Emergent long-range interaction and state-selective localization in a strongly driven $XXZ$ model

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The nonlinear effect of a driving force in periodically driven quantum many-body systems can be systematically investigated by analyzing the effective Floquet Hamiltonian. In particular, under an appropriate definition of the effective Hamiltonian, simple driving forces may result in non-local interactions. Here we consider a driven $XXZ$ model to show that four-site interactions emerge owing to the driving force, which can produce state-selective localization, a phenomenon where some limited Ising-like product states become fixed points of dynamics. We first derive the effective Hamiltonian of a driven $XXZ$ model on an arbitrary lattice for general spin $S$. We then analyze in detail the case of a one-dimensional chain with $S = 1/2$ as a special case, and find a condition imposed on the cluster of four consecutive sites as a necessary and sufficient condition for the state of the whole system to be localized. We construct such a localized state based on this condition and demonstrate it by numerical simulation of the original time-periodic model.

I. INTRODUCTION

Periodically driven quantum many-body systems often have a structure that we can solve effectively as a time-independent system. This provides a hint for designing Hamiltonians with new quantum states as stationary states, which would not be realized in bare time-independent systems. This technique is called Floquet engineering in reference to the Floquet theory [1], which is a general theory for time-periodic systems and has contributed to generation of new gauge fields in the optical lattice [2–9] and deformation of the band structure of graphene [10–17], for example.

The starting point of the Floquet engineering is to represent the periodically driven system in terms of a time-independent effective Hamiltonian. Using the Floquet theory, given a time-periodic Hamiltonian $\hat{H}(t) = \hat{H}(t + T)$, we can construct a generator of time evolution over one cycle of $\hat{H}(t)$, which is called the Floquet Hamiltonian. In the short-period limit $T \to 0$, or equivalently in the high-frequency limit $\Omega := 2\pi/T \to \infty$, the Floquet Hamiltonian coincides with the average Hamiltonian given by

$$\hat{H}_{\text{ave}} := \frac{1}{T} \int_{0}^{T} dt \hat{H}(t).$$

Therefore, the average Hamiltonian is often referred to as the effective (Floquet) Hamiltonian and is denoted by $\hat{H}_\text{eff}$.

It is important, however, to note that we can in general obtain a number of effective Hamiltonians. In other words, for a given periodic Hamiltonian $\hat{H}(t)$, there are a generally infinite number of effective Hamiltonians

$$\hat{H}_\text{eff} = \frac{1}{T} \int_{0}^{T} dt \hat{U}(t) [\hat{H}(t) - i\hbar \partial_t] \hat{U}^\dagger(t),$$

each of which is specified by the unitary transformation $\hat{U}(t)$. The average Hamiltonian in Eq. (1) is only a special case in which $\hat{U}(t)$ is the identity operator. Out of the infinitely large set $\{\hat{U}(t)\}$, we should choose one that is appropriate to the purpose of research, especially when investigating nonlinear effects of driving forces in interacting many-body systems.

One example of such nonlinear effects is correlated tunneling in interacting bosons, which was predicted theoretically by Rapp et al. [18] for the Bose-Hubbard model with time-dependent on-site interactions and was demonstrated experimentally by Meinert et al. [19] using the corresponding cold-atom system. Correlated tunneling can only be described by $\hat{H}_\text{eff}$ under a specific choice of $\hat{U}(t)$. The driven Bose-Hubbard model used in Refs. [18, 19] is given by the time-dependent Hamiltonian

$$\hat{H}(t) = -J \sum_{\langle i,j \rangle} (\hat{a}_i^\dagger \hat{a}_j + \text{h.c.}) + \sum_i \frac{U(t)}{2} \hat{n}_i(\hat{n}_i - 1),$$

where $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the bosonic creation and annihilation operators at site $i$, $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ is the number operator, and

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the on-site interaction $U(t)$ oscillates in time as $U(t) = U + \delta U \sin \Omega t$. Adopting the unitary transformation

$$\hat{U}(t) = \exp \left[ \frac{1}{2i} \frac{\delta U}{\hbar \Omega} \cos \Omega t \sum_i \hat{n}_i (\hat{n}_i - 1) \right],$$

we obtain the effective Hamiltonian

$$\hat{H}_{\text{eff}} = - J \sum_{(i,j)} \left( \hat{a}_i^\dagger \hat{J}_0 (\hat{A}_{i,j}) \hat{a}_j + \text{h.c.} \right) + \sum_i \frac{\bar{U}}{2} \hat{n}_i (\hat{n}_i - 1),$$

(5)

where $\hat{J}_0(x)$ is the zeroth-order Bessel function of the first kind and

$$\hat{A}_{i,j} := (\delta U / \hbar \Omega) \times (\hat{n}_i - \hat{n}_j)$$

(6)

is the scaled particle-number difference between sites $i$ and $j$.

In this Hamiltonian, the elementary processes describing the movement of particles are governed by effective bond operator

$$\hat{b}^{(\text{eff})}_{i,j} := \hat{a}^\dagger_i \hat{J}_0 (\hat{A}_{i,j}) \hat{a}_j + \text{h.c.}$$

(7)

for each bond $(i,j)$. Let $n_i := \langle \Psi | \hat{n}_i | \Psi \rangle$ be the expectation value of the number operator $\hat{n}_i$ on site $i$ under the state $|\Psi\rangle$. We can expect the dynamics through $\hat{A}_{i,j}$ due to the bond operator $\hat{b}^{(\text{eff})}_{i,j}$ depending on the spatial distribution of $n_i$. For example, for a bond $(i,j)$, the process of changing the particle number distribution from $(n_i, n_j) = (0, 1)$ to $(n_i, n_j) = (1, 0)$ always takes place with a constant amplitude $-J$ because $\hat{J}_0(0) = 1$. On the other hand, for the same bond, the process of changing from $(n_i, n_j) = (1, 1)$ to $(n_i, n_j) = (2, 0)$ takes place with an amplitude $-J \times \hat{J}_0(\delta U / \hbar \Omega)$. When the value of $(\delta U / \hbar \Omega)$ is set to a zero of the Bessel function $J_0(x)$, the latter process is strongly suppressed while the former process is intact. This complex state dependence is one of the features of correlated tunneling. Note that if we averaged the original Hamiltonian $\hat{H}(t)$ as in Eq. (1) by choosing the identity for $\hat{U}(t)$, we would only find a trivial time-independent part of $\hat{H}(t)$:

$$\hat{H}_{\text{ave}} = - J \sum_{(i,j)} \hat{a}_i^\dagger \hat{a}_j + \sum_i \frac{\bar{U}}{2} \hat{n}_i (\hat{n}_i - 1),$$

(8)

from which we would not be able to derive correlated tunneling. This demonstrates that the choice of $\hat{U}(t)$ is essential for the observation of target phenomena.

When the total Hamiltonian is written as $\hat{\mathcal{H}}(t) = \hat{H}_0 + \hat{\mathcal{Y}}(t)$ such that $\int_0^{2\pi / \Omega} dt \hat{\mathcal{Y}}(t) = 0$, we should utilize the unitary transformation in the form

$$\hat{U}(t) = \mathcal{T} \exp \left[ - \int_0^t \frac{dt'}{i\hbar} \hat{\mathcal{Y}}(t') \right],$$

(9)

which reduces to Eq. (4) in the case of the Hamiltonian in Eq. (3). Combining this with Eq. (2), we obtain the interaction term as

$$\hat{V}_{\text{eff}} := \frac{1}{T} \int_0^T dt [\hat{U}(t) \hat{H}_0 \hat{U}^\dagger(t) - \hat{H}_0].$$

(10)

Note that the driving term in Eq. (3) acts on each site $i$, whereas the resulting interaction in Eq. (5) acts on each bond $(i,j)$. This motivates us to investigate a possibility of finding, out of a driving two-body interaction, an effective long-range interaction of the form in Eq. (10).

Here in this study, we indeed find a four-site interaction out of an XXZ model in which the longitudinal exchange interaction is periodically driven with constant amplitude. In the resulting model, we observe a state-selective localization, i.e., a limited number of Ising-like product states becoming fixed points of the dynamics generated by the effective Hamiltonian. This means that under the basis of the Ising-like product states, dynamics depends strongly on the initial state. Such an initial-state-dependent dynamics resembles the quantum scar recently studied extensively, which is one of the mechanisms preventing the thermal equilibration of quantum many-body systems [20, 21].

The rest of this paper is organized as follows. In Sec. II, we derive the effective Hamiltonian of the driven XXZ model and show a general idea of state-selective localization through emergent long-range interactions. In Sec. III, we analyze the localization condition in detail in the case of spin-1/2 one-dimensional interactions. Finally, in Sec. IV, we conclude the paper with summary and discussion. The detailed derivation of the effective Hamiltonian is given in Appendix A.

## II. EMERGENT LONG-RANGE INTERACTIONS IN A DRIVEN XXZ MODEL

We consider an XXZ model with periodically driven longitudinal exchange interactions on an arbitrary lattice, whose Hamiltonian is given by

$$\hat{\mathcal{H}}(t) = - \frac{J_0}{2} \sum_{(i,j)} (\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+) - J_0(t) \sum_{(i,j)} \hat{S}_i^z \hat{S}_j^z,$$

(11)

where $J_0$ and $J_0(t) \equiv \bar{J} + \delta J \cos \Omega t$ are the transverse and longitudinal components of the exchange interaction, respectively, and $\{\hat{S}_i^{\pm, z}\}$ are the spin operators at site $i$, satisfying $[\hat{S}_i^+, \hat{S}_j^-] = \delta_{ij} \hbar \hat{S}_i^z$ and $[\hat{S}_i^z, \hat{S}_j^{\pm}] = \pm \delta_{ij} \hbar \hat{S}_i^\pm$. Here, the symbol $\sum_{(i,j)}$ indicates summation over all the bonds on the lattice.

In the present case, for the unitary transformation in Eq. (9) we choose

$$\hat{U}(t) = \exp \left[ -i A \sin \Omega t \sum_{(i,j)} \hat{S}_i^z \hat{S}_j^z \right],$$

(12)
where

\[ A := \frac{\delta J}{\hbar \Omega} \]  

(13)

is the dimensionless amplitude of the driving force and thus the effective Hamiltonian in Eq. (2) takes the form

\[ \hat{H}_{\text{eff}}(A) = -\frac{J_J}{2} \sum_{(i,j)} \left[ \hat{S}_{\hat{i}}^{+} J_0(A \hat{Z}_{i,j}) \hat{S}_{\hat{j}}^{-} + (+ \leftrightarrow -) \right] - \bar{J}_|| \sum_{(i,j)} \hat{S}_{\hat{i}}^{z} \hat{S}_{\hat{j}}^{z} \]  

(14)

as described in Appendix A for the derivation. Here,

\[ \hat{Z}_{i,j} = \sum_{(k,l)} \hat{S}_{k}^{z} - \sum_{(k,l)} \hat{S}_{l}^{z} \]  

(15)

is the local staggered magnetization operator around the bond \((i,j)\). Note that \( \sum_{(k,l)} \hat{O}_{k} \) denotes summation over all sites \( k \) that are connected to site \( i \).

The operator \( \hat{Z}_{i,j} \) produces a new type of long-range interaction. First, the operator \( \hat{Z}_{i,j} \) involves many spins that depend on the underlying lattice structure. In fact, if the lattice is given on a hypercubic lattice of dimension \( d \), \( \hat{Z}_{i,j} \) is written in terms of \( \hat{S}_{k}^{z} \) over 4\( d \) pieces of sites, since the coordination number is given by 2\( d \) (see Fig. 1).

Second, \( \hat{Z}_{i,j} \) has the following property. Let the value of the parameter \( A \) be one of the innumerable zeros of the Bessel function \( J_0(z) \), i.e., \( A_{\lambda} \) with \( J_0(A_{\lambda}) = 0 \). If the operator \( \hat{Z}_{i,j} \) produces the eigenvalue \( \pm 1 \) in the argument of \( J_0(A_{\lambda} \hat{Z}_{i,j}) \), when \( \hat{H}_{\text{eff}}(A_{\lambda}) \) acts on a quantum state \( |\Psi\rangle \), then it holds that

\[ \hat{H}_{\text{eff}}(A_{\lambda}) |\Psi\rangle = -\bar{J}_|| \sum_{(i,j)} \hat{S}_{\hat{i}}^{z} \hat{S}_{\hat{j}}^{z} |\Psi\rangle. \]  

(16)

Since \( \hat{Z}_{i,j} \) is written in terms of \( \hat{S}_{k}^{z} \), let us choose the set of Ising-like product states as the representation basis hereafter. We can then classify all basis states into two types depending on the eigenvalues of \( \hat{Z}_{i,j} \) states for which the eigenvalue is either 1 or \(-1\) and hence Eq. (16) holds, and the other states for which Eq. (16) does not hold.

We formulate it more specifically as follows. First, we break down the Hamiltonian into two terms:

\[ \hat{H}_{\text{eff}}(A) = \hat{H}^{XY}(A) + \hat{H}^{\text{Ising}}, \]  

(17)

where

\[ \hat{H}^{XY}(A) := -\frac{J_J}{2} \sum_{(i,j)} \left[ \hat{S}_{\hat{i}}^{+} J_0(A \hat{Z}_{i,j}) \hat{S}_{\hat{j}}^{-} + (+ \leftrightarrow -) \right] \]  

and

\[ \hat{H}^{\text{Ising}} := -\bar{J}_|| \sum_{(i,j)} \hat{S}_{\hat{i}}^{z} \hat{S}_{\hat{j}}^{z} \]  

(18)

(19)

Let \( \mathcal{H}^{\text{Ising}} \) be the set of Ising-like product states that can be realized on a given lattice [see Eq. (25) for an example of \( S = 1/2 \)]. Then an arbitrary state \( |\Psi\rangle \in \mathcal{H}^{\text{Ising}} \) can be classified into two types depending on whether or not it satisfies the vanishing condition

\[ \hat{H}^{XY}(A_{\lambda}) |\Psi\rangle = 0. \]  

(20)

Let \( \mathcal{H}_0^{\text{Ising}} \) be the set of the former, and \( \mathcal{H}_1^{\text{Ising}} \) be the set of the latter. The condition in Eq. (20) gives the decomposition into the direct sum \( \mathcal{H}^{\text{Ising}} = \mathcal{H}_0^{\text{Ising}} \oplus \mathcal{H}_1^{\text{Ising}} \). By casting the condition in Eq. (20) into the form

\[ \exp \left[ \frac{1}{\hbar} \hat{H}_{\text{eff}}(A_{\lambda}) \times t \right] |\Psi\rangle \propto |\Psi\rangle \]  

(21)

for an arbitrary \( t \in \mathbb{R} \), we realize that the state \( |\Psi\rangle \in \mathcal{H}_0^{\text{Ising}} \) is a fixed point of the dynamics generated by \( \hat{H}_{\text{eff}}(A_{\lambda}) \). In other words, \( \hat{H}_{\text{eff}}(A) \) generates dynamics such that only the states \( |\Psi\rangle \in \mathcal{H}_0^{\text{Ising}} \) are selectively localized.

### III. SPIN-1/2 ONE-DIMENSIONAL CHAIN

In order to investigate the state-selective localization more specifically, we now analyze an \( S = 1/2 \) chain of length \( L \) under the periodic boundary conditions. The two terms in Eqs. (18) and (19) constituting the effective Hamiltonian \( \hat{H}_{\text{eff}}(A) \) in Eq. (14) now reads

\[ \hat{H}_{\text{eff}}^{XY}(A) = -\frac{J_J}{2} \sum_{i=1}^{L} \left[ \hat{S}_{i}^{+} J_0(A \hat{Z}_{i,i+1}) \hat{S}_{i+1}^{-} + (+ \leftrightarrow -) \right] \]  

(22)

and

\[ \hat{H}_{\text{eff}}^{\text{Ising}} = -\bar{J}_|| \sum_{i=1}^{L} \hat{S}_{i}^{z} \hat{S}_{i+1}^{z} \]  

(23)

respectively, where \( \hat{S}_{L+1} = \hat{S}_{1} \) and

\[ \hat{Z}_{i,i+1} = \hat{S}_{i+1} - \hat{S}_{i} - \hat{S}_{i+2} - \hat{S}_{i+1}. \]  

(24)
Let us denote each element of $\mathcal{H}^{\text{sing}}$ as

\[
    |\Psi_m\rangle = \bigotimes_{i=1}^{L} |m_i\rangle = |m_1, m_2, \ldots, m_L\rangle,
\]

where $|m_j\rangle$ is either of the local eigenstates $|\uparrow\rangle$, $|\downarrow\rangle$ defined by

\[
    \hat{S}_j^z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle, \quad \hat{S}_j^z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle
\]

and $m = (m_1, m_2, \ldots, m_L)$ in the label of the state $|\Psi_m\rangle$. In this case, the condition in Eq. (20) for $|\Psi\rangle$ is equivalent to the condition

\[
    \hat{b}_{i,i+1}(A_\lambda) |\Psi_m\rangle = 0
\]

for every bond $(i, i+1)$, where we introduced the effective bond operator

\[
    \hat{b}_{i,i+1}(A) := \hat{S}_i^+ \mathcal{J}_0(A\hat{Z}_{i,i+1})\hat{S}_{i+1}^- + (+ \leftrightarrow -)
\]

Since the bond operator $\hat{b}_{i,i+1}$ acts on the four spins at sites $i-1, i, i+1,$ and $i+2$, we can focus on the 16 pieces of four-site states

\[
    |\psi_{i,i+1}\rangle = |m_{i-1}, m_i, m_{i+1}, m_{i+2}\rangle,
\]

out of the whole state $|\Psi_m\rangle$ given by

\[
    |\Psi_m\rangle = |m_1 \ldots m_{i-2}\rangle \otimes |\psi_{i,i+1}\rangle \otimes |m_{i+3} \ldots m_L\rangle,
\]

reducing the condition in Eq. (27) to

\[
    \hat{b}_{i,i+1}(A_\lambda) |\psi_{i,i+1}\rangle = 0.
\]

First, we can classify the 16 four-site states into two groups: a group of 8 states satisfying $|m_{i+1}\rangle = |\uparrow\rangle$ and a group of 8 states satisfying $|m_{i+1}\rangle = |\downarrow\rangle$. For any state $|\psi_{i,i+1}\rangle$ in the former group, it holds that

\[
    \hat{b}_{i,i+1}(A) |\psi_{i,i+1}\rangle = \hat{S}_i^+ \mathcal{J}_0(A\hat{Z}_{i,i+1})\hat{S}_{i+1}^- |\psi_{i,i+1}\rangle.
\]

In this case, since $\hat{S}_i^+ |\psi_{i,i+1}\rangle$ is also one of the 16 four-site states, it is an eigenstate of $\mathcal{J}_0(A\hat{Z}_{i,i+1})$, and thus we have

\[
    \hat{S}_i^+ \mathcal{J}_0(A\hat{Z}_{i,i+1})\hat{S}_{i+1}^- |\psi_{i,i+1}\rangle \propto \hat{S}_i^+ \hat{S}_{i+1}^- |\psi_{i,i+1}\rangle
\]

with the corresponding eigenvalue as the proportionality factor. In the same way, for any state $|\psi_{i,i+1}\rangle$ in the latter group, since it holds that

\[
    \hat{b}_{i,i+1}(A) |\psi_{i,i+1}\rangle = \hat{S}_i^- \mathcal{J}_0(A\hat{Z}_{i,i+1})\hat{S}_{i+1}^+ |\psi_{i,i+1}\rangle
\]

and since $\hat{S}_i^- |\psi_{i,i+1}\rangle$ is an eigenstate of $\mathcal{J}_0(A\hat{Z}_{i,i+1})$, we have

\[
    \hat{S}_i^- \mathcal{J}_0(A\hat{Z}_{i,i+1})\hat{S}_{i+1}^+ |\psi_{i,i+1}\rangle \propto \hat{S}_i^- \hat{S}_{i+1}^+ |\psi_{i,i+1}\rangle.
\]

Putting these two together, we obtain

\[
    \hat{b}_{i,i+1}(A) |\psi_{i,i+1}\rangle = \mathcal{J} \times \hat{b}_{i,i+1}(0) |\psi_{i,i+1}\rangle
\]

with the proportionality coefficient $\mathcal{J}$.

We then classify the 16 four-site states further into three groups (also see Fig. 2):

\[
    \mathfrak{b}_0 := \{ |\uparrow\downarrow\downarrow\downarrow\rangle, |\uparrow\downarrow\downarrow\uparrow\rangle, |\downarrow\downarrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle \},
\]

\[
    \mathfrak{b}_1 := \{ |\uparrow\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\downarrow\rangle, |\downarrow\downarrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\uparrow\rangle \},
\]

\[
    \mathfrak{b}_\infty := \{ |m_{i-1} \uparrow\uparrow m_{i+2}\rangle, |m_{i-1} \downarrow\downarrow m_{i+2}\rangle \} \cup \{ |\uparrow\downarrow m_{i-1}, m_{i+2}=\uparrow\downarrow\rangle \}.
\]

The behavior of $\mathcal{J}$, $\hat{b}_{i,i+1}(0) |\psi_{i,i+1}\rangle$, and $\mathcal{J} \times \hat{b}_{i,i+1}(0) |\psi_{i,i+1}\rangle$ on the right-hand side of Eq. (36) is shown in Table 1. Therefore, setting $A = A_\lambda$ yields

\[
    \hat{b}_{i,i+1}(A_\lambda) |\psi_{i,i+1}\rangle = \begin{cases} \mathfrak{b}_0 & |\psi_{i,i+1}\rangle \in \mathfrak{b}_0 \oplus \mathfrak{b}_\infty \setminus \mathfrak{b}_1 \\ \mathfrak{b}_1 & |\psi_{i,i+1}\rangle \in \mathfrak{b}_1 \\ 0 & \text{otherwise} \end{cases}
\]

Let us consider the following two product states:

\[
    |A_0\rangle = |\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\rangle, \quad |A_1\rangle = |\uparrow\uparrow\downarrow\downarrow\rangle
\]

on an $L = 16$ finite-size chain with the periodic boundary conditions. The former is a state with one domainwall at the bond $(8, 9)$ and is contained in $\mathfrak{b}_0^{\text{sing}}$ because $|\psi_{n,n+1}\rangle \in \mathfrak{b}_\infty \oplus \mathfrak{b}_0$ holds for every bond $(n, n+1) \in E$. The latter is a state in which the spins at the bond $(8, 9)$,
FIG. 2. Classification of $2^4 = 16$ possible Ising-like product states in four consecutive sites (a cluster). The three groups $b_0$, $b_1$, and $b_x$ shown in the figure correspond to the states given by Eqs. (37), (38), and (39), respectively. In each cluster state, the two sites enclosed in a dashed rectangle correspond to the bond $(i, i + 1)$.

TABLE I. The right-hand side of Eq. (36) for three cases of $|\psi_{i,i+1}\rangle \in b_0, b_1, b_x$.

| $J$ | $b_{i,i+1}(0) \psi_{i,i+1} \chi_{J \times b_{i,i+1}(0) \psi_{i,i+1}}$ |
|-----|---------------------------------------------------------------|
| $|\psi_{i,i+1}\rangle \in b_0$ | $J_0(A) \psi_{i,i+1} \in J_0(A) b_0$ |
| $|\psi_{i,i+1}\rangle \in b_1$ | $1 \in b_1 \in b_1$ |
| $|\psi_{i,i+1}\rangle \in b_x$ | It depends, 0 |

The position indicated by the underlines in Eqs. (41) and (42) are flipped and is contained in $S_{1}^{\text{sing}}$ because the bonds $(7,8)$ and $(9,10)$ satisfy $|\psi_7,9\rangle \in b_1$ and $|\psi_9,10\rangle \in b_1$.

These two states $|A_0\rangle$ and $|A_1\rangle$ only differ in the partial state at the bond $(8,9)$, which may be negligible in the thermodynamic limit $L \to \infty$. The slight difference, nevertheless, generates largely different time-evolution dynamics due to the fact that one state belongs to $S_0^{\text{sing}}$ and the other state belongs to $S_{1}^{\text{sing}}$. To explore this quantitatively, we calculate the value of $S_2^L(t) := \langle \Psi(t) | S_{2}^{\text{sing}} | \Psi(t) \rangle$ and the half-chain entanglement entropy per a lattice site given by

$$\sigma_{L/2}(t) := \frac{1}{L} \text{Tr} \left[ -\hat{\rho}_{L/2}(t) \log \hat{\rho}_{L/2}(t) \right].$$

where

$$\hat{\rho}_{L/2}(t) := \text{Tr}_{1 \leq n \leq L/2} \left[ |\Psi(t)\rangle \langle \Psi(t)| \right].$$

The state vector at time $t$ is given by

$$|\Psi(t)\rangle = \mathcal{T} \exp \left[ \int_0^t \frac{dt'}{i \hbar} \hat{H}(t') \right] |\Psi(0)\rangle,$$

which is numerically calculated by the package QuSpin [22, 23]. We set the parameters in the Hamiltonian $\hat{H}(t)$ for which the effective Hamiltonian $\hat{H}_{\text{eff}}(A\lambda)$ well approximates the dynamics and the condition $J_0(A\lambda) \simeq 0$ is satisfied.

Figure 3 shows the results of the spatial profile of $S^L_2(t)$ at several different times for the initial states $|A_0\rangle$ and $|A_1\rangle$ and for the frequencies $\Omega = 10, 8, 6, 4$. These results demonstrate that the slight difference of the spin configuration in the initial states $|A_0\rangle$ and $|A_1\rangle$ makes a significant difference in the time evolution; the spin configuration remains robust and almost intact for $|A_0\rangle$ at the beginning, but decays quickly for $|A_1\rangle$. Although the initial spin configuration for $|A_0\rangle$ is also eventually destroyed, the robustness of $|A_0\rangle$ is a direct consequence of the fact that the initial state $|A_0\rangle$ is at the very vicinity of a fixed point of the dynamics.

Figure 4 shows the time evolution of $\sigma_{L/2}(t)$ for the same set of the initial states and for the same frequencies $\Omega$. Figure 4(a) suggests that when the initial state satisfies $\mathcal{H}_{\text{eff}}^{XY}(A\lambda) |\Psi\rangle \simeq 0$, the initial state essentially remains intact until a certain time that depends on $\Omega$ and can be delayed as late as possible with increasing $\Omega$. This feature of $\Omega$-dependent relaxation found in Figs. 4(a) and 4(b) is similar to the prethermalization and subsequent relaxation to the infinite temperature state reported in some Floquet systems.

However, as reported in Ref. [24], the value of entanglement entropy corresponding to the infinite temperature state usually takes the universal value $\sigma_{L/2} \sim \ln 2 - 1/L$. This is different form the values of entanglement entropy estimated at the largest time available in Fig. 4, which are in fact both far from that value. Besides, there is no guarantee that these values of the entanglement entropy at the largest time in Fig. 4 is actually the value after the full relaxation. Thus, one of two possible conclusions can be drawn from our results: either a new and unprecedented relaxation phenomenon occurs, or no relaxation is occurring at all. Verifying this would require additional computational resources and more extended simulations on larger systems. Therefore, we leave this issue in a future study.

We also perform the same numerical calculations for the following two product states as the initial states:

$$|B_0\rangle = |↓↓↑↑↑↑↑↑↓↓↓↓↓↓↑↑↑↑\rangle,$$

$$|B_1\rangle = |↓↓↓↓↑↑↑↑↓↓↑↑↑↑↑↑\rangle.$$

These states also form a set of states where $|B_0\rangle$ belongs to $b_0$ and $|B_1\rangle$ belongs to $b_1$ with a slight difference at the bond $(8,9)$, indicated by the underlines in Eqs. (46) and (47). Figure 5 shows the numerical results for the spatial profile of $S_2^L(t)$ at three different times for the initial states $|B_0\rangle$ and $|B_1\rangle$ and for the frequencies $\Omega = 10, 8, 6, 4$. Figure 6 shows the time evolution of $\sigma_{L/2}(t)$ for the same set of initial states and for the same frequencies $\Omega$. The behaviors shown in Figs. 5 and 6 are qualitatively the same as those in Figs. 3 and 4, respectively.

Our results imply that by simply flipping a pair of spins on a single bond we can switch the whole state back and forth between $S_{0}^{\text{sing}}$ and $S_{1}^{\text{sing}}$. Hence, it is also possible
especially in the one-dimensional case with spin $S$ Ising-like product states are fixed points of the dynamics. We have shown that driven longitudinal exchange interactions in the Floquet picture, and transverse interactions reduce to four-site interactions in the $A$ model lead to long-range transverse exchange interactions in the Floquet picture, and result in a state-selective localization in which limited Ising-like product states are fixed points of the dynamics. Especially in the one-dimensional case with spin $S = 1/2$ and arbitrary length $L$, we have shown the long-range transverse interactions reduce to four-site interactions and specified the condition for a given Ising-like product state to be the fixed point. We have also shown some examples of such a localizable product state and numerically verified that these are actually fixed points of the dynamics by demonstrating quantitatively different dynamics for two different initial product states with only a slightly different spin configuration on a single bond.

The driving protocol presented in this study may be experimentally realized in magnetic insulators by a technique of controlling magnetism with ultrafast electric fields [25]. In addition to such experimental approaches,
it is also interesting to investigate the state-selective localization as a quantum scar [20, 21], which prevents thermal equilibration in isolated quantum many-body systems. The state-selective localization proposed in this study has a common feature to the quantum scar in that the dynamics is strongly dependent on the initial state. It is desirable that the model given in this study is properly extended to describe experimental situations and to investigate the quantum scar in more details.

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Appendix A: Derivation of $\hat{H}_{\text{eff}}(A)$ in Eq. (14)

In this appendix, we present a detailed derivation of $\hat{H}_{\text{eff}}(A)$ in Eq. (14). First, we decompose the original time-dependent Hamiltonian $\hat{H}(t)$ given in Eq. (11) into $\hat{H}(t) = \hat{H}_0 + \hat{V}(t)$ such that $\int_0^{2\pi/\Omega} dt \, \hat{V}(t) = 0$ holds. Then, for

$$\hat{H}_0 = -\frac{J_z}{2} \sum_{\langle i,j \rangle} (\hat{S}_i^+ \hat{S}_j^- + \hat{S}_i^- \hat{S}_j^+) - \vec{J}_\parallel \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z \quad (A1)$$

and

$$\hat{V}(t) = -\delta J \cos \Omega t \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z, \quad (A2)$$
we find the unitary transformation in Eq. (9) in the form
\[
\hat{U}(t) = \mathcal{T} \exp \left[ - \int^t dt' \frac{d}{dt} \hat{\mathcal{V}}(t') \right]
\]
where \( \mathcal{T} \) is the time-ordering operator. Thus, we find Eq. (14) by substituting it into the formula
\[
\hat{H}_{\text{eff}}(A) = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt \hat{U}(t) \hat{H}_{\text{eff}} \hat{U}^\dagger(t).
\]

Let us thereby evaluate \( \hat{U}(t) \hat{H}_{\text{eff}} \hat{U}^\dagger(t) \). Because of the commutation relation \([\hat{S}_z, \hat{U}(t)] = 0\), we obtain
\[
\hat{U}(t) \hat{H}_{\text{eff}} \hat{U}^\dagger(t) = -\frac{J}{2} \sum_{\langle i,j \rangle} \hat{U}(t) \left( \hat{S}_j^+ \hat{S}_j^- + \hat{S}_j^- \hat{S}_j^+ \right) \hat{U}(t) - \hat{J}_z \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z.
\]

Furthermore, using the formula
\[
e^{-i\hat{A}} \hat{B} e^{i\hat{A}} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left( \text{ad} \hat{A} \right)^n \hat{B},
\]
where \( \text{ad} \hat{A} \cdot \hat{B} := [\hat{A}, \hat{B}] \), for any pair of operators \( \hat{A} \) and \( \hat{B} \), one of the terms in the parentheses in Eq. (A7) can be written as
\[
\hat{U}(t) \hat{S}_j^+ \hat{S}_j^- \hat{U}(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left( A \sin \Omega t \right)^n \left( \text{ad} \sum_{\langle k,l \rangle} \hat{S}_k^z \hat{S}_l^z \right)^n \left( \hat{S}_j^+ \hat{S}_j^- \right)
\]
which \( \hat{H}_{\text{eff}} \) is defined in Eq. (15).

We can prove the equality between Eqs. (A10) and (A11) by induction as follows. If we accept
\[
\left( \text{ad} \sum_{\langle k,l \rangle} \hat{S}_k^z \hat{S}_l^z \right)^m \left( \hat{S}_j^+ \hat{S}_j^- \right) = \hat{S}_j^+ \left( \hat{Z}_{i,j} \right)^m \hat{S}_j^-
\]
for a non-negative integer \( m \), we obtain
\[
\left( \text{ad} \sum_{\langle k,l \rangle} \hat{S}_k^z \hat{S}_l^z \right)^{m+1} \left( \hat{S}_j^+ \hat{S}_j^- \right) = \sum_{\langle k,l \rangle} \left[ \hat{S}_k^z \hat{S}_l^z, \hat{S}_j^+ \left( \hat{Z}_{i,j} \right)^m \hat{S}_j^- \right] \]
\[
= 2x \hat{S}_j^+ \left[ \sum_{\langle k,l \rangle} \left( \delta_{l,i} - \delta_{l,j} \right) \hat{S}_k^z \right] \left( \hat{Z}_{i,j} \right)^m \hat{S}_j^- = \hat{S}_j^+ \left( \hat{Z}_{i,j} \right)^{m+1} \hat{S}_j^-.
\]

For the equality between Eq. (A15) and Eq. (A16), we use the symmetry of exchanging indices \( k \) and \( l \). For the equality between Eq. (A16) and Eq. (A17), we use the decomposition of the sum given by
\[
2 \times \sum_{\langle k,l \rangle} \hat{O}_{k,l} = \sum_k \sum_{\langle i,j \rangle} \hat{O}_{k,l},
\]
where \( \sum_{\langle i,j \rangle} \hat{O}_{k,l} \) denotes summation of \( \hat{O}_{k,l} \) for all sites \( l \) that are connected to site \( k \). Since Eq. (A13) trivially holds for \( m = 0 \), it holds for any non-negative integers.

In exactly the same way, we obtain
\[
\hat{U}(t) \hat{S}_i^x \hat{S}_i^y \hat{U}(t) = \hat{S}_i^x \exp \left[ +iA \sin \Omega t \times \hat{Z}_{i,j} \right] \hat{S}_i^y.
\]

Hence, we find
\[
\hat{U}(t) \hat{H}_{\text{eff}} \hat{U}^\dagger(t) = -\frac{J}{2} \sum_{\langle i,j \rangle} \left[ \hat{S}_j^+ e^{-iA \sin \Omega t \times \hat{Z}_{i,j}} \hat{S}_j^- + (\leftrightarrow) \right] - \hat{J}_z \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z.
\]
in Eq. (A20), we arrive at

\[ \hat{\mathcal{H}}_{\text{eff}}(A) = \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt \, \mathcal{U}(t) \hat{\mathcal{H}}_0 \mathcal{U}(t) \]

\[ = -\frac{J}{2} \sum_{\langle i,j \rangle} \left[ \hat{S}_i^+ \mathcal{J}_0(A \hat{Z}_{i,j}) \hat{S}_j^- + (+ \leftrightarrow -) \right] - J_\parallel \sum_{\langle i,j \rangle} \hat{S}_i^z \hat{S}_j^z, \]

where \( \mathcal{J}_0(x) \) is the zeroth-order Bessel function of the first kind.

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