Monte Carlo Renormalization Group calculation in $\lambda\phi^4_3$

Alex Travesset$^a$

$^a$Departament d’ECM, Facultat de Fisica, Universitat de Barcelona Diagonal 647, 08027 Barcelona

We start by discussing some theoretical issues of renormalization group transformations and Monte Carlo renormalization group technique. A method to compute the anomalous dimension is proposed and investigated. As an application, we find excellent values for critical exponents in $\lambda\phi^4_3$. Some technical questions regarding the hybrid algorithm and strong coupling expansions, used to compute the critical couplings of the canonical surface, are also briefly discussed.

1. The MCRG calculation

1.1. Introduction

The Monte Carlo renormalization group (MCRG) technique [1], consists of the numerical determination of the couplings (and their derivatives) of the RG transformed action. What is remarkable about this method is that, if the RG transformation (RGT) is sufficiently short ranged, one may compute, in a finite lattice, critical quantities as if working on an infinite lattice.

Rather heuristically, one may see that as follows; suppose that a RGT possesses a Fixed Point (FP) consisting in a sum of local operators, each one just involving fields (or spins) separated at most by $n_s$ lattice sites. Assuming periodic boundary conditions, this FP may be accommodated in a lattice as small as $(2{n_s} + 1)^d$. Let us consider it now in a $(2(2n_s + 1))^d$ volume, and apply a RGT, reducing it to a $(2n_s + 1)^d$ lattice. As the FP still fits in, one may expect the FP couplings (and their derivatives) not to feel the lattice (except possibly by tiny effects dying off exponentially with the linear lattice size). Of course, the same line of reasoning does not follow for observables, as it is then the correlation length what should be fitted in, which is $\infty$.

It appears crucial then, both for a practical implementation and a theoretical understanding of the RG, to study the properties, such as locality of the FP, convergence of eigenoperators, fast approach to the FP, etc. that different RGTs have. This is one of the reasons for this project.

In this work, as we are dealing with unbounded spins, we must determine the rescaling of the field, which at the FP is related to the conformal anomalous dimension $\eta$. This is a difficult problem [2], and we propose and investigate a variation of the Bell-Wilson criteria [3].

1.2. The RGT

RGTs in real space are (exponentially) short ranged. The simplest of those, introduced by Bell and Wilson [3], is

$$e^{-S[\phi]} = \int \prod_{i=1}^{n} d\phi(n) \left( e^{-S[\phi]} \times \right. (1) \times e^{-a_W \sum_{n_B} \left( \phi(n_B) - b \sum_{n \in n_B} \phi(n) \right)^2}$$

There are two free parameters $a_W$ and $b$.

Recall that, for unbounded spins, we may rescale the fields. Then, if a local FP exists, there is a whole line of equivalent FPs, so there is a marginal (eigenvalue is 1) redundant direction. As we stick to one of those FPs, we determine the parameter $b$ by preventing moves along this direction. Nevertheless, when one goes to a finite lattice approximation, it is not clear that such a line of FPs still exists, the reason being that it is not guaranteed that all FPs are sufficiently short.
ranged. In the other case, the parameter $a_W$ just labels different RG transformations. It is a free parameter at our disposal to play with.

Our canonical surface is

$$S[\phi] = \sum_n \frac{1}{2} \sum_{\mu} (\phi(n) - \phi(n + \mu))^2 + \frac{m^2}{2} \phi(n)^2 + \frac{\lambda}{4} \phi(n)^4,$$

and we expand the RG transformed action in Eq. 1 as a sum

$$S[\vartheta] = \sum_i O_i(\vartheta),$$

where the set of local operators are shown in fig. 1.

Figure 1. Operators included in the RG transformed action.

2. The critical surface

Following the discussion in the preceding section, a MCRG calculation must be done at the exact critical point of the infinite volume system. In Eq. 2, for each value of $\lambda$ there exists a critical value $m_c(\lambda)$. A strong coupling expansion, which for this model amounts to an expansion in the hopping parameter $\kappa$, allows to extract all critical couplings with negligible CPU time.

Table 1

| $g$ | $\kappa$ | $m^2$ | $\lambda$ |
|-----|----------|-------|----------|
| 0.5 | 0.3973(1) | -6.0000(30) | 12.672(6) |
| 0.45 | 0.3976(1) | -5.4969(26) | 11.388(6) |
| 0.39458 | 0.3973(1) | -4.9386(25) | 10.000(6) |
| 0.3 | 0.3948(1) | -3.9736(21) | 7.700(6) |
| 0.2287 | 0.3905(2) | -3.2207(20) | 6.000(5) |
| 0.15204 | 0.3820(2) | -2.3568(14) | 4.167(3) |
| 0.005 | 0.3369(-) | -0.1230(-) | 0.176(-) |
| 0.00 | 0.3333(-) | 0.0000(-) | 0.000(-) |

Table 1 Critical couplings at the region of interest, if error bars are not written, all digits are significant.

The first 11 coefficients of this expansion were tabulated in [4]. To extract critical couplings from those, presents some technical difficulties related to the appearance of antiferromagnetic singularities in loose-packed lattices, which nevertheless, may be overcome. Critical couplings used are given in table 1.

3. The algorithm

The algorithm we used in our simulation is the Hybrid algorithm [5], which has two freedoms, namely, the size of the leap frog step $\delta t$, and the number of integration steps $n_I$ before the Metropolis test is passed. While there is not much room to play with $\delta t$, big gains come in playing with $n_I$.

Indeed, in table 3 is reported a sample calculation. Gradually increasing $n_I$, autocorrelations reduce, with a minor penalty in CPU time. In a $24^3$ lattice, we took $n_I = 60$ or $n_I = 120$.

4. Results

We perform 3 RGTs for different $a_W$ values in a $24^3$ lattice including up to 27 operators. For each value of $a_W$, there exists a $b$ such that the eigenvalue of the T matrix in the second RGT is 1 (we discard the first RGT as we assume we are not yet on the linear region around the FP). If this value for $b$ makes the marginal eigenvalue stable against the last RGT, we assume we reached a local FP. So, we fix $b$ and choose the $a_W$ so that the canonical surface, Eq. 2, is optimally close to the
There are two cross checks. First, computing the flow (using of Schwinger-Dyson equations), and secondly, performing 2 RGTs in a $12^3$ lattice, and testing if expectation values of operators in the last RGT agree with the ones coming from the third transformation on a $24^3$ lattice.

For small values of $\lambda$, though strictly we are in the domain of the Wilson FP, we are on the linear region around the Gaussian FP, and indeed, results are totally compatible with those.

At the strong coupling we studied different values of $\lambda$. The most extensive one was performed at $\lambda = 12.672$, where the optimal values were $a_W = 25$ and $\eta = 0.03$. As an example, in fig. 2 we plot the value of the second eigenvalue at $\eta = 0.03$ for different values of $a_W$ clearly singling out $a_W = 25$. The couplings of the action in the last two RGTs agree within statistical error bars, though those are surprisingly large. Furthermore, the agreement between expectation of the third RGT and the ones that come from a second RGT starting in a $12^3$ differ, at most, by a 5%, which is good but not fully satisfactory yet.

Our preliminary results for the critical exponents are

\begin{align}
\eta &= 0.030(5) \quad , \quad \nu = 0.625(7) \quad , \\
\omega &= 0.77(5) \quad , \quad \lambda_{-2} = 0.28(9),
\end{align}

where error bars are just tentative. Those figures are excellent when compared with the most accepted results $\eta \sim 0.035$, $\nu \sim 0.631$, $\omega \sim 0.8$, (2th irrelevant eigenvalue is not known to us).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\textit{m} & \textit{\delta t} & \langle O_1 \rangle & \textit{CPU} \\
\hline
6 & 0.01 & 0.27449(140) & 500 & 140 \\
6 & 0.03 & 0.27425(57) & 215 & 140 \\
6 & 0.06 & 0.27506(40) & 100 & 140 \\
11 & 0.06 & 0.27527(23) & 35 & 160 \\
21 & 0.06 & 0.27510(12) & 8 & 189 \\
31 & 0.06 & 0.27511(6) & 4 & 218 \\
\hline
\end{tabular}
\caption{The results correspond to a $\lambda \phi^4$ theory in a $10^3$ lattice at $m^2 = -3.32$, $\lambda = 6.0$, $1.2 \times 10^6$ configurations, throwing away $6 \times 10^5$ for thermalization, $O_1 = \phi^2$, $\tau$ is the autocorrelation time, $CPU$ is in units of $10^5$ sec. in a Power Challenge.}
\end{table}

5. Acknowledgments

It is a pleasure to acknowledge interest and discussions with B.Alles, S. Catterall, J.Comellas, D.Espriu and R. Toral. I have benefited from a fellowship of the Generalitat of Catalunya. Part of the numerical work was done at C.E.P.B.A. This work has been supported by grants AEN95-0590 (CYCIT) and GRQ93-1047(CIRIT).

REFERENCES

1. R.H. Swendsen, Phys. Rev. Lett. 42 (1979) 859
2. D.Espriu and A. Travesset, Phys. Lett. B 356 (1995) 329
3. T.L. Bell and K.G. Wilson Phys. Rev. B 11 (1975) 3431
4. G.A. Baker and J.M. Kincaid J. of Stat. Phys. 24 (1981) 469
5. S. Duane et al. Phys. Lett. 195 B (1985) 1195
6. A. Gonzalez-Arroyo and M. Okawa Phys. Rev. D 35 (1987) 672