Grounding Game Semantics in Categorical Algebra

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1. Context: Building reliable computer systems

2. Background: Game semantics and algebraic effects

3. Result: Strategies for algebraic effects

4. Conclusion: Towards algebraic game semantics
Section 1

Context: Building reliable computer systems
Making computers reliable is hard

In modern computer systems:

- Layers upon layers of complex hardware and software components
- A bug in any one of them could prevent the system from working
- Components may break in subtle ways through their interactions
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- Layers upon layers of complex hardware and software components
- A bug in any one of them could prevent the system from working
- Components may break in subtle ways through their interactions

Thankfully, there are ways to control this:

- Precise specifications for each component
- Careful and systematic testing
- Formal verification
Formal verification of software components

To prove a program correct:

- Start with a model of the programming language
- Make the specification mathematically precise
- Write a proof showing that the program indeed meets the specification
Formal verification of software components

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Mechanizing this process in a proof assistant has many advantages:
- Almost no possibility of mistake in the proof
- Can scale up the methodology to complex programs
- The proof can easily be checked by a third-party (certified software)
State of the art and next steps

Over the past $\sim$10 years, verification has become increasingly tractable:

- Researchers have verified complex components of various kinds
- Industrial-strength verification tools exist
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The next step is *end-to-end* verification:

- Until then, bugs can sneak into the “gaps” between correctness proofs
- Solution: use formal specifications as *interfaces* to connect proofs
- Challenge: existing projects use different models and proof methods
- This is by *necessity* and not by accident
End-to-end verification using a hierarchy of models

This suggests we should organize semantic models into a *hierarchy*:

- Individual components are verified using specialized models
- Embed these models into increasingly general ones where certified component can be assembled into certified systems
End-to-end verification using a hierarchy of models

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- Individual components are verified using specialized models
- Embed these models into increasingly general ones where certified component can be assembled into certified systems

Category theory provides a unified framework to:

- Characterize existing models
- Establish connections between them
- Guide the design of more general ones
I will present a very small step in this direction, looking at connections between two important lines of work:

- *Game semantics* expresses the behavior of program components as *strategies* in games derived from their types;
- *Algebraic effects* model computations with side-effects as *terms* in an algebraic theory.
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I will show that:

- Simple strategies can be used to construct interesting models of algebraic effects
- Conversely, we can take inspiration from algebraic effects to characterize these simple strategies categorically.

I hope this correspondence can be extended in the future to formulate a more general algebraic approach to game semantics.
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Section 2

Background: Game semantics and algebraic effects
Game Semantics

Game semantics is a general approach to programming language semantics:

- Types are two-player *games* between a component and its environment.
- Programs of a given type are *strategies* for the corresponding game.
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- Games provide a very general notion of interface
- “Rely-guarantee” flavor facilitates compositional reasoning
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This approach is very compelling for heterogeneous verification:

- Games provide a very general notion of interface
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However there are challenges to overcome:

- Huge variety of constructions for games and strategies
- Often too complex to formalize in a proof assistant
- Existing work rarely focuses on specifications and verification
Algebraic effects address the narrower problem of computational side-effects:

- The available side-effects are given by an algebraic theory
- *Terms* in the theory represent computations, which proceed inwards.
- *Operations* represent effects, their arguments are the possible continuations.
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- *Terms* in the theory represent computations, which proceed inwards.
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Advantages:

- Composing effect theories is easier than in the monadic approach
- The framework is simple and systematic, grounded in categorical algebra

Limitations:

- Narrower scope than game semantics, less generality
Algebraic signatures can be read as games

In the algebraic framework, a program with side-effects:

\[
greeting(*) := (\textbf{if} \ \text{readbit} \ \textbf{then} \ \text{print} \ \text{"Hi"} \ \textbf{else} \ \text{print} \ \text{"Hello"}) \ ; \ \text{stop}
\]

is modeled in the following way:

\[
\Sigma := \{ \text{readbit} : 2, \ \text{print}[s] : 1, \ \text{done} : 0 \mid s \in \text{string} \} \\
t := \text{readbit} (\text{print}[\text{"Hi"}](\text{stop}), \ \text{print}[\text{"Hello"}](\text{stop}))
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t := \text{readbit}\left(\text{print[“Hi”]}(\text{stop}), \text{print[“Hello”]}(\text{stop})\right)
\]

Reading the signature \(\Sigma\) as a game, the term \(t\) becomes a strategy tree:

```
readbit
  /
/  
|  
print[“Hi”]  print[“Hello”]
  /
/    
|     
stop  stop
```
Definition (Effect signature)

An effect signature is a set $E$ of operations together with a map $\text{ar} : E \to \textbf{Set}$ which assigns an arity set to each one.
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**Definition (Algebraic interpretation)**

The *terms* in $E$ with variables in the set $X$ are defined by the grammar:

$$t \in E^* X ::= x \mid m(t_n)_{n \in \text{ar}(m)} \quad (m \in E, \ x \in X)$$
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Definition (Game interpretation)

The plays over an effect signature $E$ with results in $X$ are defined by the grammar:

$$s \in P_E(X) ::= x \mid m \mid mns \quad (x \in X, \ m \in E, \ n \in \text{ar}(m))$$
Categorical characterization of terms and strategies

An effect signature can be interpreted as a polynomial endofunctor $E : \textbf{Set} \to \textbf{Set}$ constructing the terms of depth one:

$$EX := \sum_{m \in E} \prod_{n \in \text{ar}(m)} X$$

An *algebra* for $E$ is a set $A$ with a function $\alpha : EA \to A$; they form a category $\textbf{Set}^E$. 
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By working in a category of directed-complete partial orders, I obtain a similar characterization for the strategies over $E$. 
Section 3

Result: Strategies for algebraic effects
Strategies over an effect signature

Definition (Coherent plays)

The coherence relation $\varnothing \subseteq \mathcal{P}_E(X) \times \mathcal{P}_E(X)$ is the smallest relation satisfying:

\[ x \varnothing x \quad m \varnothing m \quad m \varnothing m ns (n_1 = n_2 \Rightarrow s_1 \varnothing s_2) \Rightarrow m n_1 s_1 \varnothing m n_2 s_2 \]

Definition (Effect strategy)

A strategy $\sigma \in \mathcal{S}_E(X)$ over a signature $E$ with results in $X$ is a prefix-closed set $\sigma \subseteq \mathcal{P}_E(X)$ of pairwise coherent plays.
Strategies over an effect signature

**Definition (Coherent plays)**

The *coherence* relation $\prec \subseteq P_E(X) \times P_E(X)$ is the smallest relation satisfying:

\[
\begin{align*}
    x \prec x & \quad m \prec m & \quad m \prec mns \\
    (n_1 = n_2 \Rightarrow s_1 \prec s_2) & \Rightarrow \underline{m} n_1 s_1 \prec \underline{m} n_2 s_2
\end{align*}
\]
Strategies over an effect signature

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The *coherence* relation $\preceq \subseteq P_E(X) \times P_E(X)$ is the smallest relation satisfying:

- $x \preceq x$
- $m \preceq m$
- $m \preceq mns$
- $(n_1 = n_2 \Rightarrow s_1 \preceq s_2) \Rightarrow m n_1 s_1 \preceq m n_2 s_2$

**Definition (Effect strategy)**

A *strategy* $\sigma \in S_E(X)$ over a signature $E$ with results in $X$ is a prefix-closed set $\sigma \subseteq P_E(X)$ of pairwise coherent plays.
Algebraic characterization of strategies

Strategies under set inclusion form a pointed directed-complete partial order:

- The empty strategy is the least element
- Unions of directed sets of strategies preserve the coherence condition
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It turns out the strategies for $E$ can be characterized as free algebras in $\mathbf{DCPO}_\bot$, where the effect signature $E$ is interpreted as the endofunctor:

\[
\hat{E}X := \bigoplus_{m \in E} \left( \prod_{n \in \text{ar}(m)} X \right)_\bot
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$$\hat{E}X := \bigoplus_{m \in E} \left( \prod_{n \in \text{ar}(m)} X \right)_\bot$$

**Theorem**

The forgetful functor $\hat{U} : \text{DCPO}^{\hat{E}} \rightarrow \text{Set}$ has a left adjoint. The pointed dcpo $S_E(X)$ carries the corresponding $\hat{E}$-algebra.
Strategies are ideal completions of terms

The *ideal completion* $\mathcal{I}$ constructs the free dcpo on a poset:

$$
\begin{array}{c}
\text{DCPO}_\bot \cong \\
\mathcal{I} \\
\text{Pos}_\bot
\end{array}
\quad \cong \\
\begin{array}{c}
\bot \\
\cup
\end{array}
$$

If we order terms with variables in $X_\bot$ using the rules

$$
\begin{align*}
\bot & \sqsubseteq t \\
x & \sqsubseteq x \\
\forall n \in \text{ar}(m) \cdot t_n & \sqsubseteq t'_n
\end{align*}
$$

$$
\frac{}{m(t_n)_{n \in \text{ar}(m)} \sqsubseteq m(t'_n)_{n \in \text{ar}(m)}},
$$

this provides an alternative construction of strategies as $\mathcal{I}E^*(X_\bot)$. 

\[\text{Theorem}\] The following partial orders are isomorphic:

$$
S_E(X) \cong \mathcal{I}E^*(X_\bot) \cong \mu Y \cdot \hat{E}Y \oplus X_\bot
$$
Strategies are ideal completions of terms

The *ideal completion* \( \mathcal{I} \) constructs the free dcpo on a poset:

\[
\begin{align*}
\xymatrix{
\text{DCPO} \downarrow & \downarrow & \text{Pos} \downarrow \\
\mathcal{I} \ar@{<->}[rr] & & \downarrow \ar@{<->}[rr] & & \downarrow U
}
\end{align*}
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If we order terms with variables in \( X \) using the rules

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\begin{align*}
\bot & \sqsubseteq t \\
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m(t_n)_{n \in \text{ar}(m)} \sqsubseteq m(t'_n)_{n \in \text{ar}(m)}
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this provides an alternative construction of strategies as \( \mathcal{I}E^*(X) \).

**Theorem**

*The following partial orders are isomorphic:* \( S_E(X) \cong \mathcal{I}E^*(X) \cong \mu Y \cdot \hat{E}Y \uplus X \)
Section 4

Conclusion: Towards algebraic game semantics
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Algebraic effects and game semantics have themes in common, but they look at them very differently.
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For example:

- Generalizing to multi-sorted signatures allows for richer games. This could serve as a common low-level algebraic grounding for a variety of sequential game semantics models.
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For example:

- Generalizing to multi-sorted signatures allows for richer games. This could serve as a common low-level algebraic grounding for a variety of sequential game semantics models.

- Effect signatures and natural transformations $\eta_X : EX \rightarrow FX \in \textbf{Set}$ form a symmetric monoidal closed category. Endofunctor composition and the free monad construction can be defined directly on signatures. We can carry out a version of Reddy’s object-based semantics in this setting.