THE MULTIPARTITE RAMSEY NUMBER
FOR THE 3-PATH OF LENGTH THREE

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Abstract. We study the Ramsey number for the 3-path of length three and $n$ colors and show that $R(P^3_3; n) \leq \lambda_0 n + 7\sqrt{n}$, for some explicit constant $\lambda_0 = 1.97466\ldots$.

1. Introduction

Let $P^3_3$ be the 3-uniform hypergraph with the set of vertices $\{a, b, c, d, e, f, g\}$ and the set of edges $\{(a, b, c), (c, d, e), (e, f, g)\}$. The Ramsey number $R(P^3_3; n)$ is the smallest integer $N$ such that any coloring of the edges of the complete 3-uniform hypergraph $K^3_N$ on $N$ vertices with $n$ colors leads to a monochromatic copy of $P^3_3$. It is easy to see that $R(P^3_3; n) \geq n + 6$ (see [2, 4]) and it is believed that this lower bound is sharp, i.e. that $R(P^3_3; n) = n + 6$. However, so far this conjecture has been confirmed only for $n \leq 10$ (see [4, 6, 9, 10]). On the other hand, from the fact that for $N \geq 8$ each $P^3_3$-free 3-uniform hypergraph $H$ on $N$ vertices satisfies

\begin{equation}
|H| \leq \binom{N-1}{2},
\end{equation}

(see [1] and [4]), it follows that

$R(n; P^3_3) \leq 3n$.

In [7] the authors of this note improved the above upper bound to

\begin{equation}
R(n; P^3_3) \leq 2n + \sqrt{18n + 1} + 2.
\end{equation}

Our argument relied on the fact that for 2-graphs the analogous multicolored Ramsey number for a ‘usual’ 2-path of length three is know to be $2n + O(1)$, where the small hidden constant $O(1)$ depends only on divisibility of $n$ by 3 (see [3]). Thus, it seemed the method we used

2010 Mathematics Subject Classification. Primary: 05D10, secondary: 05C38, 05C55, 05C65.

Key words and phrases. Ramsey number, hypergraphs, paths.

The first author partially supported by NCN grant 2012/06/A/ST1/00261.
The main result of this note is to match our previous approach with later results from [8] and get the following estimate for $R(P_3^3; n)$.

**Theorem 1.** Let

$$f(\gamma) = (\gamma^3 - 3\gamma^2 + 6\gamma - 6)^2 - 72\gamma(2 - \gamma)(\gamma - 1)^2.$$  

and let $\lambda_0 = 1.97466\ldots$ be the solution to the equation $f(\gamma) = 0$, such that $f(\gamma) > 0$ whenever $\gamma \in (\lambda_0, 2)$. Then

$$R(P_3^3; n) \leq \lambda_0 n + 7\sqrt{n}.$$  

2. Proof of Theorem 1

Our argument is based on the following decomposition lemma proved in [8]. Before we state it we need some definitions. We call a 3-graph $H$ quasi-bipartite if one can partition its set of vertices into three sets: $X = \{x_1, x_2, \ldots, x_s\}$, $Y = \{y_1, y_2, \ldots, y_s\}$, and $Z = \{z_1, z_2, \ldots, z_t\}$ in such a way that all the edges of $H$ can be written as $\{x_i, y_i, z_j\}$ for some $i = 1, 2, \ldots, s$, and $j = 1, 2, \ldots, t$. By a star with center $v$ we mean any 3-graph in which each edge contains $v$. Then the following holds.

**Lemma 2.** For any $P_3^3$-free 3-graph $H$ there exists a partition of its set of vertices $V = V_R \cup V_T \cup V_S$, such that subhypergraphs of $H$ defined as $H_R = \{h \in H : h \cap V_R \neq \emptyset\}$, $H_T = H[V_T]$ and $H_S = H \setminus (H_R \cap H_T) = \{h \in H[V \setminus V_R] : h \cap V_S \neq \emptyset\}$ satisfy the following three conditions:

(i) $|H_R| \leq 6|V_R|$,  
(ii) $H_T$ is quasi-bipartite and $|H_T| \leq |V_T|^2/8$,  
(iii) $H_S$ is a family of disjoint stars such that centers of these stars are in $V_T$ whereas all other vertices are in $V_S$, and $|H_S| \leq \binom{|V_S|}{2}$.

The following lemma is a direct consequence of the above result.

**Lemma 3.** Let $H$ be a $P_3^3$-free 3-graph $H$ on $N \geq 95$ such that for some $s$, $(N + 3)/2 \leq s \leq N - 46$, we have

$$|H| \geq \binom{s - 1}{2} + \binom{N - s}{2},$$

and let $H = H_R \cup H_T \cup H_S$ be a decomposition of $H$ as described in Lemma 2. Then $H_S$ contains a star on at least $s$ vertices.
The Multipartite Ramsey Number for the 3-Path

Proof. Let \( V = V_R \cup V_T \cup V_S \) be a partition of the set of vertices \( H \) given by Lemma 2. Note that \( |V_S| \geq s - 1 \), since otherwise

\[
|H| \leq 6|V_R| + \frac{|V_T|^2}{8} + \left( \frac{|V_S|}{2} \right)^2 \leq \left( \frac{|V_S|}{2} \right)^2 + \left( \frac{N - |V_S|}{2} \right)^2
\]

\[
\leq \left( \frac{s - 2}{2} \right)^2 + \left( \frac{N - s + 2}{2} \right)^2 < \left( \frac{s - 1}{2} \right)^2 + \left( \frac{N - s}{2} \right)^2.
\]

Recall that \( H_S \) is a collection of disjoint stars. Suppose that the largest of these stars consists of at most \( s - 1 > N/2 \) vertices. Then one can easily verify that the number of edges in \( H_S \) is maximised if \( H_S \) consists of two stars on \( s - 1 \) and \( |V_S| - (s - 1) + 2 \) vertices respectively. Consequently

\[
|H| \leq 6|V_R| + \frac{|V_T|^2}{8} + \left( \frac{s - 2}{2} \right) + \left( \frac{|V_S| - s + 2}{2} \right)
\]

\[
\leq \left( \frac{s - 2}{2} \right) + \left( \frac{N - s + 1}{2} \right) < \left( \frac{s - 1}{2} \right) + \left( \frac{N - s}{2} \right),
\]

again contradicting the fact that \( |H| \geq \left( \frac{s - 1}{2} \right) + \left( \frac{N - s}{2} \right) \). Thus, \( H_S \) contains a star on at least \( s \) vertices. \( \square \)

Proof of Theorem 1. We show that if for given integers \( N \) and \( n \) one can find a coloring of edges of \( K_3^N \) by \( n \) colors without monochromatic copies of \( P_3 \), then \( \gamma = (N - 7\sqrt{n})/n < \lambda_0 \) where \( \lambda_0 \) is defined in Theorem 1. Some parts of our argument are quite technical and, since we aim to prove the statement for every \( n \), we start with few remarks which makes our future computations a bit easier.

Note that since \( \lambda_0 > 1.97 \), we may assume that \( \gamma > 1.9 \). Moreover, due to (2), it is enough to consider \( \gamma < 2 \). Finally, since \( R(n; P_3^3) \leq 3n \) we can restrict to the case \( n \geq 41 \) (and hence \( N > 122 \)) because otherwise \( 3n < 1.9n + 7\sqrt{n} \).

Thus, for \( n \geq 41 \) and \( 1.9 < \gamma < 2 \), let us consider a coloring of edges of \( K_3^N \), \( N = \gamma n + 7\sqrt{n} \), by \( n \) colors without monochromatic copies of \( P_3^3 \), and let \( H_i \) denote the \( P_3^3 \)-free hypergraph generated by the \( i \)-th color.

We say that the \( i \)-th color is rich if

\[
|H_i| \geq \left( \frac{n + 6\sqrt{n} - 1}{2} \right) + \left( \frac{N - n - 6\sqrt{n}}{2} \right).
\]

Claim 4. At least \( \beta n \) colors are rich, where

\[
\beta = \frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{6(\gamma - 1)}.
\]
Proof. Due to technical calculations it will be convenient to show the statement by contradiction. Denote the number of rich colors by \( \beta n \) and assume that
\[
\beta < \frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{6(\gamma - 1)} < \frac{1}{3}
\]
Since by (1), for each \( i \in [n] \) we have \(|H_i| \leq \binom{N-1}{2}\),
\[
\binom{N}{3} < \beta n \binom{N-1}{2} + n(1-\beta) \left( \frac{n+6\sqrt{n}-1}{2} + \frac{N-n-6\sqrt{n}}{2} \right).
\]
Now substituting \( N = \gamma n + 7\sqrt{n} \) and putting all leading terms on the left hand side of the equation we arrive at
\[
[(\gamma^3 - 3\gamma^2 + 6\gamma - 6) - \beta(6\gamma - 6)]n^3 < [\beta(36\gamma - 30) - (21\gamma^2 - 6\gamma - 30)]n^{5/2}
+ [(\beta(42 - 6\gamma) - (150\gamma - 3\gamma^2 - 105)]n^2
- [6\beta + 400 - 42\gamma]n^{3/2} - [2\gamma - 153]n - 14\sqrt{n}.
\]
But for \( 1.9 < \gamma < 2 \) and \( 0 \leq \beta < 1/3 \) the right hand side of the above equation is smaller than \(-19n^{5/2} - 157n^2 - 316n^{3/2} + 150n - 14\sqrt{n}\) which, in turn, is negative for all natural \( n \). Consequently,
\[
[(\gamma^3 - 3\gamma^2 + 6\gamma - 6) - \beta(6\gamma - 6)]n^3 < 0,
\]
and thus
\[
\beta > \frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{6(\gamma - 1)},
\]
contradicting (4). \( \square \)

Recall that each \( H_i \) is \( P_3 \)-free and so one can apply to it Lemma 2 to get a decomposition of \( H_i \) into three graphs, \( H_R^i \cup H_T^i \cup H_S^i \). For all \( i \in [n] \) let us ‘uncolor’ all the triples in \( H_R^i \) and mark them ‘blank’, and set \( \hat{H}_i = H_T^i \cup H_S^i \). Let \( G_i \) be the graph whose edges are pairs which belong to at least two hyperedges of \( \hat{H}_i \) and fewer than \( 6\sqrt{n} \) blank triples. Note that, because of the structure of \( \hat{H}_i \), \( G_i \) is a forest consisting of stars.

We say that an edge of \( G_i \) is private if it is not an edge of any other graph \( G_j \), \( j \neq i \), and public otherwise. By \( e_i \) and \( e'_i \) we denote the number of private and public edges of \( G_i \), respectively. The weight \( w(G_i) \) of \( G_i \) is defined as
\[
w(G_i) = e_i + e'_i/2.
\]
Since \( G_i \) is a forest we have also
\[
w(G_i) \leq e_i + e'_i < N.
\]
Note that at most
\[
\frac{3 \sum_{i=1}^{n} |H_R^i|}{6 \sqrt{n}} \leq \frac{3 \sum_{i=1}^{n} 6N}{6 \sqrt{n}} = 3 \sqrt{n}N = 3 \sqrt{n}(\gamma n + 7 \sqrt{n}) < 6n^{3/2} + 21n
\]
pairs of $K_N^2$ belong to at least $6 \sqrt{n}$ blank triples. Since by the pigeonhole principle all pairs which are contained in fewer than $6 \sqrt{n}$ blank triples are edges of at least one $G_i$, we have
\[
(6) \quad \binom{N}{2} - 6n^{3/2} - 21n \leq \sum_{i \in [n]} w(G_i).
\]

Let us make the following easy yet crucial observation.

**Claim 5.** If a color $i$ is rich, then $G_i$ contains more than $2w(G_i) - N$ private edges. All of them form a star.

**Proof.** Since $G_i$ is a forest we have $e_i + e'_i < N$. Thus,
\[
w(G_i) = e_i + e'_i/2 < e_i + N/2 - e_i/2 = (e_i + N)/2,
\]
and so $G_i$ contains more than $2w(G_i) - N$ private edges. Note also that, by Lemma 3, $H_S^i$ contains the unique largest star $F$ on at least $n + 6 \sqrt{n}$ vertices. Let us denote the center of this star by $w$. Then each edge of $G_i$ which does not contain $w$ is clearly contained in fewer than $N - n - 6 \sqrt{n}$ hyperedges of $\hat{H}_i$ and so belongs to at least $n$ triples of $\bigcup_{j \neq i} \hat{H}_j$. Thus, by the pigeonhole principle, each such edge must be public. Consequently, all private edges must contain $w$ and form in $G_i$ a large star. \qed

Let $I$ denote the set of all rich colors. As an immediate corollary of Claim 5 we get the following inequality
\[
\sum_{i \in I} (2w(G_i) - N) < \sum_{i \in I} e_i \leq \binom{|I|}{2} + |I|(N - |I|))
\]
which leads to
\[
\sum_{i \in I} w(G_i) \leq N|I| - |I|^2/4 - |I|/4.
\]
Thus, using (5) and (6) we get
\[
\binom{N}{2} - 6n^{3/2} - 21n \leq \sum_{i \in \mathbb{N}} w(G_i) = \sum_{i \in I} w(G_i) + \sum_{i \notin I} w(G_i) \\
< N|I| - |I|^2/4 - |I|/4 + \sum_{i \notin I} N \\
= N|I| - |I|^2/4 - |I|/4 + (n - |I|)N \\
\leq nN - |I|^2/4.
\]
Hence, using the estimate for the size of $I$ given by Claim 4 we arrive at
\[
\left(\frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{6(\gamma - 1)}\right)^2 \frac{n^2}{2} \leq \frac{|I|^2}{2} < 2nN - 2\binom{N}{2} + 12n^{3/2} + 42n.
\]
Now substituting $N = \gamma n + 7\sqrt{n}$ and putting all leading terms on the left hand side of the inequality results in the following formula
\[
\left(\frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{72(\gamma - 1)^2} - \gamma(2 - \gamma)\right) n^2 < (26 - 14\gamma)n^{3/2} + (\gamma - 7)n + 7\sqrt{n}.
\]
But for $1.9 < \gamma < 2$ and $n \geq 2$ we have
\[
(26 - 14\gamma)n^{3/2} + (\gamma - 7)n + 7\sqrt{n} < 0,
\]
and so
\[
\left(\frac{\gamma^3 - 3\gamma^2 + 6\gamma - 6}{72(\gamma - 1)^2} - \gamma(2 - \gamma)\right) n^2 < 0.
\]
Consequently,
\[
(\gamma^3 - 3\gamma^2 + 6\gamma - 6)^2 < 72\gamma(2 - \gamma)(\gamma - 1)^2,
\]
which implies that $\gamma$ is smaller than $\lambda_0$ defined in Theorem 1. 

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