On Homological and Homotopical Algebra of Supersymmetries and Integrability in String Theory

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Abstract

The text contains introduction and preliminary definitions and results to my talk on category theory description of supersymmetries and integrability in string theory. In the talk I plan to present homological and homotopical algebra framework for Calabi-Yau supermanifolds and stacks in open and closed string theory. In the framework we investigate supersymmetries and integrability.

1 Introduction

In this paper we plan to discuss homological and homotopical algebra description of supersymmetries and integrability in string theory. Recall that the Picard groupoid on an open set $U$ is the category whose objects are line bundles on $U$ and whose morphisms are isomorphisms. A gerbe is a stack over a topological space which is locally isomorphic to the Picard groupoid of the space. A description of a stack or a gerbe in terms of an atlas with relations is known as a presentation. Every stack has a presentation of the form of a global quotient stack $[X/G]$, for some space $X$ and some group $G$. However, presentations are not unique. A given stack can have many different presentations of the form $[X/G]$. Review very shortly some selected works in the directions. Authors of the paper[1] investigate gauged linear sigma model descriptions of toric stacks. The paper includes (i) the gauged linear sigma model (GLSM) description of toric stacks; (ii) a describing of the physics of GLSM; (iii) checking that physical predictions of those GLSM exactly match the corresponding stacks. The description of Deligne-Mumford stacks over toric varieties (toric stacks) is given with the help of stacky fans by authors of [2]. It is well known that the GLSM is closely related to toric geometry. Some ground facts can be found in the book [3]. In the absence of a superpotential, the set of supersymmetric ground states of the GLSM is a toric variety. Conversely, toric varieties can be described as the set of ground states of an appropriate gauged linear sigma model. String theories on orbifolds were intoduced by Dixon, Harvey, Vafa, and Witten in [4].

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Recently very interesting results were obtained in the local theory of current-algebraic orbifolds\[5\].

Some integrable models are connected with solitons. Bogomolnyi vortices are static, topologically stable, finite energy solution of the critically coupled Abelian Higgs model in 2 + 1 dimensions \[9, 10\]. Monopoles are solitons in three dimensions. The Yang-Mills equations are differential equations which are hyperbolic in Minkowski space-time, but have elliptic counterparts when Minkowski space-time is replaced by Euclidian space. In both (hyperbolic and elliptic) cases soliton type solutions (magnetic monopoles and instantons respectively) have been extensively studied \[11, 12\]. A class of $D$--branes for the type $IIB$ plane-wave background as well as classical description of the $D$--instanton in the light-cone gauge can be found in \[13\].

There is a correspondence between rational conformal field theories of $SU(2)$--type \[6\] and double triangle algebra associated to an ADE graph \[7, 8\].

After the introductory section and preliminary results we plan to consider next topics in the framework of Calabi-Yau supermanifolds and stacks in open and closed string theory:

- Homological algbra of connections on manifolds and variation of Hodge structure
- Homotopical categories
- PROPs and Operads
- $A_\infty$--categories
- $A_\infty$--functors

2 Examples, Definitions and Previous results

2.1 Superalgebras and their representations

One of the simplest example of superalgebra\[15\] is connected with supersymmetric oscillator. Let $n_B, n_F$ be bosonic and fermionic infill numbers. State vectors of the system are defined by

$$|n_B, n_F\rangle, n_B = 0, 1, 2, \ldots, \infty; n_F = 0, 1.$$  

Let $b^+, b^-, f^+, f^+$ be creation and annihilation bosonic and fermionic operators respectively. Operators act on state vectors $|n_B, n_F\rangle$ and vary infill numbers $n_B, n_F$ by standard manner \[16\]. Operators $b^+, b^-, f^+, f^+$ satisfy following commutator and anticommutator relations:

$$[b^+, b^-] = 1, f^+, f^- = 1, (f^+)^2 = (f^-)^2 = 0, [b, f] = 0.$$
Let $Q_+, Q_-$ operators that transform a boson to a fermion and vice versa. In previous notations

$$Q_+ = q b^+ f^-, Q_- = q b^+ f^-,$$

where $q$ is an arbitrary constant. Hermitian operators

$$Q_1 = Q_+ + Q_-, Q_2 = i(Q_+ + Q_-),$$

satisfy anticommutator relation

$$\{Q_1, Q_2\} = 0.$$

The supersymmetric Hamiltonian is defined by

$$H = Q_1^2 = Q_2^2 = \{Q_+, Q_-\},$$

It satisfies relation

$$[H, Q] = 0,$$

where $Q$ is one of $Q_+, Q_-, Q_1, Q_2$. The Lie superalgebra is defined by relations

$$\{Q_l, Q_k\} = 2 \delta_{lk} H, [Q_k, H] = 0.$$

### 2.2 Categories of algebras

Let $k$ be a field, $\mathcal{A}lg_k$ the category of algebras over $k$. Objects of $\mathcal{A}lg_k$ are $k$-algebras and morphisms are $k$-homomorphisms of the algebras. $\mathcal{A}lg_k$ contains many subcategories. Let $\mathcal{G}k$ be the category of finite groups, $\mathcal{GA}lg_k$ the category of group $k$-algebras. Let $\mathcal{M}k$ the category of all square matrices over $k$. Objects of $\mathcal{M}k$ are natural numbers and set of morphisms $Hom(n)$ is the set of all square matrices $k^{n \times n}$.

**Grassmanian algebras.**

Let $GA(k)$ be the Grassmanian algebra over $k$ generated by the set $e_1, e_2, \ldots$. Each element $l \in GA(k)$ may be represented as a finite sum of an element $a \in k$ and elements $e_1 \land \ldots \land e_r$, $r > 1$, with coefficients in $k$. The mapping $l \mapsto f(l) = a$ defines an epimorphism $f : GA(k) \to k$. The monomial $e_1 \land \ldots \land e_r$ is called even or odd depending on the parity of $r$. The monomial 1 is considered to be even. Linear combinations of even (odd) monomials with coefficients in $k$ form the set $GA_e$ of even elements (the set $GA_o$ of odd elements) of the algebra $GA(k)$.

**Hopf algebras.**

Let $\mathcal{H}$ be an algebra with unit $e$ over field $k$. Let $a, b \in \mathcal{H}$, $\mu(a, b) = ab$ the product in $\mathcal{H}$, $\alpha \in k$, $p : k \to \mathcal{H}$, $p(\alpha) = \alpha e$ the unit, $\varepsilon : \mathcal{H} \to k$, $\varepsilon(a) = 1$, the counit, $\Delta(a) = a \otimes a$ the coproduct, $S : \mathcal{H} \to \mathcal{H}$ the antipode, such that the axioms
1) \((1 \otimes \Delta) \circ \Delta = (\Delta \otimes 1) \circ \Delta\) (coassociativity);
2) \(\mu \circ (1 \otimes S) \circ \Delta = \mu \circ (S \otimes 1) \circ \Delta = p \circ \varepsilon\) (antipode).

Then the system \((H, \mu, \eta, \Delta, S)\) is called the Hopf algebra.

Let \(A\) and \(B\) be \(C^*\)-algebras. Let \(K\) be the \(C^*\)-algebra of compact operators. If \(A \otimes K\) is isomorphic to \(B \otimes K\) then \(A\) is called Morita equivalent to \(B\).

2.2.1 \(A_\infty\)–algebras

Strong homotopy algebras have been introduced by Stasheff. Strong homotopy algebras, or shortly \(A_\infty\)–algebras have been investigated in the context of algebras and superalgebras by Gugenheim, Stasheff, Penkava, Schwarz, Kontsevich, Barannikov, Merkulov and others.

2.3 SHEAVES AND MANIFOLDS

In the section we follow to [17, 18, 19, 20, 21]. At first collect some notions related to ringed spaces.

The ringed space is the pair \((X, \mathcal{O})\), where \(X\) is a topological space and \(\mathcal{O}\) is a sheaf of rings on \(X\). For a ringed space \((X, \mathcal{O}_X)\) and an open \(U \subset X\) the restriction of the sheaf \(\mathcal{O}_X\) on \(U\) defines the ringed space \((U, \mathcal{O}_X|_U)\).

Let \(\mathcal{O}_X\) be the sheaf of smooth functions on a topological space \(X\). Then any smooth manifold \(X\) with the sheaf \(\mathcal{O}_X\) is a ringed space \((X, \mathcal{O}_X)\).

**Example** : Let \(X\) be a Hausdorff topological space and \(\mathcal{O}_X\) a sheaf on \(X\). Let it satisfies conditions: (i) \(\mathcal{O}_X\) is the sheaf of algebras over \(C\); (ii) \(\mathcal{O}_X\) is a subsheaf of the sheaf of continuous complex valued functions. Let \(W\) be a domain in \(C^n\) and \(\mathcal{O}_{an}\) the sheaf of analytical functions on \(W\). The ringed space \((X, \mathcal{O}_X)\) is called the complex analytical manifold if for any point \(x \in X\) there exists a neighbourhood \(U \ni x\) such that \((U, \mathcal{O}_X|_U) \simeq (W, \mathcal{O}_{an})\) (here \(\simeq\) denotes the isomorphism of ringed spaces).

Let \(X\) be a projective algebraic variety over \(C\) and \(\text{Coh}_X\) the category of coherent sheaves on \(X\).

**Example** : If \(X\) is the algebraic curve then indecomposable objects of \(\text{Coh}_X\) are skyscrapers and indecomposable vector bundles.

2.3.1 Supermanifolds

Let \(k^{\text{dim}}\) be the linear superspace of all sequences \((z_1, \ldots, z_n, \xi_1, \ldots, \xi_m)\), where \(z_i\) and \(\xi_j\) are even and odd coordinates respectively. It is possible to represent coordinates \(z_i\) and \(\xi_j\) by different elements of sets \(GA_e\) and \(GA_o\) respectively. In the case the mapping \(f((z_1, \ldots, z_n, \xi_1, \ldots, \xi_m)) = (f(z_1), \ldots, f(z_n))\) defines
a morphism \( f : k^{n|m} \to k^n \). There are several possible ways of defining superspaces and constructing supermanifolds and supervarieties. For instance one way of constructing Riemann supervarieties is to use superspace \( H_1 = \{(z, \xi) \in \mathbb{C}^{1|1} | \text{Im}(f(z)) > 0 \} \) or superspace \( H_2 = \{(z, \xi_1, \xi_2) \in \mathbb{C}^{1|2} | \text{Im}(f(z)) > 0 \} \) and corresponding super-Fuchsian groups. As we consider mainly the complex case \( k = \mathbb{C} \), recall the P. Deligne’s definition: the complex superspace is the space \( X \) locally ringed by a sheaf of commutative superalgebras \( \mathcal{O}_X = \mathcal{O}_0 \oplus \mathcal{O}_1 \) such that the pair \( (X, \mathcal{O}_0) \) is the usual complex space and \( \mathcal{O}_1 \) is a coherent sheaf of \( \mathcal{O}_0 \)-modules.

2.3.2 Vector Bundles over Projective Algebraic Curves

Let \( X \) be a projective algebraic curve over algebraically closed field \( k \) and \( g \) the genus of \( X \). Let \( \mathcal{VB}(X) \) be the category of vector bundles over \( X \). Grothendieck have shown that for a rational curve every vector bundle is a direct sum of line bundles. Atiyah have classified vector bundles over elliptic curves. The main result is

**Theorem** [22]. Let \( X \) be an elliptic curve, \( A \) a fixed base point on \( X \). We may regard \( X \) as an abelian variety with \( A \) as the zero element. Let \( \mathcal{E}(r, d) \) denote the the set of equivalence classes of indecomposable vector bundles over \( X \) of dimension \( r \) and degree \( d \). Then each \( \mathcal{E}(r, d) \) may be identified with \( X \) in such a way that

\[
\det : \mathcal{E}(r, d) \to \mathcal{E}(1, d)
\]

corresponds to \( H : X \to X \),

where \( H(x) = hx = x + x + \cdots + x \ (h \text{ times}) \), and \( h = (r, d) \) is the highest common factor of \( r \) and \( d \).

Curve \( X \) is called a *configuration* if its normalization is a union of projective lines and all singular points of \( X \) are simple nodes. For each configuration \( X \) can assign a non-oriented graph \( \Delta(X) \), whose vertices are irreducible components of \( X \), edges are its singular and an edge is incident to a vertex if the corresponding component contains the singular point. Drozd and Greuel have proved:

**Theorem** [23]. 1. \( \mathcal{VB}(X) \) contains finitely many indecomposable objects up to shift and isomorphism if and only if \( X \) is a configuration and the graph \( \Delta(X) \) is a simple chain (possibly one point if \( X = \mathbb{P}^1 \)).

2. \( \mathcal{VB}(X) \) is tame, i.e. there exist at most one-parameter families of indecomposable vector bundles over \( X \), if and only if either \( X \) is a smooth elliptic curve or it is a configuration and the graph \( \Delta(X) \) is a simple cycle (possibly, one loop if \( X \) is a rational curve with only one simple node).

3. Otherwise \( \mathcal{VB}(X) \) is wild, i.e. for each finitely generated \( k \)-algebra \( \Lambda \) there exists a full embedding of the category of finite dimensional \( \Lambda \)-modules into \( \mathcal{VB}(X) \).
2.4 Homological algebra of connections on manifolds and variation of Hodge structure

The section follows ideas and results by Grothendieck, Griffiths, Manin, Katz, Deligne and others. Let $S/k$ be the smooth scheme over field $k$, $U$ an element of open covering of $S$, $\mathcal{O}_S$ the structure sheaf on $S$, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on $U$. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1—dimension differentials, $\mathcal{F}$ a coherent sheaf on $S$. The connection on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S)$, $g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma(U, \Theta^1_{S/k})$. The connection is the homomorphism

$$\rho : \Theta^1_{S/k} \to \text{End}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}),$$

$$\rho(\partial)(fg) = \partial(f)g + f\rho(\partial).$$

2.4.1 Integration of connection

Let $\Omega^i_{S/k}$ be the sheaf of germs of $i$—differentials,

$$\nabla^i(\alpha \otimes f) = d\alpha \otimes f + (-1)^i\alpha \wedge \nabla(f).$$

Then $\nabla$, $\nabla^i$ define the sequence of homomorphisms:

$$\mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F} \to \Omega^2_{S/k} \otimes \mathcal{F} \to \cdots,$$

(1)
The map $K = \nabla \circ \nabla^1 : \mathcal{F} \to \Omega^2_{S/k} \otimes \mathcal{F}$ is called the curvature of the connection $\nabla$.

The cochain complex

$$(K^\bullet, d) = \{K^0 \overset{d}{\to} K^1 \overset{d}{\to} K^2 \overset{d}{\to} \cdots\}$$

is the sequence of abelian groups and differentials $d : K^p \to K^{p+1}$ with the condition $d \circ d = 0$. A connection is integrable if (1) is a complex.

**Proposition.** The statements a), b), c) are equivalent:

a) the connection $\nabla$ is integrable;

b) $K = \nabla \circ \nabla^1 = 0$;

c) $\rho$ is the Lie-algebra homomorphism of sheaves of Lie algebras.
2.5 Topological and Homotopical categories

Let $X$ be a compact smooth manifold with boundary $\partial X$. Let $\partial X = \partial X_0 \cup \partial X_1$ be a partition such that $\partial X_0$ and $\partial X_1$ are integer components of $\partial X$. In the case the manifold $X$ is called the *cobordism* with the source $\partial X_0$ and target $\partial X_1$. The case $\partial X_0 = \emptyset$ and $\partial X_1 = \emptyset$ means that $X$ is closed. Let $M X$ be the class of $n$–dimensional compact smooth manifolds with boundary and $OX$ the class of the connected components of the boundaries. From these data form the category $\mathbf{CB}_n$ whose objects are connected components of the boundaries of $n$–dimensional manifolds. For any two boundaries $\partial Y_0$ and $\partial Y_1$ a morphism between them is a manifold $Y \in M X$ such that $\partial Y = \partial Y_0 \cup \partial Y_1$. If we regard two morphisms in $\mathbf{CB}_n$ as equivalent if they are homotopic then we can form the quotient category $\mathbf{hCB}_n$. Respectively for a given compact smooth manifold $X$ with boundary it is possible to form a category $\mathbf{CB}_X$ whose morphisms are $n$–dimensional submanifolds of $X$ and whose objects are connected components of boundaries of such submanifolds. The category $\mathbf{hCB}_X$ have the same objects as $\mathbf{CB}_X$ and morphisms of $\mathbf{hCB}_X$ are homotopy equivalent classes of morphisms of $\mathbf{CB}_X$.

Recall now the relevant properties of complexes, derived categories, cohomologies and quasimorphisms referring to [18, 19] for details and indication of proofs.

Let $\mathcal{A}$ be an abelian category, $\mathcal{K}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. Furthermore, there are various full subcategories of $\mathcal{K}(\mathcal{A})$ whose respective objects are the complexes which are bounded below, bounded above, bounded in both sides. The notions of homotopy morphism, homotopical category, triangulated category are described in [19]. The bounded derived category $D^b(X)$ of coherent sheaves on $X$ has the structure of a triangulated category [19].

2.5.1 The Fourier-Mukai transform

Let $A$ be an abelian variety and $\hat{A}$ the dual abelian variety which is by definition a moduli space of line bundles of degree zero on $A$. The *Poincaré bundle* $\mathcal{P}$ is a line bundle of degree zero on the product $A \times \hat{A}$, defined in such a way that for all $a \in \hat{A}$ the restriction of $\mathcal{P}$ on $A \times \{a\}$ is isomorphic to the line bundle corresponding to the point $a \in A$. This line bundle is also called the *universal bundle*. Let

$$\pi_A : A \times \hat{A} \to A,$$

$$\pi_{\hat{A}} : A \times A \to \hat{A},$$

and $\mathcal{C}_A$ be the category of $O_A$–modules over $A$, $M \in \text{Ob } \mathcal{C}_A$,

$$\hat{S}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$
Then, by definition, the Fourier-Mukai transform $\mathcal{FM}$ is the derived functor $R\hat{S}$ of the functor $\hat{S}$. Let $\mathcal{D}(A)$, $\mathcal{D}(\hat{A})$ be bounded derived categories of coherent sheaves on $A$ and $\hat{A}$ respectively.

**Theorem** (Mukai.) The derived functor $\mathcal{FM} = R\hat{S}$ induces an equivalence of categories between two derived categories $\mathcal{D}(A)$ and $\mathcal{D}(\hat{A})$.

### 2.6 PROPs, Operads, $A_{\infty}$–categories and $A_{\infty}$–functors

PROPs was introduced by Mac Lane and Adams. Their topological versions was defined by Bordman and explored by Bordman and Vogt [24]. An interval PROP $P$ is a category with objects $0, 1, 2, \ldots$ satisfying the conditions from [24] and is defined on interval manifolds. It has representation by generalized trees that are defined and take values on interval manifolds. Interval PROPs have $A_{\infty}$–category properties.

Operads was introduced by J. May [25]. Here we give a short description of interval operad. The space of continuous interval functions of $j$ variables forms the topological space $I^C(j)$. Its points are operations $I^C_0(0,1,2,\ldots)$ that are defined and take values on interval manifolds. Let $X \in I^U$ and for $k \geq 0$ let $I^E(k)$ be the space of maps $M(X^k, X)$. There is the action (by permuting the inputs) of the symmetric group $S_k$ on $I^E(k)$. The identity element $1 \in I^E(1)$ is the identity map of $X$. In the above mentioned conventions let $k \geq 0$ and $j_1, \ldots, j_k \geq 0$ be integers. Let for each choice of $k$ and $j_1, \ldots, j_k$ there is a map

$$\gamma : I^E(k) \times \cdots \times I^E(j_1) \times \cdots \times I^E(j_k) \to I^E(j_1 + \cdots + j_k)$$

given by multivariable composition. If maps $\gamma$ satisfy associativity, equivalence and unital properties then $I^E$ is the endomorphism interval operad $I^E_X$ of $X$.

Here all categories will be assumed to be *linear* over $k$ and small. A *non-unital* $A_{\infty}$–category $A$ consists of a set of objects $\text{Ob} A$, a graded vector space $\text{hom}_A(X_i, X_j)$ for any pair of objects, and composition maps of every order $d \geq 1$,

$$\mu^A_d : \text{hom}_A(X_{d-1}, X_d) \otimes \cdots \otimes \text{hom}_A(X_0, X_1) \to \text{hom}_A(X_0, X_d)[2 - d],$$

where $[s]$ means shifting the grading of a vector space down by $s \in \mathbb{Z}$. The maps must satisfy the (quadratic) $A_{\infty}$–associativity equations [24].

### 2.7 Integrability and foliations

Integrability is connected with foliations. A simplest example of foliation is a trivial $k$ dimensional foliation or a trivial codimension $n-k$ foliation of Euclidean space $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}, 0 \leq k \leq n$:

$$\mathbb{R}^n = \bigcup_{(x_{k+1}, \ldots, x_n)} \mathbb{R}^k \times (x_{k+1}, \ldots, x_n).$$
that is, $\mathbb{R}^n$ decomposes into a union of $\mathbb{R}^k \times (x_{k+1}, \ldots, x_n)$’s each of which is $C^\infty$ diffeomorphic to $\mathbb{R}^k$. $\mathbb{R}^k \times (x_{k+1}, \ldots, x_n)$ is called a leaf of the foliation. 

Still one example can be obtained from consideration of nonsingular vector fields on the torus. More generally, a nonsingular flow on a manifold corresponds to a foliation on the manifold by one dimensional leaves where the leaves are provided with Riemannian metrics and directed.

Let $(M, \mathcal{F})$ be a foliated manifolds, $(M/\mathcal{F})$ its leaf space, and $G = G(M, \mathcal{F})$ the holonomy groupoid of $(M/\mathcal{F})$ with $s, r$ the source and target maps. Recall follow to [26] some fact about foliation $C^\ast-$algebras and holonomy groupoids. Let $C_c(G)$ denote the space of continuous functions on $G$ with compact support.

It is possible define a convolution product on $C_c(G)$. Given $\gamma \in G$, $\phi \in C_c(G)$, the involution $\ast$ on $C_c(G)$ is defined as $\phi^\ast(\gamma) = \overline{\phi(\gamma^{-1})}$.

Denote by $x$ any element $M$. Let $L_x$ be the leaf that contains $x$ and $\tilde{L}_x = \{ \gamma \in G | s(\gamma) = x \}$. The target map $r$ yields the covering projection $r : L_x \to \tilde{L}_x$. Denote by $d\mu = \{ d\mu_x \}_{x \in M}$ the family of measures on $\tilde{L}_x$. Let $\mathcal{H}_x$ be the Hilbert space of $L^2-$functions on $\tilde{L}_x, \xi \in \mathcal{H}_x, \alpha, \beta \in \tilde{L}_x$. A $\ast-$representation $\pi_x$ on $\mathcal{H}_x$ is defined by

$$[\pi_x(\phi)\xi](\alpha) = \int_{\tilde{L}_x} \phi(\alpha\beta^{-1})\xi(\beta)d\mu_x(\beta).$$

Let $\| \phi \|$ be the operator norm of $\pi_x(\phi)$ on $\mathcal{H}_x$.

For a given foliated manifold $(M, \mathcal{F})$ the foliated $C^\ast-$algebra $C^\ast(M, \mathcal{F})$ is the $C^\ast-$completion of $C_c(G)$ with respect to the $\ast-$norm.

### 2.8 To TQFT

Let $M$ be an $n-$dimensional manifold and $\mathrm{hCB}_n$ the quotient category of $\mathrm{CB}_n$. A topological quantum field theory (TQFT) on $M$ is a symmetric monoidal functor from $\mathrm{hCB}_n$ to the category of vector spaces. A TQFT on $n-$dimensional manifolds is then a functor from $\mathrm{hCB}_n$ to the category of vector spaces, which takes disjoint unions of bordisms to the tensor product of corresponding vector spaces.

Let $AM$ be the associative algebra of functions on $M$ defined by the product of functions and $C^\ast(AM)$ the Hochschild complex. A $p-$cochain $C$ on $AM$ is a $p-$linear map of $AMP$ into $AM$. A 1-$cocycle is a derivation of the algebra, that is a vector field. The Hochschild coboundary of $C$ is the $(p - 1)-$cochain $\partial C$ and $\partial^2 = 0$. The complex $C^\ast(AM)$ has the structure of homotopic Gerstenhaber algebra. The proof is found in [28].

Let $H^p(AM; AM)$ be the $p-$th Hochschild cohomology space for the considered differeniat complex. By Vey’s result [29] $H^p(AM; AM)$ is isomorphic to the space of the antisymmetric contravariant $p-$tensors of $M$. 

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