{D_{\text{sg}}(A)}(q(A/r))$. By Proposition 3.2 this yields an equivalence of categories $D_{\text{sg}}(A) \simeq \text{proj } \Gamma(A)$.

By transport of structures, the shift functor $[1]$ on $D_{\text{sg}}(A)$ corresponds to an auto-equivalence $\Sigma$ on $\text{proj } \Gamma(A)$, and then $\text{proj } \Gamma(A)$ becomes a triangulated category. However, by Lemma 3.4 the semisimple abelian category $\text{proj } \Gamma(A)$ has a unique triangulated structure with $\Sigma$ the shift functor. Then this structure necessarily coincides with the transported one. Then we are done. \qed

We are interested in the auto-equivalence $\Sigma$ above. The following result characterizes it using the bimodules $K^n(A)$.

**Lemma 3.6.** Use the notation as above. Then for each $n \in \mathbb{Z}$, the auto-equivalence $\Sigma^n$ is isomorphic to $- \otimes_{\Gamma(A)} K^n(A)$: $\text{proj } \Gamma(A) \rightarrow \text{proj } \Gamma(A)$.

**Proof.** Recall that the above equivalence $\Psi$ is given by $\text{Hom}_{D_{\text{sg}}(A)}(q(A/r), -)$. The auto-equivalence $\Sigma^n$ corresponds, via $\Psi$, to $[n]$ on $D_{\text{sg}}(A)$. Then by Proposition 3.2 we have an isomorphism $\phi: K^n(A) \rightarrow \Sigma^n(\Gamma(A))$ of right $\Gamma(A)$-modules. Recall that $\Sigma^n(\Gamma(A))$ has a natural $\Gamma(A)$-$\Gamma(A)$-bimodule structure such that $\Sigma^n$ is isomorphic to $- \otimes_{\Gamma(A)} \Sigma^n(\Gamma(A))$. Thanks to Remark 3.3 the isomorphism $\phi$ is an isomorphism of bimodules. This proves the lemma. \qed

Recall that for an algebra $A$, a $\Gamma$-$\Gamma$-bimodule $K$ is invertible provided that the functor $- \otimes_{\Gamma} K$ induces an auto-equivalence on the category of right $\Gamma$-modules. For details, we refer to [7, Definition and Proposition 12.13].

We recall that for an artin algebra $A$ with radical square zero, the associated $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are defined to be $\text{lim lim } \text{Hom}_{A/r}(r^\otimes i, r^\otimes j - n)$, where $i \geq \max\{0, n\}$. Then composition of maps between the $A/r$-modules $r^\otimes j$ yields morphisms

$$\phi^{n,m}: K^n(A) \otimes_{\Gamma(A)} K^m(A) \rightarrow K^{n+m}(A)$$

of $\Gamma(A)$-$\Gamma(A)$-bimodules, for all $n, m \in \mathbb{Z}$. More precisely, let $f \in K^n(A)$ and $g \in K^m(A)$ be represented by $f': r^\otimes j - m \rightarrow r^\otimes j - n$ and $g': r^\otimes j - m \rightarrow r^\otimes j - m$ for some large $j$, respectively. Then $\phi^{n,m}(f \otimes g)$ is represented by the composite $f' \circ g'$.

The following result is Theorem A.

**Theorem 3.7.** Let $A$ be an artin algebra with radical square zero. Use the notation as above. Then for all $n, m \in \mathbb{Z}$, the $\Gamma(A)$-$\Gamma(A)$-bimodules $K^n(A)$ are invertible and the morphisms $\phi^{n,m}$ are isomorphisms.

**Proof.** By Lemma 3.6 the functor $- \otimes_{\Gamma(A)} K^n(A)$: $\text{proj } \Gamma(A) \rightarrow \text{proj } \Gamma(A)$ is an auto-equivalence for each $n$. This functor extends naturally to an auto-equivalence on the category of all right $\Gamma(A)$-modules. Then $K^n(A)$ is an invertible bimodule. The second statement follows from Lemma 3.6 and the fact that $\Sigma^n \Sigma^n$ is isomorphic to $\Sigma^{n+m}$. Here, we use [7, Proposition 12.9] implicitly. \qed

We now have Theorem B. Denote the functor $- \otimes_{\Gamma(A)} K^1(A)$: $\text{proj } \Gamma(A) \rightarrow \text{proj } \Gamma(A)$ by $\Sigma_A$.

**Theorem 3.8.** Let $A$ be an artin algebra with radical square zero. Use the notation as above. Then we have a triangle equivalence

$$D_{\text{sg}}(A) \simeq \text{proj } \Gamma(A),$$

which sends $q(A/r)$ to $\Gamma(A)$.

**Proof.** It follows from Proposition 3.5 and Lemma 3.6. \qed
Let $A$ be an artin algebra with radical square zero. For each $n \geq 1$, we consider the artin algebra $G^n = A/r \oplus r^{\otimes n}$, which is the trivial extension of the $A/r$-$A/r$-bimodule $r^{\otimes n}$ ([11, p.78]). All these algebras $G^n$ have radical square zero.

The following observation seems to be of independent interest.

**Proposition 3.9.** Use the notation as above. Then for each $n \geq 1$, we have a triangle equivalence

$$D_{sg}(G^n) \simeq (\text{proj } \Gamma(A), \Sigma^n_A).$$

In particular, we have a triangle equivalence $D_{sg}(A) \simeq D_{sg}(G^1)$.

**Proof.** Write $G^n = A'$. Then from the very definition, we have a natural identification $\Gamma(A') = \Gamma(A)$. Moreover, the $\Gamma(A')$-$\Gamma(A')$-bimodule $K^1(A')$ corresponds to the $\Gamma(A)$-$\Gamma(A)$-bimodule $K^n(A)$. Then by Lemma 3.6 $\Sigma_A'$ corresponds to $\Sigma^1_A$. Then the result follows from Theorem 3.8 immediately. □

**Remark 3.10.** We point out that for $n \geq 2$, $D_{sg}(G^n)$ might not be triangle equivalent to $D_{sg}(A)$, although the underlying categories are equivalent.

We conclude this section with an example.

**Example 3.11.** Let $k$ be a field and let $n \geq 2$. Consider the algebra $A = k[x_1, x_2, \ldots, x_n]/(x_ix_j, 1 \leq i, j \leq n)$, which is with radical square zero. We identify $A/r$ with $k$, and $r$ with the $n$-dimensional $k$-space $V = kx_1 \oplus kx_2 \oplus \cdots \oplus kx_n$. Consequently, for each $i \geq 0$, the algebra $\text{End}_{A/r}(r^{\otimes i})$ is isomorphic to $\text{End}_k(V^{\otimes i})$, which is identified with the $n^i \times n^i$ total matrix algebra $M_{n^i}(k)$. Then the associated regular algebra $\Gamma(A)$ is isomorphic to the direct limit of the following chain of algebra embeddings

$$k \rightarrow M_n(k) \rightarrow M_{n^2}(k) \rightarrow M_{n^3}(k) \rightarrow \cdots$$

Here, for each algebra $B$, $B \rightarrow M_n(B)$ is the algebra embedding sending $b$ to $bI_n$ with $I_n$ the $n \times n$ identity matrix.

We observe that $\Gamma(A)$ is a simple algebra. We point out that this construction is classical; see [19, 26 B, Example].

Let $1 \leq r, s \leq n$. Define $E_{rs}: V \rightarrow V$ to be the linear map such that $E_{rs}(x_i) = \delta_{i,r}x_s$, where $\delta$ is the Kronecker symbol. Consider, for all $i \geq 0$, the linear maps $-\otimes_k E_{rs}: \text{End}_k(V^{\otimes i}) \rightarrow \text{End}_k(V^{\otimes i+1})$. Taking the limit, we have the induced linear map $-\otimes_k E_{rs}: \Gamma(A) \rightarrow \Gamma(A)$ for each pair of $r, s$. Then we have an isomorphism $\sigma: M_n(A) \rightarrow A$ of algebras, which sends an $n \times n$ matrix $(a_{ij})$ to $\sum_{1 \leq i,j \leq n} a_{ij} \otimes_k E_{ij}$.

The associated $\Gamma(A)$-$\Gamma(A)$-bimodule $K^1(A)$ is described as follows. As a $k$-space, $K^1(A) = \Gamma(A) \oplus \Gamma(A) \oplus \cdots \Gamma(A)$ with $n$ copies of $\Gamma(A)$. The left action is given by $a(a_1, a_2, \ldots, a_n) = (aa_1, aa_2, \ldots, a_n)$, while the right action is given by $(a_1, a_2, \ldots, a_n)a = (a_1a, a_2a, \ldots, a_na)^{-1}(a)$.

We remark that the regular algebra $\Gamma(A)$ is related to a quotient abelian category studied in [18], which might relate to the singularity category $D_{sg}(A)$ via a version of Koszul duality.

4. **ONE-POINT (CO)EXTENSIONS AND CYCLICIZATIONS OF ALGEBRAS**

In this section, we prove that one-point extensions and coextensions of algebras preserve their singularity categories. We then introduce the notion of cyclicization of an algebra, which is a repeated operation to remove sources and sinks on the valued quiver. The obtained result will be used in the proof of Theorem C.

Let $A$ be an artin algebra. Let $D$ be a simple artin algebra and let $A_M D$ be an $A$-$D$-bimodule, on which $R$ acts centrally. The one-point extension of $A$ by $M$ is the upper triangular matrix algebra $A[M] = \begin{pmatrix} A & M \\ 0 & D \end{pmatrix}$. A left $A[M]$-module
is denoted by a column vector \( \begin{pmatrix} X \\ V \end{pmatrix} \), where \( X \) and \( V \) are a left \( A \)-module and \( D \)-module, respectively, and that \( \phi: M \otimes_D V \to X \) is a morphism of \( A \)-modules. We sometimes suppress the morphism \( \phi \), when it is clearly understood. For details, we refer to [13 III.2].

Recall from [13 III.1] the notion of valued quiver \( Q_A \) for an artin algebra \( A \). We observe that for the unique simple \( D \)-module \( S \), the corresponding \( A[M] \)-module \( \begin{pmatrix} 0 \\ S \end{pmatrix} \) is simple injective, which corresponds to a source in the valued quiver \( Q_{A[M]} \) of the one-point extension \( A[M] \). Indeed, this valued quiver is obtained from \( Q_A \) by adding this source together with some valued arrows starting at it.

One-point extensions of algebras preserve singularity categories. Observe the natural exact embedding \( i: A\text{-mod} \to A[M]\text{-mod} \), which sends \( A \)-module to \( A[M] \)-module, respectively, and that \( \sim \) is at most one. In particular, the left derived functor \( D_{sg}(A) \) is an isomorphism; see [13 Lemma 1.2]. In particular, the functor \( i_* \) is fully faithful. It remains to show the denseness of \( i_* \). We now view the essential image \( \text{Im } i_* \) of \( i_* \) as a full triangulated subcategory of \( D_{sg}(A[M]) \). It suffices to show that for each \( A[M] \)-module \( \begin{pmatrix} X \\ V \end{pmatrix} \), its image in \( D_{sg}(A[M]) \) lies in \( \text{Im } i_* \); see Lemma 2.2.

Observe that \( q\left( \begin{pmatrix} 0 \\ V \end{pmatrix} \right) \) lies in \( \text{Im } i \), and then by Lemma 2.2 \( q\left( \begin{pmatrix} 0 \\ V \end{pmatrix} \right) \) lies in \( \text{Im } i_* \). The following natural exact sequence induces a triangle in \( D_{sg}(A[M]) \)

\[
0 \to i(X) \to \begin{pmatrix} X \\ V \end{pmatrix} \to \begin{pmatrix} 0 \\ V \end{pmatrix} \to 0.
\]

This triangle implies that \( q\left( \begin{pmatrix} X \\ V \end{pmatrix} \right) \) lies in \( \text{Im } i_* \). Then we are done. \( \square \)

Let \( D \) be a simple artin algebra, and let \( D: N_A \) be a \( D \)-\( A \)-bimodule, on which \( R \) acts centrally. The one-point coextension of \( A \) by \( N \) is the upper triangular matrix algebra \( [N]A = \begin{pmatrix} D & N \\ 0 & A \end{pmatrix} \). A left \( N[A] \)-module is written as \( \begin{pmatrix} V \\ X \end{pmatrix} \), where \( DV \) and \( AX \) are a left \( D \)-module and \( A \)-module, respectively, and that \( \phi: M \otimes_A X \to V \) is a morphism of \( D \)-modules. The valued quiver \( Q_{[N]A} \) is obtained from \( Q_A \) by adding

\[
\begin{pmatrix} X \\ V \end{pmatrix} \]
a sink together with some valued arrows ending at it, where the sink corresponds to the simple projective \([N]A\)-module \(\begin{pmatrix} S \\ 0 \end{pmatrix}\) for a simple \(D\)-module \(S\).

For the one-point coextension \([N]A\), we have an exact embedding \(i: A\text{-mod} \to [N]A\text{-mod}\), which sends \(AX\) to \(i(X) = \begin{pmatrix} 0 \\ X \end{pmatrix}\).

The following result is similar to Proposition 4.1, while the proof is simpler. This result is closely related to [4, Theorem 4.1(1)].

**Proposition 4.2.** Let \([N]A\) be the one-point coextension as above. Then the embedding \(i: A\text{-mod} \to [N]A\text{-mod}\) induces a triangle equivalence \(D_{sg}(A) \simeq D_{sg}([N]A)\).

**Proof.** We observe that \(i(A)\) has projective dimension at most one. Then the obviously induced functor \(i_*: D^b(A\text{-mod}) \to D^b([N]A\text{-mod})\) preserves perfect complexes, and it induces the required functor \(i_*: D_{sg}(A) \to D_{sg}([N]A)\).

The functor \(i\) admits an exact left adjoint \(j: [N]A\text{-mod} \to A\text{-mod}\), which sends \(\begin{pmatrix} V \\ X \end{pmatrix}\) to \(X\); moreover, \(j\) preserves projective modules. Then it induces a triangle functor \(\tilde{j}_*: D_{sg}([N]A) \to D_{sg}(A)\), which is left adjoint to \(i_*\). Then as in the proof of Proposition 4.1 we have that \(i_*\) is fully faithful. The denseness of \(i_*\) follows from the natural exact sequence

\[
0 \to \begin{pmatrix} V \\ 0 \end{pmatrix} \to \begin{pmatrix} V \\ X \end{pmatrix} \to i(X) \to 0,
\]

for each \([N]A\)-module \(\begin{pmatrix} V \\ X \end{pmatrix}\), and that the module \(\begin{pmatrix} V \\ 0 \end{pmatrix}\) is projective. We omit the details. \(\Box\)

We use the above two propositions to reduce the study of the singularity category of arbitrary artin algebras to cyclic-like ones.

Let \(A\) be an artin algebra. Consider the valued quiver \(Q_A\). A vertex \(e\) is called cyclic provided that there is an oriented cycle containing it, and the corresponding simple \(A\)-module is called cyclic. More generally, a vertex \(e\) is called cyclic-like provided that there is a path through \(e\), which starts with a cyclic vertex and ends at a cyclic vertex, while the corresponding simple \(A\)-module is called cyclic-like. An artin algebra \(A\) is called cyclic-like provided that its valued quiver \(Q_A\) is cyclic-like. This is equivalent to that \(A\) has neither simple projective nor simple injective modules.

For an artin algebra \(A\), its cyclicization is an artin algebra \(A_c\) which is either simple or cyclic-like, such that there is a sequence \(A_c = A_0, A_1, \ldots, A_r = A\) with each \(A_{i+1}\) is a one-point (co)extension of \(A_i\).

The following is an immediate consequence of the definition.

**Lemma 4.3.** Let \(A\) be an artin algebra with its cyclicization \(A_c\). Then we have a triangle equivalence

\(D_{sg}(A_c) \simeq D_{sg}(A)\).

**Proof.** Apply Propositions 4.1 and 4.2 repeatedly. \(\Box\)

The following result seems to be well known.

**Lemma 4.4.** The following statements hold.

1. Each artin algebra has a cyclicization.
2. Let \(A_c\) and \(A_{c'}\) be two cyclicizations of \(A\). Then if \(A_c\) is simple, so is \(A_{c'}\). Otherwise, we have an isomorphism \(A_c \simeq A_{c'}\) of algebras.
Proof. (1) It follows from the well-known fact that the existence of a simple injective (resp. projective) module of \( A \) implies that \( A \) is a one-point extension (resp. coextension) of \( A' \). Moreover, the valued quiver \( Q_A' \) of \( A' \) is obtained from the one of \( A \) by deleting the relevant source (resp. sink).

(2) The first statement follows from the observation that passing from \( A \) to \( A' \) in (1), the set of cyclic-like vertices stays the same. For the isomorphism of algebras, it suffices to observe that \( A_c \)-mod is equivalent to the smallest Serre subcategory ([7, Chapter 15]) of \( A \)-mod containing the cyclic-like simple \( A \)-modules \( S \); moreover, the multiplicity of \( P_{A_c}(S) \) in the indecomposable decomposition of \( A_c \) equals the multiplicity of \( P(S) \) in the one of \( A \). Here, \( P(S) \) and \( P_{A_c}(S) \) denote the projective cover of \( S \) as an \( A \)-module and \( A_c \)-module, respectively. □

5. Hom-finiteness of singularity categories

In this section, we study the Hom-finiteness of the singularity category of an artin algebra with radical square zero, and prove Theorem C.

Throughout, \( A \) is an artin \( R \)-algebra such that its Jacobson radical \( r \) satisfies \( r^2 = 0 \). Recall that in this case, the syzygy \( \Omega(X) \) of any module \( X \) is semisimple.

Lemma 5.1. Suppose that \( A \) is cyclic-like. Then we have

1. each simple \( A \)-module has infinite projective dimension;
2. for each \( i \geq 0 \), the algebra homomorphism \( \text{End}_{A/r}(r^{\otimes i}) \to \text{End}_{A/r}(r^{\otimes i+1}) \) induced by \( r \otimes A/r - \) is injective.

Proof. (1) Recall that a cyclic-like algebra does not have simple projective modules. Then the statement follows from the observation that for a simple module \( S \) with finite projective dimension, we have that \( \text{proj.dim} \, \Omega(S) = \text{proj.dim} \, S - 1 \).

(2) We recall that \( A\text{-ssmod} \) is the full subcategory of \( A\)-mod consisting of semisimple modules. Then by (1), \( A\text{-ssmod} \) is naturally equivalent to \( A\text{-ssmod} \), and the syzygy functor \( \Omega: A\text{-ssmod} \to A\text{-ssmod} \) is faithful. Now the result follows from Lemma 3.1. □

Recall that the singularity category \( D_{sg}(A) \) is naturally \( R \)-linear. We are interested in the problem when it is Hom-finite, that is, all the Hom sets are finitely generated \( R \)-modules.

Theorem 5.2. Let \( A \) be an artin algebra with radical square zero. Then the following statements are equivalent:

1. the singularity category \( D_{sg}(A) \) is Hom-finite;
2. the associated regular algebra \( \Gamma(A) \) is semisimple;
3. the cyclicization \( A_c \) of \( A \) is either simple or isomorphic to a finite product of self-injective algebras.

We point out that the cyclicization \( A_c \) of \( A \) is necessarily with radical square zero. Recall that an indecomposable non-simple artin algebra with radical square zero is self-injective if and only if its valued quiver is an oriented cycle with the trivial valuation; see [11 Proposition IV.2.16] or the proof of [5 Corollary 1.3]. Then the statement (3) above is equivalent to the corresponding one in Theorem C.

Proof. Recall from Proposition 3.2 the isomorphism \( \Gamma(A) \simeq \text{End}_{D_{sg}(A)}(q(A/r)) \). Then we have the implication "(1) \Rightarrow (2)", since an artin regular algebra is necessarily semisimple.

For "(2) \Rightarrow (1)"", consider the cyclicization \( A_c \) of \( A \), whose Jacobson radical is denoted by \( r_c \). Then by Lemma 4.3 we have an equivalence \( D_{sg}(A_c) \simeq D_{sg}(A) \). Applying Proposition 5.5 we have an equivalence \( \text{proj} \, \Gamma(A_c) \simeq \text{proj} \, \Gamma(A) \), that is,
Γ(A_c) and Γ(A) are Morita equivalent. Then Γ(A_c) is also semisimple. Recall that Γ(A_c) = lim
\lim_{\mathbb{N}} \text{End}_{A_c}(k^{\omega_1}) / \bigoplus_{i=1}^{\infty} \text{End}_{A_c}(k^{\omega_i}). \text{By Lemma 5.1, all \text{End}_{A_c}(k^{\omega_1}) is injective. Recall that for a semisimple algebra, the number of pairwise orthogonal idempotents is bounded. Then the R-lengths of the algebras End_{A_c}(k^{\omega_1}) are uniformly bounded. Consequently, the algebra Γ(A) is an artin R-algebra. By Proposition 3.5, the singularity category D_{sg}(A_c) is Hom-finite. Then we are done by Lemma 4.3.}

Recall from [14, Theorem 2.1] that the singularity category of a self-injective algebra is equivalent to its stable category. In particular, it is Hom-finite. Then the implication “(3)⇒(1)” follows from Lemma 4.3.

It remains to show “(1)⇒(3)”.

We claim that the sysygy Ω(S) of any cyclic simple \(A\)-module \(S\) is simple. Then there is only one arrow starting at \(S\) in \(Q_A\), which is valued by \((1, b)\) for some natural number \(b\). Since \(Q_A\) is cyclic-like, this forces that \(Q_A\) is a disjoint union of oriented cycles. In each oriented cycle, every arrow has valuation \((1, b_i)\) for some \(b_i\). Then the symmetrization condition implies that all these \(b_i\)’s are necessarily one; compare the proof of [11, Proposition VIII. 6.4]. As we point out above, this implies that \(A\) is self-injective.

We prove the claim. Since by Corollary 2.4 \(D_{sg}(A)\) is idempotent split, we have that \(D_{sg}(A)\) is a Krull-Schmidt category ([6, Appendix A]). In particular, each object is uniquely decomposed as a direct sum of finitely many indecomposable objects. We observe that for each semisimple module \(X\), \(lX \leq l\Omega(X)\). Here, \(l\) denotes the composition length. Consider a cyclic simple \(A\)-module \(S\), and take a path \(S = S_1 \to S_2 \to \cdots \to S_r \to S_{r+1} = S\) in \(Q_A\). Assume on the contrary that \(l\Omega(S) \geq 2\). Then we have \(\Omega(S) = S_2 \oplus X\) for some nonzero semisimple module \(X\). Observe that \(S\) is a direct summand of \(\Omega^{-1}(S_2)\), and then we have \(\Omega^n(S) = S \oplus X'\) for a nonzero semisimple module \(X'\). Consequently, we have \(\Omega^n(S) = S \oplus X' \oplus \Omega^r(X') \oplus \cdots \oplus \Omega^{(n-1)r}(X')\). Then the lengths of the semisimple modules \(\Omega^n(S)\) tend to the infinity, when \(n\) goes to the infinity. By Lemma 5.1, \(q(T)\) is not zero for any simple \(A\)-module \(T\). Recall from Lemma 2.2 that \(q(S) \simeq q(\Omega^n(S))[nr]\). This contradicts to the Krull-Schmidt property of \(q(S)\), and we are done with the claim.

Acknowledgements. The author thanks Professor Zhaoyong Huang and Professor Yu Ye for their helpful comments.

References

[1] M. Auslander, I. Reiten, and S.O. Smal\o, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. 36, Cambridge Univ. Press, Cambridge, 1995.
[2] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization, Comm. Algebra 28(10) (2000), 4547–4596.
[3] R.O. Buchweitz, Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings, Unpublished Manuscript, 1987.
[4] X.W. Chen, Singularity categories, Schur functors and triangular matrix rings, Algebr. Represent. Theor. 12 (2009), 181–191.
[5] X.W. Chen, Algebras with radical square zero are either self-injective or CM-free, Proc. Amer. Math. Soc., in press.
[6] X.W. Chen, Y. Ye and P. Zhang, Algebras of derived dimension zero, Comm. Algebra 36 (2008), 1–10.
[7] C. Faith, Algebra: Rings, Modules and Categories I, Springer-Verlag, Berlin Heidelberg New York, 1973.
[8] C. Faith, Algebra II, Ring Thory, Springer-Verlag, Berlin Heidelberg New York, 1976.
[9] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Math. Soc., Lecture Notes Ser. 119, Cambridge Univ. Press, Cambridge, 1988.
[10] D. Happel, On Gorenstein algebras, In: Progress in Math. 95, Birkhäuser Verlag, Basel, 1991, 389–404.
[11] R. Hartshorne, Duality and Residue, Lecture Notes in Math. 20, Springer, Berlin, 1966.
[12] B. Keller and D. Vossieck, Sous les catégories dérivées, C.R. Acad. Sci. Paris, t. 305 Série I (1987) 225–228.
[13] H. Krause, The stable derived category of a noetherian scheme, Compositio Math. 141 (2005), 1128–1162.
[14] D. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, Trudy Steklov Math. Institute 204 (2004), 240–262.
[15] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (1989), 303–317.
[16] S.P. Smith, The non-commutative scheme having a free algebra as a homogeneous coordinate ring, arXiv:1104.3822v1.

Xiao-Wu Chen, Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, PR China
Wu Wen-Tsun Key Laboratory of Mathematics, USTC, Chinese Academy of Sciences, Hefei 230026, Anhui, PR China.
URL: http://mail.ustc.edu.cn/~xwchen