Faddeev Calculation of the Hypertriton using the $SU_6$ Quark-Model Nucleon-Nucleon and Hyperon-Nucleon Interactions

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Quark-model nucleon-nucleon and hyperon-nucleon interactions by the Kyoto-Niigata group are applied to the hypertriton calculation in a new three-cluster Faddeev formalism using the two-cluster resonating-group method kernels. The most recent model, fss2, gives a reasonable result similar to the Nijmegen soft-core model NSC89, except for an appreciable contributions of higher partial waves.

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I. INTRODUCTION

The QCD-inspired spin-flavor $SU_6$ quark model for the baryon-baryon interaction, proposed by the Kyoto-Niigata group, is a unified model for the complete baryon octet ($B_8 = N, \Lambda, \Sigma$ and $\Xi$), which has achieved very accurate descriptions of the nucleon-nucleon ($NN$) and hyperon-nucleon ($YN$) interactions [1–3]. In particular, the $NN$ interaction of the most recent model fss2 [2] is accurate enough to compare with the modern realistic meson-exchange models. These quark-model interactions can be used for realistic calculations of few-baryon and few-cluster systems, once an appropriate three-body equation is formulated for the pairwise interactions described by the resonating-group method (RGM) kernels. The desired three-cluster equation should be able to deal with the non-locality and the energy-dependence intrinsically involved in the quark-exchange RGM kernel. Furthermore, the quark-model description of the $YN$ and hyperon-hyperon ($YY$) interactions in the full coupled-channel formalism sometimes involves a Pauli forbidden state at the quark level, which excludes the most compact $(0s)^6$ spatial configuration, resulting in the strongly repulsive nature of the interactions in some particular channels. We have recently formulated a new three-cluster equation which uses two-cluster RGM kernels explicitly [4]. This equation exactly eliminates three-cluster redundant components by requiring the orthogonality of the total wave function to the pairwise two-cluster Pauli-forbidden states. The explicit energy-dependence inherent in the exchange RGM kernel is self-consistently treated. This equation is entirely equivalent to the Faddeev equation which uses a singularity-free $T$-matrix (which we call the RGM $T$-matrix) generated from the two-cluster RGM kernel.

We first applied this formalism to three di-neutron and three-alpha systems, and obtained complete equivalence between the Faddeev calculations and variational calculations using the translationally invariant harmonic-oscillator (h.o.) basis [4, 5]. This formalism was also applied to the Faddeev calculation of the three-nucleon bound state [6], employing the off-shell $T$-matrices which are derived from the non-local and energy-dependent RGM kernels for our quark-model $NN$ interactions, fss2 and FSS. The model fss2 yields the triton binding energy $B_t = 8.519$ MeV in the 50 channel calculation, when the $np$ interaction is employed for all the $NN$ pairs in the isospin basis [7]. The effect of the charge dependence of the two-body $NN$ interaction is estimated to be $-0.19$ MeV for the triton binding energy [8]. This implies that our result is not overbinding in comparison with the empirical value, $B_t^{\text{exp}} = 8.482$ MeV. If we attribute the difference, $0.15$ MeV, to the effect of the three-nucleon force, it is by far smaller than the generally accepted values, $0.5 \sim 1$ MeV [9], predicted by many Faddeev calculations employing modern realistic meson-theoretical $NN$ interactions. The charge rms radii for $^3$H and $^3$He are also correctly reproduced. The non-local description of the short-range repulsion in the quark model is essential to reproduce the large binding energy and the correct size of the three-nucleon bound state without reducing the $D$-state probability of the deuteron.

Here we apply our quark-model $NN$ and $YN$ interactions to the hypertriton ($\Lambda^3$H) with the small separation
energy of the Λ-particle, $B_{\Lambda}^{\text{exp}} = 130 \pm 50$ keV. Since the Λ-particle is far apart from the two-nucleon subsystem, the on-shell properties of the $\Lambda N$ and $\Sigma N$ interactions are expected to be well reflected in this system. In particular, this system is very useful to determine the relative strength of $^1S_0$ and $^3S_1$ interactions in our framework. We will be able to fine-tune the quark model interaction to the hypertriton binding energy. This enables firmer quark model predictions for the $^1S_0$ and $^3S_1$ phase shifts. In fact, Refs. [10–12] showed that it is not at all trivial to bind the hypertriton as most meson-theoretical interactions fail to bind the hypertriton, except for the Nijmegen soft-core potentials NSC89 [13], NSC97f and NSC97e [14]. It is also pointed out in Refs. [10, 12] that a small admixture of the $\Sigma NN$ components less than 1% is very important for this binding.

In the next section, the Faddeev equation of the hypertriton system, using the quark-model $NN$ and $YN$ RGM kernels, is discussed, paying a special attention to the $\Lambda N-\Sigma N \bar{T}$-matrix. The results is given in the third section and the summary in the last section. Appendix gives some essential points to derive the Faddeev equation, whose solution exactly satisfies the orthogonality conditions.

II. FORMULATION

We start from the three-cluster equation of the $\Lambda NN$-$\Sigma NN$ system interacting via two-cluster RGM kernels

$$P \left[ E - H_0 - V_{12}^{\text{RGM}} (\varepsilon_{12}) - V_{31}^{\text{RGM}} (\varepsilon_{31}) - V_{32}^{\text{RGM}} (\varepsilon_{32}) \right] \times P \Psi = 0 ,$$

where $E$ is the negative three-body energy measured from the $\Lambda NN$ threshold, and the free Hamiltonian $H_0 = H^\text{c.m.} + \Delta m_3$ is composed of the kinetic-energy operators $H^\text{c.m.} = h_{31} + h_{32}$ etc. and the mass term $\Delta m_3$. In the following, the two nucleons are numbered 1 and 2, the $\Lambda$ or $\Sigma$ is numbered 3. The equation actually implies the $2 \times 2$ matrix form and $P \Psi$ is the two-dimensional vectors composed of the upper component with $\Lambda NN$ configuration and the lower one $\Sigma NN$. The mass term $\Delta m_3$ is therefore a diagonal matrix whose matrix elements are zero for the $\Lambda NN$ channel and $\Delta M_{\Lambda \Sigma} = M_{\Lambda \Sigma} - M_{\Lambda \Lambda}$ for the $\Sigma NN$ channel. Here $M_{\Lambda \Lambda}$ and $M_{\Sigma \Sigma}$ are the $\Lambda$ and $\Sigma$ masses, respectively. The RGM kernel $V_{ij}^{\text{RGM}} (\varepsilon_{ij}) = V_{Dij} + G_{ij} + \varepsilon_{ij} K_{ij}$ consists of the direct potential $V_{Dij}$, the sum of the exchange kinetic-energy and interaction kernels, $G_{ij} = G_{ij}^{\text{K}} + G_{ij}^{\text{V}}$, and the exchange normalization kernel $K_{ij}$ multiplied with the center-of-mass energy $\varepsilon_{ij}$ of the $ij$ subsystem for the relative motion. These are also $2 \times 2$ matrices. For example,

$$\varepsilon_{31} = \begin{pmatrix} \varepsilon_{\Lambda \Lambda} & 0 \\ 0 & \varepsilon_{\Sigma \Sigma} \end{pmatrix} = \begin{pmatrix} \varepsilon_{\Lambda \Lambda} & 0 \\ 0 & \varepsilon_{\Lambda \Lambda} - \Delta M_{\Lambda \Lambda} \end{pmatrix} ,$$

$$K_{31} = \begin{pmatrix} K_{\Lambda \Lambda} & K_{\Lambda \Sigma} \\ K_{\Sigma \Lambda} & K_{\Sigma \Sigma} \end{pmatrix} .$$

The two-cluster RGM equation is expressed as

$$\left[ \varepsilon_{31} - h_{31} - V_{31}^{\text{RGM}} (\varepsilon_{31}) \right] \chi = 0 .$$

The necessity of the projection operator $P$ in Eq. (1) is related to the existence of the eigenstate of the $\Lambda NN-\Sigma N$ normalization kernel; $K_{31} |u_{31} \rangle = \gamma |u_{31} \rangle$ with the eigenvalue $\gamma = 1$. This is the most compact $(0s)^6$ spin-singlet configuration with the flavor $SU_3$ quantum number $(11)_\Lambda$; $|u_{31} \rangle = |(0s)^6; (11)_1^1S_0 \rangle$. We seek for the Pauli-allowed state of the $\Lambda NN-\Sigma NN$ system by diagonalization

$$\sum_{\lambda = 1,2} |u_{31} \rangle \langle u_{31} | \Psi_\lambda \rangle = \lambda |\Psi_\lambda \rangle ,$$

in the antisymmetric model space, $\Psi_{123} (123) = - \Psi_{123} (213)$, which we denote as $\Psi_\lambda \in [11]$ in the following. The projection operator on the Pauli-allowed space, $P$, is defined by taking the model space spanned by the eigenvectors with $\lambda = 0$:

$$P = \sum_{\lambda = 0} |\Psi_\lambda \rangle \langle \Psi_\lambda | .$$

The three-cluster Faddeev equation, which is completely equivalent to Eq. (1), is derived by using some nice properties satisfied by $P$, which is briefly discussed in Appendix. The total wave function of the hypertriton system is expressed as a superposition of two independent Faddeev components $\psi$ and $\phi$; $P \Psi = \psi (1 - P_{12}) \phi$ with $\psi \in [11]$. The coupled-channel Faddeev equation reads

$$\psi = G_0 (E) T_{12} (E, \varepsilon_{12}) (1 - P_{12}) \phi ,$$

$$\phi = G_0 (E) \tilde{T}_{31} (E, \varepsilon_{31}) (\psi - P_{12} \phi) ,$$

where $P_{12}$ is the exchange operator of particle 1 and 2, $G_0 (E) = 1/(E - H_0)$ is the three-body free Green function for the negative energy $E < 0$, $T_{12} (E, \varepsilon_{12})$ is the $NN T$-matrix in the three-body space, and $\tilde{T}_{31} (E, \varepsilon_{31})$ is the redundancy-free $\Lambda NN \bar{T}$-matrix in the coupled-channel formalism. These $T$-matrices are generated from the $NN$ and $YN$ RGM kernels $V_{NN}^{\text{RGM}} (\varepsilon_{NN})$ and $V_{YN}^{\text{RGM}} (\varepsilon_{YN})$, respectively, according to the prescription essentially given in Ref. [4]. The energy dependence involved in these kernels is treated self-consistently by calculating the matrix elements of the two-cluster Hamiltonian, which will be discussed in some details.

For the $NN$ sector, $T_{12} (E, \varepsilon_{12})$ in the $\Lambda NN$ and $\Sigma NN$ spaces are given by the two-body $T$-matrix $t(\omega, \varepsilon)$ through a simple replacement of the starting-energy argument:

$$T_{12} (E, \varepsilon_{12}) = \begin{pmatrix} t_{NN}^\Lambda & 0 \\ 0 & t_{NN}^\Sigma \end{pmatrix} ,$$

$$t_{NN}^\Lambda = t_{NN} \left( E - \frac{\hbar^2 \xi_\Lambda + 2}{M_N} q^2, \varepsilon_{NN} \right) ,$$

$$t_{NN}^\Sigma = t_{NN} \left( E - \Delta M_{\Sigma \Lambda} - \frac{\hbar^2 \xi_\Sigma + 2}{M_\Sigma} q^2, \varepsilon_{NN} \right) ,$$

(7)
where $M_N$ is the nucleon mass, $\zeta_\beta = M_\beta/M_N$ is the mass ratio for $\beta = \Lambda$ or $\Sigma$, and $q$ is the momentum of the residual $\Lambda$ or $\Sigma$. The $NN$ relative energy in the three-body space, $\varepsilon_{NN}$, is determined from

$$\varepsilon_{NN} = \langle P\Psi| h_{NN} + V_{NN}^{\text{RGM}}(\varepsilon_{NN})|P\Psi\rangle ,$$  

which is actually the sum of the $\Lambda NN$ and $\Sigma NN$ components.

For the $YN$ sector, the situation is more complicated since the $\Lambda N-$,$\Sigma N$ coupling involves a complete Pauli forbidden state $|u_{31}\rangle$ in the two-dimensional space and the difference of the momentum dependent starting energies in the $\Lambda N$ and $\Sigma N$ channels is not constant because of the two types of reduced masses between $YN$ and $N$. In fact, the equation satisfied by $T_{31}(E,\varepsilon_{31})$ can never be reduced to the two-cluster coupled-channel equation in the free space. We request

$$T_{31}(E,\varepsilon_{31}) = V_{31}^{\text{RGM}}(\varepsilon_{31}) + V_{31}^{\text{RGM}}(\varepsilon_{31})G_0(E)T_{31}(E,\varepsilon_{31}) ,$$

$$T_{31}(E,\varepsilon_{31}) = T_{31}(E,\varepsilon_{31}) - (E - H_0)|u_{31}\rangle ,$$

$$\times [E - \varepsilon_{31} - h_2 - \Delta m_3]^{-1} (u_{31}(E - H_0) .$$

(9)

However, this equation is not actually satisfied, since the derivation is based on the unrealistic assumptions

1) $[E - \varepsilon_{31} - h_2 - \Delta m_3]$ is channel independent,

2) $[\varepsilon_{31} - h_3 - V_{31}^{\text{RGM}}(\varepsilon_{31})] |u_{31}\rangle = 0$ is satisfied.

The second condition is only approximately satisfied since, in the strict RGM framework, it is not possible to use the empirical internal energies of clusters and reduced masses for the relative kinetic-energy operators. In fact, the correct expression for 2) is

$$\varepsilon_0 - h_0 - V_{31}^{\text{RGM}}(\varepsilon_0) |u_{31}\rangle = 0 ,$$

(10)

where $\varepsilon_0 = \varepsilon_{\Lambda N} - E_\text{int}$ with $E_\text{int}$ being the calculated internal energies, and $h_0$ uses the calculated reduced mass unlike $h_{31}$. We can choose $E_{\text{int}}|\Lambda = 0$ for the $\Lambda NN$ channel, but $(E_{\text{int}})_{\Sigma} = \Delta M_{\Sigma \Lambda}$ is only approximately satisfied in our quark models, fss2 and FSS. This difficulty also takes place when we try to derive the basic orthogonality condition of the $T_{31}(E,\varepsilon_{31})$

$$\langle u_{31}| \left[ 1 + G_0(\omega)T_{31}(E,\varepsilon_{31}) \right] = 0 ,$$

(11)

which is essential to yield the orthogonality of the total wave function through the second equation Eq. (6);

$$\langle u_{31}| \phi + (1 - P_{12})\phi \rangle = 0 \ (i = 1, 2) .$$

Fortunately, these problems are completely solved by simply adding a small correction term to the RGM kernel, which is a procedure developed in Ref. [15] for making it possible to use the empirical values of the internal energies and reduced masses in the RGM formalism. In Ref. [15], we have slightly modified the original two-cluster RGM equation $[\varepsilon_0 - h_0 - V_{31}^{\text{RGM}}(\varepsilon_0)] \chi = 0$ and considered the following RGM equation in the OCM (orthogonality condition model) form:

$$\Lambda [\varepsilon - h_0^{\text{exp}} - V_{\text{RGM}}(\varepsilon_0)] \chi = 0 .$$

(12)

From here on, we omit the subscript 31 or $YN$ as much as possible, in order to simplify the notations. For example, $V_{\text{RGM}}(\varepsilon_0) = V_{\text{RGM}}^{\Lambda N}(\varepsilon_0)$ and $\Lambda = \Lambda_{31} = 1 - |u\rangle\langle u|$, with $|u\rangle = |u_{31}\rangle$ in Eq. (12). Furthermore, $\varepsilon = \varepsilon_{\Lambda N} - E_\text{int}^{\text{exp}}$ in Eq. (12) uses the empirical internal energy, $E_\text{int}^{\text{exp}} = \Delta m_3$, and the relative kinetic-energy operator, $h_0^{\text{exp}} = h_{31} = (\hbar^2/2\mu^{\text{exp}})\nabla^2$, uses the empirical reduced mass, $\mu^{\text{exp}} = \mu^{\Lambda N}$ or $\mu^{\Sigma N}$.

On the other hand, we need to use $\varepsilon_0$ with the calculated internal energies in the RGM kernel. It is shown in Ref. [15] that Eq. (12) is converted to Eq. (3) by simply adding $\Delta G$ to $V_{\text{RGM}}(\varepsilon_0)$; i.e., $V_{\text{RGM}}^{\Lambda N}(\varepsilon_{31}) = V_{\text{RGM}}(\varepsilon_0) + \Delta G$. The explicit expression of $\Delta G$ is given in the paper (or $\Delta \varepsilon = 0$ case in Eq. (16) below). We use the same idea to eliminate the channel dependence of $[E - \varepsilon_{31} - h_2 - \Delta m_3]$ in 1). Let us start from the off-shell extension of the $\Lambda N -$,$\Sigma N$ coupled-channel RGM equation in the following OCM form:

$$\Lambda [\omega - h_0^{\text{exp}} - V_{\text{RGM}}(\varepsilon_0)] \chi + |u\rangle(\omega - \varepsilon)(u|\chi) = 0 .$$

(13)

The solution of this equation satisfies the orthogonality, $|u|\chi) = 0$ for $\omega \neq \varepsilon$. The various energies in Eq. (13) are usually channel dependent; namely, if we use the label $\beta = \Lambda$ or $\Sigma$ to specify the $\Lambda N$ or $\Sigma N$ channel, the diagonal matrix elements of these energies are given by

$$\omega = E - \frac{\hbar^2}{2M_N} \left( \frac{\zeta_\beta + 2}{\zeta_\beta + 1} q^2 - E_\text{int}^{\text{exp}} ,
$$

$$\varepsilon = \varepsilon_{\Lambda N} - \frac{\hbar^2}{2M_N} \left( \frac{\zeta_\beta + 2}{\zeta_\beta + 1} - \frac{\zeta_\Lambda + 2}{\zeta_\Lambda + 1} q^2 - E_\text{int}^{\text{exp}} ,
$$

$$\varepsilon_0 = \varepsilon_{\Lambda N} - E_\text{int}^{\text{exp}} ,$$

(14)

where $E$ is the negative three-body energy, $q$ this time is the momentum of the residual nucleon. We can prove that Eq. (13) is equivalent to the following Schrödinger-type RGM equation:

$$\omega \chi = [h_0^{\text{exp}} + V_{\text{RGM}}^{\text{mod}}(\varepsilon)] \chi ,$$

(15)

where a newly defined RGM kernel $V_{\text{RGM}}^{\text{mod}}(\varepsilon)$ is given by

$$V_{\text{RGM}}^{\text{mod}}(\varepsilon) = V_{\text{RGM}}(\varepsilon_0) + \Delta G ,$$

$$\Delta G = \Lambda (\Delta E_\text{int} + \Delta \varepsilon + \Delta h_0) \Lambda$$

$$- (\Delta E_\text{int} + \Delta \varepsilon + \Delta h_0) \Lambda$$

$$\Delta E_\text{int} = E_\text{int}^{\text{exp}} - E_\text{int} ,$$

$$\Delta h_0 = h_0^{\text{exp}} - h_0 ,$$

$$\Delta \varepsilon = \frac{\hbar^2}{2M_N} \left( \frac{1}{\zeta_\beta + 1} - \frac{1}{\zeta_\Lambda + 1} q^2 .$$

(16)

The RGM $T$-matrix $T_{31}(E,\varepsilon_{31})$ is therefore formulated for this modified RGM kernel $V_{\text{RGM}}^{\text{mod}}(\varepsilon)$. By repeating the same process as to derive Eq. (9) with respect to
$V_{\text{mod}}^{\text{RGM}}(\epsilon)$, we can find that Eq. (9) (and also Eq. (11)) is just valid if we replace (see Appendix for details)

$$V_{31}^{\text{RGM}}(\epsilon_{31}) \rightarrow V_{\text{mod}}^{\text{RGM}}(\epsilon),$$

$$[E - \epsilon_{31} - h_2 - \Delta m_3]^{-1} \rightarrow [E - \epsilon_{AN} - (h_2)^{-1}].$$

(17)

In order to determine $\epsilon_{AN}$ or $\epsilon_0$, we approximate $\omega$ in Eq. (15) as

$$\omega = \begin{pmatrix} E - \frac{\epsilon^2}{2\omega N} & \frac{\epsilon + 2}{2\omega N} & 0 \\ 0 & E - \frac{\epsilon^2}{2\omega N} & \frac{\epsilon + 2}{2\omega N} \\ 0 & 0 & \Delta M_{\Sigma_A} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \epsilon_{AN} \\ 0 \\ 0 \end{pmatrix},$$

(18)

with keeping the relationship $\epsilon_{\Sigma_N} = \epsilon_{AN} - \Delta M_{\Sigma_A}$ in the free space, and first calculate a $\Lambda N$-$\Sigma N$ averaged value

$$\epsilon_{YN} = \langle P \Psi | h_0 \exp (V_{\text{mod}}^{\text{RGM}}) | P \Psi \rangle.$$

(19)

Then, from $\epsilon_{YN} = \epsilon_{AN} (\langle P \Psi | \Psi \rangle^{\Lambda} + \epsilon_{\Sigma_N} (\langle P \Psi | \Psi \rangle^{\Sigma}$ and $\Delta M_{\Sigma_A} = \epsilon_{AN} - \epsilon_{\Sigma_N}$, we find

$$\epsilon_{AN} = \epsilon_{YN} + \Delta M_{\Sigma_A} P_{\Sigma},$$

$$\epsilon_{\Sigma_N} = \epsilon_{YN} - \Delta M_{\Sigma_A} (1 - P_{\Sigma}),$$

(20)

where $P_{\Sigma} = (\langle P \Psi | \Psi \rangle^{\Sigma}$ is the probability of the $\Sigma N$ component admixed in the hypertriton wave function.

### III. Result

The angular-momentum states of the $NN$-$Y$ channel is specified by $|((\lambda_1) I, (\ell_2) j)_{1/2}; ((\lambda_2) I, t) 0\rangle$, where $(\lambda_1) I$ stands for the two-nucleon state, $(\lambda_2) I$ and $t$ is the isospin value $t = 0$ for $\Lambda$ and $t = 1$ for $\Sigma$. Due to the anti-symmetric property of the two nucleons, $(-)^{\lambda_1+t+\ell_2} = -1$, we find that the $3E$ and $1O$ states contribute to the $\Lambda NN$ channel and the $1E$ and $3O$ states contribute to the $\Sigma NN$ channel. For the $YN$-$N$ channel, the angular-momentum states are specified by $|((\ell_1) I, (\ell_2) j)_{1/2}; ((\lambda_2) I, t) 0\rangle$. Since the isospin of the hypertriton is zero, only the isospin $T = 1/2$ sector of the $\Lambda N$-$\Sigma N$ interaction contributes to the hypertriton calculation. All the partial waves of the orbital angular-momentum are possible for each of the $\Lambda NN$ and $\Sigma NN$ channels, which makes the number of channels for a particular partial-wave truncation just three-times larger than in the triton Faddeev calculation. The hyperon species of the $YN$-$N$ channels are uniquely specified by the isospin value $t = 0$ or 1. For the orbital part, the parity conservation requires $(-)^{\lambda_1+t+\ell_2} = -1$. The channel truncation is specified by the maximum value of the total angular momenta of the pairwise baryons, $I$ and $I_1$, which we denote $J$. As an example, all the channels of the standard 15-channel calculation with $S$- and $D$-waves are listed in Table I.

VI. For the numerical calculation, we discretize the continuous momentum variable $p$ (or $q$) for the Jacobi coordinate vectors, using the Gauss-Legendre $n_1$- (or $n_2$-) point quadrature formula, for each of the three intervals of $0 - 1$ fm$^{-1}$, $1 - 3$ fm$^{-1}$ and $3 - 6$ fm$^{-1}$. The small contribution from the intermediate integral over $p$ beyond $p_0 = 6$ fm$^{-1}$ in the two-body $T$-matrix calculation is also taken into account by using the Gauss-Legendre $n_3$-point quadrature formula through the mapping $p = p_0 + \tan \{\pi/2 + x/4\}$. [These $n_3$ points for $p$ are not included for solving the Faddeev equation Eq. (6), since it causes a numerical inaccuracy for the interpolation.] The momentum region $q = 6$ fm$^{-1}$ - $\infty$ is also discretized by the $n_3$-point formula just as in the $p$ discretization case. We take $n_1-n_2-n_3=10-10-5$ as is used for the triton Faddeev calculation in Refs. [6, 7], for which well converged results are obtained. The partial-wave decomposition of the two-cluster RGM kernel is carried out numerically using the Gauss-Legendre 20-point quadrature formula. The modified spline interpolation technique developed in Ref. [16] is employed for simplifying the treatment of the rearrangement of the Jacobi momentum coordinates. The Faddeev formalism with two identical particles or clusters is discussed in Ref. [17], together with some formulas for calculating the matrix elements of the two-cluster Hamiltonian. For the diagonalization of the large non-symmetric matrix appearing in solving Faddeev equations, the Arnoldi-Lanczos algorithm developed in the ARPACK subroutine package [18] is very useful.

Table II shows the results of the Faddeev calcula-

| \(Y-NN\) | \(2s+1\lambda_1\) | \((\ell_2/2) j\) | \(t\) |
|---|---|---|---|
| 1 | 3S\(_1\) | \((S_{1}^{3/2})\_1\) | 0 |
| 2 | 3S\(_1\) | \((D_{1}^{3/2})\_1\) | 0 |
| 3 | 3D\(_1\) | \((S_{1}^{3/2})\_1\) | 0 |
| 4 | 3D\(_1\) | \((D_{1}^{3/2})\_1\) | 0 |
| 5 | 3S\(_1\) | \((S_{1}^{3/2})\_1\) | 1 |

| \(N-YN\) | \(2s+1(m_1)\_I\) | \((\ell_2/2) j_2\) | \(t\) |
|---|---|---|---|
| 1 | 3S\(_1\) | \((S_{1}^{3/2})\_1\) | 0 (1) |
| 2 | 3S\(_1\) | \((D_{1}^{3/2})\_1\) | 0 (1) |
| 3 | 3D\(_1\) | \((S_{1}^{3/2})\_1\) | 0 (1) |
| 4 | 3D\(_1\) | \((D_{1}^{3/2})\_1\) | 0 (1) |
| 5 | 3S\(_1\) | \((S_{1}^{3/2})\_1\) | 0 (1) |
somewhat smaller in order to reproduce the correct ex-
sult is still slightly overbound, this difference should be

\[ \delta \]

pairs with

\[ \Sigma \]

the

\[ B \]

is the \( \Lambda \) separation energy measured from the \( \Lambda NN \) threshold; \( B_A \) is the \( \Lambda \) separation energy in the \( \Lambda NN \) expectation value determined self-consistently; and \( P_\Sigma \) is the \( \Sigma NN \) probability in percent. The norm of redundant components, \( N_{red} = (|u_i|P_i^2) (i = 1, 2) \), is less than \( 10^{-9} \).

| model | No. of channels | \( E \) (MeV) | \( B_A \) (keV) | \( \varepsilon_\Sigma \) (MeV) | \( \varepsilon_\Lambda \) (MeV) | \( P_\Sigma \) (%) |
|--------|----------------|-------------|--------------|-----------------|-----------------|-------------|
| 6 (S)  | -2.362         | 137         | -1.815       | 3.548           | 0.450           |
| 15 (SD)| -2.423         | 198         | -1.762       | 5.729           | 0.652           |
| 30 (J = 1)| -2.403       | 178         | -1.786       | 5.664           | 0.615           |
| fss2  | 54 (J = 2)     | -2.498      | 273          | -1.673          | 5.974           | 0.777       |
| 78 (J = 3) | -2.510       | 285         | -1.660       | 6.014           | 0.800           |
| 102 (J = 4) | -2.513      | 288          | -1.658       | 6.022           | 0.804           |
| 126 (J = 5) | -2.514       | 289          | -1.657       | 6.024           | 0.805           |
| 150 (J = 6) | -2.514       | 289          | -1.657       | 6.024           | 0.805           |
| 6 (S)  | -2.910         | 653         | -1.309       | 3.984           | 1.022           |
| 15 (SD)| -2.967         | 710         | -1.433       | 6.171           | 1.200           |
| 30 (J = 1)| -2.947       | 691         | -1.427       | 6.143           | 1.191           |
| FSS   | 54 (J = 2)     | -3.121      | 865          | -1.323          | 6.467           | 1.348       |
| 78 (J = 3) | -3.128       | 872         | -1.320       | 6.480           | 1.357           |
| 102 (J = 4) | -3.134      | 877         | -1.317       | 6.488           | 1.360           |
| 126 (J = 5) | -3.134       | 878         | -1.317       | 6.488           | 1.361           |
| 150 (J = 6) | -3.134       | 878         | -1.317       | 6.489           | 1.361           |

In order to make sure that this extrapolation gives a good estimation, we modify the \( \kappa \)-meson mass of the model fss2 from the original value, \( m_\kappa = 936 \) MeV [2], to 1,000 MeV, and repeat the whole calculation. It is known that this modification makes the \( ^1S_0 \) \( \Lambda NN \) interaction less attractive and the \( ^3S_1 \) more attractive. We obtain \( B_A = 145 \) keV with \( P_\Sigma = 0.53 \% \). The effective range parameters of this modified fss2 interaction are \( a_s = -2.15 \) fm, \( r_s = 3.05 \) fm, and \( a_t = -1.80 \) fm, \( r_t = 2.87 \) fm. The phase-shift difference is only 1.3° and the total cross section of the \( \Lambda N \) scattering increases at most 10 mb at \( \rho = 100 \) MeV/c from 286 mb to 296 mb, which is still within the experimental error bars.

It should be kept in mind that the effective range parameters or the \( S \)-wave phase-shift values determined in this way is very much model dependent, since the \( B_A \) value is not solely determined by these quantities. It depends on how higher partial waves contribute and also on the details of the \( \Lambda N - \Sigma N \) coupling of a particular model.

A nice extrapolation shown here is based on the similarity of the models fss2 and FSS, which have a common framework for the quark sector and the effective meson-exchange potentials.

Table II also shows that the expectation value of the \( \Lambda \) separation energy \( \varepsilon_\Lambda \), determined self-consistently is rather close to the deuteron energy \( -\varepsilon_d \), especially in fss2. This feature is even marked if we decompose these energies to the kinetic-energy and potential-energy contributions. Table IV shows this decomposition with respect to fss2, FSS, and NSC89. (For this comparison, we use the definition of the kinetic-energy part of the deuteron by \( h_d = \langle \chi_d \mid h_{NN} \mid \chi_d \rangle \langle \chi_d \mid \chi_d \rangle \), where \( \chi_d \) is the RGM relative wave function between the neutron and the proton.) In fss2, the kinetic-energy of the \( NN \) subsystem is 1.88 MeV larger than that of the deuteron, which implies that the \( NN \) subsystem shrinks by the effect of the outer \( \Lambda \)-particle, in comparison with the deuteron in the free space. In NSC89, this difference is even smaller; i.e., 1.18 MeV. These results are consistent with the fact that the hypertriton in NSC89 is more loosely bound \( (B_A = 143 \) keV [12]) than in fss2 (289 keV), and the \( \Lambda \)-particle is very far apart from the \( NN \) cluster. The \( \Sigma NN \) probability in NSC89 is \( P_\Sigma = 0.5 \% \) [10, 12]. Table IV also lists the kinetic-energy and potential-energy

\[ \text{TABLE III: } ^1S_0 \text{ and } ^3S_1 \text{ effective range parameters of FSS [1], fss2 [2, 3], and NSC89 [13] } \Lambda NN \text{ interactions (Ap for NSC89) and the } \Lambda \text{ separation energies } B_A \text{ of the hypertriton. The values for NSC89 are taken from Ref. [12].} \]

| model | \( a_s \) (fm) | \( r_s \) (fm) | \( a_t \) (fm) | \( r_t \) (fm) | \( B_A \) (keV) |
|-------|--------------|--------------|--------------|--------------|--------------|
| FSS   | -5.41        | 2.26         | -1.03        | 4.20         | 878          |
| fss2  | -2.59        | 2.83         | -1.60        | 3.01         | 289          |
| NSC89 | -2.59        | 2.90         | -1.38        | 3.17         | 143          |
TABLE IV: Decomposition of the $NN$ and $YN$ expectation values ($\varepsilon_{NN}$ and $\varepsilon_{YN}$), the deuteron energy ($-\varepsilon_d$) and the total three-body energy $E$ to the kinetic-energy and potential-energy contributions. The unit is in MeV. The results for NSC89 are from [10].

| model  | $h_{NN} + V_{NN} = \varepsilon_{NN}$ | model  | $h_{YN} + V_{YN} = \varepsilon_{YN}$ |
|--------|-------------------------------------|--------|-------------------------------------|
| FSS    | 19.986 - 21.303 = -1.317            | FSS    | 10.036 - 4.602 = 5.435              |
| fss2   | 19.376 - 21.032 = -1.657            | fss2   | 8.071 - 2.671 = 5.401               |
| NSC89  | 20.48 - 22.25 = -1.77               | NSC89  | 7.44 - 3.54 = 3.90                  |

decompositions for the averaged $YN$ expectation value $\varepsilon_{YN}$ and the total energy $E$. The kinetic energies of $\varepsilon_{YN}$ are much smaller than those of $\varepsilon_{NN}$, which indicates that the relative wave functions between the hyperon and the nucleon are widely spread in the configuration space. The comparison of the total-energy decomposition shows that the wave functions of fss2 and NSC89 may be very similar. A clear difference between fss2 and NSC89 appears in the roles of higher partial waves. The energy increase due to the higher partial waves than the $S$ and $D$ waves is 91 keV in fss2 and 168 keV in FSS, respectively. On the other hand, the results in Ref. [10] imply that this is only 20 - 30 keV in the case of NSC89. This difference can originate from both of the $NN$ and $YN$ interactions. Since the characteristics of the meson-theoretical $YN$ interactions in higher partial waves are a priori unknown, more detailed analysis of the fss2 results might shed light on the adequacy of the quark-model baryon-baryon interactions.

IV. SUMMARY

In this study, we have carried out the Faddeev calculations, using the recent quark-model $NN$ and $YN$ interactions, FSS [1] and fss2 [2, 3]. These are realistic interactions which describe all the available $NN$ and $YN$ data, by incorporating the effective meson-exchange potentials at the quark level. Since these quark-model baryon-baryon interactions are formulated in the RGM framework, they are non-local, energy-dependent, and sometimes involve the Pauli-forbidden component at the quark level. The hypertriton is an appropriate place to investigate the roles of the compact $SU_3 (11)_s$ component, which is completely Pauli-forbidden in the $^1S_0 \Lambda N\Sigma N$ channel coupling with the isospin $T = 1/2$. In order to deal with this off-shell effect of the quark-model interaction precisely, we have formulated a new type of the Faddeev equation which explicitly employs the two-cluster RGM kernels [4, 5]. The energy-dependence of the RGM kernel is self-consistently treated, by calculating the expectation values of the two-cluster Hamiltonian with respect to the obtained solutions of the Faddeev equation [17]. We have especially payed attention to how to extend the microscopic description of the $\Lambda N\Sigma N$ coupling in the hypertriton system without spoiling the effect of the Pauli-principle at the quark level. The present study is the second application of this formalism to the few-baryon systems interacting via the quark-model baryon-baryon interactions, following our previous one to the triton system [6, 7]. The hypertriton is well suited to investigate the on-shell properties of the $\Lambda N$ and $\Sigma N$ interactions, since the hyperon is very far apart from the two-nucleon cluster.

We have found that our quark-model interaction fss2 gives a reasonable result for the hypertriton properties, which is rather similar to the result of the Nijmegen soft-core model NSC89 [13]. The $\Lambda$ separation energy given by fss2 is $B_\Lambda = 289 \text{ keV}$, which is a little too large in comparison with the experimental value $B_\Lambda^{\text{exp}} = 130 \pm 50 \text{ keV}$. The admixture of the $\Sigma NN$ component is $P_\Sigma = 0.80 \%$. Modifying the $\kappa$-meson mass of fss2 from the original value, $m_\kappa = 936 \text{ MeV}$, to 1,000 MeV leads to the almost correct $\Lambda$-separation energy 145 keV with $P_\Sigma = 0.53 \%$. Unlike the NSC89 result, the effects of higher partial waves up to the $G$ wave are rather important in the quark-model $NN$ and $YN$ interactions. If we use the dominant $S$-wave character of the $\Lambda N$ interaction in the hypertriton system, the $^1S_0 \Lambda N$ interaction of the model fss2 is slightly too attractive. It is a future problem to investigate whether or not a reduction in the $^1S_0 \Lambda N$ attraction like the modification $m_\kappa c^2 = 936 \text{ MeV}$ to 1,000 MeV produces a favorable feature for the level spacing of the $0^+$ and $1^+$ states of $^4_\Lambda \text{H}$ and $^4_\Lambda \text{He}$ systems. The fairly large charge symmetry breaking in these systems is also an important issue to understand the hyperon-nucleon interactions in detail.

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APPENDIX: DERIVATION OF THE THREE-CLUSTER FADDEEV EQUATION FOR THE ANN-ΣNN SYSTEM

In this appendix, we discuss some essential points to derive the Faddeev equation Eq. (6) from the three-cluster equation Eq. (1). In the derivation, we extensively use the following properties of the projection operator \( P \) defined in Eq. (5):

\[
\begin{align*}
(i) & \quad \Lambda_{3i} P = P \Lambda_{3i} = P \quad (i = 1, 2), \\
(ii) & \quad \text{when } \Psi \in [1], \forall \langle u_{3i} | \Psi \rangle = 0 \longleftrightarrow P \Psi = \Psi, \\
(iii) & \quad \text{when } \Psi \in [1], \ P \Psi = 0 \longleftrightarrow \exists | f \rangle,
\end{align*}
\]

Using the property (i), we can replace \( V_{3i}^{RGM}(\varepsilon_{3i}) \) in Eq. (1) by \( v_{3i}(\varepsilon_{3i}) = \Lambda_{3i} V_{3i}^{RGM}(\varepsilon_{3i}) \Lambda_{3i} \) or

\[
V_{3i}(E, \varepsilon_{3i}) = (E - H_0) - \Lambda_{3i}(E - H_0) \Lambda_{3i} + v_{3i}(\varepsilon_{3i}).
\]

(A.2)

We further use the property (iii) for the whole equation and introduce the ansatz for the Faddeev components, \( P \Psi = \psi + (P - P_1) \phi \), to derive a pair of equations

\[
\begin{align*}
[E - H_0 - V_{12}^{RGM}(\varepsilon_{12})] \psi &= V_{12}^{RGM}(\varepsilon_{12}) (1 - P_1) \phi , \\
[E - H_0 - V_{31}(E, \varepsilon_{31})] \phi &= V_{31}(E, \varepsilon_{31}) (\psi - P_1 \phi) + |u_{31}\rangle f_2. 
\end{align*}
\]

(A.3)

In the second equation, we note that

\[
E - H_0 - V_{31}(E, \varepsilon_{31}) = \Lambda_{31} [E - H_0 - V_{3i}^{RGM}(\varepsilon_{3i})] \Lambda_{31},
\]

(A.4)

and introduce the projected two-body Green function in the three-body space, \( G_{\Lambda_{31}}(E, \varepsilon_{31}) \), which satisfies

\[
G_{\Lambda_{31}}(E, \varepsilon_{31}) \Lambda_{31} \left[ E - H_0 - v_{31}(\varepsilon_{31}) \right] \Lambda_{31} = \Lambda_{31}. 
\]

(A.5)

This can be easily constructed through

\[
G_{\Lambda_{31}}(E, \varepsilon_{31}) = G_{v_{31}}(E, \varepsilon_{31}) - G_{v_{31}}(E, \varepsilon_{31})|u_{31}\rangle \langle u_{31}| \Lambda_{31} = \Lambda_{31},
\]

(A.6)

by using the two-body Green function \( G_{v_{31}}(E, \varepsilon_{31}) = [E - H_0 - v_{31}(\varepsilon_{31}) + i0]^{-1} \) in the three-body space. The essential equation we need for deriving the full Green function \( G_{31}(E, \varepsilon_{31}) = [E - H_0 - V_{3i}^{RGM}(\varepsilon_{3i}) + i0]^{-1} \) is the decomposition

\[
E - H_0 - V_{3i}^{RGM}(\varepsilon_{3i}) = (E - \varepsilon_{31} - h_2 - \Delta m_3) - \Lambda_{31}(E - \varepsilon_{31} - h_2 - \Delta m_3) \Lambda_{31} + \Lambda_{31} [E - H_0 - V_{3i}^{RGM}(\varepsilon_{3i})] \Lambda_{31},
\]

(A.7)

but the last equality is actually not satisfied since \( (E - \varepsilon_{31} - h_2 - \Delta m_3) \) is channel dependent. This difficulty is avoided by using \( V_{3i}^{mod}(\varepsilon) \) in Eq. (16), in place of \( V_{3i}^{RGM}(\varepsilon_{3i}) \). In fact, we find that

\[
(E - \varepsilon_0 - h_2 - \Delta m_3) - (h_{31} - h_0) + (\Delta E_{int} + \Delta \varepsilon + \Delta h_0) = E - \varepsilon_{AN} - (h_2)_{\Lambda},
\]

(A.8)

is channel independent. Here \( \varepsilon_0 \) and \( (h_{31} - h_0) \) term appear since \( |u_{31}\rangle \) actually satisfies Eq. (10) and not Eq. (3). This makes it possible to derive our basic relationship

\[
G_0(E) T_{31}(E, \varepsilon) = G_{31}(E, \varepsilon) V_{3i}^{mod}(\varepsilon) = G_{\Lambda_{31}}(E, \varepsilon) V_{31}(E, \varepsilon) - |u_{31}\rangle \langle u_{31}| + |u_{31}\rangle \langle u_{31}| \frac{1}{E - \varepsilon_{AN} - (h_2)_{\Lambda}} |u_{31}| (E - H_0),
\]

(A.9)

where all the kernels are defined by using \( V_{3i}^{mod}(\varepsilon) \). From Eq. (A.9) we can easily prove the second Faddeev equation Eq. (6) and the orthogonality condition Eq. (11).

[1] Y. Fujiwara, C. Nakamoto, and Y. Suzuki, Phys. Rev. Lett. 76, 2242 (1996); Phys. Rev. C 54, 2180 (1996).
[2] Y. Fujiwara, T. Fujita, M. Kohno, C. Nakamoto, and Y. Suzuki, Phys. Rev. C 65, 014002 (2002).
[3] Y. Fujiwara, M. Kohno, C. Nakamoto, and Y. Suzuki, Phys. Rev. C 64, 054001 (2001).
[4] Y. Fujiwara, H. Nemura, Y. Suzuki, K. Miyagawa, and M. Kohno, Prog. Theor. Phys. 107, 745 (2002).
[5] Y. Fujiwara, Y. Suzuki, K. Miyagawa, M. Kohno, and H. Nemura, Prog. Theor. Phys. 107, 993 (2002).
[6] Y. Fujiwara, K. Miyagawa, M. Kohno, Y. Suzuki, and H. Nemura, Phys. Rev. C 66, 021001(R) (2002).
[7] Y. Fujiwara, K. Miyagawa, Y. Suzuki, M. Kohno, and H. Nemura, Nucl. Phys. A721, 983c (2003).
[8] R. Machleidt, Adv. Nucl. Part. Phys. 19, 189 (1989).
[9] A. Nogga, H. Kamada, and W. Glöckle, Phys. Rev. Lett. 85, 944 (2000).
[10] K. Miyagawa, H. Kamada, W. Glöckle, and V. Stoks, Phys. Rev. C 51, 2905 (1995).
[11] K. Miyagawa, H. Kamada, W. Glöckle, H. Yamamura, T. Mart, and C. Bennhold, Few-Body Systems Suppl. 12, 324 (2000).
[12] A. Nogga, H. Kamada, and W. Glöckle, Phys. Rev. Lett. 88, 172501 (2002); A. Nogga, Ph. D. thesis, University of Bochum (2001).
[13] P. M. M. Maessen, Th. A. Rijken, and J. J. de Swart, Phys. Rev. C 40, 2226 (1989).
[14] Th. A. Rijken, V. G. J. Stoks, and Y. Yamamoto, Phys. Rev. C 59, 21 (1999).
[15] Y. Fujiwara, M. Kohno, C. Nakamoto, and Y. Suzuki, Prog. Theor. Phys. 104, 1025 (2000).
[16] W. Glöckle, G. Hasberg, and A.R. Neghabian, Z. Phys. A 305, 217 (1982).
[17] Y. Fujiwara, K. Miyagawa, M. Kohno, Y. Suzuki, D. Baye, and J.-M. Sparenberg, KUNS-1910, nucl-th/0404071, submitted to Phys. Rev. C.
[18] See ARPACK homepage, http://www.caam.rice.edu/software/ARPACK/