Inverse problem by Cauchy data on an arbitrary sub-boundary for systems of elliptic equations

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Abstract

We consider an inverse problem of determining coefficient matrices in an \(N\)-system of second-order elliptic equations in a bounded two-dimensional domain by a set of Cauchy data on an arbitrary sub-boundary. The main result of this paper is as follows. If two systems of elliptic operators generate the same set of partial Cauchy data on an arbitrary sub-boundary, then the coefficient matrices of the first-order and zero-order terms satisfy the prescribed system of first-order partial differential equations. The main result implies the uniqueness of any two coefficient matrices provided that the one remaining matrix among the three coefficient matrices is known.

1. Introduction

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with a smooth boundary, \(\tilde{\Gamma}\) be an open set of \(\partial\Omega\) and \(\Gamma_0 = \partial\Omega \setminus \tilde{\Gamma}\), and let \(\nu\) be the unit outward normal vector to \(\partial\Omega\). Consider the following boundary value problem:

\[
L(x, D)u = \Delta u + 2A\partial_u u + 2B\partial_{\nu} u + Qu = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0.
\]

(1)

Here \(u = (u_1, \ldots, u_N)\) is an unknown vector-valued function and \(A, B, Q\) be smooth \(N \times N\) matrices, \(i = \sqrt{-1}, x = (x_1, x_2) \in \mathbb{R}^2\), \(x\) is identified with \(z = x_1 + ix_2 \in \mathbb{C}\), \(\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)\) and \(\partial_{\nu} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\).

Consider the following partial Cauchy data:

\[
C_{A, B, Q} = \left\{ \left( u, \frac{\partial u}{\partial \nu} \right)|_{\tilde{\Gamma}}; L(x, D)u = 0 \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0, \quad u \in H^1(\Omega) \right\}.
\]

The paper is concerned with the following inverse problem. Using the partial Cauchy data \(C_{A, B, Q}\), determine the coefficient matrices \(A, B, Q\).

We can interpret \(C_{A, B, Q}\) as follows: as long as the trace of \(u\) on \(\partial\Omega\) belong to the space \(H^\frac{1}{2}(\partial\Omega)\) we are allowed to freely choose Dirichlet data supported on \(\tilde{\Gamma}\) and measure the...
Corollary 2. The system (1) is important in the mathematical physics and is a linearized equation, for example, for reaction–diffusion equation, and our inverse problem is concerned with the determination of coupling coefficients by boundary data on an arbitrary sub-boundary.

In one special case of $N = 1$ and $A = B = 0$, this inverse boundary value problem is related to the so-called Calderón problem (see [6]), which is a mathematical realization of electrical impedance tomography.

Similar to the case of $N = 1$ in [16], the simultaneous determination of all three coefficients $A, B, Q$ is impossible, but we can establish some equations for coefficient matrices $(A, B, Q)$ which generate the same partial Cauchy data.

Our main result is as follows.

**Theorem 1.** Let $A_j, B_j \in C^{5+\alpha}(\Omega)$ and $Q_j \in C^{4+\alpha}(\Omega)$ for $j = 1, 2$ and some $\alpha \in (0, 1)$. Suppose that $C_{A_1, B_1, Q_1} = C_{A_2, B_2, Q_2}$. Then

$$A_1 = A_2 \quad \text{and} \quad B_1 = B_2 \quad \text{on} \quad \Gamma,$$  

(2)

$$2\partial_\nu(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2) = 0 \quad \text{in} \quad \Omega$$  

(3)

and

$$2\partial_\nu(B_1 - B_2) + A_2(B_1 - B_2) + (A_1 - A_2)B_1 - (Q_1 - Q_2) = 0 \quad \text{in} \quad \Omega.$$  

(4)

In the case of $N = 1$ and two dimensions, there are many works and we refer to some of them, and here we do not intend to provide a complete list. In the case of the full Cauchy data $\Gamma = \partial \Omega$, the uniqueness in determining a potential $q$ in the two-dimensional case was proved for the conductivity equation by Nachman in [20] within $C^4$ conductivities, and later in [2] within $L^\infty$ conductivities. For a convection equation, see [8]. The case of the Schrödinger equation was discussed by Bukhgeim [4]. In the case of the partial Cauchy data on an arbitrary sub-boundary, the uniqueness was obtained in [13] for the potential $q \in C^{2+\alpha}(\Omega)$, and in [17], the regularity assumption was improved to $C^\theta(\Omega)$ in the case of the full Cauchy data and up to $W^1_p(\Omega)$ with $p > 2$ in the case of partial Cauchy data on the arbitrary sub-boundary. The case of the general second-order elliptic equation was studied in the papers [16] and [14]. The results of [13] were extended to a Riemannian surface in [11]. The case where voltages are applied and currents are measured on disjoint sub-boundaries was discussed and the uniqueness is proved in [15]. Conditional stability estimates for determining a potential are obtained in [21] and [22].

For the full Cauchy data, the system (1) was studied in connection with the inverse problem for the Lamé system in [9] and [10]. The related inverse problem for the Maxwell system was studied in [7] and for the Dirac system in [24] by Cauchy data on sub-boundaries satisfying geometric constraints. For the systems on the plane in the case of the full Cauchy data we refer to [23] and [1]. For the Calderón problem for the Schrödinger equation in dimension 3 or more, we refer to the papers [5, 18, 19, 21, 22] and [25]. To the best knowledge of the authors, there are no publications for the uniqueness by partial Cauchy data on an arbitrary sub-boundary for weakly coupled system of second-order elliptic partial differential equations, and theorem 1 is the affirmative answer.

Theorem 1 asserts that any two coefficient matrices among three are uniquely determined by the partial Cauchy data on the arbitrary sub-boundary $\Gamma$ for the system of elliptic differential equations. That is,

**Corollary 2.** Let $(A_j, B_j, Q_j) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega) \times C^{4+\alpha}(\Omega)$, $j = 1, 2$ for some $\alpha \in (0, 1)$ and be complex valued. We assume that either $A_1 \equiv A_2$ or $B_1 \equiv B_2$ or $Q_1 \equiv Q_2$ in $\Omega$. Then $C_{A_1, B_1, Q_1} = C_{A_2, B_2, Q_2}$ implies $(A_1, B_1, Q_1) = (A_2, B_2, Q_2)$ in $\Omega$. 

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Proof. Case 1: \( Q_1 = Q_2 \). Denote \( R(x, D)(w_1, w_2) = (2\partial_{x_1}w_1 + B_2w_1 + w_2A_1, 2\partial_{x_2}w_2 + A_2w_2 + w_1B_1) \) for \( N \times N \) matrices \( w_1, w_2 \). Therefore, applying theorem 1, we obtain
\[
R(x, D)(A_1 - A_2, B_1 - B_2) = 0 \quad \text{in } \Omega
\]
and
\[
(A_1 - A_2)|\bar{x} = (B_1 - B_2)|\bar{x} = 0.
\]
If a function \( \psi \in C^2(\bar{\Omega}) \) satisfies \( |\nabla \psi| > 0 \) on \( \Omega \), then it is strongly pseudo-convex (see definition 8.6.1, p 203 of [12]) with respect to the operators \( \partial_z \) and \( \partial_{\bar{z}} \) due to the fact that the set defined by formula (8.6.5), p 203 of [12] is empty. Hence, according to theorem 8.6.3 of [12], for all sufficiently large positive \( \lambda \) the function \( \phi = e^{\lambda \psi} \) satisfies conditions of theorem 8.5.2 of [12]. Then, by theorem 8.5.2 there exist constants \( \tau_0 \) and \( C \) independent of \( \tau \) such that
\[
|\tau|^{\frac{1}{2}} \| w e^{\tau \phi} \|_{L^2(\Omega)} \leq C \| (\partial_z w) e^{\tau \phi} \|_{L^2(\Omega)}, \quad \forall \tau \geq \tau_0 \quad \text{and} \quad \forall w \in H^1_0(\Omega)
\]
and
\[
|\tau|^{\frac{1}{2}} \| w e^{\tau \phi} \|_{L^2(\Omega)} \leq C \| (\partial_{\bar{z}} w) e^{\tau \phi} \|_{L^2(\Omega)}, \quad \forall \tau \geq \tau_0 \quad \text{and} \quad \forall w \in H^1_0(\Omega)
\]
Consider the boundary value problem
\[
R(x, D)(w_1, w_2) = (f_1, f_2) \quad \text{in } \Omega, \quad (w_1, w_2)|_{\partial \Omega} = 0.
\]
Applying the Carleman estimates (7) and (8) to each of \( N^2 \) equations in (9), we have
\[
|\tau|^{\frac{1}{2}} \| (w_1, w_2) e^{\tau \phi} \|_{L^2(\Omega)} \leq C \left( \sum_{j=1}^{2} \| f_j e^{\tau \phi} \|_{L^2(\Omega)} + \| (w_1, w_2) e^{\tau \phi} \|_{L^2(\Omega)} \right), \quad \forall \tau \geq \tau_0.
\]
The second term on the right-hand side of (10) can be absorbed into the left-hand side. Therefore, we have
\[
|\tau|^{\frac{1}{2}} \| (w_1, w_2) e^{\tau \phi} \|_{L^2(\Omega)} \leq C \sum_{j=1}^{2} \| f_j e^{\tau \phi} \|_{L^2(\Omega)}, \quad \forall \tau \geq \tau_0.
\]
Using (11) and repeating the arguments of theorem 8.9.1 p 224 in [12], we prove that a solution of the Cauchy problem (5), (6) is zero.

Case 2: \( B_1 = B_2 \). From equation (4), we have
\[
(A_1 - A_2)B_1 = (Q_1 - Q_2) \quad \text{in } \Omega.
\]
Hence, equation (3) can be written as
\[
2\partial_x(A_1 - A_2) + B_2(A_1 - A_2) - (A_1 - A_2)B_1 = 0 \quad \text{in } \Omega, \quad (A_1 - A_2)|\bar{x} = 0.
\]
Using (7), for the boundary value problem
\[
2\partial_x w + B_2 w - wB_1 = f \quad \text{in } \Omega, \quad w|_{\partial \Omega} = 0,
\]
we obtain the estimate
\[
|\tau|^{\frac{1}{2}} \| w e^{\tau \phi} \|_{L^2(\Omega)} \leq C \| f e^{\tau \phi} \|_{L^2(\Omega)} \quad \forall \tau \geq \tau_0.
\]
Using Carleman estimate (13) and repeating the arguments in [12], we prove that the solution of the Cauchy problem (12) is zero. Then equation (4) implies that \( Q_1 = Q_2 \).

The proof in the case \( A_1 = A_2 \) is the same. \( \square \)

Next we consider other form of elliptic systems:
\[
\tilde{L}(x, D)u = \Delta u + A\partial_z u + B\partial_{\bar{z}} u + Qu.
\]
Here, $A, B, Q$ are complex-valued $N \times N$ matrices. Let us define the following set of partial Cauchy data:

$$
\tilde{C}_{A, B, Q} = \left\{ \left( u, \frac{\partial u}{\partial \nu} \right) \mid \tilde{L}(x, D)u = \Delta u + A\partial_x u + B\partial_x u + Qu = 0 \quad \text{in} \quad \Omega, u|_{\Gamma_0} = 0, u \in H^1(\Omega) \right\}.
$$

Then one can prove the following corollary.

**Corollary 3.** Let $Q_1, Q_2 \in C^{\alpha,0}(\Omega)$ and let two pairs of complex-valued coefficient matrices $(A_1, B_1)$, $(A_2, B_2)$ be in $C^{\alpha,0}(\Omega) \times C^{\alpha,0}(\Omega)$ for some $\alpha \in (0, 1)$. We assume that $Q_1 \equiv Q_2$ in $\Omega$. Then $(A_1, B_1) \equiv (A_2, B_2)$ in $\Omega$.

**Proof.** Observe that $\tilde{L}(x, D) = \Delta + A\partial_\xi + B\partial_\xi + Q$, where $A = A + iB$ and $B = A - iB$. Therefore, applying corollary 2, we complete the proof. \(\square\)

**Remark.** Unlike corollary 2, in the two cases of $A_1 \equiv A_2$ and $B_1 \equiv B_2$, we cannot, in general, claim that $(A_1, B_1, Q_1) \equiv (A_2, B_2, Q_2)$. By the same argument as corollary 2, we can prove only

(i) $\frac{\partial B_1}{\partial x_1} = \frac{\partial B_2}{\partial x_1}$ in $\Omega$ if $A_1 = A_2$ in $\Omega$.

(ii) $\frac{\partial A_1}{\partial x_1} = \frac{\partial A_2}{\partial x_1}$ in $\Omega$ if $B_1 = B_2$ in $\Omega$.

Moreover, consider the following example:

$$
\Omega = (0, 1) \times (0, 1),
$$

and let us choose $\eta(x_2) \in C^0_0(0, 1)$ such that it is identically not equal to zero. Then the operators $L(x, D)$ and $e^{\alpha t}L(x, D)e^{-\alpha t}$ generate the same partial Cauchy data; the coefficients of the terms $\partial_\xi x^k$ are the same, but the remaining coefficient matrices are not equal.

**2. Preliminary results**

Throughout the paper, we use the following notations.

**Notations.** $i = \sqrt{-1}$, $x_1, x_2, \xi_1, \xi_2 \in \mathbb{R}^1$, $z = x_1 + ix_2$, $\xi = \xi_1 + i\xi_2$, $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. We identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. $\partial_\xi = \frac{1}{2}(\partial_{\xi_1} - i\partial_{\xi_2})$, $\partial_\xi = \frac{1}{2}(\partial_{\xi_1} + i\partial_{\xi_2})$, $\beta = (\beta_1, \beta_2)$, $|\beta| = \beta_1 + \beta_2$, $D = \left( \frac{1}{i \partial_{x_1}}, \frac{1}{i \partial_{x_2}} \right)$. Let $\chi_G$ be the characteristic function of the set $G$. The tangential derivative on the boundary is given by $\partial_\nu = v_2 \frac{\partial}{\partial x_1} - v_1 \frac{\partial}{\partial x_2}$, where $v = (v_1, v_2)$ is the unit outer normal to $\partial \Omega$, $B(\bar{\xi}, \delta) = \{ x \in \mathbb{R}^2 ; | x - \bar{\xi} | < \delta \}$. $S(\bar{\xi}, \delta) = \{ x \in \mathbb{R}^2 ; | x - \bar{x} | < \delta \}$. We set $(u, v)_{\tilde{L}^2(\Omega)} = \int_{\Gamma_0} u\overline{v} \ dx$ for functions $u, v$, while by $(a, b)$ we denote the scalar product in $\mathbb{R}^2$ if there is no fear of confusion. For $f : \mathbb{R}^2 \to \mathbb{R}^1$, the symbol $f'$ denotes the Hessian matrix with entries $\frac{\partial^2 f}{\partial x_1 \partial x_2}$, and $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear operators from a Banach space $X$ to another Banach space $Y$. Let $E$ be the $N \times N$ unit matrix. We set $\| u \|_{H^1(\Omega)} = \left[ \| u \|^2_{H^0(\Omega)} + \| \nabla u \|^2_{L^2(\Omega)} \right]^{\frac{1}{2}}$. Finally, for any $\bar{\xi} \in \partial \Omega$, we introduce the left and the right tangential derivatives as follows:

$$
D_+(\bar{\xi})f = \lim_{s \to +0} \left( \frac{f(\ell(s)) - f(\bar{\xi})}{s} \right),
$$
where \( \ell(0) = \tilde{x} \), \( \ell(s) \) is a parametrization of \( \partial \Omega \) near \( \tilde{x} \), \( s \) is the length of the curve, and we are moving clockwise as \( s \) increases

\[
D_\tau (\tilde{x}) f = \lim_{s \to 0} \frac{f(\tilde{x}(s)) - f(\tilde{x})}{s},
\]

where \( \tilde{x}(0) = \tilde{x} \), \( \tilde{x}(s) \) is the parametrization of \( \partial \Omega \) near \( \tilde{x} \), \( s \) is the length of the curve and we are moving counterclockwise as \( s \) increases. By \( \alpha_\kappa \left( \frac{1}{\tau^\kappa} \right) \), we denote a function \( f(\tau, \cdot) \) such that \( \|f(\tau, \cdot)\|_\kappa = o \left( \frac{1}{\tau^\kappa} \right) \) as \( |\tau| \to +\infty \).

The proof of theorem 1 relies on the construction of complex geometric optics solutions for the system (1), which is a solution of (1) of the form

\[
u = e^{i\Phi(z)} \sum_{j=0}^N a_j(x)\xi_j + e^{i\frac{\Phi(z)}{\kappa}} \sum_{j=0}^N b_j(x)\xi_j + e^{i\Re \Phi} \partial_{\tilde{z}}(\Omega) \left( \frac{1}{\tau^\kappa} \right),
\]

where \( \tau \) is a large parameter and the functions \( \Phi, a_j, b_j \) will be constructed below.

As a phase function for such a solution, we consider a holomorphic function \( \Phi(z) \) such that

\[
\Phi(z) = \psi(x_1, x_2) + i\psi(x_1, x_2)
\]

with real-valued \( \psi \) and \( \psi \). Let for some \( \alpha \in (0, 1) \), the function \( \Phi \) belong to \( C^{\alpha+\kappa}(\overline{\Omega}) \).

Moreover,

\[
\delta_\kappa \Phi(z) = 0 \quad \text{in} \quad \Omega, \quad \text{Im} \Phi|_{\Gamma_0} = 0. \tag{15}
\]

Denote by \( \mathcal{H} \) the set of all the critical points of the function \( \Phi \):

\[
\mathcal{H} = \left\{ z \in \overline{\Omega}; \frac{\partial \Phi}{\partial z}(z) = 0 \right\}.
\]

Assume that \( \Phi \) has no critical points on \( \overline{\Gamma} \), and that all critical points are nondegenerate:

\[
\mathcal{H} \cap \partial \Omega \subset \Gamma_0, \quad \delta_\kappa^2 \Phi(z) \neq 0, \quad \forall z \in \mathcal{H}. \tag{16}
\]

Then \( \Phi \) has only a finite number of critical points and we can set

\[
\mathcal{H} \setminus \Gamma_0 = \{ \tilde{x}_1, \ldots, \tilde{x}_\ell \}, \quad \mathcal{H} \cap \Gamma_0 = \{ \tilde{x}_{\ell+1}, \ldots, \tilde{x}_{\ell+\ell} \}. \tag{17}
\]

Let \( \partial \Omega = \bigcup_{j=0}^K \gamma_j \), where \( \gamma_j \) is a closed contour. Conclusions (3) and (4) of theorem 1 are proved pointwise in critical points of the holomorphic function \( \Phi \). In order to prove (3) and (4), for any \( \tilde{x} \in \Omega \), we have to construct a sequence of critical points of holomorphic functions which converges to \( \tilde{x} \). The following proposition asserts the convergence and was proved in [13].

**Proposition 1.** Let \( \tilde{x} \) be an arbitrary point in \( \Omega \). There exists a family of functions \( \{ \Phi_\epsilon \}_{\epsilon \in (0, 1)} \) satisfying (13) and (16) and there exists a family \( \{ \tilde{x}_\epsilon \}, \epsilon \in (0, 1) \) such that

\[
\tilde{x}_\epsilon \in \mathcal{H}_\epsilon = \left\{ z \in \overline{\Omega}; \frac{\partial \Phi_\epsilon}{\partial z}(z) = 0 \right\}, \quad \tilde{x}_\epsilon \to \tilde{x} \quad \text{as} \quad \epsilon \to +0.
\]

Moreover, for any \( j \) from \( \{ 1, \ldots, N \} \), we have

\[
\mathcal{H}_\epsilon \cap \gamma_j = \emptyset \quad \text{if} \quad \gamma_j \cap \tilde{\Gamma} \neq \emptyset, \quad \mathcal{H}_\epsilon \cap \gamma_j \subset \Gamma_0 \quad \text{if} \quad \gamma_j \cap \tilde{\Gamma} = \emptyset
\]

and

\[
\text{Im} \Phi_\epsilon(\tilde{x}_\epsilon) \notin \{ \text{Im} \Phi_\epsilon(x); x \in \mathcal{H}_\epsilon \setminus \{ \tilde{x}_\epsilon \} \} \quad \text{and} \quad \text{Im} \Phi_\epsilon(\tilde{x}_\epsilon) \neq 0.
\]

In order to obtain (3) and (4), we need a specific phase function and its existence is guaranteed by the following proposition. As for the proof, see [16].
**Proposition 2.** Let \( \hat{\Gamma}_+, \subset \hat{\Gamma} \) be an arc with the left endpoint \( x_- \) and the right endpoint \( x_+ \), oriented clockwise. For any \( \hat{x} \in \text{Int} \hat{\Gamma}_+ \), there exists a function \( \Phi(z) \) which satisfies (15) and (16), \( \text{Im} \Phi|_{\partial \Omega} = 0 \),

\[
\hat{x} \in \mathcal{G} = \left\{ x \in \hat{\Gamma}_+; \quad \frac{\partial \text{Im} \Phi}{\partial \hat{\tau}}(x) = 0 \right\}, \quad \text{card} \mathcal{G} < \infty
\]  

and

\[
\left( \frac{\partial}{\partial \hat{\tau}} \right)^2 \text{Im} \Phi(x) \neq 0, \quad \forall x \in \mathcal{G} \setminus \{x_-, x_+\}.
\]

Moreover,

\[
\text{Im} \Phi(\hat{x}) \neq \text{Im} \Phi(x), \quad \forall x \in \mathcal{G} \setminus \{\hat{x}\} \quad \text{and} \quad \text{Im} \Phi(\hat{x}) \neq 0
\]

and

\[
\mathbf{D}_-(x_-) \left( \frac{\partial}{\partial \hat{\tau}} \right)^6 \text{Im} \Phi \neq 0, \quad \mathbf{D}_+(x_+) \left( \frac{\partial}{\partial \hat{\tau}} \right)^6 \text{Im} \Phi \neq 0.
\]

Our proof is based on the stationary phase argument (e.g., [3]) and we also need the following propositions which are variants of the classical Riemann–Lebesgue lemma. As for the proof, we can refer to [13].

**Proposition 3.** Let \( \Phi \) satisfy (15) and (16). For every \( g \in L^1(\Omega) \), we have

\[
\int_\Omega g(e^{i(\Phi - \Psi)}) \, dx \to 0 \quad \text{as} \quad \tau \to +\infty.
\]

Moreover,

**Proposition 4.** Let \( \Phi \) satisfy (15) and (16), \( g \in W^1_p(\Omega) \) with some \( p > 2 \), \( g|_{\partial \Omega} = 0 \) and \( \text{supp} g \subset \Omega \). Then,

\[
\int_\Omega g(e^{i(\Phi - \Psi)}) \, dx = o \left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty.
\]

**Proof.** By the Sobolev imbedding theorem, the function \( g \) belongs to \( C^\alpha(\Omega) \) for some positive \( \alpha \). Note that by (16) and the assumption on \( g \), we have

\[
\left\| \frac{(\nabla \psi, v) e^{i(\Phi - \Psi)} }{2i|\nabla \psi|^2} \right\|_{C^\alpha(S_{\Omega, \beta})} \leq C \|g\|_{C^\alpha(S_{\Omega, \beta})} \leq \frac{C}{\delta^{1-\alpha}}.
\]

Also,

\[
\text{div} \left( g \frac{\nabla \psi}{2i|\nabla \psi|^2} \right) = \left( \nabla g, \frac{\nabla \psi}{2i|\nabla \psi|^2} \right) + g \text{div} \left( \frac{\nabla \psi}{2i|\nabla \psi|^2} \right).
\]

Since

\[
\left| \left( \nabla g, \frac{\nabla \psi}{2i|\nabla \psi|^2} \right) \right| \leq C \sum_{j=1}^{\ell+\epsilon} \frac{|g(x)|}{|x-x_j|^2}
\]

by the Hölder inequality we conclude that \( \left( \nabla g, \frac{\nabla \psi}{2i|\nabla \psi|^2} \right) \in L^1(\Omega) \). By (16) and assumption that \( g|_{\partial \Omega} = 0 \), we obtain

\[
\left| g \text{div} \left( \frac{\nabla \psi}{2i|\nabla \psi|^2} \right) \right| \leq C \sum_{j=1}^{\ell+\epsilon} \frac{|g(x)|}{|x-x_j|^2} \leq C \sum_{j=1}^{\ell+\epsilon} \frac{\|g\|_{C^\alpha(\Omega)}}{|x-x_j|^{2-\alpha}}.
\]
Therefore, \( \text{div} \left( g \frac{\nabla \psi}{2 \tau |\nabla \psi|^2} \right) \in L^1(\Omega) \). By (22), passing to the limit as \( \delta \) goes to zero, we have

\[
J = \int_\Omega g e^{\tau(\Phi - \Psi)} \, dx = \lim_{\delta \to 0} \int_{\Omega \cup \cup_{j=1}^\ell B(\tilde{x}_j, \delta)} g e^{\tau(\Phi - \Psi)} \, dx
\]

\[
= \lim_{\delta \to 0} \int_{\Omega \cup \cup_{j=1}^\ell B(\tilde{x}_j, \delta)} \frac{g (\nabla \psi, \nabla) e^{\tau(\Phi - \Psi)}}{2 \tau |\nabla \psi|^2} \, dx
\]

\[
= \lim_{\delta \to 0} \int_{\Omega \cup \cup_{j=1}^\ell B(\tilde{x}_j, \delta)} \frac{g (\nabla \psi, \nabla) e^{\tau(\Phi - \Psi)}}{2 \tau |\nabla \psi|^2} \, d\sigma - \lim_{\delta \to 0} \int_{\Omega \cup \cup_{j=1}^\ell B(\tilde{x}_j, \delta)} \text{div} \left( \frac{g (\nabla \psi, \nabla) e^{\tau(\Phi - \Psi)}}{2 \tau |\nabla \psi|^2} \right) e^{\tau(\Phi - \Psi)} \, dx
\]

\[
= - \int_\Omega \text{div} \left( \frac{g (\nabla \psi, \nabla) e^{\tau(\Phi - \Psi)}}{2 \tau |\nabla \psi|^2} \right) e^{\tau(\Phi - \Psi)} \, dx.
\]

Using proposition 3, we finish the proof. \( \square \)

The beginning terms of the complex geometric optics solution will be constructed explicitly, while the last term will be constructed implicitly using the technique based on subelliptic estimates.

Consider the boundary value problem

\[
L(x, D)u = \Delta u + 2A\partial u + 2B\partial u + Qu = f \quad \text{in} \quad \Omega, \quad u|_{\partial \Omega} = 0. \tag{23}
\]

Then, we prove a Carleman estimate with boundary terms whose weight function is degenerate.

**Proposition 5.** Suppose that \( \Phi = \varphi + i\psi \) satisfies (15) and (16), and the coefficients of the operator \( L \), matrices \( A, B, Q \) belong to \( L^\infty(\Omega) \) and \( \|A\|_{L^\infty(\Omega)} + \|B\|_{L^\infty(\Omega)} + \|Q\|_{L^\infty(\Omega)} \leq K \). Then there exist \( \tau_0 = \tau_0(K, \Phi) \) and \( C = C(K, \Phi) \), independent of \( u \) and \( \tau \), such that

\[
|\tau| \|u e^{\tau \varphi}\|_{L^2(\Omega)}^2 + \|u e^{\tau \varphi}\|_{H^1(\Omega)}^2 + \left\| \frac{\partial u}{\partial v} e^{\tau \varphi} \right\|_{L^2(\Gamma_0)}^2 + \tau^2 \left\| \frac{\partial \Phi}{\partial z} \right. \left. e^{\tau \varphi} \right\|_{L^2(\Omega)}^2 \leq C \left( \|L(x, D)u e^{\tau \varphi}\|_{L^2(\Omega)}^2 + |\tau| \int_{\Gamma_0} \left| \frac{\partial u}{\partial v} \right|^2 e^{2\tau \varphi} \, d\sigma \right) \tag{24}
\]

for all \( |\tau| > \tau_0 \) and all \( u \in H^1_0(\Omega) \).

For the scalar equation, the estimate is proved in [16]. In order to prove this estimate for the system, it is sufficient to apply the scalar estimate to each equation in the system and take advantage of the second large parameter in order to absorb the right-hand side. The proof of this proposition is given in the appendix.

Using estimate (24), we obtain the following.

**Proposition 6.** There exists a constant \( \tau_0 \) such that for \( |\tau| \geq \tau_0 \) and any \( f \in L^2(\Omega) \), there exists a solution to the boundary value problem

\[
L(x, D + i\tau \nabla \varphi)u = f \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0 \tag{25}
\]

such that

\[
\|u\|_{H^{1,1}(\Omega)} / \sqrt{|\tau|} \leq C \|f\|_{L^2(\Omega)}. \tag{26}
\]

Moreover, if \( f/\partial z \Phi \in L^2(\Omega) \), then for any \( |\tau| \geq \tau_0 \) there exists a solution to the boundary value problem (25) such that

\[
\|u\|_{H^{1,1}(\Omega)} \leq C \|f/\partial z \Phi\|_{L^2(\Omega)}. \tag{27}
\]

The constants \( C \) in (26) and (27) are independent of \( \tau \).
The proof of this proposition is exactly the same as the proof of proposition 2.5 in [16] and relies on the Carleman estimate (24). Proposition 6 provides the appropriate estimate for the last term in the complex geometric optics solution.

For constructing the complex geometric optics solution, we introduce the following operators:

\[ \partial_z^{-1} g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\xi - z} \, d\xi_1 \, d\xi_2, \quad \partial_{\bar{z}}^{-1} g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1, \xi_2)}{\bar{\xi} - \bar{z}} \, d\xi_1 \, d\xi_2. \]

Then we have (e.g., pp 47, 56, 72 in [26])

**Proposition 7.** (A) Let \( m \geq 0 \) be an integer number and \( \alpha \in (0, 1) \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(C^{m+\alpha}(\Omega), C^{m+\alpha+1}(\Omega)) \).

(B) Let \( 1 \leq p \leq 2 \) and \( 1 < \gamma < \frac{2p}{\gamma_p} \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(L^p(\Omega), L^q(\Omega)) \).

(C) Let \( 1 < p < \infty \). Then \( \partial_z^{-1}, \partial_{\bar{z}}^{-1} \in \mathcal{L}(L^p(\Omega), W^1_p(\Omega)) \).

For any matrix \( B \in C^{\mathbb{Z}+\alpha}(\Omega) \), consider the linear operators \( T_B \) and \( P_B \) such that

\[ (2\partial_z + B)T_B g = g \quad \text{in} \quad \Omega; \quad (2\partial_{\bar{z}} + B)P_B g = g \quad \text{in} \quad \Omega, \]

\[ T_B, P_B \in \mathcal{L}(H^s(\Omega), H^{s+1}(\Omega)) \cap \mathcal{L}(C^{1+s}(\partial\Omega), C^{2+s}(\partial\Omega)), \quad \forall s \in [0, 5], \quad \forall k \in \{0, 1, \ldots, 5\} \]

and

\[ T_B, P_B \in \mathcal{L}(\mathcal{V}(\omega), L^2(\Omega)), \]

where \( \mathcal{V}(\omega) = \{ g \in (H^1(\Omega))' : \text{supp } g \subset \Omega \} \). Here \( \omega \subset \subset \Omega \) is an open set to be fixed later.

Let us provide the details of construction of the operator \( T_B \). The construction of the operator \( P_B \) is similar. Let \( G \) be a bounded domain in \( \mathbb{R}^2 \) with the smooth boundary such that \( \Omega \subset \subset G \). We extend the matrix \( B \) in \( G \) in such a way that \( B \in C^{\mathbb{Z}+\alpha}(\mathbb{R}^2) \). Then the operator \( 2\partial_z + B \in \mathcal{L}(H^1(G), L^2(G)) \) and by theorem 3.25, p 83 in [27], we see that \( \text{Im} (2\partial_z + B) = L^2(G) \). Therefore, there exists a linear continuous operator \( R: L^2(G) \to H^1(G) \) such that

\[ (2\partial_z + B)Rg = g \quad \text{for all } g \text{ from } L^2(G). \]

From the general theory of elliptic operators, we have

\[ R \in \mathcal{L}(H^1(G), H^{1+1}(\Omega)) \cap \mathcal{L}(C^{k+s}(G), C^{k+1+s}(\partial G)), \quad \forall s \in [0, 5], \quad \forall k \in \{0, 1, \ldots, 5\}. \]

Set \( \mathcal{O}_\varepsilon = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq \varepsilon \} \) and \( \varepsilon \) is small enough such that \( \mathcal{O}_{2\varepsilon} \cap \omega = \emptyset \). Let \( \Pi \) be an extension operator for functions from \( \mathcal{O}_\varepsilon \) into \( G_1 = (G \setminus \Omega) \cup \mathcal{O}_\varepsilon \) such that

\[ \Pi \in \mathcal{L}(H^1(\mathcal{O}_\varepsilon), H^{1+1}(G_1)) \cap \mathcal{L}(C^{k+s}(\mathcal{O}_\varepsilon), C^{k+1+s}(\partial G_1)), \quad \forall s \in [0, 5], \quad \forall k \in \{0, 1, \ldots, 5\}. \]

Let \( \Psi(x) \) be a function such that \( \Psi|_{\mathcal{O}_\varepsilon} \equiv 1 \) and \( \Psi|_{\partial \Omega \setminus \mathcal{O}_\varepsilon} \equiv 0 \). We set \( T_B g = R(\Pi \Psi g + (1-\Psi)g) \).

From (31)–(33) we have (28) and (29). Since for any \( g \in \mathcal{V}(\omega) \) we have \( \Pi \Psi g + (1-\Psi)g = g \), the operator \( T_B \) is defined on \( \mathcal{V}(\omega) \) correctly. From the duality argument, (30) follows immediately.

Let \( \varepsilon \in C^{\mathbb{N}_0}(\Omega) \) satisfy \( |\varepsilon(\varepsilon)| \leq 1 \), the support of \( \varepsilon \) be concentrated in a small neighborhood of \( \mathcal{H} \setminus \overline{\Gamma}_0 \) and \( \varepsilon \) be identically equal to 1 in an open set \( \mathcal{O} \) which contains \( \mathcal{H} \setminus \overline{\Gamma}_0 \). We introduce the operators \( \mathcal{I}_B \) and \( \Psi_B \) by the following formula:

\[ \mathcal{I}_B = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{1}{2} \partial_z^{-1} eB \right)^j \partial_{\bar{z}}^{-1}, \quad \Psi_B = \frac{1}{2} \sum_{j=0}^{\infty} (-1)^j \left( \frac{1}{2} \partial_{\bar{z}}^{-1} eB \right)^j \partial_z^{-1}. \]
Taking the function $e$ such that $\int_{\supp e} 1 \, dx$ is sufficiently small, for any $p \in (1, +\infty)$ we have
\[
\|\partial_{\zeta}^{-1} eB\|_{L(\Omega)} < 1 \quad \text{and} \quad \|\partial_{\zeta}^{-1} eB\|_{L(\Omega)} < 1.
\] (35)

Indeed, by proposition 7 for any $p > 1$ there exists a number $q \in (1, p)$ such that the operators $\partial_{\zeta}^{-1}, \partial_{\zeta}^{-1} : L^p(\Omega) \to L^p(\Omega)$ are continuous. Therefore,
\[
\|\partial_{\zeta}^{-1} eB\|_{L(\Omega)} \leq \|\partial_{\zeta}^{-1} eB\|_{L(\Omega)} \|B\|_{L(\Omega)} \|g\|_{L(\Omega)}
\]
\[
\leq \|\partial_{\zeta}^{-1} eB\|_{L(\Omega)} \|B\|_{L(\Omega)} \left(\int_{\supp e} 1 \, dx\right)^{(p-q)/p} \|g\|_{L(\Omega)},
\]
and if $\int_{\supp e} 1 \, dx$ is small, then we easily have (35).

Hence, the operators $\pi_b$ and $\pi_{b_0}$ introduced in (34) are correctly defined.

We define two other operators:
\[
\mathcal{R}_g \, g = \frac{1}{2} e^{2\tau(\Phi_0 - \Phi)} \partial_{\zeta}^{-1} (g e^{2\tau(\Phi_0 - \Phi)}), \quad \mathcal{R}_g \, g = \frac{1}{2} e^{2\tau(\Phi_0 - \Phi)} \partial_{\zeta}^{-1} (g e^{2\tau(\Phi_0 - \Phi)}).
\] (36)

For any $N \times N$ matrix $B$ with elements from $C^1(\overline{\Omega})$, we set
\[
\mathrm{T}_B = \pi_b - T_\Phi (1 - e)B\pi_b, \quad \mathrm{P}_B = \pi_{b_0} - P_\Phi (1 - e)B\pi_{b_0},
\]
\[
\mathcal{R}_{B, g} = \pi_b \, g - e^{2\tau(\Phi_0 - \Phi)} T_\Phi (1 - e)B\pi_b \, g, \quad \mathcal{R}_{B, g} = \pi_{b_0} \, g - e^{2\tau(\Phi_0 - \Phi)} P_\Phi (1 - e)B\pi_{b_0} \, g.
\] (37)

We then for any $g \in C^\infty(\overline{\Omega})$, the functions $\mathcal{R}_{B, g}$ and $\mathcal{R}_{B, g}$ solve the following equations:
\[
(2\partial + 2\tau \partial \Phi + B) \mathcal{R}_{B, g} = \, g \quad \text{in} \quad \Omega, \quad (2\partial + 2\tau \partial \Phi + B) \mathcal{R}_{B, g} = \, g \quad \text{in} \quad \Omega.
\] (39)

For estimating the complex geometric optics solutions, we need

**Proposition 8.** Let $B \in C^1(\overline{\Omega})$, $g \in C^2(\overline{\Omega})$, $\supp g \subset \{ x | e(x) = 1 \}$ and $g|_{\Gamma} = 0$. Then for $p \in (1, \infty)$, we have
\[
\|\mathcal{R}_{B, g} - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} + \|\mathcal{R}_{B, g} - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} = o \left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to \infty.
\] (40)

**Proof.** By proposition 3.4 of [16], for any $p > 1$, we have
\[
\|\mathcal{R}_{B, g} - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} + \|\mathcal{R}_{B, g} - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} = o \left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to \infty.
\] (41)

Propositions 4 and 7 yield
\[
\|\mathcal{T}_B \, g - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} + \|\mathcal{T}_B \, g - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} = o \left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to \infty.
\] (42)

Thanks to (42) and (41), we obtain
\[
\|\mathcal{T}_B \, g - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} + \|\mathcal{T}_B \, g - \frac{g}{2\tau \partial \Phi} \|_{L(\Omega)} = o \left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to \infty.
\] (43)

By $\supp g \subset \{ x | e(x) = 1 \}$ and (29), (43), we obtain the asymptotic formula
\[
\|e^{2\tau(\Phi_0 - \Phi)} T_\Phi e^{2\tau(\Phi_0 - \Phi)} \circ (1 - e)B\pi_b \, g\|_{L(\Omega)} + \|e^{2\tau(\Phi_0 - \Phi)} P_\Phi e^{2\tau(\Phi_0 - \Phi)} \circ (1 - e)B\pi_{b_0} \, g\|_{L(\Omega)}
\]
\[
= o \left(\frac{1}{\tau}\right) \quad \text{as} \quad |\tau| \to \infty.
\]
The proof is completed. \qed
3. Proof of theorem 1

Step 1. Construction of complex geometric optics solutions.

In this step, we will construct two complex geometric optics solutions $u_1$ and $v$, respectively, for

$$u_1(x) = w_{0,\tau} e^{i\Phi} + \bar{w}_{0,\tau} e^{-i\Phi} - e^{i\Phi} \tilde{R}_{\tau,B_1}(q_1 + \bar{q}_1/\tau) - e^{-i\Phi} \tilde{R}_{\tau,A_1}(q_2 + \bar{q}_2/\tau) + e^{i\Phi} u_{-1}$$

and

$$v = w_{1,\tau} e^{-i\Phi} + \bar{w}_{1,\tau} e^{i\Phi} - e^{-i\Phi} \tilde{R}_{\tau,\tilde{A}_1}(\bar{e}_1 \left( q_{3,s} + \frac{\bar{q}_{3,s}}{\tau} \right)) - e^{i\Phi} \tilde{R}_{\tau,\tilde{A}_1}(\bar{e}_1 \left( q_{4,s} + \frac{\bar{q}_{4,s}}{\tau} \right)) - v_{-1} e^{-i\Phi}.$$

See (80) and (101) later for detailed descriptions.

Let a function $\Phi$ satisfy (15) and (16) and $\tilde{x}$ be some point from $\mathcal{H} \setminus \Gamma_0$. Without loss of generality, we may assume that $\tilde{x}$ is an arbitrary point from $\mathcal{H} \setminus \Gamma_0$. Consider the following operator:

$$L_1(x, D) = 4\partial_x \partial_\tau + 2A_1 \partial_x + 2B_1 \partial_\tau + Q_1$$

$$= (2\partial_x + B_1)(2\partial_\tau + A_1) + Q_1 - 2\partial_x A_1 - B_1 A_1$$

$$= (2\partial_x + A_1)(2\partial_\tau + B_1) + Q_1 - 2\partial_x B_1 - A_1 B_1. \quad (44)$$

Let $(w_0, \bar{w}_0) \in \mathcal{C}^{6+\alpha}(\Omega)$ be a nontrivial solution to the boundary value problem:

$$K(x, D)(w_0, \bar{w}_0) = (2\partial_x w_0 + A_1 w_0, 2\partial_\tau \bar{w}_0 + B_1 \bar{w}_0) = 0 \quad \text{in } \Omega, \quad w_0 + \bar{w}_0 = 0 \quad \text{on } \Gamma_0. \quad (45)$$

Then we have

**Proposition 9.** Let $\tilde{x}$ be an arbitrary point from $\tilde{x} \in \mathcal{H} \setminus \Gamma_0$ and $\bar{z} \in \mathcal{C}^N$ be an arbitrary vector. There exists a solution $(w_0, \bar{w}_0) \in \mathcal{C}^{6+\alpha}(\Omega)$ to problem (45) such that

$$w_0(\tilde{x}) = \bar{z}, \quad (46)$$

$$\lim_{x \to x_{\pm}} \frac{|w_0(x)|}{|x - x_{\pm}|^{98}} = \lim_{x \to x_{\pm}} \frac{|ar{w}_0(x)|}{|x - x_{\pm}|^{98}} = 0 \quad (47)$$

and

$$\partial_{x_1} \partial_{x_2} w_0(x) = \bar{a}_{x_1} \bar{a}_{x_2} \bar{w}_0(x) = 0 \quad \forall x \in \mathcal{H} \setminus \bar{z} \quad \text{and} \quad \forall \alpha_1 + \forall \alpha_2 \leq 6. \quad (48)$$

**Proof.** Let us fix a point $\tilde{x}$ from $\tilde{x} \in \mathcal{H} \setminus \Gamma_0$. By proposition 4.2 of [16] there exists a holomorphic function $a(z) \in \mathcal{C}^{2}(\Omega)$ such that $\text{Im} \ a_{|\Gamma_0} = 0$, $a(\tilde{x}) = 1$ and $a$ vanishes at each point of the set $\{x_{\pm}\} \cup \mathcal{H} \setminus \{x\}$. Let $(w_{0,0}, \bar{w}_{0,0}) \in \mathcal{C}^{6+\alpha}(\Omega)$ be a solution to problem (45) such that $w_{0,0}(\tilde{x}) = \bar{z}$. Since $(w_0, \bar{w}_0) = (a^{100} w_{0,0}, \bar{a}^{100} \bar{w}_{0,0})$ solves problem (45) and satisfies (46)-(48), the proof of the proposition is completed. □

Now we start the construction of complex geometric optics solution. Let the pair $(w_0, \bar{w}_0)$ be defined by proposition 9. Short computations and (44) yield

$$L_1(x, D)(w_0 e^{i\Phi}) = (Q_1 - 2\partial_x A_1 - B_1 A_1) w_0 e^{i\Phi},$$

$$L_1(x, D)(\bar{w}_0 e^{-i\Phi}) = (Q_1 - 2\partial_\tau B_1 - A_1 B_1) \bar{w}_0 e^{-i\Phi}. \quad (49)$$

Let $e_1$ and $e_2$ be smooth functions such that

$$\text{supp } e_1 \subset \subset \text{supp } e = 1, \quad e_1 + e_2 = 1 \quad \text{on } \Omega, \quad (50)$$

and $e_1$ vanishes in the neighborhood of $\partial \Omega$, $e_2$ vanishes in the neighborhood of the set $\mathcal{H} \setminus \Gamma_0$ and the function $e$ is determined in (34).

For any positive $\epsilon$ denote $G_{\epsilon} = \{x \in \Omega; \text{dist(supp } e_1, x) > \epsilon\}$. Now, we have
Proposition 10. Let $B, q \in C^{4,α}(\overline{\Omega})$ for some positive $α \in (0,1)$ and $\tilde{q} \in W^1_p(\overline{\Omega})$ for some $p > 2$. Suppose that $q|\mathcal{H}_e = \tilde{q}|\mathcal{H}_e = \partial^{\beta_1}_x \partial^{\beta_2}_y q|\mathcal{H}_e(\xi) = 0$ for all $α_1 + α_2 \leq 4$. There exist functions $m_+ \in C^2(\overline{\mathcal{G}}_e)$, independent of $τ$ such that for any $\mathcal{G}_e \cap \text{supp } e = \emptyset$, the asymptotic formulae hold true:

\[
\begin{align*}
\mathcal{R}_{τ,B} \left( e_1 \left( q + \tilde{q} \right) \right) &= e^{τ(\Phi - \Phi')} \left( m_+ e^{2iττ} \left( \frac{1}{τ^2} + o_C(\tau) \right) \right) \text{ as } |τ| \to +∞, \\
\mathcal{R}_{τ,B} \left( e_1 \left( q + \tilde{q} \right) \right) &= e^{τ(\Phi - \Phi')} \left( m_- e^{2iττ} \left( \frac{1}{τ^2} + o_C(τ) \right) \right) \text{ as } |τ| \to +∞.
\end{align*}
\]

Proof. Let the positive $ε$ be such that $\mathcal{G}_e \cap \text{supp } ε = \emptyset$. By the Sobolev imbedding theorem, the function $q$ belongs to the space $C^α(\overline{Ω})$ with some positive $α$. Therefore, the trace of $\tilde{q}$ on $\mathcal{H}$ is defined correctly. For all $N$ and for any domain $G_ε$ with $ε_0 > 0$ there exists a function $m_+ \in C^2(\overline{\mathcal{G}}_e)$ such that

\[
e^{2iττ} \left( \frac{1}{2} \partial^{-1}_x eB \right)^N \partial^{-1}_x e_1 = \int Ω \tilde{K}(x, ε) e_1(ε) g(ε) \, dε,
\]

where

\[
\tilde{K}(x, ε) = \frac{1}{x_1 - 1x_2 - (ξ_1 - iξ_2)}, \quad \tilde{K}_e(x, ε) ∈ C^5(\overline{Ω}) \times C^5(\overline{Ω}).
\]

Next let $x^0 = (x_1^0, x_2^0)$ be an arbitrary fixed point in $Ω$, $α^β = α^β_1 α^β_2$ and $z^0 = x^0 + ix_2^0$. Let $CV = -\frac{1}{2} \partial^{-1}_x (eVB)$ for any matrix-valued function $V(x)$. By Proposition 7 there exists $N$ such that the operator

\[
C_N ∈ L(L^1(Ω), C^5(\overline{Ω})) \quad ∀N ≥ N.
\]

We write the operator $\frac{-1}{2} \partial^{-1}_x eB$ in the form of the integral operator

\[
\begin{align*}
\mathcal{P}_N(x, ε) e_1(ε) g(ε, ξ_1, ξ_2) \, dε \, dξ_1 \, dξ_2.
\end{align*}
\]

Let us estimate the kernel $P_N$. Observe that

\[
\mathcal{P}_N(x^0_1, x^0_2, ε) = \frac{-1}{2} C_N \left( \frac{E}{2π (ξ_1 - iξ_2)} \right).
\]

Since $\sup_{ε ∈ G_ε, |β| ≤ 5} \|α^β\| L^1(Ω) + \sup_{ε ∈ G_ε} \|1\| L^1(Ω) < ∞$ there exists $r ∈ (0,1)$ independent of $N$ such that

\[
\sup_{ε ∈ Ω} \|α^{-1}_x \| L^{N-1}(Ω) \left( \frac{E}{2π (ξ_1 - iξ_2)} \right) \leq r^{N-N_r}.
\]

By (56) and (54), we obtain

\[
\|P_N(·, ·)\|_{C^α(Ω) \times L^5(Ω)} ≤ Cr^{N-N_r}.
\]
By (57) there exists a function \( P(x, \xi) \in \left( C^5(\overline{G}_r) \right) \cap L^\infty(\Omega) \times C^5(\overline{\Omega}) \) such that
\[
\sum_{j=N+2}^\infty e^{-2\tau j} \left( -1 \right)^j \left( \frac{1}{2} \partial_{\xi}^{-1} eB \right)^j \partial_{\xi}^{-1} (q + \tilde{q}) \int_\Omega P(x, \xi) e_1(\xi) g \, d\xi.
\]
Therefore, by the stationary phase argument, there exists a function \( m \in (C^2(\overline{G}_r) \cap L^\infty(\Omega)) \) such that for any positive \( \tilde{\epsilon} \):
\[
\sum_{j=N+2}^\infty e^{-2\tau j} \left( -1 \right)^j \left( \frac{1}{2} \partial_{\xi}^{-1} eB \right)^j \partial_{\xi}^{-1} \left( e^{2\tau \psi} q + \tilde{q} \right) = e^{-2\tau \psi} \left( m e^{2\tau \psi(\tilde{\epsilon})} \right) \left( \frac{1}{\tau^2} + o_{\tau^2}(G_r) \right) \left( \frac{1}{\tau^2} \right),
\]
and
\[
\sum_{j=N+2}^\infty e^{-2\tau j} \left( -1 \right)^j \left( \frac{1}{2} \partial_{\xi}^{-1} eB \right)^j \partial_{\xi}^{-1} \left( e^{2\tau \psi} q + \tilde{q} \right) = e^{-2\tau \psi} \left( m e^{2\tau \psi(\tilde{\epsilon})} \right) \left( \frac{1}{\tau^2} + o_{\tau^2}(G_r) \right) \left( \frac{1}{\tau^2} \right).
\]
By (58), (59) and (53) there exist \( \tilde{m}_\pm \in (C^2(\overline{G}_r) \cap L^\infty(\Omega)) \) such that for any positive \( \tilde{\epsilon} \) we have
\[
\tau_{B, \tau} \left( e_1 \left( q + \tilde{q} \right) \right) \bigg|_{G_r} = e^{\tau(\Phi - \Phi)} \left( \tilde{m}_+ e^{2\tau \psi(\tilde{\epsilon})} + o_{\tau^2}(G_r) \right) \left( \frac{1}{\tau^2} \right) \quad \text{as } |\tau| \to +\infty
\]
and
\[
\tau_{B, \tau} \left( e_1 \left( q + \tilde{q} \right) \right) \bigg|_{G_r} = e^{\tau(\Phi - \Phi)} \left( \tilde{m}_- e^{2\tau \psi(\tilde{\epsilon})} + o_{\tau^2}(G_r) \right) \left( \frac{1}{\tau^2} \right) \quad \text{as } |\tau| \to +\infty.
\]
Let positive \( \tilde{\epsilon} \) satisfy \( \text{supp}(1 - \epsilon) \subset G_r \) and \( \text{supp} \cap G_{\epsilon'} = \emptyset \) for \( \epsilon'' \in (\tilde{\epsilon}, \epsilon) \). Then using (60), we have
\[
e^{-2\tau \psi} \tau_B \left( e^{(\Phi - \Phi)} (1 - \epsilon) B \tau_{B, \tau} e_1 \left( q + \tilde{q} \right) \right) = e^{-2\tau \psi} \tau_B \left( (1 - \epsilon) B \tau_{B} \left( e^{(\Phi - \Phi)} e_1 \left( q + \tilde{q} \right) \right) \right) = e^{-2\tau \psi + 2\tau \psi(\tilde{\epsilon})} \tau_B \left( (1 - \epsilon) \chi_{G_r} B \frac{m_+}{\tau^2} + (1 - \epsilon) \chi_{G_r} o_{\tau^2}(\overline{G}_r) \right) + e^{-2\tau \psi + 2\tau \psi(\tilde{\epsilon})} \tau_B \left( (1 - \epsilon) (1 - \chi_{G_r}) B \frac{m_+}{\tau^2} + (1 - \epsilon) (1 - \chi_{G_r}) o_{\tau^2}(\overline{G}_r) \right) \left( \frac{1}{\tau^2} \right).
\]
Here in order to obtain the last equality, we used (60) and (50). Using (61), (62) (29) and proposition 7 we obtain (51). The proof of the second asymptotic formula is similar. □

Denote \( q_1 = P_{B_r} ((Q_1 - 2\partial_1 A_1 - B_1 A_1) w_0) - M_1, q_2 = T_{B_r} ((Q_1 - 2\partial_1 B_1 - A_1 B_1) w_0) - M_2 \in C^{5+\alpha}(\overline{\Omega}), \) where the functions \( M_1 \in \text{Ker}(\partial_1 + A_1) \) and \( M_2 \in \text{Ker}(\partial_1 + B_1) \) are taken such that \( q_1(\tilde{x}) = q_2(\tilde{x}) = 0, \quad \partial_1^a \partial_2^b \partial_1^c \partial_2^d q_1(x) = \partial_1^a \partial_2^b \partial_1^c \partial_2^d q_2(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\tilde{x}\} \) and \( \forall \alpha_1 + \alpha_2 \leq 5. \)

By proposition 10, there exist functions \( m_\pm \in C^2(\partial \Omega) \) such that
\[
\left. \tau_{\mathcal{R}, B_1} \right|_{\partial \Omega} \left( e_1 (1 - \tilde{B_1}) \right) = e^{\tau(\Phi - \Phi)} \left( \frac{m_+ e^{2\tau \psi(\tilde{\epsilon})}}{\tau^2} + o_{\tau^2}(\partial \Omega) \right) \left( \frac{1}{\tau^2} \right) \quad \text{as } |\tau| \to +\infty
\]
and
\[ \mathcal{R}_{\tau,A_1} \left( e_1 \left( q_2 + \frac{\tilde{q}_2}{\tau} \right) \right) \mid_{\partial \Omega} = e^{\i \phi_+} \left( \frac{m_- e^{-2i\psi(\tilde{\tau})}}{\tau^2} \right)
+ o_{H^1(\partial \Omega)} \left( \frac{1}{\tau^2} \right) \] as \(|\tau| \to +\infty.

(65)

Next we introduce the functions \(w_{-1}, \tilde{w}_{-1}, a_\pm, b_\pm \in C^2(\overline{\Omega})\) as a solution to the following boundary value problems:
\[ \mathcal{K}(x, D)(w_{-1}, \tilde{w}_{-1}) = 0 \quad \text{in} \quad \Omega, \quad (w_{-1} + \tilde{w}_{-1}) \mid_{\Gamma_0} = \frac{q_1}{2\partial_+ \Phi} + \frac{q_2}{2\partial_- \Phi}. \]

(66)

\[ \partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta w_{-1}(x) = \partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \tilde{w}_{-1}(x) = 0, \quad \forall x \in \mathcal{H} \quad \text{and} \quad \forall \alpha_1 + \alpha_2 \leq 2, \]
\[ \mathcal{K}(x, D)(a_\pm, b_\pm) = 0 \quad \text{in} \quad \Omega, \quad (a_\pm + b_\pm) \mid_{\Gamma_0} = m_\pm. \]

(67)

We set \(p_1 = -(Q_1 - 2\partial B_1 - A_1 B_1) \left( \frac{e^{q_1}}{2\partial_+ \Phi} - w_{-1} \right) + L_1(x, D) \left( \frac{e^{q_1}}{2\partial_+ \Phi} \right)\), \(p_2 = -(Q_1 - 2\partial A_1 - B_1 A_1) \left( \frac{e^{q_1}}{2\partial_+ \Phi} - \tilde{w}_{-1} \right) + L_1(x, D) \left( \frac{e^{q_1}}{2\partial_+ \Phi} \right), \) \(\tilde{q}_2 = T_0 p_2 - M_2, \) \(\tilde{q}_1 = P_A p_1 - M_1,\) where \(M_1 \in \text{Ker}(\partial_+ A_1)\) and \(M_2 \in \text{Ker}(\partial_+ B_1)\) are taken such that
\[ \tilde{q}_1(\tilde{x}) = \tilde{q}_2(\tilde{x}) = 0, \quad \partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \tilde{q}_1(x) = \partial_{\xi_1}^\alpha \partial_{\xi_2}^\beta \tilde{q}_2(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\tilde{x}\} \quad \text{and} \quad \forall \alpha_1 + \alpha_2 \leq 2. \]

(68)

Since by (68) the functions \(w_{-1}/2\partial_+ \Phi\) and \(\tilde{w}_{-1}/2\partial_+ \Phi\) belong to the space \(H^1(\partial \Omega)\), there exists a solution \((w_{-2}, \tilde{w}_{-2}) \in H^1(\overline{\Omega})\) to the boundary value problem
\[ \mathcal{K}(x, D)(w_{-2}, \tilde{w}_{-2}) = 0 \quad \text{in} \quad \Omega, \quad (w_{-2} + \tilde{w}_{-2}) \mid_{\Gamma_0} = \frac{\tilde{q}_1 e_2}{2\partial_+ \Phi} + \frac{\tilde{q}_2 e_2}{2\partial_- \Phi}. \]

(69)

We introduce the functions \(w_{0,\pm, \tau}, \tilde{w}_{0, \pm, \tau} \in H^1(\Omega)\) by
\[ w_{0, \pm, \tau} = w_0 + \frac{w_{-1} - e_2 q_1/2\partial_+ \Phi}{\tau} + \frac{1}{\tau^2} \left( e^{2i\psi(\tilde{\tau})} a_+ + e^{-2i\psi(\tilde{\tau})} a_- + w_{-2} - \frac{\tilde{q}_1 e_2}{2\partial_+ \Phi} \right) \]

(70)

and
\[ \tilde{w}_{0, \pm, \tau} = \tilde{w}_0 + \frac{\tilde{w}_{-1} - e_2 q_2/2\partial_- \Phi}{\tau} + \frac{1}{\tau^2} \left( e^{2i\psi(\tilde{\tau})} b_+ + e^{-2i\psi(\tilde{\tau})} b_- + \tilde{w}_{-2} - \frac{\tilde{q}_2 e_2}{2\partial_- \Phi} \right). \]

(71)

Simple computations and proposition 8 imply for any \(p \in (1, \infty)\) the asymptotic formula
\[ L_1(x, D) \left( -e^{\phi} \hat{\mathcal{K}}_{r,B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - \frac{e_2(q_1 + \tilde{q}_1/\tau)}{2\partial_+ \Phi} e^{\phi} \right)
- \frac{e_2(q_2 + \tilde{q}_2/\tau)}{2\partial_+ \Phi} e^{\phi} \right)
\]

\[ = -L_1(x, D) \left( e^{\phi} \hat{\mathcal{K}}_{r,B_1}(e_1(q_1 + \tilde{q}_1/\tau)) + \frac{e_2(q_1 + \tilde{q}_1/\tau)}{2\partial_+ \Phi} e^{\phi} \right) \]

\[ = -(Q_1 - 2\partial B_1 - A_1 B_1) e^{\phi} \hat{\mathcal{K}}_{r,B_1}(e_1(q_1 + \tilde{q}_1/\tau)) \]

\[ - \frac{e_2 q_1 + \tilde{q}_1/\tau}{2\partial_+ \Phi} e^{\phi} \right)
= -(Q_1 - 2\partial A_1 - B_1 A_1) \frac{e_2 q_1 + \tilde{q}_1/\tau}{2\partial_+ \Phi} e^{\phi} \right)
\]

\[ = -(Q_1 - 2\partial B_1 - A_1 B_1) w_0 e^{\phi} - (Q_1 - 2\partial A_1 - B_1 A_1) w_0 e^{\phi} \]

13
Similar to (49), we obtain

\[
L_1(x, D) \left( \left( Q_1 - 2 \partial \overline{B_1} - A_1 B_1 \right) \left( \frac{e^{2i q_1}}{2 \partial z \Phi} \right) + \frac{e^{2i q_1}}{2 \partial z \Phi} \right) e^\tau \Phi
\]

\[
+ \frac{1}{\tau} \left( \left( Q_1 - 2 \partial A_1 - B_1 A_1 \right) \left( \frac{e^{2i q_1}}{2 \partial z \Phi} \right) - \frac{1}{\tau} (Q_1 - 2 \partial A_1 - B_1 A_1) \frac{w_{-1}}{\partial z \Phi} \right) e^\tau \Phi
\]

\[
= - \frac{1}{\tau} (Q_1 - 2 \partial B_1 - A_1 B_1) w_{-1} e^\tau \Phi - \frac{1}{\tau} (Q_1 - 2 \partial A_1 - B_1 A_1) w_{-1} e^\tau \Phi
\]

\[
- (Q_1 - 2 \partial B_1 - A_1 B_1) \frac{w_{0}}{\partial z \Phi} \Phi (Q_1 - 2 \partial A_1 - B_1 A_1) w_{0} e^\tau \Phi + e^{2i \tau} \partial \phi (\Omega) \left( \frac{1}{\tau} \right)
\]

(72)

Using this formula, we prove the following proposition.

**Proposition 11.** For any \( p > 1 \), we have the asymptotic formula:

\[
L_1(x, D) (u_0, e^{\tau} \Phi + \overline{w}_{0, r} e^\tau \Phi - \emptyset^r \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau)))
\]

\[
= - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)) = e^{2i \tau} \partial \phi (\Omega) \left( \frac{1}{\tau} \right)
\]

as \( \tau \to \infty \). (73)

(74)

**Proof.** By (15), (45), (57), (64)–(67) and (69)–(71), we have

\[
(w_0, e^{\tau} \Phi + \overline{w}_{0, r} e^\tau \Phi - \emptyset^r \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau))) - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)))
\]

\[
\cong \overline{w}_{0, \tau} e^{2i \tau} \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau)) - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)))
\]

\[
\cong \overline{w}_{0, \tau} e^{2i \tau} \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau)) - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)))
\]

\[
\cong \overline{w}_{0, \tau} e^{2i \tau} \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau)) - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)))
\]

\[
\cong \overline{w}_{0, \tau} e^{2i \tau} \overline{\Phi}_{r, B_1} (e_1 (q_1 + \overline{q}_1 / \tau)) - e^{2i \tau} \overline{\Phi}_{r, A_1} (e_1 (q_2 + \overline{q}_2 / \tau)))
\]

(75)

By (75) and (72), we obtain (73). \( \square \)

We set \( \Omega = \{ x \in \Omega; \ \text{dist}(x, \partial \Omega) \leq \epsilon \} \). In order to construct the last term in the complex geometric optics solution, we need the following proposition.
Then we consider the function for all large

**Proposition 12.** Let $A, B \in C^{\alpha+\tau} (\Omega)$ and $Q \in C^{\alpha+\tau} (\Omega)$ for some $\alpha \in (0, 1)$, $f \in L^p (\Omega)$ for some $p > 2$, $\text{dist}(\Gamma_0, \supp f) > 0$, $q \in H^2 (\Omega)$, and $\epsilon$ be a small positive number such that $\overline{\Gamma_0} \cap (\partial \Omega \setminus \Gamma_0) = \emptyset$. Then there exists $C$ independent of $\tau$ such that for all $|\tau| > \tau_0$, there exists a solution $w \in H^1 (\Omega)$ to the boundary value problem

$$L(D, w) = f e^{i\tau} \quad \text{in } \Omega, \quad w|_{\Gamma_0} = q e^{i\tau}/\tau$$

such that

$$\sqrt{|\tau|} \|w e^{-i\tau} \|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|\nabla w e^{-i\tau} \|_{L^2(\Omega)} + \|w e^{-i\tau} \|_{H^{1,1}(\Omega)} \leq C \|f\|_{L^p(\Omega)} + \|q\|_{H^2(\Omega)}.$$  

(77)

**Proof.** First let us assume that $f$ is identically equal to zero. Let $(d, \tilde{d}) \in H^1 (\Omega) \times H^1 (\Omega)$ satisfy

$$K(D, (d, \tilde{d})) = 0 \quad \text{in } \Omega, \quad (d + \tilde{d})|_{\Gamma_0} = q.$$  

(78)

For the existence of such a solution see e.g. [27]. By (49) and (78), we have

$$L(x, D) \left( \frac{d}{\tau} e^{i\tau} + \frac{\tilde{d}}{\tau} e^{i\tau} \right) = \frac{1}{\tau} (Q - 2dA - BA) d e^{i\tau} + \frac{1}{\tau} (Q - 2dB - AB) \tilde{d} e^{i\tau}.$$  

By proposition 6, there exists a solution $\tilde{w}$ to the boundary value problem

$$L(D, \tilde{w}) = -\frac{1}{\tau} (Q - 2dA - BA) d e^{i\tau} + \frac{1}{\tau} (Q - 2dB - AB) \tilde{d} e^{i\tau}, \quad \tilde{w}|_{\Gamma_0} = 0$$  

such that there exists a constant $C > 0$ such that

$$\|\tilde{w} e^{-i\tau} \|_{H^{1,1}(\Omega)} \leq \frac{C}{\sqrt{|\tau|}} \|(Q - 2dA - BA) d e^{i\tau}\| + \frac{C}{\sqrt{|\tau|}} \|(Q - 2dB - AB) \tilde{d} e^{i\tau}\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{|\tau|}} \|q\|_{H^2(\Omega)}$$

for all large $\tau > 0$.

Then the function $\left( \frac{d}{\tau} e^{i\tau} + \frac{\tilde{d}}{\tau} e^{i\tau} \right)$ is a solution to (76) which satisfies (77) if $f \equiv 0$.

If $f$ is not identically equal to zero, without loss of generality we may assume that $q \equiv 0$. Then we consider the function $\tilde{w} = \tilde{e} e^{i\tau} \tilde{R}_{T, B}(\epsilon_{1}(q_0))$, where $\tilde{e} \in C_0^\infty (\Omega)$, $\tilde{e}|_{\text{supp } \epsilon_{1}} = 1$ and $q_0 = P_{H} f - M$, where a function $H \in C^\infty (\Omega)$ belongs to Ker $(2\tilde{e} A + A)$ and is chosen such that $q_0|_{\partial \Gamma} = 0$. Then $L(x, D) \tilde{w} = (Q - 2dB - AB) \tilde{w} + (Q - 2dA - BA) \tilde{d} e^{i\tau} + \tilde{e} e^{i\tau} (2dA + A)(\tilde{d} e^{i\tau})_{\tilde{R}_{T, B}(\epsilon_{1}(q_0)))$. Since, by proposition 8, the function $\tilde{f} (\tau, \cdot) = e^{-i\tau} L(x, D) \tilde{w} - f$ can be represented as a sum of two functions, where the first one is equal to zero in the neighborhood of $\mathcal{H}$ and is bounded uniformly in $\tau$ in the $L^2(\Omega)$ norm, the second one is $O(\frac{1}{\tau})$. Applying proposition 6 to the boundary value problem

$$L(D, \tilde{w}) = \tilde{e} e^{i\tau} \quad \text{in } \Omega, \quad \tilde{w}|_{\Gamma_0} = 0,$$

we construct a solution such that

$$\|\tilde{w} e^{-i\tau} \|_{H^{1,1}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$  

The function $w^* - \tilde{w}$ solves the boundary value problem (76) and satisfies estimate (77). \hfill \Box

Using propositions 12 and 11, we construct the last term $u_{-1}$ in the complex geometric optics solution which satisfies

$$\sqrt{|\tau|} \|u_{-1} \|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}} \|\nabla u_{-1} \|_{L^2(\Omega)} + \|u_{-1} \|_{H^{1,1}(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.$$  

(79)
Here $u_{-1}$ is the solution to
\[ L_1(x, D)u_{-1} = -L_1(x, D)u^* \text{ in } \Omega, \quad \left. u_{-1}\right|_{\partial\Omega} = -u^* \]
where
\[ u^* = w_{0,\tau} e^{i\tau} + \tilde{w}_{0,\tau} e^{i\tilde{\tau}} - e^{i\tau} \tilde{\tau} e^{-i\tau} R_{\tau,B_1}(e_1(q_1 + \tilde{q}_1/\tau)) - e^{i\tilde{\tau}} \tilde{\tau} e^{-i\tilde{\tau}} R_{\tau,A_1}(e_1(q_2 + \tilde{q}_2/\tau)). \]

Finally, we obtain a complex geometric optics solution in the form
\[ u_1(x) = w_{0,\tau} e^{i\tau} + \tilde{w}_{0,\tau} e^{i\tilde{\tau}} - e^{i\tau} \tilde{\tau} e^{-i\tau} R_{\tau,B_1}(q_1 + \tilde{q}_1/\tau) - e^{i\tilde{\tau}} \tilde{\tau} e^{-i\tilde{\tau}} R_{\tau,A_1}(q_2 + \tilde{q}_2/\tau) + e^{i\tau} u_{-1}. \]  \hspace{1cm} (80)

Obviously,
\[ L_1(x, D)u_1 = 0 \text{ in } \Omega, \quad \left. u_1\right|_{\Gamma_0} = 0. \]  \hspace{1cm} (81)

Let $u_1$ be a complex geometric optics solution as in (80).

Let $e \in C^0_0(B(0, 2))$ be a function such that $e$ is equal to 1 in a ball of small radius centered at 0. We set
\[ \eta(x, s) = e((x - \bar{x}) e^{is}). \]  \hspace{1cm} (82)

Then the operator
\[ L_2(x, s, D) = e^{-is} L_2(x, D) e^{is} = \Delta + 2(A_2 + 2s\eta)\partial_x + 2(B_2 + 2s\eta)\partial_{\eta} + Q_2 + \frac{s\Delta \eta + \nabla^2(\eta, \nabla \eta)}{2s\eta} e + 2s\eta A_2 + 2s\eta B_2 = \Delta + 2A_2, \partial_x + 2B_2, \partial_{\eta} + Q_2, \]
is of the form (1) and has the same partial Cauchy data as the operator $L_2(x, D)$ for all large $s$.

Consider the operator
\[ L_2(x, s, D)^* = 4\partial_x^2 + 2A_2, \partial_x - 2B_2, \partial_x + Q_2^*, -2\partial_x A_2, \partial_x - 2B_2, \partial_x + Q_2, \]
is of the form (1) and has the same partial Cauchy data as the operator $L_2(x, D)^*$ for all large $s$.

Similarly, we construct the complex geometric optics solutions to the operator $L_2(x, s, D)^*$.

Let $(w_1, \tilde{w}_1) \in C^{0+\alpha}(\overline{\Omega}) \times C^{0+\alpha}(\overline{\Omega})$ be a solution to the following boundary value problem:
\[ M(x, D)(w_1, \tilde{w}_1) = ((2\partial_x - B_2^*)w_1, (2\partial_x - A_2)\tilde{w}_1) = 0 \text{ in } \Omega, \quad \left. (w_1 + \tilde{w}_1)\right|_{\Gamma_0} = 0, \]  \hspace{1cm} (83)
\[ \partial_{s_1} \partial_{s_2} w_1(x) = \partial_{s_1} \partial_{s_2} \tilde{w}_1(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\bar{x}\} \text{ and } \forall s_1, s_2 \leq 2, \]
\[ \lim_{x \to \bar{x}} \frac{|w_1(x)|}{|x - \bar{x}|^\alpha} = \lim_{x \to \bar{x}} \frac{|\tilde{w}_1(x)|}{|x - \bar{x}|^\alpha} = 0. \]  \hspace{1cm} (84)

Such a pair $(w_1, \tilde{w}_1)$ exists due to proposition 9. We set $(w_{1,s}, \tilde{w}_{1,s}) = e^{is}(w_1, \tilde{w}_1)$. Observe that
\[ L_2(x, s, D)^*(w_{1,s}, e^{-is}) = (Q_2^* - 2\partial_x A_2^* - A_2 B_2^*)w_{1,s}, e^{-is}, \]
\[ L_2(x, s, D)^*(\tilde{w}_{1,s}, e^{-is}) = (Q_2^* - 2\partial_x^* B_2^* - B_2^* A_2^*)\tilde{w}_{1,s}, e^{-is}. \]

We set
\[ \rho_{-B_2} = e^{is} P_{-B_2} e^{-is}, \rho_{-A_2} = e^{is} P_{-A_2} e^{-is}, \tilde{\tau} \rho_{-A_2} = e^{is} \tilde{\tau} P_{-A_2} e^{-is}, \]
\[ \rho_{-B_2^*} = e^{is} P_{-B_2^*} e^{-is}, \]  \hspace{1cm} (86)
\[ q_3 = P_{-B_2^*}((Q_2^* - 2\partial_x A_2^* - A_2^* B_2^*)w_1) - M_3, q_4 = T_{-A_2^*}(Q_2^* - 2\partial_x B_2^* - B_2^* A_2^*)\tilde{w}_1 - M_4. \]  \hspace{1cm} (87)
Denote \( q_{3,s} = P_{B_{s}} ((Q_{3} - 2\delta B^*_2 - B^*_{2}A^*_3)w_{1,s}) - M_{3,s} = e^{\eta} q_{3}, q_{4,s} = e^{\eta} q_{4}, \) where the functions \( M_{3,s}, M_{4,s} \) are chosen such that
\[
q_{3}(\tilde{x}) = q_{4}(\tilde{x}) = 0, \quad \partial_{x_1}^{\eta} \partial_{x_2}^{2} q_{3}(x) = \partial_{x_1}^{\eta} \partial_{x_2}^{2} q_{4}(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\tilde{x}\} \text{ and } \forall \alpha_{1} + \forall \alpha_{2} \leq 5. \tag{88}
\]

By (88) the functions \( \frac{q_{3}}{2\partial^{\Phi_{-}}_{x_1}} \) and \( \frac{q_{4}}{2\partial^{\Phi_{-}}_{x_1}} \) belong to the space \( C^{2}(T_{0}) \). Therefore, we can introduce the functions \( w_{-3}, \tilde{w}_{-3} \) as a solution to the following boundary value problem:
\[
\mathcal{M}(x,D)(w_{-3}, \tilde{w}_{-3}) = 0 \quad \text{in } \Omega, \quad (w_{-3} + \tilde{w}_{-3})|_{\Gamma_{0}} = \frac{q_{3}}{2\partial^{\Phi}_{x_1}} + \frac{q_{4}}{2\partial^{\Phi}_{x_1}},
\]
\[
\partial_{x_1}^{\partial_{x_2}^{2}} w_{-3}(x) = \partial_{x_1}^{\partial_{x_2}^{2}} \tilde{w}_{-3}(x) = 0, \quad \forall x \in \mathcal{H} \text{ and } \forall \alpha_{1} + \forall \alpha_{2} \leq 2. \tag{89}
\]

Let
\[
p_{3} = (Q_{2} - 2\delta A_{1} - A_{1}^{*}B_{3}^{*}) \left( \frac{e_{1}q_{3,s} + w_{-3,s}}{2\partial^{\Phi}_{x_1}} \right) + L_{2}(x,s,D)^{*} \left( \frac{q_{3,s}e_{2}}{2\partial^{\Phi}_{x_1}} \right),
\]
\[
p_{4} = (Q_{2} - 2\delta B_{2} - B_{2}^{*}A_{3}^{*}) \left( \frac{e_{1}q_{4,s} + \tilde{w}_{-3,s}}{2\partial^{\Phi}_{x_1}} \right) + L_{2}(x,s,D)^{*} \left( \frac{q_{4,s}e_{2}}{2\partial^{\Phi}_{x_1}} \right)
\]
and
\[
\tilde{q}_{3} = e^{-\eta}(P_{B_{-2}}p_{3} - \tilde{M}_{3,s}), \quad \tilde{q}_{4} = e^{-\eta}(T_{A_{-2}}p_{4} - \tilde{M}_{4,s}),
\]
where \( \tilde{M}_{3,s} \subseteq \text{Ker}(2\partial_{x_1} - B_{-2}^{*}), \tilde{M}_{4,s} \subseteq \text{Ker}(2\partial_{x_1} - A_{-2}^{*}) \) and \( \tilde{q}_{3,s}, \tilde{q}_{4,s} = e^{\eta}(\tilde{q}_{3}, \tilde{q}_{4}) \) are chosen such that
\[
\tilde{q}_{3,s}(\tilde{x}) = \tilde{q}_{4,s}(\tilde{x}) = 0, \quad \partial_{x_1}^{\partial_{x_2}^{2}} \tilde{q}_{3,s}(x) = \partial_{x_1}^{\partial_{x_2}^{2}} \tilde{q}_{4,s}(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\tilde{x}\} \text{ and } \forall \alpha_{1} + \forall \alpha_{2} \leq 2. \tag{90}
\]

The following asymptotic formula holds true.

**Proposition 13.** There exist smooth functions \( \tilde{m}_{\pm} \in C^{2}(\partial \Omega) \), independent of \( \tau \) and \( s \), such that
\[
\mathcal{R}_{\tau_{-},A_{1}^{*}}(e_{1}(q_{3,s} + \tilde{q}_{3,s}/\tau)) \mid_{\alpha_{0}} = \frac{\tilde{m}_{\pm} e^{2\tau_{2}(\psi - \varphi(\tilde{x}))}}{\tau^{2}} + e^{2\tau_{2}} \theta(\partial \Omega) \left( \frac{1}{\tau^{2}} \right) \quad \text{as } |\tau| \to +\infty \tag{91}
\]
and
\[
\mathcal{R}_{\tau_{-},B_{2}^{*}}(e_{1}(q_{4,s} + \tilde{q}_{4,s}/\tau)) \mid_{\alpha_{0}} = \frac{\tilde{m}_{\pm} e^{-2\tau_{2}(\psi - \varphi(\tilde{x}))}}{\tau^{2}} + e^{-2\tau_{2}} \theta(\partial \Omega) \left( \frac{1}{\tau^{2}} \right) \quad \text{as } |\tau| \to +\infty. \tag{92}
\]

**Proof.** The functions \( q_{3,s}, q_{4,s} \) belong to the space \( C^{5+\alpha}(\tilde{\Omega}) \) and functions \( \tilde{q}_{3,s}, \tilde{q}_{4,s} \) belong to the space \( W^{1}_{p}(\Omega) \) for any \( p > 1 \). By (88) and (90), we have \( q_{3,s} = q_{4,s} = \tilde{q}_{3,s} = \tilde{q}_{4,s} = 0 \) on \( \mathcal{H} \). By (86) and (87), we have
\[
\mathcal{R}_{\tau_{-},A_{1}^{*}}(e_{1}(q_{3,s} + \tilde{q}_{3,s}/\tau)) = e^{\eta} \mathcal{R}_{\tau_{-},A_{1}^{*}}(e_{1}(q_{3} + \tilde{q}_{3}/\tau))
\]
and
\[
\mathcal{R}_{\tau_{-},B_{2}^{*}}(e_{1}(q_{4,s} + \tilde{q}_{4,s}/\tau)) = e^{\eta} \mathcal{R}_{\tau_{-},B_{2}^{*}}(e_{1}(q_{4} + \tilde{q}_{4}/\tau)).
\]

Then applying proposition 10 and taking into account (82), we obtain proposition 13. \( \square \)

Using the functions \( \tilde{m}_{\pm} \) we introduce functions \( \tilde{a}_{\pm}, \tilde{b}_{\pm} \in C^{2}(\Omega) \) which solve the boundary value problem
\[
\mathcal{M}(x,D)(\tilde{a}_{\pm}, \tilde{b}_{\pm}) = 0 \quad \text{in } \Omega, \quad (\tilde{a}_{\pm} + \tilde{b}_{\pm})|_{\Gamma_{0}} = \tilde{m}_{\pm}. \tag{93}
\]
By (90), there exists a pair $(w_{-4}, \tilde{w}_{-4}) \in H^1(\Omega) \times H^1(\Omega)$ which solves the boundary value problem

$$\mathcal{M}(x, D)(w_{-4}, \tilde{w}_{-4}) = 0 \quad \text{in } \Omega, \quad (w_{-4} + \tilde{w}_{-4})|_{\Gamma^4} = \frac{\tilde{q}_3}{2\partial_e \Phi} + \frac{\tilde{q}_4}{2\partial_e \Phi}. \quad (94)$$

We set $(w_{-3} - \tilde{w}_{-3}, s) = e^{\alpha}(w_{-3}, \tilde{w}_{-3}), (\tilde{a}_\pm, \tilde{b}_\pm) = e^{\alpha}(\tilde{a}_\pm, \tilde{b}_\pm)$. We introduce functions $w_{1,s,t}, \tilde{w}_{1,s,t}$ by formulae

$$w_{1,s,t} = w_{1,s} + \frac{w_{-3,s} + e_2 q_{3,s}/2\partial_e \Phi}{\tau} + \frac{1}{\tau^2} \left( e^{2i\tau \psi(\tilde{\varphi})} a_{+,s} + e^{-2i\tau \psi(\tilde{\varphi})} \tilde{a}_{-,s} + w_{-4,s} + \frac{e_2 \tilde{q}_{3,s}}{2\partial_e \Phi} \right) \quad (95)$$

and

$$\tilde{w}_{1,s,t} = \tilde{w}_{1,s} + \frac{\tilde{w}_{-3,s} + e_2 q_{4,s}/2\partial_e \Phi}{\tau} + \frac{1}{\tau^2} \left( e^{2i\tau \psi(\tilde{\varphi})} \tilde{b}_{+,s} + e^{-2i\tau \psi(\tilde{\varphi})} b_{-,s} + \tilde{w}_{-4,s} + \frac{e_2 \tilde{q}_{4,s}}{2\partial_e \Phi} \right). \quad (96)$$

By (88) and (90), the functions $w_{1,s,t}, \tilde{w}_{1,s,t}$ belong to $H^1(\Omega)$. Using (42) and proposition 8, for any $p \in (1, +\infty)$, we have

$$L_2(x, s, D)^* \left( -e^{-\tau \phi} \mathcal{R}_{-\tau, -A_1^s} (e_1 \left( q_{3,s} + \frac{\tilde{q}_3}{\tau} \right)) + \frac{e^{-\tau \phi} e_2 \left( q_{3,s} + \frac{\tilde{q}_3}{\tau} \right)}{2\partial_A \Phi} \right)$$

$$- e^{-\tau \phi} \mathcal{R}_{-\tau, -B_1^s} (e_1 \left( q_{4,s} + \frac{\tilde{q}_4}{\tau} \right)) + \frac{e^{-\tau \phi} e_2 \left( q_{4,s} + \frac{\tilde{q}_4}{\tau} \right)}{2\partial_A \Phi} \right)$$

$$= - e^{-\tau \phi} (Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s) \mathcal{R}_{-\tau, -A_1^s} (e_1 \left( q_{3,s} + \frac{\tilde{q}_3}{\tau} \right))$$

$$+ e^{-\tau \phi} L_2(x, s, D)^* \left( \frac{e_2 \left( q_{3,s} + \frac{\tilde{q}_3}{\tau} \right)}{2\partial_A \Phi} \right)$$

$$- e^{-\tau \phi} \mathcal{R}_{-\tau, -B_1^s} (e_1 \left( q_{4,s} + \frac{\tilde{q}_4}{\tau} \right)) + \frac{e^{-\tau \phi} e_2 \left( q_{4,s} + \frac{\tilde{q}_4}{\tau} \right)}{2\partial_A \Phi} \right)$$

$$+ e^{-\tau \phi} L_2(x, s, D)^* \left( \frac{e_2 \left( q_{4,s} + \frac{\tilde{q}_4}{\tau} \right)}{2\partial_A \Phi} \right)$$

$$- e^{-\tau \phi} (Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s) e_1^{q_{3,s}} \left( \frac{e_2 q_{3,s}}{2\partial_A \Phi} \right)$$

$$- e^{-\tau \phi} (Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s) \mathcal{R}_{-\tau, -B_1^s} (e_1 \left( q_{3,s} + \frac{\tilde{q}_3}{\tau} \right))$$

$$+ \left( Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s \right) \left( w_{1,s} + \frac{w_{-3,s}}{\tau} \right) e^{-\tau \phi}$$

$$- \left( Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s \right) \left( \tilde{w}_{1,s} + \frac{\tilde{w}_{-3,s}}{\tau} \right) e^{-\tau \phi}$$

$$= - \left( Q_2^s - 2\partial_B B_2^s - B_2^s A_2^s \right) \left( w_{1,s} + \frac{w_{-3,s}}{\tau} \right) e^{-\tau \phi} + e^{-\tau \psi_{DL}(\Omega)} \left( \frac{1}{\tau} \right). \quad (97)$$
Using (98) and propositions 12 and 11, we construct the last term \( v_{-1} \) in the complex geometric optics solution which solves the boundary value problem

\[
L_2(x, s, D)^*v_{-1} = L_2(x, s, D)^*v^* \quad \text{in } \Omega, \quad v_{-1}|_{\Gamma_0} = v^*
\]

and we obtain

\[
\sqrt{\tau}||v_{-1}||_{L^2(\Omega)} + \frac{1}{\sqrt{\tau}}(\nabla v_{-1})||_{L^2(\Omega)} + ||v_{-1}||_{H^{-1}(\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
\]

Finally, we have a complex geometric optics solution for the Schrödinger operator \( L_2(x, s, D)^* \) in the form

\[
v = w_{1, s, \tau} e^{-\tau \Phi} + \tilde{w}_{1, s, \tau} e^{-\tau \Psi} - e^{-\tau \Phi} \tilde{R}_{-\tau, -A_{2,s}} \left( e_1 \left( q_{3,s} + \tilde{q}_{3,s} \right) \right) - e^{-\tau \Psi} \tilde{R}_{-\tau, -B_{2,s}^c} \left( e_1 \left( q_{4,s} + \tilde{q}_{4,s} \right) \right) - v_{-1} e^{-\tau \psi}.
\]

By (101), (98) and (99), we have

\[
L_2(x, s, D)^*v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0.
\]

**Step 2. Asymptotic formula.**

Let \( \tilde{u}_2 \) be a solution to the following boundary value problem:

\[
L_2(x, D)\tilde{u}_2 = 0 \quad \text{in } \Omega, \quad \tilde{u}_2|_{\partial\Omega} = u_1|_{\partial\Omega}, \quad \frac{\partial \tilde{u}_2}{\partial v} = \frac{\partial u_1}{\partial v} |_{\partial\Omega}.
\]

Then the function \( u_2 = e^{-\tau \Phi} \tilde{u}_2 \) solves the boundary value problem

\[
L_2(x, s, D)u_2 = 0 \quad \text{in } \Omega, \quad u_2|_{\partial\Omega} = u_1|_{\partial\Omega}, \quad \frac{\partial u_2}{\partial v} = \frac{\partial u_1}{\partial v} |_{\partial\Omega}.
\]

Setting \( u = u_1 - u_2 \), we have

\[
L_2(x, s, D)u + 2(A_1 - A_{2,s})\partial_s u_1 + 2(B_1 - B_{2,s})\partial_s u_1 + (Q_1 - Q_{2,s})u_1 = 0 \quad \text{in } \Omega
\]

and

\[
u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial v} |_{\partial\Omega} = 0.
\]

Let \( v \) be a function given by (101). Taking the scalar product of (104) with \( v \) in \( L^2(\Omega) \) and using (102) and (105), we obtain

\[
0 = \mathcal{S}(u_1, v) = \int_{\Omega} (2(A_1 - A_{2,s})\partial_s u_1 + 2(B_1 - B_{2,s})\partial_s u_1 + (Q_1 - Q_{2,s})u_1, v) \, dx.
\]

Our goal is to obtain the asymptotic formula for the right-hand side of (106). We have

**Proposition 14.** There exists a constant \( C_0 \), independent of \( \tau \), such that the following asymptotic formula is valid as \( \tau \to +\infty:\)

\[
l_0 = ((Q_1 - Q_{2,s})u_1, v)_{L^2(\Omega)}
= \int_{\Omega} ((Q_1 - Q_{2,s})u_0, \tilde{w}_{1,s}) + ((Q_1 - Q_{2,s})\tilde{w}_0, \tilde{w}_{1,s}) \, dx
\]
\[ u(x) = \left( w_0 + \frac{w_{-1}}{\tau} \right) e^{\tau \Phi} + \left( \tilde{w}_0 + \frac{\tilde{w}_{-1}}{\tau} \right) e^{\tau \Phi} - q_1 e^{\tau \Phi} - \frac{q_1 e^{\tau \Phi}}{2 \tau \partial_x \Phi} + e^{\tau \Phi} \partial_x (w_{-1} \phi) \left( \frac{1}{\tau} \right) \]

Using (50), (95), (96), (88), (90), (101) and propositions 8 and 4, we obtain

\[ v(x) = \left( w_{1,s} + \frac{w_{-2,s}}{\tau} \right) e^{-\tau \Phi} + \left( \tilde{w}_{1,s} + \frac{\tilde{w}_{-2,s}}{\tau} \right) e^{-\tau \Phi} + q_{4,s} e^{-\tau \Phi} + \frac{q_{4,s} e^{-\tau \Phi}}{2 \tau \partial_x \Phi} + q_{3,s} e^{-\tau \Phi} + \frac{q_{3,s} e^{-\tau \Phi}}{2 \tau \partial_x \Phi} \]

By (108) and (109) and proposition 3, we obtain the following asymptotic formula:

\[ ((Q_1 - Q_{2,s})u_1, v) \in L^2(\Omega) \]

Applying the stationary phase phase argument (see e.g., [3]) to the last integral on the right-hand side of this formula, we complete the proof of proposition 14.

We set

\[ U = w_{0,s} e^{\tau \Phi} + \tilde{w}_{0,s} e^{\tau \Phi}, V = w_{1,s} e^{-\tau \Phi} + \tilde{w}_{1,s} e^{-\tau \Phi}. \]

By the stationary phase argument and formulae (45), (83), (70), (71), (95) and (96), short calculations yield that there exist constants \( \kappa_i, \tilde{\kappa}_i \), independent of \( \tau \), such that
\[ I_1 = 2 \left( (A_1 - A_{2, \tau}) \partial_t \mathcal{I}, \mathcal{V} \right)_{L^2(\Omega)} \]
\[ = (2(A_1 - A_{2, \tau})(\partial_t u_{0, \tau} e^{i \Phi} + \partial_\tau \tilde{w}_{0, \tau} e^{i \Phi}), u_{1, \tau, \tau} e^{-i \Phi} + \tilde{w}_{1, \tau, \tau} e^{-i \Phi})_{L^2(\Omega)} \]
\[ = \sum_{k=1}^{3} \tau^{2 - k} \kappa_k + 2 e^{2i \Phi \tilde{\psi}} \left( ((A_1 - A_{2, \tau}) \partial_t \Phi a_{+}, \tilde{w}_{1, \tau})_{L^2(\Omega)} + ((A_1 - A_{2, \tau}) \partial_\tau \Phi w_0, \tilde{b}_{+})_{L^2(\Omega)} \right) \]
\[ + 2 e^{-2i \phi \tilde{\psi}} \left( ((A_1 - A_{2, \tau}) \partial_\tau \Phi a_{-}, \tilde{w}_{1, \tau})_{L^2(\Omega)} + ((A_1 - A_{2, \tau}) \partial_t \Phi w_0, \tilde{b}_{-})_{L^2(\Omega)} \right) \]
\[ + \int_{\Omega} \left( (A_1 - A_{2, \tau}) \partial_\tau \tilde{w}_{0, \tau} \tilde{w}_{1, \tau} e^{-2i \phi \tilde{\psi}} dx \right) - \int_{\Omega} \left( 2 \partial_\tau ((A_1 - A_{2, \tau}) u_{0, \tau} \tilde{w}_{1, \tau}) e^{2i \phi \tilde{\psi}} dx \right) \]
\[ - \int_{\partial \Omega} (v_1 + iv_2) ((A_1 - A_{2, \tau}) u_{0, \tau} \tilde{w}_{1, \tau}) e^{2i \phi \tilde{\psi}} d\sigma + o\left( \frac{1}{\tau} \right) \]
\[ = \sum_{k=1}^{3} \tau^{2 - 4 \kappa_k} + 2 e^{2i \Phi \tilde{\psi}} \left( ((B_1 - B_{2, \tau}) \partial_t \Phi b_{+}, u_{1, \tau})_{L^2(\Omega)} + ((B_1 - B_{2, \tau}) \partial_\tau \Phi \tilde{w}_{0, \tau} \tilde{b}_{+})_{L^2(\Omega)} \right) \]
\[ + 2 e^{-2i \phi \tilde{\psi}} \left( ((B_1 - B_{2, \tau}) \partial_\tau \Phi b_{-}, u_{1, \tau})_{L^2(\Omega)} + ((B_1 - B_{2, \tau}) \partial_t \Phi \tilde{w}_{0, \tau} \tilde{b}_{-})_{L^2(\Omega)} \right) \]
\[ + \int_{\Omega} \left( 2 ((B_1 - B_{2, \tau}) \partial_\tau u_{0, \tau} \tilde{w}_{1, \tau}) e^{2i \phi \tilde{\psi}} dx \right) - \int_{\Omega} \left( 2 \partial_\tau ((B_1 - B_{2, \tau}) u_{0, \tau} \tilde{w}_{1, \tau}) e^{-2i \phi \tilde{\psi}} dx \right) \]
\[ - \int_{\partial \Omega} (v_1 + iv_2) ((B_1 - B_{2, \tau}) u_{0, \tau} \tilde{w}_{1, \tau}) e^{-2i \phi \tilde{\psi}} d\sigma + o\left( \frac{1}{\tau} \right) \]
Using (39) and integrating by parts, we obtain
\[ I_3 = - \int \Omega (2(A_1 - A_{2,s})\partial_t (\e^{\Phi} \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e^{\Phi} R_{t,A_1}[e_1(q_2 + \tilde{q}_2/\tau)]) + 2(B_1 - B_{2,s})\partial_t (\e^{\Phi} \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e^{\Phi} R_{t,A_1}[e_1(q_2 + \tilde{q}_2/\tau)]), \mathcal{V}) \, dx \]
\[ = - \int \Omega (2(A_1 - A_{2,s})\e^{\Phi} (-B_1 \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)) + e^{\Phi} R_{t,A_1}[e_1(q_2 + \tilde{q}_2/\tau)] + e^{\Phi} (A_1 - A_{2,s})\e^{\Phi} (B_1 - B_{2,s})\tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)), \mathcal{V}) \, dx \]
\[ = - \int \Omega (2(A_1 - A_{2,s})\e^{\Phi} (-B_1 \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)) + 2(B_1 - B_{2,s})\e^{\Phi} (A_1 - A_{2,s})\tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e^{\Phi} R_{t,A_1}[e_1(q_2 + \tilde{q}_2/\tau)] + e^{\Phi} (A_1 - A_{2,s})\tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)), \mathcal{V}) \, dx \]
\[ = - \int \Omega (2(A_1 - A_{2,s})\e^{\Phi} (-B_1 \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)) + 2(B_1 - B_{2,s})\e^{\Phi} (A_1 - A_{2,s})\tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e^{\Phi} R_{t,A_1}[e_1(q_2 + \tilde{q}_2/\tau)] + e^{\Phi} (A_1 - A_{2,s})\tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)] + e_1(q_1 + \tilde{q}_1/\tau)), \mathcal{V}) \, dx \]
\[ + (v_1 + iv_2)((B_1 - B_{2,s})\e^{\Phi} \tilde{R}_{t,B_1}[e_1(q_1 + \tilde{q}_1/\tau)], \mathcal{V}) \, d\sigma. \tag{112} \]

By (64) and (65), the boundary integrals in (112) are \( O \left( \frac{1}{\tau} \right) \). By (68) and proposition 4, we have
\[ 2 \int_{\Omega} e^{\Phi} (e_1(q_2 + \tilde{q}_2/\tau)) \cdot \nabla_{t,B_1} ((A_1 - A_{2,s})^*(\partial_\tau \tilde{w}_{1,s} - \tau \partial_\tau \Phi \tilde{w}_{1,s})) \, dx \]
\[ + 2 \int_{\Omega} e^{\Phi} (e_1(q_1 + \tilde{q}_1/\tau)) \cdot \nabla_{t,B_1} ((B_1 - B_{2,s})^*(\partial_\tau \tilde{w}_{1,s} - \tau \partial_\tau \Phi \tilde{w}_{1,s})) \, dx = o \left( \frac{1}{\tau} \right) \]
as \( \tau \to +\infty. \tag{113} \]
Applying the stationary phase argument, (113), and propositions 8 and 3, we obtain from (112) that there exists a constant $C \_1$ independent of $\tau$ such that
\[
I_3 = \frac{C \_1}{\tau} + 2 \int_\Omega e^{i(\Phi - \Phi)}(e_1 q_2, \mathbf{P}^*_{A_2})(A_1 - A_2, \mathbf{w}_1) e_{22}(-\tau \partial_2 \mathbf{w}_1) \, dx \\
+ 2 \int_\Omega e^{i(\Phi - \Phi)}(e_1 q_1, \mathbf{T}^*_{B_1})(B_1 - B_2, \mathbf{w}_1) e_{11}(-\tau \partial_1 \mathbf{w}_1) \, dx + o\left(\frac{1}{\tau}\right)
\]
as $\tau \to +\infty$.

Using (39) and integrating by parts, we obtain
\[
I_4 = \int_\Omega (2(A_1 - A_2) \partial_1 \mathcal{U} + 2(B_1 - B_2) \partial_2 \mathcal{U}, e^{i\Phi} - e^{-i\Phi \mathcal{R}_{-\tau, -A_2'}[e_1(q_3, q + \tilde{q}_3, \tau)]} - e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
= -\int_\Omega (2(A_1 - A_2) \partial_1 \tilde{w}_0 e^{i\Phi} + 2(B_1 - B_2) \partial_2 u_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
+ (2\partial_2(B_1 - B_2) \tilde{w}_0 e^{i\Phi} \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
+ \left((v_1 - iv_2)((A_1 - A_2) u_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]}) \, dx
\]

\[
+ (v_1 + iv_2)((B_1 - B_2) \tilde{w}_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
+ \int_\Omega (2(A_1 - A_2) u_0 e^{i\Phi}, \partial_2 e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]}) \, dx
\]

\[
+ \int_\Omega (2(B_1 - B_2) \tilde{w}_0 e^{i\Phi}, \partial_1 e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
= -\int_\Omega (2(A_1 - A_2) \partial_1 \tilde{w}_0 e^{i\Phi} + 2(B_1 - B_2) \partial_2 u_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -A_2'}[e_1(q_3, q + \tilde{q}_3, \tau)]) \, dx
\]

\[
\mathbf{P}^*_{B_2'}((B_1 - B_2) \partial_2 \tilde{w}_0 + \tau \partial_2 \mathbf{w}_1) e_{11}(-\tau \partial_1 \mathbf{w}_1) \, dx
\]

\[
+ \int_\Omega (2\partial_2(A_1 - A_2) u_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
+ (2\partial_2(B_1 - B_2) \tilde{w}_0 e^{i\Phi} \mathcal{R}_{-\tau, -A_2'}[e_1(q_3, q + \tilde{q}_3, \tau)]) \, dx
\]

\[
+ \left((v_1 - iv_2)((A_1 - A_2) u_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]}) \, dx
\]

\[
+ (v_1 + iv_2)((B_1 - B_2) \tilde{w}_0 e^{i\Phi}, e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]

\[
+ \int_\Omega (2(A_1 - A_2) u_0 e^{i\Phi}, \partial_2 e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]}) \, dx
\]

\[
+ \int_\Omega (2(B_1 - B_2) \tilde{w}_0 e^{i\Phi}, \partial_1 e^{-i\Phi \mathcal{R}_{-\tau, -B_2'}[e_1(q_4, q + \tilde{q}_4, \tau)]) \, dx
\]
- \int_{\Omega} \left\{ (u_1 - iv_2)((A_1 - A_{2,s})u_0, e^{v_2} \Delta \Upsilon \mathcal{R}_{t,T} - B_{2,s}^1 [e_1(q_{4,s} + \tilde{q}_{4,s}/\tau)] \right\} \, dx \\
+ \int_{\Omega} \left\{ (v_1 + iv_2)((B_1 - B_{2,s})\tilde{w}_0, e^{v_2} \Delta \Upsilon \mathcal{R}_{t,T} - B_{2,s}^1 [e_1(q_{3,s} + \tilde{q}_{3,s}/\tau)] \right\} \, dx \\
+ \int_{\Omega} \left\{ \frac{1}{\tau} \int (2(A_1 - A_{2,s})u_0 e^{v_2}, e^{-v_2} B_{2,s}^1 \mathcal{R}_{t,T} - B_{2,s}^1 [e_1(q_{4,s} + \tilde{q}_{4,s}/\tau)]) + e^{-v_2} e_1(q_{4,s} + \tilde{q}_{4,s}/\tau) \right\} \, dx \\
+ \int_{\Omega} \left\{ \frac{1}{\tau} \int (2(B_1 - B_{2,s})\tilde{w}_0 e^{v_2}, e^{-v_2} A_{2,s}^1 \mathcal{R}_{t,T} - A_{2,s}^1 [e_1(q_{3,s} + \tilde{q}_{3,s}/\tau)]) + e^{-v_2} e_1(q_{3,s} + \tilde{q}_{3,s}/\tau) \right\} \, dx.

(115)

By proposition 13, the boundary integral in (115) is \(O\left(\frac{1}{\tau}\right)\). By (90) and proposition 4, we have

\[
\frac{1}{\tau} \int_{\Omega} (2T_{A_{2,s}})((A_1 - A_{2,s})\frac{\partial}{\partial x} v_0 + \tau \frac{\partial}{\partial x} \Phi v_0), e^{(v_2)}/(e_1 q_{3,s}) \right\} \, dx \\
= \frac{1}{\tau} \int_{\Omega} (2P_{B_{2,s}})((B_1 - B_{2,s})\frac{\partial}{\partial x} \tilde{w}_0 + \tau \frac{\partial}{\partial x} \Phi \tilde{w}_0), e^{(v_2)}/(e_1 q_{4,s}) \right\} \, dx = o \left(\frac{1}{\tau}\right)

as \(\tau \to +\infty\). 

(116)

Applying the stationary phase argument, propositions 8 and 3, and (116), we obtain from (115) that there exists a constant \(C_2\), independent of \(\tau\), such that

\[
I_4 = \frac{C_2}{\tau} - \int_{\Omega} (2T_{A_{2,s}})((A_1 - A_{2,s})\tau \frac{\partial}{\partial x} \Phi v_0), e^{(v_2)}/(e_1 q_{3,s}) \right\} \, dx \\
= \int_{\Omega} (2P_{B_{2,s}})((B_1 - B_{2,s})\tau \frac{\partial}{\partial x} \tilde{w}_0, e^{(v_2)}/(e_1 q_{4,s}) \right\} \, dx + o \left(\frac{1}{\tau}\right)

as \(\tau \to +\infty\). 

(117)

**Step 3. Derivation of equations (2)–(4).**

We set

\[
U_1(x) = w_{0,0}, e^{v_2} + \tilde{w}_{0,0}, e^{v_2} - e^{v_2} R_{t,T}, e_1(q_1 + \tilde{q}_1/\tau) - e^{v_2} R_{t,T}, e_1(q_2 + \tilde{q}_2/\tau),

\]

\[
V_1(x) = w_{1,0}, e^{v_2} + \tilde{w}_{1,0}, e^{v_2} - e^{v_2} R_{t,T}, e_1(q_3 + \tilde{q}_3/\tau) - e^{v_2} R_{t,T}, e_1(q_4 + \tilde{q}_4/\tau).

We set

\[
\mathcal{G}(u, v) = (2(A_1 - A_{2,s})\frac{\partial}{\partial x} u + 2(B_1 - B_{2,s})\frac{\partial}{\partial x} v, v), L^2(\Omega).

By (79), (101) and proposition 8, we have

\[
\mathcal{G}(u_1, e^{v_2} v, v - (w_{1,0}, e^{v_2} v, e^{v_2}) = \mathcal{G}(u_1, - (w_{0,r}, e^{v_2} v, e^{v_2}), v - 1 e^{-v'})

\]

\[
= o \left(\frac{1}{\sqrt{r}}\right) \quad \text{as } \tau \to +\infty.

(118)

Then by (107), (110), (111), (114), (117), (118),

\[
\mathcal{G}(u_1, v) = \int (v_1 - iv_2)((A_1 - A_{2,s})u_0, \tilde{w}_{0,0}, e^{2iv_2} \mathcal{R}_{t,T} - \tilde{w}_{1,0}, e^{2iv_2} \mathcal{R}_{t,T} + C \\
= o \left(\frac{1}{\sqrt{r}}\right) \quad \text{as } \tau \to +\infty.

(119)
Let $\Phi$ be given in proposition 2. Then by (47) and (85) and the stationary phase argument, the asymptotic formula holds

$$
\mathfrak{G}(u_1, v) = \frac{1}{\sqrt{\tau}} \sum_{x \in \mathcal{G}} (v_1 - iv_2)((A_1 - A_{2,\tau})w_0, \bar{w}_{1,\tau}) e^{2i\tau} \\
+ (v_1 + iv_2)((B_1 - B_{2,\tau})\tilde{w}_0, \bar{w}_{1,\tau}) e^{-2i\tau} + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
$$

(120)

Since for any $\tilde{x}$ one can find $\Phi$ such that $\tilde{x} \in \mathcal{G}$ and $\text{Im}\Phi(\tilde{x}) \neq \text{Im}\Phi(x)$ for any $x \in \mathcal{G} \setminus \{\tilde{x}\}$, we have

$$
((A_1 - A_{2,\tau})w_0, \bar{w}_{1,\tau}) = ((B_1 - B_{2,\tau})\tilde{w}_0, \bar{w}_{1,\tau}) = 0 \quad \text{on } \tilde{\Gamma}.
$$

These equalities and proposition 9 imply (2).

Next we claim that

$$
\mathfrak{G}(e^{iv}u_{-1}, v) = \mathfrak{G}(u_1, e^{-iv}u_{-1}) = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
$$

(121)

Obviously, by (79) and proposition 8, we see that

$$
\mathfrak{G}(e^{iv}u_{-1}, v - \mathcal{V}) = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty.
$$

(122)

Let $\chi \in C^\infty_0(\Omega)$ satisfy $\chi|_{\Omega \cap \mathcal{O}_2} = 1$. By (79), we have

$$
\mathfrak{G}(e^{iv}u_{-1}, \mathcal{V}) = \mathfrak{G}(e^{iv}u_{-1}, \chi \mathcal{V}) + o\left(\frac{1}{\tau}\right)
$$

$$
= \int_\Omega (2(A_1 - A_{2,\tau})\partial_x(e^{iv}u_{-1}) + 2(B_1 - B_{2,\tau})\partial_x(e^{iv}u_{-1}), \chi \mathcal{V}) \, dx + o\left(\frac{1}{\tau}\right)
$$

$$
+ (2(B_1 - B_{2,\tau})\partial_x(e^{iv}u_{-1}), \chi \bar{w}_{1,\tau} e^{iv}) \, dx + o\left(\frac{1}{\tau}\right). \tag{123}
$$

Let functions $w_4$ and $w_5$ solve the equations $(-\partial_x + B_{1,\tau}^*)w_4 = 2(A_1 - A_{2,\tau})^*\bar{w}_{1,\tau}$ and $(-\partial_x + A_{1,\tau}^*)w_5 = 2(B_1 - B_{2,\tau})^*\bar{w}_{1,\tau}$.

Taking the scalar product of the equation in $u_{-1}$ (see after (79)) and the function $w_5 e^{iv\Phi} + w_4 e^{iv\bar{\Phi}}$, after integration by parts we obtain

$$
\int_\Omega ((2\partial_x(e^{iv}u_{-1}) + A_1 (e^{iv}u_{-1}), 2(A_1 - A_{2,\tau})^*\bar{w}_{1,\tau} e^{iv} \Phi)) \, dx = o\left(\frac{1}{\tau}\right). \tag{124}
$$

By (123) and (124), we obtain the first equality in (121). The proof of the second equality in (121) is the same.

By (15), (2), (82), (115), (114), (117), (111), (110) and (121), we have the asymptotic formula:

$$
2e^{2i\tau\phi(\tilde{x})}((B_1 - B_{2,\tau})\partial_x\bar{\Phi}b_+, w_{1,\tau})_{L^2(\Omega)} + ((B_1 - B_{2,\tau})\partial_x\bar{\Phi}\tilde{w}_0, \bar{\Phi})_{L^2(\Omega)}
$$

$$
+ 2e^{-2i\tau\phi(\tilde{x})}(((B_1 - B_{2,\tau})\partial_x\Phi b_-, w_{1,\tau})_{L^2(\Omega)} + ((B_1 - B_{2,\tau})\partial_x\tilde{w}_0, \Phi)_{L^2(\Omega)}
$$

$$
+ 2e^{2i\tau\phi(\tilde{x})}(((A_1 - A_{2,\tau})\partial_x\Phi a_+, \tilde{w}_{1,\tau})_{L^2(\Omega)} + ((A_1 - A_{2,\tau})\partial_x\Phi w_0, \Phi)_{L^2(\Omega)}
$$

$$
+ 2e^{-2i\tau\phi(\tilde{x})}(((A_1 - A_{2,\tau})\partial_x\Phi a_-, \tilde{w}_{1,\tau})_{L^2(\Omega)} + ((A_1 - A_{2,\tau})\partial_x\Phi w_0, \Phi)_{L^2(\Omega)}
$$

$$
- 2\pi (Q_+, \bar{w}_0, \bar{w}_{1,\tau}) e^{i\phi(\tilde{x}) + s} \frac{e^{-2i\tau\phi(\tilde{x}) + s}}{\tau |\det \psi'(\tilde{x})|^2} = 2\pi (Q_+, \bar{w}_0, \bar{w}_{1,\tau}) e^{2i\tau\phi(\tilde{x}) + s} \frac{e^{-2i\tau\phi(\tilde{x}) + s}}{\tau |\det \psi'(\tilde{x})|^2} + \mathcal{P}(\tau) + o\left(\frac{1}{\tau}\right) = 0.
$$

(125)
where $$Q_+ = 2\partial_x(A_1 - A_2) + B_2(A_1 - A_2) + (B_1 - B_2)A_1 - (Q_1 - Q_2)$$ and $$Q_- = 2\partial_x(B_1 - B_2) + A_1(B_1 - B_2) + (A_1 - A_2)B_1 - (Q_1 - Q_2)$$ and

$$P(r) = -2\int_\Omega (T_{-A_1}^r((A_1 - A_2,s)\partial_s \Phi e^{s\phi}w_0), e_i\theta_i) e^{2ir\psi} dx$$

$$- 2\int_\Omega (\tilde{P}_{-B_1}^r((B_1 - B_2,s)\partial_s \Phi e^{s\phi}\tilde{w}_0), e_i\theta_i) e^{2ir\psi} dx$$

$$- 2\int_\Omega (e_i q_2, \tilde{P}_{A_1}^r((A_1 - A_2,s)\partial_s \Phi \tilde{w}_1,s)) e^{2ir\psi} dx$$

$$- 2\int_\Omega (e_i q_1, \tilde{T}_{B_1}^r((B_1 - B_2,s)\partial_s \Phi w_1,s)) e^{2ir\psi} dx.$$ 

Observe that

$$T_{-A_1}^r((A_1 - A_2,s) e^{s\phi}w_0) - e^{s\phi}w_0 \in \text{Ker} T_{-A_1}^r, \tilde{P}_{-B_1}^r((B_1 - B_2,s) e^{s\phi}\tilde{w}_0) - e^{s\phi}\tilde{w}_0 \in \text{Ker} \tilde{P}_{-B_1}^r.$$

Thanks to proposition 4 and the above relations, there exist functions $$r_{1,s} \in \text{Ker} T_{-A_1}^r, r_{2,s} \in \text{Ker} \tilde{P}_{-B_1}^r, r_{3,s} \in \text{Ker} \tilde{P}_{A_1}^r, r_{4,s} \in \text{Ker} r_{B_1}^r$$ such that

$$P(r) = -2\int_\Omega (\partial_s \Phi w_0, e^{s\phi}e_i q_1,s) dx + \tau \int_\Omega \left(r_{1,s}, e^{s\phi}e_i q_1,s\right) dx$$

$$- 2\int_\Omega (\partial_s \tilde{w}_0, e^{s\phi}e_i q_1,s) dx + \tau \int_\Omega \left(r_{2,s}, e^{s\phi}e_i q_1,s\right) dx$$

$$- 2\int_\Omega e^{s\phi}(\partial_s \Phi e_i q_2, \tilde{w}_1,s) dx + \tau \int_\Omega e^{s\phi}(\partial_s \Phi e_i q_2, \tilde{r}_3,s) dx$$

$$- 2\int_\Omega e^{s\phi}(\partial_s \Phi e_i q_1, \tilde{r}_4,s) dx + \tau \int_\Omega e^{s\phi}(\partial_s \Phi e_i q_1, \tilde{r}_4,s) dx + o\left(\frac{1}{r}\right)$$

as $$r \to +\infty.$$ 

(126)

Integrating by parts in the above equality and using the stationary phase argument, we have

$$P(r) = 2\int_\Omega (w_0, e^{s\phi}\partial_s(e_i q_1,s)) dx + \tau \int_\Omega \left(r_{1,s}, e^{s\phi}e_i q_1,s\right) dx$$

$$+ 2\int_\Omega (\tilde{w}_0, e^{s\phi}\partial_s(e_i q_1,s)) dx + \tau \int_\Omega \left(r_{2,s}, e^{s\phi}e_i q_1,s\right) dx$$

$$- 2\int_\Omega e^{s\phi}(\partial_s(e_i q_2, \tilde{w}_1,s) dx + \tau \int_\Omega e^{s\phi}(\partial_s(e_i q_2, \tilde{r}_3,s) dx$$

$$- 2\int_\Omega e^{s\phi}(\partial_s(e_i q_1, \tilde{w}_1,s) dx + \tau \int_\Omega e^{s\phi}(\partial_s(e_i q_1, \tilde{r}_4,s) dx + o\left(\frac{1}{r}\right)$$

as $$r \to +\infty.$$ 

(127)

Applying the stationary phase argument and using the definition of the functions $$q_1, q_2, q_3, q_4, s$$ in view of (48), (63), (84), (88) we obtain that

$$P(r) = 4\pi \frac{(Q_+ w_0, \tilde{w}_1) e^{2i\phi(\Omega)}}{\tau |\det \psi''(\Omega)|^{\frac{1}{2}}} + 4\pi \frac{(Q_- \tilde{w}_0, \tilde{w}_1) e^{-2i\phi(\Omega)}}{\tau |\det \psi''(\Omega)|^{\frac{1}{2}}}$$

$$+ \frac{e^{2i\phi(\Omega)}}{\tau |\det \psi''(\Omega)|^{\frac{1}{2}}} (\Omega(\ell_4) + \Omega(\ell_2)) + \frac{e^{-2i\phi(\Omega)}}{\tau |\det \psi''(\Omega)|^{\frac{1}{2}}} (\Omega(\ell_1) + \Omega(\ell_3)).$$ 

(128)
where $\ell_1 = (q_1, r_{1,s})$, $\ell_2 = (q_2, r_{1,s})$, $\ell_3 = (r_{2,s}, \xi_2)$, $\ell_4 = (r_{1,s}, \xi_2)$ and for any smooth function $\ell(x)$ we set

$$
\mathcal{D}(\ell) = \left( \partial_x \left( \frac{\ell_x(\overline{\xi}) (\xi - \overline{\xi})}{\partial_x \Phi} \right) - \partial_x \left( \frac{\ell_x(\overline{\xi}) (\xi - \overline{\xi})}{\partial_x \Phi} \right) \right),
$$

$$
\frac{1}{2} \partial_x \left( \frac{\ell_x(\overline{\xi}) (\xi - \overline{\xi})^2}{\partial_x \Phi} \right) - \frac{1}{2} \partial_x \left( \frac{\ell_x(\overline{\xi}) (\xi - \overline{\xi})^2}{\partial_x \Phi} \right),
$$

where $\xi = \xi_1 + i\xi_2$.

Since $\psi(\overline{\xi}) \neq 0$, we obtain from (128) and (125)

$$(B_1 - B_{2,s}) \delta \Phi b_+ + w_{1,s})_{L^2(\Omega)} + 2((B_1 - B_{2,s}) \delta \Phi \tilde{w}_0, \tilde{a}_{-s})_{L^2(\Omega)}$$

$$+ 2((A_1 - A_{2,s}) \delta \Phi a_+ + \tilde{w}_1)_{L^2(\Omega)} + 2((A_1 - A_{2,s}) \delta \Phi w_0, \tilde{b}_{-s})_{L^2(\Omega)}$$

$$+ 2\pi \left( \frac{w_0, \tilde{w}_1}{\psi(\overline{\xi})} \right)^2 + \frac{\mathcal{D}(\ell_4) + \mathcal{D}(\ell_2)}{|\det \psi(\overline{\xi})|^2} = 0 \tag{129}$$

and

$$(B_1 - B_{2,s}) \delta \Phi b_- + w_{1,s})_{L^2(\Omega)} + 2((B_1 - B_{2,s}) \delta \Phi \tilde{w}_0, \tilde{a}_{-s})_{L^2(\Omega)}$$

$$+ 2((A_1 - A_{2,s}) \delta \Phi a_- + \tilde{w}_1)_{L^2(\Omega)} + 2((A_1 - A_{2,s}) \delta \Phi w_0, \tilde{b}_{-s})_{L^2(\Omega)}$$

$$+ 2\pi \left( \frac{w_0, \tilde{w}_1}{\psi(\overline{\xi})} \right)^2 + \frac{\mathcal{D}(\ell_4) + \mathcal{D}(\ell_2)}{|\det \psi(\overline{\xi})|^2} = 0 \tag{130}$$

Integrating by parts in (129) and (130), we obtain

$$((v_1 - iv_2) \delta \Phi b_+, w_1)_{L^2(\Omega)} + ((v_1 - iv_2) \delta \Phi \tilde{w}_0, \tilde{a}_{-s})_{L^2(\Omega)}$$

$$+ ((v_1 + iv_2) \delta \Phi a_+ + \tilde{w}_1)_{L^2(\Omega)} + ((v_1 + iv_2) \delta \Phi w_0, \tilde{b}_{-s})_{L^2(\Omega)}$$

$$+ 2\pi \left( \frac{w_0, \tilde{w}_1}{\psi(\overline{\xi})} \right)^2 + \frac{\mathcal{D}(\ell_4) + \mathcal{D}(\ell_2)}{|\det \psi(\overline{\xi})|^2} = 0 \tag{131}$$

and

$$((v_1 - iv_2) \delta \Phi b_-, w_1)_{L^2(\Omega)} + ((v_1 - iv_2) \delta \Phi \tilde{w}_0, \tilde{a}_{-s})_{L^2(\Omega)}$$

$$+ ((v_1 + iv_2) \delta \Phi a_- + \tilde{w}_1)_{L^2(\Omega)} + ((v_1 + iv_2) \delta \Phi w_0, \tilde{b}_{-s})_{L^2(\Omega)}$$

$$+ 2\pi \left( \frac{w_0, \tilde{w}_1}{\psi(\overline{\xi})} \right)^2 + \frac{\mathcal{D}(\ell_4) + \mathcal{D}(\ell_2)}{|\det \psi(\overline{\xi})|^2} = 0 \tag{132}$$

Observe that

$$\sum_{\ell=1}^{4} |\mathcal{D}(\ell_4)| \leq C_4 \tag{133}$$

with the constant $C_4$ independent of $s$. We prove this inequality for $\mathcal{D}(\ell_4)$. The proof for remaining terms is similar.

Let $B(\xi, \delta) \subset \subset \Omega$ and $x_0 \in C^\infty_0(B(\xi, \delta))$, $x_0|_{\partial B(\xi, \delta/2)} = 1$. By (82), the functions $(A_1 - A_{2,s}) e^{\nu_0} |x_0 w_0$ are bounded uniformly in the space $\mathcal{Y}(B(\xi, \delta))$. The functions $(A_1 - A_{2,s}) e^{\nu_0} (1 - x_0) w_0$ are bounded uniformly in the space $L^2(\Omega)$. Hence, by (29) and (30), the functions $T^{a}_{b} (\overline{\partial}_\Phi (A_1 - A_{2,s}) e^{\nu_0} w_0)$ are uniformly bounded in $L^2(\Omega)$. Then the functions $r_{1,s}$ are uniformly bounded in $L^2(\Omega)$ and belong to $\text{Ker} T^a_{b}$. Therefore, the functions $r_{1,s}$ are uniformly bounded in $C^1(K)$ for any compact $K \subset \subset \Omega$. Since $\ell_4 = (r_{1,s}, \xi_2)$, the proof of (133) is completed.

Passing to the limit in (131) and (132) as $s$ goes to infinity, we obtain $(Q_+ w_0, \overline{\tilde{w}_1}) = (Q_+ \tilde{w}_0, \overline{\tilde{w}_1}) = 0$. These equalities and (46) imply the equalities (3) and (4) at point $\hat{x}$. According to proposition 1, a point $\hat{x}$ can be chosen arbitrarily close to any point of domain $\Omega$ after an appropriate choice of the function $\Phi$. The proof of the theorem is completed. \[\square\]
Appendix

Proof of proposition 5. Let \( C \) be some smooth complex-valued function in \( \Omega \) such that
\[
2 \frac{\partial C}{\partial \bar{z}} = C_1(x) + iC_2(x) \quad \text{in} \quad \Omega, \quad \text{Im} \bar{C}[r_0] = 0,
\]
where \( \bar{C} = (C_1, C_2) \) is a smooth real-valued function in \( \Omega \) such that
\[
\text{div} \bar{C} = 1 \quad \text{in} \quad \Omega, \quad (\nu, \bar{C})[r_0] < -1.
\]

The following proposition is proved in [16].

Proposition 15. Suppose that a function \( \Phi \) satisfies (15) and (16) and \( \tilde{v} \in H^1_0(\Omega) \). Then there exist \( \tau_0 \) and \( C \) independent of \( \tau \) and \( \tau > \tau_0(N) \) such that
\[
\frac{N}{2} \|\partial_\nu \tilde{v} \|_{L^2(\Gamma)}^2 + \frac{N}{2} \|\partial_\nu \bar{z} \|_{L^2(\Gamma)}^2 \leq \|\Delta \tilde{v} \|_{L^2(\Omega)}^2 + C(N) \tau \|\partial_\nu \bar{z} \|_{L^2(\Gamma)}^2
\]
for all \( \tau > \tau_0(N) \) and all positive \( N \geq 1 \).

Applying estimate (A.3) to each equation in system (23), we have
\[
\frac{N}{2} \|2\partial_\nu \tilde{v} e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + \frac{N}{2} \|2\partial_\nu \bar{z} e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + \|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + \|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
= \sum_{k=1}^N \|2\partial_\nu u_k e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + \|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
\leq 4\|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|2\partial_\nu u + 2B\partial_\nu u + Qu e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
\leq 4\|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 20K^2 \|2\partial_\nu u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 20K^2 \|2\partial_\nu u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
+ 4\|L(x, D)u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|2\partial_\nu u + 2B\partial_\nu u + Qu e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
\leq 4\|L(x, D)u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|2\partial_\nu u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|L(x, D)u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
\leq 4\|L(x, D)u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|2\partial_\nu u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + 4\|L(x, D)u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2
\]

\[
\leq 20K^2 \|u e^{i\tau + N \text{Re} C} \|_{L^2(\Omega)}^2 + C(N) \tau \|\partial_\nu \bar{z} \|_{L^2(\Gamma)}^2.
\]

Here \( N \) is the second large parameter which is essential for the proof.
There exists $N_0(K)$ such that for all $N \geq N_0$ the second and the third terms on the right-hand side of (A.4) are absorbed into the first two terms on the left-hand side. Therefore, we have

$$
\frac{N}{2} \| \partial_z u e^{\tau \nu + NC} \|_{L^2(\Omega)}^2 + \tau \| u e^{\tau \nu + NC} \|_{L^2(\Omega)}^2 + \| u e^{\tau \nu + NC} \|_{H^1(\Omega)}^2
$$

$$
+ N \left\| \frac{\partial u e^{\tau \nu + NC}}{\partial \nu} \right\|_{L^2(\Gamma_0)}^2 + \tau \| \Phi \|_{L^2(\Omega)}^2
$$

$$
\leq 4 \left\| (L(x, D)u) e^{\tau \nu + NC} \right\|_{L^2(\Omega)}^2 + 20K \| u e^{\tau \nu + NC} \|_{L^2(\Omega)}^2 + C(N) \tau \left\| \frac{\partial \tilde{u}}{\partial \nu} e^{\tau \nu + NC} \right\|_{L^2(\Gamma)}^2.
$$

(A.5)

Then there exists $\tau_0$ such that the second term on the right-hand side of (A.5) is absorbed into the first term on the left-hand side of (A.5). The proof of the proposition is complete.

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