Online adaptive basis enrichment for mixed CEM-GMsFEM

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November 19, 2018

Abstract

In this paper, an online basis construction for constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) in mixed formulation is proposed. The online approach is based on the strategy of oversampling and makes use of the information of residual and the parameters in the partial differential equation such as the source function. The analysis presented shows that the error reduction can be made sufficiently large by suitably selecting oversampling regions and the number of oversampling layers. We show that the convergence rate is also determined by a user-defined parameter. Numerical results are provided to illustrate the efficiency of the proposed method.

1 Introduction

Many problems arising from physics and engineering involve multiple scales and high contrast, such as the Darcy’s flow in heterogeneous porous media. These problem are prohibitively costly to solve when traditional fine-scale solvers are directly applied and some type of reduced-order models are considered to avoid the high computational cost. Many model reduction techniques have been well developed in existing literature. For example, in upscaling methods [7, 20, 36] which are commonly used, one typically derives another upscaled problem related to the original model and solves it globally on a coarse grid. This can be done by solving local problems in each coarse element, namely computing the effective permeability field.

Another alternative approach to upscaling methods for the simulations in multiscale applications is the multiscale method [9, 21, 22, 25, 26]. In multiscale methods, the solution of the problem is approximated by the local basis functions, which are the solutions to a class of local problems on coarse grid. Moreover, because of the necessity of the mass conservation for velocity fields, there are many approaches have been proposed to guarantee
Preserving the property of mass conservation in multiscale simulations requires special formulation. In a mixed finite element formulation, one may consider a first-order system for pressure and velocity. The basis functions are mainly obtained to approximate the velocity field. Previous approaches developed multiscale basis functions for the velocity field using local solutions with Neumann boundary conditions and used piecewise constant to approximate the pressure field. In this case, the support of the pressure basis function consists of a single coarse block, while the support of the velocity basis function consists of more than one coarse blocks sharing a common interface. These multiscale approaches have been well developed in [1, 2, 8] and applied to various situations.

Recently, a generalized multiscale finite element method (GMsFEM) in mixed formulation has been proposed in [10, 23]. In these approaches, a systematic procedure has been proposed to construct multiple basis functions for either velocity or pressure in each local patch, which makes these methods different to previous methodology in applications. The computation of the velocity basis functions involves a construction of snapshot space and a model reduction via local spectral decomposition to identify appropriate modes to form the multiscale space. The convergence analysis in [10] addresses a spectral convergence \(1/L\), where \(L\) is the smallest eigenvalue whose modes are excluded in the multiscale space. However, this result does not include a coarse-mesh dependent convergence.

In later works [14], a variation of GMsFEM based on a constraint energy minimization (CEM) [13] for mixed formulation has been proposed, which has a better convergence rate proportional to \(H/L\) with \(H\) the size of coarse mesh. This approach makes use of the ideas of oversampling and localization [30, 32, 33] to compute the multiscale basis functions in oversampled subregions with the satisfaction of an appropriate orthogonality condition. The proposed method provides a mass conservative velocity field and allows one to identify some non-local information (depending on the contrast) by solving a specific local spectral problem for the pressure and constructing an auxiliary space.

The above considerations can be classified as offline methods since the construction of multiscale basis functions does not take into account the source term. The offline method can be tuned in various ways to achieve smaller errors; however, the error decay slows down when a certain number of degree of freedom is reached. This is due to some slow decay after certain eigenvalues. To improve the convergency, the researchers in [9, 11, 12] propose online construction for the multiscale basis functions. In the online stage of the simulation, one may construct new basis functions with local support, using the residual once a coarse-grid approximation is computed. The analysis in [11, 12] shows that the error decay is proportional to \(1 - C\Lambda\), where \(C\) is a constant independent of scales and contrast and guarantees the positivity of the convergence rate.

In this research, we investigate the online construction for CEM-GMsFEM in mixed formulation. The construction presented here is based on the residual information and the technique of oversampling, adopting the ideas in [15]. Specifically, the online basis functions are formulated in an oversampled region. Next, we present the error analysis of the proposed method. This new method has better convergence rate compared to \(1 - C\Lambda\) in online mixed GMsFEM proposed in [3], which additionally requires the online basis for
velocity is of divergence free. In particular, the convergence of the method can become faster by using larger oversampling regions provided that sufficiently many offline basis functions are included. Moreover, we show that the rate of convergence of the method is also determined by a user-defined parameter, which will be introduced in Section 4. Numerical results will be presented to illustrate the performance of the method.

The paper is organized as follows. In Section 2, we present the preliminaries of the problem. Next, we outline the framework of the multiscale method in Section 3. The online adaptive algorithm and the analysis are presented in Section 4. In Section 5, the numerical results are provided to illustrate the efficiency of the proposed method. Concluding remarks will be drawn in Section 6.

## 2 Preliminaries

In this paper, we consider a class of high contrast flow problems in the following mixed formulation over the computational domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$):

\[
\begin{align*}
\kappa^{-1}v + \nabla p &= 0 \quad \text{in } \Omega, \quad (1) \\
\text{div}(v) &= f \quad \text{in } \Omega, \quad (2) \\
v \cdot n &= g \quad \text{on } \partial\Omega, \quad (3) \\
\int_{\Omega} p \, dx &= 0, \quad (4)
\end{align*}
\]

where $n$ is the outward unit normal vector field on the boundary $\partial\Omega$. Note that the source function $f \in L^2(\Omega)$ and the function $g \in L^2(\partial\Omega)$ satisfy the following compatibility condition

\[
\int_{\Omega} f \, dx = \int_{\partial\Omega} g \, dS(x).
\]

Assume that the function $\kappa : \Omega \to \mathbb{R}$ is a heterogeneous coefficient with multiple scales and of high contrast, satisfying $0 < \kappa_{\min} \leq \kappa(x) \leq \kappa_{\max}$, where $\kappa_{\max}$ is large.

Denote $V := H(\text{div}; \Omega)$, $Q := L^2(\Omega)$ and $V_0 := \{v \in V : v \cdot n = 0 \text{ on } \partial\Omega\}$. To solve the problem, we consider the following variational system: find $u \in V_0$ and $p \in Q$ such that

\[
\begin{align*}
a(u, v) - b(v, p) &= 0 \quad \forall v \in V_0, \quad (5) \\
b(u, q) &= (f, q) \quad \forall q \in Q, \quad (6)
\end{align*}
\]

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined as follows:

\[
a(v, w) = \int_{\Omega} \kappa^{-1}v \cdot w \, dx \quad \text{and} \quad b(w, q) = \int_{\Omega} q \, \text{div}(w) \, dx.
\]

We remark that (5)-(6) are solved together with the condition $\int_{\Omega} p \, dx = 0$. Note that the following inf-sup condition holds: for all $q \in Q$ with $\int_{\Omega} q \, dx = 0$, there is a constant $C_0$ which is independent to $\kappa_{\max}$ such that

\[
\|q\|_{L^2(\Omega)} \leq C_0 \sup_{v \in V_0} \frac{b(v, q)}{\|v\|_{H(\text{div}; \Omega)}}. \quad (7)
\]
Next, we introduce the notions of fine and coarse grids. Let \( \mathcal{T}^H \) be a conforming partition of the computational domain \( \Omega \) with mesh size \( H > 0 \). We refer to this partition as the coarse grid. Subordinate to the coarse grid, define the fine grid partition (with mesh size \( h \ll H \)), denoted as \( \mathcal{T}^h \), by refining each coarse element into a connected union of fine grid blocks. We assume the above refinement is performed such that \( \mathcal{T}^h \) is a conforming partition of \( \Omega \).

Let \( N_c \) be the number of coarse elements and \( N_c \) be the number of interior coarse grid nodes of \( \mathcal{T}^H \). The basis functions used for solving the problem are constructed based on the coarse grid. In next section, we shall discuss in detail the construction of the basis functions for velocity and pressure.

3 The multiscale method

In this section, we outline the framework of the multiscale method originally proposed in [14]. The method consists of two general steps. First, we will construct a multiscale space for approximating the pressure. Next, we will use this multiscale space to construct another multiscale space for the velocity. Remark that the basis functions for pressure are local. That is, the support of each pressure basis is a coarse element. On the other hand, for each pressure basis function, we construct a corresponding velocity basis function, whose support is an oversampled region containing the support of the pressure basis function. The oversampled region is usually obtained by enlarging a coarse element by several coarse grid layers. This localized feature of the velocity basis function is the key to the proposed method.

3.1 Construction of pressure basis functions

We present the construction of the pressure basis functions for the mixed formulation. For each coarse element \( K_i \), we construct a set of auxiliary multiscale basis functions using a specific spectral problem. For a set \( S \subset \Omega \), define \( Q(S) := L^2(S) \) and \( V_0(S) := \{ v \in H(\text{div}; \Omega) : v \cdot n = 0 \text{ on } \partial S \} \). Next, we define the required spectral problem. For each coarse element \( K_i \), the spectral problem is to find \( (\phi_{j}^{(i)}, p_{j}^{(i)}) \in V_0(K_i) \times Q(K_i) \) and the eigenvalue \( \lambda_{j}^{(i)} \in \mathbb{R} \) such that

\[
a(\phi_{j}^{(i)}, v) - b(v, p_{j}^{(i)}) = 0 \quad \forall v \in V_0(K_i),
\]

\[
b(\phi_{j}^{(i)}, q) = \lambda_{j}^{(i)} s_i(p_{j}^{(i)}, q) \quad \forall q \in Q(K_i),
\]

where \( s_i \) is defined to be

\[
s_i(p, q) = \int_{K_i} \tilde{\kappa}pq \, dx, \quad \tilde{\kappa} = \kappa \sum_{i=1}^{N_c} \nabla \chi_i|^2
\]

and \( \{\chi_i\}_{i=1}^{N_c} \) is the set of standard multiscale basis functions satisfying the partition of unity property. Remark that one can also use other types of partition of unity functions. Assume that \( s_i(p_{j}^{(i)}, p_{j}^{(i)}) = 1 \) and that we arrange the eigenvalues obtained from (8)-(9) in
non-decreasing order: \( 0 = \lambda_1^{(i)} \leq \lambda_2^{(i)} \leq \cdots \). After that, we pick the first \( J_i \) eigenfunctions \( \{ p_j^{(i)} \} \) corresponding to the first \( J_i \) eigenvalues \( \lambda_j^{(i)} \) to define the local auxiliary space \( Q_{\text{aux}}(K_i) \) by

\[
Q_{\text{aux}}(K_i) := \text{span}\{ p_j^{(i)} : 1 \leq j \leq J_i \}.
\]

The global auxiliary space \( Q_{\text{aux}} \) is defined as \( Q_{\text{aux}} := \bigoplus_{i=1}^N Q_{\text{aux}}(K_i) \). Note that the space \( Q_{\text{aux}} \) will be used as the approximation space for the pressure \( p \).

### 3.2 Construction of the multiscale basis functions for velocity

In this section, we present the construction of the velocity basis function. For each pressure basis function in \( Q_{\text{aux}} \), we construct a corresponding velocity basis function, whose support is an oversampled region containing the support of the pressure basis function. Define the projection operator \( \pi : Q \to Q_{\text{aux}} \) by

\[
\pi(q) = \sum_{i=1}^N \sum_{j=1}^{J_i} s_i(q, p_j^{(i)}) p_j^{(i)}, \quad \forall q \in Q.
\]

Also, we denote \( s(\cdot, \cdot) \) as

\[
s(p, q) := \sum_{i=1}^N s_i(p, q) \quad \forall p, q \in Q.
\]

Note that \( \pi \) is the \( L^2 \) projection of \( Q \) onto \( Q_{\text{aux}} \) with respect to the inner product \( s(p, q) := \sum_{i=1}^N s_i(p, q) \). Let \( p_j^{(i)} \in Q_{\text{aux}} \) be a given pressure basis function supported in \( K_i \). Let \( K_{i,\ell} \) be an oversampled region obtained by enlarging \( \ell \) layers from \( K_i \). Namely,

\[
K_{i,0} := K_i, \quad K_{i,\ell} := \bigcup \{ K \in \mathcal{T}^H : K \cap \overline{K_{i,\ell-1}} \neq \emptyset \}, \quad \ell = 1, 2, \ldots.
\]

For simplicity, we may denote this oversampled region as \( K_i^+ \). The multiscale velocity basis function \( \psi_{j,ms}^{(i)} \in V_0(K_i^+) \) is constructed by solving the following problem: find \( (\psi_{j,ms}^{(i)}, q_{j,ms}^{(i)}) \in V_0(K_i^+) \times Q(K_i^+) \) such that

\[
a(\psi_{j,ms}^{(i)}, v) - b(v, q_{j,ms}^{(i)}) = 0 \quad \forall v \in V_0(K_i^+), \quad (10)
\]

\[
s(\pi q_{j,ms}^{(i)}, \pi q) + b(\psi_{j,ms}^{(i)}, q) = s(p_j^{(i)}, q) \quad \forall q \in Q(K_i^+). \quad (11)
\]

The multiscale space for velocity can be defined as \( V_{ms} := \{ \psi_{j,ms}^{(i)} : 1 \leq j \leq J_i, 1 \leq i \leq N \} \). We remark that the construction for velocity basis function is motivated by the unconstraint energy minimization problem (24) in [13]. One may also define the local basis function \( \psi_{j,ms}^{(i)} \) in the region \( K_i^+ \) by a localization property of the related global basis function \( \psi_j^{(i)} \). The global basis function \( \psi_j^{(i)} \in V_0 \) is defined as the solution to the following problem: find \( (\psi_j^{(i)}, q_j^{(i)}) \in V_0 \times Q \) such that

\[
a(\psi_j^{(i)}, v) - b(v, q_j^{(i)}) = 0 \quad \forall v \in V_0, \quad (12)
\]

\[
s(\pi q_j^{(i)}, \pi q) + b(\psi_j^{(i)}, q) = s(p_j^{(i)}, q) \quad \forall q \in Q. \quad (13)
\]
The global multiscale space is defined as \( V^{\text{glo}} := \text{span}\{\psi_j^{(i)} : 1 \leq j \leq J_i, 1 \leq i \leq N\} \). Note that the system (12)-(13) define a mapping \( G : Q^{\text{aux}} \rightarrow V^{\text{glo}} \times Q \) in the following manner: given \( p_{\text{aux}} \in Q^{\text{aux}} \), the image \( G(p_{\text{aux}}) = (\psi, r) \in V^{\text{glo}} \times Q \) is defined as the solution to the following system

\[
a(\psi, v) - b(v, r) = 0 \quad \forall v \in V_0, \tag{14}
\]

\[
s(\pi r, \pi q) + b(\psi, q) = s(p_{\text{aux}}, q) \quad \forall q \in Q. \tag{15}
\]

Remark. The mapping \( G_1 : p_{\text{aux}} \mapsto \psi \) is surjective and the operator \( G \) is continuous. In particular, for any \( p_{\text{aux}} \in Q^{\text{aux}} \), there exists a constant \( C > 0 \) such that

\[
\left( \|\psi\|_a^2 + \|\pi r\|_s^2 \right)^{1/2} \leq C\|p_{\text{aux}}\|_s. \tag{16}
\]

Take \( v = \psi \) in (14) and \( q = r \) in (15) and sum over the equations, one obtains

\[
\|\psi\|_a^2 + \|\pi r\|_s^2 = s(p_{\text{aux}}, r) \leq \|\tilde{\kappa}p_{\text{aux}}\|_{L^2(\Omega)}\|r\|_{L^2(\Omega)} \leq C_0B\|p_{\text{aux}}\|_s \left( \|\psi\|_a^2 + \|\pi r\|_s^2 \right)^{1/2},
\]

where \( C_0 \) is the constant in the inf-sup condition (7) and \( B = \max_{x \in \Omega} \{\kappa(x)\} \).

3.3 The method

The multiscale solution \((u_{ms}, p_{ms}) \in V_{ms} \times Q^{\text{aux}}\) is obtained by solving the following variational system:

\[
a(u_{ms}, v) - b(v, p_{ms}) = 0 \quad \forall v \in V_{ms}, \tag{17}
\]

\[
b(u_{ms}, q) = (f, q) \quad \forall q \in Q^{\text{aux}}. \tag{18}
\]

To analyze the method, we define the norms for the multiscale spaces. Given a subset \( D \subset \Omega \), for any \( v \in V \) and \( q \in Q \), define the norms \( \|\cdot\|_{a(D)}, \|\cdot\|_{s(D)} \) and \( \|\cdot\|_{V(D)} \) by

\[
\|v\|_{a(D)} = \left( \int_D \kappa^{-1}|v|^2 \, dx \right)^{1/2}, \quad \|q\|_{s(D)} = \left( \int_D \tilde{\kappa}|q|^2 \, dx \right)^{1/2},
\]

\[
\|v\|_{V(D)} = \left( \int_D \tilde{\kappa}^{-1}|\nabla \cdot v|^2 \, dx + \int_D \kappa^{-1}|v|^2 \, dx \right)^{1/2}.
\]

For the case \( D = \Omega \), we simply denote the norms as \( \|\cdot\|_a, \|\cdot\|_s \) and \( \|\cdot\|_V \).

4 Online adaptive enrichment

In this section, we will introduce an enrichment algorithm requiring the construction of new basis functions based on a pre-computed solution. These functions constructed in this manner are called online basis functions as they are built in the online stage of computations.
4.1 Construction of the online basis function

To begin, we first define a pair of residual functionals. Let \((u_{ms}, p_{ms})\) be the current numerical solution to the system \((17)-(18)\) and \(\{\chi_i\}_{i=1}^{N_c}\) be a set of multiscale partition of unity corresponding to the set of coarse neighborhoods \(\{\omega_i\}\), where

\[
\omega_i := \bigcup \{K_j \in \mathcal{T}^H : x_i \in \overline{K_j}\}
\]

and \(x_i\) is a coarse node. Define the local residuals \(\mathcal{R}_i : V \to \mathbb{R}\) and \(r_i : Q \to \mathbb{R}\) as follows: for any \(v \in V\) and \(q \in Q\)

\[
\mathcal{R}_i(v) := a(u_{ms}, \chi_i v) - b_i(v, p_{ms}) \quad \text{and} \quad r_i(q) := (f, \chi_i q) - b(u_{ms}, \chi_i q),
\]

where \(b_i(\cdot, \cdot)\) is the restriction of \(b(\cdot, \cdot)\) in the neighborhood \(\omega_i\). For \(\hat{\ell} \in \mathbb{N}\), define \(\omega_{i,\hat{\ell}}\) as follows

\[
\omega_{i,0} := \omega_i, \quad \omega_{i,\hat{\ell}} := \bigcup \{K \in \mathcal{T}^H : K \cap \overline{\omega_{i,\hat{\ell}-1}} \neq \emptyset\}, \quad \hat{\ell} = 1, 2, \ldots.
\]

We denote \(\omega_i^+ := \omega_{i,\hat{\ell}}\) for short. Write \(V(\omega_i^+) := H(\text{div}; \omega_i^+)\) and \(Q(\omega_i^+) = L^2(\omega_i^+)\). The construction of online basis function is motivated by the local residuals \(\mathcal{R}_i\) and \(r_i\). One may look for the online basis functions \((\beta^{(i)}_{on}, q^{(i)}_{on}) \in V(\omega_i^+) \times Q(\omega_i^+)\) (both supported in \(\omega_i^+)\) such that

\[
\begin{align*}
    a(\beta^{(i)}_{on}, v) - b(v, q^{(i)}_{on}) &= \mathcal{R}_i(v) \quad \forall v \in V(\omega_i^+), \quad (19) \\
    s(\pi q^{(i)}_{on}, \pi q) + b(\beta^{(i)}_{on}, q) &= r_i(q) \quad \forall q \in Q(\omega_i^+). \quad (20)
\end{align*}
\]

We remark that the above online basis functions are obtained in the local (oversampled) region \(\omega_i^+\) and this is the result of a localization of the corresponding global online basis functions \((\beta^{(i)}_{glo}, q^{(i)}_{glo}) \in V_0 \times Q\) defined by

\[
\begin{align*}
    a(\beta^{(i)}_{glo}, v) - b(v, q^{(i)}_{glo}) &= \mathcal{R}_i(v) \quad \forall v \in V, \quad (21) \\
    s(\pi q^{(i)}_{glo}, \pi q) + b(\beta^{(i)}_{glo}, q) &= r_i(q) \quad \forall q \in Q. \quad (22)
\end{align*}
\]

4.2 Online adaptive algorithm

In this section, we describe the proposed online adaptive algorithm. First, we define the operator norms of the local residuals \(\mathcal{R}_i\) and \(r_i\) corresponding to \(\omega_i \subset \Omega\) as follows

\[
\|\mathcal{R}_i\|_{a^*} := \sup_{v \in V} \frac{|\mathcal{R}_i(v)|}{\|v\|_{a(\omega_i)}} \quad \text{and} \quad \|r_i\|_{s^*} := \sup_{q \in Q} \frac{|r_i(q)|}{\|q\|_{s(\omega_i)}}.
\]

Next, we set \(m = 0\) to represent the level of online enrichment and initially define \(V_{ms} := V_{ms}\). Choose a tolerance \(\text{tol} \in \mathbb{R}_+\) and a parameter \(\theta\) such that \(0 \leq \theta \leq 1\). For each \(m \in \mathbb{N}\), assume that \(V_{ms}^m\) is given. Go to Step 1 below.

**Step 1.** Solve \((17)-(18)\) over the multiscale spaces \(V_{ms}^m \times Q_{aux}\) to obtain the current approximation \((u_{ms}^m, p_{ms}^m) \in V_{ms}^m \times Q_{aux}\).
Step 2. For each $i = 1, \ldots, N_c$, compute the norm of the local residual $\eta_i^2 := \norm{R_i}_s^2 + \norm{r_i}_p^2$. Rearrange the indices of $\eta_i$ such that $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{N_c}$. Choose the smallest integer $k \in \mathbb{N}$ such that
\[
\sum_{i=1}^k \eta_i^2 \geq \theta \sum_{i=1}^{N_c} \eta_i^2.
\]

Step 3. For each $i = 1, \ldots, k$, solve (19)–(20) over $\omega_i^+$ to obtain the online basis functions $(\beta_{on}^{(i)}, q_{on}^{(i)})$. Enrich the multiscale space by letting
\[
V_{ms}^{m+1} := V_{ms}^m \oplus \text{span}\{\beta_{on}^{(i)} : i = 1, \ldots, k\}.
\]

Step 4. If $\sum_{i=1}^{N_c} \eta_i^2 \leq \text{tol}$ or there is certain number of basis functions in $V_{ms}^{m+1}$, then Stop. Otherwise, go back to Step 1 and set $m \leftarrow m + 1$.

4.3 Analysis

In this section, we provide the convergence analysis of the proposed method. First, we recall some useful theoretical results from [14].

Lemma 4.1 (Lemma 1 in [14]). For each $p_{aux} \in Q_{aux}$ with $s(p_{aux}, 1) = 0$, there is a unique $u \in V_{gl}$ such that $(u, p) \in V_0 \times Q$ (with $\int_{\Omega} p \, dx = 0$) is the solution of the following system
\[
a(u, v) - b(v, p) = 0 \quad \forall v \in V_0, \\
b(u, q) = s(p_{aux}, q) \quad \forall q \in Q.
\]

The following lemma motivates the local multiscale basis functions defined in (10)–(14), saying that the global basis functions defined in (12)–(13) has an exponential decay outside an oversampled region. In the following, we denote $E$ as the constant of exponential decay
\[
E = C(1 + \Lambda^{-1}) \left(1 + C^{-1}(1 + \Lambda^{-1})^{-\frac{1}{2}}\right)^{1-\ell}, \quad \text{where } \Lambda := \min_{1 \leq i \leq N} \Lambda_{j_{i+1}}.
\]

Lemma 4.2 (Lemma 7 in [14]). Let $(\psi_{j}^{(i)}, q_{j}^{(i)})$ be the solution of (12)–(13) and $(\psi_{j,ms}^{(i)}, q_{j,ms}^{(i)})$ be the solution (10)–(11). For $K_i^+ = K_i, \ell$ with $\ell \geq 2$, we have
\[
\norm{\psi_{j}^{(i)} - \psi_{j,ms}^{(i)}}_V^2 + \norm{q_{j}^{(i)} - q_{j,ms}^{(i)}}_s^2 \leq E \norm{p_{j}^{(i)}}_s^2.
\]

Next, the following lemma is needed in the analysis of the main result. The proof of this lemma makes use of the cutoff function used in [13] and one may see [15] for more details about this lemma.

Lemma 4.3 (cf. Lemma 3 in [15]). Assume the same conditions in Lemma 4.2 hold. For any $\{d_{j}^{(i)}\} \subset \mathbb{R}$ and $\bar{C} = C(1 + \Lambda^{-1})(\ell + 1)^d$, we have
\[
\left\| \sum_{i=1}^N J_i \sum_{j=1}^{J_i} d_j^{(i)}(\psi_{j}^{(i)} - \psi_{j,ms}^{(i)}) \right\|_V^2 + \left\| \sum_{i=1}^N J_i \sum_{j=1}^{J_i} d_j^{(i)}(q_{j}^{(i)} - q_{j,ms}^{(i)}) \right\|_s^2
\leq \bar{C} \sum_{i=1}^N \left( \left\| \sum_{j=1}^{J_i} d_j^{(i)}(\psi_{j}^{(i)} - \psi_{j,ms}^{(i)}) \right\|_V^2 + \left\| \sum_{j=1}^{J_i} d_j^{(i)}(q_{j}^{(i)} - q_{j,ms}^{(i)}) \right\|_s^2 \right).
\]
Note that the basis functions constructed during the online stage are defined same as the multiscale one. The same localization result holds for the online basis functions and the proof of the following lemma is the same as that for Lemma 4.2 and Lemma 4.3 and is therefore omitted.

**Lemma 4.4.** Assume that $\omega_i^+$ is obtained from $\omega_i$ by enlarging $\tilde{\ell}$ coarse grid layers with $\tilde{\ell} \geq 2$. Let $(\beta_{on}^{(i)}, q_{on}^{(i)})$ be the local online basis functions in (19)-(20) and $(\beta_{glo}^{(i)}, q_{glo}^{(i)})$ be the global one in (21)-(22). Then, we have

$$
\|\beta_{glo}^{(i)} - \beta_{on}^{(i)}\|^2_V + \|q_{glo}^{(i)} - q_{on}^{(i)}\|^2_s \leq \tilde{E}(\|\beta_{glo}^{(i)}\|^2_a + \|\pi q_{glo}^{(i)}\|^2_s),
$$

where $\tilde{E} = C(1 + \Lambda^{-1}) \left(1 - C^{-1}(1 + \Lambda^{-\frac{1}{2}})\right)^{1-\tilde{\ell}}$. Furthermore, we have

$$
\left\| \sum_{i=1}^{N_c} (\beta_{glo}^{(i)} - \beta_{on}^{(i)}) \right\|^2_V + \left\| \sum_{i=1}^{N_c} \pi (q_{glo}^{(i)} - q_{on}^{(i)}) \right\|^2_s \\
\leq \bar{C} \sum_{i=1}^{N_c} (\|\beta_{glo}^{(i)} - \beta_{on}^{(i)}\|^2_V + \|\pi(q_{glo}^{(i)} - q_{on}^{(i)})\|^2_s).
$$

The main result in the research reads as follows.

**Theorem 4.5.** Assume that $u \in V$ is the solution to (5)- (6) and for $m \in \mathbb{N}$, $u_{ms}^m \in V_{ms}$ is the approximated solution to (17)-(18). Then, there are constants $C_0 = C_0(M, \ell, d, \Lambda)$, $C_1 = C_1(M, \tilde{\ell}, d, \Lambda)$ and $C_2 = C_2(M, \Lambda)$ such that

$$
\|u - u_{ms}^{m+1}\|^2_V \leq (C_0 E + C_1 \bar{E} + C_2 \bar{\theta}) \|u - u_{ms}^m\|^2_V,
$$

where $M = \max_K n_K$ with $n_K$ being the number of coarse nodes of the coarse element $K$ and $\bar{\theta}$ is the chosen adaptive parameter satisfying $\bar{\theta} = 1 - \theta$.

**Proof.** To simplify the analysis, we may assume that the test function $v$ and $\sum_i \beta_{glo}^{(i)}$ to be divergence free. Recall the definition of the global online basis function corresponding to $\omega_i$: find $(\beta_{glo}^{(i)}, q_{glo}^{(i)}) \in V_0 \times Q$ such that

$$
a(\beta_{glo}^{(i)}, v) - b(v, q_{glo}^{(i)}) = R_i(v) \quad \forall v \in V, \quad (23)
$$

$$
s(\pi q_{glo}^{(i)}, \pi q) + b(\beta_{glo}^{(i)}, q) = r_i(q) \quad \forall q \in Q. \quad (24)
$$

After summing over all the neighborhoods $\omega_i$ for $i = 1, \cdots, N_c$, one obtains

$$
a(u_{ms}^m - u + \sum_{i=1}^{N_c} \beta_{glo}^{(i)}) - b(v, \sum_{i=1}^{N_c} q_{glo}^{(i)}) = 0 \quad \forall v \in V, \quad (25)
$$

$$
b(u_{ms}^m - u + \sum_{i=1}^{N_c} \beta_{glo}^{(i)}) = s\left(\pi \left(-\sum_{i=1}^{N_c} q_{glo}^{(i)}\right), q\right) \quad \forall q \in Q, \quad (26)
$$

By Lemma 4.4, there exist a set of constants $\{c_j^{(i)}\}$ such that

$$
\eta := u_{ms}^m - u + \sum_{i=1}^{N_c} \beta_{glo}^{(i)} = \sum_{i=1}^{N_c} \sum_{j=1}^{J_i} c_j^{(i)} \psi_j^{(i)}.
$$
Denote $\xi := \sum_{i,j} c_j^{(i)} q_j^{(i)}$ and $p_{aux} := \sum_{i,j} c_j^{(i)} p_j^{(i)}$. Then, we have

$$ a(\eta, v) - b(v, \xi) = 0 \quad \forall v \in V,$$

$$ s(\pi \xi, \pi q) + b(\eta, q) = s(p_{aux}, q) \quad \forall q \in Q. $$

Note that $G(p_{aux}) = (\eta, \xi)$. Next, we estimate the error of localization, namely the term $\sum_{i,j} c_j^{(i)} (\psi_j^{(i)} - \psi_{j,ms})$. By Lemma 4.2 and Lemma 4.3, we have

$$ \left\| \sum_{i=1}^{N} \sum_{j=1}^{J_i} c_j^{(i)} (\psi_j^{(i)} - \psi_{j,ms}^{(i)}) \right\|_V^2 \leq \tilde{C} (1 + \Lambda^{-1}) (\ell + 1)^d E \sum_{i=1}^{N} \sum_{j=1}^{J_i} (c_j^{(i)})^2. $$

Applying the corollary of open mapping theorem [18] to the mapping $G_1 : Q_{aux} \to V_{glo}$, there is a constant $C_m > 0$ such that

$$ \|p_{aux}\|_s^2 = \sum_{i=1}^{N} \sum_{j=1}^{J_i} (c_j^{(i)})^2 \leq C_m \|\eta\|_a^2. $$

Take $v = \sum_{i=1}^{N_c} \beta_{glo}^{(i)}$ in (23), $q = \pi \sum_{i=1}^{N_c} q_{glo}^{(i)}$ in (23) and sum over these equations, we have

$$ \left\| \sum_{i=1}^{N_c} \beta_{glo}^{(i)} \right\|_a^2 + \left\| \sum_{i=1}^{N_c} \pi q_{glo}^{(i)} \right\|_s^2 = a \left( u - u_{ms}^m \right) \sum_{i=1}^{N_c} \beta_{glo}^{(i)} \right\|_a \leq \left\| u - u_{ms}^m \right\|_a. $$

Hence, we have

$$ \left\| \sum_{i=1}^{N} \sum_{j=1}^{J_i} c_j^{(i)} (\psi_j^{(i)} - \psi_{j,ms}^{(i)}) \right\|_V^2 \leq \tilde{C} (1 + \Lambda^{-1}) (\ell + 1)^d E \sum_{i=1}^{N} \sum_{j=1}^{J_i} (c_j^{(i)})^2 $$

$$ \leq \tilde{C} C_m (1 + \Lambda^{-1}) (\ell + 1)^d E \|\eta\|_a^2 $$

$$ \leq 2 \tilde{C} C_m (1 + \Lambda^{-1}) (\ell + 1)^d E \left( \left\| u - u_{ms}^m \right\|_a + \left\| \sum_{i=1}^{N_c} \beta_{glo}^{(i)} \right\|_a^2 \right) $$

$$ \leq 4 \tilde{C} C_m (1 + \Lambda^{-1}) (\ell + 1)^d E \|u - u_{ms}^m\|_V^2. $$

(27)

After that, we estimate the error of localization for the online basis functions. Note that, by Lemma 4.4, we have

$$ \|\beta_{glo}^{(i)} - \beta_{on}^{(i)}\|_V^2 + \|\pi (q_{glo}^{(i)} - q_{on}^{(i)})\|_s^2 \leq \tilde{E} \left( \|\beta_{glo}^{(i)}\|_a^2 + \|\pi q_{glo}^{(i)}\|_s^2 \right). $$

Take $v = \beta_{glo}^{(i)}$ in (23), $q = q_{glo}^{(i)}$ in (24) and sum these two equations up, by Cauchy-Schwartz inequality one obtains

$$ \|\beta_{glo}^{(i)}\|_a^2 + \|\pi q_{glo}^{(i)}\|_s^2 = a(u - u_{ms}^m, \chi_i \beta_{glo}^{(i)}) + b(u - u_{ms}^m, \chi_i q_{glo}^{(i)}) $$

$$ \leq \|u - u_{ms}^m\|_a |\chi_i\|_{\omega_i} + \|\tilde{k}^{-1/2} \div (u - u_{ms}^m)\|_{L^2(\omega_i)} |\chi_i q_{glo}^{(i)}|_s $$

$$ \leq \|u - u_{ms}^m\|_V(\omega_i) (|\chi_i \beta_{glo}^{(i)}|_a^2 + |\chi_i q_{glo}^{(i)}|_s^2)^{1/2} $$

$$ \leq (1 + \Lambda^{-1}) \|u - u_{ms}^m\|_V(\omega_i) (|\beta_{glo}^{(i)}|_a^2 + |q_{glo}^{(i)}|_s^2)^{1/2}. $$
It implies that
\[ \sum_{i=1}^{N_c} (\| \beta_{i}^{(i)} - \beta_{i,on}^{(i)} \|_V^2 + \| q_{i}^{(i)} - q_{i,on}^{(i)} \|_V^2) \leq \tilde{E}(1 + \Lambda^{-1}) \sum_{i=1}^{N_c} \| u - u_{ms}^m \|_V^2 (\omega_i) \]
\[ \leq M \tilde{E}(1 + \Lambda^{-1}) \| u - u_{ms}^m \|_V^2. \]  

(28)

Finally, we may prove the required convergency. Let \( I = \{1, 2, \cdots, k\} \subset \{1, 2, \cdots, N_c\} \) be the set of indices and we add online basis functions \( \beta_{ms}^{(i)} \) for \( i \in I \) into the space \( V_{ms}^m \) to form \( V_{ms}^{m+1} \). Define the following function \( w \in V_{ms}^{m+1} \)
\[ w = u_{ms}^m - \sum_{i \in I} \beta_{i, on}^{(i)} + \sum_{i=1}^{N} \sum_{j=1}^{J_r} c_j^{(i)} \psi_{j, ms}^{(i)} . \]

By the Galerkin orthogonality, we obtain
\[ \| u - u_{ms}^{m+1} \|_V^2 \leq \| u - w \|_V^2 \]
\[ = \left( \sum_{i \in I} (\beta_{i, on}^{(i)} - \beta_{i, glo}^{(i)} - \sum_{i \notin I} \beta_{i, glo}^{(i)} + \sum_{i=1}^{N} \sum_{j=1}^{J_r} c_j^{(i)} (\psi_{j}^{(i)} - \psi_{j, ms}^{(i)}) \right)^2 \]
\[ \leq 3 \left( \left\| \sum_{i \in I} \beta_{i, on}^{(i)} - \beta_{i, glo}^{(i)} \right\|_V^2 + \left\| \sum_{i \notin I} \beta_{i, glo}^{(i)} \right\|_V^2 + \left\| \sum_{i=1}^{N} \sum_{j=1}^{J_r} c_j^{(i)} (\psi_{j}^{(i)} - \psi_{j, ms}^{(i)}) \right\|_V^2 \right). \]

By Lemma 4.4 (27) and (28) we have
\[ 1 \leq \tilde{C} M \tilde{E}(1 + \Lambda^{-1}) \| u - u_{ms}^m \|_V^2, \]
\[ 3 \leq 4 \tilde{C} C_{m}(1 + \Lambda^{-1})(\ell + 1)^d E \| u - u_{ms}^m \|_V^2. \]

Next, we denote \( \tilde{\beta} = \sum_{i \notin I} \beta_{i, glo}^{(i)} \), and \( \tilde{q} = \sum_{i \notin I} q_{i, glo}^{(i)} \). By the definition of the online basis function, one may have
\[ a(\tilde{\beta}, v) - b(v, \tilde{q}) = \sum_{i \notin I} \mathcal{R}_i(v) \quad \forall v \in \tilde{V}, \]
\[ s(\pi \tilde{q}, \pi q) + b(\tilde{\beta}, q) = \sum_{i \notin I} \mathcal{R}_i(q) \quad \forall q \in Q. \]

Take \( v = \tilde{\beta}, q = \tilde{q} \) and add two equations together, we have
\[ \| \tilde{\beta} \|^2_a + \| \pi \tilde{q} \|^2_s = \sum_{i \notin I} \mathcal{R}_i(\tilde{\beta}) + r_i(\tilde{q}) \]
\[ \leq \sum_{i \notin I} (\| \mathcal{R}_i \|^2_{a^*} + \| r_i \|^2_{s^*})^{1/2}(\| \tilde{\beta} \|^2_{a(\omega_i)} + \| \pi \tilde{q} \|^2_{s(\omega_i)})^{1/2} \]
\[ \leq (1 + \Lambda^{-1})^{1/2} \sum_{i \notin I} \eta_i(\| \tilde{\beta} \|^2_{a(\omega_i)} + \| \pi \tilde{q} \|^2_{s(\omega_i)})^{1/2} \]
\[ \leq \sqrt{\tilde{\theta}(1 + \Lambda^{-1})} \left( \sum_{i=1}^{N_c} \eta_i^2 \right)^{1/2} \left( \sum_{i=1}^{N_c} \| \tilde{\beta} \|^2_{a(\omega_i)} + \| \pi \tilde{q} \|^2_{s(\omega_i)} \right)^{1/2} \]
\[ \leq \sqrt{M \tilde{\theta}} (1 + \Lambda^{-1}) \left( \sum_{i=1}^{N_c} \eta_i^2 \right)^{1/2} \left( \sum_{i=1}^{N_c} \| \tilde{\beta} \|^2_{a(\omega_i)} + \| \pi \tilde{q} \|^2_{s(\omega_i)} \right)^{1/2}, \]
where \( \eta_i = (\|R_i\|_a^2 + \|r_i\|_a^2)^{1/2} \) and \( \bar{\theta} = 1 - \theta \). It implies that

\[
\|\tilde{\beta}\|_a^2 + \|\pi\tilde{q}\|_s^2 \leq M\theta(1 + \Lambda^{-1}) \sum_{i=1}^{N_c} \eta_i^2 \leq M\theta(1 + \Lambda^{-1}) \sum_{i=1}^{N_c} \|u - u^{m}_{ms}\|_{V(\omega_i)}^2
\]

\[
\leq M^2 \bar{\theta}(1 + \Lambda^{-1}) \|u - u^{m}_{ms}\|_V^2,
\]

where we use the fact that

\[
|R_i(v)| = |a(u - u^{m}_{ms}, \chi_i v)| \leq \|u - u^{m}_{ms}\|_{a(\omega_i)} \|v\|_{a(\omega_i)} \quad \forall v \in V,
\]

\[
|r_i(q)| = |b(u - u^{m}_{ms}, \chi_i q)| \leq \|\tilde{k}\|^{-1/2} \|\text{div}(u - u^{m}_{ms})\|_{L^2(\omega_i)} \|q\|_{s(\omega_i)} \quad \forall q \in Q
\]

\[
\implies \eta_i \leq \|u - u^{m}_{ms}\|_{V(\omega_i)}.
\]

Hence, the following estimate holds

\[
2 \leq M^2 \bar{\theta}(1 + \Lambda^{-1}) \|u - u^{m}_{ms}\|_V^2.
\] (32)

Combining (29), (30) and (32), we have

\[
\|u - u^{m+1}_{ms}\|_V^2 \leq (C_0 \tilde{E} + C_1 \tilde{E} + C_2 \bar{\theta}) \|u - u^{m}_{ms}\|_V^2,
\] (33)

where \( C_0 = 12\tilde{C}C_m(1 + \Lambda^{-1})(\ell + 1)^d \), \( C_1 = 3\tilde{C}M(1 + \Lambda^{-1}) \) and \( C_2 = 3(1 + \Lambda^{-1})M^2 \). This completes the proof.

\[\square\]

Remark. The quantity \( \bar{\theta} = 1 - \theta \) is a user-defined parameter which can control the convergence rate of the method. When the parameter \( \theta \) is close to 1, the error becomes smaller during the online stage. This fact will be illustrated by our numerical experiments in next section.

Remark. The constant of exponential decay \( \tilde{E} \) is defined in the online stage. One may choose a different number of layers in the construction of online basis functions to drive the error decay faster, even with a smaller number of layer \( \ell \) in offline stage.

5 Numerical experiment

In this section, we present some numerical results to show the efficiency of the proposed method. The computational domain is \( \Omega = (0, 1)^2 \). We use a rectangular mesh for the partition of the domain dividing \( \Omega \) into \( T \times T \) equal coarse square blocks and further divide each coarse block into \( n \times n \) equal square pieces. In other words, the fine mesh contains \( Tn \times Tn \) fine rectangular elements with the mesh size \( h = \frac{1}{Tn} \). The boundary condition is set to be \( g = 0 \). We test the performance by considering uniform enrichment (\( \theta = 1 \)) and by using the online adaptive enrichment. In all the examples below, the term energy error refers to the following quantity

\[
e_{u} := \frac{\|u - u^{m}_{ms}\|_a}{\|u\|_a}
\]

where \( u \) is the reference solution solved on the fine mesh.
**Example 5.1.** In this example, we set $T = 8$ and $n = 12$. The permeability field $\kappa$ used in this example is given in Figure 1. The source function $f$ is defined as follows

$$f(x) = \begin{cases} 
1 & x \in (0, 1/8)^2, \\
-1 & x \in (7/8, 1)^2, \\
0 & \text{otherwise},
\end{cases}$$

and thus the compatibility condition holds. The number of oversampling layers is $\ell = 2$, $\tilde{\ell} = 2$ and the number of initial basis functions is $J_i = 3$.

In Table 1 we present the energy error for the case with uniform enrichment, that is $\theta = 1$. One may observe a moderately fast convergence of the method. In Table 2 we present the energy error by using the online adaptive enrichment with $\theta = 0.1$. That is, only the basis functions related to the regions which account for the largest 10% will be added in the online stage. Here, $DOF$ stands for the dimension of the multiscale space $V_m^{ms}$. From the tables, we observe that smaller parameter $\theta$ leads to slower convergence. This confirms that the user-defined parameter is useful in controlling the convergence rate of the proposed adaptive method.

Note that, one may use a larger number of layers in online stage to further reduce the error of velocity. In Table 3 we present the results with offline number of layers $\ell = 2$ and set online number of layers as $\tilde{\ell} = 4$. One may observe that the larger number of layers in online stage leads to a faster decay in error reduction with less iterations.

![Example 5.1](image1.png)

![Example 5.2](image2.png)

**Figure 1:** The permeability field $\kappa$.

| Number of offline basis | Online iterations | DOF | Energy error |
|-------------------------|------------------|-----|--------------|
| 3                       | 0                | 192 | 0.0658       |
| 3                       | 1                | 241 | 0.0022       |
| 3                       | 2                | 290 | $1.97 \times 10^{-4}$ |
| 3                       | 3                | 339 | $1.65 \times 10^{-5}$ |
| 3                       | 4                | 388 | $1.15 \times 10^{-6}$ |

Table 1: Uniform enrichment with $\theta = 1$. (Example 5.1)
| Number of offline basis | Online iterations | DOF  | Energy error  |
|------------------------|------------------|------|---------------|
| 3                      | 0                | 192  | 0.0658        |
| 3                      | 1                | 196  | 0.0289        |
| 3                      | 2                | 200  | 8.95 × 10⁻³   |
| 3                      | 3                | 204  | 6.69 × 10⁻³   |
| 3                      | 4                | 208  | 3.87 × 10⁻³   |

Table 2: Online adaptivity with $\theta = 0.1$. (Example 5.1)

| Number of offline basis | Online iterations | DOF   | Energy error  |
|------------------------|------------------|-------|---------------|
| 3                      | 0                | 192   | 0.0658        |
| 3                      | 1                | 241   | 5.52 × 10⁻⁵   |
| 3                      | 2                | 290   | 5.37 × 10⁻⁸   |

Table 3: Uniform enrichment with $\theta = 1$ and $\tilde{\ell} = 4$. (Example 5.1)

**Example 5.2.** In this example, we test the proposed method on another permeability field with high-contrast channels, see Figure 1b. The mesh parameters are $T = 10$ and $n = 16$. The source function $f$ used in this example is defined as follows

\[
f(x) = \begin{cases} 
1 & x \in (0.2, 0.4)^2, \\
-1 & x \in (0.7, 0.9)^2, \\
0 & \text{otherwise}, 
\end{cases}
\]

and thus the compatibility condition holds. In this example, we form the offline space by letting $J_i = 3$ and the number of oversampling layers is $\ell = 1$. After that, we use set $\tilde{\ell} = 2$ to construct the online basis functions.

In Table 4, we present the results with uniform enrichment. One may observe that the proposed online method can drive the energy error down fast even with a large error between the offline approximation and the fine-scale solution. Next, we test the performance of the method using online adaptivity. In Table 5, the results with parameter $\theta = 0.15$ are presented and the convergence becomes slow comparing to the case of uniform enrichment, which confirms the analytical assertion.

| Number of offline basis | Online iterations | DOF   | Energy error  |
|------------------------|------------------|-------|---------------|
| 3                      | 0                | 300   | 0.1274        |
| 3                      | 1                | 381   | 0.0170        |
| 3                      | 2                | 462   | 8.70 × 10⁻⁴   |
| 3                      | 3                | 543   | 5.81 × 10⁻⁵   |

Table 4: Uniform enrichment with $\theta = 1$. (Example 5.2)

### 6 Conclusion

In this research, we propose an online adaptive strategy for CEM-GMsFEM in mixed formulation. The CEM-GMsFEM developed in [13] provides a systematic approach to
| Number of offline basis | Online iterations | DOF  | Energy error |
|-------------------------|------------------|------|--------------|
| 3                       | 0                | 300  | 0.1274       |
| 3                       | 1                | 304  | 0.0379       |
| 3                       | 2                | 308  | 0.0113       |
| 3                       | 3                | 312  | $7.18 \times 10^{-3}$ |

Table 5: Online adaptivity with $\theta = 0.15$. (Example 5.2)

construct (offline) multiscale basis functions that give a mesh-dependent convergence rate, regardless the heterogeneities of the media. In some applications, one may need to further improve the accuracy of the approximation without additional mesh refinement. In these cases, one needs to enrich the multiscale space by adding more basis functions in the online stage. The online basis functions for mixed CEM-GMsFEM are constructed by using the oversampling technique and the information of local residuals. Moreover, an adaptive enrichment algorithm is presented to reduce error in some selected regions with large residuals. The analysis of the method shows that the convergence rate depends on the constant of exponential decay and a user-defined parameter. Numerical experiments are provided to validate the analytical estimate.

Acknowledgement

Eric Chung’s work is partially supported by Hong Kong RGC General Research Fund (Project 14304217) and CUHK Direct Grant for Research 2017-18.

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