Automatic Tabulation in Constraint Models

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Abstract

The performance of a constraint model can often be improved by converting a subproblem into a single table constraint. In this paper we study heuristics for identifying promising candidate subproblems, where converting the candidate into a table constraint is likely to improve solver performance. We propose a small set of heuristics to identify common cases, such as expressions that will propagate weakly. The process of discovering promising subproblems and tabulating them is entirely automated in the constraint modelling tool Savile Row. Caches are implemented to avoid tabulating equivalent subproblems many times. We give a simple algorithm to generate table constraints directly from a constraint expression in Savile Row. We demonstrate good performance on the benchmark problems used in earlier work on tabulation, and also for several new problem classes. In some cases, the entirely automated process leads to orders of magnitude improvements in solver performance.

Note to reviewers and editor: This paper builds on (and entirely includes) an earlier conference publication [1]. We have extended and refined the method and also performed a much more extensive experimental evaluation.

Keywords: Constraint modelling, Constraint programming, Combinatorial optimization, Constraint satisfaction problem

1. Introduction

Constraint programming provides an efficient means of solving complex combinatorial problems across a wide variety of disciplines, such as scheduling, planning, routing, and configuration [2]. In order to solve a problem using the constraint programming paradigm, it must first be modelled in a format suitable for input to a constraint solver. This involves determining the set of decision variables that represent the choices that must be made to solve the problem, and formulating a set of constraints over the variables so as to allow only valid combinations of decisions. For example, in a scheduling scenario we might employ a decision variable per task to represent the start time of that task, with
constraints to disallow a number of simultaneous tasks that would exceed the resources available.

Once a model has been chosen, a constraint solver automatically searches for a solution: a complete assignment of values to the decision variables that satisfies all of the constraints. Search is interleaved with inference known as constraint propagation, where deductions are made based on the constraints and the current set of assignments made via search. These deductions serve to narrow down the choices for the variables as yet unassigned by search, and therefore reduce the search required. Generally, there are many ways in which a given problem may be modelled, and the model chosen has a significant effect on the performance of the constraint solver in searching for solutions. Therefore, automated methods for improving constraint models are valuable [3, 4].

In order to improve the performance of a constraint model, a common step is to reformulate the expression of a subset of the problem constraints, either to strengthen the inferences made during search by the constraint solver by increasing constraint propagation, or to maintain the level of propagation while reducing the cost of propagating the constraints. One such method is tabulation, the aggregation of a set of constraint expressions into a single table constraint [5, 6, 7]. Such a table constraint explicitly lists the allowed tuples of values for the decision variables involved. This allows us to exploit efficient table constraint propagators that enforce generalised arc consistency (GAC) [8], typically a stronger level of inference than is achieved for a logically equivalent collection of separate constraints. Successful examples of this approach where the reformulation has been performed by hand include Black Hole patience [9] and Steel Mill Slab Design [10].

Recently, Dekker et al. [11] presented a method for the automation of tabulation. In their approach a predicate (a Boolean function) expressed in the MiniZinc language [12] may be annotated, requesting that it be converted into a table constraint. IBM ILOG CPLEX Optimization Studio [13] and ECLiPSe [14] have similar facilities to generate table constraints, while other approaches target alternatives such as Multi-valued Decision Diagrams (MDDs) and regular constraints [15, 16]. In all of these approaches, the crucial first step of identifying promising parts of a given model for tabulation is left to the human modeller.

In this work we present an entirely automatic tabulation method situated in the constraint modelling tool Savile Row [17, 18, 4]. A set of heuristics is employed to identify in an Essence Prime [19] model some candidate sets of constraints for tabulation, which are then tabulated automatically. In order to demonstrate the effectiveness of our approach, we first examine the same four case studies used by Dekker et al. [11] to demonstrate the utility of tabulation from explicit model annotations. We show that our automated approach can identify the same opportunities to improve the model by tabulation. We also study nine additional problem classes that show that our tabulation heuristics are effective on a wider range of problems.
1.1. Motivating Example

Consider the classic puzzle of the Knight’s Tour, studied by Euler (1759). More recently, Schwenk determined the set of board sizes (including rectangular boards) that have a knight’s tour [20]. Given a natural number $n$ and a starting square on an $n \times n$ chess board, the task is to find a sequence of moves for the knight visiting each of the remaining squares of the board exactly once. The moves of a knight are illustrated in Figure 1. We use the Hamiltonian path version of Knight’s Tour: the last square visited is not required to be a knight’s move from the starting square.

A constraint model of the Knight’s Tour is presented in Figure 2. We will refer to this as the Sequence model to distinguish it from an alternative presented in Section 5.2.3. The model in the figure is a natural formulation of the problem, using the available arithmetic and logical operators in the language to capture the legal moves of the knight. However, it is also naïve in that its performance in terms of constraint propagation is weak, as we will discuss below.

The location of the knight is encoded as a single integer $(nx + y)$ where $(x, y)$ are the coordinates of the knight on the board (from 0). The sequence model simply has a one-dimensional matrix of variables $(\text{tour})$, with \text{tour}[i] representing the location of the knight at timestep $i$. The constraints enforce that initially the knight is at the given location, the moves are all different, and each adjacent pair \text{tour}[i] and \text{tour}[i+1] corresponds to a knight’s move. As presented in the figure, the knight’s move constraint contains two location variables and is naturally expressed using integer division and modulo to obtain the $x$ and $y$ coordinates. The constraint states that the absolute difference in the $y$ coordinates is 1, and for $x$ coordinates is 2, or vice versa.

The knight’s move constraint is a relatively complex expression: a disjunction of conjunctions of reified arithmetic expressions, in which the absolute value of the difference between scaled adjacent \text{tour} variables is compared with a constant. Typically this constraint will propagate poorly for two different reasons. Firstly, some solvers implement modulo and absolute value poorly. Secondly, the constraint as a whole will propagate poorly, as most solvers will wait until the value of one side of the disjunct is known before propagating the other.
given n: int
given startCol, startRow: int(0..n-1)
find tour: matrix indexed by [int(0..n*n-1)] of int(0..n*n-1)
such that
allDiff(tour),
tour[0] = startCol + (startRow)*n,
forall i: int(0..n*n-2) .
  ((|tour[i]%n - tour[i+1]%n| = 1) \ /
   (|tour[i]/n - tour[i+1]/n| = 2)) \ /
  ((|tour[i]%n - tour[i+1]%n| = 2) \ /
   (|tour[i]/n - tour[i+1]/n| = 1))

Figure 2: The Sequence model of the Knights Tour in the constraint modelling language Essence Prime [19]. The parameters to the model (board size \(n\) and the starting square for the knight) are introduced with the keyword given. The decision variables (the single-dimensional matrix \(tour\)) are introduced with the keyword find.

There are tradeoffs involved in choosing subproblems to tabulate. Tabulating just the expression \(|tour[i]%n - tour[i+1]%n| = 1\) may well reduce search, but tabulating the entirety of the expression between two adjacent \(tour\) variables in a single step will produce a table of a similar size (as only two variables are involved), while reducing search much more. We could consider tabulating multiple adjacent moves as a single table — while this may reduce search further, the resulting tables would grow rapidly as a function of the number of variables involved. As we will see, the tabulation opportunities present in this naïve model can be identified and exploited automatically by the methods presented in this paper. Solver performance is improved by tens or hundreds of times (depending on the solver type) completely automatically, without the need to apply constraint modelling expertise.

1.2. Contributions

The primary contribution of this work is the automation of a hitherto difficult manual task: the recognition and exploitation of opportunities to tabulate parts of a constraint model in order to increase constraint propagation and therefore reduce search. In support of this primary contribution, we contribute the following:

- A set of heuristics to identify common tabulation opportunities, such as expressions that will propagate weakly.
- A caching system to avoid tabulating equivalent subproblems multiple times.
- A system of progress checks and work limits for the situation where a heuristic identifies a constraint that is too large to be tabulated.
- Empirical studies of both the frequency with which our heuristics identify tabulation opportunities, and the efficacy of the tabulated models.
1.3. Organisation

The rest of the paper is organised as follows. In Section 2, we review necessary background for the paper. In particular, we provide details of our modelling language, tools, and the abstract syntax tree data structure which our algorithms manipulate. We then discuss our approach to finding promising parts of the problem where tabulation could be applied, in Section 3. Here we cover each of our four heuristics and how they are adapted to different situations that occur in the abstract syntax tree. In Section 4 we describe the tabulation algorithm, its progress checks and work limits, and the caching system.

We then evaluate our system: in Section 5 in terms of how well the heuristics framework identifies promising subproblems; in Section 6 in terms of how overall performance is improved compared to the models without tabulation; in Section 7 with respect to how well our system performs on a large set of models where tabulation might not a priori be expected to improve performance; and finally in Section 8 we consider how tabulation scales as we allow the arity of generated tables to increase. We discuss related work in Section 9 and conclude in Section 10.

2. Background

This section gives the necessary background on constraint propagation, the constraint modelling language we employ in this paper, and the Savile Row tool in which we situate our approach.

2.1. Consistency and Constraint Propagation

Constraint propagation usually operates by establishing a consistency property on the constraints and variables. Generalised arc consistency is a common, powerful consistency property used in this paper, which we define in what follows. The scope of a constraint $c$, named $\text{scope}(c)$, is the set of variables that $c$ constrains. A literal is a decision variable-value pair (written $x \mapsto v$). A literal $x \mapsto v$ is valid iff $v$ is in the domain of decision variable $x$. A support of constraint $c$ is a set of literals containing exactly one literal for each variable in $\text{scope}(c)$, such that $c$ is satisfied by the assignment represented by these literals. A constraint $c$ is Generalised Arc Consistent (GAC) if and only if there exists a support for every valid literal of every variable in $\text{scope}(c)$. GAC is established by identifying all literals $x \mapsto v$ for which no support exists and removing $v$ from the domain of $x$.

There exist efficient constraint propagation algorithms to establish GAC on a table constraint [21, 22, 6, 7] and table constraints are widely available in constraint solvers.

2.2. Essence Prime

The importance of modelling is widely recognised in constraint programming (CP) as well as the related fields of propositional satisfiability (SAT) and integer linear programming (ILP). In CP several constraint modelling languages have
been developed, including OPL [23], MiniZinc [24], and Essence Prime [19] in order to aid in the statement of constraint models and abstract away from the details of particular constraint solvers. Herein, we focus on Essence Prime, which is comparable with OPL and MiniZinc.

Essence Prime provides the facility to model parameterised classes of problems, where an individual problem instance is specified by giving values for the class parameters, for example the integers \( n \), \( \text{startCol} \) and \( \text{startRow} \) in Figure 2. The language supports Boolean and integer finite-domain decision variables, both singly and collected into multidimensional matrices, such as the one-dimensional matrix of integers \( \text{tour} \) in Figure 2. Constraints over these variables are expressed via arithmetic and logical expressions, as can also be seen in the figure. Quantification and comprehension enable the concise statement of such expressions. Essence Prime supports a number of global constraints [2] that capture common patterns in constraint modelling, including the all-different constraint present in the figure [25], Global Cardinality Constraint (GCC) [26], and the table constraint that is the focus of this paper.

2.3. Savile Row, Tailoring, and the Abstract Syntax Tree

We investigate tabulation as a component of the constraint model reformulation tool Savile Row [4]. Savile Row is essentially a multi-pass term rewriting system. It represents a model internally using several abstract syntax trees (ASTs) representing the constraints, the objective function, the domain of each decision variable and parameter (or matrix thereof), and other statements in the model. The parser reads a model in the Essence Prime language, along with a parameter file giving a value to each of the problem class parameters. There are several backends targeting different solvers, including mature backends for CP and SAT solvers and a prototype ILP backend. The system has a number of different passes, some of which are always performed, others are required for specific backends, and others are optional reformulations intended to improve the performance of the solver. Tabulation is one optional reformulation. We refer to the entire process of transforming a model into input for a solver as tailoring the model. The early steps of tailoring prior to tabulation are as follows:

- Problem class parameters and other constants (defined by letting statements) are substituted into the model;
- All quantifiers and matrix comprehensions are unrolled;
- Matrices of decision variables are replaced with individual decision variables;
- Multi-dimensional matrix indexing is replaced with single-dimensional indexing of the flattened matrix if required;
- Global constraints are identified by simple aggregation steps (described elsewhere [4]), e.g. collecting a clique of not-equal constraints into an all-different.
In addition, simplifiers are applied after each pass to perform partial evaluation and to maintain a normal form. In particular, negation is pushed towards the leaves of the AST (similar to negation normal form), double negation is removed, and some operators are rewritten when negated (for example, negated = is rewritten to ≠). Variable domains are filtered using an external constraint solver prior to tabulation, and any assigned variables are deleted. Details of simplifiers, the normal form, and domain filtering are given elsewhere [4].

As an example, consider the Knight’s Tour problem and its Sequence model (shown in Figure 2), with an 8 × 8 board. Part of the AST for this model is shown in Figure 3. The tour matrix has been replaced with individual variables tour0, tour1, etc. The forAll quantifier has been unrolled to create $n^2 - 1$ knight’s move constraints, each containing two adjacent tour variables. A fragment of one of the knight’s move constraints is shown in the example AST. Also, the variable tour0 has been deleted because it was assigned, and its assigned value has been removed from the other variable domains.

For tabulation we consider Boolean and integer expressions, for example those represented by nodes A, B, C, and D in Figure 3 (where A, B, and C are Boolean and D is integer). We distinguish between top-level constraints and other Boolean expressions. Top-level constraints are Boolean expressions directly beneath the top And node (e.g. nodes A and B in Figure 3). We refer to other Boolean expressions as nested (e.g. node C). Integer expressions are always nested, they cannot be directly contained in the top-level conjunction.
of the AST a table constraint is a Boolean expression with two arguments: a one-dimensional matrix of decision variables, and a two-dimensional matrix representing a list of satisfying tuples. Following the optional tabulation pass, some further steps are required to tailor the model for the target solver. We use the default settings of Savile Row which can be summarised as follows (with further details elsewhere [4]):

- Decomposition of constraint types that the target solver does not support;

- Common subexpression elimination (Active CSE [4]) which factors out identical (or semantically equivalent) expressions replacing them with a new decision variable (for example, in the Knight’s Tour Sequence model in Figure 2 there are identical absolute value, division, and modulo expressions that would all be factored out by Active CSE);

- General flattening to extract nested expressions where the nesting is not allowed by the target solver (for example, in Figure 2 the absolute value operator contains a sum which (if not already extracted by CSE) would be extracted and replaced with a new decision variable).

In addition the SAT backend has a final encoding step where each integer variable is encoded using order, direct, or both as required for the constraints containing the variable. The remaining constraints (such as linear, element, and table constraints) are encoded to CNF. Others such as all-different and GCC are decomposed before reaching the SAT backend. Further details of the SAT encoding are given in the Savile Row manual [19].

3. Identifying Promising Subproblems for Tabulation

We have designed four heuristics to identify cases where expert modellers might experiment with tabulation to improve the performance of a model. The heuristics operate on the AST, identifying AST nodes that are candidates for tabulation. The heuristics are applied somewhat differently for top-level constraints, nested Boolean expressions, and integer expressions, and we describe each of these cases in sections 3.2 to 3.4 below. First we describe the types of changes tabulation can make to the AST.

Note that the heuristics to identify promising subproblems do not take account of the size of the resulting table, nor the work required to generate it. The heuristics can and frequently do identify candidates that would be impractical to tabulate. Progress checks (described in Section 4) allow tabulation to be abandoned early and play an important role in avoiding overhead.

3.1. AST Modifications

The AST that tabulation acts on is described in Section 2.3 with an example in Figure 3. Tabulation modifies the AST in one of two ways depending on the type (integer or Boolean) of the node to be replaced. When the AST node is of
type Boolean, it is directly replaced with a table constraint. For example, if node B of Figure 3 were to be replaced, the resulting tree would be Figure 4 (upper) in which B has been replaced with node E. Node B is a top-level constraint with variables \( \text{tour1} \) and \( \text{tour2} \) in scope, and its replacement is a top-level constraint with the same scope. If the node to be replaced is a nested Boolean expression, such as node C in Figure 3, then the replacement table constraint will also be nested. Some solver types do not allow nested table constraints and in this case it would be extracted and replaced with a new Boolean variable, however this occurs in another pass after tabulation is complete.

When the node to be replaced is of type integer, the node is replaced with a new variable and a top-level constraint is created to link the new variable to the variables in scope of the tabulated expression. For example, node D of Figure 3 is replaced with new auxiliary variable \( \text{aux1} \) in Figure 4 (lower), and a new table constraint G is added to the top-level conjunction with \( \text{aux1} \) and \( \text{tour1} \) in scope. The new table constraint is generated from an equality between the new variable and the expression to be tabulated, which in this example is \( \text{aux1} = \text{tour1} \mod 8 \). The domain of the auxiliary variable is generated from the expression using the extended domain filtering method [4] (the current default method for all variables introduced by Savile Row).

Finally, both examples in Figure 4 have an identifier (\( \text{aux2} \)) in place of the table of satisfying tuples; this is because matrices of constants are cached to avoid duplication.

3.2. Identifying Promising Top-Level Constraints

First we define the four heuristics as they apply to top-level constraints (i.e. constraints in the top-level conjunction), then in Section 3.3 and Section 3.4 we extend the heuristics to apply to nested Boolean and integer expressions. The heuristics are as follows, in the order that they are applied:

**Identical Scopes** identifies sets of two or more constraints whose scopes contain the same set of decision variables.

**Duplicate Variables** identifies a constraint containing at most 10 distinct variables, with at least one variable occurring more than once in the constraint.

**Large AST** identifies a constraint where the number of nodes in the AST is greater than 5 times the number of distinct decision variables in scope.

**Weak Propagation** identifies a constraint \( c_1 \) that contains at most 10 distinct variables, and that is likely to propagate weakly (i.e. less than GAC), such that there is another constraint \( c_2 \) that propagates strongly, with at least one variable in the scope of both \( c_1 \) and \( c_2 \). The method of estimating whether a constraint will propagate strongly is described in Section 3.5.

Each of the four heuristics is based on a simple rationale regarding either propagation strength or propagation speed of the constraint(s). The constants
Figure 4: Two examples of tabulation applied to the AST shown in Figure 3. Upper: Node B of Figure 3 is replaced with a table constraint labelled E. Both top-level constraints and nested Boolean expressions are directly replaced when they are tabulated. Lower: Node D of Figure 3 (an integer expression) is replaced with a new auxiliary variable (F), and a table constraint G is attached to the top-level And node. The scope of the table constraint is the scope of the tabulated expression plus the new variable (aux1 in this case).
used in these heuristics were chosen by hand based on preliminary experiments (e.g. we found that constraints with more than 10 variables are very unlikely to be successfully tabulated).

First we consider the Identical Scopes heuristic. It is well known that multiple constraints on the same scope may not propagate strongly together, even if each constraint individually does propagate strongly. The Identical Scopes heuristic is intended to collect such sets of constraints into a single table constraint that may propagate more strongly and also may be faster to propagate. An extreme example would be two contradictory constraints on the same scope (e.g. \( x \neq y + 1 \) and \( x = y + 1 \)) which would be replaced by a trivially false table constraint.

The Duplicate Variables heuristic identifies constraints that are likely to propagate weakly even when the target solver has a strong propagator for the constraint type. In most cases a GAC propagator will enforce GAC only when there are no duplicate variables. In some cases it is intractable to enforce GAC with duplicate variables. GAC on the Global Cardinality Constraint (GCC) is known to be NP-hard with duplicate variables \cite{277}, therefore Régis’s polynomial-time GAC propagator \cite{28} achieves GAC only when there are no duplicate variables. The knight’s move constraint of the Knight’s Tour Sequence model \cite{1.1} triggers the Duplicate Variables heuristic: each decision variable in scope is mentioned four times. In the evaluation we show that tabulating this constraint improves solver performance very substantially.

The Large AST heuristic identifies constraints that are not compactly represented in the AST. The knight’s move constraint of the Knight’s Tour Sequence model also triggers the Large AST heuristic. It has two decision variables in scope, and the AST representation (part of which is illustrated in Figure 3) has 43 nodes. In this case, tabulating the constraint avoids the need to create auxiliary variables during CSE and flattening \cite{2.3}, and also strengthens propagation of the constraint. In general the rationale behind the heuristic is that a table propagator may be more efficient while achieving the same or stronger propagation.

Finally we consider the Weak Propagation heuristic. It is intended to catch cases where the weak propagation of one constraint is hindering strong propagation of another. The knight’s move constraint of the Knight’s Tour Sequence model also triggers the Weak Propagation heuristic: the representation of the knight’s move constraint for a CP solver is not expected to enforce GAC (because it contains arithmetic such as sum and modulo), and it overlaps with an allDiff constraint that is expected to enforce GAC. To implement the Weak Propagation heuristic we need to estimate which constraint expressions are expected to propagate strongly. \cite{3.3} describes how we do this.

Each of the four heuristics proves to be valuable: each one is triggered on at least one of the problems that we study in the evaluation below. The heuristics are applied in the order that they are listed in this section. In the evaluation we report only the first heuristic that is triggered by a subproblem.
3.3. Identifying Promising Nested Boolean Expressions

In some cases it can be useful to tabulate an expression within a constraint without tabulating the entire top-level constraint. One example is where the top-level constraint is out of reach of tabulation (i.e. tabulation would produce an impractically large table or take too long). In this section we adapt the four heuristics to identify candidates for tabulation from the set of nested Boolean expressions. The goals remain the same: to strengthen propagation, or to replace an unwieldy expression to improve the efficiency of propagation. The heuristics are adapted as follows (in the order they are applied).

Identical Scopes (Nested) identifies a Boolean expression $c_1$ that has the same scope as a top-level constraint $c_2$, where $c_2$ does not contain $c_1$.

Duplicate Variables (Nested), and Large AST (Nested) are unchanged apart from applying to nested Boolean expressions rather than top-level constraints.

Weak Propagation (Nested) identifies a Boolean expression $c_1$ that contains at most 10 distinct variables, and that is likely to propagate weakly (Section 3.5), and there exists a top-level constraint $c_2$ that propagates strongly, with at least one variable in the scope of both $c_1$ and $c_2$.

All four heuristics except Identical Scopes (Nested) identify a single Boolean expression to be tabulated. Identical Scopes (Nested) is somewhat different: it identifies a single Boolean expression (named $c_1$) to be replaced, but the table constraint is generated from the conjunction of $c_1$ with all top-level constraints that have the same scope as $c_1$. For example, suppose that $x \neq y$ is a nested Boolean expression $c_1$, and $x \leq y$ is a top-level constraint (the only top-level constraint with scope $\{x,y\}$). $x \neq y$ would trigger the Identical Scopes (Nested) heuristic. A table would be generated for $(x \neq y) \land (x \leq y)$ and $c_1$ would be replaced with the new table constraint, while $c_2$ would remain unchanged.

The nested Boolean expression heuristics are applied after the top-level constraint heuristics. They are applied to the AST in a top-down order: all four heuristics are applied to an AST node $n$ before any of the descendants of $n$. The rationale is to tabulate the largest possible Boolean expression to get the most benefit (within the limits described in Section 4.3). If an expression is identified by one of the heuristics but tabulation fails (e.g. by exceeding a work limit), then parts of the expression (i.e. descendant nodes in the AST) may still be tabulated. This occurs in the Knight’s Tour problem for example, described in Section 5.2.3.

3.4. Identifying Promising Integer Expressions

Finally we adapt the heuristics to apply to integer expressions. The integer expression heuristics are applied last. The AST is traversed in the same order as for Boolean nested expressions (i.e. parents before children). Tabulating an integer expression has the additional step of introducing a new auxiliary variable.
To avoid the overhead of introducing unnecessary variables, we only consider expressions that would be extracted by general flattening (Section 2.3). The model may contain identical integer expressions, and to avoid tabulating multiple identical expressions (and introducing multiple auxiliary variables for them) we use a cache mapping expressions to auxiliary variables. Before applying the heuristics to an expression $e_1$, we first check the cache, and if another expression identical to $e_1$ has already been tabulated with auxiliary variable $a_1$ then $e_1$ is replaced with $a_1$.

Prior to applying the heuristics to expression $e_1$, a temporary auxiliary variable $a_{temp}$ is made, and the constraint $c_{temp}$: $a_{temp} = e_1$ is made (but not attached to the AST). In terms of $e_1$ and $c_{temp}$, the heuristics are as follows:

**Identical Scopes (Integer)** identifies an integer expression $e_1$ that contains more than one variable and has the same scope as a top-level constraint $c_2$ where $c_2$ does not contain $e_1$.

**Weak Propagation (Integer)** identifies an integer expression $e_1$ that is likely to propagate weakly (as in Section 3.5), and either (a) the top-level constraint $c_2$ that contains $e_1$ is likely to propagate strongly when $e_1$ is temporarily replaced with $a_{temp}$, or (b) the constraint $c_{temp}$ triggers the Weak Propagation heuristic.

**Duplicate Variables (Integer), and Large AST (Integer)** are identical to the heuristics Duplicate Variables and Large AST applied to $c_{temp}$.

If $e_1$ triggers any heuristic except Identical Scopes (Integer), tabulation is attempted on the constraint $c_{temp}$. If tabulation is successful, then $a_{temp}$ becomes permanent, $e_1$ is replaced with $a_{temp}$, and the new table constraint is attached to the AST (as described in Section 3.1). Identical Scopes (Integer) is similar, the only difference is that tabulation is attempted on the conjunction of $c_{temp}$ with all top-level constraints that have the same scope as $e_1$.

### 3.5. GAC Estimate

Given an expression $e$, the GAC estimate is a heuristic to estimate whether the representation of $e$ for a conventional CP solver will propagate strongly. In cases where CP solvers vary, we use Minion [29] as a reference. When $e$ is a top-level constraint, the GAC estimate is simply an estimate of whether the solver will enforce GAC on $e$. Nested Boolean expressions are treated identically to top-level constraints. When $e$ is an integer expression, the GAC estimate is applied to the constraint $a = e$ (where $a$ is a new auxiliary variable), with the rationale that $e$ will in many cases be extracted by flattening and replaced with $a$ (see Section 2.3), creating the constraint $a = e$.

The definition of the GAC estimate is recursive on the AST representing the expression $e$. At the leaves of the AST, constants and references to variables are defined to be strong. Each type of internal AST node (such as sum or product) has its own rules to define when it is weak or strong. For example, the allDiff constraint often has a GAC propagator so it is defined to be strong iff all its
children are strong. Therefore the constraint \texttt{allDiff}(x_1, x_2, x_3) is strong. A sum is defined to be strong when all its children are strong, and for each child \( c \) the interval of possible values \([a, b]\) of \( c \) satisfies \( b - a \leq 1 \). The reason is that sums are usually implemented with bound consistency propagators that are in general weaker than GAC but equivalent to GAC in this specific case. The constraint \texttt{allDiff}(x_1 - x_2, x_3 - x_4, x_5 - x_6) on integer variables \( x_1, \ldots, x_6 \in \{1, \ldots, 3\} \) is therefore defined to be weak. Its representation for a CP solver is unlikely to enforce GAC on the variables \( x_1, \ldots, x_6 \) even when \texttt{allDiff} has a GAC propagator.

4. Tabulation: Generating Tables, Caching, and Work Limits

Having identified promising candidates, the next step is to perform tabulation efficiently and with appropriate work limits to avoid impractically large tables and long tabulation times. The method to perform tabulation is quite straightforward and is described in Section 4.1. For efficiency we have developed a cache to avoid repeated generation of identical tables for expressions that are semantically equivalent (up to renaming of decision variables). The cache relies on a normal form for expressions, and both the cache and the normal form are described in Section 4.2. Finally, in Section 4.3 we describe work limits and progress checks applied to tabulation.

4.1. Generating Tables

The algorithm for generating a table is implemented entirely within \textsc{Savile Row} and operates directly on an arbitrarily nested Boolean expression, avoiding the need to tailor a candidate expression for an external solver. Given a Boolean expression \( e \) to tabulate, we first traverse the AST of \( e \) (in depth-first, left-first order) to collect a list of its variables without duplication. A table is generated by depth-first search with a static variable ordering (the order of the variable list) and \( d \)-way branching. At each node of search the expression is simplified (Section 2.3); if it evaluates to false then the search backtracks. At each leaf node of search that evaluates to true, we store the corresponding assignment as a tuple in the table.

For example, consider the knight’s move constraint of the Knight’s Tour Sequence model with \( n = 4 \). Table 1 shows the first few steps of depth-first search. Once the first variable is assigned to 0, the expression becomes much shorter and simpler. When both variables are assigned, it evaluates to either true or false. The first tuple to be added to the table is \((0, 6)\). The next would be \((0, 9)\). Tables are generated with tuples in lexicographic order, and with columns in the order of the variable list.

4.2. Caching

We use caches to avoid generating identical tables for constraints that are semantically equivalent (up to renaming of decision variables). To store or retrieve a table for an expression \( e \), we first place \( e \) into a normal form as follows.
Table 1: The first few steps of depth-first search to generate the table for the knight’s move constraint of the Knight’s Tour Sequence model, n = 4.

| tour[0] | tour[1] | Simplified expression |
|---------|---------|------------------------|
| 0       | —       | (tour[1] % 4 = 1 \ tour[1] / 4 = 2) \ (tour[1] % 4 = 2 \ tour[1] / 4 = 1) |
| 0       | 0       | false |
| 0       | 5       | false |
| 0       | 6       | true (tuple (0, 6) added to table) |
| 0       | 7       | false |

First the expression is simplified and normalised as described in Section 2.3. Then all associative and commutative k-ary expressions (such as sums) and commutative binary operators (e.g. =) within e are sorted. Alphabetical order is used because it will group together references to the same matrix (all else being equal) and place references to different matrices in a consistent order regardless of the indices. The normal form is not only used for accessing the caches. It is also applied before generating a table for an expression e to ensure that tables are generated with columns in the correct order for storing in the cache.

After applying the normal form, the expression is traversed in depth-first, left-first order to collect a sequence of decision variables (without duplication), and the variables in the sequence are then renamed to a canonical sequence of names to create e'. Thus the actual variable names in e do not affect e', only their relative positions. e' and the variable domains together are used as a key to store and retrieve tables in the caches.

There are two in-memory caches: the first contains tables, and the second stores cases where tabulation failed because it failed a progress check or reached the node limit. The in-memory caches do not persist after one tailoring process on one problem instance. We have also implemented a persistent filesystem cache but this is disabled for the evaluation because it would cause timings to change depending on the order of processes.

4.3. Tabulation Progress Checks and Work Limits

In some cases a heuristic will identify a constraint that is too large to be tabulated. Simple work limits (such as those applied in our earlier work [1] where we limited the depth-first search to generate at most 10,000 tuples, and to fail and backtrack at most 100,000 times) are not ideal because time can be wasted attempting to tabulate constraints that are far beyond reach. As an alternative we propose progress checks where the progress of the algorithm through the assignment space is compared to the total size of the space, and if the algorithm seems to be making insufficient progress then the search is stopped.
early. The depth-first search algorithm progresses through the assignment space in lexicographic order, making it straightforward to calculate the number of total assignments explored so far from the current (partial) assignment.

Suppose we have a constraint on variables $x_1, \ldots, x_r$ with domains $D_1, \ldots, D_r$, and we reached a partial assignment setting variables $\langle x_1, \ldots, x_k \rangle$ to values $\langle v_1, \ldots, v_k \rangle$ where $k \leq r$. The partial assignment is completed by filling in the minimum value of the domain for each unassigned variable:

$$\tau = \langle v_1, \ldots, v_k, \min(D_{k+1}), \ldots, \min(D_r) \rangle$$

The formulas below assume that each domain is a single contiguous range of integers $D_i = \{0\ldots\max(D_i)\}$. The implementation has an additional step to map domain values into a single contiguous range. The total assignment space $A$ and current position $C$ are:

$$A = \prod_{i=1}^{r} |D_i|$$

$$C = \sum_{i=1}^{r} \left[ \tau_i \times \prod_{j=i+1}^{r} |D_j| \right]$$

The progress check uses the current node count ($\text{nodeCount}$) as well as the parameter $\text{nodeLimit}$. It compares the progress made so far to the proportion of the search node limit that has been used so far, effectively using a linear extrapolation to estimate whether the search will complete within the node limit. The search is abandoned if:

$$C \frac{A}{\text{nodeCount}} < \frac{\text{nodeCount}}{\text{nodeLimit}}$$

The term $\frac{C}{A}$ represents the progress made so far through the search space, and $\frac{\text{nodeCount}}{\text{nodeLimit}}$ is the proportion of the search node budget used so far. The progress checks are pessimistic: if a search is expected to slightly exceed the node limit it is abandoned, even though the search algorithm is unlikely to progress through the assignment space at a constant rate. A more optimistic strategy could be obtained by adjusting the formula above (e.g. by multiplying the left-hand side by an additional parameter that is $>1$). Progress checks are carried out after 1000 and 10,000 nodes, and then after every 10,000 nodes. In addition to the progress checks, the search is terminated if it reaches $\text{nodeLimit}$ nodes.

One further limit is applied when tabulating nested Boolean expressions. For solvers or encoding backends that do not support reified or nested table constraints, we limit $A$ to be no more than $\text{nodeLimit}$. The reason is that a nested Boolean expression would be replaced with a nested table constraint by tabulation (as in Section 3.1) which would then be replaced with a new Boolean variable and a top-level reified table constraint. If the solver does not support reified table constraints, it is converted to a conventional table constraint with
5. Evaluation: Identifying Promising Subproblems

We now come to the first part of our evaluation. In this section, we show that the set of heuristics (when combined with the node limit and progress checks) can successfully identify promising subproblems of a wide range of models. Tabulation (whether performed manually or with tool support) is a well-established technique, and for some problem classes and models there are examples in the literature of subproblems that can profitably be tabulated. For these models we compare to the literature, in addition to comparing the original model to the version after tabulation. In Section 6, we will show that in many cases tabulation strongly improves the total time to tailor and solve an instance (including time taken to identify candidates and perform tabulation).

We have set the parameter nodeLimit to 100,000. The effect of this limit can be seen (for example) with the Killer Sudoku problem where some constraints of arity 5 (on variables with domain size 16) fail a progress check but others are successfully tabulated. The base models (with parameter files) are available online alongside a version of Savile Row that implements the set of heuristics and tabulation algorithm.

5.1. Baseline

Our first four case studies are the four problems presented by Dekker et al. [11]. In each case we show that our heuristics can automatically identify the same subproblems that Dekker et al. identified by hand and found to be useful. In three cases, the subproblems were also successfully tabulated, but for one problem class (Handball Tournament Scheduling) tabulation failed the first progress check. Subsequently (in Section 6.3) we will show that we obtain performance improvements that are comparable to their work but with the entire process automated.

5.1.1. Block Party Metacube Problem

The Block Party Metacube Problem is a puzzle in which eight small cubes are arranged into a larger metacube, such that the visible faces on each of the six sides of the metacube form a 'party'. Each small cube has a symbol at each corner of each of its faces (24 symbols per cube in total), and each symbol has three attributes, with each attribute in turn taking one of four values. To form a valid party (the party constraint), the four small cubes forming a visible face of the large cube must be arranged so that the four symbols in the middle of the visible face are either all different, or all the same, for each of the three attributes.

The model we use is closely based on Dekker et al. [11] and we use the same set of instances. The model has two matrices of decision variables, cubeAt and symAt. Matrix cubeAt encodes a permutation of the 8 cubes (numbered 1 to
8), representing the relative locations of the cubes in the metacube. Further, \( \text{symAt} \) represents for each of the 4 symbol positions located in the middle of each of the 6 faces of the metacube (24 symbol positions) which symbol is visible in that position. A hand-computed matrix \( pp \) encodes how the 24 positions in which symbols are placed on a cube occur together at corners of a cube.

There are some notable differences between our model and Dekker et al. They introduce a variable for each attribute at each of the 24 symbol positions, whereas in our model the expression for the attribute (one of the following: \( \text{symAt}[i]/16 \), \( \text{symAt}[i] \% 4 \), or \( (\text{symAt}[i] \% 16)/4 \) with constant \( i \) is used wherever it is needed. Also, Dekker et al introduced a rotation variable for each small cube in their non-tabulated model. The rotation variables are not present in their tabulated model. We used an existential quantifier in place of each rotation variable.

Dekker et al. tabulated the 8 channelling constraints linking cubes and symbols. The Duplicate Variables heuristic identifies the same set of channelling constraints (with arity 4) and they are all successfully tabulated. The Weak Propagation (Nested), Large AST (Nested), or Weak Propagation (Integer) heuristics identify the attribute expressions (e.g. \( \text{symAt}[i]/16 \)) or an equality of two attribute expressions, contained in the party constraints. All such expressions are successfully tabulated (and all have arity 2). Larger sub-expressions of the party constraints are also identified by the heuristics, but all fail the progress check at 1,000 nodes.

5.1.2. Black Hole

Black Hole is a single-player card game (variously called ‘patience’ or ‘solitaire’ games depending on the variety of English spoken) where cards are played one by one into the ‘black hole’ from seventeen face-up fans of three cards. All cards can be seen at all times. A card may be played into the ‘black hole’ if it is adjacent in rank to the previous card. Black Hole was modelled for a variety of solvers by Gent et al. [9] and a table constraint was used in the CP model. We use the simplest and most declarative model of Dekker et al. [11] where two variables \( a \) and \( b \) represent adjacent cards iff \(|a-b| \% 13 \in \{1,12\} \) (the adjacency constraint). The model has two matrices of variables: \( \text{blackHole} \), the sequence of cards played into the black hole; and its inverse \( \text{cardSequence} \) (the index of each card in \( \text{blackHole} \)). We post the adjacency constraint on each pair of adjacent variables in \( \text{blackHole} \). Less-than constraints on \( \text{cardSequence} \) ensure that the cards in a fan are not played out of order. Both matrices have an allDiff constraint, and they are linked by channelling constraints.

All 51 adjacency constraints trigger the Weak Propagation heuristic because they overlap with the allDiff on the \( \text{blackHole} \) matrix. In addition the Identical Scopes (Nested) heuristic is triggered by a small number of equalities (no more than 8) within the channelling constraints. No other constraint triggers any heuristic, so our set of candidates is very similar to those identified by hand, first by Gent et al. and later by Dekker et al. All candidates are successfully tabulated.
5.1.3. Handball Tournament Scheduling

The Handball Tournament Scheduling problem is to schedule matches of a tournament, while respecting the rules governing the tournament, and minimising a cost function related to the availability of venues. We use the model of Dekker et al. [11], which is simplified from the full model [31] by omitting some constraints. The problem has 14 teams (in two divisions of 7), and briefly it requires constructing one round-robin tournament for each division (in periods 1-7), followed by a round-robin tournament for all teams (periods 8-20) then its mirror image (periods 21-33). Constraints are either structural (such as balancing home and away games) or seasonal (such as respecting venue unavailability, given as a parameter). The main sets of variables represent: the home-away pattern (whether a team plays at home, away, or has a break during a period); the break period for each team; the opposing team for each team and period; and the cost of each row of the schedule (contributing to the objective).

The original model contains regular constraints (where the meaning of the constraint is defined by a deterministic finite-state automaton) and we use a manual decomposition of regular (via its finite-state automaton) because SAVILE ROW does not currently implement regular constraints. The regular constraints are applied to the home-away pattern matrices. The decomposition introduces a state variable for each step along the sequence and generates constraints of the following form (where \text{state} are the auxiliary state variables, \(M\) is a constant matrix, \(t\), \(d\) and \(k\) are constants):

\[
\text{state}[t+1] = M[d*\text{state}[t] + \text{seq}[t] - k]
\]

We use the same 20 instances (all of the same size) used by Dekker et al. [11]. Dekker et al. experimented with tabulating two types of subproblem. The first type includes a regular constraint on a row of the home-away pattern matrix (corresponding to one period) in conjunction with a sum and symmetry-breaking constraints on the same row. None of the heuristics are triggered for this type of subproblem, however Dekker et al. report no significant speedup in this case.

The second type of subproblem is a part of the objective function that calculates the cost of one row of the schedule. The Large AST heuristic triggers for this type of constraint, however it fails the progress check (described in Section 4.3) at 1,000 nodes in each case so none are tabulated. It seems that a fixed node limit may be too coarse in this case, and a more sophisticated cost-benefit calculation may be required. Dekker et al. reported that tabulating this type of subproblem produces a significant performance improvement.

The Large AST or Weak Propagation heuristic identifies matrix-indexing constraints arising from the decomposition of regular (described above), which for many solvers would not achieve GAC because of the arithmetic expression used to index \(M\). All are successfully tabulated. Several other types of constraints are successfully tabulated:

- Unary constraints that arise when some home-away variables are assigned by preprocessing, identified by the Large AST or Identical Scopes (Nested) heuristics;
Nested unary constraints on the team-of and home-away pattern variables, identified by the Large AST (Nested) heuristic; and

• Arithmetic expressions used to index matrices, identified by the Weak Propagation (Integer) and Identical Scopes (Integer) heuristics.

When a tabulated constraint is both unary and top-level, it is absorbed into the variable’s domain. Despite not tabulating the row cost constraints, the method does speed up conventional CP solvers as shown in Section 6.3.3.

5.1.4. JP Encoding Problem

The JP Encoding problem was introduced in the MiniZinc Challenge 2014. In brief, the problem is to find the most likely encoding of each byte of a stream of Japanese text where multiple encodings may be mixed. The encodings considered are ASCII, EUC-JP, SJIS, UTF-8, or unknown (this choice incurs a large penalty). Once again our model closely follows that of Dekker et al. [11]. We use all 10 instances in the MiniZinc benchmark repository. The instances are from 100 to 1900 bytes in length. Each byte has four variables: the encoding, a ‘byte status’ variable that combines the encoding with the byte’s position within a multibyte character, a ‘char start’ variable indicating whether the byte begins a new multibyte character, and the score which contributes to the objective.

Dekker et al. tabulate three subproblems. The first connects two adjacent status variables, and the Identical Scopes heuristic triggers on this. The second links status, encoding, and char start, and we found that the Identical Scopes heuristic separately links status to encoding, and status to char start. The encoding and char start variables are both functionally defined by status so no propagation is lost with two binary table constraints compared to one ternary table. Thirdly Dekker et al. tabulate the constraint linking the score to the encoding. The Duplicate Variables heuristic triggers on this. In summary, the heuristics identify almost the same set of constraints to tabulate as Dekker et al. did manually, and all candidates are successfully tabulated (creating binary tables).

5.2. New Case Studies

In this section we present ten case studies that were not featured in Dekker et al. [11]. In each case we briefly describe the model and discuss the expressions that trigger our heuristics. Tabulation appears to be helpful generally for these problem classes, with the exception of Maximum Density Still Life (Section 5.2.10) where our approach to tabulation adds overhead without strengthening propagation.

5.2.1. Accordion Patience

‘Accordion’ [32] is a single-player (patience or solitaire) card game. The game starts with the chosen cards in a sequence, each element of which we consider as a ‘pile’ of one card. Each move we make consists of moving a pile on top of either the pile immediately to the left, or three to the left (i.e. with two
piles between the source and destination) such that the top cards in the source and destination piles match by either rank (value of the card, e.g. both 7) or suit (clubs, hearts, diamonds, or spades). The result of each move is to reduce the number of piles by 1 and change the top card of the destination pile. The empty space left at the position of the source pile is deleted. The goal is to keep making moves until just one pile remains. We consider the ‘open’ variant where the positions of all cards are known before play starts, in a variant studied by Knuth \[33\] where we play with a randomly chosen subset of \( n \leq 52 \) cards.

We model accordion with a matrix called \textit{piles} of \((n - 1) \times n\) variables with domain \(0 \ldots 51\), the element \(\text{piles}[i, j]\) representing the top card of the stack in position \(j\) in the sequence after move \(i\). Only the top card in each stack is represented; others are not relevant. We have \(2(n - 1)\) decision variables for the \(n - 1\) moves, expressing which pile is moved to which other pile (named \textit{from} and \textit{to}). We also have two variables per move representing the top card of the stack that is moved (\textit{fromcard}), and the top card of the stack it is moved onto (\textit{tocard}). Frame axioms ensure that unmoved cards are copied from one timestep to the next, and that the unused slots at each timestep fill up with zeroes. A set of constraints link \textit{to} with \textit{tocard}, and \textit{from} with \textit{fromcard} by indexing the \textit{piles} matrix. Finally the move is implemented by indexing into \textit{piles} with \textit{from} and \textit{to}.

There are two key constraints on the moves. The first (Move1) is that the move is of either one or three places, written as follows:

\[
\text{forall } t : \text{int}(1..n-1) . \text{to}[t] = \text{from}[t]-1 \lor \text{to}[t] = \text{from}[t]-3
\]

The other key constraint (Move2) ensures that the top cards of the two piles are of the same rank or suit. This is expressed by stating that the relevant two cards either have the same value modulo 13 or integer-divided by 13, as follows:

\[
\text{forall } t : \text{int}(1..n-1) . \\
\text{fromcard}[t]\%13 = \text{tocard}[t]\%13 \lor \text{fromcard}[t]/13 = \text{tocard}[t]/13
\]

All Move1 and Move2 constraints are identified by the Duplicate Variables heuristic and are successfully tabulated, creating binary tables. The Weak Propagation (Integer) heuristic identifies expressions of the form \(x - c\) where \(c\) is a constant. These expressions come from indexing into the \textit{piles} matrix with \textit{from} and \textit{to}, and they are tabulated (also creating binary table constraints). Identical Scopes and Weak Propagation identify a small number of other constraints, most of which fail the progress check after 1,000 nodes.

5.2.2. Coprime Sets

Erdős and Sárközy \[34\] studied a range of problems involving coprime sets. A pair of numbers \(a\) and \(b\) are coprime if there is no integer \(n > 1\) which is a factor of both \(a\) and \(b\). The Coprime Sets problem of size \(k\) is to find the smallest \(m\) such that there is a subset of \(k\) distinct numbers from \(\{m/2\ldots m\}\) that are pairwise coprime. In our model, the set is represented as a sequence \(V\) of integer variables. Each pair of variables \(V[i]\) and \(V[j]\) has a set of coprime constraints, one for each potential factor in \(\{2\ldots m\}\):
forall d : int(2..m) . ((V[i]%d != 0) \lor (V[j]%d != 0))

Adjacent variables in the sequence are ordered with less-than constraints to break symmetry. Also, the lower bound of \(m/2\) is enforced with constraints \(V[i] >= (V[k]/2)\) for each \(i\) from 1 to \(k-1\). Finally the variable \(V[k]\) is minimised.

In the experiments, we used the instances where \(k \in \{8\ldots25\}\). For each pair of variables, the Identical Scopes heuristic is triggered by the coprime constraints, a symmetry breaking constraint if one exists, and a lower-bound constraint if one exists. Tabulation is successful for each candidate set of constraints, and all original constraints are replaced with binary table constraints.

5.2.3. Knight’s Tour Problem

The Knight’s Tour Problem was described in Section 1.1. Recall that we use the Hamiltonian path version of Knight’s Tour, i.e. the last square visited is not required to be a knight’s move from the first square visited. An instance defines \(n\) and the starting location of the knight. We experiment with instances where \(n \in \{6\ldots12,15,20,25,30,35\}\) and with two starting locations, \((0,0)\) and \((0,1)\), for 24 instances in total.

We use two models, and in both the location of the knight is encoded as a single integer \((nx+y)\) where \((x,y)\) are the coordinates of the knight on the board (from 0). The first model is the sequence model introduced in Section 1.1 (see Figure 2), in which we have a one-dimensional matrix of variables \(tour\), with \(tour[i]\) representing the location of the knight at time-step \(i\). The constraints enforce that initially the knight is at the given location, it never revisits a location (via allDiff), and each adjacent pair \(tour[i]\) and \(tour[i+1]\) corresponds to a knight’s move. The knight’s move constraint uses integer division and modulo to obtain the \(x\) and \(y\) coordinates. For convenience we re-cap this constraint here:

\[
((|tour[i] - tour[i+1]| = 1) \land (|tour[i]/n - tour[i+1]/n| = 2)) \lor
((|tour[i] - tour[i+1]| = 2) \land (|tour[i]/n - tour[i+1]/n| = 1))
\]

The second model (named successor) has a matrix of variables \(next\) which indicate the successor of each location. The \(next\) variables are constrained by allDiff. In addition, it has all variables and constraints of the sequence model and a set of channelling constraints connecting \(next\) to \(tour\), ensuring there are no cycles in \(next\). The channelling constraints are as follows:

forall i : int(0..tourLength-2). next[tour[i]] = tour[i+1]

\(^1\)The standard CP model of a Hamiltonian path problem uses a successor viewpoint and a global constraint (e.g. path in Gecode [35]). Savile Row and Minion do not have the path constraint. Instead the successor model loosely follows an example in Gecode [35] (credited to Gert Smolka) which has successor, predecessor, and jump variables that give the index of each location in the sequence. In preliminary experiments our model performed slightly better than the Gert Smolka model when using Minion. The size of both of our models and Gert Smolka’s model (as the sum of domain sizes) is \(\Theta(n^4)\).
The knight’s move constraint would trigger the Duplicate Variables, Large AST, and Weak Propagation heuristics (of which Duplicate Variables is applied first). For both models, when \( n \leq 15 \), all knight’s move constraints are tabulated (creating a binary table). When \( n = 20 \), the first 8 are tabulated (where variable domains have been reduced by preprocessing), and the number reduces further to 4 when \( n = 35 \). For the remaining knight’s move constraints, each division and modulo operator is identified by Weak Propagation (Integer), extracted and tabulated (creating a binary table constraint). Each unique division and modulo expression is tabulated once and the auxiliary variable is reused as described in Section 3.4. When tabulation is disabled, identical common subexpression elimination (CSE) (part of the default configuration \(^4\)) improves the knight’s tour constraint by adding auxiliary variables for the division, modulo, and absolute value expressions.

Finally, the channelling constraints in the successor model are translated to an element constraint that indexes from 1. If the lower bound of \( \text{tour}[i] \) is not 1 (after domain filtering) then a shifted index expression \( \text{tour}[i]+c \) is created\(^3\). The \( \text{tour}[i]+c \) expressions are identified by Weak Propagation (Integer) and tabulated for the 8 instances where \( n \leq 15 \) (also creating binary tables).

5.2.4. Killer Sudoku

Killer Sudoku \(^{36}\) is a popular puzzle similar to the classical Sudoku, where an empty \( 9 \times 9 \) grid is filled in with numbers 1...9, such that each row, column and the nine non-overlapping \( 3 \times 3 \) sub-squares take different values. In Killer Sudoku there are also clues (which differ between instances). Clues are sets of cells that sum to a given value (and also take different values). We use a straightforward model where each cell of the grid has one decision variable with domain \( \{1...9\} \). The variables of each row, column and the non-overlapping \( 3 \times 3 \) sub-squares are constrained by allDiff. Killer Sudoku instances are trivial for a constraint solver, so we use an existing model for the \( 16 \times 16 \) case, and an existing set of 100 instances \(^4\). The instances are all satisfiable but unlike conventional Killer Sudoku puzzles, they may have more than one solution.

Each clue is a set of cells (from 1 to 5 cells) that are contiguous. For each clue, the model contains both an allDiff (except for size 1 clues) and a sum equality constraint on the same scope. In a conventional constraint solver (without clause learning), the model propagates poorly and solving times can be poor. In \(^4\), it was shown that associative-commutative common subexpression elimination (AC-CSE) (when combined with implied sum constraints generated from the allDiff constraints) can improve solving times substantially by connecting the two constraints on each clue, and also connecting the clues to the rules. The constraints below represent a size 3 clue from one of the instances.

\(^3\)For instances where the starting location is \((0,1)\) the vast majority of \( \text{tour}[i] \) variables have a lower bound of 0.
For each clue of size 3 to 5, the two clue constraints are (together) identified as a candidate for tabulation by the Identical Scopes heuristic. For clues of size 2, the clue allDiff is removed prior to tabulation because it is subsumed by a larger allDiff, leaving just the sum equality constraint. The sum equality triggers the Weak Propagation heuristic. Clues of size 2 to 4 are tabulated, but in some cases clues of size 5 cannot be tabulated. For example, the first instance (named sol1) has 9 clues of size 5, of which 6 were successfully tabulated. The other 3 failed the progress check after 10,000 nodes (as described in Section 4.3). In each case the arity of the generated table is equal to the size of the clue.

5.2.5. Langford’s Problem

Langford’s problem (CSPLib problem 24 [37]) with integer parameters \( n \) and \( k \) is to find a sequence of length \( nk \) which contains \( k \) copies of each number in the set \( \{1, \ldots, n\} \). The sequence must satisfy the constraint that if the first occurrence of \( x \) is at position \( p \), then the other occurrences appear at \( p + (x + 1)i \), for \( i \in \{1, \ldots, k−1\} \). We model Langford’s with an \( n \times k \) 2D matrix \( P \), where row \( i \) represents the positions of the \( k \) occurrences of symbol \( i \) in the sequence. An allDiff constraint is applied to the entire matrix. We also break the symmetry that reverses the sequence by requiring \( P[1,1]−1 \leq (n\times k − P[1,k]) \). We post the following shift constraints to ensure that the positions of symbol \( i \) in the sequence are the correct distance apart, for each \( i \):

\[
\text{forAll } i: \text{int}(1..n) .
\text{forAll } j: \text{int}(2..k) . \ P[i,j] = P[i,j-1]+i+1
\]

The Weak Propagation heuristic triggers on all shift constraints because they overlap with the global allDiff, and all shift constraints are tabulated. Each table constraint produced is binary. We experiment with all 80 instances where \( n \in \{2\ldots17\} \) and \( k \in \{2\ldots6\} \).

5.2.6. N-Linked Sequence and Optimal N-Linked Sequence

This puzzle, proposed by Itay Bavly [38], requires arranging as many as possible out of the first 100 positive integers into a sequence, so that in every pair of adjacent numbers one is a multiple of the other. The longest possible sequence was found to consist of 77 numbers [39]. The question was also asked for 1000 numbers, and in this case a sequence with 418 numbers was constructed. We use the name *n-linked sequence* as proposed by William Gasarch [40], who also introduced the parameter \( n \) for the largest integer.

We consider two versions of the puzzle. The first version is a decision problem in which we ask whether a sequence exists of some given length, where the length \( \text{len} \) is a fraction of \( n \) chosen to be challenging (close to the unsatisfiability threshold, but still satisfiable). The second version is an optimization problem where we simply seek a longest sequence as in the original problem description.
Both versions are modelled with a one-dimensional matrix seq of variables with domain \{1 \ldots n\} representing the sequence. In the decision version, seq is of length len, which is a parameter. In the optimization version, len is a variable that is maximised. In both versions the entire matrix seq is allDiff. The divisibility constraint for the optimization problem is shown below. In the decision problem, the condition \((i<=len)\) is omitted.

\[(i<=len) \rightarrow ((seq[i]\%seq[i-1] = 0) \lor (seq[i-1]\%seq[i] = 0))\]

We experimented with instances up to size \(n = 42\) for optimization, and up to \(n = 80, \ len = 70\) for the decision version. For all instances of both problems, the Duplicate Variables heuristic identifies each divisibility constraint and they are all tabulated. The generated table constraints have arity 2 for the decision version, and arity 3 for the optimization version.

5.2.7. Peaceable Armies of Queens
This puzzle has also been discussed under the name ‘Peaceably Coexisting Armies of Queens’ \[41\]. The problem asks how to place two equal-sized armies of queens on a chessboard so that the white queens do not attack the black queens, and vice versa. On a standard 8 by 8 board, there are 71 non-isomorphic solutions and their number grows quickly with board size. Some early results and discussion are due to Stephen Ainley \[42, \text{Problem C5}\].

We use a very simple model, based on the Basic Model of Smith et al. \[41\], that takes a single parameter \(n\) for the board size (defining an \(n \times n\) board). For each square on the board we have a variable with domain \{0, 1, 2\} with the values indicating no queen, white, and black respectively. As in the third model of Smith et al., we also use one additional variable armySize to indicate the number of queens in each army, and this is maximised. There are two sum constraints stating that there are armySize occurrences of values 1 and 2 respectively. The vast majority of the constraints are to prevent an attack between a pair of queens from opposing armies. Supposing the board is named \(b\), and squares \((i,j)\) and \((k,l)\) share a row, column, or diagonal, then the attack constraint \(b[i,j]+b[k,l] \neq 3\) is posted.

The two army size constraints are identified (together) by the Identical Scopes heuristic, and tabulation fails the progress check after 1,000 nodes. Each attack constraint is identified by the Weak Propagation heuristic and all are successfully tabulated (creating a binary table constraint). The attack constraints are identical apart from the variable names, so after the first is tabulated the rest are retrieved from the cache. Without tabulation a new integer variable is introduced for each sum \(b[i,j]+b[k,l]\) in an attack constraint. We experimented with instances where \(n \in \{4 \ldots 11\}\).

5.2.8. Strong External Difference Families
A Strong External Difference Family (SEDF) is an object defined on a group, with applications in communications and cryptography \[43\]. For the purposes of
this paper, a group is a set $G$ with an associative and invertible binary operation $\times$. Also, $G$ must contain an identity element $e$, which means that $e \times g = g$ for every $g \in G$. The model of SEDF described below (with additional symmetry breaking constraints) was used to find a number of previously undiscovered SEDFs, including the first in non-Abelian groups [44].

Given a finite group $G$ on a set of size $n$, an $(n,m,k,\lambda)$ SEDF is a list $A_1,\ldots,A_m$ of disjoint subsets of size $k$ of $G$ such that, for all $1 \leq i \leq m$, the multi-set $M_i = \{xy^{-1} \mid x \in A_i, y \in A_j, i \neq j\}$ contains $\lambda$ occurrences of each non-identity element of $G$.

The parameters of the SEDF problem are $(n,m,k,\lambda)$, the group $G$ given as a multiplication table $\text{tab}$ (which is an $n \times n$ matrix of integers), and $\text{inv}$, a one-dimensional table which maps each group element to its inverse. The SEDF is represented as an $m \times k$ matrix $\text{sedf}$. The entire $\text{sedf}$ matrix is contained in a single $\text{allDiff}$ constraint. It has row symmetry (as the sets are not ordered) [43] and also each row has symmetry (as each row represents a set): any two rows may be exchanged in a solution while preserving solutionhood; also any two elements within a row may be exchanged while preserving solutionhood. The variables within each row and the first column are ordered with $<$ constraints to partially break the two kinds of symmetry.

To ensure each multi-set $M_i$ has $\lambda$ occurrences of each non-identity element, we use the $\text{gcc}$ on the comprehension below. Value 1 has cardinality 0, and all other values have cardinality $\lambda$. Each $\text{gcc}$ contains all variables in $\text{sedf}$. Note that both $\text{sedf}[i,p]$ and $\text{sedf}[j,q]$ are individual variables, and that the inner expression is equivalent to $\text{tab}[X,\text{inv}[Y]]$ for integer variables $X$ and $Y$.

\[
\text{[ tab[sedf[i,p], inv[sedf[j,q]]] | p:int(1..k), q:int(1..k), j:int(1..m), j!=i ]}
\]

The Identical Scopes heuristic identifies the $\text{allDiff}$ and all $\text{gcc}$ constraints together as a candidate, and Duplicate Variables identifies each $\text{gcc}$ individually. In each case tabulation fails the progress check at 1,000 nodes. Identical Scopes (Integer) identifies cases where $\text{tab}[X,\text{inv}[Y]]$ within a $\text{gcc}$ overlaps with $X<Y$ or $Y<X$ ordering the first column (where $X$ and $Y$ are variables in the $\text{sedf}$ matrix); Large AST (Integer) or Weak Propagation (Integer) identify the rest of the $\text{tab}[X,\text{inv}[Y]]$ expressions and all are successfully tabulated for all instances (creating table constraints of arity 3).

### 5.2.9. Sports Scheduling Completion

The Sports Scheduling problem is to construct a schedule of $n(n-1)/2$ games among $n$ teams ($n$ must be even), where each team plays every other team once. The schedule is divided into $n-1$ weeks, in each week there are $n/2$ periods, and one game is played in each period of each week. No distinction is made between home and away games. Each team plays at most twice in a period, and each team plays exactly once each week [40].

The schedule is represented explicitly with a matrix $\text{schedule}[w,p,i]$, indexed by the week $w$, period $p$, and $i$ which is 1 or 2 for the two teams in
the game. Some symmetry is broken by ordering the two teams in each game in the schedule matrix. allDiff constraints are used to ensure each team plays once each week, and a set of gcc constraints (one per period) ensure each team plays at most twice in each period. A second matrix game\([w,p]\) (also indexed by week and period) represents each game with a single integer. The two representations are channelled with the following constraints:

\[
\text{forall } w : \text{WEEKS} \ . \ \text{forall } p : \text{PERIODS} \ . \ \\
\text{game}[w,p] = n*(\text{schedule}[w,p,1]-1)+\text{schedule}[w,p,2]
\]

Finally an allDiff constraint is posted on the game matrix to ensure every team plays every other team exactly once.

In Sports Scheduling Completion, we start with a partial schedule where some of the slots in schedule are assigned. 10 instances were generated with \(n = 12\) and 10 slots assigned a team at random with uniform distribution. Trivially unsatisfiable instances were filtered out.

The Weak Propagation heuristic identifies each of the channeling constraints (on 3 variables), and all are successfully tabulated. Van Hentenryck et al. manually tabulated the same constraint in their OPL model of Sports Scheduling [46].

5.2.10. Maximum Density Still Life

The Maximum Density Still Life problem (CSPLib problem 32 [47]) is to maximise the number of live cells in an \(n \times n\) grid in such a way that applying the rules of John Conway’s Game of Life would leave the grid unchanged. Cells outside the \(n \times n\) grid are assumed to be dead. The rules state that a live cell survives if it has two or three live neighbours, but dies otherwise. A dead cell will only become alive if it has exactly three live neighbours. Maximum Density Still Life has been solved for all \(n\) [48] using a sophisticated CP model and other techniques. The CP model uses a table constraint on 9 Boolean variables (representing cells) and one ‘wastage’ variable (which is part of the objective).

We are not interested in solving Still Life for large \(n\) but in evaluating tabulation, consequently we use a simpler model. We model the problem using an \(n \times n\) matrix \(g\) of Boolean variables (where true means live), surrounded by a border of width 2 of cells that are false. The rules are implemented with two implication constraints, one setting the cell to true if exactly three of the eight neighbours are alive; the other setting the cell to false if fewer than 2 or more than 3 neighbours are alive; leaving cells with exactly two live neighbours unconstrained, as they would be unchanged in the next step of the game. The constraints are applied to each cell in the \(n \times n\) matrix and to the inner layer of border cells. We used the instances where \(n \in \{6..15\}\).

Letting \(\text{sum(neighbours)}\) abbreviate the sum of the 8 neighbours of \(g[i,j]\), the constraints for one cell \(g[i,j]\) are written as follows:

\[
\text{sum(neighbours)} = 3 \rightarrow g[i,j], \\
(\text{sum(neighbours)} > 3 \lor \text{sum(neighbours)} < 2) \rightarrow !g[i,j]
\]
For each cell within the $n \times n$ matrix, the Identical Scopes heuristic identifies the two implication constraints (together) as a candidate for tabulation, and they are successfully tabulated (with arity 9). For the cells in the inner border (and away from the corners), each cell has 3 unassigned neighbours and Savile Row simplifies the two constraints to a single sum $\neq 3$ constraint which is not a candidate for tabulation. When tabulation is disabled, the three occurrences of $\text{sum}(\text{neighbours})$ are removed by common subexpression elimination \cite{4} and replaced with a new variable.

The constraints may seem to be natural candidates for tabulation, but in fact (as we will see in Section 6.5) tabulation provides no benefit, slowing down solving in most cases. We therefore include Maximum Density Still Life as an example of successful tabulation where the result negatively affects performance.

5.3. Summary of Results

We have shown the utility of all four heuristics applied to top-level constraints, and in many cases their nested and integer versions as well. The Identical Scopes heuristic is triggered on JP Encoding, Coprime Sets, and Killer Sudoku among others, while its integer version is triggered on SEDF for example. Duplicate Variables is triggered on N-Linked Sequence, Knight’s Tour, and Accordion Patience among others. The three versions of Large AST are triggered by Handball Tournament Scheduling (top-level), SEDF (integer), and BPMP (nested). The Weak Propagation heuristic triggers for many problems including Sports Scheduling, Peaceable Armies of Queens, and Black Hole, while Weak Propagation (Integer) triggers for Accordion Patience and SEDF among others. In each of these cases the subproblems were successfully tabulated, and in the next section we show that (in almost all cases) this leads to strong improvements in solver performance.

In the first four case studies, we found that the heuristics can identify the same subproblems that Dekker et al. \cite{11} identified by hand and found to be beneficial. For three of these case studies, the subproblems were also successfully tabulated. For the remaining one (Handball Tournament Scheduling), the constraint is simply too large and tabulation fails its first progress check.

For 9 of the 16 models, the generated tables are binary. Examples include games (e.g. Accordion Patience), maths problems such as Coprime Sets, and puzzles (e.g. Peaceable Armies of Queens). There are four examples of tabulation (predominantly) producing ternary table constraints: Handball Tournament Scheduling; Optimal N-Linked Sequences; SEDF; and Sports Scheduling Completion. BPMP has two sets of expressions that are tabulated: the channelling constraints of arity 4, and division and modulo expressions where an arity 2 table is produced. Killer Sudoku has 2, 3, 4, and 5 arity tables matching the sizes of the clues in the puzzle. Finally, with Still Life tabulation produces tables of arity 9, albeit on Boolean variables. The $\text{nodeLimit}$ parameter is set to a conservative value (100,000), minimising the time cost of tabulation but also limiting the arity of generated tables. However the preponderance of arity 2 and 3 tables comes from the structure of the models rather than the effect of $\text{nodeLimit}$. 

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6. Evaluation: Search Reduction by Tabulation

Having shown that the heuristics identify promising subproblems of each of the benchmark problems, we now evaluate whether tabulation speeds up solving of these problems. We test this hypothesis with two conventional CP solvers \([29, 49]\) (taking advantage of extensive research into table propagators) and a CP solver with conflict learning \([50]\). We also test the hypothesis in a different setting, where problem instances are encoded into SAT and solved with a recent CDCL SAT solver (CaDiCaL \([51]\)). In this case we experiment with two different encodings of the table constraints, and for all other constraint types we use the default encodings of \textsc{Savile Row}. We first give experimental details and describe the statistical method, and then look at each of the problems of Section 5 in turn.

6.1. Experimental Details

We evaluate tabulation with four solvers in six configurations, as follows:

**Minion** A version of Minion \([29]\) slightly before release 1.9, with ascending value and static variable orderings. The Trie propagator is used for all table constraints \([6]\).

**Minion (Conflict)** Same as the above with Conflict variable ordering \([52]\).

**Gecode** Gecode version 6.2.0 \([49]\) (with the Compact Table propagator \([22]\)). The search order is the same as for Minion.

**Chuffed** Version 0.10.4 of the clause learning CP solver Chuffed \([50]\) with free search enabled. Chuffed encodes all table constraints into SAT internally: it uses the support encoding for binary table constraints \([53]\) and the Bacchus encoding otherwise \([54]\).

**Cadical** Cadical version 1.3.0 \([51]\) with \textsc{Savile Row}'s default SAT encodings for all constraint types. The support encoding is used for binary table constraints \([53]\) and the Bacchus encoding for non-binary tables \([54]\).

**Cadical (MDD)** Cadical 1.3.0 with \textsc{Savile Row}'s default SAT encodings of all non-table constraints. The support encoding is used for binary table constraints \([53]\). Non-binary tables are compressed into Multi-valued Decision Diagrams (MDDs) which are then encoded with the GenMiniSAT encoding \([55]\).

The solvers and configurations were chosen to include: the recent Compact Table propagator, as well as an earlier table propagator; conventional CP (Minion and Gecode), clause learning CP, and SAT solvers; and both static and dynamic search orders for the conventional CP solvers. We used \textsc{Savile Row} 1.9.1 for the experiments, extended with the new tabulation method. Each reported time is the median of five runs on a cluster containing machines with two Intel Xeon 6138 20-core 2.0 GHz processors and 192 GB RAM, jobs were
submitted requiring 1 CPU core and 6 GB RAM. A time limit of 1 hour was applied. Reported times include the time taken by Savile Row to tailor the instance and (if activated) to tabulate. Software, models and parameter files for all experiments are available online [30].

6.2. Statistical Analysis

To compare two configurations A and B of Savile Row (where A does not include tabulation and B does), we first take the median of the total time for each instance and each configuration. The total time is the time taken by Savile Row (including tabulation, if enabled) plus time taken by the backend solver. The median was chosen because it is less affected by outliers than the mean. Instances where both configurations timed out are discarded. For the remaining timeouts (i.e. median total time is >3600 s) we apply a PAR2 penalty, so their median total time is considered to be 7200 s. For each instance, we take the quotient of the two medians ($A/B$). We take the geometric mean of the set of quotients to obtain $s$, a single statistic to compare A and B. If $s > 1$ then B is considered to be better than A. The geometric mean is more appropriate than the arithmetic mean in this case [4].

Where $s$ is close to 1, it may not be clear whether the difference between A and B is statistically significant. As in [4], we use the bootstrap method to compute a 95% confidence interval of $s$ with 100,000 bootstrap samples. We consider the difference between A and B to be statistically significant when the 95% confidence interval does not include 1. Finally, some problem classes have 10 or fewer instances, in which case we report the speedup quotient for each instance instead of the 95% confidence interval.

6.3. Experimental Evaluation: Baseline Problems

First we present results for the four problem classes that were used in Dekker et al [56].

6.3.1. Block Party Metacube Problem

In the BPMP, channelling constraints between two viewpoints are tabulated, as well as some expressions containing division and modulo (as described in Section 5.1.1). Results for BPMP are shown in Figure 5. The conventional CP solvers exhibit differing speed-ups, with the largest being Gecode where $s = 11.94$ with 95% confidence interval of [10.81, 12.88]. For Minion and Minion (Conflict) the speed-ups are smaller but significant, with 95% confidence intervals of [1.31, 1.40] and [3.31, 4.61] respectively. We found no significant difference with Chuffed, where the 95% confidence interval is [0.59, 1.73]. In the results presented by Dekker et al. [56], Chuffed performs badly without tabulation. Their non-tabulated model has additional rotation variables and others (see Section 5.1.1) which may explain the difference in performance.

Tabulation substantially reduces performance of the SAT solver with both encodings. For this problem the sizes of the SAT encodings are increased by
tabulation. For example, with instance 1176, with Cadical the number of variables is increased by 9.6 times, and clauses by 4.0 times, while with Cadical (MDD) the number of variables is increased by 9.7 times and clauses by 6.5 times.

6.3.2. Black Hole

An instance of Black Hole is a permutation of the 52 cards. We experimented with the 102 randomly-generated instances from CSPLib [57]. Tabulation can speed up all six solvers as shown in Figure 5. The solvers without clause learning show very substantial speedups with tabulation and the effect is clearly significant. For example, Gecode has a geometric mean speedup of 15.28 with 95% confidence interval [9.98, 22.50]. For Minion and Minion (Conflict) the plots show that tabulation has a cost and some of the easier instances are slowed down, however this is not the case for Gecode. The speed-ups for Minion and Minion (Conflict) are significant, with 95% confidence intervals of [6.30, 15.20] and [12.79, 32.71] respectively. The solvers with learning performed much better on this problem (with or without tabulation), and the average speedup is sig-
nificant, though smaller: the 95% confidence intervals for Cadical and Chuffed are [3.37, 4.30] and [2.33, 3.19], respectively. The two SAT solver configurations are in fact identical because all table constraints are binary and the support encoding is used in both cases (see Section 6.1).

6.3.3. Handball Tournament Scheduling

Recall from Section 5.1.3 that the row cost constraints that sped up HTS elsewhere [56] were not tabulated here because they fail a progress check. Instead, our method tabulates constraints arising from decomposition of regular constraints, among others. Results are shown in Figure 5. There is a clear benefit for the conventional CP solvers, the 95% confidence intervals for Gecode, Minion, and Minion (Conflict) are [1.27, 3.89], [2.68, 34.75], and [2.29, 6.06] respectively. The dynamic variable ordering of Minion (Conflict) helps to solve HTS efficiently. The solvers with conflict learning solve HTS much faster (with no timeouts), and all show a slow-down with tabulation. The 95% confidence intervals for Cadical, Cadical (MDD), and Chuffed are [0.70, 0.87], [0.85, 1.00] (not significant), and [0.50, 0.66] respectively.

6.3.4. JP Encoding Problem

Results are shown in Figure 5. The results are somewhat positive for Chuffed, Gecode, and both Minion configurations, but the conventional CP solvers can solve only two of the ten instances. Chuffed with tabulation can solve 6 instances. The speedup quotients for each solver on each instance (except double timeouts) are shown in the following table. For both Cadical and Cadical (MDD), the encoding of the objective function (the sum of the scores for each byte) cannot be completed within 1 hour for any instance. The default encoding for sums (similar to a totalizer) is used [19], and this encoding does not scale well for sums with a large set of possible values.

| Solver       | Sorted speedup quotients for tabulation |
|--------------|----------------------------------------|
| Chuffed      | 1.43, 1.53, 1.78, 2.09, 4.42, 4.78      |
| Gecode       | 1.09, 1.41                              |
| Minion       | 1.05, 2.34                              |
| Minion (Conflict) | 0.97, 2.45          |

6.4. Experimental Evaluation: New Case Studies

In this section we give experimental results for nine of the new case studies described in Section 5.2.

6.4.1. Accordion Patience

For the three conventional CP solvers, tabulation speeds up solving considerably as shown in Figure 6. Gecode has the largest geometric mean speedup of 59.33, with 95% confidence interval [28.35, 125.91]. In some cases tabulation reduces search nodes by more than 1,000 times and this translates into very large speedups. For Chuffed the effect is small, the geometric mean speedup is 1.38 with 95% confidence interval [1.17, 1.63]. Tabulation seems to cause a
Figure 6: For each of the five problem classes: Accordion Patience, Coprime Sets, Knights Tour (with two models), Killer Sudoku, and Langford’s Problem, and the six solvers: Cadical, Cadical (MDD), Chuffed, Gecode, Minion, and Minion (Conflict), the figure plots the total time (including both Savile Row and the solver) without tabulation on the x-axis and with tabulation on the y-axis. The red dotted lines indicate the time limit of 1 hour; points appearing on the line timed out. Points appearing below the x = y diagonal were solved faster with tabulation than without. The s value on each plot is the geometric mean of speedup quotients.
slight slow-down for Cadical and Cadical (MDD) but neither reach significance (with 95% confidence intervals \([0.90, 1.00]\) and \([0.83, 1.00]\) respectively).

An informal experiment with instance \texttt{cards\_11\_01.param} and Minion (with a static variable and value ordering) showed that almost all the benefit comes from tabulating Move1 and Move2. Without tabulation, Minion takes 644,774 nodes. With just Move1 and Move2 tabulated, Minion takes 35,760 nodes, and with automatic tabulation it takes 35,712 nodes (and total time was almost identical).

6.4.2. Coprime Sets

Results are shown in Figure 6. For the conventional CP solvers, tabulation is not worthwhile for the smallest (and easiest) instances but starts to pay off for larger instances (e.g. where \(k \geq 13\) for Minion and Minion (Conflict)). Gecode shows the smallest geometric mean speedup of 1.68 with 95% confidence interval \([0.93, 3.13]\), while Minion has the largest average speedup of 3.58 with 95% confidence interval \([1.49, 9.58]\). The static variable ordering of Minion and Gecode follows the sequence from smallest to largest member of the set (branching for smallest value first). It would appear to be a natural choice, and Gecode performs well (with or without tabulation) using this search order. Both Minion configurations are slower than Gecode (both with and without tabulation). Chuffed shows very little difference overall, with a peak speedup of 2.40 and 95% confidence interval of \([0.80, 1.19]\). The two SAT solver configurations are identical because all table constraints are binary so the same encoding is used in both cases (Section 6.1). With Cadical we start to see the benefit of tabulation with the most difficult instances. The peak speedup is 2.90, average is 1.25 and the 95% confidence interval is \([0.99, 1.65]\). Only Minion and Minion (Conflict) reached the significance threshold.

6.4.3. Knight’s Tour Problem

Results for both models are shown in Figure 6. For the sequence model, tabulation causes a very clear speedup for all solvers. In this case, tabulation of instances where \(n \leq 15\) produces a quite different model with no auxiliary variables and much stronger propagation. Chuffed shows the least improvement, with a geometric mean speedup of 79.82 and 95% confidence interval \([16.88, 338.55]\). With the successor model the results are more complex. For Minion and Minion (Conflict), tabulation makes little difference except for the 8 instances described in Section 5.2.3 where shift constraints are tabulated. For these 8 instances there is a speedup of more than 100 times. Gecode is slightly slower with tabulation; it uses a GAC propagator by default for shift constraints so does not benefit from tabulation in this case. The SAT solvers and Chuffed benefit substantially from tabulation, in terms of both time and number of instances solved. For example, Cadical has a geometric mean speedup of 23.52 with confidence interval \([7.35, 80.24]\).
6.4.4. Killer Sudoku

For the three conventional CP solvers, tabulation makes a substantial difference as shown in Figure 6. It has the smallest effect on Gecode, with geometric mean speedup of 5.14 and 95% confidence interval [3.68, 7.28]. For Minion the average speedup is 12.87 with 95% confidence interval [7.70, 21.95], and Minion (Conflict) is similar. However, for the learning solvers Cadical, Cadical (MDD), and Chuffed, the geometric mean speedup is below 1 (e.g. with Cadical the average is 0.58 with 95% confidence interval [0.52, 0.63]). In this case tabulation increases SAT encoding size quite substantially. For Cadical, the number of clauses is increased by two times or more for 88 of the 100 instances (and Chuffed uses the same SAT encodings of table constraints as Cadical, as described in Section 6.1). For Cadical (MDD) the same is true for 93 instances.

6.4.5. Langford’s Problem

Results are plotted in Figure 6. For Minion, Minion (Conflict), and Chuffed, tabulation produces a modest speedup, with 95% confidence intervals of [1.08, 1.30], [1.03, 1.20], and [1.09, 1.26] respectively. Gecode is slightly slower with tabulation. Gecode uses a GAC propagator by default for the shift constraints so (for all non-timeout instances) the number of search nodes is unchanged by tabulation. The SAT solver is slowed down very slightly by replacing a specialised encoding with the support encoding of binary tables.

6.4.6. N-Linked Sequence and Optimal N-Linked Sequence

For the decision version of the problem, we used the 15 instances where \( n \in \{60, 70, 80\} \) and \( \text{len} \in \{n - 30, n - 25, n - 20, n - 15, n - 10\} \). For the optimization version, we used instances \( n = \{12, 14, 16, \ldots, 42\} \). Results of both problems are shown in Figure 6. For the decision problem, tabulation speeds up all solvers quite substantially. For example Chuffed has a geometric mean speedup of 124.46 with 95% confidence interval [30.63, 563.20]. Gecode, Minion, and Minion (Conflict) struggle to solve many instances with or without tabulation. All table constraints are binary so the two SAT configurations are the same (see Section 6.1). Cadical is affected less than the other solvers, with geometric mean speedup of 11.60 and confidence interval [6.49, 20.47].

Tabulation also appears to speed up all solvers for the optimization problem, but for the SAT solvers the gain is small (and for Cadical it does not reach significance). Cadical and Cadical (MDD) have 95% confidence intervals of [0.65, 1.75] and [1.11, 2.15] respectively. Chuffed benefits the most, with 95% confidence interval [10.18, 116.29]. The table constraints generated for the optimization problem are of arity 3 and have more tuples (i.e. the tables in the smallest optimization instance with \( n = 12 \) all have more tuples than those of the largest decision instance where \( n = 80 \) and \( \text{len} = 70 \)), shifting the tradeoff between tabulation and the default encodings or propagators.

6.4.7. Peaceable Armies of Queens

The results are plotted in Figure 7. The tabulated model propagates much better with the conventional CP solvers and they exhibit very large speedups as a
Figure 7: For each of the six problem classes: (Optimal) N-Linked Sequence, Peaceable Armies of Queens, Strong External Difference Families (SEDF), Sports Scheduling Completion, and Maximum Density Still Life, and the six solvers: Cadical, Cadical (MDD), Chuffed, Gecode, Minion (Conflict), and Minion, the figure plots the total time (including both Savile Row and the solver) without tabulation on the x-axis and with tabulation on the y-axis. The red dotted lines indicate the time limit of 1 hour; points appearing on the line timed out. Points appearing below the $x = y$ diagonal were solved faster with tabulation than without. The $s$ value on each plot is the geometric mean of speedup quotients.
result. The clause learning solvers show a modest improvement from tabulation. Cadical and Cadical (MDD) are identical since all table constraints are binary (see Section 6.1). The sorted speedup quotients for each solver on each instance (except double timeouts) are shown in the following table.

| Solver      | Sorted speedup quotients for tabulation |
|-------------|-----------------------------------------|
| Cadical     | 1.09, 1.12, 1.14, 1.56, 1.58, 1.62      |
| Chuffed     | 1.09, 1.22, 1.58, 1.62, 1.68            |
| Gecode      | 1.21, 11.57, 14.26, 439.51, 471.95      |
| Minion      | 1.07, 4.95, 31.08, 434.71, 839.78       |
| Minion (Conflict) | 1.18, 4.99, 26.48, 383.80, 692.88   |

6.4.8. **Strong External Difference Families**

The results (plotted in Figure 7) are positive for all solvers, but particularly for the conventional CP solvers on larger instances. For example, with Gecode the geometric mean speedup is 4.75 with 95% confidence interval $[1.96, 11.98]$, and a peak speedup of 57.7 times. Gecode, Minion, Minion (Conflict), Cadical, and Cadical (MDD) all have no time-outs, with and without tabulation. Chuffed solves all instances with tabulation, but times out on 4 instances of various sizes without tabulation. The 95% confidence intervals for the conflict learning solvers Chuffed, Cadical, and Cadical (MDD) are $[1.70, 4.03], [1.18, 2.37]$, and $[1.76, 2.92]$ respectively.

In SEDF all the tabulated expressions are of one kind: they are integer expressions within a $\mathtt{gcc}$ constraint. SEDF demonstrates that large speedups are possible even when no top-level constraints are tabulated.

6.4.9. **Sports Scheduling Completion**

Tabulation proves to be highly beneficial for the conventional CP solvers Gecode, Minion, and Minion (Conflict). With Chuffed the picture is mixed. Some instances are slowed by tabulation, particularly the easiest four, while some of the more difficult instances benefit from it. Results with Cadical and Cadical (MDD) are largely negative, with only three instances showing an improvement with Cadical and none with Cadical (MDD). Plots and geometric mean speedups are shown in Figure 7, and the sorted speedup quotients for each solver on each instance (except double timeouts) are shown in the following table.

| Solver      | Sorted speedup quotients for tabulation |
|-------------|-----------------------------------------|
| Cadical     | 0.25, 0.30, 0.39, 0.42, 0.56, 0.77, 0.91, 1.10, 1.91, 3.07 |
| Cadical (MDD) | 0.08, 0.09, 0.13, 0.13, 0.13, 0.15, 0.20, 0.30, 0.37, 0.87 |
| Chuffed     | 0.32, 0.55, 0.66, 0.72, 1.13, 1.26, 1.38, 1.74, 7.40, 12.05 |
| Gecode      | 12.12, 13.82, 15.91, 25.10, 35.95, 148.74 |
| Minion      | 8.60, 11.23, 19.85, 24.39, 35.05, 63.44 |
| Minion (Conflict) | 1.27, 2.93, 3.85, 14.54, 22.37, 67.79 |

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6.5. A Negative Example: Maximum Density Still Life

Still Life might seem to be an obvious candidate for tabulation, however for Gecode, Minion, and Minion (Conflict), the node counts are identical for all non-timeout instances: tabulation does not improve propagation. Minion and Minion (Conflict) are substantially slower with tabulation, as are the learning solvers that use the Bacchus encoding (Cadical and Chuffed). Gecode and Cadical (MDD) have approximately the same performance, as shown in Figure 7 and the following table.

| Solver         | Sorted speedup quotients for tabulation |
|----------------|-----------------------------------------|
| Cadical        | 0.04, 0.05, 0.17, 0.28                  |
| Cadical (MDD)  | 0.44, 0.65, 0.99, 1.14                  |
| Chuffed        | 0.03, 0.06, 0.25                         |
| Gecode         | 1.02, 1.04, 1.19, 1.20                  |
| Minion         | 0.10, 0.16, 0.23, 0.55                   |
| Minion (Conflict) | 0.17, 0.19, 0.22, 0.57              |

It is possible that we could learn from the Still Life problem and refine the heuristics. The Identical Scopes heuristic matches any set of two or more constraints with the same scopes. In this case, there is no benefit to propagation from combining these two constraints and perhaps this could be recognised by a refined heuristic. However, with the current set of heuristics, Still Life is a case where tabulation would need to be switched off for the problem class as a whole, and this could be achieved by tuning (automatically or manually) with a subset of the instances.

6.6. Summary of Results

With the conventional CP solvers (Gecode, Minion, and Minion (Conflict)) we expected tabulation to improve solving time, and it does in the vast majority of cases. Three of the four groups of heuristics (Identical Scopes, Duplicate Variables, Weak Propagation) are based on strength of propagation, i.e. identifying sub-problems where propagation is expected to be weak. Assuming the heuristics are accurate, the question then is whether the benefit of obtaining GAC outweighs the cost of generating and propagating table constraints. Gains range from minor efficiency improvements (e.g. Langford’s Problem with Minion and Minion (Conflict)) to dramatic reductions in search (e.g. Black Hole, Killer Sudoku, N-Linked Sequence). The other group of heuristics (Large AST and its integer and nested versions) aim to improve efficiency by replacing a large, cumbersome expression with a table. Large AST heuristics are triggered on the Handball Tournament Problem and SEDF, and also on BPMP for a very small proportion of the tabulated constraints. In each case, tabulation is worthwhile (particularly for the most difficult instances of SEDF which are sped up by over 20 times). We also have a few cases where tabulation slows down search: the Knight’s Tour Successor model and Langford’s with Gecode (in both cases because Gecode has an efficient GAC propagator for the shift constraint), and Still Life with Minion and Minion (Conflict) (where tabulation does not strengthen propagation and the table constraints are simply slower to propagate).
For the three solvers with conflict learning (Cadical, Cadical (MDD), and Chuffed) the picture is not as simple. These three solvers do not have efficient native table propagators. Table constraints are encoded into SAT and the size of the encoding affects solver performance. Also, conflict learning may be able to mitigate weak propagation. For some problem classes, tabulation is clearly worthwhile: for Black Hole, the solving time becomes approximately constant and the most difficult instances (without tabulation) are sped up by more than 10 times; both models of Knight’s Tour are sped up by orders of magnitude; and N-Linked Sequence exhibits large speed-ups. In each case the generated table constraints are binary and are encoded compactly without any additional Boolean variables. SEDF (where the generated table constraints have arity 3) also benefits from tabulation to a smaller degree. All problem classes where tabulation is notably worse have non-binary constraints (BPMP, Killer Sudoku, Sports Scheduling, and Still Life). The results suggest that arity, SAT encoding size, or both are important and that the heuristics could be improved by taking these factors into account.

7. Evaluation: Other Problem Classes

In this section we evaluate tabulation on a large set of models and problem instances from an existing collection. The purpose is to evaluate the effect of tabulation on models where we do not expect it to improve the model. We took all 50 models and 596 instances from Nightingale et al. [4], and removed models that were used in other experiments in this paper. The remaining 43 models (with 406 instances) are of a variety of problems including combinatorial designs (e.g. BIBD, OPD), puzzles (such as English Peg Solitaire), and industrial design problems (such as SONET).

Results are plotted in Figure 8. In this case we have one plot for each solver, and all 43 models are plotted together. Each solver has a geometric mean speedup below 1, indicating that tabulation is slowing down the solver on average. In each case the difference is significant. The geometric mean speedup ($s$) for each solver can be found in Figure 8 and 95% confidence intervals are $[0.51, 0.65]$ for Cadical, $[0.70, 0.84]$ for Cadical (MDD), $[0.45, 0.60]$ for Chuffed, $[0.73, 0.92]$ for Gecode, $[0.60, 0.71]$ for Minion, and $[0.59, 0.69]$ for Minion (Conflict).

The results for the Car Sequencing problem [58] are substantially worse with tabulation when using the solvers Chuffed, Cadical, and Cadical (MDD). In the Car Sequencing problem we are given a set of car types, with each type requiring a set of options, and the number of each type to manufacture. We are also given an integer ratio $p/q$ for each option, such that for every $q$ consecutive cars on a production line, at most $p$ may have the option installed. The problem is to find a sequence containing the correct number of each type and also satisfying the option constraints. We use a simple model with a sequence $s$ of integer variables representing the sequence of cars. For each option and each subsequence of length $q$, an option constraint is posted (where $S$ is the set of car types that require the option):
Figure 8: For each of the six solvers, the figure plots the total time (including both Savile Row and the solver) without tabulation on the $x$-axis and with tabulation on the $y$-axis for a set of 35 models. The red dotted lines indicate the time limit of 1 hour; points appearing on the line timed out. Points appearing below the $x = y$ diagonal were solved faster with tabulation than without. Each plot is labelled with $s$, the geometric mean speedup quotient.
\[(s[i] \text{ in } S) + (s[i+1] \text{ in } S) + \ldots + (s[i+q-1] \text{ in } S) \leq p\]

When two or more options have the same \(q\) then the corresponding option constraints trigger the Identical Scopes heuristic (once for each subsequence). They are successfully tabulated for all subsequences of length 3. However, the size of the SAT encoding is drastically increased by tabulation in this case. For example, on the first instance, for Cadical the number of variables is increased by 31 times and clauses by 13 times. For Cadical (MDD) the number of variables is increased by 9% and clauses by 12 times.

The Peg Solitaire (State) model of English Peg Solitaire \cite{59} also exhibits worse performance with tabulation for Cadical, and the results are mixed for Cadical (MDD) and Chuffed. The Peg Solitaire (State) model represents the board state (at each time step) with a Boolean matrix, and has an integer variable representing each move. The Identical Scopes heuristic is triggered by sets of constraints linking the move variable to state variables in the current and next timestep. Tabulation substantially increases the size of the SAT encodings. For example, for the first instance, with Cadical the number of variables is increased by 14 times and clauses by 9 times. With Cadical (MDD) the number of variables is increased by 38% and clauses by over 20 times. The results for both Car Sequencing and Peg Solitaire (State) suggest that the heuristics could be refined when targeting SAT by taking account of encoding size.

Tabulation also has a negative effect on the closely related BIBD and OPD problem classes for Minion, Minion (Conflict), and (to a smaller degree) Gecode. There are two models of BIBD in the benchmark set. All three models of BIBD or OPD are dominated by sum of product expressions where the inner product expression triggers the Weak Propagation (Integer) heuristic. Each product is successfully tabulated, and this reduces the efficiency of the conventional CP solvers without affecting the number of search nodes. In this case the products contain only Boolean variables. The GAC estimate \cite{Section 3.5} could be refined to recognise them as equivalent to a conjunction and likely to propagate strongly, avoiding triggering the Weak Propagation (Integer) heuristic in this case.

The MDD encoding of table constraints is more robust than the Bacchus encoding on these benchmarks. For Car Sequencing and Peg Solitaire (State) (as well as Maximum Density Still Life in \cite{Section 6.5}) the MDD encoding performs significantly better than the Bacchus encoding, reducing the negative effect of tabulation on these models. Similarly Gecode (with the Compact Table propagator \cite{22}) is somewhat more robust than Minion (with the older Trie propagator \cite{6}).

In summary, a small number of the 43 problem classes exhibit worse solver performance with tabulation for specific reasons (e.g. dramatically increased SAT encoding size), and we have discussed the reasons for that and how it might be mitigated. For the rest, in the vast majority of cases tabulation makes little difference to total time, and there is no substantial overhead from identifying candidates or attempting tabulation, which we view as a positive result given that we expected no benefit for any of these problem classes.
Table 2: Results of tabulating subsequences of lengths 2, 3, 4, and 5 (generating constraints of lengths 3, 4, 5, and 6) of the optimal N-Linked sequence problem with \( n = 12 \) and solving with Minion. The first line shows no tabulation. The total node count and time are reported for the tabulation process in columns 3 and 4. Savile Row total time, solver time and solver nodes are reported in columns 5-7. Finally the Auto column indicates whether tabulation would have completed with the work limits and progress checks switched on.

8. Evaluation: Scalability of Tabulation

In this section we investigate how our approach scales as the arity of a generated table increases. Even though large-arity tables may not be of practical interest in general, there might be cases where they are useful and we want our method to successfully generate such tables. For this purpose, we use the optimal n-linked sequence problem introduced in Section 5.2.6 and investigate the time and search nodes required to generate tables of various arities and also the effect of them on a CP solver.

Here we experiment with the instance where \( n = 12 \), and we scale tabulation up by increasing the number of sequence variables in the table constraint. In particular, we use 2 to 5 adjacent variables in the table, thus table arities \( r \) range from 3 to 6, and we maximize the sequence length from 6 upwards. To achieve this we have written a separate model for each value of \( r \), and in these models the divisibility constraint is extended to cover \( r - 1 \) sequence variables. For example, when \( r = 4 \) the divisibility constraint (for each \( i \) in \( \{3 \ldots n\} \)) is as follows:

\[
i > \text{len} \lor ( ((\text{seq}[i] \% \text{seq}[i-1] = 0) \lor (\text{seq}[i-1] \% \text{seq}[i] = 0)) \lor ((\text{seq}[i-1] \% \text{seq}[i-2] = 0) \lor (\text{seq}[i-2] \% \text{seq}[i-1] = 0)))
\]

We also post an allDiff constraint over the same set of seq variables:

\[
i > \text{len} \lor \text{allDiff}([\text{seq}[i-2],\text{seq}[i-1],\text{seq}[i]])
\]

The Identical Scopes heuristic identifies these two constraints together as a candidate. For each value of \( r \), 6 constraints of arity \( r \) are tabulated, and 5 binary constraints are tabulated between adjacent pairs of variables within the first 6 variables in the sequence (where \( i > \text{len} \) is false and the conjunction decomposes into \( r - 2 \) binary constraints before tabulation).

The results of the experiments are shown in Table 2. The first row corresponds to the case when no tabulation is invoked. The ‘tabulation nodes’ in the third column is the number of nodes generated to tabulate all of the constraints.
Note that each of the arity \( r \) candidates is slightly different, so the tables for those constraints cannot come from the cache. *Auto* on the last column says whether automatic tabulation could be done with the limits on. For the *No* rows, automatic tabulation would stop at the first progress check at 1,000 nodes (however in this case the limits and progress checks have been removed).

As can be witnessed from the results, the number of nodes explored by the solver decreases significantly as we generate larger tables. At the same time, tabulation explores more nodes, and tabulation, Savile Row and solver times increase. While the stronger inference comes with a cost, the incurred runtimes seem reasonable until arity 5. With arity 6, we observe a notable cost in tabulation, but still we are able to generate and use the tables.

In summary, the overhead of generating the tables scales as expected and is not prohibitive until \( r = 6 \). The best balance of propagation strength against overhead is found at \( r = 3 \), while \( r = 4 \) is the largest size allowed with the progress checks switched on.

9. Related Work

In accordance with reformulating a subset of the problem constraints for stronger and/or cheaper constraint propagation, the Globalizer tool [60] of MiniZinc helps detect opportunities to use global constraints in constraint models. For instance, it can detect that a set of disequality constraints can be converted to an all-different constraint. However, the approach is not completely automatic. First, the user provides some instances, then the tool solves them and analyzes the solutions to find properties that match global constraints. The detected global constraints are provided as a suggestion to the user because they are not guaranteed to be correct for all problem instances.

The tool proposed by Dekker et al. [11] can convert any predicate (Boolean function) in MiniZinc into a table constraint, but the user must annotate the predicates to be tabulated. In the same vein, the IBM ILOG CPLEX Optimization Studio software supports strong annotations to indicate that the solver should find a precomputed table constraint corresponding to a specified set of variables; the resulting table constraint is then added to the model as an implied constraint [13]. The Propia library performed a similar step for an annotated goal in ECLiPSe [14]. In all of these approaches, the first step of identifying promising parts of a given model for tabulation is left to the user.

With the same goals as tabulation, researchers have studied replacing a set of constraints with other constraints. De Uña et al. [15] considered alternative data structures to store the contents of tables. Specifically, they proposed the use of Multivalued Decision Diagrams (MDDs) and Deterministic Decomposable Negation Normal Forms (d-DNNFs), motivated by the fact that creating a table can be costly when the reformulated subproblem has a large number of solutions. The experimental results show that while building compact structures like MDDs or d-DNNFs can be substantially faster than building tables in certain problem classes, it is not clear which structure yields the best total solving time overall. The paper suggests that a more robust technique could
choose between the three structures (table, MDD, d-DNNF) depending on an estimate of the size of the solution set of the subproblem. In any case, the presented approach is not yet automated. The heuristics we present in this paper could be used to detect automatically opportunities for generating MDD and d-DNNF constraints.

As an alternative, Löffler et al. [16] considered transforming subproblem constraints to a regular constraint. The experimental results demonstrate that the best total solving times are achieved either by a table constraint or by the combination of regular and table constraints. Again, there is currently no specific algorithm to detect candidate subproblems automatically. The approach either relies on the heuristics presented in our earlier work [1] or is applied manually.

Our approach could also generate tables in other compact representations where ordinary tuples are replaced by compressed tuples [61, 62, 63], short tuples [64, 65, 66], or smart tuples [67]. The latter generalizes classical, compressed and short tuples, can lead to exponentially smaller tables, and can encode compactly many constraints, including a dozen well-known global constraints. Under a very reasonable assumption about the acyclicity of smart tuples, a polynomial time GAC algorithm was introduced and shown to be effective in practice [67]. In addition, Le Charlier et al. [68] proposed automatically synthesizing smart table constraints from (ordinary) table constraints. While the theoretical worst-case time complexity of the algorithm is quadratic in the size of the input table, it was shown to have quasi linear execution time on the considered benchmarks.

10. Conclusions and Future Work

In this paper we have demonstrated that a small set of heuristics can successfully identify promising sub-problems in a constraint model for tabulation, and that these opportunities can be effectively exploited. The entire process is automated in the constraint modelling system Savile Row. Our heuristics identify the same tabulation opportunities as recent work by Dekker et al. using manual annotations of a constraint model [11]. In addition we have presented nine other case studies demonstrating the efficacy of our heuristics and automated tabulation. We evaluated the method with SAT, learning CP, and conventional CP solvers, on a wide variety of models. The results vary considerably between problem classes, and between solvers on the same problem class. In some cases, the method produces orders of magnitude improvements in solving time and this is achieved completely automatically. In general we have observed more gains with CP solvers that have native, efficient table propagators. When using a SAT solver, the arity of the generated tables and the size of the encoding seem to be important to solver performance.

We have also evaluated the method on a set of 43 models (from a pre-existing collection) where we expected there to be no opportunities for useful tabulation. We found that a small number of models were slowed down, and the reasons for this suggest improvements to the set of heuristics. For example, taking account of SAT encoding size when targeting a SAT solver would avoid slow-downs for
two of the 43 models. For the other problem classes, in the vast majority of cases we found that tabulation makes very little difference to total time. In the final experiment we investigated how tabulation scales as the arity of a constraint is increased, showing (for the Optimal N-Linked Sequence problem) that table generation can scale well beyond the arity with the best trade-off between propagation strength and overhead.

There are two main avenues of future work. Firstly, there is opportunity to refine and extend our collection of heuristics. For example, heuristics could be refined to take SAT encoding size into account when targeting a SAT solver. Machine learning could be employed to help decide which expressions are promising candidates, as well as to tune the parameters of the tabulation procedure. It could also be applied to predict whether a problem class is generally amenable to tabulation. Secondly, it would be interesting to investigate generating compressed table representations such as MDDs (with a work limit and progress checks adapted to the representation). For some compressed representations (including MDDs) there already exist algorithms to generate them (surveyed in Section 9). Integrating compressed representations promises to improve the scalability of the method in some cases, allowing more of the identified candidates to be successfully tabulated.

Acknowledgements

We thank EPSRC for grants EP/P015638/1, EP/P026842/1 and EP/R513386/1. Dr Jefferson holds a Royal Society University Research Fellowship. This project was undertaken on the Viking Cluster, which is a high performance compute facility provided by the University of York. We are grateful for computational support from the University of York High Performance Computing service, Viking and the Research Computing team.

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