Symmetry breaking and the functional RG scheme

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A numerical algorithm is used to solve the Wegner-Houghton equation for the scalar $\phi^4$ model without any limiting ansatz for the local potential. Non-perturbative critical exponents are obtained in the infrared scaling regime of the symmetric theory. A precursor of the symmetry breaking, a dynamical Maxwell-cut is found which reveals itself in the approximate degeneracy of the blocked action for modes at the sufficiently low cutoff. A mechanism of Quantum Censorship is conjectured which is supposed to be operational for large enough quantum fluctuations and prevents the exact degeneracy of the Wilsonian action together with the tree-level renormalization.

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I. INTRODUCTION

The functional renormalization group method is a promising tool to study and solve strongly coupled quantum systems. It is based on the gradual elimination of degrees of freedom by decreasing a continuous parameter of the dynamics, traditionally chosen to be the cutoff, $k \rightarrow k - \Delta k$. The novelty is the appearance of a new small parameter, the relative change of the cutoff, $\Delta k/k$, which simplifies enormously the renormalization group equation.

One assumes at the derivation of the Wegner-Houghton or the Polchinski equations the availability of the loop or the perturbation expansion, respectively. But the true expansion parameter for the change of the Wilsonian action is the naive small parameter multiplied by $\Delta k/k$ and this latter factor renders the leading order expressions for the evolution equation exact in the differential equation limit, $\Delta k \rightarrow 0$. These equations are based on an approximate evaluation of the blocking transformation and their solution resums the expansion series.

There is a way to establish the renormalized trajectory by formal manipulations, without computing any path integral during the evolution. The leading order term of the evolution equation in $\Delta k$ gives a functional differential equation whose validity is not based on any expansion scheme of the physical theory. It is the effective action whose evolution is followed in this scheme and the only integration takes place in the initial condition where the theory is supposed to be perturbative. The actual evolution is simplified to a one-loop equation in the leading order in $\Delta k/k$.

The only approximation we have to make in the functional renormalization group method is to project the exact renormalization group equation into a functional space for the action where the solution is sought. This step, the choice of ansatz for the action, is guided by our intuition and our knowledge about the dynamics and provides a flexible truncation scheme.

A drawback of this scheme is its weakness of handling degrees of freedom whose dynamics, the dependence of the action on their coordinate, is weak. In fact, the methods tracing the evolution of the Wilsonian action may fail because the loop or the perturbation expansions are not reliable for a degenerate action. The functional Legendre transformation, employed in the derivation of the effective action becomes singular and renders the effective action an unreliable tool for the study of an almost degenerate system. Note that the case of the degenerate action is more complicated than those of the usual strongly coupled situation because the integrand of the path integral is constant and all field configurations contribute with equal strength.

The prototype of degenerate systems is a condensate or the spontaneous breakdown of a symmetry. The mixed phase, observed for such a system for certain values of the order parameter, is based on the competition of two different orders and the position of the domain wall, separating such realizations represents a degenerate collective mode. When equilibrium concepts are applicable like for Euclidean Quantum Field Theory, then the Maxwell-cut renders the action strictly degenerate in some collective coordinate. The degeneracy opens the possibility that completely new degrees of freedom appear, such as the gas molecules in the liquid-vapor transition. The goal of this work is to investigate this problem in its simplest realization, in the framework of the scalar $\phi^4$ theory in four dimensional Euclidean space-time by means of the reconstruction of the renormalized trajectory for the Wilsonian action, given in the local potential approximation by the Wegner-Houghton equation.

The new element of our study is a numerical algorithm to solve the evolution equation for the potential in general, without any ansatz. We find new non-perturbative results already in the symmetrical phase where fractional critical exponents in the IR scaling regime are identified. But the main results concern the symmetry broken phase, the more detailed dynamical way the condensate is built up. We can identify the asymptotic and non-asymptotic IR scaling regimes, characterized by the approximate degeneracy of the action in the former case. The issue of the survival of
the loop expansion and that of the standard evolution equation leads to the conjecture of Quantum Censorship, the realization of an approximate degeneracy for the action by quantum fluctuations.

We report here on our study of the evolution of the Wilsonian action. We have extended our work to the effective action, as well but these results, being qualitatively similar to those of the Wilsonian action, are not presented here.

The organization of this paper is the following. The evolution equation, its scaling regimes and its validity is discussed briefly in Section II. The numerical solution is presented in the main body of the paper, in Section III. First the symmetric phase is considered, followed by the more detailed study of the symmetry broken phase. The novelty of the latter phase, the building up of the condensate is identified in the IR scaling regime where the Quantum Censorship conjecture is spelled out. Finally, Section IV contains the summary of the results. Appendix A is provided to give some details of the numerical algorithm used.

II. EVOLUTION EQUATION

The Euclidean partition function of the model, investigated in this paper is given by the path integral

$$Z = \int D[\phi] e^{-\frac{1}{\hbar} S_k[\phi]}$$

where the integration extends over the field configurations with Fourier modes $|p| < k$. We follow the evolution of the blocked action, $S_k[\phi]$ when the cutoff $k$ is lowered in this section.

A. Evolution equation for the blocked potential

We lower the cutoff, $k \rightarrow k - \Delta k$, by splitting the field variable as $\phi = \phi' + \tilde{\phi}$ in such a manner that the supports of the Fourier transforms of $\phi'$ and $\tilde{\phi}$ are $|p| < k - \Delta k$ and $k - \Delta k < |p| < k$, respectively, and integrate over the UV field,

$$e^{-\frac{1}{\hbar} S_{k-\Delta k}[\phi]} = \int D[\tilde{\phi}] e^{-\frac{1}{\hbar} S_k[\phi+\tilde{\phi}]}.$$  \hfill (2)

We assume that the loop expansion is applicable, yielding

$$S_k[\phi + \tilde{\phi}_{cl}] - S_{k-\Delta k}[\phi] = -\frac{\hbar}{2} \text{Tr} \ln \left( \frac{\delta^2 S_k[\phi + \tilde{\phi}_{cl}]}{\delta \tilde{\phi} \delta \tilde{\phi}} \right) + O(\hbar^2),$$  \hfill (3)

where $\tilde{\phi}_{cl}$ denotes the saddle point in the space of the UV field configurations and $O(\hbar^2)$ stands for the higher loop contributions. The action is assumed to take the form

$$S_k[\phi] = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + V_k(\phi) \right]$$

within the local potential approximation, followed in this work. There is a single potential to keep track and the choice $\phi'_x = \Phi$ is natural. We evaluate the trace in the Fourier space and assume the vanishing of the saddle point $\tilde{\phi}_{cl} = 0$. The result is the Wegner-Houghton equation \[4\],

$$\partial_t V_k(\phi) = -\frac{\hbar k^3}{16\pi^2} \ln[k^2 + V''(\phi)],$$

in the limit $\Delta k \rightarrow 0$ where $V''(\phi) = \partial^2 \phi V(\phi)$. The higher loop contributions are suppressed by $\Delta k/k$ and are negligible in the differential equation limit. When the equation is integrated then each step $k \rightarrow k - \Delta k$ generates one further order in the loop expansion and the solution of the differential equation, obtained in the limit $\Delta k \rightarrow 0$, contains the contributions of all connected Green functions at vanishing momentum.

B. Scaling regimes

Apart of the critical theories we always have at least two scaling regimes which are separated by a crossover scale $k_{cr}$. If the theory is weakly coupled then we possess small parameters to organize an at least asymptotic expansion
to obtain the UV scaling laws displaying simple beta functions. The powers of the small parameter of the expansion generate several scales around the crossover and give rise asymptotic and non-asymptotic scaling. This is the way the multiphase structure of relativistic corrections is generated in QED. Thus we can refine our scheme and talk about asymptotic and non-asymptotic scaling laws, separated by the scales \( k_{UV} \) and \( k_{IR} \) within the UV and the IR scaling regime, respectively. The non-asymptotic scaling laws display mass corrections or non-perturbative contributions.

The theories, considered in the rest of this work will be weakly coupled close to the initial cutoff, \( k_{init} = \Lambda \), and bare perturbation expansion can be used to define the asymptotic UV scaling regime. We shall be interested in the low energy content of the theories and shall follow the evolution towards the IR direction and address no UV issues, such as triviality. A sufficiently massive theory, deeply in the symmetrical phase is perturbative on every scale. But the usual loop-corrections, responsible of the non-perturbative critical phenomena, pile up as the critical theory is approached and the UV scaling regime becomes longer.

Let us summarize briefly what kind of qualitative behavior of the running coupling constants \( g_n = \partial_n^2 V(\Phi) \) is expected. The corresponding beta functions, \( \beta_n = k \partial_k g_n \), are obtained by taking \( n \) successive derivatives of Eq. [5] with respect to \( \Phi \). The result is a sum, consisting of terms like

\[
(-1)^n c_n \prod_{j=3}^n \frac{g_j^{(s)}}{(k^2 + g_2)^{n_n}},
\]

where \( c_n \) is a dimensionless, positive number, \( \sum_j g_j^{(s)} = N_n \) and \( \sum_j (4 - j) g_j^{(s)} = 4 - n \) [3]. The dominant term of the sum, if there is one in an asymptotic scaling regime, is selected by the competition between the numerators and denominators. The alternating sign strongly reduce the size of the asymptotic scaling regimes. Consider first the most important beta function,

\[
\beta_2 = \frac{h k^4}{16 \pi^2} \left[ \frac{g_4}{k^2 + g_2} - \frac{g_2}{(k^2 + g_2)^2} \right].
\]

It makes the mass square, \( g_2 \), increasing during the evolution in a \( \phi^4 \) theory as long as \( k^2 + g_2 > 0 \). Deeply in this phase the denominator in Eq. [7] is made large by the cutoff or by the mass and the power of the propagator must be minimized to find the dominant term, giving \( \beta_2 \sim g_4/(k^2 + g_2) \). The beta function of almost critical theories is dominated by the term with the maximal power of the propagator, \( \beta_2 \sim (g_4/k^2)^{n/2} \). When we move from the inside of the symmetric phase towards the critical theory then first we loose the perturbation expansion at the crossover \( k = k_{cr} \) and the asymptotic UV scaling regime shrinks to \( k_{UV} < k \) with \( k_{cr} < k_{UV} \). The non-asymptotic scaling for \( k_{cr} < k < k_{UV} \) is governed by beta functions with competing terms. One expects the emergence of a similar scale, \( k_{IR} \), separating the asymptotic and non-asymptotic IR scaling regimes. The IR scaling in the vicinity of the critical point is highly non-perturbative and is unknown.

Our interest in the scalar model comes from the condensate building up in the symmetry broken vacuum. This process starts around the scale \( k_{IR} \) where the propagator diverges, the highest scale \( k \) for which

\[
\frac{k_{IR}^2 + g_2 k_{IR}}{k_{IR}^2} \approx 0
\]

and where the scaling laws change in an abrupt manner. The precursor of symmetry breaking shows up in the IR scaling regime, \( k_{IR} < k_{cr} \), ie. theories with symmetry broken vacuum display asymptotic UV (\( k_{UV} < k < \Lambda \)), non-asymptotic UV (\( k_{cr} < k < k_{UV} \)), non-asymptotic IR (\( k_{IR} < k < k_{cr} \)) and finally asymptotic IR (\( 0 < k < k_{IR} \)) scaling regimes. The last scale, \( k_{IR} \), depends strongly on the strength of the condensate, \( \Phi = \langle \phi \rangle \). In the symmetric phase one expects a smoother transition between the asymptotic and non-asymptotic IR scaling laws.

We shall be interested in what happens in the vicinity of \( k_{IR} \). The inverse propagator, the argument in the logarithm of Eq. [3], is an eigenvalue of the second functional derivative of the action. When a negative eigenvalue appears the integral of Eq. [2] has a non-trivial saddle point. Therefore, if the inverse propagator changes sign then the renormalization group equation has tree-level contribution. The contributions of plane waves, alias domain walls, to the evolution equation lead to the Maxwell-cut, the degeneracy of the IR local potential, \( V_{k=0}(\Phi) \), between the stable vacua, \( |\Phi| < \Phi_{vac} \) when the loop corrections to the scaling laws are ignored [9]. In addition, Eq. [8] remains satisfied for an increasing interval of \( \Phi \) around zero for scales below \( k_{IR} \). The action \( S_k[\phi] \) is therefore degenerate for modes \( p = k \) which are best described by such an effective theory. We arrive at a dynamical extension of the Maxwell-cut in this approximate solution.
C. Validity of the loop-expansion

Let us now address the question of the applicability of the loop expansion in the functional integral (2), assumed in deriving the evolution equation (5). The dimensionless small parameter of loop expansion is \( \epsilon_k(\Phi) = \hbar \Delta k \ln[1 + g_k(\Phi)/k^2]/k \). By setting the book-keeping variable to unity, \( h = 1 \), there are two ways this parameter may become large: either we reach the IR end point, \( \Delta k/k \to 1 \) or the logarithm function explodes because the action is approximately degenerate, Eq. (2) is valid. The former case requires to stop the integration of the evolution equation (5) slightly before \( k = 0 \). The results obtained in this manner are useful because they characterize the infrared end point, \( k = 0 \) if the theory has a convergent thermodynamical limit, an assumption one is usually ready to agree with. Thus the real danger for the validity of the evolution equation comes from the second mechanism, the approximate degeneracy of the action occurring already at finite scale.

The dynamical Maxwell-cut, taking place at some scale \( k_{IR} \) and field \( \Phi \), makes the "small" parameter \( \epsilon_{k_{IR}}(\Phi) \) diverging and the further integration of Eq. (5) towards smaller \( k \) is not justified. One might hope that for large enough field the mass square \( g_2(\Phi) \) is large enough to save the perturbation expansion and some information can be extracted about the system by seeking the solution of the evolution equation for \( k < k_{IR} \) for a potential given in terms of a power series around \( \Phi \). This is actually the strategy of the usual perturbative treatment of the symmetry broken vacuum of the scalar model. The application of the perturbation expansion becomes even more uncertain in the vacuum, \( \Phi = \Phi_{vac} \), because it is just at the end of the Maxwell-cut, \( c_{k=0}(\Phi_{vac}) \approx \infty \), and at least the 50% of the fluctuations, those which tend to decrease \( \int d^4 x \delta \phi_x \), are non-perturbative. The main question we embark in the following section by the use of the global, numerical solution of Eq. (5) is the range of applicability of this equation and the extent, the non-perturbative fluctuations make the action degenerate already at finite scale.

III. NUMERICAL SOLUTION

In order to avoid the need of the polynomial ansatz for the potential which can be justified by means of the perturbation expansion only, we seek the solution of Eq. (5) for a general function \( V_k(\Phi) \), represented by spline polynomials. Thus Eq. (5) is used to generate the evolution of the coefficients of the spline functions, see Appendix A for details. The fundamental difference between the expansion and the spline methods is the handling of boundary conditions. In fact, this is an implicit assumption for the expansion method but requires explicit equation for the spline representation. We have to provide two functions to make the solution of Eq. (5) unique. The range of the field variable will be restricted to the interval \( (0, \Phi_{max}) \) where \( \Phi_{max} \) is sufficiently large. One of the boundary conditions, \( V_k(0) = 0 \), expresses the \( Z_2 \) symmetry of the theory. The other condition of the form \( 0 = \alpha V_k(\Phi_{max}) + \beta V_k'(\Phi_{max}) \) will be used with the trivial choice, \( \alpha = \beta = 0 \), in the algorithm which remains well defined in this case. The error induced by this null-boundary conditions is estimated by considering the solution in larger interval, by studying the convergence of the solution in the limit \( \Phi_{max} \to \infty \). It was carefully checked that this limit indeed leaves behind a stable, convergent numerical solution.

A. Symmetric phase

Let us now consider the numerical results first in the symmetric phase. The evolution of the first three dimensionless coupling constants at \( \Phi = 0 \), \( \tilde{g}_n = k^{n-4}g_n \), together with their beta functions

\[
\tilde{\beta}_n = k^{n-4}\beta_n + (n-4)\tilde{g}_n
\]

are shown in Figs. 1-3 for a number of bare theories which share the same interactions, \( g_{nB} = g_{n,k=1} = 0.8\delta_{n,4} \) for \( n \geq 4 \) and differ in their bare mass, \(-2.52 \cdot 10^{-3} < \tilde{g}_{2B} < -2.5 \cdot 10^{-3} \). The initial value of the cutoff is chosen to be unity, \( \Lambda = 1 \).

The beta function of the dimensionless mass square, shown on the right plot of Fig. 4 indicates two asymptotic scaling regimes. The fluctuations are suppressed by the mass gap in the deep IR scaling regime. Therefore the dimensional mass is scale invariant and the beta function is dominated by the second term in (9) for \( n = 2 \), \( \tilde{\beta}_2 \sim -k^{-2} \), in this asymptotic regime. In the UV scaling regime, appearing close to the initial condition, \( k = 1 \) for smaller masses only, \( g_2 \sim k^2 \). As the mass square is decreased the first term, approximately scale independent according to perturbation expansion, starts to be visible. The horizontal lines on the right plot of Fig. 4 where the mass can be neglected extends more as the critical theory is approached. Comparison with the naive, bare perturbation expansion reveals that the UV scaling lasts until \( k \approx 0.1 \), inside the deep IR scaling regime indicated by the straight line with slope -2 on the double logarithmic beta function plot.
The absence of any irregularity in the beta function $\tilde{\beta}_2$ at the crossover scale could be interpreted as an indication that the regions of applicability of the bare and the renormalized perturbation expansions overlap for the theories displayed in this plot. But the scaling of the higher order coupling strengths, displayed in Figs. 2-3, shows that the deep IR scaling is strongly non-perturbative. We find two scaling regions on these figures, as well, but separated by an order of magnitude smaller crossover scale than for the mass. The asymptotic UV scaling law with $\tilde{\beta}_4 > 0$ in Fig. 2 indicates the weakly irrelevant nature of $g_4$. The initial negative part of $\tilde{\beta}_6$ on the right plot of Fig. 3 (the figures show the absolute magnitude) close to $k = 1$ reflects the need of blocking down the theory by a scale factor 2-5 to generate the set of irrelevant coupling constants needed for the initial cutoff-independent asymptotic UV scaling laws. The IR scaling law with fractional critical exponents, $\tilde{\beta}_4 \sim k^{3.25}$ and $\tilde{\beta}_6 \sim k^{1.75}$, is highly non-trivial and reflects a dynamical fine tuning in the competition among the different contributions to the beta functions. Such a delicate, non-analytic balance among the different higher order, irrelevant coupling constants is a non-perturbative phenomenon and the resulting fractional critical exponents are the precursors of critical scaling, approached from the symmetric phase.

It is worthwhile comment here the applicability of the loop expansion in solving the theory. The naive expansion of the complete path integral (1) is spoiled for small enough masses and large enough cutoff in either phase because the bare mass square is then negative. This makes the Euclidean propagator exploding at lower scales, indicating that the quasiparticle of the UV regime is not well suited to reproduce the IR dynamics. This problem can easily be cured by means of the renormalized perturbation or loop expansion, based on the IR propagator. But this issue underlines that the perturbation or loop expansion becomes more reliable when it is applied in a restricted window of
theories in the symmetric phase. The double logarithmic plot allows us to identify the scaling laws of the fine-tuned competition of different terms, each of them with an integer exponent. Fractional ones of Figs. 2-3 support the conjecture that the exponents of the symmetric phase are indeed the result of the fine-tuned competition of different terms, each of them with an integer exponent.

So far we followed the evolution of the local features of the potential at vanishing field. How are the interactions modified by injecting a condensate $\langle \phi_x \rangle = \Phi \neq 0$? The condensate is supposed to be stabilized either by a homogeneous, external source coupled to the field variable $\phi_x$ in the action or by keeping the value of the space-time independent component of the field $\phi_y=\Phi$ fixed in the functional integral $(1)$. We assumed that the theory is perturbative at the initial cutoff therefore $g_4(\Phi) \approx g_B$ for $k \approx \Lambda = 1$. The available states are strongly reduced if the mass, $m^2(\Phi) = m_B^2 + g_B \Phi^2/2$, is larger than the actual cutoff. Therefore no dressing is expected for $|\Phi| > \Lambda \sqrt{2/g_B} \approx 1$ and $g_4(\Phi) \approx g_B$ should remain valid at lower scales as long as $|m_B^2| < 1$. For smaller field values the scalar particles of the theory are allowed to interact and their repulsive interactions should reduce $g_4$. For $|\Phi| \approx \Lambda \sqrt{2/g_B}$ the logarithmic correction terms in the effective potential are more important in the asymptotic UV scaling regime and give some weak variation of the effective coupling strength. This can be seen in the highest curve in Fig. 4 which shows the field dependence of the quartic interaction strength $g_4$ for a strongly massive, fully perturbative theory for $k \approx 0$. There is a small bump at $\Phi \approx \Lambda = 1$ and the smaller field region where particles are allowed to interact displays a slightly screened, reduced interaction strength. The remaining curves correspond to an almost critical theory. The coupling constant establishes there the usual logarithmic dependence in $k$ but remains field-independent for small $|\Phi|$. The evolution is frozen by the cutoff in the outer zone where the coupling constant displays a logarithmic dependence in $\Phi$, a true freeze-out effect.

We can gain a more detailed insight into the different scaling regimes by considering the sensitivity matrix, the dependence of the renormalized coupling parameters on the bare one. The dependence of $g_4$ on its bare value, $10^{-4} \partial g_4 / \partial g_B$, is shown in Fig. 5 for different bare masses. In the asymptotic UV regime this quantity is a negative, mass and scale independent constant, suggesting $k_{UV} \approx 0.005$ for the theories considered. The mass dependence appears gradually in the non-asymptotic UV scaling regime. The sensitivity matrix locates the UV-IR crossover in a clear manner because the symmetric and the symmetry broken trajectories depart at this scale. When we modify the initial, bare parameters of an almost critical trajectory then the largest deviation will be observed in the region where the trajectories split and turn into different directions. This argument yields the estimate $0.001 < k_{cr} < 0.005$ for the theories in the symmetric phase. The double logarithmic plot allows us to identify the scaling laws $\partial g_4 / \partial g_B \sim k^4$ and $k^{-2}$ for IR and in the non-asymptotic UV scaling regimes. The integer critical exponents, in contrast with the fractional ones of Figs. 2-3 support the conjecture that the exponents of the symmetric phase are indeed the result of the fine-tuned competition of different terms, each of them with an integer exponent.
FIG. 4: The quartic coupling strength, $g_4$, as the function of the field $\Phi$ in the symmetric phase. The upper curve corresponds to a strongly massive theory, $\tilde{g}_{2B} = +0.25$. The remaining, lower curves are of an almost critical theory, $\tilde{g}_{2B} = -0.0025$.

FIG. 5: The logarithm of the absolute magnitude of the difference of $\beta_4$ between $g_B = 0.8001$ and $g_B = 0.8$ as the function of the scale for different mass squares, shown separately, close to the critical theory. The curves raise monotonously when $m_B^2$ is decreased and the critical theory is the separatrix.

B. Symmetry broken phase

When the critical point is approached from the symmetry broken phase the resulting trajectories are shown in Figs. 6-8. The UV scaling laws are similar in both phases but we find radically new scaling laws in the IR.

The beta function of the dimensionless mass square, shown in the right plot of Fig. 6, starts similarly to the symmetrical phase but it quickly changes sign as in the vicinity of the Gaussian fixed point. The straight lines on the right plot of Fig. 6 indicate a scale independent dimensional mass, just as in the symmetrical phase. We shall see later that this scaling regime ends at the crossover, $k_{cr}$, in a marked contrast with the symmetric phase.

The quartic coupling constant, shown in the left plot of Fig. 7, approaches zero or settles at a finite value deeply in the broken phase or in the vicinity of the critical theory, respectively. The end of the asymptotic UV scaling, $k_{UV}$ can be placed where the coupling constant starts to drop. The maximal speed of decrease, the peak in $\beta_4$, shown in the right plot of Fig. 7 locates $k_{cr}$, for instance $k_{cr} \approx 0.004898$ for $m_B^2 = -0.002545$. The higher order coupling constants, for instance the evolution of $\tilde{g}_6$, shown in Fig. 8 indicates the same crossover scale, except that the running coupling constants strongly first increase upon entering into the IR scaling regime and fall sharply onto a lower value before the program stops.

The sensitivity matrix provides a clearer separation of the two phases. In fact, the phase transition is the emergence of a singularity in the relation among bare UV and renormalized IR quantities in the thermodynamic, $k \rightarrow 0$ limit. The upper curves in Fig. 5 correspond to the broken phase and the separatrix can easily be determined by better and
FIG. 6: The $1 + \tilde{g}_2$ and the absolute magnitude of the beta function $\tilde{\beta}_2$ as the functions of the cutoff, taken at different bare masses in the symmetry broken phase.

FIG. 7: Left: $g_4$ as the functions of the cutoff, taken at different bare masses in the symmetry broken phase. Right: The beta function $\beta_4$ is shown for $m_B^2 = -0.002545$ for $k \approx k_{cr}$.

FIG. 8: $\tilde{g}_6$ as the functions of the cutoff, taken at different bare masses in the symmetry broken phase. The mass square assumes the same values as in Figs. 6 and 7 and $\tilde{g}_6$ decreases in a monotonous manner when the mass square is increased. The end of the curve corresponding to $\tilde{g}_{2B} = -0.002539$ is shown on the right. Similar, sudden fall is observed at each mass square values when better resolution is used for the peak. The magnitude of the peak and the final value of $\tilde{g}_6$ before the program stops decrease with $\tilde{g}_{2B}$ in the vicinity of the critical point.
FIG. 9: The degeneracy, $1 + \tilde{g}_2$, (left) and the quartic coupling strength $g_4$ (right) as the function of the field $\Phi$ deeply in the symmetry broken phase, $m_B^2 = -0.5$ and $g_B = 0.5$.

better accuracy if the computing power allows.

C. Asymptotic IR scaling

In order to recognize the onset of the asymptotic IR scaling laws we start with some general remark about the vicinity of the IR end point of the renormalized trajectory. The condensate of the scalar model is homogeneous, it consists of particles with vanishing momentum. Their density which is proportional to $\Phi^2$ can be controlled by an external, homogeneous source, coupled to the quantum field. The Maxwell-cut, observed at $k = 0$, assures that the vacuum energy is independent of the density of the particles in the condensate in some density range, proportional to $\Phi_{\text{vac}}^2$. Beyond this threshold density the repulsion becomes strong enough to place the additional particles at higher, non-condensed states.

Can we see a precursor of condensation, the dynamical Maxwell-cut which renders the action nearly degenerate and locates $k_{\text{IR}}$? Imagine that our scalar particles are brought into contact with a "heat bath", an ingenious device which keeps the average Euclidean wave vector length of the quasiparticles at a given scale $k$. The dependence of such a "free energy" on the background field $\Phi$ or condensate density is captured by the blocked potential $V_k(\Phi)$. When the typical particle energy is at the initial cutoff then the renormalization effects are not important. But the "cooling" of the system dresses the "free energy" of the UV modes. The question is if such a "cooling" may detect the presence of the condensate before we reach the final stage of the evolution where the potential $V_{k=0}(\Phi)$ agrees with the effective potential.

The degeneracy of the blocked action, treated within the local potential approximation requires the potential $V_k^{\text{deg}}(\Phi) = -k^2\Phi^2/2$. The dimensionless measure of this degeneracy $1 + \tilde{g}_2$, plotted in Fig. 9 shows clearly that the dressing, provided by particles with wave vector $k = k_{\text{IR}} \approx 0.73$ makes the vacuum energy almost degenerate in an interval of $\Phi$ for the bare parameters chosen in this case. Such a realization of the dynamical Maxwell-cut is the key to identify the driving force of spontaneous symmetry breaking. We have a qualitatively similar degenerate plateau, bounded by an almost discontinuous jump when we consider the potential as the function of $\Phi^2$, a variable which is proportional to the condensate density. The derivative of this function, $dV(\Phi^2)/d\Phi^2$, is proportional to the binding energy. Thus spontaneous symmetry breaking is driven by the degeneracy of the vacuum energy in the condensate density already at finite scale. It is the role of an infinitesimal external source to stabilize the condensate density at one of the end points of the degeneracy and to select a single vacuum. This scenario is further supported by the function $g_4(\Phi)$, plotted on the right of Fig. 9 showing that the strength of the two-body forces is indeed small when the vacuum is degenerate.

Higher order coupling constants shows a more involved scale dependence when the cutoff is decreased. Though their dimensionless value drops in the UV scaling regime they display a peak before $k_{\text{IR}}$ which becomes taller with the increase of $n$. The further decrease of $k$ makes these coupling strengths very small when the cutoff reaches $k_{\text{IR}}$. The large peak indicates that the particles interact with strong many-body forces at that scale. The emergence of the simple but unusual potential $V_k^{\text{deg}}(\Phi)$, the dynamical Maxwell-cut, indicates that the dynamics of the particles at that scale is already modified by the presence of the homogeneous condensate. The dramatic raise and fall of the higher order interaction strengths before $k_{\text{IR}}$ indicates that the full condensate supports radically different quasiparticles than the perturbative vacuum. This difference shows up in their "dressing", their structure in terms of the bare...
FIG. 10: The quartic coupling strength, $g_4$, shown as the function of the field $\Phi$ for a strongly broken theory, $m_B^2 = -0.00254285$ and $g_B = 0.8$. The function $g_4(\Phi)$ which decreases with the cutoff around $\Phi = 0$ is displayed only for the last few steps of the algorithm, $0.0046273 < k < 0.0046498$.

particles only. Naturally their dispersion relation which is controlled by the unbroken space-time symmetries must remain usual.

Note that the theory shown in Fig. 9 is found deeper in the symmetry broken phase than the ones considered in Figs. 6-8. We need strong symmetry breaking to observe a clear (dynamical) Maxwell-cut which gradually disappears as we approach the massless, critical theory.

D. Quantum censorship

After the onset of the asymptotic IR scaling has been located let us have a closer view on the dynamics of this scaling regime, by tracing more accurately the left hand side of Eq. (8). The renormalized trajectory, discussed so far shows that the onset of the symmetry breaking and the emergence of the condensate happens already at finite scale where the blocked action becomes approximately degenerate at the cutoff. This raises the question whether Eq. (8) may become exact and if the evolution equation (5) which assumes the loop-expansion remain valid.

The linear algebra, used in determining the evolving coefficients of the spline functions, becomes singular for degenerate action when the right hand side of the evolution equation is vanishing. The program optimizes the step size, $\Delta k$, automatically and stops when the irregularities in the interpolation of the derivatives or in the integration of the differential equation are significantly larger than the precision of the number representation in the machine. Such a dynamically adjusted step size is very small before stopping to keep the small parameter $\epsilon_k$ under control and the last value of $k$ can be taken as a good approximation for $k_{IR}$.

The program stops at finite $k$ in the symmetry broken phase and one can imagine two scenarios in the presence of unlimited computer power when the numerical accuracy can be increased without limit. One possibility is that the action does not become exactly degenerate, the dynamical readjustment of the step size $\Delta k$ in the numerical algorithm is sufficient to resolve the variation of the action and the program continues functioning down to the IR endpoint. Another scenario is that whatever computer power we employ, the evolution can not be continued beyond some finite scale $k_{IR} > 0$ because $\Delta k$ shrinks too fast. Such an accumulation point of the finite steps would indicate that the parameter of the loop expansion, $\epsilon_k$, can not be kept small and the evolution equation (5) is not reliable anymore. We need a more thorough numerical or analytic analysis of the evolution equation to decide which possibility is realized. In the latter case there is no theoretical framework to continue the evolution, a path integral with a constant integrand, the ultimate disorder, being too hard problem to tackle. The former case, the prevention of the establishment of the dynamical Maxwell-cut by non-perturbative quantum fluctuations, can be called Quantum Censorship, reminiscent of a problem in General Relativity [10].

Do the numerical results provide at least an indirect evidence in favor or against Quantum Censorship? To answer this question we have to look more closely to the short scaling regime the program can trace below $k_{cr}$. It has already been mentioned that the running coupling constants first grow below $k_{cr}$ and then drop. The potential of a theory deeply in the broken phase is shown in Fig. 10 for the last few steps before crashing the program. The cutoff is decreased gently and the eight order splines display, a regular looking potential in this final stage of the evolution
except at the edge of the degeneracy. One observes a shock wave squeezed against the beginning of field values where the action is non-degenerate and the evolution is regular. If Quantum Censorship is in action then the shock wave is dissolved in a regular manner and the potential interpolates smoothly between two analytical forms, with and without dynamical Maxwell-cut. If Quantum Censorship is violated then the solution of the evolution equation hits a real singularity and the action becomes degenerate.

One expects that stronger quantum fluctuations give more chance for Quantum Censorship to act. As a result, going deeply into the broken phase may weaken the Quantum Censorship and we may find singular renormalized trajectories. This view is supported by the behavior of $g_4$, showing that the singularity at the boundary of the regular and the irregular field intervals becomes weaker when we approach the critical point. It was found that the minimum in Fig. 10, created by the shock wave disappears and the $g_4$ has a simple maximum only for symmetry broken theories close to the critical point. Thus it seems reasonable to conjecture that the yet unknown nature of the critical theory, as seen from the symmetry broken phase is related to the presence of Quantum Censorship. If this mechanism is invalidated by the weakening of the fluctuations as we move towards theories with strongly broken symmetry leaving behind a phase transition or remains in operation everywhere in the symmetry broken phase remains a question to be settled by further studies.

IV. SUMMARY

Any attempt to evaluate functional integrals is related to some kind of expansion. This strategy is seriously hampered when the integrand is a constant, the action is degenerate. The most natural setting of this problem, the emergence of a condensate in a vacuum with a spontaneously broken symmetry, is considered in this work. Our method is the numerical construction of the renormalization of the local potential of the $\phi^4$ model, the solution of the Wegner-Houghton equation without any ansatz for the potential.

One can separate the asymptotic and non-asymptotic regimes in the UV scaling regime which are qualitatively similar in both phases. The symmetric phase serves not only to verify the numerical algorithm in the perturbative domain but we find fractional IR critical exponents for polynomial interaction vertices, characterizing the non-perturbative critical theory. The asymptotic and non-asymptotic regimes separate clearly in the IR scaling regime of the symmetry broken theory, the former being characterized by an approximate degeneracy of the action, a dynamical Maxwell-cut.

The asymptotic IR scaling poses a wonderful problem both on the analytical and numerical sides. On the analytical side it is not clear how to handle a degenerate action and if the saddle point approximation, leading to tree-level renormalization, is reliable. The challenge arising from the numerical treatment of an almost degenerate action is to overcome the errors arising from the limited precision of number representation in the computer. As a result, we had to be satisfied by a conjecture, consistent with the limited numerical data, namely that the loop-expansion method is the numerical construction of the renormalization of the local potential of the $\phi^4$ model, the solution of the Wegner-Houghton equation without any ansatz for the potential.

APPENDIX A: THE NUMERICAL ALGORITHM

Let us write the evolution equation [5] as
\[
\partial_k u = -\frac{k^3}{16\pi^2}f',
\]
where $v_k(\phi) = V''_k(\phi)$ and $r_k(\phi) = v''_k(\phi)/[k^2 + v_k(\phi)]$. The boundary conditions (i) $v_k(0) = 0$, (ii) $v_k(\Phi_{\text{max}}) = f_k$ and (iii) $v_k' (\phi) = v_{\text{h}}(\phi)$ with given functions $f_k$ and $v_{\text{h}}(\phi)$ are sufficient to have a unique solution.

The function $v_k(\phi)$ is given in a spline form [11, 12], represented as the sum of eight order Chebyshev polynomials of the field $\phi$ in order to optimize the convergence of the solution when the order of the spline polynomial and their number is increased [13]. number of sand the partial differential equation [A1] is written as a set of ordinary differential equation for the coefficients, considered as functions of $k$. The boundary conditions (i) and (iii) are easy to implement, (ii) can be replaced by an algebraic equation, written as
\[
\beta_k(\Phi_{\text{max}}) r_k(\Phi_{\text{max}}) = \gamma_k(\Phi_{\text{max}}, v_k(\Phi_{\text{max}}), v_k'(\Phi_{\text{max}}), V_k(\Phi_{\text{max}}), \partial_k V_k(\Phi_{\text{max}}))
\]
in the functional notation in terms of given functions $\beta$ and $\gamma$.

The algebraic equations for the coefficients of the Chebyshev polynomials remain well defined in the limit $\beta, \gamma \to 0$. The boundary condition [A2] is imposed in this null-equation limit. The solution is carefully monitored in a second
limit, by removing the upper limit of the equation, \( \Phi_{\text{max}} \to \infty \). The potential \( U_k(\phi) \) was found to be convergent and \( \Phi_{\text{max}} \) was finally fixed by having relative error \( 10^{-6} \) up to field values higher than \( \Lambda \).

The algorithm has been tested by the choice \( v_k(\phi) = V'_k(\phi) \) but it was found to be less stable and more time consuming.

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