ON FAITHFULLY BALANCEDNESS IN FUNCTOR CATEGORIES

JULIA SAUTER

Abstract. This is a generalization of some results of Ma-Sauter from module categories over artin algebras to more general functor categories (and partly to exact categories). In particular, we generalize the definition of a faithfully balanced module to a faithfully balanced subcategory and find the generalizations of dualities and characterizations from Ma-Sauter.

1. Introduction

For an exact category $\mathcal{E}$ in the sense of Quillen and a full subcategory $\mathcal{M}$ we define categories $\text{gen}_k^\mathcal{E}(\mathcal{M})$ (and $\text{cogen}_k^\mathcal{E}(\mathcal{M})$) of $\mathcal{E}$ (consisting of objects admitting a certain $k$-presentation in $\mathcal{M}$). We also consider the two functors $\Phi(X) := \text{Hom}_\mathcal{E}(\mathcal{M}, X) \mid_\mathcal{M}$, $\Psi(X) := \text{Hom}_\mathcal{E}(X, \mathcal{M}) \mid_\mathcal{M}$.

We give the relatively obvious but technical generalizations of results in [3] related to these categories and functors. If $\mathcal{E}$ is a functor category (of some sort) these functors have adjoints and therefore stronger results can be found. We state here two of these:

Let $\mathcal{P}$ be an essentially small additive category. We denote by $\text{Mod} \rightarrow \mathcal{P}$ the category of covariant additive functors $\mathcal{P} \rightarrow \text{Ab}$ (and we set $\mathcal{P} \rightarrow \text{Mod} := \text{Mod} \rightarrow \mathcal{P}^{\text{op}}$). We write $\text{mod}_k^{\mathcal{P}}$ for the full subcategory which admit a $k$-presentation by finitely generated projectives. We denote by $\h: \mathcal{P} \rightarrow \text{Mod}^{-\mathcal{P}}$, $\mathcal{P} \mapsto \h\mathcal{P} = \text{Hom}_{\mathcal{P}}(\mathcal{P}, \mathcal{P})$ the Yoneda embedding.

Cogen$^1$-duality: Let $k \in \mathbb{N}_0 \cup \{\infty\}$ and assume now $\mathcal{M} \subset \text{mod}_k^{\mathcal{P}}$. We shorten the notation $\text{cogen}_k^{\mathcal{P}}(\mathcal{M}) := \text{cogen}_k^{\text{mod}_k^{\mathcal{P}}}(\mathcal{M})$. We say $\mathcal{M}$ is faithfully balanced if $\h\mathcal{P} \in \text{cogen}_k^{\mathcal{P}}(\mathcal{M})$ for all $\mathcal{P} \in \mathcal{P}$.

Lemma 1.1. (cf. Lem. 3.11) (cogen$^1$-duality) If $\mathcal{M}$ is faithfully balanced, we denote by $\tilde{\mathcal{M}} = \Psi(\h\mathcal{P}) \subset \mathcal{M} \rightarrow \text{mod}_k^{\mathcal{P}}$, then $\Psi$ defines a contravariant equivalence

\[ \text{cogen}^{1}_{\text{mod}_k^{\mathcal{P}}}(\mathcal{M}) \longleftrightarrow \text{cogen}^{1}_{\mathcal{M} \rightarrow \text{mod}_k^{\mathcal{P}}}(\tilde{\mathcal{M}}) \]

The symmetry principle states as follows:

Theorem 1.2. (cf. Thm. 3.16, Symmetry principle). Let $\mathcal{E}$ be an exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$ and $k \geq 1$. The following two statements are equivalent:

1. $\mathcal{P} \subset \text{cogen}_k^\mathcal{E}(\mathcal{M})$ and $\Phi(I) = \text{Hom}_\mathcal{E}(\mathcal{M}, -) \mid_{\mathcal{M}} \in \text{mod}_k - \mathcal{M}$ for every $I \in \mathcal{I}$
2. $\mathcal{I} \subset \text{gen}_k^\mathcal{E}(\mathcal{M})$ and $\Psi(P) = \text{Hom}_\mathcal{E}(P, \mathcal{M}) \mid_{\mathcal{M}} \in \mathcal{M} \rightarrow \text{mod}_k$ for every $P \in \mathcal{P}$

A nice special case: Assume additionally that $\mathcal{E}$ is a Hom-finite $K$-category for a field $K$ and $\mathcal{M} = \text{add}(\mathcal{M})$ for an object $\mathcal{M} \in \mathcal{E}$. Then the following two statements are equivalent:

1. $\mathcal{P} \subset \text{cogen}_k^\mathcal{E}(\mathcal{M})$
2. $\mathcal{I} \subset \text{gen}_k^\mathcal{E}(\mathcal{M})$

Since: If we set $\Lambda = \text{End}_\mathcal{E}(\mathcal{M})$, then $\text{mod}_k - \mathcal{M}$, $\mathcal{M} \rightarrow \text{mod}_k$ can be identified with finite-dimensional (left and right) modules over $\Lambda$ and $\Phi(I) = \text{Hom}_\mathcal{E}(\mathcal{M}, I)$, $\Psi(P) = \text{Hom}_\mathcal{E}(P, \mathcal{M})$ are by assumption finite-dimensional $\Lambda$-modules.

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Here we want to extend Yoneda’s embedding to a bigger subcategory: Let $\mathcal{C}$ be an additive category and $\mathcal{M}$ an essentially small full additive subcategory. A right $\mathcal{M}$-module is a contravariant additive functor from $\mathcal{M}$ into abelian groups. We denote by $\text{Mod} - \mathcal{M}$ the category of all right $\mathcal{M}$-modules. This is an abelian category. We have the fully faithful (covariant) Yoneda embedding $\mathcal{M} \to \text{Mod} - \mathcal{M}$. Clearly, we can extend this functor to a functor $\Phi: \mathcal{C} \to \text{Mod} - \mathcal{M}$ defined by $M \mapsto \text{Hom}_\mathcal{M}(\_, M)$. By the Lemma of Yoneda, we have for every $\mathcal{M}$-module $F$ of all right $\mathcal{M}$-modules there is an induced map on the kernels is an isomorphism. □

Let $C \to \text{op}$ be a last notation is our shortage for the Hom functor. The aim of this section is to define a subcategory $\mathcal{M} \subset \mathcal{G} \subset \mathcal{C}$ such that $\Phi|_{\mathcal{G}}$ is fully faithful.

We define a final subcategory of $\mathcal{C}$ as follows

$$\text{gen}_1^{\text{add}}(\mathcal{M}) := \left\{ Z \in \mathcal{C} \mid \exists M_1 \xrightarrow{f} M_0 \xrightarrow{g} Z, M_i \in \mathcal{M}, g = \text{coker}(f) \text{ is an epim.} \right\}$$

We observe that $g = \text{coker}(f)$ and $g$ an epimorphism is equivalent to that we have an exact sequence of $\mathcal{C}^{\text{op}}$-modules

$$0 \to (Z, -) \to (M_0, -) \to (M_1, -)$$

Furthermore the second line in the definition is equivalent to an exact sequence in $\text{Mod} - \mathcal{M}$

$$(-, M_1) \to (-, M_0) \to (-, Z)|_{\mathcal{M}} \to 0.$$

Dually, we define $\text{cogen}_1^{\text{add}}(\mathcal{M}) := (\text{gen}_1^{\text{add}}(\mathcal{M}^{\text{op}}))^{\text{op}}$ where $\mathcal{M}^{\text{op}}$ is considered as a full additive subcategory of $\mathcal{C}^{\text{op}}$.

**Lemma 2.1.**

1. The functor $\text{gen}_1^{\text{add}}(\mathcal{M}) \to \text{Mod} - \mathcal{M}$ defined by $Z \mapsto (-, Z)|_{\mathcal{M}}$ is fully faithful. We even have for every $Z \in \text{gen}_1^{\text{add}}(\mathcal{M}), C \in \mathcal{C}$ a natural isomorphism

$$\text{Hom}_\mathcal{C}(Z, C) \to \text{Hom}_{\text{Mod} - \mathcal{M}}((-,-)|_{\mathcal{M}}, (\_, C)|_{\mathcal{M}})$$

2. The functor $\text{cogen}_1^{\text{add}}(\mathcal{M}) \to \text{Mod} - \mathcal{M}^{\text{op}}$ defined by $Z \mapsto (\_, Z)|_{\mathcal{M}}$ is fully faithful. We even have for every $Z \in \text{cogen}_1^{\text{add}}(\mathcal{M}), C \in \mathcal{C}$ a natural isomorphism

$$\text{Hom}_\mathcal{C}(C, Z) \to \text{Hom}_{\text{Mod} - \mathcal{M}^{\text{op}}}(Z, -)|_{\mathcal{M}}, (C, -)|_{\mathcal{M}})$$

**Proof.** We only prove (1), the second statement follows by passing to opposite categories. We consider the functor $\Phi: \mathcal{C} \to \text{Mod} - \mathcal{M}$ defined by $\Phi(X) := (-, X)|_{\mathcal{M}}$. Since $Z \in \text{gen}_1^{\text{add}}(\mathcal{M})$ we an exact sequence

$$0 \to (Z, C) \to (M_0, C) \to (M_1, C)$$

and $\Phi(M_1) \to \Phi(M_0) \to \Phi(Z) \to 0$ in $\text{Mod} - \mathcal{M}$. By applying $(-, \Phi(C))$ to the second exact sequence we obtain an exact sequence

$$0 \to (\Phi(Z), \Phi(C)) \to (\Phi(M_0), \Phi(C)) \to (\Phi(M_1), \Phi(C))$$

Since $\Phi$ is a functor, we find a commuting diagram

$$\begin{array}{ccc}
0 & \to & (Z, C) \\
\downarrow & & \downarrow \\
0 & \to & (\Phi(Z), \Phi(C))
\end{array} \quad \begin{array}{ccc}
0 & \to & (M_0, C) \\
\downarrow & & \downarrow \\
0 & \to & (\Phi(M_0), \Phi(C))
\end{array} \quad \begin{array}{ccc}
0 & \to & (M_1, C) \\
\downarrow & & \downarrow \\
0 & \to & (\Phi(M_1), \Phi(C))
\end{array}$$

By the Lemma of Yoneda, we have for every $F \in \text{Mod} - \mathcal{M}$ and $M \in \mathcal{M}$ that $\text{Hom}_{\text{Mod} - \mathcal{M}}(\Phi(M), F) = F(M)$. This implies that the maps $(M_1, C) \to (\Phi(M_1), \Phi(C))$ are isomorphisms of groups. and therefore, the induced map on the kernels is an isomorphism. □
Remark 2.2. If $\mathcal{M}$ is not essentially small, $\text{Hom}_{\mathcal{M} \text{-Mod}}(F,G)$ is not necessarily a set. But if one passes to the full subcategory of finitely presented $\mathcal{M}$-modules mod$_1 \mathcal{M}$, this set-theoretic issue does not arise: Observe that $Z \mapsto (Z,-)|_\mathcal{M}$ defines by definition a covariant functor

$$\Phi: \text{gen}^1_{\text{add}}(\mathcal{M}) \to \text{mod}_1 - \mathcal{M},$$

the same proof as before shows that this is fully faithful. Similarly, the functor $Z \mapsto (Z,-)|_\mathcal{M}$ defines a fully faithful contravariant functor

$$\Psi: \text{cogen}^1(\mathcal{M}) \to \text{mod}_1 - \mathcal{M}^{\text{op}}.$$  

3. IN EXACT CATEGORIES

This section is a generalization of results from [3]. For exact categories we have subcategories of $\text{cogen}^1_{\text{add}}$ such that $\Psi$ induces isomorphisms on (some) extension groups (cf. Lemma 3.3). Given an exact category $\mathcal{E}$ with a full additive subcategory $\mathcal{M}$, we define $\text{cogen}^k(\mathcal{M}) \subseteq \mathcal{E}$ to be the full subcategory of all objects $X$ such that there is an exact sequence

$$0 \to X \to M_0 \to \cdots \to M_k \to Z \to 0$$

with $M_i \in \mathcal{M}, 0 \leq i \leq k$ such that for every $M \in \mathcal{M}$ the sequence

$$\text{Hom}_{\mathcal{E}}(M_k, M) \to \cdots \to \text{Hom}_{\mathcal{E}}(M_0, M) \to \text{Hom}_{\mathcal{E}}(X, M) \to 0$$

is an exact sequence of abelian groups. We define $\text{gen}^k(\mathcal{M})$ to be the full additive category of $\mathcal{E}$ given by all $X$ such that there is an exact sequence

$$0 \to Z \to M_k \to \cdots \to M_0 \to X \to 0$$

with $M_i \in \mathcal{M}, 0 \leq i \leq k$ such that for every $M \in \mathcal{M}$ we have an exact sequence

$$\text{Hom}_{\mathcal{E}}(M, M_k) \to \cdots \to \text{Hom}_{\mathcal{E}}(M, M_0) \to \text{Hom}_{\mathcal{E}}(M, X) \to 0$$

of abelian groups. If it is clear from the context in which exact category we are working, then we leave out the index $\mathcal{E}$ and just write $\text{cogen}^k(\mathcal{M})$ and $\text{gen}^k(\mathcal{M})$.

Remark 3.1. Observe that $\text{cogen}^k(\mathcal{M}) \subseteq \text{cogen}^1_{\text{add}}(\mathcal{M}), \text{gen}^k(\mathcal{M}) \subseteq \text{gen}^1_{\text{add}}(\mathcal{M})$ for $k \geq 1$ and therefore the functor $\Psi: X \mapsto (X,-)|_\mathcal{M}$ (resp. $\Phi: X \mapsto (-,X)|_\mathcal{M}$) is fully faithful on $\text{cogen}^k(\mathcal{M})$ (resp. on $\text{gen}^k(\mathcal{M})$) by Lemma 2.1 and Remark 2.2.

Remark 3.2. Let $k \geq 1$. We denote by mod$_k - \mathcal{M}$ the category of $\mathcal{M}$-modules which admit a $k$-presentation (indexed from 0 to $k$) by finitely presented projectives. For $F \in \text{mod}_k - \mathcal{M}$ the Ext-groups $\text{Ext}^i_{\text{Mod}}(F,G)$ with $0 \leq i < k$ are sets. If $X \in \text{cogen}^k(\mathcal{M})$, then we have $\Psi(X) = (X,-)|_\mathcal{M} \in \text{mod}_k - \mathcal{M}^{\text{op}} (=: \mathcal{M} - \text{mod}_k)$. If $Y \in \text{gen}^k(\mathcal{M})$, then we have $\Phi(Y) = (-,Y)|_\mathcal{M} \in \text{mod}_k - \mathcal{M}$.

Since we are now working in exact categories, we observe the following isomorphisms on extension groups:

Lemma 3.3. Let $k \geq 1$.

(a) If $X \in \text{cogen}^k(\mathcal{M})$, then the functor $Z \mapsto \Psi(Z) = (Z,-)|_\mathcal{M}$ induces a well-defined natural isomorphism of abelian groups

$$\text{Ext}^i_{\mathcal{E}}(Y, X) \to \text{Ext}^i_{\text{Mod}}(\Psi(X), \Psi(Y)), \quad 0 \leq i < k$$

for all $Y \in \bigcap_{1 \leq i < k} \ker \text{Ext}^i_{\mathcal{E}}(-, \mathcal{M})$.

(b) If $Y \in \text{gen}^k(\mathcal{M})$, then the functor $Z \mapsto \Phi(Z) = (-,Z)|_\mathcal{M}$ induces a well-defined natural isomorphism of abelian groups

$$\text{Ext}^i_{\mathcal{E}}(Y, X) \to \text{Ext}^i_{\text{Mod}}(\Phi(Y), \Phi(X)), \quad 0 \leq i < k$$

for all $X \in \bigcap_{1 \leq i < k} \ker \text{Ext}^i_{\mathcal{E}}(\mathcal{M}, -)$.
Proof. (a) the proof is a straight forward generalization of [3], Lemma 2.4, (2) (using Rem. 3.1) and (b) follows from (a) by passing to the opposite exact category $\mathcal{E}^{op}$. □

We will later use the following simple observation:

**Remark 3.4.** Let $\mathcal{E}$ be an exact category, $\mathcal{X}$ be a fully exact category and $\mathcal{M} \subset \mathcal{X}$ an additive subcategory. We say $\mathcal{X}$ is deflation-closed if for any deflation $d: X \to X'$ in $\mathcal{E}$ with $X, X'$ in $\mathcal{X}$ it follows $\ker d \in \mathcal{X}$. The dual notion is inflation-closed.

If $\mathcal{X}$ is deflation-closed then $\text{gen}_{k}^{X}(\mathcal{M}) = \text{gen}_{k}^{X}(\mathcal{M}) \cap \mathcal{X}$. If $\mathcal{X}$ is inflation-closed then $\text{cogen}_{k}^{X}(\mathcal{M}) = \text{cogen}_{k}^{X}(\mathcal{M}) \cap \mathcal{X}$.

3.1. **Inside functor categories.** Let $\mathcal{P}$ be an essentially small additive category. We denote by $h: \mathcal{P} \to \text{Mod} - \mathcal{P}$, $P \mapsto h_{P} = \text{Hom}_{\mathcal{P}}(-, P)$ the Yoneda embedding, we write $h_{\mathcal{P}}$ for the essential image of $h$.

3.1.1. **Adjoint functors.** Let now $\mathcal{M}$ be an essentially small full additive subcategory of $\text{Mod} - \mathcal{P}$. We consider the contravariant functor

$$
\Psi: \text{Mod} - \mathcal{P} \to \mathcal{M} - \text{Mod},
$$

$$
X \mapsto \text{Hom}_{\text{Mod} - \mathcal{P}}(X, -)|_{\mathcal{M}} = (X, -)|_{\mathcal{M}}
$$

We also consider the contravariant functor

$$
\Psi': \mathcal{M} - \text{Mod} \to \text{Mod} - \mathcal{P}
$$

$$
Z \mapsto (P \mapsto \text{Hom}_{\mathcal{M} - \text{Mod}}(Z, \Psi(h_{P})))
$$

We generalize [1], Lem. 3.3..

**Lemma 3.5.** The functors $\Psi$ and $\Psi'$ are contravariant adjoint functors, i.e. the following is a (bi)natural isomorphism

$$
\chi: \text{Hom}_{\text{Mod} - \mathcal{P}}(X, \Psi'(Z)) \to \text{Hom}_{\mathcal{M} - \text{Mod}}(Z, \Psi(X))
$$

defined as follows: A natural transformation $f \in \text{Hom}_{\text{Mod} - \mathcal{P}}(X, \Psi'(Z))$, is determined by for every $P \in \mathcal{P}$, $x \in X(P)$, $M \in \mathcal{M}$ a group homomorphism

$$
f_{P,x}(M): Z(M) \to \Psi(h_{P})(M) = M(P)
$$

then, we define a natural transformation $\chi(f): Z \to \Psi(X) = \text{Hom}_{\text{Mod} - \mathcal{P}}(X, -)|_{\mathcal{M}}$ for $M \in \mathcal{M}$ as follows

$$
\chi(f)(M): Z(M) \to \text{Hom}_{\text{Mod} - \mathcal{P}}(X, M),
$$

$$
z \mapsto (X(P) \xrightarrow{f_{P,x}(z)} M(P), x \mapsto f_{P,x}(M)(z))_{P \in \mathcal{P}}
$$

**Proof.** We define $\chi': \text{Hom}_{\mathcal{M} - \text{Mod}}(Z, \Psi(X)) \to \text{Hom}_{\text{Mod} - \mathcal{P}}(X, \Psi'(Z))$ as follows: For $g: Z \to \Psi(X) = \text{Hom}_{\text{Mod} - \mathcal{P}}(X, -)|_{\mathcal{M}}$ we have for every $M \in \mathcal{M}$, $z \in Z(M)$ a natural transformation $g_{M,z}: X \to M$, i.e. for every $P \in \mathcal{P}$ a group homomorphism

$$
g_{M,z}(P): X(P) \to M(P), x \mapsto g_{M,z}(P)(x),
$$

then we define $\chi'(g)(P): X(P) \to \Psi'(Z)(P) = \text{Hom}_{\mathcal{M} - \text{Mod}}(Z, (h_{P}, -))|_{\mathcal{M}}$ as follows

$$
x \mapsto (Z(M) \to M(P), z \mapsto g_{M,z}(P)(x))_{M \in \mathcal{M}}.
$$

Then $\chi'$ is the inverse map to $\chi$. □

**Remark 3.6.** Given an adjoint pair of contravariant functors $\Psi$ and $\Psi'$, the natural isomorphisms

$$
\text{Hom}(X, \Psi(Z)) \to \text{Hom}(Z, \Psi'(X))
$$

induce natural transformations $\alpha: \text{id} \to \Psi' \Psi$ (and $\alpha': \text{id} \to \Psi \Psi'$) as follows

$$
\text{Hom}(X, X) \xrightarrow{\Psi(-)} \text{Hom}(\Psi(X), \Psi'(X)) \cong \text{Hom}(X, \Psi'(\Psi(X))), \quad \text{id}_{X} \mapsto \alpha_{X}
$$
in this case we have triangle identities

\[
\begin{split}
\text{id}_{\Psi(X)} &= (\Psi(X) \xrightarrow{\alpha'_{\Psi(X)}} \Psi'\Psi(X) \xrightarrow{\Psi(\alpha_X)} \Psi(X)) \\
\text{id}_{\Psi'(Z)} &= (\Psi'(Z) \xrightarrow{\alpha'_{\Psi'(Z)}} \Psi'\Psi'(Z) \xrightarrow{\Psi'(\alpha'_Z)} \Psi'(Z))
\end{split}
\]

In [4], section 4, a tensor bifunctor is introduced

\[- \otimes_{\mathcal{M}} : \text{Mod} \rightarrow \mathcal{M} \times \mathcal{M} \rightarrow \text{Mod} \rightarrow (Ab), (F, G) \mapsto F \otimes_{\mathcal{M}} G\]

Now, we consider the covariant functor

\[\Phi : \text{Mod} \rightarrow \text{Mod}, \quad X \mapsto \text{Hom}_{\text{Mod} - \mathcal{P}}(-, X)|_{\mathcal{M}} = (-, X)|_{\mathcal{M}}\]

and the following covariant functor

\[\Phi' : \text{Mod} \rightarrow \text{Mod} \rightarrow \mathcal{P}, \quad Z \mapsto (P \mapsto Z \otimes_{\mathcal{M}} \Phi(h_P))\]

Lemma 3.7. The functor \(\Phi\) is right adjoint to \(\Phi'\), i.e. we have a (bi)natural maps

\[\text{Hom}_{\text{Mod} - \mathcal{P}}(\Phi'(Z), X) \rightarrow \text{Hom}_{\text{Mod} - \mathcal{M}}(Z, \Phi(X))\]

Remark 3.8. If \(F : \mathcal{C} \leftrightarrow \mathcal{D} : G\) is an adjoint pair of functors (with \(F\) left adjoint to \(G\)), then we have a unit \(\eta : 1_{\mathcal{C}} \rightarrow GF\) and a counit, \(\epsilon : FG \rightarrow 1_{\mathcal{D}}.\) Let \(\mathcal{C}_a\) be the full subcategory of objects in \(\mathcal{X}\) in \(\mathcal{C}\) such that \(u(X)\) is an isomorphism. Let \(\mathcal{D}_c\) be the full subcategory of objects \(Y\) in \(\mathcal{D}\) such that \(c(Y)\) is an isomorphism. Then, the triangle identities show directly that \(F, G\) restrict to quasi-inverse equivalences \(F : \mathcal{C}_a \leftrightarrow \mathcal{D}_c : G\).

3.1.2. \(\text{cogen}^k\) Let \(k \in \mathbb{N}_0 \cup \{\infty\}\) and assume now \(\mathcal{M} \subset \text{mod}_k - \mathcal{P}\). In this subsection we study \(\text{cogen}^k(\mathcal{M}) := \text{cogen}^k_{\text{mod}_k - \mathcal{P}}(\mathcal{M}) \subset \text{mod}_k - \mathcal{P}\).

Our aim is to give a different description of the categories \(\text{cogen}^k(\mathcal{M})\) (cf. Lemma 3.9) and to introduce \textit{faithfully balancedness} which leads to the \(\text{cogen}^1\) duality (cf. Lemma 3.11).

We have the contravariant functor

\[\Psi : \text{Mod} \rightarrow \mathcal{M} \rightarrow \text{Mod}, \quad X \mapsto \text{Hom}_{\text{Mod} - \mathcal{P}}(\Psi(X), -)|_{\mathcal{M}}\]

and \(\Psi|_{\text{cogen}^k(\mathcal{M})} : \text{cogen}^k(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow \text{mod}_k\) is fully faithful for \(1 \leq k < \infty\).

The natural transformation \(\alpha : \text{id}_{\text{Mod} - \mathcal{P}} \rightarrow \Psi\Psi,\) for \(X \in \text{Mod} - \mathcal{P}\) is given by a morphism in \(\text{Mod} - \mathcal{P}, \alpha_X : X \rightarrow \Psi\Psi(X) = \text{Hom}_{\mathcal{M} - \text{Mod}}(\Psi(X), \Psi(h_-))\) which is defined at \(P \in \mathcal{P}\) via

\[X(P) = \text{Hom}_{\mathcal{M} - \mathcal{P}}(h_P, X) \rightarrow \text{Hom}_{\mathcal{M} - \mathcal{M}}(\text{Hom}_{\mathcal{M} - \mathcal{P}}(X, -)|_{\mathcal{M}}, \text{Hom}_{\mathcal{M} - \mathcal{P}}(h_P, -)|_{\mathcal{M}})\]

\[f \mapsto \text{Hom}_{\mathcal{M} - \mathcal{P}}(X, -) \xrightarrow{\alpha_X} \text{Hom}_{\mathcal{M} - \mathcal{P}}(h_P, -)|_{\mathcal{M}}\]

We observe that \(\alpha_M\) is an isomorphism for every \(M \in \mathcal{M}\) (since \((\Psi'\Psi)(M))(P) = \text{Hom}_{\mathcal{M} - \text{Mod}}(\text{Hom}_{\mathcal{M}}(M, -), \Psi(h_P)) = \Psi(h_P)(M) = \text{Hom}_{\mathcal{M} - \mathcal{P}}(h_P, M) = M(P)\) using Yoneda’s Lemma twice).

Lemma 3.9. For \(1 \leq k \leq \infty\) we have

\[
\text{cogen}^k_{\text{mod}_k - \mathcal{P}}(\mathcal{M}) = \{X \in \text{mod}_k - \mathcal{P} \mid \alpha_X \text{ isom.}, \Psi(X) \in \mathcal{M} - \text{mod}_k, \text{Ext}^i_{\mathcal{M} - \text{Mod}}(\Psi(X), \Psi(h_P)) = 0, 1 \leq i < k, \forall P \in \mathcal{P}\}
\]

Proof. The proof is a straightforward generalization of [3]. Lemma 2.2, (1) (the functor \(\text{Hom}_\Gamma(-, M)\) has to be replaced by applying \(\text{Hom}_{\mathcal{M} - \text{Mod}}(\cdot, \Psi(h_P))\) for all \(P \in \mathcal{P}\)). \(\square\)

Definition 3.10. We say \(\mathcal{M}\) is \textbf{faithfully balanced} if \(h_P \subset \text{cogen}^1(\mathcal{M})\).
Lemma 3.11. \( \text{(cogen}^1 \text{ duality)} \) If \( \mathcal{M} \) is faithfully balanced, we denote by \( \tilde{\mathcal{M}} = \Psi(h_P) \subset \mathcal{M} - \text{mod}_k \), then \( \Psi \) defines a contravariant equivalence

\[
\text{cogen}^1_{\text{mod}_1 - P}(\mathcal{M}) \leftrightarrow \text{cogen}^1_{\tilde{\mathcal{M}} - \text{mod}_1}(\tilde{\mathcal{M}})
\]

and contravariant equivalences

\[
\text{cogen}^k_{\text{mod}_k - P}(\mathcal{M}) \leftrightarrow \text{cogen}^k_{\tilde{\mathcal{M}} - \text{mod}_k}(\tilde{\mathcal{M}}) \cap \bigcap_{1 \leq i < k} \ker(\text{Ext}^i_{\tilde{\mathcal{M}} - \text{mod}_k}(\cdot, \tilde{\mathcal{M}}))
\]

Proof. Let \( k = 1 \). Since we have an adjoint pair of contravariant functors \( \Psi, \Psi' \) it follows from the triangle identities (cf. Remark 3.6): If \( \alpha_X \) is an isomorphism then also \( \alpha_{\Psi(X)}' \) and if \( \alpha_Z' \) is an isomorphism then also \( \alpha_{\Psi(Z)}' \). Now, since \( \mathcal{M} \) is faithfully balanced we have that \( \Psi \) induces an equivalence \( P^{\text{op}} \cong \mathcal{M} = \Psi(h_P) \) by Lemma 2.1. It follows from the definition of \( \Psi' \) and a right module version of Lemma 3.9 that \( \text{cogen}^1(\tilde{\mathcal{M}}) = \{ Z \in \mathcal{M} - \text{mod}_1 | \alpha_Z \text{ isom} \} \).

The rest is a straightforward generalization of the proof of [3], Lemma 2.9. \( \square \)

3.1.3. \( \text{[gen]} \). We study \( \text{gen}_k(\mathcal{M}) = \text{gen}^k_{\tilde{\text{mod}} - P}(\mathcal{M}) \subset \text{Mod} - P \). We again give a different description of these categories using tensor products of \( \mathcal{M} \)-modules (cf. Lemma 3.13). This is the main ingredient in the proof of the symmetry principle in the next subsection.

We have the covariant functor

\[
\Phi: \text{Mod} - P \to \text{Mod} - \mathcal{M}, \quad X \mapsto \text{Hom}_{\text{Mod} - P}(-, X)|_{\mathcal{M}}
\]

and \( \Phi|_{\text{gen}_k(\mathcal{M})}: \text{gen}_k(\mathcal{M}) \to \text{mod}_k - \mathcal{M} \) is fully faithful. We have an induced covariant functor

\[
\varepsilon = \Phi' \circ \Phi : \text{Mod} - P \to \text{Mod} - P, \quad X \mapsto \varepsilon_X
\]

defined for \( P \in \mathcal{P} \) as

\[
\varepsilon_X(P) := \Phi(X) \otimes_{\mathcal{M}} \Psi(h_P)
\]

and a natural transformation \( \varphi : \varepsilon \to \text{id}_{\text{Mod} - P} \), for \( X \in \text{Mod} - P \) this is given by a morphism \( \varphi_X : \varepsilon_X \to X \) which is defined at \( P \in \mathcal{P} \) via

\[
\text{Hom}_{\text{Mod} - P}(-, X)|_{\mathcal{M}} \otimes_{\mathcal{M}} \text{Hom}_{\text{Mod} - P}(h_P, -)|_{\mathcal{M}} = \text{Hom}_{\text{Mod} - P}(h_P, X) = X(P)
\]

\[
\begin{pmatrix}
g \otimes f \\
\in \text{Hom}(M, X) \otimes \text{Hom}(h_P, M)
\end{pmatrix}
\]

\( \rightarrow g \circ f \)

Remark 3.12. \( \Phi \) and is right adjoint functor of \( \Phi' \) between abelian categories therefore \( \Phi \) is left exact and \( \Phi' \) is right exact, \( \varphi \) is the counit of this adjunction. If \( M \in \mathcal{M} \), then \( \varphi_M \) is an isomorphism.

Lemma 3.13. For \( 1 \leq k \leq \infty \) we have

\[
\text{gen}^k_{\tilde{\text{mod}} - P}(\mathcal{M}) = \{ X \in \text{Mod} - P \mid \varphi_X \text{ isom, } \Phi(X) \in \text{mod}_k - \mathcal{M}, \text{ Tor}^i_{\mathcal{M}}(\Phi(X), \Psi(h_P)) = 0, 1 \leq i < k, \forall P \in \mathcal{P} \}
\]

Proof. Let \( X \in \text{gen}_k(\mathcal{M}) \), then there exists an exact sequence \( M_k \to \cdots \to M_0 \to X \to 0 \) such that \( \Phi \) preserves its exactness, this implies \( \Phi(X) \in \text{mod}_k - \mathcal{M} \). Now, we apply \( \varepsilon = \Phi' \Phi \) and consider the commutative diagram

\[
\begin{array}{ccccccc}
M_k & \longrightarrow & \cdots & \longrightarrow & M_0 & \longrightarrow & X & \longrightarrow & 0 \\
\varphi_{M_k} \downarrow & & & & \varphi_{M_0} \downarrow & & \varphi_X \downarrow & & \\
\varepsilon_{M_k} & \longrightarrow & \cdots & \longrightarrow & \varepsilon_{M_0} & \longrightarrow & \varepsilon_X & \longrightarrow & 0
\end{array}
\]

Now, since \( \Phi' \) is right exact and \( \varphi_{M_i} \) is an isomorphism for \( 0 \leq i \leq k \), we conclude that \( \varphi_X \) is an isomorphism and the lower row is exact. This implies \( \text{Tor}^i_{\mathcal{M}}(\Phi(X), \Psi(h_P)) = 0, 1 \leq i < k \).

Conversely, if we take \( X \in \text{Mod} - P \) fulfilling the assumptions in the set bracket of the lemma. We can apply \( \Phi' \) to the projective \( k \)-presentation of \( \Phi(X) \), then we can find a diagram as before
and Rem.

3.14 since the following are equivalent:

\[ P \Psi(\cdot) \in \text{gen}_k^{\text{Mod} \rightarrow \mathcal{P}}(\mathcal{M}), \]

\[ \Pi((\text{cogen}_k^{\mathcal{P}}(\mathcal{M}))^{\text{op}}) = \Pi(\text{gen}_k^{\mathcal{F}^{\text{op}}}(\mathcal{M}^{\text{op}})) = \text{Im} \cap \text{gen}_k^{\text{Mod} \rightarrow \mathcal{I}^{\text{op}}}(\Pi(\mathcal{M}^{\text{op}})) \]

cf. [2], Prop. 2.2.1, Prop. 2.2.8

Remark 3.14. For an additive category \( \mathcal{M} \) of \( \mathcal{E} \) (resp. of \( \mathcal{F} \)) we have:

\[ \mathcal{P}(\text{gen}_k^{\mathcal{F}}(\mathcal{M})) = \text{Im} \mathcal{P} \cap \text{gen}_k^{\text{Mod} \rightarrow \mathcal{P}}(\mathcal{P}(\mathcal{M})), \]

\[ \Pi((\text{cogen}_k^{\mathcal{F}}(\mathcal{M}))^{\text{op}}) = \Pi(\text{gen}_k^{\mathcal{F}^{\text{op}}}(\mathcal{M}^{\text{op}})) = \text{Im} \Pi \cap \text{gen}_k^{\text{Mod} \rightarrow \mathcal{I}^{\text{op}}}(\Pi(\mathcal{M}^{\text{op}})) \]

This follows from remark 3.4 since \( \mathcal{P} : \mathcal{E} \rightarrow \text{Im} \mathcal{P} \) is an equivalence of exact categories and \( \text{Im} \mathcal{P} \) is deflation-closed in \( \text{mod}_\infty \mathcal{P} \) and \( \text{mod}_\infty \mathcal{P} \) is deflation-closed in \( \text{Mod} \mathcal{P} \). The second statement follows by passing to the opposite category.

As before, let \( \Phi : \mathcal{E} \rightarrow \text{Mod} \mathcal{M}, \Phi(X) = \text{Hom}_\mathcal{E}(\cdot, X)|_{\mathcal{M}}, \Psi : \mathcal{E} \rightarrow \mathcal{M} - \text{Mod}, \Psi(X) = \text{Hom}_\mathcal{E}(X, \cdot)|_{\mathcal{M}} \). We have the immediate corollary:

Corollary 3.15. (of Lem. 3.13 and Rem. 3.14) (1) Let \( \mathcal{E} \) be an exact category with enough projectives \( \mathcal{P} \) and \( \mathcal{M} \) a full additive subcategory. Then the following are equivalent:

1. \( X \in \text{gen}_k^{\mathcal{F}}(\mathcal{M}) \)
2. \( \Phi(X) \in \text{mod}_k \mathcal{M} \) and for every \( P \in \mathcal{P} \):
   \[ \Phi(X) \otimes_{\mathcal{M}} \Psi(P) \rightarrow \text{Hom}_\mathcal{E}(P, X), \quad g \otimes f \mapsto g \circ f \]
   is an isomorphism, \( \text{Tor}^i_\mathcal{M}(\Phi(X), \Psi(P)) = 0, 1 \leq i < k \).

(2) If \( \mathcal{E} \) is an exact category with enough injectives \( \mathcal{I} \) and \( \mathcal{M} \) a full additive subcategory. Then the following are equivalent:

1. \( X \in \text{cogen}_k^{\mathcal{F}}(\mathcal{M}) \)
2. \( \Psi(X) \in \mathcal{M} - \text{mod}_k \) and for every \( I \in \mathcal{I} \):
   \[ \Phi(I) \otimes_{\mathcal{M}} \Psi(X) \rightarrow \text{Hom}_\mathcal{F}(X, I), \quad g \otimes f \mapsto g \circ f \]
   is an isomorphism, \( \text{Tor}^i_\mathcal{M}(\Phi(I), \Psi(X)) = 0, 1 \leq i < k \).

Theorem 3.16. (Symmetry principle). Let \( \mathcal{E} \) be an exact category with enough projectives \( \mathcal{P} \) and enough injectives \( \mathcal{I} \) and \( k \geq 1 \). The following two statements are equivalent:

1. \( \mathcal{P} \subset \text{cogen}_k^{\mathcal{F}}(\mathcal{M}) \) and \( \Phi(I) = \text{Hom}_\mathcal{E}(\cdot, I)|_{\mathcal{M}} \in \text{mod}_k \mathcal{M} \) for every \( I \in \mathcal{I} \)
2. \( \mathcal{I} \subset \text{gen}_k^{\mathcal{F}}(\mathcal{M}) \) and \( \Psi(P) = \text{Hom}_\mathcal{E}(P, \cdot)|_{\mathcal{M}} \in \mathcal{M} - \text{mod}_k \) for every \( P \in \mathcal{P} \)

Proof. We consider \( \mathcal{P}, \mathcal{I} \) as before defined for the category \( \mathcal{E} \). Then, it is straightforward from the previous Lemma to see that (1) and (2) are both equivalent to for all \( P \in \mathcal{P}, I \in \mathcal{I}, \quad \Psi(P) \in \mathcal{M} - \text{mod}_k, \Phi(I) \in \text{mod}_k \mathcal{M} \) and

\[ \Phi(I) \otimes_{\mathcal{M}} \Psi(P) \rightarrow \text{Hom}_\mathcal{E}(P, I), \quad g \otimes f \mapsto g \circ f \]

is an isomorphism, \( \text{Tor}^i_\mathcal{M}(\Phi(I), \Psi(P)) = 0, 1 \leq i < k \). Therefore (1) and (2) are equivalent.

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Julia Sauter, Faculty of Mathematics, Bielefeld University, PO Box 100 131, D-33501 Bielefeld

Email address: jsauter@math.uni-bielefeld.de