Holographic multiverse and conformal invariance

Jaume Garriga\(^1\) and Alexander Vilenkin \(^2\)

\(^1\) Departament de Física Fonamental i Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, 08193 Barcelona, Spain

\(^2\) Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, MA 02155, USA

Abstract

We consider a holographic description of the inflationary multiverse, according to which the wave function of the universe is interpreted as the generating functional for a lower dimensional Euclidean theory. We analyze a simple model where transitions between inflationary vacua occur through bubble nucleation, and the inflating part of spacetime consists of de Sitter regions separated by thin bubble walls. In this model, we present some evidence that the dual theory is conformally invariant in the UV.
I. INTRODUCTION

Defining the probability measure in an eternally inflating universe is one of the key unresolved problems of inflationary cosmology. Eternal inflation produces an infinite number of “pocket universes”, in which all possible events happen – not once, but an infinite number of times. We have to learn how to regulate and compare these infinities, since otherwise we cannot distinguish between probable and highly improbable events, and thus cannot make any predictions at all.

In “multiverse” models with a multitude of different vacua, eternal inflation gives rise to a fractal pattern, where pockets of all possible vacua are nested within one another. An important problem in this type of model is to find the probability distribution for the values of low-energy constants of nature, such as the parameters of the standard model of particle physics. Once again, we need some way of regulating infinities, and the problem here is exacerbated by the complicated spacetime structure of the multiverse.

The essence of the problem is that the numbers of all kinds of events in an eternally inflating universe are growing exponentially with time. Whatever cutoff method is used, most of the events occur near the cutoff, and the resulting probability measure depends sensitively on the cutoff prescription. This is the so-called “measure problem” of inflationary cosmology. (For a review see [1, 2].)

So far, most of the work on the measure problem has been phenomenological. Different measure proposals have been examined to check whether or not they lead to inconsistencies or to a glaring conflict with the data. (For recent discussion and references, see, e.g., [3, 4, 5, 6].) Other selection criteria have also been introduced. For example, it has been argued that the probability measure should not depend on the initial conditions at the beginning of inflation. It seems rather unlikely, however, that this kind of analysis will lead to a unique prescription for the measure.

A more satisfactory approach would be to motivate the choice of measure from some fundamental theory. Attempts in this direction have been made in [7, 8, 9, 10, 11]. In particular, in Ref. [10] we suggested that the dynamics of the inflationary multiverse could have a dual description in the form of a lower-dimensional Euclidean field theory defined at the future infinity. The measure of the multiverse can then be defined by imposing a Wilsonian ultraviolet cutoff $\xi$ in that theory. We argued that in the limit of $\xi \to 0$, the
boundary theory becomes conformally invariant, approaching a UV fixed point. We also argued that on super-horizon scales the UV cutoff $\xi$ corresponds to a scale factor cutoff in the bulk theory.

In the present paper we shall further explore the holographic duality proposed in Ref. [10], filling in some of the missing details. In Section II we consider the simple model where transitions between different vacua occur through bubble nucleation, and the inflating part of spacetime consists of de Sitter regions separated by thin bubble walls. We discuss the structure of future infinity in this model, focusing in particular on the eternal set $E$, defined by eternal timelike curves which always remain within the inflating region and never encounter terminal bubbles of negative or zero vacuum energy density. We show that (i) the boundary metric on $E$ can be chosen to be flat and (ii) the bubble distribution on the boundary is then approximately invariant under the Euclidean conformal group, with the invariance becoming exact in the UV limit. This supports the conjecture that the dual boundary theory should be conformally invariant in the UV.

By analogy with AdS/CFT correspondence [13, 14, 15], we suggested in Ref. [10] that the correspondence between the multiverse theory in the bulk and its dual on the future boundary is expressed by the relation

$$\Psi[\bar{\phi}(x)] \equiv \int D\phi \ e^{iS[\phi]} = e^{iW[\bar{\phi}]}.$$ (1)

Here, $S$ is the bulk action and the integral is over bulk fields $\phi$ approaching the prescribed $\phi = \bar{\phi}(x)$ at the boundary. The amplitude $\Psi[\bar{\phi}(x)]$ has the meaning of the wave function of the universe, and $W[\bar{\phi}]$ is the effective action for the boundary theory with the appropriate couplings to the external sources $\bar{\phi}$. In Section III we describe this proposal in more detail, with emphasis on the IR/UV connection and the implications for the measure problem.

To gain further insight into the properties of the boundary theory, in Section IV we calculate $W$ for linearized perturbations around the model of nested bubbles. First we consider tensor modes in de Sitter space. This can be used to find correlators like $\langle \bar{h}(x)\bar{h}(x') \rangle$ when the points $x, x'$ are within the same bubble. The functional form of the corresponding $W$ is consistent with that expected in a conformal field theory. Then, we consider fluctuations of the bubble walls in the approximation where the self-gravity of the bubble can be neglected. In the boundary theory, the bubble walls mark the boundaries between regions with different central charge, or different number of field degrees of freedom. Such boundaries give a con-
tribution to the trace anomaly which depends on their shape. We evaluate this contribution from the asymptotic form of the bulk wave function for the bubble wall fluctuations. Once again, the result is consistent with conformal invariance of the boundary theory.

Finally, in Section V we summarize our conclusions and discuss some open issues.

When this paper was nearly completed, we learned of the work in progress by Stephen Shenker, Douglas Stanford and Leonard Susskind, which has some overlap with the ideas presented here.

A related interesting development is the work of Freivogel and Kleban [12]. They consider a model where bubbles nucleate in a de Sitter background, and compute correlators for operators which characterize the bubble distribution at the future boundary. They find that these correlators are conformally invariant, and they also discuss a dual CFT interpretation of their results.

II. THE MODEL OF NESTED DE SITTER BUBBLES

A. Flat foliations

A diagram illustrating the causal structure of an eternally inflating spacetime is shown in Fig. 1. Bubbles of all possible types nucleate and expand, rapidly approaching the speed of light. The worldsheets of the bubble walls can therefore be approximated as light cones in the diagram. The future boundary of this spacetime includes the singular boundary corresponding to the big crunch singularities of the negative-energy anti-de Sitter (AdS) bubbles, “hats” corresponding to the future null and timelike infinities of the Minkowski bubbles, and the eternal set \( \mathcal{E} \) which is the focus of our interest in this paper. AdS and Minkowski bubbles are called “terminal bubbles”, since inflation completely terminates in their interiors. The eternal set \( \mathcal{E} \) consists of the spacelike future boundaries of the inflating de Sitter (dS) bubbles – or rather what remains of these boundaries after we remove the regions eaten up by terminal bubbles. We can think of \( \mathcal{E} \) as the set of “endpoints” of eternal timelike curves, which never encounter terminal bubbles. (A more formal definition will be given in Section III.)

We shall start by considering a simplified model in which the inflating part of spacetime consists of pure dS regions separated by thin bubble walls. This part of spacetime can be
foliated by spacelike surfaces $\Sigma_t$, labeled by some coordinate $t$. The key observation now is that in our simple model of nested dS bubbles, the spacetime region of interest can be foliated by flat Euclidean surfaces. This is not difficult to understand [16]. A dS space inflating at a rate $H$ can be represented as a hyperboloid

$$X^2 + Y^2 - T^2 = H^{-2}$$  \hfill (2)

embedded in a $5D$ Minkowski space. The flat foliation of this space is obtained by slicing it with null hyperplanes

$$Y - T = \text{const.}$$  \hfill (3)

The resulting metric can be written as.

$$ds^2 = -H^{-2}dt^2 + e^{2t}d\mathbf{x}^2,$$  \hfill (4)
where \( t \) is the scale factor time, that is, time measured in units of \( H^{-1} \). (This coordinate system covers only half of the full dS space, but this is sufficient to cover the interior of any bubble.)

The spacetime geometry for a bubble of vacuum \( \mathcal{V}' \) in a background of vacuum \( \mathcal{V} \) can be obtained by matching the hyperboloid (4) representing the parent vacuum with a similar hyperboloid for the daughter vacuum,

\[
X^2 + (Y - Y_0)^2 - T^2 = H'^{-2},
\]

where the displacement \( Y_0 \) depends on the tension of the bubble wall. The two hyperboloids are joined along a (2+1)-dimensional hyperbolic surface which represents the worldsheet of the wall (see Fig. 2).

\[\text{FIG. 2: The spacetime representing a bubble of lower-energy vacuum (shaded) expanding in a higher-energy de Sitter space. One of the flat foliation surfaces is also shown.}\]
Once again, a flat foliation can be obtained by slicing this hybrid construction with null hyperplanes \[3\]. Each slice will then consist of a spherical region of vacuum \( \mathcal{V}' \) embedded in an infinite region of vacuum \( \mathcal{V} \) (or vice versa). Since the spatial geometry of the slices is flat both inside and outside the bubble, the volume that is removed from vacuum \( \mathcal{V} \) is equal to the volume of vacuum \( \mathcal{V}' \) which replaces it. As a result, the two regions match without any discontinuity, even though there is a \( \delta \)-function curvature singularity along the domain wall in 4D.

The metric in the bubble interior can also be brought to the form \[4\], but since the expansion rates in the two vacua are different, the corresponding time variables do not match at the bubble wall. In a flat slicing, such as \[4\], the physical radius of the bubble is given by \[16\]

\[
R^2(t) = (H^{-1}e^t - D)^2 + R_0^2, \tag{6}
\]

where \( D = (H^{-2} - R_0^2)^{1/2} \) and \( R_0 \) is the constant intrinsic curvature radius of the worldsheet (this constant is determined by the bubble wall tension and the vacuum energies on both sides of the wall). The radius is the same when viewed from inside and from outside, and thus we must have

\[
t' = t + \ln(H'/H) + O(e^{-t}). \tag{7}
\]

For \( t, t' \gg 1 \), the time shifts in \( t \) and \( t' \) are related by \( \Delta t' = \Delta t \) and correspond to equal multiplicative changes in the scale factors \( a(t) = e^t \).

This argument can be extended to any number of bubbles of any kind nucleating in the parent vacuum, as well as to bubbles nucleating within those bubbles, etc. A slice \( \Sigma_t \) through the inflating region of the multiverse can then be chosen as a 3D surface having a flat Euclidean geometry. (The label \( t \) here can be chosen as the time variable in the parent vacuum.) Each bubble which is crossed by this surface is represented by a spherical region, with bubbles nucleating later having smaller images on \( \Sigma_t \). The image of an inflating bubble will generally have images of daughter bubbles nested within it. The image will therefore look like a “sponge”, whose “pores” contain vacua of different types. Some of the “pores” may correspond to other inflating vacua, in which case they will themselves look like sponges, and so forth. The “pores” corresponding to terminal vacua are represented by “holes” in \( \Sigma_t \). As we move to later and later times \( t \), we will see more and more of this structure. In the limit of \( t \to \infty \), each “sponge” becomes a fractal set, with almost all volume eaten up
by terminal bubbles.

We note that the flat foliation we have just described is not the same as the scale factor foliation (with the usual definition of the scale factor through the expansion along a geodesic congruence). The reason is that geodesics undergo some focusing at domain walls and the congruence takes some time to adjust to the new expansion rate as it crosses from one vacuum to another. As a result, the flat slices deviate from constant scale factor surfaces, typically by $\delta t \sim 1$. This deviation, however, is rather insignificant, and we shall refer to the variable $t$ as the scale factor time.

The size distribution for bubbles of different kinds on the flat slices can be found using the formalism developed in Refs. [16, 17]. Each bubble can be labeled by two indices $\{ij\}$, where $i$ refers to the vacuum inside the bubble and $j$ to the parent vacuum in which it nucleated. The number of type-$ij$ bubbles nucleated in a unit comoving volume, $V(t) = e^{3t}$, in an infinitesimal time interval $dt$ is given by

$$dN_{ij} = e^{3t} \lambda_{ij} f_j H_j^{-1} dt.$$  

(8)

Here, $\lambda_{ij}$ is the bubble nucleation rate per unit physical spacetime volume, $f_j(t)$ is the fraction of space in the 3D slice occupied by vacuum $j$, and there is no summation over $j$. The evolution equation for $f_i(t)$ can be written as

$$\frac{df_i}{dt} = \sum_j M_{ij} f_j,$$  

(9)

where

$$M_{ij} = \kappa_{ij} - \delta_{ij} \sum_m \kappa_{mi}$$  

(10)

and

$$\kappa_{ij} \equiv \lambda_{ij} \frac{4\pi}{3} H_j^{-4}$$  

(11)

is the probability for a bubble to nucleate per Hubble volume per Hubble time.

The asymptotic solution of (9) at large $t$ has the form

$$f_j(t) = f_j^{(0)} + s_j e^{-qt} + ...$$  

(12)

Here, $f_j^{(0)}$ has nonzero components only in terminal vacua, $-q < 0$ is the dominant eigenvalue of the matrix $M$ (that is, the largest nonzero eigenvalue) and $s_j$ is the corresponding
Substituting (12) into (8), we have
\[ dN_{ij} = \lambda_{ij} s_j e^{(3-q)t} H_j^{-1} dt. \] (13)

Now, consider a bubble of type \( ij \) which was formed at time \( t \). The radius of this bubble at \( t' \gg t \) is
\[ R(t', t) \approx H_j^{-1} e^{t' - t}. \] (14)
Expressing \( t \) in terms of \( R \) and substituting in (13), we obtain the bubble distribution
\[ dN_{ij} = C_{ij} R^{-(4-q)} dR, \] (15)
where \( C_{ij} \) is a constant (\( R \)-independent) coefficient. We thus see that the size distribution for all bubble types follows the same power law (15).

**B. Conformal invariance**

It follows from Eq. (14) that a forward time translation by \( \Delta t \) results in a rescaling of (large) bubble sizes by the same factor \( \exp(\Delta t) \). Since the size distribution of bubbles (15) is a power law, the form of the distribution is unchanged under the rescaling.

Factoring out the scale factor \( e^{t'} \), we can introduce the comoving bubble radius,
\[ r(t', t) = e^{-t'} R(t', t), \] (16)
which approaches a constant value at \( t' \to \infty \),
\[ r \to H_j^{-1} e^{-t}, \] (17)
where \( t \) is the time of bubble nucleation. In terms of the comoving coordinates \( x \), we thus have an asymptotically static distribution of bubbles at future infinity,
\[ dN_{ij} \propto r^{-(4-q)} dr. \] (18)
This distribution was derived using the asymptotic solution (12) for \( f_j(t) \) at large \( t \), so we expect it to apply only approximately, becoming exact in the limit \( r \to 0 \). The shape of

---

1 It has been shown in [17] that for an irreducible landscape of vacua, in which any vacuum can be reached from any other nonterminal vacuum by a sequence of transitions, the dominant eigenvalue is real, negative, and nondegenerate.
the distribution at large $r$ is not universal and is influenced by the initial conditions at the onset of inflation (e.g., the kind of parent vacuum we start with, etc.)

The distribution \[dN \propto r^{-(d+1)}dr\] is in agreement with scale invariance: in a space of dimension $d$, a scale invariant distribution has the form $dN \propto r^{-(d+1)}dr$, while the fractal dimension of the eternal set $\mathcal{E}$ is given by $d = 3 - q$ \[16, 17\].

Another asymptotic symmetry of the bubble distribution is related to the possibility of choosing different flat foliations of spacetime. These foliations are related by Lorentz transformations in the 5D embedding space, or equivalently, by de Sitter group transformations in the parent vacuum region. Each foliation defines a coordinate system and a congruence of timelike geodesics $x = \text{const}$ in the parent vacuum. Any two such congruences become asymptotically comoving (in the region where they overlap) and thus define a coordinate transformation $x \rightarrow x'$ at the future infinity. In the Appendix we show that it is a transformation of the form

\[\frac{x'^i}{x'^2} = \frac{x^i}{x^2} - b^i,\]  

accompanied by a rotation. (Here, $b^i$ is the dS boost parameter.) The transformation \[19\] is the so-called special conformal transformation (SCT). It can be described as a translation preceded and followed by inversions. An important property of SCTs is that they map spheres into spheres, and thus the spherical shape of the bubbles is preserved.

By the same argument, a change of foliation induces SCTs in the asymptotic future of all dS bubble interiors. By continuity across bubble boundaries, it follows that all transformations should be the same, so that the entire future boundary $\mathcal{E}$ is transformed by a single SCT (plus rotation).

For any flat slicing, the evolution of the bubble distribution is described by the same equations \[8, 9, 14\], although the initial conditions are generally different. However, since the late-time behavior is universal and independent of the initial state, the form of the distribution should be the same at $t \rightarrow \infty$. This means that the size distribution of the bubbles should be invariant under SCTs in the limit of $r \rightarrow 0$.

Together with dilatations, translations, and rigid rotations, SCTs comprise the Euclidean conformal group. Our conclusion is thus that the bubble distribution is conformally invariant at $r \rightarrow 0$. If indeed the multiverse has a dual description on $\mathcal{E}$, this suggests that the boundary theory should be conformally invariant in the UV.
C. Bubble collisions and the persistence of memory

So far in this discussion we have ignored bubble collisions. These are interesting in that they are sensitive to initial conditions, an effect which has been dubbed "the persistence of memory" \[18\]. This effect can be described as follows. Suppose we have some initial hypersurface of vacuum of type \(i\) which has no bubbles of any other phase. Then, memory of such initial condition persists arbitrarily far into the future. Indeed, the preferred congruence orthogonal to the initial "no bubble" hypersurface defines a rest frame. An observer in vacuum \(i\) at rest in this frame will be equally likely to be hit by a bubble of type \(j\) from any direction in space. In other words, the probability of collision with new bubbles is isotropic for this observer. However, if the observer moves with respect to the preferred congruence, then the observer is more likely to be hit head on by new bubbles.

When we look at the shapes drawn by the nucleated bubbles at the future boundary, this translates into the following effect. Let us first choose our foliation to be (locally) orthogonal to the preferred congruence. In these co-moving coordinates, any bubble which nucleates in the original vacuum \(i\) will be equally likely to be hit from any direction. At the future boundary, this will be seen as a big spherical bubble "decorated" isotropically by a froth of smaller bubbles.

However, if the foliation is boosted with respect to the preferred congruence, then the pattern of bubbles at the future boundary will be distorted by the corresponding SCT. Although the spherical shape of a bubble is preserved by inversions around an arbitrary point, the isotropic distribution of smaller spheres decorating it will be shifted preferentially in the direction determined by \(b^i\). As a result, the froth decorating the bubbles will be anisotropic.

Nonetheless, for a given value of the boost parameter \(b^i\), the effect will get smaller for bubbles nucleating at later times. The reason is that a congruence orthogonal to any flat foliation becomes asymptotically comoving to the preferred congruence. In this way, the conformal invariance is recovered in the UV.
D. More general foliations

Flat foliations of the kind we discussed so far are possible in the simple model of nested bubbles, but cannot be constructed in a general spacetime. To get an idea of what the general situation may be like, we shall now examine a wider class of surfaces.

For a general spacelike surface $\Sigma$, we can introduce a coordinate system (4) in each dS region that this surface intersects. In terms of these coordinates, the surface can be represented as

$$t = f(x).$$  \hfill (20)

The function $f(x)$ may change its form from one bubble to another, but we shall require that $\Sigma$ remains smooth at the bubble walls. We shall also assume that $f(x)$ is a slowly varying function, so that

$$e^{-f(x)}|\nabla f(x)| \ll H(x).$$ \hfill (21)

This guarantees that the typical curvature radii of $\Sigma$ are much greater than the local dS horizon. In other words, the surface is nearly flat on the horizon scale.

The induced metric on $\Sigma$ is

$$ds^2 = e^{2f(x)}dx^2 - H^{-2}\partial_i f \partial_j f dx^i dx^j.$$ \hfill (22)

The condition (21) allows us to drop the second term, so we can write

$$ds^2 \approx e^{2f(x)}dx^2$$ \hfill (23)

We thus see that the metric on $\Sigma$ is related to the flat Euclidean metric by a Weyl rescaling [10].

In a CFT, the observables have well known transformation properties under Weyl rescalings. Once we know the observables for a given metric, we can find their values for metrics in the same conformal class. In this sense, the boundary theory is Weyl covariant, and the conformal factor is a gauge redundancy. We fix the gauge by choosing a particular metric in the conformal class. For instance, the choice of a flat boundary metric has the advantage that the conformal invariance of the theory is manifest in some observables.
E. More general spacetimes

To get an idea of what the situation is in more general spacetimes, not necessarily piece-wise de Sitter, let us consider the case of a metric which is locally FRW, but with the rate of expansion varying from place to place on scales much bigger than the horizon. To be more precise, let us consider the metric of a spacetime whose expansion is isotropic. This means that the extrinsic curvature of equal time slices orthogonal to a geodesic congruence is proportional to the spatial metric on the slices. The metric can then be written as

$$ds^2 = -d\tau^2 + e^{2N} \gamma_{ij}(x)dx^i dx^j,$$

where \( N(\tau, x) \) can be interpreted as the number of e-foldings in proper time gauge. Introducing \( \eta = -e^{-N} \), we have

$$d\eta = -\eta dN = -\eta(Hd\tau + N_i dx^i),$$

where \( H = N_{,\tau} \). Hence,

$$ds^2 = \eta^{-2}[-H^{-2}(d\eta + \eta N_i dx^i)^2 + \gamma_{ij}dx^i dx^j].$$

In the limit \( \eta \to 0 \), the second term within the round brackets can be neglected,\(^2\) and we have

$$ds^2 \approx \eta^{-2}[-H^{-2}d\eta^2 + \gamma_{ij}dx^i dx^j].$$

Thus, we find that near the future boundary the metric of the inflating part of space-time is conformal to

$$ds^2 = -H^{-2}(\eta, x)d\eta^2 + \gamma_{ij}(x)dx^i dx^j.$$

This suggests that we can identify \( \mathcal{E} \) with the future conformal boundary of the metric \((27)\). Clearly, \( \gamma_{ij} \) plays the role of the metric at the conformal boundary. The arbitrariness in the choice of the congruence translates into Weyl rescalings of the metric at the conformal boundary. Indeed, in the inflating background, any two geodesic congruences become asymptotically co-moving to each other. Consider two geodesics that have the same endpoint \( x \) at the conformal boundary. By the time their relative speed becomes completely

\(^2\) In neglecting this term, we are implicitly assuming that the metric is smoothed over a fixed co-moving scale. The motivation for this smoothing will be clear from the discussion in the next Section, since the UV cut-off of the boundary theory corresponds to fixed co-moving scale.
Negligible, the number of e-foldings from the initial surface for both geodesics will be different by some $\Delta N(x)$, so in the asymptotic future we have $\eta = \eta' e^{2\Delta N(x)}$. Hence, the spatial metric on surfaces of very small constant $\eta$ will be related to that on surfaces of very small constant $\eta'$ by Weyl rescaling.

III. THE WAVE FUNCTION OF THE UNIVERSE AND THE IR/UV CORRESPONDENCE

The wave function of the universe can be expressed as a path integral

$$\Psi[\phi] = \int_{\Sigma_i}^{\Sigma_f} D\phi e^{iS},$$

(28)

where the integration is over the bulk fields $\phi(x)$ interpolating between some initial conditions on hypersurface $\Sigma_i$ and approaching the prescribed values $\bar{\phi}(x)$ on the surface $\Sigma$, which plays the role of the future boundary of the spacetime region of integration. The nature of the initial conditions on $\Sigma_i$ is the subject of some debate (for a review see [19] and references therein), but it will not be important for our discussion here. The reason is that an eternally inflating region of spacetime quickly forgets its initial conditions. The memory of the initial state is retained only on the largest comoving scales, while on smaller scales the evolution exhibits an attractor behavior. For example, we saw in Section II.A that the asymptotic bubble distribution (18) at short distances is independent of the initial conditions. This distribution is reached in any comoving region including at least one eternal geodesic.

Thus, to study the asymptotic small-scale behavior, we do not need to invoke the wave function for the entire universe. We could start, say, with a cubic region filled with vacuum $i$ and having size of a few $H_i^{-1}$. The conditions at the boundaries of this region are not important; we could, for example, impose periodic boundary conditions. In our simple model, the spacetime consists of nested regions of dS space separated by thin walls, and the evolution is nearly classical, except for occasional nucleation of bubbles. According to the discussion in Section II.A, this spacetime can be foliated by flat hypersurfaces, so the initial and final surfaces can be chosen to be flat. The future boundary can then be approximated by one of the surfaces $\Sigma_t$ and the only variables that need to be specified on that boundary are the types of vacua and the centers and radii of the corresponding bubbles. We expect that in the limit $t \to \infty$ all correlations of physical significance will not depend on the choice
of the initial vacuum on $\Sigma_i$.

The conjectured correspondence \textsuperscript{[1]} relates the wave function $\Psi[\vec{\phi}]$ for the bulk theory to the effective action of the boundary theory. The surface $\Sigma$, with regions covered by terminal bubbles removed, gives an approximate representation of the eternal set $\mathcal{E}$, as we describe below. As we shall see, the boundary theory, regularized with an appropriate UV cutoff, can also be thought of as living on that surface.

A. The discrete boundary theory and its continuum limit

A point $P \in \mathcal{E}$ can be thought of as the endpoint of an eternal timelike curve (that is, a curve that never encounters terminal bubbles). More precisely, “points” or elements of $\mathcal{E}$ are identified with chronological pasts of eternal timelike curves \textsuperscript{[20]}. Two curves with the same past define the same element of $\mathcal{E}$. Physically, an element of $\mathcal{E}$ is the equivalence class of eternal time-like curves which remain forever in mutual causal contact. An eternal curve can be pictured as the worldline of an observer and the corresponding boundary “point” $P$ is the spacetime region inside the causal horizon of that observer (see Fig. 3).

The image $P_\Sigma$ of a point $P \in \mathcal{E}$ on the surface $\Sigma$ is given by the intersection of $\Sigma$ with $P$ (see Fig. \textsuperscript{5}). In a pure dS space, $P_\Sigma$ would be a spherical region of radius equal to the dS horizon. In our model, the shape of $P_\Sigma$ will be distorted by intervening bubbles, but its size will still be roughly given by the local horizon. (If $P_\Sigma$ covers more than one vacuum, its size is typically set by the largest of the corresponding horizons.) In this way the entire eternal set $\mathcal{E}$ is mapped onto $\Sigma$, with each point represented by a roughly horizon-size region.

Now, instead of considering the boundary theory at $\mathcal{E}$, let us consider a boundary theory defined on $\Sigma$, which should approximate the theory at $\mathcal{E}$ on scales larger than a certain Wilsonian cut-off.

First of all, we note that from the point of view of the theory living on $\Sigma$, the maximum possible resolution is the size of the images $P_\Sigma$ of the points $P$ of the future boundary. This

\textsuperscript{3} Elements of $\mathcal{E}$ can also be identified with points in the spacelike part of the future conformal infinity, like the point $P$ in the conformal diagram in Fig. \textsuperscript{5}. This point is the endpoint of the eternal timelike curve $\gamma$, and the past of $\gamma$ appears in the diagram as the interior of the past light cone of $P$. This definition of $\mathcal{E}$ is possible only if the spacetime admits a conformal infinity, which may not be the case in general.
FIG. 3: Eternal curves $\gamma$ and $\gamma'$ have the same past and thus define the same point $P \in \mathcal{E}$. The intersection of this past (appearing in the figure as the past light cone of $P$) with a spacelike surface $\Sigma$ gives the image of $P$ on $\Sigma$.

resolution is position dependent,

$$l_c(x) \sim H^{-1}(x),$$

(29)

where $H^{-1}(x)$ is the local dS horizon radius. It is interesting to note that, with this choice, the number of degrees of freedom of the boundary theory on $\Sigma$ corresponds to that of the bulk theory.\(^4\) The argument here is similar to that of Susskind and Witten for AdS/CFT [21].

The number of bulk degrees of freedom in a horizon region of vacuum $i$ is given by the

\(^4\) In the present discussion, and following standard practice, number of degrees of freedom is synonymous with the logarithm of the number of quantum states that the system can be in.
Gibbons-Hawking entropy of the dS space,

\[ N_{\text{bulk}} \sim H_i^{-2}. \]  

To estimate the corresponding number \( N_{\text{boundary}} \) in the boundary theory, we note that, with the cutoff scale at \( l_c = 1/H \), the energy density of each field at the boundary cannot much exceed \( 1/H^4 \). This means that a horizon volume can contain at most \( \sim 1 \) quanta of frequency \( \sim H \). Longer wavelength modes contribute much less to the entropy, so we can disregard them. Thus, each field carries about one bit of information per horizon, and the entropy per horizon is of the order of the number of fields.\(^5\) For a conformal field theory, this number is related to the central charge of the theory. For the CFT dual to dS space, it has been estimated as \([22, 23]\)

\[ N_{\text{boundary}} \sim H_i^{-2}. \]  

(see also the discussion in Section IV.) Thus, we see that, with \( l_c \sim H_i^{-1} \), we have

\[ N_{\text{bulk}} \sim N_{\text{boundary}}. \]  

The theory on \( \Sigma \) can be thought of as a field theory only on scales larger than \( l_c \), which plays a role analogous to the lattice spacing in a discrete system. Borrowing the terminology of Ref. [8], we can call this irregular lattice “a fish net”. Each vertex of this fish net carries \( N_{\text{boundary}} \) “internal” degrees of freedom.

The continuum limit on \( \Sigma \) is obtained for physical wavelengths longer than a certain Wilsonian cutoff \( \xi_0 \), which should be much bigger than all “lattice spacings” \( H_i^{-1} \),

\[ \xi_0 \gg L_0 = \max_i H_i^{-1}, \]  

where the maximization is over all dS vacua. In the nested dS model, we can choose \( \Sigma \) to be one of the surfaces \( \Sigma_\tau \); then the Wilsonian cutoff \( \xi \) in co-moving coordinates must satisfy

\[ \xi \gg L(\tau) \equiv L_0 e^{-\tau}. \]  

Here, \( L(\tau) \) can be thought of as a physical UV cutoff, where the continuum limit breaks down. This cutoff shrinks as the bulk boundary surface \( \Sigma \) is moved to the future. As a result, at later times the eternal fractal \( \mathcal{E} \) can be seen at a greater and greater resolution.

\(^5\) We thank Raphael Bousso for clarifying this point to us.
This situation is analogous to the IR/UV correspondence in the AdS/CFT [21], with the late-time boundary playing the role of the IR cutoff in the bulk. The scale-factor time evolution in the bulk corresponds to the renormalization group flow in the boundary theory.\(^6\) The theory on \(\mathcal{E}\) is obtained in the limit \(t \to \infty\), where \(L(t) \to 0\) and there is no physical UV cutoff.

In our discussion so far we assumed the spacetime to the future of \(\Sigma\) to be well defined and classical. Of course, this is not so, even in the nested dS model. Spontaneous nucleation of bubbles allows for a multitude of possible histories, and the mapping of \(\mathcal{E}\) onto \(\Sigma\) will generally be different for different histories. We note, however, that these differences affect the mapping only on scales smaller than or comparable to the causal horizon, which in turn is smaller than \(L_0\). So, with the choice of the cutoff length satisfying (33), the mapping is insensitive to the future evolution.

The picture suggested by the above discussion is the following. The hypersurface \(\Sigma\) can be thought of as a discrete system consisting of a juxtaposition of “lumps” \(P_\Sigma\). From a bulk point of view, a lump is the interior of the causal horizon of an eternal observer. This has a finite number of internal degrees of freedom, proportional to the horizon area (it may therefore be appropriate to visualize \(P_\Sigma\) as a horizon sized closed shell embedded in \(\Sigma\), with the degrees of freedom living on the shell). At distance scales much larger than the size of the lumps, a continuum description emerges. The co-moving size of the lumps gets smaller as the reference surface \(\Sigma\) is pushed forward in time, and in the limit when it is sent to future infinity, the continuum description becomes valid all the way to the UV (where it should be conformally invariant).

B. The boundary measure

In Ref. [10] we proposed that a measure for calculating probabilities in the multiverse may be formulated at future infinity \(\mathcal{E}\). The idea was to use the Wilsonian cutoff \(\xi\) of the boundary theory to render the number of events finite. Any event occurring in the bulk,

\(^6\) For the case of Euclidean AdS, the Callan-Symanzik RG flow equations for the boundary theory can be derived from a semiclassical expansion of the Wheeler-de Witt equation in the bulk [24], with the scale factor playing the role of the renormalization scale. Similar considerations were presented in Refs. [25, 26] for the case of slow-roll inflation.
which requires for its description a bulk resolution corresponding to a physical wavelength $\lambda_{\text{min}}$, will leave an imprint on $E$. The resolution which is needed to reconstruct the event from its imprint at the future boundary will be given by

$$\xi = \lambda_{\text{min}} e^{-t_*},$$

(35)

where $t_*$ is the scale factor time of the event. For a fixed bulk resolution $\lambda_{\text{min}}$, the Wilsonian UV cutoff $\xi$ of the boundary theory corresponds to an IR scale factor cutoff at $t = t_*$ in the bulk, with $\xi$ and $t_*$ related by Eq. (35). Note that $\lambda_{\text{min}}$, which plays the role of a Wilsonian UV cut-off in the bulk, depends in principle on what kind of event we are interested in, and the cutoff time $t_*$ will have that dependence as well. An alternative approach is to set $\lambda_{\text{min}} \sim 1$, so that all events above the Planck scale will be resolved at $t < t_*$. 7

Our prescription for the boundary measure is not limited to the nested dS model or flat foliations. In the general case, we can use a scale factor foliation, starting with some surface $\Sigma_0$, which is smooth on scales larger than $L_0$ and otherwise arbitrary. The scale factor $a$ is defined as the expansion factor along the congruence of geodesics orthogonal to $\Sigma_0$, and the scale factor time is defined as $t = \ln a$. As before, each surface of constant $t$ is the site of a boundary theory with a UV cutoff $\xi$ satisfying (34), and the scale-factor time evolution corresponds to RG flow in the boundary theory.

The scale factor time is not a good foliation parameter in regions of structure formation, where geodesics converge and cross. However, these phenomena affect only sub-horizon scales, which are smoothed out by the super-horizon cutoff $\xi$. We require, therefore, that the foliation surfaces $\Sigma_t$ should also be smooth on scales $\lesssim \xi$.

Different choices of the initial surface $\Sigma_0$ are related by Weyl transformations on $E$. The freedom of choosing this surface can be used to obtain a foliation with desired properties. A flat foliation, like the one we discussed for the nested dS model, is possible only in very special cases. Its closest analogue in a more general spacetime is a foliation by surfaces

7 The correspondence between the boundary cut-off and the scale factor cut-off is only approximate. The main reason is that the scale factor time is not well defined on subhorizon scales. If one uses the density of a dust of test particles in order to define the expansion, then this becomes ill defined in regions of structure formation. One can use other definitions, but the choice is not unique. Moreover, Eq. (35) assumes that wavelengths are conformally stretched with the expansion. This is certainly the case on superhorizon scales, but on subhorizon scales the wavelengths of signals can be affected by causal processes other than the expansion of the universe.
having a vanishing Ricci scalar,

\[ R^{(3)} = 0. \]  \hspace{1cm} (36)

In a spacetime which is locally FLRW with small perturbations, this condition can always be satisfied by a position-dependent time shift. If we choose the initial 3-surface \( \Sigma_0 \) satisfying (36), it can be shown that this property is preserved (to linear order) by scale factor time evolution \cite{27}.

**IV. THE WAVE FUNCTION AND CONFORMAL INVARIANCE**

In this Section we go beyond the simple model of nested bubbles discussed in Sec.II and allow some perturbations in the inflating regions and on the spherical bubble walls. First, we consider massless fields (or metric perturbations) in the case of a single vacuum. Then, we consider the bubble walls which separate domains with different vacua. In general, the bubble wall fluctuations are entangled with the metric perturbations. Here, we shall consider bubble fluctuations in the region of parameter space where the self-gravity of the bubble can be ignored and the two types of perturbations can be disentangled. As we shall see, the effective action which is obtained through the correspondence (1) does have some of the features expected in a CFT.

For the case of massless fields in de Sitter, our discussion closely follows that of Maldacena in Ref. \cite{28}. A formal difference is in the expression of the conjectured dS/CFT correspondence. We have \( e^{iW} \) on the right-hand side of (1), while Ref. \cite{28} has \( e^{-W} \). If the effective action is expressed in terms of a boundary path integral, we propose that this should be written in the form

\[ e^{iW[\bar{\phi}]} = \int_{\mathcal{E}} D\psi e^{iS[\psi,\bar{\phi}]} . \]  \hspace{1cm} (37)

There is no time variable at the boundary, and in this sense the action \( \bar{S} \) for the boundary fields \( \psi \) is Euclidean. However, instead of using \( e^{-\bar{S}} \), as in standard Euclidean theories which are obtained by Wick rotation from Lorentzian time, here we propose using the phases \( e^{i\bar{S}} \). This is because the wave function \( \Psi \), given by the bulk path integral in (1), is complex.\(^8\) In fact, the phase of the wave function grows at late times in proportion to powers of the scale factor. In the boundary theory, these growing phases can be interpreted as ultraviolet

\(^8\) This is in contrast with the Hartle-Hawking wave function, which is real.
divergences, which can be removed by local counterterms in $\bar{S}$. With our conventions, the counterterms are real.

A. Massless fields in de Sitter

Let us consider the wave function of a free massless scalar field $h$ in de Sitter. The same wave function describes linearized gravitons in the transverse traceless gauge. For the sake of comparison with existing CFT calculations, let us work with a $(d+1)$-dimensional de Sitter space. In flat conformal coordinates, the metric is given by

$$ds^2 = a^2(\eta)[-d\eta^2 + dx^2],$$

with $a(\eta) = -1/H\eta$. Decomposing the field in Fourier components

$$h(x) = \int d^d k \frac{e^{ikx}}{(2\pi)^{d/2}} h_k,$$

the Gaussian solution of the Schrodinger equation takes the form $\Psi = e^{iW}$, with

$$W = \int d^d k \left( \frac{a^{d-1}}{2} \frac{v_k'}{v_k} |h_k|^2 + i \ln v_k \right).$$

Here a prime indicates derivative with respect to $\eta$, and $v_k(\eta)$ is a solution of the massless wave equation $v_k'' + (d-1)(a'/a)v_k' + k^2 v_k = 0$, which should be of “negative frequency”, $v_k^* v_k' - v_k v_k'^* = i a^{1-d}$, so that the Gaussian is normalizable. The Bunch-Davies vacuum corresponds to the choice

$$v_k(\eta) = \frac{\pi^{1/2}}{2H^{1/2}a^{-d/2}(\eta)} H_{d/2}^{(1)}(k\eta),$$

where $H_{d/2}^{(1)}$ is the Hankel function.

Let us consider the case $d+1 = 5$. At late times, after modes have crossed the horizon ($-k\eta \ll 1$), Eq. (40) takes the asymptotic form

$$W[h(x)] = \frac{1}{2} \int d^d k \left( \frac{-k^2 a^2}{2H^2} + \frac{k^4}{8H^3} [\ln(k^2/H^2 a^2) + i\pi + 2\gamma] + O(a^{-2}) \right) |h_k|^2 + ..., (42)$$

where the ellipsis denote terms which are independent of $h_k$. The structure of (42) is similar to the one obtained in the context of AdS/CFT by Gubser, Klebanov and Polyakov [14], except that, here, $W$ contains an imaginary part. The imaginary part should be there
because the amplitude of perturbations on superhorizon scales is determined by $|\Psi|^2 = e^{-2Im[W]}$. This has a well defined limit as $a \to \infty$,

$$|\Psi|^2 \propto \exp \left[ -\int d^4k \left( \frac{\pi}{8H^3k^4} \right) |h_k|^2 \right], \quad (43)$$
corresponding to a scale-invariant spectrum of perturbations $\langle h^*_k h_k' \rangle = (8H^3/\pi k^4)\delta(k' - k)$.

Let us now consider the real part of $W$. The first term diverges like $a^2$ as we approach the future boundary. This term is analytic in $k^2$, and its coefficient can be changed by adding boundary counterterms of the form $\int (\partial_i h)^2 d^4x$. The divergence of the kinetic term as $a \to \infty$ suggests that the field is non-dynamical at the boundary. Introducing $\mu = -\eta^{-1} = Ha$, the non-local part is written as

$$Re[W] = \frac{H^{-3}}{16} \int d^4k k^4 \ln(k^2/\mu^2) |h_k|^2 + \text{analytic}. \quad (44)$$

As mentioned above, the wave function for linearized gravitons in the traceless and transverse gauge is exactly the same as that for the massless scalar field $h$. The expression $(44)$ we obtain for $W$ does indeed take the standard form of an effective action for a conformal theory coupled to an external gravitational field in 4 dimensions. For the case of free fields, this was first discussed by Tomboulis [29].

The structure of $(44)$ is not difficult to understand. The renormalized effective action for a CFT propagating in a curved background takes the form [30] 9

$$W_{\text{ren}} = a_{d/2} \ln \mu^2 + \ldots \quad (45)$$

where the ellipsis indicate terms which are independent of $\mu$, and

$$a_{d/2} = \int \langle T \rangle \sqrt{g} \, d^d x \quad (46)$$
is the integrated trace anomaly in $d$ dimensions. When $d$ is even, $a_{d/2}$ is given in terms of an integral of geometric invariants constructed from contractions of the Riemann tensor. In $d = 4$, this takes the form

$$a_2 = \int d^4x \sqrt{g} \left[ c_1 R^2 + c_2 R_{ij} R^{ij} + c_3 R_{ijkl} R^{ijkl} \right]. \quad (47)$$

9 This can be understood as follows. A change in the renormalization scale $\mu \to \tilde{\mu}$ is equivalent to a global rescaling of the metric $g_{ij} \to \Omega^2 g_{ij}$, with $\Omega = \tilde{\mu}/\mu$. Hence $dW_{\text{ren}}/d \ln \mu = \int (\delta W_{\text{ren}}/\delta \ln \Omega) d^d x$. Since the trace anomaly $\langle T \rangle = (1/\sqrt{-g})\delta W_{\text{ren}}/\delta \ln \Omega$ should be independent of $\mu$, it follows that $W_{\text{ren}}$ is linear in $\log \mu$, which leads to the form $(45)$. We thank Igor Klebanov for a discussion of this point.
The coefficients $c_i$ are such that $a_2$ is a linear combination of the Euler number and the integral of the Weyl tensor squared. Both are invariant under Weyl rescalings (see e.g. [30] and references therein). More generally, $a_{d/2}$ is Weyl invariant in $d$ dimensions. To quadratic order in the metric perturbation $h_{ij}$, this leads to the structure $c_i k^4 |h_k|^2 \ln \mu^2$ in momentum space. The form of the non-analytic piece in (44) is recovered by noting that the argument of the logarithm must be dimensionless.

The numerical constants $c_i$ in front of the geometric invariants scale in proportion to the number of fields (central charge) in the CFT. Comparing with (44) we have

$$c \sim H^{-3}. \tag{48}$$

Parametrically, this is also the entropy of the bulk de Sitter space [22, 23, 28].

In the case of a 4 dimensional bulk ($d + 1 = 4$), substituting (41) into (40), and expanding at late times ($-k\eta \ll 1$), one obtains

$$W[\bar{h}(x)] = \frac{1}{2} \int d^3k \left(\frac{-k^2a}{H} + i \frac{k^3}{H^2} + O(a^{-1})\right) |h_k|^2 + ..., \tag{49}$$

The first term, which diverges as we approach the future boundary, is analytic, while the second one is non-local and imaginary. The second term leads to the familiar expression for the scale invariant amplitude of tensor modes $\langle h_{k}^{\ast}h_k \rangle = H^2 k^{-3} \delta(k'-k)$. In the present case, the Hankel functions are of half integer order, and do not lead to logarithmic terms in the asymptotic future. From the CFT point of view, this is in agreement with the fact that there is no trace anomaly in odd dimensions, and so there are no logarithmic divergences. The number of fields in the CFT can be estimated from the coefficient of the non-analytic piece in $W$. Again, this scales in proportion to the entropy of the bulk de Sitter space [22, 23, 28],

$$c \sim H^{-2}. \tag{50}$$

B. Bubble fluctuations.

Throughout this subsection we work in the approximation where gravity of the bubble is unimportant. In the thin wall limit, we require that

$$TR_0 \ll 1, \quad (\Delta \rho_V)R_0^2 \ll 1. \tag{51}$$

This follows from (45), by noting that the functional derivative of $W_{ren}$ with respect to $\ln \Omega$ is the anomalous trace $\langle T \rangle$, which should not depend on $\mu$. 

23
where $T$ is the tension of the bubble wall, $\Delta \rho_V$ is the energy density gap between the inside and the outside of the bubble wall, and $R_0$ is the intrinsic curvature radius of the worldsheet.

The conditions (51) mean that the geometry on the scale of the horizon is not appreciably distorted due to gravity of the bubble wall and due to the change in the vacuum energy. Under these approximations, $R_0$ is given by [31, 32]

$$R_0^2 \approx \frac{(p + 1)^2 T^2}{(p + 1)^2 H^2 T^2 + (\Delta \rho_V)^2}. \quad (52)$$

For reference, we have written the expression for arbitrary worldsheet dimension $p + 1$, although we are mostly interested in the case of ordinary membranes $p = 2$. Note that we allow the energy gap $\Delta \rho_V$ to be significant, even comparable to the energy density in the false vacuum, as long as (51) is satisfied.

The interest of this weak backreaction limit is that we can study fluctuations of the bubble wall without the need of coupling these to metric fluctuations. We will call this the “Goldstone limit”, since the only relevant degree of freedom is the one associated with normal displacements $\delta x^\mu$ of the worldsheet $x^\mu(\xi)$. In the flat chart of de Sitter space,

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2d\Omega^2) \quad (53)$$

the trajectory of a bubble wall centered at the origin of coordinates is given by

$$r_w^2(t) = H^{-2}(e^{-2Ht_0} + e^{-2Ht}) - 2H^{-1}(H^{-2} - R_0^2)^{1/2}e^{-H(t-t_0)}, \quad (54)$$

where $t_0$ is a free parameter which can be interpreted as the time of nucleation. The unit vector $n^\mu$ normal to the wall worldsheet is given by

$$n^t = R_0^{-1} \left( (H^{-2} - R_0^2)^{1/2} - H^{-1}e^{-H(t-t_0)} \right), \quad (55)$$

$$n^r = R_0^{-1} r_w(t)e^{-H(t-t_0)}. \quad (56)$$

We now parametrize the normal displacement of the wall as [33]

$$\delta x^\mu = T^{-1/2} \phi n^\mu. \quad (57)$$

The factor $T^{-1/2}$ is introduced so that $\phi$ is a worldsheet scalar with standard normalization. The linearized action for this scalar consists of a canonical kinetic term and a tachyonic mass term:

$$m_\phi^2 = -(p + 1)R_0^{-2}. \quad (58)$$
Let us now study the wave functional for this scalar field $\phi$. The worldsheet of the bubble is itself a de Sitter space of curvature radius $R_0$ and dimension $p + 1$. This worldsheet will cut a sphere at future infinity, and for that reason it is convenient to use the closed chart for the $p + 1$ dimensional de Sitter spacetime, with coordinates $\xi^i = (\tilde{\eta}, \Omega_a)$, with $a = 1, \ldots, p$, and metric given by

$$
ds^2_w = \tilde{a}^2 \left(-d\tilde{\eta}^2 + d\Omega_p^2\right).$$

Here

$$\tilde{a} = R_0/\cos \tilde{\eta},$$

and $d\Omega_p$ is the metric on a unit $p$-sphere. Expanding the normal displacement as

$$\phi = \sum_{LM} \phi_{LM} Y_{LM}(\Omega),$$

where $Y_{LM}$ are spherical harmonics, the wave functional is given by $\Psi = e^{iW}$, with

$$W = \sum_{LM} \left(\frac{\tilde{a}^{d-1}}{2} v'_L |\phi_{LM}|^2 + i \ln v_L \right).$$

The Bunch-Davies vacuum in the closed chart corresponds to the choice

$$v_L = A_L (\cos \tilde{\eta})^{p/2} \left(P_{L-1+p/2}(\sin \tilde{\eta}) + \frac{2i}{\pi} Q_{L-1+p/2}(\sin \tilde{\eta})\right),$$

where $A_L$ is a normalization constant. The index of the Legendre functions $P$ and $Q$ is given in terms of the mass of the field as

$$\nu = \left(\frac{P^2}{4} - m^2 \phi R_0^2\right)^{1/2} = \frac{p + 2}{2}.$$  

For bubbles in the 3+1 dimensional multiverse we have $p = 2$, and $\nu = 2$. Inserting (63) into (62) and expanding for late times ($\tilde{a} \to \infty$, or $\tilde{\eta} \to \pi/2$), we have

$$W = \sum_{LM} \left(\frac{\tilde{a}^2}{2R_0} + \frac{R_0 \Delta}{4} + \frac{R_0^3 \Delta (\Delta + 2)}{16 \tilde{a}^2} \left[\ln \left(\frac{R_0^2}{4\tilde{a}^2}\right) + 2 \left(\psi(L) + \frac{1}{L}\right) + i\pi + 2\gamma\right]\right) |\phi_{LM}|^2,$$

where we have omitted terms of order $\tilde{a}^{-4}$ (as well as terms independent of $\phi$). Here, $\psi$ is the digamma function, and we are using $\Delta \equiv -L(L + 1)$ to denote the Laplacian eigenvalues.

---

$^{11}$ To simplify this expression, we have used a momentum independent field redefinition at late times $\phi \equiv (\sin \tilde{\eta})^{1/2} \hat{\phi}$, when $\sin \tilde{\eta} = 1 + 0(\tilde{a}^{-2})$. In the expression above we have not written the hat on top of $\phi_{LM}$, since we will be interested in the behaviour at the future boundary, where both variables coincide.
In the limit of large $L$, $\psi(L) \approx \ln L$, so the non-analytic term in (65) is proportional to $L^4 \ln(R_0^2L^2/4\tilde{a}^2)$. The formal similarity between (42) and (65) is to be expected, since the index $\nu = 2$ of the Legendre functions which characterizes a field with $m^2 R_0^2 = -3$ in the 2+1 dimensional de Sitter worldsheet also corresponds to a massless field in 4+1 dimensional de Sitter space, considered in the previous subsection. For that reason, the mode solutions are very similar (except for an overall power of the scale factor).

Note that the imaginary part of $W$, contributing to $|\Psi|^2$, decays as $\tilde{a}^{-2}$. This is not surprising, since the normal displacement is a tachyonic field whose amplitude grows with the scale factor $\phi_{LM}(\tilde{\eta}) \propto \tilde{a}(\tilde{\eta})$ at late times.

To make contact with the boundary theory, we would like to consider the relative co-moving displacement on the flat slices of constant scale factor time. This is given by

$$\delta \equiv \frac{\delta r}{r_w} = \frac{1}{\gamma} \frac{T^{-1/2}\phi}{\tilde{a}(t)}. \quad (66)$$

The second factor in the right hand side is the proper normal displacement (in the reference frame where the worldsheet is at rest) divided by the physical radius of the bubble. The relativistic $\gamma$ factor accounts for the Lorentz contraction of the displacement in the reference frame associated to the constant scale factor hypersurfaces, and can be calculated from (56) as

$$\gamma = n^\mu \tilde{n}_\mu = R_0^{-1} r_w(t)e^{Ht_0} \rightarrow (HR_0)^{-1}. \quad (67)$$

Here, $\tilde{n}_\mu = a(t)\delta_{\mu\nu}$ is the normal to a co-moving sphere, and in the last step we have taken the late time limit.

Note that $\gamma$ will be different on both sides of the wall if the vacuum energy is different on both sides. Geometrically, this means that the surfaces of constant scale factor time on both sides of the wall meet at an angle, and the corresponding frames are not at rest with respect to each other.\(^12\) Although the unperturbed geometry of the surfaces $\Sigma_t$ is flat, the embedding of these surfaces in the bulk is non-trivial. Consequently, the relative perturbations measured by co-moving observers on both sides are related by

$$H_i^{-1}\delta_i = H_e^{-1}\delta_e, \quad (68)$$

\(^12\) Here we are neglecting the gravitational field of the domain wall, so this kinematic effect is unrelated to the jump in the extrinsic curvature as we go across the worldsheet, which would only be appreciable if the inequalities (51) were violated.
where the indices $i$ and $e$ refer to the interior and the exterior. This means that in the presence of bubble fluctuations, we cannot represent the future boundary as a smooth flat surface, because the interiors of the perturbed bubbles do not fit nicely within the holes they carve in the parent vacuum (the relative size of the wiggles differs on both sides). On the other hand, the bulk geometry is smooth, and in principle we could choose to foliate the perturbed bubble smoothly. In this case, the interior of the bubble will match the exterior, even as we approach the future boundary, but the metric on the surfaces $\Sigma'$ of the new foliation would not be flat: there would be some metric perturbations. In this sense, the situation would not be so different from the case considered in the previous subsection, where we allowed for metric perturbations on a de Sitter background.

For the remainder of this Section, we shall avoid introducing curved foliations by restricting attention to the case where the jump in energy density across the wall is sufficiently small,

$$\frac{\Delta H}{H} \ll 1.$$  \hfill (69)

In this limit, we can think of the bubble as propagating on a fixed background de Sitter space characterized by a single parameter $H$, in which case $\delta_e \approx \delta_i$. Using (66) and (67), Eq. (65) can be written as

$$W[\delta] = \frac{T R_0}{H^2} \sum_{LM} \frac{\Delta(\Delta + 2)}{16} \left[ \ln \left( \frac{R_0^2}{4\tilde{a}^2} \right) + 2 \left( \psi(L) + \frac{1}{L} \right) + i\pi + 2\gamma \right] |\delta_{LM}|^2 + ... \hfill (70)$$

where we have dropped the analytic divergent terms.

The expression vanishes both for $L = 0$ ($\Delta = 0$) and $L = 1$ ($\Delta = -2$). This is in agreement with conformal invariance, which requires that the effective action is independent of rescalings of the bubble size and linearized translations (corresponding to $L = 0$ and $L = 1$ respectively).

As mentioned in the previous subsection, for a CFT propagating in a curved background, the coefficient of the logarithmic divergence is equal to the integrated trace anomaly coefficient $a_{d/2}$. When $d$ is odd and the manifold has no boundaries, $a_{d/2}$ vanishes because we cannot build curvature invariants of odd dimension. However, if the manifold has surfaces of co-dimension 1 where the CFT fields satisfy boundary conditions, then there is a contribution to $a_{d/2}$ from geometric invariants constructed out of the intrinsic and extrinsic curvatures, $\hat{R}_{abcd}$ and $K_{ab}$, of the $p$-dimensional boundary (where $p = d - 1$), as well as contractions of the $d$-dimensional curvature $R_{ik}$ with the normal $n_i$ to the boundary surface.
If the boundary conditions are Weyl invariant, then the corresponding coefficient $a_{d/2}$ is also Weyl invariant.

Let us now argue that the locus of the bubble wall at the future boundary plays the role of such a boundary surface. The case of our interest is $p = 2$, where the invariants have to be of dimension 2, and so they must be linear in $R_{ij} n^i n^j$ or $\hat{R}$, or quadratic in the extrinsic curvature $K_{ij}$. If $g_{ij}$ is flat, as we are assuming in this subsection for the metric at the future boundary, then $R_{ij} = 0$, and so the integrated trace anomaly must be of the form

$$a_{3/2} = \int d\Sigma_2 \left[ d_1 \left( K_{ab} K^{ab} - \frac{1}{2} K^2 \right) + d_2 \hat{R} \right]. \quad (71)$$

Here, $d\Sigma_2$ is the area element on the boundary surface and $d_1$ and $d_2$ are constants. It can be shown that the term accompanying $d_1$ is Weyl invariant, while the term accompanying $d_2$ is topological. It is easy to check that in linearized theory around a spherical bubble wall, we have

$$\int d\Sigma_2 \left( K_{ab} K^{ab} - \frac{1}{2} K^2 \right) \propto \int d\Omega \, \delta \Delta (\Delta + 2) \delta, \quad (72)$$

where $\delta \equiv \delta r / r_w$ is the relative radial displacement at the future boundary, and $\Omega$ is the solid angle. Note that the coefficient of the logarithmic divergence in (70) has precisely this form. On the other hand, since the term proportional to $d_2$ is topological, it does not depend on the perturbation $\delta$.

To conclude, the logarithmic divergence in (70) takes the form expected in a CFT at the future boundary, where the domain walls play the role of surfaces where the CFT fields have to satisfy boundary conditions.

The idea that domain walls act as boundary surfaces seems very natural in the present context, since the number of field degrees of freedom $c \sim H^{-2}$ is different on both sides of the wall. The coefficient $d_1$ can be read from (70),

$$d_1 \sim \frac{TR_0}{H^2}. \quad (73)$$

Unless there are drastic cancellations, $d_1$ should be roughly equal to the number of fields satisfying boundary conditions on the wall. It follows from (51) that $d_1 \ll H^{-2}$, indicating that most of the fields pass freely through the bubble wall.

On the other hand, the number of fields satisfying nontrivial boundary conditions on the wall should at least be equal to the difference $\Delta c$ in the number of fields on the two sides of
the wall. In the absence of dramatic cancellations, this leads to

\[ d_1 \gtrsim \Delta c \sim \Delta H/H^3, \]  

(74)

Substituting (73) into (74), we obtain

\[ TR_0 \gtrsim \Delta H/H. \]  

(75)

Using (52), the above inequality can only be satisfied provided that

\[ \Delta \rho_V \lesssim HT. \]  

(76)

For such values of the parameters, the intrinsic curvature radius (or inverse proper acceleration) of the bubble walls is comparable to the inverse Hubble radius of the parent vacuum

\[ R_0 \sim H^{-1}. \]  

(77)

The necessity of the condition (76), and its possible implications, are at present unclear to us. It could be that there are unexpected cancellations in the boundary theory which allow \( d_1 \ll \Delta c \). On the other hand, it should also be noted that when (76) is violated, the proper acceleration \( R_0^{-1} \) of the bubble walls is much larger than \( H \), all the way to the future boundary. It could be that the correspondence between bulk and boundary theory in this “high energy” regime is not as straightforward as it seems to be for low energy bubbles (with \( R_0^{-1} \sim H \)). The investigation of this issue is left for future research.

V. CONCLUSIONS AND DISCUSSION

We have explored the conjecture, made in Ref. [10], that the inflationary multiverse has a dual holographic description at its future boundary, in the form of a lower dimensional theory which is conformally invariant in the UV. The duality is expressed by the relation

\[ \Psi = e^{iW}, \]  

(78)

where \( \Psi \) is the wave function of the multiverse with arguments on a spacelike hypersurface \( \Sigma \) and \( W \) is the effective action of the boundary theory on \( \Sigma \). Here, we have argued that

\[ \text{Supersymmetry will not necessarily help enforcing the cancellations which would make } d_1 \text{ much smaller than the number of fields satisfying boundary conditions at the wall. For instance, in } N=4 \text{ SYM in } d = 4 \text{ we have supersymmetry, but the coefficient of the trace anomaly is comparable to the number of fields.} \]
the boundary theory is defined on a “fish net” with characteristic spacing set by the local horizon size, and that the number of degrees of freedom in this theory is comparable to that in the bulk theory. A continuum description is obtained by imposing a Wilsonian cutoff, in the limit where the cutoff length $\xi$ is large compared to all fish net spacings.

To study the UV limit of the boundary theory, we foliate the bulk spacetime (excluding terminal bubble interiors) with surfaces of constant scale factor time $t$, starting with some initial surface $\Sigma_0$. (Geodesic crossings that may occur in structure formation regions on sub-horizon scales do not interfere with this construction, since the foliation surfaces are smoothed out with a super-horizon cutoff $\xi$.) As we go to larger values of $t$, the comoving fish net scale decreases as $e^{-t}$, so we can choose the cutoff $\xi(t) \propto e^{-t}$. Thus, renormalization group flow in the boundary theory corresponds to scale factor time evolution in the bulk, with the UV limit $\xi \to 0$ on the boundary corresponding to the IR limit $t \to \infty$ in the bulk. In this limit, the foliation surfaces approach the eternal set $\mathcal{E}$ at the future infinity. Different choices of the initial surface $\Sigma_0$ are related by Weyl rescalings in the boundary theory. They should be physically equivalent in the UV if the theory is indeed conformally invariant in that limit.

To find evidence for (or against) the conformal invariance of the boundary theory, we have studied a simple model in which the inflating bubble interiors are pure de Sitter, so the inflating part of spacetime consists of de Sitter regions separated by thin bubble walls. In this case the foliation surfaces can be chosen to be flat, and we found that the bubble distribution on these surfaces is approximately invariant under the Euclidean conformal group, with the invariance becoming exact in the limit $t \to \infty$.

We have also studied the effect of linearized perturbations about the model of nested dS bubbles. Using the duality relation (78), we have calculated the effective action $W$ for the case of linearized tensor modes in de Sitter space and for fluctuations of the bubble walls in the limit in which the gravity of the wall is unimportant. In both cases, the resulting functional form of $W$ is consistent with that expected in a conformal field theory. We interpret the locus of bubble walls at the future boundary as defining the surfaces where CFT fields must satisfy boundary conditions. The form of the logarithmically divergent terms in $W$ is in agreement with this interpretation. Altogether, our results support the conjecture that the boundary theory is conformally invariant in the UV.

A puzzle arises for the case of bubble walls whose proper acceleration $R_0^{-1}$ is much larger
than the Hubble rate $H$ of the parent vacuum. In this “high energy” regime, the numerical coefficient $d_1$ in front of the logarithmic divergence in $W$ is much smaller than the number of CFT fields which must satisfy boundary conditions at the wall. This result is somewhat counterintuitive, and its implications are left for further research.

Another important issue that has not been addressed in this paper is the treatment of terminal bubbles. We simply adopted the proposal of Ref. [10] that the interiors of such bubbles should be excised on the future boundary, with the expectation that their dynamics is holographically described by 2$d$ conformal field theories living on the bubble boundaries. Freivogel et. al. [7] provided some evidence for this in the case of Minkowski bubbles. For an AdS bubble, most of the volume in the interior is near the bubble wall, and it is natural to expect that a holographic description should apply. But as of now, this conjecture is not supported by any quantitative evidence.

An objection that has often been raised against a holographic description of de Sitter space is the anomalous behavior of massive fields in the boundary theory. Strominger [22] has studied the asymptotic behavior of the two-point function for a massive scalar field at future infinity and concluded that the field acquires a complex conformal weight if its mass is $m > 3H/2$. We note, however, that there may be a good physical reason for this behavior.

Fields with $m > 3H/2$ describe massive particles whose density is diluted as $n \propto a^{-3}$, so the comoving number of particles is conserved in the absence of interactions. The particles, however, are unstable and will decay in dS space, even if they are absolutely stable in the flat space limit: the effective Gibbons-Hawking temperature of the dS space allows decays in which the energy of the decay products is higher than that of the initial particle. Such particles cannot propagate any information to future infinity, so there is little to be learned from the asymptotic properties of their two-point function. On the other hand, the comoving number of particles with $m < 3H/2$ grows with the scale factor time as

$$N \propto e^{\beta t} \quad (79)$$

with

$$\beta = 3 \left[ 1 - \left( \frac{2m}{3H} \right)^2 \right]^{1/2}. \quad (80)$$

If their decay rate is not too high, such particles have a chance of producing an imprint at future infinity.
As we mentioned in the Introduction, one of the main motivations for studying the holographic description of the multiverse is its potential relevance for the measure problem. We have argued that a Wilsonian UV cutoff in the boundary theory corresponds to a scale factor cutoff on super-horizon scales in the bulk. The present work has also revealed some subtleties in establishing the duality, particularly in the case of bubbles which accelerate faster than the Hubble rate of the parent vacuum. Investigation of these issues may lead to a more detailed understanding of the boundary theory and its holographic relation to the bulk.

Acknowledgements

We would like to thank Raphael Bousso, Stanley Deser, Roberto Emparan, Tomeu Fiol, Daniel Freedman, Ben Freivogel, Matthew Kleban, Igor Klebanov, Jose Ignacio Latorre, Juan Maldacena, Oriol Pujolas, Stephen Shenker and Leonard Susskind for useful discussions. This work was supported in part by the Fundamental Questions Institute (JG and AV), by grants FPA2007-66665C02-02 and DURSI 2005-SGR-00082 (JG), and by the National Science Foundation grant 0353314 (AV).

Appendix: Special Conformal Transformations and their relation to boosts.

In the flat chart of a dS space of unit radius, with metric given by

$$ds^2 = \eta^{-2}(-d\eta^2 + d\mathbf{x}^2), \quad (-\infty < \eta < 0),$$

let us consider the coordinate transformation

$$\frac{x'^\mu}{x^2} = \frac{x^\mu}{x^2} - b^\mu,$$

where $x^\mu = (\eta, \mathbf{x})$ and $b^\mu = (0, \mathbf{b})$. Squares such as $x^2$ and $x'^2$, as well as the scalar products below, are constructed with the Minkowski metric, $x \cdot y = -x^0 y^0 + \mathbf{x} \cdot \mathbf{y}$. We first note that since $b^0 = 0$, we have

$$\frac{x^2}{x'^2} = \frac{\eta}{\eta'}.$$  \(83\)

Differentiating both sides of \(82\) and squaring them, we find $x'^{-4}dx'^2 = x^{-4}dx^2$. It follows that \(82\) is an isometry of dS

$$\eta'^{-2}(-d\eta'^2 + d\mathbf{x}'^2) = \eta^{-2}(-d\eta^2 + d\mathbf{x}^2).$$  \(84\)
At the future boundary $\eta = \eta' = 0$, the coordinate transformation reduces to the Special Conformal Transformation (SCT):

$$\frac{x'^i}{x'^2} = \frac{x^i}{x^2} - b^i. \tag{85}$$

Since (82) is an isometry of dS, it must correspond to boosts and rotations in the embedding Minkowski space with coordinates $(T, Y, X)$, where de Sitter space is given by the hypersurface $X^2 + Y^2 - T^2 = 1$. Let us see this in explicit form. Introducing the null coordinates $U = T - Y$ and $V = T + Y$, the embedding coordinates are given by

$$U = -\frac{x^2}{\eta}, \tag{86}$$
$$V = -\frac{1}{\eta}, \tag{87}$$
$$X = -\frac{x}{\eta}. \tag{88}$$

Using (83) we have

$$\frac{V'}{V} = \frac{x'^2}{x'^2} = 1 - 2b \cdot x + b^2 x^2. \tag{89}$$

where in the last step we have used the square of Eq. (82). After some simple algebra, we find

$$U' = U, \tag{90}$$
$$V' = V - 2b \cdot X + b^2 U, \tag{91}$$
$$X' = X - U b. \tag{92}$$

Hence, in the embedding space, the transformation can be seen as a boost and a rotation, which mixes the spatial coordinates $X$ with $Y$ and $T$.

[1] A. H. Guth, Phys. Rept. 333, 555 (2000); “Eternal inflation and its implications,” J. Phys. A 40, 6811 (2007) [arXiv:hep-th/0702178];

[2] S. Winitzki, “Predictions in eternal inflation,” Lect. Notes Phys. 738, 157 (2008) [arXiv:gr-qc/0612164];

[3] R. Bouso, B. Freivogel and I. S. Yang, “Boltzmann babies in the proper time measure,” arXiv:0712.3324 [hep-th].
[4] A. Linde, V. Vanchurin and S. Winitzki, “Stationary Measure in the Multiverse,” JCAP 0901, 031 (2009) [arXiv:0812.0005 [hep-th]].

[5] A. De Simone, A. H. Guth, M. P. Salem and A. Vilenkin, “Predicting the cosmological constant with the scale-factor cutoff measure,” arXiv:0805.2173 [hep-th].

[6] A. De Simone, A. H. Guth, A. Linde, M. Noorbala, M. P. Salem and A. Vilenkin, “Boltzmann brains and the scale-factor cutoff measure of the multiverse,” arXiv:0808.3778 [hep-th].

[7] B. Freivogel, Y. Sekino, L. Susskind and C. P. Yeh, “A holographic framework for eternal inflation,” Phys. Rev. D 74, 086003 (2006) [arXiv:hep-th/0606204].

[8] L. Susskind, “The Census Taker’s Hat,” arXiv:0710.1129 [hep-th].

[9] R. Bousso, “Holographic probabilities in eternal inflation,” Phys. Rev. Lett. 97, 191302 (2006) [arXiv:hep-th/0605263].

[10] J. Garriga and A. Vilenkin, “Holographic Multiverse,” JCAP 0901, 021 (2009) [arXiv:0809.4257 [hep-th]].

[11] R. Bousso, “Complementarity in the Multiverse,” arXiv:0901.4806 [hep-th].

[12] B. Freivogel and M. Kleban, “A Conformal Field Theory for Eternal Inflation,” arXiv:0903.2048 [hep-th].

[13] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[14] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[15] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[16] J. Garriga and A. Vilenkin, “Recycling universe,” Phys. Rev. D 57, 2230 (1998) [arXiv:astro-ph/9707292].

[17] J. Garriga, D. Schwartz-Perlov, A. Vilenkin and S. Winitzki, “Probabilities in the inflationary multiverse,” JCAP 0601, 017 (2006) [arXiv:hep-th/0509184].

[18] J. Garriga, A. H. Guth and A. Vilenkin, “Eternal inflation, bubble collisions, and the persistence of memory,” Phys. Rev. D 76, 123512 (2007) [arXiv:hep-th/0612242].

[19] A. Vilenkin, “The wave function discord,” Phys. Rev. D 58, 067301 (1998) [arXiv:gr-qc/9804051].
[20] S.W. Hawking and G.F.R. Ellis, “The large scale Structure of Spacetime”, Cambridge (1973), Sect. 6.8.

[21] L. Susskind and E. Witten, “The holographic bound in anti-de Sitter space,” arXiv:hep-th/9805114.

[22] A. Strominger, “The dS/CFT correspondence,” JHEP 0110, 034 (2001) arXiv:hep-th/0106113.

[23] A. Strominger, “Inflation and the dS/CFT correspondence,” JHEP 0111, 049 (2001) arXiv:hep-th/0110087.

[24] J. de Boer, E. P. Verlinde and H. L. Verlinde, “On the holographic renormalization group,” JHEP 0008, 003 (2000) arXiv:hep-th/9912012.

[25] J. P. van der Schaar, “Inflationary perturbations from deformed CFT,” JHEP 0401, 070 (2004) arXiv:hep-th/0307271.

[26] F. Larsen and R. McNees, “Inflation and de Sitter holography,” JHEP 0307, 051 (2003) arXiv:hep-th/0307026.

[27] See e.g. H. C. Lee, M. Sasaki, E. D. Stewart, T. Tanaka and S. Yokoyama, “A new delta N formalism for multi-component inflation,” JCAP 0510, 004 (2005) arXiv:astro-ph/0506262.

[28] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field JHEP 0305, 013 (2003) arXiv:astro-ph/0210603.

[29] E. Tomboulis, “1/N Expansion And Renormalization In Quantum Gravity,” Phys. Lett. B 70, 361 (1977).

[30] See e.g. N.D. Birrell and P.C.W. Davies, “Quantum fields in curved space”, Cambridge (1982), and references therein.

[31] V. A. Berezin, V. A. Kuzmin and I. I. Tkachev, Phys. Rev. D 36, 2919 (1987).

[32] J. Garriga, “Nucleation rates in flat and curved space,” Phys. Rev. D 49, 6327 (1994) arXiv:hep-ph/9308280.

[33] J. Garriga and A. Vilenkin, “Perturbations on domain walls and strings: A Covariant theory,” Phys. Rev. D 44, 1007 (1991); J. Garriga and A. Vilenkin, “Quantum fluctuations on domain walls, strings and vacuum bubbles,” Phys. Rev. D 45, 3469 (1992).