Border rank of powers of ternary quadratic forms

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Polynomial spaces

- $V$ vector space over $\mathbb{C}$, $\dim V = n < \infty$;
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- $\mathcal{D} = S(V^*) \simeq \mathbb{C}[y_1, \ldots, y_n]$, $\mathcal{R} = S(V) \simeq \mathbb{C}[x_1, \ldots, x_n]$. 

---

**Definition (Waring rank of a polynomial $h$)**

$$\text{rk} \, h : = \min \{ r \in \mathbb{N} : h = \sum_{j=1}^{r} a_j x_1 \ldots x_n \} : a_i \in \mathbb{C}$$

**Definition (Border rank of a polynomial $h$)**

$$\text{brk} \, h : = \min \{ r \in \mathbb{N} : \lim_{t \to 0} \sum_{j=1}^{r} a_j (x_1 + t \ldots x_n)^j \} : a_i \in \mathbb{C}$$
Polynomial spaces

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Definition (Waring rank of a polynomial $h \in \mathcal{R}_d$)

$$\text{rk}(h) = \min \left\{ r \in \mathbb{N} \mid h = \sum_{j=1}^{r} (a_{1,j}x_1 + \cdots + a_{n,j}x_n)^d : a_{i,j} \in \mathbb{C} \right\}$$
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Quadratic forms

\[ q_n = x_1^2 + \cdots + x_n^2 \]
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Problem

\[ s \in \mathbb{N} : \quad \text{rk}(q_n^s) = ? \quad \text{brk}(q_n^s) = ? \]
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Classical decompositions

\[ [q_3^2]_7 = \frac{2}{3} \sum_j^3 x_j^4 + \frac{1}{12} \sum_j^4 (x_1 \pm x_2 \pm x_3)^4 \quad \text{(E. Lucas, 1877)} \]

\[ [q_4^2]_{12} = \frac{2}{3} \sum_j^4 x_j^4 + \frac{1}{24} \sum_j^8 (x_1 \pm x_2 \pm x_3 \pm x_4)^4 \quad \text{(J. Liouville, 1859)} \]

\[ [q_n^2]_{n^2} = \frac{1}{6} \sum_{j_1 < j_2}^\binom{n}{2} (x_{j_1} + x_{j_2})^4 + \frac{1}{6} \sum_{j_1 < j_2}^\binom{n}{2} (x_{j_1} - x_{j_2})^4 + \frac{4 - n}{3} \sum_j^n x_j^4 \quad \text{(B. Reznick, 1992)} \]
Theorem (B. Reznick)

\[ \text{rk}(q^s_2) = s + 1. \]
Theorem (B. Reznick)

\[ \text{rk}(q_2^s) = s + 1. \]

Decompositions of \(q_2^s\)

\[ q_2^s = \sum_{j=1}^{s+1} \left( r(s) \cos(\tau_j)x_1 + r(s) \sin(\tau_j)x_2 \right)^{2s}, \quad \tau_j = \frac{(j - 1)\pi}{s + 1} \]
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Examples

- Decomposition of $q_2^3$ (4 points)
- Decomposition of $q_2^4$ (5 points)
Three variables

Decomposition of $q_3^2$ (6 points)

$$[q_3^2]_6 = \frac{1}{6} \sum_{j}^6 (x_j \pm \varphi x_{j-1})^4, \quad \varphi = \frac{1 + \sqrt{5}}{2}$$
Apolarity action

- Apolarity action of $\mathcal{D}_k$ on $\mathcal{R}_j$

  $\circ : \mathcal{D}_k \times \mathcal{R}_j \rightarrow \mathcal{R}_{j-k}$

  $(y^\alpha, x^\beta) \mapsto \frac{\partial}{\partial x^\alpha} (x^\beta)$
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Definition (Apolar ideal of a homogeneous polynomial $h \in \mathcal{R}_d$)

$h^\perp = \{ g \in S(V^*) \mid g \circ h = 0 \}$. 
Catalecticant map of $h \in S^d V$

\[ \text{Cat}_h : \mathcal{D} \longrightarrow \mathcal{R} \]

\[ g \mapsto g \circ h \]
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$\text{Ker}(\text{Cat}_h) = h^\perp$;
Catalecticant map of $h \in S^d V$

$$\text{Cat}_h : \mathcal{D} \longrightarrow \mathcal{R}$$

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- $\text{Ker}(\text{Cat}_h) = h^\perp$;
- $\text{Cat}_h$ graded $\implies \text{Cat}_h^j : \mathcal{D}_j \rightarrow \mathcal{R}_{d-j}$ well defined for $j \in \mathbb{N}$.
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**Proposition (J. J. Sylvester, 1851)**

For every $f \in \mathcal{R}_d$ and $0 \leq k \leq d$

$$
\text{rk } f \geq \text{brk } f \geq \text{rk}(\text{Cat}_f^k).
$$
Definition (Laplace operator)

Differential operator $\Delta: \mathcal{D}_d \to \mathcal{D}_{d-2}$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$$
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**Space of the $d$-harmonic polynomials**

$$\mathcal{H}_n^d = \text{Ker}(\Delta) \subseteq \mathcal{D}_d$$
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\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}
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**Space of the $d$-harmonic polynomials**

$$
\mathcal{H}_n^d = \text{Ker}(\Delta) \subseteq \mathcal{D}_d
$$

**Theorem**

$$
(q_n^s)^\perp = (\mathcal{H}_n^{s+1})
$$
Lower bound

\[ \text{brk}(q_3^s) \geq \text{rk}(\text{Cat}^s_{q_3^2}) = \binom{s + 2}{2} \]
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Lemma (Apolarity lemma)

Let \( f \in R_d \) and \( Z \subset \mathbb{P}^n \) a 0-dimensional scheme. Let \( \nu_d : \mathbb{P}(\mathbb{C}^n) \rightarrow \mathbb{P}(S^d \mathbb{C}^n) \) be the \( d \)-Veronese map. The following conditions are equivalent:
Lower bound

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- \( f \in \langle \nu_d(Z) \rangle; \)
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- $f \in \langle \nu_d(Z) \rangle$;
- $I(Z) \subseteq f^\perp$. 
Lie algebra of $\text{SL}_2\mathbb{C}$

\[ \mathfrak{sl}_2\mathbb{C} = \{ A \in \text{Mat}_2(\mathbb{C}) \mid \text{tr} A = 0 \}, \quad \text{dim}(\mathfrak{sl}_2\mathbb{C}) = 3. \]
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Basis of $\mathfrak{sl}_2\mathbb{C}$

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

\[
\left[ H, E \right] = 2E, \quad \left[ H, F \right] = -2F, \quad \left[ E, F \right] = H.
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Representations of $\mathfrak{sl}_2\mathbb{C}$

- $S^n(\mathbb{C}^2)$ essentially unique irreducible representation;
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Representations of $\mathfrak{sl}_2 \mathbb{C}$

- $S^n(\mathbb{C}^2)$ essentially unique irreducible representation;
- $\{x^{n-k}y^k\}_{k=0,...,n}$ set of weights;
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- $S^n(\mathbb{C}^2)$ essentially unique irreducible representation;
- $\{x^{n-k}y^k\}_{k=0,...,n}$ set of weights;
- $V_k = \langle x^{n-k}y^k \rangle$ for every $k = 0, \ldots, n$

\[
E : V_k \rightarrow V_{k+1}, \quad H : V_k \rightarrow V_k, \quad F : V_k \rightarrow V_{k-1}
\]
Proposition (R. Goodman and N. R. Wallach)

The space $\mathcal{H}_n^d$ is an irreducible $\text{SO}_n(\mathbb{C})$-module.
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Lie algebra of $SO_3(\mathbb{C})$

$$\mathfrak{so}_3 \mathbb{C} = \left\{ A \in \text{Mat}_3(\mathbb{C}) \left| A = -^tA \right. \right\} \cong \mathfrak{sl}_2 \mathbb{C}.$$
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Change of variables

$$u = \frac{y_1 + iy_2}{2}, \quad v = \frac{y_1 - iy_2}{2}, \quad z = y_3.$$
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Change of variables

$$u = \frac{y_1 + iy_2}{2}, \quad v = \frac{y_1 - iy_2}{2}, \quad z = y_3.$$

Basis of $\mathfrak{so}_3\mathbb{C}$ (with respect to $\{u, v, z\}$)

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}.$$
Laplace operator

\[ \Delta = \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{\partial^2}{\partial y_3^2} = \frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial z^2}. \]
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Notation (divided powers)

\[ u^{[k_1]} v^{[k_2]} z^{[k_3]} = \frac{1}{k_1! k_2! k_3!} u^{k_1} v^{k_2} z^{k_3}, \quad k_1, k_2, k_3 \in \mathbb{N}. \]
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\]

Basis \( B_d = \{ h_{d,k} \}_{-d \leq k \leq d} \) of \( \mathcal{H}_3^d \)

\[
h_{d,k} = \sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^j u^{[k+j]} z^{[d-k-2j]} v^{[j]}, \quad h_{d,-k} = \sum_{j=0}^{\left\lfloor \frac{d-k}{2} \right\rfloor} (-1)^j u^{[j]} z^{[d-k-2j]} v^{[k+j]}
\]
Harmonic generators of the basis $\mathcal{B}_d = \{h_{d,k}\}_{-d \leq k \leq d}$

\[
\begin{align*}
\langle \frac{1}{2} u^2 \rangle_{h_2,2} & \xrightarrow{F} \langle \frac{1}{4} uz \rangle_{h_2,1} & \xrightarrow{F} \langle \frac{1}{12} (z^2 - 2uv) \rangle_{h_2,0} & \xrightarrow{F} \langle \frac{1}{4} vz \rangle_{h_2,-1} & \xrightarrow{F} \langle \frac{1}{2} v^2 \rangle_{h_2,-2} \\
\langle \frac{1}{6} u^3 \rangle_{h_3,3} & \xrightarrow{F} \langle \frac{1}{12} u^2 z \rangle_{h_3,2} & \xrightarrow{F} \langle \frac{1}{30} u(z^2 - uv) \rangle_{h_3,1} & \xrightarrow{F} \langle \frac{1}{120} z(z^2 - 6uv) \rangle_{h_3,0} & \xrightarrow{F} \langle \frac{1}{30} v(z^2 - uv) \rangle_{h_3,-1} & \xrightarrow{F} \langle \frac{1}{12} v^2 z \rangle_{h_3,-2} & \xrightarrow{F} \langle \frac{1}{6} v^3 \rangle_{h_3,-3} \\
\vdots & & & & & & \\
\langle \frac{1}{d!} u^d \rangle_{h_{d,d}} & \xrightarrow{F} \langle \frac{1}{2d(d-1)!} u^{d-1} z \rangle_{h_{d,d-1}} & \xrightarrow{F} \langle \frac{1}{2d(d-1)!} u^{d-1} z \rangle_{h_{d,d-1}} & \cdots & \xrightarrow{F} \langle \frac{1}{2d(d-1)!} u^{d-1} z \rangle_{h_{d,-(d-1)}} & \xrightarrow{F} \langle \frac{1}{d!} v^d \rangle_{h_{d,-d}} \\
\vdots & & & & & & 
\end{align*}
\]
Candidate ideal

\[ I_{s+1} = (h_{s+1,s+1}, \ldots, h_{s+1,0}) \subset (q_3^s)^\perp \quad \sqrt{I_{s+1}} = (u, z) \quad \deg(I_{s+1}) = \binom{s + 2}{2} \]
Candidate ideal

\[ I_{s+1} = (h_{s+1,s+1}, \ldots, h_{s+1,0}) \subset (q_3^s)^\perp \ \ \ \sqrt{I_{s+1}} = (u, z) \ \ \ \deg(I_{s+1}) = \binom{s + 2}{2} \]

Proposition

The ideal \( I_{s+1} \) is saturated.
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Proposition

The ideal \( I_{s+1} \) is saturated.

Smoothable rank of \( f \in \mathcal{R}_d \)

\[ \text{smrk} \ f = \min \{ \ r \in \mathbb{N} \mid \exists \ 0\text{-dim sm. scheme } Z: \deg Z = r, \ f \in \langle v_d(Z) \rangle \} \]
Candidate ideal

\[ I_{s+1} = (h_{s+1,s+1}, \ldots, h_{s+1,0}) \subset (q_3^s)^\perp \quad \sqrt{I_{s+1}} = (u, z) \quad \deg(I_{s+1}) = \binom{s + 2}{2} \]

Proposition

The ideal \( I_{s+1} \) is saturated.

Smoothable rank of \( f \in \mathcal{R}_d \)

\[ \text{smrk } f = \min \{ r \in \mathbb{N} \mid \exists \text{ 0-dim sm. scheme } Z: \deg Z = r, f \in \langle \nu_d(Z) \rangle \} \]

Use of apolarity lemma

Every scheme \( Z \subseteq \mathbb{P}^2 \mathbb{C} \) is smoothable \( \implies \text{smrk}(q_3^s) \leq \binom{s + 2}{2} \)
Border rank

$$\left(\frac{s + 2}{2}\right) \leq \text{brk}(q_3^s) \leq \text{smrk}(q_3^s) \leq \left(\frac{s + 2}{2}\right)$$
Border rank

\[
\left(\frac{s + 2}{2}\right) \leq \text{brk}(q_3^s) \leq \text{smrk}(q_3^s) \leq \left(\frac{s + 2}{2}\right)
\]

Conclusion

Let \( f \in \mathbb{C}[x_1, x_2, x_3]_2 \). Then:
Border rank

\[
\left(\frac{s + 2}{2}\right) \leq \text{brk}(q_3^s) \leq \text{smrk}(q_3^s) \leq \left(\frac{s + 2}{2}\right)
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Conclusion

Let \( f \in \mathbb{C}[x_1, x_2, x_3]_2 \). Then:

- \( \text{rk} f = 1 \Rightarrow \text{brk}(f^s) = 1 \)
**Border rank**

\[
\binom{s+2}{2} \leq \text{brk}(q_3^s) \leq \text{smrk}(q_3^s) \leq \binom{s+2}{2}
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**Conclusion**

Let \( f \in \mathbb{C}[x_1, x_2, x_3]_2 \). Then:

- \( \text{rk} f = 1 \implies \text{brk}(f^s) = 1 \)
- \( \text{rk} f = 2 \implies \text{brk}(f^s) = s + 1 \)
Border rank

\[
\left(\frac{s + 2}{2}\right) \leq \text{brk}(q_3^s) \leq \text{smrk}(q_3^s) \leq \left(\frac{s + 2}{2}\right)
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Conclusion

Let \( f \in \mathbb{C}[x_1, x_2, x_3]_2 \). Then:

- \( \text{rk} f = 1 \implies \text{brk}(f^s) = 1 \)
- \( \text{rk} f = 2 \implies \text{brk}(f^s) = s + 1 \)
- \( \text{rk} f = 3 \implies \text{brk}(f^s) = \left(\frac{s + 2}{2}\right) \)
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