$p$–species integrable reaction–diffusion processes

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Abstract

We consider a process in which there are $p$–species of particles, i.e. $A_1, A_2, \cdots, A_p$, on an infinite one–dimensional lattice. Each particle $A_i$ can diffuse to its right neighboring site with rate $D_i$, if this site is not already occupied. Also they have the exchange interaction $A_j + A_i \rightarrow A_i + A_j$ with rate $r_{ij}$. We study the range of parameters (interactions) for which the model is integrable. The wavefunctions of this multi–parameter family of integrable models are found. We also extend the 2–species model to the case in which the particles are able to diffuse to their right or left neighboring sites.

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1 Introduction

Our understanding of nonequilibrium statistical physics is far behind that for the equilibrium theory. Even simple models may pose a formidable problem if one wants to approach them analytically. As an interesting example of stochastic models, which may be investigated analytically in some few cases, are one–dimensional reaction–diffusion processes which are of both theoretical and experimental interest in a very wide context of physics and chemistry, such as stochastic spin flip dynamics [1], traffic flow [2, 3], the kinetics of bipolymerization [4, 5], reptation of DNA in gels [6, 7], interface growth [8, 9], diffusion in zeolites [10, 11], and many other phenomena.

Asymmetric Simple Exclusion Processes (ASEP) in one dimension, is one of the simplest example of a driven diffusion system [12, 13]. For example, the totally ASEP model describes a process in which each lattice site can be occupied by at most one particle and the particles hop to their right neighboring site if they are not already occupied, with a rate which is the same for all particles, otherwise the attempted move is rejected. The dynamics of these models can be fully specified by a master equation and an appropriate boundary condition, which imposed on the probabilities appear in the master equation. Using the coordinate Bethe ansatz, the author of [14] has exactly obtained the N–particle conditional probabilities of totally ASEP, in which the particles can move to left and right with different rates.

Now the interesting point is that if one changes the boundary condition, without altering the master equation, one can model another reaction–diffusion processes even with long range interactions. For example in [15], the so called ”generalized totally ASEP model” has been exactly solved in this way. In this model the particle hops to the next right site by pushing all the neighboring particles to their next right sites, with a rate depending on the number of right neighboring particles. The partially generalized ASEP model has also been studied in [16]. Note that in all of these cases, the solvability of the models is shown by proving the factorization of N–particle S–matrices into 2–particle ones, which the latter were found exactly.

In all the above ASEPs, there is only one species of particle, that is all the particles are of the same type. But if one considers the two, or more, species problems, the situation becomes more complicated. The main complexity arises from the fact that the above mentioned factorization of N–particle systems, reduces to the condition of satisfying the two–particle S–matrices in the Quantum Yang–Baxter Equation (QYBE). In one species models, the S–matrices are not really matrices, they are c–numbers and therefore their satisfying in QYBE becomes trivial. This new condition can hardly restrict the number of solvable models with more than one–species particles. In [17], a class of two species reaction–diffusion processes with following properties has been considered: 1) the particles diffuse to their right neighboring sites, 2) they can be annihilated or created, but the total number of particles
are constant, and 3) the interaction rates are all the same. It is shown that among 4096 types of models with the above properties, which can be modeled by a master equation and a number of boundary conditions, there are only 28 independent interactions which their two–particle S–matrices satisfy the QYBE and therefore are solvable. The third condition (equality of the interaction rates) was very crucial in proof of solvability.

In this paper we want to study the effect of interaction rates in solvability of \( p \)-species reaction–diffusion processes, by considering a specific model. We begin our investigation by choosing one of the two–species interactions that has been introduced in [17], but with different interaction rates, and try to obtain the range of parameters to insure the solvability of the corresponding extended \( p \)-species model. As we show, we must restrict ourselves to a narrower and narrower range of parameters, as we go ahead, and finally arrive at a model with a specific relation between the interaction rates and also a specific range for these rates.

The plan of the paper is as following. In section 2, we begin with following two–species reaction–diffusion processes:

\[
\begin{align*}
A + \emptyset & \overset{D_A}{\rightarrow} \emptyset + A, \\
B + \emptyset & \overset{D_B}{\rightarrow} \emptyset + B, \\
A + B & \overset{s}{\rightarrow} B + A, \\
B + A & \overset{r}{\rightarrow} A + B,
\end{align*}
\]

and write down a master equation and a number of boundary conditions to describe the dynamics of these interactions. \( D_A \) and \( D_B \) are the right–diffusion rates of \( A \) and \( B \) particles, respectively, and \( s \) and \( r \) are the rates of transforming (exchanging) \( A \) and \( B \) particles to each other for \((...AB...)\) and \((...BA...)\) configurations, respectively. We will show that only for \( D_A = D_B = 1 \) case, there exists the coordinate Bethe–ansatz solution for probabilities (note that taking \( D \equiv 1 \) is in fact a choosing of time scale). Moreover, we will see that the consistency of the solutions (which will be appeared as satisfaction of 2–particle S–matrix in QYBE) restricts us to \( r = 0 \) or \( s = 0 \) cases (which are the same after relabeling \( A \leftrightarrow B \)). We therefore conclude that the solvable model (interaction) is the following one–parameter family process

\[
\begin{align*}
A + \emptyset & \overset{1}{\rightarrow} \emptyset + A, \\
B + \emptyset & \overset{1}{\rightarrow} \emptyset + B, \\
B + A & \overset{r}{\rightarrow} A + B,
\end{align*}
\]

Note that for \( r = 1 \), interactions (2) are one of the 28 interactions introduced in [17]. We then generalize the interactions (2) to a \( p \)-species model in which the particles \( A_i (i = 1, \ldots, p) \) can
diffuse to their right neighboring sites, all with equal rate one, and also they have exchange interactions with different rates:

\[ A_i + \emptyset \xrightarrow{1} \emptyset + A_i, \]
\[ A_j + A_i \xrightarrow{r_j} A_i + A_j, \quad (j > i). \]  

We label the species such that the configuration \( \ldots A_j A_i \ldots \) can go to \( \ldots A_i A_j \ldots \), only when \( j > i \), therefore \( r \) in eq.(2) is in fact \( r_{ij} \). After a lengthy calculation, we show that there must be a specific relation between \( r_{ij} \)'s until the \( p^2 \times p^2 \) two–particle S–matrix satisfying the QYBE.

In section 3, we calculate the two–particle conditional probabilities of reaction (2), and show that only for \( 0 \leq r < 2 \) we are able to calculate these probabilities by a standard superposition of the eigenfunctions with real eigenvalues. The long–time behavior of the probabilities is also discussed. Finally in section 4, we generalize the reactions (2) to the case where the particles can diffuse to both right and left, as following:

\[ A + \emptyset \xrightarrow{D_R} \emptyset + A, \]
\[ B + \emptyset \xrightarrow{D_R} \emptyset + B, \]
\[ \emptyset + A \xrightarrow{D_L} A + \emptyset, \]
\[ \emptyset + B \xrightarrow{D_L} B + \emptyset, \]
\[ B + A \xrightarrow{r} A + B, \]
\[ A + B \xrightarrow{s} B + A. \]  

We show that there must be a fine tuning of parameters if one demands the reactions (4) to be solvable.

2 \hspace{1cm} \textit{p–species exchange–diffusion processes}

2.1 \hspace{0.5cm} The master equation for 2–species case

Consider the interactions introduced in eq.(1). The basic quantities that must be calculated are the probabilities \( P_{\alpha_1 \alpha_2 \ldots \alpha_N}(x_1, x_2, \ldots, x_N; t) \) for finding at time \( t \) the particle of type \( \alpha_1 \) at site \( x_1 \), particle of type \( \alpha_2 \) at site \( x_2 \), etc. Each \( \alpha_i \) can be \( A \) or \( B \). Following [14], we take these functions to define probabilities only in the physical region \( x_1 < x_2 < \ldots < x_N \), and the regions where any two adjacent coordinates are equal, are the boundaries of the physical region. For \( x_{i-1} - x_i > 1, \forall i \), the particles can only hop to their right neighboring sites and therefore the master equation is:

\[ \frac{\partial}{\partial t} P_{\alpha_1 \alpha_2 \ldots \alpha_N}(x_1, x_2, \ldots, x_N; t) = \sum_{i=1}^{N} D_{\alpha_i}[P_{\alpha_1 \alpha_2 \ldots \alpha_N}(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N; t)]. \]
\[ P_{\alpha_1 \alpha_2 \ldots \alpha_N}(x_1, x_2, \ldots, x_N; t), \]

where the first \( N \) terms are the sources of \( P_{\alpha_1 \alpha_2 \ldots \alpha_N}(x_1, x_2, \ldots, x_N; t) \) and the second \( N \) terms are the sinks of it. It is obvious that if \( x_{i+1} = x_i + 1 \) for some \( i \)'s, then some of the probability functions in the right hand side of eq. (5) go out from the physical region. So we need to specify the boundary terms. The specification of these terms depends on the details of the interactions of the particles. For exchange interactions defined in eq. (1), the suitable boundary conditions are:

\[
\begin{align*}
D_A P_{BA}(x, x) &= sP_{AB}(x, x + 1) + (D_B - r)P_{BA}(x, x + 1), \\
D_B P_{AB}(x, x) &= rP_{BA}(x, x + 1) + (D_A - s)P_{AB}(x, x + 1), \\
P_{\alpha\alpha}(x, x) &= P_{\alpha\alpha}(x, x + 1), \quad (\alpha = A, B),
\end{align*}
\]

in which the time variable and all the other coordinates have been suppressed for simplicity. To justify these boundary conditions, it is enough to examine them in some specific cases. Let us do it for a rather complicated case, for example \( P_{ABBA}(x, x + 1, x + 2, x + 3) \).

From master equation (5), we have,

\[
\frac{\partial}{\partial t} P_{ABBA}(x, x + 1, x + 2, x + 3) = D_A P_{ABBA}(x - 1, x + 1, x + 2, x + 3) + D_B P_{ABBA}(x, x + 1, x + 2, x + 3) + D_A P_{ABBA}(x, x + 1, x + 2, x + 2) - 2(D_A + D_B)P_{ABBA}(x + 1, x + 2, x + 3). \]

If we use the relations (6) in the second, third and fourth terms of the right hand side of (7), we find

\[
\frac{\partial}{\partial t} P_{ABBA}(x, x + 1, x + 2, x + 3) = D_A P_{ABBA}(x - 1, x + 1, x + 2, x + 3) + rP_{BABA}(x, x + 1, x + 2, x + 3) + sP_{ABAB}(x, x + 1, x + 2, x + 3) - (D_A + r + s)P_{ABBA}(x, x + 1, x + 2, x + 3). \]

This equation is exactly what we expect from interactions (1), because the source terms of configuration \((\ldots \emptyset ABBA \emptyset \ldots)\) are: \((\ldots A \emptyset BBA \emptyset \ldots)\) (with rate \( D_A \)), \((\ldots \emptyset BABA \emptyset \ldots)\) (with rate \( r \)) and \((\ldots \emptyset ABAB \emptyset \ldots)\) (with rate \( s \)), and its sink terms are: \((\ldots \emptyset ABB \emptyset A \ldots)\) (with rate \( D_A \)), \((\ldots \emptyset BABA \emptyset \ldots)\) (with rate \( s \)) and \((\ldots \emptyset ABAB \emptyset \ldots)\) (with rate \( r \)). It can be shown that the boundary conditions (6) results the correct terms for any desired configuration.

### 2.2 the Bethe ansatz solution (2–species)

Now we want to solve the master equation (5) with boundary conditions (6) by the coordinate Bethe ansatz method. First we define \( \Psi_{\alpha_1 \ldots \alpha_N}(x_1, \ldots, x_N) \) through,

\[
P_{\alpha_1 \ldots \alpha_N}(x_1, \ldots, x_N; t) = e^{-\epsilon t}\psi_{\alpha_1 \ldots \alpha_N}(x_1, \ldots, x_N), \tag{9}
\]
and then substitute it in eq.(5), which results

$$\sum_{i=1}^{N} D_{\alpha_i} \Psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_N) = (\sum_{i=1}^{N} D_{\alpha_i} - \epsilon_N) \Psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_N). \quad (10)$$

To solve this equation, we use the coordinate Bethe ansatz for each of the components $\Psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_N)$:

$$\Psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_N) = \sum_{\sigma} A_{\sigma}^{(\alpha_1...\alpha_N)} e^{i\sigma(p) \cdot x}, \quad (11)$$

where $x$ and $p$ stands for $N$–tuples coordinates and momenta, respectively, and $\sigma(p)$ is a permutation of momenta. The sum is over all permutations. Inserting (11) into (10) yields:

$$\sum_{\sigma} \left[ \epsilon_N - \sum_{j=1}^{N} D_{\alpha_j} + \sum_{j=1}^{N} D_{\alpha_j} e^{-i\sigma(p_j)} \right] A_{\sigma}^{(\alpha_1...\alpha_N)} e^{i\sigma(p) \cdot x} = 0. \quad (12)$$

As $A_{\sigma}^{(\alpha_1...\alpha_N)} e^{i\sigma(p) \cdot x}$s are linearly independent for different $\sigma$’s, the only solution of eq.(12) is:

$$\epsilon_N - \sum_{j=1}^{N} D_{\alpha_j} + \sum_{j=1}^{N} D_{\alpha_j} e^{-i\sigma(p_j)} = 0, \quad \forall \sigma, \quad (13)$$

or

$$\sum_{j=1}^{N} D_{\alpha_j} e^{-i\sigma_1(p_j)} = \sum_{j=1}^{N} D_{\alpha_j} e^{-i\sigma_2(p_j)} = \ldots = \sum_{j=1}^{N} D_{\alpha_j} e^{-i\sigma_n(p_j)}, \quad (14)$$

where $n$ is the number of elements of permutation group. As these equalities must hold for an arbitrary $p$, the only nontrivial solution is:

$$D_A = D_B \equiv 1. \quad (15)$$

Now as for any group element $\sigma$ we have

$$\sum_{j=1}^{N} e^{-i\sigma(p_j)} = e^{-ip_1} + \ldots + e^{-ip_N}, \quad (16)$$

the equalities (14) satisfy satisfactorily. Therefore, the Bethe ansatz solution exists only for equal diffusion rates, which in this case the eigenvalue $\epsilon_N$ is found to be (by eq.(13)):

$$\epsilon_N = \sum_{j=1}^{N} (1 - e^{-ip_j}). \quad (17)$$

It is easier to consider $\Psi_{\alpha_1...\alpha_N}(x_1, \ldots, x_N)$ and $A_{\sigma}^{(\alpha_1...\alpha_N)}$ as the components of the tensors $\Psi$ and $A_{\sigma}$ with rank $N$, respectively. therefore eq.(11) can be written as

$$\Psi(x_1, \ldots, x_N) = \sum_{\sigma} A_{\sigma} e^{i\sigma(p) \cdot x}. \quad (18)$$
The boundary conditions of $\Psi$ can be obtained by substituting eq.(9) in (6), with $D_A = D_B \equiv 1$. The resulting equation is

$$\Psi(..., \zeta, \zeta, ...) = b_{k,k+1} \Psi(..., \zeta, \zeta + 1, ...),$$

(19)

where

$$b_{k,k+1} = 1 \otimes \cdots \otimes 1 \otimes \underbrace{b}_{k,k+1} \otimes 1 \otimes \cdots \otimes 1,$$

(20)

with

$$b = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & s & 0 \\
0 & s & 1 - r & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

(21)

The coefficients $A_\sigma$'s in eq.(18) must be found by substituting the wavefunction (18) into the boundary condition (19), which yields

$$\sum_\sigma e^{i \sum_{j \neq k,k+1} \sigma(p_j)x_j + i(\sigma(p_k) + \sigma(p_{k+1}))\xi} \left(1 - e^{i\sigma(p_{k+1})}b_{k,k+1}\right)A_\sigma = 0.$$

(22)

As the exponential part of eq.(22) is symmetric with respect to $p_k \leftrightarrow p_{k+1}$, if we also symmetrize the remaining terms with respect to this interchange, we obtain

$$(1 - e^{i\sigma(p_{k+1})}b_{k,k+1})A_\sigma + (1 - e^{i\sigma(p_k)}b_{k,k+1})A_{\sigma\sigma_k} = 0,$$

(23)

where $\sigma_k$ represent the permutation group element which only interchanges $p_k$ and $p_{k+1}$. Therefore

$$A_{\sigma\sigma_k} = S_{k,k+1}(\sigma(p_k), \sigma(p_{k+1}))A_\sigma,$$

(24)

where

$$S_{k,k+1}(z_1, z_2) = 1 \otimes \cdots \otimes 1 \otimes \underbrace{S(z_1, z_2)}_{k,k+1} \otimes 1 \otimes \cdots \otimes 1,$$

(25)

in which

$$S(z_1, z_2) = -(1 - z_1 b)^{-1}(1 - z_2 b).$$

(26)

In above equations, $z_k$ stands for $e^{ip_k}$. In this way, all the $A_\sigma$ coefficients are determined in terms of $A_1$ which is fixed by the initial conditions (the particles' positions at $t = 0$). It seems that we have solved the problem for arbitrary $b$ (i.e. interaction), but it is not true (note that we have not yet used the explicit form of $b$ in deriving $A_\sigma$). The crucial point is that $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ are equal as elements of permutation group, therefore we should impose the following condition on the corresponding $A_\sigma$'s:

$$A_{\sigma_1\sigma_2\sigma_1} = A_{\sigma_2\sigma_1\sigma_2},$$

(27)
and this highly restricts the allowed $b$ matrices (i.e. interactions). It can be shown that eq.(27) reduces to following relation for $S(z_1, z_2)$ matrices (see [7] for more details),

\[
(S(w, t) \otimes 1)(1 \otimes S(z, t))(S(z, w) \otimes 1) = (1 \otimes S(z, w))(S(z, t) \otimes 1)(1 \otimes S(w, t)), \tag{28}
\]

in which $z = e^{ip_1}$, $w = e^{ip_2}$, and $t = e^{ip_3}$. Note that eq.(28) is nothing but the Quantum Yang–Baxter equation.

Now if one calculates the $S$–matrix from eq.(26), using $b$ from (21), the QYBE (28) reduces to a $8 \times 8$ matrix with fourteen nonzero elements (after writing eq.(28) as RHS – LHS = 0) that must be equated to the zero matrix. The elements are functions of $p_1, p_2, p_3, r$ and $s$ which all must be equal to zero for arbitrary momentum values $p_1, p_2$ and $p_3$. It can be shown that the unique solutions of these fourteen equations are:

- solution 1 : $r = 0$, arbitrary $s$,
- solution 2 : $s = 0$, arbitrary $r$. \tag{29}

As these two solutions are equivalent (by relabeling $A \leftrightarrow B$), so the only integrable model is the one indicated in eq.(2).

### 2.3 The $p$–species model

Now let us generalize the reaction (2) to the case where there exist $p$ kinds of particles, which we label them by $A_1, A_2, ..., A_p$. Each particle can diffuse to its right neighboring site, and any two particle can exchange with each other. From the results obtained in previous subsection, we know that if we want the model to be integrable, we must restrict ourselves to the case where the particles’ diffusion rates are equal (scaled to one), and also for each two particles there is only one allowed exchange interaction. For example $A_2 + A_1 \rightarrow A_1 + A_2$ is allowed, but $A_1 + A_2 \rightarrow A_2 + A_1$ is forbidden. We label the particles such that $A_j + A_i \rightarrow A_i + A_j$ is allowed only for $j > i$, and denote the reaction rate of this interaction by $r_{ij}$, see eq.(3).

The master equation of this $p$–species model is again eq.(5) (with $D_{\alpha_i} = 1$), but now each $\alpha_i$ can be $A_1, A_2, ..., A_p$. The boundary conditions are the generalization of eq.(6) but now for each two particle species $A_i$ and $A_j$, i.e. $P_{AB} \rightarrow P_{ij}$, $P_{BA} \rightarrow P_{ji}$, $s = 0$, $D_A = D_B = 1$, and $r \rightarrow r_{ij}$; so

\[
\begin{align*}
P_{ij}(x, x) &= P_{ij}(x, x + 1) + r_{ij}P_{ji}(x, x + 1), & j > i, \\
P_{ji}(x, x) &= (1 - r_{ij})P_{ji}(x, x + 1), & j > i, \\
P_{ii}(x, x) &= P_{ii}(x, x + 1).
\end{align*} \tag{30}
\]

The wavefunctions can be again factorized by the eq.(9), and the coordinate Bethe ansatz solution (18) is yet valid. The boundary conditions can be rewritten as (19), but here $b$ is
the following $p^2 \times p^2$ matrix:

$$b = \sum_{i \leq j} E_{ii} \otimes E_{jj} + \sum_{i < j} r_{ij} E_{ij} \otimes E_{ji} + \sum_{i > j} (1 - r_{ji}) E_{ii} \otimes E_{jj}, \quad (31)$$

where $E_{ij}$ is a $p \times p$ matrix with elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. It can be shown that the S–matrix (26) becomes:

$$S(z, w) = \sum_{i,j} \frac{1 - w(1 - r'_{ji})}{1 - r'_{ji}} E_{ii} \otimes E_{jj} + \sum_{i,j} \frac{z - w}{z - 1}(1 - z + z r'_{ij}) E_{ij} \otimes E_{ji}, \quad (32)$$

in which,

$$r'_{ij} = \begin{cases} 
0 & \text{if } i \geq j \\
r_{ij} & \text{if } i < j . 
\end{cases} \quad (33)$$

Now expression (32) must satisfy the QYBE (28). After a lengthy calculation, it can be shown that the only nontrivial solutions of QYBE are as following (for each $i < j < k$ indices):

solution 1 : $r_{ij} = 0$, $r_{ik}$ and $r_{jk}$ arbitrary,

solution 2 : $r_{ik} = 0$, $r_{jk} = 0$, and $r_{ij}$ arbitrary, \quad (34)

solution 3 : $r_{ij} = r_{ik}$, $r_{jk}$ arbitrary.

For any set of interaction rates where each three of them satisfy any of the solutions 1, 2 or 3, with the constraint that the relations between $r_{ij}$s must be consistent in all subsets, we have an integrable $p$–species model with wavefunction (18) whose coefficients are determined by eq.(24) and S–matrix introduced in (32).

For $p = 3$, the allowed sets of interaction rates are exactly the same as the three solutions (34) with $(ijk) = (123)$. But for $p > 3$ cases, we can choose different consistent solutions for any $(ijk)$’s and therefore extracting all the allowed sets are not so easy. For example for $p = 4$, in which there are six interaction rates $r_{12}, r_{13}, r_{14}, r_{23}, r_{24},$ and $r_{34}$, the allowed sets of parameters are as following:

$$\begin{align*}
\{r_{14}, r_{24}, r_{34}\}, \\
\{r_{12}, r_{34}\}, \\
\{r_{13}, r_{23}\}, \\
\{r_{13}, r_{24}\}, \\
\{r_{14}, r_{23}\}, \\
\{r_{14}, r_{34}, r_{23} = r_{24}\}, \\
\{r_{24}, r_{34}, r_{12} = r_{14}\}, \\
\{r_{24}, r_{34}, r_{13} = r_{14}\}. 
\end{align*} \quad (35)$$

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\{r_{24}, r_{34}, r_{12} = r_{13} = r_{14}\},
\{r_{13}, r_{23} = r_{24}\},
\{r_{23}, r_{12} = r_{13}\},
\{r_{23}, r_{13} = r_{14}\},
\{r_{23}, r_{12} = r_{13} = r_{14}\},
\{r_{34}, r_{13} = r_{14}, r_{23} = r_{24}\},
\{r_{34}, r_{12} = r_{13} = r_{14}, r_{23} = r_{24}\}.

Note that in all the above allowed sets, we have only brought the free parameters and the relations that must be satisfied by them, and the zero reaction rates have not been written. In this way we find a large class of multi–parameter \(p\)–species integrable reaction–diffusion models.

### 3 two–particle conditional probabilities for 2–species model

Now for the simplest case, that is the 2–species reactions (2), let us calculate the two–particle conditional probabilities \(P(\alpha_1, \alpha_2, x_1, x_2; t|\beta_1, \beta_2, y_1, y_2; 0)\), which is the probability of finding particles \(\alpha_1\) and \(\alpha_2\) at time \(t\) at sites \(x_1\) and \(x_2\), respectively, if at \(t = 0\) we have the particles \(\beta_1\) and \(\beta_2\) at sites \(y_1\) and \(y_2\), respectively. These probabilities can be found by a linear combination of eigenfunctions \(P(x_1, x_2)\). Therefore,

\[
\begin{pmatrix}
P_{AA} \\
P_{AB} \\
P_{BA} \\
P_{BB}
\end{pmatrix}(x; t|\beta, y; 0) = \\
\int f(p_1, p_2)e^{-\epsilon_2 t} \Psi(x_1, x_2) dp_1 dp_2
\]

\[
= \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} e^{-i p \cdot y} \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right. e^{i(p_1 x_1 + p_2 x_2)} + S_{12}(p_1, p_2) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{i(p_2 x_1 + p_1 x_2)} \left. \right\} dp_1 dp_2. \tag{36}
\]

In these expansion, \(P(x; t|\beta, y; 0)\) stands for \(P(\alpha_1, \alpha_2, x_1, x_2; t|\beta_1, \beta_2, y_1, y_2; 0)\) and \(f(p_1, p_2)\) is the coefficient of expansion, where in the second equality we choose it to be \(\frac{1}{(2\pi)^2} \int e^{-i p \cdot y}\) (see \([14] - [17]\)). \(\epsilon_2 = 2 - e^{-i p_1} - e^{-i p_2}\) (see (17)) and \(\Psi\) is the two–particle wave function (18), in
which eq.(24) has been used for $A_{\sigma_1}(A_{\sigma_1} = S_{12}(p_1, p_2)A_1)$. The column matrix \( \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \) stands for $A_1$, which its components must be determined by initial condition and $S_{12}(p_1, p_2)$ is:

\[
S_{12}(p_1, p_2) = \begin{pmatrix}
    s_1 & 0 & 0 & 0 \\
    0 & s_1 & s_2 & 0 \\
    0 & 0 & s_3 & 0 \\
    0 & 0 & 0 & s_1 \\
\end{pmatrix},
\]

(37)

where

\[
s_1 = \frac{1 - e^{ip_2}}{e^{ip_1} - 1},
\]

\[
s_2 = \frac{r(e^{ip_2} - e^{ip_1})}{(1 - e^{ip_1})[1 + (r - 1)e^{ip_1}]},
\]

\[
s_3 = \frac{(1 - r)e^{ip_2} - 1}{1 + (r - 1)e^{ip_1}}.
\]

(38)

The matrix $S_{12}(p_1, p_2)$ is obtained from eq.(26) in which the matrix $b$ in eq.(21) (with $s = 0$) has been used. By inserting eq.(37) into eq.(36), we find:

\[
\begin{pmatrix}
    P_{AA} \\
    P_{AB} \\
    P_{BA} \\
    P_{BB} \\
\end{pmatrix}(x; t | \beta, y, 0) = \begin{pmatrix}
    a(F_0(t) + F_1(t)) \\
    b(F_0(t) + F_1(t)) + cF_2(t) \\
    c(F_0(t) + F_3(t)) \\
    d(F_0(t) + F_1(t)) \\
\end{pmatrix},
\]

(39)

in which,

\[
F_0(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon x^2 t} e^{i p.(x-y)} dp_1 dp_2,
\]

(40)

\[
F_i(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon x^2 t} e^{i(p - \tilde{p} - p).x} \tilde{s}_i(p, p_2) dp_1 dp_2, \quad (i = 1, 2, 3).
\]

(41)

In above equations, we have suppressed the $x$ and $y$ dependence of $F_i$'s, for simplicity, and $\tilde{p} = (p_2, p_1)$. Now at $t = 0$, the configuration of the system can be one of the $(A, A)$, $(A, B)$, $(B, A)$ or $(B, B)$, where the first particle is at site $y_1$ and the second one at $y_2$, therefore the only acceptable behavior of $F_i(0)$ ($i = 0, 1, 2, 3$) are:

\[
F_0(0) = \delta_{x_1, y_1}\delta_{x_2, y_2},
\]

\[
F_1(0) = F_2(0) = F_3(0) = 0.
\]

(42)

$F_0(0)$ is obviously correct (see eq.(40)). For other $F_i$'s, first we must set $p_1 \rightarrow p_1 + i \epsilon$ to avoid the singularity arising from $e^{ip_1} - 1$ term in denominator of $s_1$ and $s_2$ (see [14–17]). In
this way one can show that \( F_1(0) = 0 \). But \( F_2 \) and \( F_3 \) have another singularity because of 
\[ 1 + (r - 1)e^{ip_1} \text{ term in denominator of } s_2 \text{ and } s_3. \]
One can easily show that this singularity can be avoided only when 
\[ 0 \leq r < 2, \]
and for this range of interaction rates, we have \( F_2(0) = F_3(0) = 0 \). Therefore the validity of expansion (36) is restricted to range (43). At \( t \neq 0 \), we find

\[
F_0(t) = e^{-2t} \frac{t^{x_1-y_1}}{(x_1-y_1)!} \frac{t^{x_2-y_2}}{(x_2-y_2)!},
\]

\[
F_1(t) = e^{-2t} \left[ \frac{t^{x_1-y_2+1}}{(x_1-y_2+1)!} - \frac{t^{x_1-y_2}}{(x_1-y_2)!} \right] \sum_{k=0}^{\infty} \frac{t^{x_2-y_1+k}}{(x_2-y_1+k)!},
\]

\[
F_2(t) = re^{-2t} \frac{t^{x_1-y_2}}{(x_1-y_2)!} \sum_{l,k=0}^{\infty} \left[ \frac{1}{x_1-y_2+1} - \frac{1}{x_2-y_1+k+l+1} \right] (1-r)^t \frac{t^{x_2-y_1+k+l+1}}{(x_2-y_1+k+l)!},
\]

\[
F_3(t) = e^{-2t} \frac{t^{x_1-y_2}}{(x_1-y_2)!} \left[ \frac{(1-r)t}{(x_1-y_2+1)!} - 1 \right] \sum_{k=0}^{\infty} (1-r)^k \frac{t^{x_2-y_1+k}}{(x_2-y_1+k)!}.
\]

One can now obtain the two–particle conditional probabilities for different initial conditions:
1. If at \( t = 0 \), the particles \( \beta_1 = \beta_2 = A \) were at \( y_1 \) and \( y_2 \), respectively, we must take \( a = 1 \) and \( b = c = d = 0 \). So at \( t \neq 0 \), we have

\[
P_{AA}(x; t|A, A, y; 0) = F_0(t) + F_1(t),
\]

and all other \( P \)'s are zero.

2. If \( \beta_1 = A \) and \( \beta_2 = B \), we must take \( b = 1 \) and \( a = c = d = 0 \). Therefore the only nonzero probability is

\[
P_{AB}(x; t|A, B, y; 0) = F_0(t) + F_1(t).
\]

3. If \( \beta_1 = B \) and \( \beta_2 = A \), we have \( c = 1 \) and \( a = b = d = 0 \). So \( P_{AA} = P_{BB} = 0 \) and

\[
P_{AB}(x; t|B, A, y; 0) = F_2(t),
\]

\[
P_{BA}(x; t|B, A, y; 0) = F_0(t) + F_3(t).
\]

4. And finally if \( \beta_1 = \beta_2 = B \), we have \( d = 1 \) and \( a = b = c = 0 \). So the only nonzero probability is

\[
P_{BB}(x; t|B, B, y; 0) = F_0(t) + F_1(t).
\]

Note that the appearance of the above probabilities is in agreement with our reactions (2).

As another check of our results, it may be interesting to study the long time behavior of these probabilities, in special \( P_{AB}(x; t|B, A, y; 0) \) which is the only nondiagonal nontrivial case. We expect that if at \( t = 0 \) we have a \( B \) particle at site \( y_1 \) and an \( A \) particle at site \( y_2 \).
(with \(y_2 > y_1\)), we must certainly have two \(B\) particles at \(t \to \infty\) somewhere at \(y_1 \leq x_1 < x_2\) and \(y_2 \leq x_2 < \infty\) sites. In other words, we expect
\[
\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{AB}(x; t \to \infty|B, A, y; 0) \to 1. \tag{49}
\]
After some calculations, one can show that
\[
\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{AB}(x; t|B, A, y; 0) = e^{-2t} \sum_{n=0}^{\infty} \left[ 1 - (1 - r)^{n+1} \right] \left[ I_{n+y_2-y_1}(2t) + I_{n+y_2-y_1+1}(2t) \right], \tag{50}
\]
where \(I_n(x)\) is the \(n\)-th order Bessel function of the first kind. To obtain the long–time behavior of (50), one may simply use the following asymptotic form of \(I_n(x)\) at \(x \to \infty\),
\[
I_n(x) \to \frac{e^x}{\sqrt{2\pi x}}, \tag{51}
\]
in eq.(50). But it is not correct since eq.(51) is only valid for \(x > n\), but in eq.(50) we have a sum over \(n\) where for every large definite \(t\), there exist infinite number of \(n\) which are greater than \(t\) (it can be shown that if one calculates this limit without noting this point, finds infinity for eq.(50), which is obviously wrong). To calculate this limit, if one uses the identity \(\sum_{n=-\infty}^{\infty} I_n(x) = e^x\) and takes advantage of equality \(I_n(x) = I_{-n}(x)\), can show that
\[
\sum_{n=0}^{\infty} I_{n+k}(x) = \frac{1}{2} \left( e^x - \sum_{n=-k+1}^{k-1} I_n(x) \right). \tag{52}
\]
Now in the second term of the RHS of (52) \(n\) is bounded, so eq.(51) can be used for it which leads to zero in \(t \to \infty\) limit. For the third term we note that \(-1 < 1 - r < 1\) (see eq.(43)), so \((1 - r)^{n+1} \to 0\) for large \(n\). One can show that this extra \((1 - r)^{n+1}\) factor causes the third term in RHS of (52) goes also to zero at \(t \to \infty\) limit. Therefore the \(t \to \infty\) limit of RHS of (51) is equal to \(1/2\), from which eq.(49) is proved.

4 2–species model with left–right diffusion

In this section we want to study the range of parameters (reaction rates), which makes the reactions (4) integrable. In this case the master equation is
\[
\frac{\partial}{\partial t} P_{\alpha_1...\alpha_N}(x_1, ..., x_N; t) = D_R \sum_{i=1}^{N} P_{\alpha_1...\alpha_N}(x_1, ..., x_{i-1}, x_i - 1, x_{i+1}, ..., x_N; t) \\
+ D_L \sum_{i=1}^{N} P_{\alpha_1...\alpha_N}(x_1, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_N; t) \\
- N P_{\alpha_1...\alpha_N}(x_1, ..., x_N; t), \tag{53}
\]
in which we have used a time scale so that

$$D_R + D_L \equiv 1. \quad (54)$$

The boundary conditions are

$$D_R P_{AB}(x, x) + D_L P_{AB}(x + 1, x + 1) = r P_{BA}(x, x + 1) + (1 - s) P_{AB}(x, x + 1),$$

$$D_R P_{BA}(x, x) + D_L P_{BA}(x + 1, x + 1) = s P_{AB}(x, x + 1) + (1 - r) P_{BA}(x, x + 1), \quad (55)$$

$$D_R P_{\alpha\alpha}(x, x) + D_L P_{\alpha\alpha}(x + 1, x + 1) = P_{\alpha\alpha}(x, x + 1), \quad (\alpha = A, B).$$

Note that for $D_L = 0$, eqs.(53) and (55) reduce to eqs.(5) and (6), respectively, and for $r = s = 0$, these equations lead to corresponding ones in [14].

We must proceed the same steps as previous section, where in this case lead to the following relations for energy and boundary conditions:

$$\epsilon_N = \sum_{j=1}^{N} (1 - D_R e^{-ip_j} - D_L e^{ip_j}), \quad (56)$$

and

$$D_R \Psi(..., \zeta, \zeta, ...) + D_L \Psi(..., \zeta + 1, \zeta + 1, ...) = b_{k,k+1} \Psi(..., \zeta, \zeta + 1, ...), \quad (57)$$

where $b_{k,k+1}$ is defined through (20) with $b$ in eq.(21). If we substitute the coordinate Bethe ansatz (18) in eq.(57), the relation between coefficients is like eq.(24), but now with following S-matrix:

$$S(z_1, z_2) = -(D_R + z_1 z_2 D_L - z_1 b)^{-1} (D_R + z_1 z_2 D_L - z_2 b). \quad (58)$$

This S–matrix must satisfy the QYBE (28). Like the previous case, here there are also fourteen equations that must be solved for $r, s$ and an extra $D_R$ parameters ($D_L$ is fixed by $D_R$ through eq.(54)). These equations are highly nonlinear and we are not able to solve them exactly, even by using the standard programs like MAPLE. So we restrict ourselves to the cases in which

$$r + s = 1. \quad (59)$$

In this way, we can find the complete set of solutions. We believe that there are no other solutions even if the constraint (59) is removed (we have checked many other cases, but no one satisfied QYBE). The solutions with $r = 0$ or $s = 0$ are not new. They are the known models like the models introduced in (29), or the simple diffusion models introduced in [14] (in which both $r$ and $s$ are zero). There are only two new integrable models, as follows:

$$A + \emptyset \xrightarrow{D_R} A + \emptyset,$$

$$B + \emptyset \xrightarrow{D_R} B + \emptyset,$$

$$\emptyset + A \xrightarrow{D_L} A + \emptyset, \quad (60)$$
\[ \emptyset + B \xrightarrow{D_L} B + \emptyset, \]
\[ B + A \xrightarrow{D_R} A + B, \]
\[ A + B \xrightarrow{D_L} B + A, \]

and

\[ A + \emptyset \xrightarrow{D_R} \emptyset + A, \]
\[ B + \emptyset \xrightarrow{D_R} \emptyset + B, \]
\[ \emptyset + A \xrightarrow{D_L} A + \emptyset, \]
\[ \emptyset + B \xrightarrow{D_L} B + \emptyset, \]
\[ B + A \xrightarrow{D_L} A + B, \]
\[ A + B \xrightarrow{D_R} B + A, \]

in which \( D_R = 1 - D_L \). In this way we find two one–parameter family integrable models, which their time dependent probabilities can be found by eq.(9), with \( \epsilon_N \) in eq.(56), \( \Psi \) is given by (18) and (24), with \( S \)–matrix introduced in (58), and \( b \) in (21).

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