Central limit theorem for a class of one-dimensional kinetic equations

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Abstract We introduce a class of kinetic-type equations on the real line, which constitute extensions of the classical Kac caricature. The collisional gain operators are defined by smoothing transformations with rather general properties. By establishing a connection to the central limit problem, we are able to prove long-time convergence of the equation’s solutions toward a limit distribution. For example, we prove that if the initial condition belongs to the domain of normal attraction of a certain stable law \( \nu_{\alpha} \), then the limit is a scale mixture of \( \nu_{\alpha} \). Under some additional assumptions, explicit exponential rates for the convergence to equilibrium in Wasserstein metrics are calculated, and strong convergence of the probability densities is shown.

Keywords Central limit theorem · Domain of normal attraction · Stable law · Kac model · Smoothing transformations

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1 Introduction

This paper is concerned with the following kinetic-type evolution equation for a time-dependent probability measure \( \mu_t \) on the real line \( \mathbb{R} \),

\[
\partial_t \mu_t + \mu_t = Q^+(\mu_t) \quad (t \geq 0).
\]  

(1)

Here \( Q^+ \) is a generalized Wild convolution, and the probability measure \( Q^+(\mu_t) \) is defined in terms of its Fourier–Stieltjes transform,

\[
\hat{Q}^+(\mu_t)(\xi) = \mathbb{E} [\phi(t; L\xi)\phi(t; R\xi)] \quad (\xi \in \mathbb{R}),
\]  

(2)

where \( \phi(t; \xi) = \hat{\mu}_t(\xi) = \int_{\mathbb{R}} e^{i\xi v} \mu_t(dv) \) denotes the Fourier–Stieltjes transform of \( \mu_t \). Above, \((L, R)\) is a random vector defined on a probability space \((\Omega, \mathcal{F}, P)\) and \( \mathbb{E} \) denotes the expectation with respect to \( P \). Our fundamental assumption on \((L, R)\) is that there exists an \( \alpha \) in \((0, 2] \) such that

\[
\mathbb{E}[|L|^\alpha + |R|^\alpha] = 1.
\]  

(3)

Equations (1)–(3) constitute a generalization of previously studied kinetic models in one spatial dimension: The celebrated Kac equation is obtained for the particular choice \( L = \cos \Theta \) and \( R = \sin \Theta \) with a random angle \( \Theta \) that is uniformly distributed on \([0, 2\pi)\). Indeed, since \( L^2 + R^2 = \sin^2 \Theta + \cos^2 \Theta = 1 \) a.s., (3) holds with \( \alpha = 2 \). Moreover, the class of inelastic Kac models, introduced by Toscani and Pulvirenti in [21], fits into the framework of (1)–(3), letting \( L = |\cos \Theta|^{p-1} \cos \Theta \) and \( R = |\sin \Theta|^{p-1} \sin \Theta \), where \( p > 1 \) is the parameter of elasticity. With \( \alpha = 2/p \), one has

\[
|L|^\alpha + |R|^\alpha = 1. \quad \text{a.s.}
\]  

(4)

and thus also (3) holds.

The extension from the (inelastic) Kac equation satisfying (4) to the more general class of evolution equations satisfying (3) originates from special recent applications of kinetic theory. One such application is an approach to model the temporal distribution of wealth, represented by \( \mu_t \), in a simplified economy by means of (1)–(3) with \( \alpha = 1 \); see [17] and references therein. The relaxed condition (3) is a key element of the modeling, as it takes into account stochastic gains and losses due to the trade with risky investments. Indeed, wealth distributions with a Pareto tail are consistent with certain models satisfying (3), but are excluded under the stricter condition (4) of deterministic trading. Furthermore, we mention that a multi-dimensional extension of (1)–(3) with \( \alpha = 2 \) has been used recently by [6] to model a homogeneous gas with velocity distributions \( \mu_t \) under the influence of a background heat bath.

The goal of this paper is to study the asymptotic behavior of solutions to (1) under the assumption (3) by means of probabilistic methods based on central limit theorems. The general idea to represent solutions to Kac-like equations in a probabilistic way dates back at least to McKean [18]; this approach has been fully formalized and
employed in the derivation of various analytical results in the last decade. For the original Kac equation, probabilistic methods have been used to estimate the approximation error of truncated Wild sums in [3], to study necessary and sufficient conditions for the convergence to a steady state in [12], to study the blow-up behavior of solutions of infinite energy in [4], to obtain rates of convergence to equilibrium of the solutions both in strong and weak metrics, [7,8,13]. Also the inelastic Kac model has been studied by probabilistic methods, see [2].

In this paper, the aforementioned probabilistic methods are adapted and extended to the setting (1)–(3). Our main result is the proof of long-time convergence (both weak and strong) of the solutions \( \mu_t \) to a steady state, and a precise characterization of the latter. Indeed, a striking difference that we observe between the previously studied models—obeying the strict condition (4)—and the more general class of models introduced here—which are subject to (3) only—is that the stationary solutions are stable laws in the first case, and scale mixtures of stable laws in the second. For example,\(^1\) for \( \alpha = 2 \), one can easily define arbitrarily small perturbations of the Kac equation that obey (3), violate (4), and possess steady states \( \mu_\infty \) with fat tails instead of Gaussians.

The qualitative properties of the steady state, such as the fatness of its tails, are determined by the mixing distribution for the stable laws. We prove that the mixing distribution is a fixed point of a suitable smoothing transformation related to \( \hat{Q}^+ \) and \( \alpha \), see (17). Then we apply results of Durrett and Liggett [9] and of Liu [15,16], where the existence and properties of these fixed points have been investigated.

The paper is organized as follows. In Sect. 2, we derive the stochastic representation of solutions to (1)–(3). Section 3 contains the statements of our main theorems. The results are classified into those on convergence in distribution (Sect. 3.1), convergence in Wasserstein metrics at quantitative rates (Sect. 3.2) and strong convergence of the probability densities (Sect. 3.3). All proofs are collected in Sect. 4.

2 Preliminary results

Throughout the paper, \((L, R)\) is a vector of non-negative random variables with given distribution. We emphasize that the restriction to non-negative \( L \) and \( R \) is mainly made to simplify the presentation and to avoid further case distinctions. Most of the convergence results remain valid if no sign restrictions are imposed in \( L \) and \( R \), for instance if the solution is symmetric.

We assume that (3) holds for some \( \alpha \in (0, 2] \), that, in this case, becomes

\[
\mathbb{E}[L^\alpha + R^\alpha] = 1,
\]

and we will frequently refer to the stricter condition

\[
L^\alpha + R^\alpha = 1 \quad \text{a.s.,}
\]

\(^1\) A specific example is given after Theorem 3.
which may or may not be satisfied. For later reference, introduce the convex function $S : [0, \infty) \to [-1, \infty]$ by

$$S(s) = \mathbb{E}[L^s + R^s] - 1,$$  \hspace{1cm} (7)

with the convention that $0^0 = 0$. Clearly, $S(\alpha) = 0$.

A probability measure $\mu_0$ on $\mathbb{R}$ is prescribed as initial condition for (1)–(2). Denote by $F_0$ its probability distribution function, by $\phi_0 = \hat{\mu}_0$ its Fourier–Stieltjes transform, and by $X_0$ some random variable (independent of everything else) with law $\mu_0$. Let $\mu_t$ be the corresponding solution to (1)–(2), which is shown to be unique and global in time below. Finally, recall that $\phi(t; \cdot)$ denotes the Fourier–Stieltjes transform of $\mu_t$.

2.1 Probabilistic representation of the solution

A semi-explicit expression for the Fourier–Stieltjes transform of the solution of (1) is given by the Wild sum

$$\phi(t; \xi) = \sum_{n=0}^{\infty} e^{-t}(1 - e^{-t})^n \hat{q}_n(\xi) \quad (t \geq 0, \xi \in \mathbb{R})$$  \hspace{1cm} (8)

where $\hat{q}_n$ is recursively defined by

$$\begin{cases} \hat{q}_0(\xi) := \phi_0(\xi) \\ \hat{q}_n(\xi) := \frac{1}{n} \sum_{j=0}^{n-1} \mathbb{E}[\hat{q}_j(L\xi) \hat{q}_{n-1-j}(R\xi)] \quad (n = 1, 2, \ldots). \end{cases}$$  \hspace{1cm} (9)

Originally, the series in (8) has been derived in [25] for the solution of the Kac equation. Following the ideas of McKean [18, 19] and of Gabetta and Regazzini [12], the Wild sum is now rephrased in a probabilistic way. To this end, let the following be given on a sufficiently large probability space $(\Omega, \mathcal{F}, P)$:

- a sequence $(X_n)_{n \in \mathbb{N}}$ of i.i.d. random variables with distribution function $F_0$;
- a sequence $((L_n, R_n))_{n \in \mathbb{N}}$ of i.i.d. random vectors, distributed as $(L, R)$;
- a sequence $(I_n)_{n \in \mathbb{N}}$ of independent integer random variables, each $I_n$ being uniformly distributed on the indices $\{1, 2, \ldots, n\}$;
- a stochastic process $(\nu_t)_{t \geq 0}$ with $\nu_t \in \mathbb{N}$ and $P(\nu_t = n) = e^{-t}(1 - e^{-t})^{n-1}$.

We assume further that $(I_n)_{n \in \mathbb{N}}, (L_n, R_n)_{n \in \mathbb{N}}, (X_n)_{n \in \mathbb{N}}$ and $(\nu_t)_{t \geq 0}$ are stochastically independent. Define a random array of weights $[\beta_{j,n} : j = 1, \ldots, n]_{n \geq 1}$ recursively: Let $\beta_{1,1} := 1$, $(\beta_{1,2}, \beta_{2,2}) := (L_1, R_1)$ and, for any $n \geq 2$,

$$(\beta_{1,n+1}, \ldots, \beta_{n+1,n+1}) := (\beta_{1,n}, \ldots, \beta_{n-1,n}, L_n \beta_{I_n,n}, R_n \beta_{I_n,n}, \beta_{I_n+1,n}, \ldots, \beta_{n,n}).$$  \hspace{1cm} (10)
Central limit theorem for a class of one-dimensional kinetic equations

Fig. 1 Two four-leafed McKean trees, with associated weights \( \beta_{j,n} \): the left tree is generated by \( I_4 = (1, 1, 3) \) and its weights are \( \beta_{1,4} = L_1 L_2, \beta_{2,4} = L_1 R_2, \beta_{3,4} = R_1 L_3, \beta_{4,4} = R_1 R_3 \); the right tree is generated by \( I_4 = (1, 1, 2) \) and its weights are \( \beta_{1,4} = L_1 L_2, \beta_{2,4} = L_1 R_2 L_3, \beta_{3,4} = L_1 R_2 R_3, \beta_{4,4} = R_1 R_3 \).

Remark 1 In [18, 19], the Wild series is related to a random walk on a class of binary trees. For an introduction to the theory of the so-called McKean trees, we refer to the article of Carlen, Carvalho and Gabetta [3]. The construction above relates to this as follows. Each finite sequence \( I_n = (I_1, I_2, \ldots, I_{n-1}) \) corresponds to a McKean tree with \( n \) leaves. The tree associated to \( I_{n+1} \) is obtained from the tree associated to \( I_n \) upon replacing the \( I_n \)th leaf (counting from the left) by a binary branching with two new leaves. The left of the new branches is labeled with \( L_n \), and the right one with \( R_n \). The weights \( \beta_{j,n} \) are associated to the leaves of the \( I_n \)-tree: \( \beta_{j,n} \) is the product of the labels assigned to the branches along the ascending path connecting the \( j \)th leaf to the root. See Fig. 1 for an illustration.

Finally, set

\[
W_n := \sum_{j=1}^{n} \beta_{j,n} X_j \quad \text{and} \quad V_t := \sum_{j=1}^{v_t} \beta_{j,v_t} X_j. \tag{11}
\]

Proposition 1 (Probabilistic representation) The law of \( V_t \) is the unique and global solution \( \mu_t \) to equation (1)–(2) with initial condition \( \mu_0 \).

A representation of the form (11) has already been successfully employed in studying the long-time behavior of solutions to the classical and the inelastic Kac equation, for instance, in [2, 12], respectively.

2.2 Stable laws

Some further notations need to be introduced. Recall that a probability distribution \( g_\alpha \) is said to be a centered stable law of exponent \( \alpha \) (with \( 0 < \alpha \leq 2 \)) if its Fourier–Stieltjes transform is of the form

\[
\hat{g}_\alpha(\xi) = \begin{cases} 
\exp\{-k|\xi|^\alpha (1 - i \eta \tan(\pi \alpha/2) \text{ sign } \xi)\} & \text{if } \alpha \in (0, 1) \cup (1, 2) \\
\exp\{-k|\xi| (1 + 2i \eta/\pi \log |\xi| \text{ sign } \xi)\} & \text{if } \alpha = 1 \\
\exp\{-\sigma^2|\xi|^2/2\} & \text{if } \alpha = 2.
\end{cases} \tag{12}
\]

where \( k > 0 \) and \( |\eta| \leq 1 \).
By definition, a distribution function $F$ belongs to the domain of normal attraction of a stable law of exponent $\alpha$ if for any sequence of independent and identically distributed real-valued random variables $(X_n)_{n \geq 1}$ with common distribution function $F$, there exists a sequence of real numbers $(c_n)_{n \geq 1}$ such that the law of $n^{-1/\alpha} \sum_{i=1}^{n} X_i - c_n$ converges weakly to a stable law of exponent $\alpha$.

It is well-known that, provided $\alpha \neq 2$, a distribution function $F$ belongs to the domain of normal attraction of an $\alpha$-stable law if and only if $F$ satisfies

$$\lim_{x \to +\infty} x^\alpha (1 - F(x)) = c^+ < +\infty, \quad \lim_{x \to -\infty} |x|^\alpha F(x) = c^- < +\infty. \quad (13)$$

Typically, one also requires that $c^+ + c^- > 0$ in order to exclude convergence to the probability measure concentrated in 0, but here we shall include the situation $c^+ = c^- = 0$ as a special case. The parameters $k$ and $\eta$ of the associated stable law in (12) are identified from $c^+$ and $c^-$ by

$$k = \left( c^+ + c^- \right) \frac{\pi}{2\Gamma(\alpha) \sin(\pi \alpha/2)}, \quad \eta = \frac{c^+ - c^-}{c^+ + c^-}, \quad (14)$$

with the convention that $\eta = 0$ if $c^+ + c^- = 0$. In contrast, if $\alpha = 2$, $F$ belongs to the domain of normal attraction of a Gaussian law if and only if it has finite variance $\sigma^2$. See for example Chapter 17 of [10].

2.3 Definition of the mixing distribution

From Proposition 1 it is clear that the behavior of $\mu_t$, as $t \to +\infty$, is determined by the behavior of the law of $W_n$ as $n \to +\infty$. Apropos of this we note that a direct application of the central limit theorem is not suitable to investigate the weak limit of $W_n$, since the weights in (11) are not independent. However, one can apply the central limit theorem to study the conditional law of $W_n$, given the array of weights $\beta_{j,n}$. To this end we shall prove that

$$M_n^{(\alpha)} := \sum_{j=1}^{n} \beta_{j,n}^{\alpha} \quad (15)$$

converges a.s. to a limit $M^{(\alpha)}$ as $n \to +\infty$, and that the max $j=1,...,n \beta_{j,n}$ converges to zero in probability. This allows to apply a central limit theorem for triangular arrays to the conditional law of $W_n$ and to prove that the latter converges weakly to an $\alpha$-stable law rescaled by $(M^{(\alpha)}_{\infty})^{1/\alpha}$. Hence one obtains that the limit law of $W_n$ is a scale mixture of $\alpha$-stable laws.

The origin of the model’s richness in steady states under the milder condition (5) is now easily understood. Condition (6) implies $M_n^{(\alpha)} = 1$ a.s., and thus the mixing distribution is degenerate. In contrast, condition (5) only implies $\mathbb{E}[M_n^{(\alpha)}] = 1$, and if (6) is violated, then the law of $M^{(\alpha)}_{\infty}$ is not concentrated anymore, according to the following result.
Proposition 2 Under condition \((5)\),

\[
E[M_{n}^{(\alpha)}] = E[M_{t}^{(\alpha)}] = 1 \quad \text{for all } n \geq 1 \text{ and } t > 0,
\]

and \(M_{n}^{(\alpha)}\) converges almost surely to a non-negative random variable \(M_{\infty}^{(\alpha)}\).

In particular, recalling the definition of \(S\) in \((7)\),

- if \((6)\) holds, then \(S(s) \geq 0\) for every \(s < \alpha\) and \(S(s) \leq 0\) for every \(s > \alpha\). Moreover, \(M_{n}^{(\alpha)} = M_{\infty}^{(\alpha)} = 1\) almost surely;
- if \((6)\) does not hold, and if \(S(\gamma) < 0\) for some \(0 < \gamma < \alpha\), then \(M_{\infty}^{(\alpha)} = 0\) almost surely;
- if \((6)\) does not hold, and if \(S(\gamma) < 0\) for some \(\gamma > \alpha\), then \(M_{\infty}^{(\alpha)}\) is a non-degenerate random variable with \(E[M_{\infty}^{(\alpha)}] = 1\) and \(E[(M_{\infty}^{(\alpha)})^{\frac{\gamma}{\alpha}}] < +\infty\). Moreover, the characteristic function \(\psi\) of \(M_{\infty}^{(\alpha)}\) is the unique solution of

\[
\psi(\xi) = E[\exp(i\xi V_{\infty})] = E[\exp(|\xi|^{\alpha}kM_{\infty}^{(\alpha)}(1 - i\eta \tan(\pi\alpha/2) \text{ sign } \xi))] \quad (\xi \in \mathbb{R})
\]

with \(-i\psi'(0) = 1\). Finally, for any \(p > \alpha\), the moment \(E[(M_{\infty}^{(\alpha)})^{\frac{p}{\alpha}}]\) is finite if and only if \(S(p) < 0\).

3 Statement of the main results

3.1 Convergence in distribution

We recall that \(V_{t}\) is a time-dependent random variable with law \(\mu_{t}\). The latter constitutes the solution to \((1)\)–\((2)\). Recall also that \(M_{\infty}^{(\alpha)}\) has been defined in Proposition 2.

Theorem 1 Assume that \((5)\) holds with \(\alpha \in (0, 1) \cup (1, 2)\) and that \(S(\gamma) < 0\) for some \(\gamma > 0\). Moreover, let condition \((13)\) be satisfied for \(F = F_{0}\) and let \(X_{0}\) be centered if \(\alpha > 1\). Then \(V_{t}\) converges in distribution, as \(t \to +\infty\), to a random variable \(V_{\infty}\) with the following characteristic function

\[
\phi_{\infty}(\xi) = E[\exp(i\xi V_{\infty})] = E[\exp(-|\xi|^{\alpha}kM_{\infty}^{(\alpha)}(1 - i\eta \tan(\pi\alpha/2) \text{ sign } \xi))]
\]

(\(\xi \in \mathbb{R}\)), where the parameters \(k\) and \(\eta\) are defined in \((14)\). In particular, the law of \(V_{\infty}\) is \(\alpha\)-stable if and only if \((6)\) holds, and \(V_{\infty} = 0\) a.s. if and only if \(c^{+} = c^{-} = 0\) or \(\gamma < \alpha\). In all other cases, \(E[|V_{\infty}|^{p}] < +\infty\) if and only if \(p < \alpha\).

A consequence of Theorem 1 is that if \(E[|X_{0}|^{\alpha}] < \infty\), then the limit \(V_{\infty}\) is zero almost surely, since \(c^{+} = c^{-} = 0\). The situation is different in the cases \(\alpha = 1\) and \(\alpha = 2\), where \(V_{\infty}\) is non-trivial provided that the first respectively second moment of \(X_{0}\) is finite.

Theorem 2 Assume that \((5)\) holds with \(\alpha = 1\) and that \(S(\gamma) < 0\) for some \(\gamma > 0\). If the initial condition possesses a finite first moment \(m_{0} = E[X_{0}]\), then \(V_{t}\) converges in distribution, as \(t \to +\infty\), to \(V_{\infty} := m_{0}M_{\infty}^{(1)}\). In particular, \(V_{\infty} = m_{0}\) a.s. if and
only if (6) holds, and \( V_\infty = 0 \) a.s. if and only if \( \gamma < 1 \) or \( m_0 = 0 \). In all other cases, \( \mathbb{E}[|V_\infty|^p] < +\infty \) for \( p > 1 \) if and only if \( S(p) < 0 \).

We remark that under the hypotheses of the previous theorem, the first moment of the solution is preserved in time. Indeed one has,

\[
\mathbb{E}[V_t] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j=1}^{\nu_t} \beta_{j,v_t} X_j |v_t, \beta_{1,v_t}, \ldots, \beta_{\nu_t,v_t} \right] \right] = m_0 \mathbb{E} \left[ M_{v_t}^{(1)} \right] = m_0,
\]

where the last equality follows from (16).

Theorem 2 above is the most natural generalization of the results in [17], where the additional condition \( \mathbb{E}[|X_0|^{1+\epsilon}] < \infty \) for some \( \epsilon > 0 \) has been assumed. The respective statement for \( \alpha = 2 \) reads as follows.

**Theorem 3** Assume that (5) holds with \( \alpha = 2 \) and that \( S(\gamma) < 0 \) for some \( \gamma > 0 \). If \( \mathbb{E}[X_0] = 0 \) and \( \sigma^2 = \mathbb{E}[X_0^2] < +\infty \), then \( V_t \) converges in distribution, as \( t \to +\infty \), to a random variable \( V_\infty \) with characteristic function

\[
\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi V_\infty)] = \mathbb{E} \left[ \exp(-\xi^2 \frac{\sigma^2}{2} M_{v_t}^{(2)}) \right] \quad (\xi \in \mathbb{R}).
\]

In particular, \( V_\infty \) is Gaussian if and only if (6) holds, and \( V_\infty = 0 \) a.s. if \( \gamma < 2 \) or \( \sigma = 0 \). In all other cases, \( \mathbb{E}[|V_\infty|^p] < +\infty \) for \( p > 2 \) if and only if \( S(p) < 0 \).

Under the hypotheses of the theorem, \( \mathbb{E}[V_t] = 0 \) for \( t \geq 0 \) and, moreover, taking into account that the \( X_j \) are independent and centered,

\[
\mathbb{E}[V_t^2] = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j,k=1}^{\nu_t} \beta_{j,v_t} \beta_{k,v_t} X_j X_k |v_t, \beta_{1,v_t}, \ldots, \beta_{\nu_t,v_t} \right] \right] = \sigma^2 \mathbb{E}[M_{v_t}^{(2)}] = \sigma^2,
\]

where we have used (16) in the last step.

**Example 1** For illustration of the applicability of Theorem 3 above, we introduce a family of (arbitrarily small) perturbations of the Kac equation, which exhibit stationary solutions with heavy tails; recall that the stationary states of the Kac model are Gaussians.

Define \( L = \sqrt{(1 - \epsilon) + \epsilon z^2} \cos \Theta \) and \( R = \sqrt{(1 - \epsilon) + \epsilon z^2} \sin \Theta \), with a random angle \( \Theta \) that is uniformly distributed on \([0, 2\pi]\) and an independent positive random variable \( z \) satisfying \( \mathbb{E}[z^2] = 1 \), \( \mathbb{E}[z^\gamma] < \infty \), and \( \mathbb{E}[z^\xi] = +\infty \) for some real numbers \( \xi > \gamma > 2 \). One verifies by direct calculations that (5) holds with \( \alpha = 2 \), and that \( S(\gamma) < 0 \) for all \( \epsilon > 0 \) sufficiently small, while \( S(\xi) = +\infty \). Thus, the hypotheses of Theorem 3 are met, and it follows that \( \mathbb{E}[V_\infty^\gamma] < +\infty \) and \( \mathbb{E}[V_\infty^\xi] = +\infty \).

In fact, the models so constructed possess heavy tailed steady states for arbitrary \( \epsilon > 0 \) even if \( z \) is a bounded random variable; it suffices that \( \mathbb{E}[z^\gamma] \) grows exponentially as \( s \to \infty \).
The technically most difficult result concerns the situation $\alpha = 1$: when $\mu_0$ belongs to the domain of normal attraction of a 1-stable distribution with $c^++c^- > 0$, then its first moment is infinite and Theorem 2 does not apply. Theorem 1 can be generalized to this situation as follows.

**Theorem 4** Assume that (5) holds with $\alpha = 1$ and that $S(\gamma) < 0$ for some $\gamma > 0$. Moreover, let the condition (13) be satisfied for $F = F_0$. Then the random variable

$$V^*_t := V_t - \sum_{j=1}^{n} q_{j,n} \nu_t \int_{\mathbb{R}} \sin(\beta_j,n x) dF_0(x), \quad (20)$$

converges in distribution to a limit $V^*_\infty$ with characteristic function

$$\phi_{\infty}(\xi) = \mathbb{E}[\exp(i\xi V^*_\infty)] = \mathbb{E}\left[\exp \left\{-|\xi| kM^{(1)}(1 + 2i\eta/\pi \log |\xi| \text{ sign } \xi)\right\}\right] \quad (21)$$

($\xi \in \mathbb{R}$), where the parameters $k$ and $\eta$ are defined in (14). In particular, the law of $V^*_\infty$ is 1-stable if and only if (6) holds, and $V^*_\infty = 0$ a.s. if and only if $c^+ = c^- = 0$ or $\gamma < 1$. In all other cases, $\mathbb{E}[|V^*_\infty|^p] < +\infty$ if and only if $p < 1$.

### 3.2 Rates of convergence in Wasserstein metrics

Theorems 1–3 above provide weak convergence of the solution $V_t$ to a limit $V_\infty$ as $t \to \infty$. The (exponential) rate at which this convergence takes place can be quantified in suitable Wasserstein metrics.

Recall that the Wasserstein distance of order $\gamma > 0$ between two random variables $X$ and $Y$ is defined by

$$W_\gamma(X, Y) := \inf_{(X', Y')} (\mathbb{E}|X' - Y'|^\gamma)^{1/\max(\gamma, 1)}. \quad (22)$$

The infimum is taken over all pairs $(X', Y')$ of real random variables whose marginal distribution functions are the same as those of $X$ and $Y$, respectively. In general, the infimum in (22) may be finite; a sufficient (but not necessary) condition for finite distance is that both $\mathbb{E}[|X|^\gamma] < +\infty$ and $\mathbb{E}[|Y|^\gamma] < +\infty$. For more information on Wasserstein distances see, for example, [22].

**Theorem 5** Assume (5) and $S(\gamma) < 0$, for some $\gamma$ with $1 \leq \alpha < \gamma \leq 2$ or $\alpha < \gamma \leq 1$. Assume further that (13) holds if $\alpha \neq 1$, or that $\mathbb{E}[|X_0|^\gamma] < +\infty$ if $\alpha = 1$, respectively. Then

$$W_\gamma(V_t, V_\infty) \leq AW_\gamma(X_0, V_\infty)e^{-Bt|S(\gamma)|}, \quad (23)$$

with $A = B = 1$ if $\gamma \leq 1$, or $A = 2^{1/\gamma}$ and $B = 1/\gamma$ otherwise.

Clearly, (23) is meaningful only if $W_\gamma(X_0, V_\infty) < +\infty$. This is guaranteed for $\alpha = 1$, by the hypothesis $\mathbb{E}[|X_0|^\gamma] < +\infty$. In all other cases, the requirement $W_\gamma(X_0, V_\infty) <$
+∞ is non-trivial, since by Theorem 1, either $V_\infty = 0$ or $\mathbb{E}[|V_\infty|^\alpha] = +\infty$. The following lemma provides a sufficient criterion tailored to the situation at hand.

**Lemma 1** Assume, in addition to the hypotheses of Theorem 5, that $\gamma < 2\alpha$ and that $F_0$ satisfies hypothesis (13) in the more restrictive sense that there exists a constant $K > 0$ and some $0 < \varepsilon < 1$ with

$$
|1 - c^+ x^{-\alpha} - F_0(x)| < K x^{-(\alpha+\varepsilon)} \text{ for } x > 0,
$$

$$
|F_0(x) - c^- (-x)^{-\alpha}| < K (-x)^{-(\alpha+\varepsilon)} \text{ for } x < 0.
$$

Provided that $\gamma < \alpha / (1 - \varepsilon)$, it follows $\mathcal{W}_\gamma (X_0, V_\infty) < +\infty$.

### 3.3 Strong convergence of densities

Under suitable hypotheses, the probability densities of $\mu_t$ exist and converge strongly in the Lebesgue spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

**Theorem 6** For given $\alpha \in (0, 1) \cup (1, 2]$, let the hypotheses of Theorems 1 or 3 hold with $\gamma > \alpha$. Assume further that (13) holds with $c^+ + c^- > 0$ if $\alpha < 2$, so that the $V_t$ converges in distribution, as $t \to +\infty$, to a non-degenerate limit $V_\infty$. Moreover assume also that

(H1) $L^r + R^r \geq 1$ a.s. for some $r > 0$, and

(H2) $X_0$ has a density $f_0$ with finite Linnik-Fisher functional, i.e., $h := \sqrt{f_0} \in H^1(\mathbb{R})$, that is its Fourier transform $\hat{h}$ satisfies $\int_{\mathbb{R}} |\xi|^2 |\hat{h}(\xi)|^2 d\xi < +\infty$.

Then, the random variable $V_t$ possesses a density $f(t)$ for all $t \geq 0$, $V_\infty$ has a density $f_\infty$, and the $f(t)$ converges, as $t \to +\infty$, to $f_\infty$ in any $L^p(\mathbb{R})$ with $1 \leq p \leq 2$.

**Remark 2** Notice that, since $S(\alpha) = 0$, condition (H1) can be satisfied only if $r < \alpha$.

### 4 Proofs

#### 4.1 Probabilistic representation (Propositions 1 and 2)

The proof of Proposition 1 is inspired by the respective proof for the Kac case from [12].

**Proof of Proposition 1** First of all it is easy to prove, following [25] and [18], that formulas (8) and (9) produce the unique solution to problem (1); see also [23]. From the definition of $V_t$ in (11), it easily follows that

$$
\mathbb{E}[e^{i\xi V_t}] = \sum_{n=0}^{\infty} e^{-t} (1 - e^{-t})^n \mathbb{E}[e^{i\xi W_{n+1}}] \quad (t > 0, \xi \in \mathbb{R}).
$$

$\square$ Springer
Hence, comparing the latter with the Wild sum representation (8) it obviously suffices
to prove that

\[ \hat{q}_{\ell-1}(\xi) = \mathbb{E}[e^{i\xi} W_\ell], \]  

(27)

which we will show by induction on \( \ell \geq 1 \). First, note that \( \mathbb{E}[\exp(i\xi W_1)] = \mathbb{E}[\exp(i\xi X_1)] = \phi_0(\xi) = \hat{q}_0(\xi) \) and \( \mathbb{E}[e^{i\xi} W_2] = \mathbb{E}[e^{i\xi}(L_1 X_1 + R_1 X_2)] = \hat{q}_1(\xi) \), which shows (27) for \( \ell = 1 \) and \( \ell = 2 \). Let \( n \geq 3 \), and assume that (27) holds for all \( 1 \leq \ell < n \); we prove (27) for \( \ell = n \).

Recall that the weights \( \beta_{j,n} \) are products of random variables \( L_i \) and \( R_i \). Define the random index \( K_n < n \) such that all products \( \beta_{j,n} \) with \( j \leq K_n \) contain \( L_1 \) as a factor, while the \( \beta_{j,n} \) with \( K_n + 1 \leq j \leq n \) contain \( R_1 \). By induction it is easily seen that \( P\{K_n = i\} = 1/(n-1) \) for \( i = 1, \ldots, n-1 \); c.f. Lemma 2.1 in [3]. Now,

\[
A_{K_n} := \sum_{j=1}^{K_n} \beta_{j,n} X_j, \quad B_{K_n} := \sum_{j=K_n+1}^{n} \frac{\beta_{j,n}}{R_1} X_j \quad \text{and} \quad (L_1, R_1)
\]

are conditionally independent given \( K_n \). By the recursive definition of the weights \( \beta_{j,n} \) in (10), the following is easily deduced: the conditional distribution of \( A_{K_n} \), given \( \{K_n = k\} \), is the same as the (unconditional) distribution of \( \sum_{j=1}^{k} \beta_{j,k} X_j \), which clearly is the same distribution as that of \( W_k \). Analogously, the conditional distribution of \( B_{K_n} \), given \( \{K_n = k\} \), equals the distribution of \( \sum_{j=k}^{n} \beta_{j,n-k} X_j \), which further equals the distribution of \( W_{n-k} \). Hence,

\[
\mathbb{E}[e^{i\xi} W_n] = \frac{1}{n-1} \sum_{k=1}^{n-1} \mathbb{E}\left[ e^{i\xi(L_1 A_k + R_1 B_k)} \mid \{K_n = k\} \right] \\
= \frac{1}{n-1} \sum_{k=1}^{n-1} \mathbb{E}\left[ e^{i\xi L_1 W_k} \mid L_1, R_1 \right] \mathbb{E}\left[ e^{i\xi R_1 W_{n-k}} \mid L_1, R_1 \right] \\
= \frac{1}{n-1} \sum_{j=0}^{n-2} \mathbb{E}[\hat{q}_{n-2-j}(L_1 \xi) \hat{q}_j(R_1 \xi)]
\]

which is \( \hat{q}_{n-1} \) by the recursive definition in (9).

Before proving Proposition 2, we generalize a result from [11]. Denote by \( \mathcal{G}_n \) the \( \sigma \)-algebra generated by \( I_i, L_i \) and \( R_i \) for \( i = 1, \ldots, n-1 \).

**Lemma 2** For any \( s > 0 \) with \( \mathcal{S}(s) < +\infty \), one has

\[
\mathbb{E}\left[ M_n^{(s)} \right] = \mathbb{E}\left[ \sum_{j=1}^{n} \beta_{j,n} \right] = \frac{\Gamma(n + \mathcal{S}(s))}{\Gamma(n)\Gamma(\mathcal{S}(s) + 1)}
\]

\( \circledast \) Springer
Lemma 3
If prove that \( E \)
Observe that \( \sum_{n \geq 1} e^{-t} (1 - e^{-t})^{n-1} E \left[ M_n^{(s)} \right] = e^t S(s) \).

If in addition \( S(s) = 0 \), then \( M_n^{(s)} \) is a martingale with respect to \( (G_n)_{n \geq 1} \).

Proof Recall that \( (\beta_{1,1}, \beta_{1,2}, \beta_{2,2}, \ldots, \beta_{n,n}) \) is \( G_n \)-measurable, see (10). We first prove that \( E[M_{n+1}^{(s)} | G_n] = M_n^{(s)} (1 + S(s)/n) \), which implies that \( M_n^{(s)} \) is a \( (G_n)_{n \geq 1} \)-martingale whenever \( S(s) = 0 \), since \( M_n^{(s)} \geq 0 \) and, as we will see, \( E[M_n^{(s)}] < +\infty \) for every \( n \geq 1 \). To prove the claim write

\[
E[M_{n+1}^{(s)} | G_n] = E \left[ \sum_{i=1}^{n} \mathbb{I}[I_n = i] \left( \sum_{j=1}^{n} \beta_{j,n+1} + \beta_{i,n+1} + \beta_{i+1,n+1} \right) \right] | G_n \\
= E \left[ \sum_{i=1}^{n} \mathbb{I}[I_n = i] \left( \sum_{j=1}^{n} \beta_{j,n} + \beta_{i,n} (L_n^s + R_n^s - 1) \right) \right] | G_n \\
= M_n^{(s)} + S(s) \sum_{i=1}^{n} \beta_{j,n} \mathbb{E}[\mathbb{I}[I_n = i]] = M_n^{(s)} (1 + S(s)/n).
\]

Taking the expectation of both sides gives \( E[M_{n+1}^{(s)}] = E[M_n^{(s)} (1 + S(s)/n)] \). Since \( E[M_2^{(s)}] = S(s) + 1 \) it follows easily that \( E[M_n^{(s)}] = \prod_{i=1}^{n-1} (1 + S(s)/i) = \Gamma(n + S(s)) (\Gamma(n) \Gamma(S(s) + 1))^{-1} \). To conclude the proof use formula 5.2.12.17 in [20].

Lemma 3. If \( S(\gamma) < 0 \) for some \( \gamma > 0 \), then

\[
\beta_n := \max_{1 \leq j \leq n} \beta_{j,n}
\]
converges to zero in probability as \( n \to +\infty \).

Proof Observe that \( \beta_n \leq \sum_{j=1}^{n} \beta_{j,n}^{(\gamma)} \), hence for every \( \epsilon > 0 \), by Markov’s inequality and Lemma 2, one gets

\[
P \{ \beta_n > \epsilon \} \leq P \left\{ \sum_{j=1}^{n} \beta_{j,n}^{(\gamma)} \geq \epsilon \right\} \leq \frac{1}{\epsilon^{\gamma}} E[M_n^{(\gamma)}] \leq C \frac{1}{\epsilon^{\gamma}} n^{\gamma}.
\]

The last expression tends to zero as \( n \to \infty \) because \( S(\gamma) < 0 \).

Proof of Proposition 2. Since \( S(\alpha) = 0 \), the random variables \( M_n^{(\alpha)} \) form a positive martingale with respect to \( (G_n)_{n \geq 1} \) by Lemma 2. By the martingale convergence theorem, see e.g. Theorem 19 in Chapter 24 of [10], it converges a.s. to a positive random
variable $M_n^{(a)}$ with $\mathbb{E}[M_n^{(a)}] \leq \mathbb{E}[M_1^{(a)}] = 1$. The goal of the following is to determine the law of $M_n^{(a)}$ in the different cases we consider.

First, suppose that $L^a + R^a = 1$ a.s. It follows that $L^a \leq 1$ and $R^a \leq 1$ a.s., and hence $S(s) \leq S(a) = 0$ for all $s > a$. Moreover, it is plain to check that $M_n^{(a)} = 1$ a.s. for every $n$, and hence $M_n^{(a)} = 1$ a.s.

Next, assume that $S(\gamma) < 0$ for $\gamma < a$. Minkowski’s inequality and Lemma 2 give

$$\mathbb{E} \left[ (M_n^{(a)})^{\gamma/\alpha} \right] \leq \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^{\gamma} \right] = \frac{\Gamma(n + S(\gamma))}{\Gamma(n) \Gamma(S(\gamma) + 1)} \leq Cn^{S(\gamma)}.$$ 

Hence, $M_n^{(a)}$ converges a.s. to 0.

It remains to treat the case with $S(\gamma) < 0$ and $\gamma > a$. Since $S(\cdot)$ is a convex function satisfying $S(a) = 0$ and $S(\gamma) < 0$ with $\gamma > a$, it is clear that $S'(a) < 0$; also, we can assume without loss of generality that $\gamma < 2a$. Further, by hypothesis, $\mathbb{E}[(L^a + R^a)^{1/(\gamma/\alpha - 1)}] \leq 2^{\gamma/\alpha - 1} \mathbb{E}[L^a + R^a] < +\infty$. Hence, one can resort to Theorem 2(a) of [9]—see also Corollaries 1.1, 1.4 and 1.5 in [15]—which provides existence and uniqueness of a probability distribution $\nu_{\infty} \neq \delta_0$ on $\mathbb{R}^+$, whose Fourier–Stieltjes transform $\psi$ is a solution of Eq. (17), with $\int_{\mathbb{R}^+} x \nu_{\infty}(dx) = 1$. Moreover, Theorem 2.1 in [16] ensures that $\int_{\mathbb{R}^+} x^{\gamma/\alpha} \nu_{\infty}(dx) < +\infty$ and, more generally, that $\int_{\mathbb{R}^+} x^{p/\alpha} \nu_{\infty}(dx) < +\infty$ for some $p > a$ if and only if $S(p) < 0$.

Consequently, our goal is to prove that the law of $M_n^{(a)}$ is $\nu_\infty$. Let $(M_j)_{j \geq 1}$ be a sequence of independent random variables with common characteristic function $\psi$, such that $(M_j)_{j \geq 1}$ and $(G_n)_{n \geq 1}$ are independent. Recalling that $\psi$ is a solution of (17) it follows that, for every $n \geq 2$,

$$\mathbb{E} \left[ \exp \left\{ i \xi \sum_{j=1}^{n} \beta_{j,n}^{\alpha} M_j \right\} \right] = \sum_{k=1}^{n-1} \frac{1}{n-1} \mathbb{E} \left[ \exp \left\{ i \xi \left( \sum_{j=1}^{k-1} \beta_{j,n-1}^{\alpha} M_j + \beta_{k,n-1}^{\alpha} (L_{n-1}^{a} M_k + R_{n-1}^{a} M_{k+1}) \right. \right. \right. \left. \left. \left. + \sum_{j=k+1}^{n-1} \beta_{j,n-1}^{\alpha} M_{j+1} \right) \right\} \right]$$

$$= \mathbb{E} \left[ \exp \left\{ i \xi \sum_{j=1}^{n-1} \beta_{j,n-1}^{\alpha} M_j \right\} \right].$$

By induction on $n \geq 2$, this shows that $\sum_{j=1}^{n} M_j \beta_{j,n}^{\alpha}$ has the same law as $M_1$, which is $\nu_\infty$. Hence

$$\mathcal{W}^{\nu_{\infty}}_a(M_n^{(a)}, M_1) \leq \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j=1}^{n} (1 - M_j) \beta_{j,n}^{\alpha} \middle| G_n \right] \right].$$
We shall now employ the following result from [24]. Let $1 < \eta \leq 2$, and assume that $Z_1, \ldots, Z_n$ are independent, centered random variables and $\mathbb{E}|Z_j|^{\eta} < +\infty$. Then

$$\mathbb{E} \left| \sum_{j=1}^{n} Z_j \right|^{\eta} \leq 2 \sum_{j=1}^{n} \mathbb{E}|Z_j|^{\eta}. \quad (28)$$

We apply this result with $\eta = \gamma/\alpha$ and $Z_j = \beta_{j,n}^{\alpha}(1 - M_{j})$, showing that

$$\mathbb{E} \left[ \left| \sum_{j=1}^{n} (1 - M_{j}) \beta_{j,n}^{\alpha} \right|^{\gamma/\alpha} \right] \leq 2 \sum_{j=1}^{n} \beta_{j,n}^{\gamma} \mathbb{E}|1 - M_1|^{\gamma/\alpha}$$

almost surely. In consequence, using also Lemma 2,

$$W_{\gamma/\alpha}^{\gamma/\alpha}(M_{n}^{(\alpha)}, M_1) \leq 2 \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^{\gamma} \right] \mathbb{E} \left[ |1 - M_1|^{\gamma/\alpha} \right] \leq C'n^{S(\gamma)}.$$

This proves that the law of $M_{n}^{(\alpha)}$ converges with respect to the $W_{\gamma/\alpha}$ metric—and then also weakly—to the law of $M_1$. Hence, $M_{n}^{(\alpha)}$ has law $\nu_{\infty}$. The fact that $M_{n}^{(\alpha)}$ is non-degenerate, provided $L^{\alpha} + R^{\alpha} = 1$ does not hold a.s., follows immediately. □

4.2 Proof of convergence for $\alpha \neq 1$ (Theorems 1 and 3)

Denote by $\mathcal{B}$ the $\sigma$-algebra generated by $\{\beta_{j,n} : n \geq 1, j = 1, \ldots, n\}$. The proof of Theorems 1 and 3 is essentially an application of the central limit theorem to the conditional law of $W_n := \sum_{j=1}^{n} \beta_{j,n} X_j$ given $\mathcal{B}$. Set $Q_{j,n}(x) := F_0(\beta_{j,n}^{-1}x)$, where, by convention, $F_0(\cdot/0) := \mathbb{I}_{[0, +\infty)}(\cdot)$. In this subsection we will use the functions:

$$\zeta_n(x) := \mathbb{I}[x < 0] \sum_{j=1}^{n} Q_{j,n}(x) + \mathbb{I}[x > 0] \sum_{j=1}^{n} (1 - Q_{j,n}(x)) \quad (x \in \mathbb{R})$$

$$\sigma_n^2(\epsilon) := \sum_{j=1}^{n} \left\{ \int_{(-\epsilon, +\epsilon]} x^2 dQ_{j,n}(x) - \left( \int_{(-\epsilon, +\epsilon]} x dQ_{j,n}(x) \right)^2 \right\} \quad (\epsilon > 0)$$

$$\eta_n := \sum_{j=1}^{n} \left\{ 1 - Q_{j,n}(1) - Q_{j,n}(-1) + \int_{(-1, 1]} x dQ_{j,n}(x) \right\}.$$

In terms of $Q_{j,n}$, the conditional distribution function $F_n$ of $W_n$ given $\mathcal{B}$ is the convolution, $F_n = Q_{1,n} * \cdots * Q_{n,n}$. To start with, we show that the $Q_{j,n}$s satisfy the uniform asymptotic negligibility (UAN) assumption (30) below.
Lemma 4 Let the assumptions of Theorems 3 or 4 be in force. Then, for every divergent sequence \((n')\) of integer numbers, there exists a divergent subsequence \((n'') \subset (n')\) and a set \(\Omega_0\) of probability one such that

\[
\lim_{n'' \to +\infty} M_{n''}^{(\alpha)}(\omega) = M^{(\alpha)}(\omega) < \infty, \quad \lim_{n'' \to +\infty} \beta(n'')(\omega) = 0, \tag{29}
\]

holds for every \(\omega \in \Omega_0\). Moreover, for every \(\omega \in \Omega_0\) and for every \(\epsilon > 0\)

\[
\lim_{n'' \to +\infty} \max_{1 \leq j \leq n''} \left\{ 1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon) \right\} = 0. \tag{30}
\]

Proof The existence of a sub-sequence \((n'')\) and a set \(\Omega_0\) satisfying (29) is a direct consequence of Lemma 3 and Proposition 2. To prove (30) note that, for \(0 < \alpha \leq 2\) and \(\alpha \neq 1\),

\[
\max_{1 \leq j \leq n''} (1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon)) \leq 1 - F_0 \left( \frac{\epsilon}{\bar{\beta}(n'')} \right) + F_0 \left( \frac{-\epsilon}{\bar{\beta}(n'')} \right). 
\]

The claim hence follows from (29). \(\square\)

Lemma 5 Let the assumptions of Theorem 1 be in force. Then for every divergent sequence \((n')\) of integer numbers there exists a divergent subsequence \((n'') \subset (n')\) and a measurable set \(\Omega_0\) of probability one such that

\[
\lim_{n'' \to +\infty} \mathbb{E}[e^{i\xi W_{n''}}|B](\omega) = \exp[-|\xi|^{\alpha} k M^{(\alpha)}(\omega)(1 - i \eta \tan(\pi \alpha/2) \text{sign } \xi)] \tag{31}
\]

for every \(\xi \in \mathbb{R}\) and for every \(\omega\) in \(\Omega_0\).

Proof Let \((n'')\) and \(\Omega_0\) be the same as in Lemma 4. To prove (31), we apply the central limit theorem for every \(\omega \in \Omega_0\) to the conditional law of \(W_{n''}\) given \(B\).

For every \(\omega\) in \(\Omega_0\), we know that \(F_{n''}\) is a convolution of probability distribution functions satisfying the asymptotic negligibility assumption (30). Here, we shall use the general version of the central limit theorem as presented e.g., in Theorem 30 in Section 16.9 and in Proposition 11 in Section 17.3 of [10]. According to these results, the claim (31) follows if, for every \(\omega \in \Omega_0\),

\[
\lim_{n'' \to +\infty} \xi_{n''}(x) = \frac{c^+ M_{\infty}^{(\alpha)}}{x^{\alpha}} \quad (x > 0), \tag{32}
\]

\[
\lim_{n'' \to +\infty} \xi_{n''}(x) = \frac{c^- M_{\infty}^{(\alpha)}}{|x|^{\alpha}} \quad (x < 0), \tag{33}
\]

\[
\lim_{\epsilon \to 0^+} \limsup_{n'' \to +\infty} \sigma_{n''}^2(\epsilon) = 0, \tag{34}
\]

\[
\lim_{n'' \to +\infty} \eta_{n''} = \frac{1}{1 - \alpha} M_{\infty}^{(\alpha)}(c^+ - c^-) \tag{35}
\]

are simultaneously satisfied.
In what follows we assume that \( P\{L = 0\} = P\{R = 0\} = 0 \), which yields that \( \beta_{j,n} > 0 \) almost surely. The general case can be treated with minor modifications. In order to prove (32), fix some \( x > 0 \), and observe that

\[
\zeta_{n''}(x) = \sum_{j} \left[ 1 - F_0\left( \beta_{j,n''}^{-1}x \right) \right] = \sum_{j} \left[ 1 - F_0\left( \beta_{j,n''}^{-1}x \right) \right] \left( \beta_{j,n''}^{-1}x \right)^{\alpha} \frac{\beta_{j,n''}}{x^\alpha}.
\]

Since \( \lim_{y \to +\infty} (1 - F_0(y))y^\alpha = c^+ \) by assumption (13), for every \( \epsilon > 0 \) there exists \( \bar{y} = \bar{y}(\epsilon) \) such that if \( y > \bar{y} \), then \( |(1 - F_0(y))y^\alpha - c^+| \leq \epsilon \). Hence if \( x > \beta(n'') \bar{y} \), then

\[
x^{-\alpha}(c^+ - \epsilon)M_{n''}^{(\alpha)} \leq \sum_{j} \left( 1 - F_0\left( \beta_{j,n''}^{-1}x \right) \right) \leq x^{-\alpha}(c^+ + \epsilon)M_{n''}^{(\alpha)}. \quad (36)
\]

In view of (29), the claim (32) follows immediately. Relation (33) is proved in a completely analogous way.

In order to prove (34), it is clearly sufficient to show that for every \( \epsilon > 0 \)

\[
\limsup_{n'' \to +\infty} \sum_{j=1}^{n''} \int_{(-\epsilon, +\epsilon]} x^2 dQ_{j,n''}(x) \leq C M_\infty^{(\alpha)} e^{2-\alpha} \quad (37)
\]

with some constant \( C \) independent of \( \epsilon \). Recalling the definition of \( Q_{j,n} \), an integration by parts gives

\[
\int_{(0,\epsilon]} x^2 dF_0\left( \beta_{j,n}^{-1}x \right) = -\epsilon^2 \left[ 1 - F_0\left( \beta_{j,n}^{-1}\epsilon \right) \right] + 2 \int_{0}^{\epsilon} x \left[ 1 - F_0\left( \beta_{j,n}^{-1}x \right) \right] dx,
\]

and similarly for the integral from \(-\epsilon\) to zero. With

\[
K := \sup_{x > 0} x^\alpha[1 - F(x)] + \sup_{x < 0} (-x)^\alpha F(x), \quad (38)
\]

which is finite by hypothesis (13), it follows that

\[
\int_{(-\epsilon, +\epsilon]} x^2 dF_0\left( \beta_{j,n}^{-1}x \right) \leq \frac{2K\epsilon^2}{(\beta_{j,n}^{-1}\epsilon)^\alpha} + 4K\beta_{j,n}^{\alpha} \int_{0}^{\epsilon} x^{1-\alpha} dx \leq 2K \left( \frac{4 - \alpha}{2 - \alpha} \right) \beta_{j,n}^{\alpha} \epsilon^{2-\alpha}.
\]

To conclude (37), it suffices to recall that \( \sum_j \beta_{j,n''}^{\alpha} = M_{n''}^{(\alpha)} \to M_\infty^{(\alpha)} \) by (29).
In order to prove (35), we need to distinguish if $0 < \alpha < 1$, or if $1 < \alpha < 2$. In the former case, integration by parts in the definition of $\eta_{n''}$ reveals

$$
\eta_{n''} = \int_{-1}^{1} \zeta_{n''}(x) \, dx.
$$

Having already shown (32) and (33), we know that the integrand converges pointwise with respect to $x$. The dominated convergence theorem applies since, by hypothesis (13),

$$
|\zeta_{n''}(x)| \leq K |x|^{-\alpha} \sup_{n''} M_{n''}^{(\alpha)}
$$

with the constant $K$ defined in (38); observe that $|x|^{-\alpha}$ is integrable on $(-1, 1]$ since we have assumed $0 < \alpha < 1$. Consequently,

$$
\lim_{n'' \to \infty} \eta_{n''} = c^{-} M_{\infty}^{(\alpha)} \int_{-1}^{0} |x|^{-\alpha} \, dx + c^{+} M_{\infty}^{(\alpha)} \int_{0}^{1} |x|^{-\alpha} \, dx = \frac{c^{+} - c^{-}}{1 - \alpha} M_{\infty}^{(\alpha)}.
$$

It remains to check (35) for $1 < \alpha < 2$. Since $\int_{\mathbb{R}} x \, dQ_{j,n''}(x) = 0$, one can write

$$
\eta_{n''} = -\sum_{j=1}^{n''} \int_{(-\infty, -1]} (1 + x) \, dQ_{j,n''}(x) - \sum_{j=1}^{n''} \int_{(1, +\infty)} (x - 1) \, dQ_{j,n''}(x).
$$

Similar as for $0 < \alpha < 1$, integration by parts reveals that

$$
\eta_{n''} = \int_{\{|x| > 1\}} \zeta_{n''}(x) \, dx.
$$

From this point on, the argument is the same as in the previous case: (32) and (33) provide pointwise convergence of the integrand; hypothesis (13) leads to (39), which guarantees that the dominated convergence theorem applies, since $|x|^{-\alpha}$ is integrable on the set $\{|x| > 1\}$. It is straightforward to verify that the integral of the pointwise limit indeed yields the right-hand side of (35).

**Proof of Theorem 1** By Lemma 5 and the dominated convergence theorem, every divergent sequence $(n')$ of integer numbers contains a divergent subsequence $(n'') \subset (n')$ for which

$$
\lim_{n'' \to +\infty} \mathbb{E}[e^{i\xi W_{n''}}] = \mathbb{E}\left[ e^{-|\xi| |k M_{\infty}^{(\alpha)}(1 - i \eta \tan(\pi \alpha/2) \text{sign} \xi)|} \right],
$$

$$
(41)
$$
where the limit is pointwise in $\xi \in \mathbb{R}$. Since the limiting function is independent of the arbitrarily chosen sequence $(n')$, a classical argument shows that \((41)\) is true with $n \to +\infty$ in place of $n'' \to +\infty$. In view of \((26)\), the stated convergence follows.

By Proposition 2, the assertion about (non)-degeneracy of $V_\infty$ follows immediately from the representation \((18)\). To verify the claim about moments for $\gamma > \alpha$, observe that \((18)\) implies that

$$
\mathbb{E}[|V_\infty|^p] = \int |x|^p \, dF_\infty(x) = \mathbb{E}\left[\left(M_\infty^{(\alpha)}\right)^{\frac{p}{\alpha}}\right] \int |u|^p \, dG_\alpha(u), \quad (42)
$$

where $G_\alpha$ is the distribution function of the centered $\alpha$-stable law with Fourier-Stieltjes transform $\hat{G}_\alpha$ defined in \((12)\). The $p/\alpha$-th moment of $M_\infty^{(\alpha)}$ is finite at least for all $p < \gamma$ by Proposition 2. On the other hand, the $p$-th moment of $G_\alpha$ is finite if and only if $p < \alpha$.

The following lemma replaces Lemma 5 in the case $\alpha = 2$.

**Lemma 6** Let the assumptions of Theorem 3 hold. Then for every divergent sequence $(n')$ of integer numbers, there exists a divergent subsequence $(n'') \subset (n')$ and a set $\Omega_0$ of probability one such that

$$
\lim_{n'' \to +\infty} \mathbb{E}[e^{i \xi W_{n''}} | \mathcal{B}](\omega) = e^{-\xi^2 \sigma_2^2 M_\infty^{(2)}(\omega)} \quad (\xi \in \mathbb{R})
$$

for every $\omega$ in $\Omega_0$.

**Proof** Let $(n'')$ and $\Omega_0$ have the properties stated in Lemma 4. The claim follows if for every $\omega$ in $\Omega_0$,

\begin{align*}
\lim_{n'' \to +\infty} \zeta_{n''}(x) &= 0 \quad (x \neq 0), \quad (43) \\
\lim_{\epsilon \to 0^+} \lim_{n'' \to +\infty} \sigma_{n''}^2(\epsilon) &= \sigma^2 M_\infty^{(2)}, \quad (44) \\
\lim_{n'' \to +\infty} \eta_{n''} &= 0 \quad (45)
\end{align*}

are simultaneously satisfied.

By assumption, $X_0$ has finite second moment, and thus

$$
\lim_{y \to +\infty} y^2(1 - F_0(y)) = \lim_{y \to -\infty} y^2(F_0(y)) = 0.
$$

by Chebyshev’s inequality. Hence, given $\epsilon > 0$, there exists a $\tilde{y} = \tilde{y}(\epsilon)$ such that $y^2(1 - F_0(y)) < \epsilon$ for every $y > \tilde{y}$. Since

$$
\zeta_{n''}(x) = \sum_{j=1}^{n''} (\beta_{j,n''}^{-1} x)^2 (1 - F_0(\beta_{j,n''}^{-1} x)) \beta_{j,n''}^2 / x^2 \quad (x > 0),
$$
one gets $\eta_n''(x) \leq \epsilon M_n''(x^2)$ whenever $x > \beta_{(n'')\eta''}$. In view of property (29), the first relation (43) follows for $x > 0$. The argument for $x < 0$ is analogous.

We turn to the proof of (44). A simple computation reveals

$$0 \leq \sigma^2 M_n''(\epsilon) = \sum_{j=1}^{n''} \int_{(-\epsilon, \epsilon]} x^2 \, dQ_j(x) \leq M_n''(x^2) \sum_{j=1}^{n''} \int_{|\beta_{n''}x| > \epsilon} x^2 \, dF_0(x),$$

which tends to zero as $n'' \to \infty$ by property (29), for every $\epsilon > 0$. In view of the definition of $\sigma^2(\epsilon)$, which implies in particular $\sigma^2(\epsilon) \geq 0$, this gives (44).

Finally, in order to obtain (45), we use (43) and the dominated convergence theorem; the argument is the same as for (40) in the proof of Lemma 5.

**Proof of Theorem 3** Use Lemma 6 and repeat the proof of Theorem 1. A trivial adaptation is needed in the calculation of moments if $\gamma > 2$: consider (42) with $G_\alpha = G_2$, the distribution function of a Gaussian law, and note that it posses finite moments of every order. Hence $\mathbb{E}[|V_\infty|^\gamma]$ is finite if and only if $\mathbb{E}[(M_\infty^{(2)})^{\gamma/2}]$ is finite, which, by Proposition 2, is the case if and only if $S(p) < 0$.

4.3 Proof of convergence for $\alpha = 1$ (Theorems 2 and 4)

Theorem 4 is proven first. We shall apply the central limit theorem to the random variables $W_n^* = \sum_{j=1}^{\beta_{n''}} (\beta_{n,n} X_j - q_{j,n})$ with $q_{j,n}$ defined in (20). In what follows, $Q_{j,n}(x) := F_0((x + q_{j,n})/\beta_{j,n})$. The next Lemma is the analogue of Lemma 4 above.

**Lemma 7** Suppose the assumptions of Theorem 4 are in force. Then, for every $\delta \in (0, 1)$,

$$|q_{j,n}| = \left| \int x \sin(\beta_{j,n}x) \, dF_0(x) \right| \leq C_\delta \beta_{j,n}^{-1-\delta} \tag{46}$$

with $C_\delta = \int_{\mathbb{R}} |x|^{1-\delta} \, dF_0(x) < +\infty$. Furthermore, for every divergent sequence $(n')$ of integer numbers, there exists a divergent subsequence $(n'') \subset (n')$ and a set $\Omega_0$ of probability one such that for every $\omega$ in $\Omega_0$ and for every $\epsilon > 0$, the properties (29) and (30) are verified.

**Proof** First of all note that $C_\delta < +\infty$ for every $\delta \in (0, 1)$ because of hypothesis (13). Using further that $|\sin(x)| \leq |x|^{1-\delta}$ for $\delta \in (0, 1)$, one immediately gets

$$|q_{j,n}| = \left| \int x \sin(\beta_{j,n}x) \, dF_0(x) \right| \leq \beta_{j,n}^{1-\delta} \int |x|^{1-\delta} \, dF_0(x).$$
Note that, as a consequence of (46), \((\epsilon + q_{j,n})\beta_{j,n}^{-1} \geq \beta_{(n)}^{-1}(\epsilon - C_\delta \beta_{(n)}^{1-\delta})\). Clearly, the expression inside the bracket is positive for sufficiently small \(\beta_{(n)}\). Defining \((n'')\) and \(\Omega_0\) in accordance to Lemma 4, it thus follows

\[
\max_{1 \leq j \leq n''} \left(1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon)\right) \leq 1 - F_0 \left(\tilde{c}\beta_{(n''')}^{-1}\right) + F_0 \left(-\tilde{c}\beta_{(n''')}^{-1}\right)
\]

for a suitable constant \(\tilde{c}\) depending only on \(\epsilon, \delta\) and \(F_0\). An application of (29) yields (30). 

**Lemma 8** Suppose the assumptions of Theorem 4 are in force, then for every divergent sequence \((n')\) of integer numbers there exists a divergent subsequence \((n'') \subset (n')\) and a measurable set \(\Omega_0\) with \(P(\Omega_0) = 1\) such that

\[
\lim_{n'' \to +\infty} \mathbb{E}[e^{i\xi W_{n''}'\mid B}(\omega)] = \exp\{-|\xi| k_1 M_\infty^{(1)}(\omega)(1 + 2i\eta \log |\xi| \text{ sign } \xi)\}
\]

(\(\xi \in \mathbb{R}\)) for every \(\omega\) in \(\Omega_0\).

**Proof** Define \((n'')\) and \(\Omega_0\) according to Lemma 7, implying the convergences (29), and the UAN condition (30). In the following, let \(\omega \in \Omega_0\) be fixed. In view of Proposition 11 in Section 17.3 of [10] the claim (47) follows if (32), (33) and (34) are satisfied with \(a = 1\), and in addition

\[
\lim_{n'' \to +\infty} \sum_{j=1}^{n} \int_{\mathbb{R}} \chi(t) dQ_{j,n''}(t) = M_\infty^{(1)}(c^+ - c^-) \int_{0}^{\infty} \frac{\chi(t) - \sin(t)}{t^2} dt
\]

with \(\chi(t) = -\mathbb{I}[t \leq -1] + r\mathbb{I}[-1 < t < 1] + \mathbb{I}[t \geq 1]\).

Let us verify (32) for an arbitrary \(x > 0\). Given \(\epsilon > 0\), there exists some \(\tilde{y} = \tilde{y}(\epsilon)\) such that \(|y(1 - F_0(\tilde{y})) - c^+| \leq \epsilon\) for all \(y \geq \tilde{y}\) because of hypothesis (13). Moreover, in view of Lemma 7,

\[
\hat{y}_{j,n''} := \frac{x + q_{j,n''}}{\beta_{j,n''}} \geq \frac{x - C_{1/2} \beta_{(n'')}^{1/2}}{\beta_{(n'')}},
\]

which clearly diverges to \(+\infty\) as \(n'' \to \infty\) because of (29); in particular, \(\hat{y}_{j,n''} \geq \tilde{y}\) for \(n''\) large enough. It follows that for those \(n''\),

\[
\frac{c^+ - \epsilon}{x + q_{j,n''}} \beta_{j,n''} \leq 1 - F(\hat{y}_{j,n''}) \leq \frac{c^+ + \epsilon}{x + q_{j,n''}} \beta_{j,n''}.
\]

Recalling that \(\xi_{n''}(x) = \sum_{j=1}^{n''}[1 - F(\hat{y}_{j,n''})]\), from (49) and (46) one gets

\[
\frac{c^+ - \epsilon}{x + C_{1/2} \beta_{(n'')}^{1/2}} M_{n''}^{(1)} \leq \xi_{n''}(x) \leq \frac{c^+ + \epsilon}{x - C_{1/2} \beta_{(n'')}^{1/2}} M_{n''}^{(1)}
\]
if $n''$ is large enough. Finally, recall that $\beta_{(n'')}^{1/2} \to 0$ as $n'' \to \infty$, and that $M_{n''}^{(1)} \to M_{\infty}^{(1)}$ by (29). Since $\epsilon > 0$ has been arbitrary, the claim (32) follows. The proof of (33) for arbitrary $x < 0$ is completely analogous.

Concerning (34), it is obviously enough to prove that

$$ \lim_{\epsilon \to 0} \limsup_{n'' \to +\infty} s_{n''}^{2}(\epsilon) = 0 $$

(50)

where $s_{n''}^{2}(\epsilon) := \sum_{j=1}^{n''} \int_{(-\epsilon, \epsilon]} x^2 \, dQ_{j,n''}(x)$. We split the domain of integration in the definition of $s_{n''}^{2}$ at $x = 0$, and integrate by parts to get

$$ s_{n''}^{2}(\epsilon) = -\epsilon^2 \sum_{j=1}^{n''} Q_{j,n''}(-\epsilon) - \sum_{j=1}^{n''} \int_{(-\epsilon,0]} Q_{j,n''}(u) 2u \, du $$

$$ -\epsilon^2 \sum_{j=1}^{n''} (1 - Q_{j,n''}(\epsilon)) + \sum_{j=1}^{n''} \int_{(0, \epsilon]} (1 - Q_{j,n''}(u)) 2u \, du $$

$$ =: A_{n''}(\epsilon) + B_{n''}(\epsilon) + C_{n''}(\epsilon) + D_{n''}(\epsilon). $$

Having already proven (32) and (33), we conclude

$$ \lim_{\epsilon \to 0} \limsup_{n'' \to +\infty} \{|A_{n''}(\epsilon)| + |C_{n''}(\epsilon)|\} = 0. $$

(51)

Fix $\epsilon > 0$; assume that $n''$ is sufficiently large to have $|q_{j,n''}| < \epsilon / 2$ for $j = 1, \ldots, n''$. Then

$$ |B_{n}(\epsilon)| \leq \sum_{j=1}^{n''} 2 \int_{0}^{\epsilon} w \, F_{0} \left( \frac{-w + q_{j,n''}}{\beta_{j,n''}} \right) \, dw $$

$$ \leq \sum_{j=1}^{n''} \left\{ \int_{0}^{2|q_{j,n''}|} 2w \, dw + 2 \int_{2|q_{j,n''}|}^{\epsilon} w \, F_{0} \left( \frac{-w + q_{j,n''}}{\beta_{j,n''}} \right) \, dw \right\} $$

$$ \leq \sum_{j=1}^{n''} \left\{ 4|q_{j,n''}|^2 + 2 \int_{2|q_{j,n''}|}^{\epsilon} w \left( \frac{K \beta_{j,n''}}{w - q_{j,n''}} \right) \, dw \right\} $$

$$ \leq \sum_{j=1}^{n''} \left\{ 4C_{1/4}^{2} \beta_{j,n''}^{3/2} + \beta_{j,n''} \int_{0}^{\epsilon} 4K \, dw \right\} \leq \left( 4C_{1/4}^{2} \beta_{(n'')}^{1/2} + 4K \epsilon \right) M_{n''}^{(1)}.$$
with the constant $K$ defined in (38). In view of (29), it follows
\[
\lim_{\epsilon \to 0^+} \limsup_{n'' \to +\infty} |B_{n''}(\epsilon)| = 0
\] (52)
as desired. A completely analogous reasoning applies to $D_{n''}$. At this stage we can conclude (50), and thus also (34).

In order to verify (48), let us first show that
\[
\lim_{n'' \to \infty} \sum_{j=1}^{n''} \int_{\mathbb{R}} \sin(x) \, dQ_{j,n''}(x) = 0.
\] (53)

We find
\[
\left| \sum_{j=1}^{n''} \int_{\mathbb{R}} \sin(x) \, dQ_{j,n''}(x) \right| = \left| \sum_{j=1}^{n''} \int_{\mathbb{R}} \sin(t\beta_{j,n''} - q_{j,n''}) \, dF_0(t) \right| \\
\leq \sum_{j=1}^{n''} \left| \cos(q_{j,n''}) - 1 \right| q_{j,n''} + \left| q_{j,n''} - \sin(q_{j,n''}) \right| \\
+ \sin(q_{j,n''}) \left( 1 - \cos(t\beta_{j,n''}) \right) dF_0(t) \leq \sum_{j=1}^{n''} (|I_1| + |I_2| + |I_3|).
\]
The elementary inequalities $|\cos(x) - 1| \leq x^2/2$ and $|x - \sin(x)| \leq x^3/6$ provide the estimate
\[
\sum_{j=1}^{n''} \left( |I_1| + |I_2| \right) \leq \sum_{j=1}^{n''} |q_{j,n''}|^3 \leq C_1^3 \beta_{(n'')}^{1/2} M_{n''}^{(1)}.
\]
By (29), the last expression converges to zero as $n'' \to \infty$. In order to estimate $I_3$, observe that, since $|1 - \cos(x)| \leq 2x^{3/4}$ for all $x \in \mathbb{R}$,
\[
\int_{\mathbb{R}} \left( 1 - \cos(t\beta_{j,n''}) \right) dF_0(t) \leq 2\beta_{j,n''}^{3/4} \int_{\mathbb{R}} |t|^{3/4} dF_0(dt) = 2C_1^{1/4} \beta_{j,n''}^{3/4}.
\]
Consequently, applying Lemma 7 once again,
\[
\sum_{j=1}^{n''} |I_3| \leq \sum_{j=1}^{n''} \left| q_{j,n''} \right| \left( 1 - \cos(t\beta_{j,n''}) \right) dF_0(t) \leq 2C_1^2 \beta_{(n'')}^{1/2} M_{n''}^{(1)},
\]
which converges to zero due to (29).
Having proven (53), the condition (48) becomes equivalent to

\[ \sum_{j=1}^{n''} \int_{\mathbb{R}} (\chi(t) - \sin(t)) \, dQ_{j,n''}(t) \to M_{\infty}^{(1)}(c^+ - c^-) \int_{\mathbb{R}^+} \frac{\chi(t) - \sin(t)}{t^2} \, dt. \quad (54) \]

The proof of this fact follows essentially the line of the proof of Theorem 12 of [10]. Let us first prove that, if \(-\infty < x < 0 < y < +\infty\),

\[ \lim_{n'' \to +\infty} \int_{(x,y]} dv_{n''}(t) = M_{\infty}^{(1)}(c^+ y - c^- x), \quad (55) \]

where \( v_n[B] := \sum_{j=1}^{n} \int_{B} t^2 \, dQ_{j,n}(t) \) for every Borel sets \( B \subset \mathbb{R} \). For fixed \( \epsilon \in (0, y) \), one uses (32) to conclude

\[ \lim_{n'' \to +\infty} \int_{(\epsilon,y]} dv_{n''}(t) = \lim_{n'' \to +\infty} \sum_{j=1}^{n''} \left( t^2 \left( 1 - Q_{j,n''}(t) \right) \right)_{1}^{\epsilon} + 2 \int_{(\epsilon,y]} \left( t - Q_{j,n''}(t) \right) \, dt = \epsilon^2 \frac{c^+ M_{\infty}^{(1)}}{\epsilon} - y^2 \frac{c^+ M_{\infty}^{(1)}}{y} + 2 \int_{(\epsilon,y]} \frac{t^2}{t} \, dt = (y - \epsilon)c^+ M_{\infty}^{(1)}. \]

Notice that we have used the dominated convergence theorem to pass to the limit under the integral; this is justified in view of the upper bound provided by (36). In a similar way, one shows for fixed \( \epsilon \in (0, |x|) \) that

\[ \lim_{n'' \to +\infty} \int_{(x,-\epsilon]} dv_{n''}(t) = (|x| - \epsilon)c^- M_{\infty}^{(1)}. \]

Combining this with (50), one concludes

\[ \limsup_{n'' \to +\infty} \int_{(x,y]} dv_{n''}(t) = (c^+ y - c^- x) M_{\infty}^{(1)}. \]

The same equality is trivially true for \( \liminf \) in place of \( \limsup \), proving (55). Now fix \( 0 < R < +\infty \), and note that (55) yields that for every bounded and continuous function \( f : [-R, R] \to \mathbb{R} \)

\[ \lim_{n'' \to +\infty} \int_{[-R,R]} f(t) \, dv_{n''}(t) = M_{\infty}^{(1)} c^- \int_{-R}^{0} f(t) \, dt + M_{\infty}^{(1)} c^+ \int_{0}^{R} f(t) \, dt. \quad (56) \]
holds true. In particular, using $f(t) = (\chi(t) - \sin t) / t^2$, one obtains

$$
\lim_{n'' \to \infty} \sum_{j=1}^{n''} \int_{[-R,R]} (\chi(t) - \sin(t)) \, dQ_{j,n''}(t) = M^{(1)}_{\infty} (c^+ - c^-) \int_0^R \frac{\chi(t) - \sin(t)}{t^2} \, dt.
$$

(57)

Moreover, since $|\chi(t) - \sin t| \leq 2$,

$$
\left| \sum_{j=1}^{n''} \int_{[-R,R]^c} (\chi(t) - \sin(t)) \, dQ_{j,n''}(t) \right| \leq 2 [\xi_{n''}(-R) + \xi_{n''}(R)].
$$

Applying (32) and (33) one obtains

$$
\limsup_{n'' \to +\infty} \left| \sum_{j=1}^{n''} \int_{[-R,R]^c} (\chi(t) - \sin(t)) \, dQ_{j,n''}(t) \right| \leq 2 M^{(g)}_{\infty} (c^+ + c^-) \frac{1}{R},
$$

which gives

$$
\limsup_{R \to +\infty} \limsup_{n'' \to +\infty} \left| \sum_{j=1}^{n''} \int_{[-R,R]^c} (\chi(t) - \sin(t)) \, dQ_{j,n''}(t) \right| = 0.
$$

(58)

Combining (57) with (58) one gets (54).

Proof of Theorem 4 Use Lemma 8 and repeat the proof of Theorem 1.

Proof of Theorem 2 The theorem is a corollary of Theorem 4. Since $m_0 = \int_{\mathbb{R}} x \, dF_0(x) < \infty$ by hypothesis, it follows that $c^+ = c^- = 0$, and so $V_t^*$ converges to 0 in probability. Now write $V_t = m_0 M^{(1)}_{v_t} + V_t^* - R_{v_t}$, with the remainder $R_{v_t} := \sum_{j=1}^n (q_{j,n} - \beta_{j,n} m_0)$. Thanks to Proposition 2, $m_0 M^{(1)}_{v_t}$ converges in distribution to $m_0 M^{(1)}_{\infty}$. It remains to prove that $R_{v_t}$ converges to 0 in probability. Since

$$
\left| \frac{\sin(x)}{x} - 1 \right| \leq H(x) := 1/6 \left[ x^2 \mathbb{1}\{|x| < 1\} + \mathbb{1}\{|x| \geq 1\} \right] \leq 1/6,
$$

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it follows that
\[
|R_n| \leq \sum_{j=1}^{n} \beta_{j,n} \int_{\mathbb{R}} \left| \frac{\sin(\beta_{j,n}x)}{\beta_{j,n}x} - 1 \right| |x| \, dF_0(x)
\]
\[
\leq \sum_{j=1}^{n} \beta_{j,n} \int_{\mathbb{R}} H(\beta_{j,n}x) |x| \, dF_0(x) \leq M_n^{(1)} \int_{\mathbb{R}} H(\beta_{(n)}x) |x| \, dF_0(x).
\]

Recall that $M_n^{(1)}$ converges a.s. to $M_\infty^{(1)}$ and $\beta_{(n)}$ converges in probability to 0 by (29). By dominated convergence it follows that also $\int_{\mathbb{R}} H(\beta_{(n)}x) |x| \, dF_0(x)$ converges in probability to 0.

The (non-)degeneracy of $V_\infty$ and the (in)finiteness of its moments is an immediate consequence of Proposition 2. $\square$

4.4 Estimates in Wasserstein metric (Theorem 5)

Proof of Theorem 5 We shall assume that $\mathcal{W}_\gamma(X_0, V_\infty) < +\infty$, since otherwise the claim is trivial. Then, there exists an optimal pair $(X^*, Y^*)$ realizing the infimum in the definition of the Wasserstein distance,

\[
\Delta := \mathcal{W}_\gamma^{\max}(\gamma, 1)(X_0, V_\infty) = \mathcal{W}_\gamma^{\max}(\gamma, 1)(X^*, Y^*) = \mathbb{E}|X^* - Y^*|^{\gamma} \quad (59)
\]

see e.g. Chapter 6 in [1]. Let $(X^*_j, Y^*_j)_{j \geq 1}$ be a sequence of independent and identically distributed random variables with the same law of $(X^*, Y^*)$, which are further independent of $B_n = (\beta_{1,1}, \beta_{1,2}, \ldots, \beta_{n,n})$. Consequently, $\sum_{j=1}^{n} X^*_j \beta_{j,n}$ has the same law of $W_n$, and $\sum_{j=1}^{n} Y^*_j \beta_{j,n}$ has the same law of $V_\infty$. By definition of $\mathcal{W}_\gamma$,

\[
\mathcal{W}_\gamma^{\max}(\gamma, 1)(W_n, V_\infty) \leq \mathbb{E} \left[ \sum_{j=1}^{n} X^*_j \beta_{j,n} - \sum_{j=1}^{n} Y^*_j \beta_{j,n} \right]^{\gamma}
\]
\[
= \mathbb{E} \left[ \mathbb{E} \left[ \left| \sum_{j=1}^{n} (X^*_j - Y^*_j) \beta_{j,n} \right|^{\gamma} \bigg| B_n \right] \right].
\]

Now, if $0 < \alpha < \gamma \leq 1$, then Minkowski’s inequality yields

\[
\mathcal{W}_\gamma(W_n, V_\infty) \leq \mathbb{E} \left[ \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^{\gamma} |X^*_j - Y^*_j|^{\gamma} \bigg| B_n \right] \right] = \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^{\gamma} \right] \Delta;
\]

where $\Delta$ is defined in (59). If instead $1 \leq \alpha < \gamma \leq 2$, we can apply the Bahr–Esseen inequality (28) since $E(X^*_1 - Y^*_1) = E(X_1) - E(V_\infty) = 0$ and $E|X^*_1 - Y^*_1|^{\gamma} = W_\gamma^{(1)}(X_0, V_\infty) < +\infty$. Thus,
Moreover, evaluating the functions’ derivatives, one verifies that
\[
\alpha < \gamma < \alpha
\]
\[
\begin{align*}
\mathcal{W}_{\gamma}^y(W_n, V_\infty) & \leq \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^\gamma |X_j^* - Y_j^y| | F_n \right] = 2 \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^\gamma \right] \Delta .
\end{align*}
\]

By Jensen’s inequality,
\[
\mathcal{W}_{\gamma}^{\max(y,1)}(V_t, V_\infty) \leq \sum_{n \geq 1} e^{-t}(1 - e^{-t})^{n-1} \mathcal{W}_{\gamma}^{\max(y,1)}(W_n, V_\infty).
\]

Combining the previous estimates with Lemma 2, we obtain
\[
\mathcal{W}_{\gamma}^{\max(y,1)}(V_t, V_\infty) \leq a \Delta \sum_{n \geq 1} e^{-t}(1 - e^{-t})^{n-1} \mathbb{E} \left[ \sum_{j=1}^{n} \beta_{j,n}^\gamma \right] = a \Delta e^{t \mathcal{S}(\gamma)},
\]

with \( a = 1 \) if \( 0 < \alpha < \gamma \leq 1 \) and \( a = 2 \) if \( 1 \leq \alpha < \gamma \leq 2 \).

\( \square \)

**Lemma 9** Let two random variables \( X_1 \) and \( X_2 \) be given, and assume that their distribution functions \( F_1 \) and \( F_2 \) both satisfy the conditions (24) and (25) with the same constants \( \alpha > 0 \), \( 0 < \epsilon < 1 \), \( K \) and \( c^+, c^- \geq 0 \). Then \( \mathcal{W}_y(X_1, X_2) < \infty \) for all \( \gamma \) that satisfy \( \alpha < \gamma < \frac{\alpha}{1-\epsilon} \).

**Proof** Define the auxiliary functions \( H, H_+ \) and \( H_- \) on \( \mathbb{R} \setminus \{0\} \) by \( H(x) = \mathbb{I}(x > 0)(1 - c^+x^{-\alpha}) + \mathbb{I}(x < 0)c^-|x|^{-\alpha}, H_\pm(x) = H(x) \pm K|x|^{-(\alpha+\epsilon)} \), so that \( H_- \leq F_i \leq H_+ \) for \( i = 1, 2 \) by hypothesis. It is immediately seen that \( H(x), H_+(x) \) and \( H_-(x) \) all tend to one (to zero, respectively) when \( x \) goes to \( +\infty \) (\( -\infty \), respectively). Moreover, evaluating the functions’ derivatives, one verifies that \( H \) and \( H_- \) are strictly increasing on \( \mathbb{R}_+ \), and that \( H_+ \) is strictly increasing on some interval \((R_+, +\infty)\). Let \( \tilde{R} > 0 \) be such that \( H(\tilde{R}) > H_+(R_+) \); then, for every \( x > \tilde{R} \), the equation
\[
H_-(\tilde{x}) = H(x) = H_+(\tilde{x})
\]
possesses precisely one solution pair \((\tilde{x}, \tilde{x})\) satisfying \( R_+ < \tilde{x} < x < \tilde{x} \). Likewise, \( H \) and \( H_+ \) are strictly increasing on \( \mathbb{R}_- \), and \( H_- \) is strictly increasing on \((-\infty, -R_-)\). Choosing \( \tilde{R} > 0 \) such that \( H(-\tilde{R}) < H_-(R_-) \), Eq. (60) has exactly one solution \((\tilde{x}, \tilde{x})\) with \( \tilde{x} < x < \tilde{x} < -R_- \) for every \( x < -\tilde{R} \).

From the definition of the Wasserstein distance one has, for every \( \gamma > 0 \),
\[
\mathcal{W}_{\gamma}^{\max(y,1)}(X_1, X_2) \leq \int_0^1 |F_1^{-1}(y) - F_2^{-1}(y)|^\gamma \, dy,
\]
where \( F_i^{-1} : (0, 1) \to \mathbb{R} \) denotes the pseudo-inverse function of \( F_i \). We split the domain of integration \((0, 1)\) into the three intervals \((0, H(-\tilde{R})), [H(-\tilde{R}), H(\tilde{R})] \) and \((H(\tilde{R}), 1)\), obtaining:

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The middle integral is obviously finite. To prove finiteness of the first and the last integral, we show that
\[
\int_{H(\tilde{R})}^{H(-\tilde{R})} |F_1^{-1}(y) - F_2^{-1}(y)|^\gamma \, dy + \int_{\tilde{R}}^{\infty} |F_1^{-1}(x) - F_2^{-1}(x)|^\gamma H'(x) \, dx.
\]
the estimates for the remaining contributions are similar. Let some \( x \geq \tilde{R} \) be given, and let \( \tilde{x} > x > \tilde{R} \) satisfy (60). From \( H_+ < F_1 < H(x) < F_1(\tilde{x}) \), which implies further that
\[
\tilde{x} - x < F_1^{-1}(H(x)) - x < \tilde{x} - x.
\]
From the definition of \( H \), it follows that \( x = \tilde{x}(1 + \kappa \tilde{x}^{-\epsilon})^{-1/\alpha} \), with \( \kappa = K/c^+ \). Combining this with a Taylor expansion, and recalling that \( \tilde{x} > x > \tilde{R} > 0 \), one obtains
\[
\tilde{x} - x = x \left[ (1 + \kappa \tilde{x}^{-\epsilon})^{-1/\alpha} - 1 \right] < x \left[ (1 + \kappa x^{-\epsilon})^{-1/\alpha} - 1 \right] < \tilde{C} x^{1-\epsilon}, \tag{62}
\]
where \( \tilde{C} \) is defined in terms of \( \alpha, \kappa \) and \( \tilde{R} \). In an analogous manner, one concludes from \( x = \tilde{x}(1 - \kappa \tilde{x}^{-\epsilon})^{-1/\alpha} \), in combination with \( 0 < R_+ < \tilde{x} < x \) and \( 0 < \epsilon < 1 \), that
\[
\tilde{x} - x = \tilde{x} \left[ 1 - (1 - \kappa \tilde{x}^{-\epsilon})^{-1/\alpha} \right] \geq \tilde{x} \left[ 1 - (1 + \tilde{C} \tilde{x}^{-\epsilon}) \right] > -\tilde{C} x^{1-\epsilon}, \tag{63}
\]
where \( \tilde{C} \) only depends on \( \alpha, \kappa \) and \( R_+ \). Substitution of (62) and (63) into (61) yields
\[
\int_{\tilde{R}}^{\infty} |F_1^{-1}(H(x)) - x|^\gamma H'(x) \, dx < \max(\tilde{C}, \tilde{C})^\gamma \int_{\tilde{R}}^{\infty} x^{\gamma(1-\epsilon)-\alpha-1} \, dx,
\]
which is finite provided that \( 0 < \gamma < \alpha/(1 - \epsilon) \).

\textbf{Proof of Lemma 1} In view of Lemma 9, it suffices to show that the distribution function \( F_\infty \) of \( V_\infty \) satisfies (24) and (25) with the same constants \( c^+ \) and \( c^- \) as the initial condition \( F_0 \) (possibly after diminishing \( \epsilon \) and enlarging \( K \)). The proof is based on
the representation of $F_\infty$ as a mixture of stable laws. More precisely, let $G_\alpha$ be the distribution function whose Fourier–Stieltjes transform is $\hat{g}_\alpha$ as in (12), then

$$F_\infty(x) = \mathbb{E}\left[G_\alpha\left((M_\infty^{(\alpha)})^{-1/\alpha} x\right)\right],$$

see (18). Since $\alpha < \gamma < 2\alpha$, then there exists a finite constant $K > 0$ such that $|1 - c_+ x^{-\alpha} - G_\alpha(x)| \leq K x^{-\gamma}$ for $x > 0$, and similarly for $x < 0$; see, e.g. Sections 2.4 and 2.5 of [26]. Using that $\mathbb{E}[M_\infty^{(\alpha)}] = 1$ and $C := \mathbb{E}[(M_\infty^{(\alpha)})^{\gamma/\alpha}] < \infty$ (since $S(\gamma) < 0$) it follows further that

$$|1 - c_+ x^{-\alpha} - F_\infty(x)| = |1 - c_+ \mathbb{E}[M_\infty^{(\alpha)}] x^{-\alpha} - \mathbb{E}[G_\alpha((M_\infty^{(\alpha)})^{-1/\alpha} x)]| \\
\leq \mathbb{E}\left[|1 - c_+ ((M_\infty^{(\alpha)})^{-1/\alpha} x)^{-\alpha} - G_\alpha((M_\infty^{(\alpha)})^{-1/\alpha} x)|\right] \\
\leq \mathbb{E}\left[K(M_\infty^{(\alpha)})^{\gamma/\alpha} x^{-\gamma}\right] = CK x^{-\gamma}.$$

This proves (24) for $F_\infty$, with $\epsilon = \gamma - \alpha$ and $K' = CK$. A similar argument proves (25).

4.5 Proofs of strong convergence (Theorem 6)

The proof of strong convergence rests on $n$-independent a priori bounds on the characteristic functions $\hat{q}_n$. These bounds are derived in Lemma 10 and 11 below.

**Lemma 10** Under the hypotheses of Theorem 6, there exists a constant $\theta > 0$ and a radius $\rho > 0$, both independent of $n \geq 0$, such that $|\hat{q}_n(\xi)| \leq (1 + \theta|\xi|^\alpha)^{-1/r}$ for all $|\xi| \leq \rho$.

**Proof** By the explicit representation (18) or (19), respectively, we conclude that $|\phi_\infty(\xi)| \leq \Phi(\xi) := \mathbb{E}[\exp(-|\xi|^\alpha k M_\infty^{(\alpha)})], \quad (64)$

with the parameter $k$ from (14), or $k = \sigma^2/2$ if $\alpha = 2$. Notice further that, by (17), $\Phi$ satisfies

$$\Phi(\xi) = \mathbb{E}[\Phi(L\xi)\Phi(R\xi)].$$

Moreover, since $M_\infty^{(\alpha)} \neq 0$, $\mathbb{E}[M_\infty^{(\alpha)}] = 1$ and $\mathbb{E}[(M_\infty^{(\alpha)})^{\gamma/\alpha}] < +\infty$, the function $\Phi$ is positive and strictly convex in $|\xi|^\alpha$, with $\Phi(\xi) = 1 - k|\xi|^\alpha + o(|\xi|^\alpha)$. It follows that for each $\kappa > 0$ with $\kappa < k$, there exists exactly one point $\Sigma_\kappa > 0$ with $\Phi(\Sigma_\kappa) + \kappa|\Sigma_\kappa|^\alpha = 1$, and $\Sigma_\kappa$ decreases monotonically from $+\infty$ to zero as $\kappa$ increases from zero to $k$.

Since $\hat{q}_0 = \phi_0$ satisfies condition (13) by hypothesis, it follows by Theorem 2.6.5 of [14] that

$$\hat{q}_0(\xi) = 1 - k|\xi|^\alpha(1 - i\eta \tan(\pi\alpha/2) \text{sign} \xi) + o(|\xi|^\alpha),$$

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with the same \( k \) as before, and \( \eta \) determined by (14). For \( \alpha = 2 \), clearly \( \hat{q}_0(\xi) = 1 - \sigma^2 \xi^2/2 + o(\xi^2) \). By the aforementioned properties of \( \Phi \), there exists a \( \kappa \in (0, k) \) such that

\[
|\hat{q}_0(\xi)| \leq \Phi(\xi) + \kappa |\xi|^{\alpha} \tag{65}
\]

for all \( \xi \in \mathbb{R} \). This is evident, since \( |\hat{q}_0(\xi)| = |\phi_0(\xi)| = 1 - k|\xi|^{\alpha} + o(|\xi|^{\alpha}) \), for small \( \xi \), while inequality (65) is trivially satisfied for \( |\xi| \geq \varepsilon_k \), since \( |\phi_0| \leq 1 \).

Starting from (65), we shall now prove inductively that

\[
|\hat{q}_\ell(\xi)| \leq \Phi(\xi) + \kappa |\xi|^{\alpha}. \tag{66}
\]

Fix \( n \geq 0 \), and assume (66) holds for all \( \ell \leq n \). Choose \( j \leq n \). Using the invariance property (64) of \( \Phi \), as well as the uniform bound of characteristic functions by one, it easily follows that

\[
\begin{align*}
\mathbb{E}[\hat{q}_j(L\xi)\hat{q}_{n-j}(R\xi)] - \Phi(\xi) & \leq \mathbb{E} \left[ |\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| - \Phi(L\xi)\Phi(R\xi) \right] \\
& \leq \mathbb{E} \left[ (|\hat{q}_j(L\xi)| - \Phi(L\xi))|\hat{q}_{n-j}(R\xi)| \right] + \mathbb{E} \left[ \Phi(L\xi)(|\hat{q}_{n-j}(R\xi)| - \Phi(R\xi)) \right] \\
& \leq \mathbb{E} \left[ \kappa(L|\xi|)^{\alpha} \right] + \mathbb{E} \left[ \kappa(R|\xi|)^{\alpha} \right] = \kappa |\xi|^{\alpha}.
\end{align*}
\]

The final equality is a consequence of \( \mathbb{E}[L^{\alpha} + R^{\alpha}] = 1 \). By (9), it is immediate to conclude (66) with \( \ell = n + 1 \).

The proof is finished by noting that, since \( \kappa < k \), \( (1 + \theta|\xi|^{\alpha})^{-1/r} \geq \Phi(\xi) + \kappa|\xi|^{\alpha} \) holds for \( |\xi| \leq \rho \), provided that \( \rho > 0 \) and \( \theta > 0 \) are sufficiently small.

\[\boxempty\]

**Lemma 11** Under the hypotheses of Theorem 6, let \( \rho > 0 \) be the radius introduced in Lemma 10 above. Then, there exists a constant \( \lambda > 0 \), independent of \( \ell \geq 0 \), such that

\[
|\hat{q}_\ell(\xi)| \leq (1 + \lambda|\xi|^r)^{-1/r} \quad \text{for all } |\xi| \geq \rho. \tag{67}
\]

**Proof** Since the density \( f_0 \) has finite Linnik–Fisher information by hypothesis (H2), it follows that

\[
|\phi_0(\xi)| \leq \left( \int_{\mathbb{R}} |\xi|^2 |\hat{h}(\xi)|^2 d\xi \right)^{1/2} |\xi|^{-1}
\]

for all \( \xi \in \mathbb{R} \), where \( h = \sqrt{f_0} \) and \( \hat{h} \) is its Fourier transform. See Lemma 2.3 in [5].

For any sufficiently small \( \lambda > 0 \), one concludes

\[
|\phi_0(\xi)| \leq (1 + \lambda|\xi|^r)^{-1/r} \tag{68}
\]

for sufficiently large \( |\xi| \).

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Next, recall that the modulus of the Fourier–Stieltjes transform of a probability density is continuous and bounded away from one, locally uniformly in $\xi$ on $\mathbb{R}\setminus\{0\}$. Diminishing the $\lambda > 0$ in (68) if necessary, this estimate actually holds for $|\xi| \geq \rho$.

Thus, the claim (67) is proven for $\ell = 0$. To proceed by induction, fix $n \geq 0$ and assume that (67) holds for all $\ell \leq n$. In the following, we shall conclude (67) for $\ell = n + 1$.

Recall that $r < \alpha$ in hypothesis (H1); see Remark 2. Hence, defining

$$\rho_\lambda = (\lambda/\theta)^{1/(\alpha-r)},$$

it follows that $(1 + \theta|\xi|^{\alpha})^{-1/r} \leq (1 + \lambda|\xi|^{\alpha})^{-1/r}$ if $|\xi| \geq \rho_\lambda$. Taking into account Lemma 10, estimate (68) for $\ell \leq n$ extends to all $|\xi| \geq \rho_\lambda$. We assume $\rho_\lambda < \rho$ from now on, which is equivalent to saying that $0 < \lambda < \lambda_0 := \theta \rho^{a-r}$.

With these notations at hand, introduce the following “good” set:

$$M_{\lambda,\delta}^G := \{\omega : L^r(\omega) + R^r(\omega) \geq 1 + \delta^r \text{ and } \min(L(\omega), R(\omega))\rho \geq \rho_\lambda\},$$

depending on $\lambda$ and a parameter $\delta > 0$. We are going to show that if $\delta > 0$ and $\lambda > 0$ are sufficiently small, then $M_{\lambda,\delta}^G$ has positive probability. First observe that the law of $(L, R)$ cannot be concentrated in the two point set $\{(0, 1), (1, 0)\}$ because $\gamma < 0$ by the hypotheses of Theorem 6. Hence we can assume $P\{L' + R' > 1\} > 0$, possibly after diminishing $r > 0$ (recall that if (H1) holds for some $r > 0$, then it holds for any smaller $r' > 0$ as well). Moreover, notice that $L' + R' > 1$ and $L = 0$ or $R = 0$ implies $L^\alpha + R^\alpha > 1$. But since $\mathbb{E}[L^\alpha + R^\alpha] = 1$, it follows that $P\{L > 0, R > 0, L^r + R^r > 1\} > 0$. In conclusion, the countable union of sets

$$\bigcup_{k=1}^\infty M_{\lambda_0/k,1/k}^G = \{\omega : L^r(\omega) + R^r(\omega) > 1, L(\omega) > 0, R(\omega) > 0\}$$

has positive probability, and so has one of the components $M_{\lambda_0/k,1/k}^G$.

Also, we introduce a “bad” set, that depends on $\lambda$ and $\xi$,

$$M_{\lambda,\xi}^B := \{\omega : \min(L(\omega), R(\omega))|\xi| < \rho_\lambda\}. $$

Notice that $M_{\lambda,\delta}^G$ and $M_{\lambda,\xi}^B$ are disjoint provided $|\xi| \geq \rho$.

We are now ready to carry out the induction proof, for a given $\lambda$ small enough. Fix $j \leq n$ and some $|\xi| \geq \rho$. We prove that

$$|\mathbb{E}[\hat{q}_j(L\xi)\hat{q}_{n-j}(R\xi)]| \leq \mathbb{E}[|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)|] \leq (1 + \lambda|\xi|^{\alpha})^{-1/r}. \quad (70)$$

We distinguish several cases. If $\omega$ does not belong to the bad set $M_{\lambda,\xi}^B$, then $L|\xi| \geq \rho_\lambda$ and $R|\xi| \geq \rho_\lambda$ so that by induction hypothesis

$$Z_j(\xi) := |\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| \leq ((1 + \lambda L^r|\xi|^{\alpha})|q + \lambda R^r|\xi|^{\alpha})^{-1/r} \leq (1 + \lambda(L^r + R^r)|\xi|^{\alpha})^{-1/r} \leq (1 + \lambda|\xi|^{\alpha})^{-1/r};$$
Indeed, recall that $L' + R' \geq 1$ because of (H1). In particular, if $\omega$ belongs to the good set $M_{\lambda, \delta}^G$, then the previous estimate improves as follows,

$$Z_j(\xi) \leq (1 + \lambda(1 + \delta^r)|\xi|^r)^{-1/r} \leq \left( \frac{1 + \lambda r^r}{1 + \lambda(1 + \delta^r) r^r} \right)^{1/r} (1 + \lambda|\xi|^r)^{-1/r},$$

where we have used that $|\xi| \geq \rho$. Notice further that there exists some $c > 0$—depending on $\delta, \theta, \lambda_0, \rho$ and $r$, but not on $\lambda$—such that for all sufficiently small $\lambda > 0$,

$$\left( \frac{1 + \lambda r^r}{1 + \lambda(1 + \delta^r) r^r} \right)^{1/r} \leq 1 - c \lambda.$$ 

Finally, suppose that $\omega$ is a point in the bad set $M_{\lambda, \xi}^B$, and assume without loss of generality that $L \geq R$. Then $L'|\xi|^r \geq (1 - R'|\xi|^r) \geq |\xi|^r - \rho^r_{\lambda}$, and so, for sufficiently small $\lambda$ and for any $\xi \geq \rho$,

$$|\hat{q}_j(L\xi)| |\hat{q}_{n-j}(R\xi)| \leq (1 + \lambda L'|\xi|^r)^{-1/r} \leq (1 + \lambda|\xi|^r - \lambda \rho^r_{\lambda})^{-1/r} \leq (1 + \lambda \rho^r_{\lambda})^{1/r} (1 + |\lambda| |\xi|^r)^{-1/r}.$$ 

Again, there exists a $\lambda$-independent constant $C$ such that, for all sufficiently small $\lambda > 0$, $(1 + \lambda \rho^r_{\lambda})^{1/r} \leq 1 + C \lambda \rho^r_{\lambda}$. Putting the estimates obtained in the three cases together, one obtains

$$\mathbb{E}[|\hat{q}_j(L\xi)| |\hat{q}_{n-j}(R\xi)|] \leq (1 + \lambda|\xi|^r)^{-1/r} \left[ (1 - P(M_{\lambda, \delta}^G) - P(M_{\lambda, \xi}^B)) + P(M_{\lambda, \delta}^G) (1 - c \lambda) + P(M_{\lambda, \xi}^B) (1 + C \lambda \rho^r_{\lambda}) \right] \leq (1 + \lambda|\xi|^r)^{-1/r} \left[ 1 + \lambda(C \rho^r_{\lambda} - c P(M_{\lambda, \delta}^G)) \right].$$

Notice that we have used the trivial estimate $P(M_{\lambda, \xi}^B) \leq 1$ in the last step, which eliminates any dependence of the term in the square brackets on $\xi$. To conclude (70), it suffices to observe that as $\lambda$ decreases to zero, $\rho_{\lambda}$ tends to zero monotonically by (69), while the measure $P(M_{\lambda, \delta}^G)$ is obviously non-decreasing and we have already proven that $P(M_{\lambda, \delta}^G) > 0$ for $\lambda^*$ and $\delta$ suitably chosen. Hence $C \rho^r_{\lambda} \leq c P(M_{\lambda, \delta}^G)$ when $\lambda > 0$ is small enough. From (70), it is immediate to conclude (67), recalling the recursive definition of $\hat{q}_{n+1}$ in (9).

Thus, the induction is complete, and so is the proof of the lemma. \qed

**Proof of Theorem 6** The key step is to prove convergence of the characteristic functions $\phi(t) \to \phi_{\infty}$ in $L^2(\mathbb{R})$. To this end, observe that the uniform bound on $\hat{q}_n$ obtained in Lemma 11 above directly carries over to the Wild sum,

$$|\phi(t; \xi)| \leq \sum_{n=0}^{\infty} (1 - e^{-t})^n |\hat{q}_n(\xi)| \leq (1 + \lambda|\xi|^r)^{-1/r} \quad (|\xi| \geq \rho).$$
The weak convergence of $V_t$ to $V_\infty$ implies locally uniform convergence of $\phi(t)$ to $\phi_\infty$, and so also $|\phi_\infty(\xi)| \leq (1 + \lambda |\xi|^r)^{-1/r}$ for $|\xi| \geq \rho$. Let $\epsilon > 0$ be given. Then there exists a $\mathcal{E} \geq \rho$ such that

$$\int_{|\xi| \geq \mathcal{E}} |\phi(t; \xi) - \phi_\infty(\xi)|^2 d\xi \leq 2 \int_{|\xi| \geq \mathcal{E}} \left( |\phi(t; \xi)|^2 + |\phi_\infty(\xi)|^2 \right) d\xi \leq 4 \int_{|\xi| \geq \mathcal{E}} (1 + \lambda |\xi|^r)^{-2/r} d\xi \leq \frac{\epsilon}{2}. $$

Again by locally uniform convergence of $\phi(t)$, there exists a time $T > 0$ such that $|\phi(t; \xi) - \phi_\infty(\xi)|^2 \leq \epsilon/(4\mathcal{E})$ for every $|\xi| \leq \mathcal{E}$ and $t \geq T$. In combination, it follows that $\|\phi(t) - \phi_\infty\|_{L^2(x)}^2 \leq \epsilon$ for all $t \geq T$. Since $\epsilon > 0$ has been arbitrary, convergence of $\phi(t)$ to $\phi_\infty$ in $L^2(\mathbb{R})$ follows. By Plancherel’s identity, this implies strong convergence of the densities $f(t)$ of $V_t$ to the density $f_\infty$ of $V_\infty$ in $L^2(\mathbb{R})$.

Convergence in $L^1(\mathbb{R})$ is obtained by interpolation between weak and $L^2(\mathbb{R})$ convergence: Given $\epsilon > 0$, choose $M > 0$ such that $\int_{|x| \geq M} f_\infty(x) dx < \epsilon/4$. By weak convergence of $V_t$ to $V_\infty$ there exists a $T > 0$ such that $\int_{|x| \geq M} f(t; x) dx < \epsilon/2$ for all $t \geq T$. Now Hölder’s inequality implies

$$\int_{\mathbb{R}} |f(t; x) - f_\infty(x)| dx \leq (2M)^{1/2} \left( \int_{|x| \leq M} |f(t; x) - f_\infty(x)|^2 dx \right)^{1/2} + \int_{|x| > M} |f(t; x)| + |f_\infty(x)|) dx < (2M)^{1/2} \|f(t) - f_\infty\|_{L^2} + \frac{3\epsilon}{4}. $$

Increasing $T$ sufficiently, the last sum is less than $\epsilon$ for $t \geq T$.

Finally, convergence in $L^p(\mathbb{R})$ with $1 < p < 2$ follows by interpolation between convergence in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$. \hfill \Box

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