ON GROWTH OF DOUBLE COSETS IN HYPERBOLIC GROUPS

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Abstract. Let $H$ be a hyperbolic group, $A$ and $B$ be subgroups of $H$, and $gr(H, A, B)$ be the growth function of the double cosets $AhB, h \in H$. We prove that the behavior of $gr(H, A, B)$ splits into two different cases. If $A$ and $B$ are not quasiconvex, we obtain that every growth function of a finitely presented group can appear as $gr(H, A, B)$. We can even take $A = B$. In contrast, for quasiconvex subgroups $A$ and $B$ of infinite index, $gr(H, A, B)$ is exponential. Moreover, there exists a constant $\lambda > 0$, such that $gr(H, A, B) > \lambda f_H(r)$ for all big enough $r$, where $f_H(r)$ is the growth function of the group $H$. So, we have a clear dichotomy between the quasiconvex and non-quasiconvex case.

1. Introduction

Growth of groups has been a subject of research for many years. For main results and references see [4]. Growth of cosets in groups has also been investigated, [5]. However, de la Harp wrote in [4], p.209 that growth of double cosets in groups has not yet received much attention, but probably should. In this paper we investigate growth of double cosets of hyperbolic groups.

Let $H$ be a hyperbolic group and let $A$ and $B$ be finitely generated subgroups of $H$. Fix some set of generators of $H$. For any $k \geq 0$, let $gr(H, A, B)$ be the growth function for double cosets $AhB$, that is $gr(G, A, B)(r) = |\{AhB, |h| \leq r\}|$, where $|h|$ is the length of $h$.

Our first theorem shows that the class of growth rate functions of double cosets of non-quasiconvex subgroups is rather wide. Namely, let $G = \langle x_1, \cdots, x_m | r_1, \cdots, r_n \rangle$ be any finitely presented group. Let $f_G$ be the growth function of $G$, that is $f_G(r) = |\{g \in G, |g| \leq r\}|$, where the length $|g|$ is taken with respect to the set of generators $x_1, \cdots, x_m$.

Theorem 1. There exists a hyperbolic group $H$ and a finitely generated subgroup $N$ of $H$ such that $gr(H, N, N) = f_G$.

Our second theorem shows that if $A$ and $B$ are quasiconvex then the growth function of the double cosets $gr(H, A, B)$ is exponential.

Theorem 2. Let $H$ be a non-elementary hyperbolic group with a fixed set of generators. Let $A$ and $B$ be quasiconvex subgroups of $H$ of infinite index. Then there exists a constant $\lambda > 0$ such that $gr(H, A, B)(r) \geq \lambda f_H(r)$ for all big enough $r$.

The proof of Theorem 2 uses the following generalization of the Ping-Pong Lemma from [2].

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Theorem 3. Let $G$ be a non-elementary hyperbolic group, let $K$ be a quasiconvex subgroup of $G$ of infinite index, and let $t \in G$ be an element of infinite order such that $K \cap \langle t \rangle = \{1\}$. Denote by $H$ the commensurator of $\langle t \rangle$ and let $G_0 = H \cap K$.

For a big enough $N$ let $H_1 = \langle t^N, G_0 \rangle$. Then $(H_1, K) = H_1 \ast_{G_0} K$ and $(H_1, K)$ is a quasiconvex subgroup of $G$.

2. Proof of The First Theorem

By a well-known construction [1], there exists a hyperbolic group $H$ and a short exact sequence $1 \rightarrow N \xrightarrow{\alpha} H \xrightarrow{\beta} G \rightarrow 1$ such that the normal subgroup $N$ of $H$ is finitely generated as a group. Indeed, following [1], define $H$ to be a group generated by the elements $x_1, \ldots, x_m, t_1, t_2$ with defining relations:

$r_i t_1^i t_2^i (i = 1, 2, \ldots, m),$

$x_i^{-1} t_i x_i t_1^i t_2^i (i = 1, 2, \ldots, m, j = 1, 2),$

$x_i t_i x_i^{-1} t_1^i t_2^i (i = 1, 2, \ldots, m, j = 1, 2).$

We can choose the constants $a_i, b_i, c_{ij}, d_{ij}, e_{ij}, f_{ij}$ such that $H$ is a small cancellation group with arbitrarily small constant, and hence $H$ is a hyperbolic group.

Let us show that $gr(H, N, N) = f_G$.

Indeed, for every double coset $NhN$ consider the element $\beta(h) \in G$. In this way we obtain a 1-to-1 correspondence between double cosets and elements of $G$, because $Nh_1N = Nh_2N$ implies $\beta(h_1) = \beta(h_2)$ and for every $g = x_{i_1}^i x_{i_2}^j \cdots x_{i_s}^k \in G$, $\beta(g) = g$, so to $NgN$ there corresponds $\beta(g) = g \in G$.

The homomorphism $\beta : H \rightarrow G$ sends $x_i$ to $x_i$ and $t_j$ to $t_j$ for $i = 1, 2, \ldots, m, j = 1, 2$. It follows that for every $h \in H$, $|\beta(h)| \leq |h|$, where the length of $h$ is with respect to the generators $x_1, \ldots, x_m, t_1, t_2$ of $H$ and the length of $\beta(h)$ is with respect to the generators $x_1, \ldots, x_m$ of $G$.

Therefore, for any $k \geq 0$, the number of double cosets $NhN$ with $|h| \leq k$ is equal to the number of elements $g \in G$ with $|g| \leq k$. Hence, $gr(H, N, N) = f_G$, proving Theorem 1.

3. Proof of The Second Theorem

Since $A$ and $B$ are quasiconvex and of infinite index in $H$, there exist elements $c \in H$ and $d \in H$ of infinite order such that $\langle c \rangle \cap A = \{1\}$ and $\langle d \rangle \cap B = \{1\}$. We are grateful to Ilya Kapovich who explained that to us.

Let $c \in H$ and $d \in H$ be elements of infinite order such that $\langle c \rangle \cap A = \{1\}$, and $\langle d \rangle \cap B = \{1\}$. Denote by $H_1$ the commensurator of $\langle c \rangle$ and by $H_2$ the commensurator of $\langle d \rangle$. Let $G_1 = H_1 \cap A$ and $G_2 = H_2 \cap B$. By Theorem 3, for some big enough $M$ taking $C = \langle c^M, G_1 \rangle$ and $D = \langle d^M, G_2 \rangle$, we have $\langle A, C \rangle = A \ast C$ and $\langle B, D \rangle = B \ast D$. Denote $c_0 = c^M$ and $d_0 = d^M$.

According to Proposition 1 (below), there exists a constant $\mu > 0$ such that for any big enough $r$ there are words of the form $s_i = c_0^{N_i} u_i w_i v_i d_0^{N_i}$, $i = 1, 2, \ldots, m$ such that

1. $m \geq \mu f_H(r),$
2. $|s_i| \leq r,$
3. $N_i \geq N$ and $N_i' \geq N$ for some fixed big enough $N,$
4. for $i \neq j$, $s_i \neq s_j,$
5. all $s_i$ are quasigeodesic words with constants $(\eta, \epsilon)$. 


If for some $i, j$ we have $A_s B = A_s B$ then there exist elements $a \in A$ and $b \in B$ such that $s_j = a s_i b$, hence $b = s_i^{-1} a^{-1} s_j = d_0^{-N''} v_i^{-1} u_i^{-1} c_0^{-N'} a^{-1} c_0^N u_j w_j v_j d_0^N$.

If $a \notin G_1$ then according to Theorem 2, $c_0^{-N'} a^{-1} c_0^N$ is a quasigeodesic word. By 5), $d_0^{-N''} v_i^{-1} u_i^{-1} c_0^{-N'}$ and $c_0^N u_j w_j v_j d_0^N$ are quasigeodesics, so $d_0^{-N''} v_i^{-1} u_i^{-1} c_0^{-N'} a^{-1} c_0^N u_j w_j v_j d_0^N$ is also a quasigeodesic, contradicting the fact that it is equal to $b \in B$, where $B$ is quasiconvex.

Similarly, we cannot have $b \notin G_2$. So $s_j = a s_i b$ implies $a \in G_1$ and $b \in G_2$. It follows that at least $\frac{m}{|G_1| |G_2|}$ of the double cosets $A_s B$, $i = 1, \ldots, m$ are different from each other. So we have $gr(H, A, B)(r) \geq \frac{\mu}{|G_1| |G_2|} \cdot f_G(r)$. Taking $\lambda = \frac{\mu}{|G_1| |G_2|}$, we satisfy the requirements of Theorem 2.

**Corollary to the Proof of Theorem 2.**

Using the notation of the Proof of Theorem 2, for each $i = 1, 2, \ldots, m$ the subgroup $\langle A, s_i B s_i^{-1} \rangle$ is a quasiconvex subgroup of $H$, and its geodesic core (see Definition 4 in [3]) consists of a cylindrical neighbourhood of the path for $s_i$ with geodesic cores of $A$ and of $B$ attached at the beginning and at the end of the path for $s_i$ respectively.

### 4. Proof of The Third Theorem

Theorem 3 generalizes Theorem 2 of [2]. However, notice that we do not assume that $H_1$ is malnormal. The proof follows the proof of Theorem 2 in [2] with the following modifications.

1. In the decomposition $l = h_1 k_1 \cdots k_{m-1} h_m$ all $h_i$ are powers of $t^N$.
2. Lemma 8 from [2] is modified in the following way: instead of $H$ we take $H_1 = \langle t^N, G_0 \rangle$.

Note that $G_0$ is a finite group because $G_0$ is a subgroup of $H$ which contains the infinite cyclic group $\langle t \rangle$ as a subgroup of finite index. Indeed, by definition, $H$ is a commensurator of an infinite cyclic group $\langle t \rangle$ in a hyperbolic group, so $\langle t \rangle$ is of finite index in $H$ by Theorem 2 of [1], hence $H$ is infinite virtually cyclic. Therefore, it is known that either $H$ maps onto $Z$ with a finite kernel or $H$ maps onto the infinite dihedral group with a finite kernel. Consider two cases.

1. $H$ maps onto $Z$ with a finite kernel $P$.

   Let $x$ be a preimage of the generator of $Z$. Then for some integer $L \neq 0, t = x^L z$ with $z \in P$. For big enough $M, t^M$ induces a trivial automorphism of $P$, because $P$ is finite.

   Consider $y \in G_0 = H \cap K$. Then $y = x^L z_0$ with $z_0 \in P$, so $y^{LM} = x^{SL}_M z_1$ with $z_1 \in P$, hence $y^{LMN} = t^{SM} z_2$ with $z_2 \in P$. By the choice of $M, t^M$ commutes with $P$, hence for any $N, y^{LMN} = (t^{SM})^N z_2^N$. So if the order of $P$ divides $N$, we have $z_2^N = 1$ and $y^{LMN} = t^{SMN}$. But we know that $\langle t \rangle \cap K$ is trivial, so $y^{LMN} = 1$, hence $y$ is of infinite order, therefore $y \in P$. Thus, $G_0 = H \cap K$ is contained in $P$, hence it is finite.

2. $H$ maps onto an infinite dihedral group $D$ with a finite kernel $P$.

   Let $D = \langle d', x' \rangle$, where $d'$ is of order 2, $x'$ is of infinite order, and $d' x' d' = (x')^{-1}$. Let $d$ and $x$ be preimages of $d'$ and $x'$ correspondingly. Then for some integer $L \neq 0, t = x^L z$, where $z \in P$. For some big enough $M, t^M$ induces a trivial automorphism of $P$, because $P$ is finite. Consider $y \in G_0 = H \cap K$. Then the image of $y^2$ in $D$ belongs to the infinite cyclic...
Proceeding as in case (1), we show that $y^{2LMN} = 1$, so $y$ is of finite order, hence $y \in P$. Thus, $G_0 = H \cap K$ is contained in $P$, hence it is finite.

Note that $G_0$ does not contain elements of infinite order, since the intersection of $\langle t \rangle$ and $K$ is trivial.

In the notation of Lemma 8 from [2], $\text{Lab}(s)\text{Lab}(p_1p_2\overline{v_1})\text{Lab}^{-1}(s) \in H_1$ and $\text{Lab}(p_1p_2\overline{v_1}) \in H_1$. Without loss of generality, we can assume that the vertices $v_i, w_i, v_j, w_j$ belong to the image of $\langle t^N \rangle$ in $\text{Cayley}(G, H_1)$ because $\text{Lab}(p_1)$ and $\text{Lab}(p_2)$ belong to $\langle t^N \rangle$. The element $\text{Lab}(s)$ conjugates $\text{Lab}(p_1p_2\overline{v_1})$ into $H$. By our assumption, $\text{Lab}(p_1p_2\overline{v_1}) = (t^Nc)$ for some $c \neq 0$. Then $\text{Lab}(s)$ belongs to the commensurator of $\langle t^N \rangle$ and hence to the commensurator of $\langle t \rangle$ which is $t$. On the other hand, $\text{Lab}(s) \in K$, so $\text{Lab}(s) \in H \cap K = G_0$. This is the desired contradiction. The rest of the proof is as in the proof of Theorem 2 in [2].

**Proposition 1.** Let $G$ be a non-elementary hyperbolic group with a fixed set of generators. Let $f_G$ be the growth function of $G$, that is $f_G(r) = |\{g \in G; |g| \leq r\}|$. Let $c, d \in G$ be elements of infinite order. Fix some big enough $N$. Then there exists a constant $\mu > 0$ such that for any big enough $r$ there exist words $s_i$ of the form $s_i = c^{N_i}u_iw_iw_i^{N_i}$, $i = 1, 2, \cdots, m$ such that

1. $m \geq \mu f_G(r)$,
2. $|s_i| \leq r$,
3. $N \leq N_i \leq 2N$ and $N \leq N_i'' \leq 2N$,
4. for $i \neq j, s_i \neq s_j$,
5. all $s_i$ are quasigeodesic words with some constants $(\eta, \epsilon)$.

**Proof.** Consider the ball in $\text{Cayley}(G)$ of radius $r_1 = r - |c^{2N}| - |d^{2N}| - 2\delta_1 - |c| - |d|$ and let $m_1 = |B(r_1)| = f_G(r_1)$. Let $z_1, \cdots, z_{m_1}$ be geodesic words for all elements in $B(r_1)$, that is $|z_i| \leq r_1$. For each $z_i$ consider the word $c^{2N}z_id^{2N}$. As $c^{2N}$ and $d^{2N}$ are quasigeodesics, for big enough $r$ we have the following picture, where $\text{Lab}(p) = c^{2N}, \text{Lab}(q) = z_i$, and $\text{Lab}(t) = d^{2N}$.
Using $\delta$-hyperbolicity of $\text{Cayley}(G)$, we obtain paths $l_1$ and $l_2$ such that $p = p_1p_2, q = q_1q_2q_3, t = t_1t_2, |l_1| \leq \delta + |c|, |l_2| \leq \delta + |d|, \text{Lab}(p_1) = c^{N_i}, \text{Lab}(t_2) = d^{N_i}, \text{Lab}(p_2) = c^{2N - N_i}, \text{and } \text{Lab}(t_1) = d^{2N - N''_i}$. 


We leave out the cases when \( N_i' < N \) and \( N_i'' < N \). The number of such cases is bounded by 
\[
m_0 = N \cdot f_G(\delta + |c|) \cdot N \cdot f_G(\delta + |d|).
\]

We know that for any \( k \) there exists \( \rho > 0 \) such that \( f_G(r - k) \geq f_G(r) \) for all 
big enough \( r \), so we can choose \( \mu > 0 \) such that for all big enough \( r \) the following holds: 
\[
m = m_1 - m_0 = f_G(r - |c|^{2N} - |d|^{2N} - 2\delta - |c| - |d|) - n \geq \mu f_G(r),
\]
satisfying all the conditions of Proposition 1. Note that condition (5) follows from the local 
property of quasigeodesics. 

\[\square\]

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