HOMOGENEOUS TORIC BUNDLES
WITH POSITIVE FIRST CHERN CLASS

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Abstract. A simple algebraic characterization of the Fano manifolds in the class of homogeneous toric bundles over a flag manifold $G^C/P$ is provided in terms of symplectic data.

1. Introduction

In this paper we focus on a particular class of homogeneous bundles $M$, having a compact toric Kähler manifold $F$ as fiber and a generalized flag manifold $G^C/P = G/K$ as basis, where $G$ is a compact semisimple Lie group, $G^C$ its complexification and $P$ a suitable parabolic subgroup of $G^C$. More precisely, we consider a surjective homomorphism $\tau : P \to (T^m)^\mathbb{C}$, where $T^m$ is an $m$-dimensional torus acting effectively on the toric Kähler manifold $F$, $\dim_{\mathbb{C}} F = m$, by holomorphic isometries; we then define $M$ to be the compact complex manifold $M = G^C \times_{P, \tau} F$. Any manifold of this kind is a toric bundle over $G^C/P$ (see [12]) with holomorphic projection $\pi : M \to G^C/P$ and it is almost $G^C$-homogeneous (see [8]) with $G$-cohomogeneity equal to $m$. We will call any such manifold a homogeneous toric bundle.

The homogeneous toric bundles appear to be direct generalizations of the $\mathbb{C}P^1$-bundles over flag manifolds studied in [15, 10, 11, 9, 4, 13]. These $\mathbb{C}P^1$-bundles are known to be Kähler-Einstein if and only if they are Fano and their Futaki functional vanishes identically ([10, 13]); moreover they always admit a Kähler-Ricci soliton metric, provided the first Chern class is positive (see [9, 17, 18]). We also mention that for toric bundles a uniqueness result for extremal metrics in a given Kähler class is proved in [7].

Aiming at investigating the existence of Kähler-Einstein metrics and, more generally, of Kähler-Ricci solitons in the class of almost homogeneous toric bundles, in this paper we are interested in finding simple conditions on $G^C/P$, $F$ and the homomorphism $\tau$ which guarantee that the above defined manifold $M$ has positive first Chern class.

In order to state our main result, we first need to fix some notations. Let $J$ be the $G^C$-invariant complex structure on the flag manifold $G^C/P = G/K$ and let $C$ be the corresponding positive Weyl chamber in the Lie algebra $\mathfrak{g}(\mathfrak{k})$ of the center $Z(K)$ of $K$ (see e.g. §2, for the definition). We will also use the symbol $C'$ to denote the chamber in $\mathfrak{g}(\mathfrak{k})^*$, which is the image of $C$ by means of the dualizing map $X \mapsto -B(X, \cdot)$, where $B$ is the Cartan Killing form on the Lie algebra $\mathfrak{g}$ of $G$.

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It is well known that $G^C/P$ admits a unique $G$-invariant Kähler-Einstein metric $g$ with Einstein constant $c = 1$ (see e.g. [1]): we set $\mu : G^C/P \to \mathfrak{g}^*$ to be the moment map relative to the Kähler form $\omega = g(J\cdot, \cdot)$.

If $F$ is supposed to be Fano, the Calabi-Yau theorem implies that for any $T^m$-invariant Kähler form $\rho \in c_1(F)$, there exists a unique Kähler form $\omega_\rho$ in $c_1(F)$ such that $\rho$ is the Ricci form of $\omega_\rho$. In particular, also $\omega_\rho$ is $T^m$-invariant. Moreover, since $b_1(F) = 0$ there exists a moment map $\mu_\rho : F \to \mathfrak{t}^*$ relative to $\rho$, uniquely determined up to a constant. We will say that $\mu_\rho$ is metrically normalized if $\int_F \mu_\rho \omega_\rho^m = 0$. In §4, we will show that the convex polytope $\Delta_F = \mu_\rho(F)$, which is the image of a metrically normalized moment map $\mu_\rho$, is actually independent of $\rho$ and it can be explicitly determined just using the $T^m$ action.

Finally, for any given homomorphism $\tau : P \to (T^m)^C$, we set $\mu_\tau \overset{\text{def}}{=} (\tau|_{\mathfrak{t}^1})^* \circ \mu_\rho$. Notice that the map $\mu_\tau$ is a moment map for the action of $Z(K)^\ast$ on $F$ induced by $\tau$, and its image is the convex polytope $\Delta_{\tau,F} = (\tau|_{\mathfrak{t}^1})^*(\Delta_F) \subset \mathfrak{t}^1\ast$. Our main result can be now stated as follows.

**Theorem 1.1.** Let $F$ be a toric Kähler manifold of dimension $m$. Then, for any homomorphism $\tau : P \to (T^m)^C$, the manifold $M = G^C \times_{P,\tau} F$ has positive first Chern class if and only if $F$ is Fano and

$$\mu(eP) + \Delta_{\tau,F} \subset \mathcal{C}^\vee.$$  \hspace{1cm} (1.1)

Later we will also show that there is a simple algorithm to determine $\Delta_F$ and that (1.1) can be reformulated in a finite number of algebraic conditions, which are suited for computations (see §6).

We mention here that the proof of our main result originates from an idea for computing the first Chern class, which goes back to [19] and [5]. In a forthcoming paper we will provide explicit computations for the generalized Futaki functionals of homogenous toric bundles and we will attack the existence problem for Kähler-Einstein and Ricci solitons in this class of Kähler manifolds.

1.1. Notations. For any Lie group $G$, we will denote its Lie algebra by the corresponding gothic letter $\mathfrak{g}$; given a Lie homomorphism $\tau : G \to G'$, we will always use the same letter to represent the induced Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathfrak{g}'$. If $G$ acts on a manifold $M$, for any $X \in \mathfrak{g}$, we will use the symbol $\hat{X}$ to indicate the induced vector field on $M$; we recall here that $[\hat{X}, \hat{Y}] = -[X, Y]$ for every $X, Y \in \mathfrak{g}$. We denote by $M_{\text{reg}}$ the set of $G$-principal points in $M$.

The Cartan Killing form of a semisimple Lie algebra $\mathfrak{g}$ will be always denoted by $B$ and, for any $X \in \mathfrak{g}$, we set $X^\ast = -B(X, \cdot) \in \mathfrak{g}^*$; given a root system $R$ w.r.t to a fixed maximal torus, we will denote by $E_\alpha \in \mathfrak{g}^*$ the root vector corresponding to the root $\alpha$ in the Chevalley normalization and by $H_\alpha = [E_\alpha, E_{-\alpha}]$ the $B$-dual of $\alpha$.

2. Preliminaries

Throughout the following we will denote by $G$ a connected compact, semisimple Lie group and by $V = G/K$ a generalized flag manifold associated to $G$. If we fix a $G$-invariant complex structure $J_V$ on $V$, then the complexified group $G^C$ acts
holomorphically on $V$, which can be then represented as $G^C/P$ for some suitable parabolic subgroup $P$.

We recall that the Lie algebra $g$ of $G$ admits an $\text{Ad}(K)$-invariant decomposition $g = \mathfrak{k} \oplus \mathfrak{m}$ and that, for any fixed CSA $\mathfrak{h} \subset \mathfrak{g}^C$ of $g^C$, the corresponding root system $R$ admits a corresponding decomposition $R = R_o + R_m$, so that $E_\alpha \in \mathfrak{g}^C$ if $\alpha \in R_o$ and $E_\alpha \in \mathfrak{m}^C$ if $\alpha \in R_m$; furthermore, $J_V$ induces a splitting $R_m = R^+_m \cup R^-_m$ into two disjoint subset of positive and negative roots, so that the $J_V$-holomorphic and $J_V$-antiholomorphic subspaces of $m^C$ are given by

$$m^{(1,0)} = \sum_{\alpha \in R^+_m} \mathbb{C} E_\alpha, \quad m^{(0,1)} = \sum_{\alpha \in R^-_m} \mathbb{C} E_\alpha . \quad (2.1)$$

The Lie algebra $\mathfrak{p}$ of the parabolic subgroup $P$ is $\mathfrak{p} = \mathfrak{k}^C + m^{(0,1)}$. It is also well-known that $G^C$ is an algebraic group and $P$ an algebraic subgroup.

Finally, we recall that for any $G$-invariant Kähler form $\omega$ of $V$ there exists a uniquely associated element $Z_\omega \in \mathfrak{z}(\mathfrak{k})$ so that $\omega(\dot{X},\dot{Y})|_{eK} = \mathcal{B}(Z_\omega, [X,Y])$ for any $X, Y \in g$. Notice that, by (2.1) and the positivity of $\omega$, the element $Z_\omega$ has to belong to the positive Weil chamber

$$\mathcal{C} = \{ W \in \mathfrak{z}(\mathfrak{k}) : i\alpha(W) > 0 \text{, for any } \alpha \in R^+_m \} .$$

Moreover, a straightforward check shows that the moment map $\mu_\omega : V \to g^*$ relative to $\omega$ is given by $\mu_\omega(gK) = (\text{Ad}_g Z_\omega)^\vee$ for any $gK \in V$. We recall also that the $G$-invariant Kähler-Einstein form $\omega_V$ on $V$ with Einstein constant $c = 1$, is determined by the associated element (see e.g. [1, 3])

$$Z_V = - \sum_{\alpha \in R^+_m} \mathbb{C} E_\alpha . \quad (2.2)$$

We will now focus on those flag manifolds $G^C/P = G/K$ for which there exists a surjective homomorphism $\tau : P \to (T^m)^C$. Using the structure of parabolic subgroups and the fact that $(T^m)^C$ is abelian, we see that $\tau$ is completely determined by its restriction to $K$; moreover $\tau|_K$ takes value in $T^m$ and hence it is fully determined by its restriction to the connected component $Z^o(K)$ of the center of the isotropy $K$. We can therefore consider the algebraic manifold

$$M = G^C \times_{P,\tau} F = G \times_{K,\tau} F ,$$

where $P$ (or $K$) acts on $F$ by means of $\tau$.

The manifold $M$ is a fiber bundle over the flag manifold $G/K$ with holomorphic projection $\pi$; moreover $G^C$ acts almost homogeneously, i.e. with an open and dense orbit in $M$, while $G$ acts by cohomogeneity $m$ with principal isotropy type $(L)$, where $L = \ker \tau \cap G \subset K$.

We prove now the following Lemma, which will be useful and often tacitly used in the sequel.

**Lemma 2.1.** If $J$ denotes the complex structure of $M$, then

$$J\hat{m}_p = \hat{m}_p$$

for every $p \in \pi^{-1}([eK])$. 

Proof. We know that \( m^{(0,1)} \subset p \) and \( τ^C \mid_{m^{(0,1)}} = 0 \), so that \( \tilde{m}^{(0,1)} \mid_p = 0 \). The vectors \( E_α - E_{-α} \) and \( i(E_α + E_{-α}) \), \( α \in R_m^+ \), span \( m \) over the reals and

\[
J((E_α - E_{-α}) \mid_p) = J(E_α \mid_p) = iE_α \mid_p = i(E_α + E_{-α}) \mid_p \in \tilde{m}_p.
\]

Similarly, \( J(i(E_α + E_{-α}) \mid_p) \) is \( \tilde{m}_p \).

In the sequel \( (Z_1, \ldots, Z_m) \) will denote a fixed \( B \)-orthonormal basis of \( (\ker τ) \perp \cap \tilde{z}(\mathfrak{t}) \).

3. The algebraic representatives

We recall that, if \( ψ \) is a \( G \)-invariant 2-form on \( M \), for any \( p \in M \) there exists a unique \( ad_{g_p} \)-invariant element \( F_{ψ,p} \in \operatorname{Hom}(g, g) \) such that \( B(F_{ψ,p}(X), Y) = ψ_p(\tilde{X}, \tilde{Y}) \) for any \( X, Y \in g \). Moreover, if \( ψ \) is closed, it turns out that \( F_{ψ,p} \) is a derivation of \( g \) and hence of the form \( ad_{Z_p} \) for some element \( Z_p \) belonging to the centralizer in \( g \) of the isotropy subalgebra \( g_p \) (see e.g. [13] [13] [13]).

So, in the following, for any \( G \)-invariant, closed 2-form \( ψ \), we will denote by \( Z_ψ \) the \( G \)-equivariant map \( Z_ψ : M \to g \) defined by

\[
ψ_p(\tilde{X}, \tilde{Y}) = B([Z_ψ \mid_p, X], Y) = B(Z_ψ \mid_p, [X, Y]) \quad \text{for any } X, Y \in g
\]

and it will be called algebraic representative of \( ψ \).

Notice that, in case \( ψ \) is non-degenerate, the moment map \( μ_ψ : M \to g^* \) relative to \( ψ \) coincides with the \((-B)\)-dual of the algebraic representative, i.e. \( μ_ψ = Z_ψ^\vee \).

In fact, by the closure and \( G \)-invariance of \( ψ \), we have that for any vector field \( W \) on \( M \) and any \( X, Y \in g \)

\[
0 = dψ(W, \tilde{X}, \tilde{Y}) = W(ψ(\tilde{X}, \tilde{Y})) + ψ([W, [\tilde{X}, \tilde{Y}]] =
\]

\[
=W(ψ(\tilde{X}, \tilde{Y})) + ψ(W, [\tilde{X}, \tilde{Y}]) = B(W(Z_ψ), [X, Y]) + ψ(W, [X, Y]) ;
\]

since \( g = [g, g] \), it follows that \( d(Z_ψ^\vee)(W)(X) = ψ(W, \tilde{X}) \), which implies the claim.

By \( G \)-equivariance, notice that any algebraic representative \( Z_ψ \) is uniquely determined by its restriction on the fiber \( F = π^{-1}(eK) \subset M \). Such restriction satisfies the following.

Lemma 3.1. Let \( ψ \) be a \( G \)-invariant, \( J \)-invariant, closed 2-form on \( M \).

(a) If the restriction \( ψ \mid_F \) satisfies \( ψ(\tilde{Z}_j, \tilde{m}) = 0 \) for \( 1 \leq j \leq m \), then \( Z_ψ \mid_F \) is of the form \( Z_ψ \mid_F = \sum_{i=1}^m f_iZ_i + I_ψ \), where \( I_ψ \in 1 = \operatorname{Lie}(\ker τ \cap G) \) and \( f_i : F \to R \) are smooth functions. Moreover \( ψ_p(\tilde{J}_Z, \tilde{Z}_i) = JZ_i(f_i) \mid_p \) for any \( p \in F \) and \( 1 \leq i, j \leq m \) and \( I_ψ \) is constant; in particular, \( ψ \) can be completely recovered by its algebraic representative \( Z_ψ \) (and hence by its associated moment map, if \( ψ \) is non-degenerate);

(b) \( ψ \) is cohomologous to 0 if and only if \( Z_ψ \mid_F = -\sum_i JZ_i(φ)Z_i \) for some \( K \)-invariant smooth function \( φ : F \to R \) and, if this occurs, then \( ψ = dφ \).
Proof. (a) Since \( t = 1 + \text{span}\{Z_1, \ldots, Z_m\} \), \([t, m] = m\) and \( \hat{F} = 0\), we have that on \( F \)

\[
0 = \psi([t, m]) = B(Z_\psi, [t, m]) = B(Z_\psi, m);
\]

hence \( Z_\psi|_F \) takes values in \( t \) and has the claimed form. From (3.2) we have that for any \( X, Y \in g \) and any \( p \in F \),

\[
\psi_p(J\hat{Z}_i, [X, Y]) = -B \left( \sum_{j=1}^m J\hat{Z}_i(f_j)_p Z_j + J\hat{Z}_i(I_\psi)_p, [X, Y] \right).
\] (3.3)

On the other hand,

\[
\psi_p(J\hat{Z}_i, [X, Y]) = -\sum_{j=1}^m B([X, Y], Z_j) \psi_p(j\hat{Z}_i, \hat{Z}_j) - \psi_p(\hat{Z}_i, [X, Y]_m),
\] (3.4)

where we denote by \((\cdot)_m\) the \( B\)-orthogonal projection onto \( m \). Since \( \hat{m}_p \) is \( J\)-invariant and \( \psi_p\)-orthogonal to \( \text{span}\{\hat{Z}_i|_p\} \), the second term of (3.4) vanishes and, from (3.3), we obtain

\[
\sum_{j=1}^m B \left( \psi_p(j\hat{Z}_i, \hat{Z}_j), Z_j, [X, Y] \right) = B \left( \sum_{j=1}^m J\hat{Z}_i(f_j)_p Z_j + J\hat{Z}_i(I_\psi)_p, [X, Y] \right).
\]

Since \([g, g] = g\), we have that

\[
\psi(j\hat{Z}_i, \hat{Z}_j) \equiv J\hat{Z}_i(f_j), \quad J\hat{Z}_i(I_\psi) = 0.
\] (3.5)

This together with the fact that \( \hat{Z}_i(I_\psi) = 0 \), which is due to \( G\)-equivariance, implies that \( I_\psi \) is constant on \( F \) and the first claim follows. Furthermore, formula (3.3) implies that if the map \( Z_\psi|_F = \sum_{i=1}^m f_i Z_i + I_\psi : F \to \mathfrak{z}(t) \) is known, it is possible to evaluate \( \psi_p(\hat{X}, \hat{Y}) \) for any \( X, Y \in g^+ \) and \( p \in F \); then, from almost homogeneity also the last claim of (a) follows. (b) Notice that in case \( \psi \) is cohomologous to 0, then it is of the form \( \psi = dd^c \phi \) for some \( G\)-invariant real valued function \( \phi \). Then, for any \( X, Y \in m \), on \( F \) we have that

\[
dd^c \phi(\hat{X}, \hat{Y}) = -\hat{X}(J\hat{Y}(\phi)) + \hat{Y}(J\hat{X}(\phi)) + \hat{J}[\hat{X}, \hat{Y}](\phi) = J[\hat{X}, \hat{Y}](\phi) = -\sum_{i=1}^m B(Z_i, [X, Y]) J\hat{Z}_i(\phi).
\]

It follows immediately that \( Z_\psi|_F = -\sum_{i=1}^m J\hat{Z}_i(\phi) Z_i \). Conversely, if \( Z_\psi|_F = -\sum_{i=1}^m J\hat{Z}_i(\phi) Z_i \) for some \( K\)-invariant function \( \phi \in C^\infty(F) \), then by (a) and the above remarks, \( \psi = dd^c \phi \), where we consider \( \phi \) as \( G\)-invariantly extended to \( M \). ■

We want now to determine the algebraic representative of the Ricci form \( \rho \) of a given \( G\)-invariant Kähler form \( \omega \). In what follows, we denote by \((F_\alpha, G_\alpha)_{\alpha \in R^+_m}\) the basis for \( m \) given by the vectors

\[
F_\alpha = \frac{1}{\sqrt{2}} (E_\alpha - E_{-\alpha}), \quad G_\alpha = \frac{i}{\sqrt{2}} (E_\alpha + E_{-\alpha})
\]

with \( \alpha \in R^+_m \). Notice that, by definition of \( R^+_m \), \( J_\nu F_\alpha = G_\alpha \) and \( J_\nu G_\alpha = -F_\alpha \) and that the complex structure \( J \) of \( M \) induces on \( m \) the same complex structure induced by \( J_\nu \) (see proof of Lemma 2.1). We order the roots in \( R^+_m \) so to call them \( \alpha_1, \alpha_2, \) etc., and we denote by \( F_i, 1 \leq i \leq m + |R^+_m| \) the elements of \( t \) defined by
So, for any $F_i = Z_i$ if $1 \leq i \leq m$ and $F_i = F_{a_i m}$ if $m + 1 \leq i$. Notice that, at any point $p \in F \cap M_{reg}$ the vector fields $\{\hat{F}_j, J\hat{F}_j\}$ are linearly independent and span the whole $T_p M$. Finally, for any given Kähler form $\omega$, let us also denote by $h : M \to \mathbb{R}$ the function

$$h(q) = \omega^n(\hat{F}_1, J\hat{F}_1, \hat{F}_2, J\hat{F}_2, \ldots)|_q.$$  

We claim that, at any point $p \in F$, $\hat{X}(h)|_p = 0$ for any $X \in \mathfrak{g}$. In fact, using $\mathcal{L}_X \omega = 0$ and $\mathcal{L}_X J = 0$, we have

$$\hat{X}(h)|_p = -\omega^n([X, \hat{F}_1], J\hat{F}_1, \hat{F}_2, J\hat{F}_2, \ldots) - \omega^n(\hat{F}_1, J\hat{F}_1, J[X, \hat{F}_2], J\hat{F}_2, \ldots) - \ldots .$$

On the other hand, for any $i$, $[X, F_i] \in 1 + \text{span}\{F_j, j \neq i\} + \text{span}\{J_V F_j, j \geq m + 1\}$.

and this implies that

$$\omega^n(\hat{F}_1, \ldots, JF_{j-1}, [X, \hat{F}_j], JF_j, \ldots) = \omega^n(\hat{F}_1, \ldots, JF_{j-1}, F_j, J[X, F_j], \ldots) = 0$$

and hence the claim. We may now prove the following.

**Proposition 3.2.** The restriction to $F_{\text{reg}} = F \cap M_{\text{reg}}$ of the algebraic representative $Z_\rho$ of the Ricci form $\rho$ of a Kähler form $\omega$ is

$$Z_\rho = \sum_{i=1}^m J\hat{Z}_i (\log |h|) \frac{1}{2} Z_i + Z_V ,$$

where $h$ is the function (3.6) and $Z_V \in \mathfrak{z}(\mathfrak{t})$ is the element defined in (2.2). Furthermore, if $Z_\omega = \sum_i f_i Z_i + I_\omega$ is the restriction to $F$ of the algebraic representative of $\omega$, then the function $h$ is

$$h = K \cdot \det (f_{i,j}) \cdot \prod_{\alpha \in R^+_m} (\sum_{i=1}^m a^i_\alpha f_i + b_\alpha) ,$$

where $f_{i,j} \overset{def}{=} J\hat{Z}_j (f_i)$, $a^i_\alpha \overset{def}{=} \alpha (iZ_i)$, $b_\alpha \overset{def}{=} \alpha (iI_\omega)$ and $K$ is a real constant.

**Proof.** We first show that $\rho|_F(\hat{Z}_j, \hat{m}) = 0$. We recall that by Koszul formula (see e.g. [1], p. 89)

$$\rho_\rho(\hat{X}, \hat{Y}) = -\frac{1}{2} \mathcal{L}_{J(\hat{X}, \hat{Y})} \omega^n(\hat{F}_1, J\hat{F}_1, \hat{F}_2, J\hat{F}_2, \ldots)$$

for every $p \in F_{\text{reg}}$ and $X, Y \in \mathfrak{g}$. On the other hand, we claim that for any $W \in \mathfrak{m}$,

$$\mathcal{L}_{JW} \omega^n|_F = 0 .$$

In fact,

$$B([W, F_{a_k}], G_{a_k}) = B([W, F_{a_k}], G_{a_k}) = B(W, iH_{a_k}) = 0 .$$

So, for any $\hat{F}_j$ with $j \geq m + 1$, $[\hat{W}, \hat{F}_j]$ has trivial component along $J\hat{F}_j$ and $J ([\hat{W}, \hat{F}_j])$ has trivial component along $\hat{F}_j$. This implies that

$$\omega^n(\hat{F}_1, \ldots, J[\hat{W}, \hat{F}_j], \ldots) = \omega^n(\hat{F}_1, \ldots, J[\hat{W}, J\hat{F}_j], \ldots) = 0 .$$
Moreover, when $1 \leq i \leq m$, we have that $[W, F_i] = [W, Z_i] \in \mathfrak{m}$ and hence also in this case
\[
\omega^n(J[\dot{W}, F_1], J\dot{F}_1, \dot{F}_2, \ldots) = \omega^n(\dot{F}_1, J[\dot{W}, J\dot{F}_1], \dot{F}_2, \ldots) = \cdots = 0 .
\]
These facts imply that
\[
\mathcal{L}_{J\dot{W}} \omega^n(\dot{F}_1, J\dot{F}_1, \dot{F}_2, J\dot{F}_2, \ldots) = J\dot{W}(h) .
\]
Using $J\dot{m}_p = \dot{m}_p$ and the fact that $\dot{X}(h)_p = 0$ for any $X \in \mathfrak{g}$, we get (3.10).

Now, the fact that $\rho|_E(\dot{Z}_j, \dot{m}) = 0$ follows immediately from (3.9), (3.10) and from $[\mathfrak{t}, \mathfrak{m}] \subseteq \mathfrak{m}$.

By Lemma 3.21(a) and (3.9), in order to determine $Z_\rho$, it suffices to compute for $p \in F_{\text{reg}}$
\[
\rho_p(\dot{X}, J\dot{X}) = \rho_p(\dot{X}, J\dot{X}) = -\frac{1}{2} \mathcal{L}_{[\dot{X}, J\dot{X}]} \omega^n(\dot{F}_1, J\dot{F}_1, \dot{F}_2, J\dot{F}_2, \ldots) \bigg|_p .
\]

Now, given $X \in \mathfrak{m}$, we put $Y = \{X, JX\}$ and we write $Y = Y_m + \sum_{i=1}^m Y_i Z_i + Y_1$, where subscripts indicate $\mathcal{B}$-orthogonal projections onto the corresponding subspaces. Using (3.10) and $\|p = 0$, (3.11) reduces to
\[
\rho_p(\dot{X}, J\dot{X}) = -\frac{1}{2h} \sum_{i=1}^m Y_i \left( J\dot{Z}_i(h) \right) + \frac{1}{2h} \omega^n(J[\dot{Y}_1, \dot{F}_1], J\dot{F}_1, \dot{F}_2, \ldots) +
\]
\[
+ \frac{1}{2h} \omega^n(\dot{F}_1, J[\dot{Y}_1, \dot{F}_1], \dot{F}_2, \ldots) + \ldots .
\]
Now, recall that $[\mathfrak{t}, F_i] = 0$ for all $1 \leq i \leq m$ and that, for any $1 \leq j$ and any $H \in \mathfrak{h} \cap \mathfrak{g}$, where $\mathfrak{h}$ is the fixed CSA,
\[
J[H, F_{j+m}] = J[H, F_{\alpha_j}]_p = -i\alpha_j(H)J\overline{G_{\alpha_j}}_p = i\alpha_j(H)\overline{F_{\alpha_j}}_p .
\]
At the same time, for any $W \in \text{span}_R\{E_{\beta}, \beta \in R\} \cap \mathfrak{g}$, the bracket $[W, F_{\alpha_j}]$ is always orthogonal to $\text{span}_R\{F_{\alpha_j}, G_{\alpha_j}\}$. Therefore $[W, F_{\alpha_j}]_p$ has trivial component along the vector $\overline{G_{\alpha_j}}_p = J\overline{F_{\alpha_j}}_p$ and $[W, F_{\alpha_j}]_p$ has trivial component along the vector $\overline{F_{\alpha_j}}$. It follows that
\[
\frac{1}{2h} \omega^n(J[\dot{Y}_1, \dot{F}_1], J\dot{F}_1, \dot{F}_2, \ldots) + \frac{1}{2h} \omega^n(\dot{F}_1, J[\dot{Y}_1, J\dot{F}_1], \dot{F}_2, \ldots) + \cdots =
\]
\[
= -\frac{1}{2h} \omega^n(\dot{F}_1, \ldots, J\dot{F}_m, [Y_1, F_{m+1}], J\dot{F}_{m+1}, \dot{F}_{m+2}, \ldots) -
\]
\[
= -\frac{1}{2h} \omega^n(\dot{F}_1, \ldots, J\dot{F}_m, F_{m+1}, J[Y_1, F_{m+1}], \dot{F}_{m+2}, \ldots) - \ldots =
\]
\[
= -\sum_{\alpha \in R^+_m} i\alpha(Y_\alpha) = \mathcal{B}(Z, [X, JX]) .
\]
So, (3.12) reduces to $\rho_p(\dot{X}, J\dot{X}) = \mathcal{B} \left( \sum_{i=1}^m \frac{1}{2h} J\dot{Z}_i, Z, [X, JX] \right)$ and (3.7) follows.

To check (3.8), it suffices to observe that for any $j \geq 1$,
\[
\omega_p(\dot{F}_{j+m}, J\dot{F}_{j+m}) = \omega_p(\dot{F}_{\alpha_j}, \dot{G}_{\alpha_j}) = \mathcal{B}(\sum_i f_i(p) Z_i + I_\omega, [F_{\alpha_j}, G_{\alpha_j}]) =
\]
= B(\sum f_i(p)Z_i + L_\omega, iH_{\alpha_j}) = \sum f_i(p)\alpha_i^* + b_{\alpha_j}
and that \( h(p) = \omega_p^m(\hat{F}_1, J\hat{F}_1, \ldots, \hat{F}_m, J\hat{F}_m) \cdot \prod_{m+1 \leq j} \omega_p(\hat{F}_j, J\hat{F}_j) \). Then the conclusion follows from Lemma 3.1 (a). 

4. Canonical polytope of a Fano toric manifold

In the following, let \( F \) be Fano and, as considered in the Introduction, for any \( T^m \)-invariant Kähler form \( \rho \in c_1(F) \), let us denote by \( \omega_\rho \) the unique Kähler form in \( c_1(F) \) which has \( \rho \) as Ricci form. In particular, also \( \omega_\rho \) is \( T^m \)-invariant. Since \( b_1(F) = 0 \), there is a moment map \( \mu_\rho : F \to \mathfrak{t}^* \) which is uniquely determined up to a constant. We say that \( \mu_\rho \) is metrically normalized if \( \int_F \mu_\rho \omega^m_\rho = 0 \).

We now want to show that the convex polytope \( \Delta_{F, \rho} \subset \mathfrak{t}^* \), which is image of \( F \) under the metrically normalized moment map \( \mu_\rho \), is actually independent of the choice of \( \rho \) and it is canonically associated with \( F \). For any Kähler form \( \omega \) on \( F \), we may construct the map \( \delta_\omega : F \to \mathfrak{t}^* \) defined by

\[
\delta_\omega(W) \overset{\text{def}}{=} \frac{1}{2} \text{div}(J\hat{W}), \quad W \in \mathfrak{t}, \quad \rho \in F,
\]

where the divergence is defined by \( \mathcal{L}_X \omega^n = \text{div}(X)\omega^n \). Notice that the map (4.1) is well defined even when \( F \) is not Fano. Moreover, by standard facts on divergences, we have that for any point \( p \in \text{Fix}(T^m) \subset F \) and \( Z \in \mathfrak{t} \)

\[
\delta_\omega|_p(Z) = \frac{1}{2} \text{Tr}(J \circ A_Z|_p),
\]

where the \( A_Z|_p \) is the linear map \( A_Z|_p : T_pF \to T_pF \) defined by

\[
A_Z|_p(v) = \frac{d}{dt} \Phi^z_{t*}(v) \bigg|_{t=0}, \quad v \in T_pF,
\]

where \( \Phi^z_{t*} \) is the flow generated by \( \hat{Z} \). In particular \( \delta_\omega(\text{Fix}(T^m)) \) does not depend on \( \omega \) and it is uniquely determined just by the holomorphic action of \( T^m \) on \( F \). In the following, we will call the convex hull \( \Delta_F \subset \mathfrak{t}^* \) of the points of \( \delta_\omega(\text{Fix}(T^m)) \) the canonical polytope of \( (F, T^m) \).

Let us now go back to a \( T^m \)-invariant Kähler form \( \rho \in c_1(F) \) and to the Kähler form \( \omega_\rho \), which has \( \rho \) as Ricci form. If we denote by \( g_{\alpha, \beta} \) the components of the Kähler metric \( g = \omega_\rho(\cdot, J\cdot) \) in a system of holomorphic coordinates, then the Ricci form \( \omega_\rho \) of \( \omega_\rho \) is equal to

\[
\rho = -\frac{1}{2} dd^c \log(\det(g_{\alpha, \beta}))
\]

and at any \( T^m \)-regular point we have \( \det(g_{\alpha, \beta}) = f \cdot h_\rho \), where \( f \) is the squared norm of a holomorphic function and \( h_\rho = \omega_\rho^m(\check{Z}_1, J\check{Z}_1, \ldots, \check{Z}_m, J\check{Z}_m) \). Using the fact that \( \mathcal{L}_{\check{Z}_i} \omega^m_\rho = 0 \), we see that

\[
\rho(\check{Z}_i, J\check{Z}_k) = -\frac{1}{2} J\check{Z}_k \left( \frac{J\check{Z}_i(h_\rho)}{h_\rho} \right) = -\frac{1}{2} J\check{Z}_k \left( \text{div}(J\check{Z}_i) \right).
\]

From (4.4) and Stokes theorem, one may check that the map (4.1) with \( \omega = \omega_\rho \) coincides with the metrically normalized moment map \( \mu_\rho \) relative to \( \rho \). From the
previous remarks it follows immediately that $\Delta_{F,\rho} \overset{\text{def}}{=} \mu_\rho(F)$ coincides with the canonical polytope $\Delta_F$ of $F$ and hence it is independent of the chosen Kähler form $\rho \in c_1(F)$.

**Remark 4.1.** Notice that when $\omega$ is a $T^m$-invariant Kähler-Einstein form with Ricci form $\rho = \omega$, the metrically normalized moment map $\mu_\rho$ satisfies

$$\int_F \mu_\rho \rho^m = \int_F \mu_\rho \omega^m = 0,$$

so that the barycenter of $\Delta_F$ is the origin. This obstruction to the existence of Kähler-Einstein metrics is known to be equivalent to the vanishing of the Futaki invariant (see [12][5]). To the best of our knowledge, our characterization of $\Delta_F$ in terms of the $T^m$-action is new.

5. **Proof of the main theorem**

Let us fix a $G$-invariant Kähler form $\omega$ on $G/K$ and let $Z_{\omega} \in \mathfrak{g}(\mathfrak{k})$ be its associated element. Pick also a $T^m$-invariant 2-form $\rho \in c_1(F)$ and let $\omega_\rho$ be the unique $T^m$-invariant Kähler form in a fixed Kähler class, which has $\rho$ as Ricci form. Denote also by $\mu_{\omega_\rho}$ a fixed moment map relative to $\omega_\rho$.

We now fix the fiber $\pi^{-1}(eF) \cong F$ and define a 2-form $\tilde{\omega}$ on $TM|_F$ as follows: for $p \in F$, $X, Y \in T_p F$ and $A, B \in \mathfrak{m}$

$$\tilde{\omega}_p(X, Y) = \omega_\rho|_p(X, Y); \quad \tilde{\omega}_p(X, \dot{A}) = 0; \quad \tilde{\omega}_p(\dot{A}, \dot{B}) = -\mu_{\omega_\rho}(p)(\tau([A, B]_\mathfrak{t})), \quad (5.1)$$

where for every $U \in \mathfrak{g}$ we denote by $U_\mathfrak{t}$ the component along $\mathfrak{t}$ w.r.t. the decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$.

We now extend $\tilde{\omega}$ to a global $G$-invariant 2-form, still denoted by $\tilde{\omega}$. We can easily check that $\tilde{\omega}$ is $J$-invariant, using the fact that for any $A, B \in \mathfrak{m}$ we have $[A, B]_{\mathfrak{g}(\mathfrak{k})} = [J_Y A, J_Y B]_{\mathfrak{g}(\mathfrak{k})}$. We claim that $\tilde{\omega}$ is also closed. It is enough to check that $d\tilde{\omega}_p(\dot{A}, \dot{X}, \dot{Y}) = d\tilde{\omega}_p(\dot{A}, \dot{B}, \dot{Y}) = 0$ for any $p \in F$, since the condition $d\tilde{\omega}_p(\dot{A}, \dot{B}, \dot{C}) = 0$ for $A, B, C \in \mathfrak{m}$ follows immediately from the Jacobi identities in $\mathfrak{g}$. Since for any pair of vector fields $V_1, V_2$,

$$0 = \mathcal{L}_{\dot{A}} \tilde{\omega}(V_1, V_2) = d (i_{\dot{A}} \tilde{\omega}) (V_1, V_2) + d\tilde{\omega}(\dot{A}, V_1, V_2),$$

we are reduced to check that $d (i_{\dot{A}} \tilde{\omega})(X, Y)|_p = 0$ and $d (i_{\dot{A}} \tilde{\omega})(\dot{B}, Y)|_p = 0$ for every $X, Y \in T_p F$. Now, if we extend $X, Y$ as arbitrary vector fields on $F$, we have on $F$

$$d (i_{\dot{A}} \tilde{\omega})(X, Y)|_p = X \tilde{\omega}(\dot{A}, Y)|_p - Y \tilde{\omega}(\dot{A}, X)|_p - \tilde{\omega}(\dot{A}, [X, Y])|_p = 0$$

by definition of $\tilde{\omega}$ along $F$. On the other hand

$$d (i_{\dot{A}} \tilde{\omega})(\dot{B}, Y)|_p = \dot{B} \tilde{\omega}(\dot{A}, Y)|_p - Y \tilde{\omega}(\dot{A}, \dot{B})|_p - \tilde{\omega}(\dot{A}, [\dot{B}, Y])|_p =$$

$$= \tilde{\omega}([\dot{B}, \dot{A}], Y)|_p - Y \tilde{\omega}(\dot{A}, \dot{B})|_p = \tilde{\omega}([\dot{B}, \dot{A}], Y)|_p + d\mu_{\omega_\rho}(Y)(\tau([A, B]_\mathfrak{t}))|_p =$$

$$= \tilde{\omega}([\dot{B}, \dot{A}], Y)|_p + \omega_\rho(Y, [A, B]_\mathfrak{t})|_p = \tilde{\omega}([A, B]_\mathfrak{t}, Y)|_p - \tilde{\omega}([A, B]_\mathfrak{t}, Y)|_p = 0.$$
Now, for any sufficiently small \( \epsilon \in \mathbb{R}^+ \), we may consider the \( G \)-invariant closed two-form \( \omega_\epsilon \) on \( M \) given by

\[
\omega_\epsilon = \pi^* \omega + \epsilon \, \hat{\omega} ,
\]

By Lemma 3.1 and (5.1), the restriction to \( F \) of the algebraic representative of \( \omega_\epsilon \) is

\[
Z_{\omega_\epsilon} = \epsilon \sum_{i=1}^m f_i Z_i + Z_\omega , \quad \text{where} \quad f_i \overset{\text{def}}{=} \mu_\omega(\tau(Z_i)) .
\]

Using Proposition 3.2, we get that the restriction to \( F_{\text{reg}} \) of the algebraic representative of the Ricci form \( \rho_\epsilon \) of \( \omega_\epsilon \) is given by

\[
Z_{\rho_\epsilon} = \sum_{i=1}^m (\phi_i + \psi_\epsilon) Z_i + Z_V ,
\]

where

\[
\phi_i \overset{\text{def}}{=} \frac{J \hat{Z}_i(\det(f_{a,b}))}{2 \det(f_{a,b})} , \quad \psi_\epsilon \overset{\text{def}}{=} \epsilon \sum_{a \in R_m^+} \frac{a_j^i f_{j,i}}{\epsilon a_i^a f_j + b_a} ,
\]

and

\[
f_{i,j} \overset{\text{def}}{=} J \hat{Z}_j(f_i) = \omega_\rho(\hat{J} \hat{Z}_j, \hat{Z}_i) , \quad a_j^i \overset{\text{def}}{=} \alpha(iZ_j) , \quad b_a \overset{\text{def}}{=} \alpha(iZ_V) .
\]

We now notice that the map \( Z_{\psi_\epsilon}|_F = \sum_{i=1}^m \psi_\epsilon Z_i \) defines a smooth closed \( G \)-invariant 2-form \( \psi_\epsilon \) on the regular part of \( M \) by means of (3.1) and (3.5); we claim that \( \psi_\epsilon \) extends to a smooth global 2-form on \( M \), which is cohomologous to 0. In fact, \( Z_{\psi_\epsilon}|_F \) can be written as

\[
Z_{\psi_\epsilon}|_F = \sum_i J \hat{Z}_i(\hat{f}_\epsilon) Z_i , \quad \text{with} \quad \hat{f}_\epsilon \overset{\text{def}}{=} \frac{1}{2} \log \left( \prod_{\alpha \in R_m^+} \left( \epsilon a_i^a f_j + b_a \right) \right).
\]

By the fact that \( b_a > 0 \) for every \( \alpha \in R_m^+ \), if \( \epsilon \) is sufficiently small, the function \( \hat{f}_\epsilon \) is a well-defined \( K \)-invariant function on \( F \) and it extends to a \( G \)-invariant function on \( M \), which we still denote by \( \hat{f}_\epsilon \). Therefore, by Lemma 3.1(b), it follows that \( \psi_\epsilon \) coincides with the globally defined, two-form \( -dd^c \hat{f}_\epsilon \) for any \( \epsilon \) sufficiently small.

From this we immediately get also that, for \( \epsilon \) small, the Ricci form \( \rho_\epsilon \) is cohomologous to the two-form \( \rho_\epsilon \in c_1(M) \) given by

\[
\rho_\epsilon = \rho - \psi_\epsilon = \rho_\epsilon + dd^c \hat{f}_\epsilon ,
\]

whose algebraic representative on \( F_{\text{reg}} \) is equal to

\[
Z_{\rho_\epsilon} = \sum_{j=1}^m J \hat{Z}_j(\det(f_{a,b})) Z_j + Z_V .
\]

From (5.6) and (4.4), we may notice that

\[
\frac{J \hat{Z}_j(\det(f_{a,b}))}{2 \det(f_{a,b})} F_{\text{reg}} = \frac{1}{2} \text{div}(J \hat{\tau}(Z_i)) .
\]

Consider a basis \( (W_1, \ldots, W_m) \) for \( \mathfrak{t} \) and a basis \( (Z_1, \ldots, Z_m, \ldots, Z_N) \) of \( \mathfrak{z}(t) \), which is \( \mathcal{B} \)-orthonormal and extends the set \( (Z_1, \ldots, Z_m) \); let also \( (W_1^*, \ldots, W_m^*) \) and \( (Z_1^*, \ldots, Z_N^*) \) be the corresponding dual bases of \( \mathfrak{t}^* \) and \( \mathfrak{z}(t)^* \), respectively. We
set \( C = [c_j^i] \) to be the matrix defined by \( \tau(Z_j) = \sum_i c_j^i W_i \) with \( c_j^i = 0 \) for \( j \geq m + 1 \) and observe that \( \tau^\ast(W_j^\ast) = \sum_{\ell=1}^m c_j^\ell Z_\ell^\ast \). Then, by (5.8), we get that on \( F \)

\[
- B \left( \sum_{j=1}^m J\hat{Z}_j (\text{det}(f_{a,b})) Z_j \right)_{|\mathfrak{t}} = \sum_{j=1}^m J\hat{Z}_j (\text{det}(f_{a,b})) Z_j^\ast = \sum_{j=1}^m c_j J\hat{Z}_j (\text{det}(f_{a,b})) Z_j^\ast = m \sum_{\ell=1}^m \frac{c_j^\ell}{2 \text{det}(f_{a,b})} Z_j^\ast \]

where the last equality is meaningful whenever \( \rho \) is non-degenerate. Using Lemma 3.1 (a) and (4.4), we see that on \( F_{\text{reg}} \)

\[
\rho_o(J\hat{Z}_i, \hat{Z}_j)|_{F_{\text{reg}}} = J\hat{Z}_i \left( \frac{J\hat{Z}_j (\text{det}(f_{a,b}))}{2 \text{det}(f_{a,b})} \right) = \rho(\tau(Z_i), \tau(Z_j)) \tag{5.10}
\]

so that \( \rho_o|_F = \rho \). Moreover, in case \( \rho \) is non-degenerate, we also have

\[
Z_\rho^\ast|_F = Z_\rho^\ast + \tau^\ast \mu^\rho . \tag{5.11}
\]

Let us now prove the sufficiency of conditions in Theorem 1.1. Assume that (1.1) holds and that \( F \) is Fano with \( \rho > 0 \). We want to show that \( \rho_o > 0 \). Indeed, from (5.10) and the fact that the vector fields \( \hat{A}, A \in \mathfrak{m} \), are \( \rho_o \)-orthogonal to \( TF \) at all points of the fiber, we have that \( \rho_o \) is positive if and only if the matrix \( \left( \rho_o(\hat{F}_\alpha, J\hat{F}_\beta) \right) \) is positive definite at every point of \( F = \pi^{-1}(cK) \). We now observe that the functions \( \rho_o(\hat{F}_\alpha, J\hat{F}_\beta) \) vanish if \( \alpha \neq \beta \), so that we are reduced to check that

\[
\rho_o(\hat{F}_\alpha, J\hat{F}_\alpha) = i\alpha(Z_{\rho_o}) > 0 \tag{5.12}
\]

for any \( \alpha \in R^+_m \) and at any point of the fiber \( F \). From (5.11), this turns out to be equivalent to (1.1).

Now, let us assume that \( c_1(M) > 0 \) and let \( \rho_1 \in c_1(M) \) be \( G \)-invariant and positive. Being \( \rho_1 \) cohomologous to \( \rho_o \) and by (5.10), we have on \( F \)

\[
\rho_1(\hat{Z}_i, J\hat{Z}_j)|_F = \rho_o(\hat{Z}_i, J\hat{Z}_j)|_F + dd^c \phi(\hat{Z}_i, J\hat{Z}_j)|_F = \rho(\tau(Z_i), \tau(Z_j)) + dd^c (\phi|_F)(\tau(Z_i), J\tau(Z_j))
\]

for some \( G \)-invariant function \( \phi \) on \( M \). From this it follows that \( \rho_1|_F \) is a positive 2-form in \( c_1(F) \).

Now, let us assume \( \rho \) to be positive, so that the last equality in (5.9) is meaningful. Recall that, by Lemma 3.1 (b), the algebraic representatives of \( \rho_1 \) and \( \rho_o \), restricted to \( F \), differ by a map \( Z_\phi \) so that \( Z_\phi|_F = - \sum_i J\hat{Z}_i(\phi)Z_i \), for some \( \tau \)-invariant smooth function \( \phi : F \to \mathbb{R} \). In particular, at the \( T^m \)-fixed points of \( F \), the algebraic representatives of \( \rho_1 \) and \( \rho_o \) coincide. On the other hand, we note that \( Z_{\rho_1}|_F \) is the \((B)\)-dual of a moment map for the action of \( Z(K) \) on \( F \) w.r.t. \( \rho_1|_F \), and therefore \( Z_{\rho_1}(F) \subset \mathfrak{h}(\mathfrak{t}) \) is a convex polytope with vertices given by \( Z_{\rho_1}(F) \); by the previous remark we see that \( Z_{\rho_1}(F) = Z_{\rho_o}(F) \). By the fact that for any \( \alpha \in R^+_m \)

\[
\rho_1(\hat{X}_\alpha, J\hat{X}_\alpha)|_F = i\alpha(Z_{\rho_1}|_F),
\]

we have that \( Z_{\rho_1}(F) = Z_{\rho_o}(F) \subset \mathcal{C} \) and hence that \( Z_{\rho_o}(F) = Z_{\rho_o}^\ast + \Delta_{\tau,F} \subset \mathcal{C}^\ast \).
6. Examples

1. The Hirzebruch surfaces $F_n$ ($n \in \mathbb{N}$) exhaust all the homogeneous toric bundles $M$ over flag manifolds when $\dim \mathbb{C} M = 2$. The surface $F_n$ can be realized as the homogeneous bundle $SL(2, \mathbb{C}) \times_{B, \tau} \mathbb{CP}^1$ over $V = \mathbb{CP}^1 = SL(2, \mathbb{C})/B$; here $B$ is the standard Borel subgroup of $SL(2, \mathbb{C})$ given by $B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \in SL(2, \mathbb{C}) \right\}$ and $\tau : B \rightarrow \mathbb{C}^*$ is given by $\tau(\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}) = \alpha^n$, where $\mathbb{C}^*$ acts on $\mathbb{CP}^1$ by $\zeta \mapsto \alpha \zeta$ for $\alpha \in \mathbb{C}^*$ and $\zeta \in \mathbb{C} \cup \{\infty\}$.

   It is well known that $F_n$ is Fano if and only if $n = 0, 1$ (see [2]). This property can be very rapidly established also by means of our Theorem 1.1. In fact, using our notation and identifying $\mathfrak{z}(t) = t$ with $\mathbb{R}$ by means of the map $\mathbb{R} \ni x \mapsto \text{diag}(ix, -ix) \in \mathfrak{su}(2)$, we have that $Z_V = -1/4$ and $\mathcal{C} = \{ x < 0 \}$. Then, it is quite immediate to check that $-B^{-1}(\Delta_P, \mathbb{CP}^1) = [-\frac{1}{4}, \frac{5}{4}]$, so that $F_n$ is Fano if and only if $-\frac{1}{4} + [-\frac{n}{8}, \frac{n}{8}] \subset \{ x < 0 \}$, i.e. $n < 2$.

2. Let us now assume that $F = \mathbb{CP}^m$ and that $T^m$ is the standard maximal torus of $SU(m + 1)$, so that $(T^m)^\mathbb{C}$ coincides with the subgroup of diagonal matrices in $SL(m + 1, \mathbb{C})$. Let us also denote by $C = \{ c_i \}$ the matrix with $\det C \neq 0$ with entries $c_j^i$ defined by $\tau(Z_i) = \sum_{j=1}^m c_j^i W_j$, where the $W_j$ are the matrices in $t$ defined by

   $$W_j = \frac{1}{m+1} \cdot \text{diag}(-i, \ldots, _{(j+1)}\text{th place}, \ldots, -i) \in t := \mathfrak{su}(m+1).$$

   In this case, the canonical polytope $\Delta_F$ is the convex polytope with vertices

   $$Q_r = -\sum_{j=1}^m W_j^*, \quad Q_r = Q_o + (m+1)W_r^*, \quad 1 \leq r \leq m.$$  \hspace{1cm} (6.1)

   So, condition (1.1) amounts to say that all the points $P_o^\vee = Z_V^\vee - \sum_{i,j=1}^m c_j^i Z_i^\vee$ and $P_r^\vee = P_o^\vee + (m+1)\sum_{j=1}^m c_j^i Z_j^\vee, 1 \leq r \leq m$, are in $\mathcal{C}^\vee$, or, equivalently, that the points

   $$P_o = Z_V - \sum_{i,j=1}^m c_j^i Z_i, \quad P_r = P_o + (m+1)\sum_{j=1}^m c_j^i Z_j, \quad 1 \leq r \leq m.$$  \hspace{1cm} (6.2)

   are all in $\mathcal{C}$.

   Let us now construct explicitly an homogeneous toric bundle with $c_1(M) > 0$ and fiber $F = \mathbb{CP}^2$. Consider the classical group $G = \text{SO}(4n)$ with $B(X, Y) = 2(2n-1)\text{Tr}(XY)$; we denote by $h_i, 1 \leq i \leq 2n$, the elements of $g = \mathfrak{so}(4n)$ given by $h_i = E_{2i+1} - E_{2i+2}$, where $E_{ij}$ denotes the matrix whose unique non trivial entry is 1 and in position $(i, j)$. Given $J_1 = \sum_{i=1}^n h_i$ and $J_2 = \sum_{i=n+1}^{2n} h_i$, we may consider the flag manifold $G/K = \text{Ad}(G) \cdot (J_1 + 2J_2) \cong \text{SO}(4n)/U(n) \times U(n)$. Using standard notations for the roots, we have that

   $$R^+_m = \{ \omega_i + \omega_j, 1 \leq i < j \leq 2n \} \cup \{ \omega_j - \omega_i, 1 \leq i < n < j \leq 2n \}.$$  \hspace{1cm} (6.3)

   We may consider $Z_i = \frac{J_i}{2\sqrt{n(2n-1)}}$ and the homomorphism $\tau : K \rightarrow SU(3)$ with $\tau_{|_K^\mathbb{C}} = e$ and $\tau(Z_i) = c_i^j W_j$ with $c_i^j = \frac{1}{2\sqrt{n(2n-1)}}$. Since $Z_2 \in \mathcal{C}$, by (6.2) we have
that the manifold $M \overset{\text{def}}{=} G \times K, \mathbb{C}P^2$ is Fano if and only if

$$
P_o = Z_V - \sum_{i,j} c_i^j Z_i = Z_V - \frac{3n}{\sqrt{4n(2n-1)}} (Z_1 + Z_2) =
$$

$$
= \frac{1}{4(2n-1)} \left( \sum_{1 \leq i < j \leq 2n} (h_i + h_j) + \sum_{1 \leq i \leq n, n+1 \leq i \leq 2n} (h_j - h_i) - 3 \sum_{i=1}^{2n} h_i \right)
$$

and

$$
P_1 = P_o + \frac{9n}{\sqrt{4n(2n-1)}} Z_1 = P_o + \frac{9}{4(2n-1)} \sum_{i=1}^{n} h_i
$$

are both in $\mathcal{C}$. A direct inspection shows that this occurs if and only if $n \geq 5$.

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