Thin-shell wormholes in dilaton gravity

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Abstract

In this work we construct charged thin-shell Lorentzian wormholes in dilaton gravity. The exotic matter required for the construction is localized in the shell and the energy conditions are satisfied outside the shell. The total amount of exotic matter is calculated and its dependence with the parameters of the model is analysed.

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1 Introduction

Since the leading paper by Morris and Thorne [1] the study of traversable Lorentzian wormholes has received considerable attention. These objects are solutions of the equations of gravitation that have two regions (of the same universe or may be of two separate universes [1, 2]) connected by a throat. For static wormholes, the throat is defined as a two-dimensional hypersurface of minimal area that must satisfy a flare-out condition [3]. All traversable wormholes include exotic matter, which violates the null energy condition (NEC) [1, 2, 3, 4]. Interesting discussions about the energy conditions and wormholes are given in the essays by Barceló and Visser [5] and by Roman [6]. Recently, there has been a growing interest in quantifying the amount of exotic matter present around the throat. Visser et al. [7] showed that the exotic matter can be made infinitesimally small by appropriately choosing the geometry of the wormhole, and Nandi et al. [8] proposed a precise volume integral quantifier for the average null energy condition (ANEQ) violating matter. This quantifier was subsequently used by Nandi and Zhang [9] as a test of the physical viability of traversable wormholes. A well studied class of wormholes are thin-shell ones, which are constructed by cutting and pasting two manifolds [2, 10] to form a geodesically complete new one with a throat placed in the joining shell. Thus, the exotic matter needed to build the wormhole is located at the shell and the junction-condition formalism is used for its study. Poisson and Visser [11] made a linearized stability analysis under spherically symmetric perturbations of a thin-shell wormhole constructed by joining two Schwarzschild geometries. Later, Eiroa and Romero [12] extended the linearized stability analysis to Reissner–Nordström thin-shell geometries, and Lobo and Crawford [13] to wormholes with a cosmological constant. Lobo, with the intention of minimizing the exotic matter used, matched a static and spherically symmetric wormhole solution to an exterior vacuum

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solution with a cosmological constant, and he calculated the surface stresses of the resulting shell \[14\], and the total amount of *exotic matter* using a volume integral quantifier \[15\]. Cylindrically symmetric thin-shell wormhole geometries associated to gauge cosmic strings have also been treated by the authors of the present work \[16\].

In this article we study spherical thin-shell wormholes in dilaton gravity, that is, wormholes constructed by cutting and pasting two metrics corresponding to a charged black hole which is a solution of low energy string theory, with dilaton and Maxwell fields, but vanishing antisymmetric field and all other gauge fields set to zero \[17\] \[18\] \[19\]. Lorentzian four dimensional traversable wormholes in the context of low energy effective string theory or in Einstein gravity with a massless scalar field were already proposed by Kim and Kim \[20\], Vollick \[21\], Barceló and Visser \[22\], Bronnikov and Grinyok \[23\], Armendáriz-Picón \[24\], Graf \[25\], Nandi and Zhang \[26\], and Nandi et al. \[27\], but, differing from our construction, in these works the energy conditions were violated in a non compact region. Moreover, as we consider a non vanishing charge, this allows for a comparison with the thin-shell wormhole associated to the Reissner–Nordström geometry \[12\]. We focus on the geometry of these objects and we do not intend to give any explanation about the mechanisms that might supply the *exotic matter* to them; however, we shall analyse in detail the dependence of the total amount of *exotic matter* with the parameters of the model. Thin-shell charged dilaton wormholes are constructed in Sec. 2, the energy conditions are studied in Sec. 3 and the results obtained are summarized in Sec. 4. Throughout the paper we use units such as \( c = G = 1 \).

## 2 Thin-shell wormholes in low energy string gravity

Following Ref. \[17\], here we shall consider low energy string theory with Maxwell field, but with all other gauge fields and antisymmetric field set to zero \[28\]. Then the effective theory includes three fields: the spacetime metric \( g_{\mu\nu} \), the (scalar) dilaton field \( \phi \), and the Maxwell field \( F^{\mu\nu} \). The corresponding action in the Einstein frame \[29\] reads

\[
S = \int d^4x \sqrt{-g} \left( -R + 2(\nabla \phi)^2 + e^{-2b \phi} F^2 \right),
\]

where \( R \) is the Ricci scalar of spacetime. We have also included a parameter \( b \) in the coupling between the dilaton and the Maxwell field, as is done in Ref. \[17\] to allow for a more general analysis; this parameter will be restricted within the range \( 0 \leq b \leq 1 \). The interpretation of the gravitational aspects of the theory is clear in the Einstein frame: In particular, the condition \( \delta S = 0 \) imposed on the action \[11\] leads to the Einstein equations with the dilaton and the Maxwell field as the source. Within the context of our work, this is of particular interest, as it allows for a comparison with the results already known for Einstein–Maxwell theory. The field equations yielding from the action \[11\] are

\[
\nabla_\mu \left( e^{-2b \phi} F^{\mu\nu} \right) = 0,
\]

\[
\nabla^2 \phi + \frac{b}{2} e^{-2b \phi} F^2 = 0,
\]

\[
R_{\mu\nu} = 2 \nabla_\mu \phi \nabla_\nu \phi + 2e^{-2b \phi} \left( F_{\mu\alpha} F^{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F^2 \right).
\]
These equations admit black hole spherically symmetric solutions, with metric in curvature (Schwarzschild) coordinates \[ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + h(r)(d\theta^2 + \sin^2 \theta d\varphi^2),\] where

\[f(r) = \left(1 - \frac{A}{r}\right)\left(1 - \frac{B}{r}\right)^{\frac{1-b^2}{1+b^2}},\]
\[h(r) = r^2\left(1 - \frac{B}{r}\right)^{\frac{2b^2}{1+b^2}}.\] The constants \(A, B\) and the parameter \(b\) are related with the mass and charge of the black hole by

\[M = \frac{A}{2} + \left(1 - \frac{b^2}{1+b^2}\right)\frac{B}{2},\]
\[Q = \sqrt{\frac{AB}{1+b^2}}.\] In the case of electric charge, the electromagnetic field tensor has non-null components \(F_{tr} = -F_{rt} = Q/r^2\), and the dilaton field is given by \(e^{2\phi} = (1 - B/r)^{2b/(1+b^2)}\), where the asymptotic value of the dilaton \(\phi_0\) was taken as zero. For magnetic charge, the metric is the same, with the electromagnetic field \(F_{\theta\varphi} = -F_{\varphi\theta} = Q\sin \theta\) and the dilaton field obtained replacing \(\phi\) by \(-\phi\). When \(b = 0\), which corresponds to a uniform dilaton, the metric reduces to the Reissner–Nordström geometry, while for \(b = 1\), one obtains \(f(r) = 1 - 2M/r, h(r) = r^2\left[1 - Q^2/(Mr)\right]\). In what follows, we shall consider the generic form of the metric \([4]\), with \(0 \leq b \leq 1\). \(B\) and \(A\) are, respectively, the inner and outer horizons of the black hole; while the outer horizon is a regular event horizon for any value of \(b\), the inner one is singular for any \(b \neq 0\).

From the geometry given by Eqs. \([5]\) and \([6]\) we take two copies of the region with \(r \geq a\): \(\mathcal{M}^\pm = \{x/r \geq a\}\), and paste them at the hypersurface \(\Sigma \equiv \Sigma^\pm = \{x/F(r) = r - a = 0\}\). We consider \(a > A > B\) to avoid singularities and horizons. Because \(h(r)\) is an increasing function of \(r\), the flare-out condition is satisfied. The resulting construction creates a geodesically complete manifold \(\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-\) with two asymptotically flat regions connected by the throat. On this manifold we can define a new radial coordinate \(l = \pm \int_a^r g_{rr}dr\) representing the proper radial distance to the throat, which is located at \(l = 0\); the plus and minus signs correspond, respectively, to \(\mathcal{M}^+\) and \(\mathcal{M}^-\). The cut and paste procedure follows the standard Darmois-Israel formalism \([30, 31]\). Working in the orthonormal basis \(\{e_i, e_r, e_{\theta}, e_{\varphi}\}\) \((e_i = [f(r)]^{-1/2}e_t, e_r = [f(r)]^{1/2}e_r, e_{\theta} = [h(r)]^{-1/2}e_\theta, e_{\varphi} = [h(r)\sin^2 \theta]^{-1/2}e_\varphi\)\), we have that the second fundamental forms (extrinsic curvature) associated with the two sides of the shell are:

\[K^\pm_{\theta\theta} = K^\pm_{\varphi\varphi} = \pm \frac{h'(a)}{2h(a)}\sqrt{f(a)},\] and

\[K^\pm_{tt} = \pm \frac{f'(a)}{2\sqrt{f(a)}},\] where a prime stands for a derivative with respect to \(r\). Defining \([K_{ij}] \equiv K^+_{ij} - K^-_{ij}\), \(K = tr[K_{ij}]\) and introducing the surface stress-energy tensor \(S_{ij} = \text{diag}(\sigma, p_{\theta}, p_{\varphi})\), the Einstein equations on the
shell (Lanczos equations) have the form: $- [K_{ij}] + Kg_{ij} = 8\pi S_{ij}$, which in our case give an energy density $\sigma$ and a transverse pressure $p_t = p_\theta = p_\phi$:

$$\sigma = -\frac{1}{4\pi} \frac{h'(a)}{h(a)} \sqrt{f(a)},$$

$$p_t = \frac{\sqrt{f(a)}}{8\pi} \left[ \frac{f'(a)}{f(a)} + \frac{h'(a)}{h(a)} \right].$$

Using the explicit form (6) of the metric, we obtain

$$\sigma = -\frac{1}{2\pi a^2} \left( 1 - \frac{A}{a} \right)^{1/2} \left( 1 - \frac{B}{a} \right)^{1/2} \left[ \frac{b^2 B^2}{2 + 2\pi} \left[ 2a + A \left( 1 - \frac{A}{a} \right)^{-1} + B \left( 1 - \frac{B}{a} \right)^{-1} \right] \right],$$

$$p_t = \frac{1}{8\pi a^2} \left( 1 - \frac{A}{a} \right)^{1/2} \left( 1 - \frac{B}{a} \right)^{1/2} \left[ \frac{b^2 B^2}{2 + 2\pi} \left[ 2a + A \left( 1 - \frac{A}{a} \right)^{-1} + B \left( 1 - \frac{B}{a} \right)^{-1} \right] \right].$$

For $b = 0$ we recover the energy density and pressure corresponding to the thin-shell wormhole associated with the Reissner–Nordström geometry, obtained in Ref. [12] (static case). A distinctive feature of wormhole geometries is the violation of the energy conditions, which is studied in detail in the next section.

### 3 Energy conditions

The weak energy condition (WEC) states that $T_{\mu\nu} U^\mu U^\nu \geq 0$ for all timelike vectors $U^\mu$, and in an orthonormal basis it can be put in the form $\rho \geq 0$ and $\rho + p_j \geq 0 \forall j$, with $\rho$ the energy density and $p_j$ the principal pressures. If the WEC is satisfied, the local energy density will be positive for any timelike observer. The WEC implies by continuity the null energy condition (NEC), defined by $T_{\mu\nu} k^\mu k^\nu \geq 0$ for any null vector $k^\mu$, which in an orthonormal frame takes the form $\rho + p_j \geq 0 \forall j$. In the general case of the wormhole constructed above we have from Eqs. (12) and (13), as expected, $\sigma < 0$ and $\sigma + p_r < 0$, that is a shell of exotic matter which violates WEC and NEC. The sign of $\sigma + p_t$ depends on the values of the parameters.

Outside the shell, starting from the field equations (11) in the form $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$, the stress-energy tensor can be expressed as a sum of two terms: $T_{\mu\nu} = T_{\mu\nu}^{\text{dil}} + T_{\mu\nu}^{\text{EM}}$, where $T_{\mu\nu}^{\text{dil}} = \left[ \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right]/(4\pi)$ is the part corresponding to the dilaton field, and $T_{\mu\nu}^{\text{EM}} = e^{-2b\phi} \left( F_{\mu\alpha} F^\alpha_{\nu} - \frac{1}{4} g_{\mu\nu} F^2 \right)/(4\pi)$ to the electromagnetic field. Then, using the orthonormal basis defined above, the energy density $\rho = T_{tt}$, the radial pressure $p_r = T_{\phi\phi}$ and the transverse pressure $p_t = p_\theta = p_\phi = T_{\phi\phi} = T_{\theta\theta}$, for the dilaton field are given by

$$\rho^{\text{dil}} = \rho_t^{\text{dil}} = -p_t^{\text{dil}} = \frac{b^2 B^2}{8\pi (1 + b^2) r^4} \left( 1 - \frac{A}{r} \right) \left( 1 - \frac{B}{r} \right)^{\frac{1 + 3b^2}{1 + b^2}},$$

and for the electromagnetic field

$$\rho^{\text{EM}} = \rho_t^{\text{EM}} = -\rho_r^{\text{EM}} = \frac{AB}{8\pi (1 + b^2) r^4} \left( 1 - \frac{B}{r} \right)^{-\frac{2b^2}{1 + b^2}}.$$
Figure 1: Total amount of exotic matter on the shell as a function of the dilaton coupling parameter $b$. The curves corresponding to different values of charge are shown for two throat radii.

Because our construction yields a geodesically complete space where $r > a$ and we have imposed $a > A > B$ (so that there are no horizons), we have $\rho^{\text{dil}} > 0$, $\rho^{\text{dil}} + p^{\text{dil}}_r > 0$, $\rho^{\text{dil}} + p^{\text{dil}}_t = 0$, $\rho^{\text{EM}} > 0$, $\rho^{\text{EM}} + p^{\text{EM}}_r > 0$ and $\rho^{\text{EM}} + p^{\text{EM}}_t = 0$, so the NEC and WEC are satisfied outside the shell for both the dilaton and the electromagnetic fields, and therefore the exotic matter is localized only in the shell.

The total amount of exotic matter present can be quantified, following Ref. [8], by the integrals

$$
\Omega = \int \rho \sqrt{-g} \, d^3x, \\
\Omega = \int (\rho + p) \sqrt{-g} \, d^3x,
$$

where $g$ is the determinant of the metric tensor. The usual choice as the most relevant quantifier is $\Omega = \int (\rho + p_r) \sqrt{-g} \, d^3x$; in our case, introducing in $\mathcal{M}$ a new radial coordinate $R = \pm (r - a)$ ($\pm$ for $\mathcal{M}^\pm$, respectively)

$$
\Omega = 2\pi \int_0^\pi \int_0^\infty (\rho + p_r) \sqrt{-g} \, d\mathcal{R} d\theta d\varphi.
$$

(16)

Taking into account that the exotic matter only exerts transverse pressure and it is placed in the shell, so that $\rho = \delta(R) \sigma$ ($\delta$ is the Dirac delta), $\Omega$ is given by an integral of the energy density $\sigma$ over the shell:

$$
\Omega = 2\pi \int_0^\pi \int_0^\infty \sigma \sqrt{-g} \big|_{r=a} \, d\theta d\varphi = 4\pi h(a) \sigma,
$$

(17)

and using Eq. (12), we have

$$
\Omega = -2 \left[ a \left( 1 - \frac{B}{a} \right) \frac{1 + b^2}{2 + 2b^2} \right] + \frac{b^2 B}{1 + b^2} \left( 1 - \frac{B}{a} \right) \frac{1 + b^2}{2 + 2b^2} \left( 1 - \frac{A}{a} \right)^{\frac{1}{2}}.
$$

(18)

In Fig. 1, $\Omega/M$ is plotted as a function of the parameter $b$ for different values of the charge and throat radius. We observe that: (i) For stronger coupling between the dilaton and the Maxwell fields (greater $b$), less exotic matter is required for any given values of the radius of the throat, mass and charge. (ii) For large values of the parameter $b$ (closer to 1), the amount of exotic matter for given radius and mass is reduced by increasing the charge, whereas the behaviour of $\Omega/M$ is the opposite for small values of $b$. It can be noticed from Eq. (18), that for $a \gg A$ there is an approximately linear dependence of $\Omega$ with the radius $a$ ($\Omega \approx -2a$), and when $a$ takes values close
to $A$, $\Omega$ can be approximated by

$$\Omega \approx -\frac{2}{\sqrt{A}} \left[ A \left( 1 - \frac{B}{A} \right)^{\frac{1+b^2}{2+2b^2}} + \frac{b^2 B}{1+b^2} \left( 1 - \frac{B}{A} \right)^{-\frac{1+b^2}{2+2b^2}} \right] \sqrt{a - A}. \quad \text{(19)}$$

The last equation shows a way to minimize the total amount of exotic matter of the wormhole. For a given value of the coupling parameter $b$ and arbitrary radius of the throat $a$, the exotic matter needed can be reduced by taking $A$ close to $a$. The values of mass and charge required are then determined by Eq. (7). For $A$ close to $a$ we have $\sigma + p_t > 0$, with a very small energy density and a high value of the transverse pressure. A consequence of reducing the amount of exotic matter is having a high pressure at the throat.

4 Conclusions

In this work, we have constructed a charged thin-shell wormhole in low energy string gravity. We found the energy density and pressure on the shell, and in the case of null dilaton coupling parameter, recovered the results obtained in Ref. [12]. The exotic matter is localized, instead of what happened in previous papers on dilatonic wormholes, where the energy conditions were violated in a non compact region. Accordingly to the proposal of Nandi and Zhang [9], that the viability of traversable wormholes should be linked to the amount of exotic matter needed for their construction, here we have analysed the dependence of such amount with the parameters of the model. We found that less exotic matter is needed when the coupling between the dilaton and Maxwell fields is stronger, for given values of radius, mass and charge. Besides, we obtained that the amount of exotic matter is reduced, when the dilaton-Maxwell coupling parameter is close to unity, by increasing the charge; while for low values of the coupling parameter, this behaviour is inverted. We have also shown how the total amount of exotic matter can be reduced for given values of the dilaton-Maxwell coupling and the wormhole radius, by means of a suitable choice of the parameters.

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