IMPOSSIBLE CONFIGURATIONS FOR GEODESICS ON NEGATIVELY-CURVED SURFACES

BY

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ABSTRACT

Hass and Scott’s example of a 4-valent graph on the 3-punctured sphere that cannot be realized by geodesics in any metric of negative curvature is generalized to impossible configurations filling surfaces of genus $n$ with $k$ punctures for any $n$ and $k$.

1. Introduction

By a configuration we mean a surface $S$ together with a 4-valent, connected graph embedded in $S$. Going straight (neither right nor left) at each intersection decomposes such a graph canonically into a collection of closed curves (the tracks) intersecting themselves and each other transversally.

A basic question is whether or not there is a negative-curvature metric on $S$ such that the graph is isotopic to a collection of closed geodesics intersecting transversally. We will say in this case that the configuration can be realized by geodesics in a metric of negative curvature. It is an old but remarkable fact that the simple configuration shown in Figure 1 cannot be realized in this way; the phenomenon was discovered by Joel Hass and Peter Scott in 1999 [2]. As they remark, their proof of non-realizability can be replaced by an argument, due to Ian Agol, using the Gauss–Bonnet Theorem.

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In this note the Hass–Scott example and Agol’s argument are generalized to produce an infinite family of non-realizable configurations, the polygonal impossible configurations, including a graph on the surface of genus $n$ with $k$ punctures for every $n \geq 2$ and $k \geq 0$.

1.1. Preliminaries. The Hass–Scott example, where $S$ is the 3-punctured sphere, is striking because at first inspection there seems to be no reason for it to not be realizable. Locally, it looks exactly like other, realizable configurations. In particular:

C1. The curve segments do not enclose embedded or immersed contractible 1-gons or 2-gons (“monogons” or “bigons”): these are in fact outlawed in geodesic configurations in negative curvature.

C2. No curve represents a power ($> 1$) in the free homotopy group $\pi_1 S$, nor do two distinct curves represent powers ($\geq 1$) of the same element of $\pi_1 S$. Since in negative curvature each free homotopy class contains a unique geodesic, a power curve collapses to multiple tracings of a single geodesic, and two homotopic curves collapse to the same geodesic; in either case the initial configuration is destroyed.

Furthermore, the Hass–Scott example fills its surface in the sense that:

C3. The complement of the graph is a disjoint union of discs or singly punctured discs.

The configurations constructed here will satisfy the conditions C1, C2 and C3.
2. Polygonal impossible configurations

Definition 2.1: A **polygonal impossible configuration** \((S, \mathcal{P})\) is an orientable, connected 2-dimensional surface \(S\) together with an embedded, connected, 4-valent graph \(\mathcal{P}\), the pair constructed as follows:

1. Choose a number \(N \geq 3\), which will be the number of vertices in the configuration.
2. Choose a number \(p\), with \(N/4 \leq p \leq N/3\), and \(p\) polygons \(A_1, \ldots, A_p\) which together have \(N\) corners. (This is possible since \(p \leq N/3\).) At least one \(A_i\) must be a triangle: see the remark below.
3. Choose \(q = N - 2p\) even-sided polygons \(B_1, \ldots, B_q\) with a total of \(2N\) corners. (This is possible since \(p \geq N/4\) implies \(4q = 4N - 8p \leq 2N\).) Label the edges of the \(B\)-polygons alternately active and inactive.
4. Identify an edge of one of the \(A_i\) with every active edge of each \(B_j\), preserving orientations. Avoid forming a ring of squares: such a ring would lead to two parallel tracks.
5. The \(N\) inactive edges of the \(B\)s are grouped by the identifications in step 4 into a collection \(\gamma_1, \ldots, \gamma_r\) of closed curves; adding a disc \(D_i\) along each \(\gamma_i\) creates a closed orientable surface \(S\).
6. Puncture any \(D_i\) bounded by a monogon or a bigon. If \(S\) has genus 0 or 1, add an additional puncture if necessary so that \(S\) can carry a metric of negative curvature.

**Examples:** The simplest examples have \(N = 3\), so \(p = 1\) and \(A_1\) is a triangle; then \(q = N - 2p = 1\) and the polygon \(B_1\) has \(2N = 6\) corners: a hexagon. Three non-adjacent sides of \(B_1\) get identified to the three sides of \(A_1\). This can be done in two ways, as shown in Figure 2.

![Figure 2](image.png)

Figure 2. Two different polygonal impossible configurations with \(N = 3\), where * represents a puncture. a. \(S\) is a 3-punctured sphere; this is the Haas-Scott example. b. \(S\) is a punctured torus.
Remark 2.2: At least one of the $A_i$ must be a triangle. In fact, suppose first that all the $A_i$ are squares; then $N = 4p, q = N - 2p = 2p$, and $2N/q = 4$, so all the $B_i$ must also be squares; and then the configuration $P$ constructed by the algorithm will be made up of one or more sets of parallel curves, contradicting C2 above. Otherwise, if all the $A_i$ have $\geq 4$ sides and at least one has strictly more, then $N > 4p$ contradicting $p \geq N/4$.

Theorem 2.3: In a polygonal impossible configuration $(S, P)$, the surface $S$ cannot be given a metric of negative curvature so that the set of curves defined by the edges of $P$ is a set of geodesics. This justifies the name “impossible configuration.”

Proof. Suppose such a metric exists. Set $n_i$ to be the number of vertices of $A_i$, and $\alpha_{i,j}, j = 1, \ldots, n_i$ to be the interior angle at the $j$th vertex of $A_i$.

Likewise, set $m_i$ to be the number of vertices of $B_i$, and $\beta_{i,j}, j = 1, \ldots, m_i$ to be the interior angle at the $j$th vertex of $B_i$.

Assume all the edges are geodesic arcs extending smoothly from polygon to polygon, so that each of the $\alpha_{i,j}$ is complementary to exactly two of the $\beta_{i,j}$.

The Gauss–Bonnet theorem [7] gives

$$\alpha_{1,1} + \alpha_{1,2} + \cdots + \alpha_{1,n_1} < (n_1 - 2)\pi$$

$$\vdots$$

$$\alpha_{p,1} + \alpha_{p,2} + \cdots + \alpha_{p,n_p} < (n_p - 2)\pi.$$

Adding these equations yields

$$\sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j} < (N - 2p)\pi.$$  \hfill (*)

Similarly, the sum of all the $\beta$s is strictly less than $(2N - 2q)\pi$. On the other hand each $\beta$ is $\pi - \alpha$ for some $\alpha$, with each $\alpha$ occurring exactly twice.

So

$$(2N - 2q)\pi > \sum_{i=1}^{q} \sum_{j=1}^{m_i} \beta_{i,j} = 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} (\pi - \alpha_{i,j}) = 2N\pi - 2 \sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j}$$

i.e., $\sum_{i=1}^{p} \sum_{j=1}^{n_i} \alpha_{i,j} > q\pi$. Since by the construction $q = N - 2p$, this inequality contradicts (*). \hfill \blacksquare
3. The genus of a polygonal impossible configuration; minimal and unicursal configurations

Definition 3.1: The genus of a polygonal impossible configuration \((S, P)\) is the genus of the surface \(S\). With \(N, p, q, r\) as above, \(S\) has Euler characteristic \(\chi = N - 2N + (p + q + r) = -p + r\), and genus

\[
g = \frac{1}{2}(2 - \chi) = \frac{1}{2}(2 - r + p).
\]

This genus only depends on \(P\), and matches the usual definition of the genus of a graph as the genus of the simplest surface on which it can be embedded so that its complement is topologically a set of discs.

Definition 3.2: An impossible polygonal configuration \((S, P)\), where \(S\) is a surface of genus \(g\) with \(k\) punctures, is minimal if it has the smallest possible number of vertices for such a configuration.

This minimal number depends on \(g\) and \(k\):

- If \(k = 0\), since \(r \geq 1\) equation (**) implies that \(p\), the number of \(A\)-polygons, must satisfy \(p \geq 2g - 1\). Since each \(A\)-polygon is at least a triangle, the number \(N\) of vertices for a polygonal impossible configuration filling the unpunctured surface of genus \(g\) must satisfy \(N \geq 6g - 3\).
- Otherwise, since \(r \geq k\) we have \(p \geq 2g + k - 2\) and \(N \geq 6g + 3k - 6\).

Definition 3.3: A configuration \((S, P)\) is unicursal if the edges of \(P\) form exactly one track, i.e., as described in the introduction, if they can be traversed by a single smooth curve.

Examples:

![Figure 3. Impossible configurations on the punctured torus: a has one track and b has two. (In Figure 2 configuration a has one track; configuration b has three.)](image)
The examples shown in Figures 2, 3 and 5 suggest the following statement, which has an elementary proof using concepts from [4], [5] and [6].

**Theorem 3.4:** The number of tracks of a polygonal impossible configuration is congruent mod 2 to the number of vertices. In particular, such a configuration can only be unicursal if the number of vertices is odd.

4. **Unicursal, minimal impossible configurations on surfaces of higher genus**

4.1. **Surfaces of genus \( \geq 2 \).**

**Proposition 4.1:** There exists a unicursal, minimal impossible polygonal configuration \( (S_n, P_n) \) where \( S_n \) is the surface of genus \( n \).

**Proof.** For genus 2 a configuration with these properties is exhibited in Figure 4. This construction can be extended, as shown in Figure 5, to give an impossible configuration \( (S_n, P_n) \) for each \( n \geq 2 \). The configuration \( (S_n, P_n) \) is unicursal, as can be checked; also, since \( (S_n, P_n) \) has \( 2n - 1 \) triangles and hence \( 6n - 3 \) vertices, it is minimal. \( \square \)

4.2. **Surfaces with punctures.** The Hass–Scott example is minimal and unicursal. Generalizing it to surfaces with punctures, using polygonal impossible configurations, can be done preserving both of these properties if the number of punctures is odd, but only one or the other if it is even.
Figure 5. The configuration $(S_n, P_n)$, obtained by interpolating into the $(S_2, P_2)$ configuration $n - 2$ copies of the partial configuration shown next to it, has $p = 2n - 1$ triangles matched with $q = 2n - 1$ hexagons. As can be checked, it has $r = 1$ and therefore Euler characteristic $-2n + 2$ and genus $n$.

**Proposition 4.2:** The configuration $(S_n, P_n)$ can be extended to become an impossible configuration filling the surface of genus $n$ with $k$ punctures, for any $k \geq 0$. If $k$ is odd, the new configuration can be minimal and unicursal. If $k$ is even, it can be minimal or unicursal but not both.

Figure 6. The $*$ represents a puncture. These partial configurations can be spliced into $(S_n, P_n)$ to increase the number of punctures by one (a, b) or two (c). Note that partial configurations a and c must be spliced into an arm coming away from a hexagon, as shown, in order to avoid producing a 2-gon.
Proof. Initially, \((S_n, P_n)\) has one complementary region, a disc. This disc can be punctured, yielding \(k = 1\). Splicing in \(m\) copies of partial configuration \(c\) (Figure 6) gives a minimal, unicursal polygonal configuration with the same genus and \(2m + 1\) punctures. On the other hand, as remarked in Section 3.1, a minimal configuration of genus \(n\) with \(k\) punctures has \(N = 6n + 3k - 6\) vertices; if \(k\) is even, so is \(N\); and by Theorem 3.4 such a configuration cannot be unicursal.

A unicursal configuration with genus \(n\), with \(2m\) punctures and one extra vertex can be obtained from \((S_n, P_n)\) by splicing in \(m - 1\) copies of \(c\) and one of \(b\). A minimal configuration with genus \(n\), with \(2m\) punctures and two tracks can be obtained by splicing in \(m - 1\) copies of \(c\) and one of \(a\).

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