Dissipate locally, couple globally: a sharp transition from decoupling to infinite range coupling in Josephson arrays with on-site dissipation

S. Tewari\textsuperscript{1} and J. Toner\textsuperscript{2}

\textsuperscript{1} Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, MD 20742
\textsuperscript{2} Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene, OR 97403

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Abstract. – We study the $T = 0$ normal to superconducting transition of Josephson arrays with on-site dissipation. A perturbative renormalization group solution is given. Like the previously studied case of bond dissipation (BD), this is a “floating” to coupled (FC) phase transition. Unlike the BD transition, at which only nearest-neighbor couplings become relevant, here all inter-grain couplings, out to infinitely large distances, do so simultaneously. We predict, for the first time in an FC transition, a diverging spatial correlation length. Our results show the robustness of floating phases in dissipative quantum systems.

Coupling dissipation to quantum systems has interesting consequences. Quite generally, dissipation suppresses quantum fluctuations. This is evident at the level of a single macroscopic quantum ‘particle’ whose tunnelling probability out of a metastable state is suppressed by dissipation [1]. For the quantum particle in a double well [2, 3], or in a periodic [4, 5] potential, Ohmic (i.e., linear) dissipation can suppress quantum fluctuations enough to cause a dissipative ‘quantum to classical’ phase transition. In the classical phase, all quantum fluctuations are quenched and the system is spontaneously trapped into only one of the potential minima.

From the point of view of understanding effects of dissipation on extended systems, a natural question to ask is what happens when such zero-dimensional dissipative systems are spatially coupled in a lattice. Understanding such effects has important practical applications. Such dissipative effects are thought to be at the heart of the physics of granular superconductivity [6]. They are also important in the context of decoherence in a qubit, which, it has been proposed [7], can be realized in a Josephson junction array. If the coupling of dissipation remains local, then it is natural to expect that quantum fluctuations can also be locally quenched by dissipation. At zero temperature, coupling such classical systems spatially should give rise to an ordered state of the extended system. This dissipative quantum phase transition (QPT), however, is expected to retain some of its local character.
Early work on these questions [6, 8–13] was recently extended in Ref. [14] in the context of a Josephson array with dissipation coupled to the bond phases, that is, the order parameter phase differences of two consecutive superconducting grains; see also Ref. [15]. This situation can be realized when the junctions are shunted by Ohmic resistors. It was shown that the metal to superconductor dissipation-tuned QPT occurs via a floating to coupled phase transition. Interesting local behavior in the bond dissipation model has also been addressed in Ref. [16].

The case where dissipation couples to the site variables - that is, the phases of the superconducting islands - directly has also been addressed in the literature [17–19]. Such local damping of the phase happens [17] when Cooper pairs can decay into localized pools of normal electrons in the substrate. It can also be built into an artificial array with each island shunted separately to the ground [19]. Such a 'shunt to the ground' also occurs naturally in many granular systems. The model has been mapped on to effective LZW field theories [17, 19] to study the normal to superconductor phase transition.

Working directly with the order parameter phase variables, we give in this paper a second order perturbative renormalization group (RG) solution for the site dissipation problem. We show that a floating normal metallic phase, which is even more striking than in the bond-dissipation case, and a floating to coupled QPT occur in this case. In contrast to bond-dissipation, here all couplings between grains become relevant simultaneously, and the transition is controlled by a fixed, infinite-dimensional critical hypersurface with continuously varying critical exponents whose dependence on dissipation we calculate near the threshold dissipation. We also predict, for the first time in a floating to coupled phase transition (three other examples of which are known [14, 20, 21]), a diverging spatial correlation length. The RG flows and phase diagram we derive are displayed in Fig. 1 in the $V - \alpha$ plane, where $V$ is the nearest neighbor Josephson coupling and $\alpha$ is the coefficient of dissipation.

![Phase diagram](Fig. 1 - Phase diagram and perturbative RG flows in the $V - \alpha$ plane in any dimension $D > 0$. N and SC indicate normal and superconducting phases, respectively. The normal metallic phase is the floating phase. The thick dashed lines correspond to lines of fixed points. The cross indicates a special point on the $\alpha = 0$ axis denoting a $(D + 1)$-dimensional XY transition. The part of the $\alpha = 0$ axis from the cross to infinity is a line of fixed points in $D = 1$.]

Our results demonstrate the model-independence of floating phases in dissipative Joseph-
son arrays. They also indicate that dissipative quantum systems may be the best places to experimentally look for such phases, which may otherwise be very difficult to obtain [22]. Moreover, they provide a rare example in statistical mechanics of a critical fixed locus that exists in any dimension; there is no upper critical dimension in the problem.

On-site dissipation coupled to XY spins has also been treated in Refs. [23, 24]. There, dissipation is coupled locally to a $\phi^4$-theory, which retains the compactness between 0 and $2\pi$ of the phases of the order parameter. The problem is then studied using an $\epsilon$-expansion around the upper critical dimension. In our model, the compactness is explicitly broken, as is expected in Ohmic dissipation [25]. We believe the difference of our results from Refs. [23, 24] is due to this different treatment of the phase variable.

The action for the Josephson junction array coupled to on site Ohmic dissipation reads

$$ S/h = \sum_{\vec{r}} \sum_n (\frac{C}{2} \omega_n^2 + \frac{\alpha}{2\pi} |\omega_n|) |\bar{\theta}(\vec{r},\omega_n)|^2 + V \sum_{\langle \vec{r},\vec{r}' \rangle} \int d\tau [1 - \cos \Delta \theta_{\vec{r},\vec{r}'}(\tau)]. \quad (1) $$

Here $C$ is the capacitance of the grains arranged on an arbitrary $D$-dimensional lattice with lattice points $\vec{r}$. $\theta(\vec{r}, \tau)$ is the phase of the superconducting order parameter on the grain at $\vec{r}$ at imaginary time $\tau$, and $\theta(\vec{r},\omega_n)$ is its temporal Fourier transform. The bosonic Matsubara frequency $\omega_n = 2\pi n/\beta$ with $n$ an integer, and $\beta$ the inverse-temperature. $\langle \vec{r},\vec{r}' \rangle$ denotes nearest neighbor pairs, and $\Delta \theta_{\vec{r},\vec{r}'} = \theta(\vec{r}) - \theta(\vec{r}')$. The term containing $\alpha$ arises from integrating out a harmonic-oscillator bath with the linear spectral-function required to produce Ohmic resistance [1]. In $\tau$-space, this term reads $\frac{\alpha}{2\pi} \int d\tau \int d\tau' (\frac{\bar{\theta}(\vec{r},\tau) - \theta(\vec{r},\tau')}{\tau - \tau'})^2$, which is a long ranged interaction in time. We restrict the sum on the Matsubara frequencies to those satisfying $|\omega_n| < \omega_c$, where $\omega_c$ is an ultraviolet cutoff.

To study the dissipation-tuned quantum phase transition in this model, we use a renormalization group that is dissipative in the spatial coupling $V$ [6, 14]. This RG procedure starts by dividing the field $\theta(\vec{r}, \tau)$ into slow and fast components $\theta_s(\vec{r}, \tau)$ and $\theta_f(\vec{r}, \tau)$, respectively, defined by having the temporal Fourier transform of $\theta_f(\vec{r}, \tau)$ equal that of $\theta(\vec{r}, \tau)$ for “high” Matsubara frequencies obeying $\frac{\omega_c}{b} < |\omega_n| < \omega_c$, and equaling zero for all smaller Matsubara frequencies, while the temporal Fourier transform of $\theta_s(\vec{r}, \tau)$ equals that of $\theta(\vec{r}, \tau)$ for “low” Matsubara frequencies obeying $|\omega_n| < \frac{\omega_c}{b}$. We can then write the partition function $Z$ as

$$ Z = Z_0 \int \prod_{\vec{r}} D\theta_s(\vec{r}, \tau) \exp \left[ -S^0_s/\hbar + \ln(e^{-S'/\hbar})_{0f} \right]. \quad (2) $$

Here $Z_0$ is a normalization constant, $S^0_s$ is the slow-frequency component of the quadratic part of the action, $S'$ contains the spatial coupling term $V$, and $(\ldots)_{0f}$ denotes averages over the fast components. After taking the averages, we rescale $\tau$, $\tau' = \tau/b$, $b$ being the scale factor, to restore the original frequency cut-off. Finally we redefine the coupling constants to complete the renormalization. Note that we do not rescale the fields $\theta(\vec{r}, \tau)$; this is to avoid introducing a factor multiplying $\Delta \theta_{\vec{r},\vec{r}'}(\tau)$ in the argument of the cosine in Eq. (11). With this choice to not rescale $\theta(\vec{r}, \tau)$, the dissipation term $\alpha$ in Eq. (11) is dimensionless. Furthermore, because it is non-analytic in $\omega_n$, it gets no “graphical” corrections (i.e., it is unchanged by the step of integrating out the “fast” fields). Hence, it is held fixed by the RG. The term associated with $C$ has $\tau$-dimension $-1$; it is irrelevant in the RG sense. The only place where
C appears is in the frequency integrals such as,
\[ \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{C\omega^2 + \frac{\alpha}{\pi} |\omega|^\beta} = \frac{1}{\alpha} \ln \frac{\alpha \beta}{\pi C}. \] (3)

Thus, \( \frac{1}{C} \) acts as an upper frequency cut off, and we associate it with \( \omega_c, \omega_c = \frac{\alpha}{\pi C}. \)

Writing \( b = e^l \), where \( l > 0 \) is infinitesimal, we have, to one loop order [6, 14]:
\[ \frac{dV}{dl} = (1 - \frac{1}{\alpha})V \] (4)

This recursion relation implies that the \( \alpha \)-axis (\( V = 0 \)) is a line of stable fixed points for \( \alpha < 1 \) and a line of unstable fixed points for \( \alpha > 1 \). Defining \( \alpha_c = 1 \), for \( \alpha < \alpha_c \), the barrier \( V \) between the different potential minima of the cosine potential is irrelevant, and so the fields \( \theta(\vec{r}, \tau) \) on each site fluctuate quantum mechanically independently of those on all the other sites. For \( \alpha > \alpha_c \), on the other hand, the barrier \( V \) grows, quenching these fluctuations, and the system becomes a global superconductor. Therefore, at \( \alpha = 1 \), there is a metal-to-superconductor quantum phase transition.

Before we go to the higher order corrections to Eq. 4 and discuss the phase diagram, we first show that on the metallic side of \( \alpha = 1 \), the local phases \( \theta(\vec{r}, \tau) \), which are critical in the imaginary time variable, exist in a so called ‘floating’ phase. The floating phase for the array is defined as a phase where all spatial Josephson couplings, including the longer ranged ones that nominally couple spatially far separated grains, are irrelevant [14, 20]. From Eq. 4 with irrelevant \( V \), the average of the order parameter on a grain is zero at \( T = 0 \),
\[ \langle \exp(i\theta(\vec{r}, \tau)) \rangle = \frac{1}{(\omega_c \beta)^{\frac{1}{\alpha}}}. \] (5)

Now, a generic arbitrary range Josephson coupling among pairs of grains takes the form:
\[ S_J = -\int d\tau \sum_{\vec{r}, \vec{r}'} J(\vec{r}') \cos(\theta(\vec{r}, \tau) - \theta(\vec{r} + \vec{r}', \tau)), \] (6)

where \( |\vec{r}'| \) gives the spatial range of the coupling. When \( V \) is irrelevant, the average of \( S_J \) scales as \( \langle S_J \rangle = \beta \langle \exp(i\theta(\vec{r}, \tau)) \rangle^2 \sim \beta^{1 - \frac{1}{\alpha}} \). At zero temperature, this term is irrelevant for \( \alpha < 1 \). Thus, all spatial couplings, including the nearest neighbor coupling \( V \), are irrelevant throughout the metallic phase. Interestingly, they all become relevant simultaneously at \( \alpha = 1 \). Hence, for \( \alpha < 1 \), the system is in a floating critical phase.

The first order flow equation, Eq. 4, only establishes a phase transition. To know the phase diagram and the critical properties, such as correlation length and time exponents, we have to go beyond first order. To obtain higher order corrections, we perform a cumulant expansion of Eq. 2 in second order in \( V \). By the method outlined in Ref. [14], we find higher order contributions to Eq. 4 coming from the closed loops on the lattice. At higher orders in RG, longer ranged pairwise interactions \( J(\vec{r}) \) of Eq. 4 are successively generated. In a consistent treatment of the RG, all these terms should be included in the starting Hamiltonian. Since the quadratic part of Eq. 4 decouples in space, it is easy to see that all the \( J(\vec{r}) \)'s have the same linear order recursion relation as that of the nearest neighbor interaction \( V \), Eq. 4. At second order cumulant expansion of Eq. 4, \( J(\vec{r}) \) for a particular \( \vec{r} \) is generated by all possible combinations of \( J(\vec{r}') \) and a \( J(\vec{r} - \vec{r}') \) for which \( \vec{r}, \vec{r}' \) and \( \vec{r} - \vec{r}' \) lie on a triangle on the
lattice [14]. Treating $V$ as a special case of $J(\vec{r})$ (that corresponding to the nearest neighbor vectors $\vec{a}_i$, $J(\vec{a}_i) = V$), we get, up to quadratic order in the couplings,

$$\frac{dJ(\vec{r})}{dl} = (1 - \frac{1}{\alpha})J(\vec{r}) + A \sum_{\vec{r}'} J(\vec{r}')J(\vec{r} - \vec{r}') \quad (7)$$

Here $A$ is a positive number, $A \sim \int_0^\beta d(\tau - \tau') [\exp(-G_f(\tau - \tau')) - 1]$, where $G_f(\tau - \tau') = <\theta_f(0, \tau)\theta_f(0, \tau')>_0$ is the unequal time correlation function of the fast modes [14]. To avoid generation of spurious long ranged interactions, a smooth cut off prescription is adopted to calculate $G_f(\tau - \tau')$, which then makes it exponentially decaying with $|\tau - \tau'|$ [5,26]. Since $A$ depends on this cut off scheme, physical quantities such as exponents should be independent of $A$.

It is essential for our subsequent analysis that all of the factors $J(\vec{r}')J(\vec{r} - \vec{r}')$ enter the recursion relation for $J(\vec{r})$ with the same coefficient $A$. This is a simple consequence of the fact that, in contrast to the bond dissipation case, here, in the absence of the Josephson coupling terms, fluctuations of $\theta(\vec{r}, \tau)$ on any site $\vec{r}$ are completely independent of those on all other sites $\vec{r}'$. As a result, the averages over $\theta(\vec{r}', \tau)$’s on different sites that must be computed in the perturbation theory to obtain $A$ are the same, regardless of which sites $\vec{r}$ and $\vec{r} - \vec{r}'$ are being considered.

Because of the fact that $A$ is a constant that factors out of the sum on $\vec{r}'$ in (7), that formidable looking set of infinitely many coupled non-linear recursion relations decouple in Fourier space. Defining the Fourier transform $\hat{J}(\vec{q})$ of $J(\vec{r})$ via

$$J(\vec{r}) \equiv \int_{BZ} d^d q \hat{J}(\vec{q}) \exp(i\vec{q} \cdot \vec{r})$$

where the $\int_{BZ} d^d q$ is over the Brillouin zone of the lattice, and defining $\epsilon \equiv \frac{1}{\alpha} - 1$, Eq. (7) reduces to

$$\frac{d\hat{J}(\vec{q})}{dl} = -\epsilon \hat{J}(\vec{q}) + AJ^2(\vec{q}). \quad (8)$$

Let’s consider the metallic side of $\alpha_c = 1$, where $\epsilon$ is positive. Equation (8) has a stable fixed point at $\hat{J}(\vec{q}) = 0$, and a critical fixed point at $\hat{J}(\vec{q}) = \epsilon/A$. Fig. 2 shows the flow of $\hat{J}(\vec{q})$.

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It follows that as long as the maximum of $\hat{J}(\vec{q})$ in $q$-space is less than $V_c(\epsilon) = \epsilon/A$, the system is in the floating metallic phase. As shown in Fig. 3 when the maximum lies above $V_c(\epsilon)$, $\hat{J}(\vec{q})$’s for a band of $q$ values become relevant and the system becomes a global superconductor. To illustrate this, let’s consider a particular form of the bare $J(\vec{r})$, namely, that for nearest neighbor interactions only, on a $D$-dimensional hypercubic lattice, $J(\vec{r}, l = 0) = V \sum_{\vec{a}_i} \delta(\vec{r} - \vec{a}_i)$, where the $\vec{a}_i$’s are the set of nearest neighbor vectors on this lattice.

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Fig. 2 – RG flows for $\hat{J}(\vec{q})$ as implied by Eq. (8).
This gives a bare $J(\vec{q})$ of the form,

$$J(\vec{q}, l = 0) = V \sum_{\vec{a}_i} \cos(\vec{q} \cdot \vec{a}_i).$$

(9)

Fig. 3 – A typical form of the bare $J(q)$ in $q$-space. The dotted horizontal line indicates $V_c(\epsilon)$. When the maximum of $J(q)$ at $q = 0$ lies below the dotted line (solid curve), the system is in the floating metallic phase. When the maximum lies above the dotted line (dashed curve), $J(q)$’s in a band of $q$ values, $|q| < q_c$, become relevant, and the system becomes a global superconductor.

As shown in Fig. 3, as long as $zV$ is less than $V_c(\epsilon)$, where $z = 2d$ is the coordination number of the lattice, all $J(\vec{q})$’s are irrelevant. When $zV$ is greater than $V_c(\epsilon)$, $J(\vec{q})$’s in a band of $q$ values are relevant. We expand $J(\vec{q}, l = 0)$ in the neighborhood of $q = 0$,

$$J(\vec{q}, l = 0) \approx J(q = 0) - B|q|^2,$$

(10)

where $J(q = 0) = zV$ and $B = 2Va^2$ (with $a$ the length of a nearest neighbor vector) in the current example. For more general lattice types, and further than nearest neighbor interactions, the form of Eq. (10) will still hold for small $\vec{q}$, provided only that the bare Josephson couplings $J(\vec{r})$ are such as to produce a net ferromagnetic interaction (by which we mean one that favors a spatially uniform $\theta(\vec{r}, \tau)$), and short-ranged. Indeed, that $J(\vec{q})$ has its maximum at $\vec{q} = \vec{0}$, and that that maximum is a simple quadratic maximum as in Eq. (10), can be taken as a definition of ferromagnetic, short-ranged interactions. Furthermore, it will be satisfied by any system that has, in the bare Hamiltonian, only nearest-neighbor ferromagnetic interactions, whatever the lattice type.

Equating $J(\vec{q}, l = 0)$ in the neighborhood of its maximum with $V_c(\epsilon)$, we get the edges of the band of $q$ values $|q| < q_c, q_c = \frac{\sqrt{2V_c(\epsilon)}}{z} \left( \frac{V - V_c(\epsilon)}{z} \right)^{\frac{1}{2}},$ for which the $J(\vec{q})$’s are relevant. This defines a diverging correlation length $\xi \approx q_c^{-1} \approx (V - V_c(\epsilon))^{-\frac{1}{2}}$ with a mean field like exponent $\frac{1}{2}$. We have thus established, we believe for the first time, a diverging spatial correlation length at the floating to coupled (classical or quantum) phase transition. This length $\xi$ could be measured experimentally using the AC Josephson effect. Specifically, we predict that for a 1D chain of $N$ junctions, with opposite ends held at a small temporally constant voltage difference $\Delta V$, an oscillating Josephson current $I_J(t)$ will occur, whose frequency $\Omega$ is independent of $N$ for $N \ll \xi$, while being inversely proportional to $N$ for $N \gg \xi$. The crossover between these two behaviors determines $\xi$. The linear form of $V_c(\epsilon) = \frac{\epsilon}{\alpha}$, where $\epsilon = \alpha - \alpha$, determines
the RG flows and the perturbative phase diagram in the $V - \alpha$ plane as shown in Fig. 1. Finally, linearizing Eq. 8 around the critical fixed point, we get the correlation time exponent $\nu_{\tau}, \xi_{\tau} \sim (V - V_c(\epsilon))^{-\nu_{\tau}},$ with the continuously varying critical exponent $\nu_{\tau} = \frac{1}{\alpha} = \frac{1}{(\alpha - \alpha_c)}.$ This defines a dynamic exponent $z,$ defined by $\xi_{\tau} \sim \xi^z,$ to be $z = 2\nu_{\tau} = \frac{2}{\alpha_c - \alpha}.$

To reiterate the differences of the present calculation from the case of bond dissipation [14], we note that although the linear order RG equation, Eq. 4, is identical in both models, the second order equations, Eqs. (7, 8), are different. Because of the constancy of the number $A,$ we are able here to solve the set of infinitely many coupled non-linear equations exactly by going to the Fourier space. This allows us to compute important critical exponents such as $\nu$ and $\nu_{\tau}.$ We are also able here to demonstrate the divergence of the spatial correlation length, to our knowledge for the first time, in an FC transition. Note that, in the corresponding problem of the coupled XY planes [20], or the case of bond dissipation [14], this was an interesting open problem because of technical difficulties.

In systems where the ‘shunt to the ground’ is realized, either by a ground plane, or intrinsically in granular systems [17, 19], the FC transition is tunable by varying the shunt resistance (which varies $\alpha$ in our model). By measuring the value of $\alpha$ at which the transition happens for a variety of different $V$’s, it should be possible to test our prediction for $\alpha_c,$ and that the phase boundary comes into the $\alpha$-axis at $\alpha_c$ with a finite, non-zero slope. A similar phase diagram has been obtained experimentally in a dissipative array in Ref. [27], but in that experiment a bond dissipation model was more applicable. In systems where the present model is applicable, the non-universal exponent $\nu_{\tau}$ can be directly measured by measuring a characteristic temperature, $T_{ch} \propto (V - V_c(\epsilon))^{\nu_{\tau}},$ in either phase close to the phase boundary. In the superconducting side $T_{ch}$ can be the superconducting $T_c,$ while in the resistive side, it can be the temperature where the resistance shows a minimum [8]. On the resistive, metallic side of the transition, the floating character can be established by measuring the current-voltage characteristics which are expected to be a non-uniform power law depending on $\alpha$ [28]. Finally, although the correct model for capacitances of a real experimental system may also include longer ranged capacitors, in contrast to the on-site capacitance $C$ used here, we can show that this does not change any of our universal results.

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REFERENCES

[1] Caldeira A. O., and Leggett A. J., Phys. Rev. Lett., 46 (1981) 211; Caldeira A. O., and Leggett A. J., Ann. Phys., 149 (1983) 374.
[2] Chakravarty S., Phys. Rev. Lett., 49 (1982) 681.
[3] Bray A. J. and Moore. M. A., Phys. Rev. Lett., 49 (1982) 1545.
[4] Schmid A., Phys. Rev. Lett., 51 (1983) 1506.
[5] Fisher M.P.A. and Zwerger W., Phys. Rev. B, 32 (1985) 6190.
[6] Chakravarty S., Ingold G.-L, Kivelson S., and Luther A., Phys. Rev. Lett., 56 (1986) 2393; Chakravarty S., Ingold G., Kivelson S., and Zimanyi G., Phys. Rev. B, 37 (1988) 3283.
[7] Ioffe L. B., Feigel’man M. V., Ioselevich A., Ivanov D., Troyer M., and Blatter G., Nature, 415 (2002) 503.
[8] Fisher M. P. A., Phys. Rev. B, 36 (1987) 1917.
[9] Zwerger W., J. Low temp. Phys., 72 (1988) 291.
[10] Panyukov S. V., and Zaikin A. D., J. Low Temp. Phys., 75 (1989) 365; J. Low Temp. Phys., 75 (1989) 389.

[11] Zwerger W., Z. Phys. B, 78 (1990) 111.

[12] Fazio R., and Zant H., Physics Reports, 355 (2001) 235.

[13] Werner P., Vöelker K., Troyer M., and Chakravarty S., Phys. Rev. Lett., 94 (2005) 047201.

[14] Tewari S., Toner J., and Chakravarty S., Phys. Rev. B, 72 (2005) 060505(R); Tewari S., Toner J., and Chakravarty S., Phys. Rev. B, 73 (2006) 064503.

[15] Refael G., Demler E., Oreg Y., and Fisher D. S., Phys. Rev. B, 68 (2003) 214515.; Refael G., Demler E., Oreg Y., and Fisher D. S., cond-mat/0511212.

[16] Bobbert P. A., Fazio R., Schö n G, and Zimanyi G. T., Phys. Rev. B, 41 (1990) 4009.

[17] Wagenblast K.-H, Otterlo A. V., Schö n G., and Zimán yi G. T., Phys. Rev. Lett., 78 (1997) 1779.

[18] Völker K., cond-mat/9911473.

[19] Polak T. P., and Kopec T. K., cond-mat/0504752.

[20] O’Hern C. S., Lubensky T. C., and Toner J., Phys. Rev. Lett., 83 (1999) 2745.

[21] Emery V. J., Fradkin E., Kivelson S. A., and Lubensky T. C., Phys. Rev. Lett., 85 (2000) 2160.

[22] Maltseva M. and Coleman P., cond-mat/0502003.

[23] Pankov S., Florens S., Georges A., Kotliar G., and Sachdev S., Phys. Rev. B, 69 (2004) 054426.

[24] Sachdev S., Werner P., and Troyer M., Phys. Rev. Lett., 92 (2004) 237003.

[25] Schö n G., and Zaikin A. D., Physics Reports, 198 (1990) 237.

[26] Ma S.-K., Modern Theory of Critical Phenomena (Benjamin/Cummings, Reading, Massachusetts) 1976.

[27] Miyazaki H., Takahide Y., Kanda A., and Ootuka Y., Phys. Rev. Lett., 89 (2002) 197001.

[28] Kane C. L., and Fisher M. P. A., Phys. Rev. B, 46 (1992) 15233.