Arc-transitive bicirculants

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Abstract
In this paper, we characterise the family of finite arc-transitive bicirculants. We show that every finite arc-transitive bicirculant is a normal \( r \)-cover of an arc-transitive graph that lies in one of eight infinite families or is one of seven sporadic arc-transitive graphs. Moreover, each of these ‘basic’ graphs is either an arc-transitive bicirculant or an arc-transitive circulant, and each graph in the latter case has an arc-transitive bicirculant normal \( r \)-cover for some integer \( r \).

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1 | INTRODUCTION

A graph with \( 2n \) vertices is called a bicirculant if it admits an automorphism \( g \) of order \( n \) with exactly two cycles of length \( n \). The class of bicirculants includes Cayley graphs of dihedral groups, the generalised Petersen graphs \([11]\), rose-window graphs \([34]\) and Tabacjtn graphs \([3]\). The last three families are particularly nice as the two orbits of \( \langle g \rangle \) are cycles in the graph. Arc-transitive bicirculants have been classified for small valencies, for example, valency 3 \([11, 27, 29]\), valency 4 \([19–21]\) and valency 5 \([2, 3]\). Moreover, bicirculants of valency 6 were recently investigated \([16]\), while the automorphisms of bicirculants on \( 2p \) vertices for \( p \) a prime are well understood \([26]\).

We call a graph with \( n \) vertices a circulant if it has an automorphism \( g \) that is an \( n \)-cycle. Let \( \Gamma \) be a connected arc-transitive circulant which is not a complete graph. Then Kovács \([18]\) and Li \([24]\) proved that either \( \langle g \rangle \) is a normal subgroup of \( \text{Aut}(\Gamma) \), or \( \Gamma \) is the lexicographic product of an...
arc-transitive circulant and an empty graph, or \( \Gamma \) is obtained by the lexicographic product of an arc-transitive circulant \( \Sigma \) and an empty graph and then removing some copies of \( \Sigma \). In this paper, we continue the study of arc-transitive bicirculants and give a characterisation for these graphs which is of a similar style to the result of Kovács and Li.

Let \( \Gamma \) be a \( G \)-vertex-transitive graph. Let \( B = \{ B_1, ..., B_m \} \) be a \( G \)-invariant partition of the vertex set, that is, for each \( B_i \) and each \( g \in G \), either \( B_i^g \cap B_i = \emptyset \) or \( B_i^g = B_i \). Then the quotient graph \( \Gamma_B \) of \( \Gamma \) induced on \( B \) is the graph with vertex set \( \mathcal{M} \) and \( \mathcal{M}' \) are adjacent if there exist \( x \in B_i \) and \( y \in B_j \) such that \( x, y \) are adjacent in \( \Gamma \).

Theorem 1.1. Every finite connected arc-transitive bicirculant which is not equal to \( K_2 \) is a normal \( r \)-cover of one of the following graphs:

(a) \( K_{n,n} \), \( n \geq 2 \);
(b) \( K_n \), \( K_{n[2]} \), \( K_n - nK_2 \), \( n \geq 3 \);
(c) \( G(2p, e) \), where \( p \) is a prime and \( e > 1 \) divides \( p - 1 \);
(d) \( B(\text{PG}(n, q)) \), \( B'(\text{PG}(n, q)) \) where \( q \) is a prime power and \( n \geq 2 \);
(e) \( \text{Cay}(p, e) \), where \( p \) is a prime and \( e \) is an even integer dividing \( p - 1 \);
(f) \( B'(H(11)) \);
(g) Petersen graph, \( H(2, 4) \), Clebsch graph, and their complements.

Moreover, each of the graphs in (a)–(g) is either an arc-transitive bicirculant or an arc-transitive circulant that has an arc-transitive bicirculant normal \( r \)-cover.

Lemma 2.4 determines exactly which graphs listed in Theorem 1.1 are circulants and Lemma 2.5 determines which ones are bicirculants. Note that a graph can be both a circulant and a bicirculant.

A transitive permutation group \( G \leq \text{Sym}(\Omega) \) is said to be quasiprimitive, if every non-trivial normal subgroup of \( G \) is transitive on \( \Omega \), while \( G \) is said to be bi-quasiprimitive if every non-trivial normal subgroup of \( G \) has at most two orbits on \( \Omega \) and there exists one which has exactly two orbits on \( \Omega \). Quasiprimitivity is a generalisation of primitivity as every normal subgroup of a primitive group is transitive, but there exist quasiprimitive groups which are not primitive. For more information about quasiprimitive and bi-quasiprimitive permutation groups, refer to [30–32].

The proof of Theorem 1.1 proceeds as follows. Let \( \Gamma \) be a connected \( G \)-arc-transitive bicirculant with at least 3 vertices (otherwise \( \Gamma = K_2 \)). Let \( N \) be a normal subgroup of \( G \) maximal with respect to having at least three orbits. It is proved in Theorem 3.3 that \( \Gamma \) is an \( r \)-cover of the quotient graph induced by \( N \) whose automorphism group contains \( G/N \), this quotient graph is \( G/N \)-arc-transitive and is either a circulant or a bicirculant, and \( G/N \) is quasiprimitive or bi-quasiprimitive on its vertex set. The families of vertex quasiprimitive and bi-quasiprimitive arc-transitive circulants are obtained in Proposition 3.4. Then we determine precisely the vertex quasiprimitive arc-transitive bicirculants in Proposition 4.2, and a key part of the proof is Müller’s classification of primitive groups containing a cyclic subgroup with two orbits. The vertex bi-quasiprimitive arc-transitive bicirculants are given in Propositions 5.1 and 5.2. Note that the graphs in Theorem 1.1
do not necessarily have an arc-transitive quasiprimitive or bi-quasiprimitive group of automorphisms for all values of the parameters. Each of the graphs in (a)–(g) is either an arc-transitive bicirculant or an arc-transitive circulant by Lemmas 2.4 and 2.5. Moreover, the ones that are circulants have a normal cover that is an arc-transitive bicirculant by Lemma 2.6.

Finally, in Section 6, we relate Theorem 1.1 to the classifications of arc-transitive bicirculants of valencies 3, 4 and 5.

2 | PRELIMINARIES

In this section, we give some definitions about groups and graphs and also prove some results which will be used in the following discussion.

2.1 | Groups and graphs

All graphs in this paper are finite, simple, connected and undirected. For a graph \( \Gamma \), we use \( V(\Gamma) \) and \( \text{Aut}(\Gamma) \) to denote its vertex set and automorphism group, respectively. The size of the vertex set of a graph is said to be the order of the graph. For the group theoretic terminology not defined here we refer the reader to [5, 10, 33].

We denote by \( \mathbb{Z}_n \) the cyclic group of order \( n \). Let \( G \) be a permutation group on a set \( \Omega \) and \( \alpha \in \Omega \). Denote by \( G_{\alpha} \) the stabiliser of \( \alpha \) in \( G \), that is, the subgroup of \( G \) fixing the point \( \alpha \). We say that \( G \) is semiregular on \( \Omega \) if \( G_{\alpha} = 1 \) for every \( \alpha \in \Omega \) and regular if \( G \) is transitive and semiregular. An arc of a graph is an ordered pair of adjacent vertices. A graph \( \Gamma \) is said to be \( G \)-vertex-transitive or \( G \)-arc-transitive if \( G \leq \text{Aut}(\Gamma) \) is transitive on the set of vertices or on the set of arcs, respectively. Moreover, if \( G = \text{Aut}(\Gamma) \), then we drop the prefix “\( G \)-” in the definitions.

Let \( G \) be a transitive permutation group on a set \( \Omega \) and let \( B \) be a \( G \)-invariant partition of \( \Omega \). Then \( G \) induces a transitive permutation group on \( B \), denoted by \( G^B \). If the only possibilities for \( B \) are the partition into one part, or the partition into singletons then \( G \) is called primitive. The kernel of \( G \) on \( B \) is the normal subgroup of \( G \) consisting of all elements that fix setwise each \( B \in B \). We call \( B \) maximal if \( G^B \) is primitive on \( B \). Let \( B \) be a non-empty subset of \( \Omega \). Then \( B \) is called a block of \( G \) if, for any \( g \in G \), either \( B^g = B \) or \( B^g \cap B = \emptyset \). If \( N \) is an intransitive normal subgroup of \( G \), then each \( N \)-orbit is a block of \( G \), and the set of \( N \)-orbits forms a \( G \)-invariant partition of \( \Omega \). Let \( \Gamma \) be a \( G \)-vertex-transitive graph and let \( B \) be the set of \( N \)-orbits of \( V(\Gamma) \) for some normal subgroup \( N \) of \( G \); then we denote \( \Gamma_B \) by \( \Gamma_N \) and call \( \Gamma_N \) a normal quotient graph.

For a finite group \( T \), and a subset \( S \) of \( T \) such that \( 1 \notin S \) and \( S = S^{-1} \), the Cayley graph \( \text{Cay}(T, S) \) of \( T \) with respect to \( S \) is the graph with vertex set \( T \) and edge set \( \{(g, sg) \mid g \in T, s \in S\} \). In particular, \( \text{Cay}(T, S) \) is connected if and only if \( T = \langle S \rangle \). The group \( R(T) = \{\sigma_t \mid t \in T\} \) of right multiplications \( \sigma_t : x \mapsto xt \) is a subgroup of the automorphism group of \( \text{Cay}(T, S) \) and acts regularly on the vertex set. Indeed, a graph is a Cayley graph if and only if it admits a regular group of automorphisms. We note that circulants are precisely the Cayley graphs for cyclic groups.

A graph \( \Gamma \) is said to be a bi-Cayley graph over a group \( H \) if it admits \( H \) as a semiregular automorphism group with two orbits of equal size. (Some authors have used the term semi-Cayley instead; see [9, 22].) Moreover, bicirculants are exactly the bi-Cayley graphs over cyclic groups. The family of bi-Cayley graphs has been extensively studied, for example, cubic bi-Cayley graphs over abelian groups were investigated by Zhou and Feng [35] while the automorphism groups of bi-Cayley graphs were studied in [36].
For a graph $\Gamma$, its complement $\overline{\Gamma}$ is the graph with vertex set $V(\Gamma)$, and two vertices are adjacent if and only if they are not adjacent in $\Gamma$. We denote the complete graph on $n$ vertices by $K_n$.

Let $\Gamma_1$ and $\Gamma_2$ be two graphs. The lexicographic product $\Gamma_1[\Gamma_2]$, of $\Gamma_1$ and $\Gamma_2$, is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ such that $(u_1, u_2)$ is adjacent to $(v_1, v_2)$ if and only if either $\{u_1, v_1\}$ is an edge of $\Gamma_1$, or $u_1 = v_1$ and $\{u_2, v_2\}$ is an edge of $\Gamma_2$.

The following theorem is the characterisation of arc-transitive circulants by Kovács [18] and Li [24]. A normal circulant is a Cayley graph $\text{Cay}(T, S)$ such that $T$ is a cyclic group and $R(T)$ is a normal subgroup of $\text{Aut}(\Gamma)$.

**Theorem 2.1** [24, Theorem 1.3]. Let $\Gamma$ be a connected arc-transitive graph of order $n$ which is not a complete graph and with a cyclic regular group. Then either

1. $\Gamma$ is a normal circulant, or
2. there exists an arc-transitive circulant $\Sigma$ of order $m$, such that $mb = n$ with $m, b \geq 2$, and

$$\Gamma = \begin{cases} \Sigma[K_b], & \text{or} \\ \Sigma[K_b] - b\Sigma, & (m, b) = 1. \end{cases}$$

### 2.2 The graphs appearing in Theorem 1.1

For $m, n \geq 2$, we denote the complete multipartite graph with $m$ parts of size $n$ by $K_{m[n]}$, that is, the graph such that two vertices are adjacent if and only if they are in distinct parts. The graph $K_{2[n]}$ is often denoted by $K_{n,n}$.

The Hamming graph $H(d, r)$ has vertex set $\Delta^d = \{(x_1, x_2, \ldots, x_d) | x_i \in \Delta\}$, where $\Delta = \{0, 1, \ldots, r - 1\}$, and two vertices $v$ and $v'$ are adjacent if and only if they are different in exactly one coordinate. The Hamming graph $H(d, 2)$ is called a $d$-cube, and a folded $d$-cube is the quotient of $H(d, 2)$ by the partition into blocks of size 2 given by pairs of vertices at distance $d$. The complement of the folded 5-cube is the Clebsch graph.

Let $p$ be an odd prime and let $e$ be an even integer such that $e$ divides $p - 1$. Let $Z_p$ be the group of integers modulo $p$ and let $\sigma$ be a generator of $\text{Aut}(Z_p) \cong Z_{p-1}$. Suppose that $p - 1 = er$ for some positive integer $r$. Let $\tau = \sigma^r$ and $S = \{1, 1^r, 1^{r^2}, \ldots, 1^{r^{e-1}}\} \subseteq Z_p$. Then $\text{Cay}(Z_p, S)$ denotes the graph $\text{Cay}(Z_p, S)$.

Let $p$ be an odd prime and let $r$ be a positive integer dividing $p - 1$. Let $A$ and $A'$ denote two disjoint copies of $Z_p$ and denote the corresponding elements of $A$ and $A'$ by $i$ and $i'$, respectively. Let $L(p, r)$ be the unique subgroup of the multiplicative group of $Z_p$ of order $r$. We define two graphs, $G(2p, r)$ and $G(2, p, r)$, with vertex set $A \cup A'$. The graph $G(2p, r)$ has edge set $\{(x, y') | x, y \in Z_p, y - x \in L(p, r)\}$, while the graph $G(2, p, r)$ (defined only for $r$ even) has edge set $\{(x, y), (x', y), (x, y'), (x', y') | x, y \in Z_p, y - x \in L(p, r)\}$. Note that $G(2, p, r)$ is a non-bipartite graph as it contains a $p$-cycle and is a 2-cover of $\text{Cay}(p, r)$, while $G(2p, r)$ is bipartite.

For each integer $d \geq 3$ and prime power $q$, let $B(\text{PG}(d - 1, q))$ be the bipartite graph with vertices the one-dimensional and $(d - 1)$-dimensional subspaces of a $d$-dimensional vector space over $\text{GF}(q)$, and two subspaces are adjacent if and only if one is contained in the other. We denote the bipartite complement of $B(\text{PG}(d - 1, q))$ by $B'(\text{PG}(d - 1, q))$, that is, the bipartite graph with the same vertex set but a 1-subspace and a $(d - 1)$-subspace are adjacent if and only if their intersection is the zero subspace.
We define $B'(H(11))$ to be the bipartite graph with vertices the elements of $\mathbb{Z}_{11}$ and the sets $R + i$, where $i \in \mathbb{Z}_{11}$ and $R = \{1, 3, 4, 5, 9\}$, that is, the set of non-zero quadratic residues modulo 11, such that $n \in \mathbb{Z}_{11}$ is adjacent to $R + i$ if and only if $n \not\in R + i$. We note that the bipartite complement of $B'(H(11))$ is isomorphic to $G(22, 5)$; see also [7, p. 200].

2.3 Basic lemmas

We state some lemmas which will be used in the following discussion. The first lemma gives three simple observations.

**Lemma 2.2.**

1. Every circulant of even order is a bicirculant.
2. A graph is a circulant if and only if its complement is a circulant.
3. A graph is a bicirculant if and only if its complement is a bicirculant.

**Lemma 2.3.** Let $\Gamma = G(2p, r)$ with $r > 1$ such that $r$ divides $p - 1$. Then the following hold.

1. $\Gamma$ is a Cayley graph of a dihedral group and hence is a bicirculant.
2. $\Gamma$ is a circulant if and only if $r$ is even.

**Proof.** Recall that $V(\Gamma)$ consists of the elements $i$ and $i’$ for $i \in \mathbb{Z}_p$. Let

$$
\tau : V(\Gamma) \mapsto V(\Gamma), \quad i \mapsto i + 1, \quad i’ \mapsto (i + 1)’,
$$

$$
\sigma : V(\Gamma) \mapsto V(\Gamma), \quad i \mapsto (-i)’, \quad i’ \mapsto -i.
$$

Then $\tau$ is an automorphism of $\Gamma$ of order $p$ consisting of two $p$-cycles, and $\sigma$ is an automorphism of $\Gamma$ of order 2 swapping the two orbits of $\tau$. Moreover, $\sigma \tau \sigma = \tau^{-1}$, and $\langle \sigma, \rho \rangle \cong D_{2p}$ is a dihedral group of order $2p$ that acts regularly on the vertex set. Thus $\Gamma$ is a Cayley graph of $D_{2p}$, and so $\Gamma$ is a bicirculant, and (1) holds.

If $r$ is even, then $i \in L(p, r)$ if and only if $-i \in L(p, r)$. Hence

$$
\rho : V(\Gamma) \mapsto V(\Gamma), \quad i \mapsto i’, \quad i’ \mapsto i
$$

is a graph automorphism of $\Gamma$ of order 2 and $\rho \tau = \tau \rho$. Moreover, $\langle \tau, \rho \rangle \cong \mathbb{Z}_{2p}$ is regular on $V(\Gamma)$. Thus $\Gamma$ is a circulant.

Now let $r$ be an odd divisor of $p - 1$ and suppose that $\Gamma$ is a circulant of the cyclic group $T$. Then $\Gamma = \text{Cay}(T, S)$ where $S$ is a subset of $T \setminus \{1_T\}$. Hence $|S| = r$ is an odd integer. Since $\Gamma$ is undirected, it follows that $S = S^{-1}$, and so $S$ contains the unique involution of $T$. Suppose that $T < \text{Aut}(\Gamma)$ and let $u = 1_T$. Then by [13, Lemma 2.1], $\text{Aut}(\Gamma)u \leq \text{Aut}(T)$. Since $\Gamma$ is arc-transitive and $\Gamma(u) = S$, it follows that all elements in $S$ are involutions, a contradiction. Thus by Theorem 2.1, there exists an arc-transitive circulant $\Sigma$ of order $m$, such that $mb = 2p$ with $m, b \in \{2, p\}$, and

$$
\Gamma = \begin{cases}
\Sigma[K_b], & \text{or} \\
\Sigma[K_b] - b\Sigma.
\end{cases}
$$

Assume first that $\Gamma = \Sigma[K_b]$. If $(m, b) = (2, p)$, then $\Gamma \cong K_{p, p}$ has valency $p > r$, a contradiction. If $(m, b) = (p, 2)$, then each block has 2 vertices, and any two adjacent blocks induce a
subgraph $K_{2,2}$, and so $\Gamma$ has even valency, again a contradiction. Now suppose that $\Gamma = \Sigma [K_b] - b\Sigma$. If $(m, b) = (2, p)$, then $\Gamma \cong K_p, p - pK_2$ has valency $p - 1 > r$, a contradiction. If $(m, b) = (p, 2)$, then each block has 2 vertices, and any two adjacent blocks induce a subgraph $2K_2$. Thus $\Gamma$ is a cover of $\Sigma$, and $\Sigma$ has valency $r$ and order $p$. However, both $p$ and $r$ are odd integers, which is impossible. Hence $\Gamma$ is not a circulant when $r$ is an odd divisor of $p - 1$. Therefore, $\Gamma$ is a circulant if and only if $r$ is even, so that (2) holds.

**Lemma 2.4.** A graph in Theorem 1.1 (a)–(g) is a circulant if and only if it is one of the following graphs:

1. $K_{n,n}$, $n \geq 2$;
2. $K_n, K_{n[2]}$, $n \geq 3$;
3. $K_{n,n} - nK_2$, $n \geq 3$ is an odd integer;
4. $G(2p, r)$, where $p$ is a prime and even $r > 1$ divides $p - 1$;
5. $\text{Cay}(p, e)$, where $p$ is a prime and $e$ is an even integer dividing $p - 1$.

**Proof.** Clearly $K_n$, $K_{n,n}$ and $K_{n[2]}$ are circulants for all $n \geq 2$, while $\text{Cay}(p, e)$ is a circulant by definition. By Lemma 2.3, $G(2p, r)$ is a circulant if and only if $r$ is an even divisor of $p - 1$.

It can be easily checked, for example, by Magma [4], that $H(2, 4), B'(H(11))$, the Clebsch graph and the Petersen graph are not circulants, and by Lemma 2.2 (2), neither are their complements.

Let $\Gamma = K_{n,n} - nK_2$. Then $\text{Aut}(\Gamma) \cong S_n \times S_2$ and when $n$ is odd, this contains a cyclic subgroup of order $2n$ that is regular on the vertex set, and so $\Gamma$ is a circulant. Moreover, when $n \geq 4$ is even, $\Gamma$ is not a circulant; see, for example, [1, Theorem 1.1].

It remains to consider the graphs $B(\text{PG}(d - 1, q))$ and $B'(\text{PG}(d - 1, q))$, which have automorphism group $\text{Aut} (\text{PSL}(d, q))$. Suppose that these graphs are circulants. Since $\text{PGL}(d, q)$ acts primitively on each bipartite half and these graphs are neither complete bipartite nor of the form $K_{n,n} - nK_2$, the results of Kovács [18] and Li [24] imply that $\text{Aut}(\Gamma)$ contains a normal cyclic regular subgroup, a contradiction.

**Lemma 2.5.** The graphs in Theorem 1.1 (a)–(g) are bicirculants except for $\text{Cay}(p, e)$ and $K_n$ with $n$ odd.

**Proof.** Since the graphs $\text{Cay}(p, e)$ and $K_n$ with $n$ odd, have an odd number of vertices, they are not bicirculants. It remains to prove that the remaining graphs in Theorem 1.1(a)–(g) are bicirculants.

By Lemma 2.4, $K_n$ with $n$ even, $K_{n,n}$ with $n \geq 2$ and $K_{n[2]}$ with $n \geq 3$ are circulants. Moreover, all of them have an even number of vertices, and so by Lemma 2.2(1), they are bicirculants. Clearly, the Petersen graph and its complement, and $K_{n,n} - nK_2$ with $n \geq 3$ are bicirculants.

The group $\text{PGL}(d, q)$ has a cyclic subgroup of order $(q^d - 1)/(q - 1)$ that acts regularly on the set of one-dimensional subspaces and the set of hyperplane of $\text{GF}(q)^d$. Thus both $B(\text{PG}(d - 1, q))$ and $B'(\text{PG}(d - 1, q))$ are bicirculants.

By Lemma 2.3, $G(2p, r)$ is a bicirculant. Note that $B(H(11)) \cong G(22, 5)$ is a bicirculant on 22 vertices. Hence its complement, $B'(H(11))$, is also a bicirculant by Lemma 2.2(3).

If $\Gamma = H(2, 4)$, then $\text{Aut}(\Gamma) \cong S_4 \cap S_2$ has a cyclic subgroup of order 8 that is semiregular with two orbits of size 8 on the vertex set. Hence $H(2, 4)$ is a bicirculant. Similarly, if $\Gamma$ is the Clebsch graph then $\text{Aut}(\Gamma) \cong \mathbb{Z}_2^4 \cdot S_5$ has a cyclic subgroup of order 8 that is semiregular with two orbits of size 8 on the vertex set. Thus the Clebsch graph is a bicirculant.
Let $\Gamma$ be a graph with vertex set $V(\Gamma)$ and arc set $A(\Gamma)$. We define a bipartite graph from $\Gamma$ in the following way. Let $\hat{\Gamma}$ be the graph with vertex set $V(\Gamma) \times \{1, 2\}$, and 2 vertices $(x, 1)$ and $(y, 2)$ are adjacent if and only if $(x, y) \in A(\Gamma)$. Then the new graph $\hat{\Gamma}$ is called the standard double cover of $\Gamma$, and it is bipartite with bipartite halves $V(\Gamma) \times \{i\}$ for each $i = 1, 2$. Note that $\hat{\Gamma}$ is connected if and only if $\Gamma$ is not bipartite; see [12, Lemma 3.3].

The following lemma shows that all the circulants arising in Theorem 1.1 are the quotient graphs of some bicirculants.

**Lemma 2.6.** If $\Gamma$ is a circulant then the standard double cover of $\Gamma$ is a bicirculant. Moreover, if $\Gamma$ is $G$-arc-transitive then there exists $X \leqslant \text{Aut} (\hat{\Gamma})$ such that $\hat{\Gamma}$ is $X$-arc-transitive and $N < X$ such that $\hat{\Gamma}_N \cong \Gamma$.

**Proof.** Suppose that $g \in \text{Aut}(\Gamma)$. Then $g$ induces an automorphism $\hat{g}$ of $\hat{\Gamma}$ by $(x, i) \mapsto (x^g, i)$. Moreover, if $g$ is an $n$-cycle on $V(\Gamma)$ then $\hat{g}$ has two $n$-cycles on $V(\hat{\Gamma})$. Thus the first part of the lemma follows.

Note that $\tau : (x, i) \mapsto (x, 3 - i)$ is an automorphism of $\hat{\Gamma}$ and $X := \hat{G} \times \langle \tau \rangle \leqslant \text{Aut}(\hat{\Gamma})$ where $\hat{G} = \{g | g \in G\} \cong G$. If $\Gamma$ is $G$-arc-transitive then $\hat{\Gamma}$ is $X$-arc-transitive. Letting $N = \langle \tau \rangle \leqslant X$ we see that the orbits of $N$ are $\{(v, i) | i \in \{1, 2\}\}$ for each $v \in V(\Gamma)$. Moreover, $\hat{\Gamma}_N \cong \Gamma$. \hfill $\Box$

All arc-transitive graphs on $2p$ vertices for a prime $p$ are given by the following result.

**Theorem 2.7** [7, Theorem 2.4]. Let $\Gamma$ be a connected arc-transitive graph. If $|V(\Gamma)| = 2p$ for some prime number $p$, then $\Gamma$ is one of the following graphs:

1. $K_{2p}$ or $K_p, p$;
2. the Petersen graph or its complement;
3. $G(2, r)$ for some even integer $r$ dividing $p - 1$;
4. $G(2p, r)$ for some integer $r > 1$ dividing $p - 1$;
5. $B(\text{PG}(n - 1, q))$ or $B'(\text{PG}(n - 1, q))$, and $p = \frac{q^n - 1}{q - 1}$;
6. $B'(H(11))$ and $p = 11$.

We also give the following well-known lemma. Note that every connected bipartite graph has a unique bipartite partition. For a $G$-vertex-transitive bipartite graph, we use $G^+$ to denote the stabiliser in $G$ of the two bipartite halves.

**Lemma 2.8.** Let $\Gamma$ be a $G$-arc-transitive connected graph such that $G$ is bi-quasiprimitive on $V(\Gamma)$. If $G^+$ acts unfaithfully on each orbit, then $\Gamma \cong K_{n,n}$ for some $n \geqslant 2$.

**Proof.** Let $\Delta_0$ and $\Delta_1$ be the bipartite halves of $\Gamma$. Let $G^+_{\langle \Delta_0 \rangle}$ be the kernel of $G^+$ on $\Delta_0$ and $G^+_{\langle \Delta_1 \rangle}$ be the kernel of $G^+$ on $\Delta_1$. Suppose that $G^+$ acts unfaithfully on each $\Delta_i$, that is, $G^+_{\langle \Delta_i \rangle} \neq 1$ for $i = 0, 1$.

Also, let $\sigma \in G$ be an element interchanging $\Delta_0$ and $\Delta_1$ and note that $G = \langle G^+, \sigma \rangle$. Then $(G^+_{\langle \Delta_i \rangle})^\sigma = G^+_{\langle \Delta_{1-i} \rangle}$ and so $1 \neq G^+_{\langle \Delta_0 \rangle} \times G^+_{\langle \Delta_1 \rangle} < G$. Since $G$ is bi-quasiprimitive on $V(\Gamma)$ and $G^+_{\langle \Delta_0 \rangle} \times G^+_{\langle \Delta_1 \rangle}$ is not transitive, it follows that $G^+_{\langle \Delta_0 \rangle} \times G^+_{\langle \Delta_1 \rangle}$ has two orbits on $V(\Gamma)$, namely $\Delta_0$ and $\Delta_1$. Thus $G^+_{\langle \Delta_i \rangle}$ fixes each vertex in $\Delta_0$ and is transitive on $\Delta_1$. Since $G$ is arc-transitive it follows that $\Gamma = K_{n,n}$ where $n = |\Delta_i| \geqslant 2$. \hfill $\Box$
2.4 Some group theory

The classification of primitive permutation groups that contain a cyclic regular subgroup was independently obtained by Jones [17] and Li [23, Corollary 1.2]. Moreover, by [25, Theorem 1.2], every quasiprimitive group with a regular cyclic subgroup is primitive. Hence, the family of quasiprimitive groups with a regular cyclic subgroup is also completely determined in the following lemma.

Lemma 2.9 [17; 23, Corollary 1.2]. A primitive permutation group $G$ of degree $n$ contains a cyclic regular subgroup if and only if one of the following holds:

(i) $\mathbb{Z}_p \leq G \leq \text{AGL}(1, p)$, where $n = p$ is a prime;
(ii) $G = A_n$ with $n \geq 5$ odd, or $S_n$, where $n \geq 4$;
(iii) $\text{PGL}(d, q) \leq G \leq \text{PΓL}(d, q)$ and $n = (q^d - 1)/(q - 1)$;
(iv) $(G, n) = (\text{PSL}(2, 11), 11), (M_{11}, 11), (M_{23}, 23)$.

Moreover, in cases (ii)–(iv) $G$ is 2-transitive.

The following result about primitive permutation groups that contain an element with exactly two equal cycles is due to Müller.

Theorem 2.10 [28, Theorem 3.3]. Let $G$ be a primitive permutation group of degree $2n$ that contains an element with exactly 2-cycles of length $n$. Then one of the following holds, where $G_0$ denotes the stabiliser of a point.

(1) (Affine action) $\mathbb{Z}_2^m \triangleleft G \leq \text{AGL}(m, 2)$ is an affine permutation group, where $n = 2^{m-1}$. Further, one of the following holds:
   (a) $n = 2$, and $G_0 = \text{GL}(2, 2)$;
   (b) $n = 2$, and $G_0 = \text{GL}(1, 4)$;
   (c) $n = 4$, and $G_0 = \text{GL}(3, 2)$;
   (d) $n = 8$, and $G_0 \in \{\mathbb{Z}_4 : \mathbb{Z}_4, \Gamma L(1, 16), (\mathbb{Z}_3 \times \mathbb{Z}_3) : \mathbb{Z}_4, \Sigma L(3, 4), \Gamma L(2, 4), A_6, \text{GL}(4, 2), (S_3 \times S_3) : \mathbb{Z}_2, S_5, S_6, A_7\}$.

(2) (Almost simple action) $S \leq G \leq \text{Aut}(S)$ for a non-abelian simple group $S$, and one of the following holds:
   (a) $n \geq 3$ and $A_{2n} \leq G \leq S_{2n}$ in its natural action;
   (b) $n = 5$ and $A_5 \leq G \leq S_5$ in the action on the set of 2-subsets of $\{1, 2, 3, 4, 5\}$;
   (c) $n = (q^d - 1)/2(q - 1)$ and $\text{PGL}(d, q) \leq G \leq \text{PΓL}(d, q)$ for some odd prime power $q$ and $d \geq 2$ even;
   (d) $n = 11$ and $M_{22} \leq G \leq \text{Aut}(M_{22})$;
   (e) $n = 6$ and $G = M_{12}$;
   (f) $n = 12$ and $G = M_{24}$.

Lemma 2.11. Let $\Gamma$ be a $G$-arc-transitive bipartite graph of order $2n$ with $n$ not a prime, such that $G^+$ acts faithfully and 2-transitively on each bipartite half.

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† We note that [28, Theorem 3.3] gives $|\Gamma L(1, 16) : G_0| = 3$ as one of the possibilities but there is a unique such group, namely the $\mathbb{Z}_5 : \mathbb{Z}_4$ that we list here.
(1) If $G^+$ contains a cyclic subgroup that is transitive on each bipartite half then $\Gamma = K_{n,n} - nK_2$ for some $n \geq 3$, $B(PG(d - 1, q))$ or $B'(PG(d - 1, q))$ for some $d \geq 3$ and $q$ a prime power.

(2) If $G^+$ is almost simple and contains a cyclic subgroup with two equal-sized orbits on each bipartite half then $\Gamma = K_{12,12}$, $K_{n,n} - nK_2$ for some $n \geq 3$, $B(PG(d - 1, q))$ or $B'(PG(d - 1, q))$ for some even $d \geq 3$ and odd prime power $q$.

Proof. Let $\Delta_0$ and $\Delta_1$ be the bipartite halves of $\Gamma$ and let $u \in \Delta_0$. If all 2-transitive actions of $G^+$ on $n$ points are equivalent then there exists $v \in \Delta_1$ such that $(G^+)_u = (G^+)_v$. Moreover, $(G^+)_u$ is transitive on both $\Delta_1 \setminus \{v\}$ and $\Delta_0 \setminus \{u\}$. Hence $\Gamma = K_{n,n} - nK_2$ for some $n \geq 3$. It remains to consider the case where the action of $G^+$ on $\Delta_0$ and $\Delta_1$ are inequivalent 2-transitive actions on $n$ points.

Suppose first that $G^+$ contains a cyclic subgroup that is transitive on $\Delta_0$. Then recalling that $n$ is not a prime and $G^+$ is primitive on $\Delta_0$, we have from Lemma 2.9, that one of the following holds:

(1) $G^+ = S_6$ and $n = 6$;

(2) $PGL(d, q) < G^+ \leq P\Gamma L(d, q)$ and $n = (q^d - 1)/(q - 1)$ with $d \geq 3$.

In the first case, an element acting on $\Delta_1$ as a 6-cycle acts on $\Delta_2$ as a product of a 2-cycle and a 3-cycle, and so no suitable cyclic subgroup exists. In the second case, $PGL(d, q) < G^+$ and $\Delta_0$ and $\Delta_1$ correspond to the set of 1-spaces and $(d - 1)$-spaces of a $d$-dimensional vector space over $GF(q)$. Thus $\Gamma = B(PG(d - 1, q))$ or $B'(PG(d - 1, q))$.

Next suppose that $G^+$ is almost simple and contains a cyclic subgroup with two equal-sized orbits on $\Delta_0$. Then Theorem 2.10 implies that one of the following holds:

(1) $A_6 < G^+ \leq S_6$ and $n = 6$;

(2) $PGL(d, q) < G^+ \leq P\Gamma L(d, q)$ and $n = (q^d - 1)/(q - 1)$ with even $d \geq 3$ and odd $q$;

(3) $G^+ = M_{12}$ and $n = 12$.

In the first case, if $g \in G^+$ acts on $\Delta_0$ as two disjoint 3-cycles then it has three fixed points on $\Delta_2$, a contradiction. We have also already determined the graph in the second case. In the last case, $(G^+)_u$ is transitive on $\Delta_1$ and so $\Gamma = K_{12,12}$. □

3 REDUCTION RESULT AND CIRCULANTS

In this section, we give a reduction result for the family of arc-transitive bicirculants. Let $\Gamma$ be a $G$-vertex-transitive bicirculant over a cyclic subgroup $H$ of $G$. Then the first lemma presents a relationship between the two orbits of $H$ and the blocks for the action of $G$ on the vertex set.

Lemma 3.1. Let $\Gamma$ be a $G$-vertex-transitive bi-Cayley graph over the subgroup $H$ of $G$. Let $H_0$ and $H_1$ be the two orbits of $H$ on $V(\Gamma)$ and let $B$ be a $G$-invariant partition of $V(\Gamma)$. Then the following hold.

(1) Either all elements of $B$ are subsets of $H_0$ or $H_1$; or $B \cap H_0 \neq \emptyset$ and $B \cap H_1 \neq \emptyset$ for every $B \in B$.

(2) If $B \in B$ and $B \cap H_i \neq \emptyset$ for some $i$, then $B \cap H_i$ is a block for $H$ on $H_i$.

Proof. (1) Suppose that there exists some $B \in B$ such that $B \subseteq H_i$ for some $i \in \{0, 1\}$, and assume that there is another block $B' \in B$ such that $B' \cap H_0 \neq \emptyset$ and $B' \cap H_1 \neq \emptyset$. Then for each vertex...
Let $b \in B$, there exists $h \in H$ such that $b^h \in B' \cap H_i$, as $H$ acts transitively on $H_i$. Thus $b^h \in B' \cap B^h$. Since $B^h \subseteq B$ and $B$ is a block system, we get $B' = B^h \subseteq H_i$, a contradiction. Therefore either all elements of $B$ are subsets of $H_0$ or $H_1$, or the intersection of each $B \in B$ with $H_0$ and $H_1$ is non-empty.

(2) Let $B \in B$ and suppose that $B \cap H_i \neq \emptyset$ for some $i$. Let $h \in H$. Then $B^h = B$ or $B^h \cap B = \emptyset$. Since $(B \cap H_i)^h \subseteq H_i$, it follows that $(B \cap H_i)^h = B \cap H_i$ or $(B \cap H_i)^h \cap (B \cap H_i) = \emptyset$ and so $B \cap H_i$ is a block for $H$ on $H_i$.

**Lemma 3.2.** Let $\Gamma$ be a $G$-vertex-transitive bi-Cayley graph with at least 3 vertices over the abelian subgroup $H$ of $G$, and let $H_0$ and $H_1$ be the two orbits of $H$ on $V(\Gamma)$. Let $N$ be a normal subgroup of $G$ maximal with respect to having at least three orbits. Then $\Gamma_N$ is a $G/N$-vertex-transitive graph, $G/N$ is faithful on $V(\Gamma_N)$ and the following hold.

(1) If each $N$-orbit is a subset of either $H_0$ or $H_1$, then $\Gamma_N$ is a bi-Cayley graph over $HN/N$. In particular, if $H$ is a cyclic group, then $\Gamma_N$ is a bicirculant of $HN/N$.

(2) If each $N$-orbit intersects both $H_0$ and $H_1$ non-trivially, then $\Gamma_N$ is isomorphic to a Cayley graph of $HN/N$. In particular, if $H$ is a cyclic group, then $\Gamma_N$ is isomorphic to a circulant of $HN/N$.

**Proof.** Let $K$ be the kernel of the action of $G$ on $V(\Gamma_N)$. Then $N \leq K$. If $N < K$, then $K$ has one or two orbits on $V(\Gamma)$, which is impossible. Thus $N = K$. Since $\Gamma$ is $G$-vertex-transitive, $\Gamma_N$ is $G/N$-vertex-transitive. Let $B = \{B_1, \ldots, B_t\}$ be the set of $N$-orbits.

(1) Suppose that each $N$-orbit is a subset of either $H_0$ or $H_1$. Then $|B_i|$ divides $|H|$, and without loss of generality, we may assume that $H_0 = B_1 \cup \cdots \cup B_r$ and $H_1 = B_{r+1} \cup \cdots \cup B_{2r}$ where $2r = t$. Since $H$ is an abelian group, $HN/N \cong H/(H \cap N)$ is an abelian group. Since $H$ is transitive on both $H_0$ and $H_1$, it follows that $HN/N$ is regular on both sets $\{B_1, \ldots, B_r\}$ and $\{B_{r+1}, \ldots, B_{2r}\}$. Therefore, $\Gamma_N$ is a bi-Cayley graph of $HN/N$.

(2) Suppose that each $N$-orbit intersects both $H_0$ and $H_1$ non-trivially. Then by Lemma 3.1(2), for each $B \in B$ we have that $B \cap H_i$ is a block for $H$ on $H_i$ for each $i \in \{0, 1\}$. As $H$ is transitive on $H_0$, $HN/N$ is transitive on $\{B_1 \cap H_0, \ldots, B_t \cap H_0\}$. Since $N$ is the kernel of the action of $G$ on $V(\Gamma_N)$, $H \cap N$ is the kernel of the action of $H$ on $\{B_1 \cap H_0, \ldots, B_t \cap H_0\}$. Since $H$ is an abelian group, it follows that $HN/N$ acts regularly on $\{B_1 \cap H_0, \ldots, B_t \cap H_0\}$. Thus $HN/N$ acts regularly on $B$, and so $\Gamma_N$ is a Cayley graph of $HN/N$. In particular, if $H$ is a cyclic group, then $\Gamma_N$ is a circulant of $HN/N$.

We are ready to give a reduction result.

**Theorem 3.3.** Let $\Gamma$ be a connected $G$-arc-transitive bicirculant with at least 3 vertices over the cyclic subgroup $H$ of $G$. Let $N$ be a normal subgroup of $G$ maximal with respect to having at least three orbits. Then $\Gamma$ is an $r$-cover of $\Gamma_N$, where $r$ divides the valency of $\Gamma$, $\Gamma_N$ is $G/N$-arc-transitive and is either a circulant or a bicirculant over $HN/N$, and $G/N$ is faithful and either quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$.

**Proof.** Since $N$ is a normal subgroup of $G$ maximal with respect to having at least three orbits, it follows from Lemma 3.2 that $G/N$ is faithful on $V(\Gamma_N)$. Moreover, all normal subgroups of $G/N$ are transitive or have two orbits on $V(\Gamma_N)$. Thus $G/N$ is quasiprimitive or bi-quasiprimitive on $V(\Gamma_N)$. Let $B = \{B_1, \ldots, B_t\}$ be the set of $N$-orbits, so that $t \geq 3$. Since $\Gamma$ is $G$-arc-transitive, it follows that $\Gamma_N$ is a $G/N$-arc-transitive graph, and each induced subgraph $[B_i]$ is empty. For an arc $(B_1, B_2)$
of $\Gamma_N$, there exists a vertex $b_1$ of $\Gamma$ such that $b_1 \in B_1$ and $\Gamma(b_1) \cap B_2 \neq \emptyset$. For any $b'_1 \in B_1$, we have $b'_1 = b_1^n$ for some $n \in N$. Hence

$$\Gamma(b'_1) \cap B_2 = \Gamma(b_1)^n \cap B_2 = (\Gamma(b_1) \cap B_2)^n \neq \emptyset.$$ 

Thus the number of vertices in $B_2$ adjacent to a given vertex of $B_1$ is constant. If $\Gamma(b_1) \subseteq B_2$, then by the connectivity of $\Gamma$, we have $t = 2$, a contradiction. Hence $\Gamma(b_1) \nsubseteq B_2$ and so there exists $b_2 \in B \setminus \{b_1, B_2\}$ such that $B_1$ and $B_2$ are adjacent in $\Gamma_N$. Let $b_2 \in B_2$ and $b_3 \in B_3$ such that $(b_1, b_2)$ and $(b_2, b_3)$ are arcs of $\Gamma$. Since $\Gamma$ is $G$-arc-transitive, there exists $g \in G_{b_1}$ such that $b_2^g = b_3$. Since $B_1, B_2, B_3$ are blocks of $G$, it follows that $B_1^g = B_1$ and $B_2^g = B_2$. Thus $(\Gamma(b_1) \cap B_2)^g = \Gamma(b_1) \cap B_2$, and so $|\Gamma(b_1) \cap B_2| = |\Gamma(b_1) \cap B_2|$ for any $i$ such that $(B_1, B_i)$ is an arc of $\Gamma_N$. We also showed above that $|\Gamma(b_1) \cap B_2| = |\Gamma(b'_1) \cap B_2|$ for all $b_1, b'_1 \in B_1$. Therefore, $\Gamma$ is an $r$-cover of $\Gamma_N$ where $r = |\Gamma(b_1) \cap B_2|$ is a divisor of $|\Gamma(b_1)|$.

Let $H_0$ and $H_1$ be the two orbits of $H$ on $V(\Gamma)$. If $B_i \subseteq H_i$ for some $i \in \{0, 1\}$, then by Lemma 3.1(1), for each $B' \in B$, either $B' \subset H_0$ or $B' \subset H_1$. It follows from Lemma 3.2 that $\Gamma_N$ is a bicirculant over $HN/N \cong H/(H \cap N)$. Finally, suppose that $B \cap H_0 \neq \emptyset$ and $B \cap H_1 \neq \emptyset$ for some $B \subseteq B$. Then again by Lemma 3.1(1), $B' \cap H_0 \neq \emptyset$ and $B' \cap H_1 \neq \emptyset$ for every $B' \subseteq B$. Hence, by Lemma 3.2, $\Gamma_N$ is isomorphic to a circulant of $HN/N$.

Our next proposition determines the family of arc-transitive circulants that are vertex quasiprimitive or vertex bi-quasiprimitive.

**Proposition 3.4.** Let $\Gamma$ be a connected $G$-arc-transitive circulant over a cyclic subgroup $H$ of $G$. Then the following statements hold.

1. If $G$ is quasiprimitive on $V(\Gamma)$, then $\Gamma$ is one of the following two graphs:
   1.1. a complete graph;
   1.2. $\text{Cay}(p, e)$ where $e$ is an even integer dividing $p − 1$.
2. If $G$ is bi-quasiprimitive on $V(\Gamma)$, then $\Gamma$ is one of the following three graphs:
   2.1. $K_{n,n}$;
   2.2. $K_{n,n} - nK_2$ where $n$ is an odd integer;
   2.3. $G(2p, r)$ where $p$ is a prime and $r$ is an even divisor of $p − 1$.

**Proof.** (1) Suppose that $G$ is quasiprimitive on $V(\Gamma)$. Then by [25, Theorem 1.2], $G$ is primitive on $V(\Gamma)$. Moreover, $G$ is listed in Lemma 2.9: either $|V(\Gamma)| = p$ is a prime and $G \leq AGL(1, p)$, or $G$ is 2-transitive on $V(\Gamma)$. If $G$ is 2-transitive on $V(\Gamma)$, then $\Gamma$ is a complete graph, and case (1.1) holds; if $|V(\Gamma)| = p$ is a prime and $G \leq AGL(1, p)$, then by Chao [6], $\Gamma$ is a Cayley graph $\text{Cay}(p, e)$ where $e$ is an even integer dividing $p − 1$, that is, (1.2) occurs.

(2) Suppose that $G$ is bi-quasiprimitive on $V(\Gamma)$. Then $G$ has a minimal normal subgroup that has exactly two orbits on $V(\Gamma)$, say $\Delta_0$ and $\Delta_1$. Hence $|H|$ is even. Since $\Gamma$ is $G$-arc-transitive and connected, it follows that each $\Delta_i$ does not contain any edge of $\Gamma$, and so $\Gamma$ is a bipartite graph with $\Delta_0$ and $\Delta_1$ being the two bipartite halves. Let $G^+ = G_{\Delta_0} = G_{\Delta_1}$. Since $G$ is transitive on the vertex set, it follows that $G^+$ is a normal subgroup of $G$ of index 2, and so $G = \langle G^+, \sigma \rangle$ for some element $\sigma \in G \setminus G^+$ with $\sigma^2 \in G^+$. Moreover, $\Delta_0^2 = \Delta_1$ and $\Delta_1^2 = \Delta_0$. Since $H$ is cyclic and regular on $V(\Gamma)$, the group $H^+ = H \cap G^+$ is transitive and so regular on each $\Delta_i$.

Suppose that $G^+$ is not faithful on $\Delta_0$ and $\Delta_1$. Then by Lemma 2.8, $\Gamma \cong K_{|H_1/2|, |H_1/2|}$, and (2.1) holds. In the remainder, we assume that $G^+$ is faithful on $\Delta_0$ and $\Delta_1$. Suppose first that $G^+$ is
not quasiprimitive on each $\Delta_i$. Let $N$ be a maximal intransitive normal subgroup of $G^+$ on $\Delta_0$. Then $N^G$ is a maximal intransitive normal subgroup of $G^+$ on $\Delta_1$. Let $B_0$ be the set of $N$-orbits on $\Delta_0$, and let $B_1$ be the set of $N^G$-orbits on $\Delta_1$. Then $|B_0| = |B_1| \geq 2$. Let $B = B_0 \cup B_1$. Then as $N^G = N$, $B$ is a $G$-invariant partition of $V(\Gamma)$. Let $K$ be the kernel of $G$ acting on $B$. Then $K$ is a normal subgroup of $G$ with at least four orbits on $V(\Gamma)$. Since $G$ is bi-quasiprimitive, we have $K = 1$. Thus $G$ and hence $H$ act faithfully on $B$. Since $H$ is abelian and transitive on $V(\Gamma)$ it follows that $H$ is transitive and so regular on $B$. Thus $|B| = |H| = |V(\Gamma)|$, a contradiction. Hence $G^+$ is quasiprimitive on each $\Delta_i$.

Since $H$ is cyclic and transitive on $V(\Gamma)$, $H^+$ is cyclic, transitive and so regular on each $\Delta_i$. It follows from [25, Theorem 1.2] that $G^+$ is primitive on $\Delta_i$. Then by Lemma 2.9, either $|H| = 2p$ for some prime $p$ and $G^+ \leq AGL(1, p)$, or $G^+$ is 2-transitive on $\Delta_i$.

If $|H| = |V(\Gamma)| = 2p$ for some prime $p$, then as $\Gamma$ is connected $G$-arc-transitive, $\Gamma$ is one of the graphs listed in Theorem 2.7. By Lemma 2.4, $B'(H(11))$, $B(PG(n - 1, q))$ and $B'(PG(n - 1, q))$ are not circulants, while $\Delta_0$, $\Delta_1$-invariant partition of $\Omega$ and $\Delta_0$, $\Delta_1$-invariant partition of $\Omega$. Then by Lemma 2.9, either $G$ is quasiprimitive on both $\Delta_0$ and $\Delta_1$, hence $\Gamma$ is given by Lemma 2.11 (1). However, by Lemma 2.4, the graphs $B(PG(d - 1, q))$ and $B'(PG(d - 1, q))$ are not circulants, and so $\Gamma = K_{2p}$, in which case $n$ must be odd by Lemma 2.4, and case (2.3) holds.

Finally, assume that $|H| = |V(\Gamma)| \neq 2p$ for any prime $p$. Then $G^+$ is 2-transitive on both $\Delta_0$ and $\Delta_1$, hence $\Gamma$ is given by Lemma 2.11 (1). However, by Lemma 2.4, the graphs $B(PG(d - 1, q))$ and $B'(PG(d - 1, q))$ are not circulants, and so $\Gamma = K_{2p}$, in which case $n$ must be odd by Lemma 2.4, and case (2.2) holds.

### 4 VERTEX QUASIPRIMITIVE ARC-TRANSITIVE BICIRCULANTS

This section is devoted to classifying all vertex quasiprimitive arc-transitive bicirculants. The first lemma gives a useful observation on quasiprimitive imprimitive permutation groups that contain a cyclic subgroup with exactly two orbits.

**Lemma 4.1.** Let $G$ be a quasiprimitive imprimitive permutation group on a set $\Omega$. Suppose that $H$ is a cyclic subgroup of $G$ that has exactly two orbits, $H_0$ and $H_1$, on $\Omega$ and $|\Omega| = 2|H|$. Let $B$ be a non-trivial maximal $G$-invariant partition of $\Omega$. Then $G$ is faithful and primitive on $B$, and also the following hold:

1. $H$ is regular on $B$, and for each $B \in B$ we have $|B| = 2$ and $|B \cap H_0| = 1 = |B \cap H_1|$;
2. $G$ and $|H|$ are as in Lemma 2.9, and in particular, either $G$ is a 2-transitive group on $B$ or $|H| = p$ for some prime $p$.

**Proof.** Since $G$ is quasiprimitive on $\Omega$, it follows that $G$ acts faithfully on $B$. Moreover, the maximality of $B$ implies that $G$ is primitive on $B$. Suppose that for each $B \in B$, either $B \subseteq H_0$ or $B \subseteq H_1$. Then $|B|$ divides $|H|$ and we may assume that $H_0 = B_1 \cup \cdots \cup B_r$ and $H_1 = B_{r+1} \cup \cdots \cup B_{2r}$. Since $H_0$ and $H_1$ are the two orbits of $H$, it follows that $H$ is transitive on both sets $\{B_1, \ldots, B_r\}$ and $\{B_{r+1}, \ldots, B_{2r}\}$. Since $G$ is faithful on $B$, $H$ is faithful on $B$. Further, as $H$ is a cyclic group, it is regular on each orbit on $B$. Thus $|B| = 2|H| = |\Omega|$, contradicting the fact that $B$ is a non-trivial $G$-invariant partition of $\Omega$. Hence there exists $B \in B$ such that $B \cap H_0 \neq \emptyset$ and $B \cap H_1 \neq \emptyset$. By Lemma 3.1, for all $B \in B$, we have

$$B \cap H_0 \neq \emptyset \quad \text{and} \quad B \cap H_1 \neq \emptyset.$$
Let \( B = \{ B_1, \ldots, B_t \} \). For \( i \in \{0,1\} \), let \( B_i = \{ B_j \cap H_i, \ldots, B_t \cap H_i \} \). Since each \( B_j \) meets each \( H_i \) non-trivially, we have that \( |B_0| = |B_1| = t \). Moreover, as \( H \) is transitive on each \( H_i \), it is transitive on each \( B_i \). Since \( H \) is cyclic, it has a unique subgroup of each order and so the kernel of \( H \) on \( B_0 \) is equal to the kernel of \( H \) on \( B_1 \), and so is in the kernel of \( H \) on \( B \). It follows that \( H \) acts faithfully and hence regularly on each \( B_i \). Thus \( |H| = t = |B_i| \), and so each \( B \in B \) has size 2. Hence (1) holds. Furthermore, the action of \( G \) on \( B \) satisfies the conditions of Lemma 2.9, so \( G \) and \( |H| \) are as in Lemma 2.9, and hence either \( G \) is 2-transitive on \( B \) or \( |B| = p \) is a prime, so that (2) holds. \( \square \)

The following proposition determines all the vertex quasiprimitive arc-transitive bicirculants.

Proposition 4.2. Let \( \Gamma \) be a connected \( G \)-arc-transitive bicirculant over the cyclic subgroup \( H \) of order \( n \) such that \( G \) is quasiprimitive on \( V(\Gamma) \). Then one of the following holds:

(1) \( G \) is primitive on \( V(\Gamma) \) and \( \Gamma \) is one of the following graphs:
   (1.1) \( K_{2n} \), and \( G \) is a 2-transitive group of degree \( 2n \) as in Theorem 2.10;
   (1.2) Petersen graph or its complement, and \( A_5 \leq G \leq S_5 \);
   (1.3) \( H(2,4) \) or its complement, and \( G \) is a rank 3 subgroup of \( AGL(4,2) \);
   (1.4) Clebsch graph or its complement, and \( G \) is a rank 3 subgroup of \( AGL(4,2) \).

(2) \( G \) is not primitive on \( V(\Gamma) \) and \( \Gamma \) is one of the following graphs:
   (2.1) \( K_{n[2]} \) and \( G \) has rank 3 on vertices;
   (2.2) \( K_{n,n} - nK_2 \) with \( PGL(d,q) \leq G \leq PΓL(d,q) \) and \( n = (q^d - 1)/(q - 1) \).

Proof. Suppose first that \( G \) is primitive on \( V(\Gamma) \). Then \( G \) is given in Theorem 2.10. If \( G \) is 2-transitive on \( V(\Gamma) \), then \( \Gamma \) is isomorphic to a complete graph \( K_{2n} \), so that (1.1) holds. Now assume that \( G \) is not 2-transitive on \( V(\Gamma) \). Then \( n, G \) and the vertex stabiliser \( G_u \) are one of the following:

(i) \( n = 8 \), \( G \leq AGL(4,2) \) and \( G_u \in \{ Z_5 : Z_4, (Z_3 \times Z_3) : Z_4, (S_3 \times S_3) : Z_2, S_5 \} \).
(ii) \( n = 5 \), and \( A_5 \leq G \leq S_5 \) in the action on the set of 2-subsets of \( \{1,2,3,4,5\} \).

If \( \Gamma \) is in case (ii), then by [10, p.75], \( \Gamma \) is the Petersen graph or its complement, and (1.2) holds. Now we determine the graphs in case (i) by MAGMA [4]. In all cases \( G \) has rank three. If \( G_u = (S_3 \times S_3) : Z_2 \) or \( (Z_3 \times Z_3) : Z_4 \), then \( \Gamma = H(2,4) \) or its complement, and (1.3) holds. If \( G_u = Z_5 : Z_4 \) or \( S_5 \), then \( \Gamma \) is the Clebsch graph or its complement (the folded 5-cube), so (1.4) holds.

It remains to consider the case that \( G \) is not primitive on \( V(\Gamma) \). Let \( B \) be a non-trivial maximal block system of \( G \) on \( V(\Gamma) \). Then by Lemma 4.1, for each \( B \in B \), we have that \( |B \cap H_0| = 1 = |B \cap H_1| \) and \( |B| = 2 \). Moreover, \( H \) is regular on \( B \), \( G \) is faithful and primitive on \( B \), and the pair \((G,n)\) is as in Lemma 2.9.

If the pair \((G,n)\) is as in case (i) of Lemma 2.9, then \( |V(\Gamma)| = 2p \) for some prime \( p \) and \( H \) is a normal subgroup of \( G \) of order \( p \). However, this contradicts \( G \) being quasiprimitive on a set of size \( 2p \). Moreover, \((G,n) \not\in \{ (PSL(2,11),11),(M_{23},23) \} \), as in these cases \( G \) does not have a transitive action on \( 2n \) points.

It remains to assume that \((G,n)\) is either \((M_{11},11)\) or is as in cases (ii) and (iii) of Lemma 2.9. Thus \( G \) is 2-transitive on \( B \) and \( G \) has a quasiprimitive action on \( 2n \) points. Hence the quotient graph \( \Gamma_p \) is a complete graph on \( n \) vertices for some \( n \geq 4 \). Let \( B_1, B_2 \in B \). Then \( B_1 \) and \( B_2 \) are adjacent in \( \Gamma_p \). Let \( b_1 \in B_1 \) and recall that \( |B_1| = |B_2| = 2 \).

Suppose first that \( |\Gamma(b_1) \cap B_2| = 2 \). Then as \( \Gamma \) is \( G \)-arc-transitive, \( |\Gamma(b_2) \cap B_1| = 2 \) for each \( b_2 \in B_2 \), and so

\[ \{B_1 \cup B_2 \} \cong K_{2,2} \]
Since $\Gamma_B \cong K_n$, it follows that $\Gamma \cong K_{n[2]}$. In particular, for each vertex $u$ of $\Gamma$, there is a unique vertex at distance two from $u$. Since $\Gamma$ is $G$-arc-transitive, it follows that $G$ has rank three, so that (2.1) holds.

Next assume that $|\Gamma(b_1) \cap B_2| = 1$. First suppose that $|B_1 \cup B_2|$ contains exactly one edge. Then the valency of $\Gamma_B$ is twice the valency of $\Gamma$. Since $\Gamma_B \cong K_n$, the valency of $\Gamma_B$ is $n - 1$ and so $n$ is an odd integer. Let $B_1 = \{b_1, b'_1\}$ and suppose that $b_1$ is adjacent to $b_2 \in B_2$. Then $b'_1$ is not adjacent to any vertex of $B_2$. Furthermore, $b_1$ is adjacent to a unique vertex of $(n - 1)/2$ neighbours in $\Gamma_B$ of $B_1$, say $\Theta_1$, and $b'_1$ is adjacent to a unique vertex of the remaining $(n - 1)/2$ neighbours in $\Gamma_B$ of $B_1$, say $\Theta_2$. Since $G_{B_1, B_2}$ fixes $b_1$ and $b_2$, it fixes $\Theta_1$ and $\Theta_2$ setwise. Hence, as $n \geq 4$, it follows that $G$ is not 3-transitive on $V(\Gamma_B)$. Thus Lemma 2.9 implies that $\text{PGL}(d, q) \leq G \leq \text{PΓL}(d, q)$ and $n = (q^d - 1)/(q - 1)$ with $d \geq 3$ (note that $\text{PGL}(2, q)$ is 3-transitive on $q + 1$ vertices). Moreover, the elements of $V(\Gamma_B)$ can be identified with the set of one-dimensional subspaces of a $d$-dimensional vector space over $\text{GF}(q)$. Then we see that $G_{B_1, B_2}$ has two orbits on the set of remaining one-dimensional subspaces: those in the span $U$ of $B_1$ and $B_2$, and those outside $U$. Since $B_2 \in \Theta_2$, it follows that $\Theta_1$ is one of these two orbits, but neither has size $(n - 1)/2$, a contradiction.

Thus $|B_1 \cup B_2| \cong 2K_2$, and $\Gamma$ is a cover of $\Gamma_B \cong K_n$. It follows that for any vertex $v \in V(\Gamma) \setminus B_1$, $v$ is adjacent to either $b_1$ or $b'_1$, so $v \in \Gamma(b_1) \cup \Gamma(b'_1)$. Thus

$$V(\Gamma) = \{b_1\} \cup \Gamma(b_1) \cup \Gamma(b'_1) \cup \{b'_1\}.$$  

Since $\Gamma$ is a cover of $\Gamma_B$, we have that $\Gamma(b_1) \neq \Gamma(b'_1)$ and since $\Gamma$ is $G$-arc-transitive, the vertex stabiliser $G_{b_1} = G_{b'_1}$ is transitive on $\Gamma(b_1)$ and $\Gamma(b'_1)$. Thus $G_{b_1}$ has four orbits on $V(\Gamma)$. Since $\Gamma$ is a cover of $\Gamma_B$, we have $b'_1 \not\in \Gamma_2(b_1)$. Thus $\Gamma$ has diameter 3, $\Gamma_3(b_1) = \Gamma(b'_1)$ and $\Gamma_3(b_1) = \{b'_1\}$. Hence $\Gamma$ is $G$-distance-transitive and is an antipodal cover of $K_n$ with fibres of size 2. Therefore $\Gamma$ and $G$ are listed in cases (1), (3) or (6) of [14, Main Theorem]. Recall that either $(G, n) = (M_{11}, 11)$, or $n$ and $G$ as an abstract group are given in cases (ii) and (iii) of Lemma 2.9. Thus either case (1) or case (3)(e) of [14, Main Theorem] holds. If case (1) occurs then $\Gamma \cong K_{n,n} - nK_2$. However, $M_{11}$ does not have a rank four action of degree 22 while $S_n$ and $A_n$ do not have a quasiprimitive action of degree $2n$ where the blocks have size 2. Thus $(G, n)$ is as in case (iii) of Lemma 2.9, that is, $\text{PGL}(d, q) \leq G \leq \text{PΓL}(d, q)$, and $n = (q^d - 1)/(q - 1)$, so that (2.2) holds. Finally, if case (3)(e) occurs, then $\text{Aut}(\Gamma) \cong \text{PΣL}(2, p) \times S_2$ where $p = n - 1 \equiv 1$ (mod 4). Since $G$ is quasiprimitive on $V(\Gamma)$ and is transitive on the set of arcs of $K_n$, it follows that $G$ is isomorphic to a subgroup of $\text{PΣL}(2, p)$. However, here $G$ does not contain an element of order $n = p + 1$. This completes the proof.

We remark that quasiprimitive rank 3 groups have been classified in [8], and the possibilities for the group $G$ in case (2.1) of Proposition 4.2 can be determined using [8, Table 1].

## 5 VERTEX BI-QUASIPRIMITIVE ARC-TRANSITIVE BICIRCULANTS

In this section, we will complete the proof of Theorem 1.1 by determining the arc-transitive vertex bi-quasiprimitive bicirculants.

Let $\Gamma$ be a $G$-arc-transitive graph and suppose that $G$ acts bi-quasiprimitively on $V(\Gamma)$. Then $G$ has a minimal normal subgroup $M$ that has exactly two orbits on $V(\Gamma)$. Since $\Gamma$ is $G$-arc-transitive and connected, each $M$-orbit contains no edge of $\Gamma$. Thus $\Gamma$ is a bipartite graph, and the two
M-orbits form the two bipartite halves of \( \Gamma \). In particular, all intransitive normal subgroups of \( G \) have the same orbits. Recall that \( G^+ \) denotes the index two subgroup of \( G \) that is the stabiliser of each bipartite half.

The following proposition classifies arc-transitive bicirculants that are vertex bi-quasiprimitive and such that the two orbits of the cyclic subgroup are the two bipartite halves.

**Proposition 5.1.** Let \( \Gamma \) be a connected \( G \)-arc-transitive graph such that \( G \) is bi-quasiprimitive on \( V(\Gamma) \) and \( G \) contains a cyclic subgroup \( H \) of order \( n \) such that the two bipartite halves are \( H \)-orbits. Then \( \Gamma \) is one of the following graphs:

(a) \( K_{n,n} \) where \( n \geq 2 \);
(b) \( K_{n,n} - nK_2 \) where \( n \geq 3 \);
(c) \( B'(H(11)) \);
(d) \( G(2p,r) \) where \( p \) is a prime and \( r > 1 \) divides \( p - 1 \);
(e) \( B(PG(d - 1, q)) \) and \( B'(PG(d - 1, q)) \), where \( d \geq 3, q \) is a prime power.

**Proof.** Identify the two bipartite halves of \( \Gamma \) with \( H \) and denote them by \( H_0 \) and \( H_1 \). Let \( G^+ = G/H_0 = G/H_1 \). Since \( G \) is transitive on the vertex set, it follows that \( G = \langle G^+, \sigma \rangle \) for some element \( \sigma \in G \setminus G^+ \) with \( \sigma^2 \in G^+ \). If \( G^+ \) acts unfaithfully on each \( H_i \), then it follows from Lemma 2.8 that \( \Gamma = K_{n,n} \) for some \( n \geq 2 \).

From now on we suppose that \( G^+ \) acts faithfully on each \( H_i \). Assume that \( G^+ \) acts imprimitively on \( H_0 \). Take a maximal \( G^+ \)-invariant partition \( B_0 \) on \( H_0 \). Then \( B_0^G \) is a maximal \( G^+ \)-invariant partition on \( H_1 \). Let \( C = B_0 \cup B_0^G \). Then \( G \) leaves invariant this partition \( C \) of the vertex set, as \( B_0^2 = B_0 \). Since \( G \) is bi-quasiprimitive on \( V(\Gamma) \) and \( |C| \geq 4 \), it follows that \( G \) acts faithfully on \( C \). Let \( M_0 \) be the kernel of \( H \) acting on \( B_0 \) and \( M_1 \) be the kernel of \( H \) acting on \( B_0^G \). Since \( H \) acts transitively on \( B_0 \) and \( B_0^G \), it follows that \( |M_0| = |H|/|B_0| = |H|/|B_0^G| = |M_1| \). Then as \( H \) is a cyclic group and has a unique subgroup of each order, it follows that \( M_0 = M_1 \), that is, the kernel of \( H \) on \( B_0 \) is the same as the kernel on \( B_0^G \), and is hence in the kernel of \( G \) on \( C \). However, \( G \) is faithful on \( C \), and so \( H \) acts faithfully on \( B_0 \). Thus \( \{|B_0| = |H| \} \), contradicting \( G^+ \) being imprimitive on \( H_0 \).

Therefore \( G^+ \) is primitive on each \( H_i \). Since \( G^+ \) contains the cyclic subgroup \( H \) that is transitive on each \( G^+ \)-orbit, it follows from Lemma 2.9 that either \( |H| = p \) is a prime or \( G^+ \) is 2-transitive on \( H_i \). If \( |H| = p \), then as \( \Gamma \) is a bipartite graph it follows from Theorem 2.7 that \( \Gamma \) is one of the following graphs: \( K_{p,p}, G(2p,r) \) with \( r > 1 \), \( B'(H(11)) \) where \( p = 11 \), \( B(PG(d - 1, q)) \) and \( B'(PG(d - 1, q)) \) where \( p = (q^d - 1)/(q - 1), d \geq 3 \) and \( q \) is a prime power. Suppose that \( G^+ \) is 2-transitive on \( H_i \) with \( |H_i| \) not a prime. Then Lemma 2.11 (1) implies that \( \Gamma = K_{n,n} - nK_2 \) where \( n \geq 3 \), \( B(PG(d - 1, q)) \), or \( B'(PG(d - 1, q)) \).

It remains to consider the case where the two \( H \)-orbits are not the two bipartite halves.

**Proposition 5.2.** Let \( \Gamma \) be a \( G \)-arc-transitive graph such that \( G \) is bi-quasiprimitive on \( V(\Gamma) \) and \( G \) contains a cyclic subgroup \( H \) of order \( n \) such that \( H \) has two orbits of size \( n \) and these are not the bipartite halves of \( \Gamma \). Then \( n \) is even and \( \Gamma \) is one of \( K_{n,n} \), \( K_{n,n} - nK_2 \), \( B(PG(d - 1, q)) \) or \( B'(PG(d - 1, q)) \) for even \( d \geq 3 \) and odd \( q \).

**Proof.** Let \( \Delta_0 \) and \( \Delta_1 \) be the two bipartite halves of \( \Gamma \), and let \( H_0 \) and \( H_1 \) be the two \( H \)-orbits. Let \( G^+ \) be the index two subgroup of \( G \) that stabilises both \( \Delta_0 \) and \( \Delta_1 \), and let \( \sigma \in G \) such that \( G = \langle G^+, \sigma \rangle \) and \( \sigma^2 \in G^+ \). If \( |V(\Gamma)| \leq 4 \), then the only candidate for \( \Gamma \) is \( K_{2,2} \), so from now on we suppose that
\(|V(\Gamma)| > 4\). Then \(\{\Delta_0, \Delta_1\}\) is the unique \(G\)-invariant partition of \(V(\Gamma)\) into two equal-sized parts. Since the \(\Delta_i\) are not \(H\)-orbits and \(H\) has two orbits of size \(n\), it follows that \(H^+ := H \cap G^+\) has index two in \(H\) and has two equal-sized orbits on each \(\Delta_i\). Thus \(n\) is even and \(G = (G^+, H)\).

If \(G^+\) is unfaithful on each \(\Delta_i\), then it follows from Lemma 2.8 that \(\Gamma = K_{n,n}\). Thus in the remainder we assume that \(G^+\) is faithful on each \(\Delta_i\).

Suppose first that \(G^+\) is primitive on each \(\Delta_i\). Since \(H^+\) is a cyclic subgroup with exactly two orbits of size \(n/2\) on \(\Delta_i\), the possibilities for \(G^+\) are given by Theorem 2.10. In particular, either \(G^+\) is almost simple or \(\mathbb{Z}_2^m \leq G^+ \leq \text{AGL}(m, 2)\) with \(2 < m \leq 4\). If \(\mathbb{Z}_2^m \leq G^+ \leq \text{AGL}(m, 2)\) then \(G^+\) has a normal subgroup \(N \cong \mathbb{Z}_2^m\) that is regular on each \(\Delta_i\). Now \(N\) is characteristic in \(G^+\) and so is normal in \(G\). Moreover, since \(H^+\) is cyclic, either \(H \cap N = 1\) or \(H \cap N = \mathbb{Z}_2\). Further, since \(N\) is self-centralising in \(G^+\) and \(H^+\) is cyclic, either \(|H| = 4\) and \(C_H(N) = H\), or \(C_H(N) = H^+ \cap N\). If \(m \geq 3\), then \(|H| = 2^m > 8\) and \(H^+ \cap N \neq H^+, \text{ so } C_H(N) = H^+ \cap N\). Thus \(H/(H^+ \cap N)\) is isomorphic to a cyclic subgroup of \(\text{GL}(m, 2)\) of order \(2^m\) or \(2^{m-1}\). Since \(m = 3\) or \(4\), it follows that \(m = 3\), 

\[G^+ = \text{AGL}(3, 2)\quad \text{and } H^+ \cap N = 2\]. Since \(|H| = 8\) and \(G^+\) does not have an element of order \(8\), it follows that \(G = \text{Aut}(\text{AGL}(3, 2))\). In particular, given \(u \in \Delta_0\) we have that \(G^+\) is transitive on \(\Delta_1\) and so \(\Gamma = K_{8,8}\). It remains to consider the case where \(m = 2\) and \(G^+ = A_4 \cong \text{AGL}(1, 4)\) or \(S_4 \cong \text{AGL}(2, 2)\). If \(G^+ = S_4\) then \(G = S_4 \times \mathbb{Z}_2\), contradicting \(G\) being bi-quasiprimitive. Thus \(G^+ = A_4\) and either \(G = A_4 \times \mathbb{Z}_2\) or \(G \cong S_4\). The first group does not contain an element of order \(4\) (and is also not bi-quasiprimitive), while the second implies that \(\Gamma = K_{4,4} - 4K_2\).

Next suppose that \(G^+\) is almost simple. Then either \(G^+\) is 2-transitive on each \(\Delta_i\), or \(n = 10\) and \(A_5 \leq G^+ \leq S_5\). In the first case, Lemma 2.11 (2) implies that \(\Gamma = K_{n,n} - nK_2\) where \(n \geq 3\), \(K_{12,12}, B(\text{PG}(d - 1, q))\) or \(B'(\text{PG}(d - 1, q))\) where \(d \geq 3\) is an even integer and \(q\) is odd. Suppose instead that \(n = 10\) and \(A_5 \leq G^+ \leq S_5\). Then by [10, p.75], the action of \(G^+\) on \(\Delta_i\) is \(S_5\) or \(A_5\) acting naturally on the set of unordered pairs of \(\{1, 2, 3, 4, 5\}\). Let \(u \in \Delta_0\). Then \(G^+\) has only orbits of size \(1, 3\) or \(6\) on both \(\Delta_0\) and \(\Delta_1\). Moreover, there exists \(u' \in \Delta_1\) such that \(G^+_u = G^+_{u'} = G^+_{u''}\). Since \(G^+\) is transitive on \(\Gamma(u)\), it follows that \(|\Gamma(u)| = 3\) or \(6\). Thus \(\Gamma\) is either the standard double cover of the Petersen graph, or the standard double cover of the complement of the Petersen graph. In both cases \(\text{Aut}(\Gamma) = S_5 \times \mathbb{Z}_2\). If \(G \leq \text{Aut}(\Gamma)\) contains an element \(g\) of order \(10\) with 2-cycles of length \(10\) on \(V(\Gamma)\), it follows that \(g^5\) is an involution that centralises the element \(g^2 \in S_5\) of order \(5\). As \(S_5\) contains no such involution, we have \((g^5) = \text{Z}(\text{Aut}(\Gamma)) \leq G\), contradicting the fact that all normal subgroups of \(G\) have at most two orbits.

It remains to consider the case where \(G^+\) is imprimitive on \(\Delta_i\). Let \(B_0\) be a maximal \(G^+\)-invariant partition of \(\Delta_0\). Then \(B_1 = B_2^c\) is a maximal \(G^+\)-invariant partition of \(\Delta_1\). Note that \(B = B_0 \cup B_1\) is a \(G\)-invariant partition of \(V(\Gamma)\) with at least four parts. Thus the kernel of \(G\) acting on \(B\) has at least four orbits and so the bi-quasiprimitivity of \(G\) implies that \(G\) acts faithfully on \(B\).

Suppose that each \(B \in B\) is contained in either \(H_0\) or \(H_1\). Then for \(i \in \{0, 1\}\), let \(C_i = \{B \in B \mid B \subseteq H_i\}\) and let \(M_i\) be the kernel of \(H\) acting on \(C_i\). Then \(H/M_i\) is transitive and so regular on \(C_i\). Since \(|C_0| = |C_1|\), we have \(|M_0| = |M_1|\), and since \(H\) is cyclic, it follows that \(M_0 = M_1\). Thus \(M_0\) lies in the kernel of \(G\) acting on \(B\). Since \(G\) is faithful on \(B\) it follows that \(M_0 = 1\). Hence \(|B| = |H| = 1\), contradicting the fact that \(B\) is a non-trivial block system of \(G\) on \(V(\Gamma)\). Thus Lemma 3.1(1) implies that each \(B \in B\) has non-empty intersection with both \(H_0\) and \(H_1\). Hence \(H\) is transitive on \(\Delta_i\) and \(G\) is a circulant. As \(|H| = |V(\Gamma)|/2\), it follows that each \(B \in B\) has size \(2\). Moreover, \(H^+\) acts transitively on each \(B_i\).

For \(i = 0, 1\), let \(K_i\) be the kernel of \(G^+\) acting on \(B_i\). Then \(K_0^2 = K_1\) and \(K_0 \cap K_1\) is contained in the kernel of \(G\) acting on \(B\). Thus \(K_0 \cap K_1 = 1\) and so \(K_0 \times K_1 \leq G\). Since \(G\) is bi-quasiprimitive, it follows that \(K_1 = 1\) or \(K_i\) acts transitively on \(B_{1-i}\).
Suppose that $K_i$ acts transitively on $B_{1-i}$. Then $\Gamma_B \cong K_{n/2,n/2}$ is complete bipartite. Since $K_0 \times K_1 \leq G^+$, it follows that $(K_0 \times K_1)/K_0 \leq G^+/K_0 \cong (G^+)_{K_0}$. Since each block in $B_i$ has size 2, it follows that $K_i$ contains only elements of order 2. Thus $K_i$ is abelian and $K_i \cong \mathbb{Z}_2^r$. Based on this together with the fact that $K_i$ acts transitively on $B_{1-i}$, we conclude that $K_i$ is regular on $B_{1-i}$, and so $n/2 = |B_{1-i}| = |K_i| = 2$. Furthermore, $\mathbb{Z}_2^r \cong (K_0 \times K_1)/K_0 \leq (G^+)_{K_0}$. Since $B_0$ is a maximal $G^+$-invariant partition of $\Delta_0$, $(G^+)_{K_0}$ acts primitively on $B_0$. In particular, $(G^+)_{K_0}$ acts primitively on $B_0$ of affine type. Recall that the cyclic group $(H^+)_{K_0} \leq (G^+)_{K_0}$ acts transitively on $B_0$. By Lemma 2.9, $\frac{n}{2} = p$ is a prime, so $\frac{n}{2} = p = 2$. Hence $n = 4$ and $\Gamma_B \cong K_{2,2}$. Let $B_0 = \{B_1, B_2\}$ and $B_1 = \{D_1, D_2\}$. If $[B_1 \cup D_j]$ contains exactly one edge of $\Gamma$, then $\Gamma$ has valency 1, and so $\Gamma$ is disconnected, a contradiction. Suppose that $[B_1 \cup D_1] \cong 2K_2$. Then $\Gamma$ is a cover of $\Gamma_B$. Hence $\Gamma$ has valency 2 and $\Gamma$ is a cycle. Since each block in $B$ has size 2, it follows that $\Gamma \cong C_8$. However, in this case, $\text{Aut}(\Gamma)$ has a unique order 4 cyclic subgroup whose two orbits are the two bipartite halves of $\Gamma$, a contradiction. If $[B_1 \cup D_1] \cong K_{2,2}$, then $\Gamma \cong K_{4,4}$.

Thus it remains to consider the case that $G^+$ acts faithfully on each $B_i$. The maximality of $B_i$ implies that $G^+$ is primitive on $B_i$. If $|B_i| = p$ and $G^+ \leq \text{AGL}(1, p)$, then $G^+$ has a unique minimal normal subgroup $N$ of order $p$. Since $N$ is characteristic in $G^+$, it is normal in $G$. However, $|\text{V}(\Gamma)| = 4p$ and so $N$ has four orbits, contradicting $G$ being bi-quasiprimitive. Since $H^+$ is a cyclic transitive subgroup of $G^+$ in its action on $B_i$, it follows that $G^+$ is given by Lemma 2.9 and in particular is 2-transitive on $B_i$. Moreover, if $B \in B_0$, then $(G^+)_{M}$ has an index two subgroup (the stabiliser of a vertex in $B$) and so either $G^+ = S_{n/2}, M_{11}$ or $\text{PGL}(d, q) \leq G^+ \leq \text{PGL}(d, q)$. If $G^+ = M_{11}$ or $S_{n/2}$ with $n \neq 12$, then $G = G^+ \times \mathbb{Z}_2$, contradicting $G$ being bi-quasiprimitive. If $G^+ = S_6$, then $G = \text{Aut}(S_6)$. However, in this case, all the order 6 elements of $G$ are in $G^+ = S_6$, again a contradiction. Thus $\text{PGL}(d, q) \leq G^+ \leq \text{PGL}(d, q)$. Then either $|\text{C}_G(\text{PSL}(d, q))| = 2$ or $G \leq \text{Aut}(\text{PSL}(d, q))$. The first case is not possible as this would imply that $G$ has a normal subgroup of order 2, which would contradict $G$ being bi-quasiprimitive. Thus $G \leq \text{Aut}(\text{PSL}(d, q))$. If $G \not\leq \text{PGL}(d, q)$ then $\Gamma_B$ is either $B(\text{PG}(d-1, q))$ or $B'(\text{PG}(d-1, q))$. However, these graphs are not circulants (Lemma 2.4) and so $G \leq \text{PGL}(d, q)$. In this case $|B| = 2(q^d - 1)/(q - 1)$, but $\text{PGL}(d, q)$ does not contain an element of this order, contradicting $H$ being regular on $B$. 

\section{Prime and Small Valency Arc-Transitive Bicirculants}

In this section, we compare our Theorem 1.1 to the classifications of arc-transitive bicirculants of valencies 3 and 5, and also make some observations about the prime valent case in general.

We first give a corollary of Theorem 1.1 about prime valency arc-transitive bicirculants.

\begin{corollary}
Let $\Gamma$ be a $G$-arc-transitive bicirculant of prime valency $p \geq 3$. Then $\Gamma$ is a normal cover of one of the following graphs:

\begin{enumerate}
  \item[(a)] $K_{p,p}$;
  \item[(b)] $K_{p+1}, K_{p+1,p+1} - (p+1)K_2$;
  \item[(c)] $G(2q, p)$, where $q$ is a prime integer and $p$ divides $q - 1$;
  \item[(d)] $B(\text{PG}(n, q))$, where $q$ is a prime power, $n \geq 2$, and $p = \frac{q^n - 1}{q - 1}$;
  \item[(e)] Petersen graph ($p = 3$), Clebsch graph ($p = 5$).
\end{enumerate}

Moreover, examples exist in all cases.
\end{corollary}
Proof. Let Γ be a $G$-arc-transitive bicirculant of prime valency $p \geq 3$. Then by Theorem 1.1, $G$ has a normal subgroup $N$ such that $\Gamma$ is a normal $r$-cover of $\Gamma_N$ where $\Gamma_N$ is one of the graphs in Theorem 1.1. Moreover, since $r$ divides the prime number $p$, it follows that $r = 1$, and so $\Gamma$ is a normal cover of $\Gamma_N$. Thus $\Gamma_N$ has valency $p$.

Note that, for the graphs in Theorem 1.1, $K_n[2]$ has valency $2(n-1)$ which is not a prime; $\text{Cay}(q,e)$ (a prime) has even valency $e; \text{B}'(\text{PG}(n,q))$ has valency $q^n; \text{H}(2,4), \text{B}'(\text{H}(11))$ and the complement of the Petersen graph have valency 6; and the complement of the Clebsch graph has valency 10. Thus $\Gamma_N$ is as claimed. Moreover, since all graphs listed in (a)–(e) are themselves arc-transitive prime valency bicirculants by Lemma 2.5, examples trivially exist for all these listed graphs.

For the class of arc-transitive bicirculants of valency 3 or 5 that are Cayley graphs of dihedral groups we obtain the following lemma using [2, 27].

**Lemma 6.2.** Let $n \geq 11$ and $k = 3$ or 5. Let $\Gamma_{n,k} = \text{Cay}(D_{2n},\{b, ba, bar^{+1}, \ldots, ba^{k-2+r+1}\})$, where $D_{2n} = \langle a, b | a^n = b^2 = (ba)^2 = 1 \rangle$, and $r \in \mathbb{Z}^+$ such that $r(k-1) + \cdots + r^2 + r + 1 \equiv 0 \pmod{n}$. Then $\Gamma_{n,k}$ is a bipartite arc-transitive graph and the following hold.

(a) $\langle a^p \rangle \cong \mathbb{Z}_n/p$ is a normal subgroup of $\text{Aut}(\Gamma_{n,k})$, where $p$ is a prime divisor of $n$.
(b) There exists a prime divisor $p$ of $n$ such that $k \mid (p-1)$.
(c) Let $p$ be a prime divisor of $n$ such that $k \mid (p-1)$ and let $N = \langle a^p \rangle$. Then $(\Gamma_{n,k})_N \cong G(2p,k)$.

Proof. By the definition of $\Gamma_{n,k}$, we see that $\Gamma_{n,k}$ is a bipartite graph, and the two bipartite halves of $\Gamma_{n,k}$ are $\Delta_0 = \{1, a, a^2, \ldots, a^{n-1}\}$ and $\Delta_1 = \{b, ba, ba^2, \ldots, ba^{n-1}\}$ which are both $\langle a \rangle$-orbits.

If $k = 3$, then by [27, pp. 978–979], $\Gamma_{n,3}$ is arc-transitive with $\text{Aut}(\Gamma_{n,3}) = \mathbb{Z}_n : \mathbb{Z}_6$ where $\mathbb{Z}_n = \langle a \rangle$; if $k = 5$, then by Lemma 3.7 and [2, Proof of Theorem 3.11], $\Gamma_{n,5}$ is arc-transitive and $\langle a \rangle \cong \mathbb{Z}_n$ is a normal subgroup of $\text{Aut}(\Gamma_{n,5})$. Thus in both cases $\langle a \rangle \cong \mathbb{Z}_n$ is a normal subgroup of $\text{Aut}(\Gamma_{n,k})$. Since for each prime divisor $p$ of $n$, $\langle a^p \rangle$ is a characteristic subgroup of $\langle a \rangle$, it follows that $\langle a^p \rangle \cong \mathbb{Z}_n/p$ is a normal subgroup of $\text{Aut}(\Gamma_{n,k})$.

Let $n = p_1^{e_1}p_2^{e_2} \cdots p_f^{e_f}$, where $p_1$ is a prime. Since $k < n$, we know $r \neq 1$. Since $r^{k-1} + \cdots + r^2 + r + 1 \equiv 0 \pmod{n}$, it follows that $(r-1)(r^{k-1} + \cdots + r^2 + r + 1) \equiv 0 \pmod{n}$, so $r^{k-1} \equiv 1 \pmod{n}$. Let $(n,r) = t$. Since $r^k = ne + 1$ for some integer $e$, it follows that $t$ divides 1, so $t = 1$. Hence $r$ belongs to the group of units of the ring $\mathbb{Z}_n$, which has order $\phi(n)$ (Euler totient function). Therefore, since $k$ is a prime, the order of $r$ in the group of units is $k$, and so $k$ divides $\phi(n) = p_1^{e_1-1}p_2^{e_2-1} \cdots p_f^{e_f-1} \Pi(p_i - 1)$. Suppose that $k$ (which is a prime) divides $p_1^{e_1-1}p_2^{e_2-1} \cdots p_f^{e_f-1}$. Then $k^2$ divides $n$. As $r^{k-1} + \cdots + r^2 + r + 1 \equiv 0 \pmod{n}$, we have $r^{k-1} + \cdots + r^2 + r + 1 \equiv 0 \pmod{k^2}$. However, considering the possibilities for $r$ module $k^2$ and evaluating $r^{k-1} + \cdots + r^2 + r + 1$, we see that we never have $r^{k-1} + \cdots + r^2 + r + 1 \equiv 0 \pmod{k^2}$, a contradiction. Thus $k^2$ does not divide $n$, and so $k$ divides $\Pi(p_i - 1)$. Since $k$ is a prime number, there exists a prime divisor $p_1$ of $n$ such that $k$ divides $p_1 - 1$.

Let $p$ be a prime divisor of $n$ such that $k \mid (p-1)$ and let $N = \langle a^p \rangle$. Then $N$ has $p$-orbits on each bipartite half. Thus $(\Gamma_{n,k})_N$ is a bipartite arc-transitive graph of valency $k$ and with $2p$ vertices, and also its automorphism group has two blocks of size $p$. By [7, Lemma 3.9], $(\Gamma_{n,k})_N \cong G(2p,k)$.

□
6.1 Arc-transitive bicirculants of valency 3

The generalised Petersen graph $GP(n, r)$ is the graph on the vertex set

$$\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

with the adjacencies:

$$u_i \sim u_{i+1}, \quad v_i \sim v_{i+r}, \quad u_i \sim v_i, \quad i = 0, 1, \ldots, n-1.$$ 

Hence each generalised Petersen graph $GP(n, r)$ has valency 3. By [11], $GP(n, r)$ is arc-transitive if and only if $(n, r) = (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$ and $(24, 5)$.

The following classification of cubic arc-transitive bicirculants follows from [11, 27, 29].

**Theorem 6.3.** Let $\Gamma$ be a finite connected arc-transitive bicirculant of valency 3. Then $\Gamma$ is one of the following graphs:

(a) $K_4, K_{3,3}$;
(b) $GP(4, 1)$ ($K_{4,4} - 4K_2$), $GP(5, 2)$ (Petersen graph), $GP(8, 3)$ (Möbius-Kantor graph), $GP(10, 2)$, $GP(10, 3)$ (Desargues graph), $GP(12, 5)$ (Nauru graph), $GP(24, 5)$;
(c) Heawood graph ($BPG(2, 2)$);
(d) $\text{Cay}(D_{2n}, \{b, ba, ba^{r+1}\})$, where $D_{2n} = \langle a, b \mid a^n = b^2 = (ba)^2 = 1 \rangle$, $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ such that $r^2 + r + 1 \equiv 0 \pmod{n}$.

Define the following permutations of $V(GP(n, r))$.

$$\rho : u_i \mapsto u_{i+1}, \quad v_i \mapsto v_{i+1};$$
$$\delta : u_i \mapsto u_{-i}, \quad v_i \mapsto v_{-i};$$
$$\sigma : u_{4i} \mapsto u_{4i}, \quad u_{4i+2} \mapsto v_{4i-1}, \quad u_{4i+1} \mapsto u_{4i-1}, \quad u_{4i-1} \mapsto v_{4i},$$
$$v_{4i} \mapsto u_{4i+1}, \quad v_{4i-1} \mapsto v_{4i+5}, \quad v_{4i+1} \mapsto u_{4i-2}, \quad v_{4i+2} \mapsto v_{4i-6}.$$ 

The following lemma essentially follows from [11].

**Lemma 6.4.** Let $\Gamma = GP(n, r)$ where $(n, r) = (4, 1), (8, 3), (12, 5)$ or $(24, 5)$. Let $A := \text{Aut}(\Gamma)$. Then $\Gamma$ is bipartite, 2-arc-transitive and the following hold:

(a) $A = \langle \rho, \sigma, \delta \rangle$ and $A_{u_0} = \langle \sigma, \delta \rangle$;
(b) $\langle \rho^4 \rangle \cong \mathbb{Z}_{n/4}$ is a normal subgroup of $\text{Aut}(\Gamma)$;
(c) If $N = \langle \rho^4 \rangle$, then $\Gamma_N \cong K_{4,4} - 4K_2$.

**Proof.** By [15], the generalised Petersen graph $GP(n, r)$ is bipartite if and only if $n$ is even and $r$ is odd, so $\Gamma$ is bipartite. By [11, p. 217], $\Gamma$ is 2-arc-transitive, and $A = \langle \rho, \sigma, \delta \rangle$, where $\rho^n = \delta^2 = \sigma^3 = 1, \delta \rho \delta = \rho^{-1}, \delta \sigma \delta = \sigma^{-1}, \sigma \rho \sigma = \rho^{-1}, \sigma \rho^4 = \rho^4 \sigma$. Thus $\langle \rho^4 \rangle$ is a normal subgroup of $A$. Moreover, by definition, $A_{u_0} = \langle \sigma, \delta \rangle$.

Note that the two orbits $H_0$ and $H_1$ of $\langle \rho \rangle$ are not the two bipartite halves of $\Gamma$. Thus $H_i$ intersects each bipartite half with exactly $n/2$ vertices. Further, $\langle \rho^2 \rangle$ fixes the two bipartite halves of $\Gamma$ setwise. Let $N = \langle \rho^4 \rangle$. Then each $N$-orbit is in a bipartite half, $\Gamma_N$ has 8 vertices and has valency 3, and it is also 2-arc-transitive. It follows that $\Gamma_N \cong K_{4,4} - 4K_2$. 

\[\square\]
TABLE 1 Cubic arc-transitive bicirculants

| Γ          | Γ_N       | Aut(Γ)   | N |
|------------|-----------|----------|---|
| K_4        | K_4       | S_4      | 1 |
| K_{3,3}    | K_{3,3}   | S_3 ⋊ S_2 | 1 |
| K_{4,4} − 4K_2 | K_{4,4} − 4K_2 | S_4 × S_2 | 1 |
| Heawood graph | Heawood graph | PGL(3,2) | 1 |
| G(5,2)     | Petersen graph | S_5     | 1 |
| G(8,3)     | K_{4,4} − 4K_2 | GL(2, 3) ⋊ S_2 | S_2 |
| G(10,2)    | Petersen graph | A_5 × S_2 | S_2 |
| G(10,3)    | Petersen graph | S_5 × S_2 | S_2 |
| G(12,5)    | K_{4,4} − 4K_2 | S_4 × S_3 | Z_3 |
| G(24,5)    | K_{4,4} − 4K_2 | |Aut(Γ)| = 288 | Z_6 |
| Cay(D_{2n}, {b, ba, ba^{r+1}}), n \geq 11 odd, as in Theorem 6.3 | G(2p, 3), prime p|n, 3|p − 1 | Z_6 \times Z_6/p |

By [11, p. 217], G(10, 2) has automorphism group A_5 × S_2, and G(10, 3) is the Desargues graph which has automorphism group S_5 × S_2. For these two graphs, choose the normal subgroup S_2 of the automorphism group, then the corresponding quotient graph has 10 vertices and is of valency 3. Moreover, the automorphism group induces A_5 or S_5 on the quotient graph, respectively, and in both cases the quotient graph is the Petersen graph. The graphs K_4, K_{3,3}, G(4, 1) = K_{4,4} − 4K_2, G(5, 2) and G(14, 3) are in Theorem 1.1.

Lemmas 6.2 and 6.4 now allow us to show how each cubic arc-transitive bicirculant gives rise to a graph in Theorem 1.1.

**Proposition 6.5.** Let Γ be a finite connected arc-transitive bicirculant of valency 3. Then there exists a normal subgroup N of Aut(Γ) such that Γ_N is a graph in Theorem 1.1. Moreover, Γ, Γ_N, Aut(Γ) (if known), and N are as in Table 1.

Note that in Proposition 6.5, N and Γ_N are not necessarily unique. For instance, if Γ is G(12, 5), then its automorphism group is S_4 × S_3, and Γ_N is K_4 whenever N = S_3, and Γ_N is K_{3,3} when N = Z_2 × Z_2.

6.2 Arc-transitive bicirculants of valency 5

Let L, R, and M be subsets of an additive group H := \mathbb{Z}_n such that L = −L, R = −R and R ∪ L does not contain the zero element of H. Define the bicirculant BC_n[L, M, R] to have vertex set the union of the left part H_0 = \{h_0 | h ∈ H\} and the right part H_1 = \{h_1 | h ∈ H\}, and edge set the union of the left edges \{\{h_0, (h + l)_0\} | l ∈ L\}, the right edges \{\{h_1, (h + r)_1\} | r ∈ R\} and the spokes \{\{h_0, (h + m)_1\} | m ∈ M\}.

The family of arc-transitive bicirculants of valency 5 have been classified in [2, 3].

**Theorem 6.6.** Let Γ be a finite connected arc-transitive bicirculant of valency 5. Then Γ is one of the following graphs:

(a) K_6;
(b) K_{6,6} − 6K_2;
TABLE 2  Pentavalent arc-transitive bicirculants

| Γ            | Γₐ      | Aut(Γ) | N   |
|--------------|---------|--------|-----|
| K₅           | K₅      | S₆     | 1   |
| K₆,₆ − 6K₂   | K₆,₆ − 6K₂ | S₆ × S₂ | 1   |
| B(PG(2, 4))  | B(PG(2, 4)) | PΓL(3, 2) × S₂ | 1   |
| Clebsch graph | Clebsch graph | Z₄² : S₅ | 1   |
| BC₆[±1, 0, 1, 2, 3, 4] | K₆,₆ − 6K₂ | | |
| BC₂₄[∅, {0, 1, 3, 11, 20}, ∅] | K₆,₆ − 6K₂ | | |
| BC₂₄[∅, {0, 1, 3, 11, 20}, ∅] | BC₂₄[∅, {0, 1, 3, 11, 20}, ∅] | | |
| Cay(D₂ₙ, {b, ba, ba⁻¹, ba⁻²+ra⁻¹, ba⁻²⁻¹+ra⁻¹}) | G(2p, 5), prime p∤5, 5∤p−1 | | |

The complete graph K₆ is isomorphic to the graph BC₅[±1, 0, 1, 2, 3, 4] in [2, Theorem 1.1]. Remark 1.2 of [2] indicates that

K₆,₆ − 6K₂ ≅ BC₆[∅, {0, 1, 2, 3, 4}, ∅]
≈ BC₆[±1, 3, {0, 2}, {±1, 3}] ≅ BC₆[±1, {0, 2, 4}, {±1}],

and B(PG(2, 4)) ≅ BC₂₄[∅, {0, 1, 4, 14, 16}, ∅]. It can be checked by Magma [4] that the Clebsch graph is the graph BC₅[±1, 4, {0, 2}, {±3, 4}] in [2, Theorem 1.1]. The graph K₅,₅ is the Cayley graph Cay(D₁₀, {b, ba, ba², ba³, ba⁴}) in Theorem 6.6(h).

The graphs K₆, K₆,₆ − 6K₂, B(PG(2, 4)) and the Clebsch graph are in Theorem 1.1. Let Γ = BC₁₂[∅, {0, 1, 2, 4, 9}, ∅] and Γ’ = BC₂₄[∅, {0, 1, 3, 11, 20}, ∅]. Then by [2, Theorem 3.11, p. 666], Aut(Γ) has a normal subgroup M ≅ Z₂ and Aut(Γ’) has a normal subgroup N ≅ Z₄ such that Γₐ ≅ Γ’ₐ ≅ K₆,₆ − 6K₂. It can be easily checked using Magma [4] that the automorphism group of BC₆[±1, {0, 1, 5}, {±2}] is PΓL(2, 5) × S₂.

Now we use Lemma 6.2 to show how each valency 5 arc-transitive bicirculant gives rise to a graph in Theorem 1.1.

**Proposition 6.7.** Let Γ be a finite connected arc-transitive bicirculant of valency 5. Then there exists a normal subgroup N of Aut(Γ) such that Γₐ is a graph in Theorem 1.1. Moreover, Γ, Γₐ, Aut(Γ) (if known), and N are as in Table 2.

6.3 Arc-transitive bicirculants of valency 4

We conclude with a brief discussion of the valency 4 case. By Theorem 1.1, such a graph is either a cover of one of the graphs of valency 4 listed or a 2-cover of one of the graphs of valency 2 listed.
Note that a connected graph of valency 2 is a cycle and appears in Theorem 1.1 as Cay$(p, 2)$ for $p$ odd, or as $K_{2,2}$. The graphs of valency 4 listed in Theorem 1.1 are

(a) $K_{4,4}$;
(b) $K_5$;
(c) $K_{3,5} - 5K_2$;
(d) $K_{3}[2]$;
(e) $G(2p, 4)$ with $p \equiv 1$ (mod 4);
(f) $B(PG(2, 3))$;
(g) $B'(PG(2, 2))$;
(h) Cay$(p, 4)$ with $p \equiv 1$ (mod 4).

All finite arc-transitive bicirculants of valency 4 were classified in [21].

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