Anisotropic cosmology in Sáez-Ballester theory: classical and quantum solutions

J. Socorro* and M. Sabido**
Departamento de Física, DCC, Universidad de Guanajuato-Campus León,
Apartado Postal E-143, 37150, Guanajuato, México.

M.A. Sánchez and M.G. Frias Palos
Facultad de Ciencias de la Universidad Autónoma del Estado de México,
Instituto Literario No. 100, Toluca, 50000, Edo de Mex, México.

Recibido el 25 de noviembre de 2009; aceptado el 16 de marzo de 2010

We use the Sáez-Ballester theory on anisotropic Bianchi I cosmological model, with barotropic fluid and cosmological constant. We obtain the classical solution by using the Hamilton-Jacobi approach. Also the quantum regime is constructed and exact solutions to the Wheeler-DeWitt equation are found.

Keywords: Classical and quantum exact solutions; cosmology.

Usamos la teoría de Sáez-Ballester en el modelo anisotrópico Bianchi I con un fluido barotrópico y constante cosmológica. Obtenemos las soluciones clásicas usando el enfoque de Hamilton-Jacobi. El régimen cuántico también es construido y soluciones exactas a la ecuación de Wheeler-DeWitt son encontradas.

Descriptors: Soluciones clásicas y cuánticas exactas; cosmología.

PACS: 02.30.Jr; 04.60.Kz; 12.60.Jv; 98.80.Qc.

1. Introduction

In the 80’s Saez and Ballester [1] formulated a scalar-tensor theory of gravity in which the metric is coupled to a dimensionless scalar field. Several papers in the classical regime have been written [2-5], but a study of anisotropic models, where the anisotropy is introduced in the line element, has been connected. In this theory of gravity the strength of the coupling between gravity and the scalar field is determined by an arbitrary coupling function $\omega$; one particularly interesting result is the appearance of an antigravity regime, which suggests a possible connection to the missing matter problem in non-flat FRW cosmologies. In particular, Armendariz-Picon, et al., related this scenario to $K$-essence [6], which is characterized by a scalar field with a non-canonical kinetic energy. Usually $K$-essence models are restricted to lagrangian densities of the form

$$S = \int d^4x \sqrt{-g} f(\phi) (\nabla \phi)^2;$$

one of the motivations for considering this type of lagrangian originates from string theory [7] and is related to the Dark energy problem (for more details for K-essence applied to dark energy, see Ref. 8 and reference therein).

Furthermore, the quantization program of the theory is an open problem; this is related to the difficult of building the ADM formalism. In order to proceed with the quantization program, we transform this theory to a conventional one, by interpreting the dimensionless scalar field as part of the energy-momentum tensor as an exotic matter component; this is achieved by an appropriate transformation of the coupling function $\omega$, so that we can use this modified theory where the ADM formalism is well known [9].

In this paper we use this formulation to obtain classical and quantum solutions for the anisotropic Bianchi I cosmological model with cosmological constant $\Lambda$. The first step is to write Sáez-Ballester formalism in the usual manner, that is, we calculate the corresponding energy-momentum tensor to the scalar field and give the equivalent lagrangian density. Next, we proceed to obtain the corresponding canonical lagrangian for the Bianchi type I metric and calculate the classical hamiltonian constraint $H$, and finally find the Wheeler-DeWitt (WDW) equation for the model.

The simplest generalization of the lagrangian density for the Sáez-Ballester theory [1] with cosmological constant, is

$$L_{geo} = \left(R - 2\Lambda - F(\phi)\dot{\phi}\gamma\phi\phi\gamma\right),$$

where $\phi\phi\gamma = \gamma_{\alpha\beta}\phi^{\alpha}\phi^{\beta}$, $R$ is the scalar curvature, and $F(\phi)$ a dimensionless function of the scalar field. In classical field theory, this formalism corresponds to null potential for $\phi$, but with an exotic kinetic term due to the presence of $F(\phi)$.

From the lagrangian (2) we can build the complete action

$$I = \int_{\Sigma} \sqrt{-g} (L_{geo} + L_{mat}) d^4x,$$

where $L_{mat}$ is the matter lagrangian, and $g$ is the determinant of metric tensor. The field equations for this theory are

$$G_{\alpha\beta} + \frac{\Lambda}{2} F(\phi) \left(\dot{\phi}^2 - \frac{1}{2} g_{\alpha\beta} \phi^2 \phi^2\right) = 8\pi G T_{\alpha\beta},$$

$$2F(\phi)\phi^2\phi^2 + \frac{dF}{d\phi} \phi^2\phi^2 = 0,$$

where $G$ is the gravitational constant and as usual the semicolon means a covariant derivative.
This set of Eqs. (4) can be obtained using the general relativit y lagrangian with a particular energy-momentum tensor for the field \( \phi \),

\[
\mathcal{L}_\phi = F(\phi) g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi,
\]

\[
T_{\alpha \beta}(\phi) = F(\phi) \left( \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha \beta} \partial_\gamma \phi \partial^\gamma \phi \right),
\]

and we can rewrite action (3) as

\[
I = \int \sqrt{-g} \left( R - 2 \Lambda + \mathcal{L}_{\text{mat}} + \mathcal{L}_\phi \right) d^4 x;
\]

consequently, the classical equivalence between the two theories is established through the equations of motion. We can infer that this correspondence can be carried to the quantum regime, because only the hamiltonian constraint [9] is modified.

This work is arranged as follows, in Sec. 2 we construct the hamiltonian density for the cosmological model. In Sec. 3 the classical solutions using the Hamilton-Jacobi formalism are found. Here, we have used a barotropic perfect fluid as a matter content and a cosmological constant, obtaining the solutions for different epochs of evolution. In Sec. 4 the quantization scheme is presented, obtaining the corresponding Wheeler-DeWitt equation and its solutions for different quantization schemes is established through the equation of motion. We can infer that this correspondence can be carried to the quantum regime, because only the hamiltonian constraint [9] is modified.

2. The hamiltonian density

The line element for the Bianchi type I model has the form

\[
ds^2 = -N^2 d t^2 + e^{2 \Omega + 2 \beta_+ + 2 \sqrt{3} \beta_-} (d x^1)^2 + e^{2 \Omega + 2 \beta_+ + 2 \sqrt{3} \beta_-} (d x^2)^2 + e^{2 \Omega - 4 \beta_+} (d x^3)^2
\]

where \( N \) is the lapse function, \( \beta_\pm \) are the corresponding anisotropic parameters in the scale factors, and \( \Omega \) plays an analogous role as the scale factor in a Friedmann-Robertson-Walker cosmology \( (e^{\Omega} = \Omega) \). The total volume for all diagonal Bianchi cosmological models is given by \( V = e^{3 \Omega(t)} \), which appears in the solutions for this theory. The corresponding lagrangian density is

\[
\mathcal{L} = \frac{6 \dot{\Omega} e^{3 \Omega} \rho}{N} - \frac{6 \dot{\beta}_+ e^{3 \Omega}}{N} - \frac{6 \dot{\beta}_- e^{3 \Omega}}{N} + \frac{F(\phi)}{N} \dot{\phi}^2 e^{3 \Omega} + (16 N \pi G \rho - 2 N \Lambda) e^{3 \Omega}.
\]

which can be rewritten in canonical form,

\[
\mathcal{H}_{\text{can}} = \Pi_\Omega \dot{\Omega} + \Pi_{\beta_+} \dot{\beta}_+ + \Pi_{\beta_-} \dot{\beta}_- + \Pi_\phi \dot{\phi} - N \mathcal{H},
\]

with \( \mathcal{H} \) as the hamiltonian density and the momenta define the usual way: \( \Pi_i = \partial \mathcal{L}/\partial \dot{q}_i \), where \( q_i = (\Omega, \beta_+, \beta_-, \phi) \) are the field coordinates for this system,

\[
\Pi_\Omega = \frac{\partial \mathcal{L}}{\partial \dot{\Omega}} = 12 \dot{\Omega} e^{3 \Omega} \frac{N}{N}, \quad \rightarrow \quad \Pi_\Omega \frac{2}{2} = 6 \dot{\Omega} e^{3 \Omega} \frac{N}{N} \quad \rightarrow \quad \dot{\Omega} = \text{Ne}^{-3 \Omega} \frac{\Pi_\Omega}{12}.
\]

\[
\Pi_{\beta_+} = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_+} = -12 \dot{\beta}_+ e^{3 \Omega} \frac{N}{N}, \quad \rightarrow \quad \Pi_{\beta_+} \frac{2}{2} = -6 \dot{\beta}_+ e^{3 \Omega} \frac{N}{N} \quad \rightarrow \quad \dot{\beta}_+ = -\text{Ne}^{-3 \Omega} \frac{\Pi_{\beta_+}}{12},
\]

\[
\Pi_{\beta_-} = \frac{\partial \mathcal{L}}{\partial \dot{\beta}_-} = -12 \dot{\beta}_- e^{3 \Omega} \frac{N}{N}, \quad \rightarrow \quad \Pi_{\beta_-} \frac{2}{2} = -6 \dot{\beta}_- e^{3 \Omega} \frac{N}{N} \quad \rightarrow \quad \dot{\beta}_- = -\text{Ne}^{-3 \Omega} \frac{\Pi_{\beta_-}}{12},
\]

\[
\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 2 F(\phi) \dot{\phi} e^{3 \Omega} \frac{N}{N}, \quad \rightarrow \quad \Pi_\phi \frac{2}{2} = F(\phi) \dot{\phi} e^{3 \Omega} \frac{N}{N} \quad \rightarrow \quad \dot{\phi} = \text{Ne}^{-3 \Omega} \frac{\Pi_\phi}{2 F(\phi) e^{3 \Omega}}.
\]

The matter content is introduced as a barotropic perfect fluid \( P = \gamma \rho \) where \( \gamma \) is a constant between \(-1 < \gamma < 1\), with its energy-momentum tensor given by \( T_{\mu \nu} = (\rho + P) U_\mu U_\nu - g_{\mu \nu} P \) where \( U_\mu \) is the four-velocity, \( \rho \) is the energy density and \( P \) is the thermodynamic pressure in the fluid. From this tensor we obtain the differential equation \( 3 \dot{\Omega} \rho + 3 \Omega \dot{P} + \dot{\rho} = 0 \) and the solution \( \rho = \mu_t e^{-3 \Omega(1+\gamma)} \), with \( \mu_t \) a constant for the corresponding scenario. Finally we obtain the following expression for the canonical lagrangian:

\[
\mathcal{L}_{\text{can}} = \Pi_\Omega \dot{\phi} - \text{Ne}^{-3 \Omega} \frac{24}{24} \left[ \Pi_\Omega^2 - \Pi_{\beta_+}^2 + \Pi_{\beta_-}^2 + 6 \Pi_\phi^2 F(\phi) - 38 \pi G \mu_t e^{3 \Omega(1-\gamma)} + 48 \Lambda e^{6 \Omega} \right],
\]

and

\[
\mathcal{H} = \text{e}^{-3 \Omega} \frac{24}{24} \left( \Pi_\Omega - \Pi_{\beta_+} - \Pi_{\beta_-} + 6 \Pi_\phi \frac{F(\phi)}{e^{3 \Omega(1-\gamma)}} - 38 \pi G \mu_t e^{3 \Omega(1-\gamma)} + 48 \Lambda e^{6 \Omega} \right),
\]

and using the Lagrange equation for the field \( N \), we get the Hamiltonian constraint

\[
\mathcal{H} = \Pi_\Omega - \Pi_{\beta_+} - \Pi_{\beta_-} + 6 \Pi_\phi \frac{F(\phi)}{e^{3 \Omega(1-\gamma)}} - 38 \pi G \mu_t e^{3 \Omega(1-\gamma)} + 48 \Lambda e^{6 \Omega} = 0.
\]
the Hamiltonian takes the following form:

$$\frac{\partial S}{\partial \Omega} - \frac{\partial S_\beta}{\partial \beta_+} - \frac{\partial S_\beta}{\partial \beta_-} + \frac{6}{F(\phi)} \times \frac{\partial S_\phi}{\partial \phi} = 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\lambda e^{6\Omega} = 0,$$

and solving for $\Omega$ we have

$$\left(\frac{\partial S_\Omega}{\partial \Omega}\right)^2 - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\lambda e^{6\Omega} = \left(\frac{\partial S_\beta_+}{\partial \beta_+}\right)^2 + \frac{6}{F(\phi)} \left(\frac{\partial S_\beta_-}{\partial \beta_-} - \frac{\partial S_\beta_-}{\partial \beta_-}\right)^2 + \frac{6}{F(\phi)} \left(\frac{\partial S_\phi}{\partial \phi}\right)^2 = \xi^2.$$
\[
\frac{dS_\Omega}{d\Omega} = \sqrt{\xi^2 + 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} - 48\Delta e^{6\Omega}}
\]
\[
= 12e^{3\Omega} \frac{d\Omega}{d\tau}
\]
(16)
\[
\Delta \tau = \int \frac{12}{\sqrt{\xi^2 e^{-6\Omega} + 384\pi G\mu_\gamma e^{-3\Omega(1+\gamma)} - 48\Delta}} d\Omega
\]
(17)
where \(\xi\) is a separation constant. This equation does not have a general solution, but we can solve for the particular values of the \(\gamma\); these solutions are presented in the Table I.

The corresponding solutions for the anisotropic functions and the field \(\phi\) appear in Table II. For field \(\phi\) we consider
\[
\frac{6}{F(\phi)} \left( \frac{\partial S_{\phi}}{\partial \phi} \right)^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2 = \theta^2
\]
\[
\frac{dS_{\phi}}{d\phi} = \sqrt{\frac{F(\phi)}{6}} \theta = -2F(\phi) \frac{d\phi}{d\tau} e^{3\Omega}
\]
with \(\theta^2 = -\xi^2 + \kappa_+^2 + \kappa_-^2\), and with the quadrature solutions
\[
\int \sqrt{F(\phi)} d\phi = \frac{\theta}{2\sqrt{6}} \int e^{-3\Omega(\tau)} d\tau,
\]
(18)
Now considering the original Sáez-Ballester theory, \(F(\phi) = \omega \phi^m\), the corresponding solutions for all \(m\) are shown in Table II.

This set of solutions satisfies Einstein’s field equation (4) (these solutions were checked using the REDUCE package for symbolic calculations).

4. Quantum regime: Wheeler-DeWitt equation

For the quantum regime, we calculate the Wheeler-DeWitt equation
\[
\hat{\Psi} = \left\{ \hat{\Pi}_1^2 - \hat{\Pi}_2^2 - \hat{\Pi}_3^2 + \frac{6\hat{\Pi}_0^2 \phi^{-m}}{\omega} \right\} \Psi = 0
\]
\[
- 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Delta e^{6\Omega}
\]
where the momentum operators are given by \(\hat{\Pi}_q = -i\hbar \partial / \partial q\), \(\Psi\) is the wave function of the universe, and we have chosen \(\hbar = 1\); thus
\[
\hat{\Psi} = \left\{ \left( -\frac{\partial^2}{\partial \Omega^2} + Q \frac{\partial}{\partial \Omega} \right) \tilde{\phi} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} - 6\phi^{-m} \right\} \Psi = 0
\]
(19)
where we have used \((-\partial^2 / \partial \Omega^2 + Q \partial / \partial \Omega\) to address the factor ordering problem. Applying the separation method, \(\Psi = A(\Omega) B(\beta_+) C(\beta_-) D(\phi)\), we obtain
\[
\left\{ \begin{array}{l}
- \frac{1}{A} \frac{d^2 A}{d\Omega^2} + Q \frac{1}{A} \frac{dA}{d\Omega} + \frac{1}{B} \frac{d^2 B}{d\beta_+^2} + \frac{1}{C} \frac{d^2 C}{d\beta_-^2} \\
- \frac{1}{D} \frac{6\phi^{-m} \phi^{-m} d^2 D}{\omega} - 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)} + 48\Delta e^{6\Omega}
\end{array} \right\} = 0,
\]
yielding the following set of differential equations:
\[
\frac{1}{A} \frac{d^2 A}{d\Omega^2} + Q \frac{1}{A} \frac{dA}{d\Omega} = 384\pi G\mu_\gamma e^{3\Omega(1-\gamma)}
\]
\[
+ 48\Delta e^{6\Omega} = -a_1^2,
\]
(20)
\[
\frac{1}{B} \frac{d^2 B}{d\beta_+^2} = a_2^2,
\]
(21)
\[
\frac{1}{C} \frac{d^2 C}{d\beta_-^2} = a_3^2,
\]
(22)
\[
\frac{1}{D} \frac{6\phi^{-m} \phi^{-m} d^2 D}{\omega} = a_4^2,
\]
(23)
where the following relation between the separation variables arises: \(a_4^2 = -a_1^2 + a_2^2 + a_3^2\), where the sign for these constants is arbitrary.

The solution to Eqs. (21) and (22) has the generic form
\[
B = e^{\pm a_2 \beta_+}, \quad C = e^{\pm a_3 \beta_-},
\]
(24)

The solution to Eq. (23) is more complicated, due to the dependence on \(m\), which is a parameter in Sáez-Ballester theory. This equation is rewritten as \(d^2 D/d\phi^2 - \omega a_4^2 \phi^m / 6D = 0\), which is analogous to the equation found in Ref. 10, \(y'' - ax^m = 0\), with \(a = \omega a_4^2 / 6\) and \(m = n\).

1. Case \(m=2\) corresponds to the Euler equation, whose solution has the following structure [10]:
\[
D = \sqrt{\phi} \left\{ \begin{array}{l}
c_1 \phi^m + c_2 \phi^{-m} \quad \text{si } a > -\frac{1}{2} \\
c_1 + c_2 \ln \phi \quad \text{si } a = -\frac{1}{2} \\
c_1 \sin (m \ln \phi) + c_2 \cos (m \ln \phi) \quad \text{si } a < -\frac{1}{2}
\end{array} \right.
\]
(25)
where
\[
\mu = (1/2) \sqrt{1 + 4a}
\]
2. In the case \(m=4\), we introduce the following transformation: \(z = 1/\phi\) and \(\xi/\gamma = D\). The resulting equation has the following solution:
\[
D = D \left\{ \begin{array}{l}
c_1 \sinh (\sqrt{n} \phi) + c_2 \cosh (\sqrt{n} \phi) \quad \text{si } a > 0 \\
c_1 \sin (\sqrt{|n|} \phi) + c_2 \cos (\sqrt{|n|} \phi) \quad \text{si } a < 0
\end{array} \right.
\]
(26)
the case \(a=0\) is discarded as it implies \(\omega = 0\) (GR limit).
3. When \(m\) satisfies \(2/(m + 2) = 2n + 1\), where \(n\) is an integer, the general solution takes the form
where $D_6, D_7$ are integration constants and $q = (m + 2)/2 = 1/(2n + 1)$.

4. General solution for any m: the solution is expressed in terms of the Bessel and modified Bessel functions; for $\phi$,

$$D = \sqrt{\phi} Z_\nu \left( \frac{\sqrt{\alpha}}{q} \phi^q \right),$$

where $Z_\nu$ is a generic Bessel function, $\nu = 1/2q$ is the order of the Bessel function and $q = (m + 2)/2$. If $a < 0$ implies that $\omega < 0$, $Z_\nu$ becomes the modified Bessel function, $(I_\nu, K_\nu)$. When $a > 0$, $w > 0$, $Z_\nu \rightarrow (J_\nu, Y_\nu)$.

On the other hand, Eq. (20) does not have a general solution, but solutions for particular values of $\gamma$ can be found; this is achieved by rewriting in the following form:

$$\frac{d^2 A}{d\Omega^2} - Q \frac{dA}{d\Omega} + \left( 384\pi G \mu_\nu e^{3\Omega(1-\gamma)} - 48\Lambda e^{6\Omega} - a_1^2 \right) A = 0,$$

where $A$ is a generic Bessel function.

1. Any factor ordering $Q$ and the inflationary phenomenon $\gamma = -1$:

$$\frac{d^2 A}{d\Omega^2} - Q \frac{dA}{d\Omega} + \left[ b_1 e^{6\Omega} - a_1^2 \right] A = 0,$$

where $b_1 = 384\pi G \mu_\nu - 48\Lambda$;

2. Factor ordering $Q=0$ and $\gamma = 0$:

$$\frac{d^2 A}{d\Omega^2} - (48\Lambda e^{6\Omega} - b_0 e^{3\Omega} + a_1^2) A = 0,$$

3. Any factor ordering and stiff matter $\gamma = 1$:

$$\frac{d^2 A}{d\Omega^2} - Q \frac{dA}{d\Omega} + [b_1 - 48\Lambda e^{6\Omega}] A = 0,$$

where $b_1 = 384\pi G \mu_\nu - a_1^2$.

by making the transformation $z = e^{3\Omega}$ and $R = z^{-(a_1/3)} A$, the solution is constructed by the degenerate hypergeometric functions $F_1(a, b; z)$:

$$A = e^{a_1 \Omega} x exp \left[ \frac{2}{3} \sqrt{6} e^{a_1 \Omega} \right]$$

where $\lambda = -\frac{1}{2k}$, $k = \frac{2}{3}\sqrt{6} \Lambda$, $B(k) = 9k \left( \frac{a_1}{3} + 1 \right) + b_0$.

5. Conclusions

An equivalent lagrangian density was constructed in order to apply the quantum regime in the Sáez-Ballester theory, where the constant $\omega$ can be used in order to have a lorencian or pseudo-lorenzian signature for the Wheeler-DeWitt equation. The values for this parameter in the classical regime are dictated when we apply the reality conditions on the functions, and are encoded in the parameter $a$, Eqs. (25,26). In this sense, classical and quantum exact solutions were found for the cosmological Bianchi type I model in the frame of Sáez-Ballester theory, for the different values of the $\gamma$ parameter.
The presence of the exotic field $\phi$ does not delay the anisotropic behavior in this model; the classical behavior of this field for large value in the parameter $\omega$ is similar to the anisotropic parameters $\beta_{\pm}$.

**Acknowledgments**

This work was partially supported by CONACYT grants 51306, 56946 and 62253. DINPO (2009) and PROMEP grants UGTO-CA-3 and UGTO-PTC-085. This work is part of the collaboration within the Instituto Avanzado de Cosmologia. Many calculations were done by Symbolic Program REDUCE 3.8.

* e-mail: socorro@fisica.ugto.m

** e-mail: msabido@fisica.ugto.m

1. D. Saez and V.J. Ballester, *Phys. Lett. A* **113** (1986) 467.
2. T. Singh and A.K. Agrawal, *Astrophys. Space Sci.* **182** (1991) 289.
3. Shri Ram and J.K. Singh, *Astrophys. Space Sci.* **234** (1995) 325.
4. G. Mohanty and S.K. Pattanaik, *Theor. Appl. Mech.* **26** (2001) 59.
5. C.P. Singh and Shri Ram, *Astrophys. Space Sci.* **284**, (2003) 1199.
6. C. Armendariz-Picon, V. Mukhanov and P.J. Steinhardt, *Phys. Lett.* **85** (2000) 4438; *Phys. Rev. D* **63** (2001) 103510.
7. C. Armendariz-Picon, T. Damour and V. Mukhanov, *Phys. Lett. B* **458** (1999) 209; J. Garriga and V. Mukhanov, *Phys. Lett. B* **458** (1999) 219.
8. E.J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, [arXiv:hep-th 0603057].
9. M.P. Ryan, *Hamiltonian cosmology*, (Springer, Berlin, 1972).
10. A.C. Polyanin and V.F. Zaitsev, *Handbook of Exact solutions for ordinary differential equations* (Second edition, Chapman & Hall/CRC. 2003).