Strongly contracting geodesics in a tree of spaces

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Abstract. Let $X$ be a tree of proper geodesic spaces with edge spaces strongly contracting and uniformly separated from each other by a number depending on the contraction function of edge spaces. Then we prove that the strongly contracting geodesics in vertex spaces are quasiconvex in $X$. We further prove that in $X$ if all the vertex spaces are uniformly hyperbolic metric spaces, then $X$ is a hyperbolic metric space and vertex spaces are quasiconvex in $X$.

Keywords. Hyperbolic metric spaces; tree of spaces; strongly contracting subspaces.

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1. Introduction

Let $X$ be a geodesic metric space. A subset $Q$ of a geodesic metric space $X$ is said to be strongly contracting if there exists a constant $\sigma \geq 0$ such that the diameter of the projection of balls, disjoint from $Q$ into $Q$ is at most $\sigma$. For example, the geodesics and horoballs of upper half space with Poincaré metric are strongly contracting. Using strongly contracting geodesics, Charney and Sultan [4] gave the notion of ‘contracting’ boundary for a CAT(0) space which is an analogue of the Gromov boundary. The concept of contracting projection has created a lot of interest in recent times, we refer the reader to look into the work of Arzhantseva et al. [1] for more details about the other contraction properties of subspaces. Given a finite collection of groups satisfying a property $P$, the combination type problems deal with the question that then when does the fundamental group of the graph of groups formed by that collection satisfy the property $P$. One such case is the celebrated work of Bestvina and Feighn [2] in 1992, where they have found a combination theorem of finite graphs of hyperbolic groups. Given a graph of groups $(\mathcal{G}, \Lambda)$ over a finite graph $\Lambda$, there exists a tree of spaces $X$ where the underlying tree is the Bass–Serre covering tree such that the fundamental group $\pi_1(\mathcal{G}, \Lambda)$ of $(\mathcal{G}, \Lambda)$ acts properly andco-compactly
on $X$. An action of a group $G$ on a simplicial tree $T$ is called \textit{acylindrical} if there exists $k \geq 1$ such that no non-trivial element of $G$ fixes point-wise a segment of length $k$ in $T$. Kapovich in [5] showed that for a finite graph of hyperbolic groups $(G, \Lambda)$ with edge groups quasi-isometrically embedded in vertex groups if the fundamental group $\pi_1(G, \Lambda)$ acts acylindrically on the Bass–Serre covering tree then $\pi_1(G, \Lambda)$ is hyperbolic and all vertex groups are quasiconvex in the $\pi_1(G, \Lambda)$. In this article, we prove the following.

**Theorem 1.1 (see Theorem 3.2).** Let $N \geq 1$ and $\sigma \geq 0$. There exists $R = R(N, \sigma) \geq 1$ such that the following holds: Let $X$ be a tree of proper geodesic spaces where all edge spaces are $\sigma$-contracting and uniformly $R$-separated. Let $\eta$ be a strongly contracting geodesic in a vertex space. Then every $(N, 0)$-quasi-geodesic of $X$ with end points in $\eta$ lies in a bounded neighbourhood of $\eta$ where the bound depends only on $N, \sigma$ and the contraction constant of $\eta$.

A subspace $Y$ of $X$ is said to be $k$-\textit{quasiconvex} if all geodesics of $X$ with endpoints in $Y$ lie in a $k$-neighbourhood of $Y$. By taking $N = 1$ in Theorem 1.1, we have the following corollary.

**COROLLARY 1.2**

For all $\sigma \geq 0$, there exists a number $R \geq 1$ depending only on $\sigma$ such that the following holds: Let $X$ be a tree of proper geodesic spaces whose edge spaces are $\sigma$-contracting and uniformly $R$-separated. Then every strongly contracting geodesic $\gamma$ of a vertex space is quasiconvex in $X$, where the quasiconvexity constant depends only on $\sigma$ and the contraction constant of $\gamma$.

As an application of Theorem 1.1, we will prove the following theorem in the last section.

**Theorem 1.3 (see Theorem 4.3).** Let $X$ be a tree of proper geodesic spaces. If all the vertex spaces are uniformly hyperbolic metric spaces, edge spaces are uniformly quasi-convex in vertex spaces and they are sufficiently uniformly separated, then the space $X$ is a hyperbolic metric space and vertex spaces are quasiconvex in $X$.

2. Strong contraction

Let $(X, d)$ be a geodesic metric space and $Q$ be a subspace of $X$. The nearest point projection of a point $x \in X$ to $Q$ is given by $\pi_Q(x) = \{z \in Q : d(x, z) = d(x, Q)\}$. If $Q$ is closed and $X$ is proper, then $\pi_Q(x)$ is always non-empty.

**DEFINITION 2.1** (Strongly contracting subspace)

Let $\sigma \geq 0$. We say that $Q$ is $\sigma$-contracting if for all $x, y \in X$,

$$d(x, y) \leq d(x, Q) \implies \text{diam}(\pi_Q(x) \cup \pi_Q(y)) \leq \sigma.$$ 

We say that $Q$ is strongly contracting if $Q$ is $\sigma$-contracting for some $\sigma \geq 0$. 


Theorem 2.2 (Geodesic image theorem [1], Theorem 7.1). \( Q \) is a \( \sigma \)-contracting subspace of a geodesic space \( X \) if and only if there exists constants \( \kappa'_\sigma \) and \( \kappa_\sigma \) such that for any geodesic \( \gamma \) in \( X \) lying outside \( \kappa_\sigma \)-neighborhood of \( Q \), the diameter of \( \pi_Q(\gamma) \) is at most \( \kappa'_\sigma \), where \( \kappa'_\sigma \) and \( \kappa_\sigma \) depend only on \( \sigma \).

For the sake of completeness, we give here a proof of a quasification of geodesic image Theorem 2.2. The proof is adaptation of the arguments used in [1].

Theorem 2.3 (Bounded quasi-geodesic image). Let \( Q \) be a \( \sigma \)-contracting subspace of a geodesic space \( X \). Suppose \( \beta : [0,1] \to X \) is a continuous \((K,0)\)-quasi-geodesic lying outside \( D\)-neighborhood of \( Q \), where \( D = D(K, \sigma) = 2([K]+1)\sigma, l = l(\beta) \) is the length of \( \beta \) and \([0,1]\) is the arc length parameterization of \( \beta \). Then \( \text{diam}(\pi_Q(\beta(0)))\cup \pi_Q(\beta(l)) \leq 4D \).

Proof. Let \( \beta(0) = x \) and \( \beta(l) = y \).

Case (i): \( l(\beta) \leq K(d(x, Q) + d(y, Q)) \). It implies that either \( l(\beta) \leq \frac{K}{2}d(x, Q) \) or \( l(\beta) \leq \frac{K}{2}d(y, Q) \). Suppose

\[
 l(\beta) \leq \frac{K}{2} d(x, Q). \tag{1}
\]

Let us first assume the distance between \( \beta \) and \( Q \) is realised at \( x \), i.e., \( d(\beta([0, l]), Q) = d(x, Q) \). Let \( 0 = t_0 < t_1 < \ldots < t_n = l \) be a partition of \([0, l]\) such that \( l(\beta|_{[t_i, t_{i+1}]})) = d(x, Q) \) for all \( 0 \leq i \leq n - 2 \) and \( l(\beta|_{[t_n-1, t_n]})) \leq d(x, Q) \). Note that \( n \) is at most \( \lceil \frac{l}{K} \rceil + 1 \). For each \( i \), \( d(\beta(t_i), \beta(t_{i+1})) \leq l(\beta|_{[t_i, t_{i+1}]}) = d(x, Q) \leq d(\beta(t_i), Q) \). \( Q \) is \( \sigma \)-contracting implies \( \text{diam}(\pi_Q(\beta(t_i)) \cup \pi_Q(\beta(t_{i+1}))) \leq \sigma \) for all \( i \).

\[
 \text{diam}(\pi_Q(x)) \cup \pi_Q(y)) \leq \sum_{i=0}^{n-1} \text{diam}(\pi_Q(\beta(t_i)) \cup \pi_Q(\beta(t_{i+1}))) \leq \left( \left\lceil \frac{K}{2} \right\rceil + 1 \right) \sigma.
\]

If distance between \( \beta \) and \( Q \) is not realised at \( x \), then let \( s \in [0, l] \) such that \( d(\beta([0, l]), Q) = d(\beta(s), Q) \). Divide \( \beta \) in two parts \( \beta' \) and \( \beta'' \) such that \( \beta'|_{[0, s]} = \beta|_{[0, s]} \) and \( \beta'' = \beta|_{[s, l]} \). After suitable re-parametrization of \( \beta' \) and \( \beta'' \), we have that distances \( d(\beta', Q), d(\beta'', Q) \) are realized at the starting points of \( \beta', \beta'' \) respectively. As above, union of nearest point projections of end points of \( \beta', \beta'' \) have diameters at most \( \lceil \frac{K}{2} \rceil + 1 \). So, \( l(\beta) \leq \frac{K}{2} d(x, Q) \) implies \( \text{diam}(\pi_Q(x)) \cup \pi_Q(y)) \leq 2(\lceil \frac{K}{2} \rceil + 1) \). For \( l(\beta) \leq \frac{K}{2} d(y, Q) \), we take the re-parametrization \( \beta(t) = \beta(l-t), t \in [0, l] \), and repeat the same argument for \( \tilde{\beta} \) as above.

Case (ii): \( l(\beta) > K d(x, Q) + d(y, Q) \). Let \( t_1 \in [0, l] \) be such that \( l(\beta|_{[0, t_1]}) = K d(x, Q) \). Then our assumption forces \( l(\beta|_{[t_1, l]}) > K d(y, Q) \). By replacing \( \frac{K}{2} \) with \( K \) in the inequality (1) and repeating the same argument as in Case (i), we have \( \text{diam}(\pi_Q(\beta(0)) \cup \pi_Q(\beta(t_1))) \leq 2([K] + 1)\sigma \). We will define inductively the points \( t_i \) in \([0, l] \) such that they form a partition of \([0, l] \) and \( l(\beta|_{[t_{i-1}, t_i]}) = K d(\beta(t_{i-1}), Q) \). Suppose we have defined the points \( t_0, t_1, \ldots, t_i \). If \( l(\beta|_{[t_i, l]}) \leq K d(\beta(t_i), Q) + d(y, Q) \), then by Case (i), \( \text{diam}(\pi_Q(\beta(t_i)) \cup \pi_Q(\beta(l))) \leq 2(\lceil \frac{K}{2} \rceil + 1)\sigma \) and we define \( t_{i+1} = l \).
Otherwise, if \( l(\beta_{|t_i,l|}) > K(d(\beta(t_i), Q) + d(y, Q)) \), define \( t_{i+1} \in [t_i, l] \) to be the point such that \( l(\beta_{|t_i,t_{i+1}|}) = K d(\beta(t_i), Q) \). Again this will imply that \( l(\beta_{|t_{i+1},l|}) > K d(y, Q) \).

Let \( t_r \) be the last index for which \( l(\beta_{|t_r,l|}) > K d(\beta(t), Q) + d(y, Q) \) for \( t < t_r \). Then \( l(\beta_{|t_r,l|}) = K d(\beta(t_r), Q) + d(y, Q) \). Define \( t_{r+1} = l \). Let \( D = 2([K] + 1)\sigma \) and \( \beta_i = \beta_{|t_i,t_{i+1}|} \) for all \( i \in \{0, 1, \ldots, r\} \). Then

\[
\begin{align*}
\sum_{i=0}^{r} l(\beta_i) &= K d(\beta(t_i), Q) + l(\beta_r), & (2) \\
\sum_{i=1}^{r-1} K d(\beta(t_i), Q) &= K d(x, Q) + \sum_{i=1}^{r-1} K d(\beta(t_i), Q) + l(\beta_r). & (3)
\end{align*}
\]

Also, \( \beta \) being \((K, 0)\)-quasi-geodesic, we have

\[
\begin{align*}
l(\beta) &\leq K d(x, y), & (4) \\
l(\beta) &\leq K d(x, \pi_Q(x)) + K \sum_{i=0}^{r} \text{diam}(\pi_Q(\beta(t_i))) + \pi_Q(\beta(t_{i+1})) \\
&\quad + K d(\pi_Q(y), y). & (5)
\end{align*}
\]

From (3) and (5), and as \( d(x, Q) = d(x, \pi_Q(x)) \), we have

\[
\sum_{i=1}^{r-1} K d(\beta(t_i), Q) + l(\beta_r) \leq K \sum_{i=0}^{r} \text{diam}(\pi_Q(\beta(t_i))) + \pi_Q(\beta(t_{i+1})) + K d(y, Q) \leq 2KD + (r - 1)KD + K d(y, Q).
\]

Thus

\[
\sum_{i=1}^{r-1} K (d(\beta(t_i), Q) - D) \leq 2KD - (l(\beta_r) - K d(y, Q)). & (6)
\]

By construction of the point \( t_r, l(\beta_r) - K d(y, Q) \geq 0 \), and so from (6), we have

\[
\sum_{i=1}^{r-1} (d(\beta(t_i), Q) - D) \leq 2D. & (7)
\]

As per hypothesis, \( \beta \) lies outside \( 2D \)-neighborhood of \( Q \), so \( d(\beta, Q) \geq 2D \). This implies \( d(\beta(t_i), Q) \geq 2D \) and hence \( D \leq d(\beta(t_i), Q) - D \). So, from (7), we have
\[\text{diam}(\pi_Q(x) \cup \pi_Q(y)) \leq \sum_{i=0}^{r} \text{diam}(\pi_Q(\beta(t_i)) \cup \pi_Q(\beta(t_{i+1}))) \]
\[\leq (r + 1)D \]
\[\leq \sum_{i=1}^{r-1} (d(\beta(t_i), Q) - D) + 2D \]
\[\leq 2D + 2D = 4D. \]

\[\square\]

**Notation.** For a path \( \alpha : [a, b] \to X \) with \( p = \alpha(s), q = \alpha(t) \) and \( s < t \), we denote \( \alpha|^q_p \) to be the subsegment of \( \alpha \) between \( p \) and \( q \). We denote \( \ell_X(\alpha|^q_p) \) to be the length of \( \alpha|^q_p \) in \( X \).

**Lemma 2.4** Let \( Q \) be \( \sigma \)-contracting and properly embedded in a geodesic metric space \((X, d)\). Then

(i) any \((N, 0)\)-quasi-geodesic between two points of \( Q \) lies in a \( M = M(N, \sigma)\)-neighbourhood of \( Q \), where \( M(N, \sigma) = (3N + 1)D(N, \sigma) + D = D(N, \sigma) \) is the constant from Theorem 2.3, depends only on \( N \) and \( \sigma \);

(ii) there exists \( K \geq 1 \) depending only on \( \sigma \) such that \( Q \) is \((K, K)\)-quasi-isometrically embedded in \( X \).

**Proof.**

(i) Let \( x, y \in Q \) and \( \gamma \) be \((N, 0)\)-quasi-geodesic between \( x \) and \( y \). Let \( \gamma|^q_p \) be a maximal connected subsegment of \( \gamma \) which lie outside \( D \) neighbourhood of \( Q \). Then

\[d(p, Q) = D = d(q, Q) \text{ and diam } (\pi_Q(p) \cup \pi_Q(q)) \leq 4D \text{ (by Theorem 2.3)}.\]

So \( \ell_X(\gamma|^q_p) \leq N d(p, q) \leq N(2D + 4D) = 6ND \). Now for any \( z \in \gamma|^q_p \), either \( \ell_X(\gamma|^q_z) \leq 3ND \) or \( \ell_X(\gamma|^q_z) \leq 3ND \) and therefore, \( d(z, Q) \leq 3ND + D = (3N + 1)D \). This holds for any maximal connected subsegment of \( \gamma \) which lie outside \( D \)-neighbourhood of \( Q \). Hence \( \gamma \) lies in the \( M \)-neighbourhood of \( Q \).

(ii) Let \( x, y \in Q \) and \( \gamma : [0, l] \to X \) be a \( X \)-geodesic with \( \gamma(0) = x \) and \( \gamma(l) = y \). Let \( x_i = \gamma(i) \) where \( i \in \{1, \ldots, [l]\} \). Then \( d(x, x_1) = 1, d(x_i, x_{i+1}) = 1, \ldots, d(x_{[l]}, x_1) \leq 1 \). From (i), \( \gamma \) lies in \( 4D(1, \sigma) \)-neighbourhood of \( Q \), as geodesics are \((1, 0)\)-quasi-geodesics. Hence \( d(x_i, \pi_Q(x_i)) \leq 4D(1, \sigma) \). This implies that \( d(\pi_Q(x_i), \pi_Q(x_{i+1})) \leq 8D(1, \sigma) + 1 \). As \( Q \) is properly embedded in \( X \), there exists \( K = K(D(1, \sigma)) \geq 1 \) such that

\[d(\pi_Q(x_i), \pi_Q(x_{i+1})) \leq K.\]

Now consecutively joining \( \pi_Q(x_i) \)'s by a \( Q \)-geodesic, we obtain a path \( \tilde{\gamma} \) between \( x \) and \( y \) in \( Q \). And we have \( d_Q(x, y) \leq \ell_X(\tilde{\gamma}) \leq [l]K + K \leq Kd(x, y) + K \). \[\square\]

**Lemma 2.5** Let \( \sigma \geq 0 \). Let \( C_1 := 5D(1, \sigma) \) and \( D_1 := 4D(1, \sigma) \), where \( D(1, \sigma) \) is the constant from Theorem 2.3 for \( N = 1 \). Let \( Q_1 \) and \( Q_2 \) be \( \sigma \)-contracting in \( X \), then

\[d(Q_1, Q_2) > C_1 \implies \text{diam}(\pi_{Q_1}(Q_2)) \leq D_1.\]
Proof. Let \( x, y \) be any two arbitrary points in \( Q_2 \) and \( \gamma \) be a \( X \)-geodesic between \( x \) and \( y \). Then by Lemma 2.4, \( \gamma \) lies in \( 4D(1, \sigma) \)-neighbourhood of \( Q_2 \) (note that geodesics are (1, 0)-quasi-geodesics).

Now \( d(Q_1, Q_2) > C_1 = 5D(1, \sigma) \) implies \( d(\gamma, Q_1) > D(1, \sigma) \). Then by Theorem 2.3,

\[
\text{diam}(\pi_{Q_1}(x) \cup \pi_{Q_1}(y)) \leq 4D(1, \sigma).
\]

Hence \( \text{diam}(\pi_{Q_1}(Q_2)) \leq 4D(1, \sigma) = D_1 \). \( \square \)

2.1 Tree of spaces

DEFINITION 2.6

Tree of geodesic spaces: Let \((X, d_X)\) be a proper geodesic metric space. \( P : X \to T \) is said to be a tree of geodesic metric spaces if \( X \) admits a surjective map \( P : X \to T \) onto a simplicial tree \( T \), such that the following holds:

(i) For all \( s \in T \), \( X_s = P^{-1}(s) \subset X \) with the induced path metric \( d_{X_s} \) is a geodesic metric space \( X_s \).

(ii) For a vertex \( v \) in \( T \), \( X_v = P^{-1}(v) \) will be called as vertex space for \( v \). Let \( e \) be an edge of \( T \) between vertices \( v_1 \) and \( v_2 \). Let \( X_e \) be the preimage under \( P \) of the mid-point of \( e \), \( X_e \) will be called as edge space for \( e \). There exists a continuous map \( f_e : X_e \times [0, 1] \to X \), such that \( f_e|_{X_e \times (0,1)} \) is an isometry onto the preimage of the interior of \( e \) equipped with the path metric. The maps \( f_e|_{X_e \times [0]} \) and \( f_e|_{X_e \times [1]} \) are proper embeddings into \( X_{v_1} \) and \( X_{v_2} \) respectively.

Edge spaces separation: Let \( P : X \to T \) be a tree of geodesic metric spaces. For an edge \( e \) of \( T \) between vertices \( v_1 \) and \( v_2 \), let \( f_{e,v_1} : X_e \to X_{v_1} \) be defined by \( f_{e,v_1}(x) = f_e(x, 0) \) and \( f_{e,v_2} : X_e \to X_{v_2} \) be defined by \( f_{e,v_2}(x) = f_e(x, 1) \). The edge spaces in \( P : X \to T \) are said to be \( R \)-uniformly separated if for any two distinct edges \( e, e' \) of \( T \) with a common vertex \( v \), we have \( d_X(f_{e,v}(X_e), f_{e',v}(X_{e'})) \geq R \).

DEFINITION 2.7 (Strongly contracting edge spaces)

Let \( \sigma \geq 0 \). We say that the edge spaces of a tree of spaces \( P : X \to T \) are \( \sigma \)-contracting in vertex spaces if for all edge spaces \( X_e \), \( f_e(X_e \times \{0\}) \) and \( f_e(X_e \times \{1\}) \) are \( \sigma \)-contracting in respective vertex spaces.

3. Main theorem

Let \( P : X \to T \) be a tree of proper geodesic spaces. The edge space \( X_e \) corresponding to an edge \( e \) defined in Definition 2.6 are identified to respective end vertex spaces with the help of maps \( f_e|_{X_e \times \{0\}} \) and \( f_e|_{X_e \times \{1\}} \), so we will call \( f_e(X_e \times \{0\}) \) and \( f_e(X_e \times \{1\}) \) also to be edge spaces. For an edge \( e \), we denote \( f_e(X_e \times \{0\}) \) by \( Y_e^- \) and \( f_e(X_e \times \{1\}) \) by \( Y_e^+ \).

Lemma 3.1 Let \( N \geq 1 \) and \( \sigma \geq 0 \). There exists \( R = R(N, \sigma) \geq 1 \) such that the following holds: Let \( P : X \to T \) be a tree of proper geodesic spaces, where all edge spaces are \( \sigma \)-contracting and uniformly \( R \)-separated. Let \( e \) be a directed edge with terminal vertex \( v \).
Let $x, y$ be two points in the edge space $Y^+$. Let $\gamma : [0, 1] \to X$ be a $(N, \sigma)$-quasi-geodesic with $\gamma(0) = x, \gamma(1) = y$ such that $P(\gamma) \cap e = \{v\}$ and for all $t \in (0, 1)$, $\gamma(t) \notin Y^+$. Then $\gamma$ does not intersect any other edge space other than $Y^+$.

**Proof** Let $C_N = 5D(N, \sigma)$, where the constant $D(N, \sigma)$ is as in Theorem 2.3. Let $R = R(N, \sigma) = (2N + 1)C_N + 2ND_1 + 12D(N, \sigma) + 1$, where $D_1 = 4D(1, \sigma)$ is the constant from Lemma 2.5. If possible, let $\gamma$ intersect other edge spaces non-trivially. Let $e_1, \ldots, e_m$ be the edges incident on $v$ such that $\gamma$ intersects the edge spaces $Y_{e_1}, \ldots, Y_{e_m}$. For each $Y_{e_i}$ intersected by $\gamma$, let $\gamma(s_i)$ be the first entry point and $\gamma(t_i)$ be the last exit point. Replace the portion $\gamma|_{(s_i, t_i)}$ by a geodesic in $Y_{e_i}$ joining $\gamma(s_i)$ and $\gamma(t_i)$. This results in a path $\gamma_v$ in $X_v$ joining $x$ and $y$.

Consider the $C_N$-neighbourhood of $Y^+_e$ in $X_v$. If possible, let there exists a maximal connected subsegment, $\gamma'_v$, of $\gamma_v$ which lie outside the $C_N$-neighbourhood of $Y^+_e$. Let $n$ be the number of edge spaces intersected by $\gamma'_v$. Let $p$ and $q$ be the endpoints of $\gamma'_v$. Since $d_X(Y^+_e, Y_{e_i}) \geq R$, where $R > C_N > C_1$, so $p$ and $q$ will lie in $\gamma \cap X_v$. As $\gamma'_v$ is the maximal connected subsegment of $\gamma_v$ lying outside the $C_N$-neighbourhood of $Y^+_e$, we have $d_X(p, Y^+_e) = C_N = d_X(q, Y^+_e)$. As $Y^+_e$ is $\sigma$-contracting, by Lemma 2.5, the diameter of projection of $Y_{e_i}$ into $Y^+_e$ is at most $D_1$ and the diameter of projection into $Y^+_e$ of a component of $\gamma \cap X_v$ lying outside the $C_N$-neighbourhood of $Y^+_e$ is at most $4D(N, \sigma)$, by Theorem 2.2.

Now projecting $\gamma'_v$ to $Y^+_e$, we get

$$
(n - 1)R \leq \ell_X(\gamma'_v|_p) \leq N\ell_X(p, q)
\leq N\ell_X(p, q) \leq N(2C_N + nD_1 + (n + 1)(4D(N, \sigma))).
$$

**Case (i).** Suppose $n = 1$. Then $\ell_X(\gamma'_v|_p) \leq N\ell_X(p, q) \leq N\ell_X(p, q) \leq N(2C_N + D_1 + 8D(N, \sigma))$. So, the length of the portion of the quasi-geodesic $\gamma$ between $p$ and $\gamma(s_1)$ is at most $N(2C_N + D_1 + 8D(N, \sigma))$ and hence $d_X(p, Y_{e_i}^-) \leq N(2C_N + D_1 + 8D(N, \sigma))$. This implies $R \leq d_X(Y^+_e, Y_{e_i}^-) \leq d_X(p, Y^+_e) + d_X(p, Y_{e_i}^-)$

$$
\leq C_N + N(2C_N + D_1 + 8D(N, \sigma)).
$$

Thus $R \leq (2N + 1)C_N + 2ND_1 + 8D(N, \sigma)$ which is a contradiction.

**Case (ii).** Suppose $n > 1$. Then

$$
R \leq \frac{2NC_N + ND_1 + 8D(N, \sigma)}{n - 1} + ND_1 + 4D(N, \sigma)
\leq (2N + 1)C_N + 2ND_1 + 12D(N, \sigma),
$$

which is a contradiction.

Hence, $\gamma_v \subset Nbhd(Y^+_e; C_N) \subset X_v$. As edge spaces are $R$-separated and $R > C_N, \gamma_v$ does not intersect any other edge space other than $Y^+_e$. This also implies that $\gamma_v = \gamma$ and hence $\gamma$ does not intersect any other incident edge spaces.

The underlying idea of Lemma 3.1 is present in the work of [6].
**Theorem 3.2** Let \( N \geq 1 \) and \( \sigma \geq 0 \). There exists \( R = R(N, \sigma) \geq 1 \) such that the following holds: Let \( P : X \rightarrow T \) be a tree of proper geodesic spaces where all edge spaces are \( \sigma \)-contracting and uniformly \( R \)-separated. Let \( \eta \) be a strongly contracting geodesic in a vertex space. Then every \((N,0)\)-quasi-geodesic of \( X \) with end points in \( \eta \) lies in a bounded neighbourhood of \( \eta \), where the bound depends only on \( N, \sigma \) and the contraction constant of \( \eta \).

**Proof** Let \( \eta \) be a \( \rho \)-contracting geodesic of a vertex space \( X_v \) for some \( \rho \geq 0 \). Let \( x, y \in \eta \). Let \( \alpha : [0, l] \rightarrow X \) be an arc length parameterization of a \((N,0)\)-quasi-geodesic of \( X \) with \( \alpha(0) = x, \alpha(l) = y \). Without loss of generality, we assume \( \alpha(t) \notin \eta \), for all \( t \in (0, l) \), i.e., \( \alpha \) does not intersect \( \eta \) other than the end points. Then \( P(\alpha) \) will have diameter at most one, by Lemma 3.1.

Let \( s(1), s(2), \ldots, s(l) \) be the edges of \( P(\alpha) \) incident on \( v \). Let \( v_i \) be another vertex of \( s(i) \). If \( \alpha_{ik} \) is a portion of \( \alpha \) such that the end points of \( \alpha_{ik} \) lie in \( f_{s(i)}(X_{s(i)} \times \{1\}) \) and \( P(\alpha_{ik}) \cap s(i) = \{v_i\} \), then from Lemma 3.1, \( \alpha_{ik} \) does not intersect other edge spaces in \( X_{v_i} \) and \( \alpha_{ik} \) lies in \( X_{v_i} \). Let \( a_{ik}, b_{ik} \) be end points of \( \alpha_{ik} \). Then \( \ell_{X_{v_i}}(\alpha_{ik}) = \ell_X(\alpha_{ik}) \leq N d_X(a_{ik}, b_{ik}) \leq N d_{X_{v_i}}(a_{ik}, b_{ik}) \). This implies that \( \alpha_{ik} \) is a \((N,0)\)-quasi-geodesic in \( X_{v_i} \).

Similarly, if \( \alpha_{ik} \) is a portion of \( \alpha \) such that the end points of \( \alpha_{ik} \) lie in \( f_{s(i)}(X_{s(i)} \times \{0\}) \) and \( P(\alpha_{ik}) \cap s(i) = \{v\} \), then \( \alpha_{ik} \) is a \((N,0)\)-quasi-geodesic in \( X_v \). Let \( \beta_i \) be the maximal subsegment of \( \alpha \) containing all such \( \alpha_{ik} \)'s, such that the end points of \( \beta_i \) lie in \( f_{s(i)}(X_{s(i)} \times \{0\}) = Y_{s(i)} \). The Hausdorff distance between \( Y_{s(i)}^{-} \) and \( Y_{s(i)}^{+} \) is at most one and as \( Y_{s(i)}^{-} \), \( Y_{s(i)}^{+} \) are \( \sigma \)-contracting in respective vertex spaces, so by Lemma 2.4 there exists \( M > 0 \) depending on \( N, \sigma \) such that \( \beta_i \) lies in the \( M + 1 \)-neighbourhood of \( Y_{s(i)}^{-} \). Let \( M' = M + 1 \). Let \( p_i, q_i \) be end points of \( \beta_i \). Then \( p_i, q_i \in Y_{s(i)}^{-} \subset X_v \).

Now following the proof of Lemma 2.4(ii), we divide \( \beta_i \) into \( \beta_i^1 \cup \beta_i^2 \cup \cdots \cup \beta_i^n \) such that the length of \( \beta_i^j \) is one for all \( j = 1, 2, \ldots, n \) and the length of \( \beta_i^n(i) \) is at most one. Let \( x_i^j \) and \( x_i^{j+1} \) be the end points of \( \beta_i^j \). Now for each \( j \), there exists \( y_i^j \in Y_{s(i)}^{-} \) such that \( d_X(x_i^j, y_i^j) \leq M' \). Note that \( x_i^1 = y_i^1 = p_i \) and \( x_i^n(i) = y_i^{n(i)+1} = q_i \) and \( d_X(y_i^j, y_i^{j+1}) \leq 2M' + 1 \) for all \( j \). Since \( X_v \) is properly embedded in \( X \), there exists \( K' = K (M') \geq 1 \) such that \( d_X(y_i^j, y_i^{j+1}) \leq K' \), for all \( j \). Successively joining \( y_i^j \) and \( y_i^{j+1} \) by a geodesic in \( X_v \) for every \( j \) we get a path \( \tilde{\beta}_i \) between \( p_i \) and \( q_i \) in \( X_v \) such that \( \beta_i \) and \( \tilde{\beta}_i \) lie in \( M' + K' \)-neighbourhood of each other. Now

\[
\ell_{X_v}(\tilde{\beta}_i|_{y_i^j}) = \sum_{r=j}^{j'} d_X(y_i^r, y_i^{r+1}) \leq K'(j' - j) \leq K' \ell_X(\beta_i|_{x_i^j}) + K'.
\]

For any \( y_i^j, y_i^{j'} \in \tilde{\beta}_i \), there exists \( y_i^j, y_i^{j'} \in \beta_i \) such that \( \tilde{\beta}_i|_{y_i^j} \subset \beta_i|_{y_i^j}, d_X(y_i^j, y_i^{j'}) \leq K' \) and \( d_X(y_i^j, y_i^{j'}) \leq K' \). Thus \( d_X(y_i^j, x_i^j) \leq M' + K', d_X(y_i^{j'}, x_i^{j'}) \leq M' + K' \) and

\[
\ell_{X_v}(\tilde{\beta}_i|_{y_i^j}) \leq \ell_{X_v}(\tilde{\beta}_i|_{y_i^{j'}}) \leq K' \ell_{X_v}(\beta_i|_{x_i^j}) + K'.
\]

The above process results in a truncated path \( \tilde{\beta}_i \) in \( X_v \) which is concatenation of \( \tilde{\beta}_i \)'s and subsegments of \( \alpha \), say \( \xi_i \), which lies in \( X_v \) connecting two consecutive \( \tilde{\beta}_i \)'s (see Figure 1).
Figure 1. Truncated quasigeodesics.

We claim that the truncated path $\tilde{\beta}$ is quasi-geodesic in $X_v$. Let $c, d$ be two points in $\tilde{\beta}$ and $\ell_{X_v}(\tilde{\beta}|^d)$ be the length of $\tilde{\beta}$ between $c$ and $d$.

Case 1. $c \in \xi_k$ and $d \in \xi_m$ for some $k$ and $m$. Then

\[
\ell_{X_v}(\tilde{\beta}|^d) = \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k+1}^{m-1} \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k}^{m-1} (K' \ell_{X_v}(\tilde{\beta}|^d) + K') + \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k+1}^{m-1} \ell_{X_v}(\tilde{\beta}|^d)
\]

\[
\leq K' \ell_{X_v}(\xi|^{q_{k-1}}) + (m - k)K' + K' \ell_{X_v}(\xi|^{q_{m-1}}) + (m - k)K' + K'
\]

\[
\leq 2N' d_{X_v}(c, d) \leq 2N' d_{X_v}(c, d).
\]

Case 2. $c \in \tilde{\beta}_k$ and $d \in \tilde{\beta}_m$ for some $k$ and $m$. There exists $c' \in \beta_k$ and $d' \in \beta_m$ such that $d_X(c, c') \leq M' + K'$ and $d_X(d, d') \leq M' + K'$. Then

\[
\ell_{X_v}(\tilde{\beta}|^d) = \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k+1}^{m-1} \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k}^{m-1} (K' \ell_{X_v}(\tilde{\beta}|^d) + K') + \ell_{X_v}(\tilde{\beta}|^d) + \sum_{i=k+1}^{m-1} \ell_{X_v}(\tilde{\beta}|^d)
\]

\[
\leq K' \ell_{X_v}(\beta|^{q_{k-1}}) + K' + \sum_{i=k+1}^{m-1} \ell_{X_v}(\beta|^{q_{m-1}}) + \sum_{i=k}^{m-1} (K' \ell_{X_v}(\beta|^{q_{m-1}}) + K')
\]

\[
+ (m - k)K' + 2K'
\]

\[
\leq 2N' d_{X_v}(c, d') + 2K'
\]

\[
\leq 2N' d_{X_v}(c, d') + 2K'
\]
\[ \leq 2NK' (d_X(c, d) + 2(M' + K')) + 2K' \]
\[ \leq 2NK' d_X(c, d) + 4NK' (M' + K') + 2K'. \]

**Case 3.** \( c \in \beta_k \) and \( d \in \xi_m \). There exists \( c' \in \beta_k \) such that \( d_X(c, c') \leq M' + K' \).

\[
\ell_{X_v}(\tilde{\beta}^d_{\xi c}) = \ell_{X_v}(\tilde{\beta}_i|c') + \frac{m}{i=k+1} \ell_{X_v}(\xi_i) + \sum_{i=k+1}^{m-1} \ell_{X_v}(\tilde{\beta}_i) + \ell_X(\tilde{X}_m|^d_{p_{m-1}}) \\
\leq K' \ell_X(\beta_k|c') + K' \sum_{i=k+1}^{m-1} \ell_X(\xi_i) \\
+ \sum_{i=k+1}^{m-1} (K' \ell_X(\beta_i) + K') + K' \ell_X(\tilde{X}_m|^d_{p_{m-1}}) \\
\leq 2NK' d_X(c', d) + K' \\
\leq 2NK' d_X(c, d) + 2NK' (M' + K') + 2K'.
\]

Hence \( \tilde{\beta} \) is a \( (2NK', 4NK' (M' + K') + 2K') \) quasi-geodesic in \( X_v \), where \( M' = M + 1 \). Note that \( K', M \) depends on \( N, \sigma \). As \( \eta \) is \( \rho \)-contracting, there exists \( \kappa = \kappa(N, \sigma, \rho) > 0 \) such that \( \tilde{\beta} \subset \text{Nbhd}(\eta; \kappa) \) (contracting \( \Rightarrow \) Morse, Theorem 1.3 of [1]). Also, \( \alpha \subset \text{Nbhd}(\tilde{\beta}; M + 1 + K') \). Hence \( \alpha \subset \text{Nbhd}(\eta; \kappa + M + 1 + K') \).

By taking \( N = 1 \) in Lemma 3.1 and Theorem 3.2, we have the following corollary

**COROLLARY 3.3**

Let \( \sigma \geq 0 \). There exists \( R = R(\sigma) \geq 1 \) such that the following holds: Let \( X \) be a tree of proper geodesic spaces where all edge spaces are \( \sigma \)-contracting and uniformly \( R \)-separated.

(i) Let \( e \) be a directed edge with terminal vertex \( v \). Let \( x, y \) be two points in the space \( Y_e^+ \). Let \( \gamma : [0, l] \to X \) be a \( X \)-geodesic with \( \gamma(0) = x, \gamma(l) = y \) such that \( P(\gamma) \cap e = \{v\} \) and for all \( t \in (0, l), \gamma(t) \notin Y_e^+ \). Then the geodesic \( \gamma \) does not intersect any other edge space other than \( Y_e^+ \).

(ii) Every strongly contracting geodesic \( \gamma \) in a vertex space is quasiconvex in \( X \), where the quasiconvexity constant depends only on \( \sigma \) and the contraction constant of \( \gamma \).

**Example 3.4** Let \( G \) be a finitely generated group with a finitely generated subgroup \( H \). Let \( \Gamma_G \) denote the Cayley graph of \( G \) with respect to some finite generating set. Let \( K \) be a coset of \( H \) in \( G \). We construct a one dimensional simplicial complex \( \mathcal{H}(K) \) corresponding to \( K \), it is called combinatorial horoball.

- 0-skeleton of \( \mathcal{H}(K) \), \( \mathcal{H}(K)^{(0)} := K^{(0)} \times \{(0, 1, 2, \ldots)\} \),
- 1-skeleton of \( \mathcal{H}(K) \), \( \mathcal{H}(K)^{(1)} := \{(v, 0), (w, 0) \} : v, w \in K^{(0)}, [v, w] \in K^{(1)} \} \cup \{[[v, k), (w, k)] : v, w \in K^{(0)}, k > 0, d_K(v, w) \leq 2^k \} \cup \{[(v, k), (v, k + 1)] : v \in K^{(0)}, k \geq 0 \} \).

Let \( \Gamma^h_G \) be the space obtained by gluing \( \mathcal{H}(K) \) to \( K \) in the Cayley graph \( \Gamma_G \). Suppose \( G \) is hyperbolic relative to \( H \). Then the augmented space \( \Gamma^h_G \) is hyperbolic and \( G \) acts properly discontinuously on \( \Gamma^h_G \).
We consider a subcomplex $H_m^c(K)$ of $H(K)$ whose 0-skeleton is $K(0) \times \{(m + 1, m + 2, \ldots)\}$. Let $(\Gamma_G)^{(m)} = \Gamma_G^h \setminus H_m^c(K)$. Then $G$ will act properly discontinuously and co-compactly on $(\Gamma_G)^{(m)}$. Next we consider the subcomplex of $\Gamma_G^h$ whose 0-skeleton is $K(0) \times \{(m)\}$ and denote it by $H_m(K)$. It is known that peripheral subspaces are strongly contracting, for instance, see Lemma 2.3 of [7]. So, $K$’s are strongly contracting in $\Gamma_G$. The Hausdorff distance between $H_m(K)$ and $H_0(K) = K$ is bounded by $m$. Let $A$ be a subset of diameter $r$ in $H_0(K)$, then the projection of $A$ into $H_m(K)$ has diameter at most $r/2^m$ in $H_m(K)$. Thus, $H_m(K)$’s are strongly contracting in $(\Gamma_G)^{(m)}$ and if the contraction constants for $H_0(K), H_m(K)$ are respectively $\kappa_0, \kappa_m$, then $\kappa_m \leq \kappa_0$. Then we can choose large enough $m_0$ such that $H_{m_0}(K)$ and $H_{m_0}(K')$ are $R(\kappa_0)$-separated for all distinct $K$ and $K'$. Let $T$ be the Bass–Serre tree of the graph of groups $G \ast_H G$. For every vertex of $T$, take a copy of $(\Gamma_G)^{(m)}$ and glue along appropriate translations of $H_m(K)$ whenever there is an edge between two vertices. So we get a tree of spaces, say $X$, where the conjugates of $G$ in $G \ast_H G$ act properly discontinuously and co-compactly on the respective vertex spaces. Let $\alpha$ be a strongly contracting geodesic in $\Gamma_G$. Since the Hausdorff distance between $(\Gamma_G)^{(m)}$ and $\Gamma_G$ is at most $m$, $\alpha$ is strongly contracting in $(\Gamma_G)^{(m)}$. As, $X$ satisfies all the assumptions of Corollary 3.3, $\alpha$ is quasi-convex in $X$. Note that the group $G \ast_H G$ acts properly discontinuously and co-compactly on $X$, hence $G \ast_H G$ is quasi-isometric to $X$ (by Švarc–Milnor lemma). Thus, there exists $P \geq 1, \epsilon \geq 0$ such that $\alpha$ is $(P, \epsilon)$-quasi-geodesic in $G \ast_H G$.

4. Applications

Let $\delta \geq 0$. A geodesic triangle $T$ is said to be $\delta$-slim if each side of $T$ is contained in the $\delta$-neighborhood of union of other two sides of $T$. A geodesic metric space $X$ is said to be $\delta$-hyperbolic metric space if all the triangles of $X$ are $\delta$-slim. A geodesic metric space is said to be hyperbolic if it is $\delta$-hyperbolic for some $\delta \geq 0$. Here we give a characterization of hyperbolic metric space, which we will used later.

**PROPOSITION 4.1**

*Let $Y$ be a geodesic metric space. $Y$ is hyperbolic if and only if there exists some $\delta > 0$ such that all $(3, 0)$-quasi geodesic bigons are $\delta$-slim (here $\delta$-slimness of bigon means that the Hausdorff distance between the two sides is at most $\delta$).*

*Proof.*

($\Rightarrow$) Suppose $Y$ is hyperbolic. Then from the stability of quasi-geodesics (Theorem 1.7, page 401 of [3]), it follows that $(3, 0)$-quasi-geodesics bigons are slim.

($\Leftarrow$) Suppose, there exists some $\delta > 0$ such that all $(3, 0)$-quasi geodesic bigons are $\delta$-slim. Let $ABC$ be an arbitrary triangle in $Y$ with sides $a$, $b$ and $c$ (sides $a$, $b$ and $c$ are opposite to the vertices $A$, $B$ and $C$ respectively). Let $D$ be a nearest point projection of vertex $A$ to the side $a$. Then $\alpha_1 = AD \cup DB$ and $\alpha_2 = AD \cup DC$ are $(3, 0)$-quasi geodesics. So, we have two $(3, 0)$-quasi geodesic bigons, namely $b \cup \alpha_1$ and $c \cup \alpha_2$. Now from our assumption, $b$ is in the $\delta$-neighbourhood of $\alpha_1$ and $\alpha_2$ is in the $\delta$-neighbourhood of $c$. Hence, $\delta$-neighbourhood of $a \cup c$ contains $\alpha_1 \cup \alpha_2$ and so $2\delta$-neighbourhood of $a \cup c$ contains $b$. Similarly, $2\delta$-neighbourhood of $a \cup b$ contains $c$. 
Again, taking the nearest point projection of the vertex $B$ to the side $b$, we can show that $a$ is contained in $2\delta$-neighbourhood of $b \cup c$. This proves that the triangle $ABC$ is $2\delta$-slim. Therefore, $Y$ is a hyperbolic metric space. \hfill $\square$

The following theorem, due to Gromov, is a generalization of the fact that all triangles of a hyperbolic space are uniformly slim. Here we state a part of the version of the theorem given by Bestvina and Feighn in [2].

**Theorem 4.2** (see Section 3 of [2]) (Resolution of polygons of hyperbolic spaces). Let $\delta \geq 0$. Let $Z$ be a $\delta$-hyperbolic metric space. Let $\Delta : D^2 \to Z$ be a disk with boundary an $n$-sided $(K, \epsilon)$-quasi-geodesic polygon. Then there exists a function $B(\delta, n, K, \epsilon)$ (depending on $\delta, n, K$ and $\epsilon$ only) and a finite $\mathbb{R}$-tree $S$ with a map $r : D^2 \to S$ such that

1. there are $n$ number of valence one vertices in $S$,
2. $d_S(\Delta(a), \Delta(b)) \leq d_S(r(a), r(b)) + B(\delta, n, K, \epsilon)$, for all $a, b \in S$,
3. for all $s \in S$, $r^{-1}(s)$ is properly embedded finite tree in $D^2$,
4. if $e$ is an edge of $S$, then $r$ restricted to $r^{-1}$ (interior($e$)) is an $I$-bundle, where $I = [0, 1]$.

Here, note that if two points $p = \Delta(a)$ and $q = \Delta(b)$ of $\Delta$ are identified in $S$, that is, $r(a) = r(b)$, then the distance between them in $Z$ is at most $B(\delta, n, K, \epsilon)$. A fiber $r^{-1}(s)$ of a resolution is called a singular fiber if $r^{-1}(s)$ is not isomorphic to $I$.

**Theorem 4.3.** Let $\delta, \sigma \geq 0$. Then there exists $\tilde{R} = \tilde{R}(\delta, \sigma)$ such that the following holds: Let $\tilde{P} : \tilde{X} \to \tilde{T}$ be a tree of proper geodesic metric spaces. Suppose all the vertex spaces are $\delta$-hyperbolic, edge spaces are uniformly $\sigma$-contracting and $\tilde{R}$-separated, then $\tilde{X}$ is hyperbolic.

**Proof** For a tree of hyperbolic metric spaces $P : X \to T$, the vertex spaces are hyperbolic metric spaces and hence geodesics in a vertex space are uniformly strongly contracting (see Section 1.4 of [1]). Thus, by Theorem 3.2, if we take edge spaces to be $\sigma$-contracting and at least $R = R(3, \sigma)$-separated then every $(3, 0)$-quasi-geodesic of $X$ having end points in an edge space does not intersect other edge spaces. By Lemma 2.4, there exists $K_1 \geq 1$ depending only on $\sigma$ such that the edge spaces are $(K_1, K_1)$-quasi isometrically embedded in the respective vertex spaces. Also, there exists $K_2 \geq 1$ and $\epsilon_2 \geq 0$ depending only on $\sigma$ and not on the separation constant $R$ such that the portion of a $(3, 0)$-quasi geodesic of $X$ with end points in a vertex space of $P : X \to T$ gives rise to a truncated $(K_2, \epsilon_2)$-quasi-geodesic in that vertex space.

Let $K = \max\{K_1, K_2, \epsilon_2, 3\}$ and $\tilde{R} = \max\{R(3, \sigma), B(\delta, 4, K, K) + 1\}$, where $B(\delta, n, K, \epsilon)$ is the constant from Theorem 4.2. Then $K$ depends only on $\sigma$ and $\tilde{R}$ depends only on $\delta, \sigma$. We will apply the characterization given in Proposition 4.1, to prove that $\tilde{X}$ is hyperbolic. Let $x, y$ be any two points of $\tilde{X}$ and $\gamma_1, \gamma_2$ be two $(3, 0)$-quasi geodesics between $x$ and $y$ in $\tilde{X}$. So, $\gamma_1 \cup \gamma_2$ gives a $(3, 0)$-quasi geodesic bigon in $\tilde{X}$. We will show that $\gamma_1$ and $\gamma_2$ lie in a bounded neighborhood of each other where the bound depends only on $\delta$ and $\sigma$. We $\gamma_1, \gamma_2$ to be parameterized by arc-length.

For an edge $e$, the map $f_e : \tilde{X}_e \times (0, 1) \to \tilde{X}$ is an isometry onto its image. For the convenience of notation, we denote the space $f_e(\tilde{X}_e \times (0, 1))$ by $\tilde{X}_e \times (0, 1)$. Let $e_1$ be the edge with terminal vertex $v_1$ such that $x \in \tilde{X}_{e_1} \times (0, 1) \cup \tilde{X}_{e_1}$ and $e_n+1$ be the edge with initial vertex $v_n$ such that $y \in \tilde{X}_{v_n} \cup \tilde{X}_{e_{n+1}} \times (0, 1)$. Let $e_1, v_1, e_2, v_2, e_3, v_3, \ldots, v_n, e_{n+1}$...
be the all vertices and edges in order lying on the geodesic between \( \tilde{P}(x) \) and \( \tilde{P}(y) \) in \( \tilde{T} \). The paths \( \tilde{P}(\gamma_1) \) and \( \tilde{P}(\gamma_1) \) in \( \tilde{T} \) will contain \( e_1, v_1, e_{12}, v_2, e_{23}, v_3, \ldots, v_n, e_{n+1} \) of \( T \), that means \( \gamma_1 \) and \( \gamma_2 \) will intersect the spaces \( \tilde{X}_{e_1} \times (0, 1) \sqcup \tilde{X}_{v_1}, \tilde{X}_{e_{12}} \times (0, 1), \ldots, \tilde{X}_{v_n} \sqcup \tilde{X}_{e_{n+1}} \times (0, 1) \).

For \( i = 1, 2, \ldots, n \), let \( p^1_i \) (respectively \( p^2_i \)) be the first entry point and \( q^1_i \) (respectively \( q^2_i \)) be the last exit point of \( \gamma_1 \) (respectively \( \gamma_2 \)) to \( \tilde{X}_{v_i} \). If \( x \in \tilde{X}_{v_1} \), then we take \( p^1_1 = p^2_1 = x \), and if \( y \in \tilde{X}_{v_n} \), then we take \( q^1_1 = q^2_1 = y \).

By Theorem 3.2, the subsegments \( \gamma_1|_{p^1_i} \) and \( \gamma_2|_{p^2_i} \) of \( \gamma_1 \) and \( \gamma_2 \) respectively lie in a uniformly bounded neighborhood of \( \tilde{X}_{v_i} \). The truncated paths \( \gamma^1_i \) and \( \gamma^2_i \) in \( \tilde{X}_{v_i} \) obtained from \( \gamma_1 \) and \( \gamma_2 \) respectively are \((K_2, \varepsilon_2)\) quasi geodesics and the Hausdorff distance between \( \gamma^1_i \) (respectively \( \gamma^2_i \)) and \( \gamma_1|_{p^1_i} \) (respectively \( \gamma_2|_{p^2_i} \)) is uniformly bounded by a number, say \( L \), where \( L \) depends on \( \sigma \) only. We join \( p^1_i \) and \( p^2_i \) by a geodesic in \( \tilde{X}_{e_{i-1,i}} \times \{1\} \) and call it \( p^3_i \). Similarly, we join \( q^1_i \) and \( q^2_i \) by a geodesic in \( \tilde{X}_{e_{i,i+1}} \times \{0\} \) and call it \( p^4_i \). The edge spaces are \((K_1, K_1)\)-quasi isometrically embedded in the vertex spaces. Hence \( p^i_1 \) and \( p^i_2 \) are \((K_1, K_1)\)-quasi geodesics in \( \tilde{X}_{v_i} \) for all \( i \). We denote the subsegments of \( \gamma_1 \) (respectively \( \gamma_2 \)) between \( q^1_{i-1} \) and \( p^1_i \) (respectively \( q^2_{i-1} \) and \( p^2_i \)) by \( \gamma^1_{i-1,i} \) (respectively \( \gamma^2_{i-1,i} \)). Note that \( \gamma^j_{i-1,i} \) lies in \( \tilde{X}_{e_{i-1,i}} \times [0, 1] \), where \( j = 1, 2 \).

As \( K = \max\{K_1, K_2, \varepsilon_2, 3\} \), for each \( i = 1, 2, \ldots, n \), we have a disk \( \Delta_i : D^2 \to \tilde{X}_{v_i} \) with boundary a \((K, K)\)-quasi-geodesics 4-gon (quadrilateral) in \( \tilde{X}_{v_i} \), namely \( \gamma^1_i \cup \beta^2_i \cup \gamma^2_i \cup \beta^1_i \). If \( x \in \tilde{X}_{v_1} \), then the side \( \beta^1_1 \) is the single point \( x \) and if \( y \in \tilde{X}_{v_n} \), then the side \( \beta^2_n \) is the single point \( y \). Also, for \( i = 2, \ldots, n \), we have disks \( \Delta_{(i-1,i)} : D^2 \to \tilde{X}_{e_{i-1,i}} \times [0, 1] \) with boundary a \((K, K)\)-quasi geodesics 4-gon (quadrilateral) in \( \tilde{X}_{e_{i-1,i}} \times [0, 1] \), namely \( \gamma^1_{i-1,i} \cup \beta^1_i \cup \gamma^2_{i-1,i} \cup \beta^2_{i-1} \). If \( x \in \tilde{X}_{e_1} \times (0, 1) \) (respectively \( y \in \tilde{X}_{e_{n+1}} \times (0, 1) \)), we have a disk \( \Delta_{e_1} : D^2 \to \tilde{X}_{e_1} \times (0, 1) \) with boundary a \((K, K)\)-quasi geodesic triangle \( \gamma^1_{e_1} \cup \beta^1_1 \cup \gamma^2_{e_1} \) and if \( y \in \tilde{X}_{e_{n+1}} \times (0, 1) \), we have a disk \( \Delta_{e_{n+1}} : D^2 \to \tilde{X}_{e_{n+1}} \times (0, 1) \) with boundary a \((K, K)\)-quasi geodesic triangle \( \gamma^1_{e_{n+1}} \cup \beta^1_{n+1} \cup \gamma^2_{e_{n+1}} \).

The initial and terminal disks containing \( x, y \) respectively give rise to quasi-geodesic triangles where we get only single resolution for each disks, by Theorem 4.2. Apart from the initial and terminal disks, by Theorem 4.2, we can have two types of resolutions \( r_i : D^2 \to S_i \), for each quadrilateral \( \Delta_i \) (see Figure 2). In the first type, we will have two points \( p \in \beta^1_1, q \in \beta^2_1 \) which are identified in the \( S_i \) and so \( d_{\tilde{X}_{v_i}} (p, q) \leq B(\delta, 4, K, K) \). This contradicts the fact that edge spaces are \( \tilde{R} \)-separated (note that \( \tilde{R} > B(\delta, 4, K, K) \)). Therefore, we will only have the second type for \( \Delta_i \)’s.

In the second type, there are maximal segments, \( Q^1_i \) and \( Q^2_i \) of \( \gamma^1_i \) and \( \gamma^2_i \) respectively, which are identified pointwise in \( S_i \). And so Hausdorff distance between those segments is at most \( B(\delta, 4, K, K) \). Let \( u^1_i \) (respectively \( u^2_i \)) and \( v^1_i \) (respectively \( v^2_i \)) be the end points of \( Q^1_i \) (respectively \( Q^2_i \)). Note that \( u^1_i \) and \( u^2_i \) (respectively \( v^1_i \) and \( v^2_i \)) are identified in \( S_i \). Also, there exists \( w^1_i \in \beta^1_1 \) such that \( w^1_i \) is identified with \( u^1_i \) and \( u^2_i \) in \( S_i \).

Similarly, we will have resolutions for \( \Delta_{(i-1,i)} \)’s. Here two types of resolution are possible. Now consider \( \Delta_{(i-1,i)} \) and \( \Delta_i \). Note that they have a common side, namely \( \beta^1_i \). Depending on the type of resolution for \( \Delta_{(i-1,i)} \), we will have to consider two cases.
Case I. $\Delta_{[i-1,i]}$ has type-2 resolution. Take any point $u \in \gamma_i^1|_{p_i^1}$, then there exists $w_u \in \beta_i^1$ such that $d_{\tilde{X}}(u, w_u) \leq B(\delta, 4, K, K)$. From the resolution of $\Delta_{[i-1,i]}$, there exists $w_u'$ in either $\gamma_{i-1}^1$ or $\gamma_{i-1}^2$ such that $d_{\tilde{X}}(w_u, w_u') \leq B(\delta, 4, K, K)$. Hence, $d_{\tilde{X}}(u, w_u') \leq 2B(\delta, 4, K, K)$.

Let $S_{[i-1,i]}$ be the finite tree and $r_{[i-1,i]} : D^2 \to S_{[i-1,i]}$ be the map as in Theorem 4.2 corresponding to the Type-2 resolution of $\Delta_{[i-1,i]}$. There exists three points $s_{i-1,1}^1 \in \gamma_{i-1,1}^1$, $s_{i-1,1}^2 \in \gamma_{i-1,1}^2$ and $s_i^1 \in \beta_i^1$ such that the set $\{s_{i-1,1}^1, s_{i-1,1}^2, s_i^1\}$ is mapped to a single vertex of $S_{[i-1,i]}$ under the map $r_{[i-1,i]}$ and diameter of the set $\{s_{i-1,1}^1, s_{i-1,1}^2, s_i^1\}$ is at most $B(\delta, 4, K, K)$. Note that the points $u_i^1 \in \gamma_i^1, u_i^2 \in \gamma_i^2$ and $w_{u_i^1} \in \beta_i^1$ are mapped to a single vertex of the finite tree $S_i$ corresponding to the resolution of $\Delta_i$. Now two sub-cases arise.

Sub-Case 1. Suppose $w_{u_i^1}$ comes before $s_i^1$ in $\beta_i^1$ (see Figure 3). In this case $d_{\tilde{X}}(u_i^1, w_{u_i^1}) \leq 2B(\delta, 4, K, K)$. The Hausdorff distance between $\gamma_i^1$ and $\gamma_i^1|_{p_i^1}$ is uniformly bounded by $L$. Hence, for each $u \in \gamma_i^1$, there exists $\bar{u} \in \gamma_i$ such that $d_{\tilde{X}}(\bar{u}, u) \leq L \cdot \text{Hence, there exists } \bar{u}_i^1 \in \gamma_i^1 \text{ such that } d_{\tilde{X}}(\bar{u}_i^1, u_i^1) \leq L. \text{ By triangle inequality, } d_{\tilde{X}}(\bar{u}_i^1, w_{u_i^1}) \leq 2B(\delta, 4, K, K) + L.$

The path $\gamma_i$ is a $(3,0)$-quasi-geodesic. So, the length of the subsegment of $\gamma_i$ between $\bar{u}_i^1$ and $w_{u_i^1}$ is at most $3 \times (2B(\delta, 4, K, K) + L) = 6B(\delta, 4, K, K) + 3L$. The quasi-geodesic $\gamma_i$ passes through $p_i^1$, hence $d_{\tilde{X}}(u_i^1, p_i^1) \leq d_{\tilde{X}}(u_i^1, \bar{u}_i^1) + d_{\tilde{X}}(\bar{u}_i^1, p_i^1) \leq L + 6B(\delta, 4, K, K) + 3L = 6B(\delta, 4, K, K) + 4L$. As vertex spaces are properly embedded in $\tilde{X}$, there exists a positive number $L'$ depending on $K$ such that $d_{\tilde{X}_{\gamma_i}}(u_i^1, p_i^1) \leq L'$. The truncated path $\gamma_i^1|_{p_i^1}$...
between \( u_1 \) and \( p_1 \) is a \((K, K)\)-quasi-geodesic in \( \tilde{X}_{uv} \). Hence \( \ell(\gamma_{i}^{1}\,|\,u_1^{1}\,|\,p_1^{1}) \leq KL' + K \) and we have \( d_{\tilde{X}}(u, u_1^{1}) \leq KL' + K \), as \( u \in \gamma_{i}^{1}\,|\,u_1^{1}\,|\,p_1^{1} \). So, \( d_{\tilde{X}}(u, u_2^{1}) \leq KL' + K + 2B(\delta, 4, K, K) \) and \( d_{\tilde{X}}(u, \gamma_2) \leq KL' + K + 2B(\delta, 4, K, K) + L \).

**Sub-Case 2.** Suppose \( w_{u_1}^{1} \) comes after \( s_1^{1} \) in \( \beta_1^{1} \) (see Figure 4). There exists \( x_i \in \gamma_{i}^{1}\,|\,u_1^{1}\,|\,p_1^{1} \) such that \( s_1^{1} = w_{x_i} \). We take \( x_i \in \gamma_{i}^{1}\,|\,u_1^{1}\,|\,p_1^{1} \) to be the last point such that \( s_1^{1} = w_{x_i} \). Now \( w_{x_i}^{1} \in \gamma_{i-1}^{1} \) and \( d_{\tilde{X}}(x_i, w_{x_i}^{1}) \leq 2B(\delta, 4, K, K) \). The Hausdorff distance between \( \gamma_{i}^{1} \) and \( \gamma_{i}^{1}\,|\,p_1^{1} \) is uniformly bounded by \( L \). Hence, for each \( u \in \gamma_{i}^{1} \), there exists \( \tilde{u} \in \gamma_1 \) such that \( d_{\tilde{X}}(\tilde{u}, u) \leq L \). Therefore, from the fact that \( \gamma_1 \) is a \((3, 0)\)-quasi geodesic in \( \tilde{X} \), we have

\[
\ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,\tilde{x}_i^{1}) \leq \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,w_{x_i}^{1}) \leq 3d_{\tilde{X}}(\tilde{x}_i, w_{x_i}^{1}) \leq 3(d_{\tilde{X}}(\tilde{x}_i, x_i) + d_{\tilde{X}}(x_i, w_{x_i}^{1})) \leq 3L + 6B(\delta, 4, K, K).
\]

Then,

\[
d_{\tilde{X}}(x_i, p_1^{1}) \leq d_{\tilde{X}}(x_i, \tilde{x}_i) + d_{\tilde{X}}(\tilde{x}_i, p_1^{1}) \\
\leq L + \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,\tilde{x}_i^{1}) \leq L + 6B(\delta, 4, K, K) + 3L. \quad \text{(from (*))}
\]

Since, vertex spaces are uniformly properly embedded in \( X \), hence there exists \( L' > 0 \) such that \( d_{\tilde{X}}(x_i, p_1^{1}) \leq L' \). Again, since \( \gamma_{i}^{1} \) is a \((K, K)\)-quasi geodesic in \( \tilde{X}_{uv} \), hence

\[
\ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,\tilde{x}_i^{1}) = \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,w_{x_i}^{1}) \leq KL' + K.
\]

If \( u \in \gamma_{i}^{1}\,|\,u_1^{1}\,|\,x_i^{1} \), then \( w_{u}^{2} \in \gamma_{i-1}^{2} \subset \gamma_2 \) and \( d_{\tilde{X}}(u, w_{u}^{2}) \leq 2B(\delta, 4, K, K) \).

Let \( u \in \gamma_{i}^{1}\,|\,p_1^{1} \). Now,

\[
d_{\tilde{X}}(p_1^{1}, w_{x_i}^{1}) \leq \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,p_1^{1}) \leq \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,w_{x_i}^{1}) \leq 6B(\delta, 4, K, K) + 3LM \quad \text{(from (*)).}
\]

There exists \( w_{x_i}^{2} \in \gamma_{i-1}^{2} \subset \gamma_2 \) such that \( d_{\tilde{X}}(w_{x_i}^{2}, w_{x_i}^{1}) \leq B(\delta, 4, K, K) \). Therefore,

\[
d_{\tilde{X}}(u, \gamma_2) \leq d_{\tilde{X}}(u, w_{x_i}^{1}) \leq d_{\tilde{X}}(u, p_1^{1}) + d_{\tilde{X}}(p_1^{1}, w_{x_i}^{1}) + d_{\tilde{X}}(w_{x_i}^{1}, w_{x_i}^{2}) \leq \ell_{\tilde{X}}(\gamma_{i}^{1}\,|\,p_1^{1}) + 6B(\delta, 4, K, K) + 3L + B(\delta, 4, K, K) \leq KL' + K + 7B(\delta, 4, K, K) + 3L.
\]

**Case II.** \( \Delta[i-1,i] \) has Type-1 resolution. Take any point \( u \in \gamma_{i}^{1}\,|\,p_1^{1} \) then there exists \( w_u \in \beta_{i}^{1} \) such that \( d_{\tilde{X}}(u, w_u) \leq B(\delta, 4, K, K) \). From the resolution of \( \Delta[i-1,i] \), there will exists \( w_u' \) in either \( \gamma_{i-1}^{1} \) or \( \gamma_{i-1}^{2} \) or \( \beta_{i-1}^{2} \) such that \( d_{\tilde{X}}(w_u', w_u') \leq B(\delta, 4, K, K) \).

**Sub-Case 1.** Suppose \( w_u' \in \beta_{i-1}^{2} \); \( \beta_{i-1}^{2} \) is the common side of disks \( \Delta[i-1,i] \) and \( \Delta_{i-1} \). We consider the resolution of \( \Delta_{i-1} \). There will exists \( w_u'' \) in either \( \gamma_{i-1}^{1} \) or \( \gamma_{i-1}^{2} \) such that \( d_{\tilde{X}}(w_u', w_u'') \leq B(\delta, 4, K, K) \) and so by triangle inequality, we have \( d_{\tilde{X}}(u, w_u'') \leq \)
Figure 4. Combined resolution for Sub-Case 2 in Case I.

Figure 5. Combined resolution for Sub-Case 1 in Case II.

3B(δ, 4, K, K). If \( w''_u \in \gamma^2_{i-1} \) then we are done. Suppose \( w''_u \in \gamma^1_{i-1} \). There exists \( \bar{u} \in \gamma_1 \) and \( \bar{w}''_u \in \gamma_1 \) such that \( d_\tilde{X}(u, \bar{u}) \leq L \) and \( d_\tilde{X}(w''_u, \bar{w}''_u) \leq L \). The length of the subsegment of \((3, 0)\)-quasi-geodesic \( \gamma_1 \) between \( \bar{u} \) and \( \bar{w}''_u \) is at most \( 3 \times (3B(\delta, 4, K, K)+2L) \). The quasi-geodesic \( \gamma_1 \) passes through \( p^1_{i-1} \) and \( q^1_{i-1} \). Following the process as in Case I, we get a number \( L_1 \) in terms of \( B(\delta, 4, K, K) \) and \( L \) such that \( \ell(\gamma^1_i|_{p^1_{i-1}}) + \ell(\gamma^1_{i-1,i}) + \ell(\gamma^1_{i-1}|_{q^1_{i-1}}) \leq L_1 \).

If we take \( w''_u \) to be the point on singular fiber, then \( d_\tilde{X}(w''_u, \gamma^2_{i-1}) \leq B(\delta, 4, K, K) \). In general, any path \( \gamma^1_i|_{p^1_{i-1}} * \gamma^1_{i-1,i} * \gamma^1_{i-1}|_{q^1_{i-1}} \) is contained in a path of length at most \( L_1 \) and whose one end correspond to a singular fiber in the disk \( \Delta_{i-1} \). Hence, \( d_\tilde{X}(u, \gamma^2_{i-1}) \leq L_1 + B(\delta, 4, K, K) + d_\tilde{X}(u, \gamma_2) \leq L_1 + B(\delta, 4, K, K) + L \) (see Figure 5).

Sub-Case 2. \( u' \in \gamma^1_{i-1,i} \) or \( \gamma^2_{i-1,i} \). In this case, we will follow the exact same process as in Case I to find a point in \( \gamma^2_{i-1,i} \subset \gamma_2 \) which is at uniformly bounded distance from \( u \).

If \( u \in \gamma^1_{i-1,i} \subset \gamma_1 \), then as above, there exists a point \( w''_u \) in \( \gamma^2_{i-1,i} \cup \gamma^2_{i-1,i} \cup \gamma^2_i \), and hence a point in \( \gamma_2 \), which is at uniformly bounded distance from \( u \). For the initial and terminal disks, take the left (respectively right) end of Figures 3, 4 and 5 to be the point \( x \) (respectively \( y \)) and repeat the same argument as above. So, we get that each point of \( \gamma_1 \) lies in a bounded distance of \( \gamma_2 \) where the bound depends only on \( \delta \) and \( \sigma \). We can reverse the roles of \( \gamma_1 \) and \( \gamma_2 \) to conclude that they lie in a bounded neighbourhood of each other where the bound depends only on \( \delta \) and \( \sigma \). This implies \( \tilde{X} \) is hyperbolic. As vertex spaces are \( \delta \)-hyperbolic geodesic spaces, the geodesics in a vertex space are uniformly strongly contracting and hence by Corollary 3.3, vertex spaces are quasiconvex in \( \tilde{X} \).
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References

[1] Arzhantseva G N, Cashen C H, Gruber D and Hume D, Characterizations of Morse geodesics via superlinear divergence and sublinear contraction, Doc. Math. 22 (2017) 1193–1224
[2] Bestvina M and Feighn M, A combination theorem for negatively curved groups, J. Differ. Geom. 35 (1992) 85–101
[3] Bridson M and Haefliger A, Metric Spaces of Non-Positive Curvature, Vol. 319 (1999) (Springer)
[4] Charney R and Sultan H, Contracting boundaries of CAT(0) spaces, J. Topol. 8(1) (2015) 93–117
[5] Kapovich I, The combination theorem and quasiconvexity, Int. J. Algebra Comput. 11 (2001) 185
[6] Szczepanski A, Relatively hyperbolic groups, Mich. Math. J. 45 (1998) 611–618
[7] Sisto A, Quasi-convexity of hyperbolically embedded subgroups, Math. Z. 283 (2016) 649–658

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