SOME RESULTS ON NONSTATIONARY IDEAL II

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Abstract

We answer some question of [Gi]. The upper bound of [Gi] on the strength of $NS_{\mu^+}$ precipitous for a regular $\mu$ is proved to be exact. It is shown that saturatedness of $NS^{\aleph_0}_{\kappa}$ over inaccessible $\kappa$ requires at least $o(\kappa) = \kappa^{++}$. The upper bounds on the strength of $NS_{\kappa}$ precipitous for inaccessible $\kappa$ are reduced quite close to the lower bounds.
0. Introduction

The paper is a continuation of [Gi]. An understanding of [Gi] is required. However, there is one exception, Proposition 2.1. It does not require any previous knowledge and we think it is interesting on its own.

The paper is organized as follows: In Section 1 we examine the strength of $\text{NS}_{\mu^+}$ precipitous. The proof of the main theorem there is a continuation of the proof of 2.5.1 from [Gi]. Section 2 deals with saturation and answers question 3 of [Gi]. In Section 3 a new forcing construction of $\text{NS}_\kappa$ precipitous over inaccessible is sketched. It combines ideas from [Gi, Sec. 3] and [Gi1]. We assume familiarity with these papers.

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1. On the strength of precipitousness over a successor of regular

Our aim will be to improve the results of [Gi] on precipitousness of $\text{NS}_{\mu^+}$ for regular $\mu$ to the equiconsistency. Throughout the paper $\mathcal{K}(\mathcal{F})$ is the Mitchell Core Model with the maximal sequence of measures $\mathcal{F}$, under the assumption ($\neg \exists \alpha \ \sigma^\mathcal{F}(\alpha) = \alpha^{++}$). $\sigma^\mathcal{F}(\kappa)$ denotes the Mitchell order of $\kappa$ or in other words the length of the sequence $\mathcal{F}$ over $\kappa$. We refer to Mitchell [Mi1] for precise definitions.

In order to state the result let us recall a notion of $(\omega, \delta)$-repeat point introduced in [Gi].

**Definition.** Let $\alpha, \delta$ be ordinals with $\delta < \sigma^\mathcal{F}(\kappa)$. Then $\alpha$ is called a $(\omega, \delta)$-repeat point if (1) $\text{cf}\alpha = \omega$, (2) for every $A \in \cap\{\mathcal{F}(\kappa, \alpha')|\alpha \leq \alpha' < \alpha + \delta\}$ there are unboundedly many $\gamma$’s in $\alpha$ such that $A \in \cap\{\mathcal{F}(\kappa, \gamma')|\gamma \leq \gamma' < \gamma + \delta\}$.

We are going to prove the following:

**Theorem 1.1.** Suppose $\text{NS}_{\mu^+}$ is precipitous for a regular $\mu > \aleph_1$ and GCH. Then there exists an $(\omega, \mu + 1)$-repeat point over $\mu^+$ in $K(\mathcal{F})$.

**Remark.**

It is shown in [Gi] that starting with an $(\omega, \mu + 1)$-repeat point it is possible to obtain a model of $\text{NS}_{\mu^+}$ precipitous. On the other hand precipitousness of $\text{NS}^{\aleph_0}_{\mu^+}$ implies $(\omega, \mu)$-repeat point.
In what follows we will actually continue the proof of 2.5.1 of [Gi] and assuming that the $\text{NS}_{\mu^+}$ is precipitous (or even only $\text{NS}_{\mu^+}^{\aleph_0}$ and $\text{NS}_{\mu^+}^\mu$) we will obtain $(\omega, \mu + 1)$–repeat point.

**Proof:** Let $\kappa = \mu^+$. We consider the ordinal $\alpha^* < o^F(\kappa)$ of the proof of 2.5.1 [Gi]. It was shown there to be a $(\omega, \mu)$–repeat point, under the assumption of nonexistence of up-repeat point. Intuitively, one can consider $\alpha^*$ as the least relevant ordinal. Basically, an ordinal $\alpha$ is called relevant if some condition in $\text{NS}_\kappa$ forces that the measure $F(\kappa, \alpha)$ is used first in the generic ultrapower to move $\kappa$ and the cofinality of $\kappa$ changes to $\omega$.

Using a nonexistence of up-repeat point, a set $A \in F(\kappa, \alpha^*)$ such that $A \notin F(\kappa, \beta)$ for $\beta, \alpha^* < \omega$, was picked. This set $A$ was used in [Gi] and will be used here to pin down $\alpha^*$. Thus, for $\tau < \kappa$ if there exists a largest $\tau_1 < o^F(\beta)$ such that $A \cap \tau \in F(\tau, \tau_1)$ then we denote it by $\tau^*$. In this notation $\kappa^*$ is just $\alpha^*$. If $E = \{ \tau < \kappa \mid \text{there exists } \tau^* \}$ then $E \in F(\kappa, \beta)$ for every $\beta$ with $\alpha^* < \beta < o^F(\kappa)$. Also, $A \cup E$ contains all points of cofinality $\omega$ of a club, since by the definition of $\alpha^*$, $A \cup E \in \bigcap \{ F(\kappa, \alpha) \mid \alpha \text{ is a relevant ordinal} \}$.

**Claim 1.** The set of $\alpha < \kappa$ satisfying (a) and (b) below is stationary in $\kappa$.

(a) $\text{cf } \alpha = \mu$;

(b) for every $i < \mu$

\[ \{ \beta < \alpha \mid \text{cf } \beta = \aleph_0 \text{ and } o^F(\beta) \geq \beta^* + i \} \]

is a stationary subset of $\alpha$.

**Proof:** Otherwise, let $C$ be a club avoiding all the $\alpha$’s which satisfy (a) and (b). Let $N$ be a good model in the sense of 2.5.1 of [Gi], with $C \in N$. Consider $\langle \tau^*_n \mid n < \omega \rangle$, $\langle d^N_n \mid n < \omega \rangle$ and $\langle \beta^*_n \mid n < \omega \rangle$ of 2.5.1 [Gi]. Recall that $\langle \tau^*_n \mid n < \omega \rangle$ is a sequence of indiscernibles for $N$, each $\tau^*_n$ is a limit point of $C$, $d^N_n$ is an $\omega$-club in $\bigcup (N \cap \tau_n)$ consisting of indiscernibles of cofinality $\omega$ in $C$, for $\nu \in d^N_n$, $\nu^*$ exists and $\beta^*_n$ represents it over $\kappa$ (identically for every $\nu, \nu' \in d^N_n$). Also for every $\tau < \tau'$ in $d^N_n$, $\beta^N_N(\tau) < \beta^N_N(\tau')$, where $\beta^N_N(\tau)$ is the index of the measure over $\kappa$ to which $\tau$ corresponds.

Fix $n < \omega$. Then, $\tau_n \in C$. As in 2.10 or 2.14 of [Gi] we can assume that $\text{cf } \tau_n = \mu$. Since (b) fails, there are $i_n < \mu$ and $C_n$ a club of $\tau_n$ disjoint with $\{ \nu < \tau_n \mid \text{cf } \nu = \aleph_0 \}$. 

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and $o^F(\nu) \geq \nu^* + i_n$}. Using elementarity of $N$, it is easy to find such $C_n$ inside $N$. Let $\delta = \bigcup_{n<\omega} i_n$. Using 2.1.1 (or 2.15 for inaccessible $\mu$) of [Gi] we will obtain $N^* \supseteq N$ which agrees (mod initial segment) with $N$ about indiscernibles but has sets $d^N_n$ long enough to reach $\delta$, i.e. there will be a final segment of $\tau$’s in $d^N_n$ with $\beta^N_{\tau}(\tau) > \beta^*_n + \delta$. But then, for such $\tau$, $o^F(\tau) \geq \tau^* + \delta$. This is impossible, since $C_n$, $d^N_n$ are both clubs of $\tau_n$ in $N^*$ with bounded intersection. Contradiction.

□

Let $S$ denote the set of $\alpha$’s satisfying the conditions (a) and (b) of Claim 1. Now form a generic ultrapower with $S$ in the generic ultrafilter. Denote it by $M$ and let $F(\kappa, \xi)$ be the measure used to move $\kappa$. Then, in $M$ $cf \kappa = \mu$ and $S_i = \{ \beta < \kappa \mid cf \beta = \aleph_0$ and $o^F(\beta) > \beta^* + i \}$ is a stationary subset of $\kappa$ for every $i < \mu$. Hence $S_i$ is stationary also in $V$.

Claim 2. For every $i < \mu$ and $X \in \mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$, $X \in F(\kappa, \alpha^* + i)$ iff $S_i \setminus \{ \beta < \kappa \mid o^F(\beta) > \beta^* + i \} \in F(\beta, \beta + i)$ is nonstationary.

Proof: Fix $i < \mu$. $F(\kappa, \alpha^* + i)$ is an ultrafilter over $\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$ so it is enough to show that for every $X \in F(\kappa, \alpha^* + i)$ the set $S_i \setminus \{ \beta < \kappa \mid o^F(\beta) < \beta^* + i \}$ is nonstationary.

Suppose otherwise. Let $X \in F(\kappa, \alpha^* + i)$ be so that $S' = S_i \setminus \{ \beta < \kappa \mid o^F(\beta) > \beta^* + i \}$ is stationary.

Without loss of generality we may assume that $S'$ already decides the relevant measure, i.e. for some $\gamma < o^F(\kappa)$ $S'$ forces the measure $F(\kappa, \gamma)$ to be used first to move $\kappa$ in the embedding into generic ultrapower restricted to $\mathcal{K}(\mathcal{F})$. Now, $S' \subseteq \{ \beta < \kappa \mid o^F(\beta) > \beta^* + i \}$. So, $\gamma > \gamma^* + i$, where $\gamma^*$ is the largest ordinal $\gamma$ below $\gamma$ with $A \in F(\kappa, \gamma^*)$. If $\gamma^* = \alpha^*$, then $\alpha^* + i < \gamma$ and hence $X^* = \{ \beta < \kappa \mid o^F(\beta) > \beta^* + i \}$ since this is true in the ultrapower of $\mathcal{K}(\mathcal{F})$ by $F(\kappa, \gamma)$. This leads to a contradiction, since, if $j : V \to M$ is a generic embedding forced by $S'$, then $\kappa \in j(S')$ and $\kappa \in j(X^*)$, but $S' \cap X^* = \emptyset$. Contradiction.

If $\gamma^* < \alpha^*$, then also $\gamma < \alpha^*$ which is impossible since there are no relevant ordinals below $\alpha^*$. Also, $\gamma^*$ cannot be above $\alpha^*$ since $\alpha^*$ is the last ordinal $\xi$ with $A \in F(\kappa, \xi)$.

□
For $i < \mu$ and a set $X \subseteq \kappa$ let us denote by $X_i^*$ the set $\{\beta < \kappa | o^F(\beta) > \beta^* + i \}$ and $X \cap \beta \in \mathcal{F}(\beta, \beta + i)$. By $\text{Cub}_\kappa$ we denote the closed unbounded filter over $\kappa$ and let $\text{Cub}_\kappa \upharpoonright S_i$ be its restriction to $S_i$, i.e. $\{E \subseteq \kappa | E \supseteq C \cap S_i \text{ for some } C \in \text{Cub}_\kappa\}$.

**Claim 3.** For every $i < \mu$, $\mathcal{F}(\kappa, \alpha^* + i) = \{X \in (\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F}))^M | X_i^* \in (\text{Cub}_\kappa \upharpoonright S_i)^M \}$.

**Proof:** Let $X \in \mathcal{F}(\kappa, \alpha^* + i)$, then, by Claim 2, $X_i^* \in \text{Cub}_\kappa \upharpoonright S_i$ in $V$. But then, also in $M$, $X_i^* \in (\text{Cub}_\kappa \upharpoonright S_i)^M$, since $(\text{Cub}_\kappa)^M \supseteq (\text{Cub}_\kappa)^V$. Now, if $X \not\in \mathcal{F}(\kappa, \alpha^* + i)$, then $Y = \kappa \setminus X \in \mathcal{F}(\kappa, \alpha^* + i)$, assuming $X \in \mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$. By the above, $Y_i^* \in (\text{Cub}_\kappa \upharpoonright S_i)^M$. But $X \cap Y = \emptyset$ implies $X_i^* \cap Y_i^* = \emptyset$. So $X_i^* \not\in (\text{Cub}_\kappa \upharpoonright S_i)^M$.

\[ \Box \]

**Claim 4.** $o^F(\kappa) > \alpha^* + \mu$.

**Proof:** By Claim 3, $\mathcal{F}(\kappa, \alpha^* + i) \in M$ for every $i < \mu$. Hence $(o(\kappa))^M \geq \alpha^* + \mu$. But now, in $V$, $o^F(\kappa) \geq \alpha^* + \mu + 1$.

\[ \Box \]

We actually showed more:

**Claim 5.** $S \Vdash'' o(\xi) \geq \alpha^* + \mu$ and for every $i < \mu \mathcal{F}(\kappa, \alpha^* + i) = \{X \in (\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F}))^M \mid X_i^* \in (\text{Cub}_\kappa \upharpoonright S_i)^M \}$, where $\xi$ is a name of the index of the first measure $\mathcal{F}(\kappa, \xi)$ used to move $\kappa$ and $M$ is a generic ultrapower.

In order to complete the proof, we need to show that every $Y \in \mathcal{F}(\kappa, \alpha^* + \mu)$ belongs to $\mathcal{F}(\kappa, \gamma)$ for unboundedly many $\gamma$’s below $\alpha^*$. The conclusion of the theorem will then follow by [Gi, Sec. 1]. So let $Y \in \mathcal{F}(\kappa, \alpha^* + \mu)$. Consider a set $Y^* = \{\beta < \kappa \mid \beta^* \text{ exists, } o^F(\beta) > \beta^* + \mu \text{ and } Y \cap \beta \in \mathcal{F}(\kappa, \beta^* + \mu)\} \cup Y$. Then $Y^* \in \cap \{\mathcal{F}(\kappa, \alpha) | \alpha^* + \mu \leq \alpha < o^F(\kappa)\}$. It is enough to show that $Y^*$ belongs to $\mathcal{F}(\kappa, \gamma)$ for unboundedly many $\gamma$’s below $\alpha$.

**Claim 6.** $S \setminus Y^*$ is nonstationary.

**Proof:** Suppose otherwise. Let $S' \subseteq S \setminus Y^*$ be a stationary set forcing $\mathcal{F}(\kappa, \xi)$ to be the first measure used to move $\kappa$ in the ultrapower, where $\xi < o^F(\kappa)$. Then, by Claim 5, $\xi \geq \alpha^* + \mu$. Hence, $Y^* \in \mathcal{F}(\kappa, \xi)$, which is impossible, since $Y^* \cap S' = \emptyset$. Contradiction.

\[ \Box \]
Claim 7. \( \alpha^* \) is a \( \mu + 1 \)-repeat point.

Proof: Let \( Y^* \) be as above. It is enough to find \( \gamma < \alpha^* \) such that \( Y^* \in \mathcal{F}(\kappa, \alpha) \). Let \( C \subseteq \kappa \) be a club avoiding \( S \setminus Y^* \). Let \( N, \{ \tau_n | n < \omega \} \) be as in Claim 1 (i.e. as in the proof of 2.5.1 [Gi]) only with the club of Claim 1 replaced by \( C \) and with \( Y^* \in N \). Then \( \tau_n \)'s are in \( S \cap C \) and, hence in \( Y^* \), which means that for all but finitely many \( n \)'s \( Y^* \in \mathcal{F}(\kappa, \beta^N(\tau_n)) \), by [Mit 1,2], since \( \tau_n \)'s are indiscernibles for \( \beta^N(\tau_n) \)'s.

□

The claim does not rule out the possibility that some \( Y^* \) reflects only boundedly many times below \( \alpha^* \). Thus, there is probably some \( \eta < \alpha^* \) such that the \( \beta^N(\tau_n) \)'s of Claim 7 are always below \( \eta \). This means that \( \beta^*_n > \beta^N(\tau_n) \), where \( \beta^*_n \) is the stabilized value of \( (\beta(\nu))^* \) for \( \nu \in d^N_n \). We will use Claim 5 in order to show that this is impossible. Namely, the following holds:

Claim 8. In the notation of Claim 7, for all but finitely many \( n \)'s \( (\beta^N(\tau_n))^* = \beta^*_n \).

Proof: By Claim 5, for all but nonstationary many \( \nu \)'s in \( S \) the following property (*) holds:
\[
\sigma^F(\nu) \geq \nu^* + \mu \text{ and for every } i < \mu \quad \mathcal{F}(\nu, \nu^* + i) = \{ X \in \mathcal{P}(\nu) \cap \mathcal{K}(\mathcal{F}) | X^*_i \in \text{Cub}_\nu | \{ \rho < \nu | cf \rho = \aleph_0 \text{ and } \sigma^F(\rho) > \rho^* + i \} \}.
\]

Without loss of generality let us assume that (*) holds for every element of \( S \), otherwise just remove the nonstationary many points. Then, preserving notations of Claim 7, \( \tau_n \)'s satisfy (*). We now show that ultrafilters \( \mathcal{F}(\tau_n, \tau^*_n + i) \) correspond to \( \mathcal{F}(\kappa, \beta^*_n + i) \) for all but finitely many \( n < \omega \) and all \( i < \mu \).

Let \( \overline{\beta}_n \) denote \( (\beta^N(\tau_n))^* \) and we will drop the upper index \( N \) further. Then \( \tau^*_n + i = \mathcal{C} (\kappa, \overline{\beta}_n + i, \beta(\tau_n))(\tau_n) \) for every \( n < \omega \), where \( \mathcal{C} \) is the coherence function (see [Mi1] or [Gi]). Suppose that \( \beta^*_n \neq \overline{\beta}_n \) for infinitely many \( n \)'s. For simplicity let us assume that this holds for every \( n < \omega \). In the general case only the notation is more complicated. There will be \( X_n \in (\mathcal{F}(\kappa, \overline{\beta}_n) \setminus \mathcal{F}(\kappa, \beta^*_n))(\tau_n) \) for every \( n < \omega \), since \( N \) is an elementary submodel. Let \( n < \omega \) be fixed. Pick \( \mathcal{K}(\mathcal{F}) - \text{least} X_n \in \mathcal{F}(\kappa, \overline{\beta}_n) \setminus \mathcal{F}(\kappa, \beta^*_n) \). Still it is in \( N \) by elementarity. Also its support (in the sense of [Mi1,2]) will be below \( \tau_n \), i.e. \( X_n = h^N(\delta) \), for \( \delta < \tau_n \), where \( h^N \) is the Skolem function of \( N \cap \mathcal{K}(\mathcal{F}) \). The reason for this is that \( X_n \) appears once
both $\beta_n^*$ and $\beta_n^*$ appear. But $\beta_n^*$ appear below $\tau_n$ since the support of $\tau_n$ is below $\tau_n$ and $\beta_n^*$ appear before $\tau_n$ since for $\nu \in d_n \subseteq \tau_n$ $(\beta^N(\nu))^* = \beta_n^*$. Hence $X_n \cap \tau_n \in F(\tau_n, \tau_n^*)$. Then by ($\ast$), $(X_n)_0^* \in \text{Cub}_{\tau_n} \upharpoonright \{\rho < \tau_n | cf \rho = R_0 \text{ and } oF(\rho) > \rho^*\}$. This is clearly true also in $\mathbb{N}$. But then $(X_n)_0^* \cap \bigcup (\mathbb{N} \cap \tau_n)$ contains an $\omega$-club intersected with the set $\{\rho < \tau_n | cf \rho = R_0 \text{ and } oF(\rho) > \rho^*\}$. Hence $(X_n)_0^* \in F(\kappa, \beta_n^* + i)$ for some $i$, $0 < i < \mu$, which implies that $X_n \in F(\kappa, \beta_n^*)$. Contradiction.

\[ \square \]

Combining Claims 7 and 8 we obtain that $Y^* \in F(\kappa, \beta_n^* + \chi)$ for some $\chi \geq \mu$, for all but finitely many $n$’s. Now, $\beta_n^*$’s are unbounded in $\alpha^*$ by [Gi] and hence we have an unbounded reflection of $Y$ below $\alpha^*$.

\[ \square \]

2. On the strength of saturatedness of $NS_{\kappa}$

It was shown in [Gi] that saturatedness of $NS_{\kappa}$ for an inaccessible $\kappa$ implies an inner model with $\exists \alpha o(\alpha) = \alpha^{++}$. It was asked if the saturatedness of $NS^{R_0}_{\kappa}$, i.e. the nonstationary ideal restricted to cofinality $\omega$ already implies this. In this section we are going to provide an affirmative answer.

Let us start with a “ZFC variant” of Lemma 2.18 of [Gi].

**Proposition 2.1.** Let $V_1 \subseteq V_2$ be two models of ZFC. Let $\kappa$ be a regular cardinal of $V_1$ which changes its cofinality to $\Theta$ in $V_2$. Suppose that in $V_1$ there is an almost decreasing (mod nonstationary or equivalently mod bounded) sequence of clubs of $\kappa$ of length $(\kappa^+)^{V_1}$ so that every club of $\kappa$ of $V_1$ is almost contained in one of the clubs of the sequence. Assume that $V_2$ satisfies the following:

1. $cf(\kappa^+)^{V_1} \geq (2^\Theta)^+ \text{ or } cf(\kappa^+)^{V_1} = \Theta$;
2. $\kappa > \Theta^+$.

Then in $V_2$ there exists a cofinal in $\kappa$ sequence $\langle \tau_i \mid i < \Theta \rangle$ consisting of ordinals of cofinality $\geq \Theta^+$ so that every club of $\kappa$ of $V_1$ contains a final segment of $\langle \tau_i \mid i < \Theta \rangle$.

**Remark.** (1) If in $V_1$, $2^\kappa = \kappa^+$, then clearly there exists an almost decreasing sequence of clubs of $\kappa$ of length $\kappa^+$ so that every club of $\kappa$ of $V_1$ is almost contained in one of the
clubs of the sequence.

(2) M. Dzamonja and S. Shelah [D-Sh] using club guessing techniques were able to replace the condition (1) by weaker conditions.

**Proof:** If $cf(\kappa^+)^{V_1} = \Theta$ then we can simply diagonalize over all the clubs. So let us concentrate on the case $cf(\kappa^+)^{V_1} \geq (2^{\Theta})^+$. Suppose otherwise. Assume for simplicity that $\Theta = \aleph_0$. Let $C$ be a club in $\kappa$ in $V_1$. Define in $V_2$ a wellfounded tree $\langle T(C), \leq_C \rangle$. Let the first level of $T(C)$ consist of the least cofinal in $\kappa$ sequence of order type $\omega$ in some fixed for the rest of the proof well ordering of a larger enough portion of $V_2$. Suppose that $T(C)\upharpoonright n + 1$ is defined. We define $\text{Lev}_{n+1}(T(C))$. Let $\eta \in \text{Lev}_n(T(C))$. Let $\eta^*$ be the largest ordinal in $T(C)\upharpoonright n + 1$ below $\eta$. We assume by induction that it exists. If $cf\eta = \aleph_0$, then pick $\langle \eta_n \mid n < \omega \rangle$ the least cofinal sequence in $\eta$ of order type $\omega$. Let the set of immediate successors of $\eta$, $\text{Suc}_{T(C)}(\eta)$ be $\{\eta_n \mid n < \omega, \eta_n > \eta^*\}$.

If $cf\eta \geq \aleph_1$, then consider $\eta' = \cup(C \cap \eta)$. If $\eta' = \eta$, then let $\text{Suc}_{T(C)}(\eta) = \emptyset$. If $\eta' < \eta$, then let $\text{Suc}_{T(C)}(\eta) = \{\eta'\}$. Finally, if $\eta' \leq \eta^*$ then let $\text{Suc}_{T(C)}(\eta) = \emptyset$. This completes the inductive definition of $\langle T(C), \leq_C \rangle$. Obviously, it is wellfounded and countable. Let $T^*(C)$ denote the set of all endpoints of $T(C)$ which are in $C$. Notice, that by the construction any such point is of uncountable cofinality. Also, $T^*(C)$ is unbounded in $\kappa$, since $\text{otp}(C) = \kappa$ and $\kappa > \aleph_1$.

There must be a club $C_1 \subseteq C$ in $V_1$ avoiding unboundedly many points of $T^*(C)$, since otherwise the sequence $\langle \tau_i \mid i < \aleph_0 \rangle$ required by the proposition could be taken from $T^*(C)$. This means, in particular, that for every $\alpha < \kappa$ there will be

$$\mathbf{\tau} = \langle \nu_1, \ldots, \nu_n \rangle \in T(C) \cap T(C_1)$$

so that

(a) $cf\nu_n > \aleph_0$;
(b) $\text{Suc}_{T(C)}(\nu_n) = \{\nu_{n+1}\}$ for some $\nu_{n+1} \in C \setminus \alpha$;
(c) either

(c1) $\text{Suc}_{T(C_1)}(\nu_n) = \emptyset$

or

(c2) for some $\rho \in (C_1 \cap \nu_{n+1}) \setminus \alpha$ $\text{Suc}_{T(C_1)}(\nu_n) = \{\rho\}$.  

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Now define a sequence $\langle C_\alpha \mid \alpha < (2^{\aleph_0})^+ \rangle$ of clubs so that

1. $C_\alpha$ is a club in $\kappa$ in $V_1$;
2. if $\beta < \alpha$ then $C_\alpha \setminus C_\beta$ is bounded in $\kappa$;
3. $C_{\alpha+1}$ avoids unboundedly many points of $T^*(C_\alpha)$.

Since $\text{cf} (\kappa^+)^{V_1} \geq (2^{\aleph_0})^+$ and in $V_1$ there is an almost decreasing (mod bounded) sequence of $\kappa^+$-clubs generating the club filter, there is no problem in carrying out the construction of $\langle C_\alpha \mid \alpha < (2^{\aleph_0})^+ \rangle$ satisfying (1)–(3).

Shrinking the set of $\alpha$'s if necessary we can assume that for every $\alpha, \beta < (2^{\aleph_0})^+$ $\langle T(C_\alpha), \leq_{C_\alpha}, \leq \rangle$ and $\langle T(C_\beta), \leq_{C_\beta}, \leq \rangle$ are isomorphic as trees with ordered levels.

Let $\langle \kappa_m \mid m < \omega \rangle$ be the least cofinal in $\kappa$ sequence.

Let $\alpha < \beta < (2^{\aleph_0})^+$. Since $C_\beta$ is almost contained in $C_{\alpha+1}$, it avoids unboundedly many points in $T^*(C_\alpha)$. So for every $m < \omega$ there is $\mathfrak{v} = \langle \nu_1, \ldots, \nu_n \rangle \in T(C_\alpha) \cap T(C_\beta)$ so that

(a) $\text{cf} \nu_n > \aleph_0$;
(b) $\text{Suc}_{T(C_\alpha)}(\nu_n) = \{ \nu^\alpha_{n+1} \}$ for some $\nu^\alpha_{n+1} \in C_\alpha \setminus \kappa_m$;
(c) for some $\nu^\beta_{n+1} \in (C_\beta \setminus \nu^\alpha_{n+1}) \setminus \kappa_m$ $\text{Suc}_{T(C_\beta)}(\nu_n) = \{ \nu^\beta_{n+1} \}$.

Thus, pick $\ell > m$ so that $C_\beta \setminus \kappa_{\ell-1} \subseteq C_\alpha$. We consider subtrees

$$T(C_\gamma)^\ell = \{ \mathfrak{v} \in T(C_\gamma) \mid \exists k \geq \ell \mathfrak{v} \geq_{C_\gamma} \langle \kappa_k \rangle \}$$

where $\gamma = \alpha, \beta$.

Let $\pi$ be an isomorphism between $T(C_\alpha)$ and $T(C_\beta)$ respecting the order of the levels. Notice, that the first level in both trees is the same $\{ \kappa_i \mid i < \omega \}$. Hence, $\pi$ will move $T(C_\alpha)^\ell$ onto $T(C_\beta)^\ell$.

Pick the maximal $n < \omega$ such that $\pi$ is an identity on $(T(C_\alpha)^\ell) \setminus n + 1$. It exists since $T^*(C_\alpha) \setminus C_\beta$ is unbounded in $\kappa$. Now let $\nu$ be the least ordinal in $\text{Lev}_{n+1}(T(C_\alpha)^\ell)$ such that $\pi(\langle \nu_1, \ldots, \nu_n, \nu \rangle) \neq \langle \nu_1, \ldots, \nu_n, \nu \rangle$, where $\langle \nu_1, \ldots, \nu_n \rangle$ is the branch of $T(C_\alpha)^\ell$ leading to $\nu$.

Consider $\nu_n$. If $\text{cf} \nu_n = \aleph_0$, then we are supposed to pick the least cofinal in $\nu_n$ sequence $\langle \nu^*_{ni} \mid i < \omega \rangle$ and the maximal element $\nu^*_n$ of the tree $T(C_\alpha)$ below $\nu_n$. $\text{Suc}_{T(C_\alpha)}(\nu_n)$ will be $\{ \nu_{ni} \mid i < \omega \text{ and } \nu_{ni} > \nu^*_n \}$. Notice that $\nu^*_n \geq \kappa_{n-1}$ by the definition of the tree $T(C_\alpha)$. 9
Hence, either \( \nu_n^* = \kappa_{n-1} \) or \( \nu_n^* \in T(C_\alpha)|n+1 \) since elements of \( T(C_\alpha) \) which are above \( \kappa_{n-1} \) in the tree order are below it as ordinals. But since \( T(C_\alpha)|n+1 = T(C_\beta)|n+1 \) and \( \kappa_{\ell-1} \in T(C_\beta) \), the same is true about \( \text{Suc}_{T(C_\beta)}(\nu_n) \), i.e. it is \( \{\nu_{ni}|i < \omega \text{ and } \nu_{ni} > \nu_n^*\} \). Then \( \pi \) will be an identity on \( \text{Suc}_{T(C_\alpha)}(\nu_n) \) and, in particular, will not move \( \nu \). Contradiction.

So \( cf\nu_n \) should be above \( \aleph_0 \). Once again the maximal elements of \( T(C_\alpha)|n+1 \) and \( T(C_\beta)|n+1 \) below \( \nu_n \) are the same. Let \( \nu_n^* \) denote this element. Now, \( \nu \in \text{Suc}_{T(C_\alpha)}(\nu_n) \), hence \( \nu = \cup(C_\alpha \cap \nu_n) \) \( \nu_n^* < \nu < \nu_n \) and \( \text{Suc}_{T(C_\alpha)}(\nu_n) = \{\nu\} \) by the definition of the tree \( T(C_\alpha) \). \( \pi \) is an isomorphism, so \( \text{Suc}_{T(C_\beta)}(\nu_n) \neq \emptyset \). By the definition of the tree \( T(C_\beta) \), \( \nu^* < \nu' < \nu_n \) and \( \text{Suc}_{T(C_\beta)}(\nu_n) = \{\nu'\} \) where \( \nu' = \cup(C_\beta \cap \nu_n) \). By the choice of \( \nu \), \( \nu \neq \nu' \). But \( \nu, \nu' > \kappa_{\ell-1} \) and \( C_\beta \setminus \kappa_{\ell-1} \subseteq C_\alpha \), so \( \nu' \in C_\alpha \). Hence \( \nu' < \nu \) and the sequence \( \langle \nu_1, \ldots, \nu_n \rangle \) is as desired.

Let \( \langle T, \leq_T, \leq \rangle \) be a countable tree consisting of countable ordinals with the usual order \( \leq \) between them isomorphic to \( \langle T(C_\alpha), \leq_{C_\alpha}, \leq \rangle \) \( (\alpha < (2^{\aleph_0})^+) \). Define a function \( h : [(2^{\aleph_0})^+]^2 \rightarrow \omega \) as follows:

\[
f(\alpha, \beta) = \text{the minimal element of } T \text{ corresponding to some } \nu \in T(C_\alpha) \cap T(C_\beta)
\]

satisfying the conditions (a), (b) and (c).

By Erdös–Rado there exists a homogeneous infinite set \( A \subseteq (2^{\aleph_0})^+ \). Let \( \langle \alpha_n \mid n < \omega \rangle \) be an increasing sequence from \( A \). Then there is \( \nu = \langle \nu_1, \ldots, \nu_n \rangle \in \bigcap_{m<\omega} T(C_\alpha)_m \) witnessing (a), (b), (c). But by (c), \( \nu_{n+1}^{\alpha_m} > \nu_{n+1}^{\alpha_{m+1}} \) for every \( m < \omega \). Contradiction.

\( \square \)

Suppose now that there is no inner model of \( \exists \alpha \theta(\alpha) = \alpha^{++} \). The following follows easily from Proposition 2.1 and the Mitchell Covering Lemma [Mi3].

**Proposition 2.2.** The final segment of the sequence \( \langle \tau_n \mid n < \omega \rangle \) consists of indiscernibles for \( \kappa \).

**Proof:** Suppose otherwise. Then by the Mitchell Covering Lemma [Mi3] there is \( h \in \mathcal{K}(\mathcal{F}) \) and \( \delta_n < \tau_n \) \( (n < \omega) \) such that \( h(\delta_n) \geq \tau_n \) for infinitely many \( n \)'s. Define a club in
\( \mathcal{K}(\mathcal{F}) \):

\[
C = \{ \nu < \kappa | h''(\nu) \subseteq \nu \}.
\]

Then, by the choice of \( \langle \tau_n | n < \omega \rangle \), there is \( n_0 < \omega \) such that for every \( n \geq n_0 \), \( \tau_n \in C \), which is impossible. Contraction.

\( \square \)

Let us show that \( o(\kappa) = \kappa^{++} \) if \( NS^{\Theta}_\kappa \) is saturated.

**Proposition 2.3.** * Suppose that \( NS^{\Theta}_\kappa \) is saturated over an inaccessible \( \kappa \), then \( o(\kappa) = \kappa^{++} \).

**Remark.** By Mitchell [Mi1] it follows for successor \( \kappa \)'s and moreover, by Shelah [Sh] it is impossible for successor cardinal \( \kappa \) which is above \( \Theta^+ \).

**Proof:** Let for simplicity \( \Theta = \aleph_0 \). Suppose that \( o(\kappa) < \kappa^{++} \). We call an ordinal \( \alpha \) a relevant ordinal if some \( S \in (NS^{\aleph_0}_\kappa)^+ \) forces the measure \( \mathcal{F}(\kappa, \alpha) \) (of the core model \( \mathcal{K}(\mathcal{F}) \)) to be used as the first measure moving \( \kappa \) in the generic embedding restricted to \( \mathcal{K}(\mathcal{F}) \). By Mitchell [Mi1,2], such a restriction is an iterated ultrapower of \( \mathcal{K}(\mathcal{F}) \). Let us call the corresponding measure \( \mathcal{F}(\kappa, \alpha) \) – a relevant measure.

Since \( NS^{\aleph_0}_\kappa \) is saturated, the total number of relevant measures is at most \( \kappa \). Let \( \langle A_\alpha | \alpha < \chi \leq \kappa \rangle \) be a maximal antichain such that \( A_\alpha \) forces “\( \alpha \) to be a relevant ordinal”. Without loss of generality \( A_\alpha \)'s are pairwise disjoint and \( \min A_\alpha > \alpha \). Also it is possible to pick each \( A_\alpha \) in \( \mathcal{F}(\kappa, \alpha) \) using \( o(\kappa) < \kappa^{++} \), but it is not needed for the rest. Let us assume that \( \chi = \kappa \). The case \( \chi < \kappa \) is similar and even slightly simpler.

The set \( A = \bigcup_{\alpha < \kappa} A_\alpha \) contains an \( \aleph_0 \)-club. Since \( NS^{\aleph_0}_\kappa \) is precipitous, we can assume that every \( \nu \in A \) is regular in \( \mathcal{K}(\mathcal{F}) \). Otherwise just remove from \( A \) nonstationary many \( \nu \)'s of cofinality \( \aleph_0 \) which are singular in \( \mathcal{K}(\mathcal{F}) \).

Let \( \alpha_{\text{min}} \) be the least relevant ordinal. Form a generic ultrapower \( M \) with \( A_{\alpha_{\text{min}}} \) in a generic ultrafilter \( G \subseteq (NS^{\aleph_0}_\kappa)^+ \). Applying Proposition 2.1 to \( V \) and \( V[G] \), we find a cofinal in \( \kappa \) sequence \( \langle \tau_n | n < \omega \rangle \) consisting of ordinals of uncountable cofinality such that every club of \( \kappa \) of \( V \) contains its final segment. Since \( NS^{\aleph_0}_\kappa \) is saturated in \( V \),

* This result was recently improved by S. Shelah and the author to \( 0 = 1 \).
⟨τₙ | n < ω⟩ ∈ M. We are going to use ⟨τₙ | n < ω⟩ in order to recover 𝒥(κ, αₘᵢₙ) inside M, which is impossible, since it is used already to move κ and hence cannot be in 𝐾(𝒥)ˣ⁻⁺. We proceed as follows. For every α ∈ A \ {αₘᵢₙ} pick in V a function gα ∈ 𝕊κ which is forced by Aα to represent αₘᵢₙ in a generic ultrapower. By saturatedness it is possible. Under o(κ) < κ⁺⁺ we can find such gα in 𝐾(𝒥).

For a set X ∈ 𝒥(κ, αₘᵢₙ) we define in V a set

\[ C_X = \{ ν < κ \mid \text{cf} ν \neq \aleph_0 \text{ or } (\text{cf} ν = \aleph_0 \text{ and then either } ν \in A_{αₘᵢₙ} \cap X \text{ or } ν \in A_α \text{ for some } α \in A \setminus \{αₘᵢₙ\} \text{ and then } X \cap ν \in 𝒥(ν, gα(ν)) \}, \]

Then Cₓ contains a club. Cₓ is in M as well as ⟨A_α | α ∈ A⟩ and ⟨gα | α ∈ A⟩. Moreover it is has the same definition as in V.

But now, in M, we may define a set D = {X ⊆ κ | X ∈ (K)ˣ⁻⁺}, Cₓ contains a final segment of ⟨τₙ | n < ω⟩. Then D ⊇ 𝒥(κ, αₘᵢₙ), since for every X ∈ 𝒥(κ, αₘᵢₙ) X ∈ M and Cₓ is a club of V. On the other hand, if X ∈ D, then X ∈ 𝐾(𝒥)ˣ⁻⁺ ⊆ 𝐾(𝒥) ∩ 𝒫(κ). If X /∈ 𝒥(κ, αₘᵢₙ), then Y = κ \ X belongs to 𝒥(κ, αₘᵢₙ) and hence Cᵧ contains a final segment of ⟨τₙ | n < ω⟩, as does Cₓ. But this is impossible. Thus let C′ₓ, C′ᵧ be the clubs of limit points of Cₓ and Cᵧ. Let τₓ ∈ C′ₓ ∩ C′ᵧ, then there is some ν < τₓ, cf ν = ℵ₀ and ν ∈ Cₓ ∩ Cᵧ since cfτₓ > ℵ₀. But then for some unique α ∈ A, ν ∈ A_α which implies ν ∈ X ∩ Y, in the case ν ∈ A_{αₘᵢₙ}, or X ∩ ν, Y ∩ ν in 𝒥(ν, gα(ν)) otherwise. Which is impossible since Y and X are disjoint.

So D = 𝒥(κ, αₘᵢₙ). Hence 𝒥(κ, αₘᵢₙ) ∈ M. Contradiction.

We think that the methods of [Gi-Mi] can be used in order to push the strength of “NSₖ saturated” to a strong cardinal.

3. On the strength of precipitousness of a nonstationary ideal over an inaccessible

We are going to show that the assumptions used in [Gi] making NSₖ precipitous ((ω, κ⁺⁺ - repeat point) and NS₀ᵏ⁺⁺ precipitous ((ω, κ⁺⁺)–repeat point) over an inaccessible κ can be weakened to an (ω, κ + 1)–repeat point and to an (ω, κ)-repeat point, respectively. This is quite close to the equiconsistency, since by [Gi], an (ω, < κ)–repeat
point is needed for the existence of such ideals.

**Theorem 3.1.** Suppose that there exists an \((\omega, \kappa + 1)\)-repeat point over \(\kappa\). Then in a generic extension preserving inaccessibility of \(\kappa\), \(NS_\kappa\) is a precipitous ideal.

The proof combines constructions of [Gi] and [Gi1]. We will stress only the new points.

**Sketch of the Proof:** Let \(\alpha < o(\kappa)\) be an \((\omega, \kappa + 1)\)-repeat point for \(\langle F(\kappa, \alpha') \mid \alpha' < o(\kappa) \rangle\), i.e. \(cf \alpha = \aleph_0\) and for every \(A \in \cap\{F(\kappa, \alpha^* + i) \mid i \leq \kappa\}\) there are unboundedly many \(\beta\)'s in \(\alpha\) such that \(\beta + \kappa < \alpha\) and \(A \in \cap\{F(\kappa, \beta + i) \mid i \leq \kappa\}\).

As in [Gi] we first define the iteration \(P_\delta\) for \(\delta\) in the closure of \{\(\beta < \kappa \mid \beta\) is an inaccessible or \(\beta = \gamma + 1\) for an inaccessible \(\gamma\)\}. On limit stages as in [Gi] the limit of [Gi2] is used. Define \(P_{\delta+1}\). If \(o(\delta) \neq \beta + \delta\) or \(o(\delta) \neq \beta + \delta + 1\) for some \(\beta\) then \(P_{\delta+1} = P_\delta \ast C(\delta^+) \ast P(\delta, o(\delta))\) exactly as in [Gi], where \(C(\delta^+)\) is the Cohen forcing for adding \(\delta^+\) functions from \(\delta\) to \(\delta\) and \(P(\delta, o(\delta))\) is a forcing used in [Gi] for changing cofinalities without adding new bounded sets.

Now let \(o(\delta) = \beta + \delta\) for some ordinal \(\beta, \beta > \delta\). First we force as above with \(C(\delta^+)\).

**Case 1.** The value of the first Cohen function added by \(C(\delta^+)\) on 0 is not 0. Then we force as above with \(P(\delta, o(\delta))\).

**Case 2.** The value of the first Cohen function added by \(C(\delta^+)\) on 0 is 0. Then we are going to shoot a club through \(\cap\{F(\delta, \beta + i) \mid i < \delta\}\) using the forcing notion \(Q\) described below.

\[Q = \{\langle c, e \rangle \mid c \subseteq \delta \text{ closed}, |c| < \delta, e \subseteq \cap\{F(\delta, \beta + i) \mid i < \delta\}, |e| < \delta\}\]

\(\langle c_1, e_1 \rangle \leq \langle c_2, e_2 \rangle\) iff \(c_2\) is an end-extension of \(c_1, e_1 \subseteq e_2\) and for every \(A \in e_1, c_2 \setminus c_1 \subseteq A\).

Now every regular \(i < \delta\) forcing with \(P(\delta, \beta + i)\) produces a club through \(\cap\{F(\delta, \beta + j) \mid j < i\}\) changing cofinality of \(\delta\) to \(i\). Thus \(Q\) contains an \(i\)-closed dense subset in any \(P(\alpha, \beta + i)\)-generic extension of \(V^{P_\alpha \ast C(\alpha^+)}\). Based on this observation, we are going to use here the method of [Gi1]. It makes the iteration of such forcings \(Q\) possible.

If \(o(\delta) = \beta + \delta + 1\) for some \(\beta, \beta > \delta\), then we combine both previous cases together inside the Prikry sequence produced at this stage.

Namely, we proceed as follows. Let \(i : V \to M \cong Ult(V, F(\delta, \beta + \delta))\). We consider also the second ultrapower, i.e. \(N \cong Ult(M, F(i(\delta), i(\beta) + i(\delta)))\). Let \(k : M \to N\) and
\( j = k \circ i : V \to N \) be the corresponding elementary embeddings. Then, in \( N \), \( o(\delta) = \beta + \delta \) and \( o(i(\delta)) = i(\beta) + i(\delta) \). So, in \( N \), both \( \delta \) and \( i(\delta) \) are of the type of the previous cases.

We want to deal with \( \delta \) as in Case 1 and with \( i(\delta) \) as in Case 2. This can easily be arranged, since we are free to change one value of a Cohen function responsible for the switch between Cases 1 and 2. The next stage will be to define an extension \( \mathcal{F}^*(\delta, \beta + \delta) \) of \( \mathcal{F}(\delta, \beta + \delta) \times \mathcal{F}(\delta, \beta + \delta) \) in \( V[G_\delta] \), where \( G_\delta \subseteq \mathcal{P}_\delta \) is generic. For this use \([Gi1]\) where \( N \) was first stretched by using the direct limit of \( \mathcal{F}(\delta, \beta + \delta) \). Notice that the following holds:

\((*)\) if \( \langle \delta_n, \rho_n \rangle \mid n < \omega \) is such a sequence then both \( \langle \delta_n \mid n < \omega \rangle \) and \( \langle \rho_n \mid n < \omega \rangle \) are almost contained in every club of \( \delta \) of \( V \).

Simply because \( \langle \delta, i(\delta) \rangle \in j(C) \) for a club \( C \subseteq \delta \) in \( V \).

This completes the definition of \( \mathcal{P}_{\delta+1} \) and hence also the definition of the iteration.

The intuition behind this is as follows. We add a club subset to every set \( A \in \cap \{ \mathcal{F}(\kappa, \alpha + i) \mid i \leq \kappa \} \). \( \alpha \) is \( (\omega, \kappa + 1) \)-repeat point, so \( A \) reflects unboundedly many times in \( \alpha \), i.e. \( A \in \cap \{ \mathcal{F}(\kappa, \beta + i) \mid i \leq \kappa \} \) for unboundedly many \( \beta \)'s in \( \alpha \).

Reflecting this below \( \kappa \), we will have \( A \cap \delta \in \cap \{ \mathcal{F}(\delta, \gamma + 1) \mid i \leq \delta \} \), where \( o(\delta) = \gamma + \delta \).

In \([Gi, \text{Sec. 3}]\), we had \( (\alpha, \kappa^+ + 1) \)-repeat point which translates to \( \cap \mathcal{F}(\delta, \gamma + i) \mid i \leq \delta^+ \} \).

Then just the forcing \( \mathcal{P}(\delta, o(\delta)) \) will add a club through every set in \( \cap \{ \mathcal{F}(\delta, \gamma + i) \mid i \leq \delta^+ \} \).

Here our assumptions are weaker and we use the forcing \( Q \) instead. There are basically two problems with this: iteration and integration with \( \mathcal{P}(\delta, \beta) \)'s. For the first problem the method of \([Gi1]\) is used directly. The problematic point with the second is that once using \( Q \) we break the Rudin-Keisler ordering of extensions of \( \mathcal{F}(\delta, \beta) \)'s used in \( \mathcal{P}(\delta, o(\delta)) \).

In order to overcome this difficulty, we split the case \( o(\delta) = \beta + \delta \) into two. Thus in Case 1 we keep Rudin-Keisler ordering and in Case 2 force with \( Q \). Finally at stages \( \alpha \) with \( o(\delta) = \beta + \delta + 1 \) both cases are combined in the fashion described above. The rest of the proof is as in \([Gi, \text{Sec. 3}]\).

The following obvious changes needed to be made: instead of \( E \in \cap \{ \mathcal{F}(\kappa, \beta) \mid \alpha < \beta \leq \alpha + \kappa^+ \} \) we now deal with \( E \in \cap \{ \mathcal{F}(\kappa, \beta) \mid \alpha < \beta \leq \alpha + \kappa \} \) and instead of \( E(\kappa^+) \) there we use \( E(\kappa) = \{ \delta \in E \mid \text{there is } \delta \text{ s.t. } o^E(\delta) = \delta + \kappa \text{ and } \delta \cap E \in \cap \{ \mathcal{F}(\delta, \delta') \mid \delta \leq \delta' < \delta + \kappa^+ \} \) which belongs to \( \mathcal{F}(\kappa, \beta + \kappa) \) for unboundedly many \( \beta \)'s in \( \alpha \). Lemmas 3.2-3.5 of \([Gi]\) have
the same proof in the present context. The changes in the proof of Lemma 3.6 of [Gi] (actually the claim there) use the method of iteration of $Q$’s and the principal ($*$).

If we are not concerned about a regular cardinal, then the same construction starting with an $(\omega, \kappa)$–repeat point turns $NS^\text{Sing}_\kappa$ into a precipitous ideal. So the following holds:

**Theorem 3.2.** Suppose that there exists an $(\omega, \kappa)$–repeat point over $\kappa$. Then in a generic extension preserving inaccessibility of $\kappa$, $NS^\text{Sing}_\kappa$ is a precipitous ideal.

4. **Open problems**

A. **Saturatedness.**

Are the following statements consistent:

1*.

1. $NS_\kappa$ saturated over an inaccessible $\kappa$.

2*.

2. $NS^\theta_\kappa$ saturated over an inaccessible $\kappa$ for some fixed cofinality $\theta$.

3. $NS^{\kappa_+}_\kappa$ saturated for a cardinal $\kappa \geq \aleph_1$.

Known that $NS_{\aleph_1}$ can be saturated [St-Van-We], [Fo-Ma-Sh], [Sh-Wo], $NS^\theta_\kappa$ cannot for $\theta < \kappa$ [Sh]. By [Je-Wo] $NS_\kappa \upharpoonright \text{Reg}$ can be saturated over inaccessible.

Test question:

4. Let $E \subseteq \text{Reg} \cap \kappa$, $\kappa$ inaccessible, there is no sharp for a strong cardinal. Suppose $NS_\kappa \upharpoonright E$ is precipitous (or saturated), $E \| U$ is the normal measure of the extender used to move $\kappa$ by a generic embedding restricted to the core model.

Is there a nonstationary set in $U$ in a generic ultrapower?

B. **Precipitous ideals.**

1. Is the strength of $NS^{\aleph_0}_\kappa$ precipitous over an inaccessible $\kappa$ $(\omega, < \kappa)$-repeat point?

2. Can a model for $NS_\kappa$ precipitous over an inaccessible $\kappa$ be constructed from something weaker than an $(\omega, \kappa + 1)$–repeat point?

3. What is the strength of $NS_\kappa$ precipitous over the first inaccessible?

The upper bound for (3) is a Woodin cardinal, see [Sh-Wo]. If it is possible to construct a model with $NS^{\aleph_0}_\kappa$ precipitous from an $(\omega, < \kappa)$–repeat point, then we think that this assumption is also sufficient for (2) and (3).

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* No by a recent result of S. Shelah and the author.
A test question:

4. How strong is “there is a precipitous ideal over the first inaccessible”?

By [Sh-Wo] a Woodin cardinal suffices. On the other hand one can show that at least $\omega(\kappa) = \kappa$ is needed.

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