On mod $p^c$ transfer and applications

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We study a mod $p^c$ analog of the notion of transfer for automorphic forms. Instead of existence of eigenforms, such transfers yield congruences between eigenforms but, like transfers, we show that they can be established by a comparison of trace formulas. This rests on the properties of mod $p^c$ reduced multiplicities which count congruences between eigenforms. As an application we construct finite slope $p$-adic continuous families of Siegel eigenforms using a comparison of trace formulas.

(0.1) In this article we were motivated by the idea of the comparison of trace formulas as a universal principle to relate the existence of different types of automorphic representations. If such an idea holds true then the comparison of trace formulas should be applicable to a wider class of statements relating different types of automorphic representations. Probably the most important statement of this kind is the functoriality principle. This principle yields a transfer from automorphic representations on a reductive group $G$ to the set of automorphic representations on another reductive group $G'$. Thus, it relates the existence of automorphic representations on different groups and special cases of functoriality have been proven via a comparison of trace formulas.

Another example of such a statement which also is of a general nature is the theory of $p$-adic families of automorphic forms. Given an eigenform $f_0$ it predicts the existence of a $p$-adic analytic family of eigenforms passing through $f_0$, i.e. it predicts the existence of infinitely many eigenforms which are related to each other and not only to $f_0$ and which are determined by $f$ only modulo a power of $p$ (but which are all on the same group $G$). Thus, like the functoriality principle, the theory of $p$-adic families relates the existence of automorphic forms (in varying weights) but it is of a different nature. An approach based on a comparison of trace formulas therefore is not obvious but would confirm the idea of the comparison of trace formulas as a universal principle.

(0.2) In this article we describe a comparison of trace formulas which solves the problem of existence of continuous families of finite slope. Since a continuous family is defined by a system of congruences between its eigenforms we need a comparison of trace formulas which instead of existence of eigenforms yields congruences between eigenforms. We are led to such a kind of comparison by considering a mod $p^c$ analog of the notion of transfer for automorphic forms. Unlike transfers, a mod $p^c$ transfer only yields a mod $p^c$ approximation to the predicted eigenform, thus, it yields a congruence which is satisfied by an eigenform. Like transfers, mod $p^c$ transfers can be established by a comparison of trace formulas, i.e. the existence of a mod $p^c$ transfer follows from certain congruences between traces of Hecke operators. This is based on the properties of mod $p^c$ reduced multiplicities. Unlike multiplicities, reduced multiplicities count congruences between eigenforms, hence, their relation to mod $p^c$ transfers. On the other hand, like multiplicities, they can be computed as traces of certain Hecke operators, hence, their relation to the trace formula; cf. (0.3) for more details.
In part B we verify the necessary congruences for the group $\text{GSp}_{2n}$ by comparing the geometric sides of two simple topological trace formulas. This essentially comes down to a problem about eigenvalues of certain symplectic matrices (cf. section 7, in particular (7.1) Lemma).

We only construct continuous families of eigenforms but our method is very different from the ones used in the construction of analytic families. In particular, we do not make use of overconvergent cohomology, $p$-adic Fredholm theory or rigid analytic geometry and the proof is of an elementary nature. We hope that the present method also might apply to yield analyticity of families.

Since, here, we compare two trace formulas on the same group, we avoid the deep problems which have to be solved if the trace formula is applied to the functoriality principle.

\((0.3)\text{ Mod } p^c\text{-transfer.}\) We explain the mod $p^c$ transfer on which our construction of $p$-adic families is based in more detail. Let $\mathcal{H}, \mathcal{H}'$ be free commutative $\mathbb{Z}$-algebras in countably many generators (e.g. Hecke algebras attached to reductive groups $G, G'$) and denote by $\hat{\mathcal{H}}$ the set of characters $\Theta : H \to \bar{\mathbb{Q}}_p$. For any $\mathcal{H}$-module $H$ we denote by $\mathcal{E}(H) \subseteq \hat{\mathcal{H}}$ the set of eigencharacters occurring in $H$, i.e. $\mathcal{E}(H)$ consists of all characters $\Theta$ such that the corresponding generalized (simultaneous) eigenspace $H(\Theta)$ does not vanish. Let $\Phi : \mathcal{H}' \to \mathcal{H}$ be an algebra morphism and denote by $\Phi^\vee : \hat{\mathcal{H}} \to \hat{\mathcal{H}}'$ the dual map; cf. (2.1). Let $H$ resp. $H'$ be a $H$ resp. a $H'$-module. The fundamental problem is to examine if $\Phi^\vee$ defines a map on eigencharacters $\Phi^\vee : \mathcal{E}(H) \to \mathcal{E}(H')$. (If $H, H'$ are Hecke modules of automorphic forms this means to examine whether $\Phi^\vee$ defines a transfer for automorphic representations.) In this article, with the application to the theory of $p$-adic families in mind, we want to examine whether $\Phi^\vee$ defines a map on eigencharacters if we reduce modulo a given power of $p$. More precisely, we want to examine if there is a map

\[(1)\] $\Psi^{[c]} : \mathcal{E}(H) \to \mathcal{E}(H')$

satisfying $\Psi^{[c]}(\Theta) \equiv \Phi^\vee(\Theta) \pmod{p^c}$ for all $\Theta \in \mathcal{E}(H)$. We show that the existence of such a mod $p^c$ transfer $\Psi^{[c]}$ corresponding to $\Phi$ can be established by a comparison of trace formulas if we replace the notion of multiplicity by that of a mod $p^c$ reduced multiplicity. The mod $p^c$ reduced multiplicity of $\Theta \in \hat{\mathcal{H}}$ is defined as

$$m_{H}(\Theta, c) = \sum_{\mu \equiv \Theta \pmod{p^c}} \dim H(\mu),$$

where $\mu$ runs over all characters of $\hat{\mathcal{H}}$ which are congruent to $\Theta$ modulo $p^c$. Thus, $m_{H}(\Theta, c)$ counts the number of eigencharacters of $H$ which are congruent to $\Theta$ mod $p^c$. In particular, if

\[(2)\] $m_{H}(\Theta, c) = m_{H'}(\Phi^\vee(\Theta), c)$

for all $\Theta \in \hat{\mathcal{H}}$ then for any eigencharacter $\mu \in \mathcal{E}(H)$ there is an eigencharacter $\mu' \in \mathcal{E}(H')$ such that $\mu' \equiv \Phi^\vee(\mu) \pmod{p^c}$, i.e. a mod $p^c$ transfer $\Psi^{[c]}$ corresponding to $\Phi$ exists. On the other hand, it is crucial that the reduced multiplicities in (2) can be computed as traces, i.e. there is an element $e' \in \mathcal{H}'$ such that

$$\text{tr}(\Phi(e')|H) \equiv m_{H}(\Theta, c) \quad \text{and} \quad \text{tr}(e'|H') \equiv m_{H'}(\Phi^\vee(\Theta), c)$$

modulo a "high" power of $p$; cf. (2.3) Lemma and (2.4) Remark. Using this we obtain:
**Theorem** (cf. 2.4 Theorem). Let \( \dim H, \dim H' \leq \frac{M^2}{4} \) and assume that there is an \( s \) such that

\[
\text{tr} (\Phi(T'))(H) \equiv \text{tr} (T')(H') \pmod{p^s}
\]

for all \( T' \in H' \). Then for any \( \Theta \in \hat{H} \) there is \( c = c(\Theta) > \frac{1}{s^2} - (M + 2) \log_p M \) such that equation (2) holds. In particular, a mod \( p^{(M+2)\log_p M} \) transfer corresponding to \( \Phi \) exists.

(0.4) **Application to p-adic families.** The theory of \( p \)-adic continuous families is a special case of mod \( p^c \) transfers. To explain this, let \( H_\lambda \) be a family of \( H \)-modules indexed by their “weight” \( \lambda \in \mathbb{Z}^n \). If there are \( a, b \) such that a congruence \( \lambda \equiv \lambda_0 \pmod{(p-1)p^m} \) implies that there is a mod \( p^{(m+1)+b} \)-transfer

\[
\Psi_\lambda : \mathcal{E}(H_{\lambda_0}) \to \mathcal{E}(H_\lambda)
\]

corresponding to the identity map \( \Phi = \text{id} \), i.e. \( \Psi_\lambda(\Theta) \equiv \Theta \pmod{p^{(m+1)+b}} \) then the collection of transfers \( \Psi_\lambda(\Theta_0) \lambda \) is a \( p \)-adic continuous family passing through a given initial eigencharacter \( \Theta_0 \in \mathcal{E}(H_{\lambda_0}) \) (cf. (1.9) Proposition). Thus, the existence of \( p \)-adic continuous families follows from a system of congruences of the type in (3) (cf. (3.7) Proposition).

In part B we show that the family of slope subspaces of the cohomology of Siegel upper half plane satisfies these congruences by comparing trace formulas (cf. (7.5) Theorem). This the main technical work. As a consequence we obtain

**Corollary** (cf. (7.7) Corollary). Any Siegel modular eigenform \( f_0 \) of slope \( \alpha \) fits in a \( p \)-adic continuous family of eigenforms of slope \( \alpha \).

We also obtain local constancy of the dimension of the slope spaces.

(0.5) Starting with the work of Hida (cf. [H 1], [H 2]) \( p \)-adic families of automorphic eigenforms have been constructed by several authors; we mention Hida, Ash-Stevens, Buzzard, Coleman, Emerton, Tilouine, Urban, Harder ... Moreover, there is the work of Koike [K] who applies an explicit Selberg trace formula to prove mod \( p^m \) congruences between the traces of Hecke operators on the space of elliptic modular forms for varying weights \( k \). Since he only considers the Hecke operators \( T(p^m) \) at \( p \), he can not make statements concerning (existence of) eigenforms, hence, there is no kind of transfer or relation between eigenforms in different weights and, consequently, he does not have to set up a comparison of trace formulas as we described in part A (and which involves comparing traces of all Hecke operators). We also mention work of Urban who \( p \)-analytically interpolates the traces of Hecke operators for varying weight. Here, Franke’s trace formula is applied in the construction but as far as we understand in an technical way to reduce from the whole spectrum to the cuspidal spectrum; essentially, his construction is based on the work of [A-S], in particular, on their notion of overconvergent cohomology. Thus, like Koike he does not set up a comparison of trace formulas (note that Franke’s trace formula only has a spectral side and no geometric side, hence, it cannot function in a comparison of trace formulas) and his work seems to be very different from ours.

Part of this article has been described in [Ma 2,3].
A. Reduced Multiplicities

1 Mod \( p^c \) reduced Multiplicities

We define mod \( p^c \) reduced multiplicities and we describe their connection to congruences between eigenforms.

(1.1) We fix a prime \( p \in \mathbb{N} \). We denote by \( v_p \) the \( p \)-adic valuation on \( \bar{\mathbb{Q}}_p \) normalized by \( v_p(p) = 1 \). We write \( \mathcal{O} \) for the ring of integers in \( \bar{\mathbb{Q}}_p \) and we say that \( x \equiv y \pmod{p^t} \), \( t \in \mathbb{R} \), if \( v_p(x - y) \geq t \) \((x, y \in \bar{\mathbb{Q}}_p)\). Finally, \( \lceil x \rceil \) denotes the smallest integer larger than or equal to \( x \) and \( \log_p \) is the complex logarithm with base \( p \).

(1.2) We let \( \mathcal{H} = \bar{\mathbb{Q}}_p[T_\ell, \ell \in \mathcal{I}] \) be the polynomial algebra over \( \bar{\mathbb{Q}}_p \) generated by a countable number of elements \( T_\ell, \ell \in \mathcal{I} \). We set \( \mathcal{H}^\mathcal{O} = \mathcal{O}[T_\ell, \ell \in \mathcal{I}] \), hence, \( \mathcal{H} = \mathcal{H}^\mathcal{O} \otimes \bar{\mathbb{Q}}_p \) and \( \mathcal{H}^\mathcal{O} \) is an order in \( \mathcal{H} \). We denote by \( \hat{\mathcal{H}}^\mathcal{O} \) the set of all \( \bar{\mathbb{Q}}_p \)-algebra characters \( \Theta : \mathcal{H} \to \bar{\mathbb{Q}}_p \) which are defined over \( \mathcal{O} \), i.e. which satisfy \( \Theta(\mathcal{H}^\mathcal{O}) \subseteq \mathcal{O} \). In the following we will also understand by \( \Theta \in \hat{\mathcal{H}}^\mathcal{O} \) the induced character \( \Theta : \mathcal{H}^\mathcal{O} \to \mathcal{O} \) given by restriction of \( \Theta \) to \( \mathcal{H}^\mathcal{O} \). Any \( \Theta \in \hat{\mathcal{H}}^\mathcal{O} \) is determined by the collection of values \( \Theta_\ell := \Theta(T_\ell), \ell \in \mathcal{I} \), and we obtain an embedding \( \hat{\mathcal{H}}^\mathcal{O} \hookrightarrow \mathcal{O}^\mathcal{I} \).

For any character \( \Theta \) of \( \mathcal{H} \) we set

\[
v_p(\Theta) = \inf_{T \in \mathcal{H}^\mathcal{O}} v_p(\Theta(T)) \in \mathbb{Q} \cup \{-\infty\}.
\]

We note that \( \Theta \in \hat{\mathcal{H}}^\mathcal{O} \) precisely if \( v_p(\Theta) > -\infty \) and in this case we obtain

\[
v_p(\Theta) = \inf_{\ell \in \mathcal{I}} v_p(\Theta_\ell) \in \mathbb{Q}_{\geq 0}.
\]

We say that \( \Theta \) is congruent to a character \( \mu \) of \( \mathcal{H} \) modulo \( p^t \), \( t \in \mathbb{R} \), written as \( \Theta \equiv \mu \pmod{p^t} \), if

\[
v_p(\Theta - \mu) \geq t.
\]

(1.3) Let \( \mathcal{H} \) be a \( \mathcal{H} \)-module which is finite dimensional as \( \bar{\mathbb{Q}}_p \)-vector space. We assume that \( \mathcal{H} \) is defined over \( \mathcal{O} \) meaning that \( \mathcal{H} \) contains a \( \mathcal{O} \)-submodule \( \mathcal{H}^\mathcal{O} \) which is free and stable under the action of \( \mathcal{H}^\mathcal{O} \) such that \( \mathcal{H} = \mathcal{H}^\mathcal{O} \otimes \bar{\mathbb{Q}}_p \). For any \( \Theta \in \hat{\mathcal{H}}^\mathcal{O} \) we denote by

\[
\mathcal{H}(\Theta) = \{ v \in \mathcal{H} : \text{ for all } T \in \mathcal{H} \text{ there is } n_T \in \mathbb{N} \text{ such that } (T - \Theta(T))^{n_T}(v) = 0 \}
\]

the generalized simultaneous eigenspace attached to the character \( \Theta \) (or, equivalently, to the (system of) eigenvalue(s) \( \Theta = (\Theta_\ell)_{\ell \in \mathcal{I}} \)). We denote by \( \mathcal{E}(\mathcal{H}) = \mathcal{E}_\mathcal{H}(\mathcal{H}) \) the set of all eigencharacters occurring in \( \mathcal{H} \), i.e. \( \mathcal{E}(\mathcal{H}) \) consists of all characters \( \Theta \in \hat{\mathcal{H}}^\mathcal{O} \) such that \( \mathcal{H}(\Theta) \neq 0 \). Thus, elements in \( \mathcal{E}(\mathcal{H}) \) correspond to simultaneous \( \mathcal{H} \)-eigenforms in \( \mathcal{H} \). We note that any character \( \Theta \)
of \( \mathcal{H} \) which occurs in \( \mathbf{H} \) is defined over \( \mathcal{O} \) because \( \mathbf{H} \) is defined over \( \mathcal{O} \), hence, \( \Theta \in \hat{\mathcal{H}}_{\mathcal{O}} \). Since \( \mathcal{H} \) is commutative we thus obtain a decomposition

\[
\mathbf{H} = \bigoplus_{\Theta \in \mathcal{E}(\mathbf{H})} \mathbf{H}(\Theta).
\]

Let \( \mathbf{H} \) be a \( \mathcal{H} \)-module which is finite dimensional as \( \bar{\mathbb{Q}}_p \)-vector space and which is defined over \( \mathcal{O} \) with respect to the free \( \mathcal{O} \)-submodule \( \mathbf{H}_{\mathcal{O}} \).

\textbf{(1.4) Definition.} Let \( \Theta \in \hat{\mathcal{H}}_{\mathcal{O}} \) and let \( c \in \mathbb{Q} \). We define the (mod \( p^e \))-reduced multiplicity of \( \Theta \) in \( \mathbf{H} \) as

\[
m_{\mathbf{H}}(\Theta, c, p) = \sum_{\mu \in \hat{\mathcal{H}}_{\mathcal{O}} (\mod \ p^e)} \dim \mathbf{H}(\mu).
\]

If the prime \( p \) is understood we will omit "\( p \)" and write more simply \( m_{\mathbf{H}}(\Theta, c) \) instead.

\textbf{(1.5) Remark.} The reduced multiplicities are related to the multiplicity \( m_{\mathbf{H}}(\Theta) := \dim \mathbf{H}(\Theta) \) by

\[
\lim_{c \to \infty} m_{\mathbf{H}}(\Theta, c) = m_{\mathbf{H}}(\Theta).
\]

\textbf{(1.6) Mod \( p^e \) reduced multiplicities and mod \( p^e \) reduction.} We set \( \mathbf{H}_{\mathcal{O}}(\mu) = \mathbf{H}(\mu) \cap \mathbf{H}_{\mathcal{O}} \), hence,

\[
\bigoplus_{\mu \in \mathcal{E}(\mathbf{H})} \mathbf{H}_{\mathcal{O}}(\mu) \subseteq \mathbf{H}_{\mathcal{O}}.
\]

We let \( c \in \mathbb{Q}_{\geq 0} \) and define the ideal \( \mathfrak{a} = \{ x \in \mathcal{O} : v_p(x) \geq c \} \leq \mathcal{O} \). We denote by

\[
\mathcal{O} = \mathcal{O}/\mathfrak{a}, \quad \hat{\mathcal{H}}_{\mathcal{O}} = \hat{\mathcal{H}}_{\mathcal{O}}/\mathfrak{a}\hat{\mathcal{H}}_{\mathcal{O}} \quad \text{and} \quad \mathbf{H}_{\mathcal{O}} = \mathbf{H}_{\mathcal{O}}/\mathfrak{a}\mathbf{H}_{\mathcal{O}}
\]

the mod \( \mathfrak{a} \)-reductions and we denote by \( \hat{\mathbf{H}}_{\mathcal{O}}(\mu) \) the image of \( \mathbf{H}_{\mathcal{O}}(\mu) \) in \( \hat{\mathbf{H}}_{\mathcal{O}} \). Hence, \( \hat{\mathbf{H}}_{\mathcal{O}} \) and \( \hat{\mathbf{H}}_{\mathcal{O}}(\mu) \) are \( \hat{\mathcal{H}}_{\mathcal{O}} \)-modules. We assume that \( \mathbf{H}_{\mathcal{O}}(\mu) \) \( \leq \mathbf{H}_{\mathcal{O}} \) is a free \( \mathcal{O} \)-submodule (in our later applications \( \mathbf{H} \) will be defined over a finite extension \( E/\mathbb{Q}_p \), hence, we may replace \( \mathcal{O} \) by \( \mathcal{O}_E \) which is a P.I.D and the assumption holds). Since \( \mathbf{H}_{\mathcal{O}}(\mu) \) \( \leq \mathbf{H}_{\mathcal{O}} \) is saturated, \( \hat{\mathbf{H}}_{\mathcal{O}}(\mu) \) is a free \( \mathcal{O} \)-module which is annihilated by \( (T - \bar{\mu}(T))^{p^e} \) for all \( T \in \hat{\mathcal{H}}_{\mathcal{O}} \); here, \( \bar{\mu} = \mu \mod p^e : \hat{\mathcal{H}}_{\mathcal{O}} \rightarrow \bar{\mathcal{O}} \), \( T + \mathfrak{a}\hat{\mathcal{H}}_{\mathcal{O}} \rightarrow \mu(T) + \mathfrak{a} \) is the mod \( \mathfrak{a} \)-reduced character. If the decomposition of \( \mathbf{H} \) as a sum of generalized eigenspaces is defined over \( \mathcal{O} \), i.e. if \( \bigoplus_{\mu \in \mathcal{E}(\mathbf{H})} \mathbf{H}_{\mathcal{O}}(\mu) = \mathbf{H}_{\mathcal{O}} \) then

\[
(1) \quad \bigoplus_{\mu \in \mathcal{E}(\mathbf{H})} \mathbf{H}_{\mathcal{O}}(\mu) = \mathbf{H}_{\mathcal{O}}
\]

as \( \mathcal{O} \)-modules and, hence, we obtain for the multiplicity of \( \Theta \mod p^e \) in \( \hat{\mathbf{H}}_{\mathcal{O}} \)

\[
m_{\mathbf{H}_{\mathcal{O}}}(\Theta \mod p^e) := \dim_{\mathcal{O}} \sum_{\mu \in \hat{\mathcal{H}}_{\mathcal{O}} (\mod \ p^e)} \mathbf{H}_{\mathcal{O}}(\mu) = m_{\mathbf{H}}(\Theta, c, p).
\]
In general, we only obtain an inclusion of a (not necessarily direct) sum
\[ \sum_{\mu \in \mathcal{E}(\mathcal{H})} \mathbf{H}_\mathcal{O}(\mu) \subseteq \mathbf{H}_\mathcal{O}, \]
which yields an inequality
\[ m_{\mathbf{H}_\mathcal{O}}(\Theta \mod p^c) \leq m_{\mathbf{H}}(\Theta, c, p). \]
In this sense, the reduced multiplicity \( m_{\mathbf{H}}(\Theta, c, p) \) is a substitute for the multiplicity of the mod \( p^c \)-reduction of \( \Theta \) in \( \mathcal{H}_\mathcal{O} \) if the primary decomposition of \( \mathcal{H} \) is not defined over \( \mathcal{O} \).

**Higher Congruences.** Let \( \mathcal{H} = \overline{\mathbb{Q}}_p [T_\ell, \ell \in \mathcal{I}] \) and \( \mathcal{H}' = \overline{\mathbb{Q}}_p [T_\ell, \ell \in \mathcal{I}'] \) and let \( \mathbf{H} \) resp. \( \mathbf{H}' \) be a \( \mathcal{H} \) resp. \( \mathcal{H}' \)-module which is defined over \( \mathcal{O} \) with respect to the lattice \( \mathbf{H}_\mathcal{O} \) resp. \( \mathbf{H}_\mathcal{O}' \) as in (1.3). Let
\[ \Phi^\vee : \hat{\mathcal{H}}_\mathcal{O} \to \hat{\mathcal{H}}'_\mathcal{O} \]
be a map and let \( m \in \mathbb{N} \). If for any \( \Theta \in \hat{\mathcal{H}}_\mathcal{O} \) there is a rational number \( c = c(\Theta) \geq m \) such that
\[ m_{\mathbf{H}}(\Theta, c) = m_{\mathbf{H}'}(\Phi^\vee(\Theta), c) \]
then for any \( \Theta \in \mathcal{E}(\mathcal{H}) \) there is an eigencharacter \( \Theta' \in \mathcal{E}(\mathcal{H}') \) such that
\[ \Theta' \equiv \Phi^\vee(\Theta) \mod p^c. \]
In different words there is a map on eigencharacters
\[ \Psi^c : \mathcal{E}(\mathcal{H}) \to \mathcal{E}(\mathcal{H}') \]
such that \( \Psi^c(\Theta) \equiv \Phi^\vee(\Theta) \mod p^c \). Thus, by comparing reduced multiplicities we can establish congruences between eigencharacters in \( \mathcal{E}(\mathcal{H}') \) and lifts of eigencharacters in \( \mathcal{E}(\mathcal{H}) \) or, equivalently, a mod \( p^c \) transfer from \( \mathcal{E}(\mathcal{H}) \) to \( \mathcal{E}(\mathcal{H}') \). In particular, if \( \mathcal{H} = \mathcal{H}' \) and \( \Phi^\vee = \text{id} \) then the set of identities
\[ m_{\mathbf{H}}(\Theta, c) = m_{\mathbf{H}}(\Theta, c), \quad \Theta \in \hat{\mathcal{H}}_\mathcal{O}, \]
implies that for any \( \Theta \in \mathcal{E}(\mathcal{H}) \) a congruence
\[ \Theta \equiv \Theta' \mod p^m \]
holds for some \( \Theta' \in \mathcal{E}(\mathcal{H}') \).

**Systems of higher congruences.** Slightly refining the above discussion we can give a set of simple identities between mod \( p^c \) reduced multiplicities which implies the existence of \( p \)-adic continuous families of eigencharacters (note that such families are defined by a system of congruences between their members). To be more precise, we let \( \mathbf{G}/\mathbb{Q} \) be a connected reductive algebraic group with maximal split torus \( \mathbf{T} \) and we denote by \( X(\mathbf{T}) \) the group of \( \mathbb{Q} \)-characters of \( \mathbf{T} \). \( X(\mathbf{T}) \) is a finitely generated, free abelian group which we write using additive notation. For any \( \lambda \in X(\mathbf{T}) \) we denote by \( v_p(\lambda) \) the largest integer \( m \) such that \( \lambda \in p^m X(\mathbf{T}) \). Thus, if we identify \( X(\mathbf{T}) \cong \mathbb{Z}^k \) via the choice of a basis \( \{ \gamma_i \} \) of \( X(\mathbf{T}) \) then \( v_p(\sum_i z_i \gamma_i) = \inf_i v_p(z_i) \).

**Proposition.** Let \( \mathcal{R} \subset X(\mathbf{T}) \) be a subset and let \( (\mathbf{H}_\lambda), \lambda \in \mathcal{R}, \) be a family of finite dimensional \( \mathcal{H} \)-modules which are defined over \( \mathcal{O} \). Assume there are \( a, b \in \mathbb{Q} \) with the following
property: if \( \lambda \equiv \lambda' \pmod{(p-1)p^m X(T)} \), then for any \( \Theta \in \mathcal{H}_0 \) there is \( c = c(\Theta) \geq a(m+1)+b \) with
\[
m_{\mathbf{H}_\lambda}(\Theta, c) = m_{\mathbf{H}_{\lambda'}}(\Theta, c),
\]
i.e. the transfer \( \Psi^{[a(m+1)+b]} : \mathcal{E}(\mathbf{H}_\lambda) \to \mathcal{E}(\mathbf{H}_{\lambda'}) \) corresponding to \( \Phi^\vee = \text{id} \) exists. Then, any \( \Theta \in \mathcal{E}(\mathbf{H}_\lambda) \) fits in a \( p \)-adic continuous family of eigencharacters, i.e. there is a family \( (\Theta_\lambda) \), \( \lambda \in \mathcal{R} \), such that
\[
\begin{align*}
\bullet & \ \Theta_\lambda \in \mathcal{E}(\mathbf{H}_\lambda) \\
\bullet & \ \Theta_{\lambda_0} = \Theta \\
\bullet & \ \lambda \equiv \lambda' \pmod{(p-1)p^m X(T)} \text{ implies } \Theta_\lambda \equiv \Theta_{\lambda'} \pmod{p^{2(m+1)+b}}.
\end{align*}
\]

Proof. For any weight \( \mu \in X(T) \) we set \( \mathcal{R}_{\mu} = \{ \lambda \in \mathcal{R} : \lambda \equiv \mu \pmod{(p-1)X(T)} \} \). We first construct a \( p \)-adic family \( (\Theta_\lambda)_\lambda \) satisfying the above conditions with \( \lambda \) only running over \( \mathcal{R}_{\mu} \). To this end, we enumerate the weights \( \lambda \) in \( \mathcal{R}_{\lambda_0} \) in a sequence \( \lambda_0, \lambda_1, \lambda_2, \lambda_3, \ldots \). We inductively construct elements \( \Theta_\lambda_i \in \mathcal{E}(\mathbf{H}_{\lambda_i}) \), \( i = 0, 1, 2, 3, \ldots \) such that \( \Theta_{\lambda_i} = \Theta \) and \( \lambda_i \equiv \lambda_j \pmod{(p-1)p^m X(T)} \) implies \( \Theta_{\lambda_i} \equiv \Theta_{\lambda_j} \pmod{p^{2(m+1)+b}} \). Clearly, we set \( \Theta_{\lambda_0} = \Theta \). Assume that \( \Theta_{\lambda_0}, \ldots, \Theta_{\lambda_n} \) have been defined such that \( \lambda_i \equiv \lambda_j \pmod{(p-1)p^m X(T)} \) implies that \( \Theta_{\lambda_i} \equiv \Theta_{\lambda_j} \pmod{p^{2(m+1)+b}} \) for all \( i, j = 0, \ldots, n \). To define \( \Theta_{\lambda_{n+1}} \) we select \( a \in \{0, 1, 2, \ldots, n\} \) such that
\[
v_p(\lambda_{n+1} - \lambda_a) \geq v_p(\lambda_{n+1} - \lambda_i) \quad \text{for all } i = 0, \ldots, n.
\]
We set \( w_1 = v_p(\lambda_{n+1} - \lambda_a) \), hence, \( \lambda_{n+1} - \lambda_a \in (p-1)p^{w_1} X(T) \) (note that \( \lambda_a - \lambda_{n+1} \in (p-1)X(T) \) because \( \lambda_{n+1}, \lambda_a \in \mathcal{R}_{\lambda_0} \)). By (2.4) Remark there is \( \Theta \in \mathcal{E}(\mathbf{H}_{\lambda_{n+1}}) \) such that \( \Theta \equiv \Theta_{\lambda_a} \pmod{p^{2(w_1+1)+b}} \). We then set \( \Theta_{\lambda_{n+1}} \) equal to this \( \Theta \).

Let \( i \in \{0, \ldots, n\} \) be arbitrary and set \( w_3 = v_p(\lambda_{n+1} - \lambda_i) \), hence, \( \lambda_{n+1} \equiv \lambda_i \pmod{(p-1)p^{w_3} X(T)} \). We have to show that \( \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_i} \pmod{p^{2(w_3+1)+b}} \). To this end we set \( w_2 = v_p(\lambda_a - \lambda_i) \).
\[
\begin{align*}
w_1{}[ & \quad \lambda_a \bullet \\
w_2{}[ & \quad | \lambda_i \bullet \\
& \quad \{ \lambda_{n+1} \}
\end{align*}
\]
We know that \( \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_a} \pmod{p^{2(w_1+1)+b}} \) by definition of \( \Theta_{\lambda_{n+1}} \) and that \( \Theta_{\lambda_a} \equiv \Theta_{\lambda_i} \pmod{p^{2(w_3+1)+b}} \) by our induction hypotheses, hence,
\[
(2) \quad \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_i} \pmod{p^{2(\min(w_1, w_2)+1)+b}}.
\]

We distinguish cases.

Case A \( w_2 > w_1 \). In this case \( \min\{w_1, w_2\} = w_1 \) and \( w_1 = w_1 \) by the \( p \)-adic triangle inequality. Hence, equation (2) implies that \( \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_i} \pmod{p^{2(w_1+1)+b}} \).

Case B \( w_2 < w_1 \). In this case \( \min\{w_1, w_2\} = w_2 \) and \( w_1 = w_2 \). Hence, equation (2) implies that \( \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_i} \pmod{p^{2(w_2+1)+b}} \).

Case C \( w_2 = w_1 \). In this case \( \min\{w_1, w_2\} = w_1 \). On the other hand, by the choice of \( a \) we know that \( w_1 \geq w_3 \); thus equation (2) yields \( \Theta_{\lambda_{n+1}} \equiv \Theta_{\lambda_i} \pmod{p^{2(w_3+1)+b}} \).

Thus, \( (\Theta_\lambda)_\lambda, \lambda \in \mathcal{R}_{\lambda_0} \), is a \( p \)-adic continuous family. To obtain a \( p \)-adic family \( (\Theta_\lambda) \) with \( \lambda \) running through all of \( \mathcal{R} \) we denote by \( \{\mu_0 = \lambda_0, \mu_1, \ldots, \mu_r\} \) a system of representatives for
$X(T)/(p-1)X(T)$. For any $i = 1, \ldots, r$ we construct in the same way as above a $p$-adic family $(\Theta_\lambda)_\lambda$ with $\lambda$ running through $\mathcal{R}_\mu$. Since $\mu_i \neq \mu_j \pmod{(p-1)X(T)}$ if $i \neq j$ the union $(\Theta_\lambda)_\lambda$, $\lambda \in \bigcup_i \mathcal{R}_\mu$, then is a $p$-adic family satisfying the requirements of the Proposition. This completes the proof.

2 Mod $p^c$ Transfer

We show that reduced multiplicities can be computed as traces of certain operators and we use this to establish a "mod $p^c$ transfer" for eigencharacters.

(2.1) The dual map. We let $\mathcal{H} = \bar{\mathbb{Q}}_p[T_\ell, \ell \in I]$ and $\mathcal{H}' = \bar{\mathbb{Q}}_p[T_\ell, \ell \in I']$ be countably generated polynomial algebras and we set $\mathcal{H}_\mathcal{O} = \mathcal{O}[T_\ell, \ell \in I]$ and $\mathcal{H}'_\mathcal{O} = \mathcal{O}[T_\ell, \ell \in I']$. We assume that there is a morphism of $\mathcal{O}$-algebras

$$\Phi : \mathcal{H}'_\mathcal{O} \to \mathcal{H}_\mathcal{O}.$$ 

The morphism $\Phi$ induces a dual map

$$\Phi^\vee : \hat{\mathcal{H}}_\mathcal{O} \to \hat{\mathcal{H}}'_\mathcal{O}$$

by sending $\mu$ to $\Phi^\vee(\mu) = \mu \circ \Phi$, i.e. the diagram

$$\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\Phi} & \mathcal{H}' \\
\mu \searrow & & \nearrow \Phi^\vee(\mu) \\
\bar{\mathbb{Q}}_p & & \\
\end{array}$$

commutes. Thus, for any $v_\mu \in \mathcal{H}(\mu)$, $\mu \in \hat{\mathcal{H}}_\mathcal{O}$, and any $T' \in \mathcal{H}'$ we obtain

$$\Phi(T')v_\mu = \Phi^\vee(\mu)(T')v_\mu.$$ 

(2.2) Remark. Let $\mathbf{H}$ resp. $\mathbf{H}'$ be a $\mathcal{H}$ resp. $\mathcal{H}'$-module which is defined over $\mathcal{O}$ with respect to the lattice $\mathbf{H}_\mathcal{O}$ resp. $\mathbf{H}'_\mathcal{O}$. In general, $\Phi^\vee$ does not induce a mapping on eigencharacters

$$\Phi^\vee : \mathcal{E}(\mathbf{H}) \to \mathcal{E}(\mathbf{H}').$$

We want to examine whether this is the case modulo powers of $p$, i.e. whether for any $\Theta \in \mathcal{E}(\mathbf{H})$ there is a $\Theta' \in \mathcal{E}(\mathbf{H}')$ such that

$$\Phi^\vee(\Theta) \equiv \Theta' \pmod{p^c}.$$ 

This is equivalent to the existence of a map

$$\Psi^\vee : \mathcal{E}(\mathbf{H}) \to \mathcal{E}(\mathbf{H}')$$

such that $\Psi^\vee(\Theta) \equiv \Phi^\vee(\Theta) \pmod{p^c}$ for all $\Theta \in \mathcal{E}(\mathbf{H})$. By (1.7) we can establish the existence of the map $\Psi^\vee$ by comparing the mod $p^c$ reduced multiplicities of $\Theta$ and $\Phi^\vee(\Theta)$ in $\mathbf{H}$ and $\mathbf{H}'$, i.e. we need to compute reduced multiplicities. This will be based on the following Lemma which expresses the reduced multiplicites $m_{\mathbf{H}}(\Theta, c)$ and $m_{\mathbf{H}}(\Phi^\vee(\Theta), c)$ as traces of a certain element in $\mathbf{H}'$. 

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(2.3) Reduced multiplicities as traces. From now on we assume that $\Phi : \mathcal{H}_O \to \mathcal{H}_O$ is surjective.

**Lemma.** Assume that $\dim \mathcal{H}, \dim \mathcal{H}' \leq \frac{1}{2} M$ for some $M \in 2\mathbb{N}$ and let $m \in \mathbb{N}$. For any $\Theta \in \mathcal{H}_O$ there is an element $e(\Theta) \in \mathcal{H}'$ and a rational number $c = c(\Theta) \geq m - (M + \frac{3}{2}) \log p M$ such that the following holds

- $e(\Theta) \in \frac{1}{2} \mathcal{H}'_O$, where $\xi \in \mathcal{O}$ with $v_p(\xi) \leq M m$
- $\text{tr}(e(\Theta)|H') \equiv m(\Phi^\vee(\Theta), c) \pmod{p \log p M}$
- $\text{tr}(\Phi(e(\Theta))|H) \equiv m(\Theta, c) \pmod{p \log p M}$.

(We note that $\log p M \in \mathbb{R}_{\geq 0}$ and the congruence has to be understood as in (1.1).)

**Proof.** Let $\Theta \in \mathcal{H}_O$. We proceed in steps.

a.) We first define $c$. We abbreviate $l = \log p M$. We set

$$\Omega = \Omega(\Theta) = \{v_p(\mu - \Theta), \mu \in \mathcal{E}(\mathcal{H})\} \cup \{v_p(\mu' - \Phi^\vee(\Theta)), \mu' \in \mathcal{E}(\mathcal{H}')\} \subseteq \mathbb{Q}$$

and we define the interval

$$I = \{r \in \mathbb{Q} : m - (M + 3/2)l \leq r \leq m\} \subseteq \mathbb{Q}_{\geq 0}.$$

Since $I$ has length $(M + 3/2)l$ and since $|\Omega| \leq |\mathcal{E}(\mathcal{H})| + |\mathcal{E}(\mathcal{H}')| \leq \dim \mathcal{H} + \dim \mathcal{H}' \leq M$ there is a $c \in I$ such that $[c, c + l] \cap \Omega = \emptyset$.

Thus, if $\gamma \in \mathcal{E}(\mathcal{H})$ with $v_p(\gamma - \Theta) \geq c$ then we know that $v_p(\gamma - \Theta) \in \Omega \cap [c, \infty)$, hence,

$$v_p(\gamma - \Theta) > c + l.$$  \hspace{6cm} (2)

Similarly, if $\gamma' \in \mathcal{E}(\mathcal{H}')$ with $v_p(\gamma' - \Phi^\vee(\Theta)) \geq c$ then

$$v_p(\gamma' - \Phi^\vee(\Theta)) > c + l.$$  \hspace{6cm} (2')

We note that the number $c$ obviously satisfies

$$m - (M + 3/2)l \leq c \leq m.$$

b.) Next we define $e(\Theta)$. We note that for any $\mu \in \mathcal{E}(\mathcal{H})$ with $\mu \not\equiv \Theta \pmod{p^c}$, i.e. $v_p(\mu - \Theta) < c$, there is a element $T_\mu \in \mathcal{H}'_O$ such that

$$\mu(\Phi(T_\mu)) \not\equiv \Theta(\Phi(T_\mu)) \pmod{p^c}.$$  \hspace{6cm} (note that we assume $\Phi$ to be surjective); hence,

$$\Phi^\vee(\mu)(T_\mu) \not\equiv \Phi^\vee(\Theta)(T_\mu) \pmod{p^c}.$$  \hspace{6cm} (3)

Similarly, for any $\mu' \in \mathcal{E}(\mathcal{H}')$ with $\mu' \not\equiv \Phi^\vee(\Theta) \pmod{p^c}$, i.e. $v_p(\mu' - \Phi^\vee(\Theta)) < c$, there is a element $T_{\mu'} \in \mathcal{H}'_O$ such that

$$\mu'(T_{\mu'}) \not\equiv \Phi^\vee(\Theta)(T_{\mu'}) \pmod{p^c}.$$  \hspace{6cm} (3')
We set
\[
\xi = \prod_{\mu \not\equiv \Theta (\text{mod } p^c)} \Phi^\vee (\Theta)(T_{\mu}^\prime) - \Phi^\vee (\mu)(T_{\mu}^\prime) \prod_{\mu \not\equiv \Phi^\vee (\Theta) (\text{mod } p^c)} \Phi^\vee (\Theta)(T_{\mu}^\prime) - \mu'(T_{\mu}^\prime) \in \mathcal{O}
\]
and we define
\[
e(\Theta) = \frac{1}{\xi} \prod_{\mu \not\equiv \Theta (\text{mod } p^c)} T_{\mu}^\prime - \Phi^\vee (\mu)(T_{\mu}^\prime) \prod_{\mu \not\equiv \Phi^\vee (\Theta) (\text{mod } p^c)} T_{\mu}^\prime - \mu'(T_{\mu}^\prime) \in \frac{1}{\xi} \mathcal{H}_\Theta.
\]
Equations (3) and (3') imply that
\[
v_p(\xi) \leq |E(\mathcal{H})|c + |E(\mathcal{H}')|c \leq Mc \leq Mm.
\]
which in particular implies the first claim of the Lemma.

c.) We compute the trace of \(e(\Theta)\) on \(\mathcal{H}'\). We write
\[
\mathcal{H}' = \bigoplus_{\gamma' \in \mathcal{E}(\mathcal{H}')} \mathcal{H}'(\gamma').
\]
Since \(\mathcal{H}'\) is commutative, there is for any \(\gamma' \in \mathcal{E}(\mathcal{H}')\) a basis \(\mathcal{B}(\gamma')\) of \(\mathcal{H}'(\gamma')\) such that all \(T' \in \mathcal{H}'\) are represented on \(\mathcal{H}'(\gamma')\) by an upper triangular matrix:
\[
D_{\mathcal{B}(\gamma')}(T') = \begin{pmatrix}
\gamma'(T') & * \\
. & .
. & .
. & .
\gamma'(T') & .
\end{pmatrix}.
\]
Then \(e(\Theta)\) is represented on \(\mathcal{H}'(\gamma')\) by the matrix
\[
D_{\mathcal{B}(\gamma')}(e(\Theta)) = \begin{pmatrix}
x & * \\
. & .
. & .
. & .
x' & .
\end{pmatrix},
\]
where
\[
x = x(\gamma') = \frac{1}{\xi} \prod_{\mu \not\equiv \Theta (\text{mod } p^c)} \Phi^\vee (\Theta)(T_{\mu}^\prime) - \Phi^\vee (\mu)(T_{\mu}^\prime) \prod_{\mu \not\equiv \Phi^\vee (\Theta) (\text{mod } p^c)} \Phi^\vee (\Theta)(T_{\mu}^\prime) - \mu'(T_{\mu}^\prime) \in \frac{1}{\xi} \mathcal{O}.
\]
Using this we determine the trace of \(e(\Theta)\) on \(\mathcal{H}'(\gamma'), \gamma' \in \mathcal{E}(\mathcal{H}')\), as follows.

If \(\gamma' \not\equiv \Phi^\vee (\Theta) (\text{mod } p^c)\) then \(x = 0\), hence, \(\text{tr}(e(\Theta)|\mathcal{H}'(\gamma')) = 0\).

If \(\gamma' \equiv \Phi^\vee (\Theta) (\text{mod } p^c)\) then we write for any \(\mu \in \mathcal{E}(\mathcal{H}), \mu \not\equiv \Theta (\text{mod } p^c)\)
\[
\gamma'(T_{\mu}^\prime) = \Phi^\vee (\Theta)(T_{\mu}^\prime) + \delta_{\mu},
\]
where \(\delta_{\mu} \in \mathcal{O}\). Since \(v_p(\gamma' - \Phi^\vee (\Theta)) \geq c\), equation (2') implies that \(v_p(\gamma' - \Phi^\vee (\Theta)) > c + l\), hence,
\[
(4) \quad v_p(\delta_{\mu}) > c + l.
\]
Similarly, we write
\[ \gamma'(T_{\mu'}) = \Phi^\gamma(\Theta)(T_{\mu'}) + \delta_{\mu'} \]
and find
\[ (4') \quad v_p(\delta_{\mu'}) > c + l. \]
Recalling the definition of \( \xi \) we obtain
\[ x = \prod_{\mu \not\equiv \Theta \pmod{p^c}} \Phi^\gamma(\Theta)(T_{\mu'}) - \Phi^\gamma(\mu)(T_{\mu'}) + \delta_{\mu'} \prod_{\mu' \not\equiv \Phi^\gamma(\Theta) \pmod{p^c}} \Phi^\gamma(\Theta)(T_{\mu'}) - \mu'(T_{\mu'}) + \delta_{\mu'} \]
Since \( v_p(\Phi^\gamma(\Theta)(T_{\mu'}) - \Phi^\gamma(\mu)(T_{\mu'})) < c \) for all \( \mu \not\equiv \Theta \pmod{p^c} \) (cf. equation (3)) and since, \( v_p(\delta_{\mu'}) > c + l \) (cf. equation (4)) we obtain that the first factor on the right hand side of the above equation for \( x \equiv 1 \pmod{p} \). Similarly, using equations (3') and (4') we find that the second factor on the right hand side of the above equation for \( x \equiv 1 \pmod{p} \). Thus, we obtain \( x \equiv 1 \pmod{p'} \), hence,
\[ \text{tr}(e(\Theta)|H'(\gamma')) \equiv \dim H'(\gamma') \pmod{p'}. \]
Summing over all \( \gamma' \in \mathcal{E}(H') \) finally yields
\[ \text{tr}(e(\Theta)|H') \equiv m_H(\Phi^\gamma(\Theta), c) \pmod{p'}. \]
d.) Quite analogous we compute the trace of \( \Phi(e(\Theta)) \) on \( H \). We note that
\[ \Phi(e(\Theta)) = \frac{1}{\xi} \prod_{\mu \not\equiv \Theta \pmod{p^c}} - \Phi^\gamma(\Theta)(T_{\mu'}) + \Phi^\gamma(\mu)(T_{\mu'}) - \mu'(T_{\mu'}) \in \frac{1}{\xi}\mathcal{O}. \]
Let \( \gamma \in \mathcal{E}(H) \). We choose a basis \( B(\gamma) \) of \( H(\gamma) \) such that any \( T \in H \) is upper triangular on \( H(\gamma) \):
\[ D_{B(\gamma)}(T) = \begin{pmatrix} \gamma(T) & * & \cdots & * \\ & \ddots & \ddots & \ddots \\ & & \gamma(T) & \end{pmatrix}. \]
Then \( \Phi(e(\Theta)) \) is represented on \( H(\gamma) \) by the matrix
\[ D_{B(\gamma)}(\Phi(e(\Theta))) = \begin{pmatrix} x & * & \cdots & * \\ & \ddots & \ddots & \ddots \\ & & \cdots & \ddots \\ & & & x \end{pmatrix}, \]
where
\[ x = x(\gamma) = \frac{1}{\xi} \prod_{\mu \not\equiv \Theta \pmod{p^c}} - \Phi^\gamma(\Theta)(T_{\mu'}) + \Phi^\gamma(\mu)(T_{\mu'}) - \mu'(T_{\mu'}) \in \frac{1}{\xi}\mathcal{O}. \]
If \( \gamma \not\equiv \Theta \pmod{p^c} \) then \( x = 0 \), hence, \( \text{tr}(\Phi(e(\Theta))|H(\gamma)) = 0 \).
If \( \gamma \equiv \Theta \Mod{pc} \) then equation (2) implies that \( \nu_p(\gamma - \Theta) > c + l \). We write \( \gamma(\Phi(T'_\mu)) = \Theta(\Phi(T'_\mu)) + \delta'_\mu \), or, equivalently,

\[
\Phi^\gamma(\gamma(T'_\mu)) = \Phi^\gamma(\Theta(T'_\mu)) + \delta'_\mu,
\]

where

\[
\nu_p(\delta'_\mu) > c + l.
\]

Similarly, we write

\[
\Phi^\gamma(\gamma(T'_{\mu'})) = \Phi^\gamma(\Theta(T'_{\mu'})) + \delta_{\mu'},
\]

where

\[
\nu_p(\delta_{\mu'}) > c + l.
\]

Recalling the definition of \( \xi \) we obtain

\[
x = \prod_{\mu \notin \Theta \Mod{pc}} \Phi^\gamma(\Theta(T'_\mu)) \prod_{\mu' \notin \Phi^\gamma(\Theta) \Mod{pc}} \Phi^\gamma(\Theta(T'_{\mu'})) - \nu(\mu(T'_\mu)) - \delta_{\mu'}
\]

As above, using equations (3), (3'), (5), (5') we obtain \( x \equiv 1 \Mod{p^l} \), hence,

\[
\text{tr}(\Phi(\epsilon(\Theta))|H(\gamma)) \equiv \dim H(\gamma) \Mod{p^l}.
\]

Summing over all \( \gamma \in E(H) \) finally yields

\[
\text{tr}(\Phi(\epsilon(\Theta))|H) \equiv m_H(\Theta, c) \Mod{p^l}
\]

Hence, the Lemma is proven.

**Remark.**

1.) Since \( m_H(\Theta, c) \) and \( m_H(\Phi^\gamma(\Theta), c) \) are smaller than or equal to \( M \) the congruences in (2.3) Lemma uniquely determine \( m_H(\Theta, c) \) and \( m_H(\Phi^\gamma(\Theta), c) \).

2.) Since \( c \geq m - (M + \frac{1}{2}) \log p M \), by increasing \( m \) we can search for eigencharacters which are arbitrarily close to \( \Phi^\gamma(\Theta) \) or \( \Theta \); in doing so the denominators of \( e(\Theta) \) will not grow unreasonably fast because \( \nu_p(\xi) \leq Mm \).

3.) The proof is not constructive, e.g. we do not obtain the value of \( c \); in particular, we do not know whether we can choose \( c = m - (M + \frac{1}{2}) \log p M \).

**2.4** Using (2.3) Lemma we can give the following criterion for the existence of the “mod \( p^c \) transfer” \( \Psi^{[c]} \) which makes it possible to establish “mod \( p^c \) transfer” via a comparison of trace formulas.

**Theorem.** Assume that \( \dim H, \dim H' \leq \frac{1}{2} M \) for some \( M \in 2\mathbb{N} \). Assume that there is a rational number \( s \) such that

\[
\text{tr}(\Phi(T'')|H') \equiv \text{tr}(T'|H') \Mod{p^s}
\]

for all \( T' \in H'_\Theta \). Then, for any \( \Theta \in \tilde{H}_\Theta \) there is a rational number \( c = c(\Theta) \geq \frac{5}{M} - (M+2) \log p M \) such that

\[
m_H(\Theta, c) = m_H(\Phi^\gamma(\Theta), c).
\]
Hence, for any $\Theta \in \mathcal{E}(H)$ there is an element $\Theta' \in \mathcal{E}(H')$ such that
$$\Theta' \equiv \Phi^V(\Theta) \pmod{p^{c_7 - (M + 2) \log_p M}},$$
or, equivalently, there is a map on eigencharacters
$$\psi^{[c]} : \mathcal{E}(H) \to \mathcal{E}(H')$$
such that
$$\psi^{[c]}(\Theta) \equiv \Phi^V(\Theta) \pmod{p^{c_7 - (M + 2) \log_p M}}$$
for all $\Theta \in \mathcal{E}(H)$.

**Proof.** We set $m = \frac{s - \log_p M}{M}$. Let $\Theta \in \hat{H}_0$. According to (2.3) Lemma there is an element $e(\Theta) \in \frac{1}{p}H'_0$, where $v_p(\xi) \leq Mm = s - \log_p M$, and a rational number $c = c(\Theta) \geq m - \left(\frac{M}{2} + 3\right) \log_p M = \frac{s}{M} - (M + 2) \log_p M$ (note that $\frac{1}{M} \leq \frac{1}{2}$) such that $\text{tr}(\Phi(e(\Theta))|H) \equiv m_H(\Theta, c)$ and $\text{tr}(e(\Theta)|H') \equiv m_{H'}(\Phi^V(\Theta), c) \pmod{p^{c_7 \log_p M}}$. Since $v_p(\xi) \leq Mm = s - \log_p M$, the assumption of the Theorem implies
$$\text{tr}(\Phi(e(\Theta))|H) \equiv \text{tr}(e(\Theta)|H') \pmod{p^{c_7 \log_p M}},$$
hence,
$$m_H(\Theta, c) \equiv m_{H'}(\Phi^V(\Theta), c) \pmod{p^{c_7 \log_p M}}.$$ Since $m_H(\Theta, c)$ and $m_{H'}(\Phi^V(\Theta), c)$ are natural numbers which are smaller than $\frac{1}{2}M \leq p^{c_7 \log_p M}$ this implies
$$m_H(\Theta, c) = m_{H'}(\Phi^V(\Theta), c).$$ In particular, for any $\Theta \in \mathcal{E}(H)$ there is an element $\Theta' \in \mathcal{E}(H')$ such that $\Theta' \equiv \Phi^V(\Theta) \pmod{p^s}$. Hence, the proof is complete.

(2.5) **Corollary.** Assume that $\dim H$, $\dim H' < \frac{1}{2}M$ for some $M \in 2\mathbb{N}$. Assume that there is a rational number $s$ such that
$$\text{tr}(T|H) \equiv \text{tr}(T|H') \pmod{p^s}$$
for all $T \in H_0$. Then, for any $\Theta \in \hat{H}_0$ there is a rational number $c \geq \frac{s}{M} - (M + 2) \log_p M$ such that
$$m_H(\Theta, c) = m_{H'}(\Theta, c).$$ Hence, for any $\Theta \in \mathcal{E}(H)$ there is an eigencharacter $\Theta' \in \mathcal{E}(H')$ such that
$$\Theta' \equiv \Theta \pmod{p^{c_7 - (M + 2) \log_p M}}.$$ 

**Proof.** This is the special case $\mathcal{H} = H'$ and $\Phi = \text{id}$. 

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3 Systems of higher congruences in the finite slope case

Combining (1.9) Proposition with (2.5) Corollary we would obtain a trace criterion for the existence of $p$-adic continuous families of eigencharacters passing through a given eigencharacter. However, in this section we will construct certain elements which behave like an approximate idempotent attached to the slope $\alpha$ subspace of a $\mathcal{H}$-module $\mathcal{H}$; using this we then will obtain a criterion for the existence of finite slope $p$-adic continuous families of eigencharacters.

(3.1) Slope subspaces. As in (1.2) we set $\mathcal{H} = \bar{\mathbb{Q}}_p[T_\ell, \ell \in I]$ and $\mathcal{H}_\mathcal{O} = \mathcal{O}[T_\ell, \ell \in I]$ (countably generated). We fix an element $T \in \mathcal{H}_\mathcal{O}$. For any finite dimensional $\mathcal{H}$-module $\mathcal{H}$ which is defined over $\mathcal{O}$ with respect to the lattice $\mathcal{H}_\mathcal{O}$ we denote by $\mathcal{H}^\alpha, \alpha \in \mathbb{Q}_{\geq 0}$, the slope $\alpha$ subspace of $\mathcal{H}$ with respect to the operator $T$, i.e.

$$\mathcal{H}^\alpha = \bigoplus_{v_p(\mu) = \alpha} \mathcal{H}(\mu)$$

where $\mathcal{H}(\mu)$ is the generalized eigenspace attached to $T$ and $\mu$ and we set

$$\mathcal{H}^{\leq \alpha} = \bigoplus_{\beta \leq \alpha} \mathcal{H}^{\beta}.$$ 

We denote by $\Phi_\mathcal{H} \subset \mathcal{O}$ the set of eigenvalues of $T$ acting on $\mathcal{H}$ and by $\Phi^\alpha_\mathcal{H} \subseteq \Phi_\mathcal{H}$ the subset of all eigenvalues $\gamma$ of $T$ satisfying $v_p(\gamma) \leq \alpha$. Hence, $\mathcal{H}^{\leq \alpha} = \bigoplus_{\gamma \in \Phi^\alpha_\mathcal{H}} \mathcal{H}(\gamma)$.

(3.2) Construction of approximate idempotents. Let $\mathcal{H}, \mathcal{H}'$ be arbitrary $\mathcal{H}$-modules, finite dimensional and defined over $\mathcal{O}$ and select a slope $\alpha \in \mathbb{Q}_{\geq 0}$. For any $\gamma \in \Phi_\mathcal{H}$ we choose a basis $B = B(\gamma)$ of $\mathcal{H}(\gamma)$ such that the representing matrix $D_B(T|_{\mathcal{H}(\gamma)})$ of $T$ on $\mathcal{H}(\gamma)$ is upper triangular:

$$D_B(T|_{\mathcal{H}(\gamma)}) = \begin{pmatrix} \gamma & * \\ & \ddots \\ & & \gamma \end{pmatrix}.$$

Similarly, for any $\gamma \in \Phi_{\mathcal{H}'}$ we choose a basis $B' = B'(\gamma)$ of $\mathcal{H}'(\gamma)$ such that the matrix representing $T$ on $\mathcal{H}'(\gamma)$ is upper triangular:

$$D_{B'}(T|_{\mathcal{H}'(\gamma)}) = \begin{pmatrix} \gamma & * \\ & \ddots \\ & & \gamma \end{pmatrix}.$$ 

We define the element

$$e^{\leq \alpha} = e_{\mathcal{H}, \mathcal{H}'}^{\leq \alpha} = 1 - \prod_{\mu \in \Phi_{\mathcal{H}}^{\leq \alpha} \cup \Phi_{\mathcal{H}'}^{\leq \alpha}} \frac{T - \mu}{-\mu} \in \bar{\mathbb{Q}}_p[T].$$

Clearly, $e^{\leq \alpha} = p^{\leq \alpha}(T)$, where the polynomial $p^{\leq \alpha}(X)$ is given by

$$p^{\leq \alpha} = p_{\mathcal{H}, \mathcal{H}'}^{\leq \alpha} = 1 - \prod_{\mu \in \Phi_{\mathcal{H}}^{\leq \alpha} \cup \Phi_{\mathcal{H}'}^{\leq \alpha}} \frac{X - \mu}{-\mu} \in \bar{\mathbb{Q}}_p[X].$$
We want to collect some properties of $e^{\leq \alpha}$ and $p^{\leq \alpha}$. To this end we let $M(\alpha) \in 2\mathbb{N}$, $\alpha \in \mathbb{Q}_{\geq 0}$, be a collection of natural numbers such that

$$\dim H^{\leq \alpha} \leq \frac{1}{2} M(\alpha) \quad \text{and} \quad \dim H'^{\leq \alpha} \leq \frac{1}{2} M(\alpha)$$

for all $\alpha \in \mathbb{Q}_{\geq 0}$. For an arbitrary polynomial $p = \sum_{i \geq 0} a_i X^i \in \bar{\mathbb{Q}}[X]$ we define its slope as

$$S(p) = \sup \{ s \in \mathbb{Q} \cup \{-\infty\} : v_p(a_i) \geq si \text{ for all } i \geq 0 \},$$

hence, $S(p) > -\infty$ implies $v_p(a_0) \geq 0$. Easy calculation shows that

$$(3a) \quad S(pq) \geq \min\{S(p), S(q)\}$$

and

$$(3b) \quad S(p + q) \geq \min\{S(p), S(q)\}.$$ 

**Lemma.** 1.) For any $\gamma \in \Phi_H - \Phi_H^{\leq \alpha}$ we have

$$\mathcal{D}_B(e^{\leq \alpha}_{H,H'}|H(\gamma)) = \begin{pmatrix} \zeta & \ast & \cdots & \ast \\ \cdots & \cdots & \cdots & \cdots \\ \ast & \ast & \cdots & \ast \\ \zeta & \ast & \cdots & \ast \end{pmatrix}$$

where $\zeta \in \mathcal{O}$ satisfies $v_p(\zeta) \geq \frac{2}{M(\alpha + 1)}$. The analogous statement holds for $\gamma \in \Phi_H - \Phi_H^{\leq \alpha}$. 2.) For any $\gamma \in \Phi_H^{\leq \alpha}$ we have

$$\mathcal{D}_B(e^{\leq \alpha}_{H,H'}|H(\gamma)) = \begin{pmatrix} 1 & \ast & \cdots & \ast \\ \cdots & \cdots & \cdots & \cdots \\ \ast & \ast & \cdots & \ast \\ 1 & \ast & \cdots & \ast \end{pmatrix}.$$ 

Again, the analogous statement holds for $\gamma \in \Phi_H^{\leq \alpha}$.

3.)

$$\deg p^{\leq \alpha}_{H,H'} = |\Phi_H^{\leq \alpha}| + |\Phi_H'^{\leq \alpha}| \leq \dim H^{\leq \alpha} + \dim H'^{\leq \alpha} \leq M(\alpha).$$

4.)

$$p^{\leq \alpha}_{H,H'}(0) = 0$$

5.)

$$S(p^{\leq \alpha}_{H,H'}) \geq -\alpha.$$ 

In particular, $(p^{\leq \alpha})^L = \sum_{h \geq L} b_h X^h$, where $v_p(b_h) \geq -h\alpha$ for all $h \geq L$.

**Proof.** 1.) Let $\gamma \in \Phi_H - \Phi_H^{\leq \alpha}$. Equation (1) implies that with respect to $\mathcal{B} = \mathcal{B}(\gamma)$

$$\mathcal{D}_B(e^{\leq \alpha}|H(\gamma)) = \begin{pmatrix} \zeta & \ast & \cdots & \ast \\ \cdots & \cdots & \cdots & \cdots \\ \ast & \ast & \cdots & \ast \\ \zeta & \ast & \cdots & \ast \end{pmatrix}.$$
Let $\mu \in \Phi_{H}^{\leq \alpha} \cup \Phi_{H}^{\leq \alpha}$. We distinguish two cases. First, if $\gamma \notin \Phi_{H}^{\leq \alpha + 1}$, i.e. $v_{p}(\gamma) > \alpha + 1$, then we obtain $v_{p}(\gamma) > v_{p}(\mu) + 1 \geq v_{p}(\mu) + 2/M(\alpha + 1)$. Second, if $\gamma \in \Phi_{H}^{\leq \alpha + 1}$, then $\gamma$ is an eigenvalue of $T$ acting on $H^{\leq \alpha + 1}$, hence, it is a root of the characteristic polynomial of $T$ acting on $H^{\leq \alpha + 1}$ which has degree $\dim H^{\leq \alpha + 1} \leq 2M(\alpha + 1)$. We deduce that $\gamma$ is contained in an extension of $\mathbb{Q}_{p}$ of degree less than or equal to $\frac{1}{2}M(\alpha + 1)$ which implies that $v_{p}(\gamma) \in \mathbb{Z}_{\leq \frac{1}{2}M(\alpha + 1)}$. On the other hand, since $\mu \in \Phi_{H}^{\leq \alpha + 1} \cup \Phi_{H}^{\leq \alpha}$ we obtain quite similarly that $v_{p}(\mu) \in \mathbb{Z}_{\leq \frac{1}{2}M(\alpha + 1)}$.

Since $\gamma \notin \Phi_{H}^{\leq \alpha}$ we know that $v_{p}(\gamma) > v_{p}(\mu)$, hence, $v_{p}(\gamma) \geq v_{p}(\mu) + 2/M(\alpha + 1)$. Thus, in both cases we find $v_{p}(\frac{2}{\mu}) \geq 2/M(\alpha + 1)$. Since

$$\zeta = 1 - \prod_{\mu \in \Phi_{H}^{\leq \alpha} \cup \Phi_{H}^{\leq \alpha}} \frac{\gamma - \mu}{-\mu}$$

is a sum of products of the form $\pm \prod_{\mu} \frac{2}{\mu}$, where $\mu$ runs over a non-empty subset of $\Phi_{H}^{\leq \alpha} \cup \Phi_{H}^{\leq \alpha}$ (the summand "1" cancels), we deduce that $v_{p}(\zeta) \geq 2/M(\alpha + 1)$.

2.) Immediate by the definition of $e^{\leq \alpha}$.

3.) and 4.) Clear.

5.) By definition it is immediate that $S(\frac{L - \mu}{\mu}) = S(\frac{L}{\mu} + 1) = -v_{p}(\mu)$. All $\mu$ appearing in the definition of $e^{\leq \alpha}$ satisfy $v_{p}(\mu) \leq \alpha$; hence, using equation (3a) and (3b) we deduce $S(p^{\leq \alpha}) \geq \min\{0, -\alpha\} = -\alpha$. The second statement follows because $X$ divides $p^{\leq \alpha}$ and $S((p^{\leq \alpha})^{2}) \geq S(p^{\leq \alpha})$. This finishes the proof of the Lemma.

(3.4) Proposition. For any pair of finite dimensional $H$-modules $H$, $H'$ which are defined over $O$ and satisfy equation (2) and for any $T \in \mathcal{H}_{O}$ we have

$$\text{tr}(T(e^{\leq \alpha})^{L}(H)) \equiv \text{tr}(T(H^{\leq \alpha})) \pmod{p^{\frac{2L}{M(\alpha + 1)}}}.$$ 

The same congruence holds for $H'$ in place of $H$.

Proof. Let $\gamma \in \Phi_{H}$ and let $B$ be a basis of $H(\gamma)$ such that equation (1) holds. (3.3) Lemma implies that $T(e^{\leq \alpha})^{L}$ is represented on $H(\gamma)$ by the matrix

$$\begin{pmatrix}
\gamma \zeta^{L} & \ast & \cdots & \ast \\
\cdots & \ddots & \ddots & \ddots \\
\ast & \cdots & \gamma \zeta^{L}
\end{pmatrix},$$

where $\zeta \equiv 0 \pmod{\frac{2}{M(\alpha + 1)}}$ if $v_{p}(\gamma) > \alpha$ and $\zeta \equiv 1$ if $v_{p}(\gamma) \leq \alpha$. Since $v_{p}(\gamma) \geq 0$ ($T \in \mathcal{H}_{O}$) this implies the claim. The same argument works if we replace $H$ by $H'$, hence the proof is complete.

(3.5) Remark. In particular, we obtain

$$\lim_{L \to \infty} \text{tr}(T(e^{\leq \alpha})^{L}(H)) = \text{tr}(T(H^{\leq \alpha})$$

and the same holds if we replace $H$ by $H'$. Thus, $e^{\leq \alpha}$ behaves like an approximate idempotent attached to the slope $\leq \alpha$-subspaces of $H$ and $H'$. On the other hand, $e^{\leq \alpha}$ is not universal, i.e. it not only depends on $\alpha$ but also on the pair of modules $H$ and $H'$.
(3.6) A criterion for the existence of \( p \)-adic continuous families of finite slope. We give the synthesis of our results obtained so far. We use the notations from (1.8), i.e. \( G/\mathbb{Q} \) is a reductive algebraic group with maximal split torus \( T \). Let \( \mathcal{R} \subseteq X(T) \) be a subset and let \((\mathcal{H}_\lambda)\), \( \lambda \in \mathcal{R} \), be a family of finite dimensional \( \mathcal{H} \)-modules. From now on we will always assume that the following assumptions hold for the family \((\mathcal{H}_\lambda)\):

1. Any \( \mathcal{H}_\lambda \) is defined over \( \mathcal{O} \)
2. There are numbers \( M(\alpha) \in 2\mathbb{N}, \alpha \in \mathbb{Q}_{\geq 0} \), such that

\[
\dim \mathcal{H}_\lambda^{\leq \alpha} \leq \frac{1}{2} M(\alpha) \quad \text{for all } \alpha \in \mathbb{Q}_{\geq 0} \text{ and all } \lambda \in \mathcal{R}.
\]

We select a slope \( \alpha \in \mathbb{Q}_{\geq 0} \) and we put \( e_{\lambda,\lambda'} = e_{\mathcal{H}_\lambda,\mathcal{H}_{\lambda'}}^{\leq \alpha} \).

(3.7) Proposition. Let \((\mathcal{H}_\lambda)_{\lambda \in \mathcal{R}} \) be a family of \( \mathcal{H} \)-modules. Assume that there is a collection of rational numbers \( a' = a'(\alpha), a = a(a) \in \mathbb{Q}_{\geq 0} \) and \( b = b(\alpha) \in \mathbb{Q}_{<0} \) \((\alpha \in \mathbb{Q}_{\geq 0}) \) with the following property: i) \( a'/M(\alpha+1) \), \( a \) and \( b \) are decreasing in \( \alpha \) ii) for any \( \alpha \in \mathbb{Q}_{\geq 0} \) and any pair \( \lambda, \lambda' \in X(T) \) with \( \lambda \equiv \lambda' \pmod{(p-1)^mX(T)} \) there is a natural number \( L \geq a'(m+1) \) such that

\[
\text{(†) } \quad \text{tr} (e_{\lambda,\lambda'}^{L} T|\mathcal{H}_\lambda) \equiv \text{tr} (e_{\lambda,\lambda'}^{L} T|\mathcal{H}_{\lambda'}) \pmod{p^{a(m+1)+b}}
\]

for all \( T \in \mathcal{H}_\mathcal{O} \). Then

1.) \( \dim \mathcal{H}_\lambda^{\leq \alpha} \) is locally constant as a function of \( \lambda \), i.e. there is \( D = D(\alpha) \in \mathbb{N} \) only depending on \( \alpha \) such that \( \lambda \equiv \lambda' \pmod{(p-1)^mX(T)} \) implies \( \dim \mathcal{H}_\lambda^{\leq \alpha} = \dim \mathcal{H}_{\lambda'}^{\leq \alpha} \).

2.) Any \( \Theta \in \mathcal{E}(\mathcal{H}_\lambda^{\leq \alpha}) \) fits in a \( \mathcal{H} \)-adic continuous family of eigencharacters of slope \( \alpha \), i.e. there is a family \((\Theta_\lambda)_{\lambda \in \mathcal{R}} \) such that

1. \( \Theta_\lambda \in \mathcal{E}(\mathcal{H}_\lambda^{\leq \alpha}) \)
2. \( \Theta_{\lambda_0} = \Theta \)
3. \( \lambda \equiv \lambda' \pmod{(p-1)^mX(T)} \) implies

\[
\Theta_\lambda \equiv \Theta_{\lambda'} \pmod{p^{a(m+1)+b}},
\]

where \( a = \frac{1}{M(\alpha)} \min (a, \frac{2a'}{M(\alpha+1)}) \) and \( b = \frac{b}{M(\alpha)} - (M(\alpha) + 2) \log_p(M(\alpha)). \)

Proof. We select \( \alpha \in \mathbb{Q}_{\geq 0} \) and we proceed in steps.

a.) We let \( \lambda \equiv \lambda' \pmod{(p-1)^mX(T)} \). We choose \( L \geq a'(m+1) \) such that equation (†) holds. Together with (3.4) Proposition we obtain for all \( T \in \mathcal{H}_\mathcal{O} \)

\[
(4) \quad \text{tr} (T|\mathcal{H}_\lambda^{\leq \alpha}) \equiv \text{tr} (e_{\lambda,\lambda'}^{L} T|\mathcal{H}_\lambda) \equiv \text{tr} (e_{\lambda,\lambda'}^{L} T|\mathcal{H}_{\lambda'}) \equiv \text{tr} (T|\mathcal{H}_{\lambda'}^{\leq \alpha}) \pmod{p^s},
\]

where \( s = \bar{a}(m+1)+b \) with \( \bar{a} = \min (\frac{2a'}{M(\alpha+1)}, a) \) (note that \( b \leq 0 \)). We note that our assumptions imply that \( s \) is decreasing in \( \alpha \).
Thus, (2.5) Corollary implies that for all \( \Theta \in H \). Hence, by subtracting we obtain
\[
\dim H^\leq_\lambda = \text{tr}(1|H^\leq_\lambda) \equiv \text{tr}(1|H^\leq_\lambda) \pmod{p^{D(\alpha)+b}}.
\]
and since \( p^{D(\alpha)+b} = M(\alpha) > \dim H^\leq_\lambda \), \( \dim H^\leq_\lambda \) we obtain \( \dim H^\leq_\lambda = \dim H^\leq_\lambda \). Since \( D(\alpha) \) is increasing in \( \alpha \) the congruence \( \lambda \equiv \lambda' \pmod{(p-1)p^D(T)} \) even implies \( \dim H^\leq_\lambda = \dim H^\leq_\lambda \) for all \( \beta \leq \alpha \).

c.) Let \( \lambda_i \in X(T) \) and let \( \lambda_1 < \cdots < \lambda_s \leq \alpha \) be the non-trivial slopes appearing in \( H^\leq_\lambda \). Part b.) implies that for any \( \lambda \in \lambda_i + p^{D(\alpha)}X(T) \) the non trivial slopes appearing in \( H^\leq_\lambda \) again are \( \lambda_1 < \cdots < \lambda_s \) (note that \( D(\alpha) \) is increasing in \( \alpha \)). Since \( X(T) \cong \mathbb{Z}^k \) is covered by finitely many cosets \( \lambda_i + p^{D(\alpha)}X(T) \), \( i = 1, \ldots, s \) we can select non negative rational numbers \( 0 \leq \lambda_1 < \cdots < \lambda_s \leq \alpha \) such that for any \( \lambda \in \mathcal{R} \) the inequality \( H^\lambda_\alpha \neq 0 \) implies that \( \alpha \) is one of the \( \lambda_i \).

d.) We denote by \( \beta \) the largest of the numbers \( \alpha_1 < \cdots < \alpha_s \) which is strictly smaller than \( \alpha \). We obtain \( \dim H^\alpha_\lambda = \dim H^\leq_\lambda - \dim H^\leq_\lambda \beta \) for any \( \lambda \in \mathcal{R} \). Part b.) then implies that \( \dim H^\alpha_\lambda = \dim H^\alpha_\lambda \beta \) if \( \lambda \equiv \lambda' \pmod{(p-1)p^{D(\alpha)}X(T)} \).

e.) Finally, since \( s \) is decreasing in \( \alpha \) equation (4) still holds if we replace "\( \leq \alpha \)" by "\( \leq \beta \)". Hence, by subtracting we obtain
\[
\text{tr}(T|H^\leq_\lambda) \equiv \text{tr}(T|H^\leq_\lambda) \pmod{p^s}.
\]
Thus, (2.5) Corollary implies that for all \( \Theta \in \mathcal{H}_\alpha \)
\[
m_{H^\leq_\lambda}(\Theta, c) = m_{H^\leq_\lambda}(\Theta, c)
\]
for some \( c = c(\Theta) \geq a(m+1) + b \) with \( a \) and \( b \) as in the Proposition. (1.9) Proposition now implies the claim. This completes the proof of the Proposition.

Remark. Since \( H^\leq_\beta \leq H^\leq_\alpha \beta \leq \alpha \) it is natural that on \( H^\leq_\alpha \) weaker congruences for eigenvalues of operators hold, i.e. \( a \) and \( b \) are decreasing in \( \alpha \). In particular, smaller power of \( e_{\lambda, \lambda'} \) should be sufficient, i.e. \( a'/M(\alpha + 1) \) also should be decreasing in \( \alpha \).

B. Cohomology of the Siegel upper half plane

In this second part we consider an example: we will show that the family of cohomology groups of the Siegel upper half plane with coefficients in the irreducible representation of varying highest weight \( \lambda \) satisfies equation (1) in (3.7). As a consequence we obtain the existence of \( p \)-adic continuous families of Siegel eigenforms of finite slope \( \alpha \). We start by fixing some notation.

4 Notations

(4.1) The symplectic group. From now on we set \( G = \text{GSp}_{2n} \). Hence, for any \( \mathbb{Z} \)-algebra \( K \) we have
\[
G(K) = \{ g \in GL_{2n}(K) : g' \begin{pmatrix} I_n & \nu(g) I_n \\ -I_n & -I_n \end{pmatrix} g = \nu(g) \begin{pmatrix} I_n & I_n \\ -I_n & -I_n \end{pmatrix} \} \text{ for some } \nu(g) \in K^* \}. 
\]
The multiplier defines a character $\nu : G \to G_m$ and the derived group of $G$ is the symplectic group $G^0 = \text{Sp}_{2n}$ which is the kernel of $\nu$, i.e.

$$G^0(K) = \{ g \in \text{GSp}_{2n}(K) : \nu(g) = 1 \}.$$ 

Thus, $G^0(K)$ consists of all matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying

$$(1) \quad A^t B = B^t A, \quad C^t D = D^t C, \quad A^t D - B^t C = 1$$

and is simply connected. We set $Z = Z_G$, hence, $Z(K) = \{ \lambda \cdot I_{2n}, \lambda \in K^* \}$ and we denote by $T$ the maximal $\mathbb{Q}$-split torus in $G$ whose $K$-points are given by

$$(2) \quad T(K) = \{ \text{diag}(\alpha_1, \ldots, \alpha_n, \frac{\nu}{\alpha_1}, \ldots, \frac{\nu}{\alpha_n}), \alpha_i \in K^*, \nu \in K^* \}.$$ 

The intersection $T^0 = T \cap G$ is a maximal split torus in $G^0$ and

$$T^0(K) = \{ \text{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1}), \alpha_i \in K^* \}.$$ 

We denote by $g$ resp. $g^0$ resp. $h$ resp. $h^0$ the complexified Lie Algebra of $G/\mathbb{Q}$ resp. $G^0/\mathbb{Q}$ resp. $T$ resp. $T^0$. Thus,

$$g^0 = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{C}) : A = -D^t, B = B^t, C = C^t \}$$

and

$$h^0 = \{ \text{diag}(\alpha_1, \ldots, \alpha_n, -a_1, \ldots, -a_n), a_i \in \mathbb{C} \}.$$ 

We denote by $\Phi = \Phi(T^0, g^0)$ the root system of $G^0$ with respect to $T^0$; explicitly, the roots $\alpha \in \Phi$, a generator $X_\alpha$ of the corresponding root space $g_\alpha \leq g^0$ and the 1-parameter subgroup $\exp tX_\alpha$ are given as follows:

| $\alpha$ | $X_\alpha$ | $\exp tX_\alpha$ |
|---------|------------|----------------|
| $\epsilon_i - \epsilon_j$, $1 \leq j < i \leq n$ | $E_{i,j} - E_{j+i,n+i+n}$ | $1 + t(E_{i,j} - E_{j+i,n+i+n})$ positive $\Phi_1$ |
| $\epsilon_i - \epsilon_j$, $1 \leq i < j \leq n$ | $E_{i,j} - E_{j+i,n+i}$ | $1 + t(E_{i,j} - E_{j+i,n+i})$ positive $\Phi_1$ |
| $\epsilon_i + \epsilon_j$, $1 \leq i < j \leq n$ | $E_{i,n+j} - E_{j,n+i}$ | $1 + t(E_{i,n+j} + E_{j,n+i})$ positive $\Phi_2$ |
| $-\epsilon_i - \epsilon_j$, $1 \leq i < j \leq n$ | $E_{n+i,j} - E_{n+j,i}$ | $1 + t(E_{n+i,j} + E_{n+j,i})$ positive $\Phi_2$ |
| $2\epsilon_i$, $1 \leq i \leq n$ | $E_{i,n+i}$ | $1 + tE_{i,n+i}$ |
| $-2\epsilon_i$, $1 \leq i \leq n$ | $E_{n+i,i}$ | $1 + tE_{n+i,i}$ |

Here, $\epsilon_i : T^0 \to \mathbb{C}$ is defined by mapping $\text{diag}(\alpha_1, \ldots, \alpha_n, \alpha_1^{-1}, \ldots, \alpha_n^{-1})$ to $\alpha_i$ ($1 \leq i \leq n$) and $\exp : g^0 \to G(\mathbb{C})$ is the exponential. We choose as a basis of the root system

$$\Delta = \{ 2\epsilon_1, \epsilon_{i+1} - \epsilon_i, i = 1, \ldots, n - 1 \}.$$ 

We extend the roots $\alpha \in \Phi$ to $T$ by setting them equal to 1 on the center $Z \leq T$. 

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We denote by $X(T)$ the (additively written) group of $(\mathbb{Q}_p)$-characters of $T$. Since $T$ is connected, $X(T)$ is a finitely generated, free abelian group. Let $\gamma_1, \ldots, \gamma_{n+1}$ be a $\mathbb{Z}$-basis of $X(T)$. Thus, $g$ is the subgroup consisting of all matrices $g \in X(T)$ ($\lambda_i \in \mathbb{Z}$) we set

$$v_p(\lambda) = \min_i v_p(\lambda_i) = \max \{ m \in \mathbb{N}_0 : \lambda \in p^m X(T) \}.$$ 

Hence, $\lambda \equiv \lambda' \pmod{p^m X(T)}$ is equivalent to $v_p(\lambda - \lambda') \geq m$ (compare section (1.8)).

(4.3) Irreducible Representations. Let $\lambda \in X(T)$ be a dominant character, i.e. the restriction $\lambda^p = \lambda|_{T^p}$ is a dominant character of $T^p$ and $\lambda|_{\mathbb{Z}}$ is an (algebraic) character. We denote by $(\pi, L\lambda)$ the irreducible representation of $G(\bar{\mathbb{Q}}_p)$ of highest weight $\lambda$ on a $\mathbb{Q}_p$-vector space $L\lambda$. The representation $L\lambda$ is defined over $\mathbb{Z}$, i.e. there is a $\mathbb{Z}$-submodule $L\lambda(\mathbb{Z})$ in $L\lambda$ such that $L\lambda = L\lambda(\mathbb{Z}) \otimes \mathbb{Q}_p$ and which is stable under $G(\mathbb{Z})$. Thus, for any $\mathbb{Z}$-algebra $R$ we obtain a $G(R)$-module $L\lambda(R) = L\lambda(\mathbb{Z}) \otimes R$.

5 The Hecke algebra $\mathcal{H}$ (attached to $\text{GSp}_{2n}$)

We define the Hecke algebra $\mathcal{H}$ which we shall be using. Essentially $\mathcal{H}$ omits all Hecke operators at primes dividing the level and its local component at the prime $p$ is generated by one single Hecke operator $T_p$.

(5.1) The local level subgroup $\mathcal{I}$. Let $g = (g_{ij})_{ij} \in G(K)$ be arbitrary. We partition $g$ as

$$
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \\
\begin{cases}
\delta_1 & \cdots & B_1 \\
A_+ & \cdots & b_1 \\
& \ddots & \ddots \\
c_1 & \cdots & A'_+ \\
& \cdots & \delta_{n+1} \\
& C & \cdots & A'_- \\
& c_n & \cdots & \delta_{2n} \\
\end{cases}
$$

where $A_+ = (a_{ij}^+), A_- = (a_{ij}^-), B = (b_{ij}), B' = (b'_{ij})$ and so on (a prime indicates that the submatrix and its corresponding primed submatrix are connected via one of the relations in equation (1) in section 4; the positive root spaces correspond to entries in $A_+, B, B'$ and $b_1, \ldots, b_n$). We denote by $B \leq G$ the minimal parabolic subgroup attached to the basis $-\Delta$ of $\Phi$ and we define

$$\mathcal{I} \leq G(\mathbb{Z}_p)$$

as the subgroup consisting of all matrices $g$ such that $g \pmod{p\mathbb{Z}_p}$ is contained in $B(\mathbb{Z}_p/(p))$. Thus, $g \in G(\mathbb{Z}_p)$ as in equation (1) is contained in $\mathcal{I}$ precisely if

$$a_{ij}^+ \in p\mathbb{Z}_p, b_{ij} \in p\mathbb{Z}_p, b_1, \ldots, b_n \in p\mathbb{Z}_p.$$ 

(5.2) An auxiliary Lemma. For any prime $\ell$ we set

$$T(\mathbb{Q}_\ell)^+ = \{ h \in T(\mathbb{Q}_\ell) : v_\ell(\alpha(h)) \geq 0 \text{ for all } \alpha \in \Delta \}.$$
Lemma. 1.) Let $s \in T(\mathbb{Q}_p)^+$ and set

$$V_s = \{1 + t_\alpha X_\alpha, \alpha \in \Phi^-; t_\alpha \in \mathbb{Z}_p/p^{-v_p(\alpha(s))}\mathbb{Z}_p\} \quad (\subseteq B(\mathbb{Z}_p)).$$

Then, $I\mathcal{S}I = \bigcup_{v \in V_n} I\mathcal{S}v$.

2.) Let $s_1, s_2 \in T(\mathbb{Q}_p)^+$. Then

$$V_{s_1s_2} = \text{Ad}(s_2^{-1})(V_{s_1})V_{s_2}.$$  

Proof. 1.) Since $I\mathcal{S}I = \bigcup_{v \in V_n} I\mathcal{S}v$ with $v$ running over $s^{-1}\mathcal{I}s \cap \mathcal{I}\setminus \mathcal{I}$, we have to show that $V_s$ is a system of representatives for $s^{-1}\mathcal{I}s \cap \mathcal{I}\setminus \mathcal{I}$. To this end, we set $\alpha_{ij} = \alpha_i - \alpha_j$ and $\beta_{ij} = -\alpha_i - \alpha_j$, $1 \leq i < j \leq n$ and $\Phi_1 = \{\alpha_{ij}, 1 \leq i < j \leq n\}$ and $\Phi_2 = \{\beta_{ij}, 1 \leq i < j \leq n\}$; hence, $\Phi^- = \Phi_1 \cup \Phi_2$. Since $s^{-1}xs = \text{Ad}(s^{-1})(x)$ and $\text{Ad}(s)$ acts on $g_{\alpha}$ via $\alpha$ we obtain that $s^{-1}\mathcal{I}s \cap \mathcal{I}$ consists of all matrices $g = (g_{ij}) \in \mathcal{I}$ satisfying

$$3) \ g_{ij}, g_{j+n, j+n} \in p^{-v_p(\alpha_{ij}(s))}, \ 1 \leq i < j \leq n \quad \text{and} \quad g_{i+n, j}, g_{j+n, i} \in p^{-v_p(\beta_{ij}(s))}, \ 1 \leq i < j \leq n.$$  

Now let $g = (g_{ij}) \in \mathcal{I}$ be arbitrary. We will perform two sets of elementary row operations to transform $g$ into the unit matrix.

a.) Multiplying $g$ from the left by matrices of the form $1 + t(E_{ij} - E_{j+n,i+n}) = (1 + tE_{ij})(1 - tE_{n+j,n+i})$ with $1 \leq i < j \leq n$ and $t \in \mathbb{Z}_p/p^{-v_p(\alpha_{ij}(s))}\mathbb{Z}_p$, i.e. by performing (two) elementary row operations we can achieve that all entries $g_{ij}$ with $1 \leq i < j \leq n$ are contained in $p^{-v_p(\alpha_{ij}(s))}\mathbb{Z}_p$.

(First eliminate the entries in the most right column in $A_-$, then eliminate the entries in column $n-1$ and so on; note that the $\delta_i = g_{ii}$ are units in $\mathbb{Z}_p$.) Analogously, multiplying from the left by matrices $1 + t(E_{n+j,i} + E_{n+i,j}) = (1 + tE_{n+j,i})(1 + tE_{n+i,j})$ with $1 \leq i < j \leq n$ and $t \in \mathbb{Z}_p/p^{-v_p(\beta_{ij}(s))}\mathbb{Z}_p$ we can achieve that $g_{n+j,i} \in p^{-v_p(\beta_{ij}(s))}\mathbb{Z}_p$ for all $1 \leq i < j \leq n$. We note that the matrices by which we multiplied are contained in $V_s$ because they are contained in 1-parameter subgroups corresponding to negative roots (namely $\alpha_{ij}, \beta_{ij}$; cf. the table in section (4.1)).

b.) Now, multiplying further from the left - by matrices $1 + t(E_{ij} - E_{j+n,i+n})$ with $i < j$ and $t \in p^{-v_p(\alpha_{ij}(s))}$ we can achieve $A_- = 0$ - by matrices $1 + (E_{i+n,j} + E_{j+n,i})$ with $i \leq j$, $t \in p^{-v_p(\beta_{ij}(s))}$ we can achieve $C = 0$ and $c_{11} = 0, \ldots, c_{nn} = 0$ - by matrices $1 + (tE_{ij} - E_{j+n,i+n})$ with $i > j$, $t \in p^{-v_p(\alpha_{ij}(s))}$ we can achieve $A_+ = 0$ - by matrices $1 + (tE_{i+n,j} + E_{j+n,i})$ with $j \geq i$, $t \in p^{-v_p(\beta_{ij}(s))}$ we can achieve $B = 0$ and $b_{11} = 0, \ldots, b_{nn} = 0$ - by matrices $\delta_i^{-1}, \ldots, \delta_n^{-1}$, we can achieve $\delta_1 = \cdots = \delta_n = 1$.

We note that the matrices by which we multiplied are contained in $\mathcal{I} \cap s^{-1}\mathcal{I}s$ by equations (2) and (3). Since $A = 1$, the first relation in equation (1) in section 4 implies $B = B'$, hence, $B = 0$. The last relation in equation (1) in section 4 then implies $D = 1$ and the second relation in (1) in section 4 finally implies $C = C'$, hence, $C = 0$. Thus, we have found that

$$\prod_j k_j \prod_i v_i g = 1$$

with certain $k_j \in \mathcal{I} \cap s^{-1}\mathcal{I}s$ and $v_i \in V_s$. Thus, $(\prod_i v_i)g$ is contained in $s^{-1}\mathcal{I}s \cap \mathcal{I}\setminus \mathcal{I}$ which shows that $V_s$ is a system of representatives for $s^{-1}\mathcal{I}s \cap \mathcal{I}\setminus \mathcal{I}$. Taking into account that the entries of a
matrix \( g = (g_{ij}) \in s^{-1}\mathcal{I} \cap \mathcal{I} \) satisfy equation (3) it is not difficult to verify that the elements in \( \mathcal{V}_s \) are different modulo \( s^{-1}\mathcal{I} \cap \mathcal{I} \). Thus, the proof of the first part is complete.

2.) Since \( \text{Ad}(s_2)X_{\alpha} = \alpha(s_2)X_{\alpha} \) we find that

\[
\text{Ad}(s_2^{-1})(\mathcal{V}_{s_1}) = \{ 1 + t_\alpha X_{\alpha}, \alpha \in \Phi^{-}, t_\alpha \in p^{v_p(\alpha)}B_p/p^{v_p(\alpha)+v_p(\alpha(s_2^{-1}))}B_p \}.
\]

Since \( v_p(\alpha(s_2^{-1})) = -v_p(\alpha(s_2)) \) and \( v_p(\alpha(s_1)) + v_p(\alpha(s_2^{-1})) = -v_p(\alpha(s_1)) \) an easy calculation yields the claim. This finishes the proof of the Lemma.

(5.3) The local Hecke algebra at \( p \). We set

\[
D_p = \text{TT}(\mathbb{Q}_p)^+ \I \leq \text{G}(\mathbb{Q}_p).
\]

In the Proposition below we will see that \( D_p \) is a semigroup, hence, we can define the local Hecke algebra

\[
\mathcal{H}(\mathcal{I} \setminus D_p/\mathcal{I})
\]

attached to the pair \( (D_p, \mathcal{I}) \). For any \( s \in \text{T}(\mathbb{Q}_p)^+ \) we define the element \( T_s = \mathcal{I}s\mathcal{I} \), i.e. \( \mathcal{H}(\mathcal{I} \setminus D_p/\mathcal{I}) \) is the \( \mathbb{Z} \)-linear span of the elements \( T_s, s \in \text{T}(\mathbb{Q}_p)^+ \).

**Proposition.** 1.) \( D_p \leq \text{G}(\mathbb{Q}_p) \) is a semi group.

2.) For all \( s_1, s_2 \in \text{T}(\mathbb{Q}_p)^+ \) we have \( T_{s_1}T_{s_2} = T_{s_1,s_2} \). In particular, the Hecke algebra \( \mathcal{I} \setminus D_p/\mathcal{I} \) is commutative.

**Proof.** 1.) We compute

\[
\mathcal{I}s_1\mathcal{I}s_2\mathcal{I} = \bigcup_{v \in \mathcal{V}_{s_1}} \mathcal{I}s_1vs_2\mathcal{I} = \bigcup_{v \in \mathcal{V}_{s_1}} \mathcal{I}s_1s_2\text{Ad}(s_2^{-1})(v)\mathcal{I}
\]

Since \( s_2 \in \text{T}(\mathbb{Q}_p)^+ \) we know that \( v_p(\beta(s_2^{-1})) \leq 0 \) for all positive roots \( \beta \). Hence, we obtain for any \( v = 1 + t_\alpha X_{\alpha} \in \mathcal{V}_{s_1} \):

\[
\text{Ad}(s_2^{-1})(v) = 1 + t_\alpha \alpha(s_2^{-1})X_{\alpha} \in B(\mathbb{Z}_p) \subset \mathcal{I}
\]

(note that by definition of \( \mathcal{V}_{s_1} \) the root \( \alpha \) is negative). Thus, \( \text{Ad}(s_2^{-1})(v) \in \mathcal{I} \) and we obtain \( \mathcal{I}s_1\mathcal{I}s_2\mathcal{I} = \mathcal{I}s_1s_2\mathcal{I} \). Thus \( D_p \) is closed under multiplication.

2.) To prove the second claim we compute using part 2.) of the previous Lemma

\[
T_{s_1}T_{s_2} = \bigcup_{v \in \mathcal{V}_{s_1}, w \in \mathcal{V}_{s_2}} \mathcal{I}s_1vs_2w = \bigcup_{v \in \mathcal{V}_{s_1}, w \in \mathcal{V}_{s_2}} \mathcal{I}s_1s_2\text{Ad}(s_2^{-1})(v)w = \bigcup_{z \in \mathcal{V}_{s_1s_2}} \mathcal{I}s_1s_2z = T_{s_1s_2}.
\]

Thus, the proof of the Proposition is complete.

(5.4) The adelic Hecke Algebra. We fix an integer \( N \) which is not divisible by \( p \). We select a compact open subgroup \( \mathcal{U} = \prod_{\ell \neq \infty} U_\ell \) of \( \text{G}(\mathbb{Z}) \) and a sub semigroup \( D = \prod_{\ell \neq \infty} D_\ell \) of \( \text{G}(\mathbb{A}_f) \) as follows. For all primes \( \ell \) not dividing \( pN \) we set \( U_\ell = \text{G}(\mathbb{Z}_\ell) \) and \( D_\ell = \text{G}(\mathbb{Q}_\ell) \); at the prime \( p \) we define \( U_p = \mathcal{I} \) and \( D_p = \text{TT}(\mathbb{Q}_p)^+ \mathcal{I} \) as in the previous section; for primes \( \ell | N \) we only assume that \( U_\ell \leq \text{G}(\mathbb{Z}_\ell) \) is compact open and \( \text{det}(U_\ell) = \mathbb{Z}_\ell^* \) and we set \( D_\ell = U_\ell \). We denote by

\[
\mathcal{H}(U_\ell \setminus D_\ell/U_\ell) \quad \text{resp.} \quad \mathcal{H}(U \setminus D/U)
\]

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the local resp. global (adelic) Hecke algebra attached to the pair \((U_\ell, D_\ell)\) resp. \((U, D)\). We exhibit a set of generators for \(\mathcal{H}(U \setminus D/U)\) as follows. We define

\[
\Sigma^+_\ell = \{ \text{diag}(\ell^{e_1}, \ldots, \ell^{e_n}, \ell^{-e_1}, \ldots, \ell^{-e_n}) \mid e_i \in \mathbb{Z}, 0 \leq e_1 \leq e_2 \leq \cdots \leq e_n \}
\]

if \(\ell \nmid N\) and \(\Sigma^+_\ell = \{1\}\) if \(\ell \mid N\). Thus, \(\nu_\ell(\alpha(h)) \geq 0\) for all \(h \in \Sigma^+_\ell\) and all \(\alpha \in \Delta\). Any element \(T = U_\ell s U_\ell\) in \(\mathcal{H}(U \setminus D_\ell/U_\ell)\) has a representative \(s \in \Sigma^+_\ell\) (for primes \(\ell\) not dividing \(Np\) this is well known and for primes \(\ell\) dividing \(Np\) this is immediate from the definition of \(U_\ell\) and \(D_\ell\)). In particular, the \(\mathbb{Z}\)-algebra \(\mathcal{H}(U \setminus D/U)\) is generated by the following elements

\[
\mathcal{H}(U \setminus D/U) = \langle UsU, s \in \bigcup_\ell \Sigma^+_\ell \rangle.
\]

Moreover, \(\mathcal{H}(U \setminus D/U)\) is commutative, because this holds locally at all primes: for primes \(\ell \nmid Np\) again this is well known, for primes \(\ell \mid N\) this is trivial and for \(p\) this follows from (5.3) Proposition.

### (5.5) The global (non-adelic) Hecke algebra

We set

\[
\Gamma = U \cap G(\mathbb{Q}) \quad \text{and} \quad \Delta = D \cap G(\mathbb{Q}).
\]

Thus, \(\Gamma\) satisfies the following local condition at the prime \(p\):

\[
\Gamma \leq \mathcal{I}(\leq G(\mathbb{Q}_p)).
\]

We note that \(U \subset D\), hence, \(\Gamma \subset \Delta\). In the next Lemma we compare the Hecke algebras attached to the pairs \((D, U)\) and \((\Delta, \Gamma)\)

\[
\Gamma \subset \Delta \quad \cap \quad U \subset D.
\]

### (5.5.1) Lemma

The canonical map \(\Gamma \alpha \Gamma \mapsto U\alpha U\) induces an isomorphism of rings

\[
\mathcal{H}(\Gamma \setminus \Delta / \Gamma) \rightarrow \mathcal{H}(U \setminus D/U).
\]

**Proof.** According to [M], Theorem 2.7.6, p. 72, we have to show that

i) \(D = \Delta U\)

ii) \(U\alpha U = U\alpha \Gamma\) for all \(\alpha \in \Delta\)

iii) \(U\alpha \cap \Delta = \Gamma\alpha\) for all \(\alpha \in \Delta\).

i) is an immediate consequence of strong approximation which holds since \(\text{det}(U) = \hat{\mathbb{Z}}^*\). We prove ii). The inclusion “\(\supseteq\)” is obvious. To prove the reverse inclusion we note that \(U\alpha U = \bigcup \alpha U\alpha\), where \(v\) runs over a system of representatives of \(\alpha^{-1}U\alpha \cap U\setminus U\). Thus, we have to show that \(\Gamma\) contains a system of representatives of \(\alpha^{-1}U\alpha \cap U\setminus U\). Let \(u \in U\) be arbitrary. Since \(\alpha^{-1}U\alpha \cap U \leq G(\mathbb{A}_f)\) is a compact open subgroup, strong approximation yields \(u = \gamma v\) with \(\gamma \in G(\mathbb{Q})\) and \(v \in \alpha^{-1}U\alpha \cap U\). Hence, \(\gamma = uv^{-1}\) is contained in \(U \cap G(\mathbb{Q}) = \Gamma\). Thus, \(\gamma\) is a representative of the coset of \(u\) in \(\alpha^{-1}U\alpha \cap U\setminus U\). It remains to verify that strong approximation holds with respect to \(\alpha^{-1}U\alpha \cap U\), i.e. \(\text{det}(\alpha^{-1}U\alpha \cap U) = \hat{\mathbb{Z}}^*\). It is sufficient to prove this locally for all primes \(\ell\). If \(\ell \nmid N\) we know by definition of \(U_\ell, D_\ell\) that \(\alpha \in D_\ell = U_\ell\), hence, \(\text{det}(U_\ell \cap \alpha^{-1}U_\ell\alpha) = \hat{\mathbb{Z}}^*_\ell\) because the determinant is surjective on \(U_\ell\). If \(\ell \mid N\), then the Cartan decomposition in case \(\ell \nmid Np\) and
the definition of $D_p$ show that $\alpha \in D_\ell$ can be written $\alpha = u_1 t u_2$ where $u_1, u_2 \in U_\ell$ and $t \in D_\ell$ is a diagonal matrix. We obtain $\det(U_\ell \cap \alpha^{-1} U_\alpha) = \det(U_\ell \cap t^{-1} U_\ell t)$. Let $\lambda \in \mathbb{Z}_\ell^*$ arbitrary; since $s = \text{diag}(\lambda, 1, \ldots, 1) \in U_\ell$ commutes with $t$ we see that it is contained in $U_\ell \cap t^{-1} U_\ell t$. Since $\det(s) = \lambda$ we obtain $\det(U_\ell \cap \alpha^{-1} U_\alpha) = \mathbb{Z}_\ell^*$. Thus, strong approximation holds and ii) is proven. Finally, iii) is immediate since $\Gamma$ is contained in $U$ and in $\Delta$ and since $U \cap G(\mathbb{Q}) = \Gamma$.

Thus, the proof of the lemma is complete.

(5.5.2) Since the (adelic) Hecke algebra attached to $(D, U)$ is commutative (cf. section (5.4)), we obtain from the above Lemma that $\mathcal{H}(\Gamma \setminus \Delta / \Gamma)$ is a commutative algebra. For $s \in \Delta$ we set $T_s = \Gamma s \Gamma$;

equation (4) in (5.4) and (5.5.1) Lemma imply that $\mathcal{H}(\Gamma \setminus \Delta / \Gamma)$ is generated by the following elements

\[ \mathcal{H}(\Gamma \setminus \Delta / \Gamma) = \langle \Gamma s \Gamma, s \in \bigcup_{\ell} \Sigma_\ell^+ \rangle \]

(note that $\Sigma_\ell^+ \subseteq \Delta$ for all primes $\ell$). We define the element

\[ h_p = \text{diag}(p^1, p^2, \ldots, p^n, p^0, p^{-1}, \ldots, p^{-n+1}) \in \Sigma_p^+, \]

hence, $\alpha(h_p) = p$ for all $\alpha \in \Delta$ and we denote the corresponding Hecke operator by

\[ T_p = T_{h_p} = \Gamma h_p \Gamma. \]

Since $T_p$ maps to $I h_p I \in \mathcal{H}(\mathcal{I} \setminus D_p / \mathcal{I}) \leq \mathcal{H}(U \setminus D / U)$ under the isomorphism in (5.5.1) Lemma, (5.3) Proposition part 2.) implies that

(6) \[ T_p^e = T_{h_p}^e = T_{h_p}^e. \]

(5.6) The Hecke algebra $\mathcal{H}$. We define the $\mathbb{Z}$-algebra $\mathcal{H}_{\mathbb{Z}}$ as the $\mathbb{Z}$-subalgebra of $\mathcal{H}(\Gamma \setminus \Delta / \Gamma)$ which is generated by the Hecke operators $T_s$ with $s \in \bigcup_{\ell \nmid N_p} \Sigma_\ell^+ \cup \{ h_p \}$; hence,

\[ \mathcal{H}_{\mathbb{Z}} \cong \bigotimes_{\ell \nmid N_p} \mathcal{H}(U_\ell \setminus D_\ell / U_\ell) \otimes \mathbb{Z}[T_p]. \]

We set

\[ \Sigma^+ := \prod_{\ell \nmid N_p} \Sigma_\ell^+ \cup \{ h_p^m, m \in \mathbb{N}_0 \}. \]

As a $\mathbb{Z}$-module, $\mathcal{H}_{\mathbb{Z}}$ then is generated by the operators $T_s$ with $s \in \Sigma^+$. Finally, we put $\mathcal{H} = \mathcal{H} \otimes \mathbb{Q}_p$ and $\mathcal{H}_O = \mathcal{H}_{\mathbb{Z}} \otimes \mathbb{O}$.

6 The cohomology groups $H_\lambda$ (attached to the Siegel upper half plane)

We recall the normalization of the Hecke operators which leads to an action of the Hecke algebra on the $p$-adically integral cohomology. We also state the simple topological trace formula of Bewersdorff.
(6.1) Normalization of Hecke algebra representations. We keep the notations from section 5. In particular, $\Gamma = U \cap G(\mathbb{Q}) \leq G(\mathbb{Z})$ with $U \leq G(\mathbb{Z})$ defined as in (5.4) and $H_\mathbb{Z} \cong \bigotimes_{\ell \mid Np} \mathcal{H}(U_\ell \setminus D_\ell / U_\ell) \otimes \mathbb{Z}[T_\ell]$ is the Hecke algebra defined in (5.6). To ensure that the Hecke algebra later will act on $p$-adically integral cohomology, we have to normalize the action of the Hecke algebra. This depends on a choice of a dominant character $\lambda \in X(T)$. We first define a $\mathbb{Z}$-algebra morphism

$$\varphi_\lambda = \bigotimes_{\ell \mid Np} \varphi_\ell : \mathcal{H}_\mathbb{Z} \to \mathcal{H}_\mathbb{Z}$$

as follows. For all primes $\ell \nmid Np$ we denote by $\varphi_{\lambda,\ell} : \mathcal{H}(U_\ell \setminus D_\ell / U_\ell) \to \mathcal{H}(U_\ell \setminus D_\ell / U_\ell)$ the identity map. At the prime $p$ we note that $\mathcal{H}(U_p \setminus D_p / U_p) = \mathbb{Z}[T_p]$ is a polynomial algebra generated by $T_p$. We define $\varphi_{\lambda,p} : \mathcal{H}(U_p \setminus D_p / U_p) \to \mathcal{H}(U_p \setminus D_p / U_p)$ by sending $T_p = T_{h_p}$ to $(T_p)_\lambda := \lambda(h_p)T_p$. Equation (6) in (5.5) implies that $\varphi_{\lambda}(T_{h_p}) = \varphi_{\lambda}(T_p^e) = \lambda(h_p^e)T_p^e$. Moreover, since $\lambda$ is dominant and $h_p \in \Sigma_+^e$ we obtain $\lambda(h_p) \in \mathbb{Z}$, hence, $\varphi_{\lambda}(T_p) = \lambda(h_p)T_p \in \mathcal{H}_\mathbb{Z}$ and $\varphi_\lambda$ is defined over $\mathbb{Z}$. Tensoring with $\mathbb{Q}_p$ we obtain a $\mathbb{Q}_p$-algebra morphism

$$\varphi_\lambda : \mathcal{H} \to \mathcal{H}$$

which is defined over $\mathcal{O}$.

Let $H$ be a $\mathcal{H}$-module. We define the $\lambda$-normalization "$\lambda$" of the action of $\mathcal{H}$ on $H$ by composing the $\mathcal{H}$-module structure on $H$ with the $\mathbb{Q}_p$-algebra morphism $\varphi_\lambda$, i.e.

$$T \cdot \lambda v = \{T\}_\lambda v \quad (T \in \mathcal{H}, \ v \in H).$$

Thus,

$$T_h \cdot \lambda v = T_h v \quad \text{if } h \in \Sigma_+^e, \ \ell \nmid Np \quad \text{and} \quad T_p^e \cdot \lambda v = \lambda(h_p^e)T_p^e v.$$  

(6.2) The Cohomology groups. We select a maximal compact subgroup $K_\infty$ of the connected component of the identity of $G(\mathbb{R})$ and we denote by $X = G(\mathbb{R})/K_\infty \mathbb{Z}(\mathbb{R})$ the symmetric space. We denote by $\mathcal{L}_\lambda$ the sheaf on $\Gamma \setminus X$ attached to the irreducible $G(\mathbb{Q}_p)$-module $L_\lambda$ of highest weight $\lambda \in X(T)$ (cf. (4.3)). The cohomology groups

$$H_\lambda = H^d(\Gamma \setminus X, \mathcal{L}_\lambda)$$

then are modules under the Hecke algebra $\mathcal{H}$. From now on, by $d = d_\lambda$ we will always understand the middle degree of the locally symmetric space $\Gamma \setminus X$. We denote by

$$H_{\lambda, \mathcal{O}} = H^d(\Gamma \setminus X, \mathcal{L}_\lambda)_{\mathcal{O}}$$

the image of $H^d(\Gamma \setminus X, \mathcal{L}_\lambda(\mathcal{O}))$ in $H^d(\Gamma \setminus X, \mathcal{L}_\lambda)$. $H^d(\Gamma \setminus X, \mathcal{L}_\lambda)_{\mathcal{O}}$ is a $\mathcal{O}$-lattice in $H^d(\Gamma \setminus X, \mathcal{L}_\lambda)$. On the other hand, the normalized Hecke operators $\lambda(h)T_h$, $h \in \Sigma^+$, act on cohomology with integral coefficients $H^d(\Gamma \setminus X, \mathcal{L}_\lambda(\mathbb{Z}))$ (cf. e.g. the proof of (7.2) Lemma below; cf. also [Ma 1], (5.4) Lemma for more details). If $h \in \Sigma^+_{\ell}$ with $\ell \nmid Np$ then $\lambda(h) \in \mathcal{O}^*$ is a $p$-adic unit, hence, the not normalized Hecke operator $T(h)$ already acts on cohomology with $p$-adically integral coefficients $H^d(\Gamma \setminus X, \mathcal{L}_\lambda(\mathcal{O}))$. Thus, we only have to normalize the Hecke operator at the prime $p$ i.e. we obtain that w.r.t. the $\lambda$-normalized action the Hecke algebra $\mathcal{H}_{\mathcal{O}}$ acts on cohomology with $p$-adically integral coefficients and hence, acts on $H^d(\Gamma \setminus X, \mathcal{L}_\lambda)_{\mathcal{O}}$. Thus, $H_{\lambda}$ is a finite dimensional $\mathcal{H}$-module which is defined over $\mathcal{O}$ with respect to the lattice $H_{\lambda, \mathcal{O}}$.  

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(6.3) Slope subspace of cohomology. We define the slope \( \leq \beta \) subspace
\[
H^\leq_{\lambda} = H^d(\Gamma \setminus X, L_\lambda)^\leq_{\beta}
\]
with respect to the action of the normalized Hecke operator \( \{ T_p \}_\lambda = \lambda(h_p)T_p \). In [Ma 1], 6.5 Theorem it is proven using only elementary means from representation theory that there are natural numbers \( M(\beta), \beta \in \mathbb{Q}_{\geq 0} \), s. t.
\[
\dim H^\leq_{\lambda} \leq \frac{1}{2} M(\beta) \quad \text{for all dominant } \lambda \in X(T) \text{ and all } \beta \in \mathbb{Q}_{\geq 0}.
\]
Hence, the assumptions in (3.6) are satisfied.

(6.4) A simple topological Trace Formula. We denote by \( \sim_{\Gamma} \) the equivalence relation on \( G(\mathbb{Q}) \) defined by conjugation, i.e. \( x \sim_{\Gamma} y, x, y \in G(\mathbb{Q}) \), precisely if \( x, y \) are conjugate by an element \( \gamma \in \Gamma \) and we denote by \( [\Xi]_{\Gamma} = [\Xi] \) the conjugacy class of \( \Xi \in G(\mathbb{Q}) \).

Theorem (cf. [B]). Let \( h \in \Sigma^+ \). There are integers \( c_{[\Xi]} \in \mathbb{Z}, [\Xi] \in \Gamma h\Gamma/ \sim_{\Gamma} \), such that the following holds. For all irreducible representations \( L_\lambda \) we have
\[
\text{tr} (T_h|H^d(\Gamma \setminus X, L_\lambda)) = \sum_{[\Xi] \in \Gamma h\Gamma/ \sim_{\Gamma}} c_{[\Xi]} \text{tr} (\Xi^{-1}|L_\lambda).
\]
Proof. This is a direct consequence of 2.6 Satz in [B] taking into account that for regular weight \( \lambda \) the cohomology of \( \Gamma \setminus X \) vanishes in all degrees except for the middle degree \( d \).

(6.5) Remark. 1.) We would like to emphasize that the proof of the simple trace formula of Bewersdorff is elementary. The only deeper ingredient is the existence of a good compactification of \( \Gamma \setminus X \), which is the Borel-Serre compactification. Apart from that the proof only uses very general and basic principles of algebraic topology. (In [B], the formula in equation (2) only serves as a starting point for further investigations).

2.) The terms appearing on the geometric side of Bewersdorff’s trace formula are the archimedean components of orbital integrals on the symplectic group.

7 Verification of Identity (†) in section (3.7)

We show that the family of cohomology groups \( (H^d(\Gamma \setminus X, L_\lambda)) \) satisfies equation (†) in (3.7). This is the main technical work. Use of Bewersdorff’s trace formula reduces the verification of equation (†) to congruences between values of irreducible characters and use of the Weyl character formula further reduces to a problem about conjugacy of certain symplectic matrices with which we shall begin.

We keep the notations from section 6. In particular, \( \Gamma = G(\mathbb{Q}) \cap U \leq G(\mathbb{Z}) \) is an arithmetic subgroup as in (5.4), hence, \( \Gamma \leq I \leq G(\mathbb{Q}_p) \) and \( \mathcal{H}_\mathcal{O} = \mathcal{H}(\Gamma \setminus \Delta/\Gamma) \otimes \mathcal{O} \) is the Hecke algebra defined in (5.6); in particular \( T_p = T_{h_p} \), where \( h_p \) is the diagonal matrix defined in (5.5.2).

(7.1) Lemma. Let \( \gamma \in \Gamma \) and \( h \in T(\mathbb{Z}_p) \). The matrix \( h^{-1}h_p^{-e}\gamma^{-1} \in G(\mathbb{Q}_p) \leq GL_{2n}(\mathbb{Q}_p), e \in \mathbb{N}, \) is \( G(\mathbb{Q}_p) \)-conjugate to a diagonal matrix
\[
\xi = \text{diag}(\xi_1, \ldots, \xi_{2n}) \in T(\mathbb{Q}_p)
\]
satisfying \( v_p(\alpha(\xi)) = -\varepsilon \) for all \( \alpha \in \Delta \).

**Proof.** We proceed in several steps.

a.) We begin by writing \( \gamma = (\gamma_{ij}) \) and \( h = \text{diag}(h_1, \ldots, h_{2n}) \); note that the entries of \( h \) as well as the diagonal entries \( \gamma_{ii} \) of \( \gamma \) are \( p \)-adic units. We set

\[
\underline{h}_p = \text{diag}(p^{(n-1)e}, p^{(n-2)e}, \ldots, p^e, p^{n-1}e, \ldots, p^{(2n-1)e}),
\]

hence, \( \underline{h}_p \) differs from \( h_p^{-e} \) by a scalar multiple \( p^{ne} \) and has integer entries. In particular, the matrices \( h^{-1}_p h_p^{-e} \gamma^{-1} \) and \( A = h^{-1}_p \underline{h}_p^{-e} \gamma^{-1} \) differ by a scalar factor \( p^{ne} \), hence, we may replace \( h^{-1}_p h_p^{-e} \gamma^{-1} \) by \( A \). We denote by

\[
\chi(T) = T^{2n} + c_1 T^{2n-1} + \cdots + c_{2n} \in \mathbb{Q}_p[T]
\]

the characteristic polynomial of \( A \). We write \( A = (a_{ij}) \). Since \( \gamma^{-1} \in \Gamma \leq \mathcal{I} \), we obtain

\[
v_p(a_{ij}) \geq \left\{ \begin{array}{ll}
(n - i)e & i \leq n \\
(i - 1)e & i > n.
\end{array} \right.
\]

More precisely, we find

\[
v_p(a_{ii}) = \left\{ \begin{array}{ll}
(n - i)e & i \leq n \\
(i - 1)e & i > n
\end{array} \right.
\]

and

\[
v_p(a_{ij}) \geq \left\{ \begin{array}{ll}
(n - i)e + 1 & i \leq n \\
(i - 1)e + 1 & i > n
\end{array} \right.
\]

if \( a_{ij} \) is one of the entries "\( \mu_{ij}, \rho_{ij} \) or \( \rho'_{ij} \)" below, i.e. if \( j < i \leq n \) or \( i \leq n \) and \( j > n \) or \( i, j > n \) and \( j > i \)

\[
A = \left( \begin{array}{cccccc}
\tau_1 & \beta_{ij} & \cdots & \cdots & \rho_{ij} \\
\cdots & \mu_{ij} & \tau_n & \cdots & \cdots & \rho_{ij} \\
\mu_{ij} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
* & \cdots & \cdots & \cdots & \cdots & \cdots \\
\beta'_{ij} & \cdots & \cdots & \cdots & \cdots & \tau_{2n}
\end{array} \right).
\]

We recall that

\[
\chi(T) = \text{det}(A - TT) = \sum_{\pi \in S_{2n}} \text{sgn}(\pi) \prod_{i=1}^{\pi} (a_{i, \pi(i)} - \delta_{i, \pi(i)}). \]

A permutation \( \pi \in S_{2n} \) contributes to \( c_i \) (i.e. the summand corresponding to \( \pi \) contributes in degree \( T^{2n-1} \)) only if \( \pi \) has at least \( 2n - i \) fixed points. We denote by \( \text{Fix}(\pi) \) the set of fixed points of \( \pi \) and obtain

\[
c_i = \sum_{\pi \in S_{2n}} \text{sgn}(\pi) \prod_{i=1}^{2n} a_{i, \pi(i)}. \]
In particular, the coefficient \( c_i \) of \( \chi \) is a sum of terms of the form

\[
\pm \prod_{i \in J} a_{i, \sigma(i)},
\]

where \( J \subseteq \{1, \ldots, 2n\} \) is a subset of cardinality \( i \) and \( \sigma \in \text{Sym}(J) \) is a permutation of \( J \) (\( J \) is the complement of \( I \); note that \( \pi(I) = I \), hence, the complement \( J \) of \( I \) too is \( \pi \)-invariant and \( \sigma = \pi|_J \) is the restriction of \( \pi \)

b.) We select \( i \in \{1, \ldots, 2n\} \) and look closer at the coefficient \( c_i \). We first assume \( i \leq n \) and we define the subset \( J_{\text{min}} = \{n - i + 1, n - i + 2, \ldots, n\} \); note that \( |J_{\text{min}}| = i \) and let \( \sigma \in \text{Sym}(J_{\text{min}}) \). If \( \sigma = \text{id}|_{J_{\text{min}}} \) then (5) yields a term which has \( p \)-adic value \( e \sum_{k=0}^{i-1} k \) by equation (2). If \( \sigma \neq \text{id}|_{J_{\text{min}}} \) then \( \sigma \) picks at least one entry \( ^{\mu_{ij}}_{\rho_{ij}} \) below the diagonal (cf. equation (4)) which is divisible by one more \( p \) (cf. equation (3)) and, hence, the \( p \)-adic value of the term in equation (5) is larger than \( e \sum_{k=0}^{i-1} k \). Thus,

\[
\sum_{\sigma \in \text{Sym}(J_{\text{min}})} (\pm 1) \prod_{i \in J_{\text{min}}} a_{i, \sigma(i)}
\]

has \( p \)-adic value \( e \sum_{k=0}^{i-1} k \). If \( J \neq J_{\text{min}} \) with \( |J| = i \), then equation (1) implies that \( \prod_{i \in J} a_{i, \sigma(i)} \) has \( p \)-adic value bigger than \( e \sum_{k=0}^{i-1} k \) for any \( \sigma \in \text{Sym}(J) \). Thus, we obtain

\[
v_p(c_i) = e \sum_{k=0}^{i-1} k, \quad i = 1, \ldots, n.
\]

Next we assume \( i > n \) and we define the subset \( J_{\text{min}} = \{1, \ldots, n, \ldots, i\} \). We claim: if \( \sigma \in \text{Sym}(J_{\text{min}}) \) is not the identity then there is \( i_0 \in J_{\text{min}} \) such that

\[
v_p(a_{i_0, \pi(i_0)}) \geq \begin{cases} 
(n - i_0)e + 1 & \text{ if } i_0 \leq n \\
(i_0 - 1)e + 1 & \text{ if } i_0 > n.
\end{cases}
\]

To prove the claim we assume that \( \sigma \in \text{Sym}(J_{\text{min}}) \) is a permutation such that equation (6) does not hold. Equation (3) then implies that \( \sigma(i) \leq n \) for all \( i \leq n \) ("\( \rho_{ij} \)" has \( p \)-adic value greater than or equal to \( (n - i)e + 1 \)). Thus, \( \sigma \) maps \( \{1, \ldots, n\} \) to itself and also maps \( \{n + 1, \ldots, i\} \) to itself, i.e. \( \sigma \) defines permutations \( \sigma|_{\{1, \ldots, n\}} \) resp. \( \sigma|_{\{n + 1, \ldots, i\}} \) of \( \{1, \ldots, n\} \) resp. of \( \{n + 1, \ldots, i\} \). Since equation (6) does not hold, equation (3) further implies that \( \sigma|_{\{1, \ldots, n\}} \) and \( \sigma|_{\{n + 1, \ldots, i\}} \) are the identity. Hence, \( \sigma \) is the identity which proves the claim. As above we then obtain for all \( \sigma \in \text{Sym}(J_{\text{min}}), \sigma \neq \text{id} \), that

\[
v_p(\prod_{j \in J_{\text{min}}} a_{i, \sigma(j)}) > v_p(\prod_{i \in J_{\text{min}}} a_{i, i}) = e \sum_{k=0}^{i-1} k.
\]

Thus,

\[
v_p(\sum_{\sigma \in \text{Sym}(J_{\text{min}})} (\pm 1) \prod_{i \in J_{\text{min}}} a_{i, \sigma(i)}) = e \sum_{k=0}^{i-1} k.
\]

As in the case \( i \leq n \) equation (1) implies that for any \( J \neq J_{\text{min}}, |J| = i \), and any \( \sigma \in \text{Sym}(J) \)

\[
v_p(\prod_{i \in J} a_{i, \sigma(i)}) > e \sum_{k=0}^{i-1} k.
\]
Thus,

$$v_p(c_i) = e \sum_{k=0}^{i-1} k$$

for all $i = n + 1, \ldots, 2n$. Hence, this holds for all $i = 1, \ldots, 2n$.

c.) In particular, the Newton polygon of $\chi$ consists of $2n$ segments which have slopes $0, e, 2e, \ldots, (2n - 1)e$. Thus, there are $2n$ roots $\lambda'_1, \ldots, \lambda'_{2n} \in \bar{\mathbb{Q}}_p$ of $\chi$ which have $p$-adic valuations $0, e, 2e, \ldots, (2n - 1)e$. Since $h^{-1}h_p^{-e}\gamma - 1$ and $A$ differ by a scalar factor $p^{ne}$, we deduce that $h^{-1}h_p^{-e}\gamma - 1$ has $2n$ pairwise different eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ with $p$-adic values $-ne, -(n - 1)e, \ldots, (n - 1)e$.

d.) We claim that $\chi$ splits over $\mathbb{Q}_p$ as a product of linear factors. In fact, if $\chi$ does not split completely then there is an irreducible factor of $\chi$ which is an automorphism $\sigma \in \text{Aut}(\mathbb{Q}_p/\mathbb{Q}_p)$. This would imply that $\lambda_i, \lambda_j$ which are conjugate by an automorphism $\sigma \in \text{Aut}(\mathbb{Q}_p/\mathbb{Q}_p)$ have the same $p$-adic value which is a contradiction. Thus, $\chi$ is split over $\mathbb{Q}_p$, hence, all roots $\lambda_i$ are contained in $\mathbb{Q}_p$.

e.) In c.) we have seen that the image of $h^{-1}h_p^{-e}\gamma - 1$ under $G(\mathbb{Q}_p) \subseteq \text{GL}_{2n}(\mathbb{Q}_p)$ has $2n$ different eigenvalues. Thus, $h^{-1}h_p^{-e}\gamma - 1$ is a semi simple element in $\text{GL}_{2n}(\mathbb{Q}_p)$ and, hence, in $G(\mathbb{Q}_p)$. In particular, $h^{-1}h_p^{-e}\gamma - 1$ is $(G(\mathbb{Q}_p)-)$conjugate to an element

$$\xi = \text{diag}(\xi_1, \ldots, \xi_{2n})$$

in $T(\mathbb{Q}_p)$. Since conjugate matrices in $\text{GL}_{2n}(\mathbb{Q}_p)$ have the same eigenvalues, c.) implies that $\xi \in \mathbb{Q}_p$ and the $p$-adic values of the $\xi_i \in \mathbb{Q}_p$ are contained in the sequence $-ne, -(n - 1)e, \ldots, (n - 1)e$. Conjugating the regular element $\xi \in T(\mathbb{Q}_p)$ with a suitable element in the Weyl group $W$ of $G$ we may assume that $v_p(\alpha(\xi)) < 0$ for all $\alpha \in \Delta$. This then implies that

$$(v_p(\xi_1), \ldots, v_p(\xi_{2n})) = (-e, -2e, \ldots, -ne, 0, e, \ldots, (n - 1)e)$$

which shows that $v_p(\alpha(\xi)) = -e$ for all simple roots $\alpha$. This completes the proof of the Lemma.

Remark. The entries $\xi_i$ of $\xi$ are even algebraic integers (contained in $\mathbb{Q}_p$).

(7.2) Lemma. Let $\lambda \in X(T)$ be algebraic and dominant. Then, for any $\gamma \in \Gamma$ and $h \in \Sigma^+ (\subseteq T(\mathbb{Q}))$ we have

$$\lambda(\gamma h_p^e) \text{tr}(\pi(\lambda(\gamma h_p^e)\gamma - 1)|L_\lambda) \in \mathbb{Z}.$$ 

Proof. Since $hh_p^e \in G(\mathbb{Q})$ we know that $\pi(\lambda(\gamma h_p^e)\gamma - 1)$ leaves $L_\lambda(\mathbb{Q})$ invariant. The subspace $L_\lambda(\mathbb{Z})$ decomposes

$$L_\lambda(\mathbb{Z}) = \bigoplus_{\mu} L_\lambda(\mu, \mathbb{Z}),$$

where $\mu \in X(T)$ runs over all weights of the form

$$\mu = \lambda - \sum_{\alpha \in \Delta} c_\alpha \alpha$$

(7)
with \( c_\alpha \in \mathbb{N}_0 \) for all \( \alpha \in \Delta \) and where \( T(\mathbb{Q}) \) acts on \( L_\lambda(\mu, \mathbb{Z}) \otimes \mathbb{Q} \) via \( \mu \):
\[
tv_\mu = \mu(t)v_\mu
\]
for all \( t \in T(\mathbb{Q}) \) and all \( v_\mu \in L_\lambda(\mu, \mathbb{Z}) \otimes \mathbb{Q} \). In particular, if \( \mu \) is as in equation (7) then we obtain
\[
\lambda(hh_p^c)\pi_\lambda(h^{-1}h_p^{-c})v_\mu = \prod_{\alpha \in \Delta} \alpha(hh_p^c)c_\alpha.
\]
Since \( h \in \Sigma^+ \) we obtain \( \alpha(h) \in \mathbb{Z} \) and equation (8) implies that \( \lambda(hh_p^c)\pi_\lambda(h^{-1}h_p^{-c}\gamma^{-1}) \) leaves \( L_\lambda(\mathbb{Z}) \) invariant which yields the claim (note that \( \gamma \in G(\mathbb{Z}) \)). Thus, the proof of the lemma is complete.

**Lemma.** Let \( \lambda \in X(T) \) be a dominant character. For any \( w \in W \), \( w \neq 1 \), we obtain
\[
w \cdot \lambda = \lambda - \sum_{\alpha \in \Delta} \alpha \cdot c_\alpha \alpha, \text{ where } c_\alpha = c_{\alpha, w} \in \mathbb{N}_0
\]
for at least one root \( \alpha_0 \in \Delta \).

**Proof.** Since \( w\lambda \) is a weight of the irreducible \( G \)-module of highest weight \( \lambda \) we know that
\[
w\lambda = \lambda - \sum_{\alpha \in \Delta} b_\alpha \alpha
\]
for certain \( b_\alpha \in \mathbb{N}_0 \). Since \( \lambda \) is dominant we may write \( \lambda = \sum_{\alpha \in \Delta} d_\alpha \omega_\alpha \) where \( d_\alpha = \langle \lambda, \alpha \rangle \in \mathbb{N}_0 \). On the other hand, \( w \neq 1 \) implies that \( w\lambda \) is not contained in the Weyl chamber corresponding to the basis \( \Delta \), hence, \( \langle w\lambda, \alpha_0 \rangle \leq 0 \) for some root \( \alpha_0 \in \Delta \). We obtain
\[
0 \geq \langle w\lambda, \alpha_0 \rangle = \langle \sum_{\alpha \in \Delta} d_\alpha \omega_\alpha - \sum_{\alpha \in \Delta} b_\alpha \alpha, \alpha_0 \rangle = d_{\alpha_0} - \sum_{\alpha \in \Delta} b_\alpha \langle \alpha, \alpha_0 \rangle.
\]
Since \( \langle \alpha, \alpha_0 \rangle = \alpha(h_{\alpha_0}) = 2 \) if \( \alpha = \alpha_0 \) and \( \langle \alpha, \alpha_0 \rangle \leq 0 \) if \( \alpha \neq \alpha_0 \) this yields \( 0 \geq d_{\alpha_0} - 2b_{\alpha_0} \). Thus,
\[
b_{\alpha_0} \geq \frac{1}{2}d_{\alpha_0} = \frac{1}{2}\langle \lambda, \alpha_0 \rangle.
\]
Since \( w \cdot \lambda = w\lambda + wp - \rho \) and
\[
wp - \rho = -\sum_{\alpha \in \phi^+} \alpha \in \phi^- \alpha
\]
this yields the claim and the Lemma is proven.

**7.4 Congruences between characters values for different weights.** Using the Weyl character formula we obtain the following congruences between character values of irreducible algebraic representations of \( G \) for varying highest weights.
\textbf{Proposition.} Let $\lambda, \lambda' \in X(T)$ be dominant characters and let $C \in \mathbb{Q}_{>0}$ such that $\langle \lambda, \alpha \rangle > 2C$ and $\langle \lambda', \alpha \rangle > 2C$ for all $\alpha \in \Delta$. If $\lambda \equiv \lambda' \pmod{(p-1)p^mX(T)}$, then for any $\Gamma$-conjugacy class $[\Xi] \subseteq \Gamma hh_p^e \Gamma$, $e \in \mathbb{N}$, $h \in \Sigma^+$ the following congruence holds

$$\lambda(hh_p^e) \text{tr} (\Xi^{-1}|L_{\lambda}) \equiv \lambda'(hh_p^e) \text{tr} (\Xi^{-1}|L_{\lambda'}) \pmod{p^{\min(m+1,C,e)}}.$$ 

\textit{Proof.} We note that by (7.2) Lemma

$$\lambda(hh_p^e) \text{tr} (\Xi^{-1}|L_{\lambda}) \equiv \lambda'(hh_p^e) \text{tr} (\Xi^{-1}|L_{\lambda'}) \in \mathbb{Z}.\] 

We may assume that the representative $\Xi$ of the $\Gamma$-conjugacy class $[\Xi] \subseteq \Gamma hh_p^e \Gamma$ is of the form $\Xi = \gamma h_p^e h$ for some $\gamma \in \Gamma$. By definition of $\Sigma^+$ we may write $h = h_{(p)} h_p^c$ with $h_{(p)} \in \prod_{i \notin p} \Sigma_i^+ \subseteq T(\mathbb{Z}_p)$ and $c \in \mathbb{N}_0$. Hence,

$$\Xi = \gamma h_{(p)} h_p^c'$$

with $c' = c + e \geq e(> 0)$. Thus, (7.1) Proposition implies that $\Xi^{-1}$ is $G(\mathbb{Q}_p)$-conjugate to an element $\xi \in T(\mathbb{Q}_p)$ satisfying

$$v_p(\alpha(\xi)) = -e'$$

for all $\alpha \in \Delta$. Using the Weyl character formula we therefore obtain

$$\Delta(\xi) \cdot \text{tr} (\Xi^{-1}|L_{\lambda}) = \sum_{w \in W} (-1)^{\ell(w)} (w \cdot \lambda)(\xi),$$

where

$$\Delta(\xi) = \prod_{\alpha \in \Phi^+} (1 - \alpha^{-1}(\xi)).$$

Note that equation (9) implies that $\Delta(\xi)$ is a $p$-adic unit, hence, $\Delta(\xi) \neq 0$. Using (7.3) Lemma we can write

$$w \cdot \lambda = \lambda - \sum_{\alpha \in \Delta} c_{\alpha,w} \alpha$$

with $c_{\alpha,w} \in \mathbb{N}_0$ and $c_{\alpha,w} \geq \langle \lambda, \alpha_w \rangle / 2$ for some root $\alpha_w \in \Delta$. We obtain

$$\lambda(\lambda_p^e) \text{tr} (\Xi^{-1}|L_{\lambda}) = \frac{\lambda(hh_p^e) \xi}{\Delta(\xi)} \left(1 + \sum_{w \neq 1} \text{sgn}(w) \prod_{\alpha \in \Delta} \alpha(\xi)^{-c_{\alpha,w}}\right).$$

Since $v_p(\alpha^{-1}(\xi)) = e' \geq 1$ for all $\alpha \in \Delta$ we find that $\Delta(\xi) \in \mathbb{Z}_p$ is a $p$-adic unit. Moreover, for any $\alpha \in \Delta$ we have $v_p(\alpha(hh_p^e)) = e'$, hence,

$$v_p(\alpha(hh_p^e)) = -v_p(\alpha(\xi))$$

for all $\alpha \in \Delta$. In particular, this equality holds for all $\beta$ contained in the root lattice of $G$ and since a (integral) multiple of any integral weight is contained in the root lattice we obtain

$$v_p(\chi(hh_p^e)) = -v_p(\chi(\xi))$$

for all $\chi \in X(T)$.
for all $\chi \in X(T)$. Thus, $\chi(hh_p^c\xi) \in Z_p^*$ is a $p$-adic unit. In particular, $\lambda(hh_p^c\xi)$ is a $p$-adic unit. Taking into account that $c_{\alpha, w} \geq 0$ for all $\alpha \in \Delta$, $w \in W$ and that $c_{\alpha, w} \geq \langle \lambda, \alpha_w \rangle / 2 \geq C$ we thus obtain using equation (9)

$$\lambda(hh_p^c) \text{tr} (\Xi^{-1}|L_\lambda) \equiv \frac{\lambda(hh_p^c\xi)}{\Delta(\xi)} \pmod{p^{C'c'}Z_p}.$$  

Since $\lambda \equiv \lambda' \pmod{(p-1)p^mX(T)}$ there is a $\chi \in X(T)$ such that $\lambda - \lambda' = (p-1)p^m\chi$. Taking into account that $\chi(hh_p^c\xi)$ is a $p$-adic unit this yields

$$\frac{\lambda(hh_p^c\xi)}{\chi(hh_p^c\xi)} = \chi(hh_p^c\xi)(p-1)p^m \in 1 + p^{m+1}Z_p.$$  

Hence,

$$\lambda(hh_p^c\xi) \equiv \lambda'(hh_p^c\xi) \pmod{p^{m+1}Z_p}.$$  

Together with equation (10) which also holds with $\lambda$ replaced by $\lambda'$ we obtain

$$\lambda(hh_p^c) \text{tr} (\Xi^{-1}|L_\lambda) \equiv \lambda'(hh_p^c) \text{tr} (\Xi^{-1}|L_{\lambda'}) \pmod{p^{\min(m+1,C'c')}Z_p}.$$  

Since $e' \geq e$ this completes the proof.

(7.5) Congruences for Hecke operators in different weights. Let $\beta \in \mathbb{Q}_{>0}$. For any pair of dominant characters $\lambda, \lambda' \in X(T)$ we denote by $e_{\lambda, \lambda'} = e_{\beta, \beta}^{\lambda, \lambda'}$ the approximate idempotent projecting to the slope $\leq \beta$ of $H_\lambda$ and $H_{\lambda'}$ as defined in (3.2); $\{e_{\lambda, \lambda'}\}_\lambda$ resp. $\{e_{\lambda, \lambda'}\}_{\lambda'}$ then is the approximate idempotent projecting to the slope subspaces $H_{\lambda}^{\leq \beta}$ and $H_{\lambda'}^{\leq \beta}$ which are now defined with respect to the normalized action of $T_p \in \mathcal{H}$ (cf. (6.3)).

**Theorem.** Let $C \in \mathbb{Q}_{>0}$. Assume that the dominant characters $\lambda, \lambda' \in X(T)$ satisfy

- $\langle \lambda, \alpha \rangle > 2C$ and $\langle \lambda', \alpha \rangle > 2C$ for all $\alpha \in \Delta$.
- $\lambda \equiv \lambda' \pmod{(p-1)p^mX(T)}$.

Then for all Hecke operators $T \in \mathcal{H}_O$ and all slopes $\beta \in \mathbb{Q}_{>0}$ the following congruence holds:

$$\text{tr} (\{e_{\lambda, \lambda'}^{\frac{m+1}{\beta}} T\}_\lambda|H_\lambda) \equiv \text{tr} (\{e_{\lambda, \lambda'}^{\frac{m+1}{\beta}} T\}_{\lambda'}|H_{\lambda'}) \pmod{p^\square},$$

where

$$\square = (1 - \frac{\beta M(\beta)}{C})(m+1) - \beta M(\beta).$$

**Proof.** Since $\mathcal{H}_O$ is generated as $\mathcal{O}$-module by the Hecke operators $T_h$, $h \in \Sigma^+$ (cf. (5.6)), we may assume that $T = T_h$ for some $h \in \Sigma^+$. We write $h = h_{(0)}h_{(p)}$ with $f \in \mathbb{N}_0$ and $h_{(p)} \in \prod_{p \nmid \Sigma^+} \Sigma_p^{\Delta}$ and we set $L = \left[ \frac{m+1}{\beta} \right]$. We recall that $e_{\lambda, \lambda'} = p(T_p)$, where the polynomial $p = \sum_{c=1} c e^c X^c$ satisfies the following properties: its degree $t$ is bounded by $M(\beta)$, its constant term $p(0) = 0$ and $S(p) \geq -\beta$ (cf. (3.3) Lemma). Since $\{e_{\lambda, \lambda'}\}_\lambda = p(T_p)_\lambda$ we obtain

$$\{e_{\lambda, \lambda'}^L T_h\}_\lambda = \{e_{\lambda, \lambda'}\}_\lambda^L \{T_h\}_\lambda = p^L (\{T_p\}_\lambda \{T_h\}_\lambda = \sum_{c=L}^{tL} b_c \{T_p\}_\lambda^c \{T_h\}_\lambda,$$
the existence of
Recalling that $e > C > 0$ such that $E$ holds with $\lambda$ replaced by $\lambda'$ and equation (4) yield the claim.
Thus, (3.7) Proposition yields

Thus, equation (3) (which also holds with $\lambda$ replaced by $\lambda'$) and equation (4) yield the claim. This completes the proof of the Proposition.

**7.6 $p$-Adic families of Siegel eigen classes.** We select a slope $\beta \in \mathbb{Q}_{2,0}$. The above Theorem immediately implies that the family of $\mathcal{H}$-modules $(H_\lambda) = (\mathcal{H}^d(\Gamma \setminus X, L_\lambda))$ satisfies equation (†) in (3.7), where $a' = \frac{1}{\beta M(\beta)}$, $a = 1 - \frac{\beta M(\beta)}{C}$, $b = -\beta M(\beta)$. Thus, (3.7) Proposition yields the existence of $p$-adic continuous families of finite slope with (to simplify, we may omit a factor of $2$)

$$a = \frac{1}{M(\beta)} \min \left( a', \frac{a'}{M(\beta + 1)} \right) \text{ and } b = -\beta - \left( M(\beta) + 2 \right) \log_p(M(\beta)).$$

We want to choose $C > 0$ such that $a$ becomes large. To this end we set

$$C = C(\beta) = \beta M(\beta) + \frac{1}{M(\beta + 1)}$$

and obtain

$$a = \frac{1}{(1 + \beta M(\beta + 1) M(\beta)) M(\beta)}.$$
characters \( \Theta : \mathcal{H} \to \hat{\mathbb{Q}}_p \) which are defined over \( \mathcal{O} \) and such that the corresponding eigenspace \( H^d(\Gamma \backslash \mathfrak{X}, \mathcal{L}_\lambda)^\beta(\Theta) \) w.r.t the normalized action of \( \mathcal{H} \) does not vanish. We then obtain from (3.7) the following

\[ \textbf{(7.7) Corollary.} \] Let \( \beta \in \mathbb{Q}_{\geq 0} \).

1.) \( \dim H^d(\Gamma \backslash \mathfrak{X}, \mathcal{L}_\lambda)^\beta \) is locally constant, i.e. there is \( D = D(\beta) \) only depending on \( \beta \) such that \( \lambda \equiv \lambda' \pmod{(p-1)p^D \mathfrak{X}(T)} \) implies

\[ \dim H^d(\Gamma \backslash \mathfrak{X}, \mathcal{L}_\lambda)^\beta = \dim H^d(\Gamma \backslash \mathfrak{X}, \mathcal{L}_{\lambda'})^\beta \]

2.) Any \( \Theta \in \mathcal{E}(\lambda_0)^\beta \) fits in a \( p \)-adic continuous family of eigencharacters of slope \( \beta \), i.e. there are \( \Theta_\lambda \in \mathcal{E}(\lambda)^\beta \) such that \( \Theta_{\lambda_0} = \Theta \) and \( \lambda \equiv \lambda' \pmod{(p-1)p^m \mathfrak{X}(T)} \) implies

\[ \Theta_\lambda \equiv \Theta_{\lambda'} \pmod{p^{a(m+1)+b}}; \]

here, \( a \) and \( b \) as in equation (5) and (6) and \( \lambda \) runs over all dominant characters satisfying \( \langle \lambda, \alpha \rangle > 2\beta M(\beta) + 2 \) for all simple roots \( \alpha \).

Remark. The congruence between two eigencharacters \( \Theta_\lambda \) and \( \Theta_{\lambda'} \) is non trivial only if \( a(m+1) + b > 0 \), i.e. only if

\[ m + 1 > E(\beta) = (-b + M(\beta)(M(\beta) + 2) \log_p M(\beta))(1 + \beta M(\beta) + M(\beta)). \]

Thus, only the existence of the family \( \{\Theta_\lambda\} \) with \( \lambda \equiv \lambda_0 \pmod{p^{E(\beta)}} \) is a non trivial statement. Note that \( M(\beta) \to \infty \) if \( \beta \to \infty \) and that we may assume \( M(\beta) \leq M(\beta+1) \); these two statements imply that \( E(\beta) \sim M(\beta+1)^a \log_p M(\beta + 1) \) for \( \beta \to \infty \).

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