The Encoding Complexity of Network Coding with Two Simple Multicast Sessions

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Abstract—The encoding complexity for network coding with one multicast session has been intensively studied from several aspects: e.g., the time complexity, the required number of encoding links and the required field size for a linear code solution. However, these issues are less understood for the network with multiple multicast sessions. Recently, C. C. Wang and N. B. Shroff declared that polynomial time can decide the solvability of the two simple multicast network coding (2-SMNC) problem. In this paper, we prove for the 2-SMNC networks: 1) the solvability can be determined with time $O(|E|)$; 2) a solution can be constructed with time $O(|E|)$; 3) an optimal solution can be obtained in polynomial time; 4) the number of encoding links required to achieve a solution is upper-bounded by $\max(3, 2N - 2)$; and 5) the field size required to achieve a linear solution is upper-bounded by $\max(2, \sqrt{2N - 7/4 + 1/2})$, where $|E|$ is the number of links and $N$ is the number of sinks of the underlying network. The bounds are shown to be tight and the algorithms to determine the solvability, to construct a solution and to construct an optimal solution are proposed.

Index Terms—Network coding, encoding complexity, region decomposition.

I. INTRODUCTION

A communication network is described as a finite, directed, acyclic graph $G = (V, E)$, where a number of messages are generated at some nodes, named sources, and desired to receive by to some other nodes, named sinks. Messages are assumed to be independent random process with the elements taken from some fixed finite alphabet, usually a finite field. Network coding allows the intermediate nodes to “encode” the received messages before forwarding it, and has significant throughput advantages as opposed to the conventional store-and-forward scheme [1], [2]. The multicast network coding problem has been fully investigated and well understood by the network coding community. However, for the nonmulticast networks, the problem becomes even harder, and there were only a few results, for example, some deterministic results on the capacity region for some specific networks, such as single-source two-sink nonmulticast networks [3], directed cycles [6], degree-2 three-layer directed acyclic networks [8], and two simple multicast networks [4]. The outer bounds on the capacity region for general nonmulticast networks were obtained by information theoretic arguments [6]–[9] and the inner bounds were obtained by linear programming [10], [11]. In [12] it was proved that determining whether there exist linear network coding solutions for an arbitrary nonmulticast network is NP-hard.

An important issue for network coding problem is the encoding complexity, which has been intensively studied for multicast networks [14]–[19]. For nonmulticast networks, it remains challenging due to the intrinsic hardness of the nonmulticast network coding problem. In previous works the encoding complexity is generally studied from three aspects: the time complexity for constructing a network coding solution, the number of the required encoding nodes, and the required field size for achieving a network coding solution.

The time complexity is a fundamental issue. It is well known that a network coding solution can be achieved with polynomial time for multicast networks [4]. In [18], the authors first categorized the network links into two classes, i.e., the forwarding links and the encoding links. The forwarding links only forward the data received from its incoming links. While, the encoding links transmit coded packets, which need more resources due to the computing process and the equipping of encoding capabilities. It was shown that, in an acyclic multicast network, the number of encoding nodes (i.e., the tail of an encoding link) required to achieve the capacity of the network is independent of the size of the underlying network and bounded by $h^3N^2$, where $N$ is the number of the sinks and $h$ is the number of the source messages. The third aspect of encoding complexity is the required field size. As mentioned in [19], larger encoding field size may cause difficulties, i.e., either larger delays or larger bandwidths for the implementation of network coding, hence smaller alphabets are more preferred. For the multicast network, the required alphabet size to achieve a solution is upper bound by $N$, where $N$ is the number of sinks (see [14]). In [17], the authors showed that an finite field with size $\sqrt{2N - 7/4 + 1/2}$ is sufficient for achieving a solution of a multicast network with two source messages.

In this paper, we consider the encoding complexity for two simple multicast network coding problem (2-SMNC), where two unite rate messages are send from two sources and required by two sets of sinks respectively. If the two sink sets are identical, it is a multicast network coding problem, of which the solvability can be characterized by the well-known max-flow condition and its encoding complexity has been discussed as mentioned above. However, in the case the two sink sets are distinctly different, the situation becomes complicated. The recent work of C. C. Wang and N. B. Shroff [4] investigated this problem and showed that the solvability of the 2-SMNC problem can be characterized by paths with controlled edge-overlap condition under the assumption of...
sufficient large encoding fields. They also proved that to decide the solvability of a 2-SMNC networks is polynomial time complexity and linear network codes are sufficient to achieve a solution. However, they did not consider the other issues of the encoding complexity.

This paper aims at the encoding complexity of 2-SMNC networks and the main contributions are listed as follows:

• We give algorithms to determine the solvability of the 2-SMNC problem with runtime $O(|E|)$; furthermore, we show that for a solvable network, a network coding solution can be constructed with time $O(|E|)$, where $|E|$ is the number of the links of the underlying network;
• We give a polynomial time algorithm to construct the optimal solutions for the solvable 2-SMNC networks;
• We prove that the number of encoding links for achieving a solution is upper-bounded by $\max\{3, 2N - 2\}$, where $N$ is the number of sinks. Note that it is independent with the network size and only related to $N$. We also construct a instance to show this bound is tight.
• We prove that the required field size to achieve a linear solution is upper-bounded by $\max\{2, \sqrt{2N - 7/4 + 1/2}\}$, which is amazingly as the same small as the multicast case. Also, this bound is shown to be tight by construction of a instance.
• To obtain these results, we proposed a region decomposition method, which promotes the subtree decomposition method for multicast networks [17]. Unlike subtree decomposition, we need not find a subgraph at first, also the regions are not necessarily being trees. Moreover, our method yields a unique region decomposition, namely the basic region decomposition for each network. Note that the subtree decomposition is in general not unique.

The technical line of this paper is as follows: Consider the original 2-SMNC network $G$, we first obtain its line graph $L(G)$, which can be regarded as a trivial region graph of $G$. Then we preform sequential region contractions on $L(G)$, and finally we obtain the basic region decomposition of $G$, namely, $D^{**}$. The solvability information of $G$ then can be obtained from $D^{**}$ by using the region labeling operation, and the network code solution can be obtained by assigning linear independent global encoding kernels to the regions of $D^{**}$ using a decentralized manner. To give the optimal solution, we do further region contractions on $D^{**}$ (in fact, this process can start with an arbitrary feasible region graph) and finally obtain a minimal feasible region graph. The global information such as the required encoding links and the required field size can be derived by analyzing the local structure of the minimal feasible region graph.

The rest of the paper is organized as follows. In Section II, we give the network model and some basic definitions. In Section III, we introduce the method of region decomposition. We give definitions of region, region decomposition, region graph, region contraction, codes on the region graph, feasible region graph, region labeling, and etc. We also derive some basic properties for these basic notions in this section. In Section IV, we decide the time complexity for solving the 2-SMNC problem by introducing the basic region decomposition $D^{**}$. We introduce the minimal feasible region graph and give the optimal solution in Sections V, the number of required encoding links is given in the same section. The required encoding field size is obtained in Section VI. Finally, we conclude the paper in Section VII.

II. NETWORK MODEL AND NOTATIONS

We consider the two simple multicast network coding problem (2-SMNC), of which the underlying network is assumed to be a finite, directed, acyclic graph $G = (V, E)$, where $V$ is the set of nodes (vertices) and $E$ is the set of links (edges). There are two sources $s_1, s_2 \in V$ and two sets of sinks $T_1 = \{t_{1,1}, \cdots, t_{1,N_1}\}, T_2 = \{t_{2,1}, \cdots, t_{2,N_2}\} \subseteq V$, where $s_i \notin T_i (i = 1, 2)$. Two messages $X_1$ and $X_2$ are generated at $s_1$ and $s_2$ and are demanded by $T_1$ and $T_2$ respectively. Note that $T_1 \neq T_2$ generally. The messages are assumed to be independent random variables taking values from a fixed finite field and a link $e = (u, v)$ is assumed of unit capacity, i.e., it can transmit one symbol from node $u$ to $v$ per transmission.

For $e = (u, v) \in E$, node $u$ is called the tail of $e$ and node $v$ is called the head of $e$ and denoted by $u = head(e)$ and $v = tail(e)$. For $e_1, e_2 \in E$, we call $e_1$ an incoming link of $e_2$ if $head(e_1) = tail(e_2)$. Denote by $In(e)$ the set of the incoming links of $e$. We assume $|In(e)| < M$ for each $e \in E$ and for some integer $M$.

We assume that for each source $s_i$ there is one imaginary link from nowhere to $s_i$, called the $X_i$ source link, and for each sink $t_{i,j} \in T_i$ there is one imaginary link from $t_{i,j}$ to nowhere, called the $X_i$ sink link. The following terms are used in their self-evident meaning. An $X_i$ link means the $X_i$ source link or an $X_i$ sink link. A source (resp. sink) link means the $X_1$ source (resp. sink) link or the $X_2$ source (resp. sink) link. Note that the source links have no tail and the sink links have no head, but this does not affect our discussion.

We assume that $In(e) \neq \emptyset$ if $e \in E$ is not a source link. Otherwise $e$ has no impact on the network and can be removed from $G$.

Remark 2.1: Since $G = (V, E)$ is acyclic, $E$ can be sequentially indexed as $e_1, e_2, e_3, \cdots, e_{|E|}$ such that (1) $e_1$ is the $X_1$ source link and $e_2$ is the $X_2$ source link; 2) $i < j$ if $e_i$ is an incoming link of $e_j$. Note that such an index will be used in the sequel.

The network coding solutions of a 2-SMNC network are defined as follows.

Definition 2.2 (Network Coding Solution): A network coding solution (or a solution for short) of $G$ over field $\mathbb{F}$ is a collection of functions $C = \{f_e: \mathbb{F}^2 \to \mathbb{F}; e \in E\}$ such that

1) If $e$ is an $X_i$ link ($i \in \{1, 2\}$), then $f_e(X_1, X_2) = X_i$;
2) If $e$ is not a source link, then $f_e$ can be computed from $f_{p_1}, \cdots, f_{p_k}$, where $\{p_1, \cdots, p_k\} = In(e)$. This means that there is a $\mu_e: \mathbb{F}^k \to \mathbb{F}$ such that $f_e = \mu_e(f_{p_1}, \cdots, f_{p_k})$.

The function $f_e$ is called the global encoding function of $e$ and $\mu_e$ is called the local encoding function of $e$. A solution $C$ is called a linear solution if the global and local encoding functions are all linear functions over $\mathbb{F}$.

A network $G$ is said to be (linearly) solvable if $G$ has a (linear) solution over some finite field $\mathbb{F}$. 
Remark 2.3: In the linear case, the global encoding function $f_e$ of any $e \in E$ is in the form $f_e(X_1, X_2) = c_1X_1 + c_2X_2$, where $c_1, c_2 \in \mathbb{F}$. Hence $f_e$ can be identified with the vector $d_e = (c_1, c_2) \in \mathbb{F}^2$ and $C$ can be denoted by $C = \{d_e \in \mathbb{F}^2; e \in E\}$, where $d_e$ is called the global encoding kernel of $e$. (1) and (2) of Definition 2.2 are equivalent to the following two conditions, respectively.

(1') If $e$ is an $X_i$ link ($i \in \{1, 2\}$), then $d_e = \alpha_i$, where $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$;

(2') If $e$ is not a source link, then $d_e$ is an $\mathbb{F}$-linear combination of $\{d_{p_i}; p \in \text{In}(e)\}$.

Definition 2.4 (Forwarding Link and Encoding Link): Let $C = \{f_e : \mathbb{F}^2 \rightarrow \mathbb{F}; e \in E\}$ be a solution of $G$. $e$ is called a forwarding link of $C$ if $f_e = f_u$ for some $u \in \text{In}(e)$. Else, $e$ is called an encoding link of $C$.

As in [13], we define the line graph of a network $G = (V, E)$, denoted by $L(G)$, as a directed, simple graph with vertex set $E$ and edge set $\{(e_i, e_j) \in E^2; \text{head}(e_i) = \text{tail}(e_j)\}$. The line graph $L(G)$ is obviously finite and acyclic since $G$ is finite and acyclic.

III. REGION DECOMPOSITION

In this section, we introduce the region decomposition method for 2-SMNC networks.

A. Region and Region Graph

Definition 3.1 (Region and Region Decomposition): Let $R$ be a non-empty subset of $E$ and $c_0 \in R$. $R$ is called a region of $G$ generated by $c_0$ if any $r \in R\setminus\{c_0\}$ has an incoming link in $R$. Meanwhile, $c_0$ is called the head of $R$ and is denoted by $c_0 = \text{head}(R)$. If $R$ is partitioned into mutually disjoint regions $R_1, R_2, \ldots, R_K$, we say $D = \{R_1, R_2, \ldots, R_K\}$ is a region decomposition of $G$.

Let $D$ be a region decomposition of $G$ and $R$, $D$. $R$ is called the $X_i$ source region if $R$ contains the $X_i$ source link; $R$ is called an $X_i$ sink region if $R$ contains an $X_i$ sink link, $i \in \{1, 2\}$. The $X_i$ source region and $X_2$ source region are called the source region and the $X_1$ sink region and $X_2$ sink region are called the sink region. For the sake of convenience, if $R$ is not a source region, we call $R$ a non-source region.

Obviously, there are many ways to obtain regions and decompose a network into mutually disjoint regions. For example, $\forall e \in E$, let $R_e = \{e\}$. Then $R_e$ is a region and $D^* = \{R_e; e \in E\}$ is a region decomposition of $G$. We call $D^*$ the trivial region decomposition of $G$.

We now show a nontrivial region decomposition which will also be used frequently in the sequel.

Example 3.2: Let $G_1$ be the network shown in Fig. 1(a) of which the line graph $L(G_1)$ is shown in Fig. 1(b). As illustrated in Fig. 2(a), $R_1 = \{e_1, e_3, e_4, e_{10}, e_{11}\}$, $R_2 = \{e_2, e_5, e_6\}$, $R_3 = \{e_7, e_8, e_9, e_{12}, e_{19}, e_{15}\}$, $R_4 = \{e_{14}, e_{16}, e_{18}\}$, $R_5 = \{e_{17}\}$, $R_6 = \{e_{19}\}$, $R_7 = \{e_{20}\}$, $R_8 = \{e_{21}\}$ are mutually disjoint regions of $G_1$ and $D = \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8\}$ is a region decomposition of $G_1$, in which $R_1$ is the $X_1$ source region, $R_2$ is the $X_2$ source region, $R_4$ and $R_8$ are $X_1$ sink regions, and $R_6$ and $R_7$ are $X_2$ sink regions.

We now define the region graph with respect to any region decomposition of $G$.

Definition 3.3 (Region Graph): Let $D$ be a region decomposition of $G$. The region graph with respect to $D$, denoted by $RG(D)$, is a directed, simple graph with vertex set $D$ and edge set $\{(R_i, R_j) \in D^2; \text{In}(\text{head}(R_j)) \cap R_i \neq \emptyset\}$, i.e., $(R_i, R_j) \in D^2$ is an edge of $RG(D)$ if and only if $\text{head}(R_j)$ has an incoming link in $R_i$.

If $(R_i, R_j)$ is an edge of $RG(D)$, we call $R_i$ a parent of $R_j$ ($R_j$ is called a child of $R_i$). Two regions $R_i$ and $R_j$ are said to be adjacent if $R_i$ is a parent or a child of $R_j$. Denoted by $\text{In}(R_j)$ the set of all parents of $R_j$.

For network $G_1$ and its region decomposition $D$ in Example 3.2, the corresponding region graph is depicted in Fig. 2(b).

Remark 3.4: By Definition 3.3 for the trivial region decomposition $D^*$, $RG(D^*)$ coincides with the line graph $L(G)$. For any region decomposition $D$, $RG(D)$ can be viewed as being “contracted” from $RG(D^*)$.

Lemma 3.5: Let $D$ be a region decomposition of $G$, and $P, Q \in D$ such that $P$ is a parent of $Q$. Then $D' = D \cup \{P, Q\}$ is a region of $G$ with $\text{head}(P') = \text{head}(P)$ and $D' = D \cup \{P, Q\}$ is a region decomposition of $G$.

Proof: It can be easily verified by Definition 3.3.

Definition 3.6 (Region Contraction): Under the conditions of Lemma 3.5, $D'$ is called a contraction of $D$ by combining $P$ and $Q$. Correspondingly, the region graph $RG(D')$ is called a contraction of $RG(D)$ by combining $P$ and $Q$.

Consider network $G_1$ and region decomposition $D$ of Example 3.2. Clearly, $R_5 \cup R_8 = \{e_{17}, e_{21}\}$ is still a region of $G_1$ and $D' = \{R_1, R_2, R_3, R_4, R_5 \cup R_8, R_8, R_6, R_7\}$ is a contraction...
of $D$ obtained by combining $R_5$ and $R_8$. $D'$ and $RG(D')$ are illustrated in Fig. 5.

**B. Codes on the Region Graph**

**Definition 3.7 (Codes on the Region Graph):** Let $D$ be a region decomposition of $G$ and $\mathcal{C} = \{f_R : \mathbb{F}^2 \rightarrow \mathbb{F}; R \in D\}$ be a collection of functions. $\mathcal{C}$ is said to be a code of $RG(D)$ if the following two conditions hold.

1. If $R$ is an $X_i$ source region or an $X_i$ sink region ($i \in \{1, 2\}$), then $f_R(X_1, X_2) = X_i$.
2. If $R$ is a non-source region, then $f_R$ is computable from $(f_{P_1}, \ldots, f_{P_k})$, where $P_1, \ldots, P_k = In(R)$. This means that there is a $\mu_R : \mathbb{F}^k \rightarrow \mathbb{F}$ such that $f_R = \mu_R(f_{P_1}, \ldots, f_{P_k})$.

Here, $f_R$ is called the **global encoding function** of $R$ and $\mu_R$ is called the **local encoding function** of $R$.

$RG(D)$ is said to be feasible if it has a code over some finite field $\mathbb{F}$. We also say a region decomposition $D$ feasible if $RG(D)$ is feasible. A code $\mathcal{C}$ is called a **linear code** if the global and local encoding functions are all linear functions.

**Remark 3.8:** Similar to the linear solution of $G$, the global encoding function $f_R$ of a linear code can be identified with a vector $d_R = (c_1, c_2) \in \mathbb{F}^2$, called the **global encoding kernel** of $R$, and $\mathcal{C}$ can be denoted by $\overline{\mathcal{C}} = \{d_R \in \mathbb{F}^2; R \in D\}$. Accordingly, (1) and (2) of Definition 3.7 are equivalent to the following two conditions, respectively.

1. If $R$ is an $X_i$ source region or an $X_i$ sink region ($i \in \{1, 2\}$), then $d_R = \alpha_i$, where $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$.

2. If $R$ is a non-source region, then $d_R$ is an $\mathbb{F}$-linear combination of $\{d_P : P \in In(R)\}$.

**Remark 3.9:** A (linear) code of $RG(D')$ is exactly a (linear) solution of $G$, recall that $RG(D^*)$ is just the line graph of $G$. Thus, $G$ is solvable if and only if $RG(D^*)$ is feasible.

The following lemma gives further observations on the relationship between the network coding solution and the codes on the region graph.

**Lemma 3.10:** Let $D$ be a region decomposition of $G$. Then

1. Let $C = \{f_e : e \in E\}$ be a (linear) solution of $G$ such that $f_e = f_{head(R)}$ for any $R \in D$ and $e \in R$. Then $\overline{\mathcal{C}} = \{f_R; f_R = f_{head(R)}; R \in D\}$ is a (linear) code of $RG(D)$.

2. Let $C = \{f_R; R \in D\}$ is a (linear) code of $RG(D)$, and let $C' = \{f_e; e \in E\}$ such that $f_e = f_R$ for any $R \in D$ and $e \in R$. Then $C'$ is a (linear) solution of $G$.

**Remark 3.11:** In the above construction of $C$, we assign a same encoding kernel to a region, thus $e \in E$ is an encoding link of $C$ only if $e$ is the head of some non-source region.

Consider again network $G_1$ and region decomposition $D$ shown in Fig. 2(a). A solution $C$ of $G_1$ is depicted in Fig. 4(a) (by line graph $L(G_1)$). Fig. 4(b) shows the corresponding code $\overline{\mathcal{C}}$ of $RG(D)$.

The following results show that some kinds of the region contractions can maintain the feasibility of the region graphs.

**Corollary 3.12:** Suppose $D$ is a feasible region decomposition and $\overline{\mathcal{C}} = \{f_R : \mathbb{F}^2 \rightarrow \mathbb{F}; R \in D\}$ is a code on $RG(D)$. Suppose $P, Q \subseteq D$ are two adjacent regions and $f_Q = f_P$. Let $f_{P \cup Q} = f_P$ and $D'$ be the contraction of $D$ by combining $P$ and $Q$. Let $C' = \{f_R; R \in D'\}$. Then $C'$ is a code of $RG(D')$ and thus $RG(D')$ is feasible.
Reconsider $G_1$ and $D$ in Example 3.2. Fig. 5(a) depicts a code of $RG(D)$ other than that in Fig. 4(b). Note that both $R_5$ and $R_8$ are assigned $X_1$, by Corollary 3.12, $D' = \{R_1, R_2, R_3, R_4, R_5 \cup R_6, R_7\}$ is feasible. For the same reason, $D'' = \{R_1, R_2, R_3, R_4 \cup R_5 \cup R_8, R_6, R_7\}$ is feasible. Codes of $RG(D')$ and $RG(D'')$ are depicted in Fig. 5(b) and (c), respectively.

**Lemma 3.13**: Let $D$ be a region decomposition of $G$ and $P, Q \in D$ such that $In(Q) = \{P\}$, i.e., $P$ is the unique parent of $Q$ in $RG(D)$. Let $D'$ be the contraction of $D$ by combining $P$ and $Q$. If $RG(D)$ is feasible then $RG(D')$ is feasible.

**Proof**: Let $\tilde{C} = \{f_R : \mathcal{F}^2 \to \mathcal{F}; R \in D\}$ be a code of $RG(D)$. Since $In(Q) = \{P\}$, we have $f_Q = \mu_Q(f_P)$, where $\mu_Q : \mathcal{F} \to \mathcal{F}$ is the local encoding function of $R$. If $Q$ is an $X_i$ sink region, we alter $f_P$ by letting $f_P = X_i$ (By Definition 3.14 $f_Q = \mu_Q(f_P) = X_i$ is surjective. So $\mu_i$ is surjective, hence is bijective if $\mathcal{F}$ is finite.). Otherwise, we alter $f_Q$ by letting $f_Q = f_P$. It is easy to see that in both cases we obtain a code of $RG(D)$ such that $f_Q = f_P$. By Corollary 3.12 $RG(D')$ is feasible. 

**C. Feasibility and Region Labeling**

In order to decide the solvability of $G$ efficiently, we need further discussions on the feasibility of $RG(D)$. In the following, we first define two labeling operations.

**Definition 3.14 (Region Labeling)**: Let $D$ be a region decomposition of $G$. For $i \in \{1, 2\}$, the $X_i$ labeling operation on $RG(D)$ is defined recursively as follows.

1. A region which contains an $X_i$ link is labeled with $X_i$;
2. A region whose parents are all labeled with $X_i$ is labeled with $X_i$.

The $X_i$ labeling operation is well defined because $RG(D)$ is acyclic. A region labeled with $X_i$ is called an $X_i$ region. A region which is neither $X_1$ region nor $X_2$ region is called a coding region. A region which is both $X_1$ region and $X_2$ region is called a singular region.

Consider network $G_1$ and region decomposition $D$ in Example 3.2, the labeled region graph of $RG(D)$ is depicted in Fig. 6(a). Regions $R_1, R_4$ and $R_8$ are labeled with $X_1$ since $R_1$ contains $X_1$ source link $e_{18}$ and $R_8$ contain $X_1$ sink links $e_{18}$ and $e_{21}$ respectively. Regions $R_2, R_6$ and $R_7$ are labeled with $X_2$ since $R_2$ contains $X_2$ source link $e_{22}$ and $R_6$ and $R_7$ contain $X_2$ sink links $e_{19}$ and $e_{20}$ respectively.

Now, let $D''' = \{R_1, R_2, R_3 \cup R_6, R_4, R_5, R_7, R_8\}$. By Lemma 3.15 it is a region decomposition of $G_1$. Similarly, $R_1, R_3$ and $R_8$ are labeled with $X_1$, and $R_2, R_3 \cup R_6$ and $R_7$ are labeled with $X_2$. Furthermore, $R_4$ is labeled with $X_2$ since the parents of $R_4$ are all labeled with $X_2$. Likewise, $R_6$ is labeled with $X_2$ since the parents of $R_6$ are all labeled with $X_2$. The labeled region graph $RG(D''')$ is depicted in Fig. 6(b). In this case, $R_4$ and $R_8$ are singular regions of $RG(D''')$. 

![Figures](image1.png)

![Figures](image2.png)
Form this example, we see that a same region may be labeled differently according to different region decompositions. In the following, we determine the feasibility of region decomposition through the labeled region graph. Firstly, we give some lemmas.

**Lemma 3.15**: Let $D$ be a feasible region decomposition of $G$ and $\bar{C} = \{f_R : \mathbb{F}^2 \to \mathbb{F}; R \in D\}$ be a code of $RG(D)$. Then for any $X_1$ region $R$, there exists a $\lambda_R : \mathbb{F} \to \mathbb{F}$ such that $f_R(X_1, X_2) = \lambda_R(X_1)$, that is, $f_R$ depends only on $X_1$.

**Proof**: We prove this lemma by induction. Suppose the number of $X_1$ regions of $RG(D)$ is $K$. Since $RG(D)$ is acyclic, all $X_i$ regions of $RG(D)$ can be sequentially indexed as $R_1, R_2, \cdots, R_K$ such that $i < j$ if $R_j$ is a parent of $R_i$. By Definition 3.14, $R_1$ contains an $X_1$ link. So by Definition 3.7, $f_{R_1}(X_1, X_2) = X_1$.

For $2 \leq k \leq K$, suppose $f_{R_k}$ depends only on $X_i$ for all $1 \leq j \leq k - 1$. If $R_k$ contains an $X_1$ link, the result is evident. Else, by Definition 3.14, the parents of $R_k$ are all $X_i$ regions, hence $In(R_k) \subseteq \{R_1, \cdots, R_{k-1}\}$. By Definition 3.7, $f_{R_k} = \mu_{R_k}(f_{R_1}, \cdots, f_{R_{k-1}})$ depends only on $X_i$, where $\{P_1, \cdots, P_r\} = In(R_k)$ and $\mu_R : \mathbb{F}^r \to \mathbb{F}$ is the local encoding function of $R_k$.

**Lemma 3.16**: Suppose $D$ is a feasible region decomposition of $G$. Then $RG(D)$ has no singular region.

**Proof**: Suppose $RG(D)$ has a singular region. Note that $RG(D)$ is acyclic, we can always find a singular region $Q$ such that no parent of $Q$ is a singular region. We declare that $Q$ contains either an $X_1$ link or an $X_2$ link or both. (If $Q$ contains neither $X_1$ links nor $X_2$ links, by Definition 3.14 all the parents of $Q$ will be singular regions, which yields a contradiction.). Without loss of generality, we assume $Q$ contains an $X_1$ link. Let $\tilde{C} = \{f_R : \mathbb{F}^2 \to \mathbb{F}; R \in D\}$ be a code of $RG(D)$. By Definition 3.7, $f_{Q}(X_1, X_2) = X_1$. On the other hand, by Lemma 3.15, $f_{Q}(X_1, X_2)$ depends only on $X_2$ since $Q$ is also an $X_2$ region. A contradiction follows.

**Lemma 3.17**: Suppose $D$ is a region decomposition of $G$ such that $RG(D)$ has no singular region and each non-source region has at least two parents. Suppose $\bar{C} = \{d_R \in \mathbb{F}^2; R \in D\}$ be a collection of vectors such that

1. If $R$ is an $X_i$ region, $i \in \{1, 2\}$, then $d_R = \alpha_i$, where $\alpha_1 = (1, 0)$ and $\alpha_2 = (0, 1)$;
2. If $R, Q \in D$ have a common child and are not both $X_i$ regions for some $i \in \{1, 2\}$, then $d_R$ and $d_Q$ are linearly independent.

Then $\tilde{C}$ is a linear code of $RG(D)$.

**Proof**: Note that $D$ contains no singular region, $\tilde{C}$ satisfies (1)’ of Remark 3.3. Now take a non-source region $R$, we only need to prove that $d_R$ is an $\mathbb{F}$-linear combination of $\{d_P; P \in In(R)\}$. If the parents of $R$ are all $X_i$ regions for some $i \in \{1, 2\}$. By (1), $d_R = d_P = \alpha_i$ for all $P \in In(R)$.

Otherwise, note that $R$ has at least two parents, we can find two parents of $R$, say $P_1$ and $P_2$, such that $d_{P_1}$ and $d_{P_2}$ are linearly independent, and hence span $\mathbb{F}^2$. So $d_R$ is an $\mathbb{F}$-linear combination of $d_{P_1}$ and $d_{P_2}$.

**Theorem 3.18**: Let $D$ be a region decomposition of $G$ such that each non-source region in $D$ has at least two parents. Then $RG(D)$ is feasible if and only if it has no singular region. Moreover, if $RG(D)$ is feasible, it has a linear code.

**Proof**: By Lemma 3.16 if $RG(D)$ is feasible, then $RG(D)$ has no singular region.

Conversely, if $D$ contains no singular region, we can construct a linear code of $RG(D)$ as follows. Let $Q_1, \cdots, Q_J$ be the set of coding regions of $RG(D)$ and $\mathbb{F} = \{0, c_1 = 1, c_2, \cdots, c_{q-1}\}$ be a field of size $q \geq J + 1$. Let $\tilde{C} = \{d_R \in \mathbb{F}^2; R \in D\}$ such that

1. If $R$ is an $X_i$ region, $i \in \{1, 2\}$, then $d_R = \alpha_i$;
2. $d_{Q_j}$, where $\beta_j = (1, c_j), j = 1, \cdots, J$.

Note that $\alpha_1 = (1, 0), \alpha_2 = (0, 1)$ and $\beta_j = (1, c_j), j = 1, \cdots, J$ are mutually linearly independent. By Lemma 3.17, $\tilde{C}$ is a linear code of $RG(D)$. The result follows.

**IV. TIME COMPLEXITY FOR A SOLUTION**

In this section, we give $O(|E|)$ time algorithms to determine solvability and to construct network coding solutions for 2-SMNC networks. By Theorem 3.18 if one can find out a region decomposition such that each non-source region has at least two parents, then the feasibility of the region graph can be inferred by searching the singular regions. In the following, we will show that for each 2-SMNC network, such a region decomposition exists. We first introduce a definition.

**Definition 4.1 (Basic Region Decomposition)**: We call a region decomposition $D^{**}$ a basic region decomposition if the following two conditions hold:

1. For any region $R \in D^{**}$ and any link $e \in R \setminus \{head(R)\}$, $In(e) \subseteq R$;
2. Each non-source region of $D^{**}$ has at least two parents.

The following two examples demonstrate this notion.
Example 4.2: Consider the network $G_1$ in Example 3.2. See Fig. 7(a). Let $Q_1 = \{e_1, e_3, e_4, e_{10}, e_{11}\}$, $Q_2 = \{e_2, e_5, e_6\}$, $Q_3 = \{e_7, e_8, e_9\}$, $Q_4 = \{e_{12}\}$, $Q_5 = \{e_{13}\}$, $Q_6 = \{e_{14}, e_{16}, e_{18}\}$, $Q_7 = \{e_{15}\}$, $Q_8 = \{e_{17}\}$, $Q_9 = \{e_{19}\}$, $Q_{10} = \{e_{20}\}$, $Q_{11} = \{e_{21}\}$. It can be checked that $D^{**} = \{Q_1, \cdots, Q_{11}\}$ is a basic region decomposition of $G_1$.

![Figure 7](image1)

Fig 7. (a) depicts a basic region decomposition $D^{**}$ of $G_1$ and (b) is the labeled region graph $RG(D^{**})$.

Example 4.3: Let $G_2$ be a 2-SMNC network shown in Fig. 8(a). The line graph $L(G_2)$ is shown in Fig. 8(b). As depicted in Fig. 9(a), let $R_1 = \{e_1, e_3, e_4, e_7, e_8, e_9\}$, $R_2 = \{e_2, e_6\}$, $R_3 = \{e_{10}, e_{11}, e_{12}, e_{13}, e_{15}\}$, $R_4 = \{e_{14}\}$, $R_5 = \{e_{16}\}$. It can be checked that $D^{**} = \{R_1, R_2, R_3, R_4, R_5\}$ is a basic region decomposition of $G_2$.

![Figure 8](image2)

![Figure 9](image3)

Fig 8. A 2-SMNC network $G_2$: (a) is the original network. The imaginary links $e_1$ and $e_2$ are the $X_1$ source link and $X_2$ source link respectively, and the imaginary links $e_{15}$ and $e_{16}$ are the $X_1$ sink link and $X_2$ sink link respectively. (b) is the line graph $L(G_2)$.

![Figure 9](image4)

Fig 9. (a) depicts a basic region decomposition $G_2$ and (b) is the labeled region graph $RG(D^{**})$, where $G_2$ and $D^{**}$ are as in Example 4.3.

Consider the network $G_1$ and (b) is the labeled region graph $RG(D^{**})$, where $G_2$ and $D^{**}$ are as in Example 4.3.

In general, for an arbitrary 2-SMNC network $G$, we have the following result.

Theorem 4.4: $G$ has a unique basic region decomposition.

Proof: Let $E$ be indexed as in Remark 2.1. Consider Algorithm 1 in Fig. 10 with output $D^{**} = \{R_1, \cdots, R_K\}$. Clearly $D^{**}$ satisfies the two conditions of Definition 4.1. Thus a basic region decomposition exists.

Now suppose $D = \{Q_1, \cdots, Q_L\}$ and $D' = \{Q_1, \cdots, Q_L\}$ are two basic region decompositions of $G$. We prove $D = D'$.

First, we prove that any $R_i \in D$ is contained in some region in $D'$. Let $E$ be indexed as in Remark 2.1. Assume $R_i = \{e_{i_1}, e_{i_2}, \cdots, e_{i_n}\}$ such that $i_1 < i_2 < \cdots < i_n$. Without loss of generality, we assume $e_{i_1} \in Q_1$. We now prove $R_i \subseteq Q_1$. Otherwise, there exists an $e_{i_k} \in R_i$ such that $\{e_{i_1}, \cdots, e_{i_{k-1}}\} \subseteq Q_1$ and $e_{i_k} \in Q_j (j \neq 1)$. By (1) of Definition 4.1, $\text{In}(e_{i_k}) \subseteq \{e_{i_1}, \cdots, e_{i_{k-1}}\} \subseteq Q_1$. Note that $Q_1 \cap Q_j = \emptyset$ and by Definition 3.1, we can infer that $e_{i_k} = \text{head}(Q_j)$ and $Q_1$ is the only parent of $Q_j$, which contradicts (2) of Definition 4.1.

Symmetrically, we can have $Q_1 \subseteq R_i$ for some $R_i \in D$. So $R_i \subseteq R_i$. Note that $R_i \cap R_i = \emptyset$ if $R_i \neq R_i$, we have $R_i = Q_1$. Note that $R_i$ can be arbitrarily chosen from $D$, we have $D \subseteq D'$.

Similarly, we can have $D' \subseteq D$.

Thus $D' = D$ is the unique region decomposition of $G$.

In the following, we always use $D^{**}$ to denote the the unique basic region decomposition of $G$. Now, we discuss
its feasibility remains unchanged (Lemma 3.13). Thus

$$D$$ comparisons for each

$$X$$ as

$$e$$

Fig 10. The algorithm generates the basic region decomposition

4.2. The labeled region graph

feasible (Theorem 3.18). By Lemma 4.5, labeling operation

algorithm 3 is ensured by Theorem 3.18. Based on the proof

Proof: First, note that Algorithm 1 makes $$[In(e_j)]$$ times comparisons for each $$e_j \in E, j \geq 3$$. Thus, it can output $$D^{**}$$ with time $$O(|E|)$$. Second, according to Algorithm 1, $$D^{**}$$ is in fact obtained from $$D$$* by a series of region contractions, i.e., if the region $$\{e_j\}$$ has a unique parent $$R_k$$, then combine $$e_j$$ and $$R_k$$. Hence its feasibility remains unchanged (Lemma 3.13). Thus $$G$$ is solvable if and only if $$RG(D^{**})$$ is feasible (Remark 3.9) if and only if $$RG(D^{**})$$ is feasible.

Consider the basic region decomposition of $$G_1$$ in Example 12. The labeled region graph $$RG(D^{**})$$ is shown in Fig 7(b). One can see that $$D^{**}$$ has no singular region, and hence is feasible (Theorem 3.18). By Lemma 4.5, $$G_1$$ is solvable.

Consider the basic region decomposition of $$G_2$$ in Example 12. The labeled region graph $$RG(D^{**})$$ is shown in Fig 9(b). One can see that $$D^{**}$$ has a singular region $$R_3$$ and hence is not feasible (Theorem 3.18). By Lemma 4.5, $$G_2$$ is not solvable.

Lemma 4.6: Let $$D = \{R_1, \ldots, R_K\}$$ be a region decomposition of $$G$$. The feasibility of $$RG(D)$$ can be decided in time $$O(|E|)$$. Moreover, if $$D$$ is feasible, a linear solution of $$G$$ can be constructed in time $$O(|E|)$$.

Proof: Let $$E$$ be indexed as in Remark 2.1. The $$X_i$$ labeling operation ($$i \in \{1, 2\}$$) on $$RG(D)$$ can be performed by Algorithm 2 in Fig 11 and the feasibility of $$RG(D)$$ can be determined by Algorithm 3 in Fig 12. The correctness of algorithm 3 is ensured by Theorem 3.18. Based on the proof of Theorem 3.18 a linear solution of $$G$$ can be constructed by Algorithm 4 in Fig 13. Clearly $$|D| \leq |E|$$, the runtime of these three algorithms are all $$O(|E|)$$.

Algorithm 2: $$X_i$$-Labeling ($$G = (V, E), RG(D)$$)

Algorithm 3: Determining feasibility ($$RG(D)$$)

j = 1;

while $$j \leq |E|$$ do

if $$e_j \in R_k$$ is an $$X_i$$ link, then

label $$R_k$$ with $$X_i$$;

end if

j = j + 1;

end while

k = 1;

while $$k \leq K$$ do

if the parents of $$R_k$$ are all labeled with $$X_i$$, then

label $$R_k$$ with $$X_i$$;

end if

k = k + 1;

end while

return feasible;

Fig 12. The algorithm determines the feasibility of $$RG(D)$$, where $$D = \{R_1, \ldots, R_K\}$$ have been labeled by $$X_1$$ labeling operation and $$X_2$$ labeling operation.

Now, we can conclude the section by the following theorem.

Theorem 4.7: Determining the solvability of $$G$$ is an $$O(|E|)$$ time problem. Furthermore, if $$G$$ is solvable, a linear solution of $$G$$ can be constructed in time $$O(|E|)$$.

V. THE NUMBER OF ENCODING LINKS

Throughout this section, we assume that $$G = (V, E)$$ is a 2-SMNC network with two disjoint sink sets $$T_1$$ and $$T_2$$.

1If $$t \in T_1 \cap T_2$$, we can add two additional nodes $$t'$$ and $$t''$$ and two additional links $$(t, t')$$ and $$(t, t'')$$ and replace $$t$$ by $$t'$$ in $$T_1$$ and $$t''$$ in $$T_2$$ respectively. Then any network coding solution for the old graph can be mapped bijectively to a network coding solution for the new graph without changing the encoding complexity.
Algorithm 4: Code Construction (RG(D))

\[ j = 1; \]
\[ k = 1; \]
\[ \textbf{while } j \leq K \textbf{ do} \]
\[ \quad \textbf{if } R_j \text{ is labeled with } X_i \text{ for an } i \in \{1, 2\} \text{ then} \]
\[ \quad \quad f_{R_j} = X_i; \]
\[ \quad \textbf{else} \]
\[ \quad \quad f_{R_j} = X_1 + c_k X_2; \]
\[ \quad k = k + 1; \]
\[ \textbf{end if} \]
\[ \textbf{end while} \]

\[ \text{return } \{f_{e_j}; j = 1, \ldots, |E|\}; \]

Fig 13. The algorithm constructs a linear solution of G. This algorithm is based on the proof of Theorem 3.18.

hence the number of sinks is equal to the number of sink links. We shall prove the following theorem.

Theorem 5.1: Let G be a solvable 2-SMNC network with N sinks, then G has a network coding solution with at most \( \max\{3, 2N - 2\} \) encoding links. There exist instances to achieve this bound.

To obtain this result, we need the concept of minimal feasible region graph.

Definition 5.2 (Minimal Feasible Region Graph): Let D be a feasible region decomposition of G. RG(D) is said to be a minimal feasible region graph if the following two conditions hold.

1) Deleting any link of RG(D) results in a subgraph of RG(D) which is not feasible.

2) Combining any adjacent regions results in a contraction of RG(D) which is not feasible.

We say that D is a minimal feasible region decomposition if RG(D) is a minimal feasible region graph.

According to Definition 5.2, given a feasible region graph RG(D) of G, if RG(D) is not minimal feasible, one can always get a smaller feasible region graph, i.e., with less links and/or less nodes by deleting links and/or combining nodes of RG(D). Once the deleting/combining process cannot be preformed, we get a minimal feasible region graph. By the manner of Lemma 3.10 we can obtain a network coding solution of G from a code of the minimal feasible region graph. The solution derived from the minimal feasible region graph will have less (or equal) encoding links than the solution derived from the original feasible region graph. From this sense, we call a solution constructed from the minimal feasible region graph as an optimal solution of G.

Consider the feasible region graphs RG(D), RG(D') and RG(D'') of G1 in Fig. 5. It can be checked that RG(D), RG(D') are not minimal feasible region graphs and RG(D'') is minimal feasible. An optimal solution of G can be obtained from (c) of Fig. 5. The optimal solution of G1 has only 4 encoding links, i.e., the head links of R3, R4 ∪ R5 ∪ R6, R7, and R8. In fact, the information of the required number of encoding links lies in the minimal feasible region graphs. To see this clearly, we first derive some properties of the minimal feasible region graph.

Theorem 5.3: Let D be a minimal feasible region decomposition of G. The following items hold.

1) Any non-source region has exactly two parents.

2) Two regions which are adjacent or have a common child cannot be both X1 regions nor both X2 regions.

3) Two adjacent coding regions have a common child.

4) If a coding region R is adjacent to an X1 region (X2 region), then there exists an X1 region (X2 region) P such that R and P have a common child.

Proof: 1) Let Q be a non-source region of G. Suppose Q has only one parent, namely, P. By Lemma 3.13 we can contract D by combining Q and P and obtain a new feasible region graph, which contracts to that D is minimal feasible. So Q has at least two parents.

Now, suppose Q has more than two parents. Let \( \tilde{C} = \{d_R \in \mathbb{F}^2; \ \text{R} \in D\} \) be a code of RG(D) constructed as in Theorem 3.18. There must be two parents of Q, say P1 and P2, such that dQ is an F-linear combination of dP1 and dP2. Then delete the link(s) between Q and all the other parents, and we obtain a feasible subgraph with code \( \tilde{C} \), which contradicts to that D is minimal feasible. Hence 1) holds.

2) Suppose P and Q are both X1 regions (or both X2 regions) and \( \tilde{C} \) be a code of RG(D) as in Theorem 3.18. Then \( d_P = d_Q = \alpha_1 \) (or \( d_P = d_Q = \alpha_2 \)). If P and Q are adjacent, by Corollary 3.12 D can be contracted by combining Q and P without changing its feasibility. Similarly, if P and Q have a common child R, then deleting the link between Q and R results in a subgraph RG(D)' of RG(D) such that \( \tilde{C} \) is still a code of RG(D)'. In both cases we derive contradictions and hence 2) holds.

3) Suppose P, Q ∈ D are two adjacent coding regions which have no common child. Let \( \tilde{C} \) be the code of RG(D) as in Theorem 3.18. We alter \( \tilde{C} \) by assigning the same global encoding kernel \( d_P = d_Q = \alpha_1 \) (or \( d_P = d_Q = \alpha_2 \)). If P and Q have a common child R, then deleting the link between Q and R results in a subgraph RG(D)' of RG(D) such that \( \tilde{C} \) is still a code of RG(D)'. In both cases we derive contradictions and hence 3) holds.

4) Suppose R is adjacent to an X1 region (X2 region) Q and has no common child with any X1 region (X2 region). Let \( \tilde{C} \) be the code of RG(D) as in Theorem 3.18. We alter \( \tilde{C} \) by letting \( d_R = \alpha_1 \) (or \( \alpha_2 \)), and keep the rest of global encoding kernels unchanged. Since R has no common child with any X1 region (X2 region), this assignment is still a code of RG(D) (Lemma 3.17). By Corollary 3.12 D can be contracted by
combining \( R \) and \( Q \) without changing the feasibility, which is a contradiction and hence 4) holds.

For the sake of convenience, we say that a region \( Q \) is an \( X_i \)-parent (or an \( X_i \)-child) of a region \( R \) if \( Q \) is an \( X_i \) region and a parent (or a child) of \( R \). The following corollary further shows some marvelous properties of the minimal feasible region graph.

**Corollary 5.4:** Let \( D \) be a minimal feasible region decomposition of \( G \). The following items hold.

1) An \( X_i \) region is either an \( X_i \) source region or an \( X_i \) sink region \((i \in \{1, 2\})\).

2) A coding region has at least two children which are sink regions.

3) If \( R \in D \) is a coding region such that no child of \( R \) is a coding region, then \( R \) has two children, say \( R_1 \) and \( R_2 \), such that \( R_1 \) is an \( X_i \) sink region and \( R_2 \) has an \( X_j \)-parent, \( i, j \in \{1, 2\}, j \neq i \).

**Proof:** 1) Let \( R \in D \) be an \( X_i \) region. If \( R \) is neither an \( X_i \) source region nor an \( X_i \) sink region, i.e., \( R \) contains neither \( X_i \) source link nor \( X_i \) sink link, then the parents of \( R \) are all \( X_i \) regions (Definition 3.14), which contradicts to 2) of Theorem 5.3.

2) Let \( R \) be a coding region. Then by 1) of Theorem 5.3, \( R \) has two parents, say \( P_1 \) and \( P_2 \). By 2) of Theorem 5.3, they are neither both \( X_1 \) regions nor both \( X_2 \) regions. We distinguish the discussion into three cases.

Case 1: \( P_1 \) is a coding region and \( P_2 \) is an \( X_1 \) region \((i \in \{1, 2\})\). First, consider \( P_1 \) and \( R \). By 3) of theorem 5.3, \( P_1 \) and \( R \) have a common child \( Q_1 \). If \( Q_1 \) is an \( X_j \) region for some \( j \in \{1, 2\} \), we halt. Else, if \( Q_1 \) is a coding region, then by 3) of theorem 5.3, \( R \) and \( Q_1 \) have a common child, say \( Q_2 \). Similarly, either \( Q_2 \) is an \( X_j \) region for some \( j \in \{1, 2\} \) or \( R \) and \( Q_2 \) have a common child \( Q_3 \). Since \( RG(D) \) is a finite graph, we can finally find an \( X_j \)-child \( Q_m \) of \( R \). By 1), \( Q_m \) is a sink region.

Next, consider \( P_2 \) and \( R \). Without loss of generality, we assume that \( P_2 \) is an \( X_1 \) region. By 4) of theorem 5.3, there exists an \( X_1 \) region \( P \) such that \( R \) and \( P \) have a common child \( W_1 \). If \( W_1 \) is an \( X_j \) region for some \( j \in \{1, 2\} \), we halt. Else, if \( W_1 \) is a coding region, then by 3) of theorem 5.3, \( R \) and \( W_1 \) have a common child \( W_2 \). Similarly, either \( W_2 \) is an \( X_j \) region for some \( j \in \{1, 2\} \) or \( R \) and \( W_2 \) have a common child \( W_3 \). Since \( RG(D) \) is a finite graph, we can finally find an \( X_j \)-child \( W_n \) of \( R \). By 1), \( W_n \) is a sink region. (Note that \( W_n \neq Q_m \), which can be seen by sequentially comparing the parents.)

Case 2: Both \( P_1 \) and \( P_2 \) are coding regions. Similar to case 1, we can find two child of \( R \) which are sink regions.

Case 3: \( P_1 \) is an \( X_1 \) region and \( P_2 \) is an \( X_2 \) region. Similar to case 1, we can find two child of \( R \) which are sink regions.

In all cases, we can find two child of \( R \) which are sink regions.

3) By 2), \( R \) has an \( X_i \)-child \( Q \) for some \( i \in \{1, 2\} \). Without loss of generality, we assume that \( Q \) is an \( X_1 \) region. By 4) of Theorem 5.3, there is a region \( R_2 \) which is a common child of \( R \) and an \( X_1 \) region. By 2) of Theorem 5.3, \( R_2 \) is not an \( X_1 \) region. Note that no child of \( R \) is a coding region. So \( R_2 \) is an \( X_2 \) region.

Now consider \( R \) and \( R_2 \). By 4) of Theorem 5.3, there is a region \( R_1 \) which is a common child of \( R \) and an \( X_2 \) region. By 2) of Theorem 5.3, \( R_1 \) is not an \( X_2 \) region. Note that no child of \( R \) is a coding region. \( R_1 \) is an \( X_1 \) region. By 1), \( R_1 \) and \( R_2 \) are two sink regions meeting our requirements.

**Theorem 5.5:** Let \( D \) be a minimal feasible region decomposition of \( G \) with \( n \) coding regions. Then \( n \leq \max\{1, N - 2\} \), i.e., \( n \leq 1 \) when \( N = 2 \) and \( n \leq N - 2 \) when \( N \geq 3 \), where \( N \) is the number of sinks of \( G \).

**Proof:** Let \( K \) be the number of sink regions of \( D \). Obviously, \( K \leq N \) since each sink region contains at least one sink link. Let \( J \) be the number of edges from a coding region to a sink region.

Suppose \( D \) has \( n \geq 2 \) coding regions, we prove \( n \leq N - 2 \) by counting \( J \) in two different ways. Firstly, note that \( RG(D) \) is acyclic, we index \( D \) according to the upstream-to-downstream order, i.e., \( \forall R, R' \in D, R < R' \) if \( R \) is a parent of \( R' \). Let \( P \) and \( Q \) be the two coding regions with the biggest indexes and \( P < Q \). We distinguish the following two cases to discuss.

Case 1: \( Q \) is a child of \( P \).

By 1) of theorem 5.3, the \( K \) sink regions of \( RG(D) \) have exactly \( 2K \) parents (not necessarily different). We declare that including these \( 2K \) parents, at least 2 of them are not coding regions. (Noticing the index, no child of \( Q \) is a coding region. By 3) of Corollary 5.4, we can find two children \( R_1 \) and \( R_2 \) of \( Q \), such that \( R_1 \) is an \( X_1 \) sink region and has an \( X_j \)-parent \((i \neq j) \).) Thus, we have \( J \leq 2K - 2 \).
On the other hand, by 2) of Corollary 5.4 except $Q$, the $n - 1$ coding regions have at least $2(n - 1)$ children which are sink regions. We declare $Q$ has three children which are sink regions. (Note that $P$ and $Q$ are adjacent, by 3) of Theorem 5.3, $P$ and $Q$ have a common child, say $R_3$. Since $P$ and $Q$ are the coding regions with the biggest indexes, $R_3$ could not be a coding region. By 1) of Corollary 5.4 $R_3$ is a sink region. By comparing the parent set, we have $R_3 \notin \{R_1, R_2\}$. Hence, we have $2(n - 1) + 3 \leq J$.

By the discussions above, we have $2n + 1 \leq 2K - 2$. Note that $K \leq N$ and $n$ is an integer, we have $n \leq N - 2$.

Case 2: $P$ and $Q$ are not adjacent.

By 1) of theorem 5.3 the $K$ sink regions of $RG(D)$ have exactly $2K$ children (not necessarily different). We declare that including these $2K$ parents, at least 4 of them are not coding regions. (By the index, no child of $P$ and/or $Q$ is a coding region. By 3) of Corollary 5.4 $P$ has two children, say $R_1, R_2$, such that $R_i$ is an $X_i$ sink region and each of them has an $X_i$-parent ($i \neq j$). Similarly, $Q$ also has two children $W_1, W_2$, such that $W_i$ is an $X_i$ sink region and each of them has an $X_i$-parent ($i \neq j$). Thus, we have $J \leq 2K - 4$.

On the other hand, by 2) of Corollary 5.4 $n$ coding regions have at least $2n$ children of sink regions. So $2n \leq J$. Thus, $2n \leq 2K - 4$, and we have $n \leq N - 2$.

By the above discussions, we see that if the network has 2 or more coding regions, it has at least 4 sinks (note that $N \geq n + 2$). So, if $N = 2, 3$, $RG(D)$ has at most 1 coding region. The result follows.

**Theorem 5.6:** There exist instances which achieve the bound $n = \max\{1, N - 2\}$ in Theorem 5.5.

**Proof:** Fig. 14 demonstrate the instances. Note that (a) has $n \geq 2$ coding regions and $N = n + 2$ sink regions and it is feasible since it has no singular region. We can verify that this graph satisfies the two conditions of Definition 5.2 (b) shows the case of $N = 2$, a minimal feasible region graph $RG(D)$ with one coding region.

Now we can prove the main result of this section.

**Proof of Theorem 5.7** Let $D$ be a minimal feasible region decomposition of $G$. By Lemma 3.10 $G$ has a network coding solution $C$ such that $e \in E$ is an encoding link only if $e$ is the head of a non-source region in $D$. By 1) of corollary 5.4 a non-source regions is either a coding region or a sink region. Note that the number of sink regions is at most $N$. By Theorem 5.5 when $N = 2$, $n \leq 1$ and when $N \geq 3$, $n \leq N - 2$, we have that the number of encoding links is at most $\max\{3, 2N - 2\}$.

Obviously, the networks with the minimal feasible region graphs of Fig. 14 can achieve this bound.

Algorithm 5 in Fig. 15 reduces the basic region graph $RG(D^{**})$ of $G$ to a minimal feasible region graph $RG(D_m)$. The correctness of this algorithm is obvious. For each non-source region $R_j \in D^{**}$, Algorithm 5 makes $|In[head(R_j)]|$ times verifications of (1) and (2) of Definition 5.2. Each time of the verification can be done by algorithm 2 and algorithm 3 with polynomial time. Note that $|In(R_j)| \leq |In(head(R_j))|$, we have that Algorithm 5 is also a polynomial time algorithm. After we get the minimal feasible region graph by Algorithm 5, an optimal solutions of 2-SMNC networks can be constructed by Algorithm 4 in polynomial time.

**VI. Bound of Field Size**

In this section, following a same technical line as [17], we derive a tight upper bound on the required field size for the 2-SMNC problem. The result amazingly shows it is not necessary to use a larger field for 2-SMNC network than for one multicast session with two single rate flows [17].

Let $RG(D)$ be a minimal feasible region graph of a 2-SMNC network $G$ having $n$ coding regions $Q_1, \cdots, Q_n$. We first define the associated graph of $RG(D)$.

**Definition 6.1:** The associated graph of $RG(D)$, $\Omega_D$ is an undirected graph with $n + 2$ vertices $X_1, X_2, Q_1, \cdots, Q_n$ and its edge set includes the following ones.

1) $(X_1, X_2)$. It is called the red edge of $\Omega_D$.
2) $(Q_i, Q_j)$, if $Q_i$ and $Q_j$ have a common child. It is called a blue edge of $\Omega_D$.
3) $(Q_i, X_j)$, if $Q_i$ have a common child with some $X_j$ region ($j = 1, 2$). It is called a green edge of $\Omega_D$.

**Example 6.2:** For the minimal feasible region graph in Fig. 15(a), its associated graph $\Omega_D$ is shown in Fig. 15(b).

The problem of constructing a code of $RG(D)$ can be translated into a vertex coloring problem on $\Omega_D$. First, we give a lemma.

**Lemma 6.3:** A field of size $q \geq \chi(\Omega_D) - 1$ is sufficient to achieve a linear solution of $RG(D)$, where $\chi(\Omega_D)$ is the chromatic number of $\Omega_D$.

**Proof:** Let $F = \{0, c_1 = 1, c_2, \cdots, c_{q-1}\}$ be a field with size $q \geq \chi(\Omega_D) - 1$. Let $\alpha_1 = (1, 0), \alpha_2 = (0, 1)$ and $\beta_j = (1, c_j), j = 1, \cdots, q - 1$. Let $\rho : V(\Omega_D) \rightarrow \{\alpha_1, \alpha_2, \beta_1, \cdots, \beta_{q-1}\}$ be a $q + 1$-coloring of $\Omega_D$ such that $\rho(X_i) = \alpha_i$ for $i \in \{1, 2\}$. Let $C = \{d_R = \rho(R); R \in D\}$.
Let $D$ be a minimal feasible region decomposition of $G$ with $n$ coding regions and $K$ sink regions. Let $J$ be the number of edges of the associated graph, $\Omega_D$, and let $k = \chi(\Omega_D)$ be the chromatic number of $\Omega_D$. We count $J$ in two different ways.

By Lemma 6.3 and 6.5 each vertex of $\Omega_D$ has degree at least 2 and at least $k$ vertices of degree at least $k - 1$. We have $[k(k - 1) + 2(n + 2 - k)]/2 \leq J$.

On the other hand, by 1) of Theorem 6.3, a region is a common child of two regions if and only if it is a non-source region. By 1) of Corollary 5.4, a non-source region is either a coding region or a sink region. So there are $n + K$ regions which are common children of two regions. Thus, $J \leq n + K$.

Combining the two inequalities, we have:

$$[k(k - 1) + 2(n + 2 - k)]/2 \leq n + K.$$ 

Noting that $K \leq N$, and solving the inequality, we have,

$$k \leq \sqrt{2N - 7/4} + 3/2.$$ 

By Lemma 6.3, a field with size no larger than $\sqrt{2N - 7/4} + 1/2$ is sufficient for a linear solution.

In the following, we show the tightness of the bound. To do this, we first give a lemma.

Lemma 6.8: Let $F$ be a field of size $q$ and $\gamma_1, \cdots, \gamma_k$ are $k$ vectors in $F^2$ such that any two of them are linearly independent. Then $k \leq q + 1$.

Proof: Suppose $k \geq q + 2$. Suppose $\gamma_i = (a_i, b_i), i = 1, \cdots, k$. We have the following two cases.

Case 1: There exist $i_1 \neq i_2$ such that $a_{i_1} = a_{i_2} = 0$. In this case, $\gamma_{i_1}$ and $\gamma_{i_2}$ are linearly dependent.

Case 2: There exists at most one vector whose first component is zero. Without loss of generality, assume $a_i \neq 0, i = 1, \cdots, k - 1$. We have $\gamma_i = a_i(1, b_i/a_i), i = 1, \cdots, k - 1$. Since $k \geq q + 2$, there exist $i_1 \neq i_2$ such that $b_{i_1}/a_{i_1} = b_{i_2}/a_{i_2}$. Thus $\gamma_{i_1}$ and $\gamma_{i_2}$ are linearly dependent.

In both cases, we can find two linearly dependent vectors. This contradiction yields the result.

Theorem 6.9: The bound in Theorem 6.7 is tight.

Proof: We construct a minimal feasible region graph by adding some $X_2$ sink regions to Fig.14(a), as follows.

1) For $j \in \{2, \cdots, n - 1\}$, add an $X_2$ sink region as a common child of $Q_j$ and the $X_1$ source region;
2) For $Q_i$ and $Q_j$, which are not adjacent, add an $X_2$ sink region as a common child of $Q_i$ and $Q_j$.

Denote the resulted region graph by $RG(D)$ and the corresponding region set by $D$. One can check that $RG(D)$ is still a minimal feasible region graph. We now prove that the field size for any linear code of $RG(D)$ is at least $\sqrt{2N - 7/4} + 1/2$.

Note that in Fig.14(a), there are $n - 2$ coding regions not adjacent to $Q_1$, $n - 3$ coding regions not adjacent to each $Q_j, j = 2, \cdots, n - 1$ and $n - 2$ coding regions not adjacent to $Q_n$. Thus, we have added $[2(n - 2) + (n - 2) + (n^2 - 3n + 2)/2$ sink regions in step 2) and the total number of sink regions of $RG(D)$ is $N = (n + 2)(n - 2) + (n^2 - 3n + 2)/2 = (n^2 + n + 2)/2$. So, we have $\sqrt{2N - 7/4} + 1/2 = n + 1$.

Second, if $C = \{d_R; R \in D\}$ is a linear code of $RG(D)$ over $F$, then we declare that $d_{Q_1}$ and $d_{Q_2}$ are linear independent for any coding regions $Q_i \neq Q_j$. In fact, by the...
construction of $RG(D)$, $Q_i$ and $Q_j$ have a common $X_2$ child. Thus, if $d_{Q_i}$ and $d_{Q_j}$ are linear dependent, then there exist $k_1, k_2 \in F$ such that $k_1 \cdot d_{Q_i} = k_2 \cdot d_{Q_j} = (0, 1)$. Again by the construction of $RG(D)$, one can have $d_{Q_i} = k_2 \cdot (0, 1)$ for each $Q_i$ downstream $Q_I$ and/or $Q_j$, which is impossible because an $X_1$ sink region exists. By adding $(1, 0)$ and $(0, 1)$, we have totally $n + 2$ mutually linearly independent vectors in a solution. Note that $n + 2 \geq q + 1$ (Lemma 6.8) and we have $\sqrt{2N - 7/4 + 1/2} = n + 1 \leq q$.■

Fig. 17. A minimal feasible region graph with $n = 4$ coding regions and $N = (n^2 + n + 2)/2 = 11$ sink regions.

Example 6.10: Fig[17] plots a region graph $RG(D)$ constructed as in the proof of Theorem 6.9. $RG(D)$ has $n = 4$ coding regions and $N = (n^2 + n + 2)/2 = 11$ sink regions. By Theorem 6.9 $F_5$ ensure a linear solution of $RG(D)$.

VII. CONCLUSIONS AND DISCUSSIONS

We investigated the encoding complexity of the 2-SMNC problem by proposing a region decomposition method. It showed that when the network is decomposed into mutually disjointed regions, a network coding solution can be easily obtained from some simple labeling operations on the region graph and by decentralized assigning encoding kernels. All the processes of the region decomposition, the region labeling, and the code construction can be done in time $O(E)$.

We further reduced a feasible region graph into a minimal feasible one by deleting links and/or combining nodes of the region graph. It showed that the minimal feasible region graph have some marvelous local properties, from which we derived bounds on the encoding links and on the required field size.

There are some interesting issues need further investigate. For example, given a 2-SMNC network, one may get different minimal feasible region decompositions from different feasible region decompositions. We do not known if these minimal feasible region graphs are with a same topology but we did not ever find a contra example. Another valuable topic is how to apply the region decomposition method to an acyclic network with $n > 2$ multicast sessions. In this case, region decomposition can be performed similarly and the basic region decomposition can also be obtained. However, the labeling process becomes very complicated and new opinions need be introduced in order to get some valuable results. Beyond these issues, region decomposition on cyclic networks, vector linear codes for 2-SMNC problems are also interesting topics worthy of consideration.

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