Global well-posedness of three-dimensional Navier–Stokes equations with partial viscosity under helical symmetry

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Abstract. In this paper, we investigate the global well-posedness of three-dimensional Navier–Stokes equations with horizontal viscosity under a special symmetric structure: helical symmetry. More precisely, by a revised Ladyzhenskaya-type inequality and utilizing the behavior of helical flows, we prove the global existence and uniqueness of weak and strong solutions to the three-dimensional helical flows. Our result reveals that for the issue of global well-posedness of the viscous helical flows, the horizontal viscosity plays the important role. To some extent, our work can be seen as a generalization of the result by Mahalov et al. (Arch Ration Mech Anal 112(3):193–222, 1990).

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1. Introduction and main results

In present paper, we are concerned with the three-dimensional Navier–Stokes equations with horizontal viscosity, which can be read as:

\[
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta_h \mathbf{u} + \nabla p &= 0, \\
\nabla \cdot \mathbf{u} &= 0,
\end{aligned}
\]

(1.1)

in a bounded domain \( D \subset \mathbb{R}^3 \), where \( \Delta_h = \partial^2_{x_1} + \partial^2_{x_2} \), \( \mathbf{u} = (u_1, u_2, u_3) \) represents the velocity fields, \( \nu > 0 \) is the kinematic viscosity and \( p \) is a scalar pressure. Models with a vanishing anisotropic viscosity in the vertical direction are of relevance for the study of turbulent flows in geophysics. Turbulence is the time-dependent chaotic behavior seen in many fluid flows. This motivates us to study the mathematical problems of fluid flows only with horizontal viscosity. Starting from Danchin and Païcu [6], who proved the global well-posedness for the two-dimensional Boussinesq equations only with horizontal viscosity or thermal diffusivity, the topic in this field has attracted considerable attention and great progress has been achieved. One can refer to [4,5,10,14,15,17] for details.

As we know, the global well-posedness of three-dimensional Navier–Stokes equations with partial viscosity is far from being resolved. Therefore, we intend to investigate the global well-posedness of solutions to (1.1) under a special symmetric case: helical symmetry. In particular, the flows with helical symmetry are so-called helical flows, i.e., the flows keep invariant under certain one-dimensional subgroups of the group of rigid transformations in \( \mathbb{R}^3 \). These subgroups are generated by a simultaneous rotation around a symmetry axis and a translation along the same symmetry axis. Namely, the subgroup \( G^\kappa \) is a one-parameter group of isometries of \( \mathbb{R}^3 \) as follow:

\[ G^\kappa = \{ S_\theta : \mathbb{R}^3 \rightarrow \mathbb{R}^3 | \theta \in \mathbb{R} \} \]

where \( S_\theta \) is the transformation defined by
Besides, we will assume \( \kappa \equiv 1 \) for the sake of simplicity throughout the rest paper. As a matter of fact, the transformation \( S_\theta \) corresponds to the superposition of a simultaneous rotation around the \( x_3 \)-axis and a translation along the same \( x_3 \)-axis. The symmetry lines (orbits of \( G^\kappa \)) are concentric helices. We call the solutions, and more general functions, which are invariant under \( G^\kappa \) as “helical”.

By recalling that (1.1) in a helical domain \( S \), the transformation

\[
\begin{align*}
S_\theta(x) = R_\theta(x) + \begin{pmatrix} 0 \\ 0 \\ \kappa \theta \end{pmatrix} = \begin{pmatrix} x_1 \cos \theta + x_2 \sin \theta \\ -x_1 \sin \theta + x_2 \cos \theta \\ x_3 + \kappa \theta \end{pmatrix}.
\end{align*}
\]

(1.2)

The nonzero constant \( \kappa \) denotes the length scale and \( R_\theta \) is the rotation matrix by an angle \( \theta \) around the \( x_3 \)-axis, i.e.,

\[
R_\theta = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

(1.3)

Besides, we will assume \( \kappa \equiv 1 \) for the sake of simplicity throughout the rest paper. As a matter of fact, the transformation \( S_\theta \) corresponds to the superposition of a simultaneous rotation around the \( x_3 \)-axis and a translation along the same \( x_3 \)-axis. The symmetry lines (orbits of \( G^\kappa \)) are concentric helices. We call the solutions, and more general functions, which are invariant under \( G^\kappa \) as “helical”.

Due to the special behavior of helical flows, we intend to discuss our problem in a so-called helical domain, which is invariant under the action of \( G^\kappa \), i.e.,

\[
S_\theta D = D \quad \forall \theta \in \mathbb{R}.
\]

By recalling that \( S_{2\pi} \) is a translation by \( 2\pi \) in the \( x_3 \)-direction, we obviously find that helical flows inherit a periodic boundary condition in the \( x_3 \)-direction. Specially, throughout this paper, we will study system (1.1) in a helical domain \( D = \{(x', x_3) \in \mathbb{R}^3 \mid |x'| < 1, 0 \leq x_3 < 2\pi \} = B_1 \times [0, 2\pi) \) with \( x' = (x_1, x_2) \) and implement the following initial-boundary condition to (1.1)

\[
\begin{align*}
\begin{cases}
  u(x', x_3, t) = u(x', x_3 + 2\pi, t), & x' \in B_1, \\
p(x', x_3, t) = p(x', x_3 + 2\pi, t), & x' \in B_1, \\
u(x', x_3, t) = 0, & |x'| = 1, x_3 \in [0, 2\pi), \\
u(x, 0) = u_0(x), & x \in D.
\end{cases}
\end{align*}
\]

(1.4)

As common in practice as axisymmetric flows, helical flows have attracted wide mathematical attention recently. In 1990, global existence and uniqueness of strong solutions to three-dimensional Navier–Stokes equations with helical symmetry have been obtained by Mahalov et al. [12] with helical initial data. For Euler equations with helical symmetry, when the helical swirl (which is similar to the quantity in the axisymmetric case) vanished, Dutrifoy [7] obtained the global existence and uniqueness of smooth solutions in bounded domain with regular initial data. Ettinger and Titi [8] derived the global existence and uniqueness of strong solutions in bounded domain for bounded initial vorticity. In addition, for weaker initial assumptions, the existence of weak solutions to Euler equations with helical symmetry has also been discussed in [2,9]. In [1], the authors proved the stability of weak solutions to the three-dimensional Navier–Stokes with helical initial data. In [11], when the helical parameter \( \kappa \) goes to infinity, the limiting property of the incompressible flows with helical symmetry has been investigated.

In this paper, we are devoted to studying the global existence, uniqueness and stability of weak and strong solutions to (1.1)–(1.4) with helical symmetry. In our case, on account of vertical smoothing effect vanishing in the equations (1.1), the viscosity term is not enough to control the nonlinear term in the energy estimates. To overcome this difficulty, inspired by the key observation of [12], we establish a revised version of Ladyzhenskaya-type inequality:

\[
\|u\|_{L^4(D)} \leq C \left[ \|u\|_{L^4(D)}^{\frac{1}{2}} \|\nabla u\|_{L^2(D)}^{\frac{1}{2}} \right],
\]

(1.5)

which gives us the cornerstone to obtain the global existence of weak solutions. However, to prove the stability of weak solution and global existence of strong solutions, we still have some difficulties in deriving the necessary estimate of \( \|\nabla u\|_{L^2(0,t; L^2(D))} \) (independent of time \( t \)) even with the help of (1.5). To get over this problem, we make full advantage of the helically symmetric structure of the flows. First, we establish the exponential decay estimates of velocity field. Then, thanks to a novel observation [see (3.8)
below, we set up the desired estimate of \( \| \nabla u \|_{L^2(0,t; L^2(D))} \), which is uniform with respect to the time variable. At the end, we derive the estimate of \( \| u \|_{L^\infty(0,t; H^1(D))} \), which helps us to resolve the problem.

Before showing the main theorems, we would like to introduce the spaces \( H(D) \) and \( V(D) \) be the closure of \( C^\infty \) vector fields which are periodic in the vertical variable, compactly supported in the horizontal variable. And, \( \mathcal{H}_{\text{ver}}(D) = \{ u \in L^2(D), \nabla_h u \in L^2(D) \} \) norms, respectively. On this basis, we then define the inner products of \( L^2(D) \) by \( (u,v) = \sum_{i=1}^3 \int_D u_i v_i \, dx \), and denote by \( V' \) the dual space of \( V \) and the action of \( V' \) on \( V \) by \( \langle \cdot, \cdot \rangle \). Moreover, we use the following notation for the trilinear continuous form by setting

\[
b(u,v,w) = \sum_{i,j=1}^3 \int_D u_i \partial_i v_j w_j \, dx. \tag{1.6}\]

If \( u \in V \), then

\[
b(u,v,w) = -b(v,u,w), \quad \forall v,w \in V(D), \tag{1.7}\]

and

\[
b(u,v,v) = 0, \quad \forall v \in V(D). \tag{1.8}\]

Finally, we will set up the definitions of weak and strong solutions of system (1.1) and (1.4) as below.

**Definition 1.1. (Weak solution)** Suppose \( u_0 \in H(D) \) be helically symmetric, the helical vector fields \( u(x,t) \) is called a global weak solution of (1.1) and (1.4) if for any \( t > 0 \),

\[
\| u(t,\cdot) \|_{L^2(D)}^2 + 2\nu \int_0^t \| \nabla_h u(\tau,\cdot) \|_{L^2(D)}^2 d\tau \leq \| u_0 \|_{L^2(D)}^2 \tag{1.9}\]

and

\[
\int_{D} u_0 \cdot \varphi_0 \, dx + \int_0^t \int_{D} \left[ u \cdot \varphi_t + u \cdot \nabla \varphi \cdot u - \nu \nabla_h u : \nabla_h \varphi \right] \, dx \, d\tau = 0 \tag{1.10}\]

holds for any helical vector fields \( \varphi \in C^\infty_c([0,t) \times D) \) with \( \nabla \cdot \varphi = 0 \), where \( A : B = \sum_{i,j} a_{ij} b_{ij} \) is the trace product of two matrices.

**Remark 1.1.** Following standard arguments as in the theory of the Navier–Stokes equations (see e.g., [16]), we mention that system (1.10) is equivalent to the following system

\[
\frac{d}{dt} < u, \varphi > + b(u, u, \varphi) + (\nabla_h u, \nabla_h \varphi) = 0, \quad \forall \varphi \in L^2(0,t; V(D)). \]

**Definition 1.2. (Strong solution)** Let Definition 1.1 be satisfied, if in addition, it holds that

\[
u \in L^\infty(0,t; H^1(D)), \quad \nabla_h u \in L^2(0,t; H^1(D)) \tag{1.11}\]

for any \( t > 0 \), then we call the corresponding solution as a global strong solution.

Now, we are in the position to state the main results of this paper.

**Theorem 1.1.** Suppose that \( u_0 \in H(D) \), then there exists a unique global weak solution \( u \in L^\infty(0,\infty; H(D)) \cap L^2(0,\infty; V(D)) \) to (1.1) and (1.4). Moreover, \( u \) satisfies the exponential decay rate

\[
\| u(\cdot,t) \|_{L^2(D)}^2 \leq e^{-2c_0 t} \| u_0 \|_{L^2(D)}^2, \quad \forall t \geq 0, \tag{1.12}\]

where \( c_0 \) is the constant in Poincaré inequality.

Motivated by the work of Bardos et al. in [1], we also observe a corresponding stability result.
Theorem 1.2. Given that \( u_0 \in H(D) \) a helical vector field, there exists
\[
u \in L^\infty(0, \infty; H(D)) \cap L^2(0, \infty; V(D))
\]
the weak solution of helical incompressible Navier–Stokes equations (1.1) and (1.4) with initial data \( u_0 \), given in Theorem 1.1. Moreover, let \( v_0 \in H(D) \) be a general vector field and
\[
v \in L^\infty(0, \infty; H(D)) \cap L^2(0, \infty; V(D))
\]
be a Leray–Hopf weak solution of three-dimensional incompressible Navier–Stokes equations (1.1) and (1.4) with initial data \( v_0 \). Then it holds that
\[
\| u - v \|^2_{L^2(D)(t)} \leq \| u_0 - v_0 \|^2_{L^2(D)} \exp \left[ \frac{2c^2 + c^*_2c_0}{2\nu^2} \| u_0 \|^2_{L^2(D)} \right]
\]
for any \( t \geq 0 \), where \( c_0 \) and \( c_* \) are the constants from Poincaré inequality and Lemma 2.4, respectively.

Moreover, for the strong solution of (1.1) and (1.4), we also have the similar argument as follows.

Theorem 1.3. Assume that \( u_0 \in V(D) \), then there exists a unique global strong solution \( u \in L^\infty(0, t; H^1(D)) \) to the system (1.1) and (1.4) in the sense of Definition 1.2.

This paper is organized as follows. In Sect. 2, we introduce some notations and technical lemmas. Section 3 is devoted to the a priori estimates and proof of main theorems.

2. Preliminary

In this section, we will fix some notations and set down some basic definitions. As discussed in the introduction, helical flows are invariant under certain one-dimensional subgroups. In particular, we will employ the following definition.

Definition 2.1. (Helical flow)

(i) A scalar function \( f: \mathbb{R}^3 \rightarrow \mathbb{R} \) is said to be helical if
\[
f(S_\theta(x)) = f(x), \quad \forall \theta \in \mathbb{R}.
\]
(ii) A vector field \( v: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is said to be helical if
\[
v(S_\theta(x)) = R_\theta v(x), \quad \forall \theta \in \mathbb{R}.
\]

Subsequently, we would like to introduce some important properties of helical flow. By setting \( \xi = (x_2, -x_1, 1)^\top \), the following lemmas hold.

Lemma 2.1. (Claim 2.5, [8]) A smooth vector field \( v = (v_1, v_2, v_3)^\top: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is helical if and only if the following relation holds true:
\[
\partial_\xi v = v^\perp,
\]
where \( v^\perp = (v_2, -v_1, 0)^\top \).

The following lemma tells that the helical flows can be essentially viewed as an extension of the two-dimensional one in some sense (see Proposition 2.1 of [11]), which makes it possible to improve the result of [2] to the two-dimensional case [13].

Lemma 2.2. Let \( u = u(x) \) be a smooth helical vector field and let \( p = p(x) \) be a smooth helical function, where \( x = (x_1, x_2, x_3) \). Then there exist unique \( w = (w^1, w^2, w^3)^\top = (w^1, w^2, w^3)(y_1, y_2) \) and \( q = q(y_1, y_2) \) such that
\[
u(x) = R_{x_3} w(y(x)), \quad p = p(x) = q(y(x)),
\]
with $R_{\theta}$ given in (1.3), and
\[
y(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 - \sin x_3 \\ \sin x_3 \cos x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.2}
\]

Conversely, if $u$ and $p$ are defined through (2.1) for some $w = w(y_1, y_2)$, then $u$ is a helical vector field and $p$ is a helical scalar function.

In the end, we will provide a revised version of Ladyzhenskaya-type inequality, which is given in [12]. Our version would be slightly different and plays an important role in the present paper.

Lemma 2.3. (Lemma 3.1, [12]) Let $u$ be a helical function in $H^1(D)$, then it follows that
\[
\|u\|_{L^4(D)} \leq C \left[ \|u\|_{L^2(D)}^2 \|\nabla_h u\|_{L^2(D)} + \|u\|_{L^2(D)} \right], \tag{2.3}
\]
where $\nabla_h = (\partial_{x_1}, \partial_{x_2})$. If in addition, $\|u\|_{|x'|=1} = 0$, (2.3) will be replaced by
\[
\|u\|_{L^4(D)} \leq C \|u\|_{L^2(D)} \|\nabla_h u\|_{L^2(D)}^{1/2}, \tag{2.4}
\]
where $C$ is the generic constant.

Proof. According to Lemma 2.2, there exists the corresponding vector field $w$ by virtue of (2.1). Since $R_{x_3}$ is an orthogonal matrix, it is not hard to deduce
\[
|u(x_1, x_2, x_3)|^2 = |w(y_1, y_2)|^2,
\]
which also yields that for any $x_3 \in [0, 2\pi),
\[
\|u(x_3)\|_{L^2(B_1)} = \|w\|_{L^2(B_1)} = \frac{1}{\sqrt{2\pi}} \|u\|_{L^2(D)}. \tag{2.5}
\]
Then by using the two-dimensional Sobolev embedding inequality, we can derive that
\[
\|u\|_{L^4(D)}^4 = 2\pi \|w\|_{L^2(B_1)}^2 \leq C \left[ \|w\|_{L^2(B_1)}^2 \|\nabla_y w\|_{L^2(B_1)}^2 + \|w\|_{L^2(B_1)}^2 \right]. \tag{2.6}
\]

Thus, to prove (2.3), it suffices to find out the relation between $\|\nabla_y w\|_{L^2(B_1)}$ and $\|\nabla_h u\|_{L^2(D)}$. First of all, we can rewrite (2.1) as
\[
w(y) = R_{x_3}^{-1} u(x(y)), \tag{2.7}
\]
where
\[
R_{x_3}^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2.8}
\]
and
\[
x(y) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos x_3 & \sin x_3 \\ -\sin x_3 & \cos x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{2.9}
\]
Then, according to (2.7)–(2.9), each component of $\nabla_y w$ can be expressed a composition of the components of $\partial_{x_1} u$, $\partial_{x_3} u$ and the trigonometric functions about $x_3$. Without loss of generality, we take the expression of $\partial_{y_1} w_1$ for instance, i.e.,
\[
\partial_{y_1} w_1 = \partial_{x_1} u_1 \cos^2 x_3 - \partial_{x_3} u_2 \sin x_3 - \partial_{x_2} u_1 \cos x_3 \sin x_3 + \partial_{x_2} u_2 \sin^2 x_3.
\]
Because $\partial_{y_1} w_1$ is independent of $y_3$, it is clear that
\[
\|\partial_{y_1} w_1\|_{L^2(B_1)} = \frac{1}{\sqrt{2\pi}} \|\partial_{y_1} w_1\|_{L^2(D)} \leq \|\nabla_h u\|_{L^2(D)}.
\]
Similar estimates also hold for other components of $\nabla_y w$. Therefore, by summing up all these estimates, we can derive that
\[
\|\nabla_y w\|_{L^2(B_1)} \leq C \|\nabla_h u\|_{L^2(D)}. \tag{2.10}
\]
In the end, by inserting (2.5) and (2.10) into (2.6), we can finish the proof of (2.3). As for the proof (2.4), it suffices to employ the Poincaré inequality.

\[\Box\]

Lemma 2.4. Let \( u = u(x) \in H^1(D) \) be a smooth helical vector field, then there exists an absolute constant \( c_* \) such that

\[
\sup_{x_3 \in [0,2\pi)} \| \nabla u(x_3) \|_{L^2(B_1)} \leq c_* \| u \|_{H^1(D)}. \tag{2.11}
\]

Proof. Initially, thanks to (2.1), (2.2) and some direct computations, each component of \( \nabla_y u \) can be expressed as a composition of the components of \( \partial_{x_3} w, \partial_{x_2} w \) and the trigonometric functions about \( x_3 \). Then, by similar methods as the proof of (2.10), it is easy to deduce that for any \( x_3 \in [0,2\pi) \),

\[
\| \nabla_h u(x_3) \|_{L^2(B_1)} \leq C \| \nabla_y w \|_{L^2(B_1)} \leq C \| \nabla_h u \|_{L^2(D)}. \tag{2.12}
\]

Now, it suffices to prove (2.11). Thanks to the helical property of Lemma 2.1, it is clear that for any \( x_3 \in [0,2\pi) \),

\[
\| \partial_{x_3} u(x_3) \|_{L^2(B_1)} \leq \| \nabla_h u(x_3) \|_{L^2(B_1)} + \| u(x_3) \|_{L^2(B_1)},
\]

which further implies, after applying (2.5) and (2.12), that for any \( x_3 \in [0,2\pi) \),

\[
\| \partial_{x_3} u(x_3) \|_{L^2(B_1)} \leq \| \nabla_h u \|_{L^2(D)} + \| u \|_{L^2(D)}. \tag{2.13}
\]

Finally, by summing up (2.12), (2.13) and taking supremum with respect to \( x_3 \in [0,2\pi) \), we can then find a constant \( c_* \) such that

\[
\sup_{x_3 \in [0,2\pi)} \| \nabla u(x_3) \|_{L^2(B_1)} \leq c_* \| u \|_{H^1(D)},
\]

\[\Box\]

3. Global well-posedness

3.1. A priori estimates

In this subsection, we will establish the a priori estimates of velocity fields. At first, we will list the basic energy estimate with decay rate.

Lemma 3.1. Suppose \( u_0 \in L^2(D) \) be a helical function with \( \nabla \cdot u_0 = 0 \), then for a helical smooth solution \( u \) of (1.1) and any \( t \geq 0 \), there holds that

\[
\| u(\cdot,t) \|_{L^2(D)}^2 \leq e^{-\frac{2\nu t}{c_0}} \| u_0 \|_{L^2(D)}^2 \tag{3.1}
\]

and

\[
\nu \int_0^t e^{\frac{2\nu t}{c_0}} \| \nabla_h u(\tau) \|_{L^2(D)}^2 d\tau \leq \| u_0 \|_{L^2(D)}^2 \tag{3.2}
\]

where \( c_0 \) is the constant in Poincaré inequality.

Proof. Taking inner product of (1.1) with \( u \), and then integrating over \( D \), it follows that

\[
\frac{d}{dt} \| u \|_{L^2(D)}^2 + 2\nu \| \nabla_h u \|_{L^2(D)}^2 \leq 0. \tag{3.3}
\]

Moreover, by noticing that \( u|_{|x'|=1} = 0 \) and implying Poincaré inequality, we have

\[
\| u \|_{L^2(D)}^2 \leq c_0 \| \nabla_h u \|_{L^2(D)}^2 \tag{3.4}
\]
for some constant $c_0$. Then, we can rewrite (3.3) as

$$\frac{d}{dt} \|u\|_{L^2(D)}^2 + \frac{2\nu}{c_0} \|u\|_{L^2(D)}^2 \leq 0,$$

which yields

$$\|u(\cdot, t)\|_{L^2(D)}^2 \leq e^{-\frac{2\nu}{c_0} t} \|u_0\|_{L^2(D)}^2, \quad \forall t \geq 0. \quad (3.5)$$

Then, by multiplying (3.4) with $e^{\frac{\nu}{c_0} t}$ and using (3.5), we have

$$\frac{1}{2} \frac{d}{dt} (e^{\frac{\nu}{c_0} t} \|u\|_{L^2(D)}^2) + \nu e^{\frac{\nu}{c_0} t} \|\nabla_h u\|_{L^2(D)}^2 \leq \frac{\nu}{2c_0} e^{\frac{\nu}{c_0} t} \|u_0\|_{L^2(D)}^2,$$

which also yields, after integrating in time over $[0, t]$, that

$$\nu \int_0^t e^{\frac{\nu}{c_0} \tau} \|\nabla_h u(\tau)\|_{L^2(D)}^2 d\tau \leq \|u_0\|_{L^2(D)}^2. \quad (3.6)$$

Next, we make an attempt to derive the $H^1$ estimate of $u$, especially the uniform bound of $\|\nabla u\|_{L^2(0,t;L^2(D))}$. Noticing that there is no vertical smoothing effect in the system (1.1), we only have the estimate of $\|\nabla_h u\|_{L^2(0,t;L^2(D))}$ other than $\|\nabla u\|_{L^2(0,t;L^2(D))}$ in (3.2). This gives rise to the difficulty in deriving the estimate of $\|u\|_{L^2(0,t;H^1(D))}$, even with the help of revising Ladyzhenskaya-type inequality. To overcome it, we make full advantage of the helically symmetric structure, which yields the following conclusion.

**Corollary 3.1.** Let $u_0$ be as in Lemma 3.1, then for a smooth helical solution $u$ of (1.1) and any $t \geq 0$, there holds

$$\int_0^t \|\nabla u\|_{L^2(D)}^2 d\tau \leq \frac{4 + c_0}{2\nu} \|u_0\|_{L^2(D)}^2, \quad (3.7)$$

where $c_0$ is the constant in Poincaré inequality.

**Proof.** Thanks to Lemma 2.1, for any helical flow, it holds that

$$\partial_{x_3} u = x_2 \partial_{x_1} u - x_1 \partial_{x_2} u + u^\perp. \quad (3.8)$$

Then, we notice that $D$ is bounded by 1 in $x_1$ and $x_2$ direction. Thus, by applying (3.5) and (3.6), we have
\[
\int_0^t \| \nabla u \|^2_{L^2(D)} d\tau \leq \int_0^t \| \partial_x^3 u \|^2_{L^2(D)} d\tau + \int_0^t \| \nabla_h u \|^2_{L^2(D)} d\tau
\]
\[
\leq \int_0^t \| u \|^2_{L^2(D)} d\tau + 2 \int_0^t \| \nabla_h u \|^2_{L^2(D)} d\tau
\]
\[
\leq \left[ \int_0^\tau e^{-\frac{2c_0}{\nu} \tau} d\tau + 2\nu^{-1} \right] \| u_0 \|^2_{L^2(D)}
\]
\[
\leq \frac{4 + c_0}{2\nu} \| u_0 \|^2_{L^2(D)}.
\]

On the basis of Corollary 3.1, we have derived the estimate \( \| \nabla u \|_{L^2(0,t;L^2(D))} \) independent of \( t \). This gives us a good cornerstone to estimate \( \| u \|_{L^\infty(0,t;H^1(D))} \), which is necessary to prove the existence of strong solutions (Theorem 1.3).

**Lemma 3.2.** Under the assumptions of Theorem 1.3, for a smooth helical solution \( u \) of (1.1), there holds

\[
\| u \|^2_{H^1(D)} + \nu \int_0^t \| \nabla_h u \|^2_{H^1(D)} d\tau \leq C,
\]

where the constant \( C \) depends only on \( t \).

**Proof.** Taking inner product of (1.1) with \( -\Delta u \) on \( D \), integrating by parts, employing the boundary condition (1.4) and applying H"older inequalities, Lemma 2.3, we have

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(D)} + \nu \| \Delta_h u \|^2_{L^2(D)} + \nu \| \partial_x^3 \nabla_h u \|^2_{L^2(D)} = - \int_D u \cdot \nabla u \cdot \Delta u dx
\]
\[
= \int_D \sum_{i=x_1,x_2,x_3} u \cdot \nabla \partial_i u \cdot \partial_i u dx + \int_D \sum_{i=x_1,x_2,x_3} \partial_i u \cdot \nabla \cdot \partial_i u dx
\]
\[
\leq C \| \nabla u \|_{L^4(D)} \| \nabla u \|_{L^2(D)} + \| \partial_x^3 \nabla_h u \|_{L^2(D)} + C \| \nabla u \|^2_{L^2(D)}.
\]

Then by using the elliptic theory and boundary condition (1.4) again, one can derive that \( \| \nabla_h^2 u \|_{L^2(D)} \leq C \| \Delta_h u \|_{L^2(D)} \). Thanks to it and Young inequalities, one has

\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2(D)} + \nu \| \Delta_h u \|^2_{L^2(D)} + \nu \| \partial_x^3 \nabla_h u \|^2_{L^2(D)} \leq C \| \nabla u \|_{L^2(D)} \| \Delta_h u \|_{L^2(D)} + \| \partial_x^3 \nabla_h u \|_{L^2(D)} + C \| \nabla u \|^2_{L^2(D)}
\]
\[
\leq \nu \left[ \| \Delta_h u \|^2_{L^2(D)} + \| \partial_x^3 \nabla_h u \|^2_{L^2(D)} \right] + C \left[ 1 + \| \nabla u \|^2_{L^2(D)} \right],
\]

which implies the conclusion after applying Gronwall’s inequality and Corollary 3.1. \( \square \)
3.2. Proof of Theorem 1.1

The proof will be divided into two steps, that is, **existence and uniqueness.**

**Existence:** It is well known that there exists at least a weak solution of (1.1) provided that the initial velocity belongs to $H(D)$. The proof is standard (see e.g. [16]), and we only list a sketch here. For $\epsilon > 0$, let $j$ be a positive radial compactly supported smooth function whose integral equals 1 and denote $J_\epsilon$ as a Friedrichs mollifier by

$$J_\epsilon u = j_\epsilon \ast u,$$

where $j_\epsilon = \epsilon^{-2} j(\epsilon^{-1} x)$. (3.10)

Moreover, let $\mathcal{P}$ be the Leray projector over divergence free vector fields, then the following properties hold

$$J_\epsilon^2 = J_\epsilon, \quad \mathcal{P}^2 = \mathcal{P}, \quad \mathcal{P} J_\epsilon = J_\epsilon \mathcal{P}.$$ (3.11)

Thanks to (3.11), we can then construct a family of approximating solutions through the following system

$$\begin{cases}
\partial_t u_\epsilon + \mathcal{P} J_\epsilon (J_\epsilon u_\epsilon \cdot \nabla J_\epsilon u_\epsilon) - \nu \Delta_h J_\epsilon u_\epsilon = 0, \\
\nabla \cdot u_\epsilon = 0, \\
u_\epsilon(x, 0) = J_\epsilon u_0.
\end{cases} \quad (3.12)$$

For system (3.12), by the Picard theorem (see e.g., [13]), there exists a unique solution $u_\epsilon$. Similar to the standard method, we can prove that $\{u_\epsilon\}$ is a Cauchy sequence for any $\epsilon > 0$. Furthermore, by Proposition 1.1 in [13], for the Euler and Navier–Stokes equations, the transformations **Galilean invariance, Rotation symmetry, Scale invariance** also yield solutions. This shows that $\{u_\epsilon\}$ also preserves the helical symmetry. Finally, it is not hard to prove that the limit $u$ of sequence $\{u_\epsilon\}$ satisfies (1.1) in the distribution sense as $\epsilon$ tends to zero.

**Uniqueness** For any fixed $t > 0$, suppose there are two solutions $(u, p), (\tilde{u}, \tilde{p})$ of (1.1) and let $U = \tilde{u} - u$, $P = \tilde{p} - p$, then by Remark 1.1, it holds that

$$\frac{d}{dt} \langle U, \varphi \rangle + b(\tilde{u}, U, \varphi) + b(U, u, \varphi) + (\nabla_h U, \nabla_h \varphi) = 0,$$ (3.13)

$$U(x, 0) = 0, \quad (3.14)$$

for any $\varphi \in L^2(0, t; V(D))$.

Subsequently, by taking $\varphi = U$ in (3.13), making use of (1.8) and Lions–Magenes lemma (see e.g., [16]), we have

$$\frac{1}{2} \frac{d}{dt} \|U\|^2_{L^2(D)} + \|\nabla_h U\|^2_{L^2(D)} \leq - \int_D U \cdot \nabla u \cdot U \, dx,$$

$$\leq C \|U\|^2_{L^2(D)} \|\nabla u\|_{L^2(D)}$$

$$\leq C \|U\|_{L^2(D)} \|\nabla_h U\|_{L^2(D)} \|\nabla u\|_{L^2(D)}$$

$$\leq \frac{1}{2} \|\nabla_h U\|^2_{L^2(D)} + C \|\nabla u\|^2_{L^2(D)} \|U\|^2_{L^2(D)}.$$ (3.15)

In the end, by employing Corollary 3.1 and Gronwall’s inequality, it yields that

$$\|U(\tau)\|^2_{L^2(D)} \leq C \|U_0\|^2_{L^2(D)} = 0$$

for any $\tau \in [0, t]$. Thus, the proof is finished. \qed
3.3. Proof of Theorem 1.2

To begin with, for any fixed \( t > 0 \) and \( j_\epsilon \) [defined in (3.10)], we define the notation \( f^\epsilon \) as below

\[
f^\epsilon = f^\epsilon(t, x) = \int_0^t j_\epsilon(t - \tau)f(\tau, x)d\tau.
\]

Then, we choose \( \nu \) and \( u^\epsilon \) as the test function in the weak formulation for \( u \) and \( v \) respectively. Furthermore, it follows that

\[
-(u, v^\epsilon)(t) + \int_0^t (u, \partial_t v^\epsilon)d\tau - \nu \int_0^t (\nabla_h u, \nabla_h v^\epsilon)d\tau
\]

\[
= -\int_0^t (u \cdot \nabla v^\epsilon, u)d\tau - (u_0, v^\epsilon_0)
\]

(3.16)

and

\[
-(v, u^\epsilon)(t) + \int_0^t (v, \partial_t u^\epsilon)d\tau - \nu \int_0^t (\nabla_h v, \nabla_h u^\epsilon)d\tau
\]

\[
= -\int_0^t (v \cdot \nabla u^\epsilon, v)d\tau - (v_0, u^\epsilon_0).
\]

(3.17)

Then, by adding up (3.16) and (3.17) and applying the equality

\[
\int_0^t (u, \partial_t v^\epsilon)d\tau = -\int_0^t (v, \partial_t u^\epsilon)d\tau,
\]

one has

\[
-(u, v^\epsilon)(t) - (v, u^\epsilon)(t) - \nu \int_0^t [(\nabla_h u, \nabla_h v^\epsilon) + (\nabla_h v, \nabla_h u^\epsilon)]d\tau
\]

\[
= -\int_0^t [(u \cdot \nabla v^\epsilon, u) + (v \cdot \nabla u^\epsilon, v)]d\tau - (u_0, v^\epsilon_0) - (v_0, u^\epsilon_0),
\]

which implies, after letting \( \epsilon \to 0 \), that

\[
-2(u, v)(t) - 2\nu \int_0^t (\nabla_h u, \nabla_h v)d\tau
\]

\[
= \int_0^t ((v - u) \cdot \nabla(v - u), u)d\tau - 2(u_0, v_0).
\]

(3.18)

Subsequently, by summing up (3.18) and energy inequalities (1.9) for \( u \) and \( v \), there holds that

\[
\|g\|_{L^2(D)}^2(t) + \nu \int_0^t \|\nabla_h g\|_{L^2(D)}^2d\tau \leq \|g_0\|_{L^2(D)}^2 + \int_0^t (g \cdot \nabla g, u)d\tau,
\]

(3.19)
where \( g = v - u \). Until now, it suffices to deal with the nonlinear term in (3.19). In general, it should be a hard term. Initially, by integrating by parts, using the two-dimensional Ladyzhenskaya inequality in \( B_1 \) and Young inequality, one has

\[
\begin{aligned}
\int_0^t (g \cdot \nabla g, u) d\tau &= -\int_0^t (g \cdot \nabla u, g) d\tau \\
&\leq \int_0^t \int_0^{2\pi} \|g\|^2_{L^2(B_1)} \|
abla u\|_{L^2(B_1)} dx_3 d\tau \\
&\leq \sqrt{2} \int_0^t \int_0^{2\pi} \|g\|_{L^2(B_1)} \|\nabla h g\|_{L^2(B_1)} \|
abla u\|_{L^2(B_1)} dx_3 d\tau \\
&\leq \nu \int_0^t \|
abla h g\|^2_{L^2(D)} d\tau + \frac{1}{2\nu} \int_0^t \int_0^{2\pi} \|g\|^2_{L^2(B_1)} \|
abla u\|^2_{L^2(B_1)} dx_3 d\tau.
\end{aligned}
\] (3.20)

Hence, by using Lemma 2.4, we can update (3.20) as

\[
\int_0^t (g \cdot \nabla g, u) d\tau \leq \nu \int_0^t \|
abla h g\|^2_{L^2(D)} d\tau + \frac{c_2^*}{2\nu} \int_0^t \|g\|^2_{L^2(D)} \|u\|^2_{H^1(D)} d\tau,
\]

which together with Gronwall’s inequality, Lemma 3.1 and Corollary 3.1 yield that

\[
\|g\|^2_{L^2(D)}(t) \leq \|g_0\|^2_{L^2(D)} \exp\left[\frac{c_2^*}{2\nu} \int_0^t \|u\|^2_{H^1(D)} d\tau\right]
\leq \|g_0\|^2_{L^2(D)} \exp\left[\frac{2c_2^* + c_2^* c_0}{2\nu^2} \|u_0\|^2_{L^2(D)}\right].
\]

This completes the proof of Theorem 1.2. \( \square \)

3.4. Proof of Theorem 1.3

With the help of Theorem 1.1, to prove Theorem 1.3, it suffices to verify that \( u \in L^\infty(0, t; H^1(D)) \) and \( \nabla h u \in L^2(0, t; H^1(D)) \) hold for any \( t > 0 \). By applying Lemma 3.2, one can achieve the conclusion. \( \square \)

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