Dependent conditional value-at-risk for aggregate risk models

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A R T I C L E   I N F O

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A B S T R A C T

Risk measure forecast and model have been developed in order to not only provide better forecast but also preserve its (empirical) property especially coherent property. Whilst the widely used risk measure of Value-at-Risk (VaR) has shown its performance and benefit in many applications, it is in fact not a coherent risk measure. Conditional VaR (CoVaR), defined as mean of losses beyond VaR, is one of alternative risk measures that satisfies coherent property. There have been several extensions of CoVaR such as Modified CoVaR (MCoVaR) and Copula CoVaR (CCoVaR). In this paper, we propose another risk measure, called Dependent CoVaR (DCoVaR), for a target loss that depends on another random loss, including model parameter treated as random loss. It is found that our DCoVaR provides better forecast than both MCoVaR and CCoVaR. Numerical simulation is carried out to illustrate the proposed DCoVaR. In addition, we do an empirical study of financial returns data to compute the DCoVaR forecast for heteroscedastic process of GARCH(1,1). The empirical results show that the Gumbel Copula describes the dependence structure of the returns quite nicely and the forecast of DCoVaR using Gumbel Copula is more accurate than that of using Clayton Copula. The DCoVaR is superior than MCoVaR, CCoVaR and CoVaR to comprehend the connection between bivariate losses and to help us exceedingly about how optimum to position our investments and elevate our financial risk protection. In other words, putting on the suggested risk measure will enable us to avoid non-essential extra capital allocation while not neglecting other risks associated with the target risk. Moreover, in actuarial context, DCoVaR can be applied to determine insurance premiums while reducing the risk of insurance company.

1. Introduction

Risk measure forecast has been one of major interests in finance and insurance and developed by academia and practitioners. The common and widely used risk measure is Value-at-Risk (VaR), see e.g. McNeil et al. (2005), Kabaila and Mainzer (2018), Syuhada et al. (2020); Nieto and Ruiz (2016) provided latest review on VaR and its backtesting. It forecasts maximum tolerated risk at certain probability level. Basically, VaR is calculated through the quantile of its loss distribution. Whilst the widely used risk measure of VaR has shown its performance and benefit in many applications, it is in fact not a coherent risk measure.

There have been some efforts done by authors to seek an improvement of VaR, beside describing formulas of VaR and CoVaR as shown in Nadarajah et al. (2016). Their works were derived in two different directions. The first is improvement of VaR forecast accuracy i.e. the coverage probability of VaR forecast is closer to the target nominal or probability level. The example of this is an improved VaR in which the method was developed by Kabaila and Syuhada (2008, 2010) and Syuhada (2020) whilst estimating confidence region by adjusted empirical likelihood to obtain better coverage was proposed by Yan and Zhang (2016). Furthermore, Kabaila and Mainzer (2018) considered linear regression model that consists of approximate VaR and exact VaR in which the former is an unbiased estimator for the latter.

The second improvement to VaR is seeking alternative risk measure(s) that capture coherent property. The commonly used coherent risk measure is the Conditional VaR (CoVaR), defined as mean of losses beyond VaR, see e.g. Artzner et al. (1999), McNeil et al. (2005), Jadhav et al. (2009, 2013), Righi and Ceretta (2015), and Brahimi et al. (2018). Several extensions of CoVaR have been proposed. Jadhav et al. (2013) modified CoVaR by introducing a fixed boundary, instead of infinity, for values beyond VaR. They named the risk measure as Modified CoVaR (MCoVaR). Meanwhile, another extension of CoVaR, called Copula CoVaR (CCoVaR), was suggested by Brahimi et al. (2018) in which they forecast a target risk by involving another dependent risk or associate

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1 In description we use the terms loss(es) and risk(s) interchangeably.

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risk. The use of Copula in this dependent case is crucial. The application of this method may be found when we forecast risk premia (as a target risk) that depends on claim size (as an associate risk). Note that Kang et al. (2019) considered such premium and claim size dependence to forecast VaR and CoVaR by involving Copula.

Motivated by the work of Jadhav et al. (2013) and Brahim et al. (2018), in this paper, we propose an alternative coherent risk measure that is not only "considering a fixed upper bound of losses beyond VaR" but also "taking into account a dependent risk". Our proposed risk measure is called Dependent CoVaR (DCoVaR). When we compute a MCoVaR forecast, it will reduce number of losses beyond VaR and thus make this forecast smaller than the corresponding CoVaR. This is a good feature in risk modeling. We argue that this forecast must also be accompanied by a dependent risk since this risk scenario occurs in practice, see for instance Zhang et al. (2018) and Kang et al. (2019). The risk measure of DCoVaR is superior than MCoVaR, CCoVaR and CoVaR to comprehend the connection between bivariate losses and to help us exceedingly about how optimum to position our investments and elevate our financial risk protection. In other words, putting on the suggested risk measure will enable us to avoid non-essential extra capital allocation while not neglecting other risks associated with the target risk. Moreover, in actuarial context, DCoVaR can be applied to determine insurance premiums while reducing the risk of insurance company.

This paper is organized as follows. Section 2 describes our proposed risk measure of DCoVaR in which its formula relies on joint distribution either classical or Copula. Properties of DCoVaR are also stated. The DCoVaR forecast for Pareto random loss is explained in Section 3. Such forecast is computed for target risk of Pareto and associate risk of Pareto as well. Farlie-Gumbel-Morgenstern and Archimedean Copulas are employed. The target risk may be extended to an aggregate risk. Numerical simulation is carried out. Section 4 considers a real application of DCoVaR forecast for financial returns data (NASDAQ and TWIEX) in which such returns are modeled by heteroscedastic process of GARCH(1,1). Appendix contains all proofs.

2. Description of dependent CoVaR forecast

Suppose that an aggregate loss model $S_{N-k}$ is constructed by a collection of dependent random losses $X_1, X_2, \ldots, X_{N-k}$ given by $S_{N-k} = X_1 + \ldots + X_{N-k}$, for $k = 0, 1, 2, \ldots, N - 1$. The VaR forecast of $S_{N-k}$, at a probability level $a$, is obtained by the inverse of distribution function of $S_{N-k}$ i.e. $VaR_a(S_{N-k}) = F_{S_{N-k}}^{-1}(a) = Q_a$. In practice, the parameter of the model must be estimated from data. Thus, the coverage probability of this VaR forecast is bounded to $O(n^{-1})$ since it takes into account the parameter estimation error. Provided VaR forecast, $Q_a$, the mean of losses beyond VaR to infinity may be calculated, called Conditional VaR (CoVaR). Unlike VaR, the CoVaR forecast preserves subadditivity (thus satisfies coherent property) that makes diversification reasonable. Furthermore, as stated by Koji and Kijima (2003), any coherent risk measure can be represented as a convex combination of CoVaR.

We aim to find a risk measure forecast that calculates the mean of $S_{N-k}$ beyond its VaR up to a fixed value of losses and the $S_{N-k}$ depends on another dependent or associate random loss. Our proposed risk measure forecast, namely Dependent Conditional VaR (DCoVaR), calculates the mean of $S_{N-k}$ in which $Q_a \leq S_{N-k} \leq Q_{a_1}$ and $S_{N-k}$ depends on another random loss $Y$ as follows

$$
\text{DCoVaR}(S_{N-k} | Y) = E[S_{N-k} | Q_a \leq S_{N-k} \leq Q_{a_1}, Q_a(Y) \leq Y \leq Q_{a_1}(Y)],
$$

(1)

where $a = a + (1 - a)^{d+1}$ and $\delta = \delta + (1 - \delta)^{d+1}$ for specified $a$ and $d$. Here, $a$ and $\delta$ denote probability level and exceed level, respectively. Note that such random loss $Y$ may be (i) a single component of $S_{N-k}$, (ii) another aggregate risk model $S_{N-k-1}$, or (iii) a model parameter. Note also that in many applications, the distribution of $S_{N-k}$ and $Y$ may be either non-normal or not specified so that we need a Copula. In what follows, we state our proposed DCoVaR in the two propositions below.

Proposition 1. Let $S_{N-k}$ and $Y$ be two random losses with a joint probability function $f_{S_{N-k},Y}$. Let $a, \delta \in (0, 1)$. The Dependent Conditional VaR (DCoVaR) of $S_{N-k}$ given values beyond its VaR up to a fixed value of losses and a random loss $Y$ is given by

$$
\text{DCoVaR}(a, \delta)_{a, \delta}(S_{N-k} | Y) = \frac{Q_{a_1} - Q_a}{Q_{a_1} - Q_a} \int \frac{s f_{S_{N-k},Y}(s, y) ds \ dy}{Q_{a_1} - Q_a},
$$

(2)

where $Q_a = Q_a(S_{N-k}), Q_{a_1} = Q_{a_1}(Y), a_1 = a + (1 - a)^{d+1}$ and $\delta_1 = \delta + (1 - \delta)^{d+1}$.

In practice, joint probability function is difficult to find unless a bivariate normal distribution is assumed. For the case of joint exponential distribution, we may refer to Kang et al. (2019) for Sarmanov’s bivariate exponential distribution. In most cases, two or more dependent risks rely on Copula in order to have explicit formula of its joint distribution.

Proposition 2. Let $S_{N-k}$ and $Y$ be two random losses with a joint distribution function represented by a Copula $C$. The Dependent Conditional VaR (DCoVaR) of $S_{N-k}$ given values beyond its VaR up to a fixed value of losses and a random loss $Y$ is given by

$$
\text{DCoVaR}(a, \delta)_{a, \delta}(S_{N-k} | Y) = \frac{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)}{C(a_1, \delta_1) - C(a, \delta_1)},
$$

(3)

where $F_{S_{N-k}}$ and $F_Y$ denote distribution functions of $S_{N-k}$ and $Y$, $a_1 = a + (1 - a)^{d+1}$ and $\delta_1 = \delta + (1 - \delta)^{d+1}$.

Remark. According to the method of Brahim et al. (2018) to find CCoVaR, our DCoVaR formula is now represented by

$$
\text{DCoVaR}(a, \delta)_{a, \delta}(S_{N-k} | Y, C) = \frac{\int_a^{a_1} \int_\delta^{\delta_1} (u, v) C(u, v) d u d v}{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)},
$$

(4)

where $F_{S_{N-k}}$ denotes quantile function of $S_{N-k}$, $u = F_{S_{N-k}}(s), v = F_Y(y)$, $a_1 = a + (1 - a)^{d+1}$ and $\delta_1 = \delta + (1 - \delta)^{d+1}$. This formula, however, may not be obtained when no closed form expression of the quantile function is given.

The following properties apply to our proposed DCoVaR. The first property states that the DCoVaR satisfies coherent property of risk measure in particular the subadditivity i.e. the DCoVaR of aggregate loss is no more than aggregate of DCoVaR of individual loss. Meanwhile, the second property shows that the DCoVaR provides better forecast than MCoVaR and CCoVaR.

Property 1. The Dependent Conditional VaR (DCoVaR) is a coherent risk measure.

Property 2. The Dependent Conditional VaR (DCoVaR) has larger risk than or equal to MCoVaR and lower risk than or equal to CCoVaR.

3. DCoVaR forecast for Pareto random loss

Suppose that $X_i$, component for aggregate loss $S_{N-k}$, is a Pareto random loss with parameter $(\gamma_i, \beta_i)$. We consider a dependent random
loss $Y$ that follows a Pareto distribution with parameter $(\gamma_i, \beta_i)$. The distribution functions of $X_i$ and $Y$ are, respectively, $F_{X_i}(x) = 1 - (\beta_i / (x + \beta_i))^\gamma_i$, for $x_i \geq 0$, and $F_Y(y) = 1 - (\beta_i / (y + \beta_i))^\gamma_i$, for $y \geq 0$. Their inverses are easy to find and thus their VaR’s are straightforward i.e. $\text{VaR}_a(X_i) = Q_a = \beta_i [(1 - a)^{-\gamma_i} - 1]$ and $\text{VaR}_a(Y) = Q_a = \beta_i [(1 - \delta)^{-\gamma_i} - 1]$. In what follows, we provide some examples.

**Example 1. DCoVaR Forecast of a Pareto Risk with a Pareto Marginal.** Suppose that $S_1^i = X_i$, the respective risk measure of DCoVaR forecast for $S_1^i$, given $Y$, may be found by using Proposition 2 since we apply a Copula for their distribution function. Specifically, we employ the Farlie-Gumbel-Morgenstern (FGM) Copula: $C_{FGM}(u,v) = uv + \theta_i u (1-u) (1-v)$. If the joint distribution of $S_1^i$ and $Y$, defined by an FGM copula, is $F_{S_1^i,Y}(x,y) = C_{FGM}(F_{S_1^i}(x), F_Y(y))$, where $\theta_i \in [-1,1]$, $i = 1, 2, 3$. Suppose that $\theta_i = 1$ and $\theta_i = 0$. Then, the DCoVaR of $S_1^i$ at levels $\alpha$ and $\delta$, $0 < \alpha, \delta < 1$, is given by

$$D\text{CoVaR}^{(\alpha, \delta)}(S_1^i|Y; C) = \frac{\beta_i (A + 2B) - D}{C(\alpha, \delta_1) - C(\alpha, \delta_2) - C(\alpha, \delta)},$$

(5)

where

$$A = \frac{\gamma_i}{\gamma_i - 1} \left[ (1 - a)^{\frac{\gamma_i - 1}{\gamma_i}} - (1 - a_i)^{\frac{\gamma_i - 1}{\gamma_i}} \right] \left[ (\theta_i + 1)((1 - \delta)^{\gamma_i} - (1 - \delta)^{\gamma_i} - \theta_i (\delta_i^\gamma - \delta^\gamma)) \right],$$

$$B = \frac{\gamma_i}{\gamma_i - 1} \left[ (\theta_i - \delta_i)^\gamma - (1 - \delta)^\gamma \right] \times \left\{ a(1 - a)^{\frac{\gamma_i - 1}{\gamma_i}} - a_i(1 - a_i)^{\frac{\gamma_i - 1}{\gamma_i}} + \frac{\gamma_i}{\gamma_i - 1} \left[ (1 - a)^{\frac{\gamma_i - 1}{\gamma_i}} - (1 - a_i)^{\frac{\gamma_i - 1}{\gamma_i}} \right] \right\},$$

$$D = \left( (1 - \delta)^{\gamma_i} \left[ (1 + \theta_i)(1 - a_i^{\gamma_i} - \beta_i (a_i^\gamma - a^\gamma)) \right] - (\delta_i^\gamma - \delta^\gamma)\theta_i \left[ (1 - a)^{\gamma_i} - a_i^\gamma - a^\gamma \right] \right).$$

and the Copulas are $C(\alpha, \delta_1) = a_1 \delta_1 + \theta_1 a_1 \delta_1 (1 - a_1) (1 - \delta_1), C(\alpha, \delta_2) = a_2 \delta_2 + \theta_2 a_2 \delta_2 (1 - a_2) (1 - \delta_2), C(\alpha, \delta) = \alpha \delta + \theta_1 \alpha \delta (1 - a_1) (1 - \delta), C(\alpha, \delta) = \alpha \delta + \theta_2 \alpha \delta (1 - a_2) (1 - \delta_2)$.

Consider the bivariate losses $(S_1^i, Y)$, $i = 1, 2, 3$. For each couple $(S_1^i, Y)$, we set $\theta_1 = 1, \theta_2 = 0.5$, and $\theta_3 = 0.01$, respectively. The selection of parameters $\theta_i$, $i = 1, 2, 3$ corresponds respectively to the strong, medium, and weak dependences. We have in Fig. 1(a) the comparison of the riskiness of $S_1^1$, $S_1^2$, and $S_1^3$. Notice that, the risk measures of MCoVaR of $S_1^i$ at level $\alpha$ are the same in the three cases. Furthermore, note that DCoVaR coincides with MCoVaR in the independence case ($\theta = 0$) whilst DCoVaR is exactly the same as CCoVaR when $\alpha = d = 0$. The DCoVaR of the loss $S_1^i$ is higher than those of $S_1^2$ and $S_1^3$, respectively. $S_1^1$ is riskier than $S_1^2$ and $S_1^3$. In Fig. 1(b), it is clear that both DCoVaR and MCoVaR of $S_1^i$ are located between the two VaR of $S_1^i$ with different probability levels. It is interesting to note that DCoVaR is smaller than both CCoVaR and VaR but is still greater than MCoVaR (see Fig. 1(c)). This result is in line with Property 2. This fact indicates that DCoVaR is much more flexible than CCoVaR, i.e. DCoVaR can be set equal to or smaller than CCoVaR, or even smaller than VaR, by carefully determining the parameters $a$ and $d$. Note that all of the figures and computations reported in this paper are performed with programs written using MATLAB.

**Example 2.** Consider Example 1. When the shape parameter $\gamma_i = 1$, we have the formula of DCoVaR of $S_1$ at levels $\alpha$ and $\delta$, $0 < \alpha, \delta < 1$ as follows

$$D\text{CoVaR}^{(\alpha, \delta)}(S_1|Y; C) = \frac{\beta_i (\delta_i - \delta)}{C(\alpha, \delta_1) - C(\alpha, \delta_2) - C(\alpha, \delta)} \times \left\{ a_1 - a - \ln(1 - (1 - a)^{\gamma}) \right\} \left\{ a_1 - a - \ln(1 - a_i)^{\gamma} \right\},$$

(6)
Meanwhile, when applying the method of Braham et al. (2018) for finding CCoVaR, we find the DCoVaR as follows

\[
\text{DCoVaR}^{(d,\delta)}(S_N; Y; C) = \frac{\beta_1(\delta_1 - \delta)}{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)} \\
\times \left\{ \ln \left(1 - (1 - a)^\theta \right) \left[ (1 - a_1 - \alpha(1 - \delta_1 - \delta) - 1) (a_1 - a) \right] \right\}.
\]

(7)

**Example 3.** DCoVaR of Pareto risk in Example 2 may be carried out by using a Clayton Copula (which is an Archimedean Copula): \( C^\alpha_{\theta}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \). The resulting DCoVaR forecast, however, is not in a closed form expression.

\[
\text{DCoVaR}^{(d,\delta)}(S_N; Y; C) = \frac{\beta_1}{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)} \\
\times \left\{ \int u \left[ \left( (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} - (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} \right) \right] \\
\times \left[ (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} - (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} \right] \\
\times \left[ (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} - (a_1^{-\theta} + \delta_1^{-\theta} - 1)^{-1/\theta} \right] \right\}.
\]

(8)

**Example 4.** DCoVaR for multivariate risk forecast may be expressed for the case of \( N \) identical dependent Pareto random risks: \( X_1, \ldots, X_N \). Their joint probability function is given by

\[
f(x_1, \ldots, x_N; Y, N) = \frac{\Gamma(y + N)}{\Gamma(y) B^N} \left( 1 + \frac{1}{\beta} \sum_{i=1}^N x_i \right)^{y/N}.
\]

Let \( S_N = X_1 + \ldots + X_N \) and \( Y \) be another Pareto random risk with parameter \( (1, \beta_2) \). Suppose that the joint distribution of \( S_N \) and \( Y \) is defined by a bivariate FGM Copula \( F_{S_N, Y}(y, s) = C^\alpha_{\theta}(F_{S_N}(s), F_Y(y)) \), where \( \theta \in [1, 1] \). Then, for \( N \) even, the DCoVaR of \( S_N \) at levels \( a \) and \( \delta \), \( 0 < a, \delta < 1 \), is given by

\[
\text{DCoVaR}^{(d,\delta)}(S_N; Y; C) = \frac{N y(\delta_1 - \delta)}{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)} \\
\times \left\{ \delta_1 \left[ 1 + \theta x (1 - \delta_1 - \delta) \right] \times \left[ \ln \left(1 - a_1^{-1/N} \right) - \ln \left(1 - a_1^{-1/N} \right) \right] + \frac{N - 1}{N} \left[ \ln \left(1 - a_1^{-1/N} \right) \right] \right\}.
\]

(9)

whilst for \( N \) odd, the DCoVaR of \( S_N \) at levels \( a \) and \( \delta \), \( 0 < a, \delta < 1 \), is given by

\[
\text{DCoVaR}^{(d,\delta)}(S_N; Y; C) = \frac{N y(\delta_1 - \delta)}{C(a_1, \delta_1) - C(a, \delta_1) - C(a_1, \delta) + C(a, \delta)} \\
\times \left\{ \delta_1 \left[ 1 + \theta x (1 - \delta_1 - \delta) \right] \times \left[ \ln \left(1 - a_1^{-1/N} \right) - \ln \left(1 - a_1^{-1/N} \right) \right] - \frac{N - 1}{N} \left[ \ln \left(1 - a_1^{-1/N} \right) \right] \right\}.
\]

(10)

**DCoVaR Forecast for Pareto Random Loss: A Simulation Result**

We carry out a simulation study for calculating DCoVaR forecast. The parameters of Pareto distribution of \( S_1 \) and \( Y \) are, respectively, \( \gamma_1 = \gamma_2 = 1, \beta_1 = 1.5 \) and \( \beta_2 = 1.5 \). Suppose also the model parameter \( \Lambda \) is gamma distributed with shape and scale parameters \( \tau = \omega = 1 \). The probability level \( a \) and excess level \( \delta \) are set above 0.9 whilst we set \( a = d = 0.1 \). Fig. 2, 3 and 4 show the DCoVaR forecast for the above parameters set up. As for comparison, we also plot the MCoVaR and CCoVaR forecasts. For each figure, we have an associate or dependent random loss which is a Pareto random loss, an aggregate \( Y = S_1 \) of Pareto losses, and a parameter model \( Y = \Lambda \) of gamma distribution. It is shown from the figures that the DCoVaR forecast tends to increase as \( \delta \) increases whilst the MCoVaR forecast remains the same. As for the CCoVaR forecast, it is extremely larger than both the DCoVaR and MCoVaR forecasts. If the risk measure of CCoVaR is applied to a financial institution, this will force the institution to allocate a very large extra capital which is not really needed.

Note that, as for the Copula choices, we have used Archimedean Copulas. The Clayton Copula (Fig. 1) function is given by \( C^\alpha_{\theta}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta} \). \( \theta \in [-1, 0] \). Meanwhile, for other Copulas of Gumbel (Fig. 2) and Frank (Fig. 3) the functions are \( C^\alpha_{\theta}(u, v) = \exp \left\{ - \left[ (-\ln u)^\theta + (-\ln v)^\theta \right] \right\} \), \( \theta \in (0, \infty) \) and \( C^\alpha_{\theta}(u, v) = \frac{1}{\theta} \ln \left( 1 - \frac{1 - e^{-\alpha u\theta}}{1 - e^{-\alpha v\theta}} \right) \), \( \theta \in (-\infty, 0) \cup (0, \infty) \), respectively.

The probability level \( a \) (and \( \delta \)) is set above 0.9. Unlike calculating CoVaR forecast, the DCoVaR forecast computation requires two significance levels (or a probability level and an excess level). In particular, the joint significance level is given by

\[
P \left( Q_a \leq S_1 \leq Q_{a, \delta}, Q_d \leq Y \leq Q_{d, \delta} \right) = C_a(\alpha_1, \delta_1) - C_\beta(\alpha_1, \delta_1) - C_\beta(\alpha, \delta)
\]

\[
+ C_\beta(\alpha, \delta).
\]

For the case of \( a = d = 0 \), the joint significance level is \( 1 - a - \delta + C_\beta(\alpha, \delta) \). We use joint significance level to measure the number of violations of the DCoVaR forecast. We generate data of 5000 observations for each \( X_1, Y \), and \( \Lambda \). The DCoVaR forecast is computed by using Proposition 2.

Assessment of accuracy for the DCoVaR forecast is carried out by first observing joint significance level. Before calculating joint significance level, we choose probability levels \( a = 0.90, 0.925, 0.950 \) and excess levels \( \delta = 0.900, 0.950 \). For example, in Table 1 (first row, first column), 2.79% joint significance level is lower than 10%. The number 2.79% is obtained by plugging the specified numbers \( a = \delta = 0.900, \theta = 7.0 \) and \( a = d = 0.1 \) into Clayton Copula formula. This means that the DCoVaR forecast is quite accurate. Then, by calculating the number of violations of the DCoVaR\(^{(d,\delta)}\), it is obtained 1.83% (number of violations is 55, total observations 5000; 55/3000 = 0.0183). Basically, the numbers of violations are the number of sample observations located out of the critical value i.e. more than or equal to DCoVaR\(^{(d,\delta)}\) forecast.

These computations are shown in Table 1, 2 and 3, for Clayton, Gumbel, and Frank Copulas, respectively. The joint significance level 6.61% in Table 2 (first row, first column) is obtained by plugging the same specified numbers \( a, \delta, a, \delta, a, \delta \) as in the Clayton Copula but with different \( \theta = 6.63 \) into Gumbel Copula formula. Moreover, the joint significance level 4.42% in Table 3 (first row, first column) is obtained by plugging
**Fig. 2.** DCoVaR forecast of $S_1$ with various of $Y$ and using Clayton Copula; $Y$ is Pareto distributed, $Y = S_2$ is Pareto distributed, $Y = \Lambda$ is Gamma distributed; (a)-(c) such forecasts are in comparison to MCoVaR forecast only; (d)-(f) such forecasts are in comparison to MCoVaR and CCoVaR forecasts.

**Fig. 3.** DCoVaR forecast of $S_1$ with various of $Y$ and using Gumbel Copula; $Y$ is Pareto distributed, $Y = S_2$ is Pareto distributed, $Y = \Lambda$ is Gamma distributed; (a)-(c) such forecasts are in comparison to MCoVaR forecast only; (d)-(f) such forecasts are in comparison to MCoVaR and CCoVaR forecasts.
Table 1. Joint significance level and number of violations of the DCoVaR\(^{(0.01)}\) forecast of \(S_t\) associated with \(Y\) with Clayton Copula (\(\theta = 7.0\)), \(a = d = 0.1\).

| \(\alpha\) | sig. level (%) | no. violations (%) | \(\alpha\) | sig. level (%) | no. violations (%) |
|------------|----------------|-------------------|------------|----------------|-------------------|
| 0.900      | 2.79           | 55                | 0.950      | 1.43           | 35                |
| 0.925      | 2.13           | 38                | 0.950      | 1.43           | 22                |
| 0.950      | 1.43           | 22                |            |                |                   |

Table 2. Joint significance level and number of violations of the DCoVaR\(^{(0.01)}\) forecast of \(S_t\) associated with \(Y\) with Gumbel Copula (\(\theta = 6.3\)), \(a = d = 0.1\).

| \(\alpha\) | sig. level (%) | no. violations (%) | \(\alpha\) | sig. level (%) | no. violations (%) |
|------------|----------------|-------------------|------------|----------------|-------------------|
| 0.900      | 6.61           | 140               | 0.950      | 2.91           | 83                |
| 0.925      | 5.14           | 119               | 0.950      | 2.91           | 62                |
| 0.950      | 2.91           | 78                |            |                |                   |

again the same specified numbers \(a, \delta, d, a\) as in the Clayton and Gumbel Copulas but with different \(\theta = 25\) into Frank Copula formula.

4. Application to financial returns data

We carry out a numerical analysis of returns data and model it with stochastic volatility processes. In particular, we employ the Generalized Autoregressive Conditional Heteroscedastic (GARCH) model of order one. Consider two returns processes, \([X_{1t}]\) and \([X_{2t}]\). Suppose that each process follows a GARCH\((1,1)\) model defined as

\[
X_{it} = \varepsilon_i \sqrt{h_{it}}, \quad h_{it} = k_0 + k_1 X_{it-1}^2 + \eta_i h_{i,t-1}, \quad i = 1, 2, \tag{11}
\]

where \(h_{it}\) denotes conditional variance whilst \(\varepsilon_i\) denotes standardized innovations that are \(t\)-distributed with degree of freedom \(v\). The parameter restrictions are \(k_0 > 0, k_1 \geq 0, \eta_i > 0\), and \(k_1 + \eta_i < 1\). Furthermore, \(h_{i,t} = Var(X_{i,t} | \mathcal{F}_{i,t-1})\), where \(\mathcal{F}_{i,t-1}\) denotes information set at \(t-1\). Let \(S_t = X_{1t}\) and \(Y_t = X_{2t}\). The conditional DCoVaR forecast of \(S_t\) with an associate risk \(Y_t\) is given by

\[
\text{DCoVaR}^{(a,b)}(S_t | Y_t; C, \mathcal{F}_{t-1}) = E\{S_t | Q^\delta_t < S_t < Q^\alpha_t, Q^\delta_t < Y_t < Q^\alpha_t, \mathcal{F}_{t-1}\}
\]

\[
= \int_{Q^\alpha_t}^{Q^\delta_t} \int_{Q^\alpha_t}^{Q^\delta_t} s_t f(s_t, f(Y_t | \mathcal{F}_{t-1}) | f(s_t | \mathcal{F}_{t-1}) f(y_t | \mathcal{F}_{t-1}) dy_t ds_t
\]

\[
= \frac{P\{Q^\alpha_t < S_t < Q^\delta_t, Q^\alpha_t < Y_t < Q^\delta_t | \mathcal{F}_{t-1}\}}{\text{DCoVaR}^{(a,b)}(S_t | Y_t; C, \mathcal{F}_{t-1})} \tag{12}
\]

where \(f_s(\cdot | \mathcal{F}_{t-1})\) is the conditional probability function of the target risk \(S_t\) on \(\mathcal{F}_{t-1}\). The denominator of (12) is given by

```text
(2.77)
```

```text
(3.97)
```

```text
(4.67)
```

```text
(1.27)
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```text
(2.07)
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```text
(2.21)
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(3.37)
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(2.67)
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(1.91)
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(1.91)
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```text
(0.80)
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```text
(1.17)
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(25)
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(54)
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```text
(1.80)
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(52)
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```text
(1.73)
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(32)
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(1.07)
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```text
(0.40)
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```text
(0.73)
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```text
(1.43)
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(2.13)
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```text
(0.95)
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Fig. 5. Daily returns of NASDAQ and TWIEX. A stylized fact of volatility clustering may be observed for both returns.

Table 4. Parameters estimates of GARCH(1,1)-t model for NASDAQ and TWIEX.

|       | α      | ρ      | ϕ      | t         |
|-------|--------|--------|--------|-----------|
| NASDAQ| 0.0764 | 0.0666 | 0.9678 | 6.1488    |
| TWIEX | 0.0368 | 0.0643 | 0.9082 | 6.9057    |

and \( Q_t^\alpha \) as well as \( Q_t^\delta \) satisfy

\[
P(S_t \leq Q_t^\alpha | z_{t-1}) = t_\alpha \left( \frac{Q_t^\alpha}{\sqrt{k_{10} + k_{11}S_{t-1}^2 + k_9h_{t-1}}} \right)
\]

\[
P(Y_t \leq Q_t^\delta | z_{t-1}) = t_\delta \left( \frac{Q_t^\delta}{\sqrt{k_{50} + k_{51}Y_{t-1}^2 + k_9h_{t-1}}} \right)
\]

where \( t_\alpha \) is the standard \( t \) distribution parameterized by \( \nu \).

**Empirical results**

We have used the data of NASDAQ and TWIEX assets from July 3, 2000 to May 17, 2007, taken from www.yahoofinance.com for total of 1617 observations. We define loss data as the return of an asset formulated as follows

\[ X_{it} = \ln \left( \frac{P_{it}}{P_{i,t-1}} \right) \]

where \( P_{it} \) is the price of \( i \)-th asset at time \( t \), \( i = 1, 2 \). Fig. 5 shows such daily returns. In addition, we may observe that one of the stylized facts of returns, known as volatility clustering, occurs in both NASDAQ and TWIEX returns. Huang et al. (2009) argued that the GARCH(1,1)-t model were appropriate for the returns of NASDAQ and TWIEX. Accordingly, we present the maximum likelihood estimates for such model parameter of GARCH(1,1)-t as in Table 4. These estimates are then used to generate data of returns \( S_t \) and \( Y_t \) following GARCH (1,1)-t model. After obtaining the new data of both returns, we apply Copula to have the joint probability function of the two returns.

In order to calculate the DCoVaR forecast on the left tail, Fig. 6, we do in-sample forecast in which we have used 1000 first observations whilst the out-of-sample is to evaluate forecasting performance. Fig. 6 exhibits the DCoVaR plot we forecast using the Clayton and Gumbel Copulas with marginal distribution, the GARCH(1,1)-t model at significance levels \( \alpha = 0.1 \), and contraction parameters \( \alpha = d = 0 \). In this figure, the DCoVaR of target return of NASDAQ is located nearly beneath the NASDAQ returns, and depicts the expectation of investment loss nicely. The DCoVaR using Gumbel Copula is slightly lower than that using Clayton Copula. As in Table 4 above, Student’s \( t \) distribution is assumed for innovation. Meanwhile, Archimedean Copula is used for the joint distribution function. In particular, we employ Clayton and Gumbel Copulas. The parameter \( \theta \) for each Copula is estimated by maximum likelihood method. We obtain \( \hat\theta_C = 0.4938 \) and \( \hat\theta_G = 1.2905 \), respectively.

The number of violations of the DCoVaR\((0.9)\) forecast for both Clayton and Gumbel Copulas are presented in Table 5. It is the number of sample observations located out of the critical value i.e. less than or equal the DCoVaR forecast. Note that the joint significance level 3.49% is obtained by plugging the specified numbers \( 1 - \alpha = 1 - 0.1 = 0.9, \theta = 0.4938 \) and \( \alpha = d = 0 \) into Clayton Copula formula. Then, by calculating the number of violations of the DCoVaR\((0.10)\) it is obtained 0.16% (number of violations is 1, total observations 616; 1/616 = 0.16%). Moreover, the joint significance level 1.95% is obtained by plugging the same specified numbers \( 1 - \alpha = 1 - 0.1 = 0.9, \theta = 0.4938 \) and \( \alpha = d = 0 \) into Clayton Copula but with different \( \theta = 1.2905 \) into Gumbel Copula formula. Then, by calculating the percentage of violations of the DCoVaR\((0.10)\) it is also obtained 0.16%. It is shown from the table that the DCoVaR forecast with Gumbel Copula has lower joint significance level in comparison to the DCoVaR forecast with Clayton Copula. As for the number of violations, it conforms the use of Gumbel Copula. In short, it suggests that Gumbel Copula is more appropriate Copula for describing the joint distribution of NASDAQ and TWIEX returns.

**5. Conclusion**

For a better investment it is preferable to segregate the fund of investment in more than one market, but the extremely significant question is that when these markets are connected and when one of them crashes, does the remainder of interconnected market crash in the same way? Property 2 states that DCoVaR provides better forecast than MCoVaR proposed by Jadhav et al. (2013) as well as CCoVaR proposed by Brahim et al. (2018). Fig. 1 reveals that the DCoVaR becomes larger when dependence rises. MCoVaR and VaR, however, are neither rising or decreasing when dependence rises. In addition, Figs. 2, 3 and 4 assert that DCoVaR is relatively larger than MCoVaR since DCoVaR assumes that the dependence exists between target risk and associate risk. Accordingly, to minimize the risk, it is suggested that these markets to
be independent, or preferably for the investors to select the independent markets or the less dependent one to invest their fund. In addition, Yamai and Yoshida (2002) found that for a certain number of observations and a certain probability level, the accuracy of VaR and CoVaR is about the same when the loss is normally distributed, but that VaR forecasts are more accurate than CoVaR forecasts when the losses have heavy tails. This means capital calculated from CoVaR may be less stable than capital calculated from VaR. At the same probability level ρ, the DCoVaR forecast can be adjusted so that it is not too far from VaR and remains smaller than CoVaR, so that the forecast of DCoVaR can be much more stable than that of CoVaR.

In this paper, we suggest a novel risk measure called Dependent Conditional Value-at-Risk which maintains the property of coherence. This measure is more appropriate than MCoVaR, CCoVaR and CoVaR to comprehend the connection between bivariate losses and to help us exceptionally about how optimum to position our investments and elevate our financial risk protection. We realize that probably we do not need to consider all the worst observations greater than the VaR for forecasting financial risk (Jadhav et al., 2013), but we also cannot just ignore another risk (or other risks) that may greatly affect the target risk. Therefore, putting on the suggested risk measure will enable us to avoid non-essential extra capital allocation while not neglecting other risks associated with the target risk. In addition, the contraction parameters a and δ should be determined by performing DCoVaR optimization. We leave it as our future work.

In actuarial context, DCoVaR can be applied to determine insurance premiums while reducing the risk of insurance company. As for application in finance, this paper describes a model for forecasting DCoVaR by the model of GARCH-conditional Copula, in which the empirical substantiation exhibits that this approach can be completely robust in forecasting DCoVaR, namely the GARCH(1,1) model with innovation is t distributed. Moreover, Gumbel Copula is more appropriate than Clayton Copula for describing the joint distribution of NASDAQ and TWIEX returns. The use of GARCH model for marginal of asset returns may be replaced by its extensions such as ARMA-GARCH and GJR-GARCH models. In addition, any innovations may also be applied to such volatility models. Syuhada (2020) has carried out VaR forecast and compared such observable stochastic volatility process (GARCH) class of models) with the latent one i.e. the Stochastic Volatility Autoregressive (SVAR) model.

**Table 5. Joint significance level (%) and percentage of violations (%) of the DCoVaR(0.05) forecast by using Clayton and Gumbel Copulas.**

| Copula   | Parameter | Joint sig. level | No. violations | % violations |
|----------|-----------|------------------|----------------|--------------|
| Clayton  | a = δ = 0.10 | 3.49, 4.14 | 1 | 0.16 |
| Clayton  | a = δ = 0.15 | 4.41, 4.41 | 4 | 0.49 |
| Clayton  | a = 0.10, δ = 0.15 | 4.41, 4.41 | 4 | 0.49 |
| Clayton  | a = 0.15, δ = 0.10 | 4.41, 4.41 | 4 | 0.49 |
| Gumbel   | a = δ = 0.10 | 1.95, 2.74 | 3 | 0.16 |
| Gumbel   | a = δ = 0.15 | 2.74, 2.74 | 3 | 0.16 |
| Gumbel   | a = 0.10, δ = 0.15 | 2.74, 2.74 | 3 | 0.16 |
| Gumbel   | a = 0.15, δ = 0.10 | 2.74, 2.74 | 3 | 0.16 |

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**Data availability statement**

Data included in article/supplementary material/referenced in article.

**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

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**Declarations**

**Author contribution statement**

B. Josaphat: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data. K. Syuhada: Analyzed and interpreted the data; Wrote the paper.
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