Systems described by Volterra type integral operators

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Abstract

In the paper some sufficient condition for the nonlinear integral operator of the Volterra type to be a diffeomorphism defined on the space of absolutely continuous functions are formulated. The proof relies on consideration of the linearized equation together with Palais-Smale condition, thus a combination of topological and variational methods is used. The applications of the result to the control systems with feedback and to the specific nonlinear Volterra equation are presented.

Key words and phrases: diffeomorphism, Volterra operators, continuous dependence, nonlinear integral operators, robust systems.

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1 Introduction

Let $X$ be a Banach space and let $V : X \rightarrow X$ be some operator. An operator $V$ is said to be a diffeomorphism, if $V(X) = X$, there exists an inverse operator $V^{-1} : X \rightarrow X$, and the operators $V$ and $V^{-1}$ are Fréchet differentiable at any point $x \in X$.

In the paper we shall consider the nonlinear integral operators of the form

$$V(x)(t) = x(t) + \int_{\alpha}^{t} v(t, \tau, x(\tau))\,d\tau$$

where $t \in [\alpha, \beta]$, $v : P_{\Delta} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$, $n \geq 1$ and $P_{\Delta} = \{(t, \tau) \in [\alpha, \beta] \times [\alpha, \beta] : \tau \leq t\}$ and $x \in AC_{0}$. By $AC_{0} = AC_{0}([\alpha, \beta], \mathbb{R}^{n})$ we shall denote the space of absolutely continuous functions defined on the interval $[\alpha, \beta]$ satisfying conditions $x(\alpha) = 0$ and $x' \in L^{2} = L^{2}([\alpha, \beta], \mathbb{R}^{n})$ with the norm given by the formula

$$\|x\|_{AC_{0}}^{2} = \int_{\alpha}^{\beta} |x'(t)|^{2}\,dt.$$ 

Under some appropriate assumptions, to be specified in the next sections, imposed on the function $v$, it is feasible to formulate some sufficient condition for the operator $V : AC_{0} \rightarrow AC_{0}$ to be a diffeomorphism. One can rephrase, underlying the property of the continuous dependence on parameters, the definition of being a diffeomorphism as the operator $V : AC_{0} \rightarrow AC_{0}$ is a diffeomorphism if and only if

(a) the operator $V$ is Fréchet differentiable at every point $x \in AC_{0}$ and for every $y \in AC_{0}$ there exists a unique
solution \( x = x_y \in AC^2_0 \) to the equation \( V(x) = y \) and the solution \( x_y \) depends continuously on parameter \( y \in AC^2_0 \),

(b) the operator \( AC^2_0 : \mathbb{R} \to AC^2_0 \) is Fréchet differentiable.

The systems satisfying condition (a) are often referred to as stable ones and well-posed. If, additionally, the condition (b) is guaranteed then the system is called a robust one, cf. [17].

The theory of integral operators and integral equations forms an extensive area of mathematics. The applications to physics, biology and technical sciences are vast. In particular, Volterra equations find applications in mechanics, demography and epidemiology, to mention only, specific areas of: population dynamics, the spread of an epidemics, the nuclear reactor dynamics with feedback or the general control theory for the systems with feedback loops, cf. [1, 2, 3, 4, 6, 9, 10, 13, 14, 15, 16] and references therein.

The structure of the paper reads as follows. In current section the set-up of notation and terminology is presented. Section 2 contains some regularity assumptions imposed on the function \( v \) and some auxiliary lemmas. In Section 3, we focus our attention on stating a sufficient condition for the functional defined by the Volterra type operator to satisfy Palais-Smale condition. Some sufficient condition for the nonlinear integral operator of the Volterra type to be a diffeomorphism can be found in Section 4. Finally, the examples of integral operators of the Volterra type satisfying some sufficient condition for being a diffeomorphism are presented.

2 Preliminaries

In this section we impose fundamental assumptions on the function \( v \) defining the operator \( V \). In what follows, we will need the following conditions:

(A1) (a) the function \( v(\cdot, \tau, \cdot) \) is continuous on the set \( G := [\alpha, \beta] \times \mathbb{R}^n \) for a.e. \( \tau \in [\alpha, \beta] \),

(b) there exists \( v_t(\cdot, \tau, \cdot) \) and it is continuous on \( G \) for a.e. \( \tau \in [\alpha, \beta] \),

(c) there exists \( v_x(\cdot, \tau, \cdot) \) and it is continuous on set \( G \) for a.e. \( \tau \in [\alpha, \beta] \),

(d) there exists \( v_{tx}(\cdot, \tau, \cdot) \) and it is continuous on set \( G \) for a.e. \( \tau \in [\alpha, \beta] \);

(A2) (a) the function \( v(t, \tau, x) \) is measurable with respect to \( \tau \) and locally bounded with respect to \( x \), i.e. for every \( \rho > 0 \) there exist \( l_\rho > 0 \) such that for \( (t, \tau) \in P_\Delta \) and \( x \in B_\rho = \{x \in \mathbb{R}^n : |x| \leq \rho \} \) we have \( |v(t, \tau, x)| \leq l_\rho \),

(b) the function \( v_t(t, \tau, x) \) is measurable with respect to \( \tau \) and locally bounded with respect to \( x \),

(c) the function \( v_x(t, \tau, x) \) is measurable with respect to \( \tau \) and locally bounded with respect to \( x \),

(d) the function \( v_{tx}(t, \tau, x) \) is measurable with respect to \( \tau \) and locally bounded with respect to \( x \).

Before we formulate the main result of the paper we proceed with some auxiliary lemmas.

**Lemma 2.1** If the function \( v \) satisfies (A1a), (A1b), (A2a) and (A2b), then the operator \( V : AC^2_0 \to AC^2_0 \) is well-defined by \([1]\).

**Proof.** Let \( x_0 \in AC^2_0 \). It is enough to demonstrate that the function \( u \) defined by formula

\[
u(t) = \int_\alpha^t v(t, \tau, x_0(\tau)) \, d\tau
\]
is absolutely continuous and the derivative of \( u \) is square-integrable. This can be seen by observing that

\[
\sum_{i=1}^{N} |u(t_{i+1}) - u(t_{i})| = \sum_{i=1}^{N} \left| \int_{t_{i}}^{t_{i+1}} [v(t_{i+1}, \tau, x_{0}(\tau)) - v(t_{i}, \tau, x_{0}(\tau))] \, d\tau + \int_{t_{i}}^{t_{i+1}} v(t_{i+1}, \tau, x_{0}(\tau)) \, d\tau \right|
\]

\[
\leq \int_{\alpha}^{\beta} l_\rho d\tau \sum_{i=1}^{N} |t_{i+1} - t_{i}| + \sum_{i=1}^{N} \int_{t_{i}}^{t_{i+1}} l_\rho d\tau \leq l_\rho (\beta - \alpha + 1) \sum_{i=1}^{N} |t_{i+1} - t_{i}|
\]

where \( \alpha \leq t_{1} < t_{2} < \ldots < t_{i} < t_{i+1} < \ldots < t_{N} < t_{N+1} \leq \beta \). Consequently, for any \( x_{0} \in AC_{t}^{2} \) the function \( u \) is absolutely continuous and therefore, for almost any \( t \in (\alpha, \beta) \), there exists the derivative of \( u \) and

\[
\int_{\alpha}^{\beta} |u'(t)|^{2} \, dt \leq 2 \int_{\alpha}^{\beta} |v(t, t, x_{0}(t))|^{2} \, dt + 2 \int_{\alpha}^{\beta} \left( \int_{\alpha}^{t} |v(t, \tau, x_{0}(\tau))| \, d\tau \right)^{2} \, dt.
\]

By assumptions \((A2a)\) and \((A2b)\), the derivative of \( u \) belongs to \( L^{2} \). ■

**Lemma 2.2** For any \( x \in AC_{t}^{2} \) we have

\[
|x(t)| \leq (t - \alpha)^{\frac{3}{2}} \| x \|_{AC_{t}^{2}} \quad \text{for} \quad t \in [\alpha, \beta],
\]

\[
\int_{\alpha}^{\beta} |x(t)|^{2} \, dt \leq \frac{1}{2} (\beta - \alpha)^{2} \| x \|_{AC_{t}^{2}}^{2}.
\]

**Proof.** Immediately from the Schwarz inequality, for any \( t \in (\alpha, \beta) \), we get

\[
|x(t)| \leq \int_{\alpha}^{t} |x'(\tau)| \, d\tau \leq (t - \alpha)^{\frac{3}{2}} \| x \|_{AC_{t}^{2}}
\]

and subsequently

\[
\int_{\alpha}^{\beta} |x(t)|^{2} \, dt \leq \| x \|_{AC_{t}^{2}}^{2} \int_{\alpha}^{\beta} (t - \alpha) \, dt = \frac{1}{2} (\beta - \alpha)^{2} \| x \|_{AC_{t}^{2}}^{2},
\]

and this is precisely the assertion of the lemma. ■

For the operator \( V : AC_{t}^{2} \to AC_{t}^{2} \) defined by \((1)\) we have the following differentiability result:

**Lemma 2.3** The operator \( V \) defined by \((1)\) is Fréchet differentiable at every point \( x_{0} \in AC_{t}^{2} \) and for any \( t \in [\alpha, \beta] \) we have

\[
V'(x_{0}) h(t) = h(t) + \int_{\alpha}^{t} v_{x}(t, \tau, x_{0}(\tau)) \, h(\tau) \, d\tau
\]

provided that the function \( v \) satisfies \((A1a), (A1c), (A2a)\) and \((A2c)\).

**Proof.** In view of the definition of \( V \), it suffices to show that the operator

\[
V^{0}(x)(t) = \int_{\alpha}^{t} v(t, \tau, x(\tau)) \, d\tau
\]

satisfies.
is Fréchet differentiable. For any $t \in [\alpha, \beta]$, any $h \in AC_0^2$ and some $\theta \in [0, 1]$, the application of the Mean Value Theorem (cf. [3]) enables us to write

$$V^0 (x_0 + h) (t) - V^0 (x_0) (t) = \int_{\alpha}^{t} [v (t, \tau, x_0 (\tau) + h (\tau)) - v (t, \tau, x_0 (\tau))] d\tau$$

$$= \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) h (\tau) d\tau$$

$$+ \int_{\alpha}^{t} \left[ \int_{0}^{1} v_x (t, \tau, x_0 (\tau) + \theta h (\tau)) d\theta - v_x (t, \tau, x_0 (\tau)) \right] h (\tau) d\tau.$$

We see from (2) that

$$\left| \int_{\alpha}^{t} \left[ \int_{0}^{1} v_x (t, \tau, x_0 (\tau) + \theta h (\tau)) d\theta - v_x (t, \tau, x_0 (\tau)) \right] h (\tau) d\tau \right|$$

$$\leq \int_{\alpha}^{\beta} \int_{0}^{1} |v_x (t, \tau, x_0 (\tau) + \theta h (\tau)) - v_x (t, \tau, x_0 (\tau))| d\theta d\tau (\beta - \alpha)^2 \|h\|_{AC_0^2}.$$

It should be noted that the strong convergence in $AC_0^2$ implies the uniform convergence in $C$. Then, by the assumptions of the lemma, it is enough to apply the Lebesgue Theorem to obtain, if $\|h\|_{AC_0^2} \to 0$, the following convergence

$$\int_{\alpha}^{\beta} |v_x (t, \tau, x_0 (\tau) + \theta h (\tau)) - v_x (t, \tau, x_0 (\tau))| d\tau \to 0$$

and therefore

$$V^0 (x_0 + h) (t) - V^0 (x_0) (t) = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) h (\tau) d\tau + o (h)$$

where $o (h) / \|h\|_{AC_0^2} \to 0$ as $\|h\|_{AC_0^2} \to 0$. This finishes the proof. ■

The rest of this section is devoted to a close study on the linear operator derived from Fréchet differential of the operator $V$ defined by (1). Actually, let $x_0 \in AC_0^2$ be a fixed but an arbitrary function and $T : AC_0^2 \to AC_0^2$ be a linear operator defined, for any $g \in AC_0^2$ and any $t \in [\alpha, \beta]$, by

$$(4) \quad (Tg) (t) = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) g (\tau) d\tau.$$

Due to the restrictions imposed on $v$ it is plausible to consider, for any $k \in \mathbb{N}_0$, $t \in [\alpha, \beta]$ and $g \in AC_0^2$, the following sequence of iterations

$$(5) \quad (T^{k+1}g) (t) = T (T^k g) (t) = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) (T^k g) (\tau) d\tau$$

with the first term defined as

$$(6) \quad (T^0 g) (t) = g (t).$$

Now we turn to estimating the above sequence $\{T^k g\}$. Namely, we prove the following lemma.
Lemma 2.4 Let \( v \) satisfy (A1c) and (A2c). Then for \( k \in \mathbb{N}_0, t \in [\alpha, \beta] \)

\[
| (T^k g) (t) | \leq \frac{(t - \alpha)^k}{k!} l_{\rho}^k M
\]

where \( l_{\rho} > 0 \) is a constant defined by (A2c), \( M \) is a constant such that \( M = \sup_{t \in [\alpha, \beta]} |g(t)| \) and \( \{T^k g\} \) is a sequence defined by (5) and (6).

**Proof.** The estimate in (7) can be seen by observing that from (4)−(6) and the assumptions of the lemma, it follows that

\[
| (T^1 g) (t) | = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) g (\tau) d\tau \leq (t - \alpha) l_{\rho} M. 
\]

Similarly, we have

\[
| (T^2 g) (t) | = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) (T^1 g) (\tau) d\tau \leq \int_{\alpha}^{t} l_{\rho} (\tau - \alpha) l_{\rho} M d\tau = \frac{1}{2} (t - \alpha)^2 l_{\rho}^2 M,
\]

and subsequently we get

\[
| (T^3 g) (t) | = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) (T^2 g) (\tau) d\tau \leq \int_{\alpha}^{t} l_{\rho} (\tau - \alpha)^2 l_{\rho}^2 M d\tau = \frac{(t - \alpha)^3}{3!} l_{\rho}^3 M.
\]

It is enough to proceed by induction on \( k \) to obtain the estimate (7). \( \blacksquare \)

We can now formulate the problem to which the rest of this section is dedicated. For any \( t \in [\alpha, \beta] \), let us consider the linear integral equation of the form

\[
h(t) + \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) h(\tau) d\tau = g(t)
\]

where \( x_0 \in AC^2_0 \) and \( g \in AC^2_0 \) are arbitrarily fixed. Next, we prove the existence and uniqueness results for the above equation. The lemma to be proved is the following.

**Lemma 2.5** For any \( x_0, g \in AC^2_0 \), the equation (8) possesses a unique solution in \( AC^2_0 \) provided that conditions (A1) and (A2) are satisfied.

**Proof.** The equation (8) can be rewritten in the form

\[
h + Th = g
\]

where

\[
(Th)(t) = \int_{\alpha}^{t} v_x (t, \tau, x_0 (\tau)) h(\tau) d\tau.
\]

Consider the following sequence

\[
h_{k+1} = g - Th_k, \text{ for } k \in \mathbb{N}_0, \quad h_0 = 0.
\]
It is easy to observe that

\[(10) \quad h_{k+1} = g - Th_k = g - Tg + T^2g - T^3g + \ldots + (-1)^k T^k g\]

\[\quad = g + \sum_{i=1}^{k} (-1)^i T^i g, \text{ for } k \in \mathbb{N}_0,\]

where

\[(11) \quad T^{k+1} g = T (T^k g), \text{ for } k \in \mathbb{N}_0,\]

\[T^0 g = g\]

and $T^k g$ is given by the formula in (5). By assumptions (A1), (A2) and from Lemma [2.4] analysis similar to that in the proof of Lemma [2.1] leads, if we apply induction on $k$, to the fact that $T^k g \in AC^2_0$ for any $g \in AC^2_0$ and $k \in \mathbb{N}$. From Lemma [2.4] for $k \in \mathbb{N}$, we get

\[\|T^k g\|_{AC^2_0}^2 = \int \alpha^\beta \left[ v_x (t, t, x_0 (t)) (T^{k-1} g) (t) + \int \alpha^\beta \left( T^{k-1} g \right) d\tau \right]^2 dt\]

\[\leq \int \alpha^\beta \left[ l_p (\beta - \alpha)^{k-1} (k - 1)! M + (\beta - \alpha) l_p (\beta - \alpha)^{k-1} (k - 1)! A^{k-1} M \right]^2 dt\]

\[= l_p^2 \left[ (\beta - \alpha)^{k-1} (k - 1)! \right]^2 M^2 C^2\]

where $C^2 = (\beta - \alpha) (1 + \beta - \alpha)^2$. Consequently, for $k \in \mathbb{N}$, we have

\[(12) \quad \|T^k g\|_{AC^2_0} \leq CM l_p \frac{l_p^{k-1} (\beta - \alpha)^{k-1}}{(k - 1)!} = D \frac{A^{k-1}}{(k - 1)!}\]

where $D = CM l_p > 0$ and $A = l_p (\beta - \alpha) > 0$. Since the series $\sum_{k=1}^{\infty} D \frac{A^{k-1}}{(k - 1)!}$ is a convergent bound of the series $\sum_{k=1}^{\infty} (-1)^k T^k g$, it follows that the sequence \{h_{k+1}\} defined by (10) and (11) is a Cauchy sequence in $AC^2_0$. Therefore, the sequence \{h_k\} tends to some $h_0 \in AC^2_0$. Clearly, the operator $T : AC^2_0 \rightarrow AC^2_0$ defined by (5) is continuous. Hence, $h_0 + Th_0 = g$, i.e. $h_0$ is a solution to the equation (5).

What is left is to prove that $h_0$ is unique. Suppose, on the contrary, that there exists $h_1 \in AC^2_0$ such that $h_1 \neq h_0$ and satisfies the equation (5). Denote $\tilde{h} = h_1 - h_0$. From (9) we see that

\[\tilde{h} + T\tilde{h} = 0.\]

Moreover,

\[0 = T\tilde{h} + T^2\tilde{h} = -\tilde{h} + T^2\tilde{h},\]

\[0 = -T\tilde{h} + T^3\tilde{h} = \tilde{h} + T^3\tilde{h},\]

\[\ldots,\]

\[0 = (-1)^{k+1} \tilde{h} + T^k \tilde{h} \text{ for } k \in \mathbb{N}.\]

On making use of the estimate (12) we conclude that $T^k \tilde{h} \rightarrow 0$ as $k \rightarrow \infty$ and hence $\tilde{h} = h_1 - h_0 = 0$. Therefore, the equation (5) possesses exactly one solution in $AC^2_0$, and the proof of the lemma is complete. ■
3 Palais-Smale condition for the functional defined by the Volterra type operator

Let us consider the functional \( F_y : AC_0^2 \rightarrow \mathbb{R}^+ \) of the form

\[
F_y (x) = \frac{1}{2} \| V (x) - y \|_{AC_0^2}^2 = \frac{1}{2} \int_{\alpha}^{\beta} \left| \frac{d}{dt} V (x) (t) - y'(t) \right|^2 dt
\]

where the operator \( V \) is defined by \((1)\) and \( y \in AC_0^2 \) is a given function.

Differentiating the operator \( V \) and substituting the result into \((13)\), we obtain

\[
F_y (x) = \frac{1}{2} \int_{\alpha}^{\beta} \left| x'(t) + v (t, t, x (t)) + \int_{\alpha}^{t} v_1 (t, \tau, x (\tau)) d\tau - y' (t) \right|^2 dt.
\]

The task is now to obtain conditions under which, for any \( y \in AC_0^2 \), the functional \( F_y \) satisfies Palais-Smale condition. We recall what this means. A sequence \( \{x_k\}_{k \in \mathbb{N}} \) is referred to as a Palais-Smale sequence for a functional \( \varphi \) if for some \( M > 0 \), any \( k \in \mathbb{N} \), \( |\varphi (x_k)| \leq M \) and \( \varphi' (x_k) \rightarrow 0 \) as \( k \rightarrow \infty \). We say that \( \varphi \) satisfies Palais-Smale condition if any Palais-Smale sequence possesses a convergent subsequence. To accomplish the task of guaranteeing Palais-Smale condition, we find ourselves forced to introduce an extra assumption. First, we consider, for simplicity without loss of generality, the case of the function \( v \) vanishing on the diagonal of the square \([\alpha, \beta] \times [\alpha, \beta]\). This case encompasses the problems with the integral operators with kernels depending on the difference of arguments \( t \) and \( \tau \), i.e. \( t - \tau \), with value 0 at 0, and the space variable \( x \). This kind of operators appear very often in applications, for a survey of what is known up to date, see, for example, \(2, 5, 16\) and references therein.

On \( v \) we shall impose the following condition:

(A3) \( v(t, t, x) = 0 \) for any \( t \in [\alpha, \beta], x \in \mathbb{R}^n \),

(b) \( |v_1 (t, \tau, x)| \leq c_0 (t, \tau) |x| + d_0 (t, \tau) \) where \( (t, \tau) \in P_\Delta, c_0, d_0 \in L^2 \left( P_\Delta, \mathbb{R}^+ \right) \),

(c) \( c_0 \| L^2 (P_\Delta, \mathbb{R}^+) < \frac{\beta - \alpha}{16} \).

It transpires that it is possible to show that, for any \( y \in AC_0^2 \), under assumptions imposed in \((A1), (A2), (A3)\), the functional \( F_y \) is coercive, i.e. for any \( y \in AC_0^2 \), \( F_y (x) \rightarrow \infty \) as \( \| x \|_{AC_0^2} \rightarrow \infty \). Actually we have the following lemma.

**Lemma 3.1** If the function \( v \) satisfies \((A1a), (A1b), (A2a), (A2b) \) and \((A3)\), then for any \( y \in AC_0^2 \) the functional \( F_y \) defined by \((13)\) is coercive.

**Proof.** Let us observe that the functional \( F_0 \) is bounded from below, since for any \( y \in AC_0^2 \) the functional \( F_y \) is coercive if and only if the functional \( F_0 \) is coercive. Indeed, by the Schwarz inequality and the assumptions
of the lemma, we obtain

\[
F_0 (x) = \frac{1}{2} \left| \int_0^t x' (t) + \int_0^t v_t (t, \tau, x (\tau)) d\tau \right|^2 dt
\]

\[
\geq \frac{1}{2} \|x\|_{AC_0^2}^2 - \int_0^t \left| \int_0^t v_t (t, \tau, x (\tau)) d\tau \right| dt
\]

\[
\geq \|x\|_{AC_0^2}^2 \left( 1 - \frac{1}{\sqrt{2}} \|\dot{c}\|_{L^2(P_\Delta, \mathbb{R}^+)} (\beta - \alpha) \right) - \|x\|_{AC_0^2}^2 (\beta - \alpha)^\frac{3}{2} \|d_0\|_{L^2(P_\Delta, \mathbb{R}^+)}.
\]

From (A3c) and the above estimate it follows that \( F_0 (x) \to \infty \) if \( \|x\|_{AC_0^2} \to \infty \). Consequently, for any \( y \in AC_0^2 \), \( F_y (x) \to \infty \) if \( \|x\|_{AC_0^2} \to \infty \). \( \blacksquare \)

Apart from the function \( v \) vanishing on the diagonal of the square \([\alpha, \beta] \times [\alpha, \beta] \), we shall consider the case of the function \( v \) satisfying the following growth condition:

(A4) (a) \( |v (t, t, x)| \leq c_1 (t) |x| + d_1 (t) \) where \( t \in [\alpha, \beta] \), \( c_1, d_1 \in L^2 ([\alpha, \beta], \mathbb{R}^+) \),

(b) \( |v_t (t, t, x)| \leq c_2 (t, \tau) |x| + d_2 (t, \tau) \) where \( (t, \tau) \in P_\Delta \), \( c_2, d_2 \in L^2 (P_\Delta, \mathbb{R}^+) \),

(c) \( \|\dot{c}\|_{L^2([\alpha, \beta], \mathbb{R}^+)} < \frac{1}{2} \) where \( \dot{c} (t) = (t - \alpha)^\frac{3}{2} c_1 (t) + \frac{1}{\sqrt{2}} (\beta - \alpha) \left( \int_\alpha^t \frac{1}{\sqrt{2}} (\beta - \alpha) \right)^\frac{1}{2} \) for any \( t \in [\alpha, \beta] \).

If in Lemma 3.1 instead of (A3) we assume that (A4) is fulfilled, then the conclusion of the Lemma 3.1 still holds. Hence, the lemma to be proved now is the following.

**Lemma 3.2** If the function \( v \) satisfies (A1a), (A1b), (A2a), (A2b) and (A4), then for any \( y \in AC_0^2 \) the functional \( F_y \) defined by (13) is coercive.

**Proof.** For any \( t \in [\alpha, \beta] \), let us first examine the function

\[
\tilde{v} (t, x (t)) = v (t, t, x (t)) + \int_0^t v_t (t, \tau, x (\tau)) d\tau.
\]

By the Schwarz inequality and inequality (2), we get for any \( t \in [\alpha, \beta] \)

\[
|\tilde{v} (t, x (t))| \leq |v (t, t, x (t))| + \int_0^t |v_t (t, \tau, x (\tau))| d\tau
\]

\[
\leq \|x\|_{AC_0^2} \left[ c_1 (t) (t - \alpha)^\frac{3}{2} + \left( \int_\alpha^t c_2^2 (t, \tau) d\tau \right)^\frac{1}{2} \right] + d_1 (t) + \int_\alpha^t d_2 (t, \tau) d\tau
\]

\[
= \tilde{c} (t) \|x\|_{AC_0^2} + \tilde{d} (t)
\]

where \( \tilde{c} (t) \) is defined in (A4) and \( \tilde{d} (t) = d_1 (t) + \int_\alpha^t d_2 (t, \tau) d\tau \). Proceeding as in the proof of Lemma 3.1 let us
show that the functional $F_0$ is coercive. We see from (15) that

$$F_0(x) = \frac{1}{2} \int_{\alpha}^{\beta} |x'(t) + \tilde{v}(t, x(t))|^2 dt$$

$$\geq \frac{1}{2} \|x\|^2_{AC_0^2} - \int_{\alpha}^{\beta} |x'(t)| |\tilde{v}(t, x(t))| dt$$

$$\geq \left( \frac{1}{2} - \|\tilde{v}\|_{L^2([\alpha, \beta], \mathbb{R}^+)} \right) \|x\|^2_{AC_0^2} - \|\tilde{d}\|_{L^2([\alpha, \beta], \mathbb{R}^+)} \|x\|_{AC_0^2}.$$  

Consequently, from (A4c) it follows that $F_0(x) \to \infty$ if $\|x\|_{AC_0^2} \to \infty$. □

In the next lemmas we provide some conditions under which, for any $y \in AC_0^2$, the functional $F_y$ defined by (13) satisfies Palais-Smale condition.

**Lemma 3.3** Suppose that conditions (A1), (A2) and (A3) hold. Then for any $y \in AC_0^2$ the functional $F_y$ defined by (13) satisfies Palais-Smale condition.

**Proof.** By (A3a), $v(t, t, x) = 0$ for any $t \in [\alpha, \beta]$ and consequently, for any $y \in AC_0^2$, the functional $F_y$ has the form

$$F_y(x) = \frac{1}{2} \int_{\alpha}^{\beta} x'(t) + \int_{\alpha}^{t} v_1(t, \tau, x(\tau)) d\tau - y'(t) \right|^2 dt.$$  

It is worth noting that the functional $F_y$ being a superposition of two $C^1$–mappings is also of the same regularity type. Moreover, substituting $x = z + y$ into (16) we get

$$\tilde{F}(z) = F_y(z + y) = \frac{1}{2} \int_{\alpha}^{\beta} \left| z'(t) + \int_{\alpha}^{t} v_1(t, \tau, z(\tau) + y(\tau)) d\tau \right|^2 dt.$$  

Undeniably, for any $y \in AC_0^2$, the functional $F_y$ satisfies Palais-Smale condition if and only if $\tilde{F}$ satisfies Palais-Smale condition. Immediately, from (17) we get

$$\tilde{F}(z) = \frac{1}{2} \int_{\alpha}^{\beta} \left[ |z'(t)|^2 + 2 \left\langle z'(t), \int_{\alpha}^{t} v_1(t, \tau, z(\tau) + y(\tau)) d\tau \right\rangle + \left\| \int_{\alpha}^{t} v_1(t, \tau, z(\tau) + y(\tau)) d\tau \right\|^2 \right] dt$$  

and the differential of $\tilde{F}$ at $z \in AC_0^2$ is given, for any $h \in AC_0^2$, by

$$\tilde{F}'(z) h = \int_{\alpha}^{\beta} \left[ \left\langle z'(t), h'(t) \right\rangle + \left\langle h'(t), \int_{\alpha}^{t} v_1(t, \tau, z(\tau) + y(\tau)) d\tau \right\rangle \right.$$

$$\left. + \left\langle z'(t), \int_{\alpha}^{t} v_{tx}(t, \tau, z(\tau) + y(\tau)) h(\tau) d\tau \right\rangle \right.$$  

$$\left. + \left\langle \int_{\alpha}^{t} v_1(t, \tau, z(\tau) + y(\tau)) d\tau, \int_{\alpha}^{t} v_{tx}(t, \tau, z(\tau) + y(\tau)) h(\tau) d\tau \right\rangle \right] dt.$$  

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Fix $M \geq 0$ and let $\{z_k\}_{k \in \mathbb{N}} \subset AC_0^2$ be a sequence such that $|\tilde{F}(z_k)| \leq M$ and $\tilde{F}(z_k) \to 0$ as $k \to \infty$. From Lemma 3.1 it follows that $\tilde{F}$ is coercive and hence the sequence $\{z_k\}_{k \in \mathbb{N}}$ is weakly compact. Passing, if necessary to a subsequence, we can assume that $z_k$ tends weakly in $AC_0^2$ to some $z_0 \in AC_0^2$. It may be worth reminding that the weak convergence of the sequence $\{z_k\}_{k \in \mathbb{N}}$ in $AC_0^2$ implies the uniform convergence in $C$ and the weak convergence of derivatives in $L^2$ and moreover a convergent sequence of derivatives has to be bounded. What is left is to prove that the sequence $\{z_k\}_{k \in \mathbb{N}}$ converges to $z_0$ in $AC_0^2$. Because of (19), direct calculations lead to the equality

\begin{equation}
\left\langle \tilde{F}'(z_k) - \tilde{F}'(z_0), z_k - z_0 \right\rangle = \|z_k - z_0\|^2_{AC_0^2} + \sum_{i=1}^5 G_i(z_k)
\end{equation}

where

$$G_1(z_k) = \int_\alpha^\beta \left( z_k'(t) - z_0'(t) \cdot \int_\alpha^t \left[ v_i(t, \tau, z_k(\tau) + y(\tau)) - v_i(t, \tau, z_0(\tau) + y(\tau)) \right] d\tau \right) dt,$$

$$G_2(z_k) = \int_\alpha^\beta \int_\alpha^t v_i(t, \tau, z_k(\tau) + y(\tau)) (z_k(\tau) - z_0(\tau)) d\tau dt,$$

$$G_3(z_k) = \int_\alpha^\beta \int_\alpha^t v_{ix}(t, \tau, z_k(\tau) + y(\tau)) (z_k(\tau) - z_0(\tau)) d\tau dt,$$

$$G_4(z_k) = -\int_\alpha^\beta \int_\alpha^t z_0'(t) - v_{ix}(t, \tau, z_0(\tau) + y(\tau)) (z_k(\tau) - z_0(\tau)) d\tau dt,$$

$$G_5(z_k) = -\int_\alpha^\beta \int_\alpha^t v_i(t, \tau, z_0(\tau) + y(\tau)) d\tau dt.$$

Since $\tilde{F}'(z_k) \to 0$ and $z_k \to z_0$ weakly in $AC_0^2$, $\lim_{k \to \infty} \left\langle \tilde{F}'(z_k) - \tilde{F}'(z_0), z_k - z_0 \right\rangle = 0$. It is therefore enough to show that $\lim_{k \to \infty} G_i(z_k) = 0$ for $i = 1, 2, ..., 5$. It can be easily estimated

$$|G_1(z_k)|^2 \leq \int_\alpha^\beta \left[ |z_k'(t) - z_0'(t)|^2 dt \right.$$

$$\left. + \int_\alpha^t \left| v_i(t, \tau, z_k(\tau) + y(\tau)) - v_i(t, \tau, z_0(\tau) + y(\tau)) \right| d\tau \right]^2 dt.$$

Regarding the above inequality, it is clear that the first factor is bounded whereas the second one converges to zero, which is an immediate consequence of the Lebesgue Theorem, and therefore $G_1(z_k) \to 0$ as $k \to 0$. A similar argument holds for the other terms, i.e. by the Schwarz inequality and the uniform convergence of $\{z_k\}$ to $z_0$ can be demonstrated that $G_i(z_k) \to 0$ if $k \to \infty$ for $i = 2, 3, 4, 5$. Consequently, from (20) it follows that $z_k \to z_0$ in $AC_0^2$, which completes the proof.

Furthermore, we have the following lemma.

**Lemma 3.4** If assumptions (A1), (A2) and (A4) hold then for any $y \in AC_0^2$ the functional $F_y$ defined by (13) satisfies Palais-Smale condition.
Proof. The proof proceeds along the same lines as the proof of Lemma 3.3, this time the details are more intricate. Clearly, the functional $F_y$ has the form

$$\begin{align*}
F_y(x) &= \frac{1}{2} \left[ \alpha^\beta \right] \left| x'(t) + v(t, t, x(t)) + \int_a^t v(t, \tau, x(\tau)) \, d\tau - y'(t) \right|^2 \, dt.
\end{align*}$$

Subsequently, in a similar fashion as in the proof of Lemma 3.3, we obtain

$$\begin{align*}
\tilde{F}(z) &= F_y(z + y) = \frac{1}{2} \left[ \alpha^\beta \right] \left| z'(t) + v(t, t, z(t)) + y(t) + \int_a^t v(t, \tau, z(\tau) + y(\tau)) \, d\tau \right|^2 \, dt
\end{align*}$$

where $\tilde{F}$ is defined in (18). Immediately from the above, for any $h \in AC^2_0$, we obtain

$$\begin{align*}
\tilde{F}'(z) h &= \tilde{F}'(z) + \int_a^\beta \left( \langle h'(t), v(t, t, z(t) + y(t)) \rangle + \langle z'(t), v_x(t, t, z(t) + y(t)) h(t) \rangle \right) \, dt
\end{align*}$$

and

$$\begin{align*}
\langle \tilde{F}'(z_k) - \tilde{F}'(z_0), z_k - z_0 \rangle &= \|z_k - z_0\|^2_{AC^2_0} + \sum_{i=1}^5 G_i(z_k) + \sum_{i=1}^9 H_i(z_k)
\end{align*}$$

where

$$\begin{align*}
H_1(z_k) &= \int_a^\beta \langle z'_k(t) - z'_0(t), v(t, t, z_k(t) + y(t)) - v(t, t, z_k(0) + y(t)) \rangle \, dt,
H_2(z_k) &= \int_a^\beta \langle z'_k(t), v_x(t, t, z_k(t) + y(t)) (z_k(t) - z_0(t)) \rangle \, dt,
H_3(z_k) &= \int_a^\beta \langle v(t, t, z_k(t) + y(t)), v_x(t, t, z_k(t) + y(t)) (z_k(t) - z_0(t)) \rangle \, dt.
\end{align*}$$
If the function above theorem is equivalent to Theorem 3.1 in [7], while either Lemma 3.3 or Lemma 3.4 ascertains that for any \( (A4) \) holds, then the integral operator \( V \) satisfies the assumption (a2) of Theorem 4.1, it follows that for any \( x \) \in \( H \), such that \( \| V(x) \|_H \geq \alpha_x \| h \|_X \). Therefore, the functional \( F_y(x) = \frac{1}{2} \| V(x) - y \|_H^2 \) satisfies Palais-Smale condition, i.e., the assumption (a2) of Theorem 4.1 is fulfilled.

**Remark 4.1** From the bounded inverse theorem it follows that for any \( x \in X \) there exists a constant \( \alpha_x > 0 \) such that \( \| V(x) h \|_H \geq \alpha_x \| h \|_X \). Therefore, it can be easily seen that, in notation \( F_y = \phi \) and \( V = f \), the above theorem is equivalent to Theorem 3.1 in [7].

The application of Lemmas 2.5, 3.3, 3.4 and Theorem 1.1 leads to the main conclusion of this paper.

**Theorem 4.2** If the function \( v \) satisfies assumptions (A1), (A2) and one of the assumption either (A3) or (A4) holds, then the integral operator \( V : AC^2_0 \rightarrow AC^2_0 \) defined by (1) is a diffeomorphism.

**Proof.** Set \( X = H = AC^2_0 \). From Lemma 2.5 we infer that the operator \( V \) satisfies the assumption (a1) of Theorem 1.1 while either Lemma 3.3 or Lemma 3.4 ascertains that for any \( y \in AC^2_0 \) the functional \( F_y(x) = \frac{1}{2} \| V(x) - y \|_{AC^2_0}^2 \) satisfies Palais-Smale condition, i.e., the assumption (a2) of Theorem 1.1 is fulfilled. Therefore, in result \( V : AC^2_0 \rightarrow AC^2_0 \) is a diffeomorphism. ■

Theorem 4.2 can be restated as follows.

\[
H_4(z_k) = \int_\alpha^\beta (v(t, t, z_k(t) + y(t))) (z_k(t) - z_0(t)) \cdot \int_\alpha^t (v(t, \tau, z_k(\tau) + y(\tau))) (z_k(\tau) - z_0(\tau)) \, d\tau \, dt,
\]

\[
H_5(z_k) = \int_\alpha^\beta (v(t, t, z_k(t) + y(t))) \cdot \int_\alpha^t (v(t, \tau, z_k(\tau) + y(\tau))) (z_k(\tau) - z_0(\tau)) \, d\tau \, dt,
\]

\[
H_6(z_k) = -\int_\alpha^\beta (z'_0(t), v_x(t, t, z_0(t) + y(t))) (z_k(t) - z_0(t)) \, dt,
\]

\[
H_7(z_k) = -\int_\alpha^\beta (v(t, t, z_0(t) + y(t)), v_x(t, t, z_0(t) + y(t))) (z_k(t) - z_0(t)) \, dt,
\]

\[
H_8(z_k) = -\int_\alpha^\beta (v(t, t, z_0(t) + y(t)), v_x(t, t, z_0(t) + y(t))) (z_k(t) - z_0(t)) \, dt,
\]

\[
H_9(z_k) = -\int_\alpha^\beta (v(t, t, z_0(t) + y(t)), v_x(t, t, z_0(t) + y(t))) (z_k(t) - z_0(t)) \, dt.
\]

The rest of the proof runs as in the proof of Lemma 3.3 ■

**4 Main results**

We begin with the statement of the theorem on a diffeomorphism between Banach and Hilbert spaces.

**Theorem 4.1** Let \( X \) be a real Banach space, \( H \) be a real Hilbert space, \( V : X \rightarrow H \) be an operator of \( C^1 \) class. If

(a1) for any \( x \in X \), the equation \( V'(x) h = g \) possesses a unique solution for any \( g \in H \),

(a2) for any \( y \in H \), the functional \( F_y(x) = \frac{1}{2} \| V(x) - y \|_H^2 \) satisfies Palais-Smale condition,

then \( V \) is a diffeomorphism.
Theorem 4.3 If the function \( v \) satisfies the assumptions of Theorem 4.2, then for any \( a \in AC^{0}_{0} \) the nonlinear integral equation
\[
x(t) + \int_{a}^{t} v(t, \tau, x(\tau)) \, d\tau = a(t)
\]
possesses a unique solution \( x = x_{a} \in AC^{0}_{0} \). The solution \( x = x_{a} \) depends continuously on parameter \( a \) and moreover the operator \( AC^{0}_{0} \ni a \to x_{a} \in AC^{0}_{0} \) is continuously Fréchet differentiable.

5 Examples

Example 5.1 As an example of the application of Theorem 4.2 consider the following integral operator for any \( t \in [0, 1] \)
\[
V(x)(t) = x(t) + \tilde{a} \int_{0}^{t} (t - \tau)^{\frac{3}{2}} \ln \left( 1 + 2 (t - \tau)^{2} x^{2}(\tau) \right) \, d\tau
\]
with \( \tilde{a}^{2} < \frac{35}{2} \), i.e. the function \( v : P_{\Delta} \times \mathbb{R} \to \mathbb{R} \) is of the form
\[
v(t, \tau, x) = \tilde{a} \, (t - \tau)^{\frac{3}{2}} \ln \left( 1 + 2 (t - \tau)^{2} x^{2} \right).
\]
Let us notice
\[
v_{1}(t, \tau, x) = \frac{2}{3} \tilde{a} \, (t - \tau)^{-\frac{1}{2}} \ln \left( 1 + 2 (t - \tau)^{2} x^{2} \right) + \tilde{a} \, (t - \tau)^{\frac{3}{2}} \frac{4 (t - \tau)^{2} x^{2}}{1 + 2 (t - \tau)^{2} x^{2}}.
\]
Since \( \ln (1 + z^{2}) \leq |z| \), the following inequality holds
\[
|v_{1}(t, \tau, x)| \leq \frac{2 \sqrt{2}}{3} |\tilde{a}| \, (t - \tau)^{-\frac{1}{2}} |x| + 2 |\tilde{a}| \, (t - \tau)^{-\frac{3}{2}}.
\]
Let us put \( c_{0}(t, \tau) = \frac{2 \sqrt{2}}{3} |\tilde{a}| \, (t - \tau)^{\frac{3}{2}} \) and \( d_{0}(t, \tau) = 2 |\tilde{a}| \, (t - \tau)^{-\frac{3}{2}} \). In an elementary way can be checked that \( c_{0}, d_{0} \in L^{2}(P_{\Delta}, \mathbb{R}^{+}) \) and
\[
\|c_{0}\|_{L^{2}(P_{\Delta}, \mathbb{R}^{+})}^{2} = \frac{4}{35} \tilde{a}^{2}.
\]
Thus \( v \) satisfies assumptions (A1) – (A3). Consequently, Theorem 4.2 implies that, for any \( t \in [0, 1] \) and \( a \in AC^{0}_{0} \), the equation
\[
x(t) + \tilde{a} \int_{0}^{t} (t - \tau)^{\frac{3}{2}} \ln \left( 1 + 2 (t - \tau)^{2} x^{2}(\tau) \right) \, d\tau = a(t)
\]
possesses a unique solution \( x = x_{a} \in AC^{0}_{0} \) and the operator \( AC^{0}_{0} \ni a \to x_{a} \in AC^{0}_{0} \) is continuously Fréchet differentiable.

Example 5.2 For any \( t \in [0, T] \) with arbitrarily fixed \( 0 < T < \infty \), the equation
\[
x(t) + \int_{0}^{t} w(t - \tau) \, z(x(\tau)) \, d\tau = a(t)
\]
can be viewed as nonlinear control system with a feedback loop. In this case the feedback is built up by the multiplication of a nonlinear term \( z \) with no memory and a linear time invariant part with memory \( w(t - \tau) \).
Subsequently, the function \( a \) is the convolution of the input function and the inverse of some given transfer. Finally, the function \( w \) is the convolution of the inverse of the feedback transfer function and the inverse of the transfer function, while \( x \) can be interpreted as the output of the system. For details, see, among others \[1\], [2], [3].

The nonlinear feedback term \( z \in C^1(\mathbb{R}, \mathbb{R}) \) is required to be such that there exist \( A, B > 0 \) satisfying

\[
|z(x)| \leq A|x| + B
\]

for all \( x \in \mathbb{R} \). Moreover, we demand that \( w \in C^1([0,T], \mathbb{R}) \) is such that \( w(0) = 0 \) and

\[
\int_0^T \int_0^t |w'(t-\tau)|^2 \, d\tau \, dt < \frac{1}{2AT^2}. 
\]

Then the function \( v(t, \tau, x) = w(t-\tau)z(x) \) satisfies assumptions (A1) - (A3). Hence Theorem \[4,2\] implies that, for any \( t \in [0,T] \) and any \( a \in AC_0^2 \), the equation

\[
x(t) + \int_0^t w(t-\tau)z(x(\tau)) \, d\tau = a(t)
\]

possesses a unique solution \( x = x_a \in AC_0^2 \) and the operator \( AC_0^2 \ni a \rightarrow x_a \in AC_0^2 \) is continuously Fréchet differentiable.

6 Summary

Integral operators and equations involving them are most commonly considered in the space of square-integrable functions. Under suitable conditions one usually proves existence and uniqueness theorems. In this paper the integral operator \( V \) was defined on the smaller space of \( AC_0^2 \) to obtain stronger continuity results. We have shown that assumptions (A1), (A2) and either (A3) or (A4) imply some sufficient condition for the operator \( V : AC_0^2 \rightarrow AC_0^2 \) to be a diffeomorphism, cf. Theorem \[4,2\]. It should be underlined that in the proof of Lemma \[3,3\] and Lemma \[4,4\] we have used the compactness of the embedding of the space \( AC_0^2 \) into the space \( C \), not true if we replace \( AC_0^2 \) with \( L^2 \). This compact embedding implies that any weakly convergent in \( AC_0^2 \) sequence \( \{z_k\} \) is uniformly convergent, i.e. convergent in \( C \) in the sup-norm. Apparently in the case of \( L^2 \) space such an implication does not hold. Therefore, cannot be proved, at least with the method applied herein, that the operator \( V : L^2 \rightarrow L^2 \) is a diffeomorphism, yet it is likely to be a homeomorphism in that case.

References

[1] T. M. Atanackovic and S. Pilipovic, *On a class of equations arising in linear viscoelastic theory*, Zeitschrift Angew. Math. Mech., 85 (2005), 748-754.

[2] R. M. Christensen, "Theory of Viscoelasticity," Academic Press, New York, 1982.

[3] C. Corduneanu, "Integral Equations and Applications," Cambridge University Press, Cambridge, New York, Melbourne, 1991.
[4] M. G. Crandal and J. M. Nohel, *An abstract functional differential equation and a related nonlinear Volterra equation*, Israel Journal of Mathematics, 29 (1978), 313-328.

[5] G. Gripenberg, *An abstract nonlinear Volterra equation*, Israel Journal of Mathematics, 34 (1979), 198-212.

[6] G. Gripenberg, S. -O. Londen, and O. Staffans, "Volterra Integral and Functional Equations," Cambridge University Press, Cambridge, New York, Port Chester, Melbourne, 1990.

[7] D. Idczak, A. Skowron, and S. Walczak, *On the diffeomorphisms between Banach and Hilbert spaces*, Advanced Nonlinear Studies, 12 (2012), 89–100.

[8] A. D. Ioffe and V. M.Tihomirov, "Theory of Extremal problems," Studies in Mathematics and its Applications, vol. 6, North-Holland Publishing Co., Amsterdam, 1979.

[9] S. -O. Londen, *Stability analysis on nonlinear point reactor kinetics*, Adv. Sci. Tech., 6 (1972), 45-63.

[10] S. -O. Londen, *On some nonlinear Volterra integrodifferential equations*, J. Differential Equations, 11 (1972), 169-179.

[11] A. G. J. MacFarlane, "Frequency-Response Methods in Control Systems," A. G. J. MacFarlane ed., Selected Reprint Series, IEEE Press, New York, 1979.

[12] M. S. Mousa, R. K. Miller, and A. N. Michel, *Stability analysis of hybrid composite dynamical systems: descriptions involving operators and differential equations*, IEEE Trans. Automat. Control, 31 (1986), 216-226.

[13] M. Z. Podowski, *Nonlinear stability analysis for a class of differential-integral systems arising from nuclear reactor dynamics*, Trans. Automat. Control, 31 (1986), 98-107.

[14] M. Z. Podowski, *A study of nuclear reactor models with nonlinear reactivity feedbacks: stability criteria and power overshoot evaluation*, Trans. Automat. Control, 31 (1986), 108-115.

[15] J. Prüss, "Evolutionary Integral Equations and Applications," Modern Birkhäuser Classics, Springer New York, 2012.

[16] M. Renardy, W. J. Hrusa, and J. A. Nohel, "Mathematical Problems in Viscoelasticity," volume 35 of Pitman Monographs Pure Appl. Math. Longman Sci. Tech., Harlow, Essex, 1988.

[17] R. S. Sánchez-Peña and M. Sznaier, "Robust Systems Theory and Applications," Wiley-Interscience, New Jersey, 1998.

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