Low-Dimensional Life of Critical Anderson Electron

Ivan Horváth\textsuperscript{1,2,\dagger} and Peter Markoš\textsuperscript{3,\dagger}

\textsuperscript{1}Nuclear Physics Institute CAS, 25068 Újezd (Prague), Czech Republic
\textsuperscript{2}University of Kentucky, Lexington, KY 40506, USA
\textsuperscript{3}Dept. of Experimental Physics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská Dolina 2, 842 28 Bratislava, Slovakia

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We show that critical Anderson electron in 3 dimensions is present in its spatial effective support, which was recently determined to be a region of fractal dimension $\approx 8/3$, with probability 1 in infinite volume. Hence, its physics is fully confined to space of this lower dimension. Stated differently, effective description of space occupied by critical Anderson electron becomes a full description in infinite volume. We then show that it is a general feature of the effective counting dimension underlying these concepts, that its subnominal value implies an exact description by effective support.

Keywords: Anderson transition, localization, criticality, effective counting dimension, effective number theory, effective support, effective description

Disorder-induced localization of electrons, namely the Anderson transition \cite{1}, attracts the attention of theoretical physicists for over six decades. Its scaling theory \cite{2} is based on the system size and the disorder dependence of electron conductance \cite{3–5}, and enables a quantitative description of the phenomenon \cite{6,7}, including the calculation of critical exponents \cite{8}. However, the absence of self-averaging in the localized state \cite{9} suggests that calculation of critical exponents \cite{8} requires information on statistical properties of key quantities, such as conductance \cite{10,11}, and of electronic eigenstates themselves \cite{12–14}. Here we study critical wave functions that have been of interest, among other things, due to their intriguing geometric structure \cite{15–20}. Being more complex than that of a scale invariant fractal, this structure is commonly termed as multifractal. Its chief manifestation is the presence of non-trivial dimensional features.

Recently, a new type of dimension based on effective counting \cite{21–23} has been proposed \cite{24,25}. This effective counting dimension is analogous to Minkowski box-counting dimension of fixed sets \cite{26}. To explain the connection in the present setting, consider first the nominal dimension $D_{\text{IR}}$ of spatial lattice $L$, which conveys the increase of lattice volume $N$ (number of lattice sites) with linear size $L$, namely $N[L(L)] \propto L^{D_{\text{IR}}}$ for $L \to \infty$. Hence, $D_{\text{IR}}$ is based on the same scaling as the infrared version of Minkowski dimension in a special case when all existing “boxes” are counted. Next, consider a subset $S(L) \subset L(L)$ containing $N[S(L)]$ points selected by some rule at each $L$. The Minkowski dimension $d_{\text{IR}}$ of the infrared target $S$ defined by $S(L)$ then expresses the asymptotic growth of $N$ with $L$, namely

$$N[S(L)] \propto L^{d_{\text{IR}}[S]} \quad \text{for} \quad L \to \infty \quad (1)$$

In the relations above, $N = N[\ldots]$ is treated as a function on sets, namely as the ordinary counting measure.

However, if we wish to characterize wave functions describing quantum particles by such spatial dimension, we encounter a conceptual problem. Indeed, wave function informs on probabilities for a particle to reside in any region of space rather than on a unique subregion $S$ associated with the particle. Is it then even meaningful to talk about a volume-based characteristic analogous to Minkowski dimension?

Surprising answer to this question is yes \cite{25}. The schematic logic leading to the resulting prescription is as follows. Consider a probability vector $P = (p_1, p_2, \ldots, p_N)$ where $p_i = \psi^+ \psi(r_i)$ is the probability of the particle in state $\psi$ to be at lattice site $r_i$. Assume further that we can count lattice points $r_i$ effectively, based on their relevance $p_i$. In other words, that we have a function $N = N[P]$ at our disposal, returning effective count of entries in $P$. This could be used to define the effective spatial support $S[P]$ of a particle as a collection of $N[P]$ points $r_i$ with highest probabilities. If the effective number function $N$ is additive, then $S[P]$ respects relevant measure considerations \cite{25}. For IR-regularized wave functions $\psi(L)$, the scaling of $N[P(L)]$ with $L$ would then represent the IR dimension $d_{\text{IR}}$ of the target $\psi$ in this probabilistic setting. The key result of \cite{25} is that all schemes $N$ producing valid supports $S[P]$ lead to the same $d_{\text{IR}}$, which uniquely extends Minkowski dimension into the probabilistic domain. Value of $d_{\text{IR}}$ can be extracted using the intrinsic (minimal) effective scheme $N_\star$, namely \cite{21}

$$N_\star[P] = \sum_{i=1}^N n_\star(N p_i) \quad , \quad n_\star(c) = \min\{c,1\} \quad (2)$$

More explicitly, the following analog of (1)

$$N_\star[P(L)] \propto L^{d_{\text{IR}}[\psi]} \quad \text{for} \quad L \to \infty \quad (3)$$

defines $d_{\text{IR}}$ of the target wave function $\psi$.

We emphasize that the concept of $d_{\text{IR}}[\psi]$ is recent \cite{21–25} and has so far been applied to Dirac modes in quantum chromodynamics \cite{24} and states at 3d Anderson transitions \cite{27}. Despite the extensive literature on the
spatial structure in the latter context (e.g. [12–16, 18–20, 28–30]), information conveyed by \( d_{\text{ir}} \) is different and complementary to existing results due to additivity of effective counting.

Recent calculation of \( d_{\text{ir}} \) for critical wave functions at 3d Anderson transitions produced unexpected results [27]. To explain them, consider the 3d Anderson model [1] on cubic lattice (\( D_{\text{ir}} = 3 \)) with periodic boundary conditions, and the Hamiltonian (orthogonal class)

\[
\mathcal{H} = \sum_r \epsilon_r c_r^d c_r + \sum_{r,j} c_r^d c_{r-e_j} + \text{h.c.} \tag{4}
\]

Here \( r = (x_1, x_2, x_3) \) labels lattice sites, \( \epsilon_j \) (\( j = 1, 2, 3 \)) unit lattice vectors, \( \epsilon_r \in [-W/2, +W/2] \) uniformly distributed random potentials, and \( c_r \) the 1-component electron operators. Definition of \( d_{\text{ir}} = d_{\text{ir}}(E, W) \) involves averaging over states in the vicinity of energy \( E \) and over disorder \( \{\epsilon_r\} \), thus entailing \( N_\star \rightarrow (N_\star) \) in Eq. (3). At the critical point \( E = 0, W = W_c = 16.543(2) \) [31], the calculation in Ref. [27] yielded the dimension \( d_{\text{ir}} \approx 8/3 \). In fact, the computed values in orthogonal, unitary, symplectic and chiral (AIII) classes all turned out to be well within two parts per mill of \( 8/3 \), while being pairwise equal within errors. These findings led to the proposition that the critical exponent \( d_{\text{ir}} \) is super-universal in 3d Anderson transitions [27].

Here we inquire about the probability \( P \) that 3d critical Anderson electron is present in its spatial support, which conveys the extent to which its physics tied to this low-dimensional (\( d_{\text{ir}} < D_{\text{ir}} \)) space. Formal definition of \( P \) relies on that of effective support [25]. The latter starts by ordering lattice sites in \( \mathcal{L} = \{r_1, r_2, \ldots, r_N\} \) by their relevance, so that \( p_1 \geq p_2 \geq \ldots \geq p_N \) in \( P \). To realize supports involving both integer and non-integer number of sites (\( N \) is real-valued), we represent them by generalized collections \( \{r_1, \ldots, r_J \} : f \), where \( 0 < f \leq 1 \) specifies the fraction of \( r_J \) included. The effective support of \( P \) (equivalently of \( \psi \)) on \( \mathcal{L} \) in counting scheme \( N \) is \(^1\)

\[
S[P, N] = \{ r_1, \ldots, r_J : N[P] + 1 - J \} \tag{5}
\]

where \( J = \text{ceil}(N[P]) \), and \( \text{ceil} \) is the ceiling function. Note that \( J \leq N \). The probability associated with \( S \) is

\[
P[P, N] = \sum_{j=1}^{J-1} p_j + f p_J \tag{6}
\]

where \( J = J[P, N] \) and \( f = f[P, N] \) are specified by Eq. (5).

We now determine this probability for critical Anderson electron (Hamiltonian (4)) in the intrinsic scheme \( N_\star \),

\(^1\) While the distinction between full and fractional inclusion of \( r_J \) doesn’t matter for most asymptotic (\( N \rightarrow \infty \)) considerations, it affects the accuracy of extrapolations when \( d_{\text{ir}} \) is much smaller than \( D_{\text{ir}} \). Fractional approach keeps additivity relations exact [25].
that the counting scheme $N_*$ is minimal [21], the associated effective support $S_*$ is also minimal [25], together with its $P_*$. More precisely, we have

$$P_*[P] = P[P, N_*] \leq P[P, N]$$

(8)

for all distributions $P$ on a finite lattice, and all $N$ consistently defining effective support [25]. Hence, $P_* = 1$ implies $P = 1$ for all valid $N$. In other words, similarly to $d_{ir} \approx 8/3$, the property $P = 1$ is well-defined.

We now show that $P_* = 1$ follows from $d_{ir} < D_{ir}$. In fact, the corresponding feature is built into the effective dimension already at the pre-metric level: aspects involving distance, such as infrared and ultraviolet, are not crucial to it. We thus give the argument in the most general setting where collections $O = \{o_1, o_2, \ldots, o_N\}$ contain arbitrary objects ordered by $P$, with no additional structure assumed. The notions of effective support $S[P, N]$ and its probability $P[P, N]$ (Eqs. (5) and (6)) generalize trivially, with all observations made so far still holding. Effective counting dimension in this setting expresses the scaling of $N_*$ with $N$ (see [25] and below).

Our reasoning relies on the relationship between $N_*$ and $P_*$ that follows directly from the defining relations (2), (5) and (6). In particular, splitting $N_*$ into a contribution $N_*$ from objects in support $S_*$ and the rest yields

$$N_*[P] = N_*[P]^* + N(1 - P_*[P])$$

(9)

for all $P = (p_1, p_2, \ldots, p_N)$. Indeed, note first that $S_*$ contributes $N_*[P]^* = \sum_{i=1}^{N} n_*(Np_i) + f n_*(Np)$ and the rest is $(1 - f) n_*(Np) + \sum_{i=J+1}^{N} n_*(Np_i)$. Since $n_*(Np) = Np$ for $p \leq 1/N$ and $p_i \leq 1/N$ for $i = J, J + 1, \ldots, N$, we obtain the formula (9).

Effective counting dimension $\Delta$ is assigned to a sequence $O_k$ of collections with increasing number $N_k$ of objects and probability vectors $P_k = (p_{k,1}, p_{k,2}, \ldots, p_{k,N_k})$. The pair $O_k, P_k$ “regularizes” the $k \to \infty$ target defined by the sequence. Dimension $\Delta$ of the target is specified by [25]

$$N_*[P_k] \propto N_k^\Delta \quad \text{for} \quad k \to \infty$$

(10)

thus taking values in the range $0 \leq \Delta \leq 1$. The subdimensional case $\Delta < 1$ implies $P_* = 1$. To see that, write Eq. (9) for $P_k$ in the form

$$\frac{N_*[P_k] - N_*[P_k]}{N_k} = 1 - P_*[P_k]$$

(11)

If $\Delta < 1$ then lhs approaches zero at least as fast as $(1/N_k)^{1-\Delta}$ for $k \to \infty$. Hence, lim$_{k \to \infty} P_*[P_k] = P_* = 1$.

When metric is in place and regularization removal $k \to \infty$ corresponds to infinite-volume (IR) limit $L_k \to \infty$, then $d_{ir}$ of Eq. (3) and $\Delta = \Delta_{ir}$ of Eq. (10) are related by $d_{ir} = \Delta_{ir} D_{ir}$. Thus, $\Delta_{ir} < 1$ is equivalent to $d_{ir} < D_{ir}$. Hence, the latter implies $P_* = 1$ for critical Anderson electrons, as claimed.

We close with a few comments.

(a) We showed that, in the subdimensional case, the minimal effective support $S_*$ fully describes collection $O$ of $N$ objects when $N \to \infty$. Consequently, $S$ based on counting schemes other than $N_*$, and thus entail more voluminous descriptions, involve a redundancy: any excess of probability in $S$ relative to $S_*$ goes to zero in $N \to \infty$ limit when $\Delta < 1$. This further underlines the unique standing of $N_*$ and $S_*$ in effective quantitative analyses.

(b) We focused on the Anderson system with $S_*$ describing the physical space occupied by the electron. The critical case is subdimensional ($d_{ir} \approx 8/3$ [27]) and the above property implies that electron is fully confined to the corresponding subvolume in $L \to \infty$ limit. This is confirmed by our detailed numerical analysis. Since simulations of Anderson systems provide the full access to $S_*$, all aspects of critical spatial geometry and their relationship to the underlying physics can in fact be investigated.

(c) The value $d_{ir} \approx 8/3$ may be connected to anomalous scaling of critical diffusion length $\ell(t) \propto t^{1/3}$ [33]. (Possibly also to exponent 4/3 characterizing percolation clusters [34] (see also [35]).) This is plausible since anomalous diffusion is likely due to electron propagation being effectively restricted to wave function support on a sample. That, in turn, qualitatively relates to the associated diffusion properties. Such connection is also corroborated by the observed superuniversality of $d_{ir}$ [27] since the diffusion exponent is also expected to be superuniversal. The present work fortifies this possible connection. Indeed, not only is $P_* = 1$ super-universal in light of the general argument given here, we also expect it to be a necessary ingredient in any microscopic derivation of the anomalous diffusion exponent.
(d) Electron states at Anderson transitions are frequently characterized via the participation number of Bell and Dean, \( N_p[P] = 1/\sum_{i=1}^{N} p_i^2 \) [36], and the associated dimension \( d_p \) extracted from \( N_p \propto L^{d_p} \). Applying this to our data we obtained \( d_p = 1.305(2) \). Although not an effective counting scheme, \( N_p \) satisfies all required axioms except additivity [21]. Its comparison to \( N_c \) thus tests the importance of preserving the measure property. Proceeding as in the case of effective counting schemes, we define the probability \( P_p \) within the “support” \( S_p \) based on \( N_p \). These probabilities exhibit larger fluctuations at criticality than \( P_s \) and are shown in Fig. 2. In a striking difference to the additive case, \( \langle P_p [P(L)] \rangle \) decreases with increasing \( L \) and may converge to zero in \( L \to \infty \) limit. Hence, the relevance of \( S_p \) as a descriptor of space in which the physics of Anderson critical electron takes place is rather limited.

(e) Multifractal formalism (see e.g. [26, 37]) is also frequently used in analyses of Anderson criticality. Conceived in the context of turbulence and strange attractors, it describes local singularities of associated complex measures in a statistical manner. The original UV form of multifractality has been adopted for Anderson criticality [17–20] via replacement \( a \to \lambda = 1/L \) (or \( \ell/L \)) with coarse-grained scale \( \ell \), and scaling for \( \lambda \to 0 \). In physics terms, release of UV cutoff \( (a \to 0) \) is replaced by that of IR cutoff \( (L \to \infty) \). Very recently, Ref. [38] proposed the formula for average \( N_p \) in terms of multifractal parameters obtained by the moment method. Its reliability and impact has been questioned in Ref. [39].

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* ihorv2@uky.edu
† peter.markos@fmph.uniba.sk

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