Temperature induced phase transitions in four-fermion models in curved space-time

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Abstract

The large $N$ limit of the Gross–Neveu model is here studied on manifolds with constant curvature, at zero and finite temperature. Using the $\zeta$–function regularization, the phase structure is investigated for arbitrary values of the coupling constant. The critical surface where the second order phase transition takes place is analytically found for both the positive and negative curvature cases. For negative curvature, where the symmetry is always broken at zero temperature, the mass gap is calculated. The free energy density is evaluated at criticality and the zero curvature and zero temperature limits are discussed.

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Introduction

The Gross–Neveu model [1] is the simplest version of models with four-fermion interactions first introduced in [2] as examples of dynamical symmetry breaking. These models have been widely studied in the literature in many different areas of theoretical physics: for example as low energy effective theories of QCD [3], or, in the Euclidean formalism, as realistic models describing phase transitions in statistical mechanics (see for example [4], where the stability conditions for the effective action in the large N limit are seen to imply the BCS gap equation of classical superconductivity). In the cosmological framework the massive composite field, $\sigma$, which arises after dynamical symmetry breaking, has been proposed as an inflaton in inflationary universe models [5]. With respect to each of the mentioned problems it is of interest to consider manifolds other than $\mathbb{R}^n$. In facts, the compactification of one direction from $\mathbb{R}$ to $S^1$ allows to describe $n-1$ dimensional systems at finite temperature. The introduction of scale parameters in the other directions allows, in low dimensional physics, the study of the behaviour of dynamical systems under deformation of their microscopic structure. Finally, the most obvious application of the study of this model in the non-zero curvature case is the cosmological framework.

There exists a very reach literature on four fermion models. An introduction is contained in [6]. A recent and updated review may be found in [7]. The issue of critical exponents and $\beta$ function in flat space is addressed for example in [8, 9, 10]. In [11]-[15] the effects of an external electromagnetic field in flat space-time are considered. The proof of $1/N$ renormalizability in 3 dimensions was first addressed in [16, 17, 18]. The thermodynamical behaviour in flat space-time is described in [19]-[22], where the temperature induced second order phase transition is found. The influence of a classical gravitational field on the dynamical symmetry breaking has been analyzed in [23]-[27], where the existence of curvature induced phase transitions is evidentiated. The relevance of an external electromagnetic field to descriptions of the early universe is considered in a series of papers where the electromagnetic effects are studied in combination with curvature effects [28]-[31].

In this paper we study the large $N$ limit of the GN model, on 3-d manifolds of constant, non-zero curvature, in the Euclidean formalism. We focus our attention on the second order, temperature induced, phase transition and the rôle of curvature in the process of symmetry restoration. This may be formalized by considering the model on manifolds of the form $\Sigma \times S^1$, where $\Sigma$ is a 2-dimensional surface with non-zero curvature and the inverse radius of the circle plays the rôle of the temperature (the compactification of one space-time dimension to the circle is also referred as a finite size effect). To our knowledge, the combined effect of curvature and finite temperature have only been considered in [32].
(in the weak curvature approximation) and [7]. In [33] the GN model is considered on manifolds of the form $\Sigma \times S^1$, but the thermodynamical phase transition is not discussed and the coupling constant is fixed to the flat space critical value. In non-perturbative approaches it is a known result, although yet debated, that positive curvature and finite temperature enhance the process of symmetry restoration, while negative curvature favours symmetry breaking. For positive curvature, a finite temperature does not modify the qualitative features of the model but simply changes the phase transition point, whereas, for negative curvature, there is no phase transition at $T = 0$ (the symmetry is broken for any value of the coupling and of the curvature), but as soon as the temperature is switched on a symmetry restoration becomes possible, indeed it is realized, for some value of $T$.

We consider the GN model in de Sitter and anti-de Sitter backgrounds, respectively the manifolds $S^2 \times S^1$ and $H^2 \times S^1$. We apply the large $N$ approximation, and use the zeta function regularization scheme. The GN model exhibits on $\mathbb{R}^3$ a two-phase structure, the phase transition occurring for a non-trivial value of the coupling constant, the ultraviolet (UV) fixed point. When considering the model in curved backgrounds, at finite temperature, the Lagrangian turns out to be dependent on three parameters: the coupling constant, $q$, the parameter $\beta$ (the inverse temperature), and the curvature parameter, $r$. Phase transitions can in principle occur with respect to any of them. With respect to the thermodynamical characteristics, we recover the qualitative behaviour of [4], and we find a simple analytic expression for the critical surfaces $f(r, \beta, q) = 0$. We follow the conventions of [34] and [33] where the zeta function regularization is adopted to study the non-linear sigma model and the GN model, respectively, at the critical value of the coupling constant. We find some interesting common features between the GN model and the non-linear $\sigma$ model. For example, the mass gap on $H^2 \times \mathbb{R}$ at the critical coupling, for the GN model is found to be half the mass gap of the non-linear $\sigma$ model on the same manifold.

In section 1 we briefly review the properties of the GN model and the large $N$ limit. Then we recall the flat space analysis in the zeta-function regularization. The curvature and temperature effects are considered in section 2. The two appendices are devoted to calculations related to section 2.

1 The Gross–Neveu model in three dimensions

In this section, upon giving some basic definitions for the study of the model on a generic Riemannian manifold in the $1/N$ expansion, we review the zeta function regularization
scheme as discussed in [33]. Then, to fix the notation and to illustrate the method in a simple case, we analyze the model in the Euclidean flat space $R^3$.

The Gross–Neveu model on a Riemannian manifold $(\mathcal{M}, g)$ is described in terms of an $O(n)$ symmetric action for a set of $N$ massless Dirac fermions. The Euclidean partition function in 3–dimensions in the presence of a background metric $g_{\mu\nu}(x)$ is given by

$$Z[g] = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \exp \left\{ \int_{\mathcal{M}} d^3x \sqrt{g} \left[ \bar{\psi}_i(x) \nabla \psi_i(x) + \frac{q}{2}(\bar{\psi}_i \psi_i)^2 \right] \right\} ,$$

where $i = 1, 2, \ldots, N$, $\nabla$ is the Dirac operator on $\mathcal{M}$, and $q$ is the coupling constant.$^1$

The Dirac matrices are given in terms of the Pauli matrices $\sigma_a$ by the expression

$$\gamma_\mu = V_{\mu,a} \sigma_a , \quad \text{with} \quad \mu, a = 1, 2, 3$$

where $V_{\mu,a}$ denote the dreibein defined by the equation

$$g_{\mu\nu} = V_{\mu,a}(x) V_{\nu,b}(x) \delta_{ab} .$$

The covariant derivative $\nabla_\mu$ acting on a spinor field is defined as $[3]$

$$\nabla_\mu = \partial_\mu + \Gamma_\mu(x) ,$$

where $\Gamma_\mu$ is the spin connection

$$\Gamma_\mu(x) \equiv \frac{1}{8} [\sigma_a, \sigma_b] V^\nu_{\mu} (\nabla_\mu V_{\nu,b}) .$$

In even dimensions the model has a discrete chiral symmetry which prevents the addition of a fermion mass term, while in odd dimensions a mass term breaks space parity.

As it is usually done, the partition function (1.1) is rewritten by introducing an auxiliary scalar field $\sigma$, such that

$$Z[g] = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\sigma] \exp \left\{ \int_{\mathcal{M}} d^3x \sqrt{g} \left[ \bar{\psi}_i (\nabla + \sigma) \psi_i - \frac{1}{2q\sigma^2} \right] \right\} .$$

The $\sigma$ field has no dynamical effect, its introduction amounting to multiply the partition function by an overall constant. We note here that in the large $N$ approximation the generating functional (1.6) describes the Nambu-Jona-Lasinio model as well, when the space-time dimension is greater than 2. This model differs by the GN model in the presence of an extra term in the action. It reads

$$S_{NJL} = S_{GN} - \int_{\mathcal{M}} d^3x \sqrt{g} (\bar{\psi}_i \gamma_5 \psi_i)^2 .$$

$^1$According to our notation the Dirac matrices obey the following algebra: $\gamma_\mu = \gamma^\dagger_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and $\text{Tr}(\gamma_\mu) = 0$. Thus, the Dirac operator is antihertian $\nabla^\dagger = -\nabla$. 

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Following the same method as above, it can be seen that the new interaction gives rise to another auxiliary field, called \(\pi\) in the literature. In the large \(N\) limit, for \(d > 2\), \(\pi\) can be reabsorbed into the definition of the \(\sigma\) field, giving rise to the generating functional (1.6) (see for example [7] for details).

The generating functional \(\mathcal{Z}\) has to be regularized in the ultraviolet. This is usually done by introducing a cut-off, \(\Lambda\), in the momentum space. By redefining the dimensional coupling constant \(1/q(\Lambda)\) as \(\Lambda/q'(\Lambda)\), the regularized partition function is formally rewritten as

\[
\mathcal{Z}[g, \Lambda] = \int \mathcal{D}_\Lambda[\psi] \mathcal{D}_\Lambda[\bar{\psi}] \mathcal{D}_\Lambda[\sigma] \exp \left\{ \int_M d^3x \sqrt{g} \left[ \bar{\psi}_i(\nabla + \sigma)\psi_i - \frac{\Lambda}{2q}\sigma^2 \right] \right\},
\]

(1.8)

where \(\mathcal{D}_\Lambda[\psi] = \prod_{|k|<\Lambda} d\psi(k)\) and similarly for the other fields (we will write \(q\) without the prime from now on).

As for the non linear \(\sigma\) model, the existence of a non trivial UV fixed point shows that the large momentum behaviour is not given by perturbation theory above 2 dimensions, where the theory is asymptotically free (see for example [6, 7]). We use here the \(1/N\) expansion, which has the property of providing a renormalizable theory for the 3-d GN model [16, 17, 18].

In the large \(N\) limit, which means \(N \to \infty\) keeping \(Nq(\Lambda)\) fixed, the generating functional can be calculated using the saddle point approximation. For this purpose we integrate over \(N - 1\) fermion fields, rescale the remaining fields \(\psi_N, \bar{\psi}_N\) to \(\sqrt{N-1}\psi_N, \sqrt{N-1}\bar{\psi}_N\), respectively, and redefine \((N-1)q(\Lambda)\) as \(q(\Lambda)\). Thus we get

\[
\mathcal{Z}[g, \Lambda, q(\Lambda)] = \int \mathcal{D}_\Lambda[\psi_N] \mathcal{D}_\Lambda[\bar{\psi}_N] \mathcal{D}_\Lambda[\sigma] \exp \left\{ (N-1)\text{Tr} \log_\Lambda(\nabla + \sigma) \right\}
\]

\[
\times \exp \left\{ (N-1) \int_M d^3x \sqrt{g} \left[ \bar{\psi}_N(\nabla + \sigma)\psi_N - \frac{\Lambda}{2q}\sigma^2(x) \right] \right\}.
\]

(1.9)

In the limit \(N \to \infty\) the dominating contribution to the functional integral comes from the extremals of the action. For an arbitrary metric \(g_{\mu\nu}(x)\), these are obtained by extremizing the action with respect to \(\psi_N(x)\) keeping \(\sigma(x)\) and \(\bar{\psi}_N(x)\) fixed and vice–versa. Hence, a set of equations (\textit{gap equations}) is obtained

\[
\bar{\psi}_N(\nabla - \sigma) = 0 , \quad (\nabla + \sigma)\psi_N = 0 , \quad G_\Lambda(x, x; \sigma, g) + \bar{\psi}_N\psi_N - \frac{\Lambda}{q(\Lambda)}\sigma = 0 ,
\]

(1.10)

(1.11)

(1.12)
where $G_{\Lambda}(x, x; \sigma, g) \equiv \langle x | (\nabla + \sigma)^{-1} | x \rangle_{\Lambda}$ is the two-points correlation function of the $\psi$-field, evaluated for $x \rightarrow x'$. At the saddle point Eq. (1.9) reads

$$Z[g, \Lambda, q(\Lambda)] = \exp \left\{ (N - 1) \left[ \text{Tr} \log(\nabla + \sigma) - \frac{\Lambda}{2q} \int_{\mathcal{M}} d^3 x \sqrt{g} \sigma_c^2(x) \right] \right\}. \quad (1.13)$$

Then, the free energy density,

$$W[g, \Lambda, q(\Lambda)] = \frac{\log Z}{N \int d^3 x \sqrt{g}}, \quad (1.14)$$

may be calculated. In the following analysis we will look for uniform solutions of the gap equations

$$\langle \sigma \rangle = m, \quad \langle \psi_N \rangle = b, \quad \langle \bar{\psi}_N \rangle = \bar{b}. \quad (1.15)$$

The quantities $b$ and $\bar{b}$ represent the vacuum expectation value (v.e.v.) of the fermion fields, while $m$, if positive, can be regarded as the mass of the field fluctuations around the vacuum.

Substituting these values into Eq. (1.14) we have

$$W[m, g, \Lambda, q(\Lambda)] = \left[ \log \det(\nabla + m) - \frac{\Lambda}{2q(\Lambda)} m^2 \right]. \quad (1.16)$$

Phase transitions occur at the extrema of the effective potential, $V(\sigma)$, which is defined in the usual way: introducing a source $J$ for the $\sigma$ field in the generating functional (1.6), we have

$$Z[J] = \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \mathcal{D}[\sigma] \exp \left\{ -S(\psi, \bar{\psi}, \sigma) + J\sigma \right\}; \quad (1.17)$$

we then introduce the Legendre transform

$$\Gamma[\tilde{\sigma}] = -\log Z[J] + J\tilde{\sigma} \quad (1.18)$$

where $\tilde{\sigma} = \frac{\delta \log Z[J]}{\delta J(x)}$. The effective potential is defined to be

$$V(\tilde{\sigma}) = \frac{\Gamma[\tilde{\sigma}]}{\int d^3 x \sqrt{g}}. \quad (1.19)$$

When evaluating the partition function in the large $N$ limit it is found that $\tilde{\sigma} = m$ and the effective potential is equal to minus the free energy density [6]. Then, the gap equations, (1.10)-(1.12) are the extrema of the effective potential as well.

Instead than using the cutoff regularization introduced above, we use the zeta-function regularization, which seems to us easier to handle in the presence of curvature. Following [33], the equal point Green’s function $G(x, x, \Lambda)$ for the operator $\nabla + m$ is seen to be regularized as

$$G_s(x, x; m, g) = m \langle x | (\Delta_{1/2} + m^2)^{-s} | x \rangle = m \zeta(s, x) \quad , \quad (1.20)$$
so that

\[ G(x, x; m, g) = m \lim_{s \to 1} \zeta(s, x) \quad . \] (1.21)

\( \zeta(s, x) \) is the local zeta function

\[ \zeta(s, x) = \sum_n (\lambda_n^2 + m^2)^{-s} |\psi_n(x)|^2 \quad , \] (1.22)

\( \lambda_n^2 + m^2 \) are the eigenvalues of the operator \((\Delta_{1/2} + m^2)\) and \(\{\psi_n(x)\}\) is an orthonormal basis of eigenvectors \((\Delta_{1/2} \) is the squared Dirac operator). The sum over the eigenvalues includes degeneracy and in case of a continuous spectrum the sum is replaced by an integral. On homogeneous spaces such as the ones we will be considering in this paper, \( \zeta(s, x) \) turns out to be independent of \( x \). Moreover the equal point Green’s function happens to be finite for the 3-d case, as will be clear in all the situations considered below (this is not the case in \( d \neq 3 \)).

The gap equations (1.10)-(1.12) become, in this regularization scheme

\[ \bar{b}(\gamma^\mu \Gamma_\mu - m) = 0 \] (1.23)

\[ (\gamma^\mu \Gamma_\mu + m)b = 0 \] (1.24)

\[ m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\} - \bar{b}b = 0 \quad , \] (1.25)

where the regularized coupling \( \Lambda/q(\Lambda) \) has been replaced by \( 1/q(s) \). The free energy density (1.16) can be written in terms of the zeta function in its turn, recalling that

\[ \log \det(\nabla + m) = \frac{1}{2} \log \det(\Delta_{1/2} + m^2) = -\frac{1}{2} \left[ \frac{d}{ds} \zeta(s, m) \right]_{s=0} \] . (1.26)

We have then

\[ W[g, q, m] = -\frac{1}{2} \left[ \frac{d}{ds} \zeta(s, m) \right]_{s=0} - \frac{1}{2q} m^2 \quad , \] (1.27)

where \( 1/q = \lim_{s \to 1} 1/q(s) \) is the renormalized coupling, defined by Eq. (1.25).

Let us see how this scheme applies to the case of \( \mathbb{R}^3 \). Upon substituting the appropriate eigenvalues of the Dirac operator for this space, the gap equations (1.23)-(1.25) now read

\[ m\bar{b} = mb = 0 \] (1.28)

\[ m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + m^2)^s} \right\} - \bar{b}b = 0 \quad . \] (1.29)

Using the Mellin transform to analytically continue the zeta function,

\[ \zeta(s) = \frac{1}{\Gamma(s)} \int dt \ t^{s-1} h_{\mathbb{R}^3}(t; x = x')e^{-m^2t} \] (1.30)
where $h_{R^3}(t; x = x') = (4\pi t)^{-3/2}$ is the equal point heat kernel of the spin-$\frac{1}{2}$ Laplacian, and observing that $b$ is zero, be $m$ zero or not, we are left with

$$m \lim_{s \to 1} \left( \frac{1}{q(s)} - \frac{m^{3-2s}}{(4\pi)^{\frac{3}{2}}} \frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)} \right) = 0.$$  \hspace{1cm} (1.31)

Performing the limit we observe that $m = 0$ for positive values of the coupling, while it may be non zero for negative values of the coupling. At $m = 0$ the second derivative of the effective potential changes its sign, being positive when $1/q < 0$, negative when $1/q > 0$ (we recall that Eq. (1.31) is the first derivative of the effective potential, up to a minus sign). Thus, the system undergoes a second order phase transition and the critical value of the coupling is

$$\frac{1}{q_{cr}} = 0.$$  \hspace{1cm} (1.32)

The mass gap in the broken phase is given by

$$m_0 = -\frac{4\pi}{q}$$  \hspace{1cm} (1.33)

where the index zero will serve as future reference to distinguish flat space quantities. Recalling that the critical value of the coupling is independent on the curvature but dependent on the regularization, the result (1.32) will be valid, in our regularization scheme, for all the manifolds considered, while Eq. (1.33) will be taken as the definition of the renormalized coupling when it is negative. From Eq. (1.27) the free energy density is now easily calculated to be

$$\mathcal{W} = -\frac{m^2}{2} \left( \frac{m}{6\pi} + \frac{1}{q} \right)$$  \hspace{1cm} (1.34)

which is seen to be zero for $1/q \geq 0$ (m being zero), positive for $1/q$ negative; in the latter case we have

$$\mathcal{W}_0 = \frac{m^3}{24\pi}.$$  \hspace{1cm} (1.35)

## 2 The Gross-Neveu model in curved space-time

In this section we study the large $N$ limit of the Gross-Neveu model on manifolds of the type $\Sigma \times S^1$ where $\Sigma$ is a two dimensional manifold of constant curvature. The $S^1$ component can be regarded either as a compact space-time dimension, or as a way of introducing the temperature through the inverse radius of the circle. The former point of view is adopted when studying finite size effects. Here we will adopt the latter point of view.
2.1 The manifold $S^2 \times S^1$

This is a positive curvature space-time, of scalar curvature $R = \frac{2}{r^2}$ ($r$ is the radius of the sphere). As we already mentioned in the introduction, it is expected that both the positive curvature and the finite temperature favour the symmetry restoration. This result is confirmed in our approach, where we find a second order phase transition at some finite value of $T = f(r)$. When either the curvature or the temperature tend to zero, the two-phase structure persists recovering some known results.

We parameterize this space by $x^\mu \equiv (\tau, \chi, \theta)$, where $0 \leq \tau < 2\pi$, $-\pi/2 \leq \chi \leq \pi/2$, and $0 \leq \theta < 2\pi$. The metric tensor is then defined as

$$g_{\mu\nu} dx^\mu \otimes dx^\nu = r^2 \cos^2 \chi d\tau \otimes d\tau + r^2 d\chi \otimes d\chi + \beta^2 d\theta \otimes d\theta.$$  \hspace{1cm} (2.1)

The Dirac matrices are given in terms of the Pauli matrices $\sigma_a$, by

$$\gamma_1 = r \cos \chi \sigma_1 , \quad \gamma_2 = r \sigma_2 , \quad \gamma_3 = \beta \sigma_3 .$$ \hspace{1cm} (2.2)

The spin connection (1.5) results to be

$$\Gamma_\mu(x) = -i \frac{\sigma_3}{2} \sin \chi \delta_{1\mu} .$$ \hspace{1cm} (2.3)

Due to the form of the spin connection, it was seen in [33] that the vacuum expectation value of the $\psi$ fields has to be zero, for the first two gap equations to be satisfied. Hence we are left with the gap equation

$$m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\} = 0 .$$ \hspace{1cm} (2.4)

Observing that the heat kernel of the Laplacian on a product manifold is just the product of the heat kernels on the factor spaces, and observing that the equal points heat kernel Laplacians on $S^2$ and $S^1$ are respectively

$$h_{S^2}(t) = \frac{1}{2\pi r^2} \sum_{l=1}^{\infty} l \exp \left( -\frac{l^2}{r^2} t \right)$$ \hspace{1cm} (2.5)

$$h_{S^1}(t) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \exp \left[ -4\pi \left( n + \frac{1}{2} \right)^2 t \right]$$ \hspace{1cm} (2.6)

the zeta-function is obtained by Eq. (1.30) to be

$$\zeta(s, m) = \frac{\beta^{2s-1}}{2\pi^2 r^2 \Gamma(s)} \int_0^\infty dt \ t^{s-1} \sum_{n=\infty}^{\infty} \sum_{l=1}^{\infty} l \exp \left\{ - \left[ 4\pi^2 \left( n + \frac{1}{2} \right)^2 + \frac{l^2}{r^2} + m^2 \beta^2 \right] t \right\} .$$ \hspace{1cm} (2.7)
Upon evaluating the integral and taking the limit \( s \to 1 \) the gap equation (2.4) reads

\[
\frac{m}{q} = \frac{m^2}{2\pi^2} \int_0^\infty P \int_0^\infty dx \left( \cot(x) - \frac{1}{x} \right) K_1(2mxr) \]

\[
- \frac{m^2}{4\pi} + \frac{m}{2\pi r} \sum_{n,l=1}^\infty (-1)^n \frac{l}{\sqrt{l^2 + m^2r^2}} \exp \left(-\frac{n\beta}{r} \sqrt{l^2 + m^2r^2} \right), \tag{2.8}
\]

where \( P \) indicates the principal value of the integral. The details of the calculation are

given in appendix A. Let us analyze the possible solutions. We already found in [33] that

\( m = 0 \) is the only solution at the critical coupling \( 1/q_{cr} = 0 \). In facts, \( m \) is equal to zero

for \( 1/q \geq 0 \), the RHS of Eq. (2.8) being negative for any value of \( \beta, r \). Due to this, \( m \)

can be non zero only for negative values of the coupling. Let us fix \( 1/q < 0 \). On deriving

Eq. (2.8) with respect to \( m \), it may be seen that, at \( m = 0 \) the second derivative of the

effective potential changes its sign when varying \( \beta, r \) around some \( \beta_{cr}, r_{cr} \). This is where

a second order phase transition occurs [7]. We propose here a simple expression for the

critical surface \( f(\beta, r, q) = 0 \). To find the critical surface we use the fact that in a second

order phase transition the mass gap smoothly disappears as approaching criticality. Thus,

we perform the limit \( m \to 0 \) for the equation

\[
-\frac{m}{4\pi} = \frac{m^2}{2\pi^2} \int_0^\infty P \int_0^\infty dx \left( \cot(x) - \frac{1}{x} \right) K_1(2mxr) \]

\[
- \frac{m}{4\pi} + \frac{1}{2\pi r} \sum_{n,l=1}^\infty (-1)^n \frac{l}{\sqrt{l^2 + m^2r^2}} \exp \left(-\frac{n\beta}{r} \sqrt{l^2 + m^2r^2} \right), \tag{2.9}
\]

which is obtained by Eq. (2.8) by factorizing out a common \( m \) and replacing the negative

coupling with its renormalized value (1.33). Recalling that \( K_\nu(x) \to 0 \)\( 2^{\nu-1}(\nu - 1)!x^{-\nu} \), we

obtain

\[
-\frac{m}{4\pi} = \frac{1}{\pi r} \int_0^\infty P \int_0^\infty dx \left( \cot(x) - \frac{1}{x} \right) + \frac{2}{r} \sum_{n,l=1}^\infty (-1)^n \exp \left(-\frac{n\beta}{r} l \right) \]

\[
- \frac{1}{2r} + \frac{2}{r} \sum_{n=1}^\infty (-1)^n \frac{1}{e^{n\beta_{cr}/2} - 1}. \tag{2.10}
\]

The principal value of the integral may be performed, yielding

\[
r_{cr}' = \frac{1}{2} - 2 \sum_{n=1}^\infty \frac{(-1)^n}{e^{n\beta_{cr}/2} - 1}, \tag{2.11}
\]

where the critical radii have been rescaled by the flat space mass gap, \( m_0 \). The series is

convergent, indeed its sum is a hypergeometrical function. We recall that this expression

is only valid for negative values of the coupling. In Fig. 1 we plot the critical parameter

\( 1/\beta_{cr}' \) versus the critical curvature parameter \( 1/r_{cr}' \). The relation (2.11) is particularly
easy to handle when the zero curvature limit and the zero temperature limit are to be performed. For $r \to \infty$ we obtain the known result \[19\]
\[
\left( \frac{1}{\beta_{cr}} \right)_{R^2 \times S^1} = \frac{m_0}{2 \log 2}.
\]
(2.12)
For $\beta \to \infty$ we get
\[
(r_{cr})_{S^2 \times R} = \frac{1}{2m_0},
\]
(2.13)
reproducing the result of \[26\].

To evaluate the free energy density, defined in Eq. (1.27), we need to calculate the derivative of the zeta-function (2.7). This is done in appendix A in some detail. The result is
\[
W(m) = -\frac{m^2}{2q} - \frac{m^2}{4\pi^2 r} \int_0^\infty P \int_0^\infty dx \frac{1}{x} \left( \cot(x) - \frac{1}{x} + \frac{x}{3} \right) K_2(2mxr) - \frac{m^3}{12\pi} - \frac{m}{48\pi r^2} - \frac{1}{2\pi^2 r^2} \sum_{n,l=1}^\infty \frac{(-1)^n}{n} l \exp \left[ -\frac{n\beta}{r} \sqrt{l^2 + m^2 r^2} \right]
\]
(2.14)

Different limits can be performed easily (zero temperature and/or zero curvature, zero mass), reproducing some known results. In the limit of zero curvature and temperature we recover the flat space limit (1.34). Here we just consider the zero mass limit, which allows us to calculate the free energy at criticality. For $m \to 0$, Eq. (2.14) simplifies considerably. We obtain
\[
W_{S^2 \times S^1}(m = 0) = -\frac{1}{8\pi^2 r^2} \sum_{n=1}^\infty \frac{(-1)^n}{n} \cosech^2 \left( \frac{\beta}{2r} n \right).
\]
(2.15)
This result, already found in \[33\], gives the free energy at the critical line, when $\beta$ and $r$ satisfy Eq. (2.11). Moreover, it reproduces the correct result at zero curvature \[12, 33\]
\[
W(m = 0)_{R^2 \times S^1} = \frac{3}{8\pi^2} z(3)
\]
(2.16)
where $z$ is the Riemann zeta function.

2.2 The manifold $H^2 \times S^1$

We now consider the product manifold $H^2_r \times S^1$, where $H^2$ is the 2-dimensional \textit{pseudo-sphere}. This is a space with constant negative curvature. Previous calculations suggest that the negative curvature favours symmetry breaking so that we have to expect a competitive effect between curvature and temperature \[7\]. This behaviour is indeed confirmed in our analysis. We find a second order phase transition for some finite $T = f(r)$, while
the symmetry is always broken at zero T. The latter case, which may be formalized by considering the model on the manifold \( H^2 \times R \), is studied in some detail and the mass gap is evaluated.

We parameterize \( H^2 \) as \( H^2 = \{ z = (x, y), \ x \in R, \ 0 < y < \infty \} \), while the circle \( S^1_{\beta} \) is parameterized as before by \( \theta \), \( 0 \leq \theta < 2\pi \). The scalar curvature of \( H^2 \) is \( \mathcal{R} = -2/r^2 \), where \( r \) is a constant positive parameter. The metric tensor on the whole manifold is then given by

\[
g_{\mu\nu} dx^\mu \otimes dx^\nu = \frac{r^2}{y^2} \left( dx \otimes dx + dy \otimes dy \right) + \beta^2 d\theta \otimes d\theta \tag{2.17}
\]

where \( x^\mu \equiv (x, y, \theta) \) with \( \mu = 1, 2, 3 \). The Dirac matrices on \( H^2 \times S^1_{\beta} \) are given in terms of flat Dirac matrices by

\[
\gamma_1 = \frac{r}{y} \sigma_1, \quad \gamma_2 = \frac{r}{y} \sigma_2, \quad \gamma_3 = \beta \sigma_3, \tag{2.18}
\]

while the spin connection is

\[
\Gamma_\mu = \frac{i}{2y} \sigma_3 \delta_1 \mu \tag{2.19}
\]

The v.e.v. of the \( \psi \) fields being zero for this case too, we are left with the gap equation

\[
0 = m \lim_{s \to 1} \left\{ \frac{1}{q(s)} - \zeta(s, m) \right\} \tag{2.20}
\]

The equal-points heat kernel of the spin-\( \frac{1}{2} \) Laplacian on \( H^2 \) is

\[
h_{H^2}(t; z = z') = \frac{r}{2(\pi t)^{3/2}} \int_0^\infty dx \ x \ coth x \ exp \left\{ \frac{-x^2 r^2}{t} \right\}, \tag{2.21}
\]

whereas the equal-points heat kernel of the scalar Laplacian on the circle is given by Eq. (2.6). Substituting in Eq. (1.30), the zeta-function reads

\[
\zeta(s, m) = \frac{r}{2\pi^{3/2} \beta \Gamma(s)} \int_0^\infty dt \ t^{s-5/2} \left\{ \int_0^\infty dx \ x \ coth x \right.

\times \sum_{n=-\infty}^{\infty} \exp \left[ -\frac{x^2 r^2}{t} - \left( \frac{4\pi^2}{\beta^2} \left( n + \frac{1}{2} \right)^2 + m^2 \right) \right] \left\} \right\} \tag{2.22}
\]

Performing the integral in \( t \) and taking the limit \( s \to 1 \) the gap equation (2.20) becomes:

\[
m \left\{ \frac{1}{\pi \beta} \int_0^\infty \frac{dx}{x} (x \ coth x - 1) \right\} \sum_{n=0}^{\infty} \exp \left[ \frac{-2\pi r \sqrt{\frac{4\pi^2}{\beta^2} (n + \frac{1}{2})^2 + m^2}}{2\pi r} \right]

\times \frac{1}{\beta} \log(1 + e^{-\beta m}) - \frac{1}{q} \right\} = 0. \tag{2.23}
\]
For details of the calculation we refer to appendix B. Let us discuss the solutions for β finite first. The first term in Eq. (2.23) is positive definite, hence, a non zero value of the mass is possible in principle in the whole space of the parameters r, β, 1/q, even at the critical coupling 1/q = 0 (in [33] this point is missed). m = 0 is a solution as well. In this limit the gap equation (2.23) becomes indeed

\[ (\lim_{m \to 0} m) \left\{ \frac{1}{4\pi\beta} \int_{-\infty}^{\infty} \frac{dx}{x} (x \coth x - 1) \operatorname{cosech} \left( \frac{2\pi r}{\beta} x \right) - \frac{1}{2\pi\beta} \log 2 - \frac{1}{q} \right\} = 0 \]  

(2.24)

where the factor multiplying m is finite. In facts by the study of the second derivative of the effective potential (minus the first derivative of Eq. (2.23)), the value m = 0 is seen to be either a min or a max, depending on the values of the parameters, β, r, 1/q. This is the second order phase transition point. Let us try a more quantitative analysis. To find the critical surface we proceed as in the positive curvature case. The transition being of second order, the mass gap, defined by Eq. (2.23) up to a factor of m, smoothly disappears as we approach criticality. Hence, performing the limit m \to 0, we get

\[ \frac{2\pi\beta}{q} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x} (x \coth x - 1) \operatorname{cosech} \left( \frac{2\pi r}{\beta} x \right) - \log 2 \]  

(2.25)

This equation, which is the analogue of Eq. (2.11) for the negative curvature case, is numerically solved in the same way. Moreover, it can be solved analytically for specific values of the parameters. For 2\pi r = \beta, the integral in Eq. (2.25) simplifies yielding \log 2. Hence, when the coupling is zero (the flat space critical value), we have

\[ \frac{r_{cr}}{\beta_{cr}} = \frac{1}{2\pi} \]  

(2.26)

When the coupling is positive,

\[ \frac{r_{cr}}{\beta_{cr}} < \frac{1}{2\pi} \]  

(2.27)

whereas, for negative coupling

\[ \frac{r_{cr}}{\beta_{cr}} > \frac{1}{2\pi} \]  

(2.28)

Differently from what happens in the positive curvature case, we have a phase transition also for positive and zero values of the coupling. The latter result in particular, tells us that the flat space critical coupling, 1/q = 0, which is a second order phase transition point at zero curvature and temperature, maintains this property when r and β meet the condition (2.26).

In figure 2 we plot the temperature 1/β_{cr} as a function of the inverse radius 1/r_{cr} for positive, negative, and zero values of the inverse coupling. For 1/q non-zero, the two radii
are rescaled by the common factor $4\pi/|q|$. The behaviour predicted by Eqs. (2.26)-(2.28), can be verified to hold. The zero curvature limit of Eq. (2.23) may be performed yielding
\[
\frac{1}{2\pi\beta} \log 2 = -\frac{1}{q}
\] (2.29)
which coincides with Eq. (2.12) when the coupling constant is negative, whereas it has no solution in the other cases.

Let us now consider the zero temperature limit. Since the characteristics are quite different, it deserves a careful analysis. If we take the limit $\beta \to \infty$ of Eq. (2.24), we get a divergent result. This means that $m = 0$ is never a solution of the gap equation at zero temperature. In other words, the symmetry is always broken, for any value of the coupling and of the curvature parameter (this is to be compared with the positive curvature case, where at zero temperature there is a curvature-induced phase transition).

The gap equation for this case is to be obtained by Eq. (2.23) taking the limit $\beta \to \infty$, or, more directly, by the zeta function for the space $H^2 \times R$. This reads
\[
\zeta(s)_{H^2 \times R}(s, m) = \frac{r}{4\pi^2\Gamma(s)} \int_0^\infty dt \ t^{s-3} \int_0^\infty dx \ x \ \coth x \ \exp \left( -\frac{x^2r^2}{t} - m^2t \right). \tag{2.30}
\]
and it is obtained by replacing the equal points heat kernel of the scalar Laplacian on $S^1$ with the one on $R$, in the equation (2.22). Replacing Eq. (2.30) in the gap equation (2.20) and performing the integral in $t$ we easily arrive at the gap equation
\[
m \left\{ \frac{m}{2\pi^2} \int_0^\infty dx \ x \ \coth x - 1 \right\} K_1(2\pi rm) - \frac{m}{4\pi} - \frac{1}{q} \right\} = 0. \tag{2.31}
\]
The integral on the LHS being positive for any value of $r$, and divergent for $m = 0$, this equation yields a non zero value of the mass whatever are the sign and the value of the coupling constant.

The value of the mass gap is plotted in Fig. 3 as a function of the curvature parameter $r$, for different values of the inverse coupling. At the critical coupling it takes the simple expression
\[
m(q_{cr}) \simeq \frac{1}{4r}. \tag{2.32}
\]
This equation, which has the right behaviour for $r \to \infty$, has an interesting property: it states that the mass gap at the critical coupling is half the mass gap which is found in [34] for the conformal sigma model on the same space. We think that it is an interesting result because it is a manifestation of bosonization, although in the large $N$ approximation (another warning is the fact that the result (2.32) is numerical, whereas the result in [34] is analytic).
We now consider the free energy density for the case of $\beta$ finite. The calculation is performed in appendix B. Taking the derivative of the zeta function (2.22), the free energy density (1.27) is seen to be

$$\mathcal{W}(m) = \frac{1}{4\pi\beta r^2} \left(1 - \frac{d}{dr}\right) \int_0^\infty \frac{dx}{x^2} \left(\frac{1}{x} + \frac{x}{3} - \coth x\right) \sum_{n=0}^\infty \exp \left(-2\pi r \sqrt{\frac{4\pi^2}{\beta^2} (n + \frac{1}{2})^2 + m^2}\right)$$

$$- \frac{1}{2\pi\beta^3} \left(1 - \beta \frac{\partial}{\partial \beta}\right) \sum_{n=1}^\infty \frac{(-1)^n}{n^3} e^{-n\beta m}$$

$$+ \frac{m}{48\pi r^2} - \frac{m^3}{12\pi} + \frac{1}{48\pi\beta r^2} \log \left(1 + e^{-\beta m}\right) - \frac{m^2}{2q}. \quad (2.33)$$

This expression, which looks complicated at a first sight, reproduces correctly some known limits, as can be easily checked, like, for example, the flat space limit (1.34). In particular it allows to calculate easily the free energy density on the critical surface. Upon taking the limit $m \to 0$ we arrive at

$$\mathcal{W}_{H^2 \times S^1}(0) = \frac{1}{8\pi\beta r^2} \int_0^\infty \frac{dx}{x^2} \left[\frac{1}{x} + \frac{x}{3} - \coth x\right] \left[1 + \frac{2\pi r}{\beta} x \coth \left(\frac{2\pi r x}{\beta}\right)\right] \frac{1}{\sinh(2\pi r x/\beta)}$$

$$+ \frac{3\zeta(3)}{8\pi\beta^3} + \frac{\log 2}{48\pi\beta r^2}, \quad (2.34)$$

which yields the critical value of the energy density when $r, \beta, q$ meet the conditions (2.26)-(2.28). The analogous result reported in [33] contains an error in the calculation. The limit of zero curvature yields the same expression as Eq. (2.16).

The zero temperature energy is to be derived either from Eq. (2.33) in the limit $\beta \to \infty$, or directly from the zeta function (2.34). By means of the relation (A.10), which is true in this case as well, we find

$$\mathcal{W}_{H^2 \times R} = -\frac{m^2}{4\pi^2 r} \int_0^\infty \frac{dx}{x} \left[\coth x - \frac{1}{x} - \frac{x}{3}\right] K_2(2\pi r m) - \frac{m^3}{12\pi} + \frac{m}{48\pi r^2} - \frac{m^2}{2q}. \quad (2.35)$$

3 Conclusions

Using the large $N$ approximation and the zeta-function regularization to study the Gross Neveu model in 3 dimensions, we have found that it describes a system undergoing a second order phase transition when considered on spaces of constant curvature, at finite temperature. We have found the critical surface $f(r, \beta, q) = 0$, for de Sitter and anti-de Sitter spaces, where $r, \beta, q$ are respectively, the curvature parameter, the inverse temperature, and the coupling constant. In both cases (positive and negative curvature) this turns out to be a critical line, $f(r/q, \beta/q) = 0$, the coupling constant appearing as a common scale factor when finite. In the case of positive curvature this is represented by Eq. (2.14).
and phase transitions are allowed only in the negative region of the coupling constant. For the negative curvature case we have Eq. (2.23) which describes phase transitions for an arbitrary value of the coupling. In particular at the flat space critical coupling we have found an explicit solution, Eq. (2.26), which states that the product of temperature and curvature is constant on the critical line. At zero temperature the symmetry is broken for any value of the curvature and of the coupling. In this case it is possible to calculate the mass gap. We have evaluated it at the flat space critical value of the coupling, Eq. (2.32), and we have found that this is half the mass gap of the conformal sigma model when studied under the same assumptions.

**Appendix A**

This appendix contains a proof of the results (2.8) and (2.14) of the subsection 2.1.

Let us start with the definition of the zeta-function (2.7)

\[
\zeta(s,m) = \frac{\beta^{2s-1}}{2\pi^2 r^2 \Gamma(s)} \int_0^\infty dt \ t^{s-1} \sum_{n=\infty}^{\infty} \sum_{l=1}^{\infty} l \exp \left\{ - \left[ 4\pi^2 \left( n + \frac{1}{2} \right)^2 + l^2 \frac{\beta^2}{r^2} + m^2 \beta^2 \right] t \right\}.
\]

To exchange the sum over \( n \) with the integral in \( t \) we use the Poisson sum formula for the heat kernel of the spin-\( \frac{1}{2} \) Laplacian on the circle,

\[
\sum_{n=\infty}^{\infty} \exp \left[ -4\pi^2 \left( n + \frac{1}{2} \right)^2 t \right] = \frac{1}{\sqrt{4\pi t}} \left[ 1 + 2 \sum_{n=1}^{\infty} (-1)^n \exp \left( -\frac{n^2}{4t} \right) \right]
\]

so that

\[
\zeta(s,m) = \frac{\beta^{2s-1}}{4\pi^{3/2} r^2 \Gamma(s)} \left\{ \int_0^\infty dt \ t^{-3/2} \sum_{l=1}^{\infty} l \exp \left\{ - \left[ \frac{\beta^2}{r^2} l^2 + m^2 \beta^2 \right] t \right\} \right\} (A.3)
\]

\[
+ 2 \sum_{n=1}^{\infty} (-1)^n \int_0^\infty dt \ t^{-3/2} \sum_{l=1}^{\infty} l \exp \left\{ - \left[ l^2 \frac{\beta^2}{r^2} + m^2 \beta^2 \right] t - \frac{n^2}{4t} \right\} \right\} = A + B
\]

The first integral, \( A \), is performed by means of the Poisson sum formula for the heat kernel of the spin-\( \frac{1}{2} \) Laplacian on the sphere

\[
\sum_{l=1}^{\infty} l \exp \left\{ - \left[ \frac{l^2 \beta^2}{r^2} \right] \right\} = \frac{r^3 t^{-3/2}}{\beta^3 2\sqrt{\pi}} \int_{-\infty}^{\infty} dx \ x \cot x \exp \left( -\frac{x^2 r^2}{\beta^2} \right).
\]

This is a specialization of the usual Poisson sum formula. A derivation may be found for example in Appendix B of [33]. This formula allows us to exchange the sum over \( l \) with the integral in \( t \), upon extracting the spurious divergences which eventually show up. We obtain

\[
A = \frac{m^{2-s} r^{s-1}}{2\pi^2 \Gamma(s)} \int_0^\infty \int_0^\infty dx \ x^{s-1} \left( \cot x - \frac{1}{x} \right) K_{s-2}(2mxr)
\]

\[
+ \frac{m^{-2s+3} \Gamma(s - \frac{3}{2})}{8\pi^{3/2} \Gamma(s)}
\]

(A.5)
where $K_r(x)$ is the modified Bessel function. The second integral is easier to calculate. It yields

$$B = \frac{\beta^{s-1/2}2^{1/2-s}}{\pi^{3/2}r^2\Gamma(s)} \sum_{n,l=1}^{\infty} (-1)^n n^{s-1/2} \frac{l}{(\frac{r^2}{r^2} + m^2)^{s/2-1/4}} K_{s-1/2} \left( n\beta\sqrt{\frac{l^2}{r^2} + m^2} \right)$$  \hspace{1cm} (A.6)

where we have used [36]

$$\int_0^\infty dt t^{\nu-1} e^{-at-b/t} = 2\left(\frac{b}{a}\right)^{\nu/2} K_{\nu}(2\sqrt{ab}), \hspace{0.5cm} a, b > 0.$$  \hspace{1cm} (A.7)

The zeta-function is then rewritten as

$$\zeta(s,m) = \frac{m^{2-s}e^{-s-1}}{2\pi^2\Gamma(s)} \int_0^\infty P \int_0^\infty dx \; x^{s-1} \left( \cot x - \frac{1}{x} \right) K_{s-2}(2mxr)$$

$$+ \frac{m^{-2s+3}\Gamma(s-\frac{3}{2})}{8\pi^{3/2}\Gamma(s)} + \frac{\beta^{s-1/2}2^{1-s}}{\pi^{3/2}r^2\Gamma(s)} \sum_{n,l=1}^{\infty} (-1)^n n^{s-1/2} \frac{l}{(\frac{r^2}{r^2} + m^2)^{s/2-1/4}} K_{s-1/2} \left( n\beta\sqrt{\frac{l^2}{r^2} + m^2} \right)$$

Substituting in (2.4) and taking the limit $s \to 1$, we get (2.8).

To obtain the expression of the free energy density (2.14), we first rewrite the zeta-function (A.1) in the form

$$\zeta(s,m) = \frac{m^{2-s}e^{-s-1}}{2\pi^2\Gamma(s)} \int_0^\infty P \int_0^\infty dx \; x^{s-1} \left( \cot x - \frac{1}{x} + \frac{x}{3} \right) K_{s-2}(2mxr)$$

$$+ \frac{\beta^{s-1/2}2^{1-s}}{\pi^{3/2}r^2\Gamma(s)} \sum_{n,l=1}^{\infty} (-1)^n n^{s-1/2} \frac{l}{(\frac{r^2}{r^2} + m^2)^{s/2-1/4}} K_{s-1/2} \left( n\beta\sqrt{\frac{l^2}{r^2} + m^2} \right)$$

$$+ \frac{m^{-2s+3}\Gamma(s-\frac{3}{2})}{8\pi^{3/2}\Gamma(s)} - \frac{1}{48}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$  \hspace{1cm} (A.9)

This expression differs by (A.8) by terms which cancel out when $s \to 1$. It is obtained exactly in the same way as the previous one. The extra-terms are needed to regularize the expression when $s \to 0$. When taking the derivative of this expression with respect to $s$, the only contribution at $s = 0$ is given by the terms which contain the derivative of $1/\Gamma(s)$, all the others being zero (they are multiplied by a common $1/\Gamma(s)$). To be more precise,

$$\frac{d}{ds} \zeta(s)|_{s=0} = \left. \Gamma(s)\zeta(s) \left(-\frac{\psi(s)}{\Gamma(s)} \right) \right|_{s=0} = \left. (\Gamma(s)\zeta(s)) \right|_{s=0},$$  \hspace{1cm} (A.10)

where we have used

$$\left( \frac{d}{ds} \frac{1}{\Gamma(s)} \right)_{s=0} = -\left( \frac{\psi(s)}{\Gamma(s)} \right)_{s=0} = 1.$$  \hspace{1cm} (A.11)
Hence, replacing (A.10) in the definition of the free energy, (1.27), the equation (2.14) is obtained.

**Appendix B**

This appendix contains a proof of the equations (2.23) and (2.33) of subsection 2.2. Let us start from the zeta-function (2.22),

\[ \zeta(s, m) = \frac{r}{2\pi^{3/2} \beta \Gamma(s)} \int_0^\infty dt \ t^{s-5/2} \left\{ \int_0^\infty dx \ x \ \coth x \right\}. \]

This is rewritten as

\[ \zeta(s, m) = \frac{r}{2\pi^{3/2} \beta \Gamma(s)} \int_0^\infty dt \ t^{s-5/2} \left\{ \int_0^\infty dx (x \ \coth x - 1) \right\} \]

To evaluate the contribution D we use the Poisson sum formula (A.2). We have

\[ D = \frac{1}{(4\pi)^{3/2} \Gamma(s)} \int_0^\infty dt \ t^{s-5/2} e^{-m^2 t} \]

We then use the result (A.7) to perform the last integral (the first is just a \( \Gamma \) function). To evaluate the contribution C we use the result (A.7) to perform the integral in \( t \). Summing up we obtain

\[ \zeta(s, m) = \frac{2r}{\pi^{3/2} \beta \Gamma(s)} \int_0^\infty dx (x \ \coth x - 1) \]

Replacing this expression into the gap equation (2.20) and taking the limit \( s \to 1 \) we obtain (2.23).
To obtain the expression of the free energy density (2.33), we follow the same procedure as in Appendix A. We first rewrite the zeta function as

$$
\zeta(s,m) = \frac{2r}{\pi^{3/2} \beta \Gamma(s)} \int_0^\infty dx \left( x \coth x - 1 - \frac{x^2}{3} \right)
$$

$$
\times \sum_{n=0}^\infty \left( \frac{rr}{\sqrt{\frac{4\pi^2}{\beta^2} (n + \frac{1}{2})^2 + m^2}} \right)^{s-3/2} K_{s-3/2} \left( 2rr \sqrt{\frac{4\pi^2}{\beta^2} (n + \frac{1}{2})^2 + m^2} \right)
$$

$$
+ \frac{1}{(4\pi)^{3/2}} \frac{\Gamma(s - 3/2)}{\Gamma(s)} m^{3-2s} + \frac{1}{2\pi^3/2 \Gamma(s)} \sum_1^\infty (-1)^n \frac{nb}{2n} \left( \frac{nb}{2n} \right)^{s-3/2} K_{s-3/2}(n\beta m)
$$

$$
+ \frac{1}{48\pi^{3/2}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} m^{1-2s} + \frac{1}{24\pi^3/2 \Gamma(s)} \sum_1^\infty (-1)^n \frac{nb}{2m} \left( \frac{nb}{2m} \right)^{s-1/2} K_{s-1/2}(n\beta m).
$$

This expression differs from (B.4) by terms which vanish in the limit $s \to 1$. As in the calculation of the energy for the positive curvature case, these terms have been introduced to perform the limit $s \to 0$, without introducing divergences when exchanging integrals and sums.

Observing that the derivative of the zeta-function is given by (A.10) and considering the limit $s \to 0$, the wanted expression, Eq. (2.33), is obtained.

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Figure 1: The critical temperature defined by (2.11), is plotted as a function of the critical curvature parameter. They are both rescaled as $\beta' \equiv \frac{1}{\rho_{m_0}}$, $\rho' \equiv \frac{1}{r_{m_0}}$. 
Figure 2: The critical temperature defined by (2.25), is plotted as a function of the critical curvature parameter. The solid line corresponds to $\frac{1}{q} < 0$. The dashed line corresponds to $\frac{1}{q} > 0$, whereas the dotted line corresponds to $\frac{1}{q} = 0$. When $\frac{1}{q}$ is negative or positive, both the temperature and the inverse radius are rescaled to: $\beta' \equiv \frac{4\pi}{|q|} \frac{1}{\beta}$, $r' \equiv \frac{4\pi}{|q|} r$. 
Figure 3: The mass gap defined in Eq. (2.31), is plotted as a function of the inverse radius. The solid line corresponds to $\frac{1}{q} < 0$. The dashed line corresponds to $\frac{1}{q} > 0$, whereas the dotted line corresponds to $\frac{1}{q} = 0$. When $\frac{1}{q}$ is negative or positive, both the mass and the radius are rescaled to: $m' = \frac{4\pi}{|q|m}$, $r' = \frac{|q|}{4\pi}r$. 