ON ALGEBRAIC SUPERGROUPS AND QUANTUM DEFORMATIONS

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Abstract. We give the definitions of affine algebraic supervariety and affine algebraic supergroup through the functor of points and we relate them to the other definitions present in the literature. We study in detail the algebraic supergroups $GL(m|n)$ and $SL(m|n)$ and give explicitly the Hopf algebra structure of the algebra representing the functors of points. At the end we give also the quantization of $GL(m|n)$ together with its coaction on suitable quantum spaces according to Manin’s philosophy.

1. Introduction

The mathematical foundations of supergeometry were laid in the 60s by Berezin in [Be] and later by Leites [Le] Kostant [Ko] and Manin [Ma1] among many others, its origins being mainly tied up with physical problems.

A new attention to the subject came later with the study of quantum fields and superstring. In the 1999 “Notes on Supersymmetry” Deligne and Morgan [DM] give a categorical point of view on supersymmetry notions developed originally by physicists and known from a more “operational” point of view.

In the current definitions of supermanifold, the points of a supermanifold are points of an usual manifold and the adjective super refers to an additional structure on the structural sheaf of functions on the manifold. This sheaf is assumed to be a sheaf of commutative superalgebras, where a superalgebra is a $\mathbb{Z}_2$-graded algebra. When dealing with algebraic supergroups however, there are in some sense true points. In fact, for each supercommutative algebra $A$ the $A$ points of a supergroup can be viewed as a certain subset of automorphisms of a superspace $A^{m|n}$ where $A$ is a given commutative superalgebra. For this reason the point of view of functor of points is in this case most useful. It allows to associate to an affine supergroup a commutative Hopf superalgebra the same way it does in the commutative case. It is hence possible to define a quantum deformation of an algebraic supergroup in complete analogy to the non super case: it will be a deformation of the commutative Hopf superalgebra associated to it.

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The quantum supergroup $GL(m|n)$ was first constructed by Manin [Ma3] together with its coactions on suitable quantum superspaces. The construction of its Hopf superalgebra structure is in some sense implicit. In a subsequent work, Lyubashenko and Sudbery [LS] provided quantum deformations of supergroups of $GL(m|n)$ type using the universal $R$-matrix formalism. The same formalism is also used in [P] where more explicit formulas are given. In both works however the ring and its coalgebra structure do not appear explicitly, since the calculation was too involved using the $R$-matrix approach.

In the present paper we present a definition of affine algebraic supervariety and supergroup that is basically equivalent to the one of Manin [Ma1] and then we use this point of view to give a quantum deformation of the supergroup $GL(m|n)$.

The main result of the paper consists in giving explicitly the coalgebra structures for the Hopf superalgebras associated to the supergroup $GL(m|n)$ and its quantization $k_q[GL(m|n)]$ obtained according to the Manin philosophy, that is together with coactions on suitable quantum spaces. The explicit forms of the comultiplication for $GL(m|n)$ and its quantization $k_q[GL(m|n)]$ are non trivial, they rely heavily on the presence of the nilpotents of the superalgebras and do not appear in any other work. These results were announced in the proceeding [Fi] where they appeared without proof.

The organization of this paper is as follows.

In §2 we introduce the notion of affine supervariety and affine supergroup using the functor of points. These two definitions turn out to be basically equivalent to the definitions that one finds in the literature ([De], [Ma1] among many others).

In §3 we write explicitly the Hopf algebra structure of the Hopf superalgebra associated to the supergroups $GL(m|n)$ and $SL(m|n)$, where with $GL(m|n)$ we intend the supergroup whose $A$ points are the group of automorphisms of the superspace $A^{m|n}$ and with $SL(m|n)$ the subsupergroup of $GL(m|n)$ of automorphisms with Berezinian equal to 1, with $A$ commutative superalgebra.

In §4 we construct the non commutative Hopf superalgebra $k_q[GL(m|n)]$ deformation in the quantum group sense of the Hopf superalgebra associated to $GL(m|n)$. We also see that $k_q[GL(m|n)]$ admits coactions on suitable quantum superspaces.

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2. Preliminaries on algebraic supervarieties and supergroups

Let $k$ be an algebraically closed field. All algebras and superalgebras have to be intended over $k$ unless otherwise specified. Given a superalgebra $A$ we will denote with $A_0$ the even part, with $A_1$ the odd part and with $I_{odd}$ the ideal generated by the odd part.
A superalgebra is said to be commutative (or supercommutative) if

\[ xy = (-1)^{p(x)p(y)}yx, \quad \text{for all homogeneous } x, y \]

where \( p \) denotes the parity of an homogeneous element (\( p(x) = 0 \) if \( x \in A_0 \), \( p(x) = 1 \) if \( x \in A_1 \)).

In this section all superalgebras are assumed to be commutative.

Let’s denote with \( A \) the category of affine superalgebras that is commutative superalgebras such that, modulo the ideal generated by their odd part, they are affine algebras (an affine algebra is a finitely generated reduced commutative algebra).

**Definition (2.1).** Define affine algebraic supervariety over \( k \) a representable functor \( V \) from the category \( A \) of affine superalgebras to the category \( S \) of sets. Let’s call \( k[V] \) the commutative \( k \)-superalgebra representing the functor \( V \),

\[ V(A) = \text{Hom}_{k-\text{superalg}}(k[V], A), \quad A \in A. \]

We will call \( V(A) \) the \( A \)-points of the variety \( V \).

A morphism of affine supervarieties is identified with a morphism between the representing objects, that is a morphism of affine superalgebras.

We also define the functor \( V_{\text{red}} \) associated to \( V \) from the category \( A_c \) of affine \( k \)-algebras to the category of sets:

\[ V_{\text{red}}(A_c) = \text{Hom}_{k-\text{alg}}(k[V]/I_{\text{odd}}^k[V], A_c), \quad A_c \in A_c. \]

\( V_{\text{red}} \) is an affine algebraic variety and it is called the reduced variety associated to \( V \).

If the algebra \( k[V] \) representing the functor \( V \) has the additional structure of a commutative Hopf superalgebra, we say that \( V \) is an affine algebraic supergroup. (For the definition and main properties of Hopf algebras see \([Mo]\)).

**Remarks (2.2).**

1. Let \( G \) be an affine algebraic supergroup in the sense of (2.1). As in the classical setting, the condition \( k[G] \) being a commutative Hopf superalgebra makes the functor group valued, that is the product of two morphisms is still a morphism.

In fact let \( A \) be a commutative superalgebra and let \( x, y \in \text{Hom}_{k-\text{superalg}}(k[G], A) \) be two points of \( G(A) \). The product of \( x \) and \( y \) is defined as:

\[ x \cdot y = \text{def} \ m_A \cdot x \otimes y \cdot \Delta \]

where \( m_A \) is the multiplication in \( A \) and \( \Delta \) the comultiplication in \( k[G] \). One can directly check that \( x \cdot y \in \text{Hom}_{k-\text{superalg}}(k[G], A) \), that is:

\[ (x \cdot y)(ab) = (x \cdot y)(a)(x \cdot y)(b) \]
This is an important difference with the quantum case that will be treated in §4. The non commutativity of the Hopf algebra in the quantum setting does not allow to multiply morphisms(=points). In fact in the quantum (super)group setting the product of two morphisms is not in general a morphism. For more details see [Ma4] pg 13.

2. Let \( V \) be an affine algebraic supervariety as defined in (2.1). Let \( k_0 \subset k \) be a subfield of \( k \). We say that \( V \) is a \( k_0 \)-variety if there exists a \( k_0 \)-superalgebra \( k_0[V] \) such that \( k[V] \cong k_0[V] \otimes_{k_0} k \) and

\[
V(A) = \text{Hom}_{k_0-\text{superalg}}(k_0[V], A) = \text{Hom}_{k-\text{superalg}}(k[V], A), \quad A \in \mathcal{A}.
\]

We obtain a functor that we still denote by \( V \) from the category \( \mathcal{A}_{k_0} \) of affine \( k_0 \)-superalgebras to the category of sets:

\[
V(A_{k_0}) = \text{Hom}_{k_0-\text{superalg}}(k_0[V], A_{k_0}), \quad A \in \mathcal{A}_{k_0}.
\]

This allows to consider rationality questions on a supervariety. We will not pursue this further in the present work.

**Examples (2.3).**

1. The \( k \)-points of an affine supervariety \( V \) correspond to the affine variety defined over \( k \) whose functor of points is \( V_{\text{red}} \).

2. Let \( A \) be a commutative superalgebra. Let \( M(m|n)(A) \) be the linear endomorphisms of the superspace \( A^{m|n} \) (see [De] pg. 53):

\[
\begin{pmatrix}
  a_{11} & \ldots & a_{1,m} & \alpha_{1,m+1} & \ldots & \alpha_{1,m+n} \\
  \vdots & & \vdots & & & \\
  a_{m,1} & \ldots & a_{m,m} & \alpha_{m,m+1} & \ldots & \alpha_{m,m+n} \\
  \alpha_{m+1,1} & \ldots & \alpha_{m+1,m} & a_{m+1,m+1} & \ldots & a_{m+1,m+n} \\
  \vdots & & \vdots & & & \\
  \alpha_{m+n,1} & \ldots & \alpha_{m+n,m} & a_{m+n,m+1} & \ldots & a_{m+n,m+n}
\end{pmatrix}
\]

\( a_{ij} \in A_0, \alpha_{kl} \in A_1, 1 \leq i, j \leq m \) or \( m+1 \leq i, j \leq m+n, 1 \leq k \leq m, m+1 \leq l \leq m+n \) or \( m+1 \leq k \leq m+n, 1 \leq l \leq m \).

This is an affine supervariety represented by the commutative superalgebra: \( k[M(m|n)] = k[x_{ij}, \xi_{kl}] \) where \( x_{ij} \)’s and \( \xi_{kl} \)’s are respectively even and odd variables with \( 1 \leq i, j \leq m \) or \( m+1 \leq i, j \leq m+n, 1 \leq k \leq m, m+1 \leq l \leq m+n \) or \( m+1 \leq k \leq m+n, 1 \leq l \leq m \).

Observe that \( M_{\text{red}} = M(m) \times M(n) \) where \( M(l) \) is the functor corresponding to the affine variety of \( l \times l \) matrices.
We now would like to give an equivalent point of view and define again the category of affine supervarieties and affine supergroups. (See [Ma1], [De], [Be]).

**Definition (2.4).** Let $V_{\text{red}}$ be an affine algebraic variety defined over $k$ and $O_{V_{\text{red}}}$ the structural sheaf of $V_{\text{red}}$. Define **affine algebraic supervariety** $V$ the couple $(V_{\text{red}}, O_V)$ where $O_V$ is a sheaf of affine superalgebras such that its stalk is local and $O_V/I_V$ is isomorphic to $O_{V_{\text{red}}}$, where $I_V$ is the sheaf of ideals generated by the nilpotent elements.

We want to show that this definition is equivalent to the one given previously. Clearly if we have an affine supervariety according to the definition (2.4), we have a superalgebra associated to it, namely the global sections of the sheaf $O_V$. This means that we immediately have the functor of points associated to it, hence a supervariety according to the definition (2.1).

Conversely assume we have a functor of points $V$ and a commutative superalgebra $k[V]$ to which it is associated (see definition (2.1)). We need to show that it gives rise to a sheaf of superalgebras on the affine variety $V_{\text{red}}$.

Let’s look at the maps:

$$k[V]_0 \xrightarrow{\alpha} k[V] \xrightarrow{\beta} k[V]/I_{od}^{k[V]}$$

where $I_{od}^{k[V]}$ is the ideal generated by the nilpotent elements in $k[V]$. Observe that we have a surjective map $\gamma = \beta \cdot \alpha$. whose kernel consists of the nilpotent elements of $k[V]_0$. This induces a map

$$\text{Spec}k[V]/I_{od}^{k[V]} \longrightarrow \text{Spec}k[V]_0$$

that is an isomorphism since the kernel of $\gamma$ consists of nilpotents. Let’s now view $k[V]$ as an $k[V]_0$-module. This allows us to build a sheaf on $\text{Spec}k[V]_0 = \text{Spec}k[V]/I_{od}^{k[V]}$ of $k[V]_0$-modules, where the stalk coincides with the localization of $k[V]$ into the maximal ideals of $k[V]_0$. So we have obtained from a commutative superalgebra $k[V]$ a sheaf of superalgebras on $\text{Spec}k[V]/I_{od}^{k[V]}$ which corresponds to the affine variety $V_{\text{red}}$, whose global sections coincide with $k[V]$.

The next section will be devoted to construct in detail examples of affine supergroups.

**3. The affine supergroups $GL(m|n)$ and $SL(m|n)$**

In this section we intend to give explicitly the supergroup structure for the supergroup functors $GL(m|n)$ and $SL(m|n)$. For any $A \in A$ let’s define $GL(m|n)(A)$ as the group of automorphisms of the superspace $A^{m|n}$ (see [De] pg 59). Define also $SL(m|n)(A)$ as the subset of $GL(m|n)$ of automorphisms with berezinian equal to 1. The berezinian of the matrix:

$$\begin{pmatrix} G_{11} & \Gamma_{12} \\ \Gamma_{21} & G_{22} \end{pmatrix} \in SL(m|n)(A),$$
(\(G_{11}, G_{22}\) are \(m \times m, n \times n\) invertible matrices of even elements, \(\Gamma_{12}, \Gamma_{21}\) are \(m \times n, n \times m\) matrices of odd elements) is defined as:

\[
Ber = \det(G_{22})^{-1} \det(G_{11} - \Gamma_{12} G_{22}^{-1} \Gamma_{21}).
\]

(See [Be] ch. 4, [De] pg 59 for more details).

From section 2 we know that we need to give Hopf commutative superalgebras \(k[GL(m|n)]\) and \(k[SL(m|n)]\) such that for each commutative superalgebra \(A\),

\[
GL(m|n)(A) = Hom_{k-superalg}(k[GL(m|n)], A)
\]

\[
SL(m|n)(A) = Hom_{k-superalg}(k[SL(m|n)], A).
\]

We start by defining the supercommutative algebras \(k[GL(m|n)]\) and \(k[SL(m|n)]\), then we will give explicitly its coalgebra and Hopf algebra structure.

Let \(x_{ij}\) for \(1 \leq i, j \leq m\) or \(m + 1 \leq i, j \leq m + n\) be even variables and \(\xi_{kl}\) for \(1 \leq k \leq m, m + 1 \leq l \leq m + n\) or \(m + 1 \leq k \leq m + n, 1 \leq l \leq m\) be odd variables.

Denote by \(X_{11}, X_{22}, \Xi_{12},\Xi_{21}\) the following matrices of indeterminates:

\[
X_{11} = (x_{ij})_{1 \leq i,j \leq m}, \quad X_{22} = (x_{ij})_{m+1 \leq i,j \leq m+n},
\]

\[
\Xi_{12} = (\xi_{kl})_{1 \leq k \leq m, m+1 \leq l \leq m+n}, \quad \Xi_{21} = (\xi_{kl})_{m+1 \leq k \leq m+n, 1 \leq l \leq m}.
\]

**Definition (3.1).**

\[
k[GL(m|n)] =_{def} k[x_{ij}, \xi_{kl}, d_{1\ldots m}^{1\ldots m-1}, d_{m+1\ldots m+n}^{m+1\ldots m+n-1}]\]

\[
k[SL(m|n)] =_{def} \frac{k[x_{ij}, \xi_{kl}, d_{1\ldots m}^{1\ldots m-1}, d_{m+1\ldots m+n}^{m+1\ldots m+n-1}]}{(\det(S_{22}(X_{22}))\det(X_{11} - \Xi_{12} S_{22}(X_{22})\Xi_{21}) - 1)}
\]

where \(d_{1\ldots m}^{1\ldots m-1}, d_{m+1\ldots m+n}^{m+1\ldots m+n-1}\) are even variables such that

\[
d_{1\ldots m}^{1\ldots m-1} d_{1\ldots m}^{1\ldots m} = 1, \quad d_{m+1\ldots m+n}^{m+1\ldots m+n-1} d_{m+1\ldots m+n}^{m+1\ldots m+n} = 1.
\]

with \(d_{1\ldots m}^{1\ldots m} = \det(X_{11})\), \(d_{m+1\ldots m+n}^{m+1\ldots m+n} = \det(X_{22})\).

To simplify the notation we will also write \((d_{1\ldots m}^{1\ldots m-1})^t\) as \(d_{1\ldots m}^{1\ldots m-t}\) for \(t\) a positive integer.

\[
S_{11}(x_{ij}) =_{def} (-1)^{i-j} A_{1j}^{11} d_{1\ldots m}^{1\ldots m-1}, \quad 1 \leq i,j \leq m
\]

\[
S_{22}(x_{ij}) =_{def} (-1)^{i-j} A_{ji}^{22} d_{m+1\ldots m+n}^{m+1\ldots m+n-1}, \quad m+1 \leq i,j \leq m+n
\]
where $A_{ji}^{11}$ and $A_{ji}^{22}$ denote the determinants of the minors obtained by suppressing the $j^{th}$ row and $i^{th}$ column in $X_{11}$ and $X_{22}$ respectively.

Regarding $S_{11}(X_{11})$ as a matrix of indeterminates that is $(S_{11}(X_{11}))_{ij} = S_{11}(x_{ij})$, $1 \leq i,j \leq m$, we have that:

$$S_{11}(X_{11})X_{11} = X_{11}S_{11}(X_{11}) = I_{m}$$

Similarly:

$$S_{22}(X_{22})X_{22} = X_{22}S_{22}(X_{22}) = I_{n}$$

where $I_{m}$ and $I_{n}$ denote the identity matrix of order $m$ and $n$ respectively.

The expression:

$$Ber = det(S_{22}(X_{22}))det(X_{11} - \Xi_{12}S_{22}(X_{22})\Xi_{21})$$

is called the Berezinian function. (See [Be] ch. 3 for more details).

**Proposition (3.2).** In the ring $k[GL(m|n)]$ the Berezinian function is invertible.

**Proof.** Let’s write $Ber$ as:

$$Ber = det(S_{22}(X_{22}))det(I_{m} - \Xi_{12}S_{22}(X_{22})\Xi_{21}S_{11}(X_{11}))det(X_{11}).$$

It is enough to prove that $det(I_{m} - \Xi_{12}S_{22}(X_{22})\Xi_{21}S_{11}(X_{11}))$ is invertible. To simplify the notation let’s call $A = \Xi_{12}S_{22}(X_{22})$, $B = \Xi_{21}S_{11}(X_{11})$. We now prove that $I_{m} - AB$ is invertible (as matrix of indeterminates). It’s inverse is given by:

$$I_{m} + (AB) + (AB)^{2} + \ldots + (AB)^{mn+1}$$

In fact if one multiplies this matrix with $I_{m} - AB$, one obtains a telescopic sum with last term $(AB)^{m+n}$. Each element of $A$ and $B$ is of degree 1 in the odd indeterminates hence $(AB)^{mn+1}$ is of odd degree $2mn + 2$ hence zero.

We now proceed to give the coalgebra structure. We need to give the comultiplication $\Delta$ and counit $\epsilon$ for all the generators and verify that they are well defined.

In order make the formulas more readable we introduce the notation:

$$a_{ij} = \begin{cases} 
  x_{ij} & \text{if } 1 \leq i,j \leq m \text{ or } m+1 \leq i,j \leq m+n \\
  \xi_{ij} & \text{if } 1 \leq i \leq m, m+1 \leq j \leq m+n \text{ or } m+1 \leq i \leq m+n, 1 \leq j \leq m
\end{cases}$$

**Observation (3.3).** Let’s compute $\Delta(d_{1\ldots m}^{1\ldots m})$ and $\Delta(d_{m+1\ldots m+n}^{m+1\ldots m+n})$.

$$\Delta(d_{1\ldots m}^{1\ldots m}) = \sum_{\sigma \in S_{m}} (-1)^{l(\sigma)} \Delta(a_{1,\sigma(1)}) \ldots \Delta(a_{m,\sigma(m)}) =$$

$$= \sum_{\sigma \in S_{m}, 1 \leq k_{1}, \ldots, k_{m} \leq m+n} (-1)^{l(\sigma)}a_{1,k_{1}} \ldots a_{m,k_{m}} \otimes a_{k_{1},\sigma(1)} \ldots a_{k_{m},\sigma(m)} = \sum_{i=1}^{m} r_{1}$$

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with
\[ r_i = \sum_{\sigma \in S_n, (k_1, \ldots, k_m) \in \rho_i} (-1)^{l(\sigma)} a_{1,k_1} \cdots a_{m,k_m} \otimes a_{k_1,1} \cdots a_{k_m,\sigma(m)} \]
\[ \rho_i = \{(k_1, \ldots, k_m)|1 \leq k_1, \ldots, k_m \leq m+n, \quad |\{k_1, \ldots, k_m\} \cap \{1 \cdots m\}| = m-i\} \]

Similarly one can write:
\[ \Delta(d_{m+1 \cdots m+n}^{m+1 \cdots m+n}) = \sum_{i=1}^{n} s_i \]
with
\[ s_i = \sum_{\theta \in S_n, (l_1, \ldots, l_n) \in \sigma_i} (-1)^{l(\theta)} a_{1,l_1} \cdots a_{n,l_n} \otimes a_{l_1,\theta(1)} \cdots a_{l_n,\theta(n)} \]
\[ \sigma_i = \{(l_1, \ldots, l_n)|1 \leq l_1, \ldots, l_n \leq m+n, \quad |\{l_1, \ldots, l_n\} \cap \{m+1 \cdots m+n\}| = n-i\} \]

**Proposition (3.4).** \(k[GL(m|n)]\) and \(k[SL(m|n)]\) are bialgebras with comultiplication:
\[ \Delta(a_{ij}) = \sum a_{ik} \otimes a_{kl} \]
\[ \Delta(d_{1 \cdots m}^{1 \cdots m-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} d_{1 \cdots m}^{1 \cdots m-i} \otimes d_{1 \cdots m}^{1 \cdots m-i} (\Delta(d_{1 \cdots m}^{1 \cdots m}) - d_{1 \cdots m}^{1 \cdots m} \otimes d_{1 \cdots m}^{1 \cdots m})^{i-1} \]
\[ \Delta(d_{m+1 \cdots m+n}^{m+1 \cdots m+n}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} d_{m+1 \cdots m+n}^{m+1 \cdots m+n-i} \otimes d_{m+1 \cdots m+n}^{m+1 \cdots m+n-i} (\Delta(d_{m+1 \cdots m+n}^{m+1 \cdots m+n}) - d_{m+1 \cdots m+n}^{m+1 \cdots m+n} \otimes d_{m+1 \cdots m+n}^{m+1 \cdots m+n})^{i-1} \]
and counit:
\[ \epsilon(a_{ij}) = \delta_{ij}, \quad \epsilon(d_{1 \cdots m}^{1 \cdots m-1}) = 1, \quad \epsilon(d_{m+1 \cdots m+n}^{m+1 \cdots m+n-1}) = 1. \]

**Proof.** One can directly check that these maps are well defined with respect to the commutation relations among even and odd elements.

For \(k[SL(m|n)]\): \(Ber\) is a grouplike element, that is \(\Delta(Ber) = Ber \otimes Ber\). A proof of this in the quantum case is available in [LS], [P], it clearly applies also to the non quantum case, that is when \(q = 1\).

Hence we have immediately:
\[ \Delta(Ber - 1) = (Ber - 1) \otimes Ber + 1 \otimes (Ber - 1) \]
We now need to check (for both algebras the computation is the same) that:
\[ (\Delta(d_{1 \cdots m}^{1 \cdots m-1}))(\Delta(d_{1 \cdots m}^{1 \cdots m})) = 1 \otimes 1 \]
\[ (\Delta(d_{m+1 \cdots m+n}^{m+1 \cdots m+n-1}))(\Delta(d_{m+1 \cdots m+n}^{m+1 \cdots m+n})) = 1 \otimes 1 \]
By observation (3.3) we have that:

\[ \Delta(d_{1\ldots m}^1) = r_0 + \ldots + r_m, \quad \Delta(d_{m+1\ldots m+n}^1) = s_0 + \ldots + s_n, \]

\[ \Delta(d_{1\ldots m}^{m-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} r_0^{-i} (r_1 + \ldots + r_m)^{i-1} \]

\[ \Delta(d_{m+1\ldots m+n}^{m+n-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} s_0^{-i} (s_1 + \ldots + s_n)^{i-1} \]

Notice that \( r_0 = d_{1\ldots m}^1 \otimes d_{1\ldots m}^1 \) and \( s_0 = d_{m+1\ldots m+n}^{m+1\ldots m+n} \).

So:

\[
(\Delta(d_{1\ldots m}^1)) (\Delta(d_{1\ldots m}^{m-1})) = [r_0 + (r_1 + \ldots + r_m)][r_0^{-1} - r_0^{-2} (r_1 + \ldots + r_m) + \ldots ]
\]

\[
1 \otimes 1 + (r_1 + \ldots r_m) r_0 - r_0 (r_1 + \ldots r_m) - r_0^{-2} (r_1 + \ldots + r_m)^2 + \ldots \quad (\ast)
\]

We obtain a telescopic sum. The generic term is given by:

\[
g = [r_0 + (r_1 + \ldots + r_m)][\ldots + (-1)^{i-1} r_0^{-i} (r_1 + \ldots + r_m)^{i-1} + \\
\quad (+(-1)^{i} r_0^{-i-1} (r_1 + \ldots + r_m)^i + \ldots ] = \\
\quad = \ldots + (-1)^{i-1} r_0^{-i+1} (r_1 + \ldots + r_m)^i -1 + (-1)^{i-1} r_0^{-i} (r_1 + \ldots + r_m)^i + \\
\quad \quad (+(-1)^{i} r_0^{-i} (r_1 + \ldots + r_m)^i + (-1)^{i} r_0^{-i+1} (r_1 + \ldots + r_m)^{i+1} + \ldots \]
\]

Notice that the second and third term cancel out.

The last term in the sum (\( \ast \)) is given by:

\[
(-1)^{2mn+1} r_0^{-2mn-2} (r_1 + \ldots + r_m)^{2mn+1} = \\
(-1)^{2mn+1} r_0^{-2mn-2} \sum_{1 \leq i_1 \ldots i_{2mn+1} \leq m} r_{i_1} \ldots r_{i_{2mn+1}}.
\]

But

\[
r_{i_1} \ldots r_{i_{2mn+1}} = 0
\]

since it contains the product of \( 2mn + 1 \) odd indeterminates (each \( r_i \) for \( i > 1 \) contains at least one odd indeterminate). Hence we have the result.

The check:

\[
\Delta(d_{m+1\ldots m+n}^m) \Delta(d_{m+1\ldots m+n}^{m+n-1}) = 1 \otimes 1
\]

is done in the same way.

Finally one directly checks that \( \epsilon \) is a counit.

Remarks (3.5).
1. For \( n = m = 1 \) we have:

\[
\Delta((d_1^1)^{-1}) = \Delta(x_{11}^{-1}) = x_{11}^{-1} \otimes x_{11}^{-1} - x_{11}^{-2} \otimes x_{11}^{-2}(\xi_{12} \otimes \xi_{21})
\]
\[
\Delta((d_2^2)^{-1}) = \Delta(x_{22}^{-1}) = x_{22}^{-1} \otimes x_{22}^{-1} - x_{22}^{-2} \otimes x_{22}^{-2}(\xi_{21} \otimes \xi_{12})
\]

that gives precisely the formulas in example (2.3)(3).

2. Notice that if one replaces the \( \xi_{kl} \) with commuting coordinates, \( k[GL(m|n)] \) and \( k[SL(m|n)] \) are not Hopf algebras. This comes from the fact that the product of two \( m+n \) by \( m+n \) matrices whose diagonal \( m \times m \) and \( n \times n \) blocks are invertible is not a matrix of the same type. The coalgebra and Hopf algebra structures use in an essential way the supercommutativity (i.e. the presence of nilpotents).

Let’s now define the antipode \( S \).

Let \( B = (b_{ij})_{m+1 \leq i \leq m+n, 1 \leq j \leq m} \), \( C = (c_{kl})_{1 \leq k \leq m, m+1 \leq l \leq m+n} \) be the following matrices of even elements:

\[
B = X_{11} - \Xi_{12} S_{22}(X_{22}) \Xi_{21}
\]
\[
C = X_{22} - \Xi_{21} S_{11}(X_{11}) \Xi_{12}
\]

Define the matrices:

\[
S_1(B)_{ij} = S_1(b_{ij}) = \text{def} \ (-1)^{i-j} A_{ji}^B \text{det}(X_{11} - \Xi_{12} S_{22}(X_{22}) \Xi_{21})^{-1}
\]
\[
S_2(C)_{kl} = S_2(c_{kl}) = \text{def} \ (-1)^{l-k} A_{lk}^C \text{det}(X_{22} - \Xi_{21} S_{11}(X_{11}) \Xi_{12})^{-1}
\]

where \( A_{ji}^B \) and \( A_{lk}^C \) are the determinants of the minors obtained by suppressing the \( j^{th} \) row and \( i^{th} \) column in \( B \) and the \( l^{th} \) row and \( k^{th} \) column in \( C \) respectively.

**Remark (3.6)**. The determinants that appear in the definition of \( S_1 \) and \( S_2 \) are invertible in \( k[GL(m|n)] \). The fact that the determinant \( \text{det}(X_{11} - \Xi_{12} S_{22}(X_{22}) \Xi_{21}) \) is invertible is contained in the proof of Proposition (3.2). The other determinant can be proven invertible in the same way. These determinants are also invertible in \( k[SL(m|n)] \). In fact can be easily seen in \( k[SL(m|n)] \) by observing that since \( Ber = 1 \) and since:

\[
Ber = \text{det}(X_{11}) \text{det}(X_{22} - \Xi_{21} S_{11}(X_{11}) \Xi_{12})^{-1}
\]

(see [P]), we have:

\[
\text{det}(X_{11} - \Xi_{12} S_{22}(X_{22}) \Xi_{21}) = \text{det}(S_{22}(X_{22}))^{-1} = d_{m+1 \ldots m+n}^{m+1 \ldots m+n}
\]
\[
\text{det}(X_{22} - \Xi_{21} S_{11}(X_{11}) \Xi_{12}) = \text{det}((X_{11})) = d_1^{1 \ldots m}
\]
Proposition (3.7). $k[GL(m|n)]$ and $k[SL(m|n)]$ are Hopf algebras with antipode $S$:

$$S \left( \begin{array}{cc} X_{11} & \Xi_{12} \\ \Xi_{21} & X_{22} \end{array} \right) =$$

$$\left( \begin{array}{cc} S_1(X_{11} - \Xi_{12}S_{22}(X_{22})\Xi_{21}) & -S_{11}(X_{11})\Xi_{12}S_{2}(X_{22} - \Xi_{21}S_{11}(X_{11})\Xi_{12}) \\ -S_{22}(X_{22})\Xi_{21}S_1(X_{11} - \Xi_{12}S_{22}(X_{22})\Xi_{21}) & S_2(X_{22} - \Xi_{21}S_{11}(X_{11})\Xi_{12}) \end{array} \right)$$

$$S(d_{1...m}^{1...m-1}) = d_{m+1...m+n}^{m+1...m+n}$$

Proof. One can check directly that this map is well defined and that is an antipode (See [Be]).

Proposition (3.8). Let $A$ be a commutative superalgebra.

1. $\text{Hom}_{k-superalg}(k[GL(m|n)], A)$ is the group of automorphisms of $A^{m|n}$.
2. $\text{Hom}_{k-superalg}(k[SL(m|n)], A)$ is the group of automorphisms of $A^{m|n}$ with berezinian 1.

$k[GL(m|n)]$ and $k[SL(m|n)]$ are the representing objects for the functors $GL(m|n)$ and $SL(m|n)$ respectively.

Proof. Immediate.

4. The quantum $GL(m|n)$

A quantum group is an Hopf algebra which is in general neither commutative nor cocommutative. According to this philosophy we can define a quantum supergroup in the same way.

Definition (4.1). Let $A$ be a commutative (super)algebra over $k$. A formal deformation of $A$ is a non commutative (super)algebra $A_q$ over $k[q, q^{-1}]$, $q$ being an (even) indeterminate, such that $A_q/(q - 1) \cong A$. If $A$ in addition is an Hopf superalgebra we will refer to such deformation as a quantum (super)group.

We will call quantum $GL(m|n)$ a formal deformation of the supercommutative Hopf algebra $k[GL(m|n)]$ associated to the affine algebraic group $GL(m|n)$ (see §3). We cannot define quantum groups using the functor of points as we did for supergroups. This is a consequence of the non commutativity of the Hopf algebra associated to the functor of points (see remark (2.2)(1)).

We will also show that it is possible to give a deformation of the Hopf algebra $k[GL(m|n)]$ in such a way that natural coactions on quantum superspaces are preserved.

In order to define the deformed algebra $k_q[GL(m|n)]$ we need first to define the Manin matrix superalgebra $k_q[M(m|n)]$ introduced by Manin in [Ma3] and to compute the commutation rules between the generators of $k_q[M(m|n)]$ and certain quantum determinants.
Define also dual quantum superspace \( M(\mathbb{M}|n) \) by the even elements

\[
\mathbb{M}_3 = \langle x_{ij}, \xi_{kl} \rangle / \mathcal{I}_M
\]

where \( k_q^{(i)} \), \( \mathfrak{q}^{(i)} \xi_{kl} \) denotes the free algebra over \( k_q \) generated by the even variables \( x_{ij} \) for \( 1 \leq i, j \leq m \) or \( m + 1 \leq i, j \leq m + n \) and by the odd variables \( \xi_{kl} \) for \( 1 \leq k \leq m, m + 1 \leq l \leq m + n \) or \( m + 1 \leq k \leq m + n, 1 \leq l \leq m \) satisfying the relation: \( \xi_{kl}^2 = 0 \).

Let's denote as before:

\[
a_{ij} = \begin{cases} x_{ij} & \text{if } 1 \leq i, j \leq m \text{ or } m + 1 \leq i, j \leq m + n \\ \xi_{ij} & \text{if } 1 \leq i \leq m, m + 1 \leq j \leq m + n \text{ or } m + 1 \leq i \leq m + n, 1 \leq j \leq m \end{cases}
\]

The ideal \( \mathcal{I}_M \) is generated by the relations \([\mathbb{M}_3]\):

\[
a_{ij}a_{il} = (-1)^{p(i)+1} a_{il}a_{ij} \quad j < l
\]

\[
a_{ij}a_{kj} = (-1)^{p(i)+1} a_{kj}a_{ij} \quad i < k
\]

\[
a_{ij}a_{kl} = (-1)^{p(i)+1} a_{kl}a_{ij} \quad i < k, j > l \text{ or } i > k, j < l
\]

\[
a_{ij}a_{kl} = (-1)^{p(i)+1} a_{kl}a_{ij} = (-1)^{p(j)q+k} \quad i < k, j < l
\]

where \( p(i) = 0 \) if \( 1 \leq i \leq m \), \( p(i) = 1 \) otherwise and \( \pi(a_{ij}) \) denotes the parity of \( a_{ij} \).

Notice that for \( q = 1 \) this gives us the superalgebra defined in the example \((2.3)(2)\) representing the functor \( M(m|n) \).

**Definition (4.3).** Define the quantum superspace \( k_q^{m|n} \) as the ring generated over \( k_q \) by the even elements \( x_1 \ldots x_m \) and the odd elements \( \xi_1 \ldots \xi_n \) subject to the relations \([\mathbb{M}_3]\):

\[
x_i x_j - q^{-1} x_j x_i, \quad 1 \leq i < j \leq m,
\]

\[
x_i \xi_k - q^{-1} \xi_k x_i, \quad 1 \leq i \leq m, \quad m + 1 \leq k \leq n + m,
\]

\[
\xi_k^2, \quad \xi_k \xi_l + q^{-1} \xi_l \xi_k, \quad m + 1 \leq k < l \leq m + n.
\]

Define also dual quantum superspace \( (k_q^{m|n})^* \) as the ring generated over \( k_q \) by the even elements \( y_1 \ldots y_n \) and the odd elements \( \eta_1 \ldots \eta_m \) subject to the relations \([\mathbb{M}_3]\):

\[
y_i y_j - q y_j y_i, \quad 1 \leq i < j \leq m,
\]

\[
y_i \eta_k - q \eta_k y_i, \quad 1 \leq i \leq m, \quad m + 1 \leq k \leq n + m,
\]

\[
\eta_k^2, \quad \eta_k \eta_l + q \eta_l \eta_k, \quad m + 1 \leq k < l \leq m + n.
\]

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**Observation (4.4).** The superalgebra $k_q[M(m|n)]$ admits a bialgebra structure with comultiplication and counit given by:

$$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj} \quad \epsilon(a_{ij}) = \delta_{ij}$$

and a coaction on the quantum spaces $k_q^{m|n}$ and $(k_q^{m|n})^*$ ([Ma3]).

**Observation (4.5).** Let’s examine some of the relations that generate the ideal $I_M$. For $1 \leq i, j, k, l \leq m$:

- $x_{ij}x_{il} = q^{-1}x_{il}x_{ij}, \quad j < l$
- $x_{ij}x_{kj} = q^{-1}x_{kj}x_{ij}, \quad k < i$
- $x_{ij}x_{kl} = x_{kl}x_{ij}, \quad i < k, j > l$ or $i > k, j < l$

which are the usual Manin relations (see [Ma2]). We denote the two-sided ideal generated by them as $I_1^{1...m}(q)$.

For $m+1 \leq i, j, k, l \leq m+n$:

- $x_{ij}x_{il} = qx_{il}x_{ij}, \quad j < l$
- $x_{ij}x_{kj} = qx_{kj}x_{ij}, \quad k < i$
- $x_{ij}x_{kl} = x_{kl}x_{ij}, \quad i < k, j > l$ or $i > k, j < l$

These relations are the usual Manin relations where $q$ is replaced with $q^{-1}$. We denote the ideal generated by them as $I_{m+1...m+n}^{m+1...m+n}(q^{-1})$.

Let’s define:

- $D_1^{1...m} = \text{def} \sum_{\sigma \in S_m} (-q)^{-l(\sigma)}x_{1\sigma(1)} \cdots x_{m\sigma(m)}$
- $D_{m+1...m+n}^{m+1...m+n} = \text{def} \sum_{\sigma \in S_n} (-q)^{l(\sigma)}x_{m+1,m+\sigma(1)} \cdots x_{m+n,m+\sigma(n)}$

$D_1^{1...m}$ and $D_{m+1...m+n}^{m+1...m+n}$ represent respectively the quantum determinants of the quantum matrix bialgebras:

- $M_q(1,m) = k_q < x_{ij} > / I_1^{1...m}(q), \quad 1 \leq i, j \leq m$
- $M_{q^{-1}}(m+1,m+n) = k_q < x_{kl} > / I_{m+1...m+n}^{m+1...m+n}(q^{-1}), \quad m+1 \leq k, l \leq m+n$

$D_1^{1...m}$ and $D_{m+1...m+n}^{m+1...m+n}$ are central elements in $M_q(1,m)$ and $M_{q^{-1}}(m+1,m+n)$ respectively ([PW] pg 50).
Further properties of these algebras and their quantum determinants are studied in [PW] ch. 5.

We define:

\[ GL_q(1,m) = M_q(1,m) < D^{1...m-1}_{1...m} > / (D^{1...m}_{1...m} D^{m...m-1}_{m...m} - 1), \]

\[ GL_q^{-1}(m + 1,m + n) = \frac{M_{q^{-1}}(m + 1,m + n) < D^{m+1...m+n-1}_{m+1...m+n} >}{(D^{m+1...m+n}_{m+1...m+n} D^{m+1...m+n-1}_{m+1...m+n} - 1)}. \]

These are Hopf algebras, their antipodes \( S^q_{11}, S^q_{22} \) are explicitly calculated in [PW] pg 57.

**Definition (4.6). Quantum general linear supergroup.**

\[ k_q[GL(m|n)] = \text{def} k_q[M(m|n)] < D^{1...m-1}_{1...m}, D^{m+1...m+n-1}_{m+1...m+n} > \]

where \( D^{1...m-1}_{1...m}, D^{m+1...m+n-1}_{m+1...m+n} \) are even indeterminates such that:

\[ D^{1...m}_{1...m} D^{1...m-1}_{1...m} = 1 = D^{1...m-1}_{1...m} D^{1...m}_{1...m}, \]

\[ D^{m+1...m+n}_{m+1...m+n} D^{m+1...m+n}_{m+1...m+n} = 1 = D^{m+1...m+n}_{m+1...m+n} D^{m+1...m+n}_{m+1...m+n} \]

where \( X_{11} = (x_{ij})_{1\leq i,j \leq m}, X_{22} = (x_{ij})_{m+1\leq i,j \leq m+n} \) are matrices of even indeterminates and \( \Xi_{12} = (\xi_{kl})_{1\leq k\leq m,m+1\leq l\leq m+n}, \Xi_{22} = (\xi_{kl})_{m+1\leq k\leq m+n,1\leq l\leq m} \) are matrices of odd indeterminates. The \( x_{ij} \)'s, \( \xi_{kl} \)'s are the generators of \( k_q[M(m|n)] \).

\( \text{det}_q(M) \) denotes the determinant of the Manin matrix \( M \) (that is a matrix of indeterminates satisfying the Manin commutation relations). One can verify ([PW] ch. 5) that

\[ \text{det}_q(S^q_{22}(X_{22})) = D^{m+1...m+n}_{m+1...m+n}. \]

By an abuse of language we will denote with the same letters \( x_{ij}, \xi_{kl} \) and \( a_{ij} \) the indeterminates generators of the rings \( k_q[M(m|n)] \) and \( k_q[GL(m|n)] \) the context making clear where the elements sit.

We also define:

\[ \text{Ber}_q = \text{def} \text{det}_q(S^q_{22}(X_{22})) \text{det}_q(X_{11} - \Xi_{12} S^q_{22}(X_{22}) \Xi_{21}). \]

called the quantum berezinian. \( \text{Ber}_q \) is invertible in \( k_q[GL(m|n)] \), the proof is a small variation of the one in (3.2).

We now want to explicitly give the coalgebra structure for the ring \( k_q[GL(m|n)] \). We need some lemmas on commutation relations.
Lemma (4.7).

1) \( D_{1\ldots m}^{1\ldots m} \xi_{ij} = q^{-1} \xi_{ij} D_{1\ldots m}^{1\ldots m} \), \( D_{1\ldots m}^{1\ldots m} \xi_{ij} = q \xi_{ij} D_{1\ldots m}^{1\ldots m} \),

2) \( D_{m+1\ldots m+n}^{m+1\ldots m+n} \xi_{ij} = q^{-1} \xi_{ij} D_{m+1\ldots m+n}^{m+1\ldots m+n} \), \( D_{m+1\ldots m+n}^{m+1\ldots m+n} \xi_{ij} = q \xi_{ij} D_{m+1\ldots m+n}^{m+1\ldots m+n} \),

where \( 1 \leq i \leq m, m+1 \leq j \leq m+n \) or \( m+1 \leq i \leq m+n, 1 \leq j \leq m \).

**Proof.** Let’s prove (1). It is enough to prove the first commutation relation. We will first do for \( 1 \leq i \leq m, m+1 \leq j \leq m+n \).

By induction on \( m \). For \( m = 1 \) it is a direct simple check. Assume for now that \( 1 \leq i < m, m+1 \leq j \leq m+n \) (the case \( i = m \) is treated separately).

Let \( D_{1\ldots \hat{\alpha}\ldots m}^{1\ldots \hat{\alpha}\ldots m} \) denote the quantum determinant of the quantum minor obtained from the quantum matrix \( X_{11} \) by suppressing row \( \alpha \) and column \( \beta \).

By [PW] formula at pg 47 on Laplace expansion of quantum determinants we have:

\[
D_{1\ldots m}^{1\ldots m} \xi_{ij} = \sum_{s=1}^{m} (-q)^{s-m} D_{1\ldots m-1}^{1\ldots m} x_{ms} \xi_{ij} = \sum_{s=1}^{m} (-q)^{s-m} D_{1\ldots m-1}^{1\ldots m} \xi_{ij} x_{ms}.
\]

By induction

\[
D_{1\ldots m-1}^{1\ldots m-1} \xi_{ij} = q^{-1} \xi_{ij} D_{1\ldots m-1}^{1\ldots m-1}
\]

hence we have our result.

Now the case \( i = m, m+1 \leq j \leq m+n \). By [PW] formula at pg 47 we have:

\[
D_{1\ldots m}^{1\ldots m} \xi_{mj} = \sum_{s=1}^{m} (-q)^{s-1} D_{2\ldots m}^{2\ldots m} x_{1s} \xi_{mj} = \sum_{s=1}^{m} (-q)^{s-1} D_{2\ldots m}^{2\ldots m} [\xi_{mj} x_{1s} + (q^{-1} - q) \xi_{1j} x_{ms}].
\]

By induction:

\[
D_{2\ldots m}^{2\ldots m} \xi_{mj} = q^{-1} \xi_{mj} D_{2\ldots m}^{2\ldots m}.
\]

So we have:

\[
D_{1\ldots m}^{1\ldots m} \xi_{mj} = q^{-1} \xi_{mj} D_{1\ldots m}^{1\ldots m} + (q^{-1} - q) \sum_{s=1}^{m} (-q)^{s-1} D_{2\ldots m}^{2\ldots m} x_{ms} \xi_{1j}.
\]

But notice that ([PW] pg 47):

\[
\sum_{s=1}^{m} (-q)^{s-1} D_{2\ldots m}^{2\ldots m} x_{ms} = \delta_{1m} D_{1\ldots m}^{1\ldots m} = 0
\]

hence we have our result.

The case \( m+1 \leq i \leq m+n, 1 \leq j \leq m \) is done in a similar way.
The proof of (2) goes along the same lines.

**Lemma (4.8).**

1.a) \[ D_{1...m}^{1...m} a_{1,k_1} \cdots a_{m,k_m} = q^{-t}a_{1,k_1} \cdots a_{m,k_m} D_{1...m}^{1...m} \]

1.b) \[ D_{1...m}^{1...m} a_{k_1,1} \cdots a_{k_m,m} = q^{-t}a_{k_1,1} \cdots a_{k_m,m} D_{1...m}^{1...m} \]

2.a) \[ D_{m+1...m+n}^{1...m+n} a_{m+1,l_1} \cdots a_{m+n,l_n} = q^{-s}a_{m+1,l_1} \cdots a_{m+n,l_n} D_{m+1...m+n}^{1...m+n} \]

2.b) \[ D_{m+1...m+n}^{1...m+n} a_{l_1,m+1} \cdots a_{l_n,m+n} = q^{-s}a_{l_1,m+1} \cdots a_{l_n,m+n} D_{m+1...m+n}^{1...m+n} \]

where \( t \) is such that \( m - t = |\{k_1 \ldots k_m\} \cap \{1 \ldots m\}|, 1 \leq k_1 < \ldots < k_m \leq m + n \) and \( s \) is such that \( n - s = |\{l_1 \ldots l_n\} \cap \{m + 1 \ldots m + n\}|, 1 \leq l_1 < \ldots < l_n \leq m + n \).

**Proof.** Let’s prove (1.a). By lemma (4.7) since \( D_{1...m}^{1...m} \) commutes with \( x_{ij} \) ([PW] pg 50), \( 1 \leq i,j \leq m \) we have that for \( 1 \leq i \leq m \):

\[
D_{1...m}^{1...m} a_{i,k_j} = \begin{cases} 
  a_{i,k_j} D_{1...m}^{1...m} & \text{for } 1 \leq k_j \leq m \\
  q^{-1}a_{i,k_j} D_{1...m}^{1...m} & \text{otherwise}
\end{cases}
\]

Hence we have the result. The proofs of (1.b) and (2.a), (2.b) are the same.

As in the supercommutative case we can write:

\[
\Delta(D_{1...m}^{1...m}) = \sum_{i=1}^{m} R_i
\]

with

\[
R_i = \sum_{\sigma \in S_m, (k_1, \ldots, k_m) \in \rho_i} (-q)^{-l(\sigma)} a_{1,k_1} \cdots a_{m,k_m} \otimes a_{k_1,1} \cdots a_{k_m,1} D_{1...m}^{1...m} \]

\[
\rho_i = \{(k_1, \ldots, k_m)|1 \leq k_1, \ldots, k_m \leq m + n, \ |\{k_1, \ldots, k_m\} \cap \{1 \ldots m\}| = m - i\}
\]

Similarly one can write:

\[
\Delta(D_{m+1...m+n}^{m+1...m+n}) = \sum_{i=1}^{n} S_i
\]

with

\[
s_i = \sum_{\theta \in S_n, (l_1, \ldots, l_n) \in \sigma_i} (-q)^{-l(\theta)} a_{1,l_1} \cdots a_{n,l_n} \otimes a_{l_1,1} \cdots a_{l_n,1} D_{m+1...m+n}^{m+1...m+n} \]

\[
\sigma_i = \{(l_1, \ldots, l_n)|1 \leq l_1, \ldots, l_n \leq m + n, \ |\{l_1, \ldots, l_n\} \cap \{m + 1 \ldots m + n\}| = n - i\}.
\]
Lemma (4.9).
1) $R_0 R_i = q^{-2i} R_i R_0$, \hspace{0.5cm} $R_0^{-1} R_i = q^{2i} R_i R_0^{-1}$
2) $S_0 S_i = q^{-2i} S_i S_0$, \hspace{0.5cm} $S_0^{-1} S_i = q^{2i} S_i S_0^{-1}$

Proof. Immediate from lemma (4.8) noting that $R_0 = D_{1:m}^1 \otimes D_{1:m}^1$ and $S_0 = D_{m+1:m+n}^m \otimes D_{m+1:m+n}^m$.

Lemma (4.10).
1. $R_0 (R_1 + \ldots + R_m) = (q^{-2} R_1 + \ldots + q^{-2m} R_m) R_0$
2. $(R_1 + \ldots + R_m) R_0^{-1} = R_0^{-1} (q^{-2} R_1 + \ldots + q^{-2m} R_m)$
3. $S_0 (S_1 + \ldots + S_n) = (q^{-2} S_1 + \ldots + q^{-2n} S_n) S_0$
4. $(S_1 + \ldots + S_n) S_0^{-1} = S_0^{-1} (q^{-2} S_1 + \ldots + q^{-2n} S_n)$

Proof. This is an immediate application of lemma (4.9).

Lemma (4.11). Let $a_{i_1^k, j_1^k} \ldots a_{i_m^k, j_m^k} \in k_q[GL(m|n)]$, for all $s$, $1 \leq s \leq 2mn + 1$. Assume that for all $s$ at least one $(i_k^s, j_k^s)$ is such that $1 \leq i_k \leq m$, $m \leq j_k \leq m + n$ or $m + 1 \leq i_k \leq m + n$, $1 \leq j_k \leq m$, that is $a_{i_k^s, j_k^s}$ is odd. Then

$$\prod_{s=1}^{2mn+1} a_{i_1^s, j_1^s} \ldots a_{i_m^s, j_m^s} = 0$$

Proof. Let $i$ be the map:

$$k_q[M(m|n)] \xrightarrow{i} k_q[M(m|n)] < D_{1:m}^{m-1}, D_{m+1:m+n}^{m+n-1} >$$

Let $X \in k_q[GL(m|n)]$ such that there exists a $X_0 \in k_q[M(m|n)]$, $i(X_0) = X$. If one wants to prove $X = 0$ it is enough to show $X_0 = 0$.

Now let

$$X = \prod_{s=1}^{2mn+1} a_{i_1^s, j_1^s} \ldots a_{i_m^s, j_m^s}.$$ 

If $X_0 = \prod_{s=1}^{2mn+1} a_{i_1^s, j_1^s} \ldots a_{i_m^s, j_m^s} \in k_q[M(m|n)]$ we have that $i(X_0) = X$.

By the previous argument it is enough to show that $X_0 = 0$.

By [Ma3] pg 172 we have that $\{1, a_{i_1, j_1} \ldots a_{i_r, j_r}\} \{(i_1,j_1) \leq \ldots \leq (i_r,j_r), r \geq 1\}$ form a basis for $k_q[M(m|n)]$, where $\leq$ is a suitable ordering on the indeces $(i_k, j_k)$.

Since the Manin relations are homogeneous we have:

$$X_0 = \sum_{(r_1,s_1) \leq \ldots \leq (r_t,s_t), c_{(r_1,s_1) \ldots (r_t,s_t)} \in k_q} c_{(r_1,s_1) \ldots (r_t,s_t)} a_{r_1,s_1} \ldots a_{r_t,s_t} \hspace{0.5cm} (*)$$

where the sum is taken over a suitable set of $((r_1, s_1) \ldots (r_t, s_t))$ with $t = m(2mn + 1)$. \hspace{2cm} 17
By hypothesis each term in the sum (⋆) contains \(2mn + 1\) odd indeterminates and since the indeces are ordered, it is 0.

**Proposition (4.12).** Coalgebra structure for \(k_q[GL(m|n)]\).

\(k_q[GL(m|n)]\) is a coalgebra with comultiplication:

\[
\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kl}
\]

\[
\Delta(D_{1\ldots m}^{1\ldots m-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^{i-1}
\]

\[
\Delta(D_{m+1\ldots m+n}^{m+1\ldots m+n-1}) = \sum_{i=1}^{2mn+2} (-1)^{i-1} S_0^{-1} [S_0^{-1}(q^{-2} S_1 + \ldots + q^{-2n} S_n)]^{i-1}
\]

and counit:

\[
\epsilon(a_{ij}) = \delta_{ij}, \quad \epsilon(D_{1\ldots m}^{1\ldots m-1}) = 1, \quad \epsilon(D_{m+1\ldots m+n}^{m+1\ldots m+n-1}) = 1
\]

**Proof.** \(\Delta\) is well defined on all the commutation relations among the generators.

We only need to check that

\[
\Delta(D_{1\ldots m}^{1\ldots m}) \Delta(D_{1\ldots m}^{1\ldots m-1}) = 1 = \Delta(D_{1\ldots m}^{1\ldots m-1}) \Delta(D_{1\ldots m}^{1\ldots m}),
\]

\[
\Delta(D_{m+1\ldots m+n}^{m+1\ldots m+n}) \Delta(D_{m+1\ldots m+n}^{m+1\ldots m+n-1}) = 1 = \Delta(D_{m+1\ldots m+n}^{m+1\ldots m+n-1}) \Delta(D_{m+1\ldots m+n}^{m+1\ldots m+n})
\]

Let’s check the first one.

\[
\Delta(D_{1\ldots m}^{1\ldots m}) \Delta(D_{1\ldots m}^{1\ldots m-1}) =
\]

\[
[R_0 + (R_1 + \ldots + R_m)][R_0^{-1} - R_0^{-2}(q^{-2} R_1 + \ldots + q^{-2m} R_m) + \ldots] =
\]

\[
= 1 \otimes 1 + (R_1 + \ldots R_m) R_0^{-1} - R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m) +
\]

\[
- (R_1 + \ldots + R_m) R_0^{-2}(q^{-2} R_1 + \ldots + q^{-2m} R_m) + \ldots
\]
We obtain a telescopic sum. In fact, let’s see the generic terms:

\[
G = [R_0 + (R_1 + \ldots + R_m)][\ldots + (-1)^{i-1}R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)]^{i-1} + \\
+(-1)^iR_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)^i + \ldots = \\
= \ldots + (-1)^{i-1}[R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)]^{i-1} + \\
+(-1)^i(R_1 + \ldots + R_m)R_0^{-1}[R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)]^{i-1} + \\
+(-1)^i[R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)]^i + \\
+(-1)^i(R_1 + \ldots + R_m)R_0^{-1}[R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)]^i.
\]

By lemma (4.10)(2) we have that \((R_1 + \ldots + R_m)R_0^{-1} = R_0^{-1}(q^{-2}R_1 + \ldots + q^{-2m}R_m)\) so the second and third term in the sum \(G\) cancel each other.

The last term of the sum is 0 by lemma (4.11).

This completes the check for the first part of the first relation.

Let’s see the second part.

\[
\Delta(D_{1\ldots m}^{1\ldots m})\Delta(D_{1\ldots m}^{1\ldots m}) = \\
= [R_0^{-1} - R_0^{-2}(q^{-2}R_1 + \ldots + q^{-2m}R_m) + \ldots][R_0 + (R_1 + \ldots + R_m)] = \\
= 1 \otimes 1 + R_0^{-1}(R_1 + \ldots R_m) - R_0^{-2}(q^{-2}R_1 + \ldots + q^{-2m}R_m)R_0 + \\
- R_0^{-2}(q^{-2}R_1 + \ldots + q^{-2m}R_m)(R_1 + \ldots + R_m) + \ldots
\]

By lemma (4.10)(1) the second and third term cancel out. Let’s see as before the generic
terms $G'$:

$$G' = \ldots + (-1)^{i-1} R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^{i-1} +$$

$$+ (-1)^i R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^i + \ldots [R_0 + (R_1 + \ldots + R_m)] =$$

$$= \ldots + (-1)^{i-1} R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^{i-1} R_0 +$$

$$+ (-1)^i R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^i (R_1 + \ldots + R_m) +$$

$$+ (-1)^i R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^i (R_1 + \ldots + R_m) + \ldots$$

We now look at the second and third term in $G'$.

$$+ (-1)^i R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^i (R_1 + \ldots + R_m) +$$

$$+ (-1)^i R_0^{-1} [R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m)]^{i-1} R_0^{-1}(q^{-2} R_1 + \ldots + q^{-2m} R_m) R_0.$$  

By lemma (4.10)(1) we have that $(q^{-2} R_1 + \ldots + q^{-2m} R_m) R_0 = R_0 (R_1 + \ldots + R_m)$, so the second and third term cancel each other. The last term of the sum is 0 by lemma (4.11).

The second relation can be checked in the same way.

**Proposition (4.13).** $k_q[GL(m|n)]$ admits a coaction on $k_q^{m|n}$, $k_q^{m|n^*}$.

**Proof.** Immediate.

Now we need to give the antipode on $k_q[GL(m|n)]$.

**Proposition (4.14).** $k_q[GL(m|n)]$ is an Hopf algebra with antipode $S^q$ given by:

$$S^q(X_{11}) = S^q_{11}(X_{11}) + S^q_{11}(X_{11}) \Xi_{12} S^q_2(X_{22} - \Xi_{21} S^q_{11}(X_{11}) \Xi_{12}) \Xi_{21} S^q_{11}(X_{11})$$

$$S^q(\Xi_{12}) = -S^q_{11}(X_{11}) \Xi_{12} S^q_2(X_{22} - \Xi_{21} S^q_{11}(X_{11}) \Xi_{12})$$

$$S^q(\Xi_{21}) = -S^q_2(X_{22} - \Xi_{21} S^q_{11}(X_{11}) \Xi_{12}) \Xi_{21} S^q_{11}(X_{11})$$

$$S^q(X_{22}) = S^q_2(X_{22} - \Xi_{21} S^q_{11}(X_{11}) \Xi_{12})$$
\[ S^q(D_{1\ldots m}^{1\ldots m-1}) = D_{m+1\ldots m+n}^{m+1\ldots m+n+1} \]

\[ S^q(D_{m+1\ldots m+n}^{m+1\ldots m+n-1}) = D_{1\ldots m}^{1\ldots m} \]

where \( X_{22} - \Xi_{21} S^q_{11}(X_{11}) \Xi_{12} \) is a quantum matrix (see [P] §4) and \( S^q_2 \) denotes its quantum antipode.

**Proof.** See [P] §4.

**Remark (4.15).** In the ring \( k_q[GL(m|n)] \) the parameter \( q \) can be specialized to any value in \( k^\times \). Hence in the definition of \( k_q[GL(m|n)] \) \( q \) can be also taken as any element in \( k^\times \).

**Example (4.16):** \( k_q[GL(1|1)] \).

\[ k_q[GL(1|1)] = k < x_{11}, \xi_{12}, \xi_{21}, x_{22}, x_{11}^{-1}, x_{22}^{-1} > /I \]

where \( I \) is the ideal generated by the relations:

\[ x_{11} \xi_{12} = q^{-1} \xi_{12} x_{11}, \quad x_{11} \xi_{21} = q^{-1} \xi_{21} x_{11}, \]

\[ \xi_{21} x_{22} = qx_{22} \xi_{21}, \quad \xi_{12} x_{22} = qx_{22} \xi_{12}, \]

\[ x_{11} x_{22} - x_{22} x_{11} = (q - q^{-1}) \xi_{12} \xi_{21} \]

\[ \xi_{12} \xi_{21} = -\xi_{21} \xi_{12}, \quad \xi_1^2 = \xi_2^2 = 0. \]

The coalgebra structure is the following.

\[ \Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj} \]

\[ \Delta(x_{11}^{-1}) = x_{11}^{-1} \otimes x_{11}^{-1} - q^{-2} x_{11}^{-2} \xi_{12} \otimes x_{11}^{-2} \xi_{21}, \]

\[ \Delta(x_{22}^{-1}) = x_{22}^{-1} \otimes x_{22}^{-1} - q^2 x_{22}^{-2} \xi_{21} \otimes x_{22}^{-2} \xi_{12}, \]

\[ \epsilon(x_{11}^{-1}) = \epsilon(x_{22}^{-1}) = 1, \quad \epsilon \begin{pmatrix} x_{11} & \xi_{12} \\ \xi_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ S^q \begin{pmatrix} x_{11} & \xi_{12} \\ \xi_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (x_{11} - \xi_{12} x_{22}^{-1} \xi_{21})^{-1} & -x_{11}^{-1} \xi_{12} (x_{22} - \xi_{21} x_{11}^{-1} \xi_{12})^{-1} \\ -x_{22}^{-1} \xi_{21} (x_{11} - \xi_{12} x_{22}^{-1} \xi_{21})^{-1} & (x_{22} - \xi_{21} x_{11}^{-1} \xi_{12})^{-1} \end{pmatrix} \]

\[ S(x_{11}^{-1}) = x_{22} \]

\[ S(x_{22}^{-1}) = x_{11} \]
$k_q[GL(1|1)]$ admits a coaction on the quantum spaces:

\[
k_q^{1|1} = k < x, \xi > / (x\xi - q^{-1}\xi x)
\]

\[
(k_q^{1|1})^* = k < y, \eta > / (y\eta - q^{-1}\eta y)
\]

In this particular case is it immediate to construct also a deformation for the Hopf algebra $k[SL(1|1)]$. In fact since the quantum berezinian:

\[
Ber_q = x_{22}^{-1}(x_{11} - \xi_{12}x_{22}^{-1}\xi_{21})
\]

is a central element in $k_q[M(m|n)]$ and in $k_q[GL(m|n)]$ we can define:

\[
k_q[SL(1|1)] = k < x_{11}, \xi_{12}, \xi_{21}, x_{22}, x_{11}^{-1}, x_{22}^{-1} > / I'
\]

where $I'$ is the two-sided ideal generated by the same relations as $I$ together with the extra relation $Ber_q = 1$.

The comultiplication, counit, antipode are the same as $k_q[GL(1|1)]$, in fact one can check directly that they are still well defined.

We plan to construct in a forthcoming paper the deformation $k_q[SL(m|n)]$ and its relation with $k_q[GL(m|n)]$.

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