The generalised scaling function: a note

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Abstract

A method for determining the generalised scaling function(s) arising in the high spin behaviour of long operator anomalous dimensions in the planar $sl(2)$ sector of $\mathcal{N} = 4$ SYM is proposed. The all-order perturbative expansion around the strong coupling is detailed for the prototypical third and fourth scaling functions, showing the emergence of the $O(6)$ Non-Linear Sigma Model mass-gap from different SYM 'mass' functions. Remarkably, only the fourth one gains contribution from the non-BES reducible densities and also shows up, as first, NLSM interaction and specific model dependence. Finally, the computation of the $n$-th generalised function is sketched and might be easily finalised for checks versus the computations in the sigma model or the complete string theory.

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1 Introduction

The planar $sl(2)$ sector of $\mathcal{N} = 4$ SYM contains local composite operators of the form

$$\text{Tr}(\mathcal{D}^s Z^L) + ....,$$

(1.1)

where $\mathcal{D}$ is the (symmetrised, traceless) covariant derivative acting in all possible ways on the $L$ bosonic fields $Z$. The spin of these operators is $s$ and $L$ is the so-called ‘twist’. Moreover, this sector would be described – thanks to the AdS/CFT correspondence [1] – by string states on the AdS$_5 \times S^5$ spacetime with AdS$_5$ and $S^5$ charges $s$ and $L$, respectively. In addition, as far as the one loop is concerned, the Bethe Ansatz problem is equivalent to that of twist operators in QCD [2, 3].

Proper superpositions of operators (1.1) have definite anomalous dimension $\Delta$ depending on $L$, $s$ and the ’t Hooft coupling $\lambda = 8\pi^2 g^2$:

$$\Delta = L + s + \gamma(g, s, L),$$

(1.2)

where $\gamma(g, s, L)$ is the anomalous part. A great boost in the evaluation of $\gamma(g, s, L)$ in another sector has come from the discovery of integrability for the purely bosonic $so(6)$ operators at one loop [4]. Later on, this fact has been extended to the whole theory and at all loops in the sense that, for instance, any operator of the form (1.1) is associated to one solution of some Bethe Ansatz-like equations and thus any anomalous dimension becomes a function, basically the energy, of one solution [5, 6, 7]. Nevertheless, along with this host of new results an important limitation emerged as a by-product of the on-shell ($S$-matrix) Bethe Ansatz: as soon as the interaction reaches a range greater than the chain length, then it becomes modified by unpredicted wrapping effects. More precisely, the anomalous dimension is in general correct up to the $L - 1$ loop in the convergent, perturbative expansion, i.e. up to the order $g^{2L-2}$. Which in particular implies, – fortunately for us, – that the asymptotic Bethe Ansatz should give the right result whenever the subsequent limit (1.3) is applied, as in the expansion (1.4).

In fact, an important large twist and high spin scaling may be considered on both sides of the correspondence:

$$s \to \infty, \quad L \to \infty, \quad j = \frac{L}{\ln s} = \text{fixed}.$$  

(1.3)

The relevance of this logarithmic scaling for the anomalous dimension of long operators has been pointed out in [9] within the (one-loop) SYM theory and then in [8] and [10].

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1A deep reason for that may be that one loop QCD still shows up conformal invariance, albeit integrability does not seem to hold in full generality (for instance, it apparently imposes aligned partonic helicities).

2Actually, in string theory (semi-classical) calculations the $\lambda \to +\infty$ limit needs consideration before any other (cf. for instance [8] and references therein), thus implying a different limit order with respect to our gauge theory approach (cf. below for more details).
within the string theory (strong coupling $g \gg 1$). In fact, by describing the anomalous dimension through a non-linear integral equation [12] (like in other integrable theories [11]), it has been recently confirmed the Sudakov leading behaviour for $s \to +\infty$ [10, 8]

$$\gamma(g, s, L) = f(g, j) \ln s + \ldots ,$$

(1.4)

thus indeed generalising to all loops a result by [9]. Actually, this statement was argued in [12] by computing iteratively the solution of some integral equations and then, thereof, the generalised scaling function $f(g, j)$ at the first orders in $j$ and $g^2$: more precisely, the first orders in $g^2$ have been computed for the first generalised scaling functions $f_n(g)$, forming

$$f(g, j) = \sum_{n=0}^{\infty} f_n(g) j^n .$$

(1.5)

As a by-product, the reasonable conjecture has been put forward that the two-variable function $f(g, j)$ should be analytic (in $g$ for fixed $j$ and in $j$ for fixed $g$). In [13] similar results have been derived for what concerns the contribution beyond the leading scaling function $f(g) = f_0(g)$, but with a modification which has allowed to neglect the non-linearity for finite $L$ and to end-up with one linear integral equation. The latter does not differ from the BES one (which cover the case $j = 0$, cf. [7]), but for the inhomogeneous term, which is the sum of an integral on the one loop root density, a hole contribution and a known function. In this respect, we have reckoned interesting the analysis of the next-to-leading-order (nlo) term – still coming, for finite $L$, from an asymptotic Bethe Ansatz –, as the leading order $f(g)$ has been conjectured to be independent of $L$ or universal [9, 6].

For instance, this nlo term enjoys many attractive features, like its form which contains up to the $L$-linear term, i.e. $f_1(g)L + c(g)$. In particular, the linearity in $L$ and the behaviour of the next terms with increasing powers (of $L$) has furnished us the inspiration for the past [14] and present calculations, respectively. In this respect, we deem useful to briefly introduce in the next section a suitable modification of the aforementioned method such that it applies to the regime (1.3) (still for any $g$ and $j$). In fact, a suitable modification of this LIE has been already exploited and explored in [14] to derive still a LIE in the scaling (1.3) (for any $g$ and $j$). Along this path, we want to determine here the generalised scaling function $f(g, j)$ and also its constituents $f_n(g)$ for all the values of $j$ and $g$, thus interpolating from weak to strong coupling: in fact, we will see in the following why the expansion around $j = 0$ is suitable, efficient and manageable. Specifically, we will compute $f_n(g)$ for $n = 2, 3, 4$, show that $f_2(g) = 0$ and derive the strong coupling behaviour of $f_3(g)$ and $f_4(g)$. In this respect, a crucial point is the appearance in the calculation regarding $f_4(g)$ (and onwards for $n > 4$) of the higher root densities, along with the BES density, which are not expressible in terms of the BES one. This novel feature (i.e. essentially new (recursive) linear integral equations (concerning the higher densities) beside the BES one) plays an important rôle in the
comparison with the $O(6)$ Non-Linear Sigma Model (NLSM). For, at the leading large $g$ order, we will find perfect agreement with the infrared (i.e. strong coupling) expansion of the $O(6)$ Non-Linear Sigma Model (NLSM) [15], thus enforcing the ultraviolet (i.e. weak coupling) checks and predictions by Alday and Maldacena [10]. The IR results, on the one hand, explore the other regime of the asymptotically free NLSM, which develops a mass gap, on the other yield a convergent series (for $f(g, j)$, in this limit, as a function of $j$ and $\forall g$ in the coefficients $f_n(g)$) [15]. This feature, the convergence, likely extends to the full SYM theory, so bearing new interest to this expansion. We would also remark here, but we will discuss more deeply this point afterwards, that the agreement for $f_4(g)$ (and onwards for $n > 4$) is highly non-trivial since it involves the specific interacting theory of the $O(6)$ NLSM. Finally, the ideology of our method for computing the $n$-th generalised scaling function $f_n(g)$ is exposed and shown to be easily implementable by using analytical and numerical techniques.

Incidentally, in the subsequent section we will also briefly understand the corrections (represented by dots) to the leading Sudakov factor (1.4), as they are indeed formed by the terms like $c(g)$ in the aforementioned nlo term.

**Note added:** An interesting paper [16] appears today in the web archives. It seems to have some goals and equations similar to ours, although coming more directly from the approach [12], and giving the leading strong coupling behaviour of $f_2(g)$ in agreement with us and confirming $f_2(g) = 0$ as in [12], and $f_1(g)$ as in [14]. Nothing explicit we can find in it about $f_4(g)$ and a systematic plan for all the other $f_n(g)$.

## 2 Computing the generalised scaling functions

In the framework of integrability in $\mathcal{N} = 4$ SYM, we have been finding useful [17] to rewrite the Bethe equations as a unique non-linear integral equation [11] [3]. In particular, we have developed (starting in [18]) a new technique in order to cope with the frequent (in SYM theory) case where the Bethe roots are confined within a finite interval [13]. The non-linear integral equation regarding the $sl(2)$ sector involves two functions $F(u)$ and $G(u, v)$, both satisfying linear equations. It is convenient to split $F(u)$ into its one-loop and higher than one loop contributions $F_0(u)$ and $F^H(u)$ respectively, and define the quantities $\sigma_0(u) = \frac{d}{du}F_0(u)$ and $\sigma_H(u) = \frac{d}{du}F^H(u)$, which, in the high spin limit, have the meaning of one loop and higher than one loop density of Bethe roots and holes, respectively.

It is important to stress that in this paper we will follow the route initiated in [13], and further developed in [14] where the (strong coupling) behaviour of the first constituent $f_1(g)$ was studied in full detail. In this perspective, one of the main aims of the present paper is to show how we can study the generalised scaling function

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3 Only in case of nested Bethe Ansatz we may have more than one equation, but always in small number.
at all orders in \( j \). In other words, we will show how to write down recursive linear integral equations for infinite many densities, \( \sigma_H^{(n)} \), \( n = 0, 1, 2, \ldots \) (the first one being the famous BES density \([7]\)), each one yielding \( f_n(g) \), in a way which systematically furnishes the \( n \)-th one, provided those before \( 4 \) are known. Indeed, this is the main consequence of a suitable manipulation of the initial integral equation for the full density to end up with a recursive system of Fredholm (II type) linear integral equations, all with the same kernel (the BES one), but different inhomogeneous terms. The latter are the only parts entered by the previous densities and are thus responsible for the recursive derivation and solution. Eventually, we shall remark that this structure has already appeared in \([14]\), allowing us to study \( f_1(g) \): in fact, we did not even need the knowledge of the BES density \( \sigma(0)_H \), thanks to the general rule that only \( \sigma_H^{(m)} \), with \( m \leq n - 3 \), enter the inhomogeneous term regarding \( \sigma_H^{(n)} \).

Let us now analyse in detail the linear integral equations which are satisfied by the one and higher loop densities and in terms of which we may write the anomalous dimension.

For the time being, we want to extend to the scaling situation \([1.3]\) the procedure by \([13, 14]\) for computing the one loop observables (i.e. the physical quantities depending on the one loop density \( \sigma_0(v) \)). The integration on the finite range of roots, \((-b_0, b_0)\), may be computed via an infinite integration:

\[
\int_{-b_0}^{b_0} dv f(v) \sigma_0(v) = \int_{-\infty}^{\infty} dv f(v) \sigma_0^*(v) + O((\ln s)^{-\infty}),
\]  

(2.1)

where the Fourier transform of the effective density \( \sigma_0^*(v) \) obeys the integral equation

\[
\hat{\sigma}_0^*(k) = -4\pi \frac{\frac{L}{2} - e^{-\frac{|k|}{2}} \cos \frac{k_s}{2}}{2 \sinh \frac{|k|}{2}} - \frac{e^{-\frac{|k|}{2}}}{2 \sinh \frac{|k|}{2}} \int_{-\infty}^{\infty} du e^{iku} \chi_{c_0}(u) \sigma_0^*(u) - 4\pi \delta(k) \ln 2,
\]  

(2.2)

with \((-c_0, c_0)\) the interval spanned by the internal holes \([13]\). This simply means

\[
\int_{-\infty}^{\infty} du \chi_{c_0}(u) \sigma_0^*(u) = -2\pi(L - 2) + O((\ln s)^{-\infty}),
\]  

(2.3)

with the interval function \( \chi_{c_0}(u) \) equal to 1 if \(-c_0 \leq u \leq c_0\) (here the internal holes concentrate), and 0 otherwise (no internal holes outside). Within the double limit \([1.3]\), the above remainders are \( O((\ln s)^{-\infty}) \) and are exactly given by the non-linear integrals in \([13]\) we have here dropped out.

Now, we ought to briefly comment on the corrections of the leading Sudakov formula \([1.4]\), namely the dots in there. We have deduced in \([13]\) a linear integral equation

\[O((\ln s)^{-\infty})\] means a remainder which goes to zero faster that any inverse power of \( \ln s \): \( \lim_{s \to \infty} (\ln s)^k O((\ln s)^{-\infty}) = 0, \forall k > 0 \).

\[\text{Starting, actually, from } n - 3 \text{ backwards.}\]

\[\text{The parameter } b_0 \text{ depends on the total number of roots } s \text{ through the integral on } \sigma_0(v) \text{ (the normalisation condition) which counts them (cf. [13]).}\]
suitable to detail the appearance of a constant term \( O((\ln s)^0) \), denoted in the introduction as \( f_1(g)L + c(g) \). Concerning this, the first addendum is the generalised scaling function at first \( j \)-order \([14]\), whilst the second one gives rise to a \( O((\ln s)^0) \) correction to the leading Sudakov scaling, i.e. the \( j^0 \) order of some \( f^{(0)}(g,j) \) in the expansion
\[
\gamma(g, s, L) = f(g,j) \ln s + f^{(0)}(g,j) + \ldots .
\] (2.4)

Now, similarly we may imagine that the dots should be inverse integer powers of \( \ln s \), with coefficients, at each power, depending on \( g \) and \( j \). Very interestingly, these power-like corrections to the leading \([14]\) seem to come out truly from our systematic expansion of the one-loop density, – satisfying a linear integral equation –, and its subsequent insertion into the inhomogeneous term of the higher loop linear integral equation (i.e. \([2.9]\)). In fact, the last \( \delta \)-term in \([2.2]\) contributes, for instance, to \( f^{(0)}(g,j) \), namely to the mentioned \( c(g) \), whilst it does not to \( f(g,j) \). In other words, all these terms seem to be controlled by linear equations \([10]\) after neglecting the non-linear integrals. Eventually, there might also be a possible non-analytic (in the variable \( \ln s \)) correction, \( O((\ln s)^{-\infty}) \). Yet, the \( O((\ln s)^{-\infty}) \) term would face two problems: on the one side it is determined by cumbersome non-linear integrals which deny the density (linear) treatment, on the other side it may well become affected by the non-asymptotic phenomenon of wrapping.

Therefore, as we will be constraining ourselves to the leading Sudakov factor \( f(g,j) \), we can neglect the aforementioned \( \delta \)-term and we may, for convenience’ sake, Fourier-transform the equations \([2.2]\) and \([2.3]\), respectively:
\[
\hat{\sigma}_0^*(k) = -4\pi L^2 - e^{-\frac{|k|}{2}} \cos \frac{k s}{\sqrt{2}} \left\{ -e^{-\frac{|k|}{2}} \int_{-\infty}^{\infty} \frac{dh}{2\pi} \hat{\sigma}_0^*(h) \sin(k - h) c_0 \right\},
\]
\[
2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \hat{\sigma}_0^*(k) \frac{\sin k c_0}{k} = -2\pi (L - 2).
\] (2.5) (2.6)

These final relations characterise the one-loop theory in the high spin and large twist limit \([1.3]\) and need to be solved together in order to give \( \hat{\sigma}_0^*(k) \) and \( c_0 \). In specific, we may solve the first one, a Fredholm (II type) integral equation, by Neumann-Liouville (recursive) series and then expand usefully \( c_0(j) \) according to
\[
c_0(j) = \sum_{n=1}^{\infty} c_0^{(n)} j^n.
\] (2.7)

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7 At small \( g \), the same term has been derived in \([12]\) by evaluating at leading order the non-linear integrals of a traditional NLIE approach \([11]\).

8 Naturally the subsequent expansion \([2.7]\) and its higher loop analogue ought to be amended accordingly.

9 The other part of \( c(g) \) derives from the rest of the equation, i.e. the next equation \([2.5]\), but it will be neglected in the following.

10 This is similarly true for the linear integral formula of the anomalous dimension.
Already the first step of this procedure (the approximation by the inhomogeneous term) yields the first two coefficients of this series

$$c_0^{(1)} = \frac{\pi}{4}, \quad c_0^{(2)} = -\frac{\pi}{4} \ln 2.$$  (2.8)

Regarding the high spin behaviour of long operators, the approach to the higher than one loop density, $\sigma_H(u)$, moves along similar lines. Actually, the knowledge of $\sigma_0^s(u)$ concurs to find it as the solution of the linear integral equation (discarding the following $O((\ln s)^{-\infty})$)

$$\sigma_H(u) = -iL \frac{d}{du} \ln \left( \frac{1 + \frac{g^2}{2x^-(u)^2}}{1 + \frac{g^2}{2x^+(u)^2}} \right) + \frac{i}{\pi} \int_{-\infty}^{\infty} dv \chi_c(v) \left[ \frac{d}{du} \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + \frac{1}{1 + \frac{g^2}{x^+(u)x^-(v)}} \right]$$

$$+ \frac{i}{\pi} \int_{-\infty}^{\infty} dv \frac{d}{du} \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \left[ \sigma_0^s(v) + \sigma_H(v) \right]$$

$$- \frac{i}{\pi} \int_{-\infty}^{+\infty} dv \frac{1}{\pi} \frac{d}{du} \left[ \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \right] \sigma_0^s(v) +$$

$$+ \int_{-\infty}^{+\infty} dv \frac{1}{\pi} \frac{d}{du} \ln \left( \frac{1 - \frac{g^2}{2x^+(u)x^-(v)}}{1 - \frac{g^2}{2x^-(u)x^+(v)}} \right) + i\theta(u, v) \sigma_H(v) + O((\ln s)^{-\infty}),$$  (2.9)

constrained by the two conditions on $c$ and $c_0$ respectively

$$\int_{-\infty}^{\infty} dv \chi_c(u) \left[ \sigma_0^s(u) + \sigma_H(u) \right] = -2\pi(L - 2),$$  (2.10)

$$\int_{-\infty}^{\infty} dv \chi_0(u) \sigma_0^s(u) = -2\pi(L - 2),$$

with the former fixing the interval $(-c, c)$ where the all-loop internal holes concentrate as well as the latter, equivalent to (2.6), determines the range of the one-loop internal holes.\footnote{The function $\theta(u, v)$ appearing in (2.9) is the 'dressing factor', introduced on the string side in the second of [5] and finalised as meromorphic function of the coupling constant in [7].}  

As in the one loop case, in order to gain a better insight it is convenient to re-write
in terms of Fourier transforms (upon neglecting $O((\ln s)^{-\infty})$)

\[
\hat{\sigma}_H(k) = \pi L \frac{1 - J_0(\sqrt{2gk})}{\sinh \frac{|k|}{2}} + \\
+ \frac{1}{\sinh \frac{|k|}{2}} \int_{-\infty}^{\infty} \frac{dh}{|h|} \left[ \sum_{r=1}^{\infty} r(-1)^{r+1} J_r(\sqrt{2gk}) J_r(\sqrt{2gh}) \frac{1 - \text{sgn}(kh)}{2} e^{-\frac{|h|}{2}} + \\
+ \text{sgn}(h) \sum_{r=2}^{\infty} \sum_{\nu=0}^{r+1} c_{r+1}(-1)^{r+\nu} e^{-\frac{|h|}{2}} \left( J_{r-1}(\sqrt{2gk}) J_{r+2\nu}(\sqrt{2gh}) - \\
- J_{r-1}(\sqrt{2gh}) J_{r+2\nu}(\sqrt{2gk}) \right) \right] \frac{\sinh(k-p)c_0}{k-p} + \frac{e^{-\frac{|k|}{2}}}{\sinh \frac{|k|}{2}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \hat{\sigma}_0^\times(p) \frac{\sin(k-p)c_0}{k-p},
\]

as well as the previous conditions on the internal holes distributions

\[
2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{\hat{\sigma}_0^\times(k) \sin kc_0}{k} = -2\pi(L-2), \\
2 \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ \hat{\sigma}_0^\times(k) + \hat{\sigma}_H(k) \right] \frac{\sin kc}{k} = -2\pi(L-2).
\]

Eventually, the three equations (2.11) and (2.12) form the starting point of our all loop analysis, generalising the one loop statement. In fact, we may write the anomalous dimension (in the usual limit (1.3)) as

\[
\gamma(g, s, L) = -g^2 \int_{-b_0}^{b_0} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \sigma_0(v) + \\
+ g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] [\sigma_0(v) + \sigma_H(v)] - \\
- g^2 \int_{-\infty}^{+\infty} \frac{dv}{2\pi} \left[ \frac{i}{x^+(v)} - \frac{i}{x^-(v)} \right] \sigma_H(v) + O((\ln s)^{-\infty}),
\]

and realise, upon comparing this with (2.11), that

\[
\gamma(g, s, L) = \frac{1}{\pi} \lim_{k \to 0} \hat{\sigma}_H(k) + O((\ln s)^{-\infty}).
\]

This relation extends the validity of its Kotikov-Lipatov analogue [19] (valid at the leading order $\ln s$) to all finite orders $(\ln s)^{-n}$, $n \in \mathbb{N}$. A statement of this type has been already noted by [12] in connection with $f(g, j)$ only [12] and their equations, which have a different form with respect to our (2.11), though.

\[\text{For instance, we verified this for } f^{(0)}(g, j), \text{ albeit we do not show it here.}\]
Now, a very crucial point of our construction enters the stage. The key equation (2.11) concerning the all loop density $\sigma_H(u)$ and also the conditions (2.12) show up a deep, even conceptual, difficulty to be solved for generic values of the parameters $g$ and $j$ (or $s$, $L$). Yet, finding the solution becomes easier if we think of it and of the internal holes boundary $c(j)$ as an expansion in (non-negative) powers of $j$

$$
\sigma_H(u) = \left[ \sum_{n=0}^{\infty} \sigma_H^{(n)}(u)j^n \right] \ln s, \quad c(j) = \sum_{n=1}^{\infty} c^{(n)}j^n.
$$

For in this way the equation (2.11) breaks down into a recursive linear system of integral equations for the densities $\sigma_H^{(n)}$, each one yielding $f_n(g)$ according to (2.14):

$$
f_n(g) = \frac{1}{\pi} \lim_{k \to 0} \hat{\sigma}_H^{(n)}(k).
$$

Importantly, each $\sigma_H^{(n)}$ is governed by a Fredholm (II type) linear integral equation with always the same kernel (indeed the BES one [7]), but a different inhomogeneous term. This contains a part equal for any $n$ and given by the one loop density (and an additional known function) and another part which instead varies, but is still expressible - after using the conditions (2.12) - in terms of only $\sigma_H^{(m)}$, with $m \leq n-3$. Therefore, a recursive solution procedure can be contrived, at least in principle. In [14], the first step of the recursive structure has already appeared, without any reference to the BES density $\sigma_H^{(0)}$, and allowed us to analyse $f_1(g)$. From a physical point of view, the expansion in $j$ means that we are focusing our attention on the regime $j \ll g$, with fixed coupling $g$. Unfortunately, semi-classical string results concern a different regime where $g$ is very large and then $j$ scales, instead, accordingly, i.e. $j \sim g$ (in fact, the scaling variable $y = j/g$ is fixed) [8]. Nevertheless, our regime is suitable for comparison with the string reduction to the $O(6)$ NLSM which happens for $j \ll g$ as brilliantly conjectured by Alday and Maldacena on a geometric ground [10]. These authors have also checked their statement by using the perturbative renormalisation procedure in the UV regime, where the mass-gap $m(g)$ is much smaller than the energy density $j$, i.e. $j \gg m(g)$. On the contrary, our kind of expansion obliges us to consider $j$ small and then to consider the other NLSM regime, the IR one $j \ll m(g)$. In this case, the perturbative field theory methods are much less effective, but fortunately the Bethe Ansatz formulation, based on the $S$-matrix, furnishes easily the necessary data still in a systematic way [15].

The first check of the mass gap arising has been confirmed for the first time in [14] by computing $f_1(g)$, while it could not be shown exactly in the strong coupling behaviour of the cusp anomalous dimension $f_0(g) = f(g)$ because of the UV cut-off effects on the NLSM vacuum energy. Instead, $f_1(g)$ gives rise to the simplest manifestation of the mass-gap and the other scaling functions $f_n(g)$ – as we are going to see – present other different functions (of $g$), which all coincide with the mass-gap in the strong coupling regime.
In specific, we will illustrate explicitly our general construction in the following section giving as prototypical example the first fully general case, i.e. that concerning \( f_4(g) \). As a matter of fact, this is the lowest order at which this computation is not reducible to the knowledge of the BES density \( \sigma_H^{(0)} \) (at least by us and by now), as it shows up the explicit appearance of the higher order density \( \sigma_H^{(1)} \), in the inhomogeneous term. Curiously, it is also the lowest order at which the comparison with the \( O(6) \) NLSM becomes constraining, since right at this order the specific interaction starts appearing. In other words, up to the order \( j^3 \), the strong coupling expansion obtained for the \( O(6) \) NLSM only relies on the non-interacting Fermi gas theory, which has no information about the interaction and is in some sense ”universal”, being, in particular, the same for any \( N \) of the \( O(N) \). In other words, the exact leading order of \( f_n(g) \) – as it comes out from the gauge theory – is in perfect agreement with the corresponding one as worked out in [15] within the \( O(6) \) NLSM. In the following, we have decided to illustrate the computation up to the first general example, \( f_4(g) \), which teaches us the general procedure. Nevertheless, all the details and important subtleties about the latter need to be given extensively in a separate publication; for this reason we have simplified the illustration and have limited ourselves to an effective numerical presentation of the leading non-perturbative terms, which are responsible for the mass-gap. Yet, the different physical origin of the \( f_n(g) \) will clearly prove that the complete NLSM allows (with its unique mass-gap) only for their leading terms, though giving rise to an impressive exact agreement.

3 On the calculation of the generalised scaling functions: a sketch

In the present section we will sketch how to compute the coefficients of the expansion (2.15), so that, by means of the equality (2.14), we can gain all the generalised scaling functions \( f_n(g) \). In this respect, computing the first orders in the series for \( c(j) \) (2.15) is instructive to see how the procedure works and will turn out useful for expanding the subsequent forcing terms (cf. sub-section 3.3). The computation goes along the similar lines of the one loop theory and yields

\[
\begin{align*}
    c^{(1)} &= \frac{\pi}{4 - \sigma_H^{(0)}(0)}, \\
    c^{(2)} &= -\frac{4\ln 2 - \sigma_H^{(1)}(0)}{[4 - \sigma_H^{(0)}(0)]^2}.
\end{align*}
\] (3.1)

3.1 The second generalised scaling function

We now show that \( \sigma_H^{(2)}(u) = 0 \), so that we obviously have \( f_2(g) = 0 \). Let us consider the r.h.s. of (2.2). The first term is clearly proportional to \( j \ln s \), so it does not appear in the equation for \( \sigma_H^{(2)}(u) \). The second and the fifth term both have the form, with two
different functions $f(v)$,

$$\int_{-\infty}^{+\infty} dv f(v) \sigma(v) \chi_d(v) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{f}(k) \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \tilde{\sigma}(p) 2 \frac{\sin(k - p)d}{k - p}, \quad (3.2)$$

where $\sigma, d$ stand for $\sigma_0 + \sigma_H, c$ if we consider the second term and for $\sigma_0^*, c_0$ if we consider the fifth term. Using the normalization condition

$$2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \tilde{\sigma}(p) \frac{\sin pd}{p} = -2\pi L + O((\ln s)^{-\infty}), \quad (3.3)$$

one can show that

$$2 \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \tilde{\sigma}(p) \frac{\sin(k - p)d}{k - p} = [-2\pi j + O(d^3)] \ln s. \quad (3.4)$$

Since $d$ starts from order $j$ in its expansion, the second and the fifth term in the r.h.s. of (2.9) lack of the order $j^2 \ln s$ terms in their expansion. The same reasoning, applied to the second term in the rhs of (2.5) - the one containing the integral - implies that also this term lacks of the order $j^2 \ln s$. Therefore, the third term in the r.h.s. of (2.9) is missing the quadratic order as well. It follows that the equation for $\sigma_H^{(2)}(u)$ is

$$\sigma_H^{(2)}(u) = \int_{-\infty}^{+\infty} \frac{dv}{\pi} \frac{1}{1 + (u - v)^2} \sigma_H^{(2)}(v) \quad (3.5)$$

whose solution is, of course, $\sigma_H^{(2)}(u) = 0$. Therefore $f_2(g) = 0$.

### 3.2 The third generalised scaling function

In this sub-section, we concentrate our attention on the equation for $\sigma_H^{(3)}(u)$. We find convenient to pass to the Fourier transforms of all the involved quantities. Let us define the even function

$$s^{(3)}(k) = \frac{2 \sinh \frac{|k|}{2}}{2\pi |k|} \tilde{\sigma}_H^{(3)}(k), \quad (3.6)$$

which has the property

$$\lim_{k \to 0} s^{(3)}(k) = \frac{1}{2} f_3(g). \quad (3.7)$$

For future convenience, we will study the new function

$$S^{(3)}(k) = s^{(3)}(k) - \frac{\pi^2}{6} |k| e^{-\frac{|k|}{2}} \left[ \frac{1}{16} - \frac{1}{[\sigma^{(0)}(0)]^2} \right], \quad (3.8)$$

which depends on the all loops density of roots at (the zero-th) order $\ln s$ (satisfying the BES equation [7]) at $u = 0$ (real space)

$$\sigma^{(0)}(0) = -4 + \sigma_H^{(0)}(0), \quad (3.9)$$
(being the zero-th one-loop density in zero \( \sigma_0(0) = -4 \)) and still keeps the key property \(^{(2.16)}\) in the form

\[
\lim_{k \to 0} S_3^{(3)}(k) = \frac{1}{2} f_3(g). \tag{3.10}
\]

Let us also introduce the functions

\[
A_2^{(3)}(g) = \frac{\pi^2 r}{6[-4 + \sigma_H^{(0)}(0)]^2} \int_0^{+\infty} dh \frac{J_r(\sqrt{2}gh)}{\sinh \frac{h}{2}}, \tag{3.11}
\]

and restrict our analysis within the domain \( k \geq 0 \). Upon expanding in series of Bessel functions (the so-called Neumann’s expansion) as

\[
S_3^{(3)}(k) = \sum_{p=1}^{\infty} S_2^{(3)}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} S_{2p-1}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k}, \tag{3.12}
\]

introducing the quantities\(^{13}\)

\[
Z_{n,m}(g) = \int_0^{\infty} dh \frac{J_n(\sqrt{2}gh)J_m(\sqrt{2}gh)}{e^h - 1}, \tag{3.13}
\]

we finally obtain the following infinite system of linear equations:

\[
S_2^{(3)}(g) = A_2^{(3)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(3)}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(3)}(g)
\]

\[
S_{2p-1}(g) = A_{2p-1}^{(3)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}^{(3)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S_{2m-1}^{(3)}(g). \tag{3.14}
\]

The Neumann’s decomposition \(^{(3.12)}\) reveals itself useful also thanks to the special rôle of the first component in the key relation

\[
f_3(g) = \sqrt{2}g S_1^{(3)}(g), \tag{3.15}
\]

easily derived from \(^{(3.10)}\).

As a first approach, if we forget the non-perturbative contributions around \( g = +\infty \)\(^{14}\), we may seek for the solution to the equations \(^{(3.14)}\) as an asymptotic series

\[
S_{2m}^{(3)}(g) = \sum_{n=1}^{\infty} S_{2m}^{(3,2n)} \frac{1}{2^{2n}}, \quad S_{2m-1}^{(3)}(g) = \sum_{n=2}^{\infty} S_{2m-1}^{(3,2n-1)} \frac{1}{2^{2n-1}}. \tag{3.16}
\]

\(^{13}\)Their appearance comes out, also but not only, from the coefficients of the dressing phase \( e_{r,s}(g) = 2 \cos \left[ \frac{\pi}{4} (s-r-1) \right] (r-1)(s-1)Z_{r-1,s-1}(g) \) as in \([7]\).

\(^{14}\)Sometimes these are improperly called non-analytic, in the sense that they have this character, although the following series is (as a consequence!) not convergent at all.
The first two coefficients of (3.16) at the orders $1/g$ and $1/g^2$ have the same structure as the corresponding ones in the case of the first constituent $f_1(g)$ [14]. Therefore, we are naturally led to make the same Ansatz on the form of the coefficients entering (3.16):

$$S^{(3,2n)}_{2m} = 2m \frac{\Gamma(m+n)}{\Gamma(m-n+1)} (-1)^{1+n} b^{(3)}_{2n}, \quad n \geq 1, \ m \geq 1,$$

$$S^{(3,2n-1)}_{2m-1} = (2m-1) \frac{\Gamma(m+n-1)}{\Gamma(m-n+1)} (-1)^n b^{(3)}_{2n-1}, \quad n \geq 2, \ m \geq 2. \quad (3.17)$$

Analogously to the case of the first constituent, this Ansatz implies that $S^{(3,2n-1)}_{2m-1}$ and $S^{(3,2n)}_{2m}$ are different from zero only if $n \leq m$. The coefficients $b^{(3)}_n$ are unknown and they are determined by inserting the Ansatz (3.16, 3.17) into the equations (3.14).

In this way, implementing the elegant asymptotic expansion

$$A_r^{(3)}(g) = \frac{\pi^2 r}{6[\sigma^{(0)}(0)]^2} \sum_{k=0}^{\infty} \frac{2^{k-r} - 2^{k-\frac{1}{2}}}{(2k)!} B_{2k} \frac{\Gamma \left( \frac{1}{2} + \frac{r}{2} + k \right)}{\Gamma \left( \frac{1}{2} + \frac{r}{2} - k \right)} \frac{1}{g^{1+2k}},$$

we obtain that the coefficients $b^{(3)}_n$ satisfy the following (infinite) linear system

$$b^{(3)}_{2n} = \sum_{m=0}^{n-1} (-1)^{m} 2^{m+\frac{1}{2}} \frac{B_{2m}}{(2m)!} b^{(3)}_{2n-2m+1}, \quad n \geq 1,$$

$$b^{(3)}_{2n+1} = \frac{\pi^2}{6[\sigma^{(0)}(0)]^2} 4 (-1)^n \sum_{m=0}^{n} (-1)^{m+\frac{1}{2}} \frac{B_{2m}}{(2m)!} b^{(3)}_{2n-2m+2}, \quad n \geq 1. \quad (3.19)$$

Again, the solution to these equations comes from the comparison with the corresponding ones regarding the first generalised scaling function [14], in the form of the simple mapping

$$b^{(3)}_n = \frac{\pi^2}{6[\sigma^{(0)}(0)]^2} 2b^{(1)}_{n-2}, \quad n \geq 2.$$  

(3.20)

Still the generating function

$$b^{(3)}(t) = \sum_{n=2}^{\infty} b^{(3)}_n t^n,$$  

(3.21)

is simple, though, namely

$$b^{(3)}(t) = \frac{\pi^2}{6[\sigma^{(0)}(0)]^2} \frac{2t^2}{\sqrt{2}} \cos \frac{t}{\sqrt{2}} - \sin \frac{t}{\sqrt{2}}.$$  

(3.22)

This concludes what concerns the asymptotic part of the study about $f_3(g)$. Instead, the non-perturbative (or non-analytic) terms will be studied in subsection 3.4 (in comparison with those for $f_4(g)$), as they give naturally rise to the limiting mass-gap.
3.3 On the fourth and the other generalised scaling functions

Thanks to the lesson of the previous sub-section, we are now in the position to formulate a general scheme for computing the \( n \)-th generalised scaling function \( f_n(g) \) for \( n \geq 2 \) at arbitrary value of the coupling constant. In particular, we can show how the first computation, which is not reducible to the leading BES equation, is exactly that for \( f_4(g) \). Nevertheless, we will show how the computation for \( f_4(g) \) may be reduced to that for \( f_3(g) \), via a proportionality factor. The latter turns out to be the only part which is not reducible to the BES equation. Actually, similar situations would appear for the other scaling functions, as may be guessed from the results of this sub-section.

In the limit (1.3) the scaling may be thought of as governed by the function \( S(k) \), defined through

\[
S(k) \ln s = \frac{2 \sinh \frac{|k|}{2}}{2\pi |k|} [\hat{\sigma}_H(k) + \hat{\sigma}_0(k)] + \frac{e^{-\frac{|k|}{2}}}{\pi |k|} \int_{-\infty}^{\infty} \frac{dp}{2\pi} [\hat{\sigma}_0(p) + \hat{\sigma}_H(p)] \frac{\sin(k-p)c}{k-p},
\]

(3.23)

and with the crucial expansion

\[
S(k) = \sum_{n=0}^{\infty} S^{(n)}(k) j^n.
\]

(3.24)

Let us now concentrate our attention on \( S^{(n)}(k) \), with \( n \geq 2 \), since the case \( n = 1 \) is slightly different from the generic one (as shown in (14)). Restricting the domain to \( k \geq 0 \), we expand again in Neumann’s series (of Bessel functions)

\[
S^{(n)}(k) = \sum_{p=1}^{\infty} S^{(n)}_{2p}(g) \frac{J_{2p}(\sqrt{2}gk)}{k} + \sum_{p=1}^{\infty} S^{(n)}_{2p-1}(g) \frac{J_{2p-1}(\sqrt{2}gk)}{k},
\]

(3.25)

and find the following equations for the coefficients of \( S^{(n)}(k) \), with \( n \geq 2 \),

\[
S^{(n)}_{2p}(g) = \ A^{(n)}_{2p}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S^{(n)}_{2m}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S^{(n)}_{2m-1}(g)
\]

\[
S^{(n)}_{2p-1}(g) = \ A^{(n)}_{2p-1}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S^{(n)}_{2m}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S^{(n)}_{2m-1}(g)
\]

(3.26)

The forcing terms \( A^{(n)}_r \) are given by the following integrals

\[
A^{(n)}_r = r \int_0^{\infty} \frac{dh}{2\pi \hbar} \frac{J_r(\sqrt{2}gh)}{\sinh \frac{\hbar}{2}} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\sin(h-p)c}{h-p} [\hat{\sigma}_0(p) + \hat{\sigma}_H(p)] |_{j^n},
\]

(3.27)

where the symbol \( |_{j^n} \) means that we wish to keep only the coefficient of \( j^n \) and also neglect its overall factor \( \ln s \).
Thanks to (2.16) and also by making use of (2.12), we can generalise (3.10) to
\[
\lim_{k \to 0} S^{(n)}(k) = \frac{1}{2} f_n(g).
\] (3.28)

As before (cf. (3.15)), only the first component enters the expression for the generalised constituent
\[
f_n(g) = \sqrt{2} g S_1^{(n)}(g).
\] (3.29)

From the relations (2.12, 3.1), we can gain the relevant power series expansion (here limited at the order \( j^4 \) for simplicity’s sake)
\[
\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{\sin(h-p)c}{h-p} \left[ \hat{\sigma}_0(p) + \hat{\sigma}_H(p) \right] =
\left[ -2\pi j + \frac{1}{3} \left(-4 + \sigma_H^{(0)}(0)\right)^2 \right] h^2 j^3 \ln s.
\] (3.30)

From direct inspection of the second and third term in the r.h.s. of (3.30), we easily realise the proportionality between \( f_4(g) \) and \( f_3(g) \):
\[
f_4(g) = -2 \left[ -4 \ln 2 + \sigma_H^{(1)}(0) \right] f_3(g),
\] (3.31)

which allows us to use the results about \( f_3 \), provided the non-BES pre-factor be computed. In fact, this pre-factor contains explicitly the first correction to the BES approach, i.e. \( \sigma_H^{(1)}(0) \), whose computation cannot apparently be derived from the BES equation.

In general, it is thus possible to systematically expand the solution (3.28) into the form (3.24-3.26), namely order by order in \( j \) up to the desired order. Albeit this procedure seems to be in principle straightforward, it contains many intrigued and intriguing details which are to be left for a dedicated publication. Therefore, we plan to analyse in the next section the non-analytic contributions in a numerical fashion (as we reckon it quite effective for the current presentation), leaving the analytic study about them for that publication.

4 The non-analytic contributions from SYM and the \( O(6) \) NLSM from string dual

Let us show how it is possible to make contact with the results of the \( O(6) \) NLSM \[15\], which appears in the dual string theory description, by analyzing the non-analytic exponential contributions to \( f_n(g) \).

\[\text{The term proportional to } j \text{ in (3.30) needs careful consideration since it receives additional contributions to produce the final inhomogeneous equation of [14].}\]
First of all, let us focus our attention on \( f_3(g) \). From the form of the inhomogeneous term (3.11) (given by the \( j^3 \) coefficient of (3.30)), we may think of introducing a reduced form of the equations (3.14)

\[
S_{2p,\text{red}}^{(3)}(g) = A_{2p,\text{red}}^{(3)}(g) - 4p \sum_{m=1}^{\infty} Z_{2p,2m}(g) S_{2m}^{(3),\text{red}}(g) + 4p \sum_{m=1}^{\infty} Z_{2p,2m-1}(g) S_{2m-1}^{(3),\text{red}}(g)
\]

\[
S_{2p-1,\text{red}}^{(3)}(g) = A_{2p-1,\text{red}}^{(3)}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m}(g) S_{2m}^{(3),\text{red}}(g) - 2(2p-1) \sum_{m=1}^{\infty} Z_{2p-1,2m-1}(g) S_{2m-1}^{(3),\text{red}}(g),
\]  

(4.1)

by defining a new inhomogeneous term

\[
A_{r,\text{red}}^{(3)}(g) = \frac{6 [-4 + \sigma_H^{(0)}(0)]^2}{\pi^2} A_r^{(3)}(g) = r \int_0^{+\infty} dh \frac{J_r(\sqrt{2gh})}{\sinh \frac{h}{2}},
\]  

(4.2)

which, thanks to linearity, easily entails

\[
f_3(g) = \frac{\pi^2}{6 [-4 + \sigma_H^{(0)}(0)]^2} f_3^{\text{red}}(g).
\]  

(4.3)

Actually, the reduced system may be introduced for any \( n \) of the systems (3.26) and their peculiarity will be explained in a future publication, as they admit a little involved mapping into the BES equation (only as far as the computation of the first component or \( f_n(g) \) is concerned). On the contrary, the mapping could not be found for the complete systems (3.26) (but, of course, for \( n = 1, 2, 3 \)). Thanks to the proportionality (3.31), the reduced part of \( f_4(g) \) is still a very simple matter, as it virtually coincides with \( f_3(g) \), but the prefactor with \( \sigma_H^{(1)}(0) \) could not be reformulated in terms of the BES equation: this makes \( f_4(g) \) the first (simple) general case.

Besides its analytical extra-value, the matrix inhomogeneous (linear) system (4.1) – indeed equivalent to the linear integral equation at this order – may be explored numerically at first quick instance, by truncating the dimension of the matrix kernel (and vectors). Naturally, this treatment turns out into the lines by [20] for the BES case, as it shares the same (matrix) kernel: this strategy has been already efficiently exploited in [14] for the computation of the first mass-gap, namely that related to \( f_1(g) \).

Thus, we obtain a new mass-gap behaviour with the leading form

\[
f_3^{\text{red}}(g) = k_2^{\text{fit}} g^{1/4} e^{-\sqrt{2}g}.
\]  

(4.4)

A fortiori, the numerical method [20] for the BES case in the Fourier space may be directly applied to obtain the real space BES estimate

\[
\sigma_H^{(0)}(0) = 4 + k_1^{\text{fit}} g^{1/4} e^{-\sqrt{2}g},
\]  

(4.5)

16
along with the best fit estimates for $k_1$ and $k_2$

$$k_1^{fit} = -7.1166 \pm 0.0005, \quad k_2^{fit} = 5.5896 \pm 0.0005. \quad (4.6)$$

For clarity’s sake, we should recall the exact mass-gap formula for the $O(6)$ NLSM via a convergent Taylor series of the ’t Hooft coupling around $1/g = 0$

$$m(g) = k g^{1/4} e^{-\sqrt{2} g} \left(1 + \frac{a_1}{g} + \frac{a_2}{g^2} + \ldots \right), \quad (4.7)$$

with the embedding pre-factor, $k$, fixed by the weak-coupling perturbation theory (upon comparing versus the one-loop string result [8]) by [10]

$$k = \frac{\sqrt[5]{8} \pi^{1/4}}{\Gamma(5/4)}. \quad (4.8)$$

Eventually, we can easily realise how strongly the following exact values for $k_1$, $k_2$ become suggested:

$$k_1 = -\pi k, \quad k_2 = \frac{\pi^2}{4} k. \quad (4.9)$$

To conclude the part on $f_3(g)$, we may write down its numerical expression at strong coupling

$$f_3(g) = \frac{\pi^2}{6} \left(0.110366 \pm 0.000089\right) g^{-1/4} e^{\sqrt{2} g}, \quad (4.10)$$

or, alternatively, making use of the guessed exact values for $k_1$ and $k_2$, the exact (strong coupling) value

$$f_3(g) = \frac{\pi^2}{24 m(g)}. \quad (4.11)$$

This is the same result as that derived by the exact expansion of the $O(6)$ NLSM energy in the strong regime $j \ll m(g)$ as presented in [15], though this physics is explained uniquely by the free theory.

Therefore, we ought to analyse in more detail the situation regarding $f_4(g)$ to gain some physical insight into this peculiar limit. For this purpose we shall evaluate the ratio $f_4(g)/f_3(g)$, as coming from (3.31), at large values of $g$.

Let us first compute analytically the strong coupling limit of $\sigma_H^{(1)}(0)$ (indeed, a non-vanishing constant, as we will see below). Simply using the definition of $S^{(1)}(k)$ as Neumann’s expansion with coefficients $S_m^{(1)}(g)$ [14], we obtain

$$\sigma_H^{(1)}(0) = \int_0^\infty dk \frac{k}{\sinh \frac{k}{2}} S^{(1)}(k) = \int_0^\infty dk \sum_{m=1}^\infty S_m^{(1)}(g) \frac{J_m(\sqrt{2} g k)}{\sinh \frac{k}{2}}. \quad (4.12)$$

\[16\] All the other coefficients of the series, i.e. $a_1, a_2, \ldots$ become fixed by the string UV embedding, in fact realised by the massive excitations. The latter are also responsible for the quenched exponential terms, $O(e^{-3 \sqrt{2} g k})$, allowed by the SYM theory, and instead forbidden by the NLSM.
As we have recently found the exact asymptotic solution (around $g = +\infty$) for the coefficients $S_m^{(1)}(g)$ \cite{14}, we only need to sum over the index $m$ \cite{14} so that

$$\sigma^{(1)}_H(0) = \int_0^\infty \frac{dk}{\sinh \frac{k}{2}} \left[ \sum_{n=1}^{\infty} b_{2n}^{(1)}(-1)^{n+1} \frac{k^{2n}}{2^n} + \sum_{n=1}^{\infty} b_{2n-1}^{(1)}(-1)^n \frac{k^{2n-1}}{2^n - \frac{1}{2}} \right]. \tag{4.13}$$

Eventually, we know from (4.4) of \cite{14} the generating function $b^{(1)}(k)$ and thus we may integrate to obtain at leading order the constant value

$$\sigma^{(1)}_H(0) = 2 \int_0^\infty dx \frac{1}{\sinh x} \left[ 1 - \frac{\cosh x}{\cosh 2x} - \frac{\sinh x}{\cosh 2x} \right] = 3 \ln 2 - \frac{\pi}{2}. \tag{4.14}$$

On the other hand, we have already obtained above (cf. the discussion concerning $f_3(g)$) the leading behaviour

$$-4 + \sigma^{(0)}_H(0) = -\pi m(g), \tag{4.15}$$

where $m(g)$ is exactly the $O(6)$ NLSM mass gap (the dependence on the coupling $g$ by the density values in zero is omitted). Therefore, we can conclude that

$$\frac{f_4(g)}{f_3(g)} = -\frac{2 \ln 2 + \pi}{\pi m(g)}, \tag{4.16}$$

which implies an exact relation for the first strong coupling term

$$f_4(g) = -\frac{\pi(2 \ln 2 + \pi)}{24 m^2(g)}. \tag{4.17}$$

This prediction is in remarkable agreement with the computation of the ground state energy of the $O(6)$ NLSM \cite{15} and extends the agreement between the latter model and the gauge theory up to the first order ($j^4$) where the interaction goes on the stage. On the contrary, the $O(6)$ NLSM approximation would no longer be applicable for the next (exponential) order, $O(e^{-3\sqrt{2}g})$, present in the SYM theory with all the subsequent ones, as it does not show up any terms of this form. In fact, all the exponential corrections should be generated by the massive (fermionic and bosonic) excitations of the dual string. Hence, due to their origin, these corrections should be different in the different $f_n(g)$: for the time being we can state the difference at the next-to-leading order among the mass terms coming from $f_1$, $f_3$ and $f_4$ in the gauge theory. A string theory verification of this departure from the NLSM regime would be desirable.

## 5 Summary

In this work we have initiated a general method for computing all the generalised scaling function $f_n(g)$ appearing in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM when both the spin $s$ and

\footnote{Not to be confused with the mass-gap $m(g)$.}
the twist $L$ are very large, while the ratio $j = L/\ln s$ stays fixed (namely the limit \[13\]); Our method relies on solving recursively the linear equation \[2.9\] for the higher loop density of Bethe roots $\sigma_H(u)$, which in its turn is completely determined by the analogous solution of the one loop linear equation for the density $\sigma_0(u)$. In particular, we have focused our attention on the third and the fourth constituents to disentangle the emergence, in the SYM theory, of different ‘mass’ terms, which all flow to the mass-gap of the $O(6)$ Non-Linear Sigma Model: comparison with the full string theory would be highly interesting, though still missing. Very peculiarly, the convergence of the ground state energy as a Taylor series in $j$ in the NLSM \[15\] strongly suggests how its extension to the full theory, $f(g, j) = \sum_{n=0}^{\infty} f_n(g)j^n$, should be convergent too. In conclusion, we have left apart some details about the analytic calculations and the general structure, bearing in mind a more complete and systematic study for the near future.

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