The $\beta$-function over curved space-time under $\zeta$-function regularization

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Abstract

This paper generalizes the Connes-Marcolli renormalization bundle to scalar field theories over a curved space-time background, specifically looking at $\zeta$-function regularization. It further extends the idea of renormalization mass scale from a scalar change of metric to a conformal change of metric. In this context, it becomes useful to think of the renormalization mass scale as a complex 1-density over the background manifold.

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1 Introduction

Quantum field theory(QFT) is a well understood phenomenon under the assumptions that the space-time of this universe is flat. Unfortunately, this assumption is patently wrong. The existence of gravity adds curvature to the universe. QFT can adjust for this curvature to some extent by working in coordinate patches over the space-time manifold. Attempts to find a globally well defined QFT over a curved background space have yet to be successful.

This paper takes a global view of renormalizing a scalar QFT over a curved background space-time manifold. The background manifold is assumed to be compact and closed, and the metric on it is Riemannian,
and not Lorentzian. Under these non-physical assumptions, I extend the renormalization bundle created by Connes and Marcolli in [6] to a curved space-time renormalization bundle that contains the background manifold as a base. The Connes Marcolli renormalization bundle developed in [6], is a $G$ principal bundle, where $G$ is the affine group scheme associated to the Hopf algebra of Feynman integrals determined by the divergence structure of a QFT. The base space of this bundle in the flat space context is simply an infinitesimal disk parameterized by a regularization parameter. Specifically, the regularization parameter in [6] is given by the complex dimension parameter found in dimensional regularization. Based on this data, they find that the $\beta$-function for a scalar field theory under dimensional regularization is uniquely determined by the connection determined by a certain class of sections of the bundle. This bundle lends itself to finding the $\beta$-function for other regularization schemes as well. In particular, space parameterized by the regularization parameter for $\zeta$-function regularization is isomorphic to the the space parameterized by dimensional regularization. Since $\zeta$-function regularization is the preferred regularization method for a curved background space [12], one can generalize the Connes Marcolli renormalization bundle to a bundle that sits over a curved (Riemannian) background manifold and find a similarly defined global $\beta$-function for $\zeta$-function regularization. In fact, I take the generalization one step further, and develop a renormalization bundle that allows for conformal changes of metric, and show the existence of a global $\beta$-function on the entire manifold in this context.

This process is done in several steps. First I write Feynman rules over a curved space-time in terms of operators and kernels. This makes it easy to regularize the integrals using $\zeta$-function regularization. After regularization, I show

**Proposition 1.1.** The regularized Feynman rules for a graph $\Gamma$ is a Schwartz kernel that can be written as a somewhere convergent Laurent polynomial with finite poles at 0 and distribution valued coefficients.

The next step is to use this language of operators to identify regularized Feynman rules with sections of the renormalization bundle over curved space time.

**Theorem 1.2.** The regularized Feynman rules of a QFT defined by $\mathcal{L}$ are a linear map from the symmetric algebra on the space of test functions $C^\infty(M)$ to $G(\mathbb{C}\{\{z\}\})$ the $\mathbb{C}\{\{z\}\}$ points of the affine group scheme associated to the Hopf algebra $\mathcal{H}$.

Thus the Connes Marcolli renormalization bundle over a flat space-time background extends naturally to a curved space-time renormalization bundle. The connection they associate to the $\beta$-function of a Lagrangian now corresponds to a section over the background manifold that is uniquely defined by the $\beta$-function for the Lagrangian globally defined over the manifold as a function of the curvature.

**Theorem 1.3.** There is a trivial connection $\omega(x)$ defined on the fibers $P \rightarrow B$ over $M$ in the renormalization bundle $P_M \rightarrow B_M$. This connection is uniquely defined on the pullbacks $\omega_{\gamma(z,t)}(x)$ by the $\beta$-function $\beta(\gamma(z,x))$ if and only if $\gamma(z,t,x)$ satisfies certain locality conditions. If the section $\gamma(z,x)$ is globally defined over $M$, and satisfies these conditions everywhere, then $\omega_{\gamma(z,t)}(x)$ is globally defined over $M$ by $\beta(\gamma(x,z))$.

Finally, I allow the base $B_M$, which contains the renormalization mass to be non-trivial over the background space $M$. Defining $B_M$ trivially over $M$ is equivalent to only allowing for constant scalings of the metric on the manifold. Allowing for a non-trivial bundle over $M$ allows for conformal scalings of the metric. Little is known about renormalization under conformal scaling. Let $\Delta^\ast$ be the infinitesimal punctured complex disk of the regularization parameter, and $\mathbb{C}(1)$ be the line bundle of complex 1-densities over $M$.

**Theorem 1.4.** The conformal renormalization bundle is a $\mathbb{C}^\times$ equivariant $G$ principal bundle

$$
\mathcal{P} := G \times B_M \times_{GL_n(\mathbb{R})} \text{Frame}(M) \to \Delta^\ast \times \mathbb{C}(1) \times_{GL_n\mathbb{C}} M.
$$

which is diffeomorphic, but not naturally, to the renormalization bundle over curved space-time $M$.

The second section of this paper introduces the Lagrangian over a curved background space, the Feynman rules in this setting, and the regulation scheme, $\zeta$-function regulation. Section three introduces the Hopf
algebra of Feynman graphs as developed by Connes and Kreimer in [3] and shows that the geometrization
of BPHZ renormalization developed by the same in [4] can be extended to a curved background space-time.
Section four defines a $\beta$-function as it is used in the physics literature, and then geometrically develops it
as pullbacks of a particular section of a renormalization bundle over curved-space time. Finally, section five
extends this analysis to allow for conformal changes of metric.

2 Quantum field theory in curved space-time

In this section, I develop the Feynman rules for a renormalizable scalar field theory with valence 3 interactions
on a closed compact Riemannian manifold, $(M,g)$, with metric tensor $g$. Normally, Feynman diagrams are
calculated on a Lorentzian metric. There are many good references for translating between the two metrics,
such as [20] and [16]. The Lorentzian metric is mapped to an Euclidean metric by sending time to imaginary
time, $t \mapsto it$, in a process called Wick rotation. This is a local map. A global means of translating between
the two metrics is yet to be discovered.

The Lagrangian density for the type of QFT I study in this paper is of the form

$$L_M = \frac{1}{2} \phi(\Delta - m^2) \phi + \lambda \phi^3, \quad (1)$$

where $m$ is the mass of the parameter, and $\lambda$ is the coupling constant and $\Delta$ the Laplacian on the manifold.
Let $\Delta_M = \Delta - m^2$. The Lagrangian on the manifold is then given by

$$L_M = \int_M L \sqrt{|g(x)|} d^nx,$$

where $|g| = \det g_{ij}$. The Lagrangian defines a probability amplitude for the field $\phi$, $e^{iL_M}$. A measure on
the space of fields is given by $d\mu = e^{iL_M} \prod_x d\phi(x) [9]$. Integrating against this measure gives the equations
of motion that dictate particle interactions over curved space-time. The probability amplitude of a certain
type of interaction can be written as a Feynman integral, and graphically depicted as a Feynman diagram.
The Feynman rules translates between these two representations. This section develops Feynman rules and
diagrams over curved space time.

**Definition 1.** A Feynman diagram is an abstract representation of an interaction of several fields. It is drawn
as a connected, not necessarily planar, graph with possibly differently labeled edges. It is a representative
element of the equivalence class of planar embeddings of connected non-planar graphs. The types of edges,
vertices, and the permitted valences are determined by the Lagrangian density of the theory in the following
way:

1. The edges of a diagram are labeled by the different fields in the Lagrangian. For this Lagrangian, there
is only one type of edge.

2. The composition of monomial summands with degree $> 2$ in the Lagrangian density correspond to
permissible valences and composition of internal vertices of the Feynman diagrams. The $\phi^3$ term
means that all internal vertices have valence 3.

3. Vertices of valence one are called external vertices.

**Remark 1.** This is a different definition of the Feynman diagrams than what is presented in standard physics
literature for QFTs over flat space time. This is because the Feynman rules in flat space involve a Fourier
transform from configuration space to momentum space. Since Fourier transforms on general manifolds
is a local phenomenon, in this context, I use the Feynman diagrams associated to the Feynman rules in
configuration space.

The building blocks of these Feynman diagrams are called one particle irreducible (1PI) diagrams.
**Definition 2.** A one particle irreducible graph is a connected Feynman graph such that the removal of any internal edge still results in a connected graph.

All Feynman diagrams associated to a theory can be constructed by gluing together 1PI diagrams along an exterior edge.

### 2.1 Feynman rules in curved space-time

The Feynman rules are an algorithm that map Feynman diagrams to integral operators that act on a space of functions associated to the external legs of the graph, called Feynman integrals. Details about the Feynman rules and Feynman integrals over a flat space-time with a Lorentzian metric can be found in textbooks such as [17], [21] and [22]. Unlike in flat space-time, the Feynman rules over a compact closed Riemannian manifold \((M, g)\), cannot be written in phase-space, as there is no global definition of Fourier transform. The Feynman rules are written in configuration space. The properties of a scalar field theory such as the conditions for renormalizability of a QFT, that the degree of divergence of a Feynman diagram does not increase with the complexity of a graph, is defined by local criteria. Specifically, on a Riemannian manifold, as in flat space-time, a renormalizable QFT has space-time dimension 6, and the divergent 1PI Feynman diagrams only have 2 or 3 external legs. For details about degrees of divergence of Feynman diagrams, see [13] section 8.1 or [23].

Working in configuration space, a Feynman integral is an integral operator that can be written as a generalized convolution product of Green’s functions associated to the Laplacian on the manifold \(\Delta_M\). As in momentum space on a flat background, this integral operator is not well defined and requires regularization. This is discussed below. The regularized Feynman integral acts on the symmetric algebra \(S(E) = \oplus_n S^n(E)\) for \(E = C^\infty(M)\), the external vertex data. Each external vertex of a Feynman diagram is assigned a function \(f \in E\). If a Feynman diagram has \(n\) external edges, it acts on \(S^n(E)\). Each type of edge in a QFT is associated to a type of Green’s function, or propagator, \(G_M(x, y)\) where \(x\) and \(y\) are the endpoints of the edge. Each type of vertex, determined by the valence, \(n_v\) and the types of edges meeting at it is associated to a coupling constant \(\lambda_v\). In 6 dimensional \(\phi^3\) theory, there is only one type of propagator and one coupling constant. The coupling constant, as show in [11] is \(\lambda\).

The Green’s kernel associated to \(\Delta_M\) is a distribution on \((M \times M)\) defined by the equation

\[
\Delta_M G_M(x, y) = \delta(x, y),
\]

where \(\delta(x, y)\) is the Dirac delta function. Since \(\Delta_M - m^2\) is a negative (semi)-definite elliptic operator acting on \(E = C^\infty(M)\), for a compact manifold \(M\), it has a discrete spectrum that can be ordered

\[
0 \geq \lambda_1 \geq \ldots \lambda_i \geq \lambda_{i+1} \ldots
\]

counting multiplicity, and an orthonormal set of eigen-functions \(\{\phi_i\}\) such that

\[
\int_M \phi_i(x) \phi_j(x) \, dvol(x) = \delta_{ij}
\]

where \(\delta_{ij}\) is the Kronecker delta function. Write \(E = E_0 \oplus E_-\), where \(E_0 = \ker(\Delta_M)\) and \(E_- = \oplus_i E_i\) the direct sum of the negative eigenspaces of \(\Delta_M\). The Green’s function \(G_M(x, y)\) is the inverse of \(\Delta_M|_{E_-}\) and can be written

\[
G_M(x, y) = \sum_{i=1}^{\infty} \frac{\phi_i(x) \phi_i(y)}{\lambda_i}.
\]

Given this notation, the Feynman rules on a general background manifold are:
1. If a graph $\Gamma$ has $I$ edges, write down the $I$ fold product of propagators, of various types according to the type of edges,

$$\prod_1^I G_{M,i}(x_i, y_i)$$

where $x_i$ and $y_i \in M$ are the endpoints of each edge.

2. For each internal vertex, $v_i$ of valence $n_{v_i}$, define a measure on $M \times n_{v_i}$ locally

$$\mu_i = -i\lambda_{v_i} \delta(x_1, x_2) \ldots \delta(x_{n_{v_i}} - 1, x_{n_{v_i}})$$

where the $x_i$ are the endpoints of the edges incident on the vertex in question in the graph, $\lambda_{v_i}$ is the coupling constant associated to the vertex of type $v_i$, and $\delta(x_i, x_j)$ is the Dirac delta function.

3. Integrate the product of propagators from above against this measure

$$\int_{(M)^{\Sigma_{n_{v_i}}}} \prod_1^I G_{i}(x_i, y_i) \prod_i^V \mu_i .$$

(3)

4. Divide by the symmetry factors of the graph.

Remark 2. The integration in equation (3) shows that this is a type of convolution product of distributions. Given such a convolution product of the correct type of kernels, one can determine the corresponding graph. Notice that this convolution product does not correspond to a composition of operators. In fact, this convolution product is not commutative. The order in which the convolution product is taken determines the shape of the graph.

The measure defined in equation (2) introduces factors of $\text{Tr}(\Delta_M)$ in the Feynman integrals. Since $\Delta_M$ is not trace class, this means that the Feynman integrals are frequently not well defined integral operators. In order to make sense of the probability amplitudes they are supposed to represent, the integrals need to be regularized, or written in terms of an extra parameter such that the integrals are defined away from a fixed limit. The regularization scheme I present in this paper is called $\zeta$-function regularization. It is preferable because it arises naturally on a manifold and avoids the ambiguities presented by other regularization methods, such as dimensional regularization [12]. However, the arguments presented here hold for many other regularization schemes.

### 2.2 Regularized Feynman Integrals

To regularize a Feynman integral via $\zeta$-function regularization, replace the Laplacian $\Delta_M$ with $\Delta_M^{1+z}$, and write the corresponding Green’s function

$$G_M^{1+z}(x, y) = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(y)}{\lambda_i^{1+z}} ,$$

for a complex valued parameter $z$. For $M$, a 6-dimensional manifold, $\Delta_M^{1+z}$ is trace class, with $G_M^{1+z}(x, y)$ meromorphic with simple poles at $1 + z = k - 3$, where $k \in \mathbb{Z}_{\geq 0}$ [19].

The regularized Feynman integrals now contain factors of $\text{Tr}\Delta_M^{1+z}$, which has simple poles at $z \in \left\{ \frac{n-k-1}{2}, 1 \notin \mathbb{Z} \right\}$ for $n = \dim M$, and all $k \in \mathbb{Z}_{\geq 0}$. The residues of $\text{Tr}\Delta_M^{1+z}$ have been calculated by [23], [11], and [14] using the asymptotic expansion of the kernel of the heat operator $e^{t\Delta_M}$. Specifically, away from $z = 0$, the regularized Feynman rules assign to each Feynman diagram $\Gamma$ a one parameter family of integral operators with kernel

$$K^{reg}_\Gamma(x_1, \ldots, x_E) = \int_{M^E} \prod_1^I G_M^{1+z}(x_{i1}, x_{i2}) \text{dvol}(x_1, \ldots, x_E)$$
where $E$ is the number of external vertices of $\Gamma$, $V$ the number of internal vertices, $I$ the number of edges and $i_1, i_2 \in \{1 \ldots V + E\}$.

**Proposition 2.1.** The regularized Feynman rules for a graph $\Gamma$ is a Schwartz kernel $K^{reg}_\Gamma \in \mathcal{D}(M^{rE})\{\{r\}\}$ that can be written as a somewhere convergent Laurent polynomial with finite poles at 0 and distribution valued coefficients.

**Proof.** The regularized Feynman integral for a graph $\Gamma$, is a convolution product of $G^{1+z}(x,y)$ taken over the internal vertices of $\Gamma$ for a small complex parameter $z$ by Seeley’s theorem [19]. Along the diagonal, $G^{1+z}(x,x)$ is meromorphic in $z$

$$G^{1+z}(x,x) = \sum_{-1}^{\infty} g_i(x) z^i,$$

where $g_i(x)$ is a smooth function over $M$. Away from the diagonal, $G^{1+z}(x,y)$ is entire in $z$,

$$G^{1+z}(x,y) = \sum_{0}^{\infty} h_i(x,y) z^i,$$

where $h_i(x,y)$ is a smooth function over $M$. Therefore, I can write $G^{1+z}(x,y) = \sum_{-1}^{\infty} f_i(x,y) z^i$, where $f_i(x,y)$ are distribution valued coefficients.

The Schwartz kernel associated to a Feynman integral $K^{reg}_\Gamma$ is a convolution product of some number of these Greens functions. For a fixed $r \neq 0$, the kernel of $K^{reg}_\Gamma(z) \in \mathcal{D}(M^{xE})$, where $E$ is the number of external legs of $\Gamma$, is a smooth well defined quantity. It can be written as a Laurent polynomial with a finite number of poles, since each Green’s function contributes at most one pole.

**Corollary 2.2.** There is an operator, $A_{\Gamma}(z)$ associated to $K_{\Gamma}(z)$ by the Schwartz kernel theorem can be written as

$$A_{\Gamma}(z) : C^\infty(M^{xE}) \to \mathbb{C}\{\{z\}\},$$

a linear map from the external leg data to the space of somewhere convergent Laurent polynomial with finite order poles at 0 and $\mathbb{C}$ coefficients.

**Proof.** By the Schwartz kernel theorem, there is an operator, $A_{\Gamma}(z)$, associated to $K^{reg}_{\Gamma}(z)$ that defines a linear map

$$A_{\Gamma}(z) : C^\infty(M^{x+i}) \to C^\infty(M^{x+j}) ,$$

where $i + j = E_{\Gamma}$. This operator is defined on the Feynman rules on the graph $\Gamma$ as

$$A^{reg}(z)f(x_1,\ldots,x_i) = \int_{M^0} K^{reg}(z)(f(x_1,…,x_i)) \ dvol(x_1,…,x_i).$$

The inner product

$$\langle A_{\Gamma}(z)f, g \rangle = \int_{M^E} K^{reg}(r)(x_1,\ldots,x_i,y_1\ldots,y_j)g(y_1,…,y_j)f(x_1,…,x_i) \ dvol(x_1,…,x_j,y_1\ldots,y_j)$$

for $f \in S^i(E)$ and $g \in S^j(E)$ is a Laurent polynomial in $z$ with poles of finite degree that is somewhere convergent away from $z = 0$; $\langle A_{\Gamma}f, g \rangle \in \mathbb{C}\{\{r\}\}$.

By Proposition 2.1, one can view the regularized Feynman rules as maps from Feynman diagrams to $\mathcal{D}(M^{xE})\{\{z\}\}$. This is the view taken in the physics literature, where Feynman integrals are referred to as Green’s functions. In this paper, I am more interested in the corresponding integral operator given by Corollary 2.2.
Theorem 2.3. Let $A_{M, \Gamma}$ be the operator associated to the Feynman diagram $\Gamma$. Then, for $f, g \in C^\infty(M)$, $\langle A_{M, \Gamma} f, g \rangle$ depends only on the metric $g$ of $M$, and the combinatorics of the graph $\Gamma$.

Proof. Let $E_\Gamma$ be the number of external vertices of the graph $\Gamma$, and $V$ the total number of vertices. It is sufficient to regularize $\langle A_{M, \Gamma} \phi_e, \phi_f \rangle$ where $\phi_e = \prod_{c=1}^l \phi_e(x_c)$. $\phi_f = \prod_{e \in \tilde{E}} \phi_f(x_e)$ are products of eigenfunctions of $\Delta_M$. These functions correspond to the external data at vertex $e_i$ or $f_i$. Write

$$\langle A_{M, \Gamma} \phi_e, \phi_f \rangle = \int_{M^V} \phi_e \phi_f \prod_{i=1}^l \sum_{k=0}^\infty \frac{\phi_k(x_{i_1}) \phi_k(x_{i_2})}{\lambda_k^s} \text{dvol}(x_1, \ldots, x_V).$$

This can be regularized using the Mellin transform

$$\frac{1}{\lambda^s} = \frac{1}{\Gamma(1+s)} \int_0^\infty t^s e^{-t\lambda} dt$$

then

$$\langle A_{M, \Gamma}(r) \phi_e, \phi_f \rangle = \frac{1}{\Gamma(1+r)^l} \int_{M^V} \phi_e \phi_f \prod_{k_1 \ldots k_l=0}^\infty \phi_k(x_{i_1}) \phi_k(x_{i_2}) \int_0^\infty e^{-\sum_i t_i \lambda_k} \prod_{i=1}^l t_i^r dt_i \text{dvol}(x_1, \ldots, x_V).$$

Conservation of momentum is applied at each trivalent vertex by the relation

$$\int \phi_i(y) \phi_j(y) \phi_k(y) dy = \int \phi_i(y) \sum_i a^{jk}_i \phi_i(y) dy = \sum_{j,k} a^{jk}_i.$$

Since the quantity $a^{jk}_i$ is symmetric on $i, j,$ and $k$ I write it instead as $a(i, j, k)$. The quantity $a(i, j, k)$ is tensorial, and depends only on the metric of $M$. Define a function $\varepsilon'(v)$ that gives the set of edges incident on the vertex $v$. Applying conservation of momentum gives

$$\langle A_{M, \Gamma}(r) \phi_e, \phi_f \rangle = \frac{1}{\Gamma(1+r)^l} \int_{M^V} \int_0^\infty \prod_{k_1 \ldots k_l=0}^\infty e^{-\sum_i t_i \lambda_k} \prod_{i=1}^l t_i^r dt_i \prod_{v=1}^{V-E_\Gamma} a(\varepsilon'(v)) \text{dvol}(x_1, \ldots, x_V).$$

Working out the $a(\varepsilon'(v))$ re-indexes the eigenvalues in terms of the graphs loop number, $L$ and loop indices, $l_i$. From here, one can apply the Schwinger trick, and carry out calculations in a manner similar to [13], Chapter 6. The operator $A_{M, \Gamma}(r)$ is a convolution product of the $\Delta_M^{-1}$, twisted by the quantities $a(\varepsilon'(v))$. Since the trace of $\Delta_M^{-1}$ and $a(\varepsilon'(v))$ depend only on the metric of $M$, so does $\langle A_{M, \Gamma} \phi_e, \phi_f \rangle$.

Remark 3. The functions $a(i, j, k)$ are implicitly functions of the metric, $g(x)$, on $M$. Therefore the regularized operators associated to Feynman integrals depend on $g(x)$. If the metric is constant, there regularized operator is independent of the position over $M$. In the special case where $M$ is a flat manifold, then

$$a(i, j, k) = \begin{cases} 1 & \text{if } i + j + k = 0, \\ 0 & \text{else}. \end{cases}$$

In this case, $a(i, j, k) = \delta(i + j + k)$, imposes conservation of momentum at each vertex.

Corollary 2.4. There is a graph polynomial associated to each Feynman graph on $M$. The terms of the polynomial are similar to the terms found in [13], chapter 6. The coefficients of the terms, however, are functions of $a(i, j, k)$. 

7
2.3 Renormalization

Regularization of the Feynman integrals gives a one parameter family of well defined operators, parameterized by the complex regularization parameter $z$. However, these operators do not represent the particle interactions for the original Lagrangian. To interpret the results of the original theory, renormalize the regularized theory to extract finite values at the limit $z = 0$. The appropriate renormalization scheme for $\zeta$-function regularization is BPHZ renormalization, an algorithm developed by Bogoliubov, Hepp, Parasiuk and Zimmerman in the 1950s and 60s that iteratively subtracts off subdivergences associated to subgraphs. In [3], Connes and Kreimer show that the BPHZ algorithm’s process of subtracting off subdivergences off of a Feynman diagram defines a coproduct structure on Feynman graphs which is useful in defining a Hopf algebra of Feynman diagrams. Furthermore, they show that the results of BPHZ decomposition into regularized and counterterm theories corresponds to Birkhoff decompositions of loops into the Lie group associated to the Hopf algebra of Feynman diagrams. While Connes and Kreimer work with dimensional regularization, this technique can be applied to $\zeta$-function regularization because the spaces parameterized by the regulators are same. The development of the Hopf algebra and Birkhoff decomposition are the subjects of the remainder of this section and the next. For a discussion of BPHZ renormalization, see [13] Chapter 8, section 2.

For the BPHZ renormalization procedure to work on a QFT, it has to be renormalizable. This condition means that the degree of divergence of the theory are controlled, in that the degree of divergent of divergence of a Feynman diagram does not increase with the complexity of a graph. This is equivalent to the statement that the space-time dimension of the QFT associated to the Lagrangian in (1) is 6, and that the 1PI Feynman diagrams only have 2 or 3 external legs. For details about degrees of divergence of Feynman diagrams, see [13] section 8.1 or [23].

BPHZ renormalization identifies subdivergences in a Feynman diagram, $\Gamma$, associated to divergent Feynman diagrams embedded inside it, and iteratively subtracts off the subdivergences. The graphs associated to these subdivergences, $\gamma$ are called subgraphs of $\Gamma$.

**Definition 3.** Let $V(\Gamma)$ be the set of vertices of a graph $\Gamma$, $I(\Gamma)$ the set of internal edges and $E(\Gamma)$, the set of external edges. The Feynman diagram $\gamma$ is an admissible subgraph of a 1PI Feynman diagram $\Gamma$ if and only if the following conditions hold:

1. The Feynman diagram $\gamma$ is a divergent 1PI Feynman diagram, or a disjoint union of such diagrams. If $\gamma$ is connected, it is a connected admissible subgraph, otherwise it is a disconnected admissible subgraph.
   I use admissible subgraph to mean both unless otherwise specified.

2. Let $\gamma' = \gamma \setminus E(\gamma)$ be the graph of $\gamma$ without its external edges. There is an embedding $i : \gamma' \hookrightarrow \Gamma$ such that the types of each internal edge and the valence of each vertex of $\gamma$ are preserved.

3. Let $f_\gamma$ be the set of legs (internal and external) meeting the vertex $v \in V(\gamma)$. Then $f_\gamma \subseteq I(\Gamma) \cup E(\Gamma)$ has the same number of each type of leg as $f_\gamma \subseteq I(\gamma) \cup E(\gamma)$.

The last condition ensures that the external leg conditions are preserved under the embedding. Finally, I need a definition of a contracted graph to represent the divergences that remain after the subtraction of the subdivergences.

**Definition 4.** Let $\gamma$ be a disconnected admissible subgraph of $\Gamma$ consisting of the connected components $\gamma_1 \ldots \gamma_n$. A contracted graph $\Gamma//\gamma$, is the Feynman graph derived by replacing each connected component $i(\gamma_j)$, with a vertex $v_{\gamma_j} \in V(\Gamma//\gamma)$.

This subgraph and contracted graph structure gives rise to a Hopf algebra structure on the Feynman diagrams.
3 The Hopf algebra view of renormalization

3.1 The Hopf algebra

A Hopf algebra can be built out of the Feynman diagrams by assigning variables \(x_\Gamma\) to each 1PI graph \(\Gamma\) and considering the polynomial algebra on these variables \(H = \mathbb{C}\{x_\Gamma | \Gamma \text{ is 1PI} \}\). This Hopf algebra is constructed in [3]. The product of two variables in this algebra

\[
m(x_\Gamma_1 \otimes x_\Gamma_2) = x_\Gamma_1 x_\Gamma_2
\]

corresponds to the disjoint union of graphs, and the unit is given by the empty graph, \(1_H = x_\emptyset\). The coproduct of this Hopf algebra is given by the subgraph and contracted graph structure of the Feynman diagrams

\[
\Delta x_\Gamma = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum_{\gamma \subset \Gamma} x_\gamma \otimes x_{\Gamma/\gamma}
\]

where the sum is taken over all proper admissible subgraphs of \(\Gamma\). The kernel of the co-unit is generated by all \(x_\Gamma\) such that \(\Gamma\) is non-empty. The antipode is defined to satisfy the antipode condition for Hopf algebras

\[
S : H \to H, \quad x_\Gamma \mapsto -x_\Gamma - \sum_{\gamma \subset \Gamma} m(S(x_\gamma) \otimes x_{\Gamma/\gamma})
\]

This is a bigraded Hopf algebra, with one grading given by loop number and the other by insertion number. Details on the two grading structures are given in [3] and [2]. This Hopf algebra is associative, co-associative and commutative, but not co-commutative.

In general, Hopf algebras can be interpreted as a ring of functions on a group. Since the spectrum of a commutative ring is an affine space, the group in question is affine group scheme, \(G = \text{Spec } H\). The group laws on the Lie group \(G\) are covariantly defined by the Hopf algebra properties

\[
(id \otimes \Delta) \Delta = (\Delta \otimes id) \Delta \quad \leftrightarrow \quad \text{multiplication}
\]

\[
(id \otimes \varepsilon) \Delta = id \quad \leftrightarrow \quad \text{identity}
\]

\[
m(S \otimes id) \Delta = \varepsilon \eta \quad \leftrightarrow \quad \text{inverse}
\]

The group \(G\) can also be viewed as a functor from a \(\mathbb{C}\) algebra \(A\) to \(G(A) = \text{Hom}_{\text{alg}}(H, A)\). The affine group scheme \(G\) is developed in detail in [5]. The last condition above means that if \(\gamma \in G(A)\), and \(x \in H\), then \(\gamma^{-1}(x) = S(\gamma(x)) = \gamma(S(x))\).

The Lie algebra \(g\) associated to \(G\) is generated by the algebra homomorphisms given by the Kroniker \(\delta\) functions on the generators of \(H\). That is, \(\delta_\gamma\) is a generator of \(g\) if and only if

\[
\delta_{\gamma_1}(x_{\gamma_2}) = \begin{cases} 1 & \gamma_1 = \gamma_2, \\ 0 & \text{else.} \end{cases}
\]

By the Milnor-Moore theorem, the universal enveloping algebra is isomorphic to the restricted dual of \(H\)

\[
U(g) \simeq H^\vee = \bigoplus_H H^*.
\]

The restricted dual is the direct sum of the duals of each graded component of \(H\). The product is defined on \(H^\vee\) by the convolution product

\[
\alpha_1 \ast \alpha_2(x_\Gamma) = (\alpha_1 \otimes \alpha_2)(\Delta x_\Gamma) \quad \alpha_i \in H^\vee,
\]

where the grading is given by the loop number of the graph. This is described in detail in [3] and [15]. The convolution product on \(g\) acts as an insertion operator on \(H\). For two generators of \(H\), \(x_{\Gamma_1}\) and \(x_{\Gamma_2}\), define

\[
x_{\Gamma_1} \ast x_{\Gamma_2} = \sum_{\Gamma} m(\delta_{\Gamma_1} \otimes \delta_{\Gamma_2})(x_{\Gamma}) \cdot x_{\Gamma}
\]
where the sum is taken over all generators of $\mathcal{H}$. This product induces an insertion production the 1PI graphs of a theory in the same fashion that the coproduct on $\mathcal{H}$ is induced by the subgraph structure on the 1PI graphs. This convolution product includes a pre Lie structure on the generators on the 1PI graphs of a theory. The Lie bracket

$$[x_{\Gamma_1}, x_{\Gamma_2}] = x_{\Gamma_1} \star x_{\Gamma_2} - x_{\Gamma_2} \star x_{\Gamma_1}$$

follows the Jacobi identity, as can be checked. For details on this construction, see [3] and [8].

3.2 Birkhoff decomposition

In [3], Connes and Kreimer show that BPHZ renormalization can be written as the decomposition of loops in the Lie group $G$ using the Birkhoff decomposition theorem. Connes and Marcolli also explain this construction in [1].

Let $\mathcal{A} = \mathbb{C}\{(z)\}$ the the algebra of formal Laurent series in $z$ with finite degree poles. Then $\text{Spec } \mathcal{A} = \Delta^*$, the punctured infinitesimal disk around the origin in $C$. If $\gamma(z)$ is a map from a simple loop not containing the origin in $\Delta^*$ to $G$, by the Birkhoff decomposition theorem, $\gamma(z)$ decomposes as the product

$$\gamma(z) = \gamma^{-1}_\gamma(z) \gamma_+(z) ,$$

where $\gamma_+(z)$ is a well defined map in the interior of the loop (containing $z = 0$), and $\gamma^{-1}_\gamma(z)$ is a well defined map outside of the loop (away from $z = 0$). This decomposition is uniquely defined by choosing a normalization on $\gamma^{-1}_\gamma$.

Consider $K$, a trivial $G$ principle bundle over $\Delta^*$. There is a natural isomorphism between elements of the group $G(\mathcal{A})$ and the sections of $K$,

$$\gamma(z) : \Delta^* \to G .$$

Each $\gamma(z)$ can be written as a Laurent series with poles of finite order and coefficients in $G(\mathbb{C})$ convergent in $\Delta^*$. The algebra homomorphisms decompose as $\gamma(z) = \gamma^{-1}_\gamma(z) \star \gamma_+(z)$, where $\star$ is the product on $\mathcal{H}'$ and $G(\mathcal{A})$. Following [3], the normalization for the uniqueness of the Birkhoff decomposition is given by $\gamma^{-1}_\gamma(1) = 1$.

Since $\gamma_+(z)(x_{\Gamma})$ is well defined at $z = 0$, it can be written as a somewhere convergent formal power series in $z$. Rewrite $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$, where $\mathcal{A}_+ = \mathbb{C}\{(z)\}$. This is equivalent to saying that $\gamma_+(z) \in G(\mathcal{A}_+)$. That is, for $x_{\Gamma} \in \mathcal{H}$, $\gamma_+(z)(x_{\Gamma})$ is holomorphic function in $z$. By the normalization condition on the decomposition, for $x_{\Gamma} \in \mathcal{H}_0^0$, that is, if it is a constant multiple of $x_0$, $\gamma^{-1}_\gamma(z) \circ \epsilon(x_{\Gamma}) = \epsilon(x_{\Gamma})$. If $x_{\Gamma} \in \ker \epsilon$, then $\gamma^{-1}_\gamma(z)(x_{\Gamma}) \in \mathcal{A}_-$. That is, for $x_{\Gamma} \in \ker \epsilon$, $\gamma^{-1}_\gamma(z)(x_{\Gamma})$ is a Laurent series containing only negative powers of $z$.

The following theorem shows that the regularized Feynman rules associated to any Lagrangian $\mathcal{L}$ that gives rise to the Hopf algebra $\mathcal{H}$ form a class of sections of this form.

**Theorem 3.1.** The regularized Feynman rules of a QFT defined by $\mathcal{L}$ are a linear map from $S(E)$ to $G(\mathcal{A})$.

**Proof.** Corollary 2.2 shows that

$$\text{regularized Feynman rules } : \mathcal{H} \mapsto \text{Hom}_{\text{vect}}(S(E), \mathcal{A}) .$$

To see that this is an algebra homomorphism, notice that the regularized Feynman rules for a disjoint union of 1PI graphs $x$ and $y \in \mathcal{H}$ is just the product of the regularized Feynman integrals associated to the graph $x$ and to the graph $y$. Therefore

$$\text{regularized Feynman rules } \in \text{Hom}_{\text{alg}}(\mathcal{H}, \text{Hom}_{\text{vect}}(S(E), \mathcal{A})) .$$

(4)
The symmetric algebra on $E$ can be written $S(E) = \oplus_n S^n(E)$. The restricted dual of $S^\vee(E) = \oplus_n S^{n\vee}(E)$, where there is an isomorphism from $S^{n\vee}(E) \simeq S^n(E)$, and the isomorphism $S^n(E^\vee) \simeq S^{n\vee}(E)$ gives

$$S(E^\vee) \simeq S^\vee(E) \simeq \text{Hom}_{\text{vect}}(S(E), \mathbb{C}).$$

Therefore,

$$\text{Hom}_{\text{vect}}(S(E), A) \simeq S(E^\vee) \otimes A.$$

Since an algebra homomorphism is a linear map, I can write equation (4) as

regularized Feynman rules $\in \text{Hom}_{\text{lin}}(\mathcal{H}, \text{Hom}_{\text{lin}}(S(E), A))$ $\simeq$ $\text{Hom}_{\text{lin}}(\mathcal{H}, S(E^\vee) \otimes A)$$\simeq$ $\text{Hom}_{\text{lin}}(S(E), \text{Hom}_{\text{lin}}(\mathcal{H}, A)).$

Since the regularized Feynman rules are also an algebra homomorphism, we have

regularized Feynman rules $\in \text{Hom}_{\text{lin}}(S(E), \text{Hom}_{\text{alg}}(\mathcal{H}, A))$, or

regularized Feynman rules $\in \text{Hom}_{\text{lin}}(S(E), G(A))$.

The regularized Feynman rules for a Lagrangian $\mathcal{L}$ assign an operator to each element of the Hopf algebra $\mathcal{H}$ generated by the 1PI graphs of $\mathcal{L}$. This set of regularized operators define maps from $E$ to $A$. Therefore, the regularized Feynman rules define a class of sections of the bundle

$$K_M \xrightarrow{\gamma_{\mathcal{L}, f}} \Delta^*_M \xrightarrow{\gamma_{\mathcal{L}}} M,$$

where $M$ is the background manifold and $f \in S(E)$. Since the choice of external leg data does not fundamentally change the results of renormalization, the subscript $f$ can be dropped. The $\gamma_{\mathcal{L}}$ refers to any of this class of sections defined by a regularization method on the Lagrangian $\mathcal{L}$. If the background manifold is flat, as in [3], the operator defined by regularization do not depend on the position over the manifold. The space $\Delta^*_M$ is a trivial $\Delta^*$ bundle over $M$, so the notation is dropped as in [3], and $\gamma_{\mathcal{L}}$ defines sections of the $K \to \Delta^*$ bundle. However, if the background space-time is not flat, then the regularized operators depend on the curvature, and thus position, over the background metric. For this reason, $\gamma_{\mathcal{L}}$ defines a global section of the bundle $K_M \to \Delta^*_M$ in this case.

In a flat background case, Connes and Kreimer [3] show that the recursive formula for calculating $\gamma_{\mathcal{L},+}(z)(x_\Gamma)$ and $\gamma_{\mathcal{L},-}(z)(x_\Gamma)$ is the exact same as the recursive formula for calculating the renormalized and counterterm contributions respectively of a Feynman diagram $\Gamma$ to the regularized Lagrangian given by BPHZ. The same argument carries over to the curved background manifold. For $\gamma_{\mathcal{L}}$, a section associated to the Lagrangian $\mathcal{L}$, notice that for any $x \in \text{ker} \varepsilon$, $\gamma_{\mathcal{L}}(z)(x) \in A$ is a Laurent polynomial in the regulator, $\gamma_{\mathcal{L},+}(z)(x) \in A_+$ is a somewhere convergent formal power series in $z$, and thus well defined for $z = 0$, and $\gamma_{\mathcal{L},-}(z)(x) \in A_-$ is a Laurent expansion with only negative powers of $z$, and thus undefined at $z = 0$. Therefore $\gamma_{\mathcal{L}}(z), \gamma_{\mathcal{L},+}(z)$ and $\gamma_{\mathcal{L},-}(z)$ are called the unrenormalized, renormalized and counterterm parts of $x$ respectively. This method works for any regularization scheme that results in regularized Feynman integrals as operators from the external momentum data to $\mathbb{C}\{\{z\}\}$. 

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4 The $\beta$-function

The regularization process results in a Lagrangian that is a function of the regularization parameter. Prior to regularization, the Lagrangian of any theory is scale invariant. That is
\[
\int_M L(x) \, d\text{vol}(x) = \int_M L(tx) \, d\text{vol}(tx).
\]
When the Lagrangian is regularized, and written in terms of a regularization parameter, $z$, it is no longer scale invariant. Specifically, the counterterms of a theory depends on the scale of the Lagrangian. In order to preserve scale invariance in the regularized Lagrangian one introduces a regularization mass, which is also a function of the regularization parameter, to cancel out any scaling effects introduced by regularization.

For instance, under $\zeta$-function regularization, the Lagrangian
\[
\int_M \phi(x)(\Delta M)^{1+z} \phi(x) + \lambda \phi^3(x) \, d\text{vol}(x)
\]
is not scale invariant. Using the convention where $c = \hbar = 1$, and the notation $[x] = \text{length units of the physical quantity } x$, give the following unit identifications $1 = [\text{length}] = [\text{mass}]^{-1} = [\text{energy}]^{-1}$. Thus
\[
[\phi(x)] = -2, \quad [\Delta M] = -2, \quad [\lambda] = 0, \quad [d\text{vol}] = 6.
\]

The conformal dimension of the Laplacian raised to a power is $[\Delta M^{1+z}] = -2(1+z)$. This induces a scaling of the regularized Lagrangian by a scale factor $\Lambda^{2z}$ where $[\Lambda] = 1$,
\[
L(z) = \int_M \left[ \frac{1}{2} \phi(x)(-\Delta M)^{1+z} \phi(x) + \lambda \Lambda^{-2z} \phi^3(x) \right] \Lambda^{2z} \, d\text{vol}(x).
\]
The term $\Lambda^{2z} \, d\text{vol}(x) = \Lambda^{2z} \sqrt{|g|} d^6x$ corresponds to a scaling of the metric, for instance
\[
g \to \Lambda^\frac{2z}{2} g.
\]
The renormalization mass parameter is given by $\Lambda^{2z}$.

4.1 Derivation in physics

The renormalization group describes how the dynamics of Lagrangian depends on the scale at which it is probed. One expects that probing at higher energy levels reveals more details about a system than at lower energies. To go from higher energy to lower, average over the extra information at the higher energy, $\lambda$, and rewrite it in terms of a finite number of parameters at a lower energy, $\mu$. The Lagrangian at the lower energy scale is called the effective Lagrangian at $\mu$, $(L, \mu)$. For a specified set of fields and interactions the effective Lagrangian at a $\mu$ is a Lagrangian with coefficients which depend on the scale, $\mu$.

Formally, let $M \simeq \mathbb{R}_+$ be a non-canonical energy space, with no preferred element. Fix a set of fields and interactions. Call $S$ the set of effective Lagrangians for this system in the energy space, $M$. For $\lambda, \mu \in M$ such that $\lambda > \mu$, there is a map
\[
R_{\lambda, \mu} : S \to S
\]
so that the effective Lagrangian at $\mu$ is written $R_{\lambda, \mu} L$ for $L \in S$. The map in (7) can be written as an action of $(0, 1]$ on $S \times M$:
\[
(0, 1] \times (S \times M) \quad \to \quad S \times M
\]
\[
t \circ (L, \lambda) \quad \mapsto \quad (R_{\lambda, t\lambda} L, t\lambda).
\]
In the effective Lagrangian \( R_{\lambda,t} \mathcal{L}(t) \), all parameters, \( m, \phi, \) and \( g \) are function of the mass scale \( t \). The map \( R_{\lambda,\mu} \) satisfies the properties

1. \( R_{\lambda,\mu} R_{\mu,\rho} = R_{\lambda,\rho} \).

2. \( R_{\lambda,\lambda} = 1 \).

**Definition 5.** The set \( \{ R_{\lambda,\mu} \} \) forms a semi-group called the renormalization group in the physics literature.

The renormalization group equations can be derived from differentiating the action in (8) and solving

\[
\frac{\partial}{\partial t} (R_{\lambda,t} \mathcal{L}_{\alpha}) = 0. \tag{9}
\]

This differential equation gives rise to a system of differential equations that describe the \( t \) dependence of the unrenormalized parameters, \( m(t), g(t) \) and \( \phi(t) \), in \( R_{\lambda,t} \mathcal{L}(t) \). To solve the renormalization group equations, it is sufficient to solve for \( g(t) \). The \( \beta \)-function describes the \( t \) dependence of \( g \) and can be written as

\[
\beta(g(t)) = t \frac{\partial g(t)}{\partial t}.
\]

This above development of the renormalization group and renormalization group equations follows [10]. For details on the renormalization group equations for a \( \phi^4 \) theory, QED and Yang-Mills theory, see [21] chapter 21 or [18] Chapter 9.

**Example 1.** The \( \beta \)-functions listed below are calculated in terms of a power series in the coupling constant for theories over a flat background manifold. The following are the one loop approximations of the \( \beta \)-functions for various theories. [21] [23]

1. For the scalar \( \phi^4 \) theory in 4 space-time dimensions,

\[
\beta(\lambda) = \frac{3g^2}{16\pi^2}.
\]

For a scalar \( \phi^3 \) theory in 6 space-time dimensions,

\[
\beta(g) = \frac{-g^3}{128\pi^3}.
\]

2. For QED, the \( \beta \)-function has the form

\[
\beta(e) = \frac{e^3}{12\pi^2} + O(e^5)
\]

where \( e \) is the electric charge.

3. For a general Yang-Mills theory with symmetry group \( G \), the \( \beta \)-function has the form

\[
\beta(\lambda) = -\frac{11g^3}{48\pi^2}C_2(G)
\]

where \( C_2 \) is the quadratic Casimir operator.

4. For QCD, the \( \beta \)-function has the form

\[
\beta(\lambda) = \frac{-1}{48\pi^2}(33 - 2N_f)g^3
\]

where \( N_f \) is the number of fermions.
The $\beta$ functions for these same theories over a curved space-time background depend on $g$, the curvature of the background manifold. In [1], Connes and Marcolli construct a renormalization bundle, closely related to the $K \to \Delta^*$ bundle, on which $\beta$-function defines the connections related to the section $\gamma_\mathcal{L}$, for $\mathcal{L}$ the Lagrangian for a renormalizable theory. They show that these $\beta$-functions are elements of $\mathfrak{g} = \text{Lie } G$. The quantities listed above are the sums of the $\beta$-function evaluated on the one loop graphs. The next section parallels their construction and develops the $\beta$-function for a renormalizable theory over curved space-time as a geometric object.

4.2 As a geometric object

The geometric $\beta$-function requires a more general construction of the renormalization group and effective Lagrangians. In the renormalization bundle, the non-canonical energy space is given by $M \simeq \mathbb{C}^\times$. The space $S$ of effective Lagrangians is replaced by the space $G(\mathcal{A})$, the space of evaluators of a regularized effective Lagrangians. The renormalization group is a group in this generalization (not just a semi group) given by $\theta_s = e^{sY}$ for $s \in \mathbb{C}$. The action of the renormalization group can be written as a $\mathbb{C}^\times$ action that factors through $\mathbb{C}$ by setting $t(s) = e^s$

$$t^Y : \quad G(A) \to G(A)$$
$$\gamma(z) \to t^Y \gamma(z) = \gamma_t(z).$$

The space $S \times M$ becomes $\tilde{G}(A) = G(A) \times_{\theta} \mathbb{C}^\times$ in the notation of [1]. The action of $\mathbb{C}^\times$ on $\tilde{G}(A)$ is given by

$$\mathbb{C}^\times \times \tilde{G}(A) \to \tilde{G}(A)$$
$$t \circ (\gamma, \lambda) \to (t^Y \gamma, t\lambda). \quad (10)$$

This extra structure defines the renormalization bundle, $P^*_M \to B^*_M$, which is a $\tilde{G}(A)$ principle bundle, where $B_M \simeq \Delta^*_M \times \mathbb{C}^\times$. The base space $B_M$ incorporates the non-canonical energy space $\mathbb{C}^\times$. It is a $\mathbb{C}^\times$ invariant bundle, with injections from the $K_M \to \Delta_M$ bundle by the $\mathbb{C}^\times$ action in [10].

The fiber of $P_M$ over the a point $x$ of the manifold $M$ is the Connes-Marcolli renormalization bundle $P \to B$, as defined in [6]. The $\beta$-function for the renormalization bundle over curved space time can be defined on these fibers as in [6]

$$\beta(\gamma(z, x)) = \frac{d}{dt} \lim_{t \to 0} \gamma(z, x)^{t^*} \ast (t^Y \gamma(z, x))$$

for $z \in \Delta^*$ and $x \in M$. This is only well defined when $\gamma(z, x)$ satisfies condition (9). To find the derivation of the geometric $\beta$-function in the flat geometric context, see [4], [1] or [7].

**Theorem 4.1.** There is a trivial connection $\omega(x)$ defined on the fibers $P \to B$ over $M$ in the renormalization bundle $P_M \to B_M$. This connection is uniquely defined on the pullbacks $\omega_{\gamma(z, t)}(x)$ by the $\beta$-function $\beta(\gamma(z, x))$ if and only if $\gamma(z, t, x)$ satisfies condition (9). If the section $\gamma(z, x)$ is globally defined over $M$, and satisfies condition (9) for all $x \in M$, then $\omega_{\gamma(z, t)}(x)$ is globally defined over $M$ by $\beta(\gamma(z, x))$. 

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Proof. Define the connection \( \omega(x) = \gamma^* \omega(z,t,x) \) on the fibers as in [6]. The proof that this connection on the fiber is uniquely defined by the \( \beta \)-function follows for the proof of uniqueness given in the flat renormalization bundle \( P \to B \).

The connection \( \omega(x) \) on \( P \to B \) can be viewed as a section of the bundle \( g_M \to M \), with \( g_M \) a \( g \) bundle over \( M \). Since the section is defined on the pullbacks by \( \gamma(z,t,x) \), \( \omega(x) \) is a global section over \( M \) if \( \gamma(z,t,x) \) is. The correspondence of this section by the \( \beta \) function follows from the unique definition on the fibers.

Specifically, the Lagrangian considered in this paper under \( \zeta \)-function regularization corresponds to a global section \( \gamma_L \) that satisfies condition (9) on all of \( M \). Therefore, the \( \beta \)-function for this Lagrangian can be found from the pullback of the section \( \omega(x) \) along \( \gamma_L \).

5 Non-constant conformal changes to the metric

In this section, I extend the above analysis \( \zeta \)-function regularization under a non-constant regularization mass parameter. In the previous section, the regularized Lagrangian had the form

\[
L(z) = \int_M \left[ \frac{1}{2} \phi(x)(-\Delta)^{\frac{1}{2}} \phi(x) + \lambda \Lambda^{-2z} \phi(x)^3 \right] \Lambda^{2z} \text{dvol}(x) ,
\]

as in equation (6). The factor of \( \Lambda^{2z} \) corresponds to a scaling of the metric by the constant \( \Lambda^{\frac{2}{n}} \). Instead, scale the metric by a conformal factor \( e^{2f(x)} \), where \( f \in C^\infty(M) \). This changes the renormalization bundle of the previous section, as the renormalization mass parameter \( C \times \) no longer sits trivially over \( M \). To understand this new bundle, I introduce the language of densities over the manifold \( M \), and write the renormalization bundle for conformal field theories in the language of density bundles over \( M \).

5.1 Densities

While compactness and orientability are not necessary for the arguments of the following sections, I will keep with the conventions of the previous sections and let \( M \) be a smooth, compact, oriented Riemannian \( n \)-manifold. It has a principal \( \text{Gl}_n(\mathbb{R}) \)-bundle \( \text{Frame}(M) \) of frames, whose fibers \( \text{Frame}_x(M) \) are ordered bases \( \{v_1, \ldots, v_n\} \) for the tangent space \( T_xM \). The structure group \( \text{Gl}_n(\mathbb{R}) \) acts freely and transitively on the fibers by rotating the frames.

A representation \( \rho : \text{Gl}_n(\mathbb{R}) \to \text{Aut}(V) \) of the structure group as automorphisms of a vector space \( V \) defines a vector representation

\[
V \times_{\text{Gl}_n(\mathbb{R})} \text{Frame}(M) \to M
\]

over \( M \). For a general (not necessarily orientable) manifold, \( M \), the determinant of the structure group defines a bundle

\[
\det : \text{Gl}_n(\mathbb{R}) \to \mathbb{R}^\times = \text{Gl}_1(\mathbb{R}) .
\]

Definition 6. For an orientable manifold and for any \( r \in \mathbb{R} \) the representation

\[
|\det|^r/n : \text{Gl}_n(\mathbb{R}) \to \mathbb{R}^\times_+
\]

defines a line bundle which I denote \( \mathbb{R}(r) \).

The sections of the bundle \( \mathbb{R}(r) \to M \) are called \( r \) densities on \( M \). The bundle can be trivialized by choosing a metric, \( g \), for \( M \). Let \( \phi \) be a section of \( \mathbb{R}(r) \to M \). Given a choice of \( g \), it can be written uniquely as

\[
\phi = f|g|^\frac{r}{n}
\]

for some \( f \in C^\infty(M) \).
Remark 4. Because the metric is variable in this section, I write \( d\text{vol}(g) \) for the volume form to emphasize the metric in question as opposed to the notation \( d\text{vol}(x) \) from the previous section.

If \( \phi \) is a continuous section of \( \mathbb{R}(r) \), then for any \( s > 0 \), \( |\phi|^s \) is a continuous section of \( \mathbb{R}(rs) \). In particular, when \( n \geq r > 0 \),
\[
||\phi||^{n/r} := \int_M |\phi|^{n/r}
\]
defines an analog of a classical Banach norm. This becomes apparent under a trivialization
\[
||\phi||^{n/r} = \int_M (|f|g)^{n/r} = \int |f|^{n/r} \text{dvol}(g) .
\]
When \( r = 0 \), the norm is given by the classical essential supremum. I can now consider sections of \( \mathbb{R}(r) \), and write \( L^r \) for the Lebesgue space of \( r \) densities, with these norms. In this terminology, \( n \)-forms become \( n \)-densities, the Banach space dual of \( L^r \) is \( L^{n-r} \), and Hölder’s inequality becomes the assertion that the point-wise product
\[
L^r(d_0) \otimes L^r(d_1) \to L^r(d_0 + d_1)
\]
is continuous with respect to the natural norms.

Notice that sections of \( \mathbb{R}(\frac{n}{2}) \) define a Hilbert space \( L\left(\frac{n}{2}\right) \) with inner product
\[
\langle \phi, \psi \rangle = \int_M \phi \psi .
\]
This inner product is independent of the Riemannian metric. A choice of \( g \) defines an isometry with the classical Lebesgue space \( L^2(M, g) \). Let \( \phi = f|g|^{\frac{n}{4}} \) and \( \psi = h|g|^{\frac{n}{4}} \). The inner product is
\[
\langle \phi, \psi \rangle_g = \int_M f|g|^{\frac{n}{4}} h|g|^{\frac{n}{4}} dx_1 \wedge \ldots \wedge dx_n .
\]
Finally, there is a linear operator
\[
\phi \mapsto |g|^\frac{1}{4} \phi
\]
that maps smooth sections of density \( d_0 \) to those of density \( d_1 \). When \( d_1 \geq d_0 \) it defines a continuous linear map from \( L(d_0) \) to \( L(d_1) \).

5.2 Effect of conformal changes on the Lagrangian

I can use this formalism to study how the Lagrangian varies under conformal changes to the metric
\[
g \to e^{f(x)}g ; \quad f(x) \in C^\infty(M) .
\]
For ease of notation, let \( u = e^f \). The Lagrangian density for renormalizable scalar field theory on an \( n \)-dimensional Riemannian metric is given by
\[
\mathcal{L} = \frac{1}{2} \phi(x)(-\Delta_M)\phi(x) + \lambda \phi^{\frac{2n}{n-2}}(x) ,
\]
where I consider \( \phi, m \) and \( \lambda \) to be densities of different weights. Notice that \( \phi \) is raised to an integral power only when \( n \in \{3, 4, 6\} \).

To emphasize the Laplacian’s dependence on the metric \( g \) on \( M \), write
\[
\Delta_g = \text{div} \circ \nabla = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} g^{ij} \partial_j) .
\]
Yamabe’s theorem [25] states:
Theorem 5.1. [Yamabe] Let \( \phi \in C^\infty(M) \), and let \( g \) be a metric on \( M \). Then the quantity

\[
\int_M \phi \left( -\Delta_g + \frac{1}{4} \left( n - 2 \right) R(g) \right) \phi \cdot d\text{vol}(g)
\]

is invariant under the conformal rescaling \( g \mapsto \bar{g} = e^{2f(x)}g, \phi \mapsto \bar{\phi} = e^{\frac{n-2}{2}f} \phi \), where \( f \in C^\infty(M) \).

Definition 7. For ease of notation, define the conformally invariant Laplacian

\[
\Delta_{[g]} = \Delta_g - \frac{1}{4} \left( n - 2 \right) R(g).
\]

As an operator

\[
\Delta_{[g]} : \mathbb{L}(\frac{n-2}{2}) \to \mathbb{L}(\frac{n+2}{2})
\]

it is a quadratic form on \( \mathbb{L}(\frac{n-2}{2}) \). The \([g]\) subscript indicates that the Laplace operator depends only on the conformal equivalence class of \( g \).

Notice that the function \( \bar{\phi} \) is invariant under the conformal scaling \( g \mapsto u^2g \). Therefore \( \phi \) is a \((\frac{n-2}{2})\)-density trivialized by a choice of Riemannian metric so that

\[
\phi = (|g|^{|\frac{n}{2}})^{\frac{n-2}{2n}} = |g|^{\frac{n-2}{2n}} h.
\]

Then equation (13) is a linear map from \( \mathbb{L}(\frac{n-2}{2}) \to \mathbb{R} \). The \([g]\) Lagrangian for a renormalizable scalar theory in terms of this conformally invariant operator \( \Delta_{[g]} \) and densities \( \phi \) is

\[
L = \int_M \phi (-\Delta_{[g]} + m^2) \phi + \lambda \phi^{\frac{2n}{n-2}}
\]

where \( m \) is a 1-density, \( \lambda \) is a (0)-density, and \( M \) is a 6-manifold. The free part of this Lagrangian

\[
L_F = \int_M \phi (-\Delta_{[g]} + m^2) \phi
\]

is invariant under the transformations \( g \mapsto u^2g \).

Theorem 5.2. There is a meromorphic family of quadratic forms on \((\frac{n-2}{2})\)-densities,

\[
\tilde{Y}_g(r) = |g|^\frac{1}{n} Y_{g^1+r} |g|^\frac{1}{n},
\]

that defines the self-adjoint operator that represents the free term in the Lagrangian of a scalar field theory.

Proof. In order to carry out the arguments from section 2.2, \( -\Delta_{[g]} + m^2 \) must be a self-adjoint operator acting on the Hilbert space \( \mathbb{L}(\frac{n}{2}) \). By equation (12),

\[
|g|^\frac{1}{n} \phi \in \mathbb{L}(\frac{n}{2}).
\]

Rewrite

\[
L_F = \int_M \phi |g|^\frac{1}{n} |g|^\frac{1}{n} (-\Delta_{[g]} + m^2) |g|^\frac{1}{n} |g|^\frac{1}{n} \phi
\]

Now I can define an operator

\[
Y_g := |g|^{-\frac{1}{n}} (-\Delta_{[g]} + m^2) |g|^{-\frac{1}{n}}
\]
that acts on the Hilbert space $L^2_n$.

Proceeding as before, I raise $Y_g$ to a complex power. This gives a family of Lagrangians

$$L_F(z, g) = \int_M \phi|g|^{\frac{1}{2n}} Y_g^{1+z} |g|^{\frac{1}{2n}} \phi.$$ 

As before, $\text{Tr} Y_g^{1+z}$ has simple poles in $z$. Following the same arguments as in Corollary 2.2, I can expand around $z = 0$ and write this as a Laurent series

$$Y_g^{1+z} = \sum_{i=-1}^{\infty} a_i z^i$$  \hspace{1cm} (14)

where $a_i$ are operators.

However, since $\phi \in L^2_{n-2}$, the self-adjoint operator in the Lagrangian must be a quadratic form on $L^2_{n-2}$. To define such an operator, use equation (12) to get

$$\tilde{Y}_g(z) = \phi Y_g^{1+z} \phi.$$ 

**Remark 5.** Notice that by raising $Y_g$ to a complex power, the expression $\phi \tilde{Y}_g(z) \phi$ is now a $n + 2z$ density.

Under the conformal change of metric

$$g \mapsto u^2 g = \bar{g}$$

$Y_g$ transforms as

$$Y_g \rightarrow u^{-1} Y_g u^{-1} = Y_{\bar{g}}$$

The operator $\tilde{Y}_{\bar{g}}(z)$ can be written,

$$\tilde{Y}_{\bar{g}}(z) = |g|^{\frac{1}{2n}} u \left( u^{-1} Y_g u^{-1} \right)^{1+z} u |g|^{\frac{1}{2n}}.$$  \hspace{1cm} (15)

The kernel of this operator is defined by a family of pseudo-differential operators with top symbol

$$\xi \mapsto |\xi|^{2+2z}.$$ 

**Proposition 5.3.** For a general $u = e^f$, $\tilde{Y}_{\bar{g}}$ can be expanded as a Taylor series in $f$ as

$$\tilde{Y}_{\bar{g}}(f, z) = e^{-2f z} \tilde{Y}_g(z).$$

**Proof.** Recall that $u = e^{f(x)}$. The terms of the Taylor series of $\tilde{Y}_{\bar{g}}(z)$ at $f = 0$ are given as follows

**order 0** The $0^{th}$ order term is given by evaluating $\tilde{Y}_{\bar{g}}(z)$ at $f = 0$. This gives

$$|g|^{\frac{1}{2n}} Y_g^{1+z} |g|^{\frac{1}{2n}} = \tilde{Y}_g(z)$$

**order f** Taking the derivative of $\tilde{Y}_{\bar{g}}(z)$ in terms of $f$ gives

$$\left[2e^{f} |g|^{\frac{1}{2n}} (e^{-f} Y_g e^{-f})^{1+z} |g|^{\frac{1}{2n}} e^{f} + e^{f} |g|^{\frac{1}{2n}} (1 + z)(e^{-f} Y_g e^{-f})^{z} (-2e^{-f} Y_g e^{-f}) |g|^{\frac{1}{2n}} e^{f} \right].$$  \hspace{1cm} (16)

To simply matters later, write

$$(e^{-f} Y_g e^{-f})^{z} (-2e^{-f} Y_g e^{-f}) = -2(e^{-f} Y_g e^{-f})^{1+z}.$$ 

Evaluating (16) at $f = 0$ gives

$$2\tilde{Y}_g(z) - 2(1 + z) \tilde{Y}_g(z).$$
order $f^2$ Taking the derivative of (16)
\[4e^f|g|^n \left( (1+z)(e^{-f}Y_g e^{-f})^{1+z} |g|^n e^f \right)
- 4e^f|g|^n \left( (1+z)(e^{-f}Y_g e^{-f})^{1+z} |g|^n e^f + 4e^f|g|^n \right)
- 4e^f|g|^n (1+z)(e^{-f}Y_g e^{-f})^{1+z} |g|^n e^f \right].
\]

The first two terms come from the derivative of the first term of (16). The third comes from taking the derivative of outer terms of the second term of (16), and the fourth term comes from the derivative of the middle term of the second term of (16).

Evaluating (17) at $f = 0$ gives
\[4\tilde{Y}_g(z) - 8(z + 1)\tilde{Y}_g(z) + 4(1 + z)^2\tilde{Y}_g(z).
\]
This simplifies to
\[2^2(1 - (1 + z))^2\tilde{Y}_g(z).
\]

order $f^n$ Continuing along these lines shows that the $n^{th}$ derivative of $\tilde{Y}_g(z)$ with respect to $f$, evaluated at $f = 0$ is
\[2^n(1 - (1 + z))^n\tilde{Y}_g(z) = 2^n(-z)^n\tilde{Y}_g(z).
\]

Writing this out as a Taylor expansion
\[\tilde{Y}_g(z) = \sum_{n=0}^{\infty} \frac{(2zf)^n}{n!} \tilde{Y}_g(z) = \left( \sum_{n=0}^{\infty} \frac{(2zf)^n}{n!} \right) \tilde{Y}_g(r) = e^{-2zf}\tilde{Y}_g(z).
\]

This construction defines a family of effective Lagrangians
\[L(r, [g]) = \int_M u^{-2z} \left[ \phi \tilde{Y}_g(z) \phi + u^{2z} \lambda \phi \frac{2z}{n+2} \right],
\]
where $[g]$ denotes a conformal class of metric. This is the natural analog, in a flat background, of the classical family of effective Lagrangians
\[L(z) = \int_M \left[ \phi (\Delta + m^2)^{1+z} \phi + \lambda \phi \frac{2z}{n+2} \right] \Lambda^{2z} \phi^2 x,
\]
with $\tilde{Y}_g(z)$ corresponding to $(-\Delta + m^2)^{1+z}$. In the first display, $\phi$ is a $u^{n-2}$ density. In the second display, $\phi$ is a function. This Lagrangian is not conformally invariant when $u \not\in \mathbb{C}^\times$.

The regularized Lagrangian with this conformally corrected Laplacian $\tilde{Y}_g$ can be represented as a section of a conformal renormalization bundle. The conformal scaling factor $u$ is a 1-density over $M$, or a section of the bundle $\mathbb{C}(1) \times GL_n(\mathbb{C}) M$. Instead of $\mathcal{B}_M \simeq \Delta^* \times \mathbb{C}^\times \times M$, which is trivially defined over $M$, the base of this renormalization bundle becomes $\mathcal{B}_M = \Delta^* \times \mathbb{C}^\times (1) \times GL_n(\mathbb{C}) M$. The conformal renormalization bundle is a $G$ principal bundle over the base $\mathcal{B}_M$. Thus I have constructed the bundle in the following theorem.

**Theorem 5.4.** The conformal renormalization bundle is a $\mathbb{C}^\times$ equivariant $G$ principal bundle
\[\mathcal{P} := G \times \mathcal{B}_M \times GL_n(\mathbb{R}) \text{Frame}(M) \to \mathcal{B}(M).
\]
which is diffeomorphic, but not naturally, to the renormalization bundle over curved space-time $M$.

Many of the properties of the bundle renormalization bundle $\mathcal{P}_M \to \mathcal{B}_M$ extend to the bundle $\mathcal{P}_M \to \mathcal{B}_M$. The action of the rescaling group $\mathbb{C}^\times$ extends to an action of the infinite dimensional group of local conformal rescalings. The sections of $\mathcal{P}_M \to \mathcal{B}_M$ that satisfy condition (9) are still uniquely defined by the $\beta$-function of now non-conformally invariant theories. I leave the investigation and calculation of these non-conformal $\beta$-functions for future work.

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