Real Special Geometry*

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Abstract

We give a coordinate–free description of real manifolds occurring in certain four–dimensional supergravity theories with antisymmetric tensor fields. The relevance of the linear multiplets in the compactification of string and five–brane theories is also discussed.

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In four–dimensional supergravity theories scalar fields are usually embedded in linear\cite{1} or chiral multiplets, depending on whether antisymmetric tensor fields are present or not. Examples of theories with antisymmetric tensor fields naturally occur in Kaluza–Klein compactifications of certain D–dimensional supergravity theories with $D \geq 6$, or in the low energy limit of some string and p–brane compactifications. In string theory, classical examples of linear multiplets $(L, B_{\mu\nu}, \chi)$ are the dilaton multiplet in heterotic strings and the Kähler class moduli multiplets in "dual theories" such as five–brane compactifications \cite{2,4}. However, while massless chiral multiplets with an associated continuous Peccei–Quinn symmetry are classically equivalent to linear multiplets through a “duality transformation” \cite{5–7}, this may be not so at the quantum level, due to the violation of the Peccei–Quinn symmetry by quantum effects. The latter comprehend both $\alpha'$ string corrections, i.e. quantum effects on the world-sheet sigma–model, and non perturbative effects in the string coupling constant, such as Yang–Mills and gravitational instantons \cite{8–10}. On the other hand, in supersymmetric compactifications, while $\alpha'$ corrections are expected to be relevant only for configurations for which the moduli $v.e.v.'s$ are of the order of the string scale, the breaking of the space–time axion symmetry may be relevant at a much lower scale, especially if we expect that it plays a rôle in the supersymmetry breaking mechanism and in a non–trivial effective dilaton potential which stabilizes the dilaton field.

Therefore it is of interest, also for physical applications, to treat the moduli fields as “classical” and the dilaton field as having a non–trivial superpotential. This is the natural choice in the framework of five–brane theories\cite{2} when the Kähler moduli are associated to linear multiplets \cite{3,4} and the dilaton (with its pseudoscalar partner) to a chiral multiplet.

In this letter we show that under such circumstances there is a new sigma–model geometry associated to the Kähler class moduli fields $y^i$ which we call “real special geometry”. Indeed, for particular couplings of the moduli fields (but not in general), it is related to special Kähler geometry of Calabi–Yau compactifications\cite{11,12}. Like special geometry, in a coordinate free description real geometry is characterized by a condition on the curvature tensor

$$R_{ijkl} = C^m_{i[k} C_{l]mj}$$

(1)
where $C_{ijk}$ ($i = 1, \ldots, n$) is a completely symmetric tensor. In a particular set of coordinates $L^I(y)$, it turns out that

$$
C_{IJK} = \partial_I \partial_J \partial_K F(L^I) \quad G_{IJ} = \partial_I \partial_J F(L^I),
$$

where $F(L^I)$ is a real function of the scalar fields and $G_{IJ}$ is the metric tensor. If $C_{IJK}$ is constant, then this geometry is related to the geometry occurring in five-dimensional supergravity or to the special geometry of Calabi–Yau moduli spaces

$$
C_{IJK} = d_{IJK}, \quad F = d_{IJK}(T + \overline{T})^I(T + \overline{T})^J(T + \overline{T})^K,
$$

where $d_{IJK}$ are the intersection numbers of the Calabi–Yau manifold, $T^I$ are the moduli fields of the Kähler class ($I = 1, \ldots, h_{1,1}$) and $F$ is its “volume”.

If the $C_{IJK}$ are not constant, the geometry is not related to special Kähler geometry, but it is still described by the above curvature condition.

We now turn to the derivation of real special geometry. The procedure followed in the sequel is closely analogous to the derivation of a coordinate free description of special geometry in $D = 4, N = 2$ supergravity[12][13].

We introduce a set of $n$ self-interacting linear multiplets $(L^I(y), \chi^I, B^I_{\mu\nu})$, $I = 1, \ldots, n$ where $L^I(y)$ are scalar fields functions of the coordinates $y^I(x^\mu)$ of the sigma–model manifold $\mathcal{M}_n$ ($x^\mu$ being the four–dimensional space–time coordinates). $\chi^I(x)$ and $B^I_{\mu\nu}(x)$ are the dilatino and axion fields. We then promote the space–time multiplet to a superspace multiplet and we introduce the supervielbein basis on superspace $(V^a, \Psi)$ where $V^a$, $a = 1, \ldots, 4$ is the usual vielbein and $\Psi$ is the gravitino one–form. For our present purposes it is sufficient to work in global supersymmetry. In this case, the vielbein $V^a$ and the space–time spin connection satisfy the zero curvature conditions

$$
T^a = DV^a - i\overline{\Psi} \gamma^a \Psi = 0 \quad DV^a = dV^a - \omega^a_b \wedge V^b
$$

$$
R^{ab} = d\omega^{ab} - \omega^a_c \wedge \omega^c_b.
$$

We also introduce on $\mathcal{M}_n$ a basis of “internal” supervielbein $E^A = E^A_a dy^a$ such that

$$
E^A = E^A_a V^a + \overline{\Psi} \chi^A.
$$
where the supercovariant field strength $E_a^A$ is defined as

$$E_a^A = (E^A_{\mu} - \overline{\Psi}_{\mu} \chi^A)V^\mu_a, \quad E^A_{\mu} = E^A_i \frac{\partial y^i}{\partial x^\mu}$$

and $\chi^A$ is related to $\chi^I$ by some set of covariant vectors $f^I_A(y)$:

$$\chi^I = f^I_A \chi^A, \quad \chi^A = f^A_I \chi^I \quad (f^A_I = (f^I_A)^{-1})$$

Together with $E^A$, which may be thought as the curvature of the $y^i$ fields, we define the curvatures of $\chi^A$ and $B^I$ as follows

$$\nabla \chi^A \equiv d\chi^A - \Omega^A_B \chi^B, \quad H^I \equiv dB^I + iL^I(y)\overline{\Psi} \gamma_a \Psi^a,$$

where $\Omega^A_B$ is the spin connection on $\mathcal{M}_n$ and $H^I$ is a three-form on superspace.

The parametrizations of $\nabla \chi^A$ and $H^I$ are easily found from the Bianchi identities $\nabla E^A = dH^I = 0$, and they read

$$\nabla \chi^A = \nabla_a \chi^A V^a + \frac{1}{4} C^A_{BC}(\overline{\chi}^B \chi^C + \gamma_5 \overline{\chi}^B \gamma_5 \chi^C)\Psi$$

$$+ \left[ \frac{1}{4} h^I_a f^A_i + \frac{i}{2} E^A_a - \frac{1}{4} C^A_{BC} \overline{\chi}^B \gamma_5 \gamma_a \chi^C \gamma_5 \gamma_5 \Psi \right] \gamma^a \Psi$$

$$H^I = H^I_{abc} V^a V^b V^c + f^I_A \overline{\chi}^A \gamma_{ab} \Psi V^a V^b$$

with

$$f^I_A = E^i_A \partial_i L^I$$

and

$$C^A_{BC} = f^A_I \nabla_B f^I_C$$

symmetric in $BC$.

We note that the further Bianchi identity

$$\nabla^2 \chi^A \equiv R^A_B \chi^B,$$

where $R^A_B$ is the curvature two–form, is a true identity since the linear multiplet gives an off-shell representation of supersymmetry without auxiliary fields. For this reason, no constraint on the geometry can be extracted from (12). However, geometric
constraints can be most naturally obtained by constructing the superspace lagrangian in the geometric framework\textsuperscript{[13]}, which, in contrast to tensor calculus techniques\textsuperscript{[14]}, gives a lagrangian which is covariant under reparametrizations of the scalar fields manifold. Applying the set of rules of \textsuperscript{[13]}, one finds

\[ L = \text{const} \times \left\{ \tilde{E}_a^A (E^A - \bar{\chi}_a^A \Psi - \bar{\chi}_a^A \Psi) V^b V^c V^d \epsilon_{abcd} \right. \]

\[ \left. - \frac{1}{8} \tilde{E}_a^A \tilde{E}_I^A V^a V^b V^c V^d \epsilon_{abcd} - i(\bar{\chi}_a^A \gamma_a \nabla \chi^A + \bar{\chi}_a^A \gamma_a \nabla \chi^A) V^b V^c V^d \epsilon_{abcd} \right. \]

\[ \left. - \frac{3}{2} F_{IJ} \tilde{h}_c^I [H^J - f_J^I(\bar{\chi}_a^A \gamma_{ab} \Psi + \bar{\chi}_a^A \gamma_{ab} \Psi) V^a V^b] V^c \right. \]

\[ + \frac{1}{32} \tilde{h}_a^I \tilde{h}_c^J F_{IJ} V^a V^b V^c V^d \epsilon_{abcd} + 3i E^A(\bar{\chi}_a^A \gamma_{ab} \Psi - \bar{\chi}_a^A \gamma_{ab} \Psi) V^a V^b \]

\[ - 3i f_{IA} H^I(\bar{\chi}_a^A \Psi - \bar{\chi}_a^A \Psi) - 6C_{ABC} f_J^A H^I \bar{\chi}_a^B \gamma_a \chi^C V^a \]

\[ + 2i C_{ABC}(\bar{\chi}_a^A \gamma_{ab} \Psi + \bar{\chi}_a^A \gamma_{ab} \Psi)(\bar{\chi}_a^B \chi^B + \bar{\chi}_a^A \chi^B) V^a V^b V^c \epsilon_{abcd} \right. \]

\[ \left. + U_{ABCD} \bar{\chi}_a^A \chi^C \bar{\chi}_a^B \chi^D V^a V^b V^c V^d \epsilon_{abcd} \right\}, \tag{13} \]

where we have used chiral formalism for a generic spinor field \( \lambda \),

\[ \lambda = \frac{1 + \gamma_5}{2} \lambda \quad \lambda^c = \frac{1 - \gamma_5}{2} \lambda. \tag{14} \]

Above, \( \tilde{E}_a^A \) and \( \tilde{h}_c^I \) are auxiliary first order fields which are identified through their equations of motion with the physical components (along the vielbein) of \( E^A \) and \( H^I \)

\[ \tilde{E}_a^A = E_a^A \]

\[ \tilde{h}_c^I = h_a^I \equiv \epsilon_{abcd} H^I_{bcd}, \tag{15} \]

\( F_{IJ} \) is a function of the scalar fields \( L^I(y(x)) \) which appears in the kinetic term of the axion fields, \( f_{IA} = \delta_{AB} f^B_I \) and \( U_{ABCD} \) is a four–index tensor so far undetermined. The superspace equations of motion along the outer directions (\textit{i.e.} projected on \( p \)-forms containing at least one \( \Psi \)) yield that the scalar field functions satisfy

\[ \delta_{AB} = F_{IJ} f_J^A f^I_B \]

\[ C_{ABC} \text{ totally symmetric} \tag{16} \]

as well as all the numerical coefficients in (13). The form of the tensor \( U_{ABCD} \), being a term proportional to the space–time volume element, cannot be retrieved by an
outer projection, but rather it is determined by a supersymmetry transformation on the lagrangian in a particular sector. One finds

$$U_{ABCD} = R_{ABCD} - \frac{3}{2} C^M_{BC} C_{AD} + \nabla_D C_{ABC}$$

(17)

Using these results, we can take the restriction of (13) to space–time and obtain the four–dimensional supersymmetric lagrangian for the component fields

$$\mathcal{L} = \text{const} \sqrt{-g} \left\{ E^A E^A + 2i (\overline{x}^A \nabla x^A + \overline{x}^A \nabla x^A) \right. \right.$$  

$$- \frac{1}{4} F_{IJ} h^I h^J - 2E^A (\overline{x}^A \psi^A + \overline{x}^A \psi^A) +$$  

$$+ 2E^A (\overline{x}^A \gamma^A \psi^A + \overline{x}^A \gamma^A \psi^A) +$$  

$$i F_{IJ} h^I h^J A \left[ \overline{x}^A \psi^A - \overline{x}^A \psi^A - (\overline{x}^A \gamma^A \psi^A - \overline{x}^A \gamma^A \psi^A) \right]$$  

$$+ \frac{i}{2} C_{ABC} = \frac{1}{8} U_{ABCD} \overline{x}^A \overline{x}^B \overline{x}^C \overline{x}^D \right\},$$

(18)

where components along the vielbein of the various (super)covariant field–strengths have been substituted using the parametrizations (5), (9).

Let us exploit the consequences of eqs.(16). By differentiating ($\partial_C = E^i_C \frac{\partial}{\partial y^i}$) eq.(16) and using (11) we find

$$0 = \partial_C F_{IJ} f^I_A f^J_B + 2F_{IJ} \nabla_C f^I_A f^J_B$$

$$= \frac{\partial F_{IJ}}{\partial L^I} f^I_A f^J_B f^K_C + 2F_{IJ} C^D_{CA} f^I_D f^J_B$$

$$= \frac{\partial F_{IJ}}{\partial L^I} f^I_A f^J_B f^K_C + 2C_{ABC},$$

(19)

On the other hand, the first of eqs. (16) also implies

$$\frac{\partial F_{IJ}}{\partial L^K} = \frac{\partial^2 F_I}{\partial L^J \partial L^K} = \frac{\partial^3 F}{\partial L^I \partial L^J \partial L^K},$$

(20)

hence

$$C_{ABC} = \frac{1}{2} \frac{\partial^3 F}{\partial L^I \partial L^J \partial L^K} f^I_A f^J_B f^K_C.$$  

(21)

By covariant differentiation on this equation and using again (11) we also find

$$\nabla_D C_{ABC} = - \frac{1}{2} F_{IJKL} f^I_A f^J_B f^K_C f^L_D - \frac{3}{2} F_{IJK} C^L_{AD} f^I_A f^J_B f^K_C$$

(22)

$$- 5 -$$
(with $F_{IJ...} \equiv \frac{\partial F}{\partial L^I \partial L^J ...}$), and thus
\[ \nabla_{[D} C_{A]BC} = 0 . \] (23)

From the definition (11) of $C_{ABC}$ one easily deduces
\[ 0 = \nabla_{[D} C_{A]BC} = R_{BCDA} - C^M_{B[D} C_{A]CM} \] (24)
and finally we obtain the geometric constraint on the curvature
\[ R_{ABCD} = C^M_{A[C} C_{D]BM} . \] (25)

Till now we have used the flat vielbein indices $A, B, \ldots$ in the internal manifold $\mathcal{M}_n$. If we take coordinate indices $i, j, \ldots$, then in special coordinates $y^i = L^I(y)$ we have
\[ f^I_i = \partial_i L^I = \delta^I_i \] (26)
and thus eqs.(16),(21) become
\[ g_{IJ} = \frac{\partial^2 F}{\partial L^I \partial L^J} \]
\[ C_{IJK} = -\frac{1}{2} \frac{\partial^3 F}{\partial L^I \partial L^J \partial L^K} \] (27)
while the constraint on the curvature becomes
\[ R_{IJKL} = \Gamma^M_{I[K} \Gamma_{L]JM} \] (28)
$\Gamma^I_{JK}$ being the Levi–Civita connection.

It is worth to observe that the above equation is exactly the same found in ref. [15] in the construction of the sigma–model of the scalar fields in $D = 5$ supergravity coupled to $N = 2$ supermultiplets. There, eq.(28) is obtained as a condition on the curvature of an $(n - 1)$–dimensional hypersurface $F = \text{const}$ embedded in an $n$–dimensional Riemanniann space $\tilde{\mathcal{M}}_n$ and choosing a particular coordinate system. When this theory is dimensionally reduced down to $D = 4$, a new scalar field $\sigma$ appears (from the fifth component of the vielbein) and the equation for $F$ becomes $F = \text{const} \, \sigma$. By varying $\sigma$, all the space $\tilde{\mathcal{M}}_n$ is covered so that it can be identified with $\mathcal{M}_n$, while the previously constrained function $F$ becomes actually free and can be identified with our function $F$.
We also remark that we have found the geometric characterization of $\mathcal{M}_n$ using only the supersymmetric selfinteraction of the linear multiplets without coupling to supergravity, that is, in a global supersymmetric approach. One could wonder whether such characterization would change in presence of supergravity, as it happens in special Kähler geometry\[12\]. There, the curvature constraint in absence of supergravity

$$R_{ijkl} = -C_{ikm}C_{jlm}g^{m\bar{n}} \quad (29)$$

changes to

$$R_{ijkl} = g_{i\bar{k}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} - C_{ikm}C_{jlm}g^{m\bar{n}} \quad (30)$$

when supergravity is turned on. In the above formulae, $C_{ijk} \equiv e^K W_{ijk}(z)$, where $W_{ijk}$ are the holomorphic Yukawa couplings and $K(z, \bar{z})$ is the Kähler potential. However, in the case under investigation, the constraint (25) is not changed in presence of supergravity. The change in special Kähler geometry is due to the fact that the Kähler manifold of the moduli fields in the globally supersymmetric case becomes a Kähler–Hodge manifold in presence of supergravity, i.e. it acquires the structure of an holomorphic $U(1)$–bundle with $U(1)$ connection given by $\partial_i K(z, \bar{z})$. It is such $U(1)$ gauging which triggers the presence of the extra terms in (30) with respect to (29). In the present case there is no superimposed $U(1)$ bundle structure and thus the constraint (25) remains unchanged. This has been verified\[16\] by the explicit coupling of supergravity (in the new minimal framework) to the lagrangian (13).

Finally, it is interesting to see what is the characterization of the special coordinates $y^i = L^I$ with respect to which the formulas (27) hold. It is easy to show that different sets of special coordinates are related by a duality transformation, that is by a Legendre transformation with generating function $F(L)\[4,14\]$. Infact, let

$$L^I \to \tilde{L}^I \equiv \frac{\partial F}{\partial L^I} \quad (31)$$

$$\tilde{F}(\tilde{L}) = L^I \frac{\partial F}{\partial L^I} - F .$$

Then $L^I = \frac{\partial \tilde{F}}{\partial \tilde{L}^I}$, and therefore

$$\tilde{F}_{IJ} \equiv \frac{\partial^2 \tilde{F}}{\partial \tilde{L}^I \partial \tilde{L}^J} = \frac{\partial L^I}{\partial \tilde{L}^J} \equiv (\frac{\partial \tilde{L}^I}{\partial L^J})^{-1} = (F^{-1})_{IJ} , \quad (32)$$
in agreement with the transformation law of the metric $g_{IJ} \equiv F_{IJ}$ in special coordinates

$$\tilde{g}_{KL} = g_{I\bar{J}} \frac{\partial L}{\partial \bar{L} I} \frac{\partial L}{\partial \bar{L} J}.$$ \hfill (33)

Note that the transformation (31) also implies

$$\partial_i L^I \rightarrow \frac{\partial^2 F}{\partial L^I \partial L^J} \partial L^J,$$ \hfill (34)

that is

$$f_i^I \rightarrow F_{IJ} f_i^J.$$ \hfill (35)

In this respect, eq. (31) is the analogous of $Sp(2n + 2)$ transformations relating different sets of special coordinates in special Kähler geometry [17].

A possible generalization of this framework, left to future work, is to include chiral and vector multiplets and the coupling to supergravity, which is of course needed in any study of five–brane compactifications.

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