GEODESIC NETS ON NON-COMPACT RIEMANNIAN MANIFOLDS

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Abstract. A geodesic flower is a finite collection of geodesic loops based at the same point \( p \) that satisfy the following balancing condition: The sum of all unit tangent vectors to all geodesic arcs meeting at \( p \) is equal to the zero vector. In particular, a geodesic flower is a stationary geodesic net.

We prove that in every complete non-compact manifold with locally convex ends there exists a non-trivial geodesic flower.

1. Main results.

We begin with the following definition:

Definition 1.1. Let \( M \) be a complete non-compact Riemannian manifold, and \( \delta \) a positive number. Assume that there exists a finite collection of disjoint connected closed piecewise smooth hypersurfaces \( \Sigma_i, i = 1, 2, \ldots \), that divide \( M \) into a disjoint union of open submanifolds \( M_0, M_1, \ldots, M_e \) (for some integer \( e \)) so that:

1. \( M_0 \) is bounded, and \( M_i \) are unbounded for all \( i > 1 \);
2. If \( i > 0 \), then the boundary of \( M_i \) is \( \Sigma_i \). The boundary of \( M_0 \) is the union of all hypersurfaces \( \Sigma_i \);
3. (Locally convex ends condition) For each \( i \geq 1 \) there exists a positive \( \varepsilon \) such that each minimizing geodesic connecting every pair of \( \delta \)-close points of \( \Sigma_i \) in \( M \) is, in fact, in \( M_i \cup \Sigma_i \).

Then we say that \( M \) is a Riemannian manifold with \( \delta \)-locally convex ends, and \( M_0 \) its core. If \( M \) has \( \delta \)-locally convex ends for some \( \delta > 0 \), we say that \( M \) has locally convex ends.

Informally speaking, this condition means that each end of \( M \) can be cut off by a smooth bounded hypersurface that is locally convex to infinity.

We conjecture that:

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Conjecture 1.1. There exists a non-constant periodic geodesic on each complete non-compact Riemannian manifold with locally convex ends.

To put this conjecture in proper perspective note that the well-known Fet-Lyusternik theorem asserts that there exists a periodic geodesic on each closed Riemannian manifold. However, this is no longer true in the non-compact case. The most obvious counterexamples are Euclidean spaces (or, more generally, $\mathbb{R}^n$ endowed with any Riemannian metric of non-positive curvature.) However, it is easy to exhibit examples with non-trivial topology. For example, one can consider a surface of revolution in $\mathbb{R}^3$ obtained by rotating the hyperbola $z = \frac{1}{x}, x > 0$ in the $XZ$-plane about the $Z$-axis. The fundamental group of this surface is $\mathbb{Z}$, yet it is easy to see (using, for example, Clairaut’s theorem) that there are no non-constant periodic geodesics on this surface. It is easy to see why the standard existence proof developed for the compact case does not work in this case: When we take a non-contractible closed curve and start shrinking it, the curve slides to infinity, and does not converge to any limiting closed curve.

Our main result asserts that the assumptions of this conjecture imply the existence of a closed geodesic net on the manifold, and, moreover of a geodesic flower.

Recall, that geodesic nets on $M^n$ are critical points on the space of immersed (multi)graphs in $M^n$. More formally, a geodesic net $N$ is a finite collection of points $v_i$ on $M^n$ (vertices of the net) and (not necessarily different) smooth geodesics $\gamma_j$ starting and ending at vertices $v_i$ of the net, where both ends of $\gamma_j$ can be the same vertex, so that for each vertex $v_i$ the sum of the unit tangent vectors at $v_i$ to all geodesics $\gamma_j$ starting at $v_i$ is equal to the zero vector in $T_{v_i}M^n$. Here we orient all the tangent vectors to $\gamma_j$ at $v_i$ from $v_i$ towards the other end of $\gamma_j$. The geodesics $\gamma_i$ are called edges of a geodesic net. All periodic geodesics are geodesic nets. (One can choose the corresponding multigraph as the graph with one vertex and one edge.) Further, any union of a finite set of periodic geodesics is a geodesic net. Yet a geodesic loop is a geodesic net if and only if it is a periodic geodesic. Similarly to periodic geodesics, geodesic nets are rare: for a generic Riemannian metric on a closed manifold the set of geodesic nets is countable [St]. Density and equidistribution results for geodesic nets in Riemannian manifolds were proved recently in [LS] and [LiS].

Recently O. Chodosh and C. Mantoulidis showed [CM] that geodesic nets arising from Almgren-Pitts Min-Max theory on surfaces are closed geodesics. Very little is known about the existence of geodesic nets that are not unions of periodic geodesics. The only exception is a result of J. Hass and F. Morgan [HM] asserting that for each convex Riemannian metric on $S^2$ close to a round metric there exists a $\theta$-graph shaped geodesic net. (The underlying graph consists of two vertices and three edges. In the geodesic net all edges must have non-zero length, and the stationarity condition
means that all angles between edges at each of two vertices are equal to $\frac{2\pi}{3}$). On the other hand, it has been proven in [NR] that the length of a shortest geodesic net on a closed Riemannian manifold $M^n$ does not exceed $c(n)\text{vol}(M^n)^{\frac{1}{n}}$, where $\text{vol}(M^n)$ denotes the volume of $M^n$, and $c(n)$ is an (explicit) constant that depends only on the dimension. It also does not exceed $c(n)\text{diam}(M^n)$, where $\text{diam}(M^n)$ denotes the diameter of $M^n$. Later it had been proven in [R2] (see also a later improvement in [R3]) that these results also hold for a special class of geodesic nets, namely geodesic flowers. A geodesic flower is a geodesic net that consists of a finite number of geodesic loops based at the same point $p$, and the sum of all unit tangent vectors at $p$ (as usually directed from $p$) is equal to the zero vector (in the tangent space $T_pM^n$). A geodesic flower is non-trivial if at least one of these geodesic loops is non-constant.

Here is our main theorem:

**Theorem 1.2.** Let $M^n$ be a complete non-compact manifold with locally convex ends. Then there exists a non-trivial geodesic flower on $M^n$.

The literature on the existence of periodic geodesics on complete non-compact Riemannian manifold is scarce. We would like to note papers by V. Benci and F. Giannoni ([BG]) as well as L. Asselle and M. Mazzuchelli ([AM]) providing non-trivial sufficient conditions for the existence of non-constant periodic geodesic on non-compact Riemannian manifolds. These conditions prevent sliding to infinity of a cycle in the space of closed curves in a compact “core” of the manifold. In particular, they require the free loop space of the complete non-compact Riemannian manifold to be non-contractible and, moreover, have non-trivial homology classes in a high dimension.

In contrast, our result does not assume that complete non-compact Riemannian manifold has non-trivial topology and are applicable to the case of manifolds diffeomorphic to $\mathbb{R}^n$. Further, the locally convex ends condition does not prevent a sliding of cycles in the space of free loops to infinity, but prevents movement of free loops from infinity to the “compact core” of the manifold. On the other hand, the main theorem in the paper of V. Bangert, [B], asserting the existence of a periodic geodesic on each complete surface of finite area is also applicable to surfaces diffeomorphic to the plane. (An earlier paper of G. Thorbergsson [T] established this result in the case of arbitrary surfaces with at least three ends, so, in fact, [B] dealt with the case of surfaces with either two or one ends. The first step in Bangert’s proof was to observe that all ends of surfaces of finite area can be cut off by appropriate geodesic loops with angles measured at the infinite side less than $\pi$. Therefore, all surfaces of finite area have locally convex ends (in the sense of our definition). In particular, Bangert’s work also implies that the conjecture above is true for complete surfaces.) The technique of Bangert is strictly two-dimensional. Note that for any $n > 2$ it is still not known whether or not every complete $n$-dimensional Riemannian manifold of
finite volume has a non-trivial periodic geodesic. It is also interesting to notice that if $M$ is a complete surface that has a simple periodic geodesic bounding a domain with compact closure in $M$, then the convex ends condition is trivially satisfied: One can regard this geodesic as a one element set of hypersurfaces $\{\Sigma\}$.

Below we will assume that $M^n$ is a complete non-compact Riemannian manifold with locally convex ends. We will be looking for a periodic geodesic or, more generally, a geodesic flower on $M^n$.

Our proof combines several main ideas.

2. Attempting an impossible filling.

The first idea is yet another adaptation of Gromov’s technique from [Gr] based on attempting an impossible extension of the inclusion of $M$ into a pseudo-manifold $W$ such that $\partial W = M$ to $W$. In our context this idea works as follows.

To explain the argument it will be convenient to define the following spaces (the bar $\bar{U}$ over set $U$ denotes the closure of $U$):

- $C_i$ is the cone over $\Sigma_i$, $i = 1, \ldots, e$;
- $S = M_0 \cup \bigcup_i C_i$;
- $CS$ is the cone over $S$;
- $M_i^c$ is the one point compactification of $M_i$;
- $M^c = M_0 \cup M_i^c$;
- $T = M^c/(\bigcup_i M_i^c)$.

We have the following inclusion maps:

$$\phi : (\bar{M}_0, \bigcup_i \Sigma_i) \longrightarrow (M^c, \bigcup_i M_i^c)$$

$$j : (\bar{M}_0, \bigcup_i \Sigma_i) \longrightarrow (CS, \bigcup_i C_i)$$

$$\tau : \bar{M}_0/(\bigcup_i \Sigma_i) \longrightarrow T$$

Note that $\phi$ and $\tau$ induce isomorphisms on the $n$-th homology of pairs with $\mathbb{Z}_2$ coefficients. Since $CS$ is contractible, $0 = H_n(CS; \mathbb{Z}_2) = H_n(CS, \bigcup_{i=1}^e p_i; \mathbb{Z}_2) = H_n(CS, \bigcup_{i=1}^e C_i; \mathbb{Z}_2)$, where $p_i$ denote tips of the cones $C_i$.

The following lemma easily follows:

**Lemma 2.1.** Let $\phi' : (\bar{M}_0, \bigcup_i \Sigma_i) \longrightarrow (M^c, \bigcup_i M_i^c)$ be homotopic to the inclusion $\phi$. Then there exists no extension of $\phi'$ to $(CS, \bigcup_i C_i)$. In other words, there is no continuous map $\psi : (CS, \bigcup_i C_i) \longrightarrow (M^c, \bigcup_i M_i^c)$ such that $\phi' = \psi \circ j$. 
Indeed, if lemma is false, then

\[ \phi, \psi \quad \text{where both maps are regarded as defined on} \quad \bar{M}_0 \]

\[ \phi' = \psi \circ j, \quad \text{and, therefore,} \quad \theta \circ \phi' = \theta \circ \psi \circ j = \tilde{\psi} \circ j \]

is continuous, where both maps are regarded as defined on \( \bar{M}_0 \). As both maps send \( \bigcup_i \Sigma_i \) to the point \( m \) of \( T \), they can be lifted to equal maps of \( \bar{M}_0 / (\bigcup_i \Sigma_i) \). In other words, \( \theta \circ \phi' = \tau^{-1} \circ q |_{\bar{M}_0} \), where, as we know, \( \tau^{-1} \) induces the identity isomorphism of the \( n \)th homology groups with \( \mathbb{Z}_2 \) coefficients. Similarly, \( \tilde{\psi} \circ j \) can be represented as \( \mu \circ q |_{\bar{M}_0} \) for some \( \mu \), which must coincide with \( \tau^{-1} \). But note that \( \mu \) can be decomposed as \( j : \bar{M}_0 / (\bigcup_i \Sigma_i) \rightarrow CS / (\bigcup_i C_i), \) and the map \( \theta : CS / (\bigcup_i C_i) \rightarrow T = M^c / (\bigcup_i M_i^c) \) induced by \( \psi \). As the \( n \)th homology with \( \mathbb{Z}_2 \) coefficients of \( CS / (\bigcup_i C_i) \) is trivial, the induced homomorphism \( j_* \) is zero, and, therefore, \( \mu_* \) is also trivial. But \( \mu = \tau^{-1} \), and we obtain a contradiction proving the lemma.

Proof. Indeed, if lemma is false, then \( \phi' = \psi \circ j \), and, therefore, \( \theta \circ \phi' = \theta \circ \psi \circ j = \tilde{\psi} \circ j \) is continuous, where both maps are regarded as defined on \( \bar{M}_0 \). As both maps send \( \bigcup_i \Sigma_i \) to the point \( m \) of \( T \), they can be lifted to equal maps of \( \bar{M}_0 / (\bigcup_i \Sigma_i) \). In other words, \( \theta \circ \phi' = \tau^{-1} \circ q |_{\bar{M}_0} \), where, as we know, \( \tau^{-1} \) induces the identity isomorphism of the \( n \)th homology groups with \( \mathbb{Z}_2 \) coefficients. Similarly, \( \tilde{\psi} \circ j \) can be represented as \( \mu \circ q |_{\bar{M}_0} \) for some \( \mu \), which must coincide with \( \tau^{-1} \). But note that \( \mu \) can be decomposed as \( j : \bar{M}_0 / (\bigcup_i \Sigma_i) \rightarrow CS / (\bigcup_i C_i), \) and the map \( \theta : CS / (\bigcup_i C_i) \rightarrow T = M^c / (\bigcup_i M_i^c) \) induced by \( \psi \). As the \( n \)th homology with \( \mathbb{Z}_2 \) coefficients of \( CS / (\bigcup_i C_i) \) is trivial, the induced homomorphism \( j_* \) is zero, and, therefore, \( \mu_* \) is also trivial. But \( \mu = \tau^{-1} \), and we obtain a contradiction proving the lemma.

Now we will sketch the plan of the proof of our main theorem by contradiction. We are going to assume that there are no geodesic flowers on \( M \), and we will try to extend the identity map \( \phi \) to a map \( \psi \) of \( CS \) that cannot exist according to the previous lemma. Choose a very fine triangulation of \( \bigcup_i \Sigma_i \), extend it to a very fine triangulation of \( \bar{M}_0 \), and also extend it to a (not fine) triangulation of all \( C_i, i \geq 1 \) by adding new vertices \( q_i \) (= tips of cones \( C_i \)) and triangulating all \( C_i \) as cones over the chosen fine triangulations of bases of \( C_i \). Finally, choose a vertex \( P_0 \) of the cone \( CS \), and triangulate \( CS \) as the cone over just constructed triangulation of the base.

We are going to attempt the impossible extension of \( \phi \) to \( \psi \). We map \( P_0 \) to any vertex \( p \) in the fine triangulation of \( \bar{M}_0 \) in \( M_0 \), \( q_i \) to any vertex \( v_i \) of the fine triangulation of \( \Sigma_i \). Map all 1-dimensional simplices of \( S \) that connect \( q_i \) with a vertex \( v \) of \( \Sigma_i \) to a minimizing geodesic on \( \Sigma_i \) in its intrinsic metric that connects \( v_i \) and \( v \). Observe that 1-skeleta of all \( n \)-dimensional simplices of \( S \) with a vertex at \( q_i \) will be on \( \Sigma_i \). We are going to map all “new” 1-simplices of \( CS \) with one vertex at \( P_0 \) to a minimal geodesic in the intrinsic metric of \( \bar{M}_0 \) that connects \( p \) and this vertex (or \( p \) and \( v_i \), if the vertex is \( q_i \)). Observe that 1-skeleta of all \( (n + 1) \)-dimensional simplices of \( CS \) will be in \( \bar{M}_0 \).

A natural idea is to continue extending \( \psi \) inductively to 2-skeleta, then to 3-skeleta, etc. Instead, we assume that \( M_0 \) does not have any geodesic flowers and immediately extend the inclusion of the 1-skeleton of each \((n + 1)\)-dimensional simplex of \( CS \)
into \( M \) to the map of the whole simplex. We do this using the idea of “filling of cages” from [R1]. This idea will be described in details in the next two sections. The description of “filling of cages” in the next two sections will immediately imply that the image of each \( n \)-simplex with vertex at \( q_i \) in \( S \) will be in \( \bar{M}_i, \ i \geq 1 \).

Here we only note that we would like to be able to eventually extend maps of the 1-skeleton of the standard \((m+1)\)-dimensional simplex \( \Delta^{m+1} \) for each \( m \in \{1, \ldots, n\} \) to its interior. The extension must map the interior to \( M^c \). If the 1-skeleton is mapped to one of \( \bar{M}_i \) (for example, to \( \Sigma_i \)), then the extension maps the whole simplex to \( \bar{M}^c_i \). The restriction of each of these extensions to the 1-skeleton of any \((l+1)\)-dimensional face of \( \Delta^{m+1} \) must be the extension of the restriction. We want these extensions only for maps , where the length of the image of each side of \( \Delta^{m+1} \) does not exceed \( \bar{L} = \max\{\text{Diam}(M_0), \max_{j \geq 1} \text{Diam}(\partial M_j)\} \), where all the diameters are calculated with respect to the inner metrics. Finally, mention that very small cages must be filled by very small simplices.

Combining all the extensions for all \((n+1)\)-dimensional simplices of the triangulation of \( CS \), we will obtain a map \( \psi : CS \to M^c \). We would like to argue that Lemma 2.1 yields a contradiction that refutes the assumption that there are no geodesic flowers on \( M \). Yet a minor technical difficulty is that the restriction of \( \psi \) to \( \bar{M}_0 \) is not the identity map on \( \Sigma_i \subset S \) as in the conditions of Lemma 2.1. (So, the composition of \( j \) and \( \psi \) is not the inclusion \( \phi \).) This happens because the fillings of 1-skeleta of very small simplices \( \sigma^{n-1} \) of the triangulations of \( \Sigma_i \) will be not \( \sigma^{n-1} \) but very close very small \((n-1)\)-dimensional simplices contained in a small collar of \( \Sigma_i \) in \( \bar{M}_i \). For each \( i \) there will be an obvious homotopy between the restriction of \( \psi \) to \( \Sigma_i \) and the inclusion map of \( \Sigma_i \) to \( M \) that will move all points in the fillings of 1-skeleta of small simplices triangulating \( \Sigma_i \) via normals to \( \Sigma_i \) inside \( M_i \). This homotopy can be extended to a homotopy between the restriction of \( \psi' \) to \( \bar{M}_0 \) and the inclusion of \( \bar{M}_0 \) to \( M \). Hence, applying Lemma 2.1 with \( \phi = \psi'|_{\bar{M}_0} \) homotopic to the inclusion \( \phi \) we obtain the desired contradiction.

3. Cages and their fillings.

3.1. Cages, flowers, fillings. The content of this section follows [R1], adapting one of the ideas to our situation. For each \( i = 2, 3, \ldots, n + 1 \) define an \( i \)-cage in \( M \) as a map of the 1-skeleton of the standard \( i \)-dimensional simplex \( \sigma^i \) to \( M \), where each edge is mapped into a broken geodesic. If this map is constant, we call the cage a constant cage. We extend the class of all constant cages, and, thus, all cages by allowing constant cages that map the 1-skeleton of \( \sigma^i \) to one of \( e \) points at infinity of \( M^c \). Thus, formally speaking, cages map the 1-skeleta of simplices to \( M^c \), but the image of each non-constant cage is required to be in \( M \). We are assuming that the vertices of \( \sigma^i \) form a totally ordered set (the ordering might come from a numbering
of the vertices by numbers $1, \ldots, N + 1$.) Denote the space of all $i$-cages in $M$ by $\text{Cage}_i$. Note that for each $j$-dimensional face of $\sigma^i$ the restriction of an $i$-cage $k$ to the 1-skeleton of $\sigma^i$ is a $k$-cage. We will call such $j$-cages subcages or $j$-subcages of $k$.

We will also need a special class of $i$-cages that we will call flowers. Consider the vertex $v$ of an $i$-cage with the highest number. Assume that all edges ($=1$-simplices) incident to $v$ are mapped to the same point. Thus, such cages can be also defined as maps of $\frac{i(i-1)}{2}$ circles to $M$. In fact, some of these circles can also be mapped to a point. We use notation $\text{Flwr}_k$ for maps of the wedge of $k$ loops to $M$. As we just observed, there are natural inclusions $\text{Flwr}_{\frac{i(i-1)}{2}} \subset \text{Cage}_i$, and $\text{Flwr}_{j} \subset \text{Flwr}_{i}$ for $j \leq i$. Recall, that a set $C \subset M$ is $\delta$-locally convex, if for all $x, y \in C$ that are $\delta$-close in $M$ all minimizing geodesics in $M$ between $x$ and $y$ are contained in $C$. For each $L \in (0, \infty)$ let $\text{Cage}_i^L$ denote the subset of $\text{Cage}_i$ formed by all cages of length $\leq L$, $\text{Cage}_i^\infty$, by definition, coincides with $\text{Cage}_i$. Similarly, $\text{Flwr}_j^\infty = \text{Flwr}_j$, and for each $L \in (0, \infty)$ $\text{Flwr}_j^L$ denotes a subset of $\text{Flwr}_j$ formed by all $j$-flowers of length $\leq L$.

**Definition 3.1.** Given $\delta > 0$, and $L \in (0, \infty]$ a strong filling of $(n+1)$-cages of length $\leq L$ is a family of continuous maps $H_i : \text{Cage}_i^L \times [0, 1] \rightarrow \text{Cage}_i$ for $i \in \{2, \ldots, n+1\}$ such that:

1. For each $i$-cage $k$ $H_i(k, 0) = k$ and $H_i(k, 1)$ is a constant cage. In other words, $H_i$ is a contraction of $\text{Cage}_i^L$ inside $\text{Cage}_i$ to its subspace formed by all constant cages.
2. If the image of an $i$-cage $k$ is in a $\delta$-locally convex subset $V$ of $M$ (in particular, $k$ might be in the closure of $M_j$ for some $j$), then its trajectory $H(k, t)$, $t \in [0, 1]$ will be in $V$.
3. If $k$ is a constant $i$-cage, then $H_i(k, t) = k$ for all $t \in [0, 1]$.
4. If $k \in \text{Flwr}_j^L \subset \text{Flwr}_{\frac{i(i-1)}{2}}^L \subset \text{Cage}_i^L$, ($j \leq \frac{i(i-1)}{2}$), then for $H_i(k, t) \in \text{Flwr}_j$ for all $t \in [0, 1]$.

The importance of this notion lies in the following proposition:

**Proposition 3.2.** Assume that $M$ admits a strong filling of cages for some $\delta > 0$ and $L \in (0, \infty]$. Let $k$ be an $i$-cage in $M$ of length $\leq L$, $L < \infty$, for some $i = 2, 3, \ldots, n+1$. Then:

1. There exists a continuous map $\phi = \phi(k)$ of the $i$-simplex $\sigma^i$ to $M^e$ extending $k$. Further, the dependence of $\phi$ on $k$ is continuous (that is, the restriction of $\phi(k)$ to the 1-skeleton of $\sigma^i$ is equal to $k$).
2. Let $\sigma^j$ be a $j$-dimensional face of $\sigma^i$, $2 \leq j < i$, $k_j$ the corresponding $j$-subcage of $\sigma^i$. Then $\phi(k_j)$ coincides with the restriction of $\phi(k)$ to $\sigma^i$.
3. If the image of $k$ (or one of its $j$-dimensional subcages $k_j$) is in a closed $\delta$-locally convex subset of $M$ (in particular, it might be in the closure of $M_m$ for
some $m = 1, \ldots, e$), the same will be true for the image of $\phi$ (correspondingly, the restriction of $\phi$ to the $j$-dimensional face $\sigma^j$ of $\sigma^i$ corresponding to $k_j$.)

Proof. The proof uses the induction with respect to $i$. To prove the base note that 2-cages are boundaries of 2-simplices, and for each 2-cage $k$ map $H_2$ provides the extension of the map of the boundary of the 2-simplex to its interior. Moreover, the second property of $H_2$ implies that if $k$ is in the closure of $M_m$, then its whole trajectory will be in $M_m^c$.

Now we prove the induction step. Assume that the proposition holds for all $i \leq I$. In order to prove it for $i = I + 1$ consider an $i$-cage $k$ and the one-parametric family of $i$-cages $k_t = H_i(k, t), t \in [0, 1]$. Each $i$-cage $k_t$ has $(i + 1)$ of $(i - 1)$-subcages $k_t(l), l = 1, \ldots, i + 1$ corresponding to the $(i - 1)$-dimensional faces of the $i$-simplex $\sigma^i$. Applying the induction assumption extend each $k_t(l)$ to the map $\phi(k_t(l))$ of the $(i - 1)$-dimensional simplex $\sigma^{i-1}$. Together these $i + 1$ maps provide the extension of $k_t$ to the boundary of the $\sigma^i$. When $t$ varies in $[0, 1]$, these extensions provide the extension of $k$ to $\sigma^i$ minus a point $C$ at the center of $\sigma^i$, which is then mapped to $H_i(k, 1)$. The continuity of the constructed extension at $C$ follows from property (3) of cage fillings and the continuity of $H_{i-1}$. This completes the proof of (1).

Now observe that for $j = i - 1$ (2) follows immediately from the construction of $k$, and for $j = i - l$ immediately follows from this observation and the induction assumption. Finally, (3) follows from property (2) of fillings of cages. $\square$

Definition 3.3. A (weak) filling of cages in $\text{Cage}^L_{n+1}$ is defined exactly as the strong filling with the only distinction that the requirement of continuity of maps $H_i$ is replaced by a weaker requirement that only the compositions of $H_i$ with the quotient map $M^c \to T = M^c/\bigcup_i M^c_i$ are continuous.

Proposition 3.4. Assume that $M$ admits a (weak) filling of cages for some $L \in (0, \infty]$ and $\delta > 0$. Let $k \in \text{Cage}^L_i$ be an $i$-cage in $M$ for some $i = 2, 3, \ldots, n + 1$. Then:

1. There exists a map $\tilde{\phi} = \phi(k)$ of the $i$-simplex $\sigma^i$ to $M^c$ extending $k$. The composition $\tilde{\phi}$ of $\phi$ with the quotient map $M^c \to T$ is continuous. Further, the dependence of $\phi$ on $k$ becomes continuous after projecting to $T$.

2. Let $\sigma^j$ be a $j$-dimensional face of $\sigma^i$, $(2 \leq j < i)$, $k_j$ the corresponding $j$-subcage of $\sigma^i$. Then $\phi(k_j)$ coincides with the restriction of $\phi(k)$ to $\sigma^j$.

3. Moreover, if the image of $k$ (or one of its $j$-dimensional subcages $k_j$) is in a closed convex or $\delta$-locally convex subset of $M$ (in particular, it might be in the closure of $M_m$ for some $m = 1, \ldots, e$), the same will be true for the image of $\phi$ (correspondingly, the restriction of $\phi$ to the $j$-dimensional face $\sigma^j$ of $\sigma^i$ corresponding to $k_j$.)

This proposition can be proven exactly as the previous one.
Strong and weak fillings of flowers are defined exactly as strong/weak fillings of cages. In fact, the definition of fillings of cages implies that its restriction to flowers is a filling of flowers. In section 3.2 we will demonstrate that, vice versa, given a filling of flowers we can easily extend it to fillings of cages.

We would like to finish this section by observing that combining the previous proposition with the results in the previous section we obtain the following proposition:

**Lemma 3.1.** Let $M$ be a complete non-compact $n$-dimensional manifold with locally convex ends. There exists $\delta_0 > 0$, $L_0 > 0$, such that for all $\delta \in (0, \delta_0]$ and $L \in [L_0, \infty)$ there is no weak filling of cages of length $\leq L$.

**Proof.** Consider the triangulation of $CS$ and a map $\psi$ from the 1-skeleton of $CS$ to $M^c$ described in the previous section. If there exists a filling of cages, then by Proposition 3.1 we can extend $\psi$ to a map defined on all of $C$, such that the composition $q \circ \psi$ is continuous. Moreover, the restriction of $\psi$ to $\overline{M}_0$ is homotopic to the inclusion map $\phi$. By Lemma 2.1 we obtain a contradiction. $\Box$

3.2. From cages to flowers. Recall that some edges of a cage can be mapped by means of a constant map. Of course, in this case both endpoints are mapped to the same point. It can happen that a set of edges forming a spanning tree of a cage is being mapped to the same point $p$ of $M$. In this case all vertices of the cage are mapped to $p$, and all other edges become loops based at $p$. (Some of these loops can also be constant.) Recall, that we call such cages with one vertex flowers, and their non-constant loops petals.

Our next observation is that:

**Lemma 3.2.** There exists a deformation retraction of the space of $i$-cages to the space of flowers with at most $\frac{i(i-1)}{2}$ petals with the following properties:

1. The image of a cage in $M$ does not change during this deformation.
2. If the lengths of all edges of a cage do not exceed $l$, then the lengths of all edges of the cage during the deformation (including the flower at the end of the deformation) do not exceed $3l$. Therefore, the lengths of all cages during the deformation do not exceed $\frac{i(i-1)}{2}l$.

**Remark.** This deformation is unconditional, that is, it does not require any assumptions about $M$.

**Proof.** The idea is very simple. All vertices of a cage are numerated. We just move all vertices but the one with the maximal number to the vertex with the maximal number, $v_{max}$, along the corresponding edge of the cage. The speed is constant, and chosen so that the all vertices will collide with the maximal one at the moment $t = 1$. All edges between $v_i$ and $v_{max}$ shrink and become constant at $t = 1$. On the other
hand the segments \( v_i(0)v_i(t) \) and \( v_j(0)v_j(t) \) that are being eliminated from \( v_iv_{\text{max}} \) and \( v_jv_{\text{max}} \), correspondingly, are being added to the edge \( v_iv_j \) at both ends of \( v_iv_j \). So, at the moment \( t \) the edge between \( v_i(t) \) and \( v_j(t) \) will be the join of the three segments \( v_i(t)v_i, v_iv_j \) and \( v_jv_j(t) \).

We make these deformation retractions the initial “halves” of \( H_i \), and it will remain to “fill” only the resulting flowers. (This means that given an \( i \)-cage we use the interval \([0, \frac{1}{2}]\) of time to deform this cage to a flower with the same image as in the proof of the previous lemma. We are going to use the remaining time to “fill” the resulting flower.) So, we obtain the following corollary.

**Corollary 3.3.** Given a weak (correspondingly, strong) filling of flowers in \( \text{Flwr}_j \), \( j \leq \frac{n(n+1)}{2} \), there exists a weak (correspondingly strong) filling of \((n+1)\)-cages. If \( L \) is finite, then given a weak (correspondingly, strong) filling of flowers \( k \) in \( \text{Flwr}^{nL}_j \), \( j \leq \frac{n(n+1)}{2} \) via intermediate flowers \( H(k,t) \) with length \( \leq nL \), there exists a weak (correspondingly, strong) filling of \((n+1)\)-cages of length \( \leq L \) (via cages of length \( \leq nL \)).

**Remark:** Our definition of filling of cages does not contain any restrictions for lengths of intermediate cages. Yet below we will be discussing a weak length non-increasing filling of flowers of bounded length. The last assertion means that the existence of a weak filling of \((n+1)\)-cages follows from the existence of a weak filling of flowers of length bounded by a larger value \( nL \) via flowers of length bounded by the same constant. (The exact value of this larger value is not important for us, any \( c(n,L) \) instead of \( nL \) will work for us.)

Combining the previous corollary with Lemma 3.1, we see that:

**Lemma 3.4.** Let \( M \) be a complete non-compact \( n \)-dimensional manifold with locally convex ends. There exists \( \delta_0 > 0, L_0 > 0 \), such that for all \( \delta \in (0, \delta_0] \) and \( L \in [L_0, \infty) \) there does NOT exist a weak filling of flowers of length \( \leq L \).

### 3.3. Flows on cages and fillings.

Everything that we will say in this section about cages can be verbatim repeated for flowers. We will start from the following definition.

In order to prove our main theorem, we would like to consider curve-shortening flow on cages of length \( \leq L \) for some \( L \), yet we need a weaker notion. Assume that \( M \) is a complete non-compact manifold with locally convex ends (as in Definition 1.1). Recall that \( M^c \setminus M \) is a finite collection of points. Namely, it contains one point \( q_i \) for each end \( M_i \). Let \( \text{cage}^L_i \) denote \( \text{Cage}^L_i \setminus \cup_j \{q_i\} \), where \( q_j \) are regarded as constant \( i \)-cages. One can map some pairs \((k, t)\) outside of the open \( L \)-neighbourhood of \( M_0 \) to points in \( M^c \setminus M \) (regarded as constant cages not in \( \text{cage}^L_i \)) subject to the following conditions:
Definition 3.5. Assume that $M$ is as in definition 1.1, and $L$ is finite. A weak $L$-curve-shortening flow on $n$-cages on $M$ is a family of maps $F_i : cage_i^L \times [0, \infty) \rightarrow Cage_i^L = cage_i^L \cup M^c \setminus M$ for all $i \in \{2, \ldots, n\}$ such that for all $k, t, s > 0$ $F_i(F_i(k,t),s) = F_i(k,t+s)$ as long as the images of all $F_i$ in this formula are in $Cage_i^L$ and that satisfies the following properties:

1. $F_i$ is continuous on $F_i^{-1}(cage_i^L)$;
2. Assume that for some $k \in cage_i^L$ and for all positive $t$ $F_i(k,t) = q_j$ for some $j$. Then $k$ is contained in $M_j$ and has empty intersection with the closed $L$-neighbourhood of $M_0$. On the other hand, there exists $\varepsilon > 0$ such that for each $k \in cage_i^L$ contained in $M_j$, $j > 0$, such that $k$ does not intersect the closed $(L+\varepsilon_1)$-neighbourhood of $M_0$, $F_i(k,t) = q_j$ for all $t > 0$.
3. If $F_i(k,t) = q_j$, then for all $T > t$ $F_j(k,T) = q_j$.
4. Let $k$ be a non-stationary non-constant $i$-cage. Then there exists an open neighbourhood $U$ of $k$ in the space of $i$-cages and positive $\lambda$ and $\varepsilon$ such that for each $i$-cage $k_1 \in U \text{ length}(k_1) - \text{length}(F_i(k_1,\lambda)) \geq \varepsilon$.
5. If an $i$-cage $k$ is a $j$-flower, then for all $t$ $F_i(k,t)$ is a $j$-flower (possibly $q_j$ regarded as a $j$-flower).
6. If $k$ is stationary (possibly a point), then $F_i(k,t) = k$ for all $t$.
7. For each point $p$ different from points $q_j$ there exists $r = r(p) > 0$ such that if an $i$-cage $k$ is contained in the metric ball of radius $\rho \leq r$ centered at $p$, then for some $t = t(\rho,p) > 0$ $F_i(k,t)$ is a constant cage (= a point). Moreover, this point is not one of the points $q_j$, and $\lim_{\rho \rightarrow 0} t(\rho,p) = 0$.
8. If $C$ is either $M_j$, $j \geq 1$ or a convex metric disc centered at a point $x \in M_0$ of radius $< \text{conv}(x)$, and $k \in C$, then $F_i(k,t)$ is in $C$ for all values of $t$. Here $\text{conv}(x)$ denotes the convexity radius of $M$ at $x$.

Lemma 3.6. Let $F$ be a curve-shortening flow on $(n+1)$-cages of length $\leq L$ on a complete non-compact Riemannian manifold $M^n$ with locally convex ends $M_i$, $i = 1, 2, \ldots$. Assume that $M^n$ does not contain a non-trivial stationary $i$-cages for all $\leq n+1$. Then there exists a weak filling $H_i : Cage_i^L \times [0, 1] \rightarrow M^c$ for $i \in \{1, \ldots, n+1\}$.

Proof. If $k = q_j$, the $H_i(k,t) = q_j$ for all $t$. Now, it is sufficient to consider only cages $k \in cage_i^L$.

Step 1. For each $i$-cage $k$ of length $\leq L$ there exists $t = t(k)$ such that $F_i(k,t)$ is a point (possibly, one of the points $q_j$, $j \geq 1$).
Indeed, the alternative is that there exists an unbounded increasing sequence \( t_m \) such that \( F_i(k, t_m) \) is not a point. Therefore, these cages of length \( \leq L \) are in the closed \((L + \varepsilon_1)\)-neighbourhood of \( \bar{M}_0 \). Pass to the limit of an appropriate subsequence. This limit, \( k_\infty \), cannot be a point, as in this case for a sufficiently large \( F_i(k, t_m) \) will be in a small neighbourhood of this point, and the flow will contract \( F_i(k, t_m) \) to a point different from \( q_j \) in a finite time. If \( k_\infty \) is a non-trivial closed curve, then property (2) implies that the flow will simultaneously decrease the lengths of curve \( F_i(k, t_m) \) for all sufficiently large \( m \) in time \( \lambda \) by at least \( \varepsilon > 0 \). Yet lengths of \( F_i(k, t) \) decrease to the infimum equal to the length of \( k_\infty \), and we obtain a contradiction.

Step 2. For each \( i \) here exists \( T = T(i) \) such that for all \( i \)-cages \( k \) of length \( \leq L \) either \( F(k, T) \) is a point, or its image does not intersect \( \bar{M}_0 \).

Assume that there exists an infinite sequence of \( i \)-cages \( k_m \) of length \( \leq L \) and an unbounded increasing sequence of times \( t_m \) so that \( F_i(k_m, t_m) \) is not a point. This implies that all \( k_m \) are in the (compact) closed \((L + \varepsilon_1)\)-neighbourhood of \( \bar{M}_0 \). Passing to a subsequence, if necessary, we can assume that the sequence \( k_m \) converges to an \( i \)-cage \( k_* \). If \( k_* \) is a point (that cannot be one of points \( q_j \)), then all sufficiently large values of \( m k_m \) will be in a fixed small neighbourhood of this point. Now property (5) of the definition above implies that the flow contracts all of them to points in a uniformly bounded time, and we obtain a contradiction. If \( k_* \) is not a point, but the flow contracts \( k_* \) to a point \( p \) different from all \( q_j \), then at some moment of time \( t \) and all sufficiently large \( m F_i(k_m, t) \) will be in a fixed small neighbourhood of \( p \), and property (5) in the definition again yields the contradiction. It remains to consider the case when the flow contracts \( k_* \) to one of the points \( q_j \). In a finite (possibly zero) time the flow moves \( k_* \) out of the closed \( L \)-neighbourhood of \( \bar{M} \), yet we choose this moment of time \( t_* \) so that \( F_i(k_*, t_*) \) is not yet \( q_j \). Therefore, for some \( m_0 \) and all \( m > m_0 \) \( F(k_m, t_*) \) are also not in the closed \( L \)-neighbourhood of \( \bar{M}_0 \). Therefore, none of \( F_i(k_m, t_*) \) intersect \( \bar{M}_0 \). For \( l = 1, \ldots, m_0 \) choose \( t_l = t(k_l) \) so that \( F_i(k_l, t_l) \) is a point. Now define \( T \) as \( \max\{t_*, t_1, \ldots, t_{m_0}\} \).

Step 3. Now we can define the weak fillings as follows. For each \( i \) define \( H_i(k, t) \) as \( F_i(k, T(i) t) \) for all \( t < 1 \). If \( F_i(k, T(i) t) \) is a point different from all points \( q_j \), we define \( H_i(k, 1) \) as the constant cage \( p \). Finally, if \( H_i(k, T(i) t) \) does not intersect \( \bar{M}_0 \), it is contained in some \( M_j \) for some \( j = j(k) > 1 \). In this case we define \( F_i(k, 1) \) as \( q_j \).

\[ \square \]

In the next section we are going to prove that:

**Theorem 3.7.** Let \( M^n \) be a complete \( n \)-dimensional Riemannian manifold with \( \delta \)-locally convex ends for some positive \( \delta \). Assume that there is no non-trivial geodesic flower on \( M^n \). Then there for each finite \( L \) and each \( N \) there exists a weak curve-shortening flow on \( N \)-cages of length \( \leq L \) on \( M^n \).
Combining this theorem with Lemma 3.4 and the flower version of the previous lemma, we obtain Theorem 1.2. Thus, it remains only to prove Theorem 3.7. We are going to do this in the next section.

4. Weak curve-shortening flow on spaces of flowers

In this section we prove Theorem 3.7. We construct a weak curve shortening flow on the space of $L$-flowers. All flowers outside of the $(L + 2\delta)$-neighbourhood of $\bar{M}$ will be immediately sent to one of points $q_i$ at infinity in the same component of the complement of $\bar{M}_0$. Therefore, it is sufficient to consider $L$-flowers in the $(L + 2\delta)$-neighbourhood of $\bar{M}_0$. Our description of this flow will be an adaptation of the process described in [NR] for cages on closed Riemannian manifolds, and does not contain any new ideas.

Let $i_1$ denote the infimum of the injectivity radii of points of $M$ in this set, and $I$ denote $\max\{\delta/2, i_1/4\}$. Let $N$ denote the integer part of $\frac{I}{L}$. Given a flower of length $\leq L$, we observe that the length of each petal does not exceed $L$. Consider first the obvious length-nonincreasing deformation of the space of all $L$-flowers in the $(L + \delta)$-neighbourhood of $\bar{M}$ into the space of broken geodesic flowers with each petal subdivided into $N + 1$ segments of equal length $\leq I$ by exactly $N$ intermediate points: One first subdivides the petal into $N + 1$ arcs of equal length using $N$ intermediate points. Then one connects pairs of consecutive points by (unique) minimizing geodesics. If the initial curve was outside of $\bar{M}_0$, then the $\delta$-convexity of the complement of $\bar{M}_0$ implies that the new curve will be outside of $\bar{M}_0$. Note that replacing each broken geodesic segment of the original curve by the minimal geodesic segment in the new curve cannot increase the length. We can connect the old curve and the new curve by a length non-increasing homotopy by following the $N + 1$ arcs of the original petal for shorter and shorter periods of time and then following the minimal geodesic to the end point of the arc. This stage is similar to analogous stage in the Birkhoff curve shortening process. Therefore, we will call it Birkhoff deformation.

Such flowers are completely determined by the coordinates of the base point of the flower and $N \times$ the number of petals of the flowers. From now on we can identify considered flowers with a subset of $(M^n)^K$, where $K$ is $1 + N \times$ the number of the petals, where ”1” corresponds to the base point of the flower. The flow will mostly consist of stages during which we flow $K$ points that determine the flowers along trajectory of a vector field on a domain in $(M^n)^K$. Yet, we do not want distances between pairs of points that are supposed to be connected by the unique minimizing geodesic to grow too much. It will be immediately clear from the construction of this field that the speed of movement of each point will not exceed $2(n + 1)$. Therefore, we plan to stop after each time interval of length $\frac{L}{10n}$. This guarantees that the distances
between consecutive points will still be less that $\delta$, and less than $\frac{\delta}{2}$. Therefore, the corresponding points will uniquely determine a flower. Right after stopping the flow we check if the curve is outside of $L$-neighbourhood of $M_0$. If it is, it will be immediately mapped by the flow to the corresponding point at infinity $q_j$ and will stay there forever. Then we perform the Birkhoff deformation, that will again drop the distances between consecutive points on petals to less than $I$. At the end of the Birkhoff deformation we again check if the resulting curve is outside the $L$-neighbourhood of $M_0$, and map it to the corresponding point at infinity, if it is outside.

Now we are going to describe the vector field on the considered domain of $(M^n)^K$. We are going to describe it on several overlapping open subsets covering the considered domain, and then combine these vector fields into one vector field by using a subordinate partition of unity.

We are first going to describe the flow on flowers where all petals are “not too small”, where “not to small” means that the petal is not contained in $\frac{a}{2}$-neighbourhood f the base point of the flower for some small $a$ that will be defined later.

The vector field depends on positions of all $K$ points that determine the flower. For each of these points $p$ consider the adjacent geodesic segments on all petals that contain $p$. The number of these segments will be equal to $2 \times$ the number of petals for the base point and 2 for each of the remaining $K-1$ points. For each of these geodesic segments consider the unit tangent vector at $p$ directed from $p$ and sum up all these vectors. The result will be a vector $V = V(p)$ in the tangent space $T_pM^n$ of $M^n$ at $p$. These $K$ vectors will form the vector field at the point of $(M^n)^K$ that corresponds to the considered flower, and will be used to deform it. The first variation formula for the length functional implies that the time derivative of the length functional of the flower at $t = 0$ will be equal to $-\Sigma_p \|V(p)\|^2$, where we sum over all $K$ points $p$ that determine the flower. Thus, the deformation will be length non-increasing, and will be length decreasing unless all petals of the flower are geodesics, and the flower is a stationary geodesic flower. As we assumed that there are no stationary geodesic flowers, the deformation will be uniformly length decreasing. Further, assume that the base point of the flower is on the boundary of $M_0$, yet all adjacent endpoints of geodesic segments are outside of $M_0$. Then all the tangent vectors at the base point outside of $M_0$, and so is their sum. Thus, a flower outside of $M_0$ cannot, even partially, enter $M_0$ when deformed along this vector field.

Now consider all flowers where ALL petals are in a sufficiently small neighbourhood of the base point $b$. In this case the base point does not move, and the remaining $K-1$ points move along the unique minimizing geodesics to the base point. (In other world, the component of the vector field corresponding to the base point in the zero tangent vector, for all other points $p$ the components of the vector field at $p$
are unit tangent vectors to the minimizing geodesics connecting \( p \) and \( b \) and directed towards \( b \). The size of the neighbourhood does not exceed \( I \), and is chosen so that the flow is length-decreasing. It is easy to see that one can choose the uniform size of such neighbourhoods over all points \( b \) in the closed \((L + 2\delta)\)-neighbourhood of \( \bar{M}_0 \).

If some petals are very close to the base point, the remaining \( l \) petals are not, we move the flower as if it is a flower with just the \( l \) ”long” petals. We calculate the vector field at the base point and all intermediate points on long petals as above (that is, as the sums of the unit tangent vectors to all incident geodesic segments). The tangent vectors to short petals are not included into the sum at the base point. The components of the vector field at \( N \) points \( p \) on each short petal are then calculated as follows: First, consider the vector \( V(b) \) calculated for the base point \( b \) as the sum of unit tangent vectors to arcs of all long petals adjacent to \( b \). (Each long petal contributes two unit tangent vectors at both its endpoints that coincide with \( b \)). Consider the unique minimizing geodesic between \( b \) and \( p \) and parallel translate \( V(b) \) to \( p \) along this geodesic. Denote the result by \( V_1 \). Let \( V_2 \) denote the unit tangent vector at \( p \) to the geodesic between \( p \) and \( b \). We direct this vector towards \( b \). Define \( V(p) \) as \( V_1 + V_2 \). Looking at holonomies for geodesic triangles \( bp_i p_{i+1} \), where \( p_i, p_{i+1} \) are adjacent points on all short geodesic segments along ”short” petals, we see that there exists \( a > 0 \) such that if all ”short” petals are in the \( a \)-neighbourhood of the base point (and the base point is in the \( L \)-neighbourhood of \( \bar{M}_0 \)), then the first variation of the length for the flow defined by this vector field will be negative.

Now we can consider a covering of the set of all consider \( K \)-tuples of points in \( M^n \) by the open neighbourhoods of strata that correspond to degenerate flowers where some or all petals are constant as well as an open set corresponding to flowers where all petals are ”long” (or, more precisely, not contained in a small neighbourhood of the base point). Now we can use a subordinate partition of unity to combine the constructed vector fields on open sets into one vector field. The flow along this vector field will be length-decreasing.

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