Explicit lepton texture

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Abstract

An explicit form of charged–lepton mass matrix, predicting $m_\tau = 1776.80 \text{ MeV}$ from the experimental values of $m_e$ and $m_\mu$ (in good agreement with the experimental figure $m_\tau = 1777.05^{+0.29}_{-0.26} \text{ MeV}$), is applied to three neutrinos $\nu_e$, $\nu_\mu$, $\nu_\tau$ in order to correlate tentatively their masses and mixing parameters. While for charged leptons the off–diagonal mass–matrix elements turn out to be small versus its diagonal elements, it is suggested that for neutrinos the situation is inverse. Under such a conjecture, the neutrino masses, lepton Cabibbo–Kobayashi–Maskawa matrix and neutrino oscillation probabilities are calculated in the corresponding lowest (and the next to lowest) perturbative order. Then, the nearly maximal mixing of $\nu_\mu$ and $\nu_\tau$ is predicted in consistency with the observed deficit of atmospheric $\nu_\mu$’s. However, the predicted deficit of solar $\nu_e$’s is much too small to explain the observed effect, what suggests the existence of (at least) one sort, $\nu_s$, of sterile neutrinos, whose mixing with $\nu_e$ would be responsible for the observed deficit. In the last Section, promising perspectives for applying the same form of mass matrix to quarks are outlined. Two independent predictions of $|V_{ub}|/|V_{cb}| = 0.0753 \pm 0.0032$ and unitary angle $\gamma \simeq 70^\circ$ are deduced from the experimental values of $|V_{us}|$ and $|V_{cb}|$ (with the use of quark masses $m_s$, $m_c$ and $m_b$).

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1. Introduction

In this paper, the explicit form of mass matrix invented for three generations of charged leptons $e^-, \mu^-, \tau^-$, and being surprisingly good for their masses [1], is applied to three generations of neutrinos $\nu_e, \nu_\mu, \nu_\tau$, in order to correlate tentatively their masses and mixing parameters. This form reads

$$
(M_{ij}^{(f)}) = \frac{1}{29} \begin{pmatrix}
\mu^{(f)} e^{i\varphi^{(f)}} & 2\alpha^{(f)} e^{i\varphi^{(f)}} & 0 \\
2\alpha^{(f)} e^{-i\varphi^{(f)}} & 4\mu^{(f)}(80 + \varepsilon^{(f)})/9 & 8\sqrt{3}\alpha^{(f)} e^{i\varphi^{(f)}} \\
0 & 8\sqrt{3}\alpha^{(f)} e^{-i\varphi^{(f)}} & 24\mu^{(f)}(624 + \varepsilon^{(f)})/25
\end{pmatrix},
$$

(1)

where the label $f = \nu, e$ denotes neutrinos and charged leptons, respectively, while $\mu^{(f)}, \varepsilon^{(f)}, \alpha^{(f)}$ and $\varphi^{(f)}$ are real constants to be determined from the present and future experimental data for lepton masses and mixing parameters ($\mu^{(f)}$ and $\alpha^{(f)}$ are mass–dimensional). In our approach, neutrinos are assumed to carry pure Dirac masses.

Here, the form (1) of mass matrices $(M_{ij}^{(\nu)})$ and $(M_{ij}^{(e)})$ may be considered as a detailed ansatz to be compared with the lepton data. However, in the past, we have presented an argument [2,1] in favour of the form (1), based on: (i) Kähler–like generalized Dirac equations (interacting with the Standard–Model gauge bosons) whose a priori infinite series is necessarily reduced (in the case of fermions) to three Dirac equations, due to an intrinsic Pauli principle, and (ii) an ansatz for the fermion mass matrix, suggested by the above three–generation characteristics (i).

In the case of charged leptons, assuming that the off–diagonal elements of the mass matrix $(M_{ij}^{(e)})$ can be treated as a small perturbation of its diagonal terms (i.e., that $\alpha^{(e)}/\mu^{(e)}$ is small enough), we calculate in the lowest perturbative order [1]

$$
m_\tau = 1776.80 + 10.2112 \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \text{ MeV},$$

$$
\mu^{(e)} = 85.9924 \text{ MeV} + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \mu^{(e)} \right),$$

$$
\varepsilon^{(e)} = 0.172329 + O \left( \left( \frac{\alpha^{(e)}}{\mu^{(e)}} \right)^2 \right),
$$

(2)

when the experimental values of $m_e$ and $m_\mu$ [3] are used as inputs. In Eqs. (2), the first terms are given as $\tilde{m}_\tau = 6(351m_\mu - 136m_e)/125$, $\tilde{\varepsilon}^{(e)} = 320m_e/(9m_\mu - 4m_e)$ and $\tilde{\mu}^{(e)} = 29(9m_\mu -
According to the relation $\frac{\alpha(e)}{\mu(e)}$, we can take this experimental figure as another input, obtaining

$$\left(\frac{\alpha(e)}{\mu(e)}\right)^2 = 0.024^{+0.028}_{-0.025},$$

which value is not inconsistent with zero. Hence, $\alpha(e)^2 = 180^{+210}_{-190}\text{MeV}^2$ due to Eq. (2).

For the unitary matrix $(U^{(e)}_{ij})$, diagonalizing the charged–lepton mass matrix $(M^{(e)}_{ij})$ according to the relation $U^{(e)\dagger} M^{(e)} U^{(e)} = \text{diag}(m_e, m_\mu, m_\tau)$, we get in the lowest perturbative order

$$(U^{(e)}_{ij}) = \begin{pmatrix}
1 - \frac{2}{29^2} \left(\frac{\alpha(e)}{m_\mu}\right)^2 & \frac{2}{29} \frac{\alpha(e)}{m_\mu} e^{i\varphi(e)} & \frac{16\sqrt{3}}{29^2} \left(\frac{\alpha(e)}{m_\tau}\right)^2 e^{2i\varphi(e)} \\
-\frac{2}{29} \frac{\alpha(e)}{m_\mu} e^{-i\varphi(e)} & 1 - \frac{2}{29^2} \left(\frac{\alpha(e)}{m_\mu}\right)^2 & \frac{8\sqrt{3}}{29} \frac{\alpha(e)}{m_\tau} e^{-i\varphi(e)} \\
\frac{16\sqrt{3}}{29^2} \frac{\alpha(e)^2}{m_\mu m_\tau} e^{-2i\varphi(e)} & -\frac{8\sqrt{3}}{29} \frac{\alpha(e)}{m_\tau} e^{-i\varphi(e)} & 1 - \frac{96}{29^2} \left(\frac{\alpha(e)}{m_\tau}\right)^2 
\end{pmatrix}.$$  \tag{4}

2. Neutrino masses and mixing parameters

In the case of neutrinos, because of their expected tiny mass scale $\mu^{(\nu)}$, we will tentatively conjecture that the diagonal elements of the mass matrix $(M_{ij}^{(\nu)})$ can be treated as a small perturbation of its off–diagonal terms ($i.e.$, that $\mu^{(\nu)}/\alpha^{(\nu)}$ is small enough). In addition, we put $\varepsilon^{(\nu)} = 0$ $i.e.$, $M_{11}^{(\nu)} = 0$. Then, we calculate in the lowest perturbative order the following neutrino masses:

$$m_{\nu_1} = \frac{|M_{12}^{(\nu)}|^2 M_{33}^{(\nu)}}{|M_{12}^{(\nu)}|^2 + |M_{23}^{(\nu)}|^2} = \frac{1}{49} M_{33}^{(\nu)} = \frac{1}{49} |M_{12}^{(\nu)}|,$$

$$m_{\nu_2, \nu_3} = \pm \sqrt{|M_{12}^{(\nu)}|^2 + |M_{23}^{(\nu)}|^2} + \frac{1}{2} \left(\frac{48}{49} M_{33}^{(\nu)} + M_{22}^{(\nu)}\right)$$

$$= \left[\pm \frac{1}{2} \left(\frac{48}{49} \xi + \chi\right)\right] |M_{12}^{(\nu)}|,$$  \tag{5}

where

$$\xi \equiv \frac{M_{33}^{(\nu)}}{|M_{12}^{(\nu)}|} = \frac{7488}{25} \frac{\mu^{(\nu)}}{\alpha^{(\nu)}} = \frac{299.52}{\alpha^{(\nu)}},$$

$$\chi \equiv \frac{M_{22}^{(\nu)}}{|M_{12}^{(\nu)}|} = \frac{160}{9} \frac{\mu^{(\nu)}}{\alpha^{(\nu)}} = \frac{125}{2106} \xi = \frac{1}{16.848} \xi,$$  \tag{6}
are relatively small by our perturbative conjecture, while

\[ |M^{(v)}_{12}| = \frac{2}{29} \alpha^{(v)}, \quad |M^{(v)}_{23}| = \frac{8\sqrt{3}}{29} \alpha^{(v)} = \sqrt{48} |M^{(v)}_{12}|. \]  \(7\)

As seen from Eqs. (5), the actual perturbative parameters are not \( \xi \) and \( \chi \), but rather \( \xi/7 \) and \( \chi/7 \), what is confirmed later in Eqs. (9). Note that \( m_{\nu_2} < 0 \), the minus sign being irrelevant in the relativistic case, where only \( m_{\nu_2}^2 \) is measured (cf. Dirac equation): \( |m_{\nu_2}| \) may be considered as a phenomenological mass of \( \nu_2 \).

Using Eqs. (5), we can write the formula

\[ m_{\nu_3}^2 - m_{\nu_2}^2 = 14 \left( \frac{48}{49} \xi + \chi \right) |M^{(v)}_{12}|^2 = 20.721 \alpha^{(v)} \mu^{(v)}, \]  \(8\)

which will enable us to determine the product \( \alpha^{(v)} \mu^{(v)} \) from the observed deficit of atmospheric neutrinos \( \nu_\mu \), if \( \nu_\mu \to \nu_\tau \) oscillations are really responsible for this effect.

We calculate also the unitary matrix \( (U^{(v)}_{ij}) \) diagonalizing the neutrino mass matrix \( (M^{(v)}_{ij}) \) according to the relation \( U^{(v)} \dagger M^{(v)} U^{(v)} = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3}) \). In the lowest perturbative order we obtain

\[

table
\begin{align*}
U^{(v)}_{11} &= \sqrt{\frac{48}{49}} \left( 1 - \left( \frac{24}{49^3} - \frac{1}{49^4} \right) \xi^2 \right), \\
U^{(v)}_{21} &= \frac{1}{49} \sqrt{\frac{48}{49}} e^{-i\varphi^{(v)}}, \\
U^{(v)}_{31} &= -\frac{1}{7} \left[ 1 - \left( \frac{73}{49^3} - \frac{1}{49^4} \right) \xi^2 + \frac{1}{49} \xi \right] e^{-2i\varphi^{(v)}}, \\
U^{(v)}_{12} &= -\frac{1}{\sqrt{2}} \left( 1 + \frac{36}{7 \cdot 49} \xi + \frac{1}{28} \chi \right) e^{i\varphi^{(v)}}, \\
U^{(v)}_{22} &= \frac{1}{\sqrt{2}} \left( 1 + \frac{7 \cdot 49}{7 \cdot 49} \xi - \frac{1}{28} \chi \right), \\
U^{(v)}_{32} &= -\frac{1}{\sqrt{2}} \sqrt{\frac{48}{49}} \left( 1 - \frac{13}{7 \cdot 49} \xi + \frac{1}{28} \chi \right) e^{-i\varphi^{(v)}}, \\
U^{(v)}_{13} &= \frac{1}{\sqrt{2}} \left( 1 - \frac{36}{7 \cdot 49} \xi - \frac{1}{28} \chi \right) e^{2i\varphi^{(v)}}, \\
U^{(v)}_{23} &= \frac{1}{\sqrt{2}} \left( 1 - \frac{12}{7 \cdot 49} \xi + \frac{1}{28} \chi \right) e^{i\varphi^{(v)}}, \\
U^{(v)}_{33} &= \frac{1}{\sqrt{2}} \sqrt{\frac{48}{49}} \left( 1 + \frac{13}{7 \cdot 49} \xi - \frac{1}{28} \chi \right)
\end{align*}
\]

(9)

with \( \chi = (125/2106)\xi = \xi/16.848 \).

Denoting by \( \nu_\alpha = \nu_e, \nu_\mu, \nu_\tau \) and \( \nu_i = \nu_1, \nu_2, \nu_3 \) the weak–interaction and mass neutrino fields, respectively, we have the unitary transformation
\[ \nu_\alpha = \sum_i (V^\dagger)_{\alpha i} \nu_i = \sum_i V^*_{i\alpha} \nu_i, \]  
(10)

where the lepton counterpart \((V_{\alpha i})\) of the Cabibbo—Kobayashi—Maskawa matrix is given as

\[ V = U^{(\nu)} \ast U^{(\nu)} \simeq U^{(\nu)} \ast \text{or} \]

\[ V_{\alpha i} = \sum_\beta (U^{\nu})_{\alpha i} (U^{e})_{\beta \alpha} \approx (U^{\nu})_{\alpha i}, \]  
(11)

the approximate equality being valid for negligible \(\alpha(\nu)/\mu(\nu)\) when \(U_{\beta \alpha} \approx \delta_{\beta \alpha}\) due to Eq. (4).

Of course, in Eqs. (9) we wrote \(\alpha = 1, 2, 3\) for simplicity. From Eq. (10), we get the unitary transformation

\[ |\nu_\alpha \rangle = \sum_i |\nu_i \rangle V_{i\alpha}, \]  

where \(|\nu_\alpha \rangle = \nu_\alpha^\dagger |0 \rangle\) and \(|\nu_i \rangle = \nu_i^\dagger |0 \rangle\) are weak–interaction and mass neutrino states.

In the limit of \(\mu(\nu) \to 0\) (implying \(\xi \to 0\) and \(\chi \to 0\)), we obtain from Eqs. (10), (11) and (9) the following unperturbed mixing formulae for \(\nu_{1,2,3}\):

\[ \nu_e \to -\frac{1}{7} \left[ \sqrt{48} \nu_1 e^{-i \phi_{\nu}(\nu)} - \frac{1}{\sqrt{2}} \left( \nu_2 - \nu_3 e^{i \phi_{\nu}(\nu)} \right) \right] e^{i \phi_{\nu}(\nu)}, \]

\[ \nu_\mu \to \frac{1}{\sqrt{2}} \left( \nu_2 + \nu_3 e^{i \phi_{\nu}(\nu)} \right), \]

\[ \nu_\tau \to -\frac{1}{7} \left[ \nu_1 e^{-i \phi_{\nu}(\nu)} + \sqrt{\frac{48}{2}} \left( \nu_2 - \nu_3 e^{i \phi_{\nu}(\nu)} \right) \right] e^{-i \phi_{\nu}(\nu)}. \]  
(12)

These display the maximal mixing between \(\nu_2\) and \(\nu_3\) in all three cases and a smaller mixing of \(\left[ \nu_2 - \nu_3 \exp \left( i \phi_{\nu}(\nu) \right) \right]/\sqrt{2}\) with \(\nu_1\) in the cases of \(\nu_e\) and \(\nu_\tau\), giving a minor admixture to \(\nu_e\) and a dominating admixture to \(\nu_\tau\) (in \(\nu_\mu\) there is no admixture of \(\nu_1\)).

3. Neutrino oscillations

Once knowing the elements \(V_{i\alpha}\) of the lepton Cabibbo—Kobayashi—Maskawa matrix, we can calculate the probabilities of neutrino oscillations \(\nu_\alpha \to \nu_\beta\) (in the vacuum) making use of the general formula

\[ P(\nu_\alpha \to \nu_\beta) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = \sum_{ij} V_{j\beta} V_{j\alpha}^* V_{i\beta}^* V_{i\alpha} e^{2 i x_{ji}}, \]  
(13)

where \(|\nu_\alpha(t)\rangle = \exp(-iHt)|\nu_\alpha\rangle\) and

\[ x_{ji} = 1.26693 \Delta m_{ji}^2 L/E, \quad \Delta m_{ji}^2 = m_{\nu_j}^2 - m_{\nu_i}^2, \]  
(14)
if $\Delta m_{ij}^2$, $L$ and $E$ are measured in eV$^2$, km and GeV, respectively, with $L = t$ and $E = |\vec{p}|$ ($c = 1 = \hbar$) denoting the experimental baseline and neutrino energy.

It is not difficult to show that for the mass matrix $(M_{ij}^{(\nu)})$, as it is given in Eq. (1), the quartic products of $V_{i\alpha}$’s in Eq. (13) are always real (for any phase $\varphi^{(\nu)}$), if only $V_{i\alpha} = U_{i\alpha}^{(\nu)*}$ (i.e., $U_{\beta \alpha}^{(e)} = \delta_{\beta \alpha}$). This implies that $P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha)$. In general, the last relation is valid in the case of CP invariance which, under the CPT theorem, provides the time–reversal invariance. Because of the real values of quartic products of $V_{i\alpha}$’s, the formula (13) can be rewritten as

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\beta \alpha} - 4 \sum_{i<j} V_{j\beta} V_{j\alpha}^* V_{i\beta} V_{i\alpha} \sin^2 x_{ji}$$

(15)

without the necessity of introducing phases of these products.

With the lowest–order perturbative expressions (9) for $V_{i\alpha} = U_{i\alpha}^{(\nu)*}$, the formula (15) leads to the following forms of appearance oscillation probabilities:

$$P(\nu_\mu \to \nu_e) = \frac{1}{49} \sin^2 x_{32}$$

$$+ \frac{96}{7 \cdot 49^2} \xi \left[ \left( 1 + \frac{48}{7 \cdot 49} \xi \right) \sin^2 x_{21} - \left( 1 - \frac{48}{7 \cdot 49} \xi \right) \sin^2 x_{31} \right],$$

(16)

$$P(\nu_\mu \to \nu_\tau) = \frac{48}{49} \sin^2 x_{32}$$

$$+ \frac{96}{7 \cdot 49^2} \xi \left[ - \left( 1 - \frac{1}{7 \cdot 49} \xi \right) \sin^2 x_{21} + \left( 1 + \frac{1}{7 \cdot 49} \xi \right) \sin^2 x_{31} \right],$$

(17)

$$P(\nu_e \to \nu_\tau) = - \frac{48}{49^2} \sin^2 x_{32}$$

$$+ \frac{96}{49^2} \left[ \left( 1 + \frac{23}{7 \cdot 49} \xi + \frac{1}{14} \chi \right) \sin^2 x_{21} + \left( 1 - \frac{23}{7 \cdot 49} \xi - \frac{1}{14} \chi \right) \sin^2 x_{31} \right]$$

(18)

as well as of survival oscillation probabilities:

$$P(\nu_\nu \to \nu_\nu) = 1 - \frac{1}{49^2} \sin^2 x_{32}$$

$$- \frac{96}{49^2} \left[ \left( 1 + \frac{72}{7 \cdot 49} \xi + \frac{1}{14} \chi \right) \sin^2 x_{21} + \left( 1 - \frac{72}{7 \cdot 49} \xi - \frac{1}{14} \chi \right) \sin^2 x_{31} \right],$$

(19)

$$P(\nu_\mu \to \nu_\mu) = 1 - \sin^2 x_{32} - \frac{96}{49^2} \xi^2 \left( \sin^2 x_{21} + \sin^2 x_{31} \right),$$

(20)

$$P(\nu_\tau \to \nu_\tau) = 1 - \left( \frac{48}{49} \right)^2 \sin^2 x_{32}$$

$$- \frac{96}{49^2} \left[ \left( 1 - \frac{26}{7 \cdot 49} \xi + \frac{1}{14} \chi \right) \sin^2 x_{21} + \left( 1 + \frac{26}{7 \cdot 49} \xi - \frac{1}{14} \chi \right) \sin^2 x_{31} \right].$$

(21)
Thus, we get \[ P(\nu_e \to \nu_e) + P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) = 1 \] and two other obvious summation rules for probabilities. Among these probabilities, \( P(\nu_\mu \to \nu_\mu) \) displays (in the lowest perturbative order) maximal mixing between \( \nu_2 \) and \( \nu_3 \).

In the lowest perturbative order,
\[
x_{31} - x_{21} = x_{32} = 14 \left( \frac{48}{49} \xi + \chi \right) \left( \frac{1.26693 |M_{12}^{(\nu)}| L/E}{2} \right)
\]
due to Eqs. (8) and (14). Hence,
\[
\sin^2 x_{31} = \sin^2 x_{21} + x_{32} \sin 2x_{21} + x_{32}^2 \sin 2x_{21}
\]
in experiments where \( x_{32} \ll \pi/2 \). When in such cases the relation (23) is inserted into the formulae (16), (17) and (20), its \( x_{32} \) and \( x_{32}^2 \) terms can be neglected in the lowest perturbative order.

Note that the mass formulae (5) imply \( m_{\nu_1}^2 \ll m_{\nu_2}^2 \ll m_{\nu_3}^2 \), where \( m_{\nu_1}^2 / m_{\nu_2,\nu_3}^2 = \xi^2 / 49^3 + O(\xi^3) \) and \( m_{\nu_2}^2 / m_{\nu_3}^2 = 1 - (2/7)(48\xi / 49 + \chi) + O(\xi^3) \). Thus, the inequality \( x_{31} \gtrsim x_{21} \gg x_{32} \) holds in all neutrino oscillation experiments (with some given \( L \) and \( E \)).

We have calculated the neutrino masses, lepton Cabibbo—Kobayashi—Maskawa matrix and neutrino oscillation probabilities also in the next to lowest perturbative order. Then, in Eqs. (5) the mass \( m_{\nu_1} \) gets no quadratic correction, while \( m_{\nu_2} \) and \( m_{\nu_3} \) are corrected by the terms
\[
\mp \frac{1}{14} \left[ \frac{13 \cdot 48}{49^2} \xi^2 - \frac{24}{49} \xi \chi + \frac{1}{4} \chi^2 \right] |M_{12}^{(\nu)}|,
\]
respectively. Among the derived oscillation formulae, Eq. (20), for instance, is extended to the form
\[
P(\nu_\mu \to \nu_\mu) = 1 - \left( 1 - \frac{672}{49^3} \xi^2 + \frac{24}{49^2} \xi \chi \left( 1 - \frac{1}{4} \cdot 49 \chi^2 \right) \right) \sin^2 x_{32}
\]
\[
\quad - \frac{96}{49^3} \xi^2 \left( \sin^2 x_{21} + \sin^2 x_{31} \right)
\]
\[
= 1 - \left( 1 - 0.00514 \xi^2 \right) \sin^2 x_{32} - 0.000816 \xi^2 \left( \sin^2 x_{21} + \sin^2 x_{31} \right)
\]
\[
\]
displaying nearly maximal mixing between \( \nu_2 \) and \( \nu_3 \).

In the case of Super–Kamiokande atmospheric neutrino experiment [4], if \( \nu_\mu \to \nu_\tau \) oscillations are responsible for the observed deficit of atmospheric \( \nu_\mu \)’s, we have \( x_{\text{atm}} = x_{32} \ll x_{21} \approx x_{31} \), what implies that \( \sin^2 x_{21} = \sin^2 x_{31} = 1/2 \) due to averaging over many oscillation lengths. Then, Eq. (25) leads to the following effective two–flavor oscillation formula:
\[
P(\nu_\mu \rightarrow \nu_\mu) = 1 - \left(1 - 0.00350 \xi^2\right) \sin^2 \frac{x_{32}}{2},
\]
if we assume in Eq. (25) that \(0.000816 \xi^2 = 0.000816 \xi^2 (2 \sin^2 x_{32})\) effectively. Identifying the estimation (26) with the two–flavor formula fitted in the Super–Kamiokande experiment, we obtain the limits

\[
1 - 0.00350 \xi^2 \equiv \sin^2 2\theta_{\text{atm}} \sim 0.82 \text{ to } 1 , \quad \Delta m_{32}^2 \equiv \Delta m_{\text{atm}}^2 \sim (0.5 \text{ to } 6) \times 10^{-3} \text{eV}^2 .
\]

Hence, \(\xi \sim 7.17 \text{ to } 0\) and

\[
\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} \equiv 0.00334 \xi \sim 0.0239 \text{ to } 0 , \quad \alpha^{(\nu)} \mu^{(\nu)} \equiv 0.483 \Delta m_{32}^2 \sim (0.241 \text{ to } 2.90) \times 10^{-4} \text{eV}^2 ,
\]

where Eqs. (6) and (8) are used. For instance, with \(\sin^2 2\theta_{\text{atm}} \sim 0.999\) and \(\Delta m_{\text{atm}}^2 \sim 5 \times 10^{-3} \text{eV}^2\), we get \(\xi \sim 0.535\) and

\[
\frac{\mu^{(\nu)}}{\alpha^{(\nu)}} \sim 0.00178 , \quad \alpha^{(\nu)} \mu^{(\nu)} \sim 2.41 \times 10^{-4} \text{eV}^2 ,
\]
what gives the estimation

\[
\alpha^{(\nu)} \sim 0.368 \text{eV} , \quad \mu^{(\nu)} \sim 6.55 \times 10^{-4} \text{eV} .
\]

Note that \(\xi < 1\) for \(\sin^2 2\theta_{\text{atm}} > 0.9965\). As was already mentioned, our actual perturbative parameters are not \(\xi\) and \(\chi\), but rather \(\xi/7\) and \(\chi/7 = 0.0594\xi/7\).

Having estimated \(\alpha^{(\nu)}\) and \(\mu^{(\nu)}\), we can calculate neutrino masses from Eqs. (5) with (6) and (7). Making use of the values (30) (valid for \(\sin^2 2\theta_{\text{atm}} \sim 0.999\) and \(\Delta m_{\text{atm}}^2 \sim 5 \times 10^{-3} \text{eV}^2\)), we obtain

\[
m_{\nu_1} \sim 2.76 \times 10^{-4} \text{eV} , \quad m_{\nu_2} \sim -1.71 \times 10^{-1} \text{eV} , \quad m_{\nu_3} \sim 1.85 \times 10^{-1} \text{eV} .
\]

Because of the smallness of these masses, the neutrinos \(\nu_1, \nu_2, \nu_3\) are not likely to be responsible for the entire hot dark matter.

In the case of solar neutrino experiments, all three popular fits [5] of the observed deficit of solar \(\nu_e\)'s to an effective two–flavor oscillation formula require \(\Delta m_{\text{sol}}^2 \ll \Delta m_{\text{atm}}^2\) what implies
$\Delta m_{\text{sol}}^2 \ll \Delta m_{32}^2 \ll \Delta m_{21}^2 \lesssim \Delta m_{31}^2$, if $\nu_\mu \to \nu_\tau$ oscillations are responsible for the deficit of atmospheric $\nu_\mu$'s. Then, $x_{\text{sol}} \ll x_{32} \ll x_{21} \lesssim x_{31}$, giving $\sin^2 x_{32} = \sin^2 x_{21} = \sin^2 x_{31} = 1/2$ due to averaging over many oscillation lengths. In such a case, Eq. (19) leads to

$$P(\nu_e \to \nu_e) = 1 - \frac{193}{2 \cdot 49^2} = 1 - 0.0402 = 0.960,$$

predicting only a 4% deficit of solar $\nu_e$'s, much too small to explain solar neutrino observations.

An intriguing situation arises in the case of formula (16) for $P(\nu_\mu \to \nu_e)$, if $\nu_\mu \to \nu_\tau$ oscillations really cause the bulk of deficit of atmospheric $\nu_\mu$'s. Then, for a new $x_{\text{new}} = x_{32} \ll x_{21} \lesssim x_{31}$ (with some new $L$ and $E$) we may have $\sin^2 x_{21} = \sin^2 x_{31} = 1/2$ due to averaging over many oscillation lengths and so, infer from Eq. (16) that

$$P(\nu_\mu \to \nu_e) = \frac{1}{49} \sin^2 x_{32} + \frac{2 \cdot 48^2}{49^4} \xi^2 \sim 0.0204 \sin^2 x_{32} + 2.29 \times 10^{-4},$$

where $\xi^2 \sim 0.286$ (what is valid for $\sin^2 2\theta_{\text{atm}} \sim 0.999$ and $\Delta m_{\text{atm}}^2 \sim 5 \times 10^{-3}$ eV$^2$). Such a predicted oscillation amplitude $\sin^2 2\theta_{\text{new}} \sim 0.02$ would lie in the range of $\sin^2 2\theta_{\text{LSND}}$ estimated in the positive (though still requiring confirmation) LSND accelerator experiment on $\nu_\mu \to \nu_e$ oscillations [6]. However, the lower limit $\Delta m_{\text{LSND}}^2 \gtrsim 0.1$ eV$^2$ reported by this experiment is by one order of magnitude larger than the Super–Kamiokande upper limit $\Delta m_{32}^2 \lesssim 0.01$ eV$^2$. On the other hand, the small predicted oscillation amplitude $\sin^2 2\theta_{\text{new}} \sim 0.02$ would not be in conflict with the negative result of the CHOOZ long–baseline reactor experiment on $\bar{\nu}_e \to \bar{\nu}_\mu$ oscillations [7].

In conclusion, our explicit model of lepton texture displays a number of important features. (i) It correlates correctly (with high precision) the tauon mass with electron and muon masses. (ii) It predicts (without parameters) the maximal mixing between muon and tauon neutrinos in the limit $\mu^{(\nu)} \to 0$, consistent with the observed deficit of atmospheric $\nu_\mu$'s. (iii) It fails to explain the observed deficit of solar $\nu_e$'s. (iv) It predicts new $\nu_\mu \to \nu_e$ oscillations with the amplitude consistent with LSND experiment, but with a phase corresponding to the mass squared difference at least one order of magnitude smaller.

In the framework of our model, the point (iii) may suggest that in Nature there exists (at least) one sort, $\nu_s$, of sterile neutrinos (blind to the Standard–Model interactions), responsible for the observed deficit of solar $\nu_e$’s through $\nu_e \to \nu_s$ oscillations dominating the survival probability $P(\nu_e \to \nu_e) \simeq 1 - P(\nu_e \to \nu_s)$. In an extreme version of this picture, it might even happen that in Nature there would be two sorts, $\nu_s$ and $\nu'_s$, of sterile neutrinos, where $\nu'_s$ would
replace $\nu_\tau$ in explaining the observed deficit of atmospheric $\nu_\mu$'s by means of $\nu_\mu \to \nu'_s$ oscillations that should dominate the survival probability $P(\nu_\mu \to \nu_\mu) \simeq 1 - P(\nu_\mu \to \nu'_s)$. In this case, the masses of active neutrinos might be even zero, what would correspond to $\alpha(\nu) = 0$ and $\mu(\nu) = 0$ giving then $\nu_e = \nu_1$, $\nu_\mu = \nu_2$, $\nu_\tau = \nu_3$ (however, very small $\alpha(\nu)$ and $\mu(\nu)$ leading to tiny masses and mixings of $\nu_1$, $\nu_2$, $\nu_3$ would be still allowed).

For the author of the present paper the idea of existence of two sorts of sterile neutrinos is fairly appealing, since two such spin–1/2 fermions, blind to all Standard–Model interactions, do follow (besides three standard families of active leptons and quarks) [8] from the argument (i) mentioned in Introduction, based on the Kähler–like generalized Dirac equations. Note in addition that the $\nu_e \to \nu_s$ and $\nu_\mu \to \nu'_s$ oscillations caused by appropriate mixings should be a natural consequence of the spontaneous breaking of electroweak $SU(2) \times U(1)$ symmetry.

4. Perspectives for unification with quarks

In this last Section, we try to apply to quarks the form of mass matrix which was worked out above for leptons. To this end, we conjecture for three generations of up quarks $u$, $c$, $t$ and down quarks $d$, $s$, $b$ the mass matrices $(M_{i,j}^{(u)})$ and $(M_{i,j}^{(d)})$, respectively, essentially of the form (1), where the label $f = u$, $d$ denotes now up and down quarks. The only modification introduced is a new real constant $C(f)$ added to $\varepsilon(f)$ in the element $M_{33}^{(f)}$ which now reads

\[
M_{33}^{(f)} = \frac{24\mu^{(f)}}{25 \cdot 29} \left( 624 + \varepsilon(f) + C(f) \right).
\]

(34)

Since for quarks the mass scales $\mu^{(u)}$ and $\mu^{(d)}$ are expected to be even more important than the scale $\mu^{(e)}$ for charged leptons, we assume that the off–diagonal elements of mass matrices $(M_{i,j}^{(u)})$ and $(M_{i,j}^{(d)})$ can be considered as a small perturbation of their diagonal terms. Then, in the lowest perturbative order, we obtain the following mass formulae

\[
m_{u,d} = \frac{\mu^{(u,d)}}{29} \varepsilon^{(u,d)} - A^{(u,d)} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2,
\]

\[
m_{c,s} = \frac{\mu^{(u,d)}}{29} \cdot \frac{4}{9} \left( 80 + \varepsilon^{(u,d)} \right) + \left( A^{(u,d)} - B^{(u,d)} \right) \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2,
\]

\[
m_{t,b} = \frac{\mu^{(u,d)}}{29} \frac{24}{25} \left( 624 + \varepsilon^{(u,d)} + C^{(u,d)} \right) + B^{(u,d)} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2.
\]

(35)

where
\[ A^{(u,d)} = \frac{\mu^{(u,d)}}{29} \frac{36}{320 - 5\varepsilon^{(u,d)}} , \quad B^{(u,d)} = \frac{\mu^{(u,d)}}{29} \frac{10800}{31696 + 54C^{(u,d)} + 29\varepsilon^{(u,d)}} . \]  

In Eqs. (35), the relative smallness of perturbing terms is more pronounced due to extra factors. In our discussion, we will take for experimental quark masses the arithmetic means of their lower and upper limits quoted in the Review of Particle Physics [3], i.e.,

\[ m_u = 3.3 \text{ MeV} , \quad m_c = 1.3 \text{ GeV} , \quad m_t = 174 \text{ GeV} \]  

and

\[ m_d = 6 \text{ MeV} , \quad m_s = 120 \text{ MeV} , \quad m_b = 4.3 \text{ GeV} . \]  

Eliminating from the unperturbed terms in Eqs. (35) the constants \( \mu^{(u,d)} \) and \( \varepsilon^{(u,d)} \), we derive the correlating formulae being counterparts of Eqs. (2) for charged leptons:

\[ m_{t,b} = \frac{6}{125} \left( 351m_{c,s} - 136m_{u,d} \right) + \frac{\mu^{(u,d)}}{29} \frac{24}{25} C^{(u,d)} \]
\[ - \frac{1}{125} \left( 2922A^{(u,d)} - 2231B^{(u,d)} \right) \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 , \]
\[ \mu^{(u,d)} = \frac{29}{320} \left( 9m_{c,s} - 4m_{u,d} \right) - \frac{29}{320} \left( 5A^{(u,d)} - 9B^{(u,d)} \right) \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 , \]
\[ \varepsilon^{(u,d)} = \frac{29m_{u,d}}{\mu^{(u,d)}} + \frac{29}{\mu^{(u,d)}} A^{(u,d)} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 . \]  

The unperturbed parts of these relations are:

\[ \overset{\circ}{m}_{t,b} = \frac{6}{125} \left( 351m_{c,s} - 136m_{u,d} \right) + \frac{\phi^{(u,d)}}{29} \frac{24}{25} \phi^{(u,d)} C^{(u,d)} \]
\[ = \left\{ \begin{array}{c} 21.9 \\ 1.98 \end{array} \right\} \text{ GeV} + \frac{\phi^{(u,d)}}{29} \frac{24}{25} \phi^{(u,d)} , \]
\[ \overset{\circ}{\mu}^{(u,d)} = \frac{29}{320} \left( 9m_{c,s} - 4m_{u,d} \right) = \left\{ \begin{array}{c} 1060 \\ 95.7 \end{array} \right\} \text{ MeV} , \]
\[ \overset{\circ}{\varepsilon}^{(u,d)} = \frac{29m_{u,d}}{\overset{\circ}{\mu}^{(u,d)}} = \left\{ \begin{array}{c} 0.0904 \\ 1.82 \end{array} \right\} . \]

In the spirit of our perturbative approach, the “coupling” constant \( \alpha^{(u,d)} \) can be put zero in all perturbing terms in Eqs. (35) and (39), except for \( \alpha^{(u,d)} \overset{\circ}{\mu}^{(u,d)} \) in the numerator of the factor \( (\alpha^{(u,d)}/\overset{\circ}{\mu}^{(u,d)})^2 \) that now becomes \( (\alpha^{(u,d)}/\overset{\circ}{\mu}^{(u,d)})^2 \). Then, \( A^{(u,d)} \) and \( B^{(u,d)} \) are replaced by
\[
A_{(u,d)} = \frac{\mu_{(u,d)}}{29} \frac{36}{320 - 5 \bar{\varepsilon}_{(u,d)}} \quad \text{and} \quad B_{(u,d)} = \frac{\mu_{(u,d)}}{29} \frac{10800}{31696 + 54 \bar{C}_{(u,d)} + 29 \bar{\varepsilon}_{(u,d)}}.
\]

(41)

Note that the first Eq. (35) can be rewritten identically as \(m_{u,d} = \mu_{(u,d)} \bar{\varepsilon}_{(u,d)} / 29\) according to the third Eq. (40).

We shall be able to return to the discussion of quark masses after the estimation of constants \(\alpha^{(u)}\) and \(\alpha^{(d)}\) is made. Then, we shall determine the parameters \(C^{(u)}\) and \(C^{(d)}\) (as well as their unperturbed parts \(C^{(u)}\) and \(C^{(d)}\)) playing here an essential role in providing large values for \(m_t\) and \(m_b\).

At present, we find the unitary matrices \((U_{ij}^{(u,d)})\) that diagonalize the mass matrices \((M_{ij}^{(u,d)})\) according to the relations \(U_{ij}^{(u,d)} \dagger M_{ij}^{(u,d)} U_{ij}^{(u,d)} = \text{diag}(m_{u,d}, m_{c,s}, m_{t,b})\). In the lowest perturbative order, the result has the form (4) with the necessary replacement of labels:

\[(e) \rightarrow (u) \text{ or } (d), \quad \mu \rightarrow c \text{ or } s, \quad \tau \rightarrow t \text{ or } b,
\]

(42)

respectively.

Then, the elements \(V_{ij}\) of the Cabibbo—Kobayashi—Maskawa matrix \(V = U^{(u)} \dagger U^{(d)}\) can be calculated with the use of Eqs. (42) in the lowest perturbative order. Six resulting off-diagonal elements are:

\[
V_{us} = -V_{es}^\ast = \frac{2}{29} \left( \frac{\alpha^{(d)} e^{i\varphi(u)}}{m_s} - \frac{\alpha^{(u)} e^{i\varphi(d)}}{m_c} \right),
\]

\[
V_{cb} = -V_{ts}^\ast = \frac{8\sqrt{3}}{29} \left( \frac{\alpha^{(d)} e^{i\varphi(d)}}{m_b} - \frac{\alpha^{(u)} e^{i\varphi(u)}}{m_t} \right) \simeq \frac{8\sqrt{3}}{29} \frac{\alpha^{(d)} e^{i\varphi(d)}}{m_b},
\]

\[
V_{ub} \simeq -\frac{16\sqrt{3}}{841} \frac{\alpha^{(u)} \alpha^{(d)}}{m_c m_b} e^{i(\varphi(u) + \varphi(d))},
\]

\[
V_{td} \simeq \frac{16\sqrt{3}}{841} \frac{\alpha^{(d)} e^{i\varphi(d)}}{m_c m_b} e^{-2i\varphi(d)},
\]

(43)

where the indicated approximate steps were made due to the inequality \(m_t \gg m_b\) and/or under the assumption that \(\alpha^{(u)}/m_c \gg \alpha^{(d)}/m_b\) [cf. the conjecture (46)]. All three diagonal elements are real and positive in a good approximation:

\[
V_{ud} \simeq 1 - \frac{1}{2} |V_{us}|^2, \quad V_{cs} \simeq 1 - \frac{1}{2} |V_{us}|^2 - \frac{1}{2} |V_{cb}|^2, \quad V_{tb} \simeq 1 - \frac{1}{2} |V_{cb}|^2.
\]

(44)

In fact, in the lowest perturbative order,
\[
\text{arg} V_{ud} \simeq \frac{4}{841} \frac{\alpha^{(u)} \alpha^{(d)}}{m_c m_s} \sin \left( \varphi^{(u)} - \varphi^{(d)} \right) \frac{180^\circ}{\pi} \simeq - \text{arg} V_{cs} , \quad \text{arg} V_{tb} \simeq 0 , \tag{45}
\]
what gives \( \text{arg} V_{ud} = 0.88^\circ = - \text{arg} V_{cs} \), if the values (46), (49) and (52) are used.

Taking as an input the experimental value \(|V_{cb}| = 0.0395 \pm 0.0017 \) [3], we estimate from the second Eq. (43) that

\[
\alpha^{(d)} \simeq \frac{29}{8\sqrt{3}} m_b |V_{cb}| = (355 \pm 15) \text{ MeV} , \tag{46}
\]
where \( m_b = 4.3 \text{ GeV} \). In order to estimate also \( \alpha^{(u)} \), we will tentatively conjecture the approximate proportion

\[
\alpha^{(u)} : \alpha^{(d)} \simeq Q^{(u)}^2 : Q^{(d)}^2 = 4 \tag{47}
\]
to hold, where \( Q^{(u)} = 2/3 \) and \( Q^{(d)} = -1/3 \) are quark electric charges. Note that in the case of leptons we had \( \alpha^{(\nu)} : \alpha^{(e)} = 0.37 : (\sqrt{180} \times 10^6) = 2.8 \times 10^{-8} \) for the central value of \( \alpha^{(e)} \) [cf. Eqs. (3) and (30)], what is consistent with the analagous approximate proportion

\[
\alpha^{(\nu)} : \alpha^{(e)} \simeq Q^{(\nu)}^2 : Q^{(e)}^2 = 0 \, , \tag{48}
\]
where \( Q^{(\nu)} = 0 \) and \( Q^{(e)} = -1 \) are lepton electric charges. Under the conjecture (47):

\[
\alpha^{(u)} \simeq (1420 \pm 60) \text{ MeV} . \tag{49}
\]
In this case, from the second and third Eq. (43) we obtain the prediction

\[
|V_{ub}|/|V_{cb}| \simeq \frac{2}{29} \frac{\alpha^{(u)}}{m_c} \simeq 0.0753 \pm 0.0032 , \tag{50}
\]
where \( m_c = 1.3 \text{ GeV} \). This is consistent with the experimental figure \(|V_{ub}|/|V_{cb}| = 0.08 \pm 0.02 \) [3].

Now, with the experimental value \(|V_{us}| = 0.2196 \pm 0.0023 \) [3] as another input, we can calculate from the first Eq. (43) the phase difference \( \varphi^{(u)} - \varphi^{(d)} \). In fact, taking the absolute value of this equation, we get

\[
\cos \left( \varphi^{(u)} - \varphi^{(d)} \right) = \frac{1}{8} \frac{m_c}{m_s} \left[ 1 + 16 \left( \frac{m_s}{m_c} \right)^2 - \frac{841}{4} \left( \frac{m_c}{\alpha^{(d)}} \right)^2 |V_{us}|^2 \right] = -0.0301 \tag{51}
\]
with \( m_c = 1.3 \text{ GeV} \) and \( m_s = 120 \text{ MeV} \), if the proportion (47) is taken into account. Here, the central values of \( \alpha^{(d)} \) and \(|V_{us}|\) were used. Hence,

\[
\varphi^{(u)} - \varphi^{(d)} = 91.7^\circ = -88.3^\circ + 180^\circ \tag{52}
\]

so, this phase difference turns out to be near 90°. Then, calculating the argument of the first Eq. (43), we infer that

\[
\tan \left( \arg V_{us} - \varphi^{(d)} \right) = -4 \frac{m_s}{m_c} \frac{\sin \left( \varphi^{(u)} - \varphi^{(d)} \right)}{1 - 4(m_s/m_c) \cos \left( \varphi^{(u)} - \varphi^{(d)} \right)} = -0.365 , \tag{53}
\]

what gives

\[
\arg V_{us} = -20.1^\circ + \varphi^{(d)} . \tag{54}
\]

The results (52) and (54) together with the formula (43) enable us to evaluate the rephasing–invariant CP–violating phases

\[
\arg(V_{us}^*V_{cb}^*V_{ub}) = 20.1^\circ - 88.3^\circ = -68.2^\circ \tag{55}
\]

and

\[
\arg(V_{cd}^*V_{ts}^*V_{td}) = -20.1^\circ , \tag{56}
\]

which turn out to be near to -70° and -20°, respectively (they are invariant under quark rephasing equal for up and down quarks of the same generation). Note that the sum of arguments (55) and (56) is always equal to \( \varphi^{(u)} - \varphi^{(d)} - 180^\circ \). Carrying out quark rephasing (equal for up and down quarks of the same generation), where

\[
\arg V_{us} \rightarrow 0 , \ \arg V_{cb} \rightarrow 0 , \ \arg V_{cd} \rightarrow 180^\circ , \ \arg V_{ts} \rightarrow 180^\circ \tag{57}
\]

and \( \arg V_{ud} , \ \arg V_{cs} , \ \arg V_{tb} \) remain unchanged, we conclude from Eqs. (55) and (56) that

\[
\arg V_{ub} \rightarrow -68.2^\circ , \ \arg V_{td} \rightarrow -20.1^\circ . \tag{58}
\]

The sum of arguments (58) after rephasing (57) is always equal to \( \varphi^{(u)} - \varphi^{(d)} - 180^\circ \).

Thus, in this quark phasing, we predict the following Cabibbo—Kobayashi—Maskawa matrix:
\[
(V_{ij}) = \begin{pmatrix}
0.976 & 0.220 & 0.00297 e^{-i 68.2^\circ} \\
-0.220 & 0.975 & 0.0395 \\
0.00805 e^{-i 20.1^\circ} & -0.0395 & 0.999
\end{pmatrix}.
\] (59)

Here, only \(|V_{us}|\) and \(|V_{cb}|\) [and quark masses \(m_s, m_c, m_b\)] consistent with the mass matrices \((M^{(u)}_{ij})\) and \((M^{(d)}_{ij})\) are our inputs, while all other matrix elements \(V_{ij}\), partly induced by unitarity, are evaluated from the relations derived in this Section from the Hermitian mass matrices \((M^{(u)}_{ij})\) and \((M^{(d)}_{ij})\) [and the conjectured proportion (47)]. The independent predictions are \(|V_{ab}|\) and \(\arg V_{ab}\). In Eq. (59), the small phases arising from Eqs. (45), \(\arg V_{ud} = 0.9^\circ\) and \(\arg V_{cs} = -0.9^\circ\), are neglected (here, \(\arg (V_{ud}V_{cs}V_{tb}) = 0\).

The above prediction of \(V_{ij}\) implies the following values of Wolfenstein parameters [3]:

\[\lambda = 0.2196 \ , \ A = 0.819 \ , \ \rho = 0.127 \ , \ \eta = 0.319\] (60)

and of unitary–triangle angles:

\[\gamma = \arctan \frac{\eta}{\rho} = -\arg V_{ab} = 68.2^\circ \ , \ \beta = \arctan \frac{\eta}{1 - \rho} = -\arg V_{td} = 20.1^\circ.\] (61)

The predicted large value of \(\gamma\) follows the present experimental tendency.

If instead of the central value \(|V_{us}| = 0.2196\) we take as the input the range \(|V_{us}| = 0.2173\) to 0.2219, we obtain from Eq. (51) \(\varphi^{(u)} - \varphi^{(d)} = 89.8^\circ\) to 93.6\(^\circ\) (with \(|V_{cb}| = 0.0395\) giving \(\alpha^{(d)} = 355\) MeV), what implies through Eq. (53) that \(\arg V_{us} - \varphi^{(d)} = -20.3^\circ\) to \(-19.8^\circ\).

Then, after rephasing (57), \(\arg V_{ub} = -69.9^\circ\) to \(-66.6^\circ\) and \(\arg V_{td} = -20.3^\circ\) to \(-19.8^\circ\). In this case, the Wolfenstein parameters are \(\lambda = 0.2173\) to 0.2219, \(A = 0.837\) to 0.802, \(\rho = 0.119\) to 0.135 and \(\eta = 0.325\) to 0.312 (here, \(\lambda\sqrt{\rho^2 + \eta^2} = |V_{ub}|/|V_{cb}| = 0.0753\) is fixed). Thus, \(\gamma = -\arg V_{ub} = 69.9^\circ\) to 66.6\(^\circ\) and \(\beta = -\arg V_{td} = 20.3^\circ\) to 19.8\(^\circ\).

In contrast, if the central value \(|V_{cd}| = 0.0395\) (giving \(\alpha^{(d)} = 355\) MeV) is replaced by the input of the range \(V_{cd} = 0.0378\) to 0.0412 (corresponding to \(\alpha^{(d)} = 340\) to 370 MeV), we calculate from Eq. (51) that \(\varphi^{(u)} - \varphi^{(d)} = 97.3^\circ\) to 84.9\(^\circ\) (with \(|V_{us}| = 0.2196\), what leads to \(\arg V_{us} - \varphi^{(d)} = -19.3^\circ\) to \(-20.9^\circ\). Hence, after rephasing (57), \(\arg V_{ub} = -63.4^\circ\) to \(-74.6^\circ\) and \(\arg V_{td} = -19.3^\circ\) to \(-20.9^\circ\). In this case, the Wolfenstein parameters take the values \(\lambda = 0.2196, A = 0.784\) to 0.854, \(\rho = 0.149\) to 0.0951 and \(\eta = 0.298\) to 0.345. Thus, \(\gamma = -\arg V_{ub} = 63.4^\circ\) to \(74.6^\circ\) and \(\beta = -\arg V_{td} = 19.3^\circ\) to \(20.9^\circ\). Here, \(|V_{ub}| = 0.00273\) to 0.00323 and \(|V_{td}| = 0.00738\) to 0.00874.

Eventually, we may turn back to quark masses. From the third Eq. (35) we can evaluate
\[ C^{(u,d)} = 29 \frac{25}{24} m_{t,b} - 624 - \varepsilon^{(u,d)} - 29 \frac{25}{24} B^{(u,d)} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2, \quad (62) \]

what, in the framework of our perturbative approach, gives

\[ C^{(u,d)} = C^{(u,d)} + 29 \frac{25}{24} m_{t,b} - 624 - \varepsilon^{(u,d)} - 29 \frac{25}{24} B^{(u,d)} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 \]

\[ - \frac{29}{\mu^{(u,d)}} \left( \frac{\phi^{(u,d)}}{\mu^{(u,d)}} \right) \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2, \quad (63) \]

where

\[ \frac{\phi^{(u,d)}}{\mu^{(u,d)}} = \frac{29}{\phi^{(u,d)}} \frac{25}{24} m_{t,b} - 624 = \frac{4339 - 9 \phi^{(u,d)}}{733.2} = \{ 4340, 733 \}. \quad (64) \]

With the central values of \( \alpha^{(u)} \) and \( \alpha^{(d)} \) as estimated in Eqs. (46) and (49) we find from Eqs. (41)

\[ A \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 = \{ 7.39, 5.26 \} \text{ MeV}, \quad B \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 = \{ 2.66, 6.88 \} \text{ MeV}, \quad (65) \]

where

\[ \frac{\phi^{(u,d)}}{\mu^{(u,d)}} = \frac{29}{\phi^{(u,d)}} \left( \frac{\alpha^{(u,d)}}{\mu^{(u,d)}} \right)^2 = \{ 65.6, 45.4 \} \text{ MeV}. \quad (66) \]

We calculate from Eqs. (63) with the use of values (65) that

\[ C^{(u,d)} = \{ 4339 + 5.25, 733.2 - 49.5 \} = \{ 4344, 683.7 \} = \{ 4340, 684 \}. \quad (67) \]

Similarly, from the second Eq. (39), making use of the values (65), we obtain

\[ \mu^{(u,d)} = \{ 1060 - 1.18, 95.7 + 3.23 \} \text{ MeV} = \{ 1059, 98.9 \} \text{ MeV} = \{ 1060, 98.9 \} \text{ MeV}. \quad (68) \]

We can easily check that, with the values (40) for \( \phi^{(u,d)} \) and \( \varepsilon^{(u,d)} \) and the value (64) for \( C^{(u,d)} \) determined as above from quark masses, the unperturbed parts of mass formulae (35) reproduce correctly these masses. In fact,
The same is true for the unperturbed part of the first correlating formula (39). The — here omitted — corrections to Eqs. (69), arising from all perturbing terms in the mass formulae (35) (including the corrections from $\delta \mu^{(u,d)}$, $\delta \varepsilon^{(u,d)}$ and $\delta C^{(u,d)}$), are relatively small, viz.

$$\delta m_{u,d} = \left\{ \begin{array}{c} 3.3 \\ 6 \end{array} \right\} \text{MeV}, \quad \delta m_{c,s} = \left\{ \begin{array}{c} 1300 \\ 120 \end{array} \right\} \text{MeV}, \quad \delta m_{t,b} = \left\{ \begin{array}{c} 174 \\ 4.3 \end{array} \right\} \text{GeV}. \quad (69)$$

respectively.

For instance, the conjecture that the phase difference $\varphi^{(u)} - \varphi^{(d)}$ is maximal,

$$\varphi^{(u)} - \varphi^{(d)} = 90^\circ, \quad (71)$$

leads through the first equality in Eq. (51) to the condition

$$1 + 16 \left( \frac{m_s}{m_c} \right)^2 - \frac{841}{4} \left( \frac{m_s}{\alpha^{(d)}} \right)^2 |V_{us}|^2 = 0 \quad (72)$$

predicting for s quark the mass

$$m_s = 118.7 \text{MeV} = 119 \text{MeV} \quad (73)$$

Note that a conjecture about $C^{(u)}$ and $C^{(d)}$ might lead to a prediction for quark masses and so, introduce changes in the "experimental" quark masses (37) and (38) accepted here. The same is true for a conjecture about $\varphi^{(u)}$ and $\varphi^{(d)}$.
\( \alpha^{(d)} = 355 \text{ MeV} \), being only slightly lower than the value 120 MeV used previously. Here, \( m_c \) and \( m_b \) are kept equal to 1.3 and 4.3 GeV, respectively (also masses of \( u, d \) and \( t \) quarks are not changed, while \( \bar{\mu}^{(d)}, \bar{\varepsilon}^{(d)} \) and \( \bar{C}^{(d)} \) change slightly). Then, from the first equality in Eq. (53)

\[
\tan \left( \arg V_{us} - \varphi^{(d)} \right) = -4 \frac{m_s}{m_c} = -0.365 \quad \text{arg} \ V_{us} = -20.1^\circ + \varphi^{(d)} .
\]

After rephasing (57), this gives

\[
\text{arg} \ V_{ub} + \text{arg} \ V_{td} = \varphi^{(u)} - \varphi^{(d)} - 180^\circ = -90^\circ ,
\]

i.e., practically \(-70^\circ \) and \(-20^\circ \). All \(|V_{ij}|\) remain unchanged (with our inputs of \(|V_{us}| = 0.2196 \) and \(|V_{cb}| = 0.0395 \), except for \(|V_{td}|\) which changes slightly, becoming

\[
|V_{td}| = 0.00814 .
\]

Thus, in the Cabibbo—Kobayashi—Maskawa matrix predicted in Eq. (59), only \(|V_{td}|\) and the phases (75) show some changes. The Wolfenstein parameters are

\[
\rho = 0.118 \quad , \quad \eta = 0.322
\]

and \( \lambda \) and \( A \) unchanged (here, the sum \( \rho^2 + \eta^2 = 0.118 \) is also unchanged). Hence, \( \gamma + \beta = 90^\circ \) and \( \alpha = 180^\circ - \gamma - \beta = 90^\circ \), where

\[
\gamma = \arctan \frac{\eta}{\rho} = -\text{arg} \ V_{ub} = 69.9^\circ \quad , \quad \beta = \arctan \frac{\eta}{1 - \rho} = -\text{arg} \ V_{td} = 20.1^\circ .
\]

So, in the case of conjecture (71), the new restrictive relation

\[
\frac{\eta}{\rho} = \frac{1 - \rho}{\eta} \quad , \quad \rho^2 + \eta^2 = \rho
\]

holds, implying the prediction

\[
|V_{td}|/|V_{ub}| = \sqrt{\frac{(1 - \rho)^2 + \eta^2}{\rho^2 + \eta^2}} = \frac{\eta}{\rho} = 2.74 ,
\]

due to the definition of \( \rho \) and \( \eta \) from \( V_{ub} \) and \( V_{td} \). It is in agreement with our figures for \(|V_{td}|\) and \(|V_{ub}|\). Then, the new relationship
\[
\frac{1}{4} \frac{m_c}{m_s} = \frac{\alpha^{(d)} m_c}{\alpha^{(u)} m_s} = \frac{\eta}{\rho}
\]

follows for quark masses \( m_c, m_s \) and Wolfenstein parameters \( \rho, \eta \), in consequence of Eqs. (43) and the conjectured proportion (47). Both its sides are really equal for our values of \( m_c, m_s \) and \( \rho, \eta \).

Thus, summarizing, we cannot predict quark masses without an additional knowledge or conjecture about the constants \( \mu^{(u,d)}, \varepsilon^{(u,d)}, C^{(u,d)}, \alpha^{(u,d)} \) and \( \varphi^{(u,d)} \) (in particular, the conjecture (71) predicting \( m_s \) may be natural). However, we always describe them correctly. If we describe them jointly with quark mixing parameters, we obtain two independent predictions of \( |V_{ub}| \) and \( \gamma - \arg V_{ub} \): the whole Cabibbo—Kobayashi—Maskawa matrix is calculated from the inputs of \( |V_{us}| \) and of \( |V_{ub}| \) [and of quark masses \( m_s, m_c \) and \( m_b \) consistent with the mass matrices \( (M_{ij}^{(u)}) \) and \( (M_{ij}^{(d)}) \)].

Concluding this Section, we can claim that our leptonic form of mass matrix works also in a promising way for up and down quarks. But, it turns out that, in the framework of the leptonic form of mass matrix, the heaviest quarks, \( t \) and \( b \), require an additional mechanism in order to produce the bulk of their masses (here, it is represented by the large constants \( C^{(u)} \) and \( C^{(d)} \)). Such a mechanism, however, intervenes into the process of quark mixing only through quark masses, practically \( m_t \) and \( m_b \), and so, it does not modify for quarks the leptonic form of mixing mechanism.
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