Complexity and Algorithms for Semipaired Domination in Graphs

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Abstract

For a graph $G = (V,E)$ with no isolated vertices, a set $D \subseteq V$ is called a semipaired dominating set of $G$ if (i) $D$ is a dominating set of $G$, and (ii) $D$ can be partitioned into two element subsets such that the vertices in each two element set are at distance at most two. The minimum cardinality of a semipaired dominating set of $G$ is called the semipaired domination number of $G$, and is denoted by $\gamma_{pr2}(G)$. The MINIMUM SEMIPAIRED DOMINATION problem is to find a semipaired dominating set of $G$ of cardinality $\gamma_{pr2}(G)$. In this paper, we initiate the algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem. We show that the decision version of the MINIMUM SEMIPAIRED DOMINATION problem is NP-complete for bipartite graphs and split graphs. On the positive side, we present a linear-time algorithm to compute a minimum cardinality semipaired dominating set of interval graphs and trees. We also propose a $1 + \ln(2\Delta + 2)$-approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem, where $\Delta$ denote the maximum degree of the graph and show that the MINIMUM SEMIPAIRED DOMINATION problem cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME}(|V|^{O(\log \log |V|)})$.

Keywords: Domination, Semipaired Domination, Bipartite Graphs, Chordal Graphs, Graph algorithm, NP-complete, Approximation algorithm.

1 Introduction

A dominating set in a graph $G$ is a set $D$ of vertices of $G$ such that every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The MINIMUM DOMINATION problem is to find a dominating set of cardinality $\gamma(G)$. More thorough treatment of domination, can be found in the books [6, 7]. A dominating set $D$ is called a paired dominating set if $G[D]$ contains a perfect matching. The paired domination number of $G$, denoted by $\gamma_{pr}(G)$ is the minimum cardinality of paired dominating set of $G$. The concept of paired domination was introduced by Haynes and Slater in [11].
A relaxed form of paired domination called semipaired domination was introduced by Haynes and Henning \[8\] and studied further in \[12, 9, 10\]. A set $S$ of vertices in a graph $G$ with no isolated vertices is a \textit{semipaired dominating set}, abbreviated a semi-PD-set, of $G$ if $S$ is a dominating set of $G$ and $S$ can be partitioned into 2-element subsets such that the vertices in each 2-element set are at distance at most 2. In other words, the vertices in the dominating set $S$ can be partitioned into 2-element subsets such that if $\{u, v\}$ is a 2-set, then the distance between $u$ and $v$ is either 1 or 2. We say that $u$ and $v$ are \textit{semipaired}. The \textit{semipaired domination number} of $G$, denoted by $\gamma_{pr2}(G)$, is the minimum cardinality of a semi-PD-set of $G$. Since every paired dominating set is a semi-PD-set, and every semi-PD-set is a dominating set, we have the following observation.

**Observation 1.1.** (\[8\]) For every isolate-free graph $G$, $\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G)$.

By Observation 1.1 the semipaired domination number is squeezed between two fundamental domination parameters, namely the domination number and the paired domination number.

More formally, the minimum semipaired domination problem and its decision version are defined as follows:

**MINIMUM SEMIPAIRED DOMINATION problem (MSPDP)**

**Instance:** A graph $G = (V, E)$.

**Solution:** A semi-PD-set $D$ of $G$.

**Measure:** Cardinality of the set $D$.

**SEMPIAIRED DOMINATION DECISION problem (SPDDP)**

**Instance:** A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

**Question:** Does there exist a semi-PD-set $D$ in $G$ such that $|D| \leq k$?

In this paper, we initiate the algorithmic study of the semipaired domination problem. The main contributions of the paper are summarized below. In Section 2 we discuss some definitions and notations. In Section 3 we discuss the difference between the complexity of paired domination and semipaired domination in graphs. In Section 4 we show that the \textbf{SEMPIAIRED DOMINATION DECISION} problem is NP-complete for bipartite and split graphs. In Section 5 and Section 6 we propose a linear-time algorithms to solve the \textbf{MINIMUM SEMIPAIRED DOMINATION problem} in interval graphs and trees respectively. In Section 7 we propose an approximation algorithm for the \textbf{MINIMUM SEMIPAIRED DOMINATION problem} in general graphs. In Section 8 we discuss an approximation hardness result. Finally, Section 9 concludes the paper.

### 2 Terminology and Notation

For notation and graph theory terminology, we in general follow \[13\]. Specifically, let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let $v$ be a vertex in $V$. The \textit{open neighborhood} of $v$ is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the \textit{closed neighborhood} of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. Thus, a set $D$ of vertices in $G$ is a dominating set of $G$ if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V \setminus D$, while $D$ is a total dominating set of $G$ if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V$. The \textit{distance} between two vertices $u$ and $v$ in a connected graph $G$, denoted by $d_G(u, v)$, is the length of
a shortest \((u, v)\)-path in \(G\). If the graph \(G\) is clear from the context, we omit it in the above expressions. We write \(N(v), N[v]\) and \(d(u, v)\) rather than \(N_G(v), N_G[v]\) and \(d_G(u, v)\), respectively.

For a set \(S \subseteq V(G)\), the subgraph induced by \(S\) is denoted by \(G[S]\). If \(G[C]\), where \(C \subseteq V\), is a complete subgraph of \(G\), then \(C\) is a clique of \(G\). A set \(S \subseteq V\) is an independent set if \(G[S]\) has no edge. A graph \(G\) is chordal if every cycle in \(G\) of length at least four has a chord, that is, an edge joining two non-consecutive vertices of the cycle. A chordal graph \(G = (V, E)\) is a split graph if \(V\) can be partitioned into two sets \(I\) and \(C\) such that \(C\) is a clique and \(I\) is an independent set. A vertex \(v \in V(G)\) is a simplicial vertex of \(G\) if \(N_G[v]\) is a clique of \(G\). An ordering \(\alpha = (v_1, v_2, \ldots, v_n)\) is a perfect elimination ordering (PEO) of vertices of \(G\) if \(v_i\) is a simplicial vertex of \(G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]\) for all \(i, 1 \leq i \leq n\). Fulkerson and Gross \([4]\) characterized chordal graphs, and showed that a graph \(G\) is chordal if and only if it has a PEO. A graph \(G = (V, E)\) is bipartite if \(V\) can be partitioned into two disjoint sets \(X\) and \(Y\) such that every edge of \(G\) joins a vertex in \(X\) to a vertex in \(Y\), and such a partition \((X, Y)\) of \(V(G)\) is called a bipartition of \(G\). Further, we denote such a bipartite graph \(G\) by \(G = (X, Y, E)\). A graph \(G\) is an interval graph if there exists a one-to-one correspondence between its vertex set and a family of closed intervals in the real line, such that two vertices are adjacent if and only if their corresponding intervals intersect. Such a family of intervals is called an interval model of a graph.

In the rest of the paper, all graphs considered are simple connected graphs with at least two vertices, unless otherwise mentioned specifically. We use the standard notation \([k] = \{1, \ldots, k\}\). For most of the approximation related terminologies, we refer to \([1, 14]\).

### 3 Complexity difference between paired domination and semipaired domination

In this section, we make an observation on complexity difference between paired domination and semipaired domination. We show that the decision version of the MINIMUM PAIRED DOMINATION problem is NP-complete for GP4 graphs, but the MINIMUM SEMIPAIPED DOMINATION problem is easily solvable for GP4 graphs. The class of GP4 graphs was introduced by Henning and Pandey in \([15]\). Below we recall the definition of GP4 graphs.

**Definition 3.1** (GP4-graph). A graph \(G = (V, E)\) is called a GP4-graph if it can be obtained from a general connected graph \(H = (V_H, E_H)\) where \(V_H = \{v_1, v_2, \ldots, v_{n_H}\}\), by adding a path of length 3 to every vertex of \(H\). Formally, \(V = V_H \cup \{w_i, x_i, y_i, z_i \mid 1 \leq i \leq n_H\}\) and \(E = E_H \cup \{v_iw_i, w_ix_i, x_iy_i, y_iz_i \mid 1 \leq i \leq n_H\}\).

**Theorem 3.1.** If \(G\) is a GP4-graph, then \(\gamma_{pr2}(G) = \frac{2}{3}|V(G)|\).

**Lemma 3.1.** If \(G\) is a GP4-graph constructed from a graph \(H\) as in Definition 3.1, then \(H\) has a paired dominating set of cardinality \(k, k \leq n_H\) if and only if \(G\) has a semi-PD-set of cardinality \(2n_H + k\).

Since the decision version of the MINIMUM PAIRED DOMINATION problem is known to be NP-complete for general graphs \([11]\), the following theorem follows directly from Lemma 3.1.

**Theorem 3.2.** The decision version of the MINIMUM PAIRED DOMINATION problem is NP-complete for GP4-graphs.
4 NP-completeness Results

In this section, we study the NP-completeness of the SEMIPAIRED DOMINATION DECISION problem. We show that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite graphs and split graphs.

4.1 NP-completeness proof for bipartite graphs

Theorem 4.1. The SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite graphs.

Proof. Clearly, the SEMIPAIRED DOMINATION DECISION problem is in NP for bipartite graphs. To show the hardness, we give a polynomial reduction from the MINIMUM VERTEX COVER problem. Given a non-trivial graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, we construct a graph $H = (V_H, E_H)$ in the following way:

Let $V_k = \{v_i^k \mid i \in [n]\}$ and $E_k = \{e_j^k \mid j \in [m]\}$ for $k \in [2]$. Also assume that $A = \{a_i \mid i \in [n]\}$, $B = \{b_i \mid i \in [n]\}$, $C = \{c_i \mid i \in [n]\}$, and $F = \{f_i \mid i \in [n]\}$.

Now define $V_H = V_1 \cup V_2 \cup E_1 \cup E_2 \cup A \cup B \cup C \cup F$, and $E_H = \{v_i^1 f_i, v_i^2 f_i, a_i b_i, b_i c_i, a_i f_i \mid i \in [n]\} \cup \{v_i^k e_i, v_i^k e_i^k \mid k \in [2], i \in [m]\}$, $v_i^k, v_i^k$ are endpoints of edge $e_i$ in $G$. Fig. 2 illustrates the construction of $H$ from $G$.

![Figure 1: An illustration of the construction of $H$ from $G$ in the proof of Theorem 4.1.](image)

Note that the set $I_1 = V_1 \cup V_2 \cup A \cup C$ is an independent set in $H$. Also, the set $I_2 = E_1 \cup E_2 \cup F \cup B$ is an independent set in $H$. Since $V_H = I_1 \cup I_2$, the graph $H$ is a bipartite graph. Now to complete the proof, it suffices for us to prove the following claim:

Claim 4.1. The graph $G$ has a vertex cover of cardinality at most $k$ if and only if the graph $H$ has a semi-PD-set of cardinality at most $2n + 2k$.

Proof. Let $V_c = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ be a vertex cover of $G$ of cardinality $k$. Then $D_{c} = \{v_{i_1}^1, v_{i_2}^1, \ldots, v_{i_k}^1\} \cup \{v_{i_1}^2, v_{i_2}^2, \ldots, v_{i_k}^2\} \cup B \cup F$ is a semi-PD-set of $H$ of cardinality $2n + 2k$.

Conversely, suppose that $H$ has a semi-PD-set $D$ of cardinality at most $2n + 2k$. Note that $D \cap \{a_i, b_i, c_i, f_i\} \geq 2$ for each $i \in [n]$. Hence, without loss of generality, we may assume that $\{b_i, f_i \mid i \in [n]\} \subseteq D$, where $b_i$ and $f_i$ are semipaired. Hence $|D \cap (E_1 \cup E_2 \cup V_1 \cup V_2)| \leq 2k$. Let $S = (V_1 \cup E_1) \cap D$. Without loss of generality, we may also assume that $|S| \leq k$. Now, if $e_i^j \in S$ for some $i \in [m]$, and none of its neighbors belongs to $D$, then $e_i^j$ must be semipaired with some vertex $e_i^j$ where $j \in [m] \setminus \{i\}$, and
also there must exists a vertex $v_k^1$ which is a common neighbor of $e_i^1$ and $e_j^1$. In this case, we replace the vertex $e_i^1$ in the set $S$ with the vertex $v_k^1$ and so $S \leftarrow (S \setminus \{e_i^1\}) \cup \{v_k^1\}$ where $v_k^1$ and $e_j^1$ are semipaired. We do this for each vertex $e_i^1 \in S$ where $i \in [m]$ with none of its neighbors in the set $D$. For the resulting set $S$, $|S \cap V_i| \leq k$ and every vertex $e_i^1$ has a neighbor in $V_i \cap S$. The set $V_c = \{v_i \mid v_i^1 \in S\}$ is a vertex cover of $G$ of cardinality at most $k$. This completes the proof of the claim.

Hence, the theorem is proved.

4.2 NP-completeness result for split graphs

Theorem 4.2. The SEMIPAIRED Domination Decision problem is NP-complete for split graphs.

Proof. Clearly, the SEMIPAIRED Domination Decision problem is in NP. To show the hardness, we give a polynomial time reduction from the Domination Decision problem, which is well known NP-complete problem. Given a non-trivial graph $G = (V, E)$, where $V = \{v_i \mid i \in [n]\}$ and $E = \{e_j \mid j \in [m]\}$, we construct a split graph $G' = (V_G', E_G')$ as follows:

Let $V_k = \{v_i^k \mid i \in [n]\}$ and $U_k = \{u_i^k \mid i \in [n]\}$ for $k \in [2]$. Now define $V_{G'} = V_1 \cup V_2 \cup U_1 \cup U_2$, and $E_{G'} = \{uv \mid u, v \in V_1 \cup U_1, u \neq v\} \cup \{v_i^1v_j^1, u_i^2u_j^1 \mid i \in [n] \text{ and } v_j \in N_G(v_i)\}$. Note that the set $A = V_1 \cup U_1$ is a clique in $G'$ and the set $B = V_2 \cup U_2$ is an independent set in $G'$. Since $V_{G'} = A \cup B$, the constructed graph $G'$ is a split graph. Fig. 2 illustrates the construction of $G'$ from $G$.

![Figure 2: An illustration to the construction of $G'$ from $G$ in the proof of Theorem 4.2](image)

Now, to complete the proof of the theorem, we only need to prove the following claim.

Claim 4.2. $G$ has a dominating set of cardinality $k$ if and only if $G'$ has a semi-PD-set of size cardinality $2k$.

Proof. Let $D = \{v_1, v_2, \ldots, v_k\}$ be a dominating set of size atmost $k$ of $G$. Then $D_{sp} = \{v_i^1, v_i^1, \ldots, v_i^1\} \cup \{u_i^1, u_i^1, \ldots, u_i^1\}$ is a semi-PD-set of $G'$ of size atmost $2k$.

Conversely, suppose that $G$ has a semi-PD-set $D_{sp}$ of cardinality at most $2k$. Let $S_1 = (V_1 \cup V_2) \cap D_{sp}$ and $S_2 = U_1 \cup U_2 \cap D_{sp}$. Then either $|S_1| \leq k$ or $|S_2| \leq k$. Without loss of generality, let us assume that $|S_1| \leq k$. Note that if $v_i^2 \in S_1$ and none of neighbors belong to $S_1$ then we replace $v_i^2$ by some of its neighbor $v_j^1$ in the set $S_1$. So, we may assume that $S_1 \cap V_2 = \phi$. Now the set $D = \{v_i \mid v_i^1 \in S_1\}$ is a dominating set of $G$ of size atmost $k$. Hence, the result follows.

Hence, the theorem is proved.
5 Algorithm for Interval Graphs

In this section, we present a linear-time algorithm to compute a minimum cardinality semi-PD-set of an interval graph.

A linear time recognition algorithm exists for interval graphs, and for an interval graph an interval family can also be constructed in linear time [2, 5]. Let $G = (V, E)$ be an interval graph and $I$ be its interval model. For a vertex $v_i \in V$, let $I_i$ be the corresponding interval. Let $a_i$ and $b_i$ denote the left and right end points of the interval $I_i$. Without loss of generality, we may assume that no two intervals share a common end point. Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the left end ordering of vertices of $G$, that is, $a_i < a_j$ whenever $i < j$. Now we first prove the following lemmas.

Lemma 5.1. Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the left end ordering of vertices of $G$. If $v_i v_j \in E$ for $i < j$, then $v_i v_k \in E$ for every $i < k < j$.

Proof. The proof directly follows from the left end ordering of vertices of $G$. \hfill \Box

Define the set $V_i = \{v_1, v_2, \ldots, v_i\}$, for each $i \in [n]$.

Lemma 5.2. If $G$ is a connected interval graph, then $G[V_i]$ is also connected.

Proof. The proof can easily be done using induction on $i$. \hfill \Box

Let $F(v_i)$ be the least index vertex adjacent to $v_i$, that is, if $F(v_i) = v_p$, then $p = \min \{k \mid v_k v_i \in E\}$. In particular, we define $F(v_1) = v_1$. Let $L(v_i) = v_q$, where $q = \max \{k \mid v_k v_i \notin E \text{ and } k < i\}$. In particular, if $L(v_i)$ does not exist, we assume that $L(v_i) = v_0 (v_0 \notin V)$. Let $G_i = G[V_i]$ and $D_i$ denote a semi-PD-set of $G_i$ of minimum cardinality. Recall that we only consider connected graphs with at least two vertices.

Lemma 5.3. For $i \geq 2$, if $F(v_i) = v_1$, then $D_i = \{v_1, v_i\}$.

Proof. Note that every vertex in $G_i$ is dominated by $v_1$, and $d_{G_i}(v_1, v_i) = 1$. Hence, $D_i = \{v_1, v_i\}$. \hfill \Box

Lemma 5.4. For $i > 1$, if $F(v_i) = v_j$, $j > 1$ and $F(v_j) = v_1$, then $D_i = \{v_1, v_j\}$.

Proof. Note that every vertex in $G_i$ is dominated by some vertex in the set $\{v_1, v_j\}$, and $d_{G_i}(v_1, v_i) = 1$. Hence, $D_i = \{v_1, v_i\}$. \hfill \Box

Lemma 5.5. For $r < k < j < i$, let $F(v_i) = v_j$, $F(v_j) = v_k$, $F(v_k) = v_r$. If every vertex $v_l$ where $k < l < j$, is adjacent to at least one vertex in the set $\{v_j, v_r\}$, then the following holds:

(a) $\{v_j, v_r\} \subseteq D_i$.
(b) $v_j$ is semipaired with $v_r$ in $D_i$.
(c) $D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\} = \{v_j, v_r\}$.

Proof. (a) To dominate $v_i$, either $v_i \in D_i$ or $v_{i1} \in D_i$, where $j \leq i1 < i$ and $v_{i1} \in N_{G_i}(v_i)$. If $i1 \neq j$ and $v_{i1}$ is semipaired with some vertex $v_{j1}$, then $N_{G_i}(v_{i1}) \subseteq N_{G_i}(v_j)$, and $d_{G_i}(v_j, v_{j1}) \leq 2$. Hence, we can update the set $D_i$ as $D_i = (D_i \setminus \{v_{i1}\}) \cup \{v_j\}$ and semipair $v_j$ with $v_{j1}$. This proves that $v_j \in D_i$.

If $v_r$ also belongs to $D_i$, then we are done. Otherwise, if $v_j$ is semipaired with $v_{j1}$ (where $j1 \neq r$), then $j1 > r$. Also, $N_{G}[v_{j1}] \subseteq N_{G}[v_j] \cup N_{G}[v_r]$. In that case, we can update the set $D_i$ as $D_i = (D_i \setminus \{v_{j1}\}) \cup \{v_r\}$. Hence, $\{v_j, v_r\} \subseteq D_i$.\hfill \Box
is again a contradiction. Hence, \( v \) is a contradiction. If \( v \) is not semipaired with \( v_s \), assume that \( v_j \) is semipaired with \( v_{j1} \) and \( v_r \) is semipaired with \( v_{r1} \). Note that \( j1 \) must be greater than \( r \), and \( N_G(v_{j1}) \subseteq N_G(v_j) \cup N_G(v_{r1}) \). Therefore, the set \( D_i \setminus \{v_{j1}\} \) also dominates all the vertices of \( G_i \).

Suppose that \( N_G(v_{r1}) \subseteq D_i \). In this case, \( D' = D_i \setminus \{v_{j1}, v_{r1}\} \) is a semi-PD-set of \( G_i \) where \( v_j \) and \( v_r \) are semipaired. This contradicts the fact that \( D_i \) is a semi-PD-set of \( G_i \) of minimum cardinality. Hence, \( N_G(v_{r1}) \not\subseteq D_i \).

Let \( v_{r2} \in D_i \setminus N_G(v_{r1}) \). Now update the set \( D_i \) as follows: remove \( v_{j1} \) from \( D_i \), add \( v_{r2} \) in the set \( D_i \), semipair \( v_j \) with \( v_r \) and \( v_{r1} \) with \( v_{r2} \). Clearly, the updated set is also a semi-PD-set of \( G_i \) of minimum cardinality. This proves that there always exists a semi-PD-set \( D_i \) of \( G_i \) such that \( \{v_j, v_r\} \subseteq D_i \) and \( v_j \) is semipaired with \( v_r \) in \( D_i \).

(c) We know that \( \{v_j, v_r\} \subseteq D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\} \). We need to show that \( D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\} \) belongs to \( D_i \). Suppose, to the contrary, that there does not exist any \( D_i \) for which \( D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\} \) is a semi-PD-set of \( G_i \), which is again a contradiction. Hence, \( v_{r} \not\in D_i \) and \( N_G(v_{r}) \not\subseteq D_i \). Suppose \( v_{q} \in N_G(v_{r}) \cap D_i \). Then, update the set \( D_i \) as \( D_i = (D_i \setminus \{v_{p}, v_{p1}\}) \cup \{v_{s}, v_{q}\} \). Note that \( D_i \) is still a semi-PD-set of \( G_i \) of minimum cardinality, and \( |D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\}| < l \), a contradiction.

Case 2. If \( v_{s} \not\in D_i \), then the updated set \( D_i = (D_i \setminus \{v_{p}, v_{p1}\}) \cup \{v_{s}\} \) is also a semi-PD-set of \( G_i \) of minimum cardinality. If \( v_{s} \not\in D_i \) and \( N_G(v_{s}) \not\subseteq D_i \), then the updated set \( D_i = (D_i \setminus \{v_{p}, v_{p1}\}) \cup \{v_{s}\} \) is also a semi-PD-set of \( G_i \), a contradiction. If \( v_{s} \in D_i \) and \( N_G(v_{s}) \not\subseteq D_i \), let \( v_{q} \in N_G(v_{s}) \setminus D_i \). Then, update \( D_i \) as \( D_i = (D_i \setminus \{v_{p}, v_{p1}\}) \cup \{v_{s}, v_{q}\} \). Note that \( D_i \) is still a semi-PD-set of \( G_i \) of minimum cardinality, and \( |D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\}| < l \), a contradiction.

Since both Case 1 and Case 2 produce a contradiction, there exists a semi-PD-set \( D_i \) of \( G_i \) of minimum cardinality, for which the set \( D_i \cap \{v_{s1}, \ldots, v_r, v_{r1}, \ldots, v_i\} \) contains only \( v_j \) and \( v_r \).

**Lemma 5.6.** For \( r < k < j < i \), let \( F(v_{j}) = v_{j} \), \( F(v_{r}) = v_{k} \), \( F(v_{k}) = v_{r} \). If every vertex \( v_{l} \) where \( k < l < j \), is adjacent to at least one vertex in the set \( \{v_{j}, v_{r}\} \), then the following holds.

(a) \( D_i = \{v_{j}, v_{r}\} \) if \( L(v_{r}) = v_{0} \).

(b) \( D_i = \{v_{1}, v_{2}, v_{j}, v_{r}\} \) if \( L(v_{r}) = v_{1} \).

(c) \( D_i = D_s \cup \{v_{j}, v_{r}\} \) if \( L(v_{r}) = v_{s} \) with \( s \geq 2 \).

**Proof.** (a) Clearly \( D_i = \{v_{j}, v_{r}\} \).

(b) From Lemma 5.5, we know that \( \{v_{j}, v_{r}\} \subseteq D_i \). Also, other than \( v_{1} \), all vertices are dominated by the set \( \{v_{j}, v_{r}\} \). Hence, \( D_i = \{v_{1}, v_{2}, v_{j}, v_{r}\} \).

(c) Clearly \( D_s \cup \{v_{j}, v_{r}\} \) is a semi-PD-set of \( G_i \). Hence \( |D_i| \leq |D_s| + 2 \). We also know that there exists a semi-PD-set \( D_i \) of \( G_i \) of minimum cardinality such that \( D_i \cap \{v_{s1}, v_{s2}, \ldots, v_{i}\} = \{v_{j}, v_{r}\} \) (where \( v_{j} \) and \( v_{r} \) are semipaired in \( D_i \)). Hence \( D_i \setminus \{v_{j}, v_{r}\} \subseteq V(G_s) \). Also, \( \{v_{j}, v_{r}\} \) dominates the set
\(\{v_{s+1}, v_{s+2}, \ldots, v_{t}\}\), implying that the set \(\{v_1, v_2, \ldots, v_t\}\) is dominated by the vertices in \(D_i \setminus \{v_j, v_r\}\). Hence, the set \(D_i \setminus \{v_j, v_r\}\) is semi-PD-set of \(G_s\). Therefore, \(|D_s| \leq |D_i| - 2\). This proves that \(|D_i| = |D_s| + 2\). Hence, \(D_i = D_s \cup \{v_j, v_r\}\).

**Lemma 5.7.** For \(r < k < j < i\), let \(F(v_i) = v_j, F(v_j) = v_k, F(v_k) = v_r\), and \(\{v_l \mid k < l < j\} \not\in N_{G_i}(v_r) \cup N_{G_i}(v_j)\). Let \(t = \max\{l \mid k < l < j \text{ and } v_lv_j \not\in E\}\) (assume that such a \(t\) exists). Let \(F(v_t) = v_b\). Then, the following holds.

(a) \(\{v_j, v_b\} \subseteq D_i\).
(b) \(v_j\) is semipaired with \(v_b\) in \(D_i\).
(c) \(D_i \cap \{v_{s+1}, v_{b}, v_{b+1}, \ldots, v_i\} = \{v_j, v_b\}\).

**Proof.** (a) First we show that \(v_j \in D_i\). Suppose \(v_j \notin D_i\). Let \(v_p\) be the vertex dominating \(v_j\) in \(D_i\). Note that \(j < p \leq i\) and \(N_{G_i}(v_p) \subseteq N_{G_i}(v_j)\). Let \(v_q\) be the vertex semipaired with \(v_p\) in \(D_i\). Since \(N[v_p] \subseteq N[v_j]\), any vertex which is within distance 2 from \(v_p\) is also within distance 2 from \(v_j\). We can update \(D_i\) as \(D_i \setminus \{v_p\} \cup \{v_j\}\) with \(v_j\) semipaired with \(v_q\). Hence, \(D_i\) contains \(v_j\). Similarly, we can show that \(D_i\) also contains \(v_b\). So, \(\{v_j, v_b\} \subseteq D_i\).

(b) If \(v_j\) is semipaired with \(v_b\) in \(D_i\), then we are done. Suppose, to the contrary, that \(v_j\) is not semipaired with \(v_b\) in \(D_i\). So, assume that \(v_j\) is semipaired with \(v_p\) and \(v_b\) is semipaired with \(v_q\) in \(D_i\). We consider the four cases based on the values of the indices \(p\) and \(q\).

**Case 1.** \(p > b\) and \(q > b\). Here, \(N_{G_i}(v_p) \subseteq N_{G_i}(v_j) \cup N_{G_i}(v_b)\). Hence, the set \(D_i \setminus \{v_p, v_q\}\) is also a semi-PD-set of \(G_i\), a contradiction.

**Case 2.** \(p < b\) and \(q < b\). Since the distance between \(v_p\) and \(v_j\) is at most 2, \(p \geq r\). If \(q < b\) and \(d_{G_i}(v_q, v_b) \leq 2\), then \(d_{G_i}(v_q, v_p) \leq 2\). So, in the set \(D_i\), \(v_j\) can be semipaired with \(v_b\), and \(v_p\) can be semipaired with \(v_q\).

**Case 3.** \(p > b\) and \(q < b\). Here, \(N_{G_i}(v_p) \subseteq N_{G_i}(v_j) \cup N_{G_i}(v_b)\). If \(N_{G_i}(v_q) \subseteq D_i\), then the set \(D_i \setminus \{v_p, v_q\}\) is also a semi-PD-set of \(G_i\), a contradiction. If \(N_{G_i}(v_q) \nsubseteq D_i\), let \(v_x \in N_{G_i}(v_q) \setminus D_i\). Then update \(D_i\) as \(D_i = (D_i \setminus \{v_p\}) \cup \{v_x\}\), and semipair \(v_q\) with \(v_x\) and \(v_j\) with \(v_b\).

**Case 4.** \(p < b\) and \(q > b\). Since the distance between \(v_p\) and \(v_j\) is at most 2, \(p \geq r\). Also \(N_{G_i}(v_q) \subseteq N_{G_i}(v_j) \cup N_{G_i}(v_b)\). If \(N_{G_i}(v_p) \subseteq D_i\), then the set \(D_i \setminus \{v_p, v_q\}\) is also a semi-PD-set of \(G_i\), a contradiction. If \(N_{G_i}(v_p) \nsubseteq D_i\), let \(v_y \in N_{G_i}(v_p) \setminus D_i\). Then update \(D_i\) as \(D_i = (D_i \setminus \{v_q\}) \cup \{v_y\}\), and semipair \(v_p\) with \(v_y\) and \(v_j\) with \(v_b\).

By the above four cases, there always exists a semi-PD-set \(D_i\) of \(G_i\) of minimum cardinality such that \(v_j\) is semipaired with \(v_b\) in \(D_i\). This completes the proof of part (b).

(c) The proof is similar to the proof of Lemma 5.5(c), and hence is omitted.

**Lemma 5.8.** For \(r < k < j < i\), let \(F(v_i) = v_j, F(v_j) = v_k, F(v_k) = v_r\), and \(\{v_l \mid k < l < j\} \not\in N_{G_i}(v_r) \cup N_{G_i}(v_j)\). Let \(t = \max\{l \mid k < l < j \text{ and } v_lv_j \not\in E\}\) (assume that such a \(t\) exists). Let \(F(v_t) = v_b\). Then, the following holds.

(a) \(D_i = \{v_j, v_b\}\) if \(L(v_b) = v_0\).
(b) \(D_i = \{v_1, v_2, v_j, v_b\}\) if \(L(v_b) = v_1\).
(c) \(D_i = D_s \cup \{v_j, v_b\}\) if \(L(v_b) = v_s\) with \(s \geq 2\).

**Proof.** The proof is similar to the proof of Lemma 5.6, and hence is omitted.

Based on above lemmas, we present an algorithm to compute a minimum semi-PD-set of an interval graph.
Algorithm 1 SEMI-PAIRED-DOM-IG(G)

**Input:** An interval graph $G = (V, E)$ with a left end ordering $\alpha = (v_1, v_2, \ldots, v_n)$ of vertices of $G$.

**Output:** A semi-PD-set $D$ of $G$ of minimum cardinality.

1. $V' = V$;

2. while ($V' \neq \emptyset$) do
   1. Let $i = \max\{k \mid v_k \in V'\}$.
      - if ($F(v_i) = v_1$) then
        1. $D = D \cup \{v_1, v_i\}$;
        2. $V' = V' \setminus \{v_1, v_2, \ldots, v_i\}$;
      - else if ($F(v_i) = v_j$ and $F(v_j) = v_1$ where $j > 1$) then
        1. $D = D \cup \{v_1, v_j\}$;
        2. $V' = V' \setminus \{v_1, v_2, \ldots, v_i\}$;
      - else if ($F(v_i) = v_j$ and $F(v_j) = v_k$ where $k \geq 2$) then
        1. Let $F(v_k) = v_r$.
        2. if ($L(v_r) = v_0$) then
          1. $D = D \cup \{v_j, v_r\}$;
          2. $V' = V' \setminus \{v_1, v_2, \ldots, v_i\}$;
      - else if ($L(v_r) = v_1$) then
        1. $D = D \cup \{v_1, v_j, v_r\}$;
        2. $V' = V' \setminus \{v_1, v_2, \ldots, v_i\}$;
      - else
        1. Let ($L(v_r) = v_s$) where $s \geq 2$.
        2. $D = D \cup \{v_j, v_r\}$;
        3. $V' = V' \setminus \{v_{s+1}, v_{s+2}, \ldots, v_i\}$;

3. Here, we illustrate the algorithm SEMI-PAIRED-DOM-IG, with the help of an example. An interval graph $G$ and its interval model $I$ is shown in Fig. 3.

For the interval graph $G$ given in Fig. 3 the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of minimum cardinality in 3 iterations. Below, we illustrate all the 3 iterations of the algorithm.

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Figure 3: An interval model $I$ and corresponding interval graph $G$.

**INITIALLY**

$V' = \{v_1, v_2, \ldots, v_{16}\}$ and $D = \phi$.

**ITERATION 1**

- $i = 16$ and $F(v_i) = F(v_{16}) = v_{15} \neq v_1$
- $j = 15$ and $F(v_j) = F(v_{15}) = v_{13} \neq v_1$
- $k = 13$ and $F(v_k) = F(v_{13}) = v_{12}$
- $r = 12$ and $\{v_{k+1}, \ldots, v_{j-1}\} = \{v_{14}\} \subseteq N_G[v_j] \cup N_G[v_r]$
  - Since $L(v_r) = L(v_{13}) = v_{10}$ and $s = 10 > 2$,
  - $D = D \cup \{v_{13}, v_{15}\}$ and $V' = V' \setminus \{v_{11}, \ldots, v_{16}\}$.

**AFTER ITERATION 1**

$D = \{v_{13}, v_{15}\}$ and $V' = \{v_1, v_2, \ldots, v_{10}\}$

**ITERATION 2**

- $i = 10$ and $F(v_i) = F(v_{10}) = v_9 \neq v_1$
- $j = 9$ and $F(v_j) = F(v_9) = v_7 \neq v_1$
- $k = 7$ and $F(v_k) = F(v_7) = v_5$
- $r = 5$ and $\{v_{k+1}, \ldots, v_{j-1}\} = \{v_9\} \not\subseteq N_G[v_j] \cup N_G[v_r]$
  - In this case $t = \max\{l \mid k < l < j$ and $v_l \not\in N_G(v_j)\} = 8$ and $F(v_i) = F(v_8) = v_6$ (clearly $b = 6$)
  - Since $L(v_b) = L(v_6) = v_4$ and $s = 4 > 2$,
  - $D = D \cup \{v_6, v_9\}$ and $V' = V' \setminus \{v_5, \ldots, v_{10}\}$.

**AFTER ITERATION 2**

$D = \{v_6, v_9, v_{13}, v_{15}\}$ and $V' = \{v_1, v_2, v_3, v_4\}$

**ITERATION 3**

- $i = 4$ and $F(v_i) = F(v_4) = v_2 \neq v_1$
- $j = 2$ and $F(v_j) = F(v_2) = v_1$, hence
- $D = D \cup \{v_1, v_2\}$ and $V' = V' \setminus \{v_1, v_2, v_3, v_4\}$.

**AFTER ITERATION 3**

$D = \{v_1, v_2, v_6, v_9, v_{13}, v_{15}\}$ and $V' = \phi$

As $V' = \phi$ hence, loop terminates.
Our algorithm returns the set $D = \{v_1, v_2, v_6, v_9, v_{13}, v_{15}\}$, which is a minimum cardinality semi-PD-set of the interval graph $G$.

**Theorem 5.1.** Given a left end ordering of vertices of $G$, the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of $G$ of minimum cardinality in linear-time.

**Proof.** By Lemmas 5.3, 5.4, 5.6 and 5.8 we can ensure that the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of $G$ of minimum cardinality. Also, it can be easily seen that the algorithm can be implemented in $O(m + n)$ time, where $n = |V(G)|$ and $m = |E(G)|$.

### 6 Algorithm for Trees

In this section, we present a linear-time algorithm to compute a minimum cardinality semipaired dominating set in trees.

Let $T = (V, E)$ be a tree, and $\beta = (v_0, v_{n-1}, \ldots, v_1)$ be the BFS ordering of vertices of $T$ starting at a pendant vertex $v_0$. Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the reverse ordering of $\beta$. In our algorithm, we process the vertices in the order they appear in $\alpha$. Let $p(v_i)$ denote the parent of vertex $v_i$. If $v_i$ is the root vertex, we assume $p(v_i) = v_i$.

The idea behind our algorithm is the following. We start with an empty set $D$, an array $L$ and an array $M$. Initially $L[v_i] = 0$ and $M[v_i] = 0$ for all $v_i \in V$. We process the vertices one by one in the order $\alpha = (v_1, v_2, \ldots, v_n)$. During each of the iterations, we update $D$, $L$ and $M$ suitably. During the iterations, $L[v_i] = 0$ if $v_i$ is not selected in $D$, $L[v_i] = 1$ if $v_i$ is selected in $D$ but not semipaired, and $L[v_i] = 2$ if $v_i$ is selected in $D$ and semipaired. Also, $M[v_i] = k$ if $v_k$ need to be semipaired with some vertex in $N_T[v_i] \setminus D$. At the end of the algorithm $D$ becomes a minimum cardinality semi-PD-set of the given tree $T$. At the $i^{th}$ iteration, we process the vertex $v_i$. While processing $v_i$, we update $D$, $L$ and $M$ as follows.

**Case 1:** $i \neq n, n - 1$ and $v_i$ is not dominated by $D$.

**Subcase 1.1:** For every $v_r \in N_T[p(v_i)]$, $M[v_r] = 0$.

Update $D = D \cup \{p(v_i)\}$, $L[p(v_i)] = 1$ and $M[p(v_j)] = j$, where $v_j = p(v_i)$.

**Subcase 1.2:** For some $v_r \in N_T[p(v_i)]$, $M[v_r] \neq 0$.

Let $C = \{v_r \in N_T[p(v_i)] \mid M[v_r] \neq 0\}$. Let $v_k$ be the least index vertex in $C$ and $M[v_k] = m$. Update $L[p(v_i)] = L[v_k] = 2$, and $D = D \cup \{p(v_i)\}$.

**Case 2:** $i \in \{n, n - 1\}$ and $v_i$ is not dominated by $D$.

Update $L[v_{n-1}] = L[v_n] = 2$, and $D = D \cup \{v_{n-1}, v_n\}$.

**Case 3:** $v_i$ is dominated by $D$ and $M[v_i] = 0$.

No Update in $D$, $L$ and $M$ are made.

**Case 4:** $v_i$ is dominated by $D$ and $M[v_i] = k \neq 0$ (that is, $v_k$ need to be semipaired with some vertex in $N_T[v_i] \setminus D$).

**Subcase 4.1:** $L[p(v_i)] = 0$.

Update $L[p(v_i)] = L[v_k] = 2$, $M[v_i] = 0$ and $D = D \cup \{p(v_i)\}$.

**Subcase 4.1:** $L[p(v_i)] = 1$.

This case will not arrive.

**Subcase 4.3:** $L[p(v_i)] = 2$.

Update $L[v_i] = L[v_k] = 2$, $M[v_i] = 0$ and $D = D \cup \{v_i\}$.

**Theorem 6.1.** The **Minimum Semipaired Domination problem** is linear-time solvable in trees.
7 Approximation Algorithm

In this section, we present a greedy approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem in graphs. We also provide an upper bound on the approximation ratio of this algorithm. The greedy algorithm is described as follows.

Algorithm 2 : APPROX-SEMI-PAIRED-DOM-SET(G)

Input: A graph $G = (V, E)$ with no isolated vertex.
Output: A semi-PD-set $D$ of $G$.

begin
  $D = \emptyset$;
  $i = 0$; $D_0 = \emptyset$;
  while $(V \setminus (D_0 \cup D_1 \cup \ldots \cup D_i) \neq \emptyset)$ do
    $i = i + 1$;
    choose two distinct vertices $u, v \in V$ such that $d_G(u, v) \leq 2$ and $|(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \ldots \cup D_{i-1})|$ is maximized;
    $D_i = (N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \ldots \cup D_{i-1})$;
    $D = D \cup \{u, v\}$;
  return $D$;

Lemma 7.1. The algorithm APPROX-SEMI-PAIRED-DOM-SET produces a semi-PD-set of $G$ in polynomial time.

Proof. Clearly, the output set $D$ produced by the algorithm APPROX-SEMI-PAIRED-DOM-SET is a semi-PD-set of $G$. Also, each step of the algorithm can be computed in polynomial time. Hence, the lemma follows.

Lemma 7.2. For each vertex $v \in V$, there exists exactly one set $D_i$ which contains $v$.

Proof. We note that $V = D_0 \cup D_1 \cup \ldots \cup D_{|D|/2}$. Also, if $v \in D_i$, then $v \notin D_j$ for $i < j$. Hence, the lemma follows.

By Lemma 7.2, there exists only one index $i \in [\lfloor |D|/2 \rfloor]$ such that $v \in D_i$ for each $v \in V$. We now define $d_v = \frac{1}{|D_i|}$. Now we are ready to prove the main theorem of this section.

Theorem 7.1. The MINIMUM SEMIPAIRED DOMINATION problem for a graph $G$ with maximum degree $\Delta$ can be approximated with an approximation ratio of $1 + \ln(2\Delta + 2)$.

Proof. For any finite set $X \neq \emptyset$, $\sum_{x \in X} \frac{1}{|X|} = 1$. Hence, we have

$$|D| = 2 \sum_{i=1}^{\lfloor |D|/2 \rfloor} \sum_{w \in D_i} \frac{1}{|D_i|} = 2 \sum_{w \in V} d_w.$$ 

Let $D^* = \{u_1, v_1, u_2, v_2, \ldots, u_{\lfloor |D^*|/2 \rfloor}, v_{\lfloor |D^*|/2 \rfloor}\}$ be a semi-PD-set of $G$ of minimum cardinality, where $u_i$ is semipaired with $v_i$, for each $i \in [\lfloor |D^*|/2 \rfloor]$. Define $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{\lfloor |D^*|/2 \rfloor}, v_{\lfloor |D^*|/2 \rfloor}\}\}$. Note
that for each vertex \( w \), there exists a pair \( \{u_i, v_i\} \in M \) such that \( w \in N_G[u_i] \cup N_G[v_i] \). Hence, the following inequality follows.

\[
\sum_{w \in V} d_w \leq \sum_{\{u_i, v_i\} \in M} \sum_{w \in N_G[u_i] \cup N_G[v_i]} d_w.
\]

Consider a pair \( \{u, v\} \in M \) and define \( z_k = |(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup D_2 \cup \ldots D_k)| \) for \( k \in \{0\} \cup \left[\frac{|D|}{2}\right] \). Clearly, \( z_{k-1} \geq z_k \) for \( k \in \left[\frac{|D|}{2}\right] \). Suppose \( l \) is the smallest index such that \( z_l = 0 \). At the \( k^{th} \) step of the algorithm, \( D_k \) contains \( z_{k-1} - z_k \) vertices from the set \( N_G[u] \cup N_G[v] \). Hence

\[
\sum_{w \in N_G[u] \cup N_G[v]} d_w = \sum_{k=1}^{l} (z_{k-1} - z_k) \cdot \frac{1}{|D_k|}.
\]

At the \( k^{th} \) step of the algorithm, we choose the pair \( u_k, v_k \) such that \( |D_k| = |(N_G[u_k] \cup N_G[v_k]) \setminus (D_0 \cup D_1 \cup \ldots D_{k-1})| \) is maximum. Hence \( |D_k| \geq |(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \ldots D_{k-1})| = z_{k-1} \). Therefore the following inequality follows.

\[
\sum_{w \in N_G[u] \cup N_G[v]} d_w \leq \sum_{k=1}^{l} \frac{z_{k-1} - z_k}{z_{k-1}}.
\]

For all integers \( a < b \), we know that \( H(b) - H(a) \geq \frac{b-a}{b} \), where \( H(b) = \sum_{i=1}^{b} \frac{1}{i} \) and \( H(0) = 0 \). Therefore

\[
\sum_{w \in N_G[u] \cup N_G[v]} d_w \leq \sum_{k=1}^{l} H(z_{k-1}) - H(z_k) = H(z_0) = H(|N_G[u] \cup N_G[v]|) \leq H(2\Delta + 2).
\]

It follows that

\[
|D| = 2 \sum_{w \in V} d_w \leq \sum_{\{u, v\} \in M} H(2\Delta + 2) = |D^*|H(2\Delta + 2) \leq (\ln(2\Delta + 2) + 1) \cdot |D^*|.
\]

This shows that the **Minimum Semipaired Domination** problem can be approximated with an approximation ratio of \( 1 + \ln(2\Delta + 2) \).

\[\square\]

### 8 Lower bound on approximation ratio

To obtain the lower bound on the approximation ratio of the **Minimum Semipaired Domination** problem, we give an approximation preserving reduction from the **Minimum Domination** problem. The following approximation hardness result is already known for the **Minimum Domination** problem.

**Theorem 8.1.** \([3]\) For a graph \( G = (V, E) \), the **Minimum Domination** problem cannot be approximated within \( (1 - \epsilon) \ln |V| \) for any \( \epsilon > 0 \) unless \( \text{NP} \subseteq \text{DTIME} \left( |V|^{O(\log \log |V|)} \right) \).

Now, we are ready to prove the following theorem.
Theorem 8.2. For a graph $G = (V, E)$, the Minimum Semipaired Domination problem cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME} (|V|^{O(\log \log |V|)})$.

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an arbitrary instance of the Minimum Domination problem. Now, we construct a graph $H = (V_H, E_H)$, an instance of the Minimum Semipaired Domination problem in the following way: $V_H = \{v^1_1, v^2_1, w^1_1, w^2_1, z_1 \mid i \in [n]\}$ and $E_H = \{w^1_i v^1_j, w^2_i v^2_j \mid v_j \in N_G(v_i) \cup \{v^1_i v^1_j, v^2_i v^2_j, z_i z_j \mid 1 \leq i < j \leq n\} \cup \{v^1_i z_j, v^2_i z_j \mid i \in [n], j \in [n]\}$. Fig. 4 illustrates the construction of $H$ from $G$.

![Figure 4: An illustration of the construction of $H$ from $G$ in the proof of Theorem 8.2](image)

Let $V^k = \{v^k_i \mid i \in [n]\}$ and $W^k = \{w^k_i \mid i \in [n]\}$ for $k = 1, 2$. Also, assume that $Z = \{z_i \mid i \in [n]\}$. Note that $V^1 \cup Z$ is a clique in $H$. Also $V^2 \cup Z$ is a clique in $H$.

Let $D^*$ denote a minimum dominating set of $G$. Then the set $D' = \{v^1_i, v^2_i \mid v_i \in D^*\}$ is a semi-PD-set of $H$. Hence, if $D^*_{sp}$ denotes a semi-PD-set of $H$ of minimum cardinality, then $|D^*_{sp}| \leq 2|D^*|$. Suppose that the Minimum Semipaired Domination problem can be approximated within a ratio of $\alpha$, where $\alpha = (1 - \epsilon) \ln(|V_H|)$ for some fixed $\epsilon > 0$, by some polynomial time approximation algorithm, say Algorithm A. Next, we propose an algorithm, which we call APPROX-DOMINATING-SET, to compute a dominating set of a given graph $G$ in polynomial time.

**Algorithm 3 : APPROX-DOMINATING-SET(G)**

**Input:** A graph $G = (V, E)$.

**Output:** A dominating set $D$ of $G$.

**begin**

  Initialize $k = 0$;
  Construct the graph $H$;
  Compute a semi-PD-set $D_{sp}$ of $H$ using Algorithm A;
  Define $D'_{sp} = D_{sp}$;
  if $(|D'_{sp} \cap (V^1 \cup W^1)| \leq |D_{sp}|/2)$ then
    $k = 1$;
  else
    $k = 2$;
  for $i = 1$ to $n$ do
    if $(N_H(v^k_i) \cap D'_{sp} = \emptyset)$ then
      $D'_{sp} = (D'_{sp} \setminus v^k_i) \cup \{k\}$;
    $D = \{v_i \mid v^k_i \in D'_{sp} \cap V^k\}$;
  return $D$;

**end**
Next, we show that the set $D$ returned by Algorithm 3 is a dominating set of $G$. If $D_{sp}$ is any semi-PD-set of $H$, then clearly either $|D_{sp} \cap (V^1 \cup W^1)| \leq |D_{sp}|/2$ or $|D_{sp} \cap (V^2 \cup W^2)| \leq |D_{sp}|/2$. Assume that $|D_{sp} \cap (V^k \cup W^k)| \leq |D_{sp}|/2$ for some $k \in [2]$. Now, to dominate a vertex $w^k_i \in W^k$, either $w^k_i \in D_{sp}$ or $v^k_j \in D_{sp}$ where $v^k_j \in N_H(w^k_i)$. If $N_H(w^k_i) \cap D_{sp}$ is an empty set, then we update $D_{sp}$ by removing $w^k_i$ and adding $v^k_j$ for some $v^k_j \in N_H(w^k_i)$, and call the updated set $D'_{sp}$. We do this for each $i$ from 1 to $n$. Note that even for the updated set $D'_{sp}$, we have $|D'_{sp} \cap (V^k \cup W^k)| \leq |D_{sp}|/2$. Also, in the updated set $D'_{sp}$, for each $w^k_i$, $N_H(w^k_i) \cap (D_{sp} \cap V^k)$ is non-empty. Hence $|D'_{sp} \cap V^k| \leq |D_{sp}|/2$ and $D'_{sp} \cap V^k$ dominates $W^k$. Therefore the set $D = \{v_i \mid v_i \in D'_{sp} \cap V^k\}$ is a dominating set of $G$. Also $|D| \leq |D_{sp}|/2$.

By above arguments, we may conclude that the Algorithm 3 produces a dominating set $D$ of the given graph $G$ in polynomial time, and $|D| \leq |D_{sp}|/2$. Hence, $|D| \leq |D_{sp}| = \alpha |D^*| \leq |D_{sp}|/2$.

Also $\alpha = (1 - \epsilon) \ln(|V_H|) \approx (1 - \epsilon) \ln(|V|)$ where $|V_H| = 5|V|$. Therefore the Algorithm APPROX-DOMINATING-SET approximates the minimum dominating set within ratio $(1 - \epsilon) \ln(|V|)$ for some $\epsilon > 0$. By Theorem [8.1] if the minimum dominating set can be approximated within ratio $(1 - \epsilon) \ln(|V|)$ for some $\epsilon > 0$, then NP $\subseteq$ DTIME $(|V|^{O(\log \log |V|)})$. Hence, if the MINIMUM SEMIPAIRED DOMINATION problem can be approximated within ratio $(1 - \epsilon) \ln(|V_H|)$ for some $\epsilon > 0$, then NP $\subseteq$ DTIME $(|V_H|^{O(\log \log |V_H|)})$. This proves that the MINIMUM SEMIPAIRED DOMINATION problem cannot be approximated within $(1 - \epsilon) \ln(|V_H|)$ unless NP $\subseteq$ DTIME $(|V_H|^{O(\log \log |V_H|)})$.

\section{Conclusion}

In this paper, we initiate the algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem. We have resolved the complexity status of the problem for bipartite graphs, chordal graphs and interval graphs. We have proved that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite graphs and split graphs. We also present a linear-time algorithm to compute a semi-PD-set of minimum cardinality for interval graphs and trees. A $1 + \ln(2\Delta + 2)$ approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem in general graphs is given, and we prove that it cannot be approximated within any sub-logarithmic factor. It will be interesting to study better approximation algorithms for this problem for bipartite graphs, chordal graphs and other important graph classes.

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