Learning nonlinear dynamical systems from a single trajectory

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Abstract

We introduce algorithms for learning nonlinear dynamical systems of the form $x_{t+1} = \sigma(\Theta^* x_t) + \varepsilon_t$, where $\Theta^*$ is a weight matrix, $\sigma$ is a nonlinear link function, and $\varepsilon_t$ is a mean-zero noise process. We give an algorithm that recovers the weight matrix $\Theta^*$ from a single trajectory with optimal sample complexity and linear running time. The algorithm succeeds under weaker statistical assumptions than in previous work, and in particular i) does not require a bound on the spectral norm of the weight matrix $\Theta^*$ (rather, it depends on a generalization of the spectral radius) and ii) enjoys guarantees for non-strictly-increasing link functions such as the ReLU. Our analysis has two key components: i) we give a general recipe whereby global stability for nonlinear dynamical systems can be used to certify that the state-vector covariance is well-conditioned, and ii) using these tools, we extend well-known algorithms for efficiently learning generalized linear models to the dependent setting.

1 Introduction

We consider nonlinear dynamical systems of the form

$$x_{i+1} = f^*(x_i) + \varepsilon_i,$$

where $f^*: \mathbb{R}^d \to \mathbb{R}^d$ is an unknown function, $\{\varepsilon_i\}_{i=0}^n$ is an independent, mean-zero noise process in $\mathbb{R}^d$ and $x_0 = \mathbf{0}$. Dynamical systems are ubiquitous in applied mathematics, engineering, and computer science, with applications including control systems, time series analysis, econometrics, and natural language processing. The recent success of deep reinforcement learning (Mnih et al., 2015; Silver et al., 2017; Lillicrap et al., 2016) has led to renewed interest in developing efficient algorithms for learning complex nonlinear systems such as (1) from data.

In this paper, we focus on the task of estimating the dynamics $f^*$ given a single trajectory $\{x_i\}_{i=1}^{n+1}$, where $f^*$ belongs to a known function class $\mathcal{F}$. We focus on the following questions:

- What is the sample complexity of recovering the dynamics $f^*$? How is it determined by $\mathcal{F}$?
- What algorithmic principles enable computationally efficient recovery of the dynamics?

For linear dynamical systems where $f^*(x) = \Theta^* x$, subroutines for efficiently estimating dynamics from data form a core building block of certainty-equivalent control, which enjoys optimal sample complexity guarantees for this simple setting (Mania et al., 2019; Simchowitz and Foster, 2020). While linear dynamical systems have been the subject of intense recent interest (Dean et al., 2019; Hazan et al., 2017; Tu and Recht, 2018; Hazan et al., 2018; Simchowitz et al., 2018; Sarkar and Rakhlin, 2019; Simchowitz et al., 2019; Mania et al., 2019; Sarkar et al., 2019), nonlinear dynamical systems are comparatively poorly understood.
1.1 On the performance of least squares

On the algorithmic side, a natural starting point for learning the system (1) is the least squares estimator

\[ \hat{f}_n = \arg \min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \| f(x_i) - x_{i+1} \|^2, \]  

(2)

where \( \| \cdot \| \) denotes the entrywise \( \ell_2 \) norm. A basic observation is that the in-sample prediction error (or, denoising error) of this estimator is bounded by the so-called offset Rademacher complexity introduced by Rakhlin and Sridharan (2014); Liang et al. (2015):

**Proposition 1.** The least-squares estimator (2) guarantees

\[ \mathbb{E}_x \left[ \frac{1}{n} \sum_{i=1}^{n} \| \hat{f}_n(x_i) - f^*(x_i) \|^2 \right] \leq \mathbb{E}_x \sup_{g \in \mathcal{G}} \left[ \frac{1}{n} \sum_{i=1}^{n} 4(\epsilon_i, g(x_i)) - \| g(x_i) \|^2 \right] =: \mathcal{R}_n^o(\mathcal{G}), \]  

(3)

where \( \mathcal{G} = \mathcal{F} - f^* \) and \( \mathbb{E}_x \) denotes expectation with respect to \( \{\epsilon_i\}_{i=1}^{n} \).

The proof is a simple consequence of the basic inequality for least squares (van de Geer, 2000). The offset Rademacher process captures the notion of localization/self-normalization (Bartlett et al., 2005; Koltchinskii, 2006; de la Peña et al., 2008): The negative quadratic term penalizes fluctuations from the term involving the random variables \( \{\epsilon_i\}_{i=1}^{n} \), leading to fast rates for prediction error. In particular, if \( \epsilon_i \) has subgaussian parameter \( \tau^2 \) and \( f^*(x) = \Theta^* x \) is a familiar linear dynamical system, we have \( \mathcal{R}_n^o(\mathcal{G}_{\text{linear}}) \leq \tau^2 \cdot \frac{d^2}{n} \). The utility of this approach, however, lies in the fact that it easily extends beyond the linear setting. For example, if \( \mathcal{G} \) consists of a class of generalized linear dynamical systems of the form

\[ x_{i+1} = \sigma(\Theta^* x_i) + \epsilon_i, \]  

(4)

where \( \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a 1-Lipschitz link function, we enjoy a similar guarantee: \( \mathcal{R}_n^o(\mathcal{G}_{\text{glm}}) \leq \tau^2 \cdot \frac{d^2}{n} \). More generally, even though (3) has a complex dependent structure (the variables \( \epsilon_1, \ldots, \epsilon_n \) determine the evolution of \( x_1, \ldots, x_n \) via (1)), it is possible to bound the value for general function classes \( \mathcal{F} \) such as neural networks, kernels, decision trees using sequential covering numbers and chaining techniques introduced in Rakhlin et al. (2014); Rakhlin and Sridharan (2014). However, there are number of important questions that remain if one wishes to use this type of learning guarantee for real-world control applications.

- **Efficient algorithms.** Even for simple nonlinear systems such as the generalized linear model (4), computing the least-squares estimator (2) may be computationally intractable in the worst case. For what classes of interest can we obtain algorithms that are both computationally efficient and sample-efficient?

- **Out-of-sample performance.** The prediction error guarantee (3) only concerns performance on the realized sequence \( \{x_i\}_{i=1}^{n} \). For control applications such as certainty-equivalent control, it is essential to bound the performance of the estimator \( \hat{f}_n \) on counterfactual sequences in which the data generating process is \( x_{t+1} = \hat{f}_n(x_t) + \epsilon_t \) (i.e., error in simulation). For linear and generalized linear systems, a sufficient condition for such a guarantee is to recover the weight matrix \( \Theta^* \) in parameter norm. Under what conditions on the data generating process can we obtain such guarantees?

\(^1\)Note that \( x_i \) is measurable with respect to the \( \sigma \)-algebra \( \sigma(\epsilon_1, \ldots, \epsilon_{i-1}) \).

\(^2\)See Section 1.4.
1.2 Contributions.

We provide a new efficient algorithm for recovery of generalized linear systems (4). Our algorithm runs in nearly-linear time and obtains optimal \(O(\sqrt{d^2/n})\) sample complexity for recovery in Frobenius norm. Conceptually, our key technical observations are as follows:

- We provide a general recipe based on Lyapunov functions for proving that data remains well-conditioned/nearly isometric for stable dynamical systems, without assuming linearity.

- We show that efficient algorithms for learning generalized linear models in the i.i.d. setting (Kalai and Sastry, 2009; Kakade et al., 2011) cleanly port to the dependent setting. Here the key insight is that the empirical counterparts of simple non-convex losses arising from generalized linear models remain well-behaved even under dependent data.

Our algorithm improves prior work on two fronts: First, we do not require a bound on the spectral norm of \(\Theta^*\), and instead require a bound on a parameter that generalizes the notion of the spectral radius to the nonlinear setting. Second, we can recover \(\Theta^*\) even when the link function \(\sigma\) is the ReLU, eschewing invertibility assumptions from previous results.

1.3 Related work

Learning guarantees for autoregressive processes have a long history in statistics, though early results for nonlinear systems mainly concern prediction error as in (3), and do not consider algorithmic issues (Baraud et al., 2001; van de Geer, 2002).

Generalized linear systems (4) subsume linear dynamical systems, which are fundamental topic in control theory. System identification for the linear setting has been studied since the early days of control (Åström and Eykhoff, 1971; Ljung, 1998; Campi and Weyer, 2002; Vidyasagar and Karandikar, 2006), and is closely related to LQR control. We build on a recent line of work providing non-asymptotic/finite-sample guarantees for the LQR, both for system identification (Dean et al., 2019; Tu and Recht, 2018; Simchowitz et al., 2018; Sarkar and Rakhlín, 2019; Simchowitz et al., 2019; Sarkar et al., 2019) and (offline and online) control (Abbasi-Yadkori and Szepesvári, 2011; Dean et al., 2019, 2018; Mania et al., 2019). These approaches leverage the rich structure available in the linear setting (in particular, the Riccati equations). While we cannot take advantage of such structure, our Lyapunov approach to establishing well-conditioned empirical designs may be thought of as a natural extension of these structural results to the generalized linear setting (4).

Our results are closely related to recent work of (Oymak, 2019; Bahmani and Romberg, 2019; Sattar and Oymak, 2020). In particular, the concurrent work of Sattar and Oymak (2020) considers a more general setting and provides guarantees very similar to our own using complementary techniques; we provide a detailed comparison at the end of Section 3.1. The results of Oymak (2019) and Bahmani and Romberg (2019) consider a slightly different form of generalized linear dynamical system inspired by recurrent neural networks, which takes the form \(x_{t+1} = \sigma(A^* x_t + B^* u_t)\), where \(u_t\) is an observed noise process (representing a control signal) and the link function \(\sigma\) is known and invertible. The key difference between this setup and our own is that because \(\sigma\) is invertible, and because both \(\{x_t\}\) and \(\{u_t\}\) are observed, the problem reduces to noiseless linear regression: \(\sigma^{-1}(x_{t+1}) = A^* x_t + B^* u_t\). In particular, since the regression problem is noiseless, their sample complexity guarantees allow for exact recovery once the number of samples reaches a critical threshold; in contrast, for our noisy setting, only approximate recovery is possible given finite samples.

Lastly, we mention that our setting is related to the work of Hall et al. (2018) for learning sparse generalized linear autoregressive processes. These results rely on fairly strong assumptions on the
globally exponentially stable (g.e.s.) with respect to a norm $\|\cdot\|$. Well-conditioned data plays a fundamental role in statistical estimation. For linear regression, it is

The noise variables $\xi_t$ are well-behaved, it is not clear a-priori whether the empirical design matrix should enjoy favorable conditioning—indeed, the observations $\{x_i\}_{i=1}^{n+1}$ evolve from the noise process in a complex dependent fashion. In general, the behavior of the empirical design matrix will heavily depend on the system $f^*$.

Assumption 1. The noise variables $\{\xi_t\}_{t=1}^n$ are independent. Each increment is isotropic (zero-mean, with $E[\xi_t \xi_t^\top] = I$) and satisfies $\xi_t \sim \subG(\tau^2)$.\footnote{The assumption that the noise process $\xi_t$ has identity covariance serves only to keep notation compact; our results transparently extend to general covariance $\Sigma$ under the standard assumption that $\Sigma^{-1} \xi_t$ is subgaussian. Likewise, our results extend the dependent setting as long as each increment is still conditionally mean-zero and subgaussian.}

While Assumption 1 ensures that each increment $\xi_t$ is well-behaved, it is not clear a-priori whether the empirical design matrix should enjoy favorable conditioning—indeed, the observations $\{x_i\}_{i=1}^{n+1}$ evolve from the noise process in a complex dependent fashion. In general, the behavior of the empirical design matrix will heavily depend on the system $f^*$. Here we show that classical results in control theory on exponential stability of the system $f^*$ provide sufficient conditions for both upper and lower control of the spectrum of the empirical design matrix. While our guarantees apply to the noisy system (1), our assumptions depend on the behavior of the system in absence of noise:

$$x_{t+1} = f(x_t),$$

where $x_0 = 0$.

**Definition 1** (Global exponential stability). A noiseless system (6) given by map $f : \mathbb{R}^d \to \mathbb{R}^d$ is globally exponentially stable (g.e.s.) with respect to a norm $\|\cdot\|_\Sigma$ if there exist constants $C_f > 0$ and $\rho_f < 1$ depending only on $f$ such that for all $k \geq 1$,

$$\|f^k\|_{\op(\Sigma)} \leq C_f \rho_f^k,$$ (7)
where \( \| f \|_{\text{op}} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{|x|_0} \).

In this paper, we focus on systems where \( f^* \) satisfies the g.e.s. property, and where this is certified by a quadratic Lyapunov function.

**Definition 2.** A map \( f : \mathbb{R}^d \to \mathbb{R}^d \) is \((K, \rho)\)-g.e.s. if there exists a matrix \( K > 0 \) and constant \( 0 \leq \rho < 1 \) such that for all \( x \in \mathbb{R}^d \),
\[
\| f(x) \|_K^2 \leq \rho \cdot \| x \|_K^2,
\]
where \( \| x \|_K := \sqrt{(x, K x)} \).

Any \((K, \rho)\)-g.e.s. map satisfies (7) with \( \| \cdot \|_1 = \| \cdot \|_K, C_f = 1, \) and \( \rho_f = \rho^{1/2} \). The equation (8) is homogeneous under rescaling, and consequently we will assume without loss of generality that \( K \geq I \) for the remainder of the paper.

In general, finding certificates of stability for nonlinear dynamical systems is a difficult problem. Providing necessary and sufficient conditions for stability for rich classes of nonlinear dynamical systems remains an active area of research, with most development proceeding on a fairly case-by-case basis. We develop a general reduction from lower and upper isometry to \((K, \rho)\)-stability, which allows us to leverage developments in control in a black-box fashion as opposed to having to prove concentration results case-by-case. Our main result here is Theorem 1, which shows that any \((K, \rho)\)-g.e.s. system enjoys both upper and lower isometry.

**Theorem 1.** Consider the noisy system (1), and let noise process satisfy Assumption 1. Suppose the map \( f^* \) satisfies the \((K, \rho)\)-g.e.s. property Definition 2 in the absence of noise. Then for any \( \delta > 0 \), once \( n \geq \frac{c d}{(1-\rho)^{1/2}} \log \left( R_{K, \rho}/\delta + 1 \right) \), with probability at least \( 1 - \delta \) the iterates \( \{x_i\}_{i=1}^n \) of the noisy system satisfy
\[
\frac{1}{n} \cdot I \leq \frac{1}{n} \sum_{i=1}^n x_i x_i^\top \leq 4 R_{K, \rho} \cdot I,
\]
where \( R_{K, \rho} := \frac{\nu(K)}{1-\rho} \) is the effective radius of the system and \( c > 0 \) is an absolute constant.

The key feature of Theorem 1 is that we only need to assume the \((K, \rho)\)-g.e.s. property on the map \( f^* \) in the absence of noise, yet the theorem gives a guarantee on the trajectory generated by the noisy system (1) as long as Assumption 1 is satisfied.

The proof has three parts, each of which relies on the machinery of self-normalization. We first use the structure of the dynamics (1) to show that the lower isometry in (9) holds as soon as we have a weak upper bound on the covariance of the form \( \frac{1}{n} \sum_{i=1}^n x_i x_i^\top \leq \frac{B}{\delta} \cdot I \), where \( \delta \) is the failure probability. We then show that in \((K, \rho)\)-g.e.s. systems, this condition is satisfied with \( B = R_{K, \rho} \). Finally, the strong upper bound in (9) is attained by using a self-normalized inequality to boost the weak upper bound and remove the \( 1/\delta \) factor.

### 2.1 Lower isometry for generalized linear systems

We now provide sufficient conditions under which the generalized linear systems that are the focus of our main learning results satisfy the g.e.s. property. We make the following mild regularity assumption on the link function.

**Assumption 2.** The link function \( \sigma : \mathbb{R}^d \to \mathbb{R}^d \) has the form \( \sigma(x) := (\sigma_1(x_1), \ldots, \sigma_d(x_d)) \), where each coordinate function \( \sigma_i : \mathbb{R} \to \mathbb{R} \) is non-decreasing, 1-Lipschitz, and satisfies \( \sigma_i(0) = 0 \).

\(^4\text{When } f : \mathbb{R}^d \to \mathbb{R}^d, \text{ we let } f^k \text{ denote the } k\text{-times composition of } f, \text{ i.e. } f^k = f \circ \cdots \circ f. \)
With this assumption, the following constrained Lyapunov equation provides a sufficient condition under which the generalized linear system satisfies the g.e.s. property.

**Proposition 2.** Suppose there exists a diagonal matrix $K > 0$ and scalar $\rho < 1$ such that

$$\Theta^\top K \Theta \preceq \rho \cdot K.$$  

(10)

Then the map $f = \sigma \circ \Theta$ is $(K, \rho)$-g.e.s. whenever $\sigma$ satisfies Assumption 2.

**Proof.** Observe that for any $x \in \mathbb{R}^d$ we have

$$\|f(x)\|_K^2 \leq \|\Theta x\|_K^2 \leq \rho \cdot \|x\|_K^2,$$

where (i) uses that $K$ is diagonal and positive definite and that each coordinate-wise link $\sigma_i : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz with $\sigma_i(0) = 0$, and (ii) uses the Lyapunov equation (10).

Proposition 2 can be used to invoke Theorem 1 for any generalized linear system of the form $f^* = \sigma \circ \Theta$. Thus, we can ensure lower and upper isometry hold for generalized linear systems whenever their stability is certified the Lyapunov condition (10).

The equation (10) strengthens the usual Lyapunov condition for linear systems by adding the additional constraint that $K$ is diagonal. This condition is stronger than the classical spectral radius condition that $\rho(\Theta) < 1$, but it can easily be seen that some type of strengthening is necessary, as the classical condition is not sufficient for nonlinear systems. For example, the matrix $\Theta = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ has $\rho(\Theta) = 0$, but the map $x \mapsto \text{relu}(\Theta x)$ is not g.e.s. — indeed, we have $\text{relu}(\Theta e_1) = e_1$, where $e_1$ is the first standard basis vector. A sufficient condition for Proposition 2 is that $\Theta$ has spectral norm bounded by unity, but the condition (10) is a strictly weaker than this assumption. Further sufficient conditions include: 1) $\rho(\Theta) < 1$ and $\Theta$ has non-negative entries (Rantzer, 2011, Proposition 2), and 2) $\rho(|\Theta|) < 1$, where $|\Theta|$ denotes element-wise absolute value operator.

To close this section, we remark that for any fixed link function, the Lyapunov condition (10) may be overly pessimistic as a condition for exponential stability, but a classical line of research in nonlinear control establishes that this condition is actually necessary for a somewhat more general class of nonlinearities (Megretski and Treil, 1993; Shamma, 1994; Poola and Tikku, 1995). Nonetheless, finding tighter conditions for specific link functions of interest such as the ReLU remains an interesting direction for future work, as does leveraging more general (e.g., piecewise) Lyapunov functions.

### 3 Algorithms for generalized linear dynamical systems

We now leverage the isometry results of Section 2 to develop efficient algorithms with parameter recovery guarantees for generalized linear systems. Following Section 2, we make the following assumption on the generalized linear system.

**Assumption 3.** The system (1) is generalized linear ($f^* = \sigma \circ \Theta^*$) and is $(K, \rho)$-g.e.s. in the sense of Proposition 2. Furthermore, $\|\Theta^*\|_F \leq W$, where $W$ is known to the learner.\(^5\)

### Background: Learning generalized linear models

Our algorithm for learning generalized linear dynamical systems builds on developments for learning generalized linear models in statistical

\(^5\)There is no restriction on the range of the parameter $W$, but some of our sample complexity guarantees depend on it polynomially.
learning. Consider the simpler setting where we receive \( \{(x_i, y_i)\}_{i=1}^n \) i.i.d., where \( y = \sigma(\theta^*, x) + \varepsilon \) and \( \mathbb{E}[\varepsilon | x] = 0 \). For this setting the population loss \( L(\theta) := \mathbb{E}_{x,y}[\sigma(\theta(x)) - y]^2 \) is not convex. However, if the link function \( \sigma \) is strictly increasing the and the population covariance \( \mathbb{E}[xx^T] \) is well-conditioned, the loss satisfies a gradient-dominance type property, and gradient descent on the empirical loss will converge to \( \theta^* \) given sufficiently many samples (Mei et al., 2018). To provide guarantees even when \( \sigma \) is not strictly increasing, we opt to use a variant of the GLMtron algorithm introduced by Kakade et al. (2011). The GLMtron algorithm performs gradient descent using a “pseudogradient” for the empirical loss in which the derivative of \( \sigma \) is simply dropped:

\[
\theta^{(t+1)} = \theta^{(t)} - \frac{1}{n} \sum_{i=1}^n (\sigma(\theta^{(t)}, x_i) - y_i)x_i
\]

Following the pseudogradient allows the algorithm to efficiently provide prediction error guarantees even when \( \sigma \) is not strictly increasing, and this is the starting point for our approach.

### 3.1 Algorithm and guarantees

**Algorithm 1** is a natural extension of GLMtron to handle the vector-valued target variables and matrix-valued parameters that arise in our dynamical system setting.

**Algorithm 1** Parameter estimation for generalized linear systems

1. **input**: Single trajectory: \( X_{n+1} = \{x_i\}_{i=1}^{n+1} \), Learning rate schedule: \( \eta_t \).
2. **initialize**: \( \hat{\Theta}^{(1)} = 0_{d \times d} \).
3. Define \( \mathcal{M} = \{\Theta \in \mathbb{R}^{d \times d} | \|\Theta\|_F \leq W\} \).
4. for \( t = 1, \ldots, m \) do
5. \( \hat{\Theta}^{(t+1)} = \text{Proj}_{\mathcal{M}} (\hat{\Theta}^{(t)} - \eta_t \hat{G}(\hat{\Theta}^{(t)}, X_{n+1})) \). \hspace{0.5cm} // \( \hat{G} \) is the pseudogradient; see (11).
6. return: \( \hat{\Theta} = \hat{\Theta}^{(m)} \) (Option I), or \( \hat{\Theta} = \hat{\Theta}^{(t)} \) with \( t \in [m] \) uniform (Option II).

Algorithm 1 is closely related to projected gradient descent on the empirical square loss \( \hat{L}(\Theta, X_{n+1}) := \frac{1}{n} \sum_{t=1}^n \|\Theta x_t - x_{t+1}\|^2 \), but rather than following the gradient, the algorithm follows the pseudogradient

\[
\hat{G}(\Theta^{(t)}, X_{n+1}) := \frac{1}{n} \sum_{i=1}^n (\sigma(\Theta^{(t)}x_i) - x_{i+1})x_i^T,
\]

attained by dropping the link derivative \( \sigma' \) from the gradient. This modification allows for prediction guarantees without assuming a lower bound on the link function derivative, and allows for weaker dependence on the derivative lower bound for parameter recovery guarantees. In particular, we show that the algorithm obtains the best of both worlds in a certain sense. First, with only the assumption that the link function is Lipschitz, the algorithm ensures that iterates have low prediction error on average. Consequently, if we select an iterate uniformly at random (Option II in Algorithm 1), the iterate will have low prediction error (a “slow rate” of type \( 1/\sqrt{n} \)) in expectation. On the other hand, suppose the following assumption holds.

**Assumption 4.** There exists a constant \( \zeta > 0 \) such that for all \( i, |\sigma_i(x) - \sigma_i(y)| \geq \zeta |x - y| \) for all \( x, y \in \mathbb{R} \).

In this case, the algorithm enjoys linear convergence, and taking the last iterate (Option I in Algorithm 1) leads to a “fast” \( 1/n \)-type rate for prediction error, as well as a parameter recovery guarantee.
To state the performance guarantee, we let $\mathcal{E}(\Theta) := \frac{1}{n} \sum_{i=1}^{n} \| (\sigma(\Theta x_i) - \sigma(\Theta^* x_i)) \|^2$ denote the in-sample prediction error, and let $\mathbb{E}_A$ denote expectation with respect to the algorithm’s internal randomness (uniform selection of the iterate returned in line 6 under Option II).

**Theorem 2.** Let $\delta > 0$ be fixed and let Assumptions 1-3 hold. Whenever $n \geq \frac{c\tau^4 d}{1-\rho} \log \left( R_{K,\rho}/\delta + 1 \right)$, Algorithm 1 enjoys the following guarantees:

1. **Slow rate.** If $\eta_t = \frac{1}{16R_{K,\rho}}$ and $m \geq C_0 \cdot \sqrt{n}$, then with probability at least $1 - \delta$, Algorithm 1 with Option II has
   \[
   \mathbb{E}_A[\mathcal{E}(\Theta)] \leq C_1 \cdot \sqrt{\frac{d^2}{n} \log \left( 4R_{K,\rho}/\delta + 1 \right)},
   \]
   where $C_0 \leq c \cdot W B \left( \frac{\tau^2 d R_{K,\rho} \log \left( 4R_{K,\rho}/\delta + 1 \right)}{1-\rho} \right)^{-1/2}$ and $C_1 \leq c \tau W \sqrt{\frac{\sigma_{\max}(K)}{1-\rho}}$.

2. **Fast rate.** Suppose that Assumption 4 holds in addition to Assumptions 1-3. If $\eta_t = \frac{c^2}{(16R_{K,\rho})^2}$ and $m \geq C_2 \log \left( 1 + \frac{nW^2 B^2}{\tau^2 R_{K,\rho}} \right)$, then with probability at least $1 - \delta$, Algorithm 1 with Option I has
   \[
   \mathcal{E}(\Theta) \leq C_3 \cdot \frac{d^2}{n} \log \left( 4R_{K,\rho}/\delta + 1 \right), \quad \text{and} \quad \| \Theta - \Theta^* \|^2_F \leq C_4 \cdot \frac{d^2}{n} \log \left( 4R_{K,\rho}/\delta + 1 \right),
   \]
   where $C_2 \leq cB^2 \zeta^4$, $C_3 \leq c \tau^2 B^2 \zeta^6 \cdot \frac{\sigma_{\max}(K)}{1-\rho}$ and $C_4 \leq c \tau^2 \zeta^4 \cdot \frac{\sigma_{\max}(K)}{1-\rho}$.

Assumption 4 is satisfied for the so-called “leaky ReLU” $\text{relu}_\beta(x) := \max\{x, \beta x\}$ with $\beta > 0$, but not for the ReLU. Our next theorem shows that under stronger assumptions on the noise process, the algorithm succeeds at parameter recovery for the ReLU as well. We make the following assumption.

**Assumption 5.** The link function $\sigma$ is the ReLU ($\sigma_i(x_i) = \text{relu}(x_i) := \max\{x_i, 0\}$) and the noise process is Gaussian, with $\varepsilon_t \sim \mathcal{N}(0, I)$.

The gaussian assumption ensures for any pair of parameters, sufficiently large probability mass lies in the region where the ReLU is active. In particular, we use (Lemma 10) that for any pair $u, v \in \mathbb{R}^d$, $\mathbb{E}_{\varepsilon \sim \mathcal{N}(0, I)}[(\text{relu}(u, \varepsilon) - \text{relu}(v, \varepsilon))^2] \propto \|u - v\|^2$. Similar guarantees can be established for log-concave distributions using arguments in Balcan and Long (2013), but we consider only the gaussian case for simplicity.

**Theorem 3** (Parameter recovery for the ReLU). Suppose assumptions Assumptions 1-3 and Assumption 5 hold. Let $\delta > 0$ be fixed and suppose $n \geq c \tau^4 d^2 \log \left( R_{K,\rho}/\delta + 1 \right)$. Then when $\eta_t = (16R_{K,\rho})^{-2} e^{-4\rho R_{K,\rho}}$ and $m \geq C_0 \cdot \log n$, Algorithm 1 with Option I guarantees that with probability at least $1 - \delta$,
   \[
   \| \Theta - \Theta^* \|^2_F \leq C_1 \cdot \frac{d^2}{n} \cdot R_{K,\rho}^2 \log^2(2R_{K,\rho}n/\delta + 1),
   \]
   where $C_0 \leq cB^2 e^{8\rho R_{K,\rho}}$ and $C_1 \leq c \frac{\tau^2 W^2}{(1-\rho)^2} \cdot e^{8\rho R_{K,\rho}}$.

Let us discuss some key features of Theorem 2 and Theorem 3. First, in the fast rate regime where Assumption 4 holds, Theorem 2 attains the usual parametric rate $\mathbb{E} \| \Theta - \Theta^* \|^2_F \leq \sqrt{\frac{d}{n}}$, which is optimal for this setting (Tsybakov, 2008). The algorithm is also linearly convergent in this regime, and so the runtime to attain parameter recovery is nearly linear. On the other hand, the dependence on problem-dependent parameters such as $K$ and $\rho$ in the results can almost certainly be improved.
for all of the results. For example, while there are certainly systems for which \( \| x_t \| \) grows as \( \frac{1}{\sqrt{n}} \), it is not clear whether exponential dependence on this parameter in Theorem 3 is required for parameter recovery with the ReLU. More generally, the factor \( e^{RK,\rho} \) in Theorem 3 can be replaced with \( \max_i e^{\mu_i} \), where \( \mu_i \) is an upper bound on the (conditional) expected value of \( x_i \) at time \( i \). Our analysis simply bounds \( \| \mu_i \|_2^2 \) by \( R_{K,\rho} \), and any improvements to this norm bound for systems of interest will immediately lead to improved rates.

**Detailed comparison with related work** Concurrent work of Sattar and Oymak (2020) also considers the problem of learning generalized linear systems of the form (4), and provides similar guarantees to Theorem 2 in the fast rate regime. Let \( \theta_k^* \) and \( \tilde{\theta}_k \) denote the \( k^{th} \) rows of \( \Theta^* \) and \( \tilde{\Theta} \) respectively. Theorem 6.2 of (Sattar and Oymak, 2020) considers a gradient descent-based estimator and shows that once \( n \) is a sufficiently large problem-dependent constant and the number of iterations is polylogarithmic in \( n \), \( \| \tilde{\theta}_k - \theta^*_k \| \leq \tilde{O}(\frac{c}{\sqrt{d} \cdot \sqrt{\frac{d}{n}}}) \), where \( \tilde{O}(\cdot) \) hides logarithmic dependence problem parameters. Under the same conditions, Theorem 2 attains a comparable guarantee of \( \| \tilde{\Theta} - \Theta^* \|_F \leq \tilde{O}(\frac{c}{\sqrt{d} \cdot \sqrt{\frac{d}{n}}}) \). At a more conceptual level, our techniques and tools are complementary: Sattar and Oymak (2020) use mixing time arguments and analyze gradient descent, while we use martingale arguments and analyze GLMtron. Additional results we provide include i) explicit Lyapunov conditions under which stability of the generalized linear system holds ii) prediction error guarantees for non-strictly increasing link functions, and iii) parameter recovery guarantees for the ReLU.

4 Discussion

We have shown that the exponential stability, in conjunction with Lyapunov arguments, offers a simple approach to establishing isometry guarantees for data generated by nonlinear dynamical systems, and we have provided efficient algorithms for learning and parameter recovery in generalized linear systems. We hope that the analysis techniques introduced here will find use beyond the generalized linear setting, as well as for end-to-end control.

Going forward, it will be interesting to draw further connections and build stronger bridges between Lyapunov theory and empirical process theory for dependent data. For example, what properties of data generated by dynamical systems can we use Lyapunov functions to certify, going beyond lower and upper isometry?

Lastly, we remark on an extension to the non–autonomous setting. Consider a non–autonomous system of the form \( x_{i+1} = \sigma(\Theta \cdot x_i + B \cdot u_i) + \varepsilon_i \), where \( \{u_i\}_{i=1}^n \) are control inputs. This setting reduces to the autonomous case via the expression

\[
\begin{bmatrix}
  x_{i+1} \\
  u_{i+1}
\end{bmatrix}
= \sigma \left( \begin{bmatrix}
  \Theta & B \\
  0 & 0
\end{bmatrix} \begin{bmatrix}
  x_i \\
  u_i
\end{bmatrix} + \begin{bmatrix}
  \varepsilon_i \\
  u_{i+1}
\end{bmatrix} \right),
\]

and our techniques can consequently be applied as long as the control inputs have persistent excitation.

Acknowledgements

We thank Adam Klivans, Alexandre Megretski, and Karthik Sridharan for helpful discussions. We acknowledge the support of ONR award #N00014-20-1-2336 and NSF TRIPODS award #1740751.
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A Basic technical results

**Lemma 1** (Freedman’s Inequality (e.g., Agarwal et al. (2014))). Let \(\{X_i\}_{i=1}^{n}\) be a sequence of real–valued random variables such that for all \(i\), \(|X_i| \leq R\) and \(\mathbb{E}[X_i \mid X_1, \ldots, X_{i-1}] = 0\). Define \(S := \sum_{i=1}^{n} X_i\) and \(V := \sum_{i=1}^{n} \mathbb{E}[X_i^2 \mid X_1, \ldots, X_{i-1}]\). For any \(\delta \in (0, 1)\) and \(\lambda \in [0, 1/R]\), with probability at least \(1 - \delta\),

\[
|S| \leq (e - 2)\lambda V + \frac{\log(1/\delta)}{\lambda}.
\]

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Martin J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019.
Lemma 2 (Vershynin (2012), Theorem 5.39). Let $A$ be a $n \times d$ matrix whose rows, $\{A_i\}_{i=1}^n$, are independent and isotropic random vectors in $\mathbb{R}^d$, belonging to $\text{subG}(\tau^2)$. Then for every $t \geq 0$, with probability at least $1 - 2\exp(-c\tau^{-4}t^2)$, one has

$$\sqrt{n} - c'\tau^2\sqrt{d} - t \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \sqrt{n} + c'\tau^2\sqrt{d} + t,$$

where $c$ and $c'$ are numerical constants.

Lemma 3. Let $A$ be a $n \times d$ matrix whose rows, $\{A_i\}_{i=1}^n$, are independent and isotropic random vectors in $\mathbb{R}^d$, belonging to $\text{subG}(\tau^2)$. Then whenever

$$\sqrt{n} \geq c\tau^2\left(\sqrt{d} + \sqrt{\log(2/\delta)}\right),$$

we have that with probability at least $1 - \delta$,

$$\frac{3}{4} \cdot I \leq \sum_{i=1}^d A_iA_i^\top \leq \frac{5}{4} \cdot I,$$

where $c > 0$ is a numerical constant.

Proof of Lemma 3. Let constants $c$ and $c'$ be as in Lemma 2. We invoke Lemma 2 with parameter $t := c^{-1}\tau^2\sqrt{\log(2/\delta)}$, which implies that

$$\sqrt{n} - c'\tau^2\sqrt{d} - c^{-1}\tau^2\sqrt{\log(2/\delta)} \leq \sigma_{\min}(A) \leq \sigma_{\max}(A) \leq \sqrt{n} + c'\tau^2\sqrt{d} + c^{-1}\tau^2\sqrt{\log(2/\delta)}$$

with probability at least $1 - \delta$. The final bound follows because

$$\sqrt{n} \geq 16 \cdot \left(c'\tau^2\sqrt{d} + c^{-1}\tau^2\sqrt{\log(2/\delta)}\right).$$

Lemma 4 (Sarkar and Rakhlin (2019), Proposition 7.1). Let $P$ and $V$ be arbitrary positive semidefinite and postive definite matrices, respectively, and define $\bar{P} = P + V$. Let $Q$ be any matrix for which

$$\|\bar{P}^{-1/2}Q\|_{\text{op}} \leq \gamma.$$

Then for any vector $v$, we have

$$\|v^\top Q\| \leq \gamma \sqrt{v^\top P v + v^\top V v}.$$  

Lemma 5 (Sarkar and Rakhlin (2019), Proposition 3.1). Let $V > 0$ be a fixed matrix, and consider the dynamics

$$x_{i+1} = f(x_i) + \varepsilon_i,$$

where the noise process follows Assumption 1. Define $Y_{n-1} = \sum_{i=1}^{n-1} f(x_i)f(x_i)^\top + V$. Then for any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\|\left(Y_{n-1}\right)^{-1/2}\sum_{i=0}^{n-1} f(x_i)e_i^\top\right\|_{\text{op}} \leq c\tau \sqrt{d\log\left(\frac{5\det(Y_{n-1})^{1/2d}\det(V)^{-1/2d}}{\delta^{1/d}}\right)},$$

where $c$ is an absolute constant.\(^6\)

\(^6\)This lemma is stated in Sarkar and Rakhlin (2019) for linear systems, but one can easily verify that it holds for arbitrary systems as stated here.
B Proofs from Section 1

Proof of Proposition 1. Optimality of \( \hat{f}_n \) implies that \( \sum_{i=1}^{n} \| \hat{f}_n(x_i) - x_{i+1} \| \leq \sum_{i=1}^{n} \| f^*(x_i) - x_{i+1} \| \), and consequently

\[
\sum_{i=1}^{n} \| \hat{f}_n(x_i) - f^*(x_i) \|^2 \leq 2 \sum_{i=1}^{n} \left( \epsilon_i, \hat{f}_n(x_i) - f^*(x_i) \right).
\]

Rearranging, we have

\[
\sum_{i=1}^{n} \| \hat{f}_n(x_i) - f^*(x_i) \|^2 \leq 2 \frac{\| f_i \|_{\infty}}{n \sum_{i=1}^{n}} \sum_{i=1}^{n} \left( \epsilon_i, \hat{f}_n(x_i) - f^*(x_i) \right).
\]

Proof of Lemma 6. From the dynamics (1), we have deterministic identities

\[
x_{i+1} x_{i+1}^\top = f(x_i) f(x_i)^\top + f(x_i) \epsilon_i^\top + \epsilon_i f(x_i)^\top + \epsilon_i \epsilon_i^\top,
\]

and

\[
\sum_{i=1}^{n} x_{i+1} x_{i+1}^\top = \sum_{i=1}^{n} f(x_i) f(x_i)^\top + f(x_i) \epsilon_i^\top + \epsilon_i f(x_i)^\top + \epsilon_i \epsilon_i^\top.
\]

We know from Lemma 3 that for any \( \delta > 0 \), when \( n \geq c \tau^4 (d + \log(2/\delta)) \), with probability at least \( 1 - \delta \),

\[
\frac{3}{4} I \leq \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \epsilon_i^\top \leq \frac{5}{4} I.
\]

Now, define

\[
P := \sum_{i=1}^{n-1} f(x_i) f(x_i)^\top, \quad Q := \sum_{i=1}^{n-1} f(x_i) \epsilon_i^\top, \quad \text{and} \quad V := \frac{3n}{4} I.
\]
To prove the main result, we use Lemma 4 and Lemma 5 to show that the cross terms in (13) have little impact. Specifically, Lemma 5 states that for any \( \delta > 0 \), with probability at least \( 1 - \delta \), it is ensured that

\[
\left\| (P + V)^{-1/2}Q \right\|_{op} \leq cT \sqrt{d \log \left( \frac{5 \det(P + V)^{1/2} \det(V)^{-1/2}}{\delta^{1/d}} \right)}.
\]

Let \( \gamma := cT \sqrt{d \log \left( \frac{5 \det(P + V)^{1/2} \det(V)^{-1/2}}{\delta^{1/d}} \right)} \). Conditioning on the event (15) and using Lemma 4, we have that for any unit vector \( v \in \mathbb{R}^d \), \( \left\| v^\top Q \right\| \leq \sqrt{\kappa^2 + \frac{3n}{4} \gamma} \), where \( \kappa^2 = v^\top P v \). Substituting this bound into (13) and conditioning on (14) we are guaranteed that for any unit vector \( v \),

\[
v^\top \sum_{i=1}^n x_{i+1} x_i^\top v \geq \kappa^2 - 2 \sqrt{\kappa^2 + \frac{3n}{4} \gamma} + \frac{3n}{4},
\]

(16)

Selecting \( \delta = \delta_0/2 \) and conditioning on the event \( \sum_{i=1}^n f(x_i) f(x_i)^\top \leq \frac{nB}{\delta_0} I \), which happens with probability at least \( 1 - \delta_0 \), we can upper bound \( \gamma \) as

\[
\gamma \leq cT \sqrt{\log \left( \frac{5 \det \left( \frac{nB}{\delta_0} I + \frac{3n}{4} I \right)^{1/2} \det \left( \frac{3n}{4} I \right)^{-1/2}}{\delta} \right)} \leq cT \sqrt{d \log \left( \frac{2B}{\delta_0} + 1 \right) + \log \left( \frac{1}{\delta} \right)}.
\]

Simplifying (16) further, we have

\[
v^\top \sum_{i=1}^n x_{i+1} x_i^\top v \geq \kappa^2 - 2 \kappa \gamma - 2 \sqrt{\frac{3n}{4} \gamma + \frac{3n}{4}} \geq - \gamma^2 - 2 \sqrt{\frac{3n}{4} \gamma + \frac{3n}{4}}.
\]

Thus, whenever \( n \geq c\gamma^2 \), i.e.

\[
cT \sqrt{d \log \left( \frac{2B}{\delta_0} + 1 \right) + \log \left( \frac{1}{\delta} \right)} \leq \sqrt{\frac{n}{12}},
\]

(17)

we have

\[
v^\top \sum_{i=1}^n x_{i+1} x_i^\top v \geq \frac{n}{4},
\]

for any vector unit \( v \) after conditioning on the events (14), (15), and \( \sum_{i=1}^n f(x_i) f(x_i)^\top \leq \frac{nB}{\delta_0} I \). The condition on \( n \) in the lemma statement follows from the requirement in (17). Since \( v^\top \sum_{i=1}^n x_{i+1} x_i^\top v \geq \frac{n}{4} \) holds for any unit vector – fixed or depending on \( \{x_i\}_{i=1}^n \), we have

\[
\sum_{i=1}^n x_{i+1} x_i^\top \geq \frac{n}{4} I.
\]

Note that our development so far requires that events (14), (15) and \( \sum_{i=1}^n f(x_i) f(x_i)^\top \leq \frac{nB}{\delta_0} I \) occur simultaneously, which happens with probability at least \( 1 - \delta_0 - 2\delta \). To deduce the theorem statement we set \( \delta = \delta_0/2 \).

Lemma 6 requires that there is some \( B \) such that \( \sum_{i=1}^n f(x_i) f(x_i)^\top \leq \frac{nB}{\delta} I \) with probability at least \( 1 - \delta \). We first show that in \((K, \rho)\)-g.e.s. systems, this condition is satisfied with \( B = R_{K, \rho} \) (Lemma 7). This immediately gives the lower isometry bound in Theorem 1. The upper bound in Theorem 1 is attained through Lemma 8, which sharpens the upper bound in Lemma 7 by removing the \( 1/\delta \) factor.
Lemma 7 (Weak Upper Bound). Let the noise process satisfy Assumption 1. For any \((K, \rho)\)-g.e.s. map \(f^*\), with probability \(1 - \delta\),

\[
\sum_{i=1}^{n} x_i^\top K x_i \leq \frac{n \text{tr}(K)}{(1 - \rho)\delta}, \quad \text{and} \quad \sum_{i=1}^{n} x_i^\top x_i \leq \frac{n \text{tr}(K)}{(1 - \rho)\delta}.
\]

Furthermore, with probability at least \(1 - \delta\),

\[
\sum_{i=1}^{n} f^*(x_i)^\top K f^*(x_i) \leq \frac{n \rho \text{tr}(K)}{(1 - \rho)\delta}, \quad \text{and} \quad \sum_{i=1}^{n} f^*(x_i)^\top f^*(x_i) \leq \frac{n \rho \text{tr}(K)}{(1 - \rho)\delta}.
\]

Proof of Lemma 7. Let \(\{x_i\}_{i=1}^{n}\) be the state observations from the nonlinear dynamical system. Then

\[
\sum_{i=1}^{n} x_i^\top K x_i = \sum_{i=0}^{n-1} \bar{x}_i^\top K \bar{x}_i + 2 \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i + \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i,
\]

where \(\bar{x}_i := f^*(x_i)\). Using the \((K, \rho)\)-g.e.s. condition, we can upper bound the first term on the right-hand side to get

\[
\sum_{i=1}^{n} x_i^\top K x_i \leq \rho \sum_{i=0}^{n-1} x_i^\top K x_i + 2 \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i + \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i.
\]

Define \(Y_n := \sum_{i=0}^{n} x_i^\top K x_i\). Then this is equivalent to

\[
Y_n \leq \rho Y_n + 2 \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i + \sum_{i=0}^{n-1} \varepsilon_i^\top K \varepsilon_i.
\]

By assumption we have \(x_0 = 0\), and taking an expectation gives us \(\mathbb{E}[Y_n] \leq \rho \mathbb{E}[Y_n] + \text{tr}(K)\), which implies that \(\mathbb{E}[Y_n] \leq \frac{\text{tr}(K)}{(1 - \rho)}\). Then, using Markov’s inequality and the fact that \(\sigma_{\min}(K) \geq 1\), we have that with probability at least \(1 - \delta\),

\[
\sum_{i=0}^{n} x_i^\top x_i \leq \frac{n \text{tr}(K)}{(1 - \rho)\delta}.
\]

Furthermore, since \(\rho \sum_{i=1}^{n} x_i^\top K x_i \geq \sum_{i=0}^{n-1} \bar{x}_i^\top K \bar{x}_i\), we have with probability at least \(1 - \delta\)

\[
\sum_{i=1}^{n} f^*(x_i)^\top K f^*(x_i) \leq \frac{n \rho \text{tr}(K)}{(1 - \rho)\delta}.
\]

\[
\sum_{i=1}^{n} f^*(x_i)^\top f^*(x_i) \leq \frac{n \rho \text{tr}(K)}{(1 - \rho)\delta}.
\]

\[
\left\| \frac{1}{n} \sum_{i=1}^{n-1} x_i \varepsilon_i \right\|_F \leq c \left( \frac{d}{n} \cdot \frac{\text{tr}(K)}{(1 - \rho)\delta} \cdot \log \left( \frac{4 \text{tr}(K)}{(1 - \rho)} \cdot \frac{1}{\delta} + 1 \right) \right).
\]

Lemma 8 (Strong Upper Bound). Consider any system (1) for which \(f^*\) is \((K, \rho)\)-g.e.s., and let the noise process satisfy Assumption 1. For any \(\delta > 0\), as soon as \(n\) is large enough such that

\[
c \sqrt{\frac{d}{n} \log \left( \frac{4 \text{tr}(K)}{(1 - \rho)} + 1 \right)} \leq \frac{1 - \rho}{\rho},
\]

for some absolute constant \(c\), we have that with probability at least \(1 - \delta\),

\[
\text{tr} \left( \sum_{i=1}^{n} x_i K x_i \right) \leq 4n \cdot \frac{\text{tr}(K)}{(1 - \rho)}, \quad \text{and} \quad \text{tr} \left( \sum_{i=1}^{n} x_i^\top x_i \right) \leq 4n \cdot \frac{\text{tr}(K)}{(1 - \rho)},
\]

and

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} x_i \varepsilon_i \right\|_F \leq c \sqrt{\frac{d}{n} \cdot \frac{\text{tr}(K)}{(1 - \rho)\delta} \cdot \log \left( \frac{4 \text{tr}(K)}{(1 - \rho)} \cdot \frac{1}{\delta} + 1 \right)}.
\]
Proof of Lemma 8. First, observe that
\[
\sum_{i=1}^{n} x_i^\top K x_i = \sum_{i=0}^{n-1} \tilde{x}_i^\top K \tilde{x}_i + 2 \sum_{i=0}^{n-1} \varepsilon_i^\top K \tilde{x}_i + \sum_{i=1}^{n-1} \varepsilon_i^\top K \varepsilon_i,
\]
where \( \tilde{x}_i := f^*(x_i) \). As in proof of Lemma 7, we may use the \((K, \rho)\)-g.e.s. property to upper bound by
\[
\sum_{i=1}^{n} x_i^\top K x_i \leq \rho \sum_{i=0}^{n-1} x_i^\top K x_i + 2 \sum_{i=0}^{n-1} \varepsilon_i^\top K \tilde{x}_i + \sum_{i=1}^{n-1} \varepsilon_i^\top K \varepsilon_i.
\]
(19)
For the remainder of the proof, we define \( v \) be an arbitrary random vector with \( \mathbb{E}[vv^\top] = I \); this random variable is only used for analysis and is independent of the underlying data generating process. Observe that for any vector \( u \) and matrix \( A > 0 \), we have
\[
u^\top Au = \text{tr}(A^{1/2}uu^\top A^{1/2}) = \text{tr}(\mathbb{E}_v[v^\top A^{1/2}uu^\top A^{1/2}v]).
\]
Since \( x_0 = 0 \) by assumption, Eq. (19) becomes
\[
(1 - \rho) \sum_{i=0}^{n-1} \mathbb{E}_v [v^\top (K^{1/2}x_ix_i^\top K^{1/2})v] \leq \sum_{i=0}^{n-1} \mathbb{E}_v [v^\top K^{1/2}\varepsilon_i^\top K^{1/2}v] + 2 \sum_{i=0}^{n-1} \mathbb{E}_v [v^\top K^{1/2}x_i\varepsilon_i^\top K^{1/2}v].
\]
Define random variables \( \alpha_v := (1/n) \sum_{i=0}^{n} v^\top (K^{1/2}x_ix_i^\top K^{1/2})v \), \( \beta_v := (1/n) \sum_{i=0}^{n-1} v^\top (K^{1/2}\varepsilon_i^\top K^{1/2})v \). Then, we have
\[
(1 - \rho) \mathbb{E}_v [n\alpha_v] \leq 2 \sum_{i=0}^{n-1} \mathbb{E}_v [v^\top K^{1/2}x_i\varepsilon_i^\top K^{1/2}v] + \mathbb{E}_v [n\beta_v].
\]
(20)
For any \( v \in \mathbb{R}^d \), we have \( \sum_{i=0}^{n} v^\top K^{1/2}x_i\varepsilon_i^\top K^{1/2}v \leq \sqrt{n\alpha_v + n\beta_v} \gamma \) by Lemma 4, where \( \gamma \) is an upper bound on
\[
\left\| \left( \sum_{i=0}^{n} K^{1/2}x_i\varepsilon_i^\top K^{1/2} + K^{1/2}\varepsilon_i^\top K^{1/2} \right)^{-1/2} \left( K^{1/2}x_i\varepsilon_i^\top K^{1/2} \right) \right\|_{op} \leq \gamma.
\]
We proceed to bound \( \gamma \). Since the dynamics satisfy the \((K, \rho)\)-g.e.s. property we have
\[
\sum_{i=0}^{n} \tilde{x}_i^\top K \tilde{x}_i \leq \rho \sum_{i=0}^{n} x_i^\top K x_i,
\]
and by Lemma 7 and Lemma 3 it follows with probability at least \( 1 - \delta \) that
\[
\sum_{i=0}^{n} x_i^\top K x_i \leq \frac{n \text{tr}(K)}{(1 - \rho)\delta} = \frac{n \tilde{B}}{\delta} \quad \text{and} \quad \frac{n I}{2} \leq \sum_{i=0}^{n-1} \varepsilon_i^\top \varepsilon_i \leq \frac{3n I}{2}.
\]
(21)
By conditioning on (21), \( \sum_{i=0}^{n} K^{1/2}x_i\varepsilon_i^\top K^{1/2} \leq \frac{n \tilde{B}}{\delta} I \) and choosing \( V = \frac{3n I}{2} K \), we can ensure with probability at least \( 1 - \delta \) that
\[
\left\| \left( \sum_{i=0}^{n} K^{1/2}(\tilde{x}_i\varepsilon_i^\top + \varepsilon_i^\top \varepsilon_i) K^{1/2} \right)^{-1/2} \left( K^{1/2}\tilde{x}_i\varepsilon_i^\top K^{1/2} \right) \right\|_{op} \leq C \sqrt{d \log \left( \frac{\tilde{B}}{\delta} + 1 \right)}.
\]
(22)
from Lemma 5. Thus, by setting \( \gamma = C \sqrt{d \log \left( \frac{\tilde{B}}{\delta} + 1 \right)} \) in (20) we have with probability at least \( 1 - \delta \)
\[
(1 - \rho) \mathbb{E}_v [n\alpha_v] \leq \mathbb{E}_v [\sqrt{n\alpha_v + n\beta_v}] \gamma + \mathbb{E}_v [n\beta_v] \leq \mathbb{E}_v [\sqrt{n\alpha_v} + \sqrt{n\beta_v}] \gamma + \mathbb{E}_v [n\beta_v].
\]

Whenever $n$ is large enough to satisfy $\frac{\gamma}{\sqrt{n}} \leq \frac{1-\rho}{2\tau}$, this implies
\[
\mathbb{E}_v[\alpha_v] \leq \frac{4\mathbb{E}_v[\beta_v]}{(1-\rho)}.
\]
From Lemma 3, we have
\[
\frac{1}{n} \sum_{i=0}^{n-1} \varepsilon_i \varepsilon_i^\top \leq \frac{3I}{2}
\]  
(23)
with probability $1-\delta$ whenever $n \geq c\tau^4(d + \log(2/\delta))$, which in particular is satisfied when $\frac{\gamma}{\sqrt{n}} \leq \frac{1-\rho}{2\tau}$.

Then, conditioning on both (22) and (23), we have
\[
\mathbb{E}_v[\beta_v] \leq \frac{3}{2} \mathbb{E}_v[v^\top K v]
\]  
and
\[
\text{tr}\left(\sum_{i=1}^{n} x_i K x_i^\top\right) \leq \frac{4n\text{tr}(K)}{(1-\rho)}.
\]

The main claim now follows because (22) and (23) occur simultaneously with probability at least $1-2\delta$. Furthermore, we also have
\[
2 \left\| \frac{1}{n} \sum_{i=1}^{n} K^{1/2} x_i \varepsilon_i^\top K^{1/2} \right\|_F \leq \sqrt{\mathbb{E}_v\left[\left\| \frac{1}{n} \sum_{i=1}^{n} K^{1/2} x_i \varepsilon_i^\top K^{1/2} v \right\|_2^2\right]} \leq 2 \frac{\gamma}{\sqrt{n}} \sqrt{\mathbb{E}_v[(\alpha_v + \beta_v)]}
\]
with probability at least $1-2\delta$. Simplifying, this implies
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} K^{1/2} x_i \varepsilon_i^\top K^{1/2} \right\|_F \leq cT \sqrt{\frac{\text{dtr}(K)}{n(1-\rho)} \log\left(\frac{\text{dtr}(K)}{\delta(1-\rho)} + 1\right)},
\]
for some absolute constant $c > 0$. Finally, the condition on $n$ in the lemma statement follows by simplifying and expanding the condition $\frac{\gamma}{\sqrt{n}} \leq \frac{1-\rho}{2\tau}$.

**Proof of Theorem 1.** The lower isometry bound is attained by applying Lemma 6 using the upper bound from Lemma 7, and the upper isometry bound follows immediately from Lemma 8.

### D Proofs from Section 3

For the remainder of the appendix we make use of the filtration
\[
\mathcal{G}_i := \sigma(\varepsilon_0, x_1, \varepsilon_1, \ldots, x_{i-1}, \varepsilon_{i-1}, x_i).
\]  
(24)

#### D.1 Proof of Theorem 2

We first define two parameters:
\[
\mu = cT \sqrt{\frac{\text{dtr}(K)}{n(1-\rho)} \log\left(\frac{4\text{tr}(K)}{\delta(1-\rho)} + 1\right)}, \quad \text{and} \quad B = \frac{4\text{tr}(K)}{(1-\rho)}.
\]  
(25)

Let us proceed with the proof. To begin, recall from Lemma 8 that whenever
\[
cT \sqrt{\frac{d}{n} \log\left(\frac{B}{\delta} + 1\right)} \leq \frac{1-\rho}{2\tau},
\]
we have with probability at least $1 - \delta$,
\[
\text{tr} \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right) \leq \frac{4\text{tr}(K)}{(1 - \rho)}, \quad \text{and} \quad \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i x_i^T \right\|_F \leq c\sqrt{n(1 - \rho) \log \left( \frac{4\text{tr}(K)}{\delta(1 - \rho)} + 1 \right)}.
\]

From here, we condition on this good event above and split the proof into two cases. First, we handle only the prediction error/denoising guarantee (12), which requires only that the link function $\sigma$ is non-decreasing and does not require a lower bound on the link derivative. Then, in the second part of the proof, we use the assumption that the $\sigma$ is strictly increasing to strengthen this bound and provide a parameter recovery guarantee. The first part of the proof extends the arguments of Kakade et al. (2011) to the dependent setting where data is generated from a generalized linear system, while the second part uses the refined isometry guarantees developed in Section 2.

D.1.1 Proof of Theorem 2, Part I: Slow rate for prediction error

Throughout this proof we use that the projection operation in Algorithm 1 ensures that for all $t$, $\|\Theta^{(t)} - \Theta^*\|_F \leq 2W$. From Algorithm 1 we have that
\[
\left\| \Theta^{(t+1)} - \Theta^* \right\|_F = \left\| \text{Proj}_M(\Theta^{(t)} - \eta_t \hat{G}(\Theta^{(t)}, X)) - \Theta^* \right\|_F,
\]
which implies that
\[
\left\| \Theta^{(t+1)} - \Theta^* \right\|_F \leq \left\| \Theta^{(t)} - \eta_t \hat{G}(\Theta^{(t)}, X) - \Theta^* \right\|_F.
\]
Furthermore, recall that $\hat{G}(\Theta^{(t)}, X)$ is defined as $\hat{G}(\Theta^{(t)}, X) = \frac{1}{n} \sum_{i=1}^{n} (\sigma(\Theta^{(t)} x_i) - x_{i+1}) x_i^T$. Thus, by expanding the right-hand side above, we have
\[
\left\| \Theta^{(t)} - \Theta^* \right\|^2_F - \left\| \Theta^{(t+1)} - \Theta^* \right\|^2_F \geq \frac{2 \eta_t}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)} x_i), (\Theta^* - \Theta^{(t)}) x_i)
\]
\[
- \eta^2 \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)} x_i)) x_i^T \right\|^2_F.
\]
(26)

To obtain a lower bound on $\left\| \Theta^{(t)} - \Theta^* \right\|^2_F - \left\| \Theta^{(t+1)} - \Theta^* \right\|^2_F$ we need a lower bound on the right-hand side in (26) and upper bound on the norm in (27). Analyzing (26) first, we get
\[
\frac{2}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)} x_i), (\Theta^* - \Theta^{(t)}) x_i) = \frac{2}{n} \sum_{i=1}^{n} (\sigma(\Theta^* x_i) - \sigma(\Theta^{(t)} x_i), (\Theta^* - \Theta^{(t)}) x_i)
\]
\[
+ \frac{2}{n} \sum_{i=1}^{n} \langle \varepsilon_i, (\Theta^* - \Theta^{(t)}) x_i \rangle.
\]
Note that since $\left\| \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i x_i^T \right\|_F \leq 2\mu$, and $\|\Theta^* - \Theta^{(t)}\|_F \leq 2W$, we have
\[
\frac{2}{n} \sum_{i=1}^{n} \langle \varepsilon_i, (\Theta^* - \Theta^{(t)}) x_i \rangle \geq - \left\| \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i x_i^T \right\|_F \left\| \Theta^* - \Theta^{(t)} \right\|_F \geq -4W \mu.
\]
(28)

Since each $\sigma_i$ is non-decreasing and 1-Lipschitz, we also have
\[
\frac{2}{n} \sum_{i=1}^{n} (\sigma(\Theta^* x_i) - \sigma(\Theta^{(t)} x_i), (\Theta^* - \Theta^{(t)}) x_i) \geq \frac{2}{n} \sum_{i=1}^{n} \| \sigma(\Theta^* x_i) - \sigma(\Theta^{(t)} x_i) \|_2^2.
\]
(29)
Together, these lead to the following lower bound on (26):

$$
2\eta \frac{n}{m} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{(t)} x_i), (\theta^{*} - \theta^{(t)}) x_i) \geq 2\eta (\mathcal{E}(\theta^{(t)}) - 2W\mu).
$$

For (27) we have

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{(t)} x_i)) x_i \right\|_F^2 = \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{*} x_i) + \sigma(\theta^{*} x_i) - \sigma(\theta^{(t)} x_i)) x_i \right\|_F^2
$$

\[
\leq 2 \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{*} x_i)) x_i \right\|_F^2 + 2 \left\| \frac{1}{n} \sum_{i=1}^{n} (\sigma(\theta^{(t)} x_i) - \sigma(\theta^{*} x_i)) x_i \right\|_F^2
\]

Since \( \varepsilon_i = x_{i+1} - \sigma(\theta^{*} x_i) \) we have

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{*} x_i)) x_i \right\|_F^2 \leq \mu^2,
$$

and by the Cauchy Schwarz inequality

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} (\sigma(\theta^{(t)} x_i) - \sigma(\theta^{*} x_i)) x_i \right\|_F^2 \leq \left( \frac{1}{n} \sum_{i=1}^{n} \left\| (\sigma(\theta^{(t)} x_i) - \sigma(\theta^{*} x_i)) \right\|_F^2 \right) \cdot \text{tr} \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right).
$$

Combining these upper bounds, we have

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\theta^{(t)} x_i)) x_i \right\|_F^2 \leq 2\mu^2 + 2 \left( \frac{1}{n} \sum_{i=1}^{n} \left\| (\sigma(\theta^{(t)} x_i) - \sigma(\theta^{*} x_i)) \right\|_F^2 \right) \cdot \text{tr} \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right).
$$

Since \( \text{tr} \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T \right) \leq B \), we can further upper bound the right-hand side using

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} (\sigma(\theta^{(t)} x_i) - \sigma(\theta^{*} x_i)) x_i \right\|_F^2 \leq \mathcal{E}(\theta^{(t)}) \cdot B.
$$

Combining the bounds for (26) and (27) we have

$$
\left\| \theta^{(t)} - \theta^{*} \right\|_F^2 - \left\| \theta^{(t+1)} - \theta^{*} \right\|_F^2 \geq 2\eta (\mathcal{E}(\theta^{(t)}) - 2W\mu) - 2\eta^2 (\mu^2 + \mathcal{E}(\theta^{(t)})B).
$$

By choosing \( \eta = \frac{1}{4B} \) we get

$$
\left\| \theta^{(t)} - \theta^{*} \right\|_F^2 - \left\| \theta^{(t+1)} - \theta^{*} \right\|_F^2 \geq c \left( \frac{\mathcal{E}(\theta^{(t)})}{B} - \frac{5W\mu}{B} - \frac{W\mu^2}{B^2} \right), \tag{30}
$$

Summing Eq. (30) we have

$$
\frac{1}{m} \cdot \sum_{t=0}^{m-1} \left( \left\| \theta^{(t)} - \theta^{*} \right\|_F^2 - \left\| \theta^{(t+1)} - \theta^{*} \right\|_F^2 \right) \geq c \left( \frac{\sum_{t=0}^{m-1} \mathcal{E}(\theta^{(t)})}{mB} - \frac{5W\mu}{B} - \frac{W\mu^2}{B^2} \right).
$$

Observing that the left-hand side telescopes, after rearranging we have

$$
\mathbb{E}_A[\mathcal{E}(\hat{\theta})] \leq 5W\mu + \frac{W\mu^2}{B} + \frac{BW^2}{cm} \tag{31}
$$

where \( \mathbb{E}_A[\mathcal{E}(\hat{\theta})] = \frac{\sum_{t=0}^{m-1} \mathcal{E}(\theta^{(t)})}{mB} \). We choose the number of iterations, \( m \), such that \( m \geq \frac{WB}{5\mu} \) and then (31) becomes

$$
\mathbb{E}_A[\mathcal{E}(\hat{\theta})] \leq 10W\mu + \frac{W\mu^2}{B} \leq 20(W/B) \cdot (B\mu \vee \mu^2).
$$

□
D.1.2 Proof of Theorem 2, Part II: Fast rate for prediction and parameter recovery

Compared to the slow rate setting, to attain fast rates for strictly increasing link functions, we provide a tighter bound on the term $\frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t)) x_i \rangle$ in (28) so that only terms proportional to $\mu^2$ remain. This is made possible by the fact that we can lower bound the prediction error in terms of $\|\Theta(t) - \Theta^*\|$. To begin, since $|\sigma(a) - \sigma(b)| \geq \zeta |a - b| > 0$, we have

$$\frac{1}{n} \sum_{i=1}^{n} \left\| (\sigma(\Theta(t)x_i) - \sigma(\Theta^* x_i)) \right\|^2 \geq \frac{2}{n} \sum_{i=1}^{n} \left\| \Theta(t)x_i - \Theta^* x_i \right\|^2 .$$

Recall from (26) and (27) that

$$\left\| \Theta(t) - \Theta^* \right\|^2_F - \left\| \Theta(t+1) - \Theta^* \right\|^2_F \geq \frac{2\eta_t}{n} \sum_{i=1}^{n} \langle x_{i+1} - \sigma(\Theta(t)x_i), (\Theta^* - \Theta(t))x_i \rangle - \eta_t^2 \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta(t)x_i)) x_i \right\|^2_F .$$

Using the same analysis as in Appendix D.1.1, we have

$$\eta_t^2 \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta(t)x_i)) x_i \right\|^2_F \leq 2\eta_t^2 \left( \mu^2 + \mathcal{E}(\Theta(t)) B \right) \leq 2\eta_t^2 \left( \mu^2 + \left\| \Theta(t) - \Theta^* \right\|^2_F B^2 \right) ,$$

since $\mathcal{E}(\Theta(t)) \leq \left\| \Theta(t) - \Theta^* \right\|^2_F B$. Next, recall that

$$\frac{2}{n} \sum_{i=1}^{n} \langle x_{i+1} - \sigma(\Theta(t)x_i), (\Theta^* - \Theta(t))x_i \rangle = \frac{2}{n} \sum_{i=1}^{n} \langle \sigma(\Theta^* x_i) - \sigma(\Theta(t)x_i), (\Theta^* - \Theta(t))x_i \rangle$$

$$+ \frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t))x_i \rangle,$$

$$\geq \frac{2}{n} \sum_{i=1}^{n} \left\| \sigma(\Theta^* x_i) - \sigma(\Theta(t)x_i) \right\|^2 + \frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t))x_i \rangle ,$$

where the inequality follows because $\sigma(\cdot)$ is coordinate wise 1–Lipschitz and non–decreasing. Furthermore, by Lemma 6 we have that once $n$ is sufficiently large (in particular, when $n$ satisfies the assumptions of the theorem statement), with high probability, $\sum_{i=1}^{n} x_i x_i^T \succeq \frac{nf}{4}$. This in turn implies that

$$\frac{1}{n} \sum_{i=1}^{n} \left\| \sigma(\Theta(t)x_i) - \sigma(\Theta^* x_i) \right\|^2 \geq \frac{\zeta^2}{4} \sum_{i=1}^{n} \left\| \Theta(t)x_i - \Theta^* x_i \right\|^2 \geq \frac{\zeta^2}{4} \left\| \Theta(t) - \Theta^* \right\|^2_F .$$

Next, introducing a free parameter $\gamma > 0$, we lower bound $\frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t))x_i \rangle$ via Cauchy-Schwarz and the AM-GM inequality

$$\frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t))x_i \rangle \geq - \left\| \frac{2}{n} \sum_{i=1}^{n} \xi_i x_i^T \right\|_F \left\| \Theta^* - \Theta(t) \right\|_F \geq -\gamma \left\| \frac{2}{n} \sum_{i=1}^{n} \xi_i x_i^T \right\|^2_F - \frac{1}{\gamma} \left\| \Theta^* - \Theta(t) \right\|^2_F .$$

Since $\left\| \frac{2}{n} \sum_{i=1}^{n} \xi_i x_i^T \right\|^2_F \leq 4\mu^2$, we have

$$\frac{2}{n} \sum_{i=1}^{n} \langle \xi_i, (\Theta^* - \Theta(t))x_i \rangle \geq -4\gamma \mu^2 - \frac{1}{\gamma} \left\| \Theta^* - \Theta(t) \right\|^2_F .$$
With these developments, we deduce the following lower bound on the right-hand side of (32):

\[
\frac{2}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta(t)x_i), (\Theta^* - \Theta(t))x_i) \geq \frac{\zeta^2}{4} \left\| \Theta(t) - \Theta^* \right\|_F^2 - \frac{1}{\gamma} \left\| \Theta(t) - \Theta^* \right\|_F^2 - 4\gamma \mu^2.
\]  \( (36) \)

By setting \( \gamma = \frac{8}{\zeta^2} \), this becomes

\[
\frac{2}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta(t)x_i), (\Theta^* - \Theta(t))x_i) \geq \frac{\zeta^2}{8} \left\| \Theta(t) - \Theta^* \right\|_F^2 - \frac{32}{\zeta^2} \mu^2.
\]  \( (37) \)

Combining (32), (37), and (34), we have

\[
\left\| \Theta(t) - \Theta^* \right\|_F^2 - \left\| \Theta(t+1) - \Theta^* \right\|_F^2 \geq 2\eta_t \left( \frac{\zeta^2}{8} \left\| \Theta(t) - \Theta^* \right\|_F^2 - \frac{32}{\zeta^2} \mu^2 \right) - 2\eta_t^2 (\mu^2 + \left\| \Theta(t) - \Theta^* \right\|_F^2 B^2).
\]

By selecting \( \eta_t = \frac{\zeta^2}{16B^2} \), we have that for absolute constants \( c_1, c_2 \),

\[
\left\| \Theta(t) - \Theta^* \right\|_F^2 - \left\| \Theta(t+1) - \Theta^* \right\|_F^2 \geq \frac{c_1 \zeta^4}{B^2} \left\| \Theta(t) - \Theta^* \right\|_F^2 - \frac{c_2 \mu^2}{B^2}.
\]

Rearranging, we have

\[
\left\| \Theta(t+1) - \Theta^* \right\|_F^2 \leq \left( 1 - \frac{c_1 \zeta^4}{B^2} \right) \left\| \Theta(t) - \Theta^* \right\|_F^2 + \frac{c_2 \mu^2}{B^2},
\]

and by applying this bound recursively, we get

\[
\left\| \Theta(t+1) - \Theta^* \right\|_F^2 \leq \left( 1 - \frac{c_1 \zeta^4}{B^2} \right)^t W^2 + \frac{c_2 \mu^2}{c_1 \zeta^4}.
\]

Consequently, there exists an absolute constant \( c \) such that for \( t \geq \frac{B^2}{\zeta^2} \log \left( \frac{W^2 \zeta^4}{\mu^2} \right) \lor 1 \), we have

\[
\left\| \Theta(t+1) - \Theta^* \right\|_F^2 \leq \frac{2c_2 \mu^2}{c_1 \zeta^4}.
\]

We can now use the bound on \( \left\| \Theta(t+1) - \Theta^* \right\|_F^2 \), to obtain an upper bound on

\[
E(\Theta(t)) = \frac{1}{n} \sum_{i=1}^{n} \left\| \sigma(\Theta(t)x_i) - \sigma(\Theta^*x_i) \right\|^2.
\]

We continue from the following inequality:

\[
\left\| \Theta(t) - \Theta^* \right\|_F^2 - \left\| \Theta(t+1) - \Theta^* \right\|_F^2 \geq 2\eta_t \left( E(\Theta(t)) - 4\gamma \mu^2 - \frac{1}{\gamma} \left\| \Theta^* - \Theta(t) \right\|_F^2 \right)
\]

\[
- 2\eta_t^2 (\mu^2 + \left\| \Theta(t) - \Theta^* \right\|_F^2 B^2).
\]

By choosing \( t = m \), we get

\[
2\eta_m E(\Theta^{(m)}) \leq \left( 1 + \frac{2\eta_m}{\gamma} + 2\eta_m^2 B^2 \right) \left\| \Theta^{(m)} - \Theta^* \right\|_F^2 + \left( 8\eta_m \gamma + 2\eta_m^2 \right) \cdot \mu^2,
\]

from which we conclude that

\[
E(\Theta^{(m)}) \leq c \left( \frac{1}{\eta_m} + \gamma \right) \cdot \mu^2.
\]

\[\square\]
D.2 Proof of Theorem 3

Throughout this section of the appendix we overload notation and use \( \sigma(x) = \text{relu}(x) \) for \( x \in \mathbb{R} \) and \( \sigma(x) = (\text{relu}(x_1), \ldots, \text{relu}(x_d)) \) for \( x \in \mathbb{R}^d \).

D.2.1 Structural results for ReLU prediction error

**Lemma 9** (Cho and Saul (2009)). Let \( \varepsilon \sim \mathcal{N}(0, I) \). For vectors \( u \) and \( v \) denote by \( \theta_{u,v} \) the angle between \( u \) and \( v \). Assume \( \theta_{u,v} \in [0, \pi] \), then:

\[
\mathbb{E}[\sigma(\langle u, \varepsilon \rangle) \cdot \sigma(\langle v, \varepsilon \rangle)] = \frac{1}{2\pi} \|u\| \|v\| (\sin \theta_{u,v} + (\pi - \theta_{u,v}) \cos \theta_{u,v}).
\]

**Proof.** From Eq. (1) in Cho and Saul (2009), we have

\[
\mathbb{E}[\sigma(\langle u, \varepsilon \rangle) \cdot \sigma(\langle v, \varepsilon \rangle)] = k_1(u, v),
\]

where \( k_1(u, v) \) is defined in Eq. (3) in Cho and Saul (2009) as

\[
k_1(u, v) = \frac{1}{\pi} \|u\| \|v\| J_1(\theta_{u,v}),
\]

with \( J_1(\theta) \coloneqq \sin \theta + (\pi - \theta) \cos \theta \) (see Cho and Saul (2009), Eq. (6)). \( \square \)

**Proposition 3.** Let \( u, v \in \mathbb{R}^d \), and let \( \theta_{u,v} \) be the angle between \( u \) and \( v \). Suppose that \( \theta_{u,v} \in [0, \pi] \). Then

\[
\frac{2\theta_{u,v}}{\pi} \|u\| \|v\| \sin^2(\theta_{u,v}/2) \leq \frac{4}{\pi} \cdot \|u - v\|^2.
\]

**Proof of Proposition 3.** We first expand the square as

\[
\frac{1}{4} \cdot \|u - v\|^2 = \frac{1}{4} \cdot (\|u\|^2 + \|v\|^2 - 2 \|u\| \|v\| \cos \theta_{u,v}).
\]

Next, using the AM-GM inequality, we lower bound by

\[
\geq \frac{1}{2} \cdot (\|u\| \|v\| - \|u\| \|v\| \cos \theta_{u,v}).
\]

Using the half-angle identity, this is equal to

\[
= 2 \|u\| \|v\| \sin^2(\theta_{u,v}/2).
\]

Finally, we use that \( \theta_{u,v} \in [0, \pi] \):

\[
\geq \frac{2\theta_{u,v}}{\pi} \|u\| \|v\| \sin^2(\theta_{u,v}/2).
\]

\( \square \)

**Lemma 10.** Suppose that \( \varepsilon \sim \mathcal{N}(0, \gamma I) \) for some \( \gamma > 0 \). Then for any two vectors \( u \) and \( v \),

\[
\mathbb{E}[\sigma(\langle u, \varepsilon \rangle) - \sigma(\langle v, \varepsilon \rangle))^2) \geq \frac{\gamma}{4} \|u - v\|^2.
\]
Proof of Lemma 10. Observe that if $\varepsilon' \sim N(0, I)$ we have

$$
\mathbb{E}[(\sigma(u, \varepsilon) - \sigma(v, \varepsilon))^2] = \gamma \cdot \mathbb{E}[(\sigma(u, \varepsilon') - \sigma(v, \varepsilon'))^2].
$$

Thus, going forward we assume $\varepsilon \sim N(0, I)$ without loss of generality. First, observe that

$$
\mathbb{E}[(\sigma(u, \varepsilon) - \sigma(v, \varepsilon))^2] = \mathbb{E}[(\sigma(v, \varepsilon))^2] + \mathbb{E}[(\sigma(u, \varepsilon))^2] - 2\mathbb{E}[\sigma(u, \varepsilon) \cdot \sigma(v, \varepsilon)].
$$

Without loss of generality, we will assume that $\theta_{u,v} \in [0, \pi]$, where $\theta_{u,v}$ is the angle between $u$ and $v$.

Using Lemma 9, we have the identity

$$
\mathbb{E}[\sigma(u, \varepsilon) \cdot \sigma(v, \varepsilon)] = \frac{1}{2\pi} \| u \| \| v \| (\sin \theta_{u,v} + (\pi - \theta_{u,v}) \cos \theta_{u,v}).
$$

We now lower bound the right-hand side as

$$
\mathbb{E}[(\sigma(u, \varepsilon) - \sigma(v, \varepsilon))^2] = \frac{1}{2} \| u \|^2 + \frac{1}{2} \| v \|^2 - \frac{1}{\pi} \| u \| \| v \| (\sin \theta_{u,v} + (\pi - \theta_{u,v}) \cos \theta_{u,v}),
$$

where (i) uses that $\langle u, v \rangle = \| u \| \| v \| \cos \theta_{u,v}$ and (ii) uses the half-angle identity $\sin^2(\theta/2) = \frac{1 - \cos \theta}{2}$.

Now from Proposition 3, we have

$$
\frac{2\theta_{u,v}}{\pi} \| u \| \| v \| \sin^2(\theta_{u,v}/2) \leq \frac{1}{4} \| u - v \|^2.
$$

It follows that

$$
\mathbb{E}[(\sigma(u, \varepsilon) - \sigma(v, \varepsilon))^2] \geq \frac{1}{4} \| u - v \|^2.
$$

Recall that in the statement of Theorem 3 we assume that $\{\varepsilon_i\}_{i=1}^n$ are i.i.d. Gaussian with $\varepsilon_i \sim N(0, I)$. Since

$$
x_{i+1} = \sigma(\Theta^* x_i) + \varepsilon_i,
$$

this implies that $x_{i+1}$ is distributed as $N(\sigma(\Theta^* x_i), I)$ conditioned on $\mathcal{G}_i$. This leads to the following result.

Lemma 11. Define $\mu_i = \sigma(\Theta^* x_i)$ and let $c$ be an absolute constant. For any two vectors $u, v$, we have

$$
\mathbb{E}[(\sigma(u, x_{i+1})) - \sigma(v, x_{i+1}))^2 \mid \mathcal{G}_i] \geq \frac{1}{4} \cdot e^{-|\mu_i|^2} \| u - v \|^2.
$$
Together, these inequalities imply that
\[ (\sigma(\langle \bar{u}, x_{i+1} \rangle) - \sigma(\langle \bar{v}, x_{i+1} \rangle))^2 = (\sigma(\langle \bar{u}, P_{u,v}(x_{i+1}) \rangle) - \sigma(\langle \bar{v}, P_{u,v}(x_{i+1}) \rangle))^2. \]

To prove parameter convergence for the ReLU we first establish the following key lemma, which states that with high probability a certain variant of the prediction error for $\Theta$ is lower bounded by
\[ \epsilon \]

Define a gaussian vector $\varepsilon \sim \mathcal{N}(0, \frac{1}{2}I_{d \times 2})$. Observe that $\mathbb{E}[\sigma(\langle \bar{u}, \varepsilon \rangle) - \sigma(\langle \bar{v}, \varepsilon \rangle)] = (1/\pi) \cdot \int (\sigma((\bar{u}, y)) - \sigma((\bar{v}, y)))^2 e^{-|y|^2} dy$. Furthermore, from Lemma 10 we have:
\[ \mathbb{E}[\sigma(\langle \bar{u}, \varepsilon \rangle) - \sigma(\langle \bar{v}, \varepsilon \rangle)] \geq \frac{1}{4} \| \bar{u} - \bar{v} \|^2 = \frac{1}{4} \| u - v \|^2. \]

Together, these inequalities imply that
\[ \mathbb{E}[\sigma(\langle u, x_{i+1} \rangle) - \sigma(\langle v, x_{i+1} \rangle)] \geq \frac{1}{4} \epsilon - \| \mu_i \|^2 \| u - v \|^2. \quad \square \]

### D.2.2 Relating prediction error to parameter error

To prove parameter convergence for the ReLU we first establish the following key lemma, which states that with high probability a certain variant of the prediction error for $\Theta$ is lower bounded by the parameter recovery error.

**Lemma 12.** Let the assumptions of Theorem 3 hold. Then whenever
\[ n \geq \frac{c^2 d^3}{\epsilon^2} \log \left( 1 + \frac{\text{tr}(K)}{\delta(1 - \rho)} \right), \]
we have that with probability at least $1 - \delta$, for all $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$ simultaneously,
\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \sigma(\Theta^{(1)} x_i) - \sigma(\Theta^{(2)} x_i) \|^2 \mid \mathcal{G}_{i-1} \right] \geq (1/4) \epsilon \frac{\text{tr}(K)}{1 - \rho} \| \Theta^{(1)} - \Theta^{(2)} \|^2 - \frac{1}{n}. \]

**Proof.** Fix $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$. Let $u_j$ and $v_j$ denote the $j^{th}$ rows of $\Theta^{(1)}$ and $\Theta^{(2)}$ respectively. Then
\[ \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \sigma(\Theta^{(1)} x_i) - \sigma(\Theta^{(2)} x_i) \|^2 \mid \mathcal{G}_{i-1} \right] = \frac{1}{n} \sum_{j=1}^{d} \sum_{i=1}^{n} \mathbb{E} \left[ \| \sigma(\langle u_j, x_i \rangle) - \sigma(\langle v_j, x_i \rangle) \|^2 \mid \mathcal{G}_{i-1} \right]. \]

Note that $\sum_{i=1}^{n} \| \mu_i \|^2 = \sum_{i=1}^{n} \| \sigma(\Theta x_i) \|^2 \leq \sum_{i=1}^{n} \| \Theta x_i \|^2$. Since $(\Theta^*)^\top K \Theta^* \preceq \rho K$ we have
\[ x_i^\top (\Theta^*)^\top K \Theta^* x_i \leq \rho x_i^\top K x_i. \]
Since $K \geq I$, this implies that $\sum_{i=1}^{n} \| \Theta^* x_i \|^2 \leq \rho \sum_{i=1}^{n} \text{tr} (x_i^T K x_i)$, and consequently Lemma 8 guarantees that once $n \geq \frac{\epsilon^4 d^3}{1 - \rho} \log \left(1 + \frac{\text{tr}(K)}{\delta(1 - \rho)} \right)$, with probability at least $1 - \delta$

$$\sum_{i=1}^{n} \| \mu_i \|^2 \leq \frac{4n^3 \text{tr}(K)}{1 - \rho}.$$ 

Now, we know from Lemma 11 that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[\| (\sigma((u_j, x_i)) - \sigma((v_j, x_i))^2 | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} (1/4) \cdot e^{-1} \| u_j - v_j \|^2 \leq (1/4) \cdot \| u_j - v_j \|^2 e^{-\frac{1}{4} \sum_{i=1}^{n} \| \mu_i \|^2},$$

where the second inequality is simply Jensen’s Inequality applied to $x \mapsto e^{-x}$. This establishes the result for a single pair $\Theta^{(1)}, \Theta^{(2)}$. To get a statement which holds simultaneously for all $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$, we apply a union bound over a $\epsilon$-covering set of $\mathcal{M}$. In particular,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \]

where $\Theta^{(1)}$ denotes an arbitrary element of the $\epsilon$-net such that $\| \Theta^{(1)} - \Theta^{(1)} \| \leq \epsilon$. We may take the $\epsilon$-net to have size at most $\left( 1 + \frac{2W}{\epsilon} \right)^2$, and so taking a union bound over this covering ensures that for any $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$, whenever

$$n \geq \frac{c_1^3 \rho^3}{1 - \rho} \log \left(1 + \frac{W \text{tr}(K)}{\epsilon \delta(1 - \rho)} \right),$$

we have with probability at least $1 - \delta$,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \leq \frac{4^3 \text{tr}(K)}{1 - \rho} \epsilon^2. \quad (40)$$

Putting everything together, we have

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \Theta^{(1)} x_i - \Theta^{(2)} x_i \|^2 \right] | G_{i-1} \] \leq \frac{4^3 \text{tr}(K)}{1 - \rho} \epsilon^2. \quad (40)$$

Choosing $\epsilon = (1/n) \left( \frac{16 \text{tr}(K)}{1 - \rho} \right)^{-1}$ gives the desired result. The condition on $n$ in (39) comes from invoking the requirement that $n \geq \frac{c_1^3 \rho^3}{1 - \rho} \log \left(1 + \frac{\text{tr}(K)}{\delta(1 - \rho)} \right)$ for a single pair with $\delta' := \delta \left( 1 + \frac{2W}{\epsilon} \right)^{-d^2}$. \qed
Lemma 13. Define
\[
S := \frac{1}{n} \sum_{i=1}^{n} \left( \sigma(\Theta^{(t)} x_i) - \sigma(\Theta^* x_i) \right)^2 - \mathbb{E} \left[ \left( \sigma(\Theta^{(t)} x_i) - \sigma(\Theta^* x_i) \right)^2 | G_{i-1} \right],
\]
and let \( R = c \left( \frac{\tr(K)}{1 - \rho} \right) \log (nd/\delta) W^2 \), where \( c \) is a sufficiently large numerical constant. Then for any \( \lambda \in [0, 1/R] \), with probability at least \( 1 - \delta \),
\[
S \leq \frac{\lambda R^2}{W^2} \left( \Theta^{(t)} - \Theta^* \right)^2 + \frac{d^2 \left( \log (1/\delta) + \log (1 + 2n\sqrt{R}) \right)}{n\lambda} + \frac{1}{n}.
\]

Proof. We first prove that the inequality in the lemma holds with \( \Theta^{(t)} \) and \( \Theta^* \) replaced by any pair \( \Theta^{(1)}, \Theta^{(2)} \in M \) fixed a-priori. We then establish the lemma by extending this result to a uniform bound for all \( \Theta^{(1)}, \Theta^{(2)} \in M \) simultaneously, which in particular includes \( \Theta^{(t)} \) and \( \Theta^* \).

Let \( u_j \) and \( v_j \) be the \( j^{th} \) rows of \( \Theta^{(1)} \), \( \Theta^{(2)} \) respectively, so that we can write
\[
S = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d} \left( \left( \sigma(\left( u_j, x_{i+1} \right)) - \sigma(\left( v_j, x_{i+1} \right)) \right)^2 - \mathbb{E} \left[ \left( \sigma(\left( u_j, x_{i+1} \right)) - \sigma(\left( v_j, x_{i+1} \right)) \right)^2 | G_i \right] \right).
\]
To keep notation compact, for each \( i \) define \( f_i := \sum_{j=1}^{d} \left( \sigma(\left( u_j, x_i \right)) - \sigma(\left( v_j, x_i \right)) \right)^2 \), so that \( S = \frac{1}{n} \sum_{i=1}^{n} f_i - \mathbb{E} \left[ f_i | G_{i-1} \right] \). Furthermore, define \( V := \sum_{i=1}^{n} \mathbb{E} \left[ f_i^2 | G_{i-1} \right] - \left( \mathbb{E} \left[ f_i | G_{i-1} \right] \right)^2 \). Since \( \varepsilon_{i,j} \) is Gaussian, we have \( \sup_{i,j} |\varepsilon_{i,j}| \leq \sqrt{2 \log (nd/\delta)} \) with probability at least \( 1 - \delta \). Conditioning on the event \( A := \{ \sup_{i,j} |\varepsilon_{i,j}| \leq \sqrt{2 \log (nd/\delta)} \} \), we still have
\[
\mathbb{E} \left[ \varepsilon_{i,j} | G_{i-1}, A \right] = 0
\]
so we can apply Lemma 1 to \( S \). To apply the lemma, we first prove an upper bound on the magnitude of the iterates \( \{ x_i \}_{i=1}^{n} \). Condition on the event \( A \). Using that \( (\Theta^*)^T K \Theta^* \leq \rho K \), we have
\[
\|K^{1/2} x_i\| \leq \|K^{1/2} \sigma(\Theta^* x_{i-1})\| + \|K^{1/2} \varepsilon_{i-1}\| \leq \sqrt{\rho} \|K^{1/2} x_{i-1}\| + \|K^{1/2} \varepsilon_{i-1}\|.
\]
It follows that \( \|K^{1/2} x_i\| \leq \sum_{j=0}^{i-1} \rho^{i-j-1/2} \cdot \|K^{1/2} \varepsilon_j\| \) and since \( \|\varepsilon_{i-1}\|^2 \leq 2 \tr(K) \log (nd/\delta) \) we have for every \( i \),
\[
\|K^{1/2} x_i\| \leq \sqrt{2 \tr(K) \log (nd/\delta)} \leq 2 \mathbb{E} \left( \frac{\tr(K) \log (nd/\delta)}{(1 - \rho)^2} \right),
\]
where the second inequality follows from the fact that \( \frac{1}{1 - \sqrt{\rho}} = \frac{1 + \sqrt{\rho}}{1 - \rho} \leq 2 \frac{1 - \rho}{1 - \rho} \). Since \( K \succeq I \), this implies
\[
\|x_i\|^2 \leq c \left( \frac{\tr(K)}{(1 - \rho)^2} \right) \log (nd/\delta),
\]
where \( c \) is a numerical constant. Now, using that \( \sigma \) is the ReLU and that \( \sum_{i=1}^{d} \|u_j\|^2 \leq W^2 \), we have \( f_i \leq c \left( \frac{\tr(K)}{(1 - \rho)^2} \right) \log (nd/\delta) W^2 =: R \). We also have that \( V \leq \sum_{i=1}^{n} \mathbb{E} \left[ f_i^2 | G_{i-1} \right] \leq R \sum_{i=1}^{n} \mathbb{E} \left[ f_i | G_{i-1} \right] \). Since \( \sigma \) is the ReLU, we can upper bound this by
\[
\sum_{i=1}^{n} \mathbb{E} \left[ f_i | G_{i-1} \right] \leq \sum_{i=1}^{n} \sum_{j=1}^{d} \mathbb{E} \left[ \left( u_j - v_j, x_{i+1} \right)^2 | G_{i-1} \right] \leq c \frac{\tr(K)}{(1 - \rho)^2} \cdot n \log (nd/\delta) \sum_{j=1}^{d} \|u_j - v_j\|^2.
\]
We conclude that $V \leq \frac{nR^2}{W^2} \sum_{j=1}^{d} \|u_j - v_j\|^2$. Using Lemma 1, it follows that for any $\lambda \in [0, 1/R]$ and any $\delta_0 > 0$, with probability at least $1 - \delta_0$,

$$S \leq \lambda \frac{R^2}{W^2} \sum_{j=1}^{d} \|u_j - v_j\|^2 + \frac{\log (1/\delta_0)}{n\lambda}. \quad (41)$$

Since we conditioned on the event $\sup_{i,j} |\epsilon_{i,j}| \leq \sqrt{2\log(n^d/\delta)}$, the final bound (41) holds with probability at least $1 - \delta_0 - \delta$. We set $\delta_0 = \delta$ to obtain the final result.

To get a statement which holds simultaneously for all $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$, we apply a union bound over a $\epsilon$-covering set for $\mathcal{M}$. In particular, there exists an $\epsilon$-covering of $\mathcal{M}$ of size at most $(1 + \frac{2W}{\epsilon})^d$. Taking a union bound, we have for any $\Theta^{(1)}, \Theta^{(2)} \in \mathcal{M}$,

$$S \leq \lambda \frac{R^2}{W^2} \|\Theta^{(1)} - \Theta^{(2)}\|^2 + \frac{d^2 (\log (1/\delta) + \log (1 + 2W/\epsilon))}{n\lambda} + \sup_{i} \|x_i\| \epsilon. \quad (42)$$

The coefficient $\|x_i\|$ in front of $\epsilon$ in (42) is an upper bound on the Lipschitz constant, similar to (40). Since $\|x_i\| \leq \sqrt{R}/W$, setting $\epsilon = \frac{W}{n\sqrt{R}}$ leads to the desired result:

$$S \leq \lambda \frac{R^2}{W^2} \|\Theta^{(1)} - \Theta^{(2)}\|^2 + \frac{d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{n\lambda} + \frac{1}{n}. \quad \Box$$

**Proof of Theorem 3.** Define $\mu = c\tau \sqrt{\frac{\det K}{n(1-\rho)} \log \left(\frac{4\tau(K)}{\delta(1-\rho)} + 1\right)}$ and $B = \frac{4\tau(K)}{(1-\rho)}$. Recall from (26) and (27) that

$$\|\Theta^{(t)} - \Theta^*\|^2_F - \|\Theta^{(t+1)} - \Theta^*\|^2_F \geq \frac{2\eta^2}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)}x_i), (\Theta^* - \Theta^{(t)})x_i) - \frac{\eta^2}{n} \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)}x_i))x_i \right\|^2_F.$$

Recall from the proof of Theorem 2 that

$$\eta^2 \left\| \frac{1}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)}x_i))x_i \right\|^2_F \leq 2\eta^2 (\mu^2 + \mathcal{E}(\Theta^{(t)}B) \leq 2\eta^2 (\mu^2 + \|\Theta^{(t)} - \Theta^*\|^2_F \cdot B^2),$$

since $\mathcal{E}(\Theta^{(t)}) \leq \|\Theta^{(t)} - \Theta^*\|^2_F B$, and that

$$\frac{2}{n} \sum_{i=1}^{n} (x_{i+1} - \sigma(\Theta^{(t)}x_i), (\Theta^* - \Theta^{(t)})x_i) = \frac{2}{n} \sum_{i=1}^{n} (\sigma(\Theta^*x_i) - \sigma(\Theta^{(t)}x_i), (\Theta^* - \Theta^{(t)})x_i) + \frac{2}{n} \sum_{i=1}^{n} (\varepsilon_i, (\Theta^* - \Theta^{(t)})x_i) \geq \frac{2}{n} \sum_{i=1}^{n} \|\sigma(\Theta^*x_i) - \sigma(\Theta^{(t)}x_i)\|^2 + \frac{2}{n} \sum_{i=1}^{n} (\varepsilon_i, (\Theta^* - \Theta^{(t)})x_i).$$

Let $\delta > 0$, be given. Define $R = \frac{c\tau(K)}{(1-\rho)^2} \log (nd/\delta)W^2$ and let $\lambda \in [0, 1/R]$ be a free parameter. Using Lemma 13, we are guaranteed that with probability at least $1 - \delta$,

$$\frac{2}{n} \sum_{i=1}^{n} \|\sigma(\Theta^*x_i) - \sigma(\Theta^{(t)}x_i)\|^2 \geq \frac{2}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \|\sigma(\Theta^{(t)}x_i) - \sigma(\Theta^*x_i)\|^2 \mid G_{i-1} \right] - \frac{\lambda R^2}{W^2} \|\Theta^{(t)} - \Theta^*\|^2_F - \frac{d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{n\lambda} - \frac{1}{n},$$

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For any $\gamma > 0$, using the AM-GM inequality, we have
\[
\frac{2}{n} \sum_{i=1}^{n} \langle e_i, (\Theta^* - \Theta(t)) x_i \rangle \geq - \frac{2}{n} \sum_{i=1}^{n} \| \Theta^* - \Theta(t) \|_F^2 \geq - \gamma \left( \frac{2}{n} \sum_{i=1}^{n} \| e_i x_i \|_F^2 \right) - \frac{1}{\gamma} \| \Theta^* - \Theta(t) \|_F^2.
\]
It follows that
\[
\frac{2}{n} \sum_{i=1}^{n} \langle x_{i+1} - \sigma(\Theta(t) x_i), (\Theta^* - \Theta(t)) x_i \rangle \geq \frac{2}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \| \sigma(\Theta(t) x_i) - \sigma(\Theta^* x_i) \|^2 \mid \mathcal{G}_{i-1} \right] - \frac{\lambda R^2}{W^2} \| \Theta(t) - \Theta^* \|_F^2 - \frac{d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{n\lambda} - \gamma \left( \frac{2}{n} \sum_{i=1}^{n} \| e_i x_i \|_F^2 \right) - \frac{1}{\gamma} \| \Theta^* - \Theta(t) \|_F^2 - \frac{1}{n}.
\]
From Lemma 12, we have that once $n$ is sufficiently large (in particular, when the conditions of the theorem statement hold), with probability at least $1 - \delta$,
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ (\sigma(\Theta(t) x_i) - \sigma(\Theta^* x_i))^2 \mid \mathcal{G}_{i-1} \right] \geq (1/4) \cdot e^{-\frac{4\alpha t(K)}{1-\rho}} - \frac{1}{n}.
\]
Choosing $\gamma = e^{-\frac{4\alpha t(K)}{1-\rho}}$ and $\lambda = \frac{W^2 e^{-\frac{4\alpha t(K)}{1-\rho}}}{16R^2}$ we then have,
\[
\frac{2}{n} \sum_{i=1}^{n} \langle x_{i+1} - \sigma(\Theta(t) x_i), (\Theta^* - \Theta(t)) x_i \rangle \geq c_1 e^{-\frac{4\alpha t(K)}{1-\rho}} \| \Theta(t) - \Theta^* \|_F^2 - 4c_2 e^{-\frac{4\alpha t(K)}{1-\rho}} \left( \frac{2}{n} \sum_{i=1}^{n} \| e_i x_i \|_F^2 \right) - \frac{4c_3 e^{-\frac{4\alpha t(K)}{1-\rho}} R^2 d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{nW^2} - \frac{2}{n},
\]
for absolute constants $c_1 \geq \frac{1}{\sqrt{2}}, c_2 > 0, c_3 > 0$. Next, using Lemma 8, we are guaranteed that with probability at least $1 - \delta$, $\| \sum_{i=1}^{n} e_i x_i \|_F^2 \leq \mu^2$. Using this along with the choice $\eta_t = e^{-\frac{4\alpha t(K)}{1-\rho}}$, we have
\[
\| \Theta(t) - \Theta^* \|_F^2 - \| \Theta(t+1) - \Theta^* \|_F^2 \geq c_1 \cdot \eta_t \cdot e^{-\frac{4\alpha t(K)}{1-\rho}} \| \Theta(t) - \Theta^* \|_F^2 - 4c_2 \cdot \eta_t \cdot e^{-\frac{4\alpha t(K)}{1-\rho}} \mu^2 - 4c_3 \cdot \eta_t \cdot e^{-\frac{4\alpha t(K)}{1-\rho}} R^2 d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))\]
\[
- \frac{2\eta_t^2 \cdot (\mu^2 + \| \Theta(t) - \Theta^* \|_F^2 B^2)},
\]
which simplifies to
\[
\| \Theta(t) - \Theta^* \|_F^2 - \| \Theta(t+1) - \Theta^* \|_F^2 \geq \frac{c^2}{8B^2} \cdot e^{-\frac{8\alpha t(K)}{1-\rho}} \| \Theta(t) - \Theta^* \|_F^2
\]
\[
- \frac{cR^2 d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{nB^2 W^2}.
\]
Let $\alpha := \frac{c^2}{8B^2} \cdot e^{-\frac{8\alpha t(K)}{1-\rho}}$. Then the preceding equation simplifies to
\[
\| \Theta(t+1) - \Theta^* \|_F^2 \leq (1 - \alpha) \cdot \| \Theta(t) - \Theta^* \|_F^2 + \frac{cR^2 d^2 (\log (1/\delta) + \log (1 + 2n\sqrt{R}))}{nB^2 W^2}.
\]
Applying this inequality recursively yields

\[ \left\| \Theta^{(t)} - \Theta^* \right\|_F^2 \leq W^2 (1 - \alpha)^t + c \cdot e^{s_{\text{opt}}(K)} \frac{R^2 d^2 \left( \log (1/\delta) + \log (1 + 2n\sqrt{R}) \right)}{nW^2}. \]

We conclude that whenever

\[ t \geq 8cB^2 e^{s_{\text{opt}}(K)} \log \left( \frac{nW^4}{R^2 d^2 \left( \log (1/\delta) + \log (1 + 2n\sqrt{R}) \right)} \right) \lor 1, \]

we have

\[ \left\| \Theta^{(t)} - \Theta^* \right\|_F^2 \leq 2c \cdot e^{s_{\text{opt}}(K)} \frac{R^2 d^2 \left( \log (1/\delta) + \log (1 + 2n\sqrt{R}) \right)}{nW^2}. \]

\[ \square \]