EXCEPTIONAL SEQUENCES AND DERIVED AUTOEQUIVALENCES

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Abstract. We prove a general theorem that gives a non trivial relation in the group of derived autoequivalences of a variety (or stack) X, under the assumption that there exists a suitable functor from the derived category of another variety Y admitting a full exceptional sequence. Applications include the case in which X is Calabi-Yau and either X is a hypersurface in Y (this extends a previous result by the author and R. L. Karp, where Y was a weighted projective space) or Y is a hypersurface in X. The proof uses a resolution of the diagonal of Y constructed from the exceptional sequence.

1. Introduction

If X is a smooth proper variety or stack, the group Aut(Db(X)) of (isomorphism classes of) exact autoequivalences of Db(X) (the bounded derived category of coherent sheaves on X) is an interesting object of study, in particular when X is Calabi-Yau, i.e. when ωX ≃OX and H^i(X,OX) = 0 for 0 < i < dim(X). Obviously in any case the following elements are in Aut(Db(X)): shift functors (−)[n] for every integer n; pull-back functors f* for f an automorphism of X; functors LF, defined as F⊗−, for F a line bundle on X. In fact, if ωX or ω_X^{-1} is ample, then these elements generate Aut(Db(X)), which is isomorphic to Z × (Pic(X) ⋊ Aut(X)) (see [4]). On the other hand, in general Aut(Db(X)) is much bigger and its structure is rather mysterious. However, it is known that, at least if X is a smooth projective variety (see [13]) or more generally the smooth stack associated to a normal projective variety with only quotient singularities (see [11]), then every autoequivalence of Db(X) is a Fourier-Mukai functor. An exact functor F: Db(Y) → Db(X) is a Fourier-Mukai functor if there exists an object K ∈ Db(X×Y) (called kernel of F) such that F ≃ ΦK, where ΦK denotes the functor π_1*(K ⊗ π_2*(−)).

Interesting examples of non trivial elements in Aut(Db(X)) were introduced in [14], where it was proved that the Fourier-Mukai functor T_F := ΦC(F,F∨→O_Δ) (where O_Δ is the structure sheaf of the diagonal in X×X, the morphism is the natural one and C(−) denotes the cone of a morphism) is an autoequivalence of Db(X) when F ∈ Db(X) is a spherical object, i.e. when F ⊗ ωX ≃ F and

\[ \text{Hom}_{Db(X)}(F,F[k]) \cong \begin{cases} k & \text{if } k = 0, \dim(X) \\ 0 & \text{otherwise} \end{cases} \]

(T_F is then called the spherical twist associated to F). Clearly every line bundle on X is a spherical object if X is Calabi-Yau. Spherical objects can also be constructed in some common geometric settings starting with another smooth proper variety (or stack) Y and an exceptional object E ∈ Db(Y), i.e. such that

\[ \text{Hom}_{Db(Y)}(E,E[k]) \cong \begin{cases} k & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \]

2000 Mathematics Subject Classification. 18E30.
Key words and phrases. Derived categories.
(see [8] Chapter 8 for an account on spherical and exceptional objects and relations between them). For instance, if \( f: X \rightarrow Y \) is the inclusion of a hypersurface such that \( \omega_Y \cong \mathcal{O}_Y(-X) \), then \( f^*\mathcal{E} \in \mathcal{D}^b(\mathcal{K}) \) is spherical; also, if \( g: Y \rightarrow X \) is the inclusion of a hypersurface such that \( g^*\omega_X \cong \mathcal{O}_Y \), then \( g_*\mathcal{E} \in \mathcal{D}^b(\mathcal{K}) \) is spherical. The aim of this paper is to find, in similar situations, relations in \( \text{Aut}(\mathcal{D}^b(\mathcal{K})) \) between the spherical twists associated to the images of a full exceptional sequence \( (\mathcal{E}_0, \ldots, \mathcal{E}_m) \) in \( \mathcal{D}^b(\mathcal{K}) \) (this means that each \( \mathcal{E}_i \) is exceptional, that \( \text{Hom}_{\mathcal{D}^b(\mathcal{K})}(\mathcal{E}_i, \mathcal{E}_j[k]) = 0 \) for \( i > j \) and for every \( k \in \mathbb{Z} \), and that \( \mathcal{D}^b(\mathcal{K}) \) is generated by \( \{ \mathcal{E}_0, \ldots, \mathcal{E}_m \} \) as a triangulated category). We were motivated by the search for a general framework allowing to formulate and prove relations in \( \text{Aut}(\mathcal{D}^b(\mathcal{K})) \), including as particular cases those proved in [5] and [9] and those conjectured in [9] and [10], which we now recall briefly.

In [5] it was proved that if \( X \subset \mathbb{P} = \mathbb{P}(w_0, \ldots, w_n) \) is a hypersurface of degree \( |w| := w_0 + \cdots + w_n \) (hence \( X \) is Calabi-Yau), then the autoequivalence \( \mathcal{G} := \mathcal{L}_{\mathcal{O}_X(1)} \circ \mathcal{T}_{\mathcal{O}_X} \) of \( \mathcal{D}^b(\mathcal{K}) \) satisfies \( \mathcal{G}^{|w|} \cong (-)^{|w|}[2] \). This is easily seen to be equivalent to \( \mathcal{T}_{\mathcal{O}_X(1)} \circ \cdots \circ \mathcal{T}_{\mathcal{O}_X(|w|)} \cong \mathcal{L}_{\mathcal{O}_X(-|w|)[2]} \), and, since \( (\mathcal{O}_\mathbb{P}(1), \ldots, \mathcal{O}_\mathbb{P}(|w|)) \) is a full exceptional sequence in \( \mathcal{D}^b(\mathbb{P}) \), it is natural to conjecture that more generally

\[
\mathcal{T}_{f^*\mathcal{E}_0} \circ \cdots \circ \mathcal{T}_{f^*\mathcal{E}_m} \cong \mathcal{L}_{\mathcal{O}_X(-X)[2]}
\]

if, as above, \( f: X \rightarrow Y \) is the inclusion of a hypersurface such that \( \omega_Y \cong \mathcal{O}_Y(-X) \).

On the other hand, some of the relations in [9] and [10] (proved when \( Y = \mathbb{P}^1 \) and conjectured when \( Y = \mathbb{P}^2 \) or the Hirzebruch surface \( \mathbb{F}_3 \)) are equivalent to particular cases of the following statement: if \( g: Y \rightarrow X \) is the inclusion of a hypersurface such that \( g^*\omega_X \cong \mathcal{O}_Y \), then

\[
\mathcal{T}_{g_*\mathcal{E}_0} \circ \cdots \circ \mathcal{T}_{g_*\mathcal{E}_m} \cong \mathcal{L}_{\mathcal{O}_X(Y)}.
\]

In this paper we prove both (1.1) and (1.2) (cor. [5.2] and [7.6]) as consequences of the following more general result (theorem [4.2]): given a Fourier-Mukai functor \( \mathcal{F} \cong \Phi_K: \mathcal{D}^b(\mathcal{K}) \rightarrow \mathcal{D}^b(\mathcal{K}) \) satisfying condition (1.1) (which essentially says that \( \mathcal{F} \) acts faithfully on the non trivial parts of the morphisms between the terms of the exceptional sequence), we have

\[
\mathcal{T}_{\mathcal{F}(\mathcal{E}_0)} \circ \cdots \circ \mathcal{T}_{\mathcal{F}(\mathcal{E}_m)} \cong \Phi_{\mathcal{C}(\mu_K)},
\]

where \( \mu_K \) is a morphism in \( \mathcal{D}^b(\mathcal{K} \times \mathcal{K}) \) naturally induced by \( \mathcal{K} \) (see section [2] for the precise definition). Here it is enough to say that one can construct a functor \( \mathcal{F}: \mathcal{D}^b(\mathcal{K} \times \mathcal{K}) \rightarrow \mathcal{D}^b(\mathcal{K} \times \mathcal{K}) \) and that \( \mu_K \) can be identified with a natural morphism \( \mathcal{F}(\mathcal{O}_{\Delta_Y}) \rightarrow \mathcal{O}_{\Delta_Y} \).

Our strategy of proof is similar to that of [5] Theorem 1.1] and is based on the use of a suitable resolution of the diagonal for \( \Delta \). Namely, denoting by \( (\mathcal{E}_0, \ldots, \mathcal{E}_m) \) the dual exceptional sequence of \( (\mathcal{E}_0, \ldots, \mathcal{E}_m) \) (characterized by \( \text{Hom}_{\mathcal{D}^b(\mathcal{K})}(\mathcal{E}_i, \mathcal{E}_j[k]) \cong \mathbb{K}^{k,0} \) for \( 0 \leq i, j \leq m \) and for every \( k \in \mathbb{Z} \)), objects \( \mathcal{R}_k \in \mathcal{D}^b(\mathcal{K} \times \mathcal{K}) \) for

\[
0 \leq k \leq m
\]

are defined with the following properties:

\[
(1) \quad \mathcal{R}_0 \cong \mathcal{E}_0 \boxtimes \mathcal{E}_0;
\]

\[
(2) \quad \text{there is a distinguished triangle } \mathcal{R}_{k-1} \xrightarrow{\alpha_k} \mathcal{R}_k \xrightarrow{\beta_k} \mathcal{E}_k \boxtimes \mathcal{E}_k \xrightarrow{\gamma_k} \mathcal{R}_{k-1}[1] \quad \text{for } 0 < k \leq m;
\]

\[
(3) \quad \mathcal{R}_m \cong \mathcal{O}_\Delta.
\]

In [5], where \( Y = \mathbb{P} \) and \( \mathcal{E}_i = \mathcal{O}_\mathbb{P}(i) \), this is achieved by induction on \( k \), defining explicitly the morphism \( \gamma_k \) (the notation of [5] is actually different), and then the difficult part is to show that (3) holds. This approach seems hard to follow in the general case. Instead, using suitable semiorthogonal decompositions of \( \mathcal{D}^b(\mathcal{K} \times \mathcal{K}) \), in section [5] we can define directly objects \( \mathcal{R}_k \) such that (3) is automatically satisfied, and then prove that (1) and (2) hold using the fact that there is a natural way to
define $\alpha_k$. Then the idea for the proof of [1K3] is simply to show by induction on $k$
that $T_{F(\ell_k)} \circ \cdots \circ T_{F(\ell_k)} \cong \Phi(C(F(\ell_k) - \alpha_{\Delta_X})$. Here some of the difficulties come
from the fact that at several points one has to check that the morphisms involved are the “right” ones. In most cases this is just a matter of checking that some natural
diagrams commute: to this purpose in section [2] we prove some general properties
about compositions of kernels of Fourier-Mukai functors, which allow to get in an
efficient way the needed commutative diagrams. A more delicate problem is that,
due to the non functoriality of the cone, not all the morphisms are natural: to deal
with this question we use a property (proved in [3] and recalled in the appendix)
of triangulated categories satisfying a suitable finiteness condition.

Acknowledgements. It is a pleasure to thank Robert L. Karp for useful conver-
sations and for drawing my attention to his conjectures.

Notation. We work over a fixed base field $k$. As in [3] for simplicity we will call
stack a connected Deligne-Mumford stack which is smooth and proper over $k$, and
such that every coherent sheaf is a quotient of a locally free sheaf of finite rank.

We will write $\{\cdot\}$ for the stack $\text{Spec } k$. If $X$ is a stack, $D^b(X)$ will denote the
bounded derived category of coherent sheaves on $X$. Our definition of stack implies
that tensor product admits a left derived functor $- \otimes - : D^b(X) \times D^b(X) \to D^b(X)$
and that for every morphism of stacks $f : X \to Y$ there are (left and right) derived
functors $f^* : D^b(Y) \to D^b(X)$ and $f_* : D^b(X) \to D^b(Y)$ (we use this notation
since we never need to distinguish between these functors and the corresponding
underived versions). Also the functors $\text{Hom}_X$ and $\text{Hom}_X$ admit derived functors
$\text{Hom}_X (-,-) : D^b(X)^o \times D^b(X) \to D^b(\{\cdot\})$ (hence with this convention
$$\text{Hom}_X(\mathcal{F}, \mathcal{F}') \cong \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{D^b(X)}(\mathcal{F}, \mathcal{F}'[k])[-k]$$
for $\mathcal{F}, \mathcal{F}' \in D^b(X)$) and $\text{Hom}_X (-,-) : D^b(X)^o \times D^b(X) \to D^b(X)$; for $\mathcal{F} \in D^b(X)$
we set $\mathcal{F}^* := \text{Hom}_X(\mathcal{F}, \mathcal{O}_X)$.

For a stack $X$, $\delta : X \to X \times X$ will be the diagonal morphism; we will often
write $\mathcal{O}_{\Delta_X}$ (or simply $\mathcal{O}_{\Delta}$) instead of $\delta_* \mathcal{O}_X$. Denoting as usual by $\omega_X$ the dualizing
sheaf on $X$, we define $\omega'_X := \omega_X[\dim(X)]$. For a morphism of stacks $f : X \to Y$,
$f^! : D^b(Y) \to D^b(X)$ denotes the functor $f^* (-) \otimes \omega'_X \otimes f^* \omega'_Y$.

If $X$ and $Y$ are two stacks, we will denote by $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$
the projections and by $- \boxtimes -$ the exterior (derived) tensor product

$$\pi_1^*(-) \otimes \pi_2^*(-) : D^b(X) \times D^b(Y) \to D^b(X \times Y).$$

We refer to the appendix for notations and conventions about triangulated cat-
egories and exact functors between them.

2. Composition of kernels

$X$, $Y$, $Z$ and $W$ will be stacks. For $\mathcal{K} \in D^b(X \times Y)$ and $\mathcal{L} \in D^b(Y \times Z)$, we set
$$\mathcal{K} \star \mathcal{L} := \pi_{1,3*}(\pi_{1,2}^* \mathcal{K} \otimes \pi_{2,3}^* \mathcal{L}) \in D^b(X \times Z)$$
(where $\pi_{i,j}$ denotes the projection from $X \times Y \times Z$ onto the $i^{th} \times j^{th}$ factor). Notice
that, given $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$, identifying $X$ with $X \times \{\cdot\}$ and $Y$ with
$\{\cdot\} \times Y$, $\mathcal{F} \star \mathcal{G}$ coincides with $\mathcal{F} \boxtimes \mathcal{G}$.

Clearly for every $\mathcal{K} \in D^b(X \times Y)$ and for every $Z$ there are exact functors
$$\mathcal{K} \star - : D^b(Y \times Z) \to D^b(X \times Z) \quad \text{and} \quad - \star \mathcal{K} : D^b(Z \times X) \to D^b(Z \times Y).$$

\footnote{In [3] the opposite convention $- \boxtimes - := \pi_2^*(-) \otimes \pi_1^*(-)$ is used.}
In particular, when $Z = \{\cdot\}$, $\mathcal{K} \rightsquigarrow -$, respectively $- \mathcal{K}$ can be identified with the Fourier-Mukai functor with kernel $\mathcal{K}$
\[ \pi_1(\mathcal{K} \otimes \pi_i^2(-)) : D^b(Y) \to D^b(X), \text{ respectively } \pi_2(\mathcal{K} \otimes \pi_i^1(-)) : D^b(X) \to D^b(Y), \]
which will be denoted by $\Phi_{\mathcal{K}}$, respectively $\Phi_{\mathcal{K}^*}$.

The following result shows that, up to isomorphism, the operation $\ast$ is associative and the objects $\mathcal{O}_\Delta$ act as identities: since we do not know a reference for this fact, we include a proof here.

**Lemma 2.1.** For every $\mathcal{K} \in D^b(X \times Y)$ there are natural isomorphisms $\mathcal{O}_\Delta \ast \mathcal{K} \cong \mathcal{K} \ast \mathcal{O}_\Delta$. Moreover, given also $\mathcal{L} \in D^b(Y \times Z)$ and $\mathcal{M} \in D^b(Z \times W)$, there is a natural isomorphism $(\mathcal{K} \ast \mathcal{L}) \ast \mathcal{M} \cong \mathcal{K} \ast (\mathcal{L} \ast \mathcal{M})$ in $D^b(X \times W)$.

**Proof.** Applying flat base change to the cartesian square
\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\delta} & X \\
\downarrow_{\pi_1} & & \downarrow_{\pi_2,1} \\
X & \xrightarrow{\delta} & X \\
\end{array}
\]
and using projection formula for $\delta$, we obtain
\[
\mathcal{O}_\Delta \ast \mathcal{K} = \pi_{1,3}\ast (\delta_\ast \mathcal{O}_X \otimes \pi_2^3 \mathcal{K}) \cong \pi_{1,3,4}\ast (\delta_\ast \mathcal{O}_X \otimes \pi_2^3 \mathcal{K}) \cong \pi_{1,3,4}\ast (\delta_\ast \mathcal{O}_X \otimes \pi_2^3 \mathcal{K}),
\]
and the last term is isomorphic to $\mathcal{K}$ because $\pi_{1,3} \circ \tilde{\delta} = \pi_{2,3} \circ \tilde{\delta} = \text{id}_{X \times Y}$. In a completely similar way one proves that also $\mathcal{K} \ast \mathcal{O}_\Delta \cong \mathcal{K}$.

In order to prove the second statement, we will denote by $\pi_{i,j}$ and $\pi_{i,j,k}$ the obvious projections from $X \times Y \times Z \times W$ and, for $V$ one of $X$, $Y$, $Z$ and $W$, by $\pi_i^V$ the obvious projections from the product of all the four terms except $V$. Applying flat base change to the cartesian square
\[
\begin{array}{ccc}
X \times Y \times Z \times W & \xrightarrow{\pi_{1,3,4}} & X \times Z \times W \\
\downarrow_{\pi_{1,2,3}} & & \downarrow_{\pi_{1,3}} \\
X \times Y \times Z & \xrightarrow{\pi_i^V} & X \times Z,
\end{array}
\]
using projection formula for $\pi_{1,3,4}$ and taking into account that $\pi_{1,3} \circ \pi_{1,3,4} = \pi_{1,4}$, $\pi_{1,2} \circ \pi_{1,2,3} = \pi_{1,2}$, $\pi_{2,3} \circ \pi_{1,2,3} = \pi_{2,3}$ and $\pi_{1,4} \circ \pi_{1,3,4} = \pi_{1,4}$, we obtain
\[
(K \ast L) \ast M = \pi_{1,3,4}\ast (\pi_{1,2}^Y \ast \pi_{1,3,4}\ast (\pi_{1,2}^W \ast \mathcal{K} \otimes \pi_{2,3}^W \mathcal{L}) \otimes \pi_{2,3}^Y \ast \mathcal{M})
\cong \pi_{1,3,4}\ast (\pi_{1,3,4}\ast \pi_{1,2,3}^W \ast (\pi_{1,2}^W \ast \mathcal{K} \otimes \pi_{2,3}^W \mathcal{L}) \otimes \pi_{2,3}^Y \ast \mathcal{M})
\cong \pi_{1,3,4}\ast \pi_{1,2,3}^W \ast (\pi_{1,2}^W \ast \mathcal{K} \otimes \pi_{2,3}^W \mathcal{L}) \otimes \pi_{1,3,4}^\ast \pi_{2,3}^Y \ast \mathcal{M}
\cong \pi_{1,4}\ast (\pi_{1,2}^W \ast \mathcal{K} \otimes \pi_{2,3}^W \mathcal{L} \otimes \pi_{3,4}^\ast \mathcal{M}).
\]
In a completely similar way one proves that also
\[
\mathcal{K} \ast (L \ast M) \cong \pi_{1,4}\ast (\pi_{1,2}^W \ast \mathcal{K} \otimes \pi_{2,3}^W \mathcal{L} \otimes \pi_{3,4}^\ast \mathcal{M}).
\]

\[\square\]

**Remark 2.2.** When $W = \{\cdot\}$ the above result yields the well known fact (see e.g. [8, Prop. 5.10]) that $\Phi_{\mathcal{K}} \circ \Phi_{\mathcal{L}} \cong \Phi_{\mathcal{K} \ast \mathcal{L}} : D^b(Z) \to D^b(X)$.

From [2, 2.4] it is easy to deduce various results (like [3, Lemma 2.2]) about Fourier-Mukai functors with kernels of the form $\mathcal{F} \otimes \mathcal{G}$. In particular we will need the following fact.
Corollary 2.3. Given \( F, F' \in D^b(X) \) and \( G, G' \in D^b(Y) \), there are natural isomorphisms \( \Phi_{F \boxtimes G}(G') \cong F \otimes_k \text{Hom}_Y(G', G') \) and \( \Phi_{F' \boxtimes G}(F') \cong G \otimes_k \text{Hom}_X(F', F') \).

Proof. We have \( \Phi_{F \boxtimes G}(G') \cong (F * G) * G' \cong F * (G * G') \), where we consider \( G \in D^b(\{ \} \times X) \) and \( G' \in D^b(X \times \{ \}) \); it is then enough to note that in this case \( G * G' \cong \text{Hom}_Y(G', G') \in D^b(\{ \}) \). The other proof is similar. \( \square \)

From now on by abuse of notation we will often treat as equalities the natural isomorphisms given by 2.1.

For \( K \in D^b(X \times Y) \) we set \( K^! := K^\vee \otimes \pi_2^* \omega_Y^\vee \), but considering it as an object of \( D^b(X \times X) \). Notice that, given \( F \in D^b(X) \), identifying \( X \) with \( X \times \{ \} \), \( F^! \) coincides with \( F^\vee \).

Lemma 2.4. For every \( K \in D^b(X \times Y) \) and for every other stack \( Z \) the functor

\[ K * - : D^b(Y \times Z) \to D^b(X \times Z), \text{ respectively } - * K : D^b(Z \times X) \to D^b(Z \times Y) \]

is left, respectively right adjoint of the functor

\[ K^! * - : D^b(X \times Z) \to D^b(Y \times Z), \text{ respectively } - * K^! : D^b(Z \times Y) \to D^b(Z \times X). \]

Proof. By definition \( K * - \) is the composition of the functors \( \pi^*_{2,3}, \pi^!_{1,3}K \otimes - \) and \( \pi^*_{1,3} \). Remarking that, if \( f \) is a morphism of stacks, the right adjoints of \( f^* \) and \( f^! \) are respectively \( f_* \) and \( f^! \), and that the right adjoint of \( F \otimes - \) is \( F^\vee \otimes - \), we conclude that the right adjoint of \( K * - \) is

\[ \pi^*_{2,3}((\pi^*_{1,2}K)^\vee \otimes \pi^!_{1,3}(-)) \cong \pi^!_{2,3}(\pi^*_{1,2}K^\vee \otimes \omega_{X \times Y \times Z} \otimes (\pi^*_{1,3} \omega_{X \times Z}^\vee \otimes \pi^!_{1,3}(-))) \]

\[ \cong \pi^*_{2,3}(\pi^*_{1,2}K^\vee \otimes \pi^!_{1,3}(-)) \cong K^! * - \]

In a completely similar way one proves that the left adjoint of \(- * K \) is \(- * K^! \). \( \square \)

Remark 2.5. When \( Z = \{ \} \) the above result says that \( \Phi_K : D^b(Y) \to D^b(X) \), respectively \( \Phi_K^! : D^b(X) \to D^b(Y) \) is left, respectively right adjoint of \( \Phi_K^! : D^b(X) \to D^b(Y) \), respectively \( \Phi_K : D^b(Y) \to D^b(X) \) (see e.g. [8, Prop. 5.9]).

For every \( K \in D^b(X \times Y) \) by 2.4 there are natural isomorphisms

\[ \text{Hom}_{D^b(X \times X)}(K * K^!, O_{\Delta_X}) \cong \text{Hom}_{D^b(X \times Y)}(K, K) \cong \text{Hom}_{D^b(Y \times Y)}(O_{\Delta_Y}, K^! * K), \]

and we denote by \( \mu_K : K * K^! \to O_{\Delta_X} \) and \( \nu_K : O_{\Delta_Y} \to K^! * K \) the morphisms corresponding to \( \text{id}_K \). In particular, given \( F \in D^b(X) \), identifying as usual \( X \) with \( X \times \{ \} \) and \( F \boxtimes F^\vee \) with \( F \boxtimes F^! \), \( \mu_F \) coincides with the natural morphism \( F \boxtimes F^\vee \to O_{\Delta_X} \). Given also \( A \in D^b(X \times X) \), we define

\[ (2.1) \quad \bar{\mu}_F, A : \Phi_A(F) \boxtimes F^\vee \cong A \ast (F \boxtimes F^\vee) \xrightarrow{\text{id}_A \ast \mu_F} A \]

Clearly the morphism \( \bar{\mu}_F, A \) is functorial in \( A \), meaning that for every morphism \( \alpha : A \to A' \) in \( D^b(X \times X) \) there is a commutative diagram

\[ (2.2) \quad \Phi_A(F) \boxtimes F^\vee \xrightarrow{\bar{\mu}_F, A} A \]

\[ \Phi_\alpha(F \boxtimes \text{id}) \]

\[ \Phi_A(F) \boxtimes F^\vee \xrightarrow{\bar{\mu}_F, A'} A'. \]

Proposition 2.6. Given \( K \in D^b(X \times Y) \) and \( L \in D^b(Y \times Z) \), we have:

1. There is a natural isomorphism \( (K * L)^! \cong L^! * K^! \) in \( D^b(Z \times X) \).
(2) The diagram

\[
\begin{array}{ccc}
\mathcal{K} \star \mathcal{L} \star \mathcal{L} \star \mathcal{K} & \xrightarrow{\sim} & (\mathcal{K} \star \mathcal{L}) \star (\mathcal{K} \star \mathcal{L}) \\
\mu_{\mathcal{L}} \downarrow & & \downarrow \mu_{\mathcal{L}} \mu_{\mathcal{K}} \\
\mathcal{K} \star \mathcal{K} & \xrightarrow{\mu_{\mathcal{K}}} & \mathcal{O}_\Delta 
\end{array}
\]

(where the top horizontal isomorphism is given by (1)) commutes.

(3) The compositions

\[
\mathcal{K} \xrightarrow{\text{id}_{\mathcal{K}} \circ \nu_{\mathcal{L}} \circ \mu_{\mathcal{K}}} \mathcal{K} \star \mathcal{K} \quad \text{and} \quad \mathcal{K} \xrightarrow{\mu_{\mathcal{K}} \circ \text{id}_{\mathcal{K}}} \mathcal{K} \star \mathcal{K}
\]

coincide with \text{id}_{\mathcal{K}} and \text{id}_{\mathcal{K}}.

(4) The diagram

\[
\begin{array}{ccc}
\mathcal{L} \star \mathcal{L} \star \mathcal{K} & \xrightarrow{\sim} & \mathcal{L} \star (\mathcal{K} \star \mathcal{L}) \\
\mu_{\mathcal{L}} \downarrow & & \downarrow \nu_{\mathcal{L}} \circ \mu_{\mathcal{K}} \\
\mathcal{L} \star \mathcal{K} \star \mathcal{K} & \xrightarrow{\mu_{\mathcal{K}} \circ \text{id}_{\mathcal{K}}} & \mathcal{L} \star (\mathcal{K} \star \mathcal{L})
\end{array}
\]

(where the top horizontal isomorphism is given by (1)) commutes.

Proof. Denoting by \(\mathcal{F}^!\) the right adjoint of a functor \(\mathcal{F}\), it is clear that if \(\mathcal{F}\) and \(\mathcal{G}\) are two composable functors admitting a right adjoint, then \((\mathcal{F} \circ \mathcal{G})^! \cong \mathcal{G}^! \circ \mathcal{F}^!\). Evaluating this isomorphism at \(\mathcal{O}_\Delta\) when

\[
\begin{align*}
\mathcal{F} &= \mathcal{K} \star - : \mathcal{D}^b(Y \times X) \to \mathcal{D}^b(X \times X), \\
\mathcal{G} &= \mathcal{L} \star - : \mathcal{D}^b(Z \times X) \to \mathcal{D}^b(Y \times X)
\end{align*}
\]

and using [2, 4] we obtain (1). Similarly, denoting by \(\mu_{\mathcal{F}}: \mathcal{F} \circ \mathcal{G} \to \mathcal{F}^! \circ \mathcal{G}^!\) the adjunction morphisms, (2) follows from the fact that the diagram

\[
\begin{array}{ccc}
\mathcal{F} \circ \mathcal{G} \circ \mathcal{G}^! \circ \mathcal{F} & \xrightarrow{\sim} & (\mathcal{F} \circ \mathcal{G}) \circ (\mathcal{F} \circ \mathcal{G})^! \\
\mathcal{F} \circ \mu_{\mathcal{G}} \circ \mathcal{F}^! \downarrow & & \downarrow \mu_{\mathcal{F}} \circ \mathcal{G} \\
\mathcal{F} \circ \mathcal{F}^! & \xrightarrow{\mu_{\mathcal{F}}} & \id
\end{array}
\]

commutes (see [12, Theorem 1, p. 101]), whereas (3) follows from the fact that the compositions \(\mathcal{F} \xrightarrow{\text{id}_{\mathcal{F}} \circ \mathcal{F}^!} \mathcal{F} \circ \mathcal{F}^! \circ \mathcal{F}^! \xrightarrow{\mu_{\mathcal{F}} \circ \text{id}_{\mathcal{F}}} \mathcal{F} \circ \mathcal{F}^! \circ \mathcal{F}^! \xrightarrow{\text{id}_{\mathcal{F}} \circ \mathcal{F}} \mathcal{F}^! \circ \mathcal{F}^! \circ \mathcal{F}^! \) coincide with \(\text{id}_{\mathcal{F}}\) and \(\text{id}_{\mathcal{F}}^!\) (see [12, Theorem 1, p. 80]). As for (4), in the diagram

\[
\begin{array}{ccc}
\mathcal{L} \star \mathcal{L} \star \mathcal{K} & \xrightarrow{\sim} & \mathcal{L} \star (\mathcal{K} \star \mathcal{L}) \\
\mu_{\mathcal{L}} \downarrow & & \downarrow \nu_{\mathcal{L}} \circ \mu_{\mathcal{K}} \\
\mathcal{K} \star \mathcal{K} & \xrightarrow{\mu_{\mathcal{K}} \circ \text{id}_{\mathcal{K}}} & \mathcal{K} \star \mathcal{L} \star (\mathcal{K} \star \mathcal{L})
\end{array}
\]

the triangle commutes by (2). Since the other two inner quadrangles clearly commute as well, we conclude that the outer square also commutes, which yields the result, taking into account (3). \(\Box\)
For the rest of this section we fix $K \in D^b(X \times Y)$ and we set
\[ F := \Phi_K : D^b(Y) \to D^b(X), \]
\[ \tilde{\Phi} := \mathbb{K} \ast - \ast \mathbb{K} : D^b(Y \times Y) \to D^b(X \times X). \]

Note that $\tilde{\Phi}(\mathcal{O}_{\Delta_Y})$ is naturally isomorphic to $\mathbb{K} \ast \mathbb{K}$, and we will freely regard $\mu_K$ as a morphism $\mu_K : \tilde{\Phi}(\mathcal{O}_{\Delta_Y}) \to \mathcal{O}_{\Delta_X}$.

**Corollary 2.7.** For every $G, G' \in D^b(Y)$ there is a natural isomorphism
\[ \tilde{\Phi}(G' \boxtimes G'^\vee) \cong F(G') \boxtimes F(G)^\vee. \]

**Proof.** By part (1) of 2.6 we have
\[ \tilde{\Phi}(G' \boxtimes G'^\vee) \cong (\mathbb{K} \ast G') \ast (G' \ast \mathbb{K}) \ast (\mathbb{K} \ast G)^\vee \cong F(G') \boxtimes F(G)^\vee. \]
\[ \square \]

**Corollary 2.8.** For every $G \in D^b(Y)$ the diagram
\[ \tilde{\Phi}(G \boxtimes G'^\vee) \xrightarrow{\sim} F(G) \boxtimes F(G)^\vee \]
\[ \tilde{\Phi}(\mathcal{O}_{\Delta_Y}) \xrightarrow{\mu_K} \mathcal{O}_{\Delta_X} \]
commutes, where the top horizontal map is the natural isomorphism given by 2.7.

**Proof.** It follows immediately from part (2) of 2.6.
\[ \square \]

Given $A \in D^b(Y \times Y)$ and $G \in D^b(Y)$, we define the natural morphism
\[ \vartheta_{A, G} : F(\Phi_A(G)) \cong \mathbb{K} \ast A \ast G \xrightarrow{\text{id}_{\mathbb{K}} \ast A \ast \mu_K \ast \text{id}_G} \mathbb{K} \ast A \ast \mathbb{K} \ast \mathbb{K} \ast G \cong F(\tilde{\Phi}(A)(F(G))), \]
which is functorial in $A$, meaning that for every morphism $\alpha : A \to A'$ in $D^b(Y \times Y)$ there is a commutative diagram in $D^b(X)$
\[ F(\Phi_A(G)) \xrightarrow{\vartheta_{A, G}} F(\tilde{\Phi}(A)(F(G))) \]
\[ F(\Phi_{A'}(G)) \xrightarrow{\vartheta_{A', G}} F(\tilde{\Phi}(A')(F(G))). \]

**Corollary 2.9.** For every $G \in D^b(Y)$ the composition
\[ F(G) \cong F(\Phi_{\Delta_Y}(G)) \xrightarrow{\vartheta_{(\Delta_Y, G)}} F(\tilde{\Phi}(\mathcal{O}_{\Delta_Y})(F(G))) \xrightarrow{\mu_K(F(G))} F(\mathcal{O}_{\Delta_X}(F(G))) \cong F(G) \]
coincides with $\text{id}_{F(G)}$.

**Proof.** It is enough to observe that the above sequence can be identified with the image through the functor $- \ast G$ of the first sequence in part (3) of 2.6.
\[ \square \]

**Corollary 2.10.** For every $A \in D^b(Y \times Y)$ and for every $G \in D^b(Y)$ the diagram
\[ \tilde{\Phi}(\Phi_A(G) \boxtimes G'^\vee) \xrightarrow{\sim} F(\Phi_A(G)) \boxtimes F(G)^\vee \]
\[ \tilde{\Phi}(\mathcal{O}_{\Delta_Y}) \xrightarrow{\vartheta_{A, G} \boxtimes \text{id}} \tilde{\Phi}(A) \boxtimes F(\tilde{\Phi}(G))(F(G)) \boxtimes F(G)^\vee \]
commutes, where the top horizontal map is the natural isomorphism given by 2.7.
Proof. It is immediate to see that the above diagram can be identified with the image through the functor \((\mathcal{K} \star \mathcal{A}) \star -\) of the commutative diagram in part (4) of \(\mathcal{Z.6}\) with \(\mathcal{G}\) in place of \(\mathcal{L}\).

3. Resolution of the diagonal via a full exceptional sequence

Let \(Y\) be a stack and assume that \((\mathcal{E}_0, \ldots, \mathcal{E}_m)\) is a full exceptional sequence in \(D^b(Y)\); we will denote by \((\mathcal{E}'_0, \ldots, \mathcal{E}'_m)\) its dual (full exceptional) sequence (see the appendix for its definition).

For \(0 \leq k \leq m\) we define \(\mathcal{R}_k := (\mathcal{E}'_0, \ldots, \mathcal{E}'_k)\) and \(\mathcal{S}_k := (\mathcal{E}'_{k+1}, \ldots, \mathcal{E}'_m)\) (they are admissible subcategories of \(D^b(Y)\), as clearly \((\mathcal{E}'_m, \ldots, \mathcal{E}'_0)\) is a full exceptional sequence in \(D^b(Y)\)). If \(\mathcal{C}\) is a subcategory of \(D^b(Y)\), we set

\[
D^b(Y) \otimes \mathcal{C} := \{ \langle \mathcal{F} \otimes \mathcal{G} \mid \mathcal{F} \in D^b(Y), \mathcal{G} \in \mathcal{C} \} \}
\]

It follows from \(2\) prop. 2.1.18 that \((D^b(Y) \otimes \mathcal{R}_k, D^b(Y) \otimes \mathcal{S}_k)\) is a semiorthogonal decomposition of \(D^b(Y \times Y)\). Hence by there exists (unique up to isomorphism) a distinguished triangle in \(D^b(Y \times Y)\)

\[
\mathcal{R}_k \xrightarrow{\rho_k} \mathcal{O}_\Delta \xrightarrow{\sigma_k} \mathcal{S}_k \xrightarrow{\tau_k} \mathcal{R}_k[1]
\]

with \(\mathcal{R}_k \in D^b(Y) \otimes \mathcal{R}_k\) and \(\mathcal{S}_k \in D^b(Y) \otimes \mathcal{S}_k\) (in particular, \(\mathcal{S}_m = 0\) and \(\rho_m\) is an isomorphism). For \(0 \leq k \leq m\), since \(\text{Hom}_{Y \times Y}(\mathcal{R}_{k-1}, \mathcal{S}_k) = 0\), there exists a unique morphism \(\alpha_k : \mathcal{R}_{k-1} \to \mathcal{R}_k\) such that \(\rho_{k-1} = \rho_k \circ \alpha_k\).

**Proposition 3.1.** \(\mathcal{R}_0 \cong \mathcal{E}_0 \otimes \mathcal{E}_0' = \mathcal{E}_0 \otimes \mathcal{E}_0'\) and \(\mathcal{C}(\alpha_k) \cong \mathcal{E}_k' \otimes \mathcal{E}_k'\) for \(0 \leq k \leq m\).

Proof. Setting \(\tilde{\mathcal{E}}_0 := \mathcal{R}_0\) and \(\tilde{\mathcal{E}}_k := \mathcal{C}(\alpha_k)\) for \(0 \leq k \leq m\), first we claim that \(\tilde{\mathcal{E}}_k \cong \mathcal{F}_k \otimes \mathcal{E}_k'\) for some \(\mathcal{F}_k \in D^b(Y)\). Indeed, for \(k = 0\) this is clear, and for \(0 < k \leq m\) it follows from the fact that \(\tilde{\mathcal{E}}_k\) belongs both to \(D^b(Y) \otimes \mathcal{R}_k\) (because \(\alpha_k\) is a morphism of \(D^b(Y) \otimes \mathcal{R}_k\)) and to \(D^b(Y) \otimes \mathcal{S}_k\) (because by (TR4) there is a distinguished triangle

\[
\mathcal{C}(\alpha_k) = \tilde{\mathcal{E}}_k \to \mathcal{C}(\rho_{k-1}) \cong \mathcal{S}_{k-1} \to \mathcal{C}(\rho_k) \cong \mathcal{S}_k \to \mathcal{C}(\alpha_k)[1]
\]

and \(\mathcal{S}_{k-1}, \mathcal{S}_k \in D^b(Y) \otimes \mathcal{S}_{k-1}\). Therefore by \(2.3\) we have

\[
\Phi_{\mathcal{E}_k}(\mathcal{E}_k') \cong \mathcal{E}_k' \otimes_k \text{Hom}_Y(\mathcal{E}_i, \mathcal{F}_k)
\]

for \(0 \leq i, k \leq m\). On the other hand, from the distinguished triangle

\[
\Phi_{\mathcal{R}_k}(\mathcal{E}_k') \xrightarrow{\Phi_{\rho_k}(\mathcal{E}_k')} \Phi_{\mathcal{O}_\Delta}(\mathcal{E}_k') \cong \mathcal{E}_k' \to \Phi_{\mathcal{S}_k}(\mathcal{E}_k') \to \Phi_{\mathcal{R}_k}(\mathcal{E}_k')[1],
\]

since clearly \(\Phi_{\mathcal{R}_k}(\mathcal{E}_k') \in \mathcal{R}_k\) and \(\Phi_{\mathcal{S}_k}(\mathcal{E}_k') \in \mathcal{S}_k\), we deduce that \(\Phi_{\rho_k}(\mathcal{E}_k')\) is an isomorphism for \(i \leq k\), while \(\Phi_{\mathcal{R}_k}(\mathcal{E}_k') = 0\) for \(i > k\). Thus for \(0 < k \leq m\) the equality \(\Phi_{\rho_k}(\mathcal{E}_i') = \Phi_{\rho_k}(\mathcal{E}_i') \circ \Phi_{\alpha_k}(\mathcal{E}_i')\) implies that \(\Phi_{\alpha_k}(\mathcal{E}_i')\) is an isomorphism for \(i < k\). Hence from the distinguished triangle

\[
\Phi_{\tilde{\mathcal{E}}_k}(\mathcal{E}_i')[\cdot -1] \to \Phi_{\tilde{\mathcal{R}}_{k-1}}(\mathcal{E}_i') \xrightarrow{\Phi_{\rho_k}(\mathcal{E}_i')} \Phi_{\tilde{\mathcal{R}}_k}(\mathcal{E}_i') \to \Phi_{\tilde{\mathcal{E}}_k}(\mathcal{E}_i')[1]
\]

we get \(\Phi_{\tilde{\mathcal{E}}_k}(\mathcal{E}_i') \cong (\mathcal{E}_i')_{h_{i,k}}\) (notice that, by what we have already proved, this is true also for \(k = 0\)), which is equivalent, by \(3.1\), to

\[
\text{Hom}_Y(\mathcal{E}_i, \mathcal{F}_k) \cong \mathcal{E}_i'[h_{i,k}]
\]

for \(0 \leq i, k \leq m\). In order to conclude that \(\mathcal{F}_k \cong \mathcal{E}'_k\) one can proceed as follows. As \((\mathcal{E}'_m, \ldots, \mathcal{E}'_0)\) is a full exceptional sequence in \(D^b(Y)\), there exists (unique up to isomorphism) a distinguished triangle

\[
\mathcal{A}_k \to \mathcal{F}_k \to \mathcal{A}_k' \to \mathcal{A}_k[1]
\]
with $A_k \in \langle E_i', \ldots, E_{i-1}' \rangle$ and $A_k' \in \langle E_i', \ldots, E_m' \rangle$. Now, if $i = k$, Hom$_Y(E_i, A_k) = 0$ by (A.3) whereas if $i < k$, Hom$_Y(E_i, A_k') = 0$ always by (A.3) and Hom$_Y(E_i, F_k) = 0$ by (A.2), so that Hom$_Y(E_i, A_k) = 0$ also in this case. Clearly this implies $A_k = 0$ and $F_k \cong A_k' \in \langle E_i', \ldots, E_m' \rangle$. In a similar way, there exists (unique up to isomorphism) a distinguished triangle

$B_k \rightarrow F_k \rightarrow B_k' \rightarrow B_k[1]$ with $B_k \in \langle E_k' \rangle$ and $B_k' \in \langle E_{k+1}', \ldots, E_m' \rangle$. This time Hom$_Y(E_i, B_k) = 0$ (if $i \leq k$ by (A.3) and Hom$_Y(E_i, B_k') = 0$ by (A.2), whence $B_k' = 0$ and $F_k \cong B_k \in \langle E_i' \rangle$. Then, since Hom$_Y(E_k, F_k) \cong k \cong$ Hom$_Y(E_k, E_k')$, we must have $F_k \cong E_k'$.

Hence for $0 < k \leq m$ there is a distinguished triangle in $D^b(Y \times Y)$

$$\tag{3.3} \mathcal{R}_{k-1} \xrightarrow{\alpha_k} \mathcal{R}_k \xrightarrow{\beta_k} \tilde{E}_k \xrightarrow{\gamma_k} \mathcal{R}_{k-1}[1]$$

where $\tilde{E}_k := E_k' \otimes E_k'^\vee$. Notice that the pair $(\beta_k, \gamma_k)$ is not uniquely determined, but, as clearly $\tilde{E}_k$ is an exceptional object, it could only be changed by $(\lambda \beta_k, \lambda^{-1} \gamma_k)$ for some $\lambda \in k^*$. By (TR4) (or by (TR3) and (A.2) there exists (unique because Hom($\mathcal{R}_{k-1}[1], S_{k-1}$) = 0) a morphism $\gamma_k': \tilde{E}_k \rightarrow S_{k-1}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{R}_{k-1} & \xrightarrow{\alpha_k} & \mathcal{R}_k \\
\downarrow{id} & & \downarrow{\rho_k} \\
\mathcal{R}_{k-1} & \xrightarrow{\rho_{k-1}} & O_\Delta \\
\downarrow{\sigma_k} & & \downarrow{\sigma_{k-1}} \\
\mathcal{S}_{k-1} & \xrightarrow{\gamma_k'} & \mathcal{S}_{k-1} \\
\downarrow{\tau_k} & & \downarrow{\tau_{k-1}} \\
\mathcal{R}_{k-1}[1] & \xrightarrow{\gamma_k} & \mathcal{R}_{k-1}[1]
\end{array}$$

commutes and with $C(\gamma_k') \cong S_k$.

**Lemma 3.2.** For $0 < k \leq m$ there is an isomorphism $E_k' \cong \Phi_{S_{k-1}}(\tilde{E}_k)$ with the property that the induced morphism

$$\tilde{E}_k = E_k' \otimes E_k'^\vee \cong \Phi_{S_{k-1}}(E_k) \otimes E_k'^\vee \xrightarrow{\mu_{E_k, S_{k-1}}} S_{k-1}$$

(see (2.1) for the definition of $\mu_{E_k, S_{k-1}}$) coincides with $\gamma_k'$.

**Proof.** Since $\Phi_{E_k}(E_k) \cong E_k' \otimes_k$ Hom$_Y(E_i, E_k)$ by (2.3) we have $\Phi_{E_k}(E_k) = 0$ for $i > k$, while there is a natural isomorphism $\Phi_{E_k}(E_k) \cong E_k'$. It follows easily that $\Phi_{S_k}(E_k) = 0$, hence from the distinguished triangle

$$\Phi_{E_k}(E_k) \xrightarrow{\Phi_{E_k}(E_k)} \Phi_{S_{k-1}}(E_k) \rightarrow \Phi_{S_k}(E_k) \rightarrow \Phi_{E_k}(E_k)[1]$$

we see that $\Phi_{E_k}(E_k)$ is an isomorphism. We claim that the isomorphism we are looking for can be chosen to be $E_k' \cong \Phi_{E_k}(E_k)$

$$\begin{array}{ccc}
\tilde{E}_k = E_k' \otimes E_k'^\vee \xrightarrow{\sim} \Phi_{E_k}(E_k) \otimes E_k'^\vee \\
\downarrow{id} & & \downarrow{\Phi_{E_k}(E_k) \otimes \text{id}} \\
\tilde{E}_k & \xrightarrow{\Phi_{E_k}(E_k)} & \Phi_{S_{k-1}}(E_k) \otimes E_k'^\vee \\
\downarrow{\gamma_k} & & \downarrow{\Phi_{E_k, S_{k-1}}} \\
\tilde{E}_k & \xrightarrow{\mu_{E_k, E_k'}} & S_{k-1}
\end{array}$$

commutes. Now, it is not difficult to check directly that the triangle commutes, whereas the square commutes by (2.2).
4. Main Theorem

Let $X$ and $Y$ be stacks and $\mathcal{K}$ an object of $D^b(X \times Y)$. As in section 3 (whose notation will be used) we assume that $D^b(Y)$ admits a full exceptional sequence $(\mathcal{E}_0, \ldots, \mathcal{E}_m)$, and moreover that $\mathcal{F} := \Phi_\mathcal{K} : D^b(Y) \to D^b(X)$ induces isomorphisms

\begin{equation}
\text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_j) \xrightarrow{\sim} \text{Hom}_X(\mathcal{F}(\mathcal{E}_i), \mathcal{F}(\mathcal{E}_j)) \text{ for } 0 \leq i < j \leq m.
\end{equation}

Defining $\tilde{\mathcal{F}}$ as in (2.3), for $0 \leq k \leq m$ we set $\tilde{\rho}_k := \mu_\mathcal{K} \circ \tilde{\mathcal{F}}(\mathcal{R}_k) : \tilde{\mathcal{F}}(\mathcal{R}_k) \to \mathcal{O}_{\Delta_X}$ and extend it to a distinguished triangle $\tilde{\mathcal{F}}(\mathcal{R}_k) \xrightarrow{\tilde{\rho}_k} \mathcal{O}_{\Delta_X} \xrightarrow{\tilde{\sigma}_k} C(\tilde{\rho}_k) \xrightarrow{\tilde{\tau}_k} \tilde{\mathcal{F}}(\mathcal{R}_k)[1]$. Let moreover $\zeta_k : \tilde{\mathcal{F}}(\mathcal{S}_k) \to C(\tilde{\rho}_k)$ be a morphism such that the diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{F}}(\mathcal{R}_k) & \xrightarrow{\tilde{\rho}_k} & \tilde{\mathcal{F}}(\mathcal{O}_{\Delta_Y}) \\
\downarrow \text{id} & & \downarrow \mu_\mathcal{K} \\
C(\tilde{\rho}_k) & \xrightarrow{\zeta_k} & \tilde{\mathcal{F}}(\mathcal{R}_k)[1]
\end{array}
\]

commutes (such a morphism exists by (TR3) or by (TR4)).

**Lemma 4.1.** If $\mathcal{F} = \Phi_\mathcal{K}$ satisfies (4.1), then the composition

\[\Phi_{\mathcal{S}_k}(\mathcal{E}_{k+1}) \xrightarrow{\tilde{\rho}_{k_{\mathcal{S}_k}, \mathcal{E}_{k+1}}} \Phi_{\mathcal{F}(\mathcal{S}_k)}(\mathcal{F}(\mathcal{E}_{k+1})) \xrightarrow{\Phi_{\tilde{\mathcal{F}}}(\mathcal{E}_{k+1})} \Phi_{C(\tilde{\rho}_k)}(\mathcal{F}(\mathcal{E}_{k+1}))\]

(see [2.3] for the definition of $\tilde{\rho}_{k_{\mathcal{S}_k}, \mathcal{E}_{k+1}}$) is an isomorphism in $D^b(X)$ for $0 \leq k < m$.

**Proof.** Setting for brevity $G(-) := \mathcal{F}(\mathcal{E}_{k+1})$, $H(-) := \Phi_\mathcal{K}(\mathcal{F}(\mathcal{E}_{k+1}))$ and $\tilde{H} := H \circ \tilde{\mathcal{F}}$, there is a commutative diagram (by (2.5))

\[
\begin{array}{cccc}
G(\mathcal{R}_k) \xrightarrow{G(\rho_k)} G(\mathcal{O}_{\Delta_Y}) & \xrightarrow{G(\sigma_k)} G(\mathcal{S}_k) & \xrightarrow{t_\mathcal{G}(\mathcal{R}_k) \circ G(\tau_k)} G(\mathcal{R}_k)[1] \\
\downarrow \rho_{\mathcal{R}_k, \mathcal{E}_{k+1}} & & \downarrow \rho_{\mathcal{S}_k, \mathcal{E}_{k+1}} & \downarrow \rho_{\mathcal{R}_k, \mathcal{E}_{k+1}}[1] \\
\tilde{H}(\mathcal{R}_k) \xrightarrow{\tilde{H}(\rho_k)} \tilde{H}(\mathcal{O}_{\Delta_Y}) & \xrightarrow{\tilde{H}(\sigma_k)} \tilde{H}(\mathcal{S}_k) & \xrightarrow{t_\tilde{H}(\mathcal{R}_k) \circ \tilde{H}(\tau_k)} \tilde{H}(\mathcal{R}_k)[1] \\
\downarrow \text{id} & & \downarrow \mu_\mathcal{K} & \downarrow \zeta_k & \downarrow \text{id} \\
\tilde{H}(\mathcal{R}_k) \xrightarrow{\tilde{H}(\rho_k)} \tilde{H}(\mathcal{O}_{\Delta_X}) & \xrightarrow{\tilde{H}(\sigma_k)} \tilde{H}(\mathcal{S}_k) & \xrightarrow{t_\tilde{H}(\mathcal{R}_k) \circ \tilde{H}(\tau_k)} \tilde{H}(\mathcal{R}_k)[1]
\end{array}
\]

whose rows are distinguished triangles. As $H(\mu_\mathcal{K}) \circ \tilde{\mathcal{F}}_{\mathcal{O}_{\Delta_Y}, \mathcal{E}_{k+1}} = \Phi_{\mu_\mathcal{K}}(\mathcal{F}(\mathcal{E}_{k+1})) \circ \tilde{\mathcal{F}}_{\mathcal{O}_{\Delta_Y}, \mathcal{E}_{k+1}}$ is an isomorphism by (2.3), $H(\zeta_k) \circ \tilde{\mathcal{F}}_{\mathcal{O}_{\Delta_X}, \mathcal{E}_{k+1}} = \Phi_{\zeta_k}(\mathcal{F}(\mathcal{E}_{k+1})) \circ \tilde{\mathcal{F}}_{\mathcal{O}_{\Delta_X}, \mathcal{E}_{k+1}}$ is an isomorphism if (and only if) $\tilde{\rho}_{\mathcal{R}_k, \mathcal{E}_{k+1}}$ is an isomorphism. In fact it can be proved that $\tilde{\rho}_{\mathcal{R}_k, \mathcal{E}_{k+1}}$ is an isomorphism for $0 \leq i \leq k$. To this purpose it is enough to prove that $\tilde{\rho}_{\mathcal{E}_i, \mathcal{E}_{k+1}}$ is an isomorphism for $0 \leq i \leq k$, because then, remembering that $\mathcal{R}_0 \cong \tilde{\mathcal{E}}_0$ by (3.1) one can proceed by induction on $i$, using the commutative diagrams (whose rows are distinguished triangles by (3.3))

\[
\begin{array}{cccc}
G(\mathcal{R}_{i-1}) \xrightarrow{G(\alpha_i)} G(\mathcal{R}_i) & \xrightarrow{G(\beta_i)} G(\tilde{\mathcal{E}}_i) & \xrightarrow{t_\mathcal{G}(\mathcal{R}_{i-1}) \circ G(\gamma_i)} G(\mathcal{R}_{i-1})[1] \\
\downarrow \rho_{\mathcal{R}_{i-1}, \mathcal{E}_{k+1}} & & \downarrow \rho_{\mathcal{E}_i, \mathcal{E}_{k+1}} & \downarrow \rho_{\mathcal{R}_{i-1}, \mathcal{E}_{k+1}}[1] \\
\tilde{H}(\mathcal{R}_{i-1}) \xrightarrow{\tilde{H}(\alpha_i)} \tilde{H}(\mathcal{R}_i) & \xrightarrow{\tilde{H}(\beta_i)} \tilde{H}(\tilde{\mathcal{E}}_i) & \xrightarrow{t_\tilde{H}(\mathcal{R}_{i-1}) \circ \tilde{H}(\gamma_i)} \tilde{H}(\mathcal{R}_{i-1})[1]
\end{array}
\]

for $0 < i \leq k$. In order to prove that

$\tilde{\rho}_{\mathcal{E}_i, \mathcal{E}_{k+1}} : \mathcal{F}(\Phi_{\mathcal{E}_i}(\mathcal{E}_{k+1})) \to \mathcal{F}(\Phi_{\tilde{\mathcal{E}}_i}(\mathcal{E}_{k+1}))$
is an isomorphism for $0 \leq i \leq k$, notice that by 2.3
\[ F(\Phi_{\xi_i}(\mathcal{E}_{k+1})) \cong F(\mathcal{E}_i^c \otimes_k \text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_{k+1})) \cong F(\mathcal{E}_i^c) \otimes_k \text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_{k+1}) \]
and (since $\tilde{F}(\xi_i) \cong F(\mathcal{E}_i^c) \otimes F(\mathcal{E}_i)$ by 2.7)
\[ \Phi_{\tilde{F}(\xi_i)}(F(\mathcal{E}_{k+1})) \cong F(\mathcal{E}_i) \otimes_k \text{Hom}_X(F(\mathcal{E}_i), F(\mathcal{E}_{k+1})). \]

It is then easy to see that $\tilde{\nu}_{\xi_i, \mathcal{E}_{k+1}}$ can be identified with the natural map, hence it is an isomorphism by 3.1. □

For $\mathcal{F} \in \mathcal{D}^b(X)$ we set $T_{\mathcal{F}} := \Phi_{C(\mu_{\mathcal{F}}): \mathcal{F} \otimes \mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta}}: \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ (by 4.1 $T_{\mathcal{F}}$ is an equivalence if $\mathcal{F}$ is a spherical object).

**Theorem 4.2.** If $\mathcal{F} = \Phi_{\mathcal{K}}: \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ satisfies 4.1, then
\[ T_{F(\xi_0)} \circ \cdots \circ T_{F(\xi_m)} \cong \Phi_{C(\mu_{\mathcal{K}}) \circ C(\mathcal{O}_{\Delta} \to \mathcal{O}_{\Delta})}. \]

**Proof.** Defining for $0 \leq k \leq m$
\[ C_k := C(\mu_{F(\xi_k)}) \ast \cdots \ast C(\mu_{F(\xi_k)}), \]
by 2.7 it is enough to prove that $C_m \cong C(\mu_{\mathcal{K}})$. We will show that in fact $C_k \cong C(\tilde{\rho}_k)$ for $0 \leq k \leq m$: the case $k = m$ yields the thesis because $\tilde{\rho}_m = \mu_{\mathcal{K}} \circ F(\rho_m)$ and $F(\rho_m)$ is an isomorphism, whence $C(\tilde{\rho}_m) \cong C(\mu_{\mathcal{K}})$. We proceed by induction on $k$: the case $k = 0$ follows from 2.8 since $\mu_{\mathcal{K}}$ can be identified with $\rho_0$. So let $0 < k \leq m$ and assume that $C_{k-1} \cong C(\tilde{\rho}_{k-1})$; then by definition
\[ C_k = C_{k-1} \ast C(\mu_{F(\xi_k)}) : F(\mathcal{E}_k) \otimes F(\mathcal{E}_k)^\vee \to \mathcal{O}_{\Delta X} \]
\[ \cong C(\text{id} \ast \mu_{F(\xi_k)} : C(\tilde{\rho}_{k-1}) \ast (F(\mathcal{E}_k) \otimes F(\mathcal{E}_k)^\vee) \to C(\tilde{\rho}_{k-1}) \ast C(\mu_{\mathcal{K}})) \]
\[ \cong C(\mu_{F(\xi_k)}, C(\tilde{\rho}_{k-1}) : \Phi_{C(\tilde{\rho}_{k-1})}(F(\mathcal{E}_k)) \otimes F(\mathcal{E}_k)^\vee \to C(\tilde{\rho}_{k-1})). \]

Denoting by $\xi_k : \tilde{F}(\mathcal{E}_k) \to C(\tilde{\rho}_{k-1})$ the composition
\[ \tilde{F}(\mathcal{E}_k) \cong \tilde{F}(\Phi_{S_{k-1}}(\mathcal{E}_k) \otimes \mathcal{E}_k^\vee) \cong F(\Phi_{S_{k-1}}(\mathcal{E}_k)) \otimes F(\mathcal{E}_k)^\vee \]
\[ \cong \Phi_{\tilde{F}(S_{k-1})}(F(\mathcal{E}_k)) \otimes F(\mathcal{E}_k)^\vee \]
\[ \Phi_{\tilde{F}(S_{k-1})}(F(\mathcal{E}_k) \otimes F(\mathcal{E}_k)^\vee) \cong \Phi_{C(\tilde{\rho}_{k-1})}(F(\mathcal{E}_k)) \otimes F(\mathcal{E}_k)^\vee \]
\[ (the \text{isomorphisms are the natural ones induced by 3.2 and 2.7}), \text{and remembering that the composition } \Phi_{\tilde{F}(S_{k-1})}(F(\mathcal{E}_k)) \circ \tilde{\nu}_{S_{k-1}, \mathcal{E}_k} \text{is an isomorphism by 4.1 we have } C_k \cong \Phi_{C(\mu_{\mathcal{E}_k}), C(\tilde{\rho}_{k-1})} \cong C(\xi_k). \text{ Thus the conclusion will follow if we show that the diagram} \]
\[
\begin{array}{ccc}
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\alpha_k)} & \tilde{F}(R_k) \\
\downarrow \text{id} & & \downarrow \tilde{\rho}_k \\
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{\nu}_{k-1}} & \tilde{F}(R_{k-1})[1]
\end{array}
\]
\[
\begin{array}{ccc}
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\beta_k)} & \tilde{F}(\xi_k) \\
\downarrow \tilde{\rho}_{k-1} & & \downarrow \xi_k \\
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\gamma_k)} & \tilde{F}(R_{k-1})[1]
\end{array}
\]
\[
\begin{array}{ccc}
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\alpha_k)} & \tilde{F}(R_k) \\
\downarrow \text{id} & & \downarrow \tilde{\rho}_k \\
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{\nu}_{k-1}} & \tilde{F}(R_{k-1})[1]
\end{array}
\]
\[
\begin{array}{ccc}
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\beta_k)} & \tilde{F}(\xi_k) \\
\downarrow \tilde{\rho}_{k-1} & & \downarrow \xi_k \\
\tilde{F}(R_{k-1}) & \xrightarrow{\tilde{F}(\gamma_k)} & \tilde{F}(R_{k-1})[1]
\end{array}
\]
The fact that (4.1) holds can be checked as follows. In case (1), since $F \leq C$ 4.3

Remark the outer square commutes, which is true because (1), (2) and (3) commute, re-

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Remark 4.3. Theorem 4.2 can be generalized as follows (see also [5, Remark 4.7]).

Given $0 \leq n \leq m$, we set $\Xi := \langle F(\mathcal{E}_{n+1}), \ldots, F(\mathcal{E}_{m}) \rangle^\perp$. Then, assuming that (4.1) holds for $0 \leq i < j \leq n$, one can prove that

$$\Phi_{C(\rho_n)} |_\Xi \cong \Phi_{C(\mu_k)} |_\Xi.$$

(clearly 4.2 is the case $n = m$). Indeed, with the same proof one can show $C_k \cong C(\rho_k)$ for $0 \leq k \leq n$, and then it is enough to show that $\Phi_{C(\rho_n)} |_\Xi \cong \Phi_{C(\mu_k)} |_\Xi$.

As $\rho_n = \mu_k \circ F(\rho_n)$, this follows from the fact that $\Phi_{C(F(\rho_n))} |_\Xi \cong 0$, which is true because $C(F(\rho_n)) \cong C(\rho_n) \cong C(\mathcal{S}_n)$ and $\Phi_{F(\mathcal{S}_n)}(\mathcal{F}) = 0$ for $\mathcal{F} \in \Xi$.

5. Applications

Condition (4.1) is satisfied for instance if $F$ is fully faithful, but in that case 4.2 is not so useful. Indeed, $(F(\mathcal{E}_0), \ldots, F(\mathcal{E}_m))$ is then an exceptional sequence in $\mathcal{D}(X)$.

Taking into account that, if $\mathcal{E}$ is an exceptional object, $T_F$ is just projection onto $\langle \mathcal{E} \rangle^\perp$, it is clear that $T_F(\mathcal{E}_0) \circ \cdots \circ T_F(\mathcal{E}_m)$ is projection onto $\langle F(\mathcal{E}_0), \ldots, F(\mathcal{E}_m) \rangle^\perp$, and it is easy to see directly that the same is true for $\Phi_{C(\mu_k)}$.

We are more interested in the following examples.

1. $F = f^*$, where $f : X \to Y$ is a morphism such that $C(f^\#: \mathcal{O}_Y \to f_\# \mathcal{O}_X) \cong \omega_Y [c]$ where $c = \dim(Y) - \dim(X)$; in particular, $f$ can be the inclusion of a hypersurface such that $\omega_Y \cong \mathcal{O}_Y(-X)$ (hence $\omega_X \cong \mathcal{O}_X$).

2. $F = g_*$, where $g : Y \to X$ is the inclusion of a hypersurface such that $g^* \omega_X \cong \mathcal{O}_Y$ (hence $\omega_Y \cong \mathcal{O}_Y(Y)$).

The fact that (4.1) holds can be checked as follows. In case (1), since

$$\text{Hom}_X(f^* \mathcal{E}_i, f^* \mathcal{E}_j) \cong \text{Hom}_Y(\mathcal{E}_i, f_* f^* \mathcal{E}_j) \cong \text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_j \otimes f_\# \mathcal{O}_X),$$
and taking into account that $E_j \otimes f_*O_X \cong C(E_j \otimes \omega_Y[-1] \to E_j)$ by hypothesis, it is enough to note that for $i < j$ by Serre duality
\[ \text{Hom}_Y(E_i, E_j \otimes \omega'_Y) \cong \text{Hom}_Y(E_j, E_i) = 0. \]

Similarly, in case (2) we have
\[ \text{Hom}_X(g_*E_i, g_*E_j) \cong \text{Hom}_X(g^*g_*E_i, E_j) \]
so that, since $C(g^*g_*E_i \to E_j)$, we see that $E_i \otimes O_Y(-Y)[2]$ (by [3] Cor. 11.4), again we conclude from the fact that for $i < j$ \[ \text{Hom}_Y(E_i \otimes O_Y(-Y), E_j) \cong \text{Hom}_Y(E_i, E_j \otimes \omega_Y) \cong \text{Hom}_Y(E_j, E_i(\dim(Y))[1]) = 0. \]

Remark 5.1. In case (1), if moreover $\omega_X \cong O_X$, then it can be proved that \[ f^*E_i \cong \text{Hom}(k, O_X(-1)) \] or \[ [14, \text{Prop. 8.39}]. \]

Similarly, in case (2) the $g_*E_i$ are spherical objects thanks to \[ [14, \text{Prop. 3.15}]. \]

In the following, given $F \in D^b(X)$, we will denote by $L_F$ the exact functor $\Phi_{\delta, F} \cong F \otimes - : D^b(X) \to D^b(X)$ (which is an equivalence when $F$ is a line bundle).

**Corollary 5.2.** If $f : X \hookrightarrow Y$ is the inclusion of a hypersurface such that $\omega_Y \cong O_Y(-X)$, then $T_{f, \varepsilon_0} \circ \cdots \circ T_{f, \varepsilon_m} \cong L_{O_X(-X)[2]}$.

**Proof.** $f^* \cong \Phi_K$, where $K \subseteq (\text{id}_X, f), O_X \in D^b(X \times Y)$. By [14, Th. 5.1] it is then enough to show that there is a distinguished triangle in $D^b(X \times X)$
\[ \delta_*O_X(-X)[1] \to \mathcal{K} \otimes \mathcal{K}^! \to \mathcal{O}_{\Delta_X} \to \delta_*O_X(-X)[2], \]
which is done in the proof of [8, Cor. 11.4] (notice that, by [2, 5] and [2, 2] $\Phi_K \cong f_*$ and $\Phi_{\Delta, K} \cong f^* \circ f_*$).

**Remark 5.3.** More generally, if $f : X \to Y$ is a morphism such that $C(f^! \cong \omega_Y[e]$, then it can be proved that $T_{f, \varepsilon_0} \circ \cdots \circ T_{f, \varepsilon_m} \cong \Phi_{C(f^! \otimes O_Y(\Delta_X), O_Y \to \Delta_X)}$. If moreover $f$ is flat, then by flat base change $(f \times f)^*O_{\Delta_Y} \cong O_{X \times Y}$: when $Y = \mathbb{P}^n$ and $E_i = O_X(1)$ this result had already been proved in [14].

**Remark 5.4.** If $X \subset \mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ is a (Calabi-Yau) hypersurface of degree $|w| := w_0 + \cdots + w_n$ and $(\varepsilon_0, \ldots, \varepsilon_m) = (\mathcal{O}_{(1)}, \ldots, \mathcal{O}_{(|w|)}), then [5.2] reduces to $T_{\mathcal{O}_X(1)} \circ \cdots \circ T_{\mathcal{O}_X(|w|)} \cong L_{\mathcal{O}_X(-|w|)[2]}$, which is in fact equivalent to [13] Thm. 1.1, namely $(\Phi_{\mathcal{O}})^{|w|} \cong \mathcal{O}_{X \times Y}(1)$, \[ \delta_*O_X(-X)[1] \to \mathcal{K} \otimes \mathcal{K}^! \to \mathcal{O}_{\Delta_X} \to \delta_*O_X(-X)[2], \]
which is done in the proof of [8, Cor. 11.4] (notice that, by [2, 5] and [2, 2] $\Phi_K \cong f_*$ and $\Phi_{\Delta, K} \cong f^* \circ f_*$).

**Corollary 5.5.** If $F \in D^b(X)$ and $G$ is a Fourier-Mukai autoequivalence of $D^b(X)$, then $G \circ T_{\mathcal{F}} \cong T_{G(F)} \circ G$.

**Proof.** See [14], Cor. 8.21. 

**Corollary 5.6.** If $g : Y \hookrightarrow X$ is the inclusion of a hypersurface such that $g^*\omega_X \cong O_Y$, then $T_{g, \varepsilon_0} \circ \cdots \circ T_{g, \varepsilon_m} \cong L_{O_X(Y)}$.

**Proof.** $g_* \cong \Phi_K$, where $K \subseteq (g, \text{id}_Y), O_Y \in D^b(X \times Y)$. By [14, Th. 5.1] it is enough to prove that there is a distinguished triangle in $D^b(X \times X)$
\[ \delta_*O_X(Y)[1] \to \mathcal{K} \otimes \mathcal{K}^! \to \mathcal{O}_{\Delta_X} \to \delta_*O_X(Y), \]
which can be done with a technique similar to the one used for the proof of [5.1]. As an indication that this is true, observe that, always by [2, 5] and [2, 2] $\Phi_K \cong g^*$ and $\Phi_{\Delta, K} \cong g^* \circ g_*$. Now, for every $F \in D^b(X)$ there is a natural isomorphism
\[ g_*(F \otimes \omega_Y[-1]) \cong \mathcal{F} \otimes g_*\omega_Y[-1] \cong \mathcal{F} \otimes g_*O_Y(Y)[-1], \]
hence \(g_* \circ g^! \cong L_{g* \mathcal{O}_Y}([-1])\). In fact one can prove that \(\mathcal{K} \ast \mathcal{K}' \cong \delta_* g_* \mathcal{O}_Y([-1])\), and that the above triangle can be identified with the image through \(\delta_*\) of the distinguished triangle in \(D^b(X)\)

\[
\mathcal{O}_X([-1]) \rightarrow g_* \mathcal{O}_Y([-1]) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X([-1])
\]

induced by the short exact sequence \(0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X([-1]) \rightarrow g_* \mathcal{O}_Y([-1]) \rightarrow 0\). \(\square\)

Finally we want to show how \([5,6]\) can be used in a setup like that of \([9]\), respectively \([10]\), where \(X\) is a crepant resolution of the singularity \(\mathbb{C}^2/\mathbb{Z}_3\), respectively \(\mathbb{C}^3/\mathbb{Z}_5\). Actually we will assume (as always) that \(X\) is proper, but it would be not difficult to see that our arguments can be extended to the setting of varieties or stacks which are not necessarily proper, by working with derived categories of coherent sheaves with compact supports.

If \(g: Y \hookrightarrow X\) is the inclusion morphism and \(\mathcal{F} \in D^b(Y)\), in the following we will write for simplicity \(\mathcal{F}\) instead of \(g_* \mathcal{F} \in D^b(X)\).

First let \(\dim(X) = 2\) and let \(C_i\) (for \(i = 1, \ldots, 4\)) be divisors in \(X\) such that \(C_3\) and \(C_4\) are \((-2)\)-curves. Assume also the following intersection relations

\[
(5.2) \quad C_2 \cdot C_3 = C_1 \cdot C_4 = 1, \quad C_1 \cdot C_3 = C_2 \cdot C_4 = 0
\]

and the following linear equivalence relations of divisors

\[
(5.3) \quad C_3 + 2C_2 \sim C_1, \quad C_4 + 2C_1 \sim C_2.
\]

Taking into account that \((\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1))\) is a full exceptional sequence in \(D^b(\mathbb{P}^1)\), \([5,6]\) implies that for \(i = 3, 4\)

\[
(5.4) \quad T_{C_{i_1}} \circ T_{C_{i_2}}(1) \cong L_{C_{i_3}}(C_{i_4}).
\]

Setting \(M_{2i} := T_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2})\) and \(M_{2i} := T_{C_{i_4}} \circ L_{C_{i_3}}\), in \([9]\) it was proved that \(M_{2i} \cong \mathcal{O}_{C_{i_2}}\) and conjectured that \(M_{2i} \cong \mathcal{id}\) in \(\text{Aut}(D^b(X))\). We are going to show that both results follow easily from \((5.4)\). Indeed, using also \((5.5)\) \((5.2)\) and \((5.3)\), we find

\[
M_{2i} \cong T_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) \circ L_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) \cong T_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) \circ L_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) \cong T_{C_{i_3}} \circ L_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) \cong L_{C_{i_3}}(C_{i_4}).
\]

In a completely similar way one proves that \(\hat{M}_{2i} \cong T_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2})\) satisfies \(\hat{M}_{2i} \cong L_{C_{i_3}}(C_{i_4})\). Since \(M_{2i}(C_{i_2}) \cong \mathcal{O}_{C_{i_2}}\) (see \([9]\) Section 4.2), we obtain

\[
M_{2i} = M_{2i} \circ T_{C_{i_3}} \circ M_{2i} \cong M_{2i} \circ T_{C_{i_3}} \circ M_{2i} \circ M_{2i}
\]

\[
\cong T_{C_{i_3}} \circ T_{C_{i_4}} \circ M_{2i} \cong T_{C_{i_3}} \circ T_{C_{i_4}} \circ L_{C_{i_3}} \circ L_{C_{i_4}}(C_{i_2}) = T_{C_{i_3}} \circ \hat{M}_{2i}.
\]

On the other hand, \(M_{2i} \cong \hat{M}_{2i} \circ \mathcal{O}_{C_{i_2}}\) \(M_{2i} \cong \hat{M}_{2i} \circ \mathcal{O}_{C_{i_2}}\), and we conclude

\[
M_{2i} \cong T_{C_{i_3}} \circ \hat{M}_{2i} \circ M_{2i} \cong T_{C_{i_3}} \circ \mathcal{O}_{C_{i_2}} \circ M_{2i} = M_{2i} = M_{2i} \cong \mathcal{id}.
\]

Now let \(\dim(X) = 3\) and let \(D_i\) (for \(i = 1, \ldots, 5\)) be divisors in \(X\) such that \(D_4 \cong \mathbb{P}^3\), \(D_5 \cong \mathbb{P}^3\) and \(K_X \cdot D_4 = K_X \cdot D_5 = 0\). Denoting by \(h\) the class of a hyperplane section in \(D_4\), by \(f\) the class of a fibre in \(D_5\) and by \(s\) the class of the \(-3\) section in \(D_5\), assume also the following intersection relations

\[
(5.5) \quad D_1 \cdot D_1 = h, \quad D_4 \cdot D_1 = 0, \quad D_5 \cdot D_1 = f, \quad D_5 \cdot D_2 = s + 3f
\]

and the following linear equivalence relations of divisors

\[
(5.6) \quad D_4 + 3D_1 \sim D_2, \quad D_5 + 2D_2 \sim D_1.
\]
Using the full exceptional sequences \((O_{D_4}(h), O_{D_4}(2h), O_{D_4}(3h))\) in \(D^b(D_4)\) and \((O_{D_5}(-f), O_{D_5}, O_{D_5}(s+2f), O_{D_5}(s+3f))\) in \(D^b(D_5)\), by (5.3) we have

\[
\begin{align*}
(T.7) & \quad T_{O_{D_4}(h)} \circ T_{O_{D_4}(2h)} \circ T_{O_{D_4}(3h)} \cong L_{O_X(D_4)}, \\
(T.8) & \quad T_{O_{D_5}(-f)} \circ T_{O_{D_5}} \circ T_{O_{D_5}(s+2f)} \circ T_{O_{D_5}(s+3f)} \cong L_{O_X(D_5)}.
\end{align*}
\]

Setting \(N_{Z_2} := L_{O_X(-D_1)} \circ T_{O_{D_5}} \circ L_{O_X(D_1)} \circ T_{O_{D_5}} \circ L_{O_X(D_2)}, N_{Z_3} := L_{O_X(D_1)} \circ T_{O_{D_5}} \) and \(N_{Z_4} := T_{O_{D_5}} \circ N_{Z_2}\), in [10] it was conjectured that \(N_{Z_2} \cong L_{O_X(D_1)}, N_{Z_3} \cong L_{O_X(D_2)}\) and \(N_{Z_4} \cong id \in Aut(D^b(X))\). Here we prove only the first two relations (then it is not difficult to deduce also the last one, with manipulations similar to those with which \(M_{Z_2} \cong id\) was obtained above using \(M_{Z_2} \cong L_{O_X(C_1)}\)). To this purpose, observe that by (5.7) and (5.8) we have \(L_{O_X(-D_1)} \circ T_{O_{D_5}} \cong T_{O_{D_5}(-f)} \circ L_{O_X(-D_1)}\), hence \(N_{Z_2} \cong T_{O_{D_5}(-f)} \circ T_{O_{D_5}} \circ L_{O_X(D_2)}\). Then, using also (5.8) and (5.9) we get

\[
N_{Z_2} \cong T_{O_{D_5}(-f)} \circ T_{O_{D_5}} \circ L_{O_X(D_2)} \circ T_{O_{D_5}(-f)} \circ T_{O_{D_5}} \circ L_{O_X(D_2)} \\
\cong T_{O_{D_5}(-f)} \circ T_{O_{D_5}} \circ T_{O_{D_5}(s+2f)} \circ T_{O_{D_5}(s+3f)} \circ L_{O_X(D_2)} \circ L_{O_X(D_2)} \\
\cong L_{O_X(D_3)} \circ L_{O_X(2D_2)} \cong L_{O_X(D_3+2D_2)} \cong L_{O_X(D_2)}.
\]

Similarly, using (5.10) we find

\[
N_{Z_3} = L_{O_X(D_1)} \circ T_{O_{D_5}} \circ L_{O_X(D_1)} \circ T_{O_{D_5}} \circ L_{O_X(D_1)} \circ T_{O_{D_5}} \\
\cong T_{O_{D_4}(h)} \circ T_{O_{D_4}(2h)} \circ T_{O_{D_4}(3h)} \circ L_{O_X(D_1)} \circ L_{O_X(D_1)} \circ L_{O_X(D_2)} \\
\cong L_{O_X(D_4)} \circ L_{O_X(3D_2)} \cong L_{O_X(D_4+3D_2)} \cong L_{O_X(D_2)}.
\]

**APPENDIX A. TRIANGULATED CATEGORIES AND EXCEPTIONAL SEQUENCES**

In this appendix we collect some definitions and properties about triangulated categories, semiorthogonal decompositions and exceptional sequences. We refer to [15, 3] and [7] for a thorough treatment of these subjects, including proofs of the results which are only stated here.

A **triangulated category** is an additive category endowed with an additive automorphism, called **shift functor** and denoted by \([1]\) (its \(n^{th}\) power is denoted by \([n]\) for every \(n \in \mathbb{Z}\)), and with a family of **distinguished triangles** of the form \(A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]\), subject to a list of axioms, usually denoted by (TR1)-(TR4). Let \(\mathcal{T}\) be a triangulated category. We call cone of a morphism \(f: A \to B\) of \(\mathcal{T}\) an object \(C(f)\) fitting into a distinguished triangle \(A \xrightarrow{f} B \to C(f) \to A[1]\): it is a fundamental property of triangulated categories that every morphism admits a cone, which is unique up to (non canonical) isomorphism. The following simple fact is frequently used in the paper: given two morphisms \(f: A \to B\) and \(g: B \to C\) in \(\mathcal{T}\), we have \(C(g \circ f) \cong C(f)\) if \(g\) is an isomorphism and \(C(g \circ f) \cong C(g)\) if \(f\) is an isomorphism. We recall that by axiom (TR3), given a commutative diagram of continuous arrows whose rows are distinguished triangles

\[
\begin{array}{c}
A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \\
A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} A'[1]
\end{array}
\]

there exists (not unique in general) a dotted arrow \(c\) keeping the diagram commutative. Moreover, axiom (TR4) implies that, if \(f = id_A\), then \(c\) can be chosen with the additional property that \(C(c) \cong C(b)\). We ignore if (always when \(f = id_A\))
every morphism $c$ making the diagram commute satisfies $\mathcal{C}(c) \cong \mathcal{C}(b)$, but we will see later that this is true with some assumptions on $\mathfrak{T}$.

An exact functor between two triangulated categories $\mathfrak{T}$ and $\mathfrak{T}'$ is given by a functor $F: \mathfrak{T} \rightarrow \mathfrak{T}'$ together with an isomorphism of functors $\mu_F: F \circ [1] \rightarrow [1] \circ F$, such that for every distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ of $\mathfrak{T},$

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \xrightarrow{\mu_F(A) \circ F(h)} F(A)[1]$$

is a distinguished triangle of $\mathfrak{T}'$.

Given $S \subseteq \text{Ob}(\mathfrak{T})$ (the collection of objects of $\mathfrak{T}$), we denote by $\langle S \rangle$ the smallest strictly full triangulated subcategory of $\mathfrak{T}$ containing $S$, and by $S^\perp$ the (right) orthogonal of $S$, namely the full subcategory of $\mathfrak{T}$ whose objects are those $A \in \mathfrak{T}$ such that $\text{Hom}(B, A[n]) = 0$ for every $B \in S$ and for every $n \in \mathbb{Z}$ (note that $S^\perp$ is a strictly full triangulated subcategory of $\mathfrak{T}$); if $\mathfrak{C}$ is a subcategory of $\mathfrak{T}$, we write $\mathfrak{C}^\perp$ for $\text{Ob}(\mathfrak{C})^\perp$. A strictly full triangulated subcategory $\mathfrak{T}'$ of $\mathfrak{T}$ is admissible if the inclusion functor $\mathfrak{T}' \rightarrow \mathfrak{T}$ admits left and right adjoints. A sequence $(\mathfrak{T}_0, \ldots, \mathfrak{T}_m)$ of admissible subcategories of $\mathfrak{T}$ is semiorthogonal if $\mathfrak{T}_i \subseteq \mathfrak{T}_j^\perp$ for $0 \leq i < j \leq m$; if moreover $\mathfrak{T} = \langle \text{Ob}(\mathfrak{T}_0) \cup \cdots \cup \text{Ob}(\mathfrak{T}_m) \rangle$, then $(\mathfrak{T}_0, \ldots, \mathfrak{T}_m)$ is called a semiorthogonal decomposition of $\mathfrak{T}$.

**Lemma A.1.** If $(\mathfrak{T}_0, \mathfrak{T}_1)$ is a semiorthogonal decomposition of $\mathfrak{T}$, then for every object $A$ of $\mathfrak{T}$ there exists (unique up to isomorphism) a distinguished triangle $A_1 \rightarrow A \rightarrow A_0 \rightarrow A_1[1]$ with $A_i \in \mathfrak{T}_i$.

From now on we assume that $\mathfrak{T}$ is a $k$-linear triangulated category and that $\dim_k(\bigoplus_{k \in \mathbb{Z}} \text{Hom}(A, B[k])) < \infty$ for every objects $A$ and $B$ of $\mathfrak{T}$ (this condition is clearly satisfied when $\mathfrak{T} = \text{D}^b(X)$ with $X$ a stack).

**Proposition A.2.** Given a commutative diagram in $\mathfrak{T}$

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{id} & & \downarrow{h} \\
A & \xrightarrow{f'} & B' \\
\end{array} \quad \begin{array}{ccc}
B & \xrightarrow{g} & C \\
\downarrow{b} & & \downarrow{c} \\
B' & \xrightarrow{g'} & C' \\
\end{array} \quad \begin{array}{ccc}
C & \xrightarrow{h} & A[1] \\
\downarrow{e} & & \downarrow{id} \\
C' & \xrightarrow{k'} & A[1] \\
\end{array}$$

whose rows are distinguished triangles, we have $\mathcal{C}(c) \cong \mathcal{C}(b)$.

**Proof.** It follows from [3] Prop. 2.1, since $\text{Hom}_{\mathfrak{T}}(C', C')$ is a finite-dimensional $k$-algebra. \qed

**Definition A.3.** $E \in \mathfrak{T}$ is exceptional if $\text{Hom}_{\mathfrak{T}}(E, E) \cong k$ and $\text{Hom}_{\mathfrak{T}}(E, E[k]) = 0$ for $k \neq 0$. $(E_0, \ldots, E_m)$ is an exceptional sequence in $\mathfrak{T}$ if each $E_i$ is an exceptional object and $\text{Hom}_{\mathfrak{T}}(E_i, E_j[k]) = 0$ for $0 \leq i < j \leq m$ and for every $k \in \mathbb{Z}$; if moreover $\mathfrak{T} = \langle E_0, \ldots, E_m \rangle$, then $(E_0, \ldots, E_m)$ is called a full exceptional sequence.

**Remark A.4.** If $(E_0, \ldots, E_m)$ is a (full) exceptional sequence in $\mathfrak{T}$, then $(\langle E_0 \rangle, \ldots, \langle E_m \rangle)$ is an admissible subcategory of $\mathfrak{T}$ and $(\langle E_0 \rangle, \ldots, \langle E_m \rangle)$ is a semiorthogonal sequence (decomposition).

Given a full exceptional sequence $(E_0, \ldots, E_m)$, for $0 \leq j \leq i \leq m$ we define inductively objects $L^{(j)}E_i$ by $L^{(0)}E_i := E_i$ and

$$L^{(j)}E_i := C(\bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathfrak{T}}(E_{i-j}[k], L^{(j-1)}E_i) \otimes_k E_{i-j}[k]) \rightarrow L^{(j-1)}E_i$$

for $0 < j \leq i$ (the morphism being the natural one). Setting $E'_i := L^{(j)}E_i$ for $0 \leq i \leq m$, it is easy to prove that $(E'_m, \ldots, E'_0 = E_0)$ is again a full exceptional sequence, called the (right) dual of $(E_0, \ldots, E_m)$ because of the following result.
Lemma A.5. If \((E_0, \ldots, E_m)\) is a full exceptional sequence and \((E'_m, \ldots, E'_0)\) is the dual exceptional sequence, then \(\text{Hom}_T(E_i, E'_j[k]) \cong k^{\delta_{i,j}\delta_{k,0}}\) for \(0 \leq i, j \leq m\) and for every \(k \in \mathbb{Z}\).

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