Influence-free states on compound quantum systems

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Abstract

Consider two spatially separated agents, Alice and Bob, each of whom is able to make local quantum measurements, and who can communicate with each other over a purely classical channel. As has been pointed out by a number of authors, the set of mathematically consistent joint probability assignments ("states") for such a system is properly larger than the set of quantum-mechanical mixed states for the joint Alice-Bob system. Indeed, it is canonically isomorphic to the set of positive (but not necessarily completely positive) linear maps $\mathcal{L}(H_A) \to \mathcal{L}(H_B)$ from the bounded linear operators on Alice’s Hilbert space to those on Bob’s, satisfying $\text{Tr}(\phi(1)) = 1$. The present paper explores the properties of these states. We review what is known, including the fact that allowing classical communication between parties is equivalent to enforcing "no-instantaneous-signalling" ("no-influence") in the direction opposite the communication. We establish that in the subclass of “decomposable” states, i.e. convex combinations of positive states with ones whose partial transpose is positive, the extremal (i.e. pure) states are just the extremal positive and extremal partial-transpose-positive states. And we show that two such states, shared by the same pair of parties, cannot necessarily be combined as independent states (their tensor product) if the full set of quantum operations is allowed locally to each party: this sort of coupling does not mix well with the standard quantum one. We use a framework of “test spaces” and states on these, well suited for exhibiting the analogies and deviations of empirical probabilistic theories (such as quantum mechanics) from classical probability theory. This leads to a deeper understanding of some analogies between quantum mechanics and Bayesian probability theory. For example, the existence of a “most Bayesian” quantum rule for updating states after measurement, and its association with the situation when information on one system is gained by measuring another, is a case of a general proposition holding for test spaces combined subject to the no-signalling requirement.

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1 Introduction

In the simplest formulation of orthodox non-relativistic quantum mechanics, a physical system is represented by a separable, complex Hilbert space $H$. The possible outcomes of (maximal) discrete measurements on such a system are represented by unit vectors in $H$, with each orthonormal basis representing the set of mutually exclusive possible outcomes of a given such measurement. By Gleason’s theorem [15], in dimension greater than two any consistent assignment of probabilities to all of these outcomes — i.e., any assignment of values in $[0,1]$ to each unit vector that sums to 1 over each orthonormal basis — arises uniquely from a density operator, i.e., a trace-1, positive self-adjoint operator $W$, by the formula $x \mapsto \langle x | W | x \rangle$ ($x$ a unit vector in $H$).

A slightly more general formulation associates measurement outcomes to “effects,” operators belonging to the unit interval (between 0 and the identity operator $I$) in the positive semidefinite ordering, and mutually exclusive possible outcomes of a given measurement to discrete resolutions of unity into effects $e_i$: $\sum_i e_i = I$. An analogue [3, 5] of Gleason’s theorem (cf. also the account in [12, 13]), easier because its essence is the self-duality of the cone of positive semidefinite operators, tells us that consistent probability assignments still correspond to density matrices $W$, with probabilities given by $e \mapsto \text{Tr} e W$. With a little more mathematical sophistication, one can generalize this a little further to allow certain measurements having a continuum of outcomes: this is the theory of POVM’s (positive operator valued measures) and the associated generalized observables. As is common in quantum information theory, we will also refer to the discrete resolutions of unity introduced above as POVMs, and will mostly avoid the continuous case in what follows.

A number of recent papers in quantum information theory (notably ones by C. A. Fuchs [12, 13] and N. Wallach [35]) have considered situations in which measurements are made on a pair of systems, and proved “unentangled Gleason theorems” about the representation of probabilities in such situations. The joint system is represented by the tensor product $H = H_1 \otimes H_2$. But the measurements are restricted to have all of their outcomes correspond to effects that are tensor products $e \otimes f$. (In the simpler situation without effects, the analogue would be to restrict measurements to have all outcomes corresponding to tensor product vectors, $|x\rangle|y\rangle$.) [35] considers general such measurements, while [12, 13] restrict further, to a pair of (say, space-like) separated observers.
who make local quantum measurements, and share information over a classical channel. The question arises: do all mathematically consistent probability assignments to pure tensors arise from standard quantum states, i.e., density operators on $H$? The answer is no. In the case that $H_1$ and $H_2$ are finite dimensional, one can indeed find a trace-1 self-adjoint operator $W$ on $H$ such that the probability to obtain outcome $e \otimes f$ (or $|x\rangle|y\rangle$, in the simpler model) is given by $\text{Tr} \langle W(e \otimes f) \rangle$ (or $\langle y\mid \langle x\mid W\mid x\rangle \mid y\rangle$, in the simpler model). However, the operator need not be positive. Rather, it need only be positive on pure tensors (POPT):

$$\langle y\mid \langle x\mid W\mid x\rangle \mid y\rangle \geq 0 \text{ for all } x \in H_1, y \in H_2.$$

(This is equivalent to positivity of $\text{Tr} \langle W(e \otimes f) \rangle$ for all effects $e, f$.) Thus, the theory of such “local” probability assignments outruns ordinary quantum statistics. Put another way: there exist (mathematically, at any rate) non-quantum mechanical states on coupled quantum systems having quantum-mechanical marginals.

In fact, such states were also considered in essentially this same context, by Kläy, Randall and Foulis [22] (KFR) (see also [21, 37, 38]), and also by Rudolf and Maitland-Wright [32] in connection with decoherence functionals. In this paper, we present much of the available information on the structure and properties of such states and their representing operators, along with some results we have not seen elsewhere.

In Sections 2 and 3, following Foulis and Randall [10, 11] and others, we introduce a fairly general framework, that of test spaces, for discussing statistical models such as the Hilbert-space model of quantum mechanics, or classical probability theory. In particular, we review the “tensor product” of Foulis and Randall (FR), and prove a simple representation theorem for states thereon [38]. In Section 4, we give a proof of the KFR “unentangled Gleason theorem”, showing that states on the FR tensor product of two frame manuals are representable by positive maps, or equivalently, by POPT operators (“Choi matrices” of the maps). While test spaces capture the simple version of quantum mechanics in terms of “von Neumann” measurements whose outcomes correspond to projectors, they are not quite general enough to encompass POVMs. Nevertheless the content of the “unentangled Gleason’s theorems” based on test spaces is essentially the same as that of the versions based on concrete Hilbert-space effects and “unentangled POVMs.” Since the relation between test space and effect-test space versions has little bearing on the underlying physics and information-processing content we are concerned to highlight, we will say little more about it here, and discuss it in an extended version of the paper in which the relationship between test space based models and related algebraic structures, and effect-based models and their related algebraic structures, can be investigated at length.

In Section 5, we discuss decomposable states, proving that extreme decomposable states are either extreme CP or extreme co-CP states. Section 6 develops a
variant of the standard quantum teleportation protocol that shows that POPT states on subsystems do not generally extend to POPT states on full systems. To keep the main flow of the discussion clearly in view, some of the background material has been placed in an appendix.

2 States and Weights on Test Spaces

Since we’ll be venturing outside of the usual conceptual shell of quantum probability theory, it will be helpful to establish, by way of a scaffolding, a language for discussing statistical models generally. Much of what follows could be framed in terms of abstract convex sets, ordered vector spaces, etc. However, we will use the language of states on test spaces, which is almost equally general but slightly more concrete.

2.1 Test spaces. A test space (sometimes called a manual) is just a collection of classical discrete sample spaces, possibly overlapping. To be more formal, let’s agree that a test space is a pair \((X, \mathfrak{A})\) where \(X\) is a set and \(\mathfrak{A}\) is a covering of \(X\) by non-empty subsets. Each set \(E \in \mathfrak{A}\) is supposed to represent the set of possible outcomes of some experiment, measurement, operation or test. Accordingly, a state (or probability weight) on a test space \((X, \mathfrak{A})\) is a map \(\omega : X \to [0, 1]\) summing independently to 1 over each test \(E \in \mathfrak{A}\). We write the space of all states on a test space \((X, \mathfrak{A})\) as \(\Omega(X, \mathfrak{A})\).

2.2 Examples. Of course, if there is only one test – that is, if our test space has the form \((E, \{E\})\) – we recover the framework of discrete classical probability theory. If \(X\) is the set of non-empty measurable subsets of a measurable space \(S\) and \(\mathfrak{A}\) is the set of countable partitions of \(S\) into measurable subsets, we recover the framework of classical probability theory. To recover the usual framework of quantum probability theory, let \(H\) be a Hilbert space, let \(X(H)\) denote \(H\)’s unit sphere, and let \(\mathfrak{F}(H)\) denote the set of frames (maximal orthonormal subsets) of \(H\). Then \((X, \mathfrak{F})\) is a quantum test space (or frame manual); Gleason’s theorem identifies the states on \((X, \mathfrak{F})\) with density operators on \(H\).

2.3 The Space of Signed Weights. It is sometimes useful to consider states that are not normalized, and can also be useful to consider states as belonging to a vector space of real-valued functions on \(X\) (the one they generate, in fact). So we make the following definitions. A positive weight on \((X, \mathfrak{A})\) is a positive multiple of a state, i.e., a function \(f : X \to \mathbb{R}\) such that (i) \(f(x) \geq 0\) for all \(x \in X\), and (ii) there is some constant \(K \geq 0\) with \(\sum_{x \in E} f(x) = K\) for every \(E \in \mathfrak{A}\). By a signed weight, we mean a function of the form \(f - g\), where \(f\) and \(g\) are positive weights. The set of signed weights on \((X, \mathfrak{A})\), denoted by \(V(X, \mathfrak{A})\), is a subspace of \(\mathbb{R}^X\) – in fact, just the span of \(\Omega(X, \mathfrak{A})\).\(^5\) We write \(V^*(X, \mathfrak{A})\) for the vector space dual to \(V(X, \mathfrak{A})\), i.e. the space of linear functionals \(V(X, \mathfrak{A}) \to \mathbb{R}\).

\(^5\)This has a natural base-norm, with respect to which it’s complete. Since our interest here is mainly finite-dimensional, we needn’t worry too much about this.
Note that every outcome \( x \in X \) determines a positive linear evaluation functional \( \phi_x \in V^*(X, \mathfrak{A}) \) given by \( \phi_x(\omega) = \omega(x) \). Since these obviously separate points of \( V(X, \mathfrak{A}) \), we have for \( V \) finite dimensional that \( V^*(X, \mathfrak{A}) \) is spanned by the \( \phi_x \).

2.4 Vector-Valued Weights. It’s also useful to consider more general vector-valued weights on a test space \((X, \mathfrak{A})\). Let \( M \) be a Banach space. Given a function \( \omega : X \to M \), we define the \textit{variation} of \( f \) to be

\[
\|\omega\| := \sup_{E \in \mathcal{A}} \sum_{x \in E} \|\omega(x)\|.
\]

(understanding that \( \|f\| = \infty \) if the sup does not exist.) We call \( f \) a \( M \)-valued weight iff (i) \( \|f\| < \infty \), and (ii) there exists a constant \( w \in M \) such that for any \( E \), \( \sum_{x \in E} \omega(x) \) converges in norm to \( w \). The space of \( M \)-valued weights is a Banach space with respect to the variation norm \( \|\cdot\| \).

2.5 Effect-test Spaces. Although we will not delve deeply into the relation between effect-test spaces and standard test-spaces, and related objects associated with them, we will introduce some basic concepts in order to make a few remarks on these relations later on.

Effect-test spaces, or E-test spaces, are similar to test spaces, but allow for the possibility that an outcome may occur in a test \textit{with multiplicity}. For example, in the general formulation of quantum mechanics in terms of POVMs, mentioned above, this is possible, so E-test spaces rather than just test spaces are necessary if one wants to encompass it. Confining ourselves for simplicity to \textit{locally discrete} E-test spaces, (so in the quantum case we can encompass discrete, but not yet continuous POVMs) we may define an E-test space \((Z, \mathcal{D})\) as a set \( \mathcal{D} \) of (possibly overlapping) multisets of elements of \( Z \). For our purposes a multiset \( s \) is just a map from the ground set \( Z \) to the nonnegative integers. The ground set elements \( x \) on which the map takes a nonzero value \( s(x) \) are thought of as occurring in the multiset \( s(x) \) times. Thus an ordinary set (with elements drawn from the ground set) is just a multiset for which \( s = 0 \)-valued. (In this case \( s \) is what is called the characteristic function of the set.) States are maps \( \omega : Z \to [0, 1] \) such that for every \( s \in \mathcal{A} \), \( \sum_{x \in Z} s(x)\omega(x) = 1 \).

3 States on Coupled Systems; Influence-Freedom

In what follows, we consider an arbitrary pair of test spaces \((X, \mathfrak{A})\) and \((Y, \mathfrak{B})\). We’ll think of these as associated with two “agents” (Alice and Bob, say) located at different sites, but able to communicate with one another.

We’ll use a juxtapositive notation for ordered pairs of outcomes, writing \( xy \) for

\( ^6 \)Of course, we have \( V(X) \leq W(X) \). In general, the inclusion is proper: not every real-valued weight is the difference of positive weights.
(x, y), and, for sets $A \subseteq X$ and $B \subseteq Y$, $AB$ instead of $A \times B$ for \{xy|x \in A \text{ and } y \in B\}. Thus, the outcome set for the join test in which Alice performs test $E \in \mathcal{A}$ and Bob performs test $F \in \mathcal{B}$ is $EF$. The set of all such “product tests” defines a new test space:

**3.1 Definition** The Cartesian product of $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ is the test space $(XY, \mathcal{A} \times \mathcal{B})$ where $\mathcal{A} \times \mathcal{B} = \{EF|E \in \mathcal{A} \text{ and } F \in \mathcal{B}\}$.

If $\omega$ is a state on $(XY, \mathcal{A} \times \mathcal{B})$ and $E \in \mathcal{A}$, we can define the marginal state $E\omega_{\mathcal{B}}$ on $(Y, \mathcal{B})$ by

$$E\omega_{\mathcal{B}}(y) := \omega(Ey) \equiv \sum_{x \in E} \omega(x, y).$$

The marginal state $\omega_{\mathcal{A}}F$, $x \in X$ and $F \in \mathcal{B}$, is defined similarly.

**3.2 Definition** Call a state $\omega$ on $(X \times Y, \mathcal{A} \times \mathcal{B})$ influence-free iff the marginal states $E\omega_{\mathcal{B}} : y \mapsto \omega(Ey)$ and $\omega_{\mathcal{A}}F : x \mapsto \omega(xF)$ are independent of $E$ and of $F$, respectively. In this case, we write $\omega_{\mathcal{A}}$ and $\omega_{\mathcal{B}}$ for these marginals. We’ll write $\Omega_{\text{free}}(X, Y)$ for the set of influence-free states on $X \times Y$.

“Influence-free” here should emphatically *not* be thought to imply what quantum information theorists sometimes call separable, i.e., unentangled; entangled quantum states are still influence-free. Rather, influence-free states are those that do not permit the slightest fraction of a bit of instantaneous signalling: one party cannot influence the other party’s probabilities merely by his or her choice of measurement. Influence-freedom is a conjunction of two conditions, which we’ll call $\leftarrow$-influence-freedom and $\rightarrow$-influence-freedom: no influence of system 1 on system 2, and vice versa.

**3.3 Classical Communication.** Suppose that the two systems represented by $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ belong to two agents, Alice and Bob, who are located at spatially remote sites but can communicate with one another via a classical channel. Then joint experiments more complex than $EF$ are possible. For instance, Alice may perform test $E$, and, upon securing outcome $x \in E$, instruct Bob to perform a test $F_x \in \mathcal{B}$. This yields a compound test with outcome set $\bigcup_{x \in E} xF_x$.

Let $\mathcal{A} \mathcal{B}$ denote the set of all such compound experiments, and let $\mathcal{\bar{A}} \mathcal{B}$ denote the corresponding set of two-stage tests $\bigcup_{y \in F} E_y y$ in which Bob performs the initial test.

**3.4 Definition**

$$\mathcal{A} \mathcal{B} := \mathcal{\bar{A}} \mathcal{B} \cup \mathcal{\bar{B}},$$

(1)
In other words, \( \mathcal{AB} \) is the set of two-stage experiments initiated at either Alice’s or Bob’s site. Note that \( \bigcup \mathcal{AB} = \mathcal{XY} \), and also that \( \mathcal{A} \times \mathcal{B} \subseteq \mathcal{AB} \). Thus, the state-space of \((\mathcal{XY}, \mathcal{AB})\) is a convex subset of the state-space of \((\mathcal{XY}, \mathcal{A} \times \mathcal{B})\). The following was shown in [11, 31, 21], and we will review the simple proof of a proposition that implies it. (Note that in [10, 11, 21] the notation \( \mathcal{AB} \) is used for what we call \( \mathcal{A} \times \mathcal{B} \), while \( \mathcal{\rightarrow AB} \) is used for what we call \( \mathcal{AB} \); also the term “interference” is sometimes interchangeably with “influence” as we have defined it.)

3.5 Corollary Let \( \omega \) be a state on \((\mathcal{XY}, \mathcal{A} \times \mathcal{B})\). Then the following are equivalent:

(a) \( \omega \) is a state on \((\mathcal{XY}, \mathcal{AB})\);
(b) \( \omega \) is influence-free.

This is an immediate corollary of the following easy theorem, which can be interpreted as saying that allowing classical communication in one direction enforces that the states be influence-free in the opposite direction.

3.6 Theorem (Randall and Foulis [21], Lemma 2.8) Let \( \omega \) be a state on \((\mathcal{XY}, \mathcal{A} \times \mathcal{B})\). Then the following are equivalent:

(a) \( \omega \) is a state on \((\mathcal{XY}, \mathcal{\rightarrow AB})\);
(b) \( \omega \) is \( \leftarrow\)-influence-free.

Proof of Theorem 3.6

Proof: We first show “only if.” Let \( \omega \in \Omega(\mathcal{\rightarrow AB}) \). A test \( T \in \mathcal{\rightarrow AB} \) involves performing a test \( Z \in \mathcal{A} \) and proceeds, on obtaining outcome \( z \in Z \), by performing \( W_z \in \mathcal{B} \). Thus, for any chosen \( x \in Z \),

\[
1 = \omega(T) = \sum_{z \in Z - x} \sum_{y \in W_z} \omega(z, y) + \sum_{y \in W_x} \omega(x, y).
\]

(2)

In other words,

\[
\sum_{y \in W_x} \omega(x, y) = 1 - \sum_{z \in T - x} \sum_{y \in W(z)} \omega(z, y).
\]

(3)

The RHS is independent of \( W_x \), while the LHS defines \( W^* \omega_{\mathcal{B}}(x) \). Since the construction can be done (varying \( Z \) if necessary) for any \( x \) and any choice of test \( W \) as \( W_x \) this establishes that \( W^* \omega_{\mathcal{B}} \) is independent of \( W \), i.e. \( \omega \) is influence-free.

To show “if”: suppose we have an influence-free state \( \omega \) on \( \mathcal{A} \times \mathcal{B} \). We just have to show it adds up to one on tests in \( \mathcal{\rightarrow AB} \). For an arbitrary such test
$T$ consisting of performing $Z \in \mathcal{A}$ followed, conditional on result $z \in Z$, by performing $W_z \in \mathcal{B}$, we have:

$$\sum_{(x,y) \in T} = \sum_{z \in Z} \sum_{y \in W_z} \omega(x, y)$$  \hspace{1cm} (4)$$

$$= \sum_{z \in Z} W_z \omega(x)$$  \hspace{1cm} (5)$$

$$= \sum_{z \in Z} \omega(x) = 1$$  \hspace{1cm} (6)$$

The next-to-last equality is due to left-influence-freedom of $\omega$.  

The test space $\mathcal{A}\mathcal{B}$ has sometimes been called the Foulis-Randall product or bilateral tensor product of $\mathcal{A}$ and $\mathcal{B}$. We will sometimes call it the “free no-signalling,” or “FNS” product.

### 3.7 Digression: Conditional dynamics at a distance; marginalization and Bayesian updating.

We may interpret the existence of a marginal state as giving a “conditional dynamics” of Bob’s state, caused by Alice measuring and obtaining the outcome she does. This “conditional dynamics” is nothing but ordinary probabilistic conditionalization. That is, letting $\omega^{AB}$ be the initial Alice-Bob state, $x$ any Alice outcome (such as a Bell-measurement outcome), we may ascribe to Bob the post-measurement state $\omega^B_x$, defined by $\omega^B_x(y) := \omega^{AB}(xy) / \sum_{y \in T} \omega^{AB}(xy)$. The RHS refers to a test $T$, but it does not depend on the choice of this test: the fact that it does not is precisely left-no-signalling. We may therefore write this denominator as $\omega^A(x)$ (we have a well-defined marginal state $\omega^A$), and the expression for Bob’s conditional state then looks just like ordinary probabilistic conditionalization:

$$\omega^B_x(y) = \omega^{AB}(xy) / \omega^A(x).$$  \hspace{1cm} (7)$$

In [12, 13] an analogy was made between decomposing a quantum mechanical density operator $\rho$ as a convex combination

$$\sum_b p(b) \rho_b = \rho$$  \hspace{1cm} (8)$$

of density operators, and “refining one’s knowledge” of a quantum system [12] [13]. Due to the concavity of von Neumann entropy, one might say (using entropy—or indeed any other unitarily invariant extension of a Schur-concave function, cf. 14—as a measure of one’s knowledge of a system) that if one expects to learn a value of $b$, and thereupon have knowledge represented by $\rho_b$ about the system, then one expects on average to gain information about a quantum system. In fact, if $\rho$ is taken to represent the state of a quantum system before a measurement, for any additive decomposition of the form 8 there
exists a measurement with conditional dynamics—a set of completely positive maps $\mathcal{E}_b$ summing to a trace-preserving one $\mathcal{E}$—such that the post-measurement states $\mathcal{E}_b(\rho)$ conditional on outcome $b$ of the measurement are $\rho_b$. Of course, this same measurement-with-dynamics will not in general have the same property for other input density matrices $\sigma$. Conditional dynamics such that the conditional density operators sum to the initial one are by no means the general form of conditional quantum dynamics, and (as shown in [14]) the fact that one gains information (in the above sense) on average also holds for many other conditional dynamics...indeed, for any “efficient” measurement of a POVM on a quantum system. (“Efficient” here means that the post-measurement states conditional on measurement outcomes are pure, so that there is no information extracted in the interaction of system with apparatus and environment during measurement, that is not reflected in the measurement outcome.) But (as argued in [12, 13]) the dynamics

$$\rho \xrightarrow{b} \rho_b$$

$$\text{prob}(b) = \text{Tr} \rho_b$$

meaning that conditional on the measurement outcome $b$, which occurs with probability $\text{Tr} \rho_b$, $\rho$ evolves to $\rho_b$, are especially close to those obtained classically via Bayes’ rule. (There it is also shown that it is always possible, in quantum mechanics, to measure in such a way that these conditional dynamics occur, although the “instrument” (collection of completely positive maps, one for each outcome $b$ of the POVM) achieving this will depend not only on the POVM but also on the initial density operator $\rho$.) The analogy with Bayesian updating is as follows: let there be random variables $A, B$ taking values in a finite set, and consider an initial joint distribution $p(A, B)$. We have an initial distribution $p(A)$, and wish to obtain information about $A$ by observing $B$. Then, the conditional distribution $p_{\theta}(A)$ (more frequently written $p(A|b)$) is defined via

$$p_{\theta}(a) = p(ab)/\sum_a p(ab).$$

(10)

Usually one defines $p(b) := \sum_a p(ab)$ and thus it holds that

$$\sum_b p(b)p_{\theta}(A) = p(A).$$

(11)

This form of Bayes’ rule is the “classical” analogue of (8). In fact, the analogy can be sharpened: (8) implies that (calling the observable whose values are the $b$, and which is presumed to be measured with the conditional dynamics $\rho \xrightarrow{b} \rho_b$ with $\rho_b$ satisfying (9)), for any choice of post-measurement observable $Y$, there exists a joint distribution $p(B, Y)$ determined by $\rho$, such that the probabilities for the values $a$ taken by $Y$ conditional on outcome $b$, are given by (10) and thus satisfy (11) (with $Y$ substituted for $A$ in the latter, of course).

When do quantum systems exhibit such “closest-to-Bayesian” dynamics? As shown in [12, 13], one such situation is when information about a quantum
system A is obtained by measuring another system B that may have been
tangled with it. In that case the reduced density matrix of A (calculated from
the joint state \( \rho^{AB} \)) is updated (on learning the result of a measurement on B)
precisely according to a rule of the form \( \rho^{b,p(b)} \to \rho_b \) satisfying (8).

We may understand this result as a special case of a general relationship in FNS
tensor products. Although the quantum-mechanical tensor product is not the
FNS tensor product, but rather permits more measurements and fewer states,
the measurements involved in the result do belong to those permitted in the FNS
tensor product, and for these measurements and states, the quantum probabil-
ities and the FNS ones coincide.

Looking again at (7), we see that we may rewrite the effect of conditionalizing
on a B measurement result as:

\[
\omega^B \xrightarrow{a} \omega^B''
\]
\[\text{prob}(a) = \omega^A(a)\]  
(12)

where

\[
\sum_a \omega^A(a) \omega^B''(a) = \omega^B .
\]  
(13)

These are general analogues of (10) and (11) (and (13) is a general analogue of
(8)). They can be summarized (and their “Bayesian” content sharpened just
as in the quantum case) by saying that for every choice of an Alice and Bob
measurement, the joint distribution it determines conditionalizes in the usual
Bayesian (classical probabilistic) manner, according to (11). And, just as in the
quantum case this held for joint states of distinct systems, here it holds whenever
the coupling between two systems does not permit signalling. The relation
between the simultaneous “Bayesian” nature of the conditioning (on a result
of Alice) of the probabilities of all of Bob’s measurements, and no-signalling in
the Bob-to-Alice direction, is essentially just a restatement of Theorem 3.4. It
dramatizes the “least-disturbance” nature of the nearest-to-Bayesian updating
rule: other rules correspond a “more serious” or “more physical” disturbance, in
the sense that the disturbance itself could be used to carry information (signal).

The close connection of conditioning of that occurs in the FNS tensor product,
and Bayesian updating, was also noted by Randall and Foulis, who give an
“Operational Bayes Theorem:”

3.8 Theorem [11], Theorem 2.6) Let \( \omega \) be a state on \((XY, \mathfrak{B})\), \( a \in X, b \in Y, F \in \mathfrak{B}\). Then

\[
\omega_{ab}(b) = \omega_B(b)\omega_b(a)/(\sum_{c \in F} \omega_B(a)\omega_c(a)) .
\]  
(14)

3.9 Influence-Free States Linearized. The function \( y \mapsto \omega(Ey) \) is indepen-
dent of \( E \in \mathfrak{A} \) if and only if, for every fixed \( y \in Y \), the map \( \omega_y : x \mapsto \omega(xy) \) is a (non-normalized, but positive) weight on \((X, \mathfrak{A})\). If it is also the case that \( x \mapsto \omega(xF) \) is independent of \( F \in \mathfrak{B} \) – that is, if \( \omega \) is influence-free – then the map \( y \mapsto \omega_y \) can be interpreted as a vector-valued weight on \((Y, \mathfrak{B})\) with values in the space \( V(X, \mathfrak{A}) \) of “signed weights” (i.e., linear combinations of states) on \((X, \mathfrak{A})\).

If \( V(X, \mathfrak{A}) \) and \( V(Y, \mathfrak{B}) \) are finite-dimensional, we can blithely dualize the foregoing picture: every influence-free weight on \((X, \mathfrak{A})\)×\((Y, \mathfrak{B})\) determines, and is determined by, a positive linear operator \( \hat{\omega} : V^*(Y, \mathfrak{B}) \to V(X, \mathfrak{A}) \) with the property that \( \omega(1) \in \Omega(X, \mathfrak{A}) \). More generally, given an (unnormalized) positive influence-free weight \( \omega \), we obtain a positive map \( \hat{\omega} : V^*(Y, \mathfrak{B}) \to V(X, \mathfrak{A}) \), and any such map \( \phi \), conversely, determines a influence-free weight \( \omega \) via 

\[
\omega(xy) := \text{Tr} \left[ \phi(|x\rangle\langle x|)|y\rangle\langle y| \right] \equiv \langle y|\phi(|x\rangle\langle x|)|y \rangle .
\]

where \(|x\rangle\langle x|\) is the orthogonal projection operator determined by \( x \in H \).

Thus, the set of influence-free states on \( \mathfrak{F}(H) \times \mathfrak{F}(K) \) is affinely isomorphic to the space of positive linear maps on \( \mathcal{L}(K) \).\(^8\)

### 4. Operator Representations of Influence-Free States

We now specialize to the case in which \( \dim(H) = \dim(K) \). For simplicity, we assume that \( H = K \). In this setting, one can represent influence-free states on \( \mathfrak{F}(H) \times \mathfrak{F}(K) \) by operators on \( H \otimes H \), using the following useful result:

#### 4.1 Proposition (Folklore) For any linear map \( \phi : \mathcal{L}(H) \to \mathcal{L}(H) \), there exists a unique operator \( W = W_\phi \) on \( H \otimes H \) such that, for all \( x, y, u, v \in H \),

\[^8\text{All of this pretty much goes through even in the infinite-dimensional setting, as long as \( \mathfrak{A} \) and \( \mathfrak{B} \) are locally countable. \[38\].}\]
\[ \langle y|\phi(|x\rangle|x)|y\rangle = \langle y|\langle x|W|x\rangle|y\rangle. \]

Conversely, every operator \( W \) on \( H \otimes H \) arises in this way from a unique linear map \( \phi : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \).

**Proof:** For any linear operator on \( \mathcal{L}(H) \), the quantity \( \langle y|\phi(|x\rangle|x)|y\rangle \) is bi-quadratic in \( x \) and \( y \). Polarizing twice, see that \( \phi \) is uniquely determined by the form

\[ (x, u, y, v) \mapsto \langle v|\phi(|x\rangle|x)|y\rangle. \]

Note that this is linear in \( x \) and \( y \), conjugate-linear in \( u \) and \( v \). Accordingly, there is a unique sesquilinear form \( \Phi \) on \( H \otimes H \) satisfying

\[ \Phi(|x\rangle|x)|y\rangle, |u\rangle|v\rangle := \langle v|\phi(|x\rangle|x)|y\rangle. \]

By the Riesz representation theorem (cf. [20], Theorem 2.3.1; cf. also Theorem 2.4.1 for a result close to Proposition 4.1), there is a unique operator \( W \) such that \( \Phi(|x\rangle|x)|y\rangle, |u\rangle|v\rangle \) gives the result.

This immediately yields the following “unentangled Gleason theorem”:

**4.2 Corollary** ([22]; see also [12, 13, 35]): Let \( H \) be a finite-dimensional complex Hilbert space. For every influence-free state \( \omega \) on \( \mathcal{F}(H) \times \mathcal{F}(H) \), there exists a self-adjoint operator \( W \) on \( H \) such that \( \omega(xy) = \langle y|\langle x|W|x\rangle|y\rangle \) for all unit vectors \( x, y \in H \).

Evidently, the operator \( W \) must be **positive on pure tensors** (POPT), in that \( \langle y|\langle x|W|x\rangle|y\rangle \geq 0 \) for all \( x, y \in H \). However, \( W \) need not be positive:

**4.3 Example.** Let \( S(|x\rangle|y\rangle) = |x\rangle|y\rangle \), i.e., \( S \) is the unitary “swap” operator. Then \( S \) is POPT, since \( \langle y||x\rangle|S|x\rangle|y\rangle = \langle x|\langle y|, |x\rangle|y\rangle = \langle x|y\rangle\langle y|x\rangle = |\langle x|y\rangle|^2 \).

But \( S \) is certainly not positive. Indeed, if \( \tau = |x\rangle|y\rangle - |y\rangle|x\rangle \), then \( S\tau = -\tau \), whence \( \langle \tau|W|\tau\rangle = -\|\tau\|^2 \).

The question now arises: When is the POPT operator \( W_\phi \) arising from a positive linear map \( \phi : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \) in fact positive on \( H \otimes H \)?

Recall that a linear map \( \phi : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \) is **completely positive** (CP) iff the map \( \phi \otimes \operatorname{id} : \mathcal{L}(H \otimes K) \rightarrow \mathcal{L}(H \otimes K) \) remains positive for all Hilbert spaces \( K \).

In a suitable basis, the matrix for \( W_\phi \) is just the so-called Choi matrix for \( \phi \). This is discussed below.
and to Choi [6] that $\phi$ is CP if and only if it can be expressed in the form

$$\phi(X) = \sum_i A_i X A_i^\dagger$$

for operators $A_i$ on $\mathcal{H}$. Such a decomposition of $\phi$ is called a Hellwig-Kraus representation, or HK representation for short. (Many authors in quantum information theory, probably through familiarity with [26] rather than [24, 25], just call it a Kraus decomposition.)

4.4 Theorem (Choi, Hellwig and Kraus) Let $W = W_\phi$ be the operator associated with the linear map $\phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ as in Proposition 4.1. Then $W$ is positive iff $\phi$ is completely positive.

4.5 Matrix Representations. Suppose now that $E = \{e_i\}$ is an ordered orthonormal basis for $\mathcal{H}$. We can then represent a map $\phi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ by an operator-valued matrix $\Phi_{i,j} := \phi(|e_i\rangle\langle e_j|)$. If we represent each of the entries $\Phi_{i,j}$ by an $n$-by-$n$ matrix relative to the same basis $E$, we find that the $(i,j) - (k,l)$-th entry is $\Phi_{i,j,k,l} = \langle e_l|\phi(|e_i\rangle\langle e_k|)|e_j\rangle$. This is often called the Choi matrix for $\phi$. We’ll write $\text{Ch}(\phi)$ for this matrix (taking the basis $\{e_i\}$ as understood). For the $i,j$ block of it, we’ll write $\text{Ch}_i^j(\phi)$;

4.6 Representation Using a Maximally Entangled Pure State. Again let $E$ be an orthonormal basis for $\mathcal{H}$. The product basis $\{|a\rangle|b\rangle|a, b \in E\}$ yields an operator basis for $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ consisting of the operators

$$|a\rangle|b\rangle\langle c|\langle d| = |a\rangle\langle d| \otimes |b\rangle\langle c|,$$

$a, b, c$ and $d$ running over $E$. By expanding the operator $W_\phi$ defined in Proposition 4.1 in this basis, one can show that

$$W_\phi = (\text{id} \otimes \phi)(T)$$

where $T$ is the (unnormalized) pure maximally entangled state given by

$$T = \sum_{a, c \in E} |a\rangle\langle c| \otimes |a\rangle\langle c|.$$

5 Decomposable States

The structure of the full set of positive maps between $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{K})$ is very complicated, even in low dimensions. A set of maps larger than the set of CP
maps, but still tractable, is that of decomposable maps. If \( \phi \) is a CP map, then the map

\[
\phi^t : X \mapsto \phi(X^t)
\]

obtained by composing \( \phi \) with the a transposition map is said to be co-completely positive (co-CP). A map of the form \( \phi = \psi + \eta \) where \( \psi \) is CP and \( \eta \) is co-CP is said to be decomposable. The set of CP maps, the set of co-CP maps, and the set of decomposable maps are all convex cones in the space of linear operators on \( \mathcal{L}(H) \), with the cone of decomposable maps being the convex span of the CP and co-CP cones.

We wish to understand the extremal structure of the cone of decomposable maps. By an extremal point of a cone, we mean a point generating an extreme ray. Evidently, any extremal point of the convex span of two cones must be extremal in one of the two cones to begin with; in general, however, some extremal points of the original cones will be “swallowed up” in the passage to the convex span. Our aim here is to prove that this doesn’t happen here: i.e., extreme CP maps and extreme co-CP maps all remain extreme in the larger cone of decomposable maps.

**Notation:** Given an operator \( A \) on \( H \), we’ll write \( \phi_A \) for the CP map

\[
\phi_A : X \mapsto AXA^\dagger.
\]

Evidently, any extreme CP map is of this form. Notice that \( \phi_A \phi_B = \phi_{AB} \). This gives us the trivial but useful

**5.1 Lemma** Let \( \psi : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \) be CP. Then for any operator \( A \) on \( H \), \( \phi_A \circ \psi \) and \( \psi \circ \phi_A \) are likewise CP maps.

**5.2 Transpositions.** Let \( M_n \) denote the \( * \)-algebra of \( n \times n \) complex matrices. Any orthonormal basis \( E = \{ |e_i \rangle \} \) for \( H \) induces an isomorphism \( \mathcal{L}(H) \rightarrow M_n \) given by \( X \mapsto [X]_E \), where \( [X]_E^F = \langle e_i | X | e_j \rangle \). Accordingly, the map \( M \rightarrow M^t \) pulls back to an anti-linear map \( J : \mathcal{L}(H) \rightarrow \mathcal{L}(H) \) given by \( [J(X)] = [X]^t \).

Let \( \sigma \) be the anti-linear operator on \( \mathcal{L}(H) \) corresponding to transposition with respect to another orthonormal basis \( F = \{ |f_j \rangle \} \) for \( H \). Let \( U \) be the unitary map sending \( |e_i \rangle \) to \( |f_j \rangle \), so that \( [X]^F = \langle e_j | U^\dagger XU | e_i \rangle = [U^\dagger XU]^E \). Thus,

\[
\sigma(X) = J(UXU^\dagger) = J(\phi_U(X)).
\]

**5.3 Lemma** Let \( \psi \) be a linear map \( \mathcal{L}(H) \rightarrow \mathcal{L}(H) \). If \( \psi \circ J \) is CP, then so is \( \psi \circ \sigma \).

**Proof:** By the foregoing discussion, \( \psi \circ \sigma = \psi \circ J \circ \phi_U \). By assumption, \( \psi \circ J \) is CP. By Lemma A, so is \( (\psi \circ J) \circ \phi_U \). \( \square \)

It follows that \( \psi \) is co-CP iff \( \psi \circ J \) is CP for any transposition \( J \). As in the introduction, we’ll use the notation \( X^t \) for the operator \( J(X) \), where \( J \) is some
fixed but unspecified transposition; we’ll also write \( \phi^t \) for the map \( \phi \circ J : X \mapsto \phi(X^\dagger) \).

5.4 Lemma Let \( \psi \) be co-CP. Then for any operator \( A \) on \( \mathbf{H} \), \( \phi_A \circ \psi \) is again co-CP.

Proof: By assumption, \( \psi^t = \psi \circ J \) is CP. Now, \( (\phi_A \circ \psi) \circ J = \phi_A \circ (\psi \circ J) \). By Lemma 5.1, this is also CP. Hence, \( \phi_A \circ \psi \) is co-CP. □

The following lemma gathers some elementary but helpful facts about Choi matrices. As above, \( \{ |e_i \rangle \} \) is a fixed orthonormal basis for \( \mathbf{H} \) and \( \phi^t = \phi \circ J \) where \( J \) is transposition with respect to this basis.

5.5 Lemma Let \( A, B \in \mathcal{L}(\mathbf{H}) \), with \( A |e_i \rangle = |a_i \rangle \), and let \( \phi : \mathcal{L}(\mathbf{H}) \to \mathcal{L}(\mathbf{H}) \) be any linear map. Then

\( (a) \) \( \text{Ch}(\phi_A + \phi_B) = \text{Ch}(\phi_A) + \text{Ch}(\phi_B) \);

\( (b) \) \( \text{Ch}(\phi_A)^{ij} = |a_i \rangle \langle a_j | \).

\( (c) \) \( \text{Ch}(\phi^t_A) = |a_j \rangle \langle a_i | \).

\( (d) \) \( \text{Ch}(\phi^t) = \text{Ch}(\phi) \text{id} \otimes t \).

Here \( X \text{id} \otimes t \) is the partial transpose of \( X \), i.e. the bipartite state \( X \) subjected to the extension, by the identity on map the one factor, of the transpose map acting on the other factor.

5.6 Lemma Suppose that \( \text{Ch}(\phi) \) is block-diagonal. Then \( \phi \) is CP iff \( \phi \) is co-CP.

Proof: The assumption is that \( \phi( |e_i \rangle \langle e_j | ) = 0 \) for \( i \neq j \). It follows that \( \phi^t( |e_i \rangle \langle e_j | ) = \phi( |e_j \rangle \langle e_i | ) = \phi( |e_i \rangle \langle e_j | ) \) for all \( i, j \), i.e. \( \text{Ch}(\phi^t) = \text{Ch}(\phi) \). Now invoke Theorem 4.4. □

We now prove the advertised result, which we restate for convenience:

5.7 Theorem Let \( \phi \) be an extremal map in the cone of CP maps, or an extremal map in the cone of co-CP maps. Then \( \phi \) is extremal in the cone of decomposable maps.

Equivalently, the extremal quantum states and the extremal PPT states (states corresponding to Hermitian operators with positive partial transpose) remain extremal in the cone of decomposable states.

Note that the map \( \phi \mapsto \phi^t \) is an affine isomorphism from the cone of CP maps to that of co-CP maps. Thus, without loss of generality we can focus on an extreme CP map

\[ \phi : X \mapsto AXA^\dagger \]

and ask whether this can be expressed nontrivially as a sum of CP and co-CP
maps. We therefore suppose in what follows that \( \phi = \psi + \eta \) where \( \psi \) and \( \eta \) are respectively non-zero completely and co-completely positive maps with Hellwig-Kraus representations

\[
\psi = \sum_k \phi_{B_k} \quad \text{and} \quad \eta = \sum_l \phi_{C_l}^t.
\]

Our aim is to prove that \( \psi \) and \( \eta \) are in fact multiples of \( \phi \). The strategy will be to show that \( \eta \) lies in the separable cone, i.e., is both CP and co-CP: the extremality of \( \phi \) in the CP cone then yields the desired result. The key observation (along with Lemma 5.6) is the following:

5.8 Lemma Let \( \phi, \psi, \eta \) be as above. That is, \( \phi = \psi + \eta \) with \( \phi = \phi_A \), \( \psi = \sum_k \phi_{B_k} \), \( \eta = \sum_l \phi_{C_l}^t \), so that \( \phi \) is CP and \( \eta \) co-CP. Suppose \( \{ |e_i \rangle \} \) is an orthonormal basis for \( \mathcal{H} \), and let \( |a_i \rangle = A|e_i \rangle \). The Choi matrix \( \text{Ch}(\eta) \) for \( \eta \) relative to the basis \( \{ |e_i \rangle \} \) has the following form: For all \( i, j \),

(a) if \( a_i \) and \( a_j \) are linearly dependent, then \( \text{Ch}(\eta)^{ij} = z_{ij} |a_i \rangle \langle a_j | \)

for some complex coefficient \( z_{ij} \);

(b) if \( a_i \) and \( a_j \) are linearly independent, then \( \text{Ch}(\eta)^{ij} = 0 \).

Proof: (a) Let \( |b_{ki} \rangle := B_k |e_i \rangle \), and \( |c_{li} \rangle := C_l |e_i \rangle \). Since \( \phi = \psi + \eta \), we have \( \text{Ch}(\phi) = \text{Ch}(\psi) + \text{Ch}(\eta) \). Invoking Lemma 5.6, we obtain

\[
|a_i \rangle \langle a_j | = \sum_k |b_{ki} \rangle \langle b_{kj}| + \sum_l |c_{lij} \rangle \langle c_{lij} |,
\]

for all \( i, j \). With \( i = j \), we have

\[
|a_i \rangle \langle a_i | = \sum_k |b_{ki} \rangle \langle b_{ki}| + \sum_l |c_{ii} | \langle c_{ii} |.
\]

Now, \( |a_i \rangle \langle a_i |, \) \( |b_{ki} \rangle \langle b_{ki}| \) and \( |c_{ii} | \langle c_{ii} | \) are non-negative multiples of one-dimensional orthogonal projections; these are extremal in the full cone of positive operators, so we must have \( |b_{ki} \rangle = b_{ki} |a_i \rangle \) and \( |c_{ii} | = c_{ii} |a_i \rangle \) for all \( k, l \), where \( b_{ki} \) and \( c_{ii} \) are complex scalars. In particular, then,

\[
\text{Ch}(\psi)^{ij} = r_{ij} |a_i \rangle \langle a_j | \quad \text{and} \quad \text{Ch}(\eta)^{ij} = s_{ij} |a_j \rangle \langle a_i |,
\]

where

\[
r_{ij} = \left( \sum_k b_{ki} \bar{b}_{kj} \right) \quad \text{and} \quad s_{ij} = \left( \sum_l c_{lij} \bar{c}_{lij} \right).
\]

Suppose now that \( a_i \) and \( a_j \) are linearly dependent, i.e., \( a_j = k_{ji} a_i \) for some non-zero \( k_{ji} \). Then \( \text{Ch}(\eta)^{ij} = z_{ij} |a_i \rangle \langle a_i | \) where \( z_{ij} = s_{ij} k_{ji} \). This yields part (a). If \( a_i \) and \( a_j \) are independent, then so are \( |a_i \rangle \langle a_j | \) and \( |a_j \rangle \langle a_i | \). From the fact that

\[
|a_i \rangle \langle a_j | = r_{ij} |a_i \rangle \langle a_j | + s_{ij} |a_j \rangle \langle a_i |,
\]

we see that \( s_{ij} = 0 \), i.e., \( \text{Ch}(\eta)^{ij} = 0 \). □
Proof of Theorem 5.7: We first consider the case in which \( A \) is self-adjoint. Choose the orthonormal basis \( \{|e_i}\} \) of Lemma 5.8 so as to diagonalize \( A \). In this case, \( |a_i\rangle \) and \( |a_j\rangle \) are linearly dependent only if one of them is zero, or if \( i = j \). By Lemma 5.8, the only cases in which \( \text{Ch}(\eta)^{ij} \neq 0 \) are those in which \( i = j \), i.e., \( \text{Ch}(\eta) \) is itself block-diagonal. Thus, by Lemma 5.6, \( \eta \) is separable.

Suppose now that \( A \) is arbitrary. By the polar decomposition theorem (c.f. e.g. [17], Theorem 7.3.2), there exists a partial isometry \( W \) such that \( |A| = WA \) and \( W^\dagger|A| = A \). (\( |A| \) is defined as \( \sqrt{A^\dagger A} \), while a partial isometry is defined as an operator \( W \) such that \( WW^\dagger \) and hence also \( W^\dagger W \) is a projection.) Thus,

\[
\phi_{|A|} = \phi_W \circ \phi_A = \phi_W \circ \psi + \phi_W \circ \eta.
\]

By Lemmas 5.3 and 5.4, \( \phi_W \circ \psi \) and \( \phi_W \circ \eta \) are respectively CP and co-CP. Hence, \( |A| \) satisfies the same hypotheses as \( A \). In particular, we can invoke the preceding argument to conclude that \( \phi_{|A|} \) is separable. But then we can apply Lemmas 5.3 and 5.4 again to conclude that \( \phi_W^\dagger \circ \phi_{|A|} = \phi_A \) is also separable. \( \square \)

Remark: It is a classical result of Choi [6] that if \( \text{dim}(H) = 2 \), all positive linear maps \( \mathcal{L}(H) \to \mathcal{L}(H) \) are decomposable.

6 Teleportation and POPT States: Difficulties with combining free-no-signalling and quantum composition of systems

Here we consider what happens if we view the Foulis-Randall FNS product as the way to couple quantum systems distant from each other, but attempt to couple the different “local” systems with the ordinary quantum-mechanical product (reasoning that locally, measurements with “entangled outcomes” are possible). We may view the situation as similar to that in many quantum protocols illustrating “nonlocal” effects associated with entangled states, or distilling entanglement, etc...: two agents distant from each other, “Alice” and “Bob,” are viewed as each having a number of systems, say \( A_1 \ldots A_n \), \( B_1 \ldots B_n \), under their control. We show in this section that attempts to mix these two types of coupling lead to pathologies. In particular, the following is usually considered a desideratum for a notion of compound system \( AB \) composed of two subsystems \( A \) and \( B \).

6.1 Desideratum. For every pair of states \( \omega \) on \( A \), \( \lambda \) on \( B \), there exists a “product” \( (\omega\lambda) \) of these two states on the compound system, such that in that state, the probability of the pair of outcomes \( xy \) is the product of its probabilities under the pair of subsystem states: \( (\omega\lambda)(xy) = \omega(x)\lambda(y) \).

We consider \( A_1 \) and \( A_2 \) to be coupled quantum-mechanically, and similarly with \( B_1 \) and \( B_2 \). We then couple the quantum mechanical test-space \( A_1 \otimes A_2 \) with \( B_1 \otimes B_2 \), via the FNS tensor product, obtaining a test space we will call \( AB \).
We also consider the test-spaces, and states, obtained by coupling \( A_1 \) with \( B_1 \), and \( A_2 \) with \( B_2 \), via the FNS tensor product (call the resulting test spaces 1 and 2). We ask if we can view the test space \( AB \) as any reasonable kind of product of 1 and 2.

6.2 Proposition There exist pairs of states on 1 and 2 whose product is not a state of \( AB \).

That is, the two types of coupling, “local quantum mechanical” and “nonlocal free-no-signalling (FNS)”, cannot be combined in a manner consistent with Desideratum 1. The intuitive argument is simple: if Alice and Bob share both a Bell state and a POPT state that is nonpositive, Alice may use the Bell state and a local measurement in an entangled basis of Bell states to teleport her part of the POPT state to Bob; since local coupling is quantum-mechanical, Bob has available measurements with “entangled” outcomes, which is inconsistent with his possessing both parts of a POPT state. We now formalize this argument, and explain why it proves Proposition 6.2.

The setting here is similar to that of ordinary quantum teleportation protocols, except that we suppose that both Bob’s and Alice’s systems are bipartite. Thus, we have \( H_{\text{Alice}} \equiv H_A := H_{A_1} \otimes H_{A_2} \) and \( H_{\text{Bob}} \equiv H_B := H_{B_2} \otimes H_{B_1} \). We assume here that all four Hilbert spaces \( H_{A_i}, H_{B_i} \) are copies of a common finite-dimensional Hilbert space \( H \) of (finite) dimension \( n \). “Copies of a common space” rather than just “isomorphic Hilbert spaces” implies we have selected a commuting set of isomorphisms between them; equivalently, we have selected and identified a “standard” orthonormal basis in each. This means that it makes sense to speak of a given operator \( O \), acting on different systems. (The alert reader may wonder why we defined \( H_B \) as \( H_{B_2} \otimes H_{B_1} \), rather than with the more natural ordering \( H_{B_1} \otimes H_{B_2} \) used in the informal discussion above.\(^{10}\) The reason is that if we had used the other ordering, we would have had to replace the second occurrence of the state \( W \) in the first line of Theorem 6.3 with \( \bar{W} \), defined as \( \text{Swap} W \text{ Swap}^\dagger \), where \( \text{Swap} \) is the unitary operator that swaps the states of systems \( B_1 \) and \( B_2 \). In other words, the standard teleportation protocol where Alice measures in a Bell basis and tells Bob her result, and the “source” system \( H_{A_1} \) is entangled with \( H_{B_1} \), so that \( H_{A_1} \otimes H_{B_1} \) is in state \( W \), ends up, in the case where the Bell measurement gives the standard maximally entangled state \( T \) (defined below), “pivoting” \( H_{A_1} \) ’s entanglement into \( H_{B_2} \), where we can think of it as “pivoting” around \( H_{B_1} \), whose role (and state) stays constant.)

The canonical isomorphisms between Hilbert spaces given above also imply that it makes sense to speak of a given operator defined on the (ordered) tensor

\(^{10}\) Other readers are urged to treat themselves to a double cappuccino-or a maté–before continuing; a chai latte might do, if you insist, though we cannot be held responsible for the consequences of routinely indulging in such beverages, which we have heard may end in walking up and down Telegraph Avenue clad in nothing but a poncho, beads, and patchouli oil.
product of two systems, as acting on some other (ordered) tensor product. The
convention we will use here is that (given commuting canonical isomorphisms
between Hilbert spaces $H, J, K, L$) if some operator $W$ is specified as an operator
on $H \otimes J$, then $W$ on a different tensor product $K \otimes L$ is determined by the
requirement that its matrix elements in the tensor product basis $|i⟩^K|j⟩^L$ for
$K \otimes L$ are the same as its elements in the tensor product basis $|i⟩^H|j⟩^J$ for $H \otimes J$.
Here $|i⟩^X$ is the standard basis vector $|i⟩$ for system $X$. Note that the order of
the tensor product matters: $W$ acting on $K \otimes L$ is the same as $\text{Swap} W \text{Swap}^\dagger$ acting on $L \otimes K$. We use several notations to help specify which of a set of
canonically isomorphic systems an operator acts on, or a basis element belongs
to: one is to use a superscript specifying the system, as we did above with basis
vectors; another, for tensor products of operators, is to put a subscript on the
tensor product sign used between the operators, related to subscripts used to
identify Hilbert spaces: Thus, for example, $X \otimes 12 Y$ would be interpreted as
acting on $H_1 \otimes H_2$ (and similarly $X \otimes AB Y$ acts on $H_A \otimes H_B$.

As usual, we think of $H_A$ and $H_B$ as representing spatially separated, “local”
subsystems of the total system $H_{\text{Total}} := H_A \otimes H_B$. There are also two non-local
subsystems of interest to us, namely, $(H_1 := H_{A_1} \otimes H_{B_1}$, and $H_2 := H_{A_2} \otimes H_{B_2}$.
We assume that $H_1$ is in a state represented by an operator $W$ and that $H_2$ is
in a maximally entangled pure state represented (up to normalization ) by the
rank-one projection

$$T = \frac{1}{n} \sum_{e,f \in E} |e⟩⟨e| |f⟩⟨f| = \frac{1}{n} \sum_{e,f \in E} |e⟩⟨f| \otimes_{A_1,B_2} |e⟩⟨f|,$$

where $E$ is some fixed (but arbitrary) orthonormal basis for $H_{A_2} \simeq H_{B_2}$. (Note
that it was not strictly necessary to use the subscript $A_1,A_2$ on the tensor product
sign on the right, since we identified the overall system as $H_2$ and the ordering
of $A_1, A_2$ was specified in the definition of $H_2$. We will sometimes, but not
always, leave subscripts off of tensor product signs when the system is otherwise
identified. Associativity of the ordered tensor product will also be used without
further ado.) The state of the total system is thus represented by $W \otimes 12 T$.
In the protocol, Alice makes a measurement on $H_A$, having $T$ (i.e. $T^A$) as a
possible outcome. If we calculate the state that the projection postulate would
describe for the system, conditional on that measurement outcome, we obtain,
up to normalization by $\alpha := \text{Tr} ((T \otimes_{AB} 1)(W \otimes 12 T))$,

$$(T \otimes_{AB} 1)(W \otimes 12 T)(T \otimes_{AB} 1).$$

Similarly, if Bob obtains $T$ as the outcome of a measurement on his system,
then the final total state will be, up to normalization,

$$(1 \otimes_{AB} T)(W \otimes 12 T)(1 \otimes_{AB} T).$$

6.3 Theorem Let $T$ and $U$ be as described above. Then for any operator $W$
on $H_1$,
\[(T \otimes_{AB} 1)(W \otimes_{12} T)(T \otimes_{AB} 1) = \alpha T \otimes_{AB} W,\]
and
\[(1 \otimes_{AB} T)(W \otimes_{12} T)(1 \otimes_{AB} T) = \alpha W \otimes_{AB} T.\]

Remark: If in the preceding formula we replace \(H_{B_1}\) by the one-dimensional Hilbert space \(\mathbb{C}\), so that \(H_B = H_{B_2}\), then we recover the usual teleportation scheme. The present scheme makes explicit the fact (which has been observed and exploited many times before) that the standard teleportation protocol serves to teleport not only a state, but its entanglement with a system not otherwise involved in the protocol.

It will be convenient to work with the “un-normalized state”
\[Q = nT = \sum_{e,f \in E} |e\rangle\langle f| \otimes |e\rangle\langle f|,\]
instead of \(T\). We’ll normalize later. We’ll make use of the following

6.4 Lemma Let \(E\) be an orthonormal basis for \(H = H_{A_1} = H_{B_2}\), as above. For any operator of the form \(A = |x\rangle\langle y| \otimes |u\rangle\langle v|\), with \(x, y, u, v \in E\),
\[QAQ = \begin{cases} Q & \text{if } x = u \text{ and } y = v; \\ 0 & \text{otherwise} \end{cases}\]

Proof: Direct computation yields
\[QAQ = \sum_{e,f,e',f'} (|e\rangle\langle f| \otimes |e\rangle\langle f|)(|x\rangle\langle y| \otimes |u\rangle\langle v|)(|e'\rangle\langle f'| \otimes |e'\rangle\langle f'|)) = \sum_{e,f,e',f'} (|e\rangle\langle f|)(|x\rangle\langle y|)(|e'\rangle\langle f'|) \otimes (|e\rangle\langle f|)(|u\rangle\langle v|)(|e'\rangle\langle f'|)) = \sum_{e,f,e',f'} \langle f|x\rangle\langle y|e'\rangle\langle f|u\rangle\langle v|e'\rangle\langle e\rangle\langle f'| \otimes |e\rangle\langle f'|\]

The inner products here are zero except where \(x = f = u\) and \(y = e' = v\). In other words, for the result to be non-zero, the input vector \(A\) must have the form \(A = |x\rangle\langle y| \otimes |x\rangle\langle y|\), and in this case we do in fact get \(QAQ = Q\). □

Proof of Theorem 6.3: The product basis \(E \otimes E = \{|a\rangle|b\rangle |a,b \in E\}\) gives us an operator basis for \(L(H_1)\), namely,
\[\{|a\rangle|b\rangle \langle c|\langle c| = |a\rangle|d\rangle \otimes |b\rangle|c\rangle : a, b, c, d \in E\}.\]
Expanding $W$ in this basis, we have

$$W = \sum_{a,b,c,d} W_{a,b,c,d} |a⟩⟨d| \otimes |b⟩⟨c|$$

Thus,

$$(W \otimes_{12} Q) = \sum_{a,b,c,d,e,f} W_{a,b,c,d} |a⟩⟨d| \otimes |e⟩⟨f| \otimes |e⟩⟨f| \otimes |b⟩⟨c|.$$ 

Applying $Q \otimes_{AB} 1$ to both sides, we have

$$(Q \otimes_{AB} 1)U(W \otimes_{12} Q)U(Q \otimes_{AB} 1) = \sum_{a,b,c,d,e,f} W_{a,b,c,d} Q(|a⟩⟨d| \otimes |e⟩⟨f|)Q \otimes_{AB} (|e⟩⟨f| \otimes |b⟩⟨c|).$$

According to our Lemma, we obtain non-zero terms only where $|a⟩⟨d| = |e⟩⟨f|$, i.e., where $e = a$ and $f = d$, and in this case we have $Q(|a⟩⟨d| \otimes |a⟩⟨d|)Q = Q$.

Thus, we end up with

$$\sum_{a,b,c,d} Q \otimes_{AB} (W_{a,b,c,d} |a⟩⟨d| \otimes |b⟩⟨c|) = Q \otimes_{AB} W.$$

Substituting $nT$ for $Q$ throughout yields the first line in the desired result (boxed equations in Theorem 6.3; the proof of the second line is entirely analogous.

**Remark:** In this version of teleportation, Bob has some access to the state to be teleported, through $W$’s marginal on $H_{B_1}$. Note that this remains unchanged after teleportation. (Indeed, whether we place $W$ on $H_1 \equiv H_{A_1} \otimes H_{B_1}$ or on $H_{A_1} \otimes H_{B_2}$, teleportation always “pivots” $W$ about whichever component of Bob’s system partakes of $W$. In particular, if we start with $W$ on $H_{A_1} \otimes H_{B_2}$, (and switch $T$ compatibly as well) we end up, not with $W$ on $H_{B_2} \otimes H_{B_1}$, but with $W$ on $H_{B_1} \otimes H_{B_2}$. )

We have only shown what happens conditional on a particular outcome of Alice’s measurement, the standard entangled state $T$. It is straightforward to verify that if the other outcomes of Alice’s measurement are the rest of a complete basis of maximally entangled states (equivalently, as shown e.g. in [50]), if the other outcomes correspond to projectors $T_V := (V \otimes_{AB} I)T(V^† \otimes_{AB} T)$ where $V$ varies over the elements (except $I$) of an orthogonal unitary basis for the local operators), then a similar result obtains:

**6.5 Theorem** Let $T$ and $U$ be as described above. Then for any operator $W$ on $H_1$,

$$(T_V \otimes_{AB} 1)(W \otimes_{12} T)(T_V \otimes_{AB} 1) = aT \otimes_{AB} V^†WV^*.$$  
and

$$(1 \otimes_{AB} T_V)(W \otimes_{12} T)(1 \otimes_{AB} T_V) = aVWV^† \otimes_{AB} T.$$
Theorem has the following immediate consequence:

**6.6 Corollary** Let $T$, $W$ and $U$ be as above (with $W$ normalized). Suppose the state $X = W \otimes_{AB} T$ is POPT on $H_A \otimes H_B$, i.e., suppose that $\text{Tr} ((A \otimes_{AB} B)X) \geq 0$ for all positive operators $A, B$ on $H$. Then $W$ is positive.

**Proof:** Let $A = T$. Note that $T \otimes_{AB} B = (1 \otimes_{AB} B)(T \otimes_{AB} 1)$. Thus,

$$\text{Tr} ((T \otimes_{AB} B)X) = \text{Tr} ((T \otimes_{AB} B)X(T \otimes_{AB} B))$$

$$= \text{Tr} ((1 \otimes_{AB} B)(T \otimes_{AB} 1)X(T \otimes_{AB} 1)(1 \otimes_{AB} B))$$

$$= \text{Tr} ((1 \otimes_{AB} B)(T \otimes_{AB} W)(1 \otimes_{AB} B))$$

$$= \text{Tr} (WB).$$

By assumption, this is non-negative for all positive operators $B$ on $H$; hence, $W \geq 0$. (The third equality in the displayed equations used Theorem 6.3 with $\alpha = 1$ because $\text{Tr} W = 1$.)

Thus, notwithstanding that $W$ and $T$ are POPT on $H_1$ and $H_2$ respectively, if $W$ is not positive, then $W \otimes_{12} T$ is not POPT on $H_A$ and $H_B$. This reflects the fact that pure tensors in $H_A \otimes H_B$ typically involve entanglements between $H_{A_1}$ and $H_{B_1}$.

In other words, if locally, i.e. at Alice’s and Bob’s sites, systems combine according to the usual quantum rules, and in particular, measurements with “entangled outcomes” like the outcome $T$ are permitted, then POPT but non-positive states cannot occur as independent states of “nonlocal subsystems” (subsystems such as $H_1, H_2$) of the Alice-Bob system. Since specifying a Hermitian operator such as $W \otimes_{12} T$ specifies all probabilities for outcomes $|w\rangle|x\rangle|y\rangle|z\rangle$, and since the projectors onto these span the Hermitian operators, we have established Proposition 6.2.

It is worth noting that although Propositions 6.3 and 6.5 describe operators that could be interpreted as the unnormalized overall conditional state after Alice gets various Bell-measurement results ($T$ or $T_Y$), if the “projection-postulate” dynamics applies, this does not mean that one might get around the difficulty pointed out in Corollary 6.6 by supposing the actual conditional dynamics are not described by the projection postulate. The Propositions are used in the Corollary only for the purpose of calculating probabilities of certain one-shot outcomes (that exhibit no Alice-Bob entanglement), and for these the evolution of the probability state after the measurement, i.e. the probabilities of subsequent measurements, are irrelevant. The “projected” states occur only under the trace in the calculations leading to Corollary 3, and so are only used
there in calculating the probabilities of outcomes corresponding to the projectors, a proper application whatever the subsequent dynamics may be.

It is also worth noting that if only “1-LOCC” measurements and operations are permitted locally (where the “locality” of this 1-LOCC is now a “fictitious” locality with respect to the product $H_{A_1} \otimes H_{A_2}$, or the product $H_{B_2} \otimes H_{B_1}$), then Alice cannot perform the measurement whose outcomes are maximally entangled states of $H_{A_1} \otimes H_{A_2}$, so the above argument causes no problem. We conjecture (and it is probably simple to prove) that in this case the state $U(W \otimes T)U$ is a legitimate state, that is, a state on the test space $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ themselves “1-LOCC” test spaces $(X_1, \mathfrak{A}_1, \mathfrak{A}_2)$, $(Y_2, \mathfrak{B}_2, \mathfrak{B}_1)$ made by combining standard quantum E-test spaces like $(X, \mathfrak{A})$.

6.7 Remarks: further discussion of Corollary 6.6.

One can envision a variety of reactions to this, including:

(A) The argument shows that POPT but non-positive “states” are un-physical.

(In fact, if we apply the states/maps isomorphism, we have here a non-standard proof of the standard observation that the extension of a positive but not CP map $\phi : \mathcal{L}(H) \to \mathcal{L}(H)$ to a map $\mathcal{L}(H \otimes H) \to \mathcal{L}(H \otimes H)$ ($H$ finite dimensional) needn’t be positive.)

(B) The result shows that non-positive POPT states on subsystems don’t extend to POPT states on larger systems. So what? Who says states on subsystems should generally extend to states on larger systems?

As regards potential applications of the nonstandard tensor product in physics, (A) seems relevant. Of course, at least for most physical situations we know quantum mechanics does work, and while we have a reasonable amount of evidence for the existence of standard quantum entangled states, we have so far observed no statistics like those of the nonpositive but POPT states. So even without bringing in the mathematical considerations of (A), we might be disinclined to consider such states. However, the lack of evidence could, of course, just be ascribed to the fact that we haven’t looked hard enough, or that such states, for some reason, occur in exotic contexts not yet adequately physically probed. However, one then must take the attitude expressed in observation (B)—but it may be difficult to reconcile this attitude with existing physics, including the existence of standard quantum entangled states of subsystems seemingly independent from other systems.

6.8 Resumé. The above results are inspired by an intuitive argument that, within the FNS coupling of A and B, teleportation gives rise to a post-Bell-measurement state on Bob’s side that would be nonpositive if Alice and Bob
had teleported a nonpositive POPT state. In light of the results above, we can make this more precise by saying that there is no way to consistently do both of the following simultaneously:

(1) assign probabilities to the outcome-pair consisting of a Bell-outcome (such as would occur in the teleportation protocol) on Alice’s side, and a particular “locally entangled” measurement outcome (dependent on the Bell outcome, although in teleportation this dependence could be removed via classical communication and local adjustments made by Bob) on Bob’s side

and

(2) assign to all outcomes that are not only pairs of Alice and Bob outcomes, but in which Alice’s outcome is a pair of an $A_1$ and an $A_2$ outcome (i.e. “locally unentangled”), and similarly for Bob, the probabilities given by the product of a Bell state of $A_1B_1$, and a nonpositive POPT state of $A_2B_2$.

It is nice that the argument does not depend on Bob’s doing the required adjustments to actually get the nonpositive POPT state on his side, for thus we can avoid issues of dynamics, of what Bob can do in addition to measuring. Rather, Bob’s state conditional on at least one (actually, all) of Alice’s measurement outcomes is some nonpositive POPT state, though only for one of the measurement outcomes is it the POPT state that would be teleported in a full teleportation protocol (otherwise it’s an appropriate unitary transform of that state).

7 Conclusion

The results of our final section cast some doubt on the potential relevance of nonpositive POPT states as models for undiscovered physics, though they are hardly decisive against this possibility. Regardless of their relevance or lack of it as potential models for phenomena in our physical world, though, nonpositive POPT states and the test spaces they live on remain interesting and relevant for the theoretical understanding of how systems can combine, and how this can affect systems’ information-processing capabilities\(^{11}\). Indeed, they even remain relevant to our understanding of quantum mechanics, for they can be interpreted as representing the failure of two particular very natural way (and, it turned out, equivalent) of trying to obtain the quantum-mechanical rules for constructing composite systems: from local quantum mechanics and no-signalling, or from local quantum mechanics and 1-LOCC. This raises the question, which we will discuss in future publications, of what additional natural requirements (satisfied in the quantum mechanical cases) might be imposed on notions such as test space or E-test space, so that combining them and imposing no-signalling/1-LOCC

\(^{11}\)Other theoretical studies along this line include Popescu, Rohrlich, and collaborators work on nonlocal correlations \([29, 28, 1]\) though there the restriction to quantum marginals maintained in our work is absent in most cases.
LOCC gives the quantum tensor product. Moreover, while this kind of system combination apparently cannot be easily mixed with the standard quantum one (at least, not without paying the price that product states do not universally exist, endangering the interpretation of the factors as “subsystems” in the usual sense), it can still be used in a thoroughgoing way to combine systems in a fashion different from quantum mechanics, and investigation of information processing in this theory could illuminate general questions of what properties of a theory are needed to do what information-processing tasks.

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References

[1] Barrett, J., N. Linden, S. Massar, S. Pironio, S. Popescu and David Roberts, Nonlocal correlations as an information-theoretic resource, Phys. Rev. A 71, 022101, 2005.

[2] L. J. Bunce and J. D. Maitland-Wright, The Mackey-Gleason Problem, Bulletin of the AMS 26 (1992) 288-293.

[3] P. Busch, Quantum states and generalized observables: a simple proof of Gleason’s theorem, Phys. Rev. Lett. 91, 120403 (2003).

[4] Busch, P., and Lahti, P., Remarks on separability of compound quantum systems and time reversal, Found. Phys. Lett. 10 (1997), pp. 113-117.

[5] Caves, C. M. and C. A. Fuchs and K. Manne and J. M. Renes, Gleason-type derivations of the quantum probability rule for generalized measurements, Found. Phys. 34, p. 193, 2004.

[6] Choi, M-D. Completely Positive Maps on Complex Matrices, Lin. Alg. Apl. 10 285-290, 1975.

[7] Dvurečenskij, A. Tensor product of difference posets, Trans. Am. Math. Soc. 346, pp. 1043–1057.

[8] D. J. Foulis and M.K. Bennet, Tensor products of orthoalgebras, Order 10 (1993) 271-282.

[9] D. J. Foulis and M.K. Bennet, 1994. Effect algebras and unsharp quantum logics, Found. Phys. 24, pp. 1325–1346.

[10] D. J. Foulis and C. H. Randall, What are quantum logics and what ought they to be?, in Beltrametti and van Fraassen (eds.), Current Issues in Quantum Logic, pp. 35–52. Plenum, 1980.
[11] D. J. Foulis and C. H. Randall, Empirical logic and tensor products, in B. Hartkamper and H. Neumann (eds.), Interpretations and Foundations of Quantum Mechanics: Proceedings of a Conference held [sic.] in Marburg 29-30 May, 1979, pp. 9–20. Zürich: Bibliographisches Institut Wissenschaftsverlag, 1981.

[12] Fuchs, C. A., Quantum Foundations in the light of quantum information, in A. Gonis and P. E. A. Turchi (eds.), Proceedings of the NATO Advanced Research Workshop on Decoherence and its Implications in Quantum Computation and Information Transfer. IOS Pr Inc., 2001.

[13] Fuchs, C. A., Quantum mechanics as quantum information (and only a little more), quant-ph/0205039.

[14] C. A. Fuchs and K. Jacobs, Information tradeoff relations for finite-strength quantum measurements, Phys. Rev. A 63, 062305 (2001).

[15] Gleason, A. M., Measures on the Closed Subspaces of Hilbert Space, J. Mathematics and Mechanics 6, pp. 885–893 (1957).

[16] Gudder, S., 1997. Effect Test Spaces, Int. J. Theoretical Physics, 36, p. 2681.

[17] Horn, R. A. and Ch. R. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[18] M. Horodecki, P. Horodecki and R. Horodecki, Physics Letters A 223, pp. 1–8, 1996.

[19] A. Jamiołkowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rev. Mod. Phys. 3, pp. 275–278, 1978.

[20] Kadison, R. V. and J. R. Ringrose Fundamentals of the Theory of Operator Algebras, vol. 1, Graduate Studies in Mathematics vol. 15, Providence, RI: American Mathematical Society (1997).

[21] Kláy, M. Einstein-Podolski-Rosen experiments: the structure of the probability space I, Found. Phys. Lett. 1, pp. 205–244, 1988; Einstein-Podolski-Rosen experiments: the structure of the probability space II, Found. Phys. Lett. 1, pp. 305–319, 1988.

[22] Kláy, M., Randall, C., and Foulis, D., Tensor products and probability weights, Int. J. Theor. Phys. 26 (1987) 199-219

[23] Köpka, F. and F. Chovanec, 1994. D-posets, Mathematica Slovaca, 44, pp. 21–34.

[24] K. E. Hellwig and K. Kraus, Pure operations and measurements, Commun. Math. Phys., 11, 214-220 (1969).
[25] K. E. Hellwig and K. Kraus, *Operations and measurements: II*, Commun. Math. Phys. 16 142-147 (1970).

[26] Kraus, *States, Effects and Operations*, Springer Lecture Notes in Physics 190, Springer Verlag: Berlin, 1983.

[27] Namioka, I., and Phelps, R. R. *Tensor Products of Compact Convex Sets*, Pacific J. Math 9, p. 469, 1969.

[28] I. Pitowsky, *Quantum Probability/Quantum Logic*, Springer-Verlag, 1989.

[29] S. Popescu and D. Rohrlich, *Nonlocality as an axiom of quantum theory*, Annals of the Israel Physical Society, 12, pp. 152–156, 1996. (Also quant-ph/9508009, 1995).

[30] *Quantum nonlocality as an axiom*, Foundations of Physics 24, pp. 379–385, 1994.

[31] Randall and Foulis, *Operational statistics and tensor products*, in H. Neumann (ed.), *Interpretations and Foundations of Quantum Theory* pp. 21–28, B.I. Wissenschaft: Mannheim (1981).

[32] O. Rudolph and J.D. Maitland Wright, *On unentangled Gleason theorems for quantum information theory*, quant-ph/0004036

[33] Stinespring, *Positive Functions on C*-algebras*, Proc. Amer. Math. Soc. 6 211-216, 1955

[34] Störmer, *Positive Linear Maps of Operator Algebras*, Acta Math.110 233-278 1963.

[35] Wallach, N. R., *An Unentangled Gleason’s Theorem*, quant-ph/0002058

[36] , R. F. Werner, “All teleportation and dense coding schemes,” J. Phys. A: Math. Gen. 34, p. 7081, 2001.

[37] Wilce, A., *Tensor Products of Frame Manuals*, Int. J. Theor. Phys. 29 (1990) 805-814.

[38] Wilce, A., *The Tensor Product in Generalized Measure Theory*, Int. J. Theor. Phys. 31 (1992) 1915-1928.

[39] Wilce, A., *Spaces of Vector-Valued Weights on Test Spaces*, ancient preprint.

[40] Wittstock, G., *Tensor Products of Ordered Vector Spaces*, in Hartkämper and Neumann, *Foundations of Quantum-Mechanics and Ordered Linear Spaces*, Springer Lecture Notes in Physics 29, 1974.