Koszul complexes and spectral sequences associated with Lie algebroids

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Abstract
We study some spectral sequences associated with a locally free $\mathcal{O}_X$-module $\mathcal{A}$ which has a Lie algebroid structure. Here $X$ is either a complex manifold or a regular scheme over an algebraically closed field $k$. One spectral sequence can be associated with $\mathcal{A}$ by choosing a global section $V$ of $\mathcal{A}$, and considering a Koszul complex with a differential given by inner product by $V$. This spectral sequence is shown to degenerate at the second page by using Deligne’s degeneracy criterion. Another spectral sequence we study arises when considering the Atiyah algebroid $\mathcal{D}_E$ of a holomorphic vector bundle $E$ on a complex manifold. If $V$ is a differential operator on $E$ with scalar symbol, i.e, a global section of $\mathcal{D}_E$, we associate with the pair $(E, V)$ a twisted Koszul complex. The first spectral sequence associated with this complex is known to degenerate at the first page in the untwisted ($E = 0$) case.

Keywords Lie algebroids · Koszul complexes · Holomorphic equivariant cohomologies · Spectral sequences

Mathematics Subject Classification 14F05 · 14F40 · 32L10 · 55N25 · 55N91 · 55R20

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1 Introduction

In this paper we consider some spectral sequences that one can attach to a Lie algebroid. To be more precise, if $X$ is a complex manifold, or a regular noetherian scheme over an algebraically closed field $k$ of characteristic zero, we consider a locally free $\mathcal{O}_X$-module $\mathcal{A}$ having a Lie algebroid structure (definitions will be given in the next Section). One can introduce a complex $\Omega^\bullet_{\mathcal{A}} = \Lambda^\bullet \mathcal{A}^*$ which is a generalization of the (holomorphic) de Rham complex $\Omega^\bullet_X$. Now a Lie algebroid $\mathcal{A}$ comes with a morphism of sheaves of Lie $k$-algebras (the anchor morphism) to the tangent sheaf $\Theta_X$, and the kernel of the anchor is a sheaf of ideals of $\mathcal{A}$ (and a sheaf of Lie $\mathcal{O}_X$-algebras); this allows one to introduce, in analogy with the Hochschild-Serre spectral sequence [14], a filtration leading to a spectral sequence which converges to the hypercohomology $\mathbb{H}(X, \Omega^\bullet_{\mathcal{A}})$. This was already considered in [3] in the $C^\infty$ case; moreover, [16, 17] describe this spectral sequence in the case of the Atiyah algebroid of a vector bundle. In [4] and [5] this and other spectral sequences were studied in detail. Lie-Rinehart algebras can be regarded as special cases of Lie algebroids, so that we get a spectral sequence for Lie-Rinehart algebras: this generalizes the Hochschild-Serre spectral sequence for ideals in Lie algebras [14].

Other spectral sequences arise when we fix a section $V$ of $\mathcal{A}$; this yields a complex of the Koszul type, which we call a Lie-Koszul complex. Then the general machinery of homological algebra [13, 18] produces two spectral sequences. In Sect. 2, by using Deligne’s degeneracy criterion [11], we show that the second spectral sequence degenerates. The fact that this spectral sequence satisfies the condition of Deligne’s criterion means that the Lie-Koszul complex of a (holomorphic) Lie algebroid is formal (it is isomorphic, in the derived category of coherent sheaves, with the complex formed by its cohomology sheaves).

To study the first spectral sequence of a Lie-Koszul complex, we specialize to the case when $\mathcal{A}$ is the Atiyah algebroid of a holomorphic vector bundle $\mathcal{E}$ on a complex manifold $X$ (Sect. 3). Let us recall that $\mathcal{D}_\mathcal{E}$ is the bundle of first order differential operators on $\mathcal{E}$ with scalar symbol. $\mathcal{D}_\mathcal{E}$ sits in an exact sequence of sheaves of $\mathcal{O}_X$-modules

$$0 \to \text{End}(\mathcal{E}) \to \mathcal{D}_\mathcal{E} \xrightarrow{\sigma} \Theta_X \to 0$$

(1)

where $\sigma$ is the symbol map. This spectral sequence relates to the twisted holomorphic equivariant cohomology we introduced in [6]. For $\mathcal{E} = 0$ (i.e., in the case of the de Rham complex) this spectral sequence was studied by Carrell and Lieberman [8] and Bismut [2] when $X$ is Kähler manifold. In that case the spectral sequence degenerates at the first page.
2 Formality of the Lie-Koszul complexes

We consider a (holomorphic) Lie algebroid $\mathcal{A}$, over $X$, the latter being a complex manifold, or a regular noetherian scheme over an algebraically closed field $k$. We choose a global section $V$ of $\mathcal{A}$ and consider the morphism (inner product) $i_V : K_{\mathcal{A}}^\cdot \to K_{\mathcal{A}}^\cdot + 1$, where $K_{\mathcal{A}}^p = \Omega_{\mathcal{A}}^{-p}$, $p \leq 0$. We shall call $(K_{\mathcal{A}}^\cdot, i_V)$ the Lie-Koszul complex associated with the pair $(\mathcal{A}, V)$. This generalizes the Koszul complex $(\Omega_{\mathcal{A}}^{-\cdot}, i_V)$ associated with the complex of differential forms on $X$ with the differential given by the inner product by a (holomorphic) vector field $V$ on $X$. This will be called the de Rham-Koszul complex associated with the vector field $V$.

By general principles [13, 18] we can associate two spectral sequences with this complex, both converging to the hypecohomology $\mathbb{H}(K_{\mathcal{A}}^\cdot, i_V)$. In general, if $\mathcal{A}$, $\mathcal{B}$ are Abelian categories, denote by $D^+(\mathcal{A})$ the derived category of complexes of objects in $\mathcal{A}$ bounded from below, and let $F : D^+(\mathcal{A}) \to \mathcal{B}$ a cohomological functor.¹ Let $\mathcal{K}$ be an object in $D^+(\mathcal{A})$. We recall from [13, 18] that with these data one can associate two spectral sequences, both functorial in $\mathcal{K}$, and both converging to $R^\cdot F(\mathcal{K})$. The first two pages of the first spectral sequence are

$$I_1^{p,q} = R^p F(\mathcal{K}^q), \quad I_2^{p,q} = H^p(R^q F(\mathcal{K}))$$

and the differential $d_1$ coincides (perhaps up to a sign, depending on conventions) with the differential of the complex $\mathcal{K}$. The second page of the second spectral sequence is

$$II_2^{p,q} = R^p F(H^q(\mathcal{K})).$$

The degeneration of the second spectral sequence may be studied by means of Deligne’s degeneracy criterion [11]. Let us state it in generality. We shall replace the derived category $D^+(\mathcal{A})$ by the bounded derived category $D^b(\mathcal{A})$.

**Theorem 2.1** (Deligne) The following two conditions are equivalent:

(i) the spectral sequence $\Pi$, degenerates at its second page for every choice of the functor $F$;

(ii) $\mathcal{K}$ is isomorphic to $\bigoplus H^i(\mathcal{K})[-i]$ in $D^b(\mathcal{A})$.

(In the language of homological algebra, the second condition is called formality of the complex $\mathcal{K}$.)

¹ $F$ is said to be a cohomological functor if it maps every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ to a long exact sequence

$$R^i F(X) \xrightarrow{R^i F(u)} R^i F(Y) \xrightarrow{R^i F(v)} R^i F(Z) \xrightarrow{R^i F(w)} R^{i+1} F(X).$$
To apply Deligne’s criterion to our case we take $\mathcal{A} = \text{Coh}(X)$, $\mathcal{B} = \textbf{K} (\text{Ab})$ (the category of complexes of Abelian groups) and for $F$ we take the global section functor $\Gamma$. The object we fix in $D^b(X)$ is the Lie-Koszul complex $(\mathcal{H}_{\mathcal{A}}^\bullet, i_V)$. We denote by $\mathcal{H}^\bullet_{\mathcal{A}}$ the cohomology sheaves of the complex $(\mathcal{H}_{\mathcal{A}}^\bullet, i_V)$, and by $Y$ the scheme of zeroes of $V$. It is a closed, possibly nonreduced, subscheme (analytical subspace) of $X$. The sheaves $\mathcal{H}^\bullet_{\mathcal{A}}$ are supported on $Y$. Let $j : Y \to X$ be the scheme-theoretic inclusion, or the inclusion as a morphism in the category of analytic spaces (a closed immersion). The functor $j_!$ is right adjoint to $j^*$, so that there are morphisms $j^* : \mathcal{F} \to j_! j^* \mathcal{F}$ for every coherent sheaf $\mathcal{F}$ on $X$. There is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}^P_{\mathcal{A}} & \xrightarrow{j^*} & j_! j^* \mathcal{H}^P_{\mathcal{A}} \\
\downarrow i_V & & \downarrow 0 \\
\mathcal{H}^{P+1}_{\mathcal{A}} & \xrightarrow{j^*} & j_! j^* \mathcal{H}^{P+1}_{\mathcal{A}}
\end{array}
$$

(2)
i.e., $j^*$ is a morphism of complexes if we equip $j_! j^* \mathcal{H}^\bullet_{\mathcal{A}}$ with the zero morphisms. Finally, $\mathcal{H}^\bullet_{\mathcal{A}} \simeq j_! j^* \mathcal{H}^\bullet_{\mathcal{A}}$. Now we have:

**Proposition 2.2** The morphism of complexes $j^* : (\mathcal{H}^\bullet_{\mathcal{A}}, i_V) \to (j_! j^* \mathcal{H}^\bullet_{\mathcal{A}}, 0)$ is a quasi-isomorphism.

As a consequence, the objects $(\mathcal{H}^\bullet_{\mathcal{A}}, i_V)$ and $\bigoplus_i \mathcal{H}^\bullet \mathcal{A}[-i]$ are isomorphic in the derived category $D^b(X)$. By Deligne’s degeneracy criterion, we obtain that the spectral sequence $\mathcal{II}_1$ degenerates at the second page.

We can also say something about the hypercohomology $\mathbb{H}^\bullet(\mathcal{H}^\bullet_{\mathcal{A}})$. Let us denote by $\text{dim} Y$ the dimension of the highest-dimensional component of $Y$. The proof of the following result goes as in the case of the de Rham-Koszul complex treated in [8], p. 306.

**Proposition 2.3** $\mathbb{H}^m(\mathcal{H}^\bullet_{\mathcal{A}}, i_V) = 0$ for $m > \text{dim} Y$.

**Proof** Where $V \neq 0$ the Lie-Koszul complex is exact, so that the supports of the cohomology sheaves $\mathcal{H}^\bullet_{\mathcal{A}}$ are contained in $Y$; hence $\mathcal{II}^{p,q}_2 = 0$ for $p > \text{dim} Y$. Moreover, $\mathcal{H}^\bullet_{\mathcal{A}} = 0$ for $q > 0$. Thus $\mathcal{II}^{p,q}_2 = 0$ for $p + q > \text{dim} Y$. By standard homological arguments we get the thesis. $\square$

If $\text{dim} Y = 0, 1$, this gives an easy proof of the degeneration of the second spectral sequence at the second page, since $d_2 : \mathcal{II}^{p,q}_2 \to \mathcal{II}^{p+2,q-1}_2$ vanishes in that case. One also has

$$
\mathbb{H}^m(\mathcal{H}^\bullet_{\mathcal{A}}, i_V) \simeq \bigoplus_{p+q=m} H^p(X, \mathcal{H}^\bullet_{\mathcal{A}}).
$$

When $\text{dim} Y = 0$, the second page of the spectral sequence is such that $\mathcal{II}^{p,q}_2 = 0$ if $p \neq 0$.  

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3 A spectral sequence associated with Atiyah algebroids

In this section we study the spectral sequence $I$, in the special case when the Lie algebroid $\mathcal{A}$ is the Atiyah algebroid $\mathcal{D}_E$ of a holomorphic vector bundle $\mathcal{E}$ on a complex manifold $X$ (as in eq. (1)).

We fix once and for all a section $V$ in $\Gamma(\mathcal{D}_E)$. The pair $(\mathcal{E}, V)$ is called an equivariant holomorphic vector bundle. (“Equivariant” refers to the fact that $V$ covers the infinitesimal action of the vector field $\sigma(V)$ on $X$.) We consider the associated Lie-Koszul complex, i.e., the complex $(\mathcal{K}^\bullet_{\mathcal{E}}, i_V)$ where $\mathcal{K}^p_{\mathcal{E}} = \Lambda^{-p} \mathcal{D}_E^* \otimes \Omega^0_{\mathcal{E}}$ for $p \leq 0$, and $\mathcal{K}^p_{\mathcal{E}} = 0$ for $p > 0$. This twisted Koszul complex, or, to be more precise, its Dolbeault resolution, is a building block of a “twisted holomorphic equivariant cohomology” that we introduced in [6] and for which we proved a localization formula that generalizes Carrell-Lieberman’s [7, 9], Feng-Ma’s [12] and Baum-Bott’s [1] formulas.

The spectral sequence $I$ relates in this case to the double complex we introduced in [6]. For $E = 0$ this spectral sequence was studied by Carrell and Lieberman [8] (see also Bismut [2]) and in turn relates to K. Liu’s “untwisted” holomorphic equivariant cohomology [15].

We denote by $\Omega^{p,q}_X$ the sheaf of differential forms of type $(p, q)$ on $X$, and consider the complex

$$Q^k_{\mathcal{E}}(X) = \bigoplus_{q-p=k} \Gamma \left( \Lambda^p \mathcal{D}_E^* \otimes \partial_{\mathcal{E}} \Omega^0_{\mathcal{E}} \right)$$

with the differential $\delta_{\mathcal{E}, V} = \partial_{\mathcal{E}} + i_V$, where by $\partial_{\mathcal{E}}$ we collectively denote the Cauchy-Riemann operators of the bundles $\Lambda^p \mathcal{D}_E^*$. We denote by $H^k_{\mathcal{E}}(X, \mathcal{E})$ the cohomology of this complex. For $E = 0$ this reduces to the cohomology introduced by K. Liu [15] (see also Carrell and Lieberman [8] and Bismut [2].)

**Remark 3.1** If $V = 0$ then $H^k_{\mathcal{E}}(X, \mathcal{E}) = \bigoplus_{q-p=k} H^q(X, \Lambda^p \mathcal{D}_E^*)$.

**Proposition 3.2** The cohomology $H^k_{\mathcal{E}}(X, \mathcal{E})$ is isomorphic to the hypercohomology $\mathbb{H}^* (\mathcal{K}^\bullet_{\mathcal{E}}, i_V)$ of the complex $(\mathcal{K}^\bullet_{\mathcal{E}}, i_V)$.

**Proof** The double complex $\Lambda^{-} \mathcal{D}_E^* \otimes \partial_{\mathcal{E}} \Omega^0_{\mathcal{E}}$ is an acyclic resolution of the complex $\mathcal{K}^\bullet_{\mathcal{E}}$, and the total complex of $(\Lambda^{-} \mathcal{D}_E^* \otimes \partial_{\mathcal{E}} \Omega^0_{\mathcal{E}}, i_V, \partial_{\mathcal{E}})$ coincides with $(\Lambda^* \mathcal{D}_E^*, \delta_{\mathcal{E}, V})$. (This resolution is not made by coherent sheaves, but the argument works anyway, just going into the category of sheaves of Abelian groups.)

We denote

$$G^k_p = \bigoplus_{0 \leq p' \leq -p} \Gamma \left( \Lambda^{p'} \mathcal{D}_E^* \otimes \partial_{\mathcal{E}} \Omega^0_{\mathcal{E}} \right)$$

with $p \leq 0$, so that $G^k_p$ is a descending filtration of $Q^k_{\mathcal{E}}(X)$. Note that
This filtration of the complex \((Q_\partial^p(X), \tilde{\partial}, V)\) defines a spectral sequence whose zeroth page is

\[ E_0^{pq} = \frac{G_p^{p+q}}{G_{p+1}} = \Gamma \left[ \Lambda^{-p} \mathcal{D}_{\partial}^p \otimes \partial^- \Omega_{X}^{0,q} \right]. \]

The spectral sequence converges to the cohomology \(H_V^*(X, \partial)\). The differential \(d_0\) coincides with \(\tilde{\partial}\), as one easily checks. Therefore,

\[ E_1^{pq} = H^q(E_0^{pq}, d_0) = H^q(\Gamma[\Lambda^{-p} \mathcal{D}_{\partial}^p \otimes \partial^- \Omega_{X}^{0,*}], \tilde{\partial}) \cong H^q(X, \Lambda^{-p} \mathcal{D}_{\partial}). \]

It is now easy to check that this spectral sequence coincides with \(I\).

Henceforth we assume that the zero locus \(Y\) of \(V\) is a complex submanifold of \(X\). Therefore it makes sense to consider the complex (3) on \(Y\); after letting \(\tilde{\partial} = \partial|_Y\), we denote this new complex \(Q_{\tilde{\partial}}^*(Y)\). Denoting by \(j : \tilde{Y} \to X\) the embedding, we have the restriction morphism \(j^* : Q_{\tilde{\partial}}^*(X) \to Q_{\partial}^*(Y)\), which is a morphism of filtered complexes. We are going to show that, under some conditions, this is a quasi-isomorphism.

Note that there is an exact sequence

\[ 0 \to \mathcal{D}_{\tilde{\partial}} \to \mathcal{D}_{\partial|Y} \to N_{Y/X} \to 0 \quad (4) \]

where \(N_{Y/X}\) is the normal bundle to \(Y\). Since \(V\) is zero on \(Y\), the commutator \(\mathbb{L}_V(u) = [V, u]\) is well defined if \(u \in \mathcal{D}_{\partial|Y}\). This operator vanishes on \(\mathcal{D}_{\tilde{\partial}}\), so it is well defined on \(N_{Y/X}\). If it is injective, by composing with the projection \(\mathcal{D}_{\partial|Y} \to N_{Y/X}\) it yields an isomorphism, thus splitting the sequence (4).

For clarity, we stress what we are assuming:

**Assumption 3.3** The zero locus \(Y\) of \(V\) is a complex submanifold on \(X\), and the morphism \(\mathbb{L}_V : N_{Y/X} \to \mathcal{D}_{\partial|Y}\) is injective.

This implies the following preliminary result. Let \(\mathcal{H}_{\tilde{\partial}}\) be the complex of sheaves on \(Y\)

\[ \mathcal{H}_{\tilde{\partial}}^p = \Lambda^{-p} \mathcal{D}_{\tilde{\partial}}^p \]

with the zero differential.

**Lemma 3.4** \(j^* \mathcal{H}_{\partial}^p \cong \mathcal{H}_{\tilde{\partial}}^p\). In particular, \(\mathcal{H}_{\partial}^p = 0\) if \(-p > \dim Y\).

**Proof** There is a naturally defined morphism \(j^* \mathcal{H}_{\partial}^p \to \mathcal{H}_{\tilde{\partial}}^p\). We need to show that this gives an isomorphism between the stalks of the two sheaves. Considering the exact sequence (1) restricted to the stalks at a point \(y \in Y\), it splits, and one has

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Let $\tilde{V}$ be the vector field $\tilde{V} = \sigma(V)$. It vanishes on $Y$. Then one knows that the cohomology of the complex $(\Omega^{-\bullet}X, i_{\tilde{V}})$ restricted to $Y$ is isomorphic to the cohomology of the complex $(\Omega^{-\bullet}Y, 0)$ [2]. This, together with the Künneth theorem, implies the result.

The following result generalizes to the twisted case Theorem 5.1 in [2]. The proof goes as in [2], but for clarity we report it here, adapted to the present situation, and with some more details.

**Theorem 3.5** Under the Assumption 3.3, the restriction morphism $j^*: Q^\bullet(\mathcal{E}) \to Q^\bullet(\mathcal{E}|_Y)$ is a quasi-isomorphism.

**Proof** Let $\mathcal{U}$ be an open cover of $X$, and consider the Čech-Koszul complex

$$C^{(k)}(X) = \bigoplus_{p+q=k} \check{C}^p(\mathcal{U}, \mathcal{H}^q_{\mathcal{E}})$$

with differential $\tilde{\delta} = \delta + i_{\tilde{V}}$, where $\delta$ is the usual Čech differential. We define the descending filtration

$$F_q = \bigoplus_{p' \geq p} \check{C}^{p'}(\mathcal{U}, \mathcal{H}^q_{\mathcal{E}}), \quad F^q_p = F_p \cap C^{(q)}(X)$$

so that $F^p_{q+1} \subset F^p_q$,

$$F^p_{q+1}/F^p_q = \check{C}^p(\mathcal{U}, \mathcal{H}^q_{\mathcal{E}}),$$

and

$$\tilde{\delta}(F^p_q) \subset F^{p+1}_{q+1} + F^{p+1}_q = F^{p+1}_q.$$  

Let $(E_\ast(X), d_\ast)$ be the ensuing spectral sequence. The $d_0$ differential acting on the 0-th page coincides with $i_{\tilde{V}}$, so that the first page of the spectral sequence is

$$E_1(X)^{p,q} = \check{C}^p(\mathcal{U}, \mathcal{H}^q_{\mathcal{E}}).$$

The differential $d_1$ acting on this complex is the Čech differential. By Lemma 3.4, we also have

$$E_1(X)^{p,q} \simeq \check{C}^p(\tilde{\mathcal{U}}, \mathcal{H}^q_{\mathcal{E}})$$

where $\tilde{\mathcal{U}}$ is the open cover of $Y$ obtained by restricting the open sets of $\mathcal{U}$ to $Y$.  

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Consider now the complex
\[ C^{(k)}(Y) = \bigoplus_{p+q=k} \tilde{\mathbb{C}}^p(\tilde{U}, \tilde{\mathcal{E}}^q). \]

The resulting spectral sequence \( E_r(Y) \) has a vanishing \( d_0 \) differential, hence \( E_1(Y) \) coincides with the \( E_0 \) page. The restriction morphism \( j^* \) induces a morphism \( E_1(X) \to E_1(Y) \). By the commutativity of the diagram (2), this is an isomorphism and commutes with the respective differentials (which are the Čech differentials of the respective Čech complexes). By [10, Ch. XV, Thm. 3.2] the successive pages of the two spectral sequences are isomorphic, and the spectral sequences converge to the same group. Therefore, the complexes \( C^*(X) \) and \( C^*(Y) \) are quasi-isomorphic.

Via the standard Čech-Dolbeault spectral sequence, the cohomology of the complex \( C^*(Y) \) is, after taking a direct limit on the covers \( \mathcal{U} \), isomorphic to the cohomology of \( (\mathcal{O}^*_{\partial V}, \partial) \). In the same way, the cohomology of \( C^*(X) \) is isomorphic, after taking a direct limit, to the cohomology of \( (\mathcal{O}^*_{\partial V}, \partial, \partial_{\partial V}) \). This concludes the proof. \( \square \)

**Corollary 3.6** \( H^k_V(X, \mathcal{E}) \cong \bigoplus_{q-p=k} H^q(Y, \Lambda^p \mathcal{D}^a). \)

(Compare with Remark 3.1.)

**Proof** Since \( V = 0 \) on \( Y \) this follows from Remark 3.1. \( \square \)

Let us eventually consider the first spectral sequence \( I_1 \). Its first page is
\[ I_{1}^{p,q} = H^{q}(X, \Lambda^{p} \mathcal{D}^a). \]

In the untwisted \((\mathcal{E} = 0)\) case, and assuming that \( X \) is compact and Kähler, Carrell and Lieberman [8], by an argument inspired by Deligne’s degeneracy criterion, show that \( d_1 = 0 \), so that this spectral sequence degenerates at the first page.

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