On the Domain of the Four-Dimensional Sequential Band Matrix in Some Double Sequence Spaces

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Abstract: Let E represent any of the spaces \( M_\vartheta, C_\vartheta \) (\( \vartheta = \{b, bp, r\} \)), and \( L_q \) (\( 0 < q < \infty \)) of bounded, \( \vartheta \)-convergent, and \( q \)-absolutely summable double sequences, respectively, and \( \tilde{E} \) be the domain of the four-dimensional (4D) infinite sequential band matrix \( B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) \) in the double sequence space \( E \), where \( \tilde{r} = (r_m)_{m=0}^\infty \), \( \tilde{s} = (s_m)_{m=0}^\infty \), \( \tilde{t} = (t_n)_{n=0}^\infty \), and \( \tilde{u} = (u_n)_{n=0}^\infty \) are given sequences of real numbers in the set \( \mathbb{C} \). In this paper, we investigate the double sequence spaces \( \tilde{E} \). First, we determine some topological properties and prove several inclusion relations under some strict conditions. Then, we examine \( \alpha \), \( \beta(\vartheta) \), and \( \gamma \)-duals of \( \tilde{E} \). Finally, we characterize some new classes of 4D matrix mappings related to our new double sequence spaces and conclude the paper with some significant consequences.

Keywords: matrix domain; sequentially defined 4D band matrix; double sequence spaces; dual spaces; matrix transformations

1. Introduction

The set of all complex valued double sequences is expressed as \( \Omega := \{ x = (x_{mn}) : x_{mn} \in \mathbb{C}, \forall m, n \in \mathbb{N} \} \), which is a vector space that possesses scalar multiplication and coordinate-wise addition, where \( \mathbb{C} \) is the complex field, and \( \mathbb{N} = \{0, 1, 2, ...\} \). Any vector subspace of \( \Omega \) is called a double sequence space. \( M_\vartheta, C_p, C_b_p, C_r, \) and \( L_q \) denote the classical spaces of all double sequences that are bounded, convergent in the Pringsheim sense, convergent in the Pringsheim sense and bounded, regular convergent, and \( q \)-absolutely summable, respectively, where \( 0 < q < \infty \). It is well known that the space \( L_q \) becomes the space \( L_1 \) in the case \( q = 1 \). Moreover, by \( BS, CS_\vartheta, \) where \( \vartheta = \{p, bp, r\} \), we denote all bounded and \( \vartheta \)-convergent series, respectively.

Let \( E \) be any double sequence space. Then, the following dual spaces of \( E \) are defined:

\[
E^\beta(\vartheta) := \{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl} \} \in CS_\vartheta, \text{ for every } x = (x_{kl}) \in E \},
\]

\[
E^\alpha := \{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl} \} \in L_\alpha, \text{ for every } x = (x_{kl}) \in E \},
\]

\[
E^\gamma := \{ a = (a_{kl}) \in \Omega : \{a_{kl}x_{kl} \} \in BS, \text{ for every } x = (x_{kl}) \in E \}.
\]

If \( E_1 \) and \( E_2 \) are arbitrary sets of double sequences with \( E_2 \subset E_1 \), then the inclusions \( E_1^\alpha \subset E_2^\alpha \), \( E_1^\gamma \subset E_2^\gamma \), and \( E_1^{\beta(\vartheta)} \subset E_2^{\beta(\vartheta)} \) hold. However, the inclusion \( E_1^\gamma \subset E_1^{\beta(\vartheta)} \) does not hold, since there exists...
a double sequence such that it converges in the Pringsheim sense, but is unbounded. For instance, the sequence \( x = (x_{mn}) \) is defined as:

\[
x_{mn} = \begin{cases} 
n & m = 0, n \in \mathbb{N}; 
0 & m \geq 1, n \in \mathbb{N}.
\end{cases}
\]

which is obviously in \( x \in C_p \setminus M_a \).

A matrix transformation between double sequence spaces \( E_1 \) and \( E_2 \) is given by any 4D infinite matrix \( A = (a_{mnkl}) \), where \( m, n, k, l \in \mathbb{N} \). This means that \( A \) transforms any double sequence \( x = (x_{kl}) \in E_1 \) to \( Ax = \{ (Ax)_{mn} \}_{m,n \in \mathbb{N}} \in E_2 \), where:

\[
(Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \quad \text{for each } m, n \in \mathbb{N}.
\] (2)

The set \( E_A^{(\vartheta)} \), which is the \( \vartheta \)-summability domain of a 4D infinite matrix \( A \) in a double sequence space \( E \), is defined by:

\[
E_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left( \vartheta - \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } E \right\},
\] (3)

which is a sequence space. The above transformation (2) indicates that \( A \) maps \( E_1 \) into \( E_2 \) provided that \( E_1 \subset (E_2)_A^{(\vartheta)} \), and the set of all 4D matrices, transforming the space \( E_1 \) into the space \( E_2 \), is denoted by \( (E_1 : E_2) \). Thus, \( A = (a_{mnkl}) \in (E_1 : E_2) \) if and only if the double series \( \left( \sum_{k,l} a_{mnkl}x_{kl} \right)_{m,n \in \mathbb{N}} \) \( \vartheta \)-converges for each \( m, n \in \mathbb{N} \), i.e., \( A_{mn} \in E_1^{(\vartheta)} \) for all \( m, n \in \mathbb{N} \), and we have \( Ax \in E_2 \) for all \( x \in E_1 \); where \( A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}} \) for all \( m, n \in \mathbb{N} \).

It is a common method to create a new sequence space from a given sequence space \( E \) as the matrix domains \( E_A \) of matrices \( A \) in \( E \). Following Adams [1], a 4D infinite matrix \( A = (a_{mnkl}) \) is said to be a triangular matrix if \( a_{mnkl} = 0 \) for \( k > m \) or \( l > n \) or both. From [1], we can say that an infinite matrix \( A = (a_{mnkl}) \) refers to a triangle if \( a_{mnmn} \neq 0 \) for all \( m, n \in \mathbb{N} \). It was shown in [2] that every triangle has unique and equal left and right inverses. One can easily observe that, if \( A \) is a triangle, then \( E_A^{(\vartheta)} \) and \( E \) are linearly isomorphic.

The adventure of double sequences and their topological properties started by distinguished works of several mathematicians, and those core ideas helped to describe some new double sequence spaces by the summability theory. Zeltser [3] studied the theoretical and topological properties of double sequence spaces by applying the summability theory. Then, other mathematicians (see [4–10]) studied the topological properties, inclusion relations, dual spaces, and matrix transformations of some new spaces of double sequences. Moreover, many characterizations of 4D matrix classes have been achieved by several mathematicians (see [11–14]). We list the domains \( E_A \) of some 4D triangle matrices \( A \) in a certain double sequence space \( E \) in the following Table 1.
The results were much more general than the consequences derived by the
infinite matrix classes. The domains in some double sequence spaces. After carefully searching
in the existing literature, one can see clearly that another generalization of the 4D infinite matrix
B(r,s,t,u) is possible by defining this matrix sequentially. In this work, we introduce some new double
sequence spaces \( \overline{B}(\mathcal{A})_u, \overline{B}(C_p), \overline{B}(C_{bp}), \overline{B}(C_r), \) and \( \overline{B}(L_q)_q, 0 < q < \infty \) whose 4D sequential band
matrix \( \overline{B} \)-transforms are in the spaces \( \mathcal{A}, C_p, C_{bp}, C_r, \) and \( L_q, \) respectively. Moreover, the goal of the following sections of this paper is to provide extensive proofs of some topological properties, inclusion
relations, dual spaces of these new double sequence spaces, and the characterization of some 4D
infinite matrix classes.

Let \( \overline{r} = (r_m)_{m=0}^{\infty}, \overline{s} = (s_m)_{m=0}^{\infty}, \overline{t} = (t_n)_{n=0}^{\infty}, \overline{u} = (u_n)_{n=0}^{\infty} \) be given sequences of real numbers
in the set \( c \setminus \mathbb{Q} . \) We define the 4D sequential band matrix \( B(\overline{r}, \overline{s}, \overline{t}, \overline{u}) = \{ b_{mnkl}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) \} \) by:

\[
b_{mnkl}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) := \begin{cases} 
  r_m t_n, & (k,l) = (m,n), \\
  r_m u_n, & (k,l) = (m,n-1), \\
  s_m t_n, & (k,l) = (m-1,n), \\
  s_m u_n, & (k,l) = (m-1,n-1), \\
  0, & \text{elsewhere,}
\end{cases}
\]

for all \( m,n,k,l \in \mathbb{N} . \) Thus, the matrix \( B(\overline{r}, \overline{s}, \overline{t}, \overline{u}) \)-transforms a double sequence \( x = (x_{mn}) \) into the
double sequence \( y = (y_{mn}) \) as:

\[
y_{mn} := \{ B(\overline{r}, \overline{s}, \overline{t}, \overline{u}) x \}_{mn} = \sum_{k,l} b_{mnkl}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) x_{kl} \tag{4}
\]

\[
y_{mn} = s_{m-1} u_n x_{m-1,n} + s_{m-1} t_n x_{m-1,n} + r_m t_{n-1} x_{mn} + r_m u_{n-1} x_{m,n-1} + r_m t_{n-1} x_{mn}
\]

for all \( m,n \in \mathbb{N} . \) By means of the above equation in (4), the inverse \( B^{-1}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) = F(\overline{r}, \overline{s}, \overline{t}, \overline{u}) = \{ f_{mnkl}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) \} \) of \( B(\overline{r}, \overline{s}, \overline{t}, \overline{u}) \) is defined by:

\[
f_{mnkl}(\overline{r}, \overline{s}, \overline{t}, \overline{u}) := \begin{cases} 
  \frac{1}{r_m} \prod_{i=k}^{m-1} \prod_{j=l}^{n-1} \left( \frac{-s_i}{t_i} \right) \left( \frac{-u_j}{t_j} \right), & 0 \leq k \leq m, 0 \leq l \leq n, \\
  0, & \text{elsewhere,}
\end{cases}
\tag{5}
\]
for all $m, n, k, l \in \mathbb{N}$. Therefore, by using the inverse matrix $F(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ from Equation (5) and the relation between $x = (x_{mn})$ and $y = (y_{mn})$, we obtain:

$$x_{mn} = \frac{1}{r_{m+n}} \sum_{k,l=0}^{m,n} \prod_{i=j=0}^{n-1} \left( \frac{-s_i}{r_i} \right) \left( \frac{-u_i}{l_i} \right) y_{m-k,n-l}$$

for all $m, n \in \mathbb{N}$. (6)

Throughout this work, we will consider crucial relations between Equations (4) and (6) in terms of the sequences $x = (x_{mn})$ and $y = (y_{mn})$. Moreover, the sequence $x = (x_{mn})$ is called $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ convergent to $L$ provided $p - \lim\{B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})x\}_{mn} = L$ in the Pringsheim sense.

**Remark 1.** In the case of $s_n = s$, $r_n = r$ for all $n \in \mathbb{N}$ and $t_n = t$, $u_n = u$ for all $n \in \mathbb{N}$, then the matrix $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ is decomposed into the matrix $B(r, s, t, u)$. However, if we consider $s_n = -r_n = 1$ for all $n \in \mathbb{N}$ and $t_n = -u_n = -1$ for all $n \in \mathbb{N}$, then the matrix $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ reduces to the matrix $\Delta(1, -1, 1, -1)$. Therefore, the results obtained by the domain of the matrix $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ are more general than the corresponding consequences achieved by the domains of matrices $\Delta(1, -1, 1, -1)$ and $B(r, s, t, u)$.

Note that throughout this paper, the 4D sequential band matrix $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) = (b_{mnkl}(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}))$ and its inverse $F(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) = \{f_{mnkl}(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})\}$ will be denoted as $\tilde{B} = (\tilde{b}_{mnkl})$ and $\tilde{F} = (\tilde{f}_{mnkl})$, respectively.

**2. New Double Sequence Spaces**

In this section, the matrix domains of the matrix $\tilde{B}$ in the spaces $E$ are introduced and denoted by $\tilde{B}(E)$, where $E = \{C_p, M_u, C_{bp}, C_r, L_q\}$ and $1 \leq q < \infty$, that is,

$$\tilde{B}(C_p) := \left\{ x = (x_{mn}) \in \Omega : \exists L \in \mathbb{C} : p - \lim_{m,n \to \infty} \{\tilde{B}x\}_{mn} - L = 0 \right\},$$

$$\tilde{B}(M_u) := \left\{ x = (x_{mn}) \in \Omega : \sup_{m,n \in \mathbb{N}} |\tilde{B}x|_{mn} < \infty \right\},$$

$$\tilde{B}(C_{bp}) := \left\{ x = (x_{mn}) \in \Omega : \tilde{B}x \in C_{bp} \right\},$$

$$\tilde{B}(C_r) := \left\{ x = (x_{mn}) \in \Omega : \tilde{B}x \in C_r \right\},$$

$$\tilde{B}(L_q) := \left\{ x = (x_{mn}) \in \Omega : \sum_{m,n} |\tilde{B}x|_{mn}^q < \infty, \text{ where } 1 \leq q < \infty \right\}.$$

From definition (3), we can define the spaces $\tilde{B}(E)$, where $E = \{C_p, M_u, C_{bp}, C_r, L_q\}$, and $1 \leq q < \infty$ as:

$$\tilde{B}(C_p) = \{C_p\}_{\tilde{B}}, \quad \tilde{B}(M_u) = \{M_u\}_{\tilde{B}}, \quad \tilde{B}(C_{bp}) = \{C_{bp}\}_{\tilde{B}}, \quad \tilde{B}(C_r) = \{C_r\}_{\tilde{B}}, \quad \tilde{B}(L_q) = \{L_q\}_{\tilde{B}}.$$

Now, we investigate some topological properties and inclusion relations subject to some strict conditions.

**Theorem 1.** The double sequence spaces $\tilde{B}(C_p)$, $\tilde{B}(M_u)$, $\tilde{B}(C_{bp})$, $\tilde{B}(C_r)$, and $\tilde{B}(L_q)$ are linear spaces.

**Proof.** The straightforward proof is left to the reader. $\Box$

**Theorem 2.** The double sequence spaces $\tilde{B}(C_\theta)$ are linearly isomorphic to the spaces $C_\theta$, where $\theta = \{p, bp, r\}$, that is $\tilde{B}(C_\theta) \cong C_\theta$. 

Theorem 3. Assume the transformation \( T \) from \( \tilde{B}(C_\theta) \) to \( C_\phi \) is given by \( x \mapsto Tx = \tilde{B}x \). It is obvious that \( T \) is linear for each assumption of \( \tilde{B}(C_\theta) \) and \( C_\phi \). Moreover, it can be seen that \( x = \theta \) whenever \( Tx = \theta \), which says that \( T \) is injective.

By using (4), taking an arbitrary \( y = (y_{kl}) \in C_\phi \), and defining \( x = (x_{mn}) \) through the sequence \( y = (y_{kl}) \) and the relation (6) for all \( m, n \in \mathbb{N} \), we obtain the following:

\[
\{\tilde{B}x\}_{mn} = s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}t_nx_{m-1,n} + r_{m+1}t_nx_{mn} + r_{m+1}u_{m+1,n-1} + r_{m+1}t_nx_{mn}
\]

for all \( m, n \in \mathbb{N} \). Therefore, we can derive the fact that:

\[
\theta - \lim_{m,n \to \infty} \{\tilde{B}x\}_{mn} = \theta - \lim_{m,n \to \infty} y_{mn}.
\]

This shows that \( x = (x_{mn}) \in \tilde{B}(C_\theta) \) whenever \( y = (y_{mn}) \in C_\phi \), where \( \theta = \{p, bp, r\} \). Thus, \( T \) is surjective, and finally, \( T \) is a linear bijection between the spaces \( \tilde{B}(C_\theta) \) and \( C_\phi \).

Theorem 3. The double sequence space \( \tilde{B}(M_u) \) is linearly isomorphic to the space \( M_u \), that is \( \tilde{B}(M_u) \cong M_u \).

Proof. We can analogously use a similar pattern in Theorem 2 to prove this by considering the following equality:

\[
|\{\tilde{B}x\}_{mn}| = |s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}t_nx_{m-1,n} + r_{m+1}t_nx_{mn} + r_{m+1}u_{m+1,n-1} + r_{m+1}t_nx_{mn}|
\]

\[
= |y_{mn}|.
\]
It says that $\{\hat{B}x\}_{mn} \in \mathcal{M}_u$, i.e., $x = (x_{mn}) \in \hat{B}(\mathcal{M}_u)$ by:

$$\sup_{m,n \in \mathbb{N}} |(\hat{B}x)_{mn}| = \sup_{m,n \in \mathbb{N}} |y_{mn}| < \infty,$$

since $y = (y_{mn}) \in \mathcal{M}_u$. Hence, the proof is completed. □

**Theorem 4.** The double sequence space $\hat{B}(\mathcal{L}_q)$ is linearly isomorphic to the space $\mathcal{L}_q$, where $0 < q < \infty$, that is $\hat{B}(\mathcal{L}_q) \cong \mathcal{L}_q$.

**Proof.** The technique of the proof is similar to that used to prove Theorem 2. We consider the following equality:

$$|\{(\hat{B}x)_{mn}\}|^q = |s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}t_{n}x_{m-1,n} + r_{m}u_{n-1}x_{m,n-1} + r_{m}t_{n}x_{mn}|^q.$$

It shows that $\sum_{m,n} |(\hat{B}x)_{mn}|^q = \sum_{m,n} |y_{mn}|^q < \infty$ since $y = (y_{mn}) \in \mathcal{L}_q$. This is what we claimed. □

**Theorem 5.** The following statements hold.

(a) If $\sup \frac{s_{mn}}{m_{mn}r_{mn}} < 1$ and $\sup \frac{u_{mn}}{m_{mn}r_{mn}} < 1$ for all $m, n \in \mathbb{N}$, then:

(i) $C_p = \hat{B}(C_p),$

(ii) $C_\theta = \hat{B}(C_\theta),$

(iii) $C_{bp} = \hat{B}(C_{bp}).$

(b) If $\sup \frac{s_{mn}}{m_{mn}r_{mn}} \geq 1$ and $\sup \frac{u_{mn}}{m_{mn}r_{mn}} \geq 1$ for all $m, n \in \mathbb{N}$, then the following inclusions strictly hold.

(i) $C_p \subset \hat{B}(C_p),$

(ii) $C_\theta \subset \hat{B}(C_\theta),$

(iii) $C_{bp} \subset \hat{B}(C_{bp}).$

**Proof.** (a)(i). Since the matrix $\hat{B} = (\hat{b}_{mnkl})$ satisfies the conditions of $C_p$-conservative matrices (see [11], Lemma 3.2, p. 10), we have:

$$\sup_{m,n \in \mathbb{N}} \sum_{k,l} |\hat{b}_{mnkl}| \leq \sup_{m,n \in \mathbb{N}} |s_{m-1}u_{n-1}| + \sup_{m,n \in \mathbb{N}} |s_{m-1}t_{n}| + \sup_{m,n \in \mathbb{N}} |r_{m}u_{n-1}| + \sup_{m,n \in \mathbb{N}} |r_{m}t_{n}| < \infty, \quad (7)$$

$$\theta - \lim_{m,n \to \infty} \sum_{k,l} |\hat{b}_{mnkl}| = \alpha_{kl} \text{ exists for all } k, l \in \mathbb{N},$$

$$\theta - \lim_{m,n \to \infty} \sum_{k,l} |\hat{b}_{mnkl}| = \lim_{m,n \to \infty} s_{m-1}u_{n-1} + \lim_{m,n \to \infty} s_{m-1}t_{n} + \lim_{m,n \to \infty} r_{m}u_{n-1} + \lim_{m,n \to \infty} r_{m}t_{n} = \lambda$$

exists for all $k, l \in \mathbb{N}$

$\forall k \in \mathbb{N}, \exists \lambda_0 \in \mathbb{N} \ni \hat{b}_{mnkl} = 0 \text{ for all } l > \lambda_0 \text{ and } m, n \in \mathbb{N},$

$\forall l \in \mathbb{N}, \exists \lambda_0 \in \mathbb{N} \ni \hat{b}_{mnkl} = 0 \text{ for all } k > \lambda_0 \text{ and } m, n \in \mathbb{N}.$

Thus, $\hat{B} \in (C_p; C_q)$, where $\theta = \{p, bp, r\}$. Since for every sequence $x = (x_{mn}) \in C_p$, $\hat{B}x \in C_\theta$, hence $x = (x_{mn}) \in \hat{B}(C_\theta)$. This shows that $C_p \subset \hat{B}(C_\theta)$ holds, where $\theta = \{p, bp, r\}$.
Let $\sup_{m,n} s_{mn} < 1$, and $\sup_{m,n} u_{mn} < 1$ for all $m, n \in \mathbb{N}$. Since the matrix $\tilde{F} = (\tilde{f}_{mn})$ defined in (5) is the inverse of $\tilde{B} = (\tilde{b}_{mn})$, it should satisfy the following conditions:

$$\sup_{m,n} \sum_{k,l} |\tilde{f}_{mn}| \leq \frac{1}{\inf_{m,r} t_m} \sum_{k,l} \left( \frac{\sup_{m,s} s_{mn}}{\inf_{m,s} t_m} \right)^k \left( \frac{\sup_{n,u} u_{mn}}{\inf_{n,u} t_n} \right)^l < \infty,$$  

(8)

$$\vartheta - \lim_{m,n \to \infty} \tilde{f}_{mn} = \lim_{m,n \to \infty} \frac{1}{\inf_{m,r} t_m} \prod_{i=k}^{m-1} \prod_{j=l}^{n-1} \left( \frac{-s_{ij}}{r_{ij}} \right) \text{ exists for all } k, l \in \mathbb{N},$$

$$\vartheta - \lim_{m,n \to \infty} \sum_{k,l} \tilde{f}_{mn} = \frac{1}{\inf_{m,r} t_m} \lim_{m,n \to \infty} \sum_{k,l} \left( \frac{-s_{mn}}{\inf_{m,r} t_m} \right)^k \left( \frac{-u_{mn}}{\inf_{n,u} t_n} \right)^l \text{ for all } k > k_0 \text{ and } m, n \in \mathbb{N},$$

$$\forall k \in \mathbb{N}, \exists l_0 \in \mathbb{N} \ni \frac{1}{r_m t_n} \prod_{i=k}^{m-1} \prod_{j=l}^{n-1} \left( \frac{-s_{ij}}{r_{ij}} \right) \left( \frac{-u_{ij}}{t_{ij}} \right) = 0 \text{ for all } l > l_0 \text{ and } m, n \in \mathbb{N},$$

for all $k, l \in \mathbb{N}$. Thus, $\tilde{B}^{-1} = \tilde{F} \in (C_\vartheta; C_\vartheta)$, where $\vartheta = \{p, bp, r\}$. Since for every sequence $x = (x_{mn}) \in \tilde{B}(C_\vartheta)$, it follows that $y = \tilde{B}x \in C_\vartheta$ and $x = \tilde{B}^{-1}y = \tilde{F}y \in C_p$, hence the opposite inclusion $C_p \supset \tilde{B}(C_\vartheta)$ also holds, where $\vartheta = \{p, bp, r\}$. This completes the proof of $(a)(i)$.

Parts $(a)(ii)$ and $(a)(iii)$ can be proven in a similar way by applying [11], Lemma 3.4, p. 11, and [11], Lemma 3.2, p. 11, respectively. We omit the details.

Proof of (b): Suppose that $\sup_{m,n} s_{mn} = 1$, and $\sup_{m,n} u_{mn} = 1$ and $(s_m) = (r_m) = s$ for all $m \in \mathbb{N}$ and $(u_n) = (t_n) = u$ for all $n \in \mathbb{N}$. Let us consider the sequence $x = (x_{mn}) = (-1)^m(n)$ for all $m, n \in \mathbb{N}$. Therefore, the sequence $x \in \tilde{B}(C_\vartheta) \setminus C_\vartheta$, where $\vartheta = \{p, bp, r\}$, since the $\tilde{B}$-transform of $x$ is:

$$\{\tilde{B}x\}_{mn} = s_{m-1}x_{m-1} + s_{m-1}u_{m-1}n + r_{m-1}x_{m-1} + r_{m-1}u_{m-1}n + r_{m-1}t_{m-1}n = su \left[ (-1)^{m-1}(n-1) + (-1)^{m-1}(n) \right] = 0.$$

This concludes the proof of the theorem. \(\square\)

**Theorem 6.** The following statements hold.

(a) If $\sup_{m,n} s_{mn} < 1$ and $\sup_{m,n} u_{mn} < 1$ for all $m, n \in \mathbb{N}$, then $M_u = \tilde{B}(M_u)$;

(b) If $\sup_{m,n} s_{mn} \geq 1$ and $\sup_{m,n} u_{mn} \geq 1$ for all $m, n \in \mathbb{N}$, then $M_u \subset \tilde{B}(M_u)$.

**Proof.** The proof of (a) can clearly be achieved by referring to [14], Theorem 2.2, since the matrix $\tilde{B}$ satisfies the condition (7) and the inverse matrix $\tilde{F}$ satisfies the condition (8) with the assumptions $\sup_{m,n} s_{mn} < 1$ and $\sup_{m,n} u_{mn} < 1$ for all $m, n \in \mathbb{N}$. Therefore, we omit the details.

For the proof of (b), suppose that $\sup_{m,n} s_{mn} = 1$ and $\sup_{m,n} u_{mn} = 1$. We consider the sequence $x = (x_{mn})$ with:

$$x_{mn} = \frac{(-1)^{m+n}(mn)}{r_m t_n}, \text{ for all } m, n \in \mathbb{N}.$$
Theorem 7. If \( \frac{-\sup_m u_m}{\mathfrak{m}_m t_m} \geq 1 \) and \( \frac{-\sup_n u_n}{\mathfrak{m}_n t_n} \geq 1 \) for all \( m, n \in \mathbb{N} \), then the inclusion \( \mathcal{L}_q \subset \tilde{B}(\mathcal{L}_q) \) strictly holds, where \( 1 \leq q < \infty \).

**Proof.** Suppose that an arbitrary double sequence \( x = (x_{mn}) \) is in \( \mathcal{L}_q \), where \( 1 \leq q < \infty \). Thus, \( \sum_{m,n} |x_{mn}|^q < \infty \). We have to show \( \tilde{B}x \in \mathcal{L}_q \). It follows from:

\[
\left( \sum_{m,n} |\tilde{B}x_{mn}|^q \right)^{1/q} \leq \left( \sum_{m,n} |s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}u_nx_{m,n-1} + r_{m-1}u_{n-1}x_{m-1,n} + r_{m-1}t_{n}x_{mn}|^q \right)^{1/q} < \infty
\]

that \( \tilde{B}x \in \mathcal{L}_q \). Thus, \( x \in \tilde{B}(\mathcal{L}_q) \), and the inclusion \( \mathcal{L}_q \subset \tilde{B}(\mathcal{L}_q) \) holds.

Now, it is our main goal to prove the existence of a double sequence \( x = (x_{mn}) \), which is contained in the set \( \tilde{B}(\mathcal{L}_q) \setminus \mathcal{L}_q \). We define the double sequence \( x = (x_{mn}) \) by:

\[
x_{mn} = \frac{1}{r_m t_n} \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right), \quad \text{for all } m, n \in \mathbb{N}.
\]

and suppose that \( -\sup_m s_m > 1 \) and \( -\sup_n u_n > 1 \) for all \( m, n \in \mathbb{N} \). If we choose \( s_m = s, r_m = r, u_n = u \) and \( t_n = t \), then \( \left( \frac{s}{r} \right) > 1 \) and \( \left( \frac{t}{r} \right) > 1 \). Thus, \( x = (x_{mn}) \) is not in \( \mathcal{L}_q \). However, the \( \tilde{B} \)-transform of \( x = (x_{mn}) \) satisfies:

\[
\sum_{m,n} |\{\tilde{B}x_{mn}\}|^q = |s_{m-1}u_{n-1}x_{m-1,n-1} + s_{m-1}u_nx_{m,n-1} + r_{m-1}u_{n-1}x_{m-1,n} + r_{m-1}t_{n}x_{mn}|^q
\]

\[
= \sum_{m,n} \left| \frac{su}{rl} \prod_{i=0}^{m-2} \prod_{j=0}^{n-2} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) + \frac{sr}{rl} \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) \right|^q
\]

\[
+ \frac{ru}{rl} \prod_{i=0}^{m-1} \prod_{j=0}^{n-2} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) + \frac{rt}{rl} \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) \right|^q
\]

\[
\leq \sum_{m,n} \left| \frac{su}{rl} \prod_{i=0}^{m-2} \prod_{j=0}^{n-2} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) + \frac{sr}{rl} \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \left( -\frac{s_i}{r_i} \right) \left( -\frac{u_j}{l_j} \right) \right| n^q
\]

\[
+ \frac{ru}{rl} \left( \frac{-s}{r} \right)^m \left( \frac{-u}{l} \right)^{n-1} + \frac{rt}{rl} \left( \frac{-s}{r} \right)^m \left( \frac{-u}{l} \right)^n \right| \right| = 0.
\]
Theorem 8. Let \( u \in \mathcal{L}_\theta \) in \( \mathcal{B}(\mathcal{M}_u) \). Then, \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) \( C_{yp} \) as a corollary. 

Proof. We prove \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and the necessary conditions for the members in these classes in Table 3. Finally, we state some results derived from Table 2 and Lemma 1 as a corollary.

3. Dual Spaces

In this section, we start by determining the \( \alpha \)-dual of the spaces \( \mathcal{B}(\mathcal{M}_u) \) and \( \mathcal{B}(\mathcal{C}_{yp}) \). Then, we summarize the results for the characterizations of the 4D matrix classes in Table 2 and the necessary and sufficient conditions for the members in these classes in Table 3. Finally, we state some results derived from Table 2 and Lemma 1 as a corollary.

Theorem 8. Let \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \). Then, \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \). However, \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \).

Proof. We prove \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and leave the proofs of the other cases to the reader. To prove \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \), we must show both inclusions \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) and \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \). Therefore, there exists a double sequence \( y = (y_{mn}) \in \mathcal{M}_u \) with the relations (4) and (6) such that \( \sup_{m,n \in \mathbb{N}} |y_{mn}| \leq K \), where \( K \in \mathbb{R}^+ \). Then, we have:

\[
\sum_{m,n} |a_{mn}| \leq \frac{1}{\|r'\| \|r''\|} \sum_{m,n} |a_{mn}| \sum_{k,l=0}^{m,n} |m-1| \sum_{j=0}^{m,n} \left( \frac{-s_j}{r_j} \right) \left( \frac{-u_j}{t_j} \right) |y_{m-k,n-l}|
\]

Thus, \( a = (a_{mn}) \in \mathcal{B}(\mathcal{M}_u) \) and so, the inclusion \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) holds.

Now, we show that the second inclusion \( \mathcal{L}_\theta \subseteq \mathcal{L}_\theta \) holds, that is every sequence in the space \( \mathcal{L}_\theta \) is also in the space \( \mathcal{L}_\theta \). We use the method of contradiction. Therefore, we suppose that there exists a sequence \( (a_{mn}) \in \mathcal{L}_\theta \) such that \( \sum_{m,n} |a_{mn}| < \infty \) for all \( x = (x_{mn}) \in \mathcal{B}(\mathcal{M}_u) \), but \( \sum_{m,n} |a_{mn}| = \infty \). We define the double sequence \( x \) by \( x = (x_{mn}) = \{(1)^{m+n}\} \), which is in \( \mathcal{B}(\mathcal{M}_u) \) such that:

\[
\sum_{m,n} |a_{mn}| = \sum_{m,n} |a_{mn}| = \infty.
\]

This means that \( (a_{mn}) \notin \mathcal{L}_\theta \), which contradicts our assumption. Therefore, \( (a_{mn}) \) must belong to the space \( \mathcal{L}_\theta \). □
We summarize the characterizations of some 4D matrix classes as in the following useful results, which have been established in \cite{11–14}.

\[
\sup_{m,n\in\mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \tag{9}
\]

\[
\exists a_{kl} \in \mathbb{C} \ni \theta - \lim_{m,n \to \infty} a_{mnkl} = a_{kl} \text{ for all } k, l \in \mathbb{N}, \tag{10}
\]

\[
\exists l \in \mathbb{C} \ni \theta - \lim_{m,n \to \infty} \sum_{k,l} a_{mnkl} = l \text{ exists }, \tag{11}
\]

\[
\exists k_0 \in \mathbb{N} \ni \theta - \lim_{m,n \to \infty} \sum_{l} |a_{mnkl} - a_{k_l}| = 0, \tag{12}
\]

\[
\exists l_0 \in \mathbb{N} \ni \theta - \lim_{m,n \to \infty} \sum_{k} |a_{mnkl_0} - a_{k_l}| = 0, \tag{13}
\]

\[
\forall k \in \mathbb{N}, \exists l_0 \in \mathbb{N} \ni a_{mnkl} = 0 \text{ for all } l > l_0 \text{ and } m, n \in \mathbb{N}, \tag{14}
\]

\[
\forall l \in \mathbb{N}, \exists k_0 \in \mathbb{N} \ni a_{mnkl} = 0 \text{ for all } k > k_0 \text{ and } m, n \in \mathbb{N}, \tag{15}
\]

\[
\exists l_0 \in \mathbb{N} \ni \theta - \lim_{m,n \to \infty} \sum_{k} a_{mnkl_0} = u_{l_0}, \tag{16}
\]

\[
\exists k_0 \in \mathbb{N} \ni \theta - \lim_{m,n \to \infty} \sum_{l} a_{mnkl} = v_{k_0}, \tag{17}
\]

\[
\sup_{m,n,k,l \in \mathbb{N}} |a_{mnkl}|^q < \infty, \tag{18}
\]

\[
\sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}|^q < \infty, \text{ where } \frac{1}{q} + \frac{1}{q'} = 1, \text{ for } 1 < q < \infty, \tag{19}
\]

\[
\exists a_{kl} \in \mathbb{C} \ni bp - \lim_{m,n \to \infty} \sum_{k,l} |a_{mnkl} - a_{kl}| = 0, \tag{20}
\]

\[
bp - \lim_{m,n \to \infty} \sum_{k=0}^{n} a_{mnkl} \text{ exists for each } k \in \mathbb{N}, \tag{21}
\]

\[
bp - \lim_{m,n \to \infty} \sum_{k=0}^{m} a_{mnkl} \text{ exists for each } l \in \mathbb{N}, \tag{22}
\]

\[
\sum_{k,l} |a_{mnkl}| \text{ converges}. \tag{23}
\]

| From \(E_1\) \(\to\) \(E_2\) | \(\mathcal{M}_u\) | \(\mathcal{C}_p\) | \(\mathcal{C}_{bp}\) | \(\mathcal{C}_r\) | \(\mathcal{L}_q, (0 < q < 1)\) | \(\mathcal{L}_q, (1 \leq q < \infty)\) |
|---|---|---|---|---|---|---|
| \(\mathcal{M}_u\) | 1 | 2 | 3 | * | * | * |
| \(\mathcal{C}_p\) | * | 4 | 4 | 4 | * | * |
| \(\mathcal{C}_{bp}\) | 5 | 6 | 6 | 6 | * | * |
| \(\mathcal{C}_r\) | * | 7 | 7 | 7 | * | * |
| \(\mathcal{L}_q, (0 < q < 1)\) | 8 | * | 9 | * | * | 10 |
| \(\mathcal{L}_q, (1 \leq q < \infty)\) | 11 | * | 12 | * | * | * |

We list the necessary and sufficient conditions for each class in the following table. Note that * shows the unknown characterization of respective 4D matrix class.

| \(1\) iff \(2\) iff \(3\) iff \(4\) iff \(5\) iff \(6\) iff \(7\) iff \(8\) iff \(9\) iff \(10\) iff \(11\) iff \(12\) iff |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (9) | (10) | (9) | (9) | (9) | (18) | (10) | (9) | (19) | (10) | (19) | (19) |
| (14) | (10) | (10) | (10) | (10) | (11) | (11) | (11) | (11) | (11) | (16) | (17) |
| (20) | (11) | (11) | (11) | (11) | (11) | (11) | (11) | (11) | (11) | (11) | (11) |
| (21) | (14) | (12) | (12) | (12) | (12) | (12) | (12) | (12) | (12) | (12) | (12) |
| (22) | (15) | (13) | (13) | (13) | (13) | (13) | (13) | (13) | (13) | (13) | (13) |
| (23) | | | | | | | | | | | | |
We recall that the $\alpha$- and $\gamma$-duals of spaces of double sequences are uniquely determined. To determine the $\beta(\vartheta)$-dual of the spaces of double sequences, we need to consider $\vartheta$-convergence, where $\vartheta = \{p, bp, r\}$. Therefore, there are different $\beta(\vartheta)$-duals of a double sequence space. Now, we state the following basic lemma, which will enable us to determine the $\gamma$- and $\beta(\vartheta)$-duals of some new spaces of double sequences.

**Lemma 1.** Suppose that $E$ is any double sequence space, and let the 4D matrix $\hat{D} = (\hat{a}_{mnkl})$ be defined with the double sequence $a = (a_{kl}) \in \Omega$ and the inverse matrix $\hat{F} = (\hat{f}_{mnkl})$ of the matrix $\hat{B} = (b_{mnkl})$ as:

$$
\hat{a}_{mnkl} = \left\{ \begin{array}{ll}
\sum_{l=1}^{m} \sum_{k=1}^{n} \frac{1}{r^l_t} \prod_{r=k}^{m-1} \prod_{p=l}^{k-1} \left( \frac{-s_n}{r\pi} \right) \left( \frac{-u_p}{t\rho} \right) a_{ij}, & 0 \leq k \leq m, 0 \leq l \leq n; \\
0, & \text{elsewhere}
\end{array} \right.
$$

for all $m, n, k, l \in \mathbb{N}$. Then, we have:

$$
\begin{align*}
\left\{ \hat{B}(E) \right\}_{\vartheta}^\gamma &= \{ a = (a_{kl}) \in \Omega : D \in (E; M_u) \}, \\
\left\{ \hat{B}(E) \right\}_{\vartheta}^{\beta(\vartheta)} &= \{ a = (a_{kl}) \in \Omega : D \in (E; C_\vartheta) \},
\end{align*}
$$

where $\vartheta = \{p, bp, r\}$.

**Proof.** We take $a = (a_{mn}) \in \Omega$ and $x = (x_{mn}) \in B(C_\vartheta)$. Then, clearly, $y = \hat{B}x \in C_\vartheta$. Therefore, the $(m, n)^{th}$-partial sum of $\sum_{k,l} a_{kl}x_{kl}$ can be stated by the following equality:

$$
\sum_{k,l=0}^{m,n} a_{kl} x_{kl} = \sum_{k,l=0}^{m,n} a_{kl} \frac{1}{r^l_t} \sum_{i,j=0}^{k,l} \prod_{r=k}^{m-1} \prod_{p=l}^{k-1} \left( \frac{-s_n}{r\pi} \right) \left( \frac{-u_p}{t\rho} \right) y_{k-i,l-j}
$$

$$
= \sum_{k,l=0}^{m,n} \sum_{i,j=k,l}^{m,n} \frac{1}{r^l_t} \prod_{r=k}^{m-1} \prod_{p=l}^{k-1} \left( \frac{-s_n}{r\pi} \right) \left( \frac{-u_p}{t\rho} \right) a_{ij} y_{kl}
$$

where the 4D matrix $\hat{D} = (\hat{d}_{mnkl})$ is defined by:

$$
\hat{d}_{mnkl} = \left\{ \begin{array}{ll}
\sum_{i,j=k,l}^{m,n} \frac{1}{r^l_t} \prod_{r=k}^{m-1} \prod_{p=l}^{k-1} \left( \frac{-s_n}{r\pi} \right) \left( \frac{-u_p}{t\rho} \right) a_{ij}, & 0 \leq k \leq m, 0 \leq l \leq n; \\
0, & \text{elsewhere}
\end{array} \right.
$$

for all $m, n, k, l \in \mathbb{N}$. Therefore, we can easily obtain the result by the definition of the $\gamma$-dual of a double sequence space (see (1)), that is $ax \in BS$ for every $x = (x_{mn}) \in B(E)$ if and only if $Dy \in M_u$ for every $y = (y_{mn}) \in E$. This shows that $a = (a_{mn}) \in \{B(E)\}^\gamma$ if and only if $D \in (E : M_u)$. Thus, the identity in (24) holds true.

In order to prove the validity of (25), by the definition of the $\beta(\vartheta)$-dual of a space of double sequences (see (1)), we obtain that $ax \in CS_\vartheta$ for every $x = (x_{mn}) \in B(E)$ if and only if $Dy \in C_\vartheta$ for every $y = (y_{mn}) \in E$, where $\vartheta = \{p, bp, r\}$. This proves that $a = (a_{mn}) \in \{B(E)\}^{\beta(\vartheta)}$ if and only if $D \in (E : C_\vartheta)$, where $\vartheta = \{p, bp, r\}$. Thus, we have shown that the identity in (25) holds true. \qed
The following theorems are direct consequences of Table 2 by applying the above Lemma 1. We consider the following sets $\tilde{d}_i$ with $i = \{1, 2, \ldots, 15\}$ by:

\[
\tilde{d}_1 = \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} \bigg|_{q'} < \infty \right\},
\]

\[
\tilde{d}_2 = \left\{ a = (a_{kl}) \in \Omega : \exists \beta_{kl} \in \mathbb{C} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = \beta_{kl} \right\},
\]

\[
\tilde{d}_3 = \left\{ a = (a_{kl}) \in \Omega : \exists l \in \mathbb{C} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = l \exists \right\},
\]

\[
\tilde{d}_4 = \left\{ a = (a_{kl}) \in \Omega : \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = 0 \right\},
\]

\[
\tilde{d}_5 = \left\{ a = (a_{kl}) \in \Omega : \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = 0 \right\},
\]

\[
\tilde{d}_6 = \left\{ a = (a_{kl}) \in \Omega : \forall k, \exists l_0 \in \mathbb{N} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = 0 \forall l > l_0 \right\},
\]

\[
\tilde{d}_7 = \left\{ a = (a_{kl}) \in \Omega : \forall l, \exists k_0 \in \mathbb{N} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = 0 \forall k > k_0 \right\},
\]

\[
\tilde{d}_8 = \left\{ a = (a_{kl}) \in \Omega : \forall l, \exists k_0 \in \mathbb{N} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = u_{l_0} \right\},
\]

\[
\tilde{d}_9 = \left\{ a = (a_{kl}) \in \Omega : \forall l, \exists k_0 \in \mathbb{N} \ni \exists \theta - \lim_{m,n \to \infty} \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} = v_{k_0} \right\},
\]

\[
\tilde{d}_{10} = \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} \right|_{q'} < \infty \right\},
\]

\[
\tilde{d}_{11} = \left\{ a = (a_{kl}) \in \Omega : \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{r_i l_j} \prod_{k,l} \left( \frac{u_{kl}}{r_{kl}} \right) a_{ij} \right|_{q'} < \infty \right\},
\]
Theorem 9. The following characterizations of $\gamma$-duals hold:

(i) $\{B(M_{k})\}^{\gamma} = \tilde{d}_{1}$ with $q' = 1$.
(ii) $\{B(C_{bp})\}^{\gamma} = \tilde{d}_{1}$ with $q' = 1$.
(iii) $\{B(C_{q})\}^{\gamma} = \tilde{d}_{10}$ for $0 < q < 1$.
(iv) $\{B(C_{q})\}^{\gamma} = \tilde{d}_{11}$ for $1 \leq q < \infty$.

Proof. The proof of (i), (ii), (iii) and (iv) easily follows from Lemma 1 with matrix $\tilde{D} = (\tilde{a}_{mnkl})$ instead of $A = (a_{mnkl})$ in the matrix classes 1, 5, 8, and 11 in Table 2, respectively. □

Theorem 10. The following characterizations of $\beta(\theta)$-duals hold:

(i) $\{B(C_{p})\}^{\beta(\theta)} = \bigcap_{i=1}^{3} \tilde{d}_{i} \cap \tilde{d}_{6} \cap \tilde{d}_{7}$ with $q' = 1$.
(ii) $\{B(C_{bp})\}^{\beta(\theta)} = \bigcap_{i=1}^{5} \tilde{d}_{i}$ with $q' = 1$.
(iii) $\{B(C_{q})\}^{\beta(\theta)} = \bigcap_{i=1}^{3} \tilde{d}_{i} \cap \tilde{d}_{8} \cap \tilde{d}_{9}$ with $q' = 1$.

Proof. The proof is similar to that of Theorem 9, and the proof of (i), (ii) and (iii) easily follows from Lemma 1 with matrix $\tilde{D} = (\tilde{a}_{mnkl})$ instead of $A = (a_{mnkl})$ in the matrix classes 4, 6 and 7 in Table 2, respectively. □

Theorem 11. The following characterizations of $\beta(bp)$-duals hold:

(i) $\{B(M_{k})\}^{\beta(bp)} = \tilde{d}_{1} \cap \tilde{d}_{2} \cap \tilde{d}_{14} \cap d_{i}$.
(ii) $\{B(C_{q})\}^{\beta(bp)} = \tilde{d}_{2} \cap \tilde{d}_{10}$ for $0 < q \leq 1$.
(ii) $\{B(C_{q})\}^{\beta(bp)} = \tilde{d}_{1} \cap \tilde{d}_{2}$ for $1 < q < \infty$.

Proof. The proof is similar to that of Theorem 9, and the proof of (i), (ii) and (iii) easily follows from Lemma 1 with matrix $\tilde{D} = (\tilde{a}_{mnkl})$ instead of $A = (a_{mnkl})$ in the matrix classes 3, 9 and 12 in Table 2, respectively. □
Theorem 12. We have \( \tilde{B}(\mathcal{M}_u) \beta(p) = \tilde{d}_2 \cap \tilde{d}_6 \cap \tilde{d}_7. \)

Proof. The proof is similar to that of Theorem 9, and the proof of (i) easily follows from Lemma 1 with matrix \( \tilde{D} = (\tilde{a}_{mnik}) \) instead of \( A = (a_{mnik}) \) in the matrix class 2 in Table 2, respectively. \( \square \)

4. Matrix Transformations Related to the Sequence Spaces \( \tilde{B}(\mathcal{C}_p), \tilde{B}(\mathcal{M}_u), \tilde{B}(\mathcal{C}_pp), \tilde{B}(\mathcal{C}_r), \) and \( \tilde{B}(\mathcal{L}_q) \)

This section is organized as follows. First, we state two significant theorems including dual summability matrices \( A = (a_{mnik}) \) and \( \tilde{C} = (\tilde{c}_{mnik}) \), and \( A = (a_{mnik}) \) and \( \tilde{G} = (\tilde{g}_{mnik}) \), respectively. Then, we establish necessary and sufficient conditions of some new matrix classes including the spaces of the double sequence whose \( \tilde{B} \)-transforms are in the spaces \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_pp, \mathcal{C}_r, \) and \( \mathcal{L}_q \), respectively, where \( 0 < q < \infty \). Finally, we conclude this section with some significant consequences.

We note here that since \( \tilde{B}(E) = E \) for any arbitrary double sequence space \( E \), it clearly holds that \( x \in \tilde{B}(E) \) if and only if \( y = \tilde{B}x \in E \) holds. In this context, we now prove the following basic theorems:

Theorem 13. We assume that the 4D infinite matrices \( A = (a_{mnik}) \) and \( \tilde{C} = (\tilde{c}_{mnik}) \) are connected with the relation:

\[
\tilde{c}_{mnik} = \sum_{i,j=k,l}^{\infty} \frac{1}{r_{i,j}^{l_{i,j}}} \prod_{p=k}^{l} \prod_{q=p+1}^{l-j} \left( -\frac{s_{pq}}{r_{pq}} \right) \left( -\frac{u_{pq}}{l_{pq}} \right) a_{mnij} \tag{27}
\]

for any \( m, n, k, l \in \mathbb{N} \) and \( E_2 \) is any given double sequence space. Then, \( A \in (\tilde{B}(E_1) : E_2) \) if and only if \( A_{mn} \in [\tilde{B}(E_1)]^{\beta(d)} \) for any \( m, n \in \mathbb{N} \) and \( \tilde{C} \in (E_1 : E_2) \), where \( E_1, E_2 \in \{ \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_pp, \mathcal{C}_r, \mathcal{L}_q \} \).

Proof. To prove the necessity, we assume \( A \in (\tilde{B}(E_1) : E_2) \). Then, \( Ax \) exists and \( Ax \in E_2 \) for every \( x \in \tilde{B}(E_1) \). Therefore, \( A_{mn} \in [\tilde{B}(E_1)]^{\beta(d)} \) is a necessary condition for all \( m, n \in \mathbb{N} \). Thus, by the relation in Equation (27) and the connection in (6) between the terms of the sequences \( y = (y_{kl}) \) and \( x = (x_{kl}) \), we obtain the \((m, n)\)th-partial sum of the series \( \sum_{k,l=0}^{m,n} a_{mnik}x_{kl} \) as:

\[
\sum_{k,l=0}^{m,n} a_{mnik}x_{kl} = \sum_{k,l=0}^{m,n} a_{mnik} \left( \frac{1}{r_{i,j}^{l_{i,j}}} \prod_{i=0}^{k-1} \prod_{j=0}^{l-1} \left( -\frac{s_{pq}}{r_{pq}} \right) \left( -\frac{u_{pq}}{l_{pq}} \right) y_{k-i,j-l} \right) \tag{28}
\]

for all \( m, n, k, l \in \mathbb{N} \). Thus, taking the \( d \)-limit in (28) as \( m, n \to \infty \), we obtain \( Ax = \tilde{C}y \). Hence, \( \tilde{C}y \in E_2 \) whenever \( y \in E_1 \), that is, \( \tilde{C} \in (E_1 : E_2) \). This concludes the proof of the necessity.

To prove the sufficiency, suppose that \( A_{mn} \in [\tilde{B}(E_1)]^{\beta(d)} \) for all \( m, n \in \mathbb{N} \) and \( \tilde{C} \in (E_1 : E_2) \), where the matrix \( \tilde{C} = (\tilde{c}_{mnik}) \) is defined as in (27). Now, we assume that an arbitrary double sequence satisfies \( v = (v_{kl}) \in \tilde{B}(E_1) \) if and only if \( u = \tilde{B}v \in E_1 \). Then, for this sequence \( v = (v_{kl}) \), \( Av \) exists, and it is enough to show that \( Av \in E_2 \). Clearly, \( v = \tilde{B}^{-1}u = \tilde{F}u \) holds. We obtain the \((m, n)\)th-partial sum of the series \( \sum_{k,l=0}^{m,n} a_{mnik}v_{kl} \) for all \( m, n, k, l \in \mathbb{N} \) as:

\[
\sum_{k,l=0}^{m,n} a_{mnik}v_{kl} = \sum_{k,l=0}^{m,n} a_{mnik} \sum_{i,j=0}^{k,l} f_{i,j}u_{ij} = \sum_{k,l=0}^{m,n} \left( \sum_{i,j=0}^{k,l} \frac{1}{r_{i,j}^{l_{i,j}}} \prod_{p=k}^{l} \prod_{q=p+1}^{l-j} \left( -\frac{s_{pq}}{r_{pq}} \right) \left( -\frac{u_{pq}}{l_{pq}} \right) a_{mnij} \right) u_{ij}.
\]
Theorem 14. Suppose that the following connection between the 4D infinite matrices \( A = (a_{mnkl}) \) and \( \tilde{G} = (\tilde{g}_{mnkl}) \) holds:

\[
\tilde{g}_{mnkl} = \sum_{i,j=0}^{m,n} \tilde{b}_{mni} b_{ijkl} \quad \text{(29)}
\]

for all \( m,n,k,l \in \mathbb{N} \). Then, \( A \in (E_2 : \tilde{B}(E_1)) \) if and only if \( G \in (E_2 : E_1) \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q\} \).

**Proof.** Suppose that \( A \in (E_2 : \tilde{B}(E_1)) \). Then, \( Ax \) exists and is in \( \tilde{B}(E_1) \) for each sequence \( x = (x_{kl}) \in E_2 \), and clearly, \( \tilde{B}(Ax) \in E_1 \). We obtain the \((m,n)\)th-partial sum by Relation (29) as:

\[
\tilde{B}(Ax)_{mn} = s_{m-1}u_{n-1}(Ax)_{m-1,n-1} + s_{m-1}t_n(AX)_{m-1,n} + r_m u_n - 1(AX)_{m-1,n} + r_m t_n(AX)_{mn} + r_m u_{n-1} a_{m-1,n,k,l} x_{kl} + r_m t_n a_{mnkl} x_{kl}.
\]

for all \( m,n \in \mathbb{N} \). Thus, by taking the \( \theta \)-limit as \( m,n \to \infty \), we conclude \( Ax \in \tilde{B}(E_1) \) whenever \( x = (x_{kl}) \in E_2 \) if and only if \( Gx \in E_1 \) whenever \( x = (x_{kl}) \in E_2 \). Therefore, \( \tilde{G} \in (E_2 : E_1) \) as desired.

Here, we organize the following new Tables 4–7 by referring to Tables 2 and 3. The aim is to summarize all the consequences derived by Theorems 13 and 14 by avoiding the repetition in the existing literature.

Corollary 1. Let \( A = (a_{mnkl}) \) be a 4D infinite matrix and connected with the 4D matrix \( \tilde{C} = (\tilde{c}_{mnkl}) \) by (27). Then, the following characterizations of the 4D matrix classes \( (\tilde{B}(E_1) ; E_2) \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q\} \) given by:

| From \( E_1 \) To \( E_2 \) | \( \tilde{B}(M_u) \) | \( \tilde{B}(C_p) \) | \( \tilde{B}(C_{bp}) \) | \( \tilde{B}(C_r) \) | \( \tilde{B}(L_q)(0 < q < 1) \) | \( \tilde{B}(L_q)(1 \leq q < \infty) \) |
|-----------------------|---------|---------|---------|---------|-----------------|-----------------|
| \( \tilde{B}(M_u) \)  | 1       | 2       | 3       | *       | *               | *               |
| \( \tilde{B}(C_p) \)  | *       | 4       | 4       | 4       | *               | *               |
| \( \tilde{B}(C_{bp}) \)| 5       | 6       | 6       | 6       | *               | *               |
| \( \tilde{B}(C_r) \)  | *       | 7       | 7       | 7       | *               | *               |
| \( \tilde{B}(L_q)(0 < q < 1) \) | 8       | *       | 9       | *       | *               | 10              |
| \( \tilde{B}(L_q)(1 \leq q < \infty) \)| 11      | *       | 12      | *       | *               | *               |
Hold, and the necessary and sufficient conditions for each characterization of the new classes are given by:

Table 5. Characterization of the classes \( \tilde{B}(E_1); E_2 \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q \} \).

| From \( E_1 \) \( \tilde{B}(E_1) \) \( \tilde{B}(C_p) \) \( \tilde{B}(C_{bp}) \) \( \tilde{B}(C_r) \) \( \tilde{B}(L_q) \) | 1 iff | 2 iff | 3 iff | 4 iff | 5 iff | 6 iff | 7 iff | 8 iff | 9 iff | 10 iff | 11 iff | 12 iff |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | (9) | (10) | (9) | (9) | (9) | (9) | (9) | (9) | (10) | (9) | (10) | (10) |
| 2 | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (11) | (16) | (13) | (17) |
| 3 | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (11) | (16) | (13) | (17) |

Corollary 2. Let \( A = (a_{mnkl}) \) be a 4D infinite matrix and connected with the 4D matrix \( \tilde{G} = (\tilde{g}_{mnkl}) \) by (29). Then, the following characterizations of the 4D matrix classes \( (E_1; \tilde{B}(E_2)) \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q \} \) given by:

Table 6. D matrix classes \( (E_1; \tilde{B}(E_2)) \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q \} \).

| From \( E_1 \) \( \tilde{B}(E_1) \) \( \tilde{B}(M_u) \) \( \tilde{B}(C_p) \) \( \tilde{B}(C_{bp}) \) \( \tilde{B}(C_r) \) \( \tilde{B}(L_q) \) | 1 iff | 2 iff | 3 iff | 4 iff | 5 iff | 6 iff | 7 iff | 8 iff | 9 iff | 10 iff | 11 iff | 12 iff |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | * | 2 | 3 | * | * | * | * | * | * | * | * | * |
| 2 | * | 4 | 4 | 4 | * | * | * | * | * | * | * | * |
| 3 | 5 | 6 | 6 | 6 | * | * | * | * | * | * | * | * |
| 4 | * | 7 | 7 | 7 | * | * | * | * | * | * | * | * |
| 5 | 8 | * | 9 | * | * | * | * | * | * | * | * | * |
| 6 | 11 | * | 12 | * | * | * | * | * | * | * | * | * |

Hold, and the necessary and sufficient conditions for each characterization of the new classes are given by:

Table 7. Characterization of the classes \( (E_1; \tilde{B}(E_2)) \), where \( E_1, E_2 \in \{M_u, C_p, C_{bp}, C_r, L_q \} \).

| From \( E_1 \) \( \tilde{B}(E_1) \) \( \tilde{B}(M_u) \) \( \tilde{B}(C_p) \) \( \tilde{B}(C_{bp}) \) \( \tilde{B}(C_r) \) \( \tilde{B}(L_q) \) | 1 iff | 2 iff | 3 iff | 4 iff | 5 iff | 6 iff | 7 iff | 8 iff | 9 iff | 10 iff | 11 iff | 12 iff |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | (9) | (10) | (9) | (9) | (9) | (9) | (9) | (9) | (10) | (9) | (10) | (10) |
| 2 | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (11) | (16) | (13) | (17) |
| 3 | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (10) | (11) | (16) | (13) | (17) |

5. Conclusions

In this work, we first defined the sequentially defined 4D infinite band matrix \( \tilde{B} = (\tilde{b}_{mnkl}) \) and the new double sequence spaces \( \tilde{B}(M_u), \tilde{B}(C_p), \tilde{B}(C_{bp}), \tilde{B}(C_r), \text{ and } \tilde{B}(L_q) \) whose \( \tilde{B} \)-transforms are in the spaces \( M_u, C_p, C_{bp}, C_r, L_q \) respectively. Then, we proved the isomorphism between the new double sequence spaces and some double sequence spaces. We also established some inclusion relations subject to some strict conditions. Moreover, we determined the dual spaces of our new double sequence spaces, and finally, we characterized some new 4D matrix classes related to the spaces \( \tilde{B}(M_u), \tilde{B}(C_p), \tilde{B}(C_{bp}), \tilde{B}(C_r), \text{ and } \tilde{B}(L_q) \) \( 0 < q < \infty \). The results stated in this paper were much more inclusive than the consequences in the works [11,25,27,28]. As a natural continuation of this paper and [21,22,24,26], calculating the \( \tilde{B} = (\tilde{b}_{mnkl}) \) domain on almost convergent and strongly almost convergent double sequence spaces \( \tilde{B}(C_f) \) and \( \tilde{B}[C_f] \) is still an open problem.
Author Contributions: O.T. developed the project statement and theoretical formalism and performed the analytic calculations and the numerical solutions. Both V.R. and E.M. contributed to the final version of the manuscript and supervised the project. All authors have read and agreed to the published version of the manuscript.

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Abbreviations

The following abbreviations are used in this manuscript:

4D four-dimensional

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