Scalar field–perfect fluid correspondence and non-linear perturbation equations

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Abstract. The properties of dynamical dark energy (DE) and, in particular, the possibility that it can form or contribute to stable inhomogeneities have been widely debated in recent literature, and also in association with a possible coupling between DE and dark matter (DM). In order to clarify this issue, in this paper we present a general framework for the study of the non-linear phases of structure formation, showing the equivalence of two possible descriptions of DE: a scalar field $\phi$ self-interacting through a potential $V(\phi)$ and a perfect fluid with an assigned negative equation of state $w(a)$. This enables us to show that, in the presence of coupling, the mass of DE quanta may increase where large DM condensations are present, with the result that also DE may be involved in the clustering process.

Keywords: dark matter, dark energy theory, cosmological perturbation theory
1. Introduction

An important part of cosmology concerns the large scale clustering of matter in galaxies and galaxy clusters and its statistics. The study of large scale structure (LSS) can reveal many aspects of the physics of the early Universe as well as its matter content during cosmic history.

Several observations made over recent years, related to a large extension to LSS and anisotropies of the cosmic microwave background (CMB) as well as the magnitude–redshift relation for type Ia supernovae [1], have given us a convincing picture of the energy and matter density in the Universe.

Baryonic matter accounts for no more than 30% of the mass in galaxy clusters while the existence of a large clustered component of dark matter (DM) seems now firmly established, although its nature is still unknown. However, they contribute to the total energy density of the Universe only a few per cent and about 25% respectively.

No more than another few per cent could be accounted for by massive neutrinos, but only in the most favourable, but unlikely case. According to [2] the total mass of neutrinos cannot exceed the limit of 1.43 eV. A very small part ($10^{-4}$) of the total energy density is due to massless neutrinos and CMB radiation.

The model suggested by observations is only viable if the remaining 75% is ascribed to the so-called dark energy (DE) responsible for the present day cosmic acceleration.

Although strongly indicated by the observations, the existence of DE is even more puzzling than that of DM. It can be identified with a cosmological constant or with an as yet unknown dynamical component with negative pressure. On the other hand, its manifestation can be interpreted as a geometrical property of gravity on large scales, indicating a failure of general relativity (GR) on those scales (see [3] for a review).

Within the context of GR, as an alternative to the cosmological constant, DE is usually described as a self-interacting scalar field or a cosmic fluid with negative pressure (see [4] and references therein). It is usually assumed that density perturbations of DE play a negligible role in the structure formation because of the very small mass $\sim H$ ($H$ being the Hubble parameter). Accordingly, perturbations should appear only on very...
large scales and are bound to be linear, so rates of structure formation and their growth are influenced by DE only through the overall cosmic expansion [5].

Nevertheless, the clustering properties of dynamical DE have been subjects of recent debate. The behaviour of DE in the presence of high concentration of matter and in particular of DE–matter coupling [6,7] is not clear.

Then, the key question is whether DE actively participates in the clustering and virialization processes developing non-linearity on relevant scales. Some attempts to solve the problem have been made in [8,9] and an analytical expression for uncoupled DE density perturbations valid in both the linear and non-linear regimes was derived in [10].

The aim of this paper is to provide a general framework which can be useful for the study of the non-linear phases of structure formation in cosmologies where a coupling between DE and matter is present. The paper is organized as follows.

We assume a Friedmann–Robertson–Walker (FRW) background Universe. Because we are interested in the epoch when structures form, we neglect the radiation and consider a system of two coupled fluids: a pressureless fluid and one with pressure, which can be interpreted as DM and DE respectively. Equations for the background evolution are provided in section 2. Then, in section 3, starting from a general relativistic treatment, we derive the Newtonian limit of the non-linear perturbation equations for the density contrast. In section 4, following [11] (see also [12]), we review the scalar field–fluid correspondence showing that the formalism described in the previous sections applies also to a scalar field. It is then applied to the coupled DE model in section 5. We conclude in section 6.

2. Background equations

Assuming that GR holds, let us start from the background equations. We adopt the following conventions: the four spacetime coordinates are labelled with Greek letters running from 0 to 3 while italic letters run from 1 to 3, labelling the spatial coordinates. We are interested in the epoch when structures form, so radiation is neglected. We consider a Universe filled with baryons (b) and two coupled perfect fluids with vanishing and non-vanishing pressure which, in this case, can be interpreted as DM and DE respectively and are labelled with $dm$ and $de$. According to general covariance, the sum of the individual stress energy tensors $T_{(a)\mu}^\nu$ ($a = b, dm, de$) must be conserved, so we can write

$$\nabla_\mu T_{(b)\nu}^\mu = 0 \quad \nabla_\mu T_{(dm)\nu}^\mu = -C^{\nu} \quad \nabla_\mu T_{(de)\nu}^\mu = C^{\nu}. \quad (1)$$

Here $\nabla_\mu$ is the covariant derivative while the 4-vector $C^{\nu}$ parametrizes a possible interaction between DM and DE, being the energy and momentum exchanges described by $C_0$ and $C_i$ respectively. For each component, the stress energy tensor $T^{\mu}_{\nu}$ can be put in the perfect fluid form:

$$T^{\mu}_{\nu} = p\delta^{\mu}_{\nu} + (\rho + p)u^{\mu}u_{\nu}. \quad (2)$$

Here $\rho$ and $p$ are the energy density and pressure of the fluid as measured in the rest frame and $u^{\mu}$ is its 4-velocity. We also assume the coupling term $C_{\nu} \equiv C_{\nu}(\{v_1, \ldots, v_n\}_{dm}, \{v_{1}, \ldots, v_{m}\}_{de})$ to depend on the dynamical variables of the two coupled fluids that we have generically indicated with $v_a$ ($a = 1, 2, \ldots$).
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In a spatially flat FRW background with metric $d\tau^2 = a^2(\tau)(-d\tau^2 + dx^i dx_i)$ ($\tau$ is the conformal time) the Friedmann equation for the scale factor $a$ and equations (1) read

$$\mathcal{H}^2 = \frac{8\pi G}{3}(\bar{\rho}_b + \bar{\rho}_{dm} + \bar{\rho}_{de}) a^2$$

$$\dot{\bar{\rho}}_b + 3\mathcal{H}\bar{\rho}_b = 0 \quad \dot{\bar{\rho}}_{dm} + 3\mathcal{H}\bar{\rho}_{dm} = -\bar{C}_0\quad \dot{\bar{\rho}}_{de} + 3\mathcal{H}(\bar{\rho}_{de} + \bar{\rho}_{de}) = \bar{C}_0$$

where \(\cdot\) denotes the derivative with respect to $\tau$, $\mathcal{H} = \dot{a}/a$ and $\bar{\cdot}$ denotes the background quantities. For a generic function $F(v_1, v_2, \ldots)$ we have $\bar{F} = F(\bar{v}_1, \bar{v}_2, \ldots)$ so $\bar{C}_0 \equiv C_0(\{\bar{v}_1, \ldots\}_{dm}, \{\bar{v}_1, \ldots\}_{de})$. Notice that from (1) and (2) it follows that the coupling term in the FRW background is $C_\mu = (\bar{C}_0, 0, 0, 0)$.

3. Perturbation equations

According to the standard picture, the observed structures have formed via gravitational instability from the tiny inhomogeneities left by cosmic inflation.

Relativistic theory of cosmological perturbations [13,14] provides a useful tool for understanding the evolution of inhomogeneities relating the physics of inflation to CMB anisotropy and LSS.

Despite relativistic effects, the physics of the evolution of perturbations is quite simple during the period from the end of inflation to the beginning of non-linear gravitational collapse as they have small amplitudes and different wavelengths evolve independently.

Anyway, linear theory breaks down when matter inhomogeneities become significantly denser than the cosmic medium. Although the Newtonian theory is appropriate for describing the dynamics on scales small compared to the Hubble radius (but larger than the Schwarzschild radius), full numerical simulations are needed to follow the non-linear phases of structure formation (a comprehensive review of numerical techniques is provided by, e.g., [15]). Only in cases where some symmetries are present do analytic or semi-analytic schemes apply. The simplest approach is to follow a spherical density enhancement; although real fluctuations are expected to be highly irregular and random, much work has been done along these lines, starting with [16,17] where the problem was studied within the frame of pure cold DM models. Then, the results were generalized to the case of $\Lambda$CDM [18] and other DE models [5,9,19]. In spite of the lack of cogent physical motivations, spherical geometry yields results suitable for working out useful mass functions and for studying their evolutions within the Press and Schechter [20]—or a similar [21]—approach (see, e.g., [22]).

The aim of this section is to provide a general framework which can be used to study the non-linear phases of structure formation in the context of coupled DM–DE models. Without considering any specific coupling between the dark components we derive and discuss a set of equations that help us to understand the role of the DE and its possible coupling with DM in the clustering process.

Dealing with LSS it is sufficient to consider weak gravitational fields and non-relativistic gravitational sources. Note that, because of the weak fields, slow motions do not necessarily imply small density fluctuations.

Starting from a general relativistic treatment, we derive now the perturbation equations in the Newtonian limit, valid to all orders in the density contrast. We choose to work in the conformal Newtonian gauge since it most closely corresponds to Newtonian
gravity and consider only scalar perturbations. In this gauge, the metric of a perturbed flat FRW Universe reads
\[ ds^2 = a^2(\tau) \left[ - (1 + 2\Phi) d\tau^2 + (1 - 2\Psi) dx^i dx_i \right], \] (5)
where \( \Phi \) plays the role of the Newtonian potential, \( \Psi \) is the Newtonian spatial curvature and \(|\Phi|, |\Psi| \ll 1\).

Consider now a generic perfect fluid. In the non-relativistic limit its coordinate velocity \( v^i = dx^i/d\tau \) can be treated as a perturbation of the same order as the metric perturbations. Then, keeping only terms linear in \( v^i \), \( \Phi \) and \( \Psi \) and neglecting terms involving products of \( v^i \) and the metric potentials, the components of the 4-velocity \( u^\mu = dx^\mu/\sqrt{-ds^2} \) become
\[ u^0 = a^{-1}(1 - \Phi) \quad u^i = a^{-1}v^i \quad u_0 = -a(1 + \Phi) \quad u_i = av^i. \] (6)

With the same approximations, we can now write the components of the stress energy tensor (2):
\[ T^0_0 = -\rho \quad T^0_i = (\rho + p)v^i \quad T^i_0 = -(\rho + p)v^i \quad T^i_j = p\delta^i_j. \] (7)
Here \( \rho = \bar{\rho}(\tau) + \delta\rho(\tau, x^i) \) and \( p = \bar{p}(\tau) + \delta p(\tau, x^i) \) are the perturbed energy density and pressure of the fluid. Notice that we have not required \( \delta\rho \) and \( \delta p \) to be small compared to the background quantities.

The equations of motion follow from [17]:
\[ \nabla_\mu T^\mu_\nu = \frac{1}{2}(\partial_\nu g_\mu^\sigma)T^{\mu\sigma} - \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} T^\mu_\nu \right) = C_\nu \] (8)
where \( g \) is the determinant of the metric \( g_{\mu\nu} \) given by (5) and to the first order in the gravitational potentials \( \sqrt{-g} = a^4(1 - 3\Psi + \Phi) \). Possible interactions with other components are described by \( C_\mu = \bar{C}_\mu(\tau) + \delta C_\mu(\tau, x^i) \) where \( \delta C \) represents deviations from the value \( \bar{C} \) taken in a homogeneous and isotropic Universe. Taking \( C_\mu = 0, C_\mu, -C_\mu \), equations (1) are then recovered. From the component \( \nu = 0 \) of (8) it follows that
\[ \frac{\partial \rho}{\partial \tau} + 3 \left( H - \frac{\partial \Psi}{\partial \tau} \right) (\rho + p) + \nabla \cdot [(\rho + p)v] = C_0, \] (9)
while the component \( \nu = i \) gives
\[ \frac{\partial}{\partial \tau} [(\rho + p)v] + 4H(\rho + p)v + \nabla p + (\rho + p)\nabla \Phi = -C \] (10)
\((|v|^2 = v_iv^i, |C|^2 = C_iC^i)\). Then, making use of the unperturbed part of (8),
\[ \dot{\rho} + \frac{3}{a} \left( \frac{\rho}{\dot{a}} \right) (\rho + p) = \bar{C}_0, \] (11)

\( \dot{\rho} + \frac{3}{a} (\rho + p) = \frac{\rho}{\dot{a}}(1 + \omega) + \frac{\delta p(1 + \omega_p)}{\delta \rho} \), where \( \omega = \omega_p = \delta p/\delta \rho \) is the equation of state parameter and \( \omega_p = \delta p/\delta \rho \), we can extract from (9) and (10) the equations for
Therefore, in order to ensure their stability (\(c_s^2 \leq 1\)), in section 4 we show that a scalar field minimally coupled to gravity has vanishing anisotropic stresses and its stress energy tensor can be put in the perfect fluid form. It follows that a DE scalar field can be described as a perfect fluid. On the other hand, if anisotropic stresses are present, the fluid is non-barotropic, and non-adiabatic instabilities could arise in the presence of strong DM–DE couplings or on super-Hubble scales. Adiabatic and non-adiabatic instabilities have been discussed in [26].

\[ \frac{\delta}{\delta \rho} (\frac{\rho}{\rho}) = \frac{\omega}{\omega} < 0, \text{ e.g. DE fluids.} \]

In order to ensure their stability (\(c_s^2 > 0\)) and have a DE component that is phenomenologically viable, non-adiabatic modes must be also present (notice that the condition \(c_s^2 > 0\) needs to be imposed by hand if DE is assumed to be a fluid while \(c_s^2 = 1\) follows without assumptions in the case of DE scalar fields. Also notice that a scalar field has in general a non-barotropic equation of state (see [24, 25] and section 4)). Nonetheless, non-adiabatic instabilities could arise in the presence of strong DM–DE couplings or on super-Hubble scales. Adiabatic and non-adiabatic instabilities have been discussed in [26].

Equations (12) can be generalized to the case in which anisotropic stresses are present, i.e. imperfect fluids. In that case we have \(T^\mu_\nu = \rho \delta^\mu_\nu + \Pi^\mu_\nu\) where \(\Pi^\mu_\nu\) is the anisotropic stress tensor which satisfies \(\Pi^\mu_\nu = 0\) and \(\Pi^\mu_\nu u^\nu = 0\). The equation for the peculiar velocity evolution must then be modified, adding the term \(\nabla \Pi\) to the rhs (see [13] for further details). In section 4 we show that a scalar field minimally coupled to gravity has vanishing anisotropic stresses and its stress energy tensor can be put in the perfect fluid form. It follows that a DE scalar field can be described as a perfect fluid. On the other hand, if

\[
\begin{align*}
\delta + 3H(\omega - \omega)\delta - 3\Psi(1 + \omega)(1 + R\delta) + (1 + \omega)\nabla \cdot [(1 + R\delta)\mathbf{v}] &= \frac{C_0 - C_0(1 + \delta)}{\rho} \\
\dot{\mathbf{v}} + \left[ H(1 - 3\omega) + \frac{1}{1 + R\delta} \left( \frac{C_0 - C_0}{\rho} R + \dot{C}_0 \right) + \frac{\dot{\omega} + \omega \dot{\delta}}{1 + \omega} + R\delta \right] \mathbf{v} + \frac{\nabla \delta p}{\rho(1 + \omega)(1 + R\delta)} + \nabla \Phi &= - \frac{\mathbf{C}}{\rho(1 + \omega)(1 + R\delta)}. 
\end{align*}
\]

Here \(R = (1 + \omega R)/1 + \omega\) and \(\nabla\) denotes the spatial gradient.

Notice that in the small perturbation limit \((\delta \rho/\rho, \delta p/\rho \ll 1\)) the parameter \(\omega\) is related to the sound speed \(c_s^2\) in the fluid. In general, \(\dot{\delta} p = \delta p_a + \delta p_{na}\) where the two terms on the rhs are the adiabatic and non-adiabatic contributions to the pressure perturbations and \(c_s^2\) is then their propagation speed in the fluid rest frame (rf), i.e. the frame where \(T^0_i = 0\):

\[
c_s^2 = \left. \frac{\dot{\rho}}{\delta \rho} \right|_{\text{rf}}. \tag{13}
\]

After performing a gauge transformation from the rest frame gauge to the conformal Newtonian gauge one arrives at the following relation between \(\delta p\) and \(\delta \rho\) [14, 23, 24] (we assume \(c_s^2\) to depend only by \(\tau\)):

\[
\nabla \delta p = \nabla \delta p_a + \nabla \delta p_{na} = c_s^2 \nabla \delta \rho + (c_s^2 - c_s^2)(\nabla \delta \rho + \dot{\rho} \mathbf{v}) \tag{14}
\]

which can be rewritten as

\[
\nabla \delta p = c_s^2 \nabla \delta \rho - (c_s^2 - c_s^2)[3H(1 + \omega)\rho - C_0] \mathbf{v} \tag{15}
\]

showing that the coupling enters \(\delta p\) explicitly. In the above relation,

\[
c_s^2 = \frac{\dot{\rho}}{\rho} = \omega + \omega \frac{\dot{\delta}}{\delta} \tag{16}
\]

is the adiabatic sound speed which coincides with \(c_s^2\) if the fluid is barotropic (\(p = \omega (\rho) \rho\)). Pure adiabatic perturbations, however, go unstable for fluids with \(\omega < 0\), e.g. DE fluids. Therefore, in order to ensure their stability (\(c_s^2 > 0\)) and have a DE component that is phenomenologically viable, non-adiabatic modes must be also present (notice that the condition \(c_s^2 > 0\) needs to be imposed by hand if DE is assumed to be a fluid while \(c_s^2 = 1\) follows without assumptions in the case of DE scalar fields. Also notice that a scalar field has in general a non-barotropic equation of state (see [24, 25] and section 4)). Nonetheless, non-adiabatic instabilities could arise in the presence of strong DM–DE couplings or on super-Hubble scales. Adiabatic and non-adiabatic instabilities have been discussed in [26].

\[
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\]
one assumes DE to be a fluid, anisotropic stresses cannot be \textit{a priori} excluded. Anyway, as we shall see shortly, the absence of anisotropic stresses leads to $\Phi = \Psi$.

In addition to (12), one has to consider the equations of metric perturbations obtained from the perturbed part of the Einstein’s equations. Because of the weak field approximation, the Einstein’s tensor $G^\mu_\nu$ will be only linearly perturbed while the perturbations in the stress energy tensor components of each fluid $\alpha$ follow on subtracting their background values $\bar{T}^\mu_\nu$ from those obtained from (7):

$$\delta G^\mu_\nu = 8\pi G \sum_\alpha \left( T^\mu_\nu_\alpha - \bar{T}^\mu_\nu_\alpha \right). \tag{17}$$

Einstein’s equations then read

$$\nabla^2 \Psi - 3\mathcal{H} \left( \dot{\Psi} + \mathcal{H} \Phi \right) = 4\pi Ga^2 \sum_\alpha \delta \rho_\alpha$$

$$\nabla^2 \left( \dot{\Psi} + \mathcal{H} \Phi \right) = -4\pi Ga^2 \partial_i \left[ \sum_\alpha (\rho_\alpha + p_\alpha) v^i_\alpha \right]$$

$$\dot{\Psi} + \mathcal{H} \dot{\Phi} + 2 \Psi + \left( \frac{2\ddot{a}}{a} - \mathcal{H}^2 \right) \Phi - \frac{1}{3} \nabla^2 (\Psi - \Phi) = \frac{4\pi}{3} Ga^2 \sum_\alpha \delta \rho_\alpha$$

$$\partial_i \partial_j (\Psi - \Phi) = 8\pi Ga^2 \sum_\alpha \Pi^i_\alpha (\Phi). \tag{18}$$

Thus, if anisotropic stresses are absent the last equation shows that $\Psi = \Phi$. In this case, focusing on scales well below the Hubble radius, we can neglect the second term on the lhs in the first equation, because $\mathcal{H} \dot{\Phi} \sim \mathcal{H}^2 \Phi \ll \nabla^2 \Phi$. This corresponds to the Newtonian limit and the first of equations (18) reduces to the usual Poisson equation:

$$\nabla^2 \Phi = 4\pi G \sum_\alpha \delta \rho_\alpha. \tag{19}$$

In our particular case, neglecting the term proportional to $\dot{\Psi}$ in the first of (12) because of the Newtonian approximation and replacing the term $\delta \dot{\Phi}$ in the second, the perturbation equations for our three perfect fluid components are:

\textit{Baryons} ($\omega \approx 0$, \(c_s^2\) negligible after recombination, $C_\mu = 0$):

$$\dot{\delta}_b + \nabla \cdot [(1 + \delta_b) v_b] = 0$$

$$\dot{v}_b + \mathcal{H} v_b + \nabla \Phi = 0. \tag{20}$$

\textit{DM (pressureless perfect fluid)} ($\omega = 0$, $\omega_p = 0$, $C_\mu = -C_\mu$):

$$\dot{\delta}_{dm} + \nabla \cdot [(1 + \delta_{dm}) v_{dm}] = -\frac{\dot{C}_0 - \bar{C}_0 (1 + \delta_{dm})}{\rho_{dm}}$$

$$\dot{v}_{dm} + \left( \mathcal{H} - \frac{C_0}{\rho_{dm} (1 + \delta_{dm})} \right) v_{dm} + \nabla \Phi = \frac{C}{\rho_{dm} (1 + \delta_{dm})}. \tag{21}$$
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\[ \delta_{de} + 3 \mathcal{H} (\omega_p - \omega) \delta_{de} + (1 + \omega) \nabla \cdot [(1 + R \delta_{de}) \mathbf{v}_{de}] = \frac{C_0 - \bar{C}_0 (1 + \delta_{dm})}{\bar{\rho}_{de}} \]

\[ \dot{\mathbf{v}}_{de} = \left[ \mathcal{H} \left( \frac{1 - 3 \omega + (1 - 3 \omega_p) R \delta_{de}}{1 + R \delta_{de}} \right) + \frac{\dot{\omega} + \dot{\omega}_p \delta_{de}}{(1 + \omega)(1 + R \delta_{de})} \right] \mathbf{v}_{de} + \frac{\nabla \delta p_{de}}{\bar{\rho}_{de}(1 + \omega)(1 + R \delta_{de})} + \nabla \Phi = - \frac{C}{\bar{\rho}_{de}(1 + \omega)(1 + R \delta_{de})}. \]  

Equations (20)–(22) are valid to non-linear order in the density contrast since we have not assumed \( \delta \ll 1 \). They can be used to follow the evolution of a collapsing region in a non-linear regime and could be useful for studying the clustering properties of DE by using the spherical collapse approach. The problem of spherical fluctuation growth in coupled DE models has been considered in [19]. There, equations (20) and (21) were used to follow the evolution of a set of concentric shells of a spherical top-hat overdensity in baryons and DM while the DE was kept homogeneous. The shell description of the overdensity was needed because the coupling causes DM particles to have a different dynamics to the baryons. This can be seen from an inspection of the above equations. The most significant result of that analysis is the high level of segregation between DM and baryons arising during the growth of fluctuations: up to \( \sim 60\% \) of baryons can be expelled from the fluctuation in halo encounters before its final virialization.

The same approach could be used, relaxing the assumption of homogeneity for DE and including in the system an additional set of concentric shells for DE. Their evolution is then described by (22). In contrast to that of DM, the dynamics of DE shells is affected not only by the coupling but also by the non-vanishing pressure of DE itself. We shall consider this problem in a subsequent paper.

A similar set of equations was used in [27] to study non-linearities in the case of uncoupled DE. However, the authors consider the pseudo-Newtonian approach of [28] and the DE equations that they use lack some terms with respect to (22).

In the next section we show that equation (22) applies equally well when DE is described by a scalar field.

4. Scalar field

In most DE models, DE is due to a scalar field either minimally or non-minimally coupled to matter and/or gravity. In this paper we do not discuss coupling to gravity, while coupling with matter will be considered only in a subsequent section. Anyway, results of this section are not affected by the presence of this latter kind of coupling and are fully general.

Here we show that the equations deduced in the previous section are valid also for a scalar field. In particular, following [11], we show that the stress energy tensor of a scalar field, if minimally coupled to gravity, can be put in the perfect fluid form. As a consequence equations (12) as well as (22) can be applied.

Let us start with a minimally coupled scalar field \( \phi \) self-interacting through a potential \( V(\phi) \). In a spacetime described by a metric \( g_{\mu \nu} \) with signature \((-+++\)) its Lagrangian
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reads
\[ \mathcal{L} = \sqrt{-g} \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] \]  
(23)
and the stress energy tensor takes the form
\[ T^\mu_\nu = \partial^\mu \phi \partial_\nu \phi - \delta^\mu_\nu \left( \frac{1}{2} \partial_\sigma \phi \partial^\sigma \phi + V \right). \]  
(24)
The equations of motion then follow either from \( \nabla_\mu T^\mu_\nu = 0 \) or from the Euler–Lagrange equation:
\[ \left[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \phi \right) \right] \phi - V' = 0 \]  
(25)
(the prime denotes the derivative with respect to \( \phi \)).

Assuming that the vector \( \partial^\mu \phi \) is timelike, namely \( \partial^\mu \phi \partial_\mu \phi < 0 \), one can assign a 4-velocity \( u^\mu \) to the scalar field. It can be chosen as the unique timelike vector with unit magnitude constructed from \( \partial^\mu \phi \) (for more details see [11, 12]):
\[ u^\mu = -\frac{\partial^\mu \phi}{\sqrt{-\partial_\sigma \phi \partial^\sigma \phi}} \quad u^\mu u_\mu = -1. \]  
(26)
From this choice of the 4-velocity, it follows that the scalar field \( \phi \) can be represented as a perfect fluid and the stress energy tensor (24) can be put in the form (2). In order to show this, let us start from the stress energy tensor of a general fluid:
\[ T^\mu_\nu = pg^\mu_\nu + (\rho + p)u^\mu u_\nu + q^\mu u_\nu + u^\mu q_\nu + \Sigma^\mu_\nu \]  
(27)
where possible anisotropic stresses are accounted for by the heat flux vector \( q^\mu \) and the tensor \( \Sigma^\mu_\nu \), which satisfy \( u^\mu q_\mu = 0, \Sigma^\mu_\mu = 0 \) and \( \Sigma^\mu_\nu u_\nu = 0 \). We show now that for a scalar field these terms vanish. Making use of the projection tensor \( h^\mu_\nu = g^\mu_\nu + u^\mu u_\nu \), one can extract from (27) all the relevant quantities:
\[ \rho = T^\mu_\nu u^\nu u_\nu \quad p = \frac{1}{3} T^\mu_\nu h^\mu_\sigma h^\nu_\sigma \quad q_\mu = -T^\sigma_\lambda h^\sigma_\mu h^\lambda_\nu \quad \Sigma^\mu_\nu = T^\sigma_\lambda h^\sigma_\mu h^\lambda_\nu - ph^\mu_\nu. \]  
(28)
By using (24) and (26), the above quantities can be found for the scalar field \( \phi \):
\[ \rho = -\frac{1}{2} \partial^\mu \phi \partial^\mu \phi + V \quad p = -\frac{1}{2} \partial^\mu \phi \partial^\mu \phi - V \quad q_\mu = 0 \quad \Sigma^\mu_\nu = 0. \]  
(29)
Both the heat flux and the anisotropic stress tensor vanish, showing that the scalar field stress energy tensor has the usual perfect fluid form.

However, it is important to mention that, although this mathematical equivalence relates a scalar field with perfect fluid models, differences remain in their physical interpretation. In general, a fluid is a continuum model deduced from thermodynamical considerations of a system containing a significant number of particles interacting through elastic collisions. Its continuum variables (e.g., \( \rho, u^\mu \)) are defined in terms of fundamental discrete variables in a limit procedure. Once the large particle number limit is violated, the model is expected to fail. Because of the thermodynamical motivation, physical constraints on \( \rho \) and \( u^\mu \) are then imposed, namely that \( \rho > 0 \) and that \( u^\mu \) is timelike. On the other hand, a scalar field model is a continuum model and the solutions of its equations are generally not subject to the constraints mentioned above. Thus, the assumption \( \partial^\mu \phi \partial^\mu \phi < 0 \) only refers to those scalar field solutions that can be interpreted as physical fluids.
Let us now consider a perturbed FRW Universe described by the metric (5). Equation of motion (23) then reads
\[ \ddot{\phi} + \left( 2\dot{\mathcal{H}} - \dot{\Phi} - 3\dot{\Psi} \right) \dot{\phi} - (1 + 2\Psi + 2\Phi)\nabla^2 \phi + a^2 (1 + 2\Phi)V' = \nabla(\Phi - \Psi) \cdot \nabla \phi \] (30)
and the components of the fully non-linear stress energy tensor are
\begin{align*}
T^0_0 &= -\frac{\dot{\phi}^2}{2a^2} (1 - 2\Phi) + \sum_i \left( \frac{\partial_i \phi}{a^2} \right)^2 (1 + 2\Psi) - V \\
T^0_i &= -\frac{\dot{\phi}}{a^2} (1 - 2\Phi) \partial_i \phi \\
T^i_j &= \frac{\partial_i \phi \partial_j \phi}{a^2} (1 + 2\Psi) + \delta^i_j \left[ \frac{\dot{\phi}^2}{2a^2} (1 - 2\Phi) - \frac{\sum_i (\partial_i \phi)^2}{2a^2} (1 + 2\Psi) - V \right].
\end{align*}
(31)
Then, decomposing \( \phi \) as the sum of an unperturbed part \( \bar{\phi} \) and a perturbed one \( \delta \phi \):
\[ \phi = \bar{\phi}(\tau) + \delta \phi(\tau, x^i), \] (32)
the zeroth-order terms of (30) give the equation for the homogeneous part of \( \phi \):
\[ \ddot{\bar{\phi}} + 2\dot{\bar{\phi}} \ddot{\bar{\phi}} + a^2 \bar{V}' = 0 \] (33)
while, assuming no anisotropic stresses (\( \Phi = \Psi \)), the higher order terms yield for the perturbations
\[ \delta \phi + \left( 2\mathcal{H} - 4\dot{\Phi} \right) \dot{\delta \phi} - 4\dot{\Phi} \dot{\bar{\phi}} - (1 + 4\Phi)\nabla^2 \delta \phi + a^2 \left[ (1 + 2\Phi)V' - \bar{V}' \right] = 0. \] (34)
From (31) one can also work out the background energy density and pressure and the corresponding perturbations:
\[ \bar{\rho} = \frac{\dot{\bar{\phi}}^2}{2a^2} + \bar{V} \quad \delta \rho = -\frac{\dot{\bar{\phi}}^2}{a^2} \Phi + \frac{\dot{\bar{\phi}} \delta \phi}{a^2} (1 - 2\Phi) + \frac{\delta \phi^2}{2a^2} (1 - 2\Phi) - \frac{\sum_i (\partial_i \delta \phi)^2}{2a^2} (1 + 2\Psi) + \delta V \\
\bar{p} = \frac{\dot{\bar{\phi}}^2}{2a^2} - \bar{V} \quad \delta p = -\frac{\dot{\bar{\phi}}^2}{a^2} \Phi + \frac{\dot{\bar{\phi}} \delta \phi}{a^2} (1 - 2\Phi) + \frac{\delta \phi^2}{2a^2} (1 - 2\Phi) - \frac{\sum_i (\partial_i \delta \phi)^2}{2a^2} (1 + 2\Psi) - \delta V \] (35)
where \( \delta V = V - \bar{V} \) and \( \bar{V} = V(\bar{\phi}) \). The scalar field peculiar velocity \( v^i \) is obtained by comparing the expression (26), which can be rewritten as
\[ u^\mu = a^{-1} \left[ (1 + 2\Phi) - (1 - 2\Psi) \sum_j \left( \frac{\partial_j \phi}{\partial^\mu \phi} \right)^2 \right]^{-1/2} \frac{\partial^\mu \phi}{\partial^\tau \phi} \] (36)
with the fluid 4-velocity
\[ u^\mu = \frac{dx^\mu}{\sqrt{-ds^2}} = a^{-1} \left[ (1 + 2\Phi) - (1 - 2\Psi) \sum_j \left( \frac{dx^j}{d\tau} \right)^2 \right]^{-1/2} \frac{dx^\mu}{d\tau}. \] (37)
Scalar field–perfect fluid correspondence and non-linear perturbation equations

Table 1. Scalar field–perfect fluid correspondence in the small velocity and weak gravitational field limits.

|               | Perfect fluid | Scalar field |
|---------------|---------------|--------------|
| $u^0$         | $a^{-1}(1 - \Phi)$ | $a^{-1}(1 - \Phi)$ |
| $u^i$         | $\frac{\partial x^i}{\partial \tau} a^{-1}$ | $\frac{\partial \phi}{\partial \phi} a^{-1}$ |
| $v^i$         | $\frac{\partial x^i}{\partial \tau}$ | $\frac{\partial \phi}{\partial \phi}$ |
| $T_{00}$      | $-\rho$       | $\frac{a^2}{2\dot{\phi}}(1 - 2\Phi) - V(\phi)$ |
| $T_{0i}$      | $-(\rho + p)v^i$ | $\phi \partial_i \phi$ |
| $T_{ij}$      | $\delta_j \rho$ | $\delta_j \left[ \frac{a^2}{2\dot{\phi}}(1 - 2\Phi) - V(\phi) \right]$ |

Taking $\mu = i$, it then follows that

$$v^i = \frac{dx^i}{d\tau} = \frac{\partial \phi}{\partial \phi}.$$  (38)

As we are interested in the small scale and weak gravitational field limits some approximations are possible. Terms proportional to $\Phi$ can be dropped in equation (34). We can also neglect the time derivatives of the gravitational potential with respect to $H$, since $\dot{\Phi} \sim \Phi H$. We are then left with

$$\ddot{\delta \phi} + 2H \dot{\delta \phi} - \nabla^2 \delta \phi + a^2 \delta \dot{V} = 4 \dot{\Phi} \dot{\phi}.$$  (39)

The correspondence between the perfect fluid and scalar field descriptions is summarized in table 1 where the forms taken by the 4-velocity and the stress energy tensor components in the small velocity and weak gravitational field limits are given. Notice that these limits, in the scalar field description, correspond to considering only linear terms in $\partial \mu / \partial \phi$ and neglecting terms involving products of the former and the metric perturbations. It is worth mentioning that equation (39) is equivalent to the first of (22), from which it can be directly derived by setting $C_\mu = 0$. However, once the prescriptions of table 1 are taken into account, equation (39) is more simply deduced from (8) or directly from (9) after considering the small scale limit and eliminating the homogeneous part.

As mentioned in the previous section, in general, a scalar field cannot be considered as a barotropic fluid, its equation of state being given by

$$p = \rho - 2V.$$  (40)

Therefore, for small perturbations, if $V \neq \text{const}$, we have $\delta p/\delta \rho \neq \dot{p}/\dot{\rho} = c_s^2$ yielding also non-adiabatic contributions to the fluctuations. The sound speed is given again by (13) where the rest frame is defined by the hypersurfaces $\phi = \text{const}$ orthogonal to the rest frame 4-velocity (26). By definition, in the rest frame, the scalar field carries no perturbations ($\delta \phi |_{\text{rf}} = 0$) leading $\delta V |_{\text{rf}} = 0$. It follows that energy density and pressure perturbations come purely from kinetic terms, so $\delta \rho |_{\text{rf}} = \delta p |_{\text{rf}}$ and $c_s^2 = 1$ independently of the form of $V(\phi)$ (for a detailed discussion see [29]).

In the next section we introduce the coupling through a discussion on the coupled DE model.
5. Coupled dark energy

In addition to self-interaction, a scalar field can in principle be coupled to any other field present in Nature. However, if this field is the one which accounts for DE, in order to drive the cosmic acceleration, its present time mass is expected to be, at least on large scales, \( m_\phi \sim H_0 \sim 10^{-33} \text{ eV} \). Such a tiny mass gives rise to long-range interactions which could be tested with fifth-force type experiments. Couplings to ordinary particles are strongly constrained by such experiments but limits on the DM coupling are looser (constrains on coupling for specific models were obtained in \([30,31]\) from CMB, N-body simulations and matter power spectrum analysis). If present, DM coupling could have a relevant role in the cosmological evolution affecting not only the overall cosmic expansion but also modifying the DM particle dynamics with relevant consequences for the growth of the density perturbations in both linear and non-linear regimes (e.g., halo density profiles, the mass function and its evolution) \([30,32,33]\). Here we review one of the most popular models where a coupling between DM and DE is present, namely coupled DE \([32]\), and show how the formalism introduced in section 3 can be applied.

The model is quite general and includes a vast class of models also motivated by string theory as proposed in \([6,7]\). Conservation equations for baryons, DM and a DE scalar field \( \phi \) read

\[
\nabla_\mu T^\mu_{(b)\nu} = 0 \quad \nabla_\mu T^\mu_{(dm)\nu} = -c(\phi)T_{(dm)} \nabla_\nu \phi \quad \nabla_\mu T^\mu_{(\phi)\nu} = c(\phi)T_{(dm)} \nabla_\nu \phi, \tag{41}
\]

so the coupling \( C_\mu \) introduced in section 2 now takes the form

\[
C_\mu = c(\phi)T_{(dm)} \nabla_\mu \phi, \tag{42}
\]

where \( T_{(dm)} \) is the trace of the DM stress energy tensor and \( c(\phi) \) parametrizes the strength of the DM–DE interaction. Such a coupling can be derived from a scalar–tensor theory after performing a conformal transformation (see \([34]\) for a review on scalar–tensor gravity). Different couplings were also considered by various authors \([35]\).

The equation of motion of the scalar field follows from the last of (41) and (24) or equivalently from the Lagrangian

\[
\mathcal{L} = \sqrt{-g} \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - B(\phi)m\bar{\chi}\chi \right] \tag{43}
\]

in the case when \( c(\phi) = d(\ln B)/d\phi \). The last term in (43) sets the coupling between the DE and the spinor field \( \chi \) supposed here to yield DM (results are however independent of whether the DM particles are scalars or fermions). Notice that in such a theory the masses of the DM particles, \( m_{dm} = B(\phi)m \), as well as their energy density, \( \rho_{dm} = m_{dm} \langle \bar{\chi}\chi \rangle \), are \( \phi \) dependent. In this case equation (39) reads

\[
\ddot{\delta\phi} + 2H\dot{\delta\phi} - \nabla^2 \delta\phi + a^2 \delta V'_{eff} = 4\dot{\Phi}\dot{\phi}, \tag{44}
\]

where \( \delta V_{eff} = V_{eff} - \bar{V}_{eff} \) and \( V_{eff} = V + \rho_{dm} \). Notice that no assumptions about the value of \( \delta\phi \) were made in deriving the above equation which is valid also for non-linear \( \delta\phi \).

It is also worth mentioning that, unlike for the uncoupled case, the effective masses of the scalar fields, \( m_\phi = V''(\phi) = V''(\phi) + c'(\phi)\rho_{dm} + c^2(\phi)\rho_{dm} \), as well as the value of \( \phi \) depend on the local DM density \( \rho_{dm} \). It follows that, for massive objects, where the DM density is high compared to the background density, \( m_\phi \) could be very different from the value that it takes in the cosmos, where \( \rho_{dm} \approx \rho_{cr} \) (\( \rho_{cr} \) being the critical energy density).
and $\phi = \bar{\phi}$. Thus, perturbations in the DE energy density and pressure could be non-negligible on the scales relevant for the structure formation. Effects of a scale-dependent mass was considered in the context of the so-called chameleon theory [36] where, unlike for the coupled DE case, the scalar field universally couples to all the kinds of matter, e.g. DM and baryons. However, even though this theory simultaneously provides a viable cosmology and a mechanism which allows the scalar field to evade constraints from fifth-force effects, a very low value of the energy scale entering the chameleon potential is needed. Although tracker potentials were considered, it seems therefore hard to avoid fine-tuning and coincidence problems.

In order to study the evolution of DE perturbations, alternatively to equation (44), one can use the perfect fluid description and the equations (22) once the coupling vector $C_{\mu}$ is rewritten in terms of $\rho_{\text{de}}, p_{\text{de}}$ and $v_{\text{de}}$. Let us consider first the case when the parameter $c$ of (42) is constant. In this case, using the relations of table 1, the coupling vector becomes

$$C_0 = c\bar{\rho}_{\text{dm}}(1 + \delta_{\text{dm}})[\bar{\rho}_{\text{de}}(1 + \omega)(1 + R\delta_{\text{de}})]^{1/2}a$$
$$C = -c\bar{\rho}_{\text{dm}}(1 + \delta_{\text{dm}})[\bar{\rho}_{\text{de}}(1 + \omega)(1 + R\delta_{\text{de}})]^{1/2}av_{\text{de}}.$$  (45)

Problems can arise when $c$ is $\phi$ dependent. However, if the potential $V(\phi)$ is a strictly monotonic function, as in the case of an inverse power law or an exponential potential, one can invert it to obtain $\phi$ as function of $V = (\rho_{\text{de}} - p_{\text{de}})/2$. As an example, consider the Ratra–Peebles potential $V = \Lambda^{4+\alpha}/\phi^\alpha$ where $\alpha > 0$. In this case,

$$\phi = \left(\frac{\Lambda^{4+\alpha}}{V}\right)^{1/\alpha} = \left(\frac{2\Lambda^{4+\alpha}}{\bar{\rho}_{\text{de}}[(1 - \omega) + \delta_{\text{de}}(1 - \omega_p)]}\right)^{1/\alpha}.$$  (46)

so $c = c(\rho_{\text{de}} - p_{\text{de}}; \Lambda, \alpha)$. Notice that, once the energy scale $\Lambda$ is fixed, the $\alpha$ parameter follows by requiring $\bar{\rho}_{\text{de},0}$ (or $\bar{\rho}_{m,0} = \bar{\rho}_{\text{dm},0} + \bar{\rho}_{b,0}$) to have a value as inferred by observations (here the subscript 0 refers to the present time). Alternatively, one can fix $\alpha$ and $\bar{\rho}_{\text{de},0}$ (or $\bar{\rho}_{m,0}$) and then find $\Lambda$.

6. Conclusions

One of the main topics which has attracted a deep interest in recent years concerns the clustering properties of dynamical DE, in particular when a coupling with matter is present. Scalar fields are among the favourite DE candidates. However, in certain circumstances, it can be more advantageous to describe DE in terms of a perfect fluid rather than in terms of scalar fields.

This paper is a technical review meant to set up a formalism for the study of non-linearity in the presence of DM–DE coupling. To this end, we derive the Newtonian limit of the perturbation equations for DM and DE (and baryons) valid to non-linear order in the density contrast, highlighting the equivalence of the perfect fluid and the scalar field descriptions of DE. We then provide some prescriptions (summarized in table 1) for relating the two descriptions and apply them to the coupled DE case. This enables us to give the expression for the coupling vector in terms of the fluid variables and show that, in the presence of coupling, the mass of DE quanta could increase where large DM condensations are present. As a consequence, DE can also take part in the clustering process.
The framework presented here gives the DE equations of motion in the presence of coupling once the time evolution of the DE equation of state, $\omega(a)$, is assigned, without reconstructing the self-interacting potential $V(\phi)$. It can be useful to follow the evolution of a collapsing region in a non-linear regime in order to study the clustering properties of DE, e.g. by using the spherical collapse approach. This point is currently under investigation.

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