Dynamics of Noisy Quantum Systems: Application to the Stability of Fractional Charge

Armin Rahmani

1Theoretical Division, T-4 and CNLS, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
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Recent developments in atomic, molecular and optical physics have made it possible to create several important many-body Hamiltonians in optical lattices with a large degree of control and tunability [1]. Such systems exhibit remarkable isolation from a thermal environment and can undergo coherent unitary evolution in experimentally accessible time scales. These developments have motivated numerous recent studies of the nonequilibrium quantum dynamics of thermally isolated systems [2]. Such systems are, nevertheless, vulnerable to noise-induced heating, which can originate from, e.g., amplitude fluctuations of the lasers forming the optical lattice. Due to the temporal fluctuations of the Hamiltonian, the time evolution is governed by a stochastic (instead of deterministic) Schrödinger equation.

Generally, such fluctuations are undesirable as they reduce the degree of control over the quantum system. Understanding the effects of noise in such systems is thus of paramount importance both in designing experiments and interpreting their results. Several publications have investigated the heating dynamics due to such stochastic driving in specific systems such as harmonic traps [3-4], Luttinger liquids [5], transverse-field Ising chain [6, 7], and the Bose-Hubbard model [8, 9] using various approximations (see also Ref. [10] for results on energy fluctuations of generic driven systems). Here, we study such dynamics exactly focusing on the long-time behavior of observables in arbitrary quadratic fermionic systems driven by white noise.

While our Hamiltonians are quadratic for a given realization of noise, the effective action obtained by integrating out the noise contains quartic interactions, which makes it difficult to obtain analytical results from this standard effective-action approach. Equivalently, one can formulate a master equation for the noise-averaged density matrix. However, the effective noise-induced interactions once again make it difficult to make analytical progress without resorting to approximations. Langevin-type numerical simulations are also limited to short time scales and can not access the long-time limit.

Nevertheless, we argue in this paper that a Heisenberg-picture approach to the evolution of noise-averaged observables allows us to obtain the exact time dependence and the long-time-limit of the noise-averaged expectation values of all quadratic observables such as energy and the fermionic Green’s function. We apply our formalism to a simple model relevant to polyacetylene (the simplest system exhibiting topological properties and charge fractionalization) in a dimerized optical lattice [11]. The domain walls in this system bind a fractional charge. We find that the charge density profile is remarkably robust against noisy fluctuations (even when the fluctuations have spatial as well as temporal disorder) as long as the fluctuations are limited to nearest-neighbor hopping. The presence of noisy fluctuations in the second-neighbor hopping can lead to an instability, which globally distorts the charge density profile. The elimination of second-neighbor hopping processes may thus play an important role in the experimental observation of fractionalization in optical lattices.

Let us begin by discussing the general formulation of the problem. Consider a Hamiltonian $H_0$ that is perturbed by fluctuating terms

$$H(t) = H_0 + \sum_i \alpha_i(t)V_i,$$

where the perturbations $V_i$ may originate from the fluctuations of lasers forming the optical lattice. The system is then described by a stochastic Schrödinger equation

$$i\partial_t \psi(t) = \left[H_0 + \sum_i \alpha_i(t)V_i\right] \psi(t),$$

interpreted in the Stratonovich sense as we are dealing with continuous processes. (Throughout the paper $\hbar$ is set to unity.) The equation above simply describes an ensemble of quantum evolutions, each characterized by one realization of noise $\alpha_i(t)$. We focus on Gaussian white noise with zero mean and second moment

$$ \overline{\alpha_i(t)\alpha_j(t')} = \delta_{ij} W^2 \delta(t-t'),$$

where the “overline” indicates averaging over noise. We also assume that the system is initially in the ground state $|\psi_0\rangle$ of $H_0$. The goal is to find how the noise-averaged expectation value of an operator $O$ (which does not have explicit time dependence) changes after the system has evolved for a time $\tau$ with the stochastic Schrödinger equation [2]. Each
realization of noise evolves the system deterministically, resulting in a unique quantum expectation value $\langle O(\tau) \rangle$. The solution of the stochastic equation (2) corresponds to the average of these quantum expectation values over the realizations of noise. The operator $O$ can be a local density, an equal-time correlation function, or, in the case of heating, the Hamiltonian $H_0$.

Conceptually, it is easier to analyze the problem by approximating the white noise with an ensemble of discrete piece-wise constant protocols shown in Fig. 1. The time $\tau$ is divided into $N$ intervals of length $\delta t = \tau/N$ with $N \to \infty$ and each $x_n$ is assumed constant over interval $n$ and $x_n \equiv x_n^i$ drawn from a uniform distribution $\left[ -\sqrt{3} \frac{w_n^{\imath}}{\sqrt{\delta t}}, \sqrt{3} \frac{w_n^{\imath}}{\sqrt{\delta t}} \right]$ such that $x_n^i = 0$ and $w_n^{ij} = w_n^{\imath j} \delta_{mn}$. It is convenient to further define

$$w_n^j \equiv x_n^j \sqrt{\delta t}, \quad w_n^i = 0, \quad w_n^{ij} = w_n^i \delta_{mn}.$$  

Notice that while the short-time behavior of noise-averaged observables can be accessed by Langevin-type numerical simulations (generating such random protocols, evolving the system with each of them, finding the expectation values $\langle O \rangle$ for all generated realizations of noise, and averaging them), the long-time behavior cannot be easily accessed with such direct methods.

The noise-averaged expectation value of a time-independent operator $O$ at time $\tau$ is then given by

$$\langle O(\tau) \rangle = \langle \psi_0 | U(\tau) O | \psi_0 \rangle \equiv \langle \psi_0 | \overline{O(\tau)} \rangle | \psi_0 \rangle,$$

where the evolution operator $U(\tau)$ for a fixed $\{w_n^j\}$ realization of noise can be written as $U(\tau) = e^{-iH_0 \delta t} \sum_j V_j w_n^j \sqrt{\delta t} \ldots e^{-iH_0 \delta t} \sum_j V_j w_n^j \sqrt{\delta t}$, with $e^{-iH_0 \delta t} \sum_j V_j w_n^j \sqrt{\delta t}$ approximately given by

$$1 - i \sum_j V_j w_n^j \sqrt{\delta t} - i H_0 \delta t - \frac{1}{2} \sum_{jk} V_j V_k w_n^j w_n^k \delta t,$$

in the limit of $N \to \infty (\delta t \to 0)$. As the $w_n^j$ are uncorrelated for different $n$, we can compute $\overline{O(t + \delta t)}$ by acting on $\overline{O(t)}$ with Eq. (6) and its Hermitian conjugate, respectively from the left- and the right-hand side, and then performing a noise averaging over only one set of stochastic variables $w_n^j$. We obtain three nonvanishing terms (to order $\delta t$), which involve an even number of the same stochastic variable (and thus survive the noise averaging): one from taking the two $i V_j w_n^j \sqrt{\delta t}$ terms from the two sides and two others from taking one $\frac{1}{2} V_j w_n^j \sqrt{\delta t}$ from each of the two sides.

The above argument leads to the following equation of motion for $\overline{O(t)}$:

$$\frac{d}{dt} \overline{O(t)} = i \left[ H_0, \overline{O(t)} \right] + \frac{1}{2} \sum_j W_j^2 \left[ \overline{[V_j, \overline{O(t)}]}, V_j \right],$$

where $H$ is an $L \times L$ matrix $(i, j = 1 \ldots L)$, and $c_i$ is the annihilation operator for a fermion on site $i$. There are many interesting Hamiltonians that fall into this category. These include topological insulators, disordered systems, spin chains mapped to free fermions, and interacting systems treated with various mean-field schemes. We focus on the case of only one fluctuating perturbation $V = \Gamma(\mathcal{H})$ and a quadratic observable $O = \Gamma(\mathcal{O})$.

As the time evolution is quadratic for each realization of noise, the only nontrivial aspect of the problem is averaging over noise. As mentioned in the introduction, since the noise couples to quadratic terms in $c_i$, performing a Gaussian integral over noise (in the standard effective-action approach) generically results in quartic terms in $c_i$ (see, e.g., Refs. [6][13][16]). The master-equation for the noise-averaged density matrix is also plagued by the same problem. To be more explicit, consider the evolution of the density matrix with the stochastic Schrödinger equation. The initial zero-temperature density matrix can be written as $\rho(0) = e^{\Gamma(\mathcal{H})(0)}$ for an $L \times L$ matrix $\mathcal{H}(0)$. As discussed in the next paragraph, for any realization of noise, the density matrix retains the form above for a time-dependent $L \times L$ matrix $\mathcal{H}(t)$ (and consequently the Wick’s theorem and the free-fermion picture survive). However, as argued below, the noise-averaged density matrix will not retain this form. Therefore, the simplicity of quadratic Hamiltonians would not be utilized if we use a Lindblad-type master equation.

First, consider the time evolution of the density matrix $\rho(t) = e^{\Gamma(\mathcal{H})(t)}$ with a deterministic Hamiltonian $\Gamma[\mathcal{H}(t)]$: $\dot{\rho} = -i [\Gamma(\mathcal{H}), \rho]$, where the “dot” symbol represents the time
the Kronecker $\delta_{\alpha\eta} = \sum_{r} U_{\alpha r} U_{\eta r}^{\dagger}$ (and similarly for $\delta_{\psi\beta}$), we obtain $K_{\alpha i \psi j, \eta \gamma} = -\sum_{r} U_{\alpha r} U_{\psi j}^{\dagger} (D_{\sigma r} - D_{\lambda})^{2} U_{\eta \gamma}^{\dagger} U_{\gamma r}$. The matrix $e^{\hat{K}_{i}}$ can now be simply written in terms of the above diagonalized form, which leads to

$$\hat{O}(t) = \sum_{\alpha \beta \gamma} U_{\alpha r} U_{\psi j}^{\dagger} e^{(iD_{\sigma r} - D_{\lambda})^{2} t} U_{\eta \gamma}^{\dagger} U_{\gamma r} \delta_{\eta \gamma}(0).$$

Note that none of the eigenvalues $-(D_{\sigma r} - D_{\lambda})^{2}$ of $\mathcal{K}$ are positive so there are no $t \rightarrow \infty$ divergences in $\hat{O}(t)$. Assuming there are no degeneracies in the spectrum of $\mathcal{V}$, the matrix $\mathcal{K}$ has $L$ vanishing eigenvalues (corresponding to eigenvalues of $e^{\hat{K}_{i}}$, which do not decay exponentially) for $\sigma = \lambda$. The limit of long times can then be easily accessed by setting these decaying eigenvalues of $e^{\hat{K}_{i}}$ to zero. We then obtain

$$\hat{O}(t) \rightarrow \infty = \sum_{r \eta \gamma} U_{\alpha r} U_{\eta \gamma}^{\dagger} e^{(iD_{\sigma r} - D_{\lambda})^{2} t} U_{\eta \gamma}^{\dagger} U_{\gamma r} \delta_{\eta \gamma}(0).$$

We now use Eq. (12) above and its $t \rightarrow \infty$ limit (13) (with the initial condition $\hat{O}(0) = \mathcal{O}$) to respectively write the exact time dependence and the long-time limit of the noise-averaged expectation value of an operator $O = \Gamma(\mathcal{O})$ through

$$\langle O(t) \rangle = \sum_{a} \langle \hat{O}^{\dagger}(t) \mathcal{O} \rangle_{\hat{a}a} ,$$

where the “prime” symbol indicates summing over the initially occupied single-particle levels in the ground state ($\langle \psi_{0} \rangle$) of $H_{0} = \Gamma(H_{0})$, and the unitary matrix $\mathcal{O}_{\hat{a}}$ diagonalizes $H_{0}$. If we are interested in heating, we have $\Gamma = H_{0}$. For the Green’s function $\langle c_{i}^{\dagger} c_{j} \rangle$, we have $\delta_{ij} = \delta_{\alpha \beta} \delta_{\lambda \gamma}$ and a special case $i = j$ gives the local density. We note in passing that up to Eq. (14), we did not use the assumption that $H_{0}$ is quadratic. In fact, as long as $V$ and $O$ are quadratic, we can use Eqs. (12) and (13) with an interacting $H_{0}$ if we are able to compute the expectation values of the corresponding static observables in the ground state of $H_{0}$, e.g., with density-matrix renormalization group or quantum Monte Carlo.

We now focus on a simple model (relevant to polycyctelene), which exhibits an interesting topological property, namely, a fractional charge bound to domain walls [18] [19]. This model (without the domain wall at this point) has been recently implemented in optical lattices [11]. As shown in Fig. 2 (b), we consider spinless fermions hopping on a one-dimensional lattice, with the hopping amplitudes modulated as follows:

$$H_{0} = \sum_{x} [t + (-1)^{x} t'] \{ c_{x}^{\dagger} c_{x+1}^{\dagger} + c_{x+1} c_{x} \} .$$

For $t' = 0$, the low-energy effective description of the system consists of linearly dispersing massless Dirac fermions. The term proportional to $t'$ opens a gap by making the fermions massive (the sign of the mass is the same as the sign of $t'$). It is well-known that the two signs of mass correspond to two topologically distinct phases, and, thus, a domain wall [as shown
from a uniform distribution 
random hopping for up to second-nearest neighbor 
scopic details. Here, as a demonstration, we consider local 

$H = \sum_x V(x)$, with 

$W$ 

We are interested in the local charge density 

$n(x,t)$ as well as the energy 

$H$ of the double commutator of 

$\langle \sigma \sigma \rangle$ decay to zero. Initially, the system heats up 

by computing $\langle H_0(t) \rangle$. Initially, the system heats up 

very fast as the $e^{-c_{\text{free}} - D_{\text{eff}}t}$ factors for $(D_{\text{eff}} - D_{\text{lat}})^2 \sim O(1)$ 

decay quickly. The heating slows down significantly at later 
times. We expect to reach the $t \to \infty$ limit in a time of order 

$L^2$ in which all the exponential damping factors corresponding 
to nearby energy levels $D_{\text{eff}}$ (with $\frac{1}{2}$ spacing) decay to zero. 
The results are shown in Fig. 4 for $W^2/2 = 1$.

In summary, we studied, for general systems of noninter-
acting fermions, the effects of stochastic Hamiltonian noise in 
the generation of excess energy and the variations of quadratic 
observables. We argued that a Heisenberg-picture approach to 
problem is more powerful than a Schrödinger-picture master 
equation as well as the standard effective-action methods. De-
spite the fact that the effective noise-averaged theory is a com-
plex interacting one, for such quadratic observables, we were 
able to calculate the exact time dependence using free-fermion 
techniques. This provides access to the long-time limit, which 
was not previously accessible by any analytical or numerical 
methods (such as Langevin-type simulations).

We applied our formalism to the stability of fractional 
charge in a one-dimensional dimerized optical lattice with a 
domain wall, which provides a promising candidate for real-
izing the phenomenon of fractionalization in optical lattices. 
We found an instability in the charge density profile in the 
presence of fluctuating second-nearest neighbor hopping 
processes. Our findings suggest that the elimination of next-
nearest neighbor hopping processes may enhance the robust-
ness of experiments designed for the detection of fractional-
ization in optical lattices. Application to the stability of frac-
tional charge in two-dimensional quadratic systems such as 

in Fig. 2(b)] entails a change in the mass sign, resulting in a 
zero mode and fractional charge.

The fluctuating component $V$ can depend on various micro-
scopic details. Here, as a demonstration, we consider local 
random hopping for up to second-nearest neighbor $V = \Gamma(Y)$ 
with $Y_{ij}(T_1, T_2) = \sum_{b=1,2} t_b^i (\delta_{i+b,j} + \delta_{i-b,j})$, where $t_b^i$ is drawn 
from a uniform distribution $[-T_1, T_2]$. As for the operator $O$, 
we are interested in the local charge density $n_x = c_x^\dagger c_x$ at site 
$x$ as well as the energy $H_0$. Let us focus on the half-filling 
case and compute the charge distribution of the the system. In 
Fig. 3(a), we show the numerically computed charge density 
profile for a system of $L = 400$ sites with a domain wall in 
the middle. We set the hopping $t$ to unity and express all other 
hoping amplitudes as dimensionless numbers in units of $t$. In 
Fig. 3(a), we used $t' = 0.2$ but the actual value is inessential. 
As expected, the fractional charge bound to the domain wall 
is equal to $1/2$.

In Figs. 3(b) and 3(c), we show the long-time limit of the 
noise-averaged charge density profile $\langle n_x(t \to \infty) \rangle$ computed 
using Eqs. (13) and (14), respectively for the case of random 
nearest neighbor hopping $T_1 = 1$ and $T_2 = 0$ and random first 
and second neighbor hopping $T_1 = T_2 = 1$ (notice that rescal-
ing $T_0$ as well as the strength of noise $W$ can affect how fast 
the $t \to \infty$ limit is reached but do not change the steady-state 
profile, which only depends on eigenfunctions of $\mathcal{Y}$. Interest-
ingly, the fractional charge remains robust for fluctuations in 
the nearest-neighbor hopping but becomes unstable as soon as 
second-neighbour hopping processes are included.

One way to understand this marked difference between 
Figs. 3(b) and 3(c) is to consider the initial rate of change of 
$\langle n_x(t) \rangle$ in the uniform dimerized lattice (corresponding to 
regions far away from the domain wall or the boundaries). The 
initial rate of change for a generic nearest-neighbor perturba-
tion $V = \sum_x t_x (c_x^i c_{x+1} + \text{h.c.})$ is given by the expectation value 
of the double commutator of $V$ and $n_x$. The only terms ap-
pearing in the double commutator have the form $\langle n_y - n_{y'} \rangle$ and 
$\langle c_y^i c_{y'+2} \rangle$, both of which vanish in the ground state of 
the uniform system at half-filling. On the other hand, including 
second-neighbors gives rise to a disorder-dependent initial 
rate, which globally destabilizes the charge density profile. 
Our numerical calculations show that the initial robustness 
in case of nearest-neighbor perturbations survives to arbitrary 
time scales.

We also study the full time dependence of the noise-induced 
heating by computing $\langle H_0(t) \rangle$. Initially, the system heats up 
very fast as the $e^{-c_{\text{free}} - D_{\text{eff}}t}$ factors for $(D_{\text{eff}} - D_{\text{lat}})^2 \sim O(1)$ 
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tional charge in two-dimensional quadratic systems such as
graphene [20] and frustrated itinerant magnets [21] calls for future investigations. Also, extending our results to colored noise as well as to fluctuations around time-dependent Hamiltonians (for studying, e.g., the robustness of optimal-control protocols [22]–[27]) are of considerable interest.

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