Abstract

We analyse adversarial bandit convex optimisation with an adversary that is restricted to playing functions of the form \( f_t(x) = g_t((x, \theta)) \) for convex \( g_t : \mathbb{R} \rightarrow \mathbb{R} \) and unknown \( \theta \in \mathbb{R}^d \) that is homogeneous over time. We provide a short information-theoretic proof that the minimax regret is at most \( O(d\sqrt{n} \log(n \text{ diam}(\mathcal{K}))) \) where \( n \) is the number of interactions, \( d \) the dimension and diam(\( \mathcal{K} \)) is the diameter of the constraint set.

1 Introduction

Let \( \mathcal{K} \subset \mathbb{R}^d \) be a convex body (non-empty interior, compact, convex). A game proceeds over \( n \) rounds. At the start of the game, an adversary secretly chooses a vector \( \theta \in \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : \|x\| = 1 \} \) and sequence \( (f_t)_{t=1}^n \) such that for all \( t \), \( f_t : \mathcal{K} \rightarrow \mathbb{R} \) is a function that is:

(a) convex; and

(b) bounded: \( f_t(x) \in [0, 1] \) for all \( x \in \mathcal{K} \); and

(c) Lipschitz: \( f_t(x) - f_t(y) \leq \|x - y\| \) for all \( x, y \in \mathcal{K} \); and

(d) a ridge function: \( f_t(x) = g_t((x, \theta)) \) for some \( g_t : \mathbb{R} \rightarrow \mathbb{R} \).

The learner then sequentially chooses \( (x_t)_{t=1}^n \) with \( x_t \in \mathcal{K} \) and observes \( f_t(x_t) \), which means that \( x_t \) should only depend on the previous actions \( (x_s)_{s=1}^{t-1} \), observed losses \( (f_s(x_s))_{s=1}^{t-1} \) and possibly a source of randomness. The minimax regret is

\[
\mathcal{R}_n = \inf_{\text{policy}} \sup_{\text{adversary}} \max_{x \in \mathcal{K}} \mathbb{E} \left[ \sum_{t=1}^n f_t(x_t) - f_t(x) \right],
\]

where the infimum is over all policies, the supremum is over the choices of the adversary subject to the constraints (a)-(d) above, and the expectation integrates over the randomness in the policy. Note that the direction \( \theta \) of the ridge is not known to the learner, but does not change with time.
Fact 1. Suppose that $f$ satisfies (a)–(d) above and $f(x) = g(x, \theta)$ with $\theta \in S^{d-1}$. Then $g$ is convex and Lipschitz on the closed interval $\{\langle x, \theta \rangle : x \in K\}$.

A proof is given in Section 5. A proof of the following theorem is our main contribution.

Theorem 2. Assume that $S^{d-1} \subset K$. Then

$$\mathcal{R}_n \leq \text{const} d^\alpha \sqrt{n} \log(n \text{diam}(K)),$$

where const is a universal constant and $\text{diam}(K) = \max_{x, y \in K} \|x - y\|$.

Since the dependence on $\text{diam}(K)$ is only logarithmic, the Lipschitz assumption can be relaxed entirely by scaling and restricting the domain of $K$ as explained by Lattimore [2020] and Bubeck et al. [2017].

Related work Our setting is a subset of bandit convex optimisation with the adversary restricted to ridge functions. There is a long line of work without this restriction and correspondingly worse regret bounds. The study of bandit convex optimisation was initiated by Kleinberg [2005] and Flaxman et al. [2005], who introduced simple algorithms based on gradient descent with importance-weighted gradient estimates. Although these approaches are simple, they suffer from provably suboptimal dependence on the regret [Hu et al., 2016].

A major open question was whether or not $\tilde{O}(n^{1/2})$ regret is possible, which was closed affirmatively in dimension 1 by Bubeck et al. [2015] and in higher dimensions by Hazan and Li [2016] and Bubeck and Eldan [2018]. The regret of the former algorithm depends exponentially on the dimension, while in the latter the dependence is polynomial. The best known upper bound on the minimax regret for bandit convex optimisation is $\tilde{O}(d^{2.5} \sqrt{n})$ by Lattimore [2020].

Our analysis is based on the information-theoretic arguments introduced by Russo and Van Roy [2014] and used for the analysis of convex bandits by Bubeck et al. [2015], Bubeck and Eldan [2018] and Lattimore [2020]. All these works rely on minimax duality and consequently do not yield efficient algorithms. This is also true of the result presented here.

The only polynomial time algorithm with $\tilde{O}(\sqrt{n})$ regret for the general case is by Bubeck et al. [2017]. Although a theoretical breakthrough, the dependence of this algorithm’s regret on the dimension is $\tilde{O}(d^{10.5})$ and practically speaking the algorithm is not implementable except when the dimension is very small. All genuinely practical algorithms for adversarial bandit convex optimisation are gradient methods with importance-weighted gradient estimates. These algorithms have optimal dependence on the horizon for strongly convex functions [Hazan and Levy, 2014, Ito, 2020] but otherwise not [Kleinberg, 2005, Flaxman et al., 2005, Saha and Tewari, 2011, Hu et al., 2016].

As far as we know, the setting of the present paper has not been considered before. Saha et al. [2021] tackle the case where $f_t(x) = g_t(h_t(x))$ with $h_t : \mathbb{R}^d \to \mathbb{R}$ a function that is known to the learner. By choosing $f_t(x) = \langle x, \theta \rangle + \eta_t \in [0, 1]$ for some (adversarial) noise $(\eta_t)_{t=1}^n$, our setting subsumes an interesting version
of the stochastic linear setting, with the restriction being that the noise is homogeneous and bounded. The standard lower bound for this setting is $\Omega(d\sqrt{n})$ [Dani et al., 2008], but with the assumptions required here and by taking $(\eta_t)_{t=1}^n$ to be truncated Gaussian, naively this construction would yield a lower bound of $\Omega(d\sqrt{n} / \log(n))$. Nevertheless, this shows that the new result is optimal up to logarithmic factors. Note that our setting does not subsume the adversarial linear setting because the direction of the ridge is fixed. An obvious open question is whether or not this can be relaxed.

Notation The unit sphere is $S^{d-1} = \{ x \in \mathbb{R}^d : \|x\| = 1 \}$. The support function of a set $A \subset \mathbb{R}^d$ is $h_A(\theta) = \sup_{x \in A} \langle x, \theta \rangle$. Given a convex function $f : K \to \mathbb{R}$, its minimum is $f_\ast = \min_{x \in K} f(x)$. The expectation and variance of a random variable $X$ are denoted by $\mathbb{E}[X]$ and $\mathbb{V}[X]$ respectively. The underlying probability measure will always be obvious from the context. All probability measures are defined over the Borel $\sigma$-algebra on the corresponding space. Given a positive definite matrix $\Sigma$, let $\|x\|_\Sigma = \sqrt{x^\top \Sigma x}$. The convex hull of a set $A \subset \mathbb{R}^d$ is $\text{conv}(A)$.

2 Distributions with high projected variance

Let $A \subset \mathbb{R}^d$ be compact and recall that for $\theta \in S^{d-1}$, $h_A(\theta) + h_A(-\theta)$ is the width of $A$ in direction $\theta$. The next lemma asserts the existence of a distribution on $A$ for which the standard deviation in every direction is nearly as large as the width.

Lemma 3. For any compact set $A \subset \mathbb{R}^d$ there exists random element $X$ supported on $A$ such that for all $\theta \in S^{d-1}$,

$$h_A(\theta) + h_A(-\theta) \leq 2\sqrt{d \mathbb{V}[\langle X, \theta \rangle]}.$$ 

Proof. Assume without loss of generality that the affine hull of $A$ spans $\mathbb{R}^d$. If not, you may work in a suitable affine space. There exists a probability measure $\pi$ on $A$ such that with $\mu = \int_A x \, d\pi(x)$ and $\Sigma = \int_A (x - \mu)(x - \mu)^\top \, d\pi(x)$, the ellipsoid $E = \{ x : \|x - \mu\|_{S^{-1}}^2 \leq d \}$ is the minimum volume enclosing ellipsoid of $A$ [Todd, 2016, Corollary 2.11]. Therefore, when $X$ has law $\pi$,

$$h_A(\theta) + h_A(-\theta) = \sup_{x,y \in A} \langle x - \mu, \theta \rangle + \langle \mu - y, \theta \rangle$$

$$\leq 2 \sup_{x \in A} \|x - \mu\|_{S^{-1}} \|\theta\|_\Sigma$$

$$\leq 2\sqrt{d \|\theta\|_\Sigma^2}$$

$$= 2\sqrt{d \mathbb{V}[\langle X, \theta \rangle]}$$

\qed

3 Information ratio

Let us first recall the main tool, which upper bounds the minimax regret in terms of the information ratio. The following theorem is a combination of Theorem 2
and Lemma 4 by Lattimore [2020], which are proven by extending the machinery developed by Russo and Van Roy [2014], Bubeck et al. [2015] and Bubeck and Eldan [2018]. We outline the key differences in Section 6.

**Theorem 4.** Given $\theta \in S^{d-1}$, let $F_\theta$ be the space of all functions satisfying (a)-(d) in the introduction and $F = \cup_{\theta \in S^{d-1}} F_\theta$. Suppose that $\alpha, \beta \geq 0$ are reals such that for all $\bar{f} \in \text{conv}(F)$ there exist probability measures $\pi_1, \ldots, \pi_m$ on $K$ for which

$$
\sup_{f \in F} \min_{1 \leq k \leq m} \left( \int_K f \, d\pi_k - f_* - \sqrt{\beta \int_K (\bar{f} - f)^2 \, d\pi_k} \right) \leq \alpha.
$$

Then $\mathcal{R}_n \leq \text{const}(1 + n\alpha + \sqrt{dnm\beta \log(n \text{diam}(K))})$ with const a universal constant.

**Important remark** A previous version of this manuscript claimed the above theorem even when the ridge $\theta$ was time dependent, meaning that $f_t(x) = g_t(\langle x, \theta_t \rangle)$ for some adversarial sequence $(\theta_t)_{t=1}^n$. Unfortunately this result does not follow from the standard machinery. The problem is that $F$ is not closed under convex combinations. Why this causes a problem will be apparent when reading Section 6. New ideas are needed to understand whether or not Theorem 4 might continue to hold for time varying ridge directions.

Combining the following proposition with Theorem 4 yields Theorem 2.

**Proposition 5.** The conditions of Theorem 4 hold with

$$
\alpha = \frac{(2 + 16\sqrt{d})}{n} \quad \beta = 2^9 d \quad m \leq 2 + \log_2(n^2 \text{diam}(K)).
$$

**4 Proof of Proposition 5**

The proof relies on the following simple lemma.

**Lemma 6.** Suppose that $a \leq b$ and $g : [a, b] \to \mathbb{R}$ is convex and continuous. Let $X$ be a random variable supported on $[a, b]$. Then, for any $\alpha > g(b)$,

$$
\mathbb{E}[(g(X) - \alpha)^2] \geq \frac{(g(b) - \alpha)^2 \mathbb{V}[X]}{(b - a)^2}.
$$

**Proof.** If it exists, let $x \in [a, b]$ be a point such that $g(x) = \alpha$. Otherwise let $x = a$, which by continuity of $g$ means that $g(a) < \alpha$. Then let $\varphi : [a, b] \to \mathbb{R}$ be the linear function with $\varphi(b) = g(b)$ and $\varphi(x) = g(x)$, which satisfies

$$
\mathbb{E}[(g(X) - \alpha)^2] \geq \mathbb{E}[(\varphi(X) - \alpha)^2] \\
\geq \min_{\eta \in \mathbb{R}} \mathbb{E}[(\eta(X - b) + g(b) - \alpha)^2] \\
= \frac{(\alpha - g(b))^2 \mathbb{V}[X]}{\mathbb{E}[(b - X)^2]} \\
\geq \frac{(\alpha - g(b))^2 \mathbb{V}[X]}{(b - a)^2},
$$

4
Figure 1: The typical situation in the proof of Lemma 6. For all $x \in [a, b]$, the distance between the linear function $\varphi$ and the horizontal line at $\alpha$ is less than or equal to the distance between $f$ and the same horizontal line.

where the first inequality is geometrically obvious from the convexity of $g$ (Fig. 1) and the second follows by optimising over all linear functions passing through $(b, g(b))$. The last equality follows by optimising for $\eta$, while the finally inequality holds because $X$ is supported on $[a, b]$.

Proof of Proposition 5. There are two steps. First, we define a collection of probability measures $\pi_0, \ldots, \pi_m$ with $m = O(\log(n))$. In the second step we show this collection satisfies the conditions of Theorem 4.

Step 1: Exploration distributions and perturbations Let $\tilde{f} \in \text{conv}(\mathcal{F})$ and choose coordinates on $\mathcal{K}$ so that $0 = \arg \min_{x \in \mathcal{K}} \tilde{f}(x)$. Let

$$\tilde{f}(x) = \tilde{f}(x) + \|x\|_n \frac{\text{diam}(\mathcal{K})}{n^2},$$

which is close to $\tilde{f}$, minimised at $0$ and increases at least linearly with $\|x\|$. For $\epsilon > 0$, define

$$\mathcal{L}_\epsilon = \left\{ x \in \mathcal{K} : \tilde{f}(x) = \tilde{f}_* + \epsilon \right\},$$

which is a level set of $\tilde{f}$. Note that $\mathcal{L}_\epsilon$ is not convex and boundary effects mean that $0 \in \text{conv}(\mathcal{L}_\epsilon)$ is not guaranteed. Let $\pi_0$ be a Dirac at $0$ and for $k \geq 1$ let

$$\epsilon_k = \frac{2^{k-1}}{\text{diam}(\mathcal{K})n^2}$$

and $\pi_k$ be the distribution on $\mathcal{L}_k = \mathcal{L}_{\epsilon_k}$ given by Lemma 3. Consequentially, if $h_k$ is the support function of $\mathcal{L}_k$, then for all $\theta \in \mathbb{S}^{d-1}$,

$$h_k(\theta) + h_k(-\theta) \leq 2\sqrt{d \mathbb{V}_{x \sim \pi_k} \langle x, \theta \rangle}.$$  (1)
Let

\[ m = \max\{k : \epsilon_k \leq 1\} \leq 1 + \log_2(\text{diam}(\mathcal{K})n^2). \] (2)

**Step 2** Let \( f \in \mathcal{F} \) so that \( f(x) = g((x, \theta)) \) for some convex \( g \) and \( \theta \in \mathbb{S}^{d-1} \). Let \( x_* \in \mathcal{K} \) be a minimiser of \( f \) and assume without loss of generality that the sign of \( \theta \) has been chosen so that \( \langle x_*, \theta \rangle \geq 0 \). We will prove that there exists a \( \pi \in \{\pi_0, \ldots, \pi_m\} \) such that

\[
\int_{\mathcal{K}} \tilde{f} \, d\pi - f_* \leq \frac{2}{n} + 16 \sqrt{d} \int_{\mathcal{K}} (\tilde{f} - f)^2 \, d\pi. \tag{3}
\]

Let \( \epsilon = \min\{\tilde{f}(x) - \tilde{f}_*: x \in \mathcal{K}, (x-x_*, \theta) = 0\} \), which exists because \( \tilde{f} \) is continuous and \( \mathcal{K} \cap \{ x : (x - x_*, \theta) = 0 \} \) is compact. We claim that \( h_{\mathcal{L}_\epsilon}(\theta) = \langle x_*, \theta \rangle \). That \( h_{\mathcal{L}_\epsilon}(\theta) \geq \langle x_*, \theta \rangle \) is immediate. Suppose there exists an \( x \in \mathcal{L}_\epsilon \) with \( \langle x, \theta \rangle > \langle x_*, \theta \rangle \). Then by convexity of \( \mathcal{K} \) and \( \tilde{f} \), there exists a \( y \in \mathcal{K} \) with \( \langle y - x_*, \theta \rangle = 0 \) and \( \tilde{f}(y) < \tilde{f}_* + \epsilon \), which contradicts the definition of \( \epsilon \). Therefore \( h_{\mathcal{L}_\epsilon}(\theta) \leq \langle x_*, \theta \rangle \) and hence \( h_{\mathcal{L}_\epsilon}(\theta) = \langle x_*, \theta \rangle \). Next, by the definition of \( \tilde{f} \),

\[ \mathcal{L}_\epsilon \subset \{ x : \tilde{f}(x) \leq \tilde{f}_* + \epsilon \} \subset \{ x : \|x\| \leq \epsilon n \text{ \text{diam}(\mathcal{K})} \}. \]

Hence,

\[ \langle x_*, \theta \rangle = h_{\mathcal{L}_\epsilon}(\theta) \leq \epsilon n \text{ \text{diam}(\mathcal{K})}. \tag{4} \]

Suppose for a moment that \( \langle x_*, \theta \rangle \geq 1/n \). It follows from Eq. (4) that \( \epsilon \geq 1/(\text{diam}(\mathcal{K})n^2) \), in which case there exists a largest \( k \) such that \( \epsilon_k \leq \epsilon \). Let \( x \in \mathcal{L}_\epsilon \) be such that \( \langle x, \theta \rangle = \langle x_*, \theta \rangle \), which satisfies \( \tilde{f}(x) = \epsilon + \tilde{f}_* \). By continuity, there exists an \( \alpha \in [0, 1] \) such that \( \tilde{f}(\alpha x) = \tilde{f}_* + \epsilon_k \). By convexity of \( \tilde{f} \) and the fact that \( \tilde{f} \) is minimised at \( 0 \),

\[ \epsilon_k + \tilde{f}_* = \tilde{f}(\alpha x) \leq \alpha \tilde{f}(x) + (1 - \alpha) \tilde{f}_* = \alpha \epsilon + \tilde{f}_*. \]

Rearranging shows that \( \alpha \geq \epsilon_k / \epsilon \geq 1/2 \) and hence

\[ \langle x_*, \theta \rangle \geq h_k(\theta) \geq \langle \alpha x, \theta \rangle \geq \frac{\langle x_*, \theta \rangle}{2}. \tag{5} \]

We consider four cases, the last two of which are illustrated in Fig. 2:

1. \( f_*, \geq \tilde{f}_* - 2/n \).
2. \( f_* \leq \tilde{f}_* - 2/n \) and \( \langle x_*, \theta \rangle \leq 1/n \).
3. \( f_* \leq \tilde{f}_* - 2/n \) and \( \langle x_*, \theta \rangle \geq 1/n \) and \( -h_k(-\theta) \geq h_k(\theta) \).
4. \( f_* \leq \tilde{f}_* - 2/n \) and \( \langle x_*, \theta \rangle \geq 1/n \) and \( -h_k(-\theta) \leq h_k(\theta) \).
Case 1  By (1a), Eq. (3) holds with \( \pi = \pi_0 \) trivially.

Case 2  By Fact 1, \( g \) is Lipschitz on the closed interval \( \{ \langle x, \theta \rangle : x \in K \} \). Combining this with (2a) and (2b) yields

\[
\sqrt{\int_K (\hat{f} - f)^2 \, d\pi_0} = \hat{f}_* - f(0)
\]

\[
\geq \hat{f}_* - f_* - \langle x_*, \theta \rangle \quad \text{(g is Lipschitz)}
\]

\[
= \frac{1}{2} (\hat{f}_* - f_*) + \frac{1}{2} (\hat{f}_* - f_*) - \langle x_*, \theta \rangle
\]

\[
\geq \frac{1}{2} (\hat{f}_* - f_*) \quad \text{(using 2a and 2b)}
\]

\[
= \frac{1}{2} \left( \int_K \hat{f} \, d\pi_0 - f_* \right).
\]

And again, Eq. (3) holds with \( \pi = \pi_0 \).

Case 3  Let \( \Delta = \hat{f}_* - f_* \). Suppose that \( f(0) \geq \tilde{f}_* + \Delta/4 \). Then

\[
\int_K (\hat{f} - f)^2 \, d\pi_0 = \Delta \leq 4 \sqrt{\int_K (\hat{f} - f)^2 \, d\pi_0}
\]

and Eq. (3) holds with \( \pi = \pi_0 \). Suppose for the remainder of this case that \( f(0) \leq \hat{f}_* + \Delta/4 \). For all \( x \in L_k \),

\[
\langle x_*, \theta \rangle \geq h_k(\theta) \geq \langle x, \theta \rangle \geq -h_k(-\theta) \geq h_k(\theta)/2 \geq \frac{\langle x_*, \theta \rangle}{4}. \quad (6)
\]

Therefore, \( \langle x, \theta \rangle / \langle x_*, \theta \rangle \in [0, 1] \) and

\[
f(x) = f\left( \frac{\langle x, \theta \rangle}{\langle x_*, \theta \rangle} x_* + \left( 1 - \frac{\langle x, \theta \rangle}{\langle x_*, \theta \rangle} \right) 0 \right)
\]

\[
\leq \frac{\langle x, \theta \rangle}{\langle x_*, \theta \rangle} f_* + \left( 1 - \frac{\langle x, \theta \rangle}{\langle x_*, \theta \rangle} \right) \left( \hat{f}_* + \Delta/4 \right) \quad \text{(convexity of g)}
\]

\[
= \hat{f}_* + \frac{\Delta}{4} \left( 1 - \frac{5\langle x, \theta \rangle}{\langle x_*, \theta \rangle} \right)
\]

\[
\leq \hat{f}_* - \frac{\Delta}{16}. \quad \text{(by Eq. (6))}
\]

Therefore, since \( \pi_k \) is supported on \( L_k \),

\[
\int_K (\hat{f} - f)^2 \, d\pi_k \geq \int_K (\hat{f}_* + \epsilon_k - f)^2 \, d\pi_k \geq \frac{1}{16^2} (\Delta + \epsilon_k)^2 = \frac{1}{16^2} \left( \int_K \hat{f} \, d\pi_k - f_* \right)^2.
\]

Hence, Eq. (3) holds with \( \pi = \pi_k \).
Figure 2: Examples of Case 3 (left) and Case 4 (right). Case 3 only occurs when 
\(-h_k(-\theta) \geq h_k(\theta)/2\), which means that 0 is far outside the convex hull of the level 
set \(L_k\). Meanwhile, in Case 4 the width of \(L_k\) in the direction \(\theta\) is at least the 
same order of magnitude as \(h_k(\theta)\). The distance between the dotted lines in both 
illustrations is the width of \(L_k\) in direction \(\theta\), which is \(h_k(\theta) + h_k(-\theta)\). Note that 
because \(f\) is a ridge function, all points \(x\) in \(K\) with \(\langle x - x_*, \theta \rangle = 0\) minimise \(f\).
Case 4  By definition of the support function \( \langle x, \theta \rangle \in [-h_k(-\theta), h_k(\theta)] \) for all \( x \in \mathcal{L}_k \). Furthermore, by Eq. (5), \( \langle x, \theta \rangle \in [h_k(\theta), 2h_k(\theta)] \). Hence, by Lemma 6 with \( a = -h_k(-\theta), b = \langle x, \theta \rangle \leq 2h_k(\theta) \) and \( \alpha = f_* + \epsilon_k \),

\[
\int_{\mathcal{K}} (\tilde{f} - f)^2 \, d\pi_k = \int_{\mathcal{K}} (f_* + \epsilon_k - g(\langle x, \theta \rangle))^2 \, d\pi_k(x) \quad (\pi_k \text{ supported on } \mathcal{L}_k)
\]

\[
\geq \frac{(\tilde{f}_* + \epsilon_k - f_*)^2 \mathbb{1}_{x \in \mathcal{L}_k}(\langle x, \theta \rangle)}{(2h_k(\theta) + h_k(-\theta))^2} \quad (\text{By Lemma 6})
\]

\[
\geq \frac{(\tilde{f}_* + \epsilon_k - f_*)^2(2h_k(\theta) + h_k(-\theta))^2}{4d(2h_k(\theta) + h_k(-\theta))^2} \quad (\text{By Eq. (1)})
\]

\[
\geq \frac{(\tilde{f}_* + \epsilon_k - f_*)^2}{64d} \quad (\text{By } 4c)
\]

\[
= \frac{1}{64d} \left( \int_{\mathcal{K}} \tilde{f} \, d\pi_k - f_* \right)^2. \quad (\pi_k \text{ supported on } \mathcal{L}_k)
\]

where the final inequality is trivial if \( h_k(-\theta) \) is positive, while if \( h_k(-\theta) \) is negative, then using (4c), \( h_k(\theta) + h_k(-\theta) \geq h_k(\theta)/2 \geq (2h_k(\theta) + h_k(-\theta))/4 \) and the result follows.

Summary  We have shown that for all \( f \in \mathcal{F} \) there exists a policy \( \pi \in \{\pi_0, \ldots, \pi_m\} \) such that Eq. (3) holds. By the definition of \( \tilde{f} \) it follows that

\[
\int_{\mathcal{K}} \tilde{f} \, d\pi - f_* \leq \int_{\mathcal{K}} \tilde{f} \, d\pi - f_* \quad (\text{since } \tilde{f} \leq \hat{f})
\]

\[
\leq \frac{2}{n} + 16\sqrt{d} \int_{\mathcal{K}} (\tilde{f} - f)^2 \, d\pi \quad (\text{by Eq. (3)})
\]

\[
\leq \frac{2 + 16\sqrt{2d}}{n} + 16\sqrt{2d} \int_{\mathcal{K}} (\tilde{f} - f)^2 \, d\pi,
\]

where we used the inequalities \( (a + b)^2 \leq 2a^2 + 2b^2 \) and \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for reals \( a, b \geq 0 \). The result now follows from Eq. (2) and Theorem 4 and noting that in this proof the number of exploration policies is \( m + 1 \).

\[ \square \]

5 Proof of Fact 1

Convexity is obvious. We need to prove that \( g \) is Lipschitz on \( \{\langle x, \theta \rangle : x \in \mathcal{K}\} \). Elementary differentiation does not suffice; for example, if \( d = 2 \) and \( f(x) = \sqrt{2}x_1 \) and \( \mathcal{K} = \{(x, x) : x \in [0, 1]\} \subset \mathbb{R}^2 \), then \( f \) is Lipschitz on \( \mathcal{K} \), but \( g(x) = \sqrt{2}x \) is not Lipschitz. The problem is that \( \mathcal{K} \) is not a convex body because it does not have a non-empty interior. To exploit the fact that \( \mathcal{K} \) is a convex body, note that for any \( x, y \) in the interior of \( \mathcal{K} \) with \( \langle y - x, \theta \rangle \geq 0 \), there exists a path from \( x \) to \( y \) that moves either perpendicular to \( \theta \) or in the direction of \( \theta \). Since \( f \) is constant on hyperplanes \( \{z : \langle z, \theta \rangle = c\} \) for any \( c \), the Lipschitzness of \( f \) applied along the path implies that \( g \) is also Lipschitz. The extension to \( x, y \) on the boundary of \( \mathcal{K} \) follows from the continuity of \( f \) and by passing to the limit.

9
6 Proof outline of Theorem 4

The majority of the argument follows that given by [Lattimore, 2020, Appendix A]. Let $\mathcal{K}_a \subset \mathcal{K}$ and $\mathbb{S}^{d-1}_a \subset \mathbb{S}^{d-1}$ be finite covering sets such that

$$ \sup_{x \in \mathcal{K}} \inf_{y \in \mathcal{K}_a} \|x - y\| \leq \epsilon \quad \text{and} \quad \sup_{x \in \mathbb{S}^{d-1}} \inf_{y \in \mathbb{S}^{d-1}_a} \|x - y\| \leq \epsilon, $$

with $\epsilon = \text{poly}(1/n)$ suitably small. Let $\mathcal{H} = \bigcup_{\theta \in \mathbb{S}^{d-1}} \mathcal{F}_\theta^n$, which is the set of possible sequences of functions that the adversary can choose. Let $\mathcal{H}_a = \bigcup_{\theta \in \mathbb{S}^{d-1}_a} \mathcal{F}_\theta^n$.

We assume that rather than observe $f_t(x_t)$ after round $t$, the learner observes $y_t = f_t(x_t) + \eta_t$ where $\eta_t$ is a Gaussian with zero mean and variance $\sigma^2 = \text{poly}(1/n)$ and truncated in $[-1, 1]$. This can only increase the regret, since the effect can be modelled by restricting the class of policies to add the noise before making a decision. The value of $\sigma^2$ should be chosen such that the probability that a truncation occurs is $O(1/n^2)$ while $\epsilon$ should be $O(\text{poly}(\sigma^2/n))$. Standard results show that log $|\mathcal{K}_a|$ and log $|\mathbb{S}^{d-1}_a|$ are both at most $d \log(n \text{ diam}(\mathcal{K}))$.

**Policies and the Bayesian probability space** Let $\mathcal{P}(A)$ be the space of finitely supported probability measures on set $A$ with the discrete $\sigma$-algebra. Given a measure $\nu \in \mathcal{P}(\mathcal{H})$ and a policy $\pi$, let $\mathbb{P}_\nu^n$ be the probability measure on $(f_t)_{t=1}^n$ and $(x_t)_{t=1}^n$ and $(y_t)_{t=1}^n$ when $(f_t)_{t=1}^n$ are sampled from $\nu$ and the laws of the outcome are determined by the interaction between $\pi$ and the bandit. Expectations with respect to $\mathbb{P}_\nu^n$ are denoted by $\mathbb{E}_\nu^n$. The optimal action in $\mathcal{K}_a$ in hindsight is $x_* = \arg\min_{x \in \mathcal{K}_a} \sum_{t=1}^n f_t(x)$.

**Minimax duality** Let $\Pi_a$ be the set of policies that plays actions in $\mathcal{K}_a$ almost surely. Let $\mathcal{R}_n(\pi, (f_t)_{t=1}^n)$ be the regret for a given policy and sequence of functions. By minimax duality,

$$ \mathcal{R}_n \leq \inf_{\pi \in \Pi_a} \sup_{(f_t)_{t=1}^n \in \mathcal{H}} \mathcal{R}_n(\pi, (f_t)_{t=1}^n) $$

$$ \leq 1 + \sup_{\nu \in \mathcal{P}(\mathcal{H})} \inf_{\pi \in \Pi_a} \mathbb{E}_\nu^n \left[ \sum_{t=1}^n f_t(x_t) - f_t(x_*) \right] $$

$$ \leq 2 + \sup_{\nu \in \mathcal{P}(\mathcal{H}_a)} \inf_{\pi \in \Pi_a} \mathbb{E}_\nu^n \left[ \sum_{t=1}^n f_t(x_t) - f_t(x_*) \right], \quad (7) $$

where the first inequality is trivial. The second follows from minimax duality [Lattimore and Szepesvári, 2019]. The last inequality follows by choosing $\epsilon = \text{poly}(1/n)$ suitably small and using standard information-theoretic arguments that for all $(f_t)_{t=1}^n \in \mathcal{H}$ there exists an approximation in $\mathcal{H}_a$ that because of the noisy losses is statistically indistinguishable from the perspective of the policy. This is where the conditions on $\epsilon$ and $\sigma^2$ are needed, but we omit details.

**Bayesian regret** For the remainder we bound Eq. (7) for any $\nu \in \mathcal{P}(\mathcal{H}_a)$ and a carefully constructed policy $\pi$. Abbreviate $\mathbb{E} = \mathbb{E}_\nu^n$. Let $\theta \in \mathbb{S}^{d-1}_a$ be the random
element such that \( f_t(x) = g_t((x, \theta)) \) for suitably chosen \((y_t)_{t=1}^n\) and let \( Z = (x_*, \theta) \).

The next step is to bound the Bayesian regret on the right-hand side of Eq. (7). Let \( \mathbb{P}_t(\cdot) = \mathbb{P}(\cdot|x_1, y_1, \ldots, x_t, y_t) \) and \( \mathbb{E}_t \) be the expectation with respect to \( \mathbb{P}_t \). For \( z \in \mathcal{K}_n \times \mathbb{S}^{d-1}_n \), let

\[
f_{t,z}(x) = \mathbb{E}_{t-1}[f_t(x)|Z = z] \quad \text{and} \quad \bar{f}_t(x) = \mathbb{E}_{t-1}[f_t(x)].
\]

Note that \( f_{t,z} \) is a ridge function. Let \( \mu_t \) be the finitely supported measure on \( \mathcal{F} \) with \( \mu_t(\{f_{t,z}\}) = \mathbb{P}_{t-1}(Z = z) \). In previous arguments \( Z \) was defined to be the optimal action, but in our setup the resulting function would no longer be a ridge function and the analysis in Section 4 would not apply. By the assumptions of the theorem and a lemma for combining exploratory distributions [Lattimore, 2020, Lemma 4], there exists a distribution \( \pi_t \) on \( \mathcal{K}_n \) such that

\[
\int_{\mathcal{K}_n} \bar{f}_t(x) d\pi_t(x) - \int_{\mathcal{F}} f_t d\mu_t(f) \leq 1/n + \alpha + \sqrt{\beta_m \int_{\mathcal{K}_n} \int_{\mathcal{F}} (\bar{f}_t - f_t)^2 d\mu_t(f) d\pi_t(x)}.
\]

The additional constant \( 1/n \) appears because \( \pi_t \) must modified to be supported on \( \mathcal{K}_n \). We let \( \pi_t = (\pi_t)_{t=1}^n \). Then,

\[
\mathbb{E} \left[ \sum_{t=1}^n f_t(x_t) - f_t(x_*) \right] \leq \mathbb{E} \left[ \sum_{t=1}^n \left( \int_{\mathcal{K}_n} \bar{f}_t(x) d\pi_t(x) - \int_{\mathcal{F}} f_t d\mu_t(f) \right) \right]
\]

\[
\leq 1 + n\alpha + \mathbb{E} \left[ \sum_{t=1}^n \sqrt{\beta_m \int_{\mathcal{K}_n} \int_{\mathcal{F}} (\bar{f}_t - f_t)^2 d\mu_t d\pi_t} \right].
\]

Next, we relate the right-hand side above to the information gain about \( Z \). Letting \( I_t(U; V) \) be the mutual information between random elements \( U \) and \( V \) under \( \mathbb{P}_t \), by Pinsker’s inequality,

\[
\int_{\mathcal{F}} \int_{\mathcal{K}_n} (\bar{f}_t - f_t)^2 d\pi_t d\mu_t = \mathbb{E}_{t-1} \left[ (\mathbb{E}_{t-1}[y_t|x_t] - \mathbb{E}_{t-1}[y_t|Z, x_t])^2 \right]
\]

\[
\leq \mathbb{E}_{t-1} \left[ I_{t-1}(Z; x_t, y_t) \right].
\]

By the chain rule for the mutual information and letting \( H(Z) \) be the entropy of random element \( Z \),

\[
\mathbb{E} \left[ \sum_{t=1}^n I_{t-1}(Z; x_t, y_t) \right] \leq H(Z) \leq \log |\mathcal{K}_n| + \log |\mathbb{S}^{d-1}_n| \leq \text{const} d \log(n \text{diam}(\mathcal{K})).
\]

Therefore, by Cauchy–Schwarz, the Bayesian regret is bounded by

\[
\mathbb{E} \left[ \sum_{t=1}^n f_t(x_t) - f_t(x_*) \right] \leq 1 + n\alpha + \mathbb{E} \left[ \sum_{t=1}^n \sqrt{\beta_m I_{t-1}(Z; x_t, y_t)} \right]
\]

\[
\leq 1 + n\alpha + \sqrt{\beta_m n \mathbb{E} \left[ \sum_{t=1}^n I_{t-1}(Z; x_t, y_t) \right]}
\]

\[
\leq 1 + n\alpha + \text{const} \sqrt{\beta_m n \log(n \text{diam}(\mathcal{K}))}.
\]
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