The Light–Cone Effective Potential

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It is shown how to calculate simple vacuum diagrams in light–cone quantum field theory. As an application, I consider the one–loop effective potential of \( \phi^4 \) theory. The standard result is recovered both with and without the inclusion of zero modes having longitudinal momentum \( k^+ = 0 \).

I. INTRODUCTION

There has been a recent debate concerning the evaluation of certain \( S \)–matrix elements using light–cone quantum field theory where canonical commutators are postulated on null–planes tangent to the light–cone. Taniguchi et al. [1] have considered a one–loop scattering amplitude in \( \phi^4 \) theory in two dimensions, given by the finite integral,

\[
M(p^2) \equiv -i \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2}.
\]

(1)

Here, the external momentum \( p \) denotes the momentum flowing through the diagram. The authors of [1] were particularly interested in the case where the longitudinal momentum component, \( p^+ \equiv p^0 + p^3 \), vanishes, entailing that \( p^2 = p^+ p^- = 0 \). The main claim of [1] was that the associated scattering amplitude vanishes, \( M(0) = 0 \), if the techniques of discretized light–cone quantization (DLCQ) [2, 3] are used. Its application in [1] amounts to cutting off \( |k^+| \) for small values, \( |k^+| > \delta \). So the finding of [1] actually was that \( M_d(0) = 0 \), with \( M_d \) denoting the integral [1] with small–\( k^+ \) cutoff.

On the other hand, standard covariant perturbation theory yields (e.g. via Wick rotation),

\[
M(0) = -i \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - m^2 + i\epsilon} = \frac{1}{4\pi m^2}. \tag{2}
\]

Thus, there seems to be a paradox.

The conclusions of [1] were questioned shortly afterwards by the authors of [4]. They studied the behavior of \( M(p^+ p^-) \) for \( p^+ > 0 \), calculated the integral in non–covariant (Hamiltonian) light–cone perturbation theory and performed the limit \( p^+ \to 0 \) at the end. Their result was that the scattering amplitude \( M(p^2) \) does approach the correct limit [2].

If covariant and light–cone perturbation theory are equivalent, then it should be possible to reproduce [2] by evaluating the integral in LC coordinates. In this case, [4] becomes

\[
M_{LC} \equiv -i \int \frac{dk^+ dk^-}{8\pi^2} \frac{1}{(k^+ k^- - m^2 + i\epsilon)^2}. \tag{3}
\]

The crux is the \( k^- \)–integration which is somewhat tricky. Fortunately, this issue has been analysed long ago by Yan in [5] who noted the following. Obviously, one wants to use contour methods to perform the \( k^- \)–integration. Apparently, one can always choose the contour such that the double pole, \( k^- = m^2/k^+ \), can be avoided. While this is true for \( k^+ \neq 0 \), the argument breaks down for \( k^+ = 0 \). This suggests that the whole contribution to the integral comes from the ‘zero mode’, \( k^+ = 0 \), the integrand thus being proportional to \( \delta(k^+) \). This is indeed what Yan has found,

\[
\int dk^- (k^+ k^- - m^2 + i\epsilon)^{-2} = \frac{2\pi i}{m^2} \delta(k^+) \tag{4}
\]

which is most easily shown using Schwinger’s parametrization to exponentiate denominators. Plugging [1] into (3) correctly yields [2].

At this point it becomes completely obvious what is going on in DLCQ. By cutting off the small–\( k^+ \) region one is removing the support of the delta function in (3) so \( M_d(0) = 0 \) for all \( \delta > 0 \). In other words, the limit \( \delta \to 0 \) is nonuniform. In any case, the correct way to proceed is to either keep the external momentum \( p^+ \) finite as in [4] or to use Yan’s formula [5].

The discussion above might seem somewhat academic. However, light–cone zero modes are of relevance in a broad range of physical problems. For instance, they have been discussed in the matrix formulation resulting from DLCQ of M–theory (see [6] and references therein). Within phenomenological applications, zero modes show up as end–point singularities of Feynman graphs [7] which may lead to a breakdown of collinear factorization [8]. Amplitudes with vanishing external momenta arise in particular, whenever vacuum properties are of concern, for example in the phenomenon of spontaneous symmetry breaking. The elusive character of vacuum diagrams in light–cone quantum field theory (encoded in the phrase “the light–cone vacuum is trivial”), has never been quite resolved. To understand spontaneous symmetry breaking in this particular framework thus still
remains a theoretical challenge (see \[8\] \[10\] \[11\] \[12\] for reviews and \[13\] \[14\] \[15\] for recent discussions).

It turns out that a slight generalization of Yan's formula \(\|\) can be used to calculate a particularly relevant tool for studying spontaneous symmetry breaking, namely the effective potential.

\section{The Effective Potential: Ordinary Coordinates}

The effective potential \(V[\phi_c]\) is the leading term of the effective action \(\Gamma[\phi_c]\) in a derivative expansion. In other words, it is the effective action evaluated for constant classical field \(\phi_c\), divided by the volume \(\Omega\) of space–time. As \(\Gamma\), on the other hand, is the generating functional of one–particle irreducible (1PI) diagrams (proper vertices), the effective potential can be expanded in terms of 1PI diagrams with vanishing external momenta,

\[\Omega V[\phi_c] = - \sum_n \frac{1}{n!} \Gamma^{(n)}(p_1 = 0, \ldots, p_n = 0) \phi_n^2. \tag{5}\]

It is this vanishing of external momenta which makes the considerations of the introduction apply. To make this letter self–contained we recall some basic results concerning the effective potential. For \(\phi^4\)–theory in \(d\) dimensions, \(\|\) becomes, in one–loop approximation,

\[V[\phi_c] = U + i \int \frac{d^d k}{(2\pi)^d} \sum_{n \geq 1} \frac{1}{2n} \left( \frac{V''_0}{k^2 - m^2 + i\epsilon} \right)^n, \tag{6}\]

where \(U\) denotes the classical potential,

\[U[\phi_c] = \frac{1}{2} m^2 \phi_c^2 + \frac{\lambda}{4!} \phi_c^4 = \frac{1}{2} m^2 \phi_c^2 + V_0, \tag{7}\]

and the prime(s) differentiation with respect to \(\phi_c\). Usually, one evaluates \(\|\) by first summing and then integrating. The summation yields a divergent integral,

\[V[\phi_c] = U - i \int \frac{d^d k}{(2\pi)^d} \log \frac{U'' - k^2 - i\epsilon}{m^2 - k^2 - i\epsilon}, \tag{8}\]

which can be evaluated most elegantly in dimensional regularisation (see e.g. the text \(\|\)).

In principle, for noninteger dimension \(d\), all individual integrals in the sum \(\|\) are regulated as well. Using Schwinger's parametrization they can be calculated straightforwardly,

\[I_{n,d} = \int \frac{d^d k}{(2\pi)^d} \left( \frac{1}{(k^2 - m^2 + i\epsilon)^n} \right) \]

\[= \frac{i}{(4\pi)^d} \frac{1}{\Gamma(n)} \frac{1}{\Gamma(n-d/2)} \Gamma(n-d/2), \tag{9}\]

From the Gamma functions one reads off that \(I_{n,d}\) is finite for integers \(n > d/2\). In particular, for even dimension \(d\), there are \(l = d/2\) divergent integrals. Plugging \(\|\) into \(\|\) yields the one–loop effective potential according to

\[V[\phi_c] = U + \frac{i}{2} \sum_{n \geq 1} \frac{(V''_0)^n}{n} I_{n,d}. \tag{10}\]

Instead of evaluating this sum in full generality, we will content ourselves with the discussion of the effective potential in \(d = 4\) which is the most interesting case.

The integrals in the series \(\|\) are evaluated as follows. There are two divergent ones,

\[I_{1,4} = - \frac{i}{(4\pi)^2} \left( \lambda^2 - m^2 \log \frac{\lambda^2}{m^2} \right), \tag{11}\]

\[I_{2,4} = \frac{i}{(4\pi)^2} \log \frac{\lambda^2}{m^2}, \tag{12}\]

where, as usual, we have neglected terms that are small compared to the covariant cutoff \(\Lambda\). The finite integrals are given by the \(I_{n,4}\) from \(\|\) with \(n \geq 3\). Using the summation formula

\[\sum_{n \geq 3} \frac{\Gamma(n-2)}{\Gamma(n+1)} x^n = -\frac{1}{2} \left[ (1-x)^2 \log(1-x) + x - \frac{3}{4} x^2 \right], \tag{13}\]

the sum of the integrals \(\|\), \(\|\) and the \(I_{n,4}, n \geq 3\), yields the regularized one–loop effective potential

\[V[\phi_c] = U + \frac{1}{4} \left( \frac{U''}{4\pi} \right)^2 \left( 2 \frac{\lambda^2}{U''} + \log \frac{U''}{\lambda^2} - \frac{1}{2} \right). \tag{14}\]

Exactly the same result is obtained by directly evaluating \(\|\) with a covariant cutoff.

To renormalize the potential, two counterterms are required corresponding to mass and coupling constant renormalization, respectively,

\[V_R = V + \frac{A}{2} \phi_c^2 + B \frac{\lambda}{4!} \phi_c^4. \tag{15}\]

The counterterms are fixed by the renormalization conditions

\[V_R \big|_{\phi_c=0} = m^2, \tag{16}\]

\[V_R'' \big|_{\phi_c=0} = \lambda, \tag{17}\]

which finally lead to the renormalized effective potential

\[V_R[\phi_c] = U + \frac{1}{4} \left( \frac{U''}{4\pi} \right)^2 \left( 2 \frac{m^2}{U''} + \log \frac{U''}{m^2} - \frac{3}{2} \right). \tag{18}\]

In what follows we will try to reproduce \(\|\) and hence \(\|\) using light–cone methods.
III. THE EFFECTIVE POTENTIAL: LC COORDINATES

Within the LC formalism, the calculation of the effective potential is a tricky business. Some early attempts \cite{17, 18, 19, 20} did not reproduce the canonical effective potential \cite{18}. In \cite{21}, an approach based on a *bona fide* path integral has been adopted, which, however, did not properly include the LC constraints into the measure. In addition, the intricacies of regulating LC diagrams have not been addressed. The paper that comes closest to ours in spirit is \cite{22}. These authors were discussing the integrals \(I_{n,A}\) and realized that the finite ones are entirely dominated by the zero mode. Being not aware of Yan’s formula they suggested to define LC vacuum diagrams by letting some small amount \(p\) of external momentum flow through the diagrams and take the limit \(p \to 0\) after integration (cf. the discussion in the introduction).

In order to calculate the effective potential using the LC approach, two basic difficulties have to be overcome. First, one has to find a way to calculate the finite diagrams given by the integrals \(I_n, n > d/2 = l\), and second, one needs a reasonable regularization for the divergent integrals. The first problem is solved in two steps. We specialize to even dimensions, \(d = 2l\), and calculate the integral over transverse momenta,

\[
J_{n,2l} = \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2}} \frac{1}{(k^+ k^- - k_\perp^2 - m^2 + i\epsilon)^n} = \left(-\frac{1}{4\pi}\right)^{l-1} \frac{\Gamma(n - l +1)/\Gamma(n)}{(k^+ k^- - m^2 + i\epsilon)^{n-l+1}}. \tag{19}
\]

This formula is only relevant in more than two dimensions (\(l > 1\)) and yields a finite answer for integers \(n > l - 1\). The integrals defining the one–loop effective potential are then

\[
I_{n,2l} = \frac{1}{8\pi^2} \int dk^+ dk^- J_{n,2l}. \tag{20}
\]

The \(k^-\)–integration is done using a slight generalization of Yan’s formula \cite{14},

\[
\int dk^- (k^+ k^- - m^2 + i\epsilon)^{-p} = 2\pi i \delta(k^+) \left(\frac{-1}{p - 1}\right)^p \frac{1}{(m^2)^{p-1}}, \tag{21}
\]

which holds for \(p > 1\) and can be proved by differentiating \(p - 2\) times with respect to \(m^2\). The remaining \(k^+\)–integration in \(20\) immediately leads to \(19\). All finite integrals are reproduced in this way. Thus, all that remains to be discussed are the divergent integrals which, in \(d = 4\), are \(I_{1,4}\) and \(I_{2,4}\). The latter is simpler, as \(19\) applies (\(l = n = 2\)) and results in

\[
I_{2,4} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\epsilon)^2} = -\frac{1}{4\pi} I_{1,2}. \tag{22}
\]

The integral \(I_{1,2}\) is the light–cone tadpole in \(d = 2\),

\[
I_{1,2} = \frac{1}{8\pi^2} \int_{(\Lambda)} dk^+ dk^- \frac{1}{k^+ k^- - m^2 + i\epsilon}, \tag{23}
\]

where by \((\Lambda)\) we have indicated a regularization prescription (cutting off small and large values of \(k^+\)) which will be specified in a moment. In any case, with \(|k^+| > \delta\), the \(k^-\)–integration can be done by closing the contour, so that \(23\) becomes

\[
I_{1,2} = -\frac{i}{4\pi} \int_{(\Lambda)} dk^+ \frac{\theta(k^+)}{k^+}. \tag{24}
\]

It has been repeatedly noted by many people \cite{17, 23, 24, 25} that the cutoffs for small and large \(k^+\) have to be related via parity,

\[
\delta = \frac{m^2}{\Lambda} \leq k^+ \leq \Lambda, \tag{25}
\]

leading to

\[
I_{1,2} = -\frac{i}{4\pi} \int_{m^2/\Lambda}^{\Lambda} \frac{dk^+}{k^+} = -\frac{i}{4\pi} \log \frac{\Lambda^2}{m^2}, \tag{26}
\]

which reproduces the standard result. Note that zero modes have been excluded and that the entire mass dependence of the integral comes from the cutoff. Plugging \(26\) into \(22\) finally yields

\[
I_{2,4} = \frac{i}{(4\pi)^2} \log \frac{\Lambda^2}{m^2}, \tag{27}
\]

which is \(12\). The \(d = 4\) tadpole \(I_{1,4}\) is slightly more complicated as the \(k_\perp\)–integration also diverges so that \(10\) cannot be used. The problem has been solved some time ago in \cite{22}. The crucial point is to employ the transverse–momentum cutoff,

\[
\Lambda^2(x) \equiv \Lambda^2 x(1 - x) + m^2, \quad \text{where} \quad x \equiv k^+ / \Lambda. \tag{28}
\]

By means of \(28\) and the standard \(k^-\)–cutoff \(29\), the tadpole becomes

\[
I_{1,4} = -\frac{i}{(4\pi)^2} \left( \frac{\Lambda^2}{2} - m^2 \log \frac{\Lambda^2}{m^2} \right). \tag{29}
\]

Rescaling \(\Lambda \to \sqrt{2} \Lambda\) and omitting subleading terms this coincides with \(11\). Again, zero modes have not been invoked. The regularized and renormalized effective potentials, \(11\) and \(18\), respectively, follow.

We can ask ourselves what would happen if (in the spirit of \(11\)) the zero mode contributions were strictly cut off in all integrals be they finite or infinite. In this case, all *finite* integrals \(I_n\) are zero, as the support of Yan’s delta function in \(21\) is located outside the integration region. The regularized effective potential in \(d = 4\) becomes

\[
V_{nZM} = U + \frac{1}{4} \left( \frac{U''}{4\pi} \right)^2 \left( 2 \frac{\Lambda^2}{U''} + \log \frac{m^2}{\Lambda^2} \right), \tag{30}
\]

where ‘nZM’ stands for ‘no zero modes’. Note that there is no \(\phi_c\)–dependence in the log as this resummation effect
must be absent. Accordingly, the result (30) is polynomial in $\phi_c$. This has peculiar consequences. Imposing the renormalization conditions (16) and (17) we find that the renormalized effective potential coincides with the tree level expression, $V_{\text{tree}} = U$. In other words, the (renormalized) one-loop contributions vanish, a result that is clearly false. In some sense, this can be viewed as an unwanted consequence of assuming (or rather enforcing) a trivial light–cone vacuum.

There is, however, a way around this obstacle. Note that the mass dependence appearing in the log of (30) is entirely due to the cutoff. If we change this cutoff in an ad–hoc manner by replacing

$$m^2 \rightarrow U'' = m^2 + \frac{\lambda}{2} \phi_c^2 = m^2 + V_0'' ,$$  \hspace{1cm} (31)

we obtain instead of (30) the regularized potential

$$V = U + \frac{1}{4} \left( \frac{U''}{4\pi} \right)^2 \left( \frac{2\Lambda^2}{U''} + \log \frac{m^2}{\Lambda^2} + \log \frac{V_0''}{\Lambda^2} \right) ,$$  \hspace{1cm} (32)

which coincides with (14) up to a finite renormalization. The first two terms in brackets are just the contribution (30) where the zero modes had been discarded. The latter exclusively constitute the last log term. We thus can state a peculiar finding: the resummation of the zero mode contributions results in a modification of the cutoff given by the replacement (31). This replacement seems to be very much ad hoc. Its justification will be explained in what follows.

It would have been desirable to directly calculate the resummed expression (30) for the effective potential using LC coordinates. This is not straightforward. Being the log–determinant of the fluctuation operator $\Box + U''$, (8) diverges and has to be regulated. We know how to do so for tadpole type Feynman diagrams, but not for the log–determinant.

However, there is a simple trick to relate the latter to a tadpole. Taking the derivative with respect to $\phi_c$,

$$\frac{\partial}{\partial \phi_c} V = U' + \frac{i}{2} U'' \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - U'' + i\epsilon} = U' + \frac{i}{2} U'' I_{1,d}(U'') ,$$  \hspace{1cm} (33)

precisely yields a tadpole with the mass $m^2$ replaced by $U''$. For $d = 4$, the relevant integral has been calculated in (20). All that is left is to plug this into (33) and undo the differentiation by integrating. This precisely reproduces (14).

We have seen that the effective potential can be calculated both with and without the inclusion of light–cone zero modes. Omitting the latter, however, requires a sophisticated modification of the regularization prescription which need not work for arbitrary vacuum graphs (see also (26) and (27) for a similar philosophy). In order to correctly reproduce finite vacuum amplitudes, the inclusion of zero modes seems indispensable. It would be of interest to extend the techniques presented to the light–cone Hamiltonian formulation. Work in this direction is under way.

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