K-THÉORIE OF LINE BUNDLES AND SMOOTH VARIETIES

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Abstract. We give a $K$-theoretic criterion for a quasi-projective variety to be smooth. If $L$ is a line bundle corresponding to an ample invertible sheaf on $X$, it suffices that $K_q(X) \cong K_q(L)$ for all $q \leq \dim(X) + 1$.

Let $X$ be a quasi-projective variety over a field $k$ of characteristic 0. The main result of this paper gives a $K$-theoretic criterion for $X$ to be smooth. For affine $X$, such a criterion was given in [2]: it suffices that $X$ be $K_{d+1}$-regular for $d = \dim(X)$, i.e., that $K_{d+1}(X) \cong K_{d+1}(X \times \mathbb{A}^m)$ for all $m$. If $X$ is affine, we also showed that $K_{d+1}$-regularity of $X$ is equivalent to the condition that $K_i(X) \cong K_i(X \times \mathbb{A}^1)$ for all $i \leq d + 1$.

We also showed that $K_{d+1}$-regularity is insufficient for quasi-projective $X$; see [2, Thm. 0.2]. In this paper we prove:

Theorem 0.1. Let $X$ be quasi-projective over a field $k$ of characteristic 0, of dimension $d$, and let $L = \text{Spec}(\text{Sym} \mathcal{L})$ be the line bundle corresponding to an ample invertible sheaf $\mathcal{L}$ on $X$.

If $K_i(L) \cong K_i(X)$ for all $i \leq n$ then $X$ is regular in codimension $< n$.

If $K_i(L) \cong K_i(X)$ for all $i \leq d + 1$, then $X$ is regular.

For example, if $K_i(L) \cong K_i(X)$ for all $i \leq d$, then $X$ has at most isolated singularities.

In the affine case, of course, every line bundle is ample, and when $L = \mathbb{A}^1_R$ we recover our previous result, proven in [2, 0.1]:

Corollary 0.2. If $R$ is essentially of finite type over a field of characteristic 0, and $K_i(R) \cong K_i(R[t])$ for all $i \leq n$ then $R$ is regular in codimension $< n$.

The affine assumption in this corollary is critical. In [2], we gave an example of a curve $Y$ which is $K_n$-regular for all $n$, but which is not regular; no affine open $U$ is even reduced. However, $K_0(X) \neq K_0(L)$ for the line bundle associated to an ample $\mathcal{L}$; see Example 4.1 below. In Theorem 4.3 we give a surface $X$ which is $K_n$-regular for all $n$, but which is not regular and such that $K_0(X) \neq K_0(L)$ for the line bundle associated to an ample $\mathcal{L}$; it is a cusp bundle over an elliptic curve.

As in our previous papers [1, 2, 3], our technique is to compare $K$-theory to cyclic homology using cdh-descent and cyclic homology. The parts of cdh descent we need are developed in Section 1, and applied to give a formula for the cyclic

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homology of line bundles in Section 2. The main theorem is proven in Section 3, and the two examples are given in Section 4.

Notation. If $E$ is a presheaf of spectra, we write $\pi_n E$ for the presheaf of abelian groups $X \mapsto \pi_n E(X)$; we say that a spectrum $E$ is $n$-connected if $\pi_q E = 0$ for all $q \leq n$. For example, $K_n(X)$ is the homotopy group $\pi_n K(X)$ of the spectrum $K(X)$.

Similarly, if $E$ is a cochain complex of presheaves, we may regard it as a presheaf of spectra via Dold-Kan [15, ch. 10]. Thus $\pi_i E(X)$ is another notation for $H^{-i} E(X)$. We will use the cochain shift convention $E[i]^n = E_i + n$, so that the spectrum corresponding to $E[1]$ is the suspension of the spectrum of $E$, and $\pi_n E[1] = \pi_{n-1} E$. Thus if $E$ is $n$-connected then $E[1]$ is $(n + 1)$-connected.

1. Zariski and cdh descent

In this paper, we fix a field of characteristic 0, and work with the category Sch of schemes $X$ of finite type over the field. We will be interested in the Zariski and cdh topologies on Sch.

If $\tau$ is a Grothendieck topology on Sch, there is an “injective $\tau$-local” model structure on the category $\text{Psh}(\text{Ch}(\text{Ab}))$ of presheaves of cochain complexes of abelian groups on Sch. In this model structure, a map $A \to B$ is a cofibration if $A(X) \to B(X)$ is an injection for all $X$, and it is a weak equivalence if $H^n A \to H^n B$ induces an isomorphism on the associated $\tau$-sheaves. The fibrant replacement of $A$ in this model structure is written as $A \to \mathbb{H}_{\tau}(-, A)$. We say that $A$ satisfies $\tau$-descent if the canonical map $A(X) \to \mathbb{H}_{\tau}(X, A)$ is a quasi-isomorphism for all $X$. There is a parallel notion of $\tau$-descent for presheaves of spectra.

If $A$ is a sheaf then $A \to \mathbb{H}_{\tau}(-, A)$ is an injective resolution; it follows that $\mathbb{H}^n_{\tau}(X, A) = H^n \mathbb{H}_{\tau}(X, A)$ for all $n$. For a complex $A$, the hypercohomology group $\mathbb{H}_{\tau}^n(X, A)$ equals $H^n \mathbb{H}_{\tau}(X, A)$. See [1, 3.3] for these facts.

The inclusion of complexes of sheaves (for a topology $\tau$) into complexes of presheaves induces an injective $\tau$-local model structure on complexes of sheaves, and the inclusion is a Quillen equivalence; see [3, 5.9].

For the Zariski, Nisnevich and cdh topologies, there is a parallel “injective $\tau$-local” model structure on the category $\text{Psh}(\text{Ch}(\mathcal{O}_\tau))$ of presheaves of complexes of $\mathcal{O}_\tau$-modules, and the functor forgetting the module structure is a Quillen adjunction. In particular, if $A$ is a presheaf of complexes of $\mathcal{O}_\tau$-modules, the forgetful functor sends its fibrant $\mathcal{O}_\tau$-module replacement to a presheaf that is objectwise weak equivalent to $\mathbb{H}_{\tau}(-, A)$.

Example 1.1. The Hochschild complex $HH/k$ satisfies Zariski descent by [17, 0.4]. By definition, the cochain complex $HH(X/k)$ is concentrated in negative cohomological degrees and has the (quasi-coherent) Zariski sheaf $\mathcal{O}_X(X) \otimes H_{n+1}$ in cohomological degree $-n$. When $k$ is understood, we drop the '/$k$' from the notation. We sometimes regard $HH$ as a sheaf of spectra, using Dold-Kan, and use the notation $\mathbb{H}_{\text{zar}}^q(X) = \pi_q HH(X)$ for $\mathbb{H}_{\text{zar}}^q(X, HH)$. Recall from [17, 4.6] that if $X$ is noetherian then $\mathbb{H}_{\text{zar}}^q(X) = 0$ for $q < -\dim(X)$.

If $E$ is a complex of Zariski sheaves of $\mathcal{O}$-modules on Sch, we may assume that $\mathbb{H}_{\text{zar}}(-, E)$ is a complex of Zariski sheaves of $\mathcal{O}$-modules, and similarly for
This it makes sense to form the sheaf tensor product $\mathbb{H}_{cdh}(\mathcal{F}, E)$ for the flat case. Thus it makes sense to form the sheaf tensor product $\mathbb{H}_{zar}(\mathcal{F}, E) \otimes_{\mathcal{O}_X} \mathcal{L}$ with a Zariski sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules.

If $E$ is a Zariski sheaf of $\mathcal{O}_X$-modules on $X$, then there is a Zariski sheaf $E'$ of $\mathcal{O}$-modules on $Sch/X$, unique up to unique isomorphism, such that for every $f : Y \to X$ in $Sch/X$ the restriction of $E'$ to the small Zariski site of $Y$ is naturally isomorphic to the sheaf $f^* E$. In this paper we will always work with this sheaf on the big site; so for example “an invertible sheaf $\mathcal{L}$ on $X$” will indicate the sheaf on the big site associated in this way to an invertible sheaf on $X$.

**Lemma 1.2.** If $\mathcal{L}$ is an invertible sheaf on $X$, $\mathcal{L}$ is an auto-equivalence of the category $\mathcal{Sh}(Ch(\mathcal{O}_{zar}))/X$ of sheaves of complexes of $\mathcal{O}_{zar}$-modules on $Sch/X$ which preserves cofibrations, fibrations and weak equivalences.

**Proof.** The functor $\mathcal{L}^{-1}$ is a quasi-inverse to $\mathcal{L}$. Since $\mathcal{L}$ is flat, $\mathcal{L}$ preserves injections. Since $\mathcal{L}$ is locally trivial on $X$ (and hence on any $X$-scheme), and $A \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong A$, $\mathcal{L}$ preserves weak equivalences. Now suppose that $C \to D$ is a Zariski-local fibration; we want to see that $C \otimes \mathcal{L} \to D \otimes \mathcal{L}$ is a Zariski-local fibration. By invertibility, it suffices to observe that $A \otimes \mathcal{L}^{-1} \to B \otimes \mathcal{L}^{-1}$, a fact we have just verified.

**Corollary 1.3.** If $\mathcal{L}$ is an invertible sheaf on $X$, and $A$ is a complex of Zariski sheaves of $\mathcal{O}$-modules, then there is a quasi-isomorphism on $Sch/X$:

$$\mathbb{H}_{zar}(-, A) \otimes_{\mathcal{O}_X} \mathcal{L} \xrightarrow{\sim} \mathbb{H}_{zar}(-, A \otimes \mathcal{L}).$$

**Proof.** This follows immediately from Lemma 1.2.

We write $(a^*, a_*)$ for the usual adjunction between Zariski and cdh sheaves associated to the change-of-topology morphism $a : (Sch/k)_{cdh} \to (Sch/k)_{zar}$. Thus if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{cdh}$-modules on $(Sch/X)_{cdh}$, $a_* \mathcal{F}$ is the underlying sheaf of $\mathcal{O}_{zar}$-modules, and for any Zariski sheaf $E$ of $\mathcal{O}_X$-modules on $X$, we may form the Zariski sheaf $a_* \mathcal{F} \otimes_{\mathcal{O}_X} E$ on $Sch/X$.

Recall from [EGA, 01(5.4.1)] that a Zariski sheaf $E$ of $\mathcal{O}_X$-modules is locally free if each point of $X$ has an open neighborhood $U$ such that $E|_U$ is a free $\mathcal{O}_U$-module, possibly of infinite rank.

**Lemma 1.4.** If $E$ is a locally free sheaf on $X$, and $\mathcal{F}$ is a cdh sheaf of $\mathcal{O}_{cdh}$-modules, then $a_* \mathcal{F} \otimes_{\mathcal{O}_X} E$ is a cdh sheaf on $(Sch/X)$.

**Proof.** Since the question is local on $X$, we may replace $X$ by an open subscheme to assume that $E$ is free. Because the cdh-topology on $Sch/X$ is quasi-compact, and therefore arbitrary direct sums of sheaves are sheaves, we are reduced to the trivial case $E = \mathcal{O}_X$, when $a_* \mathcal{F} \otimes_{\mathcal{O}_X} E = a_* \mathcal{F}$.

**Definition 1.5.** If $\mathcal{F}$ is a cdh sheaf of $\mathcal{O}_{cdh}$-modules, we will write $\mathcal{F} \otimes_{\mathcal{O}_X} E$ for the cdh sheaf $a_* \mathcal{F} \otimes_{\mathcal{O}_X} E$.

Note that $\mathbb{H}_{zar}(X, \mathcal{F} \otimes_{\mathcal{O}_X} E) \neq \mathbb{H}_{zar}(X, \mathcal{F}) \otimes E(X)$. For example, $E(X) = 0$ does not imply that $(\mathcal{F} \otimes_{\mathcal{O}_X} E)(X) = 0$.

**Lemma 1.6.** If $E$ is locally free on $X$ then $\otimes_{\mathcal{O}_X} E$ preserves weak equivalences and cofibrations for complexes of cdh sheaves of $\mathcal{O}_{cdh}$-modules on $Sch/X$. 
Proof. As in the proof of Lemma 1.4 we may replace $X$ by an open subscheme and assume that $E$ is a sheaf of free modules. Since $A \otimes_{\text{zar}} E$ is a sum of copies of $A$, it follows that $A \mapsto A \otimes_{\text{zar}} E$ preserves weak equivalences and cofibrations. □

Definition 1.7. Given a cochain complex $A$ of presheaves of abelian groups on Sch, we write $F_A(X)$ for the homotopy fiber (the shifted mapping cone) of the canonical map $A(X) \to \mathbb{H}_{\text{cdh}}(X, A)$, so for each $X$ there is a long exact sequence

$$\cdots \mathbb{H}_{\text{cdh}}^{n-1}(X, A) \to H^n F_A(X) \to H^n A(X) \to \mathbb{H}_{\text{cdh}}^n(X, A) \to H^{n+1} F_A(X) \cdots .$$

If $A$ is a complex of sheaves (in some topology) of $\mathcal{O}$-modules, then $\mathbb{H}_{\text{cdh}}(X, A)$ can be represented by a complex of sheaves of $\mathcal{O}$-modules as well (see [8, 8.1]), and hence so can $F_A$. We also write $F_K(X)$ for the homotopy fiber of $K(X) \to KH(X)$.

It is well known that $HH$, $HC$ and $K$-theory satisfy Zariski descent; it follows that $F_{HH}$, $F_{HC}$ and $F_K$ also satisfy Zariski descent.

Proposition 1.8. If $\mathcal{L}$ is an invertible sheaf on $X$ and $A$ is a complex of Zariski sheaves of $\mathcal{O}$-modules on $\text{Sch}/X$, then:

$$\mathbb{H}_{\text{cdh}}(-, A) \otimes_{\text{zar}} \mathcal{L} \xrightarrow{\sim} \mathbb{H}_{\text{cdh}}(-, A \otimes \mathcal{L}).$$

Consequently, $F_A \otimes_{\text{zar}} \mathcal{L} \xrightarrow{\sim} F_{A \otimes \mathcal{L}}$.

Proof. Arguing as in the proof of Lemma 1.2, Lemma 1.4 shows that $\otimes_{\text{zar}} \mathcal{L}$ preserves cdh-local fibrations (in addition to cofibrations and weak equivalences). The first statement follows immediately from this. Because $\otimes_{\text{zar}} \mathcal{L}$ is exact, the second statement follows from the triangles

$$F_A \to A \to \mathbb{H}_{\text{cdh}}(-, A) \quad \text{and} \quad F_{A \otimes \mathcal{L}} \to A \otimes \mathcal{L} \to \mathbb{H}_{\text{cdh}}(-, A \otimes \mathcal{L}) \to .$$

Lemma 1.9. Let $A_i$ be cochain complexes of presheaves on $\text{Sch}/X$. Then for every $X$-scheme $Y$, the canonical maps

$$\bigoplus_i \mathbb{H}_{\text{zar}}(Y, A_i) \to \mathbb{H}_{\text{zar}}(Y, \bigoplus_i A_i)$$

and

$$\bigoplus_i \mathbb{H}_{\text{cdh}}(Y, A_i) \to \mathbb{H}_{\text{cdh}}(Y, \bigoplus_i A_i)$$

are quasi-isomorphisms.

Proof. These sites are quasi-compact, and thus cohomology in them commutes with direct limits. □

2. Homology of line bundles

Suppose that $R$ is a (commutative) noetherian algebra over a field $k$ of characteristic 0. In [3, 3.2, 4.1], we showed that $NK(R) = K(R[t])/K(R)$ is isomorphic to $NF_{HC/Q}(R)[1]$ as well as $F_{HH/Q}(R)[1] \otimes_R tR[t]$. In this section, we replace $R[t]$ by the symmetric algebra $R[L] = \text{Sym}_R(L)$ of a rank 1 projective $R$-module, and the ideal $tR[t]$ by $LR[L]$. More generally, if $\mathcal{L}$ is an invertible sheaf on a scheme $X$, we replace $X \times A^1$ by the line bundle $L = \text{Spec}(\text{Sym}_X \mathcal{L})$. 
Lemma 2.1. Let $L$ be a rank 1 projective $R$-module. Then the symmetric algebra $R[L] = \text{Sym}_R(L)$ satisfies:

$$HH(R[L]) \simeq HH(R) \otimes_R R[L] \oplus HH(R)[1] \otimes_R LR[L]$$
$$HC(R[L]) \simeq HC(R) \oplus HH(R) \otimes_R LR[L].$$

Similarly, if $X$ is a scheme over $R$ and $X[L]$ denotes $X \times_R \text{Spec}(R[L])$, then

$$HH(X[L]) \simeq HH(X) \otimes_R R[L] \oplus HH(X)[1] \otimes_R LR[L]$$
$$HC(X[L]) \simeq HC(X) \oplus HH(X) \otimes_R LR[L].$$

Note that, as an $R$-module, $LR[L] = R[L] \otimes_R L$ is just $\bigoplus_{j=1}^{\infty} L^{\otimes j}$.

Proof. The cochain complex $HH(R[L])$ ends: $\to R[L] \otimes R[L] \to R[L] \to 0$. Therefore there are natural maps from $R[L]$ and $R[L] \otimes L[1]$ to $HH(R[L])$. Using the shuffle product, we get a natural map $\mu(R)$ from the direct sum of $HH(R) \otimes_R R[L]$ and $HH(R) \otimes_R (R[L] \otimes L[1])$ to $HH(R[L])$. For each prime ideal $\mathfrak{p}$ of $R$, we have $R_\mathfrak{p}[L] \cong R_\mathfrak{p}[t]$ and $\mu(R_\mathfrak{p})$ is a quasi-isomorphism by the Künneth formula [10 9.4.1]. It follows that $\mu(R)$ is a quasi-isomorphism. The formula for $HC(R[L])$ follows by induction on the SBI sequence, just as it does for $HC(R[t])$.

Now suppose that $X$ is a scheme over $R$, the same argument applies to $\pi_*HH(O_X[L])$, the direct image along $X[L] \rightarrow X$ of the cochain complex $HH(O_X[L])$ on $X[L]$ of quasi-coherent sheaves described in Example [1.1]. Because $\pi$ is affine, we have a quasi-isomorphism

$$H_{\text{zar}}(X[L], HH(O_X[L])) \cong H_{\text{zar}}(X, \pi_*HH(O_X[L])).$$

Now the assertions about $X[L]$ follow from Corollary [1.3] and Lemma [1.9] \hfill \square

Corollary 2.2. $F_{HC}(R[L]) \cong F_{HC}(R) \oplus \bigoplus_{j=1}^{\infty} \left( F_{HH} \otimes_R L^{\otimes j} \right)(R)$.

Proof. This follows from Lemma 2.1, Proposition 1.8 and Lemma 1.9 \hfill \square

Now suppose that $X$ is a scheme of finite type over a field of characteristic 0, containing $k$, and write $HH$, $HC$, etc for $HH/k$, $HC/k$, etc.

Lemma 2.3. Let $L$ be a line bundle over $X$, and write $F_{HH}$ for the cochain complex of Zariski sheaves on $X$ associated to the complex of presheaves $U \rightarrow F_{HH}(L|_U)$. Then $F_{HH}(L) \cong H_{\text{zar}}(X, F_{HH})$.

Proof. As observed after [1.7] the presheaf of complexes $F_{HH}$ satisfies Zariski descent: $F_{HH}(L) \simeq H_{\text{zar}}(L, F_{HH})$. By [11 1.56], $H_{\text{zar}}(L, F_{HH}) \rightarrow H_{\text{zar}}(X, F_{HH})$. \hfill \square

In what follows, we write $\otimes$ for the tensor product of $O_X$-modules.

Proposition 2.4. Let $L$ be the line bundle $\text{Spec}(\text{Sym}(L))$ on $X$ associated to an invertible sheaf $L$, and $p : L \rightarrow X$ the projection. Then we have quasi-isomorphisms:

$$HC(L) \simeq HC(X) \oplus H_{\text{zar}}(X, HH \otimes \text{Sym}(L) \otimes L);$$
$$H_{\text{cdh}}(X, p_*HC) \simeq H_{\text{cdh}}(X, HC) \oplus H_{\text{cdh}}(X, HH \otimes \text{Sym}(L) \otimes L);$$
$$F_{HC}(L) \simeq F_{HC}(X) \oplus \bigoplus_{j=1}^{\infty} \left( F_{HH} \otimes L^{\otimes j} \right)(X);$$
$$K(L, X) \simeq F_{HC}(L, X)[1].$$
Proof. Using Zariski descent, we may assume that $X = \text{Spec}(R)$ for some $R$. The first two quasi-isomorphisms are immediate from Corollary $2.1$ while the third is immediate from Corollary $2.2$. By Theorem $1.6$ of $[2]$, $K(L)/K(X) \cong F_K(L)/F_K(X) \simeq F_{HC/Q}(L)[1]/F_{HC/Q}(X)[1]$. Now use the formula for $F_{HC}(L)$ to get the final quasi-isomorphism. \hfill \Box

Now suppose that $R$ is a commutative $\mathbb{Q}$-algebra. Then $K_n(R[L], R)$ is a $\mathbb{Q}$-module $[13]$, and the Adams operations give an $R$-module decomposition

$$K_n(R[L], R) \cong \bigoplus_{i=n}^{\infty} K_n^{(i)}(R[L], R)$$

with $K_n^{(0)}(R[L], R) = 0$ for all $n$. The relative terms $F_K(R) \cong F_{HC}(R)[1]$ have a similar decomposition, and $F_K^{(i)}(R[L], R) \simeq F_{HC}^{(i-1)}(R[L], R)[1]$. As in $[3]$ 5.1, we define the typical piece $TK_n(R)$ to be $H^{1-n}(F_{HH}(R))$, and set $TK_n^{(i)}(R) = H^{1-n}(F_{HH}^{(i-1)}(R))$. Since these groups were determined in $[3]$, we may rephrase the last part of Proposition $2.3$ as follows:

**Corollary 2.5.** If $R$ is a commutative $\mathbb{Q}$-algebra, $K_n(R[L], R) \cong TK_n(R) \otimes_R LR[L]$ and

$$K_n^{(i)}(R[L]) \cong K_n^{(i)}(R) \oplus TK_n^{(i)}(R) \otimes_R LR[L].$$

Moreover,

$$TK_n^{(i)}(R) \cong \begin{cases} H^{i-n}_{\text{cdh}}(R), & \text{if } i < n, \\ H^{i-n}_{\text{cdh}}(R, \Omega^{i-1}), & \text{if } i \geq n + 2. \end{cases}$$

(The formulas for $TK_n^{(n)}$ and $TK_n^{(n+1)}$ are more complicated; see loc. cit.) The following special case $n = 0$ of $[2,3]$ which is an analogue of $[3] (0.5)$, shows that we cannot twist out the example in $[2]$ Theorem 0.2

**Corollary 2.6.** Let $L$ be a rank 1 projective $R$-module, where $R$ is a $d$-dimensional commutative $\mathbb{Q}$-algebra, with seminormalization $R^+$, and $R[L]$ the twisted polynomial ring. Then

$$K_0(R[L], R) \cong \left( (R^+/R) \oplus \bigoplus_{p=1}^{d-1} \mathbb{H}^{p}_{\text{cdh}}(R, \Omega^p) \right) \otimes_R LR[L].$$

In particular, $K_n(R) = K_n(R[t])$ if and only if $K_n(R) = K_n(R[L])$.

**Proof.** This follows from the fact that $\mathbb{H}_{\text{cdh}}(X, HH^{(i)}) \cong Ra_{*}a^{*}\Omega^{i-1}[i]$, so that when $i > 1$ we have $K_0^{(i)}(R[L], R) \cong \mathbb{H}_{\text{cdh}}^{i-1}(R, \Omega^{i-1}) \otimes_R LR[L]$; see $[2]$ 2.2. \hfill \Box

**Remark 2.7.** Corollary $2.4$ shows that $K_*(R[L], R)$ is a graded $R[L]$-module. As in $[3]$, this reflects the fact that locally $R[L]$ is a polynomial ring, and $K_*(R[t], R)$ has a continuous module structure over the ring of big Witt vectors $W(R)$, compatible with the operations $V_n$ and $F_n$; when $Q \subset R$, such modules are graded $R[t]$-modules. Since $H^0(\text{Spec } R, W) = W(R)$, patching the structures via Zariski descent proves that $K_*(R[L], R)$ is a graded $R[L]$-module.

When $X$ is no longer affine, this Zariski descent argument shows that

$$K_n(L, X) = \mathbb{H}^{1-n}(X, F_{HH} \otimes_{\text{zar}} \mathcal{L}^{\otimes i})$$

is a graded module over $S = \mathbb{H}^0(X, \mathcal{L}^{\otimes i})$. This is clear from Proposition $2.4$. Previously, using $[13]$, it was only known that the $K_n(L, X)$ are continuous modules over $H^0(X, \tilde{W}) = W(k) = \prod_{i=1}^{\infty} k$. 

In order to use Proposition 2.4 we need to analyze $H^{n}_{\text{zar}}(X, F_{HH/k} \otimes \mathcal{L}^j)$. For this, we use the hypercohomology spectral sequence. (See [16, 5.7.10].)

\[
E_{2}^{p,q} = H^{p}_{\text{zar}}(X, H^q E) \Rightarrow H^{p+q}_{\text{zar}}(X, E).
\]

Here $E$ is a cochain complex which need not be bounded below and (by abuse of notation) the $E_2$ term denotes cohomology with coefficients in the Zariski sheaf associated to $H^q E$; the spectral sequence converges if $X$ is noetherian and finite dimensional. When $E = F_{HH} \otimes \mathcal{L}^j$, we have $H^q E = H^q(F_{HH}) \otimes \mathcal{L}^j$, because $\mathcal{L}^j$ is flat.

**Lemma 3.2.** If $X$ is noetherian and finite dimensional, and $E$ is a complex of Zariski sheaves such that $H^{p}_{\text{zar}}(X, H^q E) = 0$ for $1 \leq p \leq \dim(X)$ and $p + q = s, s + 1$ then $H^{s}_{\text{zar}}(X, E) \cong H^{0}_{\text{zar}}(X, H^s E)$.

**Proof.** This is immediate from the hypercohomology spectral sequence (3.1). □

In the remainder of this section, we will write $H^{p}(X, -)$ for $H^{p}_{\text{zar}}(X, -)$. By a “quasi-coherent” (or “coherent”) sheaf on $\text{Sch}/k$ we mean a Zariski sheaf whose restriction to every small Zariski site is quasi-coherent (or coherent). When discussing Hochschild homology (or cyclic homology, or differentials, etc.) relative to $\mathbb{Q}$, we will suppress the base from the notation. For example, if $X$ is a $k$-scheme then $HH_n(X)$ and $\Omega^p_\mathbb{Q}(X)$ will mean $HH_n(X/\mathbb{Q})$ and $\Omega^p_\mathbb{Q}(X/\mathbb{Q})$.

Recall that when $k \subseteq k$, the Hochschild homology complex relative to $k$ decomposes into a direct sum of weight pieces $HH^{(j)}(-/k)$; this induces decompositions on $\mathbb{H}^{\text{cdh}}(-, HH(-/k))$, the fiber $F_{HH/k}$, and on their cohomology sheaves and hypercohomology groups as well. As in [2], we use versions of a spectral sequence introduced by Kassel and Sletsjøe in [9] to obtain information about $F_{HH/k}$ from information about $F_{HH}$.

**Lemma 3.3.** (Kassel-Sletsjøe) Let $k \subseteq k$ and $p \geq 1$ be fixed, and $X$ a scheme over $k$. Then there are bounded cohomological spectral sequences of quasi-coherent sheaves on $\text{Sch}/k$ ($p + s \geq 0$):

\[
E_1^{s,t} = \Omega^s_k \otimes_k H^{2s+t-p} HH^{(p-s)}(-/k) \Rightarrow H^{s+t-p} HH^{(p)}(-/\mathbb{Q})
\]

(for $s + t \leq 0$) and

\[
E_1^{s,t} = \Omega^s_k \otimes_k H^{s+t}(Ra_* \Omega^{(p-s)}(-/k, \text{cdh})) \Rightarrow H^{s+t}(Ra_* \Omega^p_{\text{cdh}})
\]

and a morphism of spectral sequences between them. If $k$ has finite transcendence degree, then both spectral sequences are spectral sequences of coherent sheaves.

We remark that the second spectral sequence is just the sheafification of the spectral sequence in [2] 4.2.

**Proof.** If $X = \text{Spec}(R)$, the homological spectral sequence in [9] 4.3a] is

\[
p^{i+j}E_{i+j}^{0} = \Omega^s_k \otimes_k HH^{(p-i-j)}_{R/k}(R) \Rightarrow HH^{(p)}_{R\otimes \mathbb{Q}}(R)
\]

(0 $i < p$, $j \geq 0$; see [2] 4.1].

We claim that this is a spectral sequence of $R$-modules, compatible with localization of $R$. Indeed, following the construction in [9] Theorem 3.2], the exact couple underlying the spectral sequence is constructed by choosing $\mathbb{Q}$-cofibrant simplicial resolutions $P_\bullet \to k$ and $Q_\bullet \to R$ and then filtering the differential modules $\Omega^p_{Q_\bullet/\mathbb{Q}}$.
Theorem 3.6. Then they are in fact $Q$-submodules. (Although the filtration steps are defined as certain $P$-submodules in \cite{9} Section 3], they are in fact $Q$-submodules.) The identification of the associated graded via \cite{9} Lemma 3.1] is easily checked to be a $B$-module isomorphism. The whole construction commutes with localization because forming differential modules does.

Setting $\ell = i + j$, the spectral sequence is

$$pE^1_{-i,-\ell} = \Omega^i_k \otimes_k H^j_H[(p-i)](R/k) \Rightarrow H^s_H[p](R/k), \quad \ell \leq i.$$  

As this spectral sequence is a spectral sequence of $R$-modules, compatible with localization and natural in $R$, we may sheafify it for the Zariski topology to obtain a spectral sequence of quasi-coherent sheaves. Reindexing cohomologically, with $s = i$ and $t = -\ell$, we have

$$pE^1_{s,t} = \Omega^i_k \otimes_k H^{2s+t-p}(HH^{(p-s)})(-k) \Rightarrow H^{s+t-p}(HH^{(p)}).$$

This yields the first spectral sequence. If we sheafify it for the cdh topology, and use the isomorphism $HH^{(p)} \cong \Omega^i_{cdh}[p]$, we get the second spectral sequence. That it is still a spectral sequence of quasi-coherent sheaves follows from \cite{2} lemma 2.8]. The morphism between the spectral sequences is just the change-of-topology map.

Finally, if $k$ has finite transcendence degree, then the $E_1$-terms of both spectral sequences are coherent (apply \cite{2} lemma 2.8] again for the second one) and hence so are the abutments.

Corollary 3.4. There is a bounded spectral sequence of quasi-coherent sheaves

$$E^1_{s,t} = \Omega^i_k \otimes_k H^{2s+t-p}(F^{(p-s)}_{HH/k}) \Rightarrow H^{s+t-p}(F^{(p)}_{HH}).$$

If $k$ has finite transcendence degree, this is a spectral sequence of coherent sheaves.

Proof. The morphism of spectral sequences in Lemma \ref{lem:3.3} comes from a morphism $HH^{(p)} \to HH^{(p)}_{cdh}$ of filtered complexes of quasi-coherent sheaves on Sch/k. By a lemma of Eilenberg–Moore \cite{10} Ex. 5.4.4], there is a filtration on the [shifted] mapping cone $F^{(p)}_{HH}$ of $HH^{(p)} \to HH^{(p)}_{cdh}$, yielding a spectral sequence converging to $H^*(F_{HH})$. This is the displayed spectral sequence.

Proposition 3.5. Assume that $k$ has finite transcendence degree. If $L$ is an ample line bundle on $X$, then for every $n$ and $p \geq 0$ there is an $N_0 = N_0(n,p)$ such that for all $N > N_0$ the Zariski sheaf $H^n_F^{(p)}_{HH} \otimes \mathcal{O}^\otimes N$ is generated by its global sections, and $H^q(X, H^n_F^{(p)}_{HH} \otimes \mathcal{O}^\otimes N) = 0$ for all $q > 0$.

Proof. The complex $F^{(p)}_{HH}$ is quasi-isomorphic to the cone of the map from the structure sheaf $\mathcal{O}$ to $R_\alpha a^*\mathcal{O}$ and thus has coherent cohomology by \cite{11} Lemma 6.5]. If $p > 0$, then by Corollary \ref{cor:3.4} the cohomology sheaves in question are coherent as well. Now apply Serre’s Theorem B.

Let $L$ be an ample sheaf on $X$ and $L$ the line bundle $\text{Spec}(	ext{Sym} L)$. Recall that for any $Y$, $F_{HC}(Y)$ is $n$-connected if and only if $F_{HH}(Y)$ is $n$-connected; see \cite{2} 1.7]. If $L$ is a line bundle over $X$, we define $F_{HH/k}(L,X)$ to be the cokernel of the canonical split injection $F_{HH/k}(X) \to F_{HH/k}(L)$, and similarly for cyclic homology.

Theorem 3.6. If $F_{HC}(L,X)$ is $n$-connected for some ample line bundle $L$ on $X$, then $F_{HH}(L,X)$ is $n$-connected and:

1. The Zariski sheaf $F_{HH}$ is $n$-connected.
(2) $X$ is regular in codimension $\leq n$.
(3) If $F_{\text{HC}}(\mathbb{L}, X)$ is $d$-connected for $d = \dim(X)$, then $X$ is regular.

Proof. There is a finitely generated subfield $k_0$ of $k$, a $k_0$-scheme $X_0$ and an ample line bundle $\mathcal{L}_0$ such that $X = X_0 \otimes_{k_0} k$ and $\mathcal{L} = \mathcal{L}_0 \otimes_{k_0} k$. The Küneth formula for Hochschild homology implies that $F_{\text{HH}}(\mathbb{L}, X) = F_{\text{HH}}(\mathbb{L}_0, X_0) \otimes \Omega^*_k(k_0)$, whence $F_{\text{HH}}(\mathbb{L}, X)$ is $n$-connected if and only if $F_{\text{HH}}(\mathbb{L}_0, X_0)$ is. Thus we may assume that $k$ has finite transcendence degree.

(1) Recall [2, 2.1] that $F_{\text{HH}}(\mathbb{L}, X) = \prod F_{\text{HH}}(\mathbb{L}, X)$. Thus it suffices to fix $p$ and show that $F_{\text{HH}}^{(p)}(\mathbb{L}, X)$ is $n$-connected. Set $\mathcal{G}_N = \mathcal{L}^N \otimes F_{\text{HH}}^{(p)}(\mathbb{L}, X)$, and note that $H^s \mathcal{G}_N = \mathcal{L}^N \otimes H^s F_{\text{HH}}^{(p)}$. By Proposition 3.5 and Lemma 3.2, $H^s(X, \mathcal{G}_N) \cong H^0(X, H^s \mathcal{G}_N)$ for large $N$ and all $s \geq -n$.

By assumption and Lemma 2.3, the groups $\pi_n F_{\text{HH}}^{(p)}(\mathbb{L}, X) = \mathbb{H}^s_{\text{zar}}(X, F_{\text{HH}}^{(p)}(\mathbb{L}, X)) = \mathbb{H}^s_{\text{zar}}(X, F_{\text{HH}}^{(p)}(\mathbb{L})/F_{\text{HH}}^{(p)})$ vanish for $s \leq n$. By Lemma 2.3, this implies that for all $N > 0$:

$$H^0(X, H^{-s} \mathcal{G}_N) \cong H^{-s}(X, \mathcal{G}_N) = H^{-s}(X, \mathcal{L}^N \otimes F_{\text{HH}}^{(p)}) = 0, s \leq d.$$ 

Since $\mathcal{L}$ is ample, the sheaves $H^s \mathcal{G}_N = \mathcal{L}^N \otimes H^s F_{\text{HH}}^{(p)}$ are generated by their global sections $H^0(X, H^s \mathcal{G}_N)$ for large $N$ and $s \geq -n$. This implies that the sheaves $\mathcal{L}^N \otimes H^s F_{\text{HH}}^{(p)}$ vanish, and hence that the sheaves $H^s F_{\text{HH}}^{(p)}$ vanish for $s \geq -n$. This proves (1).

Given (1), the stalks $F_{\text{HH}}(\mathcal{O}_{X,x})$ are $n$-connected. We proved in [2, 4.8] that this implies that each $F_{\text{HH}}(\mathcal{O}_{X,x})$ is $n$-connected. If $\dim(\mathcal{O}_{X,x}) \geq n$, we proved in [2, 3.1] that $\mathcal{O}_{X,x}$ is smooth over $k$, and hence regular. \hfill \Box

Variant 3.7. Let $X$, $\mathcal{L}$ and $\mathbb{L}$ be as in Proposition 3.6. Suppose that $F_{\text{HC}/k}(\mathbb{L}, X)$ is $n$-connected. Then the proof of Theorem 3.4 goes through to show that:

(1) The sheaf $F_{\text{HH}/k}$ is $n$-connected.
(2) $X$ is regular in codimension $\leq n$.
(3) If $F_{\text{HH}/k}(\mathbb{L}, X)$ is $d$-connected for $d = \dim(X)$, then $X$ is regular.

Proof of Theorem 4.4. Suppose that $K_i(\mathbb{L}) \cong K_i(\mathbb{L}^d)$ for all $i \leq n$. By Proposition 2.4, $F_{\text{HC}/k}(\mathbb{L}, X)$ is $(n - 1)$-connected. By Theorem 3.6, $F_{\text{HH}/k}(\mathbb{L}, X)$ is $(n - 1)$-connected and $X$ is regular in codimension $< n$. \hfill \Box

4. Two Examples

We conclude with two quick examples. Let $E$ be an elliptic curve over $\mathbb{Q}$ with basepoint $Q$, and $P$ a point such that $P - Q$ does not have finite order in Pic($E$).

Example 4.1. Consider the non-reduced scheme $Y = \text{Spec}(\mathcal{O}_E \oplus J)$, where $J$ is the invertible sheaf $\mathcal{O}(P - Q)$. We showed in [2, 0.2] that $Y$ is $K_n$-regular for all $n$, because $K_n(Y \times \mathbb{A}^1) \cong K_0(Y) \cong K_n(E)$ for all $n$.

Let $\mathcal{L}$ be the sheaf $\mathcal{O}(Q)$ and set $\mathbb{L} = \text{Spec} \mathcal{O}(\text{Sym} \mathcal{L})$. Then $K_0(\mathbb{L}) \cong K_0(Y) \oplus \mathbb{Q}[x, y]$.

For our second example, recall that if $R$ is a regular $\mathbb{Q}$-algebra and $J$ is a rank 1 projective $R$-module and $A$ is the subring $R[J^2, J^3]$ of $R[J] = \text{Sym}_R(J)$...
then Spec$(A)$ is an affine cusp bundle over Spec$(R)$. For $n \geq 2$, set

$$V_n(R) = \begin{cases} J^6(i-1) \oplus (J^6(i-2) \otimes \Omega_R^2) \oplus \cdots \oplus (R \otimes \Omega_R^{n-2}), & n = 2i \geq 2; \\ J^6(i-1) \otimes \Omega_R^1 \oplus (J^6(i-2) \otimes \Omega_R^2) \oplus \cdots \oplus (R \otimes \Omega_R^{n-2}), & n = 2i + 1 \geq 3. \end{cases}$$

In particular, $V_2(R) = R$ and $V_3(R) = \Omega_R^3$. Let us write $\tilde{K}_n(A)$ for $K_n(A)/K_n(R)$.

**Proposition 4.2.** If $A = R[J^2, J^3]$ and $R$ is a regular $\mathbb{Q}$-algebra then

$$\tilde{K}_n(A) \cong (J^5 \oplus J^6) \otimes V_n(R) \oplus (J \otimes \Omega_R^n).$$

In particular, $\tilde{K}_0(A) \cong J$, $\tilde{K}_1(A) \cong J \otimes \Omega_R^1$ and

$$\tilde{K}_2(A) \cong (J^5 \oplus J^6) \oplus (J \otimes \Omega_R^2).$$

**Proof.** For $J = R$, this is Theorem 9.2 of [7], which holds for any regular $\mathbb{Q}$-algebra $R$ (not just for any field). In order to pass to $R[J^2, J^3]$, we need more detail. Using the classical Mayer-Vietoris sequence for $A \subset R[J]$, it is easy to see that $K_0(A)/K_0(R) \cong J$ and $K_1(A)/K_1(R) \cong J \otimes \Omega_R^1$.

For $n \geq 2$ the factors in $K_n(A)$ come from $HH_{n-1}(A)$ via the maps $HH_n(A) \to HC_n(A)$ and $\tilde{K}_n(A) \to \tilde{HC}_{n-1}(A)$. The summand $J \otimes \Omega_R^n$ of $K_n(A)$ comes from the $J \otimes \Omega_R^n$ in $K_1(A)$ (or $HH_0(A, R[J^2, J])$) by multiplication by $HH_{n-1}(R) \cong \Omega^{-1}$.

The $V_n$ factors come from the explicit description of the corresponding cyclic homology cycles (coming from cycles in Hochschild homology $HH_{n-1}(A)$) in 4.3, 4.7 and 5.8 of [7]. Locally, $J$ is generated by an element $t$; we set $x = t^2 \in J^2$, $y = t^3 \in J^3$ so that $y^2 = x^3$. The summands $J^5$ and $J^6$ of $K_2(A)$ are locally generated by the cycles $z = 2x[y] + 3y[x]$ and $tz = 2y[y] + 3x^2[x]$ in $HH_1(A)$. Multiplication by $\Omega_R^{n-2}$ gives the summands $(J^5 \oplus J^6) \otimes \Omega_R^{n-2}$ in $K_n(A)$.

Now consider the summand $J^6$ in the degree 2 part $A^\otimes 3$ of the Hochschild complex for $A$, locally generated by the element $w = [y[y] - x[x] - [x^2]].$ The product $zw^j$ is a cycle in $HH_{2j-1}(A)$, and locally generates a summand $j^5+6(i-1)$ of $HH_{2j-1}(A)$, corresponding to the factor $j^5+6(i-1)$ of the summand $J^5 \otimes V_{2j}(R)$ of $K_{2j}(A)$. As above, multiplication by $\Omega_R^n$ gives the rest of the summands.

**Remark 4.2.1.** In the spirit of Corollary 2.5 we note that $NK_n(A) \cong TK_n(A) \otimes_R LR[L]$, where

$$TK_n(A) = \tilde{K}_n(A) \oplus \tilde{K}_n(A).$$

**Theorem 4.3.** Let $J$ be the invertible sheaf $O(P-Q)$ on the elliptic curve $E$ and let $X$ denote the affine cusp bundle $Spec_E(O_E[J^2, J^3])$ over $E$. (X has a codimension 1 singular locus.) If $J$ does not have finite order in Pic$(E)$ then $X$ is $K_n$-regular for all integers $n$: for all $m \geq 0$ we have

$$K_n(X) \cong K_n(X \times \mathbb{A}^m) \cong K_n(E).$$

On the other hand, if $L = Sym_E(O(Q))$ then $K_{-1}(L) \neq K_{-1}(X)$ and $K_0(L) \neq K_0(X)$.

**Proof.** Since $\Omega_E \cong O_E$, $V_n(O_E)$ is a sum of terms $J^i$ for $i > 0$; the same is true for the pushforward of the sheaf $V_n(O_E[t_1, \ldots, t_m])$ to $E$. Recall that $H^p(E, J^r) = 0$ for all $r \neq 0$. From the Zariski descent spectral sequence $E_2^{p,q} = H^p(E, K_{-q}(O_E)[J^2, J^3][t_1, \ldots, t_m])/K_{-q}(O_E)) \Rightarrow K_{-p-q}(X \times \mathbb{A}^m)/K_{-p-q}(E)$ we see that $K_n(X \times \mathbb{A}^m) \cong K_n(E)$ for all $n$. 


On the other hand, Proposition 4.2 yields $\widetilde{K}_{-1}(L) \cong \oplus_{j \geq 1} H^1(E, J \otimes \mathcal{L}^j)$ and $\widetilde{K}_0(L) \cong \oplus_{j \geq 1} H^0(E, J \otimes \mathcal{L}^j) \oplus \widetilde{K}_{-1}(L)$. These groups are nonzero because $L$ is ample.

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