On a theory of vessels and the inverse scattering

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Abstract

In this paper we present a theory of vessels and its application to the classical inverse scattering of the Sturm-Liouville differential equation. The classical inverse scattering theory, including all its ingredients: Jost solutions, the Gelfand-Levitan equation, the tau function, corresponds to regular vessels, defined by bounded operators. A contribution of this work is the construction of models of vessels corresponding to unbounded operators, which is a first step for the inverse scattering for a wider class of potentials.

A detailed research of Jost solutions and the corresponding vessel is presented for the unbounded Sturm-Liouville case. Models of vessels on curves, corresponding to unbounded operators are presented as a tool to study Linear Differential equations of finite order with a spectral parameter and as examples, we show how the family of Non Linear Schrödinger equations and Canonical Systems arise.

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1 Introduction

The Sturm Liouville differential equation [Lio95, Stu36] is one of the oldest differential equations, studied by mathematicians. It is defined as a linear differential equation of second order

\[-\frac{d^2}{dx^2}y(x) + q(x)y(x) = s^2y(x),\]

where \(\lambda \in \mathbb{C}\) is called the spectral parameter and the coefficient \(q(x)\) is called the potential. It is the simplest linear equation, for which one can not usually find closed-form solutions.

This equation was and probably is one of the most influential with mathematical analysis, because many techniques were developed in order to solve it. For example, it was studied

1. by C. Sturm [Stu36], and R. Liouville [Lio95] in connection with the dynamics, the heat equation,
2. using the monodromy preserving deformation problem of Linear Differential Equations (LDEs) by L. Schlesinger [Sch08], R. Fuchs [Fuc07] and Garnier [Gar12],
3. using the Scattering theory by Lax–Phillips [LP67], and Gelfand-Levitan [IMG51],
4. using Riemannian transformations by A. Povzner [Pov50] and V.A. Marchenko [Mc50, Mc77].

Also M. G. Krein [Kre55] and many other famous mathematicians gave fundamental contributions to this equation. Actually, the list of the contributors and techniques can easily fill few pages.

The third theory in this list, the Scattering theory, studies asymptotic behavior of solutions of the equation (1) and compares them to the trivial ones, corresponding to the zero potential. Notice that solving SL equation (1) for \(q(x) = 0\) one obtains that the solutions are linear combinations of the exponents \(e^{isx}, e^{-isx}\). For the potential \(q(x)\), which is locally integrable and satisfies the condition [Fad63]

\[\int_{-\infty}^{\infty} x|q(x)|dx = C < \infty.\]

one can define Jost solutions, which behave asymptotically (when \(x \to \infty\)) as the trivial ones with a certain phase, depending on the spectral parameter \(s\). Following L.D. Faddeyev [Fad63] "the fundamental problem arising in the quantum theory of scattering is the solution of

\[L\psi(x, k) = -(\frac{d^2}{dx^2}\psi(x, k)) + q(x)\psi(x, k) = k^2\psi(x, k)\]

satisfying the condition \(\psi(0, k) = 0\), behaves asymptotically like \(\psi(x, k) \approx C(k)\sin(kx - \eta(k))\) provided the potential \(q(x)\) decreases sufficiently fast as \(x\) tends to infinity; to what extent does the assignment of \(\eta(k)\) determine the potential \(q(x)\) and how these functions are related".

In this paper, we will present a theory, which generalize the idea of the inverse scattering, i.e. which finds a correspondence between potentials and some (matrix-valued) functions of a complex variable in a slightly different setting, and which coincides with the classical inverse scattering in a "regular" case. The benefit is that we can unify all the approaches and apply this theory to study for example NLS equations (section 4.3) and Canonical systems (section 4.4) beyond the classical results. It is important to notice that it is a separate project by itself and the present work is a background for this future work.

Under slightly different assumption on the potential [Fad74] \(\int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx = C < \infty\) one can construct the Jost solutions \(f_1(s, x)\) and \(f_2(s, x)\) of (1) such that

\[\lim_{x \to \infty} \frac{f_1(s, x)}{e^{isx}} = \lim_{x \to -\infty} \frac{f_2(s, x)}{e^{-isx}} = 1.\]
and create the following matrix of $\lambda = \sqrt{is^2}$, where we choose $3s \geq 0$ and $\beta = \frac{1}{2} \int q$:

$$S(\lambda, x) = \begin{bmatrix}
    f_1 + f_2 & f_1 - f_2 \\
    f_1 + f_2 - \beta f_1 + f_2 & f_1 - f_2 - \beta f_1 - f_2
\end{bmatrix}
\begin{bmatrix}
    \cos(sx) & i\sin(sx) \\
    is\sin(sx) & \cos(sx)
\end{bmatrix}^{-1},$$

which has 4 defining properties

1. behaves like $I$ for $\lambda$ approaching infinity, and has a jump along a finite cut $\Gamma$ on the imaginary positive axis,

2. twice differentiable with respect to $x$,

3. symmetric (with respect to Pauli matrix $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) see [13],

4. maps solutions of the trivial SL equation to solutions of the complicated one, if one concentrates on the first row.

These properties are similar to the properties of the scattering matrix, appearing in [Fad74, section 5]. It turns out that one can realize this matrix, using [BGR90] in the form (for each $x$)

$$S(\lambda, x) = I - B^*(\mu, x)X^{-1}(x)(\lambda - A)^{-1}B(\mu, x)\sigma_1,$$

where there arises an auxiliary separable Hilbert space $\mathcal{H}$ and operators $A, X(x) : \mathcal{H} \to \mathcal{H}, B(x) : \mathbb{C}^2 \to \mathcal{H}$ satisfying certain relations (see Definition 2.3). One can also represent such a function in the form

$$S(\lambda, x) = I + \int_{it \in \Gamma} \frac{S(\mu(t), x)}{\lambda - it} dt.$$

Since $S(\lambda, x) - I$ satisfies conditions of the limiting values on an axis theorem, i.e. $S(\lambda, x)$ is represented as a Poisson integral of its limiting values on the axis. These two ideas brought us to a far reaching generalization. It turns out that one can generalize the construction of such a matrix $S(\lambda, x)$ not only for SL equation but also to a wider class of differential equations (this is done in section 2.1).

The background for this research is the work [MVC] (which was announced in [AM09]) and a realization theory of matrix-valued $p \times p$ functions of a complex variable $\lambda$, analytic and invertible (hence identity) at infinity, and $J$-contractive $(J = J^* = J^{-1})$ [MB58, AD]. On the one hand, they have a so called realization theorem (based on Theorem 2.4)

$$S(\lambda) = I - B^*X^{-1}(\lambda I - A)^{-1}BJ,$$

where $A, X$ are selfadjoint bounded operators, acting on an auxiliary Krein space $\mathcal{H}$ and $B : \mathbb{C}^n \to \mathcal{H}$ is also bounded. On the other hand one can apply a "vessel construction" (see Section 2.2) and to obtain a vessel, whose transfer function depends additionally on a real variable $x$ and is of the form

$$S(\lambda, x) = I - B^*(x)X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1(x).$$

It holds that for $x = x_0$ the function $S(\lambda, x)$ coincides with the above realization for $S(\lambda)$.

Starting from $S(\lambda)$ realized with bounded operators, from the properties of the vessel it follows that the class of the transfer functions

$$S(\lambda, x) \in \mathcal{I}_p(\sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x), 1),$$

3
of so called regular vessels, constructed in this manner, consists of functions which are \( \sigma_1(x) \) symmetric, identity around \( \lambda = \infty \) and map solutions of the input Linear Differential Equation (LDE) (14) with spectral parameter \( \lambda \)

\[
-\sigma_1(x) \frac{\partial}{\partial x} u(\lambda, x) + (\sigma_2(x) \lambda + \gamma(x)) u(\lambda, x) = 0
\]

to solutions \( y(\lambda, x) = S(\lambda, x) u(\lambda, x) \) of the output LDE (15) with the same spectral parameter:

\[
-\sigma_1(x) \frac{\partial}{\partial x} y(\lambda, x) + (\sigma_2(x) \lambda + \gamma_\ast(x)) y(\lambda, x) = 0
\]

The first important result is Theorem 2.7, which relies on a realization theorem of symmetric functions on Krein spaces \([AD]\):

**Theorem 1.1.** Given a transfer function \( S(\lambda, x) \in \mathcal{L}_c(\sigma_1(x), \sigma_2(x), \gamma(x), \gamma_\ast(x), 1) \) there exists a vessel

\[
\mathcal{V}_1 = (A, B(x), X(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_\ast(x); \mathcal{H}, E; I_0),
\]

such that the transfer function of \( \mathcal{V}_1 \) coincides with \( S(\lambda, x) \), defined probably on a smaller interval \( I_0 \subseteq 1 \).

Since the existence of the inverse of \( X(x) \) plays so important role, we define in (20) \( \tau = \det(X^{-1}(0)X(x)) \).

Since \( X(x) \) is a solution of the Lyapunov equation (7), this is a first sign, why we call this function as "tau" function.

In order to even more emphasize the name and the role of the tau function, one have to consider SL equation (1). In this case, one can uniquely reconstruct the potential from solutions \( y(\lambda, x) \) with spectral parameter

\[
\Omega(x, y) = \left[ \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right] B(x) X^{-1}(x_0) B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right],
\]

\[
K(x, y) = - \left[ \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right] B(x) X^{-1}(x) B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right],
\]

one finds that

\[
K(x, y) + \Omega(x, y) + \int_{x_0}^x K(x, t) \Omega(t, y) dt = 0,
\]

and the potential have the classical formula \( q(x) = \frac{d}{dx} K(x, x) = -2 \frac{d^2}{dx^2} \ln \tau(x) \), which again explains the name for it.

Basic ideas in this article come primarily from the work of M. Livšić [Ls01], which actually started in [Ls78]. A generalization of Livšić’ vessel was developed in [Mel09, MVa], creating a comfortable background to learn linear differential equations with a spectral parameter. It is important to notice that Livšić’ definition corresponds to dissipative vessels (see Definition 2.1) in our framework. In [Mel], there is presented an interesting research on finite dimensional vessels of the equation (1), which correspond to potentials, having purely discrete finite spectrum, along with some interesting results related to differential algebras.

In section 4.1 there are discussed asymptotic behavior of vessel objects, in the case the spectrum of \( A \) is in \( i\mathbb{R}^+ \), which correspond to the classical case of being on the negative real line. It turns out (Theorem 4.2) that the potential of such a vessel satisfies \(|q(x)| \leq \frac{Q}{(x - x_0)} \) for big enough \( x \).

This theory would be less applicable without a concrete and simple example, which would show how to construct a nontrivial example of a vessel. This task is successfully accomplished in Section
where it is constructed a transfer function, having singularities (jumps) along a given symmetric with respect to the imaginary axis curve (which may be unbounded).

Finally, in Sections 4.3, 4.4 we show that Non Linear Schrödinger equations and Canonical Systems fit in our framework.

2 Vessels

2.1 Definition of a vessel

Before we define a notion of a vessel, one needs to define a list parameters, which will be fixed in many cases, and thus is dealt separately. Then one defines a notion of a vessel, corresponding to these vessel parameters.

Definition 2.1. Let $\sigma_1, \sigma_2, \gamma, \text{ and } \gamma^*$ be operators from a finite dimensional Hilbert space $E$ to itself, locally integrable on an interval $I = [a, b]$. Suppose that $\sigma_1$ is differentiable and invertible on $I$, and that the following relations hold:

$$
\sigma_1(x) = \sigma_1^*(x), \quad \sigma_2(x) = \sigma_2^*(x)
$$

$$
\gamma(x) + \gamma(x)^* = \gamma^*(x) + \gamma^*(x)^* = -\frac{d}{dx}\sigma_1(x), \quad x \in I.
$$

Then $\sigma_1, \sigma_2, \gamma, \gamma^*$ and the interval $I$ are called vessel parameters on $E$.

Before we define a notion of a vessel which involves an auxiliary Hilbert space $H$ and operators (for $x \in I$)

$$
A, \mathcal{X}(x) : H \to H,
$$
$$
B(x) : E \to H
$$

we have to consider some regularity assumptions. We will assume that the operator $A$ may be unbounded with a domain $D(A)$. Moreover, certain algebraic and differential relations will connect these operator, and as a result, we have to determine assumptions, which will ensure that the relations between $A, \mathcal{X}(x), B(x)$ will become solvable equations.

Definition 2.2 (Regularity assumptions). Operators $A, \mathcal{X}(x), B(x)$ are said to satisfy regularity assumptions on $I$, if there exists a point $x_0 \in I$ such that

1. $B(x_0)E \in D(A^n)$ for all $n \in \mathbb{N}$ and there exists $C > 0$ such that

$$
\|A^nB(x_0)\| \leq (C\sqrt{n})^n,
$$

2. The operator $\mathcal{X}(x)$ is self-adjoint and invertible for all $x \in I$.

In order to show that such a requirement is fulfilled for some operators, we notice that operator $A$ is usually isomorphic to the operator of multiplication by $t$ on $\mathbb{R}$. Taking the initial condition $B(x_0) = e^{-t^2},$ we will obtain that

$$
\|A^{2n}B(x_0)\| = \int_{\mathbb{R}} t^{2n}e^{-t^2} dt = \frac{2}{2n+1}||A^{2(n+1)}B(x_0)||
$$

and the estimate above follows by induction. This means that in the case $A$ is a multiplication by $\mu$ on an unbounded curve $\Gamma$ and $\mathcal{H} = L^2(\Gamma)$, one can take $B(x_0)$ such that it decreases at infinity as $e^{-|\mu|^2}$. For the vessel parameters one defines a notion of a vessel:
Definition 2.3. A vessel is a collection of operators and spaces

\[ \mathcal{R}_v = (A, B(x), \mathcal{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; I), \]

where \( \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x) \) and \( I \) are vessel parameters on \( \mathcal{E} \). The spaces \( \mathcal{H} \) is Hilbert and the operators \( A, \mathcal{X}(x), B(x) \) are defined in (5) so that the regularity assumptions hold. Operators are subject to the following vessel conditions:

\[ 0 = \frac{d}{dx} (B(x)\sigma_1(x)) + AB(x)\sigma_2(x) + B(x)\gamma(x), \quad \text{(6)} \]
\[ A\mathcal{X}(x) + \mathcal{X}(x)A^* + B(x)\sigma_1(x)B^*(x) = 0, \quad \text{(7)} \]
\[ \frac{d}{dx} \mathcal{X}(x) = B(x)\sigma_2(x)B^*(x), \quad \text{(8)} \]
\[ \gamma_*(x) = \gamma(x) + \sigma_2(x)B^*(x)\mathcal{X}^{-1}(x)B(x)\sigma_1(x) - \sigma_1(x)B^*(x)\mathcal{X}^{-1}(x)B(x)\sigma_2(x). \quad \text{(9)} \]

In order to understand why it is a well defined object, it is enough to show that the equations, defining the vessel are solvable. We will see later that the equation (6) is the key point of the construction and the rest will be easily constructed from it.

Theorem 2.1. Suppose that \( B(x_0) \), \( A \) satisfy the condition (11), then there exists a solution \( B(x) \) of (6) with the value \( B(x_0) \) for \( x = x_0 \). Moreover, the estimate similar to (4) holds

\[ \| A^n B(x) \| \leq (C(x)\sqrt{n})^n \]

Proof Before solve the equation (6), notice that it is equivalent to

\[ B'(x) + AB(x)\sigma_2\sigma_1^{-1} + B(x)[\gamma(x) + \sigma_1'(x)]\sigma_1^{-1} = 0. \]

So, defining \( E(x) \) such that

\[ E'(x) = [\gamma(x) + \sigma_1'(x)]\sigma_1^{-1}E(x), \quad E(x_0) = I, \]

we will obtain the following equation

\[ \frac{d}{dx} [B(x)E(x)] + A[B(x)E(x)]E^{-1}(x)\sigma_2\sigma_1^{-1}E(x) = 0. \quad \text{(10)} \]

Denote by \( \Psi(x, \lambda) \) the solution of (substituting \( B(x)E(x) \) with \( \Psi(x, \lambda) \), \( A \) with \( \lambda \), and denoting \( E^{-1}(x)\sigma_2\sigma_1^{-1}E(x) \) by \( \tilde{E}(x) \) in the last equation)

\[ \frac{d}{dx} \Psi(x, \lambda) + \lambda \Psi(x, \lambda)\tilde{E}(x) = 0, \quad \Psi(x_0, \lambda) = I. \]

From the Peano-Baker formula it follows that

\[ \Psi(x, \lambda) = I - \lambda \int_{x_0}^{x} \tilde{E}(y)dy_1 + \lambda^2 \int_{x_0}^{x} \int_{x_0}^{y_1} \tilde{E}(y_2)dy_2\tilde{E}(y_1)dy_1 + \cdots = \sum_{n=0}^{\infty} \Psi_n(x)\lambda^n, \quad \Phi_0(x) = I, \]

and it is a well know result that the coefficient of this matrix satisfy the relation \( \Psi_{n+1}' = -\Psi_n(x)\tilde{E}(x) \) and decrease as coefficients of an exponential function \( \| \Psi_n(x) \| \leq \frac{M}{n!} \). Let

\[ B_1(x) = \sum_{n=0}^{\infty} A^n B(x_0)\Psi_n(x). \]

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Finally, $B$ satisfies (8) Lemma 2.2, which means that since the sum 

$$\sum_{n=0}^{\infty} A^n B(x_0) \Psi_n(x) = (C \sqrt{n})^n \leq C_1^n n!$$

for some $C_1 < 1$, we obtain that 

$$\|A^n B(x_0) \Psi_n(x)\| \leq MC_1^n$$

which means that the series is absolutely convergent. The same holds for the derivative. Differentiating this expression, we find that 

$$\frac{d}{dx} B_1(x) = \sum_{n=0}^{\infty} A^n B(x_0) \frac{d}{dx} \Psi_n(x) = (since \Psi_0 = I)$$

$$= -\sum_{n=1}^{\infty} A^n B(x_0) \Psi_{n-1}(x) E = -\sum_{n=0}^{\infty} A^{n+1} B(x_0) \Psi_n(x) E$$

$$= -A \sum_{n=0}^{\infty} A^n B(x_0) \Psi_n(x) E = -AB_1(x) E(x),$$

which means that $B_1(x)$ and $(B(x) E(x))$ satisfy the equation (10) with the initial condition $B(x_0)$. Finally, $B(x) = B_1(x) E^{-1}(x)$ and this prove the existence.

Let us prove now the norm estimate for this function 

$$\|A^n B(x)\| = \| \sum_{k=0}^{\infty} A^{n+k} B(x_0) \Psi_k(x) \| \leq \sum_{k=0}^{\infty} (C \sqrt{n+k})^{n+k} \frac{M}{k!} = C^n \sum_{k=0}^{\infty} (\sqrt{n+k})^{n+k} \frac{M C^k}{k!}$$

Since $k!$ behaves like $\sqrt{2\pi k(e)^k}$ asymptotically, there exists $C_1 > 0$ such that $\frac{1}{k!} \leq \frac{C_1 e^k}{k^k}$. Thus the last inequality becomes 

$$\|A^n B(x)\| \leq (C \sqrt{n})^n \sum_{k=0}^{\infty} \left( \sqrt{1 + \frac{k}{n}} \right)^n (\sqrt{n+k})^k \frac{M C_1 (e C)^k}{k^k}$$

$$\leq (C \sqrt{n})^n \sum_{k=0}^{\infty} \left( \sqrt{1 + \frac{k}{n}} \right)^n (\sqrt{1 + \frac{k}{n}})^k \frac{M C_1 (e C)^k}{k^k}$$

$$\leq (C \sqrt{n})^n \sum_{k=0}^{\infty} \frac{C_2 e^{n/2} M C_1 (e C)^k}{k^{k/2}}$$

$$\leq (C \sqrt{n})^n C_2 e^{n/2} M C_1 (e C)^k$$

$$\leq (C \sqrt{n})^n C_3 C_2 e^{n/2} M C_1 (e C)^k$$

$$\leq (C \sqrt{n})^n,$$

since the sum $\sum_{k=0}^{\infty} \frac{e^{k/2}}{k^{k/2}}$ is finite. \(\square\)

The equation (9) is also called by M. Livsić as **linkage** condition. It turns out that the so called **Lyapunov equation** (7) is partially redundant.

**Lemma 2.2 (Lyapunov condition permanence).** Suppose that $B(x)$ satisfies (6) and $X(x)$ satisfies (5), then if the Lyapunov equation (4)

$$AX(x) + X(x) A^* + B(x) \sigma_1 B^*(x) = 0$$

holds for a fixed $x_0$, then it holds for all $x$. If $X(x_0) = X^*(x_0)$ then $X(x)$ is self-adjoint for all $x$. 

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Proof: By differentiating the Lyapunov equation, we will obtain that LHS is constant. Since the derivative \( \frac{d}{dx} L(x) = B(x)\sigma_2(x)B^*(x) \) is self-adjoint, \( L(x) \) will be self-adjoint, once \( L(x_0) \) is.

We notice that \((\mathcal{H}, X(x))\) form a Krein space, which is the same as a set, but whose (Krein) inner product depends on \( x \) and is differentiable. From the system theory [Bi71, KR69] and the operator theory related to \( J \)-contractive functions [BGR90, Pot55] we borrow some of the following additional characterizations of the vessel

**Definition 2.4.** The vessel \( \mathcal{K}_0 \) (5) is called
- **dissipative**, if it is the case that \( X(x) > 0 \) for all values of \( x \in I \),
- **Pontryagin**, if \( X(x) \) has \( \kappa \in \mathbb{N} \) negative squares at the right half plane for all values of \( x \in I \),
- **regular**, if all the operators \( A, B(x), X(x) \) are bounded operators for all \( x \),
- **minimal**, if for all \( x \) it holds that
  \[
  \text{cl}\{A^n B(x)E \mid n \in \mathbb{N}\} = \mathcal{H},
  \]
where "cl" stands for the closed span of the corresponding vectors. One of the most important functions associated to the vessel is as follows [Bi71]:

**Definition 2.5.** The \( \mathcal{E} \times \mathcal{E} \) valued function \( S(\lambda, x) \) defined by
  \[
  S(\lambda, x) = I - B^*(x)X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1(x).
  \]
  (12)
is called the **transfer function** of the vessel \( \mathcal{K}_0 \).

It is extremely important and interesting case that the transfer functions and regular vessels are determined one from the other as it will be shown in the following Section 2.3 (it was first shown for regular, dissipative vessels in [Mel09, MVb]).

### 2.2 Standard construction of a vessel

Let us show that one can easily construct vessels. For this to happen, choose two Hilbert spaces \( \mathcal{H}, \mathcal{E} \) and define three operators \( \mathcal{X}_0, A : \mathcal{H} \to \mathcal{H} \) and \( B_0 : \mathcal{E} \to \mathcal{H} \) such that \( \mathcal{X}_0 \) is invertible and the following equalities hold

\[
\mathcal{X}_0^* = \mathcal{X}_0, \quad A\mathcal{X}_0 + \mathcal{X}_0 A^* + B_0 \sigma_1(x_0) B_0^* = 0.
\]

Then solve (9) using Theorem 2.1

\[
0 = \frac{d}{dx}(B(x)\sigma_1(x)) + AB(x)\sigma_2(x) + B(x)\gamma(x), \quad B(x_0) = B_0
\]

and solve the equation (9) by

\[
\mathcal{X}(x) = \mathcal{X}_0 + \int_{x_0}^{x} B(y)\sigma_2(y)B(y)^*dy.
\]

Finally, define \( \gamma_*(x) \) from \( \gamma(x) \) using (9). Thus a vessel is created (the interval I is defined in the proof):

**Lemma 2.3.** The collection

\[
\mathcal{K}_0 = (A, B(x), \mathcal{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; I)
\]
is a vessel.
Proof: The equations (6), (8), and (9) are satisfied by the construction. The Lyapunov equation (7) and the self-adjointness of $X(\mathcal{x})$ follow from Lemma 2.2. Since $X_0$ is an invertible operator, there exists a non trivial interval $I$ (of length at least $\frac{1}{\|X_0^{-1}\|}$) on which $X^{-1}(\mathcal{x})$ exists.

We can obtain in this manner a rich family of vessels, since there exist standard models, creating operators $X_0, A, B_0$:

1. Livšić model of a non selfadjoint operator \cite{MB58}, where $X = I, A + A^* + BJB^* = 0$, and $J$ is a signature matrix ($J = J^* = J^{-1}$),
2. Theory of nodes, developed in \cite{Bi71},
3. Krein space realizations for symmetric functions \cite{AD}, see the following Section 2.3.
4. Vessels on curves (see Section 4.2 in this article).

2.3 Regular vessels

We start from a realization theorem, which will enable us to construct regular vessels.

2.3.1 A realization theorem for symmetric functions using Krein spaces

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Let $X$ be a self-adjoint invertible operator on $H$. We define a sesquilinear form $[\cdot, \cdot]$ on $H$ as $[u, v] = \langle X u, v \rangle$. Define $K$ to be as a set the same Hilbert space $H$, but equipped with (indefinite) inner product: $(K, [\cdot, \cdot])$, which is called Krein space. In the most general case one do not need the invertability of the operator $X$, but we will assume it for our purposes. For any operator $T$ on $K$ we denote by $T^+$ the unique operator satisfying $[Tu, v] = [u, T^+ v]$ for all $u, v \in K$. Actually, it follows that $T^+ = X^{-1} T^* X$, where $T^*$ is the adjoint with respect to $\langle \cdot, \cdot \rangle$. The space $H$ admits the decomposition

$$H = H^+ \oplus H^-$$

such that $[u, u] > 0$ for all $x \in H^+$ and $[u, u] < 0$ for all $x \in H^-$. Moreover, the spaces $(H^+, [\cdot, \cdot]), (H^-, -[\cdot, \cdot])$ are complete with respect to the norms $[\cdot, \cdot]$ and $-[\cdot, \cdot]$ respectively.

Let $j = 1, 2$ and $(K_j, [\cdot, \cdot])$ be Krein spaces and let $U_j$ be linear operators in $K_j$. The operator $U_1, U_2$ will be called weakly isomorphic, if there exists dense subsets $L_j \subseteq K_j$ such that $\Delta_j = L_j \cap D(U_j)$ (D - domain of) is dense in $K_j$ and $U_j(\Delta_j) \subseteq L_j$ and a bijection $V$ from $L_1$ to $L_2$ whic preserves the indefinite inner product:

$$[Vu, Vv]_1 = [v, u]_2, \quad \forall u, v \in L_1$$

and has the properties $V \Delta_1 = \Delta_2, VU_1 u = U_2 V u \ (u \in \Delta_1)$.

The following theorem is taken from \cite{AD, Theorem 3}, when we use a less powerful version of it

**Theorem 2.4.** Let $S$ be a function, which is holomorphic on $\{|\lambda| \geq r \cup \{\infty\}$ with values in a Hilbert space $E$. Moreover, suppose that $S$ is symmetric with respect to the real axis:

$$S(\lambda) = S^*(\lambda)$$

Then there exist a Krein space $\mathcal{K}$, a bounded self-adjoint operator $\tilde{A} \in \mathcal{K}$ and $\tilde{\Gamma} : E \rightarrow \mathcal{K}$, such that

$$S(\lambda) = S(\infty) + \tilde{\Gamma}^+ (\tilde{A} - I\lambda)^{-1} \tilde{\Gamma}.$$
The space $K$ can be chosen minimal:

$$K = \text{cls}\{(I\lambda - \tilde{A})^{-1}\Gamma \mathcal{E} \mid \lambda \in \mathbb{C}\};$$

then $\tilde{A}$ is uniquely determined up to a weak isomorphism.

Important remark, relevant to this research is that when the matrix $\mathcal{X}$ is strictly positive (and invertible) we obtain the usual notion of the Hilbert space, equipped with the norm $\|u, v\| = \langle \mathcal{X}u, v \rangle$.

2.3.2 Regular vessels versus transfer functions

Regular vessels, defined by bounded operators have a very good realization theory for their transfer functions. Notice that in this case the functions $S(\lambda, x)$ are analytic at infinity (actually out of the spectrum of $A$) with value $I$ there. It turns out that given just a transfer function itself one can reconstruct a vessel using a theory of Krein realizations for functions, analytic at infinity (see Theorem [BGR90]). Notice also that poles and singularities of $S(\lambda, x)$ with respect to $\lambda$ are determined by $A$ only and are independent of $x$. Moreover, if a vessel is minimal (i.e. $\|A\| = 1$ holds), standard theorems [BGR90] in realization theory ensure that the singularities of $S(\lambda, x)$ occurs precisely at the spectrum of $A$.

In the next proposition, we summarize the properties of the transfer function of a regular vessel:

**Proposition 2.5.** Let $\mathcal{R}_\mathcal{E}$ be a regular vessel and let $S(\lambda, x)$ be its transfer function. Then

1. For all $x$, $S(\lambda, x)$ is an analytic function of $\lambda$ in the neighborhood of $\infty$, where it satisfies:
   $$S(\infty, x) = I.$$

2. For all $\lambda \notin \text{spec}(A)$, $S(\lambda, x)$ is a differentiable function of $x$.

3. $S(\lambda, x)$ satisfies the symmetry condition
   $$S^*(-\bar{\lambda}, x)\sigma_1(x)S(\lambda, x) = \sigma_1(x)$$
   for $\lambda$ in the domain of analyticity of $S(\lambda, x)$.

4. Multiplication by $S(\lambda, x)$ maps solutions $u(\lambda, x)$ of the input LDE with the spectral parameter $\lambda$:
   $$-\sigma_1(x)\frac{\partial}{\partial x}u(\lambda, x) + (\sigma_2(x)\lambda + \gamma(x))u(\lambda, x) = 0$$
   to solutions $y(\lambda, x) = S(\lambda, x)u(\lambda, x)$ of the output LDE with the same spectral parameter:
   $$-\sigma_1(x)\frac{\partial}{\partial x}y(\lambda, x) + (\sigma_2(x)\lambda + \gamma_*(x))y(\lambda, x) = 0$$

**Proof:** These properties are easily checked, and follow from the definition of $S(\lambda, x)$:

$$S(\lambda, x) = I - B^*(x)X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1(x).$$

The function $S(\lambda, x)$ is analytic for $\lambda > \|A(x)\|$ and since all the operators are bounded, we have $S(\infty, x) = I$. The second property follows from the differentiability assumptions on the operators $X(x), B(x)$. The third property follows from straightforward calculations using the Lyapunov equation [12]:

$$S(\mu, x)^*\sigma_1(x)S(\lambda, x) - \sigma_1(x) =$$

$$-((\bar{\mu} + \lambda)\sigma_1(x)B^*(x)(\bar{\mu}I - A^*)^{-1}X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1(x) = 0$$

for $\mu = -\bar{\lambda}$. The fourth property follows directly from the definitions by plugging $y(\lambda, x) = S(\lambda, x)u(\lambda, x)$ into [14] and using [13] for $u(\lambda, x)$, and the formula [12] for $S(\lambda, x)$, for which in turn we use vessel conditions in order to differentiate it.

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In fact, the converse of Proposition 2.5 holds. It was first proved in [Mel09, MVb, chapter 5] for the dissipative case (when \( \mathcal{X}(x) > 0 \)) and we shall see later in Theorem 2.7 that it holds for a regular vessel too. We define the class of transfer functions, corresponding to the regular vessels as follows:

**Definition 2.6 ([MVb]).** The class \( \mathcal{I}_r = \mathcal{I}_r(\sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x), I) \) is the class consisting of functions \( S(\lambda, x) \) of two variables possessing properties appearing in Proposition 2.5.

Recall (see [CL55]) that to every LDE one can associate an invertible matrix (or operator) function \( \Phi(x, x_0) \), called the fundamental solution, which takes value \( I \) at some preassigned point \( x_0 \) and such that any other solution \( u(x) \) of the LDE, with initial condition \( u(x_0) = u_0 \) is of the form

\[
u(x) = \Phi(x, x_0)u_0.\]

Let \( \Phi(\lambda, x_0) \) and \( \Phi_*(\lambda, x_0) \) be the fundamental solutions of the input LDE (14) and the output LDE (15) respectively, where we have added in the notation the dependence in \( \lambda \). Then,

\[
S(\lambda, x)\Phi(\lambda, x, x_0) = \Phi_*(\lambda, x, x_0)S(\lambda, x_0)
\]

and consequently \( S(\lambda, x) \) satisfies the following LDE

\[
\frac{\partial}{\partial x}S(\lambda, x) = \sigma^{-1}_1(x)(\sigma_2(x)\lambda + \gamma_*(x))S(\lambda, x) - S(\lambda, x)\sigma^{-1}_1(x)(\sigma_2(x)\lambda + \gamma(x)).
\]

The following properties of the fundamental matrices will be used in the sequel.

**Proposition 2.6.** The following formulas hold

\[
\Phi^*(x, -\lambda)\sigma_1\Phi_*(x, \lambda) = \sigma_1,
\]

\[
\frac{\partial}{\partial x} [\Phi^*(x, \mu)\sigma_1\Phi(\lambda, x)] = (\bar{\mu} + \lambda)\Phi^*(x, \mu)\sigma_2\Phi(\lambda, x).
\]

and the same formulas hold, substituting \( \Phi \) by \( \Phi_*. \)

**Proof:** Immediate from the definitions. \( \square \)

The next theorem shows that the class \( \mathcal{I}_r \) is well-defined in the sense, that given a function, one can also find a corresponding to it vessel.

**Theorem 2.7.** Given a transfer function \( S(\lambda, x) \in \mathcal{I}_r(\sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x), I) \), there exists a regular vessel

\[
\mathcal{R}_0 = (A, B(x), X(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; I_0),
\]

such that the transfer function of \( \mathcal{R}_0 \) coincides with \( S(\lambda, x) \), defined probably on a smaller interval \( I_0 \subseteq I \).

**Proof:** Fix a point \( x_0 \in I \) and define a function \( Q(\lambda) \) using Calley transform, which satisfies

\[
S(-i\lambda, x_0) = (I + \frac{i}{2}Q(\lambda)\sigma_1(x_0))(I - \frac{i}{2}Q(\lambda)\sigma_1(x_0))^{-1}.
\]

Actually, this function is given by

\[
Q(\lambda) = 2i\sigma^{-1}_1(x_0)(I - S(-i\lambda, x_0))(I + S(-i\lambda, x_0))^{-1}
\]

and is well-defined at the neighborhood of infinity with value 0 there. Then from the equality

\[
(I - S^*(-i\lambda, x_0))\sigma^{-1}_1(x_0)(I + S(-i\lambda, x_0)) = -(I + S^*(-i\lambda, x_0))\sigma^{-1}_1(x_0)(I - S(-i\lambda, x_0))
\]

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resulting from the symmetry condition \[ \text{(20)}, \] considered with \(-i\lambda\) instead of \(\lambda\), it follows that \(Q(\lambda)^* = Q(\bar{\lambda})\) and \(Q(\lambda)\) is zero at the neighborhood of \(\lambda = \infty\). Thus from Theorem 2.3 it follows that \(Q(\lambda, x)\) admits a following Krein space realization

\[
Q(\lambda) = \Gamma^+(\tilde{A} - \lambda I)^{-1}\Gamma
\]

for a Krein space \(\tilde{K}\) with self-adjoint operator \(\tilde{A}\) in \(\tilde{K}\), and \(\Gamma \in \mathcal{E} \rightarrow \mathcal{K}\). Inserting further this realization formula into \(\text{(20)}\) and simplifying we obtain:

\[
\begin{align*}
S(-i\lambda, x_0) &= (I + \frac{i}{2}Q(\lambda)\sigma_1(x_0))(I - \frac{i}{2}Q(\lambda)\sigma_1(x_0))^{-1} = \\
&= (2I - I + \frac{i}{2}Q(\lambda)\sigma_1(x_0))(I - \frac{i}{2}Q(\lambda)\sigma_1(x_0))^{-1} = -I + 2(I - \frac{i}{2}Q(\lambda)\sigma_1(x_0))^{-1} = \\
&= -I + 2(I - \frac{i}{2}\Gamma^+(\tilde{A} - \lambda I)^{-1}\Gamma\sigma_1(x_0))^{-1}
\end{align*}
\]

There is a simple formula \[ \text{[BGR90]} \] for evaluating the inverse of a matrix in a realized form:

\[
(I - \frac{i}{2}\Gamma^+(\tilde{A} - \lambda I)^{-1}\Gamma\sigma_1(x_0))^{-1} = I + \frac{i}{2}\Gamma^+(\tilde{A}^x - \lambda I)^{-1}\Gamma\sigma_1(x_0),
\]

where \(\tilde{A}^x = \tilde{A} - \frac{i}{2}\Gamma\sigma_1(x_0)\Gamma^+\). So, the last formula becomes

\[
\begin{align*}
S(-i\lambda, x) &= -I + 2(I - \frac{i}{2}\Gamma^+(\tilde{A} - \lambda I)^{-1}\Gamma\sigma_1(x_0))^{-1} = \\
&= -I + 2(I + \frac{i}{2}\Gamma^+(\tilde{A}^x - \lambda I)^{-1}\Gamma\sigma_1(x_0)) = I + \frac{i}{2}\Gamma^+(\tilde{A}^x - \lambda I)^{-1}\Gamma\sigma_1(x_0) = \quad \text{(21)}
\end{align*}
\]

Let us define \(A = -i\tilde{A}^x\) then we obtain that

\[
\begin{align*}
A + A^+ &= -i\tilde{A}^x + i(\tilde{A}^x)^+ = \\
&= -i(\tilde{A} - \frac{i}{2}\Gamma\sigma_1(x_0)\Gamma^+) + i(\tilde{A}^x + \frac{i}{2}\Gamma\sigma_1(x_0)\Gamma^+) = \quad \text{(22)}
\end{align*}
\]

since \(\tilde{A}\) is self-adjoint. Using the formulas for Krein space adjoint

\[
A^+ = \tilde{X}^{-1}A^*\tilde{X}, \quad \Gamma^+ = \tilde{X}\Gamma
\]

the last equation \(\text{(22)}\) is

\[
A + A^+ = -\Gamma\sigma_1(x_0)\Gamma^+ \Leftrightarrow \\
A + \tilde{X}^{-1}A^*\tilde{X} = \Gamma\sigma_1(x_0)\Gamma^+ \Leftrightarrow \\
A\tilde{X}^{-1} + \tilde{X}^{-1}A^* = \Gamma\sigma_1(x_0)\Gamma^+
\]

which is exactly the Lyapunov equation \(\text{(7)}\) at \(x_0\) after defining \(B_0 = \Gamma\) and \(\tilde{X}_0 = \tilde{X}^{-1}\). Thus we obtain that

\[
\begin{align*}
S(\lambda, x_0) &= I - B_0^*\tilde{X}_0^{-1}(\lambda I - A)^{-1}B_0\sigma_1(x_0), \quad \text{(23)} \\
A\tilde{X}_0 + \tilde{X}_0A^* + B_0\sigma_1(x_0)B_0^* &= 0, \quad \tilde{X}_0 = \tilde{X}_0^{-1}. \quad \text{(24)}
\end{align*}
\]

As a result, we can use the standard construction of a vessel (see Section 2.2), starting from \(A, \tilde{X}_0, B_0\) and to obtain a vessel, whose transfer function \(Y(x, s)\) maps solution of the input LDE \(\text{(14)}\) to solution of the output LDE \(\text{(15)}\) for some \(\gamma'_i(x)\). Thus the two functions \(S(\lambda, x)\) and \(Y(\lambda, x)\) have the same value at \(x_0\) and map solutions of the same input LDE to (possibly different) output LDEs:

\[
\begin{align*}
S(\lambda, x) &= \Phi_s(\lambda, x, x_0)S(\lambda, x_0)\Phi^{-1}_s(\lambda, x, x_0), \\
Y(\lambda, x) &= \Phi_s(\lambda, x, x_0)S(\lambda, x_0)\Phi^{-1}_s(\lambda, x, x_0).
\end{align*}
\]
Consequently, the function $S^{-1}(\lambda, x)Y(\lambda, x)$ is equal to $I$ at infinity and is entire. By Liouville’s theorem it is a constant function and is equal to $I$. Thus

$$\Phi_x(\lambda, x, x_0) = \Phi_x'(\lambda, x, x_0),$$

from where we obtain that

$$\Phi^{-1}_x(\lambda, x, x_0)\Phi_x'(\lambda, x, x_0) = I.$$ Differentiating both sides of this last equation we are led to

$$0 = \frac{\partial}{\partial x_1}[\Phi^{-1}_x(\lambda, x, x_0)\Phi_x'(\lambda, x, x_0)] = \Phi^{-1}_x(\lambda, x, x_0)\sigma_1^{-1}(x)(-\gamma(x) + \gamma'(x))\Phi_x'(\lambda, x, x_0).$$

Since the matrices $\Phi_x(\lambda, x, x_0), \Phi_x'(\lambda, x, x_0), \sigma_1(x)$ are invertible we obtain that $\gamma_x(x) = \gamma_x'(x)$. It is remained to notice that constructed $\mathcal{X}(x), B(x), \gamma_x(x)$ satisfy the vessel conditions and as a result we obtain that the collection

$$\tilde{\mathcal{R}}_0 = (A, B(x), \mathcal{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_x(x); \mathcal{H}, E; I_0),$$

is a vessel whose characteristic function $Y(\lambda, x)$ coincides with $S(\lambda, x)$ on $I_0$. 

Finally, for creating a complete picture of the correspondence between vessels and their transfer function, we have to recall the following theorem

**Theorem 2.8.** Suppose that we are given two regular minimal vessels

$$\mathcal{R}_0 = (A, B(x), \mathcal{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_x(x); \mathcal{H}, E; I)$$

$$\tilde{\mathcal{R}}_0 = (\tilde{A}, \tilde{B}(x), \tilde{\mathcal{X}}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_x(x); \tilde{\mathcal{H}}, \tilde{E}; I)$$

then the transfer functions of these vessels are equal at the neighborhood of infinity if and only if there exists an invertible densely definable operator

$$T : \mathcal{H} \to \tilde{\mathcal{H}},$$

such that

$$\tilde{A} = TAT^{-1}, \quad \tilde{B}(x) = TB(x), \quad \tilde{\mathcal{X}}(x) = V\mathcal{X}(x)V^*.$$ 

**Proof(outline):** One of the directions is simple. Suppose that there exists such an operator $T$, then the transfer function of $\tilde{\mathcal{R}}_0$ is

$$\tilde{S}(\lambda, x) = I - \tilde{B}^*(x)\tilde{\mathcal{X}}^{-1}(x)(\lambda - \tilde{A})^{-1}\tilde{B}(x)\sigma_1(x) = I - B^*(x)T^*(T\mathcal{X}(x)T^*)^{-1}(\lambda - TAT^{-1})^{-1}TB(x)\sigma_1(x) = I - B^*(x)(\mathcal{X}(x))^{-1}(\lambda - A)^{-1}B(x)\sigma_1(x) = S(\lambda, x).$$

For the converse direction, one uses the fact that this theorem holds when we fix $x_0 \in I$ (this a standard theorem in the realization theory of functions [Hel74, BCR90]). Then using the idea appearing in Theorem 2.7 (see last argument) that if two transfer functions are identical at $x_0$, then they are identical for all $x$, we will obtain the desired result. 

**Remark:** The second part of this theorem, we could probably call as the permanence of the similarity operator, in the spirit of Lemmas 2.2, 2.10 and Theorems 2.8, 2.14.
2.4 Additional properties of the transfer function of a vessel

The idea of this realization theorem may lead the reader to a conclusion that fixing $\sigma_1, \sigma_2, \gamma$ and varying the initial data $S(\lambda, x_0)$ one should obtain different $\gamma_*$, uniquely determined by the vessel condition [13]. Unfortunately, it is not true

**Proposition 2.9.** Suppose that there exists a symmetric function $Y(\lambda)$, which commutes with $\Phi(\lambda, x, x_0)$ and suppose that a function $S(\lambda, x)$ corresponds to vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$. Then the function $S(\lambda, x)Y(\lambda)$ corresponds to the same vessel parameters $\sigma_1, \sigma_2, \gamma, \gamma_*$.

**Proof:** Using formula (16) we obtain that

$$S(\lambda, x) = \Phi_*(\lambda, x, x_0)S(\lambda, x_0)\Phi^{-1}(\lambda, x, x_0).$$

Consequently,

$$S(\lambda, x)Y(\lambda) = \Phi_*(\lambda, x, x_0)S(\lambda, x_0)Y(\lambda)\Phi^{-1}(\lambda, x, x_0)$$

intertwines solutions of the input (14) and the output (15) ODEs with the spectral parameter $\lambda$, and is identity at infinity, because $S(\lambda, x)$ and $Y(\lambda)$ and their product are such. Symmetry is easy to check and by the definition the function $S(\lambda, x)Y(\lambda)$ corresponds to the same vessel parameters as $S(\lambda, x)$. \[
\]

Here are some interesting properties of the determinant of transfer functions, which will be used later

**Lemma 2.10 (Permanence of det $S$).** The determinant of the matrix-function $S(\lambda, x)$ is $x$-independent and it holds that

$$\det S(\lambda, x) = \det S(\lambda, x_0), \quad \lambda \notin \text{spec} A.$$

For $\lambda$ on the imaginary axis, it holds that $|\det S(\lambda, x_0)| = 1, \lambda \notin \text{spec} A$.

**Proof:** Using equation (17) we obtain that

$$\frac{\partial}{\partial x} \det S(\lambda, x) = \text{tr}(S^{-1}(\lambda, x) \frac{\partial}{\partial x} S(\lambda, x)) =$$

$$= \text{tr}(S^{-1}(\lambda, x) \sigma_1^{-1}(x)(\sigma_2(x)\lambda + \gamma_*)(x))S(\lambda, x) - S(\lambda, x)\sigma_1^{-1}(x)(\sigma_2(x)\lambda + \gamma(x)) =$$

$$= \text{tr}(\sigma_1^{-1}(x)(\sigma_2(x)\lambda + \gamma_*)(x)) - \sigma_1^{-1}(x)(\sigma_2(x)\lambda + \gamma(x)) = \text{tr}(\sigma_1^{-1}(x)(\gamma_*(x) - \gamma(x)) =$$

$$= \text{tr}(\sigma_1^{-1}(x)B^*(x)X^{-1}(x)B(x)\sigma_2(x) - \sigma_1^{-1}(x) \gamma_*(x) = 0,$$

we have used the property of trace $\text{tr}(AB) = \text{tr}(BA)$ and the linkage condition (9). So, $\frac{\partial}{\partial x} \det S(\lambda, x) = 0$ and the result follows.

The second part of the lemma follows from taking determinant at the symmetry condition (13) and the fact that $-\bar{\lambda} = \lambda$, if $\lambda$ is on the imaginary axis. \[
\]

Another interesting property of transfer functions is that when $X(x) > 0$, they define a Kernel of a Reproducing Kernel Hilbert Space [Aro50] as the following lemma states

**Lemma 2.11.** Suppose that $X(x) > 0$, then the Kernels

$$K_1(\lambda, \mu, x) = \frac{\sigma_1(x) - S^*(\mu, x)\sigma_1(x)S(\lambda, x)}{\mu + \lambda}, \quad K_2(\lambda, \mu, x) = \frac{\sigma_1^{-1}(x) - S(\lambda, x)\sigma_1^{-1}(x)S^*(\mu, x)}{\mu + \lambda},$$

are positive.
Proof: Using Lyapunov equation \( (7) \), one obtains that
\[
K_1(\lambda, \mu, x) = \sigma_1(x) B^*(x) (\mu I - A^*)^{-1} X^{-1}(x)(\lambda I - A)^{-1} B(x) \sigma_1(x),
\]
and
\[
K_2(\lambda, \mu, x) = B^*(x) X^{-1}(x)(\mu I - A^*)^{-1} X(x)(\lambda I - A)^{-1} X^{-1}(x) B(x),
\]
from where the positivity follows. \( \square \)

M. Livsič model of a non self-adjoint operator \( [MB58] \) uses (implicitly) the assumption \( X_0 = I \) and corresponds to a dissipative vessel by the definition. The next theorem shows that an arbitrary dissipative vessel can be brought to a Livsič form for a fixed value \( x_0 \). Then one can use the standard construction, based on the Livsič model at \( x_0 \).

Theorem 2.12. For a dissipative vessel there exists a Hilbert space similarity \( \mathcal{V} : \mathcal{H} \to \mathcal{H} \), so that the new vessel
\[
\widetilde{R}_{\mathcal{V}} = (\widetilde{A}, \widetilde{B}(x), \widetilde{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_+(x); \mathcal{H}, \mathcal{E}; I_0)
\]
where
\[
\widetilde{A} = V^{-1} A V, \quad \widetilde{B}(x) = V^{-1} B(x), \quad \widetilde{X}(x) = V^{-1} X(x) V^{-1},
\]
has the same transfer function as \( R_{\mathcal{V}} \) but satisfies additionally
\[
\widetilde{X}(x_0) = I.
\]
Proof: Define \( V = \sqrt{X(x_0)} > 0 \), which exists, since \( X(x_0) > 0 \). Then check that the transfer function of \( \widetilde{R}_{\mathcal{V}} \) coincides with that of \( R_{\mathcal{V}} \):
\[
\tilde{S}(\lambda, x) = I - \tilde{B}^*(x) \tilde{X}^{-1}(x)(\lambda I - \tilde{A})^{-1} \tilde{B}(x) \sigma_1 =
\]
\[
= I - B^*(x) V^{-1} (V^{-1} X(x) V^{-1})^{-1} (\lambda I - V^{-1} A V)^{-1} V^{-1} B(x) \sigma_1 = S(\lambda, x),
\]
after cancellations. Finally notice that
\[
\tilde{X}(x_0) = V^{-1} X(x_0) V^{-1} = I.
\]

The minimality condition turns out to be independent of \( x \) in the sense that if it holds for one \( x_0 \), then it holds for all \( x \).

Theorem 2.13 (Permanence of minimality). Suppose that we are given a vessel \( R_{\mathcal{V}} \), which is minimal at \( x_0 \). Then the vessel is minimal for all \( x \in \mathbb{R} \).

Proof: Let us show it for the regular case first. Suppose that the realization is minimal \( [11] \): \( \operatorname{cl} \{ A^n B(x_0) E \mid n \in \mathbb{N} \} = \mathcal{H} \). Using regularity assumption \( (4) \), we can represent \( B(x) \) using a fundamental matrix \( \Phi_1(\lambda, x, x_0) \):
\[
B(x) = \frac{1}{2\pi i} \oint (\lambda I - A)^{-1} B(x_0) \Phi_1(\lambda, x, x_0) d\lambda.
\]
Since \( \Phi_1(\lambda, x, x_0) \) and its inverse are entire, we can use Taylor series in \( \lambda \)
\[
\Phi_1^{-1}(\lambda, x, x_0) = \sum \phi_k(x) \lambda^k.
\]
Finally, applying \( A^{n+k} B(x) \) to matrices \( \phi_k(x) \) and taking sums, whose convergence follows from the analyticity of \( \Phi_1^{-1}(\lambda, x, x_0) \) we will obtain that
\[
\sum_{k=0}^{\infty} A^{n+k} B(x) \phi_k(x) = A^n \frac{1}{2\pi i} \sum_{k=0}^{\infty} A^k (\lambda I - A)^{-1} B(x_0) \Phi(\lambda, x, x_0) \phi_k(x) =
\]
\[
= A^n \frac{1}{2\pi i} \oint (\lambda I - A)^{-1} B(x_0) \Phi(\lambda, x, x_0) \sum_k \lambda^k \phi_k(x) d\lambda = A^n \frac{1}{2\pi i} \oint (\lambda I - A)^{-1} B(x_0) d\lambda = A^n B_0.
\]
and so
\[ \text{cl}\{A^n B(x_0)E \mid n \in \mathbb{N}\} \subseteq \text{cl}\{A^n B(x)E \mid n \in \mathbb{N}\}, \]
from where the minimality for all \( x \) follows.

In the unbounded case, immediate consequence of the formula
\[ B(x) = \sum A^k B_0 \phi_k(x) E^{-1}(x) \]
is that
\[ A^n B(x)E \subseteq \text{cl}\{A^n B_0 E\}. \]
On the other hand, solving the differential equation starting from \( x \) down to \( x_0 \), we will obtain that similar formulas hold with \( x \) and \( x_0 \) interchanged and in this case
\[ A^n B(x_0)E \subseteq \text{cl}\{A^n B(x)E\}, \]
from where it follows the permanence of the minimality.

The next theorem shows that the symmetry condition (as the Lyapunov equation) can be checked in one point. This theorem is similar to the property of a solution of a Riccati equation [Zel98, theorem 2.1]

**Theorem 2.14** (Permanence of symmetry). Suppose that the vessel parameters satisfy Livsic (not M. Livsic) condition. Suppose that \( S(\lambda, x) \) is a differentiable function of \( x \) for each \( \lambda \), analytic in \( \lambda \) for each \( x \), except for a set of singular points, and satisfies \( S(\infty, x) = I \). Suppose also that \( S(\lambda, x) \) is an intertwining function of LDEs and . Then if the symmetry condition holds for \( x_0 \), then it holds for all values of \( x \).

**Proof:** Since \( S(\lambda, x) \) intertwines solutions of and , then it satisfies the differential equation
\[ \frac{\partial}{\partial x} S(\lambda, x) = \sigma_1^{-1}(x) S^{-1}(\lambda, x) \gamma_1(x) - S(\lambda, x) \sigma_1^{-1}(x) \gamma_2(x) \lambda + \gamma(x). \]
Consequently, using properties of \( \gamma_1, \gamma \) appearing in Definition we obtain that the function \( \sigma_1^{-1}(x) S^{-1}(\lambda, x) \sigma_1(x) \) satisfies the same differential equation. If these two functions are equal at \( x_0 \), from the uniqueness of solution for a differential equation with continuous coefficients, they are also equal for all \( x \).

2.5 The tau function of a vessel

Following the ideas presented in we define the tau function of the vessel 2.3 in the following way

**Definition 2.7.** For a given vessel (see Definition 2.3)

\[ \mathcal{R}_3 = (A, B(x), \mathcal{X}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_0(x); \mathcal{H}, \mathcal{E}; I) \]
the tau function \( \tau(x) \) is defined as
\[ \tau = \det(\mathcal{X}^{-1}(x_0)X(x)) \]
for an arbitrary point \( x_0 \in I \).
Notice that using vessel condition \( \mathbb{X}(x) \) has the formula

\[
\mathbb{X}(x) = \mathbb{X}(x_0) + \int_{x_0}^{x} B^*(y)\sigma_2 B(y)dy,
\]

and as a result

\[
\mathbb{X}^{-1}(x_0)\mathbb{X}(x) = I + \mathbb{X}^{-1}(x_0)\int_{x_0}^{x} B^*(y)\sigma_2 B(y)dy.
\]

Since \( \sigma_2 \) has finite rank for \( \dim \mathcal{E} < \infty \), this expression is of the form \( I + T \), for a trace class operator \( T \) and since \( \mathbb{X}_0 \) is an invertible operator, there exists a non trivial interval \( \frac{1}{\|\mathbb{X}_0\|} \) on which \( \mathbb{X}(x) \) and \( \tau(x) \) are defined. Recall \([\text{IG69}]\) that a function \( F(x) \) from \( (a, b) \) into the group \( G \) (the set of bounded invertible operators on \( \mathcal{H} \) of the form \( I + T \), for a trace class operator \( T \)) is said to be differentiable if \( F(x) \) is differentiable as a map into the trace-class operators. In our case,

\[
\frac{d}{dx}(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)) = \mathbb{X}^{-1}(x_0)\frac{d}{dx}\mathbb{X}(x) = \mathbb{X}^{-1}(x_0)B(x)\sigma_2 B^*(x)
\]

exists in trace-class norm. Israel Gohberg and Mark Krein \([\text{IG69}]\) formula 1.14 on p. 163 proved that if \( \mathbb{X}^{-1}(x_0)\mathbb{X}(x) \) is a differentiable function into \( G \), then \( \tau(x) = \text{sp}(\mathbb{X}^{-1}(x_0)\mathbb{X}(x)) \) is a differentiable map into \( \mathbb{C}^+ \) with

\[
\frac{\tau'}{\tau} = \text{sp}((\mathbb{X}^{-1}(x_0)\mathbb{X}(x))^{-1}\frac{d}{dx}(\mathbb{X}^{-1}(x_0)\mathbb{X}(x))) = \text{sp}(\mathbb{X}(x)\mathbb{X}^{-1}(x)) = \text{sp}(B(x)\sigma_2 B^*(x)\mathbb{X}^{-1}(x)) = \text{tr}(\sigma_2 B^*(x)\mathbb{X}^{-1}(x)B(x)). \quad (27)
\]

Since any two realizations of a symmetric function are (weakly) isomorphic, one obtains from standard theorems \([\text{BGR90}]\) in realization theory of analytic at infinity functions that they will have the same tau function up to a scalar, i.e. this notion is independent of the realization we choose for the given function \( S(\lambda, x) \). And we can call this property as the permanence of the tau function.

3 Sturm-Liouville vessels

Now we are ready to consider a particular example of vessels, which corresponds to the Sturm-Liouville differential equation \( (1) \). Some definitions in the general theory of vessels (such as the tau function) are inspired by this particular example. In order to obtain a SL vessel we choose \( \mathcal{E} = \mathbb{C}^2 \), i.e., a Hilbert space of dimension 2 and make the following

**Definition 3.1.** The Sturm Liouville (SL) vessel parameters are defined to be

\[
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \gamma_*(x) = \begin{bmatrix} -i(\beta'(x) - \beta^2(x)) & -\beta(x) \\ \beta(x) & i \end{bmatrix}
\]

for a real valued function \( \beta(x) \) differentiable an interval \( I \).

Suppose now that we have a vessel \( \mathbb{H}_2 \) realizing these vessel parameters. Then it turns out that multiplication by the transfer function maps solutions of the trivial SL equation \( (q(x) = 0) \) to a more complicated one. As a result it can be considered as a Bäcklund transformation \([\text{CL55}]\). One can check that the Crum transformations \([\text{Cru55}]\) are a particular case of vessels constructions (see \([\text{Mei}]\) Section 3.2 for details).
Denoting \( u(\lambda, x) = \begin{bmatrix} u_1(\lambda, x) \\ u_2(\lambda, x) \end{bmatrix} \) we shall obtain that the input compatibility condition is equivalent to
\[
\begin{align*}
-\frac{\partial^2}{\partial x^2} u_1(\lambda, x) &= -i\lambda u_1(\lambda, x), \\
u_2(\lambda, x) &= -i \frac{\partial}{\partial x} u_1(\lambda, x).
\end{align*}
\]
The output \( y(\lambda, x) = \begin{bmatrix} y_1(\lambda, x) \\ y_2(\lambda, x) \end{bmatrix} \) = \( S(\lambda, x)u(\lambda, x) \) satisfies the output equation, which is equivalent to
\[
\begin{align*}
-\frac{\partial^2}{\partial x^2} y_1(\lambda, x) + 2\beta(x)y_1(\lambda, x) &= -i\lambda y_1(\lambda, x), \\
y_2(\lambda, x) &= -i \frac{\partial}{\partial x} y_1(\lambda, x) - \beta(x)y_1(\lambda, x).
\end{align*}
\]
Observing the first coordinates \( u_1, y_1 \) of the vector-functions \( u, y \) we can see that multiplication by \( S(\lambda, x) \) maps solution of the trivial SL equation to solutions of the more complicated one, defined by the potential
\[ q(x) = 2\beta'(x). \] (28)
If we denote the fundamental matrix of the input SL equation as following
\[
\Phi(x, \lambda) = \begin{bmatrix} \cos(sx) & \frac{is\sin(sx)}{s} \\ \frac{is\sin(sx)}{s} & \cos(sx) \end{bmatrix}, \quad s^2 = -i\lambda \quad (29)
\]
then \( S(\lambda, x)\Phi(x, s) \) is the fundamental matrix for solutions of the output SL equation corresponding to the potential \( q(x) \). Notice that the matrix \( \Phi(\lambda, x) \) is an entire function of \( \lambda \) by considering its Taylor series.

We saw in Proposition 2.9 that multiplication on a symmetric function \( Y(\lambda) \) of the variable \( \lambda \), which commute with \( \Phi(\lambda, x, x_0) \) does not change vessel parameters. In the case of Sturm-Liouville vessel parameters we can describe these functions explicitly. In order to understand which symmetric \( x \)-independent \( Y(\lambda) \) commute with \( \Phi(\lambda, x, x_0) \), it is necessary and sufficient to understand when \( Y(\lambda) \) commutes with its "generator" \( \sigma^{-1}_1(\sigma_2\lambda + \gamma) \). Extracting condition on \( Y(\lambda) \) so that
\[ Y(\lambda)\sigma^{-1}_1(\sigma_2\lambda + \gamma) = \sigma^{-1}_1(\sigma_2\lambda + \gamma)Y(\lambda), \]
we will obtain that
\[ Y(\lambda) = I + \begin{bmatrix} a(\lambda) & \frac{ic(\lambda)}{\lambda} \\ c(\lambda) & a(\lambda) \end{bmatrix} \quad (30) \]
for functions \( a(\lambda), c(\lambda) \), which are zero at infinity. It turns out that for a given \( \gamma_*(x) \) in SL case, any two functions corresponding to the same vessel parameters differ by a constant symmetric function \( Y(\lambda) \):

**Theorem 3.1.** Given SL vessel parameters \( \sigma_1, \sigma_2, \gamma, \gamma_*(x) \), there exists a unique initial value \( S(0, \lambda) \) up to multiplication from the right on a symmetric, \( x \)-independent, commuting with \( \Phi(\lambda, x, x_0) \) function.

**Proof:** Given now two functions \( S_1(\lambda, x), S_2(\lambda, x) \) as in the theorem, the function \( S^{-1}_1(\lambda, x)S_2(\lambda, x) \) will intertwine solutions of the input LDE with itself. Let us show that such a function must be \( x \)-independent and commuting with \( \Phi(\lambda, x, x_0) \).

Using \( \lambda = is^2 \) we find that
\[
\sigma^{-1}_1(\sigma_2\lambda + \gamma) = \begin{bmatrix} 0 & i \\ \lambda & 0 \end{bmatrix}, \quad \Phi(\lambda, x, x_0) = V \begin{bmatrix} e^{-s(x-x_0)} & 0 \\ 0 & e^{s(x-x_0)} \end{bmatrix} V^{-1},
\]

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where
\[ V = \begin{bmatrix} -\frac{1}{s} & 1 \\ \frac{1}{s} & 1 \end{bmatrix}. \]

Consequently, for the expression \( S(\lambda, x) = \Phi(\lambda, x, x_0)S_0(\lambda)\Phi^{-1}(\lambda, x, x_0) \) to be identity for \( \lambda = \infty \), it is necessary to "cancel" the essential singularity arising from two entire functions \( \Phi(\lambda, x, x_0) \) and \( \Phi^{-1}(\lambda, x, x_0) \) (or more precisely, let them cancel each other).

Using the formula for \( \Phi(\lambda, x, x_0) \) in
\[
\Phi(\lambda, x, x_0)S_0(\lambda)\Phi^{-1}(\lambda, x, x_0) = V \begin{bmatrix} e^{-s(x-x_0)} & 0 \\ 0 & e^{k(x-x_0)} \end{bmatrix} V^{-1} S(\lambda, x_0) V^{-1} [e^{s(x-x_0)} & 0 \\ 0 & e^{-k(x-x_0)}] V^{-1} = I
\]
and considering coefficients of the exponents, we conclude that
\[ V^{-1} S(\lambda, x_0) V = \begin{bmatrix} b(s) & 0 \\ 0 & d(s) \end{bmatrix}, \]
for some analytic in \( s \) at infinity functions \( b(s), d(s) \). From here it follows that
\[ S(\lambda, x_0) = \begin{bmatrix} -\frac{b(s) + d(s)}{s} & \frac{d(s) - b(s)}{s} \\ \frac{s}{d(s) - b(s)} & \frac{-b(s) + d(s)}{s} \end{bmatrix}. \]

so, that \( a(\lambda) = \frac{b(s) + d(s)}{s} \) and \( c(\lambda) = d(s) - b(s) \) and we shall obtain that \( S_0(\lambda) \) is of the form \( 30 \), i.e. commutes with the fundamental matrix \( \Phi(\lambda, x, x_0) \).

### 3.1 Construction of \( S(\lambda, x) \) for a given \( \gamma_* (x) \). Classical case.

Let us consider the Sturm-Liouville differential equation
\[-y''(x, s) + q(x)y(x, s) = s^2 y(x, s),\]
where the potential \( q(x) \) is a real measurable function satisfying the condition \( 31 \)
\[ \int_{-\infty}^{\infty} (1 + |x|)|q(x)|dx < \infty. \]

Consider the following solutions \( f_1(x, s), f_2(x, s) \) defined from a Volterra type equation \( 33 \)
\[ f_1(x, s) = e^{ikx} + \int_{-\infty}^{\infty} G_1(x - y, s)q(y)f_1(y, s)dy, \]
\[ f_2(x, s) = e^{-ikx} + \int_{-\infty}^{\infty} G_2(x - y, s)q(y)f_2(y, s)dy, \]
where
\[ G_1(t, s) = -Hev(-t)\frac{\sin(st)}{s}, \quad G_2(t, s) = Hev(t)\frac{\sin(st)}{s} \]
and $Hev(t)$ is the Heaviside function

$$Hev(t) = 1, t > 0, \quad Hev(t) = 0, t < 1.$$  

The functions $f_1, f_2$ behave \[ \text{[Fad74, 1.4, 1.5]} \] as $e^{ixx}$ and $e^{-ixx}$ for $x$ approaching $+\infty$ and $-\infty$ respectively. Moreover, the following bounds hold \[ \text{[Lev49]} \]

$$|f_1(x, s) - e^{ixx}| \leq C \frac{e^{3xx}}{1 + |s|} \int_{-\infty}^{\infty} (1 + |y|)|q(y)|dy,$$

(33)

$$|f_2(x, s) - e^{-ixx}| \leq C \frac{e^{3xx}}{1 + |s|} \int_{-\infty}^{\infty} (1 + |y|)|q(y)|dy.$$  

(34)

Then Z. S. Agranovich and V. A. Marchenko \[ \text{[MV60]} \] proved that the same solutions $f_1, f_2$ may be represented as \[ \text{[Fad74, 1.10, 1.11]} \]

$$f_1(x, s) = e^{ixx} + \int_{x}^{\infty} A_1(x, y) e^{iyy} dy, \quad f_2(x, s) = e^{-ixx} + \int_{x}^{\infty} A_2(x, y) e^{-iyy} dy$$

where $A_1, A_2$ are square integrable functions of $y$ for each $x$. Moreover they also showed that defining

$$\xi_1(x) = \int_{x}^{\infty} |q(y)|dy, \quad \xi_2(x) = \int_{-\infty}^{x} |q(y)|dy,$$

the functions $A_1, A_2$ satisfy Volterra type equations \[ \text{[Fad74, 1.10, 1.11]} \] and successive approximations give the following bounds for them \[ \text{[Fad74, 1.12]} \]

$$|A_1(x, y)| \leq C \xi_1(\frac{x + y}{2}), \quad |A_2(x, y)| \leq C \xi_2(\frac{x + y}{2}).$$

One can also find that

$$\left| \frac{\partial}{\partial x} A_1(x, y) + \frac{1}{4} q(\frac{x + y}{2}) \right| \leq C \xi_1(x) \xi_1(\frac{x + y}{2}),$$

(35)

$$\left| \frac{\partial}{\partial x} A_2(x, y) - \frac{1}{4} q(\frac{x + y}{2}) \right| \leq C \xi_2(x) \xi_2(\frac{x + y}{2}),$$

(36)

and

$$q(x) = -2 \frac{\partial}{\partial x} A_1(x, x) = 2 \frac{\partial}{\partial x} A_2(x, x).$$

Finally, we define

$$\beta(x) = -A_1(x, x) = -\frac{1}{2} \int_{x}^{\infty} q(y)dy$$

(37)

and

$$S(\lambda, x) = \Phi_\ast(x, \lambda) \Phi^{-1}(x, \lambda),$$

where $\Phi, \Phi_\ast$ are solutions of the input and the output LDE respectively, corresponding to the SL parameters, defined using the function $\beta(x)$. One can take

$$\Phi(x, \lambda) = \begin{bmatrix} \cos(sx) & is\sin(sx) \\ is\sin(sx) & \cos(sx) \end{bmatrix}$$

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\[ \Phi_s(x, \lambda) = \begin{bmatrix} f_1 + f_2 & f_1 - f_2 \\ f_1' - f_2' & f_1' + f_2' \end{bmatrix} \]

Note that the function \( \Phi \) depend on \( s^2 = -i\lambda \) and as a result is an entire function of \( \lambda \). Simple calculations show that

\[ S(\lambda, x) = \begin{bmatrix} f_1 e^{-i\lambda x} + f_2 e^{i\lambda x} & f_1 e^{-i\lambda x} - f_2 e^{i\lambda x} \\ (f_1' - \beta f_1) e^{-i\lambda x}/2 + (f_2' - \beta f_2) e^{i\lambda x}/2 & (f_1' - \beta f_1) e^{-i\lambda x}/2 - (f_2' - \beta f_2) e^{i\lambda x}/2 \end{bmatrix} \]

**Theorem 3.2.** For all \( \lambda = -is^2 \) with \( \Im{s} > \epsilon > 0 \) it holds that

\[ \lim_{\lambda \to \infty} S(\lambda, x) = I. \]

**Proof:** For the first row of \( S \) notice that from (33), (34) it follows that

\[ f_1(x, s)e^{-i\lambda x} = 1 + o(1), \quad f_2(x, s)e^{i\lambda x} = 1 + o(1), \]

where \( o(1) \) means a function going to zero as \( s \) goes to infinity for all \( x \). For the second row, let us concentrate first on the derivative of \( f_1(x, s) \), which can be found from (31)

\[ f_1'(x, s) = is e^{i\lambda x} - A_1(x, x)e^{i\lambda x} + \int \frac{\partial}{\partial x} A_1(x, y) e^{i\lambda y} dy. \]

From where it follows that

\[ f_1'(x, s)e^{-i\lambda x} - i\lambda + A_1(x, x) = \int \frac{\partial}{\partial x} A_1(x, y) e^{i\lambda(y-x)} dy \]

Then one can estimate using (35) that for \( \Im{s} > 0 \)

\[ \left| \int \frac{\partial}{\partial x} A_1(x, y) e^{i\lambda(y-x)} dy \right| \leq \int \left| \frac{1}{4} q(x + y/2) + C \xi_1(x) \xi_1(x/2) e^{-\Im{s}(y-x)} dy \right| \leq \int \left| q(x + y/2) \right| dy \int e^{-\Im{s}(y-x)} dy + C \xi_1(x) \xi_1(-\infty) \int e^{-\Im{s}(y-x)} dy \leq K(x) \int e^{-\Im{s}(y-x)} = K(x) \frac{1}{\Im{s}}. \]

from where it follows that when \( \Im{s} \to +\infty \) the integral \( \int \frac{\partial}{\partial x} A_1(x, y) e^{i\lambda(y-x)} dy \) approaches zero.

For the case when \( \Im{s} > 0 \) is fixed, we notice that then

\[ \int \frac{\partial}{\partial x} A_1(x, y) e^{i\lambda(y-x)} dy = \int \frac{\partial}{\partial x} A_1(x, y) e^{-\Im{s}(y-x)} e^{i\Re{s}(y-x)} dy \]
and from the above calculation it follows that \( \frac{\partial}{\partial x} A_1(x, y) e^{-\Im s(y-x)} \) is \( L^1(\mathbb{R}) \) function. By the Riemann-Lebesgue lemma as \( \Re s \to \pm \infty \), the integral approaches zero too. Thus we conclude that

\[
f_1'(x, s)e^{-is\imath} - is + A_1(x, x) = o(1),
\]

and as a result, by the definition \( \triangledown M_i(x) \) of \( \beta \)

\[
(f_1' - \beta f_1)e^{-is\imath} = is - A_1(x, x) - \beta f_1 e^{-is\imath} + o(1) = is + o(1).
\]

Similarly one can show that

\[
(f_2' - \beta f_2)e^{is\imath} = -is + A_2(x, x) - \beta f_2 e^{2is\imath} + o(1) = -is + o(1).
\]

From where it follows the statement for the second row of \( S(\lambda, x) \).

Finally, we focus on the function \( S(\lambda, x) \) at the value \( x = 0 \). Then notice that \( f_i(0, s) = \bar{f}_i(0, -\bar{s}) \) \( (i = 1, 2) \). As a result we can define a function

\[
M_i(\mu) = f_i(0, \sqrt{\mu}),
\]

where we choose the root in such a manner that \( \Re \sqrt{\mu} = s \geq 0 \). Consequently, for \( \mu = s^2 \) it holds that \( \sqrt{\mu} = -\bar{s} \), which must be at the upper half plane, and

\[
M_i(\bar{\mu}) = f_i(0, \sqrt{\bar{\mu}}) = f_i(0, -\bar{s}) = \bar{f}_i(0, s) = M_i(\mu).
\]

Consequently, the function \( M_i(\lambda) \) is bounded for all \( \lambda \), has the value 1 at infinity (for big \( \mu \)), analytic at the whole complex plane except for a cut on the positive real axis, where it has a jump. Similarly, we define the functions, corresponding to the derivatives of \( f_i(x, s) \), \( M_i^0(0, \mu) = f_i'(0, \sqrt{\mu}) \).

Finally, we substitute \( \mu = -i\lambda \) and define the function

\[
S(\lambda, 0) = \begin{bmatrix} M_1(-i\lambda) + M_2(-i\lambda) & M_1(-i\lambda) - M_2(-i\lambda) \\ M_1^0(-i\lambda) + M_2^0(-i\lambda) & 2\sqrt{-i\lambda} M_1^0(-i\lambda) - M_2^0(-i\lambda) \end{bmatrix}
\]

where again, the function \( \sqrt{-i\lambda} \) may be defined except for a cut on the imaginary axis. Notice that

\[
S(\lambda, x) = \Phi_\ast(x, \lambda) S(\lambda, 0) \Phi^{-1}\ast(\lambda, x),
\]

Where \( \Phi_\ast(x, \lambda) \) is the fundamental solution of the output LDE, attaining \( I \) at \( x = 0 \).

Let us check next the symmetric condition. Notice that from Theorem \( \triangledown \) it is enough to check the symmetry of \( S(\lambda, 0) \) only. In order to do it, we notice that all the entries of \( S(\lambda, 0) \) are created from the functions \( f_i(x, s) \) satisfying \( \bar{f}_i(x, -\bar{s}) = f_i(x, s) \). As a result, it holds that

\[
M_i(i\lambda) = M_i(\bar{\mu}) = M_i(\mu) = M_i(-i\lambda)
\]

and similarly, \( M_i^0(i\lambda) = M_i^0(-i\lambda) \). Thus denoting \( S(\lambda, 0) = \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix} \) we obtain that

\[
\bar{a}(-\lambda) = a(\lambda), \quad \bar{b}(-\lambda) = -b(\lambda)
\]

and

\[
\bar{d}(-\lambda) = d(\lambda), \quad \bar{c}(-\lambda) = -c(\lambda).
\]
As a result the symmetry condition for \( S(\lambda, 0) \) becomes

\[
S^*(-\lambda, 0)\sigma_1 S(\lambda, 0) = \begin{bmatrix} a(-\lambda) & c(-\lambda) \\ b(-\lambda) & d(-\lambda) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{bmatrix} = \begin{bmatrix} c(-\lambda)a(\lambda) + a(-\lambda)c(\lambda) & c(\lambda)b(\lambda) + a(-\lambda)d(\lambda) \\ d(-\lambda)a(\lambda) + b(-\lambda)c(\lambda) & d(\lambda)b(\lambda) + b(-\lambda)d(\lambda) \end{bmatrix} = \begin{bmatrix} -c(\lambda)a(\lambda) + a(\lambda)c(\lambda) & -c(\lambda)b(\lambda) + a(\lambda)d(\lambda) \\ d(\lambda)a(\lambda) - b(\lambda)c(\lambda) & d(\lambda)b(\lambda) - b(\lambda)d(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & -c(\lambda)b(\lambda) + a(\lambda)d(\lambda) \\ d(\lambda)a(\lambda) - b(\lambda)c(\lambda) & 0 \end{bmatrix} = [d(\lambda)a(\lambda) - b(\lambda)c(\lambda)] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Thus we obtain

**Theorem 3.3.** The function \( \frac{1}{\det S(\lambda, 0)} S(\lambda, x) \) is in the class \( \mathcal{I}_r(\sigma_1, \sigma_2, \gamma, \gamma_*(x), I = \mathbb{R}) \). In other words, it realizes the given \( \gamma_*(x) \).

The following theorem was established for the finite dimensional case in [MG3]. We present now its generalization

**Theorem 3.4.** For Sturm Liouville vessel the following formula for \( \gamma_* \) holds

\[
\gamma_* = \gamma + \begin{bmatrix} i\frac{\tau^\prime}{\tau} & \frac{\tau^\prime}{\tau} \\ -\frac{\tau^\prime}{\tau} & 0 \end{bmatrix}
\]

**Proof:** Let us take

\[
\gamma_* = \begin{bmatrix} -i(\beta^\prime(x) - \beta^2(x)) & -\beta(x) \\ \beta(x) & i \end{bmatrix}
\]

If we denote \( -\beta = \frac{\tau^\prime}{\tau} \), then \( \gamma_* = \begin{bmatrix} \frac{\tau^\prime}{\tau} & \frac{\tau^\prime}{\tau} \\ -\frac{\tau^\prime}{\tau} & i \end{bmatrix} \) and we have to prove that \( -\beta = \frac{\tau^\prime}{\tau} \). Consider now the formula (27)

\[
\frac{\tau^\prime}{\tau} = \text{tr}(\sigma_2 B^*(x)X^{-1}(x)B(x)).
\]

Notice that the expression \( B^*(x)X^{-1}(x)B(x) \) is the first moment and from the general formula of the first moment appearing in [MV3 section 5] (in this article the moments are different from the moments in this article by multiplication on \( \sigma_1^{-1} \) from the right)

\[
H_0(x) = \begin{bmatrix} -\beta & \frac{r + i(\beta^\prime - \beta^2)}{2h_0^2} \\ r - i(\beta^\prime + \beta^2) & \frac{r + i(\beta^\prime - \beta^2)}{2h_0^2} \end{bmatrix}
\]

it follows that

\[
\frac{\tau^\prime}{\tau} = \text{tr}(\sigma_2 H_0(x)) = -\beta
\]

as desired. \( \square \)

**Corollary 3.5.** The following formula for the potential holds

\[
q(x) = -2d^2dx^2 \ln \tau(x).
\]
Proof: Immediate from (28).

This notion is important in the sense that conjecturally the matrix $S(\lambda, x), q(x)$ and the solutions of (15) may be represented from it and $e^{ix\tau}$, which are solutions of the input LDE (14). We make the following conjecture (generalization of [Mel, theorem 3])

**Conjecture 1.** The entries of the matrix $S(\lambda, x)$ are linear combinations of $\frac{\tau^{(n)}(x)}{\tau(x)}$ in some $p$-norm on $I$, in other words, the entries are of the form

$$\sum \alpha_n \frac{\tau^{(n)}(x)}{\tau(x)}, \quad \alpha_n \in \mathbb{C}.$$ 

### 3.1 Scattering data versus $S(\lambda, x_0)$, Gelfand-Levitan equation.

Following [Fad63] for the case $\int_0^\infty |x|q(x)|dx < \infty$ there are introduced Jost solutions $\phi(x, s)$ and $f(x, s)$ [Jos47, Lev49]

$$\phi(x, s) : \phi(0, s) = 0, \quad \phi'(0, s) = 1, \quad (41)$$

$$f(x, s) : \lim_{x \to \infty} e^{-isx} f(x, s) = 1. \quad (42)$$

and defining further $M(k) = \phi'(x, k)f(x, s) - f'(x, s)\phi(x, k)$ one reconstructs the potential $q(x)$ using Gelfand-Levitan equation [IMG51] (or alternatively Marchenko equation [Mc50]). There are two steps, essential for this construction, namely, one considers the case when the spectrum of the operator $L$ is purely continuous and the case when this spectrum additionally contains finite number of points. For the purely continuous case, one proves that there is a solution $K(x, y)$ of the Gelfand-Levitan equation [Fad63, (8.5)]

$$K(x, y) + \Omega(x, y) + \int_0^x K(x, t)\Omega(t, y)dt = 0, \quad x > y. \quad (43)$$

where $\Omega(x, y)$ is uniquely defined from $M(k)$ by [Fad63, (8.4)]

$$\Omega(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\sin(kx)}{k} \left[ \frac{1}{M(k)M(-k)} - 1 \right] \frac{\sin(ky)}{k} k^2 dk.$$ 

The formula for the potential is [Fad63, (10.4)] $q(x) = 2 \frac{dx}{dx} K(x, x)$. Then one make a modification, so that the discrete spectrum is taken into account [Fad63, (8.14, 8.15)]. We are going to present analogues of these formulas in our setting. Suppose that we are given a vessel (5)

$$\mathcal{K}_g = (A, B(x), X(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_\ast(x); \mathcal{H}, \mathcal{E}; I),$$

and let as fix an arbitrary $x_0 \in I$. Define

$$\Omega(x, y) = \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)X^{-1}(x_0)B(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (44)$$

and

$$K(x, y) = -\begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)X^{-1}(x_0)B(y) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (45)$$
Then Gel’fand-Levitan equation holds

\[ K(x, y) + \Omega(x, y) + \int_{x_0}^x K(t, y)\Omega(t, y)dt = \]

\[ = K(x, y) + \Omega(x, y) - \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(x)X^{-1}(x)B(t) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] - \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(t)X^{-1}(x_0)B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] dt = \]

\[ = K(x, y) + \Omega(x, y) - B^*(x)X^{-1}(x)\int_{x_0}^x B(t)\sigma_2B^*(t)dtX^{-1}(x_0)B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \]

using vessel condition \[{38}\] = \[= K(x, y) + \Omega(x, y) - B^*(x)\left( X(x) - X(x_0) \right)X^{-1}(x_0)B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \]

\[ = K(x, y) + \Omega(x, y) - \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(x)X^{-1}(x_0)B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(x)X^{-1}(x_0)B(y) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = 0. \]

Finally, the formula \[{25}\] for the potential gives

\[ q(x) = 2\frac{d}{dx}\sp\left( X^{-1}(x) \frac{d}{dx}\right) = \sp\left( X^{-1}(x)B(x)\sigma_2B^*(x) \right) = \]

\[ = 2\frac{d}{dx}\sp\left( X^{-1}(x)B(x) \right) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(x) = \frac{d}{dx}\left( \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B^*(x)X^{-1}(x)B(x) \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \right) = \]

\[ = 2\frac{d}{dx}K(x, x), \]

which is identical to \[\text{Fad63} (10.4)\].

### 3.3 Jost solutions.

In the sequel we will use the following number

\[ m(A) = \max\{\exists \lambda \mid \lambda \in \text{spec}(A)\}. \quad (46) \]

**Lemma 3.6.** For all \( x \in I \) it holds that \( X(x) \geq X_0 \). The operator \( B(x) \) satisfies the following

\[ \lim_{x \to \infty} B(x)e^{ixx} = 0, \quad \exists s > m(A). \]

**Proof:** The inequality is immediate from the formula \[{31}\]

\[ X(x) = X_0 + \int_{x_0}^x B(y)\sigma_2B(y)dy, \]

since \( \sigma_2 \geq 0 \) and positive operators form a convex set inside the space of all operators. For the second part, we can use Dunford-Schwartz calculus \[{12}\]

\[ B(x)e^{ixx} = \frac{1}{2\pi i} \int (\lambda I - A)^{-1}B_0\Phi(\lambda, x, x_0)d\lambda e^{ixx} = \]

\[ = \frac{1}{2\pi i} \int (\lambda I - A)^{-1}B_0 \left[ \begin{array}{c} \cos(tx) \\ i\sin(tx) \\ \cos(tx) \end{array} \right] d\lambda e^{ixx}, \quad \lambda = it^2. \]

Taking the norm of this expression, we shall obtain that

\[ \|B(x)e^{ixx}\| \leq \frac{1}{2\pi i} \int \|(\lambda I - A)^{-1}B_0\|\max_{t} |e^{i(s\pm t)x}|d\lambda \leq C(s)e^{-((\lambda_m - m(A))x)}, \]

25
Lemma 3.7. Suppose that we are given a vessel \( R_0 \) on a half axis \( \mathbb{I} = [x_0, \infty) \), then for \( \Im s > m(A) \) it holds that

\[
\int_{x_0}^{\infty} |f_1(y, s)|^2 dy = \frac{1}{\lambda + \lambda}.
\]

Proof: We shall use the fundamental matrices \( \Phi_x = \Phi_x(s, x, x_0) \), and \( \Phi = \Phi(s, x, x_0) \) of the output and the input LDEs respectively. Then

\[
f_1(x, s)f_1(x, s) = \begin{bmatrix} 1 & \bar{s} \end{bmatrix} \Phi^*(\lambda, x) \begin{bmatrix} 0 & 1 \end{bmatrix} \Phi(\lambda, x) \begin{bmatrix} 1 & 0 \end{bmatrix} S(\lambda, x) \begin{bmatrix} 1 & 0 \end{bmatrix} f(s).
\]

Integrating the last equation and using additionally the formulas

\[
\int_{x_0}^{\infty} |f_1(y, s)|^2 dy = \frac{1}{\lambda + \lambda} \begin{bmatrix} 1 & \bar{s} \end{bmatrix} \frac{\Phi^* \sigma_1 \Phi_{x} \lambda + \lambda}{\lambda} S(\lambda, 0) \begin{bmatrix} 1 & 0 \end{bmatrix} f(s).
\]

where \( C(s) \) is a constant function, depending on \( s \) only. When \( x \) tend to infinity, we obtain the desired result.

Using the transfer function \( S(x, s) \), we will look for the Jost solutions, defined from the following formulas

\[
\phi(x, s) = \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \Phi(x, s) \begin{bmatrix} \phi_1(s) \\ \phi_2(s) \end{bmatrix}, \tag{47}
\]

\[
f(x, s) = \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \Phi(x, s) \begin{bmatrix} 1 \\ s \end{bmatrix} f(s), \tag{48}
\]

where \( \phi_1(s), \phi_2(s), f(s) \) are functions, which must be found. Solving for the \( \phi_1, \phi \) so that the condition \( \text{(47)} \) are fulfilled, one will come to the conclusion that

\[
\phi(x, s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \Phi_*(\lambda, x, x_0) \begin{bmatrix} 0 \\ -1 \end{bmatrix}.
\]

Particularly, the Jost solution \( \phi(\lambda, x) \) is an entire function of \( \lambda \) for each \( x \). The choice for the function \( f(x, s) \) comes from the following formula:

\[
f(x, s) = \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \Phi(\lambda, x, x_0) \begin{bmatrix} 1 \\ s \end{bmatrix} f(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \begin{bmatrix} e^{ixs} \\ se^{ixs} \end{bmatrix} f(s).
\]

So if we want to satisfy the condition of the Jost solution, we will demand that

\[
\lim_{x \to \infty} e^{-ixs} f(x, s) = \lim_{x \to \infty} \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \begin{bmatrix} 1 \\ s \end{bmatrix} f(s) = 1.
\]

Let us define \( f_1(x, s) = \begin{bmatrix} 1 & 0 \end{bmatrix} S(x, s) \Phi(\lambda, x, x_0) \begin{bmatrix} 1 \\ s \end{bmatrix} \), then the following lemma holds

**Lemma 3.7.** Suppose that we are given a vessel \( R_0 \) on a half axis \( \mathbb{I} = [x_0, \infty) \), then for \( \Im s > m(A) \) it holds that

\[
\int_{x_0}^{\infty} |f_1(y, s)|^2 dy = \frac{1}{\lambda + \lambda}.
\]
Using here the expression (24) for $\Phi$:

\[
\int_0^s |f_1(y, s)|^2 dy = \left[ 1 \quad \bar{s} \right] \frac{S^*(\lambda, x) \sigma_1 S(\lambda, x)}{\lambda + \lambda} \left[ 1 \quad s \right] e^{i(s-\bar{s})x} - \left[ 1 \quad s \right] \frac{S^*(\lambda, 0) \sigma_1 S(\lambda, 0)}{\lambda + \lambda} \left[ 1 \quad s \right]
\]

\[
= \left[ 1 \quad \bar{s} \right] \frac{S^*(\lambda, x) \sigma_1 S(\lambda, x) - \sigma_1}{\lambda + \lambda} \left[ 1 \quad s \right] e^{i(s-\bar{s})x} + \left[ 1 \quad \bar{s} \right] \frac{\sigma_1 e^{i(s-\bar{s})x} - S^*(\lambda, 0) \sigma_1 S(\lambda, 0)}{\lambda + \lambda} \left[ 1 \quad s \right].
\]

Let us focus on the expression

\[
\left[ 1 \quad \bar{s} \right] \frac{\sigma_1 - S^*(\lambda, x) \sigma_1 S(\lambda, x)}{\lambda + \lambda} \left[ 1 \quad s \right] = \sigma_1 B^*(x)(\lambda I - A^*)^{-1}X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1.
\]

From Lemma 3.6 for all $x \in I$ it holds that $X(x) \geq X(x_0)$ so

\[
\sigma_1 B^*(x)(\lambda I - A^*)^{-1}X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1 \leq \sigma_1 B^*(x)(\lambda I - A^*)^{-1}X^{-1}(x_0)(\lambda I - A)^{-1}B(x)\sigma_1
\]

Taking the norm, we shall obtain that

\[
\|\sigma_1 B^*(x)(\lambda I - A^*)^{-1}X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1\| \leq \|\sigma_1 B^*(x)(\lambda I - A^*)^{-1}X^{-1}(x_0)(\lambda I - A)^{-1}B(x)\sigma_1\| \leq K(\lambda)\|B(x)\|^2, \quad K(\lambda) \text{- } x \text{- independent.}
\]

Multiplying this inequality by $e^{i(s-\bar{s})}$ and using condition on $\|B(x)\|$ in Lemma 3.6 we obtain that for $\exists s > m(A)$

\[
\lim_{x \to \infty} \|\left[ 1 \quad \bar{s} \right] \frac{\sigma_1 - S^*(\lambda, x) \sigma_1 S(\lambda, x)}{\lambda + \lambda} \left[ 1 \quad s \right] e^{i(s-\bar{s})x} \| \leq 0
\]

Plugging this into equation (49) we obtain that

\[
\int_0^\infty |f_1(y, s)|^2 dy = \left[ 1 \quad \bar{s} \right] \frac{S^*(\lambda, 0) \sigma_1 S(\lambda, 0)}{\lambda + \lambda} \left[ 1 \quad s \right].
\]

Let us define the following expression, which is essential for the existence of the Jost solution $f(x, s)$

\[
h(x, s) = \frac{f(x, s)}{f(s)} e^{-ixs} = \left[ 1 \quad 0 \right] S(x, s) \left[ 1 \quad s \right]
\]

Let as also denote

\[
K_S(x, s) = \left[ 1 \quad \bar{s} \right] \frac{S^*(\lambda, x) \sigma_1 S(\lambda, x)}{\lambda + \lambda} \left[ 1 \quad s \right]
\]

then the following theorem holds:

**Theorem 3.8.** Suppose that we are given a vessel $\mathcal{B}$(\ref{B}), then the function

\[
h(x, s) = \frac{f(x, s)}{f(s)} e^{i\Theta_S(x, s)}
\]

defined in (50) has the following properties

1. $h^*(x, -\bar{s}) = h(x, s) \det S(\lambda, x_0)$,

2. $|h(x, s)|^2 = \frac{\partial}{\partial x} K_S(s, x) + i(s - \bar{s}) K_S(x)$,
3. \( \frac{2}{\partial x} \Theta_h(x, s) = -\frac{s + \bar{s}}{|h(x, s)|^2} \frac{\partial}{\partial x} K_S(x, s). \)

**Proof:** Let us denote \( \Phi = \Phi(\lambda, x, x_0), \Phi_s = \Phi_s(\lambda, x, x_0), \) then

\[
\bar{h}(x, -\bar{s}) = [1 \quad -\bar{s}] S^*(x, -\lambda) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \quad -s] \sigma_1 S^{-1}(x, \lambda) \sigma_1^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \]

\[
[\begin{array}{cc} -s & 1 \\ 1 & 0 \end{array}] S^{-1}(x, \lambda) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \text{denoting } S(x, \lambda) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

\[
[\begin{array}{cc} d & -b \\ -c & a \end{array}] \frac{1}{\det S(x, \lambda)} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \]

\[
(a + sb) \frac{1}{\det S(x, \lambda)} = h(x, s) \frac{1}{\det S(\lambda, x)} = h(x, s) \frac{1}{\det S(\lambda, x_0)}.
\]

Using the formula (29) for \( \Phi \), we can calculate that \( \Phi h(x, s) \frac{1}{h(x, s)} = e^{i \pi x}. \) As a result, using this and (16), the formula for \( h^*(x, s) h(x, s) = |h(x, s)|^2 \) becomes

\[
|h(x, s)|^2 = \frac{\partial}{\partial x} \left( \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \sigma_1 S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} e^{i(s-\bar{s})} \right) =
\]

\[
= \frac{\partial}{\partial x} \left( \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \sigma_1 S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} \right) + i(s - \bar{s}) \lambda + \lambda
\]

\[
= \frac{\partial}{\partial x} K_S(s, x) + i(s - \bar{s}) K_S(x).
\]

3. Using the formula (17) we find that

\[
\bar{h}(x, s) h'(x, s) - \bar{h}'(x, s) h(x, s) = \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \begin{bmatrix} 1 & 0 \\ \beta(x) & i \end{bmatrix} S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} =
\]

\[
- \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \begin{bmatrix} \beta(x) \quad -i \\ 1 \end{bmatrix} S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} - i(s + \bar{s}) h(x, s) h(x, s) =
\]

\[
i \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \sigma_1 S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} - i(s + \bar{s}) h(x, s) h(x, s).
\]

Dividing by \( i h(x, s) h(x, s) \), we will obtain that the last formula is

\[
2 \Theta_h(x, s) = \frac{1}{h(x, s) h(x, s)} \begin{bmatrix} 1 & \bar{s} \end{bmatrix} S^*(\lambda, x) \sigma_1 S(\lambda, x) \begin{bmatrix} 1 \\ s \end{bmatrix} - (s + \bar{s}) =
\]

\[
= \frac{\lambda + \bar{\lambda}}{h(x, s) h(x, s)} K_S(x, s) - (s + \bar{s}).
\]
Finally, using part 2 we obtain that \( \frac{|h(x,s)|^2 - \frac{\partial}{\partial x} K_S(s,x)}{i(s-\bar{s})} = K_S(x) \), which plugged into the last formula gives

\[
2\Theta'_h(x) = \frac{\lambda + \bar{\lambda}}{h(x,s)h(x,s)} \frac{|h(x,s)|^2 - \frac{\partial}{\partial x} K_S(s,x)}{i(s-\bar{s})} - (s+\bar{s}) = -\frac{s+\bar{s}}{|h(x,s)|^2} \frac{\partial}{\partial x} K_S(x,s).
\]

Important corollary of this theorem is a sort of independence of the formulas for \( h(x,s) \) on the realization we choose for the vessel:

**Corollary 3.9.** Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be two vessels realizing the same potential \( q(x) \). Suppose that 

\[
h(x,s) = |h(x,s)|e^{i\Theta_h(x,s)} \quad \text{and} \quad \tilde{h}(x,s) = |\tilde{h}(x,s)|e^{i\tilde{\Theta}_h(x,s)}
\]

are the corresponding functions, defined by (50). Then there exists a function \( H(s) \), \( x \)-independent so that 

\[
|h(x,s)|^2 = |\tilde{h}(x,s)|H(s), \quad \Theta'_h(x,s) = \tilde{\Theta}'_h(x,s).
\]

**Proof:** From Theorem 3.1 it follows that there exist two analytic functions 

\[
a(\lambda), c(\lambda),
\]

which are zero at infinity such that for (30)

\[
Y(\lambda) = I + \begin{bmatrix} a(\lambda) & ic(\lambda) \\ c(\lambda) & a(\lambda) \end{bmatrix}
\]

it holds \( \tilde{S}(\lambda,x) = S(\lambda,x)Y(\lambda) \), for the corresponding transfer functions. Then we calculate

\[
\tilde{K}_S(s,x) = \frac{1}{\lambda + \bar{\lambda}} \left[ \begin{array}{cc} 1 & s \\ \bar{s} & \lambda + \bar{\lambda} \end{array} \right] \tilde{S}^*(\lambda,x)S(\lambda,x) \left[ \begin{array}{c} 1 \\ s \end{array} \right] = \frac{1}{\lambda + \bar{\lambda}} Y^*(\lambda)S^*(\lambda,x)\sigma_1 S(\lambda,x)Y(\lambda) \left[ \begin{array}{c} 1 \\ s \end{array} \right].
\]

Notice that

\[
Y(\lambda) \left[ \begin{array}{c} 1 \\ s \end{array} \right] = \left[ \begin{array}{cc} 1 + a(\lambda) & \frac{ic(\lambda)}{c(\lambda)} \\ c(\lambda) & 1 + a(\lambda) \end{array} \right] \left[ \begin{array}{c} 1 \\ s \end{array} \right] = (1 + a(\lambda) + c(\lambda)) \left[ \begin{array}{c} 1 \\ s \end{array} \right],
\]

from where the result follows denoting \( H(s) = |1 + a(\lambda) + \frac{c(\lambda)}{s}|^2 \) and the formula

\[
\tilde{K}_S(s,x) = H(s)K_S(s,x).
\]

As a corollary of this theorem, we obtain a necessary conditions on \( K_S(x,s) \) following from the existence of the Jost solution \( f(x,s) \):

**Corollary 3.10.** Suppose that we are given a vessel \( \mathcal{V}_1 \), existing on a half-line \( I = [x_0, \infty) \). Suppose also that for some value \( s \), the Jost solution \( f(x,s) \) exists, i.e. satisfies condition (42). Then

1. \( \lim_{x \to \infty} K_S(x,s) \) exists,
2. \( \lim_{x \to \infty} \frac{\partial}{\partial x} K_S(x,s) \) exists,
**Proof:** From the Jost condition (42) it follows that there exist two limits: \( \lim_{x \to \infty} |h(x,s)| \) and \( \lim_{x \to \infty} \Theta_h(x,s) \). Then integrating part 3 of Theorem 3.8 it follows that

\[
2(\lim_{x \to \infty} \Theta_h(x,s) - \Theta_h(x_0,s)) = -\int_{x_0}^{\infty} \frac{s + \bar{s}}{|h(y,s)|^2} \frac{\partial}{\partial y} K_S(y,s)dy.
\]

Thus the improper integral on the right hand side exists. As a result its integrand satisfies the necessary condition of the convergence

\[
\lim_{x \to \infty} \frac{s + \bar{s}}{|h(x,s)|^2} \frac{\partial}{\partial x} K_S(x,s) = 0.
\]

Dividing next part 2 of Theorem 3.8 we find that

\[
1 = \frac{\partial}{\partial x} K_S(x,s) \left| \frac{s + \bar{s}}{|h(x,s)|^2} + i(s - \bar{s}) K_S(x,s) \frac{|h(x,s)|^2}{|h(x,s)|^2} \right.
\]

Taking \( x \) approaching infinity on both sides we find that

\[
\lim_{x \to \infty} i(s - \bar{s}) K_S(x,s) \frac{|h(x,s)|^2}{|h(x,s)|^2} = 1,
\]

or that (since \( h(\infty, s) \neq 0 \))

\[
\lim_{x \to \infty} K_S(x,s) = \frac{|h(\infty,s)|^2}{i(s - \bar{s})}.
\]

Finally, using again part 2 of Theorem 3.8

\[
\frac{\partial}{\partial x} K_S(s,x) = |h(x,s)|^2 - i(s - \bar{s}) K_S(x),
\]

we obtain that since the right hand side has a limits as \( x \) approaches infinity, so does the left. \( \square \)

### 4 Applications

#### 4.1 SL vessels with a spectrum on the imaginary positive axis.

When the spectrum of the operator \( A \) is on the imaginary axis, it means that \( m(A) \), defined in (46) is zero:

\[
m(A) = \max \{ \Im \lambda \mid \lambda \in \text{spec}(A) \} = 0.
\]

Let us consider a vessel (see Definition 2.3)

\[
K_V = (A,B(x),\mathcal{H}(x); \sigma_1(x), \sigma_2(x), \gamma(x), \gamma_*(x); \mathcal{H}, \mathcal{E}; 1),
\]

which has an additional restriction, identical to the classical case, namely, the operator \( A \) has all its spectrum on the imaginary positive axis:

\[
\text{spec} A \subseteq i \mathbb{R}^+.
\]

(52)

We can find more accurate bounds then in Lemma 3.10 on norms of vessel operators. Starting from

\[
B(x) = \frac{1}{2\pi i} \oint (\lambda I - A)^{-1} B_0 \Phi^i(\lambda, x, x_0) d\lambda
\]

30
and denoting \( K = \sqrt{-iA} > 0 \) (exists from 52), we obtain

\[
B(x) = \cos(Kx)B(x_0) + iK \sin(Kx)B(x_0) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (K)^{-1} \sin(Kx)B(x_0) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

\[
\mathcal{X}(x) = \mathcal{X}(x_0) + \int_{x_0}^{x} \left\{ \cos(Ky)B_0\sigma_2B_0^* \cos(Ky) + K^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_0^* \sin(Ky)K^{-1} - iK \sin(Ky)B_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_0^* \sin(Ky)K^{-1} + iK^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B^*(x_0) \cos(Ky) \right\} dy.
\]

From where we obtain the following bounds:

**Theorem 4.1.** Let \( \mathcal{A}_B \) be a vessel, existing on \([x_0, \infty)\), for which \( A \) satisfies condition 52. Then on the interval \([x_0, \infty)\) the following bounds hold

1. \( \|B(x)\| \leq B_1 \) and \( \|\mathcal{X}(x)\| \leq \|\mathcal{X}(x_0)\| + B_2(x-x_0) \) for some \( B_1, B_2 \),

2. \( \text{tr}(\mathcal{X}(x) - \mathcal{X}(x_0)) \geq T_1x + T_2 \) for some constants \( T_1 > 0, T_2 \).

**Proof:** 1. Examining the formulas above, \( B(x) \) is determined using formulas of the form \( \cos(Ky) \), \( \sin(Ky) \), which are bounded by 1 for a positive \( K \). Integrating further these expressions, we will obtain the bound for \( \|\mathcal{X}(x)\| \).

2. Consider the following calculation

\[
\text{tr}(\mathcal{X}(x) - \mathcal{X}_0) = \int_{x_0}^{x} \cos(Ky)B_0\sigma_2B_0^* \cos(Ky) + K^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_0^* \sin(Ky)K^{-1} - iK \sin(Ky)B_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} B_0^* \sin(Ky)K^{-1} + iK^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B^*(x_0) \cos(Ky) \right\} dy =
\]

\[
= \int_{x_0}^{x} \text{tr}(B_0\sigma_2B_0^* \cos(Ky) \cos(Ky) + B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_0^* \sin(Ky)K^{-1} + iK^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B^*(x_0) \cos(Ky) \right\} dy =
\]

\[
= \int_{x_0}^{x} \text{tr}(B_0\sigma_2B_0^* \cos(Ky) \cos(Ky) + B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B_0^* \sin(Ky)K^{-1} + iK^{-1} \sin(Ky)B_0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} B^*(x_0) \cos(Ky) \right\} dy =
\]

where \( T_1 > 0 \) and \( T_2(x) \) is uniformly bounded for all \( x \). So defining \( T_2 = -\inf T_2(x) \), we will obtain the desired result.

More accurate bounds are obtained in the dissipative case and are presented in the next theorem.

**Theorem 4.2.** Let \( \mathcal{A}_B \) be a dissipative vessel, existing on \([x_0, \infty)\), for which \( A \) satisfies condition 52. Then, on the interval \([x_0, \infty)\) the following bounds hold

1. \( \tau(x) \geq T_1(x-x_0) + T_2 \) for some constants \( T_1, T_2 > 0 \),

2. \( \|X^{-1}(x)\| \leq \frac{1}{\tau(x)} \leq \frac{1}{T_1(x-x_0) + T_2} \),

3. \( |q(x)| \leq \frac{Q}{(x-x_0)} \) for some \( Q \) and for big enough \( x \).
Proof: 1. Notice that using Theorem 4.1 we may suppose that $x_0 = I$. Then, using the formula
\[ \det(I + F) > 1 + \text{tr}F \]
for a trace class positive operator $F$ and part 2 of Theorem 4.1 we will obtain the bound for $\tau(x)$.

2. Follows from the following chain of inequalities
\[ \|X^{-1}(x)\| \leq \|I\| + \|X^{-1}(x) - I\| \leq 1 + \|X^{-1}(x) - I\|_1 = 1 + \text{tr}(X^{-1}(x) - I) \leq \frac{1}{\det(X(x))} = \frac{1}{\tau(x)} \leq \frac{1}{T_1(x - x_0) + T_2}. \]

3. Recall the formula (28) for the potential. Differentiating it and using vessel equations we will arrive to
\[ -\frac{1}{2}q(x) = \frac{d}{dx}\text{tr}(X(x)X^{-1}(x)) = \text{tr}\left( \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} B^*(x)X^{-1}(x)B(x) - \left( \begin{bmatrix} 1 & 0 \end{bmatrix} B^*(x)X^{-1}(x)B(x) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right)^2. \]

using part 1 of Theorem 4.1 and part 2 of this Theorem we obtain that
\[ |q(x)| \leq Q_1\|X^{-1}\|_\infty + Q_2\|X^{-1}\|_\infty \leq \frac{Q_1}{T_1(x - x_0) + T_2} + \frac{Q_2^2}{(T_1(x - x_0) + T_2)^2} \leq \frac{Q}{(x - x_0)} \]
for big enough $x$.

4.2 Models of vessels with a spectrum on a symmetric curve $\Gamma$.

Let us construct a vessel, for which the operator $A$ is diagonal.

1. Suppose that we are given a smooth curve in the complex domain, parametrized by
\[ \Gamma = \{ \mu(t) \mid a \leq t \leq b \} \]

Let us also suppose that the curve is symmetric with respect to the imaginary axis, i.e.
\[ \Gamma = -\Gamma^* \]. The reason why we need to require it is the symmetry condition (13), which means that if $\lambda_0$ is an analytic point of $S(\lambda, x)$ and $S(\lambda, x) = \sum \alpha_n(x)(\lambda - \lambda_0)^n$, then
\[ S^*(\bar{\lambda}, x) = \sum \sigma_1(x)\alpha_n(x)\sigma_1^{-1}(x)(\lambda - \lambda_0)^n, \]
and consequently, taking the conjugate
\[ S(\bar{\lambda}, x) = \sum \sigma_1^{-1}(x)\alpha_n(x)\sigma_1(x)(-1)^n(-\bar{\lambda} + \bar{\lambda}_0)^n, \]
where substituting $-\bar{\lambda} = \mu$, we will obtain that $S(\mu, x)$ is analytic at $-\bar{\lambda}_0$. Particularly, it can’t be a singular point of $S(\lambda, x)$. Since we will verify that the singularities occur on the curve $\Gamma$, this means that the curve must be symmetric. The inner space is defined as
\[ \mathcal{H} = L^2(\Gamma) = \{ f(\mu) \mid \int_a^b |f(\mu(t))|^2dt < \infty \}. \]

with the inner product
\[ \langle f(\mu), g(\mu) \rangle_{\mathcal{H}} = \int_a^b g^*(\mu(t))f(\mu(t))dt. \]
2. Define the operator $A$ as the multiplication operator on a function $\mu$:
\[ Af(\mu) = \mu f(\mu) \]

3. Then $B(x)$ is a solution of
\[ 0 = \frac{d}{dx}(B(x)\sigma_1(x)) + AB(x)\sigma_2(x) + B(x)\gamma(x). \]

For example, in the SL case, we shall obtain that $B(x)$ is an operator by multiplication on $B(\mu, x) = \left[ c_1(\mu) \quad c_2(\mu) \right] \Phi(\mu, x, x_0)$, for "good" (satisfying (11)) functions $c_1(\mu), c_2(\mu)$. In order to verify that the singularities of the final transfer function will occur exactly on the curve, we need the minimality condition, which must be satisfied at $x_0$ by Theorem 2.13. Since $A = \mu$, it is enough to demand that $B(\mu, x_0) \neq 0$ for all $\mu$, since then the condition (11) is immediate.

Notice that from the definition it follows that the adjoint of $B(x)$ is
\[ B^*(x)f(\mu) = \int_a^b B^*(\mu(t), x)f(\mu(t))dt. \]

4. Define the operator $X(x)$ as follows
\[ X(x)f(\mu) = \int_a^b \frac{B(\mu, x)\sigma_1B^*(\delta(t), x)}{\mu + \delta(t)}dt. \]

Notice that in order to obtain a well-defined operator, we have to verify that the integral converges. For this to hold, we need to verify that for values of $\delta$, where $\mu + \delta(t) = 0$, it holds that
\[ \frac{B(\mu, x)\sigma_1B^*(\delta(t), x)}{\mu + \delta(t)} \]
is integrable. One can demand for that that the Hölder condition [Mus41] is satisfied, for example. Moreover, if the curve $\Gamma$ is unbounded, we have to choose $\delta = B(\mu) = B(\mu, a)$ such that $\|\mu^*B(\mu)\| \leq C^\alpha$ for a constant $C$. For this it is enough to choose $B(\mu)$ to be a Schwartz function on the curve $\Gamma$.

**Theorem 4.3.** The following collection
\[ \mathcal{Q}_d = (A = \mu, B(\mu, x), X(x); \sigma_1, \sigma_2, \gamma, \gamma_\star(x); \mathcal{H} = L^2(\Gamma), \mathcal{E}), \]
for $\gamma_\star(x)$, defined by the linkage condition (10), is a vessel.

**Proof:** Equation (54) is satisfied by the construction of $B(\mu, x)$. Lyapunov equation (7) follows from the definitions
\[ (A_1X(x) + X(x)A^*)f(\mu) = \]
\[ = \mu \int_a^b B(\mu, x)\sigma_1B^*(\delta(t), x)\frac{f(\delta(t))dt}{\mu + \delta(t)} + \int_a^b \sigma_1B^*(\delta(t), x)\frac{A^*(\delta(t))f(\delta(t))dt}{\mu + \delta(t)} = \]
\[ = \int_a^b \frac{B(\mu, x)\sigma_1B^*(\delta(t), x)}{\mu + \delta(t)}(\mu + \delta(t))f(\delta(t))dt = \]
\[ = \int_a^b B(\mu, x)\sigma_1B^*(\delta(t), x)f(\delta(t))dt = B(x)\sigma_1B^*(x)f(\mu). \]
Similarly, equation (8) holds. Finally, \( \sigma_\ast \) serves to define \( \gamma_\ast(x) \).

Finally notice that the transfer function of this vessel is

\[
S(\lambda, x) = I - B^*(x)X^{-1}(x)(\lambda I - A)^{-1}B(x)\sigma_1(x)
\]

\[
= I - \int_a^b \frac{B^*(\mu(t), x)X^{-1}(x)B(\mu(t), x)}{\lambda - \mu(t)} \sigma_1(x) dt
\]

and has singularities (jumps) exactly on the curve \( \Gamma \) by the minimality condition.

4.3 NLS equations

The first part of the article suggests that one can construct vessels with a prescribed spectrum for a wider class of vessel parameters. For example one can study Non-Linear Schrödinger (NLS) equations presented by A.P. Fordy, P.P. Kulish in [AF83]. The classical NLS equation corresponds to the following parameters:

Definition 4.1. The Non-Linear Schrödinger equation parameters are defined to be

\[
\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
\gamma_\ast(x) = \begin{bmatrix} 0 & \beta(x) \\ -\beta^*(x) & 0 \end{bmatrix}.
\]

and the regularity assumptions can be taken as in the SL case. The output compatibility conditions take the form of the classical non-linear Schrödinger equation with the spectral parameter \( i\lambda \)

\[
\frac{\partial}{\partial x} u(x, \lambda) = (i\lambda A + Q(x))u(x, \lambda),
\]

where

\[
I = \sigma_1, \quad A = -\frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = -i\sigma_2, \quad Q(x) = -\gamma_\ast(x).
\]

A more complicated example is [AF83, 3.19] as follows

\[
\frac{\partial}{\partial x} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} i\lambda & 0 & q_1 & q_2 \\ 0 & \frac{1}{2} i\lambda & q_4 & q_3 \\ -q_1^* & -q_4^* & -\frac{1}{2} i\lambda & 0 \\ -q_2^* & -q_3^* & 0 & \frac{1}{2} i\lambda \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}
\]

and corresponds to the following vessel parameters (changing the spectral parameter \( \lambda \) to \(-i\lambda\)).

\[
\sigma_1 = I, \quad \sigma_2 = \text{diag} \left\{ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\}, \quad \gamma = 0, \quad \gamma_\ast(x) = \begin{bmatrix} 0 & 0 & q_1 & q_2 \\ 0 & 0 & q_4 & q_3 \\ -q_1^* & -q_4^* & 0 & 0 \\ -q_2^* & -q_3^* & 0 & 0 \end{bmatrix}.
\]

One can use models, presented in section 4.2 for the study of the Scattering Theory of these NLS equations.

It is important to notice that the connection between the corresponding to NLS parameters vessels and the Lie algebras appearing in [AF83] is of great interest.
4.4 Canonical Systems

Starting from a standard model of a canonical system \[\text{[Fad74]}\]

\[
\left[J \frac{d}{dx} + Q(x)\right] \phi(x,k) = k \phi(x,k)
\]

where \( J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), \( Q(x) = \begin{bmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{bmatrix} \) notice that multiplying this equation by \( i \), we will obtain a differential equation, which fits the setting of a vessel:

\[
\left[ \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \frac{d}{dx} - ik \phi(x,k) \right] - \begin{bmatrix} -ip(x) & -iq(x) \\ -iq(x) & ip(x) \end{bmatrix} \phi(x,k) = 0.
\]

**Definition 4.2.** The canonical system vessel parameters are

\[
\sigma_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma_2 = I, \quad \gamma = 0, \quad \gamma_\ast(x) = \begin{bmatrix} -ip(x) & -iq(x) \\ -iq(x) & ip(x) \end{bmatrix}.
\]

Requiring \[\text{[Fad74] 2.2}\]

\[
\int_{-\infty}^{\infty} |q(x)|dx < \infty, \quad \int_{-\infty}^{\infty} |p(x)|dx < \infty
\]

we will obtain a vessel with a spectrum on a cut of the imaginary positive axis, imitating the construction for SL case and using the formulas from \[\text{[Fad74] section 2}\]. The general case will produce an interesting class of potentials \( Q(x) \) in this case too.

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