Spectrum of kinematic fast dynamo operator in Ricci flows

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Abstract

Spectrum of kinematic fast dynamo operators in Ricci compressible flows in Einstein 2-manifolds is investigated. A similar expression, to the one obtained by Chicone, Latushkin and Montgomery-Smith (Comm Math Phys (1995)) is given, for the fast dynamo operator. The operator eigenvalue is obtained in a highly conducting media, in terms of linear and nonlinear orders of Ricci scalar. Eigenvalue spectra shows that there is a relation between the Ricci scalar and expansion of the flow. Spatial 3-Einstein manifold section of Friedmann-Robertson-Walker (FRW) is obtained in the limit of ideal plasma. If the trace of the Ricci curvature tensor is negative, a contraction of the inflationary phase of the universe takes place, and the dynamo action takes place. When the universe expands a decaying magnetic field or non-dynamo is obtained. As in Latushkin and Vishik (Comm Math Phys (2003)) the Lyapunov exponents in kinematic dynamos is also investigated. Since positive curvature scalar are preserved under Ricci flow, it is shown that fast dynamos are preserved under this same flow.
I Introduction

Investigations in the Riemannian geometry of magnetic dynamos ranged from the early investigations of Arnold, Zeldovich, Ruzmaikin and Sokoloff [1] to the more recently papers by Chicone, Latushkin and their group [2] on the fast dynamo existence. Investigation of Riemannian geometry applications to plasma dynamos and anti-fast dynamo Vishik’s theorem [3] has been performed by the Garcia de Andrade [4]. Chicone et al [2] have also shown that the fast dynamo operator spectrum for an ideally conducting fluid and the spectrum of the group acting on the associated compact Riemannian manifold. Yet more recently Latushkin and Vishik [5] investigated Lyapunov exponents in kinematic dynamos. On the more topological and dynamical system settings, Young and Klapper [6] has investigated the topological entropy systems and dynamo action. In this last paper Young and Klapper used the same concept of topological entropy used recently, by Fields medalist Grisha Perelman [7] to prove the long standing Poincare conjecture. In his important proof, Perelman has made used the concept of Ricci flows, proposed by Hamilton in 1982 [8]. In this paper a proof is given of the following theorem:

**Theorem:** Let $\mathcal{M}$ be a two dimensional Riemannian manifold $(\mathcal{M}, g)$, not necessarily closed, endowed with a metric $g(t)$, is given in the interval $t \in [a, b]$ in the field of real numbers $\mathbb{R}$ of an Einstein 3-manifold and a Ricci flow given by the equation

**Definition:**

$$\frac{\partial g}{\partial t} = -2\text{Ric} \quad (I.1)$$

where $\text{Ric}$ represents the Ricci tensor with components $R_{ij}$ on a chart $\mathcal{U}$, submanifold of the tangent space $T\mathcal{M}$ with coordinates $(i, j = 1, 2)$, thus the kinematic fast dynamo operator (where back reactions due to Lorentz forces are negligable), in ideally highly conducting fluid is

$$\Gamma_0 = \frac{\partial}{\partial t} = (\theta + \text{Ric}) \quad (I.2)$$

while the its spectrum eigenvalues are given by

$$\lambda_0 = \frac{1}{2}[(R - \theta) \pm \sqrt{\frac{3}{4}R^2 + \theta R}] \quad (I.3)$$

where $R$ is the trace of the $\text{Ric}$ and $\gamma$ is the growth rate usually found in dynamo theory, and the magnetic field obeys the rule $|B| \approx e^{\lambda t}$. In general where the diffusion constant $\eta$
does not vanish the dynamo operator is written as $\Gamma_q B = \text{curl}(v \times B) + \eta \Delta B$. The absence of shear and vorticity traces are due to the fact that they vanish in geodesic flows. Here, $g$ is the Riemann metric, over manifold $\mathcal{M}$, and the parameter $t$ in the Riemann metric $g(t)$, is given in the interval $t \in [a, b]$ in the field of real numbers $\mathbb{R}$. On this local chart, the expression (I.1) can be expressed as [7]

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \quad (I.4)$$

where $\text{Ric}$, is the Ricci tensor, whose components $R_{ij}$. From this expression, one defines the eigenvalue problem as

$$R_{ij} \chi^j = \lambda \chi^i \quad (I.5)$$

where $(i, j = 1, 2)$. Substitution of the Ricci flow equation (I.2) into this eigenvalue expression and cancelling the eigendirection $\chi^i$ on both sides of the equation yields

$$\frac{\partial g_{ij}}{\partial t} = -2\lambda g_{ij} \quad (I.6)$$

Solution of this equation yields the de Sitter-Lyapunov expression for the metric

$$g_{ij} = \exp[-2\lambda t] \delta_{ij} \quad (I.7)$$

where $\delta_{ij}$ is the Kroenecker delta. Note that in principle if $\lambda \leq 0$ the metric grows without bounds, and in case it is negative it is bounded as $t \to \infty$. Recently, Thiffeault [9] has used a similar Lyapunov exponents expression in Riemannian manifolds to investigate chaotic flows, without attention to dynamos or Ricci flow. These exponents are obtained in the metric as

$$g = \Lambda_{11} e_1 e_1 + \Lambda_{22} e_2 e_2 \quad (I.8)$$

Thus one has proven the following lemma:

**Lemma II.1:**

If $\lambda_i$ is an eigenvalue spectra of the $\text{Ric}$ tensor, the finite-time Lyapunov exponents spectra is given by

$$\lambda_i = -\gamma_i \leq 0 \quad (I.9)$$

In the next section I shall use this argument to work with the de Sitter metric

$$ds^2 = -dt^2 + e^{M}(dx^2 + dy^2) \quad (I.10)$$
In the next and last section, one shall be concerned with the proof of the above theorem through the self-induction equation in Ricci flows in Einstein 2-manifolds.
II Kinematic fast dynamo operator in Ricci flows

In this section one shall use the formalism of dynamo theory as in the book by Arnold and
Khesin [10] in chaotic flows and Anosov cat map [11]. Let us consider first the kinematic
self-induced equation in Euclidean three-dimensional space $E^3$, as dynamo equation. In
the mathematician notation, Arnold writes down the Poisson bracket

$$ [v, B] = -\text{curl}(v \times B) \quad \text{(II.11)} $$

between the flow $v$ and the magnetic field $B$. The self-induced magnetic field equation
reads

$$ \frac{\partial B}{\partial t} = -[v, B] + \text{div}(v)B + \eta \Delta B \quad \text{(II.12)} $$

where $\eta$ is the plasma resistivity and $\Delta := \nabla^2$ is the Laplacian operator. In Chicone et al
work the Riemannian manifold was confined to incompressible dynamo flows, where the
second term on the RHS of expression (II.11) vanishes, and along with the solenoidicity
of the magnetic field

$$ \text{div}B = 0 \quad \text{(II.13)} $$

they form a solenoidal vector field in Riemannian manifold. Here incompressibility of the
flow is not assumed and only solenoidal property of the magnetic field is kept. As one shall
see, is exactly this compressible flow term that is responsible for the introduction of the
Ehlers-Sachs optical parameters of shear, vorticity and expansion of the lower dimensional
universe. Earlier a two dimensional model of a dynamo flow has been numerically tested,
by Otani [12]. More recently, an example of a two dimensional Moebius dynamo flow, has
also been obtained by Shukurov, Stepanov and Sokoloff [13] to model the Perm dynamo
experiment. Let us now consider, the diffusive term given by

$$ \Delta B = \frac{1}{\sqrt{g}} \partial_i[\sqrt{g}g^{ij}\partial_j B] \quad \text{(II.14)} $$

which expanded in the frame $e_i$ with

$$ B = B^i e_i \quad \text{(II.15)} $$

yields

$$ \Delta B = [g^{ij} \partial_i \partial_j B^p + B^k[\partial_i \gamma^p_{jk}g^{ij} + \gamma^l_{jk} \gamma^p_{dl}g^{ij}] + [\gamma^p_{jk}g^{ij} \partial_i B^k]]e_p \quad \text{(II.16)} $$
Here $\gamma^l_{jk}$ represents the Ricci rotation coefficients (RRCs) analogous to the Riemann-Christoffel symbols, which is defined by

$$\partial_k e_i = \gamma^l_{ki} e_j$$ (II.17)

The Christoffel symbols

$$\Gamma^i_{jk} = g^{il}[\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}]$$ (II.18)

do not appear in the computations, since we have assumed, that the trace of the Christoffel symbols vanish. To complete the derivation of the self-induction equation it remains to obtain the diffusion free part of the self-induced equation above, which in general curvilinear coordinates $x^i \in U_i$, of the sub-chart $U_i$ of the manifold, in the rotating frame reference of the flow $e_i$, reads

$$\frac{dB}{dt} = <B, \nabla v > v - (div v) B$$ (II.19)

Before this derivation, let us now introduce the Ricci tensor into play, by considering the following trick

$$\frac{d}{dt}[g^{il}g_{lk}] = \frac{d}{dt}[\delta^i_k] = 0$$ (II.20)

which can be applied to the expression

$$\frac{dB}{dt} = \frac{d}{dt}(g^{ik}B_k e_i)$$ (II.21)

to obtain

$$\frac{d}{dt}(g^{ik}B_k) = \frac{d}{dt}(g^{ik})B_k + g^{ik}\frac{d}{dt}B_k$$ (II.22)

Now by making use of the Ricci flow equation above into this last expression, yields

$$\frac{d}{dt}g^{ik}B_k = -2R^{ik}B_k + g^{ik}\frac{d}{dt}B_k - div v B^i$$ (II.23)

Now let us consider that the Ehlers-Sachs optical scalar of the fluid appears in the gradient of the flow expression such as

$$(\nabla v)_p = \partial_p v_l = \Omega_l + \sigma_{pl} - \frac{1}{3}\theta g_{lp} - A_p v_l$$ (II.24)

where A is the acceleration of the flow. Taking the trace of this expression, one obtains the $div v$ expression as given by the gradient strain

$$Tr(\nabla v) = div v = \partial_p v^p = \sigma - \theta$$ (II.25)
where, $\sigma := Tr(\sigma_{ij})$ is the trace of the shear tensor $\sigma_{ij}$ while, $\theta$ is the expansion of the flow. In cosmology they vanish and the acceleration is orthogonal to velocity, that is the reason they shall disappear throughout from the rest of the paper computations. In the language of dynamo theory, this scalar represents the stretching of the dynamo flow. The magnetic field here is stretched by a cosmological Ricci two-dimensional flow, what Arnold has called a particle stretching in the flow. From evolution of the reference frame

$$\frac{d\mathbf{e}_i}{dt} = \omega^{ij}_i \mathbf{e}_j \quad (\text{II.26})$$

where we have used the following expression

$$\partial_k \mathbf{e}_i = \Gamma^{ji}_{ki} \mathbf{e}_j \quad (\text{II.27})$$

The eigenvalue problem

$$\Gamma_\eta \mathbf{B} = \lambda \mathbf{B} \quad (\text{II.28})$$

yields the following algebraic second order eigenvalue equation

$$\lambda^2 - 2[R - \theta] \lambda - [R - \theta]^2 = 0 \quad (\text{II.29})$$

where one has consider that in Einstein constant curvature manifold the components of the Ricci tensor components coincide as $R_{11} = R_{22} = R$. Note that a simple solution yields the result (I.2) for the eigenvalue $\lambda_0$ above and the theorem is proved. Just for comparison with the Chiconne and Latushkin [14] case and convenience of the reader, one repeats here their eigenvalue expression

$$\lambda_\eta = \frac{1}{2} \left[ -\eta(1 + \kappa^2) + \sqrt{-4\kappa + \eta(1 - \kappa^2)} \right] \quad (\text{II.30})$$

Where $\kappa$ is the surface curvature scalar similar to the Ricci scalar, $Tr\mathbf{Ric}$ used in this paper. This is in agreement with Woolgar [15] argument that in Ricci flows, Einstein gravity is similar, but cosmology departures from the more traditional relativistic cosmology. One notes that in the absence of diffusion $\eta$ vanishes, and this expression reduces to

$$\lambda_0 = \frac{1}{2} \sqrt{-4\kappa} \quad (\text{II.31})$$
Note that when second-order terms in the Ricci scalar are dropped, the eigenvalue (I.3) is given by
\[ \lambda_0 = [R - \theta](\pm \sqrt{2} - 1) \] (II.32)
which is similar to the equation (II.31). This formula shows that if the eigenvalue \( \lambda > 0 \) implies that the curvature scalar \( \kappa \) is negative as in Anosov [11] compact Riemannian of constant curvature. A corollary can be obtained by investigating the eigenvalue degeneracy and the discriminant \( \Delta \) of the second order equation given by
\[ \Delta = 8(R - \theta) \] (II.33)
Note that the degenerate eigenvalues implies that \( \theta = R = 2\Lambda \). This shows that when the universe expands, the curvature grows. This contradicts present cosmological Hubble satellite observation, which shows that based in inflationary models the actual universe would be flat, thus the degenerate eigenvalues cannot appear in the fast cosmic dynamo in Ricci flows. When the dynamo action is present say by assuming that the eigenvalue \( \lambda_- > 0 \) [16], thus expression (II.32) tells us that
\[ [\theta - TrRic] > 0 \] (II.34)
which implies that the Ricci curvature of the universe is bound by the expansion or contraction depending on the \( \theta \) sign. When \( \theta < 0 \) or a contraction phase of the universe is present, the following inequality is present
\[ TrRic < \theta < 0 \] (II.35)
Transitivity property, in turn implies that, the Ricci curvature scalar is negative, which agrees with Chicone et al result [2], that in Riemannian compact manifold of negative curvature the fast dynamo action is present. On the other hand when the expansion is effective, or \( \theta > 0 \), the decaying of the magnetic field, or non-dynamo, allows us to consider that \( \lambda_- < 0 \), which from expression (II.32) implies that
\[ TrRic > \theta \] (II.36)
Thus if the universe has a positive curvature this is compatible with the Ricci flow cosmic fast dynamo, since then the expansion factor \( \theta \) is positive. This phenomenon is physically
predicted also in general relativistic cosmology as shown, recently by Barrow and Tsagos [16]. Thus the following corollary has been proved.

**Corollary II.1:**

Expansion of the non-relativistic Ricci flow two dimensional universe cannot support dynamo action, while a contracting phase of de Sitter-like model can. Since, as shown by R S Hamilton [8], the curvature scalar is preserved under Ricci flow, here one has proven the following lemma:

**Lemma II.2:**

Since the positive curvature scalar is preserved under the Ricci flow, and the fast dynamo action is supported in this case, the fast dynamo action is preserved under the Ricci flow. When the Ricci flows obey the Einstein 2-manifold condition

$$\text{Ric} = \Lambda g$$

The value of the Ricci curvature $R$ should be simply substituted by the cosmological constant.

Taking into account the magnetic energy $\epsilon$ as

$$\epsilon = \int B^2 \mu$$

one is able to simply shown that in terms of the 2D Riemann metric components

$$\epsilon = \int B^i g_{ij} B^j \mu$$

Here $\mu$ represents the volume form. Since, by definition fast dynamo action corresponds to the growth of magnetic energy in time as $\frac{d\epsilon}{dt} \geq 0$, this amount has to be computed by performing the partial time derivative of the expression (II.39). Actually the equal sign in the last condition represents the lower limit of marginal dynamos, where the magnetic energy integral remains constant. This computation yields

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial [\int B^i g_{ij} B^j \mu]}{\partial t}$$

Expansion of the RHS of this expression shows clearly now where the Ricci flow eigenvalue effect is going to appear. A simple computation, shows that the energy integral confirms
the dynamo action. Throughout the paper the diffusion term was not explicitly computed since because we use the limit of diffusion free to check for the presence of slow dynamos, which seems not exist globally in the universe. Note that the in the de Sitter case the magnetic can be written as

\[ B^i = B^i_0 e^{[2T_v \text{Ric} - \theta]t} \]  

(II.41)

which is equivalent to the most familiar cosmologists expression

\[ B^i = B^i_0 e^{[2\Lambda - \theta]t} \]  

(II.42)

Substitution of this expression into the magnetic energy one, is possible to obtain the result

\[ \epsilon \approx e^{2[2\Lambda - \theta]t} \]  

(II.43)

which confirms the growth rate of magnetic energy of the fast dynamo as long as the following condition be fulfilled. Anti-de Sitter effective [16] spacetime, of course contributes to slow down magnetic field as negative exponents contributes to the decay of magnetic field in the effective universe.
III Conclusions

Stretching magnetic field lines by plasma flows has led to dynamo action, as shown recently by M Nunez [17] in the case of the nonlinear hydromagnetic dynamos. Here by making use of the spectrum of the fast dynamo operators it is shown that de Sitter-Lyapunov metric in Ricci flows play the same role in two space section of Einstein-de Sitter-Ricci flows cosmology. As pointed out by Woolgar [15] Einstein gravity called general relativity does not show any departure from Ricci flow gravity, however, the same does not happen in Ricci flow cosmology. The example investigated here shows that, though small differences appears in the fast dynamo operator and eigenvalue spectra, these differences actually represents no contradict with the cosmological experiments. Besides the fast dynamo action for de Sitter or closed (2+1)-spacetime Ricci flows, where the cosmological constant $\Lambda > 0$, which is a new result, one is able to reproduce the BT magnetic field decay in (2+1) real spacetime of GR cosmology. One of the main results of the paper, is that dynamo action is transported along Ricci flow, and that the eigenvalues of fast dynamo operator are also Ricci scalar dependent. For the interested mathematician, a detailed account of GR cosmological dynamos is contained in Widrow [18]. A more detailed account of Einstein manifolds can be seen in Besse book [19].

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References

[1] V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, JETP 81 (1981), n. 6, 2052. V. Arnold, Ya B. Zeldovich, A. Ruzmaikin and D.D. Sokoloff, Doklady Akad. Nauka SSSR 266 (1982) n6, 1357. V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, chapter V, Applied Mathematics Sciences 125 (1991) Springer.

[2] C. Chicone and Yu Latushkin, Evolution Semigroups in Dynamical systems and differential equations, American Mathematical Society, AMS-(1999). C. Chicone and Yu Latushkin and S. Montgomery-Smith, Comm. Math. Physics 173 379 (1995). C. Chicone and Yu Latushkin, Proc. of the American Mathematical Society 125, N. 11,3391 (1997). G Perelmann, The entropy formula for the Ricci flow and its geometric applications, Los Alamos arxives math-DG /0211159.

[3] M. Vishik, Izv Acad Science, USSR Phys Solid Earth 24 173 (1988).

[4] L C Garcia de Andrade, The role of stretching and curvature in fast dynamo plasmas in Riemannian space, Phys Plasmas 15 (2008) in press, and Phys Plasmas 14, 102902, (2007).

[5] Yu Latushkin and M Vishik, Comm Math Phys (2003).

[6] I Klapper and L S Young, Comm Math Phys (1995).

[7] G Perelmann, The entropy formula for the Ricci flow and its geometric applications, Los Alamos arxives math-DG /0211159.

[8] R S Hamilton, J Diff Geom, 17,255 (1982). B Chow and D Knopf, Ricci Flows: An introduction, (2004) AMS, New York.

[9] J L Thiffeault, Differential constraints in Chaotic Flows on Curved Manifolds, (2002) arXiv:nlin/0209042.v1.

[10] V Arnold and B Khesin, Topological methods in Hydrodynamics (1998) Springer, New York.
[11] D Anosov, Geodesic Flows on Closed Riemannian Manifolds of negative Curvature, (1967) Steklov Mathematical Institute, Moscow.

[12] M Otani, J Fluid Mech (1990).

[13] A Shukurov, R Stepanov and D D Sokoloff, Phys. Rev. E 78, 025301 (2008).

[14] C. Chicone and Yu Latushkin, Evolution Semigroups in Dynamical systems and differential equations, American Mathematical Society, AMS-(1999).

[15] E. Woolgar, Some applications of the Ricci flows in Physics, Los Alamos arxiv: 0708.2144.

[16] J D Barrow and C Tsagas, Phys Rev D 77,107302 (2008). B Bassett,G Pollifrone, S Tsujikawa and F Viniegra, Pre-heating: Cosmic dynamo?, arxiv:astro-ph/0010628v3.

[17] M Nuñez, J Phys A,8903 (2003).

[18] L Widrow, Rev Mod Phys 74 (2002) 775.

[19] A Besse, Einstein manifolds (2007) Springer, New York-Berlin.