Microstate Renormalization in Deformed D1-D5 SCFT

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Abstract

We derive the corrections to the conformal dimensions of twisted Ramond ground states in the deformed two-dimensional \(\mathcal{N} = (4, 4)\) superconformal \((T^4)^N/S_N\) orbifold theory describing bound states of the D1-D5 brane system in type IIB superstring theory. Our result holds to second order in the deformation parameter, and at the large \(N\) planar limit. The method of calculation involves the analytic evaluation of integrals of four-point functions of two R-charged twisted Ramond fields and two marginal deformation operators. We also calculate the deviation from zero, at first order in the considered marginal perturbation, of the structure constant of the three-point function of two Ramond fields and one deformation operator.

Keywords: Symmetric product orbifold of \(\mathcal{N} = 4\) SCFT, marginal deformations, twisted Ramond fields, correlation functions, anomalous dimensions.
1. Introduction

The bound states of the two-charge D1-D5 brane system describe, at certain limits, an extremal supersymmetric black hole. Its Bekenstein-Hawking entropy can be calculated by counting BPS states in a particular two-dimensional $\mathcal{N} = (4, 4)$ super-conformal field theory (SCFT$_2$) [1–3], while its classical (low-energy) description in IIB supergravity is an asymptotically flat, five-dimensional extremal black ring with a degenerate horizon of radius zero. In the near-horizon (decoupling) limit, this geometry becomes supersymmetric AdS$_3 \times S^3 \times T^4$, with large Ramond-Ramond charges [2, 4], see also Ref. [5] for an extensive review. Mathur’s fuzzball proposal [6, 7] replaces the interior of this extremal black hole (or black ring) with a fuzzy quantum average over asymptotically AdS$_3$ geometries — or, more generally, AdS$_3/Z_N$, with conic singularities [4, 8, 9] — and, according to the AdS$_3$/CFT$_2$ correspondence, such geometries can be microscopically expressed in terms of twisted Ramond states of a $\mathcal{N} = (4, 4)$ SCFT$_2$ with central charge $c = 6N$. This SCFT$_2$ is best understood at a point in moduli space where, one conjectures, it becomes a free theory with target space $(T^4)^N/S_N$; going towards the supergravity limit requires the deformation of this free $S_N$-orbifold theory by a scalar moduli marginal operator [10].

Extensive research of the orbifold SCFT$_2$ and its deformation has been able to explain essential quantum and thermodynamical properties of black holes [11–32], but the complete description of the spectra and the dynamics of the deformed SCFT$_2$ — the energies and charges of the fields and their multi-point correlation functions — is still missing. Recent progress [33–37] in understanding the rules which select protected states from those that get ‘lifted’, i.e. whose conformal dimensions change, indicates a need for more efficient methods of calculation for the energy lifts of ‘$n$-strands’ twisted states.

The problem addressed in the present letter concerns the renormalization of twisted ground-state Ramond fields in the deformed theory. We present a method for calculating an explicit analytic expression for the $\lambda^2$-correction of their conformal dimension,

$$\Delta^R_n(\lambda) = \Delta^R_n(0) + \frac{1}{2}\pi \lambda^2 |J_R(n)|, \quad (1.1)$$

where $\Delta^R_n(0)$ is the “bare” dimension of the Ramond fields in the free orbifold.
point. We calculate $J_R$ by integrating a four-point function, and exploring an analytic continuation of ‘Dotsenko-Fateev’ integrals. Our main result is an expression for $J_R(n)$, which is finite, non-vanishing, and at large $n$ seems to stabilize around definite values, as shown in Fig. 2.

2. The $(T^4)^N/S_N$ Free Orbifold and Its Deformation

The ‘free orbifold point’ theory is composed of $N$ copies of the $\mathcal{N} = (4,4)$ SCFT$_2$ with free bosons $X^{AA}(z, \bar{z})$ and free fermions $\psi^{\ddagger A}(z)$, $\tilde{\psi}^{\ddagger A}(z)$, identified under the symmetric group $S_N$, thus forming the orbifolded target space $(T^4)^N/S_N$, where there are twist operators corresponding to twisted boundary conditions [38]. The indices $A = 1, 2$ and $\ddagger A = 1, 2$ label doublets of the internal symmetry group of $T^4$, $SO(4)_1 = SU(2)_1 \times SU(2)_2$, while $\alpha = +, -$ and $\ddagger \alpha = +, -$ label doublets of the R-symmetry group $SU(2)_L \times SU(2)_R$. We work with Euclidean signature, and on the conformal plane $E_2$ or its $S^2$ compactification (in opposition to the $\mathbb{R} \times S^1$ cylinder picture).

One moves away from the free orbifold point, and towards the supergravity description, with a deformation,

$$S_{\text{def}}(\lambda) = S_{\text{orb}} + \lambda \int d^2z O^{(\text{int})}[2](z, \bar{z}),$$

(2.1)

where $\lambda$ is a dimensionless coupling constant. The scalar moduli interaction operator $O^{(\text{int})}[2](z, \bar{z})$ is marginal, with conformal dimension $\Delta_{\text{int}} = \hat{h}_{\text{int}} + \bar{\hat{h}}_{\text{int}} = 2$ protected from renormalization. It is a singlet of the R-symmetry group constructed from NS modes of the supercharges $G^{\alpha A}(z)$, and $\tilde{G}^{\ddagger A}(\bar{z})$,

$$O^{(\text{int})}[2](z, \bar{z}) = \epsilon_{AB} G^{-A} G_{-\frac{1}{2}}^{-B} \sigma_{[2]}^+[z, \bar{z}]$$

(2.2)

where $\sigma_{[2]}^+$ is a chiral primary NS field with twist 2, conformal weights $(h, \tilde{h}) = (\frac{1}{2}, \frac{1}{2})$ and R-charges $(j^3, \tilde{j}^3) = (\frac{1}{2}, \frac{1}{2})$, see e.g. [11]. Brackets around the twist of an operator $\mathcal{O}_n$ indicate an $S_N$-invariant combination of length-$n$ single-cycle “twistings” of $\mathcal{O}$, obtained by summing over the elements of the conjugate class of $(1 \cdots n) \in S_N$, as in [15, 39].

3. Four-point functions and renormalization of Ramond fields

The effect of the deformation (2.1) on the conformal data of fields is described by conformal perturbation theory [39]. In this letter we are interested in the changes to the conformal dimension of twisted Ramond fields $R^{\ddagger R}_n(z, \bar{z})$. These are charged with $(j^3, \tilde{j}^3) = (\pm \frac{1}{2}, \pm \frac{1}{2})$ under R-symmetry, have twist $n$, conformal weights $(h_n^R, \tilde{h}_n^R) = (\frac{n}{4}, \frac{n}{4})$, and conformal dimension $\Delta_n^R(0) = \frac{n}{2}$ at the free orbifold point [11, 39]. In the deformed theory (2.1), the dimension becomes a function $\Delta_n^R(\lambda)$, which can be determined order-by-order in the parameter $\lambda$ by looking at the corrections to the two-point function $(R^{\ddagger R}_n(z_1, \bar{z}_1) R^{+ R}_n(z_2, \bar{z}_2))_\lambda$. At first order, the change is proportional to
the integral of the 3-point function \( \langle R_{[n]}^{-}(\infty)O_{[2]}^{(\text{int})}(z)R_{[n]}^{+}(0) \rangle_0 \). This function however vanishes, since there is no field \( R_{[n]}^{+} \) in the OPE \( O_{[2]}^{(\text{int})}(z)R_{[n]}^{+}(0) \) at all \([39]\).

At second order in \( \lambda \), the correction to the two-point function is given by the integral

\[
\frac{\lambda^2}{2}\int d^2z_2 \int d^2z_3 \langle R_{[n]}^{-}(z_1, \bar{z}_1)O_{[2]}^{(\text{int})}(z_2, \bar{z}_2)O_{[2]}^{(\text{int})}(z_3, \bar{z}_3)R_{[n]}^{+}(z_4, \bar{z}_4) \rangle,
\]

with the \( S_N \)-invariant four-point function is evaluated at the free orbifold point. Conformal invariance implies that

\[
\langle R_{[n]}^{-}(z_1, \bar{z}_1)O_{[2]}^{(\text{int})}(z_2, \bar{z}_2)O_{[2]}^{(\text{int})}(z_3, \bar{z}_3)R_{[n]}^{+}(z_4, \bar{z}_4) \rangle = G(u, \bar{u})
\]

where \( G(u, \bar{u}) = G(u)\bar{G}(\bar{u}) \) is an arbitrary function of the anharmonic ratio \( u = (z_{12}z_{34})/(z_{13}z_{24}) \). Global \( \text{SL}(2, \mathbb{C}) \) transformations can be used to fix \( z_1 = \infty, z_2 = 1 \) and \( z_4 = 0 \); as a consequence, \( u = z_3 \) and

\[
G(u) = \langle R_{[n]}^{-}(\infty)O_{[2]}^{(\text{int})}(1)O_{[2]}^{(\text{int})}(u)R_{[n]}^{+}(0) \rangle.
\]

The standard technology for calculation of multi-point functions in the orbifolded theory is the ‘covering surface technique’ of Lunin and Mathur [11, 12]. Applied to a four-point function, the idea is to map the ‘base sphere’ \( S_{\text{base}} \), with the four twist operators, to a ‘covering surface’ \( \Sigma_{\text{cover}} \), with four branching points, on which the twist operators are trivialized and one is left with a free, non-orbifolded theory. For large \( N \), we can consider \( \Sigma_{\text{cover}} \) to have genus \( g = 0 \), i.e. to be a ‘covering sphere’ \( S_{\text{cover}} \) [40]. For the twists in (3.3), the unique map \( \Sigma_{\text{cover}} \mapsto S_{\text{base}} \) is [12, 39, 40]

\[
z(t) = \left( \frac{t}{t_1} \right)^n \left( \frac{t - t_0}{t_1 - t_0} \right) \left( \frac{t_1 - t_\infty}{t - t_\infty} \right),
\]

where \( z \in S_{\text{base}} \) and \( t \in S_{\text{cover}} \). If we label the image of \( u \) on the covering by \( x \), such that \( z(x) = u \), then the correct monodromy requires that the parameters \( t_0, t_1, t_\infty \) all be functions of \( x \), which defines a function \( u(x) \). With a convenient parameterization, this is the ‘Arutyunov-Frolov map’ [41],

\[
u(x) = \frac{x^{n-1}(x + n)^{n+1}}{(x - 1)^{n+1}(x + n - 1)^{n-1}}.
\]

After fixing the map (3.5), we can calculate the (image of the) connected part of the four-point function (3.3) on the covering surface. This will be a function \( G(x) \), in a free theory with \( N \) identical copies of the fields but no twisted boundary conditions. Using the ‘stress-energy tensor method’ pioneered in [38], cf. also [15, 16, 39], we find\(^1\)

\[
G(x) = C_R \frac{x^{5(2+n)}(x - 1)^{5(2+n)}(x + n)^{2-2n}(x + n - 1)^{2-2n}}{(x + n^{1-2})^4}.
\]

\(^1\)These results will appear in a more detailed presentation elsewhere [39].
If we take the limit of coincidence between the two deformation operators, i.e. $u \to 1$ in (3.3), we can check [39] that (3.6) does give the same result as recently found in [22] from the four-point function with two chiral NS fields. This limit also fixes the constant $C_R = 1/16n^2$.

We are prompt to proceed with the calculation of the second-order correction to $\langle R_{[n]}^- R_{[n]}^+ \rangle_\lambda$, by inserting $|G(x(u))|^2$ into the integral (3.1),

$$
\frac{1}{2} \lambda^2 \frac{1}{|z_{14}|^n} \int d^2z_3 \frac{|z_{14}|^2}{|z_{13}|^2 |z_{34}|^2} \int d^2u G(u, \bar{u})
= \lambda^2 \frac{\pi}{n} \log \Lambda \int d^2u G(u, \bar{u}).
$$

(3.7)

We have used $z_3$ and $u$ as integration variables, and introduced a cutoff $\Lambda$ to regulate the divergent integral over $z_3$. The logarithm at the r.h.s. indicates that there will be renormalization of the conformal dimension of $R_{[n]}^\pm$, given by the remaining integral over $u$.

Hence we turn to calculating the latter integral,

$$
J_R \equiv \int d^2u G(u, \bar{u})
= \int d^2x |u'(x)G(x)|^2 \equiv [n(n+1)C_R]^2 I,
$$

(3.8)

making use of the function $G(x)$. Note that after the change of variables, all we need to know is the function $G(x)$ that we found on the covering surface. By a convenient change of the variables, $y(x) = -\frac{4(x-1)(x+n)}{(n+1)^2}$, the integral $I$ defined in (3.8), becomes

$$
I = \int d^2y |y|^{2a} |1 - y|^{2b}|y - w_n|^{2c}, \quad w_n = \frac{4n}{(n+1)^2},
$$

(3.9)

a double integral over the complex plane studied in detail by Dotsenko and Fateev [42–44]. The exponents in (3.9) are

$$
a = \frac{1}{2} + \frac{1}{4}n, \quad b = -\frac{3}{2}, \quad c = \frac{1}{2} - \frac{1}{4}n
$$

(3.10)

so $I$ is clearly divergent at $y = 1$ and $y = w_n$. The integral is also divergent at $|y| = \infty$. As we now show, however, $I$ does have a well-defined, finite value, obtained through an analytic continuation.

In order to solve the Dotsenko-Fateev integral (3.9) we follow [44], and perform an analytic rotation of the axis $\text{Im}(y) \to i(1 - 2i\varepsilon)\text{Im}(y)$ with $\varepsilon$ positive and arbitrarily small. The double integral factorizes into

$$
I = \frac{i}{2} \left[ \int_{\gamma_1} f(v_+) dv_+ \int f(v_-) dv_- + \int_{w_n}^1 f(v_+) dv_+ \int_{\gamma_2} f(v_-) dv_- 
+ \int_{-\infty}^0 f(v_+) dv_+ \int f(v_-) dv_- + \int_{\gamma_3}^\infty f(v_+) dv_+ \int f(v_-) dv_- \right].
$$

(3.11)
\[ f(\zeta) = \zeta^a (\zeta - 1)^b (\zeta - w_n)^c \]

where \( f(\zeta) = \zeta^a (\zeta - 1)^b (\zeta - w_n)^c \) and the way the contours \( \gamma_k \) go around the branch points \( 0, w_n, 1 \) is determined by \( \varepsilon \) as shown in Fig.1(a). These “unidimensional” integrals diverge for the values of \( a, b, c \) given in (3.10), and here starts our regularization procedure. Assume instead that \( a, b, c \) are such that the integrals do exist, and are finite at branching points \( 0, 1, w_n \). This means, in particular, that we can deform the contours to pass on these branching points, and close them with a semi-circle as shown in Fig.1(b) for \( \gamma_1 \). Since the integrand is analytic outside of the branching points, the semi-circle can be deformed as in Fig.1(c). The same assumption about \( a, b, c \) made above implies also that the integral over the circle in Fig.1(c) vanishes for an infinite radius. Then the integrals originally over the contours \( \gamma_0 \) and \( \gamma_3 \) vanish, while the integrals originally over \( \gamma_1 \) and \( \gamma_2 \) become integrals over \( C_1 \) and \( C_2 \) in Fig.1(d), coasting the branching cuts (in red) and acquiring, each, a phase of the form \( s(\theta) \equiv \sin(\pi \theta) \). The final result is that

\[ I(a, b, c; w_n) = -s(a)\tilde{I}_1 I_2 - s(b)I_1 \tilde{I}_2, \quad (3.12) \]

where we introduce the ‘canonical integrals’

\[ I_1(a, b, c; w_n) \equiv \int_1^\infty z^a (z - 1)^b (z - w_n)^c \, dz \quad (3.13a) \]
\[ \tilde{I}_1(a, b, c; w_n) \equiv I_1(b, a; c; 1 - w_n) \quad (3.13b) \]
\[ I_2(a, b, c; w_n) \equiv \int_0^{w_n} z^a (z - 1)^b (z - w_n)^c \, dz \quad (3.13c) \]
\[ \tilde{I}_2(a, b, c; w) \equiv I_2(b, a; c; 1 - w_n) \quad (3.13d) \]

The canonical integrals (3.13) are all representations of the hypergeometric function, provided the same assumptions made above on \( a, b, c \) hold. Now the crucial point is that, represented as hypergeometrics, \( I_{1,2}(a, b, c; w_n) \) and \( \tilde{I}_{1,2}(a, b, c; w_n) \) are entire functions in the variables \( a, b, c \), which are well-defined at the values (3.10). Hence the hypergeometric representation is the unique analytic continuation of these functions, and, with Eq.(3.12), can be taken as
Figure 2: The (finite part of the) integral $J_R$ for every $n$. The horizontal axis simultaneously denotes two different numbers: $n$ for $J_R(n)$, $n \neq 4k + 2$, and $k$ for $J_{R}^{\text{reg}}(4k + 2)$.

the definition of the integral (3.9). For $a, b, c$ given in (3.10),

$$I_1(n) = \frac{\pi(4 - n^2)}{32} w_n^2 F\left(\frac{3}{2}, \frac{6+n}{4}; 3; w_n\right)$$  \hspace{1cm} (3.14a)

$$I_2(n) = \frac{1}{2} \Gamma\left(\frac{6-n}{4}\right) \Gamma\left(\frac{6+n}{4}\right) w_n^2 F\left(\frac{3}{2}, \frac{6+n}{4}; 3; w_n\right)$$  \hspace{1cm} (3.14b)

$$\tilde{I}_1(n) = -2\sqrt{\pi} \Gamma\left(\frac{6-n}{4}\right) F\left(-\frac{n-2}{4}, -\frac{1}{2}; \frac{n+4}{4}; 1 - w_n\right)$$  \hspace{1cm} (3.14c)

$$\tilde{I}_2(n) = -\frac{2\sqrt{\pi}}{(1 - w_n)^{\frac{3}{4}}} \Gamma\left(\frac{6-n}{4}\right) F\left(-\frac{n+2}{4}, -\frac{1}{2}; \frac{4-n}{4}; 1 - w_n\right)$$  \hspace{1cm} (3.14d)

where $F(\alpha, \beta; \gamma; \zeta) \equiv F(\alpha, \beta; \gamma)/\Gamma(\gamma)$, see [45]. We can now evaluate $I$ given by Eq.(3.12), then finally evaluate $J_R$ from Eq.(3.8),

$$J_R(n) = -\left(\frac{n + 1}{16n}\right)^2 \left[ \cos\left(\frac{n\pi}{4}\right) I_1(n) I_2(n) + I_1(n) \tilde{I}_2(n) \right].$$  \hspace{1cm} (3.15)

The final result, which is finite and non-vanishing, is plotted in Fig.2. When $n = 4k + 2$, a pole of the Gamma function appears in $I_2$ and $\tilde{I}_2$. One can regularize this Gamma function by taking $k + \epsilon$ with $\epsilon \to 0$, and isolating the singularity in $\Gamma(1 - k - \epsilon)$ in a way that is typical of dimensional regularization in QFT, obtaining a finite and an infinite part. The latter has to be renormalized away, see [39]. The finite result, after some manipulation which relates $\tilde{I}_1$ to $s(a) I_2$, can be expressed as

$$J_{R}^{\text{reg}}(k) = -\left(\frac{3 + 4k}{32(1 + 2k)}\right)^2 I_1(k) \left[ \tilde{I}_1(k) + \tilde{I}_2^{\text{reg}}(k) \right],$$  \hspace{1cm} (3.16)

$$\tilde{I}_2^{\text{reg}}(k) = \frac{(-1)^{k-1} 2\sqrt{\pi} (4k + 3)^{2k+1} \psi(k)}{(4k + 1)^{2k+1} (k - 1)!} \times F\left(-\frac{1}{2}, -k - 1; \frac{1 - 2k}{2}; \frac{(4k+1)^2}{(4k+3)^2}\right),$$  \hspace{1cm} (3.17)

and is also plotted in Fig.2. (Note that the argument of $J_{R}^{\text{reg}}(k)$ is taken to be $k = \frac{1}{4}(n-2)$, instead of $n$.) We should make a comment about the limit of large
As seen in Fig. 2, the expressions for $J_R(n)$ “stabilizes” around finite values. For example, for large $k \in \mathbb{N}$, we have $J_R(4(k + 1)) \approx -0.1140$, $J_R(2k + 1) \approx -0.0861$, $J_R(2k + 3) \approx -0.1419$. It is, however, hard to find an analytic expression for these limits, since $n$ enters the hypergeometric functions (3.14)-(3.17) in a complicated way. The final step in the renormalization procedure is to cancel the logarithmic divergence in Eq.(3.7) by replacing the bare Ramond fields with their renormalized counterparts $R^{\pm(\text{ren})}[n] = \Lambda \frac{1}{2} \pi \lambda^2 |J_R(n)| R^{\pm}[n]$. One can easily verify that the conformal dimension of the field $R^{\pm(\text{ren})}[n]$ (at $\lambda^2$-order and in the planar large $N$ approximation) is indeed given by Eq.(1.1).

We have thus found the renormalization of the anomalous dimension (1.1) of twisted Ramond fields $R^{\pm}[n]$ in the deformed orbifold SCFT $2$ (2.1). Our method consisted of a regularization procedure of Dotsenko-Fateev integrals (3.9) by analytic continuation, which allowed us to express them in terms of well-defined, finite hypergeometric functions. An obvious check of the validity of this method is to apply it to chiral NS fields — since these are BPS-protected, the corresponding integral $J_{NS}$ should vanish for all $n$. We can check that this is indeed true; in this case, the integral (3.9) is much simpler, and has been discussed in [16].

The same integral $J_R$ which gives the second-order correction of $\Delta_R^n$, also gives the first order correction of the specific structure constant $C_n(\lambda)$ in the three-point function $(R^{-}[n](z_1, \bar{z}_1)O_{[2]}^\text{int}(z_3, \bar{z}_3) R^{+}[n](z_2, \bar{z}_2))$. At zero order, i.e. in the free orbifold, $C_n(0) = 0$, but its correction can be easily calculated to be $C_n(\lambda) = \lambda^2 |J_R(n)|$, see [39], hence

$$\langle R^{-}[n](\infty)O_{[2]}^\text{int}(1)R^{+}[n](0) \rangle_\lambda = \lambda |J_R(n)| + \cdots ,$$

where the ellipsis indicate terms of higher order in $\lambda$.

4. Conclusion

In this letter we have studied a simple example of renormalization in the Ramond sector of the deformed orbifold SCFT $2$ (2.1). We consider this to be a hint that correlation functions involving two generic products of (composite) twisted Ramond fields (as well as of some of their descendants), and two deformation operators, can be studied with the very same methods used here. The knowledge of the explicit covering surface map seems to be sufficient for obtaining important information about the deformed orbifold D1-D5 SCFT $2$, and consequently for a more complete microstate description of the related near-extremal 3-charge black holes as well.

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