Lemaître-Tolman-Bondi cosmological models, smoothness, and positivity of the central deceleration parameter

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(Dated: May 7, 2009)

We argued in a previous paper [R. A. Vandersveld et al. 2006] that negative deceleration parameters at the center of symmetry in Lemaître-Tolman-Bondi cosmological models can only occur if the model is not smooth at the origin. Here we demonstrate explicitly the connection between non-smoothness and the failure of positivity theorems for deceleration. We also address some confusion that has arisen in the literature and respond to some recent criticisms of our arguments.

I. INTRODUCTION

In a previous paper [1] (henceforth VFW) we studied spherically symmetric, dust cosmological models, which are described by the Lemaître-Tolman-Bondi (LTB) metric. In particular we discussed the potential for such models to reproduce the apparent acceleration of the Universe, and how some models evade positiveness theorems for the central deceleration parameter. Our results were criticized in the recent paper [2] by Krasinski, Hellaby, Célérier, and Bolejko (henceforth KHCB). Most of these criticisms are due to misinterpretations and are invalid. In this short note we clarify some of the issues involved and respond to the criticisms.

II. TWO KEY MISCONCEPTIONS IN KHCB

A. Misidentification of deceleration parameter

A key issue that arises is the definition of the deceleration parameter. Throughout VFW we defined the deceleration parameter $q(z)$ as a function of redshift $z$ to be what would be computed by observers from observed luminosity distances $D_L(z)$ by assuming a spatially flat Friedmann-Robertson-Walker (“FRW” in VFW and “FLRW” in KHCB) cosmological model. This is discussed explicitly around our Eq. (2.10). Specializing to $z = 0$ gives the observed deceleration parameter at the location of the observer, $q_0 = q(0)$.

This discussion applies to spherically symmetric spacetimes, with the observer at the symmetry center. For more general inhomogeneous spacetimes, some prescription for angle averaging is required in order to define a deceleration parameter. One such definition at $z = 0$ is given in Ref. [2], which in spherical symmetry coincides with the definition of $q_0$ given above. Slightly different definitions, specialized to $z = 0$, are given in Hirata and Seljak [4] (henceforth HS), namely the quantities which they denote by $q_3$ and $q_4$. Those definitions also coincide with the above $q_0$ in spherical symmetry.

The starting point of our paper VFW was the following contradiction that existed then in the literature. Namely, the result of Ref. [6] specialized to spherical symmetry showed that $q_0 \geq 0$ always for any smooth LTB model. Similar results in HS related to their quantities $q_3$ and $q_4$, specifically the argument from their Eqs. (18) – (41), give the same result when specialized to spherical symmetry. On the other hand, the explicit LTB solutions constructed in Ref. [2] and elsewhere (see VFW for references) have $q_0 < 0$. We pointed out in VFW the resolution of this apparent contradiction: the solutions of Ref. [2] are not smooth at their center, and therefore violate the smoothness assumptions used in deriving the positivity theorems of Refs. [3, 4].

The first key error made in the comments of KHCB is a misidentification of which definition of deceleration parameter is involved, wherein they focus on another definition given in HS – related to the derivative with respect to a fluid element’s proper time of the expansion of the fluid – which HS denote by $q_1$ and KHCB denote by $q_{HS}$. One of the main criticisms in KHCB is that we were confusing different definitions of $q$ in VFW, using both $q_0$ defined above and $q_1$, but this is not the case as $q_1$ was never referred to in VFW. In addition, when we cited HS we explicitly stated which definition of acceleration parameter was involved, saying “the local expansions of . . . Hirata and Seljak show that $q_0$ is constrained to be positive.” We then proceeded to specify the definition of deceleration parameter $q_0$ [our Eq. (2.16)] and reviewed the corresponding positivity theorem.

Having focused on the incorrect deceleration parameter $q_1$, KHCB then argued, correctly, that there is no contradiction between the positivity theorem of HS for $q_1$ and the explicit models of Ref. [2]. This is because, although the definitions of $q_0$ and $q_1$ coincide for smooth, spherically symmetric models at $z = 0$, they do not coincide for models which are not smooth, and in particular the models of Ref. [2] have $q_1 > 0$ even though $q_0 < 0$. While KHCB are correct about this point, it is not rele-

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1 If one instead allows for a spatially curved geometry, the deceleration $q(z)$ at finite $z$ can be altered, but the central deceleration $q(0)$ at $z = 0$ is unchanged, as we discussed in VFW.
vant to a discussion of VFW.

We emphasize that we used only a single definition of deceleration parameter throughout our paper, and KHCB’s claim that we “intermittently used the same name...for two different quantities” is incorrect. Although KHCB give two different formulae that they claim we use, their Eqs. (2.12) and (2.13), in fact the first formula is just the second formula evaluated at \( z = 0 \).

There are further criticisms in Sec. VI of KHCB related to our computation of the deceleration parameter. After their Eq. (6.1) KHCB say that “this makes it clear that the shear and vorticity terms...are also evaluated at \( z = 0 \), though this is not clear in either paper.” In fact this is explicitly stated in our Ref. 3, where the first sentence in the paragraph containing Eq. (3) reads “...in terms of the density, 4-velocity gradients of the cosmological fluid evaluated at the observer’s location \( P \).” It is also stated explicitly in VFW, where \( q_0 \) is defined to be the “central deceleration parameter” before Eq. (2.16), so it is clear that everything is evaluated where the observer lives, at \( z = 0 \). The fact that the gradients of the velocity are evaluated at \( z = 0 \) also invalidates KHCB’s criticism in the first paragraph of their Sec. VID.

### B. Skepticism about relevance of differentiability of solution

Because there is no contradiction involving the deceleration parameter \( q_1 \), KHCB believed that our explanation in terms of non-smoothness was invalid. They did appreciate that a contradiction still existed for the deceleration parameter \( q_0 \), which they summarized in their Sec. VIA. However, they discounted our resolution of the contradiction, and instead argued that the formula for the deceleration parameter derived in the earlier paper 3 was “erroneous or erroneously interpreted in the VFW paper.”

In order to address this skepticism, we give here a detailed and explicit demonstration of the connection between smoothness of the geometry and the positivity theorems of Refs. 3, 4. Although such a demonstration is not strictly necessary since the argument of VFW is complete and self contained, it may nonetheless help to dispel confusion about this issue.

A basic ingredient of those positivity theorems is a local covariant Taylor expansion of the fluid’s 4-velocity about a given point \( x^α \), of the form

\[
 u_α(x') = g_αβ(x, x') [u_α(x) + u_αβ(x)σ_α(x, x')σ_β(x, x') + \frac{1}{2} u_αβγ(x)σ_α(x, x')σ_β(x, x')σ_γ(x, x') + \ldots], \tag{2.1}
\]

where \( u_αβ(x) = -\nabla(α)u_β(x) \) and \( u_αβγ(x) = \nabla(α)\nabla(β)u_γ(x) \). Here \( g_αβ(x, x') \) is the parallel transport bivector and \( σ(x, x') \) is Synge’s worldfunction; see Ref. 4 for details. Clearly, in order for this expansion to be valid, the coefficients \( u_α, u_αβ, \) and \( u_αβγ \) in the expansion must exist, and in particular the symmetrized second covariant derivative of the 4-velocity

\[
 ∇_α(∇_βu_γ) \tag{2.2}
\]

evaluated at \( x_0 \) must exist. We now show that this quantity fails to exist at the center of symmetry for the LTB models with \( q_0 < 0 \), which explains why the positivity theorems are inapplicable to such models.

The metric of LTB models can be written in the general form

\[
 ds^2 = -dt^2 + e^{2α(t, r)}dr^2 + e^{2β(r)}r^2(dθ^2 + \sin^2 θ dφ^2), \tag{2.3}
\]

for some functions \( α(t, r) \) and \( β(r, t) \). The fluid 4-velocity is \( u_α = -(dt)_α \). We assume the differential structure on the manifold associated with the coordinates \( (t, x, y, z) \), where \( (x, y, z) \) are given in terms of \( (r, θ, ϕ) \) by the usual formulæ for spherical polar coordinates 2. Then, it can be seen that both \( u_α \) and \( u_αβ \) are smooth tensor fields. However the metric \( g_αβ \) and connection \( ∇_α \) need not be smooth, so covariant derivatives of \( u_α \) need not exist.

A convenient piece of the second derivative to look at is \( h^α = (g^{αβ} + u_αu_β)g^{γδ}∇_β∇_γu_δ \). From the metric 4, we then obtain

\[
 h_αdx^α = \left[ \frac{4}{3r}(\dot{α} - \ddot{β})(1 + rβ') + α' + \frac{2}{3}β' \right] dr, \tag{2.4}
\]

where dots denote derivatives with respect to \( t \) and primes denote derivatives with respect to \( r \). We now expand the functions \( α \) and \( β \) as

\[
 α(t, r) = α_0(t) + α_1(t)r + α_2(t)r^2 + O(r^3), \tag{2.5}
\]

\[
 β(t, r) = β_0(t) + β_1(t)r + β_2(t)r^2 + O(r^3). \tag{2.6}
\]

For all LTB models we have \( α_0 = β_0 [\text{see Eq. (2.20) of VFW}] \), and so we get near \( r = 0 \) that

\[
 h_αdx^α = \frac{1}{3} \left[ 7α_1(t) - 2β_1(t) + O(r) \right] dr. \tag{2.7}
\]

If the quantity in square brackets is nonzero, then \( h_α \) is nonzero in the limit \( r → 0 \), and points in the radial direction. Therefore it has a direction dependent limit as \( r → 0 \), i.e., the limit does not exist. (Correspondingly, higher order derivatives of this quantity diverge.)

2 For this differential structure the four velocity is smooth \((C^∞)\) while the metric is \( C^1 \) but not \( C^2 \). It is possible to find an alternative differential structure [using a coordinate transformation of the form \( r = r + f(t)r^2 + \ldots \) for some function \( f(t) \)] for which the metric is \( C^2 \) but not \( C^3 \) and the four velocity is \( C^1 \) but not \( C^2 \). However there is no choice of differential structure for which the metric is smooth, as can be seen by computing coordinate invariants like \( ∇_α∇_αR \), where \( R \) is the Ricci scalar, which diverge as \( r → 0 \).[4]
The coefficient in Eq. (2.7) will generically be non-vanishing for the type of models in Ref. [1] with \( q_0 < 0 \). Such models were characterized by having non-vanishing linear terms in the expansions of the bang time function

\[
t_0(r) = t_{00} + t_{01} r + t_{02} r^2 + O(r^3)
\]

and curvature function

\[
k(r) = k_0 + k_1 r + k_2 r^2 + O(r^3).
\]

Smoothness requires that \( k_1 = t_{01} = 0 \). For example, for the case \( k(r) = 0 \) we have

\[
h_{\alpha\beta} dx^\alpha = \left[ \frac{8 t_{01}}{3 r^2} + O(r) \right] dr.
\]

Therefore we see explicitly that for the models with \( q_0 < 0 \) the second covariant derivative of the 4-velocity does not exist, explaining why the positivity theorem does not apply.

III. CRITICISMS RELATED TO CHOICES OF TERMINOLOGY

Many of the criticisms in KHCB are not directed at our mathematical results, but instead are related to the choice of terminology we employ to describe our results. In most cases the criticisms are based on misinterpretations and are unfounded, but some of their criticism has validity, since some of the terminology we used was poorly chosen and apt to lead to confusion. We now discuss and clarify the relevant choices of terminology:

1. “Singularity?” In VFW we discussed in detail the nature and implications of the lack of smoothness at the origin of the LTB models with \( q_0 < 0 \). We called these locations singularities, in the loose sense that there were some coordinate-invariant quantities which become singular there. This was a poor choice of terminology, since “singularity” is usually used in the general relativity literature to mean geodesic incompleteness, which does not apply here, and/or a divergence of the Riemann tensor, which also does not apply here. Although we did not claim that the singularity was a curvature singularity, our terminology was confusing, as correctly pointed out by KHCB.

KHCB also object to our describing the singularity as “weak.” While it is true that there are variety of different terminologies for classifying singularities in use in general relativity, “weak” and “strong” are now in common use as referring to whether parallel propagated components of the tidal distortion tensor diverge or not. The precise definitions of weak and strong are given, for example, in Ref. [6]. The spatial origin in the non-smooth LTB models is weak in this sense (rather trivially, since the components of the Riemann tensor are finite). Therefore our description is appropriate and the criticisms of KHCB on this point are unfounded.

2. “Unphysical?” In VFW we asserted that the non-smooth LTB models are unphysical. KHCB disagreed by arguing that one can smooth out the central singularity easily at \( z < z_s \) for some small redshift \( z_s \) without changing the predictions at larger \( z \). We agree with this point, and in fact we mentioned it in the concluding section of VFW. The possibility of performing such smoothing is well-known, see, for example, the numerical studies in Ref. [7]. However, the non-smooth models are still unphysical when the observer is placed at the center, in the following sense: Such models have considerably more fine tuning than smooth models, since it is unlikely for an observer to live exactly on such a point in the density distribution, and especially since to smooth out the singularity in a manner compatible with luminosity distance observations requires the introduction of a new, artificial, very small lengthscale.

3. “Inverse Problem?” In Sec. II of VFW we discussed the straightforward computational procedure for obtaining the luminosity distance \( D_L(z) \) from the bang time function \( t_0(r) \) of a zero-energy, LTB model. In Sec. III we discussed the what we called the “inverse problem,” by which we meant simply the inverse process of attempting to find \( t_0(r) \) from a specified \( D_L(z) \). KHCB appear to interpret the phrase “inverse problem” in a different and much more general sense, and therefore their criticism on this point is unfounded.

4. “Effective Equation of State?” In VFW we defined the effective equation of state parameter \( w_{\text{eff}}(z) \) to be the equation of state that is obtained from the data when assuming the usual framework of a flat FRW cosmology. We by no means said that it is the equation of state underlying the LTB model itself (which is of course that of pressureless dust). KHCB seem to be confused about this point in their Sec. VIB, and appear to believe that we intended a literal interpretation of \( w_{\text{eff}}(z) \).

IV. FINDING LTB MODELS THAT YIELD A SPECIFIED LUMINOSITY DISTANCE AS A FUNCTION OF REDSHIFT

In Sec. III of VFW, we explored the problem of trying to find an LTB model that would have the same luminosity distance \( D_L(z) \) as a given FRW model with or without dark energy. We called this the “inverse problem.” We found this problem to be complicated by a generic critical point in the differential equations to be solved. However we were still able to show that “transcritical” LTB models could be constructed. In Sec. V of their paper, KHCB criticize our discussion on a number of issues. Their criticism is largely unfounded.

First, KHCB appear to have misread our paper on one important point. In Sec. V, they say “VFW . . . argue that only [dust] FRW models have . . . [a] critical point,”
and “[VFW] say that [dust] FRW models provide examples of transcritical solutions... but they fail to find other viable [transcritical] solutions.” These two assertions are incorrect, since in our Sec. IIIB we explicitly constructed a variety of transcritical solutions that correspond to choices of \(D_L(z)\) other than dust FRW. The origin of KHCB’s misinterpretation might be the fact that we gave the explicit formula angular diameter distance for dust FRW models after Eq. (3.9). However, this was given only as an illustrative example.

Second, KHCB appear to misinterpret our statements about the generality of critical points. They argue that all light rays in physically reasonable cosmological models do encounter critical points (local maxima of angular diameter distance as a function of redshift). We agree with this assertion, and we never claimed otherwise in VFW. After our Eq. (3.13) we say “only the special class with this assertion, and we never claimed otherwise in favor of this assertion they presented is in the context of fully general LTB models, in which one varies both the energy function and the bang time function, a context different from that of VFW. For the context studied in VFW, where one varies only the bang time function, it is straightforward to explicitly confirm that not every \(D_A(z)\) can be realized by studying LTB models that are linear perturbations of dust FRW and using the test outlined after Eq. (3.25) of VFW. In the notation of VFW we choose the angular diameter distance to be

\[
r_{\text{FRW}}(z) = 3 \left[ 1 - \frac{1}{\sqrt{1+z}} \right] \left[ 1 + \varepsilon \delta(z) \right],
\]

where \(\varepsilon \ll 1\) is a small parameter and \(\delta(z)\) is an arbitrary function. We choose \(\delta(z)\) to vanish in a neighborhood of \(z = 0\) and in a neighborhood of the critical point at \(z = 5/4\). Then the linearized version of the differential equation (3.15) is

\[
V(z) = \gamma(z) \delta'(z),
\]

where we have parameterized the function \(V(z)\) as

\[
V(z) = 3 - 2\sqrt{1+z} + \varepsilon(1+z)^{2/3} \left[ 3 - 2\sqrt{1+z} \right]^{-1/3} V_1(z)
\]

and the function \(\gamma(z)\) is

\[
\gamma(z) = -\frac{2\sqrt{3} - 2\sqrt{z} + \gamma_1(z)}{(z+1)^{2/3} \left( 4z \sqrt{z+1} - 4 + 25 \left( \sqrt{z+1} - 1 \right) \right)}
\]

with \(\gamma_1(z) = -2z^2 + (9\sqrt{z+1} - 19) z + 20 \left( \sqrt{z+1} - 1 \right)\). The boundary conditions for the differential equation for \(V_1(z)\) are \(V_1(0) = V_1(5/4) = 0\), and it follows that a transcritical solution will be possible for a given choice of \(\delta(z)\) only if \(\int_0^{5/4} \gamma(z) \delta'(z) = 0\). Therefore there are many choices of angular diameter distance that are not realizable.

V. CONCLUSIONS

Because of recent criticism \[2\], and to prevent future misunderstandings, we have clarified some of the points made in VFW \[1\]. In doing so we have shown that many of the criticisms are unfounded.

Acknowledgments

The work of RAV was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA. EF was supported in part by NSF grant PHY-0757735 and by NASA grant NNX08AH27G. EF and IW are also supported in part by NSF contract with NASA. EF was supported in part by NSF grant PHY-0555216. Copyright 2009. All rights reserved.

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