Viscous evolution of the rapidity distribution of matter created in relativistic heavy-ion collisions*

Piotr Bożek\textsuperscript{1, 2}

\textsuperscript{1}Institute of Physics, Rzeszów University, PL-35959 Rzeszów, Poland
\textsuperscript{2}The H. Niewodniczański Institute of Nuclear Physics, PL-31342 Kraków, Poland

Longitudinal hydrodynamic expansion of the fluid created in relativistic heavy-collisions is considered taking into account shear viscosity. Both a non-vanishing viscosity and a soft equation of state make particle distributions in rapidity narrower. The presence of viscosity has dramatic consequences on the value of the initial energy density. The reduction of the longitudinal work and dissipative processes due to the shear viscosity, increase the total entropy and the particle multiplicity at central rapidities. The total energy in the collision, dominated by the longitudinal motion, is conserved. Viscous corrections make the longitudinal velocity of the fluid to stay close to the Bjorken scaling \(v_z = z/t\) through the evolution. At the freeze-out viscous corrections are the strongest for non-central rapidities.

I. INTRODUCTION

Properties of hot and dense strongly interacting matter can be studied in ultrarelativistic nuclear collisions. The modeling of the evolution of the dense collective phase is most commonly undertaken within the framework of relativistic fluid dynamics \([1, 2, 3]\). A clear evidence of a collective behavior of the system created in a collision is given by the observation of transverse elliptic flow of the produced particles. Collectives flow arises naturally during a hydrodynamic evolution. In ultrarelativistic heavy-ion collisions the movement of the matter at the initial stage is dominated by the expansion in the longitudinal direction. Most of the experimental data at the Relativistic Heavy-Ion Collider (RHIC) are restricted to the central rapidity region. Therefore, hydrodynamic models often assume a simplified geometry of the fireball with a Bjorken boost invariant flow in the longitudinal direction and concentrate on the dynamics in the transverse directions with azimuthal symmetry for central collisions \([4, 5, 6]\) or azimuthally asymmetric geometry for collisions of nuclei at non-zero impact parameters \([7, 8, 9, 10]\). Only few calculations consider a fully three-dimensional evolution of the fluid \([11, 12, 13]\). Results of the hydrodynamic evolution are sensitive to the chosen equation of state. Values of the Hanbury Brown-Twist (HBT) radii depends on the initial temperature and its profile, on the freeze-out temperature, possible final rescattering, therefore estimates of the viscosity coefficient are still under debate \([22, 23]\).

Theoretical estimates of the ratio of the shear viscosity coefficient \(\eta\) to the entropy density \(s\) range from a conjectured lower bound \(\eta/s = 1/4\pi\) \([23]\) to \(\eta \sim s\) \([26, 27]\). The role of the shear viscosity in the dynamics is most important during the early evolution of the system, when the velocity gradients are the largest. Gradients of the Bjorken flow give rise to corrections of the pressure tensor in the fluid. The transverse pressure increases, and the fluid expands faster in the transverse directions, this leads to stronger transverse flow and also to the saturation of the elliptic flow \([22, 23, 24, 25, 30, 31, 32, 33]\). All recent calculations using viscous relativistic hydrodynamics assume boost invariant Bjorken flow in the longitudinal direction and study the transverse development of the...
fluid in azimuthally symmetric or asymmetric conditions. Longitudinal pressure is reduced, and hence so is the longitudinal flow of the fluid. The fluid cools slower, at least until substantial transverse flow builds up. Weaker longitudinal expansion and entropy production due to dissipative evolution require an adjustment of the initial entropy (temperature). As a result, the lifetimes of the plasma in the viscous and ideal fluid evolutions are similar. Finally, let us note that viscous corrections to the distribution functions at the freeze-out modify the spectra, and the HBT radii\cite{32,33}.

Enhanced transverse pressure and the resulting modification of the spectra at central rapidity, raise the question of possible modifications of the fluid dynamics in the longitudinal direction due to viscosity. Since the boost invariant scaling solution is not applicable at RHIC energies, quantitative description of the energy flow and entropy production in the fireball should take into account a fully three dimensional geometry. Such a task within the dissipative fluid dynamics has not been accomplished yet. In the following we consider a simpler problem of the evolution of a longitudinally expanding non-boost invariant fluid with viscosity. One expects that reduced longitudinal work in a viscous fluid generates narrower particle distributions in rapidity. The author is aware of only one, two-decades old work, considering this problem\cite{35}, where first-order viscous hydrodynamics has been applied. At lower energies no strong effect of viscosity on the longitudinal expansion has been observed\cite{35}.

In the following we use BRAHMS data\cite{19} to constrain meson rapidity distributions after freeze-out. Solving viscous hydrodynamics in the 1+1 longitudinal geometry, we find a significant reduction of the initial energy density when viscosity is taken into account. Also the longitudinal flow is modified, viscosity reduces the flow and counteracts the acceleration due rapidity gradients of the pressure. At the Large Hadron Collider (LHC) energies we find that the modification of the longitudinal dynamics due to shear viscosity leads to an increase of the rapidity range where the Bjorken scaling flow applies.

\section{II. Longitudinal Hydrodynamic Equations with Shear Viscosity}

We consider a baryon-free fluid with non-zero shear viscosity. The energy-momentum tensor is the sum of the ideal fluid component and the shear tensor $\pi^{\mu\nu}$

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - g^{\mu\nu} + \pi^{\mu\nu}, \quad (2.1)$$

$\epsilon$ and $p$ are the local energy density and pressure of the fluid, and

$$u^\mu = \gamma(1,0,0,v) = (\cosh Y,0,0,\sinh Y) \quad (2.2)$$

is the four-velocity of the fluid element ($\gamma = 1/\sqrt{1-v^2}$), and $Y = \frac{1}{2} \ln \left( \frac{t+z}{t-z} \right)$ is its rapidity. The energy density and the pressure are related by the equation of state. The energy density $\epsilon(t,z)$ and the longitudinal velocity component $v(t,z)$ are functions of time $t$ and the beam axis coordinate $z$. Instead of the time and the $z$ coordinate it is preferable to use the proper time $\tau = \sqrt{t^2 - z^2}$ and the space-time rapidity

$$\theta = \frac{1}{2} \ln \left( \frac{t+z}{t-z} \right). \quad (2.3)$$

Hydrodynamic equations $\partial_\mu T^{\mu\nu} = 0$ can be written as

$$\epsilon + p = \nabla_\mu \pi^{\mu\nu} + \pi^{\mu\nu} \partial_\nu, \quad (2.4)$$

and

$$D\epsilon = -(\epsilon + p)\nabla_\mu u^\mu + \frac{1}{2} \pi^{\mu\nu} < \nabla_\mu u_\nu >, \quad (2.5)$$

where

$$D = u^\mu \partial_\mu = \cosh(Y - \theta)\partial_x + \frac{\sinh(Y - \theta)}{\tau} \partial_\theta, \quad (2.6)$$

$$< \nabla_\mu u_\nu > = \nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \Delta_{\mu\nu} \nabla_\alpha u^\alpha, \quad (2.7)$$

$$\nabla_\mu = \Delta_{\mu\nu} \partial_\nu = \left( -\sinh YK, -\partial_x, -\partial_\theta, -\cosh YK \right) \quad (2.8)$$

with $\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ and

$$K = \sinh(Y - \theta)\partial_x - \frac{\cosh(Y - \theta)}{\tau} \partial_\theta. \quad (2.9)$$

The equations of the second order viscous hydrodynamics\cite{23,24} are supplemented with a dynamic equation for the stress tensor\cite{29,34,37,38,39,40}

$$\tau_\pi \Delta^{\alpha}_{\beta} \Delta^{\beta}_{\gamma} \pi^{\alpha\beta} + \pi^{\mu\nu} = \eta < \nabla_\mu u_\nu > - 2\tau_\pi \pi^{\alpha}_\mu \omega^\alpha, \quad (2.10)$$

$\eta$ is the shear viscosity coefficient and $\tau_\pi$ is the relaxation time of the stress tensor. $\omega^{\mu\nu} = \Delta^{\mu\nu} \Delta^{\alpha\beta} (\partial_\alpha u_\beta - \partial_\beta u_\alpha)$ is the vorticity of the fluid; it is zero for the longitudinal flow considered here. The relaxation time and the viscosity coefficient can be estimated from microscopic models, considering equilibration processes\cite{23,26,27,41}. In this work we take several values for the ratio of the viscosity coefficient to the entropy $\eta/s$, and we drop viscosity effects for temperatures below 130MeV, the relaxation time is $26 \tau_\pi/\eta = 6/Ts$, unless specified otherwise ($T$ is the local temperature). For short relaxation times the stress tensor relaxes fast and stays close to the Navier-Stokes value $\pi^{\mu\nu} = \eta < \nabla_\mu u_\nu >$. The dependence on the initial value of the stress tensor and on the relaxation time is discussed in Section\cite{41}.

For a fluid expanding only in the longitudinal direction, with the energy density and the velocity constant in the
transverse plane, the stress tensor can be written using one scalar function \( \Pi \)

\[
\tau^{\mu\nu} = \begin{pmatrix}
-\sinh^2 Y & 0 & 0 & -\sinh Y \cosh Y \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
-\sinh Y \cosh Y & 0 & 0 & -\cosh^2 Y
\end{pmatrix} \Pi
\]

(2.11)

and viscous hydrodynamic equations take the form

\[
(\epsilon + p)DY = -Kp + \Pi D\epsilon + 3\Pi \Pi
\]

\[
D\epsilon = (\epsilon + p)KY - \Pi KY
\]

\[
D\Pi = \left( \frac{4}{3} \eta KY - \Pi / \tau_\pi \right).
\]

(2.12)

The above equations are solved numerically in the \( \tau\theta \) plane, starting from some energy density \( \epsilon(\tau_0, \theta) \) at the initial proper time of the evolution \( \tau_0 = 1.0 \text{fm/c} \). For the initial fluid rapidity we always take the Bjorken flow

\[
Y(\tau_0, \theta) = \theta.
\]

(2.13)

Assuming boost invariance, i.e.

\[
Y(\tau, \theta) = \theta, \quad \epsilon(\tau, \theta) = \epsilon(\tau), \quad p(\tau, \theta) = p(\tau)
\]

(2.14)

hydrodynamic equations simplify to

\[
\frac{d\epsilon}{d\tau} = -\frac{\epsilon + p - \Pi}{\tau},
\]

\[
\frac{d\Pi}{d\tau} = \frac{(4\eta/(3\tau)) - \Pi}{\tau_\pi}.
\]

(2.15)

III. EQUATION OF STATE

The equation of state determines the evolution of the fireball. In the following we take a parameterization of the equation of state proposed by Chojnacki and Florkowski \[42\]. It is an interpolation of the lattice data at temperatures above \( T_c = 170\text{MeV} \) and of an equation of state of noninteracting hadrons at lower temperatures. The two limiting formulas are joined smoothly with only a slight softening of the equation of state around the critical temperature (Fig. 1). This minimally softened, realistic equation of state has shown itself suitable for the description of the transverse expansion of the fluid and of the build up of the elliptic flow and gives reasonable HBT radii \[42, 43\].

In this section we consider the hydrodynamic longitudinal expansion of an ideal fluid. Equations for the fluid rapidity and energy density are obtained from Eqs. (2,12) setting \( \Pi = 0 \). This problem has been discussed recently in the context of heavy-ion collisions at RHIC \[20\]. For completeness we study the effect of the equation of state we use on the longitudinal expansion. For that purpose we compare the evolution using the equation of state of Chojnacki and Florkowski with an evolution based on a relativistic gas equation of state \( p = \frac{4}{3}\epsilon \).

### FIG. 1: Square of the velocity of sound as a function of the temperature for an equation of state interpolating between the hadron gas and the quark-gluon plasma expressions \[42\].

At the initial time \( \tau_0 = 1.0 \text{fm/c} \) the energy density is

\[
\epsilon(\tau_0, \theta) = \epsilon_0 \exp\left(-\theta^2/(2a^2)\right).
\]

(3.1)

The initial energy density \( \epsilon_0 \) and the width of the initial rapidity distribution \( a \) are parameters adjusted to reproduce final meson rapidity distributions. The freeze-out takes place at the temperature \( T_f = 165\text{MeV} \), this high freeze-out temperature is the same as the chemical freeze-out temperature \[44, 45\]. The spectrum of particles emitted with four-momentum \( q^\mu = (E, q) \) is given by the Cooper-Frye formula \[20, 46\]

\[
E \frac{d^3N}{dq^2} = \frac{1}{(2\pi)^3} \int d\Sigma \epsilon q^\mu f(q^\mu u_{\mu})
\]

(3.2)

\( f(E) = e^{-E/T_f} \) is the thermal distribution (Boltzmann distribution for simplicity), and \( y \) denotes the rapidity of the emitted particle. The element of the hypersurface of constant freeze-out temperature is \( d\Sigma^\mu = S (\tau(\theta) \sinh \theta + \tau(\theta) \cosh \theta, 0, 0, \tau(\theta) \sinh \theta - \tau(\theta) \cosh \theta) \), \( S = \pi R_{Au}^2 \) is the transverse area of the fireball in central collisions and \( \tau(\theta) \) is the line of constant temperature \( T_f \) in the \( \tau\theta \) plane. Particle distributions in rapidity are obtained by integration over the transverse momenta \( q_\perp \) in Eq. (3.2)

\[
\frac{dN}{dy} = \frac{S}{4\pi^2} \int_{\theta_{\max}}^{\theta_{\max}} \left( \tau(\theta) \cosh(y - \theta - \tau(\theta) \sinh(y - \theta)) \right)
\]

\[
(2m \xi + 2\xi^2 + m^2) \exp \left(-m \cosh(y - Y_f(\theta)) \right)
\]

\[
\frac{d\theta}{T_f \cosh(y - Y_f(\theta))}
\]

(3.3)

\( m \) is the meson mass, \( Y_f(\theta) = Y(\tau(\theta), \theta) \) is the fluid rapidity at the freeze-out hypersurface, and

\[
\xi = \frac{T_f}{\cosh(y - Y_f(\theta))}
\]

(3.4)
The above expression neglects the transverse expansion of the fluid at the freeze-out. This has only a small effect on rapidity distributions, the distribution should be narrower. Pions and kaons come to a large extend from secondary decays of resonances. The emission takes place in two stages, first an emission of a heavy resonance according to Eq. (3.3) and then the decay of the resonance into pions (kaons). The emission of resonances and their decay is also influenced by their transverse expansion. A hint on the spread in rapidity of the decay products of resonances is given by charge balance correlations [47, 48, 49]. Narrow charge balance functions indicate that decay products of a resonance are only 0.5 unit of rapidity away from the parent resonance, convoluting this distribution with the spread in rapidity of the emitted resonances one obtains a distribution of half-width similar as for the emission of direct pions in Eq. (3.3). Since 75% of pions come from resonances at $T_f = 165$ MeV [50], we multiply the distribution from (3.3) by a factor 4 to account for all pions, direct and from resonance decays (the factor is 1.7 for kaons).

The parameters $\epsilon_0$ and $\sigma$ have been adjusted for the calculation using a realistic sound velocity (Fig. 1) to reproduce the width and normalization of the observed pion distribution. The resulting meson distributions are similar as the ones observed experimentally. When using a relativistic gas equation of state ($p = \frac{1}{3} \epsilon$) one always gets a meson distribution in rapidity that is too wide. We present three calculations with different initial widths $\sigma$ and with the initial energy densities $\epsilon_0$ adjusted to reproduce $dN/dy$ for central rapidities only (Fig. 2). The final meson distribution is much wider than the initial energy density distribution in all cases. It is due to the breaking of the Bjorken scaling of the longitudinal flow, $Y(\tau, \theta) > \theta$ (Fig. 3). Fast moving fluid elements emit mesons in the far forward and backward rapidities. Concluding this section, we confirm the findings of Ref. [20].

A hard equation of state never works; a narrow initial distribution of the energy density in rapidity leads to a strong acceleration of the longitudinal flow, wider initial distributions are incompatible with the narrow final meson distributions. Only by imposing a softened equation of state, experimental pion and kaon distributions can be approximately reproduced. In the following we take the realistic equation of state from Ref. [42] and study the effect of non-negligible shear viscosity.

| EOS [42] | $\epsilon_0$ (GeV/fm$^3$) | $\sigma$ (fm/c) | $\tau(0)$ (fm/c) |
|----------|--------------------------|----------------|-----------------|
| $p = \frac{1}{3} \epsilon$ case I | 16.9 | 1.05 | 14.8 |
| $p = \frac{1}{4} \epsilon$ case II | 71.5 | 1.05 | 14.8 |
| $p = \frac{1}{4} \epsilon$ case III | 102 | 0.8 | 15.7 |
| $p = \frac{1}{4} \epsilon$ case III | 50.8 | 1.5 | 13.7 |

TABLE I: Parameters of the initial energy density distribution (3.1) for ideal fluid calculations, one using a realistic equation of state and three calculations using a relativistic gas equation of state. In the last column is shown the lifetime of the system until freeze-out.
The correction to the particle distribution function is proportional to the ratio of the viscous correction \( \Pi \) to the enthalpy \( \epsilon + p \). Viscous corrections to the distribution functions at the freeze-out modify the Cooper-Frye formula

\[
\frac{dN_{\text{visc}}}{dy} = \frac{dN}{dy} + \frac{d\delta N}{dy}.
\]  

(4.2)

The correction to the particle distribution function is proportional to the ratio of the viscous correction \( \Pi \) to the enthalpy \( \epsilon + p \). Viscous corrections to the distribution functions at the freeze-out modify the Cooper-Frye formula

\[
f(q) + \delta f(q) = f(q) \left( 1 + \frac{q\rho q_\mu \pi^{\mu\nu}}{2T^2(\epsilon + p)} \right),
\]  

(4.1)

To the expression (3.3) one has to add

\[
\frac{d\delta N}{dy} = \frac{S}{4\pi} \int_{-\theta_{\text{max}}}^{\theta_{\text{max}}} \left( \tau(\theta) \cosh(y - \theta) - \tau'(\theta) \sinh(y - \theta) \right) \left[ 12\xi^2 + 5\xi^3 m^2 + 12\xi^4 m + \xi^2 m^3 - \sinh(y - Y_f(\theta)) \right] \left[ 24\xi^5 + 12\xi^6 m^2 + 24\xi^7 m + 4\xi^2 m^3 + \xi^4 m^4 \right] \frac{\Pi}{2T^2(\epsilon + p)} \exp\left(-\frac{m \cosh(y - Y_f(\theta))}{T_f}\right) d\theta,
\]  

(4.3)

In Fig. 4 is shown the result of this procedure for the case \( \eta/s = 0.2 \). Again the direct meson spectra are multiplied by a factor 4 for pions and 1.7 for kaons, to account for the expected ratio of all mesons to directly produced pions \( \frac{A_0}{A_{12}} \), at the chosen freeze-out temperature. Pion emission at the end of the viscous hydrodynamic evolution (solid line) is similar as observed experimentally. In Fig. 4 is also shown the meson distribution obtained using the equilibrium distribution at freeze-out (Eq. 3.3) (dotted line). Deviations from the full result (Eq. 4.3) is only noticeable at rapidities 3 unit away from central rapidity. It can be understood as due to an earlier freeze-out at large rapidities, which makes the relative viscous corrections

\[
\frac{\Pi(\tau(\theta), \theta)}{\epsilon(\tau(\theta), \theta) + p(\tau(\theta), \theta)}
\]  

(4.4)

at freeze-out more important (Fig. 5). Accordingly one expects the largest correction from viscosity to particle spectra, elliptic flow, and HBT radii at large rapidities.

We have noticed (Section III) that longitudinal gradients of the pressure cause the acceleration of the motion of the fluid in the beam direction. On the other hand, viscosity reduces the longitudinal motion of the fluid, and as a consequence reduces its expansion in the longitudinal direction. Shear viscosity prevents large gradients of the velocity field to develop. One can compare

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**FIG. 4:** (color online) Rapidity distribution of mesons calculated using a realistic equation of state and viscosity \( \eta/s = 0.2 \) (solid line) and using ideal fluid hydrodynamics (dashed line). The dotted line denotes the results of a viscous hydrodynamic evolution, but neglecting the viscous corrections to the particle emission at freeze-out (Eq. 4.3). Data are from the BRAHMS Collaboration [19].

**FIG. 5:** Relative viscous corrections \( \Pi/(\epsilon + p) \) at the freeze-out as a function of space-time rapidity (Eq. 4.3).
FIG. 6: (color online) Difference between the flow rapidity of the fluid and the Bjorken value, calculated for an evolution with shear viscosity coefficient $\eta/s = 0.2$ (solid lines), for an ideal fluid with a realistic equation of state (dashed lines) and using a relativistic gas equation of state (case I) (dashed-dotted lines).

![Graph](image)

TABLE II: Parameters of the initial energy density distribution (3.1) for hydrodynamic calculations with several values of the shear viscosity coefficient. In the last column is shown the lifetime of the system until freeze-out.

| $\eta/s$ | $\epsilon_0$ [GeV/fm$^3$] | $\sigma$ [fm/c] | $\tau(0)$ [	ext{fm}/c] |
|----------|-----------------|-----------------|-----------------|
| 0        | 16.9            | 1.05            | 14.8            |
| 0.1      | 9.8             | 1.18            | 14.1            |
| 0.2      | 5.6             | 1.8             | 13.1            |
| 0.3      | 4.0             | 1.86            | 12.4            |

At the freeze-out hypersurface the flow is still Bjorken-like for $|\theta| < 1.8$ for $\eta/s = 0.2$ (Fig. 7). The acceleration from pressure gradients and the deceleration from viscosity approximately cancel for this choice of parameters. Reduced longitudinal expansion with viscosity requires smaller initial energy densities to reproduce the final meson distributions. In Table II are listed the initial energy densities and widths of rapidity distributions adjusted to reproduce the observed meson distributions for several values of the shear viscosity coefficient. The shape of the initial energy density is extremely sensitive to the dynamics of the longitudinal expansion (Fig. 8).

FIG. 7: (color online) Difference between the flow rapidity of the fluid and the Bjorken value at the freeze-out hypersurface, calculated for an evolution with shear viscosity coefficient $\eta/s = 0.2$ (solid line), for an ideal fluid with a realistic equation of state (dashed line) and using a relativistic gas equation of state (case I) (dashed-dotted line).

![Graph](image)

FIG. 8: (color online) Initial energy density distribution for the ideal fluid hydrodynamic evolution with a realistic equation of state (dashed line), for viscous hydrodynamic evolutions (solid lines), and for a relativistic gas equation of state (case I) (dashed-dotted line).

![Graph](image)
words the cooling rate from the longitudinal motion, is an important ingredient in the modeling of the transverse expansion of the fluid. Let us also note, that when reduced initial energy densities are imposed, the lifetime of the system until freeze-out is only weakly dependent on the viscosity coefficient.

V. COOLING AND ENTROPY PRODUCTION

It is instructive to compare the cooling rate in our solution and in the boost invariant scaling solution. Finite extension in space-time rapidity makes the cooling rate faster. On the other hand, reduced velocity of sound and shear viscosity reduce the longitudinal work of the pressure, and slow down the cooling. In Fig. 9 is compared the cooling of the central region of a finite system to the cooling in the boost invariant case (Eq. 2.15). To compare the solution in the 1+1 dimensional system to the boost-invariant one, the initial temperatures are fixed so as to give the same lifetimes of the two systems until freeze-out. Boost invariant solutions underestimate the values of the initial temperature (energy density) and of the cooling rate for an ideal fluid evolution. In a finite system additional cooling appears and the longitudinal flow is stronger than the Bjorken scaling flow. As the shear viscosity coefficient increases the velocity gradients in the dynamics are more and more constrained. The flow accelerates less. At η/s = 0.2 the effects of the viscosity and space-time rapidity gradients of the pressure counterbalance each other and the flow is approximately Bjorken-like. With stronger viscosity η/s = 0.3, the Bjorken flow is decelerated and the cooling is slower than for the boost-invariant solution. It means that the gradients of the viscous correction Π are larger than the gradients of the pressures itself and applicability of second order viscous hydrodynamics is questionable.

Dissipative hydrodynamics conserves the total energy but produces entropy. The expression for the total energy of the system at proper time τ is

\[ E(τ) = τS \int_{-∞}^{∞} dθ \left[ e \cosh Y \cosh(Y - θ) + (p - Π) \sinh Y \sinh(Y - θ) \right]. \]  (5.1)

It is mainly composed of the kinetic energy of the longitudinal motion of the fluid. Small changes of the energy density at large rapidities can cause significant changes of the energy density for central rapidities. Softening of the equation of state and non-zero shear viscosity modify the dynamics at large rapidities, which leads to less cooling at θ = 0, while the global energy of the fireball is unchanged. Dissipative processes driving the system locally towards equilibrium produce entropy \[ S(τ) = τS \int_{-∞}^{∞} dθ s \cosh(Y - θ) \]  (5.2)

FIG. 9: (color online) Time dependence of the temperature at the center for a longitudinally expanding ideal fluid fireball (dashed line) compared to the boost-invariant solution (dashed-dotted line). Same for the evolution with viscosities η/s = 0.1, 0.2, 0.3 (solid lines, increasing η from top to bottom), and for the boost-invariant case with viscosity (dotted lines). For η/s = 0.2 the solid and dotted curves lie on top of each other.

FIG. 10: (color online) Relative entropy production in the viscous hydrodynamic evolution (Eq. 5.2) (solid line), of the entropy density at central space-time rapidity (dashed line), and of the entropy from the boost-invariant Bjorken solution (Eq. 5.3) (dashed-dotted line). All calculation with η/s = 0.2.
increases with time, if shear viscosity is active (Fig. 11). The increase of the entropy at central rapidity $\tau s(\tau, \theta)$ is responsible for an increase of the particle multiplicity in the central rapidity region. Estimates of the entropy production $\frac{d\tau s}{d\tau} = \frac{\Pi}{T}$ (5.3) are very close to the 1+1 dimensional dynamical result, it happens if the shear viscosity is strong enough to conserve the Bjorken flow in the evolution.

VI. ROLE OF THE INITIAL STRESS TENSOR

Second order viscous hydrodynamics introduces a dynamical equation for the stress tensor. A crucial parameter is given by the relaxation time $\tau_s$. For small relaxation times the viscosity correction stays close to the Navier-Stokes value

$$\Pi_{NS}(\tau) = \frac{4\eta}{3\tau}. \quad (6.1)$$

Such a behavior of viscous corrections has been confirmed in numerical simulations of the transverse expansion in viscous hydrodynamics with small relaxation times [23, 32], unlike in Ref. [51] where a large value of the relaxation time was postulated. We check the effect of the initial value of $\Pi(\tau)$ on the evolution by comparing two scenarios, $\Pi(\tau_0) = p(\tau_0)$ which corresponds to an anisotropic momentum (pressure) $P_{\tau s}/T$ at the initial time and $\Pi(\tau_0) = 0$, which corresponds to initially locally equilibrated distributions. With the choice of the relaxation time $\tau_s = 6\eta/Ts$ the two initial conditions lead to different results. As before, for each choice of the initial conditions and parameters we readjust the initial energy density distribution to reproduce the final pion rapidity distribution.

In Fig. 11 is shown the ratio of the dynamical value of the viscous correction $\Pi(\tau)$ to the Navier-Stokes value $\Pi_{NS}(\tau)$ [23, 32]. After several $\text{fm}/c$ the stress tensor, that is set initially to zero (dashed-dotted line), relaxes to and overshoots the steady flow value (dashed line). For an initial stress tensor corresponding to the anisotropic effective pressure $P_{\tau s}/Ts$ the viscous correction overshoots the Navier-Stokes value almost immediately. Since the dissipative effects are the strongest at the early stages, the integrated entropy production is smaller for an evolution starting with nonzero stress tensor (solid line) than for an evolution starting with zero stress tensor (dashed line).

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In that case the initial value of $\Pi(\tau)$
VII. EXPECTATIONS FOR LHC

In this section we present some simple estimates of the effects of the shear viscosity on the longitudinal expansion at LHC energies. To get a rough estimate we set arbitrary the multiplicity of pions for central rapidity at twice the value observed for central collisions at RHIC. The initial energy density distribution in space-time rapidity is modified, it includes a plateau of width $\sigma_p = 3.3$ and takes all the increase of the rapidity range when going from RHIC to LHC energies. The parameter $\sigma$ is the same as at RHIC energies, and the energy density $\epsilon_0$ is adjusted to reproduce the assumed final pion density $\frac{dN_d}{dy}|_{y=0}$. The meson distributions are shown in Fig. 13. For the viscous evolution the plateau in the final meson distributions survives. For an ideal fluid one gets $\epsilon_0 = 16.9 \text{GeV/fm}^3$, while for a shear viscosity $\eta/s = 0.2$ one needs $\epsilon_0 = 12.4 \text{GeV/fm}^3$. The difference between the initial energy densities in the viscous and ideal fluid evolutions is not as big as for RHIC energies. At LHC energies, both the ideal and viscous fluid evolutions have a Bjorken scaling form in several units of central rapidity (Fig. 14). The lack of space-time rapidity gradients in the distributions makes the evolution and cooling last longer, around 20 fm/c. Even at the freeze-out the flow is Bjorken-like at central rapidities (Fig. 15). This observation justifies the use at LHC energies of thermal and hydrodynamic models assuming boost invariance [8, 54, 55]. The hydrodynamic evolution can be restricted to the 1+2 dimensional boost invariant geometry to describe matter in the 4-5 central units of rapidity. At the freeze-out hypersurface dissipative corrections to the momentum distributions have almost disappeared (compare the solid and dotted lines in Fig. 13).

Since the cooling of the fluid due to the longitudinal expansion is slow (like in the Bjorken solution), a realistic modeling of the time scales and of the freeze-out hypersurface must take into account the transverse expansion. The speed of the transverse expansion would determine the life-time of the system, while at RHIC energies longitudinal expansion (in space-time rapidity) is also important. We have also performed calculations using Gaussian initial energy distributions (3.1) with rescaled width parameters $\sigma$ for the increased LHC rapidity range. For a viscous fluid we find a broad region of rapidities where the Bjorken scaling flow survives through the evolution, $|\theta| < 2.5$. For the ideal fluid the scaling is broken, but to a significantly lesser extent than at RHIC energies.
FIG. 15: (color online) Difference between the flow rapidity of the fluid and the Bjorken value at the freeze-out hypersurface, calculated for an evolution with shear viscosity coefficient $\eta/s = 0.2$ (solid line) and for an ideal fluid with a realistic equation of state (dashed line) for LHC plateau-like initial conditions.

VIII. CONCLUSIONS

The evolution of a fireball of dense and hot matter created in heavy-ion collisions can be modelled as a hydrodynamic expansion of a viscous fluid. We analyze the effects of the shear viscosity on the longitudinal expansion of the matter. We solve numerically coupled evolution equations for the longitudinal flow, the energy density and the viscous corrections in a 1+1 dimensional geometry, corresponding to a rapid expansion in the beam direction. As a function of the space-time rapidity the distribution of matter evolves slowly with proper time. The average density drops and the distribution gets wider. The last phenomenon takes place when the flow of the fluid gets stronger than the Bjorken one. At the freeze-out temperature the hydrodynamic stage finishes and particles are emitted thermally according to the Cooper-Frye formula. Experimental measurements of the distribution of mesons in rapidity [19] constrain the allowed distribution of the longitudinal velocities of the fluid elements. Correlation between space-time and momentum rapidities of the fluid, means that the space-time rapidity extension of the fluid must be limited and that its longitudinal flow cannot deviate significantly from the Bjorken flow. Shear viscosity counteracts the gradients of the velocity field. As a consequence it slows down the longitudinal expansion. At the freeze-out the energy density distribution in space-time rapidity is narrower and the longitudinal flow gets less accelerated than for the ideal fluid hydrodynamics. Fitting the initial energy density distribution to reproduce the final meson distributions, one observes a striking effect. With increasing shear viscosity coefficient the initial energy density of the fireball decreases significantly, from 16.9GeV/fm$^3$ for an ideal fluid to 5.6GeV/fm$^3$ for $\eta/s = 0.2$. In the 1+1 dimensional longitudinal geometry, this energy density corresponds to an average over the transverse plane. Nevertheless, estimates of the maximal energy density reached in heavy-ion collisions at RHIC energies must be strongly revised down if shear viscosity is effective during the expansion of the fireball. This dramatic reduction of the initial density should also be taken into account in hydrodynamic models dealing with transverse expansion only, both in 1+1 and 1+2 dimensions.

Depending on the balance of the acceleration of the flow from pressure gradients and deceleration from viscosity, the flow gets faster or slower than the Bjorken one. For some values of the parameters, effects of the shear viscosity and pressure gradients on the longitudinal flow of the fluid cancel, i.e. the flow stays close to the Bjorken flow. This could be an argument justifying models, which combine transverse viscous expansion with a Bjorken flow in the beam direction. When the initial conditions are adjusted to reproduce the final meson distributions, we find that the freeze-out hypersurfaces are very similar, irrespective of the value of the shear viscosity coefficient, obviously the lifetime of the system is not sensitive to viscous effects either (Table II). At the freeze-out the viscous corrections (from the longitudinal flow) to the thermal distributions are small, except at large space-time rapidities. At LHC energies a substantial rapidity plateau, where Bjorken scaling applies, is expected to appear. Shear viscosity helps to preserve it in a wider rapidity interval through the evolution.

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