Gauge Theories from Orientifolds and Large $N$ Limit

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Abstract

Extending the recent work in [1], we consider string perturbative expansion in the presence of D-branes and orientifold planes imbedded in orbifolded space-time. In the $\alpha' \to 0$ limit the weak coupling string perturbative expansion maps to 't Hooft’s large $N$ expansion. We focus on four dimensional $\mathcal{N} = 1, 2, 4$ supersymmetric theories, and also discuss possible extensions to $\mathcal{N} = 0$ cases. Utilizing the string theory perturbation techniques we show that computation of any $M$-point correlation function in these theories reduces to the corresponding computation in the parent $\mathcal{N} = 4$ theory. In particular, we discuss theories (which are rather constrained) with vanishing $\beta$-functions to all orders in perturbation theory in the large $N$ limit. We also point out that in theories with non-vanishing $\beta$-functions the gauge coupling running is suppressed in the large $N$ limit. Introduction of orientifold planes allows to construct certain gauge theories with $SO$, $Sp$ and $SU$ gauge groups and various matter (only unitary gauge groups with bi-fundamental/adjoint matter arise in theories without orientifold planes).

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I. INTRODUCTION

‘t Hooft’s large \( N \) expansion \cite{2} is an attractive possibility for understanding gauge theories. In this limit the gauge theory diagrams look like Riemann surfaces with boundaries and handles. It is therefore natural to attempt to map the large \( N \) expansion of gauge theories to some kind of string world-sheet expansion.

The first concrete example of such a map was given by Witten \cite{3} for the case of three dimensional Chern-Simons gauge theory where the boundaries of the string world-sheet are “topological” D-branes.

Recently the map between the large \( N \) expansion and string expansion has been made precise \cite{1} in the context of Type II string theory. The idea of \cite{1} is to consider Type IIB string theory with a large number \( N \) of D3-branes and take a limit where \( \alpha' \to 0 \) while keeping \( \lambda = N \lambda_s \) fixed, where \( \lambda_s \) is the Type IIB string coupling. In this setup we have four dimensional gauge theories with unitary gauge groups. A world-sheet with \( b \) boundaries and \( g \) handles is weighted with

\[
(N \lambda_s)^b \lambda_s^{2g-2} = \lambda^{2g-2+b} N^{-2g+2}.
\]

Upon identification \( \lambda_s = g_{YM}^2 \), this precisely maps to ‘t Hooft’s large \( N \) expansion. This expansion is valid in the limit where \( N \to \infty \) and the effective coupling \( \lambda \) is fixed at a weak coupling value.

In \cite{1} the above idea was applied to prove that four dimensional gauge theories (including the cases with no supersymmetry) considered in \cite{4,5} are conformal to all orders in perturbation theory in the large \( N \) limit. In particular, the corresponding gauge theories were obtained from Type IIB string theory with D3-branes imbedded in orbifolded space-time. The ultraviolet finiteness of string theory (that is, one-loop tadpole cancellation conditions) was shown to imply that the resulting (non-Abelian) gauge theories where conformal in the large \( N \) limit (in all loop orders). Moreover, in \cite{1} it was also proven that computation of any correlation function in these theories in the large \( N \) limit reduces to the corresponding computation in the parent \( \mathcal{N} = 4 \) supersymmetric gauge theory.

The all-order proofs in \cite{1} were possible due to the fact that the power of string perturbation techniques was utilized. In particular, string theory perturbation expansion is a very efficient way of summing up various field theory diagrams. Thus, often a large number of field theory Feynman diagrams in a given order of perturbation theory corresponds to a single string theory diagram with certain topology. This has been successfully exploited to compute tree and loop level scattering processes in gauge theories \cite{8}. (For a recent discussion, see, e.g., \cite{9}.) The all-order proofs in \cite{1} crucially depended on the fact that the string world-sheet expansion was self-consistent. In particular, the arguments in \cite{1} would not go through if the tadpoles were not cancelled. The tadpole cancellation conditions, however, ultimately produced theories which were (super)conformal in the large \( N \) limit. In particular, the one-loop \( \beta \)-functions (for non-Abelian gauge groups) in all of those theories were zero (even at finite \( N \)).

\[1\] For other related works, see, e.g., \cite{6,7}.
It is natural to ask whether there exists a consistent string theory setup similar to that of [1] which can produce: (i) finite gauge theories with gauge groups other than unitary (in particular, \(SO\) and \(Sp\)), and with matter other than bi-fundamentals; (ii) non-finite gauge theories such that string theory perturbation techniques can be applied to learn about the gauge theories in the large \(N\) limit. In this paper we consider precisely such a setup which is a generalization of the work in [1]. Here we will consider Type IIB string theory with D3-branes as well as orientifold planes. (In certain cases string consistency will also require presence of D7-branes.) That is, we are going to study gauge theories that arise from Type IIB orientifolds. In this framework we can expect appearance of \(SO\) and \(Sp\) gauge groups as well as matter other than bi-fundamentals (in the product gauge groups). Moreover, as we will see later, non-finite gauge theories can also be obtained within a consistent perturbative string theory framework. However, in all such theories obtained via orientifolds the running of the gauge couplings is suppressed in the large \(N\) limit. (In this sense these theories are “finite” in the large \(N\) limit).

Introduction of orientifold planes changes the possible topologies of the world-sheet. Now we can have a world-sheet with \(b\) boundaries (corresponding to D-branes), \(c\) cross-caps (corresponding to orientifold planes), and \(g\) handles. Such a world-sheet is weighted with

\[
(N\lambda_s)^b\lambda_s^c\lambda_s^{2g-2} = \lambda^{2g-2+b+c}N^{-c-2g+2}.
\]  

Note that addition of a cross-cap results in a diagram suppressed by an additional power of \(N\), so that in the large \(N\) limit we can hope for simplifications (or, rather, we can hope to avoid complications with unoriented world-sheets, at least in some cases). In fact, we will show that for string vacua which are perturbatively consistent (that is, the tadpoles cancel) calculations of correlation functions in \(\mathcal{N} < 4\) gauge theories reduce to the corresponding calculations in the parent \(\mathcal{N} = 4\) oriented theory. This holds not only for finite (in the large \(N\) limit) gauge theories but also for the gauge theories which are not conformal. We will prove that certain \(\mathcal{N} = 1\) gauge theories obtained this way are superconformal to all orders in perturbation theory in the large \(N\) limit. We will also discuss possible extensions to non-supersymmetric cases.

It is very satisfying to observe that in the large \(N\) limit using the power of string theory perturbation techniques we can reduce very non-trivial calculations in gauge theories with lower supersymmetries to calculations in \(\mathcal{N} = 4\) gauge theories. In particular, this applies to multi-point correlators in gauge theory.

Here we note that the correspondence between ’t Hooft’s large \(N\) expansion and string world-sheet expansion is expected to hold only in the regime where the effective coupling \(\lambda\) is small. If \(\lambda\) is large one expects an effective supergravity description to take over. This domain has been recently studied in several papers [10–15] as well as in various related works [16]. The supergravity picture has, in particular, led to the conjectures in [4] as well as in [3] about finiteness of certain gauge theories. However, proofs of those conjectures (in the large \(N\) limit) presented in [1] were given in the weakly coupled region. Also, \(1/N\) corrections can only be reliably computed in this region but not in the strong coupling regime where a priori there is no world-sheet expansion nor ’t Hooft’s expansion is valid.

The remainder of this paper is organized as follows. In section II we review the arguments of [1] for the cases without the orientifold planes. In section III we generalize these arguments to the orientifold cases. In particular, we prove various vanishing theorems in the large \(N\)
limit, and give the relation between various correlators in $\mathcal{N} < 4$ gauge theories to those in the parent $\mathcal{N} = 4$ theory. In section IV we construct some explicit $\mathcal{N} = 2$ examples. All of these examples turn out to be superconformal. In section V we construct an example of $\mathcal{N} = 1$ gauge theory with $Sp(N)$ gauge group. This gauge theory is superconformal in the large $N$ limit to all orders in perturbation theory. We also discuss some examples of non-finite $\mathcal{N} = 1$ gauge theories.

II. LARGE $N$ LIMIT AND FINITENESS

In this section we review the discussion in [1] for the cases without orientifold planes. We will generalize these arguments to the orientifold cases in section III.

A. Setup

Consider Type IIB string theory with $N$ parallel D3-branes where the space transverse to the D-branes is $\mathcal{M} = \mathbb{R}^6/\Gamma$. The orbifold group $\Gamma = \{ g_a \mid a = 1, \ldots, |\Gamma| \}$ ($g_1 = 1$) must be a finite discrete subgroup of $Spin(6)$. If $\Gamma \subset SU(3)$ ($SU(2)$), we have $\mathcal{N} = 1$ ($\mathcal{N} = 2$) unbroken supersymmetry, and $\mathcal{N} = 0$, otherwise.

Let us confine our attention to the cases where type IIB on $\mathcal{M}$ is a modular invariant theory\footnote{This is always the case if $\Gamma \subset SU(3)$. For the non-supersymmetric cases this is also true provided that $\not\exists \mathbb{Z}_2 \subset \Gamma$. If $\exists \mathbb{Z}_2 \subset \Gamma$, then modular invariance requires that the set of points in $\mathbb{R}^6$ fixed under the $\mathbb{Z}_2$ twist has real dimension 2.}. The action of the orbifold on the coordinates $X_i$ ($i = 1, \ldots, 6$) on $\mathcal{M}$ can be described in terms of $SO(6)$ matrices: $g_a : X_i \rightarrow \sum_j (g_a)_{ij} X_j$. We need to specify the action of the orbifold group on the Chan-Paton charges carried by the D3-branes. It is described by $N \times N$ matrices $\gamma_a$ that form a representation of $\Gamma$. Note that $\gamma_1$ is an identity matrix and $\text{Tr}(\gamma_1) = N$.

The D-brane sector of the theory is described by an oriented open string theory. In particular, the world-sheet expansion corresponds to summing over oriented Riemann surfaces with arbitrary genus $g$ and arbitrary number of boundaries $b$, where the boundaries of the world-sheet correspond to the D3-branes.

For example, consider one-loop vacuum amplitude ($g = 0$, $b = 2$). The corresponding graph is an annulus whose boundaries lie on D3-branes. The one-loop partition function in the light-cone gauge is given by

$$Z = \frac{1}{2|\Gamma|} \sum_a \text{Tr} \left( g_a (1 + (-1)^F) e^{-2\pi t L_0} \right),$$

where $F$ is the fermion number operator, $t$ is the real modular parameter of the annulus, and the trace includes sum over the Chan-Paton factors.

The orbifold group $\Gamma$ acts on both ends of the open strings. The action of $g_a \in \Gamma$ on Chan-Paton charges is given by $\gamma_a \otimes \gamma_a$. Therefore, the individual terms in the sum in (3) have the following form:
where \( Z_a \) are characters corresponding to the world-sheet degrees of freedom. The “untwisted” character \( Z_1 \) is the same as in the \( \mathcal{N} = 4 \) theory for which \( \Gamma = \{1\} \). The information about the fact that the orbifold theory has reduced supersymmetry is encoded in the “twisted” characters \( Z_a, a \neq 1 \).

In [1] it was shown that the one-loop massless (and, in non-supersymmetric cases, tachyonic) tadpole cancellation conditions require that

\[
\text{Tr}(\gamma_a) = 0 \, \forall a \neq 1 .
\]

It was also shown that this condition implies that the Chan-Paton matrices \( \gamma_a \) form an \( n \)-fold copy of the \textit{regular} representation of \( \Gamma \). The regular representation decomposes into a direct sum of all irreducible representations \( r_i \) of \( \Gamma \) with degeneracy factors \( n_i = |r_i| \). The gauge group is \( (N_i \equiv nn_i) \)

\[
G = \otimes_i U(N_i) .
\]

The matter consists of Weyl fermions (and scalars) transforming in bi-fundamentals \( (N_i, \overline{N}_j) \) according to the decomposition of the tensor product of \( 4 \) (6) of Spin(6) with the corresponding representation (see [5] for details).

**B. Large \( N \) Limit**

The gauge group in the theories we are considering here is \( G = \otimes_i U(N_i) \subset U(N) \). In the following we will ignore the \( U(1) \) factors (for which the gauge couplings do run for \( \mathcal{N} < 4 \) as there are matter fields charged under them) and consider \( G = \otimes_i SU(N_i) \). In this subsection we review the arguments of [1] which show that in the large \( N \) limit this non-Abelian gauge theory is conformal.

There are two classes of diagrams we need to consider: (i) diagrams without handles; (ii) diagrams with handles. The latter correspond to closed string loops and are subleading in the large \( N \) limit. The diagrams without handles can be divided into two classes: (i) planar diagrams where all the external lines are attached to the same boundary; (ii) non-planar diagrams where the external lines are attached to at least two different boundaries. The former are subleading in the large \( N \) limit.

In the case of planar diagrams we have \( b \) boundaries with all \( M \) external lines attached to the same boundary (which without loss of generality can be chosen to be the outer boundary) as depicted in Fig.1. We need to sum over all possible twisted boundary conditions for the boundaries. The boundary conditions must satisfy the requirement that

\[
\gamma_{a_1} = \prod_{s=2}^{b} \gamma_{a_s} ,
\]
where $\gamma_{a_1}$ corresponds to the outer boundary (to which we have attached the external lines), and $\gamma_{a_s}$ ($s = 2, \ldots, b$) correspond to the inner boundaries (with no external lines). Here we have chosen the convention (consistent with the corresponding choice made for the annulus amplitude in (3)) that the outer and inner boundaries have opposite orientations. Then the above condition is simply the statement that only the states invariant under the action of the orbifold group contribute into the amplitude.

If all the twisted boundary conditions are trivial (i.e., $a_s = 1$ for all $s = 1, \ldots, b$) then the corresponding amplitude is the same as in the $\mathcal{N} = 4$ case (modulo factors of $1/\sqrt{|\Gamma|}$ coming from the difference in normalizations of the corresponding D-brane boundary states in the cases with $\mathcal{N} = 4$ (where $|\Gamma| = 1$) and $\mathcal{N} < 4$ (where $|\Gamma| \neq 1$)). Therefore, such amplitudes do not contribute to the gauge coupling running (for which we would have $M = 2$ gauge bosons attached to the outer boundary) since the latter is not renormalized in $\mathcal{N} = 4$ gauge theories due to supersymmetry.

Let us now consider contributions with non-trivial twisted boundary conditions. Let $\lambda_r$, $r = 1 \ldots M$, be the Chan-Paton matrices corresponding to the external lines. Then the planar diagram with $b$ boundaries has the following Chan-Paton group-theoretic dependence:

$$\sum \text{Tr} (\gamma_{a_1} \lambda_1 \ldots \lambda_M) \prod_{s=2}^{b} \text{Tr}(\gamma_{a_s}),$$

where the sum involves all possible distributions of $\gamma_{a_s}$ twists (that satisfy the condition (7)) as well as permutations of $\lambda_r$ factors (note that the $\lambda$'s here are the states which are kept after the orbifold projection, and so they commute with the action of $\gamma$'s). The important point here is that unless all twists $\gamma_{a_s}$ are trivial for $s = 2, \ldots, b$, the above diagram vanishes by the virtue of (5). But then from (7) it follows that $\gamma_{a_1}$ must be trivial as well. This implies that the only planar diagrams that contribute are those with trivial boundary conditions which (up to numerical factors) are the same as in the parent $\mathcal{N} = 4$ gauge theory. This establishes that computation of any $M$-point correlation faction in the large $N$ limit reduces to the corresponding $\mathcal{N} = 4$ calculation, and that these gauge theories are (super)conformal in this limit.

Here we should mention that the models of [1] are perturbatively consistent string theories at all energy scales. In particular, the Abelian factors (that run in the low energy effective theory and decouple in the infrared) are not problematic from the string theory viewpoint (although in the field theory context they would have Landau poles in the ultraviolet).

III. GENERALIZATION TO ORIENTIFOLDS

In this section we generalize the approach of [1] to theories with orientifold planes. Here we will mostly concentrate on supersymmetric cases, and briefly discuss possible generalizations to non-supersymmetric cases at the end of this section.

\[^{4}\text{In [1] it was also shown that a large class of non-planar diagrams without handles also vanish. We refer the reader to [1] for details.}\]
A. Setup

Consider Type IIB string theory on $\mathcal{M} = C^3/\Gamma$ where $\Gamma \subset SU(3)$ so that the resulting theory has some number of unbroken supersymmetries. Consider the $\Omega J$ orientifold of this theory, where $\Omega$ is the world-sheet parity reversal, and $J$ is a $Z_2$ element ($J^2 = 1$) acting on the complex coordinates $z_i$ ($i = 1, 2, 3$) on $C^3$ such that the set of points in $C^3$ fixed under the action of $J$ has real dimension $\Delta = 0$ or 4.

If $\Delta = 0$ then we have an orientifold 3-plane. If $\Gamma$ has a $Z_2$ subgroup, then we also have an orientifold 7-plane. If $\Delta = 4$ then we have an orientifold 7-plane. We may also have an orientifold 3-plane depending on whether $\Gamma$ has an appropriate $Z_2$ subgroup. Regardless of whether we have an orientifold 3-plane, we can a priori introduce an arbitrary number of D3-branes (as the corresponding tadpoles vanish due to the fact that the space transverse to the D3-branes is non-compact). On the other hand, if we have an orientifold 7-plane we must introduce 8 of the corresponding D7-branes to cancel the R-R charge appropriately. (The number 8 of D7-branes is required by the corresponding tadpole cancellation conditions.)

We need to specify the action of $\Gamma$ on the Chan-Paton factors corresponding to the D3- and/or D7-branes. Just as in the previous section, these are given by Chan-Paton matrices which we collectively refer to as $\gamma^\mu_a$, where the superscript $\mu$ refers to the corresponding D3- or D7-branes. Note that $\text{Tr}(\gamma^\mu_a) = n^\mu$ where $n^\mu$ is the number of D-branes labelled by $\mu$.

At one-loop level there are three different sources for massless tadpoles: the Klein bottle, annulus, and Möbius strip amplitudes depicted in Fig.2. The Klein bottle amplitude corresponds to the contribution of unoriented closed strings into one-loop vacuum diagram. It can be alternatively viewed as a tree-level closed string amplitude where the closed strings propagate between two cross-caps. The latter are (coherent Type IIB) states that describe the familiar orientifold planes. The annulus amplitude corresponds to the contribution of open strings stretched between two D-branes into one-loop vacuum amplitude. It can also be viewed as a tree-channel closed string amplitude where the closed strings propagate between two D-branes. Finally, the Möbius strip amplitude corresponds to the contribution of unoriented open strings into one-loop vacuum diagram. It can be viewed as a tree-channel closed string amplitude where the closed strings propagate between a D-brane and an orientifold plane.

Note that there are no Chan-Paton matrices associated with the Klein bottle amplitude since it corresponds to closed strings propagating between two cross-caps which do not carry Chan-Paton charges. The Möbius strip has only one boundary. This implies that the individual terms (corresponding to twists $g_a \in \Gamma$) in the Möbius strip amplitude are proportional to $\text{Tr}(\gamma^\mu_a)$. The annulus amplitude is the same (up to an overall factor of $1/2$ due to the orientation reversal projection) as in the oriented case discussed in the previous section. Thus, the individual terms (corresponding to twists $g_a \in \Gamma$) in the annulus amplitude are proportional to $\text{Tr}(\gamma^\mu_a)\text{Tr}(\gamma^\nu_a)$. Thus, the tadpoles can be written as

$$\sum_a \left( K_a + \sum_\mu M^\mu_a \text{Tr}(\gamma^\mu_a) + \sum_{\mu, \nu} A^{\mu\nu}_a \text{Tr}(\gamma^\mu_a)\text{Tr}(\gamma^\nu_a) \right).$$

Here terms with $K_a$, $M^\mu_a$ and $A^{\mu\nu}_a$ correspond to the contributions of the Klein bottle, Möbius strip and annulus amplitudes, respectively. In fact, the factorization property of string theory implies that the Klein bottle amplitude should factorize into two cross-caps.
connected via a long thin tube. The Möbius strip amplitude should factorize into a cross-cap and a disc connected via a long thin tube. Similarly, the annulus amplitude should factorize into two discs connected via a long thin tube. These factorizations are depicted in Fig. 3. The implication of this for the tadpoles is that they too factorize into a sum of perfect squares

\[ \sum_a \left( B_a + \sum\mu C_a^\mu \text{Tr}(\gamma_a^\mu) \right)^2 , \]  

(10)

where \( B_a^2 = K_a, \ 2B_a C_a^\mu = M_a^\mu \) and \( C_a^\mu C_a^\nu = A_a^{\mu\nu} \). Thus, the tadpole cancellation conditions read:

\[ B_a + \sum\mu C_a^\mu \text{Tr}(\gamma_a^\mu) = 0 \]  

(11)

Note that

\[ \text{Tr}(\gamma_a^\mu) = 0 \ \forall a \neq 1 \text{ only if } K_a = 0 \ \forall a \neq 1 . \]  

(12)

In the next subsection we will see that if this condition is satisfied then the corresponding (non-Abelian) gauge theories are superconformal in the large \( N \) limit. On the other hand, if not all \( K_a \) with \( a \neq 1 \) are zero, then some of the Chan-Paton matrices \( \gamma_a^\mu \) with \( a \neq 1 \) must have non-zero traces. This will generically lead to theories with non-vanishing one-loop \( \beta \)-functions.

**B. Large \( N \) Limit**

In this subsection we extend the arguments reviewed in section [I] to the cases with orientifold planes. In particular, we will study the large \( N \) behavior of \( M \)-point correlators of fields charged under the gauge group that arises from the D3-branes. In the following we will ignore the \( U(1) \) factors (if any) in the D3-brane gauge group.

There are two classes of diagrams we need to consider: (i) diagrams without handles and cross-caps; (ii) diagrams with handles and/or cross-caps. The latter are subleading in the large \( N \) limit. The diagrams without handles and cross-caps can be divided into two classes: (i) planar diagrams where all the external lines are attached to the same boundary; (ii) non-planar diagrams where the external lines are attached to at least two different boundaries. The latter are subleading in the large \( N \) limit.

In the case of planar diagrams we have \( b \) boundaries corresponding to D3- and/or D7-branes. We will attach \( M \) external lines to the outer boundary as depicted in Fig. 1. We need to sum over all possible twisted boundary conditions for the boundaries. The boundary conditions must satisfy the requirement that

\[ \gamma_{a_1}^{\mu_1} = \prod_{s=2}^{b} \gamma_{a_s}^{\mu_s} , \]  

(13)

where \( \gamma_{a_1}^{\mu_1} \) corresponds to the outer boundary, and \( \gamma_{a_s}^{\mu_s} (s = 2, \ldots, b) \) correspond to the inner boundaries.
Let us first consider the $\mathcal{N} = 4$ theories for which the orbifold group $\Gamma$ is trivial. (Note that in this case we can only have D3-branes as introduction of D7-branes would break some number of supersymmetries.) The computation of any correlation function in the orientifold theory (with $SO(N)$ or $Sp(N)$ gauge group) is reduced to the corresponding computation in the oriented $\mathcal{N} = 4$ theory (with $U(N)$ gauge group) up to factors of $1/\sqrt{2}$ (coming from the difference in normalizations of the corresponding D-brane boundary states in the oriented and unoriented cases). Such a simplification is due to the fact that the unoriented world-sheets with cross-caps give contributions suppressed by extra powers of $1/N$.

Next, consider cases where $\Gamma$ is non-trivial (and hence supersymmetry is reduced). If all the twisted boundary conditions are trivial (i.e., $a_s = 1$ for all $s = 1, \ldots, b$) then the corresponding amplitude is the same as in the $\mathcal{N} = 4$ case (modulo factors of $1/\sqrt{|\Gamma|}$). Therefore, such amplitudes do not contribute to the gauge coupling running (for which we would have $M = 2$ gauge bosons attached to the outer boundary) since the latter is not renormalized in $\mathcal{N} = 4$ gauge theories due to supersymmetry.

Let us now consider contributions with non-trivial twisted boundary conditions. Let $\lambda_r$, $r = 1 \ldots M$, be the Chan-Paton matrices corresponding to the external lines. Then the planar diagram with $b$ boundaries has the following Chan-Paton group-theoretic dependence:

$$\sum \text{Tr} \left( \gamma_{a_1}^{\mu_1} \lambda_1 \ldots \lambda_M \right) \prod_{s=2}^{b} \text{Tr}(\gamma_{a_s}^{\mu_s}) , \quad (14)$$

where the sum involves all possible distributions of $\gamma_{a_s}$ twists (that satisfy the condition (7)) as well as permutations of $\lambda_r$ factors. If the condition (12) is satisfied, i.e., if all the twisted Chan-Paton matrices are traceless, then the situation is analogous to that in the oriented cases. That is, the only planar diagrams that contribute are those with trivial boundary conditions. Such diagrams with all the boundaries corresponding to D3-branes (up to numerical factors) are the same as in the parent $\mathcal{N} = 4$ gauge theory. The diagrams with trivial boundary conditions but with some boundaries corresponding to D7-branes are subleading in the large $N$ limit as the numbers of D7-branes (that is, the traces $\text{Tr}(\gamma_{1}^{\mu})$ corresponding to D7-branes) are of order one. This establishes that computation of any $M$-point correlation function in the large $N$ limit reduces to the corresponding $\mathcal{N} = 4$ calculation in oriented theory, and that these gauge theories are superconformal in this limit\(^5\).

Now consider the cases where some of the twisted Chan-Paton matrices are not traceless. Then there are going to be corrections to the $M$-point correlators coming from planar diagrams with non-trivial twisted boundary conditions. These diagrams are subleading in the large $N$ limit as the corresponding traces are always of order one. This follows from the tadpole cancellation conditions (11) where the coefficients $B_a$ and $C_a^{\alpha}$ are of order one, so for $a \neq 1$ we have $\text{Tr}(\gamma_{a}^{\mu}) \sim 1$. This implies that even for non-finite theories computation of the correlation functions reduces to the corresponding computation in the parent $\mathcal{N} = 4$ oriented theory.

Here we should point out that “non-finiteness” of such theories is a subleading effect in the large $N$ limit. This is because the $\beta$-function coefficients grow as

\(^5\)Just as in [1], it is also straightforward to show that a large class of non-planar diagrams without handles and cross-caps also vanish.
\[ b_s = O(N^s) \, , \quad s = 0, 1, \ldots , \] (15)

instead of \( b_s = O(N^{s+1}) \) (as in, say, pure \( SU(N) \) gauge theory). This can be seen by considering planar diagrams with \( M = 2 \) external lines corresponding to gauge bosons. Note that in string theory running of the gauge couplings in the low energy effective field theory is due to infrared divergences corresponding to massless modes \[ \] . The diagrams with all the boundaries corresponding to D3-branes and with all the boundary conditions corresponding to the identity element of \( \Gamma \) are the same (up to overall numerical factors) as in the parent \( \mathcal{N} = 4 \) theory. Such diagrams, therefore, do not contain infrared divergencies, and thus do not contribute the gauge coupling running. (That is, their contributions to the \( \beta \)-function coefficients \( b_s \) vanish.) Therefore, the only diagrams that can contribute to the \( \beta \)-function coefficients \( b_s \) are those with some boundaries corresponding to D7-branes and/or having twisted boundary conditions with \( \text{Tr}(\gamma_\mu^a) \neq 0 \). These are, however, suppressed at least by one power of \( N \) since the numbers of D7-branes are of order one, and also such \( \text{Tr}(\gamma_\mu^a) \sim 1 \). This establishes (15).

Note that the estimates for the \( \beta \)-function coefficients in (13) for \( b_s > 0 \) are non-trivial from the field theory point of view as they imply infinitely many cancellations between couplings (such as Yukawas) in the gauge theory. On the other hand, within string perturbation expansion these statements become obvious once we carefully consider twisted boundary conditions and tadpole cancellation.

C. Comments

Here we would like to comment on some issues concerning the discussion in the previous subsections.

First, note that the entire argument in the previous subsection crucially depends on the assumption that there is a well defined world-sheet description of the orientifold theories at hand. Naively, it might seem that orientifolds of Type IIB on \( \mathbb{C}^3/\Gamma \) should have such world-sheet descriptions for any \( \Gamma \) which is a subgroup of \( \text{Spin}(6) \). This is, however, not the case. In particular, there are certain cases where perturbative description is inadequate as there are additional states present in the corresponding orientifolds such that they are non-perturbative from the orientifold viewpoint. In section V we will give an explicit example of this. For further details we refer the reader to [18].

The second comment is on possible extensions of the above results to \( \mathcal{N} = 0 \) theories. Note that in the cases without orientifold planes the tadpole cancellation conditions for both \( \mathcal{N} = 0 \) and supersymmetric theories are given by (3) for which there always exists a solution for any choice of \( \Gamma \). The situation in the cases with orientifold planes might not be so simple: the tadpoles are of the form (4), and the coefficients \( K_a, M_\mu^a, A_{\mu\nu}^a \) may be different for the massless and tachyonic tadpoles. (The latter are absent in the supersymmetric cases but are generically present for \( \mathcal{N} = 0 \).) It is a priori unclear whether the massless and tachyonic tadpole cancellation conditions are compatible. However, if they have a solution in a given model, then the statements about the correlation functions in subsection B persist.

\[ ^6 \text{Some examples of this were also given in [19].} \]
for such $\mathcal{N} = 0$ models in the large $N$ limit. It would therefore be interesting to construct $\mathcal{N} = 0$ models in which all the tadpoles cancel.

**IV. $\mathcal{N} = 2$ SUPERCONFORMAL GAUGE THEORIES**

In this section we construct four dimensional $\mathcal{N} = 2$ superconformal gauge theories from Type IIB orientifolds. We start from Type IIB string theory on $\mathcal{M} = \mathbb{C}^3/\Gamma$, where $\Gamma = \{g^k | k = 0, \ldots, M-1\} \approx \mathbb{Z}_M$ is the orbifold group whose action on the complex coordinates $z_i (i = 1, 2, 3)$ on $\mathbb{C}^3$ is given by $gz_1 = z_1, gz_2 = \omega z_2, gz_3 = \omega^{-1} z_3$ ($\omega = \exp(2\pi i/M)$). Next, we consider an orientifold of this theory where the orientifold action is given by $\Omega$. Here $\Omega$ is the world-sheet parity reversal, and $J$ acts on $z_i$ as $Jz_i = -z_i$. The orientifold group is given by $\mathcal{O} = \{g^k, \Omega Jg^k | k = 0, \ldots, M-1\}$.

**A. Orbifolds of Even Order**

Let us first consider the cases where $M$ is even. Then the $\Omega J$ orientifold of Type IIB on $\mathcal{M}$ is equivalent to the $\Omega \tilde{J}$ orientifold of Type IIB on $\mathcal{M}$, where $\tilde{J}$ acts on $z_i$ as $\tilde{J}z_1 = -z_1$, $\tilde{J}z_2 = z_2$, $\tilde{J}z_3 = z_3$. (This is due to the presence of the $\mathbb{Z}_2$ element $R$ in $\Gamma$ where $Rz_1 = z_1$, $Rz_2 = -z_2$, $Rz_3 = -z_3$. Note that $\tilde{J} = JR$.) These orientifolds are similar to those considered in [20–22]. In particular, calculation of massless tadpoles closely parallels that of [22].

The orientifolds (with $M \in 2\mathbb{N}$) we are considering here contain both D3- and D7-branes. The action of the orbifold group on the Chan-Paton charges carried by the D3- and D7-branes is described by matrices $\gamma_{k,3}$ and $\gamma_{k,7}$ that form a (projective) representation of $\Gamma$. Note that $\gamma_{0,3}$ and $\gamma_{0,7}$ are identity matrices, and $\text{Tr}(\gamma_{0,3}) = n_3$ and $\text{Tr}(\gamma_{0,7}) = n_7$, where $n_3$ and $n_7$ are the numbers of D3- and D7-branes, respectively.

The massless tadpoles can be computed following [22]. There are two types of massless tadpoles we need to consider. The first type of tadpoles correspond to untwisted closed string exchange in the tree-channel. Cancellation of untwisted tadpoles implies that there are $n_7 = 8$ D7-branes present (for even $M$). On the other hand, the number of D3-branes is unconstrained by the untwisted tadpole cancellation conditions. This is due to the fact that the space transverse to the D3-branes is non-compact in the theories we are considering here.

Next, consider the tadpoles corresponding to twisted closed string exchange in the tree-channel. For even $M$ the twisted massless tadpole cancellation conditions read:

\[
\text{Tr}(\gamma_{2k-1,7}) - 4 \sin^2 \left( \frac{(2k-1)\pi}{M} \right) \text{Tr}(\gamma_{2k-1,3}) = 0 , \tag{16}
\]

\[
\text{Tr}(\gamma_{2k,7}) - 4 \sin^2 \left( \frac{2\pi k}{M} \right) \text{Tr}(\gamma_{2k,3}) - 8 \cos \left( \frac{2\pi k}{M} \right) = 0 . \tag{17}
\]

\(^7\)A priori they can form a representation of the double cover of $\Gamma$. 
Note that the difference between these tadpole cancellation conditions and those of \[22\] is that in the second line we have \(8 \cos(2\pi k/M)\) instead of \(32 \cos(2\pi k/M)\). This is due to the fact that we are considering a system of D3- and D7-branes instead of a system of D5- and D9-branes.

It is not difficult to show that the above tadpole cancellation conditions have solutions only for \(M = 2, 4, 6\). The corresponding Chan-Paton matrices (up to equivalent representations) are given by:

- \(M = 2 (N = n_3/2)\):
  \[
  \gamma_{1,3} = \text{diag}(i \text{ (} N \text{ times)}, -i \text{ (} N \text{ times})) ,
  \gamma_{1,7} = \text{diag}(i \text{ (} 4 \text{ times)}, -i \text{ (} 4 \text{ times})) .
  \]

- \(M = 4 (N = n_3/4)\):
  \[
  \gamma_{1,3} = \text{diag}(\exp(\pi i/4) \text{ (} N \text{ times}), \exp(-\pi i/4) \text{ (} N \text{ times)}, \exp(3\pi i/4) \text{ (} N \text{ times}), \exp(-3\pi i/4) \text{ (} N \text{ times})) ,
  \gamma_{1,7} = \text{diag}(\exp(\pi i/4) \text{ (} 2 \text{ times}), \exp(-\pi i/4) \text{ (} 2 \text{ times)}, \exp(3\pi i/4) \text{ (} 2 \text{ times}), \exp(-3\pi i/4) \text{ (} 2 \text{ times})) .
  \]

- \(M = 6 (N = (n_3 - 2)/6)\):
  \[
  \gamma_{1,3} = \text{diag}(i \exp(2\pi i/3) \text{ (} N \text{ times}), -i \exp(2\pi i/3) \text{ (} N \text{ times)}, i \exp(-2\pi i/3) \text{ (} N \text{ times}), -i \exp(-2\pi i/3) \text{ (} N \text{ times)}),
  i ((N + 1) \text{ times}), -i ((N + 1) \text{ times})) ,
  \gamma_{1,7} = \text{diag}(i \exp(2\pi i/3), -i \exp(2\pi i/3), i \exp(-2\pi i/3), -i \exp(-2\pi i/3), i \text{ (} 2 \text{ times}), -i \text{ (} 2 \text{ times})) .
  \]

The massless spectra (including twisted closed string sectors) of these models are given in Table \[\].
Note that the one-loop $\beta$-functions for the non-Abelian subgroups of the 33 open string sector gauge groups vanish in these models\textsuperscript{10}. Thus, we have $\mathcal{N} = 2$ superconformal gauge theories living in the world-volumes of the D3-branes\textsuperscript{11}.

**B. Orbifolds of Odd Order**

Let us now discuss the $\Omega J$ orientifold of Type IIB on $\mathcal{M} = \mathbb{C}^3/\Gamma$, where the orbifold group $\Gamma \approx \mathbb{Z}_M$ has odd order ($M \in 2\mathbb{N} + 1$). Such an orientifold contains an arbitrary number of D3-branes. The number of D7-branes, however, is now zero which follows from the untwisted tadpole cancellation conditions.

To understand the twisted tadpole cancellation conditions, first consider $\Omega \tilde{J}$ orientifold of Type IIB on $\mathcal{M}$ (for odd $M$). This theory has only D7-branes and no D3-branes. The twisted tadpole cancellation conditions for the $\Omega J$ orientifold of Type IIB on $\mathcal{M}$ are isomorphic (after interchanging the corresponding D3- and D7-brane Chan-Paton matrices) to those for the $\Omega \tilde{J}$ orientifold of Type IIB on $\mathcal{M}$ provided that the tadpole cancellation conditions for the $\Omega \tilde{J}$ orientifold of Type IIB on $\mathcal{M}' = \mathbb{C}^3/\Gamma'$ can be satisfied, where $\Gamma' = \{ g^k, Rg^k | k = 0, \ldots, M - 1 \} \approx \mathbb{Z}_{2M}$. If the latter condition is not satisfied, then there is no solution to the twisted tadpole cancellation conditions for the $\Omega J$ orientifold of Type IIB on $\mathcal{M}$. The discussion in the previous subsection then implies that the twisted tadpole cancellation conditions can be satisfied only for the $\Omega J$ orientifold of Type IIB on $\mathcal{M}$ with $M = 3$.

The twisted tadpole cancellation conditions for the $\Omega \tilde{J}$ orientifold of Type IIB on $\mathcal{M}$ (for arbitrary odd $M$) can be computed following \cite{22}, and are given by:

$$\text{Tr}(\gamma_{2k,7}) - 4 \sin^2\left(\frac{2\pi k}{M}\right) \text{Tr}(\gamma_{2k,3}) - 8 \cos^2\left(\frac{\pi k}{M}\right) = 0 .$$

(Note that the difference between these tadpole cancellation conditions and those of \cite{22} is that we have $8 \cos^2(\pi k/M)$ instead of $32 \cos^2(\pi k/M)$ for the reasons discussed in the previous subsection.) The above discussion then implies the following solution to the twisted tadpole cancellation conditions for the $\Omega J$ orientifold of Type IIB on $\mathcal{M}$ with $M = 3$ ($N = (n_3 + 2\eta)/3$):

$$\gamma_{1,3} = \text{diag}(\exp(2\pi i/3) \text{ (N times)}, \exp(-2\pi i/3) \text{ (N times)}, 1 \text{ (N - 2\eta times)}) .$$

Here $\eta = -1$ if the $\Omega$ projection is of the $SO$ type, and $\eta = +1$ if it is of the $Sp$ type\textsuperscript{12}. The massless spectra (for both choices of $\eta$) of these models are given in Table I.

\textsuperscript{10}The Abelian gauge couplings run in these models so that $U(1)$'s decouple at low energies.

\textsuperscript{11}The 77 gauge groups for large enough values of $N$ are infrared free. They can therefore be treated as global symmetries (in the context of four dimensional gauge theories living on the D3-branes) at low energies.

\textsuperscript{12}In this case we have D3-branes only and (unlike in the cases with even $M$) there is a choice for the action of $\Omega$ on the Chan-paton charges.
Note that the one-loop $\beta$-functions for the non-Abelian subgroups of the 33 open string sector gauge groups vanish in these models\textsuperscript{13}. Thus, we have $\mathcal{N} = 2$ superconformal gauge theories living in the world-volumes of the D3-branes.

C. Comments

In this subsection we would like to comment on some of the issues relevant for the $\mathcal{N} = 2$ superconformal models constructed in the previous subsections.

First, note that the world-sheet parity reversal $\Omega$ maps the $g^k$ twisted closed string sector to the $g^{M-k}$ twisted closed string sector. This implies that (for $k \neq 0, M/2$) the $g^k$ and $g^{M-k}$ twisted closed string sectors together give rise to one hypermultiplet and one vector multiplet. On the other hand, the $\mathbb{Z}_2$ twisted sector (present for $M \in 2\mathbb{N}$) gives rise to one hypermultiplet only.

The second comment concerns finiteness of $\mathbb{Z}_6$ and $\mathbb{Z}_3$ theories. From our discussion in section III we (at least naively) expect that models where twisted Chan-Paton matrices are not traceless should not be conformal. Note that in the $\mathbb{Z}_2$ and $\mathbb{Z}_4$ models discussed above the twisted Chan-Paton matrices are traceless so it is not surprising that they are finite. On the other hand, some of the twisted Chan-Paton matrices in the $\mathbb{Z}_6$ and $\mathbb{Z}_3$ models are not traceless. Thus, it might appear surprising that they are still finite. Apparently, some subtle cancellation has taken place in these models\textsuperscript{14}.

We can understand the origin of this “accidental” cancellation in the light of recent results obtained in [23]. In [23] it was shown that the six dimensional orientifold of Type IIB on $T^4/\mathbb{Z}_6$ is on the same moduli as the orientifold of Type IIB on $T^4/\mathbb{Z}_2$ with an (untwisted) NS-NS antisymmetric tensor background. The twisted sector Chan-Paton matrices in the $\mathbb{Z}_2$ model are traceless (for turning on the untwisted NS-NS $B$-field does not affect the twisted tadpole cancellation conditions [23]). Since the four dimensional orientifolds we are considering here are related to the six dimensional ones (upon compactifying on $T^2$, T-dualizing and taking the dimensions of the dual $\tilde{T}^2$ to infinity), it is not surprising that the $\mathbb{Z}_6$ model is finite (that is, this explains the “accidental” cancellation mentioned above). The finiteness of the $\mathbb{Z}_3$ model then follows as $\mathbb{Z}_6 \approx \mathbb{Z}_3 \otimes \mathbb{Z}_2$, and the $\mathbb{Z}_2$ model is finite due to tracelessness of the twisted Chan-Paton matrices.

Here we should mention that there are other $\mathcal{N} = 2$ models which are not superconformal. One way of constructing such models is to consider the $\Omega J$ orientifold of Type IIB on $\mathcal{M} = \mathbb{C}^3/\Gamma$, where $\Gamma \approx \mathbb{Z}_M$ has odd order ($M \in 2\mathbb{N} + 1$). Note that in these models there is an orientifold 7-plane (which requires presence of 8 D7-branes) but no orientifold 3-plane\textsuperscript{15}. Nonetheless, we can introduce an arbitrary number of D3-branes. The gauge group

\textsuperscript{13}Just as in the previous cases, the Abelian gauge couplings run in these models so that $U(1)$’s decouple at low energies.

\textsuperscript{14}This cancellation need occur at one-loop level only since these models have $\mathcal{N} = 2$ supersymmetry.

\textsuperscript{15}These models are “T-dual” (in the non-compact limit) to some of the six dimensional models
coming from the D3-branes is (generically) a product of unitary subgroups, and contains bi-fundamental matter just as in \([\text{5,1]}\). However, there is additional matter coming from the 37 open string sector (which gives fundamentals in the 33 gauge group which also transform as fundamentals under the 77 gauge group). The twisted tadpole cancellation conditions are given by \((\text{24})\). It is not difficult to show that in these models (the non-Abelian part of) the 33 gauge theory is not superconformal (and, in particular, the corresponding one-loop \(\beta\)-functions are non-vanishing). Note that this is in accord with expectations in section III. (The “accidental” finiteness of the gauge theories discussed in the previous subsections is therefore a special feature of those models.) It would be interesting to study these non-finite gauge theories further in the present string theory framework for they may bring additional insight relevant for understanding more general non-conformal gauge theories.

\section{\(\mathcal{N} = 1\) Gauge Theories}

In this section we construct a four dimensional \(\mathcal{N} = 1\) gauge theory which has vanishing \(\beta\)-function to all orders in perturbation theory in the large \(N\) limit. This particular gauge theory is obtained by appropriately orientifolding Type IIB on \(\mathbb{C}^3/\Gamma\), where \(\Gamma = \{1, R_1, R_2, R_3\} \approx \mathbb{Z}_2 \otimes \mathbb{Z}_2\) is the orbifold group whose action on the complex coordinates \(z_i\) \((i = 1, 2, 3)\) on \(\mathbb{C}^3\) is given by \(R_i z_j = -(-1)^{\delta_{ij}} z_j\). Next, we consider an orientifold of this theory where the orientifold action is given by \(\Omega J\). The orientifold group is given by \(O = \{1, R_1, R_2, R_3, \Omega J, \Omega R_1, \Omega R_2, \Omega R_3\}\).

This model is a “T-dual” (in the non-compact limit) of the model studied in \([\text{23}]\). The untwisted tadpole cancellation conditions require presence of three sets of D7-branes with 8 D7-branes in each set. Thus, the locations of D7-branes are given by points in the \(z_i\) complex plane. The number of D3-branes is unconstrained (which is due to the fact that the space transverse to the D3-branes is non-compact). The twisted tadpole cancellation conditions imply that the corresponding Chan-Paton matrices \(\gamma_{R_i,3}\) and \(\gamma_{R_i,7}\) are traceless:

\[
\text{Tr}(\gamma_{R_i,3}) = \text{Tr}(\gamma_{R_i,7}) = 0 .
\]

A choice\[^{16}\] consistent with requirements that the Chan-Paton matrices form a (projective) representation of (the double cover) of \(\Gamma\) is given by \((N = n_3/2)\)

\[
\gamma_{R_i,3} = i \sigma_i \otimes \mathbb{I}_N ,
\]

discussed in \([\text{23}]\).

\[^{16}\]This choice is unique up to equivalent representations \([\text{25}]\).
where $\sigma_i$ are Pauli matrices, and $I_N$ is an $N \times N$ identity matrix. (The action on the $D7_i$ Chan-Paton charges is similar.) The spectrum of this model is given in Table I.

Note that the one-loop $\beta$-function for the 33 open string sector gauge group vanishes in this model\textsuperscript{17}. Moreover, all the twisted Chan-Paton matrices are traceless in this model. Following our discussion in section III we thus conclude that this four dimensional gauge theory is superconformal (to all loop orders) in the large $N$ limit.

For completeness, let us give the tree-level superpotential for this model. Let $\Phi_i$, $\Phi^i_j$, $Q^i$ and $Q^i_j$ be the matter fields in the 33, $7_i^7_i$, $37_i$ and $7_i7_j$ open string sectors, respectively. The subscript in $\Phi_i$ and $\Phi^i_j$ labels three different chiral superfields (see Table II) in the 33 and $7_i^7_i$ sectors. The superpotential can be computed as in [25] and is given by (here we suppress the actual values of the Yukawa couplings)

$$W = \epsilon_{ijk} \Phi_i \Phi^j \Phi^k + \epsilon_{ijk} \Phi^i_j \Phi^j_k + \epsilon_{ijk} \Phi^i_k \Phi^j Q^i + \Phi^i Q^i Q^i + Q^i Q^j Q^k + Q^i Q^j Q^j .$$  (28)

Here the summation over repeated indices is understood.

### B. Other $\mathcal{N} = 1$ Gauge Theories from Orientifolds

In this subsection we consider $\mathcal{N} = 1$ orientifolds with orbifold groups such that twisted Chan-Paton matrices are not traceless. Unlike the $\mathcal{N} = 2$ examples (where certain “accidental” cancellations at one-loop order led to finiteness as discussed in the previous section) the $\mathcal{N} = 1$ models no longer have vanishing one-loop $\beta$-functions (as expected from our discussion in section III).

Instead of being most general, for illustrative purposes in this subsection we will confine our attention to $\Omega J$ orientifolds of Type IIB on $\mathbb{C}^3/\Gamma$ where the orbifold group $\Gamma = \{g^k | k = 0, \ldots, M-1\} \approx \mathbb{Z}_M$ ($M$ is odd) is a subgroup of $SU(3)$ (but not of $SU(2)$). The action of $\Gamma$ on the complex coordinates $z_i$ ($i = 1, 2, 3$) on $\mathbb{C}^3$ is given by $g z_i = \omega^{\ell_i} z_i$, where $\omega = \exp(2\pi i/M)$, $\ell_i \neq 0$, and $\ell_1 + \ell_2 + \ell_3 = M$.

In these orientifolds we have D3-branes only whose number is arbitrary. The twisted tadpole cancellation conditions for these orientifolds are isomorphic (upon interchanging the corresponding D3- and D9-brane Chan-Paton matrices) to those for the $\Omega$ orientifolds of Type IIB on $\mathbb{C}^3/\Gamma$. The latter tadpole cancellation conditions were derived in [26]. Applying those results to the cases under consideration we have the following twisted tadpole cancellation conditions ($k = 1, \ldots, N-1$):

$$\text{Tr}(\gamma_{2k,3}) = -4\eta \prod_{i=1}^3 (1 + \omega^{k\ell_i}) .$$  (29)

Here $\eta = -1$ if the $\Omega$ projection is of the $SO$ type, and $\eta = +1$ if it is of the $Sp$ type.

It is not difficult to solve these tadpole cancellation conditions for the general case. For illustrative purposes, however, here we will discuss only two particular examples.

\textsuperscript{17}For $N > 4$ the $7_i7_i$ gauge groups are infrared free. They can therefore be treated as global symmetries (in the context of the four dimensional gauge theory living on the D3-branes) at low energies.
First consider the case where \( M = 3 \). This model is a “T-dual” (in the non-compact limit) of the model studied in [27]. The solution to the twisted tadpole cancellation conditions reads (\( N = (n_3 - 4\eta)/3 \)):

\[
\gamma_{1,3} = \text{diag}(\exp(2\pi i/3) \text{ (}N\text{ times)}, \exp(-2\pi i/3) \text{ (}N\text{ times)}, 1 \text{ (}N + 4\eta\text{ times})) . \tag{30}
\]

The massless spectra (for both choices of \( \eta \)) of these models are given in Table III. The non-Abelian gauge anomaly is cancelled in this model. However, there is an anomalous \( U(1) \). (The superpotential for this model can be found in [27,30].)

Next, let us consider the case where \( M = 7 \). This model is a “T-dual” (in the non-compact limit) of the model studied in [26]. The solution to the twisted tadpole cancellation conditions reads (\( N = (n_3 + 4\eta)/7 \)):

\[
\gamma_{1,3} = \text{diag}(\exp(2\pi i/7) \text{ (}N\text{ times)}, \exp(-2\pi i/7) \text{ (}N\text{ times)}, \exp(4\pi i/7) \text{ (}N\text{ times)}, \exp(-4\pi i/7) \text{ (}N\text{ times)}, \exp(6\pi i/7) \text{ (}N\text{ times)}, \exp(-6\pi i/7) \text{ (}N\text{ times)}, 1 \text{ (}N - 4\eta\text{ times})) . \tag{31}
\]

The massless spectra (for both choices of \( \eta \)) of these models are given in Table III. The non-Abelian gauge anomaly is cancelled in this model. However, there is an anomalous \( U(1) \). (The superpotential for this model can be found in [26].)

Note that none of the above two models have vanishing one-loop \( \beta \)-functions for the non-Abelian factors in the gauge group. Moreover, the presence of anomalous \( U(1) \) implies that there is a Fayet-Iliopoulos D-term which must be cancelled via a generalized Green-Schwarz mechanism [32]. Thus, some fields must acquire non-zero vevs to cancel the Fayet-Iliopoulos D-term. Using Type I-Heterotic duality [29] it was argued in [30] that the twisted closed string sector states transform non-trivially under the anomalous \( U(1) \) gauge transformation, and (together with the dilaton plus axion supermultiplet) can cancel the Fayet-Iliopoulos D-term. In this process a non-zero vev is generated for the field responsible for this cancellation.

Thus, in the \( \mathcal{N} = 1 \) cases we see that the models with non-vanishing traces of the twisted Chan-Paton matrices lead (as expected in section III) to non-finite models. In particular, the “accidental” vanishing of the one-loop \( \beta \)-functions (as in the corresponding \( \mathcal{N} = 2 \) cases) does not occur.

Here we should mention that the tadpole cancellation condition (29) is necessary not only for ultraviolet finiteness of the corresponding theories but also for non-Abelian gauge anomaly cancellation. For illustrative purposes, to see what can go wrong if we relax this condition, let us consider the above \( \mathbb{Z}_3 \) example with the following choice for the twisted Chan-Paton matrices:

\[
\gamma_{1,3} = \text{diag}(\exp(2\pi i/3) \text{ (}N\text{ times)}, \exp(-2\pi i/3) \text{ (}N\text{ times)}, 1 \text{ (}N'\text{ times})) . \tag{32}
\]

The gauge group in the 33 open string sector is \( U(N) \otimes G_\eta(N') \), and the chiral matter is given by \( 3(\mathbf{R}_\eta, 1) \) and \( 3(\overline{\mathbf{N}}, \mathbf{N}') \) (see Table III for notation). Note that \( \mathbf{R}_\eta \) of \( SU(N) \)

\footnote{Also see, e.g., [28].}
contributes as much as $N + 4\eta$ chiral superfields in $N$ of $SU(N)$ into the non-Abelian gauge anomaly. Thus, the non-Abelian gauge anomaly cancellation implies that $N' = N + 4\eta$. This is precisely the solution to the tadpole cancellation condition \( (29) \).

C. Cases without World-Sheet Description

In subsection C of section [11] we mentioned that for certain choices of the orbifold group the corresponding orientifolds may not have a world-sheet description. Here we give an explicit example of this.

Consider Type IIB on $\mathbb{C}^3/\Gamma$ with $\Gamma \approx D_N$ (non-Abelian dihedral group) where the action of $\Gamma$ on the complex coordinates $z_i (i = 1, 2, 3)$ on $\mathbb{C}^3$ is given by ($\omega = \exp(2\pi i/N)$):

\[
\begin{align*}
gz_1 &= z_1 , & gz_2 &= \omega z_2 , & gz_3 &= \omega^{-1} z_3 , \\
grz_1 &= -z_1 , & rz_2 &= z_3 , & rz_3 &= z_2 ,
\end{align*}
\]

where $g, r$ are the generators of $D_N$. Note that $g$ and $r$ do not commute: $rg = g^{-1}r$.

Now consider $\Omega J$ ($Jz_i = -z_i$) orientifold of this theory. The orientifold group is $O = \{g^k, rg^k, \Omega g^k, \Omega rg^k | k = 0, \ldots, N - 1\}$. Note that $(Jrg^k)^2 = 1$, and the set of points in $\mathbb{C}^3$ fixed under the action of $Jrg^k$ has real dimension two. This implies that there are $N$ kinds of orientifold 7-planes corresponding to the elements $\Omega Jrg^k$. Note, however, that due to non-commutativity between $g$ and $r$ (and, therefore, between different $Jrg^k$), these orientifold 7-planes (as well as the corresponding D7-branes) are mutually non-local. This implies that this orientifold does not have a world-sheet description. One way to understand such models (at least in compact cases) is to use their F-theory [33] description.

Here we should mention that there are other models with both Abelian and non-Abelian orbifold groups for which the perturbative description in terms of open strings stretched between mutually local D-branes is inadequate. We refer the reader to [18] for details.

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FIG. 1. A planar diagram.
FIG. 2. Tree-channel Klein bottle, Möbius strip and annulus amplitudes.
FIG. 3. Factorization of the Klein bottle, Möbius strip and annulus amplitudes.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Model & Gauge Group & Charged Hypermultiplets & Twisted Sector Hypermultiplets & Twisted Sector Vector Multiplets \\
\hline
$\mathbb{Z}_2$ & $U(N)_{33} \otimes U(4)_{77}$ & $2(\mathbf{A};1)_{33}$ & 1 & 0 \\
 & & $2(1;6)_{77}$ & & \\
 & & $(N;4)_{37}$ & & \\
\hline
$\mathbb{Z}_4$ & $[U(N) \otimes U(N)]_{33} \otimes [U(2) \otimes U(2)]_{77}$ & $(\mathbf{A},1;1,1)_{33}$ & 2 & 1 \\
 & & $(1,\mathbf{A};1,1)_{33}$ & & \\
 & & $(N,N;1,1)_{33}$ & & \\
 & & $(1,1;1',1)_{77}$ & & \\
 & & $(1,1,1')_{77}$ & & \\
 & & $(1,1,2,2)_{77}$ & & \\
 & & $(N,1;2,1)_{37}$ & & \\
 & & $(1,N;1,2)_{37}$ & & \\
\hline
$\mathbb{Z}_6$ & $[U(N) \otimes U(N) \otimes U(N+1)]_{33} \otimes [U(1) \otimes U(1) \otimes U(2)]_{77}$ & $(\mathbf{A},1,1;1)_{33}$ & 3 & 2 \\
 & & $(1,\mathbf{A};1,1)_{33}$ & & \\
 & & $(N,1,N+1;1)_{33}$ & & \\
 & & $(1,N,N+1;1)_{33}$ & & \\
 & & $(1,1,1;2)_{77}$ & & \\
 & & $(1,1,1;2')_{77}$ & & \\
 & & $(N,1;1,1)_{37}$ & & \\
 & & $(1,N;1,1)_{37}$ & & \\
 & & $(1,1,N+1;2)_{37}$ & & \\
\hline
$\mathbb{Z}_3$ & $[U(N) \otimes G_\eta(N-2\eta)]_{33}$ & $(\mathbf{R}_\eta,1)_{33}$ & 1 & 1 \\
 & & $(N,N-2\eta)_{33}$ & & \\
\hline
\end{tabular}
\caption{The massless spectra of $\mathcal{N}=2$ orientifolds of Type IIB on $\mathbb{C}^3/\mathbb{Z}_N \; M = 2,3,4,6$.}
\end{table}

The semi-colon in the column “Charged Hypermultiplets” separates $33$ and $77$ representations. The notation $\mathbf{A}$ stands for the two-index antisymmetric representation (which is $N(N-1)/2$ dimensional for $U(N)$) of the corresponding unitary group. The notation $\mathbf{1}'$ in the $\mathbb{Z}_4$ model stands for the antisymmetric representation of the corresponding $U(2)$ group (whose charge with respect to the $U(1)$ factor is 2). In the $\mathbb{Z}_6$ model the subscripts “1” and “2” indicate that the corresponding states are charged under the first and the second $U(1)$ factors in the $77$ gauge group. In the $\mathbb{Z}_3$ model $G_\eta = SO$ for $\eta = -1$ and $G_\eta = Sp$ for $\eta = +1$. (Here we are using the convention that $Sp(2m)$ has rank $m$.) Also, $R_{\eta} = \mathbf{A}$ for $\eta = -1$, and $R_{\eta} = \mathbf{S}$ (two-index $N(N+1)/2$ dimensional symmetric representation of $U(N)$) for $\eta = +1$. By twisted sector hypermultiplets/vector multiplets we mean those in the twisted \textit{closed} string sectors. The untwisted closed string sector states are not shown.
| Model | Gauge Group | Charged Chiral Multiplets | Twisted Sector Chiral Multiplets | Twisted Sector Vector Multiplets |
|-------|-------------|---------------------------|---------------------------------|---------------------------------|
| $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ | $Sp(N)_{33} \otimes Sp(4)_{7171} \otimes Sp(4)_{7272} \otimes Sp(4)_{7373}$ | $3(A; 1; 1; 1)_{33}$, $3(1; 6; 1; 1)_{7171}$, $3(1; 1; 6; 1)_{7272}$, $3(1; 1; 1; 6)_{7373}$, $(N; 4; 1; 1)_{371}$, $(N; 1; 4; 1)_{372}$, $(N; 1; 1; 4)_{373}$, $(1; 4; 4; 1)_{7172}$, $(1; 1; 4; 4)_{7273}$, $(1; 4; 1; 4)_{7373}$ | 3 | 0 |

**TABLE II.** The massless spectrum of $\mathcal{N}=1$ orientifold of Type IIB on $\mathbb{C}^3/(\mathbb{Z}_2 \otimes \mathbb{Z}_2)$. The notation $A$ stands for the two-index antisymmetric (reducible) representation of $Sp(N)$. The untwisted closed string sector states are not shown.

| Model | Gauge Group | Charged Chiral Multiplets | Twisted Sector Chiral Multiplets | Twisted Sector Vector Multiplets |
|-------|-------------|---------------------------|---------------------------------|---------------------------------|
| $\mathbb{Z}_3$ | $[U(N) \otimes G_\eta(N + 4\eta)]_{33}$ | $3(R_\eta, 1)(+2)_{33}$, $3(N, N + 4\eta)(-1)_{33}$ | 1 | 0 |
| $\mathbb{Z}_7$ | $[U(N) \otimes U(N) \otimes U(N) \otimes G_\eta(N - 4\eta)]_{33}$ | $(R_\eta, 1, 1, 1)(+2, 0, 0)_{33}$, $(1, R_\eta, 1, 1)(0, +2, 0)_{33}$, $(1, 1, R_\eta, 1)(0, 0, +2)_{33}$, $(N, 1, 1, N - 4\eta)(+1, 0, 0)_{33}$, $(1, N, 1, N - 4\eta)(0, +1, 0)_{33}$, $(1, 1, N, N - 4\eta)(0, 0, +1)_{33}$, $(N, N, 1, 1)(-1, +1, 0)_{33}$, $(1, N, N, 1)(0, -1, +1)_{33}$, $(N, 1, N, N)(+1, 0, -1)_{33}$, $(N, N, 1, N)(-1, -1, 0)_{33}$, $(1, N, N, N)(0, -1, -1)_{33}$, $(N, 1, N, N)(-1, 0, -1)_{33}$ | 3 | 0 |

**TABLE III.** The massless spectra of $\mathcal{N}=1$ orientifolds of Type IIB on $\mathbb{C}^3/\mathbb{Z}_M$ ($M = 3, 7$). Here we are using some of the notations from Table I. The $U(1)$ charges of the states in the 33 open string sector are given in parentheses. The untwisted closed string sector states are not shown.
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