On the connection between the solutions to the Dirac and Weyl equations and the corresponding electromagnetic four-potentials

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Abstract

In this work we study in detail the connection between the solutions to the Dirac and Weyl equations and the associated electromagnetic four-potentials. First, it is proven that all solutions to the Weyl equation are degenerate, in the sense that they correspond to an infinite number of electromagnetic four-potentials. As far as the solutions to the Dirac equation are concerned, it is shown that they can be classified into two classes. The elements of the first class correspond to one and only one four-potential, and are called non-degenerate Dirac solutions. On the other hand, the elements of the second class correspond to an infinite number of four-potentials, and are called degenerate Dirac solutions. Further, it is proven that at least two of these four-potentials are gauge-inequivalent, corresponding to different electromagnetic fields. In order to illustrate this particularly important result we have studied the degenerate solutions to the force-free Dirac equation and shown that they correspond to massless particles. We have also provided explicit examples regarding solutions to the force-free Weyl equation and the Weyl equation for a constant magnetic field. In all cases we have calculated the infinite number of different electromagnetic fields corresponding to these solutions. Finally, we have discussed potential applications of our results in cosmology, materials science and nanoelectronics.

Keywords: Dirac equation, Weyl equation, degenerate solutions, electromagnetic four-potentials, electromagnetic fields, massless particles

1. Introduction

The Dirac equation has been the first electron equation in quantum mechanics to satisfy the Lorentz covariance [1, 2], initiating the beginning of one of the most powerful theories ever formulated: the quantum electrodynamics. This equation predicted the spin and the magnetic moment of the electrons, the existence of antiparticles and was able to reproduce accurately the spectrum of the hydrogen atom. The Dirac equation plays an important role in various fields of Physics, as mathematical physics [3–5], particle physics [6–8], solid state physics [9, 10], astrophysics [11], quantum computing [12], nonlinear optics [13], etc. As it can be deduced from the cited articles the Dirac equation and its applications is still a hot topic of research.

The majority of the previously reported works focus on the determination of the wave function \( \Psi \) when the electromagnetic four-potential is given. However, the inverse problem, as formulated by Eliezer [14] is also quite interesting: ‘Given the wave function \( \Psi \), what can we say about the
electromagnetic potential \(A_{ij}\), which is connected to \(\Psi\) by Dirac’s equation? Is \(A_{ij}\) uniquely determined, and if not, what is the extent to which it is arbitrary? In his relevant work Eliezer found an expression for the magnetic vector \(A_{ij}\) and the electric scalar potential \(\phi\) as a function of \(\Psi\). In another important work by Radford et al [15], the Dirac equation is expressed in a 2-spinor form, which allows it to be (covariantly) solved for the electromagnetic four-potential, in terms of the wave function and its derivatives. This approach subsequently led to some physically interesting results [16–18] (for a review see [19]). Further, in [20] it was demonstrated that the Dirac equation is indeed algebraically invertible if a real solution for the four-potential is required. Namely, two expressions for the components \(A_{ij}\) of the electromagnetic four-potential are presented, equivalent to the one given in [15]. However, these expressions are not valid in the case that \(\Psi^{\gamma_0}\Psi, \Psi^{\gamma_1}\gamma_0\gamma_2\gamma_3\Psi\) are identically zero, where \(\gamma^\mu, \mu = 0, 1, 2, 3\), is the standard representation of the Dirac matrices, which are explicitly provided in the following section. Thus, in the above mentioned works, the existence of nonzero Dirac solutions satisfying the aforementioned conditions is not studied. Further, if it is assumed that these solutions exist, it should be investigated if one such solution corresponds to a unique real four-potential or not. Consequently, the inversion of the Dirac equation is an open problem, which will be fully solved in the present article.

The main results of this paper are the following. First, it is proven that all solutions to the Weyl equations are degenerate, in the sense that they correspond to an infinite number of electromagnetic four-potentials, which are explicitly calculated. This result is particularly important from a practical point of view as it is expected to offer new possibilities regarding the technological applications of certain materials as graphene sheets, Weyl semimetals, etc, where ordinary charged particles can collectively behave as massless [21–26].

Further, it is thoroughly proven that every Dirac spinor corresponds to one and only one mass. This result is utilized to show that the set of all Dirac spinors can be classified into two classes. The elements of the first class correspond to one and only one four-potential, and are called non-degenerate Dirac solutions, while the elements of the second class correspond to an infinite number of four-potentials, and are called degenerate Dirac solutions. In the first case the four-potential is fully defined as a function of the Dirac spinor, while in the second one explicit expressions are provided for the infinite number of four-potentials corresponding to the degenerate Dirac spinors. Further, it is proven that at least two of these four-potentials are gauge-inequivalent, corresponding to different electromagnetic fields. From a physical point of view this is quite a surprising result, meaning that particles described by a specific class of spinors can be in the same state under the influence of different electromagnetic fields.

In order to illustrate this particularly important result we study the degeneracy condition for the force-free Dirac equation and found that it is satisfied for massless particles. Thus, a massless Dirac particle can exist in the same state both in the absence of any electromagnetic field as well as under the influence of an infinite number of different electromagnetic fields, which are explicitly calculated. This remarkable result could be used to explain the rapid evolution of the Universe during the first stages of its creation, before the particles acquire mass [27, 28]. We also show that the degeneracy can be extended to massive particles if particle–antiparticle pairs are considered. Further, we provide explicit examples of Weyl spinors, and calculate the infinite number of different electromagnetic fields corresponding to these solutions. Finally, we show that the state of massless Dirac and Weyl particles does not change if a static or time-dependent electric field is applied parallel (or anti-parallel) to their direction of motion. This important result implies that Ohm’s law does not hold for massless Dirac and Weyl particles, and the current they transfer remains constant, unaffected by the applied electric field.

2. Preliminaries

The Dirac equation for a fermion of charge \(q\) and mass \(m\) described by the spinor wavefunction \(\Psi\) in the presence of an external electromagnetic four-potential \(A_{ij}\), can be written as [15]

\[
a_{ij}\gamma^\mu\Psi = -(i\gamma^\mu\partial_\mu\Psi - m\Psi), \tag{2.1}
\]

where \(a_{ij} = qA_{ij}, \gamma^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 & -a^0 \end{bmatrix}\) and \(\gamma^\mu = \begin{bmatrix} 0 & 0 \\ 0 & -a^\mu \end{bmatrix}\) \(\mu = 1, 2, 3\). Here it is assumed that we are working in the natural system of units where \(\hbar = c = 1\). We have also used the notation \((\sigma^0, \sigma^1, \sigma^2, \sigma^3) = (I_2, \sigma_x, \sigma_y, \sigma_z)\), where \(I_2\) is the two-dimensionnal identity matrix and \(\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) are the well-known Pauli matrices.

In the following we shall also use the two Weyl equations under the electromagnetic potential \(A_{ij}\), which can be written in the form [29],

\[
a_{ij}\sigma^\mu\psi = -i\sigma^\mu\partial_\mu\psi, \tag{2.2}
\]

and

\[
2a_{0}\sigma^0\psi - a_{ij}\sigma^\mu\psi = -(2i\sigma^0\partial_0\psi - i\sigma^\mu\partial_\mu\psi) \tag{2.3}
\]

describing massless particles with their spin parallel to their propagation direction (positive helicity) and anti-parallel to their propagation direction (negative helicity) respectively. In the rest of the article, we shall also assume that the electromagnetic four-potential is always real.

Definition 1. Any solution of equation (2.1) for a four-potential \(a_{ij}\) and a mass \(m\) will be called Dirac solution, and any solution of equation (2.2) or (2.4) for a four-potential \(a_{ij}\) will be called Weyl solution.

Definition 2. A Dirac solution \(\Psi\) is said to correspond to a mass \(m\), if there exists a four-potential \(a_{ij}\), such that \(\Psi\) is a solution of (2.1) for this mass \(m\).
**Definition 3.** A Dirac solution $\Psi$ is said to correspond to a four-potential $a_\mu$, if there exists a mass $m$, such that $\Psi$ is a solution of (2.1) for this four-potential $a_\mu$.

**Definition 4.** A Dirac solution $\Psi$ is said to correspond to an electromagnetic field $(E, B)$, if $\Psi$ corresponds to a four-potential $a_\mu$ associated to $(E, B)$.

**Definition 5.** A Weyl solution $\psi$ is said to correspond to a potential $a_\mu$, if $\psi$ is a solution of (2.2) or (2.4) for this four-potential $a_\mu$.

* We denote the following matrices

$$
\gamma^5 := i\gamma^0\gamma^2\gamma^3, \quad \gamma := \gamma^0 + \gamma^5.
$$

**Remark 2.1.** Since $\gamma^0$ is Hermitian and $\gamma^0\gamma^5$ is anti-Hermitian, we have that $\Psi'(0,0,0,0,0)$ is real and $\Psi'\gamma_0\gamma_5\Psi'$ is imaginary. Therefore the equation $\Psi'\gamma\Psi = 0$ is equivalent to $\Psi'\gamma^5\Psi = \Psi'\gamma_0\gamma_5\Psi = 0$.

In [20] two equivalent algebraic expressions for the four-potential in terms of the Dirac solution are derived:

$$
\alpha_\mu = \frac{1}{2} \left( \mp \gamma^\mu \dot{\Psi} = \gamma^\mu \frac{\psi}{\psi'} - 2m_j \right),
$$

(2.4)

$$
\alpha_\mu = \frac{1}{2} \left( \mp \gamma^\mu \dot{\Psi} = \gamma^\mu \frac{\psi}{\psi'} - 2m_j \right),
$$

(2.5)

where $\mp \gamma^\mu \dot{\Psi} = \gamma^\mu \frac{\psi}{\psi'}$ and $\mp \gamma^\mu \dot{\Psi} = \gamma^\mu \frac{\psi}{\psi'}$.

**Notation.** Let $f, g$ be any functions. Then $f \equiv g$ means that $f$ is identically equal to $g$.

**Remark 2.2.** Clearly, from equations (2.4), (2.5) and the remark 2.1 follows that if $\Psi'\gamma\Psi \neq 0$, then $\Psi$ corresponds to one and only one real four-potential $a_\mu$. In more detail if $\Psi'\neq 0$ and $\Psi'\gamma\Psi \neq 0$, then the potential is given by (2.4). On the other hand, if $\Psi'\neq 0$ and $\Psi'\gamma\Psi \neq 0$, then the potential is given by (2.5). Finally, if $\Psi'\neq 0$ and $\Psi'\gamma\Psi \neq 0$, then according to [20] both formulas (2.4) and (2.5) are equivalent and so the potential is given either by (2.4) or (2.5).

However, these inversion formulas are valid only in the case where $\Psi'\neq 0$ or $\Psi'\gamma\Psi \neq 0$. Therefore, a question arises here, concerning the existence of Dirac solutions $\Psi \neq 0$ satisfying the conditions $\Psi'\gamma\Psi \neq 0$, or equivalently $\Psi'\gamma\Psi \neq 0$. First, do such solutions exist, and if yes, do they correspond to one and only one real four-potential or not? The main aim of the rest of this article is to answer this question. The above considerations lead us to introduce the following definition:

**Definition 6.** A Dirac or Weyl solution $\Psi$ is said to be degenerate, if and only if $\Psi$ corresponds to more than one four-potentials.

3. **On the degeneracy of Weyl spinors**

In this section we shall prove that all solutions to the Weyl equations in the form (2.2) and (2.4) are degenerate, in the sense that they correspond to an infinite number of electromagnetic four-potentials.

**Theorem 3.1.** Any non identically zero Weyl spinor is degenerate, corresponding to an infinite number of electromagnetic four-potentials $b_\mu$, given by the formulae

$$
(b_{01}, b_{12}, b_3) = (a_0 + f_0, a_1 + f_0, a_2 + f_0),
$$

(3.1)

and

$$
(b_{01}, b_{12}, b_3) = (a_0 + f_0, a_1 + f_0, a_2 + f_0),
$$

(3.2)

for equations (2.2) and (2.4) respectively. Here $f$ is any real function of the variables $x_\mu$, and $\phi_0, \phi_0$ are real functions given by

$$
(b_0, b_1, b_2, b_3) = \left(1 - \frac{\psi\sigma^1\psi}{\psi\psi'}, \frac{\psi\sigma^2\psi}{\psi\psi'}, \frac{\psi\sigma^3\psi}{\psi\psi'}\right),
$$

and

$$
(b_0, b_1, b_2, b_3) = \left(1 - \frac{\psi\sigma^1\psi}{\psi\psi'}, \frac{\psi\sigma^2\psi}{\psi\psi'}, \frac{\psi\sigma^3\psi}{\psi\psi'}\right).
$$

**Proof.** First we will show that any solution $\psi$ of the first Weyl equation (2.2) corresponds to any potential $b_\mu$ as given by (3.1): clearly, equation (2.2) can be written as

$$
(a_\mu + f_\mu)\sigma^\mu\psi = -i\sigma^\mu\partial_\mu\psi + f_\mu\sigma^\mu\psi.
$$

(3.3)

It is easy to verify that for any two spinor $\psi$ the following identity holds:

$$
\phi_\mu \sigma^\mu\psi = 0.
$$

(3.4)

Then, from (3.3) and (3.4) we conclude that $\psi$ corresponds to $b_\mu$.

Next we will try to extract all four-potentials $b_\mu$ corresponding to $\psi$ from the equation (2.2):

Since $\psi$ corresponds also to $b_\mu$, we have

$$
(a_\mu - f_\mu)\psi = 0
$$

(3.5)

Subtracting (2.2) from (3.5) we obtain

$$
(a_{\mu}(b_{\mu} - a_{\mu})\psi = 0.
$$

(3.6)

Multiplying (3.6) from the left successively by $\psi\sigma^1, \psi\sigma^2, \psi\sigma^3$, adding the resulting three equations with their hermitian conjugates, and taking into account that the matrices $\sigma^\mu$ are hermitian and $\sigma^1, \sigma^2, \sigma^3$ are antihermitian, we obtain
the following set of equations:

\[
(b_0 - a_0)\psi^\dagger \sigma^1 \psi + (b_1 - a_1)\psi^\dagger \psi = 0,
\]
\[
(b_0 - a_0)\psi^\dagger \sigma^2 \psi + (b_2 - a_2)\psi^\dagger \psi = 0,
\]
\[
(b_0 - a_0)\psi^\dagger \sigma^3 \psi + (b_3 - a_3)\psi^\dagger \psi = 0.
\]

Finally, setting \( b_0 = a_0 + f \), we get (3.1). Further, from the hermiticity of \( \sigma^\mu \) follows that the functions \( \phi_1, \phi_2, \phi_3 \) are real. Consequently, all four-potentials given by (3.1) are real. Working exactly as above we can prove that any solution \( \psi \) of (2.4) is also solution to the second Weyl equation under any of the four-potentials \( b_a \), given by (3.2), which are also real. \( \square \)

This is a particularly interesting result not only from a theoretical, but also from a practical point of view, because it has been shown recently [21–26], that ordinary charged particles can also behave as massless in certain materials as graphene sheets, Weyl semimetals, etc. Therefore, the property of Weyl particles to be in the same state under a wide variety of different electromagnetic fields is expected to be offer new possibilities regarding the practical applications of these materials, which are already important [30–32]. In section 7 we provide explicit examples of Weyl spinors, calculating also the electromagnetic fields corresponding to these solutions.

4. Uniqueness of mass

In this section we shall prove that any Dirac spinor corresponds to one and only one value of the mass. This result is essential to prove the main theorem of this paper regarding degenerate Dirac solutions, which is presented in the next section. In [14] and [20] there was given the following formula for the mass \( m \) in terms of spinor \( \Psi \):

\[
m = \frac{-i}{2} \frac{\nu^+ \gamma^0 \gamma^1 \psi + \nu^+ \gamma^0 \gamma^2 \psi}{\bar{\nu}^+ \gamma^3 \psi},
\]

which clearly is not valid in the case where \( \psi^+ \gamma^1 \gamma^2 \gamma^3 \psi = 0 \).

In this section we prove that any not identically zero Dirac solution corresponds to one and only one mass. This result will also be used in the next section.

**Definition 7.** In the set of all Dirac solutions we define the following relation:

- \( \Psi_1 \approx \Psi_2 \) if and only if \( \Psi_1, \Psi_2 \) are gauge equivalent, that is there exists a non zero number \( c \) and a differentiable function of the spatial and temporal variables \( f: \mathbb{R}^4 \to \mathbb{R} \) such that \( \Psi_1 = ce^{i(f)} \Psi_2 \). Clearly \( \approx \) is an equivalence relation, and by \( \left[ \Psi \right] \) we will denote the equivalence class of \( \Psi \).

- We denote by \( B(\Psi) \) the set of all masses to which \( \Psi \) corresponds.

**Lemma 4.1.** Let \( \Psi \neq 0 \) be a Dirac solution, then for any \( \Psi_1 \in [\Psi] \) we have \( B(\Psi_1) = B(\Psi) \).

**Proof.** From \( \Psi_1 \in [\Psi] \) we have \( \Psi = c_1 \exp(i\phi) \Psi_1 \), for some not identically zero \( c_1 \in \mathbb{R} \) and some differentiable function \( \phi: \mathbb{R}^4 \to \mathbb{R} \). So from (2.1) we get

\[
i\gamma^\mu \partial_\mu \Psi_1 - m\Psi_1 = (a_\mu - \partial_\mu f) \gamma^\mu \Psi.
\]

Therefore, \( \Psi_1 \) corresponds to the mass \( m \) and to the four-potential

\[
b_\mu = a_\mu - \partial_\mu f.
\]

**Corollary 4.2.** Any not identically zero Dirac solution \( \Psi \) corresponds to one and only one mass.

**Proof.** Let \( \Psi \) be a Dirac solution corresponding to a four-potential \( a_\mu \) and a mass \( m \). We choose \( \Psi_1 \in [\Psi] \) defined by \( \Psi = \exp(i\phi) \Psi_1 \), where

\[
f = \int_0^x a_0(s, x_1, x_2, x_3) ds,
\]

where \( k \) is a real constant. Then according to lemma 4.1, \( \Psi_1 \) corresponds to the potential \( b_\mu = a_\mu - \partial_\mu f \) and the mass \( m \), and from (4.1) we have \( b_0 = a_0 - \partial_0 f = a_0 - a_0 = 0 \). Now we apply the formula (2.4) for the spinor \( \Psi_1 \) by writing only the first component of \( b_\mu \), and using the fact that \( b_0 = 0 \):

\[
m = \frac{i}{2} \frac{\nu^+ \gamma^1 \gamma^2 \psi}{\bar{\nu}^+ \gamma^3 \psi}.
\]

which, since \( \Psi_1 \neq 0 \), means that the above formula is well defined. Therefore \( \Psi_1 \) corresponds to a unique mass \( m \) which, according to lemma 4.1, means that \( \Psi \) also corresponds to a unique mass \( m \). \( \square \)

5. On degenerate Dirac solutions

In this section we shall prove the main theorem of this paper, according to which all Dirac solutions are classified into two classes. Degenerate solutions corresponding to an infinite number of four-potentials and non-degenerate solutions corresponding to one and only one four-potential. The four-potentials are explicitly calculated in both cases. Further, as far as the degenerate solutions are concerned, it is proven that at least two four-potentials are gauge inequivalent, and consequently correspond to different electromagnetic fields. In order to proceed with these results, the following lemmas are required:

**Lemma 5.1.** Let \( \Psi \) be any spinor. Then we have

\[
\begin{align*}
\left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \gamma^1 \gamma^2 \Psi \right) &= \left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \gamma^1 \gamma^2 \Psi \right), \\
\left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \Psi \right) &= \left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \Psi \right), \\
\left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \gamma^2 \gamma^3 \Psi \right) &= \left( \Psi^T \gamma_2 \Psi \right) \left( \Psi^T \gamma^0 \gamma^2 \gamma^3 \Psi \right).
\end{align*}
\] (5.1)
if and only if

\[(\Psi^T \gamma^2 \Psi)(\Psi^I \gamma \Psi) = 0.\] (5.2)

**Proof.** The above lemma will be proved pointwise. We set

\[
\Psi = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix},
\]

\[
= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} Y, \tag{5.3}
\]

where \(Y = [\psi_1(x) \ \psi_2(x) \ \psi_3(x) \ \psi_4(x)]^T\), and \(x\) is any element in the domain of \(Y\). After some algebra, the relations (5.1) can be written as

\[
(\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 - \zeta_2 \zeta_3) = (\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 - \zeta_2 \zeta_3),
\]

\[
(\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 + \zeta_2 \zeta_3) = (\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 + \zeta_2 \zeta_3),
\]

which by setting

\[
\zeta_1 = \omega_1 \zeta_3 \quad \text{and} \quad \zeta_4 = \omega_2 \zeta_2,
\] (5.4)

take the following form:

\[
|\zeta_2 \zeta_3|^2 (\omega_2 \omega_3 - 1) (\omega_1 \omega_2) + (\omega_2 \omega_3 - 1) (\omega_1 \omega_2) = 0,
\]

\[
|\zeta_2 \zeta_3|^2 (\omega_2 \omega_3 - 1) (\omega_1 \omega_2) + (\omega_2 \omega_3 + 1) (\omega_1 \omega_2) = 0,
\]

\[
|\zeta_2 \zeta_3|^2 (\omega_2 \omega_3 - 1) (\omega_1 \omega_2) = 0.
\]

Consequently

\[
\zeta_2 = 0 \quad \text{or} \quad \zeta_3 = 0 \quad \text{or}
\]

\[
\omega_1 \omega_2 - 1 = 0 \quad \text{or} \quad \omega_1 \omega_2 + 1 = 0.
\]

which is equivalent to

\[
\zeta_2 = 0 \quad \text{or} \quad \zeta_3 = 0 \quad \text{or}
\]

\[
\omega_1 \omega_2 - 1 = 0 \quad \text{or} \quad \omega_1 + \omega_2 = 0.
\]

or

\[
\zeta_2 = 0 \quad \text{or} \quad \zeta_3 = 0 \quad \text{or} \quad \omega_1 \omega_2 - 1 = 0 \quad \text{or} \quad \omega_1 + \omega_2 = 0,
\]

or

\[
|\zeta_2 \zeta_3|^2 (\omega_1 \omega_2 - 1) (\omega_1 + \omega_2) = 0.
\]

Using (5.4) the above relations take the form

\[
(\zeta_1 \zeta_4 - \zeta_2 \zeta_3)(\zeta_1 \zeta_4 + \zeta_2 \zeta_3) = 0,
\]

which through (5.3) can be rewritten as (5.2).

**Lemma 5.2.** Let \(\Psi\) be any spinor. Then we have,

\[
\Psi^T \gamma^2 \Psi = \Psi^I \gamma \Psi = 0,
\] (5.5)

if and only if

\[
\Psi = \begin{bmatrix} \psi \ \\ -\psi \end{bmatrix},
\]

where \(\psi\) is a two spinor.

**Proof.** The present lemma will also be proved pointwise. For any element \(x\) in the domain of \(\Psi\), setting (5.3) in (5.5) we get

\[
\zeta_1 \zeta_4 = \zeta_2 \zeta_3, \quad \zeta_1 \zeta_4 + \zeta_2 \zeta_3 = 0.
\] (5.6)

We suppose that for some \(x\) in the domain of \(\Psi\), holds that \(\zeta_1 \zeta_4 \zeta_3 = 0\). Then, from (5.6) we easily obtain

\[
|\frac{\zeta_1}{\zeta_3}|^2 = -1.
\]

Therefore for any \(x\) in the domain of \(\Psi\) holds that \(\zeta_1 \zeta_4 \zeta_3 = 0\). Hence, from (5.6) we have that for any \(x\) in the domain of \(\Psi\)

\[
\zeta_1 = \zeta_3 = 0 \quad \text{or} \quad \zeta_2 = \zeta_4 = 0,
\]

which using (5.3) can be rewritten as

\[
\psi + \psi^2 = \psi^3 + \psi = 0 \quad \text{or} \quad \psi^3 - \psi^2 = \psi - \psi^3 = 0.
\]

\[\square\]

**Lemma 5.3.** Let \(\Psi \not\equiv 0\) be a Dirac solution correspondig to a mass \(m\) and to a four-potential \(a_i\). Then we have

\[
\Psi^T \gamma^2 \Psi \equiv \Psi^I \gamma \Psi \equiv 0 \quad \text{if and only if} \quad \text{either} \quad \psi = \begin{bmatrix} \psi \ \\ \psi \end{bmatrix},
\]

\[\Psi = \begin{bmatrix} \psi \ \\ -\psi \end{bmatrix}, \quad \psi \text{ a solution to the Weyl equation in the form (2.2) or}
\]

\[\Psi = \begin{bmatrix} \psi \ \\ \psi \end{bmatrix}, \quad \psi \text{ a solution to the Weyl equation in the form (2.4), corresponding to the four-potential } a_i.
\]

**Proof.** Setting \(\Psi = \begin{bmatrix} \psi \\ \psi \end{bmatrix}\) in (2.1) we equivalently get the following two equations

\[
\sigma^i a_i \psi = i \sigma^i \partial_i \psi + m \psi,
\]

\[
\sigma^i a_i \psi = i \sigma^i \partial_i \psi - m \psi.
\]

Subtracting the second equation from the first one, and taking into account that \(\psi \not\equiv 0\), we obtain \(m = 0\). Therefore both equations are quivalent to the Weyl equation (2.2).

In a similar way, setting \(\Psi = \begin{bmatrix} \psi \\ -\psi \end{bmatrix}\) in (2.1) we get equation (2.4).

\[\square\]

Now we are ready to prove the main theorem of this paper:
Theorem 5.4. Let \( \Psi \neq 0 \) be a Dirac solution corresponding to a mass and a four-potential \( a_\mu \). Then \( \Psi \) is degenerate, if and only if \( \Psi^\dagger \gamma \Psi \equiv 0 \). Specifically we have:

1. If \( \Psi^\dagger \gamma \Psi \neq 0 \), then \( \Psi \) corresponds to one and only one real potential \( a_\mu \) given by (2.4) or (2.5).
2. If \( \Psi^\dagger \gamma \Psi \equiv 0 \), and \( \Psi^\dagger \gamma^2 \Psi \neq 0 \), then \( \Psi \) corresponds to an infinite number of real potentials \( b_\mu \) of the form
   \[
   b_\mu = a_\mu + f \theta_\mu, \tag{5.7}
   \]
   where \( f \) is any real function of the variables \( x_\mu \), while the real parameters \( \theta_\mu \) are given by the formulae
   \[
   (\theta_0, \theta_1, \theta_2, \theta_3) = \left( 1, -\frac{\Psi^\dagger \gamma^0 \gamma_0 \gamma^2 \Psi}{\Psi^\dagger \gamma^2 \Psi}, -\frac{\Psi^\dagger \gamma^0 \gamma_2 \gamma^3 \Psi}{\Psi^\dagger \gamma^2 \Psi}, -\frac{\Psi^\dagger \gamma^0 \gamma_3 \gamma^1 \Psi}{\Psi^\dagger \gamma^2 \Psi} \right). \tag{5.8}
   
3. If \( \Psi^\dagger \gamma \Psi \equiv 0 \), and \( \Psi^\dagger \gamma^2 \Psi \equiv 0 \), then either \( \Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \psi \) solution to the Weyl equation in the form (2.2) or (2.4) respectively. In this case the set of all real four-potentials \( b_\mu \) corresponding to \( \Psi \) is given by (3.1) or (3.2) respectively.

Proof. 

1. See remark 2.2.
2. First we will show that \( \Psi \) corresponds to any real or complex four-potential \( b_\mu \) as given by (5.7): clearly, equation (2.2) can be written as
   \[
   (a_\mu + f \theta_\mu) \gamma^\mu \Psi = i \partial_\mu \gamma^\mu \Psi - m \Psi + f \theta_\mu \gamma^\mu \Psi. \tag{5.9}
   
   After some algebraic calculations it is easy to verify that for any spinor \( \Psi \) the following identity holds:
   \[
   \theta_\mu \gamma^\mu \Psi \equiv 0. \tag{5.10}
   
   Now, from (5.9) and (5.10) we conclude that \( \Psi \) corresponds to \( b_\mu \).

Next we will try to extract from equation (2.1) all four-potentials \( b_\mu \) corresponding to \( \Psi \): since \( \Psi \) corresponds also to \( b_\mu \) we have that
   \[
   b_\mu \gamma^\mu \Psi = i \gamma^\mu \partial_\mu \Psi - m \Psi \tag{5.11}
   
   Substracting (2.1) from (5.11) we obtain
   \[
   \gamma^\mu (b_\mu - a_\mu) \Psi = 0. \tag{5.12}
   
   Multiplying (5.12) from the left successively by \( \Psi^\dagger \gamma^0 \gamma^2 \), \( \Psi^\dagger \gamma^2 \gamma^3 \) and using the fact that the matrices \( \gamma^0 \gamma^2 \gamma^3 \), \( \gamma^1 \) , \( \gamma^3 \) are antisymmetric, we obtain the following set of equations:
   \[
   \begin{align*}
   (b_1 - a_1) \Psi^\dagger \gamma^0 \gamma^2 \Psi &= -(b_0 - a_0) \Psi^\dagger \gamma^0 \gamma^1 \gamma^2 \Psi, \\
   (b_2 - a_2) \Psi^\dagger \gamma^2 \gamma^3 \Psi &= -(b_0 - a_0) \Psi^\dagger \gamma^0 \gamma_2 \gamma^3 \Psi, \\
   (b_3 - a_3) \Psi^\dagger \gamma^2 \gamma^3 \Psi &= -(b_0 - a_0) \Psi^\dagger \gamma^0 \gamma_3 \gamma^1 \Psi.
   \end{align*}
   
   Setting \( b_0 = a_0 + f \), in the above equations we arrive to (5.7). Finally, from lemma 5.1 it follows that the functions \( \theta_1, \theta_2, \theta_3 \) real. Consequently, all the four-potentials given by (5.7) are real. This last part of theorem 5.4 follows from theorem 3.1 and lemma 5.3.

Remark 5.5. It is known [14, 20] that any Dirac solution corresponds to infinitely many complex four-potentials. Indeed, in the case that \( \Psi^\dagger \gamma \Psi \equiv 0 \), and \( \Psi^\dagger \gamma^2 \Psi \equiv 0 \), according to lemma 5.1, the functions \( \theta_1, \theta_2, \theta_3 \) are complex and consequently the set of all complex four-potentials \( b_\mu \) corresponding to \( \Psi \) are given by (5.7), where \( f \) is an arbitrary complex function.

Remark 5.6. Let \( \Psi \) be any degenerate Dirac solution such that \( \Psi^\dagger \gamma \Psi \equiv 0 \), and \( \Psi^\dagger \gamma^2 \Psi \equiv 0 \). Then there are at least two different electromagnetic fields corresponding to \( \Psi \).

Proof. We suppose that \( \Psi \) corresponds to the four-potential \( a_\mu \). Then, if we set \( f = 1 \) and \( f = x_0 \), from theorem 5.4 follows that \( \Psi \) also corresponds to the four-potentials \( b_\mu = \alpha_\mu + \beta_\mu, c_\mu = \sigma_\mu + x_0 \theta_\mu \) respectively, where the parameters \( \theta_\mu \) are given by equation (5.8). We also suppose that the four-potentials \( \alpha_\mu, \beta_\mu, c_\mu \) are associated with a common electromagnetic field. Then the fields \( (1, \theta_1, \theta_2, \theta_3), (x_0, x_0 \theta_1, x_0 \theta_2, x_0 \theta_3) \) are conservative. Therefore, we have that
   \[
   \partial_\mu \theta_\mu = 0, \mu = 1, 2, 3, \tag{5.13}
   \]
   and
   \[
   x_0 \partial_\mu \theta_\mu + \theta_\mu = 0, \mu = 1, 2, 3. \tag{5.14}
   
   From (5.13) and (5.14) follows that \( \theta_\mu = 0, \mu = 1, 2, 3 \). Then, using equations (5.8) it can be easily shown that the spinors \( \Psi \) must be of the form:
   \[
   \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \text{ or } \Psi = \begin{pmatrix} \psi_1 \\ -\psi_1 \\ \psi_2 \\ -\psi_2 \end{pmatrix},
   \]
   which clearly satisfy: \( \Psi^\dagger \gamma^2 \Psi \equiv 0 \). This contradicts the condition that \( \Psi^\dagger \gamma^2 \Psi \neq 0 \). Therefore at least one of the two four-potentials \( b_\mu \) and \( c_\mu \) is gauge inequivalent to \( a_\mu \).

Remark 5.7. It is easy to prove, that \( \Psi^\dagger \gamma \Psi = 0 \) if and only if \( \Psi \) has the following form,
   \[
   \Psi = u \begin{pmatrix} \varpi \\ w \end{pmatrix} + v \begin{pmatrix} 1 \\ -w \end{pmatrix}, \tag{5.15}
   
   where \( u, v, w \) are arbitrary functions of \( x_\mu \).
Closing this section we should note that the property of Dirac particles, described by spinors satisfying the condition $\Psi^\dagger \gamma_5 \Psi = 0$, to be in the same state under a wide variety of different electromagnetic fields is a particularly interesting and surprising result, at least in the framework of classical Physics, where one would expect that any changes to the electromagnetic fields, and consequently to the electromagnetic forces acting on the particles, would alter their state. However, as it will be shown in the following section massless Dirac particles can exist in the same state both in the absence of electromagnetic fields and in a wide variety of different electromagnetic fields, which are explicitly calculated. The degeneracy can also be extended to massive particles, when particle–antiparticle pairs are considered.

6. Degenerate solutions to the force-free Dirac equation and the corresponding electromagnetic fields

In this section we focus on the degenerate solutions to the force-free Dirac equation providing details regarding their structure as well as their physical interpretation. We also provide explicit expressions regarding the electromagnetic fields corresponding to these solutions, as well as the electric charge and current densities required to produce these fields. This analysis leads to the surprising result that a massless Dirac particle can be in the same state either in free space (zero electromagnetic field) or in a region of space with constant electric charge and current densities.

The general solution to the force-free Dirac equation for a particle of mass $m$, energy $E$, and momentum $p = |p|(\sin \theta \cos \varphi \hat{a} + \sin \theta \sin \varphi \hat{b} + \cos \theta \hat{c})$, propagating along a direction defined by the angles $\theta, \varphi$ in spherical coordinates can be written in the form [33]

$$\Psi_p(r, t) = [c_1 u_1(E, p) + c_2 u_2(E, p)] \times \exp[-i(p_x x + p_y y + p_z z - Et)], \quad (6.1)$$

where the four-vectors $u_1(E, p)$, $u_2(E, p)$ are eigenstates of the helicity operator and are explicitly given by the formulae

$$u_1(E, p) = \begin{pmatrix} \cos \left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} \cos \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} e^{i\varphi} \sin \left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$u_2(E, p) = \begin{pmatrix} \sin \left(\frac{\theta}{2}\right) \\ e^{i\varphi} \cos \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} \sin \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} e^{i\varphi} \cos \left(\frac{\theta}{2}\right) \end{pmatrix}$$

(6.2)

corresponding to a particle with negative helicity (spin anti-parallel to its propagation direction), and

$$v_1(E, p) = \begin{pmatrix} \frac{|p|}{E + m} \sin \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} e^{i\varphi} \cos \left(\frac{\theta}{2}\right) \\ -\sin \left(\frac{\theta}{2}\right) \\ e^{i\varphi} \cos \left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_2(E, p) = \begin{pmatrix} \frac{|p|}{E + m} \cos \left(\frac{\theta}{2}\right) \\ \frac{|p|}{E + m} e^{i\varphi} \sin \left(\frac{\theta}{2}\right) \\ \cos \left(\frac{\theta}{2}\right) \\ e^{i\varphi} \sin \left(\frac{\theta}{2}\right) \end{pmatrix}$$

(6.5)

(6.6)

corresponding to a particle with positive helicity (spin parallel to its propagation direction), and

$$\Psi_0(r, t) = [c_1 v_1(E, p) + c_2 v_2(E, p)] \times \exp[-i(p_x x + p_y y + p_z z - Et)], \quad (6.4)$$

where

In order these solutions to be degenerate we apply the condition $\Psi^\dagger \gamma_5 \Psi = 0$ and find that it is valid if and only if

$$|p|^2 = (E + m)^2 \rightarrow E^2 - m^2$$

$$= E^2 + m^2 + 2Em \rightarrow m^2 + Em = 0 \rightarrow \begin{cases} m = 0 \quad E = -m \end{cases},$$

where we have also used the well-known formula of special relativity $E^2 = c^2 |p|^2 + m^2 c^4$ which in the natural system of units $(\hbar = c = 1)$ becomes $E^2 = |p|^2 + m^2$. Obviously, the condition $E = -m$, corresponding to particles or antiparticles with negative mass, or negative energy, cannot be valid.
because in this case some terms of the four-vectors $u_1, u_2, v_1, v_2$ become infinite. Therefore, we are led to the very interesting result, that the wavefunction of a free Dirac particle, or antiparticle, is degenerate if and only if the particle, or antiparticle, is massless. At this point it should be noted that, to our knowledge, massless charged particles have not been discovered yet. However, in the initial stages of the evolution of the Universe, before the particles acquire mass, there was a plethora of massless charged particles [27, 28]. Further, it has been shown recently [21–26], that in certain materials as graphene sheets, Weyl semimetals, etc ordinary charged particles can also behave as massless. Therefore, our analysis could provide new insight to the study of the initial phases of the evolution of the Universe as well as the behavior of ‘massless’ particles in new exotic materials.

In the case of massless particles, the general form of the degenerate wavefunctions is

$$
\Psi_\text{p}(E, p) = \begin{pmatrix}
    c_1 \cos \left(\frac{\theta}{2}\right) - c_2 \sin \left(\frac{\theta}{2}\right) \\
    e^{i\frac{\phi}{2}} \left(c_1 \sin \left(\frac{\theta}{2}\right) + c_2 \cos \left(\frac{\theta}{2}\right)\right) \\
    c_1 \cos \left(\frac{\theta}{2}\right) + c_2 \sin \left(\frac{\theta}{2}\right) \\
    e^{i\frac{\phi}{2}} \left(c_1 \cos \left(\frac{\theta}{2}\right) - c_2 \sin \left(\frac{\theta}{2}\right)\right)
\end{pmatrix}
$$

$$
\Psi_\text{\bar{p}}(E, p) = \begin{pmatrix}
    c_1 \sin \left(\frac{\theta}{2}\right) + c_2 \cos \left(\frac{\theta}{2}\right) \\
    e^{i\frac{\phi}{2}} \left(-c_1 \cos \left(\frac{\theta}{2}\right) + c_2 \sin \left(\frac{\theta}{2}\right)\right) \\
    -c_1 \sin \left(\frac{\theta}{2}\right) + c_2 \cos \left(\frac{\theta}{2}\right) \\
    e^{i\frac{\phi}{2}} \left(c_1 \sin \left(\frac{\theta}{2}\right) + c_2 \cos \left(\frac{\theta}{2}\right)\right)
\end{pmatrix}
$$

$$
\times \exp \left[iE (x \sin \theta \cos \varphi + y \sin \theta \sin \varphi + z \cos \theta - t)\right]
$$

(6.7)

$$
\Psi_\text{p}(E, p) = \begin{pmatrix}
    c_1 \sin \left(\frac{\theta}{2}\right) + c_2 \cos \left(\frac{\theta}{2}\right) \\
    c_1 \cos \left(\frac{\theta}{2}\right) + c_2 \sin \left(\frac{\theta}{2}\right) \\
    c_1 \cos \left(\frac{\theta}{2}\right) - c_2 \sin \left(\frac{\theta}{2}\right) \\
    c_1 \sin \left(\frac{\theta}{2}\right) - c_2 \cos \left(\frac{\theta}{2}\right)
\end{pmatrix}
$$

$$
\times \exp \left[-iE (x \sin \theta \cos \varphi + y \sin \theta \sin \varphi + z \cos \theta - t)\right]
$$

(6.8)

for particles and anti-particles respectively. These expressions are easily obtained from equations (6.1)–(6.6) setting $m = 0$ and taking into account the fact that in this case $|p| = E$ (in the natural system of units where $c = 1$). The parameters $\theta_1, \theta_2, \theta_3$, defined by equation (5.8) in the framework of theorem 5.4, corresponding to the above general degenerate solutions of the force-free Dirac equation, are given by the particularly simple expressions

$$
\theta_1 = -\sin \theta \cos \varphi, \quad \theta_2 = -\sin \theta \sin \varphi, \\
\theta_3 = -\cos \theta
$$

which are obviously the opposite to the projections on the x-, y- and z-axis respectively of a unit vector along the direction defined by the angles $(\theta, \varphi)$. Then, from theorem 5.4, it is concluded that the wavefunctions (6.7), (6.8) can describe particles and anti-particles both in space free of electromagnetic fields, as well as in space with non-zero electromagnetic fields related to the four-potentials

$$
(f, -f \sin \theta \cos \varphi, -f \sin \theta \sin \varphi, -f \cos \theta),
$$

where $g(r, t)$ is an arbitrary function of the spatial variables and time. Defining $g(r, t) = \frac{1}{\sqrt{q}} (r, t)$ the electric and magnetic potentials corresponding to the above four-potentials are

$$
\varphi(r, t) = g(r, t),
$$

$$
A(r, t) = -g(r, t) \sin \theta \cos \varphi i + g(r, t) \sin \theta \sin \varphi j - g(r, t) \cos \theta k
$$

respectively. Thus, the electric and magnetic fields (in Gaussian units) derived from the above four-potential are explicitly given by the formulae [34]

$$
E(r, t) = \nabla \varphi - \frac{\partial A}{\partial t} = -\nabla g
$$

$$
+ \frac{\partial g}{\partial t} \left(\sin \theta \cos \varphi i + \sin \theta \sin \varphi j + \cos \theta k\right)
$$

(6.9)

$$
B(r, t) = \nabla \times A = \left(-\cos \theta \frac{\partial g}{\partial y} + \sin \theta \sin \varphi \frac{\partial g}{\partial z}\right) i
$$

$$
+ \cos \theta \frac{\partial g}{\partial x} + \sin \theta \cos \varphi \frac{\partial g}{\partial y} + \sin \theta \sin \varphi \frac{\partial g}{\partial z} - \nabla \times \left(\sin \theta \sin \varphi \frac{\partial g}{\partial x} + \cos \theta \frac{\partial g}{\partial y} + \sin \theta \cos \varphi \frac{\partial g}{\partial z}\right) k,
$$

(6.10)

where we have set $c = 1$, assuming that we are working in the natural system of units.

Thus, a massless Dirac particle propagating along the direction defined by the angles $(\theta, \varphi)$ will be in the same state both in free space and in any electromagnetic field determined by equations (6.9), (6.10). At this point we can proceed a step further calculating the electric charge $\rho(r, t)$ and the current $J(r, t)$ densities required to produce the above electromagnetic fields, using Maxwell’s equations in the form [34]

$$
\nabla^2 \varphi + \frac{\partial}{\partial t} (\nabla \cdot A) = -4\pi \rho
$$

$$
\left(\nabla^2 A - \frac{\partial^2 A}{\partial t^2}\right) - \nabla \left(\nabla \cdot A + \frac{\partial \varphi}{\partial t}\right) = -4\pi J,
$$

where we have also set $c = 1$, assuming that we are working in the natural system of units.

Then, it is easy to show that the electric charge and current densities required to produce the electromagnetic fields (6.9), (6.10) are given, in cartesian coordinates, by the formulae

$$
\rho(r, t) = -\frac{1}{4\pi} \nabla^2 g + \frac{1}{4\pi} \frac{\partial}{\partial t} \left(\sin \theta \cos \varphi \frac{\partial g}{\partial x} + \sin \theta \sin \varphi \frac{\partial g}{\partial z} + \cos \theta \frac{\partial g}{\partial y}\right)
$$

(6.11)
and

\[ J_y(r, t) = \frac{1}{4\pi} \sin \theta \cos \varphi \left( \nabla^2 g - \frac{\partial^2 g}{\partial t^2} \right) + \frac{1}{4\pi} \frac{\partial}{\partial x} \left( \sin \theta \cos \varphi \frac{\partial g}{\partial x} + \sin \theta \sin \varphi \frac{\partial g}{\partial y} + \cos \theta \frac{\partial g}{\partial z} - \frac{\partial g}{\partial t} \right) \]  

(6.12)

Thus, a massless Dirac particle propagating along the direction defined by the angles \((\theta, \varphi)\) in spherical coordinates will be in the same state both in free space and in any electromagnetic field determined by equations (6.9), (6.10), which is produced by the electric charge and current densities given by equations (6.11)–(6.14). We should note that, as it is easily deduced from the above equations, if the arbitrary function \(g\) is time independent, then the electric charge and current densities become also time-independent, corresponding to a static electromagnetic field.

A special case of particular interest is when \(g(r) = c_0 |r|^2 = c_0 (x^2 + y^2 + z^2)\), where \(c_0\) is an arbitrary real constant. Then, it is easy to show that the electric charge and current densities take the particularly simple form

\[ \rho = -\frac{3c_0}{2\pi}, \]

\[ J = \frac{c_0}{\pi} \left( \sin \theta \cos \varphi \hat{i} + \sin \theta \sin \varphi \hat{j} + \cos \theta \hat{k} \right) \]

leading to the surprising result that a massless charged particle propagating along the direction defined by the angles \((\theta, \varphi)\) will be in the same state either in free space (zero electromagnetic field) or in a region of space with constant electric charge and current densities given by the above formulae.

Finally, it should be noted that in the case of particle–antiparticle pairs, one can obtain degenerate solutions even for massive particles. Indeed, it is easy to show that if \(|c_1| = |c_2|\), the spinors

\[ \Psi(r, t) = c_1 u_1(E, p) \times \exp[i(p_x x + p_y y + p_z z - Et)] + c_2 v_1(E, p) \times \exp[-i(p_x x + p_y y + p_z z - Et)] \]

\[ \times \exp[i(p_x x + p_y y + p_z z - Et)] + c_2 v_1(E, p) \times \exp[-i(p_x x + p_y y + p_z z - Et)] \]

corresponding to particle–antiparticle pairs with positive and negative helicity respectively, are degenerate, for any value of the mass \(m\). Here, \(c_1, c_2\) are arbitrary complex constants satisfying the condition \(|c_1| = |c_2|\), while \(u_1, u_2, v_1, v_2\) are defined by (6.2), (6.3), (6.5), (6.6) respectively. More details on these solutions will be provided in a future work.

7. Solutions to the Weyl equation and the corresponding electromagnetic fields

In this section we shall provide explicit expressions regarding the infinite number of electromagnetic fields corresponding to specific classes of Weyl spinors. First we consider the force-free Weyl equation in the form (2.2) where all the components of the electromagnetic four-potential are set equal to zero. In this case it is easy to show that the spinor

\[ \Psi = \cos \left( \frac{\theta}{2} \right) \exp \left[ iE(x \sin \theta \cos \varphi + y \sin \theta \sin \varphi + z \cos \theta - t) \right] \]

(7.1)

is solution to the force-free Weyl equation corresponding to a free Weyl particle of energy \(E\), propagating along a direction defined by the angles \((\theta, \varphi)\) in spherical coordinates. Then, according to theorem 3.1, the spinor (7.1) is also solution to the Weyl equation for the following set of four-potentials

\[ (f, \varphi_1 f, \varphi_2 f, \varphi_3 f), \]

where \(f, \varphi_1, \varphi_2, \varphi_3\) is an arbitrary function of the spatial coordinates and time. The parameters \(\varphi_1, \varphi_2, \varphi_3\) can be easily calculated taking the values

\[ \varphi_1 = -\sin \theta \cos \varphi, \quad \varphi_2 = -\sin \theta \sin \varphi, \quad \varphi_3 = -\cos \theta \]

which are obviously the opposites to the projections of a unit vector along the direction defined by the angles \((\theta, \varphi)\) on the \(x, y, \) and \(z\)-axis respectively. Defining the function \(g(r, t) = \frac{1}{4} f(r, t)\) the electric and magnetic potentials corresponding to the above four-potentials are

\[ \varphi(r, t) = g(r, t), \]

\[ A(r, t) = -\frac{1}{4} g(r, t) \sin \theta \cos \varphi \]

\[ -g(r, t) \sin \theta \sin \varphi - g(r, t) \cos \theta k \]

respectively. Here it is assumed that we are working in the natural system of units where \(\hbar = c = 1\). The electric and magnetic fields (in Gaussian units) derived from the above...
four-potential are explicitly given by the formulae [34]

$$E(r, t) = -\nabla \varphi - \frac{\partial A}{\partial t}$$

$$= -\nabla g + \frac{\partial g}{\partial t} \left( \sin \theta \cos \varphi i + \sin \theta \sin \varphi j + \cos \theta k \right)$$

$$B(r, t) = \nabla \times A = -\left( -\cos \theta \frac{\partial g}{\partial y} + \sin \theta \sin \varphi \frac{\partial g}{\partial z} \right) i$$

$$- \left( -\cos \theta \frac{\partial g}{\partial x} + \sin \theta \cos \varphi \frac{\partial g}{\partial z} \right) j$$

$$+ \sin \theta \left( -\sin \varphi \frac{\partial g}{\partial x} + \cos \varphi \frac{\partial g}{\partial y} \right) k.$$  

(7.2, 7.3)

Thus, a free Weyl particle propagating along the direction defined by the angles $(\theta, \varphi)$ will be in the same state both in free space (zero electromagnetic field) as well as in any electromagnetic field of the form described by equations (7.2), (7.3). Setting $g(r, t) = s(t)$ in the above formulae, where $s(t)$ is an arbitrary real function of time, the magnetic field becomes zero, while the electric field takes the simple form

$$E(r, t) = \frac{ds}{dt} \left( \sin \theta \cos \varphi i + \sin \theta \sin \varphi j + \cos \theta k \right).$$

Obviously, if $s(t) = c_0 t$, where $c_0$ is an arbitrary real constant, the above formula corresponds to a constant electric field. Consequently, a constant or time-dependent electric field parallel (or antiparallel) to the direction of motion of a Weyl particle does not alter its state, and it will keep on moving with the same momentum as if there was no electric field. Thus, Ohm’s law does not hold for Weyl particles, and the current transferred by them remains constant, independent of the applied electric field. This practically means that the resistance of a Weyl material ‘adjusts’ to the applied electric field in a way that the current remains constant, and it can even become negative if the polarity of the applied electric field changes. This non-conventional behavior of Weyl materials regarding their interaction with electric fields is expected to offer new opportunities in nanoelectronics. As it can be deduced from (6.9), (6.10), this remarkable result is also true for massless Dirac particles. Finally, it should be mentioned that the property of massless particles to maintain their state under a constant or time-dependent electric field, is closely related to the fact that they move at a constant speed, either the speed of light if moving in vacuum, or a much smaller value if moving in materials [25].

In the following we shall study the case of a Weyl particle confined in one dimension by a constant magnetic field. Specifically, it is easy to show that any spinor of the form

$$\Psi = h(y) \begin{pmatrix} 0 \\ 1 \end{pmatrix} d(z + t),$$

(7.4)

where $h(y)$ is an arbitrary real function of $y$ and $d(z + t)$ an arbitrary complex function of $z + t$, is solution to the Weyl equation for the four-potential

$$E(r, t) = \begin{pmatrix} 0, -\frac{h_y}{h}, 0, 0 \end{pmatrix},$$

where $h_y = \frac{dh(y)}{dy}$. Defining $H(y) = \frac{1}{2} \frac{h_y}{h}$ it is straightforward to show that the electromagnetic field corresponding to the above four-potential is

$$E = 0, \quad B = -\frac{\partial H}{\partial y} k = \frac{1}{q} \frac{h_y h - h^2}{h^2} k.$$  

According to theorem 3.1, the spinor (7.4) is also solution to the Weyl equation for the four-potentials

$$\begin{pmatrix} f, -\frac{h_y}{h}, 0, f \end{pmatrix},$$

where $f(r, t)$ is an arbitrary real function of the spatial coordinates and time. Defining $g(r, t) = \frac{f(r, t)}{h(y)}$ it is straightforward to show that a Weyl particle described by the spinor (7.4) will be in the same state under the influence of any of the following electromagnetic fields

$$E(r, t) = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j - \left( \frac{\partial g}{\partial z} + \frac{\partial g}{\partial t} \right) k,$$

$$B(r, t) = \frac{\partial g}{\partial i} j - \frac{\partial g}{\partial j} i - \frac{\partial H}{\partial y} k.$$  

(7.5, 7.6)

As a specific example we consider the case where $h(y)$ is a Gaussian function of the form $h(y) = \exp(-\lambda y^2)$, where $\lambda$ is an arbitrary positive real constant, and $d(z + t)$ is a wave function of the form $d(z + t) = A \exp[-iE(z + t)]$, where $A$ is an arbitrary complex constant and $E$ a positive real constant corresponding to the energy of the particle. In this case the spinor

$$\Psi = A \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp[-\lambda y^2 - iE(z + t)]$$

corresponds to a Weyl particle of energy $E$ propagating along the $z$ direction, which is confined along the $y$-axis by a constant magnetic field, given by the formula

$$B = \frac{\partial H}{\partial y} k = \frac{2\lambda}{q} k.$$  

Then, according to equations (7.5), (7.6) this particle will be in the same state under the influence of an infinite number of electromagnetic fields of the form

$$E(r, t) = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j - \left( \frac{\partial g}{\partial z} + \frac{\partial g}{\partial t} \right) k,$$

$$B(r, t) = \frac{\partial g}{\partial i} j - \frac{\partial g}{\partial j} i - \frac{2\lambda}{q} k,$$

where $g(r, t) = \frac{1}{q} f(r, t)$ is an arbitrary real function of the spatial variables and time. Finally, it should be mentioned that if the arbitrary function $g$ depends only on time, the electric and magnetic fields become

$$E(r, t) = -\frac{\partial g}{\partial t} k, B(r, t) = -\frac{2\lambda}{q} k.$$  

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Thus, the state of the Weyl particle does not change if a constant or time-dependent electric field is applied parallel to the magnetic field. Similar results can be obtained regarding the alternative form of the Weyl equation (2.4) for negative helicity particles.

From the examples discussed in sections 6 and 7 it is clear that our theory has very important physical implications as well as high potential for practical applications. Specifically, we have shown that Weyl particles—as well as massless Dirac particles—can be in the same state under a wide variety of different electromagnetic fields. This result could be utilized to explain the rapid evolution of the Universe during the first stages of its creation, where all particles were massless [27, 28]. Further, the fact that massless Dirac and Weyl particles are very resistant to electromagnetic perturbations is expected to play an important role regarding the practical applications of new exotic materials, as graphene sheets, Weyl semimetals, etc, where ordinary charged particles can behave as massless [21–26]. For example, as discussed above, one very important property of these particles is that they remain in the same state under the influence of a constant or time—dependent electric field parallel (or anti-parallel) to their direction of motion. This practically means that a material where the current carriers are massless Dirac or Weyl particles does not obey Ohm’s law, in the sense that the resistance of this material ‘adjusts’ to the applied electric field in a way that the current remains constant, independent of the applied voltage. Obviously, this non-conventional behavior could be utilized in nanoelectronics for designing novel components and devices.

8. Summary

In the present study we have first shown that all solutions to the Weyl equations are degenerate, in the sense that they correspond to an infinite number of electromagnetic four-potentials, which are explicitly provided. Further, it is thoroughly proven that every Dirac spinor corresponds to one and only one mass. Using this result we have proven the main theorem of this article, according to which all Dirac spinors can be classified into two classes. The elements of the first class correspond to one and only one four-potential, and are called non-degenerate Dirac solutions, while the elements of the second class correspond to an infinite number of four-potentials, and are called degenerate Dirac solutions. Further, it has been shown that at least two of these four-potentials are gauge-inequivalent, corresponding to different electromagnetic fields. In order to illustrate this particularly important result we have studied the force-free Dirac equation and found that the degenerate solutions correspond to massless particles. Thus, a massless Dirac particle can be in the same state both in the absence of electromagnetic fields as well as under the influence of an infinite number of different electromagnetic fields, which are calculated. It has also been shown that the degeneracy can be extended to massive particles, if particle–antiparticle pairs are considered. Specific examples of Weyl spinors are also provided, calculating the infinite number of different electromagnetic fields corresponding to these solutions. This property of massless particles to be in the same state under the influence of different electromagnetic fields could be used to explain the rapid evolution of the Universe during the first stages of its creation—before the particles acquire mass—as well as the behavior of ordinary charged particles in new exotic materials, e.g. graphene sheets, Weyl semimetals, etc, where they can collectively behave as massless. Finally, we have shown that Ohm’s law does not hold for massless Dirac and Weyl particles, and the current transferred by them is constant, independent of the applied electric field.

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