LEFT FIBRATIONS AND HOMOTOPY COLIMITS

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Abstract. For a small category $A$, we prove that the homotopy colimit functor from the category of simplicial diagrams on $A$ to the category of simplicial sets over the nerve of $A$ establishes a left Quillen equivalence between the projective (or Reedy) model structure on the former category and the covariant model structure on the latter. We compare this equivalence to a Quillen equivalence in the opposite direction previously established by Lurie.

1. Introduction and main results

Let $A$ be a small category and consider the category $sSets^A$ of functors from $A$ to the category of simplicial sets. Such functors are also frequently referred to as simplicial diagrams on $A$, or simplicial presheaves on the opposite category $A^{op}$. The well-known construction of homotopy colimits provides a functor $h! : sSets^A \rightarrow sSets/NA$ from the category of simplicial diagrams on $A$ to the category of simplicial sets over the nerve of $A$. It maps a diagram $F$ to the simplicial set $h!(F)$ with as $n$-simplices the pairs $(x, a_0 \rightarrow \cdots \rightarrow a_n)$ consisting of an $n$-simplex $a_0 \rightarrow \cdots \rightarrow a_n$ in $NA$ and an $n$-simplex $x$ in $F(a_0)$. There is also a functor $r! : sSets/NA \rightarrow sSets^A$ in the other direction, which we will refer to as the rectification functor. For a simplicial set over $NA$, say $\pi : X \rightarrow NA$, we abusively denote its image under $r!$ by $r!(X)$, leaving the map $\pi$ implicit. Its value $r!(X)(b)$ at an object $b$ of $A$ has as $n$-simplices the pairs $(x, \beta)$ with $x \in X_n$ and $\beta : \pi(x_n) \rightarrow b$ a morphism in $A$. (Here $x_n$ denotes the last vertex of $x$.) This functor is closely related to the construction of a split cofibered category out of an arbitrary category over $A$ [4].

The goal of this paper is to analyze the homotopy colimit functor $h!$ from the point of view of model categories. To this end, we consider the projective (and possibly Reedy) model structures on $sSets^A$ and the covariant model structure on $sSets/NA$. (The definitions of these are reviewed in Section 2.) We will give direct, complete and self-contained proofs of the following three statements:

**Proposition A.** (a) The homotopy colimit functor $h!$ is part of a Quillen pair (left adjoint on the left)

$$h! : sSets^A \rightleftarrows sSets/NA : h^*$$

with respect to the projective model structure on $sSets^A$ and the covariant model structure on $sSets/NA$.

(b) In case $A$ is a (generalized) Reedy category, the same is true for the Reedy model structure on $sSets^A$.

**Proposition B.** The rectification functor $r!$ is part of a Quillen pair

$$r! : sSets/NA \rightleftarrows sSets^A : r^*$$

between the covariant and projective model structures.
Note that since the identity functor on $\text{sSets}$ is a left Quillen equivalence from the projective to the Reedy model structure (if $A$ is indeed generalized Reedy), part (b) provides additional information in Proposition $[A]$ but the analogous statement for Proposition $[B]$ just follows by composing the two Quillen pairs. The same remark applies to the following theorem:

**Theorem C.** The two Quillen pairs (left adjoints on top)

$$
\text{sSets}^A \xrightarrow{h^\ast} \text{sSets}/NA \xleftarrow{r^\ast} \text{sSets}^A
$$

are Quillen equivalences between the covariant and projective model structures. Furthermore, the left derived functor of $h^\ast$ and the right derived functor of $r^\ast$ are naturally equivalent.

We will prove this theorem by exhibiting, for cofibrant simplicial diagrams $F$, a natural weak equivalence $r^\ast h^\ast (F) \to F$ and for objects $X \in \text{sSets}/NA$ a natural zigzag of weak equivalences between $h^\ast r^\ast (X)$ and $X$. This then shows that the left derived functors of $h^\ast$ and $r^\ast$ are mutually inverse and are therefore both part of Quillen equivalences. It also shows that the derived functors of $h^\ast$ and $r^\ast$ define naturally isomorphic functors on the level of homotopy categories;

$$Lh^\ast \simeq Rr^\ast : \text{Ho(sSets}^A) \to \text{Ho(sSets}/NA).$$

Finally, we will deduce two consequences of our results.

**Corollary D.** After localizing the projective model structure on $\text{sSets}^A$ with respect to all the maps $A(b, -) \to A(a, -)$ induced by morphisms $a \to b$ in $A$, the functor $h^\ast$ induces a Quillen equivalence between this model structure and the Kan-Quillen model structure on the category $\text{sSets}/NA$.

In the special case where $A$ is a groupoid, the localizing maps of Corollary $[D]$ are natural isomorphisms; therefore, localizing with respect to them does not change the model structure. Combining this observation with Theorem $[C]$ then yields the corollary below. This is a well-known fact; for example, in case $A$ is a group $G$, this reproduces the equivalence of $[3]$ between the model categories of simplicial sets with $G$-action and simplicial sets over the classifying space $BG$. See also $[6, 7]$.

**Corollary E.** If $A$ is a groupoid, the functor $h^\ast$ induces a Quillen equivalence between the projective model structure on $\text{sSets}^A$ and the Kan-Quillen model structure on $\text{sSets}/NA$.

Our main Theorem $C$ is only partially new. Indeed, at least in the special case $A = \Delta^{op}$, it is a classical fact that $h^\ast$ and $r^\ast$ are two models for the homotopy colimit, and it is proved in detail in $[2]$ that these functors produce objects which are equivalent in the classical Kan-Quillen homotopy theory of simplicial sets. The introduction to $[2]$, which uses the perhaps more familiar notation $W$ for the functor we call $r^\ast$, provides an overview of occurrences of this functor in the literature and connects it to classical topics in the homotopy theory of simplicial sets and simplicial groups.

Our result that the derived functors of $h^\ast$ and $r^\ast$ are naturally equivalent for the covariant model structure over $NA$ is sharper and gives additional significance to this model structure. The fact that $r^\ast$ constitute a Quillen equivalence is proved in a more general context by Lurie $[9]$; indeed, the functor we denote by $r^\ast$ coincides with what Lurie calls the relative nerve functor in his Section 3.2.5. However, his proof uses rather a lot of machinery and is essentially different from ours, since it does not exploit the relation between $r^\ast$ and the standard model $h^\ast$ for the homotopy colimit.
The virtue of this note, if any, thus lies in the fact that it provides a direct and self-contained proof of the stated properties and additionally yields left Quillen functors in both directions.

2. Review of several model structures

In this section we recall the details of the model categories involved in the main results stated above. As before, $\mathcal{A}$ is a fixed small category.

2.1. The projective model structure on $s\text{Sets}^\mathcal{A}$. The material in this section is standard and can for example be found in [5]. The projective model structure is determined by specifying that a map $G \to F$ of simplicial diagrams on $\mathcal{A}$ is a fibration, respectively a weak equivalence, if and only if for each object $a$ in $\mathcal{A}$ the map $G(a) \to F(a)$ is a fibration, respectively a weak equivalence, with respect to the classical Kan-Quillen model structure on simplicial sets. This model structure is cofibrantly generated, with generating cofibrations and trivial cofibrations of the form

$$\partial \Delta^n \times \mathcal{A}(a,-) \to \Delta^n \times \mathcal{A}(a,-) \quad (n \geq 0)$$

and

$$\Lambda^n_k \times \mathcal{A}(a,-) \to \Delta^n \times \mathcal{A}(a,-) \quad (n \geq 0, 0 \leq k \leq n)$$

respectively. We refer to the three distinguished classes of maps in this model structure as projective (co)fibrations and projective weak equivalences. The projective model structure is left proper (as well as right proper) and simplicial. For simplicial diagrams $F$ and $G$ on $\mathcal{A}$ and a simplicial set $M$, the simplicial structure is given by

$$(M \otimes F)(a) = M \times F(a)$$

$$\text{Map}(F,G)_n = \text{Hom}(\Delta^n \otimes F, G)$$

where Hom denotes the set of morphisms in $s\text{Sets}^\mathcal{A}$.

2.2. The Reedy model structure. If $\mathcal{A}$ is a (generalized) Reedy category [1], there is another useful model structure on $s\text{Sets}^\mathcal{A}$ with the same weak equivalences as the projective structure. We will write $\mathcal{A}^-$ and $\mathcal{A}^+$ for the subcategories given by the generalized Reedy structure, so that any arrow $b \to a$ in $\mathcal{A}$ factors as

$$b \to c \to a$$

where $b \to c$ is in $\mathcal{A}^-$ and $c \to a$ is in $\mathcal{A}^+$, in a way which is unique up to unique isomorphism. We refer to [1] for a complete list of axioms such a structure should satisfy. For an object $b$ of $\mathcal{A}$, let us write $\mathcal{A}^-(b,-)$ for the subfunctor of the representable functor $\mathcal{A}(b,-)$ whose value $\mathcal{A}^-(b,a)$ consists of those morphisms $b \to a$ which are not in $\mathcal{A}^+$, or equivalently, which admit a factorization $b \to c \to a$ with $b \to c$ in $\mathcal{A}^-$ and not an isomorphism. Then a map of simplicial diagrams $G \to F$ is a Reedy fibration if and only if for each $b \in \mathcal{A}$, it has the right lifting property with respect to the maps

$$\Lambda^n_k \times \mathcal{A}(b,-) \cup \Delta^n \times \mathcal{A}^-(b,-) \to \Delta^n \times \mathcal{A}(b,-) \quad (n \geq 0, 0 \leq k \leq n)$$

Indeed, it easily follows from the definition of a generalized Reedy category that the map

$$\lim_{\longrightarrow \atop b \to c} \mathcal{A}(c,-) \to \mathcal{A}^-(b,-)$$

is an isomorphism, where the colimit is over all non-isomorphisms $b \to c$ in $\mathcal{A}^-$. This Reedy model structure is again cofibrantly generated, left proper (as well as right proper) and simplicial. Moreover, the identity functor is a left Quillen equivalence from the projective to the Reedy model structure.
2.3. The covariant model structure on sSets/NA. This model structure is treated by Joyal [8] and Lurie [9]. Let us first recall the Joyal model structure on simplicial sets; it is uniquely determined by stating that its cofibrations are the monomorphisms, while its fibrant objects are the quasicategories (or $\infty$-categories), i.e. simplicial sets having the extension property with respect to the inner horns $\Lambda^n_k \to \Delta^n$ (for $n > 1$ and $0 < k < n$). The fibrations between fibrant objects are the maps having the right lifting property with respect to these inner horns, as well as with respect to the inclusion $\{0\} \to J$, where $J$ denotes the nerve of the contractible groupoid

\[ 0 \to 1. \]

The Joyal model structure is Cartesian, but not simplicial. Any simplicial set $B$ defines an induced Joyal model structure on the slice category sSets$/B$. The covariant model structure on sSets$/B$ is the left Bousfield localization of this Joyal structure along the left horn inclusions over $B$, i.e. maps of the form

\[
\begin{tikzcd}
\Lambda^n_0 
& \Delta^n \\
B & 
\end{tikzcd}
\]

for $n \geq 1$. We refer to the distinguished classes of maps in the covariant model structure as covariant (co)fibrations and covariant weak equivalences. The fibrant objects are the left fibrations over $B$, i.e. those maps $X \to B$ having the right lifting property with respect to the maps $\Lambda^n_k \to \Delta^n$ for $n \geq 1$ and $0 \leq k < n$. Having the right lifting property with respect to these maps also characterizes the fibrations between fibrant objects in the covariant model structure. (Although we will not use this fact, the weak equivalences between fibrant objects are precisely those maps that induce homotopy equivalences on fibers [8].)

The covariant model structure is left proper and simplicial. For a simplicial set $M$ and maps $X \to B, Y \to B$, the objects

\[ M \otimes (X \to B) \quad \text{and} \quad \operatorname{Map}(X \to B, Y \to B) \]

are the composition $M \times X \to X \to B$ and the simplicial set with $n$-simplices the maps $\Delta^n \times X \to Y$ over $B$. We will simply write $\operatorname{Map}_B(X, Y)$ for this simplicial set. Since the covariant model structure is simplicial, a weak equivalence $M \to N$ of simplicial sets (in the Kan-Quillen model structure) gives a covariant weak equivalence $M \times X \to N \times X$ over $B$.

We will be interested in the special case where $B = NA$ and use the following lemmas.

**Lemma 2.1.** Let $a$ be an object of $A$. Then the diagram

\[
\begin{tikzcd}
\Delta^0 & N(a/A) \\
& NA, \\
a & 
\end{tikzcd}
\]

given by the codomain functor $a/A \to A$, is a covariant trivial cofibration in sSets$/NA$. 

Proof. Every non-degenerate \( n \)-simplex \( \alpha \) of \( N(a \backslash A) \) is the 0'th face of a unique \((n + 1)\)-simplex \( \tilde{\alpha} \) with \( \tilde{\alpha}_0 = \text{id}_a \):

\[
\begin{array}{c}
\alpha : \\
\begin{array}{c}
a_0 \\
\vdots \\
a_n
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\alpha \\
a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
a_0 \\
\vdots \\
a_n
\end{array}
\end{array}
\]

Let us call simplices of the latter kind admissible. Let \( A^{(n)} \subseteq N(a \backslash A) \) be the union of all admissible \( k \)-simplices for \( k \leq n \). Then

\[
\Delta^0 \xrightarrow{\text{id}_a} N(a \backslash A)
\]

is the unique admissible 0-simplex and the lemma states that

\[
A^{(0)} \twoheadrightarrow N(a \backslash A) = \bigcup_{n=1}^{\infty} A^{(n)}
\]

is a covariant trivial cofibration over \( NA \). Now observe that if \( \tilde{\alpha} \) is an admissible \((n + 1)\)-simplex, then for each 0 < \( i \leq n + 1 \) the face \( d_i \tilde{\alpha} \) is an admissible \( n \)-simplex, whereas \( d_0 \tilde{\alpha} \) is not admissible (here we use that \( \alpha \) is nondegenerate) and not a face of a different admissible \((n + 1)\)-simplex. It follows that the inclusion \( A^{(n)} \to A^{(n+1)} \) is a (possibly transfinite) composition of pushouts of horn inclusions \( \Lambda_{0}^{n+1} \to \Delta_{n+1}^{n} \).

The lemma follows. \( \square \)

In a similar way, one proves the following:

**Lemma 2.2.** Let \( A = (a_0 \to \cdots \to a_n) \) be an \( n \)-simplex of \( NA \). Then

\[
\Delta^0 \xrightarrow{0} \Delta^n
\]

is a covariant trivial cofibration in \( \text{sSets}/nA \).

**Proof.** Let \( A^{(k)} \) be the union of the faces of \( \Delta^n \) of dimension less than or equal to \( k \) which contain the vertex 0. Then

\[
\Delta^0 \xrightarrow{0} \Delta^n
\]

is \( A^{(0)} \) and \( \Delta^n = A^{(n)} \), while \( A^{(k-1)} \to A^{(k)} \) is again a composition of pushouts of left horns \( \Lambda_{0}^{k} \to \Delta_{k}^{k} \).

\( \square \)

### 3. The Homotopy Colimit Functor

In this section we will consider the adjoint pair

\[
h_! : \text{sSets}^A \xrightarrow{\sim} \text{sSets}/NA : h^*
\]

in more detail and prove Proposition A from the introduction. Recall that for a diagram \( F \), the simplicial set \( h_!(F) \) is defined as follows. Let \( \int_A F \) be the degreewise category of elements of \( F \). This is a category object in \( \text{sSets} \) with an evident projection to \( A \). Then \( h_!(F) \) is the diagonal of the nerve \( N(\int_A F) \), which is a bisimplicial set. Thus, an \( n \)-simplex of \( h_!(F) \) is a pair \( (A, x) \), where \( A = (a_0 \to \cdots \to a_n) \) is an \( n \)-simplex of \( NA \) and \( x \in F(a_0)_a \). Note that \( h_! \) is compatible with the simplicial structures, in the sense that there is a natural isomorphism \( h_!(M \otimes F) \simeq M \otimes h_!(F) \) for a simplicial set \( M \) and simplicial diagram \( F \).
For an object $X \to NA$ of $\text{sSets}/NA$, the value of the right adjoint $h^*(X)$ may be described by

$$h^*(X)(b) = \text{Map}_{NA}(N(b/A), X)$$

where $b$ is any object of $A$ and $\text{Map}_{NA}$ refers to the simplicial structure specified in Section 2.

**Lemma 3.1.** The functor $h_! : \text{sSets}^A \to \text{sSets}/NA$ preserves monomorphisms.

**Proof.** This is clear from the explicit description of $h_!$. □

**Remark 3.2.** In fact, the functor $h_!$ has a left adjoint $h^+$, defined on representables $A : \Delta^n \to NA$, $A = (a_0 \to \cdots \to a_n)$ by

$$h^+(A) = \Delta^n \times A(a_0, -).$$

We will not use this left adjoint, since it is in general not left Quillen.

**Lemma 3.3.** The functor $h_!$ sends projective trivial cofibrations to covariant trivial cofibrations.

**Proof.** Observe that $h_!$ sends the generating trivial cofibration

$$\Lambda^k_n \times A(b, -) \to \Delta^n \times A(b, -)$$

to the map

$$\Lambda^k_n \times (b/A) \to \Delta^n \times (b/A),$$

which is a covariant trivial cofibration, since the covariant model structure is simplicial (see Section 2). □

**Lemma 3.4.** If $A$ is a (generalized) Reedy category, the functor $h_!$ sends trivial Reedy cofibrations to covariant trivial cofibrations.

**Proof.** Let $A^-(b, -)$ be as defined in Section 2.2 and let $N^-(b/A) = h_!A^-(b, -)$. Then we have to verify that

$$\Lambda^k_n \times (b/A) \cup \Delta^n \times N^-(b/A) \to \Delta^n \times (b/A)$$

is a covariant trivial cofibration. The map $N^-(b/A) \to N(b/A)$ is a monomorphism and so the pushout-product above is indeed a covariant trivial cofibration, again since the covariant model structure is simplicial. □

**Proof of Proposition A.** The proposition follows directly from Lemmas 3.1, 3.3 and 3.4. □

**Remark 3.5.** The Quillen pair of Proposition A is simplicial, in the sense that there is a natural isomorphism

$$\text{Map}_{NA}(h_!F, X) \simeq \text{Map}(F, h^*X)$$

for any simplicial diagram $F$ and any simplicial set $X$ over $NA$. Indeed, this is clear from the fact that $h_!$ strictly commutes with the simplicial structure; i.e. for any simplicial set $M$ there is an isomorphism $h_!(M \otimes F) \simeq M \otimes h_!(F)$. 


4. The rectification functor

In this section we will prove Proposition B from the introduction, concerning the adjoint pair

\[ r_! : \text{sSets} / \text{NA} \rightleftarrows \text{sSets}^\mathbf{A} : r^* \]

For an \(n\)-simplex \(A\) in \(\text{NA}\) of the form

\[ a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_n \]

and an object \(b\) in \(\mathbf{A}\) we have

\[ r_!(A)(b) = N(A/b). \]

Here the category \(A/b\) has as objects pairs \((i, f)\), where \(0 \leq i \leq n\) and \(f : a_i \rightarrow b\) is a morphism in \(\mathbf{A}\). Morphisms in \(A/b\) are commutative triangles

\[ \begin{array}{ccc}
  a_i & \rightarrow & a_j \\
  \alpha_{ij} & \rightarrow & \\
  f & \rightarrow & g \\
 \end{array} \]

where \(\alpha_{ij} = \alpha_j \circ \cdots \circ \alpha_i \circ 1\). The category \(A/b\) is in fact a poset and one easily verifies that \(N(A/b)\) is weakly equivalent to the (discrete) set \(\mathbf{A}(a_0, b)\). Also, one easily verifies that the value of \(r_!\) on a simplicial set \(X\) over \(\text{NA}\) is as described in Section 1.

This description of \(r_!\) also yields an explicit description of \(r^*\) for any simplicial diagram \(F\) on \(\mathbf{A}\). Indeed, the simplices \(x \in r^*(F)_A\) over an \(n\)-simplex \(A\) in \(\text{NA}\) as above are families

\[ x = (x_u)_u, \quad u : \Delta^k \rightarrow \Delta^n, \]

where \(x_u \in F(a_{u(k)})\) is a \(k\)-simplex, and these are compatible in the sense that for each commuting triangle on the left, the square on the right also commutes:

\[ \begin{array}{ccc}
  \Delta^k & \xleftarrow{u} & \Delta^n \\
  \Delta^k' & \xrightarrow{u'} & \Delta^n' \\
  \end{array} \quad \begin{array}{ccc}
  \Delta^k & \xrightarrow{x_u} & F(a_{u(k)}) \\
  \Delta^k' & \xrightarrow{x_{u'}} & F(a_{u'(k)'}). \\
  \end{array} \]

Here \(\alpha_v = \alpha_{u(k)} \circ \alpha_{u'(k)'} : a_{u(k)} \rightarrow a_{u'(k)'}\) is the appropriate composition of \(\alpha_i\)'s.

We will prove Proposition B by the following two lemmas.

**Lemma 4.1.** The functor \(r^* : \text{sSets}^\mathbf{A} \rightarrow \text{sSets}/\text{NA}\) preserves trivial fibrations.

**Proof.** Let \(G \rightarrow F\) be a map in \(\text{sSets}^\mathbf{A}\). Given \(A = (a_0 \rightarrow \cdots \rightarrow a_n)\), an \(n\)-simplex in \(\text{NA}\) as above, one uses the explicit description of \(r^*\) just given to check that a diagonal lifting in a square as on the left corresponds to a diagonal lifting on the right:

\[ \begin{array}{ccc}
  \partial \Delta^n & \xleftarrow{\partial^n} & r^*(G) \\
  \Delta^n & \xrightarrow{r^*(A)} & r^*(F) \\
  \end{array} \quad \begin{array}{ccc}
  \partial \Delta^n & \xleftarrow{\partial^n} & G(a_n) \\
  \Delta^n & \xrightarrow{G(a_n)} & F(a_n) \\
  \end{array} \]

From this the lemma is clear. \(\square\)
Lemma 4.2. The functor $r^* : sSets^A \to sSets/NA$ sends projective fibrations to left fibrations.

Proof. The proof is identical to that of Lemma 4.1 now using a correspondence between lifts in the same diagrams, but with $\partial \Delta^n$ replaced by $\Lambda^n_k$ for $0 \leq k < n$. □

Remark 4.3. Notice that the preceding proof does not work for the horns $\Lambda^n_n$. Indeed, already for $n = 1$ and $\alpha : a_0 \to a_1$, a commutative square on the left corresponds to a vertex $y_1 \in G(a_1)$, together with a 1-simplex $x \in F(a_1)$ and a lift of $x_0 = d_1 x$ to a vertex $\tilde{x}_0$ in $F(a_0)$, with $y_1$ mapped to $x_1 = d_0 x$ by $G(a_1) \to F(a_1)$. A lifting $\Delta^1 \to r^*(G)$ would consist of a lift $y \in G(a_1)_{1}$ of $x$ (which can be found if $G(a_1) \to F(a_1)$ is a Kan fibration), together with a vertex $\tilde{y}_0$ in $G(a_0)$ over $y_0 \in G(a_0)$ lifting $\tilde{x}_0$ in $F(a_0)$. In general, there is no way to obtain such a vertex $\tilde{y}_0$ from the data of a lift in the square on the right.

Proof of Proposition 5.1. The proof follows from Lemmas 4.1 and 4.2. Indeed, the first lemma ensures that $r_\tau$ preserves cofibrations. Using that the fibrant objects in the covariant model structure are the left fibrations over $NA$ and the fibrations between such fibrant objects are also precisely maps that are left fibrations, the second lemma shows that $r^*$ preserves fibrant objects and fibrations between such. Since in any model category a cofibration is a weak equivalence if and only if it has the left lifting property with respect to fibrations between fibrant objects, this ensures that $r_\tau$ preserves trivial cofibrations. □

Remark 4.4. The pair $(r_\tau, r^*)$ is also simplicial. Indeed, it follows from the explicit description of $r^*$ that for a simplicial diagram $F$ and any $n \geq 0$, there is a natural isomorphism

$$r^*(F)^{\Delta^n} \simeq r^*(F^{\Delta^n})$$

and hence by adjunction for any simplicial set $X$ over $NA$ a natural isomorphism

$$\text{Map}_{NA}(X, r^*F) \simeq \text{Map}(r_\tau X, F).$$

5. Proof of the equivalence

In this section we prove Theorem stated in the introduction. The proof will be split into two propositions.

Proposition 5.1. For any simplicial diagram $F$, there exists a natural map $\tau : r_! h_0(F) \to F$, which is a weak equivalence whenever $F$ is projectively cofibrant.

Proposition 5.2. For any simplicial set $X \to NA$ over the nerve of $A$, there exists a natural zigzag

$$X \to L(X) \longleftarrow h_1 r_\tau(X)$$

of covariant weak equivalences over $NA$.

Proof of Proposition 5.1. Recall that for an object $\pi : X \to NA$ of $sSets/NA$, the value of the diagram $r_!(X)$ at an object $b$ of $A$ is the simplicial set described by

$$r_!(X)(b)_n = \{ (x, \beta) \mid x \in X_n, \beta : \pi(x_n) \to b \},$$

where $x_n$ denotes the final vertex of the $n$-simplex $x$. Thus, for a simplicial diagram $F$ on $A$, the $n$-simplices of $r_! h_0(F)(b)$ are of the form

$$(x \in F(a_0), a_0 \overset{\alpha_1}{\to} \cdots \overset{\alpha_n}{\to} a_n \overset{\beta}{\to} b).$$

The natural map $\tau : r_! h_0(F) \to F$ is defined by sending such a simplex to $F(\beta \alpha_n \cdots \alpha_1)(x)$, an $n$-simplex of $F(b)$.
In order to prove that this map is a weak equivalence for cofibrant objects $F$, one can argue by the usual skeletal induction on $F$ and conclude that it suffices to prove that for each representable object $\Delta^a \times A(a, -)$, the map

$$\tau = \tau_{n, a} : r_{h!}(\Delta^a \times A(a, -)) \rightarrow \Delta^a \times A(a, -)$$

is a projective weak equivalence. Now consider the square

$\begin{array}{ccc}
\tau_{n, a} & \rightarrow & \Delta^a \times A(a, -) \\
\downarrow & & \downarrow \\
r_{h!}(\Delta^0 \times A(a, -)) & \rightarrow & \Delta^0 \times A(a, -)
\end{array}$

where the vertical maps are given by the inclusion of the zero’th vertex and its image under $r_{h!}$. The right vertical map is a projective trivial cofibration and since $r_1$ and $h_0$ are both left Quillen, the left vertical map is so as well. So to prove that $\tau_{n, a}$ is a weak equivalence, it suffices to prove that $\tau_{0, a}$ is.

But $r_{h!}(\Delta^0 \times A(a, -))(b)$ is the nerve of the category $A'(a, b)$ whose objects are factorizations $a \rightarrow c \rightarrow b$ of arrows in $A(a, b)$, while its arrows are commutative diagrams

$$\begin{array}{ccc}
a & \rightarrow & c \\
\downarrow & & \downarrow \\
& \rightarrow & b.
\end{array}$$

The map $\tau$ is then the nerve of the composition functor $A'(a, b) \rightarrow A(a, b)$, where the latter set is interpreted as a discrete category. This map is a weak equivalence, because each connected component of $A'(a, b)$ has an initial as well as a terminal object ($a = a \rightarrow b$ and $a \rightarrow b = b$ respectively). This proves the proposition. □

**Proof of Proposition** We first define the zigzag of the proposition. Consider a map of simplicial sets $\pi : X \rightarrow NA$. Recall that $h_{r!}(X)$ is the simplicial set over $NA$ having as $n$-simplices over $a_0 \rightarrow \cdots \rightarrow a_n$ the pairs

$$(x, \pi(x_n) \rightarrow a_0 \rightarrow \cdots \rightarrow a_n)$$

where $x \in X_n$ and $x_n$ is the final vertex of $x$ (as before). The projection $h_{r!}(X) \rightarrow NA$ maps such a pair to $a_0 \rightarrow \cdots \rightarrow a_n$. Define another functor $L : sSets/NA \rightarrow sSets/NA$, for which the $n$-simplices of $L(X)$ are pairs $(x, \lambda)$, where $x$ is an $n$-simplex of $X$ and $\lambda$ is a commutative diagram in $A$ of the form

$$\begin{array}{cccc}
\pi(x_0) & \rightarrow & \pi(x_1) & \rightarrow \cdots & \rightarrow & \pi(x_n) \\
\lambda_0 & & \lambda_1 & & \cdots & \lambda_n \\
a_0 & \rightarrow & a_1 & \rightarrow \cdots & \rightarrow & a_n
\end{array}$$

and the map to $NA$ is given by taking the bottom row. We will refer to such diagrams as ladders. Another way to describe this functor $L$ is by

$$L(X) = X \times_{NA} N(Ar(A))$$

where $Ar(A)$ is the category of arrows of $A$ (with commutative squares as morphisms) and $N(Ar(A)) \rightarrow NA$ is the map induced by the domain functor $Ar(A) \rightarrow A$. This functor $L$ preserves monomorphisms and colimits (and is in fact left Quillen, which will follow from our proof).

Let us define natural maps

$$X \xrightarrow{\iota} L(X) \xleftarrow{\gamma} h_{r!}(X).$$
The map \( \iota \) sends an \( n \)-simplex \( x \) to the pair \((x, \text{id}_{\pi(x)})\), where \( \text{id}_{\pi(x)} \) is the identity ladder

\[
\begin{array}{c}
\pi(x_0) \\
\pi(x_0) \\
\end{array} \xrightarrow{}
\begin{array}{c}
\pi(x_1) \\
\pi(x_1) \\
\end{array} \xrightarrow{}
\begin{array}{c}
\cdots \\
\cdots \\
\end{array} \xrightarrow{}
\begin{array}{c}
\pi(x_n) \\
\pi(x_n) \\
\end{array}.
\]

The map \( \gamma \) sends a simplex

\[
(x, \pi(x_n) \rightarrow a_0 \rightarrow \cdots \rightarrow a_n)
\]

to the pair given by \( x \) and the ladder

\[
\begin{array}{c}
\pi(x_0) \\
\pi(x_0) \\
\end{array} \xrightarrow{} \begin{array}{c}
\pi(x_1) \\
\pi(x_1) \\
\end{array} \xrightarrow{} \begin{array}{c}
\cdots \\
\cdots \\
\end{array} \xrightarrow{} \begin{array}{c}
\pi(x_n) \\
\pi(x_n) \\
\end{array}
\]

in which the vertical maps \( \pi(x_i) \rightarrow a_i \) are the appropriate compositions of arrows formed from the sequence

\[
\pi(x_0) \rightarrow \cdots \rightarrow \pi(x_n) \rightarrow a_0 \rightarrow \cdots \rightarrow a_n.
\]

As stated in the proposition, we claim that both \( \iota \) and \( \gamma \) are trivial cofibrations in the covariant model structure. First of all, notice that by the usual skeletal induction on \( X \), it suffices to prove this for a representable of the form \( \Delta^n \rightarrow NA \).

Indeed, if \( X' \) is obtained from \( X \) by attaching an \( n \)-simplex \( \Delta^n \rightarrow NA \), consider the cube over \( NA \)

\[
\begin{array}{c}
\partial\Delta^n \\
\Delta^n \\
\end{array} \xrightarrow{} \begin{array}{c}
X \\
X' \\
\end{array} \xrightarrow{} \begin{array}{c}
L\partial\Delta^n \\
L\Delta^n \\
\end{array} \xrightarrow{} \begin{array}{c}
LX \\
LX' \\
\end{array}
\]

in which the back and front faces are pushouts. Since the vertical maps are cofibrations (\( L \) preserves monos) and all objects are cofibrant, these pushouts are also homotopy pushouts. It follows that the component \( X' \rightarrow LX' \) of \( \iota \) from back to front is a weak equivalence whenever the other three components from back to front are weak equivalences, which establishes the inductive step. The same argument applies to the natural transformation \( \gamma \).

So now consider a simplex \( A = (a_0 \rightarrow \cdots \rightarrow a_n) : \Delta^n \rightarrow NA \) and the diagram over \( NA \)

\[
\begin{array}{c}
a_0 \\
\end{array} \xrightarrow{} \begin{array}{c}
A \\
\end{array} \xrightarrow{} \begin{array}{c}
h_1r_1(A) \\
LA \\
\end{array}
\]

where \( a_0 \) is shorthand for the map \( \Delta^0 \rightarrow NA \) given by the object \( a_0 \). The top horizontal map is a trivial cofibration by Lemma 2.2. The left vertical map can be factored into maps

\[
a_0 \rightarrow h_1r_1(a_0) \rightarrow h_1r_1(A).
\]

The second map is the image under \( h_1r_1 \) of the inclusion \( a_0 \rightarrow A \) and hence a trivial cofibration. The first map may be identified with the inclusion \( a_0 \rightarrow N(a_0/A) \) and is therefore a trivial cofibration by Lemma 2.1. Thus to prove the proposition, it suffices to prove that the diagonal map \( a_0 \rightarrow LA \) in the diagram is a weak
equivalence. Let us inspect this map more closely: $LA$ is the simplicial set with $k$-simplices of the form

$$
\begin{array}{ccccccccc}
  a_0 & \rightarrow & a_1 & \rightarrow & \cdots & \rightarrow & a_k \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  b_0 & \rightarrow & b_1 & \rightarrow & \cdots & \rightarrow & b_k
\end{array}
$$

and $a_0 \to LA$ is the vertex for which $a_0 \to b_0$ is the identity $a_0 \to a_0$. Any $k$-simplex of the form just described is the 0’th face of a uniquely determined $(k+1)$-simplex

$$
\begin{array}{ccccccccc}
  a_0 & \rightarrow & a_0 & \rightarrow & \cdots & \rightarrow & a_k \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow \\
  b_0 & \rightarrow & b_0 & \rightarrow & \cdots & \rightarrow & b_k
\end{array}
$$

Let $L^{(k+1)}A \subseteq LA$ be the subobject spanned by all non-degenerate $(k+1)$-simplices of the latter form. Then $L^{(0)}A = a_0$ and $LA$ is the union of all the $L^{(k+1)}A$ for $k \geq -1$. Moreover, each inclusion $L^{(k)}A \to L^{(k+1)}A$ is a covariant trivial cofibration. Indeed, for each non-degenerate $(k+1)$-simplex $\sigma$ that we are adjoining, its faces $d_i(\sigma)$ belong to $L^{(k)}A$ for $i > 0$, while its missing face $d_0(\sigma)$ uniquely determines $\sigma$. This shows that $L^{(k)}A \to L^{(k+1)}A$ is a (possibly transfinite) composition of pushouts of left horn inclusions $\Lambda_0^{k+1} \to \Delta^{k+1}$ and completes the proof. □

6. A localization of the projective model structure

In this final section we prove Corollary [C] from the introduction. Denote by $S$ the set of natural transformations $A(b, -) \to A(a, -)$ induced by morphisms $a \to b$ in $A$. It follows formally from Theorem [C] that $h_t$ induces a Quillen equivalence between the localization of the projective model structure with respect to $S$ and the localization of the covariant model structure with respect to $h_tS$. Denote by $T$ the class of weak equivalences in the latter (localized) model structure.

**Lemma 6.1.** The left Bousfield localization of the covariant model structure on $sSets/NA$ with respect to $T$ is the same as the localization of the covariant model structure with respect to all the maps of the form

$$
\begin{array}{ccc}
  \Delta^0 & \rightarrow & \Delta^1 \\
  \downarrow & & \downarrow \\
  NA & & NA
\end{array}
$$

where 1 indicates the inclusion of the final vertex of $\Delta^1$.

**Proof.** For a morphism $f: a \to b$ in $A$, consider the diagram

$$
\begin{array}{ccc}
  \Delta^0 & \rightarrow & \Delta^1 \\
  \downarrow^{id} & & \downarrow^{id} \\
  N(b/A) & \rightarrow & N(a/A) \\
  \downarrow^{f *} & & \downarrow^{f *} \\
  NA & & NA
\end{array}
$$

$$
\begin{array}{ccc}
  \Delta^0 & \rightarrow & \Delta^1 \\
  \downarrow^{id} & & \downarrow^{id} \\
  N(b/A) & \rightarrow & N(a/A) \\
  \downarrow^{f *} & & \downarrow^{f *} \\
  NA & & NA
\end{array}
$$
where $a/f$ denotes the 1-simplex of $N(a/A)$ given by

\[ \begin{array}{c}
\Delta^0 \\
\downarrow \\
\Delta^1 \\
\downarrow \\
\Delta^n
\end{array} \]

The map 0 is a weak equivalence in the covariant model structure, and so are the two vertical maps, by Lemma 2.1. Therefore $a/f$ is a weak equivalence in the covariant model structure as well. By two-out-of-three, it then follows that localizing with respect to $f^*$ is the same as localizing with respect to the map 1, and the lemma follows.

**Proof of Corollary 13.** It rests us to show that any weak equivalence of simplicial sets over $NA$ (in the Kan-Quillen model structure) is contained in $T$. Given the definition of the covariant model structure, it will suffice to show that any map in $sSets/NA$ of the form

\[ \Lambda^n \rightarrow \Delta^n \rightarrow NA \]

is contained in $T$. In fact, we will show the following: for any $V \subseteq \Lambda^n$ a union of faces, all containing the final vertex $\{n\}$, the two horizontal maps

\[ \Delta^0 \rightarrow V \rightarrow \Delta^n \rightarrow NA \]

are contained in $T$ (where the first horizontal map is the inclusion of the final vertex $\{n\}$). We will work by induction on $n$. In case $n = 1$, the 1-simplex $A$ corresponds to a morphism $f : a \rightarrow b$ in $A$ and we have to show that the inclusion

\[ \begin{array}{c}
\Delta^0 \\
\downarrow \\
\Delta^1 \\
\downarrow \\
\Delta^n
\end{array} \]

is in $T$. This follows from Lemma 6.1 above. If $n > 1$, we use a further induction on the number of faces in $V$. If $V$ consists of one face $d_i \Delta^n$, for $i \neq n$, then the inclusion $\{n\} \rightarrow V$ is in $T$ by induction. Over the simplex $A : \Delta^n \rightarrow NA$ we have a diagram

\[ \begin{array}{c}
\Delta^0 \\
\downarrow \\
\Delta^1 \\
\downarrow \\
\Delta^n
\end{array} \]

The maps 1 and $n$ are in $T$ by induction and both the maps labelled 0 are in $T$ by Lemma 2.2. By two-out-of-three, the maps $(0n)$ and $V \rightarrow \Delta^n$ are then also in $T$. Now suppose $V$ consists of more than one face and $i \neq n$. Over $A : \Delta^n \rightarrow NA$ we can now form the
following diagram:

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{n} & W \cap d_i \Delta^n \\
\downarrow \beta & & \downarrow \delta \\
d_i \Delta^n & \xrightarrow{\gamma} & V \\
\downarrow \epsilon & & \downarrow \zeta \\
\Delta^n. & & \\
\end{array}
\]

The simplicial set \(W \cap d_i \Delta^n\) is a face of \(d_i \Delta^n\), so that both maps labelled \(n\) are in \(T\) by the inductive hypothesis (and therefore also \(\beta\) is in \(T\)). The composite \(\alpha \circ n\) of the top two horizontal maps is in \(T\) by the inductive hypothesis on \(W\), so that \(\alpha\) too is in \(T\). Since the square is a pushout, it follows that \(\gamma\) and \(\delta\) are in \(T\). Again \(\epsilon\) is in \(T\) by induction, so that finally \(\zeta\) must be in \(T\) as well. \(\square\)

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