INFINITE-DIMENSIONAL SYMPLECTIC NON-SQUEEZING USING NON-STANDARD ANALYSIS

OLIVER F. ABERT

Abstract. Believing in the axiom of choice, we show how to deduce symplectic non-squeezing in infinite dimensions from the corresponding result for all finite dimensions. Employing methods from model theory of mathematical logic and motivated by analogous constructions in non-standard stochastic analysis, we use that every infinite-dimensional symplectic Hilbert space is contained in a symplectic vector space which behaves like a finite-dimensional symplectic vector space.

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1. An infinite-dimensional non-squeezing theorem

We start with formulating our main theorem, where we refer to the recent papers [1], [6] for further references, including partial results.

Let $H$ be a real Hilbert space which is explicitly assumed to be infinite-dimensional. A strong symplectic form on $H$ is a skew-symmetric bounded bilinear form $\omega = \omega_H : H \times H \to \mathbb{R}$ which is nondegenerate in the sense that the induced linear mapping $i_\omega : H \to H^*$ is an isomorphism. A (smooth) symplectomorphism $\varphi : H \to H$ is a diffeomorphism of $H$ which preserves the symplectic form, $\varphi^* \omega = \omega$.

Let $B(H, r) = \{ x \in H : |x| < r \}$ denote the open Hilbert ball of radius $r > 0$, and let $P : H \to \mathbb{R}^2$ denote the symplectic projector onto a two-dimensional symplectic subspace of $H$, which we directly identify with $\mathbb{R}^2$. Using methods from non-standard analysis, we give a proof of the following

Theorem 1.1. For every smooth symplectomorphism $\varphi$ on the (infinite-dimensional) symplectic Hilbert space $(H, \omega)$ we have

$$\text{area}_\omega(P \varphi(B(H, r))) \geq r^2 \pi,$$

where the area on $\mathbb{R}^2$ is measured with respect to the restriction of $\omega$ to $\mathbb{R}^2 \subset H$.

O. Fabert, VU Amsterdam, The Netherlands. Email: oliver.fabert@gmail.com.
The key idea for the proof is to deduce the non-squeezing result for infinite-dimensional symplectic spaces from the fact that the corresponding result holds for all finite dimensions, using methods from non-standard analysis. Believing in the axiom of choice it is well-known, see e.g. [3], that there exist non-standard models of mathematics in which, on one side, one can do the same mathematics as before (transfer principle) but, on the other side, all sets behave like compact sets (saturation principle). In order to construct such a new model one successively adds in new "ideal" objects such as infinitively small and large numbers. The proof of existence of the resulting poly-satured model is then performed in complete analogy to the proof of the statement that every field has an algebraic closure, by employing Zorn’s lemma.

With this the proof of our main theorem consists of two parts: First, using the saturation principle, we can show that there exists a symplectic vector space \( F \) which, on one side, is finite-dimensional in the new model of mathematics setup but, at the same time, contains the original infinite-dimensional symplectic Hilbert space \( \mathbb{H} \). Since \( F \) is finite-dimensional, it follows from the transfer principle that non-squeezing holds for \( F \). Although \( \mathbb{H} \) is not a set in the new non-standard model, and hence can not be treated like a finite-dimensional subspace itself, we follow classical ideas from non-standard analysis (more precisely, non-standard stochastics, see e.g. [4]) to show that \( F \) can be chosen to approximate \( \mathbb{H} \) sufficiently well such that non-sequeezing for \( F \) indeed implies non-squeezing for \( \mathbb{H} \).

The idea for this project came up after listing to Alberto Abbondandolo’s talk on his joint work [1] at VU Amsterdam in December 2014. Further I am deeply grateful to Horst Osswald from LMU Munich for introducing me to the fascinating world of non-standard analysis during my time as undergraduate student.

2. Non-standard models of mathematics setup

We now give a very quick introduction to the world of non-standard analysis. Here we describe the original model-theoretic approach of Robinson ([5]), outlined in the excellent exposition [3] (which shall also be consulted for details). We emphasize that it is highly non-constructive in the sense that it crucially relies on the axiom of choice. Besides the model-theoretic approach there also exist axiomatic approaches (e.g. the internal set theory of Nelson), whose compatibility with the Zermelo-Fraenkel axioms of standard set theory is however precisely proven using the above model-theoretic approach.

First it is well-known that all mathematical results can be formulated in the language of set theory of mathematical logic. A standard model of mathematics setup is a set \( V \) which is, informally speaking, large enough to do the mathematics that one has in mind. More formally, we assume that \( V \) is a superstructure \( V = (V_n)_{n \in \mathbb{N}} \), consisting of a hierarchically ordered family of sets \( V_n, n \in \mathbb{N} \), where \( V_0 \) is the set of
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Urelements and elements in $V_n$ are sets formed from elements in $V_0, \ldots, V_{n-1}$. Here $V_0$ is chosen rich enough to do the mathematics one has in mind (at least it contains all natural numbers) and we further assume that the model $V$ is full in the sense that $V_n = \mathcal{P}(V_0 \cup \ldots \cup V_{n-1})$ and that $V_0 \cap V_n = \emptyset$ for all $n \in \mathbb{N}$. Note that since $V$ still has to be a set there definitely exist (abstract) sets which do not belong to $V$. On the other hand, it is a well-known fact from mathematical logic that the standard model $V$ can always be chosen large enough for all problems one has in mind. In particular, here we can assume that the symplectic Hilbert space $H$ with its strong symplectic form $\omega: H \times H \to \mathbb{R}$ as well as the smooth symplectomorphism $\phi$ on $H$ are elements in $V$.

Using the axiom of choice it is well-known that one can construct a new non-standard model of mathematics setup $W = (W_n)_{n \in \mathbb{N}}$ and a transfer map $*: V \to W$ (respecting the filtration, $*_n: V_n \to W_n$) satisfying the following two important principles.

- **Transfer principle:** If a sentence holds in the language of the model $V$, then the same sentence holds in the language of the model $W$ after replacing the elements from $V$ by their images in $W$ under $*$. Informally speaking this means that the results in $V$ also hold in the new model $W$ after identifying elements in $V$ with their images under the $*$-map.

- **Saturation principle:** If $(A_i)_{i \in I}$ is a collection of sets in $W$ (and $I$ is a set in $V$) satisfying $A_i \cap \ldots \cap A_{i_n} \neq \emptyset$ for all $i_1, \ldots, i_n \in I$, $n \in \mathbb{N}$ (finite intersection property), then also the common intersection of all $A_i$, $i \in I$ is non-empty, $\bigcap_{i \in I} A_i \neq \emptyset$.

Denoting, as it is common, by $*_a \in W$ the image of $a \in V$ under $*$, the transfer principle immediately implies that $*: V \to W$ is indeed an embedding, since $a \neq b$ in $V$ implies that $*_a \neq*_b$ in $W$. Furthermore we have $\{*_a_1, \ldots, a_n\} = \{*_a_1, \ldots, _*a_n\}$ for all finite sets, and so on. On the other hand, if $A$ is a set in $V$ with infinitely many elements, then it easily follows from the saturation principle that $*_A$ is strictly larger than $A := A^{*} = \{*_a : a \in A\}$. In particular, while, by the transfer principle, $\mathbb{R}^2 \subset H$ implies $*\mathbb{R}^2 \subset *H$, we have that the space $*H$ is much larger than $H$. On the other hand, the restriction of $*_\omega$ to $H$ still agrees with $\omega$ by the transfer principle.

Apart from the fact that $A$ is a strict subset of $*_A$, it even holds that $A$ is not even a set in the new model $W$. In particular, this applies to our original Hilbert space $H := *[H]$, viewed as a subset of $*H$. Indeed one can show that every non-standard model is necessarily not full, that is, there exist subsets of sets in $W$ which do not belong to $W$ itself. While this destroys all obvious logical paradoxes, on the positive side the transfer principle implies that every definable subset (subset of
elements of a set in \( W \) which fulfill a sentence in the language of \( W \) still belongs to \( W \), so the lack of fullness does not cause problems either. In particular, the open Hilbert ball \( B(^* H, r) = ^* B(H, r) \subset ^* H \) is an element in \( W \).

We end this quick introduction to non-standard analysis with the proof of the existence of infinitively small (and large) real numbers in the non-standard model \( W \). For each \( r \in \mathbb{R}^+ \) define \( A_r := \{ s \in ^* \mathbb{R}^+ : s < r \} \subset ^* \mathbb{R} \), which indeed belongs to \( W \) since it is definable. Note that here we follow the standard convention in non-standard analysis and write \( a \) instead of \( ^* a \) if no confusion is likely to arise. Since \( (A_r)_{r \in \mathbb{R}^+} \) obviously satisfies the finite intersection property, there exists \( \epsilon \in ^* \mathbb{R} \) with \( \epsilon \in \bigcap_{r \in \mathbb{R}} A_r \), that is, \( \epsilon < r \) for all real numbers \( r \in \mathbb{R}^+ \).

While this, at first sight, seems to contradict the Archimedean axiom, \( \forall r \in \mathbb{R}^+ \exists n \in \mathbb{N} : 1/n < r \), and hence to violate the transfer principle, the latter is indeed not true. Indeed, in \( W \) the Archimedean axiom now reads as \( \forall r \in ^* \mathbb{R}^+ \exists n \in ^* \mathbb{N} : 1/n < r \), which holds since \( ^* \mathbb{N} \) indeed contains natural numbers \( H \) which are larger than every natural number \( n \in \mathbb{N} \subset ^* \mathbb{N} \).

### 3. Proof of the main theorem

As mentioned at the end of the first section, we have to show that, for a given smooth symplectomorphism \( \varphi \) on \( H \), the original infinite-dimensional symplectic Hilbert space \( H \) can be approximated sufficiently well by a finite-dimensional symplectic vector space \( F \) in the new model \( W \) such that non-squeezing for \( F \) implies non-squeezing for \( H \).

**Step 1:** There exists a symplectic subspace \( F \) of \( ^* H = (^* H, ^* \omega) \) in the new non-standard model \( W \) which at the same time has finite dimension \( \dim(F) = 2H \in ^* \mathbb{N} \) (in the sense of the new model) and contains the original symplectic Hilbert space, \( (H, \omega) \subset (F, ^* \omega) \).

*Proof.* We prove the existence of \( F \) in \( W \) using the saturation principle. For each finite-dimensional symplectic subspace \( E \) of \( H = (H, \omega) \) let \( A_E \) denote the set of symplectic subspaces of \( ^* H = (^* H, ^* \omega) \) which are finite-dimensional in the new model \( W \) and contain \( E \) as a symplectic subspace, 

\[
A_E := \{ F : (E, ^* \omega) \subset (F, ^* \omega) \subset (^* H, ^* \omega), \dim(F) \in ^* \mathbb{N} \}.
\]

Note that here and later we use that each finite-dimensional symplectic subspace \( E \) of \( H \) can be viewed as finite-dimensional symplectic subspace of \( ^* H \) using the image of the basis under the \( ^* \)-map.

Now it is immediate to see that the collection of sets \( (A_E)_E \) has the finite intersection property, since \( E^1 + \ldots + E^n \in A_{E^1} \cap \ldots \cap A_{E^n} \). By the saturation principle there exists \( F \in \bigcap_E A_E \), i.e., a symplectic subspace of \( ^* H = (^* H, ^* \omega) \) of finite dimension \( \dim(F) = 2H \in ^* \mathbb{N} \) (in the sense of the new model \( W \)!) which contains all finite-dimensional symplectic subspaces of \( H \) - and hence \( H \) itself. \( \square \)
If \( H \) and \( F \) were finite-dimensional symplectic vector spaces (in the standard sense), then non-squeezing for \( F \) would immediately imply non-squeezing for \( H \). For this one would just need to extend the map for \( H \) in the obvious way to a map starting from \( F \) making use of the quotient space \( F/H \).

Here however we observe that \( H \) is a set in the standard model \( V \) but not in the non-standard model \( W \), while \( F \) only exists in \( W \) but not in \( V \). Hence the quotient space \( F/H \) does not exist in any of the two models, since the inclusion \( H \subset F \) is just a sentence in the all-embracing Zermelo-Fraenkel set theory universe.

In order to show that non-squeezing for \( F \) indeed implies non-squeezing for \( H \), we have to argue more indirectly as follows.

Let \( \varphi : H \to H \) be an arbitrary smooth symplectomorphism of \( H \). By the transfer principle it follows that \( \ast \varphi : \ast H \to \ast H \) is now a smooth symplectomorphism of \( \ast H \), which extends \( \varphi \) in the sense that \( \ast \varphi(x) = \varphi(x) \) for all \( x \in H \subset \ast H \). Similarly, \( \ast P : \ast H \to \ast R^2 \) is a again a symplectic projector onto a two-dimensional symplectic subspace, extending \( P \) in the obvious way. In the same way as \( P \circ \varphi : H \to \mathbb{R}^2 \) is a surjective symplectic submersion, we see that \( \ast P \circ \ast \varphi : \ast H \to \ast \mathbb{R}^2 \) is a surjective symplectic submersion.

**Step 2:** After passing to a symplectic subspace if necessary, we can assume that the composed map \( \ast P \circ \ast \varphi : F \to \ast \mathbb{R}^2 \) is again a surjective symplectic submersion.

**Proof.** Surjectivity follows immediately from \( H \subset F \). For the submersion property we again employ the saturation principle together with the transfer principle and start with the given symplectic vector space \( F_0 := F \). For each finite-dimensional symplectic vector space \( E \subset H \) (in the standard sense) let \( A_E \) now denote the set of symplectic subspaces \( F \) of \((F_0, \ast \omega)\) which do not only contain \( E \) as a symplectic subspace, but also for which the composed linear map \( \ast P \cdot D_x \ast \varphi : F_0 \to \ast \mathbb{R}^2 \) is surjective at all points \( x \in F \).

Note that the latter surjectivity result is indeed true for all subspaces \( F \) of \( H \), i.e., which are finite-dimensional in the standard sense. Denote by \( R \subset \ast H \) the two-dimensional symplectic subspace of \( \ast H \) which gets mapped to \( \ast \mathbb{R}^2 \subset \ast H \) under \( D \). Since for all \( x \in H \) the differential \( D := D_x \ast \varphi : \ast H \to \ast H \) extends the differential \( D_x \varphi : H \to H \) in the obvious way, it follows that \( R \) is indeed a finite-dimensional subspace of \( H \), and hence lies in \( F_0 = F \) by definition. Note there here we still canonically identify vector spaces over \( \mathbb{R} \) with those over \( \ast \mathbb{R} \) making use of a basis.

With this it follows that \( (A_E)_F \) again has the finite intersection property, so that by the saturation principle we find a symplectic subspace \( F_1 \) of \( F_0 = F \) where still \( H \subset F_1 \) and \( \ast P \cdot D_x \ast \varphi : F_0 \to \ast \mathbb{R}^2 \) is surjective at all points \( x \in F_1 \). Since, in general, \( F_1 \) is a proper subspace of \( F_0 \), it still
remains to be shown that we can achieve that the two symplectic vector spaces agree. For this we observe that the above method provides with a descending sequence of symplectic subspaces $F = F_0 \supset F_1 \supset F_2 \supset \ldots \supset H$ such that for all $i = 0, 1, 2, \ldots$ we know that $*P \cdot D_x^i \varphi : F_i \to *\mathbb{R}^2$ is surjective at all points $x \in F_{i+1}$.

Since the dimensions of the $F_i$ give a descending sequence of numbers in $*N$, $\dim(F_{i+1}) \leq \dim(F_i)$ and descending sequences of standard natural numbers have to become constant (possibly equal to zero), it follows from the transfer principle that the exists a symplectic subspace covering $H$, by abuse of notation again denoted by $F$, such that $*P \cdot D_x^i \varphi : F \to *\mathbb{R}^2$ is surjective at all points $x \in F$. Note that instead of using the transfer principle we could also use the more abstract fact that a strictly descending sequence of ordinal numbers always reaches zero after a finite number of steps, since $*N$ is still an ordered set. □

**Step 3:** Symplectic non-squeezing in finite dimensions implies non-squeezing for smooth symplectomorphisms on $H$.

**Proof.** Let $\varphi : H \to H$ be a smooth symplectomorphism which maps the Hilbert ball $B(H, r) \subset H$ of radius $r > 0$ to the unit ball $B^2(1) = B(\mathbb{R}^2, 1) \subset \mathbb{R}^2$ after composing with the symplectic projection $P : H \to \mathbb{R}^2$. In order to prove the main theorem it suffices to prove that necessarily $r \leq 1$.

First it follows from the transfer principle that $*\varphi : *H \to *H$ again maps the Hilbert ball $*B(H, r) = B(*H, r) \subset *H$ of radius $r > 0$ to the unit ball $*B^2(1) = B(*\mathbb{R}^2, 1) \subset *\mathbb{R}^2$ after composing with the symplectic projection $*P : *H \to *\mathbb{R}^2$. Note that, again by the transfer principle, the composed map $*P \circ *\varphi : *H \to *\mathbb{R}^2$ is an extension of the original map $P \circ \varphi : H \to \mathbb{R}^2$.

Considering the restriction to the finite-dimensional subspace $F \subset *H$, it hence follows that $*P \circ *\varphi$ maps $B(F, r) = \{x \in F : |x| < r\}$, the ball of radius $r$ in $F$, to $*B^2(1) = B(*\mathbb{R}^2, 1) \subset *\mathbb{R}^2$, since obviously $B(H, r) \subset B(F, r) \subset B(*H, r)$.

Summarizing, we find that $*P \circ *\varphi : F \to *\mathbb{R}^2$ is a surjective symplectic submersion which furthermore maps $B(F, r)$, the ball of radius $r$ in $F$, into the unit ball $*B^2(1) \subset *\mathbb{R}^2$. But, using the transfer principle and Gromov’s non-squeezing result in finite dimensions, this is only possible if $r \leq 1$. □

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