Collective model with isovector pair and alpha-particle type correlations

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The collective Hamiltonian including isovector pairing and α-particle type correlation degrees of freedom is constructed. The Hamiltonian is applied to description of the relative energies of the ground states of even-even nuclei around $^{56}$Ni. A satisfactory description of the experimental data is obtained. A significant improvement of the agreement with the experimental data compared to our previous calculations is explained by inclusion in the Hamiltonian of the dynamical variables describing α-particle type correlations.

Keywords: isovector pairing; alpha-particle type correlations, collective Hamiltonian; isospin.

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1. Introduction

It is well known how important is the role of the pair correlations of nucleons in description of the structure of atomic nuclei. In the case of nuclei with $N \approx Z$ in which protons and neutrons fill the same single particle levels it is necessary to take into account not only pair correlations of nucleons of the same kind but also neutron-proton pair correlations. It means that at least the isovector pair correlations should be considered.

The isovector pair correlations were first considered by A. Bohr and continued in Refs. In the last years a lot of papers have been dedicated to this problem. The revival of the interest to this problem is related to the open experimental opportunity to study structure of the heavier nuclei close to the $N = Z$ line where...
the isovector pair correlations play an important role. Another important subject related to the isovector pair correlations is the masses of nuclei.\cite{8,9}

In principle, in nuclei with $N \approx Z$ both isovector $T = 1$ and isoscalar $T = 0$ pair correlations can take place. The majority of investigations have been concentrated on the study of the proton-neutron pair correlations in the framework of the BCS-type microscopic models.\cite{10,11,12,13,14,15} It was noted also that it is important in these type of studies to conserve exactly the particle number and isospin.\cite{16,17,18} The approach keeping a conservation of isospin and based on the introduction of the alpha-particle like quartets was developed in\cite{23,24,25}

In our previous paper\cite{26} isovector pair correlations have been considered in the framework of the collective model formulated in Refs.\cite{6,7} This approach was applied to consideration of the energies of the ground states of nuclei around $^{56}$Ni. The results of calculations have shown that especially large deviations from the experimental data are obtained for the ground states of nuclei with even numbers of $Z$ and $N$, i.e. for nuclei which can be considered as systems of some numbers of $\alpha$-particle type quartets. This indicates that isovector pairing forces although contribute but don’t exhaust the alpha-particle type correlations in considered nuclei. For this reason, the aim of the present paper is to construct the collective Hamiltonian which includes not only the isovector pairing degree of freedom but also those degrees of freedom which could be responsible for the additional alpha-particle type correlations. This Hamiltonian is applied below to description of the relative energies of the ground states of nuclei around $^{56}$Ni. The expressions for the amplitudes of the two-nucleon and $\alpha$-particle transfer reactions are also presented.

2. Collective Hamiltonian. Kinetic energy.

There are two sets of dynamical variables, namely, corresponding to pair addition and pair removal modes, needed to describe pair correlations in nuclei.\cite{5} Each of these two sets of variables is characterized by the three projections of isospin corresponding to neutron-neutron, neutron-proton and proton-proton correlated pairs. Thus, there are six collective variables in total needed to describe isovector pair correlations. They can be presented by the complex isovector $z_{1\mu}$ ($\mu = 0, \pm 1$). It was suggested in\cite{6} to separate in $z_{1\mu}$ the variables related to isospin invariance and gauge invariance

$$z_{1\mu}^+ = \Delta \exp(i\phi) \left( D_{\mu 0}^1(\psi_1, \psi_2, \psi_3) \cos \theta + \frac{1}{\sqrt{2}}(D_{\mu 1}^1(\psi_1, \psi_2, \psi_3) + D_{\mu -1}^1(\psi_1, \psi_2, \psi_3)) \sin \theta \right). \tag{1}$$

Here $D_{\mu 1}^1(\psi_k, \psi_2, \psi_3)$ is the Wigner function and $\psi_1, \psi_2, \psi_3$ are Euler angles characterizing orientation in isospace. The angle $\phi$ is conjugate to the operator of the number of the nucleon pairs $\hat{N} = -i\partial/\partial \phi$ added or removed from the basic nucleus. The variable $\Delta$ characterizes a strength of the isovector pair correlations and $\theta$ describes isospin structure of the pair correlations in the intrinsic frame.
In addition to $z_{1\mu}$ we introduce a dynamical variable $\alpha$ describing the $\alpha$-particle type correlations

$$\alpha = \exp(2i\phi)a. \quad (2)$$

This variable depends on the same gauge angle $\phi$, however, with factor 2 in the exponent since addition or removal of the $\alpha$-particle type quartet changes the number of nucleon pairs in the nucleus by two. The variable $a$ characterizes a strength of the $\alpha$-particle type correlations. This dynamical variable has no isospin indices since it corresponds to excitation of the isoscalar mode.

Our next task is to construct a collective Hamiltonian depending on $z_{1\mu}$ and $\alpha$. The greatest challenge is a construction of the kinetic energy operator. We begin by writing a classical expression for the kinetic energy

$$T = \frac{1}{2} B_\Delta \sum_\mu \dot{z}_{1\mu}^* \dot{z}_{1\mu} + \frac{1}{2} B_\alpha \dot{\alpha}^* \dot{\alpha} \quad (3)$$

It is assumed in (3) that inertia coefficients do not depend on dynamical variables. This Hamiltonian can be quantized using Pauli prescription. The procedure is analogous to that used in and for the isovector pairing mode is presented in Ref. We extended it by including the $\alpha$-particle correlation mode which is very important for the correct description of the energies as it will be shown below. The details of derivation are given in the Appendix.

The expression for the kinetic energy term of the collective Hamiltonian containing both the isovector pairing and isoscalar $\alpha$-particle type correlation degrees of freedom is given by expression (A.22).

The collective potential for isovector pair correlations depends on two invariants: $\Delta^2$ and $\Delta^4 \cos^2 \theta$. In collective potential for isovector pair correlations was derived using the boson representation technique. It was shown there that the minimum of the potential is located at $\theta=0$. We assume below that the amplitude of fluctuations of $\theta$ around $\theta=0$ is small for the lowest states with given $N$ and $T$. As a consequence we put $\theta=0$ in the expression for the kinetic energy term in (A.22) if it does not create singularities. In the case of the kinetic energy of $\theta$ vibrations and the moment of inertia of the isospin rotations around $z$-axes we keep the lowest order terms in $\theta$. As a result we obtain for $\hat{T}$:

$$\hat{T} = \frac{1}{2} \frac{\hat{T}_2^2 + \hat{T}_y^2}{B_\Delta \Delta_2^2} + \frac{1}{2} \frac{\hat{N}^2}{B_\Delta \Delta_2^2 + 4B_\alpha a^2}$$

$$- \frac{1}{2B_\Delta \Delta^4 \sqrt{B_\Delta \Delta_2^2 + 4B_\alpha a^2}} \frac{\partial}{\partial \Delta} \Delta^4 \sqrt{B_\Delta \Delta_2^2 + 4B_\alpha a^2} \frac{\partial}{\partial \Delta}$$

$$- \frac{1}{2B_\Delta \Delta_2^2 \theta} \frac{\partial}{\partial \theta} \theta \frac{\partial}{\partial \theta} + \frac{\hat{T}_2^2}{B_\Delta \Delta_2^2 \theta^2}$$

$$- \frac{1}{2B_\alpha \sqrt{B_\Delta \Delta_2^2 + 4B_\alpha a^2}} \frac{\partial}{\partial a} \sqrt{B_\Delta \Delta_2^2 + 4B_\alpha a^2} \frac{\partial}{\partial a} \quad (4)$$

As the next step we decouple approximately $\theta$ and $\Delta$ modes replacing $\Delta$ in the kinetic energy term for $\theta$-vibrations by the average value of $\Delta^2$, namely, by $\langle \Delta^2 \rangle$. 
This is done in analogy with the consideration of the critical point symmetries in nuclear structure physics related to the collective quadrupole excitations. For the lowest states with given \( N \) and \( T \) we can put \( T_z=0 \). As the result, the \( \theta \)-dependent part of the collective Hamiltonian, being completely separated from the other terms, will give the same contribution into the energies of the lowest states for each \( N \) and \( T \). Since we will compare with the experimental data only the relative energies of nuclei counted from the energy of the ground state of the basic nucleus, we exclude \( \theta \)-mode from consideration below.

3. Collective Hamiltonian. Potential energy

In our previous paper we based on the phenomenological approach to determination of the collective potential and considered various variants of a \( \Delta \)-dependence of the collective potential. It was found that Davidson’s potential containing one dimensional and one nondimensional parameters is the best suited to describe the energies of the collective pair excitations of nuclei in the \( ^{56}\text{Ni} \) region. By changing the value of the nondimensional parameter we can describe all intermediate cases between vibrational and rotational limits of pair correlations. We have seen in the previous section that inclusion into the Hamiltonian of the collective variable corresponding to the \( \alpha \)-particle mode leads to a significant complication of the terms in \( T \) corresponding to kinetic energies of \( \Delta \)- and \( a \)- modes. There appears the expression \( \sqrt{B_\Delta \Delta^2 + 4B_\alpha a^2} \) which changes the coefficients at the first derivatives over \( \Delta \) and \( a \). However, it is well known that by some transformation of the wave function it is possible to change a coefficient at the first derivative or even to eliminate this term in the Hamiltonian. It leads to appearance of additional terms in the expression for the potential energy. But in our case, when the potential energy is treated phenomenologically, we can ignore this effect and exclude from consideration the corresponding factors in the expression for the kinetic energy. As the result, the Hamiltonian takes the following form where we put \( T(T+1) \) instead of \( (\hat{T}_x^2 + \hat{T}_y^2) \) since for the lowest states for each set of \( N \) and \( T \), \( T_z=0 \):

\[
\hat{H} = \frac{T(T+1)}{2B_\Delta \Delta^2} + \frac{N^2}{2B_\Delta \Delta^2} + \frac{4B_\alpha a^2}{2B_\Delta \Delta^2} - \frac{1}{2B_\Delta \Delta^5} \frac{\partial}{\partial \Delta} \frac{\Delta^5}{\Delta} - \frac{1}{2B_\alpha} \frac{\partial^2}{\partial a^2} + \frac{1}{2} C \Delta \left( \frac{\Delta^4}{\Delta^2} + \Delta^2 \right) + V_\alpha(a)
\]  

(5)

In (5) \( V_\alpha \) is the potential energy of the \( \alpha \)-mode.

To determine dependence of the potential energy on \( a \) let us consider the relative energies of nuclei around \( ^{56}\text{Ni} \) which differ by some number of \( \alpha \)-particles, i.e. \( ^{48}\text{Cr} \), \( ^{52}\text{Fe} \), \( ^{60}\text{Zn} \), and \( ^{62}\text{Ge} \). There was suggested a procedure to extract from the experimental data the quantities which can be directly compared with the results of calculations based on the collective Hamiltonian. It is necessary to do since the ground state energies of different nuclei are compared. This procedure has been used in Ref. According to this procedure it is necessary to subtract from the
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experimental binding energies of nuclei under consideration those contributions that are generated by the sources other than the isovector monopole pairing and \(\alpha\)-particle type correlations. For nuclei around the basic nucleus with \(A = A_0\) and \(Z = Z_0\), which in our case is \(^{56}\text{Ni}\), the following quantity is defined.

\[
E(A, Z) = -(B_{\text{exp}}(A, Z) - B_{LD}(A, Z)) + (B_{\text{exp}}(A_0, Z_0) - B_{LD}(A_0, Z_0)),
\]

(6)

where \(B_{\text{exp}}(A, Z)\) is the experimental binding energy of the nucleus with mass number \(A\) and charge \(Z\). The quantity \(B_{LD}(A, Z)\) is defined by the liquid drop mass formula without the symmetry energy and the pairing energy terms:

\[
B_{LD}(A, Z) = a_v A - a_s A^{2/3} - a_c Z(Z + 1) A^{1/3},
\]

(7)

where \(a_v=15.75\) MeV, \(a_s=17.8\) MeV, and \(a_c=0.711\) MeV.

Fig. 1 shows the spectrum of the ground state energies of nuclei that differ from \(^{56}\text{Ni}\) by one- and two \(\alpha\)-particles. For convenience, a linear in \(A\) term has been added, in order to give the equal energies to \(^{52}\text{Fe}\) and \(^{60}\text{Zn}\). As it is seen in Fig. 1 the spectrum of the relative ground state energies of the \(\alpha\)-particle type nuclei around \(^{56}\text{Ni}\) is close to the equidistant one. For this reason we approximate an \(\alpha\)-dependence of the collective potential by the harmonic oscillator. Thus, the collective potential used below is presented by the following expression

\[
V(\Delta, \alpha) = \frac{1}{2} C_\Delta \left( \frac{\Delta^4}{\Delta^2} + \Delta^2 \right) + \frac{1}{2} C_\alpha \alpha^2.
\]

(8)

Fig. 1. Experimental relative energies of the ground states of even-even nuclei that differ from \(^{56}\text{Ni}\) by one- and two \(\alpha\)-particles. Energies are determined according to (6). Linear in \((A - 56)\) term is added so as to give equal energies to \(^{52}\text{Fe}\) and \(^{60}\text{Zn}\).
4. Results

It is seen from (9) that the pairing and $\alpha$-particle correlation modes are coupled only through the pairing rotational term which can be presented for our aims as

$$
\frac{N^2}{B_\Delta \Delta^2 + 4B_\alpha a^2} - \frac{N^2 B_\Delta \Delta^2}{4B_\alpha a^2 (B_\Delta \Delta^2 + 4B_\alpha a^2)}.
$$

(9)

Due to proportionality to $\Delta^2$ the second term in (9) can be considered as a correction to Davidson potential. As will be shown below this correction is small and can be neglected. However, we will retain this term by replacing $\Delta^2$ and $a^2$ in the denominator of the second term on the right side of (9) with their mean values $\langle \Delta^2 \rangle$ and $\langle a^2 \rangle$ in the lowest states with given $N$ and $T$. As the result $\Delta$ and $a$ will be separated in the Hamiltonian. Thus, we obtain

$$
\hat{H} = \frac{T(T+1)}{2B_\Delta \Delta^2} + \frac{N^2}{2B_\Delta \Delta^2} - \frac{1}{2B_\Delta \Delta^2} \frac{\partial}{\partial \Delta} \frac{\Delta}{\partial \Delta} - \frac{1}{2B_\alpha} \frac{\partial^2}{\partial a^2} + \frac{1}{2} C_\alpha a^2.
$$

(10)

The eigenfunctions of (10) are:

$$
\Psi_{NTT_0}(\Delta, a, \phi, \tilde{\psi}) \sim \sqrt{\frac{2T+1}{8\pi^2}} D_{T,0}^T(\tilde{\psi}) \exp(iN\phi) \exp \left(-\frac{1}{2} \sqrt{\frac{C_\Delta}{C_\Delta}} \cdot \sqrt{B_\Delta C_\Delta} \Delta^2 - \frac{1}{2} \sqrt{B_\alpha C_\alpha} a^2 \right) \times \left( (B_\Delta C_\Delta)^{1/4} \Delta \right)^{\frac{1}{2} + (1 + 1 + N/2)} \times L_n^{(1 + 1 + 2)} \left( \sqrt{\frac{C_\Delta}{C_\Delta}} \cdot \sqrt{B_\Delta C_\Delta} \Delta^2 \right) \times L_k^{(1 + 1 + N/2)} \left( \sqrt{B_\alpha C_\alpha} a^2 \right).
$$

(11)

Here $L_n(k)$ are Laguerre polynomials, where $n$ and $k$ are positive integer numbers. For each set of $N$ and $T$ for the lowest states $n = k = 0$. Also, $\rho_0^0 \equiv B_\Delta C_\Delta \Delta^4$ and

$$
\frac{C_\Delta}{C_\Delta} = 1 - \frac{N^2 B_\Delta}{4B_\alpha C_\alpha (B_\Delta \Delta^2 + 4B_\alpha a^2)}.
$$

Using the wave functions (11) we can calculate $\langle N, T | \Delta^2 | N, T \rangle$ and $\langle N, T | a^2 | N, T \rangle$. The results are

$$
\langle N, T | \Delta^2 | N, T \rangle = \frac{C_\Delta}{C_\Delta} \frac{\sqrt{T(T+1) + \rho_0^0 + 4}}{\sqrt{B_\Delta C_\Delta}}.
$$

(12)

$$
\langle N, T | a^2 | N, T \rangle = \frac{3 + \sqrt{1 + N^2}}{2\sqrt{B_\alpha C_\alpha}}.
$$

(13)
With the help of (12) and (13) we obtain the following relation for \( \tilde{C}_\Delta / C_\Delta \):

\[
\frac{\tilde{C}_\Delta}{C_\Delta} = \frac{N^2}{1 - \frac{4R^2(3 + \sqrt{1 + N^2})^2 + 2R(\tilde{C}_\Delta^\alpha / C_\Delta^\alpha)^{-1/2}(3 + \sqrt{1 + N^2})\sqrt{T(T + 1)} + \rho_0^4 + 4}}.
\]

where

\[
R = \sqrt{\frac{(C_\Delta B_\alpha)}{(B_\Delta C_\alpha)}}.
\]

Using the values of the parameters fixed below we obtain that even for \( N=4 \) \( \tilde{C}_\Delta / C_\Delta \geq 0.99 \).

For the lowest states with given \( N \) and \( T \) the energies calculated from the energy of the basic nucleus having \( N=0, T=0 \) are given by the expression

\[
E(N, T) = \frac{1}{2} \sqrt{C_\alpha / B_\alpha} \left( \sqrt{1 + N^2} - 1 \right) + \sqrt{\frac{C_\Delta}{B_\Delta}} \left( \sqrt{T(T + 1)} + \rho_0^4 + 4 - \sqrt{\rho_0^4 + 4} \right). \tag{15}
\]

The wave functions and the energies \( E(N, T) \) depend on two dimensional \( \sqrt{C_\Delta / B_\Delta} \) and \( \sqrt{C_\alpha / B_\alpha} \) and one nondimensional \( \rho_0 \) parameters. As in our previous paper, we will compare with the experimental data the ratios of the energies determined by (15). To reduce the number of parameters for calculating these quantities we fix the relative energy of the nucleus with \( N=1 \) and \( T=1 \), whose experimental value is 4.5 MeV. It gives the relation between \( \sqrt{C_\Delta / B_\Delta} \), \( \sqrt{C_\alpha / B_\alpha} \) and \( \rho_0 \)

\[
\sqrt{\frac{C_\Delta}{B_\Delta}} \left( \sqrt{\rho_0^4 + 6} - \sqrt{\rho_0^4 + 4} \right) + \frac{1}{2} \sqrt{\frac{C_\alpha}{B_\alpha}} \left( \sqrt{2} - 1 \right) = 4.5 \text{ MeV}. \tag{16}
\]

The value of \( \sqrt{C_\alpha / B_\alpha} \) is fixed as follows. Consider \( 0^+ \) excited states of nuclei around \(^{56}\text{Ni}\) having \( 4p-4h \) structure which can be considered as a quartet type excitations. There are known \( 0^+ \) state in \(^{40}\text{Ca}\) with the excitation energy \( E^* = 3.35 \) MeV, \( 0^+ \) state in \(^{48}\text{Ca}\) with \( E^* = 4.28 \) MeV, \( 0^+ \) state in \(^{56}\text{Ni}\) with \( E^* = 3.96 \) MeV, \( 0^+ \) state in \(^{58}\text{Ni}\) with \( E^* = 2.94 \) MeV, and \( 0^+ \) state in \(^{56}\text{Fe}\) with \( E^* = 2.56 \) MeV. The excitation energies in these nuclei vary from 2.5 MeV to 4.0 MeV with the average value 3.4 MeV. We take as an approximate estimation of this energy the value of \( \sqrt{C_\alpha / B_\alpha} \) which is an energy of the \( \alpha \)-particle type oscillations in harmonic approximation. This makes it possible to establish the relation between \( \sqrt{C_\Delta / B_\Delta} \) and \( \rho_0 \). We used in calculations two values of \( \sqrt{C_\alpha / B_\alpha} \), namely, 3 MeV and 4 MeV. The results of calculations with these two values did not differ significantly, but nevertheless better agreement with the experimental data was obtained with \( \sqrt{C_\alpha / B_\alpha} = 3 \) MeV and \( \rho_0 = 6 \). The results of calculations with these values of the parameters and their comparison with the experimental data are presented in Fig. 2, where the energies of the states with different \( N \) and \( T \) are counted from the energy of the state with \( N = 0, T = 0 \) and are given in units of \( E(N = 1, T = 1) - E(N = 0, T = 0) \). As it is seen from Fig. 2 the results of the present calculations of the energies of the
$T=0$ states are in a quite satisfactory agreement with the experimental data. Comparison with the results obtained in demonstrates how important is the inclusion of the $\alpha$-particle type correlations in description of the $T=0$ states. The agreement with the experimental energies of the states with $T \neq 0$ is also satisfactory. This improvement of the results of the present calculations in comparison with the previous ones for the states with different $T$ is explained by inclusion of the $\alpha$-particle type correlations into consideration. Due to this fact it becomes possible to fit the value of $\rho_0$ so as to get a better description of the energies of the states with $T \neq 0$ without worsening description of the energies of the states with $T=0$.

![Diagram](image)

**Fig. 2.** Experimental and calculated relative energies of the ground states of even-even nuclei with different $N$ and $T$. Thick solid line (black) - experimental data, dashed line (red) - the results of calculations with $\rho_0^4=6$ and $\sqrt{C_{\alpha}/B_{\alpha}}=3$ MeV, dotted line (green) - the results of calculations without inclusion of the $\alpha$-particle type correlations, taken from. Energies are given in units of $(E(N=1, N=1) - E(N=0, T=0))$.

5. Two-nucleon and $\alpha$-particle transfer reactions

The wave functions of the lowest states with given $N$ and $T$ are

$$\Psi_{NTT_z}(\Delta, a, \phi, \vec{\psi}) \sim \sqrt{\frac{2T+1}{8\pi^2}} D_{T_z,0}^T(\vec{\psi}) \exp(iN\phi) \times \exp\left(-\frac{1}{2} \sqrt{B_{\Delta} C_{\Delta}} \Delta^2 \right) \left((B_{\alpha} C_{\alpha})^{1/4} \right) \frac{1}{2} (1+\sqrt{1+N})$$

(17)

The operator corresponding to the two-nucleon transfer is

$$P_{\mu}^{(\pm 1)} = \pm i \exp(\pm i\phi) \Delta D_{\mu 0}^1$$

(18)

The operator associated with the $\alpha$-particle transfer is given by

$$S^{(\pm)} = \exp(\pm 2i\phi)a,$$

(19)
where it is assumed that \( \alpha \)-particle transfer is described by the \( \alpha \)-particle mode only. Using (17) and (18) we obtain the following result for the two-nucleon transfer amplitude

\[
\langle N + 1, T'^{\prime} z | P_{\mu}^{(+1)} | N T T z \rangle = (B_\Delta C_\delta)^{-1/4} C_{TT_0}^{T^{\prime} T^{\prime}_1} C_{0010}^{T^{\prime} T^{\prime}_0} \\
\times \Gamma \left( 1 + \frac{1}{2} \sqrt{T(T + 1) + \rho_0^3 + 4 + \frac{1}{2} \sqrt{T + 1} + \rho_0^3 + 4} \right) \\
\sqrt{\Gamma \left( \frac{1}{2} + \sqrt{T(T + 1) + \rho_0^3 + 4} \right) \Gamma \left( \frac{1}{2} + \sqrt{T + 1} + \rho_0^3 + 4 \right) \Gamma \left( \frac{3}{2} + \frac{1}{4} \sqrt{1 + N^2} \right) \Gamma \left( \frac{3}{2} + \frac{1}{4} \sqrt{1 + (N + 1)^2} \right)}
\]

\( (20) \)

It is seen from (20) that the amplitude for the \( \Delta T = 0 \) transition is equal to zero if excitations in \( \theta \)-mode are excluded from consideration. In this approximation the reaction \((^3He, p)\) to a state of an odd-odd nucleus with the same \( T \) is forbidden if even-even nucleus is a target.\(^{35}\)

For the \( \alpha \)-particle transfer amplitude we obtain

\[
\langle N + 2, TT_0 | S_{\alpha}^{(+)} | N T T z \rangle = (B_\alpha C_\alpha)^{-1/4} \\
\sqrt{\Gamma \left( \frac{3}{2} + \frac{1}{4} \sqrt{1 + N^2} \right) \Gamma \left( \frac{3}{2} + \frac{1}{4} \sqrt{1 + (N + 2)^2} \right)}
\]

\( (21) \)

This amplitude does not depend on isospin.

6. Conclusion

In the present paper we have constructed the collective Hamiltonian which includes not only the isovector pairing mode, but also the \( \alpha \)-particle degree of freedom describing the \( \alpha \)-particle type correlations. The Hamiltonian is applied to description of the relative energies of the ground states of even-even nuclei around \(^{56}\)Ni. The results obtained demonstrate a quite satisfactory agreement with the experimental energies especially for the states with \( T=0 \), i.e. for nuclei which are systems of some numbers of \( \alpha \)-particles. The agreement with the experimental energies of the states with \( T \neq 0 \) is also satisfactory. This improvement of the results of calculations in comparison with the previous one\(^{36}\) is explained by inclusion in the Hamiltonian of the dynamical variable describing \( \alpha \)-particle type correlations. Because of this it becomes possible to include into consideration a difference in description of the moments of inertia for isospin and gauge rotations.

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Appendix A. Appendices

The classical expression for the kinetic energy looks as

$$T = \frac{1}{2}B\Delta \sum_{\mu} \dot{z}_{1\mu} \dot{z}_{1\mu} + \frac{1}{2}B\theta \dot{\alpha} \dot{\alpha},$$  \hspace{1cm} (A.1)

where

$$z_{1\mu} = \exp(\nu\phi) \sum_{\nu} D_{1\mu\nu} \Delta_{\nu},$$  \hspace{1cm} (A.2)

$$\Delta_0 = \Delta \cos \theta, \quad \Delta_{\pm 1} = \frac{1}{\sqrt{2}}\Delta \sin \theta,$$

$$\alpha = \exp(2\nu\phi) a,$$  \hspace{1cm} (A.3)

$$\dot{z}_{1\mu} = \exp(\nu\phi) \sum_{\nu} \dot{D}_{1\mu\nu} \Delta_{\nu} + i \dot{\phi} \exp(\nu\phi) \sum_{\nu} D_{1\mu\nu} \Delta_{\nu} \cos \theta,$$

$$+ \exp(\nu\phi) \sum_{\nu} D_{1\mu\nu} \dot{\Delta}_{\nu} \sin \theta.$$  \hspace{1cm} (A.4)

$$\dot{\alpha} = 2\nu \dot{\phi} \exp(2\nu\phi) a + \exp(2\nu\phi) \dot{a}.$$  \hspace{1cm} (A.5)

Using the expression for $\dot{D}_{1\mu\nu}^{[28]}$ we obtain for (A.4)

$$\dot{z}_{1\mu} = -i \exp(\nu\phi) \sum_{j=1,2,3} \sum_{\mu'} D_{1\mu\mu'}^{[28]} \langle 1\mu' | \dot{\hat{L}}_j | 1\nu \rangle \omega_j \Delta_{\nu}$$

$$+ i \dot{\phi} \exp(\nu\phi) \sum_{\nu} D_{1\mu\nu} \Delta_{\nu} + \exp(\nu\phi) \sum_{\nu} D_{1\mu\nu} \dot{\Delta}_{\nu}.$$  \hspace{1cm} (A.6)

In (A.6) $\omega_j$ is the $j$th component of the angular velocity in isospace and $\langle 1\mu' | \dot{\hat{L}}_j | 1\nu \rangle$ is the matrix element in the three-dimensional representation of the $j$th component of the isospace angular momentum.

Substituting (A.3) and (A.6) into (A.1) we obtain the following expression for $T$:

$$T = T_{\text{rot}} + T' + T_\alpha,$$  \hspace{1cm} (A.7)

where

$$T_{\text{rot}} = \frac{1}{2} \sum_{k,j=1,2,3} \Im_{k,j} \omega_k \omega_j,$$  \hspace{1cm} (A.8)

$$\Im_{k,j} = B\Delta \sum_{\mu,\nu,\mu'} \langle 1\mu | \hat{L}_k | 1\nu \rangle \langle 1\mu | \dot{\hat{L}}_j | 1\nu' \rangle \Delta_{\nu} \Delta_{\nu'},$$  \hspace{1cm} (A.9)
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\[ T' = -\frac{1}{2} B_\Delta \sum_j \sum_{\mu,\nu,\nu'} D^1_{\mu\nu'} \langle 1\mu | \hat{L}_j | 1\nu \rangle \omega_j \Delta_{\nu'} \]

\[ \times \left( \phi \sum_{\nu'} D^1_{\mu\nu'} \Delta_{\nu'} - i \sum_{\nu'} D^1_{\mu\nu'} \Delta_{\nu'} \right) \]

\[ \times D^1_{\mu\nu'} \langle 1\mu' | \hat{L}_j | 1\nu' \rangle \omega_j \Delta_{\nu'} \]

\[ + \frac{1}{2} B_\Delta \sum_{\mu} \left( \phi \sum_{\nu} D^1_{\mu\nu} \Delta_{\nu} + i \sum_{\nu} D^1_{\mu\nu} \Delta_{\nu} \right) \]

\[ \times \left( \phi \sum_{\nu} D^1_{\mu\nu} \Delta_{\nu} + i \sum_{\nu} D^1_{\mu\nu} \Delta_{\nu} \right), \] (A.10)

\[ T_\alpha = \frac{1}{2} B_\alpha \left( \dot{\phi}^2 \dot{a}^2 + \dot{\theta}^2 \right). \] (A.11)

Using the standard expressions for the matrix elements of the isospin momentum operators and (A.2) we obtain

\[ \Im_{kj} = \delta_{kj} \Im_j, \]

\[ \Im_j = B_\Delta \sum_{\nu,\nu'} \langle 1\nu | \hat{L}_j^2 | 1\nu' \rangle \Delta_{\nu} \Delta_{\nu'}, \] (A.12)

and finally

\[ \Im_1 = B_\Delta \Delta^2, \]

\[ \Im_2 = B_\Delta \Delta^2 \cos^2 \theta, \]

\[ \Im_3 = B_\Delta \Delta^2 \sin^2 \theta. \] (A.13)

Thus,

\[ T_{rot} = \frac{1}{2} B_\Delta \Delta^2 \omega_1^2 + \frac{1}{2} B_\Delta \Delta^2 \cos^2 \theta \omega_2^2 + \frac{1}{2} B_\Delta \Delta^2 \sin^2 \theta \omega_3^2. \] (A.14)

In a similar way we obtain for \( T' \) the following expression

\[ T' = -\sqrt{2} B_\Delta \dot{\phi} \omega_1 \Delta^2 \sin 2\theta + \frac{1}{2} B_\Delta \Delta^2 \dot{\phi}^2 + \frac{1}{2} B_\Delta \left( \Delta^2 + \Delta^2 \dot{\theta}^2 \right) \] (A.15)

For \( T_\alpha \) we get

\[ T_\alpha = \frac{1}{2} B_\alpha \left( 4 \dot{\phi}^2 \dot{a}^2 + \dot{\theta}^2 \right). \] (A.16)

Thus, a classical expression for \( T \) is

\[ T = \frac{1}{2} B_\Delta \Delta^2 \omega_1^2 + \frac{1}{2} B_\Delta \Delta^2 \cos^2 \theta \omega_2^2 + \frac{1}{2} B_\Delta \Delta^2 \sin^2 \theta \omega_3^2 \]

\[ + \frac{1}{2} \sqrt{2} B_\Delta \Delta^2 \sin 2\theta (\omega_1 \omega_1 + \omega_1 \omega_1) \]

\[ + \frac{1}{2} \left( B_\Delta \Delta^2 + 4 B_\alpha a^2 \right) \dot{\phi}^2 \]

\[ + \frac{1}{2} B_\Delta \left( \Delta^2 + \Delta^2 \dot{\theta}^2 \right) + \frac{1}{2} B_\alpha \dot{a}^2. \] (A.17)
We can present (A.17) as follows

\[ T = \frac{1}{2} \sum_{j,k} g_{jk}(\xi) \dot{\xi}_j \dot{\xi}_k, \]  

(A.18)

where \( \xi_1 = \beta_1, \xi_2 = \phi, \xi_3 = \beta_2, \xi_4 = \beta_3, \xi_5 = \Delta, \xi_6 = \theta, \) and \( \xi_7 = a. \) Here \( \beta_j \) is the angle characterizing rotation around \( j \)th axis in isospace. In agreement with Pauli prescription the quantum expression for \( T \) is

\[ \hat{T} = -\frac{1}{2} \sum_{j,k} g^{-1/2} \frac{\partial}{\partial \xi_k} g^{1/2} (g^{-1})_{kj} \frac{\partial}{\partial \xi_j}, \]  

(A.19)

where \( g \) is determinant of the matrix \( g_{jk}, \) and \( g^{-1}_{kj} \) is the inverse of the matrix \( g_{jk}. \)

The elements of the matrix \( g_{jk} \) are presented in (A.17). Its determinant \( g \) is equal to

\[ g = \frac{1}{16} B_\Delta^5 \Delta^8 \sin^2 4\theta \cos^2 2\theta B_\alpha (B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2). \]  

(A.20)

Further, using a relation between the components of the isospin momentum operators \( \hat{T}_j \) and the derivatives over \( \beta_j; \)

\[ \hat{T}_j = -i \frac{\partial}{\partial \beta_j} \]

we obtain the following expression for \( \hat{T} \)

\[ \hat{T} = \frac{1}{2} \left( \frac{1}{\sqrt{g}} \hat{T}_x \sqrt{g} \frac{B_\Delta \Delta^2 + 4B_\alpha a^2}{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \frac{\partial}{\partial \Delta} \hat{T}_x \right) \]

\[ + \frac{1}{\sqrt{g}} \hat{T}_y \sqrt{g} \frac{1}{B_\Delta \Delta^2 \cos^2 2\theta} \hat{T}_y + \frac{1}{\sqrt{g}} \hat{T}_z \sqrt{g} \frac{1}{B_\Delta \Delta^2 \sin^2 \theta} \hat{T}_z \]

\[ + \frac{1}{\sqrt{g}} \hat{T}_x \sqrt{g} \frac{\sin 2\theta}{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \hat{N} \]

\[ + \frac{1}{\sqrt{g}} \hat{N} \sqrt{g} \frac{\sin 2\theta}{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \hat{T}_x \]

\[ + \frac{1}{\sqrt{g}} \hat{N} \sqrt{g} \frac{\sin 2\theta}{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \hat{N} \]

\[ - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \Delta} \sqrt{g} \frac{1}{B_\alpha \Delta} \frac{\partial}{\partial \Delta} - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta} \sqrt{g} \frac{1}{B_\Delta \Delta^2} \frac{\partial}{\partial \theta} \frac{1}{\sqrt{g}} \frac{\partial}{\partial a} \sqrt{g} \frac{1}{B_\alpha \Delta} \frac{\partial}{\partial a}. \]  

(A.21)

Substituting the expression for \( g \) from (A.21) and taking into account the assump-
tion made above that $B_\Delta$ and $B_\alpha$ don’t depend on dynamical variables, we obtain

\[
\hat{T} = \frac{1}{2B_\Delta \Delta^2 (B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2)} \hat{T}_x^2 \\
+ \frac{1}{2B_\Delta \Delta^2 \cos^2 \theta} \hat{T}_y^2 + \frac{1}{2B_\Delta \Delta^2 \sin^2 \theta} \hat{T}_z^2 \\
+ \frac{1}{2B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \hat{T}_x \hat{N} \\
+ \frac{1}{2B_\Delta \Delta^2 + 4B_\alpha a^2} \\
- \frac{1}{2\sqrt{B_\Delta}} \frac{\partial}{\partial \Delta} \Delta^4 \sqrt{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \partial \Delta \\
\cos 2\theta \\
- \frac{1}{2\sqrt{B_\Delta}} \sin 4\theta \sqrt{B_\Delta \Delta^2 \cos^2 2\theta + 4B_\alpha a^2} \partial \Delta \\
\sin 4\theta \\
- \frac{1}{2\sqrt{B_\Delta}} \partial \Delta^4 \sqrt{B_\Delta} \Delta^2 \cos 2\theta + 4B_\alpha a^2 \partial a .
\] (A.22)

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