Set membership with a few bit probes

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Abstract

We consider the bit-probe complexity of the set membership problem, where a set \( S \) of size at most \( n \) from a universe of size \( m \) is to be represented as a short bit vector in order to answer membership queries of the form “Is \( x \) in \( S \)?” by adaptively probing the bit vector at \( t \) places. Let \( s(m, n, t) \) be the minimum number of bits of storage needed for such a scheme. Several recent works investigate \( s(m, n, t) \) for various ranges of the parameter; we obtain the following improvements over the bounds shown by Buhrman, Miltersen, Radhakrishnan, and Srinivasan [5] and Alon and Feige [2].

For two probes \((t = 2)\):

(a) \( s(m, n, 2) = O(m^{1 - \frac{1}{4t + 1}}) \); this improves on a result of Alon and Feige that states that for \( n \leq \lg m \), \( s(m, n, 2) = O(mn / \lg m) \).

(b) \( s(m, n, 2) = \Omega(m^{1 - \frac{1}{4t + 1}}) \); in particular, \( s(m, n, 2) = \Omega(m) \) for \( n \geq \lg m \), that is, if \( s(m, n, 2) = o(m) \) (significantly better than the characteristic vector representation), then \( n = o(\lg m) \).

For three probes \((t = 3)\): \( s(m, n, 3) = O(\sqrt{mn / \lg m}) \). This improves a result of Alon and Feige that states that \( s(m, n, 2) = O(m^{\frac{2}{5}} n^{\frac{1}{3}}) \).

In general:

(a) (Non-adaptive schemes) For odd \( t \geq 5 \), there is a non-adaptive scheme using \( O(t^2 mn^{1 - \frac{1}{t + 1}} \lg \frac{2m}{n}) \) bits of space. This improves on a result of Buhrman et al. [5] that states that for odd \( t \geq 5 \), there exists a non-adaptive scheme that uses \( O(tm^{\frac{1}{2t+1}} n) \) bits of space.

(b) (Adaptive schemes) For odd \( t \geq 3 \) and \( t \leq \frac{1}{16} \lg m \) and for \( n \leq m^{1 - \epsilon} \) (\( \epsilon > 0 \)), we have \( s(m, n, t) = O(\exp(c^2) m^{\frac{1}{2t+1}} n^{1 - \frac{1}{t+1}} \lg m) \). Previously, for \( t \geq 5 \), no adaptive scheme was known that was more efficient than the non-adaptive scheme due to Buhrman et al. [5], which uses \( O(tm^{\frac{1}{2t+1}} n) \) bits of space.

(c) If \( t \geq 3 \) and \( 4^t \leq n \), then \( s(m, n, t) \geq \frac{1}{15} m^{\frac{1}{2t+1}} n^{1 - \frac{1}{t+1}} \). For \( n \leq \lg m \), this improves on the lower bound \( s(m, n, 3) = \Omega(\sqrt{mn / \lg m}) \) (valid only for \( n \geq 16 \lg m \)) and for non-adaptive schemes due to Alon and Feige; for small values of \( n \), it also improves on the lower bound \( s(m, n, t) = \Omega(tm^{\frac{1}{2t+1}} n^{1 - \frac{1}{t+1}}) \) due to Buhrman et al. [5].

Key words: Data structures, Bit-probe model, Compression, Bloom filters, Graphs of large girth, Expansion.
1 Introduction

We study the static set membership problem: given a subset $S$ of $[m]$ represent it in memory so that membership queries can be answered using a small number of bit probes (we assume random access is allowed into the memory). Standard solutions to the set membership problem can be examined in this light. (We use $\lg$ to mean logarithm to the base two.)

The characteristic vector: Sets can be represented as a bit-string of length $m$, and membership queries are answered using a single bit probe. However, this representation is not sensitive to the number of elements in the set, which can be much smaller than $m$.

The sorted table: Suppose the set $S$ has $n$ elements. Using the standard representation of elements of the universe in $\lg m$ bits, we may store $S$ in memory as a sorted table of $n \lg m$ bits. Queries can then be answered using binary search taking about $(\lg m)(\lg n)$ bit probes in the worst case.

The static membership problem in the bit probe model (in contrast to the more common cell-probe model) was already studied (in the average case) by Minsky and Papert in their 1969 book *Perceptrons* [11]. More recently, the worst-case space-time trade-off for this problem was considered by Buhrman, Miltersen, Radhakrishnan and Venkatesh [5] and in several subsequent works [2] [9] [12] [13] [14]. The set membership problem for sets where each element is included with probability $p$ was considered by Makhdoumi, Huang, Méard and Polyanskiy [13]; they showed, in particular, that no savings over the characteristic vector can be obtained in this case for non-adaptive schemes with $t = 2$.

To describe the previous results and our contributions formally, we will use the following definitions.

**Definition 1.1.** An $(m, n, s)$-storing scheme is a method for representing a subset of size at most $n$ of a universe of size $m$ as an $s$-bit string. Formally, an $(m, n, s)$-storing scheme is a map $\phi$ from $\binom{[m]}{\leq n}$ to $\{0, 1\}^s$. A deterministic $(m, n, s, t)$-query scheme is a family $\{T_u\}_{u \in [m]}$ of $m$ Boolean decision trees of depth at most $t$. Each internal node in a decision tree is marked with an index between 1 and $s$, indicating the address of a bit in an $s$-bit data structure. For each internal node, there is one outgoing edge labeled “0” and one labeled “1”. The leaf nodes of every tree are marked ‘Yes’ or ‘No’. Such a tree $T_u$ induces a map from $\{0, 1\}^s$ to $\{\text{Yes}, \text{No}\}$; this map will also be referred to as $T_u$. An $(m, n, s)$-storing scheme $\phi$ and an $(m, s, t)$-query scheme $\{T_u\}_{u \in [m]}$ together form an $(m, n, s, t)$-scheme if $\forall S \in \binom{[m]}{\leq n}, \forall u \in [m] : T_u(\phi(S)) = \text{Yes}$ if and only if $u \in S$. Let $s(m, n, t)$ be the minimum $s$ such that there is an $(m, n, s, t)$-scheme.

We say that an $(m, n, s, t)$-scheme is systematic if the value returned by each of its trees $T_u$ is equal to the last bit it reads (interpreting 0 as No/False and 1 as Yes/True).

**Remark 1.2.** Note that this definition describes a non-uniform model and ignores the important issue of uniformly representing the decision trees in the query algorithm. Furthermore, disregarding the fact that in practice memory is organized in words, it instead focuses attention on the fundamental trade-off between the compactness of information representation and the efficiency of information extraction in the context of the set membership problem. The upper bounds derived in this model are not always realistic (they sometimes rely on probabilistic existence arguments); however, lower bounds derived here are generally applicable.

The main focus of Buhrman *et al.* was the randomized version of the above schemes; they showed that membership queries can be answered correctly with probability $1 - \epsilon$ by making just one bit probe into a representation of size $O(n^t/\epsilon \lg m)$ bits. They also showed the following lower and upper bounds for deterministic schemes: $s(m, n, t) = \Omega(tm^{1/2}n^{-1/2})$ valid when

\footnote{In the literature this function is often written as $s(n, m, t)$; we list the parameters in alphabetical order.}
n \leq m^{1-\epsilon} \text{ (for } \epsilon > 0 \text{ and } t \ll \lg m \text{) and (ii) } s(m,n,t) = O(m^{\frac{1}{t+1}}n) \text{ for odd } t \geq 5. \text{ However, Buhrman et al. left open the question of whether a scheme better than the characteristic vector was possible for } t = 2, 3, 4, \text{ and } n \text{ large. Alon and Feige } [2], \text{ in their paper, “On the power of two, three and four probes,” addressed this shortcoming. Our contributions are closely related to theirs.}

For two probes, Alon and Feige [2] show the following.

**Theorem 1.3.** For \( n < \lg m \), \( s(m,n,2) = O\left( mn \lg \left[ \frac{\lg m}{n} \right] / \lg m \right) \).

Thus, \( s(m,n,2) = o(m) \), whenever \( n = o(\lg m) \).

They state:

There are still rather substantial gaps between the upper and lower bounds for the minimum required space in most cases considered here; it will be nice to get tighter estimates. In particular, it will be interesting to decide if there are adaptive \((m,n,s,2)\)-schemes with \( s < m \), for \( n > \sqrt{m}/2 \), and to identify the behavior of the largest \( n = n(m) \) so that there are adaptive \((m,n,s,2)\)-schemes with \( s = o(m) \).

In this paper, we address this by showing the following. (We assume \( m \) is large; all asymptotic claims made below hold for large \( m \).)

**Theorem 1.4** (Result 1). (a) There is a constant \( C > 0 \), such that for all large \( m \), \( s(m,n,2) \leq C \cdot m^{1-\frac{1}{4n+1}} \).

(b) Let \( 4 \leq n \). There is constant \( D > 0 \), such that for all large \( m \), \( s(m,n,2) \geq Dm^{1-\frac{1}{\lfloor \sqrt{m}/4 \rfloor}} \).

For three probes, Alon and Feige [2] show that \( s(m,n,3) = O(m^{\frac{3}{2}}n^{\frac{1}{3}}) \). Their query scheme is adaptive and based on random graphs. We show the following.

**Theorem 1.5** (Result 2), \( s(m,n,3) = O(\sqrt{mn \lg \frac{2m}{n}}) \).

This scheme is adaptive. For small values of \( n \), this result comes close to the lower bound shown below in Theorem 1.8. We further generalize this construction for large values of \( t \).

**Theorem 1.6** (Result 3, non-adaptive schemes). For \( odd \, t \geq 5 \), there is a non-adaptive scheme using \( O(tm^{\frac{2}{t+1}}n^{1-\frac{1}{t+1}} \lg \frac{2m}{n}) \) bits of space.

This improves on a result of Buhrman et al. [3] that states that for odd \( t \geq 5 \) and \( n \leq m^{1-\epsilon} \), there exists a non-adaptive scheme that uses \( O(tm^{\frac{1}{t+1}}n) \) bits of space. These schemes, as well as the non-adaptive scheme for \( t = 4 \) due to Alon and Feige [2], have implications for the problem studied by Makhdoumi et al. [4]; unlike in the case of \( t = 2 \), significant savings are possible if \( t \geq 4 \), even with non-adaptive schemes.

**Theorem 1.7** (Result 4, adaptive schemes). For odd \( t \geq 3 \) and \( t \leq \frac{1}{10} \lg \lg m \) and for \( n \leq m^{1-\epsilon} \) (\( \epsilon > 0 \)), we have \( s(m,n,t) = O(\exp(e^{2t})m^{\frac{2}{t+1}}n^{1-\frac{1}{t+1}} \lg m) \).

We observe that the two-probe lower bound shown above can be used to derive slightly better lower bounds for \( t \geq 3 \).

**Theorem 1.8** (Result 5). If \( 4^t \leq n \), then \( s(m,n,t) \geq \frac{1}{15} m^{\frac{2}{t+1}}(1-\frac{4}{n}) \).

In particular, for \( t = 3 \) and \( n \approx \lg m \), this gives an \( \Omega(\sqrt{m}) \) bound, whereas the previous best bound [5] was of the form \( \Omega(tn^2m^2) \).

\(^2\)We are grateful to Tom Courtade and Ashwin Pananjady for this observation.
What is new, what is old: As stated before, this work is closely related to the paper of Alon and Feige [2]. For two probes, they explicitly modeled their problem using graphs, and translated the high girth of the graphs to their expansion. This allowed them to use Hall’s matching theorem to avoid conflict while allocating memory locations to elements of the universe. We borrow the idea of using graphs of high-girth but we do not reduce the allocation to a matching theorem. Instead, we observe that the constraints in this case can be written down as a 2-SAT expression. Furthermore, if the graph has high girth then this 2-SAT expression must be satisfiable and we will be able to represent our set successfully. Working with 2-SAT instead of the matching problem allows us to show a stronger upper bound. For the lower bound we turn the argument on its head: we show roughly that any valid two-probe scheme must conceal a certain dense graph that avoids small cycles. Standard graph theoretic results (the Moore bound) that relate density and girth then deliver us the lower bound. We believe this approach via 2-SAT offers a better understanding of the connection between two-probe schemes and graphs of high girth.

Our three-probe scheme (Theorem 1.5) is based on the following idea. We must ensure that the data structure returns the answer ‘Yes’ for all query elements in $S$ and ‘No’ for all elements not in $R$ (in the end we would want $R = [m] \setminus S$). If $R$ is small, then this can be arranged using Hall’s theorem, by slightly extending the argument used by Alon and Feige [2] for their three-probe scheme. But we still need take care of large $R$. We notice that the last two probes of a three-probe scheme induce two-probe schemes (precisely how this comes about is not important here). We will show that whenever $R$ is large, there is always an element in it that cannot appear in a short cycle in these two-probe schemes. That is, we may peel this element away, work on the rest, and then make appropriate adjustments to accommodate this element. A form of this argument has been used in the randomized schemes of Buhrman et al. [3]; it appears in in the literature in other contexts, such as Invertible Bloom Lookup Tables [7] and graph based LDPC codes [8]. Our scheme is not explicit, for it relies on random graphs that are suitable for the peeling and matching arguments we employ.

We generalize the above arguments to more than four probes by considering appropriate random query schemes, and identifying properties of the resulting random graph that allow us to find the necessary assignment to correctly represent each possible set.

1.1 Other related work

Some recent work on the bit probe complexity of the set membership problem has focused on sets of small size. The simplest case for which tight bounds are not known is $n = 2$ and $t = 2$: an explicit scheme showing $s(m, 2, 2) = O(m^{2/3})$ was obtained by Radhakrishnan, Raman and Rao [12]. Radhakrishnan, Shah and Shamirghi [14] showed that $s(m, 2, 2) = \Omega(m^{4/7})$. They also considered the complexity $s(m, n, t)$ for $n$ small as $t$ becomes large. These latter results were significantly improved by Lewenstein, Munro, Nicholson and Raman [3], who, in particular, gave a interesting explicit adaptive schemes showing that for $t \geq 3$ we have

$$s(m, 2, t) \leq (2^t - 1)m^{1/(t-2^{t-1})}.$$ 

Thus, the exponent of $m$ in their bound for $n = 2$ is at most $(1 + \frac{4}{2^t})^{-1}$; in contrast, the lower bound of Theorem 1.8 shows that the exponent is at least $\frac{1}{t-1} \geq (1 + \frac{1}{t})^{-1}$ when the set size is much bigger than $4^t \lg m$. Furthermore, for $n \geq 2$, they obtain explicit schemes showing $s(m, n, t) = O(2^t m^{1/(t-\min \{\log n, n-3/2\})}$. 

2 Two-probe upper bound: Proof of Theorem 1.4 (a)

We assume that $n \leq \frac{1}{49} \lg m$, for otherwise, the claim follows from the trivial bound $s(m, n, 2) \leq m$ (taking $C$ large enough).
Our upper bound is based on dense graphs of high girth. The connection between graphs of high girth and two-probe schemes was first noticed by Alon and Feige. They used graphs as templates for their query schemes, and reduced the existence of a corresponding storing scheme to the existence of matchings. Exploiting the expansion properties of small sets in graphs of large girth, they then showed that the necessary matchings do exist. Our query scheme is essentially the same as theirs. However, we sharpen their analysis and observe that the storing problem reduces to a 2-SAT instance. The underlying graph’s high girth this time implies that the 2-SAT instance has the necessary satisfying assignment.

**Definition 2.1** (Query graph). An \((m, s)\)-query graph is a graph \(G\) with three sets of vertices \(A\), \(A_0\) and \(A_1\), each with \(s\) vertices. Each vertex \(v \in A\) has even degree. With each element \(x \in [m]\) we associate a triple \((i(x), i_0(x), i_1(x)) \in A \times A_0 \times A_1\) such that \(\{i(x), i_0(x)\} \subseteq E(G)\). We label both these edges with \(x\), and require that no edge receive more than one label.

An \((m, s)\)-query graph immediately gives rise to a systematic query scheme. The scheme uses three arrays \(A\), \(A_0\) and \(A_1\) each containing \(s\) bits. The query tree \(T_x\) processes the query “Is \(x\) in \(S\)?” as follows: if \(A[i(x)]\) then \(A_1[i_1(x)]\) else \(A_0[i_0(x)]\). We use \(T_G\) to refer to this query scheme. We say that the query scheme \(T_G\) is satisfiable for a set \(S \subseteq [m]\), if there is an assignment to the arrays \(A\), \(A_0\) and \(A_1\) such that all queries of the form “Is \(x\) in \(S\)?” are answered correctly by \(T_G\).

**Proposition 2.2.** If there is a \((m, s)\)-query graph such that the query scheme \(T_G\) is satisfiable for all sets \(S \subseteq [m]\) of size at most \(n\), then \(s(m, n, 2) \leq 3s\).

Our claim will thus follow immediately if we establish the following two lemmas.

**Lemma 2.3.** Let \(G\) be an \((m, s)\)-query graph and \(S \subseteq [m]\). If \(\text{girth}(G) > 4|S|\), then \(T_G\) is satisfiable for \(S\).

**Lemma 2.4.** There is an \((m, O(m^{1+\frac{1}{3n+1}}))\)-query graph with girth more than \(4n\).

**Proof of Lemma 2.3.** Fix a non-empty set \(S\) of size at most \(n\). We need to assign values to the bits of \(A\), \(A_0\) and \(A_1\) so that all queries are answered correctly. Note that since our query scheme is systematic, the only constraints we have are the following.

\(x \in S\):

\[-A[i(x)] \rightarrow A_0[i_0(x)]; \quad (2.1)\]

\[A[i(x)] \rightarrow A_1[i_1(x)]. \quad (2.2)\]

\(y \not\in S\):

\[-A[i(y)] \rightarrow \neg A_0[i_0(y)]; \quad (2.3)\]

\[A[i(y)] \rightarrow \neg A_1[i_1(y)]. \quad (2.4)\]

Let us examine the implications of the above constraints for the variables from the first array: \(A[1], A[2], \ldots, A[s]\). From (2.1) and (2.3), we conclude that whenever \(x \in S\) and \(y \not\in S\) and an edge with label \(x\) and an edge with label \(y\) meet in \(A_0\), we have the constraint

\[A[i(x)] \lor A[i(y)]. \quad (2.5)\]

Similarly, from (2.2) and (2.4), if \(x \in S\) and \(y \not\in S\), and an edge with label \(x\) and edge with label \(y\) meet in \(A_1\), we have the constraint

\[\neg A[i(x)] \lor \neg A[i(y)]. \quad (2.6)\]
Let \( \psi_S(A) \) be the 2-SAT instance on variables \( A[1], \ldots, A[s] \) consisting of all clauses of the form (2.5) and (2.6). It can be verified that a satisfying assignment for \( \psi_S(A) \) can be extended to the other arrays, \( A_0 \) and \( A_1 \), in order to satisfy all constraints in (2.1)-(2.4). So, it suffices to show that \( \psi_S(A) \) is satisfiable.

Each clause of the form \( x \lor y \) is equivalent to the \( \neg x \rightarrow y \) and \( \neg y \rightarrow x \). Furthermore, if \( \psi_S(A) \) is not satisfiable, then there must be a chain of such implications from a literal to its negation (see, e.g., Aspvall, Plass and Tarjan [1]). We now observe that since our graph has large girth, such a chain cannot exist. Suppose the shortest such chain has the form

\[
A[i_0] \rightarrow \neg A[i_1] \rightarrow A[i_2] \rightarrow \cdots \rightarrow A[i_{\ell-1}] \rightarrow \neg A[i_{\ell}],
\]

where \( i_\ell = i_0 \) and otherwise the \( i_j \)'s are distinct (if they were not distinct, there would be a shorter chain). Since each clause of \( \psi_S \) involves at least one element from \( S \), we have \( \ell \leq 2|S| \). The first implication corresponds to a path of length two in \( G \) from \( A[i_0] \) to \( A[i_1] \) via an intermediate vertex in \( A_1 \), the second to a path of length two in \( G \) from \( A[i_1] \) to \( A[i_2] \) via \( A_0 \), and so on; the last implication corresponds to a path of length two from \( A[i_{\ell-1}] \) to \( A[i_{\ell}] \) via \( A_1 \). If \( \ell = 0 \), we have a path in \( G \) from \( A[i_0] \) to itself via \( A_1 \) (consisting of two different edges, one with label in \( S \) and the other with label not in \( S \)), resulting in a cycle of length two—a contradiction. If \( \ell \geq 1 \), the first implication shows that there is a path of length two in \( G \) from \( A[i_0] \) to \( A[i_1] \) via \( A_1 \). The remaining implications show that there is a walk of length \( 2(\ell - 1) \) from \( A[i_1] \) to \( A[i_0] \) that starts with an edge from \( A[i_1] \) to \( A_0 \). Thus, \( A[i_1] \) is in a cycle in \( G \) of length at most \( 2\ell \leq 4|S| \)—a contradiction.

A similar argument shows that the shortest such chain cannot be of the form \( \neg A[i_0] \rightarrow A[i_1] \rightarrow \neg A[i_2] \rightarrow \cdots \rightarrow \neg A[i_{\ell-1}] \rightarrow A[i_0] \).

\[ \square \]

**Proof of Lemma 2.4.** Let \( G = (V_1, V_2, E) \) be a bipartite graph of girth \( g \), with \(|V_1|, |V_2| = s\), and each vertex in \( V_1 \) of even degree. (Later, we will indicate how such graphs \( G \) can be obtained.) Let

\[
V_1 = \{V_1[1], V_1[2], \ldots, V_1[s]\}; \\
V_2 = \{V_2[1], V_2[2], \ldots, V_2[s]\}.
\]

Consider the \((|E|/2, s)\)-query graph \( H \) constructed as follows. \( H \) has three vertex sets \( A, A_0 \) and \( A_1 \). \( A \) will be a copy of \( V_1 \), and \( A_0 \) and \( A_1 \) will be copies of \( V_2 \). Half the edges of \( G \) between \( V_1 \) and \( V_2 \) will be placed between \( A \) and \( A_0 \) and the rest between \( A \) and \( A_1 \). More precisely, suppose the neighbors of \( V_1[i] \) are \( V_2[j_1], V_2[j_2], \ldots, V_2[j_d] \). Then, for \( k = 1, 2, \ldots, d/2 \), we include edges \( \{A[i], A_0[j_{2k}]\} \) and \( \{A[i], A_1[j_{2k-1}]\} \) in \( H \); furthermore, these two edges will have the same label \( x \in [m] \). It is immediate that \( H \) is a \((|E(G)|/2, s)\)-query graph with girth at least \( g \). Thus, it is enough to exhibit a bipartite graph \( G \) with \(|V_1| = |V_2| = O(m^{1-\frac{1}{s+1}})\), \(|E(G)| = 2m\), girth(G) > 4m and all vertices in \( V_1 \) of even degree. We present a probabilistic argument (essentially due to Erdös) to establish the existence of such graphs.

**Dense graphs of large girth:** A probabilistic argument (due to Erdös) establishes the existence of such graphs. Let \( k = 4n \leq \frac{1}{10} \lg m \), and consider the following random bipartite graph \( G \) on vertex sets \( V_1 \) and \( V_2 \), each with \( s = \left\lceil 4m^{1-\frac{1}{s+1}} \right\rceil \) vertices each. Let \( d \) be the largest even number at most \( s^{\frac{k}{2}} \), thus \( s^{\frac{k}{2}} > d > s^{\frac{k}{2}} - 2 \geq 2 \). For each vertex \( v \in V_1 \), we assign \( d \) distinct neighbors from \( V_2 \). Then, the expected number of short cycles in \( G \) is at most

\[
\sum_{\ell=2}^{k/2} \binom{s^{2\ell}}{2\ell} \left( \frac{d}{s} \right)^{2\ell} \leq \frac{1}{4} \sum_{\ell=2}^{k/2} d^{2\ell} \leq \frac{d^k}{4} \sum_{\ell=0}^{k/2-2} \frac{1}{\ell!} \leq \frac{d^k}{2} \leq \frac{s}{2}.
\]
Thus, there is such a graph with at most $\frac{1}{3}$ short cycles. Consider each such cycle one by one, and for each pick one of its vertices in $V_1$ and delete both edges from the cycle incident on it. Then, the average degree in $V_1$ is at least $d - 1 > s^{1/k} - 3 \geq \frac{1}{2}s^{1/k}$, and $|E(G)| \geq \frac{1}{2}s^{1/k} \geq 2m$. Using this in the above construction we obtain $s(m, n, 2) \leq \left\lceil 4m^{1-\frac{1}{2m+1}} \right\rceil$.

**Explicit schemes:** Two-probe schemes can be obtained more explicitly using the following construction of graphs of large girth.

**Proposition 2.5** (see Proposition 2.1 of F. LAZEBNIK, V. A. USTIMENKO, AND A. J. WOLDAR [10]). Let $q$ be a prime power and $k \geq 1$ be an odd integer. Then, there is graph $D(k, q)$ that is

(i) $q$-regular and of order $2q^k$.

(ii) $D(k, q)$ has girth at least $k + 5$

Now assume $n \leq \frac{1}{3}(\log m)^{1/3}$. Set $k = 4n - 3$, and choose $q = 2^r$ such that $(2m)^{1/(k+1)} \leq q < 2(2m)^{1/(k+1)}$. Then, $D(k, q)$ is a bipartite graph on vertex sets $(V_1, V_2)$, with girth at least $4n + 2$, at least $2m$ edges and

$$|V_1| = |V_2| = s = q^k \leq (2(2m))^{1/(k+1)} \leq 2^{k+1}m^{-1/(k+1)}.$$ 

If $n \leq \frac{1}{3}(\log m)^{1/3}$, we have $(k + 1)^2(k + 2) \leq \log m$, and we have

$$s \leq m^{1/((k+1)(k+2))}m^{-1/(k+1)} = m^{-1/(k+2)} \leq m^{-1/(4n+1)}.$$

3 Two-probe lower bound: Proof of Theorem 1.4 (b)

Since $s(m, n, 2)$ is an non-decreasing function of $n$, it is enough to establish the claim for $n \leq \lg m$.

**Proposition 3.1.** If there is an $(m, n, s, t)$-scheme, then there is a systematic $(m, n, 2s, t)$-scheme.

So, from now on, we will assume that our schemes are systematic.

**Definition 3.2.** (Bipartite graph $H_\Phi$, pseudo-graph $G_\Phi$) Fix a systematic $(m, n, s, 2)$-scheme $\Phi$. We will associate the following bipartite graph $H_\Phi$ with such a scheme. There will be two sets of vertices, each with $s$ elements: $A_0$ and $A_1$. Each edge of $H_\Phi$ will have a color and a label. We include the edge $\{A_0[j], A_1[k]\}$ with label $x$ and color $i$, if on query “Is $x$ in $S$?” the first probe is made to location $i$, and if it returns 0, the second probe is made to location $j$ and if the first probe returns a 1, the second probe is made to location $k$. $H_\Phi$ thus has $2s$ vertices and $m$ edges, which are colored using $s$ colors.

The pseudo-graph $G_\Phi$ is a bipartite graph obtained from $H_\Phi$ as follows. $G_\Phi$ and $H_\Phi$ have the same set of vertices. The edges of $G_\Phi$ are obtained as follows. Consider the edges of color $\alpha$ in $H_\Phi$. We partition these edges into ordered pairs (excluding one edge if the number of edges of this color is odd). For each such pair we include a pseudo-edge in $G_\Phi$ as follows. Let $(e, e')$ be one such pair; suppose $e = \{u, v\}$ has label $x$, $e' = \{u', v'\}$ has label $x'$, $u, v, u', v' \in A_0$ and $v, v' \in A_1$. Then, in $G_\Phi$ we include the edge $\{u, v'\}$ with label $\{(u, x), (v', x')\}$ (we do not include the edge $\{u', v\}$). We repeat this for all colors $\alpha$. Thus $G_\Phi$ is a bipartite graph with at least $(m - s)/2$ edges. For a set of edges $P$ of $G_\Phi$, let $\text{lab}(P) \subseteq [m]$ be the set of elements of the universe that appear in the label of some edge in $P$.  

6
3.1 Forcing

Lemma 3.3 (Forcing lemma). Let $G_{\Phi}$ be a pseudo-graph associated with a systematic scheme $\Phi$. Let $C$ be a cycle in $G_{\Phi}$ starting at vertex $v$. Let $b \in \{0, 1\}$. Then there are disjoint subsets $S_0, S_1 \subseteq \text{lab}(C)$, each with at most $|C| + 1$ elements, such that in any representation under $\Phi$ of a set $S$ such that $S_1 \subseteq S \subseteq S_0$, location $v$ must be assigned $b$. Further, if $b = 0$ then $|S_1| = |C| - 1$, and if $b = 1$, then $|S_1| = |C| + 1$.

Proof. First, consider the claim with $b = 1$. Suppose the cycle is

$$v = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_{k-1}} v_{k-1} \xrightarrow{e_k} v_k = v_0,$$

and the pseudo-edge $e_i$ has label $\{(v_{i-1}, a_i), (v_i, b_i)\}$. Let $S_1 = \{a_1, a_2, \ldots, a_{k-1}, a_k, b_k\}$ (it has $k + 1$ elements) and $S_0 = \{b_1, b_2, \ldots, b_{k-1}\}$. We claim that if the scheme $\Phi$ represents a set $S$ such that $S_1 \subseteq S \subseteq S_0$, then location $v$ must be assigned 1. Suppose location $v$ is assigned 0. Recall that the scheme is systematic. Since $a_1 \in S$ and $b_1 \notin S$, we conclude from the definition of the pseudo-edge $e_1$ that $v_1$ must also be assigned 0. Using the subsequent edges in the cycle, we conclude that the locations $v_0, \ldots, v_{k-1}$ must all be assigned 0. Now, however, $a_k, b_k \in S$, so 1 must be assigned to location $v_k = v$—a contradiction, for we assumed that $v$ was assigned 0. This proves our claim for $b = 1$. For $b = 0$, we take $S_1 = \{b_1, b_2, \ldots, b_{k-1}\}$ (it has $k - 1$ elements) and $S_0 = \{a_1, a_2, \ldots, a_{k-1}, a_k, b_k\}$, and reason as before.

Corollary 3.4. If a scheme stores sets of size up to $2k$, then the pseudo-graph associated with the scheme cannot have two edge-disjoint cycles of length at most $k$ each that have a vertex in common.

Proof. Suppose there are two such edge-disjoint cycles, $C_1$ and $C_2$, both starting at $v$. We apply Lemma 3.3 with $b = 0$ and obtain sets $S_0, S_1 \subseteq \text{lab}(C_1)$. Next we apply Lemma 3.3 with $b = 1$ and obtain sets $T_0, T_1 \subseteq \text{lab}(C_2)$. Now, consider $S = S_1 \cup T_1$, a set of size at most $2k$. When the scheme stores the set $S$, then location $v$ must be assigned a 0 (because $S_1 \subseteq S \subseteq S_0$), and also 1 (because $T_1 \subseteq S \subseteq T_0$)—a contradiction.

3.2 Calculation

Our lower bound will use Corollary 3.4 as follows. We will show that if a scheme uses small space to represent sets of size up to $n$ from a universe of size $m$, then its pseudo-graph must be dense (for it must accommodate about $m/2$ edges). In such a dense graph there must be short cycles. In fact, if $m \gg s^{1-4/n}$, then we can ensure that there are two cycles of length at most $n/2$ each that have a vertex in common. But, then Corollary 3.4 states that such a scheme does not exist.

To make the above argument precise, we will use the following proposition, which is a consequence of a theorem of Alon, Hoory and Linial [3] (see also Ajesh Babu and Radhakrishnan [6]).

Proposition 3.5. Let $G$ be a bipartite graph average degree $d \geq 2$, and girth greater than $\left\lfloor \frac{n}{2} \right\rfloor$ (for positive integer $n \geq 4$). Then, $d \leq (|V(G)|/2)^{1/\left\lfloor \frac{n}{2} \right\rfloor} + 1$.

Proof. Let $n = 4p + q$, for $p = \left\lfloor \frac{n}{4} \right\rfloor$ and $q$ such that $q \in \{0, 1, 2, 3\}$. From our assumption, the girth of $G$ is greater than $\left\lfloor \frac{n}{2} \right\rfloor \geq 2p$. Since $G$ is bipartite, $G$ has girth at least $2(p+1)$. The result of Alon, Hoory and Linial then immediately implies that $|V(G)| \geq 2(d-1)^p$. Since $p = \left\lfloor n/4 \right\rfloor$, our claim follows from this.
Corollary 3.6. Suppose $G$ is a bipartite graph with $|V(G)| \geq \left\lceil \frac{n}{2} \right\rceil$ and $|E(G)| > \left(\frac{|V(G)|}{2}\right)^{1+\frac{1}{\lfloor n/4 \rfloor}} + \frac{3}{2}|V(G)|$ and $(|V(G)|/2)^{\lfloor n/4 \rfloor} \geq 2$. Then, $G$ has two edge disjoint cycles of length at most $\frac{n}{2}$ that have at least one vertex in common.

Proof. If $|E(G)| > \left(\frac{|V(G)|}{2}\right)^{1+\frac{1}{\lfloor n/4 \rfloor}} + |V(G)|/2$, then the average degree is at least $\left(\frac{|V(G)|}{2}\right)^{1+\frac{1}{\lfloor n/4 \rfloor}} + 1 \geq 2$. By the proposition above, $G$ has a cycle of length $\ell_1 \leq \left\lceil \frac{n}{2} \right\rceil$. Remove this cycle, and consider the remaining graph, which has more than $(2s)^{1+\frac{1}{\lfloor n/4 \rfloor}} + 3|V(G)|/2 - \ell_1$ edges. Again, we find a cycle of length at most $\ell_2 \leq \left\lceil \frac{n}{2} \right\rceil$. We may continue in this way, finding cycles of length $\ell_i$ ($i = 1, 2, \ldots$, until the sum of the $\ell_i$’s exceeds $|V(G)|$. At that point, two of the cycles we found must intersect (for there are only $|V(G)|$ vertices).

Proof. of Theorem 4.4 (b) Fix an $(m, n, s, 2)$-scheme. Assume $m$ is large. Using Proposition 3.1 we obtain a systematic $(m, n, 2s, 2)$-scheme, say $\Phi$, and consider the corresponding pseudo-graph $G_\Phi$ (note $|V(G_\Phi)| = 4s$ and $|E(G)| \geq (m-s)/2$). The lower bound of Buhrman et al. [5] implies that $s \geq \sqrt{m}$; thus $(2s)^{1+\frac{1}{\lfloor n/4 \rfloor}} \geq m^{\frac{3}{4}} \geq 2$; also since we assume $n \leq \lg m$, we have $4s \geq \left\lceil \frac{n}{4} \right\rceil$. By applying Corollary 4.2 and Corollary 3.1 to $G_\Phi$, we conclude that

$$\frac{m-s}{2} \leq |E(G_\Phi)| \leq (2s)^{1+\frac{1}{\lfloor n/4 \rfloor}} + 6s.$$

Thus (recall from above that $(2s)^{1+\frac{1}{\lfloor n/4 \rfloor}} \geq 2$),

$$m \leq 13s + 2(2s)^{1+\frac{1}{\lfloor n/4 \rfloor}} \leq (7s)^{1+\frac{1}{\lfloor n/4 \rfloor}} + (4s)^{1+\frac{1}{\lfloor n/4 \rfloor}} \leq (11s)^{1+\frac{1}{\lfloor n/4 \rfloor}}.$$

By raising both sides to the power $1 - \frac{1}{\lfloor n/4 \rfloor}$ and rearranging the inequality, we obtain $s \geq \frac{1}{11} m^{1-\frac{1}{\lfloor n/4 \rfloor}}$.

4 Three-probe upper bound: Proof of Theorem 1.5

As in the case of two probes, our three-probe scheme will be based on the existence of certain graphs. The framework we use will be general and also applicable to schemes that make $t \geq 4$ probes. We will present the general framework first, and specialize it to $t = 3$ when we describe our proof.

Definition 4.1. An $(m, s, t)$-graph is a bipartite graph $G$ with vertex sets $U = [m]$ and $V$ ($|V| = (2^t - 1)s$). $V$ is partitioned into $2^t - 1$ disjoint sets: $A_0, A_1, A_2, \ldots, A_{2^t}$, each $A_\sigma$ for each $\sigma \in \{0, 1\}^{t-1}$, each $A_\sigma$ has $s$ vertices. Between each $u \in U$ and each $A_\sigma$ there is exactly one edge. For $i = 1, \ldots, t$, the subgraph of $G$ induced by $U$ and $V_i = \bigcup_{\sigma : |\sigma| = i-1} A_\sigma$ will be referred to as $G_i$. An $(m, s, t)$-graph naturally gives rise to a systematic $(m, (2^t - 1)s)$-query scheme $T_G$ as follows. We view the memory (an array $L$ of $(2^t - 1)s$ bits) as being indexed by vertices in $V$. For query element $u \in U$, if the first $i - 1$ probes resulted in values $\sigma \in \{0, 1\}^{t-1}$, then the $i$-th probe is made to the location indexed by the unique neighbor of $u$ in $A_\sigma$. In particular, the $i$-th probe is made at a location in $V_i$.

We say that the query scheme $T_G$ is satisfiable for a set $S \subseteq [m]$, if there is an assignment to the memory locations $(L[v] : v \in V)$, such that $T_G$ correctly answers all queries of the form “Is $x$ in $S$?”.

We now restrict attention to $t = 3$ probes. First, we identify an appropriate property of the underlying $(m, s, 3)$-graph $G$ that guarantees that the $T_G$ is satisfiable for all sets $S$ of size at most $n$. We then show that such a graph does exist for some $s = O\left(\sqrt{mn \ln \frac{2m}{n}}\right)$.

Definition 4.2 (Admissible graph). We say that an $(m, s, 3)$-graph $G$ is admissible for sets of size at most $n$, if
Lemma 4.3. If an \((m, s, 3)\)-graph \(G\) is admissible for sets of size at most \(n\), then the \((m, 7s, 3)\)-query scheme \(\mathcal{T}_G\) is satisfiable for \((S, [m]\setminus S)\) for every \(S\) of size at most \(n\).

Lemma 4.4. There is an \((m, s, 3)\)-graph with \(s = \left\lceil 500 \sqrt{mn \log \frac{2m}{n}} \right\rceil\) that is admissible for every set \(S \subseteq [m]\) of size at most \(n\).

Proof of Lemma 4.3. Fix an \((m, s, 3)\)-graph \(G\) that is admissible for sets of size at most \(n\). Thus, \(G\) satisfies (P1) and (P2) above. Fix a set \(S\) of size at most \(n\). Suppose \(\mathcal{T}_G\) is not satisfiable for \(S\). Then, there is a minimal set \(T \subseteq [m]\setminus S\) such that \(\mathcal{T}_G\) fails to correctly answer queries for all \(u \in S \cup T\). We have two cases.

1. \(|S \cup T| \leq n + \left\lceil n \log \frac{2m}{n} \right\rceil\): We use an idea from Alon and Feige [1]. From (P1) and Hall’s theorem, we may assign to each element \(u \in S \cup T\) a set \(V_u \subseteq \Gamma_G(u)\) such that (i) \(|V_u| = 5\) and (ii) the \(V_u\)’s are disjoint. It can be verified that in a binary decision tree of depth 3 and any value \(b \in \{0, 1\}\), given any set of FIVE nodes, values can be assigned to those nodes to ensure that the tree returns the value \(b\). Thus, there is an assignment (fixing five bits for each \(u \in S \cup T\)) so that \(\mathcal{T}_G\) returns the correct answer for all \(u \in S \cup T\) — contradicting our choice of \(T\).

2. \(|S \cup T| > n + \left\lceil n \log \frac{2m}{n} \right\rceil\): From property (P2), we conclude that there is a \(y \in T\) such that one of the following holds.

   - (a) \(\Gamma_{G_3}(y) \cap \Gamma_{G_3}(S) = \emptyset\) or
   - (b) \(|\Gamma_{G_3}(y) \cap \Gamma_{G_3}(S)| = 1\) AND \(|\Gamma_{G_1 \cup G_2}(y) \cap \Gamma_{G_1 \cup G_2}((T \cup S) \setminus \{y\})| \leq 1\).

By the minimality of \(T\), there is an assignment \(\sigma \in \{0, 1\}^V\) so that \(\mathcal{T}_G\) correctly answers queries for all elements \(u \in S \cup T\setminus \{y\}\). In case (a), modify \(\sigma\) so that all locations in \(\Gamma_{G_3}(y)\) (the locations that are probed in the third step by \(T_y\)) are 0. For the new assignment \(\sigma'\), the query for \(y\) is clearly answered correctly; the operation of \(T_u\) for \(u \in S \cup T\setminus \{y\}\) is identical in \(\sigma\) and \(\sigma'\). This again contradicts the choice of \(T\).

In case (b), we again start with an assignment \(\sigma \in \{0, 1\}^V\) so that \(\mathcal{T}_G\) correctly answers queries for all elements \(u \in S \cup T\setminus \{y\}\). Now, to accommodate \(y\), we will modify \(\sigma\) to \(\sigma'\), by making changes to locations in \(V_1, V_2\) and \(V_3\). We have exactly one \(\ell \in V_3\) such that \(\ell \in \Gamma_{G_3}(y) \cap \Gamma_{G_3}(S)\); in \(\sigma'\) all locations in \(\Gamma_{G_3}(y)\) other than \(\ell\) are set to 0. Furthermore, at least two of the three locations in \(\Gamma_{G_1 \cup G_2}(y)\) are outside \(\Gamma_{G_1 \cup G_2}(S \cup T\setminus \{y\})\); so we may modify them without affecting the operation of any decision tree \(T_u\) for \(u \in (S \cup T)\setminus \{y\}\). In \(\sigma'\), we assign these values appropriately so that the third probe of \(T_y\) is not \(\ell\). We have thus ensured that under assignment \(\sigma'\), queries for all \(u \in S \cup T\) are answered correctly — again contradicting the choice of \(T\).

Proof of Lemma 4.4. We show that a suitable random \((m, s, 3)\)-graph \(G\) is admissible with positive probability, for \(s = \left\lceil 500 \sqrt{mn \log \frac{2m}{n}} \right\rceil\). The graph \(G\) is constructed as follows. Recall that \(V = \bigcup_{z \in \{0, 1\}^3} A_z\). For each \(u \in U\), one neighbor is chosen uniformly and independently from each \(A_z\).
(P1) holds. If (P1) fails, then for some non-empty \( W \subseteq U \), \((|W| \leq n + [n \log \frac{2m}{n}]\)) we have \(|\Gamma_G(W)| \leq 5|W| - 1\). Fix a set \( W \) of size \( r \geq 1 \) and \( L \subseteq V \) of size at most \( 5r - 1 \). Let \( L \) have \( \ell_z \) elements in \( A_z \). Then,
\[
\Pr[\Gamma_G(W) \subseteq L] \leq \prod_z \left( \frac{\ell_z}{|A_z|} \right)^r \leq \left( \frac{5r - 1}{7s} \right)^7 r.
\]
If \( s \geq 500\sqrt{mn \log \frac{2m}{n}} \), then we conclude, using the union bound over choices of \( W \) and \( L \), that the probability that (P1) fails is at most
\[
\sum_{r=1}^{n + [n \log \frac{2m}{n}]} \binom{m}{r} \left( \frac{7s}{5r - 1} \right)^r \left( \frac{5r - 1}{7s} \right)^7 r \\
\leq \frac{n + [n \log \frac{2m}{n}]}{s} \binom{cm}{r} \left( \frac{7es}{5r - 1} \right)^r \left( \frac{5r - 1}{7s} \right)^7 r \\
\leq \frac{n + [n \log \frac{2m}{n}]}{s} \binom{5r}{7es} \left( \frac{5^2 e^r m r}{7^2 s^2} \right)^r \\
\leq \frac{1}{3} \text{ (if } s \geq 500\sqrt{mn \log \frac{2m}{n}}\).
\]

(P2) holds. For (P2) to fail, there must be disjoint sets \( S, R \subseteq U \), where \(|S| = n' \leq n\), \(|R| = r \geq [n \log \frac{2m}{n}]\) for which the condition specified in Definition 4.2 does not hold. Then, \( R = R_1 \cup R_2 \), where \( R_1 = \{ y \in R : |\Gamma_G(y) \cap \Gamma_G(S)| = 1 \} \) and \( R_2 = \{ y \in R : |\Gamma_G(y) \cap \Gamma_G(S)| \geq 2 \} \); let \( r_1 = |R_1| \) and \( r_2 = |R_2| \). Furthermore, for \( y \in R_1 \) we have \(|\Gamma_{G_1 \cup G_2}(y) \cap \Gamma_{G_1 \cup G_2}(R \cup S \setminus \{ y \})| \geq 2\). This implies that \(|\Gamma_{G_1 \cup G_2}(R \cup S)| \leq 3(n' + r) - r_1\).
Fix \( R_1 \subseteq R \), \( R_2 = R \setminus R_1 \) and define events \( \mathcal{E}_1, \mathcal{E}_2 \) and \( \mathcal{E}_* \) as follows.
\[
\mathcal{E}_1 \equiv \forall y \in R_1 : |\Gamma_G(y) \cap \Gamma_G(S)| = 1; \\
\mathcal{E}_2 \equiv \forall y \in R_2 : |\Gamma_G(y) \cap \Gamma_G(S)| \geq 2; \\
\mathcal{E}_* \equiv |\Gamma_{G_1 \cup G_2}(R \cup S)| \leq 3(n' + r) - r_1.
\]
By the union bound, the probability that (P2) fails is bounded by the sum of \( \Pr[\mathcal{E}_1] \Pr[\mathcal{E}_2] \Pr[\mathcal{E}_*] \) taken over all valid choices of \( S, R, R_1 \) and \( R_2 \). We have \( \Pr[\mathcal{E}_1] \leq \left( \frac{4n}{s} \right)^{r_1} \) and \( \Pr[\mathcal{E}_2] \leq \left( \frac{4n}{s} \right)^{2r_2} \). To bound \( \Pr[\mathcal{E}_*] \), we proceed as we did above for (P1). We have
\[
\Pr[\mathcal{E}_*] \leq \left( \frac{3s}{3(n' + r) - r_1} \right)^{S \cup \mathcal{R}} \left( \frac{\ell_0 \ell_1}{s} \right)^{|S \cup \mathcal{R}|} \leq \left( \frac{3es}{3(n' + r) - r_1} \right)^{3(n'+r)-r_1} \left( \frac{3(n'+r) - r_1}{3s} \right)^{3(n'+r)} \\
\leq \exp(3(n'+r) - r_1) \left( \frac{3(n'+r) - r_1}{3s} \right)^{r_1}.
\]
Thus,
\[
\Pr[\mathcal{E}_1] \Pr[\mathcal{E}_2] \Pr[\mathcal{E}_*] \leq \left( \frac{4n}{s} \right)^{r_1} \left( \frac{3n}{s} \right)^{2r_2} \exp(3(n'+r) - r_1) \left( \frac{3(n'+r) - r_1}{3s} \right)^{r_1} \\
\leq \left( \frac{9e^6}{s} \right)^{r_1} \left( \frac{n'^{r_2} (n + r)^{r_1}}{s^{2r}} \right).
\]
Using the union bound, we conclude that

\[
\Pr[(P2) \text{ fails}] \\
\leq \sum_{n'=1}^{n} \sum_{r \geq \left[ \frac{n \lg 2m}{m} \right]} \left( \frac{m}{n'} \right) \left( \frac{m}{r} \right) (9e^6)^r \frac{n^r}{s^r} \sum_{r_1=0}^{r} \left( \frac{r}{r_1} \right) n^{r_2}(n+r)^{r_1}
\]

\[
\leq \sum_{n'=1}^{n} \sum_{r \geq \left[ \frac{n \lg 2m}{m} \right]} \left( \frac{em}{n'} \right) \left( \frac{em}{r} \right) (9e^6)^r \frac{n^r}{s^r} (2n+r)^r
\]

\[
\leq \sum_{n'=1}^{n} \sum_{r \geq \left[ \frac{n \lg 2m}{m} \right]} \left[ \frac{m^{1+n'}}{r} n^{1-\frac{n'}{m}} (9e^6) (2n+r) \right]^r
\]

\[
\leq \sum_{n'=1}^{n} \sum_{r \geq \left[ \frac{n \lg 2m}{m} \right]} \left[ \frac{3e^8 mn^2}{s^2} \right]^r
\]

\[
\leq \frac{1}{3} \text{ (for } s \geq \left[ \frac{500 \sqrt{mn \lg 2m}}{m} \right] \text{ and } m \text{ large}).
\]

Thus, with probability at least \( \frac{4}{5} \) the random graph \( G \) is admissible.

\[\square\]

5 Lower bound: Proof of Theorem 1.8

Our theorem follows immediately from the following lemma.

**Lemma 5.1.** Suppose \( t \geq 2 \), and \( s, m \) and \( n \) are such that (i) \( 4^t \leq n \leq m^{\frac{4^t}{4^t-t}} \) and (ii) \( s \leq \frac{1}{15} m^{\frac{1}{4^t-t}(1-\frac{4^t}{m})} \). If there is a \( t \)-probe scheme on a universe of size \( m \) that uses space at most \( s \), then there are disjoint sets \( S \) and \( T \) of size at most \( n \) each such that for every assignment to the memory, some query in \( S \) is answered with a ‘No’ or some query in \( T \) is answered with a ‘Yes’.

**Proof.** For \( t = 2 \), the claim is established in the proof of Theorem 1.4 (see the last line of the proof). We will use induction on \( t \) to generalize this claim to larger values of \( t \). Assume the claim is true for \( t = k - 1 \) and we wish to show that it holds for \( t = k \). Fix \( m, n \) and a \( k \)-probe scheme that satisfy our assumptions. We now show how the sets \( S \) and \( T \) are obtained. There is a cell to which at least \( \frac{m}{s} \) of the elements make their first probe: call this set of elements \( U' \). By fixing the value of this cell at 0, we obtain a \((k-1)\)-probe scheme for the universe \( U' \). We will verify that the assumptions needed for induction are satisfied for this scheme. We conclude by induction that there are disjoint sets \( S_0, T_0 \subseteq U' \) each of size at most \( \frac{n}{2} \). Let \( U'' = U' \setminus (S_0 \cup T_0) \). Now, assume that the cell has value 1, and apply induction to the resulting \((k-1)\)-probe scheme (for the universe \( U'' \)) to obtain sets \( S_1 \) and \( T_1 \) of size at most \( \frac{n}{2} \). Our claim for \( t = k \) then follows immediately by taking \( S = S_0 \cup S_1 \) and \( T = T_0 \cup T_1 \).

It remains to verify that the assumptions (i) and (ii) needed for the induction hypothesis in fact do hold. Since \( |U'| \geq |U''| \), it is enough to verify the conditions for \( U'' \). Now \( |U'| \geq \frac{n}{2} \geq 15m^{\frac{k-2}{4^t-t}} \geq 15m^{2(k-2)} \geq 2n \). Thus, \( m' = |U''| \geq |U'| - n \geq \frac{m}{2} \). We need to find sets of size at most \( n' = \left\lfloor \frac{n}{2} \right\rfloor \). Clearly \( n' \geq \frac{n}{4} = 4^{k-1} \), so condition (i) holds. Also, since \( m' \geq \frac{m}{2} \) and \( n' \geq \frac{n}{4} \), we have \( s \leq \frac{1}{15} m^{\frac{1}{4^t-t}(1-\frac{4^{k-1}}{m})} \), so condition (ii) holds. \[\square\]
6 General upper bound: non-adaptive (Theorem 1.6)

**Definition 6.1.** A non-adaptive \((m, s, t)\)-graph is a bipartite graph \(G\) with vertex sets \(U = [m]\) and \(V (|V| = ts)\). \(V\) is partitioned into \(t\) disjoint sets: \(V_1, \ldots, V_t\); each \(V_i\) has \(s\) vertices. Every \(u \in U\) has a unique neighbour in each \(V_i\). A non-adaptive \((m, s, t)\)-graph naturally gives rise to a non-adaptive \((m, ts, t)\)-query scheme \(T_G\) as follows. We view the memory (an array \(L\) of \(ts\) bits) to be indexed by vertices in \(V\). On receiving the query "Is \(u\) in \(S\)?", we answer "Yes" iff the Majority of the locations in the neighbourhood of \(u\) contain a 1. We say that the query scheme \(T_G\) is satisfiable for a set \(S \subseteq [m]\), if there is an assignment to the memory locations \((L[v] : v \in V)\), such that \(T_G\) correctly answers all queries of the form "Is \(x\) in \(S\)?".

We now restrict attention to odd \(t \geq 5\). First, we identify an appropriate property of the underlying non-adaptive \((m, s, t)\)-graph \(G\) that guarantees that \(T_G\) is satisfiable for all sets \(S\) of size at most \(n\). We then show that such a graph exists for some \(s = O(m^{\frac{1}{2t}}n^{1 - \frac{3}{2r}} \lg \frac{2m}{n})\).

**Definition 6.2 (Non-adaptive admissible graph).** We say that a non-adaptive \((m, s, t)\)-graph \(G\) is admissible for sets of size at most \(n\) if the following two properties hold:

(P1) \(\forall R \subseteq [m] (|R| \leq n + \lceil 2n \lg \frac{2m}{n} \rceil) : |\Gamma_G(R)| \geq \frac{t+1}{2} |R|\), where \(\Gamma_G(R)\) is the set of neighbors of \(R\) in \(G\).

(P2) \(\forall S \subseteq [m] (|S| = n) : |T_S| \leq \lceil 2n \lg \frac{2m}{n} \rceil\), where \(T_S = \{y \in [m] \setminus S : |\Gamma_G(y) \cap \Gamma_G(S)| \geq \frac{t+1}{2}\}\).

Our theorem will follow from the following claims.

**Lemma 6.3.** If a non-adaptive \((m, s, t)\)-graph \(G\) is admissible for sets of size at most \(n\), then the non-adaptive \((m, ts, t)\)-query scheme \(T_G\) is satisfiable for every set \(S\) of size at most \(n\).

**Lemma 6.4.** There is a non-adaptive \((m, s, t)\)-graph, with \(s = O(m^{\frac{1}{2t}}n^{1 - \frac{3}{2r}} \lg \frac{2m}{n})\), that is admissible for every set \(S \subseteq [m]\) of size at most \(n\).

**Proof of Lemma 6.3.** Fix an admissible graph \(G\). Thus, \(G\) satisfies (P1) and (P2) above. Fix a set \(S \subseteq [m]\) of size at most \(n\). We will show that there is a 0-1 assignment to the memory such that all queries are answered correctly by \(T_G\).

Let \(S' \subseteq [m]\) be such that \(S \subseteq S'\) and \(|S'| = n\). From (P2), we know \(|T_{S'}| \leq \lceil 2n \lg \frac{2m}{n} \rceil\). Hence, \(|S' \cup T_{S'}| \leq n + \lceil 2n \lg \frac{2m}{n} \rceil\). From (P1) and Hall’s theorem, we may assign to each element \(u \in S' \cup T_{S'}\) a set \(A_u \subseteq V\) such that (i) \(|A_u| = \frac{t+1}{2}\) and (ii) the \(A_u\)’s are disjoint. For each \(u \in S \subseteq S'\), we assign the value 1 to all locations in \(A_u\). For each \(u \in (S' \cup T_{S'}) \setminus S\), we assign the value 0 to all locations in \(A_u\). Since \(\frac{t+1}{2} > \frac{r}{2}\), all queries for \(u \in S' \cup T_{S'}\) are answered correctly.

Assign 0 to all locations in \(\Gamma_G([m]\setminus(S' \cup T_{S'}))\). For \(y \in [m]\setminus(S' \cup T_{S'})\), \(|\Gamma_G(y) \cap \Gamma_G(S)| \leq \frac{t+1}{2}\). As a result, queries for elements in \([m]\setminus(S' \cup T_{S'})\) are answered correctly, as the majority evaluates to 0 for each one of them.

**Proof of Lemma 6.4.** In the following, set

\[ s = \left\lceil 60m^{\frac{1}{2t}}n^{1 - \frac{3}{2r}} \lg \frac{2m}{n} \right\rceil. \]

We show that a suitable random non-adaptive \((m, s, t)\)-graph \(G\) is admissible for sets of size at most \(n\) with positive probability. The graph \(G\) is constructed as follows. Recall that \(V = \bigcup_i V_i\). For each \(u \in U\), one neighbor is chosen uniformly and independently in each \(V_i\).

(P1) holds. If (P1) fails, then for some non-empty \(W \subseteq U\), \(|W| \leq n + \lceil 2n \lg \frac{2m}{n} \rceil\), we have \(|\Gamma_G(W)| \leq \frac{t+1}{2} |W| - 1\). Fix a set \(W\) of size \(r \geq 1\) and \(L \subseteq V\) of size \(\frac{t+1}{2} r - 1\). Let \(L\) have \(\ell_i\) elements in \(V_i\); thus, \(\sum_i \ell_i = \frac{t+1}{2} r - 1\). Then,

\[ \Pr[|\Gamma_G(W)| \leq L] \leq \prod_{i=1}^t \left( \frac{\ell_i}{|V_i|} \right)^r \leq \left( \frac{\frac{t+1}{2} r - 1}{ts} \right)^r, \]
where the last inequality is a consequence of \( GM \leq AM \). We conclude, using the union bound over choices of \( W \) and \( L \), that (P1) fails with probability at most
\[
\sum_{r=1}^{n+\left\lceil \frac{2n \lg \frac{2m}{n}}{m} \right\rceil} \binom{m}{r} \left( \frac{ts^{t+1}}{r^{t+1}r-1} \right)^{t+1} \leq 1 - \frac{10}{2n^{n+1}}.
\]

To conclude that \( |T_S| \) is bounded with high probability, we will use the following version of Chernoff bound: if \( X = \sum_{i=1}^{N} X_i \), where each random variable \( X_i \in \{0,1\} \) independently, then if \( \gamma > 2eE[X] \), then \( \Pr[X > \gamma] \leq 2^{-\gamma} \). Then, for all large \( m \),
\[
\Pr[|T_S| > 2n \lg \frac{2m}{n}] \leq 2^{-2n \lg \frac{2m}{n}}.
\]

Using the union bound, we conclude that
\[
\Pr[(P2) \text{ fails}] \leq \left( \frac{em}{n} \right)^n 2^{-2n \lg \frac{2m}{n}} \leq \frac{1}{3}.
\]

Thus, with probability at least \( \frac{1}{3} \) the random graph \( G \) is admissible.

\[\square\]

7 General upper bound: adaptive (Theorem 1.7)

In order to show that \( s(m,n,t) \) is small, we will exhibit efficient adaptive schemes to store sets of size exactly \( n \). This will imply our bound (where we allow sets of size at most \( n \)) because we may pad the universe with \( n \) additional elements, and extend \( S \) (\( |S| \leq n \)) by adding \( n - |S| \) additional elements, to get a subset of size exactly \( n \) in a universe of size \( m + n \leq 2m \).

**Definition 7.1.** An adaptive \((m,s,t)\)-graph is a bipartite graph \( G \) with vertex sets \( U = [m] \) and \( V \) (\( |V| = (2^t-1)s \)). \( V \) is partitioned into \( 2^t-1 \) disjoint sets: \( A, A_0, A_1, A_{00}, \ldots \), that is, one \( A_\sigma \) for each \( \sigma \in \{0,1\}^{<(t-1)} \); each \( A_\sigma \) has \( s \) vertices. Between each \( u \in U \) and each \( A_\sigma \) there is exactly one edge. \( V_i := \cup_{|\sigma|=i-1} A_\sigma \). An \((m,s,t)\)-graph naturally gives rise to a systematic \((m, (2^t-1)s, t)\)-query scheme \( T_G \) as follows. We view the memory (an array \( L \) of \((2^t-1)s\) bits) to be indexed by vertices in \( V \). For query element \( u \in U \), if the first \( i-1 \) probes resulted in values \( \sigma \in \{0,1\}^{i-1} \), then the \( i \)-th probe is made to the location indexed by the unique neighbor
of \( u \) in \( A_x \). In particular, the \( i \)-th probe is made at a location in \( V_i \). We answer “Yes” iff the last bit read is 1. We refer to \( V_i \) as the leaves of \( G \) and for \( y \in [m] \), let \( \text{leaves}(y) := V_i \cap \Gamma_G(y) \). For \( R \subseteq [m] \), let \( \text{leaves}(R) := V_i \cap \Gamma_G(R) \).

We say that the query scheme \( \mathcal{T}_G \) is satisfiable for a set \( S \subseteq [m] \), if there is an assignment to the memory locations \( (L[v] : v \in V) \), such that \( \mathcal{T}_G \) correctly answers all queries of the form “Is \( x \in S \)?”.

We assume that \( t \geq 3 \) is odd and show that \( \forall \epsilon > 0 \) \( \forall n \leq m^{1-\epsilon} \) \( \forall t \leq \frac{1}{10} \lg lg m \) \( s(m, n, t) = O(\exp(e^{2t}) m^{2-t} n^{1-\frac{2t}{t+1}} \log m) \). Our \( t \)-probe scheme will have two parts: a \( t_1 \)-probe non-adaptive part and a \( t_2 \)-probe adaptive part, such that \( t_1 + t_2 = t \). The respective parts will be based on appropriate non-adaptive \((m, s, t_1)\)-graph \( G_1 \) and adaptive \((m, s, t_2)\)-graph \( G_2 \) respectively. To decide set membership, we check set membership in the two parts separately and take the AND, that is, we answer “Yes” iff all bits read in \( \mathcal{T}_G \) are 1 and the last bit read in \( \mathcal{T}_G \) is 1. We refer to this scheme as \( \mathcal{T}_G \cap \mathcal{T}_G \).

First, we identify appropriate properties of the underlying graphs \( G_1 \) and \( G_2 \) that guarantee that all queries are answered correctly for sets of size \( n \). We then show that such graphs exist with \( s = O(\exp(e^{2t} - t) m^{2-t} n^{1-\frac{2t}{t+1}} \log m) \).

We will use the following constants in our calculations: \( \alpha := 2^t - 1 \) and \( \beta := 2^t - t_2 \). Note that \( \alpha \) is the total number of nodes in a \( t_2 \)-probe adaptive decision tree. In any such decision tree, for every choice of \( \beta \) nodes and every choice \( b \in \{0, 1\} \) of the answer, it is possible to assign values to those \( \beta \) nodes so that the decision tree returns the answer \( b \).

**Definition 7.2 (admissible-pair).** We say that a non-adaptive \((m, s, t_1)\)-graph \( G_1 \) and an adaptive \((m, s, t_2)\)-graph \( G_2 \) form an admissible pair \((G_1, G_2)\) for sets of size \( n \) if the following conditions hold.

(P1) \( \forall S \subseteq [m] \) \( (|S| = n) \): \( |\text{survivors}(S)| \leq 10m \left( \frac{2}{m} \right)^{t_1} \), where \( \text{survivors}(S) = \{ y \notin S : \Gamma_G(y) \subseteq \Gamma_{G_1}(S) \} \).

(P2) For \( S \subseteq [m] \) \( (|S| = n) \), let \( \text{survivors}^+(S) = \{ y \in \text{survivors}(S) : \text{leaves}_{G_2}(S) \cap \text{leaves}_{G_2}(y) \neq \emptyset \} \). Then, \( \forall S \subseteq [m] \) \( (|S| = n) \) \( \forall T \subseteq S \cup \text{survivors}^+(S) \): \( \Gamma_{G_2}(T) \geq \beta|T| \).

**Lemma 7.3.** If a non-adaptive \((m, s, t_1)\)-graph \( G_1 \) and an adaptive \((m, s, t_2)\)-graph \( G_2 \) form an admissible pair for sets of size \( n \), then the query scheme \( \mathcal{T}_G \) is satisfiable for every set \( S \subseteq [m] \) of size \( n \).

**Lemma 7.4.** Let \( t \geq 3 \) be an odd number; let \( t_1 = \frac{4t-3}{t+1} \) and \( t_2 = \frac{1+3t}{t+1} \). Then, there exist an admissible pair of graphs consisting of a non-adaptive \((m, s, t_1)\)-graph \( G_1 \) and an adaptive \((m, s, t_2)\)-graph \( G_2 \) with \( s = O(\exp(e^{2t} - t) m^{2-t} n^{1-\frac{2t}{t+1}} \log m) \).

**Proof of Lemma 7.3.** Fix an admissible pair \((G_1, G_2)\). Thus, \( G_1 \) satisfies (P1) and \( G_2 \) satisfies (P2) above. Fix a set \( S \subseteq [m] \) of size \( n \). We will show that there is an assignment such that \( \mathcal{T}_G \cap \mathcal{T}_G \) answers all questions of the form “Is \( x \in S \)” correctly.

The assignment is constructed as follows. Assign 1 to all locations in \( \Gamma_{G_1}(S) \) and 0 to the remaining locations in \( \Gamma_{G_1}(S) \). Thus, \( \mathcal{T}_G \) answers “Yes” for all query elements in \( S \) and answers “No” for all query elements outside \( S \cup \text{survivors}(S) \). However, it (incorrectly) answers “Yes” for elements in \( \text{survivors}(S) \). We will now argue that these false positives can be eliminated using the scheme \( \mathcal{T}_G \).

Using (P2) and Hall’s theorem, we may assign to each element \( u \in S \cup \text{survivors}^+(S) \) a set \( L_u \subseteq V(G_2) \) such that \( (i) |L_u| = \beta \) and \( (ii) L_u \)’s are disjoint. Set \( b_u = 1 \) for \( u \in S \) and \( b_u = 0 \) for \( u \in \text{survivors}^+(S) \) (some of the false positives). As observed above for each \( u \in S \cup \text{survivors}^+(S) \) we may set the values in the locations in \( L_u \) such that the value returned on the query element \( u \) is precisely \( b_u \). Since the \( L_u \)’s are disjoint we may take such an action independently for each \( u \). After this partial assignment, it remains to ensure that queries for
elements \( y \in \text{survivors}(S) \setminus \text{survivors}^+(S) \) (the remaining false positives) return a “No”. Consider any such \( y \). By the definition of \( \text{survivors}^+(S) \), no location in \( \text{leaves}_{G_2}(y) \) has been assigned a value in the above partial assignment. Now, assign 0 to all unassigned locations in \( V(G_2) \). Thus \( T_{G_2} \) returns the answer “No” for queries from \( \text{survivors}(S) \setminus \text{survivors}^+(S) \). \( \square \)

**Proof of Lemma 7.4.** In the following, let

\[
s = \left\lfloor \exp(e^{2t} - t)m^{2t^2}n^{1-\frac{2t}{m}} \lg m \right\rfloor.
\]

We will construct the non-adaptive \((m, s, t_1)\)-graph \( G_1 \) and the \((m, s, t_2)\)-graph \( G_2 \) randomly, and show that with positive probability the pair \((G_1, G_2)\) is admissible. The graph \( G_1 \) is constructed as in the proof of Lemma 6.4 and the analysis is similar. Recall that \( V(G_1) = \bigcup_{i \in [k]} V_i(G_1) \).

For each \( u \in U \), one neighbor is chosen uniformly and independently from each \( V_i(G_1) \).

**(P1) holds.** Fix a set \( S \) of size \( n \). Then, \( \mathbb{E}[|\text{survivors}(S)|] \leq (m - n) \left( \frac{n}{s} \right)^{t_1} \leq m \left( \frac{n}{s} \right)^{t_1} \). As before, using the Chernoff bound, we conclude that

\[
\Pr[|\text{survivors}(S)| > 10m \left( \frac{n}{s} \right)^{t_1}] \leq 2^{-10m \left( \frac{n}{s} \right)^{t_1}}.
\]

Then, by the union bound,

\[
\Pr[\text{P1 fails}] \leq \binom{m}{n} 2^{-10m \left( \frac{n}{s} \right)^{t_1}} \leq \frac{1}{10},
\]

where the last inequality follows from our choice of \( s \).

Fix a graph \( G_1 \) such that (P1) holds. The random graph \( G_2 \) is constructed as follows. Recall that \( V(G_2) = \bigcup_{i \in [0,1]} A_{t_2} \). For each \( u \in [m] \), one neighbor is chosen uniformly and independently from each \( A_z \).

To establish (P2), we need to show that all sets of the form \( S' \cup R \), where \( S' \subseteq S \) and \( R \subseteq \text{survivors}^+(S) \) expand. To restrict the choices for \( R \), we first show in Claim 7.5 (a) that with high probability \( \text{survivors}^+(S) \) is small. Then, using direct calculations, we show that whp the required expansion is available in the random graph \( G_2 \).

**Claim 7.5.** (a) Let \( \mathcal{E}_a \equiv \forall S \subseteq [m] \, |S| = n \, : \, |\text{survivors}^+(S)| \leq 100 \cdot 2^{t_2} m \left( \frac{n}{s} \right)^{t_1+1} \); then, \( \Pr[\mathcal{E}_a] \geq \frac{9}{10} \).

(b) Let \( \mathcal{E}_b \equiv \forall R \subseteq [m] \, (|R| \leq n + [n \lg m]) \, : \, |\Gamma_{G_2}(R)| \geq \beta |R| \); then, \( \Pr[\mathcal{E}_b] \geq \frac{9}{10} \).

(c) Let \( \mathcal{E}_c \equiv \forall S \subseteq [m] \, (|S| = n), \forall S' \subseteq S, \forall R \subseteq \text{survivors}^+(S) \, (|n \lg m| \leq |R| \leq 100 \cdot 2^{t_2} m \left( \frac{n}{s} \right)^{t_1+1}) \, : \, |\Gamma_{G_2}(S' \cup R)| \geq \beta |S' \cup R| \); then, \( \Pr[\mathcal{E}_c] \geq \frac{9}{10} \).

**Proof of claim 7.5.** Part (a) follows by a routine application of Chernoff bound, as in several previous proofs. For a set \( S \) of size \( n \), we have \( \mathbb{E}[|\text{survivors}^+(S)|] \leq |\text{survivors}(S)| \leq 2^{t_2} m \left( \frac{n}{s} \right)^{t_1+1} \). Then,

\[
\Pr[\neg \mathcal{E}_a] \leq \binom{m}{n} 2^{-2^{t_2} 10m \left( \frac{n}{s} \right)^{t_1+1}} \leq \frac{1}{10},
\]

where the last inequality holds because of our choice of \( s \).
Next consider part (b). If $\mathcal{E}_b$ does not hold, then for some non-empty $W \subseteq [m]$, $|W| \leq n + \lceil n \lg m \rceil$, we have $|\Gamma_{G_2}(W)| \leq \beta|W| - 1$. Fix a set $W$ of size $r \geq 1$ and $L \subseteq V(G_2)$ of size $\beta r - 1$. Let $L$ have $\ell_z$ elements in $A_z$. Then,

$$
\Pr[\Gamma_{G_2}(W) \subseteq L] \leq \prod_z \left( \frac{\ell_z}{|A_z|} \right)^r \leq \left( \frac{\beta r - 1}{\alpha s} \right)^{ar}.
$$

We conclude, using the union bound over choices of $W$ and $L$, that the probability that $\mathcal{E}_b$ does not hold is at most

$$
\sum_{r=1}^{\lceil n \lg m \rceil} \left( \frac{m}{r} \right) \beta^r \left( \frac{\alpha s}{\beta r - 1} \right)^{ar}
$$

where the last inequality holds because of our choice of $s$.

Finally, we justify part (c). To bound the probability that $\mathcal{E}_c$ fails, we consider a set $S \subseteq [m]$ of size $n$, a subset $S' \subseteq S$ of size $i$ (say), a subset $R \subseteq \text{survivors}^+(S)$ of size $r$ (where $\lceil n \lg m \rceil \leq r \leq 100 \cdot 2^d m \left( \frac{2}{\beta} \right)^{i+1}$) and $L \subseteq V(G_2)$ of size $\ell = \beta(i + r)$ and define the event

$$
\mathcal{E}(S, S', R, L) \equiv (\forall y \in R : \text{leaves}_{G_2}(S) \cap \text{leaves}_{G_2}(y) \neq \emptyset) \land \Gamma_{G_2}(S' \cup R) \subseteq L.
$$

Then,

$$
\Pr[\mathcal{E}(S, S', R, L)] \leq \left( \frac{2^d n}{s} \right)^r \left( \frac{\ell}{(\alpha - 1)s} \right)^{(a-1)r} \left( \frac{\ell}{\alpha s} \right)^{ai} \quad (7.1)
$$

$$
\leq \left( \frac{2^d n}{s} \right)^r \left( \frac{\beta(i + r)}{(\alpha - 1)s} \right)^{(a-1)(i+r)} \left( \frac{\beta(i + r)}{\alpha s} \right)^i, \quad (7.2)
$$

where the factor $\left( \frac{2^d n}{s} \right)^r$ is justified because of the requirement that every $y \in R$ has at least one neighbour in $\text{leaves}_{G_2}(S)$; the factor $\left( \frac{\ell}{(\alpha - 1)s} \right)^{(a-1)r}$ is justified because all the remaining neighbours must lie in $L$ (we use $\text{AM} \geq \text{GM}$); the last factor $\left( \frac{\ell}{\alpha s} \right)^{ai}$ is justified because all neighbors of elements in $S$ lie in $L$ (again we use $\text{AM} \geq \text{GM}$). To complete the argument we apply the union bound over the choices of $(S, S', R, L)$. Note that we may restrict attention to $\ell = \beta(i + r)$ (because for our choice of $s$, we have $\beta(i + r) \leq |V(G_2)| = \alpha s$). Thus, the probability that $\mathcal{E}_c$ fails to hold is at most

$$
\sum_{S, S', R, L} \Pr[\mathcal{E}_c(S, S', R, L)],
$$

where $S$ ranges over sets of size $n$, $S' \subseteq S$ of size $i$, $R \subseteq \text{survivors}(S)$ of size $r$ such that $|n \lg m| \leq r \leq 100^2 n \left( \frac{2}{\beta} \right)^{i+1}$, $L$ is a subset of $V(G_2)$ of size $\beta(i + r)$. We evaluate this sum
as follows.

\[
\sum_r \sum_i \left( \frac{m}{n} \left( 10m \left( \frac{n}{r} \right)^{t_1} \right) \right) \left( \frac{n}{i} \right) \left( \frac{\alpha s}{\beta(i + r)} \right) \left( \frac{\beta(i + r)}{r} \right)^{\alpha - 1} \left( \frac{\beta(i + r)}{(\alpha - 1)s} \right) \left( \frac{\beta(i + r)}{\alpha s} \right)^i
\]

(7.3)

\[
\leq \sum_r \sum_i \left[ \left( \frac{em}{n} \right)^{\frac{n}{r+i}} \left( 10em \left( \frac{n}{r} \right)^{t_1} \right) \right] \left( \frac{n}{i} \right) \left( \frac{\beta(i + r)}{r} \right)^{\alpha - 1} \left( \frac{\beta(i + r)}{(\alpha - 1)s} \right) \left( \frac{\beta(i + r)}{\alpha s} \right)^i
\]

(7.4)

\[
\leq \sum_r \sum_i \left[ \left( \frac{em}{n} \right)^{\frac{n}{r+i}} \left( 10em \left( \frac{n}{r} \right)^{t_1} \right) \right] \left( \frac{n}{i} \right) \left( \frac{\beta(i + r)}{r} \right)^{\alpha - 1} \left( \frac{\beta(i + r)}{(\alpha - 1)s} \right) \left( \frac{\beta(i + r)}{\alpha s} \right)^i
\]

(7.5)

We will show that the quantity inside the square brackets is at most \( \frac{1}{2} \). Then, since \( r \geq n \lg m \) and \( i \geq 0 \)

\[
\Pr[-E] \leq \left( \sum_r 2^{-r} \right) \left( \sum_i 2^{-i} \right) \leq \frac{1}{10}.
\]

The quantity in the brackets can be decomposed as a product of two factors, which we will bound separately.

**Factor 1:** Consider the following contributions

\[
\left( \frac{em}{n} \right)^{\frac{n}{r+i}} \left( 10e \right)^{\frac{n}{r+i}} \left( \frac{n}{i} \right) \left( \frac{\beta(i + r)}{\alpha - 1} \right) \left( \frac{\beta(i + r)}{2^{2t_2} n} \right)^{\alpha - 1} \left( \frac{\beta(i + r)}{\alpha s} \right)^i
\]

Since \( r \geq n \lg m \) and \( i \leq n \), we have \( \frac{1}{r+i} \leq \frac{n}{n+i} \leq \frac{1}{\lg m} \leq \frac{1}{\lg e} m \). Thus, for all large enough \( m \), this quantity is at most

\[
e^2 \cdot 10e \cdot e^2 \cdot (2e)^\beta \cdot e \leq \exp(e^{2t} - t).
\]

**Factor 2:** We next bound the contribution for the remaining factors.

\[
\left( \frac{m \left( \frac{n}{r} \right)^{t_1}}{r} \right) \left( \frac{\beta(i + r)}{(\alpha - 1)s} \right)^{\alpha - 1} \left( \frac{2^{2t_2} n}{s} \right)
\]

(7.6)

\[
\leq \left( \frac{m \left( \frac{n}{r} \right)^{t_1}}{r} \right) \left( \frac{2}{s} \right)^{\alpha - 1} \left( \frac{2^{2t_2} n}{s} \right)
\]

(7.7)

\[
= \frac{mm^{t_1+1}2^{2t_2-1}r^{\alpha - \beta + t_2 - 1}r^{-\alpha - \beta}}{s^{\alpha - \beta + t_1}}
\]

(7.8)

To justify (7.4), Recall that \( r \leq 100 \cdot 2^{t_2} m \left( \frac{n}{r} \right)^{t_1+1} \) and \( s = \exp(e^{2t} - t) m^{\frac{1}{1+i}} n^{-\frac{1}{1+i}} \lg m \); thus \( \frac{m \left( \frac{n}{r} \right)^{t_1}}{r} \geq 1 \). Then, the above quantity is bounded by

\[
\frac{mm^{t_1+1}2^{2t_2-1} \left( 100 \cdot 2^{t_2} mn^{t_1+1} \right)^{\alpha - \beta - 2}}{s^{(t_1+1)(\alpha - \beta - 2)n^{-\alpha - \beta + t_1}}}
\]

(7.9)

\[
\leq \left( \frac{100 \cdot 2^{t_2} mn^{t_1+1}}{s^{t_1+2}} \right)^{\alpha - \beta - 1}
\]

(7.10)
Thus, since $s = \left\lceil \exp(e^{2t} - t)m^{\frac{2}{1 - r_s}} n^{1 - \frac{r_s}{1 - r_s}} \log m \right\rceil$, then the product of the factors is at most $\frac{1}{10}$, as required.

Acknowledgments

We are grateful to Pat Nicholson and Venkatesh Raman for their comments on these results, and for sharing with us their recent work [9].

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