Finite temperature phase transition for disordered weakly interacting bosons in one dimension

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It is commonly accepted that there are no phase transitions in one-dimensional (1D) systems at a finite temperature, because long-range correlations are destroyed by thermal fluctuations. Here we demonstrate that the 1D gas of short-range interacting bosons in the presence of disorder can undergo a finite temperature phase transition between two distinct states: fluid and insulator. None of these states has long-range spatial correlations, but this is a true albeit non-conventional phase transition because transport properties are singular at the transition point. In the fluid phase the mass transport is possible, whereas in the insulator phase it is completely blocked even at finite temperatures. We thus reveal how the interaction between disordered bosons influences their Anderson localization. This key question, first raised for electrons in solids, is now crucial for the studies of atomic bosons where recent experiments have demonstrated Anderson localization in expanding very dilute quasi-1D clouds.

The absence of finite temperature phase transitions in one-dimensional (1D) systems is considered as almost a dogma. Its justification is based on another dogma which states that any phase transition is related to the appearance/disappearance of a long-range order or at least long-range spatial correlations. Thermal fluctuations in 1D systems destroy any long-range order, lead to exponential decay of all spatial correlation functions and thus make phase transitions impossible. Non-interacting quantum particles in a one-dimensional random potential show a similar behavior in the sense that all single-particle eigenfunctions are localized, i.e. decay exponentially in space. The same statement holds for two dimensional systems without spin-orbit interaction. By contrast the single-particle states in three dimensions are either localized or extended as a result of the Anderson transition. In this paper we demonstrate that the 1D gas of weakly short-range interacting bosons in the presence of disorder exhibits a finite temperature phase transition between two distinct states, fluid and insulator, and the transition temperature depends on the disorder. None of these states is characterized by a long-range order or long-range spatial correlations. Moreover, thermodynamic functions, such as specific heat, do not have singularities at the transition point. From this point of view, the dogma is not violated. Nevertheless, this is a true albeit non-conventional phase transition, because transport and energy dissipation properties of the fluid and insulator phases are dramatically different and are singular at the transition. The difference between the fluid and insulator phases can be qualitatively understood by comparing two many-body 1D systems: interacting particles without disorder and the 1D Anderson insulator (disorder without interactions). In the fluid without disorder, the dissipation of energy of an arbitrarily slow external field and mass transport are possible. At the same time, even at finite temperatures there is no mass transport in Anderson insulators and the energy dissipation vanishes for the frequency of the external field tending to zero. Here we show that interacting 1D bosons in disorder demonstrate one of the two types of behavior and describe the phase diagram in the temperature-disorder plane. We thus provide an answer to the subtle question of how the interaction between disordered particles may suppress Anderson localization and permit them to acquire the fluid behavior. This was the key problem for charge transport in electronic systems, and it is now emerging in a new light in the studies of disordered ultracold bosons. These studies are driven by fundamental interest and by potential applications of atom waveguides on a chip. Recent remarkable experiments have demonstrated Anderson localization in expanding extremely dilute quasi-1D Bose gases, and the investigations of effects of the interparticle interaction in relatively dense clouds are underway.

I. SINGLE PARTICLE LOCALIZATION IN 1D

Let us first discuss the density of states (DoS) for a single particle with mass $m$ in 1D and introduce relevant energy and distance scales. In the absence of disorder the particle eigenstates are plane waves with energies $\epsilon > 0$. The DoS is $\nu(\epsilon) = \sqrt{m/2\pi^2\hbar^2} \epsilon$ and it diverges at $\epsilon \to 0$.

What happens in a static random potential? At large and positive $\epsilon$ the DoS is only slightly affected by disorder: $\nu(\epsilon) \approx \nu(0) \sim 1/\sqrt{\epsilon}$. At the same time, the disorder cuts off the DoS divergence at $\epsilon = 0$ and transforms it into a peak with a finite height $\nu_*$ and width $E_*$ as shown in Fig. 1. Also, the disorder creates states with negative energies, thus making the DoS finite at $\epsilon < 0$. For large negative $\epsilon$ the DoS is exponentially small, and this part of the curve $\nu(\epsilon)$ is known as Lifshitz tail.

Both $\nu_*$ and $E_*$ are determined by the statistics of the random potential $U(x)$. For simplicity we assume a short-range Gaussian potential with the amplitude
$U_0$ and correlation length $\sigma$ such that condition $U_0 \ll h^2/m\sigma^2$ hold. Then the only relevant energy and length scales are

$$E_* \sim (U_0^3/\sigma^2m/h^2)^{1/3};$$
$$\zeta_* = h/\sqrt{mE_*} = (h^3/U_0^3\sigma^2m^2)^{1/3}. \quad (2)$$

They determine the width of the DoS peak and the maximum DoS value $\nu_* \sim 1/E_*\zeta_*$. In order to obtain these scales consider a weakly bound state of a particle in the potential $U(x)$, with an extension of the wave function, $\zeta \gg \sigma$. The particle energy can be written as $E \sim (h^2/2m\sigma^2 - U_0\sqrt{\sigma/\zeta})$, where for the Gaussian disorder the potential energy term is obtained multiplying the contribution of each potential well, $U_0\sigma/\zeta$, by $\sqrt{\zeta/\sigma}$, which is the square root of the number of wells on the length scale $\zeta$. The energy $E$ reaches a minimum value $E_*$ at $\zeta \sim \zeta_*$, with $E_*$ and $\zeta_*$ given by Eqs. (1) and (2).

We thus see that the single-particle spectrum can be divided into three parts: high-energy states with $\epsilon \gg E_*$ and $\nu(\epsilon) \sim \sqrt{m/\epsilon\zeta}$, low-energy states located in the region of the DoS peak, and Lifshitz tail at negative energies. As we already noted, all single-particle eigenfunctions in 1D are localized with an energy-dependent localization length $\zeta(\epsilon)$. For high-energy states, $\epsilon \gg E_*$, we have $\zeta(\epsilon) \sim \epsilon\zeta_*/E_*$, whereas in the Lifshitz tail $\zeta(\epsilon) \sim \epsilon\zeta_*/|\epsilon|$. For the low-energy states in the DoS peak, $|\epsilon| \lesssim E_*$, the localization length is $\zeta(\epsilon) \sim \zeta_*$. 

![FIG. 1: Properties of single particle localization. In (a) the brown curve shows the disorder potential $U(x)$, with blue solid lines and dashed red lines indicating the location of tail and high-energy states, respectively. The shape of the wavefunctions of these states is shown by blue (tail) and red (high-energy) curves $\psi(x)$. In (b) the density of states $\nu$ and localization length $\zeta$ versus energy $\epsilon$ are shown in blue for the DoS peak and tail states, and in red for high-energy states. For studying the many-body localization transition, $\nu(\epsilon)$ and $\zeta(\epsilon)$ in the DoS peak (low-energy states) can be approximated by the green dotted lines.](image)

**II. MANY-BODY LOCALIZATION-DELOCALIZATION TRANSITION**

Repulsive short-range interaction between bosons in 1D gives rise to two other energy scales in addition to $E_*$: the temperature of quantum degeneracy $T_d = h^2n^2/m$ (we use the units with Boltzmann constant $k_B = 1$), and the interaction energy per particle, $ng$, where $g$ is the coupling constant for the interaction, and $n$ is the mean boson density. We focus on the weakly interacting regime where the dimensionless coupling strength is small:

$$\gamma \equiv ng/T_d \ll 1 \quad (3)$$

so that at the mean separation between the bosons their wavefunction is not influenced by the interactions. It is convenient to introduce the dimensionless temperature

$$t = T/\gamma T_d = 1/T_d. \quad (4)$$

Another dimensionless parameter characterizes the strength of the disorder:

$$\kappa \equiv E_*/ng \quad (5)$$

so that large values of $\kappa$ correspond to strong disorder. At this point we should make two important statements. If the disorder is extremely strong ($\kappa \rightarrow \infty$),
the bosons occupy only states in the Lifshitz tail. As the DoS in the tail is exponentially small, the bosons are distributed among the “lakes” located exponentially far from each other. The bosons can not hop between the lakes, and the system is in an insulating state. The reduction of $\kappa$ at $T = 0$ eventually (for $\kappa = \kappa_c \approx 1$) transforms the insulator into an algebraic superfluid (spatial phase correlations do decay, but only algebraically). This Kosterlitz-Thouless type transition was first analyzed in Ref.[13] and more recently discussed in relation to disordered Josephson chains[14] and to cold atomic gases.[15,16] However, well before the transition from insulating to superfluid state most of the particles find themselves in low-energy states where the DoS is much higher than in the Lifshitz tail. Thus, we may neglect the tail in our discussion of the fluid-insulator transition and consider only low-energy and high-energy states (green and red in Fig. 1a).

Second, although the interaction between bosons renormalizes (screens) the disordered potential $U(x)$, this does not change the picture of single-particle eigenstates in our discussion. The reason is that relevant particle energies are of the order of $E_\ast$ or larger. For $\kappa \gtrsim 1$ they exceed the interaction energy $ng$, and as we will see there is no need to consider $\kappa \ll 1$. In this respect, the main effect of the interaction on the ground state of the system is not screening the random potential but rather controlling the occupation of single-particle states.

FIG. 2: Scattering processes leading to the many-body localization-delocalization transition for a classical Bose gas $(T > T_d)$ in (a), for a degenerate thermal Bose gas $(T_0 < T < T_d)$ in (b), and for the low-temperature regime $(T < T_d)\sqrt{\kappa}$ in (c).

At a finite temperature $T$ it is crucial to take into account two-body processes that change occupation numbers and can dramatically affect the properties of the system. For example, the system of interacting localized fermions (electrons) can have a finite DC conductivity even in the absence of coupling with any outside bath[17], whereas without interactions the conductivity is exactly zero at any $T$. In the presence of the interactions the conductivity remains zero unless the temperature exceeds a critical value $T_c$. This transition can be thought of[17] as Anderson localization[13] of many-body wavefunctions. The critical temperature $T_c$ depends on the interaction strength. This is the many-body analog of the mobility edge[15,16] that separates bands of localized and extended states in the single-particle Anderson transition.

The Anderson transition is based on the fact that two quantum states belonging to different lattice sites hybridize provided that the hopping matrix element between these states exceeds the difference in onsite energies. As soon as the density of the hybridized pairs exceeds a critical value the eigenstates turn out to be extended.[13] The many-body localization-delocalization transition can be qualitatively understood by extending this physical picture to states of more than one particle.

Consider an occupied localized single-particle state $|i\rangle$ with energy $\epsilon_i$. The interaction of a particle occupying this state with a particle in the state $|k\rangle$ can transfer the $|i\rangle$-state particle to the state $|j\rangle$, transferring simultaneously the $|k\rangle$-state particle to another state $|l\rangle$ (see Fig. 2). Let $U_{ik, jl}$ be the matrix element of this process. Due to the exponential decay of the localized wave functions one may assume that $U_{ik, jl} = 0$ unless all four states are located near each other. Moreover, it turns out[20,21] that the matrix element rapidly decreases with an increase in the energy transfer $|\epsilon_i - \epsilon_k|$. We thus may confine ourselves to the case where the states $|i\rangle$ and $|k\rangle$...
are nearest neighbors in the energy space. For simplicity we replace \( U_{ik,jl} \) by a certain typical value \( U_t \) provided that the states \(|i\rangle, |j\rangle, |k\rangle, |l\rangle\) are localized nearby and pairwise are nearest neighbors in energy.

In a random system the energies of the final and initial states can not be matched exactly. As long as the energy mismatch \( \Delta_{ik,jl} = \epsilon_i + \epsilon_k - \epsilon_j - \epsilon_l \) exceeds \( U_{ik,jl} \) the effect of the interactions on the quantum state of 4 particles is negligible. One may say that single-particle excitations do not decay. Suppose that the interaction is weak and a typical mismatch \( \Delta_t \) exceeds the matrix element \( U_t \). Does it mean that single-particle excitations have an infinite lifetime? The answer depends on the number of channels, \( N_1 \), for the decay of a given excitation (more precisely, \( N_1 \) is the number of possible processes \(|i\rangle, |k\rangle \rightarrow |j\rangle, |l\rangle\) that involve a given state \(|i\rangle\)). Indeed, with probability of order unity, these processes should have a channel with the mismatch that is smaller by a factor of \( N_1 \) than the typical value \( \Delta_t \). Therefore, \( U_t \) should be compared with \( \Delta_t/N_1 \). Note that \( N_1 \) plays a role of the number of nearest neighbors in the single-particle localization problem. However, in the many-body case \( N_1 \) is determined by the density of thermal excitations and is temperature dependent. Since characteristic single-particle energies and localization lengths are determined by the temperature, both \( \Delta_t \) and \( U_t \) are also temperature dependent. As a result, there is a critical temperature \( T_c \) following from the equation

\[
\Delta_t(T_c) = U_t(T_c) N_1(T_c). \tag{6}
\]

At \( T > T_c \) many-body states are extended, i.e. they are linear combinations of one-, two-particle, etc. excitations, and the number of terms is infinite. This leads to the fluid behavior. For \( T < T_c \) the many-body localization takes place and one expects the insulating behavior.

Note that the arguments given above and Eq. \(6\) are general and independent of quantum statistics of the particles. At the same time, both \( U_t(T) \) and \( N_1(T) \) do depend on the statistics. For disordered bosons these quantities are determined by the density of single-particle states \( \nu(\epsilon) \) and by the occupation number \( N(\epsilon) \). The latter is controlled by the chemical potential \( \mu \) which is related to the mean density \( n \) and temperature \( T \) by the normalization condition \( n = \int d\epsilon N(\epsilon) \nu(\epsilon) \), with \( N(\epsilon) \) determined by the Bose-Einstein distribution. More precisely, as long as the interaction is weak \( (\gamma \ll 1) \) the occupation is \( N(\epsilon) = (\exp[\epsilon_{HF} - \mu]/T - 1)^{-1} \), where \( \epsilon_{HF} \) differs from the single-particle energy \( \epsilon \) by the Hartree-Fock corrections. We will see that these corrections become important only at sufficiently low \( T \) and are negligible at high temperatures where in the vicinity of the transition relevant particle energies greatly exceed \( n \).

### III. PHASE DIAGRAM

We now use Eq. \(6\) and analyze the phase diagram of weakly interacting disordered 1D bosons. It is convenient to represent the phase diagram in terms of the dimensionless temperature \( t = T/\sqrt{n g} \) and find the dependence of \( t_c \) on the strength of the disorder \( \kappa \). The relation \( t = t_c(\kappa) \) determines the boundary between the fluid and insulator phases in the \((t, \kappa)\) plane. Alternatively, one can speak of a temperature-dependent critical disorder \( \kappa_c(t) \).

At high temperatures \( T > T_d \), or \( t > \gamma^{-1} \), the Bose gas is not degenerate and a characteristic energy of particles is of the order of \( T \gg n g \). As will be seen, near the transition temperature we have \( E_\ast < T \) so that most of the particles are in the high-energy states and the occupation of all the states is small and described by Boltzmann distribution. In this case the typical matrix element \( U_t(T) \) in Eq. \((6)\) does not depend on the occupation and can be estimated as \( U_t(T) \sim g/(\zeta(T)) \). The typical mismatch is the mean nearest neighbor energy spacing: \( \Delta_t \sim \sqrt{T}/(\zeta(T)) \). The quantity \( N_1(T) \) is given by the number of particles localized within a distance \( \zeta(T) \) from a given state \(|j\rangle\), i.e. \( N_1(T) \approx (\zeta(T)/\sqrt{E}) \). Then, Eq. \((6)\) is reduced to \( \nu g(\zeta(T)/\sqrt{E}) \sim 1 \). Using the high-energy localization length \( \zeta(T) = \zeta(T)/E \) and the density of states \( \nu(T) = 1/\sqrt{E
\zeta^2(T)/T} \), with the help of Eqs. \((4)\) and \((5)\) we obtain the high temperature relation for the critical disorder:

\[
\kappa_c(t) \sim t^{1/3}; \quad t > \gamma^{-1}. \tag{7}
\]

Close to the transition we have \( E_\ast /T = \kappa /t \sim t^{-2/3} \ll g/\zeta^2 \ll 1 \), which justifies our initial assumption that most of the particles are in the high-energy states.

As the temperature is reduced below \( T_d \), the Bose gas becomes degenerate. In the absence of disorder the chemical potential is \( \mu = -T^2/T_d \) as long as \( T > T_d \sqrt{\gamma} \), or \( t > 1/\sqrt{\gamma} \). Characteristic energies of particles are \( \sim |\mu| < T \). However, as will be seen below, they still exceed both \( n \) and \( E_\ast \). Therefore, most of the 1D disordered interacting bosons in the temperature interval \( \gamma^{-1/2} < t < \gamma^{-1} \) occupy high-energy states and one can use the ideal Bose gas distribution \( \nu(\epsilon) \), i.e. neglect Hartree-Fock corrections to single-particle energies. The major part of particles has energies \( \epsilon \ll |\mu| \) and is characterized by a multiple occupation \( N(\epsilon) \sim T/\epsilon \gg 1 \). This manifests itself in the dependence of interaction matrix elements on \( N(\epsilon) \).

In this regime there are two energy scales \((T \text{ and } \mu \ll T)\) characterizing the distribution of particles. What are the particle energies that determine the many-body delocalization? Let us apply Eq. \((6)\) to particles with energies \( \epsilon \sim E \) in the energy interval of width \( \sim E \). A typical value of the energy spacing is \( \Delta_t = (1/\nu(E))(E) \), and the typical matrix element of the two-body interaction is enhanced due to a multiple occupation of single-particle states: \( U_t = (g/\nu(E))(E) \). The number \( N_1 \) of occupied levels at distances smaller than \( \zeta(E) \) from a given state is \( N_1(E) \sim E \nu(E)(E) \) and Eq. \((6)\) takes the form

\[
N_1 U_t/\Delta_t \sim g T_d (E)^2 \zeta(E)(E). \tag{8}
\]

Then, using the high-energy density of states \( \nu(E) = 1/\sqrt{E \zeta^2} \) and localization length \( \zeta(E) = \zeta(E)/E \), we
find a remarkable result: the criterion of delocalization does not involve the single-particle energy scale! The transition temperature follows from the relation $gT_c \sim E_0^2 \zeta_\star = \sqrt{\hbar^2 E_0^2/m}$. (The fact that this is valid for all energy scales suggests that the expression for the critical temperature/disorder can contain a prefactor logarithmic in $\gamma$, which we neglect.) In terms of the parameters $t$ and $\kappa$ the relation for the critical disorder becomes

$$\kappa_c = t^{2/3}\gamma^{1/3}; \quad \gamma^{-1/2} < t < \gamma^{-1}. \quad (8)$$

Equation (8) shows that $\kappa_c \gtrsim 1$ in the entire temperature interval $\gamma^{-1/2} < t < \gamma^{-1}$. Hence, close to the transition we have $E_\star \gtrsim ng$ and characteristic particle energies are $\sim |\mu| \gtrsim E_\star \gtrsim ng$. This justifies our assumption that the major part of particles occupies high-energy states, and the interparticle interaction affects neither the occupation numbers $N(\epsilon)$ nor the chemical potential $\mu$.

Consider now $t = 0$. For $\kappa \gg 1$ the boson density is fragmented into “lakes”. Lake number $i$ is formed by $N_i$ bosons in the single-particle eigenstate $|i\rangle$ which is characterized by energy $\epsilon_i$ and localization length $\zeta_i \approx \zeta_\star$. The energy cost $E_i$ of bringing an extra particle to this lake is enhanced by the repulsive interaction between the bosons, $E_i \approx \epsilon_i + gN_i/\zeta_\star$, and it should be equal to the global chemical potential $\mu$ measured from the lowest low-energy state $\epsilon = 0$. The bosons thus occupy only states below the chemical potential, $\epsilon_i < \mu$, with the occupation numbers $N_i \approx (\mu - \epsilon_i)/g$, and as long as $\mu < E_\star$ only low-energy states are occupied. The density of these states is $\nu_\star = 1/E_\star\zeta_\star$, and thus the mean density of bosons is related to the chemical potential by $n = \mu^2/2gE_\star$. The chemical potential can be expressed in terms of the parameter $\kappa$ as $\mu = E_\star/\sqrt{\kappa}$, i.e. the chemical potential is indeed smaller than $E_\star$ provided that $\kappa > 1$. In this regime only a small fraction ($\sim \mu/E_\star$) of low-energy states is occupied and neighboring lakes are separated by a distance $l(\kappa) \sim \zeta_\star\sqrt{\kappa}$, while their size is $\sim \zeta_\star$. Using equations (1) and (5) one can show that a typical lake $(\epsilon_i \sim \mu)$ contains $N_i \sim n\zeta_\star\sqrt{\kappa} = 1/\sqrt{\gamma}$ bosons.

The fact that $l(\kappa) \gg \zeta_\star$ implies that for $\kappa \gg 1$ the system is a strong insulator: the coupling between different lakes is exponentially small in $\kappa$. As soon as $\kappa$ is reduced to the value of the order of unity, the distance between neighboring lakes becomes of the order of their size $\zeta_\star$ and the interlake coupling drives the system to the fluid state. So, the insulator-fluid Berezinskii-Kosterlitz-Thouless transition at $T = 0$ occurs for $\kappa_c \sim 1$.

Note that at the lower bound of the temperature interval in Eq. (8), $t \sim \gamma^{-1/2}$, the critical disorder is also $\kappa_c \sim 1$. Therefore, one expects that in the entire temperature range $t < \gamma^{-1/2}$ it remains $\kappa_c \sim 1$.

Why the insulating state of bosons is stable at these temperatures as long as $\kappa > 1$, i.e. $E_\star > ng$? Let the disorder be as strong and reduce the temperature below $\sqrt{E_\star T_d} \left( t < \sqrt{\kappa/\gamma} \right)$. Under these conditions we have $|\mu| < E_\star$ and only a few bosons are hosted by high-energy states. For high-energy bosons $|\epsilon - |\mu||$ the condition $|\epsilon - |\mu|| < E_\star$ is not satisfied because their density is too small for the many-body delocalization. It turns out that the main body of the bosons, which occupy low-energy states, also forms an insulator. Indeed, for low-energy states we have $\epsilon < E_\star$ and the number of channels $N_1 \sim \nu(\epsilon)\zeta(\epsilon)$ is smaller than unity, since $\nu(\epsilon)\zeta(\epsilon) \sim E_\star^{-1}$. In other words, most of the particles occupy single-particle states which are separated from each other by distances exceeding the localization length $\zeta_\star$. This causes exponential reduction of $U_i$ and, according to Eq. (8), prevents delocalization. Finite temperature in this situation does not lead to any increase in the phase volume for available transitions. By contrast temperature fluctuations of the number of particles in each lake lead to a growth of the energy mismatch by an amount $\sim \sqrt{Tg/\zeta_\star}$, which further suppresses the probability to hybridize several states. The physical situation is somewhat similar to the one described in Ref. 23, where it was demonstrated numerically that in a finite-width band with less than one state per localization length the insulator remains stable with respect to the interactions at arbitrarily high temperatures.

If the disorder is weak, $\kappa < 1$, the chemical potential is determined by the interaction and always exceeds $E_\star$. As a result, we have $N_i(T) > 1$. The condition (6) then indicates that the insulator is unstable, and one deals with the fluid state.

The arguments presented above indicate that the phase transition line has to be almost horizontal in the region $t < 1/\sqrt{\gamma}$. We can also realize that at $t \to 0$ the line $\kappa_c(t)$ should terminate at the quantum phase transition point $\kappa = 1$. Indeed, assuming $\kappa_c(t = 0) < 1$ we arrive at a contradiction as the underlying superfluid phase at $T = 0$ has delocalized excitations (phonons) at low energies. On the other hand, the assumption of $\kappa_c > 1$ is also not consistent with the criterion (6) as all of the excitations are localized and the temperature should be finite to provide a finite density of the excitations.

$$\kappa_c = t^{2/3}\gamma^{1/3}$$

$\kappa_c(t)$ follows from Eqs. (7) and (8), and the blue part is an estimate for the low-temperature regime.

FIG. 3: Phase diagram for weakly interacting disordered bosons. The green line shows the zero temperature algebraic superfluid. The red part of the curve $\kappa_c(t)$ follows from Eqs. (7) and (8), and the blue part is an estimate for the low-temperature regime.
This completes our description of the finite temperature fluid-insulator phase transition for 1D interacting bosons. The phase diagram is presented in Fig. 3. Detailed calculations and a more accurate description of the temperature range \( t < \gamma^{-1/2} \) (blue part of the curve \( \kappa(t) \)) in Fig. 3 will be presented elsewhere.

IV. DYNAMICS OF EXPANSION

The observation of the fluid-insulator phase transition described above is feasible in experiments with cold bosonic atoms in the 1D geometry. For electrons in solids one measures e.g. the DC conductivity. In quantum (neutral-atom) gases the analysis of transport properties is based on the dynamics induced by significant external perturbations which may drive the system far from the initial state. We believe that the localization-delocalization transition can be identified in the dynamics of expansion of disordered systems released from the superimposed trapping potential.

In an array of harmonically trapped quasi-1D tubes with about a hundred of atoms per tube, the density \( n \) can be made \(~\sim~\) \( 10^4 \) cm\(^{-1}\) so that the length \( L \) of each tube is \(~\sim~\) \( 100 \mu m \) and the temperature of quantum degeneracy \( T_d \) is of the order of tens of nanokelvins. Tuning the interaction strength by Feshbach resonances or by variations of the tight transverse confinement one can achieve the interaction energy \( \hbar U_i \) of the order of nanokelvins or even larger values in the experiment\( \textsuperscript{12,13} \). The dimensionless strength of the disorder, \( \kappa \), ranges from about unity to large values and one can to study all temperature regimes of the phase diagram in Fig. 3. Note, however, that the conditions described here are quite different from those in the experiment\( \textsuperscript{12,13} \) where the interaction energy was greatly exceeding the interaction energy \( \hbar U_i \) of the order of tens of nanokelvins. Tuning the interaction strength by Feshbach resonances or by variations of the tight transverse confinement one can achieve the interaction energy \( \hbar U_i \) of the order of tens of nanokelvins.

Switching off abruptly the 1D trap but still keeping the disorder (and the transverse tight confinement) like in the experiments\( \textsuperscript{12,13} \) is expected to cause the expansion of the cloud. Not very far from the fluid-insulator transition, the localization length \( \zeta \) of single-particle states in the initial cloud is much smaller than its size \( L \). Under this condition, the size rapidly increases by an amount of \(~\sim~\) \( \zeta L \). If the entire initial cloud is in the insulator phase, i.e. locally \( \kappa > \kappa_c(t) \), the expansion then stops (see Fig. 4b). If the central part of the cloud is in the fluid phase (while the outskirts are insulators), two-body scattering processes induce further expansion of the central part. In this stage the expansion is a slow diffusive process governed by the diffusion equation:

\[
\frac{\partial n}{\partial \tau} = \frac{\partial}{\partial x} D(n,t) \frac{\partial n}{\partial x},
\]

where \( \tau \) denotes the time and \( t \) is the dimensionless temperature \( t_\gamma \). The diffusion coefficient \( D(n,t) \) strongly depends on the local density \( n(x) \) and vanishes at the border of the insulating phase, i.e. for the density \( n_c(t) = E_\gamma/(\hbar \xi_c(t)) \). Hence, the local decrease of the density stops when it becomes equal to \( n_c(t) \). Thus, the density profile of the expanded cloud represents a plateau with \( n = n_c(t) \) in the central part and the initial insulating wings with \( n(x) < n_c \) (see Fig. 4b).

Moderately far from the transition, i.e. at \((n - n_c)/n_c \sim 1 \), the diffusion coefficient can be estimated as follows. The occupation \( N \) of the state \(| \psi \rangle \) changes by \(~\sim~\) \( 1 \) on a time scale of the order of the inverse matrix element \( \hbar / U_i \sim \hbar / g N_i \). The distance of a typical hop is \(~\sim~\) \( \zeta \). Therefore, on a time scale \( \tau_0 \sim N_i \hbar / U_i \sim \hbar / g \) each boson will move by a distance \(~\sim~\) \( \zeta \), and \( D \sim \zeta^2 / \tau_0 \sim \zeta / \hbar \). For high-energy states the localization length at typical energies \( \epsilon \) is \( \zeta(\epsilon) = \zeta_c \epsilon / E_\gamma \) and the diffusion coefficient is given by \( D \sim \epsilon \zeta / \hbar E_\gamma \). Locally, the decrease of the density ceases when it reaches the critical value \( n_c(t) \). Thus, for the initial central density comparable with \( n_c(t) \), the evolution of the cloud to the final shape of the plateau and wings (see Fig. 4c) requires a characteristic time \( \tau_e \sim L^2 / D \). Since \( L \sim (\epsilon / \hbar \omega^2)^{1/2} \), \( \omega \) is the initial trap frequency, we obtain

\[
\tau_e \sim \frac{1}{\omega} \frac{\hbar E_\gamma}{m \omega \zeta_c g} \sim \frac{1}{\omega} \left( \frac{n_g}{\hbar \omega} \right)^{3/2} \frac{\gamma^3/2}{\gamma^1/2}.
\]

However, in the temperature interval \( T_d \sqrt{\gamma} < T < T_d \) most of the particles, which have energies \( \epsilon \sim |\mu| = T^2 / T_d \), and particles with \( \epsilon \sim T \) will expand with different velocities. So, the redistribution of particles in the course of the expansion may become important and it can slightly modify the above estimate for the time \( \tau_e \).

The time \( \tau_e \) by far exceeds the time \( \hbar / \omega \) of ballistic expansion of the clean thermal cloud. For example, near the lower bound of Eq. (9), i.e. for \( T \sim T_d \) and \( \kappa \sim \gamma^{-1/3} \), we have \( \tau_e \sim T_d / \hbar \omega^2 \). For typical values \( T_d \sim 50 \) nK and \( \omega \sim 10 \) Hz this estimate yields \( \tau_e \sim 1 \) s. Moving to the lower bound of Eq. (10) where \( T \sim T_d \sqrt{\gamma} \) and \( \kappa \sim 1 \), the time \( \tau_e \) reduces by a factor of \( \sqrt{\gamma} \), i.e. it is of the order of 0.1 s. As we see, these time scales are such that one can observe the evolution of the expanding cloud from the very beginning until the final density distribution is formed. The expansion of the fluid part of the cloud will also be slow in the low-temperature regime, \( T < T_d \sqrt{\gamma} \) \( t < \gamma^{-1/2} \). The analysis of the diffusion coefficient for this case is beyond the scope of the present paper.

V. PHASE DIAGRAM IN HIGHER DIMENSIONS

We conclude our discussion of the properties of disordered interacting bosons with a brief sketch of the phase diagram in higher dimensions. As is well known, in the absence of disorder bosons form a superfluid (algebraic superfluid in 2D) below a critical temperature \( T_c \). At
high temperatures, $T > \bar{T}_c$, the clean system is a normal fluid. The superfluid survives a sufficiently weak disorder, but the superfluid transition temperature $\bar{T}_c$ decreases with increasing the strength of the disorder and vanishes at a critical strength $\gamma \gg 1$ (see the black curve and brown point in Fig. 5). What is the state of the disordered system at $T > \bar{T}_c$, i.e. above the black curve in Fig. 5? It follows from our previous discussion that bosons can form either the normal fluid or the insulating state. The suggested phase diagram is presented in Fig. 5.

This sketch can be justified in the following way. First, it is safe to assume that at sufficiently strong disorder bosons are in the insulating state. The second observation is that at the critical disorder (brown point in Fig. 5) and arbitrarily small but finite $T$ one should expect the normal fluid rather than the insulator. Indeed, the zero-temperature insulator can be thought of as a system of superfluid lakes which are separated from each other and have uncorrelated phases. A typical size of such a lake increases with decreasing the strength of the disorder and diverges at the critical strength. It means that although excitations in the insulating phase at $T = 0$ are always localized, their localization length can be arbitrarily large. Therefore, at any finite temperature and interaction strength there is a vicinity of the critical disorder, where the insulator is unstable with respect to the many-body delocalization. As a result, the insulator - normal fluid phase boundary can not follow the lower blue curve on Fig. 5

On the other hand, the normal fluid can not be stable at $T = 0$. Indeed, the wavefunctions of low-energy single-particle states have to be localized, otherwise particles can not avoid Bose-Einstein condensation and the system would become superfluid. At extremely low but finite temperatures, the density of thermal excitations is vanishingly small and the interaction between them is unable to delocalize many-body wavefunctions. This rules out the phase boundary following the upper blue curve in Fig. 5. Thus, the only possible option is represented by the red solid curve $\gamma \gg 1$.

The arguments given above apply to both 2D and 3D cases. However there is a big difference, since in 2D all single-particle states are known to be localized, and the strong-disorder phase is a true insulator. This means that the diffusion constant is zero even at finite temperatures. At the same time, in three dimensions high-energy states can be extended. As soon as the extended states appear they can host thermal excitations and thus allow exponentially small but finite diffusion. From this point of view, there is no qualitative difference between the insulator and normal fluid, and the red curve in 3D represents a crossover rather than a true phase transition.

VI. CONCLUDING REMARKS

A remarkable possibility to compare disordered interacting bosons with non-interacting ones is offered by a $^7\mathrm{Li}$ atomic gas, where the coupling strength $g$ can be varied by a Feshbach resonance from practically zero to large positive values. In the 1D case, achieving the strongly interacting regime where $\gamma \gtrsim 1$, is expected to present new transparent physics. For $\gamma \to \infty$ as in the recent cesium experiment, the bosons become impenetrable and show a clear analogy with non-interacting fermions. All single-particle states are then localized irrespective of the strength of the disorder. At intermediate values of $\gamma$ one expects a peculiar interplay between the interparticle interaction and temperature. For $T \gg ng$ the situation should be the same as described in Section III for the high-temperature regime, $T \gg T_d$, and in this sense equation (7) is universal. In the other extreme, $T \lesssim ng$, an increase in the interaction strength first leads to the localization-delocalization transition at small $\gamma$, but then causes a reentrance to the insulating phase at a critical disorder-dependent value of $\gamma$. This behavior is expected because at $T = 0$ even an infinitesimally small disorder leads to the appearance of an insulating (Bose glass) phase if $\gamma$ is sufficiently large. The comparison of

![FIG. 4: Manifestation of the many-body localization-delocalization transition in the expansion of a quasi-1D cloud. In (a) the entire initial cloud is an insulator, whereas in (b) initially the central part is in the fluid and the outskirts in the insulating phase. Initial and final shapes of the cloud are shown in black and red, respectively.](image)

![FIG. 5: Phase diagram for two-dimensional weakly interacting bosons. The black curve shows the thermodynamic Berezinskii-Kosterlitz-Thouless transition. The red curve is the many-body localization-delocalization transition. The blue dashed curves indicate phase boundaries which can be ruled out on rather general grounds (see text).](image)
the insulating phases emerging at small and large $\gamma$ with each other is supposed to shed new light on the structure of Bose glasses.

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