Singularity categories of toric Gorenstein varieties with isolated singularities

Xiaojun Chen∗, Leilei Liu†, and Jieheng Zeng‡

aSchool of Mathematics, Sichuan University, Chengdu, Sichuan Province, 610064 P.R. China
bSchool of Mathematics (Zhuhai), Sun Yat-sen University, Zhuhai, Guangdong Province, 519082 P.R. China

Abstract

We study certain toric Gorenstein varieties with isolated singularities which are the quotient spaces of generic unimodular representation spaces by the one-dimensional torus, or the product of the one-dimensional torus with a finite abelian group. Based on the works of Špenko and Van den Bergh [Invent. Math. 210 (2017), no. 1, 3-67] and Mori and Ueyama [Adv. Math. 297 (2016), 54-92], we show that the singularity categories of these varieties admit tilting objects, and hence are triangle equivalent to the derived categories of some finite dimensional algebras.

Keywords: singularity category, tilting object, non-commutative crepant resolution

MSC2020: 14A22, 16G50, 18G85, 32S20

Contents

1 Introduction 1

2 Non-commutative crepant resolutions 4

3 Generators of singularity categories 10

4 Proof of Theorem 1.1 22

1 Introduction

For a Noetherian graded Gorenstein algebra $S$ with singularities, its singularity category $D^g(S) := D^b(\text{grmod } S)/\text{Perf}(S)$ is a triangulated category, which reflects many properties of the singularities of $S$. For example, Buchweitz shows in [2] that $D^g(S)$ is triangulated equivalent to the stable category $\text{CM}^Z(S)$, which is the quotient category of the category of maximal Cohen-Macaulay modules over $S$ by the full subcategory of projective modules.

∗Email: xjchen@scu.edu.cn
†Email: liuleilei@mail.sysu.edu.cn
‡Email: zengjh6620163.com
In a series of papers \[11, 12, 13\], Orlov shows that the graded singularity categories are determined by the local properties of the singularities, and have a deep relationship with the Homological Mirror Symmetry conjecture, where the category of graded D-branes of type B with a homogeneous superpotential is equivalent to the singularity category \(D_{\text{gr}}^{\text{sg}}(A)\) for some commutative algebra \(A\). If \(S\) is a Gorenstein isolated singularity and we forget the gradings of the \(S\)-modules, then \(D_{\text{gr}}(S) \cong \text{CM}(S)\) is a Calabi-Yau triangulated category, which has fruitful homological properties and applications, which have been extensively studied in recent years.

For a triangulated category, the existence of a tilting object is very important since it replaces the abstract triangulated category by the concrete derived category of modules over some algebra; see, for example, \[4\] and references therein for more details. For the case of singularities categories, finding the tilting objects is also an interesting topic, which turns out to have deep relations to representation theory and algebraic geometry.

Recall that every affine toric Gorenstein variety is in the form \(\text{Spec}(\text{Sym}(W)^G)\), where \(W\) is a generic unimodular representation of an abelian reductive group \(G\), and \(G\) is the one-dimensional torus, or a finite abelian group, or the product of them (c.f. \[15\]). When \(G\) is a finite group, Iyama and Takahashi show in \[6\] that the corresponding singularity category has a tilting object. Later in \[9\], Mori and Ueyama generalize the above result to Noetherian Koszul AS-regular algebras.

However, for the case that \(G\) contains a torus, the singularity categories are considered very rarely in the literature. The difficulty lies in two folds. On one hand, there may be infinitely many non-isomorphic irreducible maximal Cohen-Macaulay modules over \(\text{Sym}(W)^G\), and we therefore cannot apply the McKay-type correspondence, such as in \[6\], to construct the corresponding quiver and hence the tilting object. On the other hand, the non-commutative crepant resolution of the quotient singularity is not a Koszul algebra in general, and thus the method of \[9\] cannot be used to this case directly, either. Nevertheless, we prove in this paper the following.

**Theorem 1.1.** Let \(k\) be an algebraically closed field of characteristic zero and \(S\) be an isolated affine toric Gorenstein singularity over \(k\). Suppose \(\text{Spec}(S)\) is the quotient of \(k^n\), for some \(n \in \mathbb{N}\), by

1. either the one-dimensional torus \(T\),
2. or the product of \(T\) with a nontrivial abelian finite group

via a generic unimodular representation. Then the singularity category \(D_{\text{gr}}^{\text{sg}}(S)\) admits a tilting object, where the grading of \(S\) is canonically induced from \(k[x_1, \cdots, x_n]\).

The main idea of the proof of the above theorem is as follows.

**Main idea of the proof.** First, we use the method of Špenko and Van den Bergh \[14\] Theorem 1.6.2] to construct a non-commutative crepant resolution (NCCR) \(\Lambda\) of \(R^G\). By the results of Iyama and Reiten \[5\] Lemma 3.6 & Theorem 3.7] and Wemyss \[19\] Theorem 4.6.3], \(\Lambda\) is an \((n - 1)\)-Calabi-Yau algebra, and hence is an Artin-Schelter (AS-) regular algebra of dimension \(n - 1\) with Gorenstein parameter \(n\) (see Proposition 2.12 below). Therefore by
Orlov [13, Theorem 16], $D^b(\text{tail } \Lambda)$ has a tilting object which is $\bigoplus_{i=-n+1}^0 \Lambda(i)$, where for an algebra $A$, tail $A := \text{grmod } A/\text{tors } A$ with tors $A$ being the full subcategory of grmod $A$ consisting of the graded torsion $A$-modules.

Second, we have the following diagram which is constructed in [9]:

\[
\begin{array}{cccccc}
D^b(\text{grmod } \Lambda) & \xrightarrow{(-)e} & D^b(\text{grmod } R^G) & \xrightarrow{\pi} & D^b(\text{grmod } \Lambda) \\
\downarrow{\pi} & & \downarrow{\pi} & & \\
D^b(\text{tail } \Lambda) & \xrightarrow{(-)e} & D^b(\text{tail } R^G) & \xrightarrow{\Phi}\xrightarrow{\nu} & D_{sg}(R^G) \\
\end{array}
\]

where $e$ is an idempotent of $\Lambda$ (see Notation 3.4 below), $\pi$ is the natural projection, $\mu$ is the projection from $D^b(\text{tail } R^G)$ to its right admissible subcategory $D_{sg}(R^G)$, and $\Phi$ and $\nu$ are the Orlov embedding and Verdier localization respectively. Moreover, there is an identity of functors

\[(-)e \circ \pi \cong \pi \circ (-)e.\]

We shall show that $R^G$ is a graded Gorenstein algebra (see Proposition 2.11 below). By Mori and Ueyama [10, Lemma 2.7], the algebra $\Lambda/\Lambda e \Lambda$ is finite dimensional if and only if the functor

\[(-)e : D^b(\text{tail } \Lambda) \to D^b(\text{tail } R^G)\]

is an equivalence. The isolatedness of the singularity ensures that the tilting objects in $D^b(\text{tail } \Lambda)$ can be transferred to $D^b(\text{tail } R^G)$ by functor $(-)e$ (see Lemma 3.5 below).

Now to prove Theorem 1.1 we improve the method of Mori and Ueyama in [9, Lemma 4.5]. Since $\Lambda$ is an AS-regular algebra of dimension $n - 1$ with Gorenstein parameter $n$, we add a summand $(1-e)\Lambda e$ to the direct sum $\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e$ obtained by using syzygies in loc cit. Then taking the right mutation of

\[\nu\left(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e\right)\]

with respect to $e\Lambda e$, we get the tilting object

\[\nu\left(R_{e\Lambda e}\left(\bigoplus_{i=1}^{n-1}(\Omega^i_{\Lambda}(1-e)(i))e \oplus (1-e)\Lambda e\right)\right).\]

This paper is organized as follows: In §2 we introduce some often used notations and recall some properties of AS-regular algebras, graded Gorenstein algebras and non-commutative crepant resolutions. In §3 we construct an tilting object of $D_{sg}(R^T)$. In §4 we prove Theorem 1.1 and also study an example with the tilting object explicitly given.

Notations. Throughout the paper, $k$ is an algebraically closed field of characteristic zero. All modules are right modules. The dimensions of all commutative rings and varieties are more than one unless otherwise stated.

Acknowledgements. This paper was completed when the authors were visiting IASM, Zhejiang University. We would like to thank Professor Yongbin Ruan for invitation and hospitality. This work is partially supported by NSFC (Nos. 11890660 and 11890663).
2 Non-commutative crepant resolutions

In this section, we go over the definition and some properties of the NCCR of singularities, which is introduced by Van den Bergh in [17, 18]. Some materials are also taken from [14, 15, 16].

2.1 Basics of NCCR

Definition 2.1 (Van den Bergh). Let $R$ be a Gorenstein normal domain. A non-commutative crepant resolution (NCCR) of $R$ is an algebra of the form $\text{End}_R(M)$ for some reflexive $R$-module $M$ such that

1. $\text{End}_R(M)$ is a Cohen-Macaulay $R$-module, and
2. the global dimension of $\text{End}_R(M)$ is finite.

Later Wemyss in [19] replaces Condition (2) in the above definition by that the global dimension of $\text{End}_R(M)$ is equal to the Krull dimension of $R$. However, if $R$ is an equicodimensional Gorenstein normal domain, then both definitions coincide.

Definition 2.2. Let $G$ be a reductive group, $T$ be a maximal torus of $G$ and $V$ be a finite dimensional representation of $G$. The representation is said to be quasi-symmetric if for every line $\ell \subset X(T) \otimes \mathbb{Z} \mathbb{R}$ through the origin, we have

$$\sum_{\alpha_i \in \ell} \alpha_i = 0,$$

where $X(T) := \text{Hom}(T, \mathbb{G}_m)$ is the weights of $T$, that is, the set of group homomorphisms of $T$ to $\mathbb{G}_m$.

Definition 2.3. Let $G$ be an algebraic group and $X$ be a smooth affine variety with an action of $G$. We say $G$ acts generically on $X$ if the action satisfies that:

1. $X$ contains a closed point with closed orbit and its stabilizer is trivial, and
2. if $X^s \subseteq X$ is the set of points satisfying (1), then $\text{codim}(X \setminus X^s) \geq 2$.

Let $V$ be an $n$-dimensional representation of $T$. Then $T$ acts on the graded polynomial ring $R := k[x_1, x_2, \cdots, x_n]$ naturally, where $\deg(x_i) = 1$. We denote the weights of the action of $T$ on $R$ by

$$\alpha_T = (\chi_1, \chi_2, \cdots, \chi_n) \in \mathbb{Z}^n.$$

Definition 2.4. We say $\chi := (\chi_1, \chi_2, \cdots, \chi_n) \in \mathbb{Z}^n$ is effective if

1. there are at least two positive terms and two negative terms in $\chi$,
2. $\sum_{i=1}^n \chi_i = 0$, and
3. $\gcd(\chi_i, \chi_j) = 1$, for any $1 \leq i, j \leq n$ such that $\chi_i \chi_j < 0$.

Lemma 2.5. If the weights of the action of $T$ on $R := k[V]$ is effective in the sense of Definition 2.4, then $R^T$ is a Gorenstein algebra.
Proof. In [14, §1.6], Spenko and Van den Bergh show that if the action of $T$ on $\text{Spec}(R)$ satisfies that:

1) $G$ acts generically on $\text{Spec}(R)$, and

2) the natural action of $T$ on the canonical bundle of $\text{Spec}(R)$ is trivial,

then $R^T$ is a Gorenstein algebra. Thus, to prove the lemma, it suffices to show the action of $T$ on $R$ satisfies these two conditions.

First, choose a closed point $x \in \text{Spec}(R)$ such that there is at most one coordinate is zero; it is direct to check that the orbit of $x$ is closed and by Definition 2.4 (3) the stabilizer of $x$ is trivial.

Second, the triviality of the action of $T$ on the canonical bundle follows from Definition 2.4 (2).

Now, let $\Delta$ be a bounded closed interval on $\mathbb{R}$, and $\Delta_x$ be obtained from $\Delta$ by removing the left endpoint.

Let $G$ be a reductive group that acts on $k^n$ and and a finite dimension vector space $W$. Denote by $M_R^G(W)$ the $R^G$-module $(W \otimes R)^G$, where $R := k[x_1, x_2, \ldots, x_n]$ is the coordinate ring of $k^n$. Now, let $(\beta^i_n)_{i=1}^{n}$ be the $T$-weights of $W$, where $T$ is a maximal torus of $G$. Set

$$\Sigma := \left\{ \sum_i a_i \beta_i \middle| a_i \in (-1, 0] \right\} \subset X(T)_R := X(T) \otimes_\mathbb{Z} \mathbb{R}.$$

The following is proved in [14, Theorem 1.19]:

**Proposition 2.6.** Let $G = T$ be a one-dimensional torus and $W$ be a finite dimensional representation of $G$, which is quasi-symmetric and generic. Let

$$\mathcal{L} = X(T) \cap (1/2)\Sigma_c, \quad U = \bigoplus_{\chi \in \mathcal{L}} V_\chi,$$

where $V_\chi$ is the representation of $G$ with weight $\chi$. Then $\text{End}_{R^G}(M)$ is an NCCR of $R^G$, where $M := M_R^G(U)$ and $R = k[W]$.

In what follows, we use $\Lambda$ to denote the NCCR of $R^G$ constructed above.

The following theorem is contained in [3, Theorem 3.7] and is explicitly stated in [19, Theorem 4.6.3], which says that $D^b(\Lambda)$ is a Calabi-Yau triangulated category:

**Theorem 2.7.** If $R$ is an equicodimensional Gorenstein normal domain over $k$, and $\Lambda$ is an NCCR of $R$, then we have that

$$\text{Hom}_{D^b(\Lambda)}(X,Y) \cong D\text{Hom}_{D^b(\Lambda)}(Y,X)$$

for all $X \in D^b(\Lambda)$ and $Y \in D^b(\Lambda)$, where $D^b(\Lambda)$ is the full subcategory of $D^b(\Lambda)$ consisting of objects whose homologies are finite dimensional, and $D(-) := \text{Hom}_k(-, k)$. 

5
Going back to Proposition 2.6, \( k^\oplus L \) is equipped with a \( \Lambda \)-module structure given by the algebra homomorphism \( p \) such that

\[
\begin{array}{c}
\oplus_{\chi \in \Lambda} \text{End}_R(M(V_\chi)) \\
\end{array}
\]

\[
\begin{array}{c}
k^\oplus L \\
\end{array}
\]

is commutative, where

\[
\varphi_\chi : \text{End}_R(M(U)) \longrightarrow \text{End}_R(M(V_\chi))
\]

is the canonical projection from \( \text{End}_R(M(U)) \) to \( \text{End}_R(M(V_\chi)) \), and

\[
\eta_\chi : \text{End}_R(M(V_\chi)) \rightarrow k
\]

is the canonical projection from \( \text{End}_R(M(V_\chi)) \) to \( k \). That is,

\[
p = \left( \bigoplus_{\chi \in \Lambda} \eta_\chi \right) \circ \left( \bigoplus_{\chi \in \Lambda} \varphi_\chi \right).
\]

Plugging \( X = k^\oplus L \) and \( Y = \Lambda = \text{End}_R(M(U)) \) in Theorem 2.7, we have the following corollary.

**Corollary 2.8.** Under the conditions of Theorem 2.7, we have

\[
\text{Hom}_{D^b(\Lambda)}(k^\oplus L, \Lambda) \cong D\text{Hom}_{D^b(\Lambda)}(\Lambda, k^\oplus L).
\]

### 2.2 AS-regular algebras

NCCRs have a close relationship with Artin-Schelter regular algebras, whose definition we now recall.

Suppose \( A \) is a graded associative \( k \)-algebra. For \( M \in \text{grmod} A \) a finitely generated graded \( A \)-module, we write \( M = \bigoplus_i M_i \) as graded vector spaces, where \( M_i \) is the degree \( i \) component of \( M \). Let \( A(j) \) be the graded \( A \)-module such that \( A(j)_i = A_{i+j} \). For any \( M, N \in \text{grmod} A \), we denote

\[
\text{Ext}^i_A(M, N) := \bigoplus_j \text{Ext}^i_{\text{grmod} A}(M, N(j)),
\]

for \( i \in \mathbb{Z} \).

**Definition 2.9.** Suppose \( A \) is graded algebra with \( A_0 \) semi-simple over \( k \). We say \( A \) is an **Artin-Schelter (AS-) regular** algebra of dimension \( d \) with Gorenstein parameter \( a \) if the following conditions hold:

1. the global dimension \( \text{gldim}(A) = d \), and
2. there is an isomorphism

\[
\text{Ext}^i_A(A_0, A) \cong \begin{cases} 
D(A_0)(a), & i = d, \\
0, & i \neq d
\end{cases}
\]

in \( \text{grmod}(A_0) \), where \( D(-) := \text{Hom}_k(-, k) \) as before.
If an algebra $A$ satisfies (2) in above definition and has finite injective dimension in $\text{grmod}(A)$, then we call it a graded Gorenstein algebra of dimension $d$ with Gorenstein parameter $a$.

Suppose $A$ is a Noetherian Gorenstein algebra of global dimension $d$. If we endow $A$ with a $\mathbb{Z}$-grading such that $A_0$ is finite dimensional, one may ask whether $A$ is an AS-regular algebra of dimension $d$ with Gorenstein parameter $a$, for some $a \in \mathbb{Z}$. We have the following.

**Proposition 2.10.** Let $A$ be a unital Noetherian algebra of global dimension $d$ over a finite dimensional $k$-algebra $K$ with an augmentation map $\varepsilon : A \to K$ such that $A$ satisfies the Gorenstein condition

$$\text{Ext}^i_A(K, A) \cong \begin{cases} K, & i = d, \\ 0, & i \neq d. \end{cases}$$

If we endow $A$ with a $\mathbb{Z}$-grading such that $A_0 = K$ and $A_j = 0$, for $j < 0$, then there is a decomposition

$$A_0 \cong \bigoplus_{j=1}^r A_j^0$$

in $\text{grmod}(A_0)$ and a sequence of numbers $(a_1, a_2, \ldots, a_r)$ in $\mathbb{Z}$ such that $A_j^0 \cong k$ as vector spaces, and

$$\text{Ext}^i_A(A_j^0, A) \cong \begin{cases} A_j^0(a_j), & i = d, \\ 0, & i \neq d \end{cases}$$

in $\text{grmod}(A_0)$.

**Proof.** Let $P^\bullet$ be a projective resolution of $A_0$ in $\text{grmod}(A)$. Then $P^\bullet$ is bounded with length at most $d$ by [3, Theorem 12]. Now we view $P^\bullet$ as a projective resolution of $A_0$ in $\text{mod}A$, which is denote by $\widetilde{P^\bullet}$. Then we have

$$\text{RHom}^*_D(A_0, A) \cong \text{RHom}^*_D(\widetilde{P^\bullet}, A)$$

$$\cong \bigoplus_i \text{Ext}^*_D(P_i, A)$$

$$\cong \bigoplus_i \text{Ext}^*_\text{grmod} A(P_i, A(i))$$

$$\cong \bigoplus_i \text{Ext}^*_\text{grmod} A(A_0, A(i)),$$

in $\text{grmod}A_0$. Since $A$ satisfies the Gorenstein condition for $A_0$ and there is a decomposition

$$A_0 \cong \bigoplus_{j=1}^r A_j^0,$$

then there exists a sequence numbers $(b_1, b_2, \ldots, b_r)$ in $\mathbb{Z}$ such that

$$\text{Ext}^d_A\left( \bigoplus_{j=1}^r A_j^0 \bigoplus_{i=1}^r A(h_i) \right) \cong \bigoplus_{j=1}^r A_j^0.$$
in grmod$_A$. Since above isomorphism is a morphism in grmod$_A$, we have

$$\text{Ext}^d_A(A^j_0, A(b_j)) \cong A^j_0,$$

where the number $b_j \in (b_1, b_2, \ldots, b_r)$ corresponds to $A^j_0$. □

**Proposition 2.11.** With the setting of Proposition 2.6, we have that $R^T$ is a Noetherian graded Gorenstein algebra of dimension $n - 1$ with Gorenstein parameter $n$.

**Proof.** First, it is obvious that

$$\text{dim}(R^T) = \text{dim}(\text{Spec}(R^T)) = \text{dim}(\text{Spec}(R)) - \text{dim}(T) = n - 1.$$

By Orlov [13, Corollary 25 & Proposition 28], which say that the Gorenstein parameter of $R^T$ is equal to the number $r \in \mathbb{N}$ such that $O_X(-r) \cong \omega_X$, where $X = (\text{Spec}(R^T) \setminus \{0\})/\mathbb{G}_m$, and the action of $\mathbb{G}_m$ on $\text{Spec}(R^T)$ is given by the grading on $R^T$. Thus to prove the proposition, it suffices to prove that $O_X(-n) \cong \omega_X$, which is equivalent to showing that the degree of the canonical divisor of $X$ is $n$.

Observe that $X$ is isomorphic to the projective variety $((\text{Spec}(R) \setminus \{0\})/\mathbb{G}_m) \setminus H$ by $T$, where $H = H_1 \cup H_2$, where

$$H_1 := \{x = (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1} \mid x_i = 0 \text{ for } x_i < 0\},$$

and

$$H_2 := \{x = (x_1 : x_2 : \cdots : x_n) \in \mathbb{P}^{n-1} \mid x_i = 0 \text{ for } x_i > 0\}.$$ Denote $((\text{Spec}(R) \setminus \{0\})/\mathbb{G}_m) \setminus H$ by $Y$; then $\omega_Y = i^*\omega_{\mathbb{P}^{n-1}}$, where $i : Y \hookrightarrow \mathbb{P}^{n-1}$ is the embedding. Hence the degree of canonical divisor of $Y$ is $n$. Moreover, the canonical divisor of $Y$ can be written by the section $s_Y = x_1x_2\cdots x_n$ of $\omega_Y$.

By [13] Lemma 4.7, which says that

$$\omega_X \cong \omega_Y/T \cong t \otimes \omega_Y$$

in the category $\text{Coh}(Y, G)$ of $G$-equivariant coherent sheaves, where $t$ is the Lie algebra of $T$, we have $\omega_X = \rho_*\omega_Y$, where $\rho : Y \to X$ is the quotient morphism given by $T$. Thus, there is a lift of $\rho$ to the total space of the canonical line bundle, which is also a quotient morphism given by $T$, such that

$$\tilde{\rho} : \omega_Y \to \omega_X.$$

Since $\tilde{\rho} \circ s_Y$ is a $T$-equivariant morphism from the fact that the action of $T$ on vector space $k(x_1x_2\cdots x_n)$ is trivial, there is a section $s_X$ of $\omega_X$ such that the following diagram

$$\begin{array}{ccc}
Y & \xrightarrow{s_Y} & \omega_Y \\
\rho \downarrow & & \downarrow \tilde{\rho} \\
X & \xrightarrow{s_X} & \omega_X
\end{array}$$

is commutative. Thus the degree of the section $s_X$ of $\omega_X$ is $n$ and therefore the degree of the canonical divisor of $X$ is $n$. □
Now, recall that $\Lambda$ is the NCCR of $R^G$ by Proposition 2.6; we have the following.

**Proposition 2.12.** $\Lambda$ is an AS-regular algebra of dimension $n - 1$ with Gorenstein parameter $n$.

**Proof.** By Theorem 2.7 and Propositions 2.10 and 2.11 we only need to prove

$$a_1 = a_2 = \cdots = a_r = n,$$

where $(a_1, a_2, \cdots, a_r)$ is given in Proposition 2.10.

Let $\Lambda$ be the graded algebra $A$ in Proposition 2.10, and $P^•_i$ be the projective resolution of $\Lambda^i_0$ of graded $\Lambda$-modules.

From the proof of Proposition 2.10, we know that

$$\text{Ext}^{n-1}_A(\Lambda^i_0, \Lambda) \cong \Lambda^i_0(a_i).$$

Thus, we obtain that $(P^•_i)^{n-1} = e_i\Lambda(-a_i)$, where $e_i$ is the idempotent corresponding to $\Lambda^i_0$. Moreover, since $(P^•_i)^{i} \in \text{add}(\Lambda)$ for any $i$, $(P^•_i)^{i}e_i$ is a Cohen-Macaulay $R^T$-module. Observe that when $S$ is a Gorenstein algebra, $M$ is a Cohen-Macaulay $R^T$-module if and only if

$$\text{Hom}_{D^b(S)}(M, S) = 0,$$

for any $i \neq 0$. Hence we have an injective morphism

$$k(a_i) \cong \text{Ext}^{n-1}_A(\Lambda^i_0, \Lambda) \cong \text{Ext}^{n-1}_A(\Lambda^i_0, e_i\Lambda) \cong \text{Hom}^{n-1}_{C^b(\text{grmod}\, e_i\Lambda_i)}(P^•_i, e_i\Lambda) \hookrightarrow \text{Hom}^{n-1}_{C^b(\text{grmod}\, e_i\Lambda_i)}(P^•_i e_i, e_i\Lambda e_i)$$

and an isomorphism:

$$\text{Hom}^{n-1}_{C^b(\text{grmod}\, e_i\Lambda_i)}(P^•_i e_i, e_i\Lambda e_i) \cong k(n)$$

in $\text{grmod}(k)$. From these two morphisms we get that $a_i = n$. \qed

We next state several facts that will be used in the rest of this paper.

First, consider the two categories $\text{mod}(R^G)$ and $\text{mod}(R,G)$, where $\text{mod}(R,G)$ is the category of $G$-equivariant $R$-modules. In general, $\text{mod}(R^G)$ and $\text{mod}(R,G)$ are not equivalent. However, in the case of reflexive modules, we have following lemma.

**Lemma 2.13** ([14], Lemma 3.3). Under the conditions of Theorem 2.7, assume moreover that $G$ acts generically on $k^n$. Let $\text{ref}(G,R)$ be the category of $G$-equivariant $R$-modules which are reflexive as $R$-module and $\text{ref}(R^G)$ be the category of the reflexive $R^G$-modules. Then the functors

$$\text{ref}(G, R) \rightarrow \text{ref}(R^G) : M \mapsto M^G,$$

$$\text{ref}(R^G) \rightarrow \text{ref}(G, R) : N \mapsto (R \otimes_{R^G} N)^{**}$$

are mutually inverse equivalences between the two symmetric monoidal categories.

9
Thus, by above lemma, we obtain that
\[
\text{Hom}_{\text{mod}(R^G)}(M_R^G(V_1), M_R^G(V_2)) \\
\cong \text{Hom}_{\text{mod}(R,G)}(R \otimes V_1, R \otimes V_2) \\
\cong M_R^G(\text{Hom}_k(V_1, V_2))
\]
in mod(R^G).
Second, we know that
\[
M_R^G(V) = (V \otimes R)^G \cong ((V^*)^* \otimes R)^G \cong \text{Hom}_{\text{Rep}(G)}(V^*, R)
\]
for any V ∈ Rep(G), where Rep(G) is the category of representations of G. If V is an irreducible representation of G, then the elements of M_R^G(V) is in one to one correspondence with the irreducible sub-representations of G in R which are isomorphic to V^* in Rep(G). Moreover, it is direct to check that the correspondence is preserved under the action of R^G on both sides, where the action of R^G on the sub-representations of G in R is given by
\[
(f, W) \mapsto k(f) \otimes W \in \text{Rep}(G)
\]
for any f ∈ R^G, W ∈ Rep(G), and k(f) is the one-dimensional vector space spanned by f.

3 Generators of singularity categories

In this section, we study the generator of the singularity category of RT. For RG where G is the product of T with a finite abelian group, the argument is similar and will be studied in §4.2. The main results of this section are Propositions 3.9 and 3.17.

3.1 Existence of the generators

Let us start with the following lemma, which is due to Mori and Ueyama:

Lemma 3.1 (\cite{9} Lemma 3.17). Let A be a Noetherian graded algebra and e is an idempotent of A such that eAe is also a Noetherian graded algebra and Ae ∈ grmod e. Then eA is a finitely generated left eAe-module and A/⟨e⟩ ∈ tors A if and only if
\[
- \otimes_A^{L} eA : \text{tail } A \rightarrow \text{tail } eAe
\]
is an equivalence functor.

The following result also appears in \cite{10}, which we state as a lemma:

Lemma 3.2. Let A be a Noetherian AS-regular algebra, and e be an idempotent of A such that eAe is a Noetherian graded algebra. If (−)e : tail A → tail eAe is an equivalence functor, then so is (−)e : D^b(tail A) → D^b(tail eAe).

Proof. Since (−)e : tail A → tail eAe is an equivalence functor of abelian categories, we have that
\[
(−)e : C^b(\text{tail } A) \rightarrow C^b(\text{tail } eAe)
\]
is also an equivalence functor, where \( C^b(\text{tail} \, A) \) and \( C^b(\text{tail} \, eAe) \) are the bounded chain complex categories of \( \text{tail} \, A \) and \( \text{tail} \, eAe \) respectively. Thus to prove the lemma it suffices to prove that \( M^* \in C^b(\text{tail} \, A) \) is exact if and only if \( M^*e \in C^b(\text{tail} \, eAe) \) is exact.

If \( M^* \in C^b(\text{tail} \, A) \) is exact, it is easy to check that \( M^*e \in C^b(\text{tail} \, eAe) \) is also exact since \((-)e \cong (-) \otimes_A Ae \) is an exact functor on \( \text{grmod} A \).

Conversely, without loss of generality, we assume that \( N_* \) is a short exact sequence

\[
0 \longrightarrow N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \longrightarrow 0.
\]

After tensoring \( N_* \) with \( eA \), we obtain an exact sequence

\[
0 \longrightarrow \ker(f \otimes_A eA) \longrightarrow N_1 \otimes_A eA \xrightarrow{f_1 \otimes_A eA} N_2 \otimes_A eA \xrightarrow{f_2 \otimes_A eA} N_3 \otimes_A eA \longrightarrow 0
\]

since the functor \((-) \otimes_A eA \) is right exact on \( \text{grmod}(eA) \). Since

\[
(N_* \otimes_A eA) \otimes_A eA \cong N_*
\]

in \( C(\text{tail} \, eAe) \), we have

\[
(\ker(f \otimes_A eA)) \otimes_A eA \cong 0
\]

in \( \text{tail} \, eAe \). Therefore we obtain that

\[
\ker(f \otimes_A eA) \cong 0
\]

in \( \text{tail} \, A \) since \((-)e \) is an equivalent functor in \( \text{tail} \, A \). Thus

\[
0 \longrightarrow N_1 \otimes_A eA \xrightarrow{f_1 \otimes_A eA} N_2 \otimes_A eA \xrightarrow{f_2 \otimes_A eA} N_3 \otimes_A eA \longrightarrow 0
\]

is an exact sequence in \( C(\text{tail} \, A) \).

**Definition 3.3** ([10] Definition 1.1). A right Noetherian graded algebra \( A \) is called a graded isolated singularity if \( \text{gldim}(\text{tail} \, A) < \infty \), where

\[
\text{gldim}(\text{tail} \, A) := \sup \left\{ i \left| \text{Hom}^i_{D^b(\text{tail} \, A)}(M, N) \neq 0, M, N \in \text{tail} \, A \right. \right\}.
\]

Next we recall a result of Mori and Ueyama obtained in [10] Theorem 3.10], which is stated as follows: suppose a finite group \( G \) acts on an AS-regular algebra \( S \) of dimension \( d \geq 2 \) and suppose the action has homological determinant one; then \( (S \ast G)/(e) \) is finite dimensional over \( k \), where \( e = \frac{1}{|G|} \sum_{g \in G} 1 \ast g \), if and only if \( S^G \) is a graded isolated singularity and

\[
\psi : S \ast G \rightarrow \text{End}_{SG}(S); \quad s \ast g \mapsto \{ t \mapsto sg(t) \}
\]

is an isomorphism of graded algebras.

**Notation 3.4.** From now on, let \( e \) denote the idempotent of \( \Lambda \) which corresponds to the summand \( M^G_l(V_l) \), where \( l \) is the minimal number in \( \mathcal{L} \) constructed in Proposition 2.6 in the non-commutative crepant resolution \( \Lambda \).

The following lemma generalizes the above result of Mori and Ueyama to the case where \( G = T \) is the one-dimensional torus.
Lemma 3.5. Let $R = k[x_1, x_2, \ldots, x_n]$ and $T$ be the one-dimensional torus acting on $R$ with weights $\chi = (\chi_1, \chi_2, \cdots, \chi_n)$. If $R^T$ satisfies the first and second conditions of Definition 2.4 and is a graded Gorenstein algebra with the canonical grading, then the followings are equivalent:

1. $\chi$ is effective;
2. the functor $(-)_e : \text{tail} \Lambda \to \text{tail} R^T$ is an equivalence functor with inverse $(-) \otimes_{e\Lambda} e\Lambda$;
3. $R^T$ is graded isolated singularity;
4. $\text{Spec}(R^T)$ has a unique isolated singularity at the origin in usual sense.

Proof. From (2) to (1): If $R^T$ does not satisfy the third condition in Definition 2.4 without loss of generality we may assume that $\gcd(\chi_1, \chi_2) = h > 1$. From the construction of $\Lambda = \text{End}_{\mathcal{R}}(\bigoplus_{\lambda \in \mathcal{L}} M^T_{\mathcal{R}}(V_{\lambda}))$ in Proposition 2.6, we may choose an indecomposable element $e' \in \Lambda_0$ associated to the summand $M^T_{\mathcal{R}}(V_{l+1})$, where $l$ is the minimal number in $\mathcal{L}$.

Assume that $\chi_1 > 0 > \chi_2$. Since $x_1^{-\chi_2} x_2^{\chi_1} \in R^T \subset \Lambda$, we have that $e'(x_1^{-\chi_2} x_2^{\chi_1})^N \in e'R^T \subset \Lambda$, for any $N > 0$.

Now, fix $N > 0$. Since $e'(x_1^{-\chi_2} x_2^{\chi_1})^N = e'(x_1^{-\chi_2} x_2^{\chi_1})^N e'$,

$$e'(x_1^{-\chi_2} x_2^{\chi_1})^N \in e\Lambda$$

if and only if there are two elements $f_1 \in e'\Lambda e \cong (R \otimes V_{-1})^T$ and $f_2 \in e\Lambda e' \cong (R \otimes V_1)^T$ such that

$$f_1 f_2 = e'(x_1^{-\chi_2} x_2^{\chi_1})^N.$$

On the other hand

$$f_1 f_2 = e'(x_1^{-\chi_2} x_2^{\chi_1})^N$$

if and only if there are two numbers $\alpha, \beta \in \mathbb{Z}$ such that $-\chi_2 N \geq \alpha \geq 0$, $\chi_1 N \geq \beta \geq 0$ and $f_2 = e x_1^\alpha x_2^\beta e'$, which is equivalent to that

$$\chi_1 \alpha + \chi_2 \beta = -1$$

since $k(f_2) \cong V_{-1}$. However, we have assumed that $\gcd(\chi_1, \chi_2) = h > 1$, which is a contradiction.

Thus we can choose infinity many $N$ making

$$e'(x_1^{-\chi_2} x_2^{\chi_1})^N \notin e\Lambda.$$

It is contradiction to the fact that $\Lambda/e\Lambda$ is a finite dimensional vector space.

From (3) to (2): Since $R^T$ is a Noetherian graded Gorenstein algebra, the result follows from [10, Theorem 2.5].

From (4) to (3): Suppose Spec($R^T$) has a unique isolated singularity at the origin. Then Spec($R^T \setminus \{0\}$) is smooth, and so is (Spec($R^T \setminus \{0\}$)/$G_m$ by Luna slice theorem and the proof of [14, Lemma 3.8], which says that for any closed point $p \neq \{0\} \in \text{Spec}(R^T)$, suppose it is the image of $\tilde{p} \in \text{Spec}(R)^*$, then there are two étale morphisms

$$S//T_{\tilde{p}} \to N_{\tilde{p}}//T_{\tilde{p}}.$$
and

\[ S//G_\tilde{p} \rightarrow \text{Spec}(R)/\text{T} \cong \text{Spec}(R^T), \]

where \( T_\tilde{p} \) is the stabilizer of \( \tilde{p} \), \( N_\tilde{p} \) is a \( T_\tilde{p} \)-invariant complement to the inclusion of \( T_\tilde{p} \)-representations \( T_\tilde{p}(T)/T_\tilde{p}(\tilde{p}) \subseteq T_\tilde{p}(\text{Spec}(R)) \), and \( S \) is the affine \( T_\tilde{p} \)-invariant slice to the \( T \)-orbit of \( \tilde{p} \). Thus by the fact that the category of coherent sheaves \( \text{Coh}(\text{Spec}(R^T) \setminus \{0\})/\mathbb{G}_m \cong \text{tail } R^T \), we have \( \text{gldim}(\text{tail } R^T) < \infty \).

From (1) to (4): By Luna slice theorem and the proof of [14, Lemma 3.8] as above, \( \tilde{p} \) is not singular in \( \text{Spec}(R^T) \) if and only if the origin is not singular in \( N_\tilde{p}//T_\tilde{p} \), which is equivalent to that \( T_\tilde{p} \) is trivial. Since \( \tilde{p} = (x_1, x_2, \cdots, x_n) \in \text{Spec}(R)^\times \), then there are \( x_i, x_j \in (x_1, x_2, \cdots, x_n) \) such that \( \chi_i \chi_j < 0 \). Therefore \( \text{gcd}(\chi_i, \chi_j) = 1 \) by effectiveness. Thus for any \( t \in T_\tilde{p} \), we have \( t^{x_i} = 1 \) and \( t^{x_j} = 1 \), from which we obtain that \( t = 1 \in T \). Therefore \( T_\tilde{p} \) is trivial.

Suppose \( \chi \) is effective as above. Since \( \Lambda \) is an AS-regular algebra and a Cohen-Macaulay \( R^T \)-module, \( (1 - e) \) has a bounded projective resolution, denoted by \( P^e \), of graded \( \Lambda \)-modules with length \( n \) such that \( P^{1-n} = (1-e)\Lambda(-n) \), \( P^0 = (1-e)\Lambda \), \( P^i \in \text{add}_{j \in [-n+1,-1]}(\bigoplus e_i \Lambda(j)) \) for \( -n + 1 \leq i \leq -1 \), and it is given by the left approximation by graded projective \( \Lambda \)-modules.

Now we fix this resolution and denote by \( \Omega^e_\Lambda(1-e) \) the \( i \)-times syzygy of \( 1-e \) on \( P^e \).

**Notation 3.6.** Let \( Q \) be the following set of objects in \( D^g_{sg}(R^T) \):

\[ Q := \{ \nu((1-e)\Lambda e), \nu(\Omega^e_\Lambda(1-e)\Lambda e(1)), \nu(\Omega^e_\Lambda(1-e)\Lambda e(2)), \cdots, \nu(\Omega^e_\Lambda(1-e)\Lambda e(n-1)) \}, \]

and denote by \( E_Q \) the direct sum of the objects in \( Q \).

The following two lemmas are due to Mori and Orlov respectively, which will be used soon.

**Lemma 3.7 (77 Lemma 2.9).** Let \( A \) be a right Noetherian graded algebra with \( A_0 \) semi-simple over \( k \). Then the natural morphism

\[ \text{Hom}_{\text{grmod } A}(M, N) \rightarrow \text{Hom}_{\text{tail } A}(M, N) \]

is an isomorphism of vector spaces for any \( N, M \in \text{grmod } A \) with \( \text{depth}(N) \geq 2 \).

**Lemma 3.8 ([13] Theorem 16).** Suppose \( A \) is a Gorenstein algebra of dimension \( d \) with positive Gorenstein parameter \( a \). If \( A \) is Noetherian, then there is a fully faithful functor \( \Phi : D^g_{sg}(A) \rightarrow D^b(\text{tail } A) \), and a semi-orthogonal decomposition

\[ D^b(\text{tail } A) = \langle \pi A(a+1), \pi A(a+2), \cdots, \pi A, \Phi D^g_{sg}(A) \rangle. \]

Note that the functor \( \Phi \) in above lemma is \( \Phi_0 \) in [13].

Now let us remind that an action of a group, say \( G \), on a vector space \( V \) is said to be unimodular if the all elements have determinant one. In other words, we may view \( G \) as a subgroup of \( \text{SL}(V) \); in particular, an effective torus action on \( V \) in the sense of Definition 2.34 is always unimodular.
Proposition 3.9. With the setting of Theorem 1.1 in the first case, we have

\[ D_{sg}^{gr}(R^T) = \text{thick}(E_Q), \]

where \( E_Q \) is given in Notation 3.3. In other words, \( E_Q \) as an object in \( D_{sg}^{gr}(R^T) \) generates \( D_{sg}^{gr}(R^T) \) itself.

Proof. By Lemma 3.8 and Proposition 2.12 we have

\[ D^b(\text{tail } \Lambda) = \text{thick}(\bigoplus_{i=-n+1}^{-} \Lambda(i)). \]

On the other hand, by Lemma 3.5 we have

\[ D^b(\text{tail } R^T) = \text{thick}(\pi(\bigoplus_{i=-n+1}^{-} \Lambda e(i))). \]

Moreover, by Lemma 3.8 we have that

\[ D_{sg}^{gr}(R^T) = \text{thick}(\bigoplus_{i=-n+1}^{-} \mu \circ \pi((1-e)\Lambda e(i))). \]

Thus to prove the proposition it suffices to prove that

\[ \mu \circ \pi((1-e)\Lambda e(j)) \in \text{thick}(E_Q), \]

for any \(-n+1 \leq j \leq 0\). To this end, we need the following:

Claim 3.10. We have that

\[ \Phi(E_Q) \cong \Phi \circ \nu\left( R_{e\Lambda e}\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \right) \]

\[ \cong \pi\left( R_{e\Lambda e}\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \right). \]

Admit this claim for a moment. Since \( \mu \circ \Phi \) is the identify functor on \( D_{sg}^{gr}(R^T) \), we have

\[ \nu\left( R_{e\Lambda e}\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \right) \]

\[ \cong \mu \circ \Phi \circ \nu\left( R_{e\Lambda e}\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \right) \]

\[ \cong \mu \circ \pi\left( R_{e\Lambda e}\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \right), \]

where the last equality follows from Claim 3.10. Hence, we have that

\[ \nu\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right) \cong \mu \circ \pi\left( \bigoplus_{i=1}^{n-1} (\Omega_{\Lambda}^i (1-e)(i))e \oplus (1-e)\Lambda e \right). \]

Therefore the proof of the proposition is done by admitting the following claim. \( \square \)
Claim 3.11. We have that
\[ \pi((1 - e)\Lambda e(j)) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \right) \]
for any \(-n + 1 \leq j \leq 0\).

Proof. We show the proof by induction. First, observe that
\[ \pi((1 - e)\Lambda e(j)) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \right) \]
for \( j = (0, -1) \) since \( \pi\Omega^{n-1}_A(1 - e)(n - 1)e \cong \pi(1 - e)\Lambda e(-1) \).
Next, we assume the claim holds for \( l \leq j \leq 0 \); that is,
\[ \pi((1 - e)\Lambda e(j)) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=0}^{-n+1} e\Lambda e(r) \right) \right) \right) \]
for \( l \leq j \leq 0 \) and some \(-n + 1 \leq l \leq -1\). Consider the following exact sequence of graded \( \Lambda \)-modules
\[ 0 \to (1 - e)\Lambda(l - 1) \cong P^{l-n}(n + l - 1) \to P^{2-n}(n + l - 1) \to \cdots \to \Omega^{l+n-1}_A(1 - e)(n + l - 1) \to 0, \]
which implies that
\[ 0 \to \pi((1 - e)\Lambda e)(l - 1) \to \pi(P^{2-n}e(n + l - 1)) \to \cdots \to \pi(\Omega^{l+n-1}_A(1 - e)(n + l - 1)) \to 0 \]
is equal to 0 in \( D^b(\text{tail } R^T) \). Now observe that \( \pi(P^{2-n}e(n + l - 1)) \) is the direct sum of some summands of \( \pi(\Lambda e(l)) \) and \( \pi(\Lambda e(l + 1)) \). Also, \( \pi(P^{-l-n+1}e(n + l - 1)) \) is the direct sum of some summands of \( \pi(\Lambda e(-1)) \) and \( \pi(\Lambda e) \). Combining these two statements, we have
\[ \pi(P^r e(n + l - 1)) \in \text{thick}\left( \pi\left( \bigoplus_{i=l}^{0} (1 - e)\Lambda e(i), \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \right) \]
for any \( 2 - n \leq r \leq l - n - 1 \). Hence from
\[ \text{thick}\left( \pi\left( \bigoplus_{i=l}^{0} (1 - e)\Lambda e(i) \right) \right) \subseteq \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \right), \]
we have
\[ \pi(P^r e(n + l - 1)) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \right) \]
for any \( 2 - n \leq r \leq l - n - 1 \). Thus we obtain that
\[ \pi\left( (1 - e)\Lambda e(l - 1) \right) \in \text{thick}\left( \pi\left( \bigoplus_{i=1}^{n-1} \Omega^j_A(1 - e)(i) e \oplus (1 - e)\Lambda e, \pi\left( \bigoplus_{r=-n+1}^{0} e\Lambda e(r) \right) \right) \right), \]
and the claim follows. \( \square \)
Now we prove Claim 3.10. To this end, denote by grmod\((A)\) the full subcategory of grmod\((A)\) whose objects consist of \(N \in \text{grmod}(A)\) such that \(N_i = 0\) for \(i < 0\). Moreover, denote by \(D^b(\text{grmod}(A)_{\geq 0})\) the full subcategory of \(D^b(\text{grmod}(A))\) whose objects consist of \(M^* \in D^b(\text{grmod}(A))\) such that \(M^j \in \text{Ob}(\text{grmod}(A)_{\geq 0})\) for \(j \in \mathbb{Z}\). The following lemma can be found in Amiot [1].

**Lemma 3.12** ([1] Theorem 4.3). Let \(A\) be a Noetherian graded Gorenstein algebra and \(M^* \in D^b(\text{grmod}(A)_{\geq 0})\). If moreover

1. \(M^* \in D^b(\text{grmod}(A)_{\geq 0})\) and
2. \(\text{Hom}_{D^b(\text{grmod}(A))}(M^*, A(i)) = 0\) for any \(i \leq 0\),

then

\[ \pi(M^*) = \Phi \circ \nu(M^*). \]

**Proof of Claim 3.10.** The first equality holds since

\[ E_Q = \nu \left( R_{e \Lambda}(\bigoplus_{i=1}^{n-1} \Omega^i_{\Lambda}(1-e)(i)) \oplus (1-e)\Lambda e \right) \]

in \(D_{ls}^b(R^T)\).

We now use Lemma 3.12 to prove the second equality; that is, we show the following:

**Claim 3.13.** \(E_Q \in D^b(\text{grmod}(A)_{\geq 0})\).

**Claim 3.14.** \(\text{Hom}_{D^b(\text{grmod}(A))}(E_Q, A(i)) = 0\) for any \(i \leq 0\).

Once these two claims are proved, Claim 3.10 follows immediately. \(\square\)

**Proof of Claim 3.13.** First, since \(A\) is an AS-regular algebra of dimension \(n - 1\) with Gorenstein parameter \(n\), in the \((-i)\)-th and \((-i+1)\)-th positions of the resolution \(P^*(i)\) after shifting \(i\) times, which are

\[ P^{-i}(i) \rightarrow \Omega^i_{\Lambda}(1-e)(i) \rightarrow P^{-i+1}(i), \]

we have that \(P^{-i}(i)\) is the direct sum of some summands of \(\Lambda(-1)\) and \(\Lambda(0)\), and \(P^{-i+1}(i)\) is the direct sum of some summands of \(\Lambda(0)\) and \(\Lambda(1)\). Since \(\Omega^i_{\Lambda}(1-e)(i)\) is a submodule of \(P^{-i+1}(i)\), if \(\Omega^i_{\Lambda}(1-e)(i)\) is not in grmod\((A)_{\geq 0}\), then the image of \(\Omega^i_{\Lambda}(1-e)(i)\) in \(P^{-i+1}(i)\) contains a summand of \(\Lambda_0(1)\). But it is well known that the image of

\[ d_i(i) : P^{-i}(i) \rightarrow P^{-i+1}(i) \]

does not contain any summand of \(\Lambda_0(1)\) from the construction of \(P^*\). Thus

\[ \Omega^i_{\Lambda}(1-e)(i) \in \text{grmod}(A)_{\geq 0} \]

for any \(n - 1 \geq i \geq 1\).
Moreover, we know that $1 - e \Lambda \in \text{grmod}(\Lambda)_{\geq 0}$. Thus after taking the tensor product $(-) \otimes_{\Lambda} e \Lambda$ we have

$$
\bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1 - e)(i))e \oplus (1 - e)e \Lambda \in \text{grmod}(R^T)_{\geq 0}.
$$

After right mutation by $e \Lambda \in \text{grmod}(R^T)_{\geq 0}$, we also have

$$
R_{e \Lambda e} \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda}(1 - e)(i))e \oplus (1 - e)e \Lambda \right) \in \text{grmod}(R^T)_{\geq 0}.
$$

\begin{proof}[Proof of Claim 3.14] Since

$$
\text{Hom}_{D^b(\text{grmod } R^T)}((1 - e)e \Lambda, e \Lambda e(j)) = 0
$$

for $j \leq 0$, we only need to show that

$$
\text{Hom}_{D^b(\text{grmod } R^T)}(R_{e \Lambda e}(\Omega^i_{\Lambda}(1 - e)(i))e, e \Lambda e(j)) = 0
$$

for any $n - 1 \geq i \geq 1$ and $j \leq 0$.

First, we give $\Omega^i_{\Lambda}(1 - e)(i)$ the following resolution

$$
0 \to (1 - e)\Lambda(i - n) \cong P^{-n+1}(i) \to \cdots \to P^{-i-1}(i) \to P^{-i}(i) \to \Omega^i_{\Lambda}(1 - e)(i)) \to 0
$$

from $P^*$, which is denoted by $P^*_\Lambda(i)$. Since $e \Lambda$ is a Cohen-Macaulay $R^T$-module, we have

$$
\text{Hom}_{D^b(\text{grmod } R^T)}^*(\Omega^i_{\Lambda}(1 - e)(i))e, e \Lambda e(j)) = \text{Hom}_{C^b(\text{grmod } R^T)}^*(P^*_\Lambda(i))e, e \Lambda e(j))
$$

Moreover, since $\Lambda$ is an AS-regular algebra, we have

$$
\text{Hom}_{C^b(\text{grmod } R^T)}^l(P^*_\Lambda(i))e, e \Lambda e(j)) \cong \text{Hom}_{C^b(\text{grmod } R^T)}^{l+i}(P^*_\Lambda(i))e, e \Lambda e(j)) \cong \text{Hom}_{D^b(\text{grmod } R^T)}^{l+i}(P^*_\Lambda(i))e, e \Lambda e(j)) \cong \text{Hom}_{D^b(\text{grmod } R^T)}^{l+i}(1 - e) \otimes_{\Lambda} e \Lambda e(i), e \Lambda e(j)) \cong \text{Hom}_{D^b(\text{grmod } R^T)}^{l+i}(1 - e)e(i), e \Lambda e(j)) \cong 0
$$

for $l \geq 1$. Thus, we have that

$$
\text{Hom}_{D^b(\text{grmod } R^T)}^l(\Omega^i_{\Lambda}(1 - e)(i))e, e \Lambda e(j)) = 0
$$

for $l \geq 1$. After the right mutation by $e \Lambda e$, we also have

$$
\text{Hom}_{D^b(\text{grmod } R^T)}^l(R_{e \Lambda e}(\Omega^i_{\Lambda}(1 - e)(i))e, e \Lambda e(j)) = 0
$$

for $l \geq 1$.

17
Moreover, by the properties of the right mutation by $e\Lambda e$, we have

$$\Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^r(R_{e\Lambda e}(\Omega^i_\Lambda(1-e)(i)e), e\Lambda e) = 0,$$

for any $i \in [1, n-1]$ and $r \in \{0, -1\}$.

Since $P^{-i}e(i)$ is a direct sum of some summands of $\Lambda e(-1)$ and $\Lambda e(0)$,

$$\Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^{-1}(R_{e\Lambda e}(\Omega^i_\Lambda(1-e)(i)e), e\Lambda e(-1)) = 0$$

and

$$\Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0(\Omega^1_\Lambda(1-e)(1)e, e\Lambda e(-1)) \cong \Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0((1-e)\Lambda e, e\Lambda e(-1)) = 0,$$

the proof is done by admitting the following.

**Claim 3.15.** For $i \geq 2$ we have

$$\Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0(R_{e\Lambda e}(\Omega^i_\Lambda(1-e)(i)e), e\Lambda e(-1)) \cong \Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0(\Omega^i_\Lambda(1-e)(i)e, e\Lambda e(-1)) = 0.$$

**Proof.** We show the equality by contradiction. If there were a morphism

$$0 \neq s \in \Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0(\Omega^1_\Lambda(1-e)(i)e, e\Lambda e(-1)),$$

then there is a morphism

$$0 \neq \tilde{s} \in \Hom_{D^b(gr\text{-}mod\mathcal{R}T)}^0(P^{-i}(i)e, e\Lambda e(-1)),$$

which is the composition of $s$ with the morphism $P^{-i}e(i) \rightarrow \Omega^i_\Lambda(1-e)(i)e$ in $P^{-i}$. Since $P^{-i}e(i)$ is the direct sum of some summands of $\Lambda e(-1)$ and $\Lambda e(0)$, there is a summand $e\Lambda e(-1)$ of $P^{-i}e(i)$ such that the embedding

$$e\Lambda e(-1) \hookrightarrow \Lambda e(-1) \hookrightarrow P^{-i}(i)e$$

composing with $\tilde{s}$ is the identify. Now fix the above summand $e\Lambda(-1)$ of $P^{-i}(i)$. By Lemmas 3.5 and 3.7 we have

$$\Hom_{gr\text{-}mod\mathcal{R}T}(\Omega^i_\Lambda(1-e)(i)e, e\Lambda e(-1)) \cong \Hom_{\text{tail}\mathcal{R}T}(\Omega^i_\Lambda(1-e)(i)e, e\Lambda e(-1)) \cong \Hom_{\text{tail}\Lambda}(\Omega^i_\Lambda(1-e)(i), e\Lambda(-1)) \cong \Hom_{gr\text{-}mod\Lambda}(\Omega^i_\Lambda(1-e)(i), e\Lambda(-1)).$$

Denote the above isomorphism by

$$(-) : \Hom_{gr\text{-}mod\Lambda}(\Omega^i_\Lambda(1-e)(i), e\Lambda(-1)) \rightarrow \Hom_{gr\text{-}mod\mathcal{R}T}(\Omega^i_\Lambda(1-e)(i)e, e\Lambda e(-1)).$$

By the same method, we also have the following two isomorphisms with a slight abuse of notations:

$$(-) : \Hom_{gr\text{-}mod\Lambda}(e\Lambda(-1), e\Lambda(-1)) \rightarrow \Hom_{gr\text{-}mod\mathcal{R}T}(e\Lambda(-1), e\Lambda(-1)),$$
and

$$(-): \text{Hom}_{\text{grmod} \Lambda}(e \Lambda(-1), \Omega^i_{\Lambda}(1-e)(i)) \to \text{Hom}_{\text{grmod} R T}(e \Lambda e(-1), \Omega^i_{\Lambda}(1-e)(i)e).$$

We also use $p_i$ to denote the morphism $e \Lambda(-1) \hookrightarrow P^{-i}(i) \to \Omega^i_{\Lambda}(1-e)(i)$ in grmod($\Lambda$).

Then we have

$$s \circ p_i = \bar{s} \circ \bar{p}_i = id_{e \Lambda e(-1)}.$$

Since $s \circ p_i \in \text{Hom}_{\text{grmod} \Lambda}(e \Lambda(-1), e \Lambda(-1)) = e \Lambda e$, we have that $s \circ p_i$ is equal to $id_{\Lambda}$ up to a scalar. Thus $e \Lambda(-1)$ is a summand of $\Omega^i_{\Lambda}(1-e)(i)$ and $p_i$ is injective. However, this contradicts to the construction of the resolution $P^*$ and the fact that

$$\text{Hom}^r_{D^b(\text{grmod} \Lambda)}(P^*, e \Lambda(i)) = 0$$

for any $i, j \in \mathbb{Z}$. Therefore we have that

$$\text{Hom}^0_{D^b(\text{grmod} R T)}(Re \Lambda e(\Omega^i_{\Lambda}(1-e)(i)e), e \Lambda e(-1)) = 0.$$

3.2 More on the generators

In this subsection we study the Ext-groups of the generators discussed in the previous subsection. We start with a lemma due to Minamoto and Mori.

**Lemma 3.16** ([8] Proposition 4.4). Let $\Lambda$ be an AS-regular algebra of dimension $d \geq 1$ with Gorenstein parameter $a$. Then we have

$$\text{Hom}^q_{D^b(\text{tail} \Lambda)}(\Lambda(i), \Lambda(j + ma)) = 0$$

for $q \neq 0, 0 \leq i, j \leq a - 1$ and $m \geq 1$.

With this lemma, we are ready to show the main result of this subsection.

**Proposition 3.17.** Let $E_Q$ be given in Notation [3.6]. With the setting of Proposition [3.9], we have

$$\text{Hom}^r_{D^b(\text{grmod} R T)}(E_Q, E_Q) = 0$$

for any $r \neq 0$.

**Proof.** We divide the proof into three cases, namely when $r \leq -2$, $r \geq 1$ and $r = -1$ respectively.

**Case 1:** $r \leq -2$. By Lemma [3.5] and Claim [3.10] we have

$$\text{Hom}^r_{D^b_{\text{gr}}(R T)}(E_Q, E_Q) \cong \text{Hom}^r_{D^b(\text{tail} R T)}(\Phi E_Q, \Phi E_Q)$$

$$\cong \text{End}^r_{D^b(\text{tail} R T)}\left(\pi \left(Re \Lambda \left(\bigoplus_{i=1}^{n-1} \Omega^i_{\Lambda}(1-e)(i)e \oplus (1-e)\Lambda e\right)\right)\right)$$

$$\cong \text{End}^r_{D^b(\text{tail} \Lambda)}\left(\pi \left(Re \Lambda \left(\bigoplus_{i=1}^{n-1} \Omega^i_{\Lambda}(1-e)(i) \oplus (1-e)\Lambda\right)\right)\right).$$
Since
\[ \text{Hom}_{D^b(\text{grmod} \Lambda)}^j(P^*_{-i}(i), e\Lambda) = \text{Hom}_{D^b(\text{grmod} \Lambda)}^{j+i}(P^*(i), e\Lambda) = 0 \]
for any \( j \geq 1 \), there is morphism \( q^{-i} : \Omega^i_{\Lambda}(1-e)(i) \to e\Lambda^{\leq m_i} \), for some \( m_i \in \mathbb{N} \) such that \( R_{e\Lambda}(\Omega^i_{\Lambda}(1-e)(i)) \) is equal to
\[ 0 \to (1-e)\Lambda(-n) \to P^{1-n}(-n+1) \to \cdots \to \Omega^1_{\Lambda}(1-e) \xrightarrow{q^{-i}} e\Lambda^{\leq m_i} \to 0 \]
in \( D^b(\text{grmod} \Lambda) \), which denoted by \( P^*_{-i} \).

Moreover, by Lemmas 3.7 and 3.16, we have that

\[ \text{End}_{D^b(\text{grmod} \Lambda)}^r \pi \left( P^*_{-i}(i) \right) = \text{End}_{D^b(\text{grmod} \Lambda)}^r \left( P^*_{-i}(i) \right) \]
for \( n - 1 \geq i \geq 2 \) and therefore

\[ \text{End}_{D^b(\text{grmod} \Lambda)}^r \pi \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i) \right) = \text{End}_{D^b(\text{grmod} \Lambda)}^r \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i) \right). \]

This equality implies that

\[ \text{End}_{D^b(\text{grmod} \Lambda)}^r \pi \left( R_{e\Lambda} \left( \bigoplus_{i=1}^{n-1} \Omega^i_{\Lambda}(1-e)(i) \oplus (1-e)\Lambda(i) \right) \right) = \text{End}_{D^b(\text{grmod} \Lambda)}^r \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i) \right) \]
since \( \Omega^i_{\Lambda}(1-e)(1) \cong (1-e)\Lambda(1) \) in tail \( \Lambda \). Thus we get that

\[ \text{End}_{D^b(\text{grmod} \Lambda)}^r \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i) \right) = 0 \]
for \( r \leq -2 \) since the homologies of the complex \( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i) \) are concentrated at the degree 0 and 1.

**Case 2:** \( r \geq 1 \). We consider the following vector spaces

\[ \text{Hom}_{D^b(\text{grmod} \Lambda)}^0 \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i)[r] \right). \]

By the fact that

\[ \text{Hom}^r_{D^b(\text{grmod} \Lambda)}(1-e, \Lambda(j)) = 0 \]
for either \( r \neq n-1 \) and \( j \in \mathbb{Z} \), or \( r = n-1 \) and \( j \neq -n \), and by the properties of mutation, we have that these vector spaces

\[ \text{Hom}_{D^b(\text{grmod} \Lambda)}^0 \left( \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i), \bigoplus_{i=2}^{n-1} P^*_{-i}(i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda(i)[r] \right) \]

20
are all trivial.

**Case 3:** \( r = -1 \). First, since the homologies of complex \( \bigoplus_{i=2}^{n-1} P_{-i}(i) \oplus \bigoplus_{i=0}^{1}(1-e)\Lambda(i) \) are concentrated in degree 0 and 1, we have

\[
\text{Hom}_{D^b(\text{grmod}\, \Lambda)}^{-1}(\bigoplus_{i=0}^{n-1}(1-e)\Lambda(i), \bigoplus_{i=0}^{n-1} P_{-i}(i) \oplus \bigoplus_{i=0}^{1}(1-e)\Lambda(i)) = 0.
\]

Second, the vector space

\[
\text{Hom}_{D^b(\text{grmod}\, \Lambda)}^{-1}(P_{-i}(i), P_{-j}(j))
\]

is isomorphic to the commutative diagram of chain complexes

\[
\begin{array}{ccc}
0 & \rightarrow & \Omega^i_{\Lambda}(1-e)(i) \rightarrow e\Lambda^m_{-i} \rightarrow 0 \\
\downarrow & & \downarrow f \\
0 & \rightarrow & \Omega^j_{\Lambda}(1-e)(j) \rightarrow e\Lambda^m_{-j} \rightarrow 0
\end{array}
\]

in \( C^b(\text{grmod}\, \Lambda) \), for any \( i, j \in [2, n-1] \). Moreover, the above diagram is contained in the following commutative diagram of chain complexes

\[
\begin{array}{ccc}
0 & \rightarrow & P^{-i}(i) \rightarrow \eta_{-i} \rightarrow e\Lambda^m_{-i} \rightarrow 0 \\
\downarrow & & \downarrow g \\
0 & \rightarrow & P^{-j+1}(j) \rightarrow \eta_{-j+1} \rightarrow e\Lambda^m_{-j+1} \rightarrow 0
\end{array}
\]

in \( C^b(\text{grmod}\, \Lambda) \), for any \( i, j \in [2, n-1] \), where \( \eta_{-i} \) is given by composing \( q_{-i} \) with

\[
P^{-i}(i) \rightarrow \Omega^i_{\Lambda}(1-e)(i),
\]

and \( g \) is given by composing \( f \) with

\[
\Omega^j_{\Lambda}(1-e)(j) \rightarrow P^{-j+1}(j).
\]

By the definition of \( \Lambda \), the idempotent element \( e \) corresponds to the \( R^T \)-module \( M_{R}^G(V_i) \), where \( l \) is the minimal integral number in \( L \) constructed in Proposition 2.2.

Without loss of generality, we choose two indecomposable summands \( e'\Lambda(-1) \in P^{-i}(i) \) and \( e''\Lambda(1) \in P^{-j+1}(j) \) respectively. By the argument after Lemma 2.13 we know that any morphism from \( e\Lambda^m_{-i} \) to \( e''\Lambda(1) \) is given by some linear combinations of \( (x_{j_1}, x_{j_2}, \cdots, x_{j_r}) \), which is the subset of \( \{x_1, x_2, \cdots, x_n\} \) consisting of elements whose weights are all negative. Similarly, any morphism from \( e'\Lambda(-1) \) to \( e\Lambda^m_{-i} \) can be given by some linear combinations of \( (x_{j_1}, x_{j_2}, \cdots, x_{j_i}) \), which is the subset of \( \{x_1, x_2, \cdots, x_n\} \) consisting of elements whose weights are all positive.

Thus we can write the morphism

\[
e\Lambda^m_{-i} \rightarrow P^{-j+1}(j) \rightarrow e''\Lambda(1)
\]

as \( (\sum_{r=1}^{j_r} a_r^1 x_{j_r}, \cdots, \sum_{r=1}^{j_r} a_r^u x_{j_r}) \) for some \( a_r^u \in k \), where \( x_{j_r} \in (x_{j_1}, x_{j_2}, \cdots, x_{j_r}) \). In the same way, we can write the morphism

\[
e''\Lambda(-1) \rightarrow P^{-i}(i) \rightarrow e\Lambda^m_{-i}
\]
as \((\sum_{r=1}^{t} b_r^1x_{i_r}, \cdots, \sum_{r=1}^{t} b_r^m x_{i_r})\) for some \(b_r^u \in k\), where \(x_{i_r} \in (x_{i_1}, x_{i_2}, \cdots, x_{i_t})\). And therefore we have
\[
\sum_{u=1}^{m} \left( \sum_{r=1}^{s} a_r^u x_{j_r} \right) \left( \sum_{r=1}^{s} b_r^u x_{i_r} \right) = 0
\]
in \(R\). Now the above equality can be written in the form
\[
\sum_{r=1}^{s} x_{j_r} f_{j_r} = 0,
\]
where \(f_{j_r}\) is a linear combination of \((x_{i_1}, x_{i_2}, \cdots, x_{i_t})\) for any \(j_r\). However, it is impossible since \(x_{j_1} \mid \sum_{r=2}^{s} x_{j_r} f_{j_r}\). In fact, we have
\[
x_{j_1} \notin (x_{j_2}, \cdots, x_{j_s}, x_{i_1}, x_{i_2}, \cdots, x_{i_t}).
\]
In the same way, we can check that
\[
\text{Hom}_{-1}^{D^b(\text{grmod} \Lambda)} \left( \bigoplus_{i=2}^{n-1} \widetilde{P}_{-i}^s (i), \bigoplus_{i=0}^{1} (1-e)\Lambda (i) \right) = 0,
\]
and therefore we have
\[
\text{Hom}_{-1}^{D^b(\text{grmod} \Lambda)} \left( \bigoplus_{i=2}^{n-1} \widetilde{P}_{i}^s (i) \oplus \bigoplus_{i=0}^{1} (1-e)\Lambda (i) \right) = 0. \quad \square
\]

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. As stated in the theorem, the proof is divided into two cases: the first case is when \(G\) is the one dimensional torus \(T\), and the second case is when \(G\) is the product of \(T\) with a finite abelian group. First, let us recall the following definition:

**Definition 4.1** (Tilting object). Let \(C\) be triangulated category. An object \(X \in \text{Ob}(C)\) is said to be tilting if

1. \(C = \text{thick}(X)\), and
2. \(\text{Hom}_C(X, X[i]) = 0\) for any \(i \neq 0\).

An important property of tilting theory is the following (see [4] for more details and applications). If \(X\) is a tilting object of \(C\), then the following functor
\[
C \to D(\text{End}_C(X)) : M \mapsto \text{Hom}_C(X, M)
\]
is an equivalence of triangulated categories.
4.1 Proof of Theorem 1.1 Case 1

Proof of Theorem 1.1 (first case). By Propositions 3.9 and 3.17 $E_Q$ is a tilting object in $D^g(R^T)$, from which the theorem follows.

Example 4.2. Let $R = k[x_1, x_2, x_3, x_4]$. Assume the weights of the action of one dimensional tours $T$ on $R$ are $(1, 1, -1, -1)$. By Proposition 2.6 $\mathcal{L} = \{0, 1\}$ if we choose $\varepsilon = \frac{1}{2}$, and then $M = R^T \oplus M^T_R(V_1)$, and $\Lambda = \text{End}_{R^T}(M)$. Moreover, $\Lambda$ can be written as a quiver algebra $kQ_\Lambda/I$, where $Q_\Lambda$ is given as follows:

![Quiver Diagram]

where the vertex 1 corresponds to the summand $R^T$ of $M$, the vertex 2 corresponds the summand $M^T_R(V_1)$ of $M$, the arrows $\bar{x}_3, \bar{x}_4$ correspond to the morphisms 

$$x_3, x_4 \in \text{Hom}_{R^T}(R^T, M^T_R(V_1)) \cong \text{Hom}_{(T,k)}(V_0, R \otimes V_1) \subseteq R$$

respectively, the arrows $\bar{x}_1, \bar{x}_2$ correspond to the morphisms

$$x_1, x_2 \in \text{Hom}_{R^T}(M^T_R(V_1), R^T) \cong \text{Hom}_{(T,k)}(V_1, R \otimes V_0) \subseteq R$$

respectively, and $I$ is generated by elements

$$(\bar{x}_1 \bar{x}_3 \bar{x}_2 - \bar{x}_2 \bar{x}_3 \bar{x}_1, \bar{x}_1 \bar{x}_4 \bar{x}_2 - \bar{x}_2 \bar{x}_4 \bar{x}_1, \bar{x}_3 \bar{x}_1 \bar{x}_4 - \bar{x}_4 \bar{x}_1 \bar{x}_3, \bar{x}_3 \bar{x}_2 \bar{x}_4 - \bar{x}_4 \bar{x}_2 \bar{x}_3).$$

Now, we construct a projective resolution of $\Lambda_0 \cong k^2$ as graded $\Lambda$-modules as follows:

$$0 \longrightarrow \Lambda(-4) \xrightarrow{\varphi_3} \Lambda^{\oplus 4}(-3) \xrightarrow{\varphi_2} \Lambda^{\oplus 4}(-1) \xrightarrow{\varphi_1} \Lambda \xrightarrow{\varphi_0} \Lambda_0 \longrightarrow 0$$

where

1. $\varphi_0$ is given by the canonical projection $\Lambda = \bigoplus_i \Lambda_i \twoheadrightarrow \Lambda_0$;
2. $\varphi_1$ is given by $\varphi_1(a, b, c, d) = (\bar{x}_1 a + \bar{x}_2 b + \bar{x}_3 c + \bar{x}_4 d)$, for any $(a, b, c, d) \in \Lambda^{\oplus 4}(-1)$;
3. $\varphi_2$ is given by $\varphi_2(a, b, c, d) = (\bar{x}_4 \bar{x}_2 a - \bar{x}_3 \bar{x}_2 b, -\bar{x}_4 \bar{x}_1 a + \bar{x}_3 \bar{x}_1 b, \bar{x}_1 \bar{x}_4 c - \bar{x}_2 \bar{x}_4 d, -\bar{x}_1 \bar{x}_3 c + \bar{x}_2 \bar{x}_3 d)$, for any $(a, b, c, d) \in \Lambda^{\oplus 4}(-3)$;
4. $\varphi_3$ is given by $\varphi_3(1_A(-4)) = (\bar{x}_3(-4), \bar{x}_4(-4), \bar{x}_2(-4), \bar{x}_1(-4)) \in \Lambda^{\oplus 4}(-3)$,

and $1_A$ is the identity element of $\Lambda$. 23
Thus, from above projective resolution, we get the tilting object $E_Q$ of $D^b_{gr}(R^T)$ as the direct sum of the following objects:

1. $(1 - e)\Lambda = M^R_{G}(V_1)$;
2. $R_{e\Lambda e}(\Omega^1(1 - e)e(1)) = \ker(\varphi_1) = \bigoplus_{i \geq 1} (1 - e)i; e$;
3. $R_{e\Lambda e}(\Omega^1(1 - e)e(2)) = \ker(\varphi_1) = \bigoplus_{i \geq 1} (1 - e)i; e(2)$;
4. $R_{e\Lambda e}(\Omega^1(1 - e)e(3)) = \ker(\varphi_3) = e\Lambda^{(2)}$.

We also have

$$\text{End}_{D^b_{gr}(R^T)}(E_Q) = \{\text{diag}(k_1, k_2, k_3, k_4) | k_1, k_2, k_3, k_4 \in k\}.$$  

### 4.2 Proof of Theorem 1.1: Case 2

By [14, Lemma 3.15], an NCCR can be constructed for case that $G$ is not a connected group if an NCCR exists for the connected component $G_0$ containing the identity element. Based on this lemma, we construct the NCCR of $R^G$ as follows. Let

$$\Lambda := \text{End}_{R^T}\left( \bigoplus_{\chi \in L} M^T_R(V_\chi) \right) \cong \text{End}_{(R,G)}\left( \bigoplus_{\chi \in L} R \otimes V_\chi \right)$$

be the NCCR of $R^T$ constructed in §3, where $T \subset G$ is the maximal one-dimensional torus containing the identity element in $G$ and $H := G/T$ is a finite abelian group in $\text{SL}(N,k)$. Set

$$\Lambda' := \text{End}_{R^G}\left( \bigoplus_{\chi \in L} M^G_R(U_\chi) \right) \cong \text{End}_{(R,G)}\left( \bigoplus_{\chi \in L} R \otimes U_\chi \right),$$

which is an NCCR of $R^G$, where $U_\chi := \text{Ind}^G_T(V_\chi)$. We have

$$\Lambda' \cong M^G_R\left( \text{End}_{(k,G)}\left( \bigoplus_{\chi \in L} \text{Ind}^G_T(V_\chi) \right) \right).$$

For any $\chi, \chi' \in L$, we have that

$$\text{Hom}_{k,G}(U_\chi, U_{\chi'}) \cong \text{Hom}_{(k,G)}\left( \text{Ind}^G_T(V_\chi), \text{Ind}^G_T(V_{\chi'}) \right)$$

and its multiplicative structure is given by $kG \otimes \Lambda$ simultaneously.

Next, we endow $\Lambda'$ with a canonical grading from $R$ such that $\deg(g \otimes 1) = 0$ for any $g \in G$. Then we obtain the following propositions and lemma, whose proofs are completely analogous to the case that $G$ is one-dimensional torus, and hence are omitted:
Proposition 4.3 (Compare with Proposition 2.11). \( R^G \) is a Noetherian graded Gorenstein algebra of dimension \( n - 1 \) and of Gorenstein parameter \( n \).

Proposition 4.4 (Compare with Proposition 2.12). \( \Lambda' \) is an AS-regular algebra of dimension \( n - 1 \) and of Gorenstein parameter \( n \).

Lemma 4.5 (Compare with Lemma 3.5). Let \( R = k[x_1, x_2, \ldots, x_n] \), and \( T \) be the maximal one-dimensional torus in \( G \) such that \( T \) contains the identity element of \( G \), which acts on \( R \) with weights \( \chi = (\chi_1, \chi_2, \ldots, \chi_n) \). If \( R^G \) satisfies the first and second conditions in Definition 2.4, and is a graded Gorenstein algebra with grading endowed canonically, then the following are equivalent:

1. \( \chi \) is effective;
2. the functor \( (-)^{e'} : \text{tail} \Lambda \rightarrow \text{tail} R^G \) is an equivalence functor with inverse \( (-) \otimes e'\Lambda e' \);
3. \( R^G \) is graded isolated singularity;
4. \( \text{Spec}(R^G) \) has unique isolated singularity at the origin in the usual sense.

By Proposition 4.4, we construct an object

\[
E'_Q = \nu \left( R e'\Lambda e' \left( \bigoplus_{i=1}^{n-1} (\Omega^i_{\Lambda'}(1-e')(i)) e' \oplus (1-e')\Lambda' e' \right) \right)
\]

in \( D^{gr}_{sg}(R^G) \).

By Propositions 4.3 and 4.4 and Lemma 4.5, we obtain the following two propositions, whose proofs are again analogous and hence are omitted:

Proposition 4.6 (Compare with Proposition 3.9). With the setting of Theorem 1.1 in the second case, we have

\[
D^{gr}_{sg}(R^G) = \text{thick}(E'_Q).
\]

In other words, \( E'_Q \) as an object of \( D^{gr}_{sg}(R^G) \) generates \( D^{gr}_{sg}(R^G) \) itself.

Proposition 4.7 (Compare with Proposition 3.17). With the setting of Proposition 4.6 we have

\[
\text{Hom}_{D^{gr}_{sg}(R^G)}(E'_Q, E'_Q) = 0
\]

for any \( r \neq 0 \).

Proof of Theorem 1.1 (second case). The proof follows from the combination of Propositions 4.6 and 4.7.

Remark 4.8. The above method of constructing the tilting object does not work for the case that \( G \) contains a two or higher dimensional torus.

In fact, our method uses [14, Lemma 1.19], which assumes that the action of the connected subgroup \( G_0 \) of \( G \) on \( R \) is quasi-symmetric. Moreover, \( R^{G_0} \) has to be an isolated singularity. Without loss of generality, we assume that \( G_0 \) is an \( m \)-dimensional torus and the
weights of its action on $R$ are $(v_1, v_2, \ldots, v_n)$ in $\mathbb{Z}^m$. Similarly to Lemma 3.5, the isolated singularities of $R^{G_0}$ implies that, if there is a sequence of integral number $(c_1, c_2, \ldots, c_r)$ and a subset $(u_1, u_2, \ldots, u_r) \subseteq (v_1, v_2, \ldots, v_n) \subset X(T)$ such that
\[ \sum_i c_i u_i = 0, \]
then $(u_1, u_2, \ldots, u_r)$ spans the vector spaces $\mathbb{Z}^m$ as $\mathbb{Z}$-modules.

Since the action of $G_0$ on $R$ is quasi-symmetric, there is a line $\ell \subset \mathbb{Z}^m$ through the origin such that
\[ (\beta_1, \beta_2, \ldots, \beta_t) := (v_1, v_2, \ldots, v_n) \bigcap \ell \neq \emptyset \]
and $\sum_{\beta_i \in \ell} \beta_i = 0$. Thus, $(\beta_1, \beta_2, \ldots, \beta_t)$ spans the vector spaces $\mathbb{Z}^m$ as $\mathbb{Z}$-modules. But this is impossible since $(\beta_1, \beta_2, \ldots, \beta_t)$ is contained in a line which goes through the origin.

References

[1] Claire Amiot, Preprojective algebras, singularity categories and orthogonal decompositions, Algebras, quivers and representations, 1-11, Abel Symp., 8, Springer, Heidelberg, 2013.

[2] Ragnar-Olaf Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, unpublished manuscript.

[3] Samuel Eilenberg, Homological dimension and syzygies, Ann. of Math. (2) 64 (1956), 328-336.

[4] Lidia A. H"ugel, Dieter Happel and Henning Krause (Eds.), Handbook of Tilting Theory, London Math. Soc. Lect. Note Series 332, Cambridge Univ. Press, Cambridge, 2007.

[5] Osamu Iyama and Idun Reiten, Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras, Amer. J. Math. 130 (2008), no. 4, 1087-1149.

[6] Osamu Iyama and Ryo Takahashi, Tilting and cluster tilting for quotient singularities Math. Ann. 356 (2013), no. 3, 1065-1105.

[7] Izuru Mori, Mckay type correspondence for AS-Gorenstein algebras, J. Lond. Math. Soc. 88 (2013) 2071-2091.

[8] Hiroyuki Minamoto and Izuru Mori, The structure of AS-Gorenstein algebras, Adv. Math. 226 (2011), no. 5, 4061-4095.

[9] Izuru Mori and Kenta Ueyama, Stable categories of graded maximal Cohen-Macaulay modules over noncommutative quotient singularities, Adv. Math. 297 (2016), 54-92.

[10] Izuru Mori and Kenta Ueyama, Ample group action on AS-regular algebras and noncommutative graded isolated singularities, Trans. Amer. Math. Soc. 368 (2016), no. 10, 7359-7383.
[11] Dmitri Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 240-262; translation in Proc. Steklov Inst. Math. 2004, no. 3 (246), 227-248.

[12] Dmitri Orlov, *Triangulated categories of singularities, and equivalences between Landau-Ginzburg models*, (Russian) Mat. Sb. 197 (2006), no. 12, 117-132.

[13] Dmitri Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503-531, Progr. Math., 270, Birkhäuser Boston, Boston, MA, 2009.

[14] Špela Špenko and Michel Van den Bergh, *Non-commutative resolutions of quotient singularities for reductive groups*, Invent. Math. 210 (2017), no. 1, 3-67.

[15] Špela Špenko and Michel Van den Bergh, *Non-commutative crepant resolutions for some toric singularities I*, Int. Math. Res. Not. IMRN 2020, no. 21, 8120-8138.

[16] Špela Špenko and Michel Van den Bergh, *Comparing the Kirwan and Non-commutative resolutions of quotient varieties*, arXiv: 1912.01689v1.

[17] Michel Van den Bergh, *Three-dimensional flops and noncommutative rings*. Duke Math. J. 122 (2004), no. 3, 423-455.

[18] Michel Van den Bergh, *Non-commutative crepant resolutions*. The legacy of Niels Henrik Abel, 749-770, Springer, Berlin, 2004.

[19] Michael Wemyss, *Noncommutative resolutions*. Noncommutative algebraic geometry, Math. Sci. Res. Inst. Publ., 64, Cambridge Univ. Press, New York, 2016, 239-306.