THE WEIGHTED PROPERTY (A) AND THE GREEDY ALGORITHM

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Abstract. We investigate various aspects of the “weighted” greedy algorithm with respect to a Schauder basis. For a weight \( w \), we describe \( w \)-greedy, \( w \)-almost-greedy, and \( w \)-partially-greedy bases, and examine some properties of \( w \)-semi-greedy bases. To achieve these goals, we introduce and study the \( w \)-Property (A).

1. Introduction

In this paper, we investigate the operation of the “weighted” greedy algorithm, and its efficiency. Throughout, \( (X, \| \cdot \|) \) is a real Banach space with a semi-normalized Schauder basis \( B = (e_n)_{n=1}^{\infty} \), with biorthogonal functionals \( (e^*_n)_{n=1}^{\infty} \); that is,

\[
A1) \quad 0 < c_1 := \inf_n \min \{\|e_n\|, \|e^*_n\|\} \leq \sup_n \max \{\|e_n\|, \|e^*_n\|\} := c_2 < \infty,
A2) \quad e^*_i(e_j) = 1 \text{ if } i = j \text{ and } e^*_i(e_j) = 0 \text{ for } i \neq j,
A3) \quad X = \text{span} \{e_i : i \in \mathbb{N}\},
A4) \quad \|S_m\| \leq K \text{ for every } m, \text{ where } (S_m)_{m=1}^{\infty} \text{ are partial sum operators – that is, } S_m(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{m} a_i e_i.\]

We denote by \( K \) the least value of \( K \) for which the preceding inequality holds, and call it the basis constant.

We will refer to \( B \) as a basis. Of course, for every \( x \in X \), there exists a unique expansion \( x = \sum_j e^*_j(x)e_j \). As usual, \( \text{supp}(x) = \{i \in \mathbb{N} : e^*_i(x) \neq 0\} \), \( |A| \) denotes the cardinality of a set \( A \) and \( \mathbb{N}^m = \{A \subset \mathbb{N} : |A| = m\} \), \( \mathbb{N}^{<\infty} = \bigcup_{m=0}^{\infty} \mathbb{N}^m \).

Further notations will be often used: if \( a \) and \( b \) are functions of some variable, \( a \lesssim b \) means that there exists a constant \( c > 0 \) such that \( a \leq c \cdot b \); if \( A \) and \( B \) are subsets of \( \mathbb{N} \), \( A < B \) means that \( \max_{j \in A} j < \min_{j \in B} j \), \( P_A \) is the projection operator, i.e., if \( A \) is a finite set, \( P_A(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j \in A} a_j e_j \) and \( P_A^\perp = I - P_A \) is the complementary projection, \( 1_A = \sum_{n \in A} e_n e_n \) for \( e_n \in \{\pm 1\} \) and if \( e_n \equiv 1 \), we write \( 1_A \).

In 1999, S. V. Konyagin and V. N. Temlyakov introduced in [16] the Thresholding Greedy Algorithm (TGA): for in \( x \in X \) we produce the sequence of greedy approximands

\[
G_m(x) = \sum_{n=1}^{m} e^*_{\pi(n)}(x)e_{\pi(n)};
\]

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where \( \pi \) is a greedy ordering, that is, \( \pi : \{1, 2, \ldots, |\text{supp } x|\} \rightarrow \text{supp } x \) is a bijection such that \( |e_{\pi(i)}^*(x)| \geq |e_{\pi(j)}^*(x)| \) for \( i \leq j \). Alternatively we can write \( G_m(x) = \sum_{k \in A_m(x)} e_k^* e_k \), where \( A_m(x) = \{ \pi(n) : n \leq m \} \) is a greedy set of \( x \): \( \inf_{k \in A_m(x)} |e_k^*| \geq \sup_{k \notin A_m(x)} |e_k^*| \).

Also, they defined in [16] the quasi-greedy bases as those bases such that there exists a constant \( C \) such that

\[
\|G_m(x)\| \leq C\|x\|, \quad \forall x \in X, \forall m \in \mathbb{N}.
\]

P. Wojtaszczyk proved in [18] that a basis is quasi-greedy if and only if the (TGA) converges – that is,

\[
\lim_{m \to \infty} \|x - G_m(x)\| = 0, \quad \forall x \in X.
\]

Of course, (1) is equivalent to the existence of a constant \( C' \) such that

\[
\|x - G_m(x)\| \leq C'\|x\|, \quad \forall x \in X, \forall m \in \mathbb{N}.
\]

We denoted by \( C_q \) the least constant that satisfies (2), it is called the quasi-greedy constant and we say that \( B \) is \( C_q \)-quasi-greedy.

On the other hand, the (TGA) is a good candidate to obtain the best m-term approximation with regard to \( B \). In this sense, S. V. Konyagin and V. N. Temlyakov defined in [16] the greedy bases as those bases such that there exists a constant \( C \geq 1 \) such that

\[
\|x - G_m(x)\| \leq C \inf \{\|x - \sum n \in A a_n e_n\| : A \subset \mathbb{N}, |A| = m, a_n \in \mathbb{R}\}.
\]

Furthermore, they showed that \( B \) is greedy if and only if \( B \) is democratic (that is, \( \|1_A\| \lesssim \|1_B\| \), for all \( |A| \leq |B| \) and unconditional.

Some years later, G. Kerkyacharian, D. Picard and V. N. Temlyakov [15] introduced the following extension of the greedy bases: we consider a weight \( w = (w_i)_{i=1}^{\infty} \in (0, \infty)^{\mathbb{N}} \). If \( A \subset \mathbb{N} \), \( w(A) = \sum_{i \in A} w_i \) denote the \( w \)-measure of \( A \). We define the error \( \sigma_\delta^w(x) \) as

\[
\sigma_\delta^w(x, B) = \sigma_\delta^w (x) := \inf \{\|x - \sum n \in A a_n e_n\| : A \subset \mathbb{N}^{\infty}, w(A) \leq \delta, a_n \in \mathbb{R}\}.
\]

**Definition 1.1.** We say that \( B \) is \( w \)-greedy if there exists a constant \( C \geq 1 \) such that

\[
\|x - G_m(x)\| \leq C \sigma_\delta^w (A_m(x)), \quad \forall x \in X, \forall m \in \mathbb{N}.
\]

We denote by \( C_g \) the least constant that satisfies (4) and we say that \( B \) is \( C_g \)-\( w \)-greedy.

Roughly, the greedy bases are those where the greedy approximation is “as effective as m-term approximation can possibly be”.

This generalization was motivated by the work of A. Cohen, R. A. DeVore and R. Hochmuth in [8]. In their recent paper [6], the first author and Ó. Blasco characterize \( w \)-greedy bases using the best \( m \)-term error in the approximation “with polynomials of constant coefficients”. Moreover, [17] characterizes \( w \)-greedy bases in terms of their \( w \)-democracy and unconditionality.

**Definition 1.2.** We say that \( B \) is \( w \)-democratic if there exists a constant \( C \geq 1 \) such that

\[
\|1_A\| \leq C\|1_B\|,
\]

for any pair of sets \( A, B \in \mathbb{N}^{\infty} \) with \( w(A) \leq w(B) \). We denote by \( C_d \) the least constant that satisfies (5) and we say that \( B \) is \( C_d \)-\( w \)-democratic.
Recall that a basis $\mathcal{B}$ in $X$ is {unconditional} if any rearrangement of $\sum_n e_n^*(x)e_n$ converges in norm to $x$ for any $x \in X$. This is equivalent to the uniform boundedness of basis projections:

$$\|x - P_A(x)\| \leq K\|x\|, \ \forall x \in X, \forall A \subset \mathbb{N}. \quad (6)$$

We denote by $K_a$ the least constant that satisfies (6), it is called the (suppression) unconditional constant, and we say that $\mathcal{B}$ is $K_a$-(suppression) unconditional.

Other important $w$-type greedy basis in this context is the $w$-almost-greedy basis.

**Definition 1.3.** We say that $\mathcal{B}$ is $w$-{almost-greedy} if there exists a constant $C \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C\tilde{\sigma}^w_{w(A_m(x))}, \ \forall x \in X, \forall m \in \mathbb{N}, \quad (7)$$

where

$$\tilde{\sigma}^w_\delta(x, \mathcal{B}) = \tilde{\sigma}^w_\delta(x) := \inf\{|\|x - P_A(x)\|| : A \in \mathbb{N}^{<\infty}, w(A) \leq \delta\}.$$

We denote by $C_{al}$ the least constant that satisfies (7) and we say that $\mathcal{B}$ is $C_{al}$-$w$-almost-greedy.

**Remark 1.4.** If $w \equiv 1$, that is, $w(A) = |A|$, we recover the classical definition of almost-greediness (resp. greediness, democracy and Property (A)-see the definition below), and we will say that $\mathcal{B}$ is almost-greedy (resp. greedy, democratic, has the Property (A)).

In the classical sense, that is, when $w \equiv 1$, S. J. Dilworth, N. J. Kalton, D. Kutzarova and V. N. Temlyakov gave in [12] a characterization of almost-greedy bases in terms of the quasi-greedy and democracy. Recently, S. J. Dilworth, D. Kutzarova, V. N. Temlyakov and B. Wallis, in [13], gave a characterization of $w$-almost-greedy bases in terms of quasi-greedy and $w$-democratic bases.

It is well known that, even for $w \equiv 1$, the $w$-democracy and unconditionality (resp. quasi-greediness), cannot be used to determine whether a given basis is $w$-greedy (resp. $w$-almost-greedy) with constant 1. For the weight $w \equiv 1$, F. Albiac and P. Wojtaszczyk introduced in [4] the so called Property (A) (defined below) in order to obtain finer estimate for the greedy constant $C_g$ (and, in particular, to characterize bases with $C_g = 1$). The results of [4] were further generalized in [11]; in [2], the Property (A) was used to estimate the almost-greedy constant $C_{al}$.

Throughout the paper, we will be using a weighted version of Property (A):

**Definition 1.5.** We say that $\mathcal{B}$ satisfies the $w$-{Property (A)} if there exists a constant $C \geq 1$ such that

$$\|x + t1_{\varepsilon A}\| \leq C\|x + t1_{\eta B}\|, \quad (8)$$

for any $x \in X$, for any $A, B \in \mathbb{N}^{<\infty}$ such that $w(A) \leq w(B)$, $A \cap B = \emptyset$, supp $(x) \cap (A \cup B) = \emptyset$, for any $\varepsilon, \eta \in \{\pm 1\}$ and $t \geq \sup_j |e_j^*(x)|$. We denote by $C_a$ the least constant that satisfies (8) and we say that $\mathcal{B}$ has the $C_a$-$w$-Property (A).

**Remark 1.6.** The definition of $w$-Property (A) was motivated by the “classical” Property (A) (introduced in [4]), which states that

$$\|x + t1_{\varepsilon A}\| \leq C\|x + t1_{\eta B}\|,$$

whenever $|A| = |B| < \infty$, $A \cap B = \emptyset$, supp $(x) \cap (A \cup B) = \emptyset$, $\varepsilon, \eta \in \{\pm 1\}$ and $t \geq \sup_j |e_j^*(x)|$. Proposition 3.2 shows that the classical Property (A) is equivalent to the $w$-Property (A) if $0 < \inf_n w_n \leq \sup_n w_n < \infty$. 

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Another way of estimating the efficiency of greedy approximation is to compare the rate of convergence with straightforward Schauder approximation. To this end we consider \( w \)-partially-greedy bases. In [12], the authors defined the partially-greedy bases as those satisfying
\[
\|x - G_m(x)\| \leq C \|x - S_m(x)\|, \quad \forall x \in X, \forall m \in \mathbb{N},
\]
for some positive and absolute constant \( C \). Moreover, they proved that \( B \) is partially-greedy if and only if \( B \) is quasi-greedy and conservative (that is, \( \|1_A\| \leq \|1_B\| \) for all pair of finite sets \( A, B \) such that \( A < B \) and \( |A| \leq |B| \)). Here, we present the notion of \( w \)-partially-greedy bases and we characterize these bases using \( w \)-conservative bases.

The paper is structured as follows. In Section 2 we describe the \( w \)-greedy and \( w \)-almost-greedy bases in terms of their other properties (such as \( w \)-Property (A), unconditionality, or being quasi-greedy). The main results are Theorems 2.1 and 2.2.

In Section 3, we collect basic facts about the \( w \)-Property (A). In addition, we consider the \( w \)-semi-greedy bases – that is, the bases where the Chebyshev greedy approximands are optimal. It turns out (Theorem 3.7) such bases necessarily possess the \( w \)-Property (A).

Section 4 is devoted to properties (C) and (D), which arise naturally in the study of quasi-greedy bases. In particular, it is shown that \( w \)-superdemocracy and Property (C) imply \( w \)-Property (A) (Proposition 4.2). However, superdemocracy does not imply Property (C) (Example 4.8). Further, we show that any \( w \)-semi-greedy basis has Property (C) if the weight \( w \) is equivalent to a constant (Proposition 4.10).

In Section 5, we compare the efficiency of greedy approximation with that of the canonical basis projections. This gives rise to the notion of an \( w \)-partially-greedy basis; such bases are characterized in Theorem 5.7.

Finally, in Section 6 and Section 7 we state some open questions related to our results, and prove some basic lemmas used throughout the paper.

We freely use the standard “greedy” terminology. The reader can consult e.g. [17] for more information.

2. Characterization of \( w \)-greedy and \( w \)-almost-greedy bases

In this section we describe the \( w \)-(almost)-greediness of a basis in terms of its \( w \)-Property (A) and unconditionality (resp. quasi-greediness). The corresponding results for the constant weight \( w \equiv 1 \) can be found, for instance, in [17].

**Theorem 2.1.** Let \( B \) be a basis of a Banach space \( X \).

- a) If \( B \) is \( C_g \)-\( w \)-greedy, then the basis is \( K_u \)-unconditional and has the \( C_a \)-\( w \)-Property (A) with constants \( K_u \leq C_g \) and \( C_a \leq C_g \).
- b) If \( B \) is \( K_u \)-unconditional and has the \( C_a \)-\( w \)-Property (A), then the basis is \( C_g \)-\( w \)-greedy with \( C_g \leq K_u C_a \).

**Theorem 2.2.** Let \( B \) be a basis of a Banach space \( X \).

- a) If \( B \) is \( C_{al} \)-\( w \)-almost-greedy, then the basis is \( C_q \)-quasi-greedy and has the \( C_a \)-\( w \)-Property (A) with constants \( C_q \leq C_{al} \) and \( C_a \leq C_{al} \).
- b) If \( B \) is \( C_q \)-quasi-greedy and has the \( C_a \)-\( w \)-Property (A), then the basis is \( C_{al} \)-\( w \)-almost-greedy with \( C_{al} \leq C_q C_a \).

Later (Proposition 5.11) we will see examples of bases with \( w \)-Property (A) for a certain weight \( w \), but failing the “classical” Property (A).

For further use, we need the following reformulation of the \( w \)-Property (A) (inspired by [2]).
Proposition 2.3. A basis $\mathcal{B}$ has the $C_a$-w-Property (A) if and only if
\[ \|x\| \leq C_a\|x - P_A(x) + 1_{\eta B}\|, \]
for any $x \in \mathbb{X}$ with $\sup_j |e_j^*(x)| \leq 1$, $A, B \in \mathbb{N}^{<\infty}$, $w(A) \leq w(B)$, $B \cap \text{supp} \,(x) = \emptyset$ and $\eta \in \{\pm 1\}$.

The proof requires a technical result.

Lemma 2.4. Suppose $D$ is a finite subset of $\mathbb{N}$, and $x \in \mathbb{X}\backslash\{0\}$ satisfies $\text{supp} \,(x) \cap D = \emptyset$. Then for any $\varepsilon > 0$ there exists a finitely supported $y \in \mathbb{X}$, so that $\|x - y\| < \varepsilon$, $\text{supp} \,(y) \cap D = \emptyset$, and $\max_j |e_j^*(x)| = \max_j |e_j^*(y)|$.

Proof. It suffices to consider $\varepsilon < 1/(2c_2)$. By scaling, we can assume that $\max_j |e_j^*(x)| = 1$ (then $\|x\| \geq 1/c_2$). Clearly $P_D(x) = 0$, and $P_D^c(x) = x$. Now set $\delta = \varepsilon/(3c_2\|x\|)$. As span $[e_j : j \in \mathbb{N}]$ is dense in $\mathbb{X}$, there exists a finitely supported $z \in \mathbb{X}$ so that $\|x - z\| < \delta/\|P_D^c\|$. Let $u = P_D(z)$, then $\|x - u\| = \|P_D^c(x - z)\| < \delta$. For every $j$, $|e_j^*(x - u)| < c_2\delta$, hence $C = \max_j |e_j^*(x)| \in (1 - c_2\delta, 1 + c_2\delta)$. Now let $y = u/C$. Then $\max_j |e_j^*(y)| = 1$, and
\[ \|x - y\| \leq \|x - u\| + \|1 - C^{-1}\|u\| \leq \delta + \frac{c_2\delta}{1 - c_2\delta}(|x| + \delta) < \varepsilon. \]

Proof of Proposition 2.3. By Lemma 2.4 it suffices to restrict our attention to finitely supported vectors $x \in \mathbb{X}$ only. So, throughout this proof, we assume $|\text{supp} \,(x)| < \infty$.

Suppose that $\mathcal{B}$ has the $C_a$-w-Property (A), and $x, A, B, \varepsilon, \eta$ are as in the statement of the proposition with $\sup_j |e_j^*(x)| \leq 1$. Applying the definition of w-Property (A) to $P_A \cdot x$, $A$, and $B$, we obtain
\[ \|P_A \cdot x + 1_{\varepsilon A}\| \leq C_a\|P_A \cdot x + 1_{\eta B}\| = C_a\|x - P_A \cdot x + 1_{\eta B}\|. \]

To finish the proof, observe that $x$ belongs to the convex hull of the set $\{P_A \cdot x + 1_{\varepsilon A} \mid \varepsilon \in \{\pm 1\}\}$.

Now, suppose (9), and prove that the basis $\mathcal{B}$ has the w-Property (A) with the same constant. Take $x \in \mathbb{X}$ and $\sup_j |e_j^*(x)| \leq 1$, $A, B \in \mathbb{N}^{<\infty}$ such that $w(A) \leq w(B)$, $A \cap B = \emptyset$, $\text{supp} \,(x) \cap (A \cup B) = \emptyset$ and $\varepsilon, \eta \in \{\pm 1\}$. Define $x' = x + 1_{\varepsilon A}$. Using (9),
\[ \|x + 1_{\varepsilon A}\| = \|x'\| \leq C_a\|x' - P_A \cdot (x') + 1_{\eta B}\| = C_a\|x + 1_{\eta B}\|. \]

Proof of Theorem 2.7. Assume that $\mathcal{B}$ is $C_g$-w-greedy.

Unconditionality: Let $x \in \mathbb{X}$ and $A \subset \text{supp} \,(x)$. Define $y := P_A \cdot x + \sum_{n \in A} (\alpha + e_n^*(x))e_n$, where
\[ \alpha > \sup_{j \in A} |e_j^*(x)| + \sup_{j \in A^c} |e_j^*(x)|. \]

As $A$ is a greedy set of $y$,
\[ \|x - P_A \cdot x\| = \|y - P_A \cdot y\| \leq C_g \sigma_{w(A)}^w(y) \leq C_g\|y - \alpha 1_A\| = C_g\|x\|. \]

Thus, the basis is unconditioned with constant $K_u \leq C_g$.

w-Property (A): Fix $x \in \mathbb{X}$, take $t \geq \sup_n |e_n^*(x)|$. Consider $\varepsilon, \eta \in \{\pm 1\}$ and finite sets $A, B$ such that $A \cap B = \emptyset$, $w(A) \leq w(B)$, and $(A \cup B) \cap \text{supp} \,(x) = \emptyset$. Set $y := x + t 1_{\varepsilon A} + (t + \delta) 1_{\eta B}$ with $\delta > 0$. Hence,
\[ \|x + t 1_{\varepsilon A}\| = \|y - G_{\eta B}(y)\| \leq C_g \sigma_{w(B)}^w(y) \leq C_g\|y - t 1_{\varepsilon A}\| = C_g\|x + (t + \delta) 1_{\eta B}\|. \]

Taking $\delta \to 0$, we obtain that the basis satisfies the w-Property (A) with constant $C_a \leq C_g$. 

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Next we prove that if $\mathcal{B}$ is $K_u$-unconditional and has the $C_u$-$w$-Property (A), then it is $w$-greedy.

Take $x \in X$ and suppose that $A$ is a greedy set of cardinality $m$ for $x \in X$—that is, $P_A(x) = G_m(x)$. For $\varepsilon > 0$ find $y \in X$ such that $\|x - y\| < \sigma^w_{w(A)}(x) + \varepsilon$, with $\text{supp}(y) = B$ and $w(B) \leq w(A)$. Then, taking $t := \min \{ |e_j^*(x)| : j \in A \}$ and $\eta \equiv \text{sgn} \{ e_j^*(x) \}$, using the reformulation of the $w$-Property (A) and Lemma 7.1, we obtain that

$$\|x - G_m(x)\| \leq C_a \|x - P_A(x) - P_{B \setminus A}(x) + t1_{\eta(A)B}\| = C_a \|P_{(A \cup B)^c}(x - y) + t1_{\eta(A)B}\|$$

$$= C_a \|T_t(I - P_B)(x)\| = C_a \|T_t(I - P_B)(x - y)\| \leq K_u C_a \|x - y\|.$$

Consequently, for any greedy set $A$ we have $\|x - P_A x\| \leq K_u C_a \sigma^w_{w(A)}(x)$. \hfill \Box

**Proof of Theorem 2.2.** Assume that $\mathcal{B}$ is $C_{al}$-$w$-almost-greedy.

- **Quasi-greedy:** Since

$$\|x - G_m(x)\| \leq C_{al} \inf \{ \|x - \sum_{n \in B} e_n^*(x) e_n\| : w(B) \leq w(A_m(x)), B \in N^{<\infty} \},$$

we can select $B = \emptyset$. Then, we obtain that $\|x - G_m(x)\| \leq C_{al} \|x\|$, hence the basis is quasi-greedy with constant $C_q \leq C_{al}$.

- **$w$-Property (A):** We can use the same argument as in Theorem 2.1.

Now, we will prove that if $\mathcal{B}$ is $C_q$-quasi-greedy and has the $C_u$-$w$-Property (A), then it is $w$-almost-greedy.

For $x \in X$, let $A$ be a greedy set of cardinality $m$. For $\varepsilon > 0$, find $B$ such that $\|x - P_B(x)\| < \tilde{\sigma}^w_{w(A)}(x) + \varepsilon$, with $w(B) \leq w(A)$. Then, taking $t := \min \{ |e_j^*(x)| : j \in A \}$ and $\eta \equiv \text{sgn} \{ e_j^*(x) \}$, using the reformulation of the $w$-Property (A) and Lemma 7.1,

$$\|x - G_m(x)\| \leq C_a \|P_{(A \cup B)^c}(x - y) + t1_{\eta(A)B}\|$$

$$= C_a \|T_t(I - P_B)(x)\| \leq C_q C_a \|x - P_B(x)\|.$$

This gives that, for any greedy set $A$, $\|x - P_A(x)\| \leq C_q C_a \sigma^w_{w(A)}(x)$ as desired. \hfill \Box

**Remark 2.5.** In this paper, we focus on the situation when $\mathcal{B}$ is a Schauder basis. However, the $w$-Property (A) can be defined for any complete biorthogonal system satisfying the conditions (A1)-(A3); the proof of Proposition 2.3 goes through as well.

Moreover, in the definition of the $w$-Property (A), it suffices to show that (8) holds for with $\max_j |e_j^*(x)| = t$. More specifically, the following four statements are equivalent:

(a) $\mathcal{B}$ satisfies the $w$-Property (A) (see Definition 1.5).

(b) There exists a constant $C$ so that $\|x\| \leq C \|x - P_A(x) + t1_{\eta B}\|$ for any $\eta \in \{\pm 1\}$, $x \in X$, $B \cap \text{supp}(x) = \emptyset$, $w(A) \leq w(B)$ and $t \geq \sup_j |e_j^*(x)|$.

(c) There exists a constant $C'$ so that $\|x + s1_{\eta B}\| \leq C \|x + s1_{\eta B}\|$ for any $x \in X$, $w(A) \leq w(B)$, $A \cap B = \emptyset$, $\sup_j |e_j^*(x)|$. \hfill \Box

(d) There exists a constant $C''$ so that $\|x\| \leq C'' \|x - P_A(x) + s1_{\eta B}\|$ for any $x \in X$, $\eta \in \{\pm 1\}$, $B \cap \text{supp}(x) = \emptyset$, $w(A) \leq w(B)$ and $s = \sup_j |e_j^*(x)|$.

Indeed, the implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) (with $C' = C$) are immediate. The equivalence (a) $\Leftrightarrow$ (b) (with the same constant $C$) has been established in Proposition 2.3. Minor adjustments to that argument give us (c) $\Leftrightarrow$ (d).

To establish (d) $\Rightarrow$ (b), take $x, A, B, \eta$ as in (b) and $t \geq \sup_j |e_j^*(x)|$. As before, we can assume that $x$ is finitely supported. Find $k$ so that $\|e_k^*(x)\| = \sup_j |e_j^*(x)|$. By replacing $x$ by
\(-x\) if necessary, we can assume \(s = e'_k(x) \geq 0\). Let \(c = t - s\), and consider
\[
x' = x + c e_k = \sum_{j \in \text{supp}(x) \setminus \{k\}} e^*_j(x) e_j + t e_k.
\]
Note that \(\|x - x'\| \leq cc_2 \leq tc_2\). Furthermore, \(x' - P_A(x')\) equals either \(x - P_A(x)\) (if \(k \in A\)), or \(x - P_A(x) + ce_k\) (if \(k \notin A\)). In either case,
\[
\|x - P_A(x) + t1_{\eta B}\| \geq \|x' - P_A(x') + t1_{\eta B}\| - tc_2.
\]
By (d), we have \(\|x'\| \leq C'\|x' - P_A(x') + t1_{\eta B}\|\). By the above,
\[
\|x\| - tc_2 \leq C'\|x - P_A(x) + t1_{\eta B}\| + tc_2
\]
As \(\|x - P_A(x) + t1_{\eta B}\| \geq tc_2^{-1}\), we conclude that \(\|x\| \leq (C' + 2c_2^2)\|x - P_A(x) + t1_{\eta B}\|\).

3. Some Remarks on the \(w\)-Property (A)

**Definition 3.1.** Let \(v = (v_n)_{n=1}^{\infty}\) and \(w = (w_n)_{n=1}^{\infty}\) be weights. We say that \(v\) is equivalent to \(w\), written \(v \approx w\), whenever there exist positive real constants \(0 < a \leq b < \infty\) satisfying
\[
av_n \leq w_n \leq bw_n \quad \text{for all } n \in \mathbb{N}.
\]

**Proposition 3.2.** Let \(v, w\) weights and suppose that \(v \approx w\). Then every basis with the \(w\)-Property (A) also has the \(v\)-Property (A).

**Proof.** Let \(x \in X\) with \(|\text{supp}(x)| < \infty\) and \(\sum_j |e^*_j(x)| \leq 1\), \(A\) and \(B\) finite satisfying \(v(A) \leq v(B)\), \(A \cap B = \emptyset\), \(\text{supp}(x) \cap (A \cup B) = \emptyset\) and \(\varepsilon, \eta \in \{\pm 1\}\). We set
\[
\Gamma = \{n \in A : w_n \geq w(B)\}.
\]
Observe that
\[
w(A) \leq b \cdot v(A) \leq b \cdot v(B) \leq \frac{b}{a} \cdot w(B),
\]
which gives us
\[
w(A) \geq w(\Gamma) \geq |\Gamma| \cdot w(B) \geq |\Gamma| \cdot \frac{a}{b} \cdot w(A),
\]
and hence \(|\Gamma| \leq b/a\). Next, we give the following partition of \(A \setminus \Gamma\): \(A_1 < \ldots < A_m\), so that for each \(i = 1, \ldots, m\), the set \(A_i\) is a maximal such that \(w(A_i) \leq w(B)\). Due to maximality,
\[
w(B) < w(A_i) + w(A_{i+1})\] for all \(i = 1, \ldots, m - 1\).
Thus,
\[
(m - 1) \cdot w(B) < \sum_{i=1}^{m-1} [w(A_i) + w(A_{i+1})] < 2 \cdot w(A \setminus \Gamma) \leq 2 \cdot w(A) \leq \frac{2b}{a} \cdot w(B).
\]
This gives us
\[
m \leq \frac{2b}{a} + 1.
\]
Hence, using the bounds of $|\Gamma|$, $m$ and the condition of the $w$-Property (A),
\[
\|x + 1 \varepsilon A\| \leq \|1_{\Gamma}\| + \|x + \sum_{i=1}^{m} 1_{\varepsilon A_i}\| \leq \sum_{n \in \Gamma} \|e_n\| + \sum_{i=1}^{m} \|\frac{x}{m} + 1_{\varepsilon A_i}\|
\]
\[
\leq c_{2}^{2}|\Gamma|\|x + 1_{\eta B}\| + C_{a}m\|\frac{x}{m} + 1_{\eta B}\| \leq \frac{c_{2}^{2}b}{a}\|x + 1_{\eta B}\| + C_{a}\|x + m1_{\eta B}\|
\]
\[
\leq \frac{c_{2}^{2}b}{a}\|x + 1_{\eta B}\| + C_{a}m\|x + 1_{\eta B}\| + C_{a}(m - 1)\|x\|
\]
\[
\leq \frac{c_{2}^{2}b}{a}\|x + 1_{\eta B}\| + C_{a}m\|x + 1_{\eta B}\| + C_{a}^{2}(m - 1)\|x + 1_{\eta B}\|
\]
\[
\leq \left(\frac{c_{2}^{2}b + 2bC_{a}^{2}}{a}\right)\|x + 1_{\eta B}\|.
\]
\[\square\]

**Remark 3.3.** In a similar fashion, one can show that, if the weights $w$ and $v$ are equivalent, then any $w$-democratic ($w$-superdemocratic, $w$-conservative – for the definitions, see below) basis is also $v$-democratic (resp. $v$-superdemocratic or $v$-conservative).

**Remark 3.4.** The converse to Proposition 3.2 does not hold in general. For example, suppose the weights $w, v$ belong to $\ell_1$. By [13], the family of $w$-democratic (or $v$-democratic) bases consists precisely of those bases which are equivalent to the canonical basis of $c_0$. However, $w$ and $v$ need not be equivalent.

The rest of this section is motivated by the recent definition of $w$-semi-greedy bases introduced in [13]. To give this notion, we need the Chebyshev Greedy Algorithm: for $x \in X$ and $m \in \mathbb{N}$, if $A_{m}(x)$ is the greedy set of $x$ with cardinality $m$, we define the Chebyshev Greedy Approximand of order $m$ as any $G_{m}(x) \in \text{span}\{e_{i} : i \in A_{m}(x)\}$ such that
\[
\|x - G_{m}(x)\| = \min\{|x - \sum_{n \in A_{m}(x)} b_{n}e_{n}| : b_{n} \in \mathbb{R}\}.
\]

**Definition 3.5.** We say that $B$ is $w$-semi-greedy if there exists a constant $C \geq 1$ such that
\[
\|x - G_{m}(x)\| \leq C_{w}^{\sigma_{w}(A_{m}(x))}(x), \quad \forall x \in X, \forall m \in \mathbb{N}.
\] (10)

We denote by $C_{sg}$ the least constant that satisfies (10) and we say that $B$ is $C_{sg}$-$w$-semi-greedy.

By [13], any $w$-semi-greedy basis is $w$-superdemocratic.

**Definition 3.6.** We say that $B$ is $w$-superdemocratic if there exists a constant $C \geq 1$ such that
\[
\|1_{A}\| \leq C\|1_{\eta B}\|, \quad (11)
\]
for any $A, B \in \mathbb{N}^{\leq \infty}$ with $w(A) \leq w(B)$ and $\varepsilon, \eta \in \{\pm 1\}$.

We denote by $C_{s}$ the least constant that satisfies (11) and we say that $B$ is $C_{s}$-superdemocratic.

Here, we show that
\[
w - \text{semi-greedy} \Rightarrow w - \text{Property (A)} \Rightarrow w - \text{superdemocracy}. \quad (12)
\]

**Theorem 3.7.** If a basis $B$ is $w$-semi-greedy, then $B$ has the $w$-Property (A).
Proof. Assume that \( \| x - \overline{G}_m(x) \| \leq C_{sg}\sigma_{w(A_m(x))}^w \) for any \( x \in \mathbb{N} \) and \( m \in \mathbb{N} \).

We take \( \varepsilon, \eta, A, B \) and \( x \) in the conditions of the definition of the \( w \)-Property (A). In all of the following cases we consider \( x \in \mathbb{N} \) such that \( |\text{supp} (x)| < \infty \) and \( \sup_n |e_n^\ast(x)| \leq 1 \).

**Case 1:** \( \sum_{n=1}^{\infty} w_n = \infty \) and \( \sup_n w_n < \infty \).

**Case 1.1:** \( w(B) > \limsup_{n \to \infty} w_n \). Since \( \sum_n w_n = \infty \), we can choose \( E \) and \( n_0 \in \mathbb{N} \) with \( \min E > \max(A \cup B \cup \text{supp} (x)) \) and \( n_0 > \max E \) such that
\[
\begin{align*}
\inf_{n \in E} \| x + \varepsilon \| &\leq \inf_{n \in F} \| x + 1 \| + \delta \| 1_F \|.
\end{align*}
\]

Consequently, \( \| x + \varepsilon \| \leq K_b C_{sg} \| x + (1 + \delta) 1_F \| \). Taking \( \delta \to 0 \),
\[
\begin{align*}
\| x + \varepsilon \| &\leq K_b C_{sg} \| x + 1 \| + K_b C_{sg} \| e_n \| \leq K_b C_{sg} \| x + 1 \| + K_b C_{sg} \| 1_E \| \leq K_b C_{sg} \| x + 1 \| + K_b C_{sg} \| 1_E \| + K_b C_{sg} \| 1_E \|.
\end{align*}
\]

Now, we set \( y := 1_{\eta B} + (1 + \delta) 1_F \). Reasoning as before, we obtain
\[
\begin{align*}
\| 1_{\eta B} \| &\leq K_b \inf_{c_n} \| 1_{\theta B} + \sum_{n \in F} \| c_n e_n \| \leq K_b C_{sg} \sigma_{w(F)}^w(y) \leq K_b C_{sg} \| (1 + \delta) 1_F \|.
\end{align*}
\]

Sending \( \delta \to 0 \), we obtain
\[
\| 1_{\eta B} \| \leq K_b C_{sg} \| 1_F \| \leq K_b C_{sg} \| 1_E \| + K_b C_{sg} \| 1_F \|.
\]

On the other hand, taking \( s := x + (1 + \delta) 1_{\theta B} + 1_E \),
\[
\begin{align*}
\| 1_E \| &\leq (K_b + 1) \| x + \sum_{n \in B} b_n e_n + 1_E \| \leq C_{sg} (K_b + 1) \sigma_{w(B)}^w(s) \leq C_{sg} (K_b + 1) \| x + 1_{\eta B} \|.
\end{align*}
\]

Then, taking \( \delta \to 0 \),
\[
\| 1_E \| \leq C_{sg} (K_b + 1) \| x + 1_{\eta B} \|.
\]

Finally, using (13), (14) and (15), the basis satisfies the \( w \)-Property (A) with constant \( K = O(C_{sg}^3 K_{sg}^2) \).

**Case 1.2:** \( w(A) \leq w(B) \leq \lim \sup_{n \to \infty} w_n \). Using Proposition 3.5 of [13],
\[
\max \{ \| 1_{\varepsilon A} \|, \| 1_{\eta B} \| \} \leq 2 K_b C_{sg} \| 1_F \|.
\]

Since \( 1 = |e_j^\ast(x + 1_{\eta B})| \leq \| e_j^\ast \| \| x + 1_{\eta B} \| \leq c_2 \| x + 1_{\eta B} \| \) for \( j \in B \), then
\[
\begin{align*}
\| x + 1_{\varepsilon A} \| &\leq \| x + 1_{\eta B} \| + \| 1_{\eta B} \| + \| 1_{\varepsilon A} \| \\
&\leq \| x + 1_{\eta B} \| + 4 K_b C_{sg} \| 1_F \| \leq (4 K_b C_{sg}^2 + 1) \| x + 1_{\eta B} \|.
\end{align*}
\]

**Case 2:** If \( \sum_n w_n < \infty \) or \( \sup_n w_n = \infty \), using the Proposition 3.5 of [13], \( B \) is equivalent to the canonical basis of \( c_0 \) and the result is trivial.

\[\square\]
Proposition 3.8. If $\mathcal{B}$ has the $C_a$-$w$-Property (A), then $\mathcal{B}$ is $2C_a$-$w$-superdemocratic.

Proof. Take $A, B \in \mathbb{N}^\infty$ with $w(A) \leq w(B)$, and show that, for any choice of signs, $\|1_{\eta A}\| \leq 2C_a\|1_B\|$. As in [7], Subsection 4.4, it is enough to prove our inequality for $\varepsilon \equiv 1$ (otherwise, replace $\mathcal{B} = \{e_n : n \in \mathbb{N}\}$ by $\{\varepsilon_n e_n : n \in \mathbb{N}\}$). Since $1_{\eta A} \in 2S$, where $S = \{\sum_{A' \subseteq A} \theta_{A'} 1_{A'} : \sum_{A' \subset A} |\theta_{A'}| \leq 1\}$ (see [10] Lemma 6.4), it suffices to show that

$$\|1_{A'}\| \leq C_a\|1_B\|, \forall A' \subset A.$$ 

Take $A' \subset A$. Obviously, $1_{A'} = 1_{A' \setminus B} + 1_{B \cap A'}$. Then, using the $w$-Property (A),

$$\|1_{A'}\| = \|1_{A' \setminus B} + 1_{B \cap A'}\| \leq C_a\|1_{B \setminus A'} + 1_{B \cap A'}\| = C_a\|1_B\|.$$ 

We can apply the $w$-Property (A) because

$$w(A') = w(A' \setminus B) + w(A' \cap B) \leq w(A) \leq w(B) = w(B \setminus A') + w(B \cap A') \Rightarrow w(A' \setminus B) \leq w(B \setminus A').$$ 

This completes the proof. 

With these results, we have proved the implications (12).

Remark 3.9. If $w \equiv 1$, we recover the classical definition of semi-greediness (resp. superdemocracy), and we will say that $\mathcal{B}$ is semi-greedy (resp. superdemocratic).

Improving [13] Proposition 4.5, we prove that, in certain cases, any $w$-superdemocratic basis has to contain a subsequence equivalent to the canonical basis of $c_0$ ($c_0$-basis henceforth), or even to be equivalent to such basis.

Proposition 3.10. Suppose a basis $\mathcal{B} = (e_n)_{n=1}^\infty$ is $C_s$-$w$-superdemocratic.

i) If $A \in \mathbb{N}^\infty$ and $w(A) \leq \limsup_{n \to \infty} w_n$, then $\max_{\varepsilon \in \{\pm 1\}} \|1_{\varepsilon A}\| \leq c_2C_s$.

ii) If $\sup_n w_n = \infty$, then $\mathcal{B}$ is equivalent to the $c_0$-basis.

iii) If $\inf_n w_n = 0$, then there exist $i_1 < i_2 < \ldots$ so that the sequence $(e_{i_k})_{k \in \mathbb{N}}$ is equivalent to the $c_0$-basis. Moreover, if $\lim_n w_n = 0$, then for any infinite set $A \subset \mathbb{N}$ we can select $i_1, i_2, \ldots \in A$ with the properties described above.

iv) If $\sum_n w_n < \infty$, then $\mathcal{B}$ is equivalent to the $c_0$-basis.

Proof. i) Find $n \in \mathbb{N} \setminus A$ so that $w_n > w(A)$, then $\|1_{\varepsilon A}\| \leq C_s\|1_{\{n\}}\| \leq c_2C_s$.

ii) By (i), $\|1_{\varepsilon A}\| \leq c_2C_s$ for all choices of signs, which yields the desired equivalence.

iii) Suppose $\inf_n w_n = 0$, and find $i_1 < i_2 < \ldots$ so that $\sum_k w_{i_k} < \infty$. By convexity, it suffices the existence of a constant $K$ with the property that the inequality $\|1_{\varepsilon E}\| \leq K$ holds for any finite set $E \subset \{i_1, i_2, \ldots\}$. To this end, find $N \in \mathbb{N}$ so that $\sum_{j=1}^N w_j \geq \sum_k w_{i_k} < \infty$. Let $B = \{1, \ldots, N\}\setminus\{i_1, i_2, \ldots\}$, $D = \{1, \ldots, N\} \cap \{i_1, i_2, \ldots\}$, and $A = E \setminus D$. Note that $|B|, |D| \leq N$, hence, for every $\varepsilon \in \{-1, 1\}^N$, $\|1_{\varepsilon B}\|, \|1_{\varepsilon D}\| \leq c_2N$. Then $w(A) \leq w(B)$ and hence $\|1_{\varepsilon A}\| \leq C_s\|1_{\varepsilon B}\| \leq c_2NC_s$. By the triangle inequality,

$$\|1_{\varepsilon E}\| \leq \|1_{\varepsilon A}\| + \|1_{\varepsilon D}\| \leq c_2N(C_s + 1).$$ 

If $\lim_n w_n = 0$, then every infinite $A \subset \mathbb{N}$ contains $i_1 < i_2 < \ldots$ with $\sum_k w_{i_k} < \infty$. It remains to invoke the preceding result.

iv) The proof proceeds as in (iii).

From this we immediately obtain:

Corollary 3.11. If the weight $w$ is unbounded, then a basis has the $w$-Property (A) if and only if it is equivalent to the canonical basis of $c_0$. 

4. Properties (C) and (D)

Properties (C) and (D) (discussed below) naturally arise in the study of quasi-greedy bases.

**Definition 4.1.** We say that $B$ satisfies the Property (C) if for any $x \in X$, there exists a positive constant $C$ such that

$$\min_{j \in \Lambda} |e_j^*(x)| \leq C \|x\|,$$

for any greedy set $\Lambda$ of $x$ and $\varepsilon \in \{\pm 1\}$. We denote by $C_u$ the least constant that satisfies this inequality and we say that $B$ has the Property (C) with constant $C_u$.

It is well known any quasi-greedy basis has Property (C) (see [7, Lemma 2.3]). Generalizing [7, Lemma 2.2], we prove that any $w$-superdemocratic basis with the Property (C) has the $w$-Property (A).

**Proposition 4.2.** If $B$ is a $C_s$-w-superdemocratic and satisfies the Property (C) with constant $C_u$, then $B$ has the $C_a$-w-Property (A) with $C_a \leq 3C_uC_s$.

**Proof.** Take $x, A, B, \varepsilon, \eta$ as in the definition of the $w$-Property (A) and assume that $\sup_j |e_j^*(x)| \leq 1$. Then,

$$\|x + 1_{xA}\| \leq \|x + 1_{yB}\| + \|1_{yB}\| + \|1_{xA}\|.$$  \hspace{1cm} (18)

Using the $w$-superdemocracy and $w(A) \leq w(B)$, we obtain that $\|1_{xA}\| \leq C_s\|1_{yB}\|$. Now, we only have to estimate $\|1_{yB}\|$. For that, we consider the element $y := x + 1_{yB}$. It’s clear that $1_{yB}$ is a greedy sum for $y$, so

$$\min_{j \in B} |e_j^*(y)| \leq \|1_{yB}\| \leq C_u\|y\| = C_u\|x + 1_{yB}\|.$$  \hspace{1cm} (19)

Then, using (18) and (19),

$$\|x + 1_{xA}\| \leq \|x + 1_{yB}\| + 2C_sC_u\|x + 1_{yB}\| \leq 3C_sC_u\|x + 1_{yB}\|.$$  \hspace{1cm} (20)

Hence, the basis has the $w$-Property (A) with constant $C_a \leq 3C_uC_s$. \hfill \Box

**Example 4.3.** We next revisit a “pathological” basis constructed in Section 5.5 of [7] (using some ideas from [9, Example 4.8]): a basis which has the Property (A), but fails to be quasi-greedy. The initial proof of the Property (A) was unwieldy. Here we present a streamlined proof that the basis has the Property (C), and then invoke Proposition 4.2.

First recall the construction: $D_k$ denote the set of all dyadic intervals $I \subset [0, 1]$ with length $|I| = 2^{-k}$, and consider $D = \bigcup_{k \geq 0} D_k$. Now, we consider the space $\ell^q_1$ of all real sequences $a = (a_I)_{I \in D}$ such that

$$\|a\|_{\ell^q_1} = \left\| \left( \sum_I |a_I\chi_I^{(1)}|^q \right)^{1/q} \right\|_{L^1} < \infty,$$

where $\chi_I^{(1)} = |I|^{-1} \chi_I$. By [13], the canonical basis $\{e_I\}_{I \in D}$ is unconditional and democratic.

For every $N \geq 1$, we shall pick a subset $\{k_1, \ldots, k_N\} \subset \mathbb{N}_0$ and look at the finite dimensional space $F_N$ consisting of sequences supported in $\bigcup_{j=1}^N D_{k_j}$. We order the canonical basis by $\bigcup_{j=1}^N \{e_I\}_{I \in D_{k_j}}$, so we may as well write their elements as $a = (a_j)_{j=1}^{\dim F_N}$. We also consider in...
\[ F_N \text{ the James norm} \]

\[
\| (a_j) \|_{J_q} = \sup_{m_0 = 0 < m_1 < \cdots} \left( \sum_{k \geq 0} \left| \sum_{m_k < j \leq m_{k+1}} a_j \right|^q \right)^{1/q}.
\]

Now, set in \( F_N \) a new norm

\[
\| a \| = \max \{ \| a \|_{J_q}, \| a \|_{J_q} \}.
\]

Finally, we consider the Banach space \( X = \oplus_q F_N \) with \( B \) the consecutive union of the natural bases in \( F_N \).

It’s possible to show that \( B \) is superdemocratic and \( \| 1_{\varepsilon A} \| \approx |A| \approx \| 1_{\varepsilon A} \|_{q} \). To show that \( B \) satisfies the Property (C), we use that the canonical basis in \( f_q \) is unconditional: take \( a \in X \) and \( A \) a greedy set of \( a \), then

\[
\min_{n \in A} |a_n| \| 1_{\varepsilon A} \| \lesssim \min_{n \in A} |a_n| \| 1_{\varepsilon A} \|_{q} \lesssim \| a \|_{q} \leq \| a \|.
\]

Hence, the basis satisfies the Property (C). Also, since the basis is superdemocratic, using the Proposition 4.2, the basis satisfies the Property (A).

**Definition 4.4.** We say that \( B \) is bidemocratic if there exists a constant \( C \geq 1 \) such that

\[
\| 1_{\varepsilon A} \| \| 1_{\eta A} \|_{*} \leq C |A|, \quad \forall \text{ finite } A, \forall \varepsilon, \eta \in \{ \pm 1 \}.
\]

Here, \( \| \cdot \|_{*} \) is the norm of \( X^* \), and \( 1_{\eta A} = \sum_{i \in A} \eta_i e_i^* \).

**Lemma 4.5.** If \( B \) is bidemocratic, then \( B \) satisfies the Property (C).

**Proof.** Here, we prove a stronger condition than Property (C). Take \( x \in X \) and \( A \subset \text{supp} \,(x) \). Then, taking \( \eta = 1/\text{sgn} \,(e_i^*(x)) \),

\[
\min_{j \in A} |e_j^*(x)| \| 1_{\varepsilon A} \| \lesssim \min_{j \in A} |e_j^*(x)| \| 1_{\eta A} \|_{*} \leq \sum_{j \in A} |e_j^*(x)| \| 1_{\eta A} \|_{*} \leq \| 1_{\eta A}^* (x) \|_{*} \leq \| x \|.
\]

\n
**Corollary 4.6.** All bidemocratic bases satisfy the “classical” Property (A).

**Proof.** If a basis \( B \) is bidemocratic, then it is superdemocratic. Now combine the preceding lemma with Proposition 4.2. \( \square \)

Relaxing the assumptions of Definition 4.1, we consider:

**Definition 4.7.** We say that \( B \) satisfies the Property (D) if there exists a positive constant \( C \) such that

\[
\min_{n \in A} |a_n| \| 1_A \| \leq C \sum_{n \in A} |a_n e_n|,
\]

for any finite set \( A \) and scalars \( (a_n)_{n \in A} \).

It’s clear that if \( B \) satisfies the Property (C), then \( B \) satisfies the Property (D) as well.

**Example 4.8.** [Example of superdemocratic basis in a Banach space without the Property (D)] Let \( X = \ell_1 \oplus c_0 \) and \( \|(x, y)\| = \|x\|_{\ell_1} + \|y\|_{\infty} \). Let \( (e_n) \) be the canonical basis in \( \ell_1 \) and \( (f_m) \) the canonical basis in \( c_0 \). We define

\[
E_{2n-1} = \left( \frac{1}{2} e_n, -\frac{1}{2} f_n \right), \quad E_{2n} = \left( \frac{1}{4} e_n, \frac{3}{4} f_n \right), \quad n = 1, 2, \ldots,
\]
and consider $\mathcal{B} = \{E_n\}_n = \{E_{2n-1}, E_{2n}\}_n$. This basis is normalized.

To prove that this basis is superdemocratic, we show the following proposition:

**Proposition 4.9.** $D(m) \approx d(m) \approx m$, where $D(m) := \sup\{\|1_{\varepsilon A}\| : |A| \leq m, \varepsilon \in \{\pm 1\}\}$ and $d(m) := \inf\{\|1_{\varepsilon A}\| : |A| \geq m, \varepsilon \in \{\pm 1\}\}$.

**Proof.** Of course, $d(m) \leq D(m) \leq m$. We prove that $d(m) \geq \frac{1}{3}m$. To this end, given $A \subset \mathbb{N}$ finite, we write

$$A_1 = \{k \in \mathbb{N} : 2k \in A \text{ and } 2k - 1 \in A\},$$

$$A_2 = \{k \in \mathbb{N} : 2k \in A \text{ and } 2k - 1 \notin A\},$$

$$A_3 = \{k \in \mathbb{N} : 2k \notin A \text{ and } 2k - 1 \in A\}.$$

Observe that the sets $A_1, A_2, A_3$ are mutually disjoint, and $2|A_1| + |A_2| + |A_3| = |A|$. For any choice of signs,

$$\|1_{\varepsilon A}\| = \sum_{k \in A_1} \varepsilon_{2k}E_{2k} + \varepsilon_{2k-1}E_{2k-1} + \sum_{k \in A_2} \varepsilon_{2k}E_{2k} + \sum_{k \in A_3} \varepsilon_{2k-1}E_{2k-1}$$

$$= \sum_{k \in A_1} \left(\frac{1}{4}E_{2k} + \frac{1}{2}E_{2k-1}\right) + \frac{3}{4}E_{2k} - \frac{1}{2}E_{2k-1}\right) + \sum_{k \in A_3} \varepsilon_{2k-1}\left(\frac{1}{2}E_k, \frac{1}{2}E_{k-1}\right)$$

$$\geq \sum_{k \in A_1} \frac{1}{4}E_{2k} + \frac{1}{2}E_{2k-1} + \sum_{k \in A_2} \frac{1}{4} + \sum_{k \in A_3} \frac{1}{2}.$$

Therefore,

$$\|1_{\varepsilon A}\| \geq \frac{1}{4}|A_1| + \frac{1}{4}|A_2| + \frac{1}{2}|A_3| \geq \frac{1}{8}|A|.$$

This finishes the proof. \qed

Back to Example 4.8 to see that the basis does not have the Property (D), take $z = \sum_{n=1}^{N} 2E_{2n} - \sum_{n=1}^{N} E_{2n-1}$. Then,

$$\|z\| = \|\sum_{n=1}^{N} (0, 2f_n)\| = 2.$$

Write $z = \sum_{i \in A} a_iE_i$. Then, $\min_{i \in A} |a_i| = 1$ and

$$\|\sum_{i \in A} E_i\| = \|\sum_{n=1}^{N} E_{2n} + \sum_{n=1}^{N} E_{2n-1}\| = \|\sum_{n=1}^{N} (\frac{3}{4}e_n, \frac{1}{4}f_n)\| = \frac{3}{4}N + \frac{1}{4}.$$

This shows that the Property (D) fails.

Lemma 4.13 of [13] establishes that, if $w$ is equivalent to the constant, and $\mathcal{B}$ is $w$-semi-greedy, then $\mathcal{B}$ satisfies the Property (D). Here, we improve this result showing that the condition of being $w$-semi-greedy implies the Property (C).

**Proposition 4.10.** Assume that $w$ is equivalent to the constant, and the basis $\mathcal{B}$ is $w$-semi-greedy. Then $\mathcal{B}$ satisfies the Property (C).

**Proof.** By Theorem 3.7 $\mathcal{B}$ has the $w$-Property (A). By Proposition 3.2 $\mathcal{B}$ also has the “classical” Property (A). This, in turn, implies the Property (C). \qed
5. $w$-Partially-greedy bases

Partially-greedy and conservative bases were introduced in [12], in order to compare the errors of greedy approximation with those of the canonical approximation relative to Schauder basis (the “tails” of the basis expansion). In this section we define $w$-partially-greedy and $w$-conservative bases and extend the characterization of partially-greedy bases proved in [12] to this more general setting.

**Definition 5.1.** We say that $B$ is $w$-partially-greedy if for all $m$ and $r$ such that $w(\{1, \ldots, m\}) \leq w(A_r(x))$, there exists a positive constant such that

$$\|x - G_r(x)\| \leq C \sum_{n=m+1}^{\infty} e^*_n(x)e_i\|.$$  \hspace{1cm} (20)

We denote by $C_p$ the least constant that satisfies (21) and we say that $B$ is $C_p$-$w$-partially-greedy.

**Definition 5.2.** We say that $B$ is $w$-conservative if there exists a positive constant $C$ such that

$$\|1_A\| \leq C\|1_B\|,$$  \hspace{1cm} (21)

for all pair of $A, B \in \mathbb{N}^{<\infty}$ such that $A < B$ and $w(A) \leq w(B)$. We denote by $C_c$ the least constant that satisfies (21) and we say that $B$ is $C_c$-$w$-conservative.

**Remark 5.3.** If $w \equiv 1$, we recover the classical definition of partially-greediness (resp. conservativeness), and we will say that $B$ is partially-greedy (resp. conservative).

**Remark 5.4.** Note that for some choices of weight $w$, the property of $w$-conservativeness can be in some sense trivial. For instance, if $w = (2^{-n})_{n=1}^{\infty}$ then every seminormalized basis is $w$-conservative. This is because there are no nonempty $A, B \in \mathbb{N}^{<\infty}$ satisfying both $A < B$ and $w(A) \leq w(B)$.

Let us give a simple characterization of weights for which this occurs.

**Proposition 5.5.** Let $w$ be a weight and set

$$s_w := \sup\{n \in \mathbb{N}_0 : \text{there exist } A \in \mathbb{N}^n \text{ and } B \in \mathbb{N}^{<\infty} \text{ such that } A < B \text{ and } w(A) \leq w(B)\}.$$  

Then $s_w < \infty$ if and only if every seminormalized basis is $w$-conservative.

*Proof.* ($\Rightarrow$): Suppose $s_w < \infty$. Let $(e_n)_{n=1}^{\infty}$ be a seminormalized basis for a Banach space $X$, and select $A, B \in \mathbb{N}^{<\infty}$ such that $A < B$ and $w(A) \leq w(B)$. Observe that $\|1_A\| \leq c_2|A| \leq c_2s_w$. It follows immediately that $\|1_A\| \leq c_2s_w \leq c_2s_w\|1_B\|$. Hence, $(e_n)_{n=1}^{\infty}$ is $(c_2s_w)$-$w$-conservative.

($\Leftarrow$): Suppose $s_w = \infty$. Let’s inductively construct sequences $(A_n)_{n=1}^{\infty} \subset \mathbb{N}^{<\infty}$ and $(B_n)_{n=1}^{\infty} \subset \mathbb{N}^{<\infty}$ satisfying

$$A_1 < B_1 < A_2 < B_2 < A_3 < B_3 < \ldots,$$

and also satisfying $|A_n| \geq n$ and $w(A_n) \leq w(B_n)$ for all $n \in \mathbb{N}$. Let us begin by selecting $A_1 \in \mathbb{N}^{<\infty}$ and $B_1 \in \mathbb{N}^{<\infty}$ with $|A_1| = 1$, $A_1 < B_1$, and $w(A_1) \leq w(B_1)$, which is possible as $s_w \geq 1$. This is the base case; from now on, we proceed inductively. Since $s_w = \infty$, we may select $\hat{A}_{n+1} \in \mathbb{N}^{<\infty}$ and $B_{n+1} \in \mathbb{N}^{<\infty}$ with $|\hat{A}_{n+1}| > n + \max B_n$, $\hat{A}_{n+1} < B_{n+1}$, and $w(\hat{A}_{n+1}) < w(B_{n+1})$. Now set $A_{n+1} = \hat{A}_{n+1} \setminus \{1, \ldots, \max B_n\}$ so that we have $|A_{n+1}| > n$, $A_{n+1} < B_{n+1}$, and $w(A_{n+1}) < w(B_{n+1})$. This completes the inductive step, and gives us our
intertwining sequences with the desired properties. We may now define a norm on $c_{00}$ via the rule
\[ \| (a_n)_{n=1}^\infty \|_X = \| (a_n)_{n=1}^\infty \|_X \vee \sup_{k \in \mathbb{N}} \sum_{n \in A_k} |a_n| \quad \forall \ (a_n)_{n=1}^\infty \in c_{00}, \]
and denote by $\overline{X}$ the completion of $c_{00}$ under this norm. It is clear that the standard canonical basis for this space form a normalized 1-unconditional basis. However, it fails to be $w$-conservative as $\| 1_{A_k} \|_X = |A_k| \geq k$ whereas $\| 1_{B_k} \|_X = 1$ for all $k \in \mathbb{N}$. 

**Proposition 5.6.** Let $w$ be a nonincreasing weight, i.e., $w_{n+1} \leq w_n$ for all $n \in \mathbb{N}$. Then every conservative basis in a Banach space is $w$-conservative with the same constant.

**Proof.** Let $(e_n)_{n=1}^\infty$ be a conservative basis in a Banach space $X$, and select any $A, B \in \mathbb{N}^{<\infty}$ satisfying both $A < B$ and $w(A) \leq w(B)$. Now,
\[ |A| \cdot w_{\max A} \leq w(A) \leq w(B) \leq |B| \cdot w_{\min B} \leq |B| \cdot w_{\max A}, \]
so that $|A| \leq |B|$. 

**Theorem 5.7.** A basis $B$ is $w$-partially-greedy if and only if $B$ is quasi-greedy and $w$-conservative.

**Proof.** Assume that $B$ is $C_p$-w-partially-greedy.

1. **$w$-conservative:** take $A$ and $B$ such that $A < B$ and $w(A) \leq w(B)$. Let $m = \max A$ and define the set $D = [1, \ldots, m] \setminus A$. Of course,
\[ w(\{1, \ldots, m\}) = w(A \cup D) \leq w(B \cup D). \]
Define now $x := 1_A + (1 + \delta)1_{B \cup D}$. Then,
\[ \| 1_A \| = \| x - G_{B \cup D}(x) \| \leq C_p \| (1 + \delta)1_B \|. \]
Taking $\delta \to 0$, the basis is $w$-conservative.

2. **Quasi-greedy:** here, we consider two cases.

a) Assume that the index $1 \notin A_r(x)$. Define then $\tilde{x} = te_1 + \sum_{i=2}^\infty e^*_i(x)e_i = x + (t - e^*_1(x))e_1$, with $t = \max |e^*_i(x)| + \delta$ with $\delta > 0$. Then,
\[ G_{r}(\tilde{x}) = te_1 + G_{r-1}(x), \]
hence $\tilde{x} - G_r(\tilde{x}) = \sum_{i=2}^\infty e^*_i(x)e_i - G_{r-1}(x)$. Thus, using the triangle inequality and the fact that $w(\{1\}) \leq w(A_r(\tilde{x}))$,
\[ \| G_{r-1}(x) \| \leq \| \tilde{x} - G_r(\tilde{x}) \| + \| \sum_{i=2}^\infty e^*_i(x)e_i \| \leq C_p \| \sum_{i=2}^\infty e^*_i(x)e_i \| + \| \sum_{i=2}^\infty e^*_i(x)e_i \|
\leq (C_p + 1)(1 + K_b)\|x\|. \]
That’s implies that $\| G_r(x) \| \leq ((C_p + 1)(K_b + 1) + e_2^2)\|x\|.

b) Assume now that $1 \in A_r(x)$. Taking the same $\tilde{x}$ that in the above case,
\[ G_{r}(\tilde{x}) = G_{r-1}(x - e^*_1(x)e_1) + te_1, \]
so \( \bar{x} - G_r(\bar{x}) = \sum_{i=2}^{\infty} e_i^*(x)e_i - G_{r-1}(x - e_1^*(x)e_1) \). Hence, using the same argument than before,

\[
\|G_{r-1}(x - e_1^*(x)e_1)\| \leq \| \bar{x} - G_r(\bar{x}) \| + \| \sum_{i=2}^{\infty} e_i^*(x)e_i \|
\]

\[
\leq \ C_p \| \sum_{i=2}^{\infty} e_i^*(x)e_i \| + \| \sum_{i=2}^{\infty} e_i^*(x)e_i \|
\]

\[
\leq \ (C_p + 1 + K_b)\|x\|.
\]

Now, \( \|G_{r-1}(x - e_1^*(x)e_1)\| = \|G_{r-1}(x - e_1^*(x)e_1 + e_1^*(x)e_1 - e_1^*(x)e_1)\| = \|G_r(x) - e_1^*(x)e_1\| \). So, using the triangle inequality, we obtain that \( \|G_r(x)\| \leq ((C_p + 1)(K_b + 1) + c_2^2)\|x\| \).

Now, assume that \( B \) is \( C_c\)-w-conservative and \( C_q\)-quasi-greedy, and show that \( B \) is \( w \)-partially-greedy. Take \( x \in X \), \( m \), and \( r \) as in the definition of \( w \)-partially-greedy, and consider the sets

\[
D := \{ \rho(j) : j \leq r, \rho(j) \leq m \}, \quad B := \{ \rho(j) : j \leq r, \rho(j) > m \}, \quad A := [1, \ldots, m] \setminus D,
\]

where \( \rho \) is the greedy ordering. Then \( A_r(x) = B \cup D \), and \( w(A) = w(\{1, \ldots, m\}) - w(D) \leq w(A_r(x)) - w(D) = w(B) \).

\[
x - G_r(x) = \sum_{i=m+1}^{\infty} e_i^*(x)e_i - P_B(x) + P_A(x).
\]

On the one hand, \( \|P_B(x)\| \leq 2C_q\| \sum_{i=m+1}^{\infty} e_i^*(x)e_i \|. \) On the other hand, using Lemmas 7.1 and 7.2 with \( \eta \equiv \text{sgn}(e_j^*(x)) \),

\[
\|P_A(x)\| \leq 4C_q C_c \max_{i} |e_i^*(x)| \|1_{\eta B}\| \leq 4C_q C_c \min_{i} |e_i^*(x)| \|1_{\eta B}\|
\]

\[
\leq 8C_q^2 C_c \|P_B(x)\| \leq 8C_q^3 C_c \sum_{i=m+1}^{\infty} e_i^*(x)e_i \|
\]

Then, \( \|x - G_r(x)\| \leq C_q^3 C_c \| \sum_{i=m+1}^{\infty} e_i^*(x)e_i \|. \)

\[\Box\]

**Remark 5.8.** Note that if the inequality \( \|x - G_r(x)\| \leq C \|x - S_m(x)\| \) is satisfied for \( m \) and \( r \), then it is automatically satisfied – with a different constant – for any \( n < m \) and the same \( r \) (since \( C \|x - S_n(x)\| \leq (1 + K_b)C \|x - S_m(x)\| \) where \( K_b \) is the basis constant). So we only need to check the condition in the definition of \( w \)-partially-greedy for the largest \( m \) satisfying \( w(\{1, \ldots, m\}) \leq w(A_r(x)) \).

Using the constant weight \( w \equiv 1 \), we recover the usual definition of a partially-greedy basis. Indeed, for \( w \equiv 1 \), the largest \( m \) satisfying the definition is \( m = r \), which recaptures the original definition of partially-greedy given in [12].

5.1. **Example of conservative and not democratic basis.** Define the set

\[ \mathcal{S} = \{ A \in \mathbb{N}^{<\infty} : |A| \leq \sqrt{\min A} \}. \]

Observe that \( \mathcal{S} \) has the spreading property, i.e., if \( m \in \mathbb{N}, \ (f_i)_{i=1}^{m} \in \mathcal{S} \) and \( (g_i)_{i=1}^{m} \in \mathbb{N}^{m} \) with \( f_i \leq g_i \) for all \( i = 1, \ldots, n \), then \( (g_i)_{i=1}^{m} \in \mathcal{S} \). It also hereditary, i.e., if \( A \in \mathcal{S} \) and \( B \subseteq A \) then \( B \in \mathcal{S} \).
Now, let $X$ be the Banach space that we define like the completion of $c_{00}$ under the norm

$$\|(a_n)_n\| = \sup_{A \in S} \sum_{n \in A} |a_n|.$$  

Observe that this is a very slight modification of the Schreier space.

Of course, the canonical basis $(e_n)_n$ is a normalized 1-unconditional basis. Note that the hereditary property guarantees that

$$\|1_A\| = \sup_{F \in S, F \subseteq A} |F|.$$  

Now, if $A < B$ and $|A| \leq |B|$, then there is $F \in S$ with $F \subseteq A$ such that $\|1_A\| = |F|$. By the spreading property, we can “push out” $F$ to obtain a set $G \subseteq B$ such that $G \in S$ and $|G| = |F|$. Hence,

$$\|1_A\| = |F| = |G| \leq \|1_B\|.$$  

Thus, the basis is conservative with constant 1.

To prove that the basis is not democratic, we can select the sets $A = \{N^2 + 1, \ldots, N^2 + N\}$ and $B = \{1, \ldots, N\}$. Then, since $A \in S$, $\|1_A\| = N$. However, $\|1_B\| \leq \sqrt{N}$, hence the basis is not democratic: to prove this upper estimate, take a set $A_1 \in S$ such that $\|1_B\| = |A_1|$. Then, $\min A_1 \leq N$, so $|A_1| \leq \sqrt{N}$. Hence, $\|1_B\| \leq \sqrt{N}$.

**Remark 5.9.** Of course, since the canonical basis is unconditional (hence, quasi-greedy) and conservative, is partially-greedy, but not almost-greedy because is not democratic.

### 5.2. Example of a $w$-greedy basis which is not conservative (hence not greedy).

**Definition 5.10.** Fix $1 \leq p < q \leq \infty$, and consider $(e_n)^{\infty}_{n=1}$ and $(f_n)^{\infty}_{n=1}$ the respective canonical bases of $\ell_p$ and $\ell_q$ (or $c_0$ if $q = \infty$). Let $w = (w_n)^{\infty}_{n=1} \in (0, \infty)^N$. We define the **Rosenthal-Woo space** $X_{q,p,w}$ as the closed subspace $[f_n \oplus w_n e_n]^{\infty}_{n=1}$ of $\ell_q \oplus \ell_p$. For $s = (s_n)^{\infty}_{n=1}$, the summing basis of $c_0$, we can define $X_{q,s,w}$ similarly as the subspace $[f_n \oplus w_n s_n]^{\infty}_{n=1}$ of $\ell_q \oplus \infty c_0$.

It was mentioned in [13] that if $w \in (0, \infty)^N$ satisfies $w \in c_0 \setminus \ell_1$ then the basis formed by completing $c_{00}$ under the norm

$$\|(a_n)^{\infty}_{n=1}\|_\infty \vee \left(\sum_{n=1}^{\infty} |a_n|^2 w_n \right)^{1/2}, \quad (a_n)^{\infty}_{n=1} \in c_{00},$$

forms a normalized 1-$w$-greedy basis which is not greedy. In fact, this is just the canonical basis of the Rosenthal-Woo space $X_{\infty,2,w^{1/2}}$. More generally, we have the following.

**Proposition 5.11.** Fix $1 \leq p < \infty$ and $w \in (0, \infty)^N \cap (c_0 \setminus \ell_1)$. Then the canonical basis of $X_{\infty,p,w^{1/p}}$ is 1-$w$-greedy, but it is not conservative.

**Proof.** Clearly it is unconditional with constant 1. To prove that the canonical basis is $w$-greedy with constant 1, we need to show that it satisfies the $w$-Property (A) with constant 1 (Theorem 2.1). For that, take $x \in X_{\infty,p,w^{1/p}}$ with $\sup_j |e_j^x(x)| \leq 1$, and consider $A, B \subseteq \mathbb{N}$ such that $A \cap B = \emptyset$, $\text{supp}(x) \cap (A \cup B) = \emptyset$, $w(A) \leq w(B)$. Then, if $\varepsilon$ and $\eta$ are arbitrary choice
of signs,
\[
\left\| x + 1_{x \in A} \right\| = 1 \vee \left( \sum_{n \in \text{supp}(x)} |e_n^*(x)|^p w_n + w(A) \right)^{1/p} \\
\leq 1 \vee \left( \sum_{n \in \text{supp}(x)} |e_n^*(x)|^p w_n + w(B) \right)^{1/p} = \left\| x + 1_{x \in B} \right\|.
\]

Then, the basis satisfies the $w$-Property (A) with constant 1, hence, using that the basis is unconditional with constant 1, the basis is $w$-greedy with constant 1.

To see that it fails to be conservative, fix $m \in \mathbb{N}$ and set $A_m = \{1, \ldots, m\}$ and $B_{m,k} = \{k + 1, \ldots, k + m\}$ for each $k \in \mathbb{N}$. Now observe that $w \in c_0$ ensures that $w(B_{m,k}) \to 0$ when $k \to \infty$ and hence $\left\| 1_{B_{m,k}} \right\|_{w^1/p} = 1$ for sufficiently large $k$. Hence, we may select $k_m \in \mathbb{N}$ so that $B_{m,k_m} > A_m$ and $\left\| 1_{B_{m,k_m}} \right\|_{w^1/p} = 1$. On the other hand, $w \notin \ell_1$ guarantees that $w(A_m) \to \infty$. \hfill \Box

**Remark 5.12.** The above proof works for $X_{\infty,s,w}$ as well, except that the basis is no longer unconditional. It yields an example of a subspace of $c_0$ with a basis which is $w$-basic but not conservative, so long as $w \in (0, \infty)^{\mathbb{N}} \cap (c_0 \setminus \ell_1)$.

However, the situation is different for the canonical basis of $X_{q,p,w^1/p}$ when $q \neq \infty$. These spaces fail to contain any copies of $c_0$, and hence do not admit $v$-greedy bases for $v$ non-semi-normalized. Even when $v$ is semi-normalized the canonical basis may not be $v$-democratic (nor $v$-greedy), as we will see momentarily.

**Proposition 5.13.** Fix $1 \leq p < q < \infty$, and let $w \in (0, \infty)^{\mathbb{N}}$ be decreasing. Then the canonical basis of $X_{q,p,w^1/p}$ is unconditional and $w$-conservative with constants 1.

**Proof.** Select $A, B \in \mathbb{N}^{<\infty}$ with $w(A) \leq w(B)$ and $A < B$. Since $w$ is decreasing, we must have $|A| \leq |B|$. Thus,
\[
\left\| 1_A \right\|_{q,p,w^1/p} = (|A|)^{1/q} \vee w(A)^{1/p} \leq (|B|)^{1/q} \vee w(B)^{1/p} = \left\| 1_B \right\|_{q,p,w^1/p}.
\]

\hfill \Box

**Proposition 5.14.** Fix $1 \leq p < q < \infty$ and $0 < \theta < 1 - \frac{p}{q}$. Let $w = (w_n)_{n=1}^{\infty} \in (0, \infty)^{\mathbb{N}}$ be defined by $w_n = n^{-\theta}$ for $n \in \mathbb{N}$. Then the canonical basis for $X_{q,p,w^1/p}$ is not conservative, and not $w$-democratic.

**Proof.** First establish that our basis is not conservative. As in the proof of Proposition 5.11 for $k, m \in \mathbb{N}$ we set $A_m = \{1, \ldots, m\}$ and $B_{m,k} = \{k + 1, \ldots, k + m\}$, and for each $m \in \mathbb{N}$ we find $k_m \in \mathbb{N}$ large enough that $A_m < B_{m,k_m}$ and $\left\| 1_{B_{m,k_m}} \right\|_{w^1/p} = m^{1/q}$. Meanwhile,
\[
\left\| 1_{A_m} \right\|_{q,p,w^1/p} \geq w(A_m)^{1/p} = \left( \sum_{n=1}^{m} n^{-\theta} \right)^{1/p} \geq \left( \int_{1}^{m} t^{-\theta} \, dt \right)^{1/p} = \left( \frac{m^{1-\theta} - 1}{1-\theta} \right)^{1/p}
\]
so that, due to $1 - \theta - p/q > 0$,
\[
\left\| 1_{A_m} \right\|_{q,p,w^1/p} \geq \left( \frac{m^{1-\theta} - 1}{1-\theta} \right)^{1/p} m^{-1/q} = \left( \frac{m^{1-\theta-p/q} - m^{-p/q}}{1-\theta} \right)^{1/p} \to \infty.
\]
We next sketch the proof of the lack of \( w \)-democracy. To this end, consider the sets \( A_n \) and \( B_{m,k} \) as defined in the preceding paragraph. There exist universal constants \( c \) and \( C \) so that 
\[ w(A_n) \geq cn^{1-\theta} \] and 
\[ \|1_A_n\| \leq Cn^{1-\theta}. \]
Then \( w(B_{m,k}) < mk^{-\theta} \), while \( \|1_{B_{m,k}}\| \geq m^{1/q} \). For large values of \( k \), select \( m \in [cn^{1-\theta}k^\theta/2, cn^{1-\theta}k^\theta] \). Then \( w(B_{m,k}) \leq w(A_n) \), yet the ratio 
\[ \frac{\|1_{B_{m,k}}\|}{\|1_{A_n}\|} \approx \frac{m^{1/q}}{n^{1-\theta}} \approx \frac{n^{(1-\theta)/q}k^{\theta/q}}{n^{1-\theta}} = k^{\theta/q}n^{-(1-\theta)(1-1/q)} \]
can be arbitrarily large (for large \( k \)), ruling out the possibility of \( w \)-democracy.

\[ \square \]

**Corollary 5.15.** Fix \( 1 \leq p < q < \infty \) and \( 0 < \theta < 1 - \frac{p}{q} \). Let \( w = (w_n)_{n=1}^\infty \in (0, \infty)^\mathbb{N} \) be defined by \( w_n = n^{-\theta} \) for \( n \in \mathbb{N} \). Then the canonical basis for \( X_{q,p,w}^{1/p} \) is not \( v \)-democratic for any weight \( v \in (0, \infty)^\mathbb{N} \).

**Proof.** Suppose, for the sake of contradiction, that the canonical basis for \( X_{q,p,w}^{1/p} \) is \( v \)-democratic. Note that this basis contains no subsequences equivalent to the \( c_0 \)-basis. Proposition 5.10 shows that \( 0 < \inf v_n \leq \sup v_n < \infty \) – that is, the weight \( v \) is equivalent to a constant. Then, by Remark 5.3, the canonical basis for \( X_{q,p,w}^{1/p} \) has to be democratic, hence conservative. This, however, contradicts Proposition 5.14. \[ \square \]

**Remark 5.16.** Consider again the weight \( w \) from Proposition 5.14

- It follows from Propositions 5.13 and 5.14 that the canonical basis of \( X_{q,p,w}^{1/p} \) is \( w \)-partially-greedy, but not \( w \)-almost-greedy. However, the space \( X_{q,p,w}^{1/p} \) does have an almost-greedy basis. Indeed, we recall from [3, Theorem 10.7.1] that if \( X \) has a complemented subspace with a symmetric basis and finite cotype then \( X \) admits an almost-greedy basis. If \( w \in (0, \infty)^\mathbb{N} \cap (c_0 \setminus \ell_{(p,q)/(q-p)}) \) then the Woo-Rosenthal spaces \( X_{q,p,w} \) contain complemented copies of \( \ell_p \) and \( \ell_q \) (or \( c_0 \) if \( q = \infty \); see [20, Corollary 3.2]), and hence satisfy this condition.
- Just as in Proposition 5.14, one can show that the canonical basis of \( X_{q,s,w} \) is not conservative when \( 0 < \theta < 1 - \frac{1}{q} \). However, it is not quasi-greedy, either. For \( \theta = 1 - \frac{1}{q} \), this basis becomes quasi-greedy and democratic (for \( q = 2 \), this was observed in [3, Example 10.2.9], the argument is valid for all \( 1 < q < \infty \)).

6. Questions

- Does Property (D) imply Property (C)?
- Does Property (A) imply Property (D)?
- If a basis has Properties (C) and (A), is it necessarily semi-greedy?
- Is it possible to formulate a new property so that every conservative basis with this property is necessarily democratic?
- If \( w = (1, 1, \ldots) \) in Theorem 5.7, do we get the same constant as in the classic case ([12, Theorem 3.4])?

7. Appendix

The purpose of this appendix is to show two basic lemmas. The first one resembles a result from [7]. For each \( \lambda > 0 \), we define the \( \lambda \)-truncation of \( z \in \mathbb{C} \) by
\[
T_\lambda(z) = \begin{cases} 
\text{sgn}(z) & \text{if } |z| \geq \lambda \\
z & \text{if } |z| \leq \lambda
\end{cases}
\]
We extend $T_\lambda$ to an operator on $\mathbb{X}$ by

$$T_\lambda(x) = \sum_j T_\lambda(e_j^*(x))e_j = \sum_{j \in \Lambda} \lambda \text{sgn}(e_j^*(x))e_n + \sum_{j \in \Lambda^s} e_j^*(x)e_j,$$

where $\Lambda = \{ j : \lambda < |e_j^*(x)| \}$.

**Lemma 7.1.** For all $\lambda > 0$ and $x \in \mathbb{X}$, if $B$ is $C_q$-quasi-greedy, we have

$$\|T_\lambda(x)\| \leq C_q\|x\|, \quad \|(I - T_\lambda)(x)\| \leq (C_q + 1)\|x\|, \quad \alpha\|1_{\epsilon A}\| \leq 2C_q\|x\|,$$

where $\alpha = \min_{j \in \Lambda} |e_j^*(x)|$, $\Lambda$ is a greedy set of $x$ and $\epsilon \equiv \text{sgn}(e_j^*(x))$.

Moreover, if $B$ is $K_u$-unconditional, for every set $A$ with $|A| < \infty$,

$$\|T_\lambda(I - P_A)(x)\| \leq K_u\|x\|.$$

**Proof.**

$$T_\lambda(x) = \int_0^1 \left[ \sum_j \chi_{[0,|e_j^*(x)|]}(s)e_j^*(x)e_j \right] ds = \int_0^1 (I - P_{\Lambda, s})x ds,$$

where $\Lambda, s = \{ j : \frac{\lambda}{s} < |e_j^*(x)| \}$ is a greedy set of $x$ of finite cardinality. Then, using the Minkowski’s integral inequality,

$$\|T_\lambda(x)\| \leq \int_0^1 \|(I - P_{\Lambda, s})x\| ds \leq C_q\|x\|.$$

Also, since $(I - T_\lambda)x = \int_0^1 P_{\Lambda, s}(x) ds$, hence

$$\|(I - T_\lambda)(x)\| \leq (C_q + 1)\|x\|.$$

Now, since $\alpha 1_{\epsilon A} = T_\alpha(x) - P_A(x) = \int_0^1 P_A(x) - P_{\Lambda, s}(x) ds$,

$$\alpha\|1_{\epsilon A}\| \leq 2C_q\|x\|.$$

On the other hand, if $A$ is a general set with $|A| < \infty$,

$$T_\lambda(I - P_A)x = \int_0^1 (I - P_{\Lambda, s})(I - P_A)x ds = \int_0^1 (I - P_{\Lambda, s})x ds,$$

thus

$$\|T_\lambda(I - P_A)(x)\| \leq K_u\|x\|.$$

\[\square\]

The second lemma involves the concept of $w$-partially-greedy bases.

**Lemma 7.2.** If $B$ is $C_w$-$w$-conservative and $C_q$-quasi-greedy, then

$$\| \sum_{j \in A} a_j e_j \| \leq 4C_qC_w \max_{j \in A} |a_j| \|1_{\eta B}\|,$$

for any sign $\eta$ and $A, B \in \mathbb{N}^{<\infty}$ such that $w(A) \leq w(B)$, $A < B$ and any collection of scalars $(a_j)_{j \in A}$.
Proof. We prove that \( \|1_{\varepsilon A}\| \leq 4C_qC_w\|1_\eta B\| \) for any signs \( \varepsilon \) and \( \eta \). First, we can decompose \( 1_{\varepsilon A} = 1_{A^+} - 1_{A^-} \), where \( A^\pm = \{ j \in A : \varepsilon_j = \pm 1 \} \). Then,

\[
\|1_{\varepsilon A}\| \leq \|1_{A^+}\| + \|1_{A^-}\| \leq 2C_w\|1_B\|.
\]

Now, using the condition to be quasi-greedy, it is clear that \( \|1_B\| \leq 2C_q\|1_\eta B\| \), then

\[
\|1_{\varepsilon A}\| \leq 4C_qC_w\|1_\eta B\|.
\]

Now, using convexity, we are done. \( \square \)

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