Temporal and spatial chaos in the Kerr-AdS black hole in an extended phase space

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ABSTRACT

Based on the Melnikov method, we investigate chaotic behaviors in the extended thermodynamic phase space for a slowly rotating Kerr-AdS black hole under temporal and spatial perturbations. Our results show that the temporal perturbation coming from a thermal quench of the spinodal region in the phase diagram may cause the temporal chaos only when the perturbation amplitude is above a critical value, which involves the angular momentum \( J \). Under the spatial perturbation, however, it is found that the spatial chaos always occurs, which is independent of the perturbation amplitude.

1. Introduction

Since Hawking [1] firstly proved that black holes can radiate thermal energy, more and more attention has been focused on the studies of black hole thermodynamics [2-11]. These studies have showed that several black holes reveal the important properties in different dimensions, which can give new understanding of some fundamental problems in physics. Especially, charged [3] and rotating asymptotically AdS back holes [4,9] can exist a first order phase transition between small-black-hole and large-black-hole, which is analogous to the transition between liquid and gas in the Van der Waals fluid. Theoretically, this analogy can been extended to more general cases by analyzing thermodynamics in an extended phase space of the black holes, where the cosmological constant corresponds to the thermodynamic pressure and its conjugate quantity as thermodynamic volume is essential with the increasing attention of considering the variation of the cosmological constant in the first law of black hole thermodynamics. Based on these considerations,

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Kubizňák et al. investigated the $P-V$ criticality of charged AdS black holes[12]. They perfectly completed the analogy between charged AdS black holes and the liquid–gas system, which is first observed by Chamblin et al.[3]. Subsequently, $P-V$ criticality in the extended phase space (including pressure and volume as thermodynamic variables) of black holes has led a new trend of phase transition research [13-16].

Chaos is a universal physical phenomenon in those nonlinear dynamical systems. Due to the intrinsic nonlinearity of Einstein’s general relativity theory, the chaotic behavior is a universal characteristic of relativistic systems [17-21]. Especially, more and more researchers have paid attention to chaotic behavior in the black hole background [22-34]. As a relatively early work in this field, Letelier and Vieira have proved that the particles moving along timelike geodesics of the Schwarzschild black hole can present a chaotic motion when a particular class of gravitational perturbations are introduced [22]. Furthermore, some works have showed that the motion of a particle is also chaotic when it approaches very near to the black hole horizon[25-30]. Recently, Chabab et al. [31]studied chaos in the context of black hole thermodynamics and phase transitions by means of the Melnikov method[35]. They firstly revealed the deep connection between the charged AdS black hole and the Van der Waals fluid. Their results can help to understand the correlation between the chaotic behavior and the $P-V$ diagram of black holes. Subsequently, their study were extended to other black holes, including charged or neutral Gauss-Bonnet AdS black holes [32], the Born-Infeld-AdS black hole [33], and the charged dilaton-AdS black hole [34].

In this work, we will investigate temporal and spatial chaos phenomena in the Kerr-AdS black hole within the extended phase space. Gunasekaran et al. proved that a slowly rotating Kerr-AdS black hole is analogous to the Van der Walls liquid-gas system[16]. When the temperature of the black hole is below the critical value, a region of the violated stable equilibrium can appear. By the use of Maxwell’s equal area law, Zhao et al. found an isobar for the $P-V$ diagram of the Kerr-AdS black hole, which corresponds to the real two phase coexistence line [36]. Based on these results, we will try to apply the Melnikov method to investigate chaotic dynamics of the slowly rotating Kerr-AdS black hole in extended thermodynamic phase space. According to the common research scheme, we firstly review the main thermodynamic characteristics of the Kerr-AdS black hole. Secondly, we introduce a small temporal perturbation in the spinodal region of the Kerr-AdS black hole thermodynamic phase space. Then, we derive the Melnikov function for the homoclinic orbit and display when the temporal chaos can occur in the spinodal region of the thermodynamic phase space. Lastly, we recompute the Melnikov function for homoclinic or heteroclinic orbit and probe spatial chaotic behavior. More details will be displayed in the following sections.

2. Thermodynamics of the Kerr-AdS black hole in an extended phase space

Our work starts from a four-dimensional AdS rotating black hole solution, which can be describe by the following Kerr-AdS metric [9,37]
\[ ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a\sin^2\theta}{\Xi} \, d\varphi\right)^2 + \rho^2 \frac{\rho}{\Delta} \, dr^2 + \rho^2 \frac{\Sigma \sin^2\theta}{\rho^2} \left(adt - \frac{r^2 + a^2}{\Xi} \, d\varphi\right)^2 + \sum \sin^2\theta \left(\sum \sin^2\theta \, \left(adt - \frac{r^2 + a^2}{\Xi} \, d\varphi\right)^2\right) \]  

with

\[ \rho^2 = r^2 + a^2 \cos^2\theta \, , \, \Xi = 1 - \frac{a^2}{l^2} \, , \, \Sigma = 1 - \frac{a^2}{l^2} \cos^2\theta \]

\[ \Delta = (r^2 + a^2)(1 + \frac{l^2}{r^2}) - 2mr. \]  

(1)

For the thermodynamics of the rotating AdS black holes, the cosmological constant can be viewed as an independent variable. In particular, the corresponding thermodynamic quantities are as follows[9,16]:

\[ S = \frac{\pi (r_c^2 + a^2)}{\Xi}, \quad T = \frac{r_c (1 + \frac{a^2}{l^2} + \frac{3}{l^2} - \frac{a^2}{r_c^2})}{4\pi (r_c^2 + a^2)} \, , \quad \Omega_{\mu} = \frac{a\Xi}{r_c^2 + a^2} \]

(3)

The mass \( M \) and the angular momentum \( J \) are connected with parameters \( m \) and \( a \), namely,

\[ M = \frac{m}{\Xi}, \quad J = \frac{am}{\Xi}. \]

(4)

Theoretically, the cosmological constant \( \Lambda = -\frac{3}{l^2} \) can be interpret as a thermodynamic pressure \( P \), which has the following relation

\[ P = -\frac{1}{8\pi} \Lambda = \frac{3}{8\pi l^2}. \]

(5)

In this case, the first law of black hole thermodynamics and Smarr formula are respectively written as

\[ \delta M = TS + \Omega_{\mu} \delta J + V \delta P, \]

\[ \frac{M}{2} = TS + \Omega_{\mu} J - VP. \]

(6)

(7)

Here, \( V \) is the thermodynamic volume conjugate to \( P \). In the present work, we consider a slowly rotating Kerr-AdS black hole. After neglecting all higher order terms of \( J \), the equation of state reads [36,38]

\[ P = \frac{T}{V} - \frac{1}{2\pi V^2} + \frac{48J^2}{\pi V^5}, \]

where \( V \) is the specific volume satisfying the relation
\[ v = 2 \left( \frac{3V}{4\pi} \right)^{1/3} = 2r_c + \frac{12}{r_c (3r_c^2 + 8\pi r_c^4 P)} J^2. \]  

(9)

Eq. (8) suggests that some \( P - v \) critical behaviours really exist. The “real” phase diagram of the slowly rotating Kerr-AdS black hole has been investigated in the extended phase space via Maxwell’s equal area law [36]. There exists a second order transition, i.e., between large and small black hole phase transition, which is analogous to between the gas and the fluid phase transition in the Van der Waals system. This phase transition occurs at the following critical point

\[ P_c = \frac{1}{36\pi \sqrt{10}} J, v_c = 2 \times 90^{1/4} \sqrt{J}, T_c = \frac{90^{3/2}}{225\pi} \frac{1}{\sqrt{J}}. \]  

(10)

The \( P - v \) diagram of the Kerr-AdS black hole for different temperatures. The angular momentum parameter is chosen as \( J = 1 \). On the right panel with the case of \( T < T_c \), it is obvious that the curve is divided into three regions, which contains two stable regions (black lines) and one region (blue line). The red dot line represents the coexisting line of the small black hole (with specific volume \( v_m^- \)) and the large black hole (with specific volume \( v_m^+ \)) with transition pressure \( P_0 \), which satisfies Maxwell’s equal area law.

In Fig.1, we show the \( P - v \) diagram corresponding to the Kerr-AdS black hole. We can clearly see that a small-large black hole phase transition can occur for the case of \( T < T_c \). Especially, an example of this transition is directly presented on the right panel of Fig. 1. Notice that the \( P - v \) curve contains two stable regions and one unstable region. Two stable regions are \( v \in [0, \alpha] \) and \( v \in [\beta, \infty] \), which are respectively corresponding to the small black hole region and the large black hole region. The unstable region \( v \in [\alpha, \beta] \) is referred to as the spinodal region. In the unstable region, the small and large black hole phase can coexist. Furthermore, we
note that \( \frac{\partial P(v,T_0)}{\partial v} < 0 \) in two stable regions but \( \frac{\partial P(v,T_0)}{\partial v} > 0 \) in the unstable region. Mathematically, the two extreme points \( \alpha \) and \( \beta \) are determined by \( \left. \frac{\partial P(v,T_0)}{\partial v} \right|_{v=\alpha} = \left. \frac{\partial P(v,T_0)}{\partial v} \right|_{v=\beta} = 0 \), and the inflection point at \( v = v_0 \) is determined by \( \frac{\partial^2 P(v,T_0)}{\partial v^2} = 0 \).

Some works have suggested the Melnikov method is well suited for studying chaotic behavior of the black hole within the extended thermodynamic phase space[31-34]. Hence, we shall investigate the chaos phenomenon of the Kerr-AdS black hole under periodic thermal perturbations with the help of the Melnikov method.

3. Temporal chaos in a spinodal region

In this section, the effect of a weak temporally periodic perturbation will be analyzed. We assume that the Kerr-AdS black hole is quenched to the spinodal region. According to the standard procedure, we need to make use of the Kerr-AdS black hole equation of state to construct the Hamiltonian for the fluid flow using and derive the Melnikov function containing information about the occurrence of temporal chaos. With respect to the temporal perturbation, we first consider a specific volume \( v_0 \) (the inflection point) in the spinodal region corresponding to an isotherm \( T_0 \) \( (T_0 < T_c) \). Thus, the weak time-periodic fluctuation of the absolute temperature near \( T_0 \) can be written as [39]

\[
T = T_0 + \varepsilon \gamma \cos(\omega t) \cos(M) \tag{11}
\]

where \( \varepsilon << 1 \). According to Ref.[31], one can suppose that the black hole flow takes place along the \( x \) axis in a finite tube with a unit cross section, which includes a total of a mass \( 2\pi/q \) of the black hole in a volume \( (2\pi/q)v_0 \). Here, \( q \) is a positive constant.

In the present work, \( x_0 \) is denoted as the Eulerian coordinate of a reference system. Then, the mass \( M \) of a column of the Kerr-AdS black hole with the unit cross section between the reference \( x_0 \) and a general Eulerian coordinate \( x \) can be expressed in the following form

\[
M = \int_{x_0}^{x} \rho(\xi, t) d\xi, \tag{12}
\]
where $\rho(x,t)$ denotes the black hole density at the position $x$ and the time $t$. It should be pointed out that $\rho(x(M,t),t) = x_{\mu}(M,t) = v$, where $v$ represents the specific volume.

Now, let us consider the thermodynamic phase transition displayed in the one dimensional thermal compressible Kerr-AdS black hole flow, which can be depicted in Lagrangian coordinates via the following system:

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial M}, \quad \frac{\partial u}{\partial t} = \frac{\partial T}{\partial M}. \quad (13)$$

Here, $u$ stands for the velocity and $\tau$ corresponds to the stress tensor. According to Korteweig’s theory, $\tau$ is defined as

$$\tau = -P(v,T) + \mu \frac{\partial u}{\partial M} - A \frac{\partial^3 v}{\partial M^2}. \quad (14)$$

After substituting (14) into (13), we can obtain

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial P(v,T)}{\partial M} + \mu \frac{\partial^3 x}{\partial t \partial M^2} - A \frac{\partial^3 x}{\partial M^2}. \quad (15)$$

Let $\tilde{M} \rightarrow qM, \tilde{t} \rightarrow qt, \tilde{x} \rightarrow qx, \mu \rightarrow \varepsilon \mu_0$, Eq. (15) can be recast as

$$\frac{\partial^2 x}{\partial t^2} = \frac{\partial P(v,T)}{\partial M} + \varepsilon \mu_0 q \frac{\partial^3 x}{\partial t \partial M^2} - A q^2 \frac{\partial^4 x}{\partial M^4}. \quad (16)$$

Here, $\varepsilon$ and $\mu_0$ are positive constants. For convenience, "$~"$ has been omitted in Eq. (16). Then, the mass-volume constraint is given by

$$\int_0^2 \nu(M,t)dM = 2\pi v_0. \quad (17)$$

In the present case, it is not hard to construct the Hamiltonian of the system, which has the following form

$$H = \frac{1}{\pi} \int_0^{2\pi} \left[ \frac{u^2}{2} + F(v,T) + \frac{A q^2}{2} \left( \frac{\partial v}{\partial M} \right)^2 \right] dM \quad (18)$$

where $A$ is a positive constant and $F(v,T)$ is given by [32]

$$F(v,T) = -\int_{v_0}^\nu \bar{P}(\zeta,T)d\zeta. \quad (19)$$

Notice that $\bar{P}(\zeta,T) = P(\zeta,T)dV/d\zeta$ is an effective equation of state obtained by replacing $\zeta$ in terms of the thermodynamic volume $V = \pi \zeta^3/6$ before performing the integral. Considering that $v = v_0$ is the infection point in the spinodal region, thus
we have \( P_v(v_0, T_0) = 0 \), \( P_v(v_0, T_0) > 0 \), and \( P_{vv}(v_0, T_0) < 0 \). At the equilibrium point \((v_0, T_0)\) with \( v = v_0 \) and \( u = 0 \), \( v \) and \( u \) can be expand in Fourier cosine and sine series on \([0, 2\pi]\), respectively. By the use of Eq. (17), we can obtain

\[
v(M, t) = x_M(M, t) \\
\quad = v_0 + x_1(t) \cos M + x_2(t) \cos 2M + x_3(t) \cos 3M + \ldots,
\]

\[
u(M, t) = x_M(M, t) \\
\quad = u_1(t) \sin M + u_2(t) \sin 2M + u_3(t) \sin 3M + \ldots \tag{20}
\]

For the sake of carrying out the perturbation analysis, we need to expand \( \bar{P}(v, T) \) near the equilibrium point \((v_0, T_0)\) in a Taylor series and keep terms to third order. In this case, one has(32)

\[
\bar{P}(v, T) = \bar{P}(v_0, T_0) + \bar{P}_v(v_0, T_0)(v - v_0) + \bar{P}_T(v_0, T_0)(T - T_0) + \frac{1}{2} \bar{P}_{vv}(v_0, T_0)(v - v_0)^2 \\
\quad + \bar{P}_{TT}(v_0, T_0)(T - T_0)^2 + \frac{\bar{P}_{vT}(v_0, T_0)(v - v_0)(T - T_0)}{3!} \\
\quad + \frac{\bar{P}_{vTT}(v_0, T_0)(v - v_0)^2(T - T_0)}{2}.
\]

Because \( \bar{P}_{TT}(v_0, T_0), \bar{P}_{vTT}(v_0, T_0) \), and \( \bar{P}_{vTT}(v_0, T_0) \) are equal to be zero, they have been ignored in the above expression. By substituting Eqs. (19)-(21) into Eq. (18), the Hamiltonian can be rewritten in the following form

\[
H = \frac{u_1^2}{2} + \frac{u_2^2}{2} - \left( \frac{\pi}{4} T_0 - \frac{48J^2}{v_0^5} \right) x_1^2 - \left( \frac{\pi}{4} T_0 - \frac{48J^2}{v_0^5} \right) x_2^2 \\
- \frac{120J^2}{v_0^6} x_1^2 x_2 + \frac{90J^2}{v_0^4} x_1^4 + \frac{360J^2}{v_0^6} x_2^4 + \frac{90J^2}{v_0^4} x_2^4 \\
+ \frac{Aq^2 x_1^2}{2} + 2Aq^2 x_2^2 - \frac{\pi v_0}{2} \epsilon cos(\omega t) x_1 - \frac{\pi}{4} \epsilon \gamma \cos(\omega t) x_1 x_2, \tag{22}
\]

where \((x_1, x_2)\) and \((u_1, u_2)\) stands for the positions and velocities of the first two modes. Consequently, it is not difficult to drive the corresponding equations of motion, which are as follows

\[
\dot{x}_1 = u_1 \\
\dot{x}_2 = u_2
\]
\[
\begin{align*}
\dot{u}_1 &= -\frac{\partial H_z}{\partial x_i} - \varepsilon\mu q u_i \\
&= \left(\frac{\pi}{2} T_0 - \frac{96J^2}{v_0} \right) x_1 + \frac{240J^2}{v_0^2} x_1 x_2 - \frac{360J^2}{v_0^3} x_1^2 - \frac{720J^2}{v_0^4} x_1 x_2 - Aq^2 x_i \\
&\quad + \frac{\pi v_0}{2} \varepsilon\gamma \cos(\omega t) + \frac{\pi}{4} \varepsilon\gamma \cos(\omega t) x_2 - \varepsilon\mu q u_i \\
\dot{u}_2 &= -\frac{\partial H_z}{\partial x_2} - 4\varepsilon\mu q u_2 \\
&= \left(\frac{\pi}{2} T_0 - \frac{96J^2}{v_0} \right) x_2 + \frac{120J^2}{v_0^2} x_2 x_3 - \frac{720J^2}{v_0^3} x_2^2 \\
&\quad - \frac{360J^2}{v_0^4} x_2^3 - 4Aq^2 x_2 - \frac{\pi}{4} \varepsilon\gamma \cos(\omega t) x_i - 4\varepsilon\mu q u_2 \\
\end{align*}
\]

By setting \( z = (x_1, x_2, u_1, u_2)^T \), the above equations can be organized in a compact form, namely,

\[
\dot{z}(t) = g_0(z) + \varepsilon g_1(z,t) .
\]

Here, the small perturbation \( g_1(z,t) \) is periodic in time. The unperturbed system \( (\varepsilon = 0) \) is given by

\[
\dot{z}(t) = g_0(z) .
\]

By linearizing the unperturbed system about \( z = 0 \), one can obtain

\[
\dot{z}_A(t) = Az_A(t)
\]

where the Jacobian matrix \( A \) reads [39]

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-Aq^2 + \varphi & 0 & -\varepsilon\mu_0 q & 0 \\
0 & -4Aq^2 + \varphi & 0 & -4\varepsilon\mu_0 q
\end{pmatrix}
\]

with \( \varphi = \frac{\pi}{2} T_0 - \frac{96J^2}{v_0^5} \). The eigenvalues of \( A \) are as follows

\[
\begin{align*}
\lambda_{1,2} &= -\frac{\varepsilon\mu_0 q}{2} \pm \frac{1}{2} \left[ e^2 \mu_0^2 q^2 - 4(Aq^2 - \frac{\pi T_0}{2} + \frac{96J^2}{v_0^5}) \right]^{\frac{1}{2}}, \\
\lambda_{3,4} &= -2\varepsilon\mu_0 q \pm [4e^2 \mu_0^2 q^2 - (4Aq^2 - \frac{\pi T_0}{2} + \frac{96J^2}{v_0^5})]^{\frac{1}{2}}.
\end{align*}
\]
If the parameters satisfy the following constraints
\[
\frac{\pi T_0}{2} - \frac{96J^2}{v_0^2} < q^2 < \frac{\pi T_0}{2} - \frac{96J^2}{v_0^2}
\]
and \( \varepsilon(>0) \) is small enough, then the first mode is unstable with \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \) but the second and higher modes are stable.

Next, let us analyze the unperturbed (\( \varepsilon = 0 \)) system \( \dot{z}(t) = g_0(z) \) existing a two-dimensional invariant symplectic manifold, which contains a homoclinic orbit and connecting the origin to itself. The corresponding analytical solution has the following form[35]
\[
z_0(t) = \begin{cases}
\left( \frac{\beta^2v_0^7}{180J^2} \right)^{\frac{1}{2}} \sech(\beta t) \\
0 \\
-\beta^2 \left( \frac{v_0^6}{180J^2} \right)^{\frac{1}{2}} \sech(\beta t) \tanh(\beta t) \\
0
\end{cases}
\]
with
\[
\beta = \sqrt{\varphi - Aq^2}.
\]

**Figure 2.** The homoclinic orbit of the unperturbed system with \( T = 0.02894 < T_c \). The parameters are set to \( J = 1, \ q = 1, \) and \( A = 0.01 \). These arrows stand for the time flow direction.
In Fig. 2, we show the phase portrait corresponding to the homoclinic orbit of the unperturbed system, which agrees with the analytical solution (30). From this figure, we can see that the homoclinic orbit possesses the two branches, which are two wings of the butterfly-like orbit, respectively. According to Eq. (30), it is easy to deduce that \( z_0 \) tends to a saddle point (i.e., the origin point) as \( t \to \pm \infty \).

After introducing the time-periodic perturbation (10) (i.e., \( \varepsilon \neq 0 \) ), the above homoclinic orbit may be destroyed, which cause that the chaotic behavior of the system is possible. In the light of the Melnikov method, the Melnikov function of the present perturbed system can be compute by using the following formula [39]

\[
M(t_0) = \int_{-\infty}^{+\infty} g_0^r(z_0(t-t_0))J_{n=2}g_1(z_0(t-t_0),t)dt
\]

with

\[
J_{n=2} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Here, \( g_0(z_0(t-t_0)) \) and \( g_1(z_0(t-t_0),t) \) have the following forms

\[
g_0(z_0(t-t_0)) = \begin{pmatrix}
-\beta^2\left(\frac{v_0}{180J^2}\right)^{\frac{1}{2}}\text{sech}[\beta(t-t_0)]\tanh[\beta(t-t_0)] \\
0 \\
\beta^2\left(\frac{\beta^2 v_0}{180J^2}\right)^{\frac{1}{2}}\text{sech}[\beta(t-t_0)] - \frac{360J^2}{V_0^2}\left(\frac{\beta^2 v_0}{180J^2}\right)^{\frac{3}{2}}\text{sech}^3[\beta(t-t_0)] \\
\frac{2}{3}\beta v_0 \beta^2\text{sech}^2[\beta(t-t_0)]\text{sech}^2[\beta(t-t_0)]
\end{pmatrix}
\]

and

\[
g_1(z_0(t-t_0),t) = \begin{pmatrix}
0 \\
\frac{\pi}{2}\varepsilon \gamma v_0 \cos(\omega t) + \varepsilon \mu_0 q \beta^2\left(\frac{v_0}{180J^2}\right)^{\frac{1}{2}}\text{sech}[\beta(t-t_0)]\tanh[\beta(t-t_0)] \\
0 \\
\frac{\pi}{4}\left(\frac{\beta^2 v_0}{180J^2}\right)^{\frac{1}{2}}\text{sech}[\beta(t-t_0)]\varepsilon \gamma \cos(\omega t)
\end{pmatrix}.
\]

Through a direct calculation, we find that the Melnikov function can be written in a simplified form, namely,

\[
M(t_0) = N\omega \gamma \sin(\omega t_0) - q\mu_0 I
\]
with

$$N = \left( \frac{v_0^7}{180J^2} \right)^{\frac{1}{2}} \pi^2 v_0 \frac{\operatorname{sech}\left( \frac{\pi \omega}{2\beta} \right)}{2}, \quad I = \frac{v_0^7 \beta^3}{270J^2}. \quad (37)$$

Obviously, $M(t_0)$ has simple zeros for $N \omega \gamma \sin(\omega t_0) - q \mu_0 I = 0$ only when

$$\left| \frac{q \mu_0 I}{N \omega \gamma} \right| \leq 1. \quad (38)$$

Hence, it is not hard to conclude that the sufficiently small temporal perturbation may produce the chaos behavior because of the temporal thermal fluctuation. What is more, Eq. (38) can be translated into a critical value corresponding to the perturbation parameter $\gamma$, which reads

$$\gamma_c = \frac{2\sqrt{5}q \mu_0 v_0^7 \beta^3 \cosh\left( \frac{\pi \omega}{2\beta} \right)}{45J \pi^2 \omega}. \quad (39)$$

\textbf{Figure 3.} Temporal evolution in the phase space of velocity vs displacement for the perturbed system in the specific temperature $T = 0.02894 < T_c$ (a) $\gamma = 0.0188 < \gamma_c$ (b) $\gamma = 5 > \gamma_c$. Parameters are set to $J = 1$, $q = 1$, $A = 0.01$, $\varepsilon = 0.001$, $\omega = 0.01$, and $\mu_0 = 0.1$. The initial conditions are chosen as a fixed point $((\sqrt{\beta^2 v_0^7/(180J^2)}, 0)$ in the right branch of the homoclinic orbit.

From Eq. (39), we note that the small temporal perturbation with $\gamma > \gamma_c$ makes sure the transversal intersection between unstable and stable manifolds, which may give rise to the emergence of the Smale horseshoe chaotic motion[40]. In order to check this analytical condition for the chaotic threshold, numerical results for different values of $\gamma$ are shown in Fig. 3. For convenience, both $x_2$ and $u_2$ have been fixed as zero. Figs. 3(a) displays normal trajectories of the system in the
presence of the small temporal perturbation (for $\gamma < \gamma_c$). Figure 3(b) shows the occurrence of chaotic motion for $\gamma > \gamma_c$.

**Figure 4.** Dependence of the critical value $\gamma_c$ on the angular momentum $J$ for the Kerr-AdS black hole. Other parameters are set as in Fig. 2.

From Eq.(39), the critical value $\gamma_c$ should depend on the value of the angular momentum $J$. In Fig. 4, one can clearly see find that $\gamma_c$ decreases as the value of the angular momentum $J$ increases. This means that the larger $J$ makes the occurrence of the chaos behavior easier under the time-periodic thermal perturbation.

### 4. Spatial chaos in the equilibrium state

In the present section, our purpose is to investigate the effect of the small spatially periodic perturbation in the equilibrium configuration with an absolute temperature ($T_0 < T_c$) of the form [39]

$$T = T_0 + \epsilon \cos(qx) . \quad (40)$$

On the basis of the Korteweg theory, the stress tensor can be written as

$$\tau = -p(v,T) - Av'' \quad (41)$$

where $p(v,T)$ satisfies the Kerr-AdS black hole equation of state in Eq. (8), $A > 0$
is a constant, and the symbol \( \frac{d}{dx} \) has been employed. It should be pointed out that, for a static equilibrium without body forces, one can obtain \( \frac{d\tau}{dx} = 0 \) so as to \( \tau = \text{const} = -B \). Furthermore, \( B \) is the ambient pressure as \( |x| \to \infty \). Based on this fact, Eq. (41) can be changed into the following form

\[ v'' = B - p(v, T). \]  

(42)

Here, the constant \( A \) has been set to \( A=1 \) as Refs. [31,32].

First, we discuss the unperturbed system ( \( T = T_0 \) ). For arbitrary temperature \( T = T_0 < T_c \), the nonlinear systems in Eq. (42) exist three fixed points, which are respectively \( v_1 \), \( v_2 \) and \( v_3 \). In the light of the magnitude of the ambient pressure \( B \), Eq. (42) can generate three different types of portraits in the \( v-v' \) phase plane, namely,

**Case 1.** The ambient pressure \( B \) lies in the range \( 0 < B < P(\beta, T_0) \). \( P_0 \) is the phase transition pressure. Then, the values \( v_1 \), \( v_2 \) and \( v_3 \) so that \( P(v_1, T_0) = P(v_2, T_0) = P(v_3, T_0) = B \) are displayed in Fig. 5(a), meanwhile the portrait of Eq. (42) in the \( v-v' \) phase plane is displayed in Fig. 5(b).

**Case 2.** The ambient pressure \( B \) lies in the range \( P(\alpha, T_0) < B < P_0 \). Then, the values \( v_1 \), \( v_2 \) and \( v_3 \) so that \( P(v_1, T_0) = P(v_2, T_0) = P(v_3, T_0) = B \) are presented in Fig. 6(a), meanwhile the corresponding \( v-v' \) phase plane is presented in Fig. 6(b).

**Case 3.** The ambient pressure \( B \) is equal to the phase transition pressure \( P_0 \), i.e., \( B = P_0 \). In this case, the values \( v_1 \), \( v_2 \) and \( v_3 \) so that \( P(v_1, T_0) = P(v_2, T_0) = P(v_3, T_0) = P_0 \) are exhibited in Fig. 7(a), meanwhile the corresponding \( v-v' \) phase plane is exhibited in Fig. 7(b).

From these figures, one can conclude that the present unperturbed system exists a homoclinic orbit for both case 1 and case 2 but it possesses a heteroclinic orbit for the final case. It is obvious that the above characteristics can allow us to calculate the Melnikov function corresponding to these orbits.
Figure 5. Case 1: (a) $B$ and $v_1$, $v_2$, $v_3$ in $P-v$ diagram; (b) $v-v'$ phase portrait. One can see a homoclinic orbit connecting $v_3$ to itself (read dash dot line).

Figure 6. Case 2: (a) $B$ and $v_1$, $v_2$, $v_3$ in $P-v$ diagram; (b) $v-v'$ phase portrait. One can see a homoclinic orbit connecting $v_1$ to itself (blue dash dot line).

Figure 7. Case 3: (a) $B$ and $v_1$, $v_2$, $v_3$ in $P-v$ diagram; (b) $v-v'$ phase portrait. One can see a heteroclinic orbit connecting $v_1$ to $v_3$ (read dash dot line).

After adding a small spatial perturbation expressed in Eq. (40), Eq. (42) for the perturbed system can be recast as
\[ v'' = B - p(v, T_0) - \frac{\varepsilon \cos(qx)}{v}. \] (43)

For the present perturbed system, the corresponding Melnikov function can be expressed as [39]

\[ M(x_0) = \int_{-\infty}^{\infty} F^T(Z(x-x_0)) J_{n=1} G(Z(x-x_0), x) dx \] (44)

with

\[ J_{n=1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (45)

Let \( v' = u \), then Eq. (43) can be changed into two first-order equations, namely,

\[ v' = u, \]

\[ u' = B - P(v, T_0) - \frac{\varepsilon \cos(qx)}{v}. \] (46)

As in the previous section, the general solutions of homoclinic or heteroclinic orbit can be written in the following form

\[ Z(x-x_0) = \begin{pmatrix} v_0(x-x_0) \\ u_0(x-x_0) \end{pmatrix}. \] (47)

Then, the expressions of the \( F \) and \( G \) functions are respectively written as

\[ F(Z(x-x_0)) = \begin{pmatrix} w_0(x-x_0) \\ B - p(v_0(x-x_0), T_0) \end{pmatrix}, \] (48)

and

\[ G(Z(x-x_0)) = \begin{pmatrix} 0 \\ -\cos(qx) \\ -\frac{\cos(qx)}{v_0(x-x_0)} \end{pmatrix}. \] (49)

By introducing a new variable \( X = x-x_0 \), the Melnikov function can be written in the following simple form

\[ M(x_0) = -L \cos(qx_0) + K \sin(qx_0) \] (50)

with

\[ L = \int_{-\infty}^{\infty} \frac{u_0(X) \cos(qX)}{v_0(X)} dX, \quad K = \int_{-\infty}^{\infty} \frac{u_0(X) \sin(qX)}{v_0(X)} dX. \] (51)

From Eq. (50), one can find that \( M(x_0) \) always exists simple zeros for any given values of \( L \) and \( K \). Hence, it is not difficult conclude that there always exist
spatial chaos always occurs in the thermodynamic system of the Kerr-AdS black hole under the spatially periodic thermal perturbation. This result is in agreement with other AdS black holes [31-34]. In Fig. 8-Fig. 10, the numerical solutions of the perturbed dynamical equation (46) are respectively presented in the $v-v'$ plane for these three cases. The corresponding initial configurations are chosen as the homoclinic orbit or heteroclinic orbit. These figures clearly show that there exists indeed the spatial chaos under the spatially periodic thermal perturbation.

![Figure 8](image1.png)

**Figure 8.** Portrait of the perturbed equation in $v-v'$ phase plane for case 1.

![Figure 9](image2.png)

**Figure 9.** Portrait of the perturbed equation in $v-v'$ phase plane for case 2.
5. Summary

In the present study, we have investigated the occurrence of chaotic behavior under temporally and spatially periodic perturbations in the unstable spinodal region of the Kerr-AdS black hole within the extended phase space. Firstly, we found that analytical solution for the homoclinic orbit exists in the unstable spinodal region. By making use of the Melnikov method, we displayed that the zeros of the Melnikov function give a critical value of the temporal perturbation $\gamma_c$ for the occurrence of the temporal chaos. Only when the perturbation amplitude $\gamma$ is larger than the critical value $\gamma_c$, the chaotic behavior can emerge. What is more, our results showed that the larger $J$ makes the occurrence of the chaos behavior easier under the time-periodic thermal perturbation. Secondly, we also found that the spatial chaos can occur in the small/large black hole equilibrium configuration when the system is suffered from the spatial thermal perturbation.

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