S.P.Novikov, A.S.Schwarz

Discrete Lagrangian Systems on Graphs. Symplecto-Topological Properties

This work extends the results of [1, 2, 3] where the Symplectic Wronskian was constructed for the linear systems on Graphs and applied in the Scattering Theory for the Graphs with tails.

Let us consider any one-dimensional locally finite simplicial complex (Graph) \( \Gamma \) without ends (i.e. every vertex \( P \) belongs to at least two but to the finite number of edges \( R_i \) only). All set of vertices is numerated by the index \( j, P \rightarrow j(P) \). The following set of data should be given:

1. Family \( S \) of finite subsets \( \alpha \in S \) in the set of all vertices.
2. Differentiable manifolds \( N_P \) associated with every vertex \( P \) of our Graph \( \Gamma \). Let \( x_P \) be a local coordinate in the manifold \( N_P \).
3. Smooth real-valued functions \( \Lambda^\alpha \) for every set \( \alpha \); These functions are defined on the product of all manifolds \( N_{j_q} \) associated with vertices belonging to the set \( \alpha \)

\[
\Lambda^\alpha(x_{j_1}, \ldots, x_{j_{|\alpha|}}) : N_{j_1} \times \cdots \times N_{j_{|\alpha|}} \rightarrow R
\] (1)

Here \( |\alpha| \) means the number of vertices belonging to \( \alpha \).

Using this data, we introduce a Nonlinear Discrete Lagrangian Problem:

The Total Lagrangian of the system is by definition a functional \( L \) below defined on the functions (fields) \( \Psi(P) = x_P \in N_P \); The Euler–Lagrange equation is defined by this Lagrangian. So we have

\[
L\{\Psi\} = \sum_\alpha \Lambda^\alpha(x_{j_1}, \ldots, x_{j_{|\alpha|}})
\] (2)

---

1S.Novikov, Math Department and IPST, University of Maryland at College Park, MD, 20742-2431, USA and Landau Institute for Theoretical Physics, Moscow 117940, novikov@ipst.umd.edu, fax (USA)-301-3149363; A.Schwarz, Math Department, University of California, Davis, USA. Research of the first author is supported in part by the NSF Grant DMS9704613

2This work already appeared in the Russia Math Surveys, 1999, v 54 n 1
\[
\frac{\delta L}{\delta \Psi(P)} = \frac{\partial L}{\partial x_P}
\]  

(3)

We consider the Graph \( \Gamma \) as a geodesic metric space with length of all edges equal to 1. The **Diameter** \( d(\alpha) \) of any finite set \( \alpha \) is equal to the maximal distance between the points of this set.

**Definition 1** The Lagrangian Problem is called **Local** if there exists a number \( M \) such that \( d(\alpha) < M \) for all subsets \( \alpha \in S \). We say that this problem is presented in the **tree-like form** if every set of vertices \( \alpha \) is realized as a full set of vertices of the simply-connected subgraph (tree) \( \Gamma_\alpha \in \Gamma \).

**Lemma 1** Every Local Lagrangian Problem can be presented in the local tree-like form.

Proof of this lemma is easy. First of all, we add to the set \( \alpha \) the shortest paths in \( \Gamma \) transforming every set \( \alpha \) into the set of all vertices of the connected subgraph. This step does not increase the diameter of our set. We extend our function \( \Lambda^\alpha \) to the new variables (vertices) trivially. If the new subgraph is nonsimply-connected, we start to remove edges (not vertices) from its cycles one by one, destroying all cycles. Finally we are coming to the connected trees. Lemma is proved.

**Remark 1** As far as we know only nonlinear discrete time-independent Lagrangian Systems on the discretized line have been considered before where \( \Gamma = \mathbb{Z}, N_P = N, \Lambda^\alpha = \Lambda \) (see [4]).

We consider now only local tree-like presented Lagrangian Systems. For every pair of vertices \( P, Q \in \alpha \) with indices \( j, k \) we choose a unique oriented path \( l_{jk}^\alpha \subset \Gamma_\alpha \) joining these vertices. We have \( \partial [l_{jk}^\alpha] = Q - P \) for the corresponding chains. Let us introduce now a \( C_1(\Gamma; P) \)-valued 2-form on the infinite product of all manifolds \( N_P \) associated with vertices

\[
N^\infty = \prod_j N_j
\]

with local coordinates \( (\cup_j x_j) \). Here \( C_1 \) means a linear space of all \( R \)-valued 1-chains on the Graph.

\[
\Omega = \sum_\alpha \Omega^\alpha = \sum_\alpha \frac{\partial^2 \Lambda^\alpha}{\partial x_j \partial x_k} [l_{jk}^\alpha] dx_j dx_k
\]  

(4)
Theorem 1

1. The form $\Omega$ is well-defined, i.e. for every edge $R \subset \Gamma$ its coefficient is a well-defined $R$-valued 2-form on the finite-dimensional manifold;
2. This form is closed $d\Omega = 0$;
3. After restriction on the submanifold of the solutions of Euler-Lagrange equation this form is a 1-cycle on the Graph $\Gamma$

\[ \delta L = 0 \rightarrow \partial \Omega = 0 \]

Therefore this form takes values in the group $H_1^{\text{open}}(\Gamma; R)$.

Proof. The first part of this theorem follows immediately from the construction of the form and locality. For the proof of the part 2 we need to use the properties of the tree-like representation calculating the coefficient of the edge $R \subset \Gamma_\alpha$ in the form $d\Omega^\alpha$. Let $\partial R = Q - P$ with indices $j, k$. The subgraph $\Gamma_\alpha$ minus $R$ is a union of 2 disconnected parts $\Gamma_P$ and $\Gamma_Q$ containing separately the vertices $P$ and $Q$. Let $T \in \Gamma_\alpha$ be any third vertex with index $i$ and $T \in \Gamma_P$. The unique path $l_{ij}^\alpha$ joining $T$ and $P$, does not contain the edge $R$. The unique path $l_{ik}^\alpha$ joining $T$ and $Q$ contains the edge $R$. There are exactly two terms containing $[R]$ in the 3-vertex configuration $P, Q, T$. Consider the coefficient of the edge $[R]$ coming from the third derivatives in the calculation of $d\Omega^\alpha$:

\[ \left(\cdots\right)[R] \frac{\partial^3 \Lambda^\alpha}{\partial x_i \partial x_j \partial x_k} = \partial_i(\cdots) = \partial_k(\cdots) \]

We can see that this term in the coefficient appears exactly twice with opposite signs. Therefore $d\Omega^\alpha = 0$. Let us point out that we did not used the Euler-Lagrange equation until now. For the proof of the statement 3, we observe that

\[ \partial \Omega|_P = \sum_{\alpha} \sum_k \frac{\partial^2 \Lambda^\alpha}{\partial x_j \partial x_k} dx_j dx_k \]

where $P, Q \in \alpha$ and $j, k$ are the corresponding indices. Therefore we have

\[ \partial \Omega|_P = \sum_k \partial x_k \{ \sum_{\alpha} d_{x_j} \Lambda^\alpha \} dx_k \]

However, the Euler-Lagrange equation implies:

\[ \sum_{\alpha} d_{x_j} \Lambda^\alpha = 0 \]

Our theorem is proved.
Remark 2 For the pairwise interactions where all sets $\alpha \in S$ contain 2 points only, we construct a tree-like representation trivially: Add to every set $\alpha = (P,Q)$ any simple path $l_{PQ}$ joining these points. Extend the potentials $\Lambda^\alpha(P,Q)$ trivially. In this case the second statement of our theorem is obvious because all third derivatives of the Lagrangian $L$ along the variables of any triple of distinct points are equal to zero.

Remark 3 In the Appendix to the work [3] written by the present authors, the $H^1_{\text{open}}(\Gamma; R)$-valued 2-form $\Omega$ on the space of the classical solutions $\delta L = 0$ has been constructed already. We considered a real selfadjoint operator $L_\psi$ which is a linearization of our problem near the solution $\psi$. Following [3], we have a Symplectic Scalar Product-”The Symplectic Wronskian” on the tangent space to the space of solutions in the point $\psi$:

$$W(\delta\psi_2, \delta\psi_1) = -W(\delta_1\psi, \delta_2\psi), L_\psi\delta_q\psi = 0, q = 1, 2$$

This construction determines a $H^1_{\text{open}}(\Gamma; R)$-valued 2-form on the space of solutions for the Lagrangian Problem $\delta L = 0$. However, this construction is unique and canonical for the special cases only: for the case of nearest neighbors and for the case of trees. Our proof that this construction leads to the closed form, is based on identification of this form with one obtained from the present construction. After this identification (which is easy) we see that we need to invent the tree-like representation of the local variational problem as it was done in the present work. Otherwise our construction of the form $\Omega$ can lead to the nonclosed form.

Following Partial Cases have been considered already:

1. The case of discretized line $\Gamma = Z$– see [4]. The symplectic form $\Omega$ in this case has been considered as an ordinary $R$-valued form. It agrees with our theorem because $H^1_{\text{open}}(R; R) = R$. In this case our 2-form coincides with the one constructed by Veselov. However, even that is not obvious immediately.

2. The cases of linear operators on Simplicial Complex $K$ acting in the spaces of chains of different dimensions–see [4, 2, 3]. We proved in [3] that all these cases can be reduced to the operators acting on 0-cochains of the Graph $\Gamma$–the one-dimensional skeleton in the first baricentrical subdivision $K^*$ of the complex $K$. In these works beginning from [1] the properties of $\Omega$
as a symplectic $H^\text{open}_1(\Gamma, R)$-valued form have been applied to the **Scattering Theory for the Graphs with Tails**. In particular, all unitarity properties of scattering follow from Elementary Topology and Symplectic Algebra.

**Problem:** Investigate the form $\Omega$ for the pairwise nonlocal interaction $\Lambda^\alpha(P, Q), \alpha = (P, Q)$. Let all manifolds $N_P$ are equal to the sphere $S^m$ or to the same compact Lie Group $G$, and the interaction potentials are also translationally invariant. Especially interesting cases are following:

1. The Graph $\Gamma$ is equal to $Z^n$. How to make an optimal choice of the minimal paths $l_{P, Q}$? Which power of decay is necessary for the good properties of the form $\Omega$ on the space of classical solutions? Is it always of the order below?

   $$\Lambda(P, Q) \sim n^{-n-2-\epsilon}, n = d(P, Q)$$

2. The Graph $\Gamma$ is a homogeneous tree with $f$ number of edges meeting each other in every vertex. It looks like the decay should be exponential like

   $$\Lambda(P, Q) \sim \exp\{ad(P, Q)\}$$

with $a > a_0(m)$ where $a_0$ can be found easily. However, we do not know what is going to happen with our form after the restriction on the space of exact solutions.

**References**

[1] S.Novikov. *The Schrodinger Operators on Graphs and Topology*. Russia Math Surveys, 1997, v 52 n 6 pp 178-179

[2] S.Novikov. *The Schrodinger Operators on Graphs and Symplectic Geometry*. to appear in the special volume of the Fields Institute, Toronto, 1999, dedicated to the 60th birthday of V.Arnold.

[3] S.Novikov. *The Schrodinger Operators on Graphs, Topology and Symplectic Geometry*. to appear in the special volume of the Asian Math Journal, dedicated to the 70th birthday of Mikio Sato.

[4] A.Veselov. *Integrable Maps*. Russia Math Surveys, 1991, v 46 n 5 pp 3-45.