BACKWARD COMPACTNESS AND PERIODICITY OF RANDOM ATTRACTORS FOR STOCHASTIC WAVE EQUATIONS WITH VARYING COEFFICIENTS

RENhai WANG AND YANGrong LI

School of Mathematics and Statistics
Southwest University
Chongqing 400715, China

Dedicated to Professor Peter Kloeden on his 70th birthday

(Communicated by Tomas Caraballo)

Abstract. We study some time-related properties of the random attractor for the stochastic wave equation on an unbounded domain with time-varying coefficient and force. We assume that the coefficient is bounded and the time-dependent force is backward tempered, backward complement-small, backward tail-small, and then prove both existence and backward compactness of a random attractor on the universe of all backward tempered sets. By using the Egoroff and Lusin theorems, we show the measurability of the absorbing set although it is the union of some random sets over an uncountable index set. Moreover, we obtain the backward compactness of the attractor if the force is periodic, and obtain the periodicity of the attractor if both force and coefficient are periodic.

1. Introduction. We focus on the time-related property of pullback random attractors for stochastic wave equations with time-varying coefficients and forces:

\[
\begin{aligned}
&\frac{du_t}{dt} + \beta(t)u_t - \Delta u_t + \lambda u_t + f(t, x, u)dt = g(t, x)dt + \epsilon u \circ dW, \quad t \geq \tau, \\
u(\tau, x) = u_{0}(x), &\quad u_{t}(\tau, x) = u_{1}(x), \quad x \in \mathbb{R}^{3},
\end{aligned}
\]

where \( \lambda > 0, \epsilon > 0, \beta \in C(\mathbb{R}, (0, \infty)) \) and \( g \in L_{loc}^{2}(\mathbb{R}, L^{2}(\mathbb{R}^{3})). \)

The existence of an attractor for the wave equation had been investigated in various special cases. For examples, the deterministic case (\( \epsilon = 0 \)) was discussed in [1, 5, 6, 15, 29, 30], the stochastic case was studied in [14, 32, 34, 35, 36, 39, 40]. Comparing with (1), the nearest equation was given by Wang [33], in which \( \beta(t) \) is a constant.

In this paper, we investigate not only the existence of a pullback random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau, \omega) \} \) for the cocycle \( \Phi \) generated by the equation (1), but also some time-related properties of the attractor due to the nature of the non-autonomous equation.

2010 Mathematics Subject Classification. Primary: 37L55; Secondary: 35B40, 60H15.

Key words and phrases. Stochastic wave equation, random attractor, backward compactness, periodicity, varying coefficient, measurability, backward tempered universe.

Li was supported by Natural Science Foundation of China grant 11571283, Wang was supported by Postgraduate Research and Innovation Project of Chongqing grant CYB18115.

* Corresponding author: liyr@swu.edu.cn (Yangrong Li).
In particular, we wonder whether the attractor $\mathcal{A}$ is \textit{backward compact}, which means that $\bigcup_{s \leq t} \mathcal{A}(s, \omega)$ is pre-compact for each time $t \in \mathbb{R}$ and sample $\omega \in \Omega$. Such a backward compactness illustrates that the system has more concentrate attraction ability in the past.

In the deterministic case, some abstract criteria on backward compactness of a pullback attractor had been established by [7, 25, 37]. Other time-related properties of a deterministic attractor had been considered in [12, 13, 16, 17, 18, 19, 24].

These abstract results can be partly generalized to the random case. There are some different contexts, such as, variety of the sample, the time-dependence of the universe (instead of the universe of all bounded sets). However, an essential criterion for backward compactness of a random attractor still seems to be the \textit{backward asymptotic compactness} of the cocycle on an attracted universe, which means that the usual asymptotic compactness is uniform in the past.

If we take the attracted universe $\mathfrak{D}_0$ by all of tempered sets as usual (see [2, 9, 20, 21, 28]), we cannot prove that the $\mathfrak{D}_0$-pullback asymptotic compactness is uniform in the past.

In this paper, we take a new sub-universe $\mathfrak{D}$, which consists of all \textit{backward tempered sets} instead of the usual tempered sets.

We then assume that the external force $g$ is \textit{backward tempered} (Hypothesis G1). In this case, we can prove that there is a $\mathfrak{D}$-pullback absorbing set $\mathcal{K}$ such that the absorption is uniform in the past. The measurability of $\mathcal{K}$ is not obvious due to it is the union of some random sets over an uncountable index set. This difficulty will be overcome by using Egoroff and Lusin theorems (see Proposition 2).

By using the backward-uniform absorption of $\mathcal{K}$, we can establish backward flattening property of the cocycle $\Phi$ inside any bounded sub-domain if $g$ is backward complement-small (Hypothesis G2). Note that the usual complement-small property automatically holds true by using the Lebesgue controlled convergence theorem. However, the complement-small property may not be uniform in the past.

Also, the backward-uniform absorption allows us to establish the backward-uniform tail-estimates if $g$ is further assumed to be backward tail-small (Hypothesis G3). Therefore, we can prove the $\mathfrak{D}$-pullback backward asymptotic compactness of the cocycle and thus obtain a backward compact random attractor (see Theorem 6.1).

Another issue in this paper is to establish backward compactness of the attractor from a periodicity assumption only. We can prove an abstract result that a periodic random attractor is backward compact if and only if it is local compact. A random attractor may not be locally compact (see [10]), although a deterministic attractor is automatically locally compact (see [27]).

However, we can show the periodicity condition can imply three conditions G1, G2, G3. Therefore, we can obtain backward compactness of the random attractor if we only assume $g$ is periodic. Also, we obtain the periodicity of the attractor if both $\beta$ and $g$ are periodic.

2. The cocycle from the stochastic wave equation. Let $\hat{v} = u_t + \delta u$ for a suitable positive number $\delta > 0$. Then, the equation (1) can be rewritten as a system of two first-order equations:

\begin{align}
\dot{u}_t &= -\delta u + \hat{v}, \\
\dot{\hat{v}} + (\lambda + \delta^2 - \delta \beta(t))u dt - \Delta u dt &= g(t, x) dt + \varepsilon u \circ dW.
\end{align}
As usual [4], we identify the Wiener process $W(\cdot, \omega)$ with the standard process $\omega(\cdot)$ on the metric dynamical system $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$, where

$$
\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0, \quad \lim_{t \to \pm \infty} \frac{w(t)}{t} = 0 \}
$$
equipped with the Frechét metric: for $\omega_1, \omega_2 \in \Omega$,

$$
\rho(\omega_1, \omega_2) := \sum_{i=1}^{\infty} \frac{1}{2^n} \frac{\|\omega_1 - \omega_2\|_n}{1 + \|\omega_1 - \omega_2\|_n}, \quad \|\omega_1 - \omega_2\|_n := \sup_{-n \leq t \leq n} |\omega_1(t) - \omega_2(t)|,
$$

$\mathcal{F}$ is the Borel $\sigma$-algebra on $(\Omega, \rho)$, $P$ is the two-sided Wiener measure on $(\Omega, \mathcal{F})$ (in fact, $P$ can be taken by an arbitrary probability measure as given in [22]), and $\{\theta_t\}_{t \in \mathbb{R}}$ is a group defined by $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t)$ for $(\omega, t) \in \Omega \times \mathbb{R}$. Let

$$
z(\theta_t \omega) := -\delta \int_{-\infty}^{0} e^{\delta s}(\theta_t \omega)(s) ds,
$$

which is a pathwise continuous solution of the Ornstein-Uhlenbeck equation $dz + \delta z dt = dW(t)$. Let

$$
v(t, \omega) := \dot{v}(t, \omega) - \varepsilon z(\theta_t \omega) u(t, \omega), \quad t \geq \tau, \quad \omega \in \Omega.
$$

Then, the system (2a)-(2b) can be read as

$$
u_t = -\delta - \varepsilon z(\theta_t \omega) u + v, \quad (3a)
$$

$$
u_t + (\lambda + \delta^2 - \delta \beta(t)) u - \Delta u + f(x, u) + (\beta(t) - \delta) v
$$
$$
= g(t, x) - \varepsilon z(\theta_t \omega) v + \varepsilon z(\theta_t \omega)(3\delta - \beta(t) - \varepsilon z(\theta_t \omega)) u, \quad (3b)
$$

$$
u(\tau) = u_\tau := u_0, \quad v(\tau) = v_\tau := u_1 + \delta u_0 - \varepsilon z(\theta_\tau \omega) u_0. \quad (3c)
$$

2.1. An energy inequality. We need some assumptions.

Hypothesis B. The varying coefficient $\beta \in C(\mathbb{R}, \mathbb{R})$ satisfies that

$$
0 < \beta_1 \leq \beta(t) \leq \beta_2 < +\infty, \quad \forall t \in \mathbb{R}.
$$

Hypothesis F. $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies the $p$-th growth conditions:

$$
|f(x, s)| \leq \gamma_1|s|^p + h_1(x), \quad h_1 \in L^2(\mathbb{R}^3), \quad (4a)
$$

$$
f(x, s) \geq \gamma_2 F(x, s) + h_2(x), \quad h_2 \in L^1(\mathbb{R}^3), \quad (4b)
$$

$$
F(x, s) \geq \gamma_3|s|^{p+1} - h_3(x), \quad h_3 \in L^1(\mathbb{R}^3), \quad (4c)
$$

$$
|f_s(x, s)| \leq \gamma_4|s|^{p-1} + h_4(x), \quad h_4 \in L^6(\mathbb{R}^3), \quad (4d)
$$

where $F(x, s) := \int_0^s f(x, \sigma) d\sigma, \ \gamma_i > 0 \ (i = 1, 2, 3, 4)$ and $p \in [1, 3)$. The growth rate $p \in [1, 3)$ will be used in Proposition 3.

We take a $\delta > 0$ such that

$$
\delta_1 := \lambda + \delta^2 - \sqrt{\delta}/2 > 0, \quad \delta_2 := \beta_1 - \beta_2 - \beta_2^2 \sqrt{\delta}/2 > 0.
$$

Such a $\delta$ exists because $\delta^2 + \delta + \beta_2^2 \sqrt{\delta}/2 \to 0$ as $\delta \to 0$. So,

$$
\kappa_1 := \min \left\{ \frac{\delta \delta_1}{\lambda + \delta_2}, \ \delta, \ \delta_2, \ \delta_2 \sqrt{\gamma_2} \right\} > 0,
$$

$$
\kappa_2 := \max \left\{ 2(3\delta + \beta_2 + 2), \ \frac{2(3\delta + \beta_2 + 2)}{\lambda + \delta^2}, \ \frac{\gamma_1}{\gamma_3} \right\} < \infty.
$$
We then establish an energy inequality on the state space $X = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ equipped with the norm:

$$\|\varphi\|_X^2 = (\lambda + \delta^2)\|u\|^2 + \|\nabla u\|^2 + \|v\|^2, \quad \forall \varphi = (u, v) \in X.$$ 

**Lemma 2.1.** Let the hypotheses $B$, $F$ be satisfied, $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3))$ and $\varepsilon \in (0, 1)$. Then, the solution $\varphi = (u, v)$ of (3a)-(3c) satisfies

$$\frac{d}{dt} \left(\|\varphi\|_X^2 + 2 \int_{\mathbb{R}^3} F(x, u) dx \right) + (\kappa_1 - \varepsilon \kappa_2 Z(\theta_t \omega))(\|\varphi\|_X^2 + 2 \int_{\mathbb{R}^3} F(x, u) dx)
\leq c\varepsilon |z(\theta_t \omega)| + c\|g(t)\|^2. \quad (5)$$

where $Z(\theta_t \omega) = |z(\theta_t \omega)| + |z(\theta_t \omega)|^2$.

**Proof.** We take the inner product of Eq.(3a) with $v$ to see

$$\frac{d}{dt} \|v\|^2 + 2(\lambda + \delta^2)(u, v) - 2(\Delta u, v) + 2(f(x, u), v) + 2(\beta(t) - \delta)\|v\|^2
= 2\varepsilon \beta(t)(u, v) + 2(g(t), v) - 2\varepsilon z\|v\|^2 + 2\varepsilon z(3\delta - \beta(t) - \varepsilon z)(u, v). \quad (6)$$

By the products of Eq.(3a) with $u$, $\Delta u$, $f(x, u)$ respectively,

$$2(u, v) = 2(u, u_t + \delta u - \varepsilon zu) = \frac{d}{dt} \|u\|^2 + 2(\delta - \varepsilon z)\|u\|^2, \quad (7a)$$

$$-2(\Delta u, v) = -2(\Delta u, u_t + \delta u - \varepsilon zu) = \frac{d}{dt} \|\nabla u\|^2 + 2(\delta - \varepsilon z)\|\nabla u\|^2, \quad (7b)$$

$$2(f(x, u), v) = 2 \frac{d}{dt} \int_{\mathbb{R}^3} F(x, u) dx + 2(\delta - \varepsilon z)(f(x, u), u). \quad (7c)$$

Hence, the equality (6) can be rewritten as

$$\frac{d}{dt} \left(\|\varphi\|_X^2 + 2 \int_{\mathbb{R}^3} F(x, u) dx \right) + 2(\delta - \varepsilon z)(f(x, u), u) + I_1 + I_2 = I_3 + I_4 + I_5. \quad (8)$$

where, by (7a)-(7b) and $\kappa_2 \geq 4$,

$$I_1 := 2(\delta - \varepsilon z)\left((\lambda + \delta^2)\|u\|^2 + \|\nabla u\|^2\right)
\geq 2\delta(\lambda + \delta^2)\|u\|^2 + 2\delta\|\nabla u\|^2 - \frac{\kappa_2}{2} \varepsilon |z|\|\varphi\|^2_X.$$ 

By the hypothesis $B$,

$$I_2 := 2(\beta(t) - \delta)\|v\|^2 \geq 2(\beta_1 - \delta)\|v\|^2.$$ 

By the hypothesis $B$ and the Young inequality,

$$I_3 := 2\delta \beta(t)(u, v) \leq 2\delta \beta_2\|u\|\|v\| \leq \delta \sqrt{\beta}\|u\|^2 + \sqrt{\beta} \beta_2^2\|v\|^2.$$ 

So, by the definition of $\kappa_1$, we have

$$I_1 + I_2 - I_3 \geq 2(\delta \delta_1 \|u\|^2 + \delta \|\nabla u\|^2 + \delta_2 \|v\|^2) - \frac{\kappa_2}{2} \varepsilon Z\|\varphi\|^2_X
\geq 2\kappa_1 \|\varphi\|^2_X - \frac{\kappa_2}{2} \varepsilon Z\|\varphi\|^2_X.$$ 

By the Young inequality,

$$I_4 := 2(g(t), v) \leq 2\|v\|\|g(t)\| \leq \frac{\kappa_1}{2} \|\varphi\|^2_X + c\|g(t)\|^2.$$
By \( \varepsilon \in (0, 1] \) and the Young inequality,
\[
I_5 := -2\varepsilon z|v|^2 + 2\varepsilon z(3\delta - \beta(t) - \varepsilon z)(u, v)
\]
\[
\leq (2 + 3\delta + \beta_2)\varepsilon(|z| + |z|^2)(|u|^2 + |v|^2) \leq \frac{K^2}{2}\varepsilon Z\|\varphi\|_X^2.
\]
We obtain
\[
I_1 + I_2 - I_3 - I_4 - I_5 \geq \frac{3K_1}{2}\|\varphi\|_X^2 - K_2\varepsilon Z(\theta_1, \omega)\|\varphi\|_X^2 - c(\theta(t)) \geq 0.
\]
Finally, we deal with the nonlinear term in (8): 2\( \delta(f(x, u), u) - 2\varepsilon z(f(x, u), u) \).
By (4b), (4c) and \( \delta \gamma_2 \geq \kappa_1 \) and \( F(x, u) + h(x) \geq 0 \),
\[
2\delta(f(x, u), u) \geq 2\delta \gamma_2 \int_{\mathbb{R}^3} F(x, u)dx + 2\delta \int_{\mathbb{R}^3} h_2(x)dx.
\]
By (4a), (4c) and \( \gamma_1 \leq \kappa_2 \gamma_3 \),
\[
2\varepsilon|z(f(x, u), u)| \leq 2\varepsilon \gamma_1 |z| \int_{\mathbb{R}^3} |u|^{p+1}dx + 2\varepsilon |z||h_1||u||
\]
\[
\leq 2\kappa_2\varepsilon(|z| + |z|^2) \int_{\mathbb{R}^3} \gamma_3 |u|^{p+1}dx + c\varepsilon |z||u||
\]
\[
\leq 2\kappa_2\varepsilon Z \int_{\mathbb{R}^3} F(x, u)dx + 2\kappa_2\varepsilon(|z| + |z|^2)||h_3||1 + c\varepsilon |z||\varphi||X
\]
\[
\leq \frac{K_1}{2}\|\varphi\|_X^2 + 2\kappa_2\varepsilon Z \int_{\mathbb{R}^3} F(x, u)dx + c(|z| + |z|^2).
\]
We substitute (9)-(11) into (8) to obtain the energy inequality (5), where we have used the inequality \( 1 + |z| + |z|^2 \leq 2e^{2|z|} \).

2.2. Measurability of the solution operator in sample. As usual, one can show the well-posed property by using the energy inequality (5).

**Lemma 2.2.** For each \((\tau, \omega) \in \mathbb{R} \times \Omega\) and \( \varphi_\tau = (u_\tau, v_\tau) \in X \), the problem (3a)-(3c) has a unique solution
\[
\varphi(\cdot, \tau, \omega, \varphi_\tau) = (u(\cdot, \tau, \omega, u_\tau), v(\cdot, \tau, \omega, v_\tau)) \in C([\tau, \infty), X)
\]
such that it continuously depends on the initial data \( \varphi_\tau \) in \( X \).

Next, we consider the measurability of the mapping \( \omega \rightarrow \varphi(t, \tau, \omega, \varphi_\tau) \). By \( \omega(t)/t \rightarrow 0 \) as \( t \rightarrow \pm \infty \), it is easy to show \( \Omega = \bigcup_{N \in \mathbb{N}} \Omega_N \), where
\[
\Omega_N := \{ \omega \in \Omega : |\omega(t)| \leq N e^{t/2}, \forall t \in \mathbb{R} \}, \forall N \in \mathbb{N}.
\]
By [8, Corollary 22], we know that \( \omega \rightarrow z(\theta(t)) \) is continuous on each \((\Omega_N, \rho)\) and thus Lusin-continuous on \((\Omega, \rho)\) due to \( P(\Omega_N) \rightarrow 1 \) as \( N \rightarrow \infty \). More precisely,

**Lemma 2.3.** [8]. Let \( \omega_k, \omega_0 \in \Omega_N \) such that \( \rho(\omega_k, \omega_0) \rightarrow 0 \) as \( k \rightarrow \infty \). Then,
\[
\sup_{t \in [\tau, \tau + T]} |z(\theta(t)\omega_k) - z(\theta(t)\omega_0)| \rightarrow 0 \text{ as } k \rightarrow \infty,
\]
\[
\sup_{k \in \mathbb{N}} \sup_{t \in [\tau, \tau + T]} |z(\theta(t)\omega_k) + z(\theta(t)\omega_0)| \leq C = C(\tau, T, N, \omega_0).
\]
Hypothesis S. The density
\[ \rho(s, \tau, \omega) \] given in Proposition 3 and 4.

Proposition 1. For each \( N \in \mathbb{N} \), the mapping \( \omega \mapsto \varphi(t, \tau, \omega, \varphi_\tau) \) is continuous from \( (\Omega_N, \rho) \) to \( X \), uniformly in \( t \in [\tau, \tau + T] \) with \( T > 0 \).

By Lemma 2.2, we can define a mapping \( \Phi: \mathbb{R}^+ \times \Omega \times X \to X \) by
\[ \Phi(t, \tau, \omega)\varphi_\tau = \varphi(t + \tau, \theta_{-\tau} \omega, \varphi_\tau), \quad (t, \tau, \omega, \varphi_\tau) \in \mathbb{R}^+ \times \Omega \times X. \]
which is continuous in \( t \) and \( \varphi_\tau \) respectively. By Proposition 1, \( \Phi(t, \tau, \omega)\varphi_\tau \) is measurable in \( \omega \). The uniqueness of solution implies the cocycle property: \( \Phi(0, \tau, \omega) = I \),
\[ \Phi(t + s, \tau, \omega) = \Phi(t, \tau + s, \theta_s \omega) \Phi(s, \tau, \omega) \] for all \( t, s \in \mathbb{R}^+ \).

So, \( \Phi \) is a continuous cocycle in the sense of Wang [31].

We need to give the assumption of small noise.

Hypothesis S. The density \( \varepsilon \) of noise is small: \( \varepsilon \in (0, \varepsilon_0] \), where
\[ \varepsilon_0 := \min \left\{ 1, \frac{\kappa_1}{16\kappa_2(\frac{2}{\sqrt{\pi \delta}} + \frac{1}{\delta})} \right\}, \]
and the number \( 1/16 \) will be used to prove that the absorbing set \( K \in \mathcal{D} \).

By [3, 4, 11], we know
\[ \lim_{t \to \pm \infty} \frac{z(\theta_s \omega)}{t} = \lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{0} z(\theta_s \omega) ds = 0, \]
\[ \lim_{t \to \pm \infty} \frac{1}{t} \int_{-t}^{0} |z(\theta_s \omega)|^m ds = \frac{\Gamma(1+m)}{\sqrt{\pi} \delta^m}, \quad \forall m > 0, \]
where \( \Gamma \) is the Gamma function. We need some further properties of
\[ Z(\theta_s \omega) = |z(\theta_s \omega)|^2. \]

Lemma 2.4. Let the hypothesis S be true. Then, for each \( \omega \in \Omega \), there are \( T_0(\omega) > 0 \) and \( C_0(\omega) > 0 \) such that
\[ \sup_{\varepsilon \in (0, \varepsilon_0]} \varepsilon \kappa_2 \int_{-t}^{0} Z(\theta_s \omega) ds \leq \frac{1}{16} \kappa_1 t, \quad \forall t \geq T_0(\omega), \]
\[ \sup_{\varepsilon \in (0, \varepsilon_0]} \varepsilon \kappa_2 \int_{-t}^{0} Z(\theta_s \omega) ds \leq \frac{1}{16} \kappa_1 t + C_0(\omega), \quad \forall t \geq 0. \]
By taking $C_0(\omega) = \epsilon_0 e_{2} \int_{-\tau_0}^{0} Z(\theta_s \omega) d\sigma$, we obtain (19). \hfill \Box

3. Backward-uniform absorption. In order to obtain a backward-uniformly absorbing set, we need to give an assumption on the force.

**Hypothesis G1.** $g \in L_{loc}^2(\mathbb{R}, L^2(\mathbb{R}^3))$ is **backward tempered**, more precisely, there are $\alpha_0 > 0$ and $\tau_0 \in \mathbb{R}$ such that

$$G_{\alpha_0}(\tau_0) := \sup_{s \leq \tau, \sigma > 0} \int_{-\infty}^{0} e^{\alpha_0 \sigma} \|g(\sigma + s)\|^2 d\sigma < \infty. \quad (20)$$

In fact, the growth rate $\alpha_0$ can be arbitrary as given in the following lemma, the proof can be found in [26].

**Lemma 3.1.** $G_{\alpha_0}(\tau_0) < +\infty$ for some $\alpha_0 > 0$ and $\tau_0 \in \mathbb{R}$ if and only if $G_{\alpha}(\tau) < +\infty$ for all $\alpha > 0$ and $\tau \in \mathbb{R}$.

**Remark 1.** In general, if $\sigma \rightarrow \|g(\sigma)\|$ is increasing, then $g$ is backward tempered. In the last section, we will show a periodic locally integrable force is backward tempered.

**Lemma 3.2.** Let the hypotheses B, F, S be satisfied and the force $g$ satisfy G1. Then, for each $D \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is a $T := (D, \tau, \omega) > 0$ such that

$$\sup_{s \leq \tau} \sup_{\sigma > 0} \|\varphi(s, s - t, \theta_{-s} \omega, \varphi_{s-t})\|^2_X \leq c R_0(\omega) + c R_1(\tau, \omega), \quad (21)$$

uniformly in $\varphi_{s-t} \in D(s - t, \theta_{-s} \omega)$, where

$$R_0(\omega) := 1 + \int_{-\infty}^{0} e^{\epsilon_1 \sigma + |z(\theta_s \omega)|} e_{0} e_{2} \int_{0}^{\infty} Z(\theta_s \omega) d\sigma d\tau, \quad (22)$$

$$R_1(\tau, \omega) := \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\epsilon_1 \sigma + c_{\alpha_2} e_{0} e_{2}} \|g(\sigma + s)\|^2 d\sigma. \quad (23)$$

**Proof.** Let $s \leq \tau$ and $t \geq 0$. By the energy inequality (5), the function $r \rightarrow \varphi(r, s - t, \theta_{-s} \omega, \varphi_{s-t})$ $(r \geq s - t)$ satisfies

$$\frac{d}{dr} \left( \|\varphi\|^2_X + 2 \int F(x, u) dx \right) + (\kappa_1 - \epsilon \kappa_2 Z(\theta_{s-t} \omega) \left( \|\varphi\|^2_X + 2 \int F(x, u) dx \right) \leq ce^{\epsilon_1 (\theta_{-s} \omega)} + c \|g(r)\|^2. \quad (24)$$

By applying the Gronwall inequality to (24) over $[s - t, r]$, we obtain

$$\|\varphi(r, s - t, \theta_{-s} \omega, \varphi_{s-t})\|^2_X + 2 \int F(x, u(r, s - t, \theta_{-s} \omega, u_{s-t})) dx \leq e^{-\kappa_1(r-s+t) + \epsilon \kappa_2 \int_{-s}^{r-s} Z(\theta_s \omega) d\sigma} \left( \|\varphi_{s-t}\|^2_X + 2 \int F(x, u_{s-t}) dx \right)$$

$$+ c \int_{-t}^{r-s} e^{\kappa_1 \sigma + \epsilon \kappa_2 \int_{-s}^{\sigma - t} Z(\theta_s \omega) d\sigma} \|g(\sigma + s)\|^2 d\sigma.$$

$$+ c \int_{-t}^{r-s} e^{\kappa_1 \sigma + \epsilon \kappa_2 \int_{-s}^{\sigma - t} Z(\theta_s \omega) d\sigma} \|g(\sigma + s)\|^2 d\sigma.$$
By (4c), \( F(x,u) \geq -h_3(x) \) and so \( \int F(x,u)dx \geq -c \). Thereby,

\[
\|\varphi(r,s-t,\theta_{-s}\omega,\varphi_{s-t})\|_X^2 \\
\leq e^{-\kappa_1(r-s-t)+\varepsilon\kappa_2 r_{\sigma-t}^{-1-t}} Z(\theta_{s}\omega) \int_{\mathbb{R}^3} F(x,u_{s-t})dx \\
+ c \int_{-t}^{r-s} e^{\kappa_1(s+r-t)+\varepsilon\kappa_2 f_{\sigma-t}^\sigma Z(\theta_{s}\omega)ds} \\
+ c \int_{-t}^{r-s} e^{\kappa_1(s+r-t)+\varepsilon\kappa_2 f_{\sigma-t}^\sigma Z(\theta_{s}\omega)ds} \|g(s)\|_2 ds + c.
\] (25)

Letting \( r = s \) in (25) and taking the supremum over \( s \in (-\infty, \tau] \), we obtain

\[
sup_{s \leq \tau} \|\varphi(s,s-t,\theta_{-s}\omega,\varphi_{s-t})\|_X^2 \leq cR_0(\omega) + cR_1(\tau,\omega) \\
+ e^{-\kappa_1 t + \varepsilon\kappa_2 f_{0-t}^\sigma Z(\theta_{s}\omega)ds} \sup_{s \leq \tau} \left( \|\varphi_{s-t}\|_X^2 + 2 \int_{\mathbb{R}^3} F(x,u_{s-t})dx \right).
\]

By (4a)-(4b) and the Sobolev embedding \( H^1(\mathbb{R}^3) \hookrightarrow L^{p+1}(\mathbb{R}^3) \),

\[
sup_{s \leq \tau} \left( \|\varphi_{s-t}\|_X^2 + 2 \int_{\mathbb{R}^3} F(x,u_{s-t})dx \right) \\
\leq c \sup_{s \leq \tau} (\|\varphi_{s-t}\|_X^2 + \|u_{s-t}\|_{H^{p+1}}^2 + \|h_1\| + \|h_2\|_1) \\
\leq c \sup_{s \leq \tau} (\|\varphi_{s-t}\|_X^2 + \|u_{s-t}\|_{H^{p+1}}^2 + 1) \leq c \sup_{s \leq \tau} \|D(s-t,\theta_{-t}\omega)\|_{X^1}^{p+1} + c.
\]

By Lemma 2.4,

\[
e^{-\kappa_1 t + \varepsilon\kappa_2 f_{0-t}^\sigma Z(\theta_{s}\omega)ds} \sup_{s \leq \tau} \left( \|\varphi_{s-t}\|_X^2 + 2 \int_{\mathbb{R}^3} F(x,u_{s-t})dx \right) \\
\leq ce^{-\left(\frac{\kappa_1}{p+1} - \frac{p+1}{p}\right) t}\kappa_1^t \left( e^{-\frac{\kappa_1}{p+1} t} \sup_{s \leq \tau} \|D(s-t,\theta_{-t}\omega)\|_{X^1}^{p+1} \right) + ce^{-\frac{\kappa_1}{p}\kappa_1^t} \tag{26}
\]

which tends to zero as \( t \to +\infty \). □

**Proposition 2.** The cocycle \( \Phi \) has a random absorbing set \( K \in \mathcal{D} \) given by

\( K(\tau,\omega) := \{ \varphi \in X : \|\varphi\|_X^2 \leq cR_0(\omega) + cR_1(\tau,\omega) \}, \forall (\tau,\omega) \in \mathbb{R} \times \Omega \),

where \( R_0(\omega) \) and \( R_1(\tau,\omega) \) are given by (22) and (23) respectively. Moreover, the absorption is backward uniform, that is, for each \( D \in \mathcal{D} \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there is \( T = T(D,\tau,\omega) > 0 \) such that

\( \Phi(t,s-t,\theta_{-t}\omega)D(s-t,\theta_{-t}\omega) \subset K(\tau,\omega) \), \( \forall s \leq \tau, t \geq T \).

**Proof.** The backward uniform absorption follows from (21) immediately. By (17a) and Lemma 2.4, we know \( R_0(\omega) \) is a finite random variable.

We show \( R_1(\tau,\omega) < +\infty \) for each \( \tau \in \mathbb{R} \). Indeed, by Lemmas 2.4 and 3.1,

\[
R_1(\tau,\omega) = \sup_{s \leq \tau} \int_{-\infty}^0 e^{\kappa_1 \sigma + \varepsilon\kappa_2 f_{0-t}^\sigma Z(\theta_{s}\omega)ds} \|g(\sigma)\|_2 ds \\
\leq e^{C_0(\omega)} \sup_{s \leq \tau} \int_{-\infty}^0 e^{\frac{15}{16}\kappa_1 \sigma} \|g(\sigma)\|_2 ds < +\infty.
\]
Next, we show $K \in \mathcal{O}$. It suffices to show that $R_1$ is backward tempered with the growth rate $\kappa_1/15$. Note that $\tau \rightarrow R_1(\tau, \omega)$ is increasing. It follows from Lemmas 2.4 and 3.1 that for all $t \geq T_0$,

$$e^{-\frac{1}{15} \kappa_1 t} \sup_{s \leq \tau} R_1(s - t, \theta_{-t}\omega) = e^{-\frac{1}{15} \kappa_1 t} R_1(\tau - t, \theta_{-t}\omega)$$

$$= e^{-\frac{1}{15} \kappa_1 t} \sup_{s \leq \tau} \int_{-\infty}^{-t} e^{\kappa_1 (\sigma + t) + \varepsilon_0 \kappa_2} (\theta_{\sigma}\omega) d\sigma \|g(\sigma + t)\|^2 d\sigma$$

$$\leq e^{-\frac{1}{15} \kappa_1 t} \sup_{s \leq \tau} \int_{-\infty}^{-t} e^{\kappa_1 (\sigma + t) + \varepsilon_0 \kappa_2} (\theta_{\sigma}\omega) d\sigma \|g(\sigma + t)\|^2 d\sigma$$

$$\leq e^{-\frac{1}{15} \kappa_1 t} e^{C(\omega)} \sup_{s \leq \tau} \int_{-\infty}^{-t} e^{\frac{1}{15} \kappa_1 (\sigma + t) - \frac{1}{15} \kappa_1 \sigma} \|g(\sigma + t)\|^2 d\sigma$$

$$= e^{C(\omega)} e^{-\frac{1}{15} \kappa_1 t} \sup_{s \leq \tau} \int_{-\infty}^{-t} e^{\frac{1}{15} \kappa_1 \sigma} \|g(\sigma + t)\|^2 d\sigma \rightarrow 0,$$

as $t \rightarrow +\infty$, since $g$ is backward tempered with arbitrary growth rates.

Finally, we show the measurability of $K$. It suffices to show the measurability of $\omega \rightarrow R_1(\tau, \omega)$, which is the supremum of some random variables over an uncountable index set $(-\infty, \tau)$.

Indeed, by the Egoroff theorem, the convergence of (17b) is basically uniform on $\Omega$, that is, for each $N \in \mathbb{N}$, there is a $\Omega_N \subset \Omega$ such that $P(\Omega \setminus \Omega_N) < \frac{1}{N}$ and

$$\lim_{t \rightarrow \pm \infty} \sup_{\omega \in \Omega_N} \frac{1}{t} \int_{-t}^{0} |z(\theta_{\sigma}\omega)|^m d\sigma = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi} \delta^m}, \quad \forall m > 0. \quad (27)$$

Let $\Omega_N = \hat{\Omega}_N \cap \Omega_N$, where

$$\Omega_N = \{\omega \in \Omega : |\omega(t)| \leq Ne^{\frac{3}{2}|t|}, \forall t \in \mathbb{R}\}$$

is defined in (12) such that $P(\Omega_N) \rightarrow 1$ as $N \rightarrow \infty$. So,

$$\lim_{N \rightarrow \infty} P(\Omega \setminus \hat{\Omega}_N) \leq \lim_{N \rightarrow \infty} P(\Omega \setminus \hat{\Omega}_N) + \lim_{N \rightarrow \infty} P(\Omega \setminus \Omega_N) = 0.$$

Suppose that $\omega_k, \omega_0 \in \Omega_N$ such that $\rho(\omega_k, \omega_0) \rightarrow 0$ as $k \rightarrow \infty$. Then, by (27), the same method as given in Lemma 2.4 shows that there is a $T_0 > 0$ such that

$$\sup_{k \in \mathbb{N}, \{0\}} \varepsilon_0 \kappa_2 \int_{t}^{0} Z(\theta_{\sigma}\omega_k) d\sigma \leq -\frac{1}{16} \kappa_1 \sigma, \quad \forall \sigma \leq -T_0. \quad (28)$$

Given $\eta > 0$, by (28), there is a $T_1 > T_0$ such that

$$\sup_{\sigma \leq -T_1} e^{\frac{1}{15} \kappa_1 \sigma} |e^{\varepsilon_0 \kappa_2} \int_{\sigma}^{0} Z(\theta_{\sigma}\omega_k) d\sigma - e^{\varepsilon_0 \kappa_2} \int_{\sigma}^{0} Z(\theta_{\sigma}\omega_0) d\sigma| \leq 2e^{\frac{1}{15} \kappa_1 T_1} < \eta.$$

By Lemma 2.3 (or see [8, Corollary 22]), we have

$$\sup_{-T_1 \leq \sigma \leq 0} |e^{\varepsilon_0 \kappa_2} \int_{\sigma}^{0} Z(\theta_{\sigma}\omega_k) d\sigma - e^{\varepsilon_0 \kappa_2} \int_{\sigma}^{0} Z(\theta_{\sigma}\omega_0) d\sigma| \rightarrow 0 \text{ as } k \rightarrow \infty.$$
Therefore, there is a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,
\[
\sup_{\sigma \leq 0} e^{\xi_1 \sigma} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| \\
\leq \sup_{-T_1 \leq \sigma \leq 0} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| \\
+ \sup_{\sigma \leq -T_1} e^{\frac{1}{2} \xi_\omega \sigma} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| < 2\eta.
\]
In a conclusion,
\[
\sup_{\sigma \leq 0} e^{\frac{1}{2} \xi_\omega \sigma} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| \to 0 \text{ as } k \to \infty. \tag{29}
\]
Now, by (29) and the hypothesis $G_1$, we can prove the continuity of $\omega \to R_1(\tau, \omega)$ on $(\bar{\Omega}_N, \rho)$.
\[
|R_1(\tau, \omega_k) - R_1(\tau, \omega_0)| \\
\leq \sup_{s \leq \tau} \int_{-\infty}^0 e^{\xi_\omega \sigma} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| |g(\sigma + s)|^2 d\sigma \\
\leq \sup_{\sigma \leq 0} e^{\frac{1}{2} \xi_\omega \sigma} |e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_k) d\sigma - e^{\xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega_0) d\sigma| \sup_{s \leq \tau} \int_{-\infty}^0 e^{\frac{1}{2} \xi_\omega \sigma} |g(\sigma + s)|^2 d\sigma
\]
which tends to zero as $k \to \infty$. So, $\omega \to R_1(\tau, \omega)$ is continuous on $(\bar{\Omega}_N, \rho)$ for each $N \in \mathbb{N}$ and thus Lusin continuous on $(\Omega, \rho)$. It is measurable as desired. \qed

**Remark 2.** If we assume the continuity of $s \to ||g(s)||$, it is relatively easy to prove the measurability of $\omega \to R_1(\tau, \omega)$. Indeed, let $\mathbb{Q}$ be the set of all rational numbers, by the continuity,
\[
R_1(\tau, \omega) = \sup_{s \leq \tau} \int_{-\infty}^0 e^{\xi_\omega \sigma + \xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega) d\sigma |g(\sigma + s)|^2 d\sigma \\
= \sup_{s \in \mathbb{Q} \cap (-\infty, \tau]} \int_{-\infty}^0 e^{\xi_\omega \sigma + \xi_\omega k_2} \int_0^\infty Z(\theta_\omega \omega) d\sigma |g(\sigma + s)|^2 d\sigma,
\]
which is the supremum of a sequence of random variables and thus measurable.

4. **Backward flattening inside a ball.** Let $\mathcal{O}_k = \{x \in \mathbb{R}^3 : |x| < k\}$ for $k \in \mathbb{N}$. We consider the canonical projection:
\[
P_n : L^2(\mathcal{O}_k) \mapsto Y_n := \text{span}\{e_1, e_2, \cdots, e_n\}, \; P_n = P_n \times P_n, \; n \in \mathbb{N},
\]
where $e_n \in H^1_0(\mathcal{O}_k)$ is the eigenfunction for $-\Delta$ with respect to the eigenvalues:
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty \text{ as } n \to \infty.
\]
We then take a cut-off function by
\[
\xi_k(x) := \xi\left(\frac{|x|^2}{k^2}\right), \; x \in \mathbb{R}^3, k \in \mathbb{N}, \tag{30}
\]
where $\xi : \mathbb{R} \mapsto [0, 1]$ is a smooth function such that $\xi(s) \equiv 1$ on $[0, 1/4]$ and $\xi(s) \equiv 0$ on $[1, +\infty)$. 

Hypothesis G2. The force $g$ is backward complement-small: there are $\alpha_0 > 0$ and $\tau_0 \in \mathbb{R}$ such that
\[
\lim_{n \to \infty} \sup_{s \leq \tau_0} \int_{-\infty}^{0} e^{\alpha_0 \sigma} \| (I - P_n)(\xi_k g(\sigma + s)) \|^2 d\sigma = 0, \ \forall k \in \mathbb{N},
\]
where $\xi_k g(\sigma) \in L^2(O_k)$ due to $\xi_k(x)g(\sigma, x) = 0$ for $|x| \geq k$.

Lemma 4.1. Suppose $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^3))$ satisfies the hypothesis G2, then, for all $\alpha > 0$ and $\tau \in \mathbb{R},$
\[
\lim_{n \to \infty} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha \sigma} \| (I - P_n)(\xi_k g(\sigma + s)) \|^2 d\sigma = 0, \ \forall k \in \mathbb{N}. \tag{31}
\]

Proof. Let $k \in \mathbb{N}$ be fixed. For each $\alpha > 0$, $\tau \in \mathbb{R}$ and $n \in \mathbb{N}$, we have
\[
G_n(\alpha, \tau) := \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha \sigma} \| (I - P_n)(\xi_k g(\sigma + s)) \|^2 d\sigma
\]
\[
= \sup_{s \leq \tau} \sum_{i=0}^{\infty} \int_{s-i}^{s-i+1} e^{\alpha(\sigma-s)} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma
\]
\[
\leq \sum_{i=0}^{\infty} e^{-\alpha i} \sup_{s \leq \tau} \int_{s-i}^{s-i+1} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma
\]
\[
\leq \frac{1}{1 - e^{-\alpha}} \sup_{s \leq \tau} \int_{s-1}^{s} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma.
\]

We assume without lose of generality that $\tau \geq \tau_0$, then,
\[
\sup_{s \leq \tau} \int_{s-1}^{s} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma
\]
\[
\leq \sup_{s \leq \tau_0} \int_{s-1}^{s} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma + \int_{\tau_0-1}^{\tau} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma
\]
\[
\leq e^{\alpha_0} G_n(\alpha_0, \tau_0) + \int_{\tau_0-1}^{\tau} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma.
\]

By $\| I - P_n \| = 1$, $\| \xi_k \|_\infty = 1$ and $g \in L^2([\tau_0 - 1, \tau], L^2(\mathbb{R}^3))$, $\| (I - P_n)(\xi_k g(\sigma)) \|^2 \leq \| g(\sigma) \|^2$, and $\int_{\tau_0-1}^{\tau} \| g(\sigma) \|^2 d\sigma < +\infty$.

Then, the Lebesgue controlled convergence theorem gives
\[
\lim_{n \to \infty} \int_{\tau_0-1}^{\tau} \| (I - P_n)(\xi_k g(\sigma)) \|^2 d\sigma = 0,
\]
which along with $G_n(\alpha_0, \tau_0) \to 0$ implies $G_n(\alpha, \tau) \to 0$ as $n \to \infty$. 

Let $\tilde{\varphi} = (\tilde{u}, \tilde{v}) := (\xi_k u, \xi_k v) \in X(O_k) = H^1_0(O_k) \times L^2(O_k)$. By multiplying (3a)-(3b) with $\xi_k$, we obtain
\[
\tilde{u}_t = - (\delta - \varepsilon z) \tilde{u} + \tilde{v} \tag{32a}
\]
\[
\tilde{v}_t + (\lambda + \delta^2 - \delta \beta(t)) \tilde{u} - \Delta \tilde{u} + \xi_k f(x, u) + (\beta(t) - \delta) \tilde{v}
\]
\[
= \xi_k g(t, x) - \varepsilon z \tilde{v} + \varepsilon z (3\delta - \beta(t) - \varepsilon z) \tilde{u} - u \Delta \xi_k - 2 \nabla \xi_k \cdot \nabla u. \tag{32b}
\]
Note that $\bar{\varphi}$ has the orthogonal decomposition:

$$
\bar{\varphi} = P_n \bar{\varphi} + (I - P_n) \bar{\varphi} := \bar{\varphi}_1 + \bar{\varphi}_2 = (\bar{u}_n, \bar{v}_n) + (\bar{u}_2, \bar{v}_2).
$$

**Proposition 3.** Let the hypotheses B, F, S be true and the force $g$ satisfy the conditions G1 and G2. Then, for each $(r, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D}$,

$$
\lim_{n,k,t \to \infty} \sup_{s \leq r} \|(I - P_n)(\xi_k \varphi(s, s - t, \theta_{-s} \omega, \varphi_{s-1}))\|_{X(\Omega_k)} = 0.
$$

uniformly in $\varphi_{s-1} \in D(s - t, \theta_{-s} \omega)$.

**Proof.** Applying $I - P_n$ to Eq. (32b) and taking the inner product of the resulting equation with $\bar{v}_n = \bar{v}_n(r, s - t, \theta_{-s} \omega)$ $(r \geq s - t)$ in $L^2(\Omega_k)$, we obtain

$$
\frac{d}{dr} \|\bar{v}_n\|^2 + 2(\lambda + \delta^2)(\bar{u}_n, \bar{v}_n) - 2(\Delta \bar{u}_n, \bar{v}_n) + 2(\xi_k f(x, u, \bar{v}_n)) =
$$

$$
-2(\beta(t) - \delta)\|\bar{v}_n\|^2 + 2\delta \beta(t)(\bar{u}_n, \bar{v}_n) + 2(\xi_k g(t), \bar{v}_n)
$$

$$
-2\varepsilon z(\theta_{-s} \omega)(\|\bar{v}_n\|^2 + 2\varepsilon z(\theta_{-s} \omega)(3\delta - \beta(t) - \varepsilon z(\theta_{-s} \omega))(\bar{u}_n, \bar{v}_n) + I_0. \tag{33}
$$

where $I_0 := -2(u(\Delta \xi_k + 2\nabla \xi_k \cdot \nabla u, \bar{v}_n))$. To deal with the first line in (33), we apply $I - P_n$ on Eq. (32a) to obtain

$$
2(\bar{u}_n, \bar{v}_n) = \frac{d}{dr} \|\bar{u}_n\|^2 + 2(\delta - \varepsilon z)\|\bar{u}_n\|^2, \tag{34a}
$$

$$
-2(\Delta \bar{u}_n, \bar{v}_n) = \frac{d}{dr} \|\nabla \bar{u}_n\|^2 + 2(\delta - \varepsilon z)\|\nabla \bar{u}_n\|^2, \tag{34b}
$$

$$
(\xi_k f(x, u, \bar{v}_n)) = \frac{d}{dr} (\xi_k f(x, u), \bar{u}_n) - (\xi_k f_u(x, u) u_r, \bar{u}_n)
$$

$$
+ (\delta - \varepsilon z)(\xi_k f(x, u), \bar{v}_n). \tag{34c}
$$

We substitute (34a)-(34c) into (33) to find

$$
\frac{d}{dr}(\|\bar{\varphi}_n\|^2_X + 2(\xi_k f(x, u), \bar{u}_n)) - 2(\xi_k f_u(x, u) u_r, \bar{u}_n)
$$

$$
+ 2\varepsilon z(\xi_k f(x, u), \bar{u}_n) + I_6 = I_7 + I_8 + I_9 + I_0, \tag{35}
$$

where $I_6, I_7, I_8, I_9$ are defined and estimated as follow. By (34a)-(34b) and $\kappa_2 \geq 4$,

$$
I_6 := 2(\delta - \varepsilon z)((\lambda + \delta^2)\|\bar{u}_n\|^2 + \|\nabla \bar{u}_n\|^2)^2
$$

$$
\geq 2\delta(\lambda + \delta^2)\|u\|^2 + 2\delta\|\nabla u\|^2 - \frac{\kappa_2^2}{\varepsilon z}\|\bar{\varphi}_n\|^2,
$$

where

$$
\|\bar{\varphi}_n\|^2 = \|\bar{\varphi}_n\|^2_X = (\lambda + \delta^2)\|\bar{u}_n\|^2 + \|\nabla \bar{u}_n\|^2 + \|\bar{v}_n\|^2.
$$

By the hypothesis B and the Young inequality,

$$
I_7 := -2(\beta(t) - \delta)\|\bar{v}_n\|^2 + 2\delta \beta(t)(\bar{u}_n, \bar{v}_n)
$$

$$
\leq -2(\beta_1 - \delta)\|\bar{v}_n\|^2 + 2\delta \beta_2(\bar{u}_n, \bar{v}_n)
$$

$$
\leq -2(\beta_1 - \delta)\|\bar{v}_n\|^2 + \delta \sqrt{\delta}\|\bar{u}_n\|^2 + \sqrt{\delta}\beta_2\|\bar{v}_n\|^2.
$$

So, we have

$$
I_6 - I_7 \geq 2(\delta \beta_1\|\bar{u}_n\|^2 + \delta\|\nabla \bar{u}_n\|^2 + \delta_2\|\bar{v}_n\|^2) - \frac{\kappa_2^2}{\varepsilon Z}\|\bar{\varphi}_n\|^2_X
$$

$$
\geq 2\kappa_1\|\bar{\varphi}_n\|^2 - \frac{\kappa_2^2}{\varepsilon Z}\|\bar{\varphi}_n\|^2.
$$
By \((I - P_n)^2 = I - P_n\),
\[
I_8 := 2(\xi_k g(r), \bar{v}_{n2}) = 2((I - P_n)(\xi_k g(r)), (I - P_n)\bar{v})
\leq \frac{k_1}{4}\|\bar{v}_{n2}\|^2 + c\|(I - P_n)(\xi_k g(r))\|^2.
\]

By \(\varepsilon \in (0, 1)\) and the Young inequality,
\[
I_9 := -2\varepsilon z\|\bar{v}_{n2}\|^2 + 2\varepsilon z(3\delta - \beta(t) - \varepsilon z)(\bar{u}_{n2}, \bar{v}_{n2})
\leq (2 + 3\delta + \beta_2)\varepsilon(z^2)\|\bar{u}_{n2}\|^2 + \|\bar{v}_{n2}\|^2 \leq \frac{k_2}{2}\varepsilon z\|\bar{v}_{n2}\|^2.
\]

We obtain
\[
I_6 - \sum_{i=7}^9 I_i \geq \frac{7k_1}{4}\|\bar{v}_{n2}\|^2 - \kappa_2\varepsilon Z\|\bar{v}_{n2}\|^2 - c\|(I - P_n)(\xi_k g(r))\|^2. \tag{36}
\]

By \(\|\nabla \xi_k\|_{\infty} + \|\Delta \xi_k\|_{\infty} \leq \frac{c}{k}\) for all \(k \geq 1\),
\[
I_0 := -2(u\Delta \xi_k + 2\nabla \xi_k \cdot \nabla u, \bar{v}_{n2}) \leq \frac{c}{k}\|\varphi\|_{X}\|\bar{v}_{n2}\| \leq \frac{k_1}{4}\|\bar{v}_{n2}\|^2 + \frac{c}{k}\|\varphi\|_{X}^3 \tag{37}
\]

We substitute (36) and (37) into (35) to find that
\[
\frac{d}{dr}\left(\|\bar{v}_{n2}\|^2 + 2(\xi_k f(x, u, \bar{u}_{n2}) \right)
\leq \left(\kappa_1 - \varepsilon \kappa_2 Z(\theta_{r-\omega})\right)\|\bar{v}_{n2}\|^2 + 2(\xi_k f(x, u, \bar{u}_{n2})
\leq (2 + 3\delta + \beta_2)\varepsilon(z^2)(\|\bar{u}_{n2}\|^2 + \|\bar{v}_{n2}\|^2)
\leq \frac{k_1}{2}\|\bar{v}_{n2}\|^2 + \frac{c}{k}\|\varphi\|_{X}^3 + c\|(I - P_n)(\xi_k g(r))\|^2. \tag{38}
\]

We then estimate the nonlinear terms. By (4a) and \(p \in [1, 3]\),
\[
2(\kappa_1 - \varepsilon \kappa_2 Z + \varepsilon z)(\xi_k f(x, u, \bar{u}_{n2})
\leq c(1 + |z|^2)(\|u\|_{L^p}^p \|\bar{u}_{n2}\| + c\|h_1\|\|\bar{u}_{n2}\|)
\leq c(1 + |z|^2)\lambda_{n+1}^{-\frac{1}{2}}(1 + \|u\|_{H^1(\mathbb{R}^3)})\|\nabla \bar{u}_{n2}\|
\leq \frac{k_1}{4}\|\bar{v}_{n2}\|^2 + c\lambda_{n+1}^{-1}(1 + |z|^{\frac{2}{1-p}} + \|u\|_{H^1(\mathbb{R}^3)}^p)
\leq \frac{k_1}{4}\|\bar{v}_{n2}\|^2 + c\lambda_{n+1}^{-1}(\|\varphi\|_{X}^3 + \|\varphi\|_{X}^3). \tag{39}
\]

By (4d), (3a) and \(p \in [1, 3]\),
\[
2\left|\xi_k f_u(x, u)u_r, \bar{u}_{n2}\right|
\leq c(\|u_r\|_{L^6} \|u\|_{L^6} \frac{1}{\lambda_{n+1}^{\frac{2}{3}}} + \|u_r\|_{L^4} \|h_4\|_{L^6} \|\bar{u}_{n2}\|)
\leq c(1 + |z|)(\|u\|_{L^6} + \|\varphi\|_{L^6})(\|u\|_{H^1(\mathbb{R}^3)} \|\bar{u}_{n2}\| \frac{1}{\lambda_{n+1}^{\frac{2}{3}}} \|\nabla \bar{u}_{n2}\| \frac{1}{\lambda_{n+1}^{\frac{1}{2}}} + \|\bar{u}_{n2}\|^{\frac{1}{2}} \|\nabla \bar{u}_{n2}\|^{\frac{1}{2}})
\leq c(1 + |z|)(\|\varphi\|_{X}^3 \lambda_{n+1}^{\frac{2}{3}} + \|\varphi\|_{X} \lambda_{n+1}^{-\frac{1}{2}})\|\nabla \bar{u}_{n2}\|
\leq \frac{k_1}{4}\|\bar{v}_{n2}\|^2 + c(\lambda_{n+1}^{\frac{2}{3}} + \lambda_{n+1}^{-\frac{1}{2}})(\|\varphi\|_{X}^3 + \|\varphi\|_{X}^3). \tag{40}
\]
Since $\lambda_{n+1}^{-\frac{1}{2}} + \lambda_{n+1}^{-\frac{1}{2}} + \lambda_{n+1}^{-\frac{1}{2}} + \frac{1}{\kappa} \to 0$, it follows from (38), (39) and (40) that for each $\eta > 0$ there are $n_\eta, k_\eta \in \mathbb{N}$ such that
\[
\frac{d}{dt}\left(\|\bar{\varphi}_{n2}\|^2 + 2(\xi_k f(x, u), \bar{u}_{n2})\right) + (\kappa_1 - \delta_k Z(\theta_{r-s}\omega))\left(\|\bar{\varphi}_{n2}\|^2 + 2(\xi_k f(x, u), \bar{u}_{n2})\right) \\
\leq \epsilon \|I - P_n\| \|\xi_k g(r)\|^2 + \eta \epsilon^{\|\theta_{r-s}\omega\|} + \|\varphi\|_X^6.
\] (41)
for all $n \geq n_\eta, k \geq k_\eta$ and $r \geq s-t$. Applying the Gronwall lemma to (41) over $r \in [s-t, s]$ and taking the supremum over $s \in (-\infty, \tau]$, we obtain
\[
\sup_{s \leq \tau}\left(\|\bar{\varphi}_{n2}(s, s-t, \theta_{r-s}\omega, \bar{\varphi}^\bot_{s-t})\|^2 + 2(\xi_k f(x, u(s), \bar{u}_{n2}(s, s-t, \theta_{r-s}\omega, \bar{\varphi}^\bot_{s-t}))\right) \\
\leq J_1 + J_2 + \eta(J_3 + J_4).
\] (42)
where $\bar{\varphi}^\bot_{s-t} := (I - P_n)(\xi_k u_{s-t})$ such that $\|\bar{\varphi}^\bot_{s-t}\| \leq \|u_{s-t}\|_X$. All $J_i$ in (42) are defined and estimated as follows. By (18), (4a), as $t \to +\infty$,
\[
J_1 := \sup_{s \leq \tau} s^\top Z(\theta_{s-t}\omega)d\sigma \left(\|\bar{\varphi}^\bot_{s-t}\|^2 + 2(\xi_k f(x, u_{s-t}), \bar{u}_{s-t})\right) \\
\leq ce^{-\frac{R}{16}\kappa_1t} \sup_{s \leq \tau} s^\top \left(\|\varphi_{s-t}\|^2 + (\|u_{s-t}\|^p + \|h_1\|^2) + \|\bar{u}_{s-t}\|^2\right) \\
\leq ce^{-\frac{R}{16}\kappa_1t} \sup_{s \leq \tau} \|\varphi_{s-t}\|^2 + 1 + \|\bar{u}_{s-t}\|^2 \\
\leq ce^{-\left(\frac{R}{16} - \frac{1}{p}\right)\kappa_1t} \left(e^{-\frac{2R}{16}\sup_{s \leq \tau} \|D(s-t, \theta_{t-s}\omega)\|^2} + ce^{-\frac{R}{16}\kappa_1t}\right) \\
\leq 0.
\]
We remark here that the power of the initial datum $u_{s-t}$ is 2p rather than $p+1$ as given in the literature (e.g. [33, 38]), because we do not know whether the absolute value $|I - P_n)(\xi_k u_{s-t})| \leq \|u_{s-t}\|_X$.
By the hypothesis $G2$ and (19),
\[
J_2 := \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\kappa_1\sigma + \epsilon_k Z(\theta_{s-t}\omega)d\sigma} \|(I - P_n)(\xi_k g(\sigma + s))\|^2d\sigma \\
\leq e^{C_0(\omega)} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\frac{R}{16}\kappa_1\sigma} \|(I - P_n)(\xi_k g(\sigma + s))\|^2d\sigma \to 0
\]
as $n \to \infty$ for each $k \in \mathbb{N}$. Similarly, by (19) and $z(\theta_{s\omega})/\sigma \to 0$,
\[
J_3 := \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\kappa_1\sigma + \epsilon_k Z(\theta_{s-t}\omega)d\sigma} e^{\|z(\theta_{s-t}\omega)\|}d\sigma \\
\leq e^{2C_0(\omega)} \int_{-\infty}^{0} e^{\frac{2R}{16}\kappa_1\sigma}d\sigma = e^{2C_0(\omega)} \frac{8}{T\kappa_1} < +\infty.
\]
Finally, we show boundedness of the following term of the sixth power:
\[
J_4 := \sup_{s \leq \tau} \int_{s-t}^{s} e^{\kappa_1(\sigma - s) + \epsilon_k Z(\theta_{s-t}\omega)d\sigma} \|\varphi(\sigma, s-t, \theta_{s-t}\omega)\|^2d\sigma.
\] (43)
By (25), we have $J_4 \leq c(J_{4,1} + J_{4,2} + J_{4,3})$, where, by (4a) and (4b),

$$J_{4,1} := \sup_{s \leq t} \int_{s-t}^{s} e^{\kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( e^{-\kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( \|\varphi_{s-s'}\|_{X}^{2} + \int_{\mathbb{R}} |F(x, u_{s-t})| dx \right) \right)^{3} d\sigma.$$

where we have reduced the coefficient from 1 to $p$ such that $\frac{p+1}{15} < \frac{1}{16}$ for $p \geq 1$.

$$J_{4,1} \leq e^{4C_0(\omega)} \int_{-t}^{0} e^{\frac{15}{16} \kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( e^{-\frac{15}{16} \kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( \|\varphi_{s-s'}\|_{X}^{2} + \int_{\mathbb{R}} |F(x, u_{s-t})| dx \right) \right)^{3} d\sigma < \infty.$$

which is bounded due to $\frac{15}{16} > \frac{p+1}{30}$ for $p \in [1, 3)$. The number $1/15$ in the tempered set (16) was mainly used in the above inequality.

Similarly, by (17a) and Lemma 2.4,

$$J_{4,2} := \sup_{s \leq t} \int_{s-t}^{s} e^{\kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( \int_{-t}^{s-t} e^{\kappa_1(r+s-s') + \varepsilon \kappa_2 \int_{s'-r}^{s} Z(\theta_{s'} \omega) d\sigma} dr \right)^{3} d\sigma < \infty.$$

By Lemma 3.1, the force $g$ is backward tempered with arbitrary rates. So,

$$J_{4,3} := \sup_{s \leq t} \int_{s-t}^{s} e^{\kappa_1(s-s') + \varepsilon \kappa_2 \int_{s'-t}^{s} Z(\theta_{s'} \omega) d\sigma} \left( \int_{-t}^{s-t} e^{\kappa_1(r+s-s') + \varepsilon \kappa_2 \int_{s'-r}^{s} Z(\theta_{s'} \omega) d\sigma} \|g(r+s)\|^{2} dr + 1 \right)^{3} d\sigma < \infty.$$

Therefore $J_4$ is bounded as $t \to +\infty$. So, the left-hand side in (42) tends to zero.

On the other hand, by the same argument as given in (39),

$$\sup_{s \leq t} |2(\xi f(x, u(s)), \bar{u}_{n}(s, s-t, \theta_{s-t} \omega, \bar{u}_{s-t}^{+})|$$

$$\leq \frac{1}{2} \sup_{s \leq t} \left( \|\varphi_{n}(s, s-t, \theta_{s-t} \omega, \bar{\varphi}_{s-t}^{+})\|^{2} + c \lambda_{n+1}^{-1} (1 + \sup_{s \leq t} \|\varphi(s, s-t)\|^{2p}) \right). \quad (44)$$

By (21), $\sup_{s \leq t} \|\varphi(s, s-t)\|^{2p} \leq R_{0}^{p}(\tau, \omega) + R_{1}^{p}(\tau, \omega) < \infty$. So, it follows from (42) and (44) that

$$\sup_{s \leq t} \|\varphi_{n}(s, s-t, \theta_{s-t} \omega, \bar{\varphi}_{s-t}^{+})\|^{2} \rightarrow 0$$

as $n, k, t \to +\infty$. The proof is complete.
5. Backward uniform tail-estimates. We need a further assumption for $g$.

**Hypothesis G3.** The force $g$ is backward tail-small: there are $\alpha_0 > 0$ and $\tau_0 \in \mathbb{R}$ such that

$$
\lim_{k \to \infty} \sup_{s \leq \tau_0} \int_{-\infty}^{0} e^{\alpha_0 |s|} \int_{\mathcal{O}_k^c} |g(s + x)|^2 dx ds = 0,
$$

where $\mathcal{O}_k^c = \mathbb{R}^3 \setminus \mathcal{O}_k = \{x \in \mathbb{R}^3 : |x| \geq k\}$ for $k \in \mathbb{N}$.

The growth rate $\alpha_0$ can be arbitrary. The proof of the following result is similar to the proof of Lemma 4.1 and so omitted.

**Lemma 5.1.** Suppose $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3))$ satisfies the hypothesis G3, then, for all $\alpha > 0$ and $\tau \in \mathbb{R}$,

$$
\lim_{k \to \infty} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha |s|} \int_{\mathcal{O}_k^c} |g(s + x)|^2 dx ds = 0.
$$

**Proposition 4.** Let the hypotheses B, F, S be true and the force $g$ satisfy the conditions G1 and G3. Then, for each $(\tau, \omega, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathcal{D}$,

$$
\lim_{t, \lambda \to +\infty} \sup_{s \leq \tau} \sup_{\varphi, \theta \in \mathcal{D}(s-t, \varphi, \theta)} \| \varphi(s, t, \varphi, \theta) \|_{L^2(\mathcal{O}_k^c)} = 0.
$$

**Proof.** Let $\rho_k(x) = 1 - \xi_k(x)$, where $\xi_k$ is the cut-off function given in section 4. We then take the inner product of Eq. (3b) with $\rho_k v(r, s - t, \theta - \omega t)$ to obtain

$$
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^3} \rho_k |v|^2 dx + 2(\lambda + \delta^2) \int_{\mathbb{R}^3} \rho_k v u dx - \int_{\mathbb{R}^3} \rho_k \nabla u dx + 2 \int_{\mathbb{R}^3} \rho_k g(x, u) dx &= 2(\delta - \beta(t)) \int_{\mathbb{R}^3} \rho_k |v|^2 dx + 2 \epsilon \beta(t) \int_{\mathbb{R}^3} \rho_k v u dx + 2 \int_{\mathbb{R}^3} \rho_k v g(r) dx \\
&- 2 \epsilon \tau (\theta - \omega t) \int_{\mathbb{R}^3} \rho_k |v|^2 dx + 2 \epsilon \tau (3\delta - \beta(t) - \epsilon \tau) \int_{\mathbb{R}^3} \rho_k v u dx.
\end{align*}
$$

(45)

By $u_r = v - (\delta - \epsilon \tau) u$ as given in (3a), we have

$$
\begin{align*}
\int_{\mathbb{R}^3} \rho_k v u dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho_k |u|^2 dx + (\delta - \epsilon \tau) \int_{\mathbb{R}^3} \rho_k |u|^2 dx, \\
- \int_{\mathbb{R}^3} \rho_k \nabla u dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \rho_k |\nabla u|^2 dx + \int_{\mathbb{R}^3} u_r (\nabla u \cdot \nabla \rho_k) dx \\
&+ (\delta - \epsilon \tau) \int_{\mathbb{R}^3} \rho_k |\nabla u|^2 dx + (\delta - \epsilon \tau) \int_{\mathbb{R}^3} u (\nabla u \cdot \nabla \rho_k) dx, \\
\int_{\mathbb{R}^3} \rho_k f(x, u) dx &= \frac{d}{dt} \int_{\mathbb{R}^3} \rho_k F(x, u) dx + (\delta - \epsilon \tau) \int_{\mathbb{R}^3} \rho_k F(x, u) dx.
\end{align*}
$$

Substituting the above three equalities into (45) and letting $|\varphi|^2 := (\lambda + \delta^2)|u|^2 + |\nabla u|^2 + |v|^2$, we obtain

$$
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^3} \rho_k (|v|^2 + 2F(x, u)) dx + 2(\delta - \epsilon \tau) \int_{\mathbb{R}^3} \rho_k v u dx + I_{10} &= I_{11} + I_{12} + I_{13} + I_{14},
\end{align*}
$$

(46)

where $I_i$ ($i = 10, 11, 12, 13, 14$) are defined and estimated as follows.

$$
\begin{align*}
I_{10} &:= 2(\delta - \epsilon \tau)(\lambda + \delta^2) \int_{\mathbb{R}^3} \rho_k |u|^2 dx + \int_{\mathbb{R}^3} \rho_k |\nabla u|^2 dx \\
&\geq \delta(\lambda + \delta^2) \int \rho_k |u|^2 dx + 2\delta \int \rho_k |\nabla u|^2 dx - \frac{\rho_k}{2} \epsilon |z| \int \rho_k |\varphi|^2.
\end{align*}
$$
By the hypothesis B and the Young inequality,

\[ I_{11} := 2(\delta - \beta(t)) \int_{\mathbb{R}^3} \rho_k |v|^2 \, dx + 2\delta \beta(t) \int_{\mathbb{R}^3} \rho_k v \, dx \]

\[ \leq 2(\delta - \beta_1) \int_{\mathbb{R}^3} \rho_k |v|^2 + \delta \sqrt{\delta} \int_{\mathbb{R}^3} \rho_k |u|^2 + \sqrt{\delta} \beta_2 \int_{\mathbb{R}^3} \rho_k |v|^2. \]

So, we have

\[ I_{10} - I_{11} \geq 2(\delta_\varepsilon - \beta_1) \int_{\mathbb{R}^3} \rho_k |u|^2 + \delta \int_{\mathbb{R}^3} \rho_k |\nabla u|^2 + \delta \int \rho_k |v|^2) - \frac{\kappa_2}{2} \varepsilon |z| \int \rho_k |\varphi|^2 \]

\[ \geq 2\kappa_1 \int \rho_k |\varphi|^2 - \frac{\kappa_2}{2} \varepsilon Z \int \rho_k |\varphi|^2. \]

By the Young inequality,

\[ I_{12} := 2 \int_{\mathbb{R}^3} \rho_k v \, dx \leq \frac{\kappa_1}{2} \int \rho_k |\varphi|^2 + c \int_{\mathbb{R}^3} \rho_k |g(r, x)|^2 \, dx. \]

By \( \varepsilon \in (0, 1] \) and the Young inequality,

\[ I_{13} := -2\varepsilon \int_{\mathbb{R}^3} \rho_k |v|^2 + 2\varepsilon (3\delta - \beta(t) - \varepsilon) \int_{\mathbb{R}^3} \rho_k v \, dx \]

\[ \leq (2 + 3\delta + \beta_2) \varepsilon (|z| + |z|^2)(\int \rho_k |u|^2 + \int \rho_k |v|^2) \]

\[ \leq \frac{\kappa_2}{2} \varepsilon Z \int \rho_k |\varphi|^2. \]

We obtain

\[ I_{10} - \sum_{i=11}^{13} I_i \geq \frac{3\kappa_1}{2} \int \rho_k |\varphi|^2 - \kappa_2 \varepsilon Z \int \rho_k |\varphi|^2 - c \int_{\mathbb{R}^3} \rho_k |g(r, x)|^2 \, dx. \]  

(47)

By \( u_r = v - (\delta - \varepsilon z) u \) as given in (3a) and by \( \|\nabla \rho_k\|_{\infty} \leq \frac{\varepsilon}{k} \),

\[ I_{14} := -2 \int_{\mathbb{R}^3} u_r (\nabla u \cdot \nabla \rho_k) \, dx + (\delta - \varepsilon) \int_{\mathbb{R}^3} u (\nabla u \cdot \nabla \rho_k) \, dx \]

\[ \leq \frac{c}{k} (\|u_r\|_{1} \|\nabla u\|_{\infty} + (1 + |z|) \|u\|_{\infty} \|\nabla u\|_{\infty}) \]

\[ \leq \frac{c}{k} (1 + |z| \|\nabla u\|_{\infty}) \]

\[ \leq \frac{c}{k} (1 + |z|^{4} + \|\varphi\|_{X}^{4}) \leq \frac{c}{k} (e|z| + \|\varphi\|_{X}^{6}). \]  

(48)

We substitute (47)-(48) into (46) to obtain

\[ \frac{d}{dr} \int_{\mathbb{R}^3} \rho_k (|\varphi|^2 + 2 F(u, x)) \, dx + I_{15} + \frac{3\kappa_1}{2} \int \rho_k |\varphi|^2 - \kappa_2 \varepsilon Z \int \rho_k |\varphi|^2 \]

\[ \leq c \int_{\mathbb{R}^3} \rho_k |g(r, x)|^2 \, dx + \frac{c}{k} (e|z| + \|\varphi\|_{X}^{6}), \]  

(49)

where the nonlinear term is given by

\[ I_{15} := 2(\delta - \varepsilon) \int_{\mathbb{R}^3} \rho_k f(x, u) \, dx = 2\delta \int_{\mathbb{R}^3} \rho_k f(x, u) u - 2\varepsilon \int_{\mathbb{R}^3} \rho_k f(x, u) u \]
By (4b), (4c), \( \delta \gamma_2 \geq \kappa_1 \) and \( F(x, s) + h_3(x) \geq 0 \),

\[
2 \delta \int_{\mathbb{R}^3} \rho_k f(x, u) u \geq 2 \delta \gamma_2 \int \rho_k F(x, u) + 2 \delta \int \rho_k h_2 \\
\geq 2 \kappa_1 \int \| \rho_k F(x, u) + h_3 \| - 2 \delta \gamma_2 \int \rho_k h_3 - c \int \rho_k \| h_2 \| \\
\geq 2 \kappa_1 \int \rho_k F(x, u) - c \int \rho_k (|h_2| + |h_3|) \
\]

(50)

By (4a), (4c) and \( \gamma_1 \leq \kappa_2 \gamma_3 \),

\[
|2 \varepsilon z \int_{\mathbb{R}^3} \rho_k f(x, u) u dx |
\leq 2 \varepsilon \gamma_1 |z| \int_{\mathbb{R}^3} \rho_k |u| |u + 1| + 2 \varepsilon |z| \int \rho_k |h_1| |u| dx \\
\leq 2 \varepsilon \kappa_2 Z \int \rho_k \gamma_3 |u| |u + 1| + c Z \int \rho_k |h_1|^2 + \frac{\kappa_1}{2} \int \rho_k |\varphi|^2 \\
\leq 2 \varepsilon \kappa_2 Z \int \rho_k F(x, u) + c Z \int \rho_k (|h_3| + |h_1|^2) + \frac{\kappa_1}{2} \int \rho_k |\varphi|^2. 
\]

(51)

By (50) and (51),

\[
I_{15} \geq - \frac{\kappa_1}{2} \int \rho_k |\varphi|^2 + (\kappa_1 - \varepsilon \kappa_2 Z) \int \rho_k F(x, u) \\
- c e^{\varepsilon |z|} \int \rho_k (|h_1|^2 + |h_2| + |h_3|). 
\]

(52)

We substitute (52) into (49). By \( 1/k + \int \rho_k (|h_1|^2 + |h_2| + |h_3|) \to 0 \) as \( k \to \infty \), we obtain that for each \( \eta > 0 \) there is a \( k_\eta \in \mathbb{N} \) such that for all \( k \geq k_\eta \),

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \rho_k (|\varphi|^2 + 2F(x, u)) dx + (\kappa_1 - \varepsilon \kappa_2 Z (\theta_{r-s}) \int_{\mathbb{R}^3} \rho_k (|\varphi|^2 + 2F(x, u)) dx \\
\leq c \int_{\mathbb{R}^3} \rho_k |g(r, x)|^2 dx + \eta (e^{\varepsilon |z|} + \| \varphi \|_{X}^6). 
\]

(53)

Applying the Gronwall lemma to (53) over \( r \in [s - t, s] \) and taking the supremum over \( s \in (-\infty, \tau] \), we obtain

\[
\sup_{s \leq \tau} \int_{\mathbb{R}^3} \rho_k (|\varphi(s, s - t, \theta_{-s}\omega, \varphi_{s-t})|^2 + 2F(x, u)) dx \leq J_5 + J_6 + \eta (J_3 + J_4), 
\]

(54)

where \( J_3, J_4 \) are the same numbers as given in Proposition 3 and so they are bounded as \( t \to \infty \). We now estimates \( J_5 \) and \( J_6 \).

\[
J_5 := \sup_{s \leq \tau} e^{-\kappa_1 t + \varepsilon \kappa_2 \int_{-t}^{0} Z(\theta_s \omega) ds} \int_{\mathbb{R}^3} \rho_k (|\varphi_{s-t}|^2 + 2F(x, u_{s-t}) dx \\
\leq c e^{-\frac{15}{16} \kappa_1 t} \sup_{s \leq \tau} (\| \varphi_{s-t} \|_{X}^{p+1} + 1) \\
\leq c e^{-\frac{15}{16} \kappa_1 t} + c e^{-\frac{15}{16} \kappa_1 t} \left( e^{-\frac{15}{16} \kappa_1 t} \sup_{s \leq \tau} \| D(\tau - t, \theta_{-t} \omega) \|_{X}^{2} \right)^{\frac{p+1}{p}}. 
\]
which tends to zero as $t \to +\infty$. By the assumption $G3$,

$$J_6 := \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\kappa \sigma + \varepsilon \kappa^2} \int_{\mathbb{R}^3} \rho_k |g(\sigma + s, x)|^2 \, dx \, d\sigma$$

$$\leq e^{C_0(\omega)} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\frac{15}{16} \kappa_0^2} \int_{\mathcal{O}_{k/2}} |g(\sigma + s, x)|^2 \, dx \, d\sigma \to 0 \text{ as } k \to +\infty.$$  

Moreover, we see from (4c) that for all $s \leq \tau$,

$$-2 \int_{\mathbb{R}^3} \rho_k F(x, u(s)) \, dx \leq c \int \rho_k |h_3| \, dx \to 0 \text{ as } k \to +\infty.$$  

Hence, the needed result follows from (54).

\[\square\]

6. Backward compact and periodic random attractors.

6.1. Backward compact random attractor. In this subsection, we show the existence of a unique backward compact random attractor.

**Theorem 6.1.** Let the hypotheses $B,F,S$ be true and the force $g$ satisfy $G1$, $G2$ and $G3$. The cocycle $\Phi$, associated with the stochastic wave equation, has a unique pullback random attractor $A$ in $X = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ such that $A$ is backward compact and given by

$$A(\tau, \omega) = \bigcap_{t_0 > 0} \bigcup_{t \geq t_0} \Phi(t, \tau - t, \theta_{-t} \omega) \mathcal{K}(\tau - t, \theta_{-t} \omega).$$  

(55)

**Proof.** We finish the proof in three steps.

**Step 1.** By Proposition 2, the cocycle $\Phi$ has an increasing and closed random absorbing set $K \in \mathcal{D}$.

**Step 2.** We show the cocycle $\Phi$ is backward asymptotically compact, that is, for each $(\tau, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D}$, the sequence

$$\varphi_n := \Phi(t_n, s_n - t_n, \theta_{-t_n} \omega) \varphi_{n,0} = \varphi(s_n, s_n - t_n, \theta_{-s_n} \omega, \varphi_{n,0})$$

has a convergent subsequence whenever $s_n \leq \tau$, $t_n \to +\infty$ and $\varphi_{n,0} \in D(s_n - t_n, \theta_{-t_n} \omega)$.

By [23, Lemma 2.7], it suffices to verify the Kuratowski measure

$$\Upsilon_X \{ \varphi_n \mid n \geq N \} \to 0 \text{ as } N \to \infty,$$

(56)

where $\Upsilon_X(B)$ is the minimum of $d$ such that $B \subset X$ has a finite cover by sets of diameter lesser than $d$.

Given $\eta > 0$. By Proposition 4, there are $N_1 \in \mathbb{N}$ and $k_0 \geq 1$ such that

$$\|\varphi_n\|_{X(\mathcal{O}_{k_0})} \leq \eta, \quad \forall n \geq N_1.$$

(57)

By Proposition 3, we can choose $i \in \mathbb{N}$, $k \geq 2k_0$ and $N_2 \geq N_1$ such that

$$\|(I - P_i)(\xi \varphi_n)\|_{X(\mathcal{O}_k)} \leq \eta, \quad \forall n \geq N_2.$$  

(58)

By Lemma 3.2, there is an $N_3 \geq N_2$ such that

$$\sup_{n \geq N_3} \|\varphi_n\|_X^2 \leq c R_0(\omega) + c R_1(\tau, \omega) < +\infty,$$

(59)
and so \( \{ \varphi_n | n \geq N_3 \} \) is bounded in \( X \). Hence, \( P_t \{ \xi_k \varphi_n | n \geq N_3 \} \) is bounded in \( X(O_k) \). Since \( P_t \) has a finite-dimensional range, it follows that \( P_t \{ \xi_k \varphi_n | n \geq N_3 \} \) is pre-compact in \( X(O_k) \) and thus

\[
\Upsilon_{X(O_k)} P_t \{ \xi_k \varphi_n | n \geq N_3 \} = 0.
\]

This along with (58) gives

\[
\Upsilon_{X(O_k)} \{ \xi_k \varphi_n | n \geq N_3 \} \leq \Upsilon_{X(O_k)} P_t \{ \xi_k \varphi_n | n \geq N_3 \} + \Upsilon_{X(O_k)} (I - P_t) \{ \xi_k \varphi_n | n \geq N_3 \} \leq 2\eta.
\]

Note that \( \xi_k \varphi_n = \varphi_n \) on \( O_{k/2} \) (due to \( \xi \equiv 1 \) on \([0, 1/4]) \), it yields

\[
\Upsilon_{X(O_{k/2})} \{ \varphi_n | n \geq N_3 \} = \Upsilon_{X(O_{k/2})} \{ \xi_k \varphi_n | n \geq N_3 \} \leq \Upsilon_{X(O_k)} \{ \xi_k \varphi_n | n \geq N_3 \} \leq 2\eta.
\]

(59)

Note that \( k/2 \geq k_0 \), by (57) and (59), we obtain

\[
\Upsilon_X \{ \varphi_n | n \geq N_3 \} \leq \Upsilon_{X(O_{k/2})} \{ \varphi_n | n \geq N_3 \} + \Upsilon_{X(O_{k/2})} \{ \varphi_n | n \geq N_3 \} \leq 2\eta + \Upsilon_{X(O_{k_0})} \{ \varphi_n | n \geq N_1 \} \leq 4\eta,
\]

which proves the backward asymptotic compactness as desired.

**Step 3.** We show existence of a backward compact random attractor. Since the backward asymptotic compactness implies the usual asymptotic compactness, the abstract existence theorem (see e.g. Wang[31]) shows that the cocycle \( \Phi \) has a unique \( \mathcal{D} \)-pullback random attractor \( A \) as given by (55), where the measurability of the attractor \( A \) follows from the measurability of the cocycle (see Proposition 1) and the measurability of the absorbing set (see Proposition 2).

Next, we show \( A \) is backward compact. It suffices to show that \( \cup_{s \leq \tau} A(s, \omega) \) is pre-compact for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). We take an arbitrary sequence \( \{ \varphi_n \} \) from the union and choose \( s_n \leq \tau \) such that \( \varphi_n \in A(s_n, \omega) \). Also, we take a sequence \( 0 < t_n \to +\infty \). By the invariance of the attractor,

\[
\varphi_n \in A(s_n, \omega) = \Phi(t_n, s_n - t_n, \theta_{-t_n} \omega) A(s_n - t_n, \theta_{-t_n} \omega).
\]

So, there are \( \varphi_{n,0} \in A(s_n - t_n, \theta_{-t_n} \omega) \) such that

\[
\varphi_n = \Phi(t_n, s_n - t_n, \theta_{-t_n} \omega) \varphi_{n,0}.
\]

Since \( A \in \mathcal{D} \), it follows from the backward asymptotic compactness (proved in Step 2) that \( \{ \Phi(t_n, s_n - t_n, \theta_{-t_n} \omega) \varphi_{n,0} \} \) has a convergent subsequence and so is \( \{ \varphi_n \} \). \( \square \)

6.2. **Backward compactness from a periodic force.** We can obtain a backward compact random attractor if we only assume that the force is periodic.

**Hypothesis G4.** The force \( g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3)) \) is \( T \)-periodic with \( T > 0 \), i.e.,

\[
g(t + T, x) = g(t, x), \ \forall t \in \mathbb{R}, \ x \in \mathbb{R}^3.
\]

**Theorem 6.2.** Let the hypotheses B,F,S be satisfied and \( g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3)) \). We have the following two conclusions.

(i) The cocycle \( \Phi \) has a backward compact random attractor if \( g \) is \( T \)-periodic.

(ii) The cocycle \( \Phi \) has a periodic random attractor if both \( \beta \) and \( g \) are \( T \)-periodic.

**Proof.** (i) By Theorem 6.1, it suffices to show that the hypothesis G4 can imply all hypotheses G1, G2 and G3.
Let $g$ be $T$-periotic, $\alpha > 0$ and $\tau \in \mathbb{R}$. We have

$$\sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha r} \|g(r + s)\|^2 dr = \sup_{s \leq \tau} \sum_{i=0}^{\infty} \int_{s-(i-1)T}^{s-iT} e^{\alpha(r-s)} \|g(r)\|^2 dr \leq \sup_{s \leq \tau} \sum_{i=0}^{\infty} e^{-\alpha iT} \int_{s-(i-1)T}^{s-iT} \|g(r)\|^2 dr = \frac{1}{1-e^{-\alpha T}} \int_{0}^{T} \|g(r)\|^2 dr < \infty,$$ (60)

which implies Hypothesis G1.

Let $P_n$ be the projection and $\xi_k$ the cut-off function as given in Section 4. We claim that the periodicity does not change under the operator $P_n$:

$$P_n(\xi_k g(\sigma + T)) = P_n(\xi_k g(\sigma)), \ \forall \sigma \in \mathbb{R}. \ \ \ \ (61)$$

Indeed, by the expansion of the Fourier series in $L^2(O_k)$,

$$P_n(\xi_k g(\sigma + T)) = \sum_{i=1}^{n} e_i \int_{O_k} \xi_k(x) g(\sigma + T, x) e_i(x) dx = \sum_{i=1}^{n} e_i \int_{O_k} \xi_k(x) g(\sigma, x) e_i(x) dx = P_n(\xi_k g(\sigma)).$$

From (61), we know that $(I - P_n)\xi_k g(\sigma)$ is $T$-periodic in $\sigma \in \mathbb{R}$. By using this fact, we obtain

$$\sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha r} \|(I - P_n)(\xi_k g(\sigma + s))\|^2 d\sigma$$

$$= \sup_{s \leq \tau} \sum_{i=0}^{\infty} \int_{s-(i-1)T}^{s-iT} e^{\alpha(r-s)} \|(I - P_n)(\xi_k g(\sigma))\|^2 d\sigma$$

$$\leq \sup_{s \leq \tau} \sum_{i=0}^{\infty} e^{-\alpha iT} \int_{s-(i-1)T}^{s-iT} \|(I - P_n)(\xi_k g(\sigma))\|^2 d\sigma$$

$$= \frac{1}{1-e^{-\alpha T}} \int_{0}^{T} \|(I - P_n)(\xi_k g(\sigma))\|^2 d\sigma.$$ (62)

Since $g \in L^2([0, T], L^2(\mathbb{R}^3))$, $\|\xi_k\|_{\infty} = 1$ and $\|I - P_n\| = 1$, we have

$$\sup_{n \in \mathbb{N}} \int_{0}^{T} \|(I - P_n)(\xi_k g(\sigma))\|^2 d\sigma \leq \int_{0}^{T} \|g(\sigma)\|^2 d\sigma < +\infty.$$

Then, the Lebesgue controlled convergence theorem gives

$$\lim_{n \to \infty} \int_{0}^{T} \|(I - P_n)(\xi_k g(\sigma))\|^2 d\sigma = 0.$$ (63)

Therefore, both (62) and (63) imply that

$$\lim_{n \to \infty} \sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha r} \|(I - P_n)(\xi_k g(\sigma + s))\|^2 d\sigma = 0, \ \forall k \in \mathbb{N},$$ (64)

which proves the hypothesis G2.

By the same method as given in (62), we have

$$\sup_{s \leq \tau} \int_{-\infty}^{0} e^{\alpha r} \int_{O_k} |g(r + s, x)|^2 dx dr \leq \frac{1}{1-e^{-\alpha T}} \int_{0}^{T} \int_{O_k} |g(r, x)|^2 dx dr.$$

Then, the Lebesgue controlled convergence theorem shows the hypothesis G3.
(ii) Suppose both $\beta$ and $g$ are $T$-periodic, then, the uniqueness of solutions implies that the cocycle $\Phi$ is $T$-periodic: $\Phi(t, \tau + T, \omega) = \Phi(t, \tau, \omega)$.

On the other hand, we claim that the universe $\mathcal{D}$ (defined by (16)) is $T$-invariant. Let $D \in \mathcal{D}$, that is,

$$\lim_{t \to +\infty} e^{-\frac{\beta}{2} t} \sup_{s \leq \tau} \|D(s - t, \theta_{-\tau} \omega)\|_X^2 = 0, \quad \forall \tau \in \mathbb{R}.$$ 

It suffices to show $\mathcal{D}_T \in \mathcal{D}$ and $\mathcal{D}_{-T} \in \mathcal{D}$, where

$$\mathcal{D}_T(\tau, \omega) := D(\tau + T, \omega), \quad \mathcal{D}_{-T}(\tau, \omega) := D(\tau - T, \omega).$$

Indeed, we have

$$e^{-\frac{\beta}{2} t} \sup_{s \leq \tau} \|D_T(s - t, \theta_{-\tau} \omega)\|_X^2 = e^{-\frac{\beta}{2} t} \sup_{s \leq \tau} \|D(s - t + T, \theta_{-\tau} \omega)\|_X^2$$

$$= e^{-\frac{\beta}{2} t} \sup_{s \leq \tau + T} \|D(s - t, \theta_{-\tau} \omega)\|_X^2 \to 0 \text{ as } t \to +\infty.$$ 

It is similar to show $\mathcal{D}_{-T} \in \mathcal{D}$. So, the periodicity of the attractor follows from the abstract result given by Wang\cite{31} immediately.

**Remark 3.** The hypothesis $G2$ is weaker than Assumption $V$ in $\cite{38}$:

$$\lim_{n \to \infty} \sup_{s \leq \tau} \|((I - P_n)(\xi_k g(s)))\|^2 = 0, \quad \forall k \in \mathbb{N}.$$ 

**REFERENCES**

[1] J. M. Ball, Global attractors for damped semilinear wave equations, *Discrete Cont. Dyn. Syst.*, 10 (2004), 31–52.
[2] P. Bates, K. Lu and B. Wang, Attractors of non-autonomous stochastic lattice systems in weighted spaces, *Physica D*, 289 (2014), 32–50.
[3] T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss and J. Valero, Asymptotic behaviour of a stochastic semilinear dissipative functional equation without uniqueness of solutions, *Discrete Cont. Dyn. Syst. Ser. B*, 14 (2012), 439–455.
[4] I. Chueshov, *Monotone Random Systems Theory and Applications*, vol.1779, Springer Science & Business Media, 2002.
[5] I. Chueshov and I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, *Memoirs of Amer Math Soc*, 195 (2008), viii+i83 pp.
[6] I. Chueshov, I. Lasiecka and D. Toundykov, Global attractor for a wave equation with nonlinear localized boundary damping and a source term of critical exponent, *J. Dynam Differ. Equ.*, 21 (2009), 269–314.
[7] H. Cui, J. A. Langa and Y. Li, Regularity and structure of pullback attractors for reaction-diffusion type systems without uniqueness, *Nonlinear Anal.*, 140 (2016), 208–235.
[8] H. Cui, J. A. Langa and Y. Li, Measurability of random attractors for quasi strong-to-weak continuous random dynamical systems, *J. Dynam. Differ. Equ.*, 30 (2018), 1873–1898.
[9] H. Cui, M. M. Freitas and J. A. Langa, On random cocycle attractors with autonomous attraction universes, *Discrete Cont. Dyn. Syst. Ser. B*, 22 (2017), 3379–3407.
[10] H. Cui, P. E. Kloeden and F. Wu, Pathwise upper semi-continuity of random pullback attractors along the time axis, *Physica D*, 374/375 (2018), 21–34.
[11] X. Fan, Random attractors for damped stochastic wave equations with multiplicative noise, *Int. J. Math.*, 19 (2008), 421–437.
[12] A. Gu and P. E. Kloeden, Asymptotic behavior of a nonautonomous $p$-Laplacian lattice system, *Intern. J. Bifur. Chaos*, 26 (2016), 1650174, 9pp.
[13] X. Han and P. E. Kloeden, Non-autonomous lattice systems with switching effects and delayed recovery, *J. Differ. Equ.*, 261 (2016), 2986–3009.
[14] R. Jones and B. Wang, Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms, *Nonlinear Anal. RWA*, 14 (2013), 1308–1322.
[15] A. K. Khanmamedov, Global attractors for wave equations with nonlinear interior damping and critical exponents, *J. Differ. Equ.*, 230 (2006), 702–719.
[16] P. E. Kloeden and T. Lorenz, Construction of nonautonomous forward attractors, *Proc. Amer. Math. Soc.*, 144 (2016), 259–268.

[17] P. E. Kloeden and J. Simsen, Attractors of asymptotically autonomous quasi-linear parabolic equation with spatially variable exponents, *J. Math. Anal. Appl.*, 425 (2015), 911–918.

[18] P. E. Kloeden, J. Simsen and M. S. Simsen, Asymptotically autonomous multivalued cauchy problems with spatially variable exponents, *J. Math. Anal. Appl.*, 445 (2017), 513–531.

[19] P. E. Kloeden and M. Yang, Forward attraction in nonautonomous difference equations, *J. Differ. Equ. Appl.*, 22 (2016), 513–525.

[20] D. Li, K. Lu, B. Wang and X. Wang, Limiting behavior of dynamics for stochastic reaction-diffusion equations with additive noise on thin domains, *Discrete Contin. Dyn. Syst.*, 38 (2018), 187–208.

[21] D. Li, B. Wang and X. Wang, Limiting behavior of non-autonomous stochastic reaction-diffusion equations on thin domains, *J. Differ. Equ.*, 262 (2017), 1575–1602.

[22] F. Li, Y. Li and R. Wang, Regular measurable dynamics for reaction-diffusion equations on narrow domains with rough noise, *Discrete Cont. Dyn. Syst.*, 38 (2018), 3663–3685.

[23] Y. Li, A. Gu and J. Li, Existence and continuity of bi-spatial random attractors and application to stochastic semilinear Laplacian equations, *J. Differ. Eq.*, 258 (2015), 504–534.

[24] Y. Li, L. She and R. Wang, Asymptotically autonomous dynamics for parabolic equations, *J. Math. Anal. Appl.*, 459 (2018), 1106–1123.

[25] Y. Li, R. Wang and J. Yin, Backward compact attractors for non-autonomous Benjamin-Bona-Mahony equations on unbounded channels, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 2569–2586.

[26] Y. Li, L. She and J. Yin, Equi-atraction and backward compactness of pullback attractors for point-dissipative Ginzburg-Landau equation, *Acta. Math. Sci.*, 38 (2018), 591–609.

[27] Y. Li, L. She and J. Yin, Longtime robustness and semi-uniform compactness of a pullback attractor via nonautonomous PDE, *Discrete Cont. Dyn. Syst. Ser. B*, 23 (2018), 1535–1557.

[28] Y. Li and J. Yin, A modified proof of pullback attractors in a Sobolev space for stochastic Fitzhugh-Nagumo equations, *Discrete Cont. Dyn. Syst. Ser. B*, 21 (2016), 1203–1223.

[29] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equ.*, 253 (2012), 1544–1583.

[30] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

[31] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Cont. Dyn. Syst.*, 34 (2014), 269–300.

[32] B. Wang, Fractal dimension of random attractor for stochastic non-autonomous strongly damped wave equation with linear multiplicative white noise, *J. Differ. Equ.*, 264 (2013), 1–36.

[33] B. Wang, Random attractors for stochastic strongly damped damped 3D Navier-Stokes equations, *J. Differ. Equ.*, 254 (2013), 1–36.

[34] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

[35] B. Wang, Random attractors for stochastic strongly damped damped 3D Navier-Stokes equations, *J. Differ. Equ.*, 254 (2013), 1–36.

[36] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

[37] B. Wang, Random attractors for stochastic strongly damped damped 3D Navier-Stokes equations, *J. Differ. Equ.*, 254 (2013), 1–36.

[38] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

[39] B. Wang, Random attractors for stochastic strongly damped damped 3D Navier-Stokes equations, *J. Differ. Equ.*, 254 (2013), 1–36.

[40] B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

Received June 2018; revised August 2018.

E-mail address: rwang-math@outlook.com
E-mail address: liyr@swu.edu.cn