STABILITY OF THE OVERDAMPED LANGEVIN EQUATION IN DOUBLE-WELL POTENTIAL

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ABSTRACT. In this article, we discuss stability of the one-dimensional overdamped Langevin equation in double-well potential. We determine unstable and stable equilibria, and discuss the rate of convergence to stable ones. Also, we derive conditions for stability of general diffusion processes which generalize the classical and well-known results of Khasminskii ([Kha12]).

1. Introduction

The Langevin equation is a stochastic differential equation describing the dynamics of a particle immersed in a fluid, subjected to an external potential force field and collisions with the molecules of the fluid:

\[m \, dX_t = P_t \, dt - \nabla V(X_t) \, dt + \sigma(X_t) \, dB_t, \quad (X_0, P_0) \in \mathbb{R}^d \times \mathbb{R}^d.\]

Here, \(\{X_t\}_{t \geq 0}\) and \(\{P_t\}_{t \geq 0}\) denote, respectively, the position and momentum of the particle, \(m\) is particle’s mass, \(-\frac{\lambda}{m} P_t \, dt\), \(\lambda > 0\), is the velocity-proportional damping (friction) force, \(V\) is particle’s potential and \(\sigma(X_t) \, dB_t\) is the noise term representing the effect of the collisions with the molecules of the fluid, where \(\{B_t\}_{t \geq 0}\) denotes a standard \(d\)-dimensional Brownian motion. Observe that here we assume the measure of the noise strength \(\sigma\) is non-constant, meaning that the effect of collisions depends on the position of the particle (e.g. due to heterogeneity of the fluid). In this case, the function \(\sigma\) models the nature of the position-dependence.

In the case when the inertia of the particle is negligible in comparison with the damping force (due to friction), the trajectory of the particle is described by the so-called overdamped Langevin equation:

\[\lambda \, dX_t = -\nabla V(X_t) \, dt + \sigma(X_t) \, dB_t, \quad X_0 \in \mathbb{R}^d.\]

Namely, in [Nel67, Chapter 10] it has been shown that (under certain assumptions on the potential \(V\) and diffusion coefficient \(\sigma\)) the solution to (1.1) converges a.s. to the solution to (1.2), as \(m \searrow 0\).

The main purpose of this article is to discuss stability of the solution to the one-dimensional overdamped Langevin equation in double-well or Landau potential \(V(x) = -ax^2/2 + bx^4/4, \ a, b > 0:\)

\[\lambda \, dX_t = (-bX_t^3 + aX_t) \, dt + \sigma(X_t) \, dB_t, \quad X_0 \in \mathbb{R}.\]

This potential is of considerable interest in quantum mechanics and quantum field theory for the exploration of various physical phenomena or mathematical properties since it permits in many cases explicit calculation without over-simplification (see e.g. [Col79] and [LMK92]). Typical example where it occurs is in the so-called ammonia inversion phenomenon. This is a switching of the nitrogen atom from above to below the hydrogen plane. More precisely, the ammonia molecule is pyramidal shaped with the three hydrogen
atoms forming the base and the nitrogen atom at the top. The nitrogen atom sees a double-well potential with one well on either side of the hydrogen plane. Because the potential barrier is finite, it is possible for the nitrogen atom to tunnel through the plane of the hydrogen atoms, thus "inverting" the molecule (see [Leh70] for more details).

For the sake of simplicity, but without loss of generality, in the sequel we assume $a = b = \lambda = 1$. Also, we impose the following assumptions on the diffusion coefficient $\sigma$:

**A1:** $\sigma$ is locally Lipschitz continuous;

**A2:** $\limsup_{|x| \to \infty} |\sigma(x)|/|x|^2 < \sqrt{2}$.

Under (A1) and (A2), in [ABW10, Theorem 3.1 and Proposition 4.2] and [PR07, Theorem 3.1.1] it has been shown that the equation in (1.3) admits a unique non-explosive strong solution $\{X_t\}_{t \geq 0}$ which, in addition, is a temporally homogeneous strong Markov process with continuous sample paths. Furthermore, in [ABW10, Remark 2.2 and Proposition 4.3] it has been also shown that $\{X_t\}_{t \geq 0}$ is a $C_b$-Feller process and that for any $f \in C^2(\mathbb{R})$ the process

$$
M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds, \quad t \geq 0,
$$

is a local martingale, where

$$
\mathcal{L}f(x) = (-x^3 + x)f'(x) + \frac{\sigma^2(x)}{2}f''(x), \quad f \in C^2(\mathbb{R}).
$$

Recall, $C_b$-Feller property means that the semigroup $\{P_t\}_{t \geq 0}$ of $\{X_t\}_{t \geq 0}$, defined as

$$
P_tf(x) := \int_{\mathbb{R}} f(y)p_t(x,dy), \quad t \geq 0, \ x \in \mathbb{R}, \ f \in B_b(\mathbb{R}),
$$

maps $C_b(\mathbb{R}) := C(\mathbb{R}) \cap B_b(\mathbb{R})$ to $C_b(\mathbb{R})$. Here, $p_t(x,dy)$ and $B_b(\mathbb{R})$ denote, respectively, the transition kernel of $\{X_t\}_{t \geq 0}$ and the space of bounded Borel measurable functions.

1.1. **Stability of the deterministic overdamped Langevin equation (1.3).** We consider

$$
\dot{x} = -x^3 + x, \quad x(0) \in \mathbb{R}.
$$

It is easy to check that (1.6) admits three solutions: $x_1(t) \equiv 0$ (corresponding to the initial condition $x_1(0) = 0$),

$$
x_2(t) = -\frac{e^t}{\sqrt{e^{2t} - 1 + 1/x_2(0)^2}}, \quad x_2(0) < 0,
$$

and

$$
x_3(t) = \frac{e^t}{\sqrt{e^{2t} - 1 + 1/x_3(0)^2}}, \quad x_3(0) > 0.
$$

Now, recall that $x_e \in \mathbb{R}$ is called equilibrium state to the Cauchy problem

$$
\dot{x} = f(x), \quad x(0) \in \mathbb{R},
$$

if $x(0) = x_e$ implies that $x(t) \equiv x_e$ (or, equivalently, if $f(x_e) = 0$). Clearly, the only equilibria to (1.6) are $-1, 0$ and $1$. Further, an equilibrium $x_e$ to (1.7) is called stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|x(0) - x_e| < \delta$ implies $|x(t) - x_e| < \varepsilon$ for all $t \geq 0$; otherwise it is called unstable. An equilibrium $x_e$ to (1.7) is called asymptotically stable if it is stable, and if there is $\delta > 0$ such that whenever $|x(0) - x_e| < \delta$ then

$$
\lim_{t \to \infty} |x(t) - x_e| = 0,
$$
and it is called \textit{exponentially stable} if it is stable, and if there is \( \delta > 0 \) such that whenever \( |x(0) - x_e| < \delta \) then
\[
\lim_{t \to \infty} e^{\kappa t}|x(t) - x_e| = 0
\]
for some \( \kappa > 0 \). Clearly, \(-1\) and \(1\) are exponentially stable for any \( 0 < \kappa < 2 \), and \(0\) is unstable equilibrium to (1.6).

Previous discussion suggests that in the non-deterministic setting the states \(-1\), \(0\) and \(1\) might also play an important role. However, in this setting, due to the random term \( \sigma(X_t)dB_t \) which can “regularize” the equation, these points will not necessarily be equilibria of \( \{X_t\}_{t \geq 0} \): if \( \sigma \) is “regular” enough, i.e. if \( \sigma \) does not vanish at \(-1\), \(0\) and \(1\), \( \{X_t\}_{t \geq 0} \) will admit only one equilibrium which does not explicitly depend on \(-1\), \(0\) and \(1\).

1.2. \textbf{Stability of the overdamped Langevin equation (1.3).} In the non-deterministic setting the role of equilibria take invariant measures of the underlying process. A probability measure \( \pi \) on \( \mathbb{R} \) is \textit{invariant} for \( \{X_t\}_{t \geq 0} \) if
\[
\int_{\mathbb{R}} p_t(x,dy)\pi(dx) = \pi(dy), \quad t \geq 0.
\]
In other words, under \( \pi \) as an initial distribution the marginals of \( \{X_t\}_{t \geq 0} \) do not change over time, i.e. \( \{X_t\}_{t \geq 0} \) is a stationary process.

As the first main result of this article we show that \( \{X_t\}_{t \geq 0} \) admits at least one equilibrium (invariant measure).

\textbf{Theorem 1.1.} Assume \((A1)\) and \((A2)\). Then \( \{X_t\}_{t \geq 0} \) admits an invariant measure.

Furthermore, we also show that if \( \sigma \) is “regular” enough, then \( \{X_t\}_{t \geq 0} \) admits a unique equilibrium.

\textbf{Theorem 1.2.} Assume \((A1)\) and \((A2)\). If there is an open interval \( I \) containing \(-1\), \(0\), and \(1\), such that \( \inf_{x \in I} \sigma(x) > 0 \), then \( \{X_t\}_{t \geq 0} \) admits a unique equilibrium \( \pi \) such that for any \( \kappa > 0 \),
\[
\lim_{t \to \infty} e^{\kappa t}\|p_t(x,dy) - \pi(dy)\|_{TV} = 0, \quad x \in \mathbb{R},
\]
where \( \|\cdot\|_{TV} \) stands for the total variation norm on the space of signed measures.

On the other hand, if \( \sigma \) vanishes at \( x_e \in \{-1,0,1\} \), then, obviously, \( X_t = x_e, t \geq 0 \), is a solution to (1.3), i.e. \( \delta_{x_e} \) is an invariant measure for \( \{X_t\}_{t \geq 0} \). The point \( x_e \) is said to be \textit{stable in probability} if for any \( \varepsilon > 0 \),
\[
\lim_{x \to x_e} \mathbb{P}^x \left( \sup_{t \geq 0} |X_t - x_e| > \varepsilon \right) = 0;
\]
otherwise it is called \textit{unstable}. It is called \textit{asymptotically stable in probability} if it is stable in probability and
\[
\lim_{x \to x_e} \mathbb{P}^x \left( \lim_{t \to \infty} |X_t - x_e| = 0 \right) = 1.
\]
We then conclude the following.

\textbf{Theorem 1.3.} Assume \((A1)\), \((A2)\) and that \( \sigma \) has a root at \( x_e \in \{-1,0,1\} \).

(i) If \( x_e = 0 \) and
\[
\inf_{\varepsilon > 0} \inf_{\kappa > 0} \left\{ \kappa : \frac{\|\sigma(x)\|}{|x|} \leq \kappa, \ 0 < |x| < \varepsilon \right\} < \sqrt{2},
\]
then \( x_e \) is unstable. Moreover, there is \( \varepsilon > 0 \) such that \( \mathbb{P}^x (\sup_{t \geq 0} |X_t| < \varepsilon) = 0 \) for every \( 0 < |x| < \varepsilon \).
(ii) If \( x_e = 0 \) and there is \( \delta > 0 \) such that \( |\sigma(x)| = \sqrt{2}|x| \) for \( |x| < \delta \), then \( x_e \) is unstable. Also, there is \( 0 < \varepsilon < \delta \) such that \( \mathbb{P}^x(\sup_{t\geq 0}|X_t| < \varepsilon) = 0 \) for every \( 0 < |x| < \varepsilon \).

(iii) If \( x_e = 0 \) and
\[
\inf_{t>0} \sup_\kappa \left\{ \kappa : \frac{|\sigma(x)|}{|x|} \geq \kappa, \ 0 < |x| < \varepsilon \right\} > \sqrt{2},
\]
then \( x_e \) is asymptotically stable in probability.

(iv) If \( x_e \in \{-1,1\} \), then \( x_e \) is asymptotically stable in probability. Furthermore, if
\[
c := \alpha \inf_{x \in \mathbb{R}, x > 1} \left( x(x + x_e) - (\alpha - 1)\frac{\sigma(x)^2}{2}|x - x_e|^2 \right) > 0
\]
for some \( \alpha > 0 \) (which is always the case for \( 0 < \alpha \leq 1 \)), then \( \mathbb{E}^x(|X_t - x_e|^\alpha) \leq |x - x_e|^\alpha e^{-ct} \) for \( x \in \mathbb{R}, xx_e \geq 1, \) and \( t \geq 0, \) and
\[
\lim_{t\to \infty} \frac{\ln |X_t - x_e|}{t} \leq -\frac{c}{\alpha} \quad \mathbb{P}^x\text{-a.s.}
\]
for all \( x \in \mathbb{R}, xx_e \geq 1. \)

1.3. Stability of general diffusion processes. At the end we discuss stability of general multidimensional diffusion processes.

**Theorem 1.4.** Assume that equation
\[
dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R}^d,
\]
adopts a unique, non-explosive strong solution which is a strong Markov process with continuous sample paths, where \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times n} \) are continuous, and \( \{B_t\}_{t \geq 0} \) stands for a standard \( n \)-dimensional Brownian motion. Further, assume that there is \( x_e \in \mathbb{R}^d \) such that \( b \) and \( \sigma \) vanish at \( x_e \) (hence, \( X_t = x_e, t \geq 0 \), is a solution to (1.8)). If there are \( c > 0 \), and concave, continuously differentiable and strictly increasing around the origin function \( \varphi : [0, \infty) \to [0, \infty) \), with \( \varphi(0) = 0 \), such that

(i) the function \( \Phi_c(t) := c^{-1} \int_1^t ds/\varphi(s) \) maps \( (0, \infty) \) onto \( \mathbb{R} \);

(ii) \( \mathcal{L}\mathcal{V}(x) \leq -c \varphi \circ \mathcal{V}(x) \) for \( |x - x_e| > 0 \), where
\[
\mathcal{L}f(x) = \langle b(x), \nabla f(x) \rangle + \frac{1}{2} \text{Tr} \sigma(x)\sigma'(x)D^2f(x), \quad f \in \mathcal{C}^2(\mathbb{R}^d)
\]
and \( \mathcal{V}(x) := |x - x_e|^\alpha \) for some \( \alpha > 0 \),

then
\[
|X_t - x_e|^\alpha \leq \Phi_c^{-1}(\Phi_c(Y_x) - t), \quad \mathbb{P}^x\text{-a.s.}, \ x \in \mathbb{R}^d, \ t \geq 0,
\]
where \( Y_x \) is a strictly positive \( \mathbb{P}^x\)-finite random variable.

As a consequence of Theorem 1.4 we get a generalization of [Kha12, Theorem 5.15] (where it is assumed that \( \mathcal{L}\mathcal{V}(x) \leq -c\mathcal{V}(x) \) for some \( c > 0 \) and all \( |x - x_e| > 0 \)). Also, we conclude super-geometric stability result.

**Corollary 1.5.** Assume the conditions of Theorem 1.4.

(i) If
\[
\mathcal{L}\mathcal{V}(x) \leq \begin{cases} -c\mathcal{V}(x), & 0 < \mathcal{V}(x) \leq r \\ -cr, & \mathcal{V}(x) \geq r, \end{cases}
\]
for some \( c > 0 \) and \( r > 0 \), then
\[
\limsup_{t \to \infty} \frac{\ln |X_t - x_e|}{t} \leq -\frac{c}{\alpha}.
\]
(ii) If there are $c > 0$, $\beta > 1$ and $0 < r_{\beta} \leq e^{1/\beta - 1}$ such that $\mathcal{L} \mathcal{V}(x) \leq -c \varphi_{\beta} \circ \mathcal{V}(x)$ for all $|x - x_{a}| > 0$, where

$$
\varphi_{\beta}(t) := \begin{cases} 
\beta t (\ln t)^{1-1/\beta}, & 0 \leq t \leq r_{\beta} \\
\beta r_{\beta} (\ln r_{\beta})^{1-1/\beta}, & t \geq r_{\beta},
\end{cases}
$$

then

$$
\limsup_{t \to \infty} \frac{\ln |X_{t} - x_{a}|}{t^{\beta}} \leq -\frac{c^\beta}{\alpha}
$$

($r_{\beta}$ is chosen such that $\varphi_{\beta}$ is non-decreasing).

The remainder of the article is organized as follows. In Section 2, we discuss existence and uniqueness of invariant measures of $\{X_{t}\}_{t \geq 0}$, and prove Theorems 1.1, 1.2 and 1.3. In Section 3, we prove Theorem 1.4 and Corollary 1.5, and discuss super-geometric stability of general diffusion processes.

2. Stability of the overdamped Langevin equation (1.3)

In this section, we discuss existence and uniqueness of invariant measures of $\{X_{t}\}_{t \geq 0}$. For $a \in \mathbb{R}$ and $r > 0$ denote by $I_{r}(a)$ the open $r$-interval around $a$, i.e. $I_{r}(a) := (a - r, a + r)$. Also, $\bar{I}_{r}(a)$ and $\bar{I}^{c}_{r}(a)$ denote, respectively, the closure and complement of $I_{r}(a)$.

**Proof of Theorem 1.1.** According to [MT93a, Theorem 3.1] it suffices to prove that for each $x \in \mathbb{R}$ and $0 < \epsilon < 1$, there is a compact set $K \subset \mathbb{R}$ such that

$$
\liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} p^{a}(x, K) ds \geq 1 - \epsilon.
$$

In order to show this, define $\mathcal{V}(x) := x^{2}$ and observe that

$$
\mathcal{L} \mathcal{V}(x) = -2x^{4} + 2x^{2} + \sigma(x)^{2}.
$$

Now, according to (A2), there are $0 < \delta < 2$ and $r_{\delta} > 2/\sqrt{\delta}$, such that $\sigma(x)^{2} \leq (2 - \delta) |x|^{4}$ for $|x| \geq r_{\delta}$. Thus,

$$
\mathcal{L} \mathcal{V}(x) = -2x^{4} \mathbb{1}_{I_{r_{\delta}}(0)}(x) - 2x^{4} \mathbb{1}_{\bar{I}^{c}_{r_{\delta}}(0)}(x) + 2x^{2} \mathbb{1}_{I_{r_{\delta}}(0)}(x) + 2x^{2} \mathbb{1}_{\bar{I}^{c}_{r_{\delta}}(0)}(x)
$$

$$
\leq -\frac{\delta}{2} r_{\delta}^{4} + \left(\frac{\delta}{2} r_{\delta}^{4} + 2r^{2} + \sigma^{2}\right) \mathbb{1}_{I_{r_{\delta}}(0)}(x).
$$

By denoting $\sigma := \sup_{x \in \bar{I}_{r_{\delta}}(0)} |\sigma(x)|$, we get

$$
\mathcal{L} \mathcal{V}(x) \leq -\frac{\delta}{2} r_{\delta}^{4} + \left(\frac{\delta}{2} r_{\delta}^{4} + 2r^{2} + \sigma^{2}\right) \mathbb{1}_{I_{r_{\delta}}(0)}(x).
$$

Analogously, for $r > r_{\delta}$ we conclude

$$
\mathcal{L} \mathcal{V}(x) \leq -\frac{\delta}{2} r^{4} + \left(\frac{\delta}{2} r^{4} + 2r^{2} + \sigma^{2} + (2 - \delta) r^{4}\right) \mathbb{1}_{I_{r}(0)}(x),
$$

where we used the fact

$$
|\sigma(x)|^{2} \mathbb{1}_{I_{r}(0)} \leq (\sigma^{2} + (2 - \delta) r^{4}) \mathbb{1}_{I_{r}(0)}.
$$

Now, from [MT93b, Theorem 1.1] we conclude that for each $x \in \mathbb{R}$ and $r > r_{\delta}$ we have

$$
\liminf_{t \to \infty} \frac{1}{t} \int_{0}^{t} p^{a}(x, \bar{I}_{r}(0)) dt \geq \frac{(\delta/2) r^{4}}{(\delta/2) r^{4} + 2r^{2} + \sigma^{2} + (2 - \delta) r^{4}}.
$$

The assertion now follows by choosing $\delta$ close to 2 and $r$ large enough. \qed
Standard assumptions which ensure uniqueness of an invariant measure are strong Feller property and open-set irreducibility. Recall, \( \{X_t\}_{t \geq 0} \) is called

(i) **strong Feller** if \( P_t f \in C_b(\mathbb{R}) \) for any \( t > 0 \) and \( f \in B_b(\mathbb{R}) \).

(ii) **open-set irreducible** if for any \( x \in \mathbb{R} \) and open set \( O \subseteq \mathbb{R} \),

\[
\int_0^{\infty} p'(x, O) dt > 0.
\]

According to [XZ17, Theorem 5.2 and Lemma 6.1] \( \{X_t\}_{t \geq 0} \) will be strong Feller and open-set irreducible if \( \inf_{x \in \mathbb{R}} |\sigma(x)| > 0 \). Furthermore, under the same assumption, (A2) together with [MT93b, Theorem 6.1] and [Twe94, Theorems 3.2 and 5.1] (by taking \( \mathcal{V}(x) = x^2 \)) implies that \( \{X_t\}_{t \geq 0} \) admits a unique invariant measure \( \pi \) such that for any \( \kappa > 0 \),

\[
\lim_{t \nearrow \infty} e^{\kappa t} \|p'(x, dy) - \pi(dy)\|_{\mathcal{V}} = 0, \quad x \in \mathbb{R}.
\]

However, in many interesting situations the diffusion coefficient \( \sigma \) can be singular, i.e. it can vanish (see e.g. [MGL87] and [DT98]).

2.1. **Equilibria of the overdamped Langevin equation (1.3) with singular noise term.** Assume (A1), (A2) and

(A3): There is a bounded open interval \( I \subset \mathbb{R} \) such that

(i) \( \inf_{x \in I} |\sigma(x)| > 0; \)

(ii) \( \sup_{x \in K} E^x(\tau_I) < \infty \) for any compact \( K \subset \mathbb{R} \), where \( \tau_I := \inf\{t \geq 0 : X_t \in I\} \).

Then, according to [Kha12, Theorems 4.1 and 4.2, and Corollary 4.4] \( \{X_t\}_{t \geq 0} \) admits a unique invariant measure \( \pi \) such that

\[
\lim_{t \nearrow \infty} \frac{1}{t} \int_0^t f(X_s) ds = \int_{\mathbb{R}} f(y) \pi(dy), \quad \mathbb{P}^x \text{-a.s.,}
\]

for every \( x \in \mathbb{R} \) and \( f \in B_b(\mathbb{R}) \).

**Proposition 2.1.** The process \( \{X_t\}_{t \geq 0} \) will satisfy (A3) if there is an open interval \( I \), containing \(-1, 0\) and \( 1 \), such that \( \inf_{x \in I} |\sigma(x)| > 0 \).

**Proof.** By assumption, there is \( 0 < \epsilon < 1 \) such that \( I_\epsilon := (-1 - \epsilon, 1 + \epsilon) \subset I \). Thus, in particular \( \inf_{x \in I_\epsilon} |\sigma(x)| > 0 \). Let us now show that \( \sup_{x \in K} E^x(\tau_{I_\epsilon}) < \infty \) for any compact \( K \subset \mathbb{R} \). Clearly, it suffices to prove the assertion for compact subsets of \( I_\epsilon \) only. Let \( \mathcal{V} : \mathbb{R} \to \mathbb{R}_+ \), \( \mathcal{V} \in C^2(\mathbb{R}) \), be such that \( \mathcal{V}(x) = |x| \|_{I_\epsilon}(x) \). Further, for \( n \in \mathbb{N} \) define \( \tau_n := \inf\{t \geq 0 : |X_t| \geq n\} \). Clearly, since \( \{X_t\}_{t \geq 0} \) is conservative, \( \tau_n \nearrow \infty \), as \( n \nearrow \infty \). Now, due to the martingale property of the process \( \{M^y_t\}_{t \geq 0} \) (defined in (1.4)), we have

\[
\mathbb{E}^x(\mathcal{V}(X_{t \wedge \tau_{I_\epsilon} \wedge \tau_n})) - \mathcal{V}(x) = \mathbb{E}^x \left( \int_0^{t \wedge \tau_{I_\epsilon} \wedge \tau_n} \mathcal{L} \mathcal{V}(X_s) ds \right), \quad t \geq 0, \ x \in \mathbb{R}, \ n \in \mathbb{N}.
\]

In particular, for \( x \in I_\epsilon^c, x < -1 \), we have

\[
\mathcal{V}(x) \geq -\mathbb{E}^x \left( \int_0^{t \wedge \tau_{I_\epsilon} \wedge \tau_n} \mathcal{L} \mathcal{V}(X_s) ds \right) = \mathbb{E}^x \left( \int_0^{t \wedge \tau_{I_\epsilon} \wedge \tau_n} (-X^3_s + X_s) ds \right) \geq \left((1 + \epsilon)^3 - (1 + \epsilon)\right) \mathbb{E}^x(t \wedge \tau_{I_\epsilon} \wedge \tau_n), \quad t \geq 0, \ n \in \mathbb{N}.
\]
By letting \( t \nearrow \infty \) and \( n \nearrow \infty \) we conclude

\[
\mathbb{E}^\pi(\tau_{I_n}) \leq \frac{|x|}{(1+\varepsilon)^3 - (1+\varepsilon)}, \quad x \in \overline{I}_n, \ x < -1.
\]

Analogously, for \( x \in \overline{I}_n, \ x > 1 \), we have

\[
\mathbb{E}^\pi(\tau_{I_n}) \leq \frac{|x|}{(1+\varepsilon)^3 - (1+\varepsilon)},
\]

which concludes the proof.

**Remark 2.2.** Let us remark that the above results can be slightly generalized. Namely, according to [Abu00, Theorem 5.2] and [Kha12, Lemma 4.6], \( \{X_t\}_{t \geq 0} \) will satisfy (A3) if there is \( 0 < \varepsilon < 1/2 \) such that

1. \( \sigma \) does not vanish on \( I_\varepsilon(-1) \cup I_\varepsilon(0) \cup I_\varepsilon(1); \)
2. \( \sup_{x \in [-1,0]} \mathbb{E}^\pi(\tau_{I_\varepsilon(-1)} \vee \tau_{I_\varepsilon(0)}) < \infty \) and \( \sup_{x \in [0,1]} \mathbb{E}^\pi(\tau_{I_\varepsilon(0)} \vee \tau_{I_\varepsilon(1)}) < \infty. \)

**Proof of Theorem 1.2.** The first assertion follows from Proposition 2.1 and [Kha12, Corollary 4.4]. To prove the second assertion we proceed as follows. From (2.1) we automatically conclude that \( \{X_t\}_{t \geq 0} \) is \( \pi \)-irreducible, i.e.

\[
\int_0^\infty p^I(x, B)dt > 0, \quad x \in \mathbb{R},
\]

whenever \( \pi(B) > 0, \ B \in \mathcal{B}(\mathbb{R}). \) Here, \( \mathcal{B}(\mathbb{R}) \) denotes the Borel algebra on \( \mathbb{R}. \) Next, [Kha12, Lemma 4.8] implies that the support of \( \pi \) has a non-empty interior, which together with the fact that \( \{X_t\}_{t \geq 0} \) is a \( C_0 \)-Feller process, (2.1) and [Twe94, Theorems 3.4 and 7.1] implies that \( \{X_t\}_{t \geq 0} \) is positive Harris recurrent process, i.e. there is a \( \sigma \)-finite measure \( \phi \) such that

\[
\mathbb{P}^x \left( \int_0^\infty 1_B(X_t)dt = \infty \right) = 1, \quad x \in \mathbb{R},
\]

whenever \( \phi(B) > 0, \ B \in \mathcal{B}(\mathbb{R}). \) Now, according to [Twe94, Theorems 5.1 and 7.1] and [MT93b, Theorem 6.1] (by taking \( \mathcal{V}(x) = x^2 \)), the assertion will follow if we show that there is a \( \sigma \)-finite measure \( \phi \), whose support has a non-empty interior, such that

\[
\sum_{n=1}^{\infty} p^n(x, B) > 0, \quad x \in \mathbb{R},
\]

whenever \( \phi(B) > 0, \ B \in \mathcal{B}(\mathbb{R}). \) Due to [Dur96, Theorems 7.3.6 and 7.3.7] there is a function \( p^I(x, y) > 0, \ t > 0, \ x, y \in I, \) jointly continuous in \( t \) and \( x, y \), and \( C^2 \) in \( x \) on \( I, \) satisfying

\[
\mathbb{E}^\pi(f(X_t), \tau_{\mathcal{F}_t} > t) = \int_I p^I(x, y)f(y)dy, \quad t > 0, \ x \in I, \ f \in \mathcal{C}_0(\mathbb{R}),
\]

where \( \tau_{\mathcal{F}_t} := \inf\{t \geq 0 : X_t \in \mathcal{F}\}. \) Clearly, by employing dominated convergence theorem, the above relation holds also for any open interval \( J \subseteq I. \) Denote by \( \mathcal{D} \) the class of all \( B \in \mathcal{B}(I) \) (the Borel \( \sigma \)-algebra on \( I \)) such that

\[
\mathbb{P}^x(X_t \in B, \ \tau_{\mathcal{F}_t} > t) = \int_B p^I(x, y)dy, \quad t > 0, \ x \in I.
\]

Clearly, \( \mathcal{D} \) contains the \( \pi \)-system of open intervals in \( I, \) and forms a \( \lambda \)-system. Hence, by employing the famous Dynkin’s \( \pi \)-\( \lambda \) theorem we conclude that \( \mathcal{D} = \mathcal{B}(I). \) Consequently, for any \( t > 0, \ x \in I \) and \( B \in \mathcal{B}(\mathbb{R}) \) we have that

\[
p^I(x, B) \geq \int_{B \cap I} p^I(x, y)dy.
\]
Let us now take \( \phi(\cdot) := \lambda(\cdot \cap I) \) and prove (2.2), where \( \lambda \) stands for the Lebesgue measure on \( \mathbb{R} \). Let \( x \in I^c \) (for \( x \in I \) the assertion is obvious) and \( B \in \mathfrak{B}(\mathbb{R}) \), \( \phi(B) > 0 \), be arbitrary. Then,
\[
\sum_{n=1}^{\infty} p^n(x, B) \geq \int \sum_{n=1}^{\infty} p^{n-t}(x, dy)p^t(y, B), \quad 0 < t < 1.
\]
Since \( p^t(y, B) > 0 \) for \( y \in I \), it suffices to show that
\[
\sum_{n=1}^{\infty} p^{n-t}(x, I) \geq \mathbb{P}^x \left( \bigcup_{n=1}^{\infty} \{ X_{n-t} \in I \} \right) > 0
\]
for some \( 0 < t < 1 \). Assume this is not the case, i.e. that
\[
\mathbb{P}^x \left( \bigcup_{n=1}^{\infty} \{ X_{n-t} \in I \} \right) = 0, \quad 0 < t < 1.
\]
This, in particular, implies that
\[
\mathbb{P}^x \left( \bigcup_{q \in \mathbb{Q}, Z} \{ X_q \in I \} \right) = 0,
\]
which is impossible since \( \{ X_t \}_{t \geq 0} \) has continuous sample paths, \( I \) is an open set and, by assumption, \( \mathbb{P}^x(\tau_I < \infty) = 1 \) for every \( x \in \mathbb{R} \). Thus,
\[
\sum_{n=1}^{\infty} p^n(x, B) > 0, \quad x \in \mathbb{R},
\]
whenever \( \phi(B) > 0 \), which concludes the proof. \( \square \)

The crucial assumption in the above discussion was that \( \sigma \) does not vanish at the roots of \( \nabla V \), i.e. at \(-1, 0 \) and \( 1 \). Recall, if \( \sigma \) vanishes at \( x_e \in \{-1, 0, 1\} \), then \( \mathbb{P}^x( X_t = x_e, \ t \geq 0) = 1 \). In particular, \( \delta_{x_e} \) is an invariant measure for \( \{ X_t \}_{t \geq 0} \).

**Proposition 2.3.** If \( \sigma \) vanishes at \( x_e \in \{-1, 0, 1\} \), then \( \mathbb{P}^x( X_t > x_e, \ t \geq 0) = 1 \) for all \( x > x_e \), and \( \mathbb{P}^x( X_t < x_e, \ t \geq 0) = 1 \) for all \( x < x_e \).

**Proof.** Let us discuss the case when \( x_e = 0 \) and \( x > 0 \). The other five cases are treated in a similar way, simply by appropriately shifting and/or mirroring the function \( V \) defined below. We follow the proof of [LW11, Lemma 1]. For \( n \in \mathbb{N} \) define \( \tau_n := \inf \{ t \geq 0 : X_t \notin (1/n, n) \} \). Clearly, \( \{ \tau_n \}_{n \in \mathbb{N}} \) is a non-decreasing sequence of stopping times. Set \( \tau_\infty := \lim_{n \to \infty} \tau_n \). In the sequel we show that \( \mathbb{P}^x(\tau_\infty = \infty) = 1 \) for all \( x > 0 \), which automatically implies the assertion. Assume this is not the case. Then there exist \( x_0 > 0 \) and \( 0 < \varepsilon < 1 \), such that \( \mathbb{P}^{x_0}(\tau_\infty < \infty) > \varepsilon \). This automatically implies that there are \( n_0 \in \mathbb{N} \) and \( T > 0 \), such that \( x_0 \in (1/n_0, n_0) \) and \( \mathbb{P}^{x_0}(\tau_{n_0} < T) > \varepsilon \) for all \( n \geq n_0 \). Next, define \( \mathcal{V}(x) := x - 1 - \ln x \). It is elementary to check that \( \mathcal{V} : (0, \infty) \to \mathbb{R}_+ \) and \( \mathcal{V} \in \mathcal{C}^2(0, \infty) \). Also, for \( n \in \mathbb{N} \), \( \mathcal{V}_n \) is such that \( \mathcal{V}_n(x)_{[1/n_0, n]}(x) = \mathcal{V}(x) \). Now, by the martingale property of \( \{ M_{t \wedge \tau_n} \}_{t \geq 0} \) (defined in (1.4)), we have that for all \( n \in \mathbb{N} \),
\[
\mathbb{E}^{x_0}( \mathcal{V}_n(X_{T \wedge \tau_n}) - \mathcal{V}_n(x_0) ) - \mathbb{E}^{x_0}( \mathcal{V}_n(X_{T \wedge \tau_{n_0}}) - \mathcal{V}(x_0) )
\]
\[
= \mathbb{E}^{x_0} \left( \int_0^{T \wedge \tau_n} \mathcal{L} \mathcal{V}(X_s) ds \right)
\]
\[
= \mathbb{E}^{x_0} \left( \int_0^{T \wedge \tau_{n_0}} \left( (-X_s^3 + X_s)(1 - X_s^{-1}) + \frac{1}{2} \Phi^2(X_s)X_s^{-2} \right) ds \right)
\]
Lemma 2.4. (i) If \( x < 0 \), then for any \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \) we have that
\[
\sup_{x \in I_{\varepsilon_1, \varepsilon_2}(x)} \mathbb{E}^x (\tau_{I_{\varepsilon_1, \varepsilon_2}(x)}) \leq \frac{\varepsilon_2 - \varepsilon_1}{(\varepsilon_1 - \varepsilon_2)}.
\]

(ii) If \( x \in (-1, 1) \), then for any \( 0 < \varepsilon_1 < \varepsilon_2 < 1 \) we have that
\[
\sup_{x \in I_{\varepsilon_1, \varepsilon_2}(x)} \mathbb{E}^x (\tau_{I_{\varepsilon_1, \varepsilon_2}(x)}) \leq \frac{\varepsilon_2}{(\varepsilon_1 - \varepsilon_2)}.
\]

Proof. Let \( \mathcal{V} : \mathbb{R} \to \mathbb{R}_+ \), \( \mathcal{V} \in C^2(\mathbb{R}) \), be such that \( \mathcal{V}(x) = |x - x_e|_{I_{\varepsilon_1}(x_e)}(x) \). Again, due to the martingale property of \( \{M^{\varepsilon_1}_{t \wedge \tau_{I_{\varepsilon_1, \varepsilon_2}(x_e)}}\}_{t \geq 0} \), we have
\[
\mathbb{E}^x (\mathcal{V}(X_{t \wedge \tau_{I_{\varepsilon_1, \varepsilon_2}(x_e)}(x)})) - \mathcal{V}(x) = \mathbb{E}^x \left( \int_0^{t \wedge \tau_{I_{\varepsilon_1, \varepsilon_2}(x)}} \mathcal{L} \mathcal{V}(X_s) \, ds \right), \quad t \geq 0, \ x \in \mathbb{R}.
\]

(i) If \( x_e = 0 \), then for \( x \in I_{\varepsilon_1, \varepsilon_2}(x_e) \) we have
\[
\varepsilon_2 - \varepsilon_1 \geq \mathbb{E}^x (\mathcal{V}(X_{t \wedge \tau_{I_{\varepsilon_1, \varepsilon_2}(x_e)}(x)})) - \mathcal{V}(x) = \mathbb{E}^x \left( \int_0^{t \wedge \tau_{I_{\varepsilon_1, \varepsilon_2}(x)}} \mathcal{L} \mathcal{V}(X_s) \, ds \right), \quad t \geq 0, \ x \in \mathbb{R}.
\]

where \( \text{sgn} \) denotes the signum function. Finally, by letting \( t \nearrow \infty \) the assertion follows.
(ii) If \( x_e \in \{-1, 1\} \), then for \( x \in I_{\varepsilon_1, \varepsilon_2}(x_e) \) we have
\[
\varepsilon_2 \geq \mathcal{V}(x) \\
\geq -\varepsilon \left( \int \tau_0 \tau_{\varepsilon_1, \varepsilon_2}(x_e) \operatorname{sgn}(X_s - x_e)(-X_s^3 + X_s)ds \right) \\
\geq ((\varepsilon_1 - \varepsilon_1^3) \land (\varepsilon_2 - \varepsilon_2^3))\varepsilon \tau_{\varepsilon_1, \varepsilon_2}(x_e),
\]
where in the last line we used the fact that
\[
(1 + r)^3 - (1 + r) > r - r^3, \quad 0 < r < 1.
\]
Finally, by letting \( t \not< \infty \) the desired result follows.
\[
\square
\]
Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. (i) According to Lemma 2.4 and [Kha12, Theorem 5.5] it suffices to show that there are \( \varepsilon > 0 \) and non-negative \( \mathcal{V} \in \mathcal{C}^2(\mathbb{R} \setminus \{0\}) \), such that
\[
\lim_{|x| \to 0} \mathcal{V}(x) = \infty \quad \text{and} \quad \mathcal{L}\mathcal{V}(x) \leq 0 \quad \text{for} \quad 0 < |x| < \varepsilon.
\]
By assumption, there is \( 0 < \varepsilon_0 < 1 \) such that
\[
\kappa_\varepsilon := \inf \left\{ \kappa : \frac{|\sigma(x)|}{|x|} \leq \kappa, \ 0 < |x| < \varepsilon \right\} < \sqrt{2}
\]
for every \( 0 < \varepsilon \leq \varepsilon_0 \). Observe that this is well defined due to (A1). Thus, there is \( 0 < \varepsilon_1 \leq \varepsilon_0 \) such that for any \( 0 < \varepsilon \leq \varepsilon_1 \), \( \kappa_\varepsilon^2 + 2\varepsilon - \kappa_\varepsilon^2 + 2 - \kappa_\varepsilon^2 \leq 2 \). In particular, \( \kappa_\varepsilon^2 < 2(1 - \varepsilon^2) \) for \( 0 < \varepsilon \leq \varepsilon_1 \). Now, fix \( 0 < \varepsilon \leq \varepsilon_1 \), and \( \alpha > 0 \) such that
\[
\frac{1}{\alpha + 1} \geq \frac{\kappa_\varepsilon^2}{2(1 - \varepsilon^2)}.
\]
Further, define \( \mathcal{V}(x) := |x|^{-\alpha} \). For \( 0 < |x| < \varepsilon \) we have that
\[
\mathcal{L}\mathcal{V}(x) = -\alpha(-x^3 + x)|x|^{-\alpha-1}\operatorname{sgn}(x) + \frac{\alpha(\alpha + 1)}{2}\sigma^2(x)|x|^{-\alpha-2} \\
\leq -\alpha(-x^3 + x)|x|^{-\alpha-1}\operatorname{sgn}(x) + \frac{\alpha(\alpha + 1)}{2}\kappa_\varepsilon^2|x|^{-\alpha}.
\]
If \( \kappa_\varepsilon = 0 \), then \( \mathcal{L}\mathcal{V}(x) \leq 0 \) for \( 0 < |x| < \varepsilon \), and if \( \kappa_\varepsilon > 0 \), then
\[
\mathcal{L}\mathcal{V}(x) \leq \alpha|x|^{-\alpha+2} - \alpha|x|^{-\alpha} + \alpha(1 - \varepsilon^2)|x|^{-\alpha} = \alpha|x|^{-\alpha+2} - \alpha\varepsilon^2|x|^{-\alpha} < 0
\]
for \( 0 < |x| < \varepsilon \), which proves the assertion.

(ii) We use the same strategy as in (i). Let \( \mathcal{V}(x) := \ln \ln (1/|x| + e) \). For \( 0 < |x| < \delta \), we have that
\[
\mathcal{L}\mathcal{V}(x) = \frac{-(-x^3 + x)\sigma x}{(|x| + e|x|^2)\ln (1/|x| + e)} + \frac{-1 + \ln(1/|x| + e)(1 + 2e|x|)}{(1 + e|x|^2)\ln^2 (1/|x| + e)} \\
= \frac{(1 + e|x|)(|x|^2 - 1)\ln(1/|x| + e) - 1 + \ln(1/|x| + e)(1 + 2e|x|)}{(1 + e|x|^2)\ln^2 (1/|x| + e)} \\
= \frac{|x|^2(1 + e|x|)\ln(1/|x| + e) + e|x|^2\ln(1/|x| + e) - 1}{(1 + e|x|^2)\ln^2 (1/|x| + e)}
\]
Now, by observing that
\[
\lim_{|x| \to 0} |x| \ln \left( \frac{1}{|x|} + e \right) = 0,
\]
we conclude that there is \( 0 < \varepsilon < \delta \) such that \( \mathcal{L}\mathcal{V}(x) \leq 0 \) for \( 0 < |x| < \varepsilon \).
(iii) According to Lemma 2.4 and [Kha12, Theorem 5.6] it suffices to show that there are \( \epsilon > 0 \) and non-negative \( V \in C^2(\mathbb{R} \setminus \{0\}) \), such that \( V(x) > 0 \) for \( x \in \mathbb{R} \setminus \{0\} \), \( \lim_{|x| \downarrow 0} V(x) = 0 \) and \( \mathcal{L}V(x) \leq 0 \) for \( 0 < |x| < \epsilon \). By assumption, there is \( \epsilon_0 > 0 \) such that

\[
V = \mathcal{L}V = \alpha(-x^3 + x)|x|^{\alpha-1}\sigma(x) + \frac{\alpha(\alpha-1)}{2}\sigma^2(x)|x|^{\alpha-2} \\
\leq -\alpha|x|^\alpha + \frac{\alpha(\alpha-1)}{2}\kappa_\epsilon^2|x|^\alpha \\
= -\alpha|x|^{-\alpha+2},
\]

which proves the desired result.

(iv) Define \( V(x) := |x - x_e| \). Then, \( V \in C^2(\mathbb{R} \setminus \{x_e\}) \), \( V(x) > 0 \) for \( |x - x_e| > 0 \), \( \lim_{|x-x_e| \downarrow 0} V(x) = 0 \) and

\[
\mathcal{L}V(x) = (-x^3 + x)\sigma(x - x_e) \leq 0, \quad 0 < |x - x_e| < 1.
\]

Thus, the first assertion follows from Lemma 2.4 and [Kha12, Theorem 5.6].

To prove the second assertion, according to [Kha12, Theorems 5.11 and 5.15], it suffices to show that for \( V(x) := |x - x_e|^\alpha \), \( \alpha > 0 \), we have

\[
\mathcal{L}V(x) \leq -cV(x), \quad x \in \mathbb{R}, \ xx_e > 1.
\]

We have that

\[
\mathcal{L}V(x) = \alpha(-x^3 + x)|x - x_e|^{\alpha-1}\sigma(x - x_e) + \alpha(\alpha-1)\frac{\sigma(x)^2}{2}|x - x_e|^{\alpha-2} \\
= -\alpha|x - x_e|^{\alpha}(x(x + x_e) - (\alpha - 1)\frac{\sigma(x)^2}{2}|x - x_e|^{-2}) \\
\leq -c|x - x_e|^{\alpha} \\
= -cV(x), \quad x \in \mathbb{R}, \ xx_e > 1.
\]

\( \square \)

**Remark 2.5.**

(i) If \( 0 < \alpha \leq 1 \), then the constant \( c \) in Theorem 1.3 (iv) satisfies

\[
2 + (1 - \alpha)\frac{\kappa^2}{2} \geq \frac{c}{\alpha} \geq 2,
\]

where

\[
\kappa := \inf_{\epsilon > 0} \{ \rho : \frac{|\sigma(x)|}{|x-x_e|} \leq \rho, \ 0 < |x - x_e| < \epsilon \}.
\]

(ii) Assume \( 1 < \alpha \leq 2 \), and let

\[
r := \inf\{ \rho \geq 2 : \sigma(x)^2 \leq 2|x - x_e|^2 \text{ for } xx_e \geq \rho \},
\]

which is finite according to (A2). Now, we have

\[
\inf_{xx_e \geq r} \left( x(x + x_e) - (\alpha - 1)\frac{\sigma(x)^2}{2}|x - x_e|^{-2} \right) \\
\geq \inf_{xx_e \geq r} \left( (2 - \alpha)x^2 + xx_e \right)
\]
Further, let
\[ \beta := \inf \{ \gamma \geq 0 : |\sigma(x)| \leq \gamma |x - x_e| \text{ for } 1 \leq xx_e \leq r \}, \]
which is finite due to (A1), \( \sigma(x_e) = 0 \) and compactness of \([1, r]\), and assume that \( \beta < 2/\sqrt{\alpha - 1} \). Hence,
\[
\inf_{1 \leq xx_e \leq r} \left( x(x + x_e) - (\alpha - 1) \frac{\sigma(x)^2}{2} |x - x_e|^{-2} \right)
\geq \inf_{1 \leq xx_e \leq r} \left( x(x + x_e) - (\alpha - 1) \frac{\beta^2}{2} \right)
= 2 - (\alpha - 1) \frac{\beta^2}{2}.
\]

Thus,
\[
2 \geq \frac{c}{\alpha} \geq 2 - (\alpha - 1) \frac{\beta^2}{2}.
\]

(iii) Assume that \( \alpha > 2 \) and
\[
\limsup_{|x| \to \infty} |\sigma(x) |/|x|^2 < \sqrt{2/\sqrt{\alpha - 1}}.
\]
Let
\[ r := \inf \{ \rho \geq 2 : |\sigma(x)| \leq (\sqrt{2/\sqrt{\alpha - 1}})|x - x_e|x \text{ for } xx_e \geq \rho \}, \]
which is finite by assumption. Thus,
\[
\inf_{xx_e \geq r} \left( x(x + x_e) - (\alpha - 1) \frac{\sigma(x)^2}{2} |x - x_e|^{-2} \right) \geq r.
\]
Further, let \( \beta \) be as in (ii). Then again
\[
\inf_{1 \leq xx_e \leq r} \left( x(x + x_e) - (\alpha - 1) \frac{\sigma(x)^2}{2} |x - x_e|^{-2} \right) \geq 2 - (\alpha - 1) \frac{\beta^2}{2},
\]
which implies
\[
2 \geq \frac{c}{\alpha} \geq 2 - (\alpha - 1) \frac{\beta^2}{2}.
\]

3. Stability of general diffusion processes

We start with proofs of Theorem 1.4 and Corollary 1.5.

Proof of Theorem 1.4. Observe first that \( \mathbb{P}^x(X_t \neq x_e, \ t \geq 0) = 1 \) for all \( |x - x_e| > 0 \) (see [Kha12, Lemma 5.3]). Next, for \( n \in \mathbb{N} \) define \( \tau_n := \inf \{ t \geq 0 : |X_t| \geq n \} \). Clearly, under the assumptions of the theorem, the process \( \{ M^f_t \}_{t \geq 0} \) (defined in (1.4)) is a local martingale for every \( f \in C^2(\mathbb{R}^d) \). Now, define \( f : [0, \infty) \times \mathbb{R}^d \setminus \{ x_e \} \to [0, \infty) \) by
\[
f(t, x) := \Phi^{-1}_c(t + \Phi_c \circ \mathcal{V}(x)).
\]
Clearly, \( f \) is continuously differentiable with respect to the first variable on \((0, \infty)\), and twice continuously differentiable with respect to the second variable on \( \mathbb{R}^d \setminus \{ x_e \} \). Next, note that the process
\[
\left\{ \mathcal{V}(X_{t \wedge \tau_n}) + \frac{c}{t} \int_{0}^{t \wedge \tau_n} \varphi \circ \mathcal{V}(X_s)ds \right\}_{t \geq 0}
\]
is a supermartingale for any \( n \in \mathbb{N} \). Indeed, for \( t \geq s \geq 0, \ |x - x_e| > 0 \) (for \( x = x_e \) the assertion is obvious) and \( n \in \mathbb{N} \), we have that

\[
\mathbb{E}^x \left( \mathcal{V}(X_{t \wedge \tau_n}) + c \int_0^{t \wedge \tau_n} \varphi \circ \mathcal{V}(X_u) du \mid \mathcal{F}_s \right) \\
= \mathbb{E}^x \left( \mathcal{V}(X_{t \wedge \tau_n}) + c \int_0^{t \wedge \tau_n} \left( \varphi \circ \mathcal{V}(X_u) + \frac{\mathcal{L}\mathcal{V}(X_u)}{c} - \frac{\mathcal{L}\mathcal{V}(X_u)}{c} \right) du \mid \mathcal{F}_s \right) \\
= \mathcal{V}(X_{s \wedge \tau_n}) - \int_0^{s \wedge \tau_n} \mathcal{L}\mathcal{V}(X_u) du + c \int_0^{s \wedge \tau_n} \left( \varphi \circ \mathcal{V}(X_u) + \frac{\mathcal{L}\mathcal{V}(X_u)}{c} \right) du \\
+ c \mathbb{E}^x \left( \int_0^{t \wedge \tau_n} \varphi \circ \mathcal{V}(X_u) + \frac{\mathcal{L}\mathcal{V}(X_u)}{c} \right) du \mid \mathcal{F}_s \\
\leq \mathcal{V}(X_{s \wedge \tau_n}) + c \int_0^{s \wedge \tau_n} \varphi \circ \mathcal{V}(X_u) du.
\]

Now, by using this fact, [Hai16, Corollary 4.5] states that the process \( \{f(s + t \wedge \tau_n, X_{t \wedge \tau_n})\}_{t \geq 0} \) is also supermartingale for any \( s \geq 0 \) and \( n \in \mathbb{N} \). In particular

\[
\mathbb{E}^x \left( f(t \wedge \tau_n, X_{t \wedge \tau_n}) \right) \leq f(0, x) = \mathcal{V}(x), \quad t \geq 0, \ |x - x_e| > 0.
\]

Consequently, by employing Fatou’s lemma and conservativeness of \( \{X_t\}_{t \geq 0}, \ \{f(t, X_t)\}_{t \geq 0} \) is also a supermartingale. Furthermore, since it is positive, it converges \( \mathbb{P}^x \)-a.s. for all \( |x - x_e| > 0 \). Consequently,

\[
\sup_{t \geq 0} f(t, X_t) \leq Y_x \quad \mathbb{P}^x \text{-a.s.,}
\]

where \( Y_x \) is a strictly positive \( \mathbb{P}^x \)-finite random variable. Finally, we conclude that

\[
\mathcal{V}(X_t) = |X_t - x_e|^\alpha \leq \Phi_{c}^{-1}(\Phi_{c}(Y_x) - t), \quad \mathbb{P}^x \text{-a.s., } x \in \mathbb{R}^d, \ t \geq 0.
\]

\[\square\]

**Proof of Corollary 1.5.**

(i) Fix \( 0 < \varepsilon < r \) and take concave and continuously differentiable function \( \varphi \) such that

\[
\varphi(t) = \begin{cases} 
  t, & 0 \leq t \leq r - \varepsilon \\
  r, & t \geq r + \varepsilon \leq t \leq r 
\end{cases}
\]

The assertion now follows from Theorem 1.4 and by observing that

\[
\lim_{t \to \infty} \frac{\ln \Phi_{c}^{-1}(\Phi_{c}(Y_x) - t)}{t} = -c, \quad \mathbb{P}^x \text{-a.s., } x \in \mathbb{R}^d.
\]

(ii) Fix \( 0 < \varepsilon < r^r_\beta \) and take concave and continuously differentiable function \( \varphi \) such that \( \varphi(t) \leq \varphi_{\beta}(t) \) for \( t > 0 \), and \( \varphi(t) = \varphi_{\beta}(t) \) for \( t \in (r^r_\beta - \varepsilon, r^r_\beta + \varepsilon)^c \). Again, the assertion follows from Theorem 1.4 and by observing that

\[
\lim_{t \to \infty} \frac{\ln \Phi_{c}^{-1}(\Phi_{c}(Y_x) - t)}{t^{\beta}} = -c^{\beta}, \quad \mathbb{P}^x \text{-a.s., } x \in \mathbb{R}^d.
\]

\[\square\]

At the end we also conclude the following.

**Corollary 3.1.** Assume the conditions of Theorem 1.4, and assume there are \( c > 0, r > 0 \) and \( 0 < r < 1 \), such that

\[
\mathcal{L}\mathcal{V}(x) \leq \begin{cases} 
  -c(\mathcal{V}(x))^\gamma, & 0 < \mathcal{V}(x) \leq r \\
  -cr^{\gamma}, & \mathcal{V}(x) \geq r
\end{cases}
\]

Then,

\[
\limsup_{t \to \infty} \frac{\ln |X_t - x_e|}{t^\beta} \leq -\frac{c^\beta}{\alpha}
\]
for any $\beta \geq 1$ and $x \in \mathbb{R}^d$.

Proof. The assertion follows from the fact that
\[ t^\gamma \geq t \quad \text{and} \quad t^\gamma \geq \varphi_\beta(t) \]
for all $t > 0$ small enough. \qed

Proposition 3.2. In the one-dimensional case, the condition in Corollary 3.1 will hold if there are $c$, $r > 0$ and $0 < \gamma < 1$, such that
\begin{enumerate}[(i)]  
  \item $\text{sgn}(x-x_e)b(x) \leq -c|x-x_e|^{\gamma}$ for $|x-x_e| \leq r$;
  \item $\sup_{|x-x_e| \geq r} \text{sgn}(x-x_e)b(x) \leq -cr^{\gamma}$.
\end{enumerate}

Proof. Take $\mathcal{V}(x) := |x-x_e|$. Then, for $|x-x_e| > 0$, we have that
\[ \mathcal{L}\mathcal{V}(x) = \text{sgn}(x-x_e)b(x) \leq \begin{cases} -c(\mathcal{V}(x))^{\gamma}, & 0 < \mathcal{V}(x) \leq r \\ -cr^{\gamma}, & \mathcal{V}(x) \geq r. \end{cases} \]

Observe that in Proposition 3.2 we deal with diffusion processes with H"older continuous coefficients. For existence, uniqueness and structural properties of such processes see [FZ05], [LW14] and [XZ17]. \qed

Acknowledgement

Financial support through the Croatian Science Foundation (under Project 8958) is gratefully acknowledged.

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