SUPG stabilization for the nonconforming virtual element method for advection-diffusion-reaction equations

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Abstract

We present the design, convergence analysis and numerical investigations of the nonconforming virtual element method with Streamline Upwind/Petrov-Galerkin (VEM-SUPG) stabilization for the numerical resolution of convection-diffusion-reaction problems in the convective-dominated regime.

According to the virtual discretization approach, the bilinear form is split as the sum of a consistency and a stability term. The consistency term is given by substituting the functions of the virtual space and their gradients with their polynomial projection in each term of the bilinear form (including the SUPG stabilization term). Polynomial projections can be computed exactly from the degrees of freedom. The stability term is also built from the degrees of freedom by ensuring the correct scalability properties with respect to the mesh size and the equation coefficients.

The nonconforming formulation relaxes the continuity conditions at cell interfaces and a weaker regularity condition is considered involving polynomial moments of the solution jumps at cell interfaces. Optimal convergence properties of the method are proved in a suitable norm, which includes contribution from the advective stabilization terms. Experimental results confirm the theoretical convergence rates.

Keywords: Virtual Element Methods, Advection-diffusion-reaction problem, SUPG, stability, convergence

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1. Introduction

The virtual element method (VEM) was proposed in [1] as a variational reformulation of the nodal mimetic finite difference (MFD) method [2–5] for solving diffusion problems on unstructured polygonal meshes. A survey on the MFD method can be found in the review paper [6] and the research book [7]. The VEM inherits the great flexibility of the MFD method with respect to the admissible meshes, and, despite its introduction dates back to a few years ago a huge amount of development has taken place, see, for example, [8–31]. We emphasize that the VEM is not the only existing way to treat partial differential equations numerically on unstructured meshes. Other methods or families of methods that are available from the literature include the polygonal/polyhedral finite element method (PFEM) [32, 33], the BEM-based FEM [34, 35], the finite volume methods [36, 37], hybrid high-order (HHO) method [38], the discontinuous Galerkin (DG) method [39, 40], and the hybridized discontinuous Galerkin (HDG) method [41]. Many of these methods are also part of the Gradient Scheme framework recently proposed by [42, 43]. Moreover, the connection between VEM and finite elements on polygonal/polyhedral meshes is thoroughly investigated in [44–46], between VEM and BEM-based FEM method in [47].

The virtual element method is a finite element method, but is dubbed virtual because its formulation does not require the explicit knowledge of a set of shape functions and gradients of shape functions to compute the bilinear forms, e.g., mass and stiffness matrices. The global approximation space is defined over the whole domain by gluing together local elemental spaces under some regularity constraint. Each elemental space is formed by the solutions of a local Poisson problem with a polynomial right-hand side and nonhomogeneous polynomial Dirichlet (or Neumann) boundary conditions. Clearly, a subspace of polynomials up to a given degree always belongs by construction to each elemental space. The remarkable fact is that we can compute exactly the projections of the virtual functions and their first derivatives onto such polynomials by using only the degrees of freedom. Therefore, a straightforward strategy to approximate the bilinear forms is to substitute the shape functions and their derivatives in their arguments with their polynomial projections. This approach yields the so-called consistency term, to which we add a stability term that ensures the nonsingularity of the resulting discretization. The stability term is designed to be easily computable from the degrees of freedom.

The VEM was originally formulated in [1] as a conforming FEM for the Poisson problem. It was later extended to convection-reaction-diffusion problems with variable coefficients in [12, 48]. Meanwhile, the nonconforming formulation for diffusion problems was proposed in [49] as the finite element reformulation of [50] and later extended to general elliptic problems [51], Stokes problem [52], and the biharmonic equation [53, 54].

The two major differences between the conforming and nonconforming formulations are:

(i) at the elemental level the virtual space is formed by the solution of a Poisson problem with Neumann boundary conditions;

(ii) at the global level we relax the interelement conformity requirement, and the definition of the global discrete space just relays on some form of weaker regularity according to [55].
Nonconforming finite element spaces were historically proposed to approximate the velocity field of the Stokes equations on triangular meshes [55]. The functions in these finite element spaces are piecewise polynomials of degree \( k = 1 \) [55], \( k = 2 \) [56], \( k = 3 \) [57], and \( k > 3 \) [58–60]. In such formulations, continuity is required only at a discrete set of special points located at cell interfaces, which are the roots of the one-dimensional \( k^{th} \)-order Legendre polynomials defined over each edge, i.e., the nodes of the Gauss-Legendre quadrature rule of order \( k \). This minimal continuity requirement ensures the optimal convergence rate; see, for instance, [55]. Attempts to extend non-conforming finite elements to quadrilaterals, tetrahedra and hexahedra are found in [61–63]. A major issue of the nonconforming formulations is that they may strongly depend on the parity of the underlying polynomial space, the geometric shape of the element and its spatial dimensionality (2-D or 3-D). For example, on triangles the nonconforming finite element space for even \( k \geq 2 \) in [56, 58, 59] must be enriched by a one-dimensional subspace generated by a bubble function. Also, the definition of nonconforming spaces is substantially different from 2D and 3D and requires a simple geometric shape for the element (e.g., a simplex, a quadrilateral, or an hexahedral cell), and also differs from 2D to 3D. Instead, the nonconforming virtual element space proposed in [49] has the same construction for every \( k \) regardless of the parity, the space dimension, and the elemental geometric shape.

In the case of the convection-dominated regime, a stabilization must be included in the variational formulation to deal with high Péclet number situations. In finite element approximations, different strategies have been designed to such purpose, as, for example, local projections [64], bubble functions [65, 66] the SUPG method [67–72]. The SUPG stabilization in the conforming virtual element formulation was previously considered in [73]. The main goal of this work is the development of the nonconforming formulation with SUPG stabilization suitable to solve convection-dominated transport problems with a moderate reaction term. In such a situation the SUPG stabilization parameters becomes dependent on a local Péclet number and a local Karlovitz-like number. In case of a vanishing reaction the SUPG stabilization parameter converges to its expected classical definition. We prove the robustness of the method with respect to high Péclet numbers when the problem coefficients are constants and a conforming formulation is considered, whereas a weak dependence on the Péclet number is observed, due to the non-consistency of the VEM bilinear form [1] when the coefficients are variable or a non conforming formulation is considered. The presented analysis is also valid for SUPG-stabilized conforming virtual elements as presented in [73].

The outline of the paper is as follows. In Section 2 we introduce the mathematical model of the convection-reaction-diffusion problem. In Section 3 we present the non-conforming VEM with the SUPG stabilization for the convection-dominated regime. In Section 4 we carry out the convergence analysis and derive optimal a priori error estimates. In Section 5 we show the performance of the method on a set of representative problems. In Section 6 we offer our final remarks and conclusions.

1.1. Notation

The notation throughout the paper is as follows: \((\cdot, \cdot)\) and \(\|\cdot\|\) denote the \(L^2(\Omega)\) scalar product and norm, and \((\cdot, \cdot)_\omega\) and \(\|\cdot\|_\omega\) denote the \(L^2(\omega)\) scalar product and norm defined on the subdomain \(\omega \subseteq \Omega\); \(\|\cdot\|_\alpha\) and \(|\cdot|_\alpha\) denote the \(H^\alpha(\Omega)\) norm and semi-norm; \(\|\cdot\|_{\alpha, \omega}\) and \(|\cdot|_{\alpha, \omega}\) denote the \(H^\alpha(\omega)\) norm and semi-norm; \(\|\cdot\|_{W^k(\omega)}\) and \(|\cdot|_{W^k(\omega)}\) denote the
\[ W^q_p(\omega) \] norm and semi-norm, where \( p \geq 1 \) is the Lebesgue regularity index and \( q \) is the order of the Sobolev space. Moreover, \( \mathbb{P}_k(\omega) \) denotes the space of polynomial functions of degree up to the integer number \( k \geq 0 \) that are defined on the \( d \)-dimensional subset \( \omega \subseteq \Omega \) with \( d = 1, 2, 3 \). If \( \mathcal{T}_h \) is a partitioning of \( \Omega \) in a set of non-overlapping polytopal elements \( E \), i.e., the mesh, (for the formal definition see Section 3.1), by \( \mathbb{P}_k(\mathcal{T}_h) \) we denote the space of polynomial functions \( \text{of degree up to the integer number } k \geq 0 \) that are defined on the \( \mathcal{T}_h \). If \( T \) is a partitioning of \( \Omega \) in a set of non-overlapping polytopal elements \( E \), i.e., the mesh, (for the formal definition see Section 3.1), by \( \mathbb{P}_k(T_h) \) we denote the space of discontinuous functions defined on \( \Omega \) whose restriction to any element \( E \) is a polynomial of degree less than or equal to \( k \); hence, \( p \in \mathbb{P}_k(T_h) \) iff \( p|_E \in \mathbb{P}_k(E) \). Finally, \( H^t(T_h) \) for any \( t \geq 1 \) is the broken Sobolev space of globally \( L^2 \)-integrable functions on \( \Omega \) whose restriction to any mesh element \( E \) of the mesh \( T_h \) belongs to \( H^t(E) \); formally, we can write that
\[
H^t(T_h) := \left\{ v \in L^2(\Omega) : v|_E \in H^t(E), \forall E \in T_h \right\}. \tag{1}
\]

To ease the notation, since these spaces contain discontinuous functions, in the following we will intend all norms and seminorms to be “broken” on the mesh. For example:
\[
\|\nabla v\| = \left( \sum_{E \in T_h} \|\nabla v\|^2_E \right)^{\frac{1}{2}}.
\]

Furthermore, we will use the symbol \( C \) to denote a generic constant independent of the mesh size and the problem data \( K, \beta \) and \( \gamma \). In the estimates this constant may have a different value for each occurrence.

2. The variational formulation

Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \) be a polytopal domain with boundary \( \partial \Omega \) and consider the convection-diffusion-reaction problem:
\[
-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \gamma u = f \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega, \tag{2}
\]

We assume that \( K \in [L^\infty(\Omega)]^{d \times d} \) is a strongly elliptic and symmetric tensor almost everywhere (a.e.) on \( \Omega \). Hence, there exist two positive constant \( \kappa \) and \( \kappa^* \) such that \( \kappa \xi \cdot \xi \leq \xi \cdot K(x) \xi \leq \kappa^* \xi \cdot \xi \) for every \( \xi \in \mathbb{R}^d \) and almost every \( x \in \Omega \). We denote \( C_\kappa = \kappa^*/\kappa \). Moreover, we assume that \( \beta \in [L^\infty(\Omega)]^d \) with \( \nabla \cdot \beta = 0 \) and \( \gamma \in L^\infty(\Omega) \) such that \( \inf_{x \in \Omega} \gamma(x) = \gamma_0 \geq 0 \). To ease the exposition, we present the virtual element formulation and the convergence analysis assuming homogeneous Dirichlet boundary conditions. However, all the results presented in this paper can readily be extended to more general situations.

Consider the bilinear form \( B: H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R} \) defined by
\[
B(w, v) := (K \nabla w, \nabla v) + (\beta \cdot \nabla w, v) + (\gamma w, v) \quad \forall w, v \in H^1_0(\Omega), \tag{3}
\]
and the linear functional \( F: H^1_0(\Omega) \to \mathbb{R} \) defined by
\[
F(v) := (f, v) \quad \forall v \in H^1_0(\Omega).
\]
The variational formulation of (2) reads as: Find \( u \in H_0^1(\Omega) \) such that
\[
B(u, v) = F(v) \quad \forall v \in H_0^1(\Omega).
\] (4)

The bilinear form \( B \) is coercive and bounded, and the variational problem (4) has a unique solution in view of the Lax-Milgram lemma.

3. The virtual element formulation

Hereafter, we consider only the case for \( d = 2 \). However, the nonconforming virtual element formulation is almost the same for \( d = 2 \) and 3, the main substantial difference being necessarily in the mesh assumptions that for \( d = 3 \) must also consider a star-shaped condition on the faces. Therefore, most of the results presented in the next sections can easily be generalized to the three-dimensional case with minor or no changes at all.

3.1. General assumptions

Let \( \{ T_h \} \) be a sequence of meshes of \( \Omega \), i.e., a sequence of non-overlapping polygonal partitions of the domain \( \Omega \). Each \( T_h \) is labeled by the subscript \( h \), the maximum diameter of its polygonal elements \( E \). The polygonal elements can have a different number of edges and hanging node-like configurations are possible with nodes placed on an edge and forming a flat angle. We denote the set of all the mesh edges \( e \) of the polygonal cells \( T_h \) by \( E_h \). We also distinguish between the subset of internal edges \( E_h^{int} \) and the subset of the boundary edges \( E_h^{bnd} \); clearly, \( E_h = E_h^{int} \cup E_h^{bnd} \).

We assume that the members of the sequence \( \{ T_h \} \) satisfy the following regularity assumptions: There exists a global constant \( \rho > 0 \) such that for each mesh \( T_h \):

(i) every polygon \( E \in T_h \) is star-shaped with respect to a ball whose radius is greater than or equal to \( \rho h_E \), where \( h_E = \max_{x, y \in E} \| x - y \| \) is the element diameter;
(ii) \( \forall E \in T_h \), each side \( e \) of \( E \) is such that \( h_e \geq \rho h_E \), where \( h_e \) is the length of \( e \);

Remark 1. Assumption (i) implies that each element is simply connected. Assumption (ii) implies that the number of sides of each polygon of the mesh is uniformly bounded over the mesh sequence.

The restriction of \( K \) to any element \( E \in T_h \) is still a strongly elliptic tensor and its spectrum can be locally bounded by using two constants \( K_{\text{E}}^{\text{int}} \) and \( K_E \), so that for any vector-valued field \( \xi(x) \) defined on \( E \) it holds that
\[
K_{\text{E}}^{\text{int}} \xi(x) \cdot \xi(x) \leq K(x)\xi(x) \leq K_E \xi(x) \cdot \xi(x) \quad \forall x \in E.
\] (5)

We will find convenient for the next theoretical developments to assume that the inequalities \( 0 < \kappa^* \leq K_{\text{E}}^{\text{int}} \leq K_E \leq \kappa^* \) holds true for every mesh element \( E \). Since \( K \) is represented by a symmetric and positive definite matrix we consider the decomposition \( K = (\sqrt{R})^T \sqrt{R} \) and we write \( \langle K \nabla v, \nabla v \rangle_E = \langle \sqrt{R} \nabla v, \sqrt{R} \nabla v \rangle_E = \| \sqrt{R} \nabla v \|^2_E \) for any sufficiently regular function \( v \). Therefore, setting \( \xi = \nabla v \) in (5) yields
\[
K_{\text{E}}^{\text{int}} \| \nabla v \|^2_E \leq \| \sqrt{R} \nabla v \|^2_E \leq K_E \| \nabla v \|^2_E.
\]
We will use this relation extensively in the analysis of the next sections. For each element $E \in T_h$ we also set

$$
\beta_E := \sup_{x \in E} \|\beta(x)\|_{\mathbb{R}^2}, \quad \gamma_E := \|\gamma\|_{\infty,E}.
$$

Let $k \geq 0$ be an integer number and $\alpha = (\alpha_1, \alpha_2)$ a two-dimensional multi-index of order $|\alpha| = \alpha_1 + \alpha_2 \leq k$. The polynomial space $\mathbb{P}_k(E)$ is spanned by the monomials $m_\alpha \in \mathcal{M}_k(E)$ defined as

$$
m_\alpha(x) := \frac{(x - x_E)^{\alpha}}{h_E^{\alpha}} \quad \forall x \in E,
$$

where $x_E$ is the center of the ball with respect to which $E$ is star-shaped. Similarly, $\mathbb{P}_k(e)$, the space of polynomials of degree $k$ defined on edge $e$, is spanned by the monomials $m_\alpha(e) := (\xi - x_e)^{\alpha} / h_e^{\alpha} \in \mathcal{M}_k(e)$ for $0 \leq \alpha \leq k$, where $\xi$ is a local coordinate defined on $e$, $x_e$ the midpoint of $e$, and $h_e$ the length of $e$.

In the formulation of the method we will make use of the elliptic projection operator $\Pi^E_\alpha : H^1(T_h) \rightarrow \mathbb{P}_k(T_h)$, whose restriction to each element $E$ is the solution of the local problem:

$$
\begin{align*}
(\nabla \Pi^E_\alpha v, \nabla p)_E &= (\nabla v, \nabla p)_E \quad \forall p \in \mathbb{P}_k(E) \\
(\Pi^E_\alpha v, 1)_{\partial E} &= (v, 1)_{\partial E} \quad \text{if } k = 1, \\
(\Pi^E_\alpha v, 1)_E &= (v, 1)_E \quad \text{if } k > 1.
\end{align*}
$$

We will also consider the $L^2$-projection operator $\Pi^E_1 : H^1(T_h) \rightarrow \mathbb{P}_1(T_h)$ whose restriction to each element $E$ is the $L^2$-projection onto $\mathbb{P}_1(E)$. A crucial property of these projection operators, which will be discussed in the next section (see Remark 2), is that they are computable on the functions of the virtual element space using only their degrees of freedom.

### 3.2. The local nonconforming virtual element space

The local nonconforming virtual element space of order $k \geq 1$ is defined as follows:

$$
V^E_k := \left\{ v_h \in H^1(E) : \Delta v_h \in \mathbb{P}_k(E), \frac{\partial v_h}{\partial n_e} \in \mathbb{P}_{k-1}(e) \, \forall e \subset \partial E, \right. \\
\left. (v_h, p)_E = (\Pi^E_k v_h, p)_E \, \forall p \in \mathbb{P}_k(E) / \mathbb{P}_{k-2}(E) \right\},
$$

where $\mathbb{P}_k(E) / \mathbb{P}_{k-2}(E)$ is the subspace of $\mathbb{P}_k(E)$ of the polynomials that are $L^2$-orthogonal to $\mathbb{P}_{k-2}(E)$ (or, alternatively, the polynomials whose degree is exactly $k - 1$ and $k$), and for $k = 1$ we conventionally take $\mathbb{P}_{-1}(E) = \{0\}$. The definition of $V^E_k$ is based on the enhancement strategy that was introduced in [43] for the conforming case and extended to the nonconforming case in [51]. From the definition above it follows immediately that $\mathbb{P}_k(E)$ is a linear subspace of $V^E_k$.

A function $v_h \in V^E_k$ is uniquely identified by the following set of degrees of freedom:
Figure 1: Degrees of freedom of a hexagonal cell for \(k = 1, 2, 3, 4\); edge moments are marked by a circle; cell moments are marked by a square.

- for \(k \geq 1\), the moments of \(v_h\) of order up to \(k - 1\) on each mesh interface \(e\):
  \[
  \frac{1}{|e|} \int_e v_h m_\alpha d\xi \quad \forall m_\alpha \in M_{k-1}(e); \tag{7}
  \]

- for \(k > 1\), the moments of \(v_h\) of order up to \(k - 2\) inside element \(E\):
  \[
  \frac{1}{|E|} \int_E v_h m_\alpha dx \quad \forall m_\alpha \in M_{k-2}(E), \tag{8}
  \]

The unisolvency of these degrees of freedom is proved in [49]. A counting argument shows that the cardinality of this set of degrees of freedom, which is also the dimension of \(V_{E}^{h}\), is equal to \(n_E(k - 1) + k(k - 1)/2\), where \(n_E\) is the number of edges of \(E\). The degrees of freedom for an hexagonal cell are shown in Figure 1.

Remark 2. The elliptic projection \(\Pi^k v_h\) is computable from the degrees of freedom of \(v_h\). In fact, an integration by parts of the right-hand side of (6) yields:

\[
(\nabla v_h, \nabla p)_E = -(v_h, \Delta p)_E + \sum_{e \in \partial E} (v_h, n_e \cdot \nabla p)_e,
\]

The terms on the right can be expressed by using the \((k - 2)\)-order moments of \(v_h\) inside \(E\) and the \((k - 1)\)-order moments of \(v_h\) on each edge \(e \in \partial E\) and are thus computable. A similar argument shows that also \(\Pi^k v_h \nabla v_h\) are computable from the degrees of freedom of \(v_h\).

3.3. Global nonconforming virtual element spaces

For the construction of the global virtual element spaces we introduce the nonconforming functional space

\[
H^1_{k}^{nc}(\mathcal{T}_h) := \left\{ v \in H^1(\mathcal{T}_h) : \int_e [v] q \, d\xi = 0 \quad \forall q \in P_{k-1}(e) \quad \forall e \in \mathcal{E}_h \right\},
\]

where \([\cdot]\) denotes the *jump operator* \([\cdot]\) across a mesh interface, which is defined as follows. If \(e\) is an internal edge, we fix a unique unit normal vector \(n_e\) and we set \([v] := v^+ - v^-\), where \(v^\pm\) are the traces of \(v\) on \(e\) from within the two elements \(E^\pm\).
sharing the edge, being $E^+$ the element for which $\hat{n}_e$ is pointing outward. If $e$ is a boundary edge, $\hat{n}_e$ is orthogonal to $e$ and pointing out of the computational domain $\Omega$ and $[v] := v^+$.

Finally, the global nonconforming virtual element space of order $k$ is defined by
\[ V_h := \left\{ v_h \in H^{k,nc}_0(T_h) : v_h|_E \in V^E_h \quad \forall E \in T_h \right\}. \tag{9} \]
Each function $v_h$ of $V_h$ is uniquely characterized by:
- for $k \geq 1$, the moments of order up to $k - 1$ on each internal mesh edge $e \in E^\text{int}_h$:
  \[ \frac{1}{|e|} \int_{e} v_h m_{\alpha} d\xi \quad \forall m_{\alpha} \in \mathcal{M}_{k-1}(e); \tag{10} \]
- for $k > 1$, the moments of order up to $k - 2$ inside each element $E \in T_h$:
  \[ \frac{1}{|E|} \int_{E} v_h m_{\alpha} d\mathbf{x} \quad \forall m_{\alpha} \in \mathcal{M}_{k-2}(E). \tag{11} \]

The unisolvency of these degrees of freedom in $V_h$ is a direct consequence of the unisolvency of the local degrees of freedom introduced in section 3.2 and the definition of the nonconforming space $H^{k,nc}_0(T_h)$, cf. [19].

### 3.4. SUPG-VEM formulation

The discretization of the variational formulation (4) may lead to instabilities when the convective term $(\beta \cdot \nabla w, v)$ is dominant with respect to the diffusive term $(K \nabla w, \nabla v)$. Here we consider also a moderate reaction term that we assume not to be source of instabilities. In this section we recast the classical Streamline Upwind Petrov Galerkin (SUPG) approach [68] in the framework of the nonconforming VEM, showing that the optimal order of convergence can be preserved. To this end, we assume that $K \in \left[ W^{1,\infty}_0(\Omega) \right]^{d \times d}$.

Then, we introduce the functional space
\[ V := \left\{ v \in H^1_0(\Omega) : \Delta v \in L^2(E) \quad \forall E \in T_h \right\}, \tag{12} \]
the bilinear form $B_{\text{supg}} : V \times H^1_0(\Omega) \to \mathbb{R}$ given by
\[ B_{\text{supg}}(w, v) := a(w, v) + b(w, v) + c(w, v) + d(w, v), \tag{13} \]
where
\[ a(w, v) := \sum_{E \in T_h} (K \nabla w, \nabla v)_E + \tau_E (\beta \cdot \nabla w, \beta \cdot \nabla v)_E, \tag{14} \]
\[ b(w, v) := \frac{1}{2} \sum_{E \in T_h} \left[ (\beta \cdot \nabla w, v)_E - (w, \beta \cdot \nabla v)_E \right], \tag{15} \]
\[ c(w, v) := \sum_{E \in T_h} (\gamma w, v + \tau_E \beta \cdot \nabla v)_E, \tag{16} \]
\[ d(w, v) := -\sum_{E \in T_h} \tau_E (\nabla \cdot (K \nabla w), \beta \cdot \nabla v)_E. \tag{17} \]
Furthermore, let \( F_{\text{supg}} : H^1_0(\Omega) \rightarrow \mathbb{R} \) be the linear functional given by
\[
F_{\text{supg}}(v) = (f, v) + \sum_{E \in T_h} \tau_E (f, \beta \cdot \nabla v)_E. \tag{18}
\]
The real positive factor \( \tau_E \) is the local \textit{SUPG parameter} and is discussed in section 3.5.

The SUPG variational formulation of problem 2 reads as: Find \( u \in V \) such that
\[
B_{\text{supg}}(u, v) = F_{\text{supg}}(v) \quad \forall v \in H^1_0(\Omega). \tag{19}
\]

Remark 3. Under the assumptions of Section 2, the bilinear term \( b \) in (3) that corresponds to the convective flux is equivalent to the skew-symmetric term \( b \) in (15).

Remark 4. By introducing the matrix \( K_{\beta,E} = K + \tau_E \beta^T \), the bilinear form \( a \) in (14) can be reformulated as:
\[
a(w, v) := \sum_{E \in T_h} a^E(w, v) = \sum_{E \in T_h} (K_{\beta,E} \nabla w, \nabla v)_E. \tag{20}
\]
Since matrix \( K_{\beta,E} \) is positive definite, we can use the decomposition \( K_{\beta,E} = \sqrt{K_{\beta,E}} \sqrt{K_{\beta,E}} \) and prove that the bilinear form is continuous, i.e,
\[
a^E(w, v) \leq \| \sqrt{K_{\beta,E}} \nabla w \|_E \| \sqrt{K_{\beta,E}} \nabla v \|_E, \tag{21}
\]
which holds for every pair of nonconforming functions \( v, w \).

The SUPG-stabilized virtual element approximation of (4) reads as: Find \( u_h \in V_h \) such that
\[
B_{\text{supg},h}(u_h, v_h) = F_{\text{supg},h}(v_h) \quad \forall v_h \in V_h, \tag{22}
\]
where the bilinear form \( B_{\text{supg},h} : V_h \times V_h \rightarrow \mathbb{R} \) and the right-hand side \( F_{\text{supg},h} : V_h \rightarrow \mathbb{R} \) are the virtual element approximation of \( B_{\text{supg}} \) and \( F_{\text{supg}} \), respectively. The bilinear form \( B_{\text{supg},h} \) is given by
\[
B_{\text{supg},h}(w_h, v_h) := a_h(w_h, v_h) + b_h(w_h, v_h) + c_h(w_h, v_h) + d_h(w_h, v_h) \tag{23}
\]
for any \( w_h, v_h \in V_h \), where
\[
a_h(w_h, v_h) := \sum_{E \in T_h} \left( (\Pi^0_{k-1} \nabla w_h, \Pi^0_{k-1} \nabla v_h)_E + \tau_E (\beta \cdot \Pi^0_{k-1} \nabla w_h, \beta \cdot \Pi^0_{k-1} \nabla v_h)_E 
\right.
\left. + S^E \left( (I - \Pi^0_{k-1}) w_h, (I - \Pi^0_{k-1}) v_h \right) \right), \tag{24}
\]
\[
b_h(w_h, v_h) := \sum_{E \in T_h} \frac{1}{2} \left( (\beta \cdot \Pi^0_{k-1} \nabla w_h, \Pi^0_{k-1} v_h)_E - (\Pi^0_{k-1} w_h, \beta \cdot \Pi^0_{k-1} \nabla v_h)_E \right), \tag{25}
\]
\[
c_h(w_h, v_h) := \sum_{E \in T_h} \left( \gamma \Pi^0_{k-1} w_h, \Pi^0_{k-1} v_h + \tau_E \beta \cdot \Pi^0_{k-1} \nabla v_h \right)_E, \tag{26}
\]
\[
d_h(w_h, v_h) := - \sum_{E \in T_h} \tau_E (\nabla \cdot (K \Pi^0_{k-1} \nabla w_h), \beta \cdot \Pi^0_{k-1} \nabla v_h)_E. \tag{27}
\]
the local VEM stabilization term in $a_h(w_h, v_h)$ is given by

$$S^E((I - \Pi^0_{k-1}) w_h, (I - \Pi^0_{k-1}) v_h) := (K_E + \tau_E \beta_E^2) S^E_*((I - \Pi^0_{k-1}) w_h, (I - \Pi^0_{k-1}) v_h),$$

(28)

where $S^E_*((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h)$ is such that there exist two constants $\sigma^*, \sigma_* > 0$, independent of $h$ and the problem parameters satisfying, $\forall v_h \in V_h$,

$$\sigma_* \| \nabla (I - \Pi^0_{k-1}) v_h \|^2_E \leq S^E_*((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) \leq \sigma^* \| \nabla (I - \Pi^0_{k-1}) v_h \|^2_E.$$

(29)

Moreover, the linear functional $F_{\text{supg}, h}(v_h)$ is given by

$$F_{\text{supg}, h}(v_h) = (J, \Pi^0_{k-1} v_h) + \sum_{E \in T_h} \tau_E \left( f, \beta \cdot \Pi^0_{k-1} \nabla v_h \right)_E,$$

(30)

for any $v_h \in V_h$. In view of Remark 4, we can define the local bilinear form

$$a^E_h(w_h, v_h) = (K_{\beta, E} \Pi^0_{k-1} \nabla w_h, \Pi^0_{k-1} \nabla v_h)_E$$

such that

$$a_h(w_h, v_h) = \sum_{E \in T_h} \left( a^E_h(w_h, v_h) + S^E((I - \Pi^0_{k-1}) w_h, (I - \Pi^0_{k-1}) v_h) \right).$$

Notice that, by (21), (28) and (29), the stabilization term $S^E: V_h \times V_h \rightarrow \mathbb{R}$ satisfies

$$S^E((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) \geq \sigma_* (K_E + \tau_E \beta_E^2) \| \nabla v_h - \nabla \Pi^0_{k-1} v_h \|^2_E \geq \sigma_* (K_E + \tau_E \beta_E^2) \| \nabla v_h - \Pi^0_{k-1} \nabla v_h \|^2_E \geq \sigma_* \sqrt{K_{\beta, E}} \| \nabla v_h - \Pi^0_{k-1} \nabla v_h \|^2_E,$$

(31)

being $\| \nabla v_h - \Pi^0_{k-1} \nabla v_h \|^2_E \leq \| \nabla v_h - \nabla \Pi^0_{k-1} v_h \|^2_E$. According to [31], a possible choice for $S^*_E$ is given by

$$S^*_E((I - \Pi^0_{k-1}) w_h, (I - \Pi^0_{k-1}) v_h) = \sum_{i=1}^{N_E} \chi_i \left( (I - \Pi^0_{k-1}) w_h \right) \chi_i \left( (I - \Pi^0_{k-1}) v_h \right),$$

(32)

where $N_E$ is the number of degrees of freedom on the element $E$ and $\chi_i$ is the operator that selects the $i$-th degree of freedom.

The effect of the SUPG stabilization in the VEM stabilization is reflected by the term $\tau_E \beta_E^2$ that appears in the local coefficient multiplying $S^E$ in definition (28).

3.5. The SUPG parameter $\tau_E$

According to [68] [73], the stability parameter $\tau_E$ when there is no reaction term is defined by

$$\tau_E = \frac{h_E}{2 \beta_E} \min \{ \text{Pe}_E, 1 \}, \quad \text{where} \quad \text{Pe}_E := \frac{m_k \beta_E h_E}{2 K_E}.$$  

(33)
and

$$m^E_k := \begin{cases} \frac{1}{3} & \text{if } \nabla \cdot (K \nabla v_h) = 0 \forall v_h \in V^E_h, \\ 2\tilde{C}_k & \text{otherwise}, \end{cases}$$

is the mesh Péclet number of $E$; $\tilde{C}_k^E$ is the biggest constant number satisfying the following inverse inequality:

$$\tilde{C}_k^E h^2_E \|\nabla \cdot (K \nabla v_h)\|_E^2 \leq \|K \nabla v_h\|_E^2 \forall v_h \in V^E_h. \quad (34)$$

A proof of such a local inverse inequality for the virtual element space $V^E_h$ is provided in [9, Lemma 10] for constant $K$ and any function with polynomial laplacian under the current mesh regularity assumptions. For a nonconstant $K$, using standard manipulations we obtain (34) with a constant $\tilde{C}_k^E$ that may depend on the variations of $K$ on the element.

In [68, 73] no reaction term was considered. Since here we have such a term we need to modify the definition of $\tau_E$ by adding a constraint that guarantees the coercivity of $B_{supg}$ and $B_{supg,h}$. As in the proof of Lemma 1 we need that $1 - \frac{\gamma_E}{2} \geq C > 0$, we may assume that there exists a constant $C_{\tau} \in (0, 1)$ such that

$$\tau_E := \min \left\{ \frac{\tilde{C}_k^E h^2_E}{K_E}, \frac{h_E}{2\beta_E}, \frac{C_{\tau}}{\gamma_E} \right\}. \quad (35)$$

Finally, we introduce the local *Karlovitz number*, i.e., the dimensionless parameter associated with each mesh element $E$,

$$K_{aE} := \frac{2\beta_E C_{\tau}}{h_E \gamma_E},$$

and we redefine $\tau_E$ as

$$\tau_E = \frac{h_E}{2\beta_E} \min \{ Pe_E, 1, K_{aE} \}.$$

The comparison between $Pe_E$ and $K_{aE}$ determines whether the value of $\tau_E$ is dominated by the convective term, the diffusive term, or the reactive term. The two curves in Figure 2 show the behaviour of $\tau_E$ for two possible choices of the problem coefficients. The curves are parametrized by the diameter $h_E$ with decreasing values from left to right along each curve. We see that for small values of $h_E$ (right-most part of each curve), $\tau$ falls in the diffusive regime, possibly passing through the convective regime, as expected.

4. Error Analysis

In the following we assume that the problem is well written in the non-dimensional way, and consequently $K_E \leq 1$, $h_E \leq 1$, $\beta_E = O(1)$, $\gamma_E \leq O(1)$. Let $h := \max_{E \in T_h} h_E$ and define the following norms:

$$\|v\|_{K,\beta} := (a(v, v) + a_h(v, v))^\frac{1}{2}, \quad (36)$$

$$\|v\|_{K,\beta,\gamma} := \left(\|v\|_{K,\beta}^2 + \|\sqrt{\gamma} \Pi_k v\|^2\right)^\frac{1}{2}, \quad (37)$$
on the nonconforming space $H^1_{k,nc}(T_h)$. In (36), for the evaluation of $S^E(v, v)$, we assume to use the VEM interpolant of the function $v$, see [9, Theorem 11]. Clearly, $\|v\|_{K,B} \leq \|v\|_{K,B\gamma}$.

### 4.1. Discretization errors

The following Lemmas 2, 3, 4, and 5 provide a continuity bound for the discrete bilinear forms (24)-(27) and an estimate of the approximation error when compared with the corresponding continuous ones (14)-(17). Throughout the section, we use the approximation results for the local polynomial projections of a function $v \in H^{s+1}(E)$, cf. [74, Lemma 5.1], given by:

$$
\|v - \Pi_{k-1}^0 v\|_E + h_E \|v - \Pi_{k-1}^0 v\|_{1,E} \leq C h^{s+1}_E |v|_{s+1,E} \quad 1 \leq s + 1 \leq k, \quad (38)
$$

$$
\|v - \Pi_k^0 v\|_E + h_E \|v - \Pi_k^0 v\|_{1,E} \leq C h^{s+1}_E |v|_{s+1,E} \quad 1 \leq s + 1 \leq k + 1, \quad (39)
$$

which hold for every mesh element $E$ and polynomial degree $k \geq 1$ under the mesh assumptions of Section 3.1. For every internal edge $e = \partial E^+ \cap \partial E^-$ and functions $v \in H^{s+1}(\omega_e)$ with $\omega_e = E^e \cup E^-$ we will also consider the trace inequality

$$
\|v - \Pi_{k-1}^{0,e} v\|_e + h_e \|v - \Pi_{k-1}^{0,e} v\|_{1,e} \leq C h^{s+\frac{1}{2}}_e |v|_{s+1,\omega_e} \quad 1 \leq s + 1 \leq k. \quad (40)
$$

We use the error estimate for the virtual element interpolant of order $k$ of a function $\varphi \in H^{s+1}(E)$, $1 \leq s + 1 \leq k + 1$ [49]:

$$
\|\varphi - \varphi_I\|_E + h_E \|\varphi - \varphi_I\|_{1,E} \leq C h^{s+1}_E |\varphi|_{s+1,E}. \quad (41)
$$

Figure 2: Different regimes of $\tau_E$ for different values of $Pe_E$ and $Ka_E$. 

We use the error estimate for the virtual element interpolant of order $k$ of a function $\varphi \in H^{s+1}(E)$, $1 \leq s + 1 \leq k + 1$ [49]:

$$
\|\varphi - \varphi_I\|_E + h_E \|\varphi - \varphi_I\|_{1,E} \leq C h^{s+1}_E |\varphi|_{s+1,E}. \quad (41)$$
Furthermore, (13) implies that

$$\|\varphi - \varphi_I\|_{K,\beta,\gamma}^2 = \sum_{E \in T_h} \left( \left\| \sqrt{K} \nabla (\varphi - \varphi_I) \right\|_E^2 + \tau_E \|\beta \cdot \nabla (\varphi - \varphi_I)\|_E^2 + \|\sqrt{K} \Pi_{k-1}^0 (\nabla (\varphi - \varphi_I))\|_E^2 + \tau_E \|\beta \cdot \Pi_{k-1}^0 (\nabla (\varphi - \varphi_I))\|_E^2 ight)$$

for some positive constant $C$ independent of $h$ and the local problem coefficients $K$, $\beta$, $\gamma$.

**Assumption 1.** We assume that the solution $u$ to (19) belongs to $H^{s+1}(T_h) \cap V$, with $1 < s + 1 \leq k + 1$ and that $K \in [W_{\infty}^d(\Omega)]^d \times d$, $\beta \in [W_{\infty}^d(\Omega)]^d$, $\gamma \in [W_{\infty}^d(\Omega)]^d$.

The following technical lemma is needed in the upcoming proofs.

**Lemma 1.** Let $a, b \in W_{\infty}^d(\Omega)$ be given, $E \in T_h$. Then,

$$\|ab - \Pi_{k}^0(ab)\|_{W_{\infty}^d(E)} \leq \frac{3}{2} \left( \|a\|_{W_{\infty}^d(E)} \|b - \Pi_{k}^0b\|_{W_{\infty}^d(E)} + \|b\|_{W_{\infty}^d(E)} \|a - \Pi_{k}^0a\|_{W_{\infty}^d(E)} \right).$$

**Proof.** We consider the following decomposition, exploiting the fact that $\Pi_{k}^0(a\Pi_{k}^0b) = \Pi_{k}^0(a\Pi_{k}^0b) = \Pi_{k}^0(b\Pi_{k}^0a)$:

$$ab - \Pi_{k}^0(ab) = \frac{1}{2} \left( (a + b) - \Pi_{k}^0(a) - \Pi_{k}^0(b) \right)$$

$$= \frac{1}{2} \left[ (a - \Pi_{k}^0a) b + \Pi_{k}^0(a) (b - \Pi_{k}^0b) + \Pi_{k}^0(b) (a - \Pi_{k}^0a) - \Pi_{k}^0(ab) \right]$$

$$+ \left[ (a - \Pi_{k}^0a) b + a \Pi_{k}^0(b) - \Pi_{k}^0a (b - \Pi_{k}^0b) \right]$$

The proof is concluded by the triangle inequality, the Cauchy-Schwarz inequality and by exploiting the fact that $\|\Pi_{k}^0ab\|_{W_{\infty}^d(E)} = \|a\|_{W_{\infty}^d(E)} \|b\|_{W_{\infty}^d(E)} \|a\|_{W_{\infty}^d(E)} \|b\|_{W_{\infty}^d(E)}$.

We now estimate the terms inside $B_{supp, k}$ to analyse their continuity, and their consistency with respect to polynomials of order $k$.

**Lemma 2.** For every function $w \in H_{h}^{1, nc}(T_h)$ and $v_h \in V_h \subset H_{h}^{1, nc}(T_h)$,

$$a_h (w, v_h) \leq \|w\|_{K,\beta,\gamma} \|v_h\|_{K,\beta,\gamma}.$$

Moreover, if $w \in H_{h}^{1, nc}(T_h) \cap H^{s+1}(T_h)$, then

$$\left| a (\Pi_{k-1}^0w, v_h) - a_h (\Pi_{k-1}^0w, v_h) \right| \leq C \max_{E \in T_h} \left\{ C_{a, E}^\infty \right\} h^s \|w\|_{s+1} \|v_h\|_{K,\beta}.$$


where

\[ C_{a,E}^{nc} = \frac{\| K_{\beta,E} - \Pi_0^0(K_{\beta,E}) \|_{W_2(E)}}{\sqrt{K_E}}. \]

(46)

**Proof.** Regarding (44), to estimate the continuity of \(a_h\), we use the Cauchy-Schwarz and Hölder inequalities and the definition of the norm (36):

\[
|a_h(w, v_h) - (a_h(w, w)a_h(v_h, v_h))^{1/2} | \leq \| w \|_{K_{\beta}} \| v_h \|_{K_{\beta}}.
\]

To prove (45), we first notice that \(S^E((I - \Pi_0^{0}) \Pi_0^{0} w, v_h) = 0\) and \(\Pi_0^{0} \nabla \Pi_0^{0} w = \nabla \Pi_0^{0} w:\)

\[
\begin{align*}
| a((\Pi_0^{0} w, v_h) - a_h((\Pi_0^{0} w, v_h)) | & \leq \sum_{E \in T_h} |(K_{\beta,E} \nabla \Pi_0^{0} w, \nabla v_h)_E) \\
- (K_{\beta,E} \Pi_0^{0} \nabla \Pi_0^{0} w, \nabla v_h)_E | & = \sum_{E \in T_h} |(K_{\beta,E} \nabla \Pi_0^{0} w, \nabla v_h - \Pi_0^{0} \nabla v_h)_E | .
\end{align*}
\]

The local terms are bounded using the k-consistency \(a^E_h(\cdot, \cdot) = a^E(\cdot, \cdot)\) when the coefficients are constants and one of the arguments is a polynomial:

\[
\begin{align*}
(K_{\beta,E} \nabla \Pi_0^{0} w, \nabla v_h - \Pi_0^{0} \nabla v_h)_E & = (K_{\beta,E} - \Pi_0^{0}(K_{\beta,E})) \nabla \Pi_0^{0} w, \nabla v_h - \Pi_0^{0} \nabla v_h)_E \\
& = ((K_{\beta,E} - \Pi_0^{0}(K_{\beta,E})) \nabla \Pi_0^{0} w, \nabla v_h - \Pi_0^{0} \nabla v_h)_E \\
& \leq \|(K_{\beta,E} - \Pi_0^{0}(K_{\beta,E})) \nabla \Pi_0^{0} w\|_E \| \nabla v_h - \Pi_0^{0} \nabla v_h\|_E \\
& \leq C h_E \frac{\| K_{\beta,E} - \Pi_0^{0}(K_{\beta,E}) \|_{W_2(E)} }{\sqrt{K_E}} \| w \|_{s+1,E} \| v_h \|_{K_{\beta,E}}.
\end{align*}
\]

Remark 5. We can bound \(C_{a,E}^{nc}\) as follows, using (43) and (35):

\[
C_{a,E}^{nc} = \frac{\| K_{\beta,E} - \Pi_0^{0}(K_{\beta,E}) \|_{W_2(E)}}{\sqrt{K_E}}
\]

\[
\leq \frac{1}{\sqrt{K_E}} (\| K - \Pi_0^{0} K \|_{W_2(E)} + \sigma_E \| \beta^T - \Pi_0^{0}(\beta^T) \|_{W_2(E)} )
\]

\[
\leq \frac{C}{\sqrt{K_E}} (\| K - \Pi_0^{0} K \|_{W_2(E)} + h_E \sigma_E \| \beta \|_{W_2(E)} \| \beta - \Pi_0^{0}(\beta) \|_{W_2(E)} ) .
\]

Lemma 3. For every function \(w \in H^{1,nc}_k(T_h)\) and \(v_h \in V_h \subset H^{1,nc}_k(T_h)\) it holds that

\[
|b_h(w, v_h)| \leq C \left[ \max_{E \in T_h} \left( \sigma_E^{-1}, h_E^{-1} K_{b,E}^{nc} \right) \| w \| + \max_{E \in T_h} \left( C_{b,E}^{nc,1}, K_{b,E}^{nc} \right) \| \nabla w \| \right] \| v_h \|_{K_{\beta}} ,
\]

(47)
where

\[ C_{h,E}^{c,r} = h_E \frac{\beta - \Pi_0^\beta}{\|\beta - \Pi_0^\beta\|_{W^1_\infty(E)}}, \ r \geq 1, \quad (48) \]

\[ K_{h,E}^{c} = h_E \frac{\beta \cdot \hat{n}}{\|\beta \cdot \hat{n}\|_{W^1_\infty(E)}}, \quad (49) \]

Moreover, for every function \( w \in H^{1,nc}(\mathcal{T}_h) \cap H^{s+1}(\mathcal{T}_h) \), it holds that

\[ \left| b \left( \Pi_{k-1}^0 w, v_h \right) - b_h \left( \Pi_{k-1}^0 w, v_h \right) \right| \leq C \max_{E \in \mathcal{T}_h} C_{h,E}^{c,s+1} h^s \| w \|_{s+1} \| v_h \|_{K_\beta}. \quad (50) \]

**Proof.** To obtain (47), we first introduce the following decomposition:

\[ b_h \left( w, v_h \right) = \sum_{E \in \mathcal{T}_h} T_{E,1} + T_{E,2} + T_{E,3} + T_{E,4}, \]

where, \( \forall E \in \mathcal{T}_h \),

\[ T_{E,1} = \left( \beta \cdot \left( \Pi_{k-1}^0 \nabla w - \nabla w \right), \Pi_{k-1}^0 v_h \right)_E, \quad (51) \]

\[ T_{E,2} = \left( \beta \cdot \nabla w, \Pi_{k-1}^0 v_h - v_h \right)_E, \quad (52) \]

\[ T_{E,3} = \left( \beta \cdot \nabla w, v_h \right)_E, \quad (53) \]

\[ T_{E,4} = - \left( \Pi_{k-1}^0 w, \beta \cdot \Pi_{k-1}^0 \nabla v_h \right)_E. \quad (54) \]

We estimate \( T_{E,1} \) in (51) as follows:

\[ |T_{E,1}| = |(\Pi_{k-1}^0 \nabla w - \nabla w, (\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h)_E| \]

\[ = |(\nabla w, (\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h - \Pi_{k-1}^0 ((\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h))_E| \]

\[ \leq \|\nabla w\|_E \left\| ((\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h - \Pi_{k-1}^0 ((\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h)) \right\|_E \]

\[ \leq C h_E \|\nabla w\|_E \left\| ((\beta - \Pi_0^\beta)\Pi_{k-1}^0 v_h)_{1,E} \right\| \]

\[ \leq C h_E \|\nabla w\|_E \left\| \beta - \Pi_0^\beta \right\|_{W^1_\infty(E)} \| \Pi_{k-1}^0 v_h \|_{1,E} \]

\[ \leq C \frac{h_E}{\sqrt{K_E^*}} \|\beta - \Pi_0^\beta\|_{W^1_\infty(E)} \| \nabla w\|_E \| v_h \|_{K_\beta}. \]
The estimation of $T_{E,3}$ in (53) requires an application of Green's formula, as follows:

$$
\sum_{E \in T_h} T_{E,3} \leq \sum_{E \in T_h} (w, \beta \cdot \nabla v_h)_E + \sum_{E \in T_h} \int_{\partial E} (\beta \cdot \hat{n}) w v_h \\
= \sum_{E \in T_h} (w, \beta \cdot \nabla v_h)_E + \frac{1}{2} \sum_{E \in T_h} \int_{\partial E} (\beta \cdot \hat{n}) v_h .
$$

The first term is estimated locally by the Cauchy-Schwarz inequality:

$$(w, \beta \cdot \nabla v_h)_E \leq \tau_{E}^{-\frac{1}{2}} \|w\|_E \|v_h\|_{K_{\beta,E}} .$$

The boundary terms are estimated exploiting $\int_{\partial E} \beta \cdot \hat{n} = \int_E \nabla \cdot \beta = 0$, and denoting by
\( \Pi_{k-1}^0 v \) the piecewise polynomial projection of \( v \) on each \( e \subset \partial E \):

\[
\int_{\partial E} (\beta \cdot \hat{n}) [wv_h] = \sum_{R \in \omega_e} \int_{E \cap R} (\beta \cdot \hat{n}) (w|_R [v_h] + \|w\| v_h|_E)
\]

\[
= \sum_{R \in \omega_e} \int_{E \cap R} \left((\beta \cdot \hat{n}) w|_R - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) w|_R)\right) [v_h]
\]

\[
+ \sum_{R \in \omega_e} \int_{E \cap R} \|w\| \left((\beta \cdot \hat{n}) v_h|_E - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) v_h)\right)
\]

\[
= \sum_{R \in \omega_e} \int_{E \cap R} \left((\beta \cdot \hat{n}) w|_R - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) w|_R)\right) \left[v_h - \Pi_{k-1}^{0,\partial E} (v_h)\right]
\]

\[
+ \sum_{R \in \omega_e} \int_{E \cap R} \|w - \Pi_{k-1}^0 w\| \left((\beta \cdot \hat{n}) v_h|_E - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) v_h)\right)
\]

\[
= \sum_{R \in \omega_e} \left|(\beta \cdot \hat{n}) w|_R - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) w|_R)\right|_{E \cap R} \left\|v_h - \Pi_{k-1}^{0,\partial E} (v_h)\right\|_{E \cap R}
\]

\[
+ \sum_{R \in \omega_e} \left\|w - \Pi_{k-1}^0 w\|_{E \cap R} \left|(\beta \cdot \hat{n}) v_h|_E - \Pi_{k-1}^{0,\partial E} ((\beta \cdot \hat{n}) v_h)\right|_{E \cap R}
\]

\[
\leq C_1 \left(\sum_{R \in \omega_e} h_E \left|(\beta \cdot \hat{n}) w|_{1,R \cap \partial E}\right| \right) h_E^{-\frac{1}{2}} \|\nabla v_h\|_{\omega_E}
\]

\[
+ C_2 \left(\sum_{R \in \omega_e} h_E \left|(\beta \cdot \hat{n}) v_h|_{1,R \cap \partial E}\right| \right) h_E^{-\frac{1}{2}} \|\nabla w\|_{\omega_E}
\]

\[
\leq C_1 h_E^{-\frac{1}{2}} \|\beta \cdot \hat{n}\|_{W^1_{\alpha}(\partial E)} \|\nabla w\|_{\omega_E} \cdot h_E^{\frac{1}{2}} \|\nabla v_h\|_{\omega_E}
\]

\[
+ C_2 h_E^{-\frac{1}{2}} \|\beta \cdot \hat{n}\|_{W^1_{\alpha}(\partial E)} \|\nabla v_h\|_{\omega_E} \cdot h_E^{\frac{1}{2}} \|\nabla w\|_{\omega_E}
\]

\[
\leq C \frac{h_E \|\beta \cdot \hat{n}\|_{W^1_{\alpha}(\partial E)} \|\nabla w\|_{\omega_E} \|v_h\|_{K_{\beta,\omega_E}}}{\sqrt{\tau_E}}
\]

The estimate of \( \|T_{E,4}\| \) defined by \[(54)\] is obtained by the Cauchy-Schwarz inequality, the continuity of projections and the definition of the norm \[(36)\]:

\[
|T_{E,4}| = |(\Pi_{k-1}^0 w, \beta \cdot \Pi_{k-1}^0 \nabla v_h)|_E \leq C \tau_E^{-\frac{1}{2}} \|w\|_E \|v_h\|_{K_{\beta,E}}.
\]

To derive \[(50)\], we set

\[
b (\Pi_{k-1}^0 w, v_h) - b_h (\Pi_{k-1}^0 w, v_h) = \sum_{E \in T_h} \left(R_{E,1} - R_{E,2}\right),
\]

where, recalling that \( \Pi_{k-1}^0 (\nabla \Pi_{k-1}^0 w) = \nabla \Pi_{k-1}^0 w \),

\[
R_{E,1} = (\beta \cdot \nabla \Pi_{k-1}^0 w, v_h - \Pi_{k-1}^0 v_h)_E,
\]

\[
R_{E,2} = (\Pi_{k-1}^0 w, \beta \cdot (\nabla v_h - \Pi_{k-1}^0 \nabla v_h))_E.
\]
$R_{E,1}$ can be estimated as follows:

$$R_{E,1} = (\beta \cdot \nabla \Pi_{k-1}^0 w, v_h - \Pi_{k-1}^0 v_h)_E = ((\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w, v_h - \Pi_{k-1}^0 v_h)_E$$

$$= \left( (\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w - \Pi_{k-1}^0 \left( (\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w \right) , v_h - \Pi_{k-1}^0 v_h \right)_E$$

$$\leq \left\| (\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w - \Pi_{k-1}^0 \left( (\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w \right) \right\|_E \| v_h - \Pi_{k-1}^0 v_h \|_E$$

$$\leq C h_E \left\| (\beta - \Pi_{k-1}^0 \beta) \cdot \nabla \Pi_{k-1}^0 w \right\|_{1,E} \| v_h \|_E$$

$$\leq C h_E \frac{\| \beta - \Pi_{k-1}^0 \beta \|_{W^{s+1}_0,0(E)}}{\sqrt{K_E}} \| v_h \|_{K_\beta}.$$

The estimate of $R_{E,2}$ in (56) is obtained as follows:

$$R_{E,2} = (\Pi_{k-1}^0 w, \beta \cdot (\nabla v_h - \Pi_{k-1}^0 \nabla v_h))_E = ((\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w, \nabla v_h - \Pi_{k-1}^0 \nabla v_h)_E$$

$$= ((\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w - \Pi_{k-1}^0 \left( (\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w \right), \nabla v_h)_E$$

$$\leq \left\| (\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w - \Pi_{k-1}^0 \left( (\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w \right) \right\|_E \| \nabla v_h \|_E$$

$$\leq C h_E^{s+1} \left\| (\beta - \Pi_{k-1}^0 \beta) \Pi_{k-1}^0 w \right\|_{s+1,E} \| \nabla v_h \|_E$$

$$\leq C h_E \frac{\| \beta - \Pi_{k-1}^0 \beta \|_{W^{s+1}_0,0(E)}}{\sqrt{K_E}} \| v_h \|_{K_\beta}.\,$$

Remark 6. The coefficient $\mathcal{K}_{b,E}^{nc}$ can be rewritten, considering that $\overline{\beta \cdot \hat{n}} = \int_{\partial E} \beta \cdot \hat{n} = 0$, in the following way:

$$\mathcal{K}_{b,E}^{nc} = \frac{h_E \| \beta \cdot \hat{n} \|_{W^{1,0}_0(\partial E)}}{\sqrt{K_E}} = \frac{h_E \left\| \beta \cdot \hat{n} - \overline{\beta \cdot \hat{n}} \right\|_{W^{1,0}_0(\partial E)}}{\sqrt{K_E}}.$$

Lemma 4. For every function $w \in H^1_{nc}(T_h)$ and $v_h \in V_h$ it holds that

$$|c_h (w, v_h)| \leq (1 + \sqrt{C_\tau}) \| w \|_{K_\beta} \| v_h \|_{K_\beta}.$$

Moreover, for every function $w \in H^1_{nc}(T_h) \cap H^{s+1}(T_h)$, it holds that

$$|c \left( \Pi_{k-1}^0 w, v_h \right) - c_h \left( \Pi_{k-1}^0 w, v_h \right)| \leq C \max_{E \in T_h} C_{c,E} \| w \|_{s+1} \| v_h \|_{K_\beta},$$

$$C_{c,E}^{nc} = \max \left\{ \frac{h_E^2 \| \gamma - \Pi_{k-1}^0 \gamma \|_{W^{s+1}_0(E)}}{\sqrt{K_E}}, \frac{h_E T_E \| \gamma \beta - \Pi_{k-1}^0 (\gamma \beta) \|_{W^{s+1}_0(E)}}{\sqrt{K_E}} \right\}.$$
Proof. Inequality (57) follows easily from the definition of the norm (37) and the definition of \( \tau_E \) [45]:

\[
(\gamma \Pi_{k-1}^0 w, \Pi_{k-1}^0 v_h)_E + \tau_E (\gamma \Pi_{k-1}^0 w, \beta \cdot \Pi_{k-1}^0 \nabla v_h)_E \leq \| \sqrt{\gamma} \Pi_{k-1}^0 w \|_E \| \sqrt{\gamma} \Pi_{k-1}^0 v_h \|_E \\
+ \sqrt{\tau_E \gamma E} \| \sqrt{\gamma} \Pi_{k-1}^0 w \|_E \cdot \sqrt{\tau_E} \| \beta \cdot \Pi_{k-1}^0 \nabla v_h \|_E \leq \left(1 + \sqrt{C_T} \right) \| w \|_{K_{\beta \gamma}, E} \| v_h \|_{K_{\beta \gamma}, E}.
\]

To prove (58), we start with:

\[
c (\Pi_{k-1}^0 w, v_h) - c_h (\Pi_{k-1}^0 w, v_h) = \sum_{E \in T_h} \left( R_{E,1} + R_{E,2} \right),
\]

where

\[
R_{E,1} = (\gamma \Pi_{k-1}^0 w, v_h - \Pi_{k-1}^0 v_h)_E,
\]

\[
R_{E,2} = \tau_E (\gamma \Pi_{k-1}^0 w, \beta \cdot \nabla v_h - \beta \cdot \Pi_{k-1}^0 \nabla v_h)_E.
\]

The first term, given by (60), can be bounded as follows:

\[
R_{E,1} = ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w, v_h - \Pi_{k-1}^0 v_h)_E \\
= ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w - \Pi_{k-1}^0 ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w), v_h - \Pi_{k-1}^0 v_h)_E \\
\leq \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w - \Pi_{k-1}^0 ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w) \|_E \| v_h - \Pi_{k-1}^0 v_h \|_E \\
\leq C h_E^{1+1} \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w \|_{s+1,E} \| v_h \|_E \\
\leq C h_E^{1+2} \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w \|_{s+1,E} \| \nabla v_h \|_E \\
\leq C h_E^2 \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w \|_{s+1,E} \| \nabla v_h \|_E \| v_h \|_{K_{\beta \gamma}, E}.
\]

The term \( R_{E,2} \) in (59) can be bounded as follows:

\[
R_{E,2} = \tau_E (\gamma \Pi_{k-1}^0 w, \beta \cdot \nabla v_h - \beta \cdot \Pi_{k-1}^0 \nabla v_h)_E \\
= \tau_E ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w, \nabla v_h - \Pi_{k-1}^0 \nabla v_h)_E \\
= \tau_E ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w - \Pi_{k-1}^0 ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w), \nabla v_h - \Pi_{k-1}^0 \nabla v_h)_E \\
\leq \tau_E \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w - \Pi_{k-1}^0 ((\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w) \|_E \| \nabla v_h - \Pi_{k-1}^0 \nabla v_h \|_E \\
\leq C h_E^{1+1} \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w \|_{s+1,E} \| \nabla v_h \|_E \\
\leq C h_E \| (\gamma - \Pi_{k-1}^0 \gamma) \Pi_{k-1}^0 w \|_{s+1,E} \| \nabla v_h \|_E \| v_h \|_{K_{\beta \gamma}, E}.
\]

Remark 7. The second argument of the max in (59) can be bounded by (45) and (55):

\[
\frac{h_E \tau_E \| \gamma - \Pi_{k-1}^0 (\gamma) \Pi_{k-1}^0 w \|_{s+1,E}}{\sqrt{K_E}} \leq \frac{C}{\sqrt{K_E}} \left( \frac{h_E}{\gamma E} \| \gamma \Pi_{k-1}^0 w \|_{s+1,E} \| \beta - \Pi_{k-1}^0 \beta \Pi_{k-1}^0 w \|_{s+1,E} \\
+ \frac{h_E^2}{\beta E} \| \beta \Pi_{k-1}^0 w \|_{s+1,E} \| \gamma - \Pi_{k-1}^0 \gamma \Pi_{k-1}^0 w \|_{s+1,E} \right).
\]

\( \Box \)
Lemma 5. For any \( w \in H_k^{1,nc}(T_h) \) and \( \forall v_h \in V_h \),

\[
d_h(w, v_h) \leq \|w\|_{K,\beta} \|v_h\|_{K,\beta}.
\]  \hspace{1cm} (62)

Moreover, if \( w \in V \cap H^{s+1}(T_h) \), then

\[
|d(\Pi^0_{k-1} w, v_h) - d_h(\Pi^0_{k-1} w, v_h)| \leq C \max_{E \in T_h} C_{d,E}^n h_E^s \|w\|_{s+1} \|v_h\|_{K,\beta},
\]  \hspace{1cm} (63)

where

\[
C_{d,E}^n = \max \left\{ \frac{h_{k-1}^s \tau_E}{\sqrt{K_E}}, \frac{\tau_E \|((\nabla \beta)^T K - \Pi^0_k((\nabla \beta)^T K))\|_{W^s(E)}}{\sqrt{K_E}}, \frac{\tau_E \|((\nabla \beta)^T K - \Pi^0_k((\nabla \beta)^T K))\|_{W^s(E)}}{\sqrt{K_E}} \right\}.
\]  \hspace{1cm} (64)

Proof. To prove (62), we use the inverse inequality \((34)\) and the definition of the norm \((36)\); \( \forall E \in T_h \),

\[
\tau_E (\nabla \cdot (K\Pi^0_{k-1} \nabla w), \beta \cdot \Pi^0_{k-1} \nabla v_h)_E \leq \sqrt{K_E} \|\nabla \cdot (K\Pi^0_{k-1} \nabla w)\|_E \|\beta \cdot \Pi^0_{k-1} \nabla v_h\|_E \leq \frac{1}{\sqrt{K_E}} \|K\Pi^0_{k-1} \nabla w\|_E \|v_h\|_{K,\beta} \leq \|w\|_{K,\beta} \|v_h\|_{K,\beta}.
\]

Regarding (63), we proceed as follows: \( \forall E \in T_h \),

\[
\tau_E (\nabla \cdot (K\Pi^0_{k-1} \nabla w), \beta \cdot (\nabla v_h - \Pi^0_{k-1} \nabla v_h))_E = R_{E,1} + R_{E,2},
\]

where, with the notation \( E_{k-1} = I - \Pi^0_{k-1} \),

\[
R_{E,1} = \tau_E \sum_{i=1}^d \left( \nabla \cdot (\beta_i K\Pi^0_{k-1} \nabla w), E_{k-1} \left( \frac{\partial v_h}{\partial x_i} \right) \right)_E,
\]  \hspace{1cm} (65)

\[
R_{E,2} = \tau_E \left( - (\nabla \beta)^T K\Pi^0_{k-1} \nabla w, E_{k-1} (\nabla v_h) \right)_E.
\]  \hspace{1cm} (66)
The term $R_{E,1}$ in (55) can be estimated as follows, using (43):

$$R_{E,1} = \tau_E \sum_{i=1}^{d} \left( \nabla \cdot \left( \left( \beta_i K - \Pi_0^0 (\beta_i K) \right) \Pi_{k-1} \nabla w, \mathcal{E}_{k-1} \left( \frac{\partial v_h}{\partial x_i} \right) \right) \right)_E$$

$$= \tau_E \sum_{i=1}^{d} \left( \nabla \cdot \left( \mathcal{E}_{k-1} \left( \left( \beta_i K - \Pi_0^0 (\beta_i K) \right) \Pi_{k-1} \nabla w \right), \mathcal{E}_{k-1} \left( \frac{\partial v_h}{\partial x_i} \right) \right) \right)_E$$

$$\leq \tau_E \sum_{i=1}^{d} \left\| \nabla \cdot \left( \mathcal{E}_{k-1} \left( \left( \beta_i K - \Pi_0^0 (\beta_i K) \right) \Pi_{k-1} \nabla w \right) \right) \right\|_E \left\| \mathcal{E}_{k-1} \left( \frac{\partial v_h}{\partial x_i} \right) \right\|_E$$

$$\leq C h_E^{-1} \tau_E \sum_{i=1}^{d} \left\| \mathcal{E}_{k-1} \left( \left( \beta_i K - \Pi_0^0 (\beta_i K) \right) \Pi_{k-1} \nabla w \right) \right\|_E \left\| \frac{\partial v_h}{\partial x_i} \right\|_E$$

$$\leq C h_E^{-1} \tau_E h_E^s \sum_{i=1}^{d} \left\| \beta_i K - \Pi_0^0 (\beta_i K) \right\|_{W_{s,E}} \left\| \nabla w \right\|_{s,E} \left\| \nabla v_h \right\|_E$$

$$\leq C \frac{h_E^{-1} \tau_E \sum_{i=1}^{d} \left\| \beta_i K - \Pi_0^0 (\beta_i K) \right\|_{W_{s,E}}}{{\sqrt{K_E}}} h_E^s \| w \|_{s+1,E} \| v_h \|_{K_{s,E}}.$$

The term $R_{E,2}$ in (56) can be estimated as follows, using also (43):

$$|R_{E,2}| = \tau_E \left( (\nabla \beta)^T K \Pi_{k-1} \nabla w, \mathcal{E}_{k-1} \left( \nabla v_h \right) \right)_E$$

$$= \tau_E \left( \mathcal{E}_{k-1} \left( \left( (\nabla \beta)^T K - \Pi_0^0 ((\nabla \beta)^T K) \right) \Pi_{k-1} \nabla w \right), \nabla v_h \right)_E$$

$$\leq \tau_E \left\| \mathcal{E}_{k-1} \left( \left( (\nabla \beta)^T K - \Pi_0^0 ((\nabla \beta)^T K) \right) \Pi_{k-1} \nabla w \right) \right\|_E \left\| \nabla v_h \right\|_E$$

$$\leq C \frac{\tau_E \left\| (\nabla \beta)^T K - \Pi_0^0 ((\nabla \beta)^T K) \right\|_{W_{s,E}}}{{\sqrt{K_E}}} h_E^s \| w \|_{s+1,E} \| v_h \|_{K_{s,E}}.$$

**Remark 8.** The first argument of the max in (64) can be bounded as follows, using (43) and (55):

$$\frac{h_E^{-1} \tau_E \sum_{i=1}^{d} \left\| \beta_i K - \Pi_0^0 (\beta_i K) \right\|_{W_{s,E}}}{{\sqrt{K_E}}} \leq \frac{3h_E^{-1} \tau_E}{2\sqrt{K_E}} \left( 2 \left\| K \right\|_{W_{s,E}} \left\| \beta - \Pi_0^0 \beta \right\|_{W_{s,E}} + 2 \left\| \beta \right\|_{W_{s,E}} \left\| K - \Pi_0^0 K \right\|_{W_{s,E}} \right)$$

$$\leq C \frac{h_E}{\sqrt{K_E}} \left( h_E \left\| \frac{K}{K_E} \right\|_{W_{s,E}} \left\| \beta - \Pi_0^0 \beta \right\|_{W_{s,E}} + \left\| \frac{\beta}{\beta_E} \right\|_{W_{s,E}} \left\| K - \Pi_0^0 K \right\|_{W_{s,E}} \right).$$
Similarly, the second argument in (64) can be bounded as follows:

\[
\tau_E \left\| (\nabla \beta)^T K - \Pi_0^b (\nabla \beta)^T K \right\|_{W^{1,\infty}_2 (E)} \leq \frac{C}{\sqrt{K_E}} \left( h_E^2 \left\| K \right\|_{W^{1,\infty}_2 (E)} \left\| \nabla \beta \right\|_{W^{1,\infty}_2 (E)} + h_E \left\| \frac{\nabla \beta}{\beta_E} \right\|_{W^{1,\infty}_2 (E)} \left\| K - \Pi_0^b K \right\|_{W^{1,\infty}_2 (E)} \right).
\]

Finally, the following Lemma states the continuity of \( B_{\text{supg}} \), defined by (13).

**Lemma 6.** Let \( w \in H^{1,nc}_k (T_h) \cap H^2 (T_h) \) and \( v_h \in V_h \). Then,

\[
B_{\text{supg}} (w, v_h) \leq C \max_{E \in T_h} \left( \tau_E^2 \gamma_E, h_E^{-1} K_{b,E} \right) \left\| w \right\| + \frac{\sqrt{K_E}}{\sqrt{K_E}} \left\| w \right\|_{K_{\gamma_E}} + \max_{E \in T_h} K_{b,E} \left\| \nabla w \right\|
\]

\[
+ \max_{E \in T_h} \left( \sqrt{\tau_E} \left\| K \right\|_{W^{1,\infty}_2 (E)} \left\| w - \Pi_{k-1}^b w \right\|_{2,E} \right) \left\| v_h \right\|_{K_{\gamma_E}}.
\]

**Proof.** The proof of the continuity of \( a, b \) and \( c \) follows the same arguments of Lemmas [2], [3] and [4]. The proof of the continuity of \( d \) is slightly different, and can be done as follows:

\[
\tau_E (\nabla \cdot (K \nabla w), \beta \cdot \nabla v_h) = \tau_E (\nabla \cdot (K \nabla w - K \nabla \Pi_{k-1}^b w), \beta \cdot \nabla v_h)
\]

\[
+ \tau_E (\nabla \cdot (K \nabla \Pi_{k-1}^b w), \beta \cdot \nabla v_h)
\]

\[
\leq \tau_E \left\| \nabla \cdot (K \nabla (w - \Pi_{k-1}^b w)) \right\|_E \left\| \beta \cdot \nabla v_h \right\|_E + \tau_E \left\| \nabla \cdot (K \nabla \Pi_{k-1}^b w) \right\|_E \left\| \beta \cdot \nabla v_h \right\|_E
\]

\[
\leq \left( \sqrt{\tau_E} \left\| \nabla \cdot (K \nabla (w - \Pi_{k-1}^b w)) \right\|_E + \sqrt{\tau_E} \left\| K \Delta (w - \Pi_{k-1}^b w) \right\|_E
\]

\[
+ \left\| \nabla \Pi_{k-1}^b w \right\|_E \right) \sqrt{\tau_E} \left\| \beta \cdot \nabla v_h \right\|_E
\]

\[
\leq C \left( \sqrt{\tau_E} \left\| K \right\|_{W^{1,\infty}_2 (E)} \left\| w - \Pi_{k-1}^b w \right\|_{2,E} + \frac{\sqrt{K_E}}{\sqrt{K_E}} \left\| \nabla w \right\|_E \right) \left\| w \right\|_{K_{\beta}}.
\]

The above lemmas can be summarized in the following lemma, estimating the error of approximation of the exact bilinear form by the discrete bilinear form.

**Lemma 7.** For any given \( w \in H^{1,nc}_k (T_h) \cap H^{s+1} (T_h) \) and any \( v_h \in V_h \),

\[
|B_{\text{supg}} (w, v_h) - B_{\text{supg}, h} (w, v_h)| \leq C \max_{E \in T_h} \left\{ \frac{\left\| K \right\|_{W^{1,\infty}_2 (E)}}{\sqrt{K_E}}, \sqrt{h_E \beta_E}, h_E \sqrt{\tau_E}, \right. \]

\[
C^{nc}_{a,E}, C^{nc}_{b,E}, C^{nc}_{c,E}, C^{nc}_{d,E}, K^{nc}_{b,E} \left\} h^s \left\| w \right\|_{s+1} \left\| v_h \right\|_{K_{\beta}},
\]

where \( C^{nc}_{a,E}, C^{nc}_{b,E}, C^{nc}_{c,E}, C^{nc}_{d,E} \) and \( K^{nc}_{b,E} \) are defined by (46), (48), (50), (54) and (40).
Proof. Collecting the results of Lemmas 2, 3, 4, 5 and 6 and the approximation estimates on the polynomial projections, we get

\[
|B_{\text{supg}}(w, v_h) - B_{\text{supg}, h}(w, v_h)| \leq |B_{\text{supg}}(w - \nabla h, v_h)| + |B_{\text{supg}, h}(w - \nabla h, v_h)|
\]

\[
+ |B_{\text{supg}}(\nabla h, v_h) - B_{\text{supg}, h}(\nabla h, v_h)|
\]

\[
\leq C \left( \max_{E \in T_h} \left( \frac{\sqrt{k_E}}{\sqrt{E_k}} \right) \|w - \nabla h\|_{K \beta, \gamma} \right)
\]

\[
+ \max_{E \in T_h} \left( \frac{\sqrt{k_E}}{\sqrt{E_k}} \|\nabla h\|_{K \beta, \gamma} \right)
\]

\[
+ \max_{E \in T_h} \left\{ \frac{\|\nabla h\|_{K \beta, \gamma}}{\sqrt{E_k}}, h_E \sqrt{\beta + 1}_E \right\} \|\nabla h\|_{K \beta, \gamma} \]

\[
\leq C \max_{E \in T_h} \left\{ \frac{\|\nabla h\|_{K \beta, \gamma}}{\sqrt{E_k}}, h_E \sqrt{\beta + 1}_E \right\} \|\nabla h\|_{K \beta, \gamma}.
\]

Due to the non-conformity of our approach and since the functions in the global virtual element space \( V_h \) may be discontinuous, for the exact solution \( u \in H^2(T_h) \cap H^1(\Omega) \) and every \( v_h \in V_h \) it holds that

\[
B_{\text{supg}}(u, v_h) = F_{\text{supg}}(v_h) + \mathcal{N}_h(u, v_h),
\]

where

\[
\mathcal{N}_h(u, v_h) := \sum_{E, \partial E} \left( (K \nabla u) \cdot \hat{n} - \frac{1}{2} (\beta \cdot \hat{n}) u, v_h \right)_{\partial E}.
\]

(69)

is called the \textit{conformity error}. This term is a generalization of the one of the pure diffusion problem that is introduced and estimated in [49] Lemma 4.1.

Lemma 8 (Conformity error). Let \( u \in H^{s+1}(T_h) \cap H^1(\Omega) \), \( 1 \leq s \leq k \), be the solution of the variational problem (5). Let \( \beta \in W^{s+1}(\Omega) \) and suppose \( K \nabla u \in H(\text{div}, \Omega) \). Under the mesh regularity assumptions of Section 3.1, for every \( v_h \in V_h \) it holds that

\[
|\mathcal{N}_h(u, v_h)| \leq C \max_{E \in T_h} \left\{ \frac{\|K\|_{W^{s,}(E)}}{\sqrt{E_k}}, K_{N, E}^{nc} \right\} h^s \|u\|_{s+1} \|v_h\|_{K \beta, \gamma},
\]

(70)

where

\[
K_{N, E}^{nc} = \frac{h_E \sqrt{\beta \cdot \hat{n}}}{\sqrt{E_k}}.
\]

(71)

Proof. The first term in (69) is bounded following [49] Lemma 4.1, using the fact that,
by hypothesis, \( K \nabla u \cdot \hat{n} \) is continuous:

\[
\begin{align*}
\sum_{E \in \mathcal{T}_h} ((K \nabla u) \cdot \hat{n}, v_h)_{\partial E} &= \sum_{e \in \mathcal{E}_h} ((K \nabla u) \cdot \hat{n}, [v_h])_e = \sum_{e \in \mathcal{E}_h} ((K \nabla u - \Pi_{k-1}^0(K \nabla u)) \cdot \hat{n}, [v_h])_e \\
&\leq \sum_{e \in \mathcal{E}_h} \left\| (K \nabla u - \Pi_{k-1}^0(K \nabla u)) \cdot \hat{n} \right\|_e \left\| v_h - \Pi_{k-1}^0 v_h \right\|_e \\
&\leq \sum_{e \in \mathcal{E}_h} C h_e^{s+\frac{1}{2}} |K \nabla u|_{s+\frac{1}{2}, e} \cdot h_e^{\frac{1}{2}} \left\| \nabla v_h \right\|_{\omega_e} \\
&\leq \frac{\|K\|_{W_{-1}(E)} h_E^s}{\sqrt{K_E}} \|u\|_{s+1, \omega_E} \left\| v_h \right\|_{K_{\beta} \omega_E}.
\end{align*}
\]

The second term in \( \text{[69]} \) is estimated using the fact that \( v_h \in H^{1, \text{uc}}(T_h) \). Denoting by \( \Pi_{k-1}^{0, \partial E} v_h \) the piecewise polynomial projection of \( v_h \) on each \( e \subset \partial E \) and since \( \left[ \Pi_{k-1}^{0, \partial E} v_h \right]_e = \Pi_{k-1}^{0, \partial E} \left( [v_h]_e \right) = 0 \forall e \subset \partial E \) because \( \int_E [v_h] q = 0 \forall q \in P_{k-1}(e) \), and since \((\beta \cdot \hat{n}) u\) is continuous across the edges being \( \beta \) a divergence-free vector and \( u \in H^{1}_h(\Omega) \), we get

\[
\begin{align*}
\sum_{E \in \mathcal{T}_h} ((\beta \cdot \hat{n}) u, v_h)_{\partial E} &= \frac{1}{2} \sum_{E \in \mathcal{T}_h} \left((\beta \cdot \hat{n}) u, [v_h]_{\partial E}\right) \\
&= \frac{1}{2} \sum_{E \in \mathcal{T}_h} \left((\beta \cdot \hat{n}) u - \Pi_{k-1}^{0, \partial E} ((\beta \cdot \hat{n}) u), [v_h - \Pi_{k-1}^{0, \partial E} v_h]_{\partial E}\right) \\
&\leq \frac{1}{2} \sum_{E \in \mathcal{T}_h} \left\| (\beta \cdot \hat{n}) u - \Pi_{k-1}^{0, \partial E} ((\beta \cdot \hat{n}) u) \right\|_{\partial E} \left\| [v_h - \Pi_{k-1}^{0, \partial E} v_h] \right\|_{\partial E} \\
&\leq C \sum_{E \in \mathcal{T}_h} h_E^{s+\frac{1}{2}} \left\| (\beta \cdot \hat{n}) u \right\|_{s+\frac{1}{2}, \partial E} \cdot h_E^{\frac{1}{2}} \left\| \nabla v_h \right\|_{\omega_E} \\
&\leq C \sum_{E \in \mathcal{T}_h} h_E^{s+\frac{1}{2}} \frac{\|\beta\cdot\hat{n}\|_{W_{-\frac{1}{2}}(\partial E)}}{\sqrt{K_E}} \left\| u \right\|_{s+1, \omega_E} \left\| v_h \right\|_{K_{\beta} \omega_E}.
\end{align*}
\]

\( \square \)

4.2. Well-posedness of the discrete problem

The following theorem proves the well-posedness of the discrete formulation.

**Theorem 1** (Coercivity of \( B_{\text{supg}, h} \)). For any \( v_h \in V_h \),

\[
B_{\text{supg}, h} (v_h, v_h) \geq \min \left\{ \frac{1}{4}, \frac{\sigma_{\gamma}}{2} \right\} \frac{1 - C_{\tau}}{2} \left\| v_h \right\|_{K_{\beta} \gamma},
\]

where \( C_{\tau} \) is the constant introduced in \( \text{[35]} \).
Proof. Let $v_h \in V_h$. By definition (25), it holds that
\[ b(v_h, v_h) = \frac{1}{2} \sum_{E \in T_h} \left[ \langle \beta \cdot \Pi_{k-1}^0 \nabla v_h, \Pi_{k-1}^0 v_h \rangle_E - \langle \Pi_{k-1}^0 v_h, \beta \cdot \Pi_{k-1}^0 \nabla v_h \rangle_E \right] = 0. \]

Moreover, using Cauchy-Schwarz and Young inequalities we find that
\[ \tau_E \left| \gamma \Pi_{k-1}^0 v_h, \beta \cdot \Pi_{k-1}^0 \nabla v_h \right|_E \leq \sqrt{\gamma E \tau_E} \| \sqrt{\gamma} \Pi_{k-1}^0 v_h \|_E \| \beta \cdot \Pi_{k-1}^0 \nabla v_h \|_E \]
\[ \leq \frac{1}{2} \left( \sqrt{\gamma} \Pi_{k-1}^0 v_h \right)^2 + \frac{\gamma E \tau_E^2}{2} \| \beta \cdot \Pi_{k-1}^0 \nabla v_h \|_E^2, \]
which implies that
\[ \tau_E \left( \gamma \Pi_{k-1}^0 v_h, \beta \cdot \Pi_{k-1}^0 \nabla v_h \right)_E \geq -\frac{1}{2} \left( \sqrt{\gamma} \Pi_{k-1}^0 v_h \right)^2 - \frac{\gamma E \tau_E^2}{2} \| \beta \cdot \Pi_{k-1}^0 \nabla v_h \|_E^2. \] (73)

Inverse inequality (34) imply that
\[ \tau_E \| \nabla \left( \Pi_{k-1}^0 \nabla v_h \right) \|_E^2 \leq \frac{C_E h_k^2}{K_E} \| \nabla \left( \Pi_{k-1}^0 \nabla v_h \right) \|_E^2 \leq \frac{1}{K_E} \| \Pi_{k-1}^0 \nabla v_h \|_E^2 \]
\[ \leq \left( \sqrt{K} \Pi_{k-1}^0 \nabla v_h \right)_E^2, \] (74)

since $\| \Pi_{k-1}^0 \nabla v_h \|_E \leq \sqrt{K} E \| \sqrt{K} \Pi_{k-1}^0 \nabla v_h \|_E$. Using the definition of $B_{\text{supg}, h}$, cf. (23), Cauchy-Schwarz inequality, inverse inequality (34), inequalities (73), (74) and (35), we
have

\[ B_{\text{supg},h}(v_h, v_h) = \sum_{E \in T_h} \left\{ \left\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \right\|_E^2 + \tau_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E^2 \right. \]

\[ + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) \]

\[ + \left\| \sqrt{\tau} \Pi^0_{k-1} v_h \right\|_E^2 + \tau_E \left( \gamma \Pi^0_{k-1} v_h, \beta \cdot \Pi^0_{k-1} \nabla v_h \right)_E \]

\[ - \tau_E \left\{ \nabla \cdot \left( \sqrt{K} \Pi^0_{k-1} \nabla v_h \right), \beta \cdot \Pi^0_{k-1} \nabla v_h \right\}_E \left\} \right. \]

\[ \geq \sum_{E \in T_h} \left\{ \left\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \right\|_E^2 + \left( 1 - \frac{\gamma \tau E}{2} \right) \tau_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E^2 \right. \]

\[ + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) + \left( 1 - \frac{1}{2} \right) \left\| \sqrt{\tau} \Pi^0_{k-1} v_h \right\|_E^2 \]

\[ - \tau_E \left\| \nabla \cdot \left( \sqrt{K} \Pi^0_{k-1} \nabla v_h \right) \right\|_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E \left\} \right. \]

\[ \geq \sum_{E \in T_h} \left\{ \left\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \right\|_E^2 + \left( \frac{1}{2} - \frac{C^E}{2} \right) \tau_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E^2 \right. \]

\[ + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) + \frac{1}{2} \left\| \sqrt{\tau} \Pi^0_{k-1} v_h \right\|_E^2 \]

\[ - \sum_{E \in T_h} \left( \frac{1}{2} \tau_E \left\| \nabla \cdot (K \Pi^0_{k-1} \nabla v_h) \right\|_E^2 \right) \]

\[ \geq \sum_{E \in T_h} \left\{ \left( \frac{1}{2} - \frac{C^E}{2} \right) \tau_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E^2 \right. \]

\[ + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) + \frac{1}{2} \left\| \sqrt{\tau} \Pi^0_{k-1} v_h \right\|_E^2 \right\} \]

\[ \geq \frac{1 - C^E}{2} \sum_{E \in T_h} \left( \left\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \right\|_E^2 + \tau_E \left\| \beta \cdot \Pi^0_{k-1} \nabla v_h \right\|_E^2 \right. \]

\[ + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) + \left\| \sqrt{\tau} \Pi^0_{k-1} v_h \right\|_E^2 \right) . \]
Next, using the coercivity of the VEM stabilization in [31] we get $∀ E ∈ T_h$,
\[
\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \|_E^2 + \tau_E \| \beta \cdot \Pi^0_{k-1} \nabla v_h \|_E^2 + S^E ((I - \Pi^0_{k-1}) v_h, (I - \Pi^0_{k-1}) v_h) \\
\geq \frac{1}{2} a_h (v_h, v_h) + \frac{1}{2} \left( \| \sqrt{K} \Pi^0_{k-1} \nabla v_h \|_E^2 + \tau_E \| \beta \cdot \Pi^0_{k-1} \nabla v_h \|_E^2 \\
+ \sigma_*(K_E + \tau_E \beta^2) \| \nabla v_h - \nabla \Pi^0_{k-1} v_h \|_E^2 \right) \\
\geq \frac{1}{2} a_h (v_h, v_h) + \min \left\{ \frac{1}{2}, \sigma_* \right\} \left( \| \sqrt{K} \Pi^0_{k-1} \nabla v_h \|_E^2 + \tau_E \| \beta \cdot \Pi^0_{k-1} \nabla v_h \|_E^2 \\
+ (K_E + \tau_E \beta^2) \| \nabla v_h - \nabla \Pi^0_{k-1} v_h \|_E^2 \right) \\
\geq \frac{1}{2} \left( \min \left\{ \frac{1}{2}, \sigma_* \right\} \right) (a_h (v_h, v_h) + a (v_h, v_h)).
\]

In the last line we use the following inequalities:
\[
\| \sqrt{K} \Pi^0_{k-1} \nabla v_h \|_E^2 + \| \sqrt{K} (\nabla v_h - \Pi^0_{k-1} \nabla v_h) \|_E^2 \geq \frac{1}{2} \| \sqrt{K} \nabla v_h \|_E^2,
\]
\[
\tau_E \left( \| \beta \Pi^0_{k-1} \nabla v_h \|_E^2 + \| \nabla v_h - \Pi^0_{k-1} \nabla v_h \|_E^2 \right) \geq \frac{1}{2} \| \beta \cdot \nabla v_h \|_E^2.
\]

4.3. A priori error estimates

Here, we prove the a priori error estimates showing that the stabilized formulation of the problem has optimal rates of convergence. Several constants in the error inequalities are numbered to track their dependence on the local problem coefficients.

**Theorem 2.** Let $u \in H^{s+1} (T_h) \cap H^1_0 (\Omega)$, $2 \leq s + 1 \leq k + 1$, be the solution of the variational problem [10] with $f \in H^{-1} (\Omega)$, $K \in [W^{s+1}_0(\Omega)]^{d \times d}$, $\beta \in [W^{s+1}_0(\Omega)]^d$ and $\gamma \in W^{s+1}_0(\Omega)$. Let $u_h \in V_h$ be the solution of the VEM [22] under the mesh assumption of Section 3.2. Then, for $h$ sufficiently small, it holds

\[
\| u - u_h \|_{K^s} \leq C h^s \left\{ \max_{E \in T_h} \left\{ \frac{\| K \|_{W^{s+1}_0(E)}}{\sqrt{K^s_E}}, \sqrt{h_E \beta E}, h_E \sqrt{\gamma E}, C_{\alpha, E}^{a,c}, C_{\alpha, E}^{a,c+1}, C_{\alpha, E}^{c}, C_{\alpha, E}^{c+1}, \kappa_{E}^{c}, \kappa_{E}^{c+1} \right\} \right\},
\]

where $c_{\alpha, E}^{a,c}, c_{\alpha, E}^{a,c+1}, c_{\alpha, E}^{c}, c_{\alpha, E}^{c+1}, \kappa_{E}^{c}$ and $\kappa_{E}^{c+1}$ are defined by [46], [48], [59], and [64], respectively, and

\[
C_{f, E}^{c} = \max \left\{ \frac{| f \Pi^0_{k-1} f |_{s-1, E}}{\sqrt{K^s_E}}, \frac{h_E^{-1} \tau_E | f \beta - \Pi^0_{k-1} (f \beta) |_{s-1, E}}{\sqrt{K^s_E}} \right\}.
\]
Proof. First, by using the triangle inequality we have
\[
\|u - u_h\|_{K,\beta} \leq \|u - u_I\|_{K,\beta} + \|u_h - u_I\|_{K,\beta}.
\]
The first term is bounded using (42) with \(\psi = u\). We are left to estimate the norm of \(e_h := u_h - u_I\). Since \(e_h \in V_h\), by (72) we know that
\[
\alpha \|e_h\|_{K,\beta}^2 \leq B_{\text{supg},h} (u_h - u_I, e_h) = F_{\text{supg},h} (e_h) - B_{\text{supg},h} (u_I, e_h)
\]
\[
= F_{\text{supg},h} (e_h) - F_{\text{supg}} (e_h) - N_h (u, e_h) + B_{\text{supg}} (u, e_h) - B_{\text{supg},h} (u_I, e_h)
\]
\[
\leq |F_{\text{supg},h} (e_h) - F_{\text{supg}} (e_h)| + |N_h (u, e_h)| + |B_{\text{supg},h} (u - u_I, e_h)|
\]
\[
+ |B_{\text{supg}} (u, e_h) - B_{\text{supg},h} (u, e_h)|.
\]
(77)

We estimate the first term as follows:
\[
\sum_{E \in T_h} \left| (f, e_h - \Pi_{k-1}^0 e_h)_E + \tau_E (f, \beta \cdot (\nabla e_h - \Pi_{k-1}^0 \nabla e_h))_E \right|
\]
\[
= \sum_{E \in T_h} \left| (f - \Pi_{k}^0 f - \Pi_{k-1}^0 e_h)_E \right| + \tau_E \left| (f - \Pi_{k}^0 f, \nabla e_h)_E \right|
\]
\[
= \sum_{E \in T_h} \left| (f - \Pi_{k}^0 f - \Pi_{k-1}^0 (f - \Pi_{k}^0 f), e_h - \Pi_{k-1}^0 e_h)_E \right|
\]
\[
+ \tau_E \left| (f - \Pi_{k}^0 f - \Pi_{k-1}^0 (f - \Pi_{k}^0 f), \nabla e_h)_E \right|
\]
\[
\leq \sum_{E \in T_h} \left( |f - \Pi_{k}^0 f - \Pi_{k-1}^0 (f - \Pi_{k}^0 f)|_E \right) \left( \|e_h - \Pi_{k-1}^0 e_h\|_E \right)
\]
\[
+ \tau_E \left( \|f - \Pi_{k}^0 f - \Pi_{k-1}^0 (f - \Pi_{k}^0 f)\|_E \right) \left( \|\nabla e_h\|_E \right)
\]
\[
\leq Ch^s \sum_{E \in T_h} \left( \|f - \Pi_{k}^0 f\|_{s-1,E} + h_E^{-1} \tau_E \|f - \Pi_{k}^0 f\|_{s-1,E} \right) \left( \|e_h\|_{K,\beta,E} \right).
\]

Using the continuity estimate (44) to bound \(a_h\), (47) to bound \(b_h\), (47) to bound \(h\), and the estimate of the VEM interpolant (42), we estimate the third term as follows:
\[
|B_{\text{supg},h} (u - u_I, e_h)| \leq Ch^s \left\{ \left( \max_{E \in T_h} \left( \|e_h\|_{K,\beta} + \|\nabla (u - u_I)\| \right) \right) \right\}
\]
\[
\leq Ch^s \max_{E \in T_h} \left\{ \sqrt{K_E}, h_E^2, h_E \sqrt{c_{\text{nc},E}^{b_h,E} + c_{\text{nc},E}^{\text{nc},E}} \right\} \|u\|_{s+1} \|e_h\|_{K,\beta}.
\]

The proof of (75) is concluded by using the above estimates, the estimate (70) on the non-conformity term and (68). □
Remark 9. The second argument of the max in (76) can be estimated as follows, using (43):

\[
\frac{h^{-1}_E}{\sqrt{K_E}} \left| \frac{f - \Pi_0^0(f \beta)}{\beta E} \right|_{s-1,E} \leq C \frac{\|\beta - \Pi_0^0 \beta\|_{W^{1}_0(E)} \|f\|_{s-1,E} + \|\beta\|_{W^{1}_0(E)} \|f - \Pi_0^0 f\|_{s-1,E}}{\beta E \sqrt{K_E}}.
\]

Remark 10. When we consider constant coefficients and a constant right-hand side all the non-consistency terms in (75) vanish, yielding the following estimate:

\[
\|u - u_h\|_{K,\beta,\gamma} \leq C h^s \max_{E \in T_h} \left( \sqrt{K_E}, \sqrt{h_E \beta E}, h_E \sqrt{\gamma E}, K_{nc}^0, K_{nc}^1 \right) \|u\|_{s+1}.
\]

Moreover, if we consider a conforming discretization, Theorem 2 proves a robust estimate with respect to the Péclet number:

\[
\|u - u_h\|_{K,\beta,\gamma} \leq C h^s \max_{E \in T_h} \left( \sqrt{K_E}, \sqrt{h_E \beta E}, h_E \sqrt{\gamma E} \right) \|u\|_{s+1},
\]

as obtained for classical Finite Elements.

5. Numerical Results

The numerical experiments of this section are aimed at confirming the convergence rates predicted by the \textit{a priori} analysis developed in the previous sections and comparing the performance of the nonconforming VEM with that of the conforming VEM. In a preliminary stage, the consistency of the numerical method, i.e. the exactness of these methods for polynomial solutions, has been tested numerically by solving the elliptic equation with boundary and source data determined by the monomials \(u(x,y) = x^\mu y^\nu\) on different set of polygonal meshes and for all possible combinations of nonnegative integers \(\mu\) and \(\nu\) such that \(\mu + \nu \leq k\), with \(k = 1, 2, 3\). In all the cases, the error magnitude was within the arithmetic precision, thus confirming the consistency of the VEM.

To study the accuracy of the method we solve the convection-reaction-diffusion equation on the domain \(\Omega = [0,1] \times [0,1]\). The variable coefficients of the equation are given by

\[
K(x,y) = \alpha \begin{bmatrix} 1 + x^2 & xy \\ xy & 1 + y^2 \end{bmatrix}, \quad \alpha = 10^{-7};
\]

\[
\beta(x,y) = (\cos(2\pi x), \sin(2\pi y))^T;
\]

\[
\gamma(x,y) = \exp(x + y).
\]

Since the Péclet number here is in the range \([10^6, 10^7]\), all calculations are in the convection dominated regime. The forcing term and the Dirichlet boundary conditions are set such that the exact solution is

\[
u(x,y) = \sin(2\pi x) \sin(2\pi y) + x^5 + y^5 + 1.
\]
The performances of the methods presented above are investigated by evaluating the rate of convergence on four different sequences of unstructured meshes, labeled by $M_1$, $M_2$, $M_3$, and $M_4$ respectively. The top panels of Fig. 3 show the first mesh of each sequence and the bottom panels show the mesh of the first refinement.

The meshes in $M_1$ are built by partitioning the domain $\Omega$ into regular hexagonal cells. At the boundaries of $\Omega$ each mesh is completed by half hexagonal cells. The meshes in $M_2$ are built as follows. First, we determine a primal mesh by remapping the position $(\hat{x}, \hat{y})$ of the nodes of a uniform square partition of $\Omega$ by the smooth coordinate transformation:

\[
    x = \hat{x} + \frac{1}{10} \sin(2\pi \hat{x}) \sin(2\pi \hat{y}),
\]

\[
    y = \hat{y} + \frac{1}{10} \sin(2\pi \hat{x}) \sin(2\pi \hat{y}).
\]

The corresponding mesh of $M_2$ is built from the primal mesh by splitting each quadrilateral cell into two triangles and connecting the barycenters of adjacent triangular cells by a straight segment. The mesh construction is completed at the boundary by connecting the barycenters of the triangular cells close to the boundary to the midpoints of the boundary edges and these latters to the boundary vertices of the primal mesh. The meshes in $M_3$ are taken from the mesh suites of the FVCA-6 Benchmark [75], and are formed by highly skewed quadrilateral cells. The meshes in $M_4$ are obtained by filling $\Omega$ with a suitably scaled non-convex octagonal reference cell.

All the meshes are parametrised by the number of partitions in each direction. The starting mesh of every sequence is built from a $5 \times 5$ regular grid, and the refined meshes are obtained by doubling this resolution.

All errors, computed as in [51, 73], are reported in Figs. 4, 5, 6, and 7. Error values are labeled by a circle for the nonconforming VEM and by a square for the conforming VEM, that are stabilized by the method developed in [73]. Each figure shows the relative errors with respect to the maximum diameter of the discretization, in the $L^2$ norms (left
panel) and in the $H^1$ norms (right panel). In the same figures we report the slopes $k+1$ for the $L^2$-norm and $k$ for the $H^1$-norm. The numerical results confirm the theoretical rate of convergence for the $H^1$-norm. The conforming and nonconforming VEMs provide very close results on any fixed mesh, with the conforming method slightly over performing the nonconforming VEM in few cases.

To test the robustness of the approach with respect to very large Péclet numbers, we have performed some tests with values of $K$ and $\beta$ in the form of (78) and (79), with $\alpha$ spanning a wide range of orders of magnitude ($\alpha \in \{10^{-i}: i = 4, \ldots, 11\}$), with $\gamma(x, y) = 0$. In Figure 8 we display the $H^1$ approximation error plotted with respect to the values of $\alpha$, on two of the meshes previously used. We can see that, as far as the presented tests are concerned, the error is bounded independently of the values of $\alpha$, even on non convex polygons, thus confirming the robustness of the approach.

5.1. Approximation of internal and boundary layers

The second test is the classic problem from [68]. The computational domain and the boundary conditions are as shown in Figure 9. The velocity forms an angle $\theta$ with the x-axis, and propagates the non-homogeneous boundary condition $u = 1$ inside $\Omega$, thus generating an internal discontinuity, which is numerically approximated by an internal layer, a sharp transition between the constant solution states $u = 0$ and 1. The homogeneous boundary condition at the top of the computational domain produces a boundary layer. The diffusion coefficient is constant on $\Omega$ and given by $K = 10^{-6}$, while the velocity is $\beta = (\cos \theta, \sin \theta)$, and $\theta = \arctan(1)$. The Péclet number is about $10^6$.

We solve this problem using the remapped and the regular hexagonal meshes (see plots (a) – (b) of Figure 33, with resolution $40 \times 40$ (third refinement). Figures 10 and 11 show the results obtained with the conforming VEM [73] (left panels) and the nonconforming VEM (right panels) for the polynomial degrees $k = 1$ and $k = 3$.

The results are quite similar to those presented in [68, 73], and are coherent with the expected behaviour of the method. Undershoots and overshoots are present near the internal layer, as is normal for this problem. However, by increasing the accuracy order of the VEM, the numerical solution becomes smoother. A thorough inspection of these plots also reveals that the nonconforming VEM tends to provide a sharper internal layer than that of the conforming VEM at the price of a relatively bigger amplitude of the spurious oscillations in the transition region.

6. Conclusions

In this paper, we proposed a nonconforming VEM for the advection-diffusion-reaction problem in the convection-dominated regime. Due to the strong convective field with respect to the diffusion term, we introduced the SUPG stabilization by extending to the nonconforming VEM the stabilization technique proposed in [73]. The stabilization included in the virtual element formulation is a natural extension of the classical SUPG stabilization for the standard FEM. To ensure coercivity of the discrete operators, we modify the SUPG stabilization by introducing a VEM stabilization of the SUPG stabilization term. Optimal convergence rates are obtained from the convergence analysis under proper assumptions on the regularity of problem coefficients, the meshes, and the exact solution. The numerical results confirm the behaviour of the VEM that is expected
from the theory and the stabilizing effect of the additional SUPG term provides stable discrete solutions even for very large Péclet numbers in the order of $10^6$.

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Figure 5: Relative approximation errors obtained using the conforming VEM (dashed lines labeled with squares) and the nonconforming VEM (solid lines labeled with circles) for $k = 1, 2, 3$ (from top to bottom). Calculations are carried out using the remapped hexagonal meshes of Figure 3(b). Errors are measured in the $L^2$ norm (left panels) and $H^1$ norm (right panels), and plotted versus $h$.

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Figure 8: $H^1$ relative approximation error versus the viscous coefficient $\alpha \in [10^{-11}, 10^{-4}]$ using the first refined mesh of mesh families (a) (top panels) and (d) (bottom panels) for the test case with $\gamma(x,y) = 0$. The problem is solved by applying the conforming VEM (left panel) and nonconforming VEM (right panel) of degree $k = 1$ (circles), $k = 2$ (squares), $k = 3$ (diamonds).
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