SUFFICIENCY AND MIXED TYPE DUALITY FOR
MULTIOBJECTIVE VARIATIONAL CONTROL PROBLEMS
INVOLVING $\alpha$-V-UNIVEXITY

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Abstract. In this paper, we focus our study on a multiobjective variational
control problem and establish sufficient optimality conditions under the as-
sumptions of $\alpha$-V-univex function. Furthermore, mixed type duality results
are also discussed under the aforesaid assumption in order to relate the primal
and dual problems. Examples are given to show the existence of $\alpha$-V-univex
function and to elucidate duality result.

1. Introduction. Often, in real-world problems, we come across situations where
we have to optimize more than one objective functions which are conflicting in
nature. These types of problems can be solved by using multiobjective optimization
techniques.

The calculus of variation plays a significant role in many areas of pure and applied
mathematics. It is concerned with optimization of functionals, which are mappings
from a set of functions to the real numbers and are often expressed as definite
integrals involving functions and their derivatives. This technique was developed
to solve the problems of finding the best possible objects, for example, to find the
shortest distance between two points, minimal surfaces or trajectory of the fastest
travel.

Thereafter, control theory emerged as an extension of the calculus of variation.
The control problem is to optimize a cost functional that is a function of state and
control variables. It finds applications in diverse areas of science, engineering and
decision making. Hanson [7,8] was the first to observe the relationship between cal-
culus of variation, control problems and mathematical programming and obtained

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duality results for control problems under the assumption of convexity. Moreover, the most systematic applications of duality to optimal control theory were developed by Rockafellar [20, 21]. Thereafter, several researchers have been interested in the optimality conditions and duality theorems for variational control problems, see for example, [1, 2, 6, 10–12, 15–17, 22]. For details in control theory, one can refer to the book of Craven [4].

As we know, convexity is one of the most important hypothesis in optimization theory but it is not sufficient for many real world mathematical models. Thus, Mond and Smart [15] extended the work of Mond and Hanson [9], under the assumptions of invexity. Liang et al. [13, 14] defined a new class of functions, known as \((F, \alpha, \rho, d)\)-convex functions, which unifies several concepts of generalized convexity. Various new classes of functions were introduced by several authors, see [2, 10–12, 16–19, 22, 23], on account of certain limitations of convexity. Bhatia and Kumar [5] introduced multiobjective control problem and established duality conditions for multiobjective control problems under generalized invexity. Moreover, Ahmad and Gulati [1] formulated mixed type duality for multiobjective variational problems under the assumption of \((F, \rho)\)-convexity. Later, Ahmad and Gulati et al. [6] obtained Fritz John and Kuhn-Tucker type necessary optimality conditions and duality results for mixed type dual of a multiobjective variational control problem under the assumption of \(\alpha\)-V-univexity.

This paper is organized as follows: In Section 2, we recall some preliminary definitions and introduce the concept of \(\alpha\)-V-univexity. In Section 3, we derive the sufficient optimality conditions for multiobjective variational control problems under the assumption of \(\alpha\)-V-univexity functions and finally in Section 4, we prove duality results for mixed type multiobjective dual problem.

2. Notations and preliminaries. Let \(R^n\) denote the \(n\)-dimensional Euclidean space. Let \(y, z \in R^n\), we denote: \(y \leq z \Leftrightarrow y_i \leq z_i, i = 1, 2, \ldots, n; y \leq z \Leftrightarrow y \leq z\) and \(y \neq z; y < z \Leftrightarrow y_i < z_i, i = 1, 2, \ldots, n\).

Let \(I = [a, b]\) be a real interval. Let \(f_i : I \times R^n \times R^n \times R^n \times R^n \to R, i \in P = \{1, 2, \ldots, p\}, g_j : I \times R^n \times R^n \times R^n \times R^n \to R, j \in M = \{1, 2, \ldots, m\}\) and \(h_k : I \times R^n \times R^n \times R^n \to R, k \in N = \{1, 2, \ldots, n\}\) be continuously differentiable functions. Consider the function \(f(t, x(t), \dot{x}(t), u(t), \dot{u}(t))\), where \(t\) is the independent variable, \(x : I \to R^n\) is the state variable and \(u : I \to R^n\) is the control variable. \(u(t)\) is related to \(x(t)\) via the state equation \(h(t, x(t), \dot{x}(t), u(t), \dot{u}(t)) = 0\), where the dot denotes the derivative with respect to \(t\). \(f_{ix}, f_{ix}, f_{iu}\) and \(f_{iu}\) denote the partial derivatives of \(f_i\) with respect to \(x, \dot{x}, u, \dot{u}\) respectively. For instance,

\[
(f_{ix} = \left(\frac{\partial f_i}{\partial x_1}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n}\right), f_{i\dot{x}} = \left(\frac{\partial f_i}{\partial \dot{x}_1}, \frac{\partial f_i}{\partial \dot{x}_2}, \ldots, \frac{\partial f_i}{\partial \dot{x}_n}\right). 
\]

Similarly, \(g_{jx}, g_{j\dot{x}}, g_{ju}, g_{j\dot{u}}, h_{kx}, h_{k\dot{x}}, h_{ku}, h_{k\dot{u}}\) can be defined. For notational convenience, we use \(x, \dot{x}, u, \dot{u}\) in place of \(x(t), \dot{x}(t), u(t), \dot{u}(t)\), respectively.

Let \(X\) denote the space of all piecewise smooth functions \(x : I \to R^n\) with norm \(\|x\| = \|x\|_\infty + \|Dx\|_\infty\) and \(Y\) denote the space of all piecewise smooth functions...
u : I \mapsto \mathbb{R}^m \text{ with the norm } \|u\|_\infty, \text{ where the differentiation operator } D \text{ is given by}
\[ z = Dx \iff x(t) = \gamma + \int_a^t z(s)ds, \]
where \( \gamma \) is a given boundary value. Therefore \( D = d/dt \) except at discontinuities.

We consider the following multiobjective variational control problem:

\[
\begin{array}{l}
\text{(CP) Min } \int_a^b f(t, x, \dot{x}, u, \dot{u})dt = \left( \int_a^b f_1(t, x, \dot{x}, u, \dot{u})dt, \ldots, \int_a^b f_P(t, x, \dot{x}, u, \dot{u})dt \right) \\
\text{subject to} \\
\quad x(a) = \gamma, \ x(b) = \delta, \\
\quad g(t, x, \dot{x}, u, \dot{u}) \leq 0, \ t \in I, \\
\quad h(t, x, \dot{x}, u, \dot{u}) = 0, \ t \in I.
\end{array}
\]

Let \( S = \{(x, u) \in X \times Y : x(a) = \gamma, x(b) = \delta, g(t, x, \dot{x}, u, \dot{u}) \leq 0, h(t, x, \dot{x}, u, \dot{u}) = 0\} \) denote the set of all feasible solutions to (CP).

**Definition 2.1.** [5] A point \((\bar{x}, \bar{u})\) \(\in S\) is said to be an efficient solution of (CP), if there exists no other point \((x, u) \in S\) such that,
\[
\int_a^b f_i(t, x, \dot{x}, u, \dot{u})dt < \int_a^b f_i(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})dt \text{ for some } i \in P
\]
and
\[
\int_a^b f_r(t, x, \dot{x}, u, \dot{u})dt \leq \int_a^b f_r(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})dt \text{ for all } r \in P \setminus \{i\}.
\]

In the case of maximization, the signs of above inequalities are reversed.

**Definition 2.2.** A feasible point \((\bar{x}, \bar{u}) \in S\) is said to be a weakly efficient solution of (CP), if there exists no other point \((x, u) \in S\) such that
\[
\int_a^b f(t, x, \dot{x}, u, \dot{u})dt < \int_a^b f(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})dt.
\]

It follows that if \((\bar{x}, \bar{u}) \in S\) is efficient for (CP), then it is also weakly efficient for (CP).

Let \( b_0(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \in R_+^l \), \( \alpha(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \in R^l_+ \setminus \{0\} \), \( \eta(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \in R^n \), \( \xi(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \in R^m \) and \( \phi_0 : R \rightarrow R \). For notational convenience, we use \( b_0 \) for \( b_0(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \), \( \alpha_i \) for \( \alpha_i(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \), \( \eta \) for \( \eta(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \) and \( \xi \) for \( \xi(t, x, \dot{x}, u, \dot{u}, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \). Let \( \psi : I \times X \times X \times Y \times Y \mapsto R^p \).

**Definition 2.3.** A vector functional \( \int_a^b \psi(t, x, \dot{x}, u, \dot{u})dt \) is said to be (strictly) \(\alpha\)-\( V\)-univex at \((\bar{x}, \bar{u}) \in X \times Y\) with respect to the functions \( b_0, \phi_0, \alpha_i, \eta \) and \( \xi \), if for all \((x, u) \in X \times Y \) and \( i \in P \),
\[
b_0 \int_a^b \phi_0[\psi_i(t, x, \dot{x}, u, \dot{u})] - \psi(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})]dt(\rangle \geq \int_a^b \alpha_i[\psi_{ix}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})]dx + (\psi_{iu}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) - D\psi_{iu}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}))\xi]dt.
\]
Remark 2.1. If we take, $b_o = 1$, $\phi_o(a) = a$ and $D\psi_i^a = 0$, then $\alpha$-$V$-univex function reduces to $V$-invex function given in Nahak and Nanda [17].

Now, we present the following example which is $\alpha$-$V$-univex but not $V$-invex.

Example 2.1. Let $I = [0, 1]$ and let $X$ and $Y$ denote the space of all piecewise smooth functions $x : I \mapsto R_+$ and $u : I \mapsto R_+$ respectively. Let $\psi : I \times X \times X \times Y \mapsto R^2$ be defined as

$$\psi(t, x, \dot{x}, u, \dot{u}) = (-4(x^3(t) + u(t)), -x^3(t) - 3u(t)).$$

Further, let $\phi_o : R \mapsto R$ be given as $\phi_o(a) = -3a^3$. Define

$$\alpha_1 = \frac{\bar{x}(t) + \bar{u}(t) + 1}{2}, \quad \alpha_2 = \frac{\bar{x}(t) + \bar{u}(t) + 5}{2},$$

$\eta = \frac{x(t) + u^2(t)}{4}, \quad \xi = \frac{x^3(t) + u(t) + 3\bar{x}(t)\bar{u}(t)}{4}$

and take $b_o = 3$. Then, $\int_0^1 \psi(t, x, \dot{x}, u, \dot{u})dt$ is $\alpha$-$V$-univex at $(\bar{x}, \bar{u}) = (0, 0)$ but not $V$-invex as can be seen below.

Explanation: Firstly, we have to show that at $(\bar{x}, \bar{u}) = (0, 0)$,

$$b_o \int_0^1 \phi_o[\psi_1(t, x, \dot{x}, u, \dot{u}) - \psi_1(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})]dt \geq \int_0^1 \alpha_1[(\psi_1(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) - D\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}))\eta + (\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) - D\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}))\xi]dt.$$

L.H.S.

$$b_o \int_0^1 \phi_o[\psi_1(t, x, \dot{x}, u, \dot{u}) - \psi_1(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})]dt$$

$$= 3 \int_0^1 \phi_o[-4(x^3(t) + u(t)) + 4(\bar{x}^3(t) + \bar{u}(t))]dt$$

$$= 3 \int_0^1 (-3)[-4(x^3(t) + u(t)) + 4(\bar{x}^3(t) + \bar{u}(t))]^3 dt$$

$$= 576 \int_0^1 (x^3(t) + u(t))^3 dt.$$

R.H.S.

$$\int_0^1 \alpha_1[(\psi_1(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) - D\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}))\eta + (\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) - D\psi_{1u}(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}))\xi]dt$$

$$= \int_0^1 \frac{x(t) + u(t) + 1}{2} \left[-12\bar{x}^2(t) \frac{x(t) + u^2(t)}{4} - 4 \frac{x^3(t) + u(t) + 3\bar{x}(t)\bar{u}(t)}{4} \right] dt$$

$$= - \int_0^1 \frac{\bar{x}(t) + \bar{u}(t) + 1}{2} \left[3\bar{x}^2(t)(x(t) + u^2(t)) + x^3(t) + u(t) + 3\bar{x}(t)\bar{u}(t) \right] dt$$

$$= - \frac{1}{2} \int_0^1 (x^3(t) + u(t)) dt.$$

Therefore, it follows that

$$b_o \int_0^1 \phi_o[\psi_1(t, x, \dot{x}, u, \dot{u}) - \psi_1(t, \bar{x}, \dot{x}, \bar{u}, \dot{u})]dt$$
\[ \geq \int_0^1 \alpha_1 \left[ (\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}))\eta \right. \\
\left. + (\psi_{1u}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{1u}(t, \dot{x}, \ddot{u}, \dot{u}))\xi \right] dt, \text{ at } (x, u) = (0, 0). \]

Similarly, it can be shown that
\[ b_0 \int_0^1 \phi_2(t, x, \dot{x}, u, \dot{u}) dt \]
\[ \geq \int_0^1 \alpha_2 \left[ (\psi_{2x}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{2x}(t, \dot{x}, \ddot{u}, \dot{u}))\eta \right. \\
\left. + (\psi_{2u}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{2u}(t, \dot{x}, \ddot{u}, \dot{u}))\xi \right] dt, \text{ at } (x, u) = (0, 0). \]

Therefore, \( \int_0^1 \psi(t, x, \dot{x}, u, \dot{u}) dt \) is \( \alpha \)-V-univex at \( (x, u) = (0, 0) \). Again,
\[ \int_0^1 [\psi_1(t, x, \dot{x}, u, \dot{u}) - \psi_1(t, \dot{x}, \ddot{u}, \dot{u}, \ddot{u})] dt = \int_0^1 [-4(x^3(t) + u(t)) + 4(\ddot{x}(t) + \ddot{u}(t))] dt \\
= -4 \int_0^1 (x^3(t) + u(t)) dt. \]

Also,
\[ \int_0^1 \alpha_1 \left[ (\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}))\eta + \psi_{1u}(t, \dot{x}, \ddot{u}, \dot{u})\xi \right] dt \\
= \int_0^1 \frac{\ddot{x}(t) + \ddot{u}(t) + 1}{2} [-12\dot{x}^2(t) + u^2(t)] - 4x^3(t) + u(t) + 3\ddot{x}(t)\ddot{u}(t)] dt \\
= -\int_0^1 \frac{\ddot{x}(t) + \ddot{u}(t) + 1}{2} [3\dot{x}^2(t)(x(t) + u^2(t)) + x^3(t) + u(t) + \ddot{x}(t)\ddot{u}(t)] dt \\
= -\frac{1}{2} \int_0^1 (x^3(t) + u(t)) dt. \]

Hence, it follows that
\[ \int_0^1 [\psi_1(t, x, \dot{x}, u, \dot{u}) - \psi_1(t, \dot{x}, \ddot{u}, \dot{u}, \ddot{u})] dt \]
\[ \geq \int_0^1 \alpha_1 \left[ (\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}) - D\psi_{1x}(t, \dot{x}, \ddot{u}, \dot{u}))\eta + \psi_{1u}(t, \dot{x}, \ddot{u}, \dot{u})\xi \right] dt, \]

which shows that \( \int_0^1 \psi(t, x, \dot{x}, u, \dot{u}) dt \) is not \( V \)-invex at \( (x, u) = (0, 0) \).

For notational convenience, the vector \( (\int_a^b \lambda_1 f_1(t, \dot{x}, \ddot{u}, \dot{u}) dt, \ldots, \int_a^b \lambda_p f_p(t, \dot{x}, \ddot{u}, \dot{u}) dt) \) is denoted by \( \int_a^b \lambda f(t, \dot{x}, \ddot{u}, \dot{u}) dt \) and the vector \( (\int_a^b \mu_1 g_1(t, \dot{x}, \ddot{u}, \dot{u}) dt, \ldots, \int_a^b \mu_m g_m(t, \dot{x}, \ddot{u}, \dot{u}) dt) \) is denoted by \( \int_a^b \mu \bar{g}(t, \dot{x}, \ddot{u}, \dot{u}) dt \). Similarly, \( \int_a^b \phi(t, \dot{x}, \ddot{u}, \dot{u}) dt \) can be defined. Let \( \phi_0, \phi_1, \phi_2 : R \rightarrow R \) and \( b_1, b_2 \in R^+ \) are defined similar to \( b_0 \).

3. **Sufficient optimality conditions.** In this section, we establish the following sufficient optimality conditions involving \( \alpha \)-V-univexity assumptions.
Theorem 3.1. Let us suppose that for all feasible solution $(\bar{x}, \bar{u})$ to (CP), there exists scalar $\lambda \in \mathbb{R}^p$ and piecewise smooth functions $\mu : I \mapsto \mathbb{R}^m$ and $\nu : I \mapsto \mathbb{R}^n$ such that for all $t \in I$,

$$
\sum_{i \in P} \lambda_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) = 0,
$$

for all $i \in P$.

Further, assume that

(i) $\int_a^b f^\lambda(t, \cdot, \cdot, \cdot, \cdot) dt$ is $\alpha$-V-univex at $(\bar{x}, \bar{u})$ with respect to $b_0, \phi_0, \hat{\alpha}, \eta$ and $\xi$;

(ii) $\int_a^b g^{\mu(t)}(t, \cdot, \cdot, \cdot, \cdot) dt$ is $\alpha$-V-univex at $(\bar{x}, \bar{u})$ with respect to $b_1, \phi_1, \hat{\alpha}, \eta$ and $\xi$;

(iii) $\int_a^b h^{\nu(t)}(t, \cdot, \cdot, \cdot, \cdot) dt$ is $\alpha^*$-V-univex at $(\bar{x}, \bar{u})$ with respect to $b_2, \phi_2, \alpha^*, \eta$ and $\xi$;

(iv) $\hat{\alpha}_1 = \hat{\alpha}_2 = \ldots = \hat{\alpha}_p = \hat{\alpha}_1 = \hat{\alpha}_2 = \ldots = \hat{\alpha}_m = \alpha_1^* = \alpha_2^* = \ldots = \alpha_n^* = \hat{\gamma}$;

(v) $f_0(t) = 0$ and for any scalar function $p(t)$,

$$
\int_a^b p(t)dt < 0 \Rightarrow \int_a^b \phi_0(p(t))dt < 0,
$$

$$
\int_a^b \phi_1(p(t))dt > 0 \Rightarrow \int_a^b p(t)dt > 0;
$$

(vi) $b_0 > 0, b_1 > 0$. Then $(\bar{x}, \bar{u})$ is a weakly efficient solution of (CP).

Proof. Suppose, contrary to the result, that $(\bar{x}, \bar{u}) \in S$ is not a weakly efficient solution to (CP). Then there exists $(x, u) \in S$ such that

$$
\int_a^b f_i(t, x, \dot{x}, u, \dot{u})dt < \int_a^b f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt, \text{ for all } i \in P,
$$

which by $\lambda \geq 0$, $\sum_{i \in P} \lambda_i = 1$, gives

$$
\int_a^b \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u})dt < \int_a^b \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt
$$

$$
\Rightarrow \int_a^b (\sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))dt < 0.
$$
From assumptions (v), (vi) and the above inequality, it follows that
\[ b_2 \int_a^b \phi_b \left( \sum_{i \in P} \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt < 0. \] (8)

By hypotheses (i) and (iv), we have
\[ b_2 \int_a^b \phi_b \left( \sum_{i \in P} \bar{\lambda}_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \bar{\lambda}_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt \]
\[ \geq \int_a^b \gamma \left( \sum_{i \in P} \bar{\lambda}_i f_{ix}(t, x, \dot{x}, u, \dot{u}) - D \sum_{i \in P} \bar{\lambda}_i f_{z}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \eta \]
\[ + \left( \sum_{i \in P} \bar{\lambda}_i f_{iu}(t, x, \dot{x}, u, \dot{u}) - D \sum_{i \in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \xi \right) dt, \]
which on combining with inequality (8), yields
\[ \int_a^b \gamma \left( \sum_{i \in P} \bar{\lambda}_i f_{ix}(t, x, \dot{x}, u, \dot{u}) - D \sum_{i \in P} \bar{\lambda}_i f_{z}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \eta \]
\[ + \left( \sum_{i \in P} \bar{\lambda}_i f_{iu}(t, x, \dot{x}, u, \dot{u}) - D \sum_{i \in P} \bar{\lambda}_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \xi \right) dt < 0. \] (9)

Now, by hypotheses (iii) and (iv) and \( \bar{\nu} \neq 0 \), we have
\[ b_2 \int_a^b \phi_2 \left( \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt \]
\[ \geq \int_a^b \gamma \left( \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - D \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \eta \]
\[ + \left( \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - D \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \xi \right) dt. \] (10)

From the feasibility of (x, u) to (CP) and (6) and \( \bar{\nu} \neq 0 \), we have
\[ \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) = 0. \] (11)

Therefore, by assumption (v) and (11), inequality (10) gives
\[ \int_a^b \gamma \left( \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - D \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \eta \]
\[ + \left( \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - D \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \xi \right) dt \leq 0. \] (12)

On adding inequalities (9) and (12), we obtain
\[ \int_a^b \gamma \left( \sum_{i \in P} \bar{\lambda}_i f_{ix}(t, x, \dot{x}, u, \dot{u}) + \sum_{k \in N} \bar{\nu}_k(t) h_k(t, x, \dot{x}, u, \dot{u}) - D \sum_{i \in P} \bar{\lambda}_i f_{z}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \eta \]
\[ + \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})) \right) \eta + \left( \sum_{i \in P} \bar{\lambda}_i f_{iu}(t, x, \dot{x}, u, \dot{u}) + \sum_{k \in N} \bar{\nu}_k(t) h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) \xi \right) dt < 0.
The preceding inequality together with relations (3) and (4) yields
\[
\int_a^b \sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, \dot{u}, \dot{\bar{u}}) - D \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})) \eta \\
+ (\sum_{j \in M} \bar{\mu}_j(t) g_ju(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) - D \sum_{j \in M} \bar{\mu}_j(t) g_{j\bar{u}}(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})) \xi) dt > 0. \tag{13}
\]
Again by hypotheses (ii) and (iv), we have
\[
b_1 \int_a^b \phi_1\left(\sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})\right) dt \\
\geq \int_a^b \sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})) \eta \\
+ (\sum_{j \in M} \bar{\mu}_j(t) g_ju(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) - D \sum_{j \in M} \bar{\mu}_j(t) g_{j\bar{u}}(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})) \xi) dt. \tag{14}
\]
From inequalities (13) and (14), we get
\[
b_1 \int_a^b \phi_1\left(\sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})\right) dt > 0.
\]
Hence, it follows from assumptions (v), (vi) and the above inequality, that
\[
\int_a^b \sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})) dt > 0,
\]
\[
\Rightarrow \int_a^b \sum_{j \in M} \mu_j(t) g_j(t, x, \dot{x}, u, \dot{u}) dt > \int_a^b \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) dt. \tag{15}
\]
On the other hand, from the feasibility of \((x, u)\) to (CP) and (5), we have
\[
\int_a^b \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, u, \dot{u}) dt \leq \int_a^b \sum_{j \in M} \bar{\mu}_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) dt,
\]
which contradicts (15). Hence, \((\bar{x}, \bar{u})\) is a weakly efficient solution of (CP). This completes the proof. \(\Box\)

**Remark 3.1.** Theorem 3.1 also holds if \(\alpha\)-\(V\)-university of \(\int_a^b f^\lambda(t, x, \dot{x}, u, \dot{u}) dt\) is replaced by strict \(\alpha\)-\(V\)-university of \(\int_a^b f^\lambda(t, x, \dot{x}, u, \dot{u}) dt\).

4. **Duality.** Let \(J\) be a subset of \(M\) and \(K = M \setminus J\) such that \(J \cup K = M\). Let
\[
\sum_{j \in J} \mu_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) = \sum_{j \in J} \mu_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}})
\]
and
\[
\sum_{j \in K} \mu_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) = \sum_{j \in K} \mu_j(t) g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}).
\]
Now, we present the following mixed type multiobjective variational control dual program \([1]\) for (CP):
\[
\text{(MD) Max } \int_a^b (f(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j \in J} \mu_j(t) g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \xi) dt
\]
subject to
\[
\sum_{i \in P} \lambda_i f_i(x, \tilde{x}, \tilde{u}, \tilde{u}) + \sum_{j \in M} \mu_j(t) g_{j1}(x, \tilde{x}, \tilde{u}, \tilde{u}) + \sum_{k \in N} \nu_k(h_kx(t, \tilde{x}, \tilde{u}, \tilde{u})) = D(I\sum_{i \in P} \lambda_i f_i(x, \tilde{x}, \tilde{u}, \tilde{u}) + \sum_{j \in M} \mu_j(t) g_{j1}(x, \tilde{x}, \tilde{u}, \tilde{u}) + \sum_{k \in N} \nu_k(h_kx(t, \tilde{x}, \tilde{u}, \tilde{u}))],
\]
(16)
\[
\int_a^b \sum_{k \in N} \nu_k(h_kx(t, \tilde{x}, \tilde{u}, \tilde{u}))dt \geq 0,
\]
(17)
\[
\int_a^b \sum_{k \in N} \nu_k(h_kx(t, \tilde{x}, \tilde{u}, \tilde{u}))dt = 0,
\]
(18)
\[
\lambda \geq 0, \quad \lambda e = 1, \quad \mu \geq 0,
\]
where \(e = (1, \ldots, 1)\) is a \(p\)-dimensional vector.

**Remark 4.1.** If \(K = \phi\), then the dual (MD) reduces to Wolfe type dual (WD) and if \(J = \phi\), then (MD) reduces to Mond-Weir type dual (MD) given in Ahmad and Sharma [2].

**Theorem 4.1.** (Weak duality) Let \((x, u)\) and \((\tilde{x}, \tilde{u}, \lambda, \mu(t), \nu(t))\) be the feasible solutions of (CP) and (MD), respectively. Assume that \(\lambda > 0\) and
(i) \(\int_a^b f^\lambda(t, x, \tilde{x}, u, \tilde{u})dt\) is \(\tilde{\alpha}\)-\(\nu\)-univex at \((\tilde{x}, \tilde{u})\) with respect to \(b_0, \phi_0, \tilde{\alpha}, \eta, \lambda, \xi;\)
(ii) \(\int_a^b g^{\phi_1}(t, x, \tilde{x}, u, \tilde{u})dt\) is \(\tilde{\alpha}\)-\(\nu\)-univex at \((\tilde{x}, \tilde{u})\) with respect to \(b_1, \phi_1, \tilde{\alpha}, \eta, \lambda, \xi;\)
(iii) \(\int_a^b h^{\phi_2}(t, x, \tilde{x}, u, \tilde{u})dt\) is \(\tilde{\alpha}\)-\(\nu\)-univex at \((\tilde{x}, \tilde{u})\) with respect to \(b_2, \phi_2, \tilde{\alpha}, \eta, \lambda, \xi;\)
(iv) \(\tilde{\alpha}_1 = \tilde{\alpha}_2 = \ldots = \tilde{\alpha}_p = \tilde{\alpha}_1 = \tilde{\alpha}_2 = \ldots = \tilde{\alpha}_m = \tilde{\alpha}_1^* = \tilde{\alpha}_2^* = \ldots = \tilde{\alpha}_n^* = \tilde{\gamma};\)
(v) \(\phi_2(0) = 0\) and for any scalar function \(p(t),\)
\[
\int_a^b \phi_0(p(t))dt = \int_a^b p(t)dt,
\]
\[
\int_a^b \phi_1(p(t))dt = \int_a^b p(t)dt,
\]
\(\lambda > 0, \quad \mu > 0.\) Then the following cannot hold:
\[
\int_a^b f_i(t, x, \tilde{x}, u, \tilde{u})dt \leq \int_a^b (f_i(t, x, \tilde{x}, u, \tilde{u}) + \sum_{j} \mu_j(t) g_j(t, \tilde{x}, \tilde{u}, \tilde{u}))dt,
\]
for all \(i \in P,\)
(20)
\[
\int_a^b f_i(t, x, \tilde{x}, u, \tilde{u})dt < \int_a^b (f_i(t, x, \tilde{x}, u, \tilde{u}) + \sum_{j} \mu_j(t) g_j(t, \tilde{x}, \tilde{u}, \tilde{u}))dt,
\]
for some \(i \in P.\)
Proof. By hypotheses (i), (iii) and (iv) and \( \nu \neq 0 \), we have
\[
\begin{align*}
\int_a^b \phi_0 \left[ \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right] dt \\
\geq \int_a^b \gamma \left[ \left( \sum_{i \in P} \lambda_i f_{ix}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{i \in P} \lambda_i f_{ix}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \eta \right] + \left( \sum_{i \in P} \lambda_i f_{iu}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{i \in P} \lambda_i f_{iu}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt \\
\geq \int_a^b \gamma \left[ \left( \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \eta \right] + \left( \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt.
\end{align*}
\]
and
\[
\int_a^b \phi_2 \left[ \sum_{k \in N} \nu_k(t) h_{k}(t, x, \dot{x}, u, \dot{u}) - \sum_{k \in N} \nu_k(t) h_{k}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right] dt \\
\geq \int_a^b \gamma \left[ \left( \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \eta \right] + \left( \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt.
\]
Therefore, by assumption (v) and (11), the above inequality gives
\[
\begin{align*}
\int_a^b \gamma \left[ \left( \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \eta \right] + \left( \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt \leq 0. \quad (23)
\end{align*}
\]
On adding inequalities (22) and (23), we obtain
\[
\begin{align*}
\int_a^b \phi_0 \left[ \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right] dt \\
\geq \int_a^b \gamma \left[ \left( \sum_{i \in P} \lambda_i f_{ix}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) + \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) - D \left( \sum_{i \in P} \lambda_i f_{ix}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) + \sum_{k \in N} \nu_k(t) h_{kx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \right) \eta \right] + \left( \sum_{i \in P} \lambda_i f_{iu}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) + \sum_{k \in N} \nu_k(t) h_{ku}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt.
\end{align*}
\]
The above inequality together with relations (16) and (17) yields
\[
\begin{align*}
\int_a^b \phi_0 \left[ \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right] dt \\
\geq \int_a^b \gamma \left[ \left( \sum_{j \in M} \mu_j(t) g_{jx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{j \in M} \mu_j(t) g_{jx}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \eta \right] + \left( \sum_{j \in M} \mu_j(t) g_{ju}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) - D \sum_{j \in M} \mu_j(t) g_{ju}(t, \tilde{x}, \tilde{\dot{x}}, \tilde{u}, \tilde{\dot{u}}) \right) \xi \right] dt. \quad (24)
\end{align*}
\]
Again by hypotheses (ii) and (iv), we have
\[
\begin{align*}
&\quad \int_a^b \phi_1 \left( \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt \\
&\geq \int_a^b \gamma \left( \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) - D \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, \bar{u}, \dot{\bar{u}}) \right) dt \\
&\quad + \left( \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) - D \sum_{j \in M} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt,
\end{align*}
\]
which because of inequality (24) yields
\[
\int_a^b \phi_2 \left( \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt \\
\geq -b_1 \int_a^b \phi_1 \left( \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt.
\]
From assumptions (v), (vi) and the above inequality, it follows that
\[
\int_a^b \left( \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) - \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt \\
\geq - \int_a^b \left( \sum_{j \in M} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) - \sum_{j \in M} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt,
\]
\[
\Rightarrow \int_a^b \left( \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) + \sum_{j} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) \right) dt \\
\geq \int_a^b \left( \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt. \quad (25)
\]
Suppose, to the contrary, that (20) and (21) hold. Then, in view of the feasibility of \((x, u)\) for (CP) and \(\mu(t) \geq 0\), we have
\[
\int_a^b \left( f_i(t, x, \dot{x}, u, \dot{u}) + \sum_{j} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) \right) dt \\
\leq \int_a^b \left( f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt, \quad \text{for all } i \in P,
\]
and for some \(i \in P\),
\[
\int_a^b \left( f_i(t, x, \dot{x}, u, \dot{u}) + \sum_{j} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) \right) dt \\
< \int_a^b \left( f_i(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \right) dt.
\]
Since \(\lambda_i > 0\), for all \(i \in P\) and \(\lambda e = 1\), the above inequalities give
\[
\int_a^b \left( \sum_{i \in P} \lambda_i f_i(t, x, \dot{x}, u, \dot{u}) + \sum_{j} \mu_j(t)g_j(t, x, \dot{x}, u, \dot{u}) \right) dt
\]
\[
\leq \int_a^b \left( \sum_{i \in P} \lambda_i f_i(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) + \sum_j \mu_j g_j(t, \bar{x}, \dot{x}, \bar{u}, \dot{u}) \right) dt,
\]
which contradicts (25). This completes the proof. \(\Box\)

Now, we present the following example to illustrate the weak duality theorem.

**Example 4.1.** Let \(I = [0, 1]\) and let \(X = Y = R\) denote the space of all piecewise smooth functions \(x : I \mapsto R\) and \(u : I \mapsto R\) respectively. Consider the following multiobjective variational control problem:

\[\text{(CP)} \ \text{Min} \ \int_0^1 f(t, x, \dot{x}, u, \dot{u}) dt = \left( \int_0^1 2(x^2(t) + u^2(t)) dt, \int_0^1 4(x(t) + u(t) - x(t)u(t)) dt \right)\]

subject to
\[
x(0) = 0, \ x(1) = 1,
-3(x^2(t) + u^2(t)), \ -5(x(t) + u(t)) \leq 0, \ t \in I,
x(t) - u(t) = 0, \ t \in I.
\]

Then \(x(t) = t\) and \(u(t) = -t\) is a feasible solution to (CP).

The mixed type multiobjective variational control dual program for (CP) is given by:

\[\text{(MD)} \ \text{Max} \ \left( \int_0^1 (2 - 3\mu_1(t)) (\bar{x}^2(t) + \bar{u}^2(t)) dt, \int_0^1 (4(\bar{x}(t) + \bar{u}(t) - \bar{x}(t)\bar{u}(t))
-3\mu_1(t)(\bar{x}^2(t) + \bar{u}^2(t))) dt \right)\]

subject to
\[
\bar{x}(0) = 0, \ \bar{x}(1) = 1,
4\lambda_1 \bar{x}(t) + 4\lambda_2 (1-\bar{u}(t)) - 6\mu_1(t)\bar{x}(t) - 5\mu_2(t) - \nu(t) = 0,
4\lambda_1 \bar{u}(t) + 4\lambda_2 (1-\bar{x}(t)) - 6\mu_1(t)\bar{u}(t) - 5\mu_2(t) - \nu(t) = 0,
\int_0^1 (-5)\mu_2(t)(\bar{x}(t) + \bar{u}(t)) \geq 0,
\int_0^1 \nu(t)(-\bar{x}(t) - \bar{u}(t)) = 0,
\lambda \geq 0, \ \lambda e = 1, \ \mu \geq 0,
\]

where \(\lambda = (\lambda_1, \lambda_2)\), \(\mu(t) = (\mu_1(t), \mu_2(t))\), \(e = (1,1)\).

Let \(\bar{x} = 0, \ \bar{u} = 0, \ \lambda = \left(\frac{1}{2}, \frac{1}{2}\right), \ \mu(t) = \left(0, \frac{b_1}{7}\right), \ \nu(t) = 2-t\). Then \((\bar{x}, \bar{u}, \lambda, \mu(t), \nu(t))\) is a feasible solution to (MD).

Now, we define
\[
\eta = \frac{x(t) - u(t)}{4}, \ \xi = \frac{x(t) - u(t) + \bar{x}(t)\bar{u}(t)}{4},
\hat{\alpha}_1 = \hat{\alpha}_2 = \hat{\alpha}_1 = \hat{\alpha}_2 = \alpha^* = 1 + (x(t) - u(t))^2,
\phi_{\alpha} (a) = \phi_1 (a) = \phi_2 (a) = a,
\]
and take \(b_0 = 6, \ b_1 = \frac{1}{3}, \ b_2 = 2\).
We observe that
(i) \( f^\lambda(t, \bar{x}, \bar{u})dt \) is \( \phi \)-\( V \)-univex at \((\bar{x}, \bar{u})\) with respect to \( b_0, \phi, \bar{\alpha}, \eta \) and \( \xi \);
(ii) \( g^\mu(t, \bar{x}, \bar{u}) dt \) is \( \phi \)-\( V \)-univex at \((\bar{x}, \bar{u})\) with respect to \( b_1, \phi_1, \bar{\alpha}, \eta \) and \( \xi \);
(iii) \( h^\upsilon(t, \bar{x}, \bar{u}) dt \) is \( \alpha^- \)-\( V \)-univex at \((\bar{x}, \bar{u})\) with respect to \( b_2, \phi_2, \alpha^*, \eta \) and \( \xi \);
(iv) \( \phi_2(0) = 0 \) and for any scalar function \( p(t) \),
\[
\int_0^1 \phi_0(p(t)) dt = \int_0^1 p(t) dt,
\]
\[
\int_0^1 \phi_1(p(t)) dt = \int_0^1 p(t) dt;
\]
(v) \( b_0 > 0, b_1 > 0 \). Since,
\[
\int_0^1 \phi_0[\lambda_1 f_1(t, \bar{x}, \bar{u}) - \lambda_1 f_1(t, \bar{x}, \bar{u})] dt \\
= 6 \int_0^1 [\lambda_1 f_1(t, \bar{x}, \bar{u}) - \lambda_1 f_1(t, \bar{x}, \bar{u})] dt \\
= 3 \int_0^1 [2(x^2(t) + u^2(t)) - 2(\bar{x}^2(t) + \bar{u}^2(t))] dt \\
= 12 \int_0^1 t^2 dt \\
= 4
\]
and
\[
\int_0^1 \lambda_1[(\lambda_1 f_1(t, \bar{x}, \bar{u}) - D\lambda_1 f_1(t, \bar{x}, \bar{u})) \eta \\
+ (\lambda_1 f_1(t, \bar{x}, \bar{u}) - D\lambda_1 f_1(t, \bar{x}, \bar{u})) \xi] dt \\
= 2 \int_0^1 (1 + (x(t) - u(t))^2) [4x(t)(\frac{x(t) - u(t)}{4}) + 4u(t)(\frac{x(t) - u(t) + \bar{x}(t)\bar{u}(t)}{4})] dt \\
= 0.
\]
Therefore,
\[
\int_0^1 \phi_0[\lambda_1 f_1(t, \bar{x}, \bar{u}) - \lambda_1 f_1(t, \bar{x}, \bar{u})] dt \\
\geq \int_0^1 \lambda_1[(\lambda_1 f_1(t, \bar{x}, \bar{u}) - D\lambda_1 f_1(t, \bar{x}, \bar{u})) \eta \\
+ (\lambda_1 f_1(t, \bar{x}, \bar{u}) - D\lambda_1 f_1(t, \bar{x}, \bar{u})) \xi] dt.
\]
Also,
\[
\int_0^1 \phi_0[\lambda_2 f_2(t, \bar{x}, \bar{u}) - \lambda_2 f_2(t, \bar{x}, \bar{u})] dt \\
= 6 \int_0^1 [\lambda_2 f_2(t, \bar{x}, \bar{u}) - \lambda_2 f_2(t, \bar{x}, \bar{u})] dt \\
= 3 \int_0^1 [4(x(t) + u(t)) - x(t)u(t)] - 4(\bar{x}(t) + \bar{u}(t) - \bar{x}(t)\bar{u}(t))] dt \\
= 12 \int_0^1 t^2 dt \\
= 4
\]
and \[\int_0^1 \hat{\alpha}_2[(\lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) - D\lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))\eta
\]
\[
+ (\lambda_2 f_{2u}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) - D\lambda_2 f_{2u}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))\xi)dt
\]
\[
= \frac{1}{2} \int_0^1 (1 + (x(t) - u(t))^2)[4(1 - \bar{u}(t)) \frac{(x(t) - u(t))}{4}
\]
\[
+ 4(1 - \bar{x}(t)) \frac{(x(t) - u(t) + \bar{x}(t)\bar{u}(t))}{4} dt
\]
\[
= \frac{1}{2} \int_0^1 (1 + 4t^2)(2t + 2t)dt = 2 \int_0^1 (t + 4t^3)dt = 3.
\]

Thus,
\[
b_0 \int_0^1 \phi_0[(\lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) - \lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))]dt
\]
\[
\geq \int_0^1 \hat{\alpha}_2[(\lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) - D\lambda_2 f_{2x}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))\eta
\]
\[
+ (\lambda_2 f_{2u}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) - D\lambda_2 f_{2u}(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))\xi)dt.
\]

Hence, it follows that \(\int_0^1 f^*(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}})dt\) is \(\hat{\alpha}\)-V-univex at \((\bar{x}, \bar{u})\) with respect to \(b_0, \phi_0, \hat{\alpha}, \eta\) and \(\xi\).

Similarly, we can prove that \(\int_0^1 g^\mu(t, x, \dot{x}, u, \bar{u})dt\) is \(\hat{\alpha}\)-V-univex at \((\bar{x}, \bar{u})\) with respect to \(b_1, \phi_1, \hat{\alpha}, \eta\) and \(\xi\) and \(\int_0^1 h^\nu(t, x, \dot{x}, u, \bar{u})dt\) is \(\alpha^\nu\)-V-univex at \((\bar{x}, \bar{u})\) with respect to \(b_2, \phi_2, \alpha^\nu, \eta\) and \(\xi\).

Now,
\[
\int_0^1 f_1(t, x, \dot{x}, u, \bar{u})dt = 2 \int_0^1 (x^2(t) + u^2(t))dt
\]
\[
= 4 \int_0^1 t^2 dt = \frac{4}{3}
\]

and \(\int_0^1 (f_1(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) + \mu_1(t)g_1(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))dt\)
\[
= \int_0^1 (2 - 3\mu_1(t))(\bar{x}^2(t) + \bar{u}^2(t))dt = 0.
\]

Therefore,
\[
\int_0^1 f_1(t, x, \dot{x}, u, \bar{u})dt > \int_0^1 (f_1(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}) + \mu_1(t)g_1(t, \bar{x}, \hat{x}, \bar{u}, \bar{\hat{u}}))dt.
\]

Also,
\[
\int_0^1 f_2(t, x, \dot{x}, u, \bar{u})dt = 4 \int_0^1 (x(t) + u(t) - x(t)u(t))dt
\]
\[
= 4 \int_0^1 (t - t^2)dt = \frac{4}{3}
\]
and \( \int_0^1 (f_2(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \mu_1(t)g_1(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))dt \)
\[= \int_0^1 (4(\bar{x}(t) + \bar{u}(t)) - \bar{x}(t)\bar{u}(t)) - 3\mu_1(t)(\dot{\bar{x}}(t) + \dot{\bar{u}}(t)))dt \]
\[= 0. \]

Hence,
\[\int_0^1 f_2(t, \dot{x}, u, \dot{u})dt > \int_0^1 (f_2(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \mu_1(t)g_1(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}))dt.\]
This verifies the weak duality theorem.

Let us now turn our attention towards strong duality theorem. For this, we shall consider the following Proposition as an extension to the Proposition 1 given in Ahmad and Gulati [1].

**Proposition 4.1.** Let \((\bar{x}, \bar{u})\) be a weakly efficient solution for (CP) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exist \(\lambda \in \mathbb{R}^p\) and piecewise smooth functions \(\mu : I \mapsto \mathbb{R}^m\) and \(\nu : I \mapsto \mathbb{R}^n\) such that
\[
\sum_{i \in P} \lambda_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j \in M} \mu_j(t)g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k \in N} \nu_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \]
\[= D \sum_{i \in P} \lambda_i f_{ix}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j \in M} \mu_j(t)g_{jx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k \in N} \nu_k(t) h_{kx}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}), \tag{26} \]
\[
\sum_{i \in P} \lambda_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j \in M} \mu_j(t)g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k \in N} \nu_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) \]
\[= D \sum_{i \in P} \lambda_i f_{iu}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{j \in M} \mu_j(t)g_{ju}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}) + \sum_{k \in N} \nu_k(t) h_{ku}(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}}), \tag{27} \]
\[
\int_a^b \sum_{K} \mu_j(t)g_j(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt = 0, \tag{28} \]
\[
\int_a^b \sum_{k \in N} \nu_k h_k(t, \bar{x}, \dot{\bar{x}}, \bar{u}, \dot{\bar{u}})dt = 0, \tag{29} \]
\[
\lambda \geq 0, \ \lambda e = 1, \ \mu \geq 0, \tag{30} \]
where \(e = (1, 1, \ldots, 1)\) is a \(p\)-dimensional vector.

**Theorem 4.2.** (Strong duality) Let \((x^o, u^o)\) be an efficient solution of (CP). Assume that \((x^o, u^o)\) satisfies the Kuhn-Tucker constraint qualification for (CP). Then there exist piecewise smooth \(\lambda \in \mathbb{R}^p\), \(\mu : I \mapsto \mathbb{R}^m\) and \(\nu : I \mapsto \mathbb{R}^n\) such that \((x^o, u^o, \lambda, \mu(t), \nu(t))\) is feasible for (MD) along with the condition \(\int_a^b \mu(t)g(t, x^o, \dot{x}^o, u^o, \dot{u}^o)dt = 0\). Furthermore, if weak duality (Theorem 4.1) holds between (CP) and (MD), then \((x^o, u^o, \lambda, \mu(t), \nu(t))\) is an efficient solution of the problem (MD).

**Proof.** Since \((x^o, u^o)\) is an efficient solution for (CP) and every efficient solution for (CP) is also weakly efficient. Therefore by Proposition 4.1, there exists \(\lambda \in \mathbb{R}^p\) and piecewise smooth functions \(\mu : I \mapsto \mathbb{R}^m\) and \(\nu : I \mapsto \mathbb{R}^n\) satisfying (26) to (30).
Hence, \((x^o, u^o, \lambda, \mu(t), \nu(t))\) is a feasible solution for (MD).

Assume that \((x^o, u^o, \lambda, \mu(t), \nu(t))\) is not an efficient solution for (MD). Then there exists a feasible solution \((x, u, \lambda^o, \mu^o, \nu^o)\) for (MD) such that
\[
\int_a^b (f_r(t, x, \dot{x}, u, \dot{u})) \, dt > \int_a^b (f_r(t, x^o, \dot{x}^o, u^o, \dot{u}^o)) \, dt \text{ for some } r \in P \tag{31}
\]
and
\[
\int_a^b (f_i(t, x, \dot{x}, u, \dot{u})) \, dt \geq \int_a^b (f_i(t, x^o, \dot{x}^o, u^o, \dot{u}^o)) \, dt \text{ for all } i \in P \setminus \{r\}. \tag{32}
\]

By hypotheses, we have
\[
\int_a^b \sum_j \mu_j(t) g_j(t, x^o, \dot{x}^o, u^o, \dot{u}^o) \, dt = 0.
\]

Therefore, inequalities (31) and (32) reduce to
\[
\int_a^b (f_r(t, x, \dot{x}, u, \dot{u})) \, dt > \int_a^b (f_r(t, x^o, \dot{x}^o, u^o, \dot{u}^o)) \, dt \text{ for some } r \in P
\]
and
\[
\int_a^b (f_i(t, x, \dot{x}, u, \dot{u})) \, dt \geq \int_a^b (f_i(t, x^o, \dot{x}^o, u^o, \dot{u}^o)) \, dt \text{ for all } i \in P \setminus \{r\}.
\]

These inequalities contradict the conclusion of weak duality Theorem 4.1. Hence, \((x^o, u^o, \lambda, \mu(t), \nu(t))\) is an efficient solution for (MD).

5. Conclusions. We have considered mixed type vector dual variational problem for a class of differentiable multiobjective control problem. Also, weak and strong duality theorems have been proved under the assumptions of \(\alpha\)-V-univexity, using the notion of efficiency. There is a wide scope for extending these concepts to a class of non-differentiable multiobjective variational problem.

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