EVOLUTION OF STATES OF A CONTINUUM JUMP MODEL WITH ATTRACTION

YURI KOZITSKY

Abstract. We study a model of an infinite system of point particles in $\mathbb{R}^d$ performing random jumps with attraction. The system’s states are probability measures on the space of particle configurations, and their evolution is described by means of Kolmogorov and Fokker-Planck equations. Instead of solving these equations directly we deal with correlation functions evolving according to a hierarchical chain of differential equations, derived from the Kolmogorov equation. Under quite natural conditions imposed on the jump kernels – and analyzed in the paper – we prove that this chain has a unique classical sub-Poissonian solution on a bounded time interval. This gives a partial answer to the question whether the sub-Poissonianity is consistent with any kind of attraction. We also discuss possibilities to get a complete answer to this question.

1. Introduction

1.1. Setup. In this work, we deal with the model introduced and studied in [6]. It describes an infinite system of point particles placed in $\mathbb{R}^d$ which perform random jumps with attraction. To the best of our knowledge, [6] and the present research are the only works where the dynamics of an infinite particle system of this kind has been studied hitherto.

The phase space of the model is the set $\Gamma$ of all subsets $\gamma \subset \mathbb{R}^d$ such that the set $\gamma \cap \Lambda$ is finite whenever $\Lambda \subset \mathbb{R}^d$ is compact. It is equipped with a topology, see below, and thus with a $\sigma$-field of measurable subsets. Thereby, one can consider probability measures on $\Gamma$ as states of the system. Among them there are Poissonian states in which the particles are independently distributed over $\mathbb{R}^d$. Such states are completely characterized by the density of the particles. In sub-Poissonian states, the dependence between the positions of the particles is controlled in a certain way (see the next subsection), and the particles’ density is still an important characteristic of the state.

For an infinite particle system with repulsion, in [5] the evolution of the system’s states $\mu_0 \mapsto \mu_t$ in the set of sub-Poissonian measures was shown to hold for $t < T$ with some $T < \infty$. Then in [3] this result was improved by constructing the global in time evolution of states. Thus, a paramount question regarding such models is whether the sub-Poissonianity is consistent
YURI KOZITSKY

with some sort of attraction, and – if yes – for which sort and on which time intervals. In this paper, we give a partial answer to this question. Namely, we present quite a reasonable condition on the attraction, see (2.15) below, under which – as we show – the correlation functions evolve $k_0 \mapsto k_t$ and remain sub-Poissonian on a bounded time interval. This result extends the corresponding result of [6] in the following directions: (i) the evolution $k_0 \mapsto k_t$ is constructed as a classical solution of the corresponding Cauchy problem, not in a weak sense; (ii) our result is valid for much more general types of attraction (see subsections 3.3 and 3.4 below). At the same time, the following problems remain open: (a) proving that each $k_t$ is the correlation function of a unique sub-Poissonian state; (b) continuing the evolution $k_0 \mapsto k_t$ to all $t > 0$. In subsection 3.4 below, we discuss possibilities to solve them.

1.2. Presenting the result. States of an infinite particle system are usually characterized by means of their values $\mu(F)$ on observables $F : \Gamma \rightarrow \mathbb{R}$, defined as

$$\mu(F) = \int_{\Gamma} F \, d\mu.$$

The system’s evolution is supposed to be Markovian and hence described by the Kolmogorov equation

$$\frac{d}{dt} F_t = LF_t, \quad F_t|_{t=0} = F_0,$$

where the operator $L$ specifies the model. Alternatively, the evolution of states is derived from the Fokker-Planck equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0,$$

related to that in (1.1) by the duality $\mu_t(F_0) = \mu_0(F_t)$. For the model considered in this work, the operator $L$ is

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} a(x, y) [1 + \epsilon(x, y|\gamma)] [F(\gamma \setminus x \cup y) - F(\gamma)] \, dy,$$

with

$$\epsilon(x, y|\gamma) = \sum_{z \in \gamma \setminus x} b(x, y|z).$$

The quantity $b(x, y|z) \geq 0$ describes the increase of the jump rate from $x \in \gamma$ to $y \in \mathbb{R}^d$ caused by the particle located at $z \in \gamma \setminus x$. Then $\epsilon(x, y|\gamma)$ is the (multiplicative) increase of the corresponding jump rate caused by the whole configuration $\gamma$. For $\epsilon \equiv 0$, (1.3) turns into the generator of free jumps, see, e.g., [2].

As is usual for models of this kind, the direct meaning of (1.1) or (1.2) can only be given for states of finite systems, cf. [9]. In this case, the Banach space where the Cauchy problem in (1.2) is defined can be the space of signed measures with finite variation. For infinite systems, the evolution is
described by means of correlation functions, see \[5, 6, 7\] and the references quoted in these works. In the present paper, we follow this approach the main idea of which can be outlined as follows. Let $\Theta$ be the set of all compactly supported continuous functions $\theta : \mathbb{R}^d \rightarrow (-1,0]$. For a state $\mu$, its Bogoliubov functional $B_{\mu} : \Theta \rightarrow \mathbb{R}$ is set to be

$$B_{\mu}(\theta) = \int_\Gamma \prod_{x \in \gamma} (1 + \theta(x)) \mu(d\gamma), \quad \theta \in \Theta. \quad (1.5)$$

The function $\gamma \mapsto \prod_{x \in \gamma}(1+\theta(x))$ is bounded and measurable for each $\theta \in \Theta$; hence, (1.5) makes sense for each measure. For the homogeneous Poisson measure $\pi_\kappa$, $\kappa > 0$, we have

$$B_{\pi_\kappa}(\theta) = \exp \left( \kappa \int_{\mathbb{R}^d} \theta(x) dx \right).$$

In state $\pi_\kappa$, the particles are independently distributed over $\mathbb{R}^d$ with density $\kappa$. The set of sub-Poissonian states $\mathcal{P}_{\text{exp}}(\Gamma)$ is then defined as that containing all those states $\mu$ for which $B_{\mu}$ can be continued to an exponential type entire function of $\theta \in L^1(\mathbb{R}^d)$. This means that it can be written down in the form

$$B_{\mu}(\theta) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_{\mu}^{(n)}(x_1, \ldots, x_n) \theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \quad (1.6)$$

where $k_{\mu}^{(n)}$ is the $n$-th order correlation function of the state $\mu$. It is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ for which

$$\|k_{\mu}^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq C \exp(\vartheta n), \quad n \in \mathbb{N}_0, \quad (1.7)$$

with some $C > 0$ and $\vartheta \in \mathbb{R}$. Sometimes, (1.7) is called Ruelle bound, cf. \[11\] Chapter 4]. Note that (1.6) can be viewed as an analog of the Taylor expansion of the characteristic function of a probability measure. That is why, $k_{\mu}^{(n)}$ are also called moment functions. Their evolution is described by a chain of differential equations derived from that in (1.1). The central problem of this work is the existence of classical solutions of this chain satisfying (1.7) with possibly time-dependent $C$ and $\vartheta$. Its solution is given in Theorem 3.3 formulated in subsection 3.2 and proved in Section 4. In Section 2 we give some necessary information on the methods used in the paper and specify the model. In subsection 3.1, we place the mentioned chain of equations into suitable Banach spaces, that is mostly performed by defining the corresponding operators. Then we formulate Theorem 3.3 and analyze the assumptions regarding the jump kernels under which we then prove this statement. In subsection 3.4, we give some comments on the result and the assumptions, including discussing open problems related to the model, and compare our result with the corresponding result of \[6\]. Section 4 is dedicated to the proof of Theorem 3.3.
2. Preliminaries and the Model

Here we briefly recall the main notions relevant to the subject – for further information we refer to [1, 5, 6, 7] and the literature quoted in these works.

2.1. Configuration spaces. Let $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_b(\mathbb{R}^d)$ denote the sets of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively. The configuration space $\Gamma$, equipped with the vague topology, is homeomorphic to a separable metric (Polish) space, cf. [1, 8]. Let $\mathcal{B}(\Gamma)$ be the corresponding Borel $\sigma$-field. For $\Lambda \in \mathcal{B}(\mathbb{R}^d)$, the set $\Gamma_\Lambda = \{ \gamma \in \Gamma : \gamma \subset \Lambda \}$ is clearly in $\mathcal{B}(\Gamma)$, and hence $\mathcal{B}(\Gamma_\Lambda) := \{ A \cap \Gamma_\Lambda : A \in \mathcal{B}(\Gamma) \}$ is a sub-field of $\mathcal{B}(\Gamma)$. The projection $p_\Lambda : \Gamma \to \Gamma_\Lambda$ defined by $p_\Lambda(\gamma) = \gamma_\Lambda = \gamma \cap \Lambda$ is measurable. Then, for each Borel $\Lambda$ and $A_\Lambda \in \mathcal{B}(\Gamma_\Lambda)$, we have that

$$p_\Lambda^{-1}(A_\Lambda) := \{ \gamma \in \Gamma : p_\Lambda(\gamma) \in A_\Lambda \} \in \mathcal{B}(\Gamma).$$

Let $\mathcal{P}(\Gamma)$ denote the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$. For a given $\mu \in \mathcal{P}(\Gamma)$, its projection on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined as

$$\mu^\Lambda(A_\Lambda) = \mu(p_\Lambda^{-1}(A_\Lambda)), \quad A_\Lambda \in \mathcal{B}(\Gamma_\Lambda). \tag{2.1}$$

Let $\Gamma_0$ be the set of all finite $\gamma \in \Gamma$. Then $\Gamma_0 \in \mathcal{B}(\Gamma)$ as each of $\gamma \in \Gamma_0$ lies in some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$, and hence belongs to $\Gamma_\Lambda$. It can be proved that a function $G : \Gamma_0 \to \mathbb{R}$ is $\mathcal{B}(\Gamma)/\mathcal{B}(\mathbb{R})$-measurable if and only if, for each $n \in \mathbb{N}_0$, there exists a symmetric Borel function $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ such that

$$G(\eta) = G^{(n)}(x_1, \ldots, x_n), \tag{2.2}$$

for $\eta = \{x_1, \ldots, x_n\}$.

**Definition 2.1.** A measurable function $G : \Gamma_0 \to \mathbb{R}$ is said have bounded support if: (a) there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G(\eta) = 0$ whenever $\eta \cap \Lambda^c \neq \emptyset$; (b) there exists $N \in \mathbb{N}_0$ such that $G(\eta) = 0$ whenever $|\eta| > N$. Here $\Lambda^c := \mathbb{R}^d \setminus \Lambda$ and $|\cdot|$ stands for cardinality.

The Lebesgue-Poisson measure $\lambda$ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by the following formula

$$\int_{\Gamma_0} G(\eta) \lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n, \tag{2.3}$$

which has to hold for all $G \in \mathcal{B}_{bs}(\Gamma_0)$.

In this work, we use the following (real) Banach spaces of functions $g : \Gamma_0 \to \mathbb{R}$. The first group consists of the spaces $\mathcal{G}_\vartheta = L^1(\Gamma_0, w_\vartheta d\lambda)$, indexed by $\vartheta \in \mathbb{R}$. Here we have set $w_\vartheta(\eta) = \exp(\vartheta|\eta|)$. Hence the norm of $\mathcal{G}_\vartheta$ is

$$|g|_\vartheta = \int_{\Gamma_0} |g(\eta)| w_\vartheta(\eta) \lambda(d\eta). \tag{2.4}$$

Along with this norm we also consider

$$\|g\|_\vartheta := \text{ess sup}_{\eta \in \Gamma_0} \{ |g(\eta)| \exp(-\vartheta|\eta|) \}, \tag{2.5}$$
and then set $\mathcal{K}_\vartheta = \{ g : \Gamma_0 \to \mathbb{R} : \| g \|_\vartheta < \infty \}$. These spaces constitute the second group which we use in the sequel. From (2.4) and (2.5) we see that $\mathcal{K}_\vartheta$ is the dual space to $\mathcal{G}_\vartheta$ with the duality

$$(G, k) \mapsto \langle G, k \rangle := \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta),$$

(2.6)

holding for $G \in \mathcal{G}_\vartheta$ and $k \in \mathcal{K}_\vartheta$. Note that $B_{bs}(\Gamma_0)$ is contained in each $\mathcal{G}_\vartheta$ and each $\mathcal{K}_\vartheta$, $\vartheta \in \mathbb{R}$.

For $G \in B_{bs}(\Gamma)$, we set

$$(KG)(\gamma) = \sum_{\eta \subset \gamma} G(\eta),$$

(2.7)

where the sum is taken over all finite $\eta$.

2.2. Correlation functions. For a given $\mu \in \mathcal{P}_{\exp}(\Gamma)$, similarly as in (2.2) we introduce $k_\mu : \Gamma_0 \to \mathbb{R}$ such that $k_\mu(\emptyset) = 1$ and $k_\mu(\eta) = k_\mu^{(n)}(x_1, \ldots, x_n)$ for $\eta = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, cf. (1.5) and (1.6). With the help of the measure introduced in (2.3), the formulas in (1.5) and (1.6) can be combined into the following

$$B_\mu(\theta) = \int_{\Gamma_0} k_\mu(\eta)\prod_{x \in \eta} \theta(x)\lambda(d\eta) =: \int_{\Gamma_0} k_\mu(\eta)e(\eta; \theta)\lambda(d\eta)$$

$$= \int_{\Gamma} \prod_{x \in \gamma} (1 + \theta(x))\mu(d\gamma) =: \int_{\Gamma} F_\theta(\gamma)\mu(d\gamma).$$

Thereby, we can transform the action of $L$ on $F$, as in (1.3), to the action of $L^\Delta$ on $k_\mu$ according to the rule

$$\int_{\Gamma}(LF_\theta)(\gamma)\mu(d\gamma) = \int_{\Gamma_0} (L^\Delta k_\mu)(\eta)e(\eta; \theta)\lambda(d\eta).$$

(2.8)

This will allow us to pass from (1.1) to the corresponding Cauchy problem for the correlation functions, cf. (3.5) below. The main advantage here is that $k_\mu$ is a function of finite configurations.

For $\mu \in \mathcal{P}_{\exp}(\Gamma)$ and $\Lambda \in B_b(\mathbb{R}^d)$, let $\mu^\Lambda$ be as in (2.1). Then $\mu^\Lambda$ is absolutely continuous with respect to the restriction $\lambda^\Lambda$ to $\mathcal{B}(\Gamma_\Lambda)$ of the measure defined in (2.3), and hence we may write

$$\mu^\Lambda(d\eta) = R^\Lambda_\mu(d\eta)\lambda^\Lambda(d\eta), \quad \eta \in \Gamma_\Lambda.$$  

(2.9)

Then the correlation function $k_\mu$ and the Radon-Nikodym derivative $R^\Lambda_\mu$ satisfy

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R^\Lambda_\mu(\eta \cup \xi)\lambda^\Lambda(d\xi).$$

By (2.7), (2.1), and (2.9) we get

$$\int_{\Gamma}(KG)(\gamma)\mu(d\gamma) = \langle G, k_\mu \rangle.$$
holding for each $G \in B_{bs}(\Gamma_0)$ and $\mu \in \mathcal{P}_{\exp}(\Gamma)$, see \[6.6\]. Define

$$B_{bs}^*(\Gamma_0) = \{G \in B_{bs}(\Gamma_0) : (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}.$$ 

By \[8, Theorems 6.1 and 6.2 and Remark 6.3\] one can prove the next statement.

**Proposition 2.2.** Let a measurable function $k : \Gamma_0 \rightarrow \mathbb{R}$ have the following properties:

1. $\langle G, k \rangle \geq 0$, for all $G \in B_{bs}^*(\Gamma_0)$;
2. $k(\emptyset) = 1$;
3. $k(\eta) \leq C|\eta|$, with (c) holding for some $C > 0$ and $\lambda$-almost all $\eta \in \Gamma_0$. Then there exists a unique $\mu \in \mathcal{P}_{\exp}(\Gamma)$ for which $k$ is the correlation function.

2.3. **The model.** The model which we study is specified by the operator given in \[1.3\]. The jump kernel $a$ is supposed to satisfy

$$a(x, y) = a(y, x) \geq 0, \quad \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} a(x, y) dx = 1. \quad (2.10)$$

Regarding the quantities in \[1.4\] we assume

$$\sup_{x, y \in \mathbb{R}^d} \int_{\mathbb{R}^d} b(x, y|z) dz =: \langle b \rangle < \infty, \quad \sup_{x, y \in \mathbb{R}^d} b(x, y|z) =: \bar{b} < \infty, \quad (2.11)$$

Moreover, let us define

$$\phi_+(x, y) = \int_{\mathbb{R}^d} a(z, x)b(z, x|y) dz, \quad (2.12)$$
$$\phi_-(x, y) = \int_{\mathbb{R}^d} a(x, z)b(x, z|y) dz.$$ 

By \[2.10\] and \[2.11\] we have that

$$\phi_\pm(x, y) \leq \bar{b}, \quad \text{for all } x, y \in \mathbb{R}^d. \quad (2.13)$$

**Remark 2.3.** The quantities defined in \[2.12\] can be given the following interpretation: $\phi_+(x, y)$ is the rate with which the particle located at $y$ attracts other particles to jump (from somewhere) to $x$; $\phi_-(x, y)$ is the rate with which the particle located at $y$ forces that located at $x$ to jump (to anywhere). In the latter case, the particle at $y$ ’pushes out’ the one at $x$. Thus, $\phi_+(x, y)$ and $\phi_-(x, y)$ can be called attraction and repulsion rates, respectively.

Now we set

$$\Phi_\pm(\eta) = \sum_{x \in \eta} \sum_{y \in \eta \setminus x} \phi_\pm(x, y), \quad (2.14)$$
which can be interpreted as the total rates of attraction and repulsion of the configuration \( \eta \), respectively. In addition to (2.10) and (2.11) we assume that the following holds

\[
\exists \omega \geq 0 \quad \forall \eta \in \Gamma_0 \quad \Phi_+(\eta) \leq \Phi_-(\eta) + \omega |\eta|.
\]

(2.15)

Note that, for some \( c > 0 \) and all \( \eta \in \Gamma_0 \), by (2.11) it follows that

\[
\Phi_-(\eta) + \omega |\eta| \leq c |\eta|^2.
\]

(2.16)

According to the condition in (2.15), the rate of the jumps from somewhere to points close to the configuration \( \eta \) (i.e., those which make \( \eta \) denser) is in a sense dominated by the rate of the jumps to anywhere, which thin it out.

3. The Result

3.1. The operators. By means of (1.3) and (2.8) we calculate \( L^\Delta \) and present it in the form

\[
L^\Delta = A^\Delta + B^\Delta + C^\Delta + D^\Delta,
\]

(3.1)

with the entries

\[
(A^\Delta k)(\eta) = \sum_{y \in \eta} \int_{\mathbb{R}^d} a(x,y) \left( 1 + \sum_{z \in \eta \setminus y} b(x,y|z) \right) k(\eta \setminus y \cup x) dx,
\]

(3.2)

\[
(B^\Delta k)(\eta) = -\Psi(\eta)k(\eta),
\]

where

\[
\Psi(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x,y) dy + \Phi_-(\eta).
\]

(3.3)

Furthermore,

\[
(C^\Delta k)(\eta) = \sum_{y \in \eta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a(x,y) b(x,y|z) k(\eta \setminus y \cup \{x,z\}) dx dz,
\]

(3.4)

\[
(D^\Delta k)(\eta) = -\int_{\mathbb{R}^d} \left( \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x,y) b(x,y|z) dy \right) k(\eta \cup z) dz.
\]

As mentioned above, instead of directly dealing with the problem in (1.2) we pass from \( \mu_0 \) to the corresponding correlation function \( k_{\mu_0} \) and then consider the problem

\[
\frac{d}{dt} k_t = L^\Delta k_t, \quad k_t|_{t=0} = k_{\mu_0},
\]

(3.5)

with \( L^\Delta \) given in (3.1) – (3.4). Our aim now is to place this problem into the corresponding Banach space. By (1.7) we conclude that \( \mu \in \mathcal{P}_{\exp}(\Gamma) \) implies that \( k_{\mu} \in K_{\vartheta} \) for some \( \vartheta \in \mathbb{R} \). Hence, we assume that \( k_{\mu_0} \) lies in some \( K_{\vartheta_0} \). Then the formulas in (3.1) – (3.4) can be used to define an unbounded operator acting in some \( K_{\vartheta} \). Like in [5, 7] we take into account that, for each \( \vartheta'' < \vartheta' \), the space \( K_{\vartheta''} \) is continuously embedded into \( K_{\vartheta'} \), see (2.5),
and use the ascending scale of such spaces. This means that we are going
to define (3.5) in a given $K_\vartheta$ assuming that $k_{\vartheta_0} \in K_{\vartheta_0} \hookrightarrow K_\vartheta$.

For $\omega$ as in (2.15) and $\Psi$ as in (3.3), we set

$$\Psi_\omega(\eta) = \omega|\eta| + \Psi(\eta). \tag{3.6}$$

In the sequel, along with those as in (3.2) and (3.3) we use the following operators

$$(B_\vartheta \omega k)(\eta) = -\Psi_\omega(\eta) k(\eta), \tag{3.7}$$

$$(C_\vartheta \omega k)(\eta) = (C_\vartheta k)(\eta) + \omega|\eta| k(\eta).$$

Then the decomposition (3.1) can be rewritten

$$L_\vartheta = A_\vartheta + B_\vartheta \omega + C_\vartheta \omega + D_\vartheta, \tag{3.8}$$

with $A_\vartheta$ and $D_\vartheta$ being as above.

For a given $\vartheta \in \mathbb{R}$, we define $L_\vartheta$ in $K_\vartheta$ by means of the following estimates. For $k \in K_\vartheta$, by (2.5) we have that

$$|k(\eta)| \leq \|k\|_{\vartheta} e^{\vartheta|\eta|}, \quad \text{for } \lambda - \text{a.a. } \eta \in \Gamma_0. \tag{3.9}$$

By means of the latter estimate and (2.10), (2.11) we obtain from (3.2), (3.3) and (3.7) that

$$|(C_\vartheta \omega k)(\eta)| \leq (\omega + \langle b \rangle)|\eta| \exp [\vartheta(|\eta| + 1)] \cdot \|k\|_\vartheta, \tag{3.10}$$

$$|(C_\vartheta k)(\eta)| \leq \langle b \rangle|\eta| \exp [\vartheta(|\eta| + 1)] \cdot \|k\|_\vartheta,$$

$$|(D_\vartheta k)(\eta)| \leq \langle b \rangle|\eta| \exp [\vartheta(|\eta| + 1)] \cdot \|k\|_\vartheta.$$

Now we use (2.13) and (2.14) to obtain from (3.2), (3.3), (3.6) the following

$$|(A_\vartheta k)(\eta)| \leq (|\eta| + \tilde{b}|\eta|^2) e^{\vartheta|\eta|} \cdot \|k\|_\vartheta, \tag{3.11}$$

$$|(B_\vartheta k)(\eta)| \leq (|\eta| + \tilde{b}|\eta|^2) e^{\vartheta|\eta|} \cdot \|k\|_\vartheta,$$

$$|(D_\vartheta \omega k)(\eta)| \leq [(1 + \omega)|\eta| + \tilde{b}|\eta|^2] e^{\vartheta|\eta|} \cdot \|k\|_\vartheta.$$

The estimates in (3.10) and (3.11) allow us to define $(L_\vartheta^\Delta, D_\vartheta^\Delta)$, where

$$D_\vartheta^\Delta := \{ k \in K_\vartheta : |\eta|^2 k \in K_\vartheta \} \tag{3.12}$$

**Lemma 3.1.** For each $\vartheta'' < \vartheta$, it follows that $K_{\vartheta''} \subset D_\vartheta^\Delta$.

**Proof.** By means of (3.9) and the inequality $x \exp(-\sigma x) \leq 1/2e\sigma$, $x, \sigma > 0$, we get from (3.10) and (3.11) the following estimate,

$$|\eta|^2 |k(\eta)| \leq \frac{4}{e^2(\vartheta - \vartheta'')^2} \|k\|_{\vartheta''} e^{\vartheta|\eta|},$$

which yields the proof. \qed
The same estimate and (3.10), (3.11) also yield
\[ \| L^\Delta k \| \leq 2 \left( \frac{1 + \langle b \rangle}{e(\vartheta - \vartheta''')} + \frac{4\tilde{b}}{e^2(\vartheta - \vartheta''')} \right) \| k \|, \quad (3.13) \]
which allows us to define a bounded linear operator \( L^{\Delta}_{\vartheta'''}: \mathcal{K}_{\vartheta'''} \to \mathcal{K}_{\vartheta} \) the norm of which can be estimated by means of (3.13). In what follows, we consider two types of operators defined by the expression in (3.1) – (3.4):

(a) unbounded operators \( (L^{\Delta}_{\vartheta}, D^{\Delta}_{\vartheta}) \), \( \vartheta \in \mathbb{R} \), with domains as in (3.12) and Lemma 3.1; (b) bounded operators \( L^{\Delta}_{\vartheta'''} \) as just described. These operators are related to each other in the following way:
\[ \forall \vartheta'' < \vartheta \quad \forall k \in \mathcal{K}_{\vartheta'''} 
\quad L^{\Delta}_{\vartheta'''}k = L^{\Delta}_{\vartheta}k. \quad (3.14) \]

3.2. The statement. We assume that the initial state \( \mu_0 \) is fixed, which determines \( \vartheta_0 \in \mathbb{R} \) by the condition that \( k_{\mu_0} \) lies in \( \mathcal{K}_{\vartheta_0} \). Since \( \mathcal{K}_{\vartheta''} \hookrightarrow \mathcal{K}_{\vartheta'} \) for \( \vartheta'' < \vartheta' \), we take the least \( \vartheta_0 \) satisfying this condition. Then for \( \vartheta > \vartheta_0 \), we consider in \( \mathcal{K}_{\vartheta} \) the problem, cf. (3.5) and Lemma 3.1,
\[ \frac{d}{dt}k_t = L^{\Delta}_{\vartheta}k_t, \quad k_t|_{t=0} = k_{\mu_0} \in \mathcal{K}_{\vartheta_0}. \quad (3.15) \]

**Definition 3.2.** By a (classical) solution of (3.15) on a time interval, \([0, T)\), \( T \leq +\infty \), we mean a continuous map \([0, T) \ni t \mapsto k_t \in \mathcal{D}^{\Delta}_{\vartheta} \) such that the map \([0, T) \ni t \mapsto dk_t/dt \in \mathcal{D}^{\Delta}_{\vartheta} \) is also continuous and both equalities in (3.15) are satisfied.

For \( \omega \geq 0 \) as in (3.6) and (3.8), we set, cf. (2.11),
\[ T(\vartheta, \vartheta_0) = \frac{(\vartheta - \vartheta_0)e^{-\vartheta}}{\omega + 2\langle b \rangle}, \quad (3.16) \]
where \( \vartheta \) and \( \vartheta_0 \) are as in (3.15).

**Theorem 3.3.** Let the conditions in (2.10) – (2.15) be satisfied. Then, for each \( \vartheta > \vartheta_0 \), the problem in (3.15) has a unique solution on the time interval \([0, T(\vartheta, \vartheta_0))\).

The proof of this statement will be done in Section 4 below. Let us now analyze how to choose \( \vartheta \) in an optimal way. Since the length \( T(\vartheta, \vartheta_0) \) of the time interval in Theorem 3.3 depends on the choice of \( \vartheta \), we take \( \vartheta = \vartheta^* \) defined by the condition
\[ T(\vartheta^*, \vartheta_0) = \max_{\vartheta > \vartheta_0} T(\vartheta, \vartheta_0), \]
which by (3.16) yields \( \vartheta^* = 1 + \vartheta_0 \). Hence, the maximum length of the time interval is
\[ \tau(\vartheta_0) = T(1 + \vartheta_0, \vartheta_0) = \frac{e^{-\vartheta_0}}{e(\omega + 2\langle b \rangle)}. \]
Note that \( \tau(\vartheta_0) \to 0 \) as \( \vartheta_0 \to +\infty \).
3.3. Analyzing the assumptions. Our main assumption in (2.15) looks like the stability condition (with stability constant \( \omega \geq 0 \)) for the interaction potential \( \phi = \phi^- - \phi^+ \), see (2.12), used in the statistical mechanics of continuum systems of interacting particles, cf. [11, Chapter 3] and also [7, Section 3.3]. A particular case of the kernels is where they are translation invariant and \( b \) has the following form

\[
b(x, y|z) = \kappa_1(x - z) + \kappa_2(y - z),
\]

where \( \kappa_i(x) = \kappa_i(-x) \geq 0 \) belong to \( L^1(\mathbb{R}^d) \). Then

\[
\phi_+(x, y) = (\alpha * \kappa_1)(x - y) + \kappa_2(x - y),
\]

\[
\phi_-(x, y) = \kappa_1(x - y) + (\alpha * \kappa_2)(x - y),
\]

where \( \alpha(x) = a(0, x) \) and * denotes the usual convolution. By (2.10) and (2.12) \( \alpha \) and both \( \kappa_i \) are integrable. Thus, we can use their transforms

\[
\hat{\alpha}(p) = \int_{\mathbb{R}^d} \alpha(x) \exp(i(p, x)) \, dx, \quad p \in \mathbb{R}^d,
\]

\[
\hat{\phi}_\pm(p) = \int_{\mathbb{R}^d} \phi_\pm(0, x) \exp(i(p, x)) \, dx,
\]

\[
\hat{\kappa}_i(p) = \int_{\mathbb{R}^d} \kappa_i(x) \exp(i(p, x)) \, dx, \quad i = 0, 1,
\]

Note that the left-hand sides here are real. Moreover, \( \hat{\alpha}(p) \leq \hat{\alpha}(0) = 1 \) and \( \hat{\kappa}_i(p) \leq \hat{\kappa}_i(0), i = 1, 2 \). Then a sufficient condition for (2.15) to be satisfied, see [11, Section 3.2, Proposition 3.2.7], is that the following holds: (a) both \( \phi_\pm(0, x) \) are continuous; (b) \( \hat{\phi}_-(p) \geq \hat{\phi}_+(p) \) for all \( p \in \mathbb{R}^d \). The latter means that the potential \( \phi = \phi^- - \phi^+ \) is positive definite (in Bochner’s sense). In view of (3.18), (b) turns into

\[
\forall p \in \mathbb{R}^d \quad (1 - \hat{\alpha}(p)) (\hat{\kappa}_1(p) - \hat{\kappa}_2(p)) \geq 0.
\]

Thus, a sufficient condition for the latter to hold is \( \hat{\kappa}_1(p) \geq \hat{\kappa}_2(p) \) for all those \( p \) for which \( \hat{\alpha}(p) < 1 \). An example can be

\[
\kappa_i(x) = \frac{1}{(2\pi)^{d/2} \sigma_i} \exp\left(-\frac{|x|^2}{2\sigma_i^2}\right), \quad \sigma_1 < \sigma_2,
\]

cf. [7, Proposition 3.8].

3.4. Comments. First we make some comments on the result of Theorem 3.3. For the model specified in (1.3) with a particular choice of \( b \), which we discuss below, in [6, Theorem 2 and Proposition 1] there was constructed a weak solution of the problem in (3.5) on a bounded time interval. Our Theorem 3.3 yields a solution in the strongest sense – a classical one – see Definition 3.2, existing, however, also on a bounded time interval. At the same time, this solution \( k_t \) yet may not be the correlation function of a state. To prove this, one ought to develop a technique similar to that
used in [7, Section 5] and based on the use of Proposition 2.2. Noteworthy, the fact, proved in [7], that $k_t$ is a correlation function allowed there for continuing to all $t > 0$ the solution primarily obtained on a bounded time interval. For jump dynamics with repulsion, such continuation was realized in [3, 4], also by means of the corresponding property of $k_t$. However, for the model considered here for such a continuation to be done proving that the solution $k_t$ is a correlation function – and hence is positive in a certain sense – might not be enough. If this is the case, then the attraction in the form as in (1.3) is not consistent with the sub-Poissonicity of the states and hence essentially changes the dynamics of the model. We plan to clarify this in our forthcoming work.

Now let us return to discussing the conditions imposed on the model. As mentioned above, in [6] there was studied the model specified in (1.3) with the choices of $b$ (cf. [6, Eqs. (3) – (5)]) which in our notations can be presented as follows: (i) $b(x, y|z) = \kappa(x - z)$; (ii) $b(x, y|z) = \kappa(y - z)$; (iii) $b(x, y|z) = [\kappa(x - z) + \kappa(y - z)]/2$. Note that all the three are particular cases of (3.17). However, instead of our condition (2.15) there was imposed a stronger one, which in our context can be written down as

$$\forall x \in \mathbb{R}^d \quad \phi_- (0, x) \geq \phi_+(0, x).$$  \hfill (3.21)

In case (i), (3.21) turns into $\kappa(x) \geq (\alpha \ast \kappa)(x)$, which is much stronger than $\hat{\kappa}(p) \geq 0$ that follows from (3.19) in this case. E.g., the latter clearly holds for the Gaussian kernel $\kappa$, see (3.20). In case (ii), which corresponds to pure attraction, cf. Remark 2.3, (3.21) turns into $\kappa(x) \leq (\alpha \ast \kappa)(x)$, which, in fact, is equivalent to $\kappa = (\alpha \ast \kappa)$. The latter can be considered as the problem of the existence of strictly positive fixed points of the corresponding (positive) integral operator in $L^1(\mathbb{R}^d)$. In some cases, this problem has such points, e.g., if the operator is compact – by the Krein-Rutman theorem. In the symmetric case (iii), we have $\phi_+ = \phi_-$, and hence (2.15) trivially holds.

4. The Proof

The main idea of proving Theorem 3.3 is to construct the family of bounded linear operators $Q_{\vartheta_0}(t) : \mathcal{K}_{\vartheta_0} \rightarrow \mathcal{K}_\vartheta$ with $t \in [0, T(\vartheta, \vartheta_0))$ such that the solution of (3.15) is obtained in the form

$$k_t = Q_{\vartheta_0}(t)k_0.$$  \hfill (4.1)

An important element of this construction is another family of bounded operators obtained by means of a substochastic semigroup constructed in the $\mathcal{G}_\vartheta$. We obtain this semigroup in the next subsection in Proposition 4.2.

4.1. An auxiliary semigroup. For a given $\vartheta \in \mathbb{R}$, the formulas in (3.11) allows one to define the corresponding unbounded operators in $\mathcal{K}_\vartheta$. The predual space of $\mathcal{K}_\vartheta$ is $\mathcal{G}_\vartheta$ equipped with the norm defined in (2.4). For $A^\Delta$ and $B^{\Delta \omega}$, see (3.7), we introduce $\hat{A}$ and $\hat{B}^{\omega}$ by setting, cf. (2.6),

$$\langle \langle \hat{A}G, k \rangle \rangle = \langle \langle G, B^{\Delta \omega}k \rangle \rangle, \quad \langle \langle \hat{B}^{\omega}G, k \rangle \rangle = \langle \langle G, B^{\Delta \omega}k \rangle \rangle.$$
This yields

\[
(\hat{A}G)(\eta) = \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x, y) \left( 1 + \sum_{z \in \eta \setminus x} b(x, y|z) \right) G(\eta \setminus x \cup y) dy, \quad (4.2)
\]

\[
(\hat{B}^\omega G)(\eta) = -\Psi_\omega(\eta)G(\eta).
\]

Now for a given $\vartheta$, we set, cf. (3.12)

\[
\hat{D}^\omega(\vartheta) := \{ G \in \mathcal{G}_\vartheta : \Psi_\vartheta G \in \mathcal{G}_\vartheta \}. \quad (4.3)
\]

Clearly, the multiplication operator $\hat{D}^\omega(\vartheta) : \hat{D}^\omega(\vartheta) \subset \mathcal{G}_\vartheta \rightarrow \mathcal{G}_\vartheta$ defined in the second line of (4.2) is closed. Moreover, it generates a $C_0$-semigroup $\{ S^{(0)}(t) \} t \geq 0$ of bounded multiplication operators $S^{(0)}(t)G(\eta) = \exp(-t\Psi_\vartheta(\eta))G(\eta)$.

Note that each operator is a positive contraction, i.e., $S^{(0)}(t)G$ maps $\mathcal{G}^+ :\mathcal{G}^{\vartheta} := \{ G \in \mathcal{G} : G(\eta) \geq 0, \lambda - a, a. \eta \in \Gamma_0 \}$ into itself and $|S^{(0)}(t)G|_\vartheta \leq |G|_\vartheta$, see (2.4). That is, $\{ S^{(0)}(t) \} t \geq 0$ is a substochastic semigroup.

For $G \in \hat{D}^\omega(\vartheta)$ we have

\[
|\hat{A}G|_\vartheta = \int_{\Gamma_0} \left( \sum_{y \in \eta} \int_{\mathbb{R}^d} a(x, y)dx + \Phi_+(\eta) \right) G(\eta)e^{\vartheta|\eta|}\lambda(d\eta) \quad (4.4)
\]

\[
\leq \int_{\Gamma_0} \left( \sum_{x \in \eta} \int_{\mathbb{R}^d} a(x, y)dy + \omega|\eta| + \Phi_-(\eta) \right) G(\eta)e^{\vartheta|\eta|}\lambda(d\eta)
\]

\[
= -\int_{\Gamma_0} (\hat{B}^\omega G)(\eta)e^{\vartheta|\eta|}\lambda(d\eta).
\]

Likewise, for $G \in \hat{D}^\omega(\vartheta)$ we get

\[
|\hat{A}G|_{\vartheta'} \leq \int_{\Gamma_0} \Phi_\vartheta(\eta) |G(\eta)| e^{\vartheta'|\eta|}\lambda(d\eta), \quad (4.5)
\]

which means that $\hat{A}$ can be defined on $\hat{D}^\omega(\vartheta)$, see (4.3).

**Lemma 4.1.** The closure $\hat{T}_{\vartheta}$ of $(\hat{A} + \hat{B}^\omega, \hat{D}^\omega(\vartheta))$ in $\mathcal{G}_\vartheta$ is the generator of a substochastic semigroup.

**Proof.** We use the Thieme-Voigt perturbation technique [12], see also [9, Section 3]. For each $G \in \mathcal{G}^+_\vartheta$, we have that

\[
|G|_{\vartheta} = \varphi_{\vartheta}(G) := \int_{\Gamma_0} G(\eta)e^{\vartheta|\eta|}\lambda(d\eta).
\]

Clearly, $\varphi_{\vartheta}$ is a positive linear functional on $\mathcal{G}_\vartheta$, and thus the norm defined in (2.4) is additive on the cone $\mathcal{G}^+_\vartheta$. For $\vartheta' > \vartheta$, by (2.4) $\mathcal{G}_{\vartheta'}$ is densely and continuously embedded into $\mathcal{G}_\vartheta$. Moreover, the mentioned above semigroup
\{S^{(0)}_\vartheta(t)\}_{t \geq 0} has the property \(S^{(0)}_\vartheta(t) : \mathcal{G}_\vartheta \to \mathcal{G}_\vartheta, t \geq 0\), and the restrictions \(S^{(0)}_\vartheta(t)|_{\mathcal{G}_\vartheta'}\) constitute a \(C_0\)-semigroup, which is just \(\{S^{(0)}_\vartheta(t)\}_{t \geq 0}\) generated by \((\hat{\mathcal{B}}_\vartheta, \hat{\mathcal{D}}_\vartheta')\). By (4.1) we have that \(\hat{A} : \hat{\mathcal{D}}_\vartheta' \to \mathcal{G}_\vartheta\). Then according to [12, Theorem 2.7], see also [9, Proposition 3.2], the proof will be done if we show that, for some \(\vartheta' > \vartheta\), the following holds

\[ \forall G \in \hat{D}_\vartheta^{\omega^+}, \quad \varphi_\vartheta((\hat{A} + \hat{\mathcal{B}}^{\omega})G) \leq \varphi_\vartheta(G) - \varepsilon \varphi_\vartheta(\Psi_\omega G) \quad (4.6) \]

with some \(\varepsilon > 0\). Since (4.4) holds for each \(\vartheta \in \mathbb{R}\), we have that

\[ \varphi_\vartheta((\hat{A} + \hat{\mathcal{B}}^{\omega})G) \leq 0. \]

Then (4.6) turns into \(\varphi_\vartheta(\Psi_\omega G) \leq (1/\varepsilon) \varphi_\vartheta(G)\). By (2.16) the latter holds for each \(\vartheta' > \vartheta\) and the correspondingly small \(\varepsilon\).

Let \(S_\vartheta := \{S_\vartheta(t)\}_{t \geq 0}\) be the semigroup as in Lemma 4.1. The semigroup which we need is the sun-dual to \(S_\vartheta\). It is introduced as follows. Let \(T^*_\vartheta\) be the adjoint to the generator of \(S_\vartheta\) with domain \(\text{Dom}(T^*_\vartheta) \subset \mathcal{K}_\vartheta\) defined in a standard way. That is,

\[ \text{Dom}(T^*_\vartheta) = \{k \in \mathcal{K}_\vartheta : \exists q \in \mathcal{K}_\vartheta \; \forall G \in \hat{D}_\vartheta \langle \langle \hat{T}_\vartheta G, k \rangle \rangle = \langle \langle G, q \rangle \rangle \}. \]

For each \(k \in \text{Dom}(T^*_\vartheta)\), we have that

\[ (T^*_\vartheta k)(\eta) = (A^k)(\eta) + (\mathcal{B}^\omega k)(\eta), \quad (4.7) \]

see (3.2) and (3.7). By (3.11) we then get that \(\mathcal{K}_{\vartheta''} \subset \text{Dom}(T^*_\vartheta)\) for each \(\vartheta'' < \vartheta\). Let \(Q_\vartheta\) stand for the closure of \(\text{Dom}(T^*_\vartheta)\) in \(\mathcal{K}_\vartheta\). Then

\[ Q_\vartheta := \overline{\text{Dom}(T^*_\vartheta)} \supset \text{Dom}(T^*_\vartheta) \subset \mathcal{K}_{\vartheta''}. \quad (4.8) \]

For each \(t \geq 0\), the adjoint \((S_\vartheta(t))^*\) of \(S_\vartheta(t)\) is a bounded operator in \(\mathcal{K}_\vartheta\). However, the semigroup \(\{(S_\vartheta(t))^*\}_{t \geq 0}\) is not strongly continuous. For \(t > 0\), let \(S^{\omega}_{\vartheta}(t)\) denote the restriction of \((S_\vartheta(t))^*\) to \(Q_\vartheta\). Since \(S_\vartheta\) is the semigroup of contractions, for \(k \in Q_\vartheta\) and all \(t \geq 0\), we have that

\[ \|S^{\omega}_{\vartheta}(t)k\|_\vartheta = \|S^*(t)k\|_\vartheta \leq \|k\|_\vartheta. \quad (4.9) \]

**Proposition 4.2.** For every \(\vartheta'' < \vartheta\) and any \(k \in \mathcal{K}_{\vartheta''}\), the map

\[ \{0, +\infty\} \ni t \mapsto S^{\omega}_{\vartheta}(t)k \in \mathcal{K}_\vartheta \]

is continuous.

**Proof.** By [10, Theorem 10.4, page 39], the collection \(S^{\omega}_{\vartheta} := \{S^{\omega}_{\vartheta}(t)\}_{t \geq 0}\) constitutes a \(C_0\)-semigroup on \(Q_\vartheta\) the generator of which, \(T^{\omega}_{\vartheta}\), is the part of \(\overline{T^*_\vartheta}\) in \(Q_\vartheta\). That is, \(T^{\omega}_{\vartheta}\) is the restriction of \(T^*_\vartheta\) to the set

\[ \text{Dom}(T^{\omega}_{\vartheta}) := \{k \in \text{Dom}(T^*_\vartheta) : T^*_\vartheta k \in Q_\vartheta\}, \]

cf. [10, Definition 10.3, page 39]. The continuity in question follows by the \(C_0\)-property of the semigroup \(\{S^{\omega}_{\vartheta}(t)\}_{t \geq 0}\) and (4.8). \(\square\)
By (3.11) it follows that $\text{Dom}(T_\vartheta^\varphi) \supset \mathcal{K}_{\varphi''}$, holding for each $\varphi'' < \varphi'$. Hence, see [10] Theorem 2.4, page 4,

$$S_{\vartheta'}^\varphi(t)k \in \text{Dom}(T_\vartheta^\varphi),$$

and

$$\frac{d}{dt} S_{\vartheta'}^\varphi(t)k = A_{\varphi'}^\varphi S_{\vartheta'}^\varphi(t)k,$$

which holds for all $\varphi' \in (\varphi'', \varphi]$ and $k \in \mathcal{K}_{\varphi''}$.

4.2. Getting the solutions. Here we construct the family of the operators $Q_{\varphi_0\varphi}(t)$ which appear in (4.1). To this end we use the semigroup as in Proposition 4.2.

For $\varphi'' < \vartheta$, let $\mathcal{L}(\mathcal{K}_{\varphi''}, \mathcal{K}_\vartheta)$ denote the Banach space of bounded linear operators acting from $\mathcal{K}_{\varphi''}$ to $\mathcal{K}_\vartheta$. By means of the estimates in (3.11) one can introduce $A_{\varphi''}^\varphi$ and $B_{\varphi''}^\varphi$, both in $\mathcal{L}(\mathcal{K}_{\varphi''}, \mathcal{K}_\vartheta)$. Then, cf. (3.14) and (4.1),

$$\forall k \in \mathcal{K}_{\varphi''}, \quad T_\vartheta^\varphi k = \left(A_{\varphi''}^\varphi + B_{\varphi''}^\varphi\right) k.$$  \hspace{1cm} (4.11)

Let now $S_{\vartheta''}(t), t > 0$ be the restriction of $S_{\vartheta}^\varphi(t)$ to $\mathcal{K}_{\varphi''}$. Let also $S_{\vartheta''}(0)$ be the embedding $\mathcal{K}_{\varphi''} \hookrightarrow \mathcal{K}_\vartheta$. By (4.9) we have that the operator norm of such operators satisfy

$$\forall t \geq 0 \quad \|S_{\vartheta''}(t)\| \leq 1.$$  \hspace{1cm} (4.12)

By Proposition 4.2 the map

$$[0, +\infty) \ni t \mapsto S_{\vartheta''}(t) \in \mathcal{L}(\mathcal{K}_{\varphi''}, \mathcal{K}_\vartheta)$$

is continuous, and for each $\vartheta' \in (\varphi'', \vartheta)$, the following holds, see (4.10) and (4.11),

$$\frac{d}{dt} S_{\vartheta''}(t) = \left(A_{\varphi''}^\varphi + B_{\varphi''}^\varphi\right) S_{\vartheta''}(t).$$  \hspace{1cm} (4.13)

Now by means of the estimates in (3.10) one concludes that the formulas in (3.14) and (4.7) can be used to introduce $C_{\varphi''}^\varphi$ and $D_{\varphi''}^\varphi$, both in $\mathcal{L}(\mathcal{K}_{\varphi''}, \mathcal{K}_\vartheta)$. Their operator norms satisfy, cf. (3.13),

$$\|C_{\varphi''}^\varphi\| \leq \frac{\langle b \rangle e^\varphi}{e(\vartheta - \varphi'')}, \quad \|D_{\varphi''}^\varphi\| \leq \frac{\langle b \rangle e^\varphi}{e(\vartheta - \varphi'')}.$$  \hspace{1cm} (4.14)

Let $\varphi_0$ be as in (3.15). Take $\vartheta > \varphi_0$ and then define

$$A(\vartheta, \varphi_0) = \{(\vartheta_1, \vartheta_2, t) : \varphi_0 \leq \vartheta_1 < \vartheta_2 \leq \vartheta, \quad 0 \leq t < T(\vartheta_2, \vartheta_1)\},$$

where $T(\vartheta_2, \vartheta_1)$ is as in (3.10).

Lemma 4.3. For any $(\vartheta_1, \vartheta_2, t) \in A(\vartheta, \varphi_0)$, there exists $Q_{\varphi_1\varphi}(t) \in \mathcal{L}(\mathcal{K}_{\varphi_1}, \mathcal{K}_{\varphi_2})$ such that the family $\{Q_{\varphi_1\varphi}(t) : (\vartheta_1, \vartheta_2, t) \in A(\vartheta, \varphi_0)\}$ has the following properties:

(i) the map $[0, T(\vartheta_2, \vartheta_1)] \ni t \mapsto Q_{\varphi_1\varphi}(t) \in \mathcal{L}(\mathcal{K}_{\varphi_1}, \mathcal{K}_{\varphi_2})$ is continuous;
(ii) the operator norm of $Q_{\vartheta_2 \vartheta_1}(t)$ satisfies
\[
\|Q_{\vartheta_2 \vartheta_1}(t)\| \leq \frac{T(\vartheta_2, \vartheta_1)}{T(\vartheta_2, \vartheta_1) - t},
\]
(4.15)

(iii) for each $\vartheta_3 \in (\vartheta_1, \vartheta_2)$ and $t < T(\vartheta_3, \vartheta_1)$, the following holds
\[
\frac{d}{dt}Q_{\vartheta_2 \vartheta_1}(t) = L_{\vartheta_2 \vartheta_3}^\Delta Q_{\vartheta_3 \vartheta_1}(t),
\]
(4.16)

where $L_{\vartheta_2 \vartheta_3}^\Delta$ is as in (3.14).

The proof of this statement employs the following construction. For $l \in \mathbb{N}$ and $t > 0$, we set
\[
\mathcal{T}_l := \{(t, t_1, \ldots, t_l) : 0 \leq t_i \leq \cdots \leq t_1 \leq t\},
\]
fix some $\theta \in (\vartheta_1, \vartheta_2]$, and then take $\delta < \theta - \vartheta_1$. Next we divide the interval $[\vartheta_1, \theta]$ into subintervals with endpoints $\vartheta^s$, $s = 0, \ldots, 2l + 1$, as follows. Set $\vartheta^0 = \vartheta_1$, $\vartheta^{2l+1} = \theta$, and
\[
\vartheta^{2s} = \vartheta_1 + \frac{s}{l+1} \delta + s\epsilon, \quad \epsilon = (\theta - \vartheta_1 - \delta)/l, \quad s = 0, 1, \ldots, l.
\]
(4.17)

Then for $(t, t_1, \ldots, t_l) \in \mathcal{T}_l$, define
\[
\Pi_{\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) = S_{\vartheta^{2l}}(t - t_1) \left( C_{\vartheta^{2l} \vartheta^{2l-1}}^{\Delta \omega} + D_{\vartheta^{2l} \vartheta^{2l-1}}^\Delta \right) \times
\]
\[
\times \cdots \times S_{\vartheta^{2s+1} \vartheta^{2s}}(t_{l-s} - t_{l-s+1}) \left( C_{\vartheta^{2s} \vartheta^{2s-1}}^{\Delta \omega} + D_{\vartheta^{2s} \vartheta^{2s-1}}^\Delta \right) \times
\]
\[
\times \cdots \times S_{\vartheta^{2} \vartheta} t_{l-1} - t_l \left( C_{\vartheta^{2} \vartheta}^{\Delta \omega} + D_{\vartheta^{2} \vartheta}^\Delta \right) S_{\vartheta^{1} \vartheta_1}(t_l).
\]
(4.18)

**Proposition 4.4.** For each $l \in \mathbb{N}$, the operators defined in (4.18) have the following properties:

(i) for each $(t, t_1, \ldots, t_l) \in \mathcal{T}_l$, $\Pi_{\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) \in \mathcal{L}(K_{\vartheta_1}, K_{\theta})$, and the map
\[
\mathcal{T}_l \ni (t, t_1, \ldots, t_l) \mapsto \Pi_{\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) \in \mathcal{L}(K_{\vartheta_1}, K_{\theta})
\]
is continuous;

(ii) for fixed $t_1, t_2, \ldots, t_l$, and each $\varepsilon > 0$, the map
\[
(t_1, t_1 + \varepsilon) \ni t \mapsto \Pi_{\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) \in \mathcal{L}(K_{\vartheta_1}, K_{\vartheta_2})
\]
is continuously differentiable and for each $\vartheta' \in (\vartheta_1, \theta)$ the following holds
\[
\frac{d}{dt} \Pi_{\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) = \left( A_{\vartheta_1 \vartheta'}^{\Delta \omega} + B_{\vartheta_1 \vartheta'}^{\Delta \omega} \right) \Pi_{\vartheta' \vartheta_1}^{(l)}(t, t_1, \ldots, t_l).
\]
(4.19)

**Proof.** The first part of claim (i) follows by (4.18), (4.12), and (4.14). To prove the second part we apply Proposition 4.2 and (4.13), which yields (4.19). □
Proof of Lemma 4.3. Take any $T < T(\vartheta_2, \vartheta_1)$ and then pick $\theta \in (\vartheta_1, \vartheta_2]$ and a positive $\delta < \theta - \vartheta_1$ such that

$$T < T_\delta := \frac{(\theta - \vartheta_1 - \delta)e^{-\vartheta_2}}{\omega + 2(b)}.$$  

For this $\delta$, take $\Pi_{\theta\vartheta_1}^{(l)}$ as in (4.18), and then set

$$Q_{\theta\vartheta_1}^{(n)}(t) = S_{\theta\vartheta_1}(t) + \sum_{l=1}^{n} \int_{0}^{t} \int_{0}^{t_1} \cdots \int_{0}^{t_{l-1}} \Pi_{\theta\vartheta_1}^{(l)}(t, t_1, \ldots, t_l) dt_l \cdots dt_1, \quad n \in \mathbb{N}.$$  

By (4.12), (4.14), and (4.17) the operator norm of (4.18) satisfies

$$\|\Pi_{\theta\vartheta_1}^{(l)}(t, t_1, \ldots, t_l; B)\| \leq \left( \frac{l}{eT_\delta} \right)^l,$$  

holding for all $l = 1, \ldots, n$. This yields in (4.20)

$$\|Q_{\theta\vartheta_1}^{(n)}(t) - Q_{\theta\vartheta_1}^{(n-1)}(t)\| \leq \frac{1}{n!} \left( \frac{T}{eT_\delta} \right)^n,$$  

which implies

$$\forall t \in [0, T] \quad Q_{\theta\vartheta_1}^{(n)}(t) \to Q_{\theta\vartheta_1}(t) \in \mathcal{L}(K_{\vartheta_2}, K_{\vartheta_1}), \quad n \to +\infty.$$  

This proves claim (i). The estimate in (4.15) follows from that in (4.21). Now by (4.18), (4.13), and (4.19) we obtain

$$\frac{d}{dt} Q_{\vartheta_2\vartheta_1}^{(n)}(t) = \left( A_{\vartheta_2\vartheta} + B_{\vartheta_2\vartheta} \right) Q_{\vartheta_2\vartheta_1}^{(n)}(t) + \left( C_{\vartheta_2\vartheta} + D_{\vartheta_2\vartheta} \right) Q_{\vartheta_2\vartheta_1}^{(n-1)}(t), \quad n \in \mathbb{N}.$$  

Then the continuous differentiability of the limit and (4.16) follow by standard arguments. 

Now let $k_t$ be as in (4.1). Then by (3.14) and (4.16) we conclude that it has all the properties assumed in Definition 3.2 and hence solves (3.15). Then to complete the proof of Theorem 3.3 we have to show that this is a unique solution.

4.3. Proving the uniqueness. Since the problem in (3.15) is linear, it is enough to show that its version with the zero initial condition has only the zero solution. Let $u_t \in \mathcal{K}_\vartheta$ be a solution of this version. Take some $\vartheta' > \vartheta$ and then $t > 0$ such that $t < T(\vartheta, \vartheta')$. Clearly, $u_t$ solves (3.15) also in $\mathcal{K}_{\vartheta'}$. Thus, it can be written down in the following form

$$u_t = \int_{0}^{t} S_{\vartheta'\vartheta'}(t - s) \left( C_{\vartheta'\vartheta'} + D_{\vartheta'\vartheta} \right) u_s ds,$$  

where $u_t$ on the left-hand side (resp. $u_s$ on the right-hand side) is considered as an element of $\mathcal{K}_{\vartheta'}$ (resp. $\mathcal{K}_\vartheta$) and $\vartheta' \in (\vartheta, \vartheta')$. Let us show that for all $t < T(\vartheta, \vartheta')$, $u_t = 0$ as an element of $\mathcal{K}_\vartheta$. In view of the embedding $\mathcal{K}_\vartheta \hookrightarrow \mathcal{K}_{\vartheta'}$, this will follow from the fact that $u_t = 0$ as an element of $\mathcal{K}_{\vartheta'}$.  

For a given $n \in \mathbb{N}$, we set $\epsilon = (\vartheta' - \vartheta)/2n$ and $\alpha^l = \vartheta + l\epsilon$, $l = 0, \ldots, 2n$. Then we repeatedly apply (4.22) and obtain

$$u_t = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} S_{\vartheta'q_{2n-1}}(t - t_1) \left( C_{\vartheta'q_{2n-1}} + D_{\vartheta'q_{2n-1}} \right) \times \cdots \times S_{\vartheta q_1}(t_{n-1} - t_n) \left( C_{\vartheta q_1} + D_{\vartheta q_1} \right) u_{t_n} dt_n \cdots dt_1.$$ 

Like in (4.21), we then get from the latter

$$\|u_t\|_{\vartheta'} \leq \frac{t^n}{n!} \prod_{l=1}^n \|C_{\vartheta'q_{2l-1}} + D_{\vartheta'q_{2l-1}}\| \sup_{s \in [0,t]} \|u_s\|_{\vartheta'}$$

$$\leq \frac{1}{n!} \left( \frac{n}{\epsilon} \right)^n \left( \frac{2t(\omega + 2\langle b \rangle)e^{\vartheta'}}{\vartheta' - \vartheta} \right)^n \sup_{s \in [0,t]} \|v_s\|_{\vartheta'}.$$ 

This implies that $u_t = 0$ for $t < (\vartheta' - \vartheta)/2(\omega + 2\langle b \rangle)e^{\vartheta'}$. To prove that $u_t = 0$ for all $t$ of interest one has to repeat the above procedure appropriate number of times.

**Acknowledgment**

The present research was supported by the European Commission under the project STREVCOMS PIRSES-2013-612669.

**References**

[1] S. Albeverio, Yu. G. Kondratiev and M. Röckner, Analysis and geometry on configuration spaces, *J. Func. Anal.* 154 (1998), 444–500.

[2] J. Barańska and Yu. Kozitsky, Free jump dynamics in continuum, *Contemporary Mathematics* 653 (2015) 13–23.

[3] J. Barańska and Yu. Kozitsky, The global evolution of states of a continuum Kawasaki model with repulsion, Preprint [arXiv:1509.02044](http://arxiv.org/abs/1509.02044) 2016.

[4] J. Barańska and Yu. Kozitsky, A Widom-Rowlinson jump dynamics in the continuum, Preprint [arXiv:1604.07735](http://arxiv.org/abs/1604.07735) 2016.

[5] Ch. Berns, Yu. Kondratiev, Yu. Kozitsky and O. Kutoviy, Kawasaki dynamics in continuum: Micro- and mesoscopic descriptions, *J. Dyn. Diff. Equat.* 25 (2013), 1027–1056.

[6] Ch. Berns, Yu. Kondratiev and O. Kutoviy, Markov jump dynamics with additive intensities in continuum: state Evolution and mesoscopic scaling, *J. Stat. Phys.* 161 (2015), 876–901.

[7] Yu. Kondratiev and Yu. Kozitsky, The evolution of states in a spatial population model, *J. Dyn. Diff. Equat.* (2016); DOI 10.1007/s10884-016-9526-6.

[8] Yu. Kondratiev and T. Kuna, Harmonic analysis on configuration space. I. General theory, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 5 (2002), 201–233.

[9] Yu. Kozitsky, Dynamics of spatial logistic model: finite systems, in: J. Banasiak, A. Bobrowski, M. Lachowicz (Eds.), Semigroups of Operators – Theory and Applications: Beedlewo, Poland, October 2013. Springer Proceedings in Mathematics & Statistics 113, Springer 2015, pp. 197–211.

[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[11] D. Ruelle, Statistical Mechanics: Rigorous Results, W. A. Benjamin, Inc., 1969.
[12] H. R. Thieme and J. Voigt, Stochastic semigroups: their construction by perturbation and approximation, in: M. R. Weber and J. Voigt (Eds.), Positivity IV – Theory and Applications, Tech. Univ. Dresden, Dresden, 2006, pp. 135–146.

Instytut Matematyki, Uniwersytet Marii Curie-Skłodowskiej, 20-031 Lublin, Poland

E-mail address: jkozi@hektor.umcs.lublin.pl