Exact Algorithms for Computing Generalized Eigenspaces of Matrices via Annihilating Polynomials

Shinichi Tajima, Katsuyoshi Ohara, Akira Terui

Abstract
An effective exact method is proposed for computing generalized eigenspaces of a matrix of integers or rational numbers. Keys of our approach are the use of minimal annihilating polynomials and the concept of the Jordan-Krylov basis. A new method, called Jordan-Krylov elimination, is introduced to design an algorithm for computing Jordan-Krylov basis. The resulting algorithm outputs generalized eigenspaces as a form of Jordan chains. Notably, in the output, components of generalized eigenvectors are expressed as polynomials in the associated eigenvalue as a variable.

Keywords: Minimal annihilating polynomial, Generalized eigenvectors, Jordan chains, Krylov vector space

1. Introduction
Exact linear algebra plays important roles in many fields of mathematics and sciences. Over the last two decades, this area has been extensively studied, and new algorithms have been proposed for various types of computations, such as computing canonical forms of matrices (Ausrot and Cam ion 1997), Dumas et al. (2001), Havas and Wagner (1999), Moritsugu (2004), Pernet and Stei n (2010), Saunders and Wai (2004), Storjohann (2001), Storjohann and Labahn (1996), the characteristic or the minimal polynomial of a matrix (Dumas et al. 2005), Neunhöffer and Praeger (2008), LU and other decompositions and/or solving a system of linear equations (Bostan et al. 2008, Eberly et al. 2006, Jeannerod et al. 2013, Muy et al. 2007, Saunders et al. 2011). Also, the software has been developed (Albrecht 2012, Chen and Storjohann 2005, Dumas et al. 2002a, Dumas et al. 2002b, Dumas et al. 2004, Dumas et al. 2008) and comprehensive research results (Giorgi 2019) have been presented. Kreuzer and Robbiano (2016) investigate

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the interconnections between linear algebra and commutative algebra. Notably, they study, in particular, generalized eigenproblems from the point of view of commutative algebra and its applications to polynomial ideals.

In the context of symbolic computation for linear algebra, we proposed algorithms for eigenproblems including computation of spectral decomposition and (generalized) eigendecomposition (Ohara and Tajima (2009), Tajima (2013), Tajima et al. (2014), Tajima et al. (2018a)). We studied eigenproblems from the point of view of spectral decomposition and proposed an exact algorithm for computing eigenvectors of matrices over integer or rational numbers by using minimal annihilating polynomials (Tajima et al. (2018b)).

In this paper, we treat generalized eigenspaces via Jordan chains as an extension of our previous papers. In a naive method for computing generalized eigenspaces, the generalized eigenvector is obtained by solving a system of linear equations. However, in general, the method has the disadvantage of algebraic number arithmetic. It is often time-consuming for matrices of dozens of dimensions. As methods for computing eigenvectors without solving a system of linear equations, Takeshima and Yokoyama (1990) have proposed one by using the Frobenius normal form of a matrix and Moritsugu and Kurivama (2001) have extended it for the case that the Frobenius normal form has multiple companion blocks and for computing generalized eigenvectors. Their methods require computation of the Frobenius normal form of the matrix, which tends to be inefficient for matrices of large dimensions.

We introduce a concept of Jordan-Krylov basis and a new method, called Jordan-Krylov elimination in this paper, for computing a Jordan-Krylov basis. We show that the use of minimal annihilating polynomials allows us to design an effective method for computing generalized eigenspaces. Notably, the resulting algorithm does not involve linear system solving or normal form computation. Our method has the following features. First, a basis of the generalized eigenspace is computed as Jordan chains. Second, each Jordan chain has a unified representation as a vector consisting of polynomials with rational coefficients in a symbol \( \lambda \) denoting the associated eigenvalue \( \alpha \) and its conjugates. We emphasize that our representation gives a relationship among the generalized eigenspaces associated to \( \alpha \) and all its conjugates. Third, in implementing the algorithms, the implementation of arithmetic in an algebraic extension field is not necessary. In part of our implementation, we perform calculations in the quotient ring of a polynomial ring, which is related to operations in an algebraic extension field. However, these calculations can be implemented solely with operations in the polynomial ring over the field of rational numbers. All other operations are also performed in the polynomial ring over the field of rational numbers.

We believe that our algorithm could be applied to several methods that require generalized eigenspaces and their structures in the case of multiplicities. For example, it could be useful when solving a zero-dimensional system of polynomial equations as the eigenproblem (Cox et al. 2005), or finding formal power series solutions of ordinary differential equations (Turrittin 1953, Barkatou and Pf"ugel 1999).

We remark that the essential ideas in this paper, such as the use of the minimal annihilating polynomials for computing generalized eigenvectors were reported in Tajima (2013). We have reviewed the original ideas deeply and sought to optimize the efficiency of the algorithm. As a result, we have come to the concept of Jordan-Krylov basis and the algorithm of Jordan-Krylov elimination based on it for computing generalized eigenspaces efficiently.

This paper is organized as follows. In Section 2, we give a formula that represents Jordan chains of generalized eigenvectors as a polynomial with rational coefficients. In Section 3, we introduce the concept of Jordan-Krylov basis. We show that the computation of Jordan chains can be reduced to that of a Jordan-Krylov basis. In Section 4, we give a method called Jordan-Krylov
elimination, and we present the resulting algorithm that computes generalized eigenspaces from minimal annihilating polynomials. In Section 5, we discuss the time complexity of the algorithm. In Section 6, we give an example. Finally, in Section 7, we give the result of a benchmark test.

2. Generalized eigenvectors and Jordan chains

Let us begin by recalling a result given in Tajima et al. (2018b) on algebraic properties of eigenvectors. Let $K \subseteq \mathbb{C}$ be a computational subfield, and $A$ be a square matrix of order $n$ over $K$. Consider a monic irreducible factor $f(\lambda)$ in $K[\lambda]$ of the characteristic polynomial $\chi_A(\lambda)$ of $A$, and a bivariate polynomial $\psi_f(\mu,\lambda)$ associated to $f(\lambda)$, defined as

$$\psi_f(\mu,\lambda) = f(\mu) - f(\lambda) \frac{\mu}{\mu - \lambda} \in K[\mu,\lambda].$$

Since the rank of the matrix $f(A)$ is less than $n$, there exists a non-zero vector $w \in K^n$ satisfying $f(A)w = 0$. For that $w$, we define

$$p(\lambda, w) = \psi_f(A, \lambda E)w,$$

where $E$ is the identity matrix of order $n$. Then, $p(\lambda, w)$ satisfies the following lemma (Tajima et al. (2018b)).

**Lemma 1.** Let $d = \deg f$ and $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the roots of $f(\lambda)$ in $\mathbb{C}$. Let $w \in K^n$ be a non-zero vector such that $f(A)w = 0$. Then, for $i = 1, 2, \ldots, d$,

1. $p(\alpha_i, w) \neq 0$.
2. $(A - \alpha_i E)p(\alpha_i, w) = 0$.
3. $p(\alpha_1, w), \ldots, p(\alpha_d, w)$ are linearly independent over $\mathbb{C}$.

**Proof.** By the definition of $\psi_f(\mu, \lambda)$, we have

$$(A - \alpha_i E)p(\alpha_i, w) = (A - \alpha_i E)\psi_f(A, \alpha_i E)w = f(A)w = 0.$$

For an eigenvalue $\alpha_i \in \mathbb{C}$, consider a vector

$$q(\alpha_i, w) = \frac{1}{\psi_f(\alpha_i, \alpha_i)} p(\alpha_i, w).$$

Note that, since

$$\psi_f(\mu, \alpha_i) = (\mu - \alpha_1) \cdots (\mu - \alpha_{i-1})(\mu - \alpha_{i+1}) \cdots (\mu - \alpha_d),$$

we have $\psi_f(\alpha_i, \alpha_i) \neq 0$. With the Lagrange’s interpolation formula, we have

$$\sum_{i=1}^{d} \frac{\psi_f(\mu, \alpha_i)}{\psi_f(\alpha_i, \alpha_i)} = 1,$$

which derives

$$\sum_{i=1}^{d} q(\alpha_i, w) = w.$$
As a result, we have
\[ \alpha_1^i q(\alpha_1, w) + \alpha_2^i q(\alpha_2, w) + \cdots + \alpha_d^i q(\alpha_d, w) = A^i w, \quad k = 0, 1, 2, \ldots, d - 1, \]
which is rewritten as
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_d \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{d-1} & \alpha_2^{d-1} & \cdots & \alpha_d^{d-1}
\end{pmatrix}
\begin{pmatrix}
q(\alpha_1, w) \\
q(\alpha_2, w) \\
\vdots \\
q(\alpha_d, w)
\end{pmatrix}
= \begin{pmatrix}
w \\
Aw \\
\vdots \\
A^{d-1}w
\end{pmatrix}.
\]
In the above formula, the determinant of the matrix in the left-hand-side is not equal to zero. Furthermore, since \( f(p) \) is irreducible over \( K \), the vectors \( w, Aw, \ldots, A^{d-1}w \) are linearly independent over \( K \), so are \( q(\alpha_1, w), q(\alpha_2, w), \ldots, q(\alpha_d, w) \) over \( \mathbb{C} \). Accordingly, the vectors \( p(\alpha_1, w), p(\alpha_2, w), \ldots, p(\alpha_d, w) \) are also linearly independent over \( \mathbb{C} \) and \( p(\alpha, w) \neq 0 \), which completes the proof.

Notice that the vector \( p(\lambda, w) \) representing eigenvectors associated to the eigenvalues \( \alpha_1, \alpha_2, \ldots, \alpha_d \) is a polynomial in \( \lambda \) of degree \( d - 1 \). In this sense, this representation has no redundancy. In this section, we generalize the above results and give a formula that represents a Jordan chain of generalized eigenvectors.

Let \( \ell \) be the multiplicity of a monic irreducible factor \( f(\lambda) \) in \( \pi_A(\lambda) \in K[\lambda] \), where \( \pi_A(\lambda) \) is the minimal polynomial of \( A \). For \( 1 \leq \ell \leq \bar{\ell} \), let
\[ \ker f(A)^\ell = \{ u \in K^n \mid f(A)^\ell u = 0 \}, \]
with \( \ker f(A)^0 = \{ 0 \} \). There exists an ascending chain
\[ \{ 0 \} \subset \ker f(A) \subset \ker f(A)^2 \subset \cdots \ker f(A)^\ell \subset \cdots \subset \ker f(A)^\bar{\ell}, \]
of vector spaces over \( K \).

**Definition 2.** For \( 0 \neq u \in \ker f(A)^\ell \) and \( 1 \leq \ell \leq \bar{\ell} \), the rank of \( u \) with respect to \( f(\lambda) \), denoted by \( \text{rank}_\lambda u \), is equal to \( \ell \) if \( u \in \ker f(A)^\ell \setminus \ker f(A)^{\ell-1} \).

**Definition 3.** For \( 1 \leq k \leq \bar{\ell} \), let \( \psi_j^{(k)}(\mu, \lambda) = (\psi_j(\mu, \lambda))^k \mod f(\lambda) \), in which \( (\psi_j(\mu, \lambda))^k \) is regarded as an element in \( K[\lambda][\mu] \) and the coefficients in \( \psi_j^{(k)}(\mu, \lambda) \) are defined as the remainder of those in \( (\psi_j(\mu, \lambda))^k \) divided by \( f(\lambda) \).

Let \( u \in \ker f(A)^\ell \) be a non-zero vector of rank \( \lambda u = \ell \). For \( k = 1, \ldots, \ell \), we define
\[
\psi_j^{(k)}(\lambda, u) = (\psi_j(\lambda, \mu) f(\lambda))^{\ell-k} u.
\] (2)
Then, we have the following lemmas.

**Lemma 4.** Let \( d \) and \( \alpha_1, \ldots, \alpha_d \) be the same as those in Lemma[1] and assume that \( \text{rank}_\lambda u = \ell \). Then, for \( i = 1, 2, \ldots, d \) and \( 1 \leq k \leq \ell \),
\[ (A - \alpha_i E)p^{(k)}(\alpha_i, u) = p^{(k-1)}(\alpha_i, u) \]
holds.
Proof. Since

$$(A - \alpha_i E)\psi_j^{(k)}(A, \alpha_i E) = \psi_j^{(k-1)}(A, \alpha_i E)f(A),$$

we have

$$(A - \alpha_i E)\psi_j^{(k)}(A, \alpha_i E)f(A)^{\ell-k} = \psi_j^{(k-1)}(A, \alpha_i E)f(A)^{\ell-(k-1)}.$$ 

By multiplying $u$ on both sides from the right, we have

$$(A - \alpha_i E)p^{(k)}(\alpha_i, u) = p^{(k-1)}(\alpha_i, u),$$

which proves the lemma.

\[ \square \]

**Lemma 5.** Let $d$ and $\alpha_1, \ldots, \alpha_d$ be the same as those in Lemma 4 and assume that $\text{rank}_f u = \ell$. Then, for $i = 1, 2, \ldots, d$ and $1 \leq k \leq \ell$,

$$(A - \alpha_i E)^{k-1}p^{(k)}(\alpha_i, u) \neq 0 \quad \text{and} \quad (A - \alpha_i E)^k p^{(k)}(\alpha_i, u) = 0$$

hold.

**Proof.** By Lemma 4 we have

$$(A - \alpha_i E)^{k-1}p^{(k)}(\alpha_i, u) = p^{(k)}(\alpha_i, u),$$

$$(A - \alpha_i E)^k p^{(k)}(\alpha_i, u) = (A - \alpha_i E)p^{(k)}(\alpha_i, u),$$

where $p^{(k)}(\alpha_i, u) = p(\alpha_i, f(A)^{\ell-1}u)$. Let $w = f(A)^{\ell-1}u$, then $w$ satisfies that $w \neq 0$ and $f(A)w = 0$ since $\text{rank}_f u = \ell$. Thus, by Lemma 4 we have $p^{(k)}(\alpha_i, u) \neq 0$ and $(A - \alpha_i E)p^{(k)}(\alpha_i, u) = 0$. This completes the proof.

**Lemma 5** says that $p^{(k)}(\lambda, u)$ gives a representation of generalized eigenvectors of $A$ of rank $k$. Summarizing the above, we have the following theorem.

**Theorem 6.** Let $d$ and $\alpha_1, \ldots, \alpha_d$ be the same as those in Lemma 4 and assume that $\text{rank}_f u = \ell$. Then, for $i = 1, 2, \ldots, d$,

$$\{p^{(k)}(\alpha_i, u), p^{(k-1)}(\alpha_i, u), \ldots, p^{(1)}(\alpha_i, u)\}$$

(3)

gives a Jordan chain of length $\ell$.

Accordingly, $\{p^{(k)}(\lambda, u), p^{(k-1)}(\lambda, u), \ldots, p^{(1)}(\lambda, u)\}$ can be regarded as a representation of the Jordan chain \[3\]. That is, from a vector $u$ of rank $\ell$ in $\ker f(A)^\ell \subset \mathbb{K}^n$, the representation of the Jordan chain is derived in terms of $p^{(k)}(\lambda, u)$.

**Example 1.** Let $f(\lambda) = \lambda^2 + \lambda + 5$ and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -125 \\ 1 & 0 & 0 & 0 & -175 \\ 0 & 1 & 0 & 0 & -90 \\ 0 & 0 & 1 & 0 & -31 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix},$$

the companion matrix of $f(\lambda)^3$. The characteristic polynomial $\chi_A(\lambda)$ and the unit minimal annihilating polynomial $\pi_{A,j}(\lambda) (j = 1, 2, \ldots, 6)$ are $\chi_A(\lambda) = \pi_{A,j}(\lambda) = f(\lambda)^3$. We see that $e_1 \in \ker f(A)^3$.
and \( \text{rank}_f e_1 = 3 \). Now, let us compute the generalized eigenvector \( p_1(\lambda, e_1) \) associated to \( f(\lambda) \). For

\[
\psi_f(\mu, \lambda) = \mu + \lambda + 1,
\]

\( \psi_f(\mu, \lambda)^2 \) and \( \psi_f(\mu, \lambda)^3 \) are polynomials of degree 2 and 3 with respect to \( \lambda \), respectively. Let \( \psi_f(\mu, \lambda)^2 \) and \( \psi_f(\mu, \lambda)^3 \) be the remainder of the division of \( \psi_f(\mu, \lambda)^2 \) and \( \psi_f(\mu, \lambda)^3 \) divided by \( f(\lambda) \) with respect to \( \lambda \), calculated as

\[
\psi_f(\mu, \lambda)^2(\mu, \lambda) = \mu^2 + (2\lambda + 2)\mu + \lambda - 4,
\]

\[
\psi_f(\mu, \lambda)^3(\mu, \lambda) = \mu^3 + (3\lambda + 3)\mu^2 + (3\lambda - 12)\mu - 4\lambda - 9,
\]

respectively. Then, the generalized eigenvectors are computed with \( e_1 \) as

\[
p^{(3)}(\lambda, e_1) = \psi_f^{(3)}(A, \lambda E)e_1 = (-4, 3, 3, 0, 0, 0)\lambda + (-9, -12, 3, 1, 0, 0),
\]

\[
p^{(2)}(\lambda, e_1) = \psi_f^{(2)}(A, \lambda E)f(A)e_1 = (5, 11, 3, 2, 0, 0)\lambda + (-20, 6, 3, 1, 0),
\]

\[
p^{(1)}(\lambda, e_1) = \psi_f^{(1)}(A, \lambda E)f^2(A)e_1 = (25, 10, 11, 2, 1, 0)\lambda + (25, 35, 21, 13, 1),
\]

which satisfy

\[
(A - \lambda E)^2 p^{(3)}(\lambda, e_1) \neq 0,
\]

\[
(A - \lambda E)^3 p^{(3)}(\lambda, e_1) = 0,
\]

\[
(A - \lambda E)^2 p^{(2)}(\lambda, e_1) \neq 0,
\]

\[
(A - \lambda E)^3 p^{(2)}(\lambda, e_1) = 0,
\]

\[
p^{(1)}(\lambda, e_1) \neq 0,
\]

\[
(A - \lambda E)p^{(1)}(\lambda, e_1) = 0,
\]

respectively. Thus, the set \( \{p^{(3)}(\lambda, e_1), p^{(2)}(\lambda, e_1), p^{(1)}(\lambda, e_1)\} \) is a Jordan chain of length 3.

3. Jordan-Krylov basis in \( \ker f(A)^\ell \)

The discussion in the previous section shows that a vector \( u \in \ker f(A)^\ell \subset K^n \) of rank \( \ell \) gives rise to a representation of the corresponding Jordan chains of length \( \ell \). In this section, we show that there exists a finite subset \( \mathcal{B} \) of \( \ker f(A)^\ell \) such that any generalized eigenvectors can be represented as a linear combination of vectors \( p^{(k)}(\lambda, u) \) with \( u \in \mathcal{B} \).

For \( u \in K^n \), a vector space

\[
L_A(u) = \text{span}_K \{A^k u \mid k = 0, 1, 2, \ldots\}
\]

is called Krylov vector space. Since \( A \) and \( f(A) \) commute, \( \text{rank}_f u = \ell \) implies that \( L_A(u) \subset \ker f(A)^\ell \). For \( d = \text{deg} f(\lambda) \), let

\[
L_{A,d}(u) = \{u, Au, A^2u, \ldots, A^{d-1}u\}
\]

and

\[
L_{A,d}(u) = \text{span}_K L_{A,d}(u).
\]

Since \( f(\lambda) \) is irreducible, \( L_{A,d}(u) \) is linearly independent, and the rank of the non-trivial elements in \( L_{A,d}(u) \) is equal to \( \ell \). Let \( \mathcal{L}_A(u) = L_{A,d}(u) \cup L_{A,d}(f(A)u) \cup \cdots \cup L_{A,d}(f(A)^{\ell-1}u) \). Clearly, \( \mathcal{L}_A(u) \) is also linearly independent. Then, it holds that

\[
L_A(u) = L_{A,d}(u) \oplus f(A)L_{A,d}(u) \oplus \cdots \oplus f(A)^{\ell-1}L_{A,d}(u).
\]
thus \( \dim_K L_A(u) = \ell d \). (In a more straight manner, we see that \( \dim_K L_A(u) = \ell d \) by the fact that \( f(A)^{\ell} \) is the minimal annihilating polynomial of \( u \).) Note that, for \( 1 \leq \ell' \leq \ell - 1 \), the rank of the non-trivial elements in \( f(A)^{\ell'} L_{A,\ell}(u) \) is equal to \( \ell - \ell' \).

Consider the subspace spanned by a Jordan chain

\[
P_A(\alpha_i, u) = \text{span}_C \{ p^{(i)}(\alpha_i, u), p^{(\ell-1)}(\alpha_i, u), \ldots, p^{(1)}(\alpha_i, u) \},
\]

where \( \alpha_1, \alpha_2, \ldots, \alpha_d \) are the roots of \( f(\lambda) \) in \( C \). Let

\[
P_A(u) = P_A(\alpha_1, u) + P_A(\alpha_2, u) + \cdots + P_A(\alpha_d, u).
\]

We have the following property on \( P_A(u) \).

**Lemma 7.** \( P_A(u) = C \otimes_K L_A(u) \) holds.

**Proof.** Note that, for \( 1 \leq k \leq \ell \), \( p^{(k)}(\alpha_i, u) \in C \otimes_K L_A(u) \) since \( p^{(k)}(\alpha_i, u) \) is expressed as \( h(\lambda) \) for a certain polynomial \( h \) of degree less than \( df \). We also mention that \( L_A(u) \) is invariant under \( h(\lambda) \) for any polynomial \( h(\lambda) \in K[\lambda] \). By the definition of \( P_A(\alpha_i, u) \), it implies that \( P_A(u) \subseteq C \otimes_K L_A(u) \), and \( \dim_C P_A(\alpha_i, u) = \ell \) implies that \( \dim_C P_A(u) = \ell d \). On the other hand, from \( \dim_K L_A(u) = \ell d \) and \( L_A(u) \subseteq K^n \), we have \( \dim_C C \otimes_K L_A(u) = \ell d \). Since the dimensions of \( P_A(u) \) and \( C \otimes_K L_A(u) \) are equal, we have \( P_A(u) = C \otimes_K L_A(u) \). This completes the proof. \( \square \)

Notice that the decomposition \( P_A(\alpha_1, u) + P_A(\alpha_2, u) + \cdots + P_A(\alpha_d, u) \) of \( C \otimes_K L_A(u) \) can be regarded as a spectral decomposition.

A sum \( V_1 + V_2 + \cdots + V_r \) of vector spaces \( V_1, V_2, \ldots, V_r \) that satisfy the direct sum condition,

\[
(V_1 + \cdots + V_{i-1} + V_{i+1} + \cdots + V_r) \cap V_i = \{ 0 \}, \quad i = 1, 2, \ldots, r,
\]
is written as \( V_1 \oplus V_2 \oplus \cdots \oplus V_r \).

Now, we introduce notions of Krylov generating set, Jordan-Krylov independence and Jordan-Krylov basis as follows.

**Definition 8 (Krylov generating set).** Let \( W \) be a subspace of \( K^n \). A subset \( \mathcal{W} \) of \( W \) is a Krylov generating set with respect to \( A \) if \( W = \sum_{w \in \mathcal{W}} L_A(w) \) holds. Especially, the empty set is a Krylov generating set of the trivial vector space \( \{ 0 \} \).

**Definition 9 (Jordan-Krylov independence).** A finite set \( \mathcal{W} \) of vectors is Jordan-Krylov independent with respect to \( A \) if Krylov vector spaces \( \{ L_A(w) \mid w \in \mathcal{W} \} \) satisfy the direct sum condition. Especially, the empty set is regarded as Jordan-Krylov independent.

**Definition 10 (Jordan-Krylov basis).** A finite Krylov generating set \( \mathcal{W} \) of \( W \) is a Jordan-Krylov basis if it is Jordan-Krylov independent.

**Theorem 11.** There exists a Jordan-Krylov basis \( \mathcal{B} \) of \( f(A)^{\ell} \) with respect to \( A \). Furthermore, the number of elements in \( \mathcal{B} \) of each rank is uniquely determined.

A Jordan-Krylov basis is a key concept in our approach. In Section 3.1, we give a proof of Theorem 11 in a constructive manner which plays a crucial role in our algorithm for computing a Jordan-Krylov basis.

For convenience, we introduce a superscript \( (\ell) \) for a finite subset \( \mathcal{A} \subseteq \ker f(A)^{\ell} \), as \( \mathcal{A}^{(\ell)} = \{ u \in \mathcal{A} \mid \text{rank } u = \ell \} \). Then \( \mathcal{A} = \mathcal{A}^{(1)} \cup \cdots \cup \mathcal{A}^{(\ell)} \).

Theorem 11 and Lemma 7 immediately imply the following lemma.
Lemma 12. Assume that $\{b_1, b_2, \ldots, b_r\} \subset \ker f(A)^\ell$ is Jordan-Krylov independent. Then, the following holds.

1. $P_A(\alpha, b_1), P_A(\alpha, b_2), \ldots, P_A(\alpha, b_r)$ satisfy the direct sum condition for $i = 1, 2, \ldots, d$.
2. $P_A(b_1), P_A(b_2), \ldots, P_A(b_r)$ satisfy the direct sum condition.

As a result, we have the following theorem.

Theorem 13. Let $\mathcal{B} = \mathcal{B}^{(1)} \cup \mathcal{B}^{(2-1)} \cup \cdots \cup \mathcal{B}^{(1)}$ be a Jordan-Krylov basis of $\ker f(A)^\ell$. Then, for $i = 1, 2, \ldots, d$, it holds the following.

1. A direct sum $\bigoplus_{b \in \mathcal{B}} P_A(\alpha, b)$ is spanned by Jordan chains of length $\ell$.
2. A direct sum $\bigoplus_{b \in \mathcal{B}} P_A(\alpha, b)$ gives a generalized eigenspace of $A$ associated to the eigenvalue $\alpha$.
3. $\bigoplus_{b \in \mathcal{B}} P_A(b) \simeq \mathbb{C} \otimes K \ker f(A)^\ell$.

3.1. A proof of Theorem 13

We first give a lemma, which will play a key role in our approach.

Lemma 14. Assume that $\{b_1, b_2, \ldots, b_r\} \subset \ker f(A)^\ell$ is Jordan-Krylov independent and $v \in \ker f(A)^\ell$ satisfies that

$$f(A)^{\text{rank}(v)-1}v \notin \bigoplus_{i=1}^{r} P_{A}(f(A)\text{rank}(v)-1, b_i).$$

Then, $L_{A}(b_1), \ldots, L_{A}(b_r)$, $L_{A}(v)$ satisfy the direct sum condition.

Proof. By contradiction. Let $b'_1 = f(A)^{\text{rank}(b_1)-1}b_1$ and $v' = f(A)^{\text{rank}(v)-1}v$, and assume that $L_{A}(b_1), \ldots, L_{A}(b_r)$, $L_{A}(v)$ do not satisfy the direct sum condition. We have

$$L_{A}(v) \cap \bigoplus_{i=1}^{r} L_{A}(b_i) \neq \{0\},$$

thus there exists a non-zero vector $u \in L_{A}(v) \cap \bigoplus_{i=1}^{r} L_{A}(b_i)$. Then, $u' = f(A)^{\text{rank}(u)-1}u$ satisfies that $\text{rank}(u') = 1$, hence $u' \in L_{A}(v') \cap \bigoplus_{i=1}^{r} L_{A}(b_i')$. By $u' \in L_{A}(v')$, we have $L_{A}(u') \subset L_{A}(v')$. Since $f(A)$ is irreducible, $\dim_k L_{A}(w) = d$ for any $w \in \ker f(A) \setminus \{0\}$, which implies that $L_{A}(u') = L_{A}(v')$ hence $v' \in L_{A}(u')$. Now, $u' \in \bigoplus_{i=1}^{r} L_{A}(b_i')$ implies that $L_{A}(u') \subset \bigoplus_{i=1}^{r} L_{A}(b_i')$. Thus, we have $v' \in \bigoplus_{i=1}^{r} L_{A}(b_i')$, which contradicts the assumption. This completes the proof.

For simplicity of notation, for $U \subset K^n$, let $L_{A}(U) = \bigcup_{u \in U} L_{A}(u)$. Now we are ready to give a proof of Theorem 13. Let $\mathcal{G}$ be a finite generating set of $\ker f(A)^\ell$, then it is obvious that $\tilde{\ell} = \max\{\text{rank}_u u | u \in \mathcal{G}\}$. In the following, for $\tilde{\ell} = \hat{\ell}, \ldots, 2, 1$ let us construct a Jordan-Krylov basis recursively. For $\ell$ with $1 \leq \ell \leq \tilde{\ell}$, assume that we have constructed a finite generating set $L_{A}(T_{\ell}) \cup \mathcal{G}_{\ell}$ of $\ker f(A)^\ell$ satisfying

Condition A:
1. \( \mathcal{F}_\ell \) is a Jordan-Krylov independent set such that any \( \nu \in \mathcal{F}_\ell \) satisfies that \( \text{rank}_I \nu > \ell \).

2. Any \( u \in \mathcal{G}_\ell \) satisfies that \( \text{rank}_I u \leq \ell \).

Note that, for \( \ell = \ell_0 \), put \( \mathcal{F}_\ell = \emptyset \) and \( \mathcal{G}_\ell = \mathcal{G} \). Then, we see that \( \mathcal{F}_\ell \) and \( \mathcal{G}_\ell \) satisfy Condition A.

Now, we construct a finite generating set \( \mathcal{L}_A(\mathcal{F}_{\ell-1}) \cup \mathcal{G}_{\ell-1} \) of \( \ker f(A)^\ell \) as follows. Put \( \{v_1, v_2, \ldots, v_k\} = \{ u \in \mathcal{G}_\ell \mid \text{rank}_I u = \ell \} \), and let \( \mathcal{D}_{\ell,1} = \mathcal{F}_\ell \) and \( \mathcal{H}_{\ell,1} = \mathcal{G}_\ell \setminus \{v_1, v_2, \ldots, v_k\} \).

For \( i = 1, 2, \ldots, k_i \), according to the relationship between a subspace \( W_{\ell,i} = \bigoplus_{b \in \mathcal{D}_{\ell,i}} L_A(f(A)^{\text{rank}_I b-1}b) \) (6)

and the vector \( f(A)^{-1}v_i \), define \( \mathcal{D}_{\ell,i+1} \) and \( \mathcal{H}_{\ell,i+1} \), divided into the following cases:

**Case 1.** In the case \( f(A)^{-1}v_i \notin W_{\ell,i} \); Lemma 4 tells us that \( \{L_A(b) \mid b \in \mathcal{D}_{\ell,i}\} \cup \{L_A(v_i)\} \) satisfy the direct sum condition. Thus, let \( \mathcal{D}_{\ell,i+1} = \mathcal{D}_{\ell,i} \cup \{v_i\} \) and \( \mathcal{H}_{\ell,i+1} = \mathcal{H}_{\ell,i} \).

**Case 2.** In the case \( f(A)^{-1}v_i \in W_{\ell,i} \); the membership can be expressed as

\[
0 = f(A)^{-1}v_i - \sum_{b \in \mathcal{D}_{\ell,i}} \sum_{j=0}^{d-1} c_{b,j} A^j f(A)^{\text{rank}_I b-1}b.
\]

Define

\[
\mathbf{r} = v_i - \sum_{b \in \mathcal{D}_{\ell,i}} \sum_{j=0}^{d-1} c_{b,j} A^j f(A)^{\text{rank}_I b-1}b,
\]

and do one of the following:

- **Case 2-a.** In the case \( \mathbf{r} \neq 0 \), this means that \( \text{rank}_I \mathbf{r} < \ell \), thus let \( \mathcal{D}_{\ell,i+1} = \mathcal{D}_{\ell,i} \) and \( \mathcal{H}_{\ell,i+1} = \mathcal{H}_{\ell,i} \cup \{\mathbf{r}\} \).

- **Case 2-b.** In the case \( \mathbf{r} = 0 \), this means that \( v_i \in \bigoplus_{b \in \mathcal{D}_{\ell,i}} L_A(b) \) and \( v_i \) is unnecessary, thus discard it and let \( \mathcal{D}_{\ell,i+1} = \mathcal{D}_{\ell,i} \) and \( \mathcal{H}_{\ell,i+1} = \mathcal{H}_{\ell,i} \).

Note that, in any case above,

\[
\mathcal{L}_A(\mathcal{D}_{\ell,i+1}) \cup (\{v_1, v_2, \ldots, v_k\} \cup \mathcal{H}_{\ell,i+1})
\]

is a generating set satisfying Condition A. Then, after deriving a sequence of tuples of sets \( \mathcal{D}_{\ell,1}, \mathcal{H}_{\ell,1}, \mathcal{D}_{\ell,2}, \mathcal{H}_{\ell,2}, \ldots, \mathcal{D}_{\ell,k_{\ell+1}}, \mathcal{H}_{\ell,k_{\ell+1}} \), by letting \( \mathcal{F}_{\ell-1} = \mathcal{D}_{\ell,k_{\ell+1}} \), \( \mathcal{G}_{\ell-1} = \mathcal{H}_{\ell,k_{\ell+1}} \), we see that \( \mathcal{F}_{\ell-1} \) and \( \mathcal{G}_{\ell-1} \) also satisfy Condition A.

After the above computation for \( \ell = \ell_0, \ldots, 2, 1 \), a generating set \( \mathcal{L}_A(\mathcal{F}_0) \cup \mathcal{G}_0 = \mathcal{L}_A(\mathcal{F}_0) \) of \( \ker f(A)^\ell \) is computed (note that \( \mathcal{G}_0 = \emptyset \)), which shows that \( \mathcal{F}_0 \) a Jordan-Krylov basis \( \mathcal{B} \). This completes the proof.

**Remark 1.** Note that the multiplicity \( m \) of \( f(\lambda) \) in the characteristic polynomial of \( A \) is equal to the sum of the length of the linearly independent Jordan chains. In the case \( m = \ell \), Let \( v_1 \in \{u \in \mathcal{G} \mid \text{rank}_I u = \ell \} \) and \( \mathcal{F}_\ell = \{v_1\} \). Then, we have

\[
L_A(v_1) = \ker f(A)^\ell,
\]

thus \( \mathcal{F}_\ell = \{v_1\} \) gives a Jordan-Krylov basis of \( \ker f(A)^\ell \).
4. Jordan-Krylov elimination

In this section, based on the concept of Jordan-Krylov basis, we provide an algorithm for computing generalized eigenvectors. In Section 4.1, we give an effective method for computing a Jordan-Krylov basis. In Section 4.2, we look at the proof of Theorem 4.3 given in the previous section from the point of view of symbolic computation, and we introduce a method called Jordan-Krylov elimination for computing a Jordan-Krylov basis. In Section 4.3, we refine the method and give an algorithm that computes a Jordan-Krylov basis from the Krylov generating set. In Section 4.4, we present a resulting algorithms for computing the generalized eigenspace associated to the eigenvalues of an irreducible factor of the characteristic polynomial \( \chi_A(z) \) of \( A \).

4.1. Finite generating set of \( \ker f(A)^J \)

**Definition 15** (The minimal annihilating polynomial). For \( u \in K^n \), let \( \pi_{A,u}(\lambda) \) be the monic generator of a principal ideal \( \text{Ann}_K(u) = \{ b(\lambda) \in K[\lambda] \mid b(A)u = 0 \} \). Then, \( \pi_{A,u}(\lambda) \) is called the minimal annihilating polynomial \( \lambda \) with respect to \( A \).

Let \( E \) be a basis of \( K^n \) and let \( f(\lambda) \) be a monic irreducible factor of the minimal polynomial \( \pi_A(\lambda) \) of \( A \). Let \( \mathcal{P} = \{ \pi_{A,e}(\lambda) \mid e \in E \} \). Since each minimal annihilating polynomial \( \pi_{A,e}(\lambda) \) can be expressed as

\[
\pi_{A,e}(\lambda) = \lambda^{\ell_e} g_e(\lambda), \quad \gcd(f, g_e) = 1, \tag{10}
\]

for some \( \ell_e \geq 0 \) and \( g_e(\lambda) = \text{lcm}_{e \in E} \pi_{A,e}(\lambda) \), it holds that \( \ell = \max \{ \ell_e \mid e \in E \} \) and \( g_e(A)e \in \ker f(A)^J \). We define \( E_f \) and \( V \) as

\[
E_f = \{ e \in E \mid \pi_{A,e}(\lambda) = \lambda^{\ell_e} g_e(\lambda), \quad \ell_e > 0 \}, \quad V = \{ g_e(A)e \mid e \in E_f \}.
\]

We have the following proposition.

**Proposition 16.** It holds that \( \ker f(A)^J = \text{span}_K \mathcal{L}_A(V) \).

**Proof.** Since \( V \subset \ker f(A)^J \), \( L_A(V) \subset \ker f(A)^J \) holds. The opposite inclusion is shown as follows. Let \( \pi_A(\lambda) = \lambda^{\ell} g(\lambda) \), where \( g(\lambda) \) is relatively prime with \( \lambda^{\ell} \). Then, there exist \( a(\lambda), b(\lambda) \in K[\lambda] \) satisfying that \( a(\lambda)\lambda^{\ell} + b(\lambda)g(\lambda) = 1 \). Let \( v \in \ker f(A)^J \), then, \((aA^\ell + bA^\lambda)g = v \) implies that \( bA^\lambda g = v \). Now, by expressing \( bA^\lambda g = \sum_{e \in E} c_e e \) with \( c_e \in K \), we have \( bA^\lambda g = g(A)b(A)w = g(A)(\sum_{e \in E} c_e e) = v \). It implies that \( g(A)(\sum_{e \in E} c_e e) = \sum_{e \in E} c_e g(A)e = \sum_{e \in E} c_e g_A e \), because, for \( e \in E \), \( \pi_{A,e}(\lambda) \) does not have \( \lambda^{\ell} \) as a factor and \( g(A)e = 0 \). For \( e \in E_f \), by eq. (10), we have \( g_e(A)e = g(A)e \), thus there exists \( h_e(\lambda) \in K[\lambda] \) satisfying that \( g(\lambda) = g_e(\lambda)h_e(\lambda) \). Therefore, \( v = g(A)b(A)w = \sum_{e \in E} c_e h_e(\lambda)g_e(A)e \), where \( g_e(A)e \in V \). Furthermore, for \( h_e(\lambda) = \sum_{j=0}^d h_j \lambda^j \) we have \( h_e(\lambda)g_e(A)e = \sum_{j=0}^d h_j A^j g_e(A)e \in L_A(g_e(A)e) \), which proves the claim. \( \Box \)

Proposition 16 says that the set \( V \) is a Krylov generating set of \( \ker f(A)^J \). In [Tajima et al. 2018], an algorithm has been proposed for computing all the minimal annihilating polynomials \( \pi_{A,e}(\lambda) \) of the element \( e \) of a basis \( E \) of \( K^n \). Accordingly, the set \( V \) is computable as shown in Algorithm 5.1.
Remark 3. When calculating Krylov generating set, using random vectors would make it easier to find generators with rank \(\bar{\ell}\). However, in this case, we need to determine all generators of all ranks, including those with a small rank, in order to examine the structure of \(\ker f(A)^{\ell}\). To extract generators with a smaller rank from those with a larger rank, elimination by each vector will be necessary, which will likely take time. Thus, we use minimal annihilating polynomials to properly determine the Krylov generating set.

4.2. Computing a Jordan-Krylov basis of \(\ker f(A)^{\ell}\) using Jordan-Krylov elimination

By using Algorithm 1, a Krylov generating set of \(\ker f(A)^{\ell}\) can be computed as \(\mathcal{V}\). Now, we need to compute a Jordan-Krylov basis \(\mathcal{B}\) of \(\ker f(A)^{\ell}\) from \(\mathcal{V} = \bigcup_{\ell=1}^{\ell} \mathcal{V}^{(\ell)}\).

In the proof of Theorem 11, we examine a membership of a vector \(v\) in \(\mathcal{V}\). For a vector \(v\), define an augmented matrix \([W | v]\) of \(\mathcal{V}\) and the rank of every column vector in \(\mathcal{V}\). Notice that \(\mathcal{V}\) is equal to \([W | v] \rightarrow [W | 0]\).

Now, we look at the proof of Theorem 11. For a vector \(v\) of rank \(\ell\), we have solved a membership problem of the vector \(f(A)^{\ell-1}v\) in \(\ker\mathcal{A}(A)^{\ell} = \ker f(A)^{\ell}\) for \(D_{\ell,i}\). For \(b \in D_{\ell,i}\), consider a vector \(\tilde{b} = (f(A)^{\ell}b, f(A)^{\ell-1}b, \ldots, f(A)_{\text{rank}}b)^T\) and define \(\tilde{D} = [D' \ D]\) by \(D = [D'_{\ell} | b \in D_{\ell,i}]\), where \(\mathcal{A}\) is a matrix obtained by placing \(A\) diagonally. Notice that \(W_{\ell,i}\) is spanned by the column vectors in \(D'\) and the rank of every column vector in \(D\) is equal to \(\ell\). Then, we see that \(f(A)^{\ell-1}v \in W_{\ell,i}\) is equivalent to the existence of a column reduction \([D' | \tilde{v}] \rightarrow [D' \ 0 | \tilde{v}]\), where \(\tilde{v} = [v' \ v']\) with \(v' = f(A)^{\ell-1}v\). In this way, the proof of Theorem 11 yields an algorithm as shown in Algorithm 2.
Algorithm 2 Computing a Jordan-Krylov basis of \( \ker f(A)^\ell \)

**Input:** A matrix \( A \in K^{n \times n} \), an irreducible factor \( f(\lambda) \in K[\lambda] \),

- a Krylov generating set \( \mathcal{V} = \bigcup_{i=1}^{\ell} \mathcal{V}^{(i)} \) of \( \ker f(A)^\ell \),
- the multiplicity \( m \) of \( f(\lambda) \) in the characteristic polynomial of \( A \).

**Output:** A Jordan-Krylov basis \( B = \bigcup_{i=1}^{\ell} B^{(i)} \) of \( \ker f(A)^\ell \)

1: \[ \tilde{A} \leftarrow \begin{bmatrix} A & O \\ O & A \end{bmatrix} \]

2: for \( \ell = \bar{\ell}, \bar{\ell} - 1, \ldots, 1 \) do
3: \[ \check{V}^{(i)} \leftarrow \{ \check{v} \mid v \in \mathcal{V}^{(i)} \}, \quad B^{(i)} \leftarrow \emptyset \]
4: end for
5: for \( \ell = \bar{\ell}, \bar{\ell} - 1, \ldots, 1 \) do
6: if \( \ell = \bar{\ell} \) then
7: Choose \( \check{v} \in \check{V}^{(i)} \), \( \check{V}^{(i)} \leftarrow \check{V}^{(i)} \setminus \{ \check{v} \}, \quad B^{(i)} \leftarrow \{ v \} \)
8: if \( \bar{\ell} = m \) then
9: return \( B = B^{(i)} \) \( \triangleright \) Case of eq. (9)
10: else
11: Do a column reduction \( [ L_{A,d}(\check{v}) ] \rightarrow \tilde{D} \)
12: end if
13: end if
14: while \( \check{V}^{(i)} \neq \emptyset \) do
15: Choose \( \check{v} \in \check{V}^{(i)} \), \( \check{V}^{(i)} \leftarrow \check{V}^{(i)} \setminus \{ \check{v} \} \)
16: Do a column reduction of the right-most column in the augmented matrix
\[ [ \tilde{D} | \check{v} ] \rightarrow \begin{bmatrix} D' & r' \\ D & r \end{bmatrix} \]
17: if \( r' \neq 0 \) then \( \triangleright \) Case 1. in the proof of theorem 11
18: \[ B^{(i)} \leftarrow B^{(i)} \cup \{ r \} \]
19: Do a column reduction \( [ \tilde{D} | L_{A,d}(\check{v}) ] \rightarrow \tilde{R} \)
20: \[ D \leftarrow \tilde{R} \]
21: else if \( r \neq 0 \) and \( \ell > 1 \) then \( \triangleright \) Case 2. (a) in the proof
22: \[ \ell' \leftarrow \text{rank}(r), \check{V}^{(\ell')} \leftarrow \check{V}^{(\ell')} \cup \{ \check{v} \} \]
23: end if
24: end while
25: if \( \ell > 1 \) then
26: \[ \tilde{D} \leftarrow \begin{bmatrix} E_n & O \\ O & f(A) \end{bmatrix} \tilde{D} \] \( \triangleright \) For the next step in the loop
27: end if
28: return \( B = \bigcup_{i=1}^{\ell} B^{(i)} \)

4.3. A refined algorithm for computing Jordan-Krylov basis

In this section, we present Algorithm 3 as a refinement of the algorithm presented in the last section.

Since the multiplicity \( m \) of \( f(\lambda) \) in the characteristic polynomial of \( A \) is equal to the sum of
the length of linearly independent Jordan chains, it holds that

\[ m = \sum_{i=1}^{\ell} \ell \cdot \#B^{(i)}, \tag{11} \]

where \( \#B^{(i)} \) denotes the number of elements in \( B^{(i)} \). If a Jordan-Krylov independent set satisfies eq. (11), this gives a Jordan-Krylov basis. That is, eq. (11) is a terminating condition of the algorithm.

In Algorithm 2, for a vector \( v \) of rank \( \ell \), a vector membership problem is reduced to a column reduction of an augmented matrix using \( \tilde{v} \) defined by placing vectors \( f(A)^{\ell-1}v \) and \( v \) vertically. For example, in line 11 of Algorithm 2, the matrix \( [L_{A,\tilde{v}}] \) is constructed by multiplying a matrix of size \( 2n \) from the left. However, since the matrix is block diagonal, actual multiplication can be executed by multiplying a matrix of size \( n \). Thus, in Algorithm 3, \( [L_{A,\tilde{v}}] \) is divided into \( [L_{A,v}] \) and \( [L_{A,f(A)^{\ell-1}v}] \). Then, the column reduction of the matrix \( \begin{bmatrix} W & r' \\ S & r \end{bmatrix} = \begin{bmatrix} W & v' \\ S & v \end{bmatrix} C \) is divided into two column reductions \( [W | r'] = [W | v'] C \) and \( [S | r] = [S | v] C \) with the same matrix \( C \), which is called a simultaneous column reduction. Note that, in the case of \( \ell = 1 \), we have \( W = S \); hence, both parts of the simultaneous column reduction are identical.

**Remark 4.** During the Jordan-Krylov elimination, there are cases where unnecessary calculations can be eliminated based on the basis that has already been computed, which is included in lines 3 and 17 of Algorithm 3. The former case is described as in Remark 1. In the latter case, if the sum of the length of already calculated Jordan chains is equal to the multiplicity \( m \) of \( f(\lambda) \) in \( \chi_A(\lambda) \), then the algorithm can be terminated.

**Remark 5.** Algorithm 3 is usually executed with multiple-precision arithmetic over integers or rational numbers. An increase in the number of arithmetic operations may cause an increase in the number of digits of nonzero elements in computed Jordan-Krylov basis or Jordan chains, which may make the computation slow. However, keeping a “simpler” form of Krylov generating set, it is expected that the computation of Jordan-Krylov elimination and Jordan chains may be more efficient, as follows.

1. If the input vectors have a simpler form, then the computed Jordan-Krylov basis may have a simpler form and its computation may be faster.

2. If a Jordan-Krylov basis is computed in a simpler form, then the computation of the Jordan chains may be faster.

A heuristic by the reduction of Krylov generating set for obtaining the simpler form is shown in Procedures 4, which may be performed prior to the Jordan-Krylov elimination. This procedure uses a simple column reduction on the Krylov generating set.
Algorithm 3 Computing a Jordan-Krylov basis of $\ker f(A)\ell$: a refined version

**Input:** A matrix $f(A) \in K^{n \times n}$, a Krylov generating set $\mathcal{V} = \bigcup_{\ell=1}^{\ell} \mathcal{V}^{(\ell)}$ of $\ker f(A)\ell$, the multiplicity $m$ of $f(\lambda)$ in the characteristic polynomial of $A$

**Output:** A Jordan-Krylov basis $\mathcal{B} = \bigcup_{\ell=1}^{\ell} \mathcal{B}^{(\ell)}$ of $\ker f(A)\ell$

1: (Optional) Call Procedure 4 with $f(A), \bigcup_{\ell=1}^{\ell} \mathcal{V}^{(\ell)}$
2: Choose $v \in \mathcal{V}^{(\ell)}$, $\mathcal{V}^{(\ell)} \leftarrow \mathcal{V}^{(\ell)} \setminus \{v\}$, $\mathcal{B}^{(\ell)} \leftarrow \{v\}$, $m \leftarrow m - \ell$
3: if $m = 0$ then
   4: return $\mathcal{B}^{(\ell)}$  \(\triangleright\) Case eq. (9)
5: end if
6: $S \leftarrow [L_{A,d}(v)], \; W \leftarrow f(A)^{\ell-1}S$  \(\triangleright\) $W = f(A)^{\ell-1}S$ can be reduced
7: for $\ell = \ell, \ldots, 2, 1$ do
8:   if $\ell < \ell$ and $\mathcal{V}^{(\ell)} \neq \emptyset$ then
9:     (Optional) Call Procedure 4 with $f(A), \bigcup_{\ell=1}^{\ell} \mathcal{V}^{(\ell)}$
10:   end if
11: while $\mathcal{V}^{(\ell)} \neq \emptyset$ do
12:     Choose $v \in \mathcal{V}^{(\ell)}$, $\mathcal{V}^{(\ell)} \leftarrow \mathcal{V}^{(\ell)} \setminus \{v\}$
13:     $v' \leftarrow f(A)^{\ell-1}v$
14:     Simultaneous column reduction of the rightmost column in the augmented matrix
15:     \[ [W | v'] \rightarrow [W | r'], \; [S | v] \rightarrow [S | r] \]
16:     if $r' \neq 0$ then \(\triangleright\) $r' \notin \text{span}_K W$
17:     $\mathcal{B}^{(\ell)} \leftarrow \mathcal{B}^{(\ell)} \cup \{r\}$, $m \leftarrow m - \ell$
18:     if $m = 0$ then
19:     return $\mathcal{B} = \mathcal{B}^{(1)} \cup \mathcal{B}^{(2-1)} \cup \cdots \cup \mathcal{B}^{(1)}$
20:     end if
21:     else if $r \neq 0$ and $\ell > 1$ then
22:     $\ell' \leftarrow \text{rank}_K r$, $\mathcal{V}^{(\ell')} \leftarrow \mathcal{V}^{(\ell')} \cup \{r\}$
23:     end if
24: end while
25: if $\ell > 1$ then
26: $\mathcal{B}^{(\ell-1)} \leftarrow \emptyset$, $S \leftarrow f(A)S$  \(\triangleright\) For the next step in the loop
27: end if
28: end for
**Procedure 4** Reduction of Krylov generating set (See Remark 5)

**Input:** A matrix $f(A) \in \mathbb{K}^{n \times n}$, a Krylov generating set $\bigcup_{\ell=1}^{\bar{\ell}} V^{(\ell)}$ of $\ker f(A)^{\ell}$, where $1 \leq \ell \leq \bar{\ell}$

**Output:** The reduced Krylov generating set $\bigcup_{\ell=1}^{\ell'} V^{(\ell)}$

1: $T \leftarrow V^{(\ell)}$, $V^{(\ell)} \leftarrow \emptyset$
2: Reduce $[T] \rightarrow [T']$ with a column reduction
3: while $T' \neq \emptyset$ do
4: Choose $v \in T'$, $T' \leftarrow T' \setminus \{v\}$
5: if $v \neq 0$ then
6: $\ell' \leftarrow \text{rank}_1 v$, $V^{(\ell')} \leftarrow V^{(\ell')} \cup \{v\}$
7: end if
8: end while
9: return $\bigcup_{\ell=1}^{\ell'} V^{(\ell)}$

4.4. An algorithm for computing generalized eigenvectors of $A$

Algorithm 5 computes Jordan chains using the elements in a Jordan-Krylov basis of $\ker f(A)^{\ell}$. Note that $\lambda$ represents the eigenvalue and its conjugates. Finally, Algorithm 6 integrates Algorithms 1, 3 and 5 for computing the generalized eigenspace of $A$ associated to the roots of $f(\lambda)$.

**Algorithm 5** Computing Jordan chains from a Jordan-Krylov basis of $\ker f(A)^{\ell}$

**Input:** A matrix $f(A) \in \mathbb{K}^{n \times n}$, an irreducible factor $f(\lambda) \in \mathbb{K}[\lambda]$

a Jordan-Krylov basis $B = \bigcup_{\ell=1}^{\ell'} B^{(\ell)}$ of $\ker f(A)^{\ell}$

**Output:** a set of Jordan chains $\Phi$ of $A$ associated to the roots of $f(\lambda)$

1: $\Phi \leftarrow \emptyset$
2: Compute polynomials $\psi^{(1)}(\mu, \lambda), \ldots, \psi^{(\ell)}(\mu, \lambda)$
3: for $\ell = \ell', \ell - 1, \ldots, 1$ do
4: while $B^{(\ell)} \neq \emptyset$ do
5: Choose $b \in B^{(\ell)}$, $B^{(\ell)} \leftarrow B^{(\ell)} \setminus \{b\}$
6: for $k = \ell, \ell - 1, \ldots, 2$ do
7: $p^{(k)} \leftarrow \psi^{(\ell)}(A, \lambda \mu)b$
8: $b \leftarrow f(A)b$
9: end for
10: $p^{(1)} \leftarrow \psi^{(1)}(A, \lambda \mu)b$
11: $\Phi \leftarrow \Phi \cup \{\{p^{(i)} : i = 1, \ldots, \ell\}\}$
12: end while
13: end for
14: return $\Phi$
The generalized eigenspace (a set of Jordan chains)

5. The time complexity of the algorithm

In this section, we discuss the time complexity of the arithmetic operations over $K$ of the algorithms presented in the previous sections.

The computational complexity of the algorithm depends not only on the size of the input matrix but also on its structure as eigenproblems, including the number of irreducible factors in the characteristic polynomial, their degrees, multiplicities, and the way all the minimum annihilating polynomials are factored into irreducible factors. Thus, the analysis of computational complexity for a general matrix will become extremely complicated. Therefore, we treat a special but typical case with the following assumptions.

Assumption 1. Let $A \in K^{n \times n}$ be the input matrix with $\chi_A(\lambda) = f(\lambda)^{\ell+k}g(\lambda)$ and $\pi_A(\lambda) = f(\lambda)^{\ell}g(\lambda)$, where $f(\lambda) \in K[\lambda]$ and $\deg f(\lambda) = d$. Assume from 1. to 3. as follows.

1. For the eigenvalue that are roots of $f(\lambda)$ and on which we focus, there is one Jordan chain of length $\ell$ and $k$ linearly independent eigenvectors associated to that eigenvalue.

2. The polynomial $g(\lambda)$ is square-free in $K[\lambda]$.

3. Let $r = \#\{e \in E \mid f(\lambda) \text{ divides } \pi_{A,e}(\lambda)\}$ and $t = \#\{e \in E \mid f(\lambda) \text{ divides } \pi_{A,e}(\lambda) \text{ and } \deg g_e(\lambda) > 0 \}$.

Assumption 2. Assume that, for $e \in E$, the minimal annihilating polynomial $\pi_{A,e}(A)$ is given. As a result, the matrices $f(A)$ and $g_e(A)$ also are given.

Note that Assumption 1 is about the structure of the input matrix. Assumption 2 means that the complexity of calculating the minimal annihilating polynomials is not considered in this analysis. For calculating the minimal annihilating polynomials, the choice of algorithm, whether deterministic or probabilistic, can affect the overall computational complexity estimation. In this paper, we will focus on the complexity estimation of algorithms for calculating general eigenvectors, excluding the estimation of the algorithm for calculating the minimal annihilating polynomials, which can be found in our previous paper (Tajima et al. 2018a). Furthermore, since $f(A)$ and $g_e(A)$ are obtained during the computation of the minimal annihilating polynomials, they will be excluded from this evaluation.

Based on the above assumptions, we first present an estimate of the complexity of the overall algorithm as a conclusion.
Theorem 17. Under Assumptions 1 and 2, the complexity of Algorithm 6 is $O(\frac{d n^2 \tilde{\ell}}{\ell} \max\{d, \tilde{\ell}, r\})$ plus the complexity of calculating $f(A)$.

For the proof of Theorem 17 we estimate the complexity of each step as follows. First, we estimate it of Algorithm 1.

**Lemma 18.** Algorithm 1 is performed in $O(n^2 t)$.

**Proof.** The complexity of each step is as follows. In Line 3, for $e \in E_f$, calculating $g_e(A)e$ is performed in $O(n^2)$. In the for-loop, the number of $e \in E_f$ satisfying that $\deg(g_e(\lambda)) > 0$ is at most $t$. Thus, this step is performed in $O(n^2 t)$, which proves the lemma.

Next, we estimate the complexity of Algorithms 3 to 5 as well.

**Lemma 19.** Algorithm 3 is performed in $O(dnr(\tilde{\ell}n + dk))$.

**Proof.** The complexity of each step is as follows:

- **Line 6** The columns of $S$ are calculated as repeating calculation of $Av$ for $d - 1$ times, which costs $O(dn^2)$. For calculating $f(A)^{i-1}S$, the multiplication $f(A)S$ is performed in $O(dn^2)$ since the size of $S$ is $n \times d$. Then, this multiplication is repeated for $\tilde{\ell} - 1$ times, which costs $O(d\tilde{\ell}n^2)$. If the reduction of $W$ is performed, its cost is $O(d^2n)$.

- **Line 11** The while-loop, together is the for-loop in line 7, is repeated for $dr$ times, which is equal to $|V^{(1)}| + \cdots + |V^{(\ell)}|$. In line 14 the vector $v'$ is calculated in $O(\tilde{\ell}n^2)$, which is bounded by $O(n^2)$.

- **Line 26** is performed in $O(dn^2)$, thus this step is performed in $O(d\tilde{\ell}n^2)$ in the total of the for-loop in Line 7.

Summarizing above, the complexity of Algorithm 3 is $O(dnr(\tilde{\ell}n + dk))$, which proves the lemma.

**Lemma 20.** Procedure 4 is performed in $O(n(r^2 + \tilde{\ell}n))$.

**Proof.** The number of elements in the Krylov generating set of $\ker f(A)\tilde{\ell}$ is $r$, thus, line 2 is performed in $O(nr^2)$. In line 6 the complexity of calculating rank $r$ is estimated as follows. The complexity of calculating $f(A)w$ is $O(n^2)$, and the repeated calculation of $f(A)w$ for rank $r w$ is performed in $O(\tilde{\ell}n^2)$. Thus, the complexity of the whole algorithm is $O(n(r^2 + \tilde{\ell}n))$, which proves the lemma.

**Lemma 21.** Algorithm 5 is performed in $O((d + \tilde{\ell})(d^3 + n^3))$.

**Proof.** Each step is performed as follows:
We have proposed (Tajima et al. (2018a)), the minimal annihilating polynomials are calculated as $\psi(\mu, \lambda)$ is estimated as follows. Note that $\psi(\mu, \lambda)$ is regarded as a polynomial in $K[\lambda][\mu]$. For $\ell = 1$, $f(\lambda) - f(\mu)$ and $(f(\lambda) - f(\mu))/(\lambda - \mu)$ are calculated in $O(d)$. For $\ell > 1$, first we calculate $\psi(\mu, \lambda)$ in $O(\deg(\psi))$ $\psi(\mu, \lambda)$. For the coefficients in $K[\lambda]$, the coefficients are calculated as the product of two polynomials in $K[\lambda]$ of degree less than or equal to $d$, which costs $O(d^2)$, followed by the division of a polynomial of degree less than or equal to $d^2$ by $f(\lambda)$, which costs $O(d^2)$. Thus, $\psi(\mu, \lambda)$ is calculated in $O((d + \ell)d^3)$. Finally, $\psi(\mu, \lambda), \ldots, \psi(\mu, \lambda)$ are calculated in $O((d + \ell)\bar{d}d^3)$ in total.

• Line 2 By the assumption, the for-loop is repeated for $\ell = \bar{\ell}, 1$. In line 7 $p^{(k)}$ is calculated using the Horner’s rule with matrix-vector multiplications in $O(n^2(d + k))$, since $\deg(\psi^{(k)}) = d + k - 1$. In line 8 the multiplication $f(A)b$ is performed in $O(n^2)$. Thus, calculating a Jordan chain of length $\ell$ and 1 is performed in $O(\bar{\ell}n^2(d + \bar{\ell}))$ (since $p^{(k)}$ is calculated for $k = \bar{\ell}, \ldots, 1$) and $O(n^2d)$, respectively, and the complexity of calculating the Jordan chains is bounded by $O(\bar{\ell}n^2(d + \bar{\ell}))$.

Summarizing the above, the complexity of Algorithm 5 is $O((d + \bar{\ell})\bar{d}d^3 + \bar{\ell}n^2(d + \bar{\ell})) = O((d + \bar{\ell})\bar{d}d^3 + n^2d)$, which proves the lemma.

A proof of Theorem 7. Summarizing the above, we see that the complexity of Algorithm 3 dominates that of the whole algorithm, which is $O(dn^2\bar{\ell} + \max(d, \bar{\ell}, r))$.

6. An example of computation

We give an example for illustration, in which $K = \mathbb{Q}$. Note that, in the example, Procedure 4 is not performed. Let $A$ be a square matrix of order 10 defined as

$$A = \begin{pmatrix}
5 & -5 & 6 & -9 & 5 & 0 & 0 & -4 & 5 & -6 \\
-14 & 11 & -9 & 39 & -2 & -2 & 6 & 16 & -10 & 12 \\
-5 & 5 & -6 & 9 & -5 & 1 & 0 & 5 & -5 & 5 \\
5 & 2 & 1 & 7 & 7 & -4 & 6 & 3 & 5 & 2 \\
-5 & -9 & 9 & -9 & 1 & 3 & -5 & -7 & -5 & 9 \\
5 & 2 & -4 & -2 & 5 & -5 & 5 & -1 & 5 & 2 \\
5 & 9 & -14 & 0 & -3 & -4 & 3 & 4 & 5 & 9 \\
-5 & -9 & 4 & -23 & -8 & 7 & -11 & -11 & -5 & -9 \\
0 & 8 & -6 & 16 & 2 & -4 & 6 & 7 & 0 & 9 \\
4 & -7 & 4 & -25 & -3 & 3 & -6 & -11 & 0 & -8
\end{pmatrix}.$$

We have $\chi_A(\lambda) = f_1(\lambda)^4 f_2(\lambda)$, where $f_1(\lambda) = \lambda^2 + \lambda + 5$ and $f_2(\lambda) = \lambda^2 + \lambda + 4$, and $m_1 = 4$ and $m_2 = 1$. Let $E = \{e_1, e_2, \ldots, e_{10}\}$ be the standard basis of $K^{10}$. Then, with the method we have proposed (Tajima et al. (2018a)), the minimal annihilating polynomials are calculated as

$$\pi_{A, e_j}(\lambda) = \begin{cases}
f_1(\lambda) & (j = 2, 9) \\
f_1(\lambda)f_2(\lambda) & (j = 1, 10) \\
f_1(\lambda)^2 f_2(\lambda) & (j = 3) \\
f_1(\lambda)^3 f_2(\lambda) & (j = 4, 5, 7) \\
f_1(\lambda)^3 f_3(\lambda) & (j = 6, 8)
\end{cases}.$$

We see that $\bar{\ell}_1 = 3$ and $\bar{\ell}_2 = 1$. 

18
6.1. Computing the generalized eigenspace associated to the roots of \( f_1(\lambda) \)

Let us compute the Jordan chain through a Jordan-Krylov basis of \( \ker f_1(A)^3 \). First, Algorithm 3 computes

\[
\mathcal{V}_1^{(3)} = \{ v_{1,4}, v_{1,5}, v_{1,6}, v_{1,7}, v_{1,8} \}, \quad \mathcal{V}_1^{(2)} = \{ v_{1,3} \}, \quad \mathcal{V}_1^{(1)} = \{ v_{1,1}, v_{1,2}, v_{1,9}, v_{1,10} \},
\]

as outputs, where \( v_{1,j} = f_1(A) e_j \) and \( e_j \) (\( j = 1, 3, 6, 8, 10 \)) and \( e_j \) (\( j = 2, 4, 5, 7, 9 \)).

Next, Algorithm 3 computes a Jordan-Krylov basis \( B_1 = B_1^{(3)} \cup B_1^{(2)} \cup B_1^{(1)} \) as follows. Let \( m = m_1 = 4 \) and \( \ell = 1 \). For \( \ell = 1 \), choose \( v_{1,4} \in \mathcal{V}_1^{(3)} \) and assign

\[
B_1^{(3)} \leftarrow \{ v_{1,4} \}, \quad S \leftarrow \{ [L_{3,4}(\{v_{1,4}\})] = [v_{1,4}, Av_{1,4}], \quad W \leftarrow f_1(A)^2 S, \quad m \leftarrow m - \ell = 1. \quad (12)
\]

Then, we see that

\[
v_{1,5}' = v_{1,4}' + (2/5)Av_{1,4}', \quad v_{1,6}' = v_{1,4}', \quad v_{1,7}' = v_{1,4}' + (1/5)Av_{1,4}', \quad v_{1,8}' = (1/5)Av_{1,4}',
\]

where \( v_{1,j}' = f_1(A)^2 v_{1,j} \). Thus, computation of \( B_1^{(3)} \) is finished and \( S \) and \( W \) remain the same.

Next, Algorithm 4 computes a Jordan-Krylov basis \( B_1 = B_1^{(3)} \cup B_1^{(2)} \cup B_1^{(1)} \) as follows. Let \( m = m_1 = 4 \) and \( \ell = 1 \). For \( \ell = 1 \), choose \( v_{1,4} \in \mathcal{V}_1^{(3)} \) and assign

\[
B_1^{(3)} \leftarrow \{ v_{1,4} \}, \quad S \leftarrow \{ [L_{3,4}(\{v_{1,4}\})] = [v_{1,4}, Av_{1,4}], \quad W \leftarrow f_1(A)^2 S, \quad m \leftarrow m - \ell = 1. \quad (13)
\]

and let \( W \) remain the same. For all \( v_1' = f(A)v_1 \) with \( v_1 \in \mathcal{V}_1^{(2)} \), there exists a simultaneous column reduction as \([W \mid v_1'] \rightarrow [W \mid 0]\) and \([S \mid v_1] \rightarrow [S \mid r]\), thus we have \( v_{1,3} \rightarrow v_{1,15}, v_{1,11} \rightarrow v_{1,16}, v_{1,12} \rightarrow v_{1,17}, v_{1,13} \rightarrow v_{1,18}, v_{1,14} \rightarrow v_{1,19} \). Since the ranks of \( v_{1,15}, v_{1,16}, v_{1,17}, v_{1,18} \) and \( v_{1,19} \) are equal to 2, \( \mathcal{V}_1^{(2)} \) is renewed as \( \mathcal{V}_1^{(2)} \leftarrow \{ v_{1,15}, v_{1,16}, v_{1,17}, v_{1,18}, v_{1,19} \} \). As a result, \( B_1^{(2)} = \emptyset \) and \( W \) remain the same.

For \( \ell = 1 \), since \( m \geq \ell \), there exists a vector \( v_1 \in \mathcal{V}_1^{(1)} \) satisfying that a column reduction of the rightmost column outputs \([W \mid v_1] \rightarrow [W \mid r \neq 0]\). The column reduction of \( v_{1,1} \) with respect to \( W \) yields

\[
r = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0),
\]

assign \( B_1^{(1)} \leftarrow \{ r \} \) and \( m \leftarrow m - \ell = 0 \). Then, the algorithm terminates.

As a consequence, a Jordan-Krylov basis \( B_1 \) is computed as

\[
B_1 = B_1^{(3)} \cup B_1^{(2)} \cup B_1^{(1)} = \{ v_{1,4}, r \}. \quad (14)
\]

Finally, Algorithm 5 computes a set of Jordan chains. For \( i = 1, 2, 3 \), \( \psi_i^{(3)}(\mu, \lambda) \) is computed as

\[
\psi_i^{(1)}(\mu, \lambda) = \mu + \lambda + 1, \quad \psi_i^{(2)}(\mu, \lambda) = \mu^2 + (2\lambda + 2)\mu + \lambda - 4,
\]

\[
\psi_i^{(3)}(\mu, \lambda) = \mu^3 + (3\lambda + 3)\mu^2 + (3\lambda - 12)\mu + (-4\lambda + 9).
\]
For \( v_{1,4} \in B_1^{(3)} \), the Jordan chain \( \{p_1^{(3)}(\lambda, v_{1,4}), p_1^{(2)}(\lambda, v_{1,4}), p_1^{(1)}(\lambda, v_{1,4})\} \) of length 3 is computed as
\[
p_1^{(3)}(\lambda, v_{1,4}) = \psi_3^{(3)}(A, \lambda) v_{1,4}
= \lambda^3(57, -60, -57, 8, 36, -3, -66, 6, -30, 3)
+ \lambda^2(205, -755, -205, -121, 150, 54, 6, 401, -307, 455),
\]
\[
p_1^{(2)}(\lambda, v_{1,4}) = \psi_2^{(2)}(A, \lambda) f_1(A) v_{1,4}
= \lambda^2(11, 32, -11, 35, -78, 35, 78, -78, 24, -43)
+ \lambda(-175, 225, 175, -49, -191, 46, 286, -96, 126, -50),
\]
\[
p_1^{(1)}(\lambda, v_{1,4}) = \psi_1(A, \lambda) f_1(A) E_1 v_{1,4}
= 19(\lambda^0(1, 0, 1, -2, 1, 2, -2, 1, -1) + \lambda(-5, 11, 5, 1, -7, 1, 7, -7, 6, -6)).
\]

For \( r \in B_1^{(3)} \), the Jordan chain \( \{p_1^{(1)}(\lambda, r)\} \) of length 1 is computed as
\[
p_1^{(1)}(\lambda, r) = \psi_1(A, \lambda) E r
= \lambda^1(0, 0, 0, 0, 0, 0, 0, -1, 0) + \lambda(-5, 10, 5, -5, 5, -5, -5, -5, -1, 0),
\]
that is, \( p_1^{(1)}(\lambda, r) \) is an eigenvector.

**Remark 6.** In eq. (12), \( m \) is renewed as \( m = 1 \), then we see that \#\( B_1^{(3)} = 1 \), \#\( B_1^{(2)} = \emptyset \) and \#\( B_1^{(1)} = 1 \). Choose a vector \( v_{1,1} \) from \( \mathcal{V}_1^{(1)} = \{v_{1,1}, v_{1,2}, v_{1,9}, v_{1,10}\} \). Then, the result \( r \) of the rightmost column reduction \([W \mid v_{1,1}] \rightarrow [W \mid r] \) of \( v_{1,1} \) with respect to \( W \) is not equal to zero. Thus, we set \( B_1^{(1)} = \{r\} \). Accordingly, a Jordan-Krylov basis is given as \( B_1^{(3)} \cup B_1^{(1)} = \{v_{1,4} \cup v_{1,1}\} \).

### 6.2. Computing the generalized eigenspace associated to the roots of \( f_2(\lambda) \)

Let us compute a Jordan chain through a Jordan-Krylov basis of \( \text{ker} \ f_2(A) \). First, Algorithm[1] computes
\[
\mathcal{V}_2 = \mathcal{V}_2^{(1)} = \{v_{2,1}, v_{2,2}, v_{2,8}, v_{2,10}\},
\]
where \( v_{2,j} = f_1(A) e_j \) \((j = 1, 10)\), \( f_1(A) e_j \) \((j = 3)\), \( f_1(A) e_j \) \((j = 6, 8)\). Next, with Algorithm 3, choose \( v_{2,1} \in \mathcal{V}_2^{(1)} \) and let us assign \( B_2^{(3)} \leftarrow \{v_{2,1}\} \), then \( \tilde{\ell}_2 = m_2 \) implies that \( B_2 = B_2^{(3)} \). Thus, as a Jordan-Krylov basis of \( \text{ker} f_2(A) \), we have \( B_2 = B_2^{(3)} = \{v_{2,1}\} \). Finally, with Algorithm[5] using \( \psi_2^{(1)}(\mu, \lambda) = \psi_2^{(3)}(\mu, \lambda) + \mu + \lambda + 1 \) and \( \{v_{2,1}\} \in B_2 \), the eigenvector \( p_2^{(1)}(\lambda, v_{2,1}) \) is computed as
\[
p_2^{(1)}(\lambda, v_{2,1}) = \psi_2(A, \lambda) E v_{2,1}
= \lambda^1(1, 0, 0, 0, 0, 0, 0, -1, 0) + \lambda(-1, 4, 0, 0, 0, 0, 0, -1, 4).
\]

**Remark 7.** If Procedure[4] is applied on \( \mathcal{V}_2^{(1)} \) in eq. (15), linearly independent vectors are obtained as
\[
v_{2,1} = \lambda(1, 0, 0, 0, 0, 0, 0, -1, 0), \quad v_{2,6} = \lambda(0, 1, 0, 0, 0, 0, 0, 0, -1).
\]
It is expected that the use of Procedure[4] makes the computation more efficient.

We see that the matrix \( A \) is similar to
\[
\begin{pmatrix}
C(f_1) & E_2 \\
C(f_1) & C(f_1)
\end{pmatrix},
\]
where \( C(f) \) is the companion matrix of \( f \) and \( E_2 \) is the identity matrix of order 2.
7. Experiments

We have implemented the algorithms introduced above on a computer algebra system Risa/Asir (Noro 2003) and evaluated them. In Algorithm 1 it is assumed that minimal annihilating polynomials are pre-calculated by using the method in our previous paper (Tajima et al. 2018a) and given. A product \( g(A) \) of a matrix polynomial and a vector can be computed efficiently by using Horner’s rule (Tajima et al. 2018a, Tajima et al. 2018b). Similarly, in Algorithm 3, \( f(A) \) can also be calculated efficiently by using an improved Horner’s rule for matrix polynomials that reduces the number of matrix-matrix multiplications (Tajima et al. 2014). For the reduction of the Krylov generating set, Procedure 4 was used (see Remark 5).

We have executed experiments with changing matrix sizes. The test was carried out in the following environment: Apple M1 Max up to 3.2 GHz, RAM 32 GB, macOS 12.2.1, Risa/Asir version 20220309. All data are the average of measurements from 10 experiments.

7.1. Computing generalized eigenvectors of matrices with Jordan chains of several lengths

In the experiments, matrices are given as follows. Let \( f(\lambda) \) be a monic polynomial of degree \( d \) with integer coefficients. Let

\[
\begin{align*}
A' &= \begin{pmatrix}
C(f) & E_d & C(f) \\
C(f) & E_d & C(f) \\
\vdots & \vdots & \ddots \\
C(f) & E_d & C(f) \\
C(f) & C(f) & C(f) \\
E_d & C(f) & C(f) \\
C(f) & C(f) & C(f) \\
C(f) & C(f) & \end{pmatrix}
\end{align*}
\]

where \( C(f) \) denotes the companion matrix of \( f(\lambda) \) and \( E_d \) denotes the identity matrix of order \( d \). The test matrix \( A \) is calculated by applying a similarity transformation on \( A' \), as follows. Let \( T_{ij}(m) \) be the matrix obtained by adding \( m \) times the \( i \)-th row of the identity matrix to the \( j \)-th row. Then, \( A \) was obtained by multiplying \( T_{ij}(m) \) from the left and \( T_{ij}(m)^{-1} \) from the right several times until \( A \) becomes dense, where \( m \in \mathbb{Z} \) is a small random integer. Clearly, \( \chi_A(\lambda) = f(\lambda)^{10} \).

Table 1 shows the results of computation without the reduction of Krylov generating set by Procedure 4. Computing times are measured in seconds. Memory usage, measured in megabytes, represents the total amount of memory requested by the program, not necessarily the amount of memory used at one time. “Average \(|a_{ij}|\)” and “max \(|a_{ij}|\)” denote the average and the maximum of the absolute value of the element of \( A \), respectively. To make a difference in each matrix small, we fix the magnitude of the determinant by fixing the constant term of \( f(\lambda) \). For all the matrices used in the experiments, Average \(|a_{ij}| < 10^3 \) and max \(|a_{ij}| < 10^4 \), thus we expect that the difference in the test matrices does not have much effect on the computing time.

Table 2 shows the computing time of each step of the algorithm. The column “\( f(A) \)” corresponds to the computing time of \( f(A) \); “Alg. 1” corresponds to computing time of a Krylov generating set; “Alg. 3” corresponds to computing time of Jordan-Krylov bases; “Alg. 5” corresponds to the computing time of Jordan chains, calculated as shown in the example in Section 6.

Now, the calculation results were compared with those that also included the reduction of the Krylov generating set by Procedure 4. The calculation results are shown in Table 3. The columns
Table 1: Computing time and memory usage for the case in Section 7.1 without employing Procedure 4.

| deg(f) | size(A) | Time (sec.) | Memory usage (MB) | Average | max |
|-------|--------|-------------|-------------------|--------|-----|
| 2     | 20     | 0.02        | 23.89             | 33.76  | 322 |
| 4     | 40     | 0.16        | 206.51            | 74.03  | 1409|
| 6     | 60     | 0.70        | 819.13            | 43.33  | 496 |
| 8     | 80     | 1.72        | 2.51 × 10^3       | 108.49 | 2397|
| 10    | 100    | 3.86        | 6.31 × 10^3       | 66.24  | 1977|
| 12    | 120    | 11.58       | 1.59 × 10^4       | 91.78  | 4179|
| 14    | 140    | 20.59       | 3.10 × 10^4       | 87.61  | 1229|
| 16    | 160    | 40.72       | 5.83 × 10^4       | 103.40 | 4944|
| 18    | 180    | 90.07       | 1.36 × 10^5       | 195.82 | 8226|
| 20    | 200    | 151.09      | 2.13 × 10^5       | 182.82 | 5457|

Table 2: Computing time of each step of the algorithm for the case in Section 7.1 without employing Procedure 4.

| deg(f) | size(A) | f(A) | Alg. 1 | Alg. 3 | Alg. 5 |
|-------|--------|------|--------|--------|--------|
| 2     | 20     | 5.61 × 10^{-4} | 4.48 × 10^{-5} | 0.02   | 0.01   |
| 4     | 40     | 0.02 | 8.61 × 10^{-5} | 0.11   | 0.03   |
| 6     | 60     | 0.11 | 1.36 × 10^{-4} | 0.39   | 0.20   |
| 8     | 80     | 0.35 | 1.79 × 10^{-4} | 0.83   | 0.53   |
| 10    | 100    | 0.86 | 2.51 × 10^{-4} | 1.57   | 1.43   |
| 12    | 120    | 1.90 | 2.91 × 10^{-4} | 3.64   | 6.04   |
| 14    | 140    | 3.58 | 3.40 × 10^{-4} | 6.63   | 10.39  |
| 16    | 160    | 6.60 | 4.09 × 10^{-4} | 11.60  | 22.52  |
| 18    | 180    | 10.83| 4.55 × 10^{-4} | 26.78  | 52.46  |
| 20    | 200    | 14.69| 5.54 × 10^{-4} | 38.95  | 97.54  |

under “Without Proc. 4” are the same results as in Table 1. The columns under “With Proc. 4” are the results when Procedure 4 was also executed. Table 4 shows the computing time of each step of the algorithm. The column “Proc. 4” corresponds to the computing time of the reduction of Krylov generating set, and the rest of the columns are the same as those in Table 2.

7.2. Computing generalized eigenvectors associated to a specific eigenfactor

In the experiments, matrices are given as follows. Let \( f(\lambda) \) and \( g_i(\lambda) \) \((i = 1, 2, 3)\) be monic polynomials with integer coefficients. Their degrees are as follows: \( \deg(g_2) = 2d \), and the degrees of all the other polynomials are equal to \( d \). Let

\[
A' = \begin{pmatrix}
C(f) & E_d & E_d & E_d & E_d \\
C(f) & C(f) & E_d & E_d & E_d \\
C(f) & C(f) & C(f) & E_d & E_d \\
C(f) & C(f) & C(f) & C(f) & E_d \\
C(g_1) & E_{2d} & E_d & C(g_1) & C(g_3) \\
C(g_1) & C(g_1) & C(g_1) & C(g_3) & C(g_3)
\end{pmatrix},
\]

where \( C(f) \) and \( E_d \) are defined similarly as in Section 7.
Table 3: Computing time and memory usage for the case in Section 7.1.

| deg(f) | size(A) | Without Proc. 4 | With Proc. 4 |
|--------|---------|-----------------|--------------|
|        |         | Time            | Memory       |
|        |         | Time            | Memory       |
| 2      | 20      | 0.02            | 23.89        |
| 4      | 40      | 0.16            | 206.51       |
| 6      | 60      | 0.70            | 819.13       |
| 8      | 80      | 1.72            | $2.51 \times 10^3$ |
| 10     | 100     | 3.86            | $6.31 \times 10^3$ |
| 12     | 120     | 11.58           | $1.59 \times 10^4$ |
| 14     | 140     | 20.59           | $3.10 \times 10^4$ |
| 16     | 160     | 40.72           | $5.83 \times 10^4$ |
| 18     | 180     | 90.07           | $1.36 \times 10^5$ |
| 20     | 200     | 151.09          | $2.13 \times 10^7$ |

Table 4: Computing time of each step of the algorithm for the case in Section 7.1 with employing Procedure 4.

| deg(f) | size(A) | f(A)  | Alg. 1 | Alg. 3 | Proc. 4 | Alg. 5 |
|--------|---------|-------|--------|--------|---------|--------|
| 2      | 20      | $6.52 \times 10^{-3}$ | 5.01 $\times 10^{-3}$ | 0.02 | 0.01 | 9.95 $\times 10^{-3}$ |
| 4      | 40      | 0.01  | 8.89 $\times 10^{-3}$ | 0.09 | 0.04 | 0.02 |
| 6      | 60      | 0.08  | 1.25 $\times 10^{-3}$ | 0.31 | 0.13 | 0.14 |
| 8      | 80      | 0.31  | 1.72 $\times 10^{-3}$ | 0.55 | 0.24 | 0.30 |
| 10     | 100     | 0.68  | 2.15 $\times 10^{-3}$ | 1.11 | 0.39 | 0.77 |
| 12     | 120     | 1.66  | 3.29 $\times 10^{-3}$ | 2.01 | 0.61 | 1.86 |
| 14     | 140     | 3.18  | 3.52 $\times 10^{-3}$ | 3.55 | 0.84 | 3.83 |
| 16     | 160     | 6.08  | 3.78 $\times 10^{-3}$ | 6.36 | 1.16 | 7.63 |
| 18     | 180     | 9.83  | 5.12 $\times 10^{-4}$ | 13.08 | 1.67 | 18.02 |
| 20     | 200     | 15.51 | 5.36 $\times 10^{-4}$ | 24.88 | 2.20 | 45.56 |

then, the test matrix $A$ is calculated by applying a similarity transformation on $A'$, in the same manner as in Section 7.1. Clearly, $\chi_A(\lambda) = f(\lambda)g_1(\lambda)g_2(\lambda)g_3(\lambda)$. We focus on computing generalized eigenspace associated to the roots of $f(\lambda)$. Note that Procedure 4 is employed in this experiment.

**Remark 8.** When computing the Jordan chains associate to just one eigenvalue, the computation of a Krylov generating set in Algorithm 1 can be shortened by combining it with the computation of the minimal annihilating polynomials as follows. In the algorithm for computing the minimal annihilating polynomials, a pseudo minimal annihilating polynomial is computed as $\pi_{A,e}(\lambda) = f(\lambda)^r g_2'(\lambda)$. Then, after computing $g_2'(A)e$, it is tested whether $f(A)^r g_2'(A)e = 0$. If it is satisfied, $g_2'(A)e$ can be used as an element in the Krylov generating set, hence the computation in Algorithm 1 can be omitted (see Tajima et al. 2018a).

Tables 5 and 6 shows the results of computation. Note that, according to Remark 8, in the column “Time” in Table 5, the computing time of Algorithm 1 is excluded, and, in Table 6, computing time of Algorithm 1 is written in parenthesis. The rest of the contents of the tables are the same as Tables 1 to 4 respectively. Note that the computing time is only for the generalized eigenvector computations associated to the root of $f(\lambda)$.

The results show the following.
Table 5: Computing time and memory usage for the case in Section 7.2.

| $\deg(f)$ | size($A$) | Time (sec.) | Memory usage (MB) | Average $|a_{ij}|$ | max $|a_{ij}|$ |
|-----------|-----------|-------------|-------------------|----------------|-------------|
| 4         | 40        | 0.04        | 348.03            | 38.42          | 984         |
| 8         | 80        | 0.64        | $5.24 \times 10^3$ | 38.91          | 1942        |
| 12        | 120       | 3.34        | $2.72 \times 10^4$ | 63.28          | 4209        |
| 16        | 160       | 11.68       | $8.88 \times 10^4$ | 131.31         | 10578       |
| 20        | 200       | 31.22       | $2.32 \times 10^5$ | 108.55         | 6495        |

Table 6: Computing time of each step of the algorithm for the case in Section 7.2.

| $\deg(f)$ | size($A$) | $f(A)$ (Alg. 1) | $f(A)$ (Alg. 3) | Proc. 4 | Alg. 5 |
|-----------|-----------|----------------|----------------|---------|-------|
| 4         | 40        | (0.0433)       | $2.02 \times 10^{-5}$ | 0.01    | 0.02  |
| 8         | 80        | (0.6689)       | $3.24 \times 10^{-5}$ | 0.06    | 0.35  |
| 12        | 120       | (3.4451)       | $4.31 \times 10^{-5}$ | 0.05    | 1.85  |
| 16        | 160       | (11.6171)      | $5.41 \times 10^{-5}$ | 0.10    | 6.29  |
| 20        | 200       | (31.6356)      | $6.81 \times 10^{-5}$ | 0.18    | 16.71 |

1. Algorithm 1 in Section 7.1 since the minimal annihilating polynomial has only $f(A)$ as a factor, the computing time is relatively short. On the other hand, in Section 7.2, the minimal annihilating polynomial has more number of factors; thus, the computing time is relatively long.

2. Algorithm 3 in Section 7.1 since the numbers of vectors in the Jordan-Krylov basis is larger, the computing time is relatively long. On the other hand, in Section 7.2, the structure of $A'$ tells us that there is just one element in the Jordan-Krylov basis, and the computing time is very short.

3. Procedure 4 in Section 7.1. Table 3 shows that calling Procedure 4 from Algorithm 3 reduces the computing time of the algorithms. Tables 2 and 4 show that the computing time of Algorithms 3 and 5 is reduced. While Procedure 4 may not be effective in all cases, it is expected to be effective in many instances, as demonstrated by this example.

7.3. Comparison of performance with Maple

In this section, we compare the performance of the proposed algorithm with that of the computer algebra system Maple. We conducted the experiments in Sections 7.1 and 7.2 using Maple 2021 (Maplesoft, a division of Waterloo Maple Inc.) on the same computing environment as the above. Note that the results in Section 7.1 are based on the results for “With Proc. 4” in Table 3.

7.3.1. Comparison with the Frobenius normal form

There is an algorithm for computing generalized eigenvectors via the Frobenius normal form proposed by Takeshima and Yokoyama (1990) and by Moritsugu and Kuriyama (2001). Here, we compare the performance of our algorithm with only the computation of the Frobenius normal form. In this experiment, we use the function “FrobeniusForm” in the “LinearAlgebra” package of Maple. Table 7 shows the results of computation for the examples in Section 7.2 in the case
Table 7: Computing time and memory usage of the Frobenius normal form of the matrices used in Section 7.2 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3.1 for details.

| deg(f) | size(A) | Maple (FrobeniusForm) | Proposed method |
|--------|---------|-----------------------|-----------------|
|        |         | Time (sec.) | Memory usage | Time (sec.) | Memory Usage |
| 4      | 40      | 0.116       | 20.33        | 0.04        | 348.03       |
| 8      | 80      | 1.45        | 0.23 × 10³  | 0.64        | 5.24 × 10³   |
| 12     | 120     | 8.06        | 1.17 × 10³  | 3.34        | 27.2 × 10³   |
| 16     | 160     | 24.26       | 3.42 × 10³  | 11.68       | 88.8 × 10³   |
| 20     | 200     | 66.14       | 8.76 × 10³  | 31.22       | 232 × 10³    |

The experimental results show that, for matrices with multiple irreducible factors in the characteristic polynomial, the computation of the Frobenius normal form is time-consuming, although the memory utilization is minimal.

7.3.2. Computing generalized eigenvectors

In this experiment, we compare the performance of the proposed algorithm with that of Maple by executing the function “Eigenvectors” in the “LinearAlgebra” package, which computes (generalized) eigenvectors. According to the disclosed source code, the function calculates the characteristic polynomial when the components of the matrix are integers or rational numbers. Then, by sequentially solving the system of linear equations, it calculates the general eigenvectors in the algebraic extension field.

Tables 8 and 9 show the results of computation. Computing time includes the time for calculating the characteristic polynomial. The results of “Proposed method” are those shown in Tables 1 and 3 (With Proc. 4), respectively. We see that while computing time in Maple is long, the proposed method efficiently computes general eigenspace.

For reference, Tables 10 and 11 show the computing time of the characteristic polynomial of A used in Sections 7.1 and 7.2, respectively, by using Maple. Note that the computing time is denoted in milli-seconds (ms). Although the degree of \( f(\lambda) \) appearing in the experiment in Tables 8 and 9 are up to 12 and 8, respectively, we have measured the computing time of all the characteristic polynomials used in Sections 7.1 and 7.2.

7.3.3. Computing Jordan canonical form and Jordan chains

In this experiment, we compare the performance of the proposed algorithm with that of Maple by executing the function “JordanForm” in the “LinearAlgebra” package, which computes the Jordan canonical form with the Jordan chains. In our experiment, we set the option “output = ‘Q’” to obtain the Jordan chains.

Tables 12 and 13 shows the results of computation. The results of “Proposed method” are those shown in Tables 1 and 3 (With Proc. 4), respectively. We see that, in Maple, computing time is significantly lengthy even for small-seized inputs, the proposed method efficiently computes...
Table 8: Computing time and memory usage of Generalized eigenvectors for the case in Section 7.1 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3.2 for details.

| deg(f) | size(A) | Maple (Eigenvectors) | Proposed method |
|--------|---------|----------------------|-----------------|
|        |         | Time (sec.) | Memory usage | Time (sec.) | Memory Usage |
| 2      | 20      | 0.72        | 86.95        | 0.03        | 40.28        |
| 4      | 40      | 13.54       | 1.84 × 10³   | 0.17        | 0.27 × 10³   |
| 6      | 60      | 62.80       | 9.31 × 10³   | 0.65        | 0.93 × 10³   |
| 8      | 80      | 396.0       | 6.09 × 10⁴   | 1.40        | 0.25 × 10⁴   |
| 10     | 100     | 2170.8      | 21.7 × 10⁴   | 2.96        | 0.58 × 10⁴   |
| 12     | 120     | 12096.0     | 69.6 × 10⁴   | 6.14        | 1.27 × 10⁴   |

Table 9: Computing time and memory usage of Generalized eigenvectors for the case in Section 7.2 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3.3 for details.

| deg(f) | size(A) | Maple (Eigenvectors) | Proposed method |
|--------|---------|----------------------|-----------------|
|        |         | Time (sec.) | Memory usage | Time (sec.) | Memory Usage |
| 4      | 40      | 73.72       | 11.00 × 10⁴   | 0.04        | 0.35 × 10⁴   |
| 8      | 80      | 3454.80     | 389.26 × 10³  | 0.64        | 5.24 × 10³   |
| 12     | 120     | >8h         | N/A           | 3.34        | 2.72 × 10⁴   |

the Jordan chains. Especially for the example in Section 7.2, computational efficiency is achieved by restricting the eigenvalue of interest to the root of \( f(A) \).

8. Concluding remarks

In this paper, we have proposed an exact algorithm for computing generalized eigenspaces of matrices of integers or rational numbers by exact computation. The resulting algorithm computes generalized eigenspaces and the structure of Jordan chains by computing a Jordan-Krylov basis of \( \ker f(A) \) using Jordan-Krylov elimination. In Jordan-Krylov elimination, minimal annihilating polynomials are effectively used for computing a Krylov generating set.

A feature of the present method is that it computes generalized eigenvectors in decreasing order of their ranks, which is the opposite of how they are computed in a conventional manner. In the conventional method, such as the one shown in Maple’s built-in function (see Section 7.3.2), generalized eigenvectors are computed in increasing order, then it is hard to construct Jordan chains in general. In contrast, in the proposed method, Jordan chains of eigenvectors are directly computed from Jordan-Krylov basis without solving generalized eigenequations.

Another feature of the present method is that the cost of computation can be reduced by computing just a part of generalized eigenspace associated to a specific eigenvalue. In conventional methods, even if the generalized eigenspace of our interest is relatively small compared to the whole generalized eigenspace, a whole matrix gets transformed for solving the system of linear equations or computing canonical forms of the matrix, which may make the computation inefficient. On the other hand, the present method concentrates the computation on \( \ker f(A) \) using Jordan-Krylov elimination, which means that, a smaller part of generalized eigenspace can be computed with a smaller amount of computation in an effective way.

As a further extension of the proposed method, it would be beneficial to apply the proposed method to matrices with polynomial components, for efficiently computing the generalized
Table 10: Computing time and memory usage of the characteristic polynomials of the matrices used in Section 7.1 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3 for details.

| $\deg(f)$ | $\text{size}(A)$ | Time (ms) | Memory usage |
|----------|-----------------|-----------|--------------|
| 2        | 20              | 1.2       | 0.10         |
| 4        | 40              | 3.8       | 0.21         |
| 6        | 60              | 9.4       | 0.36         |
| 8        | 80              | 18.6      | 0.58         |
| 10       | 100             | 35.0      | 0.84         |
| 12       | 120             | 53.5      | 1.16         |
| 14       | 140             | 82.1      | 1.50         |
| 16       | 160             | 121.6     | 1.96         |
| 18       | 180             | 166.7     | 2.43         |
| 20       | 200             | 261.2     | 2.93         |

Table 11: Computing time and memory usage of the characteristic polynomials of the matrices used in Section 7.2 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3 for details.

| $\deg(f)$ | $\text{size}(A)$ | Time (ms) | Memory usage |
|----------|-----------------|-----------|--------------|
| 4        | 40              | 6.5       | 0.21         |
| 8        | 80              | 40.1      | 0.57         |
| 12       | 120             | 121.9     | 1.14         |
| 16       | 160             | 282.8     | 1.95         |
| 20       | 200             | 532.7     | 2.90         |

eigenspace in the algebraic extension field of the rational function field.

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Table 12: Computing time and memory usage of Jordan chains for the case in Section 7.1 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3 for details.

| $\deg(f)$ | $\size(A)$ | Maple (JordanForm) | Proposed method |
|-----------|-----------|--------------------|-----------------|
|           |           | Time (sec.) | Memory usage | Time (sec.) | Memory Usage |
| 2         | 20        | 0.17        | 22.65        | 0.03        | 40.28        |
| 4         | 40        | 420.6       | $1.54 \times 10^3$ | 0.17        | $0.28 \times 10^3$ |
| 6         | 60        | $>8h$       | N/A          | 0.65        | $0.93 \times 10^3$ |

Table 13: Computing time and memory usage of Jordan chains for the case in Section 7.2 using Maple. The memory utilization is measured in megabytes (MB). See Section 7.3 for details.

| $\deg(f)$ | $\size(A)$ | Maple (JordanForm) | Proposed method |
|-----------|-----------|--------------------|-----------------|
|           |           | Time (sec.) | Memory usage | Time (sec.) | Memory Usage |
| 4         | 40        | 420.6       | $25.8 \times 10^3$ | 0.04        | $0.35 \times 10^3$ |
| 8         | 80        | $>8h$       | N/A          | 0.64        | $5.24 \times 10^3$ |

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