Duality in the Context of
Topological Quantum Field Theory

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Abstract

We present a summary of the progress made in the last few years on topological quantum field theory in four dimensions. In particular, we describe the role played by duality in the developments which led to the Seiberg-Witten invariants and their relation to the Donaldson invariants. In addition, we analyze the fruitful framework that this connection has originated. This analysis involves the study of topological quantum field theories which contain twisted $\mathcal{N} = 2$ supersymmetric matter fields as well as theories obtained after twisting $\mathcal{N} = 4$ supersymmetry. In the latter case, we present some recent results including the generalization of the partition function of the Vafa-Witten theory for gauge group $SU(N)$ with prime $N$.

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Topological quantum field theory (TQFT) in four dimensions \([1][2][3][4]\) has become a very fruitful link between physics and mathematics. On the one hand, the progress made in the last years on the duality properties of \(\mathcal{N} = 2\) supersymmetric Yang-Mills theories has been applied to their topological counterparts to define new invariants, the Seiberg-Witten invariants \([5]\), and to show their relation with known invariants as the Donaldson invariants \([6]\). On the other hand the topological nature of twisted \(\mathcal{N} = 4\) supersymmetric theories has provided important tests of our ideas on duality symmetry. Both sides benefit from each other and certainly they will continue to do so in the forthcoming years after the consequences of the recent AdS/CFT conjecture \([7][8][9]\) in this context start being explored – see \([10][11]\) for some proposals in this direction.

Edward Witten inaugurated the field of topological quantum field theory in the beginning of 1988 with his work on Donaldson theory from a quantum field theory perspective \([1]\). He formulated a twisted version of \(\mathcal{N} = 2\) supersymmetric gauge theory, now known as Donaldson-Witten theory, whose observables were identified with the Donaldson invariants of four-manifolds \([1]\). His formulation was later reinterpreted \([2]\) from a more geometrical point of view, in terms of a representative of the Thom class of a vector bundle associated to certain moduli problem in the framework of the Mathai-Quillen formalism \([13]\). Twisted \(\mathcal{N} = 2\) supersymmetric theories, in general, are associated to certain moduli problems which, properly treated in the context of the Mathai-Quillen formalism, lead to representatives of the Thom class which become the exponential of the twisted actions on the field theory side. Both pictures of Donaldson-Witten theory have been known for some time. One important property of the resulting TQFT is that the vacuum expectation values of its observables are independent of the coupling constant. This means that these quantities could be computed in either the strong or the weak coupling limit. The weak coupling limit analysis showed the relation of the observables of the theory to the Donaldson invariants. However, in such analysis no new progress was made from the quantum field theory representation regarding the calculation of these invariants. The difficult problems that one had to face were similar to those in ordinary Donaldson theory.

The strong-coupling effective theory of \(\mathcal{N} = 2\) supersymmetric Yang-Mills theory was obtained by Seiberg and Witten in 1994 \([5]\). One would expect that the twisted version of this effective theory would be related to the Donaldson-Witten theory. Furthermore, since the observables of a TQFT are independent of the coupling constant, the weak coupling limit of the effective theory should be exact, \(i.e.,\) it would lead to Donaldson invariants. This is in fact what turns out to be the case. The twisted effective theory could be regarded as a TQFT dual to the original one. In addition, one could ask for the dual moduli problem associated to this dual TQFT. It turns out that in some of the most interesting situations \((b_2^+ > 1)\) this moduli space is an Abelian version of the moduli space of instantons modified by the presence of chiral spinors. This space is known as the moduli space of Abelian monopoles \([14]\). Being related to an Abelian gauge theory this space is simpler to analyze than the moduli space of instantons. Furthermore, for a large set of four-manifolds (of simple type), only particular classes of Abelian gauge configurations (basic classes) contribute. For these classes the moduli space of Abelian monopoles reduces to a finite set of points.

Donaldson-Witten theory has been generalized after studying its coupling to topological matter fields \([15][16][17]\). The resulting theory can be regarded as a twisted form
of $N = 2$ supersymmetric Yang-Mills theory coupled to hypermultiplets, or, in the context of the Mathai-Quillen formalism, as the TQFT associated to the moduli space of non-Abelian monopoles [18] [19]. Perturbative and non-perturbative methods have been applied to this theory for the case of gauge group $SU(2)$ and one hypermultiplet of matter in the fundamental representation [20]. In this case, again, it turns out that when $b_2^+ > 1$ the generalized Donaldson invariants can be written in terms of Seiberg-Witten invariants.

Recently, a general framework to analyze models with gauge group $SU(2)$, known as integration over the $u$-plane [21] [22] [23], has been constructed. From this new viewpoint, the presence of Seiberg-Witten invariants turns out to be rather general. They are believed to provide the only contributions to the invariants for manifolds with $b_2^+ > 1$.

Generalizations to higher-rank gauge groups have been also studied [24]. In all these examples one finds relations among different moduli spaces. In the context of TQFT in four dimensions, duality relates moduli spaces: observables which are to pological invariants of a given four-manifold can be computed using information from two different moduli spaces. The duality properties of the physical theory fix the type of moduli spaces which are involved in each case.

Not all the theories obtained after twisting extended supersymmetric theories fall into the duality pattern among moduli spaces described above. That is the case for some of the twistings which originate from $N = 4$ supersymmetric Yang-Mills theory, in particular for the twisting considered in [25] by Vafa and Witten. From the duality point of view discussed in this paper this theory can be regarded as a self-dual theory in the sense that it involves only one moduli space. In this case duality manifests itself in a different form: it becomes an $SL(2, \mathbb{Z})$ symmetry which involves the coupling constant and the dual gauge groups [25].

To begin with the description of Donaldson-Witten theory we first review some generalities concerning $N = 2$ supersymmetry in four-dimensions. The global symmetry group of $N = 2$ supersymmetry in $\mathbb{R}^4$ is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(2)_I \otimes U(1)_R$ where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group and $SU(2)_I \otimes U(1)_R$ is the internal (chiral) symmetry group. The supercharges, $Q^\alpha_i$ and $\overline{Q}_{i\dot{\alpha}}$, which generate $N = 2$ supersymmetry, have the following transformations under $\mathcal{H}$:

$$Q^\alpha_i \left( \frac{1}{2}, 0, \frac{1}{2} \right), \quad \overline{Q}_{i\dot{\alpha}} \left( 0, \frac{1}{2}, \frac{1}{2} \right)^{-1}, \quad (1)$$

where the superindex denotes the $U(1)_R$ charge and the numbers within parentheses label the representations under each of the factors in $SU(2)_L \otimes SU(2)_R \otimes SU(2)_I$. The supercharges $\mathcal{H}$ satisfy:

$$\{Q^\alpha_i, \overline{Q}_{j\dot{\alpha}}\} = \delta^i_j P_{\alpha\dot{\beta}}. \quad (2)$$

The twist consists of considering as the rotation group the group, $\mathcal{K}' = SU(2)'_L \otimes SU(2)_R$, where $SU(2)'_L$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_I$. This implies that the isospin index $i$ becomes a spinorial index $\alpha$: $Q^\alpha_i \rightarrow Q^\alpha_\beta$ and $\overline{Q}_{i\dot{\alpha}} \rightarrow G_{\alpha\dot{\beta}}$. The trace of $Q^\alpha_\beta$ is chosen as the generator of a new scalar symmetry: $Q = Q^\alpha_\alpha$. Under the new global symmetry group $\mathcal{H}' = \mathcal{K}' \otimes U(1)_R$, the symmetry generators transform as:

$$G_{\alpha\dot{\beta}} \left( \frac{1}{2}, \frac{1}{2} \right)^{-1}, \quad Q_{(\alpha\beta)} \left( 1, 0 \right)^1, \quad Q \left( 0, 0 \right)^1. \quad (3)$$
It is important to stress that as long as we stay on a flat space (or one with trivial holonomy), the twist is just a fancy way of considering the theory, for in the end we are not changing anything. However, the appearance of a scalar symmetry makes the procedure meaningful when we move to an arbitrary four-manifold. Once the scalar symmetry is found we must study if it can be written as the transformation of some quantity under $Q$. If this is the case, the vacuum expectation value of $Q$-invariant operators will be metric independent. The $\mathcal{N} = 2$ supersymmetry algebra gives a necessary condition for this to hold. Indeed, after the twisting, this algebra becomes:

$$\{Q_\alpha^i, \bar{Q}_{j\dot{\beta}}\} = \delta^i_j P_{\alpha\dot{\beta}} \rightarrow \{Q, G_{\alpha\dot{\beta}}\} = P_{\alpha\dot{\beta}},$$

where $P_{\alpha\dot{\beta}}$ is the momentum operator of the theory. Certainly (4) is only a necessary condition for the theory to be topological. However, up to date, for all the supersymmetric models whose twisting has been studied the relation on the right hand side of (4) has become valid for the whole energy-momentum tensor.

In $\mathbb{R}^4$ the original and the twisted theories are equivalent. However, for arbitrary manifolds $X$ they are certainly different due to the fact that their energy-momentum tensors are not the same. The twisting changes the spin of the fields in the theory, and therefore their couplings to the metric on $X$ become modified. This suggests an alternative way of looking at the twist. All that has to be done is: gauge the internal group $SU(2)_I$, and identify the corresponding $SU(2)$ connection with the spin connection on $X$. This process changes the spin connection and therefore the energy-momentum tensor of the theory, which in turn modifies the couplings to gravity of the different fields of the theory. This alternative point of view to the twisting procedure has been reviewed in this context in [26].

As mentioned above, the Donaldson-Witten theory can be constructed by twisting the pure $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $SU(2)$. This theory contains a gauge field $A$, a pair of chiral spinors $\lambda_i$ and a complex scalar field $B$. Under the twist, this field content is modified as follows:

$$A_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2}, 0 \right)^0 \rightarrow A_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2} \right)^0,$$

$$\lambda_{\dot{\alpha}} \left( \frac{1}{2}, 0, \frac{1}{2} \right)^{-1} \rightarrow \chi_{\alpha\beta} (1,0)^{-1}, \ \eta(0,0)^{-1},$$

$$\bar{\lambda}_{\dot{\alpha}} \left( 0, \frac{1}{2}, \frac{1}{2} \right)^1 \rightarrow \psi_{\alpha\dot{\alpha}} \left( \frac{1}{2}, \frac{1}{2} \right)^1,$$

$$B \ (0,0,0)^{-2} \rightarrow \lambda \ (0,0)^{-2},$$

$$B^* \ (0,0,0)^2 \rightarrow \phi \ (0,0)^2.$$

In the process of twisting, the $U(1)_R$ symmetry becomes the $U(1)$-like symmetry associated to the ghost number of the topological theory. The ghost number anomaly is thus naturally related to the chiral anomaly of $U(1)_R$. The twisted action has the form:

$$\int_X d^4x \sqrt{g} \text{Tr} \left( F^{+2} - i\chi^{\mu\nu}D_\mu \psi_\nu + i\eta D_\mu \psi^\mu + \frac{1}{4} \phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} + \frac{i}{4} \lambda\{\psi_\mu, \psi^\mu\} - \lambda D_\mu D^\mu \phi \right. \left. + \frac{i}{2} \phi\{\eta, \eta\} + \frac{1}{8}[\lambda, \phi]^2 \right).$$

$$\int_X d^4x \sqrt{g} \text{Tr} \left( F^{+2} - i\chi^{\mu\nu}D_\mu \psi_\nu + i\eta D_\mu \psi^\mu + \frac{1}{4} \phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} + \frac{i}{4} \lambda\{\psi_\mu, \psi^\mu\} - \lambda D_\mu D^\mu \phi \right.$$
It is invariant under the transformations generated by $Q$ which from now on will be denoted as $\delta$-transformations:

$$
\begin{align*}
\delta A_\mu &= \psi_\mu, & \delta \chi_{\mu\nu} &= G_{\mu\nu}, \\
\delta \psi &= d A \phi, & \delta \eta &= i [\lambda, \phi], \\
\delta \phi &= 0, & \delta \lambda &= \eta.
\end{align*}
$$

(7)

In these transformations, $\delta^2$ is a gauge transformation with gauge parameter $\phi$. Observables are therefore related to the $G$-equivariant cohomology of $\delta$ (that is, the cohomology of $\delta$ restricted to gauge invariant operators). Of course, auxiliary fields can be introduced so that the action (6) is $\delta$-exact [4].

To construct the observables of the theory we begin by pointing out that for each independent Casimir of the gauge group $G$ it is possible to construct a highest-ghost-number operator $W_0$, from which lower ghost-number operators $W_i$ can be defined recursively through the descent equations $\delta W_i = dW_{i-1}$. For example, for the quadratic Casimir this operator is:

$$
W_0 = \frac{1}{8\pi^2} \text{Tr} \left( \phi^2 \right),
$$

(8)

and it generates the following family of operators:

$$
W_1 = \frac{1}{4\pi^2} \text{Tr} \left( \phi \psi \right), \quad W_2 = \frac{1}{4\pi^2} \text{Tr} \left( \frac{1}{2} \psi \wedge \psi + \phi \wedge F \right), \quad W_3 = \frac{1}{4\pi^2} \text{Tr} \left( \psi \wedge F \right).
$$

(9)

From them one defines the following observables:

$$
O^{(k)} = \int_{\gamma_k} W_k,
$$

(10)

where $\gamma_k \in H_k(X)$. The descent equations imply that they are $\delta$-invariant and that they only depend on the homology class of $\gamma_k$. According to (5), the ghost numbers of these operators are $U(O^{(k)}) = 4 - k$.

The functional integral corresponding to the topological invariants of the theory has the form:

$$
\left\langle O^{(k_1)}O^{(k_2)}\cdots O^{(k_p)} \right\rangle = \int O^{(k_1)}O^{(k_2)}\cdots O^{(k_p)} \exp(-S/g^2),
$$

(11)

where the integration has to be understood on the space of field configurations modulo gauge transformations, and $g$ is a coupling constant. Standard arguments show that due to the $\delta$-exactness of the action $S$, the quantities obtained in (11) are independent of $g$. This implies that the observables of the theory can be obtained either in the limit $g \to 0$, where perturbative methods apply, or in the limit $g \to \infty$, where one is forced to consider a non-perturbative approach.

Let us consider first the theory in the weak coupling limit $g \to 0$. The previous argument affirms that the semiclassical approximation is exact. In the weak coupling limit the contributions to the functional integral are dominated by the bosonic field configurations which minimize $S$. These turn out to be given by the equations:

$$
F^+ = 0, \quad D_\mu D^\mu \phi = 0.
$$

(12)
Let us assume that in the situation under consideration there are only irreducible connections (this is true in the case $b^+ = \dim H^{2,+}(X) > 1$). In this case the contributions from the bosonic part of the action are given entirely by the solutions of the equation $F^+ = 0$, i.e., by instanton configurations. Since the connection is irreducible, there are no non-trivial solutions to the second equation in (12).

The zero modes of the field $\psi$ come from the solutions to the equations

$$(D_\mu \psi_\nu)^+ = 0, \quad D_\mu \psi^\mu = 0,$$

which define precisely the tangent space to the space of instanton configurations. The number of independent solutions of these equations determine the dimension of the instanton moduli space $M_{\text{ASD}}$. For $SU(2)$ this dimension is $d_{M_{\text{ASD}}} = 8k - 3(\chi + \sigma)/2$, where $k$ is the instanton number, while $\chi$ and $\sigma$ are the Euler characteristic and the signature of the manifold $X$, respectively.

The fundamental contribution to the functional integral (11) is given by the elements of $M_{\text{ASD}}$ and by the zero-modes of the solutions to (13). Once these have been obtained they must be introduced in the action and an expansion up to quadratic terms in non-zero modes must be performed. The fields $\phi$ and $\lambda$ are integrated out originating a contribution which is equivalent to the replacement of the field $\phi$ in the operators $O^{(k)}$ by

$$\phi^a \longrightarrow \int d^4 y \sqrt{g} G^{ab}(x,y)[\psi_\mu(y),\psi^\mu(y)]^b,$$

where $G^{ab}(x,y)$ is the inverse of the Laplace operator,

$$D_\mu D^\mu G^{ab}(x,y) = \delta^{ab}\delta^4(x-y).$$

These are the only relevant terms in the limit $g \to 0$. The resulting Gaussian integrations then must be performed. Due to the presence of the $\delta$ symmetry these come in quotients whose value is $\pm 1$. The functional integral (11) takes the form:

$$\langle O^{(k_1)}O^{(k_2)}\cdots O^{(k_p)} \rangle = \int_{M_{\text{ASD}}} da_1 \cdots da_{d_{M_{\text{ASD}}}} d\psi_1 \cdots d\psi_{d_{M_{\text{ASD}}}} O^{(k_1)}O^{(k_2)}\cdots O^{(k_p)}(-1)^{\nu(a_1,\ldots,a_{d_{M_{\text{ASD}}}})} \nu_i, \quad \nu_i = 0, 1,$$

where $\nu(a_1,\ldots,a_{d_{M_{\text{ASD}}}}) = 0, 1$. The integration over the odd modes leads to a selection rule for the product of observables. This selection rule is better expressed making use of the ghost numbers of the fields. For the operators in (10) the selection rule can be written as $d_{M_{\text{ASD}}} = \sum_i U(O^{(k_i)}) = \sum_i (4 - k_i)$.

In the case in which $d_{M_{\text{ASD}}} = 0$, the only observable is the partition function, which takes the form:

$$\langle 1 \rangle = \sum_i (-1)^{\nu_i},$$

where the sum is over isolated instantons, and $\nu_i = 0, 1$. In general, the integration over the zero-modes in (16) leads to an antisymmetrization in such a way that one ends up with the integration of a $d_{M_{\text{ASD}}}$-form on $M_{\text{ASD}}$. The resulting real number is a topological invariant of the four-manifold $X$. Notice that in the process a map

$$H_k(X) \longrightarrow H^k(M_{\text{ASD}}),$$

is obtained.
has been constructed. Hence, the correlation functions of the topological theory give polynomials in $H_k_1(M_{\text{ASD}}) \times H_k_2(M_{\text{ASD}}) \times \cdots \times H_k_p(M_{\text{ASD}})$, which are precisely the Donaldson polynomial invariants of $X$.

The vacuum expectation values (16) can be collected in a very convenient way by introducing a generating function. Let us assume that we consider only smooth, compact, oriented four-manifolds which are simply connected. In this case only $W_0$ in (8) and $W_2$ in (9) are relevant operators. Given a basis $\{\Sigma_a\}_{a=1, \ldots, b_2(X)}$ of $H_2(X)$, we define, following [14][21][27], the generating function

$$F(\lambda, \alpha_1, \alpha_2, \ldots) = \left\langle e^{\sum_a \alpha_a I(\Sigma_a) + \lambda O}\right\rangle,$$

where $O = W_0$ as in (8), and $I(\Sigma_a) = f_{\Sigma_a} W_2$ as in (10). In (19) the quantities $\alpha_a$, $a = 1, \ldots, b_2(X)$ and $\lambda$ are constant parameters. The expectation values (16) can be easily extracted from the power expansion of (19).

In the weak coupling limit one finds for the function $F(\lambda, \alpha_1, \alpha_2, \ldots)$ in (19) an expression similar to (16). This expression does lead to the computation difficulties inherent in the Donaldson theory. In the weak coupling limit one proves that the Donaldson theory has a quantum field theory interpretation but this interpretation does not provide new insights to compute the Donaldson invariants. Nevertheless, the field theory connection is very important since in this theory the strong and weak coupling limits are exact, and therefore the door is open to find a strong coupling description which could lead to a new, simpler representation for the Donaldson invariants. The strong coupling realization of the Donaldson-Witten theory was found by Witten [14] after using the results on the strong coupling behavior of $\mathcal{N} = 2$ supersymmetric gauge theories which he and N. Seiberg [5] had discovered. The key ingredient used by Witten was to assume that the strong coupling limit of Donaldson-Witten theory is equivalent to the “sum” over the twisted effective low energy descriptions of the corresponding $\mathcal{N} = 2$ physical theory. This “sum” is not entirely a sum, as in general it has a part which contains a continuous integral. The “sum” is now known as integration over the $u$-plane after the work by Moore and Witten [21]. The reasons for this will become clear below. Witten’s assumption in [14] can be simply stated as saying that the weak-strong coupling limit and the twist commute. In other words, to study the strong coupling limit of the topological theory, first one untwists, then one works out the strong coupling limit of the physical theory and, finally, one twists back.

In order to implement the duality picture among moduli spaces we need to know two important pieces of information: the low energy description of $\mathcal{N} = 2$ extended supersymmetric Yang-Mills theories and the map of the topological observables (10) into their strong coupling counterparts. The first of these issues was addressed by Seiberg and Witten in 1994 [5] and its basic structure is by now well-known in rather general situations. The second was discussed in several works [14][20][27] but only recently it has been systematized using the canonical solution to the descent equations used for the first time, though in a different TQFT, in [28].

From the work by Seiberg and Witten [5] follows that at low energies $\mathcal{N} = 2$ supersymmetric Yang-Mills theories behave as Abelian gauge theories. For the case of gauge group $SU(2)$, which will be the case considered in this discussion, the effective low energy theory is parametrized by a complex variable $u$ which labels the vacuum structure of the
theory. At each value of \( u \) the effective theory is an \( \mathcal{N} = 2 \) supersymmetric Abelian gauge theory coupled to \( \mathcal{N} = 2 \) supersymmetric matter fields. One of the most salient features of the low-energy description is that there are points in the complex \( u \)-plane where some matter fields become massless. These points are singular points of the effective theory and they are located at \( u = \pm 1 \). At \( u = 1 \) the effective theory consists of an \( \mathcal{N} = 2 \) supersymmetric Abelian gauge theory coupled to a massless monopole, while at \( u = -1 \) it is coupled to a dyon. The effective theories at each singular point are related by a chiral \( \mathbb{Z}_2 \) symmetry which exists on the \( u \)-plane. This symmetry relates the behavior of the theory around one singularity to its behavior around the other.

One of the most important features of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theory is that its lagrangian can be written in terms of a single holomorphic function, the prepotential \( F \). This prepotential is holomorphic in the sense that it depends only on an \( \mathcal{N} = 2 \) chiral superfield \( \Psi \) which defines the theory, and not on its complex conjugate. The microscopic theory is defined by a classical quadratic prepotential:

\[
F_{cl}(\Psi) = \frac{1}{2} \tau_{cl} \Psi^2, \quad \tau_{cl} = \frac{\theta_{\text{bare}}}{2\pi} + \frac{4\pi i}{g_{\text{bare}}^2}.
\]

In terms of this prepotential the lagrangian is given by the following expression in \( \mathcal{N} = 1 \) superspace:

\[
\mathcal{L} = \frac{1}{4\pi} \text{ImTr} \left[ \int d^4\theta \frac{\partial F(A)}{\partial A} \bar{A} + \int d^2\theta \frac{1}{2} \frac{\partial^2 F(A)}{\partial A^2} W^\alpha W_\alpha \right],
\]

where \( A \) is a chiral \( \mathcal{N} = 1 \) superfield containing the fields \( (\phi, \psi) \), and \( W \) is a constrained chiral spinor superfield containing the non-Abelian gauge field and its \( \mathcal{N} = 1 \) superpartner \( (A_\mu, \lambda) \). All the fields take values in the adjoint representation of the gauge group, which we take to be \( SU(2) \). The potential term for the complex scalar \( \phi \) is:

\[
V(\phi) = \text{Tr} \left( [\phi, \phi]^2 \right).
\]

The minimum of this potential is attained at field configurations of the form \( \phi = \frac{1}{2} a \sigma^3 \), which define the classical moduli space of vacua. A convenient gauge invariant parametrization of the vacua is given by \( u = \text{Tr} \phi^2 \), which equals \( \frac{1}{2} a^2 \) semiclassically. For \( u \neq 0 \), \( SU(2) \) is spontaneously broken to \( U(1) \). The spectrum of the theory splits up into two massive \( \mathcal{N} = 2 \) vector multiplets, which accommodate the massive \( W^\pm \) bosons together with their superpartners, and an \( \mathcal{N} = 2 \) Abelian multiplet which accommodates the \( \mathcal{N} = 2 \) photon together with its superpartners. For \( u = 0 \), the full \( SU(2) \) symmetry is (classically) restored.

To study the quantum vacua Seiberg and Witten analyzed the structure of the low energy theory, whose effective lagrangian up to two derivatives is given, after integrating out the massive modes, by an expression like (21) but with a new effective prepotential depending only on an Abelian multiplet. The result of their analysis has some important features. First of all, it turns out that at the quantum level the \( SU(2) \) symmetry is never restored, i.e., the theory stays in the Coulomb phase throughout the \( u \)-plane. The moduli space of vacua (\( u \)-plane) is a complex one-dimensional Kähler manifold in which the prepotential \( F \) has singularities at the points \( u = \pm 1 \). These singularities correspond to the presence of a massless monopole (at \( u = 1 \)) and a massless dyon (at \( u = -1 \)). Near
each of the singularities, the complete non-singular effective action should include together
with the $\mathcal{N} = 2$ Abelian vector multiplet, a massless monopole or a dyon hypermultiplet.

As we did in the perturbative approach, we will consider the theory on manifolds $X$ with $b^+_2 > 1$. In the limit $g \to 0$ one has to take into account the classical moduli space. Since for $b^+_2 > 1$ there are not Abelian instantons the only contribution comes from $u = 0$ and one has to go through the analysis carried out in our discussion of the perturbative approach. As described there, one is led to the standard approach to Donaldson invariants via integration over the moduli space of non-Abelian instantons. On the other hand, in the limit $g \to \infty$, since the supersymmetric theory is asymptotically free, we are in the infrared regime, and the contributions come from the quantum moduli space. In the case under consideration ($b^+_2 > 1$) there are no Abelian instantons. Since the Abelian gauge field is the only massless field away from the singularities, the only contributions come from the singular points, $u = \pm 1$, where there are additional massless fields. Near each of these points, $\mathcal{N} = 2$ supersymmetry dictates the form of the weakly coupled effective theory. Since the observables of the twisted theory are independent of the coupling constant, one expects that Donaldson invariants can be expressed in terms of vevs of some operators in the twisted effective theories around each singular point.

The theory around the monopole singularity is an $\mathcal{N} = 2$ supersymmetric Abelian gauge theory coupled to a massless hypermultiplet. This theory has a twisted version which has been constructed in [15][16] from the point of view of twisting $\mathcal{N} = 2$ supersymmetry, and in [18] using the Mathai-Quillen formalism. It has been addressed in other works [17][29]. The structure of this theory is similar to that of the Donaldson-Witten theory. The resulting action is $\delta$-exact and therefore one can study the theory in the weak coupling limit, which, as the theory is Abelian, corresponds to the low energy limit.

Let us describe the structure of the twisted $\mathcal{N} = 2$ supersymmetric Abelian gauge theory coupled to a twisted hypermultiplet. We will assume that the four-dimensional manifold $X$ is a spin manifold. The analysis naturally extends to the case of manifolds which are not spin as shown in [14]. A hypermultiplet is built out of two chiral $\mathcal{N} = 1$ superfields, $Q$ and $\tilde{Q}$,

\begin{align*}
Q(q^1, \psi_{q\alpha}), & \quad Q^\dagger(q^1, \bar{\psi}_{\bar{q}\dot{\alpha}}), \\
\tilde{Q}(q^2, \psi_{\tilde{q}\alpha}), & \quad \tilde{Q}^\dagger(q^2, \bar{\psi}_{\bar{\tilde{q}}\dot{\alpha}}).
\end{align*}

(23)

After the twisting these fields become:

\begin{align*}
q^i \left(0, 0, \frac{1}{2}\right)^0 & \quad \mapsto \quad M^\alpha \left(\frac{1}{2}, 0\right)^0, \\
\psi_{q\alpha} \left(\frac{1}{2}, 0, 0\right)^1 & \quad \mapsto \quad \mu_\alpha \left(\frac{1}{2}, 0\right)^1, \\
\bar{\psi}_{\bar{q}\dot{\alpha}} \left(0, \frac{1}{2}, 0\right)^{-1} & \quad \mapsto \quad \nu_{\dot{\alpha}} \left(0, \frac{1}{2}\right)^{-1}, \\
q^i_\dagger \left(0, 0, \frac{1}{2}\right)^0 & \quad \mapsto \quad \bar{M}_\alpha \left(\frac{1}{2}, 0\right)^0, \\
\bar{\psi}_{\bar{q}\dot{\alpha}} \left(0, \frac{1}{2}, 0\right)^{-1} & \quad \mapsto \quad \bar{\nu}_{\dot{\alpha}} \left(0, \frac{1}{2}\right)^{-1},
\end{align*}

8
The twisted fields $M_\alpha, \mu_\alpha$, and $\nu_\alpha$ belong, respectively, to $\Gamma(S^+ \otimes L)$ and $\Gamma(S^- \otimes L)$, where $S^\pm$ are the positive/negative chirality spin bundles and $L$ is a complex line bundle. The action of the twisted Abelian effective theory can be found in \cite{13}. In the weak-coupling limit, the main contribution to the functional integral comes from a bosonic configuration given by the solutions to the equations:

$$F^+_{\alpha\beta} + \frac{i}{2} M_{(\alpha M_{\beta})} = 0, \quad D_{\alpha\dot{\alpha}} M^{\alpha} = 0.$$  \hfill (25)

These equations are known as monopole equations \cite{14}. The tangent space to the moduli space, $\mathcal{M}_{AM}$, defined by these equations is given by the linearization of (25), which happen to be the field equations:

$$(d\psi)_{\alpha\beta}^+ + \frac{i}{2} (\overline{M}_{(\alpha \mu_{\beta})} + \overline{\mu}_{(\alpha M_{\beta})}) = 0,$$

$$D_{\alpha\dot{\alpha}} \mu^{\alpha} + i\psi_{\alpha\dot{\alpha}} M^{\alpha} = 0.$$ \hfill (26)

The dimension of the moduli space can be calculated from (26) by means of an index theorem, and turns out to be \cite{14},

$$d_{\mathcal{M}_{AM}} = (c_1(L))^2 - \frac{2\chi + 3\sigma}{4}.$$ \hfill (27)

The only contributions to the partition function come from $d_{\mathcal{M}_{AM}} = 0$ (isolated monopoles). Introducing the shorthand notation, $x = -2c_1(L)$, we have:

$$d_{\mathcal{M}_{AM}} = 0 \Leftrightarrow x^2 = 2\chi + 3\sigma.$$ \hfill (28)

As in the Donaldson-Witten theory, the integration over the quantum fluctuations around the background (25) gives an alternating sum over the different monopole solutions for a given class $x$:

$$n_{x} = \sum_{i} \epsilon_{i,x}, \quad \epsilon_{i,x} = \pm 1.$$ \hfill (29)

The $n_x$ are the partition functions of the twisted Abelian theory for a fixed class $x$ (compare to (17)). Those classes such that (28) holds and $n_x \neq 0$ are called basic classes. The quantities $n_x$ turn out to constitute a new set of topological invariants for four-manifolds known as Seiberg-Witten invariants. Other observables different than the $n_x$ could be studied in the theory of Abelian monopoles. We have restricted our attention only to the partition function of this theory because it turns out that the Seiberg-Witten invariants, $n_x$, are the fundamental quantities that enter in the computation of the generating function (19) for the case of manifolds of simple type.

To compute the partition function of the full theory we must sum over classes and take into account the contribution from each of the singularities on the $u$-plane. Instead of computing the partition function we will concentrate our attention on the more general generating function \cite{13}. In computing this we must address the question of what is the
form of the observables of Donaldson-Witten theory in terms of operators of the effective Abelian theory. To answer this question we will use the expansion of the observables in the untwisted, physical theory, together with the canonical solution to the descent equations in the topological Abelian theory. We follow the argument presented in [4][21]. Near the monopole singularity, the $u$ variable has the expansion [5]:

$$u(a_D) = 1 + \left(\frac{du}{da_D}\right)_0 a_D + \text{higher order},$$

where $(du/da_D)_0 = -2i$, while “higher order” stands for operators of higher dimensions in the expansion. The field $a_D$ corresponds to the field $\phi_D$ of the topological Abelian theory [18], while the gauge-invariant parameter $u$ corresponds to the observable (8). Therefore, the map among operators between the microscopic and the macroscopic descriptions must be such that $O$ becomes $u$ up to a factor:

$$O \rightarrow \langle O \rangle u,$$

where $\langle O \rangle$ is a constant parameter. Once the operator $O$ has been identified one can use the canonical solution to the descent equations of the macroscopic description to obtain the operator which must correspond to $I(\Sigma)$ in (19). Actually we will concentrate on $I(v) = \sum a \alpha_a I(\Sigma_a)$ and we will think of $v$ as the formal sum $v = \sum a \alpha_a \Sigma_a$. The starting point is the operator $u$ and the procedure is described in [21]. One finds:

$$I(v) \rightarrow -\frac{i}{\pi \sqrt{2}} \int v \left( \frac{1}{32 d a_D^2} \psi \wedge \psi - \frac{\sqrt{2}}{4} \frac{du}{da_D} (F_{D-} + G_-) \right),$$

where $F_{D-}$ is the anti-self-dual part of the Abelian field strength $F_D$ and $\psi$ and $G_-$ are the Abelian versions of the anticommuting vector and the auxiliary field, respectively, in (3). As in the case of (31), the quantity $\langle V \rangle$ is a constant parameter. Both $\langle O \rangle$ and $\langle V \rangle$ can be reabsorbed by a rescaling of the parameters $\alpha_a$ and $\lambda$ in (19). They will be fixed later comparing the prediction from the TQFT to known mathematical results for a given manifold assuming that the parameters $\alpha_a$ and $\lambda$ in (19) can be identified with the ones used in the referred mathematical results.

Once the mapping has been established, one must study the vacuum expectation value (19) in the effective Abelian theory near the singularity at $u = 1$. For this theory the operator associated to $O$ in (31) becomes a $c$-number ($u = 1$) and we can make the replacement:

$$O \rightarrow \langle O \rangle.$$

The contribution from the operator associated to $I(v)$ is more subtle. First of all, for simply connected manifolds one can ignore the $\psi^2$ terms. In the part containing $F_{D-}$ in (32) one can replace $F_{D-}$ by $F_D$ since the difference is $Q$-exact. The integration of the term containing an auxiliary field $G_+$ can be easily done and it turns out that $\exp(I(v))$ is then mapped to [21]:

$$\exp(I(v)) \rightarrow \exp \left( -\frac{i}{4\pi} \int v \left( \frac{du}{da_D} F_D + \langle V \rangle^2 \frac{(du/da_D)^2}{8\pi \text{Im}\tau} v_+ \cdot v_+ \right) \right),$$

where $F_D$ is the field strength of the Abelian theory and $\langle V \rangle$ is a constant parameter. This is then mapped to (31):
where \( v_+ \cdot v_+ \) denotes the intersection pairing on \( H_2(X) \).

The term \( v_+ \cdot v_+ \) in (34) is a first sign of the presence of a contact term in the operator associated to \( \exp(I(v)) \) in the macroscopic description. On general grounds one expects that if a \( I(v) \) is mapped to \( \tilde{I}(v) \), the product \( I(\Sigma_1) I(\Sigma_2) \cdots I(\Sigma_n) \) is not mapped to the product \( \tilde{I}(\Sigma_1) \tilde{I}(\Sigma_2) \cdots \tilde{I}(\Sigma_n) \). Rather, one would expect contributions at the intersections of the surfaces \( \Sigma_a \). Thus, it is natural to expect a term containing \( v_+ v_+ = v^2 \) in the operator associated to \( \exp(I(v)) \). The explicit form of this term has been worked out in [21] using arguments based on the modular invariance of the low energy description. It turns out that the correct mapping is:

\[
\exp(I(v)) \rightarrow \exp \left( -\frac{i \langle V \rangle}{4\pi} \int_v \left( \frac{du}{da_D} F_D \right) + \langle V \rangle^2 v^2 T(u) \right),
\]

(35)

where

\[
T(u) = -\frac{1}{24} \left( E_2(\tau) \left( \frac{du}{da_D} \right)^2 - 8u \right),
\]

(36)

with \( E_2(\tau) \) the Eisenstein series of weight 2. In the limit \( u \to 1 \), one finds \( du/da = -2i \), \( T(1) = 1/2 \), and, therefore, the operator \( \exp(I(v)) \) becomes the \( c \)-number:

\[
\exp(I(v)) \rightarrow \exp \left( \langle V \rangle x \cdot v + \frac{\langle V \rangle^2}{2} v^2 \right),
\]

(37)

where \( x = -c_1(L^2) = -F_D / 2\pi \) is the class which entered (24).

We are now in a position to evaluate the correlation function (19) at the monopole singularity. To do this we must take into account (31) and (35), integrate over the space of monopole solutions \( \mathcal{M}_{AM} \), and then take the limit \( a_D \to 0 \). The factors (31) and (35) can be accompanied by some other terms that could be present due to the fact that the form of the twisted theory is not unique. Terms involving the \( F_D \wedge F_D \) or the Euler characteristic \( \chi \) and the signature \( \sigma \) of the manifold \( X \) could be added to the Lagrangian. The most general form of the possible terms present has been analyzed in [21]. Their concrete form can be computed using the wall crossing techniques described there. These terms are now known and one possesses an explicit expression for the vacuum expectation value (19) which involves the integration over the moduli spaces of monopole solutions and the consideration of the limit \( a_D \to 0 \). The solution notably simplifies if one assumes that the only contributions are the ones corresponding to \( d_{MA} = 0 \). Manifolds which satisfy this condition are called of simple type. In this situation the calculation simplifies because then one must consider the limit \( a_D \to 0 \) of the operators (31) and (35) which just corresponds to (33) and (37). The integration over the zero-dimensional moduli space leads to the Seiberg Witten invariants (29), and the global coefficient can be easily obtained considering the limit \( a_D \to 0 \) of the extra terms fixed imposing wall crossing conditions.

After summing over classes \( x \), the contribution from the point corresponding to the monopole singularity becomes:

\[
C_1 \exp \left( \frac{\langle V \rangle^2}{2} v^2 + \lambda \langle O \rangle \right) \sum_x n_x e^{\langle V \rangle v \cdot x},
\]

(38)
where $C_1$ is a constant which turns out to be:

$$C_1 = 2^{1 + \frac{7}{4} + \frac{1}{4\pi}}. \quad (39)$$

Next, we must work out the contribution from the dyon singularity at $u = -1$. This contribution is related to the one from $u = 1$ by a $\mathbb{Z}_2$ transformation, which is the anomaly-free symmetry on the $u$-plane which remains after the breaking of the chiral symmetry $U(1)_R$. Let us begin by recalling the transformations of the fields entering the observables under the $U(1)_R$-transformations:

$$
\begin{align*}
\psi_1^1 \alpha &\rightarrow e^{-i\varphi} \psi_1^1 \alpha, \\
\psi_2^2 \alpha &\rightarrow e^{-i\varphi} \psi_2^2 \alpha, \\
B &\rightarrow e^{-2i\varphi} B.
\end{align*}
$$

Instanton effects break this symmetry down to $\mathbb{Z}_8$ (4$N_c - 2N_f$ in the general case of $SU(N_c)$ gauge group with $N_f$ hypermultiplets in the fundamental representation). Under this anomaly-free $\mathbb{Z}_8$,

$$
B \rightarrow e^{-2i(\frac{2\pi}{8})} B = e^{-i\pi} B, \quad (40)
$$

and therefore,

$$
u = \text{Tr}(B^2) \rightarrow e^{-i\pi} u = -u, \quad (41)$$

which gives a $\mathbb{Z}_2$ symmetry on the $u$-plane. This $\mathbb{Z}_2$ symmetry relates the contributions to the vevs from $u = 1$ to those from $u = -1$. Under the $\mathbb{Z}_8$ symmetry, the observables transform as follows:

$$
\begin{align*}
I(\Sigma_a) &= \frac{1}{4\pi^2} \int_{\Sigma_a} \text{Tr} \left( \phi F + \frac{1}{2} \psi \wedge \psi \right) \rightarrow e^{-i\pi} I(\Sigma_a) = -iI(\Sigma_a), \\
O &= \frac{1}{8\pi^2} \text{Tr}(\phi)^2 \rightarrow e^{-i\pi} O = -O, \quad (42)
\end{align*}
$$

hence, using (38) one finds:

$$
u = 1, \quad C_1 \exp \left( \frac{\langle V \rangle^2}{2} v^2 + \lambda \langle O \rangle + \langle V \rangle v \cdot x \right),$$

$$u = -1, \quad C_2 \exp \left( -\frac{\langle V \rangle^2}{2} v^2 - \lambda \langle O \rangle - i\langle V \rangle v \cdot x \right). \quad (43)$$

The quantities $C_2$ and $C_1$ are related because on a curved background the $\mathbb{Z}_8$ transformation, while being preserved by gauge instantons, picks anomalous contributions from the measure due to gravitational anomalies. The contribution is of the form $\exp i\pi \Delta$, where $\Delta = \frac{\chi + \sigma}{4}$. Notice that for a basic class $x$, $\dim M_{AM} = 0$, and therefore, from (27), $(c_1(L))^2 = \frac{2\chi + 3\sigma}{4}$, so the index of the Dirac operator $D : \Gamma(S^+ \otimes L) \rightarrow \Gamma(S^- \otimes L)$ is precisely $\Delta$,

$$\text{Index } (D) = -\frac{\sigma}{8} + \frac{1}{2} (c_1(L))^2 = \frac{\chi + \sigma}{4} = \Delta \in \mathbb{Z}. \quad (44)$$

Then,

$$C_2 = i^{\Delta} C_1. \quad (45)$$
Finally, we take both contributions and sum over basic classes. The final form of the generating function (19) turns out to be:

$$F(\lambda, \alpha_1, \alpha_2, \ldots) = C_1 \left[ e^{\left(\frac{(\langle V \rangle)^2}{2} - (\langle O \rangle)\lambda\right)} \sum_x n_x e^{i\langle V \rangle \cdot x} + i^\Delta e^{\left(-\frac{(\langle V \rangle)^2}{2} - (\langle O \rangle)\lambda\right)} \sum_x n_x e^{-i\langle V \rangle \cdot x} \right].$$

(46)

By comparing to known results by Kronheimer and Mrowka [30] the constants $\langle O \rangle$ and $\langle V \rangle$ in (46) are fixed to be:

$$\langle O \rangle = 2, \quad \langle V \rangle = 1.$$  

(47)

These quantities are universal, i.e., entirely independent of the manifold $X$. This turns out to be the case according to the values (47), a very important test of construction.

We are now in a position to write down the final expression for the generating function of the Donaldson invariants:

$$F(\lambda, \alpha_1, \alpha_2, \ldots) = 2^{1+\frac{1}{2}(7\chi+11\sigma)} \left[ e^{\left(\frac{x^2}{2} + 2\lambda\right)} \sum_x n_x e^{x \cdot x} + i^\Delta e^{\left(-\frac{x^2}{2} - 2\lambda\right)} \sum_x n_x e^{-x \cdot x} \right].$$

(48)

The expression above verifies the so-called simple type condition:

$$\left(\frac{\partial^2}{\partial \lambda^2} - 4\right) F(\lambda, \alpha_1, \alpha_2, \ldots) = 0.$$  

(49)

All simply-connected four-manifolds with $b^+_2 > 1$ for which (48) is known verify this property.

So far we have discussed two different moduli problems in four-dimensional topology, one defined by the ASD instanton equations and another one defined by the Seiberg-Witten monopole equations. There is a natural generalization of these moduli problems which involves a non-Abelian gauge group and also includes spinor fields. It is the moduli problem defined by the non-Abelian monopole equations, introduced in Ref. [19] in the context of the Mathai-Quillen formalism and as a generalization of Donaldson theory. It has also been considered in Ref. [17][31], as well as in the mathematical literature [32][33][34][35].

In order to introduce these equations in the case of $G = SU(N)$ and the monopole fields in the fundamental representation $N$ of $G$, let us consider a Riemannian four-manifold $X$ together with a principal $SU(N)$-bundle $P$ and a vector bundle $E$ associated to $P$ through the fundamental representation. Suppose for simplicity that the manifold is spin, and consider a section $M_\alpha$ of $S^+ \otimes E$. The non-Abelian monopole equations read in this case:

$$F^{+ij}_{\alpha\beta} + i\left(\overline{M}_\alpha^j M_\beta^i - \delta^{ij}_N \overline{M}^k_{(\alpha} M_{\beta)}^l \right) = 0,$$

$$(D_E^{\hat{\alpha}} M_\alpha)^i = 0.$$  

(50)
Starting from these equations it is possible to build the associated TQFT within the Mathai-Quillen formalism. Not surprisingly, the resulting theory is the non-Abelian version of the topological theory of Abelian monopoles, that is, a twisted version of $\mathcal{N}=2$ super Yang-Mills coupled to one massless hypermultiplet. The field content is just the non-Abelian version of that of the Abelian monopole theory. The model can be extended by considering more than one hypermultiplet ($N_f > 1$), as proposed in [21]. Let us briefly describe in this note the case $N_f = 1$. We will follow the presentation in [3][4][19][20]. For a review, see [36].

From the monopole equations (50) follows that the appropriate geometric setting is the following. The field space is $\mathcal{A} \times \Gamma(X, S^+ \otimes E)$, which is the space of gauge connections on $P$ and positive chirality spinors in the representation $N$ of $G$. The vector bundle has as fiber, $\mathcal{F} = \Omega^{2,+}(X, \text{ad}P) \oplus \Gamma(X, S^+ \otimes E)$, as dictated by the quantum numbers of the monopole equations. The dimension of the moduli space of non-Abelian monopoles, $\mathcal{M}_{NA}$, is provided by a suitable index theorem [19]. It takes the form:

$$
\text{dim } \mathcal{M}_{NA} = \text{dim } \mathcal{M}_{ASD} + 2 \text{ index } D_E = (4N - 2) c_2(E) - \frac{N^2 - 1}{2} (x + \sigma) - \frac{N}{4} \sigma.
$$

(51)

Notice that $\mathcal{M}_{ASD} \subset \mathcal{M}_{NA}$. In addition to this, the usual conditions to have a well-defined moduli problem (like the reducibility) are essentially the same as in Donaldson theory.

The observables of the theory are the same as in the Donaldson-Witten theory since no non-trivial observables involving matter fields have been found. The topological invariants are then given by correlation functions of the form (14). In the perturbative regime, $g \to 0$, one finds the same pattern as in ordinary Donaldson-Witten theory. There is a map like in (18), $H_k(X) \longrightarrow H^k(\mathcal{M}_{NA})$, which implies that the vevs of the theory give a new set of polynomials in $H_{k_1}(\mathcal{M}_{NA}) \times H_{k_2}(\mathcal{M}_{NA}) \times \ldots \times H_{k_p}(\mathcal{M}_{NA})$. As in the case of the Donaldson-Witten theory, the perturbative approach does not provide any further insight into the precise form of these topological invariants. Fortunately, it is again possible to apply the results of Seiberg and Witten on $\mathcal{N}=2$ supersymmetric theories to analyze the model at hand in the non-perturbative regime.

To carry out the analysis at strong coupling one can follow the same strategy as in the case of Donaldson-Witten theory. The physical theory underlying the theory of non-Abelian monopoles is an $\mathcal{N}=2$ supersymmetric Yang-Mills theory coupled to one massless hypermultiplet in the fundamental representation of the gauge group, which we take to be $SU(2)$. This theory is asymptotically free. Hence, it is weakly coupled ($g \to 0$) in the ultraviolet, and strongly coupled ($g \to \infty$) in the infrared. The infrared behavior of this theory has been also determined by Seiberg and Witten [5]. As in the previous case the quantum moduli space of vacua is a one-dimensional complex Kähler manifold (the $u$-plane) and for any $u$ there is an unbroken $U(1)$ gauge symmetry (Coulomb phase). At a generic point on the $u$-plane the only light degree of freedom is the $U(1)$ gauge field (together with its $\mathcal{N}=2$ superpartners). However, in this case there are three singularities at finite values of $u$. These values are: $u_1 = -1$, $u_2 = e^{-\frac{i}{3} \pi}$ and $u_3 = e^{\frac{i}{3} \pi}$. Near each of these singularities a magnetic monopole or dyon becomes massless and weakly coupled to a dual $U(1)$ gauge field.

For $X$ such that $b^+_2 > 1$ (there are no Abelian instantons) the only contributions come
from the three singularities. Following the same arguments as in the previous case, and assuming that the manifold \( X \) is of simple type, one finds that the contribution from the singularity \( u_1 \) takes the form:

\[
\left\langle e^{(\sum \alpha_a I(\Sigma_a) + \lambda \mathcal{O})} \right\rangle_{u_1} = C_1 \sum_x n_x \exp \left( \lambda \mathcal{O} u_1 + \frac{i\langle V \rangle}{2} v \cdot x \left( \frac{du}{da} \right)_{u_1} + \langle V \rangle^2 v^2 \bar{T}(u_1) \right)
\]

\[
= C_1 \sum_x n_x \exp \left( -\lambda \mathcal{O} + i\langle V \rangle \sqrt{2} v \cdot x - \frac{2}{3} \langle V \rangle^2 v^2 \right), \quad (52)
\]

where,

\[
\bar{T}(u) = -\frac{1}{24} \left( \left( \frac{du}{da} \right)^2 - 8u \right), \quad (53)
\]

and the quantities \( C, \langle \mathcal{O} \rangle \) and \( \langle V \rangle \) are constants. In (52) we have used the fact that

\[
\bar{T}(u_j) = \frac{2}{3} u_j; \quad \left( \frac{du}{da} \right)_{u_j} = -8u_j. \quad (54)
\]

The contributions from the other singular points are obtained using the broken \( U(1)_R \) symmetry which in this case is \( \mathbb{Z}_6 \). This symmetry generates a \( \mathbb{Z}_3 \) symmetry on the \( u \)-plane which acting on the observables takes the form:

\[
I(\Sigma_a) \rightarrow e^{-\frac{2\pi i}{3}} I(\Sigma_a),
\]

\[
\mathcal{O} \rightarrow e^{\frac{2\pi i}{3}} \mathcal{O}, \quad (55)
\]

under the action of the generator of \( \mathbb{Z}_3 \). The contribution from each singularity possess a relative global factor which is obtained from the form of the gravitational anomalies [4] [20]. The final form of the generating function (19) for manifolds of simple is [20] [21]:

\[
F(\lambda, \alpha_1, \alpha_2, \ldots) = C \left( \exp \left( -\frac{2}{3} \langle V \rangle^2 v^2 - \lambda \langle \mathcal{O} \rangle \right) \sum_x n_x \exp(i\sqrt{2}\langle V \rangle v \cdot x) 
\right.
\]

\[
+ e^{-i\frac{\pi}{3}} \exp \left( -\frac{\pi}{3} \langle V \rangle^2 v^2 + \lambda \langle \mathcal{O} \rangle \right) \sum_x n_x \exp(e^{\frac{i\pi}{3}} \sqrt{2}\langle V \rangle v \cdot x),
\]

\[
+ e^{-i\frac{\pi}{3}} \exp \left( e^{\frac{i\pi}{3}} \langle V \rangle^2 v^2 + \lambda \langle \mathcal{O} \rangle \right) \sum_x n_x \exp(e^{-\frac{i\pi}{3}} \sqrt{2}\langle V \rangle v \cdot x),
\]

\[
(56)
\]

where unknown constants appear as in the pure Donaldson-Witten case. The generating function (56) verifies a generalized form of the simple type condition (43):

\[
\left( \frac{\partial^3}{\partial \lambda^3} - \langle \mathcal{O} \rangle^3 \right) \left\langle \exp \left( \sum_a \alpha_a I(\Sigma_a) + \lambda \mathcal{O} \right) \right\rangle = 0. \quad (57)
\]

Unfortunately, the left-hand side of (57) is not known for any manifold \( X \). Thus we can not fix the unknown constants as we did in the case of Donaldson theory (but see [37], where a general recipe to partly determine these constants is proposed.)
Generalized Donaldson-Witten theory for $N_f > 1$ has been considered in [21][37][38]. For $N_f < 4$ results similar to (56) are obtained for the case of manifolds of simple type. We will not review these cases in this paper. Instead we will turn our attention to the case of some of the twisted theories which emerge from $\mathcal{N} = 4$ supersymmetric theories.

Unlike the $\mathcal{N} = 2$ supersymmetric case, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is unique once the gauge group $G$ is fixed. The microscopic theory contains a gauge or gluon field, four chiral spinors (the gluinos) and six real scalars. All the above fields are massless and take values in the adjoint representation of the gauge group. As in the $\mathcal{N} = 2$ case, the $\mathcal{R}$-symmetry group of the $\mathcal{N} = 4$ algebra can be twisted to obtain a topological model. But since the $\mathcal{R}$-symmetry is now $SU(4)$, this topological twist can be performed in three inequivalent ways, so one ends up with three different TQFTs [25][39][40]. The twisted theories are topological in the sense that the partition function as well as a selected set of correlation functions are independent of the metric which defines the background geometry. In the short distance regime, computations in the twisted theory are given exactly by a saddle-point calculation around a certain bosonic background or moduli space, and in fact the correlation functions can be reinterpreted as describing intersection theory on this moduli space. This correspondence can be made more precise through the Mathai-Quillen construction [40]. Unfortunately, it is not possible to perform explicit computations from this viewpoint: the moduli spaces one ends up with are generically non-compact, and no precise recipe is known to properly compactify them.

As explained above, a complementary approach which sheds more light on the structure of the twisted theories and allows explicit computations involves the long-distance regime (or strong coupling regime in the asymptotically free theories), where one expects that a good description should be provided by the degrees of freedom of the vacuum states of the physical theory on $\mathbb{R}^4$. We have seen above how this program works for the $\mathcal{N} = 2$ theories, and it would be interesting to see whether similar constructions work for the $\mathcal{N} = 4$ theories.

While for the TQFTs related to asymptotically free $\mathcal{N} = 2$ theories the interest lies in their ability to define topological invariants for four-manifolds, for the twisted $\mathcal{N} = 4$ theories the topological character is used as a tool for performing explicit computations which might shed light on the structure of the physical $\mathcal{N} = 4$ theory. This theory is finite and conformally invariant, and is conjectured to have a symmetry exchanging strong and weak coupling and exchanging electric and magnetic fields, which extends to a full $SL(2,\mathbb{Z})$ symmetry acting on the microscopic complexified coupling $\tau_0$ [41]. It is natural to expect that this property should be shared by the twisted theories on arbitrary four-manifolds. This was checked by Vafa and Witten for one of the twisted theories and for gauge group $SU(2)$ [25], and it was clearly mostly interesting to extend their computation to higher rank groups and to the other twisted theories.

In [42] the $u$-plane approach was applied to the twisted mass-deformed $\mathcal{N} = 4$ SYM theory with gauge group $SU(2)$. This theory is obtained by twisting the $\mathcal{N} = 4$ SYM theory with bare masses for two of the chiral multiplets. It is a non-Abelian monopole theory as the one described above but with the monopole multiplets taking values in the adjoint representation of the gauge group [10]. The physical theory preserves $\mathcal{N} = 2$ supersymmetry, and its low-energy effective description for gauge group $SU(2)$ was given by Seiberg and Witten [3], and later extended to $SU(N)$ in [13].
The ghost-number symmetry of the twisted theory for gauge group $SU(2)$ has an anomaly $-3(2\chi + 3\sigma)/4$ on gravitational backgrounds. Topological invariants are thus obtained by considering the vacuum expectation value of products of observables with ghost-numbers adding up to $-3(2\chi + 3\sigma)/4$. The relevant observables for this theory and gauge group $SU(2)$ or $SO(3)$ are precisely the same as in the Donaldson-Witten theory [3] and [4]. In addition to this, since all the fields in the theory take values in the adjoint representation of the gauge group, it is possible to enrich the theory by including non-Abelian electric and magnetic ‘t Hooft fluxes [44] which should behave under $SL(2,\mathbb{Z})$ duality in a well-defined fashion [25] [44].

The generating function for these correlation functions is given [12] as an integration over the moduli space of vacua (the $u$-plane) of the physical theory. At a generic vacuum, the only contribution comes from a twisted $\mathcal{N} = 2$ Abelian vector multiplet. The effect of the massive modes is contained in appropriate measure factors, which also incorporate the coupling to gravity, and in contact terms among the observables [21] [22] [23].

The total contribution to the generating function thus consists of an integration over the moduli space with the singularities removed – which is non-vanishing for $b_2^+ (X) = 1$ [21] only – plus a discrete sum over the contributions of the twisted effective theories at each of the three singularities of the low-energy effective description [3]. The effective theory at a given singularity contains, together with the appropriate dual photon multiplet, one charged hypermultiplet, which corresponds to the state becoming massless at the singularity. The complete effective action for these massless states contains as well certain measure factors and contact terms among the observables, which reproduce the effect of the massive states which have been integrated out. How to determine these a priori unknown functions was explained in [21]. The idea is as follows. At those points on the $u$-plane where the (imaginary part of the) effective coupling diverges, the integral is discontinuous at antiself-dual Abelian gauge configurations. This is commonly referred to as “wall crossing”. Wall crossing can take place at the singularities of the moduli space – the appropriate local effective coupling $\tau$ diverges there – and, in the case of the asymptotically free theories, at the point at infinity – the effective electric coupling diverges owing to asymptotic freedom.

On the other hand, the final expression for the invariants can exhibit a wall-crossing behavior at most at $u \to \infty$, so the contribution to wall crossing from the integral at the singularities at finite values of $u$ must cancel against the contributions coming from the effective theories there, which also display wall-crossing discontinuities. Imposing this cancellation fixes almost completely the unknown functions in the contributions to the topological correlation functions from the singularities. The final result for the contributions from the singularities (which give the complete answer for the correlation functions when $b_2^+ > 1$) is written explicitly and completely in terms of the periods and the discriminant of the Seiberg-Witten solution for the physical theory. For simply-connected spin four-manifolds of simple type the generating function is given by:

$$
\left\langle e^{\nu \mathcal{O} + I(S)} \right\rangle_u = 2^{\frac{2\chi + 3\sigma}{2}} m^{-(3\nu + \sigma/4)} (\eta(\tau_0))^{-12\nu} \left\{ (\kappa_1)^\nu \left( \frac{da}{du} \right)^{-\nu + \frac{2\chi}{2}} e^{2p a_u + S^2 T_1} \sum_x \delta_{\left[ T_1 \right], v} n_x e^x (du/da)_x - S \right\}
$$
\begin{align}
+2 \frac{b}{\pi} (-1)^{\sigma/8} (\kappa_2)^{\nu} \left( \frac{du}{da} \right)^{-\nu+\frac{7}{2}} e^{2p_{a_2}+S^2T_2} \sum_x (-1)^{v_x} n_x e^{(du/da)x} \cdot S \\
+2 \frac{b}{\pi} i^{-v^2} (\kappa_3)^{\nu} \left( \frac{du}{da} \right)^{-\nu+\frac{7}{2}} e^{2p_{a_3}+S^2T_3} \sum_x (-1)^{v_x} n_x e^{(du/da)x} \cdot S \right),
\end{align}

where \( x \) is a Seiberg-Witten basic class, \( \nu = (\chi + \sigma)/4, v \in H^2(X, \mathbb{Z}_2) \) is a 't Hooft flux, \( \eta(\tau_0) \) is the Dedekind function, \( \kappa_i = (du/dq)_{u=u_i} - \) with \( q = \exp(2\pi i \tau) \) – and the contact terms \( T_i \) have the form

\[
T_i = -\frac{1}{12} \left( \frac{du}{da} \right)^2_{u=u_i} + E_2(\tau_0) \frac{u_i}{6} + \frac{m^2}{72} E_4(\tau_0)
\]

being \( E_2 \) and \( E_4 \) are the Eisenstein series of weights 2 and 4 respectively\(^2\). Evaluating the quantities in (58) gives the final result as a function of the physical parameters \( \tau_0 \) and \( m \), and of topological data of \( X \) as \( \chi, \sigma \) and the basic classes \( x \).

The formula has nice properties under the modular group. For the partition function \( Z_v \),

\[
Z_v(\tau_0 + 1) = (-1)^{\sigma/8} i^{-v^2} Z_v(\tau_0), \\
Z_v(-1/\tau_0) = 2^{-b_2/2} (-1)^{\sigma/8} \left( \frac{\tau_0}{i} \right)^{-\chi/2} \sum_w (-1)^{w-v} Z_w(\tau_0).
\]

Also, with \( Z_{SU(2)} = 2^{-1} Z_{v=0} \) and \( Z_{SO(3)} = \sum_v Z_v \),

\[
Z_{SU(2)}(\tau_0 + 1) = (-1)^{\sigma/8} Z_{SU(2)}(\tau_0), \\
Z_{SO(3)}(\tau_0 + 2) = Z_{SO(3)}(\tau_0), \\
Z_{SU(2)}(-1/\tau_0) = (-1)^{\sigma/8} 2^{-\chi/2} \tau_0^{-\chi/2} Z_{SO(3)}(\tau_0).
\]

Notice that the last of these three equations corresponds precisely to the strong-weak coupling duality transformation conjectured by Montonen and Olive [11].

As for the correlation functions, one finds the following behavior under the inversion of the coupling

\[
\langle \frac{1}{8\pi^2} \text{Tr} \phi^2 \rangle_{\tau_0}^{SU(2)} = \langle \mathcal{O} \rangle_{\tau_0}^{SU(2)} = \frac{1}{\tau_0^2} \langle \mathcal{O} \rangle_{-1/\tau_0}^{SO(3)}, \\
\langle \frac{1}{8\pi^2} \int_S \text{Tr} (2\phi F + \psi \wedge \psi) \rangle_{\tau_0}^{SU(2)} = \langle I(S) \rangle_{\tau_0}^{SU(2)} = \frac{1}{\tau_0^2} \langle I(S) \rangle_{-1/\tau_0}^{SO(3)}, \\
\langle I(S) I(S) \rangle_{\tau_0}^{SU(2)} = \left( \frac{\tau_0}{i} \right)^{-4} \langle I(S) I(S) \rangle_{-1/\tau_0}^{SO(3)} + \frac{i}{2\pi \tau_0^3} \langle \mathcal{O} \rangle_{-1/\tau_0}^{SO(3)} \#(S \cap S).
\]

Therefore we see that, as expected, the partition function of the twisted theory transforms as a modular form, while the topological correlation functions turn out to transform

\(^2\)Notice that we have changed the notation used in [13]. The parameter \( \lambda \) has been replaced by \( p \) and the formal sum \( v = \sum \alpha \Sigma \alpha \) by \( S \). In the rest of the paper \( v \) will denote a 't Hooft flux.
covariantly under $SL(2, \mathbb{Z})$, following a pattern which can be reproduced with a far more simple topological Abelian model [12].

The second example we will consider is the Vafa-Witten theory [25], which corresponds to another non-equivalent twist of the $\mathcal{N} = 4$ theory. The twisted theory does not contain spinors, so it is well-defined on any compact, oriented four-manifold. The ghost-number symmetry of this theory is anomaly-free, and therefore the only non-trivial topological observable is the partition function itself. As in the above example, it is possible to consider non-trivial gauge configurations in $G/\text{Center}(G)$ and compute the partition function for a fixed value of the 't Hooft flux $v \in H^2(X, \pi_1(G))$. In this case, however, the Seiberg-Witten approach is not available, but, as conjectured by Vafa and Witten, one can nevertheless compute in terms of the vacuum degrees of freedom of the $\mathcal{N} = 1$ theory which results from giving bare masses to all the three chiral multiplets of the $\mathcal{N} = 4$ theory.

As explained in detail in [25][45][46], the twisted massive theory is topological on Kähler four-manifolds with $h^{2,0} \neq 0$, and the partition function is actually invariant under the perturbation. In the long-distance limit, the partition function is given as a finite sum over the contributions of the discrete massive vacua of the resulting $\mathcal{N} = 1$ theory. In the case at hand, it turns that for $G = SU(N)$, the number of such vacua is given by the sum of the positive divisors of $N$ [43]. The contribution of each vacuum is universal (because of the mass gap), and can be fixed by comparing to known mathematical results [25]. However, this is not the end of the story. In the twisted theory the chiral superfields of the $\mathcal{N} = 4$ theory are no longer scalars, so the mass terms can not be invariant under the holonomy group of the manifold unless one of the mass parameters be a holomorphic two-form $\omega$. This spatially dependent mass term vanishes where $\omega$ does, and we will assume as in [25][27] that $\omega$ vanishes with multiplicity one on a union of disjoint, smooth complex curves $C_i$, $i = 1, \ldots, n$ of genus $g_i$ which represent the canonical divisor $K$ of $X$. The vanishing of $\omega$ introduces corrections involving $K$ whose precise form is not known a priori. In the $G = SU(2)$ case, each of the $\mathcal{N} = 1$ vacua bifurcates along each of the components $C_i$ of the canonical divisor into two strongly coupled massive vacua. This vacuum degeneracy is believed to stem [25][27] from the spontaneous breaking of a $\mathbb{Z}_2$ chiral symmetry which is unbroken in bulk.

The structure of the corrections for $G = SU(N)$ – see (63) below – suggests that the mechanism at work in this case is not chiral symmetry breaking. Indeed, near any of the $C_i$ there is an $N$-fold bifurcation of the vacuum. A plausible explanation for this degeneracy could be found in the spontaneous breaking of the center of the gauge group (which for $G = SU(N)$ is precisely $\mathbb{Z}_N$). For further details and speculations, we refer the reader to [17]. In any case, the formula for $SU(N)$ can be computed (at least when $N$ is prime) along the lines explained in [25] and assuming that the resulting partition function satisfies a set of non-trivial constraints which are described below.

Then, for a given 't Hooft flux $v \in H^2(X, \mathbb{Z}_N)$, the partition function for gauge group

\footnote{A similar approach was introduced by Witten in [27] to obtain the first explicit results for the Donaldson-Witten theory just before the far more powerful Seiberg-Witten approach were available.}

\footnote{Incidentally, this is the origin of the constraint $h^{(2,0)} \neq 0$ mentioned above.}
SU(N) (with prime N) is [47]:

\[ Z_v = \left( \sum_{\tilde{\varepsilon}} \delta_{v,w}(\tilde{\varepsilon}) \prod_{i=1}^{n} \prod_{\lambda=0}^{N-1} \left( \frac{\chi_{\lambda}}{\eta} \right)^{(1-g_i)\delta_{\varepsilon_i,\lambda}} \right) \left( \frac{1}{N^2} G(q^N) \right)^{\nu/2} \]

\[ + N^{1-b_v} \sum_{m=0}^{N-1} \left[ \prod_{i=1}^{n} \left( \sum_{\lambda=0}^{N-1} \left( \frac{\chi_{m,\lambda}}{\eta} \right)^{1-g_i} e^{2\pi i N \lambda [C_i]_N} \right) \right] e^{i\pi \frac{N-1}{N} m^2} \left( \frac{1}{N^2} G(\alpha^m q^{1/N}) \right)^{\nu/2}, \]

(63)

where \( \alpha = \exp(2\pi i/N) \), \( G(q) = \eta(q)^{24} \) (with \( \eta(q) \) the Dedekind function), \( \chi_{\lambda} \) are the \( SU(N) \) characters at level 1 [48] and \( \chi_{m,\lambda} \) are certain linear combinations thereof. \([C_i]_N\) is the reduction modulo \( N \) of the Poincaré dual of \( C_i \), and

\[ w_N(\tilde{\varepsilon}) = \sum_{i=1}^{n} \varepsilon_i [C_i]_N, \]

(64)

where \( \varepsilon_i = 0, 1, \ldots N - 1 \) are chosen independently.

The formula (63) does not apply directly to the \( N = 2 \) case. For \( N = 2 \) there are some extra relative phases \( t_i \) – see equations (5.45) and (5.46) in [25] – which are absent for \( N > 2 \) and prime. Modulo these extra phases, (63) is a direct generalization of Vafa and Witten’s formula. It reduces on \( K3 \) to the formula of Minahan, Nemeschansky, Vafa and Warner [49] and generalizes their result to non-zero ’t Hooft flux. In addition to this, the formula has the expected properties under the modular group [25]

\[ Z_v(\tau_0 + 1) = e^{i\pi N(2\chi + 3\tau)} e^{-i\pi N \chi_1} Z_v(\tau_0), \]

\[ Z_v(-1/\tau_0) = N^{-b_v/2} \left( \frac{\tau_0}{i} \right)^{-\chi/2} \sum_u e^{2i\pi b u} Z_u(\tau_0), \]

(65)

and also, with \( Z_{SU(N)} = N^{b_v-1} Z_0 \) and \( Z_{SU(N)/Z_N} = \sum_v Z_v \),

\[ Z_{SU(N)}(-1/\tau_0) = N^{-\chi/2} \left( \frac{\tau_0}{i} \right)^{-\chi/2} Z_{SU(N)/Z_N}(\tau_0), \]

(66)

which is, up to some correction factors which vanish in flat space, the original Montonen-Olive conjecture.

There is a further property to be checked which concerns the behavior of (63) under blow-ups. This property was heavily used in [25] and demanding it in the present case was essential in deriving the above formula. Blowing up a point on a Kähler manifold \( X \) replaces it with a new Kähler manifold \( \hat{X} \) whose second cohomology lattice is \( H^2(\hat{X}, \mathbb{Z}) = H^2(X, \mathbb{Z}) \oplus I^- \), where \( I^- \) is the one-dimensional lattice spanned by the Poincaré dual of the exceptional divisor \( B \) created by the blow-up. Any allowed \( \mathbb{Z}_N \) flux \( \hat{v} \) on \( \hat{X} \) is of the form \( \hat{v} = v \oplus r \), where \( v \) is a flux in \( X \) and \( r = \lambda B \), \( \lambda = 0, 1, \ldots N - 1 \). The main result concerning (63) is that under blowing up a point on a Kähler four-manifold with canonical divisor as above, the partition functions for fixed ’t Hooft fluxes have a factorization as

\[ Z_{\hat{X},\hat{v}}(\tau_0) = Z_{X,v}(\tau_0) \frac{\chi_\lambda(\tau_0)}{\eta(\tau_0)}. \]

(67)
Precisely the same behavior under blow-ups of the partition function (63) has been proved by Yoshioka [50] for the generating function of Euler characteristics of instanton moduli space on Kähler manifolds. This should not come out as a surprise since it is known that, on certain four-manifolds, the partition function of Vafa-Witten theory computes Euler characteristics of instanton moduli spaces [25][49]. Therefore, (63) can be seen as a prediction for the Euler numbers of instanton moduli spaces on those four-manifolds.

In the light of the AdS/CFT correspondence [7][8][9], it would be mostly interesting to investigate what the large $N$ limit of (63) corresponds to on the gravity side, and to extend the computation to all $N$. We expect to address these topics in the near future.

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