STOCHASTIC GRADIENT DESCENT FOR BARYCENTERS IN WASSERSTEIN SPACE

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ABSTRACT. We present and study a novel algorithm for the computation of 2-Wasserstein population barycenters of absolutely continuous probability measures on Euclidean space. The proposed method can be seen as a stochastic gradient descent procedure in the 2-Wasserstein space, as well as a manifestation of a Law of Large Numbers therein. The algorithm aims at finding a Karcher mean or critical point in this setting, and can be implemented "online", sequentially using i.i.d. random measures sampled from the population law. We provide natural sufficient conditions for this algorithm to a.s. converge in the Wasserstein space towards the population barycenter, and we introduce a novel, general condition which ensures uniqueness of Karcher means and moreover allows us to obtain explicit, parametric convergence rates for the expected optimality gap. We furthermore study the mini-batch version of this algorithm, and discuss examples of families of population laws to which our method and results can be applied. This work expands and deepens ideas and results introduced in an early version of [9], in which a statistical application (and numerical implementation) of this method is developed in the context of Bayesian learning.

Keywords: Wasserstein distance, Wasserstein barycenter, Fréchet means, Karcher means, gradient descent, stochastic gradient descent.

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1. Introduction

Let \( \mathcal{P}(X) \) denote the space of Borel probability measures over a Polish space \( X \). Given \( \mu, \nu \in \mathcal{P}(X) \), denote by

\[
\Gamma(\mu, \nu) := \{ \gamma \in \mathcal{P}(X \times X) : \gamma(dx, X) = \mu(dx), \gamma(X, dy) = \nu(dy) \},
\]

the set of couplings (transport plans) with marginals \( \mu \) and \( \nu \). For a fixed, compatible complete metric \( d \), and given a real number \( p \geq 1 \), we define the \( p \)-Wasserstein space by

\[
W_p(X) := \{ \eta \in \mathcal{P}(X) : \int_X d(x_0, x)^p \eta(dx) < \infty, \text{ some } x_0 \}.
\]

Accordingly, the \( p \)-Wasserstein distance between measures \( \mu, \nu \in W_p(X) \) is given by

\[
W_p(\mu, \nu) = \left( \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{X \times X} d(x, y)^p \gamma(dx, dy) \right)^{1/p}.
\]

When \( p = 2 \), \( X = \mathbb{R}^q \), \( d \) is the Euclidean distance and \( \mu \) is absolutely continuous, which is the setting which we will soon adopt for the remainder of the paper, Brenier’s theorem [51, Theorem 2.12(ii)] establishes the uniqueness of a minimizer for the r.h.s. of eq. (1.1). Furthermore, this optimizer is supported on the graph of the gradient of a convex function. See [6] and [52] for further general background on optimal transport.

We recall now the definition of the Wasserstein population barycenter:

**Definition 1.1.** Given \( \Pi \in \mathcal{P}(\mathcal{P}(X)) \), denote

\[
V_p(\hat{m}) := \int_{\mathcal{P}(X)} W_p(m, \hat{m})^p \Pi(dm).
\]

Any measure \( \hat{m} \in W_p(X) \) which is a minimizer of the problem

\[
\inf_{\hat{m} \in W_p(X)} V_p(\hat{m}),
\]

is called a \( p \)-Wasserstein population barycenter of \( \Pi \).

Wasserstein barycenters were first introduced and analyzed by Carlier and Agueh in [1], in the case when the support of \( \Pi \in \mathcal{P}(\mathcal{P}(X)) \) is finite and \( X = \mathbb{R}^q \). More generally, Bigot and Klein [12] and Le Gouic and Loubes [39] considered the so-called population barycenter, i.e., the general case in Definition 1.1 where \( \Pi \) may have infinite support. These works addressed among others, the basic questions of existence and uniqueness of solutions. See also Kim and Pass [36] for the Riemannian case. The concept of Wasserstein
barycenter has been extensively studied from both theoretical and practical perspectives during the last decade: we refer the readers to Panaretos and Zemel’s overview [44] for statistical applications and to the works of Cuturi, Peyré and coauthors [55, 28, 27, 18] for computational aspects of optimal transport and applications in machine learning.

In this article we develop a stochastic gradient descent (SGD) algorithm for the computation of 2-Wasserstein population barycenters. The method inherits features of the SGD rationale, in particular:

- It exhibits a reduced computational cost compared to methods based on the direct formulation of the barycenter problem using all the available data, as SGD only considers a limited number of samples of \( \Pi \in \mathcal{P}(\mathcal{P}(\mathcal{X})) \) per iteration.
- It is online (or continual, as referred to in the machine learning community), meaning that it can incorporate additional datapoints sequentially to update the barycenter estimate, whenever new observations becomes available. This is relevant even when \( \Pi \) is finitely supported.
- Conditions ensuring convergence towards the Wasserstein barycenter as well as finite dimensional convergence rates for the algorithm can be provided. Moreover, the variance of the gradient estimators can be reduced by using mini-batches.

From now on we assume that

(A1) \( \mathcal{X} = \mathbb{R}^d \), \( d \) is the squared Euclidean metric and \( p = 2 \).

We denote by \( \mathcal{W}_{2,ac}(\mathcal{X}) \) the subspace of \( \mathcal{W}_2(\mathcal{X}) \) of absolutely continuous measures with finite second moment, and for any \( \mu \in \mathcal{W}_{2,ac}(\mathcal{X}) \) we write \( T^\mu_\nu \) for the (\( \mu\)-a.s. unique) gradient of a convex function such that \( T^\mu_\nu(\mu) = \nu \).

Recall that a set \( B \subseteq \mathcal{W}_{2,ac}(\mathcal{X}) \) is geodesically convex, if for every \( \mu, \nu \in B \) and \( t \in [0, 1] \) we have \( ((1 - t)I + tT^\mu_\nu)(\mu) \in B \), with \( I \) denoting the identity operator. We will also assume the following condition for most of the article:

(A2) \( \Pi \) gives full measure to a geodesically convex \( \mathcal{W}_2 \)-compact set \( K_\Pi \subseteq \mathcal{W}_{2,ac}(\mathcal{X}) \).

In particular, under (A2) we have \( \int_{\mathcal{P}(\mathcal{X})} \int_{\mathcal{X}} d(x, x_0)^2 m(dx) \Pi(dm) < \infty \) for all \( x_0 \). Moreover, for each \( \nu \in \mathcal{W}_2(\mathcal{X}) \) and \( \Pi(dm) \) a.e. \( m \), there is a unique optimal transport map \( T^\mu_\nu \) from \( m \) to \( \nu \) and, by [39, Proposition 6], the 2-Wasserstein population barycenter is unique.

**Definition 1.2.** Let \( \mu_0 \in K_\Pi \), \( m_k \overset{\text{id}}{\sim} \Pi \), and \( \gamma_k > 0 \) for \( k \geq 0 \). We define the stochastic gradient descent (SGD) sequence by

\[
\mu_{k+1} := \left(1 - \gamma_k I + \gamma_k T^\mu_{T^m_\nu} \right)(\mu_k), \quad \text{for } k \geq 0.
\]

(1.2)

The reasons as to why we can truthfully refer to the above sequence as stochastic gradient descent will become apparent in Sections 2 and 3. We stress that the sequence is a.s. well-defined, as one can show by induction that \( \mu_k \in \mathcal{W}_{2,ac}(\mathcal{X}) \) a.s. thanks to Assumption (A2). We also refer to Section 3 for remarks on the measurability of the random maps \( T^\mu_{T^m_\nu} \) and sequence \( \{\mu_k\}_k \).

Throughout the article, we assume the following conditions on the steps \( \gamma_k \) in eq. (1.2), commonly required for the convergence of SGD methods:

\[
\sum_{k=0}^{\infty} \gamma_k^2 < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} \gamma_k = \infty.
\]

(1.3)

(1.4)

In addition to barycenters, we will need the concept of Karcher means (c.f. [56]), which, in the setting of the optimization problem in \( \mathcal{W}_{2,ac}(\mathcal{X}) \) considered here, can be intuitively understood as an analogue of a critical point of a smooth function in Euclidean space (see the discussion in Section 4).

**Definition 1.3.** Given \( \Pi \in \mathcal{P}(\mathcal{P}(\mathcal{X})) \), we say that \( \mu \in \mathcal{W}_{2,ac}(\mathcal{X}) \) is a Karcher mean of \( \Pi \) if

\[
\mu \left( \{ x : x = \int_{\Pi} T^0_\nu(x) \Pi(dm) \} \right) = 1.
\]

It is known that any 2-Wasserstein barycenter is a Karcher mean, though the latter is in general a strictly larger class, see [4] or Example 2.2 below. However, if there is a unique Karcher mean, then it must coincide with the unique barycenter. We can now state the main result of the article.

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1 Notice that \( x \mapsto T^\mu_\nu(x) \) is a \( \mu\)-a.s. defined map and that we use throughout the notation \( T^\mu_\nu(\mu) \) for the image measure/law of this map under the measure \( \mu \).
Theorem 1.4. Assume (A1), (A2), conditions (1.3) and (1.4), and that the 2-Wasserstein barycenter \( \bar{\mu} \) of \( \Pi \) is the unique Karcher mean. Then, the SGD sequence \( \{\mu_k\} \) in eq. (1.2) is a.s. \( W_2 \)-convergent to \( \bar{\mu} \in K_1 \).

An interesting aspect of Theorem 1.4 is that it hints at a Law of Large Numbers (LLN) on the 2-Wasserstein space. Indeed, in the conventional LLN, for iid samples \( X_i \) the summation \( S_k := \frac{1}{k} \sum_{i \leq k} X_i \) can be expressed as

\[
S_{k+1} = \frac{1}{k+1} X_{k+1} + \left( 1 - \frac{1}{k+1} \right) S_k.
\]

Therefore, if we rather think of sample \( X_k \) as a measure \( m_k \) and of \( S_k \) as \( \mu_k \), we immediately see the connection with

\[
\mu_{k+1} = \left( \frac{1}{k+1} \right) T_{\mu_k}^m + \left( 1 - \frac{1}{k+1} \right) \left( \frac{1}{\gamma_k} \right) \mu_k,
\]

obtained from eq. (1.2) when we take \( \gamma_k = \frac{1}{k+1} \). The convergence in Theorem 1.4 can thus be interpreted as an analogy to the convergence of \( S_k \) to the mean of \( X_i \) (we thank Stefan Schrott for this observation).

In order to state our second main result, Theorem 1.5, we introduce a new concept:

Definition 1.5. Given \( \Pi \in \mathcal{P}(\mathcal{P}(X)) \) we say that a Karcher mean \( \mu \in W_{2,ac}(X) \) of \( \Pi \) is pseudo-associative if there exists \( C_\mu > 0 \) such that for all \( \nu \in W_{2,ac}(X) \), one has

\[
W_2^2(\mu, \nu) \leq C_\mu \int_{\mathcal{X}} \left| \mathcal{P}(X) \left( T_{\nu}^m(x) - x \right) \right| \Pi(dm) \nu(dx).
\]

Since term on the r.h.s. of (1.5) vanishes for any Karcher mean \( \nu \), existence of a pseudo-associative Karcher mean implies \( \mu = \nu \), hence uniqueness of Karcher means. We will see that moreover the existence of a pseudo-associative Karcher mean implies a Polyak-Lojasiewicz inequality for the functional minimized by the barycenter, see (1.3). This in turn can be utilized to obtain convergence rates for the expected optimality gap in a similar way as in the Euclidean case. While the pseudo-associativity condition is a strong requirement, in the following result we are able to weaken Assumption (A2) into the milder:

(A2') \( \Pi \) gives full measure to a geodesically convex set \( K_1 \subseteq W_{2,ac}(X) \) which is \( W_2 \)-bounded (i.e. \( \sup_{m \in K_1} \int |x|^2 m(dx) < \infty \)).

Clearly this condition is equivalent to requiring that the support of \( \Pi \) be \( W_2 \)-bounded and consist of absolutely continuous measures. Our second main result is:

Theorem 1.6. Assume (A1), (A2') and that the 2-Wasserstein barycenter \( \bar{\mu} \) of \( \Pi \) is a pseudo-associative Karcher mean. Then, \( \bar{\mu} \) is the unique barycenter of \( \Pi \), and the SGD sequence \( \{\mu_k\} \) in eq. (1.2) is a.s. weakly convergent to \( \bar{\mu} \in K_1 \) as soon as (1.3) and (1.4) hold. Moreover, for every \( a > C_\mu^{-1} \) and \( b \geq a \) there exists an explicit constant \( C_{a,b} > 0 \) such that, if \( \gamma_k = \frac{a}{b+k} \) for all \( k \in \mathbb{N} \), the expected optimality gap satisfies:

\[
\mathbb{E} \left[ F(\mu_k) - F(\bar{\mu}) \right] \leq C_{a,b}.
\]

Let us explain the reason for the terminology “pseudo-associative” we have chosen. This comes from the fact that the inequality in Definition 1.5 holds true (with equality and \( C_\mu = 1 \)) as soon as the following associativity property holds:

\[
T_{\mu_k}^m \circ T_{\nu}^m(x) = T_{\nu}^m \circ T_{\mu_k}^m(x)
\]

\( \mu(dx) \) a.s., for each pair \( \nu, m \in W_{2,ac}(X) \), see Remark 4.1 below. The previous identity was assumed to hold true for the results proved in [12], and is always valid in \( \mathbb{R} \), since the composition of monotone functions is monotone. Further examples where the associativity property holds are discussed in Section 6. Thus, in all those settings, (1.5) and Theorem 1.6 hold as soon as assumption (A2’) is granted, and we further have the explicit LLN-like expression \( \mu_{k+1} = \left( \frac{1}{k+1} \right) \sum_{i=1}^k T_{\mu_i}^m \bar{\mu}_0 \). As regards the more general pseudo-associativity property, we will see that it holds for instance in the Gaussian framework studied in [20], and in certain classes of scatter-location families, which we discuss in Section 6.

The paper is organized as follows:

- In Section 2 we recall basic ideas and results on gradient descent algorithms for Wasserstein barycenters.
- Section 3 we prove Theorem 1.4 previously providing technical elements required to that end.
- In Section 4 we discuss the notion of pseudo-associative Karcher means and its relations to the so-called Polyak-Lojasiewicz and variance inequalities, and we prove Theorem 1.6.
- In Section 5 we introduce the mini-batch version of our algorithm, discuss how this improves the variance of gradient-type estimators, and state extensions of our previous results to that setting.
• In Section 6 we consider closed-form examples and explain how in these cases the existence of pseudo-associative Karcher means required in Theorem 1.6 can be guaranteed. We also explore certain properties of probability distributions which are “stable” under the operation of taking their barycenters.

In the remainder of this introduction, we provide independent discussions on various aspects of our results (some of them suggested by the referees) and on related literature.

On the assumptions. Although Assumption (A2) might appear as strong at first sight, it can be guaranteed in suitable parametric situations (e.g., Gaussian, or the scatter-location setting recalled in Section 5.3) or, more generally, under moment and density constraints on the measures in \( K_{11} \). For instance if \( \Pi \) is supported on a finite set of measures with finite Boltzmann entropy, then Assumption (A2) is guaranteed. More generally, if the support of \( \Pi \) is 2-Wasserstein compact set and the Boltzmann entropy is uniformly bounded on it, then Assumption (A2) is fulfilled too: see Lemma 3.5 below.

Conditions (A2) and of uniqueness of a Karcher mean in Theorem 1.4 are natural substitutes of, respectively, the compactness of SGD sequences and the uniqueness of critical points, which hold true under usual sets of assumptions ensuring the convergence of SGD in Euclidean space to a minimizer, e.g. some growth control and certain strict convexity-type condition at the minimum. The usual reasonings underlying the convergence analysis of stochastic gradient descent in Euclidean spaces seem however not applicable in our context, as the functional \( W_2(\mu) \equiv \mu \mapsto F(\mu) := \int W_2(m, \mu)^2 \Pi(\text{d}m) \) is not convex, in fact not even \( \alpha \)-convex for some \( \alpha \in \mathbb{R} \), when \( W_2(\cdot, \cdot) \) is endowed with its almost Riemannian structure induced by optimal transport (see [6, Ch.7.2]). The function \( F \) is neither \( \alpha \)-convex for some \( \alpha \in \mathbb{R} \), when we use generalized geodesics (see [6, Ch.9.2]). In fact, SGD in finite-dimensional Riemannian manifolds could provide a more suitable framework to draw inspiration from, see e.g. [13]. Ideas useful in that setting seem not straightforward to leverage since that work either assumes negative curvature (while \( W_2(\cdot, \cdot) \) is positively curved), or that the functional to be minimized be rather smooth and have bounded derivatives of first and second order.

The assumption that \( K_{11} \) is contained in a subset of \( W_2(\mathcal{X}) \) of absolutely continuous probability measures, and hence the existence of optimal transport maps between elements of \( K_{11} \), appears as a more structural requirement, as it is needed to construct the iterations (1.2). Indeed, by dealing with an extended notion of Karcher mean, it is in principle also possible to define an analogous iterative scheme in a general setting including in particular the case of discrete laws, and thus to relax to some extent the absolute continuity requirement. However, this introduces additional technicalities, and we unfortunately were not able to provide conditions ensuring the convergence of the method in reasonably general situations. For completeness of the discussion, we sketch the main ideas of this possible extension in the Appendix.

Uniqueness of Karcher means. Regarding situations where uniqueness of Karcher means can be granted, we refer to [56, 44, 12] for sufficient conditions when the support of \( \Pi \) is finite, based on the regularity theory of optimal transport, and to [12] for the case of an infinite support, under rather strong assumptions. We remark also that in one dimension, the uniqueness of Karcher means is known to hold without further assumptions. A first counterexample where this uniqueness is not guaranteed is given in [2], see also the simplified Example 2.2 below. A general understanding of the uniqueness of Karcher means remains however an open and interesting challenge. This is not only relevant for the present work, but also for the (non-stochastic) gradient descent method of Zemel and Panaretos [56] and the fixed-point iterations of Álvarez-Esteban, del Barrio, Cuesta- Albertos and Matrán [4]. The notion of pseudo-associativity introduced in Definition 1.5 provides an alternative viewpoint on this question, complementary to the aforementioned ones, which might deserve being further explored.

Gradient descents in Wasserstein space. Gradient descent (GD) in Wasserstein space was introduced as a method to compute barycenters in the mentioned works [4, 56]. The SGD method we develop here was introduced in an early version of the work [9] (available on arXiv:1805.10833), as a way to compute Bayesian estimators based on 2-Wasserstein barycenters. More recently, Chewi et al. [20] obtained convergence rates for the expected optimality gap for these GD and SGD methods in the case of Gaussian families of distributions with uniformly bounded, uniformly positive definite covariance matrices. To that end they relied on proving a Polyak-Lojasiewicz inequality, and also derived quantitative convergence.

2 In view of the independent, theoretical interest of this SGD method, we decided to separately present this algorithm here, along with a deeper analysis and more complete results on it, and devote [9] exclusively to its statistical application and implementation.
bounds in $W_2$ for the SDG sequence, relying on a variance inequality, which they showed to hold under general though strong conditions on the dual potential of the barycenter problem (verified under their assumptions). We refer to the recent work [17] for more on the variance inequality.

The Riemannian-like structure of the Wasserstein space and its associated gradient have been utilized in various other ways with statistical or machine learning motivations in recent years; see e.g. [29] for particle-like approximations, [35] for a sequential first order method, and [40] [19] for information geometry perspectives.

**Computational aspects.** Implementing our SGD algorithm requires, in general, computing or approximating optimal transport maps between two given absolutely-continuous distributions (some exceptions where explicit closed form maps are available are given in Section 6). During the last two decades, considerable progress has been made on the numerical resolution of the latter problem through PDE methods [41] [13] [7] [14] and, more recently, through entropic regularization approaches [26] [50] [28] [45] [42] [30] crucially relying on the Sinkhorn algorithm [48] [49], which significantly speeds up the approximate solution of the problem. It is also possible to approximate optimal transport maps via estimators built from samples (based on the Sinkhorn algorithm, plug-in estimators or using stochastic approximations) [34] [11] [46] [29] [43]. We also refer to [38] for an overview and comparison of sample-free methods for continuous distributions (for instance based on neural networks).

**Applications.** The proposed method to compute Wasserstein barycenters is well suited to situations where a population law $\Pi$ on infinitely many probability measures is considered. The Bayesian setting addressed in [9] provides a good example of such situation, i.e. where one needs to compute the Wasserstein barycenter of a prior / posterior distribution $\Pi$ on models which has a possibly infinite support.

Further instances of population laws $\Pi$ with infinite support arise in the context of Bayesian deep learning [55], in which a neural network’s (NN) weights are sampled from a given law (e.g., Gaussian, uniform, or Laplace); in the case these NNs parametrize probability distributions, the collection of resulting probability laws is distributed according to a law $\Pi$ with possibly infinite support. Another example is Variational AutoEncoders [37], in which case one models the parameters of a law by a simple random variable (usually Gaussian) which is then passed to a decoder NN. In both cases, the support of $\Pi$ is infinite naturally. Furthermore, sampling from $\Pi$ is straightforward in these cases, which eases the implementation of the SGD algorithm.

**Possible extensions.** It is in principle possible to define and study stochastic gradient descent methods similar to (1.2) for the minimization on the space of measures of other functionals than the barycentric objective function, or with respect to other geometries than the $2$-Wasserstein one. For instance, the recent paper [18] introduces the notion of barycenters of probability measures based on weak optimal transport [33] [8] [32] and extends the ideas and algorithm developed here to that setting. A further, natural example of functionals to consider are the entropy-regularized barycenters, dealt with for numerical purposes in some of the aforementioned works and most recently in the articles by Chizat et al. [21] [23].

On a different vein, it would be interesting to study conditions ensuring the convergence of the algorithm (1.2) to stationary points (Karcher means), when the latter is a class that strictly contains the minimum (barycenter). Under suitable conditions, such convergence can be expected to hold by analogy with the behavior of the Euclidean SGD algorithm in general (non necessarily convex) settings [13]. In fact, we believe this question can also be linked to the pseudo-associativity of Karcher means. A deeper study is left for future work.

2. **Gradient descent in Wasserstein space: a review**

We first survey the gradient descent method for the computation of 2-Wasserstein barycenters. This method will serve as a motivation for the subsequent development of the SGD in Section 3. For simplicity, we take $\Pi$ finitely supported. Concretely, we suppose in this section that

$$\Pi = \sum_{i \leq L} \lambda_i \delta_{m_i}, \text{ with } L \in \mathbb{N}, \lambda_i \geq 0 \text{ and } \sum_{i \leq L} \lambda_i = 1.$$  

We define the operator over absolutely continuous measures

$$G(m) := \left( \sum_{i=1}^{L} \lambda_i T_{m_i}^m \right)(m).$$

Notice that the fixed-points of $G$ are precisely the Karcher means of $\Pi$ presented in the introduction. Thanks to [24] the operator $G$ is continuous for the $W_2$ distance. Also, if at least one of the $L$ measures $m_i$ has a
bounded density, then the unique Wasserstein barycenter $\hat{m}$ of $\Pi$ has a bounded density as well and satisfies $G(\hat{m}) = \hat{m}$. This suggests to define, starting from $\mu_0$, the sequence

$$\mu_{n+1} := G(\mu_n), \quad n \geq 0. \tag{2.2}$$

The next result was proven by Álvarez-Esteban, Barrio, Cuesta-Albertos, Matrán in [4, Theorem 3.6] and independently by Zemel and Panaretos in [56, Theorem 3, Corollary 2]:

**Proposition 2.1.** The sequence $\{\mu_n\}_{n \geq 0}$ in eq. (2.2) is tight and every weakly convergent subsequence of $\{\mu_n\}_{n \geq 0}$ converges in $W_2$ to an absolutely continuous measure in $W_2(\mathbb{R}^q)$ which is also a Karcher mean. If some $m_n$ has a bounded density, and if there exists a unique Karcher mean, then $\hat{m}$ is the Wasserstein barycenter of $\Pi$ and we have that $W_2(\mu_0, \hat{m}) \to 0$.

For the reader’s convenience, we present next a counterexample to the uniqueness of Karcher means:

**Example 2.2.** In $\mathbb{R}^2$ we take $\mu_1$ as the uniform measure on $B((-1, M), \varepsilon) \cup B((1, -M), \varepsilon)$ and $\mu_2$ the uniform measure on $B((-1, -M), \varepsilon) \cup B((1, M), \varepsilon)$, with $\varepsilon$ a small radius and $M >> \varepsilon$. Then, if $\Pi = \frac{1}{2}(\delta_{\mu_1} + \delta_{\mu_2})$, the uniform measure on $B((-1, 0), \varepsilon) \cup B((1, 0), \varepsilon)$ and the uniform measure on $B((0, M), \varepsilon) \cup B((0, -M), \varepsilon)$ are two distinct Karcher means.

Thanks to the Riemannian-like geometry of $W_2(\mathbb{R}^q)$ one can reinterpret the iterations in eq. (2.2) as a gradient descent step. This was discovered by Panaretos and Zemel in [56, 44]. In fact, in [56, Theorem 1] the authors show that the functional

$$F(m) := \frac{1}{2} \sum_{i=1}^{L} A_i W^2_{\mu}(m_i, m), \quad m \in \mathcal{W}_2(\mathbb{R}^q),$$

defined on $\mathcal{W}_2(\mathbb{R}^q)$, has Fréchet derivative at each point $m \in \mathcal{W}_2(\mathbb{R}^q)$, given by

$$F'(m) = -\sum_{i=1}^{L} A_i (T^m_{m_i} - I) = I - \sum_{i=1}^{L} A_i T^m_{m_i} \in L^2(m),$$

where $I$ is the identity map in $\mathbb{R}^q$. More precisely, for such $m$ one has when $W_2(\hat{m}, m)$ goes to zero that

$$\frac{F(\hat{m}) - F(m) - \int_{\mathbb{R}^q} F'(m)(x), T^\hat{m}_{\hat{m}}(x) - x) m(dx)}{W_2(\hat{m}, m)} \to 0,$$

thanks to [6, Corollary 10.2.7]. It follows from Brenier’s theorem [51, Theorem 2.12(ii)] that $\hat{m}$ is a fixed point of $G$ defined in eq. (2.1) if and only if $F'(\hat{m}) = 0$. The gradient descent sequence with step $\gamma$ starting from $\mu_0 \in \mathcal{W}_2(\mathbb{R}^q)$ is then defined by (c.f. [56])

$$\mu_{n+1} := G_\gamma(\mu_n), \quad n \geq 0, \tag{2.6}$$

where

$$G_\gamma(m) := \left[ I + \gamma F'(m) \right](m) = \left[ (1 - \gamma)I + \gamma \sum_{i=1}^{L} A_i T^m_{m_i} \right](m) = \left[ I + \gamma \sum_{i=1}^{L} A_i (T^m_{m_i} - I) \right](m).$$

Note that, by (2.4), the iterations (2.6) truly correspond to a gradient descent in $\mathcal{W}_2(\mathbb{R}^q)$ for the function in eq. (2.3). We remark also that, if $\gamma = 1$, the sequence in eq. (2.6) coincides with that in eq. (2.2), i.e., $G_1 = G$. These ideas serve us as inspiration for the stochastic gradient descent iteration in the next part.

### 3. Stochastic Gradient Descent for Barycenters in Wasserstein space

The method presented in Section 2 is well suited for calculating the empirical barycenter. For the estimation of a population barycenter (i.e. when $\Pi$ does not have finite support) we would need to construct a convergent sequence of empirical barycenters, which can be computationally expensive. Furthermore, if a new sample from $\Pi$ arrives, the previous method would need to recompute the barycenter from scratch. To address these challenges, we follow the ideas of stochastic algorithms [47], widely adopted in machine learning [15], and define a stochastic version of the gradient descent sequence for the barycenter of $\Pi$.

Recall that for $\mu_0 \in K_\mu$ (in particular, $\mu_0$ absolutely continuous), $m_k \overset{iid}{=} \Pi$ defined in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\gamma_k > 0$ for $k \geq 0$, we constructed the stochastic gradient descent (SGD) sequence as

$$\mu_{k+1} := \left( 1 - \gamma_k \right) I + \gamma_k T^m_{m_k} \left( \mu_k \right), \quad \text{for } k \geq 0. \tag{3.1}$$

The key ingredients for the convergence analysis of the above SGD iterations are the functions:

$$F(\mu) := \frac{1}{2} \int_{\mathcal{F}(X)} W^2_2(\mu, m) d\Pi(m), \tag{3.1}$$

$$F'(\mu)(x) := -\int_{\mathcal{F}(X)} (T^m_{\mu} - I) d\Pi(m)(x), \tag{3.2}$$
the natural analogues to the eponymous objects (2.3) and (2.4). In this setting, one can formally (or rigorously, under additional assumptions) check that $F'$ is the actual Frechet derivative of $F$, and this justifies naming $(\mu_k)_k$ a SGD sequence. In the sequel, the notation $F$ and $F'$ always refers to the functions defined in eqs. (3.1) and (3.2), respectively.

Observe that the population barycenter $\hat{\mu}$ is the unique minimizer of $F$. The following lemma justifies that $F'$ is well defined and that $\|F'(\hat{\mu})\|_{L(\nu)} = 0$, and in particular $\hat{\mu}$ is a Karcher mean. This is a generalization of the corresponding result in [9] where only the case $\text{supp}(\Pi) < \infty$ is covered.

**Lemma 3.1.** Let $\tilde{\Pi}$ be a probability measure concentrated on $W_{2,\text{as}}(\mathbb{R}^d)$. There exists a jointly measurable function $\mathcal{W}_{2,\text{as}}(\mathbb{R}^d) \times \mathcal{W}_{2,\text{as}}(\mathbb{R}^d) \ni (\mu, m, x) \mapsto T^m_\mu(x)$ which is $\mu(dx)\Pi(dm)\tilde{\Pi}(d\mu)$-a.s. equal to the unique optimal transport map from $\mu$ to $m$ at $x$. Furthermore, letting $\tilde{\mu}$ be a barycenter of $\Pi$, we have $x = \int x T^m_\mu(x)\Pi(dm), \tilde{\mu}(dx)$-a.s.

**Proof.** The existence of a jointly measurable version of the unique optimal maps is proved in [31]. Let us prove the last assertion. Letting $T^m_\mu = : T^m, \Phi$ we have by Brenier’s theorem [51, Theorem 2.12(ii)] that

$$\int W^2_{2}(\tilde{\mu}, m^2)\Pi(dm) = \int \int |x - T^m(x)|^2\tilde{\mu}(dx)\Pi(dm) = \int \int |x - T^\mu(x)|^2\tilde{\mu}(dx)\Pi(dm) - \int \int |x - T^\mu(x)|^2\tilde{\mu}(dx)\Pi(dm) + \int \int |x - T^\mu(x)|\tilde{\mu}(dx)\Pi(dm)$$

where we used the fact that $2 \int (x - T^\mu(x))\tilde{\mu}(dx), \int T^\mu(x)\Pi(dm) - \int T^\mu(x)\Pi(dm)\tilde{\mu}(dx) = 0$. The second term in the last line is an upper bound for

$$\int W^2_{2}(\mu, m^2)\Pi(dm) \geq \int W^2_{2}(\tilde{\mu}, m^2)\Pi(dm)$$

We conclude that $\int |x - T^\mu(x)|\tilde{\mu}(dx)$ as required.

Notice that Lemma 3.1 ensures that the SGD sequence $(\mu_k)_k$ is well defined as a sequence of (measurable) $\mathcal{W}_{2}$-valued random variables. More precisely, denoting by $\mathcal{F}_0$ the trivial sigma-algebra and $\mathcal{F}_{k+1}, k \geq 0$, the sigma-algebra generated by $m_0, \ldots, m_k$, one can inductively apply the first part of Lemma 3.1 with $\tilde{\Pi} = \text{law}(\mu_0)$ to check that $T^m_\mu(x)$ is measurable with respect to $\mathcal{F}_{k+1} \otimes \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}$ stands for the Borel sigma-field. This implies that both $\mu_k$ and $\|F'(\mu_k)\|_{L^2(\nu)}$ are measurable w.r.t. $\mathcal{F}_k$.

The next proposition suggests that, in expectation, the sequence $(F(\mu_k))_k$ is essentially decreasing. This is a first insight into the behaviour of the sequence $(\mu_k)_k$.

**Proposition 3.2.** The SGD sequence in eq. (1.2) verifies (a.s.) that

$$\mathbb{E}[F(\mu_{k+1}) - F(\mu_k)|\mathcal{F}_k] \leq \gamma_k^2 F(\mu_k) - \gamma_k\|F'(\mu_k)\|_{L^2(\nu)}.$$  

**Proof.** Let $\nu \in \text{supp}(\Pi)$. Clearly $\left(\int (1 - \gamma_k I + \gamma_k T^m_\mu), T^m_\mu, \mu_k\right)$ is a feasible (not necessarily optimal) coupling with first and second marginals $\mu_{k+1}$ and $\nu$ respectively. Denoting $O_m := T^m_\mu - I$, we have

$$W^2_2(\mu_{k+1}, \nu) \leq \|1 - \gamma_k I + \gamma_k T^m_\mu - T^m_\mu\|_2^2 \leq \|O_m\|_{L^2(\mu)}^2$$

Evaluating $\mu_{k+1}$ on the functional $F$ and thanks to the previous inequality, we have

$$F(\mu_{k+1}) = \frac{1}{2} \int W^2_{2}(\mu_{k+1}, \nu)\Pi(dv) \leq \frac{1}{2} \|O_m\|_{L^2(\mu)}^2 \Pi(dv) - \gamma_k \left(\int O_m\Pi(dv), O_m\right)_{L^2(\nu)} + \frac{\gamma_k^2}{2}\|O_m\|_{L^2(\nu)}^2$$

Taking conditional expectation with respect to $\mathcal{F}_k$, and as $m_k$ is independently sampled from this sigma-algebra, we conclude

$$\mathbb{E}[F(\mu_{k+1})|\mathcal{F}_k] \leq F(\mu_k) + \gamma_k \left(\int O_m\Pi(dm), O_m\right)_{L^2(\nu)} + \frac{\gamma_k^2}{2}\int\|O_m\|_{L^2(\nu)}^2\Pi(dm) = (1 + \gamma_k^2) F(\mu_k) - \gamma_k\|F'(\mu_k)\|_{L^2(\nu)}.$$  

□
Next Lemma states key continuity properties of the functions $F$ and $F'$. For this result, assumption (A2) can be dropped and it is only required that $\int \|x\|^2m(dx)\Pi(dm) < \infty$.

**Lemma 3.3.** Let $(\rho_n) \subseteq \mathcal{W}_{2,\infty}(\mathbb{R}^d)$ be a sequence converging w.r.t. $\mathcal{W}_2$ to $\rho \in \mathcal{W}_{2,\infty}(\mathbb{R}^d)$. Then, as $n \to \infty$, we have i): $F(\rho_n) \to F(\rho)$ and ii): $\|F'(\rho_n)\|_{L^2(\rho_n)} \to \|F'(\rho)\|_{L^2(\rho)}$.

**Proof.** We prove both convergence claims using Skorokhod’s representation theorem. Thanks to that result, in a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$ one can simultaneously construct a sequence of random vectors $(X_n)$ of laws $(\rho_n)$, and a random variable $X$ of law $\rho$, such that $(X_n)$ converges $\mathbb{P}$-a.s. to $X$. Moreover, by Theorem 3.4 in [24], the sequence $(T_{\rho_n}^m(X_n))$, converges $\mathbb{P}$-a.s. to $T_{\rho}^m(X)$. Notice that, for all $n \in \mathbb{N}$, $T_{\rho_n}^m(X_n)$ distributes according to the law $m$, and the same holds true for $T_{\rho}^m(X)$.

We now enlarge the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ (maintaining the same notation for simplicity) with an independent random variable $\mathbf{m}$ in $\mathcal{W}_2(\mathbb{R}^d)$ with law $\Pi$ (thus independent of $(X_n)$ and $X$). Applying Lemma 3.1 with $\Pi = \delta_\rho$, for each $n$, or with $\Pi = \delta_\rho$, one can show that $(X_n, T_{\rho_n}^m(X_n))$ and $(X, T_{\rho}^m(X))$ are random variables in $(\Omega, \mathcal{G}, \mathbb{P})$. By conditioning on $\{\mathbf{m} = m\}$ one further obtains that $(X_n, T_{\rho_n}^m(X_n))$ converges $\mathbb{P}$-a.s. to $(X, T_{\rho}^m(X))$.

Notice that sup$_n E\left(\|X_n\|^2 1_{|X_n|^2 \geq M}\right) = \sup_n \int_{|x|^2 \geq M} \|x\|^2 \rho_n(dx) \to 0$ as $M \to \infty$ since $(\rho_n)$ converges in $\mathcal{W}_2(X)$, while, upon conditioning on $\mathbf{m}$, $\sup_n E\left(\|T_{\rho_n}^m(X_n)\|^2 1_{|T_{\rho_n}^m(X_n)|^2 \geq M}\right) = \sup_n \int_{|y|^2 \geq M} \|y\|^2 m(dy) \Pi(dm) \to 0$

by dominated convergence, since $\int_{|x|^2 \geq M} \|x\|^2 m(dx) \leq \int \|x\|^2 m(dx) = W_2^2(m, \delta_\rho)$ and $\Pi$ in $\mathcal{W}_2(\mathbb{W}_2(X))$. By Vitali’s convergence Theorem, we deduce that $(X_n, T_{\rho_n}^m(X_n))$ converges to $(X, T_{\rho}^m(X))$ in $L^2(\Omega, \mathcal{G}, \mathbb{P})$. In particular, as $n \to \infty$ we have $W_2^2(\rho_n, m) \Pi(dm) \to E(X_n - T_{\rho_n}^m(X_n))^2 = \mathbb{E}[X_n - T_{\rho_n}^m(X_n)]^2 = \int_{\Omega} W_2^2(\rho_n, m) \Pi(dm)$.

which proves convergence i). Denoting now by $\mathcal{G}_\infty$ the sigma-field generated by $(X_1, X_2, \ldots)$ we also obtain that

$$\mathbb{E}(X_n - T_{\rho_n}^m(X_n)|\mathcal{G}_\infty) \to \mathbb{E}(X - T_{\rho}^m(X)|\mathcal{G}_\infty)$$

in $L^2(\Omega, \mathcal{G}, \mathbb{P})$.

Observe now that the following identities hold:

$$F'(\rho_n)(X_n) = \mathbb{E}(X_n - T_{\rho_n}^m(X_n)|\mathcal{G}_\infty)$$

$$F'(\rho)(X) = \mathbb{E}(X - T_{\rho}^m(X)|\mathcal{G}_\infty).$$

The convergence ii) follows from ~\ref{3.4}, that $\mathbb{E}(F'(\rho_n)(X_n))^2 = \|F'(\rho_n)\|_{L^2(\rho_n)}$, and $\mathbb{E}(F'(\rho)(X))^2 = \|F'(\rho)\|_{L^2(\rho)}$. \hfill \(\square\)

We now proceed to prove the first of our two main results.

**Proof of Theorem 1.1** Let us denote $\hat{F}$ the unique barycenter, write $\hat{F} := F(\hat{\mu})$ and introduce $h_t := F(\mu_t) - \hat{F} \geq 0$, and $\alpha_t := \prod_{i=1}^{t+1} \frac{1}{1 + \gamma_i}$. Thanks to the condition in eq. (1.3) the sequence $(\alpha_t)$ converges to some finite $\alpha_\infty > 0$, as it can be verified simply by applying logarithm. By Proposition 3.2 we have

$$\mathbb{E}[h_{t+1} - 1 + \gamma_t \gamma_t h_{t+1} h_t | F_t] \leq \gamma_t \hat{F} - \gamma_t F'(\mu_t)^2_{L^2(\mu_t)} \leq \gamma_t \hat{F},$$

from where, after multiplying by $\alpha_{t+1}$, the following bound is derived:

$$\mathbb{E}[\alpha_{t+1} h_{t+1} - \alpha_t h_t | F_t] \leq \alpha_{t+1} \gamma_t \hat{F} - \alpha_t \gamma_t F'(\mu_t)^2_{L^2(\mu_t)} \leq \alpha_{t+1} \gamma_t \hat{F}.$$  

Defining now $\tilde{h}_t := h_t - \sum_{i=1}^{t} \alpha_{t+1} \gamma_{t+1} \hat{F}$, we deduce from eq. (3.6) that $\mathbb{E}[\tilde{h}_{t+1} - \tilde{h}_t | F_t] \leq 0$, namely that $(\tilde{h}_t)_{t \geq 0}$ is a supermartingale w.r.t $(F_t)$. The fact that $(\alpha_t)$ is convergent, together with the condition in eq. (1.3), ensures that $\sum_{i=1}^{\infty} \alpha_{t+1} \gamma_{t+1} \hat{F} < \infty$, thus $\tilde{h}_t$ is uniformly lower bounded by a constant. Therefore, the supermartingale convergence theorem [54, Corollary 11.7] implies the existence of $\tilde{h}_\infty \in L^1$ such that $h_t \to \tilde{h}_\infty$ a.s. But then, necessarily, $h_t \to h_\infty$ a.s. for some nonnegative random variable $h_\infty \in L^1$.

Thus, our goal now is to prove that $h_\infty = 0$ a.s. Taking expectations in eq. (3.6) and summing over $t$ to obtain a telescopic summation, we obtain

$$\mathbb{E}[\alpha_t h_{t+1}] - \mathbb{E}[h_0 \alpha_0] \leq \hat{F} \sum_{i=1}^{t} \alpha_{t+1} \gamma_{t+1} - \sum_{i=1}^{t} \alpha_{t+1} \gamma_{t} \mathbb{E} \left[\|F'(\mu_i)\|_{L^2(\mu_i)}^2\right].$$
Then, taking liminf, applying Fatou on the l.h.s. and monotone convergence on the r.h.s. we obtain:

\[-\infty < \mathbb{E}[\alpha_r h_\infty] - \mathbb{E}[h_0 a_0] \leq C - \mathbb{E} \left[ \sum_{r=1}^\infty \alpha_r \gamma_r \| F'(\mu_t) \|_{L^2(\mu_t)}^2 \right].\]

In particular, since \( (\alpha_r) \) is bounded away from 0, we have

\[\sum_{r=1}^\infty \gamma_r \| F'(\mu_t) \|_{L^2(\mu_t)}^2 < \infty \text{ a.s.} \tag{3.7}\]

Note that \( \mathbb{P}(\liminf_{r \to \infty} \| F'(\mu_t) \|_{L^2(\mu_t)}^2 > 0) > 0 \) would be at odds with the conditions in eqs. (3.7) and (1.4), so

\[\lim_{r \to \infty} \| F'(\mu_t) \|_{L^2(\mu_t)}^2 = 0, \text{ a.s.} \tag{3.8}\]

Observe also that, from Assumption (A2) and Lemma 3.3 we have

\[\forall \epsilon > 0, \text{ inf}_{\{\rho : F(\rho) \geq \hat{F} + \epsilon\}} K_{\Pi} \ni \hat{\mu} \implies \| F'(\hat{\mu}) \|_{L^2(\mu)}^2 \geq \epsilon, \text{ a.s.} \tag{3.9}\]

Indeed, one can see that the set \( \{\rho : F(\rho) \geq \hat{F} + \epsilon\} \cap K_{\Pi} \) is \( W_2 \)-compact, using part i) of Lemma 3.3, and then check that the function \( \rho \mapsto \| F'(\rho) \|_{L^2(\mu)}^2 \) attains its minimum on it, using part ii) of that result. That minimum cannot be zero, as otherwise we would have obtained a Karcher mean that is not equal to the barycenter (contradicting the uniqueness of the Karcher mean). Remark also that a.s. \( \hat{\mu} \in K_{\Pi} \) for each \( t \), by the geodesic convexity part of Assumption (A2). We deduce the a.s. relationships between events:

\[\{ h_\infty \geq 2\epsilon \} \subseteq \{ \mu_t \in \{\rho : F(\rho) \geq \hat{F} + \epsilon\} \cap K_{\Pi} \forall \text{ large enough} \}
\subseteq \bigcup_{t \in \mathbb{N}} \left\{ \| F'(\mu_t) \|_{L^2(\mu_t)}^2 > 1/\ell : \forall \text{ large enough} \right\} \subseteq \{ \liminf_{t \to \infty} \| F'(\mu_t) \|_{L^2(\mu_t)}^2 > 0 \},\]

where eq. (3.9) was used to obtain the second inclusion. It follows using eq. (3.8) that \( \mathbb{P}(h_\infty \geq 2\epsilon) = 0 \) for every \( \epsilon > 0 \), hence \( h_\infty = 0 \) as required. To conclude, we use the fact that sequence \( \{\mu_t\} \) is a.s. contained in the \( W_2 \)-compact \( K_{\Pi} \), by assumption (A2), and the first convergence in Lemma 3.5 to deduce that the limit \( \hat{\mu} \) of any convergent subsequence satisfies \( F(\hat{\mu}) = \hat{F} = h_\infty = 0 \), and then \( F(\hat{\mu}) = \hat{F} \). Hence \( \hat{\mu} = \hat{\mu} \) by uniqueness of the barycenter. This implies that \( \mu_t \to \hat{\mu} \) in \( W_2(X) \) a.s. as \( t \to \infty \).

**Remark 3.4.** Observe that, for a fixed \( \mu \in W_2(\mathbb{R}^d) \) and a random \( m \sim \Pi \), the random variable \( -(T_m - I)(x) \) is an unbiased estimator of \( F'(\mu)(x) \), for \( \mu(dx) \) a.e. \( x \in \mathbb{R}^d \). A natural way to jointly quantify the pointwise variances of these estimators is through the **integrated variance**: \( \mathbb{V}(-(T_m - I)) := \int \var_{m=\Pi}[T_m(x) - x]\mu(dx), \) which is the equivalent (for unbiased estimators) of the mean integrated square error (MISE) from non-parametric statistics (53). Simple computations yield the following expression for it, which will be useful in the next two sections:

\[\mathbb{V}(-(T_m - I)) = \mathbb{E} \left[ -\mathbb{E} \left[ -(T_m - I) \right] \right] = \mathbb{E} \left[ -\mathbb{E} \left[ -(T_m - I) \right] - \mathbb{E} \left[ -(T_m - I) \right]^2 \right] = 2F(\mu) - \| F'(\mu) \|_{L^2(\mu)}^2. \tag{3.10}\]

We close this section with the promised statement of Lemma 3.5 referred to in the introduction, giving us a sufficient condition for Assumption (A2):

**Lemma 3.5.** If the support of \( \Pi \) is \( W_2(\mathbb{R}^d) \)-compact and there is a constant \( C_1 < \infty \) such that

\[\Pi(\{m \in W_2(\mathbb{R}^d) : \int \log(dm/d\mu)dm(dx) \leq C_1\}) = 1,\]

then Assumption (A2) is fulfilled.

**Proof.** Let \( K \) be the support of \( \Pi \), which is \( W_2(\mathbb{R}^d) \)-compact. By de la Vallée Poussin criterion, there is \( V : \mathbb{R}_+ \to \mathbb{R}_+ \) increasing, convex and super-quadratic (i.e. \( \lim_{r \to \infty} V(r)/r^2 = +\infty \)) such that \( C_2 := \sup_{x \in K} \int V(|x|)m(dx) < \infty \) and \( \rho(dx) := \exp[-V(|x|)]dx \) is a finite measure (wlog a probability measure). Moreover, for the relative entropy with respect to \( \rho \) we have \( H(m|p) := \int \log(dm/d\rho)dm = \int \log(dm/d\rho)m(dx) + \int V(|x|)m(dx) \) if \( m \ll dx \) and \( +\infty \) otherwise. Define now \( K_{\Pi} := \{m \in W_2(\mathbb{R}^d) : H(m|p) \leq C_1 + C_2, \int V(|x|)m(dx) \leq C_2\} \) so that \( \Pi(K_{\Pi}) = 1 \). Clearly \( K_{\Pi} \) is \( W_2 \)-closed, since the relative entropy is weakly lower semicontinuous, and also \( W_2 \)-relatively compact, since \( V \) is super-quadratic. Finally \( K_{\Pi} \) is geodesically convex by (51) Proposition 5.15. \( \Box \)
For instance if all \( m \) in the support of \( \Pi \) is of the form \( m(dx) = \frac{e^{-V_0(y)}}{\int e^{-V_0(y)}dy} \) with \( V_m \) bounded from below, then one way to guarantee the conditions in Lemma 3.3 assuming w.l.o.g. that \( V_m \geq 0 \), is to ask that \( \int e^{-V_0(y)}dy \geq A \) and \( \int |y|^2 e^{-V_0(y)}dy \leq B \), for some fixed \( A, B > 0 \). In words: tails are controlled and the measures cannot be too concentrated.

4. A condition granting uniqueness of Karcher means and convergence rates

The aim of this section is to prove Theorem 1.6, a refinement of Theorem 1.4 under the additional assumption that the barycenter is a pseudo-associative Karcher mean. Notice that, with the notation introduced in Section 3, Definition 1.5 of pseudo-associative Karcher mean \( \mu \) simply reads as

\[
W^2_2(\mu, v) \leq C_\mu \|F'(v)\|_{L^2(v)}^2
\]

for all \( v \in \mathcal{W}_{2,\infty}(\mathcal{X}) \).

Remark 4.1. Suppose \( \mu \) is a Karcher mean of \( \Pi \) such that, for all \( v \in \mathcal{W}_{2,\infty}(\mathcal{X}) \) and \( \Pi(dm) \) a.e. \( m \) one has

\[
T_m^\mu(x) = T_{T_m^\mu(x)} \circ T_m^\mu(x), \quad \mu(dx) \text{ a.e. } x \in \mathbb{R}^d.
\]

Then, \( \mu \) is pseudo-associative, with \( C_\mu = 1 \) and equality holding in Definition 1.5. Indeed, in that case we have:

\[
\|F'(v)\|_{L^2(v)}^2 = \int_{\mathcal{X}} \left[ \int_{\mathcal{X}\times\mathcal{X}} (T_m^\mu \circ T_m^\mu(x) - T_m^\mu(x)) \Pi(dm) \right]^2 \mu(dx)
\]

\[
= \int_{\mathcal{X}} \left[ \int_{\mathcal{X}\times\mathcal{X}} T_m^\mu(x) \Pi(dm) - T_m^\mu(x) \right]^2 \mu(dx)
\]

\[
= \int_{\mathcal{X}} [x - T_m^\mu(x)]^2 \mu(dx) = W^2_2(\mu, v)
\]

We thus will say that the Karcher mean \( \mu \) is associative if simply (4.2) holds.

The following is an analogue in Wasserstein space, of a classical property implying convergence rates in gradient-type optimization algorithms:

Definition 4.2. We say that \( \mu \in \mathcal{W}_{2,\infty}(\mathcal{X}) \) satisfies a Polyak-Lojasiewicz inequality if

\[
F(v) - F(\mu) \leq \frac{C_\mu}{\alpha} \|F'(v)\|_{L^2(v)}^2
\]

for some \( \tilde{C}_\mu > 0 \) and every \( v \in \mathcal{W}_{2,\infty}(\mathcal{X}) \).

We next state some useful properties:

Lemma 4.3. Suppose \( \mu \) is a Karcher mean of \( \Pi \). Then:

i) For all \( v \in \mathcal{W}_{2,\infty}(\mathcal{X}) \) we have

\[
F(v) - F(\mu) \leq \frac{1}{2} W^2_2(\mu, v).
\]

ii) If \( \mu \) is pseudo-associative, then it is the unique barycenter, and it satisfies the Polyak-Lojasiewicz inequality (4.4) with \( C_\mu = C_\mu \).

iii) If the associativity relation (4.2) holds, then we have

\[
F(v) - F(\mu) = \frac{1}{2} W^2_2(\mu, v) = \frac{1}{2} \|F'(v)\|_{L^2(v)}^2.
\]

Proof. i) We notice that this is a particular case of Theorem 7 in [20], which we can prove by an elementary argument based on the notion of Karcher mean. Indeed, by definition of function \( F \) and the fact that \( (T_m^\mu, T_{T_m^\mu}) \) is a coupling of \( v \) and \( m \) we have

\[
F(v) - F(\mu) \leq \frac{1}{2} \int_{\mathcal{X}\times\mathcal{X}} \left[ |T_m^\mu(x) - T_m^\mu(x)|^2 - |x - T_m^\mu(x)|^2 \right] \mu(dx) \Pi(dm)
\]

\[
= \frac{1}{2} \int_{\mathcal{X}\times\mathcal{X}} \left[ |T_m^\mu(x)|^2 - 2(T_m^\mu(x), T_m^\mu(x)) - x^2 + 2x, T_m^\mu(x) \right] \mu(dx) \Pi(dm)
\]

\[
= \frac{1}{2} \int_{\mathcal{X}} |T_m^\mu(x) - x|^2 \mu(dx)
\]

where in the second equality we used twice the fact that \( \int_{\mathcal{X}\times\mathcal{X}} T_m^\mu(x) \Pi(dm) = x \) for \( \mu(dx) \) a.e. \( x \).

ii) The claim is obvious in view of the previous and of (4.2).

iii) Taking into account Remark 4.1 it is enough to notice that

\[
F(v) = \int_{\mathcal{X}} |y - T_m^\mu(y)|^2 v(dy) \Pi(dm) = \int_{\mathcal{X}} |T_m^\mu(x) - T_m^\mu(x)|^2 \mu(dx) \Pi(dm)
\]
Remark 4.4. Recall that \( \Pi \) is said to satisfy a variance inequality \([21]\) if for some constant \( C > 0 \),

\[
F(\nu) - F(\hat{\nu}) \geq \frac{C}{2} \mathbb{E} \left[ \| F'(\mu_k) \|_{L^2(\mu_k)}^2 \right]
\]

with \( \hat{\nu} \) the barycenter of \( \Pi \). It readily follows that \( \hat{\nu} \) is a pseudo-associative Karcher mean as soon as a variance inequality and the Polyak-Lojasiewicz inequality \([4.4]\) hold. That was the case in \([20]\), and so the barycenter in the Gaussian setting considered therein is a pseudo-associative Karcher mean too. Notice that if the barycenter is associative, by Lemma \([4.3]\)(iii) the variance inequality holds; this is the case of the examples discussed in Sections \([6.1][6.3]\) below. Notice also that, if a variance inequality holds, bounds on the optimality gap for the SDG sequence as in Theorem \([1.6]\) immediately yield similar bounds for the expected squared Wasserstein distance \( \mathbb{E}[W^2_2(\mu, \hat{\nu})] \).

Proof of Theorem \([1.4]\). If we assume the pseudo-associativity condition \([1.5]\) holds, uniqueness of Karcher means (hence equal to the barycenter) is immediate, as noted in Section \([0]\). Under that assumption, the inequality \([4.4]\) holds thanks to part ii) of Lemma \([4.3]\) and convergence estimates can then be deduced adapting classic arguments of the SGD algorithm in an Euclidean setting, see e.g. \([16]\). Indeed, by Proposition \([3.2]\) and formulae \([3.11]\) and \([4.4]\), we get that

\[
\mathbb{E} \left[ F(\mu_{k+1}) - F(\mu_k) \right] \leq \gamma_k \| F'(\mu_k) \|_{L^2(\mu_k)}^2 = \frac{\gamma_k^2}{2} \left[ -(T^m_{\mu_k} - I) - (1 - \frac{\gamma_k}{2}) \gamma_k \| F'(\mu_k) \|_{L^2(\mu_k)} \right] \leq \gamma_k \hat{F} - \left( 1 - \frac{\gamma_k}{2} \right) \gamma_k \| F'(\mu_k) - \hat{F} \| \leq \gamma_k \hat{F} - \gamma_k C^{-1}_\rho \| F'(\mu_k) - \hat{F} \|
\]

for some finite (deterministic) upper bound \( \hat{F} > 0 \) of the sequence \( F(\mu_k) \) under (A2'). In the last line we used the fact that \( \gamma_k \leq \gamma_0 \leq 1 \) for the choice \( \gamma_k = \frac{a}{2(\mu^g)} \) with fixed \( b \geq a > 0 \). It follows that

\[
\mathbb{E} \left[ F(\mu_{k+1}) - \hat{F} \right] \leq \gamma_k \hat{F} + \left( 1 - \gamma_k C^{-1}_\rho \right) \mathbb{E} \left[ F(\mu_k) - \hat{F} \right] \quad \text{for all } k \in \mathbb{N}.
\]

Assume now that \( a > C_\rho \) and that, for certain \( c \geq a^2 \hat{F}/(C_\rho a - 1) > 0 \), it holds for a given \( k \in \mathbb{N} \) that

\[
\mathbb{E} \left[ F(\mu_k) - \hat{F} \right] \leq c/(b + k).
\]

Let us show then that \( \mathbb{E} \left[ F(\mu_{k+1}) - \hat{F} \right] \leq c/(b + k + 1) \) too. By \([4.6]\) we have

\[
\mathbb{E} \left[ F(\mu_{k+1}) - \hat{F} \right] \leq \frac{(b+1)(b+k)}{b(2b+k)} c + \frac{c(1-a^2/(C_\rho a - 1))}{(2b+k)} \leq \frac{c}{(b+k)}
\]

where we used the choice of \( c \) in the second inequality. Taking \( c = \max \{ b \mathbb{E} \left[ F(\mu_0) - \hat{F} \right], a^2 \hat{F}/(C_\rho a - 1) \} \), inequality \([4.7]\) holds for \( k = 0 \), hence for all \( k \in \mathbb{N} \) by induction.

Concerning the a.s. weak-convergence of \( \mu_k \) to \( \hat{\mu} \), we can follow the proof of Theorem \([1.4]\) up to Equation \([3.9]\) and then derive in the present setting \([3.9]\), i.e.

\[
\forall \rho > 0, \inf_{\mu : F(\mu) \leq \hat{F}} \rho \mathbb{E} \left[ \| F'(\mu) \|_{L^2(\rho)}^2 \right] > 0,
\]

from the Polyak-Lojasiewicz inequality \([4.4]\), which holds thanks to Lemma \([4.3]\)(ii). As before, this is then used to obtain that \( F(\mu_k) \rightarrow \hat{F} \) almost surely. Since \( K_\Pi \) is \( W_2 \)-bounded, it is in particular tight. The fact that \( \nu \mapsto F(\nu) \) is weakly l.s.c. and the uniqueness of minimizers of \( F \) entail now that a.s. \( \mu_k \rightarrow \hat{\mu} \) weakly.

\square

Remark 4.5. In addition to concluding that a.s. \( \mu_k \rightarrow \hat{\mu} \) weakly, the above proof also establishes the a.s. existence of a subsequence \( n_k \) such that \( \mu_{n_k} \rightarrow \hat{\mu} \) in \( W_2 \). Indeed, this follows from \([3.8]\) together with \([4.1]\).

The above proof shows that we may relax the definition of pseudo-associativity by requiring \([1.5]\) to hold for \( \nu \in K_\Pi \) (or more precisely, for \( \nu \in \{ \mu_k : k \in \mathbb{N} \} \) only).
5. Variance reduction via batch SGD

Paralleling the Euclidean setting, the function \(-T_{\mu}^m - I\) can be seen as an estimator of the gradient in the \(k\)-th step of the SGD scheme. Hence, conditionally on \(m_0, \ldots, m_{k-1}\), its integrated variance given by 
\[2 F(\mu_k) - \|F'(\mu_k)\|^2_{L^2(\mu)}\] could be large for some sampled \(\mu_k\), which would yield a slow convergence if the steps \(\gamma_k\) are not small enough. To cope with this issue, as customary in stochastic algorithms, we are lead to consider alternative estimators of \(F'(\mu)\) with less (integrated) variance:

**Definition 5.1.** Let \(\mu_0 \in \mathcal{K}, m_k \overset{iid}{\sim} \Pi, \) and \(\gamma_k > 0\) for \(k \geq 0\) and \(i = 1, \ldots, S_k\). The batch stochastic gradient descent (BSGD) sequence is given by

\[
\mu_{k+1} := \left(1 - \gamma_k I + \gamma_k \frac{1}{S_k} \sum_{i=1}^{S_k} T_{\mu_i}^m \right)(\mu_k).
\]

Notice that \(\frac{1}{S_k} \sum_{i=1}^{S_k} T_{\mu_i}^m - I\) is an unbiased estimator of \(-F'(\mu_k)\). Proceeding as in Proposition 3.2, with \(\mathcal{F}_{k+1}\) now denoting the sigma-algebra generated by \(\{m_\ell : \ell \leq k, i \leq S_k\}\), and writing \(\Pi\) also for the law of an i.i.d. sample of size \(S_k\), we now have

\[
\mathbb{E} \left[ F(\mu_{k+1}) \mid \mathcal{F}_k \right] = F(\mu_k) + \gamma_k (F'(\mu_k), \int \left(\frac{1}{S_k} \sum_{i=1}^{S_k} T_{\mu_i}^m - I\right) \Pi(\text{d}m_1 \cdots \text{d}m_{S_k}))_{L^2(\mu_k)}
\]

\[+ \gamma_k \frac{1}{S_k} \int \left[ \left(\frac{1}{S_k} \sum_{i=1}^{S_k} T_{\mu_i}^m - I\right)^2 \right] \Pi(\text{d}m_1 \cdots \text{d}m_{S_k}).\]

\[
\leq F(\mu_k) - \gamma_k \|F'(\mu_k)\|^2_{L^2(\mu)} + \gamma_k \frac{1}{S_k} \int \left[ \left(\frac{1}{S_k} \sum_{i=1}^{S_k} T_{\mu_i}^m - I\right)^2 \right] \Pi(\text{d}m_1 \cdots \text{d}m_{S_k})
\]

\[
= (1 + \gamma_k^2) F(\mu_k) - \gamma_k \|F'(\mu_k)\|^2_{L^2(\mu)}.
\]

The arguments in the proof of Theorem 1.4 can then be easily adapted to get:

**Theorem 5.2.** Under the assumptions of Theorem 1.4, the BSGD sequence \(\{\mu_k\}_{k \geq 0}\) in equation (5.1) converges a.s. to the 2-Wasserstein barycenter of \(\Pi\).

The supporting idea for using mini-batches is reducing the noise of the estimator of \(F'(\mu)\), namely:

**Proposition 5.3.** The integrated variance of the mini batch estimator of fixed batch size \(S\) for \(F'(\mu)\), given by \(-\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I)\), decreases linearly in the sample size. More precisely, we have

\[
\mathbb{V}[-\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I)] = \frac{1}{S} \mathbb{V}[-(T_{\mu_i}^m - I)] = \frac{1}{S} \left[ 2 F(\mu) - \|F'(\mu)\|^2_{L^2(\mu)} \right].
\]

**Proof.** In a similar way as shown in eq. (5.11), the integrated variance of the mini-batch estimator is

\[
\mathbb{V}[-\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I)] = \mathbb{E} \left[ \left\| -\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I) \right\|^2_{L^2(\mu)} \right] - \mathbb{E} \left[ -\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I) \right]^2_{L^2(\mu)}.
\]

The first term can be expanded as

\[
\left\| -\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I) \right\|^2_{L^2(\mu)} = \frac{1}{S^2} \left( \sum_{i=1}^{S} (T_{\mu_i}^m - I) \right)^2_{L^2(\mu)} = \frac{1}{S^2} \sum_{i=1}^{S} (T_{\mu_i}^m - I)_{L^2(\mu)}^2 = \frac{1}{S^2} \sum_{i=1}^{S} \|T_{\mu_i}^m - I\|^2_{L^2(\mu)}.
\]

By taking expectation, as the samples \(m_i \sim \Pi\) are independent, we get

\[
\mathbb{E} \left[ \left\| -\frac{1}{S} \sum_{i=1}^{S} (T_{\mu_i}^m - I) \right\|^2_{L^2(\mu)} \right] = \frac{1}{S^2} \sum_{i=1}^{S} \mathbb{E} \left[ \|T_{\mu_i}^m - I\|^2_{L^2(\mu)} \right] = \frac{1}{S^2} \sum_{i=1}^{S} \mathbb{E} \left[ \|T_{\mu_i}^m - I\|^2_{L^2(\mu)} \right].
\]

Subtracting \(\|F'(\mu)\|^2_{L^2(\mu)}\) yields the asserted identities. \(\square\)

The mini-batch implementation is easily seen to inherit convergence estimates established in Theorem 1.6.
Theorem 5.4. Under the assumptions of Theorem 1.6, the BSGD sequence \( \{\mu_k\}_k \) in eq. (5.1) is a.s. convergent to \( \hat{\mu} \in K_0 \) as soon as (1.5) and (1.3) hold. Moreover, for every \( a \geq C_{\mu}^{-1} \) and \( b \geq a \) there exists an explicit constant \( C_{a,b} > 0 \) such that, if \( \gamma_k = \frac{a - b}{b + 1} \) for all \( k \in \mathbb{N} \), the expected optimality gap satisfies:

\[
\mathbb{E} [F(\mu_k) - F(\hat{\mu})] \leq \frac{C_{a,b}}{b + k}.
\]

6. SGD for closed-form Wasserstein barycenters

As discussed in the introduction, recent developments enable the approximated computation of optimal transport maps between two given absolutely-continuous distributions, and hence the practical implementation of the SGD in fairly general cases is in principle feasible. We will not address this issue in the present work, but rather content ourselves with analysing some families of models considered in [25, 5], for which this additional algorithmic aspect can be avoided, their optimal transport maps being explicit and easy to evaluate. Further, we will examine some of their closure properties which are preserved under the operation of taking barycenter. This is important, for instance, in the context of the statistical application [9], wherein \( \Pi \) really represents a posterior distribution on models and its barycenter is postulated as a useful summary in a way that e.g. the model average does not.

In the settings that will be presented, the pseudo-associativity condition (1.5) is either satisfied or conditions ensuring it can be given explicitly. Convergence bounds as in Theorem 1.6 can be easily deduced in those cases for each specification of \( \Pi \) satisfying the required assumptions.

6.1. Univariate distributions. We assume that \( m \) is a continuous distribution over \( \mathbb{R} \), and denote respectively by \( F_m \) and \( Q_m := F_m^{-1} \) its cumulative distribution function and its right-continuous quantile function. The increasing transport map from a continuous \( m_0 \) to \( m \), also known as the monotone rearrangement, is given by

\[
T_{m_0}^m(x) = Q_m(F_{m_0}(x)),
\]

and is known to be \( p \)-Wasserstein optimal for \( p \geq 1 \) (see [51, Remark 2.19(iv)]). Given \( \Pi \), the barycenter \( \hat{m} \) is also independent of \( p \) and characterized via its quantile, i.e.,

\[
(6.1)
\]

In words: the quantile function of the barycenter is equal to the average quantile function. Our SGD iteration, starting from a distribution function \( F_\mu(x) \), sampling some \( m - \Pi \), and with step \( \gamma \), produces the measure

\[
\nu = ((1 - \gamma)I + \gamma T_{m_0}^m)(\mu).
\]

This is characterized by its quantile function

\[
Q_\nu(\cdot) = (1 - \gamma)Q_\mu(\cdot) + \gamma Q_m(\cdot).
\]

The batch stochastic gradient descent iteration equals \( Q_\nu(\cdot) = (1 - \gamma)Q_\mu(\cdot) + \frac{\gamma}{\delta} \sum_{i=1}^\delta Q_{m_i}(\cdot) \).

As explained in Remark 4.1, the barycenter is automatically pseudo-associative, since transport maps (i.e. increasing functions) are in this case associative in the sense discussed therein, hence Theorem 1.6 applies. Moreover by Remark 4.4 it entails convergence bounds in \( W_2 \) for the SGD sequence itself.

Interestingly, the model average \( \hat{m} := \int m \Pi(dm) \) is characterized by the averaged cumulative distribution function, i.e.,

\[
F_\hat{m}(\cdot) := \int F_m(\cdot) \Pi(dm).
\]

The model average does not preserve intrinsic shape properties from the distributions such as symmetry or unimodality. For example, if \( \Pi = 0.3 \ast \delta_{m_1} + 0.7 \ast \delta_{m_2} \), with \( m_1 = \mathcal{N}(1, 1) \) and \( m_2 = \mathcal{N}(3, 1) \), the model average is an asymmetric bimodal distribution with modes on 1 and 3, while the Wasserstein barycenter is the Gaussian distribution \( \hat{m} = \mathcal{N}(2.4, 1) \).

The following reasoning illustrates the fact that Wasserstein barycenters preserve geometric properties in a way that e.g. the model average does not. A continuous distribution \( m \) on \( \mathbb{R} \) is unimodal with a mode on \( \hat{x}_m \) if its quantile function \( Q(y) \) is concave for \( y < \hat{x}_m \) and convex for \( y > \hat{x}_m \), where \( Q(\hat{x}_m) = \frac{1}{2} \). Likewise, \( m \) is symmetric w.r.t. \( x_m \in \mathbb{R} \) if \( Q(\frac{1}{2} + y) = 2x_m - Q(\frac{1}{2} - y) \) for \( y \in [0, \frac{1}{2}] \) (note that \( x_m = Q_{\Pi}(\frac{1}{2}) \)). These properties can be analogously described in terms of the function \( F_m \). Let us show that the barycenter preserves unimodality/symmetry:

Proposition 6.1. If \( \Pi \) is concentrated on continuous symmetric (resp. symmetric unimodal) univariate distributions, then the barycenter \( \hat{m} \) is symmetric (resp. symmetric unimodal).
Proof. Using the quantile function characterization (6.1), we have for $y \in [0, \frac{1}{2}]$ that
\[ Q_m\left(\frac{1}{2} + y\right) = \frac{1}{2} Q_m\left(\frac{1}{2} + y\right) \Pi(dm) = 2x_m - Q_m\left(\frac{1}{2} - y\right), \]
where $x_m := \int x_m \Pi(dm)$. In other words $\hat{m}$ is symmetric w.r.t. $x_m$. Now, if each $m$ is unimodal in addition to symmetric, its mode $\tilde{x}_m$ coincides with its median $Q_m(\frac{1}{2})$, and $Q_m(\lambda y + (1 - \lambda)z) \geq (\text{resp.} \leq)\lambda Q_m(y) + (1 - \lambda)Q_m(z)$ for all $y, z < (\text{resp.} >)\frac{1}{2}$ and $\lambda \in [0, 1]$. Integrating these inequalities w.r.t. $\Pi(dm)$, and using (6.1) we deduce that $\hat{m}$ is unimodal (with mode 1/2) too. □

Unimodality is not preserved in general non-symmetric cases. Still, for some families of distributions unimodality is maintained after taking barycenter, as we show in the next:

**Proposition 6.2.** If $\Pi \in \mathcal{W}_p(W_{ac}(\mathbb{R}))$ is concentrated on log-concave univariate distributions, then the barycenter $\hat{m}$ is unimodal.

**Proof.** If $f(x)$ is a log-concave density, then $-\log(f(x))$ is convex so is $\exp(-\log(f(x))) = \frac{1}{f(x)}$. Some computations reveal that $f(x)dx$ is unimodal for some $\tilde{x} \in \mathbb{R}$, so its quantile $Q(y)$ is concave for $y < \tilde{y}$ and convex for $y > \tilde{y}$ where $Q(\tilde{y}) = \tilde{x}$. Since $\frac{1}{f(Q(y))}$ is convex decreasing for $x < \tilde{x}$ and convex increasing for $x > \tilde{x}$, then $\frac{dQ}{dy} = \frac{1}{f(Q(y))}$ is convex, positive and with minimum at $\tilde{y}$. Given $\Pi$, its barycenter $\hat{m}$ fulfills
\[ \frac{dQ}{dy} = \int \frac{dQ}{dy} \Pi(dm), \]
so if all $\frac{dQ}{dy}$ are convex, then $\frac{dQ}{dy}$ is convex and positive, with a minimum at some $\tilde{y}$. Thus, $Q_m(y)$ is concave for $y < \tilde{y}$ and convex for $y > \tilde{y}$ and $\hat{m}$ is unimodal with mode at $Q_m(\tilde{y})$. □

Useful examples of log-concave distribution families include the general normal, exponential, logistic, Gumbel, chi-square and Laplace laws, as well as the Weibull, power, gamma and beta families when their shape parameters are larger than one.

6.2. Distributions sharing a common copula. If two multivariate distributions $P$ and $Q$ over $\mathbb{R}^d$ share the same copula, then their $W_p$ distance to the $p$-th power is the sum of the $W_p(\mathbb{R})$ distances between their marginals raised to the $p$-power. Furthermore, if the marginals of $P$ have no atoms, then an optimal map is given by the coordinate-wise transformation $T(x) = (T^1(x), \ldots, T^d(x))$ where $T^i(x)$ is the monotone rearrangement between the marginals $P^i$ and $Q^i$ for $i = 1, \ldots, q$. This setting allows us to easily extend the results from the univariate case to the multidimensional case.

**Lemma 6.3.** If $\Pi \in \mathcal{W}_p(W_{ac}(\mathbb{R}^d))$ is concentrated on a set of measures sharing the same copula $C$, then the $p$-Wasserstein barycenter $\hat{m}$ of $\Pi$ has copula $C$ as well, and its $i$-th marginal $\hat{m}^i$ is the barycenter of the $i$-th marginal measures of $\Pi$. In particular the barycenter does not depend on $p$.

**Proof.** It is known ([25],[3]) that for two distributions $m$ and $\mu$ with $i$-th marginals $m^i$ and $\mu^i$ for $i = 1, \ldots, q$ respectively, the $p$-Wasserstein metric satisfies
\[ W_p^i(m, \mu) \geq \sum_{i=1}^q W_p^i(m^i, \mu^i), \]
where equality is reached if $m$ and $\mu$ share the same copula $C$ (abused notation denoting $W_p$ the $p$-Wasserstein distance on $\mathbb{R}^q$ as well as on $\mathbb{R}$). Thus,
\[ \int W_p^i(m, \mu) \Pi(dm) \geq \int \sum_{i=1}^q W_p^i(m^i, \mu^i) \Pi(dm) = \sum_{i=1}^q \int W_p^i(m^i, \mu^i) \Pi(dm), \]
where $\Pi'$ is defined via the identity $\int f(y) \Pi'(dv) = \int f(m^i) \Pi(dm)$. The infimum for the lower bound is reached on the univariate measures $\hat{m}^1, \ldots, \hat{m}^q$ where $\hat{m}^i$ is the $p$-barycenter of $\Pi^i$, and means that $\hat{m}^i = \text{argmin} \int W_p^i(m^i, \mu^i) \Pi(dm^i)$. It is plain that the infimum is reached on the distribution $\hat{m}$ with copula $C$ and $i$-th marginal $\hat{m}^i$ for $i = 1, \ldots, q$, which then has to be the barycenter of $\Pi$ and is independent of $p$. □

A Wasserstein SGD iteration, starting from a distribution $\mu$, sampling $m \sim \Pi$, and with step $\gamma$, both $\mu$ and $m$ having copula $C$, produces the measure $\nu = ((1 - \gamma)I + \gamma T^\mu_m)(\mu)$ characterized by having copula $C$ and the $i$-th marginal quantile functions
\[ Q^i(\cdot) = (1 - \gamma)Q^i_m(\cdot) + \gamma Q^i_{cm}(\cdot), \]
for $i = 1, \ldots, q$. The batch stochastic gradient descent iteration works analogously. Alternatively, one can perform (batch) stochastic gradient descent component-wise (with respect to the marginals $\Pi^i$ of $\Pi$) and
then make use of the copula $C$. As in the one-dimensional case, the barycenter is in this case automatically pseudo-associative since it is associative. Bounds for the expected optimality gap and in $W^2$ for the SGD sequence can be similarly deduced in this case too.

6.3. Spherically equivalent distributions. We denote here by $\mathcal{L}(\cdot)$ the law of a random vector, so $m = \mathcal{L}(x)$ and $x \sim m$ are synonyms. Following [25], another multidimensional case is constructed as follows: Given a fixed measure $\tilde{m} \in \mathcal{W}_{2,\text{uc}}(\mathbb{R}^q)$, its associated family of spherically equivalent distributions is

$$S_\alpha := \{ \{ \tilde{m} \} \mid \mathcal{L}(\tilde{m}) = \nu \in \mathcal{N}(\mathbb{R}^q), \tilde{x} - \tilde{m} \} ,$$

where $||x||_2$ is the Euclidean norm and $\mathcal{N}(\mathbb{R}^q)$ is the set of non-decreasing non-negative functions of $\mathbb{R}_+$. These type of distributions include the simplicially contoured distributions, and also elliptical distributions. For every $\alpha$, stochastic gradient iteration produces convergence bounds for the optimality gap and the SGD sequence can be established.

$C_{\alpha}$ can assume that $\tilde{m}$ is the Euclidean norm and $\mathcal{N}(\mathbb{R}^q)$ is the distribution function of the norm of $\tilde{m}$. The family $\Pi_{\alpha}$ is supported on $\mathbb{R}^q$.

6.4. Scatter-location family. We borrow here the setting of [5], where another useful multidimensional case is defined as follows: Given a fixed distribution $\tilde{m} \in \mathcal{W}_{2,\text{uc}}(\mathbb{R}^q)$, referred to as generator, the generated scatter-location family is given by

$$\mathcal{F}(\tilde{m}) := \{ \mathcal{L}(A\tilde{x} + b) \mid A \in \mathcal{M}_{+}^{q\times q}, b \in \mathbb{R}^q, \tilde{x} \sim \tilde{m} \},$$

where $\mathcal{M}_{+}^{q\times q}$ is the set of symmetric positive definite matrices of size $q \times q$. Without loss of generality we can assume that $\tilde{m}$ has zero mean and identity covariance. A clear example is the multivariate Gaussian family $\mathcal{F}(\tilde{m})$, with $\tilde{m}$ a standard multivariate normal distribution.

The optimal transport between one generator and any target in $\mathcal{M}_{+}^{q\times q}$ is explicit. If $m_1 = \mathcal{L}(A_1 \tilde{x} + b_1)$ and $m_2 = \mathcal{L}(A_2 \tilde{x} + b_2)$ then the optimal map from $m_1$ to $m_2$ is given by $T^m_{m_1}(x) = A_2(x-b_2)+b_2$ where $A = A_1^{-1}(A_2A_1)^{1/2}A_1^{-1} \in \mathcal{M}_{+}^{q\times q}$. The family $\mathcal{F}(\tilde{m})$ is therefore geodesically closed.

If $\Pi$ is supported on $\mathcal{F}(\tilde{m})$, then its 2-Wasserstein barycenter $\tilde{m}$ belongs to $\mathcal{F}(\tilde{m})$. Call its mean $\tilde{b}$ and its covariance matrix $\tilde{\Sigma}$. Since the optimal map from $\tilde{m}$ to $m$ is $T^m_{\tilde{m}}(x) = A_0^{-1}(x-b)$ with $A_0 = \tilde{\Sigma}^{-1/2}\tilde{\Sigma}^{-1/2}/\tilde{\Sigma}^{-1/2}$ and we know that $\tilde{m}$-almost surely $\int T^m_{\tilde{m}}(x)(\Pi(dm)) = x$, then we must have that $\int A_0^{-1}(\Pi(dm)) = I$, since clearly $\tilde{b} = \int b_0 \Pi(dm)$. As a consequence, we have $\tilde{\Sigma} = \int (\tilde{\Sigma}^{-1/2} \tilde{\Sigma}^{-1/2}) \Pi(dm)$.

A stochastic gradient descent iteration, starting from a distribution $\mu = \mathcal{L}(A_0\tilde{x} + b_0)$, sampling some $m = \mathcal{L}(A_0\tilde{x} + b_0)$, with step $\gamma$, produces the measure $\nu = T^m_\gamma(\mu) := (1-\gamma)I + \gamma T^m_{\tilde{m}}(\mu)$. If $\tilde{x}$ has a multivariate distribution $\tilde{F}(\tilde{x})$, then $\nu$ has distribution $F_\gamma(x) = \tilde{F}(A_0^{-1}(x-b_0))$ with mean $b_0$ and covariance $\Sigma_0 = A_0^{-1}$. We have that $T^m_{T^m_\gamma}(x) = (1-\gamma)I + \gamma A_0^{-1}(x-b_0) + \gamma b_0 + (1-\gamma)b_0$ with $A_\nu = A_0^{-1}(A_0A_0^T)^{1/2}A_0^{-1}$.

Then $F_\gamma(x) = F_{\gamma}(x - F_{\gamma}([T^m_\gamma]^{-1}(x))) = \tilde{F}((1-\gamma)A_0 + \gamma A_0^{-1}(x - \gamma b_0 - (1-\gamma)b_0)),$
with mean $b_1 = (1 - \gamma)b_0 + \gamma b_m$ and covariance
\[
\Sigma_1 = A_1^2 = [(1 - \gamma)A_0 + \gamma A_0^1 (A_0 A_0^2 A_0^1)^{1/2} \gamma A_0^2 (A_0 A_0^2 A_0^1)^{1/2} A_0^{-1}] \\
= A_0^2 [(1 - \gamma)A_0^2 + \gamma (A_0 A_0^2 A_0^1)^{1/2} \gamma A_0^2 (A_0 A_0^2 A_0^1)^{1/2} A_0^{-1}] \\
= A_0^2 [(1 - \gamma)A_0^2 + \gamma (A_0 A_0^2 A_0^1)^{1/2} A_0^{-1}]
\]
The batch stochastic gradient descent iteration is characterized by
\[
b_1 = (1 - \gamma)b_0 + \frac{2}{n} \sum_{i=1}^n b_i \text{ and } A_1^2 = A_0^2 [(1 - \gamma)A_0^2 + \frac{2}{n} \sum_{i=1}^n (A_0 A_0^2 A_0^1)^{1/2} A_0^{-1}].
\]

Notice that, if all the matrices $A_m : m \in \text{supp}(\Pi)$ share the same eigenspaces, then the barycenter is pseudo associative as it is in fact associative. In case these matrices do not necessarily share the same eigenspaces, and is pseudo associative as it is in fact associative.

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**Appendix I. Some remarks in the case of general measures**

We discuss a natural counterpart to the SGD sequence in the case where $\Pi$ is not necessarily supported on absolutely continuous measures. However in this case we neither have a verifiable result guaranteeing the convergence of the SGD method, nor a verifiable result saying that the accumulation points of the SGD sequence are Karcher means necessarily. The goal of this section is to introduce the objects needed for an analysis of the general case, to advance the convergence analysis as much as possible, and to illustrate the difficulties that we encounter which make us stop short of obtaining general convergence results.

While we retain Assumption (A1), we modify Assumption (A2) into
\[
(A2') \quad \Pi = \sum_{j=1}^f \lambda_j \delta_{v_j}, \text{ with each } v_j \in \mathcal{W}_2(X) \text{ and where the } \lambda_j \in (0, 1) \text{ sum to 1}.
\]

**Remark 1.1.** We can find $V : X \to [0, \infty)$ convex, continuous, and super-quadratic (i.e. $\lim_{|y| \to \infty} V(y)/|y|^2 = +\infty$) and $C \in (0, \infty)$ s.t. $\int V d\nu_j \leq C$ for each $j$. Defining
\[
K_\Pi = \left\{ \mu : \int V d\mu \leq C \right\},
\]
it follows that $K_\Pi$ is compact in $\mathcal{W}_2(X)$ and $\Pi(K_\Pi) = 1$. Moreover, if $(X, Y)$ is any optimal coupling with marginals $\mu$ and $\nu$, with $\mu$ and $\nu$ in $K_\Pi$, then also Law$(tX + (1 - t)Y) \in K_\Pi$ for each $t \in (0, 1)$. In the sequel we denote by $K_\Pi$ a set with these properties which may or may not be given explicitly as in $[I.1]$.

Throughout we denote by $\text{Opt}(\mu, \nu)$ the set of optimal couplings attaining the infimum defining $W_2(\mu, \nu)$, which is non-empty and may contain multiple elements, and we fix
\[
(1.2) \quad \mathcal{W}_2(X)^2 \ni (\mu, \nu) \mapsto G(\mu, \nu) \in \text{Opt}(\mu, \nu),
\]
a measurable selection of optimal couplings. In practical terms an algorithm for computing or approximating optimal couplings would be automatically measurable.

**Definition 1.2.** Let $\mu_0 \in K_\Pi$, $m_k \overset{\text{def}}{=} \Pi$, and $\gamma_k > 0$ for $k \geq 0$. We define the stochastic gradient descent (SGD) sequence by
\[
\mu_{k+1} := \text{Law} \left[ (1 - \gamma_k)X + \gamma_k Y \right],
\]
where Law$(X, Y) = G(\mu_k, m_k)$.

By Remark 1.1 we have $\{\mu_k : k \in \mathbb{N}\} \subseteq K_\Pi$ (a.s.).

**Definition 1.3.** $\mu \in \mathcal{W}_2(X)$ is a Karcher mean of $\Pi$ if
\[
(\mu(dx) - \text{a.s.}) \times = \sum_{j=1}^f \lambda_j [\mathbb{E}[Y|X = x]],
\]
where (for each $j$) Law$(X, Y_j)$ is some coupling in $\text{Opt}(\mu, v_j)$.
We first observe that if $\mu$ is absolutely continuous, then it is a Karcher mean according to Definition 1.3. On the other hand, if $\mu$ is a barycenter of $\Pi$ then it is also a Karcher mean according to Definition 1.3. Indeed, if $\text{Law}(X, Y_j) \in \text{Opt}(\mu, \nu_j)$ and we couple all these random variables in the same probability space, then

$$\sum_{i=1}^n \lambda_i W_{2}^2(\mu, \nu_i) = \mathbb{E} \left[ \sum_{i=1}^n \lambda_i |X - Y_i|^2 \right] \geq \mathbb{E} \left[ \sum_{i=1}^n \lambda_i |Y_j - \sum_{i=1}^n \lambda_i Y_i|^2 \right] \geq \sum_{i=1}^n \lambda_i W_{2}^2(\bar{\mu}, \nu_i),$$

so the inequalities above must be actual equalities and we have $\mu = \text{Law}(\sum_i \lambda_i Y_i) : = \bar{\mu}$ as well as $X = \sum_i \lambda_i Y_i$ (a.s.). Taking conditional expectation w.r.t. $X$ in the latter, we conclude.

In direct analogy to the absolutely continuous case, we denote

$$F(\mu) := \frac{1}{2} \sum_{i=1}^n \lambda_i W_{2}^2(\mu, \nu_i),$$

and we introduce

$$F'(\mu)(x) := x - \sum_{i=1}^n \lambda_i \mathbb{E}[Y_i | X = x],$$

where $(X, Y_j)$ $\sim$ $\mu$. With this notation we have the implication: $\|F'(\mu)\|_{L^2(\mu)} = 0 \implies \mu$ is a Karcher mean. As before we denote by $\mathcal{F}_0$ the trivial sigma-algebra and $\mathcal{F}_{k+1}, k \geq 0$, the sigma-algebra generated by $m_0, \ldots, m_k$.

**Proposition 1.4.** The SGD sequence in eq. (1.3) verifies (a.s.) that

$$\mathbb{E} |F(\mu_{k+1}) - F(\mu_k)|^2 \leq \gamma_k^2 \|F'(\mu_k)\|_{L^2(\mu_k)}^2.$$  

**Proof.** We can define a coupling $(X_k, Y_k, Z_k)$ such that $\text{Law}(X_k, Y_k) \in G(\mu_k, m_k)$, $\text{Law}(X_k, Z_k) \in G(\mu_k, \nu_k)$, and where $Y_k$ and $Z_k$ are independent conditionally on $\mathcal{F}_k$. Then the coupling $((1 - \gamma_k)X_k + \gamma_k Y_k, Z_k)$ has first marginal $\mu_{k+1}$ and second marginal $\nu_k$. We compute

$$W_{2}^2(\mu_{k+1}, \nu_k) \leq \mathbb{E}[(1 - \gamma_k)|X_k| + \gamma_k|Y_k|] = \mathbb{E}[(X_k - Z_k)^2] - 2\gamma_k \mathbb{E}[X_k - Z_k, X_k - Y_k] + \gamma_k^2 \mathbb{E}[X_k - Y_k]^2].$$

Hence

$$F(\mu_{k+1}) = \frac{1}{2} \int W_{2}^2(\mu_{k+1}, \nu) d\Pi(v) \leq \frac{1}{2} \sum_{i=1}^n \lambda_i \mathbb{E}[X_k - Z_k] - \gamma_k \mathbb{E}[X_k - \sum_{i=1}^n \lambda_i Z_k, X_k - Y_k] + \frac{1}{2} \gamma_k^2 \mathbb{E}[X_k - Y_k]^2]$$

$$= F(\mu_k) - \gamma_k \mathbb{E}[(X_k - \sum_{i=1}^n \lambda_i Z_k, X_k - Y_k)] + \frac{1}{2} \gamma_k^2 W_{2}^2(\mu_k, m_k).$$

Taking conditional expectation with respect to $\mathcal{F}_k$, and as $m_k$ is independently sampled from this sigma-algebra, we see that

$$\mathbb{E} [W_{2}^2(\mu_k, m_k) | \mathcal{F}_k] = \sum_{i=1}^n \lambda_i W_{2}^2(\mu_k, \nu_i) = 2F(\mu_k).$$

Similarly, we have

$$\mathbb{E} \left[ (X_k - \sum_{i=1}^n \lambda_i Z_k, X_k - Y_k) | \mathcal{F}_k \right] = \mathbb{E} \left[ X_k - \sum_{i=1}^n \lambda_i Z_k | X_k \right] X_k - \sum_{i=1}^n \lambda_i \mathbb{E} [Y_i | X_k]).$$

We conclude that

$$\mathbb{E} [F(\mu_{k+1}) | \mathcal{F}_k] \leq (1 + \gamma_k)F(\mu_k) - \gamma_k \|F'(\mu_k)\|_{L^2(\mu_k)}^2.$$  

\hfill \square

The next lemma is the only part where we use that $\Pi$ is supported in finitely many measures. With more refined arguments one could surely avoid this limitation.

**Lemma 1.5.** Let $(\rho_n)_n \subseteq W_2(\mathbb{R}^d)$ be a sequence converging w.r.t. $W_2$ to $\rho \in W_2(\mathbb{R}^d)$. Then, as $n \to \infty$, we have i) $F(\rho_n) \to F(\rho)$ and ii) $\|F'(\rho_n)\|_{L^2(\rho_n)} \to 0$ implies that $\rho$ is a Karcher mean.

**Proof.** Part (i) is immediate since $W_2(\rho_n, \nu_j) \to W_2(\rho, \nu_j)$ for each $j$. For part (ii), we first observe that $G(\rho_n, \nu_j)$ is tight, for each $j$. Passing to a subsequence if necessary, Skorokhod representation theorem yields, on some probability space, a sequence of random variables $X^n \sim \rho_n$ and $Y^n_j \sim \nu_j$, as well as $X \sim \rho$ and $Y_j \sim \nu_j$, such that $X^n \to X$ and for each $j$ also $Y^n_j \to Y_j$ (convergence is a.s. and in $L^2$). The sequence $\{\mathbb{E}[Y^n_j | X^n] : n \in \mathbb{N}\}$ is $L^2$-bounded, and hence by repeatedly applying Komlos theorem, we find yet another subsequence (relabelled so it is indexed by $\mathbb{N}$) such that

$$\frac{1}{n} \sum_{r \leq n} \mathbb{E}[Y^n_j | X^n] \to Z_j,$$

almost surely and in $L^2$, for each $j$. 
We check that $\mathbb{E}[Z_j|X] = \mathbb{E}[Y_j|X]$. Indeed, if $g$ is bounded and continuous
\[
\mathbb{E}[g(X)Z_j] = \lim_{n} \mathbb{E}[g(X) \left( \frac{1}{n} \sum_{r \leq n} E[Y_j|X'] \right)]
= \lim_{n} \frac{1}{n} \sum_{r \leq n} \mathbb{E}[g(X)E[Y_j|X']] + \frac{1}{n} \sum_{r \leq n} \mathbb{E}[(g(X) - g(X'))E[Y_j|X']]
= \lim_{n} \frac{1}{n} \sum_{r \leq n} \mathbb{E}[g(X')Y_j] = \mathbb{E}[g(X)Y_j],
\]
by dominated convergence. Also
\[
0 = \lim_{n} ||F'(\rho_n)||_{L^2(\rho_n)} = \lim_{n} \mathbb{E} \left[ \left| X^n - \sum_{j=1}^{\nu} \lambda_j \mathbb{E}[Y_j|X'] \right|^2 \right]
\geq \lim_{n} \mathbb{E} \left[ \left| X^n - \sum_{j=1}^{\nu} \lambda_j \mathbb{E}[Y_j|X'] \right|^2 \right]
\geq \mathbb{E} \left[ \left| X - \sum_{j=1}^{\nu} \lambda_j Z_j \right|^2 \right]
\geq \mathbb{E} \left[ \left| X - \sum_{j=1}^{\nu} \lambda_j \mathbb{E}[Z|X] \right|^2 \right],
\]
and we conclude that $X = \sum_{j=1}^{\nu} \lambda_j \mathbb{E}[Y_j|X]$ (a.s.). Finally we remark that $\text{Law}(X, Y_j) \in \text{Opt}(\rho, Y_j)$, and hence we conclude that $\rho$ is a Karcher mean. \qed

We can now provide a convergence result. The hypotheses of this result are implied by the hypotheses of Theorem 1.4 in case that $\Pi$ is concentrated on absolutely continuous measures. However we stress that we do not know of any example where Theorem 1.6 is applicable and $\Pi$ is concentrated on non-absolutely continuous measures. For this reason we rather think that this result and its proof are a template of what could happen or could be done in the general case, and use it really to illustrate the difficulties present in the general case.

**Theorem 1.6.** Assume (A1), (A2’), conditions (1.3) and (1.4), that $\Pi$ admits a unique Karcher mean (in the sense of Definition 1.2) in $K_0$, and that $K_0$ contains a (hence, exactly one) 2-Wasserstein barycenter $\check{\mu}$. Then, the SGD sequence $\{\mu_k\}_k$ in eq. (1.3) is a.s. convergent to $\check{\mu}$.

**Proof.** The proof of Theorem 1.4 can be followed verbatim up to Equation (3.8), namely
\[
\lim_{\epsilon \to 0} \inf \|F'(\mu)\|_{L^2(\mu)}^2 = 0, \text{ a.s.}
\]
We observe that if $\rho_n \to \rho$ with $\{\rho_n\}_n \subseteq K_0$ is s.t. $F(\rho_n) \geq F(\check{\mu}) + \epsilon$, for $\epsilon > 0$, and $\|F'(\rho_n)\|_{L^2(\rho_n)}^2 \to 0$, then by Lemma 1.5, $\rho$ is a Karcher mean in $K_0$ which is necessarily different from $\check{\mu}$. This would contradict the uniqueness of Karcher means in $K_0$. Thus we establish
\[
\forall \epsilon > 0, \inf_{(\rho, F(\rho)) \in \mathbb{F} + \epsilon} \mathbb{E} \left[ ||F'(\rho)\|_{L^2(\rho)}^2 \right] > 0.
\]
From here on we can follow again the proof of Theorem 1.4. \qed

In the absolutely continuous case, covered in the rest of this paper, we required the Karcher mean to be absolutely continuous. Moreover in this case there is a unique barycenter (automatically absolutely continuous). Once we leave the absolutely continuous world we have to be more careful, as illustrated by the following example:

**Example 1.7.** Let $Z$ be a $X$-valued random variable with finite second moment, and let $b_j \in X$ with $\sum_j \lambda b_j = 0$. Define $v_j := \text{Law}(Z + b_j)$. Then $X \sim \mu$ is a Karcher mean if $X = \mathbb{E}[Z|X]$ for some optimal coupling of $X$ and $Z$. Hence $X := Z$ and $X := \mathbb{E}[Z]$ are both Karcher means per Definition 1.3 even if $Z$ (and so each $v_j$) is absolutely continuous. On the other hand one can check that the barycenter of the $v_j$’s is unique and given by $\text{Law}(Z)$.

**Example 1.8.** We continue in the the setting of the previous example and choose $X = \mathbb{R}^2$. Here $Z = (B, 0)$ with $B \in [-1,1]$ with equal probabilities, and $v_1 = \text{Law}((B, 1))$, $v_2 = \text{Law}((B, -1))$. In this case we check that $\mu^0 := p(\delta_{1,0}) + \delta_{-1,0} + (1 - 2p)\delta_{0,0}$ is a Karcher mean for each $p \in [0,1/2]$. In particular the barycenter, given by $\mu^1$, is not isolated in the sense that any ball around $\mu^1$ contains other Karcher means (taking $p$ close to 0.5), and moreover those Karcher means might be more ‘regular’ than the barycenter in the sense of having a strictly larger support. Also remark that defining $K := \{\text{Law}((B, r)) : r \in I\}$ with $I$
some interval containing ±1, then K is compact, contains the ν_i’s, and transport maps between elements in K are associative.

The two examples above show that uniqueness of Karcher means (in the sense of Definition [13]), or even uniqueness of these within a nice set Π, is an assumption which can be hardly verified in the general case. Furthermore we encounter the same problem with a localized version of this assumption (for example: “uniqueness of Karcher means in a small ball intersected with Π”). To complicate the matter further, it is known that in the general case multiple barycentrers may exist. For all these reasons we are inclined to believe that the correct object of study in the general case is the regularized barycenter problem (or more are associative.

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STOCHASTIC GRADIENT DESCENT FOR BARYCENTERS IN WASSERSTEIN SPACE

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