Microarticle

Discrete families of Saffman–Taylor fingers with exotic shapes

Bennett P.J. Gardiner, Scott W. McCue * , Timothy J. Moroney

Mathematical Sciences, Queensland University of Technology, QLD 4000, Australia

Abstract

The mathematical problem of determining the shape of a steadily propagating Saffman–Taylor finger in a rectangular Hele-Shaw cell is known to have a countably infinite number of solutions for each fixed surface tension value. For sufficiently large surface tension values, we find that fingers on higher solution branches are non-convex. The tips of the fingers have increasingly exotic shapes as the branch number increases.

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Introduction

When an inviscid fluid displaces a viscous fluid in a rectangular Hele-Shaw channel [1], the interface is unstable and develops into a single long finger shape (a Saffman–Taylor finger) in the long-time limit [2,3]. Previous analytical [4,5] and numerical [8] studies have shown that, for a fixed dimensionless surface tension value, there is a countably infinite number of finger shapes, each with a different dimensionless width, c. Here we show that for sufficiently large surface tension values, finger solutions on higher order branches have increasingly exotic non-convex shapes at the tips.

By applying conformal mapping and boundary integral methods, the governing equations for computing the shape of a Saffman–Taylor finger are [3–5,8]

\[ q = \gamma q \frac{d}{d\xi} \left\{ q^2 \frac{d\theta}{d\xi} \right\} + \cos \theta, \quad 0 < \xi < 1, \]  
\[ \log q = -\frac{\xi}{\pi} \int_0^1 \frac{\theta(\zeta)}{\zeta(\zeta - \xi)} d\zeta', \quad 0 < \xi < 1, \]  
\[ \theta(0) = 0, \quad q(0) = 1, \quad \theta(1) = -\frac{\pi}{2}, \quad q(1) = 0, \]  

where \( \theta(\zeta) \) is the tangent angle of the finger, \( q(\zeta) \) is the tangent velocity, and \( \xi \) is the real part of the conformally-mapped variable. Given a solution for \( \theta \) and \( q \), the finger shape and width can be computed by inverting the mapping. Solutions are found by placing an uneven grid over \( \xi \in [0, 1] \) and solving the discretised system of nonlinear algebraic equations numerically [3,8,6,7]. In order to employ a large number of grid points, we utilise a Jacobian free Newton-Krylov method as described in Ref. [7].

Typically, with an initial guess given by the exact solution for \( \gamma = 0 \) [2], a simple scheme converges to a solution on the primary branch \((m = 0)\) [3]. Then, using the computed solution as an initial guess, one can compute solutions on the primary branch for increasing values of \( \gamma \). The relationship between the finger width, \( c \) and \( \gamma \) for this branch is the lowermost plot in Fig. 1. Note the well-known result that \( c \to 1/2^* \) as \( \gamma \to 0 \) [3–5,8].

A representative solution on the primary branch \((m = 0)\) is given by the black curve in Fig. 2(a), while the corresponding finger shape is shown in Fig. 2(b). Note that \( \theta(\zeta) \) is monotonically decreasing in \( \zeta \), and therefore the finger is convex. This property is common to all solutions on the primary branch.

New results

In order to compute solutions on higher branches \((m \geq 1)\), Vanden-Broeck [8] solves a modified problem for which the finger width, \( \lambda \) is given as an input, and the angle at the nose of the finger, \( \theta(1) \) is an output. Solutions for \( \theta \) and \( \lambda \) that have \( \theta(1) = -\pi/2 \) are solutions to the original problem \((1)–(3)\). We were able to verify Vanden-Broeck’s results with a similar but independent method (see Supplementary Information). Once a solution on a higher branch is obtained, then the original numerical scheme can be used to obtain further solutions on that branch.

Fig. 1 shows the dependence of the finger width on the surface tension parameter \( \gamma \) for the first nine branches \( m = 0, 1, \ldots, 8 \). We have computed solutions on many more branches \((up to m = 70)\), but the results are not included in this figure. For all branches from about \( m = 4 \) and above, the results agree extremely well with the analytical prediction \( \lambda^{1/2}(1 + \gamma)^{1/2} - 1 = \gamma(m + 4/7)^2 \), where...
the value of surface tension such that, for \( \gamma > \gamma_{m,n} \), the solution \( \theta(\xi) \) on the \( m \)th branch has at least \( n \) critical points. We find there is a value \( \gamma_{1,1} \approx 1.4 \) and that for \( \gamma > \gamma_{1,1} \) all solutions on the branch \( m = 1 \) have a single critical point. Thus for \( \gamma < \gamma_{1,1} \) the solution on this branch are convex (‘round-nosed’), while for \( \gamma > \gamma_{1,1} \) the fingers have negative curvature at \( \xi = 1 \) and could be referred to as ‘double-tipped’. This behaviour is reminiscent of the non-convex steadily propagating bubbles computed by Tanveer [9] and the recently observed fingers propagating down a Hele-Shaw cell with spatially varying depth profile [10,11].

Fig. 1. The finger width \( \lambda \) against surface tension \( \gamma \) for the first nine branches \( m = 0, 1, \ldots, 8 \). The markers indicate a change in the number of critical points in \( \theta \). For example, the blue section of the eighth branch to the right of 0 (the tail) to 0 (the nose) shows the critical points vanishing in pairs, sequentially from the tail towards the nose. A bifurcation diagram representing this behaviour is presented in the Supplementary Information.

More generally, we find there is a value \( \gamma_{m,1} \), for odd numbered branches \( \gamma_{m,2} \) for even branches) such that for \( \gamma > \gamma_{m,1} (\gamma > \gamma_{m,2} \) for even branches), the solutions \( \theta \) are non-monotonic in \( \xi \). Furthermore, there appears to be a value \( \gamma_{m,n} \) such that for all \( \gamma > \gamma_{m,n} \) solutions for \( \theta \) have \( m \) critical points. This behaviour is demonstrated in Fig. 2(a), which shows solutions for \( \gamma = 2.03 \), a representative value for which all upper branches have non-monotonic solutions. The corresponding fingers are shown in Fig. 2(b).

The critical points shown in Fig. 2(a) correspond to a sign change in the curvature of the fingers, shown in Fig. 2(b). As \( \gamma \) decreases from 2.03, the number of critical points on each branch diminishes as shown in Fig. 1. We observe that the critical points vanish in pairs, sequentially from the tail towards the nose. A bifurcation diagram representing this behaviour is presented in the Supplementary Information.

Summary

We have computed a variety of non-convex Saffman–Taylor fingers. As these occur on higher order branches, they are known to be unstable [12]. However, these solutions may correspond to intermediate-time solutions to the time-dependent problem with carefully chosen initial conditions (see Supplementary Information) or be related to experimentally observed bubbles propagating in Hele-Shaw cells of varying depth [10,11].

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.rinp.2015.04.002.

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