The Inviscid, Compressible and Rotational, 2D Isotropic Burgers and Pressureless Euler-Coriolis Fluids; Solvable models with illustrations

Ph. Choquard\textsuperscript{a}, M. Vuffray\textsuperscript{b}

\textsuperscript{a}Ecole Polytechnique Fédérale de Lausanne, ITP-SB-EPFL, Station 8, CH-1015 Lausanne, Switzerland

\textsuperscript{b}Center for Nonlinear Studies and Theoretical Division T-4, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Abstract

The coupling between dilatation and vorticity, two coexisting and fundamental processes in fluid dynamics \cite{1} pp. 3, 6 is investigated here, in the simplest cases of inviscid 2D isotropic Burgers and pressureless Euler-Coriolis fluids respectively modeled by single vortices confined in compressible, local, inertial and global, rotating, environments. The field equations are established, inductively, starting from the equations of the characteristics solved with an initial Helmholtz decomposition of the velocity fields namely a vorticity free and a divergence free part \cite{1} Sects. 2.3.2, 2.3.3 and, deductively, by means of a canonical Hamiltonian Clebsch like formalism \cite{2}, \cite{3}, implying two pairs of conjugate variables. Two vector valued fields are constants of the motion: the velocity field in the Burgers case and the momentum field per unit mass in the Euler-Coriolis one. Taking advantage of this property, a class of solutions for the mass densities of the fluids is given by the Jacobian of their sum with respect to the actual coordinates. Implementation of the isotropy hypothesis entails a radial dependence of the velocity potentials and of the stream functions associated to the compressible and to the rotational part of the fluids and results in the cancellation of the dilatation-rotational cross terms in the Jacobian. A simple expression is obtained for all the radially symmetric Jacobians occurring in the theory. Representative examples of regular and singular solutions are shown and the competition between dilatation and vorticity is illustrated. Inspired by thermodynamical, mean field theoretical analogies, a genuine variational formula is proposed.

\textit{Email addresses:} philippe.choquard@epfl.ch (Ph. Choquard), vuffray@lanl.gov (M. Vuffray)

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which yields unique measure solutions for the radially symmetric fluid densities investigated. We stress that this variational formula, unlike the Hopf-Lax formula, enables us to treat systems which are both compressible and rotational. Moreover in the one-dimensional case, we show for an interesting application that both variational formulas are equivalent.

**Keywords:** Inviscid, compressible, isotropic, cylindrical vortices, Euler fluids, critical behavior, variational formula

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### 1. Introduction

Consider the inviscid Burgers and pressureless Euler-Coriolis equations in 2D, with \( \omega = 0 \) and \( \omega \neq 0 \) respectively, with initial velocity fields consisting of the gradients of scalar potentials and of the orthogonal gradients of stream functions, the said Helmholtz decomposition, and associated, here, to the models of single vortices confined in compressible, local, inertial and global, rotating environments. Our purpose is to study in these situations, the coupling between dilatation and vorticity, two coexisting and fundamental processes in fluid dynamics as emphasized by Wu et al. [1, pp. 3,6]

The questions concerning the existence and uniqueness of explicit solutions of these equations in general, are treated separately for the rotational but incompressible case, in the monographs of Lions [4], of Majda and Bertozzi [5], of Marchioro and Pulvirenti [6] and in the article of Shnirelman [7], and for the compressible but irrotational case, in the monograph of Lions [8] and in the article of Chen and Wang [9], but, to our knowledge, not for the combined, Helmholtz like situation, thus motivating the present paper. It is nevertheless worth quoting, in particular, [5, Sect. 2.2 and Ch. 8] and [9, Sect. 10] and the explicit solutions given by LI [10] for the inviscid and incompressible Euler fluid in 2D and by Yuen [11] for some anisotropic blowup solutions of the inviscid, compressible and pressureless Euler equation in nD.

This paper is organized as follows. Section 2 presents theoretical preambles. It consists of four subsections. The first one gives an inductive derivation of the relevant Burgers and Euler equations in the sense of starting from the equations of their characteristics with initial Helmholtz decomposition of their velocity fields expressed in terms of Lagrangian variables, and moving up into the Eulerian ones. Then, the formal, implicit solutions for the two models are given. This is followed by a deductive derivation of the field equations, a canonical Hamiltonian one, starting from Clebsch Ansatz for the momentum field density of the fluid expressed in terms of
two pairs of canonically conjugated variables, one for the compressible part and one for the rotational part of the fluid and from which the equations of the characteristics are deduced. The next subject concerns the fluid densities. Here, we exploit the fact that the Jacobian of constant vector fields satisfies a continuity equation. Thus, starting with our Helmholtz type constant velocity fields we choose our densities to be proportional to the Jacobian of their sum, with a proportionality constant having the dimension \( \text{gcm}^{-2} \text{sec}^2 \), in 2D. It remains to implement our isotropy hypothesis, i.e. a radial dependence of the potentials of the compressible and of the rotational part of our fluids. The unexpected results are an additive contribution of their compressible and rotational part with vanishing cross terms and a simple algebraic expression for them namely: the derivative of the square the vectorial velocity fields with respect to the square of vectorial coordinates at time \( t \), a result generalized to the cases of \( n \)-dimensional, radially symmetric, at least twice differentiable functions.

Section 3 consists of four subsections: two on solvable models, one on a variational formula for these models and one on a one dimensional version of the variational formula. The first illustration is that of a cylindrical vortex in a local, compressible and inertial environment, i.e. a Burgers case. The competition between dilatation and vorticity is clearly demonstrated: according to the chosen initial conditions, the density profile is regular for times larger than a critical one, at which the density explodes, this critical time going to \(-\infty\) for vanishing compressibility, as expected from the regularity properties stated in the review articles cited above. In the second illustration, an Euler-Coriolis case, we consider a cylindrical vortex in a global, compressible and rotating environment. Here, the spiraling form of the characteristics and their variation with the frequency of rotation adds a second parameter leading to a new singularity, given analytically but not illustrated numerically in this paper. However, for a given frequency, the behavior of the density is similar to that of the first illustration. It is the purpose of the third subsection to propose a genuine variational formula inspired by a thermodynamical analogy discovered between the present 2D isotropic situations and Weiss mean field theory of Magnetism in 1D and which gives rise to unique measure solutions for the densities investigated. As a fallout of the content of this section, it is shown that the variational formula that we propose, and based on a Maupertuis action instead of a Lagrangian one, is equivalent to that of Hopf-Lax [12 Sect. 3.3.2 and p.123] for the Burgers equation in 1D. This equivalence is exemplified at hand of the one-dimensional Ising model in the mean field approximation.

Lastly Section 4 is devoted to the presentation of further developments.
2. Theoretical Preambles

2.1. Equations of the characteristics and of the fluids

Let \( r = (r_1, r_2) \), be the coordinates of a test particle of mass \( m \) in a 2 dimensional inertial reference frame, \( x = (x_1, x_2) \), its coordinates in the rotating reference frame with frequency \( \omega \) and let \( O(\omega t) = \begin{pmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{pmatrix} \) be the 2D orthogonal matrix. Let \( \langle a, b \rangle \) be the scalar product of the vectors \( a \) and \( b \), and \( \langle a, a \rangle := a^2 \), for simplicity, and let \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) be the anti-symmetric matrix such that the 2D vector product \( a \wedge b = aAb \). With \( x(t) = O(\omega t)r(t) \), with \( \dot{O}(\omega t) = -\omega AO(\omega t) \) and \( \dot{x}(t) = -\omega AOr + O\dot{r} \), we have

\[
\dot{r} = O^{-1}(\dot{x} + \omega Ax),
\]

and the Lagrangian become

\[
L(\dot{x}, x) = \frac{m}{2} \langle (\dot{x} + \omega Ax), (\dot{x} + \omega Ax) \rangle = \frac{m}{2} \langle \dot{r}, \dot{r} \rangle = L^*(\dot{r}).
\]

With the momenta \( p = m(\dot{x} + \omega Ax) \), and \( \pi = m\dot{r} \) the Hamiltonian \( H(p, x) = \langle p, \dot{x} \rangle - L(\dot{x}, x) = \frac{1}{2m}p^2 - \omega pAx \) and \( H^*(\pi) = \pi\dot{r} - L^*(\dot{r}) \) become

\[
H = \frac{1}{2m} \langle (p - m\omega Ax), (p - m\omega Ax) \rangle - \frac{1}{2}m\omega^2x^2.
\]

and

\[
H^*(\pi) = \frac{1}{2m} \langle \pi, \pi \rangle.
\]

The resulting equations of motion are

\[
\ddot{x} + 2\omega A\dot{x} - \omega^2 x = 0,
\]

and

\[
\ddot{r} = 0.
\]

Inspection of the equation for \( x(t) \) shows that its eigenvalues are degenerate thus explaining the spiraling nature of the solutions. If \( y = (y_1, y_2) \) are the initial coordinates, if \( \phi(y) \) is the scalar potential associated to the compressible part of the fluid and \( \psi(y) \), the stream function associated to its vorticity, the initial velocity field \( u(y) \), compatible with Helmholtz decomposition \([1] \) Sects. 2.3.2, 2.3.3] then reads, with \( A\nabla := \text{orthogonal gradient} \),

\[
u(y) = \nabla\phi(y) + A\nabla\psi(y).
\]
The momentum per unit mass, $\pi/m$, a constant vector in the Coriolis case is denoted by
\[ v(y,\omega) := \frac{\pi}{m} = u(y) + \omega Ay. \] (8)

It follows that
\[ r(y,\omega,t) = y + tv(y,\omega), \] (9)

and that
\[ x(y,\omega,t) = O(\omega t)(y + tv(y,\omega)), \] (10)
an explicit construction of the spirals.

It remains to pass from the Lagrangian to the Eulerian coordinates. With
\[ \dot{x} = u := u(x,\omega,t), \] \[ \ddot{x} = \partial_t u + \langle u, \nabla \rangle u, \]
we get, for the inviscid and pressureless Euler-Coriolis fluid,
\[ \partial_t u + \langle u, \nabla \rangle u + 2\omega Au - \omega^2 x = 0. \] (11)

For the Burgers cases, compressible and rotational, the terms containing $\omega$ are omitted.

Let us point out here that, whereas the above derivation of the Euler-Coriolis equation has followed an inductive path, in the sense that $\dot{x} \rightarrow u(x,\omega,t)$, and $\ddot{x} \rightarrow \partial_t u + \langle u, \nabla \rangle u$, a deductive one is also feasible by means of a canonical Hamiltonian Clebsch-like formalism [2], [3] implying two pairs of canonically conjugated field variables (3 in 3D), one pair for the compressible part and one for the rotational part of the fluid, a version presented in the next subsection.

It is appropriate to give, here, the formal, implicit solutions of the equations corresponding to our two models. If $u_0(y)$ and $v_0(y,\omega)$ designate, for simplicity and for reasons of dimensionality, the two initial velocity fields, (strictly speaking: initial velocity field and initial momentum field per unit mass) and recalling that $x(t) = O(\omega t)r(t)$, then we have
\[ u(x,t) = u_0(y) = u_0(x - u(x,t)t); \] (12)
for the Burgers case ($\omega = 0$) and
\[ v(r,t,\omega) = v_0(y,\omega) = v_0(r - v(r,t,\omega)t), \] (13)
for the Euler-Coriolis one ($\omega \neq 0$).

Let us conclude this subsection in recalling that, with $\Delta$ being the Laplace operator, the dilatation field, $\Theta$, is
\[ \Theta = \langle \nabla, v \rangle = \Delta \phi, \] (14)
and the $z$ component of the vorticity field, $\Omega$, consisting of an intrinsic part and an extrinsic one, is

$$\Omega = \nabla \wedge \mathbf{v} = \Delta \psi + 2\omega.$$  \hfill (15)

2.2. Fluid-Mechanical Formulation

In 2D, two canonically conjugate pairs of variables come into play: $(\rho, \sigma)$ for the compressible part and $(\beta, \kappa)$ for the rotational one. Clebsch’s Ansatz for the total mass current, also momentum density, is

$$\mathbf{j} = \rho \nabla \sigma + \kappa \nabla \beta.$$ \hfill (16)

The pressureless Euler-Coriolis Hamiltonian is

$$\hat{H}(\sigma, \beta; \rho, \kappa) = \int_{R^2} d^2x \left( \frac{1}{2} \frac{\langle \mathbf{j}, \mathbf{j} \rangle}{\rho} - \rho \omega \langle \mathbf{j}, \mathbf{A}\mathbf{x} \rangle \right)$$

$$= \int_{R^2} d^2x \rho \left\langle \left( \nabla \sigma + \frac{\kappa}{\rho} \nabla \beta - \omega \mathbf{A}\mathbf{x} \right), \left( \nabla \sigma + \frac{\kappa}{\rho} \nabla \beta - \omega \mathbf{A}\mathbf{x} \right) \right\rangle$$

$$- \frac{1}{2} \int_{R^2} d^2x \rho \omega^2 x^2.$$ \hfill (17)

The equations of motion are, in identifying $\mathbf{j}/\rho - \omega \mathbf{A}\mathbf{x} = \mathbf{u}$, the velocity field, and in setting $\frac{\kappa}{\rho} \nabla \beta := \mathbf{w}$,

$$\partial_t \sigma + \delta \hat{H}/\delta \rho = \partial_t \sigma + \frac{1}{2} \mathbf{u}^2 - \langle \mathbf{u}, \mathbf{w} \rangle - \frac{1}{2} \omega^2 x^2 = 0,$$ \hfill (18)

$$\partial_t \rho - \delta \hat{H}/\delta \sigma = \partial_t \rho + \langle \nabla, \rho \mathbf{u} \rangle = 0,$$ \hfill (19)

$$\partial_t \beta + \delta \hat{H}/\delta \kappa = \partial_t \beta + \langle \mathbf{u}, \nabla \beta \rangle = 0,$$ \hfill (20)

$$\partial_t \kappa - \delta \hat{H}/\delta \beta = \partial_t \kappa + \langle \nabla, \kappa \mathbf{u} \rangle = 0.$$ \hfill (21)

Observe that $\beta$ is a constant of the motion and that $\kappa$, in addition to $\rho$, satisfies the equation of continuity; then, setting $\kappa/\rho := \alpha$ and $\mathbf{w} := \alpha \nabla \beta$ we notice that $\mathbf{w}$ can be identified with $A \nabla \psi$ and we have that

$$\partial_t \alpha + \langle \mathbf{u}, \nabla \rangle \alpha = 0.$$ \hfill (22)

Thus, $\alpha$ and $\beta$ are two constants of the motion, also called Clebsch parameters and the intersection of the surfaces $\alpha = e^{t\epsilon}$ and $\beta = e^{t\epsilon}$ gives vortex lines parallel to the $z$ axes in our cylindrical symmetry. In the illustrations, we have $\alpha \nabla \beta = A \nabla \psi(y) =$
\( Ay\psi'(y)/y, \) i.e \( \beta(y) = \arctan(y_2/y_1) = \varphi \) and \( \alpha = y\psi'(y) \) for \( \psi(y) = \psi_B(y) \) and \( \psi(y) = \psi_{EC}(y) \).

We compute next the gradient of \( \sigma \) with the aim of deriving the pressureless Euler-Coriolis equation, an elementary operation in the absence of intrinsic and extrinsic vorticity. We start from

\[
\partial_t \nabla \sigma + \langle u, \nabla \rangle u + u \wedge \nabla \wedge u - \nabla \langle u, w \rangle - \omega^2 x = 0.
\] (23)

On the one hand, we have

\[
\nabla \langle u, w \rangle = \langle u, \nabla \rangle w + \langle w, \nabla \rangle u + u \wedge \nabla \wedge w + w \wedge \nabla \wedge u,
\] (24)

and, on the other hand, we have

\[
\partial_t w := \partial_t (\alpha \nabla \beta) = \partial_t \alpha \nabla \beta + \alpha \nabla \partial_t \beta = -\langle u, \nabla \alpha \rangle \nabla \beta - \alpha \nabla \langle u, \nabla \beta \rangle = -\langle u, \nabla \alpha \rangle \nabla \beta - \alpha \langle \nabla \beta, \nabla \rangle u - \alpha u \wedge \nabla \wedge \nabla \beta - \alpha \nabla \beta \wedge \nabla \wedge u
\] (25)

having noticed that the term \( \alpha u \wedge \nabla \wedge \nabla \beta = 0 \). The important result is that

\[
\partial_t w = -\nabla \langle u, w \rangle + u \wedge \nabla \wedge w.
\] (26)

It follows that

\[
\partial_t (\nabla \sigma + w) + \langle u, \nabla \rangle u + u \wedge \nabla (u - w) - \omega^2 x = 0,
\] (27)

or, with \( \nabla \sigma + w = u + \omega \mathbf{A} x \) and \( u \wedge \nabla (u - w) = u \wedge \nabla (\sigma - \omega \mathbf{A} x) = -\omega u \wedge \nabla \mathbf{A} x = 2\omega \mathbf{A} u \), the final result is

\[
\partial_t (u + \omega \mathbf{A} x) + \langle u, \nabla \rangle u + 2\omega \mathbf{A} u - \omega^2 x = 0.
\] (28)

In fact, it is a generalization of the equation established in the first subsection since it applies to the cases where \( \omega = \omega(t) \), a situation not considered in this paper. Other applications of Clebsch canonical formalism to 2 and 3 D, compressible, self-interacting, neutral and charged systems are given f.i. in [13], [14, Sect. 2], and more, for magnetic and electromagnetic systems, leading, e.g. to Euler-Lorentz and Euler-Maxwell equations are also possible.

So far we have been and still are able to use the elegant Clebsch canonical formalism in our applications although the transformation from Lagrangian to Eulerian
variables is not canonical and our equations of motion are not in a canonical form. Now, the fundamental fact that Euler variables ($\rho$, $u$ and $v$ in our illustrations) are not canonical variables has triggered an extraordinary rich development of non canonical Hamiltonian theories, involving Lie-Poisson algebraic structures and pioneered by Morrison and Greene in the eighties. Here, it is most appropriate to quote Morrison’s recent and very synthetic presentation in [15] where an exhaustive list of applications and also a pertinent list of references are given.

2.3. A class of solutions for the densities

Let us recall that, if $\xi(x,t)$ is a 2 dimensional vector valued application such that each component is a constant of the motion, i.e. $d\xi/dt = \partial_t \xi + \langle u, \nabla \rangle \xi = 0$, then its Jacobian $J = \det(\partial \xi_i/\partial x_j)$ satisfies a continuity equation [14, Section 3.3 and Appendix 5.2]. Furthermore, if $x = x(y,t)$ is the equation of a characteristic, then we have that

$$J = \det(\partial \xi_i/\partial y_j) \det(\partial x_k/\partial y_l)^{-1}. \quad (29)$$

In our cases, we have two constant fields: $u$ and $v$. Considering first the Burgers case, and up to a proportionality constant, of dimension $gcm^{-2} \text{sec}^2$, set $= 1$, (the apparent dimension of $\rho$ being sec$^{-2}$), we define for the Burgers case,

$$\rho_B(x,t) = \det(\partial u_i/\partial x_j) = \det(\partial v_i/\partial y_j) \det(\partial x_k/\partial y_l)^{-1}. \quad (30)$$

In the Euler-Coriolis case, and since $\det(O(\omega t)) = 1$, we define

$$\rho_{EC}(x,\omega,t) = \rho_{EC}(r,\omega,t) = \det(\partial v_i/\partial r_j) = \det(\partial v_i/\partial y_j) \det(\partial r_k/\partial y_l)^{-1}. \quad (31)$$

This class of densities might be identified as the Gelfand class [16]. Whenever the method of characteristics is employed, more general solutions, for any $\rho_0(y)$ of positive type, would be

$$\rho(x,t) = \rho_0(y) \det(\partial x_k/\partial y_l)^{-1}. \quad (32)$$

The said Gelfand class will be used in what follows. It means that the initial densities are proportional to $\det(\partial u_i/\partial y_j)$ and to $\det(\partial v_i/\partial y_j)$.

2.4. Isotropic cases

In this subsection we consider radially-symmetric densities. Let $y = |y|$, $r = |r|$, $x = |x|$, $u = |u|$, and $v = |v|$. Thus, $\phi = \phi_B(y)$ or $\phi_{EC}(y,\omega)$, $\psi = \psi_B(y)$ or $\psi_{EC}(y,\omega)$ and for the Euler-Coriolis case we introduce the effective potential $\Psi_{EC}(\omega,y) = \frac{1}{2} \omega y^2 + \psi_{EC}(y,\omega)$. With $'$ and $''$ designating the first and second derivative with
respect to \( y \), we determine first the initial effective density. It suffices to consider 
\( \rho_{EC}(y, \omega) \). With \( \partial^2_{y,y_j} f(y) := f_{ij}(y) \) for any \( f(y) \) twice differentiable, we have

\[
\rho_{EC}(y, \omega) = v_{11} v_{22} - v_{12} v_{21}
= (\phi_{EC, 11} - \Psi_{EC, 21})(\phi_{EC, 22} + \Psi_{EC, 12}) - (\phi_{EC, 12} - \Psi_{EC, 22})(\phi_{EC, 21} + \Psi_{EC, 11}) .
\]

(33)

A detailed calculation shows that, in the isotropic case, the cross terms in \( \phi_{EC, ij} \),\( \Psi_{EC, kl} \) cancel out. Thus we get

\[
\rho_{EC}(y, \omega) = \det(\phi_{EC, ij}) + \det(\Psi_{EC, ij}) .
\]

(34)

It remains to calculate one of these determinants, for example and generically,

\[
\det(\phi_{ij}(y)) = \left( \frac{y_2^2}{y^3} \phi'(y) + \frac{y_1^2}{y^3} \phi''(y) \right) \left( \frac{y_1^2}{y^3} \phi'(y) + \frac{y_2^2}{y^3} \phi''(y) \right) - \left( \frac{y_1 y_2}{y^2} \right) \left( - \frac{\phi'(y)}{y} + \phi''(y) \right)^2
= \left( \frac{y_1^4 + 2y_1^2 y_2^2 + y_2^4}{y^5} \right) \phi'(y) \phi''(y)
= \frac{1}{y} \phi'(y) \phi''(y) = \frac{d(\phi'(y))}{d(y^2)} .
\]

(35)

It follows that

\[
\rho_B(y) = \frac{d(\phi_B'(y))^2}{d(y^2)} + \frac{d(\psi'_B(y))^2}{d(y^2)} = \frac{d(u^2(y))}{d(y^2)} .
\]

(36)

and that

\[
\rho_{EC}(y, \omega) = \frac{d(\phi_{EC}'(y, \omega))^2}{d(y^2)} + \frac{d(\Psi_{EC}'(y, \omega))^2}{d(y^2)} = \frac{d(v^2(y, \omega))}{d(y^2)} .
\]

(37)

In summary, the isotropy hypothesis and the symmetries of the velocity fields result in the fact that the Jacobian of a sum equals the sum of the Jacobians and in a simple formula for them. In fact this formula can be generalized to the cases of \( n \)-dimensional radially-symmetric, at least twice-differentiable functions \( f(y) \). Indeed, let \( e_i = \frac{y_i}{y} \), \( i = 1, \ldots, n \), be the director cosines of the vector \( y \) and let the matrix elements \( f_{ij} = \frac{f'(y)}{y} \delta_{ij} + \left( -\frac{f''(y)}{y} + f''(y) \right) e_i e_j \), then the Jacobian is \( J = \det(f_{ij}(y)) = \left( \frac{f'(y)}{y} \right)^{n-1} f''(y) = \frac{d}{d(y^n)} (f'(y))^n \), owing to the vanishing of all sub-determinants with elements \( e_i e_j \).
We proceed with the evaluation of the Jacobian

\[
\det(\frac{\partial x_k}{\partial y_l}) = 1 + t \Delta \phi(y) + \frac{t^2}{y} (\phi'(y)\phi''(y) + \psi'(y)\psi''(y)),
\]  

(38)

and

\[
\det(\frac{\partial r_k}{\partial y_l}) = 1 + t \Delta \phi(y) + \frac{t^2}{y} (\phi'(y)\phi''(y) + \omega^2 + \omega \Delta \psi(y) + \psi'(y)\psi''(y)).
\]  

(39)

We find

\[
\det(\frac{\partial x_k(y,t)}{\partial y_l}) = \frac{\partial (x^2(y,t))}{\partial (y^2)},
\]  

(40)

and

\[
\det(\frac{\partial r_k(y,\omega,t)}{\partial y_l}) = \frac{\partial (r^2(y,\omega,t))}{\partial (y^2)}.
\]  

(41)

It follows that

\[
\rho_B(x,t) = \frac{d(u^2(y))}{d(y^2)} \left( \frac{\partial (x^2(y,t))}{\partial (y^2)} \right)^{-1} = \frac{\partial (u^2(x,t))}{\partial (x^2)},
\]  

(42)

and, similarly, that

\[
\rho_{EC}(r,\omega,t) = \frac{d(v^2(y,\omega))}{d(y^2)} \left( \frac{\partial (r^2(y,\omega,t))}{\partial (y^2)} \right)^{-1} = \frac{\partial (v^2(r,\omega,t))}{\partial (r^2)}.  
\]  

(43)

3. Illustrations and variational formulation

Two illustrations are proposed in this section and their results interpreted in terms of a variational formula inspired by thermodynamical analogies.

3.1. A cylindrical vortex in a compressible, finite and inertial, environment: a Burgers case

The initial velocity fields are split into a compressible and a rotational part. The corresponding mass densities are such that the total mass is prescribed. With

\[-\pi/2 \leq \tau \leq \pi/2, 0 < \lambda < \mu,\]

for the range parameters, we choose, in omitting in the functions an explicit quotation of the parameters, unless otherwise convenient and re-calling that r = x, since \(\omega = 0\),

\[
\phi'_B(y) = \frac{\lambda y}{\sqrt{1 + (\lambda y)^2}} \cos \tau,
\]  

(44)
and, allowing negative amplitude for the rotational part,

$$\psi'_B(y) = \frac{\mu y}{\sqrt{(1 + (\mu y)^2)}} \sin \tau. \quad (45)$$

Designating the corresponding initial field densities by $\rho_{\phi_B}(y)$ and $\rho_{\psi_B}(y)$ and their sums by $\rho_B(y)$, we have

$$\rho_B(y) := \rho_{\phi_B} + \rho_{\psi_B} = \frac{(\lambda \cos \tau)^2}{(1 + (\lambda y)^2)^2} + \frac{(\mu \sin \tau)^2}{(1 + (\mu y)^2)^2}. \quad (46)$$

They are displayed in Figure 1 for $\lambda = \mu = 1$ and $\tau = 0.2$, a particular case considered later.

This model represents a cylindrical vortex with positive or negative amplitude, in a compressible, finite and inertial environment. Clearly, the total mass is, per unit height,

$$\pi \int_0^\infty dy^2 \rho_B(y, 0) = \pi \left( \phi'_B(\infty)^2 + \psi'_B(\infty)^2 \right) = \pi. \quad (47)$$

We have also

$$u(y)^2 = \phi'_B(y)^2 + \psi'_B(y)^2. \quad (48)$$
\[x(y, t)^2 = (y + t\phi_B'(y))^2 + (t\psi_B'(y))^2. \tag{49}\]

Consider the domain of regularity for the densities. Since \(\rho_B(y) > 0\) for \(y < \infty\), the condition is, in omitting again the explicit dependence upon the parameters \(\lambda, \mu, \tau\),
\[\frac{\partial}{\partial (y^2)} (x^2(y, t)) = (1 + t\phi_B'(y)/y) (1 + t\phi_B''(y)) + t^2\psi_B'(y)\psi_B''(y)/y > 0. \tag{50}\]

The two roots in the time variable of the above relation set equal to zero are, with \(\phi_B'(y)/y + \phi_B''(y) = \Delta \phi_B(y)\),
\[t_{\pm}(y) = \left( -\Delta \phi_B(y) \pm (\Delta \phi_B(y))^2 - 4\rho_B(y) \right)^{1/2} / 2\rho_B(y), \tag{51}\]
and the positivity condition implies that the discriminant \(D_B(y) = (\Delta \phi_B(y))^2 - 4\rho_B(y, 0) < 0\).

As illustration, let \(\mu = \lambda = 1\), i.e. a one-dimensional subspace \((\tau)\) of the three dimensional parameter space \((\lambda, \mu, \tau)\). In this case, \(u^2 = y^2(1 + y^2)^{-1}\) or \(y^2 = u^2(1 - u^2)^{-1}\), \(0 \leq u \leq 1\) and
\[x^2(u^2, t, \tau) = u^2 (1 - u^2)^{-1} + 2t(\cos \tau)u^2 (1 - u^2)^{-1/2} + t^2u^2. \tag{52}\]

It follows that in investigating the inverse density \(\rho_B(u^2, t, \tau)^{-1} = \partial(x^2)/\partial(u^2)\) and requiring that \(\rho_B(u^2, t, \tau)^{-1} > 0\), we can determine the domain of regularity of the solutions, i.e.
\[\rho_B(u^2, t, \tau)^{-1} = (1 - u^2)^{-2} + t(\cos \tau)(2 - u^2)(1 - u^2)^{-3/2} + t^2 > 0. \tag{53}\]

There is a one parameter family of critical points \(t_c(\tau) < 0\) at which the two roots of this quadratic equation in \(t\) (set = 0) coincide. Setting the discriminant of this equation = 0 gives an equation for the critical values of the velocity, namely
\[(\cos \tau)^2 = 4 - \frac{1 - u_c^2}{(2 - u_c^2)^2}. \tag{54}\]

Introducing the angular variable \(\theta\) such that \(1 - u_c^2 = (\tan \theta)^2\) results in \((\cos \tau)^2 = (\sin (2\theta))^2\). The relevant solution is \(2\theta = \pi/2 - |\tau|\). It follows that \(0 \leq \theta \leq \pi/4\) and thus, \(u_c^2(\tau) = 1 - (\tan (\pi/4 - |\tau|/2))^2\).

For the critical times we have, in setting the discriminant = 0, \(t_c = -1/(\tan (\pi/4 - |\tau|/2))^2\), or
\[t_c(\tau) = -1/ (\tan (\pi/4 - |\tau|/2))^2. \tag{55}\]
It is interesting to notice that $t_c(0) = -1$ and $u_c(0) = 0$, for the purely compressible case, whereas $t_c(\pm \pi/2) = -\infty$ and $u_c(\pi/2) = 1$ for the purely rotational one: this means that, in the coexisting processes of dilatation and vorticity, the domain of regularity increases with increasing vorticity \[1\].

Let us take the example considered above. For $\lambda = \mu = 1$ and $\tau = 0$, one finds $t_c \approx -1.4$, $y_c \approx 0.7$, $x_c \approx 0.2$ and $u_c^2 \approx 0.4$. The density is regular for $t \geq t_c$, with its maximum at the origin until $t \approx -0.83$, when its curvature changes sign, from concave to convex, signaling the onset of a maximum emerging from the origin and culminating to its blow up at $t = t_c$. Figure 2 shows $\rho_B(x^2, t_c)$. Algebraic singularities in $x^2$ and in $u^2$ are found to be an asymmetric one for $\rho_B(x^2, t_c)^{-1} \sim (\alpha_+ \chi(x^2 - x_c^2) + \alpha_- \chi(x_c^2 - x^2))$, $\chi(\xi)$ being the characteristic function, $\alpha_+, \alpha_-$, positive amplitudes, and a quadratic one $\sim (u^2 - u_c^2)^2$ for $\rho_B(u^2, t_c)^{-1}$.

The analysis of the behavior of this model for times $< t_c(\tau)$ will be postponed after the presentation of the second illustration. As a hint, Figure 3 shows the graphs of $u^2(x^2, t)$ for $t = t_c = -3$, and, as before, $\lambda = \mu = 1$, $\tau = 0.2$ and also $\tau = 0$, corresponding to the purely compressible case. The analogy $u^2 \sim (\text{magnetization})^2$, $x^2 \sim (\text{magnetic field})^2$, $-t \sim (\text{coupling const./temperature})$, with Weiss’s theory of magnetism is particularly striking if, for the $\tau = 0$ case, we fold in the first quadrant the curve: magnetization versus magnetic field \[17\, \text{Eqn. 23}\). Alternatively, the graph of $x^2(u^{-2}, t_c)$, shown in Figure 4, with $x^2 \sim \text{pressure}$, $u^{-2} \sim \text{volume}$, evokes a van der Waals loop, with $u^{-2} \geq 1$ mimicking a hard core repulsion. However, no
Figure 3: Weiss analogy of the Burgers case for subcritical densities at $\tau = 0$ (red) and $\tau = 0.2$ (blue).

Figure 4: Van der Waals analogy of the Burgers case.
coexistence of phases being expected here, the first analogy is favored.

3.2. A cylindrical vortex in a compressible, infinite and rotating environment: an Euler-Coriolis case

This illustration describes the evolution of a cylindrical vortex in an infinite compressible and rotating environment. The amplitudes and radial dependence of the compressible and rotational fields are chosen in such a way that if $Y$ is the radius of a large circle having the vortex at his center, if $M(Y, \lambda, \mu, \omega)$ is the mass per unit height contained in this cylinder and if $\nu^2$ is the mass density in the limit $Y \to \infty$, then, omitting exponentially decreasing contributions, we have

$$M(Y, \lambda, \mu, \omega)/\pi Y^2 = \nu^2 + O(\omega/Y) + O((\lambda Y)^{-4})$$

where $O$ denotes the asymptotic notation.

The fields chosen and compatible with the above prescription are

$$\phi'_{EC}(y, \omega) = \frac{\lambda \sqrt{\nu^2 - \omega^2} y^2}{\sqrt{1 + \lambda^2 y^2}}$$

and

$$\psi'_{EC}(y, \omega) = \frac{s \sqrt{\nu^2 - \omega^2}}{\lambda} (1 - e^{-\mu y})$$

the parameter $s = +/ - 1$ meaning that the amplitude of the rotational part of the fluid can have both signs.

At this point, let us check the prescription concerning the fields chosen. We have indeed

$$M(Y)/\pi Y^2 = ((\phi'_{EC}(y, \omega))^2 + (\omega Y + \psi'_{EC}(Y, \omega))^2)Y^{-2}$$

$$= \left( \left( \frac{\lambda \sqrt{\nu^2 - \omega^2} Y^2}{\sqrt{1 + \lambda^2 Y^2}} \right)^2 + \left( \omega Y + \frac{s \sqrt{\nu^2 - \omega^2}}{\lambda} (1 - e^{-\mu Y}) \right)^2 \right) Y^{-2}$$

$$= \nu^2 - \omega^2 + \omega^2 + 2s \frac{\sqrt{\nu^2 - \omega^2}}{\lambda} (1 - e^{-\mu Y}) \omega Y^{-1}$$

$$+ (\nu^2 - \omega^2) \left( (1 - e^{-\mu Y})^2 - 1/(1 + (\lambda Y)^{-2}) \right) (\lambda Y)^{-2}$$

as claimed. It is worth giving the densities associated to the two fields. They are

$$\rho_{\phi_{EC}}(y, \omega) = (\nu^2 - \omega^2) \left( 1 - \frac{1}{1 + (\lambda y)^2} \right)$$
and

\[ \rho_{\psi EC}(y, \omega) = \frac{(v^2 - \omega^2) \mu e^{-\mu y} (1 - e^{-\mu y})}{\lambda^2 y}. \]  
(61)

These field densities at zero frequency and their sum are plotted on Figure 5. We have next

\[ v(y, \omega)^2 = (\phi_{EC}'(y, \omega))^2 + (\omega y + \psi_{EC}'(y, \omega))^2, \]  
(62)

and

\[ r(y, \omega, t)^2 = (y + t\phi_{EC}'(y, \omega))^2 + t^2(\omega y + \psi_{EC}'(y, \omega))^2. \]  
(63)

Consider next the regularity conditions. Defining the initial effective density

\[ \rho_{EC}(y, \omega) = \partial \left( v(y, \omega, s)^2 \right) / \partial (y^2) = \omega^2 + \omega \Delta \psi_{EC}(y, \omega) + \rho_{\phi EC}(y, \omega) + \rho_{\psi EC}(y, \omega), \]  
(64)

we have

\[ \rho_{EC}(y, \omega) > 0, \]  
(65)

and

\[ \rho_{EC}(y, \omega, t) = \rho_{EC}(y, \omega)(1 + t\Delta\phi_{EC}(y, \omega) + t^2\rho_{EC}(y, \omega))^{-1} > 0. \]  
(66)

The roots in the time variable of \( \rho_{EC}(y, \omega, t)^{-1} = 0 \) are

\[ t_{\pm}(y, \omega) = -\frac{\Delta\phi_{EC}(y, \omega) \pm D_{EC}(y, \omega)^{1/2}}{2ho_{EC}(y, \omega)}, \]  
(67)
with the discriminant

\[ D_{EC}(y, \omega) = \frac{\Delta \phi_{EC}(y, \omega)^2 - 4 \rho_{EC}(y, \omega)}{(2\rho_{EC}(y, \omega))^2}. \] (68)

The regularity condition implied is

\[ D(y, \omega, \nu, \lambda, \mu, s) < 0. \] (69)

This condition ensures that the roots \( t_+,-(y, \omega, \nu, \lambda, \mu, s) \) be complex conjugate. In summary, there are two conditions: \( \rho_{EC}(y, \omega, 0, \nu, \lambda, \mu, s) > 0 \) and \( D(y, \omega, \nu, \lambda, \mu, s) < 0 \).

Focusing on a small 0-dimensional subspace of the complete, 5 dimensional, parameter space, which exhibits singular solutions, we choose the following illustration: \( \lambda = 0.5, \mu = 1.5, \nu^2 = 2, \omega = -1 \) and \( s = 1 \). In Figure 6 is plotted the initial effective density \( \rho_{EC}(y, -1, 0) \).

Numerical analysis of the critical density \( \rho_{EC}(y, -1, t_c) \), shown in Figure 7 of \( \rho_{EC}(y, -1, t_c)^{-1} \), \( \rho_{EC}(r, -1, t_c) \) and \( \rho_{EC}(r, -1, t_c)^{-1} \) gives \( y_c \approx 1.5, r_c \approx 0.8 \) and \( t_c \approx -0.826 \). For \( t > t_c \), the solutions are regular. For the sub-critical time \( t < t_c = -1.68 < t_c, \lambda = 0.5, \mu = 1.5, \nu^2 = 2, s = 1, \omega = -1 \), for instance, the function \( v^2(r^2, -1, t_c) \) and \( r^2(v^{-2}, -1, t_c) \) show a behavior similar to that of the functions \( u^2(x^2, t) \) and \( x^2(u^{-2}, t) \) of the previous illustration. This is shown in Figure 8 and Figure 9.
Figure 7: Critical density of the Euler-Coriolis case.

Figure 8: Weiss analogy of the Euler-Coriolis case.
3.3. Variational formulation

The thermodynamical analogies evoked in the subsection [3.2] suggest to give the following variational formulation. For the first illustration, we introduce the potential function:

$$G_B(u^2, t) := \int_0^{u^2} d\tilde{u}^2 x^2(\tilde{u}^2, t),$$

and define the Legendre Transform (turns out to be minus a “Free Energy” functional):

$$F_B(x^2, t) = \sup_{u^2} [x^2 u^2 - G_B(u^2, t)]$$

$$= \int_0^{x^2} dx^2 u^2_{\text{sup}}(\tilde{x}^2, t).$$

This transform implies the construction of the convex envelope of $G_B(u^2, t = -3)$, shown in Figure 10 and the operation $\sup_{u^2} [.]$ which generates the $C_0$ function $F(x^2, t = -3)$, shown in Figure 11. Then, $\partial F_B(x^2, t) / \partial (x^2) = u^2_{\text{sup}}(x^2, t)$ introduces a discontinuity, a vertical cut, in the graph of of $u^2$ versus $x^2$ at $t = -3$, as shown in Figure 12 and which satisfies the equal area rule familiar in Thermodynamics. The final result is a density profile given by $\partial (u^2_{\text{sup}}(x^2, t)) / \partial (x^2)$ and composed of a Dirac distribution and of two tails, as shown in Figure 13. If $x^2$ is the coordinate at separation, $u^2_{\text{sup}, \pm}$, the corresponding velocities squared and $\chi(\xi)$, the characteristics...
Figure 10: Convex envelope of the Burgers case.

Figure 11: Legendre transform of the Burgers case.
Figure 12: Maxwell construction of the Burgers case.

Figure 13: Sub-critical density of the Burgers case, $t = -3$.  

21
function, we have the measure, sub-critical \((t = -3)\), solution
\[
\rho_B(x^2, t) = \rho_{B, -}(x^2, t) \chi(x^{\ast 2} - x^2) + (u_{\sup, +}^{x^2} - u_{\sup, -}^{x^2}) \delta(x^2 - x^{\ast 2}) + \rho_{B, +}(x^2, t) \chi(x - x^{\ast 2}).
\]
(72)

It is clear that by construction the total mass is conserved.

For the second illustration we proceed similarly. We define the potential
\[
G_{EC}(v^2, \omega, t) = \int_{0}^{v^2} d\tilde{v}^2 \tilde{r}^2(\tilde{v}^2, \omega, t),
\]
(73)
and the Legendre transform (another minus of a “Free Energy” Functional),
\[
F_{EC}(r^2, \omega, t) = \sup_{v^2} [r^2 v^2 - G_{EC}(v^2, \omega, t)]
\]
\[
= \int_{0}^{r^2} d\tilde{r}^2 v_{\sup, \ast}^{2 \ast} (\tilde{r}^2, \omega, t).
\]
(74)

We draw \(G_{EC}(v^2, \omega, t)\) and construct its convex envelope (Figure 14), the ensuing \(v_{\sup, \ast}^{2 \ast}(r^2, \omega, t)\) (Figure 15), the \(C_0\) function \(F_{EC}(r^2, t, \omega)\), (Figure 16) and the density profile (Figure 17). With \(t_\ast = -1.68\) and \(\omega = -1\), we obtain figures similar but not identical to those of the first illustration.

3.4. One-dimensional version of the variational formula

As a fallout of the content of the Subsection 3.3 it is natural to present a one-dimensional version of the isotropic 2D variational formula presented there. For this
Figure 15: Maxwell construction of the Euler-Coriolis case.

Figure 16: Legendre transform of the Euler-Coriolis.
purpose, consider the Burgers equation in 1D where \( u, x, t \in \mathbb{R} \)
\[
\partial_t u(x, t) + u(x, t)\partial_x u(x, t) = 0, \quad (75)
\]
its solution, with initial coordinate \( y \in \mathbb{R} \) and velocity field \( u_0(y) \)
\[
x = y + u_0(y)t, \quad (76)
\]
and its implicit solution
\[
u(x, t) = u_0(x - u(x, t)) = u_0(y). \quad (77)
\]
Assume the invertibility condition
\[
y = u_0^{-1}(u). \quad (78)
\]
Then
\[
x(u, t) = u_0^{-1}(u) + tu. \quad (79)
\]
Our variational formula can now be presented. Indeed, let
\[
G_1(u, t) = \int_0^u d\bar{x}(\bar{u}, t), \quad (80)
\]
and let
\[
F_1(x, t) = \sup_u (ux - G_1(u, t)). \quad (81)
\]
24
It follows that
\[ F_1(x, t) = \int_0^x d\bar{x} u_{\sup}(\bar{x}, t), \quad (82) \]
is a kind of Maupertuis action per unit mass and the correct, measurable mass density is
\[ \rho_1(x, t) = d(u_{\sup}(x, t))/dx. \quad (83) \]
An example is given in [17, sect.4], with \( u_0(y) = \alpha + \tanh(\gamma y) \), \( \alpha \) and \( \gamma \) being two parameters. This gives
\[ x(u, t) = \frac{1}{2\gamma} \ln \left( \frac{1 + u - \alpha}{1 - u + \alpha} \right) + tu. \quad (84) \]

With a critical time \( t_c = -1/\gamma \), this example is treated in details in [17, p. 850] where several illustrations are shown. It is interesting to compare this formulation with that of Hopf-Lax [17, Sect. 3, p.847] and [12, Sect. 2.3.2.b. p.123]. As expected our variational formulation is equivalent to that of Hopf-Lax for the one-dimensional problems. As illustration consider the former example with \( \gamma = 1 \) and \( \alpha = 0 \). Then
\[
F_1(x, t) = \sup_{u} (ux - G_1(u, t)) \\
= \sup_{u} \left( ux - \frac{tu^2}{2} + h_2 \left( \frac{1 + u}{2} \right) - \ln 2 \right), \quad (85)
\]
where \( h_2 \) is identified to be the binary entropy function, namely
\[ h_2(p) := -p \ln p - (1 - p) \ln (1 - p). \quad (86) \]
It is interesting to point out the following analogy with a one dimensional Ising model treated in the mean-field approximation. Let \( J \) be the coupling constant, \( H \) be the magnetic field, \( M(J, H) \) the magnetization and \( \Phi(J, H) \) its free energy. It is well is known that
\[ \Phi(J, H) = \inf_m \left( -Hm + \frac{J}{2} m^2 - \left( h_2 \left( \frac{1 + m}{2} \right) - \ln 2 \right) \right), \quad (87) \]
and that
\[ M(J, H) = \frac{\partial}{\partial H} \Phi(J, H). \quad (88) \]
It transpires that the analogy reads \( J = -t, H = x, M = -u \) and \( \Phi = -F_1 \) thus explaining the designation of \( F_1 \) as minus a “Free Energy”.

25
At this point it is worth comparing the variational formula (85) with the one deriving from the Hopf-Lax principle

\[ u_{HL}(x, t) = \frac{\partial}{\partial x} \inf_y \left( \frac{(x - y)^2}{2t} + \ln(\cosh(y)) \right). \quad (89) \]

We observe that

\[ \inf_y \left( \frac{(x - y)^2}{2t} + \ln(\cosh(y)) \right) = \sup_u \left( ux - \frac{tu^2}{2} + h_2 \left( \frac{1 + u}{2} \right) - \ln 2 \right), \quad (90) \]

as claimed above.

We notice that in higher dimensions the Hopf-Lax formula can be generalized for compressible systems but not for the combined compressible and rotational ones, whereas our variational formulas (71) and (74) does it for radially-symmetric systems.

4. Further developments

If the problems raised in the title of this work is solved in what concerns the measure solutions of the mass densities of the models considered, several aspects going beyond its scope have not been touched, f.i. the time evolution of the vorticity and of the dilatation of these fluids as well as that of their regular and singular spiraling solutions, the question concerning the applicability of the adhesion model not speaking of the multi-parametric description of the regimes displayed by our models.

Further work will imply i) generalization for anisotropic models in two and three dimensions ii) qualification of entropy solutions with respect to measure solutions of the 2D compressible and rotational isotropic models iii) analysis of axis-symmetric flows involving gravitational forces iv) extension of the strategy of the Jacobian represented densities for isothermal and isobaric axis-symmetric Burgers and Euler-Coriolis fluids with Riemann invariants coming into play, and this in view of meteorological applications.

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