In this paper, we investigate the existence and uniqueness of a coupled system of nonlinear fractional Langevin equations with nonseparated type integral boundary conditions. We use Banach’s and Krasnoselskii’s fixed point theorems to obtain the results. Lastly, we give two examples to show the effectiveness of the main results.

1. Introduction

In the recent few decades, fractional differential equations have been studied by many researchers, and this is due to the importance of this field and its applications in many problems of physics, chemistry, biology, and economy (for more details, we refer the readers to [1–6] and many other references therein).

In particular, fractional Langevin differential equations have been one of the important subjects in the field of fractional differential equations for their rich history (for more information, see [7–9]). Fractional Langevin equations are applied widely in many domains like engineering, physics, and biology (for more details, we give the following references [10–13]).

On the other hand, coupled systems of fractional differential equations are very important to study because they appear naturally in many problems (see [14–18]).

Recently, in [19], the existence and uniqueness of solutions for a coupled system of Riemann–Liouville and Hadamard fractional derivatives of Langevin equation with fractional integral conditions were proved. The existence and uniqueness of the coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary conditions have been studied in [20].

So, in this current article, we study the existence and uniqueness of solutions for a coupled system of fractional Langevin equation as follows:

\[
\begin{cases}
\frac{cD^\beta}{\alpha} \left( \frac{cD^\alpha}{\beta} + \lambda_1 \right) x_1 (t) = f_1 \left( t, x_1 (t), x_2 (t), I^\beta x_2 (t) \right), & t \in [0, 1], \\
\frac{cD^\beta}{\alpha} \left( \frac{cD^\alpha}{\beta} + \lambda_2 \right) x_2 (t) = f_2 \left( t, x_1 (t), x_2 (t), I^\beta x_1 (t) \right), & t \in [0, 1],
\end{cases}
\]
subject to the fractional nonseparated integral boundary conditions:

\[
\begin{align*}
x_1 (0) + \mu_1 x_1 (1) &= \sigma_{11} \int_0^1 g_1 (s, x_1 (s))ds, \\
c^D^\alpha x_1 (0) + \mu_1 c^D^\alpha x_1 (1) &= \sigma_{21} \int_0^1 h_1 (s, x_1 (s))ds, \\
c D^{2\alpha} x_1 (0) + \mu_1 c D^{2\alpha} x_1 (1) &= \sigma_{31} \int_0^1 k_1 (s, x_1 (s))ds, \\
x_2 (0) + \mu_2 x_2 (1) &= \sigma_{12} \int_0^1 g_2 (s, x_2 (s))ds, \\
c^D^\alpha x_2 (0) + \mu_2 c^D^\alpha x_2 (1) &= \sigma_{22} \int_0^1 h_2 (s, x_2 (s))ds, \\
c D^{2\alpha} x_2 (0) + \mu_2 c D^{2\alpha} x_2 (1) &= \sigma_{32} \int_0^1 k_2 (s, x_2 (s))ds,
\end{align*}
\]

where \(0 < \alpha_i < 1, 1 < \beta_i \leq 2, \beta_i > 0, \lambda_i, \mu_i, \sigma_{ij}, \sigma_{ij} \in \mathbb{R}^+ \) with \( \mu_i \neq -1 \) for \( i = 1, 2, \) \( c^D^\alpha \), \( D^{2\alpha} \) are Caputo’s fractional derivatives, and \( f_1, f_2: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \) \( g_1, h_1, k_1, g_2, h_2, k_2: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R} \) are given continuous functions.

To our knowledge, coupled fractional Langevin equations involving nonseparated type integral boundary conditions have not been extensively investigated yet. The main results shown in this paper can be viewed as the extension of the results in [21].

This paper is organized as follows. In Section 2, we recall some notations and several known results. In Section 3, we show the existence and uniqueness of solutions to problems (1) and (2). In Section 4, we give two examples to demonstrate the application of our main results.

2. Preliminaries and Notations

In this section, we introduce some notations, definitions, and lemmas that we need in our proofs later.

**Definition 1** (see [3]). The fractional integral of order \(\alpha > 0\) with the lower limit zero for a function \(f\) can be defined as

\[
I^\alpha f (t) = \frac{1}{\Gamma (\alpha)} \int_0^t (t-s)^{\alpha-1} f (s)ds.
\]

**Definition 2** (see [3]). The Caputo derivative of order \(\alpha > 0\) with the lower limit zero for a function \(f\) can be defined as

\[
c^D^\alpha f (t) = \frac{1}{\Gamma (n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)} (s)ds,
\]

where \(n \in \mathbb{N}, 0 \leq n - 1 < \alpha < n, t > 0\).

**Theorem 1** (see [22]). Let \(M\) be a bounded, closed, convex, and nonempty subset of a Banach space \(X\). Let \(A\) and \(B\) be operators such that

(i) \(Ax + By \in M\) whenever \(x, y \in M\).
(ii) \(A\) is compact and continuous.
(iii) \(B\) is a contraction mapping.

Then, there exists \(z \in M\) such that \(z = Az + Bz\).

**Lemma 1** (see [3]). Let \(\alpha, \beta \geq 0\); then, the following relations hold:

(1) \(I^\alpha p^\beta = (\Gamma (\beta + 1) / \Gamma (\alpha + \beta + 1)) t^{\alpha \beta} \)
(2) \(c^D^\beta p^\alpha = (\Gamma (\beta + 1) / \Gamma (\beta - \alpha + 1)) t^{\beta - \alpha} \)

**Lemma 2** (see [3]). Let \(n \in \mathbb{N}\) and \(n - 1 < \alpha < n\). If \(f\) is a continuous function, then we have

\[I^\alpha c^\alpha f (t) = f (t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}.\]

**Lemma 3.** Let \(y_1, y_2 \in C ([0, 1], \mathbb{R})\); the coupled system

\[
\begin{align*}
c^D^\alpha (c^D^\alpha + \lambda_1) x_1 (t) &= y_1 (t), \quad t \in [0, 1], \\
 c^D^\alpha (c^D^\alpha + \lambda_2) x_2 (t) &= y_2 (t), \quad t \in [0, 1],
\end{align*}
\]

subject to the boundary conditions (2) has a solution given by
\[ x_1(t) = \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (t-s)^{a_1-1} y_1(s) ds - \frac{\lambda_1}{\Gamma(a_1)} \int_0^t (t-s)^{a_1-1} x_1(s) ds + A_{11}(t) \int_0^t h_1(s, x_1(s)) ds \\
+ A_{21}(t) \int_0^1 g_1(s, x_1(s)) ds + A_{31}(t) \int_0^1 k_1(s, x_1(s)) ds + \frac{\lambda_1}{\Gamma(\beta_1 - a_1)} \int_0^1 (1-s)^{\beta_1-1} y_1(s) ds \\
+ \frac{A_{41}(t)}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1-1} y_1(s) ds + \frac{\mu_1 \lambda_1}{(1 + \mu_1) \Gamma(\alpha_1)} \int_0^1 (1-s)^{a_1-1} x_1(s) ds \\
- \frac{H_1}{(1 + \mu_1) \Gamma(\alpha_1 + \beta_1)} \int_0^1 (1-s)^{a_1+\beta_1-1} y_1(s) ds, \tag{7} \]

where

\[ A_{11}(t) = \frac{t^\alpha \sigma_2(1 - \lambda_1(2 - a_1))}{\Gamma(\alpha_1 + 1)(1 + \mu_1)} + \frac{t^\alpha \lambda_1 \sigma_2(2 - a_1)}{\Gamma(2 + a_1) \mu_1} + \frac{\mu_1}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} \Gamma(2 - a_1) \lambda_1 \sigma_2 - \frac{\mu_1 \sigma_2}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} \]

\[ A_{21}(t) = \frac{t^\alpha \lambda_1 \sigma_2}{\Gamma(\alpha_1 + 1)(1 + \mu_1)} - \frac{\sigma_2}{1 + \mu_1} + \frac{\mu_1 \lambda_1 \sigma_2}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} \]

\[ A_{31}(t) = -\frac{t^\alpha \Gamma(2 - a_1) \sigma_2}{\Gamma(\alpha_1 + 1)(1 + \mu_1)} + \frac{t^\alpha \Gamma(2 - a_1) \sigma_2}{\Gamma(\alpha_1 + 2) \mu_1} + \frac{\mu_1 \Gamma(2 - a_1) \sigma_2}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} - \frac{\Gamma(2 - a_1) \sigma_2}{(1 + \mu_1) \Gamma(\alpha_1 + 2)} \]

\[ A_{41}(t) = \frac{t^\alpha \Gamma(2 - a_1) \mu_1}{\Gamma(\alpha_1 + 1)(1 + \mu_1)} - \frac{t^\alpha \Gamma(2 - a_1)}{(1 + \mu_1) \Gamma(\alpha_1 + 2)} + \frac{\mu_1 \Gamma(2 - a_1) \mu_1}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} - \frac{\mu_1^2}{(1 + \mu_1)^2 \Gamma(\alpha_1 + 1)} \]

\[ A_{51}(t) = -\frac{t^\alpha \mu_1}{\Gamma(\alpha_1 + 1)(1 + \mu_1)} - \frac{\mu_1^2}{(1 + \mu_1)^2 \Gamma(\alpha_1 + 1)}, \text{ for } i = 1, 2. \tag{8} \]

**Proof.** Using Lemma 2, we obtain

\[ cD^{\alpha_1} x_1(t) = I^{\alpha(t)} y_1(t) + a_{01} + a_{11} t, \]

\[ cD^{\alpha} x_1(t) = I^{\alpha} y_1(t) + a_{01} + a_{11} t - \lambda_1 x_1(t), \]

\[ x_1(t) = I^{a_1 + \beta_1} y_1(t) + t^{a_1} a_{01} + t^{a_1} a_{11} t - I^{\alpha_1} \lambda_1 x_1(t) + a_{21}, \tag{9} \]

where \( a_{01}, a_{11}, a_{21} \in \mathbb{R} \).

According to the condition \( cD^{\alpha_1} x_1(0) + \mu_1 cD^{\alpha_1} x_1(1) = \sigma_{31} \int_0^1 k_1(s, x_1(s)) ds \), we find that

\[ a_{11} = \Gamma(2 - a_1) \left( \frac{\sigma_{31}}{\mu_1} \int_0^1 k_1(s, x_1(s)) ds + \frac{\lambda_1 \sigma_{21}}{\mu_1} \right) \left( 1 - s \right)^{\beta_1-a_1-1} y_1(s) ds \].

\[ h_1(s, x_1(s)) ds - \frac{1}{\Gamma(\beta_1 - a_1)} \int_0^1 (1-s)^{\beta_1-a_1-1} y_1(s) ds \].

\[ 0 \leq t \leq 1, \quad x_1(0) = x_1(1) = 0, \quad x_1(t) \geq 0. \]
Using the facts that
\[ D^\alpha x_1(0) + \mu_1 D^\alpha x_1(1) = \sigma_{21} \int_0^1 h_i(s, x_1(s))ds \]
and
\[ x_1(0) + \mu_1 x_1(1) = \sigma_{11} \int_0^1 g_i(s, x_1(s))ds, \]
we have

\[ a_{01} = \frac{-\Gamma(2 - \alpha_1)s_{31}}{1 + \mu_1} \int_0^1 k_1(s, x_1(s))ds + \left(\frac{1 - \lambda_1 \Gamma(2 - \alpha_1)}{1 + \mu_1}\right) \int_0^1 h_1(s, x_1(s))ds + \frac{\lambda_1 \sigma_{11}}{1 + \mu_1} \int_0^1 g_1(s, x_1(s))ds \]

\[ + \frac{\Gamma(2 - \alpha_1)\mu_1}{(1 + \mu_1)\Gamma(\beta_1 - \alpha_1)} \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} y_1(s)ds - \frac{\mu_1}{(1 + \mu_1)\Gamma(\beta_1)} \int_0^1 (1 - s)^{\beta_1 - 1} y_1(s)ds, \]

\[ a_{21} = \frac{\mu_1\lambda_1}{(1 + \mu_1)\Gamma(\alpha_1)} \int_0^1 (1 - s)^{\alpha_1 - 1} x_1(s)ds + \left(\frac{\sigma_{11}}{1 + \mu_1} - \frac{\mu_1\lambda_1\sigma_{11}}{\Gamma(\alpha_1 + 1)(1 + \mu_1)^2}\right) \int_0^1 g_1(s, x_1(s))ds \]

\[ - \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1)} \int_0^1 (1 - s)^{\alpha_1 + \beta_1 - 1} y_1(s)ds + \frac{\mu_1^2}{(1 + \mu_1)^2\Gamma(\alpha_1 + 1)} \int_0^1 (1 - s)^{\beta_1 - 1} y_1(s)ds \]

\[ + \Gamma(2 - \alpha_1) \left(\frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + 2)} - \frac{\mu_1^2}{(1 + \mu_1)^2\Gamma(\alpha_1 + 1)}\right) \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} y_1(s)ds \]

\[ + \left[\left(\frac{\mu_1^2}{(1 + \mu_1)^2\Gamma(\alpha_1 + 1)} - \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + 2)}\right) \Gamma(2 - \alpha_1)\lambda_1 \sigma_{21} - \frac{\mu_1 \sigma_{21}}{\Gamma(\alpha_1 + 1)}\right] \int_0^1 h_1(s, x_1(s))ds. \]

Substituting the value of \( a_{01}, a_{11}, \) and \( a_{21}, \) we obtain

\[ x_1(t) = \frac{1}{\Gamma(\alpha_1 + \beta_1)} \int_0^t (t - s)^{\alpha_1 + \beta_1 - 1} y_1(s)ds - \frac{\lambda_1}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1 - 1} x_1(s)ds + A_{31}(t) \int_0^1 h_1(s, x_1(s))ds \]

\[ + A_{21}(t) \int_0^1 g_1(s, x_1(s))ds + A_{31}(t) \int_0^1 k_1(s, x_1(s))ds + \frac{A_{41}(t)}{\Gamma(\beta_1 - \alpha_1)} \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} y_1(s)ds \]

\[ + \frac{A_{31}(t)}{\Gamma(\beta_1)} \int_0^1 (1 - s)^{\beta_1 - 1} y_1(s)ds + \frac{\mu_1 \lambda_1}{(1 + \mu_1)\Gamma(\alpha_1)} \int_0^1 (1 - s)^{\beta_1 - 1} x_1(s)ds \]

\[ + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1)} \int_0^1 (1 - s)^{\alpha_1 + \beta_1 - 1} y_1(s)ds. \]

Analogously, we can deduce that
\[ x_2(t) = \frac{1}{\Gamma(\alpha_2 + \beta_2)} \int_0^t (t-s)^{\alpha_2+\beta_2-1} f_2(s, x_2(s)) ds - \frac{\lambda_2}{\Gamma(\alpha_2)} \int_0^t (t-s)^{\alpha_2-1} x_2(s) ds + A_{12}(t) \int_0^t h_2(s, x_2(s)) ds \\
+ A_{22}(t) \int_0^t g_2(s, x_2(s)) ds + A_{32}(t) \int_0^t k_2(s, x_2(s)) ds + \frac{A_{42}(t)}{\Gamma(\beta_2 - \alpha_2)} \int_0^t (1-s)^{\beta_2-1} x_2(s) ds \\
+ \frac{\mu_2 \lambda_2}{(1 + \mu_2) \Gamma(\alpha_2)} \int_0^t (1-s)^{\alpha_2-1} x_2(s) ds \\
- \frac{\mu_2}{(1 + \mu_2) \Gamma(\alpha_2 + \beta_2)} \int_0^t (1-s)^{\alpha_2+\beta_2-1} y_2(s) ds. \]
\[ r_{11} = \max \left\{ \left[ q_{11} \left( \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 + 1)} + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right) \right] \right\} \\
+ \frac{\lambda_1}{\Gamma(\alpha_1 + 1)} + \frac{\mu_1\lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} + A_{11}q_{51} + A_{21}q_{41} + A_{31}q_{61} \right) \right]\)

\[ r_{12} = \max \left\{ \left[ q_{12} \left( \frac{1}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{A_{42}}{\Gamma(\beta_2 - \alpha_2 + 1)} + \frac{A_{52}}{\Gamma(\beta_2 + 1)} + \frac{\mu_2}{(1 + \mu_2)\Gamma(\alpha_2 + \beta_2 + 1)} \right) \right] \right\} \\
+ \frac{\lambda_2}{\Gamma(\alpha_2 + 1)} + \frac{\mu_2\lambda_2}{(1 + \mu_2)\Gamma(\alpha_2 + \beta_2 + 1)} + A_{12}q_{52} + A_{22}q_{42} + A_{32}q_{62} \right) \right]\)

\[ A_{ij} = \max_{t \in [0, 1]} \left| A_{ij}(t) \right|, \text{ for } i = 1, 2, \ldots, 5 \text{ and } j = 1, 2. \]

Before introducing the main results, we impose some assumptions:

(H1) \( f_1, f_2: [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( h_1, q_1, k_1, h_2, m_1, \kappa_1, k_2, \kappa_2, m_2, \kappa_2 \) are continuous functions.

(H2) There exist positive constants \( q_{11}, q_{21}, q_{31}, q_{22}, q_{32} \) such that for all \( t \in [0, 1] \) and \( x_1, y_1, z_1, y_2, z_2 \in \mathbb{R} \), we have \( |f_1(t, x_1, y_1, z_1) - f_1(t, x_2, y_2, z_2)| \leq q_{11}|x_1 - x_2| + q_{21}|y_1 - y_2| + q_{31}|z_1 - z_2| \).

(H3) There exist positive constants \( q_{41}, q_{51}, q_{61}, q_{42}, q_{52}, q_{62} \) such that \( |g_1(t, x_1) - g_1(t, x_2)| \leq q_{41}|x_1 - x_2|, |g_2(t, x_1) - g_2(t, x_2)| \leq q_{42}|x_1 - x_2|, |h_1(t, x_1) - h_1(t, x_2)| \leq q_{51}|x_1 - x_2|, |h_2(t, x_1) - h_2(t, x_2)| \leq q_{52}|x_1 - x_2|, |k_1(t, x_1) - k_1(t, x_2)| \leq q_{61}|x_1 - x_2|, |k_2(t, x_1) - k_2(t, x_2)| \leq q_{62}|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}. \)

Theorem 2. Suppose that (H1) – (H3) are satisfied.

Then, there exists a unique solution for systems (1) and (2) provided that \( r_{11} + r_{12} < 1. \)

Proof. Define \( \sup_{[0,1]} |f_i(t, 0, 0)| = M_{0i}, \sup_{[0,1]} |g_i(t, 0)| = M_{1i}, \sup_{[0,1]} |h_i(t, 0)| = M_{2i}, \sup_{[0,1]} |k_i(t, 0)| = M_{3i}, \) for \( i = 1, 2. \)

Let \( B_r = \{(x_1, x_2) \in \mathbb{X} \times \mathbb{X} : \|\xi(x_1, x_2)\| \leq r\} \) with

\[ r \geq \frac{r_{21} + r_{22}}{1 - (r_{11} + r_{12})}. \]

where

\[ r_{21} = \frac{M_{01}}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{M_{01}A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} + A_{11}M_{21} + A_{21}M_{11} + A_{31}M_{31} + \frac{A_{51}M_{61}}{\Gamma(\beta_1 + 1)} + \frac{\mu_1M_{01}}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)}, \]

\[ r_{22} = \frac{M_{02}}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{M_{02}A_{42}}{\Gamma(\beta_2 - \alpha_2 + 1)} + A_{12}M_{22} + A_{22}M_{12} + A_{32}M_{32} + \frac{A_{52}M_{62}}{\Gamma(\beta_2 + 1)} + \frac{\mu_2M_{02}}{(1 + \mu_2)\Gamma(\alpha_2 + \beta_2 + 1)}. \]

We prove that \( TB_r \subseteq B_r \).

For \((x_1, x_2) \in B_r, t \in [0, 1], \) we have
\[
\|U_1(x_1, x_2)\| \leq \frac{1}{\Gamma(\alpha_1 + \beta_1)} \int_0^t (t - s)^{\alpha_1 + \beta_1 - 1} \left[ |f_1(s, x_1(s), x_2(s), I^{p_2}x_2(s)) - f_1(s, 0, 0, 0)| + |f_1(s, 0, 0, 0)| \right] ds
\]
\[
+ \frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^t (t - s)^{\alpha_1 - 1} |x_1(s)| ds + |A_{11}(t)| \int_0^1 |h_1(s; x_1(s)) - h_1(s; 0)| + |h_1(s; 0)| ds
\]
\[
+ |A_{21}(t)| \int_0^1 |g_1(s; x_1(s)) - g_1(s; 0)| + |g_1(s; 0)| ds
\]
\[
+ |A_{31}(t)| \int_0^1 |k_1(s; x_1(s)) - k_1(s; 0)| + |k_1(s; 0)| ds + \frac{|A_{41}(t)|}{\Gamma(\beta_1 - \alpha_1)}
\]
\[
\times \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} \left[ |f_1(s, x_1(s), x_2(s), I^{p_2}x_2(s)) - f_1(s, 0, 0, 0)| + |f_1(s, 0, 0, 0)| \right] ds
\]
\[
+ \frac{|\lambda_1|}{\Gamma(\alpha_1)} \int_0^1 (1 - s)^{\alpha_1 - 1} |x_1(s)| ds + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} |x_1|
\]
\[
\leq q_{11}(\|x_1\| + q_{21}\|x_2\| + q_{31}\|x_2\| + M_{01}) + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \|x_1\|
\]
\[
+ A_{11}[q_{31}\|x_1\| + M_{21}] + A_{21}[q_{41}\|x_1\| + M_{11}] + A_{31}[q_{61}\|x_1\| + M_{31}]
\]
\[
+ \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} \left( q_{11}\|x_1\| + q_{21}\|x_2\| + q_{31}\|x_2\| + M_{01} \right)
\]
\[
+ \frac{A_{51}}{\Gamma(\beta_1 + 1)} \left( q_{11}\|x_1\| + q_{21}\|x_2\| + q_{31}\|x_2\| + M_{01} \right) + \frac{\mu_1\lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \|x_1\|
\]
\[
\leq q_{11}\left( \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right)
\]
\[
+ \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)}
\]
\[
+ A_{11}q_{31} + A_{21}q_{41} + A_{31}q_{61}
\]
\[
\|x_1\| \left( \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 - \alpha_1 + 1)} \right) + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)}
\]
\[
\|x_2\| + \frac{M_{01}}{\Gamma(\alpha_1 + \beta_1 + 1)} \leq r_{11}r + r_{21}.
\]

Consequently,

\[
\|U_1(x_1, x_2)\| \leq \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} \left( q_{11}\|x_1\| + q_{21}\|x_2\| + q_{31}\|x_2\| + M_{01} \right) + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} \|x_1\|
\]
\[
+ A_{11}[q_{31}\|x_1\| + M_{21}] + A_{21}[q_{41}\|x_1\| + M_{11}] + A_{31}[q_{61}\|x_1\| + M_{31}]
\]
\[
+ \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1 + 1)} \left( q_{11}\|x_1\| + q_{21}\|x_2\| + q_{31}\|x_2\| + M_{01} \right) + \frac{\mu_1\lambda_1}{(1 + \mu_1)\Gamma(\alpha_1 + 1)} \|x_1\|
\]
\[
\leq q_{11}\left( \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)} \right)
\]
\[
+ \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)} + \frac{|\lambda_1|}{\Gamma(\alpha_1 + 1)}
\]
\[
+ A_{11}q_{31} + A_{21}q_{41} + A_{31}q_{61}
\]
\[
\|x_1\| \left( \frac{1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{A_{41}}{\Gamma(\beta_1 + 1)} + \frac{A_{51}}{\Gamma(\beta_1 - \alpha_1 + 1)} \right) + \frac{\mu_1}{(1 + \mu_1)\Gamma(\alpha_1 + \beta_1 + 1)}
\]
\[
\|x_2\| + \frac{M_{01}}{\Gamma(\alpha_1 + \beta_1 + 1)} \leq r_{11}r + r_{21}.
\]
\[ \|U_1(x_1, x_2)\| \leq r_{12} + r_{22}. \] 

Therefore, we have
\[ \|U(x_1, x_2)\| = \|U_1(x_1, x_2)\| + \|U_2(x_1, x_2)\| \leq (r_{11} + r_{12}) + r_{21} + r_{22} = r. \]

Now, for \((x_1, x_2), (x_1', x_2') \in X \times X\) and for \(t \in [0, 1]\), we get
\[ |U_1(x_1, x_2)(t) - U_1(x_1', x_2')| \leq \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (t - s)^{\alpha - 1} |f_1(s, x_1(s), x_2(s), I^{\beta_2}x_2(s)) - f_1(s, x_1'(s), x_2'(s), I^{\beta_2}x_2'(s))| ds \]
\[ + |A_{11}(t)| \int_0^t |g_1(s, x_1(s)) - g_1(s, x_1'(s))| ds + |A_{31}(t)| \times \int_0^t |k_1(s, x_1(s)) - k_1(s, x_1'(s))| ds \]
\[ + \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (1 - s)^{\alpha - 1 - 1} |f_1(s, x_1(s), x_2(s), I^{\beta_2}x_2(s)) - f_1(s, x_1'(s), x_2'(s), I^{\beta_2}x_2'(s))| ds \]
\[ + \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (1 - s)^{\alpha - 1 - 1} |g_1(s, x_1(s)) - g_1(s, x_1'(s))| ds + \frac{1}{\Gamma(a_1 + \beta_1)} \times \int_0^t (1 - s)^{\alpha - 1} |k_1(s, x_1(s)) - k_1(s, x_1'(s))| ds \]
\[ \leq \frac{1}{\Gamma(a_1 + \beta_1)} \left( q_{11} \|x_1 - x_1'\| + \left( q_{21} + \frac{q_{31}}{\Gamma(P_2 + 1)} \right) \|x_2 - x_2'\| \right) + \frac{1}{\Gamma(a_1 + \beta_1)} \left( q_{11} \|x_1 - x_1'\| + \left( q_{21} + \frac{q_{31}}{\Gamma(P_2 + 1)} \right) \|x_2 - x_2'\| \right) \]
\[ + \frac{A_{41}}{\Gamma(\beta_1 - a_1 + 1)} \left( q_{11} \|x_1 - x_1'\| + \left( q_{21} + \frac{q_{31}}{\Gamma(P_2 + 1)} \right) \|x_2 - x_2'\| \right) \]
\[ + \frac{A_{51}}{\Gamma(\beta_1 + 1)} \left( q_{11} \|x_1 - x_1'\| + \left( q_{21} + \frac{q_{31}}{\Gamma(P_2 + 1)} \right) \|x_2 - x_2'\| \right) + \frac{1}{\Gamma(a_1 + \beta_1)} \left( q_{11} \|x_1 - x_1'\| + \left( q_{21} + \frac{q_{31}}{\Gamma(P_2 + 1)} \right) \|x_2 - x_2'\| \right) \]
\[ \leq \frac{1}{\Gamma(a_1 + \beta_1 + 1)} \left( q_{11} \|x_1 - x_1'\| + \frac{A_{41}}{\Gamma(\beta_1 + 1)} \left( \frac{1}{\Gamma(a_1 + \beta_1 + 1)} \right) \|x_1 - x_1'\| \right) \]
\[ + \frac{A_{51}}{\Gamma(\beta_1 + 1)} \left( \frac{1}{\Gamma(a_1 + \beta_1 + 1)} \right) \left( \frac{1}{\Gamma(\beta_1 + 1)} \right) \|x_2 - x_2'\| \]
\[ \leq r_{11} \left( \|x_1 - x_1'\| + \|x_2 - x_2'\| \right). \]
Analogously, we can also have
\[ |U_2(x_1, x_2)(t) - U_2(x'_1, x'_2)(t)| \leq r_{12}(\|x_1 - x'_1\| + \|x_2 - x'_2\|), \]
which leads to
\[ \|U(x_1, x_2) - U(x'_1, x'_2)\| \leq (r_{11} + r_{12})(\|x_1 - x'_1\| + \|x_2 - x'_2\|). \]

As \( r_{11} + r_{12} < 1 \), the operator \( U \) is a contraction mapping. Then, we deduce that systems (1) and (2) have a unique solution.

\[ \square \]

**Theorem 3.** Assume that \((H_1), (H_4)\) hold. Then, systems (1) and (2) have at least one solution on \([0, 1]\) if \( R < 1 \), where
\[ R = \max \left\{ \frac{\|\lambda_1\|}{\Gamma(a_1 + 1)} + \frac{\|\mu_1\lambda_1\|}{(1 + \mu_1)^2 \Gamma(a_1 + 1)} \right\}. \]

Proof. We define a bounded closed and convex ball \( B_r = \{(x_1, x_2) \in X \times X : \|(x_1, x_2)\| \leq r\} \) with \( r \geq (r_1/1 - R) \), where

\[ r_1 = \frac{\|m_1\|}{\Gamma(a_1 + \beta_1 + 1)} + \frac{A_{11}\|\beta_1\| + A_{21}\|\phi_1\| + A_{31}\|\psi_1\|}{\Gamma(a_1 + \beta_1 + 1)} \]
\[ + \frac{A_{41}\|m_2\|}{\Gamma(\beta_1 - a_1 + 1)} \frac{A_{51}\|m_2\|}{\Gamma(\beta_1 + 1)} \]
\[ + \frac{\|\mu_1\| m_1\|}{(1 + \mu_1)^2 \Gamma(a_1 + \beta_1 + 1)} \frac{\|\mu_2\| m_2\|}{(1 + \mu_2)^2 \Gamma(a_2 + \beta_2 + 1)} \]
\[ + A_{12}\|\beta_2\| + A_{22}\|\phi_2\| + A_{32}\|\psi_2\| \]
\[ + \frac{A_{42}\|m_2\|}{\Gamma(\beta_2 - a_1 + 1)} \frac{A_{52}\|m_2\|}{\Gamma(\beta_2 + 1)} \frac{\|\mu_2\| m_2\|}{(1 + \mu_2)^2 \Gamma(a_2 + \beta_2 + 1)} \].

Let us introduce the decomposition
\[ U(x_1, x_2)(t) = W_1(x_1, x_2)(t) + W_2(x_1, x_2)(t), \]
where
\[ W_1(x_1, x_2)(t) = (T_1(x_1, x_2), R_1(x_1, x_2))(t), \]
\[ W_2(x_1, x_2)(t) = (T_2(x_1, x_2), R_2(x_1, x_2))(t), \]
with

\[ T_1(x_1, x_2)(t) = \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (t - s)^{a_1 \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s)) + A_{11}(t) \int_0^t h_1(s; x_1(s))ds \]
\[ + A_{21}(t) \int_0^t g_t(s; x_1(s))ds + A_{31}(t) \int_0^t k_t(s; x_1(s))ds \]
\[ + \frac{A_{41}(t)}{\Gamma(\beta_1 - a_1)} \int_0^t (1 - s)^{\beta_1 - a_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds \]
\[ + \frac{A_{51}(t)}{\Gamma(\beta_1)} \int_0^t (1 - s)^{\beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds \]
\[ - \frac{\mu_1\lambda_1}{(1 + \mu_1)^2 \Gamma(a_1 + \beta_1)} \int_0^t (1 - s)^{\alpha_1 \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds, \]
\[ T_2(x_1, x_2)(t) = \frac{1}{\Gamma(a_2 + \beta_2)} \int_0^t (t - s)^{a_2 \beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s)) + A_{12}(t) \int_0^t h_2(s; x_2(s))ds \]
\[ + A_{22}(t) \int_0^t g_t(s; x_2(s))ds + A_{32}(t) \int_0^t k_t(s; x_2(s))ds \]
\[ + \frac{A_{42}(t)}{\Gamma(\beta_2 - a_2)} \int_0^t (1 - s)^{\beta_2 - a_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds \]
\[ + \frac{A_{52}(t)}{\Gamma(\beta_2)} \int_0^t (1 - s)^{\beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds \]
\[ - \frac{\mu_2\lambda_2}{(1 + \mu_2)^2 \Gamma(a_2 + \beta_2)} \int_0^t (1 - s)^{\alpha_2 \beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds. \]

\[ R_1(x_1, x_2)(t) = \frac{1}{\Gamma(a_1 + \beta_1)} \int_0^t (t - s)^{a_1 \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s)) + A_{11}(t) \int_0^t h_1(s; x_1(s))ds \]
\[ + A_{21}(t) \int_0^t g_t(s; x_1(s))ds + A_{31}(t) \int_0^t k_t(s; x_1(s))ds \]
\[ + \frac{A_{41}(t)}{\Gamma(\beta_1 - a_1)} \int_0^t (1 - s)^{\beta_1 - a_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds \]
\[ + \frac{A_{51}(t)}{\Gamma(\beta_1)} \int_0^t (1 - s)^{\beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds \]
\[ - \frac{\mu_1\lambda_1}{(1 + \mu_1)^2 \Gamma(a_1 + \beta_1)} \int_0^t (1 - s)^{\alpha_1 \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s))ds, \]
\[ R_2(x_1, x_2)(t) = \frac{1}{\Gamma(a_2 + \beta_2)} \int_0^t (t - s)^{a_2 \beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s)) + A_{12}(t) \int_0^t h_2(s; x_2(s))ds \]
\[ + A_{22}(t) \int_0^t g_t(s; x_2(s))ds + A_{32}(t) \int_0^t k_t(s; x_2(s))ds \]
\[ + \frac{A_{42}(t)}{\Gamma(\beta_2 - a_2)} \int_0^t (1 - s)^{\beta_2 - a_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds \]
\[ + \frac{A_{52}(t)}{\Gamma(\beta_2)} \int_0^t (1 - s)^{\beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds \]
\[ - \frac{\mu_2\lambda_2}{(1 + \mu_2)^2 \Gamma(a_2 + \beta_2)} \int_0^t (1 - s)^{\alpha_2 \beta_2 - 1} f_2(s, x_1(s), x_2(s), I^{\beta_2} x_1(s))ds. \]
For \((x_1, x_2) \in B_r\), we have

\[
|T_1(x_1, x_2)(t) + T_2(x_1)(t)| \leq \frac{1}{\Gamma(\alpha + \beta_1)} \int_0^t (t-s)^{\alpha + \beta_1 - 1} |f_1(s, x_1(s), x_2(s), I^\beta x_2(s))| ds + |A_{11}(t)| \tag{29}
\]

\[
\times \int_0^1 |h_1(s; x_1(s))| ds + |A_{21}(t)| \int_0^1 |\phi_1(s; x_1(s))| ds + |A_{31}(t)| \int_0^1 |k_1(s; x_1(s))| ds
\]

\[
+ \left| \frac{A_{41}(t)}{\Gamma(\beta_1 - \alpha_1)} \right| \int_0^1 (1-s)^{\alpha + \beta_1 - 1} |f_1(s, x_1(s), x_2(s), I^\beta x_2(s))| ds
\]

\[
+ \left| \frac{A_{51}(t)}{\Gamma(\beta_1)} \right| \int_0^1 (1-s)^{\beta_1 - 1} |f_1(s, x_1(s), x_2(s), I^\beta x_2(s))| ds
\]

\[
+ \left| \frac{1}{1 + \mu_1} \right| \int_0^1 (1-s)^{\alpha + \beta_1 - 1} |x_1(s)| ds + \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^1 (1-s)^{\alpha + \beta_1 - 1} |x_1(s)| ds
\]

\[
\leq \frac{1}{\Gamma(\alpha + \beta_1 + 1)} \int_0^t (t-s)^{\alpha + \beta_1 - 1} m_1(s) ds
\]

\[
+ A_{11} \int_0^1 \rho_1(s) ds + A_{21} \int_0^1 \phi_1(s) ds + A_{31} \int_0^1 \psi_1(s) ds
\]

\[
+ \frac{A_{41}}{\Gamma(\beta_1 - \alpha_1)} \int_0^1 (1-s)^{\beta_1 - \alpha_1 - 1} m_1(s) ds + \frac{A_{51}}{\Gamma(\beta_1)} \int_0^1 (1-s)^{\beta_1 - 1} m_1(s) ds
\]

\[
+ \left| \frac{\mu_1}{1 + \mu_1} \right| \int_0^1 (1-s)^{\alpha + \beta_1 - 1} m_1(s) ds
\]

\[
+ \left| \frac{\lambda_1}{\Gamma(\alpha_1)} \right| \int_0^1 (t-s)^{\alpha - 1} |x_1(s)| ds + \left| \frac{\mu_1 \lambda_1}{1 + \mu_1} \right| \int_0^1 (1-s)^{\alpha - 1} |x_1(s)| ds
\]

\[
\leq \left| \frac{m_2}{\Gamma(\alpha_1 + \beta_1 + 1)} \right| + A_{11} \| \rho_1 \| + A_{21} \| \phi_1 \| + A_{31} \| \psi_1 \| + \frac{A_{41} \| m_1 \|}{\Gamma(\beta_1 - \alpha_1 + 1)} + \frac{A_{51} \| m_1 \|}{\Gamma(\beta_1 + 1)}
\]

\[
+ \left| \frac{\mu_1 \lambda_1}{1 + \mu_1} \right| \| m_1 \| + \left| \frac{\lambda_1}{\Gamma(\alpha_1 + \beta_1 + 1)} \right| \| x_1 \| + \| \frac{\lambda_1}{\Gamma(\alpha_1 + 1)} \| \| x_1 \|.
\]

In a similar manner, we have

\[
|R_1(x_1, x_2)(t) + R_2(x_1)(t)| \leq \frac{m_2}{\Gamma(\alpha_2 + \beta_2 + 1)} + A_{12} \| \rho_2 \| + A_{22} \| \phi_2 \| + A_{32} \| \psi_2 \| + \frac{A_{42} \| m_2 \|}{\Gamma(\beta_2 - \alpha_2 + 1)} + \frac{A_{52} \| m_2 \|}{\Gamma(\beta_2 + 1)}
\]

\[
+ \left| \frac{\mu_2 \lambda_2}{1 + \mu_2} \right| \| m_2 \| + \left| \frac{\lambda_2}{\Gamma(\alpha_2 + \beta_2 + 1)} \right| \| x_2 \| + \left| \frac{\lambda_2}{\Gamma(\alpha_2 + 1)} \right| \| x_2 \|.
\]

Further, we obtain

\[
\| W_1(x_1, x_2)(t) + W_2(x_1, x_2) \| \leq R' + r_2 \leq r'. \tag{31}
\]
Hence, \( W_1(x_1, x_2)(t) + W_2(x_1, x_2)(t) \in B_r \).

For \( (x_1, x_2), (x'_1, x'_2) \in B_r \) and \( t \in [0, 1] \), we have

\[
|T_2(x_1) - T_2(x'_1)| \leq \left( \frac{\lambda_1}{\Gamma(\alpha_1 + 1)} + \frac{\mu_1 \lambda_1}{(1 + \mu_1) \Gamma(\alpha_1 + 1)} \right) \|x_1 - x'_1\|, \tag{32}
\]

\[
|R_2(x_2) - R_2(x'_2)| \leq \left( \frac{\lambda_2}{\Gamma(\alpha_2 + 1)} + \frac{\mu_2 \lambda_2}{(1 + \mu_2) \Gamma(\alpha_2 + 1)} \right) \|x_2 - x'_2\|.
\]

Therefore,

\[
\|W_2(x_1, x_2) - W_2(x'_1, x'_2)\| \leq R\|x_1 - x'_1\| + R\|x_2 - x'_2\|
\leq R\|\langle x_1 - x'_1, x_2 - x'_2 \rangle\|, \tag{33}
\]

As \( R < 1 \), then \( W_2 \) is a contraction.

Next, we prove that \( W_1 \) is compact and continuous. The continuity of \( f_1, f_2, h_1, h_2, g_1, g_2, k_1, k_2 \) implies that the operator \( W_1 \) is continuous. Moreover, \( W_1 \) is uniformly bounded on \( B_r \).

Suppose that \( 0 \leq t_1 < t_2 \leq 1 \). We have

\[
|T_1(x_1, x_2)(t_2) - T_1(x_1, x_2)(t_1)| \leq \frac{1}{\Gamma(\alpha_1 + \beta_1)} \left( \int_0^{t_2} (t_2 - s)^{\alpha_1 + \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\alpha_1} x_2(s)) ds - \int_0^{t_1} (t_1 - s)^{\alpha_1 + \beta_1 - 1} f_1(s, x_1(s), x_2(s), I^{\alpha_1} x_2(s)) ds \right) + |A_{11}(t_2) - A_{11}(t_1)|
\]

\[
\times \left( \int_0^1 h_1(s; x_1(s)) ds + |A_{21}(t_2) - A_{21}(t_1)| \right) \int_0^1 g_1(s; x_1(s)) ds + A_{31}(t_2) - A_{31}(t_1) \right) \right) \left( \int_0^1 k_1(s; x_1(s)) ds \right)
\]

\[
+ \frac{|A_{41}(t_2) - A_{41}(t_1)|}{\Gamma(\beta_1 - \alpha_1 + 1)} \left( \int_0^1 (1 - s)^{\beta_1 - \alpha_1 - 1} f_1(s, x_1(s), x_2(s), I^{\beta_1} x_2(s)) ds \right)
\]

\[
+ \frac{\|m_1\|}{\Gamma(\beta_1 - \alpha_1 + 1)} \left( \|A_{41}(t_2) - A_{41}(t_1)\| + \|A_{51}(t_2) - A_{51}(t_1)\| \right).
\tag{34}
\]

Similarly, we obtain that

\[
|R_1(x_1, x_2)(t_2) - R_1(x_1, x_2)(t_1)| \leq \frac{\|m_2\|}{\Gamma(\alpha_2 + \beta_2 + 1)} \left( t_2^{\alpha_2 + \beta_2} - t_1^{\alpha_2 + \beta_2} \right) + \|\psi_2\| \|A_{12}(t_2) - A_{12}(t_1)\|
+ \|\phi_2\| \|A_{22}(t_2) - A_{22}(t_1)\| + \|\psi_2\| \|A_{32}(t_2) - A_{32}(t_1)\|
+ \frac{\|m_2\|}{\Gamma(\beta_2 - \alpha_2 + 1)} \left( |A_{42}(t_2) - A_{42}(t_1)| \right) \|A_{52}(t_2) - A_{52}(t_1)\| + \frac{\|m_2\|}{\Gamma(\beta_2 + 1)} \left( |A_{52}(t_2) - A_{52}(t_1)| \right).
\tag{35}
\]
Therefore, the operator \( W_1 \) is equicontinuous. Thus, \( W_1 \) is relatively compact on \( B_r \). Then by Arzela–Ascoli theorem, the operator \( W_1 \) is compact on \( B_r \). In conclusion, all terms of Krasnoselskii’s theorem have been applied perfectly. Hence, (1) and (2) have at least one solution on \( B_r \).

4. Examples

Example 1. Consider the following system of fractional Langevin:

\[
\begin{aligned}
&cD^{4/3}(cD^{1/3} + \frac{1}{300})x_1(t) = \frac{1}{500 + t^2} \left( \sin(x_1(t)) + \frac{|x_2(t)|}{1 + |x_2(t)|} + \frac{1}{\Gamma(15/2)} \int_0^t (t-s)^{13/2} x_2(s)ds \right), \quad t \in [0, 1], \\
&cD^{8/7}(cD^{1/7} + \frac{1}{400})x_2(t) = \frac{1}{250 + t^2} \left( \frac{1}{2}\sin(x_1(t)) + \frac{|x_2(t)|}{2 + 2|x_2(t)|} + \frac{1}{\Gamma(15/2)} \int_0^t (t-s)^{13/2} x_2(s)ds \right), \quad t \in [0, 1], \\
x_1(0) + x_1(1) = \frac{1}{200} \int_0^1 \frac{|x_1(s)|}{300 + |x_1(s)|} ds, \\
cD^{1/3}x_1(0) + cD^{1/3}x_1(1) = \frac{1}{200} \int_0^1 \left( \frac{1}{s+2} \right)^3 \frac{|x_1(s)|}{300 + |x_1(s)|} ds, \\
cD^{2/3}x_1(0) + cD^{2/3}x_1(1) = \frac{1}{200} \int_0^1 \left( \frac{1}{s+4} \right)^2 \frac{|x_1(s)|}{300 + |x_1(s)|} ds, \\
x_2(0) + x_2(1) = \frac{1}{200} \int_0^1 \frac{|x_2(s)|}{300 + |x_2(s)|} ds, \\
cD^{1/7}x_2(0) + cD^{1/7}x_2(1) = \frac{1}{200} \int_0^1 \left( \frac{1}{s+8} \right) \frac{|x_2(s)|}{300 + |x_2(s)|} ds, \\
cD^{2/7}x_2(0) + cD^{2/7}x_2(1) = \frac{1}{200} \int_0^1 \left( \frac{1}{s+16} \right) \frac{|x_2(s)|}{300 + |x_2(s)|} ds,
\end{aligned}
\]
where

\[ \begin{align*}
\beta_1 &= \frac{4}{3}, \\
\alpha_1 &= \frac{1}{3}, \\
\beta_2 &= \frac{8}{7}, \\
\alpha_2 &= \frac{1}{7}, \\
P_1 &= P_2 = \frac{15}{2}, \\
\lambda_1 &= \frac{1}{300}, \\
\lambda_2 &= \frac{1}{400}, \\
\mu_1 &= 1, \\
\mu_2 &= 1, \\
\sigma_{11} &= \sigma_{21} = \sigma_{12} = \sigma_{22} = \sigma_{32} = \frac{1}{200}
\end{align*} \]

\[ \begin{align*}
f_1(t, x, y, z) &= \frac{1}{500 + t^2} \left( \sin(x(t)) + \frac{|y(t)|}{1 + |y(t)|} + z(t) \right), \\
f_2(t, x, y, z) &= \frac{1}{250 + t^2} \left( \frac{1}{2} \sin(x(t)) + \frac{|y(t)|}{2 + 2|y(t)|} + \frac{z(t)}{2} \right), \\
g_1(t, x) &= \frac{|x(t)|}{300 + |x(t)|}, \\
g_2(t, x) &= \frac{|x(t)|}{300 + |x(t)|}, \\
h_1(t, x) &= \left( \frac{1}{t + 2} \right)^3 \frac{|x(t)|}{30 + |x(t)|}, \\
h_2(t, x) &= \left( \frac{1}{t + 8} \right)^3 \frac{|x(t)|}{30 + |x(t)|}, \\
k_1(t, x) &= \left( \frac{1}{t + 4} \right)^2 \frac{|x(t)|}{30 + |x(t)|}, \\
k_2(t, x) &= \left( \frac{1}{t + 16} \right) \frac{|x(t)|}{30 + |x(t)|}
\end{align*} \]
Clearly, \( q_{11} = q_{21} = q_{12} = q_{22} = q_{31} = q_{32} = (1/500), \)
\( q_{41} = q_{42} = (1/300), \quad q_{51} = q_{52} = (1/240), \)
and \( q_{61} = q_{62} = (1/480); \) furthermore, we have
\[
r_{11} + r_{12} = \max(0.007421, 0.0128) + \max(0.012255, 0.007583) = 0.026 < 1.
\] (38)

Thus, by Theorem 2, system (36) has a unique solution.

**Example 2.** Consider the following problem:

\[
\begin{aligned}
\mathcal{D}^{3/2}\left(\mathcal{D}^{1/2} + \frac{1}{600}\right)x_1(t) &= \frac{1}{t^2 + 4} \left( \frac{t^2|x_1(t)|}{3|x_1(t)|} + \frac{|x_2(t)|}{6|x_2(t)| + 10} \right) + \frac{1}{\Gamma(4/3)} \int_0^t (t-s)^{1/3} \frac{ds}{1 + x_2^2(s)}, \quad t \in [0, 1], \\
\mathcal{D}^{4/3}\left(\mathcal{D}^{1/3} + \frac{1}{700}\right)x_2(t) &= \frac{1}{1 + t^4} \left( \frac{t^4|x_1(t)|}{4|x_1(t)| + 10} + \frac{|x_2(t)|}{4|x_2(t)| + 6} \right) + \frac{1}{\Gamma(4/3)} \int_0^t (t-s)^{1/3} \frac{ds}{1 + x_1^2(s)}, \quad t \in [0, 1], \\
x_1(0) + x_1(1) &= \frac{1}{300} \int_0^1 \frac{1}{1 + 1000} \frac{|x_1(s)|}{100 + |x_1(s)|} ds, \\
\mathcal{D}^{1/2}x_1(0) + \mathcal{D}^{1/2}x_1(1) &= \frac{1}{300} \int_0^1 \frac{1}{1 + 1000} \frac{|x_1(s)|}{100 + |x_1(s)|} ds, \\
x_2(0) + x_2(1) &= \frac{1}{300} \int_0^1 \frac{1}{1 + 2000} \frac{|x_2(s)|}{200 + |x_2(s)|} ds, \\
\mathcal{D}^{1/3}x_2(0) + \mathcal{D}^{1/3}x_2(1) &= \frac{1}{300} \int_0^1 \frac{1}{1 + 8000} \frac{|x_2(s)|}{8000 + |x_2(s)|} ds,
\end{aligned}
\] (39)

where
\[
\beta_1 = \frac{3}{2}, \\
\alpha_1 = \frac{1}{2}, \\
\beta_2 = \frac{4}{3}, \\
\alpha_2 = \frac{1}{3}, \\
p_1 = p_2 = \frac{4}{3}, \\
\lambda_1 = \frac{1}{600}, \\
\lambda_2 = \frac{1}{700}, \\
\mu_1 = 1, \\
\mu_2 = 1, \\
\sigma_{11} = \sigma_{21} = \sigma_{12} = \sigma_{22} = \frac{1}{300}, \text{ and } (40)
\]

\[
f_1(t, x, y, z) = \frac{1}{t^2 + 4} \left( \frac{t^2|x(t)|}{(3|x(t)| + 1)} + \frac{|y(t)|}{(6|y(t)| + 10)} + \frac{1}{1 + z^2(t)} \right),
\]

\[
f_2(t, x, y, z) = \frac{1}{1 + t^4} \left( \frac{t^4|x(t)|}{(4|x(t)| + 10)} + \frac{|y(t)|}{(4|y(t)| + 10)} + \frac{1}{1 + z^2(t)} \right),
\]

\[
g_1(t, x) = \frac{1}{t + 100} \frac{|x(t)|}{300 + |x(t)|},
\]

\[
g_2(t, x) = \frac{1}{t + 200} \frac{|x(t)|}{300 + |x(t)|},
\]

\[
h_1(t, x) = \left( \frac{1}{t + 20} \right)^3 \frac{|x(t)|}{30 + |x(t)|},
\]

\[
h_2(t, x) = \left( \frac{1}{t + 8000} \right) \frac{|x(t)|}{30 + |x(t)|},
\]

\[
k_1(t, x) = \left( \frac{1}{t + 40} \right)^2 \frac{x(t)}{30 + |x(t)|},
\]

\[
k_2(t, x) = \left( \frac{1}{t + 1600} \right) \frac{|x(t)|}{30 + |x(t)|}.
\]
After calculating, we obtain $R \approx 0.0029 < 1$.
So, by Theorem 3, problem (39) has at least one solution.

5. Conclusion

In this paper, we have investigated the existence and uniqueness results for a coupled system of nonlinear fractional Langevin equations supplemented with nonseparated integral boundary conditions by using the Banach contraction principle and Krasnosel’skii’s fixed point theorem. Finally, we gave two examples to prove the validity of our results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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