Normal approximation for sums of discrete $U$-statistics - application to Kolmogorov bounds in random subgraph counting

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June 15, 2018

Abstract

We derive normal approximation bounds in the Kolmogorov distance for sums of discrete multiple integrals and $U$-statistics made of independent Bernoulli random variables. Such bounds are applied to normal approximation for the renormalized subgraphs counts in the Erdős-Rényi random graph. This approach completely solves a long-standing conjecture in the general setting of arbitrary graph counting, while recovering and improving recent results derived for triangles as well as results using the Wasserstein distance.

Keywords: Normal approximation; central limit theorem; Stein-Chen method; Malliavin-Stein method; Berry-Esseen bound; random graph; subgraph count; Kolmogorov distance.

Mathematics Subject Classification: 60F05, 60H07, 60G50, 05C80.

1 Introduction

The Mallavin approach to the Stein method for discrete Bernoulli sequences has recently been developed in [9], [4], [3], [12], [5], as an extension of the Malliavin approach to the Stein method introduced in [8] for Gaussian fields.

In this paper we develop the use of multiple stochastic integral expansions for the derivation of bounds on the distances between probability laws by the Malliavin approach to the Stein and Stein-Chen methods. Using results of [5] for general functionals of discrete i.i.d.

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renormalized Bernoulli sequences \((Y_n)_{n \in \mathbb{N}}\), we derive a Kolmogorov distance bound to the normal distribution for sums of \(U\)-statistics (or multiple stochastic integrals) of the form
\[
\sum_{k=1}^{n} \sum_{i_1, \ldots, i_k \in \mathbb{N}, 1 \leq r \neq s \leq k} f_k(i_1, \ldots, i_k) Y_{i_1} \cdots Y_{i_k},
\]
where \((Y_k)_{k \in \mathbb{N}}\) is a normalized sequence of Bernoulli random variables, see Theorem 3.1. We note that on the Erdős-Rényi random graph \(G_n(p_n)\) constructed by independently retaining any edge in the complete graph \(K_n\) on \(n\) vertices with probability \(p_n \in (0, 1)\), various random functionals admit such representations as sums of multiple integrals. This includes the number of vertices of a given degree, and the count of subgraphs that are isomorphic to an arbitrary graph.

Our second goal is to apply such results to the normal approximation of the renormalized count of the subgraphs in \(G_n(p_n)\) which are isomorphic to an arbitrary graph. Necessary and sufficient conditions for the asymptotic normality of the renormalization

\[
\tilde{N}_n^G := \frac{N_n^G - \mathbb{E}[N_n^G]}{\sqrt{\text{Var}[N_n^G]}},
\]
where \(N_n^G\) is the number of graphs in \(G_n(p_n)\) that are isomorphic to a fixed graph \(G\), have been obtained in [14] where it is shown that

\[
\tilde{N}_n^G \xrightarrow{D} \mathcal{N} \text{ iff } np_n^\beta \to \infty \text{ and } n^2(1 - p_n) \to \infty,
\]
as \(n\) tends to infinity, where \(\mathcal{N}\) denotes the standard normal distribution,

\[
\beta := \max\{e_H/v_H : H \subset G\},
\]
and \(e_H, v_H\) respectively denote the numbers of edges and vertices in the graph \(H\).

Those results have been made more precise in [1] by the derivation of explicit convergence rates in the Wasserstein distance

\[
d_W(F, G) := \sup_{h \in \text{Lip}(1)} |\mathbb{E}[h(F)] - \mathbb{E}[h(G)]|,
\]
between the laws of random variables \(F, G\), where \(\text{Lip}(1)\) denotes the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1. In the particular case
where the graph $G$ is a triangle, such bounds have been recently strengthened in [13] using the Kolmogorov distance

$$d_K(F, G) := \sup_{x \in \mathbb{R}} |P(F \leq x) - P(G \leq x)|,$$

which satisfies the bound $d_K(F, N) \leq \sqrt{d_W(F, N)}$. Still in the case of triangles, Kolmogorov distance bounds had also been obtained by the Malliavin approach to the Stein method for discrete Bernoulli sequences in [5] when $p_n$ takes the form $p_n = n^{-\alpha}, \alpha \in [0, 1)$.

In this paper we refine the results of [1] by using the Kolmogorov distance instead of the Wasserstein distance. As in [1] we are able to consider any graph $G$, and therefore our results extend those of both [5] and [13] which only cover the case where $G$ is a triangle. Instead of using second order Poincaré inequalities [6], our method relies on an application of Proposition 4.1 in [4] to derive Stein approximation bounds for sums of multiple stochastic integrals.

Our second main result Theorem 4.2 is a bound for the Kolmogorov distance between the normal distribution and the renormalized graph count $\tilde{N}_G$. Namely, we show that when $G$ is a graph without isolated vertices it holds that

$$d_K(\tilde{N}_G, N) \leq C_G \left( (1 - p_n) \min_{H \subseteq G} \left\{ n^v_H p_n^e_H \right\} \right)^{-1/2},$$

see Theorem 4.2, where $C_G > 0$ is a constant depending only on $e_G$, which improves on the Wasserstein estimates of [1], see Theorem 2 therein. This result relies on the representation of combined subgraph counts as finite sums of multiple stochastic integrals, see Lemma 4.1, together with the application of Theorem 3.1 on Kolmogorov distance bounds. In the sequel, given two positive sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ we write $x_n \approx y_n$ whenever $c_1 < x_n/y_n < c_2$ for some $c_1, c_2 > 0$ and all $n \in \mathbb{N}$, and for $f$ and $g$ two positive functions we also write $f \lesssim g$ whenever $f \leq C_G g$ for some constant $C_G > 0$ depending only on $G$.

Using the equivalence

$$\text{Var} [N_G^n] \approx (1 - p_n) \max_{H \subseteq G} \left\{ n^{2v_G - v_H} p_n^{2e_G - e_H} \right\}$$

as $n$ tends to infinity, see Lemma 3.5 in [2], the bound (1.2) can be rewritten in terms of the
variance \( \text{Var}[N^G_n] \) as

\[
d_K(\tilde{N}^G_n, \mathcal{N}) \lesssim \frac{\sqrt{\text{Var}[N^G_N]}}{(1 - p_n)n^{1/2}p_n^G},
\]

(1.4)

Note that when \( p_n \) is bounded away from 0, the bound (1.2) takes the simpler form

\[
d_K(\tilde{N}^G_n, \mathcal{N}) \lesssim \frac{1}{n\sqrt{1 - p_n}}.
\]

(1.5)

In Corollaries 4.4, 4.5 and 4.6 we deal with examples of subgraphs such as cycle graphs and complete graphs, which include triangles as particular cases, and trees.

In the particular case where the graph \( G \) is a triangle, the next consequence of (1.2) and (1.5) recovers the main result of [13], see Theorem 1.1 therein.

**Corollary 1.1** For any \( c \in (0, 1) \), the normalized number \( \tilde{N}^G_n \) of the subgraphs in \( G_n(p_n) \) that are isomorphic to a triangle satisfies

\[
d_K(\tilde{N}^G_n, \mathcal{N}) \lesssim \begin{cases} 
\frac{1}{n\sqrt{1 - p_n}} & \text{if } c < p_n < 1, \\
\frac{1}{n\sqrt{p_n}} & \text{if } n^{-1/2} < p_n \leq c, \\
\frac{1}{(np_n)^{3/2}} & \text{if } 0 < p_n \leq n^{-1/2}.
\end{cases}
\]

When \( p_n \) takes the form \( p_n = n^{-\alpha}, \alpha \in [0, 1), \) Corollary 1.1 similarly improves on the convergence rates obtained in Theorem 1.1 of [5].

This paper is organized as follows. In Section 2 we recall the construction of random functionals of Bernoulli variables, together with the construction of the associated finite difference operator and their application to Kolmogorov distance bounds obtained in [4]. In Section 3 we derive general Kolmogorov distance bounds for sums of multiple stochastic integrals. In Section 4 we show that graph counts can be represented as sums of multiple stochastic integrals, and we derive Kolmogorov distance bounds for the renormalized count of subgraphs in \( G_n(p_n) \) that are isomorphic to a fixed graph.

## 2 Notation and preliminaries

In this section we recall some background notation and results on the stochastic analysis of Bernoulli processes, see [10] for details. Consider a sequence \( (X_n)_{n \in \mathbb{N}} \) of independent
identically distributed Bernoulli random variables with $P(X_n = 1) = p$ and $P(X_n = -1) = q$, $n \in \mathbb{N}$, built as the sequence of canonical projections on $\Omega := \{-1, 1\}^\mathbb{N}$. For any $F : \Omega \to \mathbb{R}$ we consider the $L^2(\Omega \times \mathbb{N})$-valued finite difference operator $D$ defined for any $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$ by

$$D_k F(\omega) = \sqrt{pq} (F(\omega^k_+) - F(\omega^k_-)), \quad k \in \mathbb{N},$$

(2.1)

where we let

$$\omega^k_+ := (\omega_0, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots) \quad \text{and} \quad \omega^k_- := (\omega_0, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots), \quad k \in \mathbb{N},$$

and $DF := (D_k F)_{k \in \mathbb{N}}$. The $L^2$ domain of $D$ is given by

$$\text{Dom}(D) = \{ F \in L^2(\Omega) : E[\|DF\|_{L^2(\mathbb{N})}^2] < \infty \}.$$  

We let $(Y_n)_{n \geq 0}$ denote the sequence of centered and normalized random variables defined by

$$Y_n := \frac{q - p + X_n}{2\sqrt{pq}}, \quad n \in \mathbb{N}. $$

Given $n \geq 1$, we denote by $\ell^2(\mathbb{N})^\otimes n = \ell^2(\mathbb{N}^n)$ the class of square-summable functions on $\mathbb{N}^n$, we denote by $\ell^2(\mathbb{N})^\circ n$ the subspace of $\ell^2(\mathbb{N})^\otimes n$ formed by functions that are symmetric in $n$ variables. We let

$$I_n(f_n) = \sum_{(i_1, \ldots, i_n) \in \Delta_n} f_n(i_1, \ldots, i_n)Y_{i_1} \cdots Y_{i_n}$$

denote the discrete multiple stochastic integral of order $n$ of $f_n$ in the subspace $\ell^2_s(\Delta_n)$ of $\ell^2(\mathbb{N})^\circ n$ composed of symmetric kernels that vanish on diagonals, i.e. on the complement of

$$\Delta_n = \{ (k_1, \ldots, k_n) \in \mathbb{N}^n : k_i \neq k_j, \ 1 \leq i < j \leq n \}, \quad n \geq 1.$$  

The multiple stochastic integrals satisfy the isometry and orthogonality relation

$$E[I_n(f_n)I_m(g_m)] = 1_{(n=m)}n! \langle f_n, g_m \rangle_{\ell^2_s(\Delta_n)},$$

(2.2)

$f_n \in \ell^2_s(\Delta_n)$, $g_m \in \ell^2_s(\Delta_m)$, cf. e.g. Proposition 1.3.2 of [11]. The finite difference operator $D$ acts on multiple stochastic integrals as follows:

$$D_k I_n(f_n) = nI_{n-1}(f_n(*, k)1_{\Delta_n}(*, k)) = nI_{n-1}(f_n(*, k)),$$

$k \in \mathbb{N}$, $f_n \in \ell^2_s(\Delta_n)$, and it satisfies the finite difference product rule

$$D_k(FG) = FD_k G + GD_k F - \frac{X_k}{\sqrt{pq}} D_k FD_k G, \quad k \in \mathbb{N}. $$

(2.3)
for $F, G : \Omega \to \mathbb{R}$, see Propositions 7.3 and 7.8 of [10].

Due to the chaos representation property of Bernoulli random walks, any square integrable $F$ may be represented as $F = \sum_{n \geq 0} I_n(f_n)$, $f_n \in \ell_2^2(\Delta_n)$, and the $L^2$ domain of $D$ can be rewritten as

$$\text{Dom}(D) = \left\{ F = \sum_{n \geq 0} I_n(f_n) : \sum_{n \geq 1} n n! \|f_n\|_{\ell_2^2(\mathbb{N})}^2 < \infty \right\}.$$  

The Ornstein-Uhlenbeck operator $L$ is defined on the domain

$$\text{Dom}(L) := \left\{ F = \sum_{n \geq 0} I_n(f_n) : \sum_{n \geq 1} n^2 n! \|f_n\|_{\ell_2^2(\mathbb{N})}^2 < \infty \right\}$$

by

$$LF = -\sum_{n=1}^{\infty} n I_n(f_n).$$  

The inverse of $L$, denoted by $L^{-1}$, is defined on the subspace of $L^2(\Omega)$ composed of centered random variables by

$$L^{-1}F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n),$$

with the convention $L^{-1}F = L^{-1}(F - E[F])$ in case $F$ is not centered. Using this convention, the duality relation (2.5) shows that for any $F, G \in \text{Dom}(D)$ we have the covariance identity

$$\text{Cov}(F, G) = E[G(F - E[F])] = E\left[\langle DG, -DL^{-1}F \rangle_{\ell_2^2(\mathbb{N})}\right].$$  \hspace{1cm} (2.4)$$

The divergence operator $\delta$ is the linear mapping defined as

$$\delta(u) = \delta(I_n(f_{n+1}(\cdot, \cdot))) = I_{n+1}(\tilde{f}_{n+1}), \quad f_{n+1} \in \ell_2^2(\Delta_n) \otimes \ell^2(\mathbb{N}),$$

for $(u_k)_{k \in \mathbb{N}}$ of the form

$$u_k = I_n(f_{n+1}(\cdot, k)), \quad k \in \mathbb{N},$$

in the space

$$\mathcal{U} = \left\{ \sum_{k=0}^{n} I_k(f_{k+1}(\cdot, \cdot)), \quad f_{k+1} \in \ell_2^2(\Delta_k) \otimes \ell^2(\mathbb{N}), \quad k = n \in \mathbb{N} \right\} \subset L^2(\Omega \times \mathbb{N})$$

of finite sums of multiple integral processes, where $\tilde{f}_{n+1}$ denotes the symmetrization of $f_{n+1}$ in $n + 1$ variables, i.e.

$$\tilde{f}_{n+1}(k_1, \ldots, k_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} f_{n+1}(k_1, \ldots, k_{k-1}, k_{k+1}, \ldots, k_{n+1}, k_i).$$
The operators $D$ and $\delta$ are closable with respective domains $\text{Dom}(D)$ and $\text{Dom}(\delta)$, built as the completions of $\mathcal{S}$ and $\mathcal{U}$, and they satisfy the duality relation

$$
\mathbb{E}[\langle DF, u \rangle_{\ell^2(N)}] = \mathbb{E}[F \delta(u)], \quad F \in \text{Dom}(D), \ u \in \text{Dom}(\delta),
$$

(2.5)

see e.g. Proposition 9.2 in [10], and the isometry property

$$
\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[||u||^2_{\ell^2(N)}] + \mathbb{E} \left[ \sum_{k,l=0}^{\infty} D_k u_l D_l u_k - \sum_{k=0}^{\infty} (D_k u_k)^2 \right] \leq \mathbb{E}[||u||^2_{\ell^2(N)}] + \mathbb{E} \left[ \sum_{k,l=0}^{\infty} D_k u_l D_l u_k \right], \quad u \in \mathcal{U},
$$

(2.6)

cf. Proposition 9.3 of [10] and Satz 6.7 in [7]. Letting $(P^t)_{t \in \mathbb{R}^+} = (e^{tL})_{t \in \mathbb{R}^+}$ denote the Ornstein-Uhlenbeck semi-group defined as

$$
P^t F = \sum_{n=0}^{\infty} e^{-nt} I_n(f_n), \quad t \in \mathbb{R}^+,
$$
on random variables $F \in L^2(\Omega)$ of the form $F = \sum_{n=0}^{\infty} I_n(f_n)$, the Mehler formula states that

$$
P^t F = \mathbb{E}[F(X(t)) | X(0)], \quad t \in \mathbb{R}^+,
$$

(2.7)
where $(X(t))_{t \in \mathbb{R}^+}$ is the Ornstein-Uhlenbeck process associated to the semi-group $(P^t)_{t \in \mathbb{R}^+}$, cf. Proposition 10.8 of [10]. As a consequence of the representation (2.7) of $P^t$ we can deduce the bound

$$
\mathbb{E}[|D_k L^{-1} F|^\alpha] \leq \mathbb{E}[|D_k F|^\alpha],
$$

(2.8)
for every $F \in \text{Dom}(D)$ and $\alpha \geq 1$, see Proposition 3.3 of [5]. The following Proposition 2.1 is a consequence of Proposition 4.1 in [5], see also Theorem 3.1 in [4].

**Proposition 2.1** For $F \in \text{Dom}(D)$ with $\mathbb{E}[F] = 0$ we have

$$
d_K(F, N) \leq |1 - \mathbb{E}[F^2]| + \sqrt{\text{Var}[\langle DF, -DL^{-1} F \rangle_{\ell^2(N)}]}
$$

$$
+ \frac{1}{\sqrt{pq}} \left[ \sum_{k=0}^{\infty} \mathbb{E}[(D_k F)^4] \left( \sqrt{\mathbb{E}[F^2]} + \sqrt{\mathbb{E}[(FD_k L^{-1} F)^2]} \right) + \sup_{x \in \mathbb{R}} \mathbb{E}[\langle DF_{1_{\{F>x\}}}, DF|DL^{-1} F| \rangle_{\ell^2(N)}] \right],
$$
Proof. By Proposition 4.1 in [5] we have

\[ d_K(F, N) \leq \mathbb{E}[1 - \langle DF, -DL^{-1}F \rangle_{\ell^2(N)}] \]

\[ + \frac{\sqrt{2 \pi}}{8} (pq)^{-1/2} \mathbb{E}[\langle |DF|^2, |DL^{-1}F| \rangle_{\ell^2(N)}] \]

\[ + \frac{1}{2} (pq)^{-1/2} \mathbb{E}[\langle |DF|^2, |FDL^{-1}F| \rangle_{\ell^2(N)}] \]

\[ + (pq)^{-1/2} \sup_{x \in \mathbb{R}} \mathbb{E}[\langle D1_{\{F \geq x\}}, DF|DL^{-1}F| \rangle_{\ell^2(N)}]. \]

On the other hand, the covariance identity (2.4) shows that \( \mathbb{E}[\langle DF, -DL^{-1}F \rangle_{\ell^2(N)}] = \text{Var} F \), hence by the Cauchy-Schwarz and triangular inequalities we get

\[ \mathbb{E}
\left[ 1 - \langle DF, -DL^{-1}F \rangle_{\ell^2(N)} \right] \leq \left\| 1 - \langle DF, -DL^{-1}F \rangle_{\ell^2(N)} \right\|_{L^2(\Omega)} \]

\[ \leq |1 - \|F\|^2_{L^2(\Omega)}| + \left\| \langle DF, -DL^{-1}F \rangle_{\ell^2(N)} - \|F\|^2_{L^2(\Omega)} \right\|_{L^2(\Omega)} \]

\[ = |1 - \text{Var}[F]| + \sqrt{\text{Var}[\langle DF, -DL^{-1}F \rangle_{\ell^2(N)}]} \]

Next, we have

\[ \mathbb{E}[\|DL^{-1}I_n(f_n)\|^2_{\ell^2(N)}] = \sum_{k=0}^{\infty} \mathbb{E}[(I_{n-1}(f_n(k, \cdot)))^2] \]

\[ = (n - 1)! \sum_{k=0}^{\infty} \|f_n(k, \cdot)\|^2_{\ell^2(N) \otimes (n-1)} \]

\[ = (n - 1)! \|f_n\|^2_{\ell^2(N) \otimes n} \]

\[ \leq n! \|f_n\|^2_{\ell^2(N) \otimes n} \]

\[ = \mathbb{E} \left[ |I_n(f_n)|^2 \right], \]

and consequently, by the orthogonality relation (2.2) we have

\[ \mathbb{E}[\|DL^{-1}F\|^2_{\ell^2(N)}] \leq \mathbb{E}[F^2] \]

for every \( F \in L^2(\Omega) \), hence (2.9) is bounded by

\[ \mathbb{E}[\langle |DL^{-1}F|, |DF|^2 \rangle_{\ell^2(N)}] \leq \mathbb{E} \left[ \sum_{k=0}^{\infty} |D_kL^{-1}F|^2 \sum_{k=0}^{\infty} |D_kF|^4 \right] \]

\[ \leq \sqrt{\mathbb{E} \left[ \sum_{k=0}^{\infty} |D_kL^{-1}F|^2 \right]} \sqrt{\mathbb{E} \left[ \sum_{k=0}^{\infty} (D_kF)^4 \right]} \]

\[ = \sqrt{\mathbb{E}[\|DL^{-1}F\|^2_{\ell^2(N)}]} \sqrt{\mathbb{E} \left[ \sum_{k=0}^{\infty} (D_kF)^4 \right]} \]
Eventually, regarding the third term (2.10), by the Cauchy-Schwarz inequality we find
\[
\mathbb{E}[\langle (DF)^2, |FDL^{-1}F| \rangle_{\ell^2(\mathbb{N})}] \leq \sqrt{\mathbb{E}[(DF)^2]} \sqrt{\mathbb{E}[(L^{-1}F)^4]} \leq \sqrt{\mathbb{E}[(L^{-1}F)^4]} \frac{\mathbb{E}[(DF)^2]}{\sqrt{\mathbb{E}[(L^{-1}F)^4]}}.
\]

Finally, given \( f_n \in \ell^2_s(\Delta_n) \) and \( g_m \in \ell^2_s(\Delta_m) \) we have the multiplication formula
\[
I_n(f_n)I_m(g_m) = \sum_{s=0}^{2 \min(n,m)} I_{n+m-s}(h_{n,m,s}), \tag{2.11}
\]
see Proposition 5.1 of [12], provided that the functions
\[
h_{n,m,s} := \sum_{s \leq 2i \leq 2 \min(s,n,m)} i! \left( \begin{array}{c} n \\ i \end{array} \right) \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} i \\ s-i \end{array} \right) \left( \frac{q-p}{2\sqrt{pq}} \right)^{2i-s} f_n^{s-i} f_m^s g_m
\]
belong to \( \ell^2_s(\Delta_{n+m-s}) \), \( 0 \leq s \leq 2 \min(n,m) \), where \( f_n^{s-i} f_m^s g_m \) is defined as the symmetrization in \( n + m - k - l \) variables of the contraction \( f_n^{s-i} f_m^s g_m \) defined as
\[
f_n^{s-i} f_m^s g_m(a_{i+1}, \ldots, a_n, b_{k+1}, \ldots, b_m) = \mathbb{1}_{\Delta_{n+m-k-1}}(a_{i+1}, \ldots, a_n, b_{k+1}, \ldots, b_m)
\]
\[
\times \sum_{a_1, \ldots, a_l \in \mathbb{N}} f_n(a_1, \ldots, a_l) g_m(a_1, \ldots, a_l, b_{k+1}, \ldots, b_m),
\]
\( 0 \leq l \leq k \), and the symbol \( \sum_{s \leq 2i \leq 2 \min(s,n,m)} \) means that the sum is taken over all the integers \( i \) in the interval \([s/2, \min(s,n,m)]\). We close this section with the following Proposition 2.2.

**Proposition 2.2** Let \( f_n \in \ell^2_s(\Delta_n) \) and \( g_m \in \ell^2_s(\Delta_m) \) be symmetric functions. For \( 0 \leq l \leq k \leq \min(n,m) \) we have
\[
\| f_n^{s-i} f_m^s g_m \|_{\ell^2(\mathbb{N})^{\otimes (m+n-k-l)}}^2 \leq \frac{1}{2} \| f_n^{s+i} f_n^{-l} f_n \|_{\ell^2(\mathbb{N})^{\otimes (k-l)}}^2 + \frac{1}{2} \| g_m^{l+m-k} g_m \|_{\ell^2(\mathbb{N})^{\otimes (k-l)}}^2, \tag{2.12}
\]
and
\[
\| f_n^{s-i} f_m^s g_m \|_{\ell^2(\mathbb{N})^{\otimes (m+n-2k)}}^2 \leq \frac{1}{2} \| f_n^{s-i} f_n^{-l} f_n \|_{\ell^2(\mathbb{N})^{\otimes 2k}}^2 + \frac{1}{2} \| g_m^{l+m-k} f_m \|_{\ell^2(\mathbb{N})^{\otimes 2k}}^2. \tag{2.13}
\]
**Proof.** Hölder’s inequality applied twice gives us
\[
\| f_n^{s-i} f_m^s g_m \|_{\ell^2(\mathbb{N})^{\otimes (m+n-k-l)}}^2 \leq \sum_{z_1 \in \mathbb{N}^{m-k}} \sum_{z_2 \in \mathbb{N}^{m-k}} \sum_{y \in \mathbb{N}^{k-l}} \left( \sum_{x \in \mathbb{N}^l} f_n(x, y, z_1) g_m(x, y, z_2) \right)^2.
\]
Theorem 3.1

Wasserstein bounds have been obtained for discrete multiple stochastic integrals in Theorem 4.1 of [9] in the symmetric case \( p = q \) and in Theorems 5.3-5.5 of [12] in the possibly nonsymmetric case, and have been extended to the Kolmogorov distance in the symmetric case \( p = q \) in Theorem 4.2 of [4]. The following result provides a Kolmogorov distance bound which further extends Theorem 4.2 of [4] from multiple stochastic integrals to sums of multiple stochastic integrals in the nonsymmetric case.

**Theorem 3.1** For any finite sum

\[
F = \sum_{k=1}^{n} I_k(f_k)
\]
of discrete multiple stochastic integrals with \( f_k \in \ell^2_s(\Delta_k) \), \( k = 1, \ldots, n \), we have

\[
d_K(F, \mathcal{N}) \leq C_n \left( |1 - \text{Var}[F]| + \sqrt{R_F} \right),
\]

for some constant \( C_n > 0 \) depending only on \( n \), where

\[
R_F := \sum_{0 \leq i < t \leq n} (pq)^{t-i} \| f_i \ast \eta_f f_t \|_{\ell^2(\mathbb{N}) \otimes (i-t)}^2 + \sum_{1 \leq i < t \leq n} \left( \| f_i \ast \eta_f f_t \|_{\ell^2(\mathbb{N}) \otimes (i-t)}^2 + \| f_i \ast \eta_f f_t \|_{\ell^2(\mathbb{N}) \otimes 2(t-i)}^2 \right).
\]

**Proof.** We introduce

\[
R'_F := \sum_{1 \leq i \leq j \leq n} \sum_{k=1}^{i} \sum_{l=0}^{k} \mathbf{1}_{\{i=j=k=l\}} (pq)^{l-i} \| f_i \ast \eta_f f_j \|_{\ell^2(\mathbb{N}) \otimes (i+j-k-l)}^2.
\]

Since it holds that \( R'_F \lesssim R_F \), it is enough to prove the required inequality with \( R'_F \) instead of \( R_F \). Indeed, by the inequality (2.12), all the components of \( R'_F \) for \( 0 \leq l < k \leq i, j \), are dominated by those for \( 0 \leq l < k = i = j \), and also, by the inequality (2.13), the ones where \( 1 \leq k = l < i \leq j \), are dominated by the components where \( 1 \leq l = k < i = j \). Finally, the components for \( 1 \leq k = l = i < j \) remain unchanged.

We will estimate components in the inequality from Proposition 2.1. We have

\[
D_t F = (i + 1) \sum_{i=0}^{n-1} I_i \left( f_{i+1}(r, \cdot) \right), \quad \text{and} \quad D_t L^{-1} F = \sum_{i=0}^{n-1} I_i \left( f_{i+1}(r, \cdot) \right), \quad r \in \mathbb{N},
\]

hence by the multiplication formula (2.11) we find

\[
(D_t F)^2 = \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} c_{i,j,l,k} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( f_{i+1}(r, \cdot) \ast \eta_f f_{j+1}(r, \cdot) \right)
\]

and

\[
D_t F D_t L^{-1} F = \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} d_{i,j,l,k} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( f_{i+1}(r, \cdot) \ast \eta_f f_{j+1}(r, \cdot) \right),
\]

for some \( c_{i,j,l,k}, d_{i,j,l,k} \geq 0 \). Applying the isometry relation (2.2) to (3.2) and using the bound \( \| \tilde{f}_n \|_{\ell^2(\mathbb{N})^\otimes n} \leq \| f_n \|_{\ell^2(\mathbb{N})^\otimes n}, f_n \in \ell^2(\mathbb{N})^\otimes n \), we get, writing \( f \lesssim g \) whenever \( f < C_n g \) for some constant \( C_n > 0 \) depending only on \( n \),

\[
\sum_{r=0}^{\infty} \mathbb{E} \left[ |D_t F|^4 \right] \lesssim \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} \sum_{r=0}^{\infty} \left( \frac{q-p}{\sqrt{pq}} \right)^{2k-2l} \| f_{i+1}(r, \cdot) \ast \eta_f f_{j+1}(r, \cdot) \|_{\ell^2(\mathbb{N}) \otimes (i+j-k-l)}^2
\]

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\[
\begin{align*}
&= \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} \left( \frac{q-p}{\sqrt{pq}} \right)^{2k-2l} \| f_{i+1} \ast_{k+1} f_{j+1} \|_{\ell^2(\mathbb{N}) \otimes (i+j-k-l+1)}^2 \\
&= \sum_{1 \leq i \leq j \leq n} \sum_{k=1}^{i} \sum_{l=0}^{k-1} \left( \frac{q-p}{\sqrt{pq}} \right)^{2k-2l-2} \| f_i \ast_k f_j \|_{\ell^2(\mathbb{N}) \otimes (i+j-k-l)}^2 \\
&\leq pq R'_F.
\end{align*}
\]

Furthermore, by (3.3) it follows that

\[
\langle DF, DL^{-1} F \rangle - \mathbb{E} \left[ \langle DF, DL^{-1} F \rangle \right] \\
= \sum_{r=0}^{\infty} \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} c_{i,j,l,k} \mathbf{1}_{\{i=j=k=l\}^c} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( f_{i+1}(r, \cdot) \right) \ast_{k+1} f_{j+1}(r, \cdot)
\]

\[
= \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} c_{i,j,l,k} \mathbf{1}_{\{i=j=k=l\}^c} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( \sum_{r=0}^{\infty} f_{i+1}(r, \cdot) \right) \ast_{k+1} f_{j+1}(r, \cdot)
\]

\[
= \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} c_{i,j,l,k} \mathbf{1}_{\{i=j=k=l\}^c} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( f_{i+1} \ast_{k+1} f_{j+1} \right),
\]

thus we get

\[
\text{Var} \left[ \langle DF, DL^{-1} F \rangle \right] \lesssim \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{i} \sum_{l=0}^{k} \mathbf{1}_{\{i=j=k=l\}^c} \left( \frac{1}{pq} \right)^{k-l} \| f_{i} \ast_{k} f_{j} \|^{2}_{\ell^2(\mathbb{N}) \otimes (i+j-k-l)}
\]

\[
= \sum_{1 \leq i \leq j \leq n} \sum_{k=1}^{i} \sum_{l=1}^{k} \mathbf{1}_{\{i=j=k=l\}^c} \frac{1}{(pq)^{k-l}} \| f_i \ast_k f_j \|^{2}_{\ell^2(\mathbb{N}) \otimes (i+j-k-l)}
\]

\[
\leq R'_F.
\]
\[ R_F' + \sum_{i=1}^{n} \| f_i \|^4_{\ell^2(N)^{\otimes i}} + \sum_{1 \leq i < j \leq n} \| f_i \|^2_{\ell^2(N)^{\otimes i}} \| f_j \|^2_{\ell^2(N)^{\otimes j}} \leq R_F' + (\text{Var}[F])^2, \]

while as in (3.2) and (3.3) we have

\[ \mathbb{E} \left[ \left( \sum_{k=0}^{\infty} (D_k L^{-1} F)^2 \right)^2 \right] \]
\[ = \mathbb{E} \left[ \left( \sum_{k=0}^{\infty} \sum_{0 \leq i \leq j \leq n-1} \sum_{i=0}^{k} d_{i,j,l,k} \left( \frac{q-p}{\sqrt{pq}} \right)^{k-l} I_{i+j-k-l} \left( f_{i+1}(k, \cdot) \right)^2 \right)^2 \right] \]
\[ \lesssim \sum_{0 \leq i \leq j \leq n-1} \sum_{k=0}^{\infty} \sum_{i=0}^{k} (pq)^{l-k} \| f_{i+1} \|^2_{\ell^2(N)^{\otimes (i+j-k-l)}} \]
\[ = \sum_{1 \leq i \leq j \leq n} \sum_{k=1}^{n} (pq)^{l-k} \| f_i \|^2 \| f_j \|^2_{\ell^2(N)^{\otimes (i+j-k-l)}} \]
\[ \lesssim R_F' + \sum_{i=1}^{n} \| f_i \|^4_{\ell^2(N)^{\otimes i}} + \sum_{1 \leq i < j \leq n} \| f_i \|^2_{\ell^2(N)^{\otimes i}} \| f_j \|^2_{\ell^2(N)^{\otimes j}} \]
\[ = R_F' + (\text{Var}[F])^2, \]

de hence we get

\[ \sum_{k=0}^{\infty} \mathbb{E}[ (FD_k L^{-1} F)^2 ] \lesssim R_F' + (\text{Var}[F])^2. \tag{3.6} \]

We now deal with the last component in Proposition 2.1 similarly as it is done in proof of Theorem 4.2 in [4]. Precisely, by the integration by parts formula (2.5) and the Cauchy-Schwarz inequality we have

\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left[ \langle D1_{(F > x)}, DF|DL^{-1} F| \rangle_{\ell^2(N)} \right] = \sup_{x \in \mathbb{R}} \mathbb{E} \left[ 1_{(F > x)} \delta (DF|DL^{-1} F|) \right] \]
\[ \leq \sqrt{\mathbb{E} \left[ (\delta (DF|DL^{-1} F|))^2 \right]} \tag{3.7} \]

Then, by the bound (2.6), the Cauchy-Schwarz inequality and the consequence (2.8) of Mehler’s formula (2.7), we have

\[ \mathbb{E} \left[ (\delta (DF|DL^{-1} F|))^2 \right] \]
\[ \begin{align*}
&\leq \mathbb{E}[\|DF|DL^{-1}F\|_{\ell^2(N)}^2] + \mathbb{E}\left[ \sum_{k,l=0}^{\infty} |D_k (D_l F | D_l L^{-1} F) | D_l (D_k F | D_k L^{-1} F) |\right] \\
&\leq \sqrt{\mathbb{E}[\|DF\|_{\ell^2(N)}^4] \mathbb{E}[\|DL^{-1}F\|_{\ell^2(N)}^4]} + \mathbb{E}\left[ \sum_{k,l=0}^{\infty} (D_k (D_l F | D_l L^{-1} F))^2 \right] \\
&\leq \mathbb{E}[\|DF\|_{\ell^2(N)}^4] + \sum_{k,l=0}^{\infty} \mathbb{E}[ (D_k (D_l F | D_l L^{-1} F))^2 ].
\end{align*} \]

The first term in the last expression in bounded by \( pqR_F \) as shown in (3.4), and it remains to estimate the last expectation. By the product rule (2.3) and the bound \(|D_k F| \leq |D_k F| \) obtained from the definition (2.1) of \( D \) and the triangle inequality, we get

\[ \begin{align*}
\mathbb{E}[ (D_r (D_s F | D_s L^{-1} F))^2 ] &= \mathbb{E}\left[ \left((D_r D_s F | D_s L^{-1} F) + (D_s F D_r | D_s L^{-1} F) - \frac{X_r}{\sqrt{pq}} (D_r D_s F D_r | D_s L^{-1} F) \right)^2 \right] \\
&\leq \mathbb{E}\left[ (D_r D_s F)^2 (D_s L^{-1} F)^2 + (D_s F) (D_r D_s L^{-1} F)^2 + \frac{1}{pq} (D_r D_s F)^2 (D_r D_s L^{-1} F)^2 \right],
\end{align*} \]

(3.8)

\( r, s \in \mathbb{N} \). By the Cauchy-Schwarz inequality we get

\[ \begin{align*}
\sum_{r,s=0}^{\infty} \mathbb{E}[ (D_r D_s F)^2 (D_s L^{-1} F)^2 ] &= \mathbb{E}\left[ \sum_{s=0}^{\infty} (D_s L^{-1} F)^2 \sum_{r=0}^{\infty} (D_r D_s F)^2 \right] \\
&\leq \sqrt{\mathbb{E}\left[ \sum_{s=0}^{\infty} (D_s L^{-1} F)^4 \right] \mathbb{E}\left[ \sum_{r=0}^{\infty} \left( \sum_{s=0}^{\infty} (D_r D_s F)^2 \right)^2 \right]}.
\end{align*} \]

The term \( \mathbb{E}\left[ \sum_{s=0}^{\infty} (D_s L^{-1} F)^4 \right] \) can be bounded by \( pqR_F^2 \) as in (3.4). To estimate the other term we use the multiplication formula (2.11) as in (3.2) to obtain

\[ \begin{align*}
\mathbb{E}\left[ \sum_{s=0}^{\infty} \left( \sum_{r=0}^{\infty} (D_r D_s F)^2 \right)^2 \right] &\lesssim \sum_{s=0}^{\infty} \mathbb{E}\left[ \left( \sum_{r=0}^{\infty} \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} \frac{q-p}{\sqrt{pq}} |I_{i+j-k-l} (f_{i+2}(s,r,\cdot) f_{j+2}^{k+l} f_{j+2}(s,\cdot))|^{2} \right)^2 \right] \\
&= c \sum_{s=0}^{\infty} \mathbb{E}\left[ \left( \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} \frac{q-p}{\sqrt{pq}} |I_{i+j-k-l} (f_{i+2}(s,\cdot) f_{j+2}^{k+l+1} f_{j+2}(s,\cdot))|^{2} \right)^2 \right] \\
&\lesssim \sum_{s=0}^{\infty} \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} (pq)^{l-k} \|f_{i+2}(s,\cdot) f_{j+2}^{k+l+1} f_{j+2}(s,\cdot)\|_{\ell^2(\mathbb{N})}^{2}.
\end{align*} \]
\[ \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} (pq)^{l-k} \| f_{i+2} \ast_{k+2} f_{j+2} \|^2 \|_{\ell_2(\mathbb{N}) \otimes (i+j-k-l+1)}^2 \]

\[ \sum_{2 \leq i \leq j \leq n} \sum_{k=2}^{i} \sum_{l=1}^{k-1} (pq)^{l+1-k} \| f_i \ast_k f_j \|^2 \|_{\ell_2(\mathbb{N}) \otimes (i+j-k-l)}^2 \]

\[ \leq pq R'_F. \]

The term \( \sum_{r,s=0}^{\infty} \mathbb{E}[((D_r F)(D_s D_s^{-1} F))^2] \) from (3.8) is similarly bounded by \( pq R'_F \). Regarding the last term, we have

\[ \sum_{r,s=0}^{\infty} \mathbb{E}[((D_r D_s F)(D_s D_s^{-1} F))^2] \leq \sqrt{\sum_{r,s=0}^{\infty} \mathbb{E}[(D_r D_s F)^4] \sum_{r,s=0}^{\infty} \mathbb{E}[(D_r D_s D_s^{-1} F)^4]}. \]

Using the multiplication formula (2.11), both sums inside the above square root can be estimated as

\[ \sum_{r,s=0}^{\infty} \mathbb{E} \left[ \left( \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} \left| \frac{q-p}{pq} \right|^{k-l} I_{i+j-k-l} \left( f_{i+2}(s,r) \ast_k f_{j+2}(s,r) \right) \right)^2 \right] \]

\[ \leq \sum_{r,s=0}^{\infty} \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} (pq)^{l-k} \| f_{i+2} \ast_k f_{j+2} \|^2 \|_{\ell_2(\mathbb{N}) \otimes (i+j-k-l+2)}^2 \]

\[ = \sum_{0 \leq i \leq j \leq n-2} \sum_{k=0}^{i} \sum_{l=0}^{k} (pq)^{l-k} \| f_{i+2} \ast_k f_{j+2} \|^2 \|_{\ell_2(\mathbb{N}) \otimes (i+j-k-l+2)}^2 \]

\[ \leq (pq)^2 R'_F. \]

Combining this together we get

\[ \sum_{r,s=0}^{\infty} \mathbb{E}\left[(D_r(D_s F)D_s^{-1} F))^2] \right. \leq pq R'_F. \]

and consequently, by (3.7) we find

\[ \sup_{x \in \mathbb{R}} \mathbb{E} \left[ (D1_{\{F \geq x\}}, DF|DL^{-1} F|) \right] \leq pq R'_F. \]  \hspace{1cm} (3.9)

Applying (3.4)-(3.6) and (3.9) to Proposition 2.1, we get

\[ d_K(F,N) \leq |1 - \text{Var}[F]| + \sqrt{R'_F} (1 + \text{Var}[F] + \sqrt{\text{Var}[F]} + \sqrt{R'_F}). \]

If \( R'_F \geq 1 \), or if \( R'_F \leq 1 \) and \( \text{Var} |F| \geq 2 \), it is clear that \( d_K(F,N) \leq |1 - \text{Var}[F]| + \sqrt{R'_F} \) since \( d_K(F,N) \leq 1 \) by definition. If \( R'_F \leq 1 \) and \( \text{Var}[F] \leq 2 \), we estimate \( \text{Var}[F] + \sqrt{\text{Var}[F]} + \sqrt{R'_F} \) by a constant and also get the required bound. \( \Box \)
4 Application to random graphs

In the sequel fix a numbering \((1, \ldots, e_G)\) of the edges in \(G\) and we denote by \(E_G \subset \mathbb{N}^{e_G}\) the set of sequences of (distinct) edges that create a graph isomorphic to \(G\), i.e. a sequence \((e_{k_1}, \ldots, e_{k_{e_G}})\) belongs to \(E_G\) if and only if the graph created by edges \(e_{k_1}, \ldots, e_{k_{e_G}}\) is isomorphic to \(G\). The next lemma allows us to represent the number of subgraphs as a sum of multiple stochastic integrals, using the notation \(P(X_k = 1) = p, P(X_k = -1) = 1 - p = q, k \in \mathbb{N}\).

Lemma 4.1 We have the identity

\[
\hat{N}_G = \frac{N_G - \mathbb{E}[N_G]}{\sqrt{\text{Var}[N_G]}} = \sum_{k=1}^{e_G} I_k(f_k),
\]

(4.1)

where

\[
f_k(b_1, \ldots, b_k) := \frac{q^{k/2} p^{e_G - k/2}}{(e_G - k)!k! \sqrt{\text{Var}[N_G]}} \sum_{(a_1, \ldots, a_{e_G - k}, b_1, \ldots, b_k) \in E_G} 1_{(a_1, \ldots, a_{e_G - k}, b_1, \ldots, b_k) \in E_G}.
\]

Proof. We have

\[
N_G = \frac{1}{e_G!2^{e_G}} \sum_{b_1, \ldots, b_{e_G} \in \mathbb{N}} 1_{(b_1, \ldots, b_{e_G}) \in E_G} (X_{b_1} + 1) \cdots (X_{b_{e_G}} + 1)
\]

\[
= \frac{1}{e_G!2^{e_G}} \sum_{m=0}^{e_G} \binom{e_G}{m} \sum_{b_1, \ldots, b_m \in \mathbb{N}} g_m(b_1, \ldots, b_m) X_{b_1} \cdots X_{b_m}
\]

\[
= \frac{1}{e_G!2^{e_G}} \sum_{m=0}^{e_G} \binom{e_G}{m} \sum_{k=0}^{m} \binom{m}{k} (p - q)^{m-k} \sum_{b_1, \ldots, b_k \in \mathbb{N}} g_k(b_1, \ldots, b_k) (X_{b_1} + q - p) \cdots (X_{b_k} + q - p)
\]

\[
= \frac{1}{e_G!2^{e_G}} \sum_{m=0}^{e_G} \binom{e_G}{m} \sum_{k=0}^{m} \binom{m}{k} I_k(g_k)(2\sqrt{pq})^k (p - q)^{m-k}
\]

\[
= \frac{1}{e_G!2^{e_G}} \sum_{k=0}^{e_G} \binom{e_G}{k} (2\sqrt{pq})^k I_k(g_k) \sum_{m=k}^{e_G} \binom{e_G - k}{m-k} (p - q)^{m-k}
\]

\[
= \frac{1}{2^{e_G}} \sum_{k=0}^{e_G} \frac{(2\sqrt{pq})^k}{(e_G - k)!k!} I_k(g_k)(1 + p - q)^{e_G-k}
\]

\[
= \sum_{k=0}^{e_G} \frac{q^{k/2} p^{e_G - k/2}}{(e_G - k)!k!} I_k(g_k),
\]

where \(g_k\) is the function defined as

\[
g_k(b_1, \ldots, b_k) := \sum_{(a_1, \ldots, a_{e_G - k}) \in \mathbb{N}^{e_G - k}} 1_{E_G}(a_1, a_{e_G - k}, b_1, \ldots, b_k), \quad (b_1, \ldots, b_k) \in \mathbb{N}^k,
\]

(4.2)
which shows \((4.1)\) with
\[
\begin{align*}
f_k(b_1, \ldots, b_k) := \frac{q^{k/2}p^{e_kG-k/2}}{(e_kG - k)!k! \sqrt{\text{Var}[N_G]}} g_k(b_1, \ldots, b_k).
\end{align*}
\]

Next is the second main result of this paper.

**Theorem 4.2** Let \(G\) be a graph without isolated vertices. Then we have
\[
d_K(\tilde{N}_G, N) \lesssim \left( (1 - p) \min_{H \subseteq G} \{n^{e_H}p^{e_H} \} \right)^{-1/2}.
\]

**Proof.** By \((4.1)\) and Theorem 3.1 we have
\[
d_K(\tilde{N}_G, N) \lesssim \frac{\sqrt{R_G}}{\text{Var}[N_G]},
\]
where, taking \(g_k\) as in \((4.2)\), by \((3.1)\) we have
\[
R_G = \sum_{0 \leq l < k \leq e_G} p^{4e_G-3l+1} q^{l+k} \|g_k \ast_k g_k\|_{\ell^2(N)^{(k-l)}}^2 + \sum_{1 \leq l < k \leq e_G} p^{4e_G-2k} q^{2k} \|g_k \ast_k g_k\|_{\ell^2(N)^{(k-l)}}^2
\]
\[
\leq q \left( \sum_{0 \leq l < k \leq e_G} p^{4e_G-3l+1} \|g_k \ast_k g_k\|_{\ell^2(N)^{(k-l)}}^2 + \sum_{1 \leq l < k \leq e_G} p^{4e_G-2k} \|g_k \ast_k g_k\|_{\ell^2(N)^{(k-l)}}^2 \right)
\]
\[
= (1 - p)(S_1 + S_2 + S_3).
\]

It is now sufficient to show that
\[
S_1 + S_2 + S_3 \lesssim \max_{H \subseteq G} n^{4e_G-3e_H} p^{4e_G-3e_H}.
\]

Indeed, applying \((1.3)\) and \((4.4)\) to \((4.3)\) we get
\[
\frac{\sqrt{R_G}}{\text{Var}[N_G]} \lesssim \frac{(1 - p) \max_{H \subseteq G} n^{4e_G-3e_H} p^{4e_G-3e_H}}{(1 - p) \max_{H \subseteq G} n^{2e_G-v_H} p^{2e_G-v_H}}
\]
\[
= \left( \min_{H \subseteq G} n^{e_H}p^{e_H} \right)^{-3/2}
\]
\[
\sqrt{1 - p} \left( \min_{H \subseteq G} n^{e_H}p^{e_H} \right)^{-1}.
\]
In order to estimate \( S_1 \), let us observe that

\[
\|g_k * g_k\|_{L^2(\mathbb{N}^{(k)})}^2 = \sum_{a'' \in \mathbb{N}^{k-l}} \left( \sum_{a' \in \mathbb{N}^l} \left( \sum_{a \in \mathbb{N}^{k}} 1_{E_{G \setminus a'}} (a, a', a'') \right)^2 \right) \]

\[
\approx \sum_{A \subset K \setminus \epsilon_{K}^{k-l}} \left( \sum_{A \subset B \subset K \setminus \epsilon_{K}^{k-l}} \left( \sum_{B \subset C \subset K \setminus \epsilon_{K}^{k-l}} 1_{G \setminus a'} \right)^2 \right) \]

\[
\approx \sum_{K \subset G \setminus \epsilon_{K}^{k-l}} n^{v_K} \left( \sum_{K \subset H \subset G \setminus \epsilon_{H}^{k}} n^{v_H - v_K} (n^{v_G - v_H})^2 \right) \]

\[
\approx \max_{K \subset H \subset G \setminus \epsilon_{K}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 2v_H - v_K}.
\]

Hence we have

\[
S_1 \lesssim \sum_{0 \leq l < k \leq e_G} p^{4e_G - 3k + l} \max_{K \subset H \subset G \setminus \epsilon_{K}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 2v_H - v_K}
\]

\[
= \sum_{0 \leq l < k \leq e_G} \max_{K \subset H \subset G \setminus \epsilon_{K}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 2v_H - v_K} p^{4e_G - 2e_H - e_K}
\]

\[
\lesssim \max_{K \subset H \subset G \setminus \epsilon_{K}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 2v_H - v_K} p^{4e_G - 2e_H - e_K}.
\]

For a fixed \( p \), let \( H_0 \subset G, \epsilon_{H_0} \geq 1 \), be the subgraph of \( G \) such that

\[
n^{v_{H_0}} p^{e_{H_0}} = \min_{H \subset G, \epsilon_H \geq 1} n^{v_H} p^{e_H}.
\]

Then it is clear that

\[
S_1 \lesssim \max_{K \subset H \subset G \setminus \epsilon_{K}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 2v_H - v_K} p^{4e_G - 2e_H - e_K}
\]

\[
= n^{4v_G - 3v_{H_0}} p^{4e_G - 3e_{H_0}}
\]

\[
= \max_{H \subset G \setminus \epsilon_{H}^{k-l}, \epsilon_{H}^{k}} n^{4v_G - 3v_H} p^{4e_G - 3e_H}.
\]
as required. We proceed similarly with the sum $S_2$. For $1 \leq l < k \leq n$ we have

$$\|g_l \ast l g_k\|^2_{L^2(N^{k-l})} \approx \sum_{c \in N^{k-l}} \left( \sum_{b \in N} \left( \sum_{a \in N^{c-l}} 1_{E_G} (a, b) \sum_{a' \in N^{c-k}} 1_{E_G} (a', b, c) \right) \right)^2$$

$$\approx \sum_{A \subset C \subset K_n} \left( \sum_{b \in B} \left( \sum_{A \subset C \subset K_n} 1_{B \subset C \subset K_n} \right) \left( \sum_{H \subset K \subset K_n} 1_H \right) \right)^2$$

$$\lesssim \sum_{K \subset G} \sum_{e_K = k-l} n^{v_K} n^{v_{K'}} \left( \sum_{K \subset H \subset G, H' \subset G} n^{v_H - v_K} \left( n^{v_{H'} - v_K} n^{v_{H'} - v_H} \right)^2 \right)$$

(4.9)

$$\lesssim \max_{K, H' \subset G, e_K = k-l, e_{H'} = l} n^{4v_G - 2v_{H'} - v_K}$$

(4.10)

where $H'$ in (4.9) stands for $B \setminus A$ in (4.8), whereas in (4.10) the sum over $H'$ extends to all $H' \subset G$ such that $e_{H'} = l$. It follows that

$$S_2 \lesssim \sum_{1 \leq l < k \leq e_G} p^{4e_G - k-l} \max_{K, H' \subset G} n^{4v_G - 2v_{H'} - v_K}$$

$$= \sum_{1 \leq l < k \leq e_G} \max_{e_K = k-l, e_{H'} = l} n^{4v_G - 2v_{H'} - v_K} p^{4e_G - 2v_{H'} - e_K}$$

$$\lesssim \max_{K', H' \subset G} n^{4v_G - 2v_{H'} - v_K} p^{4e_G - 2v_{H'} - e_K}$$

$$= \max_{n^{4v_G - 3v_{H}} \geq 1} n^{4v_G - 3v_{H}} p^{4e_G - 3e_H}$$

where $H_0$ is defined in (4.5). Finally, we pass to estimates of $S_3$. For $1 \leq l < k \leq n$ we have

$$\|g_k \ast l g_l\|^2_{L^2(N^{k-l})} \approx \sum_{c' \in N^{k-l}} \left( \sum_{b \in N} \left( \sum_{a \in N^{c-l}} 1_{E_G} (a, b, c') \right) \left( \sum_{a' \in N^{c-k}} 1_{E_G} (a', b, c') \right) \right)^2$$

$$\approx \sum_{A \subset C \subset K_n} \sum_{b \in B} \sum_{e_A = e_{A} = k-l} \left( \sum_{A \subset C \subset K_n} \left( \sum_{G' \subset G} 1 \right) \left( \sum_{G'' \subset G} 1 \right) \right)^2$$

$$= \sum_{A \subset C \subset K_n} \sum_{e_A = e_{A} = k-l} \left( \sum_{G' \subset G} 1 \right) \left( \sum_{G'' \subset G} 1 \right)$$

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Next, we note that given $A, A' \subset K_n$, it takes
\[v_B - v_{A \cap B} - v_{A' \cap B} + v_{A \cap A' \cap B} = v_H - v_{K \cap H} - v_{K' \cap H} + v_{A \cap A' \cap B}\]
vertices to create any subgraph $B \sim H$ such that $A \cap B \sim K \cap H$ and $A' \cap B \sim K' \cap H$,
with the bound
\[v_{A \cap A' \cap B} \leq \frac{1}{2} v_{A \cap A'} + \frac{1}{2} v_{A' \cap B} = \frac{1}{2} (v_{A \cap A'} + v_{K \cap H}).\]
Hence we have
\[
\left\|g_k \ast_1 g_k\right\|^2_{\mathcal{B}(\mathbb{N})^{(k-l)}} \leq \sum_{K,K',H \subset G} \sum_{A,A' \subset K_n} \left( n^{v_G - v_{A \cup B}} \right) \left( n^{v_G - v_{A' \cup B}} \right).
\]

In order to estimate the above sum using powers of $n$, we need to consider the possible intersections $A \cap A'$ for $A, A' \subset K_n$, as follows:
\[
\sum_{K,K',H \subset G} n^{4v_G + 2v_H - 2v_{K \cap H} - v_{K' \cap H} + v_{A \cap A'\cap H} - 2v_{K \cap H} - 2v_{K' \cap H}} \leq \sum_{i=0}^{v_K + v_{K'} - i} n^{i(4v_G + 2v_H - 2v_{K \cap H} - v_{K' \cap H} + i - 2v_{K \cap H} - 2v_{K' \cap H})}.
\]
Furthermore we have
\[v_K + v_{K'} + 4v_G + 2v_H - 2v_{K \cap H} - v_{K' \cap H} - 2v_{K \cup H} - 2v_{K' \cup H} \]
\[ 4v_G - v_K - v_H - v_{K \cup H}, \]

so the sum (4.11) can be estimated as

\[ \sum_{K,K',H \subseteq G} n^{4v_G - v_K - v_H - v_{K \cup H}} \lesssim \max_{K,H,L \subseteq G} n^{4v_G - v_K - v_H - v_L}, \]

from which it follows

\[ S_3 \lesssim \sum_{1 \leq l < k \leq e_G} p^{4e_G - 2k} \max_{K,H,L \subseteq G} n^{4v_G - v_K - v_H - v_L} \]

\[ = \sum_{1 \leq l < k \leq e_G} \max_{K,H,L \subseteq G} n^{4v_G - v_K - v_H - v_L} p^{4e_G - e_K - e_H - e_L} \]

\[ \lesssim \max_{K,H,L \subseteq G} n^{4v_G - v_K - v_H - v_L} p^{4e_G - e_K - e_H - e_L} \]

\[ \leq n^{4v_G - 3v_H} p^{4e_G - e_H} \]

\[ = \max_{H \subseteq G} n^{4v_G - 3v_H} p^{4e_G - e_H}, \]

which ends the proof. \[ \square \]

In the next corollary we note that Theorem 4.2 simplifies if we narrow our attention to \( p_n \) depending of the complete graph size \( n \) and close to 0 or to 1.

**Corollary 4.3** Let \( G \) be a graph without separated vertices. For \( p_n < c < 1, n \geq 1 \), we have

\[ d_K \left( \tilde{N}_n^G, \mathcal{N} \right) \lesssim \left( \min_{H \subseteq G} \left\{ n^{v_H} p_n^{e_H} \right\} \right)^{-1/2}. \] (4.12)

On the other hand, for \( p_n > c > 0, n \geq 1 \), it holds

\[ d_K \left( \tilde{N}_n^G, \mathcal{N} \right) \lesssim \frac{1}{n \sqrt{1 - p_n}}. \] (4.13)

As a consequence of Corollary 4.3 it follows that if

\[ np_n^\beta \to \infty \quad \text{and} \quad n^2(1 - p_n) \to \infty, \]

where \( \beta := \max \{ e_H / v_H : H \subseteq G \} \), then we have the convergence of the renormalized subgraph count \( (\tilde{N}_n^G)_{n \geq 1} \) to \( \mathcal{N} \) in distribution as \( n \) tends to infinity, which recovers the sufficient condition in [14]. When \( p \approx n^{-\alpha}, \alpha > 0 \), Corollary 4.3 also shows that

\[ d_K \left( \tilde{N}_n^G, \mathcal{N} \right) \lesssim \left( \min_{H \subseteq G} \left\{ n^{v_H - \alpha e_H} \right\} \right)^{-1/2}, \] (4.14)
and in order for the above bound (4.14) to tend to zero as \( n \) goes to infinity, we should have

\[
\alpha < \min_{H \subseteq G} \frac{\nu_H}{e_H} =: \frac{1}{\beta}.
\] (4.15)

The next Corollary 4.4 of Theorem 4.2 and (4.13) deals with cycle graphs with \( r \) vertices, \( r \geq 3 \). When \( G \) is a triangle it recovers the Kolmogorov bounds of [13] as in Corollary 1.1 above.

**Corollary 4.4** Let \( G \) be a cycle graph with \( r \) vertices, \( r \geq 3 \), and \( c \in (0, 1) \). We have

\[
d_K(\tilde{N}_n^G, \mathcal{N}) \lesssim \begin{cases} 
\frac{1}{n \sqrt{1 - p_n}} & \text{if } 0 < c < p_n, \\
\frac{1}{n^{1/p_n}} & \text{if } n^{-(r-2)/(r-1)} < p_n \leq c, \\
\frac{1}{(np_n)^{r/2}} & \text{if } 0 < p_n \leq n^{-(r-2)/(r-1)}. 
\end{cases}
\]

*Proof.* The smallest number of vertices of subgraphs \( H \) of \( G \) having \( k \) edges, \( k < r \), is realised for a linear subgraph having \( k + 1 \) vertices, which yields

\[
\min_{H \subseteq G} \{n^{e_H} p_n^{e_H}\} = \min_{1 \leq k < r} n^{k+1} p_n^k = n \min_{1 \leq k < r} (np_n)^k = \min(n^2 p_n, (np_n)^{r-1}),
\]

hence

\[
\min_{H \subseteq G} \{n^{e_H} p_n^{e_H}\} = \min \{n^2 p_n, (np_n)^{r-1}, (np_n)^r\} = \begin{cases} 
n^2 p_n & \text{if } n^{-(r-2)/(r-1)} < p_n \leq c, \\
(np_n)^r & \text{if } 0 < p_n \leq n^{-(r-2)/(r-1)},
\end{cases}
\]

which concludes the proof by (4.12) and (4.13). \( \square \)

In case \( p_n \approx n^{-\alpha} \) we should have \( \alpha \in (0, 1) \) by (4.15), Corollary 4.4 also shows that

\[
d_K(\tilde{N}_n^G, \mathcal{N}) \lesssim \begin{cases} 
n^{-1+\alpha/2} \approx \frac{1}{n \sqrt{p_n}} & \text{if } 0 < \alpha \leq \frac{r-2}{r-1}, \\
n^{-r(1-\alpha)/2} \approx \frac{1}{(np_n)^{r/2}} & \text{if } \frac{r-2}{r-1} < \alpha < 1.
\end{cases}
\]

when \( G \) is a cycle graph with \( r \) vertices, \( r \geq 3 \). In the particular case \( r = 3 \) where \( G \) is a triangle, this improves on the Kolmogorov bounds in Theorem 1.1 of [5].

In the case of complete graphs, the next corollary also covers the case of triangles.

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Corollary 4.5  Let $G$ be a complete graph with $k$ vertices, $r \geq 3$, and $c \in (0, 1)$. We have

$$d_K(\tilde{N}_n^G, N) \lesssim \begin{cases} 
\frac{1}{n\sqrt{1 - p_n}} & \text{if } c < p_n < 1, \\
\frac{1}{n\sqrt{p_n}} & \text{if } n^{-2/(r+1)} < p_n \leq c, \\
\frac{1}{n^{r/2} p_n^{(r-1)/4}} & \text{if } 0 < p_n \leq n^{-2/(r+1)}.
\end{cases}$$

Proof. The greatest number of edges of subgraphs of $G$ having $k$ vertices, $2 \leq k \leq v_G$, is realised for a complete graph having $\binom{k}{2}$ edges, which shows that

$$\min_{H \subseteq G} \left\{ n^{v_H} p_n^{e_H} \right\} = \min_{1 \leq k \leq r} n^{k} p_n^{\binom{k}{2}}.$$ 

On the other hand, from the equality

$$\frac{n^{k+1} p_n^{\binom{k}{2}}}{n^{k} p_n^{\binom{k}{2}}} = n p_n^k,$$ 

we note that if the minimum was realised with $k$ vertices where $1 < k < r$, we would have $n p_n^{k-1} \leq 1$ and $n p_n^k \geq 1$, which would lead to $p_n \geq 1$, which is not possible. Therefore we have

$$\min_{H \subseteq G} \left\{ n^{v_H} p_n^{e_H} \right\} = \min \left\{ n^{2} p_n, n^{r} p_n^{\binom{r}{2}} \right\} = \begin{cases} 
n^{2} p_n & \text{if } n^{-2/(r+1)} < p_n \leq c, \\
n^{r} p_n^{(r-1)/2} & \text{if } 0 < p_n \leq n^{-2/(r+1)},
\end{cases}$$

and we conclude the proof by (4.12) and (4.13). \hfill \Box

When $p_n \approx n^{-\alpha}$ with $\alpha \in (0, 2/(r-1))$ by (4.15), Corollary 4.5 shows that

$$\min_{H \subseteq G} \left\{ n^{v_H - \alpha e_H} \right\} = \min \left\{ n^{2-\alpha}, n^{r-\binom{r}{2}\alpha} \right\} = \begin{cases} 
n^{2-\alpha/2} & \text{if } 0 < \alpha \leq \frac{2}{r+1}, \\
n^{r-r(r-1)\alpha/2} & \text{if } \frac{2}{r+1} \leq \alpha < \frac{2}{r-1},
\end{cases}$$

hence by (4.12) we find

$$d_K(\tilde{N}_n^G, N) \lesssim \begin{cases} 
n^{-1+\alpha/2} \approx \frac{1}{n^{v_n}} & \text{if } 0 < \alpha \leq \frac{2}{r+1}, \\
n^{-r/2+r(r-1)\alpha/4} \approx \frac{1}{n^{r/2} p_n^{(r-1)/4}} & \text{if } \frac{2}{r+1} \leq \alpha < \frac{2}{r-1}.
\end{cases}$$

Finally, the next corollary deals with the important class of graphs which have a tree structure.
Corollary 4.6 Let $G$ be any tree (a connected graph without cycles) with $r$ edges, and $c \in (0, 1)$. We have

$$d_K(\tilde{N}_n^G, \mathcal{N}) \lesssim \begin{cases} 
\frac{1}{n\sqrt{1-p_n}} & \text{if } c < p_n < 1, \\
\frac{1}{n\sqrt{p_n}} & \text{if } \frac{1}{n} < p_n \leq c, \\
\frac{1}{n^{(r+1)/2}p_n^{r/2}} & \text{if } 0 < p_n \leq \frac{1}{n}. 
\end{cases}$$

Proof. We have

$$\min_{H \subseteq G, e_H \geq 1} \left\{ n^{v_H}p_n^{e_H} \right\} = \min_{1 \leq k \leq r} n^{k+1}p_n^k = n \min_{1 \leq k \leq r} (np_n)^k.$$ 

The smallest number of vertices for a subgraph of a tree $G$ having $k$ edges, $k \leq r$, is realised for a subtree having $k+1$ vertices, hence since $np_n$ can be either less or greater than 1, which gives

$$\min_{H \subseteq G, e_H \geq 1} \left\{ n^{v_H}p_n^{e_H} \right\} = n \min\left\{ np_n, (np_n)^r \right\} = \begin{cases} 
n^2p_n & \text{if } \frac{1}{n} < p_n \leq c, \\
n^{r+1}p_n^r & \text{if } 0 < p_n \leq \frac{1}{n}, 
\end{cases}$$

as required, and we conclude by (4.12) and (4.13).

In case $p_n \approx n^{-\alpha}$ with $\alpha \in (0, 1+1/r)$, we have $\beta = \max\{e_H/v_H : H \subseteq G\} = r/(r+1)$ hence

$$\min_{H \subseteq G, e_H \geq 1} \left\{ n^{v_H-\alpha e_H} \right\} = n \min\left\{ n^{-\alpha}, \left(n^{-\alpha}\right)^r \right\} = \begin{cases} 
n^{2-\alpha} & \text{if } 0 < \alpha \leq 1, \\
n^{r+1-r\alpha} & \text{if } 1 \leq \alpha < 1 + \frac{1}{r}, 
\end{cases}$$

which shows by (4.12) that

$$d_K(\tilde{N}_n^G, \mathcal{N}) \lesssim \begin{cases} 
n^{-1+\alpha/2} \approx \frac{1}{n\sqrt{p_n}} & \text{if } 0 < \alpha \leq 1, \\
n^{-(r+1-r\alpha)/2} \approx \frac{1}{n^{(r+1)/2}p_n^{r/2}} & \text{if } 1 \leq \alpha < 1 + \frac{1}{r}. 
\end{cases}$$

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