Calculation of the Self Force using the Extended-Object Approach

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Abstract

We present here the extended-object approach for the explanation and calculation of the self-force phenomenon (often called also "radiation-reaction force"). In this approach, one considers a charged extended object of a finite size $\epsilon$ that accelerates in a nontrivial manner, and calculates the total force exerted on it by the electromagnetic field (whose source is the charged object itself). We show that at the limit $\epsilon \to 0$ this overall electromagnetic field yields a universal result, independent on the object’s shape, which agrees with the standard expression for the self force acting on a point-like charge. This approach has already been considered by many authors, but previous analyses ended up with expressions for the total electromagnetic force that include $O(1/\epsilon)$ terms which do not have the form required by mass-renormalization. (In the special case of a spherical charge distribution, this $\propto 1/\epsilon$ term was found to be 4/3 times larger than the desired quantity.) We show here that this problem was originated from a too naive definition of the notion of "total electromagnetic force" used in previous analyses. Based on energy-momentum conservation combined with proper relativistic kinematics, we derive here the correct notion of total electromagnetic force. This completely cures the problematic $O(1/\epsilon)$ term, for any object’s shape, and yields the correct self force at the limit $\epsilon \to 0$. In particular, for a spherical charge distribution, the above "4/3 problem" is resolved. We recently presented an outline of this analysis [1], focusing on the special case of a dumbbell-like charge distribution (i.e. two discrete point charges). Here we provide full detail of the analysis, and also extend it to more general charge distributions: extended objects with an arbitrary number of point charges, as well as continuously-charged objects.
I. INTRODUCTION AND SUMMARY

When an electrically-charged particle accelerates (non-uniformly) in flat spacetime, it exerts a force on itself. This force, known as the self force (or "radiation-reaction force"), results from the particle’s interaction with its own electromagnetic field. Early investigations by Abraham \cite{2} and Lorentz \cite{3}, in the case of non-relativistic motion, showed that the self force is proportional to the time-derivative of the acceleration. Later, Dirac \cite{4} obtained the covariant relativistic expression for the self force:

$$f_{\text{self}}^\mu = \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu),$$

(1)

where $q$ is the electrical charge, $u^\mu$ and $a^\mu$ denote the four-velocity and four-acceleration, respectively, an overdot denotes a proper-time derivative, and $a^2 \equiv a^\mu a_\mu$. Dirac derived this expression by considering the momentum flux through a "world-tube" surrounding the particle’s worldline, and demanding energy-momentum conservation.

The fact that a particle can exert a force on itself is obviously intriguing. One of the ways to make sense of this phenomenon is by considering a charged, rigid, extended object of finite size $\epsilon$. A model of a continuously-charged, finite-size object has the obvious advantage that the electromagnetic field is everywhere regular, allowing (in principle) an almost straightforward calculation of all electromagnetic forces involved (this is of course not the case when a point-like charge is considered, as the field is singular at the particle’s location). On physical grounds, one would expect that an extended object of a sufficiently small size will behave like a point-like particle. (One should expect finite-size correction terms, which may depend on the object’s shape, but one would hope these corrections would become negligible for a sufficiently small object.)

For a finite-size extended object, each charge element exerts an electromagnetic force on each other charge element. Then, roughly speaking, the overall electromagnetic force that the charged object exerts on itself is the sum of the contributions of all mutual forces between all pairs of the object’s charge elements. One might naively expect that this sum will always vanish, by virtue of Newton’s third law; Indeed, this would be the case if the charged object were static: In this case, the sum of the Coulomb forces vanishes for each pair separately. It turns out, however, that if the object accelerates the sum of the contributions

\footnote{Throughout the paper we use the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and units where $c = 1$.}
of all mutual forces does not vanish (this has been established by many authors, as we discuss below, and it is also demonstrated explicitly later in this paper). This nonvanishing overall electromagnetic force does not conflict with the momentum conservation law, because the electromagnetic field itself contains a time-varying amount of momentum and energy; A non-vanishing overall electromagnetic force acting on the object is thus a manifestation of a momentum transfer between the charged object and the surrounding electromagnetic field. In particular, the electromagnetic radiation field carries energy and momentum away from the object to infinity (hence the name "radiation-reaction force").

Recognizing that the overall mutual electromagnetic force does not vanish, one is tempted to identify this overall force with the notion of the self force acting on a charged particle. Thus, one would hope that at the limit were the object’s size is taken to zero, a universal result (independent of the object’s size and shape) will be obtained, which will coincide with Eq. (1). Many attempts have been made to derive this extended-object total force. Two types of models have been considered: objects that are continuously charged \[3, 5, 6, 7\], and objects with a finite number of discrete charges \[7\]. The simplest model of a discretely-charged rigid object is the “dumbbell”, i.e. a fixed-length rod with two point charges located at its two edges. The previous analyses of both the continuous and discrete models revealed that indeed the overall electromagnetic force does not generally vanish. But these analyses also indicated a fundamental difficulty (which we shortly explain), that made it impossible to derive the universal small-size limit of this force. The goal of this paper is to provide a simple resolution to this difficulty.

Let \( f^\mu_{\text{sum}} \) denote the sum of (or, in the continuous model, the double-integral over) all mutual electromagnetic forces, acting on all charge elements at a particular moment. (By "particular moment" we refer here to a hypersurface of simultaneity in the particle’s rest frame; see below.) We would like to explore how \( f^\mu_{\text{sum}} \) depends on \( \epsilon \). For all types of electrically charged objects, the small-\( \epsilon \) dependence of \( f^\mu_{\text{sum}} \) is found to be of the form

\[
 f^\mu_{\text{sum}} = c_{-1}^\mu / \epsilon + c_0^\mu + O(\epsilon). \tag{2}
\]

The \( O(\epsilon) \) term will not concern us here, as it vanishes at the limit \( \epsilon \to 0 \). The coefficients \( c_{-1}^\mu \) and \( c_0^\mu \) depend on the object’s worldline, but are (by definition) independent of \( \epsilon \).

The \( O(\epsilon^{-1}) \) term is the problematic term, as it diverges at the limit of interest, i.e. \( \epsilon \to 0 \). Obviously, the small-object limit does not make sense if we do not know how to handle the
problematic term $c_{-1}/\epsilon$.

Now, there is a standard procedure of mass-renormalization, often used for eliminating such $O(\epsilon^{-1})$ terms. However, the very nature of this procedure requires that the undesired $O(\epsilon^{-1})$ term will be of the form $-ca^\mu$, where $c$ is a parameter that is independent of the time and the state of motion (though it may depend on the object’s size and shape): A force term of the form $a^\mu \cdot \text{const}$ can be dropped, because it is experimentally indistinguishable from an inertial term in the equation of motion (see next section). In order for the whole theory to make sense (assuming that interaction energy equally contributes to the inertial mass), the constant $c$ must be equal to the object’s electrostatic energy, which we denote $E_{es}$. (Recall that the latter scales like $\epsilon^{-1}$.)

The problem is that, the term $c_{-1}\mu/\epsilon$ is actually not equal to $-E_{es}a^\mu$ — and, furthermore, generally it is not in the form $-ca^\mu$, nor it is even in the direction of $a^\mu$. In the special case of a spherical charge distribution, several authors found [3, 5, 6, 7] that the term $c_{-1}\mu/\epsilon$ indeed takes the form $-ca^\mu$, but with $c = (4/3)E_{es}$. This is the well-known "4/3 problem". In this special case the mass-renormalization procedure still makes sense from the operational point of view (because any force term of the form $-ca^\mu$ is experimentally indistinguishable from an inertial term), though the logical consistency of the theory may be questioned.

The situation is worse, however, when the charge distribution is asymmetric. In such a situation one generally finds that the problematic term $c_{-1}\mu/\epsilon$ is not even in the direction of $a^\mu$. Clearly, this type of divergent term cannot be removed by mass renormalization. A simple demonstration of this situation was given by Griffiths and Owen [7], who considered a one-directional motion of a dumbbell. They found that when the dumbbell is oriented perpendicularly to the direction of motion, then $c_{-1}\mu/\epsilon = -E_{es}a^\mu$ as desired. However, if the dumbbell is co-directed with the motion, then $c_{-1}\mu/\epsilon = -ca^\mu$ with $c = 2E_{es}$. Furthermore, if the dumbbell is oriented in any other direction, the term $c_{-1}/\epsilon$ will not be co-directed with $a^\mu$. Clearly, in such a generic situation the problematic term $c_{-1}\mu/\epsilon$ cannot be removed by mass renormalization. As a consequence, the limit $\epsilon \to 0$ of $f_{\text{sum}}^\mu$ does not make a physical sense. We note that this problem arises even if the object’s motion is treated in a fully-relativistic manner [8] — as long as the quantity $f_{\text{sum}}^\mu$ is considered.

It should be noted that in the case of a spherical charge distribution there is another "4/3 problem": When the object is in slow motion, the electromagnetic-field momentum turns out to be 4/3 times the electromagnetic-field energy times the velocity [3]. We may
refer to this problem as the "inertial 4/3 problem" (as opposed to the "mass-renormalization 4/3 problem"). We shall not address this problem here. The relation between these two "4/3 problems" is not completely clear. Poincaré [9] introduced the non-electromagnetic internal stresses in order to resolve the inertial 4/3 problem. On the other hand, the analysis presented below clearly indicates that no consideration of the non-electromagnetic internal forces is required for solving the mass-renormalization 4/3 problem [10].

In this paper we shall provide a simple and natural solution to the above mass-renormalization problem. We shall show that the overall mutual electromagnetic force is not the quantity \( f^\mu_{\text{sum}} \) (i.e. the naive sum or integration over all mutual forces); By employing simple energy-momentum considerations we show that the overall mutual electromagnetic force, which we denote \( f^\mu_{\text{mutual}} \), is the sum (or integral) over all mutual forces, each multiplied by a certain kinematic factor representing the proper-time lapse of each charge element between two "moments" (i.e. between two neighboring hypersurfaces of simultaneity; see next section). This kinematic factor is of the form \( 1 + O(\epsilon) \); and the \( O(\epsilon) \) correction (when multiplying the mutual forces \( \propto \epsilon^{-2} \)) leads to a difference between \( f^\mu_{\text{sum}} \) and \( f^\mu_{\text{mutual}} \), proportional to \( \epsilon^{-1} \), which is exactly the amount required to correct the problematic term \( c^\mu_{-1}/\epsilon \).

Namely, when \( f^\mu_{\text{mutual}} \) is expanded in powers of \( \epsilon \), it takes a form similar to Eq. (2), but with an \( O(1/\epsilon) \) term which is precisely of the form \(-E_{\text{es}} a^\mu \). This \( O(1/\epsilon) \) term is naturally removed by mass renormalization.

After we have eliminated the problematic \( O(\epsilon^{-1}) \) term in Eq. (2), we are left with the regular term \( c^\mu_0 \). It is this term which should yield the desired expression for the self force. With the anticipation that the self force should be universal, one would expect \( c^\mu_0 \) to depend only on the object’s total charge \( q \), and not on the way it is distributed. In fact, the very nature of the self-force phenomenon — the force that a charge exerts on itself — suggests that the self force must be proportional to \( q^2 \). For continuous charge distributions, the term \( c^\mu_0 \) is indeed found to be \( \propto q^2 \) and it can be brought to the form [11]. However, for discrete charge distributions \( c^\mu_0 \) is found to depend on the charge distribution. This is best demonstrated in the simplest discrete model, the dumbbell. In this case, \( c^\mu_0 \) (like the mutual forces) is proportional to the product \( q_+ q_- \), rather than to \( q^2 = (q_+ + q_-)^2 \), where \( q_+ \) and \( q_- \) denote the two edge charges. This apparent inconsistency has an obvious origin: The overall force exerted on the dumbbell by the electromagnetic field includes not only the mutual forces between different charges, but also the forces that each of the individual charges exerts on
itself (to which we shall refer as the “partial self force”, to distinguish it from the ”overall self force” acting on the dumbbell). Obviously, it would be inconsistent to neglect these partial self forces: By universality considerations, one may view each of the point charges as a very small extended charged object; And, the result of our analysis, namely \( c_0^\mu \neq 0 \), should apply to each of these individual charged objects as well, therefore, these forces cannot be ignored. Note that the partial self forces do not depend on the other charges in the extended object, so they are by definition independent of \( \epsilon \). Therefore they do not affect the divergent term \( O(\epsilon^{-1}) \), but merely add to the term \( c_0^\mu \). The need to include these partial self forces might appear disturbing, as these quantities are initially unknown. However, basic considerations imply that the self force acting on each charged object must be proportional to the square of its charge. This observation provides us with the required expression for the partial self forces (more precisely, the relation of the latters to the overall self forces). The inclusion of the partial self forces leads to a universal expression for the overall electromagnetic force acting on the dumbbell (or on any other discrete charge distribution), which is indeed proportional to the square of the total charge as desired, and which coincides with Eq. (1). We further show in Sec. [IV] that for a continuous charge distribution the contribution of the partial self forces vanishes. Therefore, in the continuous case the quantity \( c_0^\mu \) (i.e. the properly weighted integral of the mutual electromagnetic forces) directly yields the desired universal expression for the self force, Eq. (1).

The overall mutual force \( f_{\text{mutual}}^\mu \) may naturally be viewed as the sum (or double-integral) of the contributions of all pairs of charge element. The contribution of each such pair is the sum of the two mutual forces, each weighted by the above mentioned kinematic factor. In summing these two forces, the dominant \( O(\epsilon^{-2}) \) term always cancels out (leaving a weaker divergence \( \propto \epsilon^{-1} \) that is in turn handled by mass-renormalization). This leading-order cancellation occurs for each pair separately, suggesting that the fundamental element in any extended-object model is the single pair of charges. Once the single-pair system is well understood, the analysis of any charge distribution will follow quite immediately — essentially by summing (or double-integrating) over all pairs of charge elements. We shall therefore start by analyzing the dumbbell model, i.e. a pair of point-like charges separated by a fixed-length rod. Then we shall consider a discrete system with an arbitrary number \( N \) of charges. Then, taking the infinitesimal limit (in which \( N \to \infty \)), we shall analyze the case of continuous charge distribution. In all cases the object (and the charge distribution)
is regarded as rigid, and we allow it to move (non-rotationally) along an arbitrary worldline. For both the discrete and continuous cases, we obtain the same universal result: After calculating the overall mutual electromagnetic force $f_{\text{mutual}}$, mass-renormalizing it, and then taking the limit $\epsilon \to 0$, we recover the desired expression (11) for the self force.

We should mention here previous analyses which seemingly overcame the mass-renormalization $4/3$ problem in the spherical case. First, Fermi [15] carried out an extended-object analysis of a different type: Instead of summing the contributions of all mutual forces, he constructed an effective relativistic Hamiltonian of a charged rigid body, and derived the equation of motion from this Hamiltonian. It seems that no ”4/3 problem” is encountered in this method. Later, Nodvick used a similar method [16] and obtained the correct expression for the self force (note, however, that these analyses [15, 16] only considered spherically symmetric distributions, whereas we are treating here an arbitrary charge distribution). Also, after this work was completed, we became aware of a previous work by Pearle [11], in which he analyzed the case of a spherically-symmetric charged object. In this analysis he took into account the above mentioned kinematic weighting factor which expresses the proper-time lapse of each charge element. Then, in a fairly complicated calculation he obtained the correct $O(1/\epsilon)$ term, namely $-E_{es}a^\mu$, thereby overcoming the 4/3 problem in the case of spherical charge distribution. We believe that our analysis is simpler, more transparent, and it is also much more general; In particular, the analysis presented here resolves the mass-renormalization problem for any type of charge distribution.

We point out that self force calculations were also carried out in the context of quantum theory, see for example [17]. This however does not diminish the need for a consistent classical treatment of the self-force problem. Indeed in some situations the classical framework is certainly the most natural one. Consider for example the self force acting on an electrically-charged satellite orbiting the earth. Obviously quantum effects are irrelevant in this case, and it is therefore desired to treat this situation from the purely classical point of view. An example of much greater current interest is that of a compact object orbiting a black hole of much larger mass, see e.g. [18]. The compact object will gradually inspiral towards the black hole, due to the gravitational self force. Again, there is no reason to use the quantum theory in this case. Note that in this case one must use the formalism of gravitational self force in curved spacetime. Whereas the present paper only deals with flat space, it appears that the extended-object approach can naturally be extended to curved space as well. For
example, Ori recently applied the extended-object approach to the case of radial free-fall in Schwarzschild spacetime [19], and used it to calculate the regularization parameters [18].

An outline of the analysis given here was published recently, focusing on the case of dumbbell-like charge distribution [1]. Here we present the full calculations, and also analyze in detail extended objects with an arbitrary number of point charges, as well as continuously-charged extended objects (these cases were only briefly mentioned in Ref. [1]).

In Sec. II we analyze the dumbbell model, i.e. the case of two point-like charges. We first formulate the dumbbell’s relativistic kinematics. Then we calculate the mutual forces, obtain their sum $f_{\text{sum}}$, and (following Griffiths and Owen [7]) demonstrate the severe mass-renormalization directionality problem discussed above. Then we use energy-momentum considerations to construct the correct expression for the overall mutual electromagnetic force $f_{\text{mutual}}$. We show that the latter is free of the mass-renormalization problem. In Sec. III we extend the analysis to a system with an arbitrary number of point-like charges. Finally, in Sec. IV we consider the case of a continuous charge distribution. In all three cases we obtain, at the limit $\epsilon \to 0$, the universal result (1).

II. A CHARGED DUMBELL

A. The general approach

We consider a dumbbell made of a rigid rod with two point charges located at its two edges. The forces acting on the dumbbell (or on its parts) may be schematically divided into several types:

- Electromagnetic forces: the forces exerted on the two charges by the electromagnetic field they produce;

- The ”other internal forces”: the inter-atomic (or ”elastic”) forces that are responsible for the dumbbell’s rigidity;

- External forces: forces exerted on the dumbbell by external fields.\(^2\)

\(^2\) Throughout this paper, by ”electromagnetic forces” we shall always refer to the forces associated with the interaction of the two charges with the electromagnetic field they themselves produce, and not to external forces (or the ”other internal forces”) even if the latters are of electromagnetic origin. Namely,
The electromagnetic forces acting on the two charges are divided into two types: (i) Mutual electromagnetic forces, i.e. forces that one charge exerts on the other one [more precisely, it is the force that the electromagnetic field produced by one charge (in the sense of the retarded Lienard-Wiechert solution) exerts on the other charge]; and (ii) the self forces that each of the two charges exerts on itself, to which we shall refer as the "partial self forces" (to be distinguished from the overall self force acting on the dumbbell). The justification and necessity of including the partial self forces in our analysis is discussed below; but two remarks should be made already at this stage: First, the partial self forces are not relevant to the mass-renormalization problem, as they only affect the term \( c_0^\mu \) (above), not the problematic term \( c_1^\mu / \epsilon \). Second, these partial self forces are very significant for a system of two charges (they contribute at least as much as the mutual forces do), but they become less important in a system including a large number \( N \) of point charges (assuming that the magnitude of the individual charges scales like \( 1/N \)). This is because the number of mutual forces scales like \( N^2 \), whereas the number of partial self forces scales like \( N \). Most importantly, the contribution of partial self forces vanishes at the continuum limit, as we discuss in Sec. IV.

In Newtonian theory it is usually presumed that the sum of any pair of mutual forces will always vanish; However, when electromagnetic interactions are concerned, this presumption does not hold. Its failure may be attributed to the long range of the electrodynamical interaction between two charges. It is this long range which is responsible for the electromagnetic radiative phenomena (which transport energy and momentum away from the interacting charges). On the other hand, the non-electromagnetic internal forces are assumed here to be of "short range". Hence, it will be assumed that upon summation these forces will always cancel out (except for a "mass-renormalization like" term, which is the interaction energy associated with these forces, multiplied by the four-acceleration). For this reason, the non-external forces that are relevant to the calculations below are only the electromagnetic forces. Our terminology is based on the simplified picture according to which the external forces and the "other internal forces" are non-electromagnetic. However, this presumption is only made for simplifying the terminology, and the analysis below is valid even if these forces are of electromagnetic character.

For our discussion it is sufficient to assume that the range of the "other internal forces" is small compared to the dumbbell's length. This assumption perfectly holds for e.g. the inter-atomic forces that are responsible for the rod's rigidity. It is also sufficient to assume that this range is small compared to the time scales charactering the world line \( z^\mu(\tau) \). e.g. \( 1/a, a/\dot{a} \) etc.
ones, namely, (i) the two mutual forces between the two charges, and (ii) the two "partial self forces”.

\section*{B. Dumbbell’s structure and kinematics}

The dumbbell consists of two point charges situated at the edges of a rigid rod of a proper length $2\epsilon$. We shall assume that $\epsilon$ is small compared to $1/a$, where $a$ denotes the norm of the acceleration vector. Throughout this section we shall use the subscripts "+" or "−" to denote the quantities associated with the two dumbbell’s edges. The two electric charges are therefore denoted $q_+$ and $q_-$, respectively, and the total charge is $q \equiv q_+ + q_-$. We do not require the two charges to be equal. We assume that $\epsilon$ is time-independent and that the dumbbell moves in a non-rotational manner (see below).

We take the dumbbell’s central point (i.e. half the way between the two edges) to represent the dumbbell’s motion. The worldline of this representative point is denoted $z^\mu(\tau)$, where $\tau$ is the proper time along the central worldline. The four-velocity and four-acceleration of the central worldline are defined in the usual manner, $u^\mu \equiv \dot{z}^\mu$ and $a^\mu \equiv \dot{u}^\mu$, where an overdot denotes differentiation with respect to $\tau$. We denote the two rod’s edges by $z^\mu_+(\tau)$ and $z^\mu_-(\tau)$. At any given moment (by "moment" we mean here a hypersurface of simultaneity in the momentary rest frame of the central point) the two rod’s edges are located at (see Fig. 1)

\begin{equation}
z^\mu_\pm(\tau) \equiv z^\mu(\tau) \pm \epsilon w^\mu(\tau) ,
\end{equation}

where $w^\mu(\tau)$ is a unit spatial vector, satisfying

\begin{equation}
w_\mu w^\mu = 1 , \quad w^\mu u_\mu = 0 .
\end{equation}

The time evolution of $w^\mu$ is subjected to two restrictions: First, as a unit vector, its norm is time-independent (corresponding to a rod of fixed length). Second, it is non-rotating in the momentary rest frame of $z^\mu(\tau)$. These restrictions correspond to a Fermi-Walker transport \cite{20} of $w^\mu$ along $z^\mu(\tau)$, given by

\begin{equation}
\dot{w}^\mu = (u^\mu a^\nu - u^\nu a^\mu)w_\nu = u^\mu a_w ,
\end{equation}

where the scalar $a_w$ denotes the projection of $a^\mu$ on the rod’s direction: $a_w \equiv a_\lambda w^\lambda$. This transport rule guarantees that if Eq. (4) is initially satisfied, it will hold at all subsequent times.
Next, we calculate the four-velocities and accelerations of the dumbbell’s edges. We denote the proper times along the worldlines of the two rod’s edges $z^\mu_\pm$ by $\tau_\pm$, respectively. Note that generally $\tau_+$ and $\tau_-$ differ from $\tau$ (and from each other). The four-velocities of the two charges are defined in the usual manner, $u^\mu_\pm \equiv dz^\mu_\pm/d\tau_\pm$. Differentiating Eq. (3), with respect to $\tau_\pm$, we obtain

$$u^\mu_\pm = (u^\mu \pm \epsilon \dot{u}^\mu) \frac{d\tau}{d\tau_\pm} = \left[(1 \pm \epsilon a_w) \frac{d\tau}{d\tau_\pm}\right] u^\mu.$$

(6)

Taking the norm of the two sides of this equation, recalling that both $u^\mu$ and $u^\mu_\pm$ are of unit norm, we find that the term in squared brackets is just unity, namely

$$\frac{d\tau_\pm}{d\tau} = 1 \pm \epsilon a_w.$$

(7)

It now immediately follows that

$$u^\mu_\pm = u^\mu.$$

(8)

These are the two key features of the rod’s kinematics.

Eq. (8) indicates that in the rest frame of the dumbbell’s central point, the two edges (and similarly any other point on the dumbbell) are at rest as well. We can therefore identify this reference frame as the rest frame of the entire dumbbell. Since at any moment there exists a reference frame in which the entire dumbbell is momentarily at rest, it is justified to view this type of motion as a rigid motion.

We denote by $a^\mu_\pm$ the four-accelerations of the two edge points, namely $a^\mu_\pm = du^\mu_\pm/d\tau_\pm$. From Eqs. (7) and (8) it immediately follows that

$$a^\mu_\pm = \frac{a^\mu}{1 \pm \epsilon a_w}.$$

(9)

C. Mutual forces

At the heart of the dumbbell’s model are the mutual forces acting between the two charges. To determine these forces, we need an expression for the retarded electromagnetic field tensor $F_{\mu\nu}$ that a single point charge $q$ moving on an arbitrary worldline $z^\mu(\tau)$ produces at a nearby point $z^\mu + \hat{\epsilon}\hat{w}^\mu$, where $\hat{\epsilon}$ is a small positive number ($\hat{\epsilon} = 2\epsilon$), and $\hat{w}^\mu$ is a unit spatial vector satisfying $\hat{w}^\mu\hat{w}_\mu = 1, \hat{w}^\mu u_\mu = 0$. Later we shall apply the limit $\epsilon \to 0$, and
therefore we shall only need an expression for $F_{\mu\nu}$ valid up to zero order in $\dot{\epsilon}$. Such an expression was derived by Dirac \[4\]:

$$F_{\mu\nu} \approx \frac{q}{\sqrt{(1 + \epsilon a_{\dot{w}})}} \left[ \frac{u_{\mu} \dot{w}_{\nu}}{\epsilon^2} + \frac{a_{\mu} u_{\nu}}{2\epsilon} + \frac{a^2 u_{\mu} \dot{w}_{\nu}}{8} - \frac{\dot{a}_{\mu} \dot{w}_{\nu}}{2} - \frac{a_{\dot{w}} a_{\mu} u_{\nu}}{2} - \frac{2}{3} \dot{a}_{\mu} u_{\nu} \right]$$

\[\approx q \left[ \frac{u_{\mu} \dot{w}_{\nu}}{\epsilon^2} + \frac{a_{\mu} u_{\nu} - a_{\dot{w}} u_{\mu} \dot{w}_{\nu}}{2\epsilon} - \frac{2}{3} \dot{a}_{\mu} u_{\nu} + \hat{\mathcal{Z}}_{\mu\nu} \right], \quad (\mu \leftrightarrow \nu) \tag{10}$$

where $a_{\dot{w}} \equiv a_{\lambda} \dot{w}_{\lambda}$,

$$\hat{\mathcal{Z}}_{\mu\nu} \equiv \frac{a^2 u_{\mu} \dot{w}_{\nu}}{8} - \frac{\dot{a}_{\mu} \dot{w}_{\nu}}{2} + \frac{3a_{\dot{w}} a_{\mu} u_{\nu}}{8} - \frac{3a_{\dot{w}} a_{\mu} u_{\nu}}{4}, \quad (11)$$

$a^2 \equiv a_{\mu} a^\mu$, and throughout this paper the "$\approx$" symbol denotes an equality up to $O(\epsilon)$ correction terms. $\hat{\mathcal{Z}}_{\mu\nu}$ is the collection of all terms that are proportional to $\epsilon^0$ and to an odd power of $\dot{w}$. Such terms will cancel out when summing the contributions of the two charges (see below). The electromagnetic field $F^\mu_{\tau}$ that the charge $q_-$ produces at the location of charge $q_+$ is obtained by substituting in Eqs. 10,11 $q \to q_-$, $a \to a_-$, $\dot{a} \to da_-/d\tau_-$, $\dot{w}^\mu \to w^\mu$, $a_{\dot{w}} \to a_{\lambda}^\lambda w_{\lambda}$, and $\dot{\epsilon} \to 2\epsilon$ (the four-velocity is unchanged, as $u^\mu_+ = u^\mu$). The electromagnetic field $F^\mu_{\nu}$ that the charge $q_+$ produces at the location of the charge $q_-$ is obtained in a similar manner, by substituting in these equations $q \to q_+$, $a \to a_+$, $\dot{a} \to da_+/d\tau_+$, $\dot{w}^\mu \to -w^\mu$, $a_{\dot{w}} \to -a_{\lambda}^\lambda w_{\lambda}$, and $\dot{\epsilon} \to 2\epsilon$. Let us denote by $f_\pm^\mu (f^\mu_\pm)$ the Lorentz force that the charge "-" ("+") exerts on the other charge "+", ("-"):

$$f_\pm^\mu = q_\pm F^\mu_{\tau} u_\nu.$$  By virtue of Eq. 10 this becomes

$$f_\pm^\mu \approx q_+ q_- u_\nu \left[ \frac{u^\mu w^\nu}{4\epsilon^2} + \frac{a^\mu_{\tau} u^\nu}{4\epsilon} - \frac{2}{3} \dot{a}_{\tau}^\mu u^\nu \pm Z^\mu_{\tau} \right], \quad (\mu \leftrightarrow \nu) \tag{12}$$

where

$$Z^\mu_{\tau} \equiv \frac{a^2 u^\mu w^\nu}{8} - \frac{\dot{a}_{\tau}^\mu u^\nu}{2} + \frac{3(a_{\lambda}^\lambda w_{\lambda})^2 u^\mu w^\nu}{8} - \frac{3(a_{\lambda}^\lambda w_{\lambda}) a_{\tau}^\mu u^\nu}{4},$$

and $\dot{a}_{\tau}^\mu \equiv da_{\tau}^\mu / d\tau_\tau$. Next we re-express $f_{\pm}^\mu$ in terms of the acceleration $a^\mu$ and proper time $\tau$ of the central point (rather than those of the source charges). To this end we use Eq. 2, and expand $a_{\tau}^\mu$ in $\epsilon$. Since the acceleration does not appear in the $O(\epsilon^{-2})$ term, it is sufficient to carry out this expansion up to first order in $\epsilon$:

$$a_{\tau}^\mu = a^\mu (1 \pm \epsilon a_w) + O(\epsilon^2).$$

12
Note that \( \dot{a}_\pm^\mu \) only appears in the \( O(\epsilon^0) \) term, hence it can be replaced by \( \dot{a}^\mu \). [The same holds for all factors \( a_\pm^\mu \) and \( a_w^2 \) that appear in the \( O(\epsilon^0) \) term.] We find

\[
f_\pm^\mu \approx q_+ q_- u_\nu \left[ \left( \pm \frac{w^\mu w^\nu}{4\epsilon^2} + \frac{a^\mu w^\nu - a_w w^\mu w^\nu}{4\epsilon} - \frac{2}{3} \dot{a}^\mu u^\nu \pm Z^{\mu\nu} \right) - (\mu \leftrightarrow \nu) \right],
\]

where

\[
Z^{\mu\nu} \equiv Z^{\mu\nu}_\pm (a_\pm^\mu \rightarrow a^\mu) + \frac{a_w a^\mu u^\nu - a_w^2 w^\mu w^\nu}{4}.
\]

Note that \( Z^{\mu\nu} \) is \( O(\epsilon^0) \), and is the same for the two charges. Recalling that \( w^\nu u_\nu = -1 \), \( u^\nu u_\nu = a^\nu u_\nu = 0 \), and \( \dot{a}^\nu u_\nu = -a^2 \) (the latter identity is obtained by differentiating \( a^\nu u_\nu = 0 \)), we find

\[
f_\pm^\mu \approx q_+ q_- \left[ \pm \frac{w^\mu}{4\epsilon^2} - \frac{a^\mu + w^\mu a_w}{4\epsilon} + \frac{2}{3} (\dot{a}^\mu - a^2 u^\mu) \right] \pm Z^\mu,
\]

D. Naive sum of the mutual forces

Next we calculate the sum of the two mutual forces, i.e. the quantity \( f_{\text{sum}}^\mu \):

\[
f_{\text{sum}}^\mu \equiv f_+^\mu + f_-^\mu \approx -\frac{q_+ q_-}{2\epsilon} (a^\mu + w^\mu a_w) + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu).
\]

This quantity would be the simplest candidate for the dumbbell's self force; However, as already discussed in the previous section, it suffers from a serious problem: The first term on the right-hand side is proportional to \( 1/\epsilon \), and hence diverges at the limit of interest, \( \epsilon \rightarrow 0 \). The usual way to eliminate such an undesired \( O(\epsilon^{-1}) \) term is by the procedure of mass renormalization (see below); However, from the very nature of this procedure, it will only be applicable if the term to be removed is of the form \( a^\mu \cdot \text{const} \) (a constant that scales like \( 1/\epsilon \)). Instead, in Eq. (15) the term \( a^\mu + w^\mu a_w \) is orientation-dependent. Furthermore, this term is not co-directed with \( a^\mu \). This difficulty was observed by Griffiths and Owen \[7\].

4 [Note that adding the two ’’partial self forces’’ would not change this situation, as it does not affect the \( O(\epsilon^{-1}) \) term – see below.]

4 When a spherical charge distribution is considered, after integrating \( f_{\text{sum}}^\mu \) over the charge distribution, one obtains an overall force which is co-directed with \( a^\mu \), due to the symmetry. In this case the only imprint of the problem in the \( O(1/\epsilon) \) term of \( f_{\text{sum}}^\mu \) is the ”4/3 problem”. However, for generic non-symmetric distributions, the integrated force will not be co-directed with \( a^\mu \).
E. Energy-momentum balance

The above pathology of the $O(1/\epsilon)$ term clearly indicates that $f^\mu_{\text{sum}}$ is not a valid candidate for the dumbbell’s self force. The reason is that, $f^\mu_{\text{sum}}$ does not correctly represent the overall mutual force. To understand the reason for this, we shall now employ simple considerations of energy-momentum conservation. These considerations will indicate the appropriate way to sum the two mutual forces, in order to obtain the correct expression for the overall mutual force.

Let us denote the total dumbbell’s four-momentum, at a given moment $\tau$, by $p^\mu(\tau)$. This quantity is to be obtained by integrating the appropriate components of the dumbbell’s stress-energy tensor over the hypersurface of simultaneity, which we denote $\sigma$. Recalling that $u^\nu$ is normal to $\sigma$, we may write this integral as

$$p^\mu \equiv - \int_\sigma T^{\mu\nu}_{(\text{dumb})} u_\nu d^3 \sigma.$$  \hspace{1cm} (16)

Here $d^3 \sigma$ is a volume element, and $T^{\mu\nu}_{(\text{dumb})}$ denotes the dumbbell’s stress-energy tensor, not including the electromagnetic field. The integration is performed over the entire volume of the dumbbell (the integrand vanishes off the dumbbell).

It is worth emphasizing two points here: First, the integration is carried out over a hypersurface of simultaneity, defined at each moment by the dumbbell’s motion, and not over a hypersurface $t = \text{const}$ of some fixed Lorentz frame. This is the natural covariant way to define the time-dependent four-momentum of a rigid body. Secondly, we choose not to include the electromagnetic stress-energy tensor in $p^\mu$, because the electromagnetic contribution is not well localized: It is partly scattered throughout the space in the form of electromagnetic waves. The non-electromagnetic part, however, is by assumption well-localized, and hence monitoring $p^\mu(\tau)$ will provide us with the desired information concerning the dumbbell’s motion. Note that the external field (i.e. the above mentioned ”external force”) is also not included in $T^{\mu\nu}_{(\text{dumb})}$.

From energy-momentum conservation it follows that $p^\mu(\tau)$ will only change due to external forces acting on the dumbbell (if such exist), and due to energy-momentum exchange

---

5 Note that an integration over a hypersurfaces $t = \text{const}$ of the Lorentz frame in use would produce a quantity $p^\mu(\tau)$ that transforms in a complicated, non-covariant manner in a Lorentz transformation. On the other hand, our $p^\mu(\tau)$ (defined by integration over the hypersurface of simultaneity) transforms like a four-vector, as desired.
between the dumbbell and the electromagnetic field. The electromagnetic energy-momentum exchange is manifested by the electromagnetic forces acting on the two charges. In an infinitesimal time interval \( d\tau \), the change in \( p^\mu(\tau) \) will be given by

\[
dp^\mu = dp_+^\mu + dp_-^\mu + dp_{\text{ext}}^\mu,
\]

where \( dp_{\text{ext}}^\mu \) is the contribution of the external force, and \( dp_+^\mu \) denote the contributions from the electromagnetic forces acting on the two charges \(^6\). Let us denote these electromagnetic forces by \( f_{(em)}^\mu \). As discussed above, \( f_{(em)}^\mu \) includes both the mutual electromagnetic force \( f_\pm^\mu \), and the partial self force acting on the \( \pm \) charge, which we denote \( \hat{f}_\pm^\mu \):

\[
f_{(em)}^\mu = f_\pm^\mu + \hat{f}_\pm^\mu.
\]

Note that simple consistency considerations require us to include the partial self forces in the analysis: Our calculation shows (as many previous analyses did) that there is a nonvanishing self force acting on a charged object (the dumbbell, in our specific model); This force is found to be universal (at the limit of small \( \epsilon \)), namely it is independent of the object’s size and orientation. It must therefore apply to any sufficiently-small charged object – and, in particular, to the two point charges \( q_+ \) and \( q_- \). Later we shall employ a simple argument to quantitatively relate the two partial self forces \( \hat{f}_\pm^\mu \) to the overall self force acting on the dumbbell. (It should be emphasized that the calculation below yields a nonvanishing overall self force even if one does not take into account the partial self forces; Nevertheless the resultant expression for the self force would be incorrect in such a case, due to the inconsistency.) Note that the need for adding the partial self forces is also made obvious from the following observation: Without the partial self forces, the overall mutual electromagnetic force is proportional to the product \( q_+ q_- \), whereas the overall self force of the dumbbell (like that of any charged particle) must be proportional to \( q^2 = (q_+ + q_-)^2 \).

Adding the partial self forces compensates for this difference exactly, as we show below.

Let us now calculate \( dp_+^\mu \), the energy-momentum exchange of the ”+” charge with the electromagnetic field, between the two hypersurfaces of simultaneity \( \tau \) and \( \tau + d\tau \). An observer located at the ”+” charge will measure a proper-time interval \( d\tau_+ \) between these two hypersurfaces. Therefore, the amount of electromagnetic energy-momentum transfer

\(^6\) By ”electromagnetic forces” we refer here to the forces exerted on the \( \pm \) charges by the electromagnetic fields produced by these two charges, as explained above.
is \[ dp_\mu^\pm = f_\mu^{(em)} d\tau_\pm. \] Similar considerations will apply of course to the other charge "]"; therefore,

\[ dp_\mu^\pm = f_\mu^{(em)} d\tau_\pm. \] (19)

Combining equations (17), (19) and (18), we obtain

\[ dp_\mu = (f_\mu^+ + \hat{f}_\mu^+) d\tau_+ + (f_\mu^- + \hat{f}_\mu^-) d\tau_- + dp_\mu^{ext}. \] (20)

Defining the overall force acting on the system to be \( f_\mu \equiv dp_\mu / d\tau \), we find

\[ f_\mu \approx \left\{ f_\mu^+ \frac{d\tau_+}{d\tau} + f_\mu^- \frac{d\tau_-}{d\tau} \right\} + (\hat{f}_\mu^+ + \hat{f}_\mu^-) + f_\mu^{ext}. \] (21)

Note that since the external force is presumably regular (i.e. it is well-behaved at the limit of small \( \epsilon \)), and \( d\tau_\pm / d\tau \to 1 \) at the limit \( \epsilon \to 0 \), we can simply take \( dp_\mu^{ext} \approx f_\mu^{ext} d\tau \). For the same reason, since the partial self forces are presumably regular too, we can ignore the factors \( d\tau_\pm / d\tau \) multiplying \( \hat{f}_\mu^\pm \). It is only the mutual force \( f_\mu^\pm \), which includes negative powers of \( \epsilon \), that requires one to make the distinction between \( d\tau \) and \( d\tau_\pm \).

The overall mutual electromagnetic force \( f^{\mu}_{\text{mutual}} \) is the term in curly brackets in Eq. (21):

\[ f^{\mu}_{\text{mutual}} = f^{\mu^+} \frac{d\tau_+}{d\tau} + f^{\mu^-} \frac{d\tau_-}{d\tau}. \] (22)

Using Eqs. (7) and (14), and again neglecting terms that vanish as \( \epsilon \to 0 \), we find

\[ f^\pm_d \frac{d\tau_\pm}{d\tau} = (1 \pm \epsilon a_w) f^{\pm}_d \approx q_+ q_- \left[ \frac{w^\mu}{4\epsilon} - \frac{a^\mu}{4\epsilon} + \frac{2}{3}(\dot{a}^\mu - a^2 u^\mu) \pm \tilde{Z}^\mu \right], \]

where

\[ \tilde{Z}^\mu = Z^\mu - \frac{a_w(a^\mu + w^\mu a_w)}{4}. \]

It now follows that

\[ f^{\mu}_{\text{mutual}} \approx -q_+ q_- \frac{a^\mu}{2\epsilon} + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu). \] (23)

The overall electromagnetic contribution to the total force \( f^\mu \) acting on the dumbbell (not including the external force) is the term in squared brackets in Eq. (21), i.e. the sum of \( f^{\mu}_{\text{mutual}} \) and the two partial self forces. We shall refer to it as the "bare self force" (because subsequently we shall apply to it the mass-renormalization procedure, to obtain the "renormalized self force"), and denote it \( f^{\mu}_{\text{bare}} \). It is given by
\[ f^\mu_{\text{bare}} = f^\mu_{\text{mutual}} + (\hat{f}^\mu_+ + \hat{f}^\mu_-) = -\frac{q_+ q_-}{2\epsilon} a^\mu + \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) + (\hat{f}^\mu_+ + \hat{f}^\mu_-) + O(\epsilon). \] 

(24)

F. Mass-renormalization and the renormalized self force

In Eq. (24) [like in Eq. (23)] the \(O(1/\epsilon)\) term has the desired form \(-E_{es} a^\mu\), where \(E_{es}\) is the dumbbell’s electrostatic energy (at rest):

\[ E_{es} \equiv \frac{q_+ q_-}{2\epsilon}. \]

This is exactly the type of \(O(1/\epsilon)\) term that is cured by mass renormalization, as we now briefly discuss.

The expression for the self force is to be used for predicting the dumbbell’s motion, through an equation of motion of the form \(m_{\text{bare}} a^\mu = f^\mu\), where \(f^\mu\) refers to the total force acting on the dumbbell, i.e. \(f^\mu = f^\mu_{\text{bare}} + f^\mu_{\text{ext}}\). (Below we shall further discuss the justification to this equation of motion.) Similarly, \(m_{\text{bare}}\) refers to the so-called ”bare mass”, i.e. the total dumbbell’s energy (in the momentary rest frame) not including the electromagnetic/electrostatic interaction energy. We now add the term \(E_{es} a^\mu\) to both sides of the equation of motion. Defining the ”renormalized mass” \(m_{\text{ren}}\) and ”renormalized self force” \(f^\mu_{\text{ren}}\) by

\[ m_{\text{ren}} \equiv m_{\text{bare}} + E_{es}, \quad f^\mu_{\text{ren}} \equiv f^\mu_{\text{bare}} + E_{es} a^\mu, \] 

(25)

the equation of motion now takes the form

\[ m_{\text{ren}} a^\mu = f^\mu_{\text{ren}} + f^\mu_{\text{ext}}. \]

This is the ”renormalized equation of motion”. Note that \(m_{\text{ren}}\) is nothing but the total dumbbell’s energy (including the electrostatic interaction) while at rest. This is in fact the measured physical mass of the dumbbell. To simplify the notation, we shall hereafter omit the suffix ”ren”, denoting the renormalized mass by \(m\) and the ”renormalized self force” by \(f^\mu_{\text{self}}\). The equation of motion now reads

\[ m a^\mu = f^\mu_{\text{self}} + f^\mu_{\text{ext}}, \]
where

\[ f_{\text{self}}^\mu \equiv f_{\text{bare}}^\mu + E_{es} a^\mu = \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) + (\dot{f}_+^\mu + \dot{f}_-^\mu) + O(\epsilon). \]  

(26)

Now that we eliminated the problematic \( O(1/\epsilon) \) term, we can safely take the limit \( \epsilon \rightarrow 0 \). It is at this limit where we expect to obtain the universal expression \( f_{\text{self}}^\mu \) for the self force. In this limit all the \( O(\epsilon) \) correction terms vanish, and we find

\[ f_{\text{self}}^\mu = \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) + (\dot{f}_+^\mu + \dot{f}_-^\mu). \]  

(27)

As it stands, Eq. (27) provides a single relation for three unknowns, \( \dot{f}_\pm^\mu \) and \( f_{\text{self}}^\mu \). In order to extract from it the expression for \( f_{\text{self}}^\mu \), we need to relate the latter to the two partial self forces \( \dot{f}_\pm^\mu \). Since the self-force is the force that a charge experiences due to its own field, it must be proportional (for a prescribed worldline) to \( q^2 \), where \( q \) is the particle’s charge. In the limit of interest, \( \epsilon \rightarrow 0 \), the trajectories of the two charges \( \pm \), and also that of the dumbbell itself (i.e. the representative point), all converge to the same worldline. Therefore, the two partial self forces \( \dot{f}_\pm^\mu \) will be given by \( \dot{f}_\pm^\mu = (q_\pm^2/q^2)f_{\text{self}}^\mu \), where \( q = q_+ + q_- \) is the dumbbell’s total charge. Substituting this in Eq. (27), rewriting it as

\[ \frac{4}{3} q_+ q_- (\dot{a}^\mu - a^2 u^\mu) = f_{\text{self}}^\mu - (\dot{f}_+^\mu + \dot{f}_-^\mu) = \left[ 1 - \frac{q_\pm^2}{q^2} - \frac{q_-^2}{q^2} \right] f_{\text{self}}^\mu, \]  

(28)

and noting that the term in squared brackets is nothing but \( 2q_+ q_- / q^2 \), we finally obtain the desired expression for the self force:

\[ f_{\text{self}}^\mu = \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu). \]  

(29)

This agrees with Dirac’s expression (1).

To summarize, let us formulate all elements of the above construction of \( f_{\text{self}}^\mu \) by a single mathematical expression. This expression takes the form

\[ f_{\text{self}}^\mu = \frac{q^2}{2q_+ q_-} \lim_{\epsilon \rightarrow 0} \left[ (1 + \epsilon a_w) f_+^\mu + (1 - \epsilon a_w) f_-^\mu + \frac{q_+ q_- a^\mu}{2\epsilon} \right]. \]  

(30)

This involves the following manipulations, which are all justified (and necessitated) by simple physical considerations: (i) the proper-time weighting of the two mutual force (the factors \( 1 \pm \epsilon a_w \)); (ii) mass-renormalization (the last term in the squared brackets); (iii) the inclusion of the partial self forces (the factor \( q^2/2q_+ q_- \)); and (iv) taking the limit \( \epsilon \rightarrow 0 \). This expression yields a universal, orientation-independent, result, which conforms with the well known expression (1) for the self force.
Finally, we briefly discuss the justification of the ("bare") equation of motion \( m_{\text{bare}} a^\mu = f^\mu \) in our case. We have defined the total force \( f^\mu \) as the proper-time derivative of the dumbbell’s non-electromagnetic energy-momentum \( p^\mu \). Let us transform to a Lorentz frame in which the dumbbell is momentarily at rest. In this frame Eq. (16) reads

\[
p^\mu \equiv \int_{t=\text{const}} T^\mu_{(dumb)} d^3x^i ,
\]

(31)

where \( x^i \) denotes the three spatial Cartesian coordinates. For simplicity let us approximate the dumbbell’s stress-energy by that of a continuous matter (plus, possibly, arbitrary number of point masses situated at fixed locations on the dumbbell). Since the matter that composes each element of the dumbbell is momentarily at rest, \( T^0_{(dumb)} \) vanishes, and hence \( p^i = 0 \). The dumbbell’s energy in the rest frame is

\[
p^0 \equiv \int_{t=\text{const}} T^{00}_{(dumb)} d^3x^i .
\]

(rest frame) (32)

This is by definition the dumbbell’s bare mass. Thus, in the momentary rest frame we have \( p^\mu = (m_{\text{bare}}, 0, 0, 0) \). Rewriting this in a covariant form (valid in any Lorentz frame), we obtain

\[
p^\mu = m_{\text{bare}} u^\mu .
\]

Since the dumbbell is approximated as rigid, its composition does not change in time, hence \( m_{\text{bare}} \) is time-independent. Differentiating now \( p^\mu \) with respect to proper time, we obtain the desired equation of motion

\[
f^\mu = m_{\text{bare}} a^\mu .
\]

Recall that this is the "bare" equation of motion. After mass renormalization, we obtain the equation of motion in its final, renormalized form [21]:

\[
ma^\mu = f^\mu_{\text{self}} + f^\mu_{\text{ext}} = \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu) + f^\mu_{\text{ext}} .
\]

(33)

III. EXTENDED OBJECT WITH \( N \) POINT CHARGES

In this section we shall consider a rigid extended object with an arbitrary number \( N \) of point charges located on it. The charges are denoted \( q_i \), where hereafter roman indices like \( i, j, ... \) run from 1 to \( N \). The total charge is \( q = \sum_i q_i \). We shall calculate the overall self force acting on the object, by a natural extension of the method used above in the dumbbell case.
A. Extended Object Kinematics

We start by describing the extended object kinematics. We choose (quite arbitrarily) a representative point inside this object and denote its worldline by $z^\mu(\tau)$, and its four-velocity and four-acceleration by $u^\mu \equiv dz^\mu/d\tau$ and $a^\mu \equiv du^\mu/d\tau$, respectively, where $\tau$ is the proper time along this worldline.

The location of a charge $i$ at each moment $\tau$ is given by
\[ z^\mu_i(\tau) \equiv z^\mu(\tau) + \epsilon_i w^\mu_i(\tau), \tag{34} \]
where $\epsilon_i \geq 0$ is the distance of the charge $i$ from the representative point, and $w^\mu_i(\tau)$ is a unit spatial vector normal to $u^\mu(\tau)$. We denote the proper time of this worldline by $\tau_i$ and its four-velocity and four-acceleration by $u^\mu_i \equiv dz^\mu_i/d\tau_i$ and $a^\mu_i \equiv du^\mu_i/d\tau_i$, respectively.

Since the object is rigid, and it moves in a non-rotational manner, the time evolution of the spatial vectors $w^\mu_i$ is given by the Fermi-Walker transport,
\[ \dot{w}^\mu_i = (u^\mu a_\nu - u_\nu a^\mu) w^\nu_i = u^\mu a_\nu w^\nu_i. \tag{35} \]

Repeating the above dumbbell kinematic calculations, we again find that
\[ \frac{d\tau_i}{d\tau} = 1 + \epsilon_i a_\mu w^\mu_i, \tag{36} \]
and
\[ u^\mu_i = u^\mu. \tag{37} \]

Again, the last equality implies that in the momentary rest frame of the representative point, all charges are (momentarily) at rest too. One also finds that
\[ a^\mu_i = \frac{a^\mu}{1 + \epsilon_i a_\nu w^\nu_i}. \tag{38} \]

We shall be interested in the limit in which the object’s size is taken to be arbitrarily small, but its shape (including the location of the charges) is unchanged in this limiting process. To describe this limit mathematically, let $\epsilon > 0$ denote the object’s size, e.g. its "radius" (i.e. half the maximal distance between pairs of object’s points). We now define
\[ \epsilon_i \equiv \epsilon \alpha_i. \]

\[ ^7 \text{Here and below there is no sum over repeated Latin indexes (such as i,j) unless explicitly indicated.} \]
The parameters $\alpha_i$ are thus dimensionless numbers of order unity or smaller. The above limiting process is thus described by $\epsilon \to 0$ with all parameters $\alpha_i$ kept fixed.

In the calculations below we shall make use of the results that were obtained in Sec. III in the dumbbell case. Recall, however, that in the latter case the representative point was chosen at half the distance between the two charges. This cannot be done in the present case (as long as $N > 2$). In order to allow the implementation of the dumbbell results to our case, we shall also need to consider, for each pair of charges $i, j$, the worldline of the central point between these two charges, which we denote $z_{ij}^\mu$:

$$z_{ij}^\mu(\tau) \equiv \frac{1}{2} \left[ z_i^\mu(\tau) + z_j^\mu(\tau) \right] = z^\mu(\tau) + \frac{1}{2} \left[ \epsilon_i w_i^\mu + \epsilon_j w_j^\mu \right].$$

Obviously there exists a number $\epsilon_{ij} \geq 0$ and a unit vector $w_{ij}^\mu$ such that $\epsilon_{ij} w_{ij}^\mu = (1/2)(\epsilon_i w_i^\mu + \epsilon_j w_j^\mu)$. Then $\epsilon_{ij}$ is the distance of this central point from the representative point, and (for $\epsilon_{ij} > 0$) $w_{ij}^\mu$ is a vector normal to $u^\alpha$ which satisfies the Fermi-Walker transport law, as one can easily verify. We denote the proper time along the worldline $z_{ij}^\mu(\tau)$ by $\tau_{ij}$, and the four-velocity and four-acceleration by $u_{ij}^\mu \equiv dz_{ij}^\mu/d\tau_{ij}$ and $a_{ij}^\mu \equiv du_{ij}^\mu/d\tau_{ij}$, respectively. Obviously all the above kinematic relations satisfied by the point charge $z_i^\mu(\tau)$, e.g. Eqs. (36-38), are also satisfied by a central point $z_{ij}^\mu(\tau)$. Of particular importance for the analysis below is the relation

$$a_{ij}^\mu \frac{d\tau_{ij}}{d\tau} = a^\mu, \quad (39)$$

which follows from Eqs. (36) and (38) (with "i" replaced by "ij").

Let us finally emphasize that, for a particular pair $i, j$, the three points $z_i^\mu$, $z_j^\mu$, and $z_{ij}^\mu$ satisfy all the dumbbell’s kinematic relations satisfied by the three dumbbell’s points $z_i^\mu$, $z_j^\mu$, and $z^\mu$, correspondingly. This will allow us to apply all the above dumbbell results to any pair $i, j$, though with the dumbbell’s central point $z^\mu$ replaced by $z_{ij}^\mu$ (and $\tau$ by $\tau_{ij}$, etc.). The dumbbell’s length $\hat{\epsilon} = 2\epsilon$ is of course replaced by the distance between the charges $i$ and $j$, which we denote $\hat{\epsilon}_{ij}$.

**B. Calculation of the self Force**

To derive the self force acting on the extended object we shall use energy-momentum considerations similar to those of Sec. III. The four-momentum of the extended object $p^\mu(\tau)$ is defined just as in the dumbbell case, by the integral (16) over a hypersurface of
simultaneity. In analogy with Eq. (17), we now have

$$dp^\mu = \sum_{i=1}^{N} dp^\mu_i + dp^\mu_{ext},$$

(40)

where $dp^\mu_i$ denotes the contribution from all the electromagnetic forces (sourced by all object’s charges) acting on the $i$’th charge, and $dp^\mu_{ext}$ denotes the contribution from the overall external force. The electromagnetic energy-momentum exchange with the charge $i$ is

$$dp^\mu_i = f^\mu_{(em)i} d\tau_i,$$

where $f^\mu_{(em)i}$ is the overall electromagnetic forces acting on the charge $i$, given by

$$f^\mu_{(em)i} = \hat{f}^\mu_i + \sum_{j=1}^{N} f^\mu_{j \rightarrow i}.$$

Here $f^\mu_{j \rightarrow i}$ denotes the electromagnetic force that the charge $j$ exerts on the charge $i$, and $\hat{f}^\mu_i$ denotes the partial self force acting on this charge. Therefore,

$$dp^\mu = dp^\mu_{ext} + \left( \sum_{i=1}^{N} \hat{f}^\mu_i d\tau_i \right) + \left( \sum_{i=1}^{N} \sum_{j=1}^{N} f^\mu_{j \rightarrow i} d\tau_i \right).$$

Since the external force is presumably well behaved as $\epsilon \rightarrow 0$ (it is essentially independent of $\epsilon$), we can use $dp^\mu_{ext} \approx f^\mu_{ext} d\tau$, without bothering which proper time exactly one should use. For the same reason we may replace $d\tau_i$ multiplying the partial self force by $d\tau$. Defining the overall force acting on the object to be

$$f^\mu = \frac{dp^\mu}{d\tau},$$

we obtain

$$f^\mu \approx \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{d\tau_i}{d\tau} f^\mu_{j \rightarrow i} + \sum_{i=1}^{N} \hat{f}^\mu_i \right] + f^\mu_{ext}. \quad (41)$$

The overall mutual force is the term including the double-sum over $i$ and $j$:

$$f^\mu_{mutual} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{d\tau_i}{d\tau} f^\mu_{j \rightarrow i} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{d\tau_{ij}}{d\tau} \left[ \frac{d\tau_i}{d\tau_{ij}} f^\mu_{j \rightarrow i} + \frac{d\tau_j}{d\tau_{ij}} f^\mu_{i \rightarrow j} \right]. \quad (42)$$
Consider the last term in squared brackets, for a particular pair of charges $i, j$. This pair satisfies a "dumbbell kinematics"; namely the kinematic relations between the worldlines of the three points $z_i^\mu$, $z_j^\mu$, and $z_{ij}^\mu$ are exactly the same as those satisfied by the three dumbbell’s points $z_+^\mu$, $z_-^\mu$, and $z^\mu$, correspondingly. This allows us to apply the dumbbell’s results to this new two-charges system. In particular, Eqs. (22,23) now yield

$$\frac{d\tau_i}{d\tau_{ij}} f_{j\rightarrow i}^\mu + \frac{d\tau_j}{d\tau_{ij}} f_{i\rightarrow j}^\mu \approx -\frac{q_i q_j}{\hat{\epsilon}_{ij}} a_{ij}^\mu + \frac{4}{3} q_i q_j (\dot{a}_{ij}^\mu - a_{ij}^2 u^\mu),$$

where $\hat{\epsilon}_{ij}$ is the distance between the two charges, and $\dot{a}_{ij}^\mu \equiv da_{ij}^\mu / d\tau_{ij}$. Note that $\hat{\epsilon}_{ij}$, like all other object’s distances, scales like $\epsilon$ (the object’s size). Since the last term at the right-hand side is of order $\epsilon^0$, we are allowed to replace $\tau_{ij}$ and $a_{ij}^\mu$ by the corresponding representative-point quantities, $\tau$ and $a^\mu$ (which we cannot do when treating the other term, the one proportional to $1/\hat{\epsilon}_{ij}$ ). With the aid of Eq. (39) we obtain

$$\frac{d\tau}{d\tau} \left[ \frac{d\tau_i}{d\tau_{ij}} f_{j\rightarrow i}^\mu + \frac{d\tau_j}{d\tau_{ij}} f_{i\rightarrow j}^\mu \right] \approx -\frac{q_i q_j}{\hat{\epsilon}_{ij}} \left( a_{ij}^\mu \frac{d\tau_{ij}}{d\tau} \right) + \frac{4}{3} q_i q_j (\dot{a}^\mu - a^2 u^\mu)$$

$$= -\frac{q_i q_j}{\hat{\epsilon}_{ij}} a^\mu + \frac{4}{3} q_i q_j (\dot{a}^\mu - a^2 u^\mu).$$

Notice that in the last expression all kinematic quantities are those associated with the representative point, and the only reference to the two charges is through $q_i$, $q_j$, and $\hat{\epsilon}_{ij}$. Substituting this result back in Eq. (42) we obtain

$$f_{\mu}^\text{mutual} \approx -E_{es} a^\mu + \frac{2}{3} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j \left( \dot{a}^\mu - a^2 u^\mu \right) \right),$$

where

$$E_{es} \equiv \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j \frac{q_i q_j}{\hat{\epsilon}_{ij}}.$$

This last expression is exactly the electrostatic energy of the system of $N$ charges (the factor $1/2$ corresponds to the fact that every pair $i, j$ appears twice in this sum).

The overall (bare) self force $f_{\mu}^\text{bare}$ is the term in squared brackets in Eq. (41), which we
write as
\[ f_{\text{bare}}^\mu = f_{\text{mutual}}^\mu + \sum_{i=1}^{N} \hat{f}_i^\mu \]
\[ = -E_{es}a^\mu + \frac{2}{3} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j \right) \left( \dot{a}^\mu - a^2 u^\mu \right) + \sum_{i=1}^{N} \hat{f}_i^\mu + O(\epsilon). \quad (45) \]

Implementing now the mass-renormalization procedure, given by Eq. (25), and then taking the limit \( \epsilon \to 0 \), we obtain the renormalized self force:
\[ f_{\text{self}}^\mu = \frac{2}{3} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j \right) \left( \dot{a}^\mu - a^2 u^\mu \right) + \sum_{i=1}^{N} \hat{f}_i^\mu. \quad (46) \]

To factor out the partial self forces, we again use the fact that the self force is quadratic in the charge, namely
\[ \hat{f}_i^\mu = \left( q_i^2 / q^2 \right) f_{\text{self}}^\mu, \]
where \( q \equiv \sum_i q_i \) is the total charge. Transferring all partial self forces to the left-hand side and then multiplying by \( q^2 \), we obtain
\[ \left[ q^2 - \sum_{i=1}^{N} q_i^2 \right] f_{\text{self}}^\mu = \frac{2}{3} q^2 \left( \sum_{i=1}^{N} \sum_{j=1}^{N} q_i q_j \right) \left( \dot{a}^\mu - a^2 u^\mu \right). \]

Noting that the two terms in squared brackets are equal, we obtain the self force in its final form:
\[ f_{\text{self}}^\mu = \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu). \quad (47) \]

The equation of motion is given by Eq. (33), just as in the dumbbell case.

**IV. CONTINUOUSLY-CHARGED EXTENDED OBJECT**

In this section we shall consider a rigid extended object which is continuously charged. Again, we denote the object’s size (e.g. its "radius") by \( \epsilon \). Let \((X,Y,Z)\) be a system of comoving Cartesian coordinates that parametrize the three-dimensional hypersurface of simultaneity, and let \( \bar{R} \equiv (X,Y,Z) \). The representative point (an arbitrary point of the object) is taken to be e.g. at \( \bar{R} = 0 \). Note that the worldline of any point of fixed \( \bar{R} \) satisfies
all the kinematic relations described in the previous section. The charge distribution is denoted \( \rho(X,Y,Z) \). We assume that the charge distribution is fixed (in the object’s frame), i.e. \( \rho(X,Y,Z) \) is independent of the proper time \( \tau \).

The calculation of the self force proceeds in full analogy with the discrete case discussed in the previous section, with the discrete charge \( q_i \) replaced by the infinitesimal charge element \( dq \equiv \rho dX dY dZ \), and with the summations replaced by integrals. There is a remarkable difference between the two cases, though: In the discrete case, the demand for consistency required us to take into account the partial self forces. No such partial self forces appear in the continuous case (see below). This makes the continuous case simpler and more elegant.

One can follow all the considerations and calculations of the previous section, up to Eq. (45). In the continuous variant of this equation, the double-sum becomes a double-integral:

\[
\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} q_i q_j \rightarrow \int \int \rho(\vec{R}_1)\rho(\vec{R}_2) d^3\vec{R}_1 d^3\vec{R}_2 = q^2,\tag{48}
\]

where \( q \equiv \int \rho(\vec{R}) d^3\vec{R} \) is the total charge. On the other hand, the term including the partial self forces has only one summation, so it would become a single integral. However, \( \dot{\mathbf{j}}_{\mu} \) is proportional to \( q_i^2 \), and at the continuous limit this becomes \( (\rho dq) d^3\vec{R} \). That is, the "integrand" is proportional to \( dq \), which means that it actually vanishes at the infinitesimal limit. We conclude that no partial self forces appear in the continuous limit. This has a simple intuitive explanation: At the limit \( N \to \infty \) in which each charge is split into many smaller charges (such that the total charge is conserved), the magnitude of the individual partial self forces scales like \( q_i^2 \propto 1/N^2 \), whereas their number only scales like \( N \). Therefore, the overall contribution of the partial self forces scales like \( 1/N \) and hence vanishes at the continuous limit \( \text{8} \). (This is to be contrasted with the situation of the mutual electromagnetic forces: The magnitude of the mutual forces scales like \( 1/N^2 \) too, but their number scales like \( N^2 \), so the overall mutual force attains a non-vanishing value at the limit \( N \to \infty \).) The integral analog of Eq. (45) is thus

\[
f_{\text{bare}}^\mu = -E_{\text{es}} a^\mu + \frac{2}{3} q^2 (\dot{a}^\mu - a^2 u^\mu) + O(\epsilon),\tag{49}
\]

\text{8} This property was also used in Eq. (48) where we have replaced the double sum, which differs from \( q^2 \), by a double integral=equal to \( q^2 \). We use this property again below, when we replace the double sum in Eq. (44) by the double integral in Eq. (50).
where $E_{es}$ is the integral analog of Eq. (44):

$$E_{es} \equiv \frac{1}{2} \int \int \frac{\rho(\vec{R}_1)\rho(\vec{R}_2)}{|\vec{R}_1 - \vec{R}_2|} d^3\vec{R}_1 d^3\vec{R}_2.$$  \hspace{1cm} (50)

Note that $E_{es}$ is the electrostatic energy of the continuous charge distribution.

The mass-renormalization (25) now removes the irregular term $E_{es}a^\mu$ in Eq. (49), and (after taking the limit $\epsilon \to 0$) one arrives at the final expression for the self force:

$$f^\mu_{\text{self}} = \frac{2}{3}(\ddot{a}^\mu - a^2 u^\mu).$$  \hspace{1cm} (51)

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FIG. 1: A spacetime diagram describing the dumbbell’s kinematics. $t$ is the time coordinate (in some inertial reference frame), and $z$ schematically represents a spatial coordinate. The dumbbell is represented by a straight bold line, with the black points representing the two edge points $z_\mu^\pm$. Two such bold lines are shown, representing the dumbbell’s location in spacetime at two moments separated by an infinitesimal time interval $d\tau$. The three thin solid lines are the worldlines of the central point $z^\mu$ and the two edge points $z_\pm^\mu$. 