ON THE CONJECTURES OF BRAVERMAN-KAZHDAN

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ABSTRACT. In this article we prove a conjecture of Braverman and Kazhdan in [BK1] on acyclicity of \( \rho \)-Bessel sheaves on reductive groups in both \( \ell \)-adic and de Rham settings. We do so by establishing a vanishing conjecture proposed in [C1]. As a corollary, we obtain a geometric construction of the non-linear Fourier kernels for finite reductive groups as conjectured by Braverman and Kazhdan. The proof of the vanishing conjecture relies on the techniques developed in [BFO] on Drinfeld center of Harish-Chandra bimodules and character D-modules, and a construction of a class of character sheaves in mixed-characteristic.

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1. INTRODUCTION

This paper is a sequel to [C1]. In loc.cit. it was shown that the (generalized) Braverman-Kazhdan conjecture on acyclicity of \( \rho \)-Bessel sheaves on reductive groups follows from a certain vanishing conjecture. The goal of this paper is to give a proof of this vanishing conjecture.

In the introduction, we would like to recall the statement of the vanishing conjecture in both \( \ell \)-adic and de Rham settings, explain its applications to the Braverman-Kazhdan conjectures, and outline a proof of the vanishing conjecture.

1.1. The vanishing conjecture. Let \( k \) be an algebraically closure of a finite field \( \mathbb{F}_q \) with \( q \)-element of characteristic \( p > 0 \) or \( k = \mathbb{C} \). We fix a prime number \( \ell \) different from \( p \). We set \( F = \overline{\mathbb{Q}}_\ell \) in the case char \( k = p \) and \( F = \mathbb{C} \) in the case \( k = \mathbb{C} \). We will consider the following two geometric/sheaf-theoretic contexts: (1) \( \ell \)-adic sheaves on schemes over \( k \) of characteristic \( p \) and (2) holonomic \( D \)-modules on schemes over \( k = \mathbb{C} \). We refer context (1) as the \( \ell \)-adic setting and context (2) as the de Rham setting. We will fix a non-trivial character \( \psi : \mathbb{F}_q \rightarrow \overline{\mathbb{Q}}_\ell^\times \). Depending on the setting, let \( \mathcal{L}_\psi \) be the Artin-Schreier sheaf on the additive group \( \mathbb{G}_a \) corresponding to \( \psi \) in the \( \ell \)-adic setting or the exponential \( D \)-module in the de Rham setting.
The vanishing conjecture proposed in [C1] is a generalization of the well-known acyclicity 
\[(1.1)\quad H_c^*(G_\alpha, L_\psi) = 0.\]
of $L_\psi$ to general reductive groups. The starting point is the observation that \[(1.1)\] can be restated as acyclicity of a certain local system on $SL_2$ over certain affine subspaces. Namely, let $\text{tr} : SL_2 \to G_\alpha$ be the trace map and let $U$ be the unipotent radical of the the standard Borel subgroup $B$ of $SL_2$. Then \[(1.1)\] is equivalent to the following acyclicity of the local system $\Phi = \text{tr}^*L_\psi$ over $U$-orbits on the open Bruhat cell: for any $x \in SL_2 \setminus B$ we have 
\[(1.2)\quad H^*_c(xU, i^*\Phi) = 0.\]
Here $i : xU \to SL_2$ is the embedding. Indeed, it follows from the fact that for any $x \in SL_2 \setminus B$, the trace map restricts to an isomorphism $\text{tr} : xU \cong G_\alpha$ between the $U$-orbit through $x$ and $G_\alpha$.

To state a generalization of \[(1.2)\] to general reductive groups, let me first recall some notations and definitions. Let $G$ be a connected reductive group over $k$. Let $T$ be a maximal torus of $G$ and $B$ be a Borel subgroup containing $T$ with unipotent radical $U$. Denote by $W = N_G(T)/T$ the Weyl group, where $N_G(T)$ is the normalizer of $T$ in $G$. Depending on the setting, we denote by $\pi_1(T)$ the tame étale fundamental group of $T$ if $\text{char} \, k > 0$, or the topological fundamental group of $T$ if $k = \mathbb{C}$. We denote by $\mathcal{O}(T)(F)$ the set of continuous $F$-valued characters of $\pi_1(T)$.

For any character $\chi \in \mathcal{O}(T)(F)$, we write $L_\chi$ for the corresponding rank one $\ell$-adic/de Rham local system on $T$. The Weyl group $W$ acts naturally on $\mathcal{O}(T)(F)$ and for any $\chi \in \mathcal{O}(T)(F)$, we denote by $W'_\chi$ the stabilizer of $\chi$ in $W$ and $W_\chi \subset W'_\chi$, the subgroup of $W'_\chi$ generated by those reflections $s_\alpha$ such that the pull-back $(\tilde{\alpha})^*L_\chi$ is isomorphic to the trivial local system, where $\tilde{\alpha} : G_m \to T$ is the coroot associated to $\chi$.

Denote by $\mathcal{D}_W(T)$ the $W$-equivariant bounded derived category of sheaves on $T$. For any $\mathcal{F} \in \mathcal{D}_W(T)$ and $\chi \in \mathcal{O}(T)(F)$, the $W$-equivariant structure on $\mathcal{F}$ together with the natural $W'_\chi$-equivariant structure on $L_\chi$ give rise to an action of $W'_\chi$ on the cohomology groups $H^*_c(T, \mathcal{F} \otimes L_\chi)$ (resp. $H^*(T, \mathcal{F} \otimes L_\chi)$). In particular, we get an action of the subgroup $W_\chi \subset W'_\chi$ on the cohomology groups above. Denote by $\text{sign}_W : W \to \{\pm 1\}$ the sign character of $W$.

The key in formulating the generalization of \[(1.2)\] to general $G$ is the following definition of central complexes on $T$ introduced in [C1, Definition 1.1]:

**Definition 1.1.** A $W$-equivariant complex $\mathcal{F} \in \mathcal{D}_W(T)$ is called central (resp. $*$-central) if for any $\chi \in \mathcal{O}(T)(F)$, the group $W_\chi$ acts on

\[H^*_c(T, \mathcal{F} \otimes L_\chi) \quad (\text{resp. } H^*(T, \mathcal{F} \otimes L_\chi))\]

via the sign character $\text{sign}_W$. It is called strongly central (resp. strongly $*$-central) if the stabilizer $W'_\chi$ acts on the cohomology groups above by the sign character.

**Remark 1.1.** If the center of $G$ is connected, then it is known that $W_\chi = W'_\chi$ for all $\chi \in \mathcal{O}(T)(F)$ (see, for example, [DL, Theorem 5.13]), thus the notions of central complexes (resp. $*$-central complexes) and strongly central complexes (resp. strongly $*$-central complexes) are the same. In general, the two notions are different (see Example \[(1.2)\]).

**Example 1.2.** Consider the case $G = SL_2$. Let $\chi \in \mathcal{O}(T)(F)$ be the quadratic character associated to the covering $T \to T, x \to x^2$. We have $W_\chi = e$, $W'_\chi = W$, and it is easy to see that the $W$-equivariant local system $\mathcal{F} = L_\chi$ corresponding to $\chi$ is central but not strongly central.

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1 The group $W_\chi$ plays an important role in the study of representations of finite reductive groups and character sheaves (see, e.g., [Lu]).
Consider the induction functor $\text{Ind}_{T \subset B}^G : \mathcal{D}(T) \to \mathcal{D}(G)$ between bounded derived category of sheaves on $T$ and $G$. For $\mathcal{F} \in \mathcal{D}_W(T)$, the $W$-equivariant structure on $\mathcal{F}$ defines a $W$-action on $\text{Ind}_{T \subset B}^G(\mathcal{F})$ and we denote by

$$\Phi_{\mathcal{F}} := \text{Ind}_{T \subset B}^G(\mathcal{F})^W$$

the $W$-invariant factor in $\text{Ind}_{T \subset B}^G(\mathcal{F})$. In [C1, Conjecture 1.2], we proposed the following conjecture on acyclicity of $\Phi_{\mathcal{F}}$ over certain affine subspaces in $G$, called the vanishing conjecture:

**Conjecture 1.2.** Assume $\mathcal{F} \in \mathcal{D}_W(T)$ is central (resp. $*$-central). For any $x \in G \setminus B$, we have the following cohomology vanishing

$$H^*_c(xU, i^*\Phi_{\mathcal{F}}) = 0 \quad (\text{resp. } H^*(xU, i^!\Phi_{\mathcal{F}}) = 0)$$

where $i : xU \to G$ is the natural inclusion map. Equivalently, the derived push-forward $\pi_!(\Phi_{\mathcal{F}})$ (resp. $\pi_*(\Phi_{\mathcal{F}})$) is supported on the closed subset $T = B/U \subset G/U$. Here $\pi : G \to G/U$ is the quotient map.

**Example 1.3.** Let $G = \text{SL}_2$ and let $\text{tr} : T \simeq \mathbb{G}_m \to \mathbb{G}_a, t \to t + t^{-1}$ be the trace map. Then it is easy to show that the pull-back $\mathcal{F} = \text{tr}^*T\mathcal{L}_\psi$ together with the canonical $W$-equivariant structure is central, moreover, we have

$$\Phi_{\mathcal{F}} \simeq \text{tr}^*L_\psi$$

where $\Phi_{\mathcal{F}}$ is the trace map (see, e.g., [C1, Example 1.2 and 1.7]). It follows that Conjecture [1.2] (in this case) is equivalent to (1.2), and hence is also equivalent to the acyclicity of Artin-Schreier sheaf (1.1).

### 1.2. Braverman-Kazhdan conjectures

Assume $k = \overline{\mathbb{F}}_q$ and $G$ is defined over $\mathbb{F}_q$. In [BK1, BK2], Braverman and Kazhdan associated to each representation $\rho : \tilde{G} \to \text{GL}(V_\rho)$ of the complex dual group, a $\overline{\mathbb{Q}}_\ell$-valued function

$$\gamma_{G,\rho,\psi} : \text{Irr}(G(\mathbb{F}_q)) \to \overline{\mathbb{Q}}_\ell$$

on the set of irreducible representation of the finite group $G(\mathbb{F}_q)$, satisfying the following remarkable properties:

1. it is constant on Deligne-Lusztig packets, that is, we have $\gamma_{G,\rho,\psi}(\pi) = \gamma_{G,\rho,\psi}(\pi')$ if $\pi$ and $\pi'$ appear in the same Deligne-Lusztig representation $R_{T,\theta}$;
2. if $\pi$ appears in $R_{T,\theta}$, then the value $\gamma_{G,\rho,\psi}(\pi)$ is given by a certain explicit Gauss-type sum associated to the character $\theta$.

They called $\gamma_{G,\rho,\psi}$ the $\gamma$-function associated to $\rho$.

The function $\gamma_{G,\rho,\psi}$ on $\text{Irr}(G(\mathbb{F}_q))$ gives rise to a $\overline{\mathbb{Q}}_\ell$-valued class function $\phi_{G,\rho,\psi}$ on $G(\mathbb{F}_q)$ characterized by the property that the operator

$$\mathcal{F}_\rho : \text{Func}(G(\mathbb{F}_q)) \to \text{Func}(G(\mathbb{F}_q))$$

on the space of functions on $G(\mathbb{F}_q)$ given by convolution with $\phi_{G,\rho,\psi}$ satisfies

$$\mathcal{F}_\rho(\chi_\pi) = \gamma_{G,\rho,\psi}(\pi)\chi_\pi,$$

where $\chi_\pi$ is the character of $\pi \in \text{Irr}(G(\mathbb{F}_q))$. In the case $G = \text{GL}_n$ and $\rho = \text{std}$ is the standard representation of $\tilde{G} = \text{GL}_n(\mathbb{C})$, the function $\phi_{G,\rho,\psi}$ is given by $\psi \circ \text{tr}$ (up to some power of $q$) and the operator $\mathcal{F}_\rho$ is the linear Fourier transform on the space of functions on $\text{GL}_n(\mathbb{F}_q)$ (or rather, the restriction of the linear Fourier transform on $\text{Func}(\mathfrak{g}_{\mathbb{F}_q})$ to the subspace $\text{Func}(\text{GL}_n(\mathbb{F}_q))$). Thus, one can view $\mathcal{F}_\rho$ as a kind of non-linear Fourier transform and $\phi_{G,\rho,\psi}$ as the corresponding Fourier kernel.
In loc. cit. Braverman and Kazhdan proposed a geometric construction of $\phi_{G,\rho,\psi}$ using the theory of $\ell$-adic sheaves. To explain their construction, let us fix a $F$-stable maximal torus $T \subset G$ where $F : G \to G$ is the geometric Frobenius morphism and consider the restriction of $\rho$ to the dual maximal torus $\bar{T} \subset \bar{G}$. Then there exists a collection of weights

$$ \lambda = \{\lambda_1, ..., \lambda_r\} \subset X^*(\bar{T}) := \text{Hom}(\bar{T}, \mathbb{C}^\times) $$

such that there is an eigenspace decomposition $V_\rho = \bigoplus_{i=1}^r V_{\lambda_i}$ of $V_\rho$, where $\bar{T}$ acts on $V_{\lambda_i}$ via the character $\lambda_i$. One can regard $\lambda$ as collection of co-characters of $T$ using the the canonical isomorphism $X^*(T) \simeq X^*_s(T)$ and define

$$ \Phi_{T,\rho,\psi} = \text{pr}\lambda^\ast \text{tr}^\ast \mathcal{L}_\psi[r] \quad \Phi_{T,\rho,\psi}^r = \text{pr}\lambda^\ast \text{tr}^\ast \mathcal{L}_\psi[r], $$

where

$$ \text{pr}\lambda := \prod_{i=1}^r \lambda_i : G_m^r \to T, \quad \text{tr} : G_m^r \to \mathbb{G}_a, \quad (x_1, ..., x_r) \to \sum_{i=1}^r x_i. $$

It is shown in \cite{BK2} that both $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^r$ carry natural $W$-equivariant structures and the resulting objects in $\mathscr{D}_W(T)$, denote again by $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^r$, are called the $\rho$-Bessel sheaves\footnote{In \cite{BK1, BK2}, the authors called $\Phi_{T,\rho,\psi}$ $\gamma$-sheaves on $T$. However, based on the fact that the classical $\gamma$-function is the Mellin transform of the Bessel function, we follow \cite{N} and use the term $\rho$-Bessel sheaves instead of $\gamma$-sheaves}. The $\rho$-Bessel sheaves on $G$, denoted by $\Phi_{G,\rho,\psi}$ and $\Phi_{G,\rho,\psi}^r$, are defined as

$$ \Phi_{G,\rho,\psi} = \text{Ind}_{T \subset B}^G(\Phi_{T,\rho,\psi})^W, \quad \Phi_{G,\rho,\psi}^r = \text{Ind}_{T \subset B}^G(\Phi_{T,\rho,\psi}^r)^W. $$

It is shown in \cite{BK2} Theorem 4.2 and \cite{CN} Appendix B that, if $\rho$ satisfies certain positivity assumption (see \cite{BK2} Section 1.4), then the $\rho$-Bessel sheaves $\Phi_{T,\rho,\psi}$ and $\Phi_{T,\rho,\psi}^r$ on $T$ are in fact local systems on the image of $\text{pr}\lambda^\ast$, moreover, we have $\Phi_{T,\rho,\psi} \simeq \Phi_{T,\rho,\psi}^r$. This is a generalization of Deligne’s theorem on Kloosterman sheaves \cite{Dei}.

Braverman and Kazhdan showed that one can endow the $\rho$-Bessel sheaf $\Phi_{G,\rho,\psi}$ with a Weil structure $F^\ast \Phi_{G,\rho,\psi} \simeq \Phi_{G,\rho,\psi}$ and they proposed the following conjecture:

**Conjecture 1.3.** Let $\text{Tr}(\Phi_{G,\rho,\psi}) : G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$ is the function corresponding to $\Phi_{G,\rho}$ via the functions-sheaves correspondence. We have

$$ \text{Tr}(\Phi_{G,\rho,\psi}) = \phi_{G,\rho,\psi} $$

Conjecture [1.3] gives a geometric construction of the non-linear Fourier kernel $\phi_{G,\rho,\psi}$. They also showed that Conjecture [1.3] follows from the following conjecture on acyclicity of $\rho$-Bessel sheaves:

**Conjecture 1.4.** \cite{BK1} Conjecture 9.12] For any $x \in G \setminus B$, we have the following cohomology vanishing

$$ H^r_c(xU, i^\ast \Phi_{G,\rho,\psi}) = 0 \quad (\text{resp. } H^r(xU, i^! \Phi_{G,\rho,\psi}^r) = 0) $$

where $i : xU \to G$ is the natural inclusion map. Equivalently, the derived push-forward $\pi_!(\Phi_{G,\rho,\psi})$ (resp. $\pi_*(\Phi_{G,\rho,\psi})$) is supported on the closed subset $T = B \subset G/U$. Here $\pi : G \to G/U$ is the quotient map.

The goal of this paper is to give a proof of Conjecture [1.4] and hence Conjecture [1.3].

Note that the construction of $\rho$-Bessel sheaves and Conjecture [1.4] are entirely geometric and have obvious counterparts in the de Rham setting. Moreover, it is shown in \cite{C1} that the $\rho$-Bessel sheaves on $T$ are in fact strongly central (see Definition [1.1]). Thus the vanishing conjecture contains Conjecture [1.4] as a special case and what we actually prove here is the vanishing conjecture (or rather, Conjecture [1.2] for strongly central complexes).
Remark 1.4. Conjecture 1.3 and Conjecture 1.4 are (slightly) generalized versions of the original conjectures of Braverman and Kazhdan. The original conjectures require that the representation \( \rho \) satisfies the positivity assumption mentioned earlier. In Corollary 1.6 and Corollary 1.7 below, we will prove that their conjecture holds without any assumption on \( \rho \).

1.3. The main result. The following theorem is the main result of the paper which establishes acyclicity of \( \Phi_F \) when \( F \) is a strongly central complex (resp. strongly \( * \)-central complex), and hence the vanishing conjecture for reductive groups with connected center (see Remark 1.1), for almost all characteristics:

**Theorem 1.5.** (Theorem 7.1) There exists a positive integer \( N \) depending only on the type of the group \( G \) such that the following holds. Assume \( k = \mathbb{C} \) or \( \text{char } k = p \) is not dividing \( N\ell \). Let \( F \in \mathcal{D}_W(T) \) be a strongly central complex (resp. strongly \( * \)-central complex) on \( T \) and let \( \Phi_F = \text{Ind}_{T \subset B}^G (F)^W \in \mathcal{D}(G) \). Then for any \( x \in G \setminus B \), we have the following cohomology vanishing

\[
\begin{align*}
H_c^*(xU, i^* \Phi_F) &= 0 \quad (\text{resp. } H^*(xU, i^* \Phi_F) = 0).
\end{align*}
\]

(1.5)

Here \( i : xU \to G \) is the embedding. Equivalently, the derived push-forward \( \pi_!(\Phi_F) \) (resp. \( \pi_*(\Phi_F) \)) is supported on the closed subset \( T = B/U \subset G/U \). Here \( \pi : G \to G/U \) is the quotient map.

In particular, the vanishing conjecture (Conjecture 1.3) holds for reductive groups with connected center.

Remark 1.5. The assumption on the characteristic of \( k \) comes from a spreading out argument used in the proof (see Section 1.6).

Remark 1.6. In [C1], we proved the vanishing conjecture in the case \( G = \text{GL}_n \) using mirabolic subgroups (note that \( \text{GL}_n \) has connected center). The argument in loc. cit. was inspired by the work of Cheng and Ngô [CN] on Braverman-Kazhdan conjectures for \( G = \text{GL}_n \). The proof of Theorem 1.5 for general \( G \) uses different methods (see Section 1.6).

1.4. Applications. In this subsection we assume the characteristic of \( k \) is either zero or not dividing \( N\ell \), where \( N \) is the positive integer in Theorem 1.5.

**Corollary 1.6.** Conjecture 1.4 holds.

**Proof.** It is shown in [C1] Theorem 1.4 that the Braverman-Kazhdan’s \( \rho \)-Bessel sheaf \( \Phi_{T, \rho, \psi} \) (resp. \( \Phi_{T, \rho, \psi}^* \)) on \( T \) is strongly central (resp. strongly \( * \)-central). Thus Theorem 1.5 immediately implies the corollary.

**Corollary 1.7.** Conjecture 1.3 holds.

**Proof.** It is shown in [BK2] Corollary 6.7 that Conjecture 1.4 implies Conjecture 1.3. Thus Corollary 1.6 implies Corollary 1.7.

Conjecture 1.4 was proved by Braverman and Kazhdan [BK2] Theorem 6.9 in the case when the semi-simple rank of \( G \) is less or equal to one, and by Cheng and Ngô [CN] Theorem 2.4 in the case \( G = \text{GL}_n \).

Conjecture 1.3 was proved by Braverman and Kazhdan [BK2] Theorem 1.6 when the semi-simple rank of \( G \) is less or equal to one or \( G = \text{GL}_n \) under some assumption on \( \rho \). In a recent work, G. Laumon and E. Letellier established Conjecture 1.3 via a different method [LL] Theorem 1.0.2. It is interesting to note that in loc. cit. they also made no assumption on the representation \( \rho \).
Remark 1.7. In [LL, Theorem 1.0.1], Laumon-Letellier also proved a formula for the non-linear Fourier kernel \( \phi_{G,\rho,\psi} \) in terms of Deligne-Lusztig inductions. It will be interesting to prove a similar result in the de Rham setting, that is, write down an explicit formula for the \( \rho \)-Bessel \( D \)-module \( \Phi_{G,\rho,\psi} \), or rather, the corresponding system of differential equations on \( G \). We expect applications of such formula to the Braverman-Kazhdan-Ngô’s approach to functional equation of automorphic \( L \)-functions [BKL, Ng].

1.5. Whittaker sheaves on \( G \). In the course of proving Theorem 1.5 we establish several characterizations of \(*\)-central \( D \)-modules on \( T \) using the Mellin transform, see Theorem 1.3. On the other hand, in their works on Whittaker \( D \)-modules, nil Hecke algebras, and quantum Toda lattices, Ginzburg [Gi2] and Lonergan [L] prove that the category of Whittaker \( D \)-modules on \( G \) is equivalent to a certain full subcategory of the category of \( W \)-equivariant \( D \)-modules on \( T \). It turns out that, as an immediate corollary of Theorem 1.3, the latter full subcategory is equivalent to the category of central \( D \)-modules on \( T \). Thus, combining the results in loc. cit. we obtain:

**Theorem 1.8.** ([Theorem 4.5]) The category of Whittaker \( D \)-modules on \( G \) is equivalent to the abelian category of central \( D \)-modules on \( T \).

It was mentioned in the introduction of [Gi2] that Drinfeld asked the question of finding a description of an \( \ell \)-adic counterpart of the category of Whittaker \( D \)-modules on \( G \) in terms of \( W \)-equivariant sheaves on \( T \). Theorem 1.8 suggests that the category of \( \ell \)-adic central perverse sheaves on \( T \) might provide an answer to Drinfeld’s question (see Section 1.4).

Remark 1.8. In a recent work [BZG], Ben-Zvi and Gunningham constructed a remarkable functor from the category of Whittaker \( D \)-modules on \( G \) to the category of \( G \)-conjugation equivariant \( D \)-module on \( G \), they called it Ngô functor, and they conjectured that the objects in the essential image of the Ngô functor satisfy the cohomology vanishing properties in [1.3], see [BZG, Conjecture 2.9 and 2.14]. Our Theorem 1.5 and Theorem 1.8 might be useful for studying their conjectures.

1.6. Outline of the proof. The proof of Theorem 1.5 consists of three steps:

Step 1. We construct for each \( W \)-orbit \( \theta \) in \( \mathcal{E}(T)(F) \) a remarkable \( W \)-equivariant local system \( \mathcal{E}_\theta \) on \( T^A \) and consider the equivariant perverse sheaf or \( D \)-module \( M_\theta = \text{Ind}_{F \subset B}(\mathcal{E}_\theta)^W \) on \( G \). A key observation is that to prove Theorem 1.5 it suffices to prove the acyclicity of \( M_\theta \), that is, the cohomology vanishing properties (1.5) for \( M_\theta \). This follows from a computation of the convolution of \( M_\theta \) with \( \Phi_F \) (Proposition 7.3), where \( F \in \mathcal{D}_W(T) \) is a strongly \(*\)-central complex, and a result of Laumon on the conservativity of the Mellin transform (Lemma 7.5).  

Step 2. We use the techniques developed in [BFO] on character \( D \)-modules and Drinfeld center of Harish-Chandra bimodules to prove acyclicity of \( M_\theta \) in the de Rham setting. An important point here is that \( \mathcal{E}_\theta \) is a tame local system on \( T \), and hence the associated \( D \)-module \( M_\theta \) on \( G \) is a character \( D \)-module and the results in loc. cit. are applicable. A key step in the proof is to show that the Harish-Chandra bimodule corresponding to \( \mathcal{E}_\theta \), under the Beilinson-Bernstein localization theorem, has a canonical central structure (Proposition 6.7).

Step 3. We construct a mixed characteristic lifting \( M_{\theta,A} \) of \( M_\theta \) over a strictly Henselian ring \( A \) with residue field \( k \) of characteristic not dividing \( N\ell \), where \( N \) is a positive integer depending only on the type of \( G \). We prove that \( M_{\theta,A} \) is universally locally acyclic with respect to the quotient map \( \pi_A : G_A \to G_A/U_A \) (here \( G_A \) and \( U_A \) are models of \( G \) and \( U \) over \( A \)). This allows us to deduce acyclicity of \( M_\theta \) in the \( \ell \)-adic setting from the de Rham setting. This completes the proof of Theorem 1.5.

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3The author learned the existence of \( \mathcal{E}_\theta \) from R. Bezrukovnikov.
The results established in this paper reveal interesting connections between the Braverman-Kazhdan conjectures, Whittaker $D$-modules, character sheaves, and categorical center of Hecke categories. We plan to explore those connections in a future work [C2].

Remark 1.9. The proof of acyclicity of $M_\theta$ in the de Rham setting makes use of Harish-Chandra bimodules, and hence is algebraic. It would be interesting to have a geometric proof which treats the cases of various ground fields and sheaf theories uniformly. Presumably, such a proof will provide an explicit bound of the integer $N$ in Theorem 1.5.

Remark 1.10. Using a similar but more involved argument, one can show that the vanishing conjecture holds for reductive groups with disconnected center. Such a generalization is not needed for the proof of Braverman-Kazhdan conjectures, so the details will be worked out in a forthcoming paper [C2].

1.7. Organization. We briefly summarize here the main goals of each section. In Section 2 we collect standard notation in algebraic groups, $\ell$-adic sheaves, and $D$-modules. In Section 3 we study induction and restriction functors. In Section 4, we study characterizations of central and $\ast$-central complexes using Mellin transforms. In Section 5 we introduce the $\ast$-central local systems $E_\theta$ and establish some basic properties of them. In Section 6 we prove acyclicity of character sheaves $M_\theta$ (Theorem 6.1). In Section 7 we prove Theorem 1.5.

Acknowledgement. I especially like to thank Gérard Laumon for useful discussions. The argument using character sheaves in mixed-characteristic to prove the Braverman-Kazhdan conjectures in the $\ell$-adic setting, follows a suggestion of his. I also would like to thank Roman Bezrukavnikov, Ngô Bao Châu, Victor Ginzburg, and Zhiwei Yun for useful discussions. I am grateful for the support of NSF grant DMS-1702337.

2. Notations

2.1. We denote by $\mathcal{B} = G/B$ the flag variety. We denote by $g, b, t, n$ the Lie algebras of $G,B,T,U$. We denote by $G_{rs}$ (resp. $T_{rs}$) the open subset consisting of regular semi-simple elements in $G$ (resp. $T$). We denote by $G_{reg}$ the open subset consisting of regular elements in $G$. We denote by $\mathbb{G}_a$ the additive group and $\mathbb{G}_m$ the multiplicative group. We denote by $\hat{G}$ the complex dual group of $G$ and $\hat{T}$ the dual maximal torus. We denote by $W_{\text{ex}} = W \ltimes \Lambda$ the extended affine Weyl group and $W_\alpha = W \ltimes R$ the affine Weyl group. Here $\Lambda = \mathbb{X}_*(\hat{T}) = \text{Hom}(\mathbb{G}_m, \hat{T})$ is the co-character lattice and $R \subseteq \Lambda$ is the set of co-roots of $\hat{G}$.

2.2. For an algebraic stack $\mathcal{X}$ over $k$, we denote by $\mathcal{D}(\mathcal{X})$ the bounded derived category of $\ell$-adic sheaves on $\mathcal{X}$ in the $\ell$-adic setting or the bounded derived category of holonomic $D$-modules on $\mathcal{X}$ in the de Rham setting.

For a smooth scheme $X$, we will write $1_X \in \mathcal{D}(X)^\bigodot$ for the constant perverse sheaf $\mathcal{O}_X[\dim X]$ on $X$ in the $\ell$-adic setting or the structure sheaf $\mathcal{O}_X$ in the de Rham setting.

For a representable morphism $f : \mathcal{X} \to \mathcal{Y}$, the six functors $f^*, f_*, f'_!, g^!, \otimes, \text{Hom}$ are understood in the derived sense. For a smooth map $f : X \to Y$ of relative dimension $d$ we write $f^o = f^*[d] = f^![d]$. For an algebraic group $H$ over $k$ acting on a $k$-scheme $X$, we denote by $X/H$, the corresponding quotient stack and $X//H$ the geometric invariant quotient (if exists). We will write $H/_{\text{ad}} H$ for the quotient stack of $H$ with respect to the adjoint action. Consider the case when $H$ is a finite group. Then the pull-back along the quotient map $X \to X/H$ induces an equivalence between $\mathcal{D}(X/H)$ and the (naive) $H$-equivariant derived category on $X$, denoted by $\mathcal{D}_H(X)$, whose objects consist of pair
(F, φ), where F ∈ D(X) and φ : a∗F ∼ pr∗F is an isomorphism satisfying the usual compatibility conditions (here a and pr are the action and projection map from H × X to X respectively).\footnote{This holds in a more general situation when the neutral component of H is unipotent.} We will call an object (F, φ) in D_H(X) a H-equivariant complex and φ a H-equivariant structure on F. For simplicity, we will write F = (F, φ) for an object in D_H(X).

We denote by τ_{≤ n}, τ_{≥ n} the truncation functors corresponding to the standard t-structure on D(X). For any F ∈ D(X), we denote by H^m(F) the m-th cohomology sheaf. In the ℓ-adic setting, we denote by pτ_{≤ n}, pτ_{≥ n} the truncation functors corresponding to the perverse t-structure. For any F ∈ D(X), the n-th perverse cohomology sheaf is defined as \( p^*H^n(F) = p^*\tau_{≥ n}p^*\tau_{≤ n}(F)[n] \).

Depending on the setting, we write D(X)_♥ for the heart corresponding to the perverse t-structure and the heart corresponding to the standard t-structure in the de Rham setting.

For any stack X over k, we denote by Coh(X) and QCoh(X) the categories of coherent and quasi-coherent sheaves on X, and Db_coh(X) and Db_geoh(X) the corresponding bounded derived categories.

Let F be an quasi-coherent sheaf or D-module on a scheme. We will write \( Γ(F) \) and \( RΓ(F) \) for the global section and derived global section of F as an quasi-coherent sheaf. For any scheme X we will write \( O_X \) for the structure sheaf of X and \( O(X) = Γ(O_X) \) the ring of global functions on X.

Assume \( k = \mathbb{C} \). For any smooth scheme X we denote by \( D_X \) the sheaf of differential operators on X. Let f : X → Y be a principal T-bundle over a smooth scheme Y. A D-module F on X is called T-monodromic if it is weakly T-equivariant (see [BB, Section 2.5]). A object F ∈ D(X) is called T-monodromic if \( H^i(F) \) is T-monodromic for all i. Let F ∈ D(X) be a T-monodromic object. For any \( μ ∈ \mathfrak{t} \simeq \mathfrak{t}^* \), we denote \( Γ^μ(F) \) (resp. \( RΓ^μ(F) \)) the maximal summand of \( Γ(F) \) (resp. \( RΓ(F) \)) where \( \mathfrak{t} \), acting as infinitesimal translations along the action of T, acts with the generalized eigenvalue μ.

## 3. Induction and restriction functors

In this section we collect some known facts about induction and restriction functors.

### 3.1. Recall the Grothendieck-Springer simultaneous resolution of the Steinberg map c : G → T//W:

\[
\begin{array}{ccc}
\tilde{G} & \overset{\tilde{q}}{\longrightarrow} & T \\
\downarrow{\tilde{e}} & & \downarrow{q} \\
G & \overset{c}{\longrightarrow} & T//W
\end{array}
\]

where \( \tilde{G} \) is the closed subvariety of \( G × G/B \) consisting of pairs \( (g, xB) \) such that \( x^{-1}gx ∈ B \). The map \( \tilde{e} \) is given by \( (g, xB) → g \), and the map \( \tilde{q} \) is given by \( (g, xB) → x^{-1}gx \mod U ∈ B/U = T \).

The induction functor \( \text{Ind}^G_{T⊂B} : D(T) → D(G) \) is given by

\[
\text{Ind}^G_{T⊂B}(F) = \tilde{c}_*q^*(F).
\]

We have the following equivalent constructions of \( \text{Ind}^G_{T⊂B} \). Consider the fiber product

\[
Z = G ×_{T//W} T.
\]
It is known that the map \( h : \tilde{G} \to Z \) induced from (3.1) is a small map and it follows that IC(Z) = \( h_1S \in \mathcal{D}(Z) \) is the IC-complex of Z. We have

\[
(3.2) \quad \text{Ind}_{T\subset B}^G(\mathcal{F}) \simeq (p_G)_*(p_T^*(\mathcal{F}) \otimes \text{IC}(Z))|_{\dim G - \dim T}
\]

where \( p_T : Z \to T \) and \( p_G : Z \to G \) are the natural projection map. Let \( X \) be a scheme acted on by an algebraic group \( H \) and let \( H' \subset H \) be a subgroup. There is an averaging functor \( \text{Av}^H_{H'} = \pi_* : \mathcal{D}(X/H') \to \mathcal{D}(X/H) \) (resp. \( \text{Av}^H_{H'} = \pi_* : \mathcal{D}(X/H') \to \mathcal{D}(X/H) \)) which is the right adjoint (resp. left adjoint) of the forgetful functor \( \text{oblv}^H_{H'} : \mathcal{D}(X/H) \to \mathcal{D}(X/H') \). Here \( \pi : X/H' \to X/H \) is the natural map. If \( H' = e \) is trivial we will omit \( H' \) and simply write \( \text{Av}^H = \text{Av}^H_{H/e} \) (resp. \( \text{Av}^H = \text{Av}^H_{H/e} \)) and \( \text{oblv}^H = \text{oblv}^H_{H/e} \).

Note that, for any \( \mathcal{T} \in \mathcal{D}(T) \), its \( * \)-pull back along \( B \to T = B/U \), denoted by \( \mathcal{T}_B \), can be regarded as an object in \( \mathcal{D}(G/_{\text{ad}}B) \) and there is a canonical isomorphism

\[
\text{oblv}^G_{/B}(\mathcal{T}_B) \simeq \text{Ind}_{T\subset B}^G(\mathcal{T}).
\]

The functor \( \text{Ind}_{T\subset B}^G \) admits a right adjoint \( \text{Res}_{T\subset B}^G : \mathcal{D}(G) \to \mathcal{D}(T) \), called the restriction functor, and is given by

\[
\text{Res}_{T\subset B}^G(\mathcal{F}) = (q_B)_*i_B^!(\mathcal{F})
\]

where \( i_B : B \to G \) is the natural inclusion and \( q_B : B \to T = B/U \) is the quotient map. More generally, one could define \( \text{Res}_{L\subset P}^G : \mathcal{D}(G) \to \mathcal{D}(L) \), for any pair \((L, P)\) where \( L \) is a Levi subgroup of a parabolic subgroup \( P \) of \( G \).

We have the following exactness properties of induction and restriction functors:

**Proposition 3.1.** (1) The functor \( \text{Ind}_{T\subset B}^G \) maps perverse sheaves on \( T \) to perverse sheaves on \( G \).
(2) The functor \( \text{Res}_{T\subset B}^G \) maps \( G \)-conjugation equivariant perverse sheaves on \( G \) to \( L \)-conjugation equivariant perverse sheaves on \( L \).

**Proof.** This is [B-Y, Theorem 5.4]. \( \square \)

### 3.2. W-action

Let \( \mathcal{T} \in \mathcal{D}_W(T) \). Since the map \( p_T : S \to T \) and the IC-complex \( \text{IC}(S) \) are \( W \)-equivariant, it follows from (3.2) that the \( W \)-equivariant structure on \( \mathcal{T} \) gives rise to a \( W \)-action on \( \text{Ind}_{T\subset B}^G(\mathcal{T}) \). We denote by

\[
(3.3) \quad \Phi_\mathcal{T} := \text{Ind}_{T\subset B}^G(\mathcal{T})^W
\]

the \( W \)-invariant factor of \( \text{Ind}_{T\subset B}^G(\mathcal{T}) \).

In the case when \( \mathcal{T} \) is a \( W \)-equivariant perverse local system on \( T \), we have the following description of \( \Phi_\mathcal{T} \): Let \( q^r: T^r \to T^r/\!/W \) and \( e^r: G \to T^r/\!/W \) be the restriction of the maps in (5.1) to the regular semi-simple locus. As \( q^r \) is an étale covering, the restriction of \( \mathcal{T} \) to \( T^r \) descends to a perverse local system \( \mathcal{T}^r \) on \( T^r/\!/W \) and we have

\[
\Phi_\mathcal{T} = \text{Ind}_{T\subset B}^G(\mathcal{T})^W \simeq j^*(e^r)^*\mathcal{T}^r|_{\dim G - \dim T}.
\]

Let \( \mathcal{T} \in \mathcal{D}_W(T) \) and let \( F[W] \to \text{End}_{\mathcal{D}(G)}(\text{Ind}_{T\subset B}^G(\mathcal{T})) \) be the map coming from the \( W \)-action. By adjunction, we get a map

\[
F[W] \to \text{End}_{\mathcal{D}(G)}(\text{Ind}_{T\subset B}^G(\mathcal{T})) \simeq \text{Hom}_{\mathcal{D}(G)}(\mathcal{T}, \text{Res}_{T\subset B}^G \circ \text{Ind}_{T\subset B}^G(\mathcal{T}))
\]

which gives rise to

\[
(3.4) \quad F[W] \otimes \mathcal{T} \to \text{Res}_{T\subset B}^G \circ \text{Ind}_{T\subset B}^G(\mathcal{T}).
\]
We have the following generalization of [Gu, Theorem 4.6] to the group setting:

**Proposition 3.2.** Let $\mathcal{F} \in \mathcal{D}(T)^\wedge$. (1) There is a canonical isomorphism $\bigoplus_{w \in W} w^*\mathcal{F} \simeq \text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(\mathcal{F})$. (2) Assume further that $\mathcal{F} \in \mathcal{D}(W(T)^\wedge)$. Then the composition

\[
(3.5) \quad F[W] \otimes \mathcal{F} \simeq \bigoplus_{w \in W} w^*\mathcal{F} \simeq \text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(\mathcal{F}),
\]

is equal to (3.4). Here $\phi$ is the $W$-equivariant structure of $\mathcal{F}$.

**Proof.** We follow the argument in loc. cit.. We shall give a proof in the de Rham setting. The same proof works for the $\ell$-adic setting. Consider the product $S_G = G \times_G B$. There are two natural maps $q_S : S_G \to T$ and $c_S : S_G \to T$ coming from $\tilde{q} : \tilde{G} \to T$ and $\tilde{c} : \tilde{G} \to T$ in (3.1). Let $S_G := \mathcal{H}^0(q_S \times c_S)(\omega_{S_G})$. Using base changes formulas, it is easy to see that the functor $\text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(-)$ is given by the kernel $S_G$, that is, we have, $\text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(\mathcal{F}) \simeq \text{pr}_I^*(\text{pr}_R^*\mathcal{F} \otimes S_G)$, here $\text{pr}_I : \text{pr}_R : T \times T \to T$ are the left and right projection maps. Now the same proof as in [Gu, Section 4.3], replacing $t$ by $T$, shows that there is a canonical isomorphism of monads $S_G \simeq \bigoplus_{w \in W} \mathcal{O}_{\Gamma_w}$, here $\Gamma_w = \{ x, wy \mid x \in T \} \subset T \times T$. It follows that $\text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(\mathcal{F}) \simeq \text{pr}_I^*(\text{pr}_R^*\mathcal{F} \otimes S_G) \simeq \bigoplus_{w \in W} w^*\mathcal{F}$. This completes the proof of (1). Assume $\mathcal{F}$ is $W$-equivariant. It follows from the construction that the isomorphism in (1) intertwines the $W$-action on $\text{Res}^G_{T \subset B} \circ \text{Ind}^G_{T \subset B}(\mathcal{F})$ with the one on $\bigoplus_{w \in W} w^*\mathcal{F}$ given by the map

\[
a_r : \bigoplus_{w \in W} w^*\mathcal{F} \xrightarrow{w^*(\delta_r)} \bigoplus_{w \in W} w^*r^*\mathcal{F} \simeq \bigoplus_{w \in W} (rw)^*\mathcal{F} = \bigoplus_{w \in W} w^*\mathcal{F},
\]

for any $r \in W$. Part (2) follows. \hfill $\square$

We will need the following properties of induction functors. Let $m_G : G \times G \to G$ and $m_T : T \times T \to T$ be the multiplication maps. For any $M, M' \in \mathcal{D}(G)$, we define $M*M' := m_G^*(M \boxtimes M') \in \mathcal{D}(G)$. Similarly, for any $\mathcal{F}, \mathcal{F}' \in \mathcal{D}(T)$, we define $\mathcal{F} * \mathcal{F}' = m_T^*(\mathcal{F} \boxtimes \mathcal{F}')$.

**Proposition 3.3.** Let $M \in \mathcal{D}(G_{/\text{ad}}G)^\wedge$ and $\mathcal{F} \in \mathcal{D}(T)$. Assume $\text{Av}^U_\cdot(M)$ is supported on $T = B/U \subset G/U$. Then we have a natural isomorphism in $\mathcal{D}(G)$

\[
\text{Ind}^G_{T \subset B}(\mathcal{F}) * M \simeq \text{Ind}^G_{T \subset B}(\mathcal{F} * \text{Res}^G_{T \subset B}(M))
\]

which is functorial with respect to $\mathcal{F}$.

**Proof.** This is proved in [BK2, Proposition 2.9]. Let us recall the construction in loc.cit.. For any $\mathcal{H} \in \mathcal{D}(G_{/\text{ad}}B)$ and $M \in \mathcal{D}(G_{/\text{ad}}G)^\wedge$, there is a natural isomorphism

\[
\text{Av}^G_{*B}(\mathcal{F} * \text{oblv}^G/B M) \simeq \text{Av}^G_{*B}(\mathcal{H}) * M.
\]

When $\mathcal{H} = \mathcal{F}_B = q^*_B \mathcal{F}$ is the pull back of $\mathcal{F}$ along $q_B : B \to T = B/U$, the assumption that $\text{Av}^U_\cdot(M)$ is supported on $T$ implies that

\[
\mathcal{F}_B * \text{oblv}^G/B M \simeq (\mathcal{F} * \text{Res}^G_{T \subset B}(M))_B \in \mathcal{D}(G_{/\text{ad}}B)
\]

and it follows that

\[
\text{Ind}^G_{T \subset B}(\mathcal{F}) * M \simeq \text{oblv}^{G/e} \circ (\text{Av}^G_{*B}(\mathcal{F}_B) * M) \simeq \text{oblv}^{G/e} \circ (\text{Av}^G_{*B}(\mathcal{F}_B * \text{oblv}^G/B M)) \simeq \text{oblv}^{G/e} \circ (\text{Av}^G_{*B}((\mathcal{F} * \text{Res}^G_{T \subset B}(M))_B)) \simeq \text{Ind}^G_{T \subset B}(\mathcal{F} * \text{Res}^G_{T \subset B}(M)).
\]

\hfill $\square$
4. Mellin transform and characterizations of central and ∗-central complexes

4.1. The scheme of tame characters. Let \( \pi_1^\ell(T) \) be the tame étale fundamental group of \( T \) in the \( \ell \)-adic setting or the topological fundamental group in the de Rham setting. For any continuous character \( \chi : \pi_1^\ell(T) \to F^\times \), we denote by \( \mathcal{L}_\chi \) the corresponding rank one local system on \( T \).

We first consider the de Rham setting. Set \( \mathcal{C}(T) = \text{Spec}(\mathbb{C}[\pi_1^\ell(T)]) \). Then the \( \mathbb{C} \)-points of \( \mathcal{C}(T) \) are in bijection with characters of \( \pi_1^\ell(T) \). Note that, under the isomorphism \( \pi_1^\ell(T) = \mathbb{X}_\bullet(T) \), characters of \( \pi_1^\ell(T) \) correspond to elements in the dual torus \( \tilde{T} \).

In the \( \ell \)-adic setting, in [GL], a \( \overline{\mathbb{Q}}_\ell \)-scheme \( \mathcal{C}(T) \) is defined, whose \( \overline{\mathbb{Q}}_\ell \)-points are in bijection with continuous characters of \( \pi_1^\ell(T) \). There is decomposition

\[
\mathcal{C}(T) = \bigsqcup_{\chi \in \mathcal{C}(T)_f} \{ \chi \} \times \mathcal{C}(T)_f
\]

into connected components, where \( \mathcal{C}(T)_f \subset \mathcal{C}(T) \) is the subset consisting of tame characters of order prime to \( \ell \) and \( \mathcal{C}(T)_f \) is the connected component of \( \mathcal{C}(T) \) containing the trivial character. It is shown in loc. cit. that \( \mathcal{C}(T) \) is Noetherian and regular and there is an isomorphism

\[
\mathcal{C}(T)_f \simeq \text{Spec}(\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[x_1, \ldots, x_r]]).
\]

In addition, the \( \overline{\mathbb{Q}}_\ell \)-points of \( \mathcal{C}(T)_f \) are in bijection with pro-\( \ell \) characters of \( \pi_1(T) \) (i.e. characters of the pro-\( \ell \) quotient \( \pi_1(T)_f \) of \( \pi_1(T) \)).

4.2. Mellin transforms. We give a review of Mellin transforms in both de Rham and \( \ell \)-adic settings and establish some basic facts about them.

We first recall the Mellin transform of \( \mathcal{D} \)-modules on \( T \). Let \( x_i \in \Lambda = \mathbb{X}_\bullet(\tilde{T}) \simeq \text{Hom}(T, \mathbb{G}_m) \) be a basis and consider the regular function \( \mathcal{O}(T) \simeq \mathbb{C}[x_i^{\pm 1}] \) and the algebra of differential operators \( \Gamma(\mathcal{D}_T) \simeq \mathbb{C}[x_i^{\pm 1}] \langle v_i \rangle/\{v_i x_j = x_j (\delta_{ij} + v_i)\} \) where \( v_i = x_i \partial_{x_i} \in \mathfrak{t} \) are a basis for the \( T \)-invariant vector fields. Recall that for any \( \mathcal{D} \)-module \( \mathcal{F} \) on \( T \), the tensor product \( \mathcal{F} \otimes \omega_T \) with the canonical line bundle \( \omega_T \) on \( T \), carries a natural right \( \mathcal{D} \)-module structure. Note also that, if we consider \( \Gamma(\mathcal{D}_T) \) as the algebra of difference operators \( \mathbb{C}[v_i] \langle x_i^{\pm 1} \rangle/\{v_i x_j = x_j (\delta_{ij} + v_i)\} \), then there is a canonical equivalence between the category \( \text{Q Coh}(\mathfrak{t}/\Lambda) \) of \( \Lambda \)-equivariant quasi-coherent sheaves on \( \mathfrak{t} \) and the category of right \( \Gamma(\mathcal{D}_T) \)-modules. The Mellin transform functor is defined as

\[
\mathcal{M} : \mathcal{D} \text{-mod}(T) \to \text{Q Coh}(\mathfrak{t}/\Lambda), \quad N \to \Gamma(N \otimes \omega_T).
\]

We have the following properties:

1. The functor \( \mathcal{M} \) is an equivalence.
2. Let \( \chi \in \tilde{T}(\mathbb{C}) \simeq \mathcal{C}(T)(\mathbb{C}) \) and let \( \lambda \in \mathfrak{t}(\mathbb{C}) \) be a lift of \( \chi \) along the universal covering \( \exp : \mathfrak{t} \to \tilde{T} \). Then for any \( \mathcal{F} \in \mathcal{D} \text{-mod}(T) \) we have

\[
H^\ast(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i_\lambda^\ast \mathcal{M}(\mathcal{F})
\]

where \( i_\lambda : \text{pt} \to \tilde{t} \) is the embedding given by \( \lambda \).
3. Consider the bounded derived category \( \mathcal{D}_q^{b}(\tilde{\mathfrak{t}}/\Lambda) \) of \( \Lambda \)-equivariant quasi-coherent sheaves on \( \tilde{\mathfrak{t}} \) with the monoidal structure given by the (derived) tensor product. We have

\[
\mathcal{M}(\mathcal{F} \ast \mathcal{F}') \simeq \mathcal{M}(\mathcal{F}) \otimes \mathcal{M}(\mathcal{F}').
\]

\[\text{For any right } \Gamma(\mathcal{D}_T)\text{-module } N, \text{ the action of } \mathbb{C}[v_i] = \mathcal{O}(\mathfrak{t}) \text{ on } N \text{ gives a } \mathcal{O}_\mathfrak{t}\text{-module structure on } \mathcal{O}_\mathfrak{t} \otimes_{\mathcal{O}(\mathfrak{t})} N \in \text{Q Coh}(\mathfrak{t}) \text{ and the action of } x_i \in \Lambda \text{ defines a } \Lambda \text{-equivariant structure.}\]
(4) Let $\mathcal{F}$ be a $W$-equivariant $D$-module $\mathcal{F}$ on $T$. Then $W$-equivariant structure on $\mathcal{F}$ gives rise to a $W_a^{\text{ex}} = W \times \Lambda$-equivariant structure on $\mathcal{M}(\mathcal{F})$. Let $\chi$ and $\lambda$ be as in (1) and let $W_{a,\lambda}$ and $W_{a,0}$ be the stabilizers of $\lambda$ in $W_a^{\text{ex}}$ and $W_a$ respectively. Then $H^*(T, \mathcal{F} \otimes \mathcal{L})$ (resp. $i^*_\lambda \mathcal{M}(\mathcal{F})$) carries a natural action of $W'_\chi$ (resp. $W_{a,\lambda}^{\text{ex}}$) such that, under the isomorphism

$$W'_\chi \simeq W_{a,\lambda}^{\text{ex}}, \quad w \mapsto (w, w^{-1}\lambda - \lambda)$$

the isomorphism (4.1) intertwines those actions.

We now consider the $\ell$-adic setting. In [GL], the authors constructed the Mellin transform

$$\mathcal{M} : \mathcal{D}(T) \rightarrow D^b_{\text{coh}}(\mathbb{C}(T))$$

with the following properties:

1. Let $\chi \in \mathbb{C}(T)\overline{\mathbb{Q}_\ell}$ and $i_\chi : \text{pt} \rightarrow \mathbb{C}(T)$ be the embedding given by $\chi$. We have

$$H^*(T, \mathcal{F} \otimes \mathcal{L}_\chi) \simeq i^*_\chi \mathcal{M}(\mathcal{F}).$$

2. For any $\chi \in \mathbb{C}(T)\overline{\mathbb{Q}_\ell}$ we have

$$\mathcal{M}(\mathcal{F} \otimes \mathcal{L}_\chi) \simeq m^*_\chi \mathcal{M}(\mathcal{F}).$$

3. The functor $\mathcal{M}$ is $t$-exact with respect to the perverse $t$-structure on $\mathcal{D}(T)$ and the standard $t$-structure on $D^b_{\text{coh}}(\mathbb{C}(T))$. Moreover, for any $\mathcal{F} \in \mathcal{D}(T)$, $\mathcal{F}$ is perverse if and only if $\mathcal{M}(\mathcal{F})$ is a coherent complex in degree zero.

4. We have

$$\mathcal{M}(\mathcal{F} \ast \mathcal{F}') \simeq \mathcal{M}(\mathcal{F}) \otimes \mathcal{M}(\mathcal{F}')$$

5. The Mellin transforms restricts to an equivalence

$$\mathcal{M} : \mathcal{D}(T)_{\text{mon}} \simeq D^b_{\text{coh}}(\mathbb{C}(T))_f$$

between the full subcategory $\mathcal{D}(T)_{\text{mon}}$ of monodromic $\ell$-adic complexes on $T$ and the full subcategory $D^b_{\text{coh}}(\mathbb{C}(T))_f$ of coherent complexes on $\mathbb{C}(T)$ with finite support.

6. For $\mathcal{F} \in \mathcal{D}_W(T)$, the $W$-equivariant structure on $\mathcal{F}$ gives rise to a $W$-equivariant structure on $\mathcal{M}(\mathcal{F})$ such that for any $\chi \in \mathbb{C}(T)\overline{\mathbb{Q}_\ell}$ the isomorphism in (1) above is compatible with the natural $W'_\chi$-actions.

Remark 4.1. In the de Rham setting, a non-zero invariant section $\sigma$ of $\omega_T$ gives rise to an isomorphism $\Gamma(\mathcal{M}(\mathcal{F})) = \Gamma(\mathcal{F} \otimes \omega_T) \simeq \Gamma(\mathcal{F})$, here $v_i$ acts on $\Gamma(\mathcal{F})$ by the formula $v_i \cdot m = -v_i m$. Since the $W$-action on invariant sections of $\omega_T$ is given by the sign character, we obtain an isomorphism of $\mathcal{O}(t)$-modules

$$\Gamma((-1)^* \mathcal{M}(\mathcal{F} \otimes \text{sign})) \simeq \Gamma(\mathcal{F})$$

cOMPATIBLE with the natural $W_a^{\text{ex}}$-actions. Here $-1 : \mathfrak{t} \rightarrow \mathfrak{t}, x \rightarrow -x$.

The properties of Mellin transforms above imply the following:

**Lemma 4.1.** Let $\mathcal{F} \in \mathcal{D}_W(T)$. (1) Assume $\text{char } k > 0$. Then $\mathcal{F}$ is $*$-central (resp. strongly $*$-central) if and only if, for any $\chi \in \mathbb{C}(T)\overline{\mathbb{Q}_\ell}$, the action of $W'_\chi$ (resp. $W_{a,\lambda}^{\text{ex}}$) on $\mathcal{H}^n(i^*_\chi \mathcal{M}(\mathcal{F} \otimes \text{sign}))$ for all $n \in \mathbb{Z}$ is trivial. (2) Assume $k = \mathbb{C}$. Then $\mathcal{F}$ is $*$-central (resp. strongly $*$-central) if and only if, for any $\lambda \in \mathfrak{t}(\mathbb{C})$, the action of $W_{a,\lambda}$ (resp. $W_{a,\lambda}^{\text{ex}}$) on $\mathcal{H}^n(i^*_\lambda \mathcal{M}(\mathcal{F} \otimes \text{sign}))$ for all $n \in \mathbb{Z}$ is trivial.

We finish this section with a lemma to be used in Section [Loc.cit.]

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\[ \text{It is denoted by } \mathcal{M}, \text{ in loc.cit.} \]
Lemma 4.2. Assume $k = \mathbb{C}$. Let $\lambda \in \mathfrak{t}(\mathbb{C})$ and let $\check{i}_\lambda$ be the completion of $\check{i}$ at $\lambda$. For any holonomic complex $\mathcal{F} \in \mathcal{D}(T)$, the restriction $\mathcal{M}(\mathcal{F})|_{\check{i}_\lambda}$ is a coherent complex on $\check{i}_\lambda$.7

Proof. By induction argument on the (finite) number of non vanishing cohomology sheaves of $\mathcal{F}$, we can assume $\mathcal{F} \in \mathcal{D}(T)^{\mathbb{C}}$. Let $I$ be the maximal ideal corresponding to $\lambda$ and let $R = \mathcal{O}(\check{i})$, $M = \Gamma(\mathcal{M}(\mathcal{F}))$. We shall show that the natural injection $M \otimes_R R_{\lambda} = \lim M/I^kM$ is an isomorphism and $M_{\lambda}$ is a finitely generated $R_{\lambda}$-module. Note that, since $\mathcal{F}$ is holonomic, the non-derived fiber $\mathcal{H}^0(\check{i}_{\lambda}^*\mathcal{M}(\mathcal{F})) \simeq M/I^0M$ is finite dimensional. Choose a finitely generated submodule $N \subset M$ such that natural map $N/I_\lambda N \to M/I_\lambda M$ is onto. This implies $N/I_\lambda N \to M/I_\lambda M$ is onto for all $n > 0$. Indeed, let $N'$ be the image of the map above. Then we have $M/I^nM = N' + IM/I^nM$. By iterating the equality above, we obtain $M/I^nM = N' + I(N' + IM/I^nM) = N' + I^2(M/I^nM) = \cdots = N' + I^k(M/I^nM)$ for any $k > 0$, and take $k = n$, we get $M/I^nM = N'$. We have shown that $N_{\lambda} \to M_{\lambda}$ is surjective and, as $N$ is finitely generated, it follows that $M_{\lambda}$ is finitely generated and $M \otimes_R R_{\lambda} \to M_{\lambda}$ is an isomorphism.

4.3. Characterization of *-central $D$-modules. In the de Rham setting, we have the following characterization of *-central $D$-modules:

Theorem 4.3. Let $\mathcal{F}$ be a $W$-equivariant holonomic $D$-module on $T$. The following are equivalent

1. $\mathcal{F}$ is *-central.
2. for any $\lambda \in \mathfrak{t}(\mathbb{C})$, the action of $W_{a,\lambda}$ on $\mathcal{H}^n(\check{i}_{\lambda}^*\mathcal{M}(\mathcal{F} \otimes \text{sign}))$, $n = 0, -1$, is trivial.
3. for any $\lambda \in \mathfrak{t}(\mathbb{C})$, the Mellin transform $\mathcal{M}(\mathcal{F} \otimes \text{sign})$, regarding as a $W_{a,\lambda}$-equivariant quasi-coherent sheaf on $\check{i}$, descends to $\check{i}/W_{a,\lambda}$.
4. for any $\lambda \in \mathfrak{t}(\mathbb{C})$, the natural map $\mathcal{O}(\check{i}) \otimes_{\mathcal{O}(\check{i})} W_{a,\lambda} \Gamma(\mathcal{F})^{W_{a,\lambda}} \to \Gamma(\mathcal{F})$ is a bijection.

A similar result in the $\ell$-adic setting was proved in [CT, Proposition 4.2].

Remark 4.2. Note that in the de Rham setting the definition of *-central $D$-modules makes sense for arbitrary $W$-equivariant $D$-modules on $T$, and the proof of Theorem 4.3 below shows that the above characterization remains true without the holonomicity assumption on $\mathcal{F}$. Very similar results are proved in [G1, L].

We begin the following lemma.

Lemma 4.4. Let $\Gamma$ be a finite reflection group with reflection representation $V$ over $\mathbb{C}$. Let $\mathcal{F}$ be a $\Gamma$-equivariant quasi-coherent sheaf on $V$. Then $\mathcal{F}$ descends to $V//\Gamma$ if and only if for any $\lambda \in V(\mathbb{C})$ the actions of the stabilizer $\Gamma_{\lambda}$ of $\lambda$ in $\Gamma$ on $\mathcal{H}^n(\check{i}_{\lambda}^*\mathcal{F})$ and $\mathcal{H}^{-1}(\check{i}_{\lambda}^*\mathcal{F})$ are trivial. Here $i_{\lambda} : \text{pt} \to V$ is the embedding given by $\lambda$.

Proof. Assume $\mathcal{F}$ descends to $V//\Gamma$. Then we have $\mathcal{F} \simeq \pi^*(\pi_*\mathcal{F})$ where $\pi : V \to V//\Gamma$ is the quotient map and it implies $i_{\lambda}^*\mathcal{F} \simeq i_{\lambda}^*(\pi_*\mathcal{F})$, where $\lambda = \pi(\lambda)$ and $i_{\lambda} : \text{pt} \to V//\Gamma$ is the inclusion. As $\Gamma$ acts trivially on $\mathcal{H}^n(i_{\lambda}^*(\pi_*\mathcal{F}))$ for all $n \in \mathbb{Z}$, it follows that $\Gamma$ acts trivially on $\mathcal{H}^n(i_{\lambda}^*\mathcal{F})$ for all $n \in \mathbb{Z}$, in particular, for $n = 0, -1$.

Assume $\Gamma$ acts trivially on $\mathcal{H}^n(i_{\lambda}^*\mathcal{F})$ for $n = 0, -1$. We would like to show that $\mathcal{F}$ descends to $V//\Gamma$. By [L Theorem 1.3.2], it suffices to show that $\mathcal{F}$ descends to $V//\sigma$ for any simple reflection $\sigma \in \Gamma$. So we could assume $\Gamma = \langle \sigma \rangle$ is generated by a reflection $\sigma$. Let $\mathcal{F}^\sigma = \{ h \in \mathcal{F} | \sigma(h) = h \}$ and

7Since $\mathcal{M}(\mathcal{F})$ is in general not coherent, the claim is not automatic.
$\mathcal{F}^{\sigma=-1} = \{ h \in \mathcal{F} | \sigma(h) = -h \}$. Choose a coordinate $(x_1, \ldots, x_n)$ of $V$ such that $\sigma(x_1) = -x_1$ and $\sigma(x_i) = x_i$ for $i \geq 2$. Let $\lambda \in V(\mathbb{C})$ be the origin with coordinate $x_i = 0$.

We claim that $\Gamma = \Gamma_\lambda$ acts trivially on $\mathcal{H}^0(i_\lambda^* \mathcal{F}) = \mathcal{F}/(x_1, \ldots, x_n)\mathcal{F}$ implies the natural map

\begin{equation}
\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F}
\end{equation}

is surjective.

Step 1. We first show that $\Gamma$ acts trivially on $\mathcal{F}/(x_1)\mathcal{F}$ implies (4.4) is surjective. Indeed, the assumption implies that the image of $f \in \mathcal{F}^{\sigma=-1}$ in the quotient $\mathcal{F}/(x_1)\mathcal{F}$ is zero, that is, $f \in (x_1)\mathcal{F}$. Since $\sigma(x_1) = -x_1$ and $\mathcal{F} = \mathcal{F}^\sigma \oplus \mathcal{F}^{\sigma=-1}$, it follows that $f = x_1f'$ for some $f' \in \mathcal{F}^\sigma$ and it implies (4.4) is surjective.

Step 2. We show that $\Gamma$ acts trivially on $\mathcal{H}^0(i_\lambda^* \mathcal{F})$ implies $\Gamma$ acts trivially on $\mathcal{F}/(x_1)\mathcal{F}$. The case $n = 1$ is trivial as $\mathcal{H}^0(i_\lambda^* \mathcal{F}) = \mathcal{F}/(x_1)\mathcal{F}$. Assume $n > 1$. Consider the exact sequence

\begin{equation}
\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F} \to \mathcal{M} \to 0
\end{equation}

where $\mathcal{M}$ is the cokernel. The quotient $\mathcal{F}' = \mathcal{F}/(x_2, \ldots, x_n)\mathcal{F}$ is a $\Gamma$-equivariant quasi-coherent sheaf on $V' = \text{Spec}(\mathbb{C}[x_1])$ such that $\Gamma$ acts trivially on $\mathcal{F}'/(x_1)\mathcal{F}' = \mathcal{F}/(x_1, \ldots, x_n)\mathcal{F}$, thus Step 1 implies $\mathcal{O}(V') \otimes (\mathcal{F}')^\sigma \to \mathcal{F}'$ is surjective. It follows that the first arrow in the exact sequence

$$\mathcal{O}(V)/(x_2, \ldots, x_n)\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F}/(x_2, \ldots, x_n)\mathcal{F} \to \mathcal{M}/(x_2, \ldots, x_n)\mathcal{M} \to 0$$

induced from (4.3) is surjective, and hence $\mathcal{M}/(x_2, \ldots, x_n)\mathcal{M} = 0$. It implies $\mathcal{M}/(x_1)\mathcal{M} = \mathcal{M}/(x_1, x_2, \ldots, x_n)\mathcal{M}$, which is an quotient of $\mathcal{H}^0(i_\lambda^* \mathcal{F}) = \mathcal{F}/(x_1, \ldots, x_n)\mathcal{F}$, and thus $\Gamma$ acts trivially on $\mathcal{M}/(x_1)\mathcal{M}$. Consider the exact sequence

$$\mathcal{O}(V)/(x_1)\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F}/(x_1)\mathcal{F} \to \mathcal{M}/(x_1)\mathcal{M} \to 0$$

induced from (4.5). As $\Gamma$ acts trivially on $\mathcal{O}(V)/(x_1)\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma$ and $\mathcal{M}/(x_1)\mathcal{M}$, it follows that $\Gamma$ acts trivially on $\mathcal{F}/(x_1)\mathcal{F}$. This completes the proof of Step 2. The desired claim follows.

Consider the short exact sequence

$$0 \to \mathcal{N} \to \mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F} \to 0,$$

where $\mathcal{N}$ is the kernel of (4.4). It gives rise to an exact sequence

\begin{equation}
\mathcal{H}^{-1}(i_\lambda^* \mathcal{F}) \to \mathcal{N}/(x_1, \ldots, x_n)\mathcal{N} \to \mathcal{O}(V)/(x_1, \ldots, x_n)\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F}/(x_1, \ldots, x_n)\mathcal{F} \to 0.
\end{equation}

Since $\Gamma$ acts trivially on $\mathcal{H}^{-1}(i_\lambda^* \mathcal{F})$ (by assumption) and $\mathcal{O}(V)/(x_1, \ldots, x_n)\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma$, (4.6) implies that $\Gamma$ acts trivially on $\mathcal{N}/(x_1, \ldots, x_n)\mathcal{N}$, and the claim above implies that the natural map $\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{N} \to \mathcal{N}$ is surjective. All together, we obtain a $\Gamma$-equivariant presentation

$$\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{N} \to \mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma \to \mathcal{F} \to 0$$

where $\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{N}$ and $\mathcal{O}(V) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{F}^\sigma$ are free $\Gamma$-equivariant sheaves that descends to $V//\Gamma$ and, by [Ne] Lemma 3.1, it implies $\mathcal{F}$ descends to $V//\Gamma$.

\[\square\]

Remark 4.3. The lemma above is inspired by [Ne] Theorem 1.2. In loc. cit., the author proved a similar descent criterion in the case when $\Gamma$ is a reductive algebraic group and $\mathcal{F}$ is a $\Gamma$-equivariant coherent sheaf. The argument in loc. cit. used the coherence assumption of $\mathcal{F}$ hence can not be applied directly to the case of quasi-coherent sheaves. The proof above uses some special features of finite reflection groups.
Proof of Theorem 4.3. (1) implies (2) is clear. (2) implies (3) follows from the fact that $\Gamma = \mathcal{W}_{a,v}$ is generated by affine reflections passing through $\lambda \in \ell(\mathcal{C})$ and, for any $\mu \in \ell(\mathcal{C})$, the stabilizer $\Gamma_\mu = \mathcal{W}_{a,\lambda} \cap \mathcal{W}^\text{ex}_{a,\mu}$ is a subgroup of $\mathcal{W}_{a,\mu}$, hence acts trivially on $\mathcal{M}^j(\mathfrak{g} \otimes \text{sign})$, $j = 0, -1$. (3) implies (1) follows from the first paragraph of the proof of Lemma [1.3]. The equivalence between (3) and (4) follows from Remark [1] and the fact $\mathcal{M}(\mathfrak{g} \otimes \text{sign})$ descends to $\ell//\mathcal{W}_{a,v}$ if and only if $(-1)^* \mathcal{M}(\mathfrak{g} \otimes \text{sign})$ descends to $\ell//\mathcal{W}_{a,v}$.

4.4. Whittaker sheaves on $G$. Consider the de Rham setting. Fix a generic character $f : \mathfrak{n} \to \mathbb{C}$. A holonomic $D$-modules $\mathcal{F}$ on $G$ is called a Whittaker $D$-module if for any vector $n \in \mathfrak{n}$, the actions of $a_i(n) - f(n)$ and $a_r(n) - f(n)$ on $\Gamma(\mathcal{F})$ are locally nilpotent. Here $a_i$ (resp. $a_r$) is the embedding of $\mathfrak{g}$ in $\Gamma(\mathcal{D}_G)$ as left-invariant (resp. right-invariant) differential operators. In [Gi2, Theorem 1.5.1] and [1, Theorem 1.2.2], Ginzburg and Lonergan proved that the category of Whittaker $D$-modules on $G$ is equivalent to the category of $\mathcal{W}$-equivariant holonomic $D$-module on $T$ satisfying condition (3) in Theorem [1.3]. Since the Verdier duality induces an equivalence between the category of central $D$-modules and the category of $\ast$-central $D$-modules, Theorem [1.3] together with the results in loc. cit. imply:

**Theorem 4.5.** There is an equivalence between the category of Whittaker $D$-modules on $G$ and the category of central (resp. $\ast$-central) $D$-modules on $T$.

**Remark 4.4.** In loc.cit. Ginzburg and Lonergan also proved that the category of Whittaker $D$-modules on $G$ is equivalent to the category of holonomic modules over the quantum Toda lattice of $G$, and is also equivalent to the category of holonomic modules over nil-Hecke algebra of $G$.

Consider the $\ell$-adic setting. Fix a non-degenerate homomorphism $\chi : U \to \mathbb{G}_a$, that is, the restriction of $\chi$ to each root subgroup $U_\alpha \subset U$ is nontrivial for each simple root $\alpha$. An $\ell$-adic Whittaker sheaf on $G$ is a perverse sheaf $\mathcal{F}$ on $G$ together with an isomorphism $\chi^* \mathcal{F} \simeq \chi^* \mathcal{L}^\psi \boxtimes \mathfrak{g}^\circ \mathcal{F}$ satisfying the usual cocycle condition. Here $a : U \times U \times G \to G$, $a(u_1, u_2, g) = u_1 gu_2^{-1}$. We conjecture the following $\ell$-adic counterpart of Theorem [1.5]:

**Conjecture 4.6.** There is an equivalence between the category of $\ell$-adic Whittaker sheaves on $G$ and the category of central perverse sheaves on $T$.

5. Local systems $\mathcal{E}_\theta$

5.1. Tame local systems $\mathcal{E}_\theta$. In this subsection we attach each $\mathcal{W}$-orbit $\theta = \mathcal{W}_\chi$ in $\mathcal{C}(T)(F)$ a $\mathcal{W}$-equivariant tame local system $\mathcal{E}_\theta$ on $T$.

We first consider the de Rham setting. Define $S$ to be the completion of the group algebra $\mathbb{C}[[\mathfrak{g}(T)]]$ with respect to the the kernel of the map $\mathbb{C}[[\mathfrak{g}(T)]] \to \mathbb{C}$ sending $\gamma \in \mathfrak{g}(T)$ to $1 \in \mathbb{C}$ and let $S_\chi$ be the argumentation ideal. The Weyl group $\mathcal{W}$ acts naturally on $S$ and we define $S_\chi = S/\langle S^W_\chi \rangle$, where $\langle S^W_\chi \rangle$ is the ideal generated by $S^W_\chi$. Since $W_\chi$ is normal in $W^W_\chi$, $S_\chi$ carries an action of $W^W_\chi$ and we define $\rho_\chi^{uni}$ to be the representation of $W^W_\chi \rtimes \mathfrak{g}(T)$ in the space $S_\chi$ by setting $\rho_\chi^{uni}(w, \gamma)v = w(\gamma)v$, where $(w, \gamma) \in W^W_\chi \rtimes \mathfrak{g}(T)$ and $v \in S_\chi$. Since $W^W_\chi$ is the stabilizer of $\chi$ in $W$, one can twist $\rho_\chi^{uni}$ by the character $\chi$ and obtain a representation $\rho_\chi := \rho_\chi^{uni} \otimes \chi$ of $W^W_\chi \rtimes \mathfrak{g}(T)$ in $S_\chi$ and its induced representation $\rho_\theta := \text{Ind}_{W^W_\chi \rtimes \mathfrak{g}(T)}^{\mathcal{W}_\chi \rtimes \mathfrak{g}(T)} \rho_\chi$ of $\mathcal{W}^\text{ex}_{a,\theta} = \mathcal{W} \rtimes \mathfrak{g}(T)$. We define $\mathcal{E}_\chi$ and $\mathcal{E}_\theta$ to be the $W^W_\chi$ and $\mathcal{W}$-equivariant local systems on $T$ corresponding to $\rho_\chi$ and $\rho_\theta$.

We now consider the $\ell$-adic setting. Choose a finite extension $K$ of $\mathbb{Q}_\ell$ such that elements in $\theta$ are defined over the ring of integer of $O_K$. Let $S_K = O_K[[\mathfrak{g}(T)]]$ be the completed group algebra of
the pro-$\ell$ quotient $\pi_1(T)_{\ell}$ and let $S_{K,+}$ be the argumentation ideal. We define $S_{X,K} = S_K/(S_{K,+}^W)$. Let $\rho_{\chi}^{uni}$ be the $\ell$-adic representation of $W_{\chi} \ltimes \pi_1(T)$ in $S_{X,K} \otimes_{R_K} \overline{\mathbb{Q}}_\ell$ given by $\rho_{\chi}^{uni}(w, \gamma)(v) = w(\gamma v)$, where $\gamma$ is the image of $\gamma$ under the quotient $\pi_1(T) \to \pi_1(T)_{\ell}$. Applying the same construction as in the de Rham setting, we obtain $\ell$-adic representations $\rho_{\chi} = \rho_{\chi}^{uni} \otimes \chi$ and $\rho_{\theta} = \text{Ind}_{W_{\chi} \ltimes \pi_1(T)}^W \rho_{\chi}$ of $W_{\chi} \ltimes \pi_1(T)$ and $W \ltimes \pi_1(T)$ in $S_{X,K} \otimes_{S_K} \overline{\mathbb{Q}}_\ell$. We define $\mathcal{E}_\chi$ and $\mathcal{E}_\theta$ to be the $W_{\chi}$ and $W$-equivariant $\ell$-adic local systems on $T$ corresponding to $\rho_{\chi}$ and $\rho_{\theta}$.

5.2. Mellin transform of $\mathcal{E}_\theta$. In this subsection we study the Mellin transform of $\mathcal{E}_\theta$.

We first consider the de Rham setting. For any $\chi \in \mathcal{C}(T)(\mathbb{C})$, consider the projection map $\pi_\chi : \mathfrak{i} \to \mathfrak{i}/W_{\chi}$. To every $\mu \in \mathfrak{i}$, let $m_\mu : \mathfrak{i} \to \mathfrak{i}$, $v \to v + \mu$. We define

$$
\mathcal{R}_\chi = \pi_\chi^* \delta, \quad \mathcal{R}_\mu = m_\mu^* \mathcal{R}_\chi
$$

where $\delta = \delta_0$ the the skyscraper sheaf supported at $0 \in \mathfrak{i}/W_{\chi}$. Let $\mu$ be a lift of $\chi$. Then we have the following cartesian diagram

$$
\begin{array}{ccc}
\mathfrak{i} & \xrightarrow{m_\mu} & \mathfrak{i} \\
\downarrow{\pi_\mu} & & \downarrow{\pi_\chi} \\
\mathfrak{i}/W_{a,-\mu} & \xrightarrow{m_\mu} & \mathfrak{i}/W_{\chi}
\end{array}
$$

where $\pi_\mu$ is the quotient map and $m_\mu$ is induced by the isomorphism $W_{a,-\mu} \simeq W_{\chi,-1} = W_{\chi}$ in $\mathfrak{i} = W_{\chi}$.

It follows that $\mathcal{R}_\mu = \pi_\chi^* \mathcal{R}_\chi$. The equality $l_{w(\mu)} \circ w = w \circ l_\mu$ gives rise to an isomorphism $\mathcal{R}_\mu \simeq w^* \mathcal{R}_{\mu}^{w(\mu)}$ for $w \in W_{\chi}$. The isomorphisms $\mathcal{R}_\chi^\mu + \lambda \simeq l_\lambda^* \mathcal{R}_\chi, \lambda \in \Lambda$ and $\mathcal{R}_\mu^\mu \simeq w^* \mathcal{R}_{\chi}^{w(\mu)}$, $w \in W_{\chi}$, define a $W_{\ell} \ltimes \Lambda$-equivariant structure on

$$
\mathcal{S}_\chi = \bigoplus_{\mu \in \Lambda + \Lambda} \mathcal{R}_\mu^\chi.
$$

The induced $W \ltimes \Lambda$-equivariant sheaf $\text{Ind}_{W_{\chi} \ltimes \Lambda}^W \mathcal{R}_\chi$ on $\mathfrak{i}$ depends only on the $W$-orbit $\theta \subset \mathcal{C}(T)(\mathbb{C})$ of $\chi$ and we denote it by $\mathcal{S}_\theta$. Note that a choice of a lift $\mu$ of $\chi$ gives rise to isomorphisms

$$
(5.1) \quad \mathcal{S}_\chi \simeq \text{Ind}_{W_{\chi} \ltimes \Lambda}^W (\mathcal{R}_\mu^\chi), \quad \mathcal{S}_\theta \simeq \text{Ind}_{W_{\chi} \ltimes \Lambda}^W \text{Ind}_{W_{a,-\mu} \ltimes \Lambda}^{W_{\chi}} (\mathcal{R}_\mu^\mu) \simeq \text{Ind}_{W_{a,-\mu} \ltimes \Lambda}^{W_{\chi}} (\mathcal{R}_\mu^\mu).
$$

We now consider the $\ell$-adic setting. Consider the quotient map $\pi_\chi : \mathcal{C}(T) \to \mathcal{C}(T)/W_{\chi}$. Let $0 \in \mathcal{C}(T)(\overline{\mathbb{Q}}_\ell)$ be the trivial character and let $\pi_\chi(0)$ be its image in $\mathcal{C}(T)/W_{\chi}$. Let $\mathcal{O}_{\pi_\chi(0)}$ be the structure sheaf of the point $\pi_\chi(0)$ and we define

$$
(5.2) \quad \mathcal{R}_\chi := \pi_\chi^* \mathcal{O}_{\pi_\chi(0)}
$$

which a $W_{\chi}$-equivariant coherent sheaf on $\mathcal{C}(T)$. Define $\mathcal{S}_\chi := m_\chi^* \mathcal{R}_\chi$ where $m_\chi : \mathcal{C}(T) \to \mathcal{C}(T)$ be the morphism of translation by $\chi$. Since $m_\chi$ intertwines the $W_{\chi}$-action on $\mathcal{C}(T)$, $\mathcal{S}_\chi$ is $W_{\chi}$-equivariant, moreover, there is natural isomorphism

$$
(5.3) \quad w^* \mathcal{S}_\chi \simeq \mathcal{S}_{w^{-1} \chi}
$$

for any $w \in W$. For any $W$-orbit $\theta$ in $\mathcal{C}(T)$, we define the following $W$-equivariant coherent sheaf with finite support

$$
\mathcal{S}_\theta = \bigoplus_{\chi \in \theta} \mathcal{S}_\chi
$$

where the $W$-equivariant structure is given by the isomorphisms in $(5.3)$. 


Note that $S_\theta$ is set theoretically supported on $\theta^{-1} = \{ \chi^{-1} | \chi \in \theta \} \subset \mathcal{C}(T)(\overline{\mathbb{Q}_\ell})$ in the $\ell$-adic setting, and on $\{-\mu | \exp(\mu) \in \theta \} \subset \mathfrak{t}$ in the de Rham setting.

**Lemma 5.1.** There is an isomorphism  
\[ \mathcal{M}(\mathcal{E}_\theta \otimes \text{sign}) \simeq S_\theta \]
compatible with the $W$-equivariant structures in the $\ell$-adic setting, and the $W_{\text{ex}}$-equivariant structures in the de Rham setting.

**Proof.** The $\ell$-adic setting. Pick a $\chi \in \theta$. It suffices to show that there is an isomorphism $\mathcal{M}(\mathcal{E}_\chi \otimes \text{sign}) \simeq S_\chi$ compatible with the $W'_\chi$-equivariant structures. The $\overline{\mathbb{Q}_\ell}[[\pi_1(T)_\ell]]$-module corresponding to $\mathcal{E}_\chi \otimes \mathcal{L}_{\chi^{-1}}$ is isomorphic to $\mathcal{R}_\chi$. Since $\mathcal{R}_\chi \simeq \text{inv}^* \mathcal{R}_\chi$ (here inv the inverse map on $\mathcal{C}(T)$), by [GL, Corollary 4.2.2.4], there is an isomorphism  
\[ \mathcal{M}(\mathcal{E}_\chi) \simeq m^*_{\chi^{-1}} \mathcal{M}(\mathcal{E}_\chi \otimes \mathcal{L}_{\chi^{-1}}) \simeq m^*_{\chi}(\mathcal{R}_\chi \otimes_{\mathbb{Z}_\ell} \wedge_{\mathbb{Z}_\ell}^{\text{top}}(\pi_1(T)_\ell)^\vee) \simeq S_\chi \otimes_{\mathbb{Z}_\ell} \wedge_{\mathbb{Z}_\ell}^{\text{top}}(\pi_1(T)_\ell)^\vee, \]
compatible with the $W'_\chi$-actions. By choosing a generator of $\wedge_{\mathbb{Z}_\ell}^{\text{top}}(\pi_1(T)_\ell)^\vee$, we obtain a $W$-equivariant isomorphism $\wedge_{\mathbb{Z}_\ell}^{\text{top}}(\pi_1(T)_\ell)^\vee \simeq \mathbb{Z}_\ell \otimes \text{sign}$. The desired claim follows.

The de Rham setting. We have an isomorphism of $\mathcal{O}(\mathfrak{t})$-modules $\Gamma(\mathcal{E}_\theta) \simeq \Gamma(S_{\theta-1})$ compatible with the $W_{\text{ex}}$-actions. Thus, by Remark 4.1, there is an isomorphism $\mathcal{M}(\mathcal{E}_\theta \otimes \text{sign}) \simeq (-1)^* S_{\chi^{-1}} \simeq S_\chi$ intertwines the $W_{\text{ex}}$-equivariant structures.

**Corollary 5.2.** $\mathcal{E}_\theta$ is $*$-central.

**Proof.** It is proved in [CI, Corollary 5.2] in the $\ell$-adic setting. In the de Rham setting, Lemma 5.1 and the construction of $S_\theta$ imply the Mellin transform $\mathcal{M}(\mathcal{E}_\theta \otimes \text{sign})$ satisfies condition (3) in Theorem 4.3 hence is $*$-central.

5.3. Convolution with $\mathcal{E}_\theta$. Recall the following descent criterion for coherent complexes [Ne]:

**Lemma 5.3.** Let $X = \text{Spec}(A)$ be an affine Noetherian scheme over $F$. Let $H$ be a finite group acting on $X$. Let $\mathcal{F} \in D^b_{\text{coh}}(X/H)$ be a $H$-equivariant complex of coherent sheaves on $X$. The following are equivalent

1. For any closed point $x \in X$, the action of the stabilizer $G_x$ of $x$ in $G$ on the derived fiber $i_x^* \mathcal{F}$ is trivial. Here $i_x : x \to X$ is the embedding.
2. $\mathcal{F}$ descends to $X//H = \text{Spec}(A^H)$.

**Proof.** By [MPK, Theorem 1.1], $X \to X//G$ is a universal geometric quotient, in particular, a good quotient in the sense of [Ne]. Now the result follows from [Ne, Theorem 1.3].

**Proposition 5.4.** Let $\mathcal{F}$ be a strongly $*$-central complex and let $\theta = W_\chi$ be a $W$-orbit of a tame character $\chi$. There is an isomorphism  
\[ \mathcal{F} \ast (\mathcal{E}_\theta \otimes \text{sign}) \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathcal{D}_W(T) \]

8In loc. cit. the Theorem is proved for schemes of finite type over $F$, but the same argument works for Noetherian schemes over $F$.
Proof. The ℓ-adic setting. For any tame character χ, let $D_\chi$ (resp. $D_\chi'$) be the fiber of $\pi_\chi : \mathcal{C}(T) \to \mathcal{C}(T)/W_\chi$ over $\chi^{-1}$ (resp. $\pi_\chi'(\chi^{-1})$). For any W-orbit $\theta = W_\chi$, let $D_\theta = \bigsqcup_{\chi \in \theta} D_\chi$ and $D_\theta' = \bigsqcup_{\chi \in \theta} D_\chi'$. We have closed embeddings $D_\chi \to D_\chi'$ and $D_\theta \to D_\theta'$, which are isomorphisms if $W_\chi = W_\chi'$.

By Lemma 5.1, we have

(5.4) $\mathcal{M}(\mathcal{E}_\theta \otimes \text{sign}) \simeq S_\theta \simeq \mathcal{O}_{D_\theta} \in \mathcal{D}_W(T)$.

Thus there is an isomorphism

(5.5) $\mathcal{M}(\mathcal{F} \otimes \mathcal{E}_\theta) \simeq \mathcal{M}(\mathcal{F}) \otimes \mathcal{M}(\mathcal{E}_\theta) \simeq \mathcal{M}(\mathcal{F})|_{D_\theta} \simeq \mathcal{M}(\mathcal{F})|_{D_\theta}$.

Since $\chi^{-1}$ is the unique closed point of $D_\chi'$, by Lemma 4.1 and Lemma 5.3, the restriction $\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{D_\chi}$ descends to $D_\chi'/W_\chi$ and it implies there is an isomorphism

(5.6) $\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{D_\chi} \simeq \mathcal{M}(\mathcal{F})|_{\chi} \otimes \mathcal{O}_{D_\chi} \simeq \mathcal{O}_{D_\chi} \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{O}_{D_\chi}$

compatible with the $W_\chi'$-equivariant structure. It follows that there is an isomorphism

(5.7) $\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{D_\chi} \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{O}_{D_\chi} \in \mathcal{D}_W(T)$.

All together, we obtain

$\mathcal{M}(\mathcal{F} \otimes \mathcal{E}_\theta) \simeq (\mathcal{M}(\mathcal{F})|_{D_\theta}) \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{O}_{D_\theta} \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{M}(\mathcal{E}_\theta)$.

Since the Mellin transform restricts to an equivalence on monodromic sheaves [4.3], the isomorphism above comes from an isomorphism

(5.8) $\mathcal{F} \otimes (\mathcal{E}_\theta \otimes \text{sign}) \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathcal{D}_W(T)$.

The de Rham setting. Choose a lifting $\lambda \in \hat{\mathcal{I}}$ of $\chi$ and write $D_\lambda$ and $D_\lambda'$ for the fiber of $\pi_{-\lambda} : i \to i/W_{a,-\lambda}$ and $\pi'_{\lambda} : i \to i/W_{a,-\lambda}$ over $\pi_{-\lambda}(-\lambda)$ and $\pi'_{\lambda}(-\lambda)$, and let $D_\theta = \bigsqcup_{\lambda \in \theta} D_\lambda$ and $D_\theta' = \bigsqcup_{\lambda \in \theta} D_\lambda'$. Now the same argument as in the ℓ-adic setting, replacing $D_\chi$, $D_\chi'$, $W_\chi$, and $W_\chi'$ by $D_\lambda$, $D_\lambda'$, $W_{a,-\lambda}$, and $W_{a,-\lambda}$, gives the desired isomorphism (5.8).

Remark 5.1. Note that if we only assume $\mathcal{F}$ is $*$-central, then the isomorphism in (5.6) is only compatible with the $W_\chi'$-equivariant structures. As a result, the isomorphisms (5.8) is not compatible with the W-equivariant structure.

The proposition above can be reformulated as follows:

Proposition 5.5. Let $\mathcal{F}$ be a strongly $*$-central complex and let $\theta = W_\chi$ be a W-orbit of a tame character $\chi$. For each $w \in W$, there is a canonical isomorphism in $\mathcal{D}(T)$

(5.9) $a_w : w^* \mathcal{F} \otimes \mathcal{E}_\theta \simeq \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes w^* \mathcal{E}_\theta$,

such that the following diagram is commutative

(5.10) $\begin{array}{ccc}
w^* \mathcal{F} \otimes \mathcal{E}_\theta & \xrightarrow{a_w} & \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes w^* \mathcal{E}_\theta \\
\Downarrow & & \Downarrow \\
\mathcal{F} \otimes \mathcal{E}_\theta & \xrightarrow{a_w} & \mathcal{H}^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta 
\end{array}$

where the vertical arrows are the isomorphism induced from the W-equivaraint structures on $\mathcal{F}$ and $\mathcal{E}_\theta$.
Let $\mathcal{E}_\theta$ be the + central local system in Section 5. We define $\mathcal{M}_\theta := \Phi \mathcal{E}_\theta = \text{Ind}_{G \subset B}^G (\mathcal{E}_\theta)^W$. Recall the averaging functor $\text{Av}_G^U := \pi_* : \mathcal{D}(G) \to \mathcal{D}(G/U)$, where $\pi : G \to G/U$ is the quotient map. The goal of this section is to prove the following theorem:

**Theorem 6.1.** There exists a positive integer $N$ depending only on the type of the group $G$ such that the following holds. Assume $k = \mathbb{C}$ or $\text{char } k = p$ is not dividing $N\ell$. We have

$$\text{Av}_G^U(M_\theta) \simeq \mathcal{E}_\theta.$$ 

In particular, $\text{Av}_G^U(M_\theta)$ is supported on $T = B/U \subset G/U$.

We will first establish Theorem 6.1 in the case $k = \mathbb{C}$ using the results in \cite{BFO}, \cite{BG}, and \cite{MS} on Harich-Chandra bimodules, categorical centers of Hecke categories, and Character $D$-modules. Then we construct a mixed characteristic lifting $\mathcal{M}_{\theta,A}$ of $\mathcal{M}_\theta$ over a strictly Henselian ring $A$ with residue field $k$ of characteristic $p$ not dividing $N\ell$, which is universal local acyclic with respect to the quotient map $G_A \to G_A/U_A$. This allows us to use spreading out arguments to prove Theorem 6.1 in positive characteristic case.

We will assume $k = \mathbb{C}$ until Section 6.8.

6.1. **Hecke categories.** Consider the left $G$ and right $T \times T$ actions on $Y = G/U \times G/U$. To every $\chi, \chi' \in \hat{T}$ we denote by $M_{\chi,\chi'}$ the category of $G$-equivariant $D$-modules on $G/U \times G/U$ which are $T \times T$-monodromic with generalized monodromy $(\chi, \chi')$, that is, $U(t) \otimes U(t)$ (acting as infinitesimal translations along the right action of $T \times T$) acts locally finite with generalized eigenvalues in $(\chi, \chi')$. Consider the quotient $Y/T$ where $T$ acts diagonally from the right. The group $T$ acts on $Y/T$ via the formula $t(xU,yU) \bmod T = (xU,ytU) \bmod T$. To every $\chi \in \hat{T}$ we denote by $M_\chi$ the category of $G$-equivariant $T$-monodromic $D$-modules on $Y/T$ with generalized monodromy $\chi$. We write $\mathcal{D}(M_{\chi,\chi'})$ and $\mathcal{D}(M_\chi)$ for the corresponding $G$-equivariant monodromic derived categories.

The groups $B$ and $T \times T$ act on $X = G/U$ by the formula $b(xU) = bx^{-1}U$, $(t,t')(xU) = txt'U$. For any $(\chi_1, \chi_2) \in \hat{T} \times \hat{T}$ we write $H_{\chi_1,\chi_2}$ for the category of $U$-equivariant $T \times T$-monodromic $D$-modules on $X$ with generalized monodromy $(\chi_1, \chi_2)$. For any $\chi \in \hat{T}$ we write $H_\chi$ for the category of $B$-equivariant $T$-monodromic $D$-modules on $X$ with generalized monodromy $\chi$, where $B$ acts on $X$ by the same formula as before and $T$ acts on $X$ by the formula $t(xU) = txU$. We denote by $\mathcal{D}(H_\chi)$ (resp. $\mathcal{D}(H_{\chi_1,\chi_2})$) the corresponding $B$-equivariant (resp. $U$-equivariant) monodromic derived category.

6.2. **The Harish-Chandra functor.** Consider the following correspondence

$$G \overset{p}{\longrightarrow} G \times G/B \overset{q}{\longrightarrow} Y/T = (G/U \times G/U)/T$$

where $p(g,xB) = g$ and $q(g,xB) = (gxU,xU) \bmod T$. The group $G$ acts on $G$, $G \times B$ and $Y/T$ by the formulas $a \cdot g = aga^{-1}$, $a \cdot (g,xU) = (aga^{-1}, axB)$, $a(xU,yU) = (axU, ayU)$. One can check that $p$ and $q$ are compatible with those $G$-actions.

Following \cite{MV}, we consider the functor

$$\text{HC} = qp^\circ : \mathcal{D}(G) \to \mathcal{D}(Y/T).$$

The functor above admits a right adjoint $\text{CH} = p_* q^\circ : \mathcal{D}(Y/T) \to \mathcal{D}(G)$. We use the same notations for the corresponding functors between $G$-equivariant derived categories $\mathcal{D}(G/_{\text{ad}} G)$ and $\mathcal{D}(G \bmod Y/T)$.

Consider the embedding $i : X \to Y, gU \to (eU,gU)$ and the projection map $\pi : G \to X = G/U$. 

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Lemma 6.2. [MV] (1) The functor \( i^0 = i^\dim X : \mathcal{D}(G\setminus Y) \to \mathcal{D}(U\setminus X) \) is an equivalence of categories with inverse given by \((i^0)^{-1} := \text{Ind}_{T \subset B}^G \circ i_* \) \([\dim G - \dim B]\). (2) We have \( i^0 \circ \text{HC} \simeq \pi_* \).

We have the convolution product \( \mathcal{D}(G\setminus Y) \times \mathcal{D}(G\setminus Y) \to \mathcal{D}(G\setminus Y) \) given by \((\mathcal{F}, \mathcal{F}') \to (p_{12})_* (p_1^* \mathcal{F} \otimes p_2^* \mathcal{F}')\). Here \( p_{ij} : G\setminus (G/U \times G/U \times G/U) \to G\setminus Y \) is the projection on the \((i,j)\)-factors. The convolution product on \( \mathcal{D}(G\setminus Y) \) restricts to a convolution product on \( \mathcal{D}(M_{\chi, \chi^{-1}}) \). The equivalence \( i^0 : \mathcal{D}(G\setminus Y) \simeq \mathcal{D}(U\setminus X) \) above induces convolution products on \( \mathcal{D}(U\setminus X) \) and \( \mathcal{D}(H_{\chi, \chi}) \). In addition, there is an action of \( \mathcal{D}(U\setminus X) \) on \( \mathcal{D}(X) \) by right convolution. The convolution operation will be denoted by \( \ast \).

We will need the following lemma. Let \( X \) be an algebraic variety with an action of an affine algebraic group \( G \). Denote the action map by \( a : G \times X \to X \).

Lemma 6.3 (Lemma 2.1 [BFO]). For any \( A \in \mathcal{D}(G), F \in \mathcal{D}(X) \) we have a canonical isomorphism

\[
\Gamma(a_*(A \boxtimes F)) \simeq \Gamma(A) \otimes_{U(\mathfrak{g})}^L \Gamma(F).
\]

Example 6.1. Consider the action map \( a : G \times G/U \to G/U, a(x, gU) = xgU \). Let \( \delta \in \mathcal{D}(G/U) \) be the delta \( D \)-module supported at the base point \( eU \in G/U \). For any \( D \)-module \( F \) on \( G \), there is a canonical isomorphism \( \text{Av}_U^* (F) \simeq a_* (F \boxtimes \delta) \) and lemma above implies that

\[
\Gamma(\text{Av}_U^* (F)) \simeq \Gamma(a_*(F \boxtimes \delta)) \simeq \Gamma(F) \otimes_{U(\mathfrak{g})}^L \Gamma(F).
\]

Note that \( \Gamma(F) = \Gamma(\mathcal{F}) \) (since \( G \) is affine) and \( \Gamma(\delta) \simeq U(\mathfrak{g})/U(\mathfrak{g}) \mathfrak{n} \), and it follows that

\[
\Gamma(\text{Av}_U^* (F)) \simeq \Gamma(\mathcal{F}) \otimes_{U(\mathfrak{g})}^L U(\mathfrak{g})/U(\mathfrak{g}) \mathfrak{n}
\]

6.3. Character \( D \)-modules. We denote by \( CS(G) \) the category of finitely generated \( G \)-equivariant \( D \)-modules on \( G \) such that the action of the center \( Z \subset U(\mathfrak{g}) \), embedding as left invariant differential operators, is locally finite. To every \( W \)-orbit \( \theta \subset \mathfrak{t} \), we denote by \( CS_\theta(G) \) the category of finitely generated \( G \)-equivariant \( D \)-modules on \( G \) such that the action of the center \( Z \subset U(\mathfrak{g}) \) is locally finite and has generalized eigenvalues in \( \{ \lambda \in \mathfrak{t} | \exp(\lambda) \in \theta \} \). We denote by \( \mathcal{D}(CS(\mathfrak{g})) \) (resp. \( \mathcal{D}(CS_\theta) \)) the minimal triangulated full subcategory of \( \mathcal{D}(G/\text{ad} G) \) containing all objects \( M \in \mathcal{D}(G/\text{ad} G) \) such that \( \mathcal{H}^i(M) \in CS(G) \) (resp. \( \mathcal{H}^i(M) \in CS_\theta(G) \)). We call \( CS(G) \) and \( CS(G)_\theta \) (resp. \( \mathcal{D}(CS(\mathfrak{g})) \) and \( \mathcal{D}(CS_\theta) \)) the category (resp. derived category) of character \( D \)-modules on \( G \) and character \( D \)-modules on \( G \) with generalized central character \( \theta \).

Proposition 6.4. We have the following:

1. Let \( \mathcal{S} \in CS(G)_\theta \). Then

\[
\text{HC}(\mathcal{S}) \in \bigoplus_{\chi \in \theta} \mathcal{D}(M_\chi), \quad (\text{resp. } \text{Av}_U^* (\mathcal{S}) \in \bigoplus_{\chi \in \theta} \mathcal{D}(H_\chi).)
\]

2. The functors \( \text{Ind}_{T \subset B}^G \) and \( \text{Res}_{T \subset B}^G \) preserve the derived categories of character \( D \)-modules.

The resulting functors \( \text{Ind}_{T \subset B}^G : \mathcal{D}(CS(T)) \to \mathcal{D}(CS(G)) \), \( \text{Res}_{T \subset B}^G : \mathcal{D}(CS(G)) \to \mathcal{D}(CS(T)) \) are independent of the choice of the Borel subgroup \( B \) and \( t \)-exact with respect to the natural \( t \)-structures on \( \mathcal{D}(CS(G)) \) and \( \mathcal{D}(CS(T)) \). Moreover, for any \( \mathcal{S} \in CS(G) \) we have \( \text{Res}_{T \subset B}^G (\mathcal{S}) \simeq (\iota_T)_* (\mathcal{S}|_{T^\theta}), \) here \( \iota_T : T^\theta \to T \) is the embedding.

3. Let \( \mathcal{S} \in CS(G) \). There is a canonical \( W \)-equivariant structure on \( \text{Res}_{T \subset B}^G (\mathcal{S}) \). Let \( j : G^W \to G \) be the open embedding. If \( \mathcal{S} = j_* j^* (\mathcal{S}), \) then we have

\[
\mathcal{S} \simeq \text{Ind}_{T \subset B}^G (\text{Res}_{T \subset B}^G (\mathcal{S}))^W.
\]
Proof. Part (1) and (2) are proved in [Gi1, L1]. We now prove part (3). We first show that $\mathcal{F} = \text{Res}^G_{T \subset B}(\mathcal{G})$ is canonically $W$-equivariant. Let $x \in N(T)$ and $w \in N(T)/T = W$ be its image in the Weyl group. Denote $B_x := \text{Ad}_x B$. Consider the following commutative diagram

$$
\begin{array}{ccc}
T & \xleftarrow{w} & B \\
\downarrow & & \downarrow \text{Ad}_x \\
T & \xleftarrow{w} & B_x \\
\end{array}
\Rightarrow
\begin{array}{ccc}
T & \rightarrow & G \\
\downarrow & & \downarrow \text{Ad}_x \\
T & \rightarrow & G \\
\end{array}
$$

where $w : T \to T$ is the natural action of $w \in W$ on $T$ and the horizontal arrows are the natural inclusion and projection maps. The base change theorems and the fact that the functors $\text{Res}^G_{T \subset B}$ and $\text{Res}^G_{T \subset B_x}$ are canonical isomorphic (see part (2)) imply

\begin{equation}
\text{Res}^G_{T \subset B}(\text{Ad}_x \mathcal{G}) \simeq w^* \text{Res}^G_{T \subset B}(\mathcal{G}) \simeq w^* \text{Res}^G_{T \subset B}(\mathcal{G}).
\end{equation}

Since $\mathcal{G}$ is $G$-conjugation equivariant, we have a canonical isomorphism $c_x : \mathcal{G} \simeq \text{Ad}_x \mathcal{G}$. Applying $\text{Res}^G_{T \subset B}$ to $c_x$ and using (6.2) we get

\begin{equation}
\mathcal{F} = \text{Res}^G_{T \subset B}(\mathcal{G}) \simeq \text{Res}^G_{T \subset B}(\text{Ad}_x \mathcal{G}) \simeq w^* \text{Res}^G_{T \subset B}(\mathcal{G}) = w^* \mathcal{F}.
\end{equation}

We claim that the isomorphism above depends only the image $w$ and we denote it by

\begin{equation}
c_w : \mathcal{F} \simeq w^* \mathcal{F}.
\end{equation}

To prove the claim it is enough to check that for $x \in T$ the restriction of the isomorphism (6.3) to $T^\text{rs}$ is equal to the identity map. By [Gi1], the restriction $\mathcal{F}|_{T^\text{rs}}$ is canonically isomorphic to $\mathcal{G}|_{T^\text{rs}}$ and the map in (6.3) is equal to the restriction of $c_x$ to $T^\text{rs}$. Since the adjoint action $\text{Ad}_x : G \to G$ is trivial on $T$, the claim follows from the fact that any $T$-equivariant structure of a local system on $T$ is trivial. The $G$-conjugation equivariant structure on $\mathcal{G}$ implies $\{c_w\}_{w \in W}$ satisfies the required cocycle condition, hence, the data $(\mathcal{F}, \{c_w\}_{w \in W})$ defines a $W$-equivariant structure on $\mathcal{F} = \text{Res}^G_{T \subset B}(\mathcal{G})$. We shall prove $\text{Ind}^G_{T \subset B}(\mathcal{F})^W \simeq \mathcal{G}$. Let $c_{\text{rs}} : G^{\text{rs}} \to T^\text{rs}/W$ the restriction of the Chevalley map $c : G \to T/W$ to $G^{\text{rs}}$. Since $\mathcal{G} \simeq j_*(\mathcal{G}|_{G^{\text{rs}}})$ and $\text{Ind}^G_{T \subset B}(\mathcal{F})^W \simeq j_*(c_{\text{rs}}(\mathcal{F}))$, where $\mathcal{F} \in \mathcal{D}(T^\text{rs}/W)$ is the descent of $\mathcal{F}|_{T^\text{rs}}$ along the map $q_{\text{rs}} : T^\text{rs} \to T^\text{rs}/W$, it suffices to show $\mathcal{G}|_{G^{\text{rs}}} \simeq c_{\text{rs}}^*(\mathcal{F}) \in \mathcal{D}(G^{\text{rs}}/G)$. This follows again from the fact that $\mathcal{G}|_{T^\text{rs}} \simeq \mathcal{F}|_{T^\text{rs}} \in \mathcal{D}(T^\text{rs}/W) \simeq \mathcal{D}(G^{\text{rs}}/G)$.

\end{proof}

6.4. Drinfeld center of Harish-Chandra bimodules and character $D$-modules. We give a review of the work [BG] [BFO] on Drinfeld center of Harish-Chandra bimodules and character $D$-modules.

Write $U = U (g)$ for the universal enveloping algebra of $g$. Let $Z = Z(U)$ be the center of $U$. Consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ on $\mathfrak{t}$. Denote by $\tilde{W}$ the group $W$ acting via the dot action $\tilde{w}$. We have the Harish-Chandra isomorphism $hc : Z \simeq O(\mathfrak{t})^W$ such that for any $\lambda \in \mathfrak{t}$ the center $Z$ acts on the Verma module associated to $\lambda$ via $z \mapsto hc(z)(\lambda)$. For any $\lambda \in \mathfrak{t}$ we write $m_\lambda$ for the corresponding maximal ideal and denote by $I_\lambda$ the maximal ideal of $Z$ corresponding to $m_\lambda$ under the Harish-Chandra isomorphism. Consider the extended universal enveloping algebra $\tilde{U} = U \otimes Z O(\mathfrak{t})$, where $Z$ acts on $O(\mathfrak{t})$ via the Harish-Chandra isomorphism. We denote by $U_\lambda = U / U I_\lambda$, $U_\lambda = \lim_{\leftarrow} U / U I_{\lambda}^n$, $\tilde{U}_\lambda = \tilde{U} / \tilde{U} m_\lambda$, and $\tilde{U}_\lambda = \lim_{\leftarrow} (\tilde{U} / \tilde{U} m_\lambda^n)$. The action of $\tilde{W}$ on $O(\mathfrak{t})$ gives rise to an action of $\tilde{W}$ on $\tilde{U}$ such that $\tilde{U}^W = U$. In addition, the stabilizer $\tilde{W}_\lambda$ of $\lambda \in \mathfrak{t}$ in $\tilde{W}$ acts naturally on $\tilde{U}_\lambda$ and the natural inclusion $U \to \tilde{U}$ induces an isomorphism $U_\lambda \simeq \tilde{U}_\lambda^W$ (see, e.g., [BG] Section 1).
We denote by \( \mathcal{H}C_\lambda \) the category of finitely generated Harish-Chandra bimodules over \( U_\lambda \), that is, finitely generated continuous \( U_\lambda \)-bimodules such that the diagonal action of \( g \) is locally finite. We denote by \( \mathcal{D}(\mathcal{H}C_\lambda) \) the corresponding derived category. The tensor product \( M \otimes_U M', M, M' \in \mathcal{H}C_\lambda \) (resp. \( M \otimes_U^L M', M, M' \in \mathcal{D}(\mathcal{H}C_\lambda) \)) defines a monoidal structure on \( \mathcal{H}C_\lambda \) (resp. \( \mathcal{D}(\mathcal{H}C_\lambda) \)).

Recall that \( \lambda \in \mathfrak{i} \) is called regular if \( \hat{W}_\lambda = 0 \), that is, \( \lambda \) does not lie on any coroot hyperplane shifted by \(-\rho\), and it is called dominant if the value of \( \lambda \) at any positive coroot is not a negative integer.

**Proposition 6.5.** (BFO) Proposition 3.1] Let \( \chi \in \hat{T} \) and \( \lambda \in \mathfrak{i} \) be a dominate regular lifting of \( \chi \). The functor

\[
RG_{\lambda, \lambda - \rho} : (\mathcal{D}(M_{\chi, \lambda - 1}), \ast) \simeq (\mathcal{H}C_\lambda, \otimes_U^L)
\]

is an equivalence of monoidal categories.

Let \( \chi \in \hat{T} \) and \( \lambda \in \mathfrak{i} \) be as in Proposition 6.5] and consider the equivalence of monoidal categories

\[
(6.5) \quad M : (\mathcal{D}(H_{\chi, \lambda}), \ast) \simeq (\mathcal{D}(M_{\chi, \lambda - 1}), \ast) \overset{R_{\lambda, \lambda - \rho}}{\simeq} (\mathcal{H}C_\lambda, \otimes_U^L).
\]

We denote by \( \mathcal{H}C_\chi \) the category of finitely generated Harish-Chandra bimodules over \( U \) (no restriction on the action of the center \( Z \)). We denote by \( \mathcal{H}C_\chi^\mu \) the category of finitely generated Harish-Chandra bimodules over the product \( \prod_{\mu \in \Lambda^+ / W^\mu} U_\mu \) (here \( W^\mu \) acts on \( \mu \in \Lambda^+ / \Lambda \) via the dot action). We denote by \( Z(\mathcal{H}C_\chi, \otimes_U) \) (resp. \( Z(\mathcal{H}C_\chi, \otimes_U), \mathcal{Z}(\mathcal{H}C_\chi, \otimes_U), \mathcal{Z}(H_{\chi, \chi}^\mu, *, *) \)) the Drinfeld center of the monoidal category \( \mathcal{H}C_\chi = \mathcal{H}C_\chi^\mu \) (resp. \( \mathcal{H}_\chi, \mathcal{H}_\chi^\mu, \mathcal{H}_{\chi, \chi}^\mu, *, * \)). Recall an element in \( Z(\mathcal{H}C_\chi, \otimes_U) \) consists of an element \( M \in \mathcal{H}C_\chi \) together with family of compatible isomorphisms \( b_{\mathfrak{G}} : M \otimes_U F \simeq M \otimes_U^L F \). Recall the notion of translation functor \( \theta^\mu_\lambda : \mathcal{H}C_\lambda \rightarrow \mathcal{H}C_\mu \) where \( \mu \in \Lambda^+ / \Lambda \) (see, e.g., [BGG]).

**Theorem 6.6.** (1) [BFO] Lemma 3.7] There is a lifting \( \theta^\mu_\lambda : Z(\mathcal{H}C_\lambda, \otimes_U) \rightarrow Z(\mathcal{H}C_\mu, \otimes_U) \) such that the functor \( F : Z(\mathcal{H}C_\lambda, \otimes_U) \rightarrow Z(\mathcal{H}C_\chi, \otimes_U), L \rightarrow \bigoplus_{\mu \in \Lambda^+ / \Lambda / W^\mu} \theta^\mu_\lambda(L) \) defines an equivalence of braided monoidal categories.

(2) [BFO] Theorem 3.6, Lemma 3.8] For any \( \mathcal{M} \in \mathcal{D}_G(G)^{\otimes} \), the global section \( \Gamma(\mathcal{M}) \) is naturally a Harish-Chandra bimodule, with a canonical central structure and the resulting functor \( \Gamma : \mathcal{D}_G(G)^{\otimes} \rightarrow Z(\mathcal{H}C_\chi, \otimes_U) \) is an equivalence of abelian categories. Moreover, the equivalence above restricts to an equivalence \( CS_\theta \simeq Z(\mathcal{H}C_\chi, \otimes_U) \) and the composed equivalence

\[
CS_\theta \simeq Z(\mathcal{H}C_\chi, \otimes_U) \overset{F^{-1}}{\simeq} Z(\mathcal{H}C_\chi, \otimes_U)
\]

is isomorphic to \( RG_{\lambda, \lambda - \rho} \circ \pi_\circ HC \). Here \( \pi : Y \rightarrow Y/T \) is the projection map.

6.5. **Harish-Chandra bimodules** \( Z_\chi \). In this subsection we attach to each dominant regular lift \( \lambda \in \mathfrak{i} \) of \( \chi \in \hat{T} \) an element \( Z_\chi \in Z(\mathcal{H}C_\lambda, \otimes_U) \) in the Drinfeld center of Harish-Chandra bimodules, and identify it with the local system \( \mathcal{E}_\chi \) under the equivalence in (6.5).

Let \( R = \mathcal{O}(\mathfrak{i}) \) and let \( \lambda \in \mathfrak{i} \) be a lift of \( \chi \). We have a canonical identification \( W_{\lambda + \rho} \simeq \hat{W}_\lambda, w \rightarrow \hat{w} = w \) such that the translation map \( m_{-\lambda} : \mathfrak{i} \rightarrow \mathfrak{i}, m_{-\lambda}(\mu) = \mu - \lambda \) satisfies \( m_{-\lambda}(wv) = \hat{w}(m_{-\lambda}(v)) \) for \( w \in W_{\lambda + \rho} \) and \( v \in \mathfrak{i} \). It induces an isomorphism \( m^*_{-\lambda} : R_0^{W_{\lambda + \rho}} \simeq R_0^{W_{\lambda}}, f \rightarrow (x \rightarrow f(x - \lambda)) \). Here \( R_0^{W_{\lambda + \rho}} \) (resp. \( R_\lambda^{W_{\lambda + \rho}} \)) is the \( W_{\lambda + \rho} \)-invariant of the completion \( R_0 \) of \( R \) at \( 0 \) (resp. \( \hat{W}_\lambda \)-invariant of the completion \( R_\lambda \) of \( R \) at \( \lambda \)). Consider the following quotient \( R_{\chi, \lambda + \rho} = R_0^{W_{\lambda + \rho}} / (R_0^{W_{\lambda + \rho}}) \) (here
\( R^W_\lambda \) is the ideal generated by the augmentation ideal \( R^W_{\lambda, +} \) of \( R^W_0 \). We introduce the following Harish-Chandra bimodule

\[
\mathcal{Z}_\lambda = U_\hat{\lambda} \otimes_{R^W_0} R^W_{\lambda, +} \in \mathcal{H}(\mathcal{L}_\lambda).
\]

Here \( R^W_{\lambda, +} \) acts on \( U_\hat{\lambda} \) via the the map

\[
b_\lambda : R^W_{\lambda, +} \to Z(U_\hat{\lambda}) \to Z(U_\hat{\lambda}),
\]

where the last map is induced from the isomorphism \( \tilde{U}_\hat{\lambda} \simeq U_\hat{\lambda} \). Equivalently, consider the \( R^W_{\lambda, +} \)-module \( R^W_{\lambda, +} \) given by the base change of the \( R^W_{\lambda, +} \)-module \( R^W_{\lambda, +} \) along the isomorphism

\[
\tilde{U}_\hat{\lambda} \simeq R^W_{\lambda, +}.
\]

Then we have

\[
\mathcal{Z}_\lambda \simeq U \otimes Z \hat{R}^W_{\lambda, +},
\]

where \( Z \) acts on \( \hat{R}^W_{\lambda, +} \) via the Harish-Chandra isomorphism \( hc : Z \simeq R^W \).

**Proposition 6.7.** Let \( \lambda \in \mathfrak{l} \) be a dominant regular lift of \( \chi \in \hat{G} \).

1. To every \( M \in \mathcal{H}(\mathcal{L}_\lambda) \) there is a canonical isomorphism

\[
b_M : \mathcal{Z}_\lambda \otimes_U M \simeq M \otimes_U \mathcal{Z}_\lambda
\]

such that the data \( (\mathcal{Z}_\lambda, b_M) \in \mathcal{H}(\mathcal{L}_\lambda) \) defines an element in the Drinfeld center \( Z(\mathcal{H}(\mathcal{L}_\lambda), \otimes) \).

2. We have \( M(\mathcal{E}_\chi) \simeq \mathcal{Z}_\lambda \).

**Proof.** Proof of (1). We have \( W_{\lambda, +} = \hat{W}_\lambda = e \) since \( \lambda \) is regular. To every \( M \in \mathcal{H}(\mathcal{L}_\lambda) \), the map \( b_\lambda \) in (6.7) gives rise to an action of \( R_0 \otimes \mathbb{C} R_0 = R^W_{\lambda, +} \otimes \mathbb{C} R^W_{\lambda, +} \) on \( M \) and it follows from [MS, Theorem 4.1] that this action factors through \( R_0 \otimes \mathbb{C} R_0 \to R_0 \otimes R^W_{\lambda, +} \) and it is shown in loc. cit. that for every finite dimensional representation \( V \) of \( g \), the action of \( R_0 \otimes \mathbb{C} R_0 \) on the bimodule \( pr_\lambda(V \otimes \mathbb{C} U / U I_\lambda) \in \mathcal{H}(\mathcal{L}_\lambda) \), \( n \in \mathbb{Z}_{\geq 0} \) factors through \( R_0 \otimes R^W_{\lambda, +} \). Here \( pr_\lambda(-) \) is the projection of the summand on which the action of \( I_\lambda \) is locally nilpotent. Since every object in \( \mathcal{H}(\mathcal{L}_\lambda) \) is isomorphic to a quotient of \( pr_\lambda(V \otimes \mathbb{C} U / U I_\lambda) \) for some \( V \) and \( n \), the claim follows. Therefore, for every \( M \in \mathcal{H}(\mathcal{L}_\lambda) \), we have a canonical isomorphism

\[
b_M : \mathcal{Z}_\lambda \otimes_U M \simeq M \otimes_U \mathcal{Z}_\lambda.
\]

It follows from the construction that those isomorphisms satisfy the required compatibility conditions and the data \( (\mathcal{Z}_\lambda, b_M) \) defines an element in \( Z(\mathcal{H}(\mathcal{L}_\lambda), \otimes) \).

Proof of (2). Let \( \tilde{\mathcal{E}}_\chi \in \mathcal{M}_\chi \) be the image of \( \mathcal{E}_\chi \) under the equivalence \((i^0)^{-1} : H_\chi \simeq M_\chi \) in Lemma [6.2]. Then by definition we have \( M(\mathcal{E}_\chi) \simeq R^\chi_{\lambda, -2} \Delta(\pi \circ \tilde{\mathcal{E}}_\chi) \), where \( \pi : Y \to Y / T \). Consider the map

\[
a : T \times (G/U \times G/U) / T \to (G/U \times G/U) / T, (t, gU, g'U) \to (gt^{-1}U, g'U).
\]

Then it follows from the definition of \((i^0)^{-1} \) that \( \tilde{\mathcal{E}}_\chi = a_\chi(\mathcal{E}_\Theta \otimes \Delta_n \Theta G / B) \), here \( \Delta : G / B \to (G/U \times G/U) / T \) is the embedding \( gB \to (gU, g'U) \mod T \). It is shown in [BFO, Proposition 3.1] that, for any \( \lambda \in \mathfrak{l} \) and any finite dimensional representation \( V \), the action of \( R_0 \otimes \mathbb{C} R_0 \) on \( pr_\lambda(V \otimes \mathbb{C} U / U I_\lambda) \) factors through \( R_0 \otimes R^W_{\lambda, +} \). But the proof in loc. cit. actually shows that, if we fix \( \lambda \in \mathfrak{l} \), the action factors through \( R_0 \otimes R^W_{\lambda, +} \).
Hence by Proposition 6.7, we have
\[
\text{Applying the equivalence } i \circ \chi, \chi \mapsto \chi, \text{ we get }
\]
\[
\Gamma^\lambda(\hat{\chi} \otimes \pi^\rho_E) = \Gamma^\lambda(\Delta(\hat{\chi} \otimes \pi^\rho_E)) \simeq 
\]
\[
\simeq \Gamma^\lambda(\Delta(\otimes \pi^\rho_E)) \simeq \hat{\chi} \otimes \Gamma^\lambda(\otimes_E).
\]
Since \( \Gamma^\lambda(\otimes_E) \simeq \hat{\chi} \otimes \Gamma^\lambda(\otimes_E) \), the theorem follows.

\[\square\]

We finish this section with a lemma to be used in the next section.

**Lemma 6.8.** Let \( \lambda \in \hat{t} \) be regular dominant weight and let \( \mu \) be a dominant weight in \( \lambda + \Lambda \). We have \( \theta^\mu_\lambda(\mathcal{Z}_\lambda) \simeq \mathcal{Z}_\mu \).

**Proof.** Note that \( \theta^\mu_\lambda : \mathcal{H}_\lambda \to \mathcal{H}_\mu \) is monoidal, so we have \( \theta^\mu_\lambda(U_\lambda) \simeq U_\mu \). Note also that, by Proposition 1.4, for any \( r \in \mathcal{R}_\lambda \) and \( M \in \mathcal{H}_\lambda \), we have \( \theta^\mu_\lambda(b_\lambda(r) \cdot m) = b_\lambda(r) \cdot \theta^\mu_\lambda(m) \), where \( m \in M \) and \( b_\nu : \mathcal{R}_\nu \to Z(U_\nu) \), \( \nu \in \hat{t} \) is the map in (6.7). Since \( \mathcal{Z}_\lambda \) (resp. \( \mathcal{Z}_\mu \)) is isomorphic to the coinvariant algebra of \( U_\lambda \) (resp. \( U_\mu \)) with respect to the action of \( \mathcal{R}_\lambda \) via the map \( b_\lambda \) (resp. \( b_\mu \)), it follows that \( \theta^\mu_\lambda(\mathcal{Z}_\lambda) \simeq \mathcal{Z}_\mu \). \[\square\]

### 6.6. Proof of Theorem 6.1 in the de Rham setting.

Let \( \mathcal{Z}_\lambda \in Z(\mathcal{H}(\lambda), \otimes \mathcal{U}) \), \( \mathcal{E}_\lambda \in \mathcal{M}_\lambda \) be as in Proposition 6.7. Define \( \mathcal{E}_\theta \simeq \bigoplus_{\chi \in \theta} \mathcal{E}_\lambda \). By the discussion above there exists a character \( D \)-module \( \mathcal{M}_\theta \in \mathcal{C}(\mathcal{M}_\theta) \) such that
\[
\Gamma^\lambda(\mathcal{E}_\theta) = \mathcal{Z}_\lambda.
\]
Hence by Proposition 6.7, we have
\[
\Gamma^\lambda(\mathcal{E}_\theta) = \mathcal{Z}_\lambda.
\]
for any regular dominant \( \lambda \in \hat{t} \) mapping to \( \xi \). Since \( \mathcal{M}_\theta \simeq \mathcal{D}(\mathcal{C}(\mathcal{M}_\theta)) \) and \( \mathcal{M}_\theta \simeq \mathcal{D}(\mathcal{M}_\lambda) \), it follows that
\[
\mathcal{H}(\mathcal{M}_\theta) \simeq \mathcal{E}_\theta.
\]
Applying the equivalence \( i^\circ : \mathcal{D}(\mathcal{M}_\xi) \simeq \mathcal{H}(\mathcal{H}_\xi) \) and using Lemma 6.2, we get
\[
(6.8) \quad \text{Av}_U(\mathcal{M}_\theta) \simeq \mathcal{E}_\theta.
\]
The isomorphism above implies \( \mathcal{E}_\theta \simeq \text{Av}_U(\mathcal{M}_\theta) \simeq \text{Res}_{\mathcal{F}_{\lambda \subset B}}(\mathcal{M}_\theta) \), therefore, by part (3) of Proposition 6.4, there is a canonical \( W \)-equivariant structure on \( \mathcal{E}_\theta \) such that \( \mathcal{M}_\theta \simeq \text{Ind}_{\mathcal{F}_{\lambda \subset B}}(\mathcal{E}_\theta)^W \). We claim that the above \( W \)-equivariant structure on \( \mathcal{E}_\theta \) is isomorphic to the one constructed in Section 5.1. Thus we have \( \mathcal{M}_\theta \simeq \hat{\mathcal{M}}_\theta \). The theorem follows.

### 6.7. Proof of the claim.

\[\text{The group } W_\lambda \text{ in [BG] is the group } \hat{W}_\lambda \text{ in the present paper}\]
6.7.1. We first give a description of (6.8) at the level of global sections (see (6.12) below). By Lemma 6.8, Example 6.1, and Theorem 6.6 the global sections of $M_g'$, $A^U_\ast (M'_g)$, and $\iota_\ast \mathcal{E}_g$ (here $\iota : T = B/U \to G/U$) are given by

$$\Gamma (A^U_\ast (M'_g)) = (U / U \cap U) \otimes_R \Gamma (M'_g), \quad \Gamma (M'_g) \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \theta^\mu_\chi (Z_\lambda),$$

$$\Gamma (\iota_\ast \mathcal{E}_g) = (U / U \cap U) \otimes_R \Gamma (\mathcal{E}_g).$$

By Lemma 6.8, $\theta^\mu_\chi (Z_\lambda) \simeq Z_\mu$, for any dominant $\mu \in \lambda + \Lambda$. Since every $W'_\chi$-orbit in $\lambda + \Lambda$ (under the dot action) contains a unique dominant weight, we have

\begin{equation}
(6.9) \quad \Gamma (M'_g) \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \theta^\mu_\chi (Z_\lambda) \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} Z_\mu \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} U \otimes Z \hat{R}^{\mu \rho}_\chi,
\end{equation}

which gives rise to

\begin{equation}
(6.10) \quad \Gamma (A^U_\ast (M'_g)) \simeq (U / U \cap U) \otimes_R \Gamma (M'_g) \simeq (U / U \cap U) \otimes_R \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} U \otimes Z \hat{R}^{\mu \rho}_\chi \simeq (U / U \cap U) \otimes_R \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R \otimes Z \hat{R}^{\mu \rho}_\chi.
\end{equation}

Since the map $\mathfrak{i}/\mathfrak{W}_\mu \to \mathfrak{i}/\mathfrak{W}$ is étale at the image of the $\mathfrak{W}$-orbit $\mathfrak{W}_\mu \subset \mathfrak{i}$ along the projection $\mathfrak{i} \to \mathfrak{i}/\mathfrak{W}_\mu$, it follows that

$$R \otimes Z \hat{R}^{\mu \rho}_\chi \simeq R \otimes R^W \hat{R}^{\mu \rho}_\chi \simeq \bigoplus_{\nu \in \hat{W} / W_{\chi_{\nu}}} R \otimes R^W \hat{R}^{\mu \rho}_\chi.$$

Since $R \otimes R^W \hat{R}^{\mu \rho}_\chi \simeq m_{-\nu} (R / (R^W_{\chi_{\nu}})) \simeq \Gamma (\mathcal{E}_g)$, it implies

\begin{equation}
(6.11) \quad \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R \otimes Z \hat{R}^{\mu \rho}_\chi \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R \otimes R^W \hat{R}^{\mu \rho}_\chi \simeq \bigoplus_{\nu \in \hat{W} / W_{\chi_{\nu}}} \Gamma (\mathcal{E}_g) \simeq \Gamma (\mathcal{E}_g).
\end{equation}

Combining (6.10) and (6.11), we obtain the description of (6.8) on global sections:

\begin{equation}
(6.12) \quad \Gamma (A^U_\ast (M'_g)) \simeq (U / U \cap U) \otimes_R \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R \otimes Z \hat{R}^{\mu \rho}_\chi \simeq (U / U \cap U) \otimes_R \Gamma (\mathcal{E}_g) \simeq \Gamma (\iota_\ast \mathcal{E}_g).
\end{equation}

Remark 6.2. Note that if we identify $\mathfrak{W} = W_{a_- \rho}$ as subgroup in $W_a$, then $Z = R^W \simeq R^W_{a_- \rho}$ and the isomorphism (6.11) restricts to an isomorphism

\begin{equation}
(6.13) \quad \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \hat{R}^{\mu \rho}_\chi \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} R^W_{a_- \rho} \hat{R}^{\mu \rho}_\chi \simeq \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \hat{R}^{\mu \rho}_\chi W_{a_- \rho} \simeq \Gamma (\mathcal{E}) W_{a_- \rho}
\end{equation}

such that the map

\begin{equation}
(6.14) \quad R \otimes R^W_{a_- \rho} \Gamma (\mathcal{E}) W_{a_- \rho} \simeq R \otimes R^W_{a_- \rho} \bigoplus_{\mu \in \lambda + \Lambda / W'_\chi} \hat{R}^{\mu \rho}_\chi \simeq \Gamma (\mathcal{E})
\end{equation}

is the isomorphism in Theorem 4.3 (4).
Let $x \in N(T)$ and $w \in W$ its image in the Weyl group. Let $U_w = xUx^{-1}$ with Lie algebra $\mathfrak{n}_w$. Consider the following commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\pi} & G/U \\
\downarrow \text{Ad}_x & & \downarrow \text{Ad}_x \\
G & \xrightarrow{\pi_{tw}} & G/U_w \end{array}
$$

where the horizontal maps are the natural quotient maps or embeddings. Let

$$
\phi_w : E_\theta \simeq w^*E_\theta
$$

and

$$
\phi_x : M'_\theta \simeq \text{Ad}_x^*M'_\theta
$$

be the isomorphisms coming from the $G$-conjugation equivariant structures on $M'_\theta$ and the $W$-equivariant structure on $E_\theta$, and let

$$
\Gamma(\phi_w) : \Gamma(E) \to \Gamma(w^*E) \simeq \Gamma(E)
$$

$$
\Gamma(\phi_x) : \Gamma(M'_\theta) \to \Gamma(\text{Ad}_x^*M'_\theta) \simeq \Gamma(M'_\theta)
$$

be the induced maps on global sections. Then the diagram (6.15) gives rise to natural isomorphisms

$$
n_*\phi_w : n_*E \simeq n_*w^*E \simeq \text{Ad}_x^*(n_w)_*E,
$$

$$
\text{Av}_U^U(\phi_x) : \text{Av}_U^U M'_\theta \simeq \text{Av}_U^U \text{Ad}_x^*M'_\theta \simeq \text{Ad}_x^* \text{Av}_U^U M'_\theta,
$$

such that the induced maps on global sections are given by

$$
\text{Ad}_x(-) \otimes \Gamma(\phi_w) : \Gamma(n_*E) = U / U \otimes \mathfrak{s} \Gamma(E) \to \Gamma(\text{Ad}_x^*(n_w)_*E) = U / U \otimes \mathfrak{r} \Gamma(E)
$$

$$
\text{Ad}_x(-) \otimes \Gamma(\phi_x) : \Gamma(\text{Av}_U^U M'_\theta) = U / U \otimes U \Gamma(M'_\theta) \to \Gamma(\text{Ad}_x^* \text{Av}_U^U M'_\theta) = U / U \otimes U \Gamma(M'_\theta).
$$

To prove the claim, we need to show that the following diagram commute

$$
\begin{array}{ccc}
\Gamma(\text{Av}_U^U M'_\theta) & \xrightarrow{6.17} & \Gamma(\text{Av}_U^U w M'_\theta) \\
\downarrow \text{Ad}_x & & \downarrow \text{Ad}_x \\
\Gamma(n_*E) & \xrightarrow{(6.16)} & \Gamma((n_w)_*E).
\end{array}
$$

where the right vertical arrow is the map (6.12) in the case when the unipotent radical is $U_w$.

6.7.3. By Theorem 6.6 isomorphism (6.9) intertwines the map $\Gamma(\phi_x) : \Gamma(M'_\theta) \to \Gamma(M'_\theta)$ with

$$
\text{Ad}_x(-) \otimes \text{id} : U \otimes \bigoplus_{\mu \in \lambda + \Lambda \cap W'_\chi} \hat{H}_{\chi}^{\mu + \rho} \to U \otimes \bigoplus_{\mu \in \lambda + \Lambda \cap W'_\chi} \hat{R}_{\chi}^{\mu + \rho}.
$$

Using Remark 6.2 it follows that we have the following commutative diagram

$$
\begin{array}{ccc}
\Gamma(\text{Av}_U^U M'_\theta) & \xrightarrow{6.17} & \Gamma(\text{Av}_U^U w M'_\theta) \\
\downarrow \text{Ad}_x & & \downarrow \text{Ad}_x \\
\Gamma(n_*E) & \xrightarrow{(6.16)} & \Gamma((n_w)_*E).
\end{array}
$$

11 Recall that there are canonical identifications $\Gamma(\text{Ad}_x^*M'_\theta) \simeq \Gamma(M'_\theta)$, $\Gamma(w^*E_\theta) \simeq \Gamma(E_\theta)$.
where the map (6.21) is given by

\[
\begin{align*}
\Gamma(\iota_*\mathcal{E}) = (U / U n) \otimes_R \Gamma(\mathcal{E}) & \xrightarrow{(6.21)} \Gamma((\iota_w)_*\mathcal{E}) = (U / U n_w) \otimes_R \Gamma(\mathcal{E}) \\
(U / U n) \otimes_R R \otimes_{R^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} & \xrightarrow{(6.14)} (U / U n_w) \otimes_R R \otimes_{R^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}
\end{align*}
\]

where the bottom arrow is the map $\text{Ad}_x(-) \otimes w \otimes \Gamma(\phi_w)$ \footnote{Note that the map is well-defined since $wW_{a,-\rho}^{-1} = W_{a,-w(\rho)}$}

On the other hand, by Theorem 4.3, we have the following commutative diagram

\[
\begin{align*}
\Gamma(\mathcal{E}) & \xrightarrow{\Gamma(\phi_w)} \Gamma(\mathcal{E}) \\
R \otimes_{R^{W_{a,-\rho}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-\rho}} & \xrightarrow{w \otimes \Gamma(\phi_w)} R \otimes_{R^{W_{a,-w(\rho)}}} \Gamma(\mathcal{E}_\theta)^{W_{a,-w(\rho)}}
\end{align*}
\]

were the vertical maps are the isomorphisms in Theorem 4.3 (4). Thus the map (6.21) is equal to (6.16), and hence (6.20) is equal to (6.18). The commutativity of (6.18) follows. This finishes the proof of the claim.

6.8. Mixed characteristic lifting of $\mathcal{E}_\theta$ and $M_\theta$. Let $G_\mathbb{Z}$ be a split reductive group over $\mathbb{Z}$. Let $T_\mathbb{Z}$ be a maximal torus of $G_\mathbb{Z}$ and let $B_\mathbb{Z}$ be a Borel subgroup containing $T_\mathbb{Z}$ with unipotent radical $U_\mathbb{Z}$. For any ring $R$ (resp. any scheme $S$), we denote by $G_R$, $B_R$, etc., (resp. $G_S$, $B_S$, etc.) the base change of $G_\mathbb{Z}$, $B_\mathbb{Z}$, etc., along $\text{Spec}(R) \to \text{Spec}(\mathbb{Z})$ (resp. $S \to \text{Spec}(\mathbb{Z})$).

Let $A \subset \mathbb{C}$ be a strict Henselization of a closed point of $\mathbb{Z}[1/\ell]$ with residue field $k$. Let $\chi \in \mathcal{O}(T_k)(\overline{\mathbb{Q}_\ell})$ and $\theta = W\chi$ be the $W$-orbit of $\chi$. Let $\mathcal{E}_\theta$ be the $W$-equivaraint $\ell$-adic local system on $T_k$ in Section 5.1.

**Lemma 6.9.** There exists a $W$-equivaraint $\ell$-adic local system $\mathcal{E}_{\theta,A}$ on $T_A$ which, after base change $A \to k$, becomes $\mathcal{E}_\theta$.

**Proof.** Let $\rho_\theta$ be the $\ell$-adic representaiton of $W \times \pi_1^i(T_k)$ associated to $\mathcal{E}_\theta$. The specialization isomorphism $\text{sp} : \pi_1^i(T_k) \to \pi_1^i(T_A)$ (see \textsuperscript{Gr}) induces an isomorphism $W \times \pi_1^i(T_k) \simeq W \times \pi_1^i(T_A)$, thus we can indentify $\rho_\theta$ as a $\ell$-adic representaiton of $W \times \pi_1^i(T_A)$ and we denote by $\mathcal{E}_{\theta,A}$ the corresponding local system on $T_A$. It is obvious that $\mathcal{E}_{\theta,A}$ satisfies the desired property. \hfill $\square$

Let $M_\theta$ be the character sheaf associated to $\theta$. Our next goal is construct a mixed characteristic version of $M_\theta$. For this we observe that for any flat group scheme $\mathcal{G}$ finite type over $\mathbb{Z}$ and a closed subgroup scheme $\mathcal{H} \subset \mathcal{G}$ flat over $\mathbb{Z}$, the universal geometric quotient $\mathcal{G} / \mathcal{H}$ exists \textsuperscript{[A]}. Note also that the Chevalley isomorphism holds $G_R / G_R \simeq T_R / W_R$ for any ring $R$ and the formation commutes with arbitrary base change $R \to R'$ \textsuperscript{[Le]}. It follows that the quotient map $\pi : G \to G / U$ and the Grothendieck-Springer simultaneous resolution in (3.1) makes sense over any ring $R$. Moreover, the formation commutes with arbitrary base change $R \to R'$. We denote them by $\pi_R : G_R \to G_R / U_R$.
and

\[(6.22) \quad \begin{array}{ccc}
G_R & \xrightarrow{q_R} & T_R \\
\downarrow{\bar{e}_R} & & \downarrow{q_R} \\
G_R & \xrightarrow{e_R} & T_R/W_R
\end{array}\]

Denote by \(h_R : \bar{G}_R \to Z_R := G_R \times_{T_R/W} T_R\) the induced map. We define

\[(6.23) \quad \text{IC}(Z_R) := (h_Z)\overline{\mathcal{Q}}_\ell[\dim G_c] \in \mathcal{D}(Z_R).\]

When \(R = k\) is an algebraically closed field of characteristic not equal to \(\ell\), \(\text{IC}(Z_k)\) is the IC-complex of \(Z_k\) and there is a canonical \(W\)-equivariant structure \((\text{IC}(Z_k), \phi_k) \in \mathcal{D}_W(Z_k)\) (see Section 3.4).

**Lemma 6.10.** There exists a positive integer \(N\), depends on the group \(G_Z\), satisfies the following. Let \(A\) be the strict Henselization of \(\mathbb{Z}[1/N\ell]\) at a closed point with residue field \(k\). One can endow the \(\ell\)-adic complex \(\text{IC}(S_A)\) in \((6.23)\) with a \(W\)-equivariant structure \((\text{IC}(Z_A), \phi_A) \in \mathcal{D}_W(Z_A)\) which, under the base change \(A \to k\), becomes \((\text{IC}(Z_k), \phi_k) \in \mathcal{D}_W(S_k)\).

**Proof.** According to [BD, Section 6.1] (or the discussion in [Dr, Section 4]), one can choose a (large enough) positive integer \(N\), a stratification \(T\) of \(S_{\mathbb{Z}[1/N\ell]}\), and for each \(T \in T\) a (finite) collection \(\mathcal{L}(T)\) of \(\ell\)-adic local systems on \(T\), satisfy the following: (1) We have \(w^*\text{IC}(Z_{\mathbb{Z}[1/N\ell]}) \in \mathcal{D}_{T,W}(Z_{\mathbb{Z}[1/N\ell]}(w) \in W\), (2) Let \(A, k\) be as above. Let \(i : Z_k \to Z_A\) be the imbedding. The functor \(i^* : \mathcal{D}_{T,W}(Z_A) \to \mathcal{D}_{T,W}(Z_k)\) is an equivalence. Here \(\mathcal{D}_{T,W}(Z_A)\) (resp. \(\mathcal{D}_{T,W}(Z_k)\)) is the full subcategory of \(\mathcal{D}(Z_A)\) (resp. \(\mathcal{D}(Z_k)\)) generated by the \(*\)-restriction of \(\mathcal{L}(T), T \in T\) to \(Z_A\) (resp. \(Z_k\)).

Note that, by (1) above, we have \(w^*\text{IC}(Z_A) \in \mathcal{D}_{T,W}(Z_A)\) and \(w^*\text{IC}(Z_k) \in \mathcal{D}_{T,W}(Z_k)\). Let \(\phi_{k,w} : \text{IC}(S_k) \simeq w^*\text{IC}(Z_k)\) be the isomorphism coming from the \(W\)-equivariant structure \(\phi_k\) on \(\text{IC}(Z_k)\). Since \(i^*\text{IC}(Z_A) \simeq \text{IC}(Z_k)\), it follows that there exists an isomorphism \(\phi_{A,w} : \text{IC}(Z_A) \simeq w^*\text{IC}(Z_A)\) which, under the base change \(A \to k\), becomes \(\phi_{k,w}\). Now the collection \(\{\phi_{A,w}|w \in W\}\) defines a \(W\)-equivariant structure \(\phi_A\) on \(\text{IC}(Z_A)\) satisfying the required property. \(\square\)

**Proposition 6.11.** There exists a positive integer \(N\), depends on the group \(G_Z\), satisfies the following: Let \(A\) be the strict Henselization of \(\mathbb{Z}[1/N\ell]\) at a closed point with residue field \(k\). There is a \(\ell\)-adic complex \(M_{\theta,A}\) on \(G_A\) which, under the base change \(A \to k\), becomes \(M_{\theta}\).

**Proof.** Let \(N, A, k\) be as in Lemma 6.10. Let \(\mathcal{E}_{\theta,A} \in \mathcal{D}_W(T_A)\) be the lift of \(\mathcal{E}_{\theta}\) in Lemma 6.9. Define

\[\text{Ind}_{T_A \subset B_A}^G (\mathcal{E}_{\theta,A}) := (\bar{e}_A)!(\bar{q}_A)^*(\mathcal{E}_{\theta,A})[\dim G_c - \dim T_c] \simeq p_{G,A}(\text{IC}(Z_A)) \otimes [\text{IC}(Z_A)] [\dim G_c - \dim T_c].\]

Here \(p_{T,A}\) and \(p_{G,A}\) are the natural projections from \(Z_A\) to \(T_A\) and \(G_A\) respectively. The \(W\)-equivariant structures on \(\mathcal{E}_{\theta,A}\) and \(\text{IC}(Z_A)\) give rise to a \(W\)-action on \(\text{Ind}_{T_A \subset B_A}^G (\mathcal{E}_{\theta,A})\) and we define \(M_{\theta,A} = \text{Ind}_{T_A \subset B_A}^G (\mathcal{E}_{\theta,A})^W\) to be the \(W\)-invariant factor. Since the base change of \(\mathcal{E}_{\theta,A}\) along \(A \to k\) is isomorphic to \(\mathcal{E}_{\theta}\), it follows that the base change of \(M_{\theta,A}\) along \(A \to k\) is isomorphic to \(M_{\theta} = \text{Ind}_{T_k \subset B_k}^G (\mathcal{E}_{\theta})^W\). \(\square\)
6.9. **ULA property of the averaging functor.** We first review the notion of universal local acyclicity (ULA) following [De2] [Z].

Let $S$ be a Noetherian scheme. Let $f : X \to S$ be a morphism of finite type and let $\mathcal{F} \in \mathcal{D}(X)$. Let $s$ be a geometric point of $S$ and let $S_{(s)}$ be the strict Henselisation at $s$. We recall the following definition in [De2]:

**Definition 6.12.** A $\ell$-adic complex $\mathcal{F} \in \mathcal{D}(X)$ is called locally acyclic with respect to $f : X \to S$ if for every geometric point $x \in X$ and every geometric point $t \in S_{(f(x))}$, the natural map $\mathcal{H}^r(X_x, \mathcal{F}) \to \mathcal{H}^r((X_x)_t, \mathcal{F})$ is an isomorphism, where $(X_x)_t = (X_x) \times_{S_{(f(x))}} t$. It is called universally locally acyclic (ULA) if it is locally acyclic after arbitrary base change $S' \to S$.

One can reformulate local acyclicity as follows. Let $t$ be a geometric point of $S_{(s)}$. Denote by $j_t : X \times_S t \to X$ and $i_s : X \times_S s \to X$ the natural maps. We write

$$(6.24) \quad \Psi_{t\to s}(\mathcal{F}) := i_s^* (j_t)_* j_t^* (\mathcal{F}).$$

It is shown in [Z, Lemma A 2.2] that $\mathcal{F}$ is locally acyclic with respect to $f$ if and only if the natural map

$$i_s^* \mathcal{F} \to \Psi_{t\to s}(\mathcal{F})$$

is an isomorphism.

Here are some properties of ULA complexes that we need.

**Theorem 6.13.**

1. Let $f : X \to Y$ be a proper morphism over $S$, and let $\mathcal{F}$ be a $\ell$-adic complex on $X$ ULA with respect to $X \to S$. Then $f_* \mathcal{F}$ is ULA with respect to $Y \to S$.
2. Let $f : X \to Y$ be a smooth morphism over $S$, and let $\mathcal{F}$ be a $\ell$-adic complex on $Y$. Then $\mathcal{F}$ is ULA with respect to $Y \to S$ if and only if $f^* \mathcal{F}$ is ULA with respect to $X \to S$.
3. Let $f_i : X \to S, i = 1, 2$ and let $\mathcal{F}_i$ be a $\ell$-adic complex on $X_i$ ULA with respect to $X_i \to S$. Then $\mathcal{F}_1 \boxtimes_S \mathcal{F}_2$ is ULA with respect to $X_1 \times_S X_2 \to S$.
4. Let $f : X \to S$ be a morphism of finite type and let $\mathcal{F}$ be a $\ell$-adic complex on $X$. Then there is an open dense subset $U$ of $S$ such that the restriction of $\mathcal{F}$ to $X_U = X \times_S U$ is ULA with respect to $X_U \to U$.
5. Let $f : X \to S$ be a smooth morphism. Then any $\ell$-adic local system $\mathcal{F}$ on $X$ is ULA with respect to $f : X \to S$.

Consider the open embedding $j_S : U_S \times_S \bar{B}_S \to G_S$, where $\bar{B}_S$ is the opposite Borel. Recall the quotient map $\pi_S : G_S \to G_S/U_S$. We have the following ULA property of the averaging functor.

**Proposition 6.14.** Assume $j_{S,*}(\overline{\mathcal{Q}})$ is ULA with respect to $G_S \to S$, of formation compatible with arbitrary change of base on $S$. Let $\mathcal{F}$ be a $\ell$-adic complex on $G_S$ ULA with respect to $G_S \to S$. Then $\pi_S^* (\mathcal{F})$ is ULA with respect to $G_S/U_S \to S$, of formation compatible with arbitrary change of base on $S$.

**Proof.** Consider the following Cartesian diagram

$$
\begin{array}{ccc}
U_S \times_S G_S & \xrightarrow{a} & G_S \\
\downarrow{\text{pr}} & & \downarrow{\pi_S} \\
G_S & \xrightarrow{\pi_S} & G_S/U_S
\end{array}
$$

where $\text{pr}$ is the projection map and $a$ is the left action map. As $\pi_S$ is smooth surjective morphism, it suffices to show that

$$(\pi_S)^* \pi_{S,*}(\mathcal{F}) \simeq a_* \text{pr}^*(\mathcal{F}) \simeq a_*(\overline{\mathcal{Q}} \boxtimes_S \mathcal{F})$$
is ULA w.r.t $G_S \to S$, of formation compatible with arbitrary change of base on $S$. Note that $a$ admits the following factorization

$$U_S \times_S G_S \xrightarrow{h} G_S \times_{\bar{B}_S} G_S \xrightarrow{\bar{a}} G_S$$

where $h$ is the open embedding $U_S \times_S G_S \simeq (U_S \times_S \bar{B}_S) \times_{\bar{B}_S} G_S \subset G_S \times_{\bar{B}_S} G_S$ and $\bar{a}$ is the left action map. As $\bar{a}$ is proper, to show that

$$a_*(\bar{Q} \boxtimes S F) \simeq \bar{a}_* h_*(\bar{Q} \boxtimes S F)$$

is ULA w.r.t $G_S \to S$, of formation compatible with arbitrary change of base on $S$, it suffices to show that

$$h_*(\bar{Q} \boxtimes S F)$$

is ULA w.r.t $G_S \times_{\bar{B}_S} G_S \to S$, of formation compatible with arbitrary change of base on $S$. Consider the following Cartesian diagram

$$
\begin{align*}
(U_S \times_S \bar{B}_S) \times_S G_S & \xrightarrow{j_\mathbf{id}} G_S \times_S G_S \\
U_S \times_S G_S & \xrightarrow{h} G_S \times_{\bar{B}_S} G_S
\end{align*}
$$

where $j_\mathbf{id}$ is the open embedding $U_S \times_S \bar{B}_S \to G_S$ is the open embedding. Since $q$ is a smooth surjective morphism and $j_\mathbf{id}$ is ULA w.r.t $G_S \to S$ (by assumption), we have

$$q^* h_*(\bar{Q} \boxtimes S F) \simeq j_\mathbf{id}* (\bar{Q} \boxtimes S F),$$

which is ULA w.r.t $G_S \times_S G_S \to S$, of formation compatible with arbitrary change of base on $S$.

The lemma follows

$\square$

Remark 6.3. By Theorem 6.13 (4) and Deligne’s generic base change theorem [De2, Corollary 2.9], the assumptions on $j_\mathbf{id}(\bar{Q}_\ell)$ holds after a base change to an open dense subset $U \subset S$.

Let $A$ be a strictly Henselian local ring and let $S = \text{Spec} A$. Let $s$ be the closed point of $S$ and let $t$ be a geometric point of $S$. Let $f : X \to S$ be a scheme over $S$ and let $Y \subset X$ be an open subscheme over $S$.

Lemma 6.15. Let $\mathcal{F}$ be a $\ell$-adic complex on $X$ ULA with respect to $X \to S$. Let $\mathcal{F}_s$ and $\mathcal{F}_t$ be the restriction of $\mathcal{F}$ to the fiber $X_s$ and $X_t$ respectively. Then $\mathcal{F}_t|Y_t \simeq 0$ implies $\mathcal{F}_s|Y_s \simeq 0$.

Proof. Indeed, since $Y \to X$ is smooth and the functor $\Psi_{t \to s}$ in (6.24) commutes with smooth pull back, the isomorphism $\mathcal{F}_s \simeq \Psi_{t \to s}(\mathcal{F})$ (coming from the ULA property) implies $\mathcal{F}_s|Y_s \simeq \Psi_{t \to s}(\mathcal{F})|Y_s \simeq \Psi_{t \to s}(\mathcal{F}|Y) \simeq i_\mathbf{t,s}^* (\mathcal{F}|Y_t) \simeq 0$.

$\square$

6.10. Proof of Theorem 6.1 in the $\ell$-adic setting. We shall show that there exists a positive integer $N$, depending only on $G_{\mathbb{Z}}$, such that for any algebraically closed field $k$ of positive characteristic not dividing $N\ell$ and a $W$-orbit $\theta = \Phi \chi$ of a tame character $\chi \in \mathfrak{C}(T_k)(\mathbb{Q}_\ell)$, the averaging $\text{Av}_{k}^{U_k}(M_\theta)$ is supported on $T_k = B_k/U_k \subset G_k/U_k$. Equivalently, the restriction of $\text{Av}_{k}^{U_k}(M_\theta)$ to the open complement $Y_k = (G_k/U_k) \setminus (B_k/U_k)$ is zero.

Let $N, A, M_\theta, A, \mathcal{C}_{\theta, A}$ be as in Proposition 6.11 such that $k$ is the residue field of $A$. According to Remark 6.3 by replacing $N$ with a larger positive integer, we can assume $(j_{\text{Spec}(\mathbb{Z}[1/N])})^* \mathcal{C}_{\theta, A}$
is ULA w.r.t \( G_{\mathbb{Z}[1/N]} \to \text{Spec}(\mathbb{Z}[1/N]) \), of formation compatible with arbitrary base change on \( \text{Spec}(\mathbb{Z}[1/N]) \).

We will write \( \mathcal{E}_{\theta,A}', M_{\theta,A}' \) for the base change of \( \mathcal{E}_{\theta,A}, M_{\theta,A} \) along \( A \to A' \). Note that, by Lemma 6.9 and Proposition 6.11, we have \( \mathcal{E}_{\theta,k} \simeq \mathcal{E}_\theta \) and \( \mathcal{M}_{\theta,k} \simeq \mathcal{M}_\theta \).

**Lemma 6.16.** \( (\pi_A)_* M_{\theta,A} \) is ULA with respect to \( G_A/U_A \to \text{Spec}(A) \), of formation compatible with arbitrary base change on \( \text{Spec}(A) \).

**Proof.** Since the map \( \tilde{c}_A \) (resp. \( \tilde{q}_A \)) in (6.22) is proper (resp. smooth) and \( \mathcal{E}_{\theta,A} \) is a \( \ell \)-local system, by Theorem 6.13 the induction \( \text{Ind}_{T_A \subset B_A}^G (\mathcal{E}_{\theta,A}) := (\tilde{c}_A\tilde{q}_A)^* (\mathcal{E}_{\theta,A})[\dim G_C - \dim T_C] \) is ULA w.r.t \( G_A \to \text{Spec}(A) \). As \( M_{\theta,A} \) is the W-invariant direct factor of \( \text{Ind}_{T_A \subset B_A}^G (\mathcal{E}_{\theta,A}) \) it implies \( M_{\theta,A} \) is ULA w.r.t \( G_A \to \text{Spec}(A) \), of formation compatible with arbitrary base change on \( \text{Spec}(A) \). By Proposition 6.14, we conclude that \( (\pi_A)_* M_{\theta,A} \) is ULA with respect to \( G_A/U_A \to \text{Spec}(A) \), of formation compatible with arbitrary base change on \( \text{Spec}(A) \). \( \square \)

**Lemma 6.17.** \( (\pi_C)_* M_{\theta,C} \) is supported on \( T_C = B_C/U_C \subset G_C/U_C \).

**Proof.** Let \( D^b_b(G_C, \overline{\mathbb{Q}_C}) \) be the bounded derived category of constructible \( \ell \)-adic complexes on \( G_C \) and let \( D^b_b(G_C(C), \mathbb{C}) \) be the usual bounded derived category of \( \mathbb{C} \)-constructible complexes on the complex Lie group \( G_C(C) \). We fix an isomorphism \( \iota : \overline{\mathbb{Q}}_\ell \simeq \mathbb{C} \). Then according to [BBD, Section 6.1], there is a comparison functor

\[
e^* : D^b_b(G_C, \overline{\mathbb{Q}}_\ell) \to D^b_b(G_C(C), \mathbb{C})
\]

which is fully-faithful and commutes with six functor formalism. Let \( \mathcal{E}_{\theta,C} \) be the base change of \( \mathcal{E}_{\theta,A} \) along \( A \to C \). We claim that there exists a character \( \chi_C \) of the topological fundamental group \( \pi_1(T(C)) \) such that, under the Riemann-Hilbert correspondence (R.H. for short), \( e^* \mathcal{E}_{\theta,C} \) corresponds to the de Rham local system \( \mathcal{E}_{\theta,C} \) in Section 5.1. Here \( \theta_C = W_{\chi_C} \) is the W-orbit of \( \chi_C \). This will imply

\[
e^* M_{\theta,C} \simeq e^* ((\text{Ind}_{T_C \subset B_C}^G (\mathcal{E}_{\theta,C}))^W) \simeq (\text{Ind}_{T_C \subset B_C}^G (e^* \mathcal{E}_{\theta,C}))^W \simeq \text{Ind}_{T_C \subset B_C}^G (\mathcal{E}_{\theta,C})^W = M_{\theta,C}
\]

and the lemma follows from Theorem 6.1 in the de Rham setting.

To prove the claim, we observe that the constructions of \( \mathcal{E}_{\theta} \) and \( \mathcal{E}_{\theta,A} \) in Section 5.1 and Lemma 6.9 imply that \( \mathcal{E}_{\theta,C} \) corresponds to a \( \ell \)-adic representation \( \text{Ind}_{W_{\chi_C,\ell}}^W (\rho_{\chi_C,\ell} \otimes \chi_{C,\ell}) \) where \( \chi_{C,\ell} \) is a \( \ell \)-adic character of \( \pi_1(T_C) \) with \( W_{\chi_{C,\ell}} = W_{\chi} \), and \( \rho_{\chi_C,\ell} = \text{Ind}_{W_{\chi_C,\ell}}^W \pi_1(T_C) \) in \( \overline{\mathbb{Q}}_\ell[[\pi_1(T_C)]]/\overline{\mathbb{Q}}_\ell[[\pi_1(T_C)]]^W \) given by the \( \overline{\mathbb{Q}}_\ell[[\pi_1(T_C)]] \)-module structure. Here \( \overline{\mathbb{Q}}_\ell[[\pi_1(T_C)]] \) is the completed group algebra of the pro-\( \ell \) part of the étale fundamental group \( \pi_1(T_C) \). Note that the restriction of the functor \( e^* \) to the subcategory of \( \ell \)-adic local systems on \( T_C \) is induced by the natural embedding

\[
\pi_1(T(C)) \to \pi_1(T(C))_\ell \simeq \pi_1(T(C)_\ell).
\]

Note also that (6.25) induces an isomorphism

\[
S/\langle S^W_{+\chi_{C,\ell}} \rangle \simeq \overline{\mathbb{Q}}_\ell[[\pi_1(T(C)_\ell)]]/\overline{\mathbb{Q}}_\ell[[\pi_1(T(C)_\ell)]^W]
\]

compatible with the \( W_{\chi_{C,\ell}} \)-action, here \( S = \mathbb{C}[\pi_1(T(C))] \) is the completion of the group algebra \( \mathbb{C}[\pi_1(T(C))] \) at \( 1 \in \pi_1(T(C)) \). Let \( \rho_{\chi_{C,\ell}}^{uni,\ell} \) be the representations of the topological fundamental group \( \pi_1(T(C)) \) given by pull back of \( \rho_{\chi_C,\ell}^{uni} \) and \( \chi_{C,\ell} \) along (6.25). It follows that \( \rho_{\chi_{C,\ell}}^{uni,\ell} \) is isomorphic to the representation \( \rho_{\chi_{C,\ell}}^{uni} \) in Section 5.1 (in the de Rham setting) and hence the pull back
Lemma 4.1, we conclude that $p$ is isomorphic to $\text{Ind}_{W_{X_C}}^W(p_{X_C}^{\text{uni}} \otimes \chi_C)$. Since, by construction, $\mathcal{E}_{\theta C}$ corresponds to $\text{Ind}_{W_{X_C}}^W(p_{X_C}^{\text{uni}} \otimes \chi_C)$ under the Reimann-Hilbert correspondence, we conclude that $\epsilon^*\mathcal{E}_{\theta C} \cong \mathcal{E}_{\theta C}$. The claim follows.

\[\square\]

Applying Lemma 6.15 to the case $\mathcal{F} = (\pi_A)_!*\mathcal{M}_{\theta,A}$, $X = G_A/U_A$, $Y = Y_S = (G_A/U_A) \setminus (B_A/U_A)$, and using Lemma 6.16 and Lemma 6.17, we conclude that $\text{Av}_*^U(M_\theta) \cong (\pi_k)_!*\mathcal{M}_{\theta,k}$ is supported on $T_k$. This finished the proof of Theorem 6.1 in the $\ell$-adic setting.

7. Proof of Theorem 1.5

In this section we prove the vanishing conjecture (Conjecture 1.2) for strongly central complex (resp. strongly $\ast$-central complex):

Theorem 7.1. There exists a positive integer $N$ depending only on the type of the group $G$ such that the following holds. Assume $k = \mathbb{C}$ or $\text{char} k = p$ is not dividing $|\mathcal{N}|$. Let $\mathcal{F} \in \mathcal{D}_W(T)$ be a strongly central complex (resp. strongly $\ast$-central complex) on $T$ and let $\Phi_T = \text{Ind}_{T \subset B}^G(\mathcal{F})^W \in \mathcal{D}(G)$.

For any $x \in G \setminus B$, we have the following cohomology vanishing

$$H^*_x(xU,i^1\Phi_T) = 0 \quad \text{(resp. } H^*(xU,i^1\Phi_T) = 0)$$

where $i : xU \to G$ is the natural inclusion map. Equivalently, $\text{Av}_1^U(\Phi_T)$ (resp. $\text{Av}_*^U(\Phi_T)$) is supported on the closed subset $T = B/U \subset G/U$.

In particular, the vanishing conjecture holds for reductive groups with connected center.

7.1. Reduction to perverse sheaves.

Lemma 7.2. If Conjecture 1.2 holds for central perverse sheaves (resp. $\ast$-central perverse sheaves), then it holds for arbitrary central complexes (resp. $\ast$-central complex). The same is true for strongly central complexes (resp. strongly $\ast$-central complexes).

Proof. It is enough to verify the lemma for $\ast$-central complexes and strongly $\ast$-central complexes. Let $\mathcal{F}$ be a $\ast$-central complex.

The $\ell$-adic setting. We shall show that $p_{\tau \leq b}(\mathcal{F})$ and $p_{\mathcal{H}^b}(\mathcal{F})$, $b \in \mathbb{Z}$, are $\ast$-central. Let $\chi \in \mathcal{C}(T)(\mathbb{Q}_\ell)$ and let $I_\chi$ be the maximal ideal corresponding to $\chi$. Write $\mathcal{C}(T)_\chi$ be the completion of at $\chi$. Let $q_\chi : \mathcal{C}(T)_\chi \to \mathcal{C}(T)_\chi/\mathcal{W}_\chi$ be the quotient map. Since $\chi$ is the unique closed point of $\mathcal{C}(T)_\chi$, and the action of $\mathcal{W}_\chi$ on the fiber $i^*_\chi(\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{\mathcal{C}(T)_\chi}) \cong i^*_\chi \mathcal{M}(\mathcal{F} \otimes \text{sign})$ is trivial, by Lemma 5.3 there exists $\mathcal{G} \in \mathcal{D}_{\text{coh}}(\mathcal{C}(T)_\chi/\mathcal{W}_\chi)$ such that there is an isomorphism

$$\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{\mathcal{C}(T)_\chi} = q^*_\chi \mathcal{G} \cong D_{\text{coh}}(\mathcal{C}(T)_\chi/\mathcal{W}_\chi).$$

Since $\mathcal{M}$, $q^*_\chi$ and taking completions $(-)|_{\mathcal{C}(T)_\chi}$ are $\text{t}$-exact functors, we have

$$\mathcal{M}(p_{\tau \leq b}(\mathcal{F}) \otimes \text{sign})|_{\mathcal{C}(T)_\chi} \simeq \tau_{\leq b}(\mathcal{M}(\mathcal{F} \otimes \text{sign})|_{\mathcal{C}(T)_\chi}) \simeq \tau_{\leq b}(q^*_\chi \mathcal{G}) \simeq q^*_\chi(\tau_{\leq b}(\mathcal{G})).$$

Similarly, we have

$$\mathcal{M}(p_{\mathcal{H}^b}(\mathcal{F}) \otimes \text{sign})|_{\mathcal{C}(T)_\chi} \simeq q^*_\chi \mathcal{H}^b(\mathcal{G}).$$

It follows that $\mathcal{W}_\chi$ acts trivially on the fibers $i^*_\chi \mathcal{M}(p_{\tau \leq b}(\mathcal{F}) \otimes \text{sign})$ and $i^*_\chi \mathcal{M}(p_{\mathcal{H}^b}(\mathcal{F}) \otimes \text{sign})$, and by Lemma 4.1 we conclude that $p_{\tau \leq b}(\mathcal{F})$ and $p_{\mathcal{H}^b}(\mathcal{F})$ are $\ast$-central. Now an induction argument on the (finite) number of non vanishing perverse cohomology sheaves of $\mathcal{F}$ in [CT, Lemma 6.6] implies the Lemma.
The de Rham setting. Let \( \lambda \in t(\mathbb{C}) \) with maximal ideal \( I_\lambda \) and let \( \breve{t}_\lambda \) be the completion at \( \lambda \). By Lemma 4.2, the restriction \( \mathfrak{M}(\mathcal{F})|_{\breve{t}_\lambda} \) is a coherent complex on \( \breve{t}_\lambda \). Since \( \mathcal{F} \) is \(+\)-central and \( \lambda \) is the unique closed point of \( \breve{t}_\lambda \), Lemma 5.3 implies

\[
\mathfrak{M}(\mathcal{F} \otimes \text{sign})|_{\breve{t}_\lambda} \simeq q_\lambda^* \mathcal{F} \in D_{\text{con}}^b(\breve{t}/W_{a,\lambda}),
\]

where \( q_\lambda : \breve{t} \rightarrow \breve{t}/W_{a,\lambda} \) and \( \mathcal{F} \) is a coherent complex on \( \breve{t}/W_{a,\lambda} \). Now we can conclude by applying the same argument as in the \( \ell \)-adic setting where \( \mathcal{C}(T)_\chi \) and \( W_\chi \) are replaced by \( \breve{t}_\lambda \) and \( W_{a,\lambda} \).

The case of strongly \(+\)-central complexes can be proved in the same way: applying the argument above where \( W_\chi \) and \( W_{a,\lambda} \) are replaced by \( W'_\chi \) and \( W'^{\text{ex}}_{a,\lambda} \). This completes the proof.

7.2. Convolution with \( M_\theta \).

**Proposition 7.3.** Let \( \mathcal{F} \in \mathscr{D}_W(T)^{\odot} \) be a strongly \(+\)-central perverse sheaf and let \( \theta \) be a \( W \)-orbit through a tame character \( \chi \in \mathcal{C}(T)(F) \). There is an isomorphism

\[
\Phi_{\mathcal{F}} \ast M_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes M_\theta.
\]

**Proof.** By Proposition 5.5 we have

\[
(7.1) \quad \mathcal{F} \ast \mathcal{E}_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta \in \mathscr{D}(T).
\]

Since \( \text{Av}_x^U(M_\theta) \simeq \mathcal{E}_\theta \) is supported on \( T = B/U \subset G/U \), Proposition 3.3 implies

\[
(7.2) \quad \text{Ind}^G_{T \subset B}(\mathcal{F}) \ast M_\theta \simeq \text{Ind}^G_{T \subset B}(\mathcal{F} \ast \mathcal{E}_\theta) \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \text{Ind}^G_{T \subset B}(\mathcal{E}_\theta).
\]

We claim that the isomorphism above is compatible with the natural \( W \)-actions. Taking \( W \)-invariant on both sides of (7.2), we get

\[
\Phi_{\mathcal{F}} \ast M_\theta \simeq \text{Ind}^G_{T \subset B}(\mathcal{F})^W \ast M_\theta \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \text{Ind}^G_{T \subset B}(\mathcal{E}_\theta)^W \simeq H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes M_\theta.
\]

The proposition follows.

7.2.1. **Proof of the claim.** Let us write \( \text{Res} = \text{Res}^G_{T \subset B}, \text{Ind} = \text{Ind}^G_{T \subset B}, \mathcal{M} = M_\theta, \mathcal{E} = \mathcal{E}_\theta, \) and \( V = H^*(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \). Using Proposition 3.3 and the adjunction between \( \text{Res} \) and \( \text{Ind} \), it is straightforward to check that the following diagram commute

\[
\begin{array}{ccc}
F[W] & \xrightarrow{(1)} & \text{Hom}(\text{Ind}(\mathcal{F}), \text{Ind}(\mathcal{F})) & \xrightarrow{s^M} & \text{Hom}(\text{Ind}(\mathcal{F}) \ast \mathcal{M}, \text{Ind}(\mathcal{F}) \ast \mathcal{M}) \\
\downarrow \text{Id} & & \downarrow \text{Hom}(\text{Ind}(\mathcal{F} \ast \mathcal{E}), \text{Ind}(\mathcal{F}) \ast \mathcal{M}) & \xrightarrow{(4)} & \downarrow \text{End}(\mathcal{F} \ast \mathcal{E}, \text{Res} \circ \text{Ind}(\mathcal{F} \ast \mathcal{E})) \\
F[W] & \xrightarrow{(3)} & \text{Hom}(\mathcal{F}, \text{Res} \circ \text{Ind}(\mathcal{F})) & \xrightarrow{s^E} & \text{End}(\mathcal{F} \ast \mathcal{E}, \text{Res} \circ \text{Ind}(\mathcal{F} \ast \mathcal{E}))
\end{array}
\]

Here (1) is given by the \( W \)-action on \( \text{Ind}(\mathcal{F}) \), (2) is the adjunction map, (3) is the composition \( (2) \circ (1) \), (4) is induced by the isomorphism \( \text{Ind}(\mathcal{F}) \ast \mathcal{M} \simeq \text{Ind}(\mathcal{F} \ast \mathcal{E}) \) in Proposition 3.3 and (5) is induced by the canonical isomorphism

\[
(7.4) \quad \text{Res}(\text{Ind}(\mathcal{F}) \ast \mathcal{M}) \simeq \text{Res} \circ \text{Ind}(\mathcal{F} \ast \mathcal{M}) \simeq \text{Res} \circ \text{Ind}(\mathcal{F}) \ast \mathcal{E}^{\text{ex}}
\]

\(^{13}\)The first isomorphism follows from the fact that, for any \( \mathcal{F}_1, \mathcal{F}_2 \in \mathscr{D}(G/\text{ad}G) \) such that \( \text{Av}_x^G \mathcal{F}_2 \) is supported on \( T \), we have canonical isomorphism \( \text{Res}(\mathcal{F}_1 \ast \mathcal{F}_2) \simeq \text{Res}(\mathcal{F}_1) \ast \text{Res}(\mathcal{F}_2) \).
and the adjunction map. On the other hand, the canonical map
\[
\text{Res} \circ \text{Ind}(\mathcal{F}) \ast \mathcal{E} \cong \text{Res}(\text{Ind}(\mathcal{F}) \ast M) \cong \text{Res}(V \otimes \text{Ind}(\mathcal{E})) \cong V \otimes \text{Res} \circ \text{Ind}(\mathcal{E})
\]
is compatible with the natural W-actions. Indeed, Proposition 3.2 and Proposition 5.5 imply that there is a commutative diagram
\[
(F[W] \otimes \mathcal{F}) \ast \mathcal{E} \quad \overset{\ast \mathcal{E}}{\longrightarrow} \quad V \otimes (F[W] \otimes \mathcal{E})
\]
\[
\text{Res} \circ \text{Ind}(\mathcal{F}) \ast \mathcal{E} \quad \overset{\ast \mathcal{E}}{\longrightarrow} \quad V \otimes (\text{Res} \circ \text{Ind}(\mathcal{E}))
\]
where the vertical arrows and the upper horizontal arrow are compatible with the natural W-actions. It follows that the following diagram commute
\[
\begin{array}{ccc}
F[W] & \overset{(3)}{\longrightarrow} & \text{Hom}(\mathcal{F}, \text{Res} \circ \text{Ind}(\mathcal{F})) \\
\downarrow \text{Id} & & \downarrow \text{Hom}(V \otimes \mathcal{E}, V \otimes \text{Res} \circ \text{Ind}(\mathcal{E})) \\
F[W] & \overset{(8)}{\longrightarrow} & \text{Hom}(\mathcal{E}, \text{Ind}(\mathcal{E}))
\end{array}
\]
\[
\begin{array}{ccc}
& & \text{Id} \\
(4) & & (6) \\
& & \downarrow \text{Id} \\
& & (7) \\
\end{array}
\]
where (6) is induced by (7.5), (7) is the adjunction map, and (8) is the W-action map. Note that the composition of (4), (5), (6), (7) gives rise to a map
\[
\text{End}(\text{Ind}(\mathcal{F}) \ast M) \to \text{End}(V \otimes \text{Ind}(\mathcal{E}))
\]
which is equal to the map induced by the isomorphism (7.2), thus the commutativity of (7.3) and (7.6) implies that the following diagram commute
\[
\begin{array}{ccc}
F[W] & \longrightarrow & \text{End}(\text{Ind}(\mathcal{F}) \ast M) \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
F[W] & \longrightarrow & \text{End}(V \otimes \text{Ind}(\mathcal{E}))
\end{array}
\]
where the horizontal arrows are the W-action maps. The claim follows.

7.3. **Proof of Theorem 7.1** Since the Verdier duality interchanges strongly central complexes with strongly *-central complexes, it suffices to verify Theorem 7.1 for strongly *-central complexes. Let \( \mathcal{F} \in \mathcal{D}(T) \) be a strongly *-central complex. We need to show that the natural map
\[
r : \text{Res}^G_{T \subset B}(\Phi_{\mathcal{F}}) \to \text{Av}^U_{\ast}(\Phi_{\mathcal{F}})
\]
is an isomorphism. By Lemma 7.2, we can assume \( \mathcal{F} \) is a perverse sheaf. We claim that, for any W-orbit \( \theta = W \chi \subset \mathcal{C}(T) \), the convolution of \( r \) with \( \mathcal{E}_\theta \) is an isomorphism
\[
\text{Res}^G_{T \subset B}(\Phi_{\mathcal{F}}) \ast \mathcal{E}_\theta \cong \text{Av}^U_{\ast}(\Phi_{\mathcal{F}}) \ast \mathcal{E}_\theta.
\]
For this, it is enough to show that \( \text{Av}^U_{\ast}(\Phi_{\mathcal{F}}) \ast \mathcal{E}_\theta \) is supported on \( T \) and this follows from Theorem 6.1 and Proposition 7.3. Indeed, we have
\[
\text{Av}^U_{\ast}(\Phi_{\mathcal{F}}) \ast \mathcal{E}_\theta \overset{\text{Thm } 6.1}{=} \text{Av}^U_{\ast}(\Phi_{\mathcal{F}}) \ast \text{Av}^U_{\ast}(M_\theta) \cong \text{Av}^U_{\ast}(\Phi_{\mathcal{F}} \ast M_\theta) \overset{\text{Prop } 7.3}{=} \text{H}^\ast(T, \mathcal{F} \otimes L_{\chi^{-1}}) \otimes \text{Av}^U_{\ast}(M_\theta) \overset{\text{Thm } 6.1}{=} \text{H}^\ast(T, \mathcal{F} \otimes \mathcal{L}_{\chi^{-1}}) \otimes \mathcal{E}_\theta.
\]
Since $E_\theta \simeq \bigoplus \chi \in \Theta E_\chi$, the isomorphism (7.9) implies that the cone of the map in (7.8), denoted by cone($r$), satisfies cone($r$)*$E_\chi = 0$ for all $\chi \in \Theta(T)(F)$. As $E_\chi$ is a local system on $T$ with generalized monodromy $\chi$, that is, $E_\chi \otimes L_\xi$ is an unipotent local system, Lemma 7.4 and Lemma 7.5 below imply cone($r$) = 0. The theorem follows.

7.4. Vanishing lemmas. Let $X$ be a smooth variety with a free $T$ action $a : T \times X \to X$. For $L \in D(T)$ and $F \in D(X)$ we define $L \ast F := a_*(L \boxtimes F) \in D(X)$.

Lemma 7.4. Let $L$ be a local system on $T$ with generalized monodromy $\chi \in \Theta(T)(F)$, that is, $L \otimes L_\chi$ is an unipotent local system. Let $F \in D(X)$ and assume $L \ast F = 0$. Then we have $L_\chi \ast F = 0$.

Proof. There is a filtration $0 = L(0) \subset L(1) \subset \cdots \subset L(k) = L$ such that

$$0 \to L^{(i-1)} \to L^{(i)} \to L^{(i)}/L^{(i-1)} \simeq L_\chi \to 0.$$ 

Assume $L_\chi \ast F \neq 0$ and let $m$ be the smallest number such that $H^{\geq m}(L_\chi \ast F) = 0$. We claim that $H^{\geq m}(L^{(i)} \ast F) = 0$ for $i = 1, \ldots, k$. The case $i = 1$ is automatic since $L^{(1)} = L_\chi$. For $1 \leq i \leq k$, we consider the distinguished triangle

$$L^{(i-1)} \ast F \to L^{(i)} \ast F \to L_\chi \ast F \to L^{(i-1)} \ast F[1]$$

induced from above short exact sequence. Then for any $n \geq m$ we obtain an exact sequence

$$H^n(L^{(i)} \ast F) \to H^n(L^{(i)} \ast F) \to H^n(L_\chi \ast F) \to$$

By induction, the first and third terms are zero, and hence $H^n(L^{(i)} \ast F) = 0$. The claim follows.

Now since $L \ast F = 0$, the distinguished triangle

$$L^{(k-1)} \ast F \to L \ast F \to L_\chi \ast F \to L^{(k-1)} \ast F[1]$$

implies

$$L_\chi \ast F \simeq L^{(k-1)} \ast F[1].$$

Therefore we have $H^{m-1}(L_\chi \ast F) \simeq H^{m-1}(L^{(k-1)} \ast F[1]) = H^m(L^{(k-1)} \ast F) = 0$ which contradicts the fact that $m$ is the smallest number such that $H^{\geq m}(L_\chi \ast F) = 0$. We are done. 

Lemma 7.5. Let $F \in D(X)$. If $L \ast F = 0$ for all $L \in \Theta(T)(F)$, then $F = 0$.

Proof. Since $T$ acts freely on $X$ we have an embedding $a_x : T \to X, t \to t \cdot x$. Moreover, by base change formulas, we have

$$H^1(T, L \chi^{-1} \otimes i_x^1 F) \simeq i_x^1(L_\chi \ast F) = 0$$

for all $\chi \in \Theta(T)(F)$. Here $i_x : x \to X$ is the natural inclusion map. By a result of Laumon [GL Proposition 3.4.5], it implies $i_x^1 F = 0$ for all $x$. The lemma follows.

\[
\square
\]

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