Exact logarithmic four-point functions in the critical two-dimensional Ising model

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Based on conformal symmetry we propose an exact formula for the four-point connectivities of FK clusters in the critical Ising model when the four points are anchored to the boundary. The explicit solution we found displays logarithmic singularities. We check our prediction using Monte Carlo simulations on a triangular lattice, showing excellent agreement. Our findings could shed further light on the formidable task of the characterization of Logarithmic Conformal Field Theories and on their relevance in physics.

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Introduction – Conformal symmetry in two dimensions [1] has been of extraordinary usefulness to study classical statistical mechanics models at criticality since the 80’s. It has notably found also extensive applications in the quantum realm, spanning from gapless one-dimensional systems [2], quantum Hall effect [3] and entanglement [4]. Two dimensional Conformal Invariant Quantum Field Theories (CFTs) and in particular Liouville theory are moreover the cornerstone of world-sheet geometry in string theory [5]. The simplest CFTs that capture the critical behaviour of lattice models and quantum spin chains are unitary. Moreover, when their Hilbert space splits into a finite number of representations of the conformal (or some larger) symmetry are usually termed rational. CFTs are classified according to their central charge $c$; for instance for a free massless boson $c = 1$.

However unitary and rational theories are far from exhausting the physically relevant conformally invariant theories. In the beginning of the 90’s the groundbreaking discovery [6] of an exact formula for the crossing probability in critical percolation forced to analyze theories violating these two assumptions. Percolation is a simple stochastic process where bonds or sites on a lattice can be occupied independently with a certain probability. Since the partition function is not affected by finite-size effects, the central charge of a putative CFT describing critical percolation is zero [7]. The existence of a non-trivial formula for the crossing probability makes it a prominent example of a non-unitary CFT (the only unitary CFT with $c = 0$ is trivial). For subsequent developments leading to the formulation of the Stochastic Lowener Evolution we refer to the reviews [8, 9]. At the same time, it was recognized that as far as the conditions of unitarity and rationality are relaxed, CFT correlation functions can display striking logarithmic singularities that are actually the signatures of intricate realizations of the conformal symmetry [10, 11]. The class of non-unitary and generally non-rational CFTs where these unconventional features show up, was christened Logarithmic CFTs. Such theories were promptly argued to play a fundamental role in the characterization of disordered systems in two dimensions [12–14], for instance by means of supersymmetry [15, 16].

With these motivations, Logarithmic CFTs were investigated in greater detail in the last decade, either from a purely algebraic point of view [17], either constructing lattice regularizations [18, 20] or generalizing the study of crossing formulas in critical percolation [19]. Recently [21], it has also been suggested that an analytic non-unitary extension of Liouville theory [22] might describe the connectivity properties of critical bulk percolation and more generally the Q-state Potts model. The domain of applicability of Liouville theory in statistical mechanics is currently an important open problem see [26, 27] and also [28].

Despite these huge advances, a satisfactory understanding of Logarithmic CFTs is still a long way off. Moreover, it is fair to say that few examples of explicit logarithmic singularities have been found in familiar statistical models: notably only in percolation [29, 30] ($c = 0$), dense polymers [31–33]; see also [34, 35] for applications to disordered systems. An exact Coulomb gas approach to CFT correlation functions [36–38] closely related to those considered in this letter reveals an infinite number of logarithmic cases. These arise from operator mixing as is the case here. These results suggest the possibility of logarithmic behavior in multiple Self Avoiding Random Walks (SAWs) [38].

In this paper we show how a logarithmic singularity due to operator mixing also arises in the context of arguably the best-known model of statistical mechanics: the two-dimensional Ising model; see [39, 40] for a pedagogical introduction to the richness of the model. This is remarkable since it shows unambiguously that critical properties of Ising clusters are ruled by a Logarithmic CFT. Moreover the four-point function we consider here, a four point connectivity in the Fortuin and Kasteleyn (FK) representation of the Ising model, is a natural observable that can be easily simulated with Monte Carlo (MC) methods. Logarithmic singularities in Ising connectivities were also argued to exist in [41, 42], in this letter we demonstrate it explicitly using CFT.

Finally, our findings could shed further light on the extremely challenging problem of the characterization of
Logarithmic CFTs and on their applications to physics.

Four points connectivities in the Ising model – It is convenient to introduce the Ising FK clusters, starting from the ferromagnetic $Q$-state Potts model [43]. The Ising case is recovered by setting $Q = 2$. The model is defined on a finite simply-connected domain $D$ of the plane, see Fig. 1 and the choice of the underlying lattice is irrelevant at the critical point. To introduce the notion of cluster connectivities, we should first recall the Fortuin-Kasteleyn (FK) representation [22]. The $Q$-state Potts model is defined in terms of spin variables $s(x)$ taking $1, \ldots, Q$ different values; its partition function can be expressed as a product over the lattice edges as

$$Z = \sum_{\{s(x)\}} \prod_{\langle x,y \rangle} \left[ (1 - p) + p\delta_{s(x),s(y)} \right], \quad (1)$$

where $p$ is a parameter related to the temperature and the product in (1) extends only to next-neighboring sites. Suppose then to expand such a product: Each term in the expansion can be represented graphically by drawing a bond between $x$ and $y$ if the factor $p\delta_{s(x),s(y)}$ is selected and leaving empty the bond if such a factor is absent. The set of non-empty bonds in each term of the expansion then defines a graph $G$ on the underlying lattice that is called FK graph. Such a graph might contain $N_c$ different connected components (including isolated points), dubbed FK clusters. Moreover, on each cluster the spin values are constrained to be the same because of the Kronecker delta in (1). Summing over their possible $Q$ values leads to the following rewriting of the $Q$-state Potts model partition function as a sum over graphs: $Z = \sum_G (1 - p)^{n_b} p^{n_n} Q^{N_c}$, where $n_b$ and $n_n$ are the number of occupied and empty bonds in the graph $G$. For arbitrary non-negative $Q$, the graph representation for $Z$ is a generalized percolation problem known as Random Cluster Model, where bonds occupied with probability $p$ are not independent random variables. The fundamental observables in the Random Cluster Model are the connectivities and they offer a purely geometrical interpretation of the magnetic Potts model phase transition. Connectivities represent the different probabilities with which $n$ points of the plane can be partitioned into FK clusters. If the points are on the boundary of the domain $D$, the total number of $n$-point connectivities is clearly equal to the number of non-crossing partitions of a set of $n$ elements, i.e. the Catalan number $C_n$; for example if $n = 4$ there are $C_4 = 14$ of them. These functions are however not linearly independent, since they satisfy sum rules; for instance the sum over all $n$-point connectivities has to be one. Following [43], it is possible to show that a valid choice of $n$-point linearly independent connectivities is given by all the probabilities associated to configurations where no point is disconnected from all the others (non-singleton partitions). In the specific example of $n = 4$ and $x_1, x_2, x_3, x_4$ on the boundary of $D$, see again Fig. 1 a possible choice of linearly independent connectivities is: $P_{(1234)}$, $P_{(12)(34)}$ and $P_{(14)(23)}$. The function $P_{(1234)}$ denotes the probability that all the four points $x_1, x_2, x_3$ and $x_4$ are on the same FK cluster; $P_{(12)(34)}$ is instead the probability that $x_1$ and $x_2$ are in the same cluster, $x_3$ and $x_4$ in the same cluster but these two are now different and analogously for $P_{(14)(23)}$. We also omitted for simplicity the explicit spacial dependence. Notice that when the points $x_1, x_2, x_3$ and $x_4$ are anchored to the boundary the function $P_{(13)(24)}$ does not appear since two clusters cannot cross.

Exact solution – We turn now to the exact determination of these three functions in the critical Ising model, using arguments inspired by the seminal work [6]. At the critical point, conformal invariance allows one to map any simply connected domain $D$ of the plane by the Riemann mapping theorem into the unit disk. Moreover, the points $x_1, x_2, x_3, x_4$ are mapped to points $w_1, w_2, w_3, w_4$ lying at the boundary (circumference) of such a disk. The three connectivities $P_{(1234)}$, $P_{(12)(34)}$ and $P_{(13)(24)}$ can be singled out by computing Potts partition functions with specific boundary conditions for the dual Potts spins [43]. As an example let us suppose to fix the values of the dual Potts spins at the boundary of the disk to be $1, 2, 3$ and $4$ as in Fig. 2 left and to compute the Potts partition function in this case. Notice that this assignment will require at least four available colors, i.e. $Q \geq 4$, and it would be non-physical for the Ising model. It has however certainly sense if we assume $Q$ real and imagine to compute connectivities in the Random Cluster Model at any values of $Q$ and take eventually the limit $Q \to 2$. Configurations of dual FK clusters with such a particular choice of boundary conditions cannot contain clusters that cross from regions with boundary conditions $\alpha$ to regions with boundary conditions $\beta$ if $\alpha \neq \beta$. Dual FK clusters are represented schematically by blue dashed curves in Fig. 2. Applying a duality transformation to the Potts model partition function [43], these configura-
tions are in one-to-one correspondence with configurations where a single FK cluster, continuous red curve in Fig. 2 connects the four boundary points. The reasoning above allows to compute then connectivities as Potts partition functions with insertion of local operators $\phi_{(\alpha|\beta)}$ that switch the values of the dual spins at the boundary from $\alpha \to \beta$, $\alpha, \beta = 1, \ldots, Q$. In the jargon of CFT, the fields $\phi_{(\alpha|\beta)}$ are called boundary-condition-changing operators. In this way, we can argue for example that $F_{(1234)}$ has to be proportional to the correlation function of $\langle \phi_{(14)}(w_1)\phi_{(12)}(w_2)\phi_{(23)}(w_3)\phi_{(34)}(w_4)\rangle$.

Let us briefly recall that in the simplest case, the scaling fields $h_{r,s}$ of any CFT can be classified by two positive integers $r,s$ such that their scaling dimensions are

$$h_{r,s} = \frac{r(m+1) - sm^2 - 1}{4m(m+1)}, \quad m \in \mathbb{R}. \quad (2)$$

The parameter $m$ is related to the central charge $c$ of the CFT through $c(m) = 1 - \frac{6}{m(m+1)}$, and in turns for the Potts model $Q = 4 \cos^2(\pi/(m+1))$. The values $h_{r,s}$ can be represented into a lattice, dubbed the Kac table; for a CFT with $c = 1/2$ as the Ising model, the Kac table is represented in Fig. 2 on the right.

The boundary condition changing operator $\phi_{(\alpha|\beta)}$ was identified in [6] for any values of $Q$ as the field $\phi_{1,3}$. Notice that at $c = 1/2$, the dimension of $\phi_{1,3}$ is $h_{1,3} = 1/2$ and coincides with the one of the Ising order parameter $\sigma$, when inserted at the boundary [10]; in this case the spin operator $\sigma$ transforms as the field $\phi_{2,1}$. In the construction of the simplest conformal field theory describing the $\mathbb{Z}_2$ universality class these two fields can be actually identified and consequently the operator product algebra of $\{\phi_{1,1}, \phi_{2,1}, \phi_{1,2}\}$ closes. The self-consistent closure of the operator product algebra was used as a criterion in [11] to build the whole family of minimal conformal models, where only a finite numbers of Virasoro representations are in one-to-one correspondence with configurations that fix the values of the dual boundary spins to 1, 2, 3 and 4 are chosen. On the right, the Kac table, obtained from the scaling dimension $h_{r,s}$ in (2) for $Q = 2$ corresponding to the Ising model.

more allows to classify all the possible conformal boundary conditions [10]. However when analyzing the connectivity properties of the Ising FK clusters, the identification of $\phi_{1,3}$ with $\phi_{2,1}$ is no longer possible. According to the general theory [11], see also [17], the four-point function of $\phi_{1,3}$ satisfies a linear differential equation of degree 3. If we map the unit disk to the upper half plane $\mathbb{H}$ and call $z_1, \ldots, z_4$ the images on the real axis of the boundary points $w_1, \ldots, w_4$ we have

$$\langle \prod_{i=1}^4 \phi_{1,3}(z_i) \rangle_{\mathbb{H}} = \left[ \frac{z_{42}z_{31}z_{21}z_{34}}{z_{21}z_{34}z_{32}z_{14}} \right]^{2h_{1,3}} F(\eta); \quad (3)$$

where $z_{ij} = z_i - z_j$ and $\eta = \frac{z_{21}z_{34}z_{32}z_{14}}{z_{21}z_{34}z_{32}z_{14}}$ is the harmonic ratio ($0 < \eta < 1$). For the Ising model, the function $F(\eta)$ is the solution of the differential equation [50]

$$[2\eta(1-\eta)^2 F'' - 3(1 - \eta + \eta^2) F' + 3(2\eta - 1) F = 0. \quad (4)$$

The equation (4) has three linear independent solutions $F_{1,1}(\eta), F_{1,3}(\eta), F_{1,5}(\eta)$. The behaviour for small $\eta$ of each of the functions $F_{r,s}$ is of the form $\eta^{h_{r,s}}$ and the exponent $h_{r,s}$ coincides with the scaling dimension of the field $\phi_{r,s}$ that is produced in the operator product algebra [11]. $\phi_{1,3} \times \phi_{1,3} = \phi_{1,1} + \phi_{1,3} + \phi_{1,5}$. Although there is not a general procedure to solve the differential equation (4), we can proceed as follows. Firstly, we observe that function $F_{1,1}(\eta)$ has to coincide apart from the prefactor in (3) with the four point function of the boundary spin $\sigma$ and such a function [11] is the simple monodromy invariant [51] polynomial $1 - \eta + \eta^2$. It is also easy to understand what $F_{1,1}$ should be in terms of connectivities. Since the four-point function of the boundary spin operator can be fully defined in the minimal Ising model, it has to correspond to the unique partition function that requires only two colors to be construct, namely $\langle \phi_{(21)}(w_1)\phi_{(12)}(w_2)\phi_{(21)}(w_3)\phi_{(12)}(w_4)\rangle$. This in turn, is proportional to the sum $P = P_{(1234)} + P_{(1423)} + P_{(12)(34)}$ of the three linearly independent connectivities. Using the known solution $F_{1,1}(\eta)$ we can reduce the degree of the differential equation (4) by substituting $F(\eta) = F_{1,1}(\eta) \int_0^\eta df G(f')$. The function $G(\eta)$ is finally obtained through a rational pull-back of the Gauss hypergeometric function [45], see also [50]. One gets two linearly independent solutions $G_{1,2}$ for $G(\eta)$, related by the transformation $\eta \to 1 - \eta$: $G_1(\eta) = f(\eta)$ and $G_2(\eta) = f(1 - \eta)$. The function $f(\eta)$ is

$$f(\eta) = \frac{p(\eta)E(\eta) + q(\eta)K(\eta)}{\sqrt{(1-\eta)\eta}}, \quad (5)$$

with $K(\eta)$ and $E(\eta)$ the elliptic integrals of first and second kind respectively, and $p(\eta)$ and $q(\eta)$ rational functions of $\eta$ [50]. The behaviour for small $\eta$ finally fixes, up to an overall constant, $F_{1,5}(\eta) = F_{1,1}(\eta) \int_0^\eta df f'(f')$. The third linear independent solution to (4) can be chosen to be $F_{1,5}(1-\eta)$ that is actually a linear combination [49] of

\[ \text{FIG. 2. On the left, schematic representation of allowed dual FK clusters (dashed blue curves) in the Potts model when boundary conditions that fix the values of the dual boundary spins to 1, 2, 3 and 4 are chosen. On the right, the Kac table, obtained from the scaling dimension } h_{r,s} \text{ in (2) for } Q = 2 \text{ corresponding to the Ising model.} \]
all the $F_i$’s. Coming back to the connectivities we observe that in the limit $w_1 \rightarrow w_2$, $P_{(14)(23)}$ contains configurations where two FK clusters are separated by a dual line. These configurations are realized by the insertion of the operator $\phi_{1,5}$ [52] at the boundary and it was argued in [44] that in this case logarithmic singularities should arise. We conjecture then the following identification for the universal probability ratio which we denote with $R(\eta)$

$$R(\eta) = \frac{P_{(14)(23)}(\eta)}{P_1(\eta)} = A \int_0^\eta d\eta f(\eta') \quad (6)$$

where the constant $A = [\int_0^1 d\eta f(\eta)]^{-1}$ is chosen to ensure that $R(1) = 1$. The conjecture [44] can be easily tested on an arbitrary geometry by applying a conformal mapping $z'$. Since all the dimensionful parameters in (3) cancel when computing (6), one has only to express $\eta$ in the new coordinates $z'$. Finally we observe that denoting $1 - \eta = \varepsilon$ one obtains [50] the small $\varepsilon$ expansion for the ratio in (6)

$$R = 1 - \varepsilon^{1/2}[a_0 + a_1 \varepsilon + a_2 \varepsilon^2(1 + b \log \varepsilon) + O(\varepsilon^3)]. \quad (7)$$

The logarithmic singularity arises from the mixing of the level two descendants of $\phi_{1,5}$ with the field $\phi_{1,5}$ that have at $c = 1/2$ the same conformal dimension $h_{1,5} = 5/2$. This is the first example where a logarithmic singularity is explicitly calculated in the context of the critical Ising model. The logarithmic behaviour in (7) has a completely different origin respect to the well-known logarithmic divergence of the specific heat at the critical temperature [29, 40]. It shows that the phenomenon of mixing of scaling fields and non-diagonalizability of the conformal dilation operator could arise potentially at any rational value of the central charge. A circumstance that was already recognized in [41, 42] and [38, 53–55] but for which in the Ising model no explicit result was available. In [56, 57] a possible source of logarithmic behaviour was also identified but appeared to be ruled out by numerical data.

**Numerical results**— Simulations have been carried out on the Ising model at the exactly known critical temperature on a triangular lattice in triangles of sides of lengths $L = 9, 17, 33, 65, 129, 257$ with open boundary conditions collecting a number of samples up to $10^{10}$. The random number generator employed is given in [59]. We implemented the efficient Swendsen-Wang algorithm [58] that provides direct access to the FK clusters [44].

In order to use our results for the upper half plane [9] in the triangle geometry a Schwarz-Christoffel is in order. Given a $z$ in $\mathbb{H}$ and a $z'$ belonging to the interior of an equilateral triangle with vertices $(-1, 1, i\sqrt{3})$ the mapping reads $z' = \frac{2\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{\sqrt{\pi}F_1\left(\frac{1}{2}, \frac{1}{2} ; \frac{3}{2} ; -z^2\right)}$, $2F_1$ being the Gauss hypergeometric function. In the simulations the three points $z'_1$, $z'_3$ and $z'_4$ have been fixed in the midpoint of each side while the point $z'_2$ takes any position on the boundary between $z'_1$ and $z'_3$. Since the problem is symmetric under rotation of $2\pi/3$ and $4\pi/3$ around the center of the triangle also the configurations obtained with these rotations have been measured to enhance the statistics. An example of the simulated system together with a realization of FK clusters is presented in Figure 3.

![FIG. 3. (Left) Triangular lattice ($L = 9$) with the four points $z'_1$, $z'_2$, $z'_3$ and $z'_4$ highlighted. (Right) a realization of FK clusters contributing to the probability $P_{(12)(34)}$.](image)

The ratios $P_{(12)(34)}/P_1, P_{(14)(23)}/P_1, P_{(1234)}/P_1$, because of the symmetry $P_{(12)(34)}(\eta) = P_{(14)(23)}(1 - \eta)$ are not independent and only one function suffices to specify all of them, that is $R(\eta)$ as defined in (6). In Figure 4 we show the simulation results together with the CFT prediction for $R$ for the four largest lattices. In the inset of Figure 4 we show the deviations from the exact result.

**Conclusion**— In this letter we have calculated the four point connectivities of FK clusters in the critical Ising model and show that they can display logarithmic branch cuts. This is a first explicit example where such type...
of singularities are determined exactly for a theory that also has a non-trivial sector belonging to the series of unitary minimal models. Previous exact CFT studies focused on percolation and SAW. Our findings are fully corroborated by numerical simulations, showing excellent agreement. Similar structures, expected in many other important two-dimensional models including critical percolation, fully deserve the attention of future investigations. It would be also of clear interest to analyze whether logarithmic singularities could be found in higher dimensions, for instance in the three-dimensional critical Ising model.

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SOME TECHNICAL DETAILS ON THE SOLUTION OF THE DIFFERENTIAL EQUATION

We give here an account on how the differential equation for the four point function of the field $\phi_{1,3}$ is solved. The field $\phi_{1,3}$ with scaling dimension $h_{1,3}$ is degenerate and has a null vector at level three. The highest weight state associated to such a field is $|h_{1,3}\rangle = \phi_{1,3}(0)|0\rangle$ and let us denote by $L_n$ the Virasoro algebra generators

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} \delta_{n+m,0} n(n^2-1),$$

being $c$ the central charge. The null vector $|\chi_{1,3}\rangle$ at level three of $|h_{1,3}\rangle$ can be determined by writing the level three Gram matrix generated by $L_{-1}|h_{1,3}\rangle$, $L_{-1}L_{-2}|h_{1,3}\rangle$, $L_{-3}|h_{1,3}\rangle$ and finding its kernel. One finds using the commutation relations (S1)

$$|\chi_{1,3}\rangle = \left[ L_{-3} - \frac{2}{h_{1,3}} L_{-1} + \frac{1}{(h_{1,3}+2)(h_{1,3}+1)} L_{-1}^3 \right] |h_{1,3}\rangle.$$  (S2)

Through the Ward identity, the action of the Virasoro algebra generators on the highest weight state can be converted into the action of differential operators on correlation functions of the primary field $\phi_{1,3}$ [1, 2]. In particular the operator $L_{-n}$ is equivalent to the linear differential operator

$$L_{-n} = -\sum_{i\neq 1} \frac{(1-n)h_{1,3}}{(z_j - z_i)^n} - \frac{1}{(z_j - z_i)^{n-1}} \frac{\partial}{\partial z_j};$$

(S3)

when acting on the four point function $\langle \phi_{1,3}(z_4)\phi_{1,3}(z_3)\phi_{1,3}(z_2)\phi_{1,3}(z_1) \rangle$. Global conformal invariance fixes the form of this four point function to be

$$\langle \phi_{1,3}(z_4)\phi_{1,3}(z_3)\phi_{1,3}(z_2)\phi_{1,3}(z_1) \rangle = \left[ \frac{z_{42}z_{31}}{z_{21}z_{43}z_{32}} \right]^{2h_{1,3}} F(\eta);$$

(S4)

where $z_{ij} = z_i - z_j$ and $\eta = \frac{z_{21}z_{43}}{z_{24}z_{31}}$ is the harmonic ratio. We can then plug the form (S4) into the differential equation obtained substituting (S3) into (S2). After factoring out the prefactor multiplying the function $F(\eta)$ in (S4) we can take advantage of global conformal invariance setting $z_1 = 0$, $z_3 = 1$ and $z_4 = \infty$; in those limits the harmonic ratio $\eta$ coincides...
with the coordinate $z_2$. Notice that when the four points are ordered on the real axis $z_1 < z_2 < z_3 < z_4$, the harmonic ratio $\eta$ can be taken in the domain $0 < \eta < 1$.

At $c = 1/2$ we have $h_{1,3} = 1/2$ and we can then derive the following differential equation for the function $F$

$$[2\eta(1-\eta)^2 F'' - 3(1-\eta+\eta^2) F' + 3(2\eta-1) F = 0.]$$  \hspace{1cm} (S5)

For other values of $c$ associated to the critical $Q$-state Potts model a similar equation can be derived, we leave this analysis for future work. It is easy to see that the polynomial $F_{1,1}(\eta) = 1 - \eta + \eta^2$ is a solution of (S5). Actually the same polynomial is a also a solution of the linear differential equation

$$3\eta(1-\eta) F''(\eta) - 2(1-2\eta) F'(\eta) - 2 F(\eta) = 0$$ \hspace{1cm} (S6)

that through (S4) analogously expresses the four-point function of the operator $\phi_{2,1}$ at $c = 1/2$. The other solution of (S6) is the Legendre $Q$ function of the second kind $Q_{1/3}^{5/3}(2\eta-1)$ that apparently has not yet a physical interpretation. Coming back to the equation (S5) and substituting $F(\eta) = F_{1,1}(\eta) \int_{0}^{\eta} d\eta' G(\eta')$ we can lower the degree obtaining a second order linear differential equation for the function $G(\eta)$

$$3(-1+2\eta+5\eta^2-14\eta^3+7\eta^4)G(\eta) + 4(1-\eta)^2 \eta^2 [3(2\eta-1)G'(\eta) + (1-\eta+\eta^2)G''(\eta)] = 0$$  \hspace{1cm} (S7)

The equation (S7) is a linear differential equation of degree 2 with coefficients that are polynomials with rational coefficients ($\in \mathbb{Q}[\eta]$) and it is of the general form recently analyzed by E. Imamoglu and M. Van Hoeij in [3]. In particular, in [3] the authors developed an algorithm to find (if existing) hypergeometric solutions for this class of differential operators. The algorithm looks for solutions that can be expressed as

$$A(\eta) \ 2F_1(a,b,c,f(\eta)) + B(\eta) \ 2F_1'(a,b,c,f(\eta)),$$  \hspace{1cm} (S8)

where $A, B$ are in general algebraic functions, $a, b, c \in \mathbb{Q}$ and the algebraic function $f$ is called the pull-back. One of author made available an implementation for Maple of the algorithm that can be freely downloaded at [4]. We ran then the code on the differential equation (S7) finding the two linearly independent [5] solutions $f(\eta)$ and $f(1-\eta)$, where

$$f(\eta) = \frac{16}{21\pi} \left( \frac{(2-\eta)(1+\eta)(-1+2\eta)E(\eta) + (2+\eta(-4+\eta+\eta^2))K(\eta))}{\sqrt{(1-\eta)\eta(1+(-1+\eta)^2)} \right)$$  \hspace{1cm} (S9)
with $K(\eta)$ and $E(\eta)$ the elliptic integrals of first and second kind respectively. Notice that $K(\eta) \propto 2F_1(1/2,1/2,1,\eta^2)$ whereas $E(\eta) \propto 2F_1(-1/2,1/2,1,\eta^2)$. It is likely that using Gauss contiguous relations one could express (S9) in the form considered in [3]; we don’t have pursued this calculation.

According to the main text the ratio $R(\eta)$ on the upper half plane can be expressed as

$$R(\eta) = A \int_0^\eta d\eta' f(\eta'),$$

(S10)

where the constant $A$ ensures $R(1) = 1$. If we set $\eta = 1 - \varepsilon$, we can rewrite

$$R(\eta) = 1 - A \int_0^\varepsilon d\eta' f(1 - \eta').$$

(S11)

In the limit $\varepsilon \to 0$ the integral is convergent and the domain of integration inside the radius of convergence of the series representation of the integrand. We can then integrate each term of the series for $f(1 - \eta')$ separately obtaining

$$R = 1 - A \varepsilon^{1/2} \left[ \frac{64}{21\pi} + \frac{16}{21\pi} \varepsilon - \frac{941 - 840 \log 2}{525\pi} \varepsilon^2 \left( 1 + \frac{210 \log \varepsilon}{941 - 840 \log 2} \right) + O(\varepsilon^3) \right],$$

(S12)

that shows explicitly the logarithmic singularity in the probability ratio. The coefficients in (S12) can be evaluated up to 5 digits: $R \simeq 1 - 1.1636 \varepsilon^{1/2} - 0.29090 \varepsilon^{3/2} + 0.26091 \varepsilon^{5/2} + 0.15272 \varepsilon^{5/2} \log \varepsilon$.

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[2] P. Di Francesco, P. Mathieu and D. Senechal, Conformal Field Theory, Springer (1997), ISBN: 0-387-94785-X.
[3] E. Imamoglu, M. van Hoeij, Computing Hypergeometric Solutions of Second Order Linear Differential Equations using Quotients of Formal Solutions and Integral Bases, arXiv:1606.01576, (2016).
[4] http://www.math.fsu.edu/ eimamogl/hypergeometricsols/.
[5] One can show that the Wronskian constructed out of these two functions is never zero in the interval $\eta \in [0, 1]$.