Quantum information approach to normal representation of extensive games

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Abstract

We modify the concept of quantum strategic game to make it useful for extensive form games. We prove that our modification allows to consider the normal representation of any finite extensive game using the fundamental concepts of quantum information. The Selten’s Horse game and the general form of two-stage extensive game with perfect information are studied to illustrate a potential application of our idea. In both examples we use Eisert-Wilkens-Lewenstein approach as well as Marinatto-Weber approach to quantization of games.

Keywords
Extensive game · Normal representation · Quantum game · Nash equilibrium

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1 Introduction

Over the period of twelve years of research on quantum games [1], the idea of quantum strategic $2 \times 2$ game [2] has been well established. From mathematical point of view, the quantum information approach to a $2 \times 2$ game is described by the four-tuple:

$$(\mathcal{H},|\psi_{\text{in}}\rangle,\{U_i\},\{E_i\}).$$

(1)

The Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$ is the place of the game and the sets of unitary operators $U_1,U_2 \subseteq \text{SU}(2)$ play the role of strategy sets for the first and the second player, respectively. Given a unit vector $|\psi_{\text{in}}\rangle \in \mathcal{H}$ (the initial state), the players each choose a unitary operator $U_i \in U_i$ changing the vector $|\psi_{\text{in}}\rangle \in \mathcal{H}$ into the vector $|\psi_{\text{fin}}\rangle := (U_1 \otimes U_2)|\psi_{\text{in}}\rangle$ (the final state). The last components are the functionals $E_i: \mathcal{H} \to \mathbb{R}$ for $i = 1,2$. They imitate payoff functions for the players assigning a real number to the final state $|\psi_{\text{fin}}\rangle$. It turns out that the four tuple (1) generalizes playing a classical $2 \times 2$ game. The two well-known ways based on the framework (1): the Eisert-Wilkens-Lewenstein (EWL) scheme [3] and the Marinatto-Weber (MW) scheme [4] show that a $2 \times 2$ game can be successfully written in the form (1): it is possible to set each of the four components, so that a $2 \times 2$ game and the corresponding game defined by the four-tuple (1) are the same with respect to a game-theoretic analysis. Another key feature is that the protocol (1) allows to achieve results unavailable in the game played classically (see, for example, [5] and [6]). In our paper we are going to deal with extensive games in quantum domain.
In spite of quite many researches connected with this issue (for instance, concerning quantum Stackelberg duopoly [7] and [8]), there is no generally accepted framework for playing quantum extensive games by now. Interestingly, we have shown in [9] that (1) may indeed be useful for extensive games with imperfect information. In this paper, we extend our previous idea. We prove that the slight modification of $(H, |\psi_i\rangle, \{U_i\}, \{E_i\})$ allows to obtain normal representation of extensive games in the quantum domain.

2 Preliminaries to game theory

Definitions in the preliminaries are based on [10]. This section starts with a definition of a finite extensive game (without chance moves).

Definition 2.1 Let the following components be given:

- A finite set $N = \{1, 2, \ldots, n\}$ of players.

- A finite set $H$ of finite sequences that satisfies the following two properties:
  
  - the empty sequence $\emptyset$ is a member of $H$;
  
  - if $(a_k)_{k=1,2,\ldots,K} \in H$ and $L < K$ then $(a_k)_{k=1,2,\ldots,L} \in H$.

Each member of $H$ is a history and each component of a history is an action taken by a player. A history $(a_1, a_2, \ldots, a_K) \in H$ is terminal if there is no $a_{K+1}$ such that $(a_1, a_2, \ldots, a_K, a_{K+1}) \in H$. The set of actions available after the nonterminal history $h$ is denoted $A(h) = \{a: (h, a) \in H\}$ and the set of terminal histories is denoted $Z$.

- The player function $P: H \setminus Z \to N$ that points to a player who takes an action after the history $h$.

- For each player $i \in N$ a partition $I_i$ of $\{h \in H \setminus Z: P(h) = i\}$ with the property that for each $I_i \in I_i$ and for each $h, h' \in I_i$ an equality $A(h) = A(h')$ is fulfilled. Every information set $I_i$ of the partition corresponds to the state of player’s knowledge. When the player makes move after certain history $h$ belonging to $I_i$, she knows that the course of events of the game takes the form of one of histories being part of this information set. She does not know, however, if it is the history $h$ or the other history from $I_i$.

- For each player $i \in N$ a utility function $u_i: Z \to \mathbb{R}$ which assigns a number (payoff) to each of the terminal histories.

A five-tuple $(N, H, P, \{I_i\}, \{u_i\})$ is called a finite extensive game.

Our deliberations focus on games with perfect recall (although Def. [21] defines extensive games with imperfect recall as well) - this means games in which at each stage every player remembers all the information about a course of the game that she knew earlier (see [10] and [11] to learn about formal description of this feature).

The notions: action and strategy mean the same in static games, because players choose their actions once and simultaneously. In the majority of extensive games a player can make her decision about an action depending on all the actions taken previously by herself and also by all the other players. In other words, players can make some plans of actions at their disposal such that these plans point out to a specific action depending on the course of a game. Such a plan is defined as a strategy in an extensive game.
Definition 2.2 A pure strategy \( s_i \) of a player \( i \) in a game \((N, H, P, \{I_i\}, \{u_i\})\) is a function that assigns an action in \( A(I_i) \) to each information set \( I_i \).

Like in the theory of strategic games, a mixed strategy \( t_i \) of a player \( i \) in an extensive game is a probability distribution over the set of player \( i \)'s pure strategies. Therefore, pure strategies are of course special cases of mixed strategies and from this place whenever we shall write strategy without specifying that it is either pure or mixed, this term will cover both cases. Let us define an outcome \( O(s) \) of a pure strategy profile \( s = (s_1, s_2, \ldots, s_n) \) in an extensive game without chance moves to be a terminal history that results if each player \( i \in N \) follows the plan of \( s_i \). More formally, \( O(s) \) is the history \((a_1, a_2, \ldots, a_K) \in Z\) such that for \( 0 \leq k < K \) we have \( s_{P(a_1, a_2, \ldots, a_k)}(a_1, a_2, \ldots, a_k) = a_{k+1} \).

Definition 2.3 Let an extensive game \( \Gamma = (N, H, P, \{I_i\}, \{u_i\}) \) be given. The normal representation of \( \Gamma \) is a strategic game \((N, \{S_i\}, \{u'_i\})\) in which for each player \( i \in N \):

- \( S_i \) is the set of pure strategies of a player \( i \) in \( \Gamma \);
- \( u'_i : \prod_{i \in N} S_i \to \mathbb{R} \) defined as \( u'_i(s) \) := \( u_i(O(s)) \) for every \( s \in \prod_{i \in N} S_i \) and \( i \in N \).

One of the most important notions in game theory is a notion of an equilibrium introduced by John Nash in [12]. A Nash equilibrium is a profile of strategies where the strategy of each player is optimal if the choice of its opponents is fixed. In other words, in the equilibrium none of the players has any reason to unilaterally deviate from an equilibrium strategy. A precise formulation is as follows:

Definition 2.4 Let \((N, \{S_i\}, \{u_i\})\) be a game in strategic form. A profile of strategies \((t_1^*, t_2^*, \ldots, t_n^*)\) is a Nash equilibrium if for each player \( i \in N \) and for all \( s_i \in S_i \):

\[
    u_i(t_i^*, t_{-i}^*) \geq u_i(s_i, t_{-i}^*) \quad \text{where} \quad t_{-i}^* = (t_1^*, \ldots, t_{i-1}^*, t_{i+1}^*, \ldots, t_n^*). \tag{2}
\]

A Nash equilibrium in an extensive game with perfect recall is a Nash equilibrium of its normal representation, hence Def. 2.4 applies to strategic games as well as to extensive ones.

3 Preliminaries to quantum computing

In this section we give a brief overview of the Dirac notation and basic terms of quantum information. The preliminaries are based on [13] and are sufficient to study the paper. Nonetheless, we encourage the reader unfamiliar with techniques from theory of quantum information to consult [13] and, for example, [14].

First of all we adopt the convention that instead of denoting vectors by boldface letters, e.g. \( v \), they are denoted as kets: \( |v\rangle \).

Let \( \mathbb{C}^{m+1} \) be a vector space with the fixed basis \( \{|v_0\rangle, |v_1\rangle, \ldots, |v_m\rangle\} \) and let

\[
    |\phi\rangle = a_0|v_0\rangle + a_1|v_1\rangle + \cdots + a_m|v_m\rangle, \quad \text{where} \quad a_j \in \mathbb{C}. \tag{3}
\]

The vector \( |\phi\rangle \) can be also written in the column matrix notation

\[
    |\phi\rangle = \begin{pmatrix} a_0 & a_1 & \cdots & a_m \end{pmatrix}^T \tag{4}
\]

Let \( \mathbb{C}^{m+1} \) be now regarded as a Hilbert space and \( |\phi\rangle, |\chi\rangle \in \mathbb{C}^{m+1} \). The inner product of the vector \( |\phi\rangle \) with the vector \( |\chi\rangle \) will be denoted by \( \langle\phi|\chi\rangle \). The notation \( \langle\phi| \) is used
for the dual vector to $|\phi\rangle$. The dual vector $\langle \phi |$ (also called bra) is a linear operator 
$\langle \phi : \mathbb{C}^{m+1} \rightarrow \mathbb{C}$ defined by $\langle \phi |(|\chi\rangle) := \langle \phi |\chi\rangle$. Thus, the inner product requirements
imply that

$$\langle \phi | = a_0^* \langle v_1 | + a_1^* \langle v_2 | + \cdots + a_m^* \langle v_m |. \quad (5)$$

The common assumption in quantum computing is to consider Hilbert space $\mathbb{C}^{m+1}$ with an orthonormal basis. Let us denote the basis as $\{|x\rangle\}_{x=0,1,\ldots,m}$ (also called computational basis). Let $|\phi\rangle = \sum_x b_x |x\rangle$ and $|\chi\rangle = \sum_x c_x |x\rangle$ be the vectors with respect to the basis $\{|x\rangle\}_{x=0,1,\ldots,m}$. Then the inner product $\langle \phi |\chi\rangle$ can be expressed in terms of matrix multiplication:

$$\langle \phi |\chi\rangle = \left( b_0^* \quad b_1^* \quad \cdots \quad b_m^* \right) \left( c_0 \quad c_1 \quad \cdots \quad c_m \right)^T. \quad (6)$$

In this case, the dual vector $\langle \phi |$ has a row matrix representation whose entries are complex conjugates of the corresponding entries of the column matrix representation of $|\phi\rangle$.

The fundamental concept of quantum information is quantum bit (qubit) described mathematically as a unit vector $|\varphi\rangle$ in a Hilbert space $\mathbb{C}^2$. According to the notation explained above:

$$|\varphi\rangle = d_0 |0\rangle + d_1 |1\rangle, \quad \text{where} \quad d_0, d_1 \in \mathbb{C} \quad \text{and} \quad |d_0|^2 + |d_1|^2 = 1. \quad (7)$$

The measurement of a qubit with respect to an orthonormal basis $\{|w_0\rangle, |w_1\rangle\}$ (not necessarily in the computational basis) yields the result $w_0$ or $w_1$ with probability $|\langle w_j |\varphi\rangle|^2$ leaving the qubit in the corresponding state $|w_0\rangle$ or $|w_1\rangle$. In particular, measuring the qubit given by (7) with respect to $\{|0\rangle, |1\rangle\}$ results in the outcome 0 with probability $|d_0|^2$ and the outcome 1 with probability $|d_1|^2$, with post-measurement states $|0\rangle$ and $|1\rangle$, respectively.

Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with orthonormal bases $\{|x\rangle\}_{x=0,1,\ldots,m_1}$ and $\{|y\rangle\}_{y=0,1,\ldots,m_2}$, respectively. Then the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a Hilbert space of $(m_1 + 1)(m_2 + 1)$ dimensionality with the orthonormal basis $\{|x\rangle \otimes |y\rangle\}$. The matrix representation of an element $|x\rangle \otimes |y\rangle$ is the Kronecker product of respective matrix representations of $|x\rangle$ and $|y\rangle$. In the further part of the paper we use the abbreviated notation $|x\rangle|y\rangle$ or $|x, y\rangle$ for the tensor product $|x\rangle \otimes |y\rangle$.

A system of $n$ qubits $|\varphi_i\rangle$ is described as a unit vector $|\psi\rangle$ in the tensor product space $\bigotimes_{j=1}^n \mathbb{C}^2$ that has $2^n$-element computational basis

$$\{|x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle\}_{x_j=0,1.} \quad (8)$$

Thus, it is described by the vector

$$|\psi\rangle = \sum_{x_1,x_2,\ldots,x_n} d_{x_1,x_2,\ldots,x_n} |x_1, x_2, \ldots, x_n\rangle,$$

where $d_{x_1,x_2,\ldots,x_n} \in \mathbb{C}$ and $\sum_{x_1,x_2,\ldots,x_n} |d_{x_1,x_2,\ldots,x_n}|^2 = 1. \quad (9)$

We say that the state (9) is separable if it can be written as $|\psi\rangle = \bigotimes_{i=1}^n |\varphi_i\rangle$ for some $|\varphi_i\rangle \in \mathbb{C}^2$, $i = 1, 2, \ldots, n$. The dual vector $\langle \psi |$ is defined in the same way as in (5). Similarly, the measurement of the state given by (9) with respect to an orthonormal basis $\{|w_j\rangle\}_{j=1}^m$ yields the result $w_j$ with probability $|\langle w_j |\psi\rangle|^2$. Otherwise, the state $|\psi\rangle$ is called entangled.

We use the Dirac notation throughout the whole paper. However, each of the results below can be easily reconstructed using the matrix notation.
4 Normal representation of extensive games in quantum domain

From that moment on, we will consider extensive games with two available actions at each information set so that we could use only qubits for convenience. Any game richer in actions can be transferred to quantum domain by using quantum objects of higher dimensionality.

Let us extend the protocol (1) to include components making it useful for extensive games. Such a quantum game is specified by a six-tuple:

$$\Gamma^\text{QI} = (\mathcal{H}, N, |\psi_{\text{in}}\rangle, \xi, \{U_j\}, \{E_i\})$$

where the components are defined as follows:

- $\mathcal{H}$ is a complex Hilbert space $\bigotimes_{j=1}^{m} C^2$ with an orthonormal basis $B$.
- $N$ is a set of players with the property that $|N| \leq m$.
- $|\psi_{\text{in}}\rangle$ is the initial state of a quantum system of $m$ qubits $|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_m\rangle$.
- $\xi : \{1, 2, \ldots, m\} \to N$ is a surjective mapping. A value $\xi(j)$ indicates a player who carries out a unitary operation on a qubit $|\varphi_j\rangle$.
- For each $j \in \{1, 2, \ldots, m\}$ the set $U_j$ is a subset of unitary operators from $SU(2)$ that are available for a qubit $j$. A (pure) strategy of a player $i$ is a map $\tau_i$ that assigns a unitary operation $U_j \in U_j$ to a qubit $|\varphi_j\rangle$ for every $j \in \xi^{-1}(i)$. The final state $|\psi_{\text{fin}}\rangle$ when the players have performed their strategies on corresponding qubits is defined as:

$$|\psi_{\text{fin}}\rangle := \left(\tau_1, \tau_2, \ldots, \tau_n\right)|\psi_{\text{in}}\rangle = \bigotimes_{i \in N} \bigotimes_{j \in \xi^{-1}(i)} U_j |\psi_{\text{in}}\rangle.$$  

- For each $i \in N$ the map $E_i$ is a utility (payoff) functional that specifies a utility for the player $i$. The functional $E_i$ is defined by the formula:

$$E_i = \sum_{|b\rangle \in B} v_i(b) |b\rangle \langle b| |\psi_{\text{in}}\rangle|^2, \ \text{where} \ v_i(b) \in \mathbb{R}.$$  

There are only two additional components in (10): $N$ and $\xi$, in comparison with (1). They completely specify qubits to which a player is permitted to apply her unitary operator. Notice also that the protocol of quantization of strategic games according to [2] is obtained from $(\mathcal{H}, N, |\psi_{\text{in}}\rangle, \xi, \{U_i\}, \{E_i\})$ by putting $|N| = m = 2$. We claim that such addition together with appropriate fixed values $v_i(b)$ in (12) are sufficient for considering an extensive game in quantum domain (of course, if the assumption that the tuple $(\mathcal{H}, |\psi_{\text{in}}\rangle, \{U_i\}, \{E_i\})$ correctly describes strategic games in quantum domain is true). The line of thought is as follows. Any strategic game can be considered as a special case of an extensive game where players move sequentially but each of them does not have any knowledge about actions taken by the other players. In other words, each player in a strategic game has exactly one information set in which she takes an action. Thus, in a simple case of $2 \times 2$ bimatrix game, the scheme (11), in fact, identifies an operation on a qubit with player’s move made at her unique information set, and
then the individual game outcomes are assigned to appropriate measurement results. An extensive game can have many information sets, and more than one of them can be assigned to the same player. Therefore, our extension of \((\mathcal{H}, |\psi_{\text{in}}\rangle, \{U_i\}, \{E_i\})\) is aimed at similar identification for extensive games. As a result, we obtain that we are able to write in (10) the normal representation of an extensive game.

Before we formulate the formal statement, notice first that the tuple (10), in fact, determines some game in strategic form in the sense of classical game theory. If \(\mathcal{H}\) and \(|\psi_{\text{in}}\rangle\) are fixed, each player \(i \in N\) chooses her strategy from a set \(\bigotimes_{j \in \xi^{-1}(i)} U_j\) and then the associated utility \(E_i\) is determined. Therefore, it always makes sense to associate (10) with some \((N, \{S_i\}, \{u_i\})\). Secondly, let us specify a sufficient condition of equivalence for two strategic games \((N, \{S_i\}, \{u_i\})\) and \((N, \{S_i\}', \{u_i\}')\). Namely, if there is a bijective mapping \(g_i: S_i \rightarrow S_i'\) for each \(i \in N\) such that for each profile \(s \in \prod_{i \in N} S_i\) we have \(u(s) = u'(g(s))\) where \(g = \prod_{i \in N} g_i\), then the games are isomorphic (to find out more about isomorphisms of strategic games see [15]). Now, we can formulate the following proposition:

**Proposition 4.1** Let \(\Gamma = (N, H, P, \{I_i\}, \{u_i\})\) be a finite extensive game with two available actions at each information set. Then there exists a six-tuple (10) that specifies a game isomorphic to the normal representation of \(\Gamma\).

**Proof.** Let us consider an \(n\)-player extensive game \(\Gamma\) with \(m\) information sets. In addition, let us assume two-element set of available actions \(A(I_i)\) in each information set \(I_i\). We specify components of the tuple \(\Gamma^\text{fi}\) as follows. Let \(\mathcal{B}\) be the computational basis of \(\bigotimes_{j=1}^m \mathbb{C}^2\) and let the initial state \(|\psi_{\text{in}}\rangle\) be of the form \(|b\rangle\), where \(|b\rangle\) is some fixed state of \(\mathcal{B}\). Let us restrict the set of available operators on \(\mathbb{C}^2\) to the set of two operators \(\{\sigma_0, \sigma_1\}\) where \(\sigma_0\) is the identity operator and \(\sigma_1\) is the bit-flip Pauli operator. This specification implies that for any mapping \(\xi: \{1, 2, \ldots, m\} \rightarrow N\) specified in (10) each strategy profile is an operator of the form \(\bigotimes_{j=1}^m \sigma_j\), where \(\sigma_j \in \{\sigma_0, \sigma_1\}\). Thus, for each strategy profile \(\tau\) there is some \(|\psi\rangle \in \mathcal{B}\) such that

\[
|\psi_{\text{in}}\rangle = \bigotimes_{j=1}^m \sigma_j |\psi_{\text{in}}\rangle = |b\rangle \langle b| \quad \text{and} \quad E_i(\tau) = v_i(\langle b|) \quad \text{for} \quad i \in N. \tag{13}
\]

Let us fix a bijective mapping \(\zeta\) between \(\{1, 2, \ldots, m\}\) and the set \(\{I_i\}\) of all players’ information sets of \(\Gamma\). Since for each player \(i\) and history \(h \in I_i\) we have \(P(h) = i\) we can simply take \(P(I_i) = i\). Then the correspondence \(\xi := P \circ \zeta\) associates each information set of each player with exactly one qubit. As \(|A(I_i)| = |\{\sigma_0, \sigma_1\}| = 2\) and a number of information sets of \(\Gamma\) is equal to a number of qubits, a set of strategies \(\{S_i\}_i\) of the normal representation of \(\Gamma\) and a strategy set \(\mathcal{T}_i\) of quantum game defined by the tuple (10) are equinumerous (with cardinality equal \(2^{\ell^{-1}(i)}\) each) for each \(i \in N\). Therefore, for each \(i \in N\), we can define a bijective mapping \(g_i: S_i \rightarrow \mathcal{T}_i\). These mappings induce the following bijection between the sets of strategy profiles:

\[
g = (g_i)_{i \in N} : \prod_{i \in N} S_i \rightarrow \prod_{i \in N} \mathcal{T}_i. \tag{14}
\]

The equations in (13) imply that for all \(i\) we can select numbers \(v_i(b) \in u_i(Z)\) in (12) in a way that \(u_i(s) = E_i(g(s))\) where \(u_i\) is the utility function of the normal representation of \(\Gamma\). Such specification of (10) makes it isomorphic to the normal representation of \(\Gamma\). \(\blacksquare\)

Many researches on quantum games played via scheme (11) are based on appropriately fixed basis for a space, the initial state, and a range of available unitary operators, in
order to obtain interesting properties of a quantum game. We will use the two best-known configurations of \( (H, |\psi_{in}\rangle, \{U_i\}, \{E_i\}) \): the Marinatto-Weber (MW) scheme \[4\] and the Eisert-Wilkens-Lewenstein (EWL) scheme \[3\] to examine extensive games via the protocol (10) (see also \[6\] and \[9\] for other applications of these schemes). In the former scheme players are allowed to use only the identity operator and the bit-flip Pauli operator. The results superior to classical results are obtained by manipulating the initial state \( |\psi_{in}\rangle \). The later scheme allows to use broader range of unitary operators (including also the whole set \( SU(2) \)). The following examples concern both settings.

To convert the following games into quantum ones, we use the same reasoning as in the proof of Proposition \[4.1\]. The first example deals with a case where each player operates on one qubit.

**Example 4.2** Let us consider a three player extensive game:

\[
\Gamma_1 = (\{1, 2, 3\}, H, P, \{I_i\}_{i\in\{1,2,3\}}, \{u_i\}_{i\in\{1,2,3\}})
\]

(15)
determined by the following components:

- \( H = \{\emptyset, (a_0), (a_1), (a_0, c_0), (a_0, c_1), (a_1, b_0), (a_1, b_1), (a_1, b_0, c_0), (a_1, b_1, c_1)\} \);
- \( P(\emptyset) = 1, P(a_1) = 2, P(a_0) = P(a_1, b_0) = 3 \);
- \( I_1 = \{\emptyset\}, I_2 = \{(a_1)\}, I_3 = \{(a_0), (a_1, b_0)\} \);
- \( u_{1,2}(a_0, c_0) = 3, u_{1,2}(a_0, c_1) = u_{1,2}(a_1, b_0, c_1) = 0, u_{1,2}(a_1, b_1) = 2, u_{1,2}(a_1, b_1, c_0) = 5, u_3(a_0, c_0) = u_3(a_1, b_0, c_1) = 1, u_3(a_0, c_1) = u_3(a_1, b_0, c_0) = 0, u_3(a_1, b_1) = 2 \).

The game is depicted in Fig.[1] It is the Selten’s Horse game \[16\] with modified payoffs. Since each of the players has one information set, their sets of strategies are \( \{a_0, a_1\}, \{b_0, b_1\}, \) and \( \{c_0, c_1\} \), respectively. Profiles: \( (a_0, b_1, c_0) \) and \( (a_1, b_1, c_1) \) are the only pure Nash equilibria in this game and indeed each of them could be equally likely chosen as a scenario of the game. The utilities for players 1 and 2 assigned to \( (a_0, b_1, c_0) \) are higher than the utilities corresponding to \( (a_1, b_1, c_1) \) - a desirable profile for player 3. The uncertainty of a result of the game follows from the peculiar strategic position of player 3. She could try to affect the decision of others by announcing before the game
starts, that she is going to take an action $c_1$. If the statement of player 3 is credible enough then the history $(a_1, b_1)$ might occur.

The MW approach. Let us examine the Selten’s Horse game via the protocol (11). It turns out that among quantum realizations of the game $\Gamma_1$ there exist ones that provide the players with a unique reasonable solution. One of these realizations is constructed, according to the idea of the MW scheme, as follows:

$$\Gamma_1^{\text{MW}} = (\mathcal{H}_c, \{1, 2, 3\}, |\psi_{\text{in}}(\gamma)\rangle, \text{id}_{\{1,2,3\}}, \{\{\sigma_0, \sigma_1\}_i\}, \{E_i\}),$$

where:

- $\mathcal{H}_c$ is a Hilbert space $\bigotimes_{j=1}^3 \mathbb{C}^2$ with the computational basis $\{|x_1\rangle|x_2\rangle|x_3\rangle\}$, where $x_j \in \{0, 1\}$ for $j = 1, 2, 3$;
- the initial state $|\psi_{\text{in}}(\gamma)\rangle$ takes the form:
  $$|\psi_{\text{in}}(\gamma)\rangle = \cos \frac{\gamma}{2}|000\rangle + i \sin \frac{\gamma}{2}|111\rangle \text{ and } \gamma \in (0, \pi);$$

$$|\psi_{\text{in}}(\gamma)\rangle = \cos \frac{\gamma}{2}|000\rangle + i \sin \frac{\gamma}{2}|111\rangle \text{ and } \gamma \in (0, \pi);$$

- $\text{id}_{\{1,2,3\}}$ is an identity mapping defined on $\{1, 2, 3\}$;
- the payoff functionals $E_i$ are defined as follows:
  $$E_{1,2} = 3 \sum_{x_2} |\langle 0, x_2, 0|\psi_{\text{in}}\rangle|^2 + 2 \sum_{x_3} |\langle 11, x_3|\psi_{\text{in}}\rangle|^2 + 5|\langle 100|\psi_{\text{in}}\rangle|^2;$$
  $$E_3 = \sum_{x_2} |\langle 0, x_2, 0|\psi_{\text{in}}\rangle|^2 + 2 \sum_{x_3} |\langle 11, x_3|\psi_{\text{in}}\rangle|^2 + |\langle 101|\psi_{\text{in}}\rangle|^2.$$

Let us first determine the utilities $E_i$ associated with any profile $\sigma_{\kappa_1} \otimes \sigma_{\kappa_2} \otimes \sigma_{\kappa_3}$, where $\kappa_j \in \{0, 1\}$ for $j = 1, 2, 3$. The final state $|\psi_{\text{in}}\rangle$ after the operation $\sigma_{\kappa_1} \otimes \sigma_{\kappa_2} \otimes \sigma_{\kappa_3}$ takes the form:

$$(\sigma_{\kappa_1} \otimes \sigma_{\kappa_2} \otimes \sigma_{\kappa_3})|\psi_{\text{in}}\rangle = \cos \frac{\gamma}{2}|\kappa_1, \kappa_2, \kappa_3\rangle + i \sin \frac{\gamma}{2}|\overline{\kappa_1}, \overline{\kappa_2}, \overline{\kappa_3}\rangle,$$

where $\overline{\kappa_j}$ in the negation of $\kappa_j$. Using the last equation and formula (12) the expected utilities, for example, for $(\sigma_1 \otimes \sigma_0 \otimes \sigma_1)$ become:

$$E_{1,2}(\sigma_1 \otimes \sigma_0 \otimes \sigma_1) = 3 \sin^2 \frac{\gamma}{2}, \quad E_3(\sigma_1 \otimes \sigma_0 \otimes \sigma_1) = 1.$$
Figure 2: A game in strategic form induced by the MW approach to the normal representation of $\Gamma_1$.

Now, we examine the six-tuple (16) to find a reasonable solution for $p$ layers. Let us determine pure Nash equilibria in the game $\Gamma_{MW}^1$ by solving for each profile $(\sigma_\kappa_1, \sigma_\kappa_2, \sigma_\kappa_3)$, where $\kappa_j \in \{0, 1\}$ the system of inequalities imposed by the condition (2). Using values of $E_i(\sigma_\kappa_1, \sigma_\kappa_2, \sigma_\kappa_3)$ placed in Fig. 2, we find that, for example, the profile $(\sigma_0, \sigma_0, \sigma_0)$ constitutes the Nash equilibrium if and only if $2\cos^2(\gamma/2) \leq 1$. Further investigation shows that the profile $(\sigma_1, \sigma_1, \sigma_1)$ also fulfills (2) with the requirement $2\sin^2(\gamma/2) \leq 1$ and that there are no other pure Nash equilibria. Taking into consideration $\gamma \in (0, \pi)$ we conclude that

$$\text{NE}_{\text{pure}}(\gamma) = \begin{cases} (\sigma_1, \sigma_1, \sigma_1), & \text{if } 0 < \gamma \leq \pi/2; \\ (\sigma_0, \sigma_0, \sigma_0), & \text{if } \pi/2 \leq \gamma < \pi. \end{cases}$$ (21)

Let us assume results of the games $\Gamma_1$ and $\Gamma_{MW}^1$ to be an equilibrium in pure strategies. Then formula (21) shows that each player can gain from playing game $\Gamma_{MW}^1$. In classical case players 1 and 2 can assure themselves 2 utility units and player 3 can get 1 unit for sure by playing pure equilibria. All these payoffs are strictly less than the payoffs corresponding to pure Nash equilibria in $\Gamma_{MW}^1$, irrespectively of what is a value of $\gamma$. Moreover, notice that there is the unique equilibrium in the game $\Gamma_{MW}^1$ if and only if $2\cos^2(\gamma/2) \leq 1$ and the same utilities are assigned to both equilibria in the case $\gamma = \pi/2$. This implies that in the game $\Gamma_{MW}^1$ the strategy profile $(\sigma_0, \sigma_0, \sigma_0)$ for $\gamma \in (\pi/2, \pi)$ and the strategy profile $(\sigma_1, \sigma_1, \sigma_1)$ for $\gamma \in (0, \pi/2)$ are reasonable profiles for all players.

An interesting fact worth pointing out is that for $\gamma$ arbitrary close to 0 or $\pi$ (i.e. for angles defining the classical game) the equilibrium is unique. This discontinuity implies possible applications of quantum games to classical game theory. Namely, the MW approach may serve as a Nash equilibrium refinement by considering only Nash equilibria that hold out some slight perturbation of $|\psi_\text{in}(0)\rangle$. In fact, the profile $(\sigma_1, \sigma_1, \sigma_1)$ is the unique pure trembling hand perfect equilibrium in $\Gamma_1$ [16] (see also [10], example 252.1). Although a further investigation is required, we believe there is a strong connection between the above method and the Selten’s concept of trembling hand equilibrium.

The EWL approach. The second quantum realization of $\Gamma_1$ is in the spirit of the EWL protocol. Contrary to the previous one, where the number of reasonable Nash equilibria was reduced to the unique one, we focus this time on improving strategic position of
only one of the players. Namely, let us modify the previous quantum game as follows:

\[ \Gamma^\text{EWL}_1 = (\mathcal{H}, \{1, 2, 3\}, |\psi_{\text{in}}\rangle, \text{id}_{\{1,2,3\}}, \{U_{1,2}(\theta, 0), U_3(\theta, \alpha)\}, \{E_i\}) \]  \hspace{1cm} (22)

where:

- \( \mathcal{H} \) is a Hilbert space \( \bigotimes_{j=1}^{3} \mathbb{C}^2 \) with the basis \( \{|\psi_{1,x_2,x_3}\rangle\}_{x_j \in \{0,1\}} \) of entangled states defined as follows:

\[ |\psi_{x_1,x_2,x_3}\rangle = \frac{|x_1, x_2, x_3\rangle + i|x_1, x_2, \overline{x}_3\rangle}{\sqrt{2}}; \]  \hspace{1cm} (23)

- \( |\psi_{\text{in}}\rangle := |\psi_{000}\rangle \);

- the unitary strategies are elements of \( \{U(\theta, \alpha) : \theta \in [0, \pi], \alpha \in [0, \pi/2]\} \) whose matrix representation with respect to the computational input and output basis is the following:

\[ U(\theta, \alpha) = \begin{pmatrix} e^{i\alpha} \cos(\theta/2) & i \sin(\theta/2) \\ i \sin(\theta/2) & e^{-i\alpha} \cos(\theta/2) \end{pmatrix}; \]  \hspace{1cm} (24)

- the payoff functional \( E_i \) is derived from \( \Gamma^\text{MW}_1 \) with respect to basis states \( |\psi_{\text{in}}\rangle \) :

\[ E_{1,2} = 3 \sum_{x_2} |\langle \psi_{0,x_2,0} | \psi_{\text{in}} \rangle|^2 + 2 \sum_{x_3} |\langle \psi_{11,3} | \psi_{\text{in}} \rangle|^2 + 5 |\langle \psi_{100} | \psi_{\text{in}} \rangle|^2; \]

\[ E_3 = \sum_{x_2} |\langle \psi_{0,x_2,0} | \psi_{\text{in}} \rangle|^2 + 2 \sum_{x_3} |\langle \psi_{11,3} | \psi_{\text{in}} \rangle|^2 + |\langle \psi_{101} | \psi_{\text{in}} \rangle|^2. \]  \hspace{1cm} (25)

In this case only the third player is allowed to use unitary strategies beyond the set of one-parameter operators of the game \( \Gamma^\text{EWL}_1 \) by using the additional parameter \( \alpha \). We demonstrate now that such extended strategy set of player 3 significantly improves her strategic position. In order to see this, let us determine the expected utility \( E_i \) for each player \( i \) that corresponds to a profile of strategies \( \tau = (\theta_1, \theta_2, (\theta_3, \alpha_3)) \) (since angles specify strategies of player 1, 2, and 3, we denote them, for convenience, as \( \theta_1, \theta_2 \) and \( (\theta_3, \alpha_3) \), respectively). Using formula (11) the final state \( |\psi_{\text{in}}\rangle \) associated with a profile \( \tau \) is the following:

\[ |\psi_{\text{in}}\rangle = U(\theta_1, 0) \otimes U(\theta_2, 0) \otimes U(\theta_3, \alpha_3) |\psi_{\text{in}}\rangle = \frac{1}{\sqrt{2}} \sum_{x \in \{0,1\}^3} \lambda_x |x\rangle, \]  \hspace{1cm} (26)

where

\[ \lambda_{x_1,x_2,x_3} = i^{\sum x_j} e^{i\pi/2} \prod_j \cos \left( \frac{x_j \pi - \theta_j}{2} \right) + (-i)^{\sum x_j} e^{-i\pi/2} \prod_j \cos \left( \frac{x_j \pi - \theta_j}{2} \right). \]  \hspace{1cm} (27)

Putting (20) into formulae (25) we obtain the utility outcomes:

\[ E_{1,2}(\tau) = 2 \left( \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \sin^2 \frac{\theta_3}{2} + \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \cos^2 \frac{\theta_3}{2} \sin^2 \alpha_3 \right) \]

\[ + \cos^2 \frac{\theta_3}{2} \cos^2 \alpha_3 \left[ 3 \cos^2 \frac{\theta_1}{2} + \sin^2 \frac{\theta_1}{2} \left( \frac{2}{3} \cos^2 \frac{\theta_2}{2} \right) \right]; \]  \hspace{1cm} (28)

\[ E_3(\tau) = \cos^2 \frac{\theta_3}{2} \cos^2 \alpha_3 \left( \frac{1}{2} \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_1}{2} \right) \]

\[ + \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_3}{2} \sin^2 \alpha_3 \left( 1 + \cos^2 \frac{\theta_2}{2} \right) + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_3}{2} \left( 1 + \sin^2 \frac{\theta_2}{2} \right). \]
As it should be expected in the EWL protocol, we get the classical game $\Gamma_1$ when the player 3 is also restricted to use only the strategies $(\theta_3, 0)$. Namely, let us put $\alpha = 0$, $p := \cos^2(\theta_1/2)$, $q := \cos^2(\theta_2/2)$ and $r := \cos^2(\theta_3/2)$ to Eq. (28). Then we obtain

$$
E_{1,2}(\theta_1, \theta_2, (\theta_3, 0)) = 3pr + 5(1-p)qr + 2(1-p)(1-q);
$$

$$
E_3(\theta_1, \theta_2, (\theta_3, 0)) = pr + (1-p)q(1-r) + 2(1-p)(1-q).
$$

Formulae (29) are exactly the players expected payoffs in the classical game $\Gamma_1$ if they choose their actions $a_0, b_0$ and $c_0$ with probability $p, q,$ and $r$, respectively. Thus, in the particular case, if $p, q, r \in \{0, 1\}$, we obtain payoffs corresponding to pure strategy profiles in $\Gamma_1$.

Now we solve a problem how player 3 can gain from using 2-parameter operators as her strategies. An interesting feature is that the game $\Gamma_{1}^{\text{EWL}}$ keeps the pure Nash equilibria of the game $\Gamma_1$, i.e., the profiles: $(0, \pi, (0, 0))$ and $(\pi, \pi, (0, 0))$ - equivalents for the respective Nash equilibria profiles $(a_0, b_1, c_0)$ and $(a_1, b_1, c_1)$ in $\Gamma_1$ - are Nash equilibria profiles also in $\Gamma_{1}^{\text{EWL}}$. However, unlike in the game $\Gamma_1$, there is another pure equilibrium $\tau^* = (0, 0, (0, \pi/2))$ where $E_i(\tau^*) = 2$ for each player $i$. This non-equivalence to the classical profile is essential for strategic position of player 3. She can force the other players to play strategies from the profile $\tau^*$ (instead of $(0, \pi, (0, 0))$ - their the most preferred equilibrium) by making an announcement that she is going to play $\tau_3^* = (0, \pi/2)$. The other players know that this threat is credible enough as the player 3 does not suffer a loss when she deviates from $(0, 0)$ to $(0, \pi/2)$, since $E_3(0, \pi, (0, 0)) = E_3(0, \pi, (0, \pi/2)) = 1$. However, the opponents of the third player lose 3 utilities, since $E_{1,2}(0, \pi, 0, \pi/2) = 0$. Furthermore, given $(\theta_3, \alpha_3) = (0, \pi/2)$ fixed, rationality demands that they play strategies dictated by $\tau^*$ as we have $\arg\max_{\theta_1, \theta_2} E_{1,2}(\theta_1, \theta_2, (0, \pi/2)) = \{(0, 0)\}$. This argumentation allows to treat the profile $\tau^*$ as reasonable solution of $\Gamma_{1}^{\text{EWL}}$. Thus, the strategic position of the player 3 has been significantly improved in comparison to her classical strategies.

The second example is aimed at showing that the proposed scheme of playing extensive games based on the six-tuple (10) can be applied to extensive games in which some of players have more than one information set. Unlike in Example 4.2 we now focus only on converting an extensive game into the form described by (10) without considering a specific strategic situation.

**Example 4.3** Let the following extensive game be given:

$$\Gamma_2 = (\{1, 2\}, H, P, \{I_1\}, u), \quad (30)$$

where

- $H = \emptyset, a_0, a_1, (a_0, b_0), (a_0, b_1), (a_1, c_0), (a_1, c_1)$;
- $P(\emptyset) = 1, P(a_0) = P(a_1) = 2$;
- $I_1 = \{(\emptyset)\}, I_2 = \{(a_0), (a_1)\}; \ u(a_{i_1}, b_{i_2}) = O_{i_1, i_2}, \ i_1, i_2 = 1, 2$.

Like in the previous example, the game $\Gamma_2$ has three information sets in which two actions are available. However, in this case, two information sets represent the knowledge of player 2. Thus, she specifies an action at each of them. The game is illustrated in Fig. 3. Since there will not be a need to use individual payoffs for players, we assign an outcome $O_{i_1, i_2}$ to each of terminal histories in $H$ for convenience.
The EWL approach. Let us first put the game (30) into the form described by the six-tuple (10) via the EWL approach. Using the same line of reasoning as in the proof of Proposition 4.1, the number of qubit on which a player is allowed to operate has to agree with the number of her information sets. To define the outcome functional $E$ for the quantum game we associate particular outcomes $O_\iota, \iota = 1, 2$ with appropriate basis states of $\{|\psi_{x_1,x_2,x_3}\rangle\}$. For example, we identify the outcome $O_{10}$ of the game $\Gamma_2$ with the basis states $\{|\psi_{1,x_2,0}\rangle\}_{x_2=0,1}$ measured on $|\psi_{\text{fin}}\rangle$, and the outcome $O_{00}$ with the basis states $\{|\psi_{00,x_3}\rangle\}_{x_3=0,1}$. Formally, the EWL approach to (30) is a six-tuple

$$
\Gamma_{\text{EWL}}^2 = (H_\iota, \{1, 2\}, |\psi_{\text{in}}(\pi/2)\rangle, \xi, \{U_i\}, E) \quad (31)
$$

defined by the following components:

- the map $\xi$ on $\{1, 2, 3\}$ given by the formula: $\xi(j) = \begin{cases} 
1, & \text{if } j = 1; \\
2, & \text{if } j \in \{2, 3\}.
\end{cases}$
- the set $U_i$ of unitary operators such that $\{U(\theta, 0)\} \subseteq U_i \subseteq SU(2)$ for $i = 1, 2$;
- $E$ is the outcome functional of the form:

$$
E = \sum_{i_2, x_3} O_{0,i_2} |\langle \psi_{0,i_2,x_3} | \psi_{\text{fin}} \rangle|^2 + \sum_{i_2, x_2=0,1} O_{1,i_2} |\langle \psi_{1,x_2,i_2} | \psi_{\text{fin}} \rangle|^2. \quad (32)
$$

Let us prove that (31) generalizes (30). Following the definition of $\Gamma_{\text{EWL}}^2$, the strategy set of player 1 is simply $U_1$, and player 2 chooses her strategies from the set $U_2 \otimes U_2$ since she operates on the second and the third qubit. Therefore, the final state $|\psi_{\text{fin}}\rangle$ in the game $\Gamma_{\text{EWL}}^2$ takes the form of $\bigotimes_{j=1}^3 U_j |\psi_{\text{in}}\rangle$, where $U_j \in U(\iota(j))$. Let us assume that the players apply unitary operators form the set $\{U(\theta, 0)\}$. Then the final state is represented by Eq. (26) and Eq. (27) for $\alpha_3 = 0$. It implies that the expected outcome $E(\theta_1, (\theta_2, \theta_3))$ equals

$$
E(\theta_1, (\theta_2, \theta_3)) = \left( O_{00} \cos^2 \frac{\theta_2}{2} + O_{01} \sin^2 \frac{\theta_2}{2} \right) \cos^2 \frac{\theta_1}{2} + \left( O_{10} \cos^2 \frac{\theta_3}{2} + O_{11} \sin^2 \frac{\theta_3}{2} \right) \sin^2 \frac{\theta_1}{2}. \quad (33)
$$

By substitution $p := \cos^2 \theta_1/2$, $q := \cos^2 \theta_1/2$, and $r := \cos^2 \theta_1/2$, Eq. (33) shows the expected outcome in game $\Gamma_2$ when player 1 chooses $a_0$ with probability $p$ and player 2
chooses $b_0$ and $c_0$ with probability $q$ and $r$, respectively. To sum up, the six-tuple $(31)$ indeed allows to describe the quantum extension of the game $\Gamma_2$ within the EWL approach.

The MW approach. In a similar way, we rewrite the game $\Gamma_2$ using the MW approach by replacing the components $\mathcal{H}_c$, $\{|\psi_{x_1, x_2, x_3}\rangle\}$, and $\{U_i\}$ with $\mathcal{H}_c$, $\{|x_1, x_2, x_3\rangle\}$, and $\{U(0, 0), U(\pi, 0)\}_i$, respectively. Then similar analysis to that in Example 4.2 shows that the game given by $\Gamma_2^{MW}$ coincides with $\Gamma_2$ if $|\psi_{\text{fin}}\rangle = |\psi_{\text{fin}}(0)\rangle$.

5 Conclusion

We have shown that the six-tuple $(\mathcal{H}, N, \rho_{\text{in}}, \xi, \{U_j\}, \{E_i\})$ allows to study extensive games using quantum information language. Although proposed scheme is suitable only for a normal representation in which some features of corresponding classical game in extensive form are lost, it yields on valuable information about how passing to quantum domain influences a course of extensive games. The examples we studied have shown that an extensive game played with the use of both the MW approach and the EWL approach substantially differs from this game played classically. Furthermore, the quantum schemes may yield to the players’ significant advantages in the form of better strategic positions and pointing out reasonable solutions, as it often happens in the area of strategic games.

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