A finite-dimensional approximation for partial differential equations on Wasserstein space∗

Mehdi Talbi†

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Abstract

This paper presents a finite-dimensional approximation for a class of partial differential equations on the space of probability measures. These equations are satisfied in the sense of viscosity solutions. The main result states the convergence of the viscosity solutions of the finite-dimensional PDE to the viscosity solutions of the PDE on Wasserstein space, provided that uniqueness holds for the latter, and heavily relies on an adaptation of the Barles & Souganidis monotone scheme [1] to our context, as well as on a key precompactness result for semimartingale measures. We illustrate our convergence result with the example of the Hamilton-Jacobi-Bellman and Bellman-Isaacs equations arising in stochastic control and differential games, and propose an extension to the case of path-dependent PDEs.

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1 Introduction

Since Lasry & Lions [38] and Caines, Huang & Malhamé [34] introduced mean field games, partial differential equations on the space of probability measures have become a popular tool to study systems of interacting agents. In [11], Cardaliaguet, Delarue, Lasry & Lions

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†Laboratoire de Probabilités, Statistique et Modélisation, Université Paris-Cité, France, talbi@lpsm.paris
used the master equation to prove the convergence of the $N$-players game to the corresponding mean field problem, provided that the value function of the problem is smooth. Since then, an extensive part of the literature has been dedicated to these convergence issues: see e.g. Cardaliaguet [9], Bayraktar, Cecchin, Cohen & Delarue [4, 3], Cecchin & Pelino [17], Cecchin, Dai Pra, Fischer & Pelino [18], Laurière & Tangpi [39], Djete [22] or Doncel, Gast & Gaujal [24].

In the context of mean field control, an extensive literature focuses on the convergence issue, see e.g. Bayraktar, Cecchin & Chakraborty [2], Cardaliaguet, Daudin, Jackson & Souganidis [10], Cardaliaguet & Souganidis [12], Cavagnari, Lisini, Orrieri & Savare [15], Cecchin [16], Djete, Possamai & Tan [23], Fischer & Livieri [30], Fornasier, Lisini, Orrieri & Savare [31], Germain, Pham & Warin [33], and Lacker [37]. In particular, by utilizing some strong regularity of the value function, [33, 10] obtained certain rate of convergence.

The contributions the most relevant to our work are the ones of Gangbo, Mayorga & Swiech [32], Mayorga & Swiech [40] and Talbi, Touzi & Zhang [47]. In the first two papers, the authors develop a finite-dimensional approximation for Hamilton-Jacobi-Bellman equations with uncontrolled volatility, in the sense of viscosity solutions (defined via lifting on the Hilbert space of square integrable random variables). The second paper introduces an approximation of the obstacle problem on Wasserstein space, which characterizes the mean field optimal stopping problem (see [48, 49]).

Our objective is to find a finite-dimensional approximation for a general class of PDEs on Wasserstein space, satisfied in the sense of viscosity. We use the notion of viscosity solutions developed by Wu & Zhang [51], which is intrinsic and allows for path-dependent PDEs (i.e., the solutions of the equations depend on the time and on a probability measure on the space of continuous paths of $\mathbb{R}^d$). This class of equations covers in particular equations arising in mean field stochastic control (including the case of controlled volatility). We intend to find a finite-dimensional PDE whose viscosity solutions converge to viscosity solutions of the equation on Wasserstein space. More precisely, we follow the methodology of [47] and adapt Barles & Souganidis’ monotone scheme [1] to our context, by proving that the semi-relaxed limits of the viscosity super/subsolutions of the finite-dimensional PDE are viscosity super/subsolutions of the PDE on Wasserstein space as the dimension goes to infinity. If uniqueness holds for the latter equation, this implies the convergence of viscosity solutions.

We propose an application of this result to the convergence of the value functions of finite dimensional control problem to the value function of mean field control problems. In particular, we show that this convergence holds under quite weak regularity assumption on the coefficients and rewards of the problem.
An important feature that ensures the convergence relies on the choice of the set of the test functions for the finite dimensional problem. Similarly to the viscosity theory developed for path-dependent PDEs (see Ekren, Keller, Ren, Touzi & Zhang [26, 28, 29, 43]; see also Guo, Zhang & Zhuo [52] and Ren & Tan [42]), we only require test functions to be tangent to the super/subsolution through the mean, whereas the tangency is pointwise in the standard literature. This tangency in expectation can be seen as the finite-dimensional counterpart of our set of test functions on Wasserstein space. We emphasize that this choice of test functions is crucial even in the Markovian case: indeed, by considering only points along the trajectories of the state process, we are able to apply a key propagation of chaos-like result and to derive essential estimates to prove the main convergence theorem.

One of the main advantages of our methodology is that it is able to handle very similarly the Markovian and path-dependent cases. Although we discuss in more detail the Markovian equations for the sake of clarity, we insist on the fact that the proofs for non-Markovian equations are (almost) the same, and that most change only lie in some definitions and notations. We also emphasize that our results apply to a large class of equations, exceeding the scope Hamilton-Jacobi-Bellman equations on Wasserstein space and including for examples equations with non-convex Hamiltonians (such as Bellman-Isaacs equations).

The paper is organized as follows. In Section 2, we present the class of equations on Wasserstein space and the notion of viscosity solutions. In Section 3, we introduce the finite-dimensional approximation and state the main result of the paper, the convergence theorem, which we apply in Section 4 to mean field stochastic control. Section 5 extends our results to the case of path-dependent PDEs. Sections 6 and 7 are respectively dedicated to the proof of the convergence theorem and of the precompactness result.

**Notations.** Let \((E, \mathcal{A})\) be a measurable space endowed with a metric \(d\). We denote by \(\mathcal{P}(E, \mathcal{A})\) the set of probability measures on \((E, \mathcal{A})\), and by \(\mathcal{P}_p(E, \mathcal{A})\) its subset of \(p\)-integrable probability measures, \(p \geq 1\), equipped with the Wasserstein distance defined by

\[
W_p(\mu, \nu) := \inf_{Q \in \Pi(\mu, \nu)} \left( \int_{E \times E} d^p(x, y) Q(x, y) \right)^{1/p}
\]

for all \((\mu, \nu) \in \mathcal{P}_p(E, \mathcal{A})^2\), where \(\Pi(\mu, \nu)\) is the set of couplings of \(\mu\) and \(\nu\). When \(\mathcal{A} = \mathcal{B}(E)\), the Borel \(\sigma\)-algebra of \(E\), we simply write \(\mathcal{P}(E)\) and \(\mathcal{P}_p(E)\). We denote by \(\text{supp}(\mu)\) the support of \(\mu \in \mathcal{P}(E, \mathcal{A})\), defined as the smallest closed set \(C \subset E\) s.t. \(\mu(C^c) = 0\). Given a random variable \(Z\) and a probability \(\mathbb{P}\), we denote by \(\mathbb{P}_Z := \mathbb{P} \circ Z^{-1}\) the law of \(Z\) under \(\mathbb{P}\). We shall sometimes write \(\langle \mu, f \rangle := \int f d\mu\). The space of the \(d \times d\) real valued symmetric matrices is denoted by \(\mathbb{S}_d\), and \(\mathbb{S}_d^{D_{d \times N}}\) denotes the set of blockwise diagonal matrices of the form \(\text{Diag}(A_1, \ldots, A_N)\), where each \(A_i \in \mathbb{S}_d\). For vectors \(x, y \in \mathbb{R}^d\) and matrices \(A, B\), denote \(x \cdot y := \sum_{i=1}^d x_i y_i\) and
A : B := tr(AB\top). Given x := (x_1, \ldots, x_N) \in E^N, we denote by \mu^N(x) := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in \mathcal{P}(E), and \mathcal{P}^N(E) := \{\mu^N(x) : x \in E^N\}. For p \geq 1, we also write

\|x\|_p := \left(\frac{1}{N} \sum_{i=1}^N |x_i|^p\right)^{1/p} \quad \text{and} \quad \|\mu\|_p := \left(\int_{\mathbb{R}^d} |x|^p \mu(dx)\right)^{1/p} \quad \text{for all } x \in \mathbb{R}^{d \times N} \text{ and } \mu \in \mathcal{P}_p(\mathbb{R}^d).

We shall also write “LSC” (resp. “USC”) for “lower (resp. upper) semi-continuous”.

2 Viscosity solutions of partial differential equations on Wasserstein space

2.1 Differentiability on Wasserstein space

For t \in [0, T), we denote

Q_t := [t, T) \times \mathcal{P}_2(\mathbb{R}^d) \quad \text{and} \quad \overline{Q}_t := [t, T] \times \mathcal{P}_2(\mathbb{R}^d).

Definition 2.1 Fix t \in [0, T).

(i) u : \overline{Q}_t \rightarrow \mathbb{R} has a functional linear derivative if there exists \delta_m u : \overline{Q}_0 \times \mathbb{R}^d \rightarrow \mathbb{R} satisfying, for any s \in [t, T] and \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),

\begin{align*}
\text{u}(t, \nu) - \text{u}(t, \mu) = \int_0^1 \int_{\mathbb{R}^d} \delta_m u(s, \lambda \nu + (1 - \lambda)\mu, x)(\nu - \mu)(dx)d\lambda,
\end{align*}

and \delta_m u has quadratic growth in x \in \mathbb{R}^d, locally uniformly in (s, m) \in \overline{Q}_t.

(ii) We denote by C^{1,2}_b(\overline{Q}_t) the set of bounded functions u : \overline{Q}_t \rightarrow \mathbb{R} such that \partial_t u, \delta_m u, \partial_x \delta_m u, \partial_{xx} \delta_m u exist and are continuous and bounded in all their variables.

2.2 Partial differential equation on Wasserstein space

Let F : [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \times \mathbb{L}_0^1(\mathbb{R}^d, \mathbb{R}^d) \times \mathbb{L}_2(\mathbb{R}^d, \mathbb{S}_d) \rightarrow \mathbb{R}, where \mathbb{L}_0^1(\mathbb{R}^d, \mathbb{R}^d) (resp. \mathbb{L}_2(\mathbb{R}^d, \mathbb{S}_d)) denotes the set of Borel measurable functions from \mathbb{R}^d to \mathbb{R}^d (resp. \mathbb{S}_d) with quadratic growth, and g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}. We consider the following equation:

\begin{align*}
-\partial_t u(t, \mu) - F(t, \mu, u(t, \mu), \partial_\mu \delta_m u(t, \mu, \cdot), \partial_{\mu \mu} \delta_m u(t, \mu, \cdot)) = 0, \quad u|_{t=T} = g, \quad (t, \mu) \in Q_0, (2.1)
\end{align*}

The following assumptions on F will be crucial to guarantee the existence of a finite-dimensional approximation to the solution of (2.1):
Assumption 2.2 (i) $F$ is continuous in the following sense:

$$F(t^n, \mu^n, y^n, Z^n, \Gamma^n) \to F(t, \mu, y, Z, \Gamma)$$

as $(t^n, \mu^n, y^n, Z^n, \Gamma^n) \to (t, \mu, y, Z, \Gamma)$,

where the convergence of $\mu^n$ to $\mu$ is in $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ and the convergence of $(Z^n, \Gamma^n)$ to $(Z, \Gamma)$ is pointwise for $Z, Z^n \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ and $\Gamma, \Gamma^n \in C^0(\mathbb{R}^d, S_d)$ with quadratic growth uniformly in $n$.

(ii) For all $(t, \mu, y, Z, \Gamma) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \times L^0_2(\mathbb{R}^d, \mathbb{R}^d) \times L^0_2(\mathbb{R}^d, S_d)$, we have

$$F(t, \mu, y, Z, \Gamma) = F(t, \mu, y, Z', \Gamma')$$

for all $Z', \Gamma'$ s.t. $Z'|_{\text{supp}(\mu)} = Z|_{\text{supp}(\mu)}, \Gamma'|_{\text{supp}(\mu)} = \Gamma|_{\text{supp}(\mu)}$.

2.3 Viscosity solutions

Let $L$ be a positive constant, $\Omega := C^0([0, T], \mathbb{R}^d)$, $X$ be the canonical process on $\Omega$ and $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0, T]}$ be the corresponding filtration.

Definition 2.3 We denote by $\mathcal{P}_L$ the set of measures $\mathbb{P} \in \mathcal{P}_2(\Omega, \mathcal{F}_T)$ s.t. $X$ is a $\mathbb{P}$-semimartingale with drift and diffusion characteristics uniformly bounded by $L$. For $(t, \mu) \in \mathcal{Q}_0$, we also define $\mathcal{P}_L(t, \mu) := \{\mathbb{P} \in \mathcal{P}_L : \mathbb{P} X_t = \mu\}$.

We define the viscosity neighborhood of $(t, \mu) \in \mathcal{Q}_0$ by

$$\mathcal{N}_\delta(t, \mu) := \{(s, \mathbb{P} X_s) : s \in [t, t + \delta], \mathbb{P} \in \mathcal{P}_L(t, \mu)\},$$

Note that the constant $L$ may be chosen arbitrarily large, and has for unique purpose to ensure the $\mathcal{W}_2$-compactness of $\mathcal{P}_L(t, \mu)$ for all $(t, \mu)$ and, therefore, the one of $\mathcal{N}_\delta(t, \mu)$ (see Wu & Zhang [51, Lemma 4.1]). We may then introduce the sets of test functions:

$$\overline{\mathcal{A}} u(t, \mu) := \left\{ \varphi \in C^{1,2}_b(\mathcal{T}_Q) : (\varphi - u)(t, \mu) = \max_{\mathcal{N}_\delta(t, \mu)} (\varphi - u) \text{ for some } \delta > 0 \right\},$$

$$\underline{\mathcal{A}} u(t, \mu) := \left\{ \varphi \in C^{1,2}_b(\mathcal{T}_Q) : (\varphi - u)(t, \mu) = \min_{\mathcal{N}_\delta(t, \mu)} (\varphi - u) \text{ for some } \delta > 0 \right\}.$$
which are crucial to the proofs of our main convergence result.

- We can show (see Proposition 7.1) that the points lying in the “viscosity neighborhood” of the finite dimensional semi-solutions converge (up to a subsequence) to points in the viscosity neighborhood of semi-solutions of the mean field equations.

Although it could be possible to resort to other notions of viscosity solutions, we believe that this one is particularly adapted to our methodology. Requiring global tangency of test functions on the Wasserstein space (as in [19]) would make difficult the use of semi-jets (see Proposition 2.5), which greatly simplify the proof of our main result. The notion of [8] allows to use semi-jets but would require an extension to the case of path-dependent PDEs to be able to provide a unified approach for both Markovian and non-Markovian PDEs.

Definition 2.4 Let $u : Q_0 \to \mathbb{R}$.

(i) $u$ is a viscosity supersolution of (2.1) if, for all $(t, \mu) \in Q_0$ and $\varphi \in \mathcal{A} u(t, m)$,

$$-\partial_t \varphi(t, \mu) - F(t, \mu, u(t, \mu), \partial_x \delta_m \varphi(t, \mu, \cdot), \partial_{xx} \delta_m \varphi(t, \mu, \cdot)) \geq 0.$$ 

(ii) $u$ is a viscosity subsolution of (2.1) if, for all $(t, \mu) \in Q_0$ and $\varphi \in \mathcal{A} u(t, m)$,

$$-\partial_t \varphi(t, \mu) - F(t, \mu, u(t, \mu), \partial_x \delta_m \varphi(t, \mu, \cdot), \partial_{xx} \delta_m \varphi(t, \mu, \cdot)) \leq 0.$$ 

(iii) $u$ is a viscosity solution of (2.1) if it is a viscosity supersolution and subsolution.

Definition of viscosity solutions via semi-jets. Fix $(t, \mu) \in Q_0$ and $\delta > 0$. For $(v, a, f) \in \mathbb{R} \times \mathbb{R} \times C^2_b(\mathbb{R}^d)$, introduce

$$\psi^{v, a, f}(s, \nu) := v + a(s - t) + \langle \nu - \mu, f \rangle$$

for all $(s, \nu) \in N_{\delta}(t, \mu)$. (2.2)

We then have the equivalence result:

Proposition 2.5 Let $u : Q_0 \to \mathbb{R}$.

(i) $u$ is a viscosity supersolution of (2.1) if and only if it satisfies the viscosity supersolution property for all test functions in $\cup_{(t, \mu) \in Q_0} \mathcal{A} u(t, \mu)$ of the form (2.2).

(ii) $u$ is a viscosity subsolution of (2.1) if and only if it satisfies the viscosity subsolution property for all test functions in $\cup_{(t, \mu) \in Q_0} \mathcal{A} u(t, \mu)$ of the form (2.2).

Proof We only provide the argument for (i). If $u$ is a viscosity supersolution of (2.1), then it satisfies the supersolution property for all $\varphi \in \mathcal{A} u(t, \mu)$, in particular for those of the form (2.2).
Assume now that the supersolution property is verified for all $\psi^{v,a,f} \in \overline{Au}(t,\mu)$, $(v, a, f) \in \mathbb{R} \times \mathbb{R} \times C^2_b(\mathbb{R}^d)$. Let $\varphi \in \overline{Au}(t,\mu)$ and $\varepsilon > 0$, and fix $(v, a, f) := (\varphi(t,\mu), \partial_t\varphi(t,\mu) - \varepsilon, \delta_m\varphi(t,\mu,\cdot))$. We have, for $(s, \nu) \in \mathcal{N}_\delta(t,\mu)$:

$$(\varphi - \psi^{v,a,f})(s, \nu) = \varphi(s, \nu) - \varphi(t,\mu) - a(s-t) - \langle \nu - \mu, f \rangle$$

$$= (\varphi(s, \nu) - \varphi(t,\nu) - a(s-t)) + (\varphi(t, \nu) - \varphi(t,\mu) - \langle \nu - \mu, f \rangle)$$

$$= (s-t)\left(\partial_t \varphi(t,\nu) + \eta(s-t) - \partial_t \varphi(t,\mu) + \varepsilon \right)$$

$$+ \int_0^1 \langle \nu - \mu, \delta_m \varphi(t,\lambda\mu + (1-\lambda)\nu,\cdot) - \delta_m \varphi(t,\mu,\cdot) \rangle d\lambda$$

where $\eta(s-t) \xrightarrow{s \to t} 0$. As $(s, \nu) \in \mathcal{N}_\delta(t,\mu)$, there exists $\mathbb{P} \in \mathcal{P}(t,\mu)$ s.t. $\nu = \mathbb{P}_{X_s}$. Thus, introducing $h^\lambda_{t,s} := \delta_m \varphi(t,\lambda\mu + (1-\lambda)\nu,\cdot) - \delta_m \varphi(t,\mu,\cdot)$, we have

$$\langle \nu - \mu, \delta_m \varphi(t,\lambda\mu + (1-\lambda)\nu,\cdot) - \delta_m \varphi(t,\mu,\cdot) \rangle = \mathbb{E}^\mathbb{P}\left[h^\lambda_{t,s}(X_s) - h^\lambda_{t,s}(X_t) \right].$$

As $\varphi$ is smooth, we may apply Itô’s formula to $h^\lambda_{t,s}$, and thus

$$\mathbb{E}^\mathbb{P}\left[h^\lambda_{t,s}(X_s) - h^\lambda_{t,s}(X_t) \right] = \mathbb{E}^\mathbb{P}\left[\int_t^s \partial_x h^\lambda_{t,s}(X_r) \cdot dX_r + \frac{1}{2} \partial_{xx} h^\lambda_{t,s}(X_r) : d\langle X \rangle_r \right] \geq -(s-t)\frac{\varepsilon}{2},$$

for all $s \in [t, t+\delta]$ and $\delta$ sufficiently small, given the boundedness of the characteristics of $X$ under $\mathbb{P}$, the boundedness and continuity of the derivatives of $\varphi$ and the continuity of the flow $s \mapsto \mathbb{P}_{X_s}$. Finally, as we also have $\partial_t \varphi(t,\nu) + \eta(s-t) - \partial_t \varphi(t,\mu) \geq -\frac{\varepsilon}{2}$ for $\delta$ sufficiently small, we have

$$(\varphi - \psi^{v,a,f})(s, \nu) \geq (s-t)(-\frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2}) = 0 \quad \text{for all } (s, \nu) \in \mathcal{N}_\delta(t,\mu),$$

which implies since $\psi^{v,a,f}(t,\mu) = \varphi(t,\mu)$ that $\psi^{v,a,f} \in \overline{Au}(t,\mu)$. Then, the supersolution property writes

$$-(\partial_t \varphi(t,\mu) - \varepsilon) - F(t,\mu, u(t,\mu), \partial_x \delta_m \varphi(t,\mu,\cdot), \partial_{xx} \delta_m \varphi(t,\mu,\cdot)) \geq 0,$$

and we obtain the desired result by letting $\varepsilon \to 0$. $\blacksquare$

**Comparison principle** Under additional assumptions on $F$ (see [51, Assumption 3.1]), viscosity solutions satisfy the usual properties of consistency with the classical solution and stability. However, there is (at our knowledge) no uniqueness result for the general equation (2.1). Wu & Zhang proved it in our setting for some specific cases (see [51, Theorem 4.13]). Over the past few years, many efforts have been made to obtain more general comparison results, see e.g. Burzoni, Ignazio, Repping & Soner [8], Soner & Yan
[45], Bertucci [6], Bayraktar, Ekren & Zhang [5], Daudin & Seeger [21]. In this paper, we shall particularly refer to the recent work of Cosso, Gozzi, Kharroubi, Pham & Rosestolato [19], who established the comparison principle for Hamilton-Jacobi-Bellman equations on Wasserstein space in a quite general framework, assuming continuity of the semi-solutions. We refer to Remark 2.7 below for more detail about how their result relates to our notion of viscosity solution. In the statement of our main results, we shall use the comparison principle as a standing assumption:

**Assumption 2.6 (Comparison principle)** Let \( u, v \) be respectively continuous viscosity subsolution and supersolution of (2.1) such that \( u(T, \cdot) \leq v(T, \cdot) \). Then \( u \leq v \).

**Remark 2.7** In the setting of [19], the tangency property of the test functions consists in global maxima/minima on \( \overline{Q}_t \); thus, since a global extremum is a fortiori an extremum on \( N_\delta(t, \mu) \), any viscosity subsolution (resp. supersolution) in the sense of Definition 2.4 is a viscosity subsolution (resp. supersolution) in the sense of [19], as long as we allow test functions to have derivatives with quadratic growth in \( x \) instead of bounded ones (our requirement for boundedness has for purpose to provide a unified approach for both Markovian and path-dependent cases; however, our convergence result still holds under the quadratic growth requirement in the Markovian case, see Remark 6.2). Therefore, under the assumptions of [19, Theorem 5.1], comparison holds for our notion of viscosity solutions.

### 3 Finite-dimensional approximation

#### 3.1 The approximating equation

Let \( N \geq 1 \). We shall write in bold character the elements \( x = (x_1, \ldots, x_N) \in \mathbb{R}^{d \times N}, z = (z_1, \ldots, z_N) \) and \( \gamma = \text{Diag}(\gamma_1, \ldots, \gamma_N) \in \mathbb{S}^d_{d \times N} \). Introduce \( F^N : [0, T] \times \mathbb{R}^{d \times N} \times \mathbb{R} \times \mathbb{R}^{d \times N} \times \mathbb{S}^{d \times N}_{d \times N} \rightarrow \mathbb{R} \) such that:

\[
F^N(t', x, y', \frac{\varphi(x)}{N}, \frac{\varphi'(x)}{N}) \rightarrow F(t, \mu, y, \varphi, \varphi') \quad \text{as} \quad (N, t', \mu^N(x), y') \rightarrow (+\infty, t, \mu, y) \quad (3.1)
\]

for all \( \varphi \in C^1_b(\mathbb{R}^d, \mathbb{R}) \), where we denote:

\[
\varphi(x) := (\varphi(x_1), \ldots, \varphi(x_N)) \quad \text{and} \quad f'(x) = \text{Diag}(\varphi'(x_1), \ldots, \varphi'(x_N)).
\]
A natural approximation. Let us explain how to construct an approximating operator satisfying the consistency requirement (3.1). Introduce, for \((t, x, y, z, \gamma) \in [0, T] \times \mathbb{R}^{d \times N} \times \mathbb{R} \times \mathbb{R}^{d \times N} \times \mathbb{S}_{d \times N}^D\),

\[
F^N(t, x, y, z, \gamma) := F(t, \mu^N(x), y, Nz \cdot 1_x, N\gamma \cdot 1_x),
\]

where \(z \cdot 1_x(x) := \sum_{k=1}^N z_k 1_{x_k}(x)\) and \(\gamma \cdot 1_x(x) := \sum_{k=1}^N \gamma_k 1_{x_k}(x)\) for all \(x \in \mathbb{R}^d\). Then:

**Proposition 3.1** Let Assumption 2.2 hold. Then \(F^N\) defined in (2.2) satisfies (3.1).

**Proof** Let \(\varphi \in C^1_b(\mathbb{R}^d, \mathbb{R})\). We have:

\[
F_N(t, x, y, \frac{\varphi(x)}{N}, \frac{\varphi'(x)}{N}) = F(t, \mu^N(x), y, \sum_{k=1}^N \varphi(x_k) 1_{x_k}, \sum_{k=1}^N \varphi'(x_k) 1_{x_k}) = F(t, \mu^N(x), y, \varphi, \varphi')
\]

by Assumption 2.2 (ii). Then (3.1) is comes from the continuity assumption 2.2 (i). \(\blacksquare\)

We now introduce the PDE on \([0, T] \times \mathbb{R}^{d \times N}\):

\[
-\partial_t u(t, x) - F^N(t, x, u(t, x), \partial_x u(t, x), \partial_{xx}^2 u(t, x)) = 0, \quad u|_{t=T} = g^N,
\]

with \(g^N(x) := g(\mu^N(x)), \partial_x u(t, x) := (\partial_{x_1} u, \ldots, \partial_{x_N} u)(t, x) \in \mathbb{R}^{d \times N}\) and \(\partial_{xx}^2 u(t, x) := \text{Diag}(\partial_{x_1x_1}^2 u, \ldots, \partial_{x_Nx_N}^2 u)(t, x) \in \mathbb{S}_{d \times N}\).

### 3.2 Viscosity solutions

We define viscosity solutions for the equation (3.3), as in the non-Markovian PDEs. We refer to Ren, Touzi & Zhang [43] for a general overview of viscosity solutions for such equations. Let \(X := (X^1, \ldots, X^N)\) be the canonical process on \(\Omega^N\) and \(\mathbb{F}^N = \{\mathcal{F}^N_t\}_{t \in [0, T]}\) the corresponding filtration. For \(t \in [0, T]\), define

\[
\Lambda^N_t := [t, T] \times \mathbb{R}^{d \times N}, \quad \text{and} \quad \bar{\Lambda}^N_t := [t, T] \times \mathbb{R}^{d \times N}.
\]

**Definition 3.2** For \((t, x) \in \Lambda^N_0\), let \(\mathcal{P}^N_L(t, x)\) be the set of \(\mathbb{P} \in \mathcal{P}_2(\Omega^N, \mathcal{F}^N_t)\) such that

- \(X_t = x, \mathbb{P}\)-a.s.,
- there exist \((b^\mathbb{P}, \sigma^\mathbb{P}) : [0, T] \times \Omega^N \to \mathbb{R}^{d \times N} \times \mathbb{S}_{d \times N}^D, \mathbb{F}^N\text{-measurable}, \text{bounded by } L\) coordinate-wisely, s.t.

\[
dX_s = b^\mathbb{P}_s dW^\mathbb{P}_s + \sigma^\mathbb{P}_s dW^\mathbb{P}_s,
\]

where \(W^\mathbb{P}\) is a \(d \times N\)-dimensional \(\mathbb{P}\)-Brownian motion.
Lemma 3.3 The set $\mathcal{P}_L^N(t, x)$ is weakly compact.

Proof Let $\bar{\mathcal{P}}_L^N(t, x)$ be defined as $\mathcal{P}_L^N(t, x)$, without requiring that $\sigma^p$ is blockwise diagonal. Clearly $\mathcal{P}_L^N(t, x) \subset \bar{\mathcal{P}}_L^N(t, x)$, and we know from Zheng [53, Theorem 3] that $\bar{\mathcal{P}}_L^N(t, x)$ is weakly compact. Therefore, we only need to prove that $\mathcal{P}_L^N(t, x)$ is closed under the weak convergence.

Let $(\mathbb{P}^n)_{n \geq 1}$ be sequence in $\mathcal{P}_L^N(t, x)$ converging weakly to some $\mathbb{P}$, and denote $t \mapsto VT_t(Y)$ the total variation process associated with a process $Y$. Clearly, the family $\{\mathbb{P}^n \circ (VT_t (\int_0^T b^n_s \, ds))^{-1}\}_{n \geq 1}$ is tight for all $t \in [0, T]$ as the $s^n$ are uniformly bounded. Therefore, we may apply Jacod & Shiryaev [35, Theorem 6.26] to deduce that $\mathbb{P}(X)$ converges weakly to $\mathbb{P}(X)$. This implies in particular that $\sigma^p$ still takes its values in $\mathbb{S}_D^{\phi}$, and therefore that $\mathcal{P}_L^N(t, x)$ is closed under the weak convergence. $\blacksquare$

Let $\mathcal{T}_{t,T}^N$ denote the set of $[t, T]$-valued $\mathbb{F}^N$-stopping times, and $\mathcal{T}_{t,T}^{N,+} := \{H \in \mathcal{T}_{t,T}^N : H > t\}$.

We define the sets of test functions:

$$\bar{\mathcal{A}}^N u(t, x) := \{\phi \in C_b^{1,2}(\overline{\Lambda}_t^N) : \exists H \in \mathcal{T}_{t,T}^{N,+} \text{ s.t. } (\phi - u)(t, x) = \max_{\theta \in \mathcal{T}_{t,T}^{N,+}} \mathbb{E}_t^N [\phi - u(\theta \wedge H, X_{\theta \wedge H})]\},$$

$$\mathcal{A}^N u(t, x) := \{\phi \in C_b^{1,2}(\overline{\Lambda}_t^N) : \exists H \in \mathcal{T}_{t,T}^{N,+} \text{ s.t. } (\phi - u)(t, x) = \min_{\theta \in \mathcal{T}_{t,T}^{N,+}} \mathbb{E}_t^N [\phi - u(\theta \wedge H, X_{\theta \wedge H})]\},$$

where $\mathbb{E}^N_{t,x}[\cdot] := \sup_{\mathbb{P} \in \mathcal{P}_L^N(t, x)} \mathbb{E}_t^\mathbb{P}[\cdot]$, $\mathcal{E}_t^N_{t,x}[\cdot] := \inf_{\mathbb{P} \in \mathcal{P}_L^N(t, x)} \mathbb{E}_t^\mathbb{P}[\cdot]$, and $C_b^{1,2}(\overline{\Lambda}_t^N)$ denotes the set of bounded functions of $C^{1,2}(\overline{\Lambda}_t^N)$ with bounded derivatives.

Definition 3.4 Let $u : \overline{\Lambda}_0^N \rightarrow \mathbb{R}$.

(i) $u$ is a viscosity supersolution of (3.3) if, for all $(t, x) \in \Lambda_0^N$ and $\phi \in \bar{\mathcal{A}}^N u(t, x)$,

$$-\partial_t \phi(t, x) - F^N (t, x, \phi(t, x), \partial_x \phi(t, x), \partial_{xx} \phi(t, x)) \geq 0. \quad (3.6)$$

(ii) $u$ is a viscosity subsolution of (3.3) if, for all $(t, x) \in \Lambda_0^N$ and $\phi \in \mathcal{A}^N u(t, x)$,

$$-\partial_t \phi(t, x) - F^N (t, x, \phi(t, x), \partial_x \phi(t, x), \partial_{xx} \phi(t, x)) \leq 0. \quad (3.7)$$

(iii) $u$ is a viscosity solution of (3.3) if it is a viscosity supersolution and subsolution.

Remark 3.5 The reader might find surprising our choice to resort to test functions for path-dependent PDEs in the context of Markovian equations. The reason is the following:
this notion only involves points of the space writing as $X_{\theta \wedge H}$ for some stopping time $\theta$. Then, we prove in Proposition 7.1 that the sequence $\{\mu^N(X_{\theta \wedge H})\}_{N \geq 1}$ is tight, and that all its accumulation points lie in a (infinite dimensional) viscosity neighborhood $\mathcal{N}_\delta(t, \mu)$. This property is crucial in the proof of our main results in Section 6.

3.3 Main results

Let $S^N$ be the set of functions $h : \mathbb{A}_0^N \rightarrow \mathbb{R}$ such that $h(t, x) = h^N(t, \mu^N(x))$ for some $h^N : [0, T] \times \mathcal{P}^N(\mathbb{R}^d) \rightarrow \mathbb{R}$.

Definition 3.6 We say $\{h^N\}_{N \geq 1} \in \prod_{N \geq 1} S^N$ is locally uniformly bounded if, for all $(t, \mu) \in \mathbb{T}_0$, there exist $\delta, M > 0$ s.t., for all $(s, x^N) \in [0, T] \times \mathbb{R}^{d \times N}$ s.t. $|s - t| + \mathcal{W}_2(\mu^N(x^N), \mu) \leq \delta$ and $N \geq 1$, we have $|h^N(s, x^N)| \leq M$.

We now state the main result of the paper:

Theorem 3.7 Let $\{V^N \in S^N\}_{N \geq 1}$ be a sequence of continuous and locally uniformly bounded viscosity solutions of (3.3) s.t. $V^N|_{t=T} = g^N$, and introduce for all $(t, \mu) \in \mathbb{T}_0$

$$
V(t, \mu) := \liminf_{N \to \infty, s \to t} V^N(s, \mu^N(x^N)), \quad V(t, \mu) := \limsup_{N \to \infty, s \to t} V^N(s, \mu^N(x^N)).
$$

If Assumption 2.6 holds, $V$ and $V$ are continuous and $V|_{t=T} = V|_{t=T} = g$, then $V^N$ converges to the unique continuous viscosity solution $V$ of (2.1), i.e., the following limit exists:

$$
V(t, \mu) = \lim_{N \to \infty, s \to t} V^N(s, \mu^N(x^N)) \quad \text{for all } (t, \mu) \in \mathbb{T}_0
$$

and is the unique viscosity solution of (2.1).

The proof of this Theorem relies heavily on the following result, which corresponds to an adaptation of the Barles & Souganidis [1] monotone scheme to our context:

Theorem 3.8 (i) Let $\{v^N \in S^N\}_{N \geq 1}$ be a sequence of continuous and locally uniformly bounded viscosity supersolutions of (3.3). The relaxed semi-limit defined by

$$
v(t, \mu) := \liminf_{N \to \infty, s \to t} v^N(s, \mu^N(x^N)) \quad \text{for all } (t, \mu) \in \mathbb{T}_0
$$
is finite and is a LSC viscosity supersolution of (2.1).

(ii) Let \( \{ u^N \in S^N \}_{N \geq 1} \) be a sequence of continuous and locally uniformly bounded viscosity subsolutions of (3.3). The relaxed semi-limit defined by

\[
\pi(t, \mu) := \limsup_{N \to \infty, s \to t} u^N(s, \mu_N(x^N)) \quad \text{for all } (t, \mu) \in \overline{Q}_0
\]

is finite and is a USC viscosity subsolution of (2.1).

The proof of this result is relegated to Section 6.

**Remark 3.9 (Comparison with the Barles-Souganidis monotone scheme)** (i) Our finite-dimensional approximation shares strong similarities with the numerical scheme of [1] for second order PDEs. The condition (3.1) can be seen as the *consistency* condition, and the existence of locally uniformly bounded solutions to (3.3) as the *stability* condition. However, the *monotonicity* condition seems less obvious at first sight. Note that our “scheme” is defined as viscosity solutions to a PDE, rather than as a classical solution to an approximating equation such as (2.1) in [1]. We then believe that the monotonicity condition lies in the very fact that \( V^N \), for \( N \geq 1 \), is a viscosity solution to (3.3); thus, for test functions tangent to \( V^N \) from above, we have the inequality (3.7), and the converse inequality (3.6) for test functions tangent from below.

(ii) The main motivation of the scheme of [1] is to derive numerical approximations for PDEs. It is then natural to wonder how the present result could be used to achieve this objective, i.e. finding numerical approximations of the solution of the PDE on Wasserstein space (2.1). Since our approximating function \( V^N \) is defined as a viscosity solution to a (finite-dimensional) second order PDE, one natural idea is the following: \( V^N \) can be approximated by the monotone scheme of [1]. The numerical approximation is then a function \( V^N_{\rho} \), \( \rho > 0 \), with \( V^N_{\rho} \to V^N \) as \( \rho \to 0 \). The question then boils down to finding an efficient numerical scheme to approximate \( V^N \) for large \( N \). This could legitimately be attempted via deep learning methods, see e.g. [44] or [14].

### 4 Application to stochastic control

#### 4.1 Mean field control

Let \( k \geq 1 \), \( A \subset \mathbb{R}^k \) and \( (b, \sigma) : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, S_d \), continuous in \( (t, a) \in [0, T] \times A \), Lipschitz-continuous in \( (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \) uniformly in \( (t, a) \) and uniformly
bounded by $L$. For $(t, \mu) \in \mathcal{Q}_0$ and $\alpha : [0, T] \times \Omega \rightarrow A$, let $\mathbb{P}^{t, \mu, \alpha}$ be s.t. $X$ is a controlled McKean-Vlasov diffusion with drift and diffusion coefficients $b$ and $\sigma$, i.e.

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}^{t, \mu, \alpha}_{X_r}, \alpha_r)dr + \int_t^s \sigma(r, X_r, \mathbb{P}^{t, \mu, \alpha}_{X_r}, \alpha_r)dW^\alpha_r, \quad \mathbb{P}^{t, \mu, \alpha}\text{-a.s.},$$

(4.1)

where $W^\alpha$ is a standard $d$-dimensional $\mathbb{P}^{t, \mu, \alpha}$-Brownian motion and $\mathbb{P}^{t, \mu, \alpha} \equiv \mu$. Let $\mathcal{A}_t$ be the set of $\mathbb{F}$-progressively measurable processes $\alpha : [t, T] \times \Omega \rightarrow A$ such that (4.1) has a unique weak solution. We consider the mean field control problem

$$V(t, \mu) := \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}[\mathbb{P}^{t, \mu, \alpha}_{X_T}] \left[ \int_t^T f(r, X_r, \mathbb{P}^{t, \mu, \alpha}_{X_r}, \alpha_r)dr + g(\mathbb{P}^{t, \mu, \alpha}_{X_T}) \right],$$

with $f : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \rightarrow \mathbb{R}$ and $g : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$. We know from Wu & Zhang [51, Theorem 5.8] that, if $V$ is continuous, then it is a viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation on Wasserstein space:

$$-\partial_t u(t, \mu) - F_{\text{HJB}}(t, \mu, \partial_x \delta_m u(t, \mu, \cdot), \partial^2_{xx} \delta_m u(t, \mu, \cdot)) = 0, \quad u(T, \cdot) = g,$$

(4.2)

where

$$F_{\text{HJB}}(t, \mu, Z, \Gamma) := \left\langle \mu, \sup_{a \in A} \left\{ b(t, \cdot, \mu, a) : \Gamma(\cdot) + f(t, \cdot, \mu, a) \right\} \right\rangle.$$

(4.3)

**Proposition 4.1** Assume $A$ is compact. Then $F_{\text{HJB}}$ satisfies Assumption 2.2.

**Proof** Let $(\mu^n, Z^n, \Gamma^n)$ be a sequence converging to some $(\mu, Z, \Gamma) \in \mathcal{Q}_0 \times C^0(\mathbb{R}^d, \mathbb{R}^d) \times C^0(\mathbb{R}^d, \mathcal{S}^d)$ in the sense of Assumption 2.2 (i). Observe that:

$$\left| \int_{\mathbb{R}^d} \sup_{a \in A} b(t^n, x, \mu^n, a) Z^n(x) \mu^n(dx) - \int_{\mathbb{R}^d} \sup_{a \in A} b(t, x, \mu, a) Z(x) \mu(dx) \right|$$

$$\leq \left| \int_{\mathbb{R}^d} \sup_{a \in A} b(t^n, x, \mu^n, a) Z^n(x) (\mu^n - \mu)(dx) \right| + \left| \int_{\mathbb{R}^d} \sup_{a \in A} b(t^n, x, \mu^n, a) (Z^n - Z)(x) \mu(dx) \right|$$

$$+ \int_{\mathbb{R}^d} \sup_{a \in A} \left( b(t^n, x, \mu^n, a) - b(t, x, \mu, a) \right) Z(x) \mu(dx)$$

$$\leq C \int_{\mathbb{R}^d} (1 + x^2)(\mu^n - \mu)(dx) + L \int_{\mathbb{R}^d} |Z^n(x) - Z(x)| \mu(dx)$$

$$+ \int_{\mathbb{R}^d} \sup_{a \in A} \left( b(t^n, x, \mu^n, a) - b(t, x, \mu, a) \right) Z(x) \mu(dx),$$

for some constant $C$ independent from $n$. The first term of the right-hand side converges to 0 because $W_2(\mu^n, \mu) \rightarrow 0$; the second term converges to 0 by the dominated convergence theorem, as $Z^n$ converges pointwise to $Z$ and the functions $\{Z^n\}_{n \geq 0}$ and $Z$ have quadratic growth uniformly in $n$; finally, the third term converges to 0 because it is continuous in
(\(t^n, \mu^n\)), by continuity of \(b\) and compactness of \(A\). Using similar estimates to handle terms in \(\sigma\) and \(f\), we deduce that \(F_{\text{HJB}}\) satisfies Assumption 2.2 (i). As to (ii), it is clearly satisfied as \(F_{\text{HJB}}\) is an integral w.r.t. \(\mu\).

\(\blacksquare\)

**Remark 4.2** We observe that, in the case of \(F_{\text{HJB}}\), \(Z\) and \(\Gamma\) may belong to \(L^1(\mu)\). However, we chose to restrict them to sets of bounded functions when we introduced the operator \(F\) in order to have a more general framework and avoid possible integrability issues. \(\blacksquare\)

**Finite-dimensional approximation** For \((t, x) \in [0, T] \times \mathbb{R}^{d \times N}\) and a given control \(\alpha = (\alpha^1, \ldots, \alpha^N): [0, T] \times \Omega^N \rightarrow A^N\), let \(\mathbb{P}^{t,x,\alpha}\) be such that, for all \(i \in [N] := \{1, \ldots, N\}\),

\[
X^i_t = x_i + \int_t^s b(r, X^i_r, \mu^N(X_r)), \alpha^i_r)dr + \int_t^s \sigma(r, X^i_r, \mu^N(X_r), \alpha^i_r)dW^i,\alpha_r, \mathbb{P}^{t,x,\alpha}\text{-a.s.},
\]

where \(W^\alpha := (W^{1,\alpha}, \ldots, W^{N,\alpha})^\top\) is a standard \(d \times N\)-dimensional \(\mathbb{P}^{t,x,\alpha}\)-Brownian motion. Let \(A_t^N\) be the set of \(\mathbb{R}^N\)-progressively measurable processes \(\alpha: [t, T] \times \Omega^N \rightarrow A^N\) s.t. (4.4) has a unique weak solution. We define the control problem

\[
V^N(t, x) := \sup_{\alpha \in A_t^N} \sum_{i=1}^N \mathbb{E}^{t,x,\alpha} \left[ \int_t^T f^{i,N}(r, X_r, \alpha_r)dr + g_N(X_T) \right],
\]

with \(f^{i,N}(t, x, a) := f(r, x_i, \mu^N(x), a)\). We know from standard stochastic control theory that, if \(V^N\) is continuous, then it is a viscosity solution of

\[
-\partial_t u(t, x) - \sup_{a \in A^N} \left\{ b(t, x, a) \cdot \partial_x u(t, x) + \frac{1}{2} \sigma^2(t, x, a) : \partial^2_{xx} u(t, x) + f(t, x, a) \cdot e \right\} = 0, \quad 0.5
\]

where \(b(t, x, a) := \left(b(t, x_i, \mu^N(x), a_i)\right)^\top\right)^{1 \leq i \leq N}\), \(\sigma(t, x, a) := \text{Diag} \left( \sigma(t, x_i, \mu^N(x), a_i) \right)_{1 \leq i \leq N}\),

\(f(t, x, a) := \left(f^{i,N}(t, x, a_i)\right)^\top\right)^{1 \leq i \leq N}\), and \(e := (1, \ldots, 1)^\top \in \mathbb{R}^N\).

**Proposition 4.3** Assume that:

- \(f\) and \(g\) are bounded and continuous on \(\mathcal{P}_2(\mathbb{R}^d)\), and extend continuously on \(\mathcal{P}_1(\mathbb{R}^d)\);
- \(b, \sigma\) and \(f\) are \(\beta\)-Hölderian in \(t\), uniformly in \((x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A\), for some \(\beta \in (0, 1]\);
- \(\sigma\) does not depend on \(\mu\), and satisfies \(\sigma(\cdot, a) \in C^{1,2}(\mathbb{R}^d)\) for all \(a \in A\), with all its derivatives uniformly bounded.

Then \(V^N\) converges to \(V\), i.e.,

\[
V(t, \mu) = \lim_{N \to \infty, s \to t} \lim_{W_2(\mu^N(x^N), \mu) \to 0} V^N(s, x^N) \quad \text{for all } (t, \mu) \in \mathbb{Q}_0.
\]
Proof We first show that (4.5) is the finite-dimensional approximation of (4.2), i.e.,

$$F_{\text{HJB}}^N(t, x, z, \gamma) = \sup_{a \in A^N} \left\{ b(t, x, a) \cdot z + \frac{1}{2} \sigma^2(t, x, a) : \gamma + f(t, x, a) \cdot e \right\},$$

where $F_{\text{HJB}}^N$ is the finite-dimensional approximation of $F_{\text{HJB}}$ defined by (3.2). We compute

$$F_{\text{HJB}}^N(t, x, z, \gamma) = F_{\text{HJB}}(t, \mu^N(x), N z \cdot 1_x, N \gamma \cdot 1_x)$$

$$= \left\{ \mu^N(x), N \sup_{a \in A} \left\{ b(t, \cdot, \mu^N(x), a) \cdot z + \frac{1}{2} \sigma^2(t, \cdot, \mu^N(x), a) : \gamma + f(t, \cdot, \mu^N(x), a) \right\} \right\}$$

$$= \sum_{i=1}^N \sup_{a \in A} \left\{ b(t, x_i, \mu^N(x), a) \cdot z_i + \frac{1}{2} \sigma^2(t, x_i, \mu^N(x), a) : \gamma_i + f(t, x_i, \mu^N(x), a) \right\}$$

$$= \sup_{a \in A^N} \sum_{i=1}^N \left\{ b(t, x_i, \mu^N(x), a_i) \cdot z_i + \frac{1}{2} \sigma^2(t, x_i, \mu^N(x), a_i) : \gamma_i + f(t, x_i, \mu^N(x), a_i) \right\}$$

$$= \sup_{a \in A^N} \left\{ b(t, x, a) \cdot z + \frac{1}{2} \sigma^2(t, x, a) : \gamma + f(t, x, a) \cdot e \right\}.$$

Moreover, as $f$ and $g$ are bounded, the $V^N$ are uniformly bounded, and $\mathcal{V}$ and $\mathcal{V}$ (defined similarly to (3.8)) are bounded.

We now prove that $\mathcal{V}$ and $\mathcal{V}$ are continuous. Let $(\Omega^0, \mathcal{F}^0, \mathbb{P}^0)$ be a probability space such that we can construct for all $(t, x, \alpha) \times [0, T] \times \mathbb{R}^{d \times N} \times A_1^N$ a diffusion process $X_t^{x, \alpha}$ such that $\mathbb{P}^0 \circ (X_t^{x, \alpha})^{-1} = \mathbb{P}^{t, x, \alpha} \circ X^{-1}$. Fix $x, x'$ and $R > 0$ s.t. $\|x\|_2, \|x'\|_2 < R$, and $R' > R$ to be determined later. Observe that $D_{R'} := \{ m \in \mathcal{P}_2(\mathbb{R}^d) : \|m\|_2 \leq R' \}$ is bounded in $\mathcal{P}_2(\mathbb{R}^d)$, and therefore is $\mathcal{W}_1$-compact. Thus, there exists a continuity modulus $\rho_{R'}$ for $g$ on this set, and then:

$$\mathbb{E}^{\mathbb{P}^0} \left[ g(\mu^N(X_t^{x, \alpha})) - g(\mu^N(X_t^{x', \alpha})) \right]$$

$$\leq \mathbb{E}^{\mathbb{P}^0} \left[ \rho_{R'} (W_1(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right]$$

$$+ \left\{ g(\mu^N(X_t^{x, \alpha})) - g(\mu^N(X_t^{x', \alpha})) \right\} \left( 1_{D_{R'}}(\mu^N(X_t^{x, \alpha})) + 1_{D_{R'}}(\mu^N(X_t^{x', \alpha})) \right)$$

$$\leq \mathbb{E}^{\mathbb{P}^0} \left[ \rho_{R'} (W_1(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right] + C \mathbb{P}^0 \left( \|X_t^{x, \alpha}\|_2 \geq R' \right) + C \mathbb{P}^0 \left( \|X_t^{x', \alpha}\|_2 \geq R' \right)$$

$$\leq \mathbb{E}^{\mathbb{P}^0} \left[ \rho_{R'} (W_1(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right] + C' \frac{R'}{R} (1 + R')$$

with $C' > 0$ is independent from $N$ and $\alpha$, due to the uniform boundedness of the drift and diffusion coefficients, and to Markov’s inequality. Note also that

$$\mathbb{E}^{\mathbb{P}^0} \left[ \rho_{R'} (W_1(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right]$$

$$\leq \rho_{R'} (\eta) + \frac{1}{\sqrt{\eta}} \sqrt{\mathbb{E}^{\mathbb{P}^0} \left[ \rho_{R'}^2 (W_1(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right] \mathbb{E}^{\mathbb{P}^0} \left[ W_1^2(\mu^N(X_t^{x, \alpha}), \mu^N(X_t^{x', \alpha}))) \right]}$$

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for all $\eta > 0$. Fix $\varepsilon > 0$. Choosing $R' = R_{\varepsilon} := \frac{2C'(1+R^2)}{\varepsilon}$, we can find $\delta > 0$ and $\eta > 0$ (possibly depending on $\varepsilon$ but not on $N$) such that

$$
\mathbb{E}^{\mathcal{F}_0} \left[ \left| g(\mu^N(T,x,\alpha)) - g(\mu^N(T,x',\alpha)) \right| \right] \leq \varepsilon \quad \text{whenever } \|x - x'\|_2 \leq \delta.
$$

We may prove a similar estimate with $f$, and finally, by arbitrariness of $\alpha \in \mathcal{A}^N$:

$$
\left| V^N(t,x) - V^N(t,x') \right| \leq \varepsilon \quad \text{whenever } \|x - x'\|_2 \leq \delta.
$$

Using similar estimates, we may also prove that

$$
\left| V^N(t,x) - V^N(t',x) \right| \leq \varepsilon \quad \text{whenever } |t - t'| \leq \delta,
$$

which implies that $\underline{V}$ and $\overline{V}$ are continuous and satisfy $\underline{V}|_{t=T} = \overline{V}|_{t=T} = g$. Note also that, by symmetry of the problem, $V^N \in \mathcal{S}^N$. Since we are under the assumptions of [19, Theorem 5.1], by Remark 2.7, Assumption 2.6 holds. We may then apply Theorem 3.7 to derive the convergence result.

**Remark 4.4** We emphasize that the assumptions on $b$ and $\sigma$ and the H"older regularity assumption on $f$ in Proposition 4.3 are solely used to apply the comparison principle of [19]. Therefore, any availability of a less restrictive comparison principle in the literature would automatically relax the assumptions we need to ensure this convergence result.

### 4.2 Zero-sum stochastic differential games

We present in a more informal way a second example. We consider the following control problem, arising in zero-sum games:

$$
V_+ (t, \mu) := \inf_{\alpha^1 \in \mathcal{A}^t_1} \sup_{\alpha_2 \in \mathcal{A}^t_2} \mathbb{E}^{\mathcal{F}_t,\mu,\alpha^1,\alpha^2} \left[ \int_t^T f(s, X_s, \mathbb{P}^{t,\mu,\alpha^1,s^2,\alpha^1,s^2}_s) ds + g(\mathbb{P}^{T,\mu,\alpha^1,s^2}) \right],
$$

where the measures $\{\mathbb{P}^{t,\mu,\alpha^1,s^2} : \alpha_1 \in \mathcal{A}^t_1, \alpha_2 \in \mathcal{A}^t_2\}$ are such that $X$ has the dynamics (4.1), substituting $(\alpha^1,\alpha^2)$ to $\alpha$. By Cosso & Pham [20], $V_+$ is a viscosity solution of (2.1), with operator

$$
F_+ (t, \mu, Z, \Gamma) := \left\langle \mu, \inf_{a_2 \in \mathcal{A}^t_2} \sup_{a_1 \in \mathcal{A}^t_1} \left\{ b(t, \cdot, \mu, a_1, a_2) \cdot Z + 1 + \frac{1}{2} \sigma^2(t, \cdot, \mu, a_1, a_2) : \Gamma + f(t, \cdot, \mu, a_1, a_2) \right\} \right\rangle.
$$

Although [20] uses another notion of viscosity solutions, we may consider in the context of our discussion that $V_+$ is also a viscosity solution in the sense of Definition 2.4. Assuming
\(A_1\) and \(A_2\) are compact, the corresponding finite-dimensional approximation is then given by (3.3), with operator 
\[
F^N_+(t, x, z, \gamma) := \inf_{a_1 \in A_1^N} \sup_{a_2 \in A_2^N} \left\{ b(t, x, a_1, a_2) \cdot z + \frac{1}{2} \sigma^2(t, x, a_1, a_2) : \gamma + f(t, x, a_1, a_2) \cdot e \right\}.
\]

As in the case of mean field control, we may show that this corresponds to the PDE satisfied by the control problem 
\[
V^N_+(t, x) := \inf_{\alpha^1 \in (A_1^N)^N} \sup_{\alpha^2 \in (A_2^N)^N} \sum_{i=1}^N \mathbb{E}^{\bar{p}^t,x,\alpha^1,\alpha^2} \left[ \int_t^T f^{i,N}(r, \bar{X}_r, \bar{\alpha}^1_r, \bar{\alpha}^2_r) dr + g(\bar{X}_T) \right],
\]
where the measures \(\{\bar{p}^{t,x,\alpha^1,\alpha^2} : \alpha^1 \in A_1^1, \alpha^2 \in A_2^1\}\) are such that \(X\) has the dynamics (4.4), substituting \((\alpha^1, \alpha^2)\) to \(\alpha\).

**Proposition 4.5** If Assumption 2.6 holds for (2.1) with \(F = F_+\), then \(V^N_+\) converges to \(V_+\), i.e., 
\[
V_+(t, \mu) = \lim_{N \to \infty, s \to t} V^N_+(s, \mu^N) \quad \text{for all } (t, \mu) \in Q_0.
\]

### 4.3 The uncontrolled case

The purpose of this paragraph is to recover the classical propagation of chaos result for diffusion processes, whose first instance was given by Snitzman [46] for some special models. \(b\) and \(\sigma\) do no longer depend on the variable \(a\). We consider the equation 
\[
-\partial_t u(t, \mu) - \left\langle \mu, b(t, \cdot, \mu) \cdot \partial_x \delta_m u(t, \mu, \cdot) + \frac{1}{2} \sigma^2(t, \cdot, \mu) : \partial_{xx} \delta_m u(t, \mu, \cdot) \right\rangle = 0, \quad u(T, \cdot) = g, \quad (4.6)
\]
where \(g \in C^0_b(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})\), the set of continuous and bounded functions from \(\mathcal{P}_2(\mathbb{R}^d)\) to \(\mathbb{R}\).

For \((t, \mu, \mu^N) \in \overline{Q}_0 \times \mathbb{R}^{d \times N}\), let \((\bar{p}^t, \mu, \bar{x}^N) \in \mathcal{P}_L(t, \mu) \times \mathcal{P}_L^N(t, \mu^N)\) be such that \(X\) and \(\bar{X}\) are the uncontrolled versions of (4.1) and (4.4), respectively \(\bar{p}^{t, \mu}\text{-a.s.}\) and \(\bar{p}^{t, x^N}\text{-a.s.}\). As \(g \in C^0_b(\mathcal{P}_2(\mathbb{R}^d), \mathbb{R})\), we know that, under some smoothness assumptions on \(b\) and \(\sigma\) (see Talbi, Touzi & Zhang [49, Lemma 3.6]), we have 
\[
u(t, \mu) = g(\bar{p}^t, X_T), \quad u^N(t, \bar{x}^N) = \mathbb{E}^{\bar{p}^t, x^N} [g^N(\bar{X}_T)] = \mathbb{E}^{\bar{p}^t, x^N} [g(\mu^N(\bar{X}_T))],
\]
for all \((t, \mu) \in \overline{Q}_0\) and \(\bar{x}^N \in \mathbb{R}^{d \times N}\). Thus, applying Proposition 4.3, we have \(u^N(0, \bar{x}^N) \to u(0, \mu)\) as \(N \to \infty\) and \(\mathcal{W}_2(\mu^N(\bar{x}^N), \mu) \to 0\), hence 
\[
\mathbb{E}^{\bar{p}^0, x^N} [g(\mu^N(\bar{X}_T))] \to_{N \to \infty} g(\bar{p}^0, X_T),
\]
which exactly means that $\bar{\mathbb{P}}^{0,\mathbb{X}} \circ (\mu^N(X^N_T))^{-1}$ converges weakly to $\bar{\mathbb{P}}^{0,\mu}_{X_T}$, as it is true for all $g \in C^0_b(\mathcal{P}_2(\mathbb{R}^d),\mathbb{R})$. This corresponds to the propagation of chaos result proved by Oelschlager [41].

5 Extension to path-dependent PDEs

5.1 Pathwise derivatives

For $t \in [0,T)$, we adapt our previous notations to the path-dependent case:

$$Q_t := [t,T) \times \mathcal{P}_2(\Omega) \quad \text{and} \quad \overline{Q}_t := [t,T] \times \mathcal{P}_2(\Omega).$$

For $(t,\mu) \in \overline{Q}_0$, we denote by $\mu_{[0,t]}$ the law of the stopped process $X_{\cdot\wedge t}$ under $\mu$. We shall use the notion of pathwise derivative of Ekren, Keller, Touzi & Zhang [26], which is tailor-made for continuous semimartingales. In particular, it allows to introduce a notion of derivative that is intrinsic to the space of continuous paths, whereas the notion of Dupire [25] which requires to include càdlàg paths.

Definition 5.1 (i) Given a metric space $E$, we denote by $C^0([0,T] \times \Omega, E)$ the set of $\mathbb{F}$-progressively measurable and continuous functions from $[0,T] \times \Omega$ to $E$, where $\Omega$ is equipped with the norm $|\omega| := \sup_{t \in [0,T]}|\omega_t|$ for all $\omega \in \Omega$.

(ii) We denote by $u \in C^{1,2}([0,T] \times \Omega)$ the set of functions $u : [0,T] \times \Omega \to \mathbb{R}$ such that there exist $\partial_t u \in C^0([0,T] \times \Omega, \mathbb{R})$, $\partial_\omega u \in C^0([0,T] \times \Omega, \mathbb{R}^d)$ and $\partial^2_{\omega\omega} u \in C^0([0,T] \times \Omega, S_d)$ such that, for all $\mathbb{P} \in \bigcup_{t \geq 0} \mathcal{P}_L$, $u$ satisfies

$$du(t, X) = \partial_t u(t, X)dt + \partial_\omega u(t, X) \cdot dX_t + \frac{1}{2} \partial^2_{\omega\omega} u(t, X) : d\langle X \rangle_t, \quad \mathbb{P}\text{-a.s.}$$

Definition 5.2 Fix $t \in [0,T]$.

(i) We denote by $C^0(\overline{Q}_t)$ the set of functions $u : \overline{Q}_t \to \mathbb{R}$ continuous for the pseudo-metric:

$$\overline{W}_2((s,\mu),(r,\nu)) := \left(|s-r|^2 + W^2_2(\mu_{[0,s]},\nu_{[0,r]})\right)^{\frac{1}{2}} \quad \text{for all } (s,\mu),(r,\nu) \in \overline{Q}_t. \quad (5.1)$$

(ii) We denote by $C^{1,2}_b(\overline{Q}_t)$ the set of bounded functions $u : \overline{Q}_t \to \mathbb{R}$ such that $\partial_t u$, $\delta_m u$, $\partial_\omega \delta_m u$, $\partial^2_{\omega\omega} \delta_m u$ exist, are bounded in all their variables and continuous in $(t,\mu)$ in the sense of (i), where the functional linear derivative takes the form $\delta_m u : [t,T] \times \mathcal{P}_2(\Omega) \times \Omega \to \mathbb{R}$ satisfying, for any $s \in [t,T]$ and $\mu,\mu' \in \mathcal{P}_2(\Omega)$,

$$u(s,\mu') - u(s,\mu) = \int_0^1 \int_0^1 \delta_m u(s,\lambda\mu' + (1-\lambda)\mu,\omega)(\mu' - \mu)(d\omega)d\lambda.$$

Note that, if $u \in C^0(\overline{Q}_t)$, then $u(s,\mu) = u(s,\mu_{[0,s]})$ for all $(s,\mu) \in \overline{Q}_t$. 18
5.2 Path-dependent equation on Wasserstein space

Let \( F : [0, T] \times \mathcal{P}_2(\Omega) \times \mathbb{R} \times L^0(\Omega, \mathbb{R}^d) \times L^0(\Omega, \mathbb{S}_d) \rightarrow \mathbb{R} \), where \( L^0(\Omega, \mathbb{R}^d) \) (resp. \( L^0(\Omega, \mathbb{S}_d) \)) denotes the set of Borel measurable functions from \( \Omega \) to \( \mathbb{R}^d \) (resp. \( \mathbb{S}_d \)) with quadratic growth, and \( g : \mathcal{P}_2(\Omega) \rightarrow \mathbb{R} \). We consider the following equation:

\[
-\partial_t u(t, \mu) - F(t, \mu, u(t, \mu), \partial_\omega \delta_m u(t, \mu, \cdot), \partial^2_{\omega, \omega} \delta_m u(t, \mu, \cdot)) = 0, \quad u|_{t=T} = g, \quad (t, \mu) \in Q_0. \tag{5.2}
\]

We define semijets similarly to (2.2) and straightforwardly adapt Proposition 2.5 and Assumption 2.2 to the path-dependent setting.

5.3 Viscosity solutions

We redefine, for all \((t, \mu) \in Q_0\), \( \mathcal{P}_L(t, \mu) := \{ P \in \mathcal{P}_L : P_{X_t^n} = \mu_{[0,t]} \} \), as well as the neighborhood

\[ N_{\delta}(t, \mu) := [t, t + \delta] \times \mathcal{P}_L(t, \mu), \]

which is compact under \( \tilde{W}_2 \) (see again Wu & Zhang [51, Lemma 4.1]). We then introduce the sets of test functions:

\[
\mathcal{A}u(t, \mu) := \left\{ \varphi \in C^{1,2}_b(\overline{Q}_t) : (\varphi - u)(t, \mu) = \max_{N_{\delta}(t, \mu)} (\varphi - u) \text{ for some } \delta > 0 \right\},
\]

\[
\mathcal{A}u(t, \mu) := \left\{ \varphi \in C^{1,2}_b(\overline{Q}_t) : (\varphi - u)(t, \mu) = \min_{N_{\delta}(t, \mu)} (\varphi - u) \text{ for some } \delta > 0 \right\}.
\]

**Definition 5.3** Let \( u : \overline{Q}_0 \rightarrow \mathbb{R} \).

(i) \( u \) is a viscosity supersolution of (5.2) if, for all \((t, \mu) \in Q_0\) and \( \varphi \in \mathcal{A}u(t, \mu) \),

\[
-\partial_t \varphi(t, \mu) - F(t, \mu, u(t, \mu), \partial_\omega \delta_m \varphi(t, \mu, \cdot), \partial^2_{\omega, \omega} \delta_m \varphi(t, \mu, \cdot)) \geq 0.
\]

(ii) \( u \) is a viscosity subsolution of (5.2) if, for all \((t, \mu) \in Q_0\) and \( \varphi \in \mathcal{A}u(t, \mu) \),

\[
-\partial_t \varphi(t, \mu) - F(t, \mu, u(t, \mu), \partial_\omega \delta_m \varphi(t, \mu, \cdot), \partial^2_{\omega, \omega} \delta_m \varphi(t, \mu, \cdot)) \leq 0.
\]

(iii) \( u \) is a viscosity solution of (5.2) if it is a viscosity supersolution and subsolution.

5.4 Finite-dimensional approximation

Let \( N \geq 1 \). We shall write in bold character the elements \( \omega = (\omega_1, \ldots, \omega_N) \in \Omega^N \). Introduce \( F^N : [0, T] \times \Omega^N \times \mathbb{R} \times \mathbb{R}^{d \times N} \times \mathbb{S}^{D}_{d \times N} \rightarrow \mathbb{R} \) such that:

\[
F^N \left( t', \omega, y', \frac{\varphi(\omega)}{N}, \frac{\varphi'(\omega)}{N} \right) \rightarrow F(t, \mu, y, \varphi, \varphi') \text{ as } (N, t', \mu^N(\omega), y') \rightarrow (+\infty, t, \mu, y) \tag{5.3}
\]
for all $\varphi \in C^1_b(\mathbb{R}^d, \mathbb{R})$, where we denote:

$$
\varphi(\omega) := (\varphi(\omega^1), \ldots, \varphi(\omega^N)) \quad \text{and} \quad f'(x) = \text{Diag}(\varphi'(\omega^1), \ldots, \varphi'(\omega^N)).
$$

As in the Markovian case, we may guarantee the existence of an approximation. Introduce, for $(t, \omega, y, z, \gamma) \in [0, T] \times \Omega^N \times \mathbb{R} \times \mathbb{R}^{d \times N} \times \mathcal{F}_{d \times N}$,

$$
F^N(t, \omega, y, z, \gamma) := F(t, \mu^N(\omega), y, Nz \cdot 1_\omega, N\gamma \cdot 1_\omega),
$$

where $z \cdot 1_\omega := \sum_{k=1}^{N} z_k 1_{\omega_k}(\omega)$ and $\gamma \cdot 1_\omega(\omega) := \sum_{k=1}^{N} \gamma_k 1_{\omega_k}(\omega)$ for all $\omega \in \Omega$. Then, if Assumption 2.2 holds, then $F^N$ satisfies 5.3.

We now introduce the path-dependent PDE on $[0, T] \times \Omega^N$:

$$
-\partial_t u(t, \omega) - F^N(t, \omega, u(t, \omega), \partial_\omega u(t, \omega), \partial^2_{\omega \omega} u(t, \omega)) = 0, \quad u|_{t=T} = g^N, \tag{5.5}
$$

with $g^N(\omega) := g(\mu^N(\omega))$, $\partial_\omega u(t, \omega) := (\partial_{\omega_1} u, \ldots, \partial_{\omega_N} u)(t, \omega) \in \mathbb{R}^{d \times N}$ and $\partial^2_{\omega \omega} u(t, \omega) := \text{Diag}(\partial^2_{\omega_1 \omega_1} u, \ldots, \partial^2_{\omega_N \omega_N} u)(t, \omega) \in \mathcal{S}_{d \times N}$.

We now define viscosity solutions for (5.5). We adapt the notations of Section 3.2 to the path-dependent case. For $t \in [0, T)$, define

$$
\Lambda^N_t := [t, T) \times \Omega^N, \quad \text{and} \quad \bar{\Lambda}^N_t := [t, T) \times \Omega^N.
$$

For $(t, \omega) \in \Lambda^N_t$, we define $\mathcal{P}^N_L(t, \omega)$ similarly to $\mathcal{P}^N_L(t, \mathbf{x})$, with the condition $X_{t \wedge} = \omega_{t \wedge}$, $\mathbb{P}$-a.s., for all $\mathbb{P} \in \mathcal{P}^N_L(t, \omega)$. Similarly to Lemma 3.3, we have the following result:

**Lemma 5.4** The set $\mathcal{P}^N_L(t, \omega)$ is weakly compact.

We define the sets of test functions:

$$
\mathcal{T}^N u(t, \omega) := \left\{ \phi \in C^{1,2}_b(\bar{\Lambda}^N_t) : \exists \mathcal{H} \in \mathcal{T}^N_{t, T} \text{ s.t. } (\phi - u)(t, \omega) = \max_{\theta \in \mathcal{T}^N_{t, T}} \mathcal{E}^N_{t, \omega} \left[ (\phi - u)(\theta \wedge \mathcal{H}, \mathbf{X}_{\theta \wedge} \wedge \mathcal{H}) \right] \right\},
$$

$$
\mathcal{A}^N u(t, \omega) := \left\{ \phi \in C^{1,2}_b(\bar{\Lambda}^N_t) : \exists \mathcal{H} \in \mathcal{T}^N_{t, T} \text{ s.t. } (\phi - u)(t, \omega) = \min_{\theta \in \mathcal{T}^N_{t, T}} \mathcal{E}^N_{t, \omega} \left[ (\phi - u)(\theta \wedge \mathcal{H}, \mathbf{X}_{\theta \wedge} \wedge \mathcal{H}) \right] \right\},
$$

where $\mathcal{E}^N$ and $\mathcal{E}^N$ are the nonlinear expectations defined by

$$
\mathcal{E}^N_{t, \omega} [\cdot] := \sup_{\mathbb{P} \in \mathcal{P}^N_L(t, \omega)} \mathbb{E}^\mathbb{P} [\cdot], \quad \mathcal{E}^N_{t, \omega} [\cdot] := \inf_{\mathbb{P} \in \mathcal{P}^N_L(t, \omega)} \mathbb{E}^\mathbb{P} [\cdot], \tag{5.6}
$$

and $C^{1,2}_b(\bar{\Lambda}^N_t)$ denotes the bounded elements of $C^{1,2}(\bar{\Lambda}^N_t)$ (defined similarly to $C^{1,2}([0, T] \times \Omega)$) with bounded derivatives.
Definition 5.5 Let $u : \tilde{\Lambda}_0^N \to \mathbb{R}$.

(i) $u$ is a viscosity supersolution of \(2.1\) if, for all $(t, \omega) \in \Lambda_0^N$ and $\phi \in \mathcal{A}^N u(t, \omega)$,

$$-\partial_t \phi(t, \omega) - F^N(t, \omega, \phi(t, \omega), \partial_\omega \phi(t, \omega), \partial^2_{\omega,\omega} \phi(t, \omega)) \leq 0.$$  

(ii) $u$ is a viscosity subsolution of \(2.1\) if, for all $(t, \omega) \in \Lambda_0^N$ and $\phi \in \mathcal{A}^N u(t, \omega)$,

$$-\partial_t \phi(t, \omega) - F^N(t, \omega, \phi(t, \omega), \partial_\omega \phi(t, \omega), \partial^2_{\omega,\omega} \phi(t, \omega)) \geq 0.$$  

(iii) $u$ is a viscosity solution of \(2.1\) if it is a viscosity supersolution and subsolution.

In this paragraph, $\mathcal{S}^N$ denotes the set of functions $h : \tilde{\Lambda}_0^N \to \mathbb{R}$ s.t. $h(t, \omega) = h^N(t, \mu^N(\omega))$ for some $h^N : [0, T] \times \mathcal{P}^N(\Omega) \to \mathbb{R}$.

Theorem 5.6 Let $\{V^N \in \mathcal{S}^N\}_{N \geq 1}$ be a sequence of uniformly continuous for \(5.1\) and locally bounded, uniformly in $N$, viscosity solutions of \(5.5\) s.t. $V^N(T, \cdot) = g^N$, and introduce

$$\underline{V}(t, \mu) := \liminf_{N \to \infty, s \to t} V^N(s, \mu^N(\omega^N)), \quad \overline{V}(t, \mu) := \limsup_{N \to \infty, s \to t} V^N(s, \mu^N(\omega^N)).$$  

If Assumption 2.6 holds and $\underline{V}|_{t=\tau} = \overline{V}|_{t=\tau} = g$, then $u^N$ converges to the unique continuous viscosity solution $V$ of \(5.2\), i.e.,

$$V(t, \mu) = \lim_{N \to \infty, s \to t} V^N(s, \mu^N(\omega^N)) \quad \text{for all } (t, \mu) \in \mathbb{Q}_0.$$  

Theorem 5.7 (i) Let $\{v^N \in \mathcal{S}^N\}_{N \geq 1}$ be a sequence of uniformly continuous for \(5.1\) and locally bounded, uniformly in $N$, viscosity supersolutions of \(5.5\). Then, the relaxed semi-limit defined by

$$\underline{\pi}(t, \mu) := \liminf_{N \to \infty, s \to t} v^N(s, \mu^N(\omega^N)) \quad \text{for all } (t, \mu) \in \mathbb{Q}_0$$  

is finite and is a LSC viscosity supersolution of \(2.1\).

(ii) Let $\{u^N \in \mathcal{S}^N\}_{N \geq 1}$ be a sequence of uniformly continuous for \(5.1\) and locally bounded, uniformly in $N$, viscosity subsolutions of \(5.5\). Then, the relaxed semi-limit defined by

$$\overline{\pi}(t, \mu) := \limsup_{N \to \infty, s \to t} u^N(s, \mu^N(\omega^N)) \quad \text{for all } (t, \mu) \in \mathbb{Q}_0$$  

is finite and is a USC viscosity subsolution of \(2.1\).

These results are proved in Section 6.
6 Proof of the main results

6.1 The Markovian setting

**Proof of Theorem 3.7** (given Theorem 3.8) By Theorem 3.8, \( \underline{V} \) and \( \overline{V} \) are respectively continuous viscosity supersolution and subsolution of (2.1). As \( \underline{V}_{|t=T} = \overline{V}_{|t=T} = g \), the comparison principle implies \( \underline{V} \geq \overline{V} \). By (3.8), we also have the converse inequality, and thus \( \underline{V} = \overline{V} \), and these functions are viscosity solutions of (2.1). Given the comparison principle, (2.1) has a unique continuous viscosity solution, and thus \( \underline{V} = \overline{V} = V \) and the two semi-limits (3.8) are equal to the limit.

**Proof of Theorem 3.8** We only prove the convergence of the viscosity supersolutions, as the case of the subsolutions is handled similarly. It is clear that \( v \) is finite as \( \{v^N\}_{N \geq 1} \) is locally bounded, uniformly in \( N \). Fix \((t, \mu) \in Q_0 \) and \( \varphi \in \overline{A}v(t, \mu) \) with corresponding \( \delta_0 \in (0, T - t) \). By Proposition 2.5, we may assume w.l.o.g. that \( \varphi \) is a semijet of the form (2.2), with characteristics \((v, a, f) \in \mathbb{R} \times \mathbb{R} \times C^2_0(\mathbb{R}^d) \). We also introduce, for all \( N \geq 1 \), the functions \( \phi^N(s, x) := \varphi(s, \mu_N(x)) \) for all \((s, x) \in \mathbb{R}_+N \). Finally, let \((t^N, x^N)\) be a sequence such that \( t^N \rightarrow t, \mu^N(x^N) \xrightarrow{W_2} \mu \), and \( v^N(t^N, \mu^N(x^N)) \rightarrow v(t, \mu) \) as \( N \rightarrow \infty \).

The main idea is the following: we shall approximate the \( t^N, x^N \) with “good” points in which we may apply the viscosity supersolution property of \( v^N \), and then deduce the one of \( v \) by passing to the limit as \( N \rightarrow \infty \). Such good points are given by the following lemma:

**Lemma 6.1** There exists a family \( \{t^N \delta, x^N \delta\}, \delta > 0 \) and \( N \geq 1 \), such that \( \phi^N \in \overline{A}v^N(t^N \delta, x^N \delta) \) and \( (t^N \delta, \mu^N(x^N \delta)) \xrightarrow{\delta \rightarrow 0} (t^N, x^N) \) for all \( N \geq 1 \).

Let \((t^N \delta, x^N \delta)\) be as in the above lemma. The supersolution property of \( v^N \) provides:

\[
-\partial_t \phi^N(q^N_\delta) - F^N(q^N_\delta, v^N(q^N_\delta), \partial_x \phi^N(q^N_\delta), \partial^2_{xx} \phi^N(q^N_\delta)) \geq 0.
\]

(6.1)

where \( q^N_\delta := (t^N \delta, x^N \delta) \). By the equalities (6.5) below, we have:

\[
F^N(q^N_\delta, v^N(q^N_\delta), \partial_x \phi^N(q^N_\delta), \partial^2_{xx} \phi^N(q^N_\delta)) = F^N(q^N_\delta, v^N(q^N_\delta), \frac{f'(x^N \delta)}{N}, \frac{f''(x^N \delta)}{N})
\]

Note that

\[
(t^N \delta, \mu^N(x^N \delta), v^N(t^N \delta, \mu^N(x^N \delta))) \xrightarrow{\delta \rightarrow 0} (t^N, \mu^N(x^N), v^N(t^N, \mu^N(x^N)))
\]

by Lemma 6.1 and continuity of \( v^N \), and thus

\[
(t^N \delta, \mu^N(x^N \delta), v^N(t^N \delta, \mu^N(x^N \delta))) \xrightarrow{(\delta, N) \rightarrow (0, \infty)} (t, \mu, v(t, \mu)).
\]
We then deduce from the consistency property (3.1) that
\[
F_N(q_N^\delta, v_N^\delta, \partial_x \phi_N(q_N^\delta), \partial_{xx} \phi_N(q_N^\delta)) \xrightarrow{\delta \to 0, N \to \infty} F(t, \mu, v(t, \mu), f', f'').
\]

Finally, as \(\partial_t \phi_N(q_N^\delta) = a = \partial_t \phi(t, \mu)\), sending \((\delta, N) \to (0, \infty)\) in (6.1) provides the viscosity supersolution property of \(v\).

**Proof of Lemma 6.1** Replacing \(\phi\) with \(\tilde{\phi}(s, \cdot) := \phi(s, \cdot) - (s - t)^2\), we may also assume w.l.o.g. that \((t, \mu)\) is a strict maximum of \((\phi - v)\) on \(\mathcal{N}_{\delta_0}(t, \mu)\).

**Step 1:** Fix \(\delta \in (0, \delta_0^3)\), and introduce the stopping time
\[
H_N^\delta := \inf \{ s \geq t^N : W_2(\mu_N^x(X_s), \mu_N(x^N)) = 2\delta \} \wedge (t^N + 2\delta).
\]

By Lemma A.1 and pathwise continuity of \(X\), we may choose \(\delta\) sufficiently small so that the domain of \(H_N^\delta\) is a convex set (namely, the Euclidian ball centered in \(x^N\) with radius \(2\delta\)).

Then, by Ekren, Touzi & Zhang [27, Theorem 3.5] and weak compactness of \(\mathcal{P}_L^N(t, x^N)\), there exists \((\theta_N^\delta, \mathbb{P}^{N,*}) \in \mathcal{T}_{tN, T}^N \times \mathcal{P}_L^N(t, x^N)\) s.t.
\[
\mathbb{E}^{\mathbb{P}^{N,*}}\left[ (\phi_N - v_N)(\theta_N^\delta \wedge H_N^\delta, X_{\theta_N^\delta \wedge H_N^\delta}) \right] = \sup_{\theta \in \mathcal{T}_{tN, T}^N} \mathbb{E}^{\mathbb{P}^{N,*}}\left[ (\phi_N - v_N)(\theta \wedge H_N^\delta, X_{\theta \wedge H_N^\delta}) \right],
\]
where \(\mathbb{E}^{\mathbb{P}^{N,*}}\) is defined by (3.5). Indeed, by continuity of \(\phi_N - v_N\), we easily see that the Markov process \(s \mapsto (\phi_N - v_N)(s, X_s)\) is bounded and uniformly continuous on \(\{(t, \omega) : t \leq H_N^\delta(\omega)\}\), and that \(\mathcal{P}_L^N(t, x^N)\) satisfies [27, Assumption 3.4]. Also, note that, since \(\{v_N\}_{N \geq 1}\) is locally bounded, uniformly in \(N\), and \(\mu_N(x^N) \xrightarrow{W_2} \mu\), we may assume w.l.o.g. that, after passing to an appropriate subsequence and for \(\delta\) small enough, \((\phi_N - v_N)(\theta \wedge H_N^\delta, X_{\theta \wedge H_N^\delta})\) is uniformly bounded for all \(\theta \in \mathcal{T}_{tN, T}^N\).

**Step 2:** We now justify that
\[
\limsup_{N \to \infty} \mathbb{P}^{N,*}(\theta_N^\delta < H_N^\delta) > 0.
\]
Indeed, assume to the contrary that \(\limsup_{N \to \infty} \mathbb{P}^{N,*}(\theta_N^\delta < H_N^\delta) = \lim_{N \to \infty} \mathbb{P}^{N,*}(\theta_N^\delta < H_N^\delta) = 0\).
We have:

\[
(\varphi - \underline{\nu})(t, \mu) = \lim_{N \to \infty} (\varphi - \nu^N)(t^N, \mu^N(x^N)) \\
\leq \liminf_{N \to \infty} \mathbb{E}^\mathbb{P}_{\nu^N, x} \left[ (\varphi - \nu^N)(\theta^N_\delta \land H^N_\delta, \mu^N(X_{\theta^N_\delta \land H^N_\delta})) \right] \\
\leq \limsup_{N \to \infty} \mathbb{E}^\mathbb{P}_{\nu^N, x} \left[ (\varphi - \nu^N)(\theta^N_\delta \land H^N_\delta, \mu^N(X_{\theta^N_\delta \land H^N_\delta})) \right] \\
= \limsup_{N \to \infty} \mathbb{E}^\mathbb{P}_{\nu^N, x} \left[ \left\{ (\varphi - \nu^N)(H^N_\delta, \mu^N(X_{H^N_\delta}))(1 - 1_{\theta^N_\delta < H^N_\delta}) \right. \right. \\
\quad \left. \quad + (\varphi - \nu^N)(\theta^N_\delta, \mu^N(X_{\theta^N_\delta}))1_{\theta^N_\delta < H^N_\delta} \left\} \right. \right. \\
= \limsup_{N \to \infty} \mathbb{E}^\mathbb{P}_{\nu^N, x} \left[ (\varphi - \nu^N)(H^N_\delta, \mu^N(X_{H^N_\delta})) \right],
\]

where we used the fact that \{(\varphi - \nu^N)(\theta^N_\delta \land H^N_\delta, \mu^N(X_{\theta^N_\delta \land H^N_\delta}))\}_{N \geq 1} is uniformly bounded and \(\limsup_{N \to \infty} \nu^N, x(\theta^N_\delta < H^N_\delta) = \lim_{N \to \infty} \mathbb{P}_{\nu^N, x}(\theta^N_\delta < H^N_\delta) = 0\).

Since \((t^N, \mu^N(x^N)) \to (t, \mu)\), by Proposition 7.1 and compactness of \([t, T]\), there exists a subsequence \(\nu^N \in \mathbb{P}_{\nu^N, x} \circ (H^N_\delta, \mu^N(X))^{-1}\) that converges weakly to some \(\nu \in \mathbb{P}_2([t, T] \times \mathcal{P}_2(\Omega))\) supported on \([t, t + \delta] \times \mathcal{P}_L(t, \mu)\). Thus, denoting by \((\tau, m)\) the canonical mapping on \([0, T] \times \mathcal{P}_2(\Omega)\), we have by upper semicontinuity of \(\varphi - \nu^N\) and continuity of \((\tau, m) \mapsto m_{X, r}\),

\[
\limsup_{N \to \infty} \mathbb{E}^\mathbb{P}_{\nu^N, x} \left[ (\varphi - \nu^N)(H^N_\delta, \mu^N(X_{H^N_\delta})) \right] = \limsup_{N \to \infty} \mathbb{E}^\nu \left[ (\varphi - \nu)(\tau, m_{X, r}) \right] \\
\leq \mathbb{E}^\nu \left[ (\varphi - \nu)(\tau, m_{X, r}) \right] \leq (\varphi - \underline{\nu})(\bar{\tau}, m_{X_{\bar{\tau}^N}(ar{\omega})}), \quad (6.4)
\]

for some \(\bar{\omega} \in [0, T] \times \mathcal{P}_2(\Omega)\). We also observe that \(\tau > t\), \(\nu\text{-a.s.}\). Indeed, given that, by definition of \(H^N_\delta\), we have

\[
\mathcal{W}_2(m_{X, r}, \mu^N(x^N)) \lor (\tau - t^N) = 2\delta \text{ or } \mathcal{W}_2(m_{X, r}, \mu^N(x^N)) + (\tau - t^N) \geq 2\delta, \quad \nu^N\text{-a.s.},
\]

for all \(N \geq 1\), and therefore, for \(N\) sufficiently large,

\[
\mathcal{W}_2(m_{X, r}, \mu) \lor (\tau - t) \leq 3\delta < \delta_0 \text{ or } \mathcal{W}_2(m_{X, r}, \mu) + (\tau - t) \geq \delta, \quad \nu\text{-a.s.}
\]

These inequalities define a fixed closed support for \(\nu^N\), which is inherited by \(\nu\) by weak convergence and continuity of \(\mathcal{W}_2\) and \(\omega \mapsto m_{X_{\tau^N}(\omega)}(\omega)\). Thus, \(\bar{\omega}\) in (6.4) may be chosen s.t.

\[
\mathcal{W}_2(m_{X_{\tau^N}(\omega)}, \mu) \lor (\tau(\omega) - t) \leq 3\delta < \delta_0 \text{ or } \mathcal{W}_2(m_{X_{\tau^N}(\omega)}, \mu) + (\tau(\omega) - t) \geq \delta.
\]

The first inequality shows that \((\tau(\omega), m_{X_{\tau^N}(\omega)}(\omega)) \in \mathcal{N}_{\delta_0}(t, \mu)\), and the second one that \((\tau(\omega), m_{X_{\tau^N}(\omega)}(\omega)) \neq (t, m)\). Therefore, (6.4) contradicts the fact that \((t, \mu)\) is a strict maximum on \(\mathcal{N}_{\delta}(t, \mu)\). Thus, (6.3) holds true, and we may find a subsequence \{\(\omega^{\delta, N}\)\}_{N \geq 1} s.t.

\[
i^N_\delta := \theta^N_\delta(\omega^{\delta, N}) < H^N_\delta(\omega^{\delta, N}) \text{ for all } N \geq 1.
\]
Step 3: We now prove that $\phi^N$ is a test function for $v^N$ in some well chosen point. Introduce $x^{\delta,N} := X_{\theta^N}^N(\omega^{\delta,N})(\omega^{\delta,N})$ and $Y^N$, the nonlinear Snell envelop of $s \mapsto (\phi^N - v^N)(s, X_s)$, i.e.,

$$Y_s^N(\omega) := \sup_{\theta \in T_{t,N}^T} E^N_{s,\omega}[(\phi^N - v^N)(\theta \wedge H^N_{\delta}, X_{\theta \wedge H^N_{\delta}})],$$

which satisfies $Y_s \geq E^N_{s,Y_s}[Y_{\theta \wedge H^N_{\delta}}]$ for all $\theta \in T_{s,t}$. Then we have, for all $\theta \in T_{t,N}^T$, $\theta^{N,N}, x^{\delta,N} := X_{\theta^N}(\omega^{\delta,N})$ and $Y^N_{\theta \wedge H^N_{\delta}}$:

$$(\phi^N - v^N)(t^{N}_N, X^{\delta,N}) = Y^N_{t^{N}_N, \omega^{\delta,N}}(\omega^{\delta,N}) \geq E^N_{t^{N}_N, \omega^{\delta,N}}[(\phi^N - v^N)(\theta \wedge H^N_{\delta}, X_{\theta \wedge H^N_{\delta}})]$$

and therefore, as we are in the Markovian case,

$$(\phi^N - v^N)(t^{N}_N, x^{\delta,N}) = \max_{\theta \in T_{t^{N}_N}^T} E^N_{t^{N}_N, x^{\delta,N}}[(\phi^N - v^N)(\theta \wedge H^N_{\delta}, X_{\theta \wedge H^N_{\delta}})].$$

Observe that $\phi^N \in C^{1,2}_{b}(\Lambda^{N,N}_{t^{N}_N})$. Indeed, since $\varphi = \psi^{v,a,f}$ is a semijet, we have

$$\partial_t \phi^N(s,x) = \partial_t \varphi(s,\mu^N(x)) = a,$$

$$\partial_x \phi^N(s,x) = \frac{1}{N} f'(x_i),$$

$$\partial_{x_i}^2 \phi^N(s,x) = \frac{1}{N} f''(x_i),$$

for all $i \in [N]$. As $H^N_{\delta} > t^{N}_N$ on $\{X \in \Lambda^{N,N}_{t^{N}_N} \}$, we have $\phi^N \in \Lambda^{N,N}_{t^{N}_N, x^{\delta,N}}$. Finally, the fact that $(t^{N}_N, \mu(x^{\delta,N})) \rightarrow (t,N, x^{N})$ simply comes from the definition of $H^N_{\delta}$. 

Remark 6.2 If we allow our test functions on Wasserstein space to have derivatives with quadratic growth in $x$ (similarly to [19]), we may not use our semijets, and therefore the computations in (6.6) must be done differently. In the case of state-dependent functions, this can be done by using general formulas for smooth functions on the space of measures, see e.g. Carmona & Delarue [13, Vol. 1, Propositions 5.35 & 5.91].

6.2 The path-dependent setting

Proof of Theorem 5.6 (Given Theorem 5.7). Identical to the proof of Theorem 3.7, using Theorem 5.7 instead of Theorem 3.8.

Proof of Theorem 5.7 This follows the same arguments as the proof of Theorem 3.8. We use the notations of Section 5. Let $(t^N, \omega^N)$ be a sequence s.t. $(t^N, \mu^N(\omega^N))$ → $(t, \mu)$ and...
$v^N(t^N, \mu^N(\omega^N)) \rightarrow v(t, \mu)$ as $N \rightarrow \infty$. We also introduce, for all $N \geq 1$, the functions $\phi^N(s, \omega) := \varphi(s, \mu^N(\omega))$ for all $(s, \omega) \in \Lambda^N_0$. Fix $\delta \in (0, \delta_0)$, and define the stopping time

$$
H^N_\delta := \inf \{ s \geq t^N : W_2(\mu^N(X_{\Lambda^N_\delta}), \mu^N(\omega_{\Lambda^N_\delta})) \geq \delta \} \wedge (t^N + \delta).
$$

Observe that, for all $s \geq t^N, \omega \in \Omega^N$ s.t. $s \leq H^N_\delta(\omega)$, we have

$$
|v^N(s, \omega)| \leq v^N(t^N, \omega^N) + \rho_N(|s - t^N| + W_2(\mu^N(\omega_{\Lambda^N_\delta}), \mu^N(\omega_{\Lambda^N_\delta}))) \leq v^N(t^N, \omega^N) + \rho_N(2\delta),
$$

where $\rho_N$ is continuity modulus of $v^N$. Furthermore, $\phi^N$ is Lipschitz-continuous as it has bounded derivatives. Thus $\phi^N - v^N$ is bounded and uniformly continuous on $\{(s, \omega) : s \leq H^N_\delta(\omega)\}$, and by [27] again, there exists $(\theta^N_\delta, \nu^N_\delta) \in \mathcal{T}^N_{t^N, T} \times \mathcal{P}^N(t, \omega^N)$ s.t.

$$
\mathbb{E}^{\nu^N_\delta, \omega^N}[\phi^N - v^N(\theta^N_\delta \wedge H^N_\delta, X_{\Lambda^N_\delta \wedge H^N_\delta})] = \sup_{\theta \in \mathcal{T}^N_{t^N, T}} \tau^N_{t^N, \omega^N}[\phi^N - v^N(\theta \wedge H^N_\delta, X_{\Lambda^N_\delta \wedge H^N_\delta})],
$$

where $\tau^N_{t^N, \omega^N}$ is defined by (5.6). Similarly to the Markovian setting, we may find a subsequence $\{\omega^{\delta, N}\}_{N \geq 1}$ s.t. $t^N_\delta := \theta^N_\delta(\omega^{\delta, N}) < H^N_\delta(\omega^{\delta, N})$ for all $N \geq 1$, and satisfying

$$
(\phi^N - v^N)(t^N_\delta, \omega^{\delta, N}) = \max_{\theta \in \mathcal{T}^N_{t^N_\delta, T}} \tau^N_{t^N_\delta, \omega^{\delta, N}}[\phi^N - v^N(\theta \wedge H^N_\delta, X_{\Lambda^N_\delta \wedge H^N_\delta})].
$$

Observe that $\phi^N \in C^{1,2}_b(\Lambda^N_\delta)$. Indeed, since $\varphi = \psi^v, a^f$ is a semijet, we have

$$
\partial_t \phi^N(s, \omega) = \partial_t \varphi(s, \mu^N(\omega)) = a,
\partial_{\omega^i} \phi^N(s, \mathbf{x}) = \frac{1}{N} \partial_{\omega^i} f(\omega^i),
\partial_{\omega^i \omega^j} \phi^N(s, \omega) = \frac{1}{N} \partial_{\omega^i \omega^j} f(\omega^i),
$$

for all $i \in [N]$. As $H^N_\delta > t^N_\delta$ on $\{X_{t^N_\delta, \Lambda^N_\delta} = \omega^{\delta, N}_{t^N_\delta, \Lambda^N_\delta}\}$, we have $\phi^N \in \mathcal{A}^N v^N(t^N_\delta, \omega^{\delta, N})$ and the supersolution property provides

$$
-\partial_t \phi^N(q^N_\delta) - F^N(q^N_\delta, v^N(q^N_\delta), \partial_{\omega^i} \phi^N(q^N_\delta), \partial_{\omega^i \omega^j} \phi^N(q^N_\delta)) \geq 0.
$$

where $q^N_\delta := (t^N_\delta, \omega^{\delta, N})$. We conclude similarly to (i).

7 A precompacness result

In this section, we state and prove our propagation of chaos-like result for continuous semimartingale with bounded characteristics, which plays a crucial role in the contradiction argument used to prove Lemma 6.1.

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Our objective is to prove that the empirical measure associated with a $N$-dimensional continuous semimartingales with characteristics bounded by some constant $L$ converges in law (up to a subsequence) to an element supported on $\mathcal{P}_L$, i.e., a measure on $\Omega$ under which the canonical process is also almost surely a continuous semimartingale with characteristics bounded by the same constant $L$.

**Proposition 7.1** Let $\{\omega^N\}_{N \geq 1} \in \prod_{N \geq 1} \Omega^N$ and $\mu \in \mathcal{P}_2(\Omega)$ s.t. $\mu^N(\omega^N_{X}) \xrightarrow{W_2} \mu_{|[0,1]}$, and $\{\mathbb{P}^N \in \mathcal{P}^N_L(t, \omega^N)\}_{N \geq 1}$. Then, the sequence $\{\mathbb{P}^N \circ (\mu^N(X))^{-1}\}_{N \geq 1}$ is tight, and all its accumulation points are supported on $\mathcal{P}_L(t, \mu)$.

**Definition 7.2** (i) Denote $Y := (A, M)$ the canonical process on $\Omega^2 := \Omega \times \Omega$. Let $\tilde{\mathcal{P}}_L$ be the set of probability measures $\mathbb{P}$ on $\Omega^2$ such that:

- $A$ is absolutely continuous w.r.t. to the Lebesgue measure on $[0, T]$, with $|\frac{dA}{dt}| \leq L$, $\mathbb{P}$-a.s.,
- $M$ is a $\mathbb{P}$-martingale on $[0, T]$, with $\sqrt{\frac{d(M)_{s}}{ds}} \leq L$, $\mathbb{P}$-a.s.

(ii) Denote $Y := (A, M) = \{(A^k, M^k)\}_{k \in [N]}$ the canonical process on $\Omega^{N,2} := \Omega^N \times \Omega^N$. Let $\tilde{\mathcal{P}}^N_L$ be the set of probability measures $\mathbb{P}$ on $\Omega^{N,2}$ s.t.:

- $A$ is absolutely continuous w.r.t. to the Lebesgue measure on $[0, T]$, with $|\frac{dA^k}{dt}| \leq L$, for all $k \in [N]$, $\mathbb{P}$-a.s.,
- $M$ is a $\mathbb{P}$-martingale on $[0, T]$, with $\sqrt{\frac{d(M^k)_{s}}{ds}} \leq L$ and $\langle M^k, M^l \rangle = 0$, for all $k \neq l \in [N]$, $\mathbb{P}$-a.s.

Since a semimartingale is defined as the sum of a finite variation process $A$ and a local martingale $M$, it is more convenient to show first the tightness of the sequence of empirical measures associated with the pair $(A, M)$ rather than handling the sum $A + M$ directly, as it is simpler to show that their properties “propagate” independently.

**Lemma 7.3** For all $\{\mathbb{P}^N \in \tilde{\mathcal{P}}^N_L\}_{N \geq 1}$, the sequence $\{\mathbb{P}^N \circ (\mu^N(Y))^{-1}\}_{N \geq 1}$ is tight, and all its accumulation points are supported on $\tilde{\mathcal{P}}_L$.

**Proof** Step 1: We first prove the existence of a converging subsequence. For all $N \geq 1$, denote $\nu^N := \mathbb{P}^N \circ (\mu^N(Y))^{-1} \in \mathcal{P}(\mathcal{P}_2(\Omega^2))$. By Lacker [36, Corollary B.1], we have to prove that

- (i) $\{\nu^N\}_{N \geq 1}$ is uniformly integrable, i.e., $\lim_{R \to \infty} \sup_{N \geq 1} \mathbb{E}^{\nu^N}[W_2^2(\lambda, \delta_0)\mathbb{1}_{W_2(\lambda, \delta_0) \geq R}] = 0$, where $\lambda$ is the identity map on $\mathcal{P}_2(\Omega^2)$.
- (ii) the sequence of mean measures $\{\mathbb{E}^{\nu^N}[\mu^N(Y)]\}_{N \geq 1}$ is tight, where, for all $\mathbb{P} \in \mathcal{P}_2(\Omega^{N,2})$ and $\tilde{\mu} : \Omega^{N,2} \to \mathcal{P}_2(\Omega^2)$, the mean measure $\mathbb{E}^{\mathbb{P}}[\tilde{\mu}] \in \mathcal{P}(\Omega^2)$ is defined by $\langle \mathbb{E}^{\mathbb{P}}[\tilde{\mu}], \varphi \rangle := \mathbb{E}^{\mathbb{P}}[(\tilde{\mu}, \varphi)]$ for all $\varphi \in C^0_0(\Omega^2)$.
Let $R > 0$. We have

$$
\mathbb{E}^{\nu_N} \left[ W_2(\lambda, \delta_0) 1_{W_2(m, \delta_0) \geq R} \right] = \mathbb{E}^{\nu_N} \left[ W_2(\mu^N(Y), \delta_0) 1_{W_2(\mu^N(Y), \delta_0) \geq R} \right]
$$

$$
\leq \frac{1}{R} \mathbb{E}^{\nu_N} \left[ W_2^2(\mu^N(Y), \delta_0) \right] \leq \frac{2}{R} \mathbb{E}^{\nu_N} \left[ \frac{1}{N} \sum_{i=1}^{N} |A_i|^2 + |M_i|^2 \right].
$$

For each $i \in [N]$, we have $\mathbb{E}^{\nu_N} \left[ |A_i|^2 \right] \leq (LT)^2$ and $\mathbb{E}^{\nu_N} \left[ |M_i|^2 \right] \leq 4L^2T$, the latter by Doob’s inequality. Thus, there exists a constant $C_{T,L}$ independent from $N$ and $R$ s.t.

$$
\mathbb{E}^{\nu_N} \left[ W_2(\lambda, \delta_0) 1_{W_2(\lambda, \delta_0) \geq R} \right] \leq \frac{C_{T,L}}{R} \quad \text{for all } N \geq 1 \text{ and } R \geq 0,
$$

and therefore $\lim_{R \to \infty} \sup_{N \geq 1} \mathbb{E}^{\nu_N} \left[ W_2(\lambda, \delta_0) 1_{W_2(\lambda, \delta_0) \geq R} \right] = 0$ and (i) is proved.

To show that $\{\mathbb{E}^{\nu_N} \left[ \mu^N(Y) \right] \}_{N \geq 1}$ is tight, we prove Aldous’ criterion (see Billingsley [7, Theorem 16.10]), i.e.,

$$
\sup_{N \geq 1} \sup_{\tau \in \mathcal{T}_{0,T}} \left\langle \mathbb{E}^{\nu_N} \left[ \mu^N(Y) \right], |A_{(r+\delta) \land T} - A_\tau|^2 + |M_{(r+\delta) \land T} - M_\tau|^2 \right\rangle \delta \to 0, \quad (7.1)
$$

where $\mathcal{T}_{0,T}$ denotes the set of $[0,T]$-valued $\mathbb{F}$-stopping times. Yet, for fixed $N, \tau$ and $\delta$,

$$
\left\langle \mathbb{E}^{\nu_N} \left[ \mu^N(Y) \right], |M_{(r+\delta) \land T} - M_\tau|^2 \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}^{\nu_N} \left[ |M_i^{(r+\delta) \land T} - M_i^\tau|^2 \right] \leq L^2 \delta
$$

by Itô’s isometry. We obtain a similar estimate for $A$, and this implies (7.1) and consequently (ii), and thus $\{\nu^N\}_{N \geq 1}$ admits a subsequence converging to some $\nu \in \mathcal{P}(\mathcal{P}_2(\Omega^2))$.

**Step 2:** We show that all the accumulations points are supported on $\tilde{\mathcal{P}}_L$, i.e. that $\nu$ is supported on $\tilde{\mathcal{P}}_L$. Observe that, by definition of $\mathcal{P}^N$, we have

$$
|A^k_s - A^k_r| \leq L|s - r|, \quad \mathbb{P}^N\text{-a.s., for all } k \in [N] \text{ and } s, r \in [0,T],
$$

and thus

$$
|A_s - A_r| \leq L|s - r|, \quad \mu^N(Y)\text{-a.s., } \mathbb{P}^N\text{-a.s., for all } k \in [N] \text{ and } s, r \in [0,T],
$$

and finally

$$
\nu^N \left[ \lambda \left( |A_s - A_r| \leq L|s - r| \right) \right] = 1, \quad \text{for all } N \geq 1 \text{ and } s, r \in [0,T].
$$

Since $\{|A_s - A_r| \leq L|s - r|\}$ is closed in $\Omega^2$, $\left\{ \lambda \left( |A_s - A_r| \leq L|s - r| \right) \right\}$ is closed in $\mathcal{P}_2(\Omega^2)$, and thus the weak convergence of $\nu^N$ to $\nu$ implies

$$
1 = \lim_{N \to \infty} \nu^N \left[ \lambda \left( |A_s - A_r| \leq L|s - r| \right) \right] \leq \nu \left[ \lambda \left( |A_s - A_r| \leq L|s - r| \right) \right] = 1 \leq 1,
$$

and therefore $\nu$ is supported on $\tilde{\mathcal{P}}_L$. Therefore, we have shown that $\nu$ is the weak limit of $\mathbb{E}^{\nu_N}$.
that is, \( \nu \left( |A_s - A_r| \leq L|s - r| \right) = 1 \). Since \( s \) and \( r \) are arbitrary, this implies that \( A \) is absolutely continuous w.r.t. the Lebesgue measure on \([0, T]\) with \( \left| \frac{dA_s}{ds} \right| \leq L \), \( \lambda \)-a.s., \( \nu \)-a.s.

We now prove that \( M \) is a \( \lambda \)-martingale on \([0, T]\), \( \nu \)-a.s. Fix \( r \leq s \) in \([0, T]\), and \( h_r := h(Y_r) \), where \( h \in C_b^0(\Omega^2) \). We compute:

\[
\mathbb{E}^\nu \left[ \langle \lambda, h_r(M_s - M_r) \rangle^2 \right] = \mathbb{E}^\nu \left[ \left( \sum_{i=1}^N h(Y_{r_i}) (M^i_s - M^i_r) \right)^2 \right] \\
\leq \frac{1}{N^2} \sum_{i=1}^N |h|^2 \mathbb{E}^\nu \left[ |M^i_s - M^i_r|^2 \right] \leq \frac{|h|^2 L^2 T}{N} \xrightarrow{N \to \infty} 0,
\]

where we used the fact that \( \langle M^k, M^l \rangle 1_{k \neq l} = 0 \), and the \( \sigma(Y_r) \)-measurability of \( h_r \) to derive the first inequality. Thus, as \( \nu^N \) converges weakly to \( \nu \),

\[
0 \leq \mathbb{E}^\nu \left[ \langle \lambda, h_r(M_s - M_r) \rangle^2 \right] \leq \liminf_{N \to \infty} \mathbb{E}^\nu \left[ \langle \lambda, h_r(M_s - M_r) \rangle^2 \right] = 0,
\]

hence \( \mathbb{E}^\nu \left[ \langle \lambda, h_r(M_s - M_r) \rangle^2 \right] = 0 \), which implies that

\[
\langle \lambda, h_r(M_s - M_r) \rangle = 0, \quad \nu \text{-a.s.,}
\]

which by the arbitrariness of \( s, r \) and \( h \) means that \( M \) is a \( \lambda \)-martingale, \( \nu \)-a.s. We prove similarly to \( A \) that \( \sqrt{\frac{d(M_s)}{ds}} \leq L \), \( \lambda \)-a.s., \( \nu \)-a.s.

We eventually prove Proposition 7.1 by deriving the tightness of the processes \( \{\mu^N(A + M)\}_{N \geq 1} \) from the one of the processes \( \{\mu^N(A, M)\}_{N \geq 1} \).

**Proof of Proposition 7.1** Introduce

\[
A^N_s := X_t + 1_{s \geq t} \int_t^s b^N_r \, dr, \quad M^N_s := 1_{s \geq t} \int_t^s \sigma^N_r \, dW^N_r,
\]

where \( b^N, \sigma^N \) and \( W^N \) are as in (3.4). Then, we clearly have

\[
\hat{P}_L^N := \mathbb{P}^N(A^N, M^N) \in \hat{P}_L^N.
\]

Therefore, by Lemma 7.3, \( \nu^N := \hat{P}_L^N \circ (\mu^N(Y))^{-1} \) converges weakly to some \( \nu \) supported on \( \hat{P}_L(t, \mu) \). Define \( \hat{\mu}^N := \mathbb{P}^N \circ (\mu^N(X))^{-1} \) and fix \( \varphi \in C_b^0(\mathcal{P}_2(\Omega)) \). We have:

\[
\langle \hat{\mu}^N, \varphi \rangle = \mathbb{E}^\nu \left[ \varphi(\mu^N(X)) \right] = \mathbb{E}^\nu \left[ \varphi(\mu^N(Y) \circ (A + M)^{-1}) \right] \\
= \mathbb{E}^\nu \left[ \varphi(\lambda \circ (A + M)^{-1}) \right] \xrightarrow{N \to \infty} \mathbb{E}^\nu \left[ \varphi(\lambda \circ (A + M)^{-1}) \right]
\]

by weak convergence of \( \nu^N \) to \( \nu \), since \( \lambda \mapsto \varphi(\lambda \circ (A + M)^{-1}) \in C_b^0(\mathcal{P}_2(\Omega^2)) \). Thus we have

\[
\langle \hat{\mu}^N, \varphi \rangle \xrightarrow{N \to \infty} \langle \hat{\mu}, \varphi \rangle,
\]

where \( \hat{\mu} := \nu \circ (\lambda \circ (A + M)^{-1})^{-1} \). Therefore, by arbitrariness of \( \varphi \), \( \mu^N \) converges weakly to \( \hat{\mu} \), which is clearly supported on \( \mathcal{P}_L(t, \mu) \) as \( \nu \) is supported on \( \hat{P}_L \) and \( \mu^N(\omega^N) \xrightarrow{W_2} \mu \). ■
A Technical lemma

Lemma A.1 Fix $x \in \mathbb{R}^{d \times N}$, and introduce for all $h > 0$:

$$D_h := \{y \in \mathbb{R}^{d \times N} : W_2(\mu^N(y), \mu^N(x)) \leq h\}.$$  

Then, for $h$ sufficiently small, $D_h$ is a disjoint union of convex sets.

Proof By Birkhoff’s theorem (see e.g. Villani [50, p. 5]), we have for all $y$:

$$W_2(\mu^N(y), \mu^N(x)) = \min_{\pi \in \mathcal{S}_N} ||y - \pi(x)||_2,$$

where $\mathcal{S}_N$ is the symmetric group of $[N]$, and where we denote by $\pi(x) := (x_{\pi(1)}, \ldots, x_{\pi(N)})$. Then we have:

$$D_h = \{y \in \mathbb{R}^{d \times N} : \exists \pi \in \mathcal{S}_N, \text{ s.t. } W_2(\mu^N(y), \mu^N(\pi(x))) \leq h\},$$

and $D_h$ writes therefore as a finite union of balls with centers in $\{\pi(x)\}_{\pi \in \mathcal{S}_N}$. Thus, for $h$ small enough, $D_h$ is a disjoint union of balls, which are convex sets. 

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