Log-Gamma Polymer Free Energy Fluctuations via a Fredholm Determinant Identity

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Abstract: We prove that under \( n^{1/3} \) scaling, the limiting distribution as \( n \to \infty \) of the free energy of Seppäläinen’s log-Gamma discrete directed polymer is GUE Tracy-Widom. The main technical innovation we provide is a general identity between a class of \( n \)-fold contour integrals and a class of Fredholm determinants. Applying this identity to the integral formula proved in Corwin et al. (Tropical combinatorics and Whittaker functions. http://arxiv.org/abs/1110.3489v3 [math.PR], 2012) for the Laplace transform of the log-Gamma polymer partition function, we arrive at a Fredholm determinant which lends itself to asymptotic analysis (and thus yields the free energy limit theorem). The Fredholm determinant was anticipated in Borodin and Corwin (Macdonald processes. http://arxiv.org/abs/1111.4408v3 [math.PR], 2012) via the formalism of Macdonald processes yet its rigorous proof was so far lacking because of the nontriviality of certain decay estimates required by that approach.

1. Introduction and Main Results

The log-Gamma polymer was introduced and studied by Seppäläinen [16]. It is an example of a more general class of discrete directed random polymers in 1+1 dimensions, which have received considerable attention in the last decade due to their connections with random matrix theory, integrable systems and the Kardar-Parisi-Zhang universality class [13]. We will focus on the log-Gamma polymer, and refer the reader to the review [10] for more details on the general model.

Definition 1.1. Let \( \theta \) be a positive real. A random variable \( X \) has inverse-Gamma distribution with parameter \( \theta > 0 \) if it is supported on the positive reals where it has distribution

\[
P(X \in dx) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-\frac{1}{x}} dx.
\]

We abbreviate this \( X \sim \Gamma^{-1}(\theta) \).
Definition 1.2. The log-Gamma polymer partition function with parameter $\gamma > 0$ is given by

$$Z(n, N) = \sum_{\pi:(1,1)\to(n,N)} \prod_{(i,j)\in\pi} d_{i,j},$$

where $\pi$ is an up/right directed lattice path from the Euclidean point $(1, 1)$ to $(n, N)$ and where the random variables $d_{i,j}$ are i.i.d, $d_{i,j} \sim \Gamma^{-1}(\gamma)$.

In [16] it was proved that

$$\lim_{n\to\infty} \frac{\log Z(n, n)}{n} = \bar{f}_\gamma, \quad \limsup_{n\to\infty} \frac{\text{var} \log Z(n, n)}{n^{2/3}} \leq C,$$

where $\bar{f}_\gamma = -2\Psi(\gamma/2)$ and $C$ is a large constant. Here $\Psi(x) = [\log \Gamma]'(x)$ is the digamma function. The scale of the variance upper-bound is believed to be tight, since directed polymers at positive temperature should have KPZ universality class scalings (see e.g. the review [10]). Moreover, it is believed that, when centered by $n\bar{f}_\gamma$ and scaled by $n^{1/3}$, the distribution of the free energy $\log Z(n, n)$ should limit to the GUE Tracy-Widom distribution [17].

We presently prove this form of KPZ universality for the log-Gamma polymer for $\gamma < \gamma^*$ for some $\gamma^* > 0$. This assumption is purely technical and comes from the asymptotic analysis. It is likely that this assumption can be removed following the approach of [8], where a similar assumption was removed in the case of the semi-discrete polymer. For this model $\gamma$ plays a role akin to temperature.

Theorem 1. There exists $\gamma^* > 0$ such that the log-Gamma polymer free energy with parameter $\gamma \in (0, \gamma^*)$ has limiting fluctuation distribution given by

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z(n, n) - n\bar{f}_\gamma}{n^{1/3}} \leq r\right) = F_{\text{GUE}}\left(\left(\frac{\bar{g}_\gamma}{2}\right)^{-1/3} r\right),$$

where $\bar{f}_\gamma = -2\Psi(\gamma/2)$, $\bar{g}_\gamma = -2\Psi''(\gamma/2)$ and $F_{\text{GUE}}$ is the GUE Tracy-Widom distribution function.

We give the proof of this theorem in Sect. 2. There are two ingredients in the proof, and then some asymptotic analysis. The first ingredient is the $n$-fold integral formula given in [11] for the Laplace transform of the polymer partition function. This is given below as Proposition 1.4. The second ingredient in the proof is a general identity between a class of $n$-fold contour integrals and a class of Fredholm determinants. This is given below as Theorem 2. Applying this identity to Proposition 1.4 yields Corollary 1.8 which is a new Fredholm determinant expression for the Laplace transform of the log-Gamma polymer partition function. This formula lends itself to straightforward asymptotic analysis, as is done in Sect. 2.

The log-Gamma polymer may be generalized, as done in [11], so that the distributions of the $\gamma_{i,j}$ depend on two collections of parameters.

Discrete directed polymer partition functions, under intermediate disorder scaling [2,14], converge to the solution of the multiplicative stochastic heat equation (whose logarithm is the KPZ equation). If the two collections of parameters determining the