An approximate model for slow free-surface flows of a fluid through a highly porous medium accounting for the effect of a capillary fringe

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Abstract. The paper proposes a model for unconfined creeping flows of a thin fluid layer through a highly porous medium accounting for the effect of a capillary fringe at the free surface. To derive the model, we used the thin-layer approximation for the Brinkman equation with the Navier slip conditions at both the impermeable underlying surface and the free surface. The equation for the fluid layer thickness appears to be a special type of the non-linear heat conduction equation. We present similarity solutions that describe fluid propagation over a horizontal substrate.

1. Introduction
In this paper we propose an equation for unconfined slow flows through a highly porous medium. In deriving the equation, the Navier slip condition heuristically accounts for the effect of a capillary fringe at the free surface. The model proposed may be considered to be a generalization of models known in lubrication theory (e.g., equations for thin viscous films [1]) and in groundwater hydrology (the Boussinesq equation for unconfined flows through low-porosity media [2]). The model may be of interest to those working on certain industrial problems (polymer impregnation, composite material processing, etc.).

2. Equation for thin layer flows through a highly porous medium
We consider a slow (low Reynolds number) flow of an incompressible non-massless fluid through a highly porous medium over a horizontal impermeable wall. The governing equations include the continuity equation

\[
\text{div} \vec{u} = 0,
\]

where \( \vec{u} \) is the Darcy velocity, and the momentum balance equation (Brinkman equation)

\[
-\text{grad} p - \frac{\mu}{k} \vec{u} + \mu_1 \nabla \vec{u} + \rho \vec{g} = 0,
\]

where \( p \) is the pressure, \( \rho, \mu \) are the density and the viscosity of the fluid, \( k \) is the permeability of the porous medium, \( \mu_1 \) is a coefficient with the dimensions of viscosity, \( \vec{g} \) is the gravitational acceleration.
For the sake of simplicity we will assume the flow to be plane. The free surface of the fluid layer is described by the function $y = h(x,t)$, where $x$, $y$ are the horizontal and vertical coordinates (the $x$ axis is directed along the substrate and the $y$ axis, upwards), $t$ is the time.

The layer thickness $h$ is supposed to be small compared with the characteristic horizontal length $L$ of the flow, $h(x,t) \ll L$, and furthermore the free surface is assumed to be a gently sloping one, $|\partial h/\partial x| \ll 1$.

At the free surface of the fluid within the porous medium an unsaturated zone (capillary fringe) may appear that offers additional resistance to the flow. This retarding action of the capillary fringe is qualitatively analogous to the well-known damping effect of foams [3] or floats at the free fluid surface in a tank.

As a first approximation, for a gently sloping free surface we suppose that the pressure variation across the capillary fringe is a fixed constant, the capillary fringe effect is accounted for by the Navier slip condition, and the vertical projection of the Darcy velocity at the surface is negligible

$$
\left. \left( u_x + a \frac{\partial u_x}{\partial y} \right) \right|_{y=h(x,t)} = 0, \quad u_y|_{y=h(x,t)} = 0, \quad p|_{y=h(x,t)} = p_0 = \text{const},
$$

where $a$ is a parameter with the dimensions of length (effective slip length), $a \to \infty$ meaning that the capillary fringe is absent. Note that these boundary conditions are motivated by high matrix porosity and are, broadly speaking, heuristic. We must also emphasize that in this approximation the mass flux conservation condition does not hold exactly (the same is true of the hydraulic theory of unconfined flows through porous media or shallow water theory).

At the impermeable underlying surface we also pose the Navier slip condition for the tangential component $u_x$ of the Darcy velocity and the zero-flux condition

$$
\left. \left( u_x - b \frac{\partial u_x}{\partial y} \right) \right|_{y=0} = 0, \quad u_y|_{y=0} = 0,
$$

where $b$ is a constant slip length. It should be stressed that the use of the Navier slip condition is not due to the presence of molecular-scale slip or superhydrophobic nature of the underlying surface [4].

In the thin-layer approximation [5], the local Darcy velocity distribution (at a fixed vertical cross-section at a fixed time) $U(y) = u_x(x,y,t)$ is determined by the solution to the boundary value problem

$$
-\rho g \frac{\partial h}{\partial x} - \frac{\mu}{k} \cdot U(y) + \mu_1 U''(y) = 0, \quad bU'(0) = U(0), \quad aU'(h) = -U(h),
$$

and then the differential equation for the fluid layer thickness is derived from the integral mass conservation equation

$$
m \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \quad Q = \int_0^h U(y) \, dy,
$$

where $m$ is the porosity.

On introducing the dimensionless variables

$$
\varepsilon^2 = \frac{\mu_1}{\mu} \cdot \frac{k}{l^2}, \quad a_1 = \frac{a}{l}, \quad b_1 = \frac{b}{l}, \quad h_1 = \frac{h}{l}, \quad x_1 = \frac{x}{L}, \quad t_1 = \frac{\rho g l k}{m \mu L^2} \cdot t,
$$

where $l, l \sim h$, is a characteristic vertical length scale of the problem (characteristic layer width), the equation for the fluid layer thickness $h$ takes the form of a non-linear heat conduction
equation (for the sake of simplicity we omit the subscript "1" denoting dimensionless quantities)
\[
\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left( F(h) \frac{\partial h}{\partial x} \right), \quad F(h) = h + \frac{4\varepsilon^3 \exp(h/\varepsilon) - \varepsilon^2((b + a + 2\varepsilon) \exp(2h/\varepsilon) + 2\varepsilon - a - b)}{(\varepsilon^2 + \varepsilon(a + b) + ab) \exp(2h/\varepsilon) - \varepsilon^2 + \varepsilon(a + b) - ab}.
\]

This equation involves three dimensionless parameters \( \varepsilon, a, \) and \( b \) that define respectively the ratio of the Darcy and Brinkman terms in the momentum equation, the effect of capillary fringe slip, and the effect of underlying surface slip.

For a small layer thickness the series expansion of the function \( F(h) \) is
\[
F(h) = \frac{ab}{(b + a)\varepsilon^2} h^2 + \frac{\varepsilon^2(a^2 + b^2 - ab) - 3a^2b^2}{3\varepsilon^4(a + b)^2} h^3 + \ldots, \quad h \to 0,
\]
and for a clear fluid \((m = 1, \mu_1 = \mu)\) with a no-slip condition at the substrate \((b = 0)\) in the absence of the capillary fringe \((a \to \infty)\) the equation for \( h \) transforms into the well-known thin-film equation [5] that has the following dimensional form
\[
\frac{\partial h}{\partial t} = \frac{\rho g}{3\mu} \frac{\partial}{\partial x} \left( h^3 \frac{\partial h}{\partial x} \right).
\]

On the other hand, for a large \( h \) the function \( F(h) \) can be expanded as
\[
F(h) = h - \frac{2\varepsilon^3 + \varepsilon^2(a + b)}{\varepsilon^2 + \varepsilon(a + b) + ab} + \ldots, \quad h \to \infty,
\]
so asymptotically we get the Boussinesq equation for unconfined flows through a low-porosity medium [2] with the dimensional form
\[
\frac{\partial h}{\partial t} = \frac{k \rho g}{m \mu} \frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right).
\]

The conditions at the discontinuities are derived in a standard way from the integral form of the mass conservation law, and for constant porosity we have
\[
[h]D + \left[ F(h) \cdot \frac{\partial h}{\partial x} \right] = 0, \quad D = \frac{dx_s(t)}{dt},
\]
where \( x_s(t) \) is the discontinuity position, and \( D \) is the discontinuity velocity, the square brackets denoting a jump across the discontinuity surface.

It is significant that this derivation is valid on condition that the form of mass fluxes is not changed as we approach the discontinuity. Physically, for example, in the vicinity of the leading edge of a fluid drop propagating over a dry surface this condition ceases to be true, so the thin-layer approximation fails to be valid.

3. Similarity solutions
As an example of the application of our model, we will describe several similarity solutions to the problem of fluid propagating from a variable-discharge source.

We seek the solutions that depend on the similarity variable \( \xi = x/\sqrt{t} \), and for the case of plane flows we obtain an ordinary differential equation
\[
\xi \frac{dh(\xi)}{d\xi} + 2 \frac{d}{d\xi} \left( F(h(\xi)) \frac{dh(\xi)}{d\xi} \right) = 0.
\]
Figure 1. Plots of fluid layer thickness $h(\xi)$ for plane similarity solutions for $\varepsilon = 0.1$, $a = 0.2$, $b = 0.2$. Line 1 corresponds to propagation over a zero background, 2, over a non-zero one.

Here, different solutions are possible depending upon the fluid layer width at the infinity. For a non-zero $h(+\infty)$ we have fluid propagation over an initially wetted porous matrix. For a zero $h(+\infty)$ the similarity solution describes propagation over a dry substrate (so we have the solutions with compact support), the mass conservation condition at the leading edge holding automatically. As an illustration, figure 1 presents two numerical examples corresponding to either case.

Analogous axisymmetric similarity solutions of the form $h = h(\xi)$ with $\xi = r/\sqrt{t}$, where $r$ is the radius of the cylindrical coordinate system, are of obvious practical significance. In this case we have an ordinary differential equation

$$\xi^2 \frac{dh(\xi)}{d\xi} + 2 \frac{d}{d\xi} \left( \xi \cdot F(h(\xi)) \frac{dh(\xi)}{d\xi} \right) = 0,$$

and two similar cases — propagation over a wet and a dry surface — are possible. In contrast to the plane case, in the axisymmetric case the width of the fluid layer near the origin tends to infinity, $h \sim (\ln \xi)^{1/2}$ as $\xi \to 0+$ (the same situation arises in similar axisymmetric problems, for example for unconfined flows near a wellbore in a low-porosity medium [2]). Two examples of $h(\xi)$ curves for both cases are presented in figure 2.

Figure 2. Axisymmetric similarity solutions $h(\xi)$ for $\varepsilon = 0.1$, $a = 0.2$, $b = 0.2$. Line 1 corresponds to propagation over a zero background, 2, over a non-zero one.

4. Conclusion
In this paper we proposed a heuristic model that accounts for the effect of a capillary fringe on free-surface flows through a highly porous medium. Within the framework of the model, simple similarity solutions are considered that may be used to determine model parameters (e.g., slip lengths for the top and bottom surfaces). The model may be evidently generalized to describe
more complex problems, for example the flows over an inclined substrate [5]. Furthermore, considering more complicated problems is of practical interest. In particular, of some interest is the problem of finite fluid mass propagation [6] that allows for determining the parameters of the model through observing the asymptotic behaviour of the leading edge [7].

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