GEODESICALLY COMPLETE SPACES WITH AN UPPER CURVATURE BOUND

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ABSTRACT. We study geometric and topological properties of locally compact, geodesically complete spaces with an upper curvature bound. We control the size of singular subsets, discuss homotopical and measure-theoretic stratifications and regularity of the metric structure on a large part.

1. Introduction

1.1. Object of investigations. Metric spaces with one-sided curvature bounds were introduced by A.D. Alexandrov in [Ale57]. After the revival of metric geometry in the eighties, properties and applications of such spaces have been investigated from various points of view, we refer to [Bal95], [BH99], [BBI01], [BS07], [AKP16] and the bibliography therein. Starting with [BGP92], a structure theory of locally compact spaces with a lower curvature bound and finite dimension, so-called Alexandrov spaces, was developed, see [AKP16] for the huge bibliography.

Due to the local uniqueness of geodesics in spaces with upper curvature bounds, the derivation of basic topological and geometric properties is simpler than in the case of Alexandrov spaces. However, finer structural features can be much more intricate. Even a compact tree, hence a topologically 1-dimensional space of non-positive curvature, can have infinite Hausdorff dimension and may not contain any kind of "manifold charts". [Kle99], [AB07]. Also the global topological structure of spaces with upper curvature bounds can be much more complicated than in the case of lower curvature bounds: for instance, any finite-dimensional simplicial complex carries a metric with an upper curvature bound [Ber83].

Without additional assumptions it seems impossible to detect some general regular structures beyond a theorem of B. Kleiner, [Kle99], claiming that the topological dimension coincides with the maximal dimension of a Euclidean ball topologically embedded into the space. In order to obtain some control, one needs an assumption, which provides a close relation between the local geometry near a point and the geometry of its tangent cone. Such a natural assumption is (local) geodesic completeness, also known as geodesic extension property. It says that any compact geodesic can be extended as a local geodesic beyond its endpoints. This condition is stable under natural metric operations and can often be detected topologically. For example, it holds if small punctured neighborhoods of all points are non-contractible.

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Finally, geodesic completeness plays an important role in geometric group theory, see, for instance, [CM09], [GST17].

The present paper is devoted to the description of basic measure-theoretic, homotopic and analytic properties of such spaces, recovering analogues of most results of [BGP92], [Per94] and [OS94]. Applications to topological questions, geometric group theory and sphere theorems will be discussed in forthcoming papers. Results and ideas of preliminary versions of this work which we have circulated in the last 10 years have already been used, for instance in [Kap07], [Kra11], [BK16], [KK17].

There has been one systematic investigation of the theory of geodesically complete spaces with upper curvature bounds by Otsu and Tanoue, [OT99], announced in [Ots97] (and continued in [Nag02]). Since [OT99] has never been published and is rather difficult to read, we do not use it. In fact, some of our theorems provide improvements and simplifications of the main results from [OT99].

The special case of two-dimensional topological surfaces has been intensively studied, cf. [Res93]. Some results from [Res93], definitely out of reach in the general case, have been generalized to two-dimensional polyhedra in [BB98].

### 1.2. Main results

From now on, we say that $X$ is GCBA, if $X$ is a locally compact, separable, locally geodesically complete space with curvature bound above.

GCBA spaces have indeed many structural similarities with Alexandrov spaces, see Section 5. Any GCBA space $X$ is locally doubling. For any $x$ in $X$, the tangent space $T_x X$ and the space of directions $\Sigma_x X$ are again GCBA. Any compact part of any GCBA admits a biLipschitz embedding into a Euclidean space.

The following theorem is implicitly contained in [OT99].

**Theorem 1.1.** Let $X$ be GCBA. The topological dimension $\dim(X)$ of $X$ coincides with the Hausdorff dimension. It equals the maximal dimension of an open subset of $X$ homeomorphic to a Euclidean ball.

The local dimension might be non-constant on $X$, as one observes by looking at simplicial complexes. But the local dimension can be understood by looking at tangent spaces. For $k = 0, 1, 2, ...$, we call the $k$-dimensional part of $X$, denoted by $X^k$, the set of all points $x \in X$ with $\dim(T_x X) = k$. In general, $X^k$ is neither open nor closed in $X$. However, $X^k$ contains large ”regular subsets”, open in $X$, as shown by the next result.

**Theorem 1.2.** Let $X$ be GCBA. A point $x$ is contained in the $k$-dimensional part $X^k$ if and only if all sufficiently small balls around $x$ have dimension $k$. The Hausdorff measure $\mathcal{H}^k$ is locally finite and locally positive on $X^k$. There is a subset $M^k$ of $X^k$, which is open in $X$, dense in $X^k$ and locally biLipschitz equivalent to $\mathbb{R}^k$. Moreover, the complement $X^k \setminus M^k$ of $M^k$ in the closure $\overline{X^k}$ of $X^k$ has Hausdorff dimension at most $k - 1$.

We refer to Section 11 for a stronger statement. The open manifold $M^k$ should be thought of as the regular $k$-dimensional part of $X$. Its finer geometry is described by the following theorem. We refer to [Per94], [KMS01], [ABL15] and Section 13 below for a discussion of the notions of DC-functions, DC-manifolds and functions of bounded variation used in the next theorem.

**Theorem 1.3.** For $k = 0, 1, ...$, the manifold $M^k \subset X^k$ in Theorem 1.2 can be chosen to satisfy the following properties. The set $M^k$ contains the set $R_k$ of all points in $x \in X$ with tangent space $T_x X$ isometric to $\mathbb{R}^k$ and $M^k \setminus R_k$ has Hausdorff
The manifold $M^k$ has a unique DC-atlas such that all convex functions on $M^k$ are DC-functions with respect to this atlas. The distance in $M^k$ can locally be obtained from a Riemannian metric tensor $g$, well-defined and continuous on $\mathcal{R}_k$. The tensor $g$ locally is of bounded variation on $M^k$.

The $k$-dimensional Hausdorff measure is the natural measure on the $k$-dimensional part $X^k$ of $X$. We put these measures together and define the canonical measure $\mu_X$ of the space $X$ to be the sum of the restrictions of $\mathcal{H}^k$ to $X^k$, thus

$$\mu_X := \sum_{k=0}^{\infty} \mathcal{H}^k \downharpoonright X^k.$$ 

By Theorem 1.2 and Theorem 1.3, the restriction of $\mu_X$ to $M^k$ is (the Riemannian measure) $\mathcal{H}^k$, and $\mu_X$ vanishes on the complement of the open submanifold $\bigcup_{k \geq 0} M^k$. The "canonicity" of $\mu_X$ is confirmed by the following two theorems.

**Theorem 1.4.** Let $X$ be GCBA. The canonical measure is positive and finite on any open relatively compact subset of $X$.

The second theorem tells us that the canonical measure is continuous with respect to the Gromov–Hausdorff topology. We formulate it here for compact spaces and refer to Section 12 for the general local statement.

**Theorem 1.5.** Let $X_1$ be a sequence of compact GCBA spaces of dimension, curvature and diameter bounded from above and injectivity radius bounded from below by some constants. The total measures $\mu_{X_i}(X_1)$ are bounded from above by a constant if and only if, upon choosing a subsequence, $X_1$ converge to a compact GCBA space $X$ in the Gromov–Hausdorff topology. In this case, $\mu_{X_i}(X_1)$ converge to $\mu_X(X)$.

Having described the regular parts of $X$ we turn to a stratification of the singular parts $X^k \setminus M^k$ neglected by the canonical measure. The following stratification of $X$, a weak surrogate of the topological stratification, is motivated by the example of skeletons of a simplicial complex.

For a natural number $k$, we say that a point $x \in X$ is $(k, 0)$-strained if its tangent space $T_xX$ admits the Euclidean space $\mathbb{R}^k$ as a direct factor. We denote by $X_{k, 0}$ the set of all $(k, 0)$-strained points in $X$.

**Theorem 1.6.** Let $X$ be GCBA and $k \in \mathbb{N}$. Then $X \setminus X_{k, 0}$ is a countable union of subsets, which are biLipschitz equivalent to some compact subsets of $\mathbb{R}^{k-1}$.

In particular, the set of not $(k, 0)$-strained points is countably $(k-1)$-rectifiable. For similar rectifiable stratifications on different classes of metric spaces we refer, for instance, to [MN17] and the literature therein.

**Remark 1.7.** Parts of Theorems 1.2, 1.3, 1.5, 1.6 have analogues in [OT99] and [Nag02]. We refer to Remarks 10.7, 11.9, 12.2, and 14.13 for a comparison.

**Remark 1.8.** Theorem 1.4 provides an answer to a question from [CL16]. Moreover, Theorem 1.4 implies the validity of the property $(U)$ from [Leu06] on any cocompact GCBA space. Hence, it shows the validity of the main theorem in [Leu06] for all such spaces, see the discussion in [CL16].
1.3. Main tool and further results. We are going to introduce the main tool of the paper and a more informal description of further central results. The set $X_{k,0}$ is usually not open in $X$. As in the theory of Alexandrov spaces developed in [BGP92], there is a natural way to open up the condition of being $(k,0)$-strained.

For any $\delta > 0$, we define an open subset $X_{k,\delta}$ of a GCBA space $X$ which consists of $(k,\delta)$-strained points. While the definition of being $(k,\delta)$-strained is slightly technical, see Sections 6 and 7, the meaning is very simple:

A point $x \in X$ is $(k,\delta)$-strained, for a small $\delta$, if its tangent space $T_x X$ is sufficiently close to a space which splits off a direct $\mathbb{R}^k$-factor, see Proposition 6.4. In other words, a point $x \in X$ is $(k,\delta)$-strained if and only if there exist $k$ points $p_1, \ldots, p_k \in X \setminus \{x\}$, close to $x$, such that the following holds true. The geodesics $p_i x$ meet in $x$ pairwise at an angle close to $\pi/2$ and the possible branching angles of the geodesics $p_i x$ at $x$ are small ("small" and "close" is expressed in terms of $\delta$).

The subsets $X_{k,\delta}$ are open in $X$ and decrease for fixed $k$ and decreasing $\delta$. The set $X_{k,0}$ of $(k,0)$-strained points is the countable intersection $X_{k,0} = \bigcap_{\delta > 0} X_{k,\delta}$. Each point $x \in X_{k,\delta}$ comes along with natural maps, so-called $(k,\delta)$-strainer maps $F : V \to \mathbb{R}^k$, defined on a neighborhood $V$ of $x$. Strainer maps are analogues of the orthogonal projection onto a face, defined in a neighborhood of that face in a simplicial complex. The coordinates of $F$ are distance functions to points $p_i$ in $X \setminus \{x\}$, for a $k$-tuple $(p_i)$ as in the above definition of $(k,\delta)$-strained points. In other words, a point $x \in X$ is $(k,\delta)$-strained if and only if there exists a $(k,\delta)$-strainer map $F$ on a neighborhood $V$ of $x$.

The basic example of a strainer map, responsible for their abundance, is given by the following observation. For any point $p$ in a GCBA space $X$, and any $\delta > 0$, the distance function to $p$ is a $(1,\delta)$-strainer map on a small punctured neighborhood $V$ of $p$, Proposition 7.3.

For $\delta$ small enough, any $(k,\delta)$-strainer map $F$ is similar to a Riemannian submersion, see Sections 8, 9. In particular, $k$ is not larger than $\dim(X)$. The following technical result is the base for all further investigations on singular sets:

**Theorem 1.9.** Let $F : V \to \mathbb{R}^k$ be a $(k,\delta)$-strainer map on a sufficiently small open subset $V$ of a GCBA space $X$. Then the set $V \setminus X_{k+1,12,\delta}$ is a union of a countable family of compact subsets $K_i$ such that $F : K_i \to F(K_i)$ is biLipschitz.

The biLipschitz constant of the restrictions $F_i : K_i \to F(K_i)$ and the total measure $H^k(V \setminus X_{k+1,12,\delta})$ in Theorem 1.9 are bounded in terms of $\delta$ and $V$, see Theorem 10.3 below. The theorem allows, by a reverse induction on $k$, a good control of the measures of singular sets. We refer to Section 10 for quantitative versions of the volume estimates, leading to proofs (and more precise versions) of Theorems 1.6, 1.7, 1.8.

The strainer construction is stable under Gromov–Hausdorff limits, Section 7. This provides us the basic tool for the proof of Theorems 1.4 and 1.5.

The relation to Theorem 1.2 is established by defining $M^k$ to be the intersection of the $k$-dimensional part $X^k$ of $X$ with $X_{k,\delta}$ for sufficiently small $\delta$. The DC-atlas on $M^k$ in Theorem 1.3 is provided by the $(k,\delta)$-strainer maps.

**Remark 1.10.** If $k = \dim(X)$ then $X_{k,\delta}$ is closely related to sets of not $\delta'$-branch points used in [OT99] to analyze the regular part of a GCBA space.

From the point of view of homotopy theory strainer maps are similar to fibrations. We refer to Section 9 for a more general and more precise version of the following...
Theorem 1.11. Let $F : V \to \mathbb{R}^k$ be a $(k, \delta)$-strainer map with $\delta \leq \frac{1}{20k}$. Then, for any compact subset $V'$ of $V$, there is some $\epsilon > 0$ with the following property. For any $x \in V'$ and any $0 < r < \epsilon$, the open ball of radius $r$ around $x$ in the fiber $F^{-1}(F(x))$ is contractible.

Using Theorem 1.11 we can apply general results from [Pet90] and obtain homotopical stability of fibers. We refer to Section 13 for exact results and state here the following illuminating special case (originating from the convergence of the rescaling of the given space to the tangent cone at a point).

Theorem 1.12. Let $X$ be GCBA. For each point $x \in X$ there is some $r_x > 0$ such that for all $r < r_x$ the metric sphere $\partial B_r(x)$ of radius $r$ around $x$ is homotopy equivalent to the space of directions $\Sigma x X$.

If the metric sphere in Theorem 1.12 is replaced by a punctured ball, the result is simpler and the extendibility of geodesics does not need to be assumed. This has been observed by Kleiner (unpublished) and appeared in [Kra11]. The term homotopy equivalent in Theorem 1.12 cannot be replaced by "homeomorphic" (Example 15.2 below), as it were the case for Alexandrov spaces, [Per91], [Kap07]. This example shows that there is no hope of obtaining a local conicality theorem or topological stability as in [Per91]. In the continuation [LN18] of this paper, we prove, starting with Theorem 1.9, that the local conicality theorem holds for GCBA spaces which are homology manifolds. As a consequence, GCBA spaces which arise as limits of Riemannian manifolds can be very well understood, similarly to [Kap02].

1.4. Structure of the paper. We are going to describe the contents and main results of the sections of the paper.

In the auxiliary Sections 2, 3, 4 we collect preliminaries about general metric spaces, spaces with upper curvature bounds and the geodesic extension property.

In Section 5 we begin the investigation of the central object of this paper and discuss all properties of GCBA spaces not based on the notion of strainers. We localize all discussions by introducing the notion of a tiny ball of a GCBA space, a relatively compact ball of a radius very small in comparison to the curvature bound. All later results are proven first inside tiny balls and then by covering the whole space by tiny balls. It is shown that tiny balls are doubling and that the bound on the doubling constant (the size of the tiny ball) is essentially equivalent to the precompactness in the Gromov–Hausdorff topology, Propositions 5.1, 5.10. We show that tiny balls admit almost isometric embeddings into finite dimensional normed spaces, Proposition 5.3. We show that tangent spaces of GCBA spaces are GCBA, Corollary 5.7, and describe a natural semicontinuity of tangent spaces under convergence, Lemma 5.13.

In Sections 6, 7 we define $(k, \delta)$-strained points, $(k, \delta)$-strainer and $(k, \delta)$-strainer maps, Definitions 6.2, 6.4, 7.1, 7.2. We confirm that $(k, \delta)$-strained points are exactly the points whose tangent space is "close" to a space with an $\mathbb{R}^k$-factor, Proposition 6.6. We discuss the abundance of strainers, Proposition 7.3, and prove that the notions are stable under small perturbations.

In Section 8 we show that $(k, \delta)$-strainer maps with small $\delta$ are almost submersions, Lemma 8.2, Corollary 8.3.

In Section 9 we prove Theorem 1.11 and discuss an application.
In Section 10 we show that no tiny ball contains arbitrary large subsets such that no point of this subset is a strainer of some other point of this subset, Proposition 10.4. Based on this result we prove generalized versions of Theorems 1.6, 1.9.

In Section 11 we prove generalized versions of Theorems 1.1, 1.2 and 1.4. Section 12 is devoted to the proof of a generalized version of Theorem 1.5.

In Section 13 we prove a generalized version of Theorem 1.12.

In Section 14 we follow [Per94] to prove that a $(k, \delta)$-strainer map on a subset of $X^k$ is a local DC-isomorphism. From this we deduce Theorem 1.3 and show that any DC-function is twice differentiable almost everywhere, Proposition 14.11. Of special interest, in particular, for volume rigidity, [Li15], might be general stability of length under convergence of DC-curves, Proposition 14.9.

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2. Preliminaries

2.1. Spaces and maps. [BH99], [BBI01] and [Bal04] are general references for this section. By $d$ we denote distances in metric spaces. For a subset $A$ of a metric space $X$ and $r > 0$, we denote by $B_r(A)$ the open tubular neighborhood of radius $r$ around $A$, hence the set of all points with distance less than $r$ from $A$. By $r \cdot X$ we denote the set $X$ with the metric rescaled by $r$. A space is proper if its closed bounded subsets are compact.

A subset of a metric space is called $r$-separated if its elements have pairwise distances at least $r$. A metric space $X$ is doubling (more precisely, $L$-doubling) if no ball of radius $r$ in $X$ has an $(r/2)$-separated subset with more than $L$ elements. Equivalently, any $r$-ball is covered by a uniform number of balls of radius $r/2$.

The length of a curve $\gamma$ in a metric space is denoted by $\ell(\gamma)$. A geodesic is an isometric embedding of an interval. A triangle is a union of three geodesics connecting three points. A local geodesic is a curve $\gamma : I \to X$ in a metric space $X$ defined on an interval $I$, such that the restriction of $\gamma$ to a small neighborhood of any $t \in I$ is a geodesic. $X$ is a geodesic metric space if any pair of points of $X$ is connected by a geodesic.

A map $F : X \to Y$ between metric spaces is called $L$-Lipschitz if $d(F(x), F(\bar{x})) \leq L \cdot d(x, \bar{x})$, for all $x, \bar{x} \in X$. A map $F : X \to Y$ is called an $L$-biLipschitz embedding if, for all $x, \bar{x} \in X$, one has $\frac{1}{L} \cdot d(x, \bar{x}) \leq d(F(x), F(\bar{x})) \leq L \cdot d(x, \bar{\bar{x}})$.

Let $Z$ be a metric space and $C > 0$. A continuous map $F : Z \to Y$ is called $C$-open if the following condition holds. For any $z \in Z$ and any $r > 0$ such that the closed ball $\bar{B}_{Cr}(z)$ is complete, we have the inclusion $B_r(F(z)) \subset F(B_{Cr}(z))$.

A function $f : X \to \mathbb{R}$ on a metric space $X$ is convex if its restriction $f \circ \gamma$ to any geodesic $\gamma : I \to X$ is a convex function on the interval $I$.

2.2. Convergence. On the set of isometry classes of compact metric spaces we will use the Gromov–Hausdorff distance. By an abuse of definition we will identify spaces and their isometry classes. Whenever spaces $X, Y$ at Hausdorff distance
smaller $\delta$ appear, we will implicitly assume that isometric embeddings $f : X \to Z$ and $g : Y \to Z$ into some metric space $Z$ are fixed such that the Hausdorff distance between $f(X)$ and $g(Y)$ is smaller than $2\delta$.

On the set of isometry classes of pointed proper metric spaces we will consider the pointed Gromov–Hausdorff topology (abbreviated as GH-topology), and denote by $(X_i, x_i) \to (X, x)$ a convergent sequence. Each sequence of doubling spaces with a uniform doubling constant has a subsequence converging in the GH-topology. The limit space is doubling with the same doubling constant.

It is often simpler to work with ultralimits instead of GH-limits. There are several advantages: the ultralimits are always defined, the limit object is a space and not just an "isometry class" and there is no need to consider subsequences, see [AKP16] for details. We fix a non-principal ultrafilter $\omega$ for just an "isometry class" and there is no need to consider subsequences, see [AKP16] for details. We fix a non-principal ultrafilter $\omega$ for details. We fix a non-principal ultrafilter $\omega$ for details.

3. Spaces with an upper curvature bound

3.1. Definitions and notations. For $\kappa \in \mathbb{R}$, let $R_\kappa$ be the diameter of the complete, simply connected surface $M^2_\kappa$ of constant curvature $\kappa$. A complete metric space is called CAT($\kappa$) if any pair of its points with distance $< R_\kappa$ is connected by a geodesic and if all triangles with perimeter $< 2R_\kappa$ are not thicker than the comparison triangle in $M^2_\kappa$. A metric space is called a space with an upper curvature bound $\kappa$ if any point has a CAT($\kappa$) neighborhood. We refer to [BH99], [BB01], [Bal04] for basic facts about such spaces.

Any CAT($\kappa$) space is CAT($\kappa'$) for $\kappa' \geq \kappa$. By rescaling we may always assume that the curvature bound $\kappa$ equals 1. Then $R_\kappa = \pi$.

For any CAT($\kappa$) space $X$, the angle between each pair of geodesics starting at the same point is well defined. The space of directions $\Sigma_x = \Sigma_x X$ at each point $x$, which is the completion of the set of geodesic directions equipped with the angle metric, is a CAT(1) space. The Euclidean cone over $\Sigma_x$ is a CAT(0) space. It is denoted by $T_x = T_x X$ and called the tangent space at $x$ of $X$. The element $w$ in $T_x$ will be written as $w = tv = (t, v) \in T_x = [0, \infty) \times \Sigma_x / \{0\} \times \Sigma_x$, and its norm is defined as $|w| = |tv| := t$.

Let $x, y, z$ be three points at pairwise distance $< R_\kappa$ in a CAT($\kappa$) space $X$. Whenever $x \neq y$, the geodesic between $x$ and $y$ is unique and will be denoted by $xy$. Its starting direction in $\Sigma_x$ will be denoted by $(xy)'$ if no confusion is possible. If $y, z \neq x$ the angle at $x$ between $xy$ and $xz$, hence the distance in $\Sigma_x$ between $(xy)'$ and $(xz)'$ will be denoted by $\angle xyz$.

For $r < R < R_\kappa/2$ we consider the contraction map $c_{R,r} : B_R(x) \to B_r(x)$ centered at $x$, that sends the point $y$ to the point $\gamma(\frac{r}{R} \cdot d(x, y))$, where $\gamma$ is the...
unique geodesic from $x$ to $y$. Due to the CAT($\kappa$) property, the map $c_{R,\kappa}$ is $(2 \cdot \frac{1}{\kappa})$-Lipschitz, compare [Nag02 Section 2.1], for the optimal estimate. We define the logarithmic map $\log_x : B_{\frac{1}{2}R,\kappa}(x) \to T_x$ by $\log_x(x) = 0$ and by sending any $y \neq x$ to $tv \in T_x$, with $t = d(x, y)$ and $v = (xy)'$. The CAT($\kappa$) property implies that $\log_x$ is $2$-Lipschitz.

3.2. Basic topological properties. On spaces with an upper curvature bound, there is a notion of geometric dimension invented by Kleiner [Kle99]. The geometric dimension satisfies $\dim X = 1 + \sup_{x \in X} \dim \Sigma_x$. The geometric dimension $\dim X$ is equal to the topological dimension if $X$ is separable [Kle99].

Convexity of all small balls in spaces with upper curvature bounds imply that any space $X$ with an upper curvature bound is an absolute neighborhood retract, see [Ont05], [Kra11]. In particular, each open subset of $X$ is homotopy equivalent to a simplicial complex.

For any CAT($\kappa$) space $X$ the map $\log_x : (B_{\frac{1}{2}R,\kappa}(x) \setminus \{x\}) \to T_x \setminus \{0\}$ is a homotopy equivalence, [Kra11].

3.3. Convergence and semicontinuity. Let $(X_i, x_i)$ be a sequence of pointed CAT($\kappa_i$) spaces with $\lim_{i \to \infty} \kappa_i = \kappa$. Then $(X, x) = \lim_{\kappa_i}(X_i, x_i)$ is CAT($\kappa$), BH99. Moreover, $\lim_{\kappa_i} \dim(X_i, x_i) \geq \dim(X, x)$, [Lyt05b Lemma 11.1], thus, the geometric dimension does not increase under convergence.

Let $y_i, z_i \in X_i$ be points such that $\epsilon \leq d(x_i, y_i), d(x_i, z_i) \leq \frac{4\kappa}{\kappa_i}$, for some $\epsilon > 0$. The points $y = \lim_{\kappa_i}y_i, z = \lim_{\kappa_i}z_i$ in $X$ and the angles $\angle y_i x_i z_i$ and $\angle y x z$ are well-defined. In this situation, we have the following semicontinuity of angles $\lim_{\kappa_i} \angle y_i x_i z_i \leq \angle y x z$, see [BH99], [Lyt05b p.748].

4. Geodesic extension property

4.1. Definition. Let $X$ be a space curvature $\leq \kappa$. We call $X$ locally geodesically complete if any local geodesic $\gamma : [a, b] \to X$, for any $a < b$, extends as a local geodesic to a larger interval $[a - \epsilon, b + \epsilon]$. If any local geodesic in $X$ can be extended to a local geodesic defined on the real line then $X$ is called geodesically complete.

In BH99, local geodesic completeness is called the geodesic extension property.

For any local geodesic $\gamma : [a, b] \to X$ in a space $X$ with curvature $\leq \kappa$, we can use Zorn’s lemma to find a maximal extension of $\gamma$ to a local geodesic $\gamma : I \to X$ defined on an interval $I \subset \mathbb{R}$. If $X$ is locally geodesically complete, then such a maximal interval $I$ is open in $\mathbb{R}$. Assume that $t = \sup(I)$ is finite. For any $t_i \in I$ converging to $t$, the sequence $\gamma(t_i)$ is a Cauchy sequence in $X$. If $\gamma(t_i)$ converge to a point $x$ in $X$ then the final part $\gamma : [t - \epsilon, t) \to X$ is contained in a CAT($\kappa$) neighborhood $U$ of $x$. Since local geodesics of length $\leq R_{\kappa}$ in $U$ are geodesics, the unique extension of $\gamma$ by $\gamma(t) = x$ is a geodesic on $\gamma : [t - \epsilon, t] \to X$. But then, contrary to our assumption, $I$ is not a maximal interval of definition of $\gamma$. Thus we have shown that $\gamma(t_i)$ cannot converge in $X$. From this we conclude:

Lemma 4.1. Let $X$ be a locally geodesically complete space with an upper curvature bound. Let the closed ball $B_r(x)$ be complete. Then any local geodesic $\gamma$ in $X$ with $\gamma(0) = x$ can be extended to a local geodesic $\gamma : (-t^-, t^+) \to X$ with $t^\pm > r$.

Proof. Extend $\gamma$ to a maximal interval of definition $I = (-t^-, t^+)$. If $t^+ \leq r$ then $\gamma([0, t^+)) \subset B_r(x)$. Thus $\lim_{t \to t^+} \gamma(t)$ exists in $B_r(x)$ in contradiction to the observation preceding the lemma. Thus, $t^+ > r$. Similarly, $t^- > r$. 

\qed
In particular, a complete metric space with an upper curvature bound is geodesically complete if it is locally geodesically complete.

Let the space $X$ with curvature at most $\kappa$ be locally geodesically complete. Let $x \in X$ be arbitrary. Then some closed ball $K = \bar{B}_{2r}(x)$ with $4r < R_\kappa$ is $\text{CAT}(\kappa)$. Note that any geodesic in $K$ is uniquely determined by its endpoints and any local geodesic in $K$ is a geodesic. Due to Lemma 4.1, for any $y \in B_r(x)$ any geodesic $\gamma$ starting in $y$ can be extended inside $K$ to a geodesic $\gamma : [-r, r] \to X$.

4.2. **Examples.** The following example shows that (local) geodesic completeness without further compactness assumption is not of much use.

**Example 4.2.** Starting with any $\text{CAT}(\kappa)$ space $X$ we glue to all points $x \in X$ a line $\mathbb{R} = \mathbb{R}_x$. The arising "hairy" space $\hat{X}$ is still $\text{CAT}(\kappa)$, geodesically complete and contains $X$ as a convex subset.

Let $X$ be a Euclidean simplicial complex with a finite number of isometry classes of simplices and curvature at most 0. Then $X$ is locally geodesically complete if and only if any face of any maximal simplex is a face of at least one other simplex. For any $\text{CAT}(1)$ space $\Sigma$, the Euclidean cone $C\Sigma$ over $\Sigma$ is geodesically complete if and only if $\Sigma$ is geodesically complete and not a singleton. The direct product of two $\text{CAT}(0)$ spaces is geodesically complete if and only if both factors are geodesically complete.

There is a simple topological condition implying local geodesic completeness, cf. [LS07] Theorem 1.5. Namely, if $X$ is a space with an upper curvature bound and if at all points $x \in X$ the local homology $H_s(X, X \setminus \{x\})$ does not vanish then $X$ is locally geodesically complete. In particular, any space with an upper curvature bound which is a (homology) manifold is locally geodesically complete.

Geodesic completeness is preserved under gluings: Let $X_1, X_2$ be two spaces of curvature $\leq \kappa$ and let $A_i \subset X_i$ be locally convex and are isometric to each other. The space $X$ which arises from gluing of $X_1$ and $X_2$ along $A_i$ has curvature $\leq \kappa$, by a theorem of Reshetnyak. It is a direct consequence of the structure of geodesics in $X$, that if $X_1$ and $X_2$ are (locally) geodesically complete then so is $X$.

Finally, geodesic completeness is preserved under ultralimits:

**Example 4.3.** Let $(X_i, x_i)$ be locally geodesically complete spaces with curvature $\leq \kappa_i$. Assume that the balls $\bar{B}_{r_i}(x_i) \subset X_i$ are $\text{CAT}(\kappa_i)$, with $2r_i \leq R_{\kappa_i}$. Assume, finally, that $\lim_{i \to \infty}(\kappa_i) = \kappa$ and $\lim_{i \to \infty}r_i > r > 0$. Consider the ultralimit $(X, x) = \lim_{i}(X_i, x_i)$. Then the closed ball $B_r(x) \subset X$ is $\text{CAT}(\kappa)$ as an ultralimit of $\text{CAT}(\kappa_i)$ spaces. We claim, that the open ball $B_r(x)$ is locally geodesically complete.

Indeed, any geodesic $\gamma$ in $B_r(x)$ is an ultralimit of the corresponding geodesics in $B_r(x_i)$. Since the latter admit extensions of a uniform size to longer geodesics, we obtain an extension of $\gamma$ as the corresponding ultralimit.

5. **GCBA**

5.1. **GCBA spaces and their tiny balls.** Now we turn to the main subject of this paper, the structure of locally compact, locally geodesically complete, separable spaces with upper curvature bounds. As in the introduction, we will denote such spaces as GCBA. Then any open subset of a GCBA space is GCBA as well.

We call an open ball $U = B_{r_0}(x_0)$ in a GCBA space $X$ of curvature $\leq \kappa$ a tiny ball if the following holds true. The radius $r_0$ of $U$ is at most $\min\{1, \frac{1}{100} \cdot R_\kappa\}$ and the closed ball $\bar{U} = \bar{B}_{10\cdot r_0}(x_0)$ with the same center and radius $10 \cdot r_0$ is compact.
5.2. Doubling property. Tiny balls turn out to be doubling.

**Proposition 5.1.** Let $U = B_{r_0}(x_0)$ be a tiny ball of radius $r_0$ in a GCBA space. Let $N$ denote the maximal number of $r_0$-separated points in the compact ball $\tilde{U} = \tilde{B}_{10 \cdot r_0}(x_0)$. Then $B_{5 \cdot r_0}(x_0)$ is $N$-doubling.

**Proof.** It suffices to prove that for any $t > 0$ and any $y \in B_{5 \cdot r_0}(x_0)$, any $\frac{t}{2}$-separated subset $S$ of $B_t(y)$ has at most $N$ elements.

The statement is clear for $t \geq 2r_0$, by the definition of $N$.

For $t < 2r_0$, consider the $\frac{1}{t \cdot r_0}$-Lipschitz map $c_{4 \cdot r_0, t} : B_{4 \cdot r_0}(y) \to B_t(y)$. The map is surjective, since $B_{4 \cdot r_0}(y)$ is contained in $\tilde{U}$. Hence, taking arbitrary preimages of points in $S$ under this contraction map, we obtain an $r_0$-separated subset of $B_{4 \cdot r_0}(y)$ with as many elements as in $S$. Since $B_{8 \cdot r_0}(y) \subset \tilde{U}$, we deduce that $S$ has at most $N$ elements. \qed

**Definition 5.2.** For a tiny ball $U = B_{r_0}(x_0)$ of a GCBA space, we say that $U$ has size bounded by $N$ if $B_{5 \cdot r_0}(x_0)$ is $N$-doubling.

Let $X$ be GCBA. Let $U \subset X$ be a tiny ball of radius $r_0$ and size bounded by $N$. Then any open ball contained in $U$ is a tiny ball in $X$ of size bounded by $N$. Moreover, for any point $x \in U$, the ball $B_s(x)$ is a tiny ball of size bounded by $N$, for any $s \leq \frac{r_0}{2}$. Finally, for every $s \leq \frac{1}{10} \cdot r_0$ the rescaled space $s \cdot U$ is a tiny ball in the GCBA space $s \cdot X$ with size bounded by the same $N$.

5.3. Distance maps and a biLipschitz embedding. Let $U \subset \tilde{U}$ be a tiny ball of radius $r_0$ and size bounded by $N$ as above.

For $p \in \tilde{U}$ we denote by $d_p : \tilde{U} \to \mathbb{R}$ the distance function $d_p(x) = d(p, x)$. The function $d_p$ is 1-Lipschitz and convex on $\tilde{U}$. For any $m$-tuple of points $(p_1, ..., p_m)$ in $\tilde{U}$ the distance map defined by the $m$-tuple is the map $F : \tilde{U} \to \mathbb{R}^m$ with coordinates $f_i(x) = d_{p_i}(x)$. Since any distance function $d_p$, is 1-Lipschitz, any distance map $F : \tilde{U} \to \mathbb{R}^m$ is $\sqrt{m}$-Lipschitz. Moreover, if we equip $\mathbb{R}^m$ with the sup-norm, then $F$ becomes a 1-Lipschitz map $F : \tilde{U} \to \mathbb{R}^m$.

Let $\gamma : [a, b] \to \tilde{U}$ be a geodesic starting at $x = \gamma(a)$. Let $p \neq x$ and $f = d_p$. The derivative of $f \circ \gamma$ at time $a$ is computed by the first variation formula $(f \circ \gamma)'(a) = -\cos(\alpha)$, where $\alpha \in [0, \pi]$ denotes the angle between $\gamma$ and the geodesic $xp$. In particular, $|(f \circ \gamma)'(a)| < \delta$ if $|\alpha - \frac{\pi}{2}| < \delta$. Moreover, $(f \circ \gamma)'(a) > 1 - \delta$ if $\alpha > \pi - \delta$ and $(f \circ \gamma)'(a) < -1 + \delta$ if $\alpha < \delta$.

Denote by $A \subset \tilde{U}$ the compact subset of all points $p \in \tilde{U}$ with $r_0 = d(p, U)$, thus a distance sphere with radius $2r_0$ around the center of $U$. Due to the assumptions on $r_0$ and the curvature bound, for all $\delta > 0$ the following holds true. For every pair of points $p, q \in A$ with $d(p, q) \leq \delta \cdot r_0$ and any $x \in U$ we have $\angle pxq < \delta$. 

As seen at the end of Subsection [1.1] any geodesic $\gamma$ with $\gamma(0) \in U$ can be extended to a geodesic $\gamma : [-9 \cdot r_0, 9 \cdot r_0] \to \tilde{U} \subset X$. For any ball $B_r(x)$ contained in $\tilde{U}$ and any $r' < r$, the contraction map $c_{r, r'} : B_r(x) \to B_{r'}(x)$ is surjective. Any point in $X$ is contained in a tiny ball. Since $X$ is separable, we can write it as a countable union of tiny balls. Any relatively compact subset of $X$ is covered by finitely many tiny balls. All theorems from the introduction will follow once we prove them for all tiny balls in $X$. 


For all $\delta > 0$, we choose a maximal $\delta \cdot r_0$-separated subset $A_3$ in $A$. Due to the doubling property, the number of elements in $A_3$ is bounded by some $m = m(N, \delta)$. Now we obtain:

**Proposition 5.3.** For every $\delta > 0$ there exists some natural $m = m(N, \delta)$ and $m$ points $p_1, ..., p_m \in \tilde{U}$ such that the corresponding distance map $F : \tilde{U} \to \mathbb{R}^m$ is a $(1 + \delta)$-biLipschitz embedding. Here $\mathbb{R}^m_\infty$ denotes $\mathbb{R}^m$ with the sup-norm.

**Proof.** Consider as above the maximal $\delta \cdot r_0$-separated subset $A_3 = \{p_1, ..., p_m\}$ in the distance sphere $A$ and note, that $m$ is bounded in terms of $N$ and $\delta$.

Consider the corresponding distance map $F : \tilde{U} \to \mathbb{R}^m_\infty$. As all distance maps, $F : \tilde{U} \to \mathbb{R}^m_\infty$ is 1-Lipschitz.

Given arbitrary $x, y \in \tilde{U}$, we extend $xy$ beyond $y$ to a point $q \in A$. We find some $p_j \in A_3$ such that $d(p_j, q) \leq \delta \cdot r_0$ hence $\angle p_j y q < \delta$. Then $\angle p_j y x > \pi - \delta$.

From the first variation formula the derivative of the distance function $d_{p_j}$ on the geodesic $y x$ at $y$ is at least $(1 - \delta)$.

Then $d(p_j, x) - d(p_j, y) \geq (1 - \delta) \cdot d(x, y)$, due to the convexity of $d_{p_j}$. Hence, $|F(x) - F(y)|_\infty \geq (1 - \delta) \cdot d(x, y)$.

This finishes the proof. \qed

We let $\delta = 1$ in Lemma 5.3 and obtain a refinement of Proposition 5.1.

**Corollary 5.4.** For some $n_0 = n_0(N)$, there exists a biLipschitz embedding $F : U \to \mathbb{R}^{n_0}$. The Hausdorff and the topological dimensions of $U$ are at most $n_0$.

5.4. **Almost Euclidean triangles.** The diameter of $\tilde{U}$ is smaller than $\frac{1}{4} R_\kappa$. Hence $\angle xyz + \angle yxz \leq \pi$ for any triangle $xyz$ in $\tilde{U}$. If $d(x, z) = d(y, z)$ then $\angle xyz < \frac{\pi}{2}$.

The following lemma shows that triangles in $U$ with one side fixed and the other side sufficiently small have almost Euclidean angles.

**Lemma 5.5.** Let $x \in U$ and $p \in \tilde{U}$ be arbitrary. For any $\epsilon > 0$ there is some $\delta > 0$ such that for any $y \in B_\delta(x)$ we have $\angle pxy + \angle pyx > \pi - \epsilon$.

**Proof.** Assume the contrary and take a sequence $y_i$ converging to $x$ and satisfying $\angle pxy_i + \angle py_i x \leq \pi - \epsilon$. Extend the geodesic $xy_i$ beyond $y_i$ up to a point $z_i$ with $d(x, z_i) = r_0$. Choosing a subsequence we may assume that $z_i$ converges to a point $z$. The semicontinuity of angles gives us

$$\lim \angle pxz_i = \angle p x z \geq \lim \sup \angle p y_i z_i.$$

This contradicts $\angle p y i x \geq \pi - \angle p y_i z_i$ and finishes the proof. \qed

5.5. **Tangent spaces and spaces of directions.** We fix an arbitrary $x \in U$ and claim that every $v \in \Sigma_x$ is the starting direction of a geodesic of length $5r_0$. Thus, for any $r \leq 5r_0$, the map $\log_x : B_r(x) \to T_x$ has the ball $B_r(0) \subset T_x$ as its image.

Indeed, write $v$ as a limit of starting directions $(x y_i)'$ of geodesics. We extend $x y_i$ to geodesics $x z_i$ of length $5r_0$ and find a subsequence converging to a geodesic $xz$ with starting direction $v$.

The restriction of $\log_x$ to small balls is an almost isometry:

**Lemma 5.6.** For any $\epsilon > 0$ there is some $\delta > 0$ (depending on the point $x$), such that for all $r < \delta$ and all $y_1, y_2 \in B_r(x)$ we have

$$|d(y_1, y_2) - d(\log_x(y_1), \log_x(y_2))| \leq \epsilon \cdot r.$$
Proof. We find some finite $\epsilon \cdot r_0$-dense subset $\{p_1, \ldots, p_m\}$ in $U = B_{r_0}(x)$. Then the union of geodesics $x_{p_i}$ is $2\epsilon \cdot r$ dense in $B_r(x)$, for any $r < r_0$.

By the definition of angles, we find a sufficiently small $\delta > 0$ such that (5.1) holds true for all $y_1, y_2$ which lie on the union of the finitely many geodesics $x_{p_i}$. Since the logarithmic map is 2-Lipschitz, we conclude (5.1) with $\epsilon$ replaced by $9\epsilon$, for arbitrary $y_1, y_2 \in B_r(x)$. \hfill $\square$

Thus, the logarithmic map provides an almost isometry between rescaled small balls in $X$ and corresponding balls in the tangent space. From the definition of GH-convergence this implies:

Corollary 5.7. For any sequence $t_i \to 0$ the rescaled spaces $(\frac{1}{t_i} \bar{U}, x)$ converge in the pointed GH-topology to the tangent space $(T_x, 0)$.

From the stability of the geodesic extension property discussed in Example 4.3 and the doubling property of $U$ we see:

Corollary 5.8. For any $x \in U$ the tangent space $T_x$ is an $N$-doubling, geodesically complete CAT(0) space.

We derive:

Corollary 5.9. For any $x \in U$ the space of directions $\Sigma_x$ is a compact, geodesically complete CAT(1) space. $\Sigma_x$ is $N_1$-doubling with $N_1$ depending on $N$. If $U$ is not a singleton then $\Sigma_x$ has diameter $\pi$.

Proof. If $U$ is a singleton then $\Sigma_x$ is empty. Otherwise, there exists at least one geodesic passing through $x$, hence $\Sigma_x$ is not empty and has diameter at least $\pi$. By the definition of the angle metric, the diameter of $\Sigma_x$ cannot be larger than $\pi$. The doubling property follows from Corollary 5.8 since $T_x$ is the Euclidean cone over $\Sigma_x$ and the embedding of $\Sigma_x$ into $T_x$ is 2-biLipschitz. \hfill $\square$

5.6. Precompactness and setting for convergence. A bound on the numbers and sizes of small balls in a covering is equivalent to precompactness in the GH-topology, once the bounds on the curvature and injectivity radius are fixed:

Proposition 5.10. Let $\kappa, t > 0$ be fixed. Let $X_i$ be GCBA and let $K_i \subset X_i$ be compact and connected. Assume that for any $x_i \in K_i$ the ball $B_r(x_i)$ in $X_i$ is a compact CAT($\kappa$) space. Then the following are equivalent:

1. There exists $r > 0$ such that closed tubular neighborhoods $\bar{B}_r(K_i)$ are uniformly compact, i.e., each one is compact and they constitute a precompact set in the Gromov–Hausdorff topology.
2. There are $r, N > 0$, such that the closed tubular neighborhoods $\bar{B}_r(K_i)$ are compact, have diameter $\leq N$ and are $N$-doubling.
3. There are some $r, N > 0$ and a covering of $\bar{B}_r(K_i)$ by at most $N$ tiny balls of size bounded by $N$.

Proof. The implication (2) to (1) is clear.

Under the assumptions of (3), $\bar{B}_r(K_i)$ is $N^3$-doubling by the definition of size. Moreover, the diameter of $\bar{B}_r(K)$ can be at most $2 \cdot N$, since the diameter of any tiny ball is at most 2 and $\bar{B}_r(K)$ is connected, at least for all $r \leq t$. Thus (3) implies (2).

Assume (1). We find some $s < \frac{\sqrt{5}}{50}$ such that for any $x \in K_i$ the open ball $B_{2s}(x)$ is tiny in $X_i$. By the assumption of uniform compactness, there is some $N > 0$ such...
that the maximal $s$-separated subset in $\bar{B}_r(K_i)$ has at most $N$ elements. Hence, we can cover $\bar{B}_r(K_i)$ by at most $N$ open balls of radius $2s$ and each of these tiny balls has size at most $N$, due to Proposition 5.11. This implies (3).

As a consequence of Example 4.3, we see:

Corollary 5.11. Under the equivalent conditions of Proposition 5.10, the compact subsets $K_i \subset X_i$ converge, upon choosing a subsequence, in the GH-topology to a compact subset $K$ of a GCBA space $X$. There is some $s > 0$ such that the compact neighborhoods $B_{10s}(K_i) \subset X_i$ converge in the GH-topology to the compact neighborhood $B_{10s}(K) \subset X$.

We can choose $s$ in Corollary 5.11 to be much smaller than 1 and the injectivity radius $t$. Then all balls with radius $s$ centered in $K_i$ or in $K$ are tiny balls in $X_i$ and $X$ respectively. Therefore, in all local questions concerning convergence, we can restrict ourselves to a convergence of tiny balls in some GCBA spaces to a tiny ball in some other GCBA space, as described in the following.

Definition 5.12. As the standard setting for convergence we will denote the following situation. The sequence $U_i \subset \tilde{U}_i$ of tiny balls in GCBA spaces $X_i$ have the same radius $r_0$ and the same bound on the size $N$. The sequence $\tilde{U}_i$ converges in the GH-topology to a compact ball $\tilde{U}$ of radius $10 \cdot r_0$ in a GCBA space $X$. The closures $\tilde{U}_i$ converge to the closure $\tilde{U}$ of a tiny ball $U \subset \tilde{U}$ of radius $r_0$ in $X$.

5.7. Semicontinuity of tangent spaces. For GCBA spaces, semicontinuity of angles discussed in Subsection 3.3 has the following nice formulation.

Lemma 5.13. Under the standard setting of the convergence as in Definition 5.12, let $x_i \in U_i \subset \tilde{U}_i$ converge to $x \in U \subset \tilde{U}$. Then the sequence of the spaces of directions $\Sigma_{x_i} U_i$ is precompact in the GH-topology. For every limit space $\Sigma'$ of this sequence there exists a surjective 1-Lipschitz map $P : \Sigma_x U \to \Sigma'$.

Proof. Corollary 5.9 implies that the sequence $\Sigma_{x_i} U_i$ is uniformly doubling, hence precompact.

In order to prove the second statement, we may replace our sequence $U_i$ by a subsequence and assume that $\Sigma_{x_i}$ converge to $\Sigma'$.

For any direction $v \in \Sigma_x U$ we take a point $y \in \tilde{U}$ with $(xy)' = v$ and $d(x, y) = r_0$. Consider a sequence $y_i \in \tilde{U}_i$ converging to $y$ and put $v_i := (x_i y_i)' \in \Sigma_{x_i} U_i$. Then we choose the limit point $w = \lim_{i}(v_i)$ in $\Sigma'$ of the sequence $(v_i)$ and set $P(v) := w$.

The semicontinuity of angles discussed in Subsection 3.3 is exactly the statement that the map $P$ is 1-Lipschitz. The surjectivity of $P$ follows from the construction and the fact that any direction $w \in \Sigma'$ is a limit direction of some directions $v_i \in \Sigma_{x_i} U_i$, which are starting directions of geodesics of uniform length $r_0$ in $U_i$.

6. Almost suspensions

6.1. Spherical and almost spherical points. In this section let $\Sigma$ be a compact, geodesically complete CAT(1) space with diameter $\pi$. Note, that any space of directions $\Sigma_x$ of any GCBA space $X$ satisfies this assumption by Corollary 5.30.

Definition 6.1. Let $\Sigma$ be a compact CAT(1) space which is GCBA and has diameter $\pi$. For $v \in \Sigma$, an antipode of $v$ is a point $\bar{v}$ with $d(v, \bar{v}) = \pi$. A point $v \in \Sigma$ is called spherical if it has only one antipode.
Consider the subset \( \Sigma^0 \) of all spherical points \( v \in \Sigma \). Then \( \Sigma^0 \) is a convex subset isometric to some unit sphere \( S^k \) and \( \Sigma \) is a spherical join \( \Sigma = \Sigma^0 * \Sigma' \), see, for instance, \cite[Corollary 4.4]{Lyt05b}. The Euclidean cone \( C \Sigma \) has an \( \mathbb{R}^k \)-factor if and only if \( \Sigma \) is decomposable as a spherical join of \( S^{k-1} \) and another space. Moreover, the maximal Euclidean factor is \( C \Sigma^0 \subset C \Sigma \).

**Definition 6.2.** Let \( \Sigma \) be as above and let \( \delta > 0 \) be arbitrary. We call a point \( v \in \Sigma \) a \( \delta \)-spherical point, if there exists some \( \bar{v} \in \Sigma \) such that for any \( w \in \Sigma \)
\[
d(v, w) + d(w, \bar{v}) < \pi + \delta.
\]
Moreover, we say that \( v \) and \( \bar{v} \) are opposite \( \delta \)-spherical points.

The triangle inequality and extendability of geodesics to length \( \pi \) directly imply:

**Lemma 6.3.** Let \( \Sigma \) be as above. The points \( v, \bar{v} \in \Sigma \) are opposite \( \delta \)-spherical points if and only if \( d(v, w) < \delta \) for any antipode \( w \) of \( v \). In particular, in this case \( d(v, \bar{v}) > \pi - \delta \) and the set of all antipodes of \( v \) has diameter less than \( 2\delta \). Finally, for every antipode \( v' \) of \( v \), the pair \( (v, v') \) are opposite \( 2\delta \)-spherical points.

### 6.2. Tuples of \( \delta \)-spherical points

We define special positions of pairs of almost spherical points:

**Definition 6.4.** Let \( \Sigma \) be as above. Let \( (v_1, ..., v_k) \) be a \( k \)-tuple of points in \( \Sigma \). We say that \( (v_i) \) is a \( \delta \)-spherical \( k \)-tuple if there exists another \( k \)-tuple \( (\bar{v}_i) \) in \( \Sigma \) with the following two properties.

1. For \( 1 \leq i \leq k \) : \( v_i \) and \( \bar{v}_i \) are opposite \( \delta \)-spherical points.
2. For \( 1 \leq i \neq j \leq k \) : \( d(v_i, \bar{v}_j) < \frac{\pi}{2} + \delta \); \( d(v_i, v_j) < \frac{\pi}{2} + \delta \); \( d(\bar{v}_i, \bar{v}_j) < \frac{\pi}{2} + \delta \).

Moreover, \((\bar{v}_i)\) and \((v_i)\) are called opposite \( \delta \)-spherical \( k \)-tuples.

From **Lemma 6.3** and the triangle inequality we deduce:

**Corollary 6.5.** Let \( \Sigma \) be as above. Let \( v_1, ..., v_k \in \Sigma \) be \( \delta \)-spherical points. If \( (v_1, ..., v_k) \) is a \( \delta \)-spherical \( k \)-tuple then, for all \( i \neq j \),
\[
\frac{\pi}{2} - 2\delta < d(v_i, v_j) < \frac{\pi}{2} + \delta.
\]

On the other hand, assume that, for all \( i \neq j \),
\[
\frac{\pi}{2} - \delta < d(v_i, v_j) < \frac{\pi}{2} + \delta.
\]

Then, for arbitrary antipodes \( \bar{v}_i \) of \( v_i \), the tuples \((v_i)\) and \((\bar{v}_i)\) are opposite \( 2\delta \)-spherical \( k \)-tuples.

It is important to notice that all definitions above only use upper bounds on distances. Thus, due to the semicontinuity of angles, they are suitable to provide open conditions on spaces of directions.

### 6.3. Connection with GH-topology

The existence of almost spherical \( k \)-tuples is equivalent to a small distance from a \( k \)-fold suspension:

**Proposition 6.6.** Let \( C \) be a compact set in the GH-topology of (isometry classes of) compact, geodesically complete CAT(1) spaces with diameter \( \pi \). Let \( k \) be a natural number. The following are equivalent for any sequence \( \Sigma_i \) in \( C \).

1. Any accumulation point \( \Sigma \in C \) of the sequence \( \Sigma_i \) is isometric to a \( k \)-fold suspension \( S^{k-1} * \Sigma' \), with possibly empty \( \Sigma' \).
(2) For any \( \delta > 0 \) and all sufficiently large \( l \), the space \( \Sigma_l \) admits a \( \delta \)-spherical \( k \)-tuple.

Proof. Choosing a subsequence we may restrict ourselves to the case that \( \Sigma_l \) converges to a space \( \Sigma \).

A sequence of \( \delta_l \)-spherical \( k \)-tuples in \( \Sigma_l \) with \( \delta_l \to 0 \) converges to a \( k \)-tuple of spherical points in \( \Sigma \) with pairwise distance \( \frac{\pi}{2} \). This spherical \( k \)-tuple determines a splitting \( \Sigma = S^{k-1} \star \Sigma' \), hence (2) implies (1).

On the other hand, if \( \Sigma = S^{k-1} \star \Sigma' \), we choose the standard coordinate directions \( e_1, \ldots, e_k \in S^{k-1} \subset \Sigma \) and consider in \( \Sigma_l \) tuples of points converging to the \( k \)-tuple \( (e_i) \). These \( k \)-tuples satisfy the condition of (2), finishing the proof. \( \square \)

7. Strainers

7.1. Strained points. The following definition, translated from [BGP92] to our setting, is central for all subsequent considerations.

Definition 7.1. Let \( X \) be GCBA, \( k \) an integer and \( \delta > 0 \). A point \( x \in X \) is \((k, \delta)\)-strained if the space of directions \( \Sigma_x \) contains some \( \delta \)-spherical \( k \)-tuple.

As in the introduction, we denote by \( X_{k, \delta} \) the set of \((k, \delta)\)-strained points in \( X \). We have \( X_{k, \delta} \subset X_{k-1, \delta} \subset X_1, \delta \subset X_0, \delta = X \). Due to Proposition 6.6, \( X_{k, 0} = \bigcap_{\delta > 0} X_{k, \delta} \) is the set of all points \( x \in X \), for which the tangent space \( T_xX \) splits off the Euclidean space \( \mathbb{R}^k \) as a direct factor.

7.2. Strainers. As in Section 5 we fix a tiny ball \( U \subset \bigcup_x X \).

Definition 7.2. Let \( x \in U \) be a point and let \( \delta > 0 \) be arbitrary. A \( k \)-tuple of points \( p_i \in U \setminus \{x\} \) is a \((k, \delta)\)-strainer at \( x \) if the \( k \)-tuple of the starting directions \( (x p_i)^\circ \) is \( \delta \)-spherical in \( \Sigma_x \).

Two \((k, \delta)\)-strainers \((p_i)\) and \((q_i)\) at \( x \) are opposite if the \( \delta \)-spherical \( k \)-tuples \((x p_i)^\circ\) and \((x q_i)^\circ\) are opposite in \( \Sigma_x \).

For a set \( V \subset U \), a \( k \)-tuple \((p_i)\) of points in \( V \) is a \((k, \delta)\)-strainer in \( V \) if \((p_i)\) is a \((k, \delta)\)-strainer at all \( x \in V \). If \((k, \delta)\)-strainers \((p_i)\) and \((q_i)\) are opposite at all points \( x \in V \), we say that \((p_i)\) and \((q_i)\) are opposite \((k, \delta)\)-strainers in \( V \).

A point \( p \) is a \((1, \delta)\)-strainer at \( x \) if and only if there is some \( v \in \Sigma_x \) such that any continuation of \( p x \) beyond \( x \) as a geodesic encloses an angle smaller than \( \delta \) with \( v \). The following observation is the most fundamental source of strainers.

Proposition 7.3. For any \( \delta > 0 \) and \( p \in U \), there is a neighborhood \( O \) of \( p \) such that the point \( p \) is a \((1, \delta)\)-strainer in \( O \setminus \{p\} \).

Proof. Otherwise we find points \( x_i \neq p \) arbitrary close to \( p \) such that \((x_i p)^\circ\) is not \( \delta \)-spherical. Set \( s_i = d(x_i, p) \) and extend \( p x_i \) by different geodesics to points \( y_i, z_i \) with \( d(y_i, x_i) = d(z_i, x_i) = s_i \) and \( \angle y_i x_i z_i \geq \delta \).

By construction, \( d(y_i, p) = d(z_i, p) = 2 \cdot s_i \) and \( \log_p(y_i) = \log_p(z_i) \). On the other hand \( d(y_i, z_i) \geq \rho \cdot s_i \), where \( \rho > 0 \) depends only on \( \delta \) and the curvature bound \( \kappa \). For \( s_i \to 0 \), this contradicts Lemma 5.6. \( \square \)

Remark 7.4. An observation similar to Proposition 7.3 can be found in [OT99].

For any \( \delta > \pi \) and any \( x \in U \), any \( k \)-tuple \((p_i)\) of points in \( U \setminus \{x\} \) is a \((k, \delta)\)-strainer at \( x \). On the other hand, we have:
Lemma 7.5. There exists a number $k_0(N)$ with the following property. For any tiny ball $U$ of size bounded by $N$ and any $1 \geq \delta > 0$, there do not exist $(k, \delta)$-strained points in $U$ with $k > k_0$.

Proof. Let $x$ be a $(k, \delta)$-strained point in a tiny ball $U$ of size bounded by $N$. By definition, we find in $\Sigma_x$ a $(\frac{N}{2} - \delta)$-separated subset with $k$ points. From the bound on the doubling constant (Corollary \ref{cor:doubling}) and the assumption $\frac{N}{2} - \delta > \frac{N}{2} > 0$, we deduce that $k$ is bounded from above in terms of $N$. \hfill \Box

7.3. Almost Euclidean triangles. The existence of strainers implies the existence of many almost Euclidean triangles. We will only use the following special case:

Lemma 7.6. Let $p, q \in \tilde{U}$ be opposite $(1, \delta)$-strainers at points $x \neq y$ in a tiny ball $U$. Then the following hold true.

1. $\pi - 2 \cdot \delta < \angle pxy + \angle qyx < \pi$.
2. If $d(p, x) = d(p, y)$ then $\frac{\pi}{2} - 2 \cdot \delta < \angle pxy < \frac{\pi}{2}$.
3. If $\frac{\pi}{2} - 2 \cdot \delta < \angle pxy < \frac{\pi}{2}$ then $\frac{\pi}{2} - 2 \cdot \delta < \angle qyx < \frac{\pi}{2} + 2 \cdot \delta$.

Proof. From the assumption on the upper curvature bound and diameter of $\tilde{U}$ we deduce the right hand side inequalities in (1) and in (2). By the same reason

\begin{equation}
\angle qxy + \angle qyx < \pi.
\end{equation}

On the other hand, by the definition of opposite strainers we have

$\angle pxy + \angle qxy > \pi - \delta$ and $\angle pxy + \angle qyx > \pi - \delta$.

Hence the sum of these four angles is at least $2\pi - 2\delta$. Combining with (7.1), we deduce the left hand side of (1).

The remaining statements are direct consequences of (1). \hfill \Box

7.4. Stability of strainers. If $(p_i)$ is a $(k, \delta)$-stainer at $x \in U$ and $\hat{p}_i \in \tilde{U} \setminus \{x\}$ is any point on an extension of $xp_i$ beyond $p_i$ then $(\hat{p}_i)$ is still a $(k, \delta)$-strainer at $x$.

From Corollary \ref{cor:opposite}, we obtain:

Lemma 7.7. Let $p_1, \ldots, p_k \in \tilde{U}$ be arbitrary points lying on an extension of the geodesic $p_ix$ beyond $x$. If $|\angle px_i - \frac{\pi}{2}| < \delta$, for all $i \neq j$, then $(p_i)$ and $(q_i)$ are opposite $(k, 2\delta)$-strainers at $x$.

The definition of strainers is designed to satisfy the following openness condition:

Lemma 7.8. Let $U_1 \subset \tilde{U}_1 \subset X_1$ converge to $U \subset \tilde{U} \subset X$ as in our standard setting in Definition \ref{def:convergence}. Let $(p_i)$ and $(q_i)$ be opposite $(k, \delta)$-strainers at $x \in U$. Let, for

\begin{center}
$i = 1, \ldots, k$,
\end{center}

the sequences $p_i^l, q_i^l, x^l \in \tilde{U}_l$ converge to $p_i, q_i$ and $x$, respectively.

Then the $k$-tuples $(p_i^l)$ and $(q_i^l)$ in $\tilde{U}_l$ are opposite $(k, \delta)$-strainers at the point $x^l$, for all $l$ large enough.

Proof. The claim is a consequence of the semicontinuity of angles under convergence, Subsection \ref{subsec:angles} (see also Lemma \ref{lem:angles}), and the definition of $\delta$-spherical $k$-tuples, which only involves strict upper bounds on distances. \hfill \Box

Restricting to the case $U_l = U$, for all $l$, we see from Lemma \ref{lem:opposite}.

Corollary 7.9. For $k$-tuples $(p_i)$ and $(q_i)$ in $\tilde{U}$, the set of points $x \in U$ at which $(p_i)$ and $(q_i)$ are opposite $(k, \delta)$-strainers is open.

The set of points $x \in U$ at which $(p_i)$ is a $(k, \delta)$-strainer is open.
Lemma 7.10. Let \((p_i)\) be a \((k, \delta)\)-strainer at \(x \in U\). Then there is some number \(0 < \epsilon_x < \frac{1}{l} \cdot d(x, \partial U)\) with the following property. If \(y \in B_{\epsilon_x}(x)\) is arbitrary and \(q_i \in \tilde{U}\), with \(d(q_i, y) = \rho_0\), lies on an arbitrary continuation of \(p_i\) beyond \(y\), then the \(k\)-tuples \((q_i)\) and \((p_i)\) are opposite \((k, 2\delta)\)-strainers in the ball \(B_{\epsilon_x}(y)\).

Proof. In order to prove the statement, we assume the contrary and find contradicting sequences \(y_i, z_i \to x\) and \(k\)-tuples \((q^i_l)\). Thus, \(d(y_i, q^i_l) = \rho_0\), the point \(y_i\) is on the geodesic \(p_i q^i_l\), and \((p^i_l)\) and \((q^i_l)\) are not opposite \((k, 2\delta)\)-strainers at \(z_i\). Taking limit points we find a \(k\)-tuple \((q_i) \in \tilde{U}\) such that \(x\) is an inner point of the geodesic \(p_i q_i\) for any \(i\).

Due to stability of strainers, Lemma 7.8, \((q_i)\) and \((p_i)\) cannot be opposite \((k, 2\delta)\)-strainers at \(x\). But this contradicts Lemma 7.7.

We will call the maximal number \(\epsilon_x\) as in Lemma 7.10 above the straining radius at \(x\), of the \((k, \delta)\)-strainer \((p_i)\). By definition, \(\epsilon_y \geq \epsilon_x - d(x, y)\), for all \(x, y\) strained by \((p_i)\). In particular, the map \(x \to \epsilon_x\) is continuous.

Note finally, that the proof above literally transfers to the convergence setting from Lemma 7.8. Thus, the proof shows:

Lemma 7.11. Under the assumptions of Lemma 7.8 let \(\epsilon_x\) and \(\epsilon_{x_i}\) be the straining radius of \((p_i)\) and \((p^i_l)\) at \(x\) and \(x_i\), respectively. Then \(\lim \inf_{i \to \infty} \epsilon_{x_i} \geq \epsilon_x\).

8. Strainer maps

8.1. Differentials of distance maps and a criterion for openness. As always in our setting, let \(U \subset \tilde{U} \subset X\) be a tiny ball. As before, denote by \(d_p : \tilde{U} \to \mathbb{R}\) the distance function to the point \(p \in \tilde{U}\).

For any point \(x \in \tilde{U}\) we collect the directional derivatives of \(f = d_p\) to a differential \(D_x f : T_x \to \mathbb{R}\). If \(x \neq p, v \in \Sigma_x\) and \(t \geq 0\) then the differential \(D_x f(tv)\) is given by the first variation formula as \(D_x f(tv) = -t \cdot \cos(\alpha)\), where \(\alpha\) is the distance in \(\Sigma_x\) between \(v\) and the starting direction of the geodesic \(xp\).

If \(F = (d_{p_1}, ..., d_{p_k}) : \tilde{U} \to \mathbb{R}^k\) is a distance map, we denote as its differential \(D_x F : T_x \to \mathbb{R}^k\) the map whose coordinates are the differentials of \(d_{p_i}\) at \(x\).

The following criterion is essentially taken from [BGP92, Section 11.5].

Lemma 8.1. Set \(\rho = \frac{1}{l} \cdot \rho_0\). Let \((p_i)\) be a \(k\)-tuple in \(\tilde{U}\) and let \(f_i = d_{p_i}\) be the corresponding distance functions. Assume that for every \(x\) in an open subset \(V\) of \(U\) and any \(1 \leq i \leq k\) there are directions \(v_i^\pm \in \Sigma_x\) with

\[
\pm D_x f_i (v_i^\pm) > 1 - \rho \quad \text{and} \quad |D_x f_j (v_i^\pm)| < 2 \cdot \rho, \; i \neq j.
\]

Then the distance map \(F = (f_1, ..., f_k) : V \to \mathbb{R}^k\) is 2-open if we equip \(\mathbb{R}^k\) with the \(L^1\)-norm \(|(t_i)|_1 = \sum_{i=1}^k |t_i|\).

Proof. Let \(x \in V\) be arbitrary and \(r > 0\) such that \(\tilde{B}_{2r}(x)\) is complete. Let \(t = (t_1, ..., t_k) \in \mathbb{R}^k\) with \(s := |t - F(x)|_1 < r\) be fixed. In order to find \(y \in \tilde{B}_{2r}(x) \cap F^{-1}(t)\) we consider the function \(h : V \to \mathbb{R}\) given by

\[
h(z) := |t - F(x)|_1 - |t - F(z)|_1 = s - |t - F(z)|_1.
\]
Then $h(x) = 0$ and we are looking for $y \in B_{2r}(x)$ with $h(y) = s$.

For every $z \in V$ with $h(z) < s$, there is some $i = 1, \ldots, k$ such that $t_i \neq f_i(z)$. On the geodesic $\gamma$ starting at such $z$ in the direction $v_i^k$ (depending on the sign of $t_i - f_i(z)$), the value of $|t_j - f_j(z)|$ decreases (infinitesimally) with velocity larger than $1 - \rho$, while the values of $|t_j - f_j(z)|$ for $j \neq i$ increase with velocity less than $2\rho$. Therefore the norm of the gradient of $h$ at any $z \in V \setminus F^{-1}(t)$ satisfies

$$\left| \nabla z h \right| := \limsup_{y \to z} \frac{h(y) - h(z)}{d(y, z)} \geq (1 - \rho) - (k - 1) \cdot (2 \cdot \rho) > 1 - 2 \cdot k \cdot \rho \geq \frac{1}{2}.$$

Due to [Lyt05a Lemma 4.1], for any $s' < s$ we find some $z \in B_{2s}(x)$ with $h(z) = s'$. We let $s'$ go to $s$ and use compactness of $B_{2r}(x)$ to find the desired point $y$.

This shows $B_r(F(x)) \subset F(B_{2r}(x))$ and finishes the proof. $\square$

8.2. Strainer maps. Let $V \subset U$ be a subset, let $(p_i)$ be a $k$-tuple in $U$, and let $F = \left( d_{p_1}, \ldots, d_{p_k} \right) : V \to \mathbb{R}^k$ be the corresponding distance map. We say that $F$ is a $(k, \delta)$-strainer map in $V$ if $(p_i)$ is a $(k, \delta)$-strainer in $V$. In this case, we define the straining radius of $F$ at $x \in V$ to be the straining radius of the $(k, \delta)$-strainer $(p_i)$ at $x$. We say that distance maps $F, G$ are opposite $(k, \delta)$-strainer maps in $V$ if their defining $k$-tuples are opposite $(k, \delta)$-strainers in $V$.

Let $F : V \to \mathbb{R}^k$ be a $(k, \delta)$-strainer map with coordinates $f_i = d_{p_i}$ defined on an open subset $V$ of $U$. For any $v \in V$, we find some distance map $G = (d_{q_i})$ such that $F$ and $G$ are opposite $(k, \delta)$-strainer maps at $v$. Choose $v_i^k \in \Sigma_v$ to be the starting directions of $x_{p_i}$ and $x_{q_i}$, respectively. By the definition of strainers and the first variation formula, we see that (8.1) hold true at the point $x$ with $\rho$ replaced by $\delta$. Replacing the $L^1$-norm by the Euclidean norm we get:

**Lemma 8.2.** If $\delta \leq \frac{1}{k}$ then any $(k, \delta)$-strainer map $F : V \to \mathbb{R}^k$ on any open non-empty set $V \subset U$ is $L$-Lipschitz and $L$-open with $L = 2\sqrt{k}$. In particular, the Hausdorff dimension of $V$ is at least $k$.

**Proof.** The Lipschitz property is true for any distance map. The openness constant follows from Lemma (8.1). The bound on the Hausdorff dimension follows, since the image $F(V)$ is open in $\mathbb{R}^k$. $\square$

As in [BGP92 Section 11.1], one can derive from Lemma (8.2) that the intrinsic metric on the fibers of $F$ is locally biLipschitz equivalent to the induced metric. Since it is not used in the sequel, we do not provide the proof.

8.3. Convergence of maps and an improvement of constants. We are going to prove that for small $\delta$, the constant $L$ in Lemma (8.2) can be chosen arbitrary close to 1. These results will only be used in Subsection 12.2 where one could rely on results from [Nag02] instead. For this reason, the argument in this subsection will be sketchy. Readers not familiar with ultralimits may restrict to the case of tiny balls with uniformly bounded size and replace ultralimits by GH-limits of a subsequence, using Corollary (11.8).

Let $F : V \to \mathbb{R}^k$ be a $(k, \delta)$-strainer map on an open subset $V$ of some tiny ball $U$. For $\delta \leq \frac{1}{k}$, the map $F$ is $L$-open and $L$-Lipschitz on $V$ with $L = 2\sqrt{k}$. Therefore, the differential $D_x F : T_x \to \mathbb{R}^k$ which is a limit of rescalings of $F$ is $L$-Lipschitz and $L$-open.

Let $F_i : V_i \to \mathbb{R}^k$ be a sequence of $(k, \delta_i)$-strainer maps with $\lim_{i \to \infty} \delta_i = 0$. Let $x_i \in V_i$ be arbitrary and consider the sequence of differentials $D_{x_i} F_i : T_{x_i} \to \mathbb{R}^k$. 
As in the proof of Proposition 9.6, we see that the ultralimit \( T = \lim\omega_T x \) is a Euclidean cone which splits as \( T = \mathbb{R}^k \times T' \). Moreover, the ultralimit \( P = \lim\omega D_x F_i : T \to \mathbb{R}^k \) is just the projection of \( T \) onto the direct factor \( \mathbb{R}^k \).

Since the maps \( D_x F_i \) are all \( L \)-open, any fiber \( P^{-1}(w) \) with \( w \in \mathbb{R}^k \) coincides with the ultralimit of fibers \( \lim\omega(D_x F_i)^{-1}(w_i) \) for any sequence \( w_i \) converging to \( w \). For any unit vector \( w \in S^{k-1} \), the fiber \( P^{-1}(w) \) has distance 1 to the origin of the cone \( T \). Therefore, for any \( \epsilon > 0 \), the distances of infinitely many of the fibers \( (D_x F_i)^{-1}(w_i) \) to the origin must be between \( 1 - \epsilon \) and \( 1 + \epsilon \).

Arguing by contradiction we conclude:

**Lemma 8.3.** For every \( k \in \mathbb{N} \) and \( L > 1 \) there exists some \( \delta = \delta(L, k) > 0 \) such that the following holds true. For any \( (k, \delta) \)-strainer map \( F \) at a point \( x \) in a tiny ball \( U \) of a GCBA space \( X \) the differential \( D_x F : T_x \to \mathbb{R}^k \) satisfies:

1. \( |D_x F(v)| < L \), for any \( v \in \Sigma_x \subset T_x \).
2. For any \( u \in S^{k-1} \subset \mathbb{R}^k \), there exists \( v \in T_x \) with \( D_x F(v) = u \) and \( |v| < L \).

The infinitesimal characterization of L-Lipschitz and \( L \)-open maps, [Lyt05a Theorem 1.2], now directly implies:

**Corollary 8.4.** For any \( L > 1 \) there is some \( \delta = \delta(L, k) > 0 \) such that the following holds true. Any \( (k, \delta) \)-strainer map \( F : V \to \mathbb{R}^k \) is \( L \)-open and \( L \)-Lipschitz, whenever \( V \) is an open convex subset of a tiny ball of size bounded by \( N \).

9. Fibers of strainer maps

9.1. Local contractibility. We are going to prove:

**Theorem 9.1.** Let \( F : \tilde{U} \to \mathbb{R}^k \) be a distance map and \( 0 < \delta \leq \frac{1}{20k} \). Assume that \( F \) is a \((k, \delta)\)-strainer map at \( x \in U \) with straining radius \( \epsilon_x \). Let \( Q = \prod_{i=1}^{k}[a_i^-, a_i^+] \) be a rectangular box in \( \mathbb{R}^k \) which contains \( F(x) \). Then, for any \( 0 < r < \epsilon_x \), the ball \( B_r(x) \cap F^{-1}(Q) \) is contractible.

More precisely, for \( W = B_r(x) \) there exists a homotopy \( \Phi : W \times [0, 1] \to W \) with the following properties.

1. For all \( y \in W \) the curve \( \gamma_y(t) := \Phi(y, t) \) starts in \( y \) and ends on \( F^{-1}(Q) \).
2. The diameter of the curve \( \gamma_y \) is bounded from above by \( 8 \cdot k \cdot d(F(y), Q) \).
3. For all \( t \in [0, 1] \) we have \( d(\gamma_y(t), x) \leq d(x, y) \).

**Proof.** Assume that a map \( \Phi \) with the properties (1), (2), (3) exists. By (2), the homotopy \( \Phi \) fixes \( W \cap F^{-1}(Q) \) pointwise. For every \( r < \epsilon_x \), \( \Phi \) provides a homotopy retraction from the contractible ball \( B_r(x) \) to \( B_r(x) \cap F^{-1}(Q) \), proving that the latter is contractible. Thus it remains to prove the existence of \( \Phi \).

We set \( \Pi := F^{-1}(Q) \). Let \( f_i = d_{p_i} \) be the coordinates of \( F \). By definition of \( \epsilon_x \) we find \( q_i \in \tilde{U} \) for \( i = 1, \ldots, k \), such that \( x \) lies on the geodesic \( p_i q_i \) and such that \((p_i), (q_i) \) are opposite \((k, 2\delta)\)-strainers in \( W \).

For \( i = 1, \ldots, k \), let \( \Pi_i \) be the set of all points \( z \in U \) with \( a_i^- \leq f_i(z) \leq a_i^+ \). First, we define flows \( \phi_i : W \times [0, 1] \to U \). For a point \( y \in W \) with \( f_i(y) \geq f_i(x) \), the flow \( \phi_i \) moves \( y \) with velocity 1 along the geodesic \( yp_i \) until it reaches \( \Pi_i \) and then the flow stops for all times. For a point \( y \) with \( f_i(y) \leq f_i(x) \), the flow moves \( y \) along the geodesic \( yq_i \) until it reaches \( \Pi_i \) and stops there.

Since \( x \) is on the geodesic \( p_i q_i \), the CAT(k) condition implies that the flow \( \phi_i \) does not increase the distance to \( x \).
By the first variation formula, the value of \( f_i \) changes along the flow lines of \( \phi_i \) with velocity at least \( 1 - 2\delta \) until the point reaches the set \( \Pi_i \). Moreover, for \( j \neq i \), the value of \( f_j \) changes along the flow lines of \( \phi_j \) with velocity at most \( 4\delta \).

Consider the function \( M_i, M : U \to \mathbb{R} \) defined by

\[
M_i(y) := \max\{0, f_i(y) - a_i^+, a_i^- - f_i(y)\} \quad \text{and} \quad M(y) := \max_{1 \leq i \leq k} M_i(y).
\]

Note that \( M(y) = 0 \) if and only if \( y \in \Pi \) and \( M(y) \leq d(F(y), Q) \), for all \( y \in U \).

The above observation shows that the flow line \( \phi_i(y, t) \) reaches \( \Pi_i \) at latest at \( t = (1 - 2\delta)^{-1} \cdot M_i(y) \). Due to the first variation formula and \( \delta \leq \frac{1}{20} \), we have for all \( j \neq i \) and all \( 1 \geq t \geq 0 \):

\[
M_j(\phi_i(y, t)) \leq M_j(y) + \frac{4\delta}{1 - 2\delta} \cdot M_i(y) \leq M_j(y) + 5\delta \cdot M(y).
\] (9.1)

Consider the concatenation \( \Psi \) of the flows \( \phi_1, ..., \phi_k \). Thus \( \Psi : W \times [0, k] \to W \) is a homotopy which moves on the time interval \( [j - 1, j] \) the point \( \Psi(y, j - 1) \) along the flow lines of \( \phi_j \) to \( \Pi_j \). We apply \( k \) times the inequality (9.1) and conclude

\[
M(\Psi(y, t)) \leq (1 + 5\delta)^k \cdot M(y),
\]

for all \((y, t) \in W \times [0, k]\). By construction \( M_j(\Psi(y, j)) = 0 \). Applying (9.1) again, for all \( j \), we improve the last inequality to

\[
M(\Psi(y, k)) \leq k \cdot 5 \cdot \delta \cdot (1 + 5 \cdot \delta)^k \cdot M(y) \leq \frac{1}{4} \cdot (1 + \frac{1}{4k})^k \cdot M(y).
\]

Since \( (1 + \frac{1}{4})^x \) is increasing and converges to the Euler number \( e \), we see

\[
M(\Psi(y, k)) \leq \frac{1}{4} \cdot e^{\frac{x}{4}} \cdot M(y) < \frac{1}{4} \cdot 2 \cdot M(y) = \frac{1}{2} \cdot M(y).
\]

Moreover, the flow line of the homotopy \( \Psi \) of a point \( y \) has length at most

\[
k \cdot \frac{1}{1 - 2\delta} \cdot (1 + 5 \cdot \delta)^k \cdot M(y) \leq 4 \cdot k \cdot M(y).
\]

Putting the last two observations together, we inductively arrive at the following conclusion about the \( m \)-fold concatenation \( \Psi_m : W \times [0, k \cdot m] \to W \) of the homotopy \( \Psi \). For any \( y \in W \), we have \( M(\Psi_m(y, k \cdot m)) \leq 2^{-m} \cdot M(y) \). Moreover, the \( \Psi_m \)-flow line of \( y \) has length at most \( 8k \cdot M(y) \). Therefore, reparametrizing \( \Psi_m \) we obtain a limit homotopy \( \Phi = \Psi^\infty \) with the required properties.

In the special case that \( Q \) just consists of one point \( F(x) \), we deduce that all small balls in all fibers of \( F \) are contractible.

**Proof of Theorem 9.11** Under the assumptions of Theorem 9.11 let \( V' \subseteq V \) be compact. Since the straining radius depends continuously on the point, we find some \( \epsilon > 0 \) smaller than the straining radius at any \( x \in V' \) and smaller than \( d(V', \partial V) \). By Theorem 9.1 for \( x \in V' \) and \( r < \epsilon \) the set \( B_r(x) \cap F^{-1}(F(x)) \) is contractible.

\[ \square \]

**9.2. Dichotomy.** The openness of strainer maps and local connectedness of their fibers implies a dichotomy in the behavior of strainer maps. First a local result:

**Lemma 9.2.** Let \( F \) be a \((k, \delta)\)-strainer map at \( x \in U \) with \( \delta \leq \frac{1}{20r} \). Let \( 3r \) be not larger than the the straining radius of \( F \) at \( x \). Then either

- \( F : B_r(x) \to \mathbb{R}^k \) is injective, or

**Proof.** \[ \square \]
• For all $y \in B_r(x)$ the fiber $\Pi := F^{-1}(F(y)) \cap B_r(y)$ is a connected set of diameter at least $r$.

**Proof.** Fix $y \in B_r(x)$ and the fiber $\Pi := F^{-1}(F(y)) \cap B_r(y)$. Due to Theorem 9.1, $\Pi$ is connected. Assume that $\Pi$ is not a singleton.

If the diameter of $\Pi$ is smaller than $r$ we find a point $z \in \Pi$ which has in $\Pi$ maximal distance $s < r$ from $y$. Consider a point $z'$, such that $z$ is on the geodesic $\eta = yz'$ with sufficiently small $l := d(z, z')$.

Let $(p_1, ..., p_k)$ be the $k$-tuple of points defining $F$. For $i = 1, ..., k$, we get from the assumption on $r$ and Lemma 7.6

$\pi/2 - 4\delta < \angle p_i y z < \pi/2$.

Another application of Lemma 7.6 shows that for any $y' \neq y$ on $\eta$

$\pi/2 - 4\delta < \angle p_i y' y < \pi/2 + 4\delta$.

From the first variation formula we obtain

$|F(z') - F(z)| \leq 4\delta \cdot \sqrt{k} \cdot l$.

Since the map $F$ is $2 \cdot \sqrt{k}$-open, we find a point $z_0$ with $F(z_0) = F(z)$ and $d(z_0, z) \leq 2 \cdot \sqrt{k} \cdot 4 \cdot \delta \cdot \sqrt{k} \cdot l < l$.

Therefore, $d(y, z_0) > d(y, z) = s$. If $l$ has been small enough, then $z_0$ is contained in $B_r(y)$ in contradiction to the choice of $z$.

Hence, for any $y \in B_r(x)$, the fiber $\Pi_y = F^{-1}(F(y)) \cap B_r(y)$ is a connected set that is either a point or has diameter at least $r$.

Since the map $F$ is open, we deduce that the set of points $y$ at which fiber $\Pi_y$ is a singleton is an open and closed subset of $B_r(x)$. Therefore, this set is either empty or the whole ball $B_r(x)$. This finishes the proof. □

As a direct consequence of this local statement, the openness of strainer maps and a standard connectedness argument we get the following global statement:

**Proposition 9.3.** For any $(k, \delta)$-strainer map $F : V \to \mathbb{R}^k$ with $\delta \leq \frac{1}{20 \cdot k}$ and connected, open $V$ the following dichotomy holds true. Either no fiber of $F$ in $V$ contains an isolated point, or all fibers of $F$ in $V$ are discrete.

10. Finiteness results

10.1. Notations. As before, let $U \subset \bar{U} \subset X$ be a tiny ball of radius $r_0 \leq 1$ and size bounded by $N$. Let $\delta > 0$ be arbitrary.

As in Subsection 5.3, we denote by $\mathcal{A}$ the distance sphere of radius $r_0$ around $U$ and by $\mathcal{A}_\delta$ a fixed maximal $\delta \cdot r_0$-separated subset of $\mathcal{A}$. Let $m = m(N, \delta)$ be an upper bound on the number of elements in $\mathcal{A}_\delta$.

Let $k$ be a natural number. Denote by $\mathcal{F}_\delta$ the set of distance maps $F : \bar{U} \to \mathbb{R}^k$, whose coordinates are distance functions to points $p_j \in \mathcal{A}_\delta$. The number of elements in $\mathcal{F}_\delta$ is bounded from above by the constant $m^{k^2}$ depending on $N, \delta$ and $k$. 
10.2. **Bounding straining sequences.** For the investigations of \((k, \delta)\)-strained points we may restrict the attention to the finitely many maps from \(F_\delta: \)

**Lemma 10.1.** Let \(F : \tilde{U} \to \mathbb{R}^k\) be a distance map which is a \((k, \delta)\)-strainer map at \(x \in U\). Then there exist maps \(F_1, F_2 \in F_\delta\) such that the pairs \((F, F_2)\) and \((F_1, F_2)\) are opposite \((k, 3 \cdot \delta)\)-strainer maps at \(x\).

**Proof.** Let \(F\) be given by the \(k\)-tuple \((p_i)\). Find an opposite \((k, \delta)\)-strainer \((q_i)\) at the point \(x\). By the definition of \(A_3\), we find \(k\)-tuples \((p'_i)\) and \((q'_i)\) in \(A_\delta\) such that

\[\angle p'_i x p_i < \delta\] and \[\angle q'_i x q_i < \delta.\]

Due to the triangle inequality and the definition of strainers, the distance maps \(F_1, F_2 \in F_\delta\) given by the \(k\)-tuples \((p'_i)\) and \((q'_i)\) have the required properties. \(\square\)

10.3. **Bounding bad sequences.** First, a simple lemma ([BGP92, Lemma 10.3]):

**Lemma 10.2.** For all \(N, L \geq 1\) and natural number \(M\) there exists \(K(N, M, L) > 0\) with the following property. Let \(Y\) be an \(N\)-doubling metric space. Then every subset \(T\) of \(Y\) with at least \(K\) elements contains an \(M\)-tuple \((x_1, ..., x_M)\) such that \(d(x_i, x_{i+1}) \geq L \cdot d(x_i, x_k)\), for all \(1 \leq k \leq i \leq M - 1\).

**Proof.** Fix \(N, L \geq 1\). We find \(C = C(N, L)\) such that any set of diameter \(D > 0\) in any \(N\)-doubling space is covered by at most \(C\) subsets with diameter at most \(\frac{D}{L}\).

We are going to prove by induction on \(M\) that \(K(N, M, L) = C^{M-1}\) satisfies the claim of the lemma. The case \(M = 1\) is clear.

Assume the claim is true for \(M - 1\) and consider a subset \(T\) of \(Y\) with at least \(C^{M-1}\) elements. Replacing \(T\) by a finite subset, we can assume that the diameter \(D > 0\) of \(T\) is finite. Cover \(T\) by at most \(C\) subsets with diameter at most \(\frac{D}{L}\). At least one of this subsets, say \(T_1\), has at least \(C^{M-2}\) elements. By the inductive assumption, we find a tuple \((x_1, ..., x_{M-1})\) in \(T_1\) as in the statement of the lemma.

Take an arbitrary point \(x_M\) in \(T\) such that \(d(x_M, x_{M-1}) \geq \frac{D}{L}\). By construction, the extended \(M\)-tuple \((x_1, ..., x_M)\) satisfies the statement of the lemma. \(\square\)

The following defines a counterpart of straining sequences:

**Definition 10.3.** A subset \(T\) of a tiny ball \(U\) is called \(\delta\)-bad if no point \(x \in T\) is a \((1, \delta)\)-strainer of another point \(y \in T\).

We derive the following uniform bound:

**Proposition 10.4.** There is a number \(C_0 = C_0(N, \delta)\) such that each \(\delta\)-bad subset of \(U\) has at most \(C_0\) elements.

**Proof.** The claim is scale invariant. Rescaling \(U\), we may assume that \(r_0 = 1\). Hence the curvature bound \(\kappa\) is at most \(\frac{1}{17}\). Moreover, we may assume \(\delta \leq \pi\).

Using comparison of quadrangles, we find some \(r_1 > 0\) depending only on \(\delta\) such that the following holds true for all \(y_1, x_1, x_2, y_2 \in \tilde{U}\). If \(d(x_1, y_1) = d(x_2, y_2) = 1\) and the angles satisfy \(\angle y_1 x_1 x_2 \geq \delta/2\) and \(\angle y_2 x_2 x_1 \geq \pi - \delta/4\) then the distance between \(y_1\) and \(y_2\) is at least \(r_1\).

We fix some number \(L_0\) depending only on \(\delta\) (and the curvature bound \(\frac{1}{17}\)), such that for all triangles \(xyz\) in \(U\), the inequality \(d(x, z) \geq L_0 \cdot d(x, y)\) implies \(\angle x y z \leq \frac{\pi}{4}\).

Assume now that the Proposition does not hold. Then there are arbitrary large \(\delta\)-bad subsets, possibly in different tiny balls \(U\) (in different GCBA spaces), but of the same bound on the size \(N\).
By Lemma 10.2 we then find \( \delta \)-bad sets \( \{x_1, \ldots, x_M\} \) with arbitrary large \( M \), such that \( d(x_i, x_{i+1}) \geq L_0 \cdot d(x_i, x_k) \) for all \( 1 \leq k \leq i \leq M - 1 \).

We fix this \( M \)-tuple \( x_1, \ldots, x_M \). Denote by \( v_{i,j} \in \Sigma_{x_i} \) the starting direction of the geodesic \( x_i x_j \). For each \( i \geq 2 \), we use that \( x_1 \) is not a \((1, \delta)\)-strainer at \( x_i \) to find antipodes \( w_i^+, w_i^- \in \Sigma_{x_i} \) of \( v_{i,1} \) such that \( d(w_i^+, w_i^-) \geq \delta \).

We proceed as follows. For each \( i \geq 3 \) the distance in \( \Sigma_{x_2} \) between \( v_{2,i} \) and either \( w_2^+ \) or \( w_2^- \) is at least \( \delta/2 \). Hence we can find a subsequence \( x_1, x_2, x_3, x_4, \ldots, x_k \) of the tuple \( (x_i) \) with at least \( M/2 \) elements such that for one of the directions \( w_2^+ \), say \( w_2^+ \), and for each \( i \geq 3 \) we have \( d(w_2^+, v_{2,i}) \geq \delta/2 \). Denote this direction \( w_2^+ \) by \( w_2 \) and replace our original tuple \( x_1, \ldots, x_M \) by this subsequence.

We repeat the procedure at \( x_3 \) and continue inductively. In this way we obtain a \( \delta \)-bad sequence \( x_1, \ldots, x_s \) with \( s \geq \log_2 M \) and, for each \( i \geq 2 \), a direction \( w_i \in \Sigma_{x_i} \), such that the following two conditions hold:

1. \( d(x_i, x_{i+1}) \geq L_0 \cdot d(x_i, x_k) \) for all \( 1 \leq k \leq i < s \);
2. The direction \( w_i \) is antipodal to \( v_{i,1} \). For all \( j > i \), we have \( d(v_{i,j}, w_i) \geq \delta/2 \).

For \( 2 \leq i \leq s \) choose a geodesic \( \gamma_i \) in \( \bar{U} \) of length \( 1 \) starting at \( x_i \) in the direction \( w_i \) and set \( y_i = \gamma_i(1) \). Thus, \( d(y_i, x_i) = 1 \).

Let \( 2 \leq i < j \leq s \) be arbitrary. By construction, \( \angle y_i x_j x_j \geq \delta/2 \). On the other hand, by the choice of \( L_0 \), we have \( \angle x_i x_j x_j \leq \delta/4 \) and therefore, \( \angle y_i x_j x_j \geq \pi - \delta/4 \). Due to the first statement in the proof, we have \( d(y_j, y_i) \geq r_1 \).

Therefore, the doubling constant of \( B_1(U) \) (and hence the size of \( U \)) bounds the number \( s \) in our sequence, providing a contradiction. \( \square \)

10.4. Extension of strainer maps. We now prove the following central result:

**Theorem 10.5.** There exists \( C_1 = C_1(N, \delta) > 0 \) with the following properties.

Let \( F : V \to \mathbb{R}^k \) be a \((k, \delta)\)-strainer map on an open subset \( V \) of size bounded by \( N \). Let \( E \) denote the set of points in \( V \) at which \( F \) cannot be extended to a \((k + 1, 12 \cdot \delta)\)-strainer map \( \bar{F} = (F, f) \) using some distance function \( f = d_{p + 1} \), as last coordinate.

Then \( E \) intersects each fiber \( \Pi \) of \( F \) in \( V \) at most \( C_1 \) points. \( E \) is a countable union of compact subsets \( E_j \), such that the restriction \( F : E_j \to \bar{F}(E_j) \) is \( 1 \)-biLipschitz. Moreover,

\[
\mathcal{H}^k(E) \leq C_1^{k+1} \cdot \mathcal{H}^k(F(E)) \leq C_1^{2k+1} \cdot 10 \cdot r_k^k.
\]

**Proof.** If \( \delta > \frac{\pi}{12} \), then \( E \) is empty, and the statement is clear. Thus, we may assume \( \delta \leq \frac{\pi}{12} \). Due to Lemma 7.3 there is a number \( k_0 = k_0(N) \) such that \( k \leq k_0 \).

Let \( F \) be defined by a \( k \)-tuple \((p_1, \ldots, p_k)\). By Lemma 10.1 there is a finite set \( \mathcal{F}_\delta \) of distance maps \( G : U \to \mathbb{R}^k \) with at most \( C = C(N, \delta) \) elements and the following property. If \( V_G \) denotes the set of points in \( V \) at which \( F \) and \( G \) are opposite \((k, 3 \cdot \delta)\)-strainer maps, then the open set \( \bigcup \{ V_G \mid G \in \mathcal{F}_\delta \} \) contains \( V \). Since \( \mathcal{F}_\delta \) has at most \( C \) elements, we may replace \( V \) by one of the sets \( V_G \) and assume that on the whole set \( V \) there exists an opposite \((k, 3 \cdot \delta)\)-strainer map \( G \) to \( F \).

Let \( \Pi \) be a fiber of the map \( F \) on \( V \). For any pair of points \( x, y \in V \cap \Pi \) we deduce from Lemma 7.4 that \( |\langle p, x - y \rangle| < 6\delta \). Therefore, if \( x \) were a \((1, 6 \cdot \delta)\)-strainer at \( y \) then the \((k + 1)\)-tuple \((p_1, \ldots, p_k, x)\) is a \((k + 1, 12 \cdot \delta)\)-strainer at \( x \), as follows from Corollary 6.3.

Hence, the subset \( E \cap \Pi \) must be \( 6\delta \)-bad. Due to Proposition 10.3 \( E \cap \Pi \) can have at most \( C_0(N, 6 \cdot \delta) \) elements. This proves the first statement of the theorem.
We claim, for any sequence \( x_l \in E \) converging to any \( x \in E \), the inequality
\[
\liminf_{l \to \infty} \frac{\|F(x) - F(x_l)\|}{d(x, x_l)} \geq \delta.
\]
Assume that (10.2) is violated. Replacing \( x_l \) by a subsequence and applying the first variation formula we deduce, for any \( i = 1, \ldots, k \) and all large \( l \), \( |\angle p_l x_l \cdot x - \overline{x}| < 2 \delta \).

Fix an opposite \((k, \delta)\)-strainer \((q_i)\) to \((p_l)\) at \( x \). Then \((q_i)\) and \((p_l)\) are opposite \((k, \delta)\)-strainers at \( x_l \), for all \( l \) large enough, Corollary \ref{cor:co-area} Applying Lemma \ref{lem:co-area} we deduce that \( |\angle p_l x_l \cdot x - \overline{x}| < 4 \delta \), for all sufficiently large \( l \) and all \( 1 \leq i \leq k \). But, due to Proposition \ref{prop:co-area} the point \( x \) is a \((1, 4 \cdot \delta)\)-strainer at \( x_l \), for all \( l \) large enough. Hence, \((p_1, \ldots, p_k, x)\) is a \((k + 1, 4 \delta)\)-strainer at \( x_l \) (Corollary \ref{cor:co-area} in contradiction to the assumption \( x_l \in E \)). This finishes the proof of (10.2).

The remaining claims are consequences of this infinitesimal property. We set \( C_1 := \max\{\frac{1}{2}, 2 \sqrt{K_0}, C_0\} \), where \( K_0 \) is a bound on \( k \) and \( C_0 \) is a bound on the number of elements of \( E \) in fibers of \( F \). The restriction of \( F \) to \( E \) is \( 2 \sqrt{K} \)-Lipschitz, as any distance map. The set \( E \) is closed in \( V \), hence locally complete. The implication that \( E \) is a union of compact subsets \( E_j \) to which \( F \) restricts as a \( C_1 \)-bilipschitz map is shown in \cite{Lyt05a}, Lemma 3.1, as a consequence of (10.2).

The set \( E \) is a union of a countable number of Lipschitz images of compact subsets of \( \mathbb{R}^k \), hence \( E \) is countably \( k \)-rectifiable, \cite{AK00}. An application of the co-area formula, \cite{AK00}, together with (10.2) proves the first inequality in (10.1). The second inequality in (10.1) follows from the fact that \( F(E) \) is contained in a Euclidean \( k \)-dimensional ball of radius \( C_1 \cdot r_0 \), and the fact that the volumes of Euclidean unit balls in any dimension are smaller than 10.

This finishes the proof. \( \square \)

10.5. **Conclusions.** Note that Theorem \ref{thm:finite} is a quantitative version of Theorem \ref{thm:finite}. Thus the proof of Theorem \ref{thm:finite} is finished as well.

In order to derive Theorem \ref{thm:finite} we prove the following localized more precise version of it. Let again \( U \) be a tiny ball of radius \( r_0 \) and size bounded by \( N \) as above. As before, \( U_{k, \delta} \) denotes the set of all \((k, \delta)\)-strained points.

**Proposition 10.6.** There exists a number \( C_2 = C_2(N, \delta) > 0 \) with the following properties. The set \( U \setminus U_{k, \delta} \) is a union of countably many images of biLipschitz maps \( G_j : A_j \to U \), with \( A_j \) compact in \( \mathbb{R}^{k-1} \). Moreover, \( \mathcal{H}^{k-1}(U \setminus U_{k, \delta}) < C_2 \cdot r_0^{k-1} \).

**Proof.** If \( \delta \) decreases, the sets \( U_{k, \delta} \) increase, thus in all subsequent considerations we may assume that \( \delta \) is sufficiently small.

We proceed by induction on \( k \). The set \( U \setminus U_{1, \delta} \) has at most \( C_0(\delta, N) \) elements, due to Proposition \ref{prop:finite} This proves the statement for \( k = 1 \).

Assuming the result is true for \( k \), we are going to prove it for \( k + 1 \). By the inductive assumption, the set \( U \setminus U_{k, \delta/50} \) is a countable union of images of biLipschitz maps defined on compact subsets of \( \mathbb{R}^{k-1} \).

Thus it suffices to represent \( K := U_{k, \delta/50} \setminus U_{k+1, \delta} \) as a union of biLipschitz images and to estimate its \( k \)-dimensional Hausdorff measure.

Any point \( x \in K \) admits a \((k, \delta/12)\)-strainer map \( F \in \mathcal{F}_{\delta/12} \), due to Lemma \ref{lem:co-area} Thus, we have a finite number of \((k, \delta/12)\)-strainer maps \( F_j : V_j \to \mathbb{R}^k \) defined on open subsets \( V_j \subset U \) such that the union of \( V_j \) covers \( K \) and such that the number of \( V_j \) is bounded by some \( C_3(N, \delta) \).
Applying now Theorem [10.5] to the maps $F_j : V_j \to \mathbb{R}^k$ and observing that $K_j := K \cap V_j$ is contained in the set $E$ from the formulation of Theorem [10.5] we deduce the following.

Each $K_j$ is a countable union of biLipschitz images of compact subsets of $\mathbb{R}^k$ and $\mathcal{H}^k(K_j)$ is bounded by $C_4 \cdot r_0^k$ for some $C_4 = C_4(N, \delta)$. Summing up, we deduce the required bound on the volume $\mathcal{H}^k(K)$ and the fact that $K$ is a union of countably many images of biLipschitz maps defined on compact subsets of $\mathbb{R}^k$.

Now we obtain:

**Proof of Theorem 1.6.** We cover $X$ by a countable number of tiny balls $U$, using the separability of $X$. The set $U \setminus X_{k,0}$ of not $(k, 0)$-strained points in $U$ is the union of the complements $U \setminus U_{k,\delta}$ where $\delta$ runs over all sufficiently small rational numbers. Applying Proposition [10.6] we deduce that $X \setminus X_{k,0}$ is a countable union of compact subsets biLipschitz equivalent to subsets of $\mathbb{R}^{k-1}$.

**Remark 10.7.** Theorem 1.6 and Theorem 1.3 strengthen the separability of $X$, any tiny ball $U$ of radius $\delta$ by Corollary 5.4 and set $\delta_0 = \delta_0(N) := \frac{1}{50 \cdot n_0^k}$. We can now relate the dichotomy observed in Proposition 9.3 to the dimension.

11. Dimension

11.1. **Topological and Hausdorff dimension.** We can now prove a quantitative version of the first part of Theorem 1.1.

**Proposition 11.1.** There is some $C(N) > 0$ such that the following holds true for any tiny ball $U$ of radius $r_0$ and size bounded by $N$.

If $n$ is the topological dimension of $U$ then $0 < \mathcal{H}^n(U) < C \cdot r_0^n$. In particular, the Hausdorff dimension of $U$ equals $n$. Moreover, $n$ is the largest number such that some tangent cone $T_x U$ is isometric to $\mathbb{R}^n$. Finally, $n$ is the largest number, such that there are $(n, \frac{1}{n})$-strained points in $U$.

**Proof.** We already know that the topological dimension $n$ of $U$ is finite. Then $\mathcal{H}^n(U) > 0$, by general results in dimension theory, compare [Edg08].

By [Kle99] the geometric dimension of $U$ is $\dim(U) = n$ as well. Therefore, there are no points in $U$ at which the tangent space $T_x U$ contains an $(n + 1)$-dimensional Euclidean space. Hence, $U$ is contained in $X \setminus X_{n+1,0}$.

Due to Theorem 1.6, $U$ is a countable union of biLipschitz images of subsets of $\mathbb{R}^n$. Therefore, the Hausdorff dimension of $U$ is at most $n$.

Due to Lemma [5.2] there are no $(n + 1, \frac{1}{(n + 1)})$-strained maps defined on subsets of $U$. Thus, there are no $(n + 1, \frac{1}{(n + 1)})$-strained points in $U$. Due to Proposition 10.6 $\mathcal{H}^n(U) < C \cdot r_0^n$, for some $C$ depending only on $N$.

Applying Theorem 1.6 again, we find a point $x \in X$ such that the tangent space $T_x$ has $\mathbb{R}^n$ as a direct factor. If $T_x$ is not equal to $\mathbb{R}^n$ then it contains $\mathbb{R}^n \times [0, \infty)$. But this is impossible, since the geometric dimension of $X$ is $n$. Therefore, $T_x = \mathbb{R}^n$.

This finishes the proof.

From now on, we fix some bound $n_0 = n_0(N)$ on the dimension on $U$ provided by Corollary 5.4 and set $\delta_0 = \delta_0(N) := \frac{1}{50 \cdot n_0^k}$. We can now relate the dichotomy observed in Proposition 9.3 to the dimension.
Corollary 11.2. Let $F : V \to \mathbb{R}^k$ be a $(k, \delta)$-strainer map on a connected open subset $V$ of a tiny ball $U$. If $\delta \leq \delta_0(N)$ then one of the following possibilities occurs:

1. No fiber of $F$ in $V$ has isolated points. Then $\dim(W) > k$, for every open subset $W \subset V$.

2. $V$ is a $k$-dimensional topological manifold. Then for every $x \in V$ and every $r$, such that $3r$ is smaller than the straining radius of $F$ at $x$, the map $F : B_r(x) \to F(B_r(x))$ is $L$-biLipschitz, where $L$ goes to 1 as $\delta$ goes to 0.

Proof. By Proposition 9.3 either no fiber of $F$ has isolated points or the map $F$ is locally injective.

In the second case, for any $x \in V$ and $r > 0$ as in the statement above, we deduce from Lemma 9.2 and Lemma 8.2 that $F : B_r(x) \to F(B_r(x))$ is $L$-biLipschitz with $L = 2\sqrt{k}$. Due to Corollary 8.3, we can choose $L$ close to 1 if $\delta$ goes to 0. Since $F(V)$ is open in $\mathbb{R}^k$, we see that $V$ is a $k$-dimensional manifold.

In the first case, any $x \in V$ is a non-isolated point in the fiber $F^{-1}(F(x))$. We apply Theorem 11.5 and find $(k + 1, 12 \cdot \delta)$-strainer points arbitrary close to $x$. Then, by Proposition 8.2, the dimension of any ball around $x$ is at least $k + 1$. □

Now we can finish

Proof of Theorem 11.1. Given any GCBA space $X$, we cover $X$ by a countable number of tiny balls $U$ and reduce all statements to the case of tiny balls. For any tiny ball $U$, the topological dimension $n$ equals the Hausdorff dimension by Proposition 11.1. Moreover, by Proposition 11.1 there exists an $(n, \delta)$-strainer map $F : V \to \mathbb{R}^n$ for arbitrary small $\delta$ and some $V \subset U$. Applying Corollary 11.2, we see that $V$ is a topological manifold. Hence, $n$ equals the maximal dimension of a Euclidean ball which embeds into $U$ as an open set. □

11.2. Lower bound on the measure. The Euclidean spheres are the smallest GCBA spaces with the same dimension and curvature bound:

Proposition 11.3. Let $\Sigma$ be a compact GCBA space, which is CAT(1) and of dimension $n$. Then there exists a 1-Lipschitz surjection $P : \Sigma \to S^n$.

Proof. By Proposition 11.1, we find a point $x \in \Sigma$ with $T_x$ isometric to $\mathbb{R}^n$. Then one can define a surjective 1-Lipschitz map $P : \Sigma \to S^0 * \Sigma_x = S^n$ as the "spherical logarithmic map", i.e., the composition of the logarithmic map in $\Sigma$ and the exponential map in $S^n$, see [Lyt05b, Lemma 2.2]. □

Remark 11.4. This observation is related to the volume minimality of constant curvature spaces proved in [Nag02, Sections 6, 7] along with rigidity statements.

11.3. Dimension and convergence. We are going to describe the behavior of dimension under convergence.

Lemma 11.5. Let $U_l$ converge to $\bar{U}$ as in the standard setting for convergence. Let $x \in U$ be a limit point of $x_l \in U_l$. If $\dim(T_x) = n$ then there exists some $\epsilon > 0$ and $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ the ball $B_l(x_l)$ has dimension $n$.

In particular, $\dim(T_{x_l}) \leq n$, for all $l$ large enough.

Proof. First, assume $\dim(B_l(x_l)) < n$, for some $\epsilon > 0$ and infinitely many $l$. Due to the semicontinuity of the geometric dimension under convergence (cf. [Lyt05b, Lemma 11.1]), we conclude $\dim(B_r(x)) < n$, for any $r < \epsilon$. But then $\dim(T_x) < n$, by the definition of geometric dimension, in contradiction to our assumption.
Assuming that the statement of the lemma is wrong, we can therefore choose a subsequence and assume that \( \dim(B_{\epsilon}(x)) = m + 1 > n \), for some fixed \( m \) (since the dimensions in question are bounded, Proposition \[5.10\]). Since the dimension equals the geometric dimension, we find some \( y \in B_{\epsilon}(x) \) with \( \dim(\Sigma_{y_i}) = m \).

Due to Proposition \[11.3\] any \( \Sigma_{y_i} \) and then also any limit space \( \Sigma' \) of this sequence, admits a surjective 1-Lipschitz map onto \( S^m \). Therefore, the Hausdorff dimension of \( \Sigma' \) is at least \( m \). Due to Lemma \[5.13\] \( \Sigma_x \) admits a surjective 1-Lipschitz map onto \( \Sigma' \), since \( y \) converge to \( x \). Hence the Hausdorff dimension of \( \Sigma_x \) is at least \( m \) as well. But this contradicts \( \dim(T_x) = n \leq m \).

This contradiction finishes the proof. \( \square \)

Let \( X \) again be GCBA. As in the introduction we consider the \( k \)-dimensional part \( X^k \) of \( X \) as the set of all points \( x \in X \) with \( \dim(T_x) = k \). Applying Lemma \[11.5\] to the constant sequence \( X_l = X \) we directly see:

**Corollary 11.6.** A point \( x \in X \) is contained in \( X^k \) if and only if there is some \( \epsilon > 0 \), such that for all \( r < \epsilon \) we have \( \dim(B_r(x)) = k \). The closure of \( X^k \) in \( X \) does not contain points from \( X^m \) with \( m < k \).

In the strained case we get more stability:

**Lemma 11.7.** In the notations of Lemma \[11.3\] above, assume that the point \( x \in U \) is \((k, \delta)\)-strained. Then, for all sufficiently large \( l \), we have \( \dim(T_{x_l}) \geq k \).

If, in addition, \( \dim(T_{x_l}) = k \) for all \( l \) large enough, then \( n = k \). Hence \( \dim(T_{x_l}) = \dim(T_x) = \dim(T_z) \), for all \( l \) large enough.

**Proof.** We find some \((k, \delta)\)-strainer map \( F \) in a neighborhood of \( x \), defined by a \( k \)-tuple \( (p_i) \). We approximate this tuple by \( k \)-tuples in \( U \) l and obtain distance maps \( F_l : U \to \mathbb{R}^k \) converging to \( F \). Moreover, for all \( l \) large enough, \( F_l \) is a \((k, \delta)\)-strainer map at \( x_l \) with a uniform lower bound \( 3r \) on the straining radii of \( F_l \) at \( x_l \). Lemma \[7.8\] and Lemma \[7.11\]. Due to Lemma \[8.2\] the dimension of any ball around \( x_l \) must be at least \( k \), hence \( \dim(T_{x_l}) \geq k \).

Assume \( \dim(T_{x_l}) = k \), for all \( l \) large enough. Due to Corollary \[11.2\] the restriction of the strainer maps \( F_l \) to the ball \( B_r(x_l) \) is \( L \)-biLipschitz. Therefore, so is the restriction of \( F \) to \( B_r(x) \). Applying Corollary \[11.2\] again, we see that \( B_r(x) \) is a \( k \)-dimensional manifold, hence \( n = k \). \( \square \)

**11.4. Regular parts.** We fix now some \( \delta \leq \delta_0 \). By the \( k \)-regular part of \( U \) we denote the set of \((k, \delta)\)-strained points \( x \in U \) with \( \dim(T_x) = k \).

**Corollary 11.8.** Let \( U \) be a tiny ball of radius \( r_0 \) and size bounded by \( N \). Let \( k \) be a natural number. The set \( \text{Reg}_k(U) \) of \( k \)-regular points is open in \( U \), dense in \( U^k \) and locally biLipschitz homeomorphic to \( \mathbb{R}^k \). The topological boundary \( \partial \text{Reg}_k(U) := U \cap (U^k \setminus \text{Reg}_k(U)) \) of \( \text{Reg}_k(U) \) in \( U \) does not contain \((k, \delta)\)-strained points. Moreover,

\[
\mathcal{H}^{k-1}(U^k \setminus \text{Reg}_k(U)) < C \cdot r_0^{k-1} \text{ and } \mathcal{H}^k(U^k) < C \cdot r_0^k,
\]

for some constant \( C \) depending only on \( N \) and the choice of \( \delta \).

**Proof.** Any point \( x \) in \( \text{Reg}_k(U) \) admits a \((k, \delta)\)-strainer map \( F \). Due to Corollary \[11.2\] the restriction of \( F \) to a small ball around \( x \) is biLipschitz onto an open subset of \( \mathbb{R}^k \). Hence, this ball is contained in \( U^k \) and consists of \((k, \delta)\)-strained points. Therefore, \( \text{Reg}_k(U) \) is open in \( X \) and locally biLipschitz to \( \mathbb{R}^k \).
Let $x \in U^k$ be arbitrary. By Corollary 11.6 any sufficiently small ball $W$ around $x$ has dimension $k$. Hence, $W$ contains $(k, \delta)$-strained points, therefore points from $\text{Reg}_k(U)$. Thus, $\text{Reg}_k(U)$ is dense in $U^k$.

Assume that $x \in \hat{U}^k$ is $(k, \delta)$-strained. Writing $x$ as a limit of points $x_l \in \text{Reg}_k(U)$ and applying Lemma 11.7 we see $\dim(T_x) = k$. Hence $x \in \text{Reg}_k(U)$.

No point in $U^k$ is $(k + 1, \delta)$-strained, due to Lemma 8.2. Thus the bounds on measures are contained in Theorem 10.5. \hfill $\square$

11.5. Conclusions. We finish the proofs of two theorems from the introduction.

Proof of Theorem 1.2. Thus, let $X$ be GCBA and $k$ a natural number. As we have seen in Corollary 11.6 a point $x \in X$ is in the $k$-dimensional part $X^k$ if and only if all sufficiently small balls around $x$ have dimension $k$.

Cover $X$ by a countable collection of tiny balls. For each of these tiny balls $U$ consider its $k$-regular part and let $M^k \subset X^k$ denote the union of these $k$-regular parts. Due to Corollary 11.8 this subset $M^k$ is open in $X$, dense in $X^k$ and locally biLipschitz to $\mathbb{R}^k$. Moreover, $X^k \setminus M^k$ is a countable union of subsets of finite $(k - 1)$-dimensional Hausdorff measure.

Every nonempty $V \subset X^k$ which is open in $X^k$, contains an open non-empty subset of $M^k$ hence $\mathcal{H}^k(V) > 0$. From Corollary 11.8 we deduce that the measure $\mathcal{H}^k(X^k \cap U)$ is finite for every tiny ball $U$.

This finishes the proof. \hfill $\square$

Remark 11.9. In [OT99, Main Theorem 1(1)] one finds the statement that $\mathcal{H}^k$ is locally positive and non-zero on $X^k$. From [OT99, Section 4] one can conclude that the set $X^k$ is a Lipschitz manifold up to a subset of $\mathcal{H}^k$-measure 0.

Recall from the introduction that the canonical measure $\mu_X$ on $X$ is the sum over all $k = 0, 1, \ldots$ of the restrictions of $\mathcal{H}^k$ to $X^k$.

Proof of Theorem 1.4. Let $X$ again be GCBA. If $x \in X$ satisfies $\dim(T_x) = k$, thus $x \in X^k$, then the measure $\mathcal{H}^k \cap X^k$ is positive on any neighborhood $V$ of $x$, due to Theorem 1.2. Hence $\mu_X(V) > 0$.

On the other hand, the dimension of any tiny ball $U$ in $X$ is finite, hence only finitely many of the measures $\mathcal{H}^k \cap X^k$ can be non-zero on $U$. Due to Corollary 11.8 the measure $\mathcal{H}^k(X^k \cap U)$ is finite, hence so is $\mu_X(U)$.

Therefore, the measure $\mu_X$ is finite on any relatively compact subset of $X$. \hfill $\square$

12. Stability of the canonical measure

12.1. Setting and preparations. We are going to prove here Theorem 1.5 and its local generalization. First we recall the notion of measured Gromov–Hausdorff convergence, sufficient for our purposes, compare [HKST15] for details.

Let $Z_l$ be a sequence of compact spaces GH-converging to a compact set $Z$. Let $\mathcal{M}_l$ be a Radon measure on $Z_l$ and let $\mathcal{M}$ be a Radon measure on $Z$. The measures $\mathcal{M}_l$ converge to $\mathcal{M}$ if for any compact sets $K_l \subset Z_l$ converging to $K \subset Z$ the following holds true:

$$\lim_{\epsilon \to 0} \liminf_{l \to \infty} \mathcal{M}_l(B_{\epsilon}(K_l)) = \lim_{\epsilon \to 0} \limsup_{l \to \infty} \mathcal{M}_l(B_{\epsilon}(K_l)) = \mathcal{M}(K).$$

By general results, any sequence of Radon measures $\mathcal{M}_l$ on $Z_l$ contains a converging subsequence if the total measures $\mathcal{M}_l(Z_l)$ are uniformly bounded.
We continue working in the standard setting for convergence as in Definition 5.12. We fix some \( k = 0, 1, \ldots \) and restrict our attention to the \( k \)-dimensional part \( \mu_U^k = \mathcal{H}^k \cdot U^k \) of \( \mu \). The aim of this section is the following:

**Theorem 12.1.** Under the GH-convergence \( \hat{U}_1 \to \hat{U} \) the \( k \)-dimensional parts of the canonical measures \( M_1 := \mu_{U_1}^k \) converge to \( M := \mu_U^k \) locally on \( U \). Thus, \( \|M_U^k - M_{U_1}^k\|_{\text{loc}} \) holds for all compact \( K \subset U \).

We know that \( M_t(\hat{U}_t) \) is uniformly bounded by a constant \( C \), Corollary 11.8. Thus, by general compactness of measures, we may choose a subsequence and assume that the measures \( M_t \) converge to a finite Radon measure \( \mathcal{N} \) on \( \hat{U} \). We need to verify that \( \mathcal{M} = \mathcal{N} \) on \( U \).

It suffices to prove that \( \mathcal{N} \) coincides with \( \mathcal{H}^k \) on the regular part \( \text{Reg}_k(U) \) and that \( \mathcal{N} \) vanishes on the complement \( U \setminus \text{Reg}_k(U) \). We fix \( \delta \) as in Subsection 11.4.

12.2. **Regular part.** In order to prove that \( \mathcal{N} \) and \( \mathcal{H}^k \) coincide on the regular part \( \text{Reg}_k(U) \), we note that \( \mathcal{N} \) satisfies \( \mathcal{N}(B_r(x)) \leq C \cdot r^k \), whenever \( B_r(x) \subset U \). Indeed, this inequality is true for all the approximating measures, by Corollary 11.8. Thus, \( \mathcal{N} \) is absolutely continuous with respect to \( \mathcal{H}^k \) on the Lipschitz manifold \( \text{Reg}_k(U) \). By the Lebesgue-Radon-Nikodym differentiation theorem (see [HKST15]), it suffices to prove that for \( \mathcal{H}^k \)-almost every point \( x \in \text{Reg}_k(U) \) the density

\[
b(x) := \lim_{r \to 0} \frac{\mathcal{N}(B_r(x))}{\mathcal{H}^k(B_r(x))},
\]

exists and is equal to 1.

Due to Theorem 1.6, \( \mathcal{H}^k \)-almost every point in \( \text{Reg}_k(U) \) has as tangent space \( T_x = \mathbb{R}^k \). Let \( x \) be such a point and let \( x_l \) be a sequence of points in \( U_1 \) converging to \( x \). We take points \( p_1, \ldots, p_k \in \hat{U} \) such that the directions \( (xp_l)_l \) are pairwise orthogonal in \( T_x = \mathbb{R}^k \). Then the distance map \( F : U \to \mathbb{R}^k \) defined by the \( k \)-tuple \( (p_l) \) is a \((k, \delta)\)-strainer map at \( x \), for any \( \delta > 0 \). Consider a sequence of distance maps \( F_l : U_l \to \mathbb{R}^k \) converging to \( F \).

For any \( \delta > 0 \), we find some \( r > 0 \) and some \( l_0 > 0 \) such that \( F \) and \( F_l \), for \( l \geq l_0 \), are \((k, \delta)\)-strainer maps with straining radius at least \( 3r \) at \( x \) and \( x_l \), respectively. Then the maps \( F : B_r(x) \to \mathbb{R}^k \) and \( F_l : B_{r_l}(x_l) \to \mathbb{R}^k \) are \( L \)-biLipschitz onto their images and \( L \) goes to 1 as \( \delta \) goes to 0, due to Lemma 11.7 and Corollary 11.2. Moreover, by Corollary 5.3 the images contain balls with radius \( r/2 \) around \( F(x) \) and \( F_l(x_l) \), respectively.

Thus, for any \( s < \frac{r}{10} \) and all sufficiently large \( l \), the volumes \( \mathcal{H}^k(B_s(x)) \), \( \mathcal{H}^k(B_s(x_l)) \) are bounded between \( L^{-2k} \cdot \omega_k \cdot s^k \) and \( L^{2k} \cdot \omega_k \cdot s^k \), where \( \omega_k \) denotes the volume of the \( k \)-dimensional Euclidean unit ball.

Since \( L \) goes to 1, as \( \delta \) goes to 0, we conclude \( b(x) = 1 \).

12.3. **Singular part.** The support \( S \) of \( \mathcal{N} \) in \( U \) is contained in the limit set of the supports of \( \mu_{U_1}^k \). Thus, \( S \) is contained in the set of all points \( x \in U \), which are limits of a sequence of \( k \)-regular points \( x_l \in U_1 \). Due to Lemma 11.7 any such point \( x \) which is not in \( \text{Reg}_k(U) \) cannot be \((k, \delta)\)-strained.

Therefore, \( T := S \setminus \text{Reg}_k(U) \) is a closed subset of \( U \setminus \hat{U}_{k, \delta} \) of points which are not \((k, \delta)\)-strained. Note that \( \mathcal{H}^k(T) = 0 \), by Theorem 1.6. It is enough to prove that \( \mathcal{N}(K) = 0 \), for any compact subset \( K \) of \( T \).
Fix a compact subset $K$ in $T$ and a sequence of compact $K_l \subset U_l$ converging to $K$. Let finally $t > 0$ be arbitrary. It suffices to find some $s = s(t) > 0$ such that

$$\mu^{k_l}_{U_l}(B_s(K_l)) = \mathcal{H}^k(B_s(K_l) \cap \text{Reg}_{k}(U_l)) < t$$

for all sufficiently large $l$.

As in Subsection 5.3, denote by $\mathcal{A}_\delta \subset \tilde{U}$ some maximal $\delta \cdot r_0$-separated subset in the distance sphere $A$ of radius $r_0$ around $U$. Numerate the elements of $\mathcal{A}_\delta$ as $\mathcal{A}_\delta = \{p_1, ..., p_m\}$ and approximate any $p_i$ by points $p_i^l$ in the distance sphere of radius $r_0$ around $U_l$ in $\tilde{U}_l$.

For all $l$ large enough, the points $\{p_1^l, ..., p_m^l\}$ are $\delta \cdot r_0$-dense in the distance sphere of radius $r_0$ around $U_l$. Denote by $\mathcal{F}_\delta$ the set of distance maps $F : \tilde{U} \to \mathbb{R}^k$ defined by $k$-tuples in $\mathcal{A}_\delta$. Denote by $\tilde{\mathcal{F}}_\delta^l$ the corresponding lifts to distance maps $F_l : \tilde{U}_l \to \mathbb{R}^k$. We numerate the elements of $F_\delta$ and $\tilde{\mathcal{F}}_\delta^l$ as $G_1, ..., G_j, ...$ and $G^{l}_1, ..., G^{l}_j, ...$, respectively. These are finite sets (with $m^k$ elements). For any $j$, the distance maps $G^{l}_j$ converge to $G_j$.

The argument in Lemma 10.1 shows, that for all $l$ large enough the following holds true: If a point $x_l$ in $U_l$ is $(k, \delta)$-strained then there exists some $G^{l}_j$ which is a $(k, 3 \cdot \delta)$-strained map at $x_l$.

From Theorem 10.5 and the finiteness of the elements in $\mathcal{F}_\delta$, we get a number $C > 0$ such that for any measurable subset $Y \subset \text{Reg}_{k}(U_l)$ we have

$$\mathcal{H}^k(Y) \leq C \cdot \max_{j} \mathcal{H}^k(G^{l}_j(Y)).$$

Since all the maps $G^{l}_j$ are $2\sqrt{k}$-Lipschitz, the image $G^{l}_j(B_s(K_l))$ is contained in the $2\sqrt{k} \cdot s$-tubular neighborhood around $G^{l}_j(K_l)$. Thus, for all $l$ large enough, $G^{l}_j(B_s(K_l))$ is contained in the $3\sqrt{k} \cdot s$-tubular neighborhood around $G_j(K)$. But $\mathcal{H}^k(K) = 0$, hence $\mathcal{H}^k(G_j(K))$ is 0, for all $j$. Thus, for all sufficiently small $s_0$, the $3 \cdot \sqrt{k} \cdot s_0$-tubular neighborhood around the compact set $G_j(K)$ has $\mathcal{H}^k$-measure less than $t$.

By the previous considerations, for such $s_0$ we have $\mu^{k_l}_{U_l}(B_{s_0}(K_l)) \leq C \cdot t$. Since $t$ was arbitrary, this proves the claim.

12.4. Conclusions. We can now finish the

Proofs of Theorem 12.1 and Theorem 1.5. The proof of Theorem 12.1 follows from the combination of the two Subsections above.

In order to prove Theorem 15.5, assume that $X_i$ are compact GCBA spaces with uniform bounds on dimension, curvature and injectivity radius. If $X_i$ converge in the GH-topology to a space $X$ then the spaces $X_i$ are covered by a uniform number of uniformly bounded tiny balls. Proposition 5.10. Thus the total measures $\mu_{X_i}(X_i)$ are uniformly bounded by Corollary 11.8. Hence, upon choosing a subsequence, we may assume that $\mu_{X_i}$ converges to a measure $\mathcal{M}$ on the limit space $X$. Applying the local statement of Theorem 12.1 we see that $\mathcal{M}$ coincides with $\mu_X$.

Therefore, it remains to show that a uniform upper bound on the total measures $\mu_{X_i}(X_i)$ implies precompactness of the sequence $X_i$ in the GH-topology.

Assume the contrary. Then, applying Proposition 5.10 we find tiny balls $U_l$ in $X_i$ of the same radius $r_0$ such that $\mu_{U_l}(U_l)$ converges to 0.

Due to the uniform upper bound on the dimension, the 2-Lipschitz property of the logarithmic maps and Proposition 11.3, we find for any $s > 0$ some $\epsilon > 0$ such
that the following holds true, for any \( l \) and any \( x_l \in U_l \). If \( T_{z_l} \) is \( k \)-dimensional and if the ball \( B_r(x_l) \) is contained in \( U_l \) then \( H^k(B_r(x_l)) \geq \epsilon \).

Set \( s = \frac{r_0}{2^n} \), where \( n \) is an upper bound on the dimensions of \( X_l \). With \( \epsilon \) as above, our assumption implies \( \mu_{U_l}(U_l) < \epsilon \), for all \( l \) large enough. By the above estimate, for any \( k \) and any point \( x \in U_l^k \) in the \( k \)-dimensional part of \( U_l \), the following holds true. If \( B_s(x) \subset U_l^k \) then \( B_s(x) \) contains points from \( U_l^{k'} \) with \( k' > k \).

Starting in the center of \( U_l \), we now construct by induction points \( z_1, ..., z_{n+1} \) in \( U_l \), such that \( d(z_{i+1}, z_i) < s \) and \( \dim(T_{z_i}) < \dim(T_{z_{i+1}}) \), for all \( 1 \leq i \leq n \).

Thus, \( \dim(T_{z_{n+1}}) > n \), in contradiction to our assumption.

\[ \square \]

**Remark 12.2.** Theorem 12.2 is a generalization of [Nag02, Theorem 1.1], where Theorem 12.1 is proved for the maximal \( k = \dim(X_l) = \dim(X) \).

### 12.5. Additional comments on the measure-theoretic structure of GCBA spaces

Let \( X \) be a GCBA space and \( k \) a natural number.

For any point \( x \in X \) we consider a tiny ball \( U \) around \( x \) and apply Theorem 12.1 to the convergence of rescaled spaces \((\frac{1}{d(U,x)} T_x, x) \to T_x \). We deduce that \( \mu^k \) has a well-defined \( k \)-dimensional density at the point \( x \), compare [Nag02, Theorem 1.4],

\[
\lim_{r \to 0} \frac{\mu^k_r(B_r(x))}{r^k} = \mu^k_{T_x}(B_1(0)).
\]

Let now \( x \) be a point in \( X^k \) and let \( \epsilon > 0 \) be as in Corollary 11.6. For every \( z \in B_r(x) \cap X^k \) and every \( r < \epsilon - d(x, z) \), the measure \( \mu^k(B_r(z)) \) is bounded from below by \( C(k) \cdot r^k \), due to the Lipschitz property of the logarithmic map and Proposition 11.3. Here \( C(k) \) is a positive constant depending only on \( k \). Together with Corollary 11.8, we see that the restriction of \( \mu^k \) to \( X^k \) is locally Ahlfors \( k \)-regular, see [Nag02, Section 6] for related statements.

Finally, for any open relatively compact subset \( U \) in any GCBA space \( X \) the Hausdorff dimension and the rough dimension coincide, see [AB07] for the definition of the rough dimension (also known as the Assouad dimension) and a discussion of this question. Indeed, a thorough look into the proof of Proposition 10.4 and Theorem 11.9 reveals the following claim, for any tiny ball \( U \): For fixed \( \delta > 0 \) and for \( \epsilon \to 0 \), any \( \epsilon \)-separated subset of \( U \setminus U_{k, \delta} \) has at most \( O(\epsilon^{1-k}) \) elements. Hence, the rough dimension of \( U \setminus U_{k, \delta} \) is at most \( k - 1 \).

### 13. Homotopic stability

#### 13.1. Homotopic stability of fibers

Let \( U_l \subset \bar{U_l} \) and \( U \subset \bar{U} \) be as in our standard setting for convergence, Definition 5.12. We have the following general stability result:

**Theorem 13.1.** Under the standard setting for convergence, let the distance maps \( F_l : \bar{U_l} \to \mathbb{R}^k \) converge to the distance map \( F : \bar{U} \to \mathbb{R}^k \). Assume that the restriction of \( F \) to an open set \( V \subset U \) is a \((k, \delta)\)-strainer map. Let \( t_l \to t \) be a converging sequence in \( \mathbb{R}^k \) and assume that the fiber \( \Pi_l := F^{-1}(t) \subset V \) is compact. Let, finally, \( K_l \subset U_l \) be compact sets converging to \( \Pi \).

Then there exists \( r > 0 \) such that the following holds true, for all \( l \) large enough. The restriction of \( F_l \) to \( V_l = B_r(K_l) \) is a \((k, \delta)\)-strainer map, the fibers \( \Pi_l := F_l^{-1}(t_l) \subset V_l \) are compact and converge to \( \Pi \).

Finally, \( \Pi_l \) is homotopy equivalent to \( \Pi \), for all \( l \) large enough.
Proof. By the compactness of $\Pi$, we find some $r > 0$, such that for any $x \in \Pi$ the straining radius of $x$ with respect to $F$ is larger than $2r$. Due to Lemma 7.8, $F_l$ is a $(k, \delta)$-strainer in $V_l$, for all $l$ large enough. Moreover, for all $l$ large enough and any $x_l \in V_l$, the straining radius of $F_l$ at $x_l$ is at least $r$, Lemma 7.11. The maps $F_l$ are $2\sqrt{k}$-open on $V_l$. This implies that $\Pi_l$ converges to $\Pi$. For all $l$ large enough, all balls in $\Pi_l$ of all radii $s < r$ are contractible, due to Theorem 9.1. The homotopy equivalence of $\Pi_l$ and $\Pi$ is now a direct consequence of the general homotopy stability theorem [Pet90, Theorem A].

We discuss two special cases. The first one is an immediate application of the theorem in the case of a constant sequence $X_l = X$.

Corollary 13.2. Let $F : V \to \mathbb{R}^k$ be a $(k, \delta)$-strainer map defined on an open subset $V$ of the tiny ball $U$. Assume that a fiber $\Pi$ of $F$ is compact. Then all fibers of $F$, sufficiently close to $\Pi$, are homotopy equivalent to $\Pi$.

As a second application we obtain:

Proof of Theorem 1.12. Indeed, for any point $x$ in a GCBA space $X$, we find a tiny ball $U \subset \tilde{U}$ containing $x$. Consider an arbitrary sequence of positive numbers $t_l$ converging 0 and corresponding metric spheres $\partial B_{t_l}(x)$.

Consider the convergence $(\frac{1}{t_l} \tilde{U}, x) \to (T_x, 0)$, provided by Corollary 5.7. In the tangent cone $T_x$ the origin is a $(1, \delta)$-strainer at any point of $T_x \setminus \{0\}$, for any $\delta > 0$. Moreover, the fiber $F^{-1}(1)$ of the distance function $F$ to the vertex of the cone is exactly $\Sigma_x \subset T_x$. The sphere $\partial B_{t_l}(x)$ is the fiber $d^{-1}_x(1)$ of the distance functions $d_x$ on $\frac{1}{t_l} \cdot \tilde{U}$. Thus, by Theorem 7.1, $\partial B_{t_l}(x)$ and $\Sigma_x$ are homotopy equivalent, for all $l$ large enough. Since the sequence $(t_l)$ was arbitrary, this finishes the proof.

Using in addition Lemma 5.10, one could observe, that a homotopy equivalence in Theorem 1.12 is provided by the logarithmic map.

13.2. Homotopy types of spaces of directions. We are going to discuss a homotopy property of the spaces of directions. Due to Theorem 1.12 this also determines the homotopy types of small distance spheres.

In CAT(1) spaces all balls of radius less than $\pi$ are contractible. Applying the stability theorem [Pet90, Theorem A] and Proposition 5.10, we see that on any compact set $C$ of isometry classes of compact, geodesically complete CAT(1) spaces the following holds true. For any $\Sigma \in C$ the set of spaces in $C$ with the homotopy type of $\Sigma$ is open and closed in $C$. From Lemma 6.6 we immediately derive:

Corollary 13.3. For every $N > 0$ there exists some $\epsilon(N) > 0$ such that for any tiny ball $U$ of size bounded by $N$, for any natural $k$, any point $x \in U_{k, \epsilon}$, the space of directions $\Sigma_x$ is homotopy equivalent to a $k$-fold suspension.

In particular, only for finitely many points $x \in U$ the space of directions $\Sigma_x$ is not homotopy equivalent to a suspension of some other space.

14. The differentiable structure

This section is essentially a rewording of [Per94]. The only result without a direct analogue in [Per94] is Proposition 14.9.
14.1. Setting. We are going to prove Theorem 1.3. The statement is local, so we may restrict ourselves to a tiny ball $U$ and assume that $U$ coincides with its set of $k$-regular points. Hence we may assume that $U$ is a $k$-dimensional manifold and that every point $x \in U$ is $(k, \delta)$-strained, for a sufficiently small $\delta = \delta(k) \leq \frac{1}{50k^2}$.

14.2. Euclidean points. For any point $x \in U$, a neighborhood of $x$ in $U$ is biLipschitz to a $k$-dimensional Euclidean ball. Therefore, $T_xU$ is biLipschitz to $\mathbb{R}^k$. If $T_xU$ is a direct product $T_xU = \mathbb{R}^{k-1} \times Y$ for some space $Y$ then $Y$ must be $\mathbb{R}$. (Indeed, $Y$ must be a 1-dimensional cone over a finite set. By homological considerations, it must be the cone over a two-point space). Therefore, a point $x \in U$ whose tangent cone $T_x$ has $\mathbb{R}^{k-1}$ as a direct factor satisfies $T_x = \mathbb{R}^k$.

We call $x \in U$ with $T_x = \mathbb{R}^k$ a Euclidean point. Let $\mathcal{R} = \mathcal{R}_k$ denote the set of all Euclidean points in $U$. From Theorem 1.3 and the previous conclusion, we deduce that $U \setminus \mathcal{R}$ has Hausdorff dimension at most $k - 2$.

14.3. Charts, differentials, Riemannian metric. Let now $V \subset U$ be an open, convex subset and assume that $F$ and $G$ are opposite $(k, \delta)$-strainer maps in $V$. Let $F$ be defined by the $k$-tuple $(p_i)$. Due to Corollary 11.2 the map $F : V \rightarrow \mathbb{R}^k$ is locally $L$-biLipschitz with $L \leq 2\sqrt{k}$. Moreover, $L$ goes to 1 as $\delta$ goes to 0. Thus, for any $x \in V$, the differential $D_xF : T_x \rightarrow \mathbb{R}^k$ is an $L$-biLipschitz map.

If $F(x) = F(y)$ for some $x, y \in V$ then, using Lemma 7.4 and the first variation formula, we see

$$D_xF(v) \leq \sqrt{k} \cdot \delta \leq \frac{1}{20\sqrt{k}},$$

where $v \in \Sigma_x \subset T_x$ is the starting direction of the geodesic $xy$. This contradicts the fact that $D_xF$ is $L$-biLipschitz. Therefore, $F : V \rightarrow F(V)$ is injective.

The preimage $F^{-1}$ is (directionally) differentiable at all points of $\hat{V} := F(V)$ with differentials being the inverse maps of the differentials of $F$. This differentiability just means, that compositions of any differentiable curve with $F^{-1}$ has well-defined directions in all points, compare [Lyt04].

For any Euclidean point $x \in \mathcal{R}$, the differential $D_xF : T_x \rightarrow \mathbb{R}^k$ is a linear map, again by the first variation formula. We have the following continuity property in $\mathcal{R}$. Let $v_i \in \Sigma_x$, be the starting direction of a geodesic $\gamma_i$. Let $x_i \in \mathcal{R}$ converge to a point $x \in \mathcal{R}$ and the geodesics $\gamma_i$ converge to a geodesic $\gamma$ with starting direction $v \in T_x$. Then $D_{x_i}F(v_i)$ converge in $\mathbb{R}^k$ to $D_xF(v)$. Indeed, this is just the reformulation of the statement that angles between $x_i; v_i$, for $i = 1, ..., k$, and $\gamma_i$ converge to the angle between $x; v$ and $\gamma$. The last statement can easiest seen as a consequence of Lemma 5.13 and the fact that the concerned spaces of directions are all unit spheres.

We call the image $\hat{V} = F(V)$ together with the map $F : V \rightarrow \hat{V}$ a metric chart. On this metric chart, we have the subset $\hat{\mathcal{R}} := F(\mathcal{R})$ whose complement in $\hat{V}$ has Hausdorff dimension at most $k - 2$. For any point $y \in \hat{\mathcal{R}}$ we get a scalar product on its tangent space $T_y\mathbb{R}^k$, given by the pullback (via the linear map $D_yF^{-1}$) of the scalar product on $T_{F^{-1}(y)}U$. Due to the previous considerations, this Riemannian metric $g_F$ is continuous on $\hat{\mathcal{R}}$.

Expressing the length of a Lipschitz curve as an integral of its pointwise velocities, we see that for any Lipschitz curve $\gamma$ in $\mathcal{R}$ the length of $\gamma$ coincides with the length of
of \( \bar{\gamma} := F \circ \gamma \) with respect to the Riemannian metric \( g_F \), hence
\[
\ell(\gamma) = \int |\bar{\gamma}'(t)|_{g_F} \, dt.
\]

14.4. DC-maps in Euclidean spaces. We refer the reader to [Per94] and [AB15] for more details.

A function \( f : V \to \mathbb{R} \) on an open subset \( V \) of \( \mathbb{R}^m \) is called a DC-function if in a neighborhood of each point \( x \in V \) one can write \( f \) as a difference of two convex functions. The set of DC-functions contains all functions of class \( C^{1,1} \) and it is closed under addition and multiplication.

A map \( F : V \to \mathbb{R}^l \) is called a DC-map if its coordinates are DC. The composition of DC-maps is again a DC-map. In other words, a map \( F : V \to \mathbb{R}^l \) is DC if and only if for every DC-function \( g : W \subset \mathbb{R}^l \to \mathbb{R} \), the composition \( g \circ F \) is a DC-function on \( F^{-1}(W) \).

14.5. DC-maps on metric spaces. The following definition is meaningful only if the metric spaces in question are (locally) geodesic.

Definition 14.1. Let \( Y \) be a metric space. A function \( f : Y \to \mathbb{R} \) is called a DC-function if it can be locally represented as difference of two Lipschitz continuous convex functions.

Due to the corresponding statements about DC-functions on intervals, the set of DC functions on \( Y \) is closed under addition and multiplication.

Remark 14.2. We refer to [Pet07] for the definition and properties of semi-convexity. Assume that in \( Y \) each point \( x \) admits a Lipschitz 1-convex function in a small neighborhood \( V \) of \( x \). Then each semi-convex function on \( Y \) is DC. Such strongly convex functions exist on Alexandrov spaces with lower curvature bound, [Pet07]. On any CAT(\( \kappa \)) space \( X \) we get such a function as a scalar multiple of \( d_x^2 \).

We use compositions to define DC-maps between metric spaces.

Definition 14.3. A locally Lipschitz map \( F : Z \to Y \) between metric spaces \( Z \) and \( Y \) is called a DC-map if for each DC-function \( f : U \to \mathbb{R} \) defined on an open subset \( U \) of \( Y \) the composition \( f \circ F \) is DC on \( F^{-1}(U) \). If \( F \) is a biLipschitz homeomorphism and its inverse is DC, then we call \( F \) a DC-isomorphism.

A composition of DC-maps is a DC-map. For a map \( F : Z \to \mathbb{R}^l \) we recover the old definition: \( F \) is DC if and only if the coordinates of \( F \) are DC.

14.6. Crucial observation. Let now \( U \subset \tilde{U} \subset X \) be again a tiny ball consisting of \( k \)-regular points as above. Since all distance functions to points in \( \tilde{U} \) are convex, each \( (k, \delta) \)-strainer map is a DC-map by definition. These strainer maps turn out to be DC-isomorphisms, in direct analogy with [Per91], see also [AB15]. The proof of the following observation is taken from [Per91].

Proposition 14.4. Let \( F \) and \( G \) be opposite \( (k, \delta) \)-strainer maps in an open subset \( V \) of a tiny ball \( U \). Then \( F : V \to F(V) \subset \mathbb{R}^k \) is a DC-isomorphism, if \( \delta \leq \frac{1}{50 \cdot k^2} \).

Proof. Denote by \( f_i = d_{p_i} \) the coordinates of \( F \). We already know that the map \( F \) is a locally biLipschitz DC-map. It remains to prove that the inverse map \( F^{-1} : F(V) \to V \) is DC too. Thus, given an open subset \( O \subset V \) and a convex function \( g : O \to \mathbb{R} \), we have to show that the function \( \bar{g} = g \circ F^{-1} \) is DC on \( F(O) \).
We introduce the following auxiliary notion. We say that a convex, Lipschitz continuous function \( g : O \to \mathbb{R} \) on an open subset \( O \subset V \) is \( \alpha \)-special for some \( \alpha \geq 0 \) if the following holds true. For any \( x \in O \) and any unit vector \( v \in T_x \) such that \( D_x f_i(v) \geq 0 \) for all \( i = 1, \ldots, k \) we have \( D_x g(v) \leq -\alpha \).

If \( g \) is \( \alpha \)-special then, for any Lipschitz curve \( \eta : [a, b] \to O \) parametrized by arclength and such that all \( f_i \) are non-decreasing on \( \eta \), the composition \( g \circ \eta : [a, b] \to \mathbb{R} \) decreases at least with velocity \( \alpha \).

The proof of the Proposition will follow from two auxiliary statements:

**Lemma 14.5.** There is a 1-Lipschitz \( \alpha \)-special function \( g \) on \( V \) with \( \alpha = \frac{1}{4 \cdot k^2} \).

**Lemma 14.6.** If \( g \) is a 0-special function in \( O \) then the composition \( \tilde{g} = g \circ F^{-1} \) is a convex function on \( F(O) \).

Indeed, assuming Lemma 14.5 and Lemma 14.6 to be true, we derive:

**Corollary 14.7.** In the notations above let \( h : O \to \mathbb{R} \) be an \( L_0 \)-Lipschitz convex function. Then \( h \) can be represented as \( h = h_1 - h_2 \) with \( h_1 \) and \( h_2 \) being 0-special \( L_0 \cdot (1 + \frac{1}{\alpha}) \)-Lipschitz functions.

Moreover, \( \tilde{h} := h \circ F^{-1} \) is the difference of two \( C \cdot L_0 \)-Lipschitz convex functions, with some \( C \) depending only on \( k \).

**Proof.** Indeed, choosing \( g \) as in Lemma 14.5, we set \( h_2(x) := \frac{4 a_i}{\alpha} \cdot g(x) \). Then \( h_2 \) is \( L_0 \)-special. Since \( h \) is convex and \( L_0 \)-Lipschitz, we deduce that the function \( h_1 = g + h_2 \) is convex and 0-special. The statement about the Lipschitz constants of \( h_1 \) and \( h_2 \) is clear.

Due to Lemma 14.6, the compositions \( \tilde{h}_i := h_i \circ F^{-1} \) are convex on \( F(O) \). The Lipschitz constants of \( \tilde{h}_i \) are bounded from above by the product of the Lipschitz constants of \( h_i \) and \( F^{-1} \).

Thus assuming Lemma 14.5 and Lemma 14.6 to be true, we finish the proof of the proposition. \( \square \)

We turn to the auxiliary lemmas used in Proposition 14.4.

**Proof of Lemma 14.5** Let \( f_i \) be the coordinates of \( F \) and let \( g_i \) be the coordinates of \( G \). The functions \( g_i \) are convex and 1-Lipschitz, for \( i = 1, \ldots, k \), hence so is \( g(x) = \frac{1}{k} \sum_{i=1}^k g_i(x) \). We claim that \( g \) is \( \frac{1}{4 \cdot k^2} \)-special.

Indeed, let \( x \in V \) be arbitrary and let \( v \in \Sigma_x \) be such that \( D_x f_i(v) \geq 0 \), for all \( i = 1, \ldots, k \). Then \( D_x g_i(v) < \delta \) for all \( i = 1, \ldots, k \), as follows directly from the first variation formula and the definition of opposite strainer maps.

The map \( D_x G : T_x \to \mathbb{R}^k \) is \( 2 \sqrt{k} \)-biLipschitz, thus \( D_x G(v) \) has norm at least \( \frac{1}{2 \sqrt{k}} \). Therefore, for at least one \( 1 \leq j \leq k \), we must have \( |D_x g_j(v)| \geq \frac{1}{2k} \).

For this \( j \) we get, \( D_x g_j(v) \leq -\frac{1}{2k} \). Summing up, we obtain

\[
D_x g(v) \leq \frac{1}{k} \cdot \left( -\frac{1}{2k} + (k-1) \cdot \delta \right) \leq \frac{1}{k} \cdot \left( -\frac{1}{2k} + \frac{1}{4k} \cdot k \right) \leq -\frac{1}{4 \cdot k^2}. 
\]

This finishes the proof. \( \square \)

**Proof of Lemma 14.6** It follows word by word as in [Per94], since the proof in [Per94] only uses convexity and differentiability and no curvature bounds. \( \square \)
14.7. The Riemannian metric revisited. As in [Per94], we have:

**Lemma 14.8.** For any metric chart \( F : V \to \mathbb{R}^k \) as above, the Riemannian metric \( g_F \) defined and continuous on the subset \( \mathcal{R} = F(R) \) is locally of bounded variation. Moreover, \( g_F \) is differentiable almost everywhere in \( F(R) \).

The proof literally follows from [Per94] (see also [AB15]). The idea is to take a sufficiently large set of generic points \( q_j \) in \( \hat{U} \). The distance functions \( h_j := h_j \circ F^{-1} \) are DC-functions by Proposition [14.4]. On the other hand, since \( g \) the gradients of \( h_j \) functions by Proposition 14.4. On the other hand, since \( g \)

Moreover, \( h \) the gradients of \( h_j \) have the following property. The compositions \( \tilde{h}_j := h_j \circ F^{-1} \) are DC-functions by Proposition [14.4]. On the other hand, since \( h_j \) are distance functions, the gradients of \( h_j \) at all points of \( \mathcal{R} \) have norm 1 with respect to the Riemannian metric \( g_F \). One obtains an equation for the coordinates of \( g_F \) and shows that they can be expressed through the first derivatives of the DC-functions \( \tilde{h}_j \).

14.8. DC-curves in GCBA spaces. In order to prove that the Riemannian structure on the set \( \mathcal{R} \) determines the metric, we will need a stability statement about variations of DC-curves, which might be of independent interest. In the following definition and Proposition [14.9] we work in general GCBA spaces, and not only in their regular parts as in the rest of this section.

Let \( U \subset \hat{U} \) be a tiny ball. We say that a curve \( \gamma : I \to \hat{U} \) on a compact interval \( I \) is a DC-curve of norm bounded by \( A \) if \( \gamma \) is \( A \)-Lipschitz and for any 1-Lipschitz convex function \( f : \hat{U} \to \mathbb{R} \) the restriction \( f \circ \gamma \) can be (globally) written as a difference of two \( A \)-Lipschitz convex functions on \( I \).

The following statement is closely related to the well-known fact [AR89], that the length is continuous under convergence of curves of uniformly bounded turn in the Euclidean space.

**Proposition 14.9.** Let \( \gamma_l : I \to \hat{U} \) be DC-curves with a uniform bound on the norms. If \( \gamma_l \) converges to \( \gamma \) pointwise then \( \lim_{l \to \infty} \ell(\gamma_l) = \ell(\gamma) \).

**Proof.** Assuming the contrary and choosing a subsequence, we find \( \epsilon > 0 \) with

\[
(1 + 2 \cdot \epsilon)^2 \cdot \ell(\gamma) < \lim_{l \to \infty} \ell(\gamma_l).
\]

Due to Proposition [5.3] we find a distance map \( F_\epsilon : \hat{U} \to \mathbb{R}_\infty^m \), which is a \((1 + \epsilon)\)-biLipschitz embedding, if \( \mathbb{R}_\infty^m \) is equipped with the sup-norm \( | \cdot |_\infty \). Set \( \eta_l = F_\epsilon \circ \gamma_l \) and \( \eta = F_\epsilon \circ \gamma \). From the biLipschitz property we obtain a contradiction, once we show that the lengths of \( \eta_l \) converge to the length of \( \eta \) in \( \mathbb{R}_\infty^m \).

The \( i \)-th coordinate of \( \eta_l \) is the composition of \( \gamma_l \) and a convex distance function \( d_{p_i} \). Thus, this \( i \)-th coordinate is a difference of two convex \( A \)-Lipschitz functions \( h_{i+}^l \) and \( h_{i-}^l \) on \( I \). Adding a constant we may assume that \( h_{i+}^l \) equals 0 at some fixed point on \( I \).

Going to subsequences, we may assume that \( h_{i+}^l \) and \( h_{i-}^l \) converge to \( h^+ \) and \( h^- \) such that \( h^+ - h^- \) is the corresponding coordinate of \( \eta \). Due to the standard results about convergence of convex functions, we see that at almost every \( t \in I \), the differentials of \( h_{i+}^l \) and \( h_{i-}^l \) exist at \( t \) and converge to the differentials of \( h^+ \), \( h^- \) at \( t \). Taking again all coordinates together, we see that for almost every \( t \in I \), the differentials \( \eta(t) \in \mathbb{R}_\infty^m \) exist and converge to \( \eta'(t) \).

Expressing the length of \( \eta \) and \( \eta_l \) as integrals of \( | \cdot |_\infty \)-norms of \( \eta' \) and \( \eta_l' \) over \( I \) we finish the proof of the convergence. This finishes the proof of the Proposition. \( \square \)

Coming back to the regular part, we can use this result to prove:
Corollary 14.10. Let $F : V \to \mathbb{R}^k$ be a metric chart as in Subsection 14.3 with convex $V \subset U$. Let $S$ be a subset of $V$ with $\mathcal{H}^{k-1}(S) = 0$. Then every pair of points $x, y \in V \setminus S$ is connected in $V \setminus S$ by curves of lengths arbitrary close to $d(x, y)$.

Proof. The statement is well-known and easy to prove for open convex subsets $V$ in $\mathbb{R}^k$, connecting $x$ and $y$ by concatenations of two segments.

Since the statement is true in $F(V)$ and the map $F : V \to \hat{V} = F(V)$ is biLipschitz, it suffices to prove the following claim. Let $\gamma : I \to V$ be a geodesic. Then there exist curves $\gamma_1 : I \to V \setminus S$ converging to $\gamma$ and such that $\ell(\gamma_1)$ converges to $\ell(\gamma)$. (Once such $\gamma_1$ is constructed we obtain, the desired curves by connecting the endpoints of $\gamma_1$ with $x$ and $y$ within $V \setminus S$, using that $F$ is biLipschitz). 

In order to find such $\gamma_1$ we consider the curve $\eta := F \circ \gamma$ in $\hat{V}$. Note that the differentials of $\eta$ at different points have distance at most $2 \cdot k \cdot \delta$ from each other, as follows from Lemma 7.6. Take a small ball $B$ around the origin in the hyperplane $\mathbb{R}^k$ orthogonal to the starting direction of $\eta$. Then we observe that the map $Q : B \times I \to \mathbb{R}^k$ given by $Q(x, t) = x + \eta(t)$ is a biLipschitz embedding.

This implies that for almost every $x_0 \in B$ the curve $t \to \eta(t) + x_0$ does not meet the set $F(S)$ with vanishing $\mathcal{H}^{k-1}$-measure. Letting $x_0$ going to 0, we find a sequence of translates $\eta(t) = \eta(t) + x_t$ converging to $\eta$ and disjoint from $F(S)$.

We set $\gamma_1 = F^{-1} \circ \eta$. It suffices to prove that $\ell(\gamma_1)$ converge to $\ell(\gamma)$.

Clearly, the curves $\gamma_1$ are uniformly Lipschitz. Let $f$ be a convex 1-Lipschitz function on $V$. We have $f \circ \gamma_1 = f \circ F^{-1} \circ \eta$.

Due to Corollary 14.7, $f \circ F^{-1}$ is the difference of two convex $A$-Lipschitz functions $h_1$ and $h_2$ on $F(V)$, where $A$ is independent of $f$. On the other hand, the curve $\eta$ is a DC-curve of bounded norm, since its coordinates are convex 1-Lipschitz functions. The curves $\eta_l$ are then also DC-curves with the same bound on the norm. Together, this implies that $f \circ \gamma_1$ can be written as a difference of two convex $B$-Lipschitz functions, with some $B$ independent of $l$.

Hence $\gamma_1$ are DC-curves of uniformly bounded norm and the claim follows from Proposition 14.9

14.9. Conclusions. Now we can summarize the results to the

Proof of Theorem 1.3. Define as above $M^k$ to be the set of all $(k, \delta)$-strained points in the $k$-dimensional part $X^k$, with $\delta \leq \frac{1}{90 \cdot k^2}$. We have seen in Theorem 1.2 that $M^k$ is a Lipschitz manifold. By construction, every point in $\mathcal{R} = R_k$ with tangent space isometric to $\mathbb{R}^k$ is contained in $M^k$.

For any open convex set $V$ with opposite $(k, \delta)$-strainer maps $F, G : V \to \mathbb{R}^k$, the map $F : V \to F(V)$ is a DC-isomorphism onto an open subset of $\mathbb{R}^k$, Proposition 14.4. Thus, the set of all such charts provides $M^k$ with a DC-atlas.

On the set of Euclidean points $\mathcal{R}$ in $M^k$ we get a Riemannian metric $g_{\mathcal{R}}$ in any chart. Moreover, $M^k \setminus \mathcal{R}$ has Hausdorff dimension at most $k - 2$ as shown in Subsection 14.2. Due to the intrinsic definition, this metric is globally well defined on $\mathcal{R}$. As shown in Subsection 14.3 the Riemannian tensor is continuous on $\mathcal{R}$ and due to Lemma 14.3 it is locally of bounded variation.

The length of all curves contained in $\mathcal{R}$ is computed via the Riemannian metric. Finally, the length of all curves in $\mathcal{R}$ locally determines the metric in $M^k$, due to Corollary 14.10.
14.10. Second order differentiability of DC-functions. Let $X$ be an arbitrary GCBA space. By Theorem 1.2, $\mu_X$-almost all of $X$ is the union of the (different dimensional) regular parts $M^k$ of $X$. Due to Theorem 1.3, $\mu_X$-almost every point of $X$ is a point with a Euclidean tangent space.

Applying the classical theorem of Rademacher in the metric charts of the regular part, we deduce that every Lipschitz function $f : X \to \mathbb{R}$ has a linear differential $\mu_X$-almost everywhere, (in the sense of Stolz, as in Proposition 14.11 below.)

As in the case of Alexandrov spaces described in [Per94], all DC-functions are almost everywhere twice differentiable, as stated in the following Proposition.

Proposition 14.11. Let $X$ be a GCBA space and let $f : X \to \mathbb{R}$ be a DC-function. Then, for $\mu_X$-almost all $x$, there exists a bilinear form $H_x = H_x(f) : T_x \times T_x \to \mathbb{R}$, called the Hessian of $f$ at $x$, such that the following holds true for any tiny ball $U$ around $x$. The remainder $R_x : U \to \mathbb{R}$ in the Taylor formula
\begin{equation}
R_x(y) := f(y) - (f(x) + D_x f(v) + H_x f(v, v)),
\end{equation}
where $v := \log_x(y)$, satisfies
\begin{equation}
\lim_{y \to x} \frac{R_x(y)}{d(x, y)^2} = 0.
\end{equation}

We only sketch the proof and refer for details to [Per94] and [AB15, Section 7.2].

The claim is local and $\mu_X$-almost all of $X$ consists of regular points. Thus, we may replace $X$ by a tiny ball $U$ which coincides with its set of regular points $U = M^k$. Now we can use the DC-structure provided by Theorem 1.3.

Using a coordinate change to “normal coordinates” as in [Per94] and [AB15], Proposition 14.11 follows directly from the corresponding theorem of Alexandrov in $\mathbb{R}^n$, [EG15 Theorem 6.9], once the following lemma is verified. In the formulation of the lemma and later on, we denote by $o$ as usual the Landau symbol.

Lemma 14.12. Let $G : V \to \mathbb{R}^k$ be a DC-isomorphism on an open subset $V \subset U$, given by a composition of a metric chart $F$ and a diffeomorphism of $\mathbb{R}^k$. Let $x \in \mathcal{R}$ be a Euclidean point with $G(x) = 0$. Assume that the metric tensor $g$ of $V$ expressed on $W = G(V)$ via $G$ satisfies, for all $y \in V \cap \mathcal{R},$
\begin{equation}
||g(G(y)) - g(G(x))|| = o(d(x, y)).
\end{equation}

Then, for all $y \in V$ and the corresponding direction $v = \log_x(y) \in T_x$, we have
\begin{equation}
||G(y) - D_x G(v)|| = o(d(x, y)^2).
\end{equation}

Proof. We sketch the proof, referring for details to [Per94] and [AB15, Section 7.2].

From 14.3, and the fact that the Riemannian tensor on $\mathcal{R}$ determines the metric in $V$, Corollary 14.10 we obtain, for all small $r$, and all $y, z \in \bar{B}_r(x)$, the estimate
\begin{equation}
|d(y, z) - ||G(y) - G(z)||| = o(r^2).
\end{equation}

Hence, it suffices to prove, that for all $y \in \bar{B}_r(x)$ the angle $\beta(y)$ between $G(y)$ and $D_x G(v)$, (with $v = \log_x(y)$ as in the formulation) satisfies the estimate $\beta(y) = o(r)$.

In order to prove this estimate, it is sufficient to show that for the midpoint $m$ of the geodesic $xy$ the angle $\beta_1(y)$ between $G(m)$ and $G(y)$ satisfies $\beta_1(y) = o(r)$.

(Relying only on 14.3, one can show $\beta_1(y) = o(\sqrt{r})$ and as a consequence that $\beta(y) = o(\sqrt{r})$, as done in the course of the proof of [AB15, Proposition 7.8 (d)].) In order to prove the required stronger estimate $\beta_1(y) = o(r)$, we will rely on the curvature bound, similarly to [Per94].
We say that the triangle $xyz$ in $\tilde{B}_r(x)$ is sufficiently non-degenerated, respectively very non-degenerated, if all of its comparison angles are at least $\frac{\pi}{10}$, respectively at least $\frac{\pi}{10}$. For any sufficiently non-degenerated triangle $xyz$ in $\tilde{B}_r(x)$, we deduce from (14.13), that the comparison angle $\angle yxz$ differs from the angle between $G(y)$ and $G(z)$ in $\mathbb{R}^k$ by at most $o(r)$.

Given a very non-degenerated triangle $xyz$ in $\tilde{B}_r(x)$, we find a point $w \in \tilde{B}_r(x)$ such that the triangles $xyw$ and $xzw$ are sufficiently non-degenerated and such that
\[
\angle yxz + \angle yzw + \angle wxx = 2\pi.
\]
Since the corresponding comparison angles are not smaller and since the three angles between pairs of different vectors in $\{G(y), G(z), G(w)\}$ sum up to at most $2\pi$, we arrive at the following conclusion:

For any very non-degenerated triangle $xyz$ in $\tilde{B}_r(x)$ the angle $\angle yxz$ differs from the angle in $\mathbb{R}^k$ between $G(y)$ and $G(z)$ by at most $o(r)$.

Let now $y \in \tilde{B}_r(x)$ be arbitrary and let $m$ be the midpoint of $xy$. We find a point $z$ with $d(x, y) = d(x, z)$, such that $G(z)$ lies in the same plane as $G(y)$ and $G(m)$ and such that $G(z)$ is orthogonal to $G(y)$. Then the difference of the angle between $G(z)$ and $G(y)$ and the angle between $G(z)$ and $G(m)$ is exactly the angle between $G(y)$ and $G(m)$. On the other hand, due to the previous considerations, the angle between $G(z)$ and $G(y)$ (respectively, between $G(z)$ and $G(m)$) coincides with $\angle zxy$ (respectively, with $\angle zxm$) up to $o(r)$. But, by construction, $\angle zxy = \angle zxm$.

Therefore, we have verified the estimate $\beta_1(y) = o(r)$, thus finishing the proof of the Lemma and of Proposition 14.11. \hfill \square

Remark 14.13. The second order differentiability of distance functions, a special case of Proposition 14.17 appears in [OT99 Main Theorem 1(3)].

15. Topological counterexamples

Example 15.1. Let $X_n$ denote a unit circle $S$ with two other unit circles $S_n^\pm$ attached to $S$ at points $p_n^\pm$ at distance $1/n$ from each other. The sequence $X_n$ converges to the wedge of three unit circles $X$. Thus, $X_n$ is not homeomorphic to $X$ for no $n$. This shows that there is no topological stability even in dimension 1.

Example 15.2. Proposition 14.15 implies that 1-dimensional GCBA spaces are locally isometric to finite graphs. On the other hand, Kleiner constructs in [Kle99] a 2-dimensional GCBA space $X$ that does not admit a triangulation, see also [Nag00] Example 2.7. This space $X$ contains a point $x$, such that no neighborhood of it is homeomorphic to a cone. Moreover, there are arbitrary small $r_1, r_2 > 0$ such that the distance spheres $\partial B_{r_1}(x)$ and $\partial B_{r_2}(x)$ are not homeomorphic.

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