Constraint-preserving boundary conditions in the Z4 numerical relativity formalism

C Bona\textsuperscript{1}, T Ledvinka\textsuperscript{2}, C Palenzuela-Luque\textsuperscript{3} and M Žáček\textsuperscript{2}

\textsuperscript{1} Departament de Física, Universitat de les Illes Balears, Palma de Mallorca, Spain
\textsuperscript{2} Institute of Theoretical Physics, Charles University, Prague, Czech Republic
\textsuperscript{3} Department of Physics and Astronomy, Louisiana State University, LA, USA

Received 22 December 2004, in final form 31 March 2005
Published 13 June 2005
Online at stacks.iop.org/CQG/22/2615

Abstract

The constraint-preserving approach is discussed in parallel with other recent developments with the goal of providing consistent boundary conditions for numerical relativity simulations. The case of the first-order version of the Z4 system is considered, and constraint-preserving boundary conditions of the Sommerfeld type are provided. The stability of the proposed boundary conditions is related to the choice of the ordering parameter. This relationship is explored numerically and some values of the ordering parameter are shown to provide stable boundary conditions in the absence of corners and edges. Maximally dissipative boundary conditions are also implemented. In this case, a wider range of values of the ordering parameter is allowed, which is shown numerically to provide stable boundary conditions even in the presence of corners and edges.

PACS numbers: 04.25.Dm, 04.20.Ex

1. Introduction

The relevance of the initial boundary value problem (IBVP) for numerical relativity has been pointed out many times since the ground-breaking work of Stewart \[1\].

The origin of the problem is the well-known fact that Einstein’s equations

\[ G^{\mu\nu} = 8\pi T^{\mu\nu}, \]  

when interpreted as second-order field equations for the metric components $g_{\mu\nu}$, provide only six evolution equations for the space components $g_{ij}$, whereas the remaining four Einstein’s equations

\[ G^{00} = 8\pi T^{00} \]  

are just second-order constraints on $g_{ij}$ \[2\]. The evolution equations are then a reduction of the full Einstein’s system. Note that this reduction is not uniquely defined, as far as one can
add in multiples of the constraints to the original evolution system: this freedom is at the root of the diversity of the proposed evolution formalisms (for a review, see [3]).

The current approach in numerical relativity is to use one of these reductions of Einstein’s field equations, plus four coordinate conditions, as the main evolution system in order to compute the full set of metric components. This is the unconstrained, or free evolution [4], approach in which the constraint equations (2) are mainly used for monitoring the accuracy of the simulations. As a consequence, the evolution system has an extended space of solutions which contains, in addition to the true ones, constraint-violating solutions that do not verify the full set of Einstein’s equations.

This poses the question of which are the requirements in order to get, in the free evolution approach, true Einstein’s solutions in a consistent and stable way, avoiding any drifting towards extended, constraint-violating solutions [1]. The key point is to analyse the subsidiary system

$$\nabla_\nu (G^\mu_\nu - \frac{8\pi}{c^2} T^\mu_\nu) = 0,$$

which follows from the contracted Bianchi identities and the conservation of the stress–energy tensor. As is well known, the subsidiary system ensures that the constraints (2) are first integrals of the main evolution system. But it can also be interpreted as providing evolution equations for the constraint deviations. Of course, the same is true for any variant obtained by combining (3) with space derivatives of either evolution or constraint equations: this freedom can be easily used in order to get a variant of the subsidiary system (3) with a strongly hyperbolic [5], even symmetric hyperbolic, principal part:

$$\partial_0 (G^{00} - \frac{8\pi}{c^2} T^{00}) + \partial_k (G^{0k} - \frac{8\pi}{c^2} T^{0k}) = \cdots$$

$$\partial_0 (G^{k0} - \frac{8\pi}{c^2} T^{k0}) + \partial_k (G^{00} - \frac{8\pi}{c^2} T^{00}) = \cdots$$

(normal coordinates).

In the case of the pure initial value problem (IVP), the question about the consistency and stability for free-evolution true Einstein’s solutions has been given a precise answer in [1].

- **Consistency.** The initial data must verify the constraint equations.
- **Stability.** The principal part of both the main evolution system and the subsidiary system must be strongly hyperbolic.

Our former considerations suggest that the strong hyperbolicity requirement on the subsidiary system is always fulfilled in the cases in which the only constraints are the energy–momentum ones (2).

In the case of the IBVP, which is the main subject of this paper, a number of developments are currently under way. Let us briefly summarize the main ones.

### 1.1. Constraint-preserving boundary conditions

The original work in [1] was focused on the Frittelli–Reula evolution system [6, 7]. More recently [8–11], the constraint-preserving boundaries approach has been extended to other symmetric-hyperbolic systems of the Kidder–Scheel–Teukolsky (KST) type [12, 13]. The full programme consists of the following steps.

- **Writing down the subsidiary system** (it can be (3) or any variant of it) as a first-order evolution system for constraint deviations, with symmetric-hyperbolic principal part.
- **Providing algebraic (Dirichlet) boundary conditions** for the incoming modes of the subsidiary system in such a way that the total amount of constraint deviation (as measured with a suitable energy estimate) remains bounded.
Interpreting these algebraic boundary conditions of the subsidiary system as differential (Neumann) boundary conditions for the constraint-related incoming modes of the main evolution system.

Completing the resulting subset of boundary conditions by adding suitable conditions for the remaining incoming modes.

Checking the stability of the final set of the main system boundary conditions. In [1], the Majda–Osher theory [14] is applied and the uniform Kreiss condition [5] is obtained as a result.

1.2. Einstein’s boundary conditions

An alternative approach has been proposed by Frittelli and Gómez [15–18]. For a better understanding, let us start by writing down the constraint equations (2) in a covariant form

\[ n_\nu(G^{\mu\nu} - 8\pi T^{\mu\nu}) = 0, \]  

where \( n_\nu \) is the normal to the constant-time hypersurfaces. In local adapted coordinates, we have

\[ n_\nu = a\delta^0_\nu, \]  

so that the original form (2) is recovered. The absence of second time derivatives in (2) can now be rephrased as the absence of second derivatives normal to the spacelike constant-time hypersurfaces.

Now let us invoke the general covariance of Einstein’s theory. We will realize that the same kind of result must hold for other kinds of hypersurfaces, not just the spacelike ones. We can then start again from the covariant form (6) but with \( n_\nu \) being now the normal to any timelike hypersurface, like the ones corresponding to the boundaries of our computational domain. In local adapted coordinates, we could take, for instance,

\[ n_\nu \sim \delta_z^\nu, \]  

so that no second derivatives normal to the constant-\( z \) hypersurfaces would appear in (6).

If the main evolution system is written in the first-order form, the absence of second-order normal derivatives ensures that four combinations of the first-order dynamical fields can be consistently computed at the boundaries without any recourse to outside information. The Frittelli–Gómez idea is to use these combinations in order to get consistent boundary conditions for a subset of four incoming modes (Einstein’s boundaries). Of course, suitable conditions for the remaining incoming modes must also be provided and the stability of the full set must be checked.

1.3. Harmonic coordinates

Still another approach is due to Szilágyi, Winicour and co-workers [19–22]. Their formulation looks quite different from the preceding ones, so that we will need to rephrase some statements in order to point out the underlying similarities.

For instance, instead of the reduction of Einstein’s equations, we will consider the equivalent extension of the solution space. In the harmonic coordinates approach, this extension is achieved by writing down the principal part of Einstein’s equations as a set of generalized wave equations with some extra terms (de Donder–Fock decomposition [23, 24]) and then getting rid of these extra terms by requiring the four spacetime coordinates to be harmonic functions, that is

\[ \Box x^\mu = 0 \]  

\((x^\mu \) is considered as a set of four scalar functions here).
Let us compare now the harmonic coordinates approach with the preceding ones.

- The resulting (relaxed) system is used as the main evolution system for the full set of metric components. By construction, its principal part amounts to a set of wave equations, so that symmetric hyperbolicity is ensured.

- True Einstein’s solutions are recovered only when imposing the coordinate conditions (9), which play here the role of the constraints. The extra principal terms that were suppressed from the original Einstein’s system contained first derivatives of these coordinate constraints (second derivatives of the metric components) [21].

- The subsidiary system can again be obtained from (3). Terms containing the main evolution system equations, or their derivatives, vanish separately so that only the contribution of the extra principal terms remains. Note that the resulting subsidiary system is of second order in the coordinate constraints (9), in contrast with the former approaches.

- The principal part of the (second-order) subsidiary system is symmetric hyperbolic: it amounts again to a set of wave equations on the coordinate constraints (9).

In [21], a boundary condition derived from the local reflection symmetry requirement is analysed. Constraint preservation is explicitly shown and a theorem of Secchi [25] is used in order to show that the resulting IBVP is well posed. This theoretical result is checked by means of the numerical robust-stability test [26], which is adapted so that reflection boundary conditions are applied along just one space axis, while keeping periodic boundary conditions along the other two.

Due to the limited use of reflection symmetry boundary conditions in practical applications, a proposal is made [22] for extending these results to boundary conditions of the Sommerfeld type, as was done previously in a different framework [27].

1.4. The Z4 case

The Z4 approach [28, 29] uses an extra dynamical 4-vector along the track of previous formulations which contained extra dynamical quantities [30–33]. It has strong similarities with the harmonic coordinates approach, while providing much more flexibility regarding the gauge choices.

- The main evolution system is obtained by modifying Einstein’s equations with the help of the extra 4-vector $Z^\mu$, namely
  \[
  G_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu - \left( \nabla_\rho Z^\rho \right) g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{10}
  \]
  This system provides ten evolution equations for the set formed by the six space components of the metric plus the four $Z^\mu$ components. A first-order version can be easily obtained which, when supplemented with suitable gauge conditions, has been shown to have a strongly hyperbolic principal part [29].

- True Einstein’s solutions can be recovered by requiring the vanishing of the extra 4-vector
  \[
  Z^\mu = 0, \tag{11}
  \]
  so that this condition can be considered as a set of four algebraic constraints. Note that the main evolution system (10) is of a mixed type: it contains second-order derivatives of the metric, but only first-order derivatives of the extra 4-vector.

- The subsidiary system can be obtained from the covariant divergence of the main system (10): it will be of third order in the metric and of second order on the constraint variables
Z\mu. Allowing for (3), the Einstein’s tensor contribution vanishes separately, so that only the contribution of the extra terms remains, namely
\begin{equation}
\nabla_\nu [\nabla^\mu Z^\nu + \nabla^\nu Z^\mu - (\nabla_\rho Z^\rho) g^{\mu\nu}] = 0,
\end{equation}
which can also be expressed in the equivalent form [29]
\begin{equation}
\Box Z_\mu + R_{\mu\nu} Z^\nu = 0.
\end{equation}

It follows from (13) that the subsidiary system, when considered as a second-order system for the algebraic constraint deviations Z_\mu, has a symmetric-hyperbolic principal part: it amounts here again to an uncoupled set of wave equations.

The fact that the subsidiary system is of second order means that the vanishing of both Z_\mu and its first time derivative must be imposed on the initial data if one wants to ensure a priori that the resulting solution will be a true Einstein’s one. This amounts to imposing the usual energy and momentum constraints (2) on the initial data hypersurface, so that it can seem that the Z4 formalism is not being much help. But on the other hand, if one is checking a given solution a posteriori, the vanishing of Z_\mu in a given spacetime domain ensures that the same is true for its derivatives, so that this solution is necessarily a true Einstein’s one. This is why one can monitor constraint violations by looking just at the values of Z_\mu and, more important, this is why one can devise constraint-preserving strategies by aiming at the vanishing of Z_\mu.

Here, the Z4 formalism shows its main advantages.

In what follows, we will consider the fully first-order version of the Z4 system [29], as summarized in section 2. In section 3, we will apply the constraint-preserving boundary conditions programme, obtaining as a result conditions of the Sommerfeld type for the main evolution system. The stability of these conditions is studied in section 4, including the use of the robust-stability numerical test. As a result, our Sommerfeld-like conditions will be shown to behave in the same way as the reflection symmetry ones proposed in [21, 22].

### 2. First-order Z4 system

The general-covariant equations (10) can be written in the equivalent 3+1 form [28] as
\begin{align}
(\partial_t - L_\beta) \gamma_{ij} &= -2\alpha K_{ij}, \\
(\partial_t - L_\beta) K_{ij} &= -\nabla_i \omega_j + \alpha [^{(3)} R_{ij} + \nabla_i Z_j + \nabla_j Z_i - 2K_{ij}^2 + (\text{tr} K - 2\Theta) K_{ij} \\
&- S_{ij} + \frac{1}{2}(\text{tr} S - \tau) \gamma_{ij}], \\
(\partial_t - L_\beta) \Theta &= \frac{\alpha}{2} [^{(3)} R + 2\nabla_k Z_k^2 + (\text{tr} K - 2\Theta) \text{tr} K - \text{tr}(K^2) - 2\tau] - Z^k \omega_k, \\
(\partial_t - L_\beta) Z_i &= \alpha \left[ \nabla_j \left( K_i^j - \delta_i^j \text{tr} K \right) + \partial_i \Theta - 2K_i^j Z_j - S_i \right] - \Theta \omega_i,
\end{align}
where we have noted τ ≡ 8\pi α \gamma^0, \quad S_i ≡ 8\pi \alpha T_i^0, \quad S_{ij} ≡ 8\pi T_{ij}, \quad Θ ≡ \alpha Z^0, \quad \alpha_i ≡ \partial_i \alpha.

In the form (14)–(17), it is evident that the Z4 evolution system is fully relaxed: it consists only of evolution equations. The original constraints (11), which can be translated into
\begin{equation}
Θ = 0, \quad Z_\mu = 0,
\end{equation}
are algebraic so that the full set of field equations (10) is actually used during evolution, as in the harmonic coordinates case.
But now we do not have to impose the harmonic coordinate conditions (9). We will consider instead a wider class of gauge conditions in which the time slicing will be of the form

\[(\partial_t - L_\beta) \ln \alpha = -f \alpha (\text{tr } K - m \Theta)\]  

(generalized harmonic slicing). Although more general cases can be considered [34], we will use here normal coordinates (zero shift) for simplicity.

A first-order version of the Z4 evolution system (14)–(17) can be obtained by introducing the first space derivatives

\[A_k \equiv \alpha_k / \alpha, \quad D_{kij} \equiv \frac{1}{2} \partial_k \gamma_{ij}\]  

as independent dynamical quantities, so that the full set of dynamical fields can be given by

\[u = \{\alpha, \gamma_{ij}, K_{ij}, A_k, D_{kij}, \Theta, Z_k\}\]  

(38 independent fields).

Of course, one must provide evolution equations for the new quantities (21): the simplest way is to take

\[\partial_t A_k + \partial_k \left[ f \alpha (\text{tr } K - m \Theta) \right] = 0, \quad (23)\]

\[\partial_t D_{kij} + \partial_k \left[ \alpha K_{ij} \right] = 0. \quad (24)\]

Note that one could add to (23) and (24) a number of terms involving first derivatives of either \(\Theta\) or \(Z_k\). This would amount to introducing coupling terms with either the energy or the momentum constraints, as in the KST system [12, 13], each one with its own free parameter.

We have chosen instead to keep the simplest form (23) and (24) because the first-order constraints (21) evolve in a trivial way, that is

\[\partial_t [A_k - \partial_k \ln \alpha] = 0, \quad (25)\]

\[\partial_t [D_{kij} - \partial_k \gamma_{ij}] = 0, \quad (26)\]

so that the relationship between the first- and second-order versions of the evolution system is more transparent. We are losing in this way the possibility of playing with a number of extra free parameters.

Care must be taken, however, when expressing the Ricci tensor \((3)R_{ij}\) in (15) in terms of the derivatives of \(D_{kij}\), because as far as definitions (21) are no longer enforced, the identities

\[C_{kl} \equiv \partial_k A_l = 0, \quad C_{klij} \equiv \partial_k D_{lj} = 0\]  

cannot be taken for granted in first-order systems. As a consequence of these ordering ambiguities, the principal part of the evolution equation (15) leads to a one-parameter family of non-equivalent first-order versions, namely

\[\partial_t K_{ij} + \partial_k \left[ \alpha \lambda^k_{ij} \right] = \cdots, \quad (28)\]

where

\[\lambda^k_{ij} = D^k_{ij} - \frac{1 + \xi}{2} \left( D^k_{ij} + D^k_{ji} - \delta^k_l E_j - \delta^k_j E_i \right) + \frac{1}{2} \delta^k_l (A_j - D_j + 2V_j) + \frac{1}{2} \delta^k_j (A_i - D_i + 2V_i), \quad (29)\]

and we have noted

\[D_j \equiv \gamma^{rs} D_{rsj}, \quad E_i \equiv \gamma^{rs} D_{rsi}, \quad V_k \equiv D_k - E_k - Z_k. \quad (30)\]
Note that the parameter choice $\zeta = +1$ corresponds to the standard Ricci decomposition
\begin{equation}
(3)R_{ij} = \partial_i \Gamma^k_{ij} - \partial_j \Gamma^k_{kj} + \Gamma^r_{rk} \Gamma^k_{ij} - \Gamma^k_{ri} \Gamma^r_{kj},
\end{equation}
whereas the opposite choice $\zeta = -1$ corresponds to the de Donder–Fock [23, 24] decomposition
\begin{equation}
(3)R_{ij} = -\partial_k D^k_{ij} + \partial_i \Gamma^k_{jk} - 2D^r_{rk} D_{kij} + 4D^r_{rs} D_{rlij} = \Gamma^r_{ri} \Gamma^s_{js} - \Gamma^r_{rij} \Gamma^s_{rk},
\end{equation}
which is most commonly used in numerical relativity formalisms. The ordering ambiguities do not affect the principal part of equation (16), namely
\begin{equation}
\Phi_t + \partial_k \left[ \alpha V^k \right] = \cdots.
\end{equation}

The resulting first-order system has been shown to be strongly hyperbolic [29] provided that the first gauge parameter $f$ is greater than zero. In the harmonic slicing case ($f = 1$), the second gauge parameter is fixed ($m = 2$). The full list of eigenvectors is given in appendix A.

3. Constraint-preserving boundary conditions

We have seen in the introduction that the simple equation (13) provides the subsidiary system for the deviations of the algebraic constraints (11). This would be the whole story if we were planning to use the second-order version (10) of the evolution system. But we prefer to focus here on the first-order in-space version, as described in the previous section. The reason is that the mathematical theory of first-order systems seems to be more developed, both at the continuum and at the discrete level, so more powerful tools are available: energy methods, total-variation-diminishing algorithms and so on (see, for instance, [5, 35]).

There is a price to pay for this. We have found in the previous section new constraints, such as (27), arising from ordering ambiguities in the space derivatives. The ordering parameter $\zeta$ appeared precisely from the coupling of these ordering constraints with the evolution system. The original subsidiary system (13) must then be extended in order to include both these coupling terms and the evolution of the ordering constraints themselves.

3.1. First-order subsidiary system

The easiest way of obtaining the full subsidiary system in the first-order case is just by computing the time derivative of the full Z4 first-order system. We give here (the principal part of) the resulting subsidiary system
\begin{equation}
\partial_t C_{li} = 0,
\end{equation}
\begin{equation}
\partial_t C_{lij} = 0,
\end{equation}
\begin{equation}
\frac{1}{\alpha^2} \partial_t^2 \Theta - \Delta \Theta = \cdots,
\end{equation}
\begin{equation}
\frac{1}{\alpha^2} \partial_t^2 Z_i = \Delta Z_i = \gamma^{ij} \partial_k [C_{ij} + \gamma^{rs} (C_{irs} + (\zeta - 1) C_{rjis} + (\zeta + 1) C_{rij})] + \cdots
\end{equation}
(the dots stand for non-principal terms).

The subsidiary system (34)–(37) can be put in the first-order form in the usual way, by considering the first derivatives of $(\Theta, Z_i)$ as new independent variables. The following evolution conditions
\begin{equation}
\partial_t (\partial_i \Theta) - \partial_i [\partial_t \Theta] = 0,
\end{equation}
\begin{equation}
\begin{split}
\partial_t(\partial_k \partial_t Z_i) - \partial_k [\partial_t Z_i] = 0
\end{split}
\tag{39}
\end{equation}

could be added then to complete (the first-order version of) the subsidiary system.

Note that the evolution equations (34) and (35) for the ordering constraints are trivial. This means that the ordering constraints themselves are eigenfields of the full subsidiary system (34)–(39) with zero characteristic speed. Moreover, the evolution equations (36) and (38) for (the derivatives of) \( \Theta \) form a separate subsystem with the structure of the wave equation. Concerning the remaining equations (37) and (39), one can express them in terms of the quantities

\begin{equation}
Z_{ki} \equiv \partial_k Z_i + C_{ik} + \gamma^{rs}[C_{ikrs} + (\zeta - 1)C_{rksi} + (\zeta + 1)C_{risk}],
\tag{40}
\end{equation}

so that they read

\begin{equation}
\begin{split}
\frac{1}{\alpha^2} \partial_t(\partial_t Z_i) - \partial_k [\partial_t Z_i] = \cdots,
\end{split}
\tag{41}
\end{equation}

\begin{equation}
\begin{split}
\partial_t Z_{ki} - \partial_k [\partial_t Z_i] = \cdots,
\end{split}
\tag{42}
\end{equation}

and we get again the structure of the wave equation.

It follows that the principal part of the subsidiary system (34)–(39) can be put in a symmetric-hyperbolic form. The characteristic speeds are either zero or light speed. A simple energy estimate is provided by

\begin{equation}
E \equiv \frac{1}{\alpha^2} [\theta(\partial_t Z_i)^2 + \gamma^{ij}(\partial_t Z_i)(\partial_t Z_j)] + (\partial_t \Theta)(\partial_t \Theta) + Z_{ij} Z_{ij}.
\tag{43}
\end{equation}

We are now in a position to take the second step in the constraint-preserving boundary conditions programme. We will impose the vanishing of all the incoming modes of (34)–(39) at the boundaries, that is

\begin{equation}
\begin{split}
\frac{1}{\alpha} \partial_t \Theta + n^k \partial_k \Theta = 0, \\
\frac{1}{\alpha} \partial_t Z_i + n^k Z_{ki} = 0,
\end{split}
\tag{44}
\tag{45}
\end{equation}

where \( \vec{n} \) stands for the outwards-pointing unit normal to the boundary surface.

Equations (44) and (45) meet the two requirements we were looking for.

- They provide maximally dissipative algebraic boundary conditions for the subsidiary system (34)–(39). In this way, no constraint-violating modes are allowed to enter across the selected boundary.
- They will provide, as we will see in what follows, four boundary conditions of the Sommerfeld type for the evolution system (14)–(17), which can be consistently imposed in order to obtain true solutions of Einstein’s field equations. Note that the extra terms in definition (40) consist in ordering constraints, which would not appear in a second order in space formulation.

### 3.2. Boundary conditions implementation

The third step in the programme is to use the resulting values \( (\Theta)^{\text{bound}}, Z_i^{\text{bound}} \), as computed from (44) and (45), in order to obtain four of the main system’s incoming fields at the boundary. This process is not free from ambiguities, like the choice of a suitable basis for the dynamical fields.

For a symmetric-hyperbolic evolution system, one could find a (positive definite) quadratic form which would provide a metric for the space of dynamical fields. The natural choice would
be then to build an orthogonal basis of dynamical fields containing both $\Theta$ and $Z_i$ (or some equivalent combinations). Imposing boundary conditions would then consist in prescribing the values (44) and (45) for these fields, while leaving the remaining ones unchanged.

But the evolution system (14)–(17) is not symmetric hyperbolic. This means that we do not have a unique prescription for imposing the boundary conditions, as far as we have many ways of selecting an appropriate set of dynamical fields at the boundary. A convenient starting point in this case is to replace the original basis

$$(\Theta, K_{ij}, Z_i, D_{kij}, A_i)$$ (46)

by one which is more adapted to the characteristic decomposition at the boundary, namely

$$(\Theta, \tilde{K}_{ij}, Z_i, D_{ij}, \tilde{D}_{n\perp}, V_i, D_i, A_i),$$ (47)

where the symbol $\perp$ replacing an index means the projection orthogonal to $\tilde{n}$. We have denoted $\tilde{D}_{n\perp}$ as the traceless part of $D_{n\perp}$ and

$$\tilde{K}_{ij} \equiv K_{ij} - \frac{\Theta}{2} \gamma_{ij}.$$ (48)

Note that the quantities $D_{n\perp}$ and $\text{tr}(D_{n\perp})$ do not appear explicitly in the new basis. These components must be computed instead from $(Z + V)_i$ and the $D_{ij}$ components. Allowing for definition (30), we actually get

$$D_{nn\perp} = D_{\perp} - h^{rs} D_{rs\perp} - (Z + V)_\perp,$$ (49)

$$h^{rs} D_{nrs} = h^{rs} D_{rsn} + (Z + V)_n,$$ (50)

where $h^{rs}$ stands for the (inverse) metric on the boundary surface, namely

$$h^{rs} \equiv \gamma^{rs} - n^r n^s.$$ (51)

The new basis (47) has been chosen in such a way that, as we can easily verify, the values of $(\Theta, Z_i)$ appear in only eight eigenfields (four characteristic cones), namely

$$E^\pm = \Theta \pm V^n,$$ (52)

$$L_{n\perp}^\pm = \tilde{K}_{n\perp} \pm \left[ \frac{1}{2} (A_n + D_{\perp} - 2Z_{\perp}) - \frac{\xi + 1}{2} D_{n\perp} + \frac{\xi - 1}{2} h^{rs} D_{rs\perp} \right],$$ (53)

$$L^\pm = h^{rs} \tilde{K}_{rs} \pm [Z_n - \xi h^{rs} D_{rsn}]$$

where we have noted

$$L_{n\perp}^\pm \equiv h^{rs} L_{rsn}^\pm - E^\pm.$$ (54)

In order to set up the required four boundary conditions, we will simply replace the original values for $(\Theta, Z_i)$ by $(\Theta^{\text{bound}}, Z_i^{\text{bound}})$, while leaving the other fields in the basis (47) unchanged. To be more specific,

- the original values for $(\Theta, Z_i)$ are replaced by $(\Theta^{\text{bound}}, Z_i^{\text{bound}})$, as computed from (44) and (45), respectively. This amounts, modulo some linear combinations with tangent fields (transverse derivatives), to prescribing the first term in $E^\pm$ and the second terms in $(L_{n\perp}^\pm, L^\pm)$;
- the values of their ‘counterpart’ fields $(V_i, \tilde{K}_{n\perp}, h^{rs} \tilde{K}_{rs})$ are not changed by the boundary conditions.
It is clear then that the values of the four characteristic cones (52)–(54) have been prescribed in such a way that the four equations (44) and (45) hold true at the selected boundary.

A further source of ambiguity comes from the prescription of the remaining incoming eigenfields (the gauge and the transverse traceless ones). We will use here a convenient generalization of the maximally dissipative boundary conditions, namely

\[
\partial_t G^- = 0, \quad \partial_t \left[ L_{\perp \perp}^- - \frac{1}{2} (h^{rs} L_{rs}^-) \gamma_{\perp \perp} \right] = 0,
\]

although we are aware that more sophisticated choices could be required in physical applications.

### 4. Constraint stability

The final step in the proposed programme is to check the stability of the constraint-preserving boundary conditions (44), (45) and (55).

Note, however, that the main evolution system is just strongly hyperbolic, but not symmetric hyperbolic (at least not in the generic case [28]). This means that the Majda–Osher theory [14] cannot be directly applied, and the same is true for the Secchi theorems [25]. This is why we will check the stability of (44), (45) and (55) by other methods, both at the theoretical and the numerical level.

From the theoretical point of view, the well-known Fourier–Laplace method [5] could provide necessary conditions for stability [10]. We will prefer here a simpler approach, by analysing the system of equations verified by the dynamical fields at the boundary. We will call it the modified system in order to distinguish it from the original evolution system, which is being used at the interior points. We will see that this approach provides some insight about the behaviour at the boundary points. The drawback is that boundary points form just the outermost layer of the computational domain. It follows that the modified system analysis has to be considered at this stage just as an heuristic approach, so that the stability of the boundary conditions must be confirmed by other means. More details are provided in [36].

#### 4.1. The modified system approach

For the sake of clarity, let us focus first on the subset of dynamical fields spanned by \((\Theta, V_i)\). As stated in the previous section, the boundary conditions are not affecting any of the \(V_i\) components. This means that the boundary values of \(V_i\) verify the main evolution system equations, namely

\[
\frac{1}{\alpha} \partial_t V_i + \partial_i \Theta = \cdots.
\]

The original equation (16) for \(\Theta\), however, no longer holds at the boundary, where one is imposing instead the advection equation (44). This means that, even at the continuum level, the evolution system is being modified at the boundaries. The modified system for the subset of dynamical fields \((\Theta, Z_i)\) is given by (44) and (56).

The modified subsystem (44) and (56) has real non-negative characteristic speeds along any direction \(\vec{r}\), oblique to \(\hat{n}\). They are actually

\[
[0, \alpha(\vec{n} \cdot \vec{r})].
\]

It follows that (the principal part of) the modified subsystem (44) and (56) can be interpreted on physical grounds as describing the outwards propagation of both \(\Theta\) and \(V_i\) at the boundary.
We can push our analysis one step further by considering the particular case in which \( \vec{r} \) is tangent to the boundary, that is orthogonal to \( \vec{n} \). In this case, the speeds (57) are fully degenerate, and a non-diagonal coupling term remains in (56), so that the modified subsystem is just weakly hyperbolic. This has some relevant consequences. Let us assume, for instance, that \( \vec{n} \) is aligned with the x-coordinate axis and that we get a static profile for \( \Theta_1 \) of the form

\[
\Theta = g(y, z),
\]

which trivially satisfies equation (44). The derivative coupling in (56) then allows modes in the \( V_x \) and \( V_z \) components, which grow in time in a linear way. These linearly growing modes will actually show up in numerical tests, as we will see below.

The analysis of the full modified system can be simplified by writing down (the principal part of) the modified evolution equations for the combinations corresponding to incoming modes of the original system. Allowing for (44) and (45), we have

\[
\frac{1}{\alpha} \partial_t E^- = 0,
\]

\[
\frac{1}{\alpha} \partial_t L^- _{n\perp} = h^{rs} \partial_t \left[ D_{n\perp} - D_{n\perp} + \frac{\zeta - 1}{2} \tilde{K}_{s\perp} \right] - \partial_\perp \left[ \frac{\zeta + 1}{2} \tilde{K}_{nn} - \frac{f + 1}{2} \text{tr} \tilde{K} + A_n + V_n - \frac{(3 - 2m)f + 1}{4} \right],
\]

\[
\frac{1}{\alpha} \partial_t L^- = -h^{rs} \partial_t \left[ D_{ns} - D_{ns} + \zeta \tilde{K}_{ns} + A_s + V_s \right],
\]

plus the trivial evolution equations (55) for the gauge and the transverse traceless incoming modes.

Note that only derivatives tangent to the boundary appear on the modified system equations (59)–(61) for the incoming modes. This means that all the characteristic speeds along the longitudinal direction \( \vec{n} \) are real and non-negative: they are actually

\[
0, \alpha, \alpha \sqrt{f}.
\]

The corresponding eigenvectors are either standing fields \( (v = 0) \)

\[
A_{\perp}, D_{\perp ij}, A_k = f D_k + f m V_k, E^-, L^-_{ij}, G^-
\]

or outgoing fields \( (v = \alpha, \alpha \sqrt{f}) \)

\[
\Theta, Z_i, \tilde{L}^-_{\perp}, G^+.
\]

These fields span the whole dynamical space; the modified system is then strongly hyperbolic along the direction \( \vec{n} \) normal to the boundary.

Computing the characteristic speeds along a generic direction \( \vec{r} \), oblique to \( \vec{n} \), and for an arbitrary value of the ordering parameter, is a much harder task, even using an algebraic computing program. We have just checked the particular cases

\[
\zeta = 0, \pm 1
\]

and we have found that the modified system is at least weakly hyperbolic (real characteristic speeds) only in the \( \zeta = 0 \) case. This suggests that the \( \zeta = 0 \) case, corresponding to a symmetric ordering of the space derivatives, could be free of boundary instabilities, as we will confirm below.
4.2. The robust-stability test

The robust-stability numerical test [26] amounts to considering small perturbations of Minkowski spacetime which are generated by taking random initial data for every dynamical field in the system. The level of the random noise must be small enough to make sure that we will remain in the linear regime even for hundreds of crossing times (the time that a light ray will take to cross the longest way along the numerical domain). We are taking advantage in this way of the peculiar nature of the Einstein’s equations, where the principal part is quasilinear and the non-principal (source) terms are quadratic in the dynamical fields. Checking the linear regime of Einstein’s equations amounts then to testing the behaviour of their principal part.

This test has been previously used [29] to check numerically the stability properties of the Z4 evolution system for interior points. In order to avoid boundary effects, the grid had the topology of a 3-torus, with periodic boundaries along every axis. We will now open the $x$ faces and impose the constraint-preserving boundary conditions there, while keeping periodic boundary conditions along the other two axes.

We show in figure 1 the $L_{\infty}$ norm of $\text{tr} K$ for different values of the ordering parameter $\zeta$. A spacing $\Delta \zeta = 0.5$ is used in the plot for the sake of clarity, although the numerical survey has been made with a finer spacing of $\Delta \zeta = 0.1$. Our results show that the constraint-preserving boundary conditions (44) and (45) are stable if and only if $\zeta$ is in the range $[-0.5, 0]$. The behaviour is the same in all the stable regions: the different values of $\zeta$ just determine when the instabilities (if any) will appear.

Note that the robust-stability analysis predicted the arising of linearly growing modes related to the transverse $V_i$ components. In terms of the original basis, one can expect to see these modes in the quantities $D_{xxx}$ and $D_{xzz}$, which are derived from $V_x$ and $V_z$ by the relationships (49) and (50), respectively. We can see, for instance, in figure 2 a growing linear mode in the $L_{\infty}$ norm of $D_{xxx}$. This confirms that the modified system analysis can be useful to anticipate the behaviour of the boundary conditions under the robust-stability test. Further
evidence in this direction is provided in appendix B, where maximally dissipative boundary conditions are considered.

The robust-stability test is also useful for checking the constraint-preserving character of the proposed boundary conditions (44) and (45). As far as true Einstein’s solutions are actually recovered by setting both $\Theta$ and $Z_i$ to zero, the values of these quantities can be considered to be good indicators of constraint violations. We can monitor the norm of these quantities to check whether constraint violations are being injected into the computational domain through the open boundaries.

We can see in figure 3 that this is actually not the case. The values of $\Theta$ and $Z_i$ are not growing at all, contrary to what happens to the $D_{xxy}$ components, as seen in figure 2. Moreover, their norm is diminishing: we can understand this decreasing by noticing that
the boundary conditions (44) and (45) are, modulo some coupling with ordering constraints, advection equations. This means that the values of (44) and (45) are just flowing out of the computational domain through the open boundary.

5. Discussion and outlook

One can wonder about whether the multi-dimensional character of the problem is lost when applying boundary conditions to just one face. This is not the case: the \( x = \) constant boundary surfaces are actually surfaces, not just points, so that oblique modes are still present and can lead to instabilities, as actually occurs when \( \zeta = \pm 1 \). The only thing we are avoiding in this way is applying constraint-preserving boundary conditions to corner points (which are assigned here to the \( y \) and \( z \) boundary surfaces, where periodic boundary conditions are applied). If we try to apply conditions (44) and (45) along every space direction, then instabilities appear even for the \( \zeta = 0 \) choice of the ordering parameter.

One could argue as well that the difficulties with corners and edges could be related to the fact that the main evolution system has just a strongly hyperbolic, but not symmetric hyperbolic, principal part. This is not the case, as is shown in appendix B, where boundary conditions of the maximally dissipative type are successfully implemented and applied to all the boundary points of a Cartesian-like numerical grid, including corners and edges.

Our results are very similar to those of [21], where the same test is applied to the reflection boundary condition: a stable linear-growing mode is detected, which becomes unstable only when the boundary conditions are applied also to the other faces, so that the numerical grid gets corners and edges. Our work can then be understood as an extension (in a different formalism) of the results of [21] to boundary conditions of the Sommerfeld type.

In our opinion, the main problem at corner points comes from the inconsistency inherent to the choice of a (unique) normal direction there. Different faces get different normal vectors, but corner points belong to two different faces at the same time. This is not just a theoretical caveat: corners and edges pose a real problem in practical applications, where more work should be done along any of the following lines.

- Devising a specific treatment for corner points. The correct implementation of constraint-preserving boundary conditions in the presence of corners is still a unsolved issue. For symmetric-hyperbolic evolution systems, using finite-difference operators satisfying the summation by parts rule with respect to a diagonal scalar product leads to stable schemes with maximally dissipative boundary conditions [37, 38]. In our case, with a strongly hyperbolic evolution system, we have got the same result, as presented in appendix B. But these results do not extend to constraint-preserving boundary conditions. A major difficulty is that compatibility conditions between boundary data at adjacent faces need to be satisfied if one wishes to obtain smooth solutions. Necessary conditions for continuous solutions have been derived in [9] for a symmetric-hyperbolic evolution system. But more conditions are needed in order to obtain smooth solutions. Compatibility issues are also present at corner points between initial and boundary data (see, for instance, chapter 9 in [35]).

- Building numerical grids with smooth boundaries (without corners and edges), so that the constraint-preserving boundary conditions (44) and (45) can be applied consistently in an stable way, as we have confirmed numerically by means of the robust-stability test. The construction of ‘multi-patch’ numerical schemes, which would allow for smooth boundaries, has become a major research topic in numerical relativity (see, for instance, [39, 40]).
Acknowledgments

This work has been supported by the EU Programme ‘Improving the Human Research Potential and the Socio-Economic Knowledge Base’ (Research Training Network Contract HPRN-CT-2000-00137) by the Spanish Ministry of Science and Education through the research project number FPA2004-03666 and by a grant from the Department of Innovation and Energy of the Balearic Islands Government.

Appendix A. Characteristic decomposition of the Z4 first-order system

Let us consider the propagation of perturbations with wavefront surfaces given by the unit (normal) vector $n_i$, we can write the principal part of the Z4 first-order system in matrix form

$$\frac{1}{\alpha} \partial_t \mathbf{u} + M n_k \partial_k \mathbf{u} = \cdots,$$

where $\mathbf{u}$ is the full array of dynamical fields (22). Note that derivatives tangent to the wavefront surface play no role here.

A straightforward analysis of the characteristic matrix $M$ provides the following list of eigenfields [29].

- Standing eigenfields (zero eigenvalues)

$$\alpha, \gamma_{ij}, A_\perp, D_{\perp ij}, A_k - fD_k + fmV_k,$$

where the symbol $\perp$ replacing an index means the projection orthogonal to $n_i$

$$D_{\perp ij} \equiv D_{kij} - n_k n_r D_{rij}.$$

- Light-cone eigenfields (eigenvalues $\pm \alpha$)

$$L^\pm_{ij} \equiv [K_{ij} - n_i n_j \text{tr} K] \pm [\lambda_{n}^{ij} - n_i n_j \text{tr} \lambda_{n}],$$

where the symbol $n$ replacing the index means the contraction with $n_i$

$$\lambda_{n}^{ij} \equiv n_k \lambda_{ij}^k,$$

$$V^n \equiv n_k V^k.$$

- Gauge eigenfields (eigenvalues $\pm \alpha \sqrt{f}$)

$$G^\pm \equiv \sqrt{f} [\text{tr} K - \mu \Theta] \pm [A^n + (2 - \mu)V^n].$$

where we have denoted for short

$$\mu \equiv \frac{fm - 2}{f - 1}.$$

In the degenerate case ($f = 1$), one must have $m = 2$, so that the value of $\mu$ is not fixed. The degeneracy allows for any combination with (A.5), as expected.

Appendix B. Maximally dissipative boundary conditions

A convenient generalization of the maximally dissipative boundary conditions can be implemented by just imposing the vanishing of (the time derivatives of) all the incoming
modes, that is
\[ \partial_t E^- = \partial_t L_{ij}^- = \partial_t G^- = 0. \] (B.1)

The principal part of the modified system is much simpler than that in the constraint-preserving case. It is, by construction, strongly hyperbolic along the direction \( \vec{n} \) normal to the boundary, with characteristic speeds given again by (62).

We will compute the characteristic speeds along a generic direction \( \vec{r} \), oblique to \( \vec{n} \), where the vector \( \vec{r} \) is related to \( \vec{n} \) by
\[ \vec{r} = \vec{n} \cos \varphi + \vec{s} \sin \varphi, \] (B.2)
and we have taken
\[ \vec{n}^2 = \vec{s}^2 = 1, \quad \vec{n} \cdot \vec{s} = 0. \] (B.3)

The hyperbolicity requirement amounts to demanding that all the resulting characteristic speeds be real for any value of the angle \( \varphi \).

The trivial equations (B.1) provide seven (remember that \( L_{ij}^\pm \) is traceless) standing eigenfields (zero characteristic speed) of the modified system. Another set of 17 standing eigenfields is given by
\[ A_p, D_{pij}, A_k - f D_k + f m V_k, \] (B.4)
where \( \vec{p} \) is the direction orthogonal to both vectors \( \vec{n} \) and \( \vec{s} \).

The remaining 14 dynamical fields can be grouped into the following sectors.

- **Energy sector** \( \{E^+, V_s\} \). The corresponding evolution equations are (principal part only)
  \[ \frac{1}{\alpha} \partial_t V_s = - \sin \varphi \partial_s \Theta = - \frac{1}{2} \sin \varphi \partial_r E^+ \cdots, \]
  \[ \frac{1}{\alpha} \partial_t E^+ = - \partial_r [V_s + \Theta \cos \varphi] = - \partial_r [E^+ \cos \varphi + V_s \sin \varphi + \cdots], \]
where the dots stand for coupling terms with the standing eigenfields (which are irrelevant for the eigenvalues calculation). It follows that the characteristic speeds are given by the solutions of the algebraic equation
\[ \lambda (\lambda - \alpha \cos \varphi) = \frac{1}{2} \alpha^2 \sin^2 \varphi, \] (B.5)
so that real characteristic speeds are obtained for every value of \( \varphi \).

- **Gauge sector** \( \{G^+, A_s\} \). The corresponding evolution equations are (principal part only)
  \[ \frac{1}{\alpha} \partial_t A_s = - \sin \varphi \partial_s [f (\text{tr} K - m \Theta)] = - \frac{1}{2} \sqrt{f} \sin \varphi \partial_r G^+ \cdots, \]
  \[ \frac{1}{\alpha} \partial_t G^+ = - \partial_r [\sqrt{f} A_s + f \cos \varphi \text{tr} K] = - \sqrt{f} \partial_r [G^+ \cos \varphi + A_s \sin \varphi + \cdots], \]
where the dots stand for coupling terms with the previous sectors. It follows that the characteristic speeds are given by the solutions of the algebraic equation
\[ \lambda (\lambda - \alpha \sqrt{f} \cos \varphi) = \frac{1}{2} f \alpha^2 \sin^2 \varphi, \] (B.6)
so that, allowing for the positivity of the gauge parameter \( f \), real characteristic speeds are obtained again for every value of \( \varphi \).
Metric sector \( \{ L_{ij}^+, D_{ij} \} \). The corresponding evolution equations can be written as (principal part only)

\[
\begin{align*}
\frac{1}{\alpha} \partial_t D_{ij} &= - \sin \phi \partial_s K_{ij} = - \frac{1}{2} \sin \phi \partial_t [L_{ij}^+ + \cdots], \\
\frac{1}{\alpha} \partial_t L_{ij}^+ &= - \partial_r \left[ \lambda_{ij}^+ \cos \phi K_{ij} + \cdots - \frac{1 + \zeta}{2} (r_i K_{nj} + r_j K_{ni} - n_i K_{rj} - n_j K_{ri}) \right] \\
&= - \partial_r \left[ L_{ij}^+ \cos \phi + D_{ij} \sin \phi + \cdots - \frac{1 + \zeta}{2} \sin \phi (s_i K_{nj} + s_j K_{ni}) \\
&- n_i K_{sj} - n_j K_{si} \right] - \frac{1 + \zeta}{2} \sin \phi (D_{ij} + D_{ji} - s_i E_j - s_j E_i) 
\end{align*}
\]

(B.7)

where the dots stand again for coupling terms with the previous sectors.

The evolution equation (B.8) for these outgoing ‘metric’ fields contains (unless \( \zeta = -1 \)) crossed coupling terms that complicate the analysis. One gets three variants of the same algebraic equation

\[
\begin{align*}
\lambda (\lambda - \alpha \cos \phi) &= \frac{1}{2} \alpha^2 \sin^2 \phi, \\
\lambda (\lambda - \alpha \cos \phi) &= \frac{1}{2} \alpha^2 \sin^2 \phi \left[ 1 - \left( \frac{1 + \zeta}{2} \right)^2 \right], \\
\lambda (\lambda - \alpha \cos \phi) &= \frac{1}{2} \alpha^2 \sin^2 \phi (1 - (1 + \zeta)^2).
\end{align*}
\]

(B.8), (B.9), (B.10)

depending on the particular set of components considered (the last two equations appear twice, so that one gets ten characteristic speeds that complete the full set of 38). The most restrictive is the last one (B.10): it implies that we get complex characteristic speeds for some values of \( \phi \) unless \( \zeta \leq 0 \),

so that the standard ordering case (\( \zeta = +1 \)) is excluded.

We can check out these results by using again the robust-stability test bed. We will enforce the maximally dissipative conditions (B.1) along every axis in a Cartesian-like numerical grid, including corners and edges. We will survey the values of the \( \zeta \) parameter in the interval \([-1, 1]\), with a spacing \( \Delta \zeta = 0.1 \).

We show in figure 4 the time evolution of the maximum of the absolute value of \( \text{tr} K \) (a spacing \( \Delta \zeta = 0.5 \) is used in the plot for the sake of clarity). Our results show that the positive choices of the ordering parameter are actually unstable, whereas the choices in the range \([-1, 0]\) are stable and behave in the same way. Note that the norm of \( \text{tr} K \) in figure 4 is decreasing, in the stable cases, much faster than in the constraint-preserving case (figure 1). This cannot be explained just by the fact that boundary conditions are now being applied along the three coordinate axes: the boundary conditions are actually dissipating all the dynamical fields.

We show in figure 5 the time evolution of the maximum of the absolute value of both \( \Theta \) and \( Z_i \) for the symmetric ordering (\( \zeta = 0 \)) case. As far as their values are diminishing, one can conclude that no constraint-violating modes are being produced at the boundaries. But note that this is at the price of dissipating all the dynamical fields. One cannot conclude then that maximally dissipative boundary conditions are constraint-preserving: constraint-related
Figure 4. Same as figure 1, but now with maximally dissipative boundary conditions enforced along every axis. Again, the values of ζ are shown with an interval of 0.5 for the sake of clarity, although the survey has been made with a finer interval of 0.1. It confirms that the ζ < 0 choices are stable, as expected from the modified system analysis. Note that, in the stable cases, the decrease after 100 crossing times is more than one order of magnitude greater than in figure 1. This shows the effect of the maximally dissipative boundary conditions: they are actually dissipating all the dynamical fields.

Figure 5. Same as in the previous figure, but now the norms of Θ and Zx are plotted in order to monitor constraint violations. Their decay indicates that the constraint-violating modes are diminishing, even much faster than in figure 3. But now the reason is a different one: the boundary conditions are dissipating all the dynamical fields.

fields are flowing out through the boundaries in the same way as the remaining degrees of freedom.

References

[1] Stewart J M 1998 Class. Quantum Grav. 15 2865
[2] Arnowit R, Deser S and Misner C W 1962 Gravitation: An Introduction to Current Research ed L. Witten (New York: Wiley)
Constraint-preserving boundary conditions in the Z4 numerical relativity formalism

[3] Reula O 1998 Living Rev. Rel. 3 http://www.livingreviews.org/lrr-1998-3
[4] Centrella J 1980 Phys. Rev. D 21 2776
[5] Kreiss H O and Lorentz J 1989 Initial-Boundary Problems and the Navier–Stokes Equations (New York: Academic)
[6] Frittelli S and Reula O A 1994 Commun. Math. Phys. 166 221
[7] Frittelli S and Reula O A 1996 Phys. Rev. Lett. 76 4667
[8] Calabrese G, Lehner L and Tiglio M 2002 Phys. Rev. D 65 104031
[9] Calabrese G, Pullin J, Reula O A, Sarbach O and Tiglio M 2003 Commun. Math. Phys. 240 377
[10] Calabrese G and Sarbach O 2003 J. Math. Phys. 44 3888
[11] Calabrese G 2004 Class. Quantum Grav. 21 4025
[12] Kidder L E, Scheck M A and Teukolsky S A 2001 Phys. Rev. D 64 064017
[13] Sarbach O and Tiglio M 2002 Phys. Rev. D 66 064025
[14] Majda A and Osher S 1975 Commun. Pure Appl. Math. 28 607
[15] Frittelli S and Gómez R 2003 Class. Quantum Grav. 20 2379
[16] Frittelli S and Gómez R 2003 Phys. Rev. D 68 044014
[17] Frittelli S and Gómez R 2004 Phys. Rev. D 70 064008
[18] Frittelli S and Gómez R 2004 Phys. Rev. D 70 064008
[19] Szilágyi B, Gómez R, Bishop N T and Winicour J 2000 Phys. Rev. D 62 104006
[20] Szilágyi B, Schmidt B and Winicour J 2002 Phys. Rev. D 65 064015
[21] Szilágyi B and Winicour J 2003 Phys. Rev. D 68 041501(R)
[22] Babuć M, Szilágyi B and Winicour J 2004 Preprint gr-qc/0404092
[23] De Donder T 1921 La Gravifique Einsteinienne (Paris: Gauthier-Villars)
[24] Fock V A 1959 The Theory of Space, Time and Gravitation (London: Pergamon)
[25] Secchi P 1996 Arch. Ration. Mech. Anal. 134 155
[26] Alcubierre M et al 2004 Class. Quantum Grav. 21 589
[27] Friedrich H and Nagy G 1999 Commun. Math. Phys. 201 619
[28] Bona C, Ledvinka T, Palenzuela C and Žáček M 2003 Phys. Rev. D 67 104005
[29] Bona C, Ledvinka T, Palenzuela C and Žáček M 2004 Phys. Rev. D 69 064036
[30] Bona C and Massó J 1992 Phys. Rev. Lett. 68 1097
[31] Bona C, Massó J, Seidel E and Stela J 1995 Phys. Rev. Lett. 75 600
[32] Shibata M and Nakamura T 1995 Phys. Rev. D 52 5428
[33] Baumgarte T W and Shapiro S L 1999 Phys. Rev. D 59 024007
[34] Bona C and Palenzuela C 2004 Phys. Rev. D 69 104003
[35] Gustafsson B, Kreiss H O and Oliger J 1995 Time Dependent Problems and Difference Methods (New York: Wiley)
[36] Bona C and Palenzuela-Luque C 2005 Elements of Numerical Relativity (Lecture Notes in Physics vol 673) (Heidelberg: Springer) at press
[37] Calabrese G, Lehner L, Reula O, Sarbach O and Tiglio M 2004 Class. Quantum Grav. 21 5735
[38] Calabrese G and Neilsen D 2004 Phys. Rev. D 69 044020
[39] Thorne K J 2004 Class. Quantum Grav. 21 3665
[40] Calabrese G and Neilsen D 2004 Preprint gr-qc/0412109