Proof of a conjecture of Bauer, Fan and Veldman

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Abstract
For a 1-tough graph $G$ we define $\sigma_3(G) = \min\{\deg(u) + \deg(v) + \deg(w) : \{u, v, w\} \text{ is an independent set of vertices}\}$ and $NC2(G) = \min\{|N(u) \cup N(v)| : d(u, v) = 2\}$. D. Bauer, G. Fan and H.J. Veldman proved that $c(G) \geq \min\{n, 2NC2(G)\}$ for any 1-tough graph $G$ with $\sigma_3(G) \geq n \geq 3$, where $c(G)$ is the circumference of $G$ (D. Bauer, G. Fan and H.J. Veldman, Hamiltonian properties of graphs with large neighborhood unions, Discrete Mathematics, 1991). They also conjectured a stronger upper bound for the circumference: $c(G) \geq \min\{n, 2NC2(G) + 4\}$. In this paper, we prove this conjecture.

Keywords: 1-tough graphs; dominating cycle; longest cycle; k-connected graphs

1. Introduction.

We consider only finite undirected graphs without loops and multiple edges. A graph $G$ is said to be $k$-connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph. A graph $G$ is 1-tough if for every nonempty $S \subset V(G)$ the graph $G - S$ has at most $|S|$ components. All paths and cycles in this paper are simple paths and simple cycles, respectively. A set of vertices of the graph $G$ is independent if no two of its elements are adjacent. The independence number of $G$, denoted by $\alpha(G)$, is defined by setting $\alpha(G) = \max\{|U| : U \subseteq V(G) \text{ independent}\}$. A cycle $C$ of $G$ is called a dominating cycle if the vertices of graph $G - C$ is independent. A graph which contains a dominating cycle is called a dominating graph.
The smallest union degree of order 3 of \( G \), denoted by \( \sigma_3(G) \), is defined by setting

\[
\sigma_3(G) := \begin{cases} 
\min\left\{ \sum_{i=1}^{3} \deg(v_i) : \{v_1, v_2, v_3\} \text{ is independent if } \alpha(G) \geq 3; \right. \\
3(n-1), \left. \quad \text{otherwise,} \right. 
\end{cases}
\]

where \( \deg(v) \) is the degree of vertex \( v \) in \( G \). Denote by \( N(v) \) the set vertices adjacent to \( v \), called the neighbor set of \( v \). We denote \( N(x, y) := N(x) \cup N(y) \), for any two distinct vertices \( x \) and \( y \). Next, we define

\[
NC^2(G) := \begin{cases} 
\min\{|N(u) \cup N(v)| : d(u, v) = 2\}, \quad \text{if } G \text{ is not complete;} \\
n - 1, \quad \text{otherwise,} 
\end{cases}
\]

where \( d(u, v) \) is the distance between \( u \) and \( v \).

Denote by \( c(G) \) the circumference of \( G \), i.e. the length of the longest cycle in \( G \). We have the following lower bound for the circumference of a 1-tough graph due to Bauer, Fan and Veldman.

**Theorem 1** ([2], Theorem 26). If \( G \) is 1-tough and \( \sigma_3(G) \geq n \geq 3 \), then

\[
c(G) \geq \min\{n, 2NC^2(G)\}. \tag{3}
\]

We have a better lower bound on \( c(G) \) due to Hoa in the theorem below.

**Theorem 2** ([6], Theorem 1). If \( G \) is 1-tough graph and \( \sigma_3(G) \geq n \geq 3 \), then there exists an independent set of \( \sigma_3 - n + 5 \) elements \( \{v_0, v_1, \ldots, v_{\sigma_3(G)-n+4}\} \) such that the distance between \( v_0 \) and \( v_i \) equals 2, for any \( 1 \leq i \leq \sigma_3(G) - n + 4 \), and

\[
c(G) \geq \min\left\{ n, 2 \left| \bigcup_{i=0}^{\sigma_3(G)-n+4} N(v_i) \right| + 2 \right\}. \tag{4}
\]

One readily sees that Hoa’s result implies

\[
c(G) \geq \min\{n, 2NC^2(G) + 2\}. \tag{5}
\]

Bauer, Fan and Veldman also conjectured in [2] that

**Conjecture 3** ([2], Conjecture 27). Assume that \( G \) is a 1-tough graph with \( \sigma_3(G) \geq n \geq 3 \). Then

\[
c(G) \geq \min\{n, 2NC^2(G) + 4\}. \tag{6}
\]
Our goal in the paper is to prove this conjecture. We divide the proof of the conjecture into two steps: we show first that \( c(G) \neq 2NC2(G) + 3 \), then show that \( c(G) \neq 2NC2(G) + 2 \) (then the Conjecture 3 follows (5)).

We note that, in [5] and [6], Hoa defined \( \sigma_3(G) := k(n - \alpha(G)) \) or \( \infty \) when \( \alpha(G) \leq 2 \) (as opposed to being \( k(n - 1) \) in our definition above). However, in all definitions we always have the key condition \( \sigma_3(G) \geq n \) when \( \alpha(G) \leq 2 \) and \( n \geq 3 \). This guarantees that we can use the results in [5] and [6] without redefining \( \sigma_3(G) \). We have the same situation in the definition of \( NC2(G) \).

Hoa defined \( NC2(G) = n - \alpha(G) \) when \( G \) is the complete graph \( K_n \) (as opposed to being \( n - 1 \) in our definition). However, in this case, we have \( c(G) = n \), then the Conjecture 3 is obviously true. Again, we do not need to worry about this minor difference in the definitions of \( NC2(G) \).

2. Preliminaries

Let \( C \) be a longest cycle in the graph \( G \) on \( n \geq 3 \) vertices. Denote by \( \overrightarrow{C} \) the cycle \( C \) with a given orientation. Denote by \( x^- \) and \( x^+ \) the predecessor and successor of \( x \) on \( \overrightarrow{C} \), respectively. Further define, \( x^{+i} := (x^{+(i-1)})^+ \) and \( x^{-i} := (x^{-(i-1)})^- \), for \( i \geq 2 \). If \( A \subseteq V(C) \), we define two sets \( A^+ := \{ x : x^- \in A \} \), and \( A^- := \{ x : x^+ \in A \} \).

If \( u \) and \( v \) are on the cycle \( C \), then \( u\overrightarrow{C}v \) denotes the set of consecutive vertices on \( C \) from \( u \) to \( v \) in the direction specified by \( \overrightarrow{C} \). The same vertices, in reverse order, are given by \( v\overleftarrow{C}u \). We consider \( u\overrightarrow{C}v \) and \( v\overleftarrow{C}u \) both as paths and as vertex sets, and we call them \( C \)-paths. In this paper, we will use the notations system in Diestel [4] to represent paths and cycles.

If \( G \) is a non-hamiltonian graph, then we define
\[
\mu(C) := \max \{ \deg(v) : v \in V(G - C) \},
\]
for every cycle \( C \) in \( G \), and
\[
\mu(G) := \max \{ \mu(C) : C \text{ is a longest cycle in } G \}.
\]

**Lemma 4** ([1], Theorem 5). Let \( G \) be a 1-tough graph on \( n \) vertices such that \( \sigma_3(G) \geq n \). Then every longest cycle in \( G \) is a dominating cycle.

**Lemma 5** ([5], Lemma 2). Let \( G \) be a 1-tough graph on \( n \) vertices with \( \sigma_3(G) \geq n \geq 3 \). If \( G \) is non-hamiltonian, then \( G \) has a longest cycle \( C \) such that \( \mu(C) \geq \frac{1}{3}n \).
We get the following lemma by modifying the proof of Theorem 26 in [2]

**Lemma 6.** Let $G$ be a non-hamiltonian 1-tough graph on $n$ vertices with $\sigma_3(G) \geq n \geq 3$. We can find a longest cycle $\overrightarrow{C}$, a vertex $u \notin V(C)$, and a vertex $v$ on $\overrightarrow{C}$, such that $v^+, v^- \in N(u)$. Moreover, for $B := N(u) \cup N(v)$ we have $B \subseteq V(C)$ and $B \cap B^+ = B \cap B^- = \emptyset$.

**Proof.** Follow the argument in the proof of Theorem 26 in [2]. The only difference is that the condition “$G$ is 2-connected and $\sigma_3 \geq n + 2$” is replaced by “$G$ is 1-tough and $\sigma_3 \geq n$”, and we use Lemma 5 above instead of Lemma 22 in [2]. \hfill \square

Let $G$ be a graph satisfying the hypothesis in Lemma 6. Assume that $B := N(u) \cup N(v) = \{b_1, b_2, \ldots, b_m\}$, where the vertices $v^+ \equiv b_1, b_2, \ldots, b_m \equiv v^-$ appear successively on $\overrightarrow{C}$. A $C$-path connecting two successive vertices of set $B$, i.e. having the form $b_i \overrightarrow{C} b_{i+1}$, is called an interval. An interval consisting of $k$ edges is called a $k$-interval, for $k \geq 2$ (by Lemma 6 there no “1-interval”). Let $\mathcal{P}$ be a path in $G$. A vertex $x \in \mathcal{P}$ is called an inner vertex of $\mathcal{P}$ if $x$ is different from the ends of $\mathcal{P}$. We say that two $C$-paths $\mathcal{P}$ and $\mathcal{P}'$ are inner-connected if some inner vertex $x$ of $\mathcal{P}$ is incident to some inner vertex $y$ of $\mathcal{P}'$. If $\mathcal{P}$ and $\mathcal{P}'$ are not inner-connected, then we say they are inner-disconnected.

**Lemma 7** ([6], Lemma 4). Let $G$ be a non-hamiltonian 1-tough graph on $n$ vertices with $\sigma_3(G) \geq n \geq 3$. Let $C$ be a longest cycle in $G$, $u$ a vertex not in $C$, and $v$ a vertex in $C$ so that $v^+, v^- \in N(u)$. Assume in addition that $A$ is the set of all inner vertices of 2-intervals whose end points are in $N(u)$. Then $V(G − C) \cup N(u)^+ \cup N(A)^+$ and $V(G − C) \cup N(u)^- \cup N(A)^-$ are two independent sets.

We have two corollaries of Lemma 7 as follows.

**Corollary 8.** If $G$ satisfies the hypothesis of Lemma 7 and $B := N(u) \cup N(v)$, then $B^+ \cup V(G − C)$ and $B^- \cup V(G − C)$ are two independent sets.

**Proof.** Note that $v \in A$, so we have

\[
B^+ \cup V(G − C) = N(u)^+ \cup N(v)^+ \cup V(G − C)
\subseteq V(G − C) \cup N(u)^+ \cup N(A)^+.
\tag{9}
\]

By Lemma 7, $V(G − C) \cup N(u)^+ \cup N(A)^+$ is independent, so $B^+ \cup V(G − C)$ is also independent. Analogously, $B^- \cup V(G − C)$ is independent. \hfill \square
Corollary 9. If $G$ satisfies the hypothesis of Lemma 7, a $k$-interval, for $k = 2$ or 3, does not inner-connect to any other 2-intervals in $G$.

Proof. An inner vertices $x$ in the $k$-interval is in $B^+ \cup B^-$, for $k = 2, 3$, where $B := N(u) \cup N(v)$. Thus by Corollary 8, $x$ is not adjacent to any inner vertex $y$ of a 2-interval, since $y$ is in $B^+ \cap B^-$. \hfill \Box

The following lemma was also proved in [6].

Lemma 10 ([6], Lemma 9). Assume that $G$ is a non-hamiltonian 1-tough graph on $n$ vertices with $\sigma_3(G) \geq n \geq 3$. Then $G$ contains a longest cycle $C$ avoiding a vertex $u$ with $\deg(u) = \mu(G)$ and $s \geq \sigma_3(G) - n + 4$, where $s$ is the number of 2-intervals whose end points are in $N(u)$.

Note that if $G$ is hamiltonian, then the inequality (6) in Conjecture 3 is obviously true. Therefore, we only need to consider the case $G$ is non-hamiltonian. Follow the light of Lemmas 4, 5, 6, 7, and 10, we assume from now on a setup (S) as follows.

**SETUP (S)**

(S1) $G$ is a non-hamiltonian 1-tough graph on $n$ vertices with $\sigma_3(G) \geq n \geq 3$.

(S2) $\overrightarrow{C}$ is a longest cycle of $G$, $u$ is a vertex not in $V(C)$, and $v$ is a vertex in $V(C)$ such that $v^+, v^- \in N(u)$. Let $B := N(u) \cup N(v) = \{b_1, b_2, \ldots, b_m\}$, where the vertices $v^+ \equiv b_1, b_2, \ldots, b_m \equiv v^-$ appear successively on $\overrightarrow{C}$.

(S3) There at least four 2-intervals whose ends are all in $N(u)$ or are all in $N(v)$.

Given a graph $G$ satisfying assumption (S1), then a setup (S) in $G$ is determined uniquely by a vertices-cycle triple $(u, v, \overrightarrow{C})$.

Remark 11 (Reversing Orientation Trick). If we reverse the orientation of $C$, then the set $B^-$ on $\overrightarrow{C}$ is now the set $B^+$ on $\overrightarrow{C}$. In the proof of Corollary 8, the independence of the set $B^- \cup V(G - C)$ follows the independence of $B^+ \cup V(G - C)$ on the reverse-orientation of $C$. Reversing the orientation (of $C$) is a useful trick that will be used frequently in this paper.
Lemma 12. (Assume a setup \((S)\)) Assume that all intervals on \(\vec{C}\) are pairwise inner-disconnected. Then there are at least two intervals of length greater than 3.

Proof. Recall from the setup \((S)\) that \(B := \{b_1, \ldots, b_m\}\), for some positive integer \(m\).

Assume otherwise that there is at most one interval of length greater than 3. Then other intervals have length at most 3, so their inner vertices are in \(B^+ \cup B^-\). We will show that \(G\) is not a 1-tough graph.

Consider the graph \(G - B\). We have two facts stated below.

(1) Two vertices \(b_i^+\) and \(b_j^+\) are not in the same component of \(G - B\), for any \(1 \leq i \neq j \leq m\).

Indeed, assume otherwise that we can find a path \(P = v_1v_2 \ldots v_t\) in \(G - B\) connecting \(b_i^+\) and \(b_j^+\), i.e. \(v_1 \equiv b_i^+\) and \(v_t \equiv b_j^+\). Since we have at most one interval of length greater than 3, we can assume that \(b_j^+\) is in an interval of length at most 3. By Corollary 8 and the inner-disconnectedness of the intervals, \(v_{t-1}\) must be an inner vertex of the interval \(b_j \vec{C} b_{j+1}\) (besides the inner vertex \(v_t \equiv b_j^+\)). Thus, \(b_j \vec{C} b_{j+1}\) must have length 3, and \(v_{t-1} \equiv b_j^{+2} \equiv b_{j+1} \in B^-\). By the Corollary 8 and the inner-disconnectedness of the intervals again, \(v_{t-2}\) in turn must be an inner vertex of the interval \(b_j \vec{C} b_{j+1}\). However, this implies that \(b_j \vec{C} b_{j+1}\) must have length at least 4, a contradiction.

(2) Two vertices \(b_i^+\) and \(u\) are not in the same component of \(G - B\), for any \(i = 1, 2, \ldots, m\).

Indeed, assume otherwise that there is a path \(P = v_1 \ldots v_t\) in \(G - B\) connecting them, i.e. \(v_1 \equiv b_i^+\) and \(v_t \equiv u\). Then \(v_{t-1}\) is in \(N(u) \subseteq B\), a contradiction to the fact that \(v_{t-1}\) is a vertex in \(G - B\).

From (1) and (2), the graph \(G - B\) has at least \(m+1\) distinct components, so that each of them contains at most one vertex in the set \(\{u\} \cup B^+\). This implies that \(G\) is not 1-tough, which contradicts our setup \((S)\).

We have two new definitions stated below.

Definition 13. A pair of vertices \((x, y)\) in graph \(G\) is called a small pair if \(|N(x, y)| \leq |B| - 1\) and \(d(x, y) = 2\).

Definition 14. Assume \(b_i\) is a vertex in \(B := N(u) \cup N(v)\) such that \(1 < i < m\), i.e. \(b_i \neq v^+, v^-\). A path \(P\) in \(G\) is called a bad path if it has one of the following two forms:
(i) \( \mathcal{P} \) consists of all vertices in \( v^+ \xrightarrow{\tilde{C}} b_i^- \), and the ends of \( \mathcal{P} \) are \( v^+ \) and some vertex \( b_j \in B \), for \( 1 \leq j < i \).

(ii) \( \mathcal{P} \) consists of all vertices in \( b_i^+ \xrightarrow{\tilde{C}} v^- \), and the ends of \( \mathcal{P} \) are \( v^- \) and some \( b_j \in B \), for \( i < j \leq m \).

We have two key results about small pairs and bad paths shown below.

**Proposition 15.** If \( |B| = NC2(G) \), then there are no small pairs.

*Proof.* Assume otherwise that \( |B| = NC2(G) \) and \((x, y)\) is a small pair. Then \( |N(x, y)| \leq |B| - 1 \). By definition of \( NC2(G) \), we have \( |B| = NC2(G) \leq |N(x, y)| \leq |B| - 1 \), a contradiction. \( \square \)

**Proposition 16.** There are no bad paths in \( G \).

*Proof.* Suppose otherwise that there is a bad path \( \mathcal{P} \) of form (i). Assume that \( v^+ \) and \( b_j \) are two ends of \( \mathcal{P} \). We will construct a cycle \( \mathcal{C}' \) that is longer than \( \mathcal{C} \), and then get a contradiction. There are four possible cases as follows:

1. If \( b_i u \) and \( b_j u \in E(G) \), then let \( \mathcal{C}' := v^+vv^-\xrightarrow{\tilde{C}} b_i ub_j \mathcal{P} v^+ \) (see Figure 1(a)).
2. If \( b_i v \) and \( b_j v \in E(G) \), then let \( \mathcal{C}' := v^+uv^-\xrightarrow{\tilde{C}} b_i vb_j \mathcal{P} v^+ \) (see Figure 1(b)).
3. If \( b_j v \) and \( b_i u \in E(G) \), then let \( \mathcal{C}' := v^+ub_i \xrightarrow{\tilde{C}} v^-vb_j \mathcal{P} v^+ \) (see Figure 1(c)).
4. If \( b_j u \) and \( b_i v \in E(G) \), then let \( \mathcal{C}' := v^+vb_i \xrightarrow{\tilde{C}} v^-ub_j \mathcal{P} v^+ \) (see Figure 1(d)).

This completes the proof for the case where \( \mathcal{P} \) is a bad path of form (i). The case where \( \mathcal{P} \) is a bad path of form (ii) follows similarly and is omitted. \( \square \)

**Remark 17** (*Interchanging roles trick*). Consider a new longest cycle \( \tilde{C} := uv^+\xrightarrow{\tilde{C}} v^-u \) (the orientation of \( \tilde{C} \) follows the order of vertices in its representation). It is easy to see that the triple \((v, u, \tilde{C})\) determines a new setup \((S')\) in \( G \) satisfying three conditions \((S_1), (S_2)\) and \((S_3)\), moreover it has the same set \( B \) as \((S)\) does. In particular situations, we need to consider two cases that are the same, except for the roles of \( u \) and \( v \) are interchanged (for example cases (1)-(2) and cases (3)-(4) in the proof of Propositions 16). Then we only need to consider the first case, the second case is obtained...
Figure 1: Illustrating the proof of Proposition 16. Path $\mathcal{P}$ is the dotted one.
by applying again the argument in the first case to the new setup \((S')\). Intuitively, the second case in the original setup \((S)\) becomes the first case in the new setup \((S')\). Thus, for example, in the proof of Proposition 16, we only need to consider two cases (1) and (3), and then the other cases follow naturally from the trick above. We call this trick the **interchanging roles trick**. Together with the reversing orientation trick in Remark 11, the interchanging roles trick is the main ingredient of our case-by-case proofs.

We finish this section by quoting a result due to Woodall, sometimes called Hopping Lemma (see [3] and [7]).

**Lemma 18.** Let \(\vec{C}\) be a cycle of length \(m\) in a graph \(G\). Assume that \(G\) contains no cycle of length \(m + 1\) and no cycle \(C'\) of length \(m\) with \(\omega(G - V(C')) < \omega(G - V(C))\), and \(u\) is an isolated vertex of \(G - V(G)\). Set \(Y_0 = \emptyset\) and, for \(i \geq 1\),

\[
X_i = N(Y_{i-1} \cup \{u\}), \\
Y_i = (X_i \cap V(C))^+ \cap (X_i \cap V(C))^-. 
\]

Set \(X = \bigcup_{i=1}^{\infty} X_i\), and \(Y = \bigcup_{i=1}^{\infty} Y_i\). Then

- (a) \(X \subseteq V(C)\);
- (b) if \(x_1, x_2 \in X\), then \(x_1^+ \neq x_2^+\);
- (c) \(X \cap Y = \emptyset\);
- (d) \(Y\) is independent.

**Proof.** Parts (a), (b), (c) were proved by Woodall [7], and part (d) follows from part (c). Indeed, assume otherwise that there are two vertices \(x, y \in Y\) so that \(xy \in E(G)\). Since \(Y = \bigcup_{i=1}^{\infty} Y_i\), we have \(x \in Y_i\) and \(y \in Y_j\), for some positive integers \(i, j\). Then by definition, we have \(x \in X_{i+1}\) and \(y \in X_{j+1}\), this contradicts part (c).

We note that in the Woodall’s Hopping Lemma 18, \(X_1 \subseteq X_2 \subseteq \ldots \subseteq X\) and \(Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y\).

3. **Step 1: Prove** \(c(G) \neq 2NC2(G) + 3\)

Assume otherwise that \(c(G) = |V(C)| = 2NC2(G) + 3\). By Lemma 6, we have

\[
|V(C)| \geq |B \cup B^+| = |B| + |B^+| = 2|B|.
\]
Moreover, \(|B| = |N(u) \cup N(v)| \geq NC2(G)|, so \(|B|\) must be \(NC2(G) + 1\) or \(NC2(G)\).

Note that a \(k\)-interval contains exactly \(k - 2\) vertices that are not in \(B \cup B^+\). In particular, the \(k\)-interval \(b_i \overrightarrow{C} b_{i+1}\) has \(k - 2\) vertices \(b_i^{+2}, b_i^{+3}, \ldots, b_i^{+(k-1)}\) that are not in \(B \cup B^+\).

If \(|B| = NC2(G) + 1\), then \(|V(C)| = 2|B| + 1\). Thus, there is only one vertex in \(V(C) - (B \cup B^+)\), say \(x\). Clearly, \(x\) must be in a 3-interval \((x^{-2} \overrightarrow{C} x^+),\) and all other intervals have length 2. By Corollary 9, the intervals are pairwise inner-disconnected. Then we have a contradiction from Lemma 12.

Therefore, we have \(|B| = NC2(G)\), and thus \(|V(C)| = 2|B| + 3 = |B \cup B^+| + 3\). It means that there are exactly 3 vertices of the cycle \(\overrightarrow{C}\) that are not in \(B \cup B^+\). We have 3 possibilities for the arrangement of these 3 vertices on the cycle \(\overrightarrow{C}\) as follows.

I. They are in the same interval.
II. They are in two different intervals.
III. They are in three different intervals.

For the sake of contradiction, we will show that all three cases above do not happen.

3.1. Case I

Before investigating this case, we present several lemmas stated below.

**Lemma 19.** If \(x\) is a vertex on \(\overrightarrow{C}\) such that \(x^-x^+ \in E(G)\) and \(x^{+2} \in B\), then \(x\) is not adjacent to any vertex \(y \in B^- - \{x^-, x, x^+\}\). Analogously, if \(x^-x^+ \in E(G)\) and \(x^{-2} \in B\), then \(x\) is not adjacent to any vertex \(y \in B^+ - \{x^-, x, x^+\}\)

**Proof.** Note that \(v^+\) and \(v^-\) are not in \(B^- \cup B^+\), so if \(y \in B^- \cup B^+\) then \(y \notin \{v^-, v^+\}\). We only prove the first statement (then the second statement follows by reversing the orientation of \(\overrightarrow{C}\) as in Remark 11).

Suppose otherwise that \(x^-x^+ \in E(G)\) and \(x\) is adjacent to some vertex \(y \in B^- - \{x^-, x, x^+\}\). If \(y \in x^{+2} \overrightarrow{C} v^{-2}\), then \(v^+ \overrightarrow{C} x^- x^+ xy \overrightarrow{C} x^{+2}\) is a bad path (see Figure 2(a)). If \(y \in v^{+2} \overrightarrow{C} x^{-2}\), then \(v^+ \overrightarrow{C} y x x^- x^- \overrightarrow{C} y^+\) is a bad path (illustrated in Figure 2(b)).
We have a variant of Lemma 19.

**Lemma 20.** Assume that $x, y$ are two distinct vertices on $\overrightarrow{C}$ so that $x^+ = b_i$ and $y^- = b_j$, for some $1 \leq j < i \leq m$, and $xy \in E(G)$. Then $x^-$ and $y^+$ are not adjacent to any inner vertices of 2-intervals on the $C$-path $x^2 \overrightarrow{C} y^{-2}$.

**Proof.** We only need to prove the statement for the vertex $x^-$, then the statement for $y^+$ follows naturally from reversing orientation trick.

Suppose otherwise that there is $x$ is adjacent to some vertex $a \in B^+ \cap B^-$ on $x^2 \overrightarrow{C} y^{-2}$. Apply Hopping Lemma 18 to the graph $G$ with cycle $\overrightarrow{C}$ and vertex $u$ as in our setup. We have

\begin{align*}
X_1 &= N(u), \tag{12a} \\
Y_1 &= N(u)^+ \cap N(u)^- \ni v, \tag{12b} \\
X_2 &= N(Y_1 \cup \{u\}) \supseteq B, \tag{12c} \\
Y_2 &\supseteq B^+ \cap B^- . \tag{12d}
\end{align*}

In particular, $a \in Y_2$, so $x^- \in N(a) \subseteq N(Y_2 \cup \{u\}) = X_3$. By definition, $x \in Y_3$, so $y \in N(x) \subseteq N(Y_3 \cup \{u\}) = X_4 \subseteq X$. However, we already have $y^- \in B \subseteq X_2 \subseteq X$, this contradicts the part (b) of the Hopping Lemma 18. \hfill \Box
Lemma 21. Assume that $x$ is a vertex on $\overrightarrow{C}$ such that $x$ and $x^+$ are on $v^+\overrightarrow{C}v^-$. Then there are no two vertices $a$ and $b$ in $(B^+ \cap B^-) - \{x, x^+\}$ such that $xa$ and $x^+b \in E(G)$, where $a$ and $b$ are not necessarily distinct.

Proof. Suppose otherwise that $xa$ and $x^+b \in E(G)$, for some vertices $a$ and $b$ in $B^+ \cap B^-$. Apply Hopping Lemma 18 to the graph $G$. Similar to Lemma 20, we have (12a)–(12d), and $a, b \in Y_2$. Thus by definition, $x, x^+ \in N(a) \cup N(b) \subseteq N(Y_2 \cup \{u\}) = X_3 \subseteq X$, contradicting the part (b) of the Hopping Lemma 18.

Next, we show that Case I does not happen by contradiction. Suppose otherwise that three vertices of $V(C) - B \cup B^+$ stay in the same interval. Then they are $x_{0-}, x_0$ and $x_{0+}$, for some vertex $x_0 \in V(C)$. Arguing similarly to the case when $|B| = NC2(G) + 1$ in the beginning of Section 3, the cycle $\overrightarrow{C}$ contains one 5-interval and all remaining intervals have length 2. One readily sees that the 5-interval is $x_{0-}^3\overrightarrow{C}x_{0+}^+ = b_i\overrightarrow{C}b_{i+1}$, for some $1 \leq i \leq m - 1$. From Corollary 8, $x_{0-}^2$ and $x_{0+}^+$ are not adjacent to any inner vertices of 2-intervals.

By Lemma 20, if $x_{0-}^2x_{0+}^+ \in E(G)$, then $x_{0-}$ and $x_0$ are not adjacent to any inner vertices of 2-intervals. This implies that the 5-interval $b_i\overrightarrow{C}b_{i+1}$ does not inner-connect to any 2-intervals. Thus, all the intervals are pairwise inner-disconnected (any two 2-intervals are inner-disconnected), contradicting Lemma 12.

Moreover, from Lemma 19, if $x_{0-}^2x_0$ and $x_0x_{0+}^+$ are in $E(G)$, then we have also $x_{0-}$ and $x_0$ are not adjacent to any inner vertices of 2-intervals, and we get the same contradiction as in the previous paragraph.

Finally, if exactly one of $x_{0-}^2x_0$ and $x_0x_{0+}^+$ is in $E(G)$, say $x_{0-}^2x_0 \in E(G)$ and $x_0x_{0+}^+ \notin E(G)$, then $G$ is not 1-tough, a contradiction to our setup (S). Indeed, we consider the graph $G - (B \cup \{x_0\})$. Arguing similarly to the proof of Lemma 12, the vertices of the set

$$\{x_{0-}^2, x_0^+, u\} \cup (B^+ \cap B^-)$$

are in distinct components of $G - (B \cup \{x_0\})$. Thus, $G - (B \cup \{x_0\})$ has at least $|B| + 2$ components, so $G$ is not 1-tough.

From the contradictions in the three previous paragraphs, all $x_{0-}^2x_0^+$, $x_{0-}^2x_0$ and $x_0x_{0+}^+$ are not in $E(G)$. Then at least one of $x_{0-}$ and $x_0$ is adjacent to an inner vertex of some 2-interval (otherwise, all intervals are pairwise inner-disconnected, then we have a contradiction to Lemma 12). On the
other hand, by Lemma 21, exact one of two vertices $x_0^-$ and $x_0$ is adjacent to the inner vertex of some 2-interval, say $x_0$. Again, we have the graph $G - (B \cup \{x_0\})$ has at least $|B| + 2$ components, so $G$ is not 1-tough, a contradiction. This implies that Case I does not happen.

3.2. Case II

We present several supporting lemmas before investigating Case II.

**Lemma 22.** Assume that $a = b_i^+$ and $c = b_j^-$ are two distinct vertices, for some $1 \leq i < j \leq m$, so that $ac \in E(G)$.

(1) If there exists a vertex $x \in \{b_i^{+1}, \ldots, b_j^{-1}\}$, then both $xa^-$ and $xc^+$ are not in $E(G)$. Analogously, we have the same conclusion if there exists a vertex $x \in \{b_i^{-1}, \ldots, b_j^{+1}\}$.

(2) If there exists a vertex $x \in a^+ \overrightarrow{C} b^-$ such that $xv^+2 \in E(G)$, then both $x^+v^+$ and $x^-v^+$ are not in $E(G)$. Analogously, if $xv^-2 \in E(G)$, then both $x^+v^-$ and $x^-v^-$ are not in $E(G)$.

**Proof.** (1) Assume otherwise that there is a vertex $x \in \{b_i^{-1}, \ldots, b_j^{+1}\}$ so that $xa^- \in E(G)$ or $xc^+ \in E(G)$.

If $xa^- \in E(G)$, then $v^+ \overrightarrow{C} a^+ \overleftarrow{C} x^- \overleftarrow{C} ac \overleftarrow{C} x^+$ is a bad path (see Figure 3(a)), which contradicts Proposition 16. If $xc^+ \in E(G)$, then $v^- \overleftarrow{C} c^+ \overrightarrow{C} ac \overleftarrow{C} x^+$ is...
also a bad path (shown in Figure 3(b)), again from Proposition 16 we have a contradiction. This finishes the proof of the first statement.

The case of $x \in \{b_{i+1}^+, \ldots, b_{j-1}^+\}$ is obtained similarly by reversing orientation trick in Remark 11.

(2) We only prove the first statement, the second one follows from reversing orientation trick.

Assume otherwise that there is a vertex $x \in a^+ \overrightarrow{C} c^-$ so that $xv^+2 \in E(G)$. If $x^+v^+ \in E(G)$, then $v^+x^+ \overrightarrow{C} ca \overrightarrow{C} xv^+2 \overrightarrow{C} a^-$ is a bad path (illustrated by Figure 4(a)). If $x^-v^+ \in E(G)$, then $v^+x^- \overrightarrow{C} ac \overrightarrow{C} xv^+2 \overrightarrow{C} a^-$ is a bad path (shown in Figure 4(b)). Thus, by Proposition 16 again, we have a contradiction. Then the first statement follows.

\begin{lemma}
Assume that $a = b_i^+$ and $c = b_j^-$ are two distinct vertices, for some $1 \leq i < j \leq m$, so that $ac \in E(G)$. If there exists a vertex $x \in N(u) \cap N(v)$ on the $C$-path $a^+ \overrightarrow{C} c^-$, then all $x^+v^+$, $x^+v^-$, $x^-v^+$ and $x^-v^-$ are not in $E(G)$.
\end{lemma}

\begin{proof}
Let $x$ be a vertex in $N(u) \cap N(v) \cap a^+ \overrightarrow{C} c^-$. If otherwise $x^+v^+ \in E(G)$, we construct a cycle $C'$ longer than $C$ as follows.

If $a^-u$ and $c^+u \in E(G)$, then let $C' := x^+v^+ \overrightarrow{C} a^-uc^+ \overrightarrow{C} vx \overrightarrow{C} ac \overrightarrow{C} x^+$ (see Figure 5(a)). If $a^-v$, $c^+u \in E(G)$, then let $C' := x^+v^+ \overrightarrow{C} c^-vx \overrightarrow{C} ac \overrightarrow{C} x^+$ (shown in Figure 5(b)). By interchanging roles trick in Remark 17, we get

\end{proof}
Figure 5: Illustrating the proof of Lemma 23.
also a cycle longer than $C$ when $a^-$ and $c^+$ are adjacent to $v$, and when $a^-$ is adjacent to $u$ and $c^+$ is adjacent to $v$.

Thus, in all cases, we get a cycle longer than $C$, a contradiction. This implies that $x^+v^+ \notin E(G)$.

If $x^+v^- \in E(G)$, we may construct a cycle $C''$ longer than $C$, then we have a contradiction. By the interchanging roles trick, we only need to consider two following cases. If $a^-u$ and $c^+u \in E(G)$, then let $C'' := x^+v^-\overset{a^-}{\longrightarrow}^{C} \overset{c^+}{\longrightarrow}^{u} \overset{a^+}{\longrightarrow}^{x} \overset{c^-}{\longrightarrow}^{v} \overset{a^-}{\longrightarrow}^{u}$ (illustrated in Figure 5(c)). If $a^-v$ and $c^+u \in E(G)$, then $C'' := x^+v^-\overset{a^-}{\longrightarrow}^{C} \overset{c^+}{\longrightarrow}^{u} \overset{a^+}{\longrightarrow}^{x} \overset{c^-}{\longrightarrow}^{v} \overset{a^-}{\longrightarrow}^{u}$ (see Figure 5(d)). This shows that $x^+v^- \notin E(G)$.

Finally, by reversing orientation trick in Remark 11, we get $x^-v^+, x^-v^- \notin E(G)$. $\square$

**Lemma 24.** Assume that $a = b_p^-$ and $c = b_q^+$ are two distinct vertices, for some $1 \leq p < q \leq m$, so that $ac \in E(G)$. Then

1. Either $b_p, b_q \in N(u) \setminus N(v)$ or $b_p, b_q \in N(v) \setminus N(u)$.
2. All $b_p v^{-2}, b_p v^{+2}, b_q v^{-2}$ and $b_q v^{+2}$ are not in $E(G)$.

**Proof.** (1) Suppose otherwise that one of $b_p$ and $b_q$ is adjacent to $u$ and the other is adjacent to $v$. We only consider the case $b_p v, b_q u \in E(G)$ (the other case follows from interchanging roles trick). We have the cycle $C' := v b_p \overset{c^-}{\longrightarrow}^{CB} b_q u v^- \overset{C}{\longrightarrow}^{u} \overset{C}{\longrightarrow}^{a} \overset{C}{\longrightarrow}^{v}$ is longer than $C$ (see Figure 6(a)), a contradiction, and the statement follows.

(2) By part (1), we have $b_p u$ and $b_q u$ are both in $E(G)$, or $b_p v$ and $b_q v$ are both in $E(G)$. Again, we only need to consider the first case (the latter case can be obtained by applying interchanging roles trick).
If $b_p v^+ v^2 \in E(G)$, the cycle $C'' := b_p v^+ v^2 \overrightarrow{C} a \overrightarrow{C} v^- u b_q \overrightarrow{C} b_p$ is longer than $C$, a contradiction (see Figure 6(b)). If $b_p v^- v \in E(G)$, the cycle $C'' := b_p v^- v \overrightarrow{C} a \overrightarrow{C} v^- u b_q \overrightarrow{C} b_p$ is longer than $C$, contradicting the choice of $C$ (see Figure 6(c)). This implies that $b_p v^+ v^2, b_p v^- v \notin E(G)$.

By reversing orientation of $\overrightarrow{C}$ as in Remark 11, we get also $b_q v^+ v^2, b_q v^- v \notin E(G)$. \hfill \Box

Next, we show that Case II does not happen by contradiction. Assume otherwise that three vertices of $V(C) - B \cup B^+$ are falling into two different intervals. Then they are $x_0, x_0^+, y_0^-$, for some vertices $x_0$ and $y_0$ on $\overrightarrow{C}$ so that $y_0 \notin \{x_0^-, x_0^+, y_0^-, y_0^+, y_0\}$. Arguing similarly to Case I, the cycle $\overrightarrow{C}$ consists of a 3-interval $(y_0 \overrightarrow{C} y_0^+ = b_j \overrightarrow{C} b_{j+1})$, a 4-interval $(x_0 \overrightarrow{C} x_0^+ = b_i \overrightarrow{C} b_{i+1})$, and several 2-intervals. Without loss of generality, we assume that $1 \leq i < j \leq m$ (the case $i > j$ is obtained from this case by reversing orientation trick). Note that in this case we have $V(C) - B = B^- \cup B^+ \cup \{x_0\}, B^- - B^+ = \{x_0^+, y_0^+\}$, and $B^- - B^- = \{x_0^-, y_0\}$.

**Proposition 25.** $x_0$ is not adjacent to any inner vertices of 2-intervals.

**Proof.** Assume otherwise that $x_0$ is adjacent to some inner vertex $a$ of a 2-interval. Apply Hopping Lemma 18 to $G$. Similar to Lemma 20, we have (12a)–(12d), and $a \in Y_2$. We have also $x_0 \in N(a) \subseteq N(Y_2 \cup \{u\}) = X_3$. By definition, $x_0^+, x_0^- \in Y_3$. In particular, by part (d) of the Hopping Lemma, we have $x_0^+ x_0^- \not\in E(G)$.

If $x_0^+ y_0 \in E(G)$, then $y_0 \in N(x_0^+) \subseteq N(Y_3 \cup \{u\}) = X_4 \subseteq X$. However, we also have $y_0^- \in B \subseteq X_2 \subseteq X$. It means that we have two consecutive vertices on $\overrightarrow{C}$ lying in $X$, this contradicts the part (b) of the Hopping Lemma 18. Thus, we have $x_0^+ y_0 \not\in E(G)$. Similarly, we obtain $x_0^+ y_0 \not\in E(G)$.

Finally, by Corollary 8, $x_0^+ y_0^+$ and $x_0^- y_0$ are not in $E(G)$. Therefore, similar to the second last paragraph of Section 3.1, the graph $G - (B \cup \{x_0\})$ has at least $|B| + 2$ components, i.e. $G$ is not 1-tough, a contradiction. Then the proposition follows. \hfill \Box

We also have the following fact about the inner vertices of the 3-interval and the 4-interval.

**Proposition 26.** $x_0^+ x_0^- \notin E(G).$ Moreover, exactly one of $y_0^+ x_0^-$ and $y_0 x_0^+$ is in $E(G)$.  

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Proof. By Lemma 19, if $x_0^+ x_0^- \in E(G)$, then $x_0$ is not adjacent to any inner vertices of 2-intervals and the 3-interval. By Corollary 9, the intervals on $\vec{C}^b$ are pairwise inner-disconnected, by Lemma 12, we have a contradiction. Therefore, $x_0^+ x_0^- \notin E(G)$.

If both $y_0$ and $y_0^+$ are not adjacent to $x_0^-$ and $x_0^+$, then by arguing similarly to the second last paragraph in Section 3.1, the graph $G - (B \cup \{x_0\})$ has at least $|B| + 2$ components. Hence, $G$ is not 1-tough, a contradiction. This implies that at least one of $y_0$ and $y_0^+$ is adjacent to $x_0^-$ or $x_0^+$. Moreover, by Corollary 8, $y_0 x_0^-$, $y_0^+ x_0^+ \notin E(G)$, so at least one of $y_0^+ x_0^-$ and $y_0 x_0^+$ is in $E(G)$.

However, if both $y_0^+ x_0^-$ and $y_0 x_0^+$ are in $E(G)$, then $b_{j+1} \vec{C}^b y_0 x_0^+ x_0^- y_0^+ \vec{C}^b v^-$ is a bad path, which contradicts what we got from Proposition 16. This implies that exactly one of $y_0^+ x_0^-$ and $y_0 x_0^+$ is in $E(G)$.

**Proposition 27.** $y_0^+ x_0^- \notin E(G)$.

Proof. Assume otherwise that $y_0^+ x_0^- \in E(G)$. We have the following claims.

**Claim 28.** $b_{j+1} \equiv b_j$.

Proof. Suppose otherwise that $b_{j+1} \neq b_j$, then $b_{j+1} \in B^+ \cap B^-$. Recall that we have $V(C) - B = B^- \cup B^+ \cup \{x_0\}$, $B^- - B^+ = \{x_0^+, y_0^+\}$, and $B^+ - B^- = \{x_0^-, y_0\}$.

By Corollary 8 and Proposition 25, $N(b_{j+1}^+) \subseteq V(C)$ and $N(b_{j+1}^+) \cap (B^+ \cup B^- \cup \{x_0\}) = \emptyset$. Thus, $N(b_{j+1}^+) \subseteq B$.

By Proposition 26, $y_0 x_0^- \notin E(G)$. Since $x_0^+ \in B^-$, Corollary 8 implies that $N(x_0^+) \cap (B^- \cup V(G - C)) = \emptyset$. Thus, $N(x_0^+) \subseteq B \cup \{x_0\}$.

Finally, from Lemma 22, all $b_{j+1}^- b_i$, $b_{j+1}^- b_{j+1}$, $x_0^- b_i$, $x_0^+ b_{j+1}$ are not in $E(G)$. Thus, $N(b_{j+1}^+, x_0^-) \subseteq B \cup \{x_0\} - \{b_i, b_{j+1}\}$. This deduces that $(b_{j+1}^+, x_0^-)$ is a small pair, contradicting Proposition 15. □

We have now all 2-intervals stay in the $C$-path $b_{j+1} \vec{C}^b b_i$. The assumption $(S_3)$ of our setup implies that the later $C$-path contains at least four good 2-intervals.

**Claim 29.** $b_j \notin N(u) \cap N(v)$.

Proof. Suppose otherwise that $b_j$ is adjacent to both $u$ and $v$. We show that $(x_0^+ b_{j+1}^-)$ is still a small pair. Note that we have now $b_{j+1}^+ \equiv y_0$.  □
We have $y_0x_0 \notin E(G)$ (otherwise $b_j \overrightarrow{C} x_0y_0y_0^+x_0^- \overrightarrow{C} v^+$ is a bad path, a contradiction to Proposition 16), and $y_0x_0^+ \notin E(G)$ (by Proposition 26). Since $y_0 \in B^+$, we have $N(y_0) \subseteq B \cup \{y_0^+\}$.

Moreover, from the proof of Claim 28, $N(x_0^+) \subseteq B \cup \{x_0\}$, and $x_0^+, b_{j+1} \equiv y_0$ are not adjacent to $b_i, b_{j+1}$.

Finally, by Lemma 23, all $y_0v^+, y_0v^-, x_0^+v^+$ and $x_0^-v^-$ are not in $E(G)$. Therefore,

$$N(y_0, x_0^+) \subseteq B \cup \{y_0^+, x_0\} - \{b_i, b_{j+1}, v^+, v^-=\}. \quad (13)$$

Since $b_{j+1} \overrightarrow{C} b_i$ contains at least four (good) 2-intervals, we have $b_i \neq v^+$ or $b_{j+1} \neq v^-$. Thus, $|\{b_i, b_{j+1}, v^+, v^-=\}| \geq 3$, so $|N(y_0, x_0^+)\mid \leq |B| - 1$. This implies that $(y_0, x_0^+)$ is a small pair, a contradiction to Proposition 15, and the claim follows.

Claim 30. If $b_{j+1} \neq v^-$, then $v^-x_0 \notin E(G)$. Analogously, if $b_i \neq v^+$, then $v^+x_0 \notin E(G)$.

**Proof.** Assume that $b_{j+1} \neq v^-$, then $v^-x_0$ is the inner vertex of some 2-interval. Therefore, the first statement follows from Proposition 25. Similarly we have the second statement.

Claim 31. If $v^+ \neq b_i$, then $v^+b_{i+1} \notin E(G)$. Analogously, if $v^- \neq b_{j+1}$, then $v^-b_{i+1} \notin E(G)$.

**Proof.** We only need to prove the first statement (then the second one follows from the reversing orientation trick).

Suppose otherwise that $v^+ \neq b_i$ and $v^+b_{i+1} \in E(G)$. From Lemma 22(2), we have $x_0^+v^+, y_0v^+ \notin E(G)$. Similar to (13), we have

$$N(x_0^+, y_0) \subseteq B \{y_0^+, x_0\} - \{b_i, b_{j+1}, v^+=\}.$$ 

Since $v^+ \neq b_i$, we obtain $|\{b_i, b_{j+1}, v^+=\}| = 3$. Thus, $(x_0^+, y_0)$ is a small pair, which contradicts Proposition 15.

As mentioned in the proof of Claim 29, we have $v^+ \neq b_i$ or $v^- \neq b_{j+1}$ (due to the assumption $(S_3)$ of our setup on the number of (good) 2-intervals). We have $v^+ \in B^+ \cap B^-$ or $v^- \in B^+ \cap B^-$, respectively, i.e. $\{v^+, v^-=\} \cap (B^+ \cap B^-) \neq \emptyset$. Assume that $w_1$ is a vertex in the later intersection. Moreover, by Claim 29, we can denote by $w_2$ the vertex in $\{u, v\}$ which is adjacent to $b_{i+1} \equiv b_j$. 

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By Claims 29, 30 and 31, we have $N(w_1, w_2) \subseteq B – \{b_{i+1}\}$. Thus $(w_1, w_2)$ is a small pair, a contradiction. Hence, the proposition follows.

From Propositions 26 and 27, we have $x_0^+y_0 \in E(G)$.

**Proposition 32.** \{b_{i+1}, b_j\} ∉ N(u) and \{b_{i+1}, b_j\} ∉ N(v).

*Proof.* Assume otherwise that $b_{i+1}$ and $b_j$ are both adjacent to $u$, or both adjacent to $v$, say $b_{i+1}v$, $b_jv \in E(G)$ (the other case follows from interchanging roles trick).

By Lemma 24(1), we have $ub_{i+1}$, $ub_j \notin E(G)$. By Lemma 24(2), all $v^+b_{i+1}$, $v^-b_{i+1}$, $v^+b_j$, and $v^-b_j$ are not in $E(G)$.

Moreover, if $b_{j+1} \neq v^-$, then $v^-$ is the inner vertex of some 2-interval, so is in $B^+ \cap B^-$. By Corollary 8, $v^-$ is not adjacent to any vertices in $B^- \cup B^+$, and by Proposition 25, $v^-x_0 \notin E(G)$. Thus, $N(u, v^-) \subseteq B – \{b_{i+1}, b_j\}$, so $(u, v^-)$ is a small pair, a contradiction to Proposition 15. This implies that $b_{j+1} \equiv v^-$, so $v^- \equiv y_0^+$. Similarly, we have $v^+ \equiv b_i$.

We have now $N(u, v^-) = N(u, y_0^+) \subseteq (B \cup \{y_0\}) – \{b_{i+1}, b_j\}$. If $b_{i+1} \neq b_j$, then $|N(u, v^-)| \leq |B| – 1$, so $(u, v^-)$ is still a small pair, a contradiction. Thus, $b_{i+1} = b_j$. However, there is only one 2-interval on $\overrightarrow{C}$, contradicting the assumption $(S_3)$ in our setup, which completes the proof of the proposition.

One readily sees that Lemma 24(1) contradicts Proposition 32. This implies that Case II does not happen.

3.3. Case III

We prove the following supporting lemma.

**Lemma 33.** Assume that all intervals on $\overrightarrow{C}$ have length 2 or 3.

(a) Assume in addition that $v_0$ is a vertex different from $v$, so that $v_0^+, v_0^- \in N(u)$. Then $(u, v_0, \overrightarrow{C})$ determines a new setup $(S^*)$ of $G$. Moreover, $N(u) \cup N(v_0) = N(u) \cup N(v) = B$.

(b) Assume in addition that $v_0$ is a vertex different from $v$, so that $v_0^+, v_0^- \in N(v)$. Then $(v, v_0, \overrightarrow{C})$ determines a new setup $(S^{**})$. Moreover, $N(v) \cup N(v_0) = N(u) \cup N(v) = B$.

*Proof.* Since part (b) is obtained from part (a) by applying interchanging roles trick, we only present the proof of part (a).
One readily verifies that \((S^\ast)\) satisfies \((S_1), (S_2)\) and \((S_3)\). Next, we show that \(B_0 := N(u) \cup N(v_0) = B\).

Since all intervals have length at most 3, the inner vertices of the intervals are in \( B^+ \cup B^- = V(C) - B \). By Corollary 8, \( v_0 \) is not adjacent to any vertices of set \( B^+ \cup B^- \cup V(G - C) \). Thus, \( N(v_0) = N(v_0) \cap V(C) \subseteq N(u) \cup N(v) \), so \( B_0 = N(u) \cup N(v_0) \subseteq N(u) \cup N(v) = B \). Apply the same argument to the setup \((S^\ast)\), we have also \( B \subseteq B_0 \). Thus, \( B = B_0 \).

We call a 2-interval having both ends in \( N(u) \) or \( N(v) \) a good 2-interval. Intuitively, Lemma 33 allows us to “refine” our setup by relocating the vertex \( v \) to any other inner vertices of good 2-intervals.

**Remark 34.** Let \( b_p \vec{C} b_{p+1} \) and \( b_q \vec{C} b_{q+1} \) be two 3-intervals on \( \vec{C} \), for \( 1 \leq p < q \leq m \), whose inner vertices are \( x_1, x_1^+ \) and \( x_2, x_2^+ \), respectively. If these 3-intervals are inner-connected, then exactly one of \( x_1x_2^+ \) and \( x_1^+x_2 \) is in \( E(G) \). Indeed, from Corollary 8, \( x_1x_2 \) and \( x_1^+x_2^+ \not\in E(G) \). However, if both \( x_1x_2^+ \) and \( x_1^+x_2 \) are in \( E(G) \), then \( b_q \vec{C} x_1^+x_2^+x_1 \vec{C} v^+ \) is a bad path, a contradiction.

Again, we show that Case III does not occur. Assume otherwise that the three vertices of \( V(C) - (B^+ \cup B^-) \) stay in three different intervals. Then they are \( x_0^+, y_0^+, \) and \( z_0^+ \), for some vertices \( x_0, y_0, \) and \( z_0 \) on \( \vec{C} \) so that \( y_0 \not\in \{ x_0^+, x_0^- \} \) and \( z_0 \not\in \{ x_0^+, x_0^-, y_0^+, y_0^- \} \). Arguing similarly to the Case I and Case II, \( \vec{C} \) consists of three 3-intervals \( (x_0^+ \vec{C} x_0^-)^{i+2} = b_i \vec{C} b_{i+1}, y_0^- \vec{C} y_0^+ = b_j \vec{C} b_{j+1}, \) and \( z_0^- \vec{C} z_0^+ = b_k \vec{C} b_{k+1} \), for \( 1 \leq i < j < k \leq m - 1 \) and several 2-intervals.

We notice that all 2-intervals fall into three \( C \)-paths \( b_{i+1} \vec{C} b_j \), \( b_{j+1} \vec{C} b_k \), and \( b_{k+1} \vec{C} b_i \).

Note that we have in this case \( V(C) - B = B^+ \cup B^- \), \( B^+ - B^- = \{ x_0^+, y_0^+, z_0^+ \} \), and \( B^- - B^+ = \{ x_0, y_0, z_0 \} \).

We have the following result by arguing similarly to the proofs of Propositions 27 and 32 in Case II.

**Proposition 35.** \( b_i \vec{C} b_{i+1} \) and \( b_k \vec{C} b_{k+1} \) are inner-disconnected.

**Proof.** Assume otherwise that \( b_i \vec{C} b_{i+1} \) inner-connects to \( b_k \vec{C} b_{k+1} \). By Remark 34, we only need to consider 2 cases as follows.

**Case 1.** \( x_0^+ z_0^+ \in E(G) \).

Similar to Claim 28 we have

**Claim 36.** \( b_{i+1} \equiv b_j \) and \( b_{j+1} \equiv b_k \).
Proof. Since the proofs of two statements are essentially the same, we only prove that \( b_{i+1} = b_i \). Assume otherwise that \( b_{i+1} \neq b_i \). Similar to the proof of Claim 28, we consider the pair of vertices \( (x_0^+, b_{i+1}^-) \).

We have \( x_0^+ z_0 \notin E(G) \) (by Remark 34), \( x_0^+ y_0 \notin E(G) \) (otherwise we have a bad path \( v^+ \overrightarrow{C} x_0^+ \overrightarrow{z^+} \overrightarrow{C} y_0 x_0^+ \overrightarrow{C} b_j \), a contradiction); \( x_0^+ b_i, x_0^+ b_{k+1}, b_{i+1}^+ b_i, b_{i+1}^- b_{k+1} \notin E(G) \) (by Lemma 22).

Similar to Claim 28, we have

\[
N(x_0^+, b_{i+1}^-) \subseteq B \cup \{x_0\} - \{b_i, b_{k+1}\}.
\]

(14)

This implies that \( (x_0^+, b_{i+1}^-) \) is a small pair, contradicting Proposition 15. Then the first statement of the claim follows.

All 2-intervals are now in the \( C \)-path \( b_{k+1} \overrightarrow{C} b_i \). Thus, by the assumption \((S_3)\) in our setup, the later \( C \)-path contains at least four good 2-intervals.

Claim 37. \( v^+ \equiv b_i \) or \( v^- \equiv b_{k+1} \).

Proof. Assume otherwise that \( v^+ \neq b_i \) and \( v^- \neq b_{k+1} \).

If \( b_{i+1} \) is adjacent to both \( u \) and \( v \), then by Lemma 23, we have \( x_0^+ \) and \( y_0 \) are not adjacent to \( v^+ \) and \( v^- \). Moreover, \( x_0^+ z_0 \notin E(G) \) (by Remark 34), \( x_0^+ y_0 \notin E(G) \) (otherwise \( v^+ \overrightarrow{C} x_0^+ \overrightarrow{z^+} \overrightarrow{C} y_0 x_0^+ \overrightarrow{C} b_j \) is a bad path, a contradiction); \( x_0^+ b_i, x_0^+ b_{k+1}, y_0 b_i, y_0 b_{k+1} \notin E(G) \) (by Lemma 22). Therefore,

\[
N(x_0^+, y_0) \subseteq B \cup \{x_0, y_0^+, z_0^+\} - \{b_i, b_{k+1}, v^+, v^-\}
\]

(15)

(the vertex \( b_{i+1}^+ \equiv y_0 \) may be adjacent to \( y_0^+ \) and \( z_0^+ \)). Since \( v^+ \neq b_i \) and \( v^- \neq b_{k+1} \), we have \(|\{b_i, b_{k+1}, v^+, v^-\}| = 4\), so \( (x_0^+, y_0) \) is a small pair, a contradiction to Proposition 15. Thus, one of \( u \) and \( v \) is not adjacent to \( b_{i+1} \).

If \( b_{i+1} v^{+2}, b_{i+1} v^{-2} \in E(G) \), then \( x_0^+ \) and \( b_{i+1}^+ \) are not adjacent to \( v^+ \) and \( v^- \) (by Lemma 22(2)). It means that we still have (15), so have a contradiction. Hence, at least one of \( v^{+2} \) and \( v^{-2} \) is not adjacent to \( b_{i+1} \).

Finally, from the facts in the previous two paragraphs, we can denote by \( w_1 \) (resp., \( w_2 \)) the vertex in \( \{u, v\} \) (resp., in \( \{v^{+2}, v^{-2}\} \)) which is not adjacent to \( b_{i+1} \). By definition \( N(w_1, w_2) \subseteq B - \{b_{i+1}\} \), so \( (w_1, w_2) \) is a small pair, a contradiction.

Since \( b_{k+1} \overrightarrow{C} b_i \) contains at least four good 2-intervals, we can find an inner vertex \( v_0 \) of a good 2-interval so that \( v_0^+ \neq b_i \) and \( v_0^- \neq b_{k+1} \). Consider the setup \((S^*)\) determined by \( (u, v_0, \overrightarrow{C}) \) or the setup \((S^{**})\) determined by
\((v, v_0, \overrightarrow{C})\) as in Lemma 33, depending on whether \(v_0^+, v_0^-\) are in \(N(u)\) or \(N(v)\). Apply the argument in Claim 37 above to the new setup, we get \(v_0^+ \equiv b_i\) or \(v_0^- \equiv b_{k+1}\), that contradicts the choice of the vertex \(v_0\). This contradiction implies \(x_0z_0^+ \notin E(G)\).

Case 2. \(x_0^+ z_0 \in E(G)\).

By Lemma 24, one of \(u\) and \(v\) is not adjacent to \(b_{i+1}\) and \(b_k\), say \(b_{i+1}u\), \(b_jv \notin E(G)\) (the other case follows from interchanging roles trick). Arguing similarly to the proof of Proposition 32. We get \(v^+ \equiv b_i\) and \(v^- \equiv b_{k+1}\).

Then \(v^2 \equiv x_0\) and \(v^2 \equiv z_0^+\). By Lemma 24(2), \(v^- \equiv y_0^+\) is not adjacent to \(b_{i+1}\) and \(b_k\). Moreover, we have \(x_0y_0^+ \notin E(G)\) (otherwise we have a bad path \(v^+ \overrightarrow{C} x_0y_0^+ \overrightarrow{C} z_0^+ x_0 \overrightarrow{C} b_j\), a contradiction). Therefore,

\[
N(u, v^-) = N(u, y_0^+) \subseteq B \cup \{y_0\} - \{b_{i+1}, b_k\}.
\] (16)

However, we have now \((u, y_0^+)\) is a small pair, a contradiction to Proposition 15. This finishes the proof of the proposition. \(\square\)

Given \(b_i \overrightarrow{C} b_{i+1}\) and \(b_k \overrightarrow{C} b_{k+1}\) are inner-disconnected by Proposition 35, we get a similar result to Proposition 35.

**Proposition 38.** \(b_i \overrightarrow{C} b_{i+1}\) and \(b_j \overrightarrow{C} b_{j+1}\) are inner-disconnected. Analogously, \(b_k \overrightarrow{C} b_{k+1}\) and \(b_j \overrightarrow{C} b_{j+1}\) are inner-disconnected.

The proof of Proposition 38 is omitted, since it is almost identical to the proof of Proposition 35.

However, by Propositions 35 and 38, together with Corollary 9, all the intervals on \(\overrightarrow{C}\) are now pairwise inner-disconnected, so by Lemma 12 we have a contradiction. This finishes the proof that \(|V(C)| \neq 2 NC2(G) + 3\).

4. Step 2: Prove \(c(G) \neq 2 NC2(G) + 2\).

Assume that \(|V(C)| = 2 NC2(G) + 2\). By Lemma 6, we have \(|V(C)| \geq |B \cup B^+| = |B| + |B^+| = 2|B|\). Moreover, \(|B| = |N(u) \cup N(v)| \geq NC2(G)\), so \(|B| = NC2(G)\) or \(NC2(G) + 1\). If \(|B| = NC2(G) + 1\), then \(V(C) = B \cup B^+\).

It implies that the vertices of \(B\) divide \(C\) into 2-intervals. From Lemma 12 and Corollary 9, we have a contradiction. Hence, \(|B| = NC2(G)\), so \(|V(C)| = |B \cup B^+| + 2\). It means that there are two vertices in \(V(C) - B \cup B^+\).

We have two cases to distinguish, depending on the arrangement of these vertices on \(\overrightarrow{C}\):
I They are in the same interval.
II They are in two different intervals.

By the sake of contradiction, we will show that all of these cases do not happen.

Case I.

Assume that the two vertices of \( V(C) - B \cup B^+ \) stay in the same interval. Then the vertices are \( x_0 \) and \( x_0^+ \), for some vertex \( x_0 \in V(C) \). The cycle \( \overrightarrow{C} \) consists of one 4-interval \( (x_0^{-2} \overrightarrow{C} x_0^{+2} = b_i \overrightarrow{C} b_{i+1}, \) for some \( 1 \leq i \leq m - 1 ) \) and several 2-intervals.

From Lemma 19, if \( x_0^{-} x_0^{+} \in E(G) \) then \( x_0 \) is not adjacent to any inner vertices of 2-intervals. Then by Corollary 8, all intervals are pairwise inner-disconnected, and from Lemma 12 we have a contradiction. Thus, \( x_0^{-} x_0^{+} \notin E(G) \). However, similar to the second last paragraph of Section 3.1, the graph \( G - (B \cup \{x_0\}) \) has at least \( |B| + 2 \) components, so \( G \) is not 1-tough, a contradiction. Thus, the Case I does not happen.

Case II.

Assume that the two vertices of \( V(C) - B \cup B^+ \) fall into two different intervals. Then the vertices are \( x_0 \) and \( z_0 \), where \( x_0 \) and \( z_0 \) are some vertices in \( V(C) \) and where \( z_0 \notin \{x_0^{-}, x_0^{+}\} \). The cycle \( \overrightarrow{C} \) consists of now two 3-intervals \( (x_0^{-} \overrightarrow{C} x_0^{+2} = b_i \overrightarrow{C} b_{i+1} \) and \( z_0^{-} \overrightarrow{C} z_0^{+2} = b_k \overrightarrow{C} b_{k+1} ) \) and several 2-intervals. Without loss of generality, we assume that \( 1 \leq i < k \leq m - 1 \).

Similar to Proposition 35 we have the following fact.

**Proposition 39.** \( b_i \overrightarrow{C} b_{i+1} \) and \( b_k \overrightarrow{C} b_{k+1} \) are inner-disconnected.

Since the proofs Propositions 35 and 39 are almost identical, we omit the proof of Proposition 39.

By Propositions 39 and Corollary 9, all the intervals on \( \overrightarrow{C} \) are pairwise inner-disconnected, so by Lemma 12 we have a contradiction. Then we deduce that Case II does not hold, and also finish the proof that \( |V(C)| \neq 2NC2(G) + 2 \).

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