SCHWARTZ FUNCTIONS ON QUASI-NASH VARIETIES

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Abstract. We introduce a new category called Quasi-Nash, unifying Nash manifolds and algebraic varieties. We define Schwartz functions, tempered functions and tempered distributions in this category. We show that properties that hold on affine spaces, Nash manifolds and algebraic varieties, also hold in this category.

1. Introduction

Schwartz functions are named after Laurent Schwartz, who defined them in $\mathbb{R}^n$. In $\mathbb{R}^n$ they are usually defined as smooth functions which decay to zero with all their derivatives faster than the inverse of any polynomial when reaching infinity. We say $f$ is a Schwartz function on $\mathbb{R}$, for example, if for any $n, k \in \mathbb{N} \cup \{0\}$ we have $|x^n f^{(k)}| < \infty$ where $f^{(k)}$ is the $k$'th derivative of $f$. The space of all Schwartz functions on $\mathbb{R}^n$ will be denoted by $\mathcal{S}(\mathbb{R}^n)$. Schwartz functions on $\mathbb{R}^n$ have some nice properties such as: $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space, $\mathcal{S}(\mathbb{R}^n)$ is invariant under Fourier transform and every function in $\mathcal{S}(\mathbb{R}^n)$ is integrable.

Later, in [dC, AG], Schwartz functions were defined on Nash manifolds, which are smooth semi-algebraic varieties. For a Nash manifold $M$, $f$ is said to be Schwartz on $M$ if for any Nash differential operator $D$ we have $||Df||_\infty < \infty$. Nash differential operator on $M$ means an element of the algebra generated by multiplying by Nash functions and by deriving along Nash sections of the tangent bundle.

Lately, in [ES] we defined Schwartz functions on real algebraic varieties which might have singularities, and showed how the affine algebraic varieties share the properties of Schwartz functions on Nash manifolds. We also showed that some of the results hold in the general case.

In this paper we define a category such that both the Nash manifolds and the algebraic varieties are subcategories of the new one. Moreover, this category enables us to prove the rest of the claims about Schwartz functions on general algebraic varieties. We call this new category Quasi-Nash, or QN, where its affine objects correspond to semi-algebraic subsets of $\mathbb{R}^n$ and the morphisms are locally restrictions of Nash maps. A general variety is defined as a glueing of open affine varieties. This new category slightly extends the category of Nash varieties.

The definitions of this category, and some useful lemmas appear in section 3.

The main results about Schwartz functions in this paper include:

Lemma 1.1. Isomorphic QN varieties $X_1 \cong X_2$, imply an isomorphism of the Fréchet spaces $\mathcal{S}(X_1) \cong \mathcal{S}(X_2)$ where $\mathcal{S}(X_i)$ is the space of Schwartz functions on $X_i$ (Lemma 4.26).

The next Proposition deals with tempered functions. Informally, a tempered function on $\mathbb{R}^n$ is a smooth function bounded by a polynomial, and so does any of its derivatives. We define a tempered function on a QN variety later on.
Proposition 1.2. (Tempered partition of unity) Let \( \{V_i\}_{i=1}^m \) be a finite open cover of a QN variety \( X \). Then, there exist tempered functions \( \{\beta_i\}_{i=1}^m \) on \( X \), such that \( \text{supp}(\beta_i) \subset V_i \) and \( \sum_{i=1}^m \beta_i = 1 \) (Proposition 4.32).

Theorem 1.3. For a QN variety \( X \) and a closed subset \( Z \subset X \), define \( U := X \setminus Z \) and \( W_Z := \{\phi \in \mathcal{S}(X) \mid \phi \text{ is flat on } Z\} \). Then \( W_Z \) is a closed subspace of \( \mathcal{S}(X) \) (and so it is a Fréchet space), and extension by zero \( \mathcal{S}(U) \rightarrow W_Z \) is an isomorphism of Fréchet spaces, whose inverse is the restriction of functions (Theorem 4.36).

It should be noted that the proof of Theorem 1.3 was the hardest to prove. Usually, a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be flat at a point \( p \in \mathbb{R}^n \) if \( f \), and all its derivatives, vanish at \( p \). We had to make clear what it means for a function on a singular variety to be flat on a singular point \( p \). For subvarieties, of \( \mathbb{R}^n \) we defined it to be a restriction of a smooth function on \( \mathbb{R}^n \) which is flat at \( p \). There are several other approaches to do so (see [BMP2, F] for example), but [BMP2] shows they are all equivalent in our case. Then, Theorem 1.3 turns out to be a Whitney type extension problem. We proved it using subanalytical geometry results in [BM1, BM2, BMP1, BMP2]. Theorem 1.3 also enables us to define Schwartz functions by a local condition rather then the global ones we used. Instead of demanding a function that decays “fast at infinity”, we just have to demand a smooth function that is flat on the points “added at infinity” in some compactification process.

Furthermore, Theorem 1.3 gives us one more important result regarding tempered distributions. The space of tempered distributions is the space of linear continuous functionals on \( \mathcal{S}(X) \). It is denoted by \( \mathcal{S}^*(X) \). Theorem 1.3 implies that for any open \( U \subset X \), the restriction morphism of tempered distributions \( \mathcal{S}^*(X) \rightarrow \mathcal{S}^*(U) \) is onto. This is not the case for general distributions. E.g. take the compactification of \( \mathbb{R} \) into a circle. The distribution \( e^x dx \) on \( \mathbb{R} \) cannot be extended to the circle.

Corollary 1.4. Let \( X \) be a QN variety. Then the assignment of the space of Schwartz functions (respectively tempered functions, tempered distributions) to any open \( U \subset X \), together with the extension by zero \( \text{Ext}_U^V \) from \( U \) to any other open \( V \supset U \) (restriction of functions, restrictions of functionals from \( \mathcal{S}^*(V) \) to \( \mathcal{S}^*(U) \)), form a flabby cosheaf (sheaf, flabby sheaf) on \( X \) (Corollaries 5.4, 5.2, 5.6).

We would like to emphasize we proved Proposition 1.2, Theorem 1.3, and Corollary 1.4 also for non-affine varieties in this category, what we could not do in the algebraic category.

Structure of this paper. In Section 2 we give preliminary definitions and results we use in this paper. Most of them concern with Nash manifolds and Schwartz functions on them, and basic properties of Fréchet spaces.

In Section 3 we define the new category and show some nice properties it has.

In Section 4 we define Schwartz functions, tempered functions and tempered distributions in this category, and prove some claims about them.

In Section 5 we show the co-sheaf structure of Schwartz functions, and the sheaf structures of tempered functions and distributions.

In Section 6 we define vector bundles over QN varieties and show some properties that hold on those bundles.

Finally, in Appendix A we build some tools needed for tempered partition of unity.
**Conventions.** Throughout this paper, we use the restricted topology of semi-algebraic sets over $\mathbb{R}^n$, unless otherwise stated. For the definition of restricted topology see [2.8].

We also use the convention that for two varieties of some kind $X \subset M$, we will denote by $I_{\text{Sch}}^M(X)$ the ideal of Schwartz functions on $M$ that vanish identically on $X$.

We say a function $f$ is smooth over $M$ if $f \in C^\infty(M)$.

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2. Preliminaries

We shall dedicate this section to definitions and results used in this paper. They will include a definition of a semi-algebraic set, and the algebraic Alexandrov compactification (2.1-2.3), Fréchet spaces (2.5-2.7), Nash manifolds and Schwartz functions on Nash manifolds (2.8-2.10), Schwartz and tempered functions over Nash manifolds (2.11-2.19).

**Definition 2.1.** [BCR] A semi-algebraic subset of $\mathbb{R}^n$ is a subset of the form

$$\bigcup_{i=1}^n \bigcap_{j=1}^{m_i} \{ x \in \mathbb{R}^n | p_{i,j}(x) > 0 \text{ or } p_{i,j}(x) = 0 \}$$

where $p_{i,j} \in \mathbb{R}[x_1, \ldots, x_n]$.

**Remark 2.2.** An affine algebraic variety is called complete if any regular function on it is bounded (cf. [BCR 3.4.9 and 3.4.10]). Thus, a closed embedding of a complete affine algebraic variety is compact in the Euclidean topology on $\mathbb{R}^n$.

**Proposition 2.3.** (Algebraic Alexandrov compactification [BCR, Proposition 3.5.3]). Let $X$ be an affine algebraic variety that is not complete, then there exists a pair $(\tilde{X}, i)$ such that:

1. $\tilde{X}$ is a complete affine algebraic variety.
2. $i : X \rightarrow \tilde{X}$ is an algebraic isomorphism from $X$ onto $i(X)$.
3. $\tilde{X} \setminus i(X)$ consists of a single point.

The chain rule for deriving composite functions can be extended to higher derivatives and higher dimensions. A relevant result is as follows:

**Lemma 2.4.** [CS, Theorem 2.1] Let $x_0 \in \mathbb{R}^d$, $V \subset \mathbb{R}^d$ be some open neighborhood of $x_0$ and $g : V \rightarrow \mathbb{R}^m$, $g \in C^k(V, \mathbb{R}^m)$, for some $k \in \mathbb{N}$. Let $U \subset \mathbb{R}^n$ be some open neighborhood of $g(x_0)$ and $f : U \rightarrow \mathbb{R}$, $f \in C^k(U)$. Assume $f$ is $k$-flat at $g(x_0)$, i.e. its Taylor polynomial of degree $k$ at $g(x_0)$ is zero. Then $f \circ g : g^{-1}(U) \rightarrow \mathbb{R}$ is $k$-flat at $x_0$.

**Fréchet spaces**
Proposition 2.5. [T, Chapter 10]. A closed subspace of a Fréchet space is a Fréchet space (in the induced topology).

Theorem 2.6. (Banach open mapping - [T, Chapter 17, Corollary 1]). A bijective continuous linear map from a Fréchet space to another Fréchet space is an isomorphism.

Theorem 2.7. (Hahn-Banach - [T, Chapter 18]). Let $F$ be a Fréchet space, and $K \subset F$ a closed subspace. By Proposition 2.5 $K$ is a Fréchet space (with the induced topology). Define $F^*$ (respectively $K^*$) to be the space of continuous linear functionals on $F$ (on $K$). Then the restriction map $F^* \to K^*$ is onto.

Nash manifolds

Definition 2.8. A restricted topological space $M$ is a set $M$ equipped with a family of subsets of $M$, including $M$ and the empty set, called the set of open subsets of $M$, that is closed with respect to finite unions and finite intersections.

Therefore, we will consider only finite open covers in restricted topology.

Definition 2.9. An $\mathbb{R}$-space is a pair $(M, O_M)$ where $M$ is a restricted topological space and $O_M$ a sheaf of $\mathbb{R}$-algebras over $M$ which is a subsheaf of the sheaf $C_M$ of all continuous real-valued functions on $M$.

A continuous map $\varphi : (M, O_M) \to (N, O_N)$ is called a morphism of $\mathbb{R}$-spaces if for any open subset $U \subset N$ and any $f \in O_N(U)$, we have $f \circ \varphi (\varphi^{-1}(U)) \in O_M\left(\varphi^{-1}(U)\right)$.

Definition 2.10. (1) A Nash submanifold $M$ of $\mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ which is a smooth submanifold. A Nash function on $M$ is a smooth semi-algebraic function.

(2) An affine Nash manifold is an $\mathbb{R}$-space which is isomorphic to an $\mathbb{R}$-space associated to a closed Nash submanifold of $\mathbb{R}^n$.

(3) A Nash manifold is an $\mathbb{R}$-space $(M, \mathcal{N}_M)$ with a sheaf of Nash functions, which has a finite open cover $(M_i)_{i=1}^n$ such that each $\mathbb{R}$-space $(M_i, \mathcal{N}_M|_{M_i})$ is an affine Nash manifold.

Schwartz and tempered functions on Nash manifolds

Proposition 2.11. [AG - Proposition 3.3.3] (1) Any open (semi-algebraic) subset $U$ of an affine Nash manifold $M$ with the induced $\mathbb{R}$-space structure is an affine Nash manifold.

(2) Any open (semi-algebraic) subset $U$ of a Nash manifold $M$ with the induced $\mathbb{R}$-space structure is a Nash manifold.

Definition 2.12. (1) Nash differential operator on an affine Nash manifold is an element of the algebra with 1 generated by multiplication by Nash functions and derivations along Nash sections of the tangent bundle (Nash vector fields).
(2) The space of **Schwartz functions on an affine Nash manifold** $M$ is $\mathcal{S}(M) := \{ \phi \in C^\infty(M) | D\phi$ is bounded for any Nash differential operator$\}$.  
(3) The topology on $\mathcal{S}(M)$ is defined by the semi norms $\|\phi\|_D := \sup_{x \in M} |D\phi(x)|$.  
(4) Let $M$ be as in (2.10), and $\phi : \bigoplus_{i=1}^k \mathcal{S}(M_i) \to C^\infty(M)$ defined by extension by zero and summing. Then $\mathcal{S}(M) := \text{Im}(\phi)$. 

**Corollary 2.13.** [corollary of AG - Corollary 4.1.2] Let $M$ be a Nash manifold. Then $\mathcal{S}(M)$ is a Fréchet space. 

**Definition 2.14.** [AG - Definition 4.2.1 and Theorem 4.6.2] A function $t : \mathbb{R}^n \to \mathbb{R}$ is called **tempered** if it is a smooth function such that for any $\alpha \in (\mathbb{N} \cup \{0\})^n$ there exists a polynomial $p_\alpha \in \mathbb{R}[x_1,\ldots,x_n]$ such that $|\frac{\partial^{\alpha}t}{\partial x^\alpha}(x)| < p_\alpha(x)$ for any $x \in \mathbb{R}^n$. 

**Proposition 2.15.** [corollary of AG - Proposition 4.2.1] Let $M$ be a Nash manifold and $\alpha$ be a tempered function on $M$. Then $\alpha \mathcal{S}(M) \subset \mathcal{S}(M)$.  

**Theorem 2.16.** [corollary of AG - Theorem 4.6.1] Let $M$ be a Nash manifold and $Z \hookrightarrow M$ be a closed Nash submanifold. The restriction $\mathcal{S}(M) \to \mathcal{S}(Z)$ is defined, continuous and onto.  

**Proposition 2.17.** [AG - Proposition 5.1.3] Let $M$ be a Nash manifold. The assignment of the space of tempered functions on $U$, to any open $U \subset M$, together with the usual restriction maps, define a sheaf of algebras on $M$.  

**Theorem 2.18.** [AG - Theorem 5.2.1] (Partition of unity for Nash manifold). Let $M$ be a Nash manifold, and let $(U_i)_{i=1}^n$ be a finite open cover. Then  
(1) there exist tempered functions $\alpha_1,\ldots,\alpha_n$ on $M$ such that $\text{supp}(\alpha_i) \subset U_i$, $\sum_{i=1}^n \alpha_i = 1$.  
(2) Moreover, we can choose $i$ in such a way that for any $\phi \in \mathcal{S}(M)$, $\alpha_i \phi \in \mathcal{S}(U_i)$. 

**Theorem 2.19.** [AG - Theorem 5.4.1] (Characterization of Schwartz functions on open subset) Let $M$ be a Nash manifold, $Z$ be a closed (semi-algebraic) subset and $U = M \setminus Z$. Let $W_Z$ be the closed subspace of $\mathcal{S}(M)$ defined by $W_Z := \{ \phi \in \mathcal{S}(M) | \phi$ vanishes with all its derivatives on $Z \}$. Then restriction and extension by $0$ give an isomorphism $\mathcal{S}(U) \cong W_Z$. 

3. **Geometry** 

**Definition 3.1.** “Naïve” Quasi Nash category - NQN - Let $X$ be a locally closed semi-algebraic subset of $\mathbb{R}^n$. A morphism in this category is a map from an NQN set $X$ to an NQN set $Y \subset \mathbb{R}^m$ which is a restriction of a Nash map on an open
semi-algebraic neighborhood $U$ of $X$ to $\mathbb{R}^m$ such that $X$ is closed in $U$ and $X$ is mapped into $Y$. i.e, $\varphi : X \to Y$, $\varphi := g|_X$ where $X \subset U$, and $g : U \to \mathbb{R}^m$ is Nash.

**Lemma 3.2.** Let $X, Y$ be NQN, and let $\varphi : X \to Y$ a continuous map. Then $\varphi$ is an NQN morphism if and only if for any NQN function $f : Y \to \mathbb{R}$, $f \circ \varphi : X \to \mathbb{R}$ is an NQN function. (NQN function is an NQN map from an NQN set to an NQN morphism if and only if for any NQN function $f : Y \to \mathbb{R}$ results in Nash functions on neighborhoods of $Y$.)

**Proof.** Let $\varphi$ be an NQN morphism. Take $f$ as required. As $f$ is a restriction of a Nash function on a neighborhood of $Y$, and $\varphi$ is a restriction of a Nash map on a neighborhood of $X$, we get a composition of Nash maps, which is Nash. Thus, $f \circ \varphi$ is an NQN function.

Now let $\varphi$ pullback NQN functions to NQN functions. As $Y \subset \mathbb{R}^m$, $\varphi$ can be presented as $\varphi = (\varphi_1, ..., \varphi_m)$. For each $i$ take the Nash function $f_i = y_i$. This results in Nash functions on neighborhoods $U_i$ of $X$ where on each $U_i$: $f_i \circ \varphi = y_i \circ \varphi = \varphi_i$ for any $i$, what makes all $\varphi$'s coordinates NQN maps.

Take the intersection $U = \bigcap_{i=1}^n U_i$ of all those neighborhoods to get a neighborhood of $X$ where all of $\varphi$'s coordinates are NQN simultaneously and therefore $\varphi$ is an NQN map itself.

Now let's take a broader category -

**Definition 3.3.** (1) Let $X$ be a locally closed semi-algebraic subset of $\mathbb{R}^n$. Define the **$\mathbb{R}$-space corresponding to $X$** to be the pair $(X, QN_X)$ where $QN_X$ is defined to be a sheaf as follows: For each open $U \subset X$, we say $f \in QN_X(U)$ if there exists a collection $\{V_i\}_{i=1}^m$ of open semi-algebraic subsets of $\mathbb{R}^n$ and Nash functions $f_i$ from these sets, such that $U_i := V_i \cap X$ is an open cover of $U$, for any $i$, $f|_{U_i} = f_i|_{U_i}$, and for any $i, j$, $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. This is a sheafification of the NQN presheaf in definition 3.1.

(2) An **affine QN variety** $X$ is defined to be an $\mathbb{R}$-space isomorphic to an $\mathbb{R}$-space obtained by sheafification from some closed NQN set $Y \subset \mathbb{R}^n$. i.e. $(X, O_X)$ is an affine QN variety if $(X, O_X) \cong (Y, O_Y)$ where $Y$ is a closed NQN set, and $O_Y$ is the sheafification of the NQN functions on $Y$.

**Remark 3.4.** Let $X$ be an affine QN variety. Then $X$ is NQN isomorphic to a closed set in $\mathbb{R}^n$, which by abuse of notation is denoted by $X$ as well. Let $V \subset \mathbb{R}^n$ be some open semi-algebraic subset such that $U = V \cap X$ is an open subset of $X$. Then $V$ is Nash diffeomorphic to some closed affine Nash submanifold $V'$ in $\mathbb{R}^N$, by [BCR - Theorems 8.4.6, 2.4.5]. Nash diffeomorphism is a QN isomorphism (i.e. an isomorphism of ringed spaces) so $V$ is NQN isomorphic to $V'$. Denote the image of $U \subset V$ by $U'$, and note that it is closed in $V'$. This makes $U'$ a closed subset of $\mathbb{R}^N$, and so an NQN object. A Nash function on $V$ can be pulled-back to a Nash function on $V'$, and a Nash function on $V'$ can be extended to a Nash function on an open neighborhood of $V'$ (e.g. by [AG - Thm 3.6.2 - Nash Tubular Neighborhood]). Thus, we can say that a restriction of a Nash function from $V$ to $U$ is QN isomorphic to an NQN function on the NQN set $U'$, which is QN isomorphic to $U$. This means a locally closed semi-algebraic set of $\mathbb{R}^n$ has a natural structure of an affine QN variety, and its $\mathbb{R}$-space functions are restrictions of Nash functions from an open semi-algebraic sets in which the affine QN-variety is closed. The same goes for locally closed subsets of a QN variety.
Lemma 3.5. Affine Nash manifolds form a full subcategory of affine QN varieties. Affine algebraic varieties form a subcategory, not a full subcategory.

Proof. An affine Nash manifold is an $\mathbb{R}$-space isomorphic to an $\mathbb{R}$-space associated to a smooth closed semi-algebraic subset of $\mathbb{R}^n$. The sheaf on a Nash manifold is made of Nash functions - smooth semi-algebraic functions. In the QN category the sheaf is made of locally restrictions of Nash functions. Due to the Nash tubular neighborhood, the sheaves are the same on Nash manifolds. Let $\varphi : X \to Y$ be a QN morphism between the Nash manifolds $X, Y$. Take a Nash function $f$ on $Y$. So there exists a cover $\bigcup_{i=1}^n X_i = X$ s.t. $g_i := f \circ \varphi|_{X_i}$ is a Nash function. As Nash functions on the Nash manifold $X$ form a sheaf, $\exists g \in \mathcal{N}(X)$ s.t. $g|_{X_i} = g_i$. So for any QN map $\varphi$, and any $f \in \mathcal{N}(Y)$, we get $f \circ \varphi \in \mathcal{N}(X)$.

Affine algebraic variety is isomorphic to an algebraic subset of $\mathbb{R}^n$. But as morphisms in this category are rational maps, and not any Nash map is such a map (e.g. $\sqrt{1+x^2}$), this is not a full subcategory. □

Lemma 3.6. Let $X, Y$ be affine QN varieties and $\tilde{X} \subset \mathbb{R}^n$, $\tilde{Y} \subset \mathbb{R}^m$ the corresponding closed QN sets. Let $\varphi : X \to Y$ be a continuous map and $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ the corresponding map. Then $\varphi$ is a QN morphism if and only if there exists an open cover $\bigcup_{i=1}^N \tilde{X}_i = \tilde{X}$ such that $\tilde{\varphi}|_{\tilde{X}_i}$ is an NQN map for any $i$.

Proof. Assume $\varphi$ is an affine QN morphism. For any $f \in QN_{\tilde{Y}}$, $\tilde{\varphi}^*f \in QN_{\tilde{X}}$. As $\tilde{Y} \subset \mathbb{R}^n$, $\tilde{\varphi}$ can be presented as $\tilde{\varphi} = (\tilde{\varphi}_1, ..., \tilde{\varphi}_m)$. For each $j \in \{1, ..., m\}$ take the Nash function $f_j = \tilde{\varphi}_j$. When pulling it back, we get an open cover $\tilde{X} = \bigcup_{k=1}^N \tilde{X}_k$ such that for any open subset $\tilde{X}_k$, the function $f_j \circ \tilde{\varphi} = \tilde{\varphi}_j \circ \tilde{\varphi} = \tilde{\varphi}_j$ is a restriction of a Nash function, i.e. $\tilde{\varphi}_j|_{\tilde{X}_k}$ is an NQN map for any $k$. Now take the refinement of those $j$ covers of $\tilde{X}$, and denote it by $\tilde{X} = \bigcup_{i=1}^N \tilde{X}_i$. As a restriction of a Nash function is a Nash function, we get that for any $i$, $(\tilde{\varphi}_1, ..., \tilde{\varphi}_m)|_{\tilde{X}_i}$ is an NQN map, what makes $\tilde{\varphi}|_{\tilde{X}_i}$ an NQN map.

Now the other direction:

Assume $\tilde{\varphi}|_{\tilde{X}_i}$ is an NQN map for any $i$ and let $f \in QN_{\tilde{Y}}(U)$ for an open $U \subset Y$. Those $f$ and $U$ correspond to some $\tilde{f}$ and $\tilde{U}$ respectively. This means there is an open cover $\tilde{U} = \bigcup_{j=1}^M \tilde{U}_j$ and for each $j$, $\tilde{f}|_{\tilde{U}_j}$ is a restriction of a Nash function. The map $\tilde{\varphi}$ is continuous, so $\tilde{\varphi}|_{\tilde{X}_i \cap \tilde{\varphi}^{-1}(\tilde{U}_j)} : \tilde{X}_i \cap \tilde{\varphi}^{-1}(\tilde{U}_j) \to \tilde{U}$ is an NQN map, i.e. it is a restriction of a Nash map from an open neighborhood of $\tilde{X}_i \cap \tilde{\varphi}^{-1}(\tilde{U}_j)$. Therefore $\tilde{f} \circ \tilde{\varphi}|_{\tilde{X}_i \cap \tilde{\varphi}^{-1}(\tilde{U}_j)}$ is a restriction of a Nash function for any $j$ and any $i$. Thus, $f \circ \varphi \in QN_{X}(\varphi^{-1}(U))$. □

Definition 3.7. A QN variety is an $\mathbb{R}$-space $(X, QN_X)$ where $X$ is a restricted topological space, which has a finite cover $\bigcup_{i=1}^m X_i = X$ of open sets such that the $\mathbb{R}$-spaces $(X_i, QN_X|_{X_i})$ are affine QN varieties.
Lemma 3.8. Let $X, Y$ be general QN varieties. Let $\varphi : X \to Y$ be a continuous map. Then $\varphi$ is a QN morphism if and only if there exists an open affine cover $Y = \bigcup_{i=1}^{M} Y_i$, and an open affine cover $\bigcup_{j=1}^{N_i} X_{ij} = X_i := \varphi^{-1}(Y_i)$, where each $X_{ij}$ corresponds to a locally closed semi-algebraic set $\tilde{X}_{ij}$ such that the induced map $\tilde{\varphi}|_{\tilde{X}_{ij}}$ is an NQN map to the closed semi-algebraic set $\tilde{Y}_i$ isomorphic to $Y_i$.

Proof. Assume $\varphi$ is a QN morphism. Take an open affine cover $Y = \bigcup_{i=1}^{M} Y_i$, and for each $i$ take the QN variety $X_i := \varphi^{-1}(Y_i)$. Cover it by some open affine subvarieties $\bigcup_{j=1}^{N_i} X_{ij} = X_i$. Now we have a restricted morphism $\varphi|_{X_{ij}} : X_{ij} \to Y_i$ of affine QN varieties and we can use Lemma 3.6. Refining the covers will yield the proof for the first direction.

To prove the other direction we start with a function $f \in QN_Y(U)$ for an open $U \subset Y$. By remark 3.3 each $X_{ij}$ corresponds to an affine QN variety, so by Lemma 3.6 we have QN morphisms $\varphi|_{X_{ij}} : X_{ij} \to Y_i$. Thus, $f|_{Y_i \cap U} \in QN_Y(Y_i \cap U)$ is pulled back to $f \circ \varphi|_{X_{ij} \cap \varphi^{-1}(U)} \in QN_X(X_{ij} \cap \varphi^{-1}(U))$ for any $i, j$. As QN functions form a sheaf, we get that $f \circ \varphi|_{\varphi^{-1}(U)} \in QN_X(\varphi^{-1}(U))$. \hfill \Box

4. Schwartz Functions, Tempered Functions and Tempered Distributions

4.1. Naive Quasi-Nash.

Claim 4.1. Let $X$ be a QN set, and $U, V \subset \mathbb{R}^n$ be open sets containing $X$ as a closed subset. Then $S(U)/I_{Sch}^U(X) \cong S(V)/I_{Sch}^V(X)$.

Proof. We start by showing $S(U)/I_{Sch}^U(X) \cong S(U \cap V)/I_{Sch}^{U \cap V}(X)$. By Theorem 2.19 $S(U \cap V)$ is isomorphic to a closed subspace of $S(U)$. Thus, it is enough to check that $S(X) := S(U)/I_{Sch}^U(X)$ and $S(U \cap V)/I_{Sch}^{U \cap V}(X)$ are equal as sets, i.e. that a Schwartz function on $X$ is a restriction of a Schwartz function on $U$ if and only if it is a restriction of a Schwartz function on $U \cap V$. Let $f \in S(U \cap V)/I_{Sch}^{U \cap V}(X)$. There exists $f \in S(U \cap V)$ such that $F|_{X} = f$. By Theorem 2.19 extending $F$ by zero to a function on $U$ (denote it by $\tilde{F}$) is a function in $S(U)$. Then $f = \tilde{F}|_{X}$ and so $f \in S(U)/I_{Sch}^U(X)$. For the other direction, let $f \in S(U)/I_{Sch}^U(X)$. There exists $F \in S(U)$ such that $F|_{X} = f$. Denote $U' = U \setminus X$. $\{U', U \cap V\}$ form an open cover of $U$ and so, by Theorem 2.18 there exist tempered functions $\alpha_1, \alpha_2$ such that $\text{supp} (\alpha_1) \subset U \cap V$, $\text{supp} (\alpha_2) \subset U'$ and $\alpha_1 + \alpha_2 = 1$ as a real valued function on $U$. Moreover, $\alpha_1$ and $\alpha_2$ can be chosen such that $(\alpha_1 \cdot F)|_{U \cap V} \in S(U \cap V)$. As $\alpha_1|_{X} = 1$, it follows that $((\alpha_1 \cdot F)|_{U \cap V})|_{X} = (\alpha_1 \cdot F)|_{X} = F|_{X} = f$, and so $f \in S(U \cap V)/I_{Sch}^{U \cap V}(X)$. As we can use the same proof exactly to show $S(V)/I_{Sch}^V(X) \cong S(U \cap V)/I_{Sch}^{U \cap V}(X)$, we end up having the result. \hfill \Box

Claim 4.2. Let $X$ be a QN set, and $U, V \subset \mathbb{R}^n$ be open sets containing $X$ as a closed subset. Then for any $T(U)|_{X} = T(V)|_{X}$.
Proof. As \( U, V \) are open Nash submanifolds (Proposition 2.11), as tempered functions form a sheaf on \( U \cup V \) (by Proposition 2.17) and as we may use tempered partition of unity on \( U \cup V \) (Theorem 2.18), the claim easily follows. \( \square \)

**Definition 4.3.** Let \( X \) be an NQN set. A Schwartz function on \( X \) is a restriction of a Schwartz function from an open semi-algebraic subset of \( \mathbb{R}^n \) in which \( X \) is closed, to \( X \). Equivalently, we can define the space of Schwartz functions on \( X \) as \( \mathcal{S}(X) := \mathcal{S}(U)/I^{U}_{\text{Sch}}(X) \).

A tempered function on \( X \) is a restriction of a tempered function from an open semi-algebraic subset of \( \mathbb{R}^n \) in which \( X \) is closed, to \( X \). The space of tempered functions on \( X \) is denoted by \( T(X) \).

**Lemma 4.4.** Let \( X \) be an NQN set. Then \( \mathcal{S}(X) \) is a Fréchet space.

Proof. \( \mathcal{S}(U) \) is a Fréchet space (see [Proposition 2.11 Proposition 2.13]) and as \( I^{U}_{\text{Sch}} \) is a closed subset of \( \mathcal{S}(U) \), we get that their quotient is a Fréchet space as well (by Proposition 2.3 and [T - Proposition 7.9]). \( \square \)

**Lemma 4.5.** Let \( \varphi : X_1 \to X_2 \) be an NQN isomorphism, i.e. a bijective morphism whose inverse is also an NQN morphism. Then \( \varphi^{\ast}|_{\mathcal{S}(X_2)} : \mathcal{S}(X_2) \to \mathcal{S}(X_1) \) is an isomorphism of Fréchet spaces.

Proof. By definition we have an open semialgebraic neighborhood \( U_1 \) of \( X_1 \) and a Nash map \( g_1 : U_1 \to \mathbb{R}^{n_2} \) such that \( g_1 |_{X_1} = \varphi \). Note that \( U_1 \) is an affine Nash manifold.

Similarly to the construction of \( U_1 \) and \( g_1 \) above, we may construct an open \( U_2 \subset \mathbb{R}^{n_2} \) and a map \( g_2 : U_2 \to \mathbb{R}^{n_1} \) such that \( g_2 |_{X_2} = \varphi^{-1} \). Note that \( g_2 \neq g_1^{-1} \): in general \( g_1 \) is not a bijection and \( U_1 \not\cong U_2 \).

Consider the following diagram, where \( \alpha \) is defined by \( \alpha(x, y) := (x, y + g_1(x)) \).

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\text{Id} \times 0} & U_1 \\
\downarrow & \circlearrowleft & \downarrow \\
\text{Id} \times g_1 & & U_1 \times \mathbb{R}^{n_2} \\
\end{array}
\]

Clearly \( U_1 \times \{0\} \) is an affine Nash manifold isomorphic to \( U_1 \). Denote \( \hat{U}_1 := \alpha(U_1 \times \{0\}) \), then \( \alpha \) restricted to \( U_1 \times \{0\} \) is an isomorphism of the affine Nash manifolds \( U_1 \times \{0\} \) and \( \hat{U}_1 \) – the inverse map is given by \( \alpha^{-1}(x, y) := (x, y - g_1(x)) \).

Thus we have:

\[
\mathcal{S}(X_1) \cong \mathcal{S}(U_1)/I^{U}_{\text{Sch}}(X_1) \cong \mathcal{S}(\hat{U}_1)/I^{\hat{U}_1}_{\text{Sch}}(\alpha(X_1 \times \{0\})) = \mathcal{S}(\hat{U}_1)/I^{\hat{U}_1}_{\text{Sch}}((\text{Id} \times \varphi)(X_1)),
\]

where the first equivalence is by definition, the second is due the fact that \( U_1 \cong U_1 \times \{0\} \cong \hat{U}_1 \) and \( \mathcal{S}(U_1) \cong \mathcal{S}(U_1 \times \{0\}) \cong \mathcal{S}(\hat{U}_1) \), and the third follows from the fact that \( g_1 |_{X_1} = \varphi \). As always \( I^{\hat{U}_1}_{\text{Sch}}(X_1) \) is the ideal in \( \mathcal{S}(U_1) \) of Schwartz functions identically vanishing on \( X_1 \).

As \( \hat{U}_1 \) is closed in \( U_1 \times \mathbb{R}^{n_2} \) (as it is defined by Nash map on \( U_1 \times \mathbb{R}^{n_2} \)), then by Theorem 2.18 we get that

\[
\mathcal{S}(\hat{U}_1)/I^{\hat{U}_1}_{\text{Sch}}((\text{Id} \times \varphi)(X_1)) \cong \mathcal{S}(U_1 \times \mathbb{R}^{n_2})/I^{U_1 \times \mathbb{R}^{n_2}}_{\text{Sch}}((\text{Id} \times \varphi)(X_1)).
\]
Applying claim 4.1 for the open subset $U_1 \times U_2 \subset U_1 \times \mathbb{R}^{n_2}$ we get that 
\[ S(U_1 \times \mathbb{R}^{n_2})/I_{Sch}^{U_1 \times \mathbb{R}^{n_2}}((Id \times \varphi)(X_1)) \cong S(U_1 \times U_2)/I_{Sch}^{U_1 \times U_2}((Id \times \varphi)(X_1)), \]
and thus we obtain 
\[ S(X_1) \cong S(U_1 \times U_2)/I_{Sch}^{U_1 \times U_2}((Id \times \varphi)(X_1)). \]

Repeating the above construction using the following diagram:

\[
\begin{array}{c}
X_2 \leftarrow U_2 \\
\downarrow \varphi \\
\mathbb{R}^{n_2} \times U_2 \\
\downarrow \text{Id} \\
Y_2 \leftarrow \text{Id}
\end{array}
\]

yields:
\[ S(X_2) \cong S(U_1 \times U_2)/I_{Sch}^{U_1 \times U_2}((\varphi^{-1} \times Id)(X_2)). \]

Clearly $(Id \times \varphi)(X_1) = (\varphi^{-1} \times Id)(X_2)$, and so $S(X_1) \cong S(X_2)$. Note that the isomorphism constructed is in fact the pull back by $\varphi$ from $S(X_2)$ onto $S(X_1)$. This proves the lemma. □

**Remark 4.6.** A similar claim about tempered functions can be proved similarly, by replacing Theorem 2.16 with a similar claim about tempered functions ([AG - 4.6.2]).

**Proposition 4.7.** Let $X$ be an NQN set, $s \in S(X)$, and $t \in T(X)$. Then $t \cdot s \in S(X)$.

**Proof.** Take some $U \subset \mathbb{R}^n$ such that $X \subset U$ as a closed subset. Then there exist some Schwartz function $S \in S(U)$ and a tempered function $T \in T(U)$ such that $s = S|_X$ and $t = T|_X$. As $T \cdot S \in S(U)$ (by Proposition 2.15), we get that $(T \cdot S)|_X \in S(X)$. □

### 4.2. Affine QN.

**Lemma 4.8.** Let $X, Y$ be affine QN varieties, and $\varphi : X \to Y$ a QN isomorphism. Let $\bar{X}, \bar{Y}$ be closed NQN sets corresponding to $X, Y$ respectively, and $\bar{\varphi}$ the corresponding map. Then a Schwartz function on $\bar{Y}$ is pulled-back by $\bar{\varphi}$ to a Schwartz function on $\bar{X}$.

**Proof.** Let $f \in S(\bar{Y})$. By Lemma 4.6, there exist open covers $\bigcup_{i=1}^N \bar{X}_i = \bar{X}, \bigcup_{i=1}^N \bar{Y}_i = \bar{Y}$ such that $\bar{\varphi}|_{\bar{X}_i} : \bar{X}_i \to \bar{Y}_i$ are NQN isomorphisms. As Schwartz functions form a cosheaf, every Schwartz function on $\bar{Y}$ is a sum of extensions of Schwartz functions on open subsets of $\bar{Y}$. Let those subsets be $\bar{Y}_i$. According to Lemma 4.8, a Schwartz function $s_i \in S(\bar{Y}_i)$ is pulled back to $\varphi^*s_i \in S(\bar{X}_i)$, and thus, we get 
\[ \varphi^*s = \sum_{i=1}^N \text{Ext}^{S \bar{X}_i}_{X_i} (\varphi^*s_i) \in S(\bar{X}). \]
Together with the NQN isomorphism on each such subset, we get the result. □

This lemma enables us to use the following definition:
Remark 4.10. Let $X$ be a locally-closed subset in $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open neighborhood of $X$ such that $X$ is closed in $U$. $U$ is an affine Nash manifold, so there is a Nash diffeomorphism $\phi$ between $U$ and a closed Nash submanifold of $\mathbb{R}^N$. Denote $\tilde{X} := \phi(X)$ and $\tilde{U} := \phi(U)$. Nash diffeomorphism is a closed map, so $\tilde{X} \subset \tilde{U}$ is a closed subset. Thus, we may define the Schwartz space $S(\tilde{X}) := S(\mathbb{R}^N) |_{\tilde{X}} = (S(\mathbb{R}^N) |_{\tilde{U}}) |_{\tilde{X}} = S(\tilde{U}) |_{\tilde{X}}$. As Nash diffeomorphisms pull back Schwartz functions, the space $S(\tilde{U})$ is pulled back to $S(U)$, and $S(\tilde{X})$ is pulled back to $S(X)$ and vice versa. This enables us to understand a Schwartz function on a locally-closed subset $X$ of $\mathbb{R}^n$ as a restriction of a Schwartz function from an open set $U$ in which $X$ is closed. I.e.: $S(X) = S(U) |_{\tilde{X}}$.

A similar claim for tempered functions can be proven the same way.

Theorem 4.11. Let $M$ be an affine QN variety, and let $X \subset M$ be a closed subset. Then the restriction of a Schwartz function from $M$ to $X$ defines an isomorphism $S(X) = S(M) / I_{Sch}^M(X)$ (with the quotient topology), where $I_{Sch}^M(X)$ is the ideal in $S(M)$ of functions identically vanishing on $X$.

Proof. Take the closed corresponding sets of $X$ and $M$, $\tilde{X}$ and $\tilde{M}$ correspondingly. As $\tilde{X} \subset \tilde{M} \subset U$ where $\tilde{M}$ is closed in some open $U \subset \mathbb{R}^n$, we get that $S(M) / I_{Sch}^M(X) \cong (S(U) / I_{Sch}^U(M)) / I_{Sch}^M(X) \cong S(U) / I_{Sch}^U(X) \cong S(X)$. □

Definition 4.12. Let $X$ be an affine QN variety corresponding to some closed subset $\tilde{X} \subset \mathbb{R}^n$. A function $f : \tilde{X} \to \mathbb{R}$ is called flat at $\tilde{p} \in \tilde{X}$ if there exists an open semi-algebraic neighborhood $\tilde{U} \subset \mathbb{R}^n$ of $\tilde{p}$ and a function $F \in C^\infty(\tilde{U})$ such that $f|_{\tilde{U}\cap \tilde{X}} = F|_{\tilde{U}\cap \tilde{X}}$ and $F$'s Taylor series is identically zero at $\tilde{p}$. If $f$ is flat at all $\tilde{p} \in \tilde{Z}$ for some $\tilde{Z} \subset \tilde{X}$, $f$ is called flat at $\tilde{Z}$.

Claim 4.13. Let $X$ and $Y$ be affine QN varieties, and $\varphi : X \to Y$ a QN isomorphism. Let $\tilde{X} \subset \mathbb{R}^n$ and $\tilde{Y} \subset \mathbb{R}^m$ be the corresponding closed subsets, and $\tilde{\varphi} : \tilde{X} \to \tilde{Y}$ the corresponding map. Let $f$ be a function on $\tilde{Y}$ that is flat at some $p \in \tilde{Y}$. Then $\tilde{\varphi}^* f$ is flat at $\tilde{\varphi}^{-1}(p)$.

Proof. By definition there is some open neighborhood $U \subset \mathbb{R}^m$ of $p$ and a smooth function $F \in C^\infty(U)$ which is flat on $p$ and such that $f = F|_{\tilde{Y}\cap U}$. As $\tilde{\varphi}$ is an isomorphism, and by Lemma 2.4, $\tilde{\varphi}^{-1}(p)$ has an open neighborhood $W \subset \mathbb{R}^n$ such that $\tilde{\varphi}|_{\tilde{X}\cap W}$ is a restriction of a Nash map $G : W \to \mathbb{R}^m$. Take an open subset $W' \subset W$ such that $p \in W'$ and $G(W') \subset U$. By sheaf properties of Nash maps, $G|_{W'}$ is Nash as well. Thus, $F \circ G|_{W'}$ is a smooth function on $W'$, and by [Lemma 4.14] it is flat on $\tilde{\varphi}^{-1}(p)$. We conclude that $F \circ G|_{\tilde{X}\cap W'} = F \circ \tilde{\varphi}|_{\tilde{X}\cap W'} = f \circ \tilde{\varphi}|_{\tilde{X}\cap W'}$, which means $\tilde{\varphi}^* f$ is flat at $\tilde{\varphi}^{-1}(p)$. □

Definition 4.14. Let $X$ be an affine QN variety. A function $f : X \to \mathbb{R}$ is called flat at $p \in X$ if the corresponding map $f$ is flat at the corresponding point $\tilde{p}$. If $f$ is flat at all $p \in Z$ for some $Z \subset X$, $f$ is called flat at $Z$. 

\vspace{1cm}

\textbf{Schwartz Functions on Quasi-Nash Varieties} 11
Lemma 4.15. (Extension by zero - the affine case) Let $X$ be an affine QN variety, and let $V \subset X$ be an open subset. Then any $f \in \mathcal{S}(V)$ can be extended to a Schwartz function on $X$ which is flat on $X \setminus V$.

Proof. First, let us take the closed set in $\mathbb{R}^n$ corresponding to $X$, and denote it, by abuse of notation, by $X$. By remark 4.10 $f$ is a restriction of a Schwartz function $\hat{f}$ on some open neighborhood $\hat{V} \subset \mathbb{R}^n$. By Theorem 2.19 $\hat{f}$ may be extended by zero to a Schwartz function on $\mathbb{R}^n$ which is flat on $\mathbb{R}^n \setminus V$. Restricting this function to $X$ yields the desired result.

In fact, for any open $V \subset X$, any restriction to $V$ of a Schwartz function on $X$ which is flat on $X \setminus V$, is a Schwartz function on $V$. In order to prove that, we need the following lemmas.

The following lemmas and proposition are required for the proof of Theorem 4.10

Lemma 4.16. Let $X$ be a QN variety corresponding to some compact set $\hat{X} \subset \mathbb{R}^n$ closed in some $U \subset \mathbb{R}^n$. Let $Z \subset \hat{X}$ be a closed subset. Define $V := \hat{X} \setminus Z$,

$$W_Z := \left\{ \phi : \hat{X} \to \mathbb{R} | \exists \hat{\phi} \in C^\infty(U) \text{ such that } \hat{\phi}|_{\hat{X}} = \phi \text{ and } \phi \text{ is flat at } Z \right\},$$

and

$$(W^U_Z)^{\text{comp}} := \left\{ \phi \in C^\infty(U) | \phi \text{ is compactly supported and is flat at } Z \right\}.$$  

Then, for any $f \in W_Z$ there exists $\hat{f} \in (W^U_Z)^{\text{comp}}$ such that $\hat{f}|_{\hat{X}} = f$.

Proof. The proof of Lemma 4.16 is exactly the same as the proof of [ES - Lemma 3.13], which deals with algebraic varieties. In the case of $Z = \{p\}$, this extension is trivial. In the case $Z$ consists of more than one point, we need to use results on Whitney’s extension theorem. We used [BM1], [BM2], [BMP2], dealing with subanalytic geometry, to prove this extension exists in the algebraic case in [ES - Lemma 3.13 and Appendix A]. As subanalytic geometry covers semi-algebraic sets as well, the proof is valid in our case with these minor changes:

Semi-algebraic sets replace algebraic sets, affine QN varieties replace affine algebraic varieties, and open\closed semi-algebraic sets replace Zariski open\closed sets.

Restricting a Schwartz function to an affine QN variety from an open neighborhood is given in Remark 4.10 which replaces [ES - Theorem 3.7].

The Uniformization theorem - [BM1, 5.1] should be replaced by the more general Theorem [BM1, Theorem 0.1].

Lemma 4.17. Let $X$ be an affine QN variety corresponding to some compact set $\hat{X} \subset \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ such that $\hat{X}$ is closed in $U$. Then

$$\mathcal{S}(\hat{X}) = \left\{ f : \hat{X} \to \mathbb{R} | \exists \hat{f} \in C^\infty(U) \text{ such that } \hat{f}|_{\hat{X}} = f \right\}.$$ 

Proof. The inclusion $\subset$ is trivial as $\mathcal{S}(U) \subset C^\infty(U)$. For the other direction consider some $g : \hat{X} \to \mathbb{R}$ and assume it extends to some $\hat{g} \in C^\infty(U)$ such that $\hat{g}|_{\hat{X}} = g$. Let $\rho \in C^\infty(U)$ be a compactly supported function such that $\rho|_{\hat{X}} = 1$. Then $\rho \cdot \hat{g}$ is a smooth compactly supported function on $U$, so $\rho \cdot \hat{g} \in \mathcal{S}(U)$. Moreover, $(\rho \cdot \hat{g})|_{\hat{X}} = \hat{g}|_{\hat{X}} = g$. Thus $g \in \mathcal{S}(\hat{X})$.  

□
Proposition 4.18. Let $X$ be an affine QN variety corresponding to some closed $\tilde{X}$, and let $Z \subset \tilde{X}$ be some closed subset. Define $V := \tilde{X} \setminus Z$ and

$$W_Z := \{ \phi \in \mathcal{S}(\tilde{X}) \mid \text{\phi is flat on } Z \}.$$ 

Then restriction from $\tilde{X}$ to $V$ of a function in $W_Z$ is a Schwartz function on $V$, i.e. $\text{Res}_{\tilde{X}}^V(W_Z) \subset \mathcal{S}(V)$.

Proof. The proof is divided into two parts. First we show the case where $X$ corresponds to a compact subset $\tilde{X} \subset \mathbb{R}^n$. Then, we deduce the general case.

Assume $\tilde{X}$ is compact. Define $V^{\mathbb{R}^n} := \mathbb{R}^n \setminus Z$ and

$$W_Z^{\mathbb{R}^n} := \{ \phi \in \mathcal{S}(\mathbb{R}^n) \mid \phi \text{ is flat on } Z \}.$$ 

As $Z$ is closed in $\mathbb{R}^n$, $V^{\mathbb{R}^n}$ is open. As $V = V^{\mathbb{R}^n} \cap \tilde{X}$, $V$ is closed in $V^{\mathbb{R}^n}$. The claim follows from the existence of these three maps:

\[
\begin{array}{ccc}
W_Z^{\mathbb{R}^n} & \xrightarrow{\text{Res}^{\mathbb{R}^n}_{\tilde{X}}} & \mathcal{S}(V) \\
\downarrow & & \downarrow \\
W_Z & \xrightarrow{\text{Res}_V} & \mathcal{S}(U)
\end{array}
\]

The existence of map (1) is clear. It is onto due to Lemma 4.16 and Lemma 4.17. Let $g \in W_Z^{\mathbb{R}^n}$. Then we get map (2) by Theorem 2.19, $g|_{V^{\mathbb{R}^n}} \in \mathcal{S}(V^{\mathbb{R}^n})$. Map (3) is obtained as for any $h \in \mathcal{S}(V^{\mathbb{R}^n})$ we get $h|_V \in \mathcal{S}(V)$ by Remark 4.10.

Now assume $\tilde{X}$ is not compact. But $\tilde{X}$ is closed in $\mathbb{R}^n$. By Proposition 2.8 we get an algebraic map $i : \mathbb{R}^n \rightarrow \hat{\mathbb{R}}^n$ where $\hat{\mathbb{R}}^n$ is an affine algebraic variety which is a one point compactification of $\mathbb{R}^n$, i.e. $\hat{\mathbb{R}}^n = i(\mathbb{R}^n) \cup \{ \infty \}$. We also get that $\hat{\mathbb{R}}^n$ and $i(\mathbb{R}^n)$ are algebraically isomorphic, which means they are also QN isomorphic. Thus, $\tilde{X}$ is QN isomorphic to some locally closed subset $i(\tilde{X})$ of the compact variety $\hat{\mathbb{R}}^n$. As $i(\tilde{X}) \cup \{ \infty \} =: \hat{\tilde{X}}$ is closed in $\hat{\mathbb{R}}^n$, it is compact. Now take some $f \in W_Z \subset \mathcal{S}(\tilde{X})$, and get that $i_* f := f \circ i^{-1} \in \mathcal{S}(i(\tilde{X}))$. By Lemma 4.15 as $i(\tilde{X})$ is open in $\hat{\tilde{X}}$, there exist $\hat{f} \in \mathcal{S}(\hat{\tilde{X}})$ such that $i_*f = \hat{f}|_{i(\tilde{X})}$. Now define $\hat{V} := \hat{\tilde{X}} \setminus (i(Z) \cup \{ \infty \})$. As $i$ is a QN isomorphism, $\hat{V}$ is open in $\hat{\tilde{X}}$. By the compact case, $\text{Res}_{\hat{\tilde{X}}}^\hat{V}(\hat{f}) \in \mathcal{S}(\hat{V})$. Note that $\hat{V}$ is QN isomorphic to $V$ by $i^{-1}|_{\hat{V}}$. Thus, $(i^{-1}|_{\hat{V}})_*\text{Res}_{\hat{\tilde{X}}}^\hat{V}(\hat{f}) \in \mathcal{S}(V)$. Finally, $(i^{-1}|_{\hat{V}})_*\text{Res}_{\hat{\tilde{X}}}^\hat{V}(\hat{f}) = (i^{-1}|_{\hat{V}})_*(i_*f)|_V = f|_V$ and thus $f|_V \in \mathcal{S}(V)$. \hfill $\square$

Theorem 4.19. (Characterization of Schwartz functions on open subset - the affine case) Let $X$ be an affine QN variety, and let $Z \subset X$ be some closed semi-algebraic subset. Define $V := X \setminus Z$ and $W_Z := \{ \phi \in \mathcal{S}(X) \mid \phi \text{ is flat on } Z \}$. Then extension by zero $\text{Ext}^X_V : \mathcal{S}(V) \rightarrow W_Z$ is an isomorphism of Fréchet spaces, whose inverse is $\text{Res}_V^X : W_Z \rightarrow \mathcal{S}(V)$. 

Proof. By 2.5 as \( W_Z = \bigcap_{z \in Z} \{ \phi \in S(X) | \phi \text{ is flat on } z \} \) is a closed subspace of \( S(X) \), as an intersection of closed subsets, it is a Fréchet space. Lemma 4.13 shows that for any \( f \in S(V) \), we get \( Ext^X_V (f) \in S(X) \) and \( Ext^X_V (f) \) is flat on \( Z \), i.e. \( Ext^X_V (S(V)) \subset W_Z \). This extension is a continuous map. To show that, consider the closed set \( \hat{X} \) corresponding to \( X \). Define \( W := \mathbb{R}^n \setminus Z \). This is an open semi-algebraic set, thus a Nash variety. \( V = W \cap \hat{X} \) so \( V \) is closed in \( W \). Take some closed embedding \( W \to \mathbb{R}^N \). Then, by Remark 4.10 \( S(V) \cong S(W) / I_{Sch}^W (V) \). \( W \) is open in \( \mathbb{R}^n \) so by Theorem 2.19 \( Ext^W_{Sch} \) is a closed embedding, and thus continuous, \( S(W) \to S(\mathbb{R}^n) \). Therefore, the map
\[
S(V) \cong S(W) / I_{Sch}^W (V) \to S(\mathbb{R}^n) / I_{Sch}^{\mathbb{R}^n} (\hat{X}) = S(\hat{X})
\]
is continuous, i.e. \( Ext^X_V \) is continuous, and \( Ext^X_V \) is continuous as well.

We saw in Proposition 4.18 that \( Res^X_V (W_Z) \subset S(V) \).
As \( Res^X_V \circ Ext^X_V : S(V) \to S(V) \) is the identity operator by definition, and so is \( Ext^X_V \circ Res^X_V : W_Z \to W_Z \), we get that \( Ext^X_V \) is a continuous bijection. By Theorem 2.6 this means \( Ext^V_X \) is an isomorphism of Fréchet spaces. \(\Box\)

Corollary 4.20. Let \( X \) be an affine QN variety. A Schwartz function \( f \in S(X) \) is flat at \( p \in X \) if and only if \( f|_{X \setminus \{p\}} \in S(X \setminus \{p\}) \).

Proof. apply Theorem 4.19 to \( Z = \{p\} \). \(\Box\)

Remark 4.21. By the same argument for an arbitrary function \( f \in C^\infty(X) \) (i.e. a function that is a restriction of a smooth function from an open set in which the set corresponding to \( X \) is closed) and any \( p \in X \), the following conditions are equivalent:

1. \( f \) is flat at \( p \).
2. There exists a smooth compactly supported function \( \rho \) on \( \mathbb{R}^n \), such that \( \rho \) is identically 1 on some open neighborhood of \( p \) and \( (f \cdot \rho)|_{X \setminus \{p\}} \in S(X \setminus \{p\}) \).

Definition 4.22. Let \( X \) be an affine QN variety. Define the space of tempered distributions on \( X \) as the space of continuous linear functionals on \( S(X) \). Denote this space by \( S^* (X) \).

4.3. General QN.

Definition 4.23. A Schwartz function on a (general) QN variety \( X \) with cover \( C \):
let \( X \) be a QN variety, and let \( C \) be an open affine QN cover of \( X \) - i.e. \( \bigcup_{i=1}^m X_i = X \).
Denote by \( Func(X, \mathbb{R}) \) the space of all real valued functions on \( X \). There is a natural map \( \psi : \bigoplus_{i=1}^m Func(X_i, \mathbb{R}) \to Func(X, \mathbb{R}) \). Define the space of Schwartz functions on \( X \) associated with the cover \( C \) by \( S_C(X) := \psi \left( \bigoplus_{i=1}^m S(X_i) \right) \).

Lemma 4.24. Let \( X \) be a QN variety, and let \( C \) be an open affine QN cover of \( X \). Then the space \( S_C(X) \) is a Fréchet space.
Proof. First note that \( \psi \left( \bigoplus_{i=1}^{m} S(X_i) \right) \cong \bigoplus_{i=1}^{m} S(X_i) / \text{Ker} \left( \psi \big|_{\bigoplus_{i=1}^{m} S(X_i)} \right) \) with the natural quotient topology. A direct sum of Fréchet spaces is a Fréchet space. The kernel of \( \psi \big|_{\bigoplus_{i=1}^{m} S(X_i)} \) is a closed subspace, as \( \bigoplus_{i=1}^{m} s_i \in \text{Ker} \left( \psi \big|_{\bigoplus_{i=1}^{m} S(X_i)} \right) \) if and only if for any \( x \in X \), \( \sum_{i \in J_x} s_i (x) = 0 \), where \( J_x := \{ 1 \leq i \leq m | x \in X_i \} \), i.e. the kernel is given by infinitely many "closed conditions". So by Proposition 2.9 and [T - Proposition 7.9], the quotient is a Fréchet space as well.

\[ \square \]

**Lemma 4.25.** Let \( X \) be a QN variety and let \( C, D \) be two open QN covers of \( X \). Then \( S_C(X) \cong S_D(X) \) as Fréchet spaces.

**Proof.** To prove this lemma, we will show the spaces have a continuous bijective map between them. Thus, by Theorem 2.6 they are isomorphic as Fréchet spaces.

We start with bijectiveness. Let the open affine cover \( X = \bigcup_{i=1}^{m} Y_i \) and the open affine cover \( D = \bigcup_{j=1}^{n} X_j \). By definition, \( s \in S_D(X) \iff s = \sum_{j=1}^{m} \text{Ext} \hat{X}_j s_j \) where for each \( j \), \( s_j \in S(X_j) \). Fix one such affine QN variety \( X_j \). \( X_j \) can be covered by the open QN covering \( X_j = \bigcup_{i=1}^{n} (Y_i \cap X_j) =: \bigcup_{i=1}^{n} X_{ij} \). As \( X_j \) is affine, for any \( j \), denote by \( \hat{X}_j \subset \mathbb{R}^{n_j} \) a closed set corresponding to \( X_j \). Denote by \( \hat{X}_{ij} \) the open subsets of \( \hat{X}_j \) corresponding to \( X_{ij} \). For any \( j \), \( s_j \in S \left( \hat{X}_j \right) \) is a restriction of a Schwartz function \( S_j \in S(U_j) \) to \( \hat{X}_j \), where \( U_j \subset \mathbb{R}^{n_j} \) is an open set in which \( \hat{X}_j \) is closed. As \( \hat{X}_{ij} \subset \hat{X}_j \) are open semi-algebraic, we may find open semi-algebraic subsets \( U_{ij} \subset U_j \) such that \( U_{ij} \cap \hat{X}_j = \hat{X}_{ij} \). As \( U_j \) and the \( U_{ij}'s \) are Nash manifolds, we may add the open Nash set \( W_j := U_j \setminus \hat{X}_j \) to get a Nash open cover of \( U_j \), and use Theorem 2.13(partition of unity) to get the Schwartz functions \( S_{ij} \in S(U_{ij}) \), \( S_{W_j} \in S(W_j) \), such that extending those functions by zero sum up to \( S_j \), i.e. \( \sum_{i=1}^{n} \text{Ext}^{U_{ij}}_{U_j} S_{ij} + \text{Ext}^{W_j}_{W_j} S_{W_j} = S_j \). Thus, after restricting those functions to \( \hat{X}_j \), we can pull them back to get \( \sum_{i=1}^{n} \text{Ext}^{X_{ij}}_{X_j} s_{ij} = s_j \).

So \( s \in S_D(X) \iff s = \sum_{j=1}^{m} \sum_{i=1}^{n} \text{Ext}^{X_{ij}}_{X_j} s_{ij} \). As both covers are finite, the sums may commute, and we get that \( s \in S_D(X) \iff s \in S_C(X) \).

To prove the map is continuous, let \( \psi : \bigoplus_{j=1}^{n} \text{Func} \left( X_j, \mathbb{R} \right) \rightarrow \text{Func} \left( X, \mathbb{R} \right) \) be the natural map from Definition 4.23. We begin with the claim that \( S(X_{ij}) \rightarrow S(X_j) \) is a continuous map. Take the closed sets \( X_j \subset \mathbb{R}^{n_j} \), and \( X_{ij}'s \) as before. Each function \( f_{ij} \in S \left( \hat{X}_{ij} \right) \) is a restriction of some \( F_{ij} \in S(U_{ij}) \), where \( U_{ij} \subset \mathbb{R}^{n_j} \) is an open subset such that \( U_{ij} \cap \hat{X}_j = \hat{X}_{ij} \). Denote \( U_j := \bigcup_{i=1}^{n} U_{ij} \). By Theorem 2.19, the extension by zero of Schwartz functions \( \text{Ext}^{U_{ij}}_{U_j} F_{ij} = F_j \in S(U_j) \) is a closed embedding, thus continuous. As \( \hat{X}_j \subset U_j \) is a closed subset, restricting
together with the NQN isomorphism on each such subset, we get the result.

Lemma 4.27. Let \( \phi \subset X \). Then \( \text{Res}_{\phi} \sim \) functions on open subsets of \( \sim \) form a cosheaf, every Schwartz function on \( \sim \) is continuous. Thus, by Theorem 2.6, \( \bigoplus_{i,j} \text{Func}(X_{ij}, \mathbb{R}) \to \text{Func}(X_j, \mathbb{R}) \) is the natural map from Definition 4.23. Thus, we get that

\[
\mathcal{S}_C(X) = \psi \left( \bigoplus_{i,j} \mathcal{S}(X_{ij}) \right) \cong \psi \left( \bigoplus_{i,j} \phi_j \left( \bigoplus_{i,j} \mathcal{S}(X_{ij}) \right) \right) \cong \phi \left( \bigoplus_{i,j} \mathcal{S}(X_{ij}) \right),
\]

where \( \phi : \bigoplus_{i,j} \text{Func}(X_{ij}, \mathbb{R}) \to \text{Func}(X, \mathbb{R}) \) is the natural map. The same can be done with the cover \( D \) to get the result.

In view of this lemma, we will denote the space of Schwartz functions on a QN variety \( X \) just by \( \mathcal{S}(X) \) without specifying the cover.

Lemma 4.26. Let \( \varphi : X \to Y \) be a QN isomorphism. Then \( \varphi^*|_{\mathcal{S}(Y)} : \mathcal{S}(Y) \to \mathcal{S}(X) \) is an isomorphism of Fréchet spaces.

Proof. Let \( f \in \mathcal{S}(\tilde{Y}) \). By Lemma 3.3, there exist open covers \( \bigcup_{i=1}^{N} \tilde{X}_i = \tilde{X}, \bigcup_{i=1}^{N} \tilde{Y}_i = \tilde{Y} \) such that \( \tilde{\varphi}|_{\tilde{X}_i} : \tilde{X}_i \to \tilde{Y}_i \) are NQN isomorphisms. As Schwartz functions form a cosheaf, every Schwartz function on \( \tilde{Y} \) is a sum of extensions of Schwartz functions on open subsets of \( \tilde{Y} \). Let those subsets be \( \tilde{Y}_i \). According to Lemma 1.3, a Schwartz function \( s_i \in \mathcal{S}(\tilde{Y}_j) \) is pulled back to \( \varphi^*s_i \in \mathcal{S}(\tilde{X}_i) \), and thus, we get \( \varphi^*s = \sum_{i=1}^{n} \text{Ext}_{\tilde{X}_i}^{\tilde{Y}_i} (\varphi^*s_i) \). By definition and Lemma 4.26, this means \( \varphi^*s \in \mathcal{S}(\tilde{X}) \).

Together with the NQN isomorphism on each such subset, we get the result.

Lemma 4.27. Let \( X \) be a QN variety, and \( Z \subset X \) be some semi-algebraic closed subset. Then \( \text{Res}_{\mathcal{S}^2}(\mathcal{S}(X)) = \mathcal{S}(Z) \).

Proof. Let \( s \in \mathcal{S}(X) \), and let \( X = \bigcup_{i=1}^{m} X_i \) be some open affine QN cover, such that \( s = \sum_{i=1}^{m} \text{Ext}_{X_i}^{X} (s_i) \) for some \( s_i \in \mathcal{S}(X_i) \). \( Z \cap X_i \) is open in \( Z \) and closed in \( X_i \). By theorem 4.21, \( s|_{Z \cap X_i} \in \mathcal{S}(Z \cap X_i) \), and thus \( s|_{Z} = \sum_{i=1}^{m} \text{Ext}_{Z \cap X_i}^{X} (s|_{Z \cap X_i}) \in \mathcal{S}(Z) \).

Now let \( h \in \mathcal{S}(Z) \), and let \( Z = \bigcup_{i=1}^{m} (Z \cap X_i) \) where \( X_i \) are as before. This is an open affine cover of \( Z \) so \( h = \sum_{i=1}^{m} \text{Ext}_{Z \cap X_i}^{Z} (h_i) \) for some \( h_i \in \mathcal{S}(Z \cap X_i) \). Each \( Z \cap X_i \) is closed in \( X_i \) so by theorem 4.1.1 there exist functions \( H_i \in \mathcal{S}(X_i) \) such that \( H_i|_{Z \cap X_i} = h_i \). Thus, \( \sum_{i=1}^{m} \text{Ext}_{X_i}^{X} (H_i) \in \mathcal{S}(X) \).

Lemma 4.28. Let \( X \) be an affine QN variety. The assignment of the space of tempered functions to any open \( V \subset X \), together with the restriction of functions, form a sheaf on \( X \).
Proof. First let us show that tempered functions restricted to an open subset remain tempered. Take a closed subset $\tilde{X} \subset \mathbb{R}^n$ corresponding to $X$. Tempered functions on $\tilde{X}$ are defined as restrictions of tempered functions on some open neighborhood $U$ of $\tilde{X}$. As tempered functions on $U$ form a sheaf, and by Remark 4.10 we get that a restricted tempered function remains tempered. It is now clear the above forms a presheaf. The proof of the glueing property follows [ES - Proposition 4.3]. It uses again the definition of functions $f_i \in \mathcal{T}(V_i)$ on subsets $V_i$ of $\tilde{X}$ as restrictions of functions $f_i \in \mathcal{T}(U_i)$ from neighborhoods $U_i$ of the $V_i$’s. Then, it uses tempered partition of unity on the $f_i$’s to create function $\hat{f}$ on $\bigcup_i U_i$ such that $\hat{f}|_{X} = f$.

Finally, it proves that $\hat{f}|_{U_i} \in \mathcal{T}(U_i)$ for any $i$, in order to show $\hat{f}$ is tempered, what implies $f$ is tempered.

Lemma 4.29. Let $X$ be a QN variety, and let $t : X \to \mathbb{R}$ be some function. Then the following conditions are equivalent:

1. There exists an open affine QN cover $X = \bigcup_{i=1}^{k} X_i$ such that for any $1 \leq i \leq k$, $t|_{X_i} \in \mathcal{T}(X_i)$.

2. For any open affine QN cover $X = \bigcup_{i=1}^{k} X_i$ and any $1 \leq i \leq k$, $t|_{X_i} \in \mathcal{T}(X_i)$.

Proof. Clearly (2) implies (1). For the other side assume there exist two open affine QN covers $X = \bigcup_{i=1}^{k} X_i = \bigcup_{j=k+1}^{l} X_j$ such that for any $k+1 \leq j \leq l$, $t|_{X_j} \in \mathcal{T}(X_j)$.

Fix some $1 \leq i \leq k$. Note that $\{X_i \cap X_j\}_{j=k+1}^{l}$ is an open cover of $X_i$. $t|_{X_i \cap X_j}$ is a restriction of the tempered function $t|_{X_j}$ to the open subset $X_i \cap X_j \subset X_i$. By remark 4.10 we get that $t|_{X_j}$ is a restriction to $X_j$ of a tempered function $T$ on an open neighborhood $U$ in which $X_j$ is closed. As tempered functions on Nash manifolds form a sheaf, take the open neighborhood $V \subset U$ of $X_i \cap X_j$ in which $X_i \cap X_j$ is closed, and get that $t := T|_{V}$ is a tempered function, and so, by remark 4.10 again, we get that $t|_{X_i \cap X_j} = \hat{t}|_{X_i \cap X_j} \in \mathcal{T}(X_i \cap X_j)$. By 4.28 these functions can be glued to a unique tempered function on $X_i$ as they form a sheaf. Thus, we get that $t|_{X_i} \in \mathcal{T}(X_i)$.

Definition 4.30. Let $X$ be a QN variety. A real valued function $t : X \to \mathbb{R}$ is called a tempered function on $X$ if it satisfies the equivalent conditions of Lemma 4.29. Denote the space of all tempered functions on $X$ by $\mathcal{T}(X)$.

Lemma 4.31. Let $X$ be a QN variety, $t \in \mathcal{T}(X)$ and $s \in \mathcal{S}(X)$. Then $t \cdot s \in \mathcal{S}(X)$.

Proof. Let $X = \bigcup_{i=1}^{k} X_i$ be some open affine QN cover such that $s = \sum_{i=1}^{k} \text{Ext}^{X}_X(s_i)$ for some $s_i \in \mathcal{S}(X_i)$. Then $t|_{X_i} \in \mathcal{T}(X_i)$ and by proposition 4.27 $t|_{X_i} \cdot s_i \in \mathcal{S}(X_i)$. Thus, $t \cdot s = \sum_{i=1}^{k} \text{Ext}^{X}_X(s_i \cdot t|_{X_i}) \in \mathcal{S}(X)$.

Proposition 4.32. (tempered partition of unity) - Let $X$ be a QN variety, and let $\{V_i\}_{i=1}^{m}$ be a finite open cover of $X$. Then:

1. There exist tempered functions $\{\alpha_i\}_{i=1}^{m}$ on $X$, such that supp$(\alpha_i) \subset V_i$ and $\sum_{i=1}^{m} \alpha_i = 1$. 


(2) We can choose \(\{\alpha_i\}_{i=1}^n\) in such a way that for any \(\varphi \in \mathcal{S}(X)\), \((\alpha_i\varphi)|_{V_i} \in \mathcal{S}(V_i)\).

The proof for the affine case is similar to that in [ES - Proposition 3.11], with the corresponding claims here (e.g. Lemma 4.27 replaces Theorem 3.7 in [ES]). The idea behind this proof is to take the corresponding set in the Nash manifold \(\mathbb{R}^n\), and extend the \(V_i\)'s to some open semi-algebraic sets covering \(\mathbb{R}^n\). By Theorem 2.18 there is a tempered partition of unity on those extending sets, so we can reduce to our \(V_i\)'s to get the result. For the general case we use claims in the Appendix as follows:

**Proof.** (1) By definition of \(X\), there exists an open affine cover \(X = \bigcup_{j=1}^n V_j\). Thus, for each \(i\), there exists an open affine cover \(V_i = \bigcup_{j=1}^n V_{ij}\) where \(V_{ij} := V_i \cap V_j\). By Proposition A.5 there exists a cover of \(V\) by \(\bigcup_{j} V_{ij} = V\) where \(V_{ij} \subset V_{ij}^j \subset V_j\). By Corollary A.7, there exist a finite collection of continuous functions \(G_{ij} : V_j \to \mathbb{R}\) and open sets \(V_{ij}^j\) such that \(V_{ij}^j = \{x \in V_j | G_{ij}(x) \neq 0\}\); \(G_{ij}|_{V_{ij}^j}\) is positive and QN and \(\bigcup_{j} V_{ij}^j = V_j\). It gives a finite cover of \(X\) which is a refinement of \(U_i\).

In order to have a unified system of indices we denote \(V_{ijk} := V_{ij}\). We re-index it to one index cover \(V_i\). By the same re-indexation we get \(G_l\) and \(V_{ij}^l\). Extend \(G_l\) by zero to a function \(\tilde{G}_l\) on \(X\). It is continuous. Denote \(G := \left(\sum \tilde{G}_l\right)/ (2n)\) where \(n\) is the number of values of the index \(l\). Consider \(G|_{V_i}\). This is a strictly positive continuous semi-algebraic function on an affine QN variety. Lemma A.8 shows that continuous strictly positive semi-algebraic function on an affine QN variety can be bounded from below by a strictly positive QN function. Thus, \(G|_{V_i}\) can be bounded from below by a strictly positive QN function \(g'_l\). Denote \(H_l := G_l/g'_l\). Extending \(H_l\) by zero outside \(V_i\) to \(X\) we obtain a collection of continuous semi-algebraic functions \(F_l\). Note that \(F_l\) is not smooth. It is easy to see that \(X_{F_l}\) is a refinement of \(V_i\).

Now, let \(\rho : \mathbb{R} \to [0, 1]\) be a smooth function such that
\[
\rho((-\infty, 0.1]) = \{0\}, \quad \rho([1, \infty)) = \{1\}
\]

Denote \(\beta_l := \rho \circ F_l\) and \(\gamma_l = \frac{\beta_l}{\sum \beta_l}\). It is easy to see that \(\gamma_l\) are tempered. For every \(l\) we choose \(i(l)\) such that \(X_{F_l} \subset U_{i(l)}\). Define \(\alpha_i := \sum_{l|l(l) = i} \gamma_l\). It is easy to see that \(\alpha_i\) is a tempered partition of unity.

(2) Take some \(\varphi \in \mathcal{S}(X)\). By definition \(\varphi = \sum_{j=1}^n \text{Ext}^1_{X_j}\varphi_j\) where the \(X_j\)'s are the affine open cover of \(X\) and \(\varphi_j \in \mathcal{S}(X_j)\). By definition, \(\alpha_i|_{X_j} \in \mathcal{T}(X_j)\), and by Lemma 4.7 \(\varphi_j \cdot \alpha_i|_{X_j} \in \mathcal{S}(X_j)\). By part (1), \(\text{supp}(\alpha_i) \subset V_i\) so \(\text{supp}(\varphi_j \cdot \alpha_i|_{X_j}) \subset V_i \cap X_j\) and \(\varphi_j \cdot \alpha_i|_{X_j}\) is flat on \(X_j \setminus V_i\). By Theorem 4.18 \((\varphi_j \cdot \alpha_i|_{X_j})|_{V_i \cap X_j} \in \mathcal{S}(V_i \cap X_j)\). Note that \(V_i \cap X_j\) is an open affine QN cover of \(V_i\), so
\[
\sum_{j=1}^n \text{Ext}^1_{V_i \cap X_j}(\varphi_j \cdot \alpha_i|_{X_j})|_{V_i \cap X_j} \in \mathcal{S}(V_i).
\]
Finally, by the definition of the $\varphi_j$’s we get that \[ \sum_{j=1}^{n} \text{Ext}^{\text{V}_i}_{\text{V}_i \cap X_j} (\varphi_j \cdot \alpha_i|_{X_j}) |_{\text{V}_i \cap X_j} = (\varphi \cdot \alpha_i)|_{\text{V}_i}. \]

\begin{proof}
Let \( X \) be a QN variety, and let \( U \subset X \) be some open subset. Define \( V \) as an open affine QN cover of \( U \). Then \( U = \bigcup_{i=1}^{k} \bigcup_{i=1}^{k} (U \cap X_i) \) is an open affine QN cover of \( U \), what makes \( U \) a QN variety as well. Take some \( s \in \mathcal{S}(U) \). Then \( s = \bigcup_{i=1}^{k} \text{Ext}^{\text{V}_i}_{\text{V}_i \cap X_i} (s_i) \) for some \( s_i \in \mathcal{S}(U \cap X_i) \). The set \( U_i := U \cap X_i \) is open in \( X_i \), so take the closed corresponding set \( X_i \) in \( \mathbb{R}^n \), and its corresponding open subset \( X_i \). Then there exists an open semi-algebraic set \( X_i \subset \mathbb{R}^n \) such that \( \tilde{U}_i = V_i \cap X_i \). By Remark 4.33, \( s_i \) corresponds to some \( \tilde{s}_i = s_i|_{\tilde{U}_i} \), where \( s_i \in \mathcal{S}(V_i) \). By Theorem 4.36, \( s_i \) can be extended by zero to the whole of \( \mathbb{R}^n \) and this extension is flat outside \( V_i \), and in particular in \( X_i \). This extension can be reduced to \( \tilde{X}_i \) to make a Schwartz function on \( X_i \) which extends \( \tilde{s}_i \) to \( \tilde{X}_i \) by zero, and thus is flat on \( \tilde{X}_i \). Thus, \( s_i \) can be extended to a Schwartz function on \( X_i \) which is flat on \( X_i \). So \( \text{Ext}^{\tilde{X}_i \cap U_i}_{\tilde{X}_i \cap U_i} (s_i) = \sum_{i=1}^{k} \text{Ext}^{\text{V}_i \cap U_i}_{\text{V}_i \cap U_i} (s_i) = \sum_{i=1}^{k} \text{Ext}^{\tilde{X}_i \cap U_i}_{\tilde{X}_i \cap U_i} (s_i) \) which, by definition, is a Schwartz function on \( X \) which is flat on \( X \). \end{proof}

\begin{theorem}
(Extension by zero for non affine varieties. Let \( X \) be a QN variety, and \( U \) an open subset of \( X \). Then the extension by zero to \( X \) of a Schwartz function on \( U \) is a Schwartz function on \( X \), which is flat at \( X \backslash U \).

\begin{proof}
As for the first part, \( W_Z = \bigcap_{z \in \mathbb{Z}} \{ \phi \in \mathcal{S}(X) \mid \phi \text{ is flat on } z \} \) is an intersection of closed sets, it is a closed subspace of \( \mathcal{S}(X) \), and thus a Fréchet space. For the second part, by Proposition $4.35$, the extension of a function in \( \mathcal{S}(U) \) by zero to \( X \) is a function in \( \mathcal{S}(X) \) that is flat at \( Z \), i.e. \( \text{Ext}^{\tilde{X}}_U (\mathcal{S}(U)) \subset W_Z \). Furthermore, we will claim further on that \( \text{Ext}^{\tilde{X}}_U \) is continuous. Before that, we will show the opposite direction - that \( \text{Res}^{\tilde{X}}_U (W_Z) \subset \mathcal{S}(U) \). Let \( f \in W_Z \). We want to show \( f|_{U_i} = \sum_{i=1}^{n} \text{Ext}^{U_i}_{U_i} s_i \) where \( X = \bigcup_{i=1}^{n} X_i \) is an open affine QN cover of \( X \), \( U_i := U \cap X_i \), (this makes \( U = \bigcup_{i=1}^{n} U_i \) an open affine cover of \( U \)) and \( s_i \in \mathcal{S}(U_i) \). Note that
\end{proof}

\end{theorem}
functions to any open $U$ form a sheaf on $X$. Let \(\alpha\) and $X$ be an affine QN open cover of $X$, we can use Proposition 4.32 to get tempered functions $\alpha_i$ such that $\alpha_i \cdot f \in \mathcal{S}(X_i)$. As $f \in W_\mathbb{Z}$ is flat at any $x \in Z$, $\alpha_i \cdot f$ is flat at any $x \in Z_i := Z \cap X_i$. Thus, $\alpha_i \cdot f \in W_{Z_i}$, and by Theorem 4.19 we get $\alpha_i \cdot f \in \mathcal{S}(U_i)$ where $U_i := U \cap X_i$. \(\bigcup U_i\) is an open affine QN cover of $U$, so \(\sum_{i=1}^n \text{Ext}^n_{U_i}(\alpha_i \cdot f) \in \mathcal{S}(U)\). Furthermore, by the definition of the $\alpha$'s, $\sum_{i=1}^n \text{Ext}^n_{U_i}(\alpha_i \cdot f) = f|_U$.

By Theorem 4.19 for any $i = 1, \ldots, n$ we have the continuous map $\text{Ext}^n_{U_i} : \mathcal{S}(U_i) \to \mathcal{S}(X_i)$. As $n < \infty$, we get the continuous map $\text{Ext}^n_{X_i} : \bigoplus_{i=1}^n \mathcal{S}(U_i) \to \bigoplus_{i=1}^n \mathcal{S}(X_i)$. Recall that $\mathcal{S}(X) := \psi\left(\bigoplus_{i=1}^n \mathcal{S}(X_i)\right) \cong \bigoplus_{i=1}^n \mathcal{S}(X_i) / \text{Ker}\left(\psi|_i \otimes \mathcal{S}(X_i)\right)$ and get the continuous map $\text{Ext}^n_{X} : \mathcal{S}(U) \to \mathcal{S}(X)$. As this map is bijective, we get by the Theorem 2.6 \(\text{Ext}^n_X\) is an isomorphism of Fréchet spaces.

**Definition 4.37.** Let $X$ be a QN variety. Define the space of tempered distributions on $X$ as the space of continuous linear functionals on $\mathcal{S}(X)$. Denote this space by $\mathcal{S}^*(X)$.

**Theorem 4.38.** Let $X$ be a QN variety, and let $U \subset X$ be some semi-algebraic open subset. Then $\text{Ext}^n_X : \mathcal{S}(U) \to \mathcal{S}(X)$ is a closed embedding, and the restriction morphism $\mathcal{S}^*(X) \to \mathcal{S}^*(U)$ is onto.

**Proof.** As $W_\mathbb{Z} = \bigcap_{z \in Z} \{ \phi \in \mathcal{S}(X) | \phi \text{ is flat on } z \}$ is an intersection of closed subsets, it is a closed subspace of $\mathcal{S}(X)$ and thus, by Theorem 4.39 the first part is proved. The second part follows from the fact that $\mathcal{S}(X)$ is a Fréchet space and from Theorem 2.7.

5. **Sheaves and Cosheaves**

The aim of this section is to prove that tempered functions and tempered distributions form sheaves and that Schwartz functions form a cosheaf. Unlike the algebraic case, this can be done to both the affine and the general case. The proofs for the affine cases, and for the general case of tempered functions, are the same as the those in the [ES] regarding algebraic varieties. Thus we give short sketches of the proofs together with relevant statements in this paper replacing statements in [ES].

First, let us recall Lemma 4.28

**Lemma 5.1.** Let $X$ be an affine QN variety. The assignment of the space of tempered functions to any open $V \subset X$, together with the restriction of functions, form a sheaf on $X$.

**Corollary 5.2.** Let $X$ be a QN variety. The assignment of the space of tempered functions to any open $U \subset X$, together with the restriction of functions, form a sheaf on $X$. 
Proof: The proof follows the proof of [ES - Proposition 5.11]. By the definition of tempered functions on QN varieties and by Lemma 4.28, they form a presheaf. Using induction on the number of the covering open subsets, it is enough to show the following:

Let $X$ be a QN variety and let $U_1 \cup U_2 = X$ be an open cover of $X$. Assume we are given $t_i \in T(U_i)$ such that $t_1|_{U_1 \cap U_2} = t_2|_{U_1 \cap U_2}$. Then, there exists a unique function $t \in T(U_1 \cap U_2)$ such that $t|_{U_i} = t_i$.

The existence of a function $t : U_1 \cup U_2 \to \mathbb{R}$ such that $t|_{U_i} = t_i$ is clear. We shall now show it is tempered. Consider some affine open cover $X = \bigcup_{j=1}^{k} X_j$. Then $U_i = \bigcup_{j=1}^{k} (U_i \cap X_j)$ is an affine open cover of $U_i$, and $U_1 \cup U_2 = \bigcup_{j=1}^{k} ((U_1 \cup U_2) \cap X_j)$ is an affine open cover of $U_1 \cup U_2$. As $t_i \in T(U_i)$, we get that $t_1|_{U_1 \cap X_j} \in T(U_i \cap X_j)$. As $(U_1 \cup U_2) \cap X_j$ is affine, and $\bigcup_{i=1}^{2} (U_i \cap X_j)$ is an affine open cover of it, and as $t_1|_{U_i \cap X_j} = t_2|_{U_i \cap X_j}$, we get by Lemma 4.28 that $t|_{(U_1 \cup U_2) \cap X_j} \in T((U_1 \cup U_2) \cap X_j)$. Thus $t \in T(U_1 \cup U_2)$.

Before dealing with cosheaves in the category of real vector spaces, let us recall their definition:

First, define the category $Top(X)$ to be such that its objects are the open sets of $X$, and its morphisms are the inclusion maps. A pre-cosheaf $F$ on a topological space $X$ is a covariant functor from $Top(X)$ to the category of real vector spaces. A cosheaf on a topological space $X$ is a pre-cosheaf, such that for any open $V \subset X$ and any open cover $\{V_i\}_{i \in I}$ of $V$, the following sequence is exact:

$$
\bigoplus_{(i,j) \in I^2} F(V_i \cap V_j) \xrightarrow{Ext_1} \bigoplus_{i \in I} F(V_i) \xrightarrow{Ext_2} F(V) \to 0,
$$

where the $k$-th coordinate of $Ext_1(\bigoplus_{(i,j) \in I^2} \xi_{i,j})$ is $\sum_{i \in I} Ext^V_{V_i \cap V_j} (\xi_{i,j} - \xi_{i,k})$, and $Ext_2(\bigoplus_{i \in I} \xi_i) := \sum_{i \in I} Ext^V_{V_i}(\xi_i)$. When exactness will be proven in Proposition 

below all calculations will be quickly reduced to finite subcovers. A cosheaf is flabby if for any two open subsets $U, V \subset X$ such that $V \subset U$, the morphism $Ext^V_U : F(V) \to F(U)$ is injective.

Lemma 5.3. Let $X$ be an affine QN variety. The assignment of the space of Schwartz functions to any open $U \subset X$, together with the extension by zero $Ext^U_V$ from $U$ to any other open $V \supset U$, form a flabby cosheaf on $X$.

Proof: The proof follows [ES - Proposition 4.5] - By extension by 0 (see Theorem 4.19), $X$ is a pre-cosheaf. Now we shall prove the exactness:

The $\bigoplus_{j=1}^{l} F(U_j) \to F(U) \to 0$ part follows immediately from partition of unity. The second part uses induction on the number of the covering sets. The base step uses the fact that the two functions sum to zero everywhere, and that their extensions are flat outside the sets’ intersection. Thus, by the characterization property given in Theorem 4.19, their restriction to the intersection is Schwartz. In the inductive step, we use partition of unity on $k+1$ sets, to create some new Schwartz functions on the first $k$ sets whose extensions sum to zero. We then define
and prove the claim for the $k + 1$'th function, using the fact that each of the $k$
functions is flat at each point they are solely defined .

The precise proof is the same as in [ES], with the following replacements:
Theorem 4.38 replaces [ES - Theorem 3.25],
Remark 3.34 replaces [ES - Cor. 3.10],
Proposition 4.32 replaces [ES - Prop. 3.11],
Proposition 4.33 replaces [ES - Prop. 3.16],
Theorem 4.19 replaces [ES - Thm. 3.20],
Remark 4.33 replaces [ES - Cor. 3.22],
Corollary 4.20 replaces [ES - Cor. 3.21],
Proposition 4.28 replaces [ES - Prop. 4.3], and
Lemma 4.31 replaces [ES - Prop. 3.9].

\[ \square \]

\textbf{Corollary 5.4.} Let $X$ be a QN variety. The assignment of the space of Schwartz functions to any open $U \subset X$, together with the extension by zero $\text{Ext}^k_U$ from $U$ to any other open $V \supset U$, form a flabby cosheaf on $X$.

\textbf{Proof.} By Proposition 4.35, the Schwartz functions form a pre-cosheaf on $X$.

Let $\bigcup_{i=1}^k X^i = X$ be a finite open cover such that for any $i$, $X^i$ is affine. Let $U \subset X$ be some open set, $\bigcup_{j=1}^l U_j = U$ is some open cover, and let $s \in \mathcal{S}(U)$. Then for any $i$, $U^i := U \cap X^i$ is an open subset of the affine $X^i$, and $\bigcup_{j=1}^l U^i_j = U^i$ where $U^i_j := U^i \cap U_j$ is an open cover of $U^i$. By definition, $s = \sum_{i=1}^k \text{Ext}^k_U(s^i)$ where $s^i \in \mathcal{S}(U^i)$. By Lemma 5.3 we know that $s^i = \sum_{j=1}^l \text{Ext}^k_{U^i_j}(s^i_j)$ where $s^i_j \in \mathcal{S}(U^i_j)$.

So we get that $s = \sum_{i=1}^k \text{Ext}^k_U \sum_{j=1}^l \text{Ext}^k_{U^i_j} s^i_j$ and as the sums are finite, we may write

$s = \sum_{j=1}^l \bigoplus_{i=1}^k \text{Ext}^k_{U^i_j} s^i_j = \sum_{j=1}^l \text{Ext}^k_{U^i_j} s^i_j$ where $s_j := \sum_{i=1}^k \text{Ext}^k_{U^i_j} s^i_j$ and $s_j \in \mathcal{S}(U_j)$ by definition. This proves the part

$$\bigoplus_{j=1}^l F(U_j) \rightarrow F(U) \rightarrow 0$$

of the cosheaf definition.

Now let us have $\{s_j\} \subset \mathcal{S}(U_j)$ such that $\sum_{j=1}^l \text{Ext}^k_{U^i_j} s_j = 0$. As $\bigcup_{i=1}^k U^i = U$ is an open (affine) cover of $U$, we may use Proposition 4.32 to get some tempered functions $\{\alpha_i\}_{i=1}^k$ on $U$, such that $\text{supp} \ (\alpha_i) \subset U^i$ and $\sum_{i=1}^k \alpha_i = 1$, and for any $\varphi \in \mathcal{S}(U)$, $(\alpha_i \varphi)|_{U^i} \in \mathcal{S}(U^i)$.

Now, $S_j := \text{Ext}^k_{U^i_j} s_j \in \mathcal{S}(U)$ by Proposition 4.32, so $(\alpha_i \cdot S_j)|_{U^i} \in \mathcal{S}(U^i)$. We get an affine QN variety $U^i$ with $\sum_{j=1}^l ((\alpha_i \cdot S_j)|_{U^i}) = 0$.

Notice that for each $j$, the function $\alpha_i \cdot S_j$ is flat on $U^i \setminus U_j$, so we may use Theorem
4.19 to get \((\alpha_i \cdot S_j) |_{V_j} \in \mathcal{S} (U_j)\). Thus, we have \(\sum_{j=1}^l \text{Ext}^i_{U_j} \left((\alpha_i \cdot S_j) |_{V_j}\right) = 0\) and by Lemma 5.3 we know there exists \(s_{jm}^i \in \mathcal{S} (U_j \cap U_m)\) satisfying \((\alpha_i \cdot S_j) |_{V_j} = \sum_{m < j} \text{Ext}^i_{U_j \cap U_m} (s_{jm}^i) - \sum_{m > j} \text{Ext}^i_{U_j \cap U_m} (s_{mj}^i)\). Sum the extensions by zero to \(U_j\) of both sides to get

\[
\sum_{i=1}^k \text{Ext}^i_{U_j} \left((\alpha_i \cdot S_j) |_{V_j}\right) = \sum_{i=1}^k \text{Ext}^i_{U_j} \left(\sum_{m < j} \text{Ext}^i_{U_j \cap U_m} (s_{jm}^i) - \sum_{m > j} \text{Ext}^i_{U_j \cap U_m} (s_{mj}^i)\right),
\]

but

\[
\sum_{i=1}^k \text{Ext}^i_{U_j} \left((\alpha_i \cdot S_j) |_{V_j}\right) = S_j |_{V_j} = s_j
\]

and

\[
\sum_{i=1}^k \text{Ext}^i_{U_j} \left(\sum_{m < j} \text{Ext}^i_{U_j \cap U_m} (s_{jm}^i) - \sum_{m > j} \text{Ext}^i_{U_j \cap U_m} (s_{mj}^i)\right) = \sum_{m < j} \text{Ext}^i_{U_j \cap U_m} (s_{jm}^i) - \sum_{m > j} \text{Ext}^i_{U_j \cap U_m} (s_{mj}^i) = s_j = \sum_{m < j} \text{Ext}^i_{U_j \cap U_m} (s_{jm}^i) - \sum_{m > j} \text{Ext}^i_{U_j \cap U_m} (s_{mj}^i)
\]

what proves the part

\[
\bigoplus_{j > m} F (U_j \cap U_m) \rightarrow \bigoplus_{j=1}^l F (U_j) \rightarrow F (U)
\]

of the cosheaf definition. \(\square\)

**Lemma 5.5.** Let \(X\) be an affine QN variety. The assignment of the space of tempered distributions to any open \(U \subset X\), together with restrictions of functionals from \(\mathcal{S}^* (U)\) to \(\mathcal{S}^* (V)\), for any other open \(V \subset U\), form a flabby sheaf on \(X\).

**Proof.** The proof follows [ES - Proposition 4.4]. The proof uses extension by zero to show presheaf structure. To show uniqueness, we use partition of unity which enables us to write \(s \in \mathcal{S} (U)\) as \(s = \sum_{i=1}^k (\beta_i \cdot s)\) where \(\text{supp} (\beta_i) \subset U_i\). Then, as \((\beta_i \cdot s) |_{U_i} \in \mathcal{S} (U_i)\), and as the functionals \(\xi, \zeta \in \mathcal{S}^* (U)\) agree on each subset.
and by the linearity of the functionals, we get \( \xi(s) - \zeta(s) = \xi \left( \sum_{i=1}^{k} (\beta_i \cdot s) \right) - \zeta \left( \sum_{i=1}^{k} (\beta_i \cdot s) \right) = 0 \). This means the uniqueness is achieved.

The existence is proven by partition of unity on \( U = \bigcup_{i=1}^{k} U_i \), and defining a functional \( \xi \in \mathcal{S}^*(U) \) in the following way: \( \xi(s) = \xi \left( \sum_{i=1}^{k} (\beta_i \cdot s) \right) := \sum_{i=1}^{k} \xi_i (\beta_i \cdot s) \) where \( \xi_i \in \mathcal{S}^*(U_i) \), \( s \in \mathcal{S}(U) \) and \( \beta_i \) are the tempered functions obtained by the partition of unity. For any \( U_\alpha \subset U \) open, where \( \alpha \in \{1, \ldots, k\} \) and \( s_\alpha \in \mathcal{S}(U_\alpha) \), we get \( (\beta_i|_{U_\alpha \cap U_i} \cdot s_\alpha)|_{U_\alpha \cap U_i} \in \mathcal{S}(U_\alpha \cap U_i) \) by Lemma \ref{lem:partition_of UNITY} and Proposition \ref{prop:partition_ofUNITY}. As \( \xi_\alpha|_{\mathcal{S}(U_\alpha \cap U_i)} = \xi_i|_{\mathcal{S}(U_\alpha \cap U_i)} \), and \( s_\alpha \) may be extended to a Schwartz function on \( U \), we get \( \xi_\alpha(s_\alpha) = \xi_\alpha \left( \sum_{i=1}^{k} (\beta_i \cdot s_\alpha) \right) = \sum_{i=1}^{k} \xi_i (\beta_i \cdot s_\alpha) = \sum_{i=1}^{k} \xi_i (\beta_i \cdot s_\alpha) = \xi(s_\alpha) \). This means that for any \( \alpha \), \( \xi|_{\mathcal{S}(U_\alpha)} = \xi_\alpha \) and the existence holds.

Proposition \ref{prop:flabby} replaces [ES - Prop. 3.16], Proposition \ref{prop:flabby} replaces [ES - Prop. 3.16], and Remark \ref{rem:flabby} replaces [ES - Cor. 3.22].

**Corollary 5.6.** Let \( X \) be a QN variety. The assignment of the space of tempered distributions to any open \( U \subset X \), together with restrictions of functionals from \( \mathcal{S}^*(U) \) to \( \mathcal{S}^*(V) \), for any other open \( V \subset U \), form a flabby sheaf on \( X \).

**Proof.** The claim that tempered distributions form a sheaf is dual to the claim that Schwartz functions form a cosheaf, which we proved in Corollary \ref{cor:flabby}.

6. **SCHWARTZ, TEMPERED AND TEMPERED “DISTRIBUTIONS” OVER QN VECTOR BUNDLES**

We begin this section with definitions of QN bundles and their sections. Most of the definitions and results in this chapter follow [AG] with light adjustments to our category.

**Definition 6.1.** Let \( \pi : X \to B \) be a morphism of QN varieties. It is called a QN **locally trivial fibration** with fiber \( Z \) if the following holds:

- \( Z \) is a QN variety.
- There exists a finite cover \( B = \bigcup_{i=1}^{n} U_i \) by open QN sets and QN isomorphisms \( \nu_i : \pi^{-1}(U_i) \to U_i \times Z \) such that \( \pi \circ \nu_i^{-1} \) is the natural projection.

**Definition 6.2.** Let \( X \) be a QN variety. A **QN vector bundle** \( E \) over \( X \) is a QN locally trivial fibration with linear fiber and such that the trivialization maps \( \nu_i \) are fiberwise linear. By abuse of notation, we use the same letters to denote bundles and their total spaces.

**Definition 6.3.** Let \( X \) be a QN variety and \( E \) a QN bundle over \( X \). A **QN section** of \( E \) is a section of \( E \) which is a QN morphism.

Now we can use the above definitions to define the more specific QN bundles relevant to our work.
Definition 6.4. Let $X$ be a QN variety, and $E$ be a QN bundle over it. Let $X = \bigcup_{i=1}^{k} X_i$ be an affine QN trivialization of $E$. A global section $s$ of $E$ over $X$ is called **tempered** if for any $i$, all the coordinate components of $s|_{X_i}$ are tempered functions. The space of global tempered sections of $E$ is denoted by $T(X, E)$.

Remark 6.5. As tempered functions on QN varieties form sheaves, the definition above does not depend on the cover.

Definition 6.6. A Schwartz section on a QN bundle $X$: let $X$ be a QN variety, and let $E$ be a QN bundle over $X$. Let $\bigcup_{i=1}^{m} X_i = X$ be affine QN trivialization of $E$. Denote by $\text{Func}(X, E, \mathbb{R})$ the bundle of all real valued sections of $E$ over $X$. There is a natural map $\psi : \bigoplus_{i=1}^{m} S(X_i)^n \to \text{Func}(X, E, \mathbb{R})$. Define the space of global Schwartz sections of $E$ by $S(X, E) := \text{Im}\psi$.

Remark 6.7. We define the topology on the space of global Schwartz sections of $E$ by the quotient topology, i.e. by the isomorphism $S(X, E) \cong \bigoplus_{i=1}^{m} S(X_i)^n / \text{Ker}\psi$.

Remark 6.8. The definition above does not depend on the cover. See the proof of Lemma 4.25.

Now we will define the local sections:

Definition 6.9. Let $X$ be a QN variety, and let $E$ be a QN bundle over it. We define the **cosheaf** $S^E_X$ of Schwartz sections of $E$ by $S^E_X(U) := \text{S}(U, E|_U)$. We define in a similar way the sheaf $T^E_X$ of tempered sections of $E$ by $T^E_X(U) := \mathcal{T}(U, E|_U)$.

Lemma 6.10. Let $X$ be a QN variety. For an open semi-algebraic subset $U \subset X$, $S^E_X|_U = S^E_U$, $T^E_X|_U = T^E_U$.

Proof. The lemma holds by definitions. \hfill \Box

Theorem 6.11. (Characterization of Schwartz sections on open subset - the bundle case) Let $X$ be a QN variety, and let $Z \subset X$ be some closed subset. Define $U := X \setminus Z$ and $W_Z := \{\phi \in \text{S}(X, E) \mid \phi \text{ vanish with all its derivatives on } Z\}$. Then extension by zero $\text{Ext}^X_U : S^E_X(U) \to W_Z$ is an isomorphism of Fréchet spaces, whose inverse is $\text{Res}^X_U : W_Z \to S^E_X(U)$.

Proof. This theorem follows from Theorem 4.36 - characterization of Schwartz functions on open subset for general QN variety, and Proposition 4.32 - tempered partition of unity for general QN variety. \hfill \Box

Appendix A. Preperations For Partition Of Unity

In order to prove partition of unity we will follow the proof of Theorem 2.18 using some definitions and lemmas, lightly adapted to our case. We will start with some definitions, and then show there is a certain refinement for the cover we will need later on.
Definition A.1. 1) Let $M$ be a QN variety and $F$ be a continuous semi-algebraic function on $M$. We denote $M_F := \{x \in M | F(x) \neq 0\}$.

2) Let $M$ be a QN variety. A continuous semi-algebraic function $F$ on $M$ is called basic if $F|_{M_F}$ is a positive QN function.

3) A collection of continuous semi-algebraic functions $\{F_i\}$ is called basic collection if every one of them is basic, and in every point of $M$ one of them is larger than 1.

Theorem A.2. ([S III.1.1]) Let $r < \infty$. A $C^r$ Nash manifold is affine. Thus, by [AG - A.2.4], a QN variety $M$ can be continuously embedded in $\mathbb{R}^n$ by a semi-algebraic map, where $M$ and its image are homeomorphic.

Corollary A.3. Let $M$ be a QN manifold. Then there exists a semi-algebraic continuous metric $d : M \times M \to \mathbb{R}$.

Definition A.4. A cover $M = \bigcup_{j=1}^{m} V_j$ is called a proper refinement of the cover $M = \bigcup_{i=1}^{n} U_i$ if for any $j$ there exists $i$ such that $V_j \subset U_i$.

Proposition A.5. Let $M = \bigcup_{i=1}^{n} U_i$ be a finite open (semi-algebraic) cover of an affine QN variety $M$. Then there exists a finite open (semi-algebraic) cover $M = \bigcup_{j=1}^{m} V_j$ which is a proper refinement of $\{U_i\}$.

Proof. Let $d$ be the metric from corollary A.3. If a set $A$ is closed in the classical topology, then the distance $d(x, A) := \inf_{y \in A} d(x, y)$ is strictly positive for all points $x$ outside $A$. Now define $F_i : M \to \mathbb{R}$ by $F_i(x) = d(x, M\setminus U_i)$. It is semi-algebraic by the Tarski-Seidenberg principle. Define $G = \left(\sum_{i=1}^{n} F_i\right)/2n$ and $V_i = \{x \in M | F_i(x) > G(x)\}$. It is easy to see that $V_i$ is a proper refinement of $U_i$. \hfill $\square$

Theorem A.6. (finiteness) Let $X \subset \mathbb{R}^n$ be a semi-algebraic set. Then every open semi-algebraic subset of $X$ can be presented as a finite union of sets of the form

$$\{x \in X | p_i(x) > 0, \ i = 1, \ldots, n\},$$

where $p_i$ are polynomials in $n$ variables.

Corollary A.7. Let $M$ be an affine QN variety. Then it has a basis of open sets of the form $M_F$ where $F$ is a basic function.

Lemma A.8. ([AG - Lemma A.2.1 + Lemma 2.2.11]) Let $M$ be an affine QN variety. Then any continuous semi-algebraic function on it can be majorated by a QN function, and any continuous strictly positive semi-algebraic function on it can be bounded from below by a strictly positive QN function.

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