Quantum Feedback Control of Atomic Motion in an Optical Cavity

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We study quantum feedback cooling of atomic motion in an optical cavity. We design a feedback algorithm that can cool the atom to the ground state of the optical potential with high efficiency despite the nonlinear nature of this problem. An important ingredient is a simplified state-estimation algorithm, necessary for a real-time implementation of the feedback loop. We also describe the critical role of parity dynamics in the cooling process and present a simple theory that predicts the achievable steady-state atomic energies.

The control of quantum systems is a problem that lies at the heart of several fields, including atom optics and nanomechanics. Many of the techniques developed thus far for controlling quantum systems in an initially unknown state involve the use of dissipative processes to achieve a well-defined final state, as in the laser cooling methods in atom optics [1]. A different paradigm for cooling involves modifying the system Hamiltonian based on information provided by measurements (i.e., quantum feedback control). Such a strategy for control is generally applicable to quantum as well as classical systems [2, 3]. As quantum-limited measurement techniques become more available, quantum feedback control is likely to soon be an essential element in atom-optical and nanomechanical toolkits.

One of the most promising avenues for the study of quantum feedback control centers on experiments in cavity quantum electrodynamics (CQED), where a single quantum system may be monitored continuously in real time, as has been demonstrated in a series of pioneering experiments [4, 10, 11, 12]. Furthermore, the optical potential due to the cavity light provides an easily adjustable means for applying forces on the atom. Feedback has been shown to increase the storage time for atoms in such an experimental system [13]. Despite these efforts, however, CQED feedback cooling has not yet been conclusively demonstrated, highlighting the need for a deeper theoretical understanding of this system as well as more sophisticated, higher performance cooling algorithms. Our goal here is to develop approximate estimation and control algorithms for cooling the atomic motion in such a CQED system—an “active cooling” approach that is distinct from passive cavity cooling methods [14]. We will then analyze the performance of these algorithms both analytically and through numerical simulations. This problem is particularly interesting due to the nonlinear form of the measurement operator, a situation distinct from previous studies of nonlinear quantum feedback control [13, 14]. An important issue that we address here is whether the (Gaussian) estimation methods developed for linear systems are still effective in this problem.

The setup we are considering is an atom in a microcavity in the strong-coupling regime, as in the aforementioned CQED experiments [3, 10, 11, 12], but where the output light is monitored via homodyne detection [10], which gives information about the atomic position in the cavity [17]. We consider the case where the light is resonant with the cavity, but the detuning from atomic resonance is much larger than the natural atomic linewidth. We thus analyze the coupled atom-cavity system after adiabatic elimination of the cavity and internal atomic dynamics [18]. The resulting stochastic master equation (SME) in Itô form [10] for the atomic motion conditioned on the homodyne measurement, describing the observer’s best estimate for the atomic state, is

\[
\dot{\rho} = -i[H_{\text{eff}}, \rho] dt - \Gamma [\cos^2(kX), [\cos^2(kX), \rho]] dt - \sqrt{2\eta \Gamma} \{[\cos^2(kX), \rho] + 2(\cos^2(kX))\rho\} dW. \tag{1}
\]

The effective atomic Hamiltonian is

\[
H_{\text{eff}} = \pi P^2 - V_{\text{max}} \cos^2(kX), \tag{2}
\]

and \(dW\) is given in terms of the scaled photodetector current (measurement record) \(d\tilde{r}\) by

\[
dW = d\tilde{r}(t)/\sqrt{\eta} + \sqrt{8\eta \Gamma} (\cos^2(kX)) dt. \tag{3}
\]

The physical interpretation of Eqs. (1) and (2) is that the photocurrent is a direct measurement of \(\cos^2(kX)\) within the adiabatic approximation. In these expressions, we are using scaled units such that position is measured in units of \(\sqrt{\hbar/(m\omega_{\text{ho}})}\), momentum is measured in units of \(\sqrt{\hbar/m\omega_{\text{ho}}}\), and time is measured in units of \(2\pi/\omega_{\text{ho}}\), where \(\omega_{\text{ho}} := \alpha g k \sqrt{2\hbar/(m\Delta)}\) is the oscillation frequency in the harmonic approximation for one of the optical potential wells, \(g\) is the CQED coupling constant, \(m\) is the atomic mass, \(\Delta\) is the detuning from the atomic resonance, \(\eta\) is the photodetection efficiency, \(\alpha\) is the mean cavity photon number in the absence of the atom,
The atomic parameters, we consider cesium driven on the optical potential that specify how the optical potential is modulated. For those used in recent CQED experiments \[9, 10, 11\], yielding

\[ \Gamma = 2 \kappa \]

\( k \) is the optical spatial frequency, \( \tilde{k} := k \sqrt{\hbar/(m \omega_{HO})} \), \( V_{\text{max}} := \pi/k^2 \), \( \Gamma := 2 \alpha^2 g^4/(\Delta^2 \kappa \omega_{HO}) \) is the effective measurement strength, \( \kappa \) is the energy decay rate of the cavity, and \( d\tau = \sqrt{\kappa/\omega_{HO}} d\tilde{\tau} \) is the physical photocurrent. The model thus has only three independent parameters, \( \tilde{k} \), \( \Gamma \), and \( \eta \), in addition to any parameters that specify how the optical potential is modulated. For the atomic parameters, we consider cesium driven on the D\(_2\) line, so that \( m = 2.21 \times 10^{-25} \text{ kg} \) and \( k = 11732 \text{ cm}^{-1} \). For the cavity parameters, we choose values similar to those used in recent CQED experiments \[8, 10, 11\], yielding \( \kappa/2\pi = 40 \text{ MHz} \), \( g/2\pi = 120 \text{ MHz} \), and \( \alpha = 1 \). We also assume an optical detuning of \( \Delta/2\pi = 4 \text{ GHz} \) to the red of the atomic resonance. The corresponding scaled parameters are \( \Gamma = 23.6 \) and \( \tilde{k} = 0.155 \). The scaled depth of the optical potential is \( V_{\text{max}} = 131 \), so that the lowest two band energies are within 1% of their values in the harmonic approximation. We also choose the idealized detection efficiency of \( \eta = 1 \). However, cooling performance similar to that shown below is achievable even in the presence of detector inefficiency (\( \eta \) well below 0.5) as well as other real-world limitations such as hardware propagation and computation delays \[20\].

While integration of the SME in order to obtain a state estimate is sensible in principle, in practice it is much too difficult to integrate the SME in real time. To solve this problem we must simplify the estimator. We make a Gaussian ansatz for the atomic Wigner function (which, equivalently, prescribes how the Weyl-ordered moments of the atomic state factorize). In doing so, we reduce the effort of integrating a partial differential equation to the much more manageable effort of integrating a set of five coupled ordinary differential equations for the two means \( \langle X \rangle \) and \( \langle P \rangle \) and the three variances \( V_x, V_p, \) and \( C_{xp} := \langle XP + PX \rangle - \langle X \rangle \langle P \rangle \) \[21\]. The tracking behavior of the Gaussian estimator is illustrated in Fig. 1, which compares the evolution of the position mean and variance for an atom evolving according to Eq. \(1\) with the same quantities from a Gaussian estimator evolution beginning with a different initial condition. The estimator rapidly locks on to the true solution. The ability of the estimator to track both quantities well is crucial to the success of the control algorithm that we present here.

Intuitively, an algorithm that responds to the wave-packet centroid, raising the optical potential while the atom is climbing one side of a well, should cool the atom. However, such a strategy eventually breaks down due to uncontrolled pumping of energy into the wave-packet variances. The key to obtaining an effective control algorithm is to consider the energy evolution due to the Hamiltonian part of the SME: \( \partial_t (H_{\text{eff}}) = -\langle \partial_t V_{\text{max}} \rangle \cos^2(\tilde{k}X) \). From this expression, it is clear that a “bang-bang” strategy is optimal for cooling if we consider only cyclic modulations of the potential amplitude within a limited range. The potential should be switched to the extreme low value when the quantity \( -\langle \cos^2(\tilde{k}X) \rangle \) is maximized and to the extreme high value when this quantity is minimized. We denote these extreme values of the potential by \( (1 - \varepsilon)^2 V_{\text{max}} \) and \( (1 + \varepsilon)^2 V_{\text{max}} \). Such a control strategy is sensitive to energy in both the centroids and variances of the atom. Of course, it is not possible to tell whether \( -\langle \cos^2(\tilde{k}X) \rangle \) is at an extremum based only on information at the current time. Thus, to complete the feedback algorithm, we fit a quadratic curve to the history of this quantity at each time step and trigger on the slope of this curve at the current time.

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**FIG. 1:** Tracking behavior of the Gaussian estimator for the position centroid and variance for a sample trajectory. The expectation values for the true wave packet (heavy lines) are shown with the corresponding Gaussian estimator quantities (narrow lines). The cooling algorithm was activated at \( t = 2 \).

**FIG. 2:** Evolution of the mean effective energy \( \langle H_{\text{eff}} \rangle \) (relative to the minimum potential energy), illustrating the cooling-algorithm performance. Shown are: cooling via the Gaussian estimator (heavy solid), cooling based on the photocurrent signal (narrow solid), and free atomic evolution with no cooling (dashed). Each curve is an average of 128 trajectories.
The cooling algorithm is able to counteract the measurement heating, and the atoms settle to a steady-state energy much lower than the initial energy. Also shown in Fig. 2 is the cooling performance of a much simpler algorithm, which arises by noting that the photocurrent is a direct measure of $-\langle \cos^2(kX) \rangle$ (plus noise), and that it should be possible to cool the atoms by triggering on the photocurrent signal itself (an approach similar in spirit to the “differentiating feedback” strategy in Ref. [13]). In this case, the same quadratic curve-fitting procedure is used to temper the measurement noise. Although cooling to a steady state still occurs in this case, the cooling performance is substantially worse than for the Gaussian estimator. This indicates that the Gaussian estimator acts as a nearly optimal filter (in the same sense as a Kalman filter [22]) for the relevant information in the photodetector signal. The results shown were generated assuming initial conditions and curve-fit details as follows, although the results do not sensitively depend on these values: The atoms begin in a coherent state $(V_x = V_p = 1/2)$, and each has an initial centroid energy corresponding to a displacement from the well center of 58% of the distance to the edge of the well, chosen so the atoms are clearly trapped but not particularly cold. The initial locations of the atoms are distributed uniformly between the bottom of the well and this maximum distance. The Gaussian estimator begins with the (impure-state) initial conditions $\langle X \rangle = 6$, $\langle P \rangle = 0$, $V_x = V_p = 1/\sqrt{2}$, and $C_x = 0$. The estimator is evolved in steps of $\Delta t = 0.0005$, and at each step the quadratic curve is fitted to the values of $-\langle \cos^2(kX) \rangle$ computed from the 300 most recent estimator steps.

The cooling performance is further illustrated in Fig. 3, which shows the populations in the lowest four bands of the optical potential. Most of the steady-state population is in the lowest two bands, accounting for 94% of the population, and these two bands are almost equally populated. We now show that this behavior can be understood in terms of the parity of the atomic state. The atomic parity is invariant under Hamiltonian evolution in the atomic potential (and hence the influence of the control algorithm). However, the measurement term in the SME causes the atomic parity to diffuse according to

$$d\langle P \rangle = -\sqrt{8\eta \Gamma} \left[ \langle P \cos(2kX) \rangle - \langle P \rangle \langle \cos(2kX) \rangle \right] dW,$$

(4)

where $P$ is the parity operator. In fact, this is an unbiased, nonstationary diffusion process that causes $\langle P \rangle$ to “purify” at late times to the extreme values $\pm 1$ of pure parity, so that the homodyne measurement acts like a QND measurement of the atomic parity. Since the atomic initial condition described above corresponds to $\langle P \rangle = 0$, the measurement process drives the atoms to states with either pure even or pure odd parity with equal probability. The obvious consequence of this effect for cooling is that only half of the atoms can be cooled to the ground band, while the remaining half can be cooled at best to the first excited band [21]. However, due to this purification effect (which means that the observer knows the final state of the atom on each individual run), cooling to the first excited band is essentially equivalent to cooling to the ground state, since the atom can be driven between these two bands with an additional coherent process (e.g., modulation via an additional classical optical potential).

Much of the residual energy (excitation out of the lowest two bands) is due to the inability of the Gaussian estimator to perfectly track the true atomic state. In this light, the cooling performance is quite impressive, especially considering the highly non-Gaussian nature of the first excited state, toward which half the atoms are

FIG. 3: Evolution of the band populations for the lowest four energy bands of the optical lattice, averaged over 128 trajectories. Top: cooling based on the Gaussian estimator. Bottom: cooling based on perfect knowledge of the actual wave function.
cooling. To show this behavior explicitly, we can remove
the estimator from the problem, and repeat the same
ensemble of simulations where the cooling algorithm op-
erates based on the value of $-\langle \cos^2(\tilde{k}X) \rangle$ computed
with respect to the actual atomic wave function. These results
are also shown in Fig. 3 and in this case the lowest two
bands account for 98% of the atomic population.

The steady-state atomic energies can be well described
by a simple analytic theory, which we now derive. We will
assume that the atomic energy is near the ground-state
energy $E_0$, so that we can make the harmonic approxi-
mation for the potential. We will further assume for the
moment that all the energy $\Delta E$ in excess of $E_0$ is asso-
ciated with the wave-packet centroid (i.e., the wave packet
is not squeezed). Then the quantity $-V_{\text{max}}(\cos^2(\tilde{k}X))$
exhibits oscillations of peak-to-peak magnitude $\Delta E$. As-
suming that the switching occurs at the extremal times,
the energy-evolution expression above leads to a coarse-
grained cooling rate of

$$\frac{\langle H_{\text{eff}} \rangle_{\text{ss}}}{\langle H_{\text{eff}} \rangle_{\text{crit}}} = \frac{E_0 + E_1}{2(1 - \beta)}.$$  

(5)

Here, $\beta := \Gamma \tilde{k}^4/2\varepsilon$, and we have made the replacement
$E_0 \rightarrow (E_0 + E_1)/2$ in view of the above parity consider-
ations, where $E_1$ is the energy of the first excited band.

We can also derive a theory for the other ex-
treme, where we assume all the excess energy is asso-
ciated purely with squeezing of the wave packet
and not with the centroid. In this case, the oscilla-
tions of $-V_{\text{max}}(\cos^2(\tilde{k}X))$ are of peak-to-peak magnitude
$\sqrt{\langle H_{\text{eff}} \rangle^2 - E_0}$. Then the $1 - \beta^{-1}$ dependence in Eq. (5)
is replaced by $(1 - \beta^2)^{-1/2}$. In the regime $\beta \ll 1$ where

this theory should be valid, this prediction is always be-
low that of Eq. (4), and thus the centroid-only theory
more faithfully represents the physical cooling limit.

Eq. (5) is compared with simulated late-time energies
in Fig. 4. The simulated energies are in good agree-
ment with the simple theory, although they are generally
higher than the simple theoretical prediction. Again,
much of this discrepancy is due to the imperfect tracking
behavior of the Gaussian estimator, and the simulated
energies from cooling based on perfect knowledge of the
actual wave function are in better agreement. Qualita-
tively, we see a transition between ineffective and effective
cooling with $\varepsilon$; essentially, $\varepsilon$ must be large enough that
the cooling rate exceeds the heating rate, ensuring that
the system is controllable. Beyond this transition, the
system displays a remarkable insensitivity to the value of
the switching amplitude; we observe similar robustness
to the changes in the other parameters $\Gamma$, $\tilde{k}$, and $\eta$.

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[1] H. J. Metcalf and P. van der Straten, Laser Cooling and
Trapping (Springer, New York, 1999).
[2] M. G. Raizen et al., Phys. Rev. A 58, 4757 (1998).
[3] V. P. Belavkin, Rep. Math. Phys., 43, 405 (1999).
[4] A. C. Doherty et al., Phys. Rev. A 62, 012105 (2000).
[5] M. A. Armen et al., Phys. Rev. Lett. 89, 133602 (2002).
[6] D. Vitali, S. Mancini, L. Ribichini, and P. Tombesi, J.
Opt. Soc. Am. B 20, 1054 (2003).
[7] A. Hopkins, K. Jacobs, S. Habib, and K. Schwab, Phys.
Rev. B 80, 235328 (2003).
[8] JM Geremia, J. K. Stockton, and H. Mabuchi, arXiv.org
preprint quant-ph/0309034.
[9] C. J. Hood, M. S. Chapman, T. W. Lynn, and H. J.
Kimble, Phys. Rev. Lett. 80, 4157 (1998).
[10] H. Mabuchi, J. Ye, and H. J. Kimble, Appl. Phys. B 68, 1095
(1999).
[11] C. J. Hood et al., Science 287, 1447 (2000).
[12] P. W. H. Pinkse, T. Fischer, P. Maunz, and G. Rempe,
Nature (London) 404, 365 (2000).
[13] T. Fischer et al., Phys. Rev. Lett. 88, 163002 (2002).
[14] P. Horak et al., Phys. Rev. Lett. 79, 4974 (1997).
[15] H. M. Wiseman, S. Mancini, and J. Wang, Phys. Rev. A
66, 013807 (2002).
[16] H. M. Wiseman and G. J. Milburn, Phys. Rev. A 47, 642
(1993).
[17] R. Quadt, M. Collett, and D. F. Walls, Phys. Rev. Lett.
74, 351 (1995).
[18] A. C. Doherty, A. S. Parkins, S. M. Tan, and D. F. Walls,
Phys. Rev. A 57, 4804 (1998).
[19] P. E. Kloeden and E. Platen, Numerical Solution
of Stochastic Differential Equations (Springer, Berlin,
1992).
[20] D. A. Steck et al., in preparation.
[21] See EPAPS Document No. E-PRLTAO-92-053419 for sample animations of two quantum trajectories and details regarding the Gaussian estimator. A direct link to this document may be found in the online reference section for this article at http://link.aps.org/abstract/PRL/v92/e223004.

[22] A. C. Doherty and K. Jacobs, Phys. Rev. A 60, 2700 (1999).