A rapidity-independent parameter
in the star-triangle relation

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Abstract

The normalization factor in the star-triangle relation can be evaluated in a simple form by taking determinants. If we combine this with the rotation symmetries, then we can show that a certain simple quantity $I$ has to be independent of the rapidities. In this sense it is an invariant. We evaluate it for several particular models and find it is one for self-dual models, and is related to the modulus $k$ (or $k'$) for the Ising, Kashiwara-Miwa and chiral Potts models.

This paper is intended as a contribution to the volume dedicated to the sixtieth birthday of Barry M. McCoy, which event was marked in Okayama and Kyoto in February 2001.

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1 Introduction

I have known Barry McCoy since 1972, when I spent six months at Stony Brook. His enthusiasm for theoretical physics is infectious, and is coupled with a great tenacity in tackling long and extremely hard calculations. I am still impressed by the evaluation he did then, with Johnson and Krinsky, of the correlation length of the eight-vertex model.\[1\] In this calculation it is not enough merely to find the next-largest eigenvalue of the transfer matrix. Instead one has to determine a whole band of eigenvalues, then perform the sum over this band. In the thermodynamic limit this sum becomes an integration, and one evaluates it by a saddle-point method. A tour de force!

Much of the work in solvable models stems from the Yang-Baxter relation, in particular its star-triangle form. Here I shall show how one can quite easily use this relation to show that a certain expression $I$ is invariant, by which I mean that it is independent of the rapidities. It is therefore a temperature-like variable, being a function only of the modulus and crossing parameter. For self-dual $\mathbb{Z}_N$ invariant models $I$ is one.

We evaluate $I$ for the Ising, self-dual Potts, self-dual Ashkin-Teller, Kashiwara-Miwa, chiral Potts and Fateev-Zamolodchikov models. In all cases it is either one, or is related to the modulus $k$ (or $k'$).

2 Star-triangle relation

Consider a model on the square lattice where spins take values 0, ..., $N - 1$. Two adjacent spins $a$ and $b$ interact with a Boltzmann weight function $W_{pq}(a, b)$ if $b$ is horizontally to the right of $a$, and $\overline{W}_{pq}(a, b)$ if $b$ is vertically above $a$, as in the figure. As usual, $p$ and $q$ are rapidity variables, associated with the broken lines.

\[ W_{pq}(a, b) \quad \overline{W}_{pq}(a, b) \]
To be as general as possible, we should also allow a rapidity-independent field weight $S(a)$ for each spin $a$ on the lattice. Then the star-triangle relation is:

$$\sum_d S(d) W_{qr}(a,d) W_{pr}(c,d) W_{pq}(d,b) = R_{pqr} W_{pq}(a,c) W_{pr}(b,a) W_{qr}(b,c)$$

(1)

$$\sum_d S(d) W_{qr}(d,a) W_{pr}(d,c) W_{pq}(b,d) = R_{pqr} W_{pq}(a,c) W_{pr}(b,a) W_{qr}(b,c)$$

where $R_{pqr}$ is some factor independent of the spins $a, b, c$.

It is helpful to consider a honeycomb lattice with edge weights $W_{qr}, W_{pr}, W_{pq}$, and a triangular lattice with weights $W_{qr}, W_{pr}, W_{pq}$, both with site weights $S$. Then the two relations (1) are the conditions for them to be related, working from either down-pointing or up-pointing stars or triangles.

For most of the solved two-dimensional models the weight functions $W_{pq}(a,b)$ and $W_{pq}(a,b)$ are symmetric, i.e. unchanged by interchanging the spin arguments $a$ and $b$. Clearly the two relations (1) are then equivalent. The only asymmetric model satisfying the star-triangle relation known to the author is the chiral Potts model, but for this model $S(a) = 1$ and $W_{pq}(a,b), W_{pq}(a,b)$ depend on $a, b$ only via their difference $a - b$, modulo $N$. It follows that one can convert one of the relations (1) into the other by negating (modulo $N$) all spins.

Thus the two relations (1) seem to be equivalent for all known models. We shall not in fact use this, but we shall use a property that is certainly true of all the models considered here, namely that there exist spin-independent quantities $P_{pq}, \overline{P}_{pq}$ such that

$$\prod_b W_{pq}(a,b) = \prod_b W_{pq}(b,a) = P_{pq},$$
$$\prod_b W_{pq}(a,b) = \prod_b W_{pq}(b,a) = \overline{P}_{pq},$$

(2)

for all values of the spin $a$. Let $Y_{pq}$ be the $N$ by $N$ matrix with entry $W_{pq}(a,b)$ in position $(a,b)$, and $\overline{Y}_{pq}$ the matrix with corresponding entry $\overline{W}_{pq}(a,b)$.

*It may be possible to absorb $S(a)$ into $W_{pq}(a,b)$ and/or $\overline{W}_{pq}(a,b)$, but to the author it seems clearer not to do so.

†There are other solvable planar models, derived from the three-dimensional Zamolodchikov model, which are asymmetric, but these satisfy the “star-star” relation, rather than the star-triangle.
Then the properties (2) state that the row and column products of \( Y \) are equal to one another and are the same for all rows and columns. Similarly for \( \overline{Y} \). It is probably not essential to assume these properties: it may be that they are implied by (1), to within a gauge transformation. However, they are in fact satisfied for our specific models, and they do greatly simplify the following argument.

In addition to the matrices \( Y_{pq}, \overline{Y}_{pq} \), we shall also need the diagonal matrices \( S, X_{pq|c}, X'_{pq|c}, \overline{X}_{pq|c}, \overline{X}'_{pq|c} \) with entries \( \delta(a, b) \), \( \delta(a, b) \), \( \delta(a, b) \), \( \delta(a, b) \), \( \delta(a, b) \) in position \( (a, b) \), respectively. Note that \( X_{pq|c}, X'_{pq|c}, \overline{X}_{pq|c}, \overline{X}'_{pq|c} \) depend on the spin \( c \).

We also define

\[
\begin{align*}
  s & = [S(0) \cdots S(N-1)]^{1/N} . 
\end{align*}
\]

3 The factor \( R_{pqr} \)

For the chiral Potts model, Baxter, Perk and Au-Yang [3] conjectured the form of the factor \( R_{pqr} \). This conjecture was verified by Matveev and Smirnov [4]: their argument generalizes at once to any model satisfying (1), as we shall now show.

Regard the spin \( c \) as fixed and think of each side of (1) as the element \( (a, b) \) of some matrix. Then the relations take the matrix form

\[
\begin{align*}
  \overline{Y}_{qr} S X_{pr|c} \overline{Y}_{pq} & = R_{pqr} X_{pq|c} \overline{Y}_{pr} X_{qr|c} \\
  \overline{Y}^T_{qr} S X'_{pr|c} \overline{Y}^T_{pq} & = R_{pqr} X'_{pq|c} \overline{Y}^T_{pr} X'_{qr|c} ,
\end{align*}
\]

\( \overline{Y}^T \) being the transpose of \( Y \). Since \( c \) takes \( N \) values, (4) consists of \( 2N \) separate matrix equations.

One can now obtain \( R_{pqr} \) by taking determinants. Because of (2) we obtain not \( 2N \) results, but only one, namely

\[
R_{pqr} = f_{qr} f_{pq} / f_{pr} 
\]

(to within an undetermined factor of an \( N \)th root of unity).

Here \( f_{pq} \) is defined by

\[
f_{pq}^N = P_{pq}^{-1} \det (S \overline{Y}_{pq}) .
\]

When \( S(a) = 1 \) we regain the formulae of [3, 4].
4 The invariant $I$

Define also $\overline{f}_{pq}$ by

$$\overline{f}_{pq}^N = \overline{f}_{pq}^{-1} \det (SY_{pq}) \quad .$$  \hfill (7)

We obtained (5) by holding $c$ fixed and regarding each side of the star-triangle relations as elements $(a, b)$ of a matrix product. Suppose instead we hold $a$ fixed and think of them as elements $(c, b)$. Then the resulting matrix relation is

$$Y_{pr}S\overline{X}_{qr|a} \overline{Y}_{pq} = \mathcal{R}_{pqr} X_{pq|a} Y_{qr} \overline{X}_{pr|a} \quad ,$$
$$Y_{pr}^T S\overline{X}_{qr|a}^T \overline{Y}_{pq}^T = \mathcal{R}_{pqr} X_{pq|a} Y_{qr}^T \overline{X}_{pr|a}^T \quad .$$  \hfill (8)

Taking determinants, these give (for all $a$ and to within an $N$th root of unity)

$$\mathcal{R}_{pqr} = f_{pq} \overline{f}_{pr}/\overline{f}_{qr} \quad .$$  \hfill (9)

Finally, holding $b$ fixed and regarding the relations as elements $(c, a)$ of a matrix, we get

$$Y_{pr} S\overline{X}_{pq|b} \overline{Y}_{qr}^T = \mathcal{R}_{pqr} X_{pq|b}^T Y_{pq} \overline{X}_{pr|b} \quad ,$$
$$Y_{pr}^T S\overline{X}_{pq|b}^T \overline{Y}_{qr} = \mathcal{R}_{pqr} X_{pq|b} Y_{qr} \overline{X}_{pr|b}^T \quad ,$$  \hfill (10)

$$\mathcal{R}_{pqr} = f_{qr} \overline{f}_{pr} / \overline{f}_{pq} \quad .$$  \hfill (11)

The relations (5), (9), (11) are mutually consistent if and only if

$$f_{pq} \overline{f}_{pq} = f_{pr} \overline{f}_{pr} = f_{qr} \overline{f}_{qr} \quad .$$  \hfill (12)

The only way this can happen is for the quantity

$$I = N^{-1} f_{pq} \overline{f}_{pq} \quad .$$  \hfill (13)

to be independent of $p$ and $q$ . (We have introduced the factor $1/N$ for later convenience.)

Thus it follows, for any model satisfying the star-triangle relations and having the properties (2), that $I$ is an invariant, independent of the rapidities $p$ and $q$. It is independent of the normalization of $W_{pq}(a, b)$, $\overline{W}_{pq}(a, b)$, i.e. it is unchanged by multiplying them by factors independent of the spins $a, b$.!
5 Inversion and rotation relations

All of the solvable models we shall discuss can be normalized so as to have the properties

\[ W_{pq}(a, b) = 1, \quad \overline{W}_{pq}(a, b) = \delta(a, b) / S(a) . \]  

(14)

In particular, \( W_{pp}(a, b) = P_{pp} = f_{pp} = 1. \) Setting \( r = p \) in (4), then interchanging \( p \) with \( q \) and using (5), it follows that

\[ \overline{Y}_{pq} S Y_{qp} = (f_{pq} f_{qp}) S^{-1} . \]  

(15)

Further, there always exists mapping \( R : p \rightarrow Rp \) such that

\[ \overline{W}_{pq}(a, b) = W_{q,Rp}(a, b), \quad W_{pq}(a, b) = \overline{W}_{q,Rp}(b, a) . \]  

(16)

This means that replacing \( p, q \) by \( q, Rp \) is equivalent to rotating the lattice through 90°. For all the models except the chiral Potts model, the mapping is very simple: \( Rp = p + \lambda \), where \( \lambda \) is a fixed “crossing parameter”. These relations (16) imply that

\[ \overline{f}_{pq} = f_{q,Rp} . \]  

(17)

If \( \kappa_{pq} \) is the partition function per site in the large-lattice limit, then it follows that

\[ \kappa_{pq} = \kappa_{q,Rp} = f_{pq} f_{qp} / \kappa_{qp} . \]  

(18)

Together with an appropriate analyticity assumption about \( \kappa_{pq} \) in a domain containing the inversion points \( (p, p) \) and \( (p, Rp) \), these relations can be used (at least for the models other than the chiral Potts model) to obtain \( \kappa_{pq} \) [5, 6].

6 \( Z_N \)-invariant models: duality

Three of the models we shall discuss, namely the Ising, self-dual Potts and chiral Potts, are \( Z_N \) invariant. That is, \( S(a) = 1 \) and

\[ W_{pq}(a, b) = W_{pq}(a - b), \quad \overline{W}_{pq}(a, b) = \overline{W}_{pq}(a - b) , \]  

(19)

the spin-difference functions \( W_{pq}(i), \overline{W}_{pq}(i) \) being periodic in \( i \) of period \( N \). The matrices \( Y_{pq}, \overline{Y}_{pq} \) are therefore cyclic (Toeplitz).
In this case there is a simple duality property \([7, \S 6.2], [8, \S 5]\). To within boundary conditions, the partition function of the square lattice model is unchanged by replacing \(W_{pq}(j), \overline{W}_{pq}(j)\) by the Fourier transforms

\[
W^{(d)}_{pq}(j) = N^{-1/2} \sum_{n=0}^{N-1} \omega^{jn} \overline{W}_{pq}(n),
\]

\[
\overline{W}^{(d)}_{pq}(j) = N^{-1/2} \sum_{n=0}^{N-1} \omega^{-jn} W_{pq}(n),
\]

where \(\omega = \exp(2\pi i/N)\) is the primitive \(N\)th root of unity.

Using the fact that the determinant of a matrix is the product of its eigenvalues, which for the Toeplitz matrices \(Y_{pq}, Y_{pq}\) are the above sums, it follows that

\[
I^N = P^{(d)}_{pq} \overline{P}^{(d)}_{pq} / (P_{pq} \overline{P}_{pq}),
\]

\(P^{(d)}_{pq}, \overline{P}^{(d)}_{pq}\) being defined by (2) with \(W, \overline{W}\) replaced by the duals \(W^{(d)}, \overline{W}^{(d)}\).

Making appropriate choices of \(N\)th roots, it follows that \(I\) is inverted by such a duality transformation, and at the self-dual point

\[
I = 1.
\]

### 7 Particular Models

#### Ising model

For the Ising model with horizontal and vertical interaction coefficients \(J, \overline{J}\), \(N = 2\) and

\[
Y_{pq} = \begin{pmatrix} e^K & e^{-K} \\
       e^{-K} & e^K \end{pmatrix}, \quad \overline{Y}_{pq} = \begin{pmatrix} e^{-K} & e^{-K} \\
e^{-K} & e^K \end{pmatrix},
\]

where \(K = J/k_B T, \overline{K} = \overline{J}/k_B T\). Hence

\[
I^2 = \sinh 2K \sinh 2\overline{K}.
\]

This is indeed the basic invariant of the Ising model - being the modulus \(1/k\) or \(k\) of its elliptic function parametrization \([4, \text{eqns. } (2.1a) \text{ and } (2.1b)], [10]\)
The Star-Triangle Relation

For \( I^2 > 1 \) the model is ferromagnetically ordered, for \( I^2 < 1 \) it is disordered, and when \( I^2 = 1 \) it is critical. The Boltzmann weights \( e^{-2K}, e^{-2\overline{K}} \) are functions of the rapidity variables \( p, q \), in fact elliptic functions of \( p - q \). We shall not pursue this further as the Ising model is the \( N = 2 \) case of the Kashiwara-Miwa and chiral Potts models discussed below.

Self-dual Potts model

The ordinary Potts model \([11]\) is an \( N \) (or \( q \)) - state model with weights \( S(a) = 1, \)

\[
W(a, b) = e^{-K} + (1 - e^{-K})\delta_{ab}, \quad \overline{W}(a, b) = e^{-\overline{K}} + (1 - e^{-\overline{K}})\delta_{ab} .
\]

It is solvable at its self-dual point, i.e. when

\[
(e^K - 1) (e^{\overline{K}} - 1) = N ,
\]

when it is equivalent to the six-vertex model \([7, \S 12.3]\). At this point the rapidity parametrization is

\[
e^{K} = \frac{\sin(\mu + q - p)}{\sin(\mu - q + p)} , \quad e^{\overline{K}} = \frac{\sin(2\mu - q + p)}{\sin(q - p)} ,
\]

where \( \mu \), the crossing parameter, is defined by

\[
N^{1/2} = 2 \cos \mu .
\]

For \( N < 4 \), \( \mu, p, q \) are real. For \( N > 4 \), they are pure imaginary. As \( N \rightarrow 4 \), they all become small but enter the equations only via their ratios.

It follows that

\[
f_{pq} = e^K(1 - e^{-\overline{K}}) = 2 \cos \mu \sin(\mu + q - p) / \sin(2\mu - q + p)
\]

\[
\overline{f}_{pq} = e^{\overline{K}}(1 - e^{-K}) = 2 \cos \mu \sin(2\mu - q + p) / \sin(\mu + q - p) ,
\]

and hence

\[
I = 1 ,
\]

in agreement with \([22]\).
Ashkin-Teller model

The Ashkin-Teller model \([12]\) is usually formulated in terms of a pair of two-valued spins at each site, interacting with neighbouring pairs via Ising and four-spin interactions \([7, \S 12.9]\). This means that it is a four-state model, with \(S(a) = 1\) and interaction matrices

\[
Y_{pq} = \begin{pmatrix}
\omega_0 & \omega_1 & \omega_2 & \omega_3 \\
\omega_1 & \omega_0 & \omega_3 & \omega_2 \\
\omega_2 & \omega_3 & \omega_0 & \omega_1 \\
\omega_3 & \omega_2 & \omega_1 & \omega_0
\end{pmatrix}, \quad Y_{pq} = \begin{pmatrix}
\overline{\omega}_0 & \overline{\omega}_1 & \overline{\omega}_2 & \overline{\omega}_3 \\
\overline{\omega}_1 & \overline{\omega}_0 & \overline{\omega}_3 & \overline{\omega}_2 \\
\overline{\omega}_2 & \overline{\omega}_3 & \overline{\omega}_0 & \overline{\omega}_1 \\
\overline{\omega}_3 & \overline{\omega}_2 & \overline{\omega}_1 & \overline{\omega}_0
\end{pmatrix}.
\] (31)

These matrices are not Toeplitz, but they are cyclic in two-by-two blocks. The restrictions (2) are obviously satisfied. The weights \(\omega_0, \ldots, \omega_3\) can be expressed in terms of interaction coefficients, but we shall just take them to be given parameters.

By performing a duality transformation on one of the two sets of Ising spins, Wegner \([13]\), \([7, \S 12.9]\) showed that this model can be converted to a staggered square-lattice eight-vertex model, with weights

\[
(a, b, c, d) = (\omega_0 + \omega_1, \omega_2 - \omega_3, \omega_2 + \omega_3, \omega_0 - \omega_1) / \sqrt{2}
\] (32)

on one sub-lattice, and weights

\[
(\overline{a}, \overline{b}, \overline{c}, \overline{d}) = (\overline{\omega}_0 + \overline{\omega}_1, \overline{\omega}_2 - \overline{\omega}_3, \overline{\omega}_2 + \overline{\omega}_3, \overline{\omega}_0 - \overline{\omega}_1) / \sqrt{2}
\] (33)

on the other.

This staggered model has not been solved and we know of no star-triangle relation relevant to it. However, it can be solved when the two sets of eight-vertex weights are proportional to one another, as it is then the regular model: if

\[
(\overline{a}, \overline{b}, \overline{c}, \overline{d}) = \xi (a, b, c, d)
\] (34)

then \(a, b, c, d\) can be parametrized as elliptic functions of \(p - q\) so as to satisfy \((1)\). However, we do not need this parametrization to note from \((31) - (34)\) that

\[
\det Y_{pq} = 16 \xi^4 P_{pq}, \quad \det Y_{pq} = 16 \xi^{-4} \overline{P}_{pq},
\] (35)

and hence (to within choices of \(N\)th roots) \(f_{pq} = 2\xi, \overline{f}_{pq} = 2/\xi\), and

\[
I = 1.
\] (36)
Thus \( I = 1 \), as for the self-dual \( Z_N \)-invariant models. In fact, this solvable case of the Ashkin-Teller model is self-dual [13, 14], but it is not critical. We know that the associated regular eight-vertex model can be naturally parametrized in terms of elliptic functions of some modulus \( k \) [15, 16]. Only when this \( k \) is \( \pm 1 \), 0 or \( \infty \) can the model be critical.

The explanation of this apparent contradiction is that the general Ashkin-Teller model has two critical temperatures, on either side of the self-dual temperature defined by (34), which map into one another by duality [13, 14]. (The duality relation is easily obtained from the above remarks: two Ashkin-Teller models are dual to one another if they transform into staggered eight-vertex models differing only in interchanging the two sub-lattices. This means that the \( \omega \)-weights of one are proportional to the eigenvalues of the \( Y, \tilde{Y} \) matrices of the other.)

Indeed, when the four-spin interaction vanishes the Ashkin Teller model becomes two independent Ising models. The solvability condition (34) then implies that these two models are dual to one another, and \( I \) is the product of the individual \( I \)s given by (24). This product is necessarily one.

**Kashiwara-Miwa model**

In 1986 Kashiwara and Miwa [17] presented an \( N \)-state generalization of the Fateev-Zamolodchikov model that breaks the \( Z_N \)-symmetry, but retains the reflection symmetry and the rapidity difference property. This model has also been studied by Hasegawa and Yamada [18], and by Gaudin [19]. Let \( K, K' \) be the complete elliptic integrals of the first kind of moduli \( k, k' \), and let \( \tilde{q} = \exp(-\pi K'/K) \) be the corresponding “nome”. Then the elliptic theta functions \( H(u), \Theta(u) \) of argument \( u \) and modulus \( k \), as defined in section 8.181 of [20], are

\[
H(u) = 2\tilde{q}^{1/4} \sin(\pi u/2K) \prod_{n=1}^{\infty} (1 - 2\tilde{q}^{2n} \cos(\pi u/K) + \tilde{q}^{4n})(1 - \tilde{q}^{2n})
\]

\[
\Theta(u) = \prod_{n=1}^{\infty} (1 - 2\tilde{q}^{2n-1} \cos(\pi u/K) + \tilde{q}^{4n-2})(1 - \tilde{q}^{2n-2})
\]  

(37)

We also use the function \( H_2(u) = H(u)\Theta(u) \). One can write \( K, K', k, k' \)
in terms of infinite products involving $\tilde{q}$, in particular

$$\frac{2k'K}{\pi} = \prod_{m=1}^{\infty} \left( \frac{1 - \tilde{q}^m}{1 + \tilde{q}^m} \right)^2.$$  \hspace{1cm} (38)

Let $\zeta$ be some arbitrary integer and define two functions

$$r_v(n) = \prod_{j=1}^{n} \frac{H[K(2j - 2 + v)/N]}{H[K(2j - v)/N]} \quad \text{and} \quad t_v(n) = \prod_{j=1}^{n} \frac{\Theta[K(2j - 2 + v)/N]}{\Theta[K(2j - v)/N]},$$

where $v$ is real and the argument $n$ is an integer. These functions are periodic:

$$r_v(n) = r_v(n + N), \quad t_v(n) = t_v(n + N).$$

Then the weights of the Kashiwara-Miwa model are (for integers $a, b$)

$$S(a) = \Theta[2K(2a + \zeta)/N]/\Theta(0),$$

$$W_{pq}(a, b) = r_{1-q+p}(a - b) t_{1-q+p}(a + b + \zeta),$$

$$\overline{W}_{pq}(a, b) = r_{q-p}(a - b) t_{q-p}(a + b + \zeta).$$

Note that for this model, unlike all the others we consider, the site weight function $S(a)$ is not unity. Curiously, $W_{pq}(a, b)$ and $\overline{W}_{pq}(a, b)$ are products of functions of the spin difference $a - b$ and of the spin sum $a + b$. Because $r_v(n) = r_v(-n)$ is an even function of $n$, $W_{pq}(a, b)$ and $\overline{W}_{pq}(a, b)$ are each unchanged by interchanging the spins $a, b$. The model is therefore reflection-symmetric, i.e. non-chiral.

Define also

$$g = \prod_{m=1}^{\infty} \frac{1 - \tilde{q}^{Nm}}{1 + \tilde{q}^{Nm}} \frac{1 + \tilde{q}^m}{1 - \tilde{q}^m}.$$  \hspace{1cm} (42)

Then, based on numerical calculations, the known $N = 2$ Ising case and the Fateev-Zamolodchikov result (62) below, I conjecture that

$$f_{pq} = N^{1/2} g \prod_{j=1}^{N'} \frac{H_2[K(q - p + 2j - 1)/N]}{H_2[K(2j + p - q)/N]}.$$  \hspace{1cm} (43)

where $N' = [N/2]$ is the integer part of $N/2$. (The constant $g$ is determined by the requirement that $f_{pp} = 1$.)

Since $f_{pq}$ is obtained from $\overline{f}_{pq}$ by replacing $q - p$ by $1 + p - q$, it then follows that

$$I = g^2 = k_N^2 K_N/(k'K) \quad \text{subject to} \quad k'K = \frac{2k'K}{\pi}.$$  \hspace{1cm} (44)
where $k'_N, K_N$ are the $k', K$ corresponding to the elliptic nome $\tilde{q}^N$. This explicit expression for $I$ is indeed independent of the rapidities $p, q$. Note also that $f_{pq}$ is a single-valued meromorphic function of $q-p$, despite the $N$th root implied by the definition of $f_{pq}$; this is consistent with the observation that $f_{pq}$ is defined by (3) and (1) as a single-valued function of the Boltzmann weights (with no $N$th root), to within factors that are functions of either $p$ or $q$ only.

The factor $f_{pq}f_{qp}$ in (15) also simplifies. Define the elliptic function

$$G(u, \tilde{q}) = \frac{H_2(2Ku/\pi)}{H_2(K + 2Ku/\pi)} = \tan u \prod_{m=1}^\infty \frac{1 - 2\tilde{q}^m \cos 2u + \tilde{q}^{2m}}{1 + 2\tilde{q}^m \cos 2u + \tilde{q}^{2m}} \ . \quad (45)$$

Then

$$f_{pq}f_{qp} = N g^2 G(w, \tilde{q})/G(Nw, \tilde{q}^N) \ , \quad (46)$$

where $w = \pi(q-p)/(2N)$ (c.f. equation 13 above and proposition 2 of Ref. [17]).

**Chiral Potts model**

In 1987 Au-Yang, McCoy, Perk and others found solutions of the star-triangle relations for the three- and four-state chiral Potts model [21, 22]. These were generalized to an arbitrary number $N$ of states in 1988 [3]. In this model, the rapidity is a point $(a_p, b_p, c_p, d_p)$ on the homogeneous curve

$$a_p^N + k'b_p^N = k'd_p^N \ , \quad k'a_p^N + b_p^N = kc_p^N \ , \quad (47)$$

where $k$ and $k'$ are two given constants, related by

$$k^2 + k'^2 = 1 \ . \quad (48)$$

Related quantities [23] are $x_p, y_p, \mu_p, t_p$, defined by

$$x_p = a_p/d_p \ , \quad y_p = b_p/c_p \ , \quad \mu_p = d_p/c_p \ , \quad (49)$$

$$t_p = x_p y_p = a_p b_p/c_p d_p \ .$$

They satisfy

$$x_q^N + y_q^N = k(1 + x_q^N y_q^N) \ , \quad kx_q^N = 1 - k'\mu_q^{-N} \ , \quad ky_q^N = 1 - k'\mu_q^N \ . \quad (50)$$
The Star-Triangle Relation

The chiral Potts model is $Z_N$-invariant, so we can write the Boltzmann weight functions as in (19). Setting $\omega = e^{2\pi i/N}$, the functions $W_{pq}(a - b)$, $\overline{W}_{pq}(a - b)$ are

$$W_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^n \frac{y_q - \omega^j x_p}{y_p - \omega^j x_q}, \quad \overline{W}_{pq}(n) = \left(\frac{\mu_p}{\mu_q}\right)^n \prod_{j=1}^n \frac{\omega x_p - \omega^j x_q}{y_q - \omega^j y_p}. \quad (51)$$

These Boltzmann weights are positive real if $x_p, x_q, y_p, y_q, \omega x_p$ all lie on the unit circle and are arranged sequentially in the widdershins direction.

The function $f_{pq}$ has been evaluated \[24, 25\]. In the present normalization, with $W_{pq}(0) = \overline{W}_{pq}(0) = 1$, it is

$$f_{pq}^N = \prod_{j=1}^{N-1} \left\{ \frac{\mu_q(1 - \omega^j)(t_p - \omega^j t_q)(x_q - \omega^j y_p)}{\mu_p(x_p - \omega^j x_q)(y_p - \omega^j y_q)(x_p - \omega^j y_q)} \right\}^j. \quad (52)$$

When $N = 2$ it reduces to the Ising model.

Let $R$ be the automorphism that takes the point $(a_p, b_p, c_p, d_p)$ to $(a_{R_p}, b_{R_p}, c_{R_p}, d_{R_p}) = (b_p, \omega a_p, d_p, c_p)$. Then the rotation relations (14) are satisfied. Using (13), (17) and (52), we deduce that

$$I = f_{pq}f_{qp} = 1/k'/(N-1)/N. \quad (53)$$

Thus $I$ is indeed rapidity-independent for the chiral Potts model, being simply a power of $k'$. It is inverted by the duality relation $k' \to 1/k' [8, §5].$

The result (53) is not new, being given in equation (2.47) of Ref. [21]. What we have done here is show that its structure is a direct consequence of the star-triangle relation, and is shared by other solvable models.

We can also deduce that

$$f_{pq}f_{qp} = \frac{N(t_p^N - t_q^N)(x_p - x_q)(y_p - y_q)}{(x_p^N - x_q^N)(y_p^N - y_q^N)(t_p - t_q)}, \quad (54)$$

in agreement with (2.48) of Ref. [25].

It is not obvious from (52) that the rhs is the $N$th power of a single-valued function (to within factors that depend only on $p$, or only on $q$). However, the only singularities whose location depends on both $p$ and $q$ occur at the zeros and poles of the bracketted expressions. These can only occur at certain points $\mathcal{P}$ on the $(x_q, y_q)$ or $(x_p, y_p)$ Riemann surface, namely when $x_q^N = x_p^N$. 


and \( y_q^N = y_p^N \), or when \( x_q^N = y_p^N \) and \( y_q^N = x_p^N \). We can count these zeros and poles by postulating the existence of functions \( \Theta_{ij}, \bar{\Theta}_{ij} \) of \( p \) and \( q \) [20], such that \( \Theta_{ij} \) has simple zeros when

\[
x_q = \omega^i y_p \quad \text{and} \quad y_q = \omega^j x_p ,
\]

and \( \bar{\Theta}_{ij} \) has simple zeros when

\[
x_q = \omega^i x_p \quad \text{and} \quad y_q = \omega^j y_p .
\]

Then, to within factors that are analytic and non-zero at all such points \( \mathcal{P} \)

\[
x_q - \omega^i y_p = \prod_{m=0}^{N-1} \Theta_{i,m} , \quad y_q - \omega^j x_p = \prod_{m=0}^{N-1} \Theta_{m,j}
\]

\[
x_q - \omega^i x_p = \prod_{m=0}^{N-1} \bar{\Theta}_{i,m} , \quad y_q - \omega^j y_p = \prod_{m=0}^{N-1} \bar{\Theta}_{m,j}
\]

\[
t_q - \omega^i t_p = \prod_{m=0}^{N-1} \Theta_{m,i-m} \bar{\Theta}_{m,i-m} .
\]

Formally substituting these expressions into (52), we find that each \( \Theta_{ij} \) or \( \bar{\Theta}_{ij} \) function occurs with a power \( N \), 0 or \(-N\) on the rhs, so we can take the \( N \)th root and obtain

\[
f_{pq} = \prod_{i=1}^{N-1} \prod_{j=1}^{N-1} \Theta_{N-i,i+j} / \bar{\Theta}_{ij} ,
\]

to within factors that are analytic and non-zero at all points \( \mathcal{P} \). So \( f_{pq} \) certainly has no branch point singularities at the points \( \mathcal{P} \); at worst it has simple poles or zeros.

**Fateev-Zamolodchikov model**

When \( k = 0 \) we can choose the parameters of the chiral Potts model so that

\[
x_p = e^{i\pi p/N} , \quad y_p = e^{i\pi (p+1)/N} , \quad \mu_p = 1 ,
\]
where $p$ is the rapidity variable. Making the corresponding choices for $x_q, y_q, \mu_q$, (51) becomes

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{\sin[\pi(p - q + 2j - 1)/2N]}{\sin[\pi(q - p + 2j - 1)/2N]}, \quad (60)$$

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{\sin[\pi(q - p + 2j - 2)/2N]}{\sin[\pi(p - q + 2j)/2N]} . \quad (61)$$

We obtain precisely the same result if we set $k = 0$ (and hence $\tilde{\eta} = 0$) in the Kashiwara-Miwa model, so the two models then coincide.

The resulting model is the Fateev-Zamolodchikov model [27]. Its weights (60) are real and positive if $0 < q - p < 1$. It is a $Z_N$ model, self-dual and critical. When $N = 3$ it is equivalent to the self-dual Potts model (with $\mu = \pi/6$ and $p, q$ scaled by $\mu$).

From both (61) and (63) we deduce that

$$I = 1 , \quad (61)$$

in agreement with (22).

We find that the right-hand side of the chiral Potts result (52) is now indeed a perfect $N$th power, and that

$$f_{pq} = N^{1/2} \prod_{j=1}^{N'} \frac{\sin[\pi(q - p + 2j - 1)/2N]}{\sin[\pi(p - q + 2j)/2N]} , \quad (62)$$

where $N' = [N/2]$. This agrees with, and is part of the evidence for, the conjecture (13) for the Kashiwara-Miwa model.

8 Summary

The method of Matveev and Smirnov [4] can be used to show that the quantity $I$ defined by (3), (7), (13) is independent of the rapidities $p$ and $q$. For particular models this is not necessarily a new observation, but the realisation that it is a direct consequence of the star-triangle relation does provide a unifying feature for the known solvable edge-interaction models. We hope this is a small step forward in the subject to which Barry McCoy has made so many outstanding contributions.
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