Linear and Nonlinear Anderson Localization in a Curved Potential*

Claudio Conti**

Department of Physics, University Sapienza, Piazzale Aldo Moro 5, Rome 00185, Italy

(Received 4 November 2013)

Disorder induced localization in the presence of nonlinearity and curvature is investigated. The time-resolved three-dimensional expansion of a wave packet in a bent cigar shaped potential with a focusing Kerr-like interaction term and Gaussian disorder is numerically analyzed. A self-consistent analytical theory, in which randomness, nonlinearity and geometry are determined by a single scaling parameter, is reported, and it is shown that curvature enhances localization.

PACS: 05.45.Yv, 42.65.T
DOI: 10.1088/0256-307X/31/3/030501

Nonlinearity, disorder and geometry contribute to various forms of wave localization. Various experiments and theoretical models have been reported on Anderson localization in Bose–Einstein condensation,[1–7] even beyond perturbation theory[6] and including nonlocality.[9] The effect of curvature on Anderson localization and the role of nonlinearity in disordered three-dimensional (3D) manifolds have not been considered so far. In this Letter, we investigate Anderson localization in a curved potential by the nonlinear Schrödinger (NLS), or Gross–Pitaevskii (GP) equation. We show that a deformed geometry enhances the degree of localization due to randomness. Specifically, we consider a cigar-shaped potential in the GP equation, which is bent in the presence of a Gaussian random potential. We numerically investigate the 3D+1 spreading of a wave packet, and trapping due to disorder and nonlinearity. The problem is analyzed, beyond perturbation theory, by resorting to a 1D disorder averaged variational formulation. This analysis is valid for an ordered trap superimposed to some disorder, or for a randomly varying shape. Geometry, nonlinearity, and randomness, are included in a self-consistent way, and results concerning the wave shape, the length of localization and the nonlinear eigenvalue are derived in closed form.

In dimensionless units, the GP equation reads

\[ i\Psi_t = -\nabla^2 \Psi + V_{3D}(r) \Psi - \chi |\Psi|^2 \Psi, \tag{1} \]

with \( N \equiv \int |\Psi|^2 |dr| \) and \( \chi = \pm 1 \). We consider a Cartesian system \( r = (x, y, z) \) containing a random curved potential \( V_{3D} = V_1(q_1) + V_L(q_2, q_3) + V_{RAD}(q_1, q_2, q_3) \), where \( q_1 = q \) is the longitudinal curvilinear coordinate along the axis of a cigar-shaped potential, and \( q_{2,3} \) are transverse curvilinear coordinates locally orthogonal to the potential axis.\[10\] \( V_1 \) determines the transverse confinement, \( V_1(q_1) \) is a weak longitudinal trap, \( V_{RAD}(q_1, q_2, q_3) \) is a Gaussian random potential with \( \langle V_{RAD}(r)V_{RAD}(r') \rangle = 2D\delta(r-r') \) and \( D \) measures the strength of disorder. We specifically investigate an elongated quadratic potential \( V_1 = w(q_2^2 + q_3^2) \), \( V_1(q_1) = w_1q_1^2 \), with \( w \gg w_1 \), corresponding to a cigar-shaped trap in Bose–Einstein condensation (BEC). The bending is modeled by a parabolic profile of the longitudinal axis of the trap \( y = kx^2 \); local curvature is given by \( K(q) = 4k^2/[1 + 4k^2x(q)^2]^{3/2} \), with \( x(q) \) the inverse of \( 4kq = 2kx/T + 4k^2x^2 + \sinh^{-1}(2kx) \); \( R_{\text{min}} = 1/(2k) \) is the minimal radius of curvature at \( q = 0 \).[11]

We numerically solve Eq.(1), following the reported experimental investigations,[1,5] we consider the expansion of a Gaussian wave-packet initially positioned at the bending point \( r = 0 \), \( \Psi(r, t = 0) = N^{1/2}(2\mu^2\pi)^{3/4}\exp(-r^2/w_0^2) \) and width \( w_0 = 0.4 \), corresponding to the ground state of the transverse trap spreading along the curved potential \( (w = 100, w_1 = 10^{-2}) \). Note that \( N \) in the linear case \( (\chi = 0) \) is just the norm of the wave functions (we use \( N = 1 \) in the simulations for \( \chi = 0 \)) and does not affect the dynamics; in the nonlinear case \( \chi = 1 \), \( N \) determines the strength of the nonlinearity and corresponds to the number of atoms for BEC. We first consider the linear case \( (\chi = 0) \) with low curvature \( (k \equiv 0.1) \), and investigate the dynamics without (Figs.1(a), 1(b), 1(c)) and with disorder (Figs.1(d), 1(e), 1(f), 1(g)). The degree of localization is quantified by the inverse participation ratio \( \lambda_p = (\int |\Psi|^2 |dr|)(\int |\Psi|^4 |dr|)^{1/2} \). Figures 1(a), 1(b), and 1(c) show that, for \( D = 0 \), the wave spreads along the curved geometry; with disorder (without nonlinearity) the spreading is slowed down by the generation of localizations (Figs.1(d), 1(e), 1(f), 1(g)). For small \( D \), \( \lambda_p \) grows with time, but tends to a stationary profile when \( D \) increases (Fig.2(a)). The localization length is further reduced, and the disorder-induced trapping enhanced, when increasing curvature (Fig.2(b)).

With reference to BEC, choosing a spatial scale \( R_{\text{min}} = 1 \) corresponding to a radius of curvature \( R_{\text{MKS}} = 10 \mu \text{m} \), and taking an interaction length \( \alpha_{\text{MKS}} = 5 \text{nm} \), we have for the number of atoms

---

*Supported by the ISCRRA High Performance Computing Initiative, and the Humboldt Foundation.

**Corresponding author. Email: claudio.conti@uniroma1.it

© 2014 Chinese Physical Society and IOP Publishing Ltd
$N_A = 10^4 N$, and ground state transversal size of the condensate $l_{\text{MKS}} = 4 \mu m$.

For $\chi = 1$, we show in Fig. 3 various profiles $|\Psi|^2$ taken after the expansion has occurred ($t = 1$, see Fig. 1), for two curvatures (top and bottom panels), and in the linear and nonlinear cases (left and right panels). In the presence of a focusing nonlinearity $\chi = 1$, disorder-induced and nonlinear self-trappings are both enhanced by the bent geometry. Figures 2(c) and 2(d) show that the wave displays a quasi-stationary regime, and it is more localized when the nonlinearity increases. In the defocusing case (not reported), we find that, at sufficiently high nonlinearity, any form of localization is destroyed.

Fig. 1. $|\Psi|^2$ in the absence of disorder ($D = 0$, linear case, $\chi = 0$) for (a) $t = 0$, (b) $t = 0.5$, (c) $t = 1$; (d,e,f) as in (a,b,c) with $D = 0.05$; (g) 3D surface corresponding to (f). The size of the insets in (a)-(f) are $\Delta y = \Delta z = 4$.

Fig. 2. (Color online) Inverse participation ratio $l_P$ versus time: (a) $k = 0.1$ linear, and $D = 0, 0.05, 1, 0.5$; (b) as in (a) with $k = 1$; (c) $\chi = 0$ (linear), $N = 0.1$ ($N_A = 1000$), and $N = 1$ ($N_A = 10000$) with $D = 1.5$ and $k = 0.1$; (d) as in (c) with $k = 1$.

In the low density regime, the 3D NLS equation can be reduced to 1D with an effective potential due to curvature:

$$\psi(q_1, q_2, q_3) = \sum_{l=1}^{3} \psi_l |q_l| \exp(-iE_l t),$$

where $V = V_I + V_G + V_R$, $V_G(q) = -K^2(q)/4$, and $V_R(q)$ a 1D Gaussian random potential such that $\langle V_R(q)V_R(q') \rangle = V_0^2 \delta(q - q')$ with $V_0^2 = 2D/l_E^2$.

For $\chi = 0$, simple perturbation theory shows that the curvature enhances the degree of localization. In order to account for nonlinearity, we use the variational approach developed in Ref. [8]. Under the so-called annealed approximation, we obtain the following nonlinear equation for the disorder-averaged bound-state,

$$-\varphi_{qq} + V_G(q)\varphi - \left(1 + \frac{12}{P l_0}\right)\varphi^3 = E\varphi,$$

with $E = E(P)$ determined by the condition $\int \varphi^2 dq = P$. Equation (4) shows that disorder enhances nonlinearity by a term $\Delta \chi = 12/(P l_0)$, and hence favors localization; on the contrary, the larger the localization length $l_0$ is (weaker disorder), the smaller the $\Delta \chi$ is. In addition, $\Delta \chi$ is negligible when $P \to \infty$, corresponding to a dominant nonlinearity with respect to disorder. To solve Eq. (4), we consider the limit of strong curvature $V_G(q) = -\delta(q)/(3R_{\text{min}})$, and we have

$$\varphi(q) = \frac{\sqrt{-2E(P)}}{\cosh[\sqrt{-2E/|P|}(q + \delta l)]},$$

with $\delta l$ determined by $\sqrt{-E} \tanh(\sqrt{-E} \delta l) = 1/(6R_{\text{min}})$. Equation (5) shows that with the increasing curvature, there is a transition from a solitary wave to an exponentially localized state (see Fig. 4(a)). Given $E_C(P) = -(P^2/16)[1 + 12/(l_0 P)]$ and $l_C(P) = 3/\sqrt{-E_C(P)}$, as in Ref. [3], we have

$$E(P) = E_C(P)\left[1 + \frac{l_C(P)}{18R_{\text{min}}}\right]^2,$$

and the energy is reduced ($E_C < 0$) by an amount which grows with $l_C(P)/R_{\text{min}}$. At the lowest order in $1/R_{\text{min}}$, for the localization length we have

$$l(P) = l_C(P)\left[1 - \frac{5l_C(P)}{36R_{\text{min}}}\right].$$
Fig. 4. (Color online) (a) Disorder-averaged profile of the nonlinear Anderson localization for $R_{\text{min}}(P + 12/\ell_0) \approx 4$ (continuous line) and $R_{\text{min}}(P + 12/\ell_0) \approx 21$ (dashed); (b) nonlinear localization length $l$ versus $R_{\text{min}}(P + 12/\ell_0)$.

Equations (5)–(7) give the bound state including all the effects, namely curvature, nonlinearity and disorder. They imply that the geometrically induced reduction of the localization length is given by the ratio between the localization length and the radius of curvature. At variance with the ordered case, where relevant localization is attained for a radius of curvature comparable with the wavelength, with randomness it is sufficient that the radius of curvature is comparable with the localization length to affect the spatial extension of the states. As the localization length in 1D and 2D may be much larger than the wavelength, topological effects in the presence of disorder are enhanced. In the general case, the disorder-averaged bound state can be found by the numerical solution of Eq. (4), as shown in Fig. 4(a). Scaling arguments in Eq. (4) show that the eigenvalue $E(P)R_{\text{min}}^2$ and localization length $l(P)/R_{\text{min}}$ can be written as functions of $R_{\text{min}}(P + 12/\ell_0)$, as illustrated in Fig. 4(b). Disorder (measured by $1/\ell_0$), nonlinearity ($P$) and curvature ($R_{\text{min}}$) enter as a single parameter $R_{\text{min}}(P + 12/\ell_0)$, which simultaneously accounts for their effects on localization. These arguments also apply to the defocusing case.

References

[1] Billy J, Josse V, Zuo Z, Bernard A, Hambrecht B, Lugan P, Clement D, Sanchez-Palencia L, Bouyer P and Aspect A 2008 Nature 453 891
[2] Conti C and Leuzzi L 2011 Phys. Rev. B 83 134204
[3] Kivshar Y S, Gredeskul S A, Sánchez A and Vázquez L 1990 Phys. Rev. Lett. 64 1693
[4] Paul T, Schlagheck P, Leboeuf P and Pavloff N 2007 Phys. Rev. Lett. 98 210602
[5] Roati G, D’Errico C, Fallani L, Fattori M, Fort C, Zaccanti M, Modugno G, Modugno M and Inguscio M 2008 Nature 453 895
[6] Sanchez-Palencia L, Clement D, Lugan P, Bouyer P, Shlyapnikov G V and Aspect A 2007 Phys. Rev. Lett. 98 210401
[7] Skipetrov S E, Minguzzi A, van Tiggele B A and Shapiro B 2008 Phys. Rev. Lett. 100 165301
[8] Conti C 2012 Phys. Rev. A 86 061801
[9] Folli V and Conti C 2012 Opt. Lett. 37 332
[10] da Costa R C T 1981 Phys. Rev. A 23 1982
[11] Conti C 2013 arXiv:1302.3866
[12] Carretero-Gonzalez R, Frantzeskakis D J and Kevrekidis P G 2008 Nonlinearity 21 R139