Darboux transformations for a twisted derivation and quasideterminant solutions to the super KdV equation

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This paper is concerned with a generalized type of Darboux transformations defined in terms of a twisted derivation $D$ satisfying $D(AB) = D(A) + \sigma(A)B$ where $\sigma$ is a homomorphism. Such twisted derivations include regular derivations, difference and $q$-difference operators and superderivatives as special cases. Remarkably, the formulae for the iteration of Darboux transformations are identical with those in the standard case of a regular derivation and are expressed in terms of quasideterminants. As an example, we revisit the Darboux transformations for the Manin–Radul super KdV equation, studied in Liu and Mañas (Liu & Mañas 1997a Phys. Lett. B 396, 133–140 (doi:10.1016/S0370-2693(97)00134-2)). The new approach we take enables us to derive a unified expression for solution formulae in terms of quasideterminants, covering all cases at once, rather than using several subcases. Then, by using a known relationship between quasideterminants and superdeterminants, we obtain expressions for these solutions as ratios of superdeterminants. This coincides with the results of Liu and Mañas in all the cases they considered but also deals with the one subcase in which they did not obtain such an expression. Finally, we obtain another type of quasideterminant solution to the Manin–Radul super KdV equation constructed from its binary Darboux transformations. These can also be expressed as ratios of superdeterminants and are a substantial generalization of the solutions constructed using binary Darboux transformations in earlier work on this topic.

Keywords: Darboux transformations; twisted derivations; super KdV equation; quasideterminants; superdeterminants

1. Introduction

There have been many papers written recently on noncommutative versions of soliton equations, such as the KP equation, the KdV equation, the Hirota–Miwa equation and the two-dimensional Toda lattice equation (Kupershmidt 2000; Paniak 2001; Hamanaka 2003; Hamanaka & Toda 2003; Wang & Wadati 2003a,b, 2004; Sakakibara 2004; Dimakis & Müller-Hoissen 2005; Nimmo 2006; Gilson & Nimmo 2007; Gilson et al. 2007; Li & Nimmo 2008; Li et al. 2009).

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It has been shown that noncommutative integrable systems often have solutions expressed in terms of quasideterminants (Gelfand & Retakh 1991). In Gilson & Nimmo (2007), for example, two families of solutions of the noncommutative KP equation were presented which were termed quasi-Wronskians and quasi-Grammians. The origin of these solutions was explained by Darboux and binary Darboux transformations. The quasideterminant solutions were then verified directly using formulae for derivatives of quasideterminants (see also Dimakis & Müller-Hoissen 2005).

Supersymmetric integrable systems are a particular noncommutative extension of integrable systems, and have attracted much attention because of their applications in physics. Some well-known examples are supersymmetric versions of the KdV, KP, sine-Gordon, nonlinear Schrödinger equation, AKNS and Harry Dym equations (Chaichian & Kulish 1978; Kupershmidt 1984; Manin & Radul 1985; Roelofs & Kersten 1992; Liu 1995; Morosi & Pizzocchero 1996; Mathieu 1998; Siddiq et al. 2006). Among these, the Manin–Radul super KdV (MRSKdV) equation is perhaps the best known and has been studied extensively and a number of interesting properties have been established. We mention here the existence of an infinite number of conservation laws, a bi-Hamiltonian structure (Figueroa-O'Farrill et al. 1991; Oevel & Popowicz 1991), a bilinear form (McArthur & Yung 1993; Carstea et al. 2001) and Darboux transformations (Liu & Mañas 1997a, b; Siddiq et al. 2006).

Partly motivated by the properties of superderivatives, which we describe in §2, we consider a generalized derivation which has regular derivations, difference operators, \(q\)-difference operators and superderivatives as some of its special cases. We call this a twisted derivation, following the terminology used in Hartwig et al. (2006), De Concini & Procesi (1993) and Dimakis & Müller-Hoissen (2006). We show that one can formulate Darboux transformations for such twisted derivations and the iteration formulae are expressed in terms of quasideterminants in which one simply replaces the derivative with the twisted derivation. A special case of this result can be used to construct solutions to supersymmetric equations in terms of quasideterminants.

In Liu & Mañas (1997a), solutions for the MRSKdV equation were constructed by iterating Darboux transformations by considering the cases of an even number and an odd number of iterations separately. All but one of the formulae obtained by the authors expressed the solutions in terms of superdeterminants.

In this paper, we use an alternative approach to the Darboux transformations using quasideterminants. This is successful in obtaining unified formulae for the solutions, not depending on the parity of the number of iterations. From these quasideterminant solutions, we are not only able to recover superdeterminant solutions given in Liu & Mañas (1997a) but also get the superdeterminant representation in the one case they did not.

The paper is organized as follows. In §2, we give a brief review on superdeterminants, quasideterminants and the relationship between them. In §3, we define a twisted derivation and related Darboux transformation. We obtain a formula for iteration of this twisted Darboux transformation in terms of quasideterminants. Section 4 gives applications of both Darboux and binary Darboux transformations to the MRSKdV system. In this section, we obtain two general solution formulae in terms of quasideterminants, obtained using iterated Darboux and binary Darboux transformations, respectively, and then
show how these can be expressed in terms of superdeterminants. On the other hand, we present quasideterminant solutions constructed from its binary Darboux transformations to the MRSKdV system. In §5, we present conclusions.

2. Superdeterminants and quasideterminants

In this section, we collect together some basic facts about supersymmetric objects, such as superderivatives, supermatrices, supertransposes and superdeterminants (DeWitt 1984; Berezin 1987), about quasideterminants (Gelfand & Retakh 1991; Etingof et al. 1997; Gelfand et al. 2005) and about the relationship between superdeterminants and quasideterminants (Bergvelt & Rabin 1999). The reader is referred to the above-mentioned literature for more details.

(a) Superderivatives, supertransposes and superdeterminants

Let $\mathcal{A}$ be a supercommutative, associative, unital superalgebra over a (commutative) ring $K$. There is a standard $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $\mathcal{A}_i \cdot \mathcal{A}_j \subseteq \mathcal{A}_{i+j}$. Elements of $\mathcal{A}$ that belong to either $\mathcal{A}_0$ or $\mathcal{A}_1$ are called homogeneous; those in $\mathcal{A}_0$ are called even and those in $\mathcal{A}_1$ are called odd. The parity $|a|$ of a homogeneous element $a$ is 0 if it is even and 1 if it is odd. It follows that if $a, b$ are homogeneous then $|ab| = |a| + |b|$. Supercommutativity means that all homogeneous elements $a, b$ satisfy $ba = (-1)^{|a||b|}ab$, i.e. even elements commute with all elements and odd elements anticommute. In particular, this implies that $a_i^2 = 0$, for all $a_i \in \mathcal{A}_1$.

(i) Grade involution and superderivatives

The homomorphism $\hat{\cdot} : \mathcal{A} \to \mathcal{A}$ satisfying $\hat{a}_i = (-1)^ia_i$ for $a_i \in \mathcal{A}_i$ is called the grade involution. For general $a \in \mathcal{A}$, expressed as $a = a_0 + a_1$ where $a_i \in \mathcal{A}_i$, we have $\hat{a} = a_0 - a_1$. Also for any matrix $M = (m_{ij})$ over $\mathcal{A}$, $\hat{M} := (\hat{M}_{ij})$. It is easy to see that $\hat{\hat{a}} = a$.

A superderivative $D$ is a linear mapping $D : \mathcal{A} \to \mathcal{A}$ such that $D(K) = 0$ and $D(\mathcal{A}_i) \subseteq \mathcal{A}_{i+1}$ and satisfying $D(ab) = D(a)b + \hat{a}D(b)$. One way to obtain a superderivative is as $D = \partial_\theta + \theta \partial_x$ where $x$ is an even variable and $\theta$ is an odd (Grassmann) variable. For such a superderivative $D^2 = \partial_x$. Note that this is the only place in this article where $\theta$ is used to denote a Grassmann variable. Below, $\theta$ will be used to denote eigenfunctions, as is commonly done in connection with Darboux transformations.

Note that since $D(\mathcal{A}_0) \subseteq \mathcal{A}_1$ and $D(\mathcal{A}_1) \subseteq \mathcal{A}_0$, it follows that $D(\hat{a}) = D(a_0) - D(a_1) = -\hat{D}(a)$ and so grade involution and superderivatives anticommute.

(ii) Even and odd supermatrices

A block matrix $M = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ over $\mathcal{A}$ where $X$ is $r \times m$, $Y$ is $r \times n$, $Z$ is $s \times m$ and $T$ is $s \times n$ for integers $r, s, m$ and $n$ with $r, m \geq 1$ and $s, n \geq 0$ is called an $(r|s) \times (m|n)$ supermatrix. It is said to be even, and has parity 0, if $X$ and $T$ (if not empty) have even entries and $Y$ and $Z$ (if non-empty) have odd entries.
the other hand, if \( X \) and \( T \) have odd entries and \( Y \) and \( Z \) have even entries then \( \mathcal{M} \) is said to be odd, and has parity 1. It is said to be homogeneous if it is either even or odd.

(iii) Supertransposes

The supertranspose of a homogeneous supermatrix, \( \mathcal{M} \), is defined to be

\[
\mathcal{M}^{st} = \begin{pmatrix}
X^t & (-1)^{1/|\mathcal{M}|} Y^t & (-1)^{|\mathcal{M}|} Z^t \\
(\mathcal{M})^t & \mathcal{T}^t
\end{pmatrix},
\]

(2.1)

where \(^t\) denotes the normal matrix transpose. In particular, an even \((m|n)\)-row vector has the form \((a_{01}, a_{02}, \ldots, a_{0m}, a_{11}, a_{12}, \ldots, a_{1n})\), where \( a_{ij} \in \mathcal{A}_i \), and its supertranspose is

\[
(a_{01}, a_{02}, \ldots, a_{0m}, a_{11}, a_{12}, \ldots, a_{1n})^{st} = (a_{01}, a_{02}, \ldots, a_{0m}, -a_{11}, -a_{12}, \ldots, -a_{1n})^t.
\]

(2.2)

On the other hand, an odd \((m|n)\)-row vector has the form \((a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})\), and the supertranspose

\[
(a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})^{st} = (a_{11}, a_{12}, \ldots, a_{1m}, a_{01}, a_{02}, \ldots, a_{0n})^t.
\]

(2.3)

For homogeneous supermatrices \( \mathcal{L} \), \( \mathcal{M} \) and \( \mathcal{N} \), it is known that

\[
(\mathcal{MN})^{st} = (-1)^{|\mathcal{M}|} \mathcal{N}^{st} \mathcal{M}^{st},
\]

(2.4)

\[
(\mathcal{M}^{st})^{st} = (-1)^{|\mathcal{M}|} \mathcal{M}^{st}
\]

(2.5)

and

\[
(\mathcal{LMN})^{st} = (-1)^{|\mathcal{L}|+|\mathcal{M}|+1} \mathcal{N}^{st} \mathcal{M}^{st} \mathcal{L}^{st}.
\]

(2.6)

Also, the supertranspose commutes with the grade involution but not with a superderivative; for a homogeneous matrix \( \mathcal{M} \),

\[
(D(\mathcal{M}))^{st} = (-1)^{|\mathcal{M}|} D(\mathcal{M}^{st}).
\]

(2.7)

(iv) Superdeterminants

Consider an even \((m|n) \times (m|n)\) supermatrix \( \mathcal{M} = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix} \) in which \( X \) and \( T \) are non-singular. The superdeterminant, or Berezinian, of \( \mathcal{M} \) is defined as

\[
\text{Ber}(\mathcal{M}) = \frac{\det(X - YT^{-1}Z)}{\det(T)} = \frac{\det(X)}{\det(T - ZX^{-1}Y)}.
\]

It is also convenient to define

\[
\text{Ber}^*(\mathcal{M}) = \frac{1}{\text{Ber}(\mathcal{M})}.
\]
(b) Quasideterminants

An \( n \times n \) matrix \( M = (m_{ij}) \) over a ring \( R \) (noncommutative, in general) has \( n^2 \) quasideterminants written as \( |M|_{i,j} \) for \( i, j = 1, \ldots, n \), which are also elements of \( R \). They are defined recursively by

\[
|M|_{i,j} = m_{i,j} - r^j_i (M^{i,j})^{-1} c^i_j, \quad M^{-1} = (|M|_{i,j}^{-1})_{i,j=1,\ldots,n}.
\] (2.8)

In the above \( r^j_i \) represents the \( i \)th row of \( M \) with the \( j \)th element removed, \( c^i_j \) the \( j \)th column with the \( i \)th element removed and \( M^{i,j} \) the submatrix obtained by removing the \( i \)th row and the \( j \)th column from \( M \). Quasideterminants can also be denoted as shown below by boxing the entry about which the expansion is made

\[
|M|_{i,j} = \begin{vmatrix} M^{i,j} & c^i_j \\ r^j_i & m_{i,j} \end{vmatrix}.
\]

Note that if the entries in \( M \) commute then

\[
|M|_{i,j} = (-1)^{i+j} \frac{\text{det}(M)}{\text{det}(M^{i,j})}.
\] (2.9)

(i) Invariance under row and column operations

The quasideterminants of a matrix have invariance properties similar to those of determinants under elementary row and column operations applied to the matrix. Consider the following quasideterminant of an \( n \times n \) matrix:

\[
\begin{vmatrix} E & 0 \\ F & g \end{vmatrix} \begin{vmatrix} A & B \\ C & d \end{vmatrix} = \begin{vmatrix} EA & EB \\ FA + gC & FB + gd \end{vmatrix} = g(d - CA^{-1}B) = g \begin{vmatrix} A & B \\ C & d \end{vmatrix}.
\] (2.10)

The above formula can be used to understand the effect on a quasideterminant of certain elementary row operations involving multiplication on the left. This formula excludes those operations which add left-multiples of the row containing the expansion point to other rows since there is no simple way to describe the effect of these operations. For the allowed operations, however, the results can be easily described; left-multiplying the row containing the expansion point by \( g \) has the effect of left-multiplying the quasideterminant by \( g \) and all other operations leave the quasideterminant unchanged. There is analogous invariance under column operations involving multiplication on the right.

(ii) Noncommutative Jacobi identity

There is a quasideterminant version of Jacobi identity for determinants, called the noncommutative Sylvester’s theorem by Gelfand & Retakh (1991).
The simplest version of this identity is given by
\[
\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & C \\ E & i \end{vmatrix} - \begin{vmatrix} A & B \\ E & h \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & f \end{vmatrix} - 1 = \begin{vmatrix} A & C \\ D & g \end{vmatrix},
\] (2.11)
where \( f, g, h, i \in \mathbb{R} \), \( A \) is an \( n \times n \) matrix and \( B \) and \( C \) (respectively \( D, E \)) are column (respectively row) \( n \)-vectors over \( \mathbb{R} \).

(iii) Quasi-Plücker coordinates

Given an \((n + k) \times n\) matrix \( A \), denote the \( i \)th row of \( A \) by \( A_i \), the submatrix of \( A \) having rows with indices in a subset \( I \) of \( \{1, 2, \ldots, n + k\} \) by \( A_I \) and \( A_{\{1,\ldots,n+k\}\setminus\{i\}} \) by \( A_i \). Given \( i, j \in \{1, 2, \ldots, n + k\} \) and \( I \) such that \( |I| = n - 1 \) and \( j \notin I \), one defines the (right) quasi-Plücker coordinates
\[
r^I_{ij} = r^{I}_{ij}(A) := \begin{vmatrix} A_I \\ A_i \\ 0 \end{vmatrix}_{ns}^{-1} \begin{vmatrix} A_i \\ A_J \\ 0 \end{vmatrix}_{ns} = -\begin{vmatrix} A_I \\ A_i \\ 0 \end{vmatrix} = -\begin{vmatrix} A_i \\ A_j \\ 1 \end{vmatrix},
\] (2.12)
for any column index \( s \in \{1, \ldots, n\} \). The final equality in equation (2.12) comes from an identity of the form (2.11) and proves that the definition is independent of the choice of \( s \).

(iv) Solving linear systems

Solutions of systems of linear systems over an arbitrary ring can be expressed in terms of quasideterminants.

**Theorem 2.1.** Let \( A = (a_{ij}) \) be an \( n \times n \) matrix over a ring \( \mathbb{R} \). Assume that all the quasideterminants \( |A|_{ij} \) are defined and invertible. Then the system of equations
\[
x_1 a_{1i} + x_2 a_{2i} + \cdots + x_n a_{ni} = b_i, \quad 1 \leq i \leq n
\] (2.13)
has the unique solution
\[
x_i = \sum_{j=1}^{n} b_j |A|_{ij}^{-1}, \quad i = 1, \ldots, n.
\] (2.14)

Let \( A_i(b) \) be the \( n \times n \) matrix obtained by replacing the \( l \)th row of the matrix \( A \) with the row \((b_1, \ldots, b_n)\). Then we have the following noncommutative version of Cramer’s rule.

**Theorem 2.2.** In notation of theorem 2.1, if the quasideterminants \( |A|_{ij} \) and \( |A_i(b)|_{ij} \) are well defined, then
\[
x_i |A|_{ij} = |A_i(b)|_{ij}.
\]

(c) Relationship between quasideterminants and superdeterminants

The basic formulae connecting quasideterminants of even supermatrices with their Berezinians are given in Bergvelt & Rabin (1999).
Theorem 2.3. Let $\mathcal{M}$ be an $(m|n) \times (m|n)$-supermatrix. Then

$$|\mathcal{M}|_{i,j} = \begin{cases} 
(-1)^{i+j} \frac{\operatorname{Ber}(\mathcal{M})}{\operatorname{Ber}(\mathcal{M}^{i,j})}, & 1 \leq i, j \leq m, \\
(-1)^{i+j} \frac{\operatorname{Ber}^*(\mathcal{M})}{\operatorname{Ber}^*(\mathcal{M}^{i,j})}, & m + 1 \leq i, j \leq m + n
\end{cases} \quad (2.15)$$

(cf. equation (2.9)).

Roughly speaking, a quasideterminant with indices in one of the even blocks of $\mathcal{M}$ is given as a ratio of Berezinians. A quasideterminant with its indices in one of the odd blocks is not well defined.

3. Darboux transformations for twisted derivations

Consider now a more general setting in which $\mathcal{A}$ is an associative, unital algebra over ring $K$, not necessarily graded. Suppose that there is a homomorphism $\sigma: \mathcal{A} \to \mathcal{A}$ (i.e. for all $\alpha \in K$, $a, b \in \mathcal{A}$, $\sigma(\alpha a) = \alpha \sigma(a)$, $\sigma(a + b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(a)\sigma(b)$) and a twisted derivation $D: \mathcal{A} \to \mathcal{A}$ satisfying $D(K) = 0$ and $D(ab) = D(a)b + \sigma(a)D(b)$ (De Concini & Procesi 1993; Dimakis & M"uller-Hoissen 2006; Hartwig et al. 2006). It may also be possible to understand twisted derivations in the broader framework of bidifferential graded algebras given in Dimakis & Müller-Hoissen (2009), in which Darboux transformations are also considered.

Simple examples arise in the case that elements $a \in \mathcal{A}$ depend on a variable $x$, say.

Derivative. Here $D = \partial/\partial x$ satisfies $D(ab) = D(a)b + aD(b)$ and $\sigma$ is the identity mapping.

Forward difference. The homomorphism is the shift operator $T$, where $T(a(x)) = a(x + 1)$ and the twisted derivation is

$$\Delta(a(x)) = \frac{a(x + h) - a(x)}{h},$$

satisfying $\Delta(ab) = D(a)b + T(a)D(b)$.

Jackson derivative. The homomorphism is a $q$-shift operator defined by $S_q(a(x)) = a(qx)$ and the twisted derivation is

$$D_q(a(x)) = \frac{a(qx) - a(x)}{(q - 1)x},$$

satisfying $D_q(ab) = D_q(a)b + S_q(a)D_q(b)$.

Superderivative. As described in §2.1, for $a, b \in \mathcal{A}$, a superalgebra, $D(ab) = D(a)b + \hat{a}D(b)$ where $\hat{a}$ is the grade involution.

There are a number of simple properties of such a twisted derivation, which are summarized in the following lemma.
**Lemma 3.1.**

(i) Let $A, B$ be matrices over $A$. Whenever $AB$ is defined, $\sigma(AB) = \sigma(A)\sigma(B)$ and $D(AB) = D(A)B + \sigma(A)D(B)$.

(ii) Let $A$ be an invertible matrix over $A$. Then $\sigma(A)^{-1} = \sigma(A^{-1})$ and $D(A^{-1}) = -\sigma(A)^{-1}D(A)A^{-1}$.

(iii) Let $A, B$ and $C$ be matrices over $A$ such that $AB^{-1}C$ is well defined. Then

$$D(AB^{-1}C) = D(A)B^{-1}C + \sigma(A)\sigma(B)^{-1}(D(C) - D(B)B^{-1}C).$$

(a) Darboux transformations

Let $\theta_0, \theta_1, \theta_2, \ldots$ be a sequence in $A$. Consider the sequence $\theta[0], \theta[1], \theta[2], \ldots$ in $A$, generated from the first sequence by Darboux transformations of the form

$$G_0 = \sigma(\theta)D\theta^{-1} = D - D(\theta)\theta^{-1},$$

where $D$ and $\sigma$ are the twisted derivation and homomorphism defined above. To be specific, $\theta[0] = \theta_0$ and $G[0] = G_{\theta[0]}$, then let

$$\theta[1] = G[0](\theta_1) = D(\theta_1) - D(\theta_0)\theta^{-1}_0\theta_1$$

and $G[1] = G_{\theta[1]}$, $\theta[2] = G[1] \circ G[0](\theta_2)$ and $G[2] = G_{\theta[2]}$ and so on. In general, for $k \in \mathbb{N}$,

$$\theta[k] = G[k - 1] \circ G[k - 2] \circ \cdots \circ G[0](\theta_k), \quad G[k] = \sigma(\theta[k])D\theta[k]^{-1},$$

and we require that each $\theta[k]$ is invertible.

In the standard case, $D = \partial$ and $\sigma = \text{Id}$, it is well known that the terms in the sequence of Darboux transformations have closed form expressions in terms of the original sequence. In the case that $A$ is commutative, they are expressed as ratios of Wronskian determinants (Crum 1955),

$$\theta[n] = \begin{vmatrix}
\theta_0 & \cdots & \theta_{n-1} & \theta_n \\
\theta_0^{(1)} & \cdots & \theta_{n-1}^{(1)} & \theta_n^{(1)} \\
\vdots & & \vdots & \vdots \\
\theta_0^{(n)} & \cdots & \theta_{n-1}^{(n)} & \theta_n^{(n)} \\
\end{vmatrix}, \quad n \in \mathbb{N},$$

$\theta[n]$,
where $\theta_j^{(i)}$ denotes $\partial^i(\theta_j)$. In the case that $A$ is not commutative, the terms in the sequence are expressed as quasideterminants (Etingof et al. 1997),

$$
\theta[n] = \begin{vmatrix}
\theta_0 & \ldots & \theta_{n-1} & \theta_n \\
\theta_0^{(1)} & \ldots & \theta_{n-1}^{(1)} & \theta_n^{(1)} \\
\vdots & \ddots & \vdots & \vdots \\
\theta_0^{(n-1)} & \ldots & \theta_{n-1}^{(n-1)} & \theta_n^{(n-1)} \\
\theta_0^{(n)} & \ldots & \theta_{n-1}^{(n)} & \theta_n^{(n)}
\end{vmatrix}, \quad n \in \mathbb{N}.
$$

(3.5)

The following theorem gives a generalization of this formula to the case of general $D$ and $\sigma$. Note in particular that the expressions do not depend on $\sigma$ and are obtained simply by replacing $v$ with $D$.

**Theorem 3.2.** Let $\phi[0] = \phi$ and for $n \in \mathbb{N}$ let

$$
\phi[n] = D(\phi[n - 1]) - D(\theta[n - 1])\theta[n - 1]^{-1}\phi[n - 1],
$$

where $\theta[n] = \phi[n]_{|\phi \rightarrow \theta_n}$. Then, for $n \in \mathbb{N}$,

$$
\phi[n] = \begin{vmatrix}
\theta_0 & \ldots & \theta_{n-1} & \phi \\
D(\theta_0) & \ldots & D(\theta_{n-1}) & D(\phi) \\
\vdots & \ddots & \vdots & \vdots \\
D^{n-1}(\theta_0) & \ldots & D^{n-1}(\theta_{n-1}) & D^{n-1}(\phi) \\
D^n(\theta_0) & \ldots & D^n(\theta_{n-1}) & D^n(\phi)
\end{vmatrix}.
$$

(3.6)

**Proof.** (By induction). The case $n = 1$ follows from the definitions (2.8) and (3.2). Now assume that equation (3.6) holds for $n = k$. Then

$$
\phi[k] = D^k(\phi) - \left[D^k(\theta_0) \quad \ldots \quad D^k(\theta_{k-1})\right] \Theta^{-1} \left[\begin{array}{c}
\phi \\
D(\phi) \\
\vdots \\
D^{k-1}(\phi)
\end{array}\right],
$$

where

$$
\Theta = \begin{vmatrix}
\theta_0 & \ldots & \theta_{k-1} \\
D(\theta_0) & \ldots & D(\theta_{k-1}) \\
\vdots & \ddots & \vdots \\
D^{k-1}(\theta_0) & \ldots & D^{k-1}(\theta_{k-1})
\end{vmatrix}.
$$

To complete the proof we must show that $\phi[k + 1] = D(\phi[k]) - D(\theta[k])\theta[k]^{-1}\phi[k]$ can be written in the form (3.6) for $n = k + 1$. 

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Using lemma 3.1, one obtains

\[
D(\phi[k]) = \begin{vmatrix}
\theta_0 & \cdots & \theta_{k-1} & \phi \\
D(\theta_0) & \cdots & D(\theta_{k-1}) & D(\phi) \\
\vdots & \cdots & \vdots & \vdots \\
D^{k-2}(\theta_0) & \cdots & D^{k-2}(\theta_{k-1}) & D^{k-2}(\phi) \\
D^{k-1}(\theta_0) & \cdots & D^{k-1}(\theta_{k-1}) & D^{k-1}(\phi) \\
D^{k+1}(\theta_0) & \cdots & D^{k+1}(\theta_{k-1}) & D^{k+1}(\phi)
\end{vmatrix}
\]

\[
+ \sigma \begin{pmatrix}
\theta_0 & \cdots & \theta_{k-1} & 0 \\
D(\theta_0) & \cdots & D(\theta_{k-1}) & 0 \\
\vdots & \cdots & \vdots & \vdots \\
D^{k-2}(\theta_0) & \cdots & D^{k-2}(\theta_{k-1}) & 0 \\
D^{k-1}(\theta_0) & \cdots & D^{k-1}(\theta_{k-1}) & 1 \\
D^{k}(\theta_0) & \cdots & D^{k}(\theta_{k-1}) & 0
\end{pmatrix} \phi[k],
\]

and, so,

\[
D(\theta[k]) = \begin{vmatrix}
\theta_0 & \cdots & \theta_{k-1} & \theta_k \\
D(\theta_0) & \cdots & D(\theta_{k-1}) & D(\theta_k) \\
\vdots & \cdots & \vdots & \vdots \\
D^{k-2}(\theta_0) & \cdots & D^{k-2}(\theta_{k-1}) & D^{k-2}(\theta_k) \\
D^{k-1}(\theta_0) & \cdots & D^{k-1}(\theta_{k-1}) & D^{k-1}(\theta_k) \\
D^{k+1}(\theta_0) & \cdots & D^{k+1}(\theta_{k-1}) & D^{k+1}(\theta_k)
\end{vmatrix}
\]

\[
+ \sigma \begin{pmatrix}
\theta_0 & \cdots & \theta_{k-1} & 0 \\
D(\theta_0) & \cdots & D(\theta_{k-1}) & 0 \\
\vdots & \cdots & \vdots & \vdots \\
D^{k-2}(\theta_0) & \cdots & D^{k-2}(\theta_{k-1}) & 0 \\
D^{k-1}(\theta_0) & \cdots & D^{k-1}(\theta_{k-1}) & 1 \\
D^{k}(\theta_0) & \cdots & D^{k}(\theta_{k-1}) & 0
\end{pmatrix} \theta[k].
\]

Then, using equation (2.11), \(D(\phi[k]) - D(\theta[k])\theta[k]^{-1}\phi[k]\) can be expressed in the required form.

As an application of this theorem, in the following section we will use it to construct solutions of the super KdV equation.

### 4. The Manin–Radul super KdV equation

The MRSKdV system (Manin & Radul 1985) is

\[
\alpha_t = \frac{1}{4}(\alpha_{xx} + 3\alpha D(\alpha) + 6\alpha u)_x, \quad u_t = \frac{1}{4}(u_{xx} + 3u^2 + 3\alpha D(u))_x,
\]

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where $u$ and $\alpha$ are even and odd dependent variables, respectively, $x$ and $t$ are even independent variables and $D$ is the superderivative satisfying $D^2 = \partial_x$. This system has the Lax pair

$$L = \partial^2_x + \alpha D + u$$

(4.2)

and

$$M = \partial^2_x + \frac{3}{4}((\alpha \partial_x + \partial_x \alpha)D + u\partial_x + \partial_x u),$$

(4.3)

in the sense that $L_t + [L, M] = 0$ implies equation (4.1). Eigenfunctions satisfy

$$L(\phi) = \lambda \phi, \quad \phi_t = M(\phi),$$

(4.4)

for eigenvalue $\lambda$.

(a) Darboux transformations

A Darboux transformation for this system (Liu & Mañas 1997a) is

$$\phi \to D(\phi) - D(\theta)\theta^{-1} \phi,$$

(4.5)

$$\alpha \to -\alpha + 2(D(\theta)\theta^{-1})_x$$

(4.6)

and

$$u \to u + D(\alpha) - 2D(\theta)\theta^{-1}(\alpha - (D(\theta)\theta^{-1})_x),$$

(4.7)

where $\theta$ is an invertible, and hence necessarily even, solution of equation (4.4). Note that it is an example of the general type of Darboux transformation discussed in §3.1. As discussed there, this transformation may be iterated by taking solutions $\theta_0, \theta_1, \theta_2, \ldots$ of equation (4.4) to obtain

$$\phi[k + 1] = D(\phi[k]) - D(\theta[k])\theta[k]^{-1}\phi[k]$$

(4.8)

and

$$\theta[k] = \phi[k]|_{\phi \to \theta_k}. $$

(4.9)

The requirement that each $\theta[k]$ is invertible means that it must be even and consequently that $\theta$, must have parity $i$. The corresponding solutions of MRSKdV are $\alpha[0] = \alpha$, $u[0] = u$ and

$$\alpha[k + 1] = -\alpha[k] + 2(D(\theta[k])\theta[k]^{-1})_x$$

(4.10)

and

$$u[k + 1] = u[k] + D(\alpha[k]) - 2D(\theta[k])\theta[k]^{-1}(\alpha[k] - (D(\theta[k])\theta[k]^{-1})_x).$$

(4.11)
Theorem 3.2 gives a closed-form expression (3.6) for $\phi[n]$ as a quasideterminant. We will obtain the corresponding expressions for $\alpha[n]$ and $u[n]$ in terms of quasideterminants of the form

$$Q_n(i,j) = \begin{vmatrix} \theta_0 & \cdots & \theta_{n-1} & 0 \\ D(\theta_0) & \cdots & D(\theta_{n-1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-j-2}(\theta_0) & \cdots & D^{n-j-2}(\theta_{n-1}) & 0 \\ D^{n-j-1}(\theta_0) & \cdots & D^{n-j-1}(\theta_{n-1}) & 1 \\ D^{n-j}(\theta_0) & \cdots & D^{n-j}(\theta_{n-1}) & 0 \\ \vdots & \ddots & \vdots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_{n-1}) & 0 \\ D^{n+i}(\theta_0) & \cdots & D^{n+i}(\theta_{n-1}) & 0 \end{vmatrix}$$

for any $s = 1, \ldots, n$ (see equation (2.12)).

The following lemma records some useful properties of the $Q_n(i,j)$.

**Lemma 4.1.**

(i) $\widehat{Q_n(i,j)} = (-1)^{i+j+1} Q_n(i,j)$, i.e. $Q_n(i,j)$ has the parity $(-1)^{i+j+1}$,

(ii) $D(\theta_0)\theta_0^{-1} = -Q_1(0,0)$ and $D(\theta[k])\theta[k]^{-1} = -Q_k(0,0) - Q_{k+1}(0,0)$ for $k \geq 1$,

(iii) $Q_{n+1}(0,1) = Q_n(1,0) + Q_{n+1}(0,0) Q_n(0,0)$.

**Proof.**

(i) This follows from the facts that $\theta_i$ has parity $i$ and $D$ is odd, and from the invariance properties of quasideterminants (2.10).

(ii) It is obvious from the definition (4.12) that $Q_1(0,0) = -D(\theta_0)\theta_0^{-1}$. The second result follows by using the same method used in the proof of theorem 3.2 with $\sigma = \wedge$. 

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Expanding $Q_{n+1}(0,1)$, written in the form (4.12), using equation (2.11)
gives the terms on the right-hand side. □

**Theorem 4.2.** After $n$ repeated Darboux transformations, the MRSKdV system
has new solutions $a[n]$ and $u[n]$ expressed in terms of $Q_n(0,0)$ and $Q_n(0,1)$,

$$a[n] = (-1)^n a - 2Q_n(0,0)_x$$ (4.13)

and

$$u[n] = u - 2Q_n(0,1)_x - 2Q_n(0,0)((-1)^n a - Q_n(0,0)_x) + \frac{1 - (-1)^n}{2} D(a).$$ (4.14)

**Proof.** (By induction.) First consider equation (4.13). For $n = 1$ we must show that

$$a[1] = -\alpha + 2(D(\theta_0)\theta_0^{-1})_x.$$ This is clear from equation (4.10) for $k = 0$ and lemma 4.1(ii).

By using equation (4.10) in general case and lemma 4.1(ii), assuming that equation (4.13) holds for $n = k$, we have

$$a[k + 1] = -\alpha[k] + 2(D(\theta[k])\theta[k]^{-1})_x$$

$$= (-1)^{k+1} a - 2Q_{k+1}(0,0)_x,$$

as required.

The proof of equation (4.14) is very similar and makes use of lemma 4.1(iii). □

**Remark 4.3.** The quasideterminant solutions (4.13) and (4.14) could also be obtained by solving a noncommutative linear system. In Liu & Mañas (1997a), the solutions $a[n]$ and $u[n]$ for the MRSKdV system were given as

$$a[n] = (-1)^n a - 2\partial a_{n,n-1}$$ (4.15)

and

$$u[n] = u - 2\partial a_{n,n-2} - 2a_{n,n-1}((-1)^n a - \partial a_{n,n-1}) + \frac{1 - (-1)^n}{2} D\alpha,$$ (4.16)

where $a_{n,n-1}, a_{n,n-2}, \ldots, a_{n,0}$ satisfy the linear system

$$(D^n + a_{n,n-1}D^{n-1} + \cdots + a_{n,0})\theta_j = 0, \quad j = 0, \ldots, n - 1.$$ (4.17)

By using theorem 2.2 and equation (2.12), we can solve the above linear system to obtain

$$a_{n,n-i} = Q_n(0, i-1), \quad i = 1, 2, \ldots, n.$$ It is obvious that solutions $a[n]$ and $u[n]$ obtained here coincide with those given by equations (4.13) and (4.14).

**(b) From quasideterminants to superdeterminants I**

It is natural for a supersymmetric system to express the solutions in terms of superdeterminants and this was done, in all but one case, in Liu & Mañas (1997a). In this section, we will show systematically that the solutions given in theorem 4.2, expressed in terms of quasideterminants $Q_n(0,0)$ and $Q_n(0,1)$,
can be reexpressed in terms of superdeterminants. The expressions we will obtain coincide with the superdeterminant solutions found in Liu & Mañas (1997a) and we also find the superdeterminant expressions in the case that they did not.

Recall the property that $\theta_i$ has parity $i$. Let us therefore introduce the relabelling

$$\theta_{2k} = E_k, \quad \theta_{2k+1} = O_k. \quad (4.18)$$

Also, we write $D^{2j}(\theta) = \theta^{(j)}$ and $D^{2j+1}(\theta) = D(\theta^{(j)})$, where $(j)$ denotes the $j$th derivative with respect to $x$.

Consider the matrix

$$W_n = \begin{bmatrix} \theta_0 & \cdots & \theta_n \\ \vdots & \ddots & \vdots \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_n) \end{bmatrix}, \quad (4.19)$$

appearing in the definition (4.12) of $Q_n(i, j)$. There is a natural reordering of the rows and columns

$$W_n \rightarrow W_n' = \begin{bmatrix} X_n & Y_n \\ Z_n & T_n \end{bmatrix}, \quad (4.20)$$

which gives an even matrix $W_n$. This reordering does not change the value of any associated quasideterminant, as long as the expansion point in each refers to the same element. In the case that $n$ is even,

$$X_{2k} = \begin{bmatrix} E_0 & \cdots & E_{k-1} \\ \vdots & \ddots & \vdots \\ E^{(k-1)}_0 & \cdots & E^{(k-1)}_{k-1} \end{bmatrix}, \quad Y_{2k} = \begin{bmatrix} O_0 & \cdots & O_{k-1} \\ \vdots & \ddots & \vdots \\ O^{(k-1)}_0 & \cdots & O^{(k-1)}_{k-1} \end{bmatrix}$$

and $Z_{2k} = D(X_{2k})$ and $T_{2k} = D(Y_{2k})$ are all $k \times k$ matrices. In the case that $n$ is odd, $X_{2k+1}$ is $(k+1) \times (k+1)$, $Y_{2k+1}$ is $(k+1) \times k$, $Z_{2k+1}$ is $k \times (k+1)$ and $T_{2k+1}$ is a $k \times k$ matrix whose precise form can be easily deduced from the above description.

Similarly, consider the matrix

$$W'_n = \begin{bmatrix} \theta_0 & \cdots & \theta_n \\ \vdots & \ddots & \vdots \\ D^{n-3}(\theta_0) & \cdots & D^{n-3}(\theta_n) \\ D^{n-1}(\theta_0) & \cdots & D^{n-1}(\theta_n) \\ D^n(\theta_0) & \cdots & D^n(\theta_n) \end{bmatrix}, \quad (4.21)$$

appearing in the definition (4.12) of $Q_n(0, 1)$. A similar reordering of this matrix

$$W'_n \rightarrow W'_n = \begin{bmatrix} X'_n & Y'_n \\ Z'_n & T'_n \end{bmatrix}, \quad (4.22)$$

gives another even matrix $W'_n$ where, for example,

$$X'_{2k} = \begin{bmatrix} E_0 & \cdots & E_{k-1} \\ \vdots & \ddots & \vdots \\ E^{(k-2)}_0 & \cdots & E^{(k-2)}_{k-1} \\ E^{(k)}_0 & \cdots & E^{(k)}_{k-1} \end{bmatrix}.$$
Theorem 4.4. For $n \in \mathbb{N}$,

\[ Q_n(0, 0) = D(\log(B(W_n))), \quad Q_n(0, 1) = -\frac{B(W_n^\prime)}{B(W_n^{-1})}, \tag{4.23} \]

where $B = \text{Ber}$ if $n$ is even, and $B = \text{Ber}^*$ if $n$ is odd.

Proof. Using a straightforward generalization of the method of differentiating quasideterminants given in Gilson & Nimmo (2007), one can show that

\[ Q_n(0, 0) = -D(|W_n|_{n,n}^1 - Q_{n-1}(0, 0)) = -D(\log |W_n|_{n,n}) - Q_{n-1}(0, 0). \tag{4.24} \]

Note that $|\hat{W}_n|_{n,n} = |W_n|_{n,n}$ and so it is even.

If $n = 2k$, then by equation (2.15)

\[ |W_n|_{n,n} = |W_{2k}|_{2k,2k} = \frac{\text{Ber}^*(W_n)}{\text{Ber}^*(W_{n-1})} = \frac{\text{Ber}(W_n)}{\text{Ber}(W_{n-1})}. \]

Similarly, if $n = 2k + 1$, then

\[ |W_n|_{n,n} = |W_{2k+1}|_{k+1,k+1} = \frac{\text{Ber}(W_n)}{\text{Ber}(W_{n-1})}. \]

By repeated use of equation (4.24), noting that $Q_1(0, 0) = -D(\log \theta_0)$, it follows that when $n$ is even

\[ Q_n(0, 0) = D(\log(\text{Ber}(W_n))), \]

and when $n$ is odd

\[ Q_n(0, 0) = D(\log(\text{Ber}^*(W_n))), \]

as required.

Also, from equation (4.12), we have

\[ Q_n(0, 1) = -|W_n|_{n,n-1}^1|W_{n-1}|_{n-1,n-1}^{-1} = -\left\{ \begin{array}{ll}
|W_{2k}|_{k,k} & n = 2k, \\
|W_{2k+1}|_{2k+1,2k+1}^1 & n = 2k + 1.
\end{array} \right. \tag{4.25} \]

The two quasideterminants in these expressions can be expressed, by equation (2.15), as a ratio of Berezinians in which a common factor cancels. The result is that

\[ Q_n(0, 1) = -\left\{ \begin{array}{ll}
\frac{\text{Ber}(W_{2k}^\prime)}{\text{Ber}(W_{2k})}, & n = 2k, \\
\frac{\text{Ber}^*(W_{2k+1}^\prime)}{\text{Ber}^*(W_{2k+1})}, & n = 2k + 1.
\end{array} \right. \tag{4.26} \]

as required.

(c) Binary Darboux transformations

Binary Darboux transformations for the MRSKdV system were discussed in Liu & Mañas (1997b) and Shaw & Tu (1998). In these articles, solutions expressed in terms of determinants were obtained. As discussed in connection with Darboux
transformations it is to be expected that solutions for this supersymmetric system should be superdeterminants in general. In this section, we will construct a more general type of binary Darboux transformation which will be shown to give these superdeterminants solutions and includes the solutions found in Liu & Mañas (1997b) and Shaw & Tu (1998) as a special case.

First we recall the definition of the adjoint for supersymmetric linear operators. For a linear operator $P$, $|P|$ denotes its parity. For example, $|D| = 1$ and $|\partial| = 0$, where $\partial$ denotes any derivative with respect to an even variable, and the parity of multiplication by a homogeneous element is the parity of that element (in the usual sense). The rules defining the superadjoint are

$$D^\dagger = -D, \quad \partial^\dagger = -\partial \quad \text{and} \quad M^\dagger = M^\text{st}, \quad (4.27)$$

where $M$ denotes any matrix over $A$, together with the product rule

$$(PQ)^\dagger = (-1)^{|P||Q|} Q^\dagger P^\dagger, \quad (4.28)$$

where $P$ and $Q$ are operators (cf. equation (2.4) for the case of matrices). In particular, this gives $(D^n)^\dagger = (-1)^n(n+1)/2 D^n$ and, consistently, $(\partial^n)^\dagger = (-1)^n \partial^n$.

The Lax pair (4.2) and (4.3) has the adjoint form

$$L^\dagger = \partial_x^2 + D\alpha + u \quad (4.29)$$

and

$$M^\dagger = -\partial_x^3 - \frac{3}{4} \left( D(\alpha \partial_x + \partial_x \alpha) + u \partial_x + \partial_x u \right), \quad (4.30)$$

and adjoint eigenfunctions satisfy

$$L^\dagger(\psi) = \bar{\xi} \psi, \quad -\psi_t = M^\dagger(\psi), \quad (4.31)$$

for eigenvalue $\xi$. Given an (eigenfunction, adjoint) eigenfunction pair $(\theta, \rho)$, the binary Darboux transformation (Liu & Mañas 1997b; Shaw & Tu 1998) is given by

$$\phi \rightarrow \phi - \theta \Omega(\theta, \rho)^{-1} \Omega(\phi, \rho), \quad (4.32)$$

$$\psi \rightarrow \psi - \rho \Omega(\theta, \rho)^{-1} \Omega(\theta, \psi), \quad (4.33)$$

$$\alpha \rightarrow \alpha + 2(\theta \Omega(\theta, \rho)^{-1} \hat{\rho})_x \quad (4.34)$$

and

$$u \rightarrow u - 2(\alpha + (\theta \Omega(\theta, \rho)^{-1} \hat{\rho})_x) \theta \Omega(\theta, \rho)^{-1} \hat{\rho} + 2(\theta \Omega(\theta, \rho)^{-1} D(\rho))_x, \quad (4.35)$$

where eigenfunction $\theta$ and adjoint eigenfunction $\rho$ have opposite parities. Since $D(\Omega(\phi, \psi)) = \psi \phi$, $\Omega$ is even and assumed to be invertible. When iterating this transformation, both previous papers (Liu & Mañas 1997b; Shaw & Tu 1998) on this topic considered the case that all eigenfunctions are even and all adjoint eigenfunctions are odd. We will show that this is not the most general possibility however.

Consider an even $(m|n)$-row vector eigenfunction $\mathcal{E} = (\theta_0, \ldots, \theta_{m+n-1})$ and an odd $(m|n)$-row vector adjoint eigenfunction $\mathcal{O} = (\rho_0, \ldots, \rho_{m+n-1})$, where $\theta_i$ for
Darboux transformations for super KdV

\[ i = 0, \ldots, m - 1 \text{ and } \rho_{m+j} \text{ for } j = 0, \ldots, n - 1 \text{ are even and } \rho_i \text{ for } i = 0, \ldots, m - 1 \text{ and } \theta_{m+j} \text{ for } j = 0, \ldots, n - 1 \text{ are odd. These row vectors satisfy} \]

\[ L(\mathcal{E}) = \mathcal{E} \Lambda, \quad \mathcal{E}_t = M(\mathcal{E}) \quad (4.36) \]

and

\[ L^\dagger(\mathcal{O}) = \mathcal{O} \Xi, \quad -\mathcal{O}_t = M^\dagger(\mathcal{O}), \quad (4.37) \]

where \( \Lambda \) and \( \Xi \) are constant \((m+n) \times (m+n)\) diagonal matrices containing the eigenvalues. Then \( \mathcal{O} = \Omega(\mathcal{E}, \mathcal{O}) \) is an even \((m|n) \times (m|n)\)-supermatrix defined up to a constant by

\[ D(\mathcal{O}) = \mathcal{O}^\dagger \mathcal{E}, \quad \Omega \Lambda - \Xi \Omega = D(\mathcal{O}^\dagger \mathcal{E} - \mathcal{O}^\dagger \mathcal{E}) - \hat{\Omega}^\dagger \alpha \mathcal{E} \quad (4.38) \]

and

\[ \Omega_t = D(\mathcal{O}^\dagger_{xx} \mathcal{E} - \mathcal{O}^\dagger_{xx} \mathcal{E} + \mathcal{O}^\dagger \mathcal{E}) + \frac{3}{2} \hat{\Omega}^\dagger \alpha \mathcal{E} + \frac{3}{4} \hat{\Omega}^\dagger \alpha \mathcal{E} + \frac{3}{2} D(\mathcal{O}^\dagger \alpha D(\mathcal{E})) + \frac{3}{2} D(\mathcal{O}^\dagger u \mathcal{E}). \quad (4.39) \]

The closed form expressions for the results of iterated binary Darboux transformations are stated in the following theorems.

**Theorem 4.5.** *Iterating the binary Darboux transformations (4.32) and (4.33) for \( m + n \geq 1 \), one obtains*

\[
\phi[m + n] = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O}) & \Omega(\phi, \mathcal{O}) \\ \mathcal{E} & \phi \end{vmatrix},
\]

\[
\psi[m + n] = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O})^\dagger & \Omega(\mathcal{E}, \psi)^\dagger \\ \mathcal{O} & \psi \end{vmatrix} = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O}) & \Omega(\phi, \mathcal{O}) \\ \Omega(\mathcal{E}, \psi) & \Omega(\phi, \psi) \end{vmatrix}.
\]

with

\[ \Omega(\phi[m + n], \psi[m + n]) = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O}) & \Omega(\phi, \mathcal{O}) \\ \Omega(\mathcal{E}, \psi) & \Omega(\phi, \psi) \end{vmatrix}. \quad (4.41) \]

**Proof.** (By induction.) Let \( k = m + n \). The formulae (4.40) and (4.41) evidently hold for \( k = 1 \). Now suppose that they hold for given \( k \) and consider another binary Darboux transformation defined in terms of

\[ \theta[k] = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O}) & \Omega(\theta, \mathcal{O}) \\ \mathcal{E} & \theta \end{vmatrix}, \quad \rho[k] = \begin{vmatrix} \Omega(\mathcal{E}, \mathcal{O}) & \mathcal{O}^\dagger \\ \Omega(\mathcal{E}, \rho) & \rho \end{vmatrix}. \quad (4.42) \]
Here, let \((\theta, \rho)\) be another eigenfunction, adjoint eigenfunction pair of opposite parity, so that \((\mathcal{E}, \theta)\) and \((\mathcal{O}, \rho)\) are even and odd row vectors, respectively. Then, using equation (2.11),

\[
\phi[k + 1] = \phi[k] - \theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} \mathcal{O}(\phi[k], \rho[k]) = \begin{bmatrix} \mathcal{O}(\mathcal{E}, \mathcal{O}) & \mathcal{O}(\theta, \mathcal{O}) & \mathcal{O}(\phi, \mathcal{O}) \\ \mathcal{O}(\mathcal{E}, \rho) & \mathcal{O}(\theta, \rho) & \mathcal{O}(\phi, \rho) \end{bmatrix} \begin{bmatrix} \mathcal{E} \\ \theta \\ \rho \end{bmatrix},
\]

verifying the first part in equation (4.40) for \(k + 1\). The second part is proved in a similar way. The alternative expression for \(\psi[m + n]\) follows from equation (2.5).

Finally, it is straightforward to show that \(D\) applied to the right-hand side of equation (4.41) gives

\[
(\psi - \hat{\mathcal{O}}(\mathcal{E}, \psi) \mathcal{O}(\mathcal{E}, \mathcal{O})^{-1} \mathcal{O}^\dagger)(\phi - \mathcal{E} \mathcal{O}(\mathcal{E}, \mathcal{O})^{-1} \mathcal{O}(\phi, \mathcal{O})) = \psi[k] \phi[k],
\]

(4.43)
as required.

**Theorem 4.6.** Let \((\alpha, u)\) be a solution of MRSKdV and let \(\mathcal{E}\) and \(\mathcal{O}\), respectively, be even and odd \((m|n)\)-row vectors satisfying equations (4.36) and (4.37). Then for any integers \(m + n \geq 0\)

\[
\begin{align*}
\alpha[m + n] &= \alpha - 2A[m + n]_x, \\
u[m + n] &= u + 2(\alpha - A[m + n]_x)A[m + n] - 2U[m + n]_x,
\end{align*}
\]

(4.44)

where

\[
A[m + n] = \begin{bmatrix} \mathcal{O}(\mathcal{E}, \mathcal{O}) & \mathcal{O}^\dagger \\ \mathcal{E} & 0 \end{bmatrix}, \quad U[m + n] = \begin{bmatrix} \mathcal{O}(\mathcal{E}, \mathcal{O}) & D(\mathcal{O}^\dagger) \\ \mathcal{E} & 0 \end{bmatrix},
\]

(4.45)

are also solutions of MRSKdV.

**Proof.** Let \(k = m + n\). The formulae (4.45) are true by definition for \(k = 0\); \(\alpha[0] = \alpha\) and \(u[0] = u\). The formulae will be proved by finding certain expressions involving \(\alpha[k]\) and \(u[k]\) that are independent of \(k\).

As in the proof of theorem 4.5, suppose that a binary Darboux transformation is determined by

\[
\theta[k] = \begin{bmatrix} \mathcal{O}(\mathcal{E}, \mathcal{O}) & \mathcal{O}(\theta, \mathcal{O}) \\ \mathcal{E} & \theta \end{bmatrix}, \quad \rho[k] = \begin{bmatrix} \mathcal{O}(\mathcal{E}, \mathcal{O}) & \mathcal{O}^\dagger \\ \mathcal{O}(\mathcal{E}, \rho) & \rho \end{bmatrix},
\]

where \((\theta, \rho)\) are an eigenfunction, adjoint eigenfunction pair of opposite parity, so that \((\mathcal{E}, \theta)\) and \((\mathcal{O}, \rho)\) are even and odd row vectors, respectively. Then, by equations (4.34) and (4.35),

\[
\alpha[k + 1] = \alpha[k] + 2(\theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} \hat{\rho}[k])_x, \quad (4.46)
\]

and

\[
u[k + 1] = u[k] - 2(\alpha[k] + (\theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} \hat{\rho}[k])_x \theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} \hat{\rho}[k]) \theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} \hat{\rho}[k] + 2(\theta[k] \mathcal{O}(\theta[k], \rho[k])^{-1} D(\rho[k]))_x. \quad (4.47)
\]
Using the notation introduced in equation (4.45), we have

\[
A[k + 1] = \begin{bmatrix}
\Omega(\mathcal{E}, \mathcal{O}) & \Omega(\theta, \mathcal{O}) & \hat{\mathcal{O}}^i \\
\mathcal{E} & \theta & 0
\end{bmatrix},
\]

\[
U[k + 1] = \begin{bmatrix}
\Omega(\mathcal{E}, \mathcal{O}) & \Omega(\theta, \mathcal{O}) & D(\mathcal{O}_1^i) \\
\mathcal{E} & \theta & 0
\end{bmatrix}.
\]

Then using equation (2.11), and noting that \(A[k]\) and \(A[k + 1]\) are odd, it is straightforward to show that

\[
\theta[k] \Omega(\theta[k], \rho[k])^{-1} \rho[k] = A[k] - A[k + 1],
\]

and

\[
\theta[k] \Omega(\theta[k], \rho[k])^{-1} D(\rho[k]) = U[k] - U[k + 1] - A[k + 1]A[k].
\]

Substituting these in equations (4.46) and (4.47) one obtains \(k\) independent expressions

\[
\alpha[k + 1] + 2A[k + 1]_x = \alpha[k] + 2A[k]_x = \alpha
\]

and

\[
u[k + 1] - 2(\alpha - A[k + 1]_x)A[k + 1] + 2U[k + 1]_x = u[k] - 2(\alpha - A[k]_x)A[k] + 2U[k]_x = u,
\]

and hence equations (4.44) hold for all \(k\).

\((d)\) From quasideterminants to superdeterminants II

In this section, we will show how the quasideterminant solutions \((A[m + n], U[m + n])\) obtained using binary Darboux transformations can be expressed in terms of superdeterminants. To do this, it is necessary to introduce a more detailed notation for row vector eigenfunctions and adjoint eigenfunctions. Recall that for the general transformation we use \((m|n)\)-row vectors \(\mathcal{E}\) and \(\mathcal{O}\) which are even and odd with entries \(\theta_i\) and \(\rho_i\), respectively. Here we will also write \(\mathcal{E}^i = (\theta_0, \ldots, \theta_{i-1})\) and \(\mathcal{O}^i = (\rho_0, \ldots, \rho_{i-1})\) for the row vectors containing the first \(i\) entries of \(\mathcal{E}\) and \(\mathcal{O}\), respectively, and denote by subscript 0 and 1 the even and odd element parts of \(\mathcal{E}\) and \(\mathcal{O}\), respectively. Thus \(\mathcal{E} = (\mathcal{E}_0, \mathcal{E}_1)\) and \(\mathcal{O} = (\mathcal{O}_0, \mathcal{O}_1)\).

**Theorem 4.7.** The expressions \((A[m + n], U[m + n])\) can be expressed as

\[
A[m + n] = D(\log \text{Ber}(\mathcal{G}_{(m|n)})), \quad U[m + n] = \frac{\text{Ber}(\mathcal{G}_{(m+1|n)})}{\text{Ber}(\mathcal{G}_{(m|n)})},
\]

where

\[
\mathcal{G}_{(m|n)} = \begin{pmatrix}
\Omega(\mathcal{E}_0, \mathcal{O}_1) & \Omega(\mathcal{E}_1, \mathcal{O}_1) \\
\ldots & \ldots \\
\Omega(\mathcal{E}_0, \mathcal{O}_0) & \Omega(\mathcal{E}_1, \mathcal{O}_0)
\end{pmatrix},
\]

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in an even \((m|n) \times (m|n)\)-supermatrix, and
\[
\mathcal{G}'_{(m+1|n)} = \begin{pmatrix}
\Omega(\mathcal{E}_0, \mathcal{O}_1) & D(\mathcal{O}_1^\dagger) & \Omega(\mathcal{E}_1, \mathcal{O}_1) \\
\mathcal{E}_0 & 0 & \mathcal{E}_1 \\
\cdots & \cdots & \cdots \\
\Omega(\mathcal{E}_0, \mathcal{O}_0) & D(\mathcal{O}_0^\dagger) & \Omega(\mathcal{E}_1, \mathcal{O}_0)
\end{pmatrix},
\]
in an even \((m+1|n) \times (m+1|n)\)-supermatrix.

**Proof.** The formula for \(U[m+n]\) in equation (4.53) follows directly from equations (4.45) and (2.15).

For any \(i = 0, \ldots, m+n\),
\[
A[i] = \begin{vmatrix} \Omega(\mathcal{E}^i, \mathcal{O}^i) & \hat{\mathcal{O}}^i \\ \mathcal{E}^i & 0 \end{vmatrix} (4.54)
\]
and
\[
\theta[i] = \begin{vmatrix} \Omega(\mathcal{E}^i, \mathcal{O}^i) & \Omega(\theta, \mathcal{O}^i) \\ \mathcal{E}^i & \theta \end{vmatrix}, \quad \rho[i] = \begin{vmatrix} \Omega(\mathcal{E}^i, \mathcal{O}^i) & \mathcal{O}^i \\ \Omega(\mathcal{E}^i, \rho) & \rho \end{vmatrix}, (4.55)
\]
where \(A[0] = U[0] = 0\) and \(\theta[0] = \theta_0, \rho[0] = \rho_0\). Also define
\[
\mathcal{O}[i] = \mathcal{O}(\theta[i], \rho[i]) = \begin{vmatrix} \Omega(\mathcal{E}^i, \mathcal{O}^i) & \Omega(\theta, \mathcal{O}^i) \\ \Omega(\mathcal{E}^i, \rho) & \Omega(\theta, \rho) \end{vmatrix} (4.56)
\]
Using equation (2.15),
\[
\mathcal{O}[i] = \begin{cases} 
\frac{\text{Ber}(\mathcal{O}(\mathcal{E}^{i+1}, \mathcal{O}^{i+1}))}{\text{Ber}(\mathcal{O}(\mathcal{E}^i, \mathcal{O}^i))} & 0 \leq i \leq m - 1, \\
\frac{\text{Ber}^*(-}(\mathcal{O}(\mathcal{E}^{i+1}, \mathcal{O}^{i+1}))}{\text{Ber}^*(-}(\mathcal{O}(\mathcal{E}^i, \mathcal{O}^i))} & m \leq i \leq m + n - 1,
\end{cases} (4.57)
\]
where \(\text{Ber}(\mathcal{O}(\mathcal{E}^0, \mathcal{O}^0)) = 1\). In fact, for \(1 \leq i \leq m\) then \(\text{Ber}(\mathcal{O}(\mathcal{E}^i, \mathcal{O}^i)) = \det(\mathcal{O}(\mathcal{E}^i, \mathcal{O}^i))\).

Since all pairs \((\theta_j, \rho_j)\) have opposite parity, \(\theta[i]\) and \(\rho[i]\) also have opposite parity and so commute with each other. Hence equation (4.49) may be written for arbitrary \(i = 0, \ldots, m + n - 1\) as
\[
A[i + 1] = A[i] - \frac{\rho[i] \theta[i]}{\mathcal{O}[i]} (4.58)
\]
Now, the parity of \(\rho[i]\) is the same as that of \(\rho_i\) and hence
\[
\hat{\rho}[i] = \begin{cases} 
-\rho[i] & 0 \leq i \leq m - 1, \\
\rho[i] & m \leq i \leq m + n - 1.
\end{cases} (4.59)
\]
Finally, this gives the recurrence relation

\[ A[i + 1] = A[i] + \begin{cases} 
D(\log \Omega[i]) & 0 \leq i \leq m - 1, \\
-D(\log \Omega[i]) & m \leq i \leq m + n - 1,
\end{cases} \quad A[0] = 0, \tag{4.60} \]

and so

\[ A[m + n] = D \left( \log \frac{\Omega[0]\Omega[1] \cdots \Omega[m-1]}{\Omega[m]\Omega[m+1] \cdots \Omega[m+n-1]} \right) = D(\log \text{Ber}(\Omega(\mathcal{E}, \mathcal{O}))), \tag{4.61} \]

as required.

**Remark 4.8.** The earlier papers on this topic (Liu & Mañas 1997b; Shaw & Tu 1998) take \( n = 0 \) only. In this case, \( \mathcal{E} = \mathcal{E}_0 \) and \( \mathcal{O} = \mathcal{O}_1 \) and we obtain solutions expressed in terms of determinants, not superdeterminants,

\[ A[m] = D(\log \det \Omega(\mathcal{E}_0, \mathcal{O}_1)) \]

and

\[ U[m] = \frac{\det \begin{pmatrix} \Omega(\mathcal{E}_0, \mathcal{O}_1) & D(\mathcal{O}_1) \\ \mathcal{E}_0 & 0 \end{pmatrix}}{\det(\Omega(\mathcal{E}_0, \mathcal{O}_1))}. \]

5. Conclusions

In this paper, we have considered a twisted derivation which includes normal derivative, forward difference operator, \( q \)-difference operator and superderivatives as special cases. We showed that a Darboux transformation defined in terms of such a twisted derivation has an iteration formula written in terms of a quasideterminant. This result opens up the opportunity for an unified approach to Darboux transformations for differential, superdifferential difference and \( q \)-difference operators. In this paper, we showed how this was achieved for one example, the MRSKdV equation.

In the Darboux transformation approach to this equation given in Liu & Mañas (1997a), the authors were forced to consider two different cases, for an odd and an even number of iterations, in order to obtain solutions expressed in terms of superdeterminants. Using the same Darboux transformation, we have obtained quasideterminant solutions in a unified manner, irrespective of the parity of the number of iterations, and then we were able in all cases to reexpress the solutions in terms of superdeterminants using the known relationship between quasideterminants and superdeterminants. This illustrates the advantage of using quasideterminants over using superdeterminants from the start and comes about because quasideterminants need no assumption about the nature of the noncommutativity whereas for superdeterminants restrictive assumptions about parity of matrix elements are required. The same advantage holds in connection with iterated binary Darboux transformations. In this case, we were able to construct much more general types of solutions as well. In earlier work (Liu & Mañas 1997b; Shaw & Tu 1998) only solutions expressed in terms of determinants were found. However, it is to be expected that for a supersymmetric integrable system the most general solutions have expressions in terms of superdeterminants.
rather than determinants. This deficiency was remedied in this paper and we have obtained a much wider class of solutions expressed in terms of both quasideterminants and superdeterminants.

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