Is minimal coupling procedure compatible with minimal action principle?

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Abstract

When space-time is assumed to be non-Riemannian the minimal coupling procedure (MCP) is not compatible, in general, with minimal action principle (MAP). This means that the equations gotten by applying MCP to the Euler-Lagrange equations of a Lagrangian $\mathcal{L}$ do not coincide with the Euler-Lagrange equations of the Lagrangian obtained by applying MCP to $\mathcal{L}$. Such compatibility can be restored if the space-time admits a connection-compatible volume element. We show how these concepts can alter qualitatively the predictions of the Einstein-Cartan theory of gravity.

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Minimal coupling procedure (MCP) provides us with a useful rule to get the equations for any physical field on non-Minkowskian manifolds starting from their versions of Special Relativity (SR). When studying classical fields on a non-Minkowskian manifold $\mathcal{X}$ we usually require that the equations of motion for such fields have an appropriate SR limit. There are, of course, infinitely many covariant equations on $\mathcal{X}$ with the same SR limit, and MCP solves this arbitrariness by saying that the relevant equations should be the “simplest” ones. MCP can be heuristically formulated as follows. Considering the equations of motion for a classical field in the SR, one can get their version for a non-Minkowskian space-time $\mathcal{X}$ by changing the partial derivatives by the $\mathcal{X}$ covariant ones and the Minkowski metric tensor by the $\mathcal{X}$ one.

MCP is also used for the quantum analysis of gauge fields, where the gauge field is to be interpreted as a connection, and it is in spectacular agreement with experience for QED.

Suppose now that the SR equations of motion for a classical field follow from an action functional via minimal action principle (MAP). It is natural to expect that the equations obtained by using MCP to the SR equations coincide with the Euler-Lagrange equations of the action obtained via MCP of the SR one. This can be better visualized with the help of the following diagram

\[
\begin{array}{c}
\mathcal{C}_{\mathcal{L}_{\text{SR}}} \xrightarrow{\text{MCP}} \mathcal{C}_{\mathcal{L}_{\mathcal{X}}} \\
\downarrow \text{MAP} \quad \downarrow \text{MAP} \\
E(\mathcal{L}_{\text{SR}}) \xrightarrow{\text{MCP}} E(\mathcal{L}_{\mathcal{X}})
\end{array}
\]

where $E(\mathcal{L})$ stands to the Euler-Lagrange equations for the Lagrangian $\mathcal{L}$, and $\mathcal{C}_{\mathcal{L}}$ is the equivalence class of Lagrangians, $\mathcal{L}'$ being equivalent to $\mathcal{L}$ if $E(\mathcal{L}') = E(\mathcal{L})$. The diagram (1) is verified when MCP is used for gauge
fields and for General Relativity. We say that MCP is compatible with MAP if (1) holds. We stress that if (1) does not hold we have another arbitrariness to solve, one needs to choose one between two equations, as we will shown with a simple example.

It is not difficulty to check that MCP is not compatible with MAP, in general, when space-time is assumed to be non-Riemannian, as for example in the Einstein-Cartan theory of gravity\cite{1}, where the linear connection $\Gamma^\alpha_{\mu\nu}$ is not symmetrical in its lower indices, but is metric-compatible, $D_\alpha g_{\mu\nu} = 0$. Let us examine for simplicity the case of a massless scalar field $\varphi$ in the frame of Einstein-Cartan gravity\cite{4}. The equation for $\varphi$ in SR is

$$\partial_\mu \partial^\mu \varphi = 0, \quad (2)$$

which follows from the minimization of the action

$$S_{SR} = \int d\text{vol} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi. \quad (3)$$

Using MCP to (3) one gets

$$S_X = \int d\text{vol} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi, \quad (4)$$

and using the canonical volume element for $X$, $d\text{vol} = \sqrt{g} d^n x$, we get the following equation from the minimization of (4)

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \varphi = 0. \quad (5)$$

It is clear that (5) does not coincide in general with the equation obtained via MCP of (2)

$$\partial_\mu \partial^\mu \varphi + \Gamma^\mu_{\mu\alpha} \partial^\alpha \varphi = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \varphi + 2 \Gamma^\mu_{[\mu\alpha]} \partial^\alpha \varphi = 0. \quad (6)$$

We have here an ambiguity, the equations (3) and (4) are in principle equally acceptable ones, to choose one of them corresponds to choose as more fundamental the equations of motion or the action formulation from MCP point
of view. As it was already said, we do not have such ambiguity when MCP is used to gauge fields and when space-time is assumed to be a Riemannian manifold. This is not a feature of massless scalar fields, all matter fields have the same behaviour in the frame of Einstein-Cartan gravity. The incompatibility of MCP and MAP for fermionic fields in the Einstein-Cartan gravity is well known[2].

An accurate analysis of the diagram (1) reveals that the source of the problems of compatibility between MCP and MAP is the volume element of $X$. It turns out that if $X$ admits a connection-compatible volume element, the diagram (1) holds for all matter fields. A connection-compatible volume element $d\text{vol} = j(x) d^nx$ is such that

$$D_\alpha j(x) = 0.$$  \hfill (7)$$

It is easy to check that the canonical volume element $d\text{vol} = \sqrt{g}d^nx$ is compatible with the connection for a Riemannian manifold and that it is not for a Riemann-Cartan manifold, the space-time of the Einstein-Cartan gravity. A Riemann-Cartan manifold admits a connection-compatible volume element if $5$

$$\Gamma_{\mu}^{\nu} = \partial_\alpha \Theta(x),$$  \hfill (8)$$
in this case the connection-compatible volume element is $d\text{vol} = e^{2\Theta} \sqrt{g}d^nx$.

It is not usual to find in the literature applications where volume elements different from the canonical one are used. In our case the new volume element appears naturally, in the same way that we require compatibility conditions between the metric tensor and the linear connection we can do it for the connection and volume element. If we remember that in a manifold with torsion there are no infinitesimal parallelograms and so there are no infinitesimal parallel cubes, there are no reasons a priori to expect that the notion of volume of Riemannian geometry be preserved in the presence of
torsion. It is also important to stress that any volume element that differs from the canonical one by the multiplication of a positive function defines, in principle, an acceptable notion of volume\[^{3}\].

With the use of the connection-compatible volume element in the action formulation for Einstein-Cartan gravity we can have qualitatively different predictions. The scalar of curvature for a Riemann-Cartan manifold, present in the Hilbert-Einstein action, is given by the Riemannian scalar of curvature plus terms quadratic in the torsion. Due to (8) such quadratic terms will provide a differential equation for $\Theta$, what will allow for non-vanishing torsion solutions for the vacuum. Torsion can propagate if the space-time admits a connection-compatible volume element, the torsion mediated interactions loose their contact aspect. As to the matter fields, the use of the connection-compatible volume element, besides of guarantee that the diagram (1) holds, brings also qualitative changes. For example, it is possible to have a minimal interaction between Maxwell fields and torsion preserving gauge symmetry. Another point of interest is that the peculiar $\Theta$-dependance of the connection-compatible volume element shows that such notion of volume can have relevance to the study of dilaton gravity.

It is important to note that the restriction that space-time should admit a connection-compatible volume element arises even in the case where MAP is not used. It appears as integrability condition for the Maxwell equations obtained by using MCP to the SR ones in the differential forms formulation\[^{1}\].
References

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