Quantum stochastic models of two-level atoms and electromagnetic cross sections.

Alberto Barchielli  
Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano, Italy  
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano. E-mail: barchielli@mate.polimi.it

Giancarlo Lupieri  
Dipartimento di Fisica, Università degli Studi di Milano, Via Celoria 16, I-20133 Milano, Italy  
and Istituto Nazionale di Fisica Nucleare, Sezione di Milano  
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Quantum stochastic differential equations have been used to describe the dynamics of an atom interacting with the electromagnetic field via absorption/emission processes. Here, by using the full quantum stochastic Schrödinger equation proposed by Hudson and Parthasarathy fifteen years ago, we show that such models can be generalized to include other processes into the interaction. In the case of a two-level atom we construct a model in which the interaction with the field is due either to absorption/emission processes either to direct scattering processes, which simulate the interaction due to virtual transitions to the levels which have been eliminated from the description.

To see the effects of the new terms, we study various types of cross sections for the scattering of monochromatic coherent light. We obtain formulas giving the total, the elastic and the inelastic cross sections as functions of the frequency and the intensity of the stimulating laser and the fluorescence spectrum as a function also of the frequency of the scattered light. The total cross section, as a function of the frequency of the stimulating laser, can present not only a Lorentzian shape, but the full variety of Fano profiles; intensity dependent widths and shifts are obtained. The fluorescence spectrum can present complicated shapes, according to the values of the various parameters; when the direct scattering is not important the usual symmetric triplet structure of the Mollow spectrum appears (for high intensity of the stimulating laser), while a strong contribution of the direct scattering process can distort such a triplet structure or even make it disappear.

I. INTRODUCTION

Quantum stochastic calculus (QSC) [1–4], a noncommutative analog of the classical Ito’s stochastic calculus, revealed to be a powerful tool to construct mathematical models of quantum optical systems [5–12] and to develop a theory of photon detection [13–16]. Just at the beginning of QSC, Hudson and Parthasarathy proposed a quantum stochastic Schrödinger equation for quantum open systems [6,15,3]. Such an equation has found applications in quantum optics, but not in its full generality [6,15,3]. It has been used to give, at least approximately, the dynamics of photoemissive sources such as an atom absorbing and emitting light, or matter in an optical cavity, which exchanges light with the surrounding free space. But in these cases the possibility of introducing the so called gauge (or number) process in the dynamical equation has not been considered; roughly speaking, the gauge process is a quadratic expression in the field operators which preserves the number of quanta, but changes their wave functions. In this paper we want to show, in the case of the simplest photoemissive source, namely a two-level atom stimulated by a laser, how the full Hudson-Parthasarathy equation allows to describe in a consistent way the scattering of the light by the atom not only through the absorption/emission channel, but also through another process which can be called “direct scattering”, which can be included in the interaction via a term containing the gauge process. When the atom is approximated by a two-level system, the introduction of an interaction term which preserves the number of photons allows to simulate also the scattering processes involving virtual transitions to states different from the two ones responsible of the real absorption/emission process.

So, a first aim is to show how the full Hudson-Parthasarathy equation is able to give a reasonable and rich model for the dynamics of an atom interacting with the electromagnetic field. A second one will be the study of the elastic, inelastic and total cross sections for the scattering of monochromatic coherent light by the atom. The resulting line-shapes are very interesting. For instance, the dependence of the total cross section on the frequency of the stimulating laser can present not only a Lorentzian shape, but the full variety of Fano profiles [18, pp. 61–63]. Moreover, the dependence of the line shape on the intensity of the stimulating laser is computed and power broadening and intensity dependent shifts are found. The study of the inelastic cross section, instead, shows possible modifications to the known triplet structure of the fluorescence spectrum [19]. Some preliminary results on the total cross sections where reported in [20].
Let us recall some notions of QSC and the Hudson-Parthasarathy equation; this is just to fix our notations, while for the proper mathematical definitions and the rules of QSC we refer to the book by Parthasarathy [1]. We denote by $F = F(X)$ the Boson Fock space over the “one-particle space” $X = \mathcal{Z} \otimes L^2(\mathbb{R}_+ ; \mathcal{Z})$, where $\mathcal{Z}$ is another separable complex Hilbert space. A vector $f$ in $X$ is a function from $\mathbb{R}_+$ into $\mathcal{Z}$; we fix a c.o.n.s. $\{e_i, \ i \geq 1\}$ in $\mathcal{Z}$ and we denote by

$$f_j(t) = \langle e_j | f(t) \rangle$$

(1.1)

the components of a vector $f(t)$ in $\mathcal{Z}$. The Fock space $F$ is spanned by the exponential vectors $E(f)$, whose components in the $0, 1, \ldots , k, \ldots$ particles spaces are

$$E(f) = \left(1 , f , (2!)^{-1/2} \otimes f , \ldots , (k!)^{-1/2} \otimes f \otimes k , \ldots \right),$$

(1.2)

$f \in X$; the inner product between two exponential vectors is given by

$$\langle E(g) | E(f) \rangle = \exp \left( \langle g | f \rangle \right) = \exp \left[ \int_{-\infty}^{+\infty} \langle g(t) | f(t) \rangle \ dt \right]$$

$$= \exp \left[ \sum_j \int_{-\infty}^{+\infty} g_j(t) f_j(t) \ dt \right],$$

(1.3)

where an overline means complex conjugation, and we get normalized vectors by defining

$$e(f) = \exp \left( -\frac{1}{2} \| f \|^2 \right) E(f).$$

(1.4)

The annihilation, creation and gauge (or number) processes are defined by

$$A_j (t) E(f) = \int_0^t f_j (s) \ ds \ E(f),$$

$$\langle E(g) | A_j^\dagger (t) E(f) \rangle = \int_0^t g_j (s) \ ds \ \langle E(g) | E(f) \rangle,$$

$$\langle E(g) | \Lambda_{ij} (t) E(f) \rangle = \int_0^t g_{ij} (s) f_j (s) \ ds \ \langle E(g) | E(f) \rangle.$$  

(1.5)

Eqs. (1.5) allow to write formally

$$A_j (t) = \int_0^t a_j (s) \ ds,$$

$$A_j^\dagger (t) = \int_0^t a_j^\dagger (s) \ ds,$$

$$\Lambda_{ij} (t) = \int_0^t a_j^\dagger (s) a_j (s) \ ds,$$

(1.6)

where $a_j (t)$, $a_j^\dagger (t)$ are usual Boson fields, satisfying the canonical commutation rules

$$[a_j (t) , a_i (s)] = 0,$$

$$[a_j (t) , a_i^\dagger (s)] = \delta_{ji} \delta(t - s),$$

(1.7)

and whose coherent vectors are the normalized exponential vectors:

$$a_j (t) e(f) = f_j (t) e(f).$$

(1.8)

In particular the vector $e(0) \equiv E(0)$ is the Fock vacuum.

The Bose fields introduced here represent a good approximation of the electromagnetic field in the so called quasi-monochromatic paraxial approximation [21, 22]. Now, $F$ is interpreted as the Hilbert space of the electromagnetic field; $A_j^\dagger (t)$ creates a photon with state $e_j$ in the time interval $[0, t]$, $A_j (t)$ annihilates it, $\Lambda_{ij} (t)$ is the selfadjoint operator representing the number of photons with state $e_j$ in the time interval $[0, t]$ and

$$N(t) = \sum_j \Lambda_{jj} (t)$$

(1.9)

is the observable “total number of photons entering the system up to time $t$”. Moreover, in the approximation we are considering, the fields behave as monodimensional waves, so that a change of position is equivalent to a change of time and viceversa. If we forget polarization, the one-particle space $\mathcal{Z}$ has to contain only the degrees of freedom linked to the direction of propagation $\mathbb{S}^2$, so that we can take

$$\mathcal{Z} = L^2(\mathcal{Y}, \sin \theta \ d\theta \ d\phi),$$

$$\mathcal{Y} = \{ 0 \leq \theta \leq \pi, \ 0 \leq \phi < 2\pi \};$$

(1.10)

the angular coordinates $(\theta, \phi)$ represent the direction of propagation. Now, a vector $f$ in the one-particle space $X$ can be identified with a function $f(\theta, \phi, t)$ such that

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^{\infty} dt \ d\theta f(\theta, \phi, t)^2 < +\infty.$$  

In QSC integrals of “Ito type” with respect to $dA_j (t)$, $dA_j^\dagger (t)$, $d\Lambda_{ij} (t)$ are defined. The main practical rules to manipulate “Ito differentials” are the facts that $dA_j (t)$, $dA_j^\dagger (t)$, $d\Lambda_{ij} (t)$ commute with anything contain the fields only up to time $t$ and that the products of the fundamental differentials satisfy

$$dA_j (t) dA_j^\dagger (t) = \delta_{ji} \ dt, $$

$$dA_j (t) d\Lambda_{kj} (t) = \delta_{jk} dA_k (t),$$

$$dA_j (t) dA_j^\dagger (t) = \delta_{jk} dA_k^\dagger (t),$$

$$d\Lambda_{ij} (t) = d\Lambda_{jk} (t) = dA_k (t),$$

$$dA_j^\dagger (t) dA_j (t) = d\Lambda_{kj} (t) dA_j (t) = dA_k^\dagger (t) d\Lambda_{ji} (t) = 0;$$

(1.11)

all the products of $dA_j (t)$, $dA_j^\dagger (t)$ or $d\Lambda_{ij} (t)$ with $dt$ vanish.
The evolution equation

Let $\mathcal{H}$ be a separable complex Hilbert space (the system space) and let $R_i$, $i \geq 1$, $S_{ij}$, $i,j \geq 1$, $H$ be bounded operators in $\mathcal{H}$ such that $H^\dagger = H$, $\sum_i R_i^\dagger R_i$ is strongly convergent to a bounded operator, and $\sum_{i,j} S_{ij} \otimes |e_i\rangle \langle e_j| = S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{Z})$ (unitary operators in $\mathcal{H} \otimes \mathcal{Z}$); we set also

$$K = H - \frac{1}{2} \sum_j R_j^\dagger R_j. \quad (1.12)$$

Then (Theor. 27.8 p. 228) there exists a unique unitary operator-valued adapted process $U(t)$ satisfying $U(0) = 1$ and

$$dU(t) = \left\{ \sum_j R_j \, dA_j^\dagger(t) + \sum_{i,j} (S_{ij} - \delta_{ij}) \, d\Lambda_{ij}(t) - \sum_i R_i^\dagger S_{ij} \, dA_j(t) - iK \, dt \right\} U(t). \quad (1.13)$$

The operator $U_t$ will be the evolution operator for the atom-field system, in the interaction picture with respect to the free dynamics of the field. In order to describe a two-level atom, we take $\mathcal{H} = \mathbb{C}^2$; then, to fix the model, we have to determine the atomic operators $H$, $R_i$, $S_{ij}$ on the basis of physical considerations. In the next section we shall require: (a) the existence of a ground state to which the atom decays by emitting at most one photon when it is not stimulated, (b) a balance equation [Eq. (2.1)] between the numbers of ingoing and outgoing photons when there is some coherent source. This suffices to determine the structure of the atomic operators [Eqs. (2.12), (2.17)].

Contents

As said before, in Section IV we fix the model by physical considerations; here, a central role is played by a balance equation saying that the mean number of outgoing photons plus the mean number of photons stored in the atom is equal to the mean number of ingoing photons. In Section VII we consider the case of a spherically symmetric atom stimulated by monochromatic coherent light, we obtain the master equation which gives the time evolution of the reduced atomic density matrix and we study the large-time behavior of its solutions. In Section VIII we study the differential (with respect to the angle) and total cross sections for the scattering of laser light by the atom, as a function of the frequency of the stimulating laser; in this section such cross sections are obtained from the direct detection scheme. In Section IX, starting from the balanced heterodyne detection scheme, we obtain the power spectrum of the fluorescence light and the elastic and inelastic cross sections. Section X1 is devoted to a discussion of the main features of the integral cross sections and of the power spectrum.

II. THE MODEL AND THE BALANCE EQUATION FOR THE NUMBER OF PHOTONS

First of all we want a model for an atom stimulated by a laser (coherent light, not necessarily monochromatic); this means to choose as initial state $\Psi(\xi, f) \in \mathcal{H} \otimes \mathcal{F}$ a generic state for the atom and a coherent vector for the field $|\xi\rangle$, i.e.

$$\Psi(\xi, f) = \xi \otimes e(f), \quad (2.1)$$

$$\xi \in \mathcal{H}, \quad \|\xi\| = 1, \quad f \in L^2(\mathbb{R}_+; \mathcal{Z}).$$

Then, the atomic reduced statistical operator $\rho(t; \xi, f)$ is defined by the partial trace over the Fock space

$$\rho(t; \xi, f) = \text{Tr}_F \{ U(t) |\Psi(\xi, f)\rangle \langle \Psi(\xi, f)| U(t)^\dagger \}. \quad (2.2)$$

Moreover, the quantity

$$\langle N(t) \rangle_f = \langle U(t) |\Psi(\xi, f)\rangle N(t) U(t)^\dagger |\Psi(\xi, f)\rangle$$

represents the mean number of photons up to time $t$, after the interaction with the atom, while

$$\langle N(t) \rangle_f^0 = \langle \Psi(\xi, f)\rangle N(t) \Psi(\xi, f) \rangle = \int_0^t \| f(s) \|^2 ds \quad (2.4)$$

is the same quantity before such an interaction [17]; we can also say that Eq. (2.4) gives the mean number of ingoing photons entering the system in the time interval $[0, t]$ and that Eq. (2.3) gives the mean number of outgoing photons leaving the system in the same time interval.

Proposition 1 The reduced statistical operator $\rho(t; \xi, f)$ satisfies the master equation

$$\frac{d}{dt} \rho(t; \xi, f) = \mathcal{L}(f(t)) [\rho(t; \xi, f)], \quad (2.5)$$

where

$$\mathcal{L}(f(t))[\rho] = -i \left[ H(f(t)) , \rho \right] + \frac{1}{2} \sum_j \left( \left[ R_j(f(t)) \rho, R_j(f(t))^\dagger \right] + \left[ R_j(f(t))^\dagger, \rho R_j(f(t)) \right] \right), \quad (2.6a)$$
$R_j(f(t)) = R_j + \sum_i S_{ji} f_i(t)$, \hfill (2.6b)

\[
H(f(t)) = H - \frac{1}{2} \sum_{ij} \left( R_j^\dagger S_{ji} f_i(t) - \overline{f_j(t)} S_{ij}^\dagger R_j \right);
\]

moreover, we have

\[
\langle N(t) \rangle_f = \int_0^t \text{Tr}_\mathcal{H} \left\{ \sum_j R_j(f(s)) \dagger R_j(f(s)) \rho(s;\xi,f) \right\} ds.
\] \hfill (2.7)

\textbf{Proof.} By using the rules of QSC, it is possible to differentiate $\langle N(t) \rangle_f$ and $\langle U(t)\Psi(\xi,f)\mid a U(t)\Psi(\xi,f) \rangle$, where $a$ is a generic system operator. Then, one gets the results by recalling that the increments of the field operators commute with $U(t)$ and that $dA_j(t)\Psi(\xi,f) = f_j(t)dt \Psi(\xi,f)$ and by using the definition of $\rho(t;\xi,f)$ given in Eq. (2.23).

In order to formulate physical requirements, let us start by considering the case when no photon is injected into the system, i.e. $f = 0$. In these conditions it is natural to ask that the atom can emit at most one photon; moreover, we require the existence of a unique equilibrium state which we denote by $\rho_g$. We take as canonical basis $\{|+,\rangle\} \in \mathcal{H}$ an orthonormal basis which diagonalises $\rho_g$, so that we can write

\[
\rho_g = p P_+ + (1 - p) P_-
\] \hfill (2.8)

for some $p$ in $[0,1]$, where $P_\pm$ are the orthogonal projections over the vectors $|\pm\rangle$. We shall use also the Pauli matrices

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

(2.9a)

by which the two orthogonal projections $P_\pm$ can be written as

\[
P_+ = \frac{1}{2}(1 + \sigma_z) = \sigma_+ \sigma_- , \quad P_- = \frac{1}{2}(1 - \sigma_z) = \sigma_- \sigma_+ .
\]

(2.9c)

\textbf{Proposition 2} We require

\[
\langle N(t) \rangle_{f=0} \leq 1 , \quad \forall \xi, \forall t , \quad (2.10a)
\]

\[
\rho(t;\xi,0) \xrightarrow{t \to +\infty} \rho_g , \quad \forall \xi . \quad (2.10b)
\]

Then, apart from an exchange of roles between the two states $|+\rangle$ and $|\rangle$, we obtain

\[
\rho_g = P_- , \quad (2.11)
\]

\[
H = \frac{1}{2} \omega_0 \sigma_z , \quad \omega_0 \in \mathbb{R} , \quad (2.12a)
\]

\[
R_j = \langle e_j | \alpha \rangle \sigma_- , \quad \alpha \in \mathbb{Z} , \quad \alpha \neq 0 . \quad (2.12b)
\]

Viceversa, Eqs. (2.12a) and (2.12b) imply Eqs. (2.10) and (2.11).

\textbf{Proof.} By Eqs. (2.7) and (2.10a), $\langle N(t) \rangle_f$ is a bounded and non decreasing function of $t$, so $\lim_{t\to +\infty} \langle N(t) \rangle_f$ exists; then, Eqs. (2.7) and (2.10b) give

\[
\sum_j \text{Tr}_\mathcal{H} \left\{ R_j^\dagger R_j \rho_g \right\} = 0 .
\]

By the cyclic property of the trace and the positivity of $\rho_g$ and of $R_j \rho_g R_j^\dagger$, we get that this condition is equivalent to $R_j \rho_g = 0, \forall j$.

Now, let us set $R_j = x_j \mathbb{1} + y_j \sigma_z + z_j \sigma_+ + \alpha_j \sigma_- \quad$ (every operator on $\mathbb{C}^2$ can be written in this way). Then, Eq. (2.8) and $R_j \rho_g = 0$ give $p(x_j + y_j) = 0, (1 - p)(x_j - y_j) = 0, (1 - p)z_j = 0, \quad \forall \alpha \in \mathbb{Z}$.

For $p \in (0,1)$ this system of equations gives $R_j = 0$, which is not acceptable because in this case the equilibrium state is not unique. For $p = 0$ we get $x_j = y_j$ and $z_j = 0$; we need also

\[
\sum_j |\alpha_j|^2 \neq 0 \quad (2.13)
\]

to have decay to an equilibrium state. We do not consider the case $p = 1$, because it is analogous to the previous one, apart from the exchange of $|+\rangle$ and $|\rangle$. Therefore we have Eq. (2.11) and

\[
R_j = \alpha_j \sigma_- + \beta_j P_+ , \quad (2.14)
\]

with $\beta_j = 2x_j$; by the convergence of $\sum_j R_j^\dagger R_j$, the complex numbers $\alpha_j$ and $\beta_j$ can be seen as the components of two vectors $\alpha$ and $\beta$ in $\mathbb{Z}$.

Eq. (2.5) and (2.10b) give $\mathcal{L}(0) |\rho_g \rangle = 0$; by Eqs. (2.6), (2.11) and (2.14) this condition reduces to $[H,\rho_g] = 0$.

Because $H$ is selfadjoint and defined up to a constant, we obtain Eq. (2.12a).
Finally, let us choose $\xi = \mid + \rangle$. By using the relation $\sum_j R_j^\dagger R_j = (\|\alpha\|^2 + \|\beta\|^2) P_+$ and by differentiating $\langle N(t) \rangle_{f=0}$ two times, we obtain
\[
\frac{d^2}{dt^2} \langle N(t) \rangle_{f=0} + \|\alpha\|^2 \frac{d}{dt} \langle N(t) \rangle_{f=0} = 0,
\]
together with the initial conditions
\[
\langle N(0) \rangle_{f=0} = 0,
\]
\[
\frac{d}{dt} \langle N(0) \rangle_{f=0} = \|\alpha\|^2 + \|\beta\|^2.
\]
This gives
\[
\langle N(t) \rangle_{f=0} = \left( 1 + \frac{\|\beta\|^2}{\|\alpha\|^2} \right) (1 - e^{-\|\alpha\|^2 t});
\]
then, condition (2.10a) implies $\beta = 0$ and Eqs. (2.13) and (2.14) give Eq. (2.12b).

The last statement of the proposition follows by direct computations. \(\square\)

Now we have to find some physical restrictions on the possible forms of the operator $S \in \mathcal{U}(\mathcal{H} \otimes \mathcal{Z})$. In \([28]\), the case $f(t) = \lambda(t) \equiv \exp(-i\omega t)\theta(T - t)\lambda$ is considered, where $\theta(x)$ is the usual step function and $\lambda \in \mathcal{Z}$; for $T \to +\infty$, $\lambda(t)$ represents a monochromatic coherent wave. Then, in \([21]\) we asked
\[
\lim_{t \to +\infty} \lim_{T \to +\infty} \langle N(t) \rangle_{\lambda} = 1, \quad \forall \lambda \in \mathcal{Z}, \ \forall \omega, \quad (2.15)
\]
which is a form of flux conservation in the mean: if the possible physical processes are absorption/emission of single photons and direct scattering without change of atomic state, for large times the mean number of injected photons $\langle N(t) \rangle_{\lambda} = \|\lambda\|t$ should be equal to the mean number of outgoing photons $\langle N(t) \rangle_{\lambda}$.

The same restrictions on $S$ are obtained by requiring a balance equation on the number of photons: the mean number of outgoing photons up to time $t$ plus the mean number of photons stored in the atom must be equal to the mean number of ingoing photons.

**Proposition 3** Under assumptions (2.12), the balance equation
\[
\langle N(t) \rangle_f + \frac{1}{2} \text{Tr}_\kappa \left\{ \sigma_z [\rho(t; \xi, f) - \rho(0; \xi, f)] \right\} = \langle N(t) \rangle_f^0
\]
holds $\forall t$, $\forall \xi$, $\forall f$ if and only if one has
\[
S = P_+ \otimes S^+ + P_- \otimes S^-, \quad S^\pm \in \mathcal{U}(\mathcal{Z}). \quad (2.17)
\]

**Proof.** Any bounded operator on $\mathcal{H} \otimes \mathcal{Z}$, like $S$, can always be decomposed as
\[
S = P_+ \otimes S^+ + P_- \otimes S^- + \sigma_+ \otimes F^+ + \sigma_- \otimes F^-,
\]
where $S^\pm, F^\pm$ are bounded linear operators on $\mathcal{Z}$; the unitarity of $S$ implies some simple relations among $S^\pm, F^\pm$.

By using Eqs. (2.5), (2.9), (2.12), we compute the time derivative of $\text{Tr}_\kappa \{ \sigma_z \rho(t; \xi, f) \}$. Then, we insert Eq. (2.6b) into Eq. (2.7) and, by using also Eq. (2.18), we get
\[
\langle N(t) \rangle_f - \langle N(t) \rangle_f^0
\]
\[
\quad + \frac{1}{2} \text{Tr}_\kappa \left\{ \sigma_z [\rho(t; \xi, f) - \rho(0; \xi, f)] \right\}
\]
\[
\quad = \int_0^t ds \text{Tr}_\kappa \left\{ \left[ \| F^+ f(s) \| P_+ - \| F^- f(s) \| P_- \right] \right. \langle S^+(f(s)) | S^+(f(s)) \rangle \sigma_+
\]
\[
\quad + \langle F^+(f(s)) | S^+(f(s)) \rangle \sigma_- \langle \rho(s; \xi, f) \rangle.
\]

By the arbitrariness of $t$, $f$ and $\xi$, condition (2.16) is equivalent to $F^\pm = 0$ and Eq. (2.17) is proved; the unitarity of $S^\pm$ follows from the unitarity of $S$. \(\square\)

From now on we assume Eqs. (2.1), (2.12), (2.17) to hold and, always for physical reasons, we take
\[
\omega_0 > 0. \quad (2.19)
\]

In order to have an atom stimulated by a monochromatic coherent wave we take
\[
f(t) = \lambda(t) \equiv e^{-i\omega t} \theta(T - t)\lambda, \quad \lambda \in \mathcal{Z}, \quad \omega > 0. \quad (2.20)
\]
The step function $\theta$ is defined by $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$, so that $\lambda(t)$ represents a monochromatic wave for $T \to +\infty$.

Quantities like $\omega_0, \alpha, S^\pm$ are phenomenological parameters, or, better, they have to be computed from some more fundamental theory, such as some approximation to quantum electrodynamics. The whole model is meaningful only for $\omega$ not too “far” from $\omega_0$ and $\omega_0$ must include the Lamb shifts. In the final results one can admit a slight $\omega$-dependence in the direct scattering matrices $S^\pm$.

### III. THE MASTER EQUATION

In this section we study the master equation (2.3) and the long time behavior of the atom; the relations (2.6), (2.12), (2.17), (2.19), (2.20) hold.
The reduced statistical operator

First of all, by setting
\[
\rho_\lambda(t) = \lim_{T \to +\infty} \exp \left\{ \frac{1}{2} \sigma_z (\beta + \omega t) \right\} \rho(t; \xi, \lambda)
\]
\[
\times \exp \left\{ -\frac{1}{2} \sigma_z (\beta + \omega t) \right\},
\]
\[
\beta = \arg \left\{ -\langle S^- \lambda | \alpha \rangle \right\},
\]
(3.1)

exactly as in Proposition [3], we obtain the master equation
\[
\frac{d}{dt} \rho_\lambda(t) = L_\lambda [\rho_\lambda(t)]
\]
(3.2)

with the time independent Liouvillian
\[
L_\lambda [\rho] = -i[H_\lambda, \rho] + \frac{1}{2} \sum_j \left( \left[ R_j^\lambda, \rho \right], R_j^{\lambda\dagger} \right)
\]
(3.3a)
\[
R_j^\lambda = e^{-i\beta (e_j |\alpha\rangle \sigma_+ + \langle e_j |S^+ \lambda \rangle P_+}
\]
\[
+ \langle e_j |S^- \lambda \rangle P_-, \quad (3.3b)
\]

The equilibrium state is given by
\[
H_\lambda = \frac{1}{2} (\omega_0 - \omega) \sigma_z - \frac{1}{2} |\langle S^- \lambda | \alpha \rangle| \sigma_y.
\]
(3.3c)

The general master equation for a two-level system is studied in [23]. In the following we shall use similar techniques, apart from a different parametrization of the statistical operator, which turns out to be more convenient in our case. By setting
\[
\rho_\lambda(t) = \begin{pmatrix} u(t) & v(t) \\ \bar{v}(t) & 1 - u(t) \end{pmatrix},
\]
(3.4a)
\[
\begin{cases}
0 \leq u(t) \leq 1, \\
u(t) \geq u^2(t) + |v(t)|^2,
\end{cases}
(3.4b)

where the conditions (3.4b) express the fact that \( \rho_\lambda(t) \) is a statistical operator, we obtain from the master equation
\[
\frac{d}{dt} \mathbf{u}(t) = -G \mathbf{u}(t) + \begin{pmatrix} 0 \\ \Omega/2 \end{pmatrix},
\]
(3.5)

where
\[
\mathbf{u}(t) = \begin{pmatrix} u(t) \\ v(t) \\ \bar{v}(t) \end{pmatrix},
\]
(3.6a)
\[
G = \begin{pmatrix}
|\alpha|^2 & -\Omega/2 & -\Omega/2 \\
-\bar{e}^{i\beta} \langle \alpha |S\lambda \rangle + \Omega & b & 0 \\
-\bar{e}^{-i\beta} \langle S\lambda |\alpha \rangle + \Omega & 0 & \bar{b}
\end{pmatrix},
\]
(3.6b)

The quantity \( \Omega \) can be interpreted as the bare Rabi frequency. Moreover, we have
\[
\det G = |\alpha|^2 \left( (\Delta \omega)^2 + \Gamma^2/4 \right),
\]
(3.7)

with
\[
\Gamma^2 = \kappa^4 |\alpha|^4 + 4\kappa^2 |\alpha|^2 \Re \langle S^- \lambda | P_\perp (S^+ + S^-) \lambda \rangle
\]
\[
- 4 \left( \Im \langle S^+ \lambda | P_\perp S^- \lambda \rangle \right)^2
\]
\[
\equiv \left( |\alpha|^2 + \| P_\perp \Delta S \lambda \|^2 + 2 |\langle \alpha | S^- \lambda \rangle|^2
\]
\[
- 2 \Re \langle S^+ \lambda | P_\perp S^- \lambda \rangle \right)^2 + |\langle \alpha | (S^+ + S^-) \lambda \rangle|^2
\]
\[
\times \left[ |\alpha|^2 (1 + \kappa^2) + \| P_\perp \Delta S \lambda \|^2 \right].
\]
(3.8)

Let us note that \(|\alpha| > 0\) implies \( \det G > 0 \) and \( \Gamma^2 > 0 \).

The equilibrium state and the general solution of the master equation

The equilibrium state is given by
\[
\lim_{t \to +\infty} \rho_\lambda(t) = \rho_{\lambda,eq} = \begin{pmatrix} u(\infty) & v(\infty) \\ \bar{v}(\infty) & 1 - u(\infty) \end{pmatrix},
\]
(3.9)

where \( u(\infty) \) and \( v(\infty) \) are computed by equating to zero the time derivative in Eq. (3.3). Then, we have
\[
\mathbf{u}(\infty) = G^{-1} \mathbf{w},
\]
which gives
\[
u(\infty) = \frac{\kappa^2 \Omega^2/4}{(\Delta \omega)^2 + \Gamma^2/4},
\]
(3.10a)
\[
\frac{\kappa^2}{2} \left[ |\alpha|^2 + i \Delta \omega + i \Im \langle S^+ \lambda | P_\lambda S^- \lambda \rangle \right].
\]
(3.10b)
For the computation of the fluorescence spectrum in Section V, we shall have to solve the master equation also when the initial condition is not a statistical operator. If

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

is a generic $2 \times 2$ matrix, we can always write

$$e^{\mathcal{L}t}[\sigma] = (\sigma_{11} + \sigma_{22})\rho^0 + \left( \begin{array}{c} d_1(t) \\ d_2(t) \end{array} \right) \left( \begin{array}{c} d_1(t) \\ -d_1(t) \end{array} \right),$$

(3.11)

where

$$d_1(0) = \sigma_{11} - (\sigma_{11} + \sigma_{22})u(\infty),$$

$$d_2(0) = \sigma_{12} - (\sigma_{11} + \sigma_{22})v(\infty),$$

$$d_3(0) = \sigma_{21} - (\sigma_{11} + \sigma_{22})u(\infty).$$

\[\text{Spherically symmetric atom stimulated by a collimated laser}\]

We end this section by particularizing our model to the case of a spherically symmetric atom stimulated by a well collimated laser.

Let us recall that the Hilbert space $\mathcal{Z}$ contains the directions of propagation of the electromagnetic field [see Eq. (1.10)]. So, in order to describe a laser beam propagating along the direction $\theta = 0$, we have to take

$$\lambda = \eta \|\alpha\| e^{i\vec{\sigma}}, \quad \eta > 0, \quad \delta \in [0, 2\pi),$$

(3.14a)

$$\overline{\lambda}(\theta, \phi) = \frac{1_{[0, \Delta \theta]}(\theta)}{\Delta \theta \sqrt{2\pi(1 - \cos \Delta \theta)}},$$

(3.14b)

where $1_{[0, \Delta \theta]}(\theta) = 1$ for $0 \leq \theta \leq \Delta \theta$, $1_{[0, \Delta \theta]}(\theta) = 0$ elsewhere; in all the physical quantities the limit $\Delta \theta \downarrow 0$ will be taken. Note that the power of the laser $\hbar \omega \|\lambda\|^2 = \hbar \omega \|\alpha\|^2 \eta^2 / (\Delta \theta)^2$ diverges for $\Delta \theta \downarrow 0$, because we need a not vanishing atom-field interaction in the limit.

Let us denote by $Y_{lm}(\theta, \phi)$ the spherical harmonic functions; then, the spherical symmetry of the atom requires

$$\overline{\alpha}(\theta, \phi) = Y_{00}(\theta, \phi) = 1 / \sqrt{4\pi},$$

(3.15)

$$S^\pm = \sum_{lm} e^{2i\delta^\pm} |Y_{lm}\rangle \langle Y_{lm}|,$$

(3.16)

where the quantities $\delta^+$ and $\delta^-$ are the phase shifts for the direct scattering in the up and down atomic states respectively. Let us note that we have

$$\lim_{\Delta \theta \downarrow 0} \langle Y_{lm}|\overline{\lambda}\rangle = \delta_{m,0} \frac{1}{2} \sqrt{2l + 1},$$

(3.17a)

$$\lim_{\Delta \theta \downarrow 0} \langle Y_{lm}|(S^\pm - \mathbb{1})\overline{\lambda}\rangle = \delta_{m,0} z \sqrt{2l + 1} e^{i\delta^\pm} \sin \delta^\pm.$$

(3.17b)

Let us recall that $Y_{00}(\theta, \phi) = \sqrt{2l + 1} P_l(\cos \theta)$, where the functions $P_l(\xi)$ are the Legendre polynomials.

Now, we set

$$g^\pm(\theta) = \lim_{\Delta \theta \downarrow 0} \left( (S^\pm - \mathbb{1})\overline{\lambda} \right)(\theta, \phi)$$

$$= i \sum_{l=0}^{\infty} \frac{2l + 1}{4\pi} e^{i\delta^\pm} \sin \delta^\pm P_l(\cos \theta),$$

(3.18a)

$$\Delta g = g_+ - g_-, \quad s = \delta^+ - \delta^-,$$

(3.18b)

$$z = \frac{2\Delta \omega}{\|\alpha\|^2}, \quad y = \frac{\eta^2}{2} \sin 2s,$$

(3.18c)

$$\varepsilon = -\frac{\|\alpha\|^2}{4} \sum_{l=1}^{\infty} (2l + 1) \sin 2(\delta^+_l - \delta^-_l),$$

(3.18d)

$$\zeta^2 = \left( 1 + \eta^2 \left\| P_\perp \Delta g \right\|^2 \right)^2$$

$$+ \eta^2 \left( 1 + \kappa^2 + \eta^2 \left\| P_\perp \Delta g \right\|^2 \right),$$

(3.18e)

$$b' = \kappa^2 - i \left( z + \frac{\eta^2}{2} \sin 2s \right),$$

(3.18f)

$$G' = \begin{pmatrix} 2\eta e^{i\varepsilon} \cos s & b' & 0 \\ 2\eta e^{-i\varepsilon} \cos s & 0 & \overline{\theta} \end{pmatrix},$$

(3.18g)

In order that all these quantities be finite, we require also

$$\sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta^+_l < +\infty,$$

(3.19a)

$$\sum_{l=0}^{\infty} (2l + 1) \sin 2(\delta^+_l - \delta^-_l) \sin \delta^+_l < +\infty.$$

(3.19b)

Then, we have

$$\beta = \pi - \delta - 2\delta^-,$$

(3.20a)

$$b = \frac{\|\alpha\|^2}{2} b',$$

(3.20b)

$$G = \frac{\|\alpha\|^2}{2} G',$$

(3.20c)

$$\Delta \omega = \omega - (\omega_0 + \eta^2 \varepsilon),$$

(3.20d)

$$\Omega = \eta \|\alpha\|^2,$$

(3.20e)

$$\Gamma^2 = \zeta^2 \|\alpha\|^4.$$  

(3.20f)
\( \kappa^2 = 1 + \eta^2 \| \Delta g \|^2, \)

\( u(\infty) = \frac{\eta^2 \kappa^2}{\sqrt{\xi^2 + \zeta^2}}, \) \hfill (3.20g)

\( v(\infty) = \frac{\eta}{\sqrt{\xi^2 + \zeta^2}} (\kappa^2 + iy), \) \hfill (3.20i)

\( \Delta g(\theta) = i \frac{e^{i(\delta^+_g + \delta^-_g)}}{\sqrt{4\pi}} \sin s + (P_\perp \Delta g)(\theta), \) \hfill (3.20j)

\( (P_\perp \Delta g)(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{\infty} \frac{2l + 1}{\sqrt{2\pi}} e^{i(\delta^+_g + \delta^-_g)} \times \sin (\delta^+_g - \delta^-_g) P_l(\cos \theta), \) \hfill (3.20k)

\( \| \Delta g \|^2 = \sin^2 s + \| P_\perp \Delta g \|^2 \) \hfill (3.20l)

\( \| P_\perp \Delta g \|^2 = \sum_{l=1}^{\infty} (2l + 1) \sin^2 (\delta^+_g - \delta^-_g), \) \hfill (3.20m)

\( \| g_\pm \|^2 = \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta^+_g, \) \hfill (3.20n)

**Low intensity laser.** For future use, it is useful to particularize the previous quantities to the case of a laser of vanishing intensity, i.e. \( \eta = 0: \)

\[ \begin{align*}
   b' &= 1 - iz, \\
   \kappa^2 &= 1, \\
   \zeta &= 1, \\
   \Omega &= 0, \\
   \Gamma &= \| \alpha \|^2, \\
   \Delta \omega &= \omega - \omega_0. 
\end{align*} \]  

(3.21a)

\[ \begin{align*}
   \beta &= \pi - \delta, \\
   \kappa^2 &= 1, \\
   \zeta &= \sqrt{1 + 2\eta^2}, \\
   \Gamma^2 &= \| \alpha \|^4 + 2\eta^2, \\
   \Delta \omega &= \omega - \omega_0. 
\end{align*} \]  

(3.22a)

\[ \begin{align*}
   u(\infty) &= \frac{\eta^2}{\sqrt{\xi^2 + \zeta^2}}, \\
   v(\infty) &= \frac{\eta}{\sqrt{\xi^2 + \zeta^2}} (1 + iz), \\
   G' &= \begin{pmatrix} 2 & -\eta & -\eta \\
                          2\eta & 1 - iz & 0 \\
                          2\eta & 0 & 1 + iz \end{pmatrix}. 
\end{align*} \]  

(3.22b)

**IV. DIRECT DETECTION AND TOTAL CROSS SECTION**

By direct detection, it is possible to measure the intensity of the light (or to count the photons) propagating in a small solid angle \( \Delta \mathbf{Y} \) around some direction, which we take different from the direction \( \theta = 0 \) of the incoming beam. The observable “number of photons in \( \Delta \mathbf{Y} \) up to time \( t \)” is represented by

\[ N(t; \Delta \mathbf{Y}) = \sum_{ij} (e_i | 1_{\Delta \mathbf{Y}} e_j \rangle \Lambda_{ij}(t), \]  

(4.1)

where \( \Lambda_{ij}(\theta, \phi) = 1 \) for \( (\theta, \phi) \in \Delta \mathbf{Y} \) and \( \Lambda_{ij}(\theta, \phi) = 0 \) elsewhere. The fact that the direction of detection is different from the beam direction is expressed by

\[ 1_{\Delta \mathbf{Y}} \lambda = 0. \]  

(4.2)

Then, the mean number of photons up to time \( t \) per unit of solid angle around \( (\theta, \phi) \) is given by

\[ \langle n(\theta, \phi; t) \rangle = \frac{1}{|\Delta \mathbf{Y}|} \langle U(t) \Psi(\xi, \lambda)|N(t; \Delta \mathbf{Y})U(t)\Psi(\xi, \lambda) \rangle, \]  

(4.3)

where \( \Psi, \lambda(t), \lambda \) are given by Eqs. (2.1), (2.20), (3.14) and \( |\Delta \mathbf{Y}| = \int_\Delta \mathbf{Y} \sin \theta d\theta d\phi; \) the limits \( T \rightarrow +\infty, \Delta \mathbf{Y} \downarrow 0, T \rightarrow +\infty, \Delta \mathbf{Y} \downarrow 0 \) are understood.

The (angular) differential cross section is proportional to the outgoing flux per unit of solid angle \( \langle n(\theta, \phi; t) \rangle/t \) divided by the incoming flux \( \langle \Psi(\xi, \lambda)|N(t)\Psi(\xi, \lambda) \rangle/t \); so we have

\[ \sigma(\theta, \phi) = A_0 \lim_{t \rightarrow +\infty} \frac{\langle n(\theta, \phi; t) \rangle}{\langle \Psi(\xi, \lambda)|N(t)\Psi(\xi, \lambda) \rangle} = \frac{A_0}{\| \lambda \|^2} \lim_{t \rightarrow +\infty} \frac{1}{\Gamma} \langle n(\theta, \phi; t) \rangle, \]  

(4.4)

where \( A_0 \) is a kinematical factor to be determined with dimensions of an area. To determine \( A_0 \) let us consider the cross section for direct photon scattering by the up or down atomic state, for which the Bohr-Peierls-Placzek formula (or optical theorem) gives \( \sigma(\theta, \phi) = |q(\theta)|^2, \sigma_{TOT} = 2 \frac{2\pi}{\omega} \text{Im} q(0); \) the total cross section is the integral of the differential one on the whole solid angle.

In our case we have to take \( \alpha = 0 \) and from Eq. (4.4) we get \( \sigma(\theta, \phi) = A_0 \| (S^z - \mathbb{1}) \lambda \| (\theta, \phi) \|^2 / \| \lambda \|^2 \) and, by the unitarity of \( S^z, \)

\[ \sigma_{TOT} = \frac{A_0}{\| \lambda \|^2} \| (S^z - \mathbb{1}) \lambda \|^2 = -\frac{2A_0}{\| \lambda \|^2} \frac{\Delta \theta}{\text{Im} e^{-is}} \left( (S^z - \mathbb{1}) \lambda \right)(0, 0). \]  

Then, we must have \( q(\theta) = -i\sqrt{A_0} \Delta \theta g_\perp(\theta) \) and, by imposing the optical theorem, we get \( A_0 = \frac{2\pi}{\omega} \frac{2}{\pi(\Delta \theta)^2}. \)

Up to now we have not taken into account the polarization degrees of freedom. If they are taken into account and the cross section for not polarized light is considered, a \( 3/2 \) extra-factor is obtained ([18] pp. 532–533) and Eq. (4.4) becomes

\[ \sigma(\theta, \phi) = \left( \frac{2\pi c}{\omega} \right)^2 \frac{3}{2\pi \eta^2 \| \alpha \|^2} \lim_{t \rightarrow +\infty} \frac{1}{\Gamma} \langle n(\theta, \phi; t) \rangle. \]  

(4.5)

To compute \( \sigma(\theta, \phi) \) we differentiate Eq. (4.3) by using the rules of QSC, using Eq. (1.3) and then we apply the transformation (3.1); the final result is
\[
\frac{d}{dt} (n(\theta, \phi; t)) = \text{Tr} \left\{ R(\theta, \phi)^{\dagger} R(\theta, \phi) \rho_{\lambda}(t) \right\}, \tag{4.6}
\]


\[
R(\theta, \phi) = e^{-i\frac{\|\alpha\|}{\sqrt{4\pi}} \sigma_- + e^{i\frac{\|\alpha\|}{\sqrt{4\pi}}} [g_+(\theta)P_+ + g_-(\theta)P_-].
\tag{4.7}
\]

Then, Eq. (4.5) gives

\[
\sigma(\theta, \phi) = \frac{6\pi e^2}{\eta^2 \|\alpha\|^2 \omega^2} \text{Tr} \left\{ R(\theta, \phi)^{\dagger} R(\theta, \phi) \rho_{\lambda_{\text{tot}}} \right\}.
\tag{4.8}
\]

Finally, by computing the trace and by using the results of the previous section we obtain the differential cross section and, by integrating it, the total one:

\[
\sigma(\theta, \phi) = \frac{6\pi e^2}{\omega^2} \left\{ \|g_-(\theta)\|^2 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right\} \times \left[ \frac{1}{4\pi} + \frac{1}{\varrho^2 + \zeta^2} \right] \tag{4.9}
\]

\[
- \frac{2}{\sqrt{4\pi} (\varrho^2 + \zeta^2)} \text{Re} \left\{ e^{-2i\delta_-} g_-(\theta) (\kappa^2 - i\eta) \right\},
\]

\[
\sigma_{\text{tot}} = \frac{6\pi e^2}{\omega^2} \left\{ \|g_-\|^2 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right\} \times \left[ 1 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right] \tag{4.10}
\]

\[
- \frac{1}{\varrho^2 + \zeta^2} (g \sin 2\delta_0 + 2\kappa^2 \sin^2 \delta_0),
\]

Let us note that the angular dependence in \(\sigma(\theta, \phi)\) is entirely due to \(g_\pm(\theta)\) \((3.18a)\) and, so, to the presence of the \(\Lambda\)-term in Eq. \((1.13)\).

By some algebraic manipulations \(\sigma_{\text{tot}}\) can be rewritten in a more perspicuous form:

\[
\frac{\omega^2}{6\pi e^2} \sigma_{\text{tot}} = \frac{\left( \sin \delta_0^\prime - \cos \delta_0^\prime \right)^2 + \eta^2 A}{\varrho^2 + \zeta^2} + \|P_\perp g_-\|^2 \frac{\varrho^2 + B}{\varrho^2 + \zeta^2}, \tag{4.11}
\]

\[
A = \sin^2 \delta_0^\prime + \kappa^2 \|g_\pm\|^2 + \|P_\perp \Delta g\|^2 \times \left[ 1 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right] \sin^2 \delta_0^\prime, \tag{4.12}
\]

\[
B = \left( 1 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right) \left( 1 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right) \left( 1 + \frac{\kappa^2}{\varrho^2 + \zeta^2} \right). \tag{4.13}
\]

According to the values of the various coefficients different line shapes appear, which are known as Fano profiles \((13)\) pp. 61–63). These shapes are typical of the interference among various channels, when one of them has an amplitude with a pole near the real axis in the complex energy plane (see also Eq. \((4.13)\) below); in our case the channels are direct scattering in the up state, direct scattering in the down state and fluorescence. Some plots of \(\frac{\omega^2}{6\pi e^2} \sigma_{\text{tot}}\) are given in Fig. \((1)\); the independent variable is the “reduced” detuning \(\tilde{\omega} = (\omega - \omega_0)/\|\alpha\|^2\), the other parameters are given in the caption of Fig. \((1)\); the same figure contains plots of elastic and inelastic cross sections, which will be discussed in Sections \((5)\) and \((6)\).

Whenever the line shape be, there is a strong variation of the cross section for \(\omega\) around \(\omega_0 + \eta^2 \tilde{\varepsilon}\) [see Eqs. \((3.18b), (3.18d), (3.20d)\)]. The intensity dependent shift \(\eta^2 \tilde{\varepsilon}\) of the resonance frequency has received various names in the literature; a very suggestive one is lamp shift, a name suggested by A. Kastler in \(24\) . Note that in our two-level system the lamp shift is not vanishing only if the two states respond differently to direct scattering; moreover, only the contributions different from the \(s\)-wave ones do matter. Let us stress that also the width \(\Gamma\) of the resonance and the whole line shape are intensity dependent.

*No direct scattering.* Let us also note that when the direct scattering is negligible, i.e. when Eqs. \((3.22)\) hold, Eq. \((4.10)\) reduces to

\[
\sigma_{\text{tot}} = \frac{6\pi e^2}{\omega^2} \left\{ \|\alpha\|^4/4 \right\} \tag{4.14}
\]

For a laser with negligible intensity, i.e. when \(\eta \downarrow 0\), Eq. \((4.14)\) reduces to the cross section for resonant scattering, given in \((13)\) pp. 530–533; for \(\eta \neq 0\), we have a power broadening (see Eq. \((2.22c)\)) of the resonance line, which maintains a Lorentzian shape \((25)\).

By comparing the general case \((4.11)\) with the usual one \((4.14)\), we see that the main differences are that in the general case we have lamp shift, asymmetric line shape and bigger power broadening.

*Low intensity laser.* For \(\eta = 0\) Eqs. \((2.21)\) hold and Eqs. \((4.9)\) and \((4.11)\) reduce to

\[
\frac{\omega^2}{6\pi e^2} \sigma(\theta, \phi) = \left| g_-(\theta) - \frac{1}{\sqrt{4\pi} (\tilde{z} + 1)} \right|^2, \tag{4.15}
\]

\[
\frac{\omega^2}{6\pi e^2} \sigma_{\text{tot}} = \|P_\perp g_-\|^2 + \frac{\left( \sin \delta_0^\prime - \cos \delta_0^\prime \right)^2}{\varrho^2 + 1}. \tag{4.16}
\]

**V. HETERODYNE DETECTION**

**A. Power spectrum**

The best way to obtain the spectrum of our stimulated atom is by means of the balanced heterodyne detection scheme; the output current of the detector is represented by the operator \((13)\)

\[
I(\nu; t) = \int_0^t F_0(t_s) j(\nu, h; ds) \, \mathrm{d}s, \tag{5.1}
\]

\[
F(t - s) \]
where $F(t)$ is the detector response function, say

$$F(t) = k_1 \sqrt{\frac{\gamma}{4\pi}} \exp\left(-\frac{\gamma}{2} t\right), \quad \gamma > 0,$$

(5.2)

$k_1 \neq 0$ has the dimensions of a current, $j$ is essentially a field quadrature

$$j(\nu, h; ds) = i\mathcal{E}_0 \mathbf{c} dA_h(s) + \text{h.c.},$$

(5.3)

d$A_h(t) = \sum_j (\hbar |e_j) dA_j(t)$, 

(5.4)

$q$ is a phase factor, $q \in \mathbb{C}$, $|q| = 1$, $\nu$ is the frequency of the local oscillator and $h \in \mathbb{Z}$, $\|h\| = 1$; $h$ contains information on the localization of the detector, say

$$h(\theta', \phi') = \frac{1}{\sqrt{|\Delta \mathcal{Y}|}} 1_{\Delta \mathcal{Y}}(\theta', \phi'),$$

(5.5)

where $\Delta \mathcal{Y}$ is again the small solid angle around $(\theta, \phi)$ introduced in the previous section.

From the canonical commutation relations for the fields one has

$$[I(\nu_1, h_1; t_1), I(\nu_2, h_2; t_2)] = \int_0^{\min(t_1, t_2)} ds F(t_1 - s)$$

$$\times \left\{ e^{i(\nu_1 - \nu_2)s} \langle h_1 | h_2 \rangle - \text{c.c.} \right\};$$

(5.6)

so, $I(\nu_1, h_1; t_1)$ and $I(\nu_2, h_2; t_2)$ are compatible observables for any choice of the times either if $\nu_1 = \nu_2$ and $h_1 = h_2$ or if $\langle h_1 | h_2 \rangle = 0$. Under the same conditions also the $j$’s commute.

In the following for the quantum expectation of any operator $B$ we shall use the notation

$$\langle B \rangle^T_{\lambda} = \langle U(T) \Psi(\xi, \lambda) | BU(T) \Psi(\xi, \lambda) \rangle.$$ 

(5.7)

In the long run the output mean power is given by

$$P(\nu, h) = \lim_{T \rightarrow +\infty} \frac{k_2}{4\pi T} \int_0^T \langle (I(\nu, h; t))^2 \rangle^T_{\lambda} dt;$$

(5.8)

$k_2 > 0$ has the dimensions of a resistance, it is independent of $\nu$, but it can depend on the other features of the detection apparatus. In this section $\lambda(t)$ is given by Eq. (2.20): the limit case of (3.12) will be considered in the next one. As a function of $\nu$, $P(\nu, h)$ gives the power spectrum observed in the “channel” $h$; in the case of the choice (5.3) it is the spectrum observed around the direction $(\theta, \phi)$. Proposition 3 relates $P(\nu, h)$ to normal ordered quantum expectations of products of field operators and gives a sum rule which relates $P(\nu, h)$ to $\|\lambda\|^2$; let us note that $\hbar \omega_0 \|\lambda\|^2$ is the total power of the input monochromatic state $\lambda(t)$ (2.21). Proposition 3 identifies an elastic and an inelastic contribution to the power and reduces the computation of $P(\nu, h)$ to the solution of the master equation (3.3). For the use of QSC in the computation of the spectrum of a two-level atom see also Ref. 22.

**Proposition 4** The mean power $P(\nu, h)$ can be expressed as

$$P(\nu, h) = \frac{k}{4\pi} + \lim_{T \rightarrow +\infty} \frac{k}{2\pi T}$$

$$\times \left\{ \left( \int_0^T dA^I_h(t) \int^t_0 dA_h(s) e^{i(2 + i\nu)(t - s)} \right)^T_{\lambda} + \text{c.c.} \right\},$$

(5.9)

where $k = k_2^2 k_2$; Eq. (5.3) holds almost everywhere in $\nu$.

We have also

$$\int_{-\infty}^{+\infty} \left[ P(\nu, h) - \frac{k}{4\pi} \right] d\nu = \lim_{T \rightarrow +\infty} \frac{k}{T} \langle \lambda_{hh}(T) \rangle^T_{\lambda},$$

(5.10)

where $\lambda_{hh}(T) = \sum_{ij} \langle e_i | h \rangle \lambda_{ij} \langle h | e_j \rangle$; moreover, for any c.o.n.s. $\{h_j \}$ in $\mathbb{Z}$, the following sum rule holds:

$$\sum_j \int_{-\infty}^{+\infty} \left[ P(\nu, h_j) - \frac{k}{4\pi} \right] d\nu = k \|\lambda\|^2.$$ 

(5.11)

**Proof.** By inserting Eqs. (5.4) and (5.5) into the definition (5.8) and by changing order of integration, one gets

$$P(\nu, h) = \lim_{T \rightarrow +\infty} \frac{k}{4\pi T} \int_0^T \int_0^T \left( e^{-2|t-s|} - e^{-\gamma(T - \frac{t + s}{2\pi})} \right)$$

\[ \times \langle j(\nu, h; dt) j(\nu, h; ds) \rangle^T_{\lambda} \]

The term containing the factor $\exp \left[ -\gamma \left(T - \frac{t + s}{2\pi}\right)\right]$ vanishes for $T \rightarrow +\infty$ and one obtains

$$P(\nu, h) = \lim_{T \rightarrow +\infty} \frac{k}{4\pi T} \int_0^T \int_0^T e^{-2|t-s|}$$

\[ \times \langle j(\nu, h; dt) j(\nu, h; ds) \rangle^T_{\lambda} \]

(5.12)

By using the canonical commutation relations and normal ordering, we have

$$P(\nu, h) - \frac{k}{4\pi} = \lim_{T \rightarrow +\infty} \frac{k}{2\pi T} \int_{t \in (0, T)} \int_{s \in (0, t)} e^{i\nu(t-s)}$$

\[ \times \left\{ \langle e^{i\nu(t-s)} dA^I_h(t) dA_h(s) + \mathcal{E}_0 c dA^I_h(t) dA_h(s) \rangle \right\}^T_{\lambda} + \text{c.c.} \]

The factor $\exp[i\nu(t + s)]$, when integrated over $\nu$ from $\nu_1$ to $\nu_2$, gives rise to $\left\{ \exp[i\nu_2(t + s)] - \exp[i\nu_1(t + s)] \right\} / \{i(t + s)\}$, which is not singular for $t > 0$ and $s > 0$; then, the integral containing this factor vanishes for $T \rightarrow +\infty$ and Eq. (5.9) is proved.

Now let us observe that
\[ \int_0^T dA_h^\dagger(t) \int_0^T dA_h(s) \delta(t-s) = \Lambda_{hh}(T). \]

By integrating over \( t \) the second term in the r.h.s. of Eq. (5.3) a Dirac delta comes out and by adding the complex conjugated term a double integral for \( s \in (0, T) \) and \( t \in (0, T) \) is obtained; then, by the previous observation Eq. (5.10) is obtained. By Eqs. (1.9), (2.3), (5.10), we obtain

\[ \sum_j \int_{-\infty}^{+\infty} \left[ P(\nu, h_j) - \frac{k}{4\pi} \right] d\nu = \lim_{T \to +\infty} \frac{k}{T} \langle N(T) \rangle^T. \]

Finally, by Eqs. (2.17), (2.4), (2.20), the sum rule (5.11) is obtained.

**Proposition 5** The mean power can be decomposed as the sum of three positive contributions

\[ P(\nu, h) = \frac{k}{2\pi} P_{el}(\nu, h) + P_{mel}(\nu, h), \quad (5.13) \]

where

\[ P_{el}(\nu, h) = k |r(h)|^2 \frac{1}{\pi} \frac{\gamma/2}{(\nu - \omega)^2 + \gamma^2/4}, \quad (5.14) \]

\[ P_{mel}(\nu, h) = \frac{k}{2\pi} \int_0^{+\infty} dt \exp \left[ -\left( \frac{\gamma}{2} + i(\nu - \omega) \right) t \right] \times \text{Tr} \left\{ D(h)^\dagger \left[ \rho_{eq}^\lambda \right] \right\} + \text{c.c.}, \quad (5.15) \]

\[ D(h) = R(h) - r(h), \quad (5.16a) \]

\[ r(h) = \text{Tr} \left\{ R(h) \rho_{eq}^\lambda \right\}, \quad (5.16b) \]

\[ R(h) = e^{-i\nu |h|} P_+ + \langle h | S^\dagger \rangle P_+ + |h_2 - \langle h | S^- \rangle P_-. \]

\[ \sum_j \langle h | e_j \rangle R_j^\lambda. \quad (5.16c) \]

**Proof.** Let us start from Eq. (5.9). We can write

\[ \langle dA_h^\dagger(t) dA_h(s) \rangle^T = \langle \Psi(\xi, \lambda) U(T)^\dagger dA_h^\dagger(t) U(T) \times U(T)^\dagger dA_h(s) U(T) \Psi(\xi, \lambda) \rangle. \]

with \( T > t > s \). By the rules of QSC (see the “output fields” in [13], Section 3), we obtain

\[ U(T)^\dagger dA_h(t) U(T) = U(t)^\dagger \left\{ \langle h | \alpha | \sigma_- \right\} dt + \sum_j \left\{ \langle h | S^\dagger e_j \rangle P_+ + \langle h | S^- e_j \rangle P_- \right\} dA_j(t) U(t). \]

By using this result we can write

\[ \left\langle \int_0^T dA_h^\dagger(t) \int_0^t dA_h(s) e^{-\frac{i}{2} \nu (t-s)} \right\rangle^T_\lambda \quad (5.17) \]

\[ = \int_0^T dt \int_0^t ds e^{-\frac{i}{2} (\nu - \omega) (t-s)} \langle \Psi(\xi, \lambda) \rangle \times \tilde{U}(t)^\dagger R(h)^\dagger \tilde{U}(s)^\dagger R(h) \tilde{U}(s) e^{\frac{i}{2} \sigma_2 \beta \Psi(\xi, \lambda)}. \]

where \( R(h) \) is defined by Eq. (5.16d) and

\[ \tilde{U}(t) = e^{\frac{1}{2} \sigma_2 \beta \Psi(\xi, \lambda)} U(t) e^{-\frac{1}{2} \sigma_2 \beta}. \]

By the quantum regression theorem, which holds for a dynamics like \( \tilde{U}(t) \) [28], we have

\[ \langle e^{\frac{1}{2} \sigma_2 \beta \Psi(\xi, \lambda)} \tilde{U}(t)^\dagger R(h)^\dagger \tilde{U}(s)^\dagger R(h) \tilde{U}(s) e^{\frac{i}{2} \sigma_2 \beta} \times \Psi(\xi, \lambda) \rangle = \text{Tr} \left\{ R(h)^\dagger e^{\frac{i}{2} \beta (t-s)} \left[ R(h) \sigma_2 \lambda \sigma_2 \beta \rho_{eq} \right] \right\}, \]

where \( \rho_{0} = \exp \left( \frac{1}{2} \sigma_2 \beta |\xi| \exp \left( -\frac{i}{2} \sigma_2 \beta \right) \right) \). By recalling that \( \lim_{t \to +\infty} e^{\frac{i}{2} \beta (t-s)} = \rho_{eq} \) for any state \( \rho \), we obtain

\[ \lim_{T \to +\infty} \frac{k}{2\pi} \int_0^T dA_h^\dagger(t) \int_0^t dA_h(s) e^{-\frac{i}{2} (\nu - \omega) (t-s)} \left\{ \text{Tr} \left\{ R(h)^\dagger e^{\frac{i}{2} \beta (t-s)} \left[ R(h) \sigma_2 \lambda \sigma_2 \beta \rho_{eq} \right] \right\} \right\} \quad (5.18) \]

By inserting \( R(h) = D(h) + r(h) \) into Eq. (5.18) and this equation into Eq. (5.4), we obtain the decomposition (5.13)-(5.13).

The positivity of \( k/(4\pi) \) and \( P_{el}(\nu, h) \) is apparent from their definitions, while to prove the positivity of \( P_{mel}(\nu, h) \) requires some transformations.

By repeating in the reverse order the steps from Eq. (5.17) to Eq. (5.18), we obtain from Eq. (5.15)

\[ P_{mel}(\nu, h) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_0^T dt \int_0^t ds \exp \left( -\frac{i}{2} \nu (t-s) \right) \left\{ \langle \phi(t) | \phi(s) \rangle \right\} \]

where

\[ \phi(t) = \sqrt{\frac{k}{2\pi}} e^{i(\nu - \omega)t} \tilde{U}(t)^\dagger D(h) \tilde{U}(t) e^{\frac{i}{2} \sigma_2 \beta \Psi(\xi, \lambda)}. \]

By exchanging the order of integration and the names of the variables \( s \) and \( t \) in the second term, we get

\[ P_{mel}(\nu, h) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_0^T dt \int_0^t ds \exp \left( -\frac{i}{2} \nu (t-s) \right) \left\{ \langle \phi(t) | \phi(s) \rangle \right\}, \]

which is positive because \( \exp \left( -\frac{i}{2} \nu |t| \right) \) is a positive-definite function, i.e. the Fourier transform of a positive function.

Notice that in the decomposition (5.13) the term \( k/(4\pi) \), independent of \( \nu \), is apparently a white noise contribution to the power; \( P_{el}(\nu, h) \) is the elastic contribution, as one sees from Eq. (5.14) which gives \( P_{el}(\nu, h) \propto \).
\(\delta(\nu - \omega)\) for \(\gamma \downarrow 0\); finally, \(P_{\text{inel}}(\nu, h)\) is the inelastic contribution (from Eq. (5.15) one can see that no delta term develops for \(\gamma \downarrow 0\)).

By Eqs. (5.9), (3.11)-(3.13), (5.14)-(5.16), we obtain
\[
r(h) = \langle h|S^{-}\lambda\rangle + \langle h|\Delta S\lambda\rangle u(\infty)
+ e^{-i\lambda}\langle h|\alpha\rangle v(\infty),
\]
(5.19)
\[
P_{\text{inel}}(\nu, h) = \frac{k}{2\pi} e^{b^\dagger} \frac{1}{G + \frac{\gamma}{2} + i(\nu - \omega)} d^b + \text{c.c.},
\]
(5.20)
\[
c^b = \begin{pmatrix}
\langle h|\Delta S\lambda\rangle \\
0
\end{pmatrix}
- e^{-i\lambda}\langle h|\alpha\rangle
\]
(5.21)
\[
\begin{aligned}
d^b_1 &= [\langle h|\Delta S\lambda\rangle (1 - u(\infty)) \\
&- e^{-i\lambda}\langle h|\alpha\rangle v(\infty)] u(\infty),
\end{aligned}
\]
(5.22a)
\[
\begin{aligned}
d^b_2 &= [\langle h|\Delta S\lambda\rangle (1 - u(\infty)) \\
&- e^{-i\lambda}\langle h|\alpha\rangle v(\infty)] v(\infty),
\end{aligned}
\]
(5.22b)
\[
\begin{aligned}
d^b_3 &= e^{-i\lambda}\langle h|\alpha\rangle (u(\infty) - \nu v(\infty))^2 \\
&- \langle h|\Delta S\lambda\rangle u(\infty) v(\infty).
\end{aligned}
\]
(5.22c)

B. Elastic and inelastic cross sections

Let us consider now the case of the spherically symmetric atom, stimulated by a well collimated laser beam, for which Eqs. (3.14)-(3.20) hold. We also assume that the detector spans a small solid angle, so that Eqs. (5.11)-(5.13) hold. We also assume that the detector, i.e., \(\theta > 0\) and so
\[
\langle h|\lambda\rangle = 0.
\]
(5.23)
From Eqs. (5.14), (5.19)-(5.22) we obtain the elastic and inelastic contributions to the power (per unit of solid angle)
\[
\begin{aligned}
\frac{1}{|\Delta Y|} P_{\text{el}}(\nu, h) &\simeq P_{\text{el}}(\nu; \theta, \phi),
\end{aligned}
\]
(5.24a)
\[
\begin{aligned}
\frac{1}{|\Delta Y|} P_{\text{inel}}(\nu, h) &\simeq P_{\text{inel}}(\nu; \theta, \phi),
\end{aligned}
\]
(5.24b)
where
\[
P_{\text{el}}(\nu; \theta, \phi) = k\eta^2|\alpha|^2|a(\theta)|^2 \frac{\gamma/(2\pi)}{\nu - \omega)^2 + \gamma^2/4},
\]
(5.25a)
\[
P_{\text{inel}}(\nu; \theta, \phi) = \frac{k\eta^2|\alpha|^2}{2\pi} c(\theta) \frac{1}{G + \frac{\gamma}{2} + i(\nu - \omega)} d(\theta)
+ \text{c.c.},
\]
(5.25b)
\[
a(\theta) = g_-(\theta) + \Delta g(\theta) \frac{\eta^2 \kappa^2}{\frac{z^2}{4} + \zeta^2}
- e^{2i\delta^0} \frac{\kappa^2 + iy}{\sqrt{4\pi} (z^2 + \zeta^2)},
\]
(5.26)
\[
c(\theta) = \frac{\eta \Delta g(\theta)}{-e^{2i\delta^0} / \sqrt{4\pi}},
\]
(5.27)
\[
d_1(\theta) = \frac{\eta^2}{z^2 + \zeta^2} m(\theta),
\]
(5.28a)
\[
d_2(\theta) = \frac{m(\theta)}{z^2 + \zeta^2} (\kappa^2 + iy),
\]
(5.28b)
\[
d_3(\theta) = -\frac{\eta^2}{(z^2 + \zeta^2)} \left[\frac{e^{2i\delta^0}}{\sqrt{4\pi}} \left[\|\Delta g\|^2 (y^2 + \kappa^4)
+ \kappa^2 y \sin 2s + 2\kappa^4 \cos^2 s\right]
+ \Delta g(\theta) \kappa^2 (\kappa^2 - iy)\right],
\]
(5.28c)
\[
m(\theta) = \Delta g(\theta) \left(1 - \frac{\eta^2 \kappa^2}{z^2 + \zeta^2}\right)
+ e^{2i\delta^0} \frac{\kappa^2 + iy}{\sqrt{4\pi} (z^2 + \zeta^2)}.
\]
(5.29)

For the elastic and inelastic cross sections we shall have \(\sigma_{\text{el}}(\nu; \theta, \phi) \propto P_{\text{el}}(\nu; \theta, \phi), \sigma_{\text{inel}}(\nu; \theta, \phi) \propto P_{\text{inel}}(\nu; \theta, \phi)\). To find the constant of proportionality, let us observe that, from Eqs. (5.10), (5.24), (5.27), (5.31), (4.4), we get
\[
\begin{aligned}
\int_{-\infty}^{+\infty} [P_{\text{el}}(\nu; \theta, \phi) + P_{\text{inel}}(\nu; \theta, \phi)] d\nu
= \lim_{t \to +\infty} \frac{k}{\nu} \langle n(\theta, \phi); t \rangle
= \frac{\eta^2 k |a| \sqrt{2}}{6\pi c^2} \sigma(\theta, \phi).
\end{aligned}
\]
(5.30)
Therefore, taking into account Eqs. (5.25), we obtain the expressions for the cross sections
\[
\sigma_{\text{el}}(\nu; \theta, \phi) = \frac{3c^2}{\omega^2} \left[\frac{a(\theta)}{\gamma} \frac{\gamma}{(\nu - \omega)^2 + \gamma^2/4}\right],
\]
(5.31)
\[
\sigma_{\text{inel}}(\nu; \theta, \phi) = \frac{3c^2}{\omega^2} \left[\frac{1}{G + \frac{\gamma}{2} + i(\nu - \omega)} d(\theta) + \text{c.c.}\right],
\]
(5.32)
and the relation
\[
\int_{-\infty}^{+\infty} [\sigma_{\text{el}}(\nu; \theta, \phi) + \sigma_{\text{inel}}(\nu; \theta, \phi)] d\nu = \sigma(\theta, \phi),
\]
(5.33)
where $\sigma(\theta, \phi)$ is given by Eq. (4.9).

Finally, let us introduce the integral cross sections

$$\sigma_{\text{el}}(\nu) = \int_0^{2\pi} \, \sin \theta \, \int_0^{2\pi} \, d \phi \, \sigma_{\text{el}}(\nu; \theta, \phi),$$  \hspace{1cm} (5.34a)

$$\sigma_{\text{el}} = \int_{-\infty}^{+\infty} \sigma_{\text{el}}(\nu) \, d\nu,$$  \hspace{1cm} (5.34b)

$$\sigma_{\text{inel}}(\nu) = \int_0^{2\pi} \, \sin \theta \, \int_0^{2\pi} \, d \phi \, \sigma_{\text{inel}}(\nu; \theta, \phi),$$  \hspace{1cm} (5.34c)

$$\sigma_{\text{inel}} = \int_{-\infty}^{+\infty} \sigma_{\text{inel}}(\nu) \, d\nu;$$  \hspace{1cm} (5.34d)

the relation (5.33) becomes

$$\sigma_{\text{el}} + \sigma_{\text{inel}} = \sigma_{\text{TOT}},$$  \hspace{1cm} (5.35)

where $\sigma_{\text{TOT}}$ is given by Eqs. (4.12), (4.11).

VI. CROSS SECTIONS AND FLUORESCENCE SPECTRUM

In this section we want to discuss the behavior of the integral cross sections and of the fluorescence spectrum.

From Eqs. (5.26), (5.31), (5.34a), (5.34b) we obtain

$$\sigma_{\text{el}}(\nu) = \frac{\gamma}{(2\pi)} \, \frac{\omega^2}{\eta^2 + \gamma^2/4},$$  \hspace{1cm} (6.1)

$$\frac{\omega^2}{6\pi c^2} \sigma_{\text{el}} = \frac{1}{(z^2 + \zeta^2)^2} \left[ \left| P_\perp \left[ (z^2 + B) g_- + \eta^2 \kappa^2 g_+ \right] \right|^2 \right] + \left| e^{-i\delta_-} \sin \delta_+ + \frac{\eta^2 \kappa^2 e^{i\gamma} \sin s - y + i \kappa^2 y}{z^2 + \zeta^2} \right|^2,$$  \hspace{1cm} (6.2)

while from Eqs. (5.27), (5.29), (5.32), (5.34c), (5.34d) we obtain

$$\frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}} = \frac{\eta^2}{(z^2 + \zeta^2)^2} \left[ 1 + \kappa^2 \right] \frac{E(y)}{2},$$  \hspace{1cm} (6.3)

$$E(y) = \left( y \sin s + \kappa^2 \cos s \right)^2 + \left| P_\perp \Delta g \right|^2 \left( y^2 + \kappa^4 \right);$$  \hspace{1cm} (6.4)

one can check that the relation (5.33) holds true.

Let us recall that the various quantities appearing in the previous formulas are given by Eqs. (6.31), (6.18a), (6.18b), (6.18c), (6.20a), (4.13) and that $\eta$ is the natural line width, $\omega_0$ is the atomic resonance frequency, $\Omega = \eta/|\alpha|^2$ is the bare Rabi frequency, $\eta^2 \bar{z}$ is the intensity dependent shift, $\delta_+ = \left\| P_\perp g_\pm \right\|^2$, $\left\| P_\perp \Delta g \right\|^2$ are parameters linked to the $S_\pm$ scattering matrices, satisfying

$$\left\| P_\perp g_\pm \right\| - \left\| P_\perp g_- \right\| \leq \left\| P_\perp \Delta g \right\| \leq \left\| P_\perp g_\pm \right\| + \left\| P_\perp g_- \right\|.$$

We introduce also a “reduced” detuning $\bar{z}$

$$\bar{z} = \frac{\omega - \omega_0}{\left\| \alpha \right\|^2};$$  \hspace{1cm} (6.5)

we have also $z = 2\bar{z} - 2\eta^2 \varepsilon/\left\| \alpha \right\|^2$, $s = \delta_+ - \delta_-$, $y = \frac{s^2 - \eta^2}{2} \sin 2s$.

As an example, in Fig. 1 we plot $\frac{\omega^2}{6\pi c^2} \sigma_{\text{TOT}}$, $\frac{\omega^2}{6\pi c^2} \sigma_{\text{el}}$, $\frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}}$ as functions of the detuning $\bar{z}$ in the four cases $\eta^2 = 10, 18, 28, 40$; the other parameters are $\delta_+ = -0.03$, $\delta_- = 0.13$, $\left\| P_\perp g_\pm \right\|^2 = 0.005$, $\left\| P_\perp \Delta g \right\|^2 = 0.02$, $\varepsilon/\left\| \alpha \right\|^2 = -0.001$. Let us note the strong asymmetry in $\bar{z}$ of the cross sections and the fact that $\left. \lim_{\bar{z} \to \pm \infty} \frac{\omega^2}{6\pi c^2} \sigma_{\text{TOT}} \right| = \left. \lim_{\bar{z} \to \pm \infty} \frac{\omega^2}{6\pi c^2} \sigma_{\text{el}} \right| = \left. \left\| P_\perp g_\pm \right\|^2 \right| + \sin^2 \left( \delta_- / \left\| \alpha \right\|^2 \right)$, which is about 0.0218 with our parameters.

Let us recall that the usual model with only the absorption/emission process corresponds to $\delta_+ = 0$, $\left\| P_\perp g_\pm \right\|^2 = \left\| P_\perp \Delta g \right\|^2 = 0$, $\varepsilon = 0$, $z = 2\bar{z}$; in this case, from Eqs. (4.14), (5.2), (5.3), we have easily

$$\frac{\omega^2}{6\pi c^2} \sigma_{\text{TOT}} = \frac{1}{4 \bar{z}^2 + 1 + 2\eta^2},$$  \hspace{1cm} (6.6a)

$$\frac{\omega^2}{6\pi c^2} \sigma_{\text{el}} = \frac{4 \bar{z}^2 + 1}{(4 \bar{z}^2 + 1 + 2\eta^2)^2},$$  \hspace{1cm} (6.6b)

$$\frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}} = \frac{2\eta^2}{(4 \bar{z}^2 + 1 + 2\eta^2)^2}. $$  \hspace{1cm} (6.6c)

Now the cross sections are symmetric in $\bar{z}$ and $\left. \lim_{\bar{z} \to \pm \infty} \frac{\omega^2}{6\pi c^2} \sigma_{\text{TOT}} \right| = \left. \lim_{\bar{z} \to \pm \infty} \frac{\omega^2}{6\pi c^2} \sigma_{\text{el}} \right| = 0$.

Then, we introduce the normalized inelastic spectrum

$$\Sigma_{\text{inel}}(x) = \frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}}(\nu),$$  \hspace{1cm} (6.7)

and the total one

$$\Sigma_{\text{TOT}}(x) = \frac{\omega^2}{6\pi c^2} \frac{\gamma}{(2\pi)} \frac{1}{x^2 + (\gamma/2)^2} + \Sigma_{\text{inel}}(x),$$  \hspace{1cm} (6.8)

where we have introduced the “reduced” frequency $x$ and the “reduced” instrumental width $\gamma$

$$x = \frac{\mu - \omega}{\left\| \alpha \right\|^2}, \quad \gamma = \frac{\gamma}{\left\| \alpha \right\|^2};$$  \hspace{1cm} (6.9)

the normalization we have chosen is

$$\int_{-\infty}^{+\infty} \Sigma_{\text{TOT}}(x) \, dx = \frac{\omega^2}{6\pi c^2} \sigma_{\text{TOT}},$$  \hspace{1cm} (6.10a)

$$\int_{-\infty}^{+\infty} \Sigma_{\text{inel}}(x) \, dx = \frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}}.$$  \hspace{1cm} (6.10b)
Proposition 6 The inelastic spectrum is given by

\[ \Sigma_{\text{inel}}(x) = \frac{\eta^2}{\pi (z^2 + \zeta^2)^2} \left( c^t \frac{1}{G + 2ix} d' + \|P \Delta g\| \frac{1}{G + 2ix} d'' + \text{c.c.} \right), \quad (6.11) \]

where

\[ c' = \begin{pmatrix} i e^{is} \sin s \\ 0 \\ 1 \end{pmatrix}, \quad c'' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (6.12) \]

\[ d'_1 = \kappa^2 m', \quad (6.13a) \]
\[ d'_2 = (\kappa^2 + i\gamma) m', \quad (6.13b) \]
\[ d'_4 = \|\Delta g\|^2 \left( y^2 + \kappa^4 \right) + \kappa^2 y \sin 2s \]
\[ + 2\kappa^4 \cos 2s + i\kappa^2 \left( \kappa^2 - i\gamma \right) e^{is} s, \quad (6.13c) \]
\[ m' = \kappa^2 + i\gamma + i \left( z^2 + \zeta^2 - \eta^2 \kappa^2 \right) e^{is} s, \quad (6.13d) \]
\[ d''_{1} = \kappa^2 \left( z^2 + \zeta^2 - \eta^2 \kappa^2 \right), \quad (6.13e) \]
\[ d''_{2} = \kappa^2 \left( z^2 + \zeta^2 - \eta^2 \kappa^2 \right), \quad (6.13f) \]
\[ d''_{3} = \kappa^2 \left( \kappa^2 - i\gamma \right), \quad (6.13g) \]

\[ \bar{G} = \begin{pmatrix} 2 + \gamma & -1 & \eta^2 \\ 2\eta^2 e^{is} \cos s & b' + \gamma & 0 \\ -2e^{-is} \cos s & 0 & \bar{b} + \gamma \end{pmatrix}; \quad (6.14) \]

\( b' \) is given by Eq. \((7.18)\).

Proof. From Eqs. \((8.20c), (5.27), (5.28), (5.29), (5.32), (5.33), (5.34), (5.35), (5.36), (5.37), (5.38), (5.39), \) we have

\[ \det \left( \bar{G} + 2ix \right) = (2 + \gamma + 2ix) \left[ (\kappa^2 + \gamma + 2ix)^2 + \left( z + \frac{\eta^2}{2} \sin 2s \right)^2 \right] \]
\[ + 4\eta^2 \cos s \left[ (\kappa^2 + \gamma + 2ix) \cos s - \left( z + \frac{\eta^2}{2} \sin 2s \right) \sin s \right], \quad (6.16) \]

\[ D_{11}(x) = (\kappa^2 + \gamma + 2ix)^2 + \left( z + \frac{\eta^2}{2} \sin 2s \right)^2, \quad (6.17a) \]
\[ D_{12}(x) = \kappa^2 + \gamma + \frac{\eta^2}{2} \sin 2s, \quad (6.17b) \]
\[ D_{13}(x) = -\eta^2 \left[ \kappa^2 + \gamma + i \left( 2x - z - \frac{\eta^2}{2} \sin 2s \right) \right], \quad (6.17c) \]
\[ D_{31}(x) = 2e^{-is} \cos s \left[ \kappa^2 + \gamma + i \left( 2x - z - \frac{\eta^2}{2} \sin 2s \right) \right], \quad (6.17d) \]
\[ D_{32}(x) = 2e^{-is} \cos s, \quad (6.17e) \]
\[ D_{33}(x) = (2 + \gamma + 2ix) \left[ \kappa^2 + \gamma + i \left( 2x - z - \frac{\eta^2}{2} \sin 2s \right) \right] + 2\eta^2 e^{is} \cos s. \quad (6.17f) \]
The matrix elements \( D_{2j}(x) \) are not needed in formula (6.11).

One can check that the inelastic spectrum is asymmetric, but it is invariant under the transformation: \( x \rightarrow -x, s \rightarrow -s, z \rightarrow -z \).

The formula for the inelastic spectrum given in Proposition \( \text{[6]} \) becomes significantly simpler in the usual case \( (g_{\pm} = 0) \) and when the intensity of the stimulating laser is low.

The case \( g_{\pm} = 0 \)

Let us consider the usual model, when the direct scattering terms are negligible, i.e. \( g_{\pm} = 0 \), which gives also \( s = 0, \kappa^2 = 1 + 2\eta^2, \gamma = z = 2\tilde{z} \); in this case the integral cross sections are given by Eqs. \( \text{(6.6)} \) and, with some computations, the inelastic spectrum is obtained from Proposition \( \text{[6]} \):

\[
\Sigma_{\text{inel}}(x) = \frac{4\eta^2 p(x)}{\pi q(x)(z^2 + 1 + 2\eta^2)^2}, \quad (6.18a)
\]

\[
p(x) = (2 + \gamma) \left[ (1 + \gamma)^2 + 2\eta^2 + z^2 \right] \times \left[ (2 + \gamma)^2 + 2\eta^2 + 4x^2 \right] \quad (6.18b)
\]

\[
q(x) = \left\{ (2 + \gamma) \left[ (1 + \gamma)^2 + z^2 \right] + 4 (1 + \gamma) \eta^2 - 4 (4 + 3\gamma) x^2 \right\} \frac{1}{2} \quad (6.18c)
\]

\[+ 4x^2 \left( 3\gamma^2 + 8\gamma + 5 + z^2 + 4\eta^2 - 4x^2 \right)^2. \]

Now the inelastic spectrum is invariant either under the transformation \( x \rightarrow -x \) either under the transformation \( z \rightarrow -z \).

If we put also \( \tilde{\gamma} = 0 \), which means that the instrumental width is negligible, then one can check that the fluorescence spectrum \( \Sigma_{\text{rot}}(x) \), given by Eqs. \( \text{(6.3)}, \text{(6.6)}, \text{(6.18)} \), coincides exactly (apart from the different normalization) with the spectrum computed by Mollow \( \text{[19]}, \text{Eq. (4.15)} \). Eq. \( \text{(6.18)} \) is simply the convolution of the inelastic part of the Mollow spectrum with a Lorentzian of width \( \tilde{\gamma} \).

If also \( z = 0 \) (no detuning), the eigenvalues of \( \tilde{G} \) can be computed and, by using them, the denominator in Eq. \( \text{(6.18)} \) can be factorized. In the case \( \eta^2 \leq 1/16, \tilde{G} \) has real eigenvalues and \( \Sigma_{\text{inel}}(x) \) has a single peak in \( x = 0 \), while for \( \eta^2 > 1/16 \) two complex eigenvalues appear; therefore, \( \Sigma_{\text{inel}}(x) \) has a three-peak structure for \( \eta^2 \) sufficiently larger than \( 1/16 \). For \( \eta \) very large Eq. \( \text{(6.18)} \) gives three peaks in \( \nu \simeq \omega - \Omega, \nu \simeq \omega = \omega_0, \nu \simeq \omega + \Omega \) with height ratio \( 1 : 2 : 2 \) and widths \( \frac{\eta}{\sqrt{2}} \| \omega \|^2 + 2 \gamma, \| \omega \|^2 + \gamma, \frac{\eta}{\sqrt{2}} \| \omega \|^2 + \gamma \) (see Ref. \[19\] or Ref. [18] pp. 387, 423-426, 437-441 for the case \( \gamma = 0 \)).

Low intensity laser

From Eq. \( \text{(6.3)} \) we see that the inelastic cross section vanishes in the limit of vanishing intensity of the laser; however, the first correction, proportional to \( \eta^2 \), presents some interesting aspects. We have immediately

\[
\frac{\omega^2}{6\pi c^2} \sigma_{\text{inel}} \simeq \frac{2\eta^2 E_0(\tilde{z})}{(4\tilde{z}^2 + 1)^2}; \quad (6.19)
\]

\[
E_0(\tilde{z}) = E(y)\big|_{y=0} = (2\tilde{z}\sin s + \cos s)^2 + \|P_1 \Delta g\| (4\tilde{z}^2 + 1). \quad (6.20)
\]

The computation of the spectrum is straightforward, but long; the final result is: for small \( \eta \) we have

\[
\Sigma_{\text{inel}}(x) \simeq \frac{\eta^2}{\pi} \left[ \frac{|P_1 \Delta g|^2}{2\tilde{z}^2 + 1/4} + \frac{2\gamma (2\tilde{z}\sin s + \cos s)^2}{4(\tilde{z}^2 + 1/4)^2} \right] \left[ \frac{1}{4(x + \tilde{z})^2 + (1 + \gamma)^2} + \frac{1}{4(x - \tilde{z})^2 + (1 + \gamma)^2} \right] \quad (6.21)
\]

\[
\left| \Delta \omega \right| \text{ sufficiently large, the inelastic spectrum presents two peaks (see also Ref. \[18\], pp. 106-108, 386). The structure given by Eq. \( \text{(6.21)} \) is similar also for } s \neq 0, \left| \|P_1 \Delta g\| \neq 0 \right\text{: again two symmetric peaks appear for } \left| \Delta \omega \right| \text{ sufficiently large.}
\]

In this case the inelastic spectrum is invariant either under the transformation \( x \rightarrow -x \) either under the transformation \( s \rightarrow -s \) and \( \tilde{z} \rightarrow -\tilde{z} \).

The usual case \( (g_{\pm} = 0) \) was already discussed by Mollow \( \text{[19]}, \text{Eq. (4.30)} \) for \( \tilde{\gamma} = 0 \) and can be obtained from Eqs. \( \text{(6.18)} \) by letting \( \eta^2 \) vanish or from Eq. \( \text{(6.21)} \) by taking \( s = 0 \) and \( \|P_1 \Delta g\| = 0 \). In the Mollow case, for
In the general case the total spectrum is given by Eqs. (6.8), (6.11), (6.17); the analytic expression is involved, but plots can be easily obtained by numerical computations. According to the values of the various parameters, a well resolved triplet structure can appear, but also single-maximum structures can be shown. With the choice of parameters of Fig. 1 and with an instrumental width $\gamma = 0.6$, the on resonance spectrum for $\eta^2 = 10, 18, 28, 40$ is given in Fig. 2 (solid lines); the dashed lines give the Mollow spectrum for the same values of $\eta^2$ and $\gamma$. The parameters in Fig. 2 have been chosen in such a way that a triplet structure appears, not too different from the usual one, but with a well visible asymmetry in the frequency $x$. Experiments confirm essentially the triplet structure; some asymmetry has been found, whose origin has been attributed to various causes. In this connection it has also been observed that calculations for multilevel atoms indicate some asymmetry. Indeed, the introduction in our model of the interaction term containing the gauge process simulates the presence of other levels and the virtual transitions to them.

Finally, in Fig. 3 we show some out of resonance spectra (detunings $\tilde{z} = -4, -2, 3, 6$) for $\eta^2 = 28$ and the other parameters as in Figs. 1 and 2 (solid lines); again, the dashed lines give the Mollow spectrum. Now, a strong difference from the usual case is shown, consistent with the strong asymmetry in $\tilde{z}$ shown by the total and the elastic cross sections in Fig. 1.

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FIG. 1. $\frac{\omega_2^2}{6\pi c^2}$ × the integral cross sections as functions of the detuning $\tilde{z}$ for $\delta_0^+ = -0.03$, $\delta_0^- = 0.13$, $\|P_{\perp} g \pm \|^2 = 0.005$, $\|P_{\perp} \Delta g\|^2 = 0.02$, $\varepsilon/\|\alpha\|^2 = -0.001$, and $\eta^2 = 10, 18, 28, 40$.

FIG. 2. Total spectrum as a function of the frequency $x$ for $\bar{z} = 0$, $\bar{\gamma} = 0.6$ and $\eta^2 = 10, 18, 28, 40$; solid line: $\delta_0^+ = -0.03$, $\delta_0^- = 0.13$, $\|P_{\perp} g \pm \|^2 = 0.005$, $\|P_{\perp} \Delta g\|^2 = 0.02$, $\varepsilon/\|\alpha\|^2 = -0.001$; dashed line: $\delta_0^+ = 0$, $\|P_{\perp} g \pm \|^2 = \|P_{\perp} \Delta g\|^2 = 0$, $\varepsilon/\|\alpha\|^2 = 0$. 

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FIG. 3. Total spectrum as a function of the frequency $x$ for $\eta^2 = 28$, $\tilde{\gamma} = 0.6$ and $\tilde{z} = -4, -2, 3, 6$; solid line: $\delta_0^+ = -0.03$, $\delta_0^- = 0.13$, $\|P_\perp g_{\pm}\|^2 = 0.005$, $\|P_\perp \Delta g\|^2 = 0.02$, $\varepsilon/\|\alpha\|^2 = -0.001$; dashed line: $\delta_0^+ = 0$, $\|P_\perp g_{\pm}\|^2 = \|P_\perp \Delta g\|^2 = 0$, $\varepsilon/\|\alpha\|^2 = 0$. 