HETEROTIC APPROACH TO THE NUCLEON
DISTRIBUTION AMPLITUDE

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Abstract

We give an in-depth analysis of the determination procedure of the recently proposed heterotic nucleon distribution amplitude which hybridizes the best features of the Chernyak-Ogloblin-Zhitnitsky and the Gari-Stefanis models. With respect to the QCD sum-rule constraints, optimized versions of these amplitudes are derived in terms of which a "hybridity" angle can be introduced to systematically classify all models.

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Large-momentum transfer exclusive reactions comprise two mechanisms based on the factorization theorem \[1,2\]: the hard-gluon exchanges that describe short-range interactions via perturbation theory and the long-range (nonperturbative) soft processes that ensure confinement in the hadronic states. Thus, such processes depend in detail on the composition of the hadron wave function or more precisely, the hadron distribution amplitude: \( \Phi(x_i, Q^2) \).

In a physical gauge as the light-cone gauge \( A^+ = A^0 + A^3 = 0 \), this is the fundamental covariant amplitude describing the correlations between massless valence quarks with fractional momenta \( 0 \leq x_i = \frac{k_i^+}{p^+} \leq 1 \) \((\sum_i x_i = 1)\) and relative transverse momenta \( k_{\perp}^{(i)} \), \( p^+ \) being the longitudinal momentum of the incoming hadron. Then, apart from logarithmic scaling violations,

\[
\Phi(x_i, Q^2) = \int [d^2k_{\perp}] \psi(x_i, k_{\perp}^{(i)}),
\]

where \([d^2k_{\perp}] = 16\pi^3 \delta^{(2)}(\sum_{i=1}^{3} k_{\perp}^{(i)}) \prod_{i=1}^{3} \left[ \frac{d^2k_{\perp}^{(i)}}{16\pi^3} \right]\) and \( \psi(x_i, k_{\perp}^{(i)}) \) is the lowest-twist Fock-space projection amplitude of the hadron bound state. The actual calculation of \( \Phi(x_i, Q^2) \) from QCD requires nonperturbative methods such as QCD sum rules \[3\], lattice gauge theory \[4,5\] or the direct diagonalization of the light-cone hamiltonian within a discretized light-cone setup (see, eg., \[6\]).

In the context of QCD sum rules, useful theoretical constraints on the nucleon distribution amplitude have been derived by a number of authors \[7–9\]. They have been used to obtain model distribution amplitudes for the nucleon \[7–10\] and the \( \Delta^+(1232) \) isobar \[11–13\] in terms of the eigenfunctions of the evolution equation \[1\] represented by Appell polynomials \[14\]. However, the inevitable model purposes lead to emphasis on different facets of the complete picture. As a result, the Chernyak-Ogloblin-Zhitnitsky (COZ) model \[9\] predicts \( R \equiv |G_{nM}^u|/G_{pM}^p \approx 0.5 \), whereas the Gari-Stefanis (GS) model \[10\], inspired by a semiphenomenological analysis \[15\], gives by construction a small \( R \), viz., \( R = 0.097 \).

One would like the sum rules eventually to force the choice between COZ-type and GS-type models on grounds of mathematical, rather than phenomenological, necessity. In a recent publication \[16\] (see also \[17\]) we have suggested that there is a third distinct
possibility for the nucleon distribution amplitude, which actually hybridizes features of both types of models into a single mold. What is novel is the requirement that these models should be regarded as different aspects of a more fundamental unifying structure, we termed the "heterotic" solution. In the present work our interest in the model is primarily theoretic. We shall show that the heterotic solution is uniquely determined in the parameter space spanned by the expansion coefficients on the orthonormalized eigenfunctions of the evolution equation. With respect to a $\chi^2$ criterion to describe the deviations from the sum rules, it corresponds to that local minimum which is associated with the smallest possible value of $R$, notably 0.1, compatible with the sum-rule constraints [9,8].

In [16,17] we have given evidence that the heterotic model is the single best way to promote agreement between theory and the data on a variety of exclusive processes including elastic electron-nucleon scattering and charmonium decays into $p\bar{p}$.

In searching for a starting point from which to develop a credible nucleon distribution amplitude, we employ the ansatz described in [18]. Then the mixed-symmetry distribution amplitude $\Phi_N(x_i) = V(x_i) - A(x_i) \ [20]$ at fixed scale $Q^2$ can be reconstructed from its moments

$$\Phi_{N}^{(n_1 n_2 n_3)} = \int_0^1 [dx] x_1^{n_1} x_2^{n_2} x_3^{n_3} \Phi_N(x_i).$$

To this end, consider

$$(z \cdot p)^{-(n_1 + n_2 + n_3)} \prod_{i=1}^3 \left(iz \cdot \frac{\partial}{\partial z_i}\right)^{n_i} \Phi_N(z_i \cdot p)|_{z_i=0} = \Phi_{N}^{(n_1 n_2 n_3)},$$

where $z$ is an auxiliary lightlike vector ($z^2 = 0$). To determine the moments, a short-distance operator-product expansion is performed at some spacelike momentum where quark-hadron duality is supposed to be valid. Such a computation involves correlation functions of the form $(-q^2 = Q^2)$

$$I^{(n_1 n_2 n_3, m)}(q, z) = i \int d^4x e^{iqx} < \Omega | T(O_\gamma^{(n_1 n_2 n_3)}(0) \hat{O}_\gamma^{(m)}(x)) | \Omega > (z \cdot \gamma)_{\gamma'\gamma},$$

$$= (z \cdot q)^{n_1 + n_2 + n_3 + m + 3} I^{(n_1 n_2 n_3, m)}(q^2),$$

$$3$$
where $O^{(n_1n_2n_3)}_\gamma$ are appropriate three-quark operators interpolating between the proton state and the vacuum. Since they contain derivatives, their matrix elements are related to moments of the covariant amplitudes $V$, $A$, and $T$:

$$
<\Omega|O^{(n_1n_2n_3)}_\gamma(0)|P(p)> = f_N(z \cdot p)^{n_1+n_2+n_3+1} N_\gamma O^{(n_1n_2n_3)}. \tag{5}
$$

The factor $(z \cdot \gamma)\gamma\gamma'$ serves to separate out the leading twist structure in the correlator; $N_\gamma$ is the proton spinor, and $f_N$ denotes the "proton decay constant".

Because $\Phi_N(x_i)$ must be a solution of the evolution equation, it can be expressed in the form

$$
\Phi_N(x_i, Q^2) = \Phi_{as}(x_i) \sum_{n=0}^{\infty} B_n \tilde{\Phi}_n(x_i) \left( \frac{\alpha_s(Q^2)}{\alpha_s(\mu^2)} \right)^{\gamma_n}, \tag{6}
$$

in which $\{\Phi_n\}_{0}^{\infty}$ are the eigenfunctions of the interaction kernel of the evolution equation, orthonormalized within a basis of Appell polynomials of degree $M$, and $\Phi_{as}(x_i) = 120x_1x_2x_3$ (see [1]). [The conventions and analytical expressions given in [18] are used.] The corresponding eigenvalues $\gamma_n$ are identical with the anomalous dimensions of multiplicatively renormalizable baryonic operators of twist three [19]. Because the $\gamma_n$ are positive fractional numbers increasing with $n$, higher terms in the expansion (6) are gradually suppressed.

The expansion coefficients $B_n$ can be determined by means of moments inversion on account of the orthogonality of the eigenfunctions $\{\tilde{\Phi}_n(x_i)\}$:

$$
B_n(\mu^2) = \frac{N_n}{120} \int_0^1 [dx] \tilde{\Phi}_n(x_i) \Phi_N(x_i, \mu^2). \tag{7}
$$

Evaluating the correlator in Eq. (4) for $n_1 + n_2 + n_3 \leq 3$ and $m = 1$, the coefficients $B_n$ for $n = 0, 1, \ldots, 5$ (i.e., $max(M) = 2$) can be computed upon imposing on the moments of $\Phi_N$ the sum-rule constraints estimated by COZ in [8].

The starting point of our method is to consider the hyperspace induced by the expansion coefficients $B_n$ and look for solutions of the form given by Eq. (6) complying with the COZ sum-rule requirements. The deviations from the sum rules are parametrized by a $\chi^2$ criterion which accounts for the order of the moments (corresponding to the superior stability of lower moments relatively to the higher ones [18]). For every moment $m_k$ ($k = 1, \ldots, 18$), we define
\[ \chi^2_k = (\chi^2_{k,(a)} + \chi^2_{k,(b)}) \left[ 1 - \Theta(m_k - M_k^{\text{min}})\Theta(M_k^{\text{max}} - m_k) \right] \] (8)

with

\[ \chi^2_{k,(a)} = \text{min}(|M_k^{\text{min}} - m_k|, |m_k - M_k^{\text{max}}|)N_k^{-1}, \] (9)

where \( N_k = |M_k^{\text{min}}| \) or \( |M_k^{\text{max}}| \), whether \( m_k \) lies on the left or on the right hand side of the corresponding sum-rule interval (\( \chi^2_{\text{tot}} = \sum_k \chi^2_k \)).

Deviations from lower-order moments are weighted by a larger penalty factor than those from higher-order moments:

\[ \chi^2_{k,(b)} = \begin{cases} 100, & 1 \leq k \leq 3 \\ 10, & 4 \leq k \leq 9 \\ 1, & 10 \leq k \leq 18. \end{cases} \] (10)

In this way we were able to filter out a COZ-type solution (designated as \( \text{COZ}^{\text{opt}} \)) which is the absolute minimum of \( \chi^2 \) and thus provides the best possible agreement with the sum rules. Evolving that solution towards lower values of \( R \), we generated a series of local minima of \( \chi^2 \), plotted in Fig. 1(a) in the \((B_4, R)\) plane. [The assignments of models to symbols are given in Table 1.] The dashed line is a convenient fit expressed by \( R = 0.436415 - 0.005374B_4 - 0.000197B_4^2 \). The ultimate local minimum accessible from \( \text{COZ}^{\text{opt}} \) on that \( \chi^2 \) orbit corresponds to \( R = 0.1 \)—the heterotic solution. This solution, although degenerated with respect to \( R \), is distinct from the GS one. The latter, as well as its optimized versions we determined, constitute an isolated region (an ”island”) in the parameter space. Technically this means that they correspond to local minima of \( \chi^2 \) at considerably lower levels of accuracy so that they are separated from the COZ-Het orbit by a large \( \chi^2 \) barrier. Without considering here applications to physical processes, we only mention that the predictions extracted from the heterotic model are dramatically different compared to those following from the GS model [16,7,21]. Also, the CZ and KS models, although in the vicinity of the COZ-Het orbit, are actually isolated points because they correspond to much larger \( \chi^2 \) values (see Fig. 1(b) in correspondence with Table 1). One has to exercise a certain amount of care to
be sure that these are the only regions in the parameter space contributing to the sum rules at the desired level of accuracy.

The general picture which emerges is a pattern of nucleon distribution amplitudes which develops into several regions of different $\chi^2$ dependence (Fig. 1(b)). The new heterotic amplitude and the original COZ one correspond to similar $\chi^2$ values. In contrast, the previous models (CZ, GS, and KS) are characterized by large deviations from the sum rule requirements. Optimum consistency with the sum rules is provided by the amplitude $COZ^{opt}$, but this amplitude is not favored because it fails to predict hadronic observables in agreement with the existing data [21]. It is remarkable that the heterotic solution [16] matches the King-Sachrajda [8] sum-rule constraints better than the original COZ amplitude. This is particularly important because the KS results have been independently verified in [11].

The variation with shape of the nucleon distribution amplitude with $R$ is shown graphically, obtained by interpolating between the calculated version of the optimized COZ amplitude and the heterotic one, in Fig 2. Note that although the heterotic solution was continuously evolved from $COZ^{opt}$, its geometrical characteristics are something of a hybrid between COZ-type and GS-type amplitudes. [The profile of $\Phi^N_{Het}$ is shown in [13,[17].] This heterotic character becomes apparent by considering the corresponding covariant distribution amplitudes $V$, $A$, and $T$ [20], shown in Fig. 3. While $V_{Het}$ has a symmetry pattern similar to that of $V_{GS}$ (one main maximum), $T_{Het}$ is characterized by two maxima, as $T_{COZ}$. The inverse heterotic combination does not belong to the COZ-Het $\chi^2$-orbit. This type of solution can be realized by the following expansion coefficients: $B_1 = 4.3025$, $B_2 = 1.5920$, $B_3 = 1.9675$, $B_4 = -19.6580$, and $B_5 = 3.3531$, corresponding to $\vartheta = 24.44^\circ$ (see Eq. (11)) and $\chi^2 = 30.634$. Remarkably, this solution yields $R = 0.448$, i.e., it is COZ-like (albeit in a different region of the parameter space). The dynamic implications of our solutions for various physical observables will be discussed elsewhere.

To put these remarks on mathematical grounds, we propose to consider a classification scheme of the various nucleon distribution amplitudes based on the observation that the op-
timized versions of the COZ-type and GS-type amplitudes we derived are almost orthogonal to each other with respect to the weight $w(x_i) = \Phi_{as}(x_i)/120$ [14]. Their normalized inner product $(COZ_{opt}, GS_{opt})$ yields 0.1607, which corresponds to an angle of 80.8°. Thus they form a quasi-orthogonal basis and can be used to classify the nucleon distribution amplitudes in terms of a "hybridity" angle $\vartheta$, defined by

$$\vartheta = \arctan \left( \frac{\kappa_1(i)}{\kappa_2(i)} \right),$$

where $\kappa_1(i) = (GS_{opt}, i)$ and $\kappa_2(i) = (COZ_{opt}, i)$ and the index $i$ denotes one of the nucleon distribution amplitudes listed in Table 1. The hybridity angle parametrizes the mingling of geometrical characteristics attributed to COZ-like and GS-like amplitudes and provides a quantitative measure for their presence in any solution conforming with the sum-rule constraints (Fig. 3(c)). The superimposed dashed line is a fit given by $\vartheta/[deg] = 8.5693 + 0.0160B_4 + 0.0073B_4^2 - 0.00067B_4^3$. The mixing of different geometrical characteristics is particularly relevant for the heterotic solution, which generically amalgamates features of both types of amplitudes. In this role, the heterotic model has the special virtue of simultaneously fitting the twin hopes for making reliable predictions with respect to the experimental data while being in agreement with the sum-rule constraints [13,16,21]. The other considered models can be tuned to fit some aspects of the data, but never all aspects simultaneously.

From the above considerations we conclude that a key clue to the determination of nucleon distribution amplitudes on the basis of QCD sum rules is the resulting pattern of order effected in Fig. 1, the ordering parameters being the expansion coefficient $B_4$ and the hybridity angle $\vartheta$. This classification scheme can valuably supplement the standard fitting procedure to the sum rules when enlarging the basis of eigenfunctions of the evolution kernel to include higher-order Appell polynomials. On the phenomenological side, fixing the value of the ratio $R$ by experiment, one could use Fig. 1(a) to extract the corresponding nucleon distribution amplitude consistent with the sum-rule constraints.
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TABLE I. Expansion coefficients $B_n$, hybridity angle $\vartheta$, and values of the ratio $R = |G^s_M|/G^p_M$, in correspondence with those for $\chi^2$ for the various nucleon distribution amplitudes discussed in the text.

| Model     | $B_1$   | $B_2$   | $B_3$   | $B_4$   | $B_5$   | $\vartheta$[deg] | R   | $\chi^2$ | Symbol |
|-----------|---------|---------|---------|---------|---------|-------------------|-----|-----------|--------|
| Het.      | 3.4437  | 1.5710  | 4.5937  | 29.3125 | -0.1250 | -1.89             | .104| 33.48     | •      |
| COZ$^{opt}$ | 3.5268  | 1.4000  | 2.8736  | -4.5227 | 0.8002  | 9.13              | .465| 4.49      | ▼      |
| GS$^{opt}$ | 3.9501  | 1.5273  | -4.8174 | 3.4435  | 8.7534  | 80.87             | .095| 54.95     | ▲      |
| GS$^{min}$ | 3.9258  | 1.4598  | -4.6816 | 1.1898  | 8.0123  | 80.19             | .035| 54.11     | ▼      |
| CZ        | 3.4050  | 1.9250  | 2.2470  | -3.4650 | 0.0180  | 13.40             | .487| 250.07    | •      |
| COZ       | 3.6750  | 1.4840  | 2.8980  | -6.6150 | 1.0260  | 10.16             | .474| 24.64     | □      |
| KS        | 3.2550  | 1.2950  | 3.9690  | 0.9450  | 1.0260  | 2.47              | .412| 116.35    | ⬤      |
| GS        | 4.1045  | 2.0605  | -4.7173 | 5.0202  | 9.3014  | 78.87             | .097| 270.82    | △      |

| Samples   | $B_1$   | $B_2$   | $B_3$   | $B_4$   | $B_5$   | $\vartheta$[deg] | R   | $\chi^2$ | +      |
|-----------|---------|---------|---------|---------|---------|-------------------|-----|-----------|--------|
| 0         | 3.3125  | 1.4644  | 3.1438  | -1.0000 | 0.8750  | 7.67              | .441| 4.63      | +      |
| 1         | 3.2651  | 1.4032  | 3.5466  | 2.8685  | 1.7954  | 8.94              | .405| 5.11      | +      |
| 2         | 3.4026  | 1.4917  | 3.0629  | 7.3430  | 0.6719  | 8.75              | .385| 16.07     | +      |
| 3         | 3.7225  | 1.5030  | 3.6592  | 10.7265 | 1.5154  | 9.29              | .355| 17.78     | +      |
| 4         | 3.8407  | 1.4968  | 3.2142  | 14.4093 | 0.8757  | 10.49             | .325| 19.41     | +      |
| 5         | 3.6544  | 1.4000  | 3.0993  | 15.5614 | -0.1329 | 6.35              | .305| 18.15     | +      |
| 6         | 3.8607  | 1.4000  | 3.2375  | 19.8571 | -0.1635 | 6.32              | .255| 20.57     | +      |
| 7         | 3.9783  | 1.4000  | 3.2706  | 22.4194 | -0.4805 | 5.29              | .225| 21.69     | +      |
| 8         | 4.1547  | 1.4000  | 3.3756  | 26.1305 | -0.5855 | 5.02              | .175| 23.53     | +      |
| 9         | 3.4044  | 1.5387  | 4.3094  | 25.5625 | 0.0625  | .01               | .153| 30.80     | +      |
FIGURES

FIG. 1. Classification scheme of nucleon distribution amplitudes conforming with the sum rules (Table 1). (a) The ratio $R = |G_M^p|/G_M$ as a function of the expansion coefficient $B_4$. The positions of the $\chi^2$ minima are indicated. (b) Distribution of local minima of $\chi^2$ (on a logarithmic scale) plotted vs. the expansion coefficient $B_4$. (c) Pattern of nucleon distribution amplitudes parametrized by the hybridity angle $\vartheta$, defined in Eq. (11), vs. the expansion coefficient $B_4$. The dashed curves are the fits described in the text.

FIG. 2. A two-dimensional example showing how the profile of the nucleon distribution amplitude $\Phi_N$ changes along the COZ-Het $\chi^2$ orbit. The amplitudes are labeled by the corresponding $R$-value.

FIG. 3. Covariant distribution amplitudes $V, A$, and $T$ associated with the optimized versions of the COZ and GS models in comparison with the heterotic model.
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