Zero-Hopf Bifurcations of 3D Quadratic Jerk System

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Abstract: This paper is devoted to local bifurcations of three-dimensional (3D) quadratic jerk system. First, we start by analysing the saddle-node bifurcation. Then we introduce the concept of canonical system. Next, we study the transcritical bifurcation of canonical system. Finally we study the zero-Hopf bifurcations of canonical system, which constitutes the core contributions of this paper. By averaging theory of first order, we prove that, at most, one limit cycle bifurcates from the zero-Hopf equilibrium. By averaging theory of second order, third order, and fourth order, we show that, at most, two limit cycles bifurcate from the equilibrium. Overall, this paper can help to increase our understanding of local behaviour in the jerk dynamical system with quadratic non-linearity.

Keywords: jerk system; limit cycle; zero-Hopf equilibrium; averaging theory

MSC: Primary 34C05; 34C07; 34C23; 34C29; 37G10; 37G15

1. Introduction

Consider the following system of ordinary differential equations

\[
\frac{dx}{dt} = f(x, \mu), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^s,
\]

where \( f \) is sufficiently smooth. Let us assume that \( x = 0, \mu = 0 \) is a bifurcation point of the system. The corresponding linear variational equation is

\[
\frac{dx}{dt} = Ax, \quad A = Df(0,0),
\]

where \( Df \) denotes the Jacobian matrix of the vector field \( f \).

When \( n = 3 \), system (1) is of particular interest for bifurcation analysis and chaos, see [1]. Assume that \( A \) has a pair of purely imaginary eigenvalues \( \lambda_{1,2} = \pm \omega i \). The other eigenvalue \( \lambda_3 \) must be real. Thus, limit cycles may be found from the system under appropriate conditions. Recall that a limit cycle is an isolated closed orbit in the set of all periodic orbits of the system. In the case of \( \lambda_3 = 0 \), small-amplitude limit cycles may be found in some neighborhood of the origin. This phenomenon is called the zero-Hopf bifurcation.

A jerk equation is a differential equation of the form

\[
\frac{d^3x}{dt^3} = J(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}).
\]
Letting $y = \frac{dx}{dt}$, $z = \frac{d^2x}{dt^2}$, the jerk equation can be transformed into

$$\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = z, \\
\frac{dz}{dt} = f(x, y, z),
\end{cases}$$

which is called the jerk system. In physics, the first three derivatives $\frac{dx}{dt}$, $\frac{d^2x}{dt^2}$, $\frac{d^3x}{dt^3}$ are called velocity, acceleration, and jerk, respectively. The system can exhibit both regular and irregular or chaotic dynamical behaviour. It is shown in [2] that both Lorenz system and Rössler system could be written in jerk form.

Consider the general three-dimensional (3D) quadratic jerk system

$$\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = z, \\
\frac{dz}{dt} = a_0 + a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 xy + a_6 xz + a_7 y^2 + a_8 yz + a_9 z^2,
\end{cases}$$

which has attracted intense interest. Some examples of Hopf bifurcation analysis can be found in [3,4]. Some examples of zero-Hopf bifurcation analysis can be found in [5–7]. Complex dynamics, such as self-excited and hidden chaotic attractors, can be found in [7–9]. In these studies, the qualitative features of equilibria play an important role in determining the complex behaviour of the system. For convenience, we call this system the general system from now on.

The rest of the paper is organized, as follows. In Section 2, we study the presence of saddle-node bifurcation in the general system. The next section is about the canonical system, which will play an important role in bifurcation analysis. In Sections 4 and 5, for the canonical system, transcritical bifurcation and zero-Hopf bifurcations are studied in detail.

2. Saddle-Node Bifurcation

**Lemma 1.** Consider the quadratic function $\varphi(\lambda) = \lambda^2 + s_1 \lambda + s_2$. It has all roots with non-zero real parts if and only if

$$s_1^2 - 4s_2 > 0, s_2 \neq 0;$$

or

$$s_1^2 - 4s_2 \leq 0, s_1 \neq 0.$$

**Proof.** Let $\Delta = s_1^2 - 4s_2$. There are three possibilities:

1. If $\Delta > 0$, then the function has two distinct real roots $\lambda_1, \lambda_2$. Because $\lambda_1 \lambda_2 = s_2$, both roots are non-zero if and only if $s_2 \neq 0$.
2. If $\Delta = 0$, then the function has one real double root $\lambda_0 = -\frac{s_1}{2}$. The root is non-zero if and only if $s_1 \neq 0$.
3. If $\Delta < 0$, then the function has two complex conjugated roots $\lambda_1, \lambda_2$. Because the real parts of these roots are $-\frac{s_1}{2}$, $\varphi(\lambda)$ has no roots with zero real parts if and only if $s_1 \neq 0$.

Summing up, we have proved this lemma. □
Lemma 2 (Sotomayor’s theorem ([10], page 338–339)). Consider system (1) with \( s = 1 \). When \( \mu = \mu_0 \), assume that there is an equilibrium \( x_0 \), for which the following hypotheses are satisfied:

(a) The Jacobian matrix \( M = Df(x_0, \mu_0) \) has a simple eigenvalue \( \lambda = 0 \) with an eigenvector \( v \), and \( M^T \) has an eigenvector \( w \) corresponding to \( \lambda = 0 \).

(b) \( M \) has \( k \) eigenvalues with negative real parts, and \( n - 1 - k \) eigenvalues with positive real parts, where \( 1 \leq k \leq n - 1 \).

(c) \( \alpha \triangleq w^T f_{\mu}(x_0, \mu_0) \neq 0 \).

(d) \( \beta \triangleq w^T [D^2 f(x_0, \mu_0) (v, v)] \neq 0 \).

Subsequently, system (1) exhibits a saddle-node bifurcation at \( x_0 \) as \( \mu \) passes through \( \mu = \mu_0 \).

1. There is a smooth curve of equilibria in \( \mathbb{R}^n \times \mathbb{R} \) passing through \( (x_0, \mu_0) \) and tangent to the hyperplane \( \mathbb{R}^n \times \{ \mu_0 \} \).

2. If \( \alpha \beta < 0 \) (resp. \( \alpha \beta > 0 \)), there are no equilibria near \( x_0 \) when \( \mu < \mu_0 \) (resp. \( \mu > \mu_0 \)), and two equilibria near \( x_0 \) when \( \mu > \mu_0 \) (resp. \( \mu < \mu_0 \)).

3. The two equilibria near \( x_0 \) are hyperbolic and they have stable manifolds of dimensions \( k \) and \( k + 1 \), respectively.

Let \( \mu = a_1^2 - 4a_0a_4 \) with \( a_4 \neq 0 \), then system (3) becomes

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= z, \\
\frac{dz}{dt} &= \frac{a_1^2 - \mu}{4a_4} + a_1x + a_2y + a_3z + a_4x^2 + a_5yx + a_6zx + a_7y^2 + a_8zy + a_9z^2.
\end{align*}
\]

It has two equilibria if \( \mu > 0 \); one equilibrium at \( E = (-\frac{a_1}{a_4}, 0, 0) \) if \( \mu = 0 \); and, no equilibria if \( \mu < 0 \).

Lemma 3. For system (4), a saddle-node bifurcation occurs at \( E \) as the parameter \( \mu \) passes through \( \mu = 0 \).

Proof. When \( \mu = 0 \), the Jacobian matrix of system (4) at the equilibrium \( E \) is

\[
M := Df(E, 0) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & a_2 - \frac{a_5 a_3}{2a_4} & a_3 - \frac{a_4 a_6}{2a_4}
\end{pmatrix}.
\]

The matrix \( M = (m_{ij})_{3 \times 3} \) has a simple eigenvalue \( \lambda_1 = 0 \) with eigenvectors \( v = (1, 0, 0)^T \), and the matrix \( M^T \) has an eigenvector

\[
w = \begin{pmatrix}
-m_{32} \\
-m_{33} \\
1
\end{pmatrix}
\]
corresponding to \( \lambda_1 = 0 \).

The other eigenvalues \( \lambda_{2,3} \) of \( M \) are the roots of \( \phi(\lambda) := \lambda^2 + s_1 \lambda + s_2 \), where

\[
s_1 = -m_{33}, \ s_2 = -m_{32}.
\]

Note that \( m_{32} \) depends on \( a_2 \), and \( m_{33} \) depends on \( a_3 \). Thus, according to Lemma 1, we can chose \( a_2, a_3 \), such that the eigenvalues \( \lambda_{2,3} \) can not have zero real parts.
Following Lemma 2, we have

\[ \alpha = w^T f_\mu(E, 0) = - \frac{1}{4a_4} \neq 0, \]
\[ \beta = w^T [D^2 f(E, 0)(v, v)] = 2a_4 \neq 0. \]

Therefore, system (4) experiences a saddle-node bifurcation at the equilibrium \( E \) as the parameter \( \mu \) passes through \( \mu = 0 \). Because \( \alpha \beta < 0 \), there are no equilibria near \( E \) when \( \mu < 0 \), and two equilibria near \( E \) when \( \mu > 0 \).

**Theorem 1.** For the jerk system (3), a saddle-node bifurcation occurs at \( E \) as the parameter \( a_0 \) passes through \( a_0 = \frac{a_1^2}{4a_4} \).

**Proof.** Recall that \( \mu = a_1^2 - 4a_0a_4 \), thus \( a_0 = \frac{a_1^2}{4a_4} - \frac{\mu}{4a_4} \). This theorem is a direct consequence of Lemma 3.

**3. Canonical System**

**Theorem 2.** Suppose that system (3) has an equilibrium, then it can be transformed into the following system

\[
\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= z, \\
\frac{dz}{dt} &= (c_1y - c_2z) + c_3(x + z) + c_4(y^2 + z^2) + c_5(y^2 - z^2) - 2c_6yz \\
&\quad + (c_7y - c_8z)(x + z) + c_9(x + z)^2,
\end{aligned}
\]

where the coefficients \( c_i \), \( i = 1, 2, \cdots, 9 \) can be derived from the original system and equilibrium.

**Proof.** Assume that \( E : (x_0, 0, 0) \) is the equilibrium. Applying the translation \( x \rightarrow x + x_0, y \rightarrow y, z \rightarrow z \), the system becomes

\[
\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= z, \\
\frac{dz}{dt} &= (2a_4x_0 + a_1)x + (a_5x_0 + a_2)y + (a_6x_0 + a_3)z + a_4x^2 + a_5xy + a_6xz \\
&\quad + a_7y^2 + a_8yz + a_9z^2.
\end{aligned}
\]

After a little algebra, it is easy to see that system (6) can be arranged as system (5) with

\[
\begin{aligned}
c_1 &= a_5x_0 + a_2, \\
c_2 &= 2a_4x_0 - a_6x_0 + a_1 - a_3, \\
c_3 &= 2a_4x_0 + a_1, \\
c_4 &= \frac{a_4 - a_6 + a_7 + a_9}{2}, \\
c_5 &= \frac{a_6 + a_7 - a_4 - a_9}{2}, \\
c_6 &= \frac{a_5 - a_8}{2}, \\
c_7 &= a_5.
\end{aligned}
\]
Thus we complete the proof. □

For convenience, from now on, we call system (5) the canonical system.

4. Transcritical Bifurcation

Lemma 4 (Sotomayor’s theorem ([10], page 338–339)). Consider system (1) with \( s = 1 \) and assume that there is a point \( x_0 \in \mathbb{R}^n \), such that \( f(x_0, \mu) = 0 \) for all \( \mu \). Furthermore, when \( \mu = \mu_0 \) suppose that the following hypotheses hold:

(a) The Jacobian matrix \( M = Df(x_0, \mu_0) \) has a simple eigenvalue \( \lambda_1 = 0 \) with an eigenvector \( v \), and \( M^T \) has an eigenvector \( w \) corresponding to \( \lambda_1 = 0 \).

(b) \( M \) has \( k \) eigenvalues with negative real parts, and \( n - 1 - k \) eigenvalues with positive real parts, where \( 1 \leq k \leq n - 1 \).

(c) \( w^T f_{\mu}(x_0, \mu_0) = 0 \).

(d) \( w^T [Df_{\mu}(x_0, \mu_0) v] \neq 0 \).

(e) \( w^T [D^2 f(x_0, \mu_0) (v, v)] \neq 0 \).

Subsequently, system (1) exhibits a transcritical bifurcation at the equilibrium \( x_0 \) as \( \mu \) passes through \( \mu = 0 \).

Theorem 3. Consider the canonical system (5) with \( c_9 \neq 0 \), a transcritical bifurcation occurs at the origin as \( c_3 \) passes through \( c_3 = 0 \).

Proof. When \( c_3 = 0 \), the Jacobian matrix of this system at the origin is

\[
Df(0, 0) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & c_1 & -c_2 \\
\end{pmatrix}.
\]

It has a simple eigenvalue \( \lambda_1 = 0 \) with eigenvector \( v = (1, 0, 0)^T \), and its transpose has an eigenvector

\[
w = \begin{pmatrix}
-c_1 \\
c_2 \\
1 \\
\end{pmatrix}.
\]

corresponding to \( \lambda_1 = 0 \).

The other eigenvalues \( \lambda_{2,3} \) of the matrix are the roots of \( \varphi(\lambda) := \lambda^2 + c_2 \lambda - c_1 \). Thus, according to Lemma 1, one can choose \( c_1, c_2 \), such that these eigenvalues cannot have zero real parts.

Following Lemma 4, we have

\[
\begin{align*}
w^T f_{c_3}(0, 0) &= 0, \\
w^T [Df_{c_3}(0, 0) v] &= 1, \\
w^T [D^2 f(0, 0) (v, v)] &= 2c_9 \neq 0.
\end{align*}
\]

Therefore, the canonical system experiences a transcritical bifurcation at the origin as \( c_3 \) passes through \( c_3 = 0 \). □
5. Zero-Hopf Bifurcations

For the canonical system, we are interested in the number of small limit cycles bifurcate from the zero-Hopf equilibrium. There is no general theory of this problem. By perturbing an equilibrium inside the canonical system and using averaging theory up to fourth order, we give a partial answer to the problem. For averaging theory of higher order, see Appendix A.

5.1. The Perturbed System in Cartesian Coordinates

The Jacobian matrix of canonical system at the origin is

\[ Df(0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_3 & c_1 & -c_2 + c_3 \end{pmatrix}, \]

whose characteristic polynomial is

\[ \varphi(\lambda; c_1, c_2, c_3) := \lambda^3 + (c_2 - c_3) \lambda^2 - c_1 \lambda - c_3 \]

(7)

Let \( I_1(c_1, c_2, c_3) = c_2 - c_3, I_2(c_1, c_2, c_3) = -c_1, I_3(c_1, c_2, c_3) = -c_3 \) be some coefficients of the polynomial in \( \lambda \).

Recall that a zero-Hopf equilibrium of a 3D system is an isolated equilibrium of the system, whose linear part at the equilibrium has a pair of purely imaginary eigenvalues and a zero eigenvalue. In the next lemma, we characterize when the equilibrium localized at the origin of canonical system is a zero-Hopf equilibrium.

**Lemma 5.** For the canonical system, the origin is a zero-Hopf equilibrium if and only if

\[ c_2 = c_3 = 0, c_1 < 0. \]

**Proof.** Suppose that the canonical system has a zero-Hopf equilibrium at the origin. According to Lemma 5, we have

\[ c_2 = c_3 = 0, c_1 < 0. \]

which is equivalent to (8). Hence, we have proved this lemma. \( \square \)

**Lemma 6.** Suppose that the canonical system has a zero-Hopf equilibrium at the origin. Subsequently, it can be transformed into the following system

\[
\begin{align*}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= z, \\
\frac{dz}{dt} &= -y + b_4 \left( y^2 + z^2 \right) + b_5 \left( y^2 - z^2 \right) - 2 b_6 y z + (b_7 y - b_8 z) (x + z) + b_9 (x + z)^2,
\end{align*}
\]

(9)

where \( b_i, i = 4, 5, \ldots, 9 \) depend on \( c_i, i = 1, 4, 5, \ldots, 9 \).

**Proof.** Suppose that the canonical system has a zero-Hopf equilibrium at the origin. According to Lemma 5, we have

\[ c_2 = c_3 = 0, c_1 < 0. \]
Setting $\omega = \sqrt{-c_1}$, the canonical system becomes

$$
\begin{align*}
\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = z, \\
\frac{dz}{dt} = -\omega^2 y + c_4 \left( y^2 + z^2 \right) + c_5 \left( y^2 - z^2 \right) - 2 c_6 y z + (c_7 y - c_8 z) (x + z) + c_9 (x + z)^2.
\end{cases}
\end{align*}
$$

With the following linear scaling

$$
x \to \frac{1}{\omega} x, \quad z \to \omega z, \quad t \to \frac{1}{\omega} t,
$$

system (10) becomes

$$
\begin{align*}
\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = z, \\
\frac{dz}{dt} = -y + \text{quadratic terms}.
\end{cases}
\end{align*}
$$

Similar to the proof of Theorem 2, it is easy to see that system (12) can be arranged as system (9). Hence, we have completed the proof. \(\Box\)

Consider the following system

$$
\begin{align*}
\begin{cases}
\frac{dx}{dt} = y, \\
\frac{dy}{dt} = z, \\
\frac{dz}{dt} = -y + f(x, y, z) + \sum_{k=1}^{3} \epsilon^k \left( g_{2k-1}(x, y, z) + g_{2k}(x, y, z) \right) + \epsilon^4 g_7(x, y, z) \\
&+ \epsilon^4 \mathcal{O}(|x, y, z|^2) + \mathcal{O}(\epsilon^5),
\end{cases}
\end{align*}
$$

where

\[
\begin{align*}
 f(x, y, z) &= b_4 (y^2 + z^2) + b_5 (y^2 - z^2) - 2b_6 y z + (b_7 y - b_8 z) (x + z) + b_9 (x + z)^2, \\
g_1(x, y, z) &= (b_1 y - b_2 z) + b_3 (x + z), \\
g_2(x, y, z) &= b_{10} (y^2 + z^2) + b_{11} (y^2 - z^2) - 2b_{12} y z + (b_{13} y - b_{14} z) (x + z) + b_{15} (x + z)^2, \\
g_3(x, y, z) &= (b_{16} y - b_{17} z) + b_{18} (x + z), \\
g_4(x, y, z) &= b_{19} (y^2 + z^2) + b_{20} (y^2 - z^2) - 2b_{21} y z + (b_{22} y - b_{23} z) (x + z) + b_{24} (x + z)^2, \\
g_5(x, y, z) &= (b_{25} y - b_{26} z) + b_{27} (x + z), \\
g_6(x, y, z) &= b_{28} (y^2 + z^2) + b_{29} (y^2 - z^2) - 2b_{30} y z + (b_{31} y - b_{32} z) (x + z) + b_{33} (x + z)^2, \\
g_7(x, y, z) &= (b_{34} y - b_{35} z) + b_{36} (x + z),
\end{align*}
\]

and $\epsilon$ is sufficiently small.

5.2. Linear Analysis

When $\epsilon = 0$, then the Jacobian matrix of system (13) at the origin has three eigenvalues $\lambda_{1,2} = \pm i$ and $\lambda_3 = 0$. Under appropriate conditions, several limit cycles can bifurcate from the origin of the system for $|\epsilon| > 0$ sufficiently small. This type of bifurcation can be referred to as a zero-Hopf bifurcation.
The Jacobian matrix of system (13) at the origin has characteristic polynomial in $\lambda$:

$$
\varphi(\lambda; \varepsilon) := \lambda^3 + \varepsilon (b_2 - b_3) \lambda^2 + (1 - b_1 \varepsilon) \lambda - b_3 \varepsilon + O(\varepsilon^2).
$$

(14)

Let $\lambda_{1,2,3}(\varepsilon)$ be the roots of this polynomial, $\lambda_{1,2}(\varepsilon)$ be the continuous extension of $\pm i$, and $\lambda_3(\varepsilon)$ be the continuous extension of 0.

Lemma 7.

$$
\frac{d \text{Re} \lambda_{1,2}}{d \varepsilon} \bigg|_{\varepsilon=0} = -\frac{b_2}{2}.
$$

Proof. By applying the implicit function theorem to (14), we have

$$
\frac{d \lambda_{1,2}}{d \varepsilon} \bigg|_{\varepsilon=0} = -\frac{\partial \varphi}{\partial \lambda} \bigg|_{\varepsilon=0, \lambda=\pm i} = -\frac{b_2}{2} \pm \frac{b_1}{2} i.
$$

(15)

Taking the real part of (15), we obtain the conclusion.

Lemma 8.

$$
\frac{d \lambda_3}{d \varepsilon} \bigg|_{\varepsilon=0} = b_3.
$$

Proof. By using the implicit function theorem, we have

$$
\frac{d \lambda_3}{d \varepsilon} \bigg|_{\varepsilon=0} = -\frac{\partial \varphi}{\partial \lambda} \bigg|_{\varepsilon=0, \lambda=0} = b_3.
$$

(16)

Thus the conclusion follows.

The following theorem is about the stability of the origin for $|\varepsilon| > 0$ sufficiently small.

Theorem 4. The origin is asymptotically stable if one of the following conditions holds:

(1) $\varepsilon < 0$, $b_2 < 0$, $b_3 > 0$;

(2) $\varepsilon > 0$, $b_2 > 0$, $b_3 < 0$,

where $|\varepsilon|$ is sufficiently small.

Proof. Let us consider the first case. According to Lemmas 7 and 8, we have

$$
\frac{d \text{Re} \lambda_{1,2}}{d \varepsilon} \bigg|_{\varepsilon=0} > 0, \quad \frac{d \lambda_3}{d \varepsilon} \bigg|_{\varepsilon=0} > 0.
$$

Thus, for $\varepsilon < 0$ and near 0, we have

$$
\text{Re} \lambda_{1,2} < 0, \quad \lambda_3 < 0.
$$

Hence the origin is asymptotically stable.

The proof for the other case is similar. Thus, we complete the proof.

5.3. The Perturbed System in Cylindrical Coordinates

By using the linear transformation

$$
\begin{cases}
  x = \varepsilon v + \varepsilon w, \\
  y = \varepsilon u, \\
  z = -\varepsilon v,
\end{cases}
$$

(17)
system (13) becomes
\[ \begin{align*}
\frac{du}{dt} &= -v, \\
\frac{dv}{dt} &= u - G(\varepsilon, u, v, w), \\
\frac{dw}{dt} &= G(\varepsilon, u, v, w),
\end{align*} \tag{18} \]

where
\[ G(\varepsilon, u, v, w) = \sum_{k=1}^{4} \varepsilon^k G_k(u, v, w) + \mathcal{O}(\varepsilon^5), \]

and
\[ G_1(u, v, w) = b_1 u + b_2 v + b_3 w + b_4(u^2 + v^2) + b_5(u^2 - v^2) + 2b_6 uv + (b_7 u + b_8 v) w + b_9 w^2, \]
\[ G_2(u, v, w) = b_{16} u + b_{17} v + b_{18} w + b_{19}(u^2 + v^2) + b_{21}(u^2 - v^2) + 2b_{14} uv + (b_{13} u + b_{14} v) w + b_{15} w^2, \]
\[ G_3(u, v, w) = b_{25} u + b_{26} v + b_{27} w + b_{29}(u^2 + v^2) + b_{31}(u^2 - v^2) + 2b_{21} uv + (b_{22} u + b_{23} v) w + b_{24} w^2, \]
\[ G_4(u, v, w) = b_{34} u + b_{35} v + b_{36} w + b_{37}(u^2 + v^2) + b_{39}(u^2 - v^2) + 2b_{30} uv + (b_{31} u + b_{32} v) w + b_{33} w^2. \]

In cylindrical coordinates
\[ u = r \cos \theta, v = r \sin \theta, w = w, \tag{19} \]

system (18) becomes
\[ \begin{align*}
\frac{dr}{dt} &= -H(\varepsilon, \theta, r, w) \sin \theta, \\
\frac{d\theta}{dt} &= 1 - H(\varepsilon, \theta, r, w) \frac{\cos \theta}{r}, \\
\frac{dw}{dt} &= H(\varepsilon, \theta, r, w),
\end{align*} \tag{20} \]

where \( H(\varepsilon, \theta, r, w) = G(\varepsilon, r \cos \theta, r \sin \theta, w). \) Thus
\[ H(\varepsilon, \theta, r, w) = \sum_{k=1}^{4} \varepsilon^k h_k(\theta, r, w) + \mathcal{O}(\varepsilon^5), \]

where
\[ h_1(\theta, r, w) = (b_1 \cos \theta + b_2 \sin \theta) r + b_3 w + (b_4 + b_5 \cos 2\theta + b_6 \sin 2\theta) r^2 + (b_7 \cos \theta + b_8 \sin \theta) rw + b_9 w^2, \]
\[ h_2(\theta, r, w) = (b_{16} \cos \theta + b_{17} \sin \theta) r + b_{18} w + (b_{19} + b_{20} \cos 2\theta + b_{21} \sin 2\theta) r^2 + (b_{13} \cos \theta + b_{14} \sin \theta) rw + b_{15} w^2, \]
\[ h_3(\theta, r, w) = (b_{25} \cos \theta + b_{26} \sin \theta) r + b_{27} w + (b_{29} + b_{30} \cos 2\theta + b_{31} \sin 2\theta) r^2 + (b_{22} \cos \theta + b_{23} \sin \theta) rw + b_{24} w^2, \]
\[ h_4(\theta, r, w) = (b_{34} \cos \theta + b_{35} \sin \theta) r + b_{36} w + (b_{37} + b_{38} \cos 2\theta + b_{39} \sin 2\theta) r^2 + (b_{31} \cos \theta + b_{32} \sin \theta) rw + b_{33} w^2. \]

5.4. Standard Form of Fourth Order

Let \( x = (r, w). \) Subsequently, in the region \( r > 0, \) system (20) is equivalent to the following system
\[ \frac{dx}{d\theta} = \sum_{k=1}^{4} \varepsilon^k F_k(\theta, x) + \mathcal{O}(\varepsilon^5), \tag{21} \]

where \( F_k \) and the remainder \( \mathcal{O}(\varepsilon^5) \) are \( 2\pi \)-periodic in the first variable \( \theta. \)

Let us write the vector functions \( F_k(\theta, x) \) in terms of components, i.e.,
\[ F_k(\theta, x) = (F_{k,1}(\theta, r, w), F_{k,2}(\theta, r, w))^T, \quad k = 1, 2, 3, 4, \]

where
\[ F_{k,1}(\theta, r, w) = -\sin \theta F_{k,2}(\theta, r, w), \]
and
\[ F_{1,2}(\theta, r, w) = h_1, \]
\[ F_{2,2}(\theta, r, w) = h_2 + \frac{\cos \theta}{r} h_1^2, \]
\[ F_{3,2}(\theta, r, w) = h_3 + \frac{2 \cos \theta}{r} h_1 h_2 + \left( \frac{\cos \theta}{r} \right)^2 h_1^3, \]
\[ F_{4,2}(\theta, r, w) = h_4 + \frac{\cos \theta}{r} \left( h_2^2 + 2h_1 h_3 \right) + 3 \left( \frac{\cos \theta}{r} \right)^2 h_1^3 h_2 + \left( \frac{\cos \theta}{r} \right)^3 h_1^4. \]

5.5. First Order Averaging

Let \( b = (b_1, b_2, \ldots, b_{36}) \) and
\[ C^{(1)} = \left\{ b \in \mathbb{R}^{36} \mid b_2 b_4 (b_3 b_8 - b_2 b_9) > 0, b_8 \neq 0 \right\}. \]

**Theorem 5.** Suppose that \( b \in C^{(1)}. \) Subsequently, for any \( \epsilon \neq 0 \) sufficiently small, system (13) has one limit cycle (denote it by \( \Gamma_\epsilon \)) bifurcating from the origin.

**Proof.** Let us recall the averaging theory of higher order in the Appendix A. The first order averaged function of system (21) is
\[ f_1(x) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, x) d\theta = \left( -\frac{b_3 w + b_2}{2}, b_2 r^2 + b_9 w^2 + b_3 w \right). \]

By solving \( f_1(x) = 0, \) we obtain a unique solution \( x = (r^*, w^*), \) with
\[ r^* = \frac{1}{|b_8|} \sqrt{\frac{b_2 (b_3 b_8 - b_2 b_9)}{b_4}} > 0, \quad w^* = -\frac{b_2}{b_8}, \tag{22} \]
where \( b \in C^{(1)}. \) By computation, we find that \( \det \left[ D_x f_1(r^*, w^*) \right] = \frac{b_2 (b_3 b_8 - b_2 b_9)}{b_8} \neq 0. \) Thus, according to Theorem A1 of the Appendix A, for \( \epsilon \neq 0 \) sufficiently small, there exists a limit cycle \((r(\theta, \epsilon), w(\theta, \epsilon))\) for system (21), which converges to \((r^*, w^*)\) as \( \epsilon \to 0. \)

Going back through the transformations of coordinates (19) and (17), for \( \epsilon \neq 0 \) sufficiently small, system (13) has a limit cycle
\[ \Gamma_\epsilon : (x(\theta, \epsilon), y(\theta, \epsilon), z(\theta, \epsilon)) = \left( \epsilon r(\theta, \epsilon) \sin \theta + w(\theta, \epsilon), \epsilon r(\theta, \epsilon) \cos \theta, -\epsilon r(\theta, \epsilon) \sin \theta \right) \]

near the origin. \( \square \)

**Corollary 1.** Let
\[ p(\epsilon) = \frac{b_2 b_8 - 2 b_2 b_9}{b_8}, \quad q = \frac{b_2 (b_3 b_8 - b_2 b_9)}{b_8}. \]
Assuming that the condition of Theorem 5 holds. Subsequently, for \( |\epsilon| > 0 \) sufficiently small, we have the following results about the limit cycle \( \Gamma_\epsilon \) in the Theorem 5.

(a) If \( p(\epsilon) > 0, q > 0, \) then the limit cycle is a local repeller.
(b) If \( p(\epsilon) < 0, q > 0, \) then the limit cycle is a local attractor.
(c) If \( q < 0 \), then the limit cycle has two invariant manifolds, one stable and the other unstable, which are locally formed by two two-dimensional cylinders.

**Proof.** The Jacobian matrix of the first order averaged system

\[
\frac{dx}{dt} = \varepsilon f_1(x)
\]

at the equilibrium \((r^*, w^*)\) is

\[
J := \begin{pmatrix}
0 & -\frac{b_8 r^*}{2} \\
\frac{2b_4 r^* \varepsilon}{p(\varepsilon)} & 0
\end{pmatrix}.
\]

In view of the expression for \( r^* \) in (22), the trace and determinant of \( J \) are \( p(\varepsilon) \) and \( q \varepsilon^2 \), respectively. Therefore, we can determine the stability and instability of the equilibrium \((r^*, w^*)\). Thus, according to Llibre et al. [12] (Theorem 1.2.1), the three statements of this Corollary 1 hold. This completes the proof.

To end this subsection, let us introduce a notation for later use. Let \( D \) be a non-empty subset of \( \mathbb{R}^m \). Suppose that \( g : D \rightarrow \mathbb{R}^n \) is a function, depending on a parameter \( b \in \Lambda \subset \mathbb{R}^s \). Let \( \Lambda_0 \) be a non-empty subset of \( \Lambda \), we can use \( g\big|_{\Lambda_0}(x) \) to denote the function of \( g \) restricted to the parameter domain \( \Lambda_0 \). It is acceptable to omit the subscript if there is no risk of confusion.

For an example, let

\[
B^{(1)} = \left\{ b \in \mathbb{R}^{36} \big| f_1(x) = f_1(r, w) \equiv 0 \right\}
\]

\[
= \left\{ b \in \mathbb{R}^{36} \big| b_2 = b_3 = b_4 = b_8 = b_9 = 0 \right\}.
\]

Subsequently, we have \( f_1\big|_{B^{(1)}}(x) \equiv 0 \).

### 5.6. Second Order Averaging

Let

\[
\begin{aligned}
a_0 &= -\frac{b_{17} (b_{14}b_{18} - b_{15}b_{17})}{b_{14}^2}, \\
a_2 &= \frac{2b_1 b_5 b_{14}^2 - 2b_5 b_7 b_{14} b_{17} - b_5 b_7 b_{14} b_{18} + 2b_5 b_7 b_{15} b_{17} + 4b_{10} b_{14}^2}{4b_{14}^2}, \\
a_4 &= \frac{b_5^2 b_7^2 (b_{15} - 2b_{14})}{16b_{14}^2}.
\end{aligned}
\]

Define

\[
C^{(2)} = \left\{ b \in B^{(1)} \big| a_2 a_4 < 0, a_0 a_4 > 0, a_2^2 - 4a_0 a_4 > 0, b_{14} \neq 0 \right\}.
\]

**Lemma 9.** The functions \( a_0, a_2, a_4 \) are linearly independent.

**Proof.** By direct computation, we have

\[
\det \left( \frac{\partial (a_0, a_2, a_4)}{\partial (b_5, b_{10}, b_{15})} \right) \neq 0.
\]

Thus, these functions are linearly independent.

**Theorem 6.** Assume that \( b \in B^{(1)} \). Subsequently, for \( |\varepsilon| > 0 \) sufficiently small, system (13) can have, at most, two limit cycles by averaging theory of second order. Moreover, if \( b \in C^{(2)} \), then the bound can be reached.
Proof. According to Appendix A, we can compute the second order averaged function of system (21), as follows

\[ f_2(x) = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ F_2(\theta, x) + D_x F_1(\theta, x) y_1(\theta, x) \right] d\theta = (f_{2,1}, f_{2,2})^T, \]

where

\[ y_1(\theta, x) = \int_{\theta}^{0} F_1(s, x) ds = (y_{1,1}, y_{1,2})^T. \]

Suppose that \( b \in B^{(1)} \), we have

\[
\begin{align*}
  y_{1,1} & = \frac{b_1 (-1 + \cos 2\theta)}{4} r + \left( \frac{b_5}{3} - \frac{b_5 \cos \theta}{2} + \frac{b_5 \cos 3\theta}{6} + \frac{b_6 \sin 3\theta}{6} - \frac{b_6 \sin \theta}{2} \right) r^2 + \frac{b_7 (-1 + \cos 2\theta)}{4} w, \\
  y_{1,2} & = b_1 r \sin \theta + \left( \frac{b_6}{2} + \frac{b_5 2\theta}{2} - \frac{b_6 \cos 2\theta}{2} \right) r^2 + b_7 w \sin \theta.
\end{align*}
\]

Thus, we obtain

\[
\begin{align*}
  f_{2,1} &= -\frac{b_1}{2} r - \frac{b_5}{2} w r - \frac{b_5 b_7}{8} r^3, \\
  f_{2,2} &= b_{18} w + \left( \frac{b_1 b_5}{2} + b_{10} \right) r^2 + b_{15} w^2 + \frac{b_5 b_7}{2} w^2.
\end{align*}
\]

Note that we are interested in the common solutions of these two polynomials for \((r, w)\) with \(r > 0\).

Solving \( f_{2,1} = 0 \) for \( w \), we obtain

\[ w = \frac{b_{17}}{b_{14}} - \frac{b_5 b_7}{4 b_{14}} r^2. \]

Substituting (26) into (25), we get

\[ P(r^2) := a_4 r^4 + a_2 r^2 + a_0, \]

where \( P \) is a quadratic polynomial in the variable \( r^2 \). Therefore \( f_2(x) = 0 \) can have at most two isolated solutions with \( r > 0 \). Hence system (13) can have at most two limit cycles bifurcating from the origin.

By Lemma 9, the functions \( a_0, a_2, a_4 \) are linearly independent. Therefore \( C^{(2)} \neq \emptyset \). Assume that \( b \in C^{(2)} \), i.e., the discriminant of \( P \) with respect to \( r^2 \) is positive and the number of changes in sign of the coefficients of the terms of \( P(r^2) \) is two. According to Descartes’ rule of sign [13], \( P \) has two distinct positive roots (simple roots) for \( r^2 \), which implies two positive roots \( r = r_{1,2} \) for \( r \). Therefore, the original system \( f_{2,1} = f_{2,2} = 0, r > 0 \) can have two solutions \((r, w) = (r_i, w_i), i = 1, 2\). It is easy to check that \( \det \left( \frac{\partial (f_{2,1}, f_{2,2})}{\partial (r, w)} \right) \bigg|_{(r, w) = (r_i, w_i)} \neq 0, i = 1, 2 \). For the general argument, see Appendix B.

Summing up, according to Theorem A1, there exist two limit cycles in the system (21). The correspondingly system (13) has two limit cycles bifurcating from the origin. \( \square \)

Before ending the subsection, we present some notations for later use. Let

\[
\begin{align*}
  B^{(2)} & = \left\{ b \in B^{(1)} \left| f_2(x) = f_2(r, w) = 0 \right. \right\}, \\
  B_1^{(2)} & = \left\{ b \in B^{(1)} \left| b_5 = b_{10} = b_{14} = b_{15} = b_{17} = b_{18} = 0 \right. \right\}, \\
  B_2^{(2)} & = \left\{ b \in B^{(1)} \left| b_7 = b_{14} = b_{15} = b_{17} = b_{18} = 0, b_{10} = -\frac{b_1 b_5}{2} \right. \right\}.
\end{align*}
\]

It can be checked that

\[ B^{(2)} = B_1^{(2)} \cup B_2^{(2)}. \]
5.7. Third Order Averaging

According to Appendix A, the third order averaged function \( f_3(x) \) of system (21) is defined to be

\[
f_3(x) = \frac{1}{2\pi} \int_0^{2\pi} [F_3(\theta, x) + D_x F_2(\theta, x)y_1(\theta, x) + \frac{1}{2} D_x^2 F_1(\theta, x)(y_1(\theta, x), y_1(\theta, x)) + \frac{1}{2} D_x F_1(\theta, x)y_2(\theta, x)] d\theta	ag{28}
\]

where

\[
y_2(\theta, x) = 2 \int_0^\theta [F_2(s, x) + D_x F_1(s, x)y_1(s, x)] ds.
\]

Let

\[
\begin{align*}
\beta_0^{(1)} &= -\frac{b_{26}}{b_{23}} (b_{23}b_{27} - b_{24}b_{26}), \\
\beta_2^{(1)} &= \frac{2b_1b_{21}b_{23}b_{25} - 2b_7b_{21}b_{23}b_{27} - 2b_7b_{21}b_{23}b_{27} + 2b_7b_{21}b_{23}b_{27} + 4b_9b_{23}^2}{4b_{23}^2}, \\
\beta_4^{(1)} &= \frac{b_12b_7^2(b_{24} - 2b_{23})}{16b_{23}^2}.
\end{align*}
\]

Define

\[C_1^{(3)} = \left\{ b \in B_1^{(2)} \mid \beta_2^{(1)} \beta_4^{(1)} < 0, \beta_0^{(1)} \beta_4^{(1)} > 0, (\beta_2^{(1)})^2 - 4\beta_0^{(1)} \beta_4^{(1)} > 0, b_{23} \neq 0 \right\}.
\]

It is easy to check that \( \beta_0^{(1)}, \beta_2^{(1)}, \beta_4^{(1)} \) are linearly independent, thus \( C_1^{(3)} \neq \emptyset \).

**Theorem 7.** Assume that \( b \in B_1^{(2)} \). Subsequently, for \( |e| > 0 \) sufficiently small, system (13) can have, at most, two limit cycles by averaging theory of third order. Moreover, if \( b \in C_1^{(3)} \), then the bound can be reached.

**Proof.** By direct computation, from (28) we have

\[
\begin{align*}
f_{3,1}|_{B_1^{(2)}} &= -\frac{b_{26}}{2} r - \frac{b_{23}}{2} wr - \frac{b_7b_{11}}{8} r^3, \\
f_{3,2}|_{B_1^{(2)}} &= b_{27} w + \left( \frac{b_1b_{11}}{2} + b_{19} \right) r^2 + b_{24} w^2 + \frac{b_7b_{11}}{2} wr^2.
\end{align*}
\]

Note that we are interested in the common solutions of these two polynomials for \((r, w)\), with \( r > 0 \).

Solving (30) for \( w \), we obtain

\[
w = -\frac{b_{26}}{b_{23}} - \frac{b_7b_{11}}{4b_{23}} r^2.	ag{32}
\]

Substituting (32) into (31), we get

\[P(r^2) := \beta_4^{(1)} r^4 + \beta_2^{(1)} r^2 + \beta_0^{(1)},
\]

where \( P \) is a quadratic polynomial in the variable \( r^2 \).

The rest of proof is similar to the proof of Theorem 6. \( \square \)
Let
\[
\begin{aligned}
\beta_0^{(2)} &= -\frac{b_{26}(b_{23}b_{27} - b_{24}b_{26})}{b_{23}^2}, \\
\beta_2^{(2)} &= \frac{b_1^2b_5b_{23}^2 + 2b_1b_{11}b_{23}^2 - 2b_5b_{13}b_{23}b_{26} - b_5b_{13}b_{23}b_{27} + 2b_5b_{13}b_{24}b_{26} + 2b_5b_{16}b_{23}^2}{4b_{23}^2} \\
+ b_{19}, \\
\beta_4^{(2)} &= \frac{b_{13}^2b_5^2(b_{24} - 2b_{23})}{16b_{23}^2}.
\end{aligned}
\]  

(34)

Define
\[
C_2^{(3)} = \left\{ b \in B_2^{(2)} \mid \beta_2^{(2)}\beta_4^{(2)} < 0, \beta_0^{(2)}\beta_4^{(2)} > 0, (\beta_2^{(2)})^2 - 4\beta_0^{(2)}\beta_4^{(2)} > 0, b_{23} \neq 0 \right\}.
\]

It is easy to check that \(\beta_0^{(2)}, \beta_2^{(2)}, \beta_4^{(2)}\) are linearly independent, thus \(C_2^{(3)} \neq \emptyset\).

**Theorem 8.** Assume that \(b \in B_2^{(2)}\). Subsequently, for \(|c| > 0\) sufficiently small, system (13) can have, at most, two limit cycles by averaging theory of third order. Moreover, if \(b \in C_2^{(3)}\), then the bound can be reached.

**Proof.** By direct computation, from (28) we have
\[
\begin{aligned}
f_{3,1}|_{B_2^{(2)}} &= \frac{1}{2}b_{26}r - \frac{1}{2}b_{23}wr - \frac{1}{8}b_{13}b_5r^3, \\
f_{3,2}|_{B_2^{(2)}} &= b_{27}w + \left(\frac{1}{4}b_1^2b_5 + \frac{1}{2}b_1b_{11} + \frac{1}{2}b_5b_{16} + b_{19}\right)r^2 + b_{24}w^2 + \frac{1}{2}b_1b_5wr^2.
\end{aligned}
\]

(35)

(36)

Note that we are interested in the common solutions of these two polynomials for \((r, w)\) with \(r > 0\).

Solving (35) for \(w\), we get
\[
w = -\frac{b_{26}}{b_{23}} - \frac{b_5b_{13}r^2}{4b_{23}}.
\]

(37)

Substituting (37) into (36), we obtain
\[
P(r^2) := \beta_4^{(2)}r^4 + \beta_2^{(2)}r^2 + \beta_0^{(2)}.
\]

(38)

where \(P\) is a quadratic polynomial in the variable \(r^2\).

The rest of proof is similar to the proof of Theorem 6. \(\square\)

Before ending this subsection, we introduce some notations for later use. Let
\[
B^{(3)} = \left\{ b \in B^{(2)} \mid f_3(x) = f_3(r, w) \equiv 0 \right\},
\]
\[
B_{1,1}^{(3)} = \left\{ b \in B_1^{(2)} \mid b_7 = b_{23} = b_{24} = b_{26} = b_{27} = 0, b_{19} = -\frac{b_1b_{11}}{2} \right\},
\]
\[
B_{1,2}^{(3)} = \left\{ b \in B_1^{(2)} \mid b_{11} = b_{19} = b_{23} = b_{24} = b_{26} = b_{27} = 0 \right\},
\]
\[
B_{2,1}^{(3)} = \left\{ b \in B_2^{(2)} \mid b_5 = b_{23} = b_{24} = b_{26} = b_{27} = 0, b_{19} = -\frac{b_1b_{11}}{2} \right\},
\]
\[
B_{2,2}^{(3)} = \left\{ b \in B_2^{(2)} \mid b_{13} = b_{23} = b_{24} = b_{26} = b_{27} = 0, b_{19} = -\frac{1}{4}b_1^2b_5 - \frac{1}{2}b_1b_{11} - \frac{1}{2}b_5b_{16} \right\}.
\]
It is easy to check that
\[ B^{(3)} = B_{1,1}^{(3)} \cup B_{1,2}^{(3)} \cup B_{2,1}^{(3)} \cup B_{2,2}^{(3)} \].

### 5.8. Fourth Order Averaging

For system (21), the averaged function of fourth order is

\[
f_4(x) = \frac{1}{2\pi} \int_0^{2\pi} \left[ f_4(\theta, x) + D_x F_3(\theta, x) y_1(\theta, x) + \frac{1}{2} D_x F_2(\theta, x) y_2(\theta, x) \\
+ \frac{1}{4} D_x^2 F_2(\theta, x)(y_1(\theta, x), y_1(\theta, x)) + \frac{1}{2} D_x F_1(\theta, x) y_3(\theta, x) \\
+ \frac{1}{2} D_x^2 F_1(\theta, x)(y_1(\theta, x), y_2(\theta, x)) + \frac{1}{4} D_x^3 F_1(\theta, x)(y_1(\theta, x), y_1(\theta, x), y_1(\theta, x)) \right] d\theta
\]

(40)

where

\[
y_3(\theta, x) = \int_0^x \left( 6F_3(s, x) + 6D_x F_2(s, x)y_1(s, x) + 3D_x F_1(s, x)y_2(s, x) \\
+ 3D_x^2 F_1(s, x)(y_1(s, x), y_1(s, x)) \right) ds.
\]

By direct computation, from (40), we have

\[
f_{4,1} \big|_{g_{1,3}^{(3)}} = \frac{1}{8} b_{11} b_{13} r^3 - \frac{1}{2} b_{35} r - \frac{1}{2} b_{32} r w,
\]

\[
f_{4,2} \big|_{g_{1,3}^{(3)}} = \left( b_{28} + \frac{1}{4} b_{1} b_{11} + \frac{1}{2} b_{1} b_{20} + \frac{1}{2} b_{11} b_{16} \right) r^2 + \frac{1}{2} b_{11} b_{13} r^2 w + b_{33} w^2 + b_{36} w,
\]

\[
f_{4,1} \big|_{g_{1,2}^{(3)}} = \frac{1}{8} b_{1} b_{20} r^3 - \frac{1}{2} b_{35} r - \frac{1}{2} b_{32} r w,
\]

\[
f_{4,2} \big|_{g_{1,2}^{(3)}} = \left( b_{28} + \frac{1}{4} b_{1} b_{20} \right) r^2 + \frac{1}{2} b_{1} b_{20} r^2 w + b_{33} w^2 + b_{36} w,
\]

\[
f_{4,1} \big|_{g_{2,3}^{(3)}} = \frac{1}{8} b_{11} b_{13} r^3 - \frac{1}{2} b_{35} r - \frac{1}{2} b_{32} r w,
\]

\[
f_{4,2} \big|_{g_{2,3}^{(3)}} = \left( b_{28} + \frac{1}{4} b_{1} b_{11} + \frac{1}{2} b_{1} b_{20} + \frac{1}{2} b_{11} b_{16} \right) r^2 + \frac{1}{2} b_{11} b_{13} r^2 w + b_{33} w^2 + b_{36} w,
\]

\[
f_{4,1} \big|_{g_{2,2}^{(3)}} = -\frac{1}{8} b_{1} b_{22} r^3 - \frac{1}{2} b_{35} r - \frac{1}{2} b_{32} r w,
\]

\[
f_{4,2} \big|_{g_{2,2}^{(3)}} = \left( b_{28} + \frac{1}{4} b_{1} b_{22} + \frac{1}{2} b_{1} b_{20} + \frac{1}{2} b_{11} b_{16} + \frac{1}{2} b_{1} b_{22} + \frac{1}{2} b_{3} b_{25} + \frac{1}{8} b_{1}^3 b_{3} \right) r^2
\]

\[+ \frac{1}{2} b_{3} b_{22} r^2 w + b_{33} w^2 + b_{36} w.
\]

For \( b \in B_{1,1}^{(3)} \), we introduce

\[
\gamma_0^{(1)} = \frac{b_{13} b_{35}^2}{b_{32}^2} - \frac{b_{36} b_{35}}{b_{32}},
\]

\[
\gamma_1^{(1)} = b_{28} + \frac{1}{4} b_{1}^2 b_{11} + \frac{1}{2} b_{1} b_{20} + \frac{1}{2} b_{11} b_{16} - \frac{1}{4} b_{11} b_{13} \left( \frac{2 b_{35} + b_{36}}{b_{32}} \right) + \frac{1}{2} b_{13} b_{35} b_{13} b_{35},
\]

\[
\gamma_2^{(1)} = \frac{b_{11}^2}{8 b_{32}} + \frac{b_{33} b_{11} b_{13}^2}{16 b_{32}^2}.
\]
For \( b \in B_{1,2}^{(3)} \), we introduce

\[
\begin{align*}
\gamma_{0}^{(2)} &= \frac{b_{33}b_{35}^2}{b_{32}^2} - \frac{b_{36}b_{35}}{b_{32}}, \\
\gamma_{2}^{(2)} &= b_{28} + \frac{1}{2} b_{1}b_{20} - \frac{b_{7}b_{20} (2 b_{35} + b_{36})}{4 b_{32}} + \frac{b_{33}b_{7}b_{20}b_{35}}{2 b_{32}^2}, \\
\gamma_{4}^{(2)} &= -\frac{b_{7}^2b_{20}^2}{8 b_{32}} + \frac{b_{33}b_{7}^2b_{20}^2}{16 b_{32}^2}.
\end{align*}
\]

For \( b \in B_{2,1}^{(3)} \), we introduce

\[
\begin{align*}
\gamma_{0}^{(3)} &= \frac{b_{33}b_{35}^2}{b_{32}^2} - \frac{b_{36}b_{35}}{b_{32}}, \\
\gamma_{2}^{(3)} &= b_{28} + \frac{1}{4} b_{1}^2b_{11} + \frac{1}{2} b_{1}b_{20} + \frac{1}{2} b_{1}b_{11}b_{16} - \frac{b_{11}b_{13} (2 b_{35} + b_{36})}{4 b_{32}} + \frac{b_{33}b_{11}b_{13}b_{35}}{2 b_{32}^2}, \\
\gamma_{4}^{(3)} &= -\frac{b_{11}^2b_{13}^2}{8 b_{32}} + \frac{b_{33}b_{11}^2b_{13}^2}{16 b_{32}^2}.
\end{align*}
\]

Finally, for \( b \in B_{2,2}^{(3)} \), we introduce

\[
\begin{align*}
\gamma_{0}^{(4)} &= \frac{b_{33}b_{35}^2}{b_{32}^2} - \frac{b_{36}b_{35}}{b_{32}}, \\
\gamma_{2}^{(4)} &= b_{28} + \frac{1}{4} b_{1}^2b_{11} + \frac{1}{2} b_{1}b_{20} + \frac{1}{2} b_{1}b_{11}b_{16} + \frac{1}{2} b_{1}b_{5}b_{16} + \frac{1}{2} b_{5}b_{25} + \frac{1}{8} b_{1}^3b_{5} - \frac{b_{5}b_{22} (2 b_{35} + b_{36})}{4 b_{32}} + \frac{b_{33}b_{5}b_{22}b_{35}}{2 b_{32}^2}, \\
\gamma_{4}^{(4)} &= -\frac{b_{5}^2b_{22}^2}{8 b_{32}} + \frac{b_{33}b_{5}^2b_{22}^2}{16 b_{32}^2}.
\end{align*}
\]

Proceed as we did in previous subsections, we have the following result.

**Theorem 9.** Assume that \( b \in B^{(3)} \). Subsequently, for \( |e| > 0 \) sufficiently small, system (13) has, at most, two limit cycles by averaging theory of fourth order. Furthermore, the bound can be reached if

\[
b \in \bigcup_{s=1}^{4} C^{(4)}_s
\]

where

\[
C^{(4)}_1 = \left\{ \begin{array}{l}
b \in B^{(3)}_{1,1} \mid (\gamma_{2}^{(1)} \gamma_{4}^{(1)} < 0, \gamma_{0}^{(1)} \gamma_{4}^{(1)} > 0, (\gamma_{2}^{(1)})^2 - 4 \gamma_{0}^{(1)} \gamma_{4}^{(1)} > 0, b_{32} \neq 0 \end{array} \right\},
\]

\[
C^{(4)}_2 = \left\{ \begin{array}{l}
b \in B^{(3)}_{1,2} \mid (\gamma_{2}^{(2)} \gamma_{4}^{(2)} < 0, \gamma_{0}^{(2)} \gamma_{4}^{(2)} > 0, (\gamma_{2}^{(2)})^2 - 4 \gamma_{0}^{(2)} \gamma_{4}^{(2)} > 0, b_{32} \neq 0 \end{array} \right\},
\]

\[
C^{(4)}_3 = \left\{ \begin{array}{l}
b \in B^{(3)}_{2,1} \mid (\gamma_{2}^{(3)} \gamma_{4}^{(3)} < 0, \gamma_{0}^{(3)} \gamma_{4}^{(3)} > 0, (\gamma_{2}^{(3)})^2 - 4 \gamma_{0}^{(3)} \gamma_{4}^{(3)} > 0, b_{32} \neq 0 \end{array} \right\},
\]

\[
C^{(4)}_4 = \left\{ \begin{array}{l}
b \in B^{(3)}_{2,2} \mid (\gamma_{2}^{(4)} \gamma_{4}^{(4)} < 0, \gamma_{0}^{(4)} \gamma_{4}^{(4)} > 0, (\gamma_{2}^{(4)})^2 - 4 \gamma_{0}^{(4)} \gamma_{4}^{(4)} > 0, b_{32} \neq 0 \end{array} \right\}.
\]
6. Conclusions

In this work, some types of bifurcations of equilibria are studied for quadratic jerk system. First, we study the saddle-node bifurcation for the general jerk system. Subsequently, for convenience, we introduce the concept of canonical system. Finally, the other bifurcations: transcritical bifurcation and zero-Hopf bifurcation are also studied for canonical system. By using the averaging theory up to fourth order, we prove that at most two limit cycles bifurcate from the zero-Hopf equilibrium, and this bound is sharp.

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Appendix A. Higher Order Averaging Theory

In the section we recall higher order averaging theory for finding periodic solutions via Brouwer degree, see Llibre et al. [14]. For some other studies, we refer to [12,15–19].

Consider the differential system

\[
\frac{dx}{dt} = \sum_{i=1}^{k} \epsilon_i F_i(t,x) + \epsilon^{k+1} R(t,x,\epsilon),
\]

where \(F_i : \mathbb{R} \times D \rightarrow \mathbb{R}^n\), \(R : \mathbb{R} \times D \times (-\epsilon_0, \epsilon_0) \rightarrow \mathbb{R}^n\) are \(T\)-periodic in the first variable and Lipschitz in the second variable. Here, \(D\) is a bounded open subset of \(\mathbb{R}^n\). As usual \(\epsilon\) is a small parameter.

The averaging method introduces a list of functions \(f_i : D \rightarrow \mathbb{R}^n\), called averaged function of order \(i\), \(1 \leq i \leq k\). Using these functions, we can determine the limit cycles of system (A1) for \(|\epsilon| > 0\) sufficiently small. It is shown in [14] that

\[
f_i(z) = \frac{y_i(T,z)}{\epsilon^T},
\]

where \(y_i : \mathbb{R} \times D \rightarrow \mathbb{R}^n\), for \(i = 1, 2, \ldots, k\), are defined recursively as follows:

\[
y_1(t,z) = \int_0^t F_1(s,z)ds,
\]

\[
y_i(t,z) = i! \int_0^t \left( F_i(s,z) + \sum_{i=1}^{i-1} \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 \cdots \mathbf{b}_l \frac{1}{l!} \mathbf{b}_l D^i F_{i-l}(s,z) \bigcup_{j=1}^l y_j(s,z)^{b_j} \right) ds, \quad i \neq 1,
\]

where \(L = b_1 + b_2 + \cdots + b_l\), and \(S_l\) denotes the set of all \(l\)-tuples of non-negative integers \((b_1, b_2, \ldots, b_l)\) such that \(b_1 + 2b_2 + \cdots + lb_l = l\). Here, \(D^l F(t,x)\) denotes the \(L\)-th order Fréchet derivative of \(F\) with respect to the variable \(x\).

Theorem A1 (Llibre-Novaes-Teixeira ([14], Theorem A)). For the functions of (A1), we assume the following conditions.

(a) For each \(i \in \mathbb{R}\), \(F_i(t,\cdot) \in C^{k-i}\) for \(i = 1, 2, \ldots, k\); \(D^{k-i} F_i\) is locally Lipschitz in the second variable for \(i = 1, 2, \ldots, k\); and \(R\) is a continuous function locally Lipschitz in the second variable.

(b) There exists an integer \(r \in \{1, 2, \ldots, k\}\), such that \(f_i \equiv 0\), \(i = 1, 2, \ldots, r-1\), and \(f_r \neq 0\).

(c) For some \(\zeta \in D\) with \(f_r(\zeta) = 0\), there exists a neighbourhood \(V \subset D\) of \(\zeta\) such that \(f_r(z) \neq 0\) for all \(z \in \overline{V} \setminus \{\zeta\}\), and that \(d_B(f_r, V, 0) \neq 0\), where \(d_B(f_r, V, 0)\) is the Brouwer degree of \(f_r\) at 0 in the set \(V\).
Then, for \(|c| > 0\) sufficiently small, there exists a \(T\)-periodic solution \(x(\cdot, c)\) of (A1) such that \(x(0, c) \to \xi\) as \(c \to 0\).

**Remark A1.** Suppose that \(f_r \in C(\overline{D}) \cap C^1(D)\) and \(J_{f_r}(z) \neq 0\) for every \(z \in D\) satisfying \(f_r(z) = 0\), where \(J_{f_r}(z)\) is the Jacobian determinant of \(f\) at \(z\). Then the Brouwer degree of \(f_r\) at 0 is given by

\[
d_B(f_r, D, 0) := \sum_{\xi(0) = 0} \text{sgn} J_{f_r}(z).
\]

Thus \(J_{f_r}(\xi) \neq 0\) is a sufficient condition for \(d_B(f_r, V, 0) \neq 0\). For the general definition of Brouwer degree, we refer to [20].

**Appendix B. The Jacobian Determinant of Two Functions**

Let \(D \subset \mathbb{R}^2\) be an open set. Let \(P, Q\) are functions of \((x, y)\) defined on \(D\) and having continuous partial derivatives there. Suppose that \(\frac{\partial P}{\partial y} \neq 0\) on \(D\). By the implicit function theorem, we can find a unique solution \(y = y(x)\) to the equation \(P(x, y) = 0\). Substituting it into \(Q(x, y)\), we get a composite function \(R(x) := Q(x, y(x))\).

Suppose that \(R(x)\) has a simple root at \(x_0\). Then we have

\[
R'(x_0) = \frac{\partial Q}{\partial x}(x_0, y_0) + \frac{\partial Q}{\partial y}(x_0, y_0)y'(x_0)
\]

\[
= \frac{\partial Q}{\partial x}(x_0, y_0) - \frac{\partial Q}{\partial y}(x_0, y_0) \frac{\partial P}{\partial y}(x_0, y_0) \frac{\partial P}{\partial x}(x_0, y_0)
\]

\[
\neq 0, \quad (A5)
\]

where \(y_0 = y(x_0)\). From (A5), we can concluded that

\[
\text{det} \left( \frac{\partial (P, Q)}{\partial (x, y)} \right)_{(x,y)=(x_0,y_0)} \neq 0.
\]

**References**

1. Zeraoulia, E.; Sprott, J.C. 2-D Quadratic Maps and 3-D ODE Systems: A Rigorous Approach; World Scientific: Singapore, 2010.
2. Linz, S.J. Nonlinear dynamical models and jerky motion. *Am. J. Phys.* 1997, 65, 523–526. [CrossRef]
3. Innocenti, G.; Tesi, A.; Genesio, R. Complex behavior analysis in quadratic jerk systems via frequency domain hopf bifurcation. *Int. J. Bifurc. Chaos* 2010, 20, 657–667. [CrossRef]
4. Sang, B. Hopf Bifurcation Formulae and Applications to the Genesio-Tesi System. *J. Nonlinear Funct. Anal.* 2019, ID 34, 1–16.
5. Llibre, J.; Makhlouf, A. Zero-Hopf periodic orbit of a quadratic system of differential equations obtained from a third-order differential equation. *Differ. Equ. Dyn. Syst.* 2019, 27, 75–82. [CrossRef]
6. Sang, B. Zero-Hopf bifurcations and chaos of quadratic jerk systems. *J. Nonlinear Funct. Anal.* 2020, 2020, 25.
7. Wei, Z.C.; Zhang, W.; Yao, M.H. On the periodic orbit bifurcating from one single non-hyperbolic equilibrium in a chaotic jerk system. *Nonlinear Dyn.* 2015, 82, 1251–1258. [CrossRef]
8. Wang, X.; Chen, G.R. Constructing a chaotic system with any number of equilibria. *Nonlinear Dyn.* 2013, 71, 429–436. [CrossRef]
9. Zhang, S.; Zeng, Y.C. A simple Jerk-like system without equilibrium: Asymmetric coexisting hidden attractors, bursting oscillation and double full Feigenbaum remerging trees. *Chaos Solitons Fractals* 2019, 120, 25–40. [CrossRef]
10. Perko, L. *Differential Equations and Dynamical Systems*; Springer: New York, NY, USA, 2001.
11. Bosi, S.; Desmarchelier, D. A simple method to study local bifurcations of three and four-dimensional systems: Characterizations and economic applications. *FAERE Work. Pap.* 2017, 2017, No. 16, 1–23.

12. Llibre, J.; Moeckel, R.; Simó, C. *Central Configurations, Periodic Orbits, and Hamiltonian Systems*; Birkhauser: Basel, Switzerland, 2015.

13. Berck, P.; Sydsæter, K. *Economists’ Mathematical Manual*; Springer: Berlin, Germany, 1991.

14. Llibre, J.; Novaes, D.D.; Teixeira, M.A. Higher order averaging theory for finding periodic solutions via Brouwer degree. *Nonlinearity* 2014, 27, 563–583. [CrossRef]

15. Buica, A.; Llibre, J. Averaging methods for finding periodic orbits via Brouwer degree. *Bull. Sci. Math.* 2004, 128, 7–22. [CrossRef]

16. Cândido, M.R.; Llibre, J. New results on averaging theory and applications. *Z. Angew. Math. Phys.* 2016, 67, No. 106, 1–11. [CrossRef]

17. Huang, B. On the limit cycles for a class of discontinuous piecewise cubic polynomial differential systems. *Electron. J. Qual. Theory Differ. Equ.* 2020, 2020, No. 25, 1–24. [CrossRef]

18. Sanders, J.A.; Verhulst, F.; Murdock, J. *Averaging Methods in Nonlinear Dynamical Systems*; Springer: New York, NY, USA, 2007.

19. Verhulst, F. *Nonlinear Differential Equations and Dynamical Systems*; Springer: Berlin, Germany, 1990.

20. Krawcewicz, W.; Wu, J.H. *Theory of Degrees, With Applications to Bifurcations and Differential Equations*; Wiley: New York, NY, USA, 1997.

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