On Time-Reversal Imaging by Statistical Testing

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Abstract—This letter is focused on the design and analysis of computational wideband time-reversal imaging algorithms, designed to be adaptive with respect to the noise levels pertaining to the frequencies being employed for scene probing. These algorithms are based on the concept of cell-by-cell processing and are obtained as theoretically-founded decision statistics for testing the hypothesis of single-scatterer presence (absence) at a specific location. These statistics are also validated in comparison with the maximal invariant statistic for the proposed problem.

Index Terms—Composite hypothesis testing, computational time-reversal, imaging functions, invariant detection.

I. PROBLEM FORMULATION AND RELATED LITERATURE

Time-reversal (TR) techniques exploit the invariance of wave equation (in lossless and stationary media) to provide focusing on a scattering object (or radiating source). This is achieved by re-transmitting a time-reversal version of the scattered (or radiated) field collected over an array and can be achieved either physically [1] or synthetically [2]. In the latter case (i.e., computational TR, C-TR) the time-reversing procedure consists in back-propagating numerically the field data by using the Green’s function of the medium in which the propagation takes place.

Accordingly, C-TR provides a powerful tool to achieve scatterer detection and localization and represents the building rationale for many imaging procedures in different application contexts, such as radar imaging [3], subsurface prospecting [4], through-the-wall imaging [5], and breast cancer detection [6]. For this reason, theoretical limits on localization accuracy of multiple-scatterers by means of C-TR, based on the Cramér–Rao lower bound, were obtained in [7] both for (linear) Born and FL models, as there is no mutual interaction effects among scatterers.

Key entity in TR-imaging is the so-called multistatic data matrix (MDM), whose entries consist of the scattered field due to each Tx-Rx pair for each probed frequency. Specifically, in this letter, we consider C-TR based localization in a multifrequency (with L frequencies) multistatic setup. We assume that M point-like scatterers are located at unknown positions \( \{ \mathbf{r}_m \}_{m=1}^M \) in \( \mathbb{R}^p \) with unknown scattering potentials \( \{ \tau_{m, i} \}_{i=1}^{M} \) at \( i \)-th angular frequency (denoted with \( \omega_i \)) in \( \mathbb{C} \). The Tx (resp. Rx) array consists of \( N_T \) (resp. \( N_R \)) isotropic point elements (resp. receivers) located at \( \mathbf{r}_i \in \mathbb{R}^p, i \in \{1, \ldots, N_T \} \) (resp. \( \mathbf{r}_j \in \mathbb{R}^p, j \in \{1, \ldots, N_R \} \)). The illuminators first send signals to the probed scenario (in a known homogeneous background) and the transducer array records the received signals.

Thus, the model at \( \omega_i \) is [8]

\[
X_{\ell} = A_{R,\ell}(\mathbf{r}_{1:M}) M_{\ell}(\mathbf{r}_{1:M}, \tau_{\ell}) A_{T,\ell}^T(\mathbf{r}_{1:M}) + W_{\ell}
\]

(1)

where \( X_{\ell} \in \mathbb{C}^{N_T \times N_T} \) denotes the measured MDM at \( \omega_i \) and \( W_{\ell} \in \mathbb{C}^{N_T \times N_T} \) is a noise matrix such that \( \mathbf{w}_\ell \triangleq \text{vec}(W_{\ell}) \sim \mathcal{C}\mathcal{N}(0_N, \sigma^2_w I_N) \), where \( N \triangleq N_T N_R \). The matrices \( W_{\ell}, \ell = 1, \ldots, L \) are also assumed to be mutually uncorrelated. Here, \( \sigma^2_w \) is assumed known (resp. unknown) in the nonadaptive (resp. adaptive) case. Also, we have denoted: (a) the vector of scattering coefficients at \( \omega_i \) as \( \tau_{\ell} \triangleq [\tau_{1,\ell}, \ldots, \tau_{M,\ell}]^T \in \mathbb{C}^M \), (b) the Tx (resp. Rx) array matrix at \( \omega_i \) as \( A_{R,\ell}(\mathbf{r}_{1:M}) \in \mathbb{C}^{N_T \times M} \) (resp. \( A_{T,\ell}(\mathbf{r}_{1:M}) \in \mathbb{C}^{N_T \times M} \)), whose \( i, j \)-th entry equals \( G_{\ell}(\mathbf{r}_i, \mathbf{r}_j) \) (resp. \( G_{\ell}(\mathbf{r}_i, \mathbf{r}_j) \)), where \( G_{\ell}(\cdot, \cdot) \) denotes the relevant (scalar) background Green function at \( \omega_i \) [9]. Finally, in (1) the scattering matrix is defined as \( M_{\ell}(\mathbf{r}_{1:M}, \tau_{\ell}) \triangleq \text{diag}(\tau_{\ell}) \) for BA model [10], whereas for FL model [7] \( M_{\ell}(\mathbf{r}_{1:M}, \tau_{\ell}) \triangleq \text{diag}^{-1}(\tau_{\ell}) - S_r(\mathbf{r}_{1:M})^{-1} \) holds, where the \((m,n)\)-th entry of \( S_r(\mathbf{r}_{1:M}) \) equals \( G_r(\mathbf{r}_m, \mathbf{r}_n) \) when \( m \neq n \) and zero otherwise.

Two popular methods for TR-imaging are the decomposition of time reversal operator (DORT) [10] and the TR multiple signal classification (TR-MUSIC) [8], [9], [11]. DORT method exploits the MDM spectrum so that its imaging is obtained by back-propagating each single eigenvector belonging to the signal subspace; this allows to selectively focus on each single scatterer if they are well-resolved by the array. Differently, TR-MUSIC offers a dual viewpoint with respect to (w.r.t.) DORT, and the orthogonal (viz. noise) subspace is employed for imaging purposes. Both methods, unfortunately, require knowledge of the number of scatterers \( M \) in the scene, which is typically obtained via model-order selection techniques [12].

Alternatively, suboptimal (simpler) imaging functions can be designed based on a single scatterer model,\(^1\) being capable of providing an image also in the case of multiple scatterers in the scene, while not requiring their exact number \( M \) (i.e., circumventing the model-order selection issue) [13]–[15]. For this reason, in what follows, we focus at design stage on a measured MDM (at \( \omega_i \)) in the form \( X_{\ell} = A_{R,\ell}(\mathbf{r}_1) \tau_{\ell} A_{T,\ell}^T(\mathbf{r}_1) + W_{\ell} \) (i.e., the Tx and Rx array matrices collapse to column vectors) which, after \( \text{vec}(\cdot) \), can be rewritten as

\[
x_{\ell} = a_{T,\ell}(\mathbf{r}_1) \otimes a_{R,\ell}(\mathbf{r}_1) \tau_{1,\ell} + w_{\ell} = b_{\ell}(\mathbf{r}_1) \tau_{1,\ell} + w_{\ell}
\]

(2)

where \( x_{\ell} \triangleq \text{vec}(X_{\ell}) \in \mathbb{C}^N \) and we have adopted the (short-hand) notation \( b_{\ell}(\mathbf{r}_1) \triangleq a_{T,\ell}(\mathbf{r}_1) \otimes a_{R,\ell}(\mathbf{r}_1) \) (resp. \( b_{\ell} \)).

\(^1\)It is worth noticing that the single-target model is coincident for both BA and FL models, as there is no mutual interaction effects among scatterers.
Based on this model, a common approach for single ($t$th) frequency imaging is the so-called TR matched filter (MF) [13], formulated as

$$ I_t(r, \ell) = \left| a_{R, \ell}(r) X \cdot a_{T, \ell}^*(r) \right|^2 $$

(3)

where $r$ generically denotes the (single) scatterer probed location. Successively, in [15] Shi and Nehorai proposed a modification of the above imaging algorithm, based on a likelihood-maximization inspired argument

$$ I_{ml}(r, \ell) = \left| a_{R, \ell}(r) X \cdot a_{T, \ell}^*(r) \right|^{2/\left(\|a_{R, \ell}(r)\|^2 \|a_{T, \ell}(r)\|^4\right)} \cdot $$

(4)

Indeed, the above imaging function can be interpreted as $I_{ml}(r, \ell) = |\hat{\tau}_r(r)|^2$, where $\hat{\tau}_r(r)$ is the maximum likelihood (ML) estimate of $\tau_r$ (assuming $r$ known) for the model in (2). Similarly, in [14] a multifrequency (wideband) imaging algorithm, based on true concentrated (w.r.t. $\tau_r$’s) likelihood-maximization (termed “likelihood imaging”) was proposed

$$ I_{li}(r) = \prod_{\ell=1}^{L} \frac{1}{\|P_{b_{i}(r)} x_{\ell}\|^2} \cdot $$

(5)

In this letter, we start from the same rationale as the aforementioned works, by considering a single-source model. However, we depart from the aforementioned approaches by constructing imaging functions based on decision statistics which test the presence of a single scatterer located at $r$. Precisely, the aforementioned statistics originate from theoretically-founded approaches for the following (composite) hypothesis testing:

$$ \begin{align*}
H_0 : x_{\ell} &= w_{\ell}, \quad \ell = 1, \ldots, L \\
H_1 : x_{\ell} &= b_{i}(r) \tau_r + w_{\ell}, \quad \ell = 1, \ldots, L.
\end{align*} $$

(6)

By denoting a generic statistic with $t(x_{1:L}, r)$, the corresponding image is then formed by varying $r$. It is worth remarking that the proposed approach is widely used in radar-related applications, and it naturally arises from a cell-by-cell processing rationale [16]. One of the main contributions of this letter is the design of imaging functions based on the well-known generalized likelihood ratio (GLR), Rao, and Wald statistics [17] for the C-TR imaging problem. The scope of the present study also includes their statistical characterization. Both adaptive ($\tau_r$ is unknown, $\sigma_r^2$ is a nuisance parameter) and nonadaptive ($\tau_r$ is unknown, $\sigma_r^2$ is known) cases are analyzed (as opposed to [13]–[15]), in order to draw interesting comparisons with the imaging functions reported in (3)–(5), respectively. In the adaptive case, also the constant false-alarm rate (CFAR) behavior of the building decision statistics is thoroughly investigated, by means of statistical invariance tools [18]–[20].

The rest of the paper is organized as follows: Section II tackles imaging task as a composite hypothesis test and theoretically-founded decision statistics are proposed; Section III analyzes their CFAR behavior via invariance theory, while in Section IV, a theoretical performance analysis of the corresponding imaging functions is provided. Finally, conclusions are in Section V.

II. NONADAPTIVE VERSUS ADAPTIVE DECISION STATISTICS

In this section, we develop decision statistics based on theoretically-founded criteria, which will be used as the basis for the development of corresponding imaging functions, focusing on the specific instances of the GLR, Rao, and Wald statistics [17] for the problem investigated. With this intent, we define, for notational compactness: $\theta_{s, \ell} \triangleq [\Re\{\tau_r\} \Im\{\tau_r\}]^T$ (unknown signal parameters at $\omega_r$); $\theta_{s, \ell} \triangleq \sigma_r^2$ (nuisance parameter at $\omega_r$); $\theta_{l} \triangleq [\theta_{l,1}^T, \theta_{l,2}^T] \sigma_r^2$ (unknown parameters set at $\omega_r$); $\theta_{r} \triangleq [\theta_{r,1}^T, \cdots, \theta_{r,L}^T] \sigma_r^2$ (overall set of unknown signal parameters); $\theta_{s} \triangleq [\theta_{s,1}^T, \cdots, \theta_{s,L}^T] \sigma_r^2$ (overall set of nuisance parameters); $\theta \triangleq [\theta^T \sigma_r^2] \sigma_r^2$ (overall set of unknown parameters). Based on these definitions, in the nonadaptive case $\theta_s = \theta_{s, \ell} = \{0\}$ and $\theta = \theta_s$ hold, respectively.

Additionally, the pdf of the $L$ MDMs is expanded as

$$ f_1(x_{\ell}; \theta_{s, \ell}) = \prod_{\ell=1}^{L} f_1(x_{\ell}; \theta_{s, \ell}) $$

(7)

due to independence of noise matrices $W_{\ell}$ among frequencies, where

$$ f_1(x_{\ell}; \theta_{s, \ell}) = (\pi \sigma_r^2)^{-N} \exp \left(-\|x_{\ell} - b_{i}(r) \tau_r \|^2 / \sigma_r^2\right) $$

(7)

where the corresponding pdf under $H_0$ is $f_0(x_{1:L}; \theta_s) = \prod_{\ell=1}^{L} f_0(x_{\ell}; \theta_{s, \ell})$, with $f_0(x_{\ell}; \theta_{s, \ell})$ obtained by setting $\tau_r = 0$ in (7). We remark that the composite hypothesis testing tackled in what follows is based on assumption of a known target position $r$. Consequently, $a_{R, \ell}$ and $a_{T, \ell}$ (and so $b_{i}$) are assumed to be known. Also, due to MDMs independence among frequencies, the GLR, Rao, and Wald statistics assume the simplified expressions [17] shown in (8) at the bottom of next page, where $I_{GLR} (\theta) \triangleq E\left[\frac{\partial \ln f_1(x_{\ell}; \theta_{s, \ell})}{\partial \theta_{s, \ell}} \ln f_1(x_{\ell}; \theta_{s, \ell})\right]$ denotes the contribution of Fisher information matrix pertaining to $\omega_r$ and $I_{RAO} \triangleq [\theta_{s, \ell}]^{T} [\theta_{s, \ell}]^{T}$ denotes the submatrix obtained by selecting from its inverse only the elements corresponding to $\theta_{s, \ell}$.

Additionally, we have defined $\hat{\theta}_{l, \ell} \triangleq [\theta_{l,1, \ell}^T, \theta_{l,2, \ell}^T]^T$ (resp. $\hat{\theta}_s \triangleq [\theta_{s,1, \ell}^T, \cdots, \theta_{s,L, \ell}^T]^T$) and $\hat{\theta}_s \triangleq [\hat{\theta}_{0, r, \ell}, \hat{\theta}_{0, s, \ell}] \sigma_r^2$ (resp. $\hat{\theta}_{0, s, \ell}$) denoting the ML estimate of $\theta_{s, \ell}$ under $H_l$ and $\hat{\theta}_{0, s, \ell}$ denoting the ML estimate of $\theta_{s, \ell}$ under $H_0$ (resp. under $H_l$).

Based on this simplification and exploiting the key result provided in the literature (e.g., [17]) regarding the closed form of the $l$th term of each statistic in (8), we obtain the final statistic:

$$ t_{glr} = \sum_{\ell=1}^{L} \left( x_{\ell}^T P_{b_{i}} x_{\ell} \right) / \sigma_r^2 $$

(9)

in the nonadaptive case, while in the adaptive case it can be shown that

$$ t_{glr} = \frac{1}{\|x_{\ell} P_{b_{i}} x_{\ell}\|} \left( x_{\ell}^T P_{b_{i}} x_{\ell} \right) $$

(9)

where we have exploited the definitions

$$ \Xi_{\ell} \triangleq \left( x_{\ell}^T P_{b_{i}} x_{\ell} \right), \quad \ell = 1, \ldots, L. $$

(10)

The proposed imaging functions are then obtained by varying $r$. Some remarks are now in order. First, it is apparent that in the nonadaptive case the imaging function $t_{na}(r)$ is a weighted sum (via $1/\sigma_r^2$) of $L$ terms, each corresponding to a single frequency. Interestingly, for $L = 1$, $t_{na}(r)$ assumes an expression which is

Indeed, it can be shown that all the three tests are statistically equivalent to one based on $t_{glr}$.

It is worth noticing that other well-founded decision statistics could be considered as well for the present composite hypothesis testing. These include the Durbin and Terrell (Gradient) statistics, as recently employed in [21]. However, it can be readily shown (the proof is left to the reader for brevity) that in this peculiar case they are both statistically equivalent to Rao test.
comparable with the imaging functions in (3) and (4). Indeed, for the generic \( \omega_{\ell}, t_{\text{na}}(r) \) simplifies to

\[
t_{\text{na}}(r) = \frac{\left(x_{\ell} P_{b_\ell}(r) x_{\ell} \right)^{\text{T}}}{\sigma_t^2} \left( P_{T,\ell}(r) X_\ell P_{T,\ell}^{\text{T}}(r) \right)^{\text{T}} F \left( P_{T,\ell}(r) X_\ell P_{T,\ell}^{\text{T}}(r) \right)^{\text{T}}}
\]

(11)

\[
t_{\text{na}}(r) = \left| a_{R,\ell}^\dagger(r) X_\ell a_{T,\ell}^\dagger(r) \right|^2 / \left\{ \left( a_{R,\ell}^\dagger(r) \right)^2 \left( a_{T,\ell}(r) \right)^2 \right\}
\]

enforcing the use of \( \text{unit-norm} \) Tx-Rx Green vector functions \( a_{T,\ell}/\|a_{T,\ell}\| \) and \( a_{R,\ell}/\|a_{R,\ell}\| \), as opposed to (3) (where unnormalized counterparts are employed) and (4) (where \( a_{T,\ell}/\|a_{T,\ell}\|^2 \) and \( a_{R,\ell}/\|a_{R,\ell}\|^2 \) are used instead). Such choice agrees with the intuitive rationale discussed in [22].

Second, it is interesting to compare the likelihood imaging in (5) with \( t_{\text{glr}}(r) \) (adaptive case), which is rewritten as

\[
t_{\text{glr}}(r) = \prod_{\ell=1}^{L} \left( \|x_{\ell}\|^2 / \| P_{b_\ell}(r) x_{\ell} \|^2 \right)
\]

(12)

where the sole difference observed is the introduction of the numerator terms \( \|x_{\ell}\|^2, \ell = 1, \ldots, L \).

III. INVARIANCE AND CFARNESS IN ADAPTIVE CASE

When referring to the adaptive case, it is desirable to build imaging functions, which are insensitive to the unknown nuisance parameters \( \sigma_t^2, \ell = 1, \ldots, L \), when no scatterer is present in the scene. To this end, we resort to the principle of invariance [18]–[20] applied to the hypothesis testing problem in (6), representing an elegant way for obtaining decision statistics (resp. tests) which ensure invariance (resp. a CFAR). To this end, hereinafter we will search for data functions sharing invariance w.r.t. those parameters (namely, the nuisances \( \sigma_t^2, \ell = 1, \ldots, L \)) which are irrelevant for the decision problem considered. The preliminary step consists in finding transformations that properly cluster data without altering:

1) the structure of the hypothesis testing problem, i.e., \( \mathcal{H}_0 : |\tau_\ell| = 0, \mathcal{H}_1 : \exists |\tau_\ell| > 0 (\ell = 1, \ldots, L) \);

2) the Gaussian assumption for the measured MDMs under each hypothesis;

3) the scaled-identity structure of the noise covariance matrix \( \Sigma_N \) and the signal subspace.

Of course, the group invariance requirement leads to a lossy data reduction, with the least compression embodied by the \textit{maximal invariant statistic} (MIS), organizing the original data into equivalence classes. Hence, every invariant statistic can be expressed in terms of the MIS [18]–[20]. For this reason, before proceeding, we first individuate a suitable group fulfilling the above requirements.

To this end, we take the (vectorized) single-source model in (2) and rewrite it in the so-called canonical form

\[
\bar{x}_{\ell} = e_{\ell} \bar{r}_{\ell} + \bar{w}_{\ell}, \ \ell = 1, \ldots, L
\]

(13)

where \( e_{\ell} \triangleq [1 \ 0^T_{N-1}]^T, \bar{r}_{\ell} \in \mathbb{C} \) and \( \bar{w}_{\ell} \sim \mathcal{C} \mathcal{N}(0, \sigma_t^2 I_N) \). The group of transformations leaving the hypothesis testing problem in (6) unaltered is represented by \( G \triangleq G_1 \times \ldots \times G_L \), where \( G_{\ell} \) is defined as follows:

\[
G_{\ell} \triangleq \{ g_{\ell} : \bar{x}_{\ell} \rightarrow \gamma \bar{V}_{\ell} \bar{x}_{\ell}, \ \gamma_{\ell} \in \mathbb{R}^+ \}
\]

(14)

\[
V_{\ell} \triangleq \left[ \begin{array}{c} e^{j\phi_{\ell}} \alpha_{0,\ell} \alpha^*_{0,\ell} \\ 0_{N-1} \end{array} \right], \phi_{\ell} \in (0, 2\pi), \ V_{1,\ell} \in \mathcal{U}(N-1) \}
\]

After defining the partitioning \( \bar{x}_{\ell} = [\bar{x}_{a,\ell} \ \bar{x}_{b,\ell}]^T \), where \( \bar{x}_{a,\ell} \in \mathbb{C} \) and \( \bar{x}_{b,\ell} \in \mathbb{C}^{N-1} \), respectively, we are able to state the proposition providing the MIS for the problem at hand.

\textbf{Proposition 1}: The MIS for the hypothesis testing in (6) under the group \( G \) is the \( L \)-dimensional vector:

\[
t \triangleq [t_1 \cdots t_L]^T, \ t_{\ell} \triangleq |\bar{x}_{a,\ell}|^2 / \| \bar{x}_{b,\ell} \|^2
\]

(15)

\textbf{Proof}: The proof is readily obtained by extending the known result developed in the literature (see, e.g., [19]) for the simpler case \( L = 1 \) and then exploiting the separability of the problem among the frequencies \( \omega_1, \ldots, \omega_L \).

For completeness, we remark that the MIS depends on the unknown parameters only through the corresponding induced maximal invariant [18], which for this specific case is \( \delta \triangleq [\delta_1 \cdots \delta_L] \), where \( \delta^2_{\ell} \triangleq \|b_{\ell} \|^2 / \sigma_t^4 \), corresponding to the signal-to-noise ratio (SNR) experienced on \( \omega_{\ell} \).

Two important considerations are now in order. First, since the MIS in (15) is vector-valued no uniformly most powerful invariant (UMPI) test exists for the hypothesis testing in (6). Such negative result differs from that of the single-frequency \( L = 1 \) case, where \( t_1 \) (i.e., a scalar-valued statistic) in (15) is also the UMPI decision statistic [17], [18].

Second, it can be shown (the proof is omitted for brevity) that the equality \( t_{\ell} = \Xi_{\ell} \) holds. Therefore, the MIS in (15) can be rewritten as \( t = [\Xi_1 \cdots \Xi_L]^T \triangleq \Xi \). Then, from direct comparison of MIS with statistics in (9) and, exploiting the theory in [18], it is readily deduced that the tests built on the aforementioned statistics, being functions of the original data \( x_{1:L} \), solely through the MIS \( \Xi \), are invariant and, therefore, they all

\footnote{The canonical form representation is obtained by rotating each vector \( x_{\ell} \) as \( x_{\ell} \triangleq U_{\ell} \bar{x}_{\ell} \), where \( U_{\ell} \triangleq [b_{\ell} \ldots U_{\ell,c}] \in \mathcal{U}(N) \), that is, a unitary matrix whose first column is aligned toward the direction of \( b_{\ell} \).}
ensure a CFAR w.r.t. the noise levels on the probed frequencies \( \sigma_{\ell}^2, \ell = 1, \ldots, L \).

Additionally, following the first consideration, since no UMPI test exists, nothing can be said in advance on the relative performance of the aforementioned tests. For this reason, it is useful analyzing the structural properties of the (clairvoyant) MPI statistic.\(^5\)

\[
 t_{\text{mpi}} \triangleq \frac{f_1(\Xi; \delta)}{f_0(\Xi; \delta)} = \prod_{\ell=1}^{L} \frac{f_1(\Xi_{\ell}; \delta_{\ell})}{f_0(\Xi_{\ell}; \delta_{\ell})} = 0 \quad \text{for} \quad \ell = 1, \ldots, L.
\]

Clearly, since the corresponding MPI test depends on \( \delta \), it cannot be implemented. However, since \( \Xi_{\ell}[H_0] \sim \mathcal{C}\mathcal{F}_{1, N-1}(\delta_{\ell}) \) and \( \Xi_{\ell}[H_1] \sim \mathcal{C}\mathcal{F}_{1, N-1}(\delta_{\ell}) \) (these results are given, for example, in [23]) each ratio \( f_1(\Xi_{\ell}; \delta_{\ell})/f_0(\Xi_{\ell}; \delta_{\ell}) = 0 \), \( \ell = 1, \ldots, L \), is monotone with \( \Xi_{\ell} \) (since the complex noncentral F-distribution is a totally-positive kernel of order 2, see [24]). Thus, it immediately follows that for each \( \delta_{\ell} \geq \delta_{\ell} \), the MPI detector is an increasing function of the vector \( \Xi \).

Such key result allows to claim that the space of monotone functions of \( \Xi \) forms a complete class of decision statistics [25]. It thus follows that the GLR, Rao, and Wald statistics in the multifrequency case are all “meaningful” candidates, as they belong to the aforementioned class [cf., (9)]. Remarkably, such consideration also applies to any other statistic built as an increasing function of \( \Xi \) (such as the geometric or harmonic means of the elements of \( \Xi \)), which also represents a good candidate for a “stable” C-TR imaging.

IV. ADAPTIVE STATISTICS AS IMAGING PROCEDURES

This section first provides a statistical characterization of \( \Xi \) in the presence of a mismatch of the \( b_\ell \)’s. This preliminary result will be useful to obtain the theoretical performance of the proposed statistics in the realistic case of multiple \( (M > 1) \) scatterers (possibly with mutual interaction effect, i.e., a FL model). To this end, we assume a generic signal form \( x_\ell \sim \mathcal{N}_C(\xi_{\ell}, \sigma_{\ell}^2) I_N \), \( \ell = 1, \ldots, L \). In this case, it holds \( \{x_\ell P_b, x_\ell/\sigma_{\ell}^2\} \sim \mathcal{C}N_{1-M}(\delta_{N, \ell}, \delta_{M, \ell}) \) and \( \{x_\ell P_b, x_\ell/\sigma_{\ell}^2\} \sim \mathcal{C}N_{1-M}(\delta_{N, \ell}, \delta_{M, \ell}) \), where \( \delta_{N, \ell} \triangleq \{\xi_{\ell} P_b, \xi_{\ell}/\sigma_{\ell}^2\} - \delta_{N, \ell}^2 \) and \( \delta_{M, \ell} \triangleq \{\xi_{\ell} P_b, \xi_{\ell}/\sigma_{\ell}^2\} - \delta_{M, \ell}^2 \). Consequently, in the mismatched case, it readily follows that \( \Xi_{\ell} \sim \mathcal{C}\mathcal{F}_{1, N-1}(\delta_{N, \ell}, \delta_{M, \ell}, \ell) \).

Based on the above result, we now characterize statisticially the obtained imaging functions for \( M > 1 \) scatterers in the scene. Indeed, by vectorizing the model in (1), we get (using the short-hand notation \( M_\ell \) for \( M_\ell(x_\ell; \tau_\ell) \))

\[
x_\ell = [A_{\ell}(r_1; M) \otimes A_{\ell}(r_1; M)] \vec{M}(r_\ell), \quad \ell = 1, \ldots, L.
\]

Let be \( \xi_{\ell} = \{A_{\ell}(r_1; M) \otimes A_{\ell}(r_1; M)] \vec{M}(r_\ell) \} \), the mismatched analysis of Section III allows to conclude that the relevant information is summarized through \( \delta_{N, \ell} \) and \( \delta_{M, \ell} \). Also, it can be shown that for the model in (17), \( \delta_{N, \ell}^2 \) assumes the expression (when varying the probed location \( r_\ell \))

\[
\delta_{N, \ell}^2(r) = \sigma_{\ell}^2 \text{vec}(M_\ell)^T [A_{\ell}(r_1; M) \otimes A_{\ell}(r_1; M)]^T P_{b_\ell}(r) \times [A_{\ell}(r_1; M) \otimes A_{\ell}(r_1; M)] \text{vec}(M_\ell)
\]

whereas \( \delta_{M, \ell}^2(r) \) is obtained from (18) when replacing \( P_{b_\ell}(r) \) with \( P_{b_\ell}(r) \). After tedious manipulations, these quantities can be rewritten in the more intuitive form

\[
\delta_{N, \ell}^2(r) = \left\| h_{R, \ell}(r) M_{\ell} h_{R, \ell}^T(r) \right\|^2 / \sigma_{\ell}^2
\]

\[
\delta_{M, \ell}^2(r) = \left\{ \left\| A_{\ell}(r_1; M) M_{\ell} A_{\ell}^T(r_1; M) \right\|^2 / \sigma_{\ell}^2 \right\} - \delta_{N, \ell}^2(r)
\]

where we have defined the vector of normalized point-spread functions of the Tx (resp. Rx) array as \( h_{\ell}(r) \triangleq \{A_{\ell}(r_1; M) a_{T, \ell}(r_1)/\|a_{T, \ell}(r_1)\|^2 \} \) (resp. \( h_{R, \ell}(r) \)).

The pdfs of GLR, Rao, and Wald imaging functions can be obtained by transformation of vector \( \Xi_\ell \), see (9). The preceding analysis is also employed to provide a theoretical characterization of the imaging functions in (3)–(5) and (11), as shown hereinafter.

**MF imaging:** The function \( I_{\text{mf}}(r, \ell) \) in (3) can be rewritten by exploiting \( a_{T, \ell}(r) X_\ell a_{R, \ell}(r) b_\ell(r) x_\ell \) (achieved with the use of vec(·) and Kronecker product properties), thus leading to \( I_{\text{mf}}(r, \ell)/\|b_\ell(r)\|^2 \sim \mathcal{C}x_\ell(\delta_{N, \ell}(r), \delta_{M, \ell}(r)) \), which in turn provides

\[
I_{\text{mf}}(r, \ell) \sim \mathcal{C}x_\ell(\delta_{N, \ell}(r), \delta_{M, \ell}(r), \|b_\ell(r)\|^2).
\]

**ML imaging:** By similar reasoning as MF imaging, \( I_{\text{ml}}(r, \ell) \) in (4) is distributed as (recall that \( \|b_\ell(r)\|^2 = \|a_{T, \ell}(r)\|^2 \|a_{R, \ell}(r)\|^2 \))

\[
I_{\text{ml}}(r, \ell) = \mathcal{C}x_\ell(\delta_{N, \ell}(r), \delta_{M, \ell}(r), \|b_\ell(r)\|^2).
\]

**Likelihood imaging:** The imaging function \( I_{\text{li}}(r) \) in (5) can be rewritten as \( I_{\text{li}}(r) = \left( \prod_{\ell=1}^{L} \delta_{N, \ell}(r) \right)^{-1} \), where \( f_\ell \sim \mathcal{C}x_\ell(\delta_{N, \ell}(r), \ell = 1, \ldots, L \).

**Nonadaptive imaging:** It can be easily shown that \( t_{\text{na}}(r) \) in (11) is distributed as \( t_{\text{na}}(r) \sim \mathcal{C}x_\ell(\delta_{N, \ell}(r)) \). We recall that such function requires the knowledge of \( \delta_{N, \ell}(r) \) and \( \delta_{M, \ell}(r) \), which may not be known in an adaptive scenario.

V. CONCLUSION

In this letter, imaging functions for wideband C-TR have been devised based on GLR, Rao, and Wald statistics under the single-source model. Both nonadaptive and adaptive (with a supporting CFAR analysis through invariance principle) have been analyzed. The proposed imaging functions have been also compared with other alternatives proposed in the literature. For all these functions a theoretical characterization for the multiplescatterers case (possibly with mutual interaction) has been derived and shown to depend only on the noncentrality parameter functions \( \delta_{N, \ell}(r) \) and \( \delta_{M, \ell}(r) \) (19).

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\(^5\) The MPI statistic is the likelihood-ratio after reduction by invariance (i.e., based on \( \Xi \)).

\(^6\) The notation \( \delta_{\ell} \geq \delta_{\ell} \) means that each element of \( \delta_{\ell} \) is greater or equal than the corresponding element of \( \delta_{\ell} \), and at least one element of \( \delta_{\ell} \) is strictly greater than the corresponding element of \( \delta_{\ell} \).
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