Conditional Interior and Conditional Closure of Random Sets

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Abstract
In this paper, we introduce two new types of conditional random set taking values in a Banach space: the conditional interior and the conditional closure. The conditional interior is a version of the conditional core, as introduced by A. Truffert and recently developed by Lépinette and Molchanov, and may be seen as a measurable version of the topological interior. The conditional closure is a generalization of the notion of conditional support of a random variable. These concepts are useful for applications in mathematical finance and conditional optimization.

Keywords  Conditional random set · Conditional optimization · Super-hedging problem · European option · Mathematical finance

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1 Introduction
The conditional essential supremum and infimum of a real-valued random variable have been introduced in [1]. A generalization is then proposed in [2] for vector-valued random variables with respect to random preference relations. Actually, these two concepts are related to the notion of conditional core as first introduced in [3]1 and developed in [4] for random sets in separable Banach spaces, with respect to a complete

1 E. Lépinette apologizes to A. Truffert for not having quoted her paper [4].

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σ-algebra \( \mathcal{H} \). A conditional core of a set-valued mapping \( \Gamma(\omega) \), \( \omega \in \Omega \), is defined as the largest \( \mathcal{H} \)-graph measurable random set \( \Gamma'(\omega) \) such that \( \Gamma'(\omega) \subset \Gamma(\omega) \). This concept provides a natural conditional risk measure that generalizes the concept of essential infimum for multi-asset portfolios in mathematical finance. Applications are deduced for geometrical market models with transaction costs: see [4,5] and the theory with transaction costs developed in [6].

In this paper, we first introduce the open version of the conditional core as proposed in [3,4]. Precisely, if \( \mathcal{H} \) is a complete sub-σ-algebra on a probability space, the conditional interior (or open conditional core) of a set-valued mapping \( \Gamma(\omega) \), \( \omega \in \Omega \), is defined as the largest \( \mathcal{H} \)-measurable random open set \( \Gamma'(\omega) \) such that \( \Gamma'(\omega) \subset \Gamma(\omega) \) \( \mathbb{P} \)-almost every \( \omega \in \Omega \). It may be seen as a measurable version of the classical interior in topology. One of our main contributions is to show the existence and uniqueness of such a conditional interior for an arbitrary random set in a separable Banach space. Then, the dual concept, the conditional closure, is introduced as a generalization of the conditional support of a real-valued random variable to a family of vector-valued random variables. A numerical application of the latter is deduced in conditional random optimization: We show that an essential supremum is a pointwise supremum on a conditional closure.

The paper is organized as follows. In Sect. 2, we recall the definition and usual properties of measurable random sets in Banach spaces. Then, we introduce the notion of conditional interior and show the existence of such sets in Banach spaces. In Sect. 3, the conditional closure is introduced and an application in conditional optimization is formulated in the next section. At last, we present an application in mathematical finance.

### 2 Conditional Interior

In all the paper, we consider a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a complete sub-σ-algebra \( \mathcal{H} \) of \( \mathcal{F} \). Set-valued mappings we consider take their values in the family of all subsets of a separable Banach space \( \mathcal{X} \) equipped with its Borel σ-algebra \( \mathcal{B}(\mathcal{X}) \). In the following, we first recall the concept of graph measurable random set. We then introduce the concept of conditional interior of a random set. Existence is deduced from the existence of the conditional core of a closed random set, see [4], even if it is not always non-empty.

Recall that a random set \( \Gamma(\omega) \), \( \omega \in \Omega \), is a set-valued mapping that assigns to each \( \omega \in \Omega \) a subset \( \Gamma(\omega) \) of \( \mathcal{X} \). We say that \( \Gamma \) is \( \mathcal{H} \)-measurable if, for any open set \( O \subseteq \mathcal{X} \),

\[
\{ \omega \in \Omega, \Gamma(\omega) \cap O \neq \emptyset \} \in \mathcal{H}.
\]

We say that \( \Gamma \) is \( \mathcal{H} \)-graph measurable, if

\[
\text{graph } \Gamma := \{(\omega, x) \in \Omega \times \mathcal{X} : x \in \Gamma(\omega)\} \in \mathcal{H} \otimes \mathcal{X}.
\]
When the sets $\Gamma(\omega)$ are closed (resp. open) $\mathbb{P}$-almost for all $\omega \in \Omega$, we shall say that $\Gamma$ is closed (resp. open).

**Remark 2.1** If $\Gamma$ is closed-valued and the $\sigma$-algebra is complete, then $\Gamma$ is $\mathcal{H}$-graph measurable, if and only if it is $\mathcal{H}$-measurable: see [7, Definition 1.3.1].

We say that a $\mathcal{F}$-measurable random variable $\xi : \Omega \to \mathcal{X}$ is an $\mathcal{F}$-measurable selection of $\Gamma$ if $\xi(\omega) \in \Gamma(\omega)$ for almost all $\omega \in \Omega$. The set of such selections is denoted by $L^0(\Gamma, \mathcal{F}) \subseteq L^0(\mathbb{R}^d, \mathcal{F})$ where the set $L^0(\mathcal{X}, \mathcal{F})$ of all $\mathcal{X}$-valued $\mathcal{F}$-measurable random variables is equipped with the metric of convergence in probability.

The following result may be found in [8, Th. 4.4] and, when applied, it is usually mentioned as measurable selection argument, see also [7, Theorem 2.3].

**Theorem 2.1** (Measurable selection argument) If $\Gamma$ is an $\mathcal{F}$-graph measurable random set which is non-empty and closed a.s., then $L^0(\Gamma, \mathcal{F}) \neq \emptyset$.

The following result is proven in [4, Proposition 2.7] whenever the random set we consider is closed or not. We denote by $\text{cl}(X)$ the closure of any subset $X \subseteq \mathcal{X}$ with respect to the norm topology. The closure of any subset of $L^0(\mathcal{X}, \mathcal{F})$ is taken with respect to the metric topology of $L^0(\mathcal{X}, \mathcal{F})$.

**Proposition 2.1** Suppose that $\mathcal{F}$ is complete. Suppose that $\Gamma$ is an $\mathcal{F}$-graph measurable random set which is non-empty a.s. Then, $\text{cl}(\Gamma)$ is $\mathcal{F}$-graph measurable and admits a Castaing representation, i.e., there exists a countable family $(\xi_n)_{n \geq 1}$ of measurable selections of $\Gamma$ such that

$$
\text{cl}(\Gamma(\omega)) = \text{cl}(\xi_n(\omega) : n \geq 1), \quad \omega \in \Omega.
$$

Moreover, we have $\text{cl}(L^0(\Gamma, \mathcal{F})) = L^0(\text{cl}(\Gamma), \mathcal{F})$.

**Corollary 2.1** If $\Gamma$ is an $\mathcal{F}$-graph measurable random set, then its topological interior $\text{int}(\Gamma(\omega))$, defined for each $\omega \in \Omega$, and its boundary $\partial \Gamma(\omega)$ are also $\mathcal{F}$-graph measurable.

**Proof** Indeed, consider $Y(\omega) = \mathcal{X} \setminus \Gamma(\omega)$. It is clear that $Y$ is $\mathcal{F}$-graph measurable. Moreover, we have

$$
\text{int}(\Gamma(\omega)) = \text{int}(\mathcal{X} \setminus Y(\omega)) = \mathcal{X} \setminus \text{cl}(Y(\omega)),
$$

where $\text{cl}(Y(\omega))$ is $\mathcal{F}$-graph measurable by the proposition above, hence, the complement set int $(\Gamma(\omega))$ is $\mathcal{F}$-graph measurable. We also deduce that the boundary $\partial \Gamma(\omega) = \text{cl}((X(\omega)) \setminus \text{int}(X(\omega))$ is $\mathcal{F}$-graph measurable. \(\square\)

For $x \in \mathcal{X}$ and $r \geq 0$, $B(x, r)$ denotes the open ball in $\mathcal{X}$ of center $x$ and radius $r$ and $\bar{B}(x, r)$ is its closure. For any $A \subseteq \mathcal{X}$ and $\lambda \in \mathbb{R}$, we use the convention that $\lambda \times A = \{\lambda a : a \in A\}$. In particular, $0 \times A = \{0\}$. Recall the following definition, see the paper by Truffert [3,4].

**Definition 2.1** The $\mathcal{H}$-conditional core $\text{m}(\Gamma|\mathcal{H})$, of a set-valued mapping $\Gamma$, is the largest $\mathcal{H}$-graph measurable random set $\Gamma'$ such that $\Gamma'(\omega) \subseteq \Gamma(\omega)$ a.s.
Note that $m(\Gamma|\mathcal{H})$ is the largest subset in the sense that, if $\tilde{\Gamma}$ is another $\mathcal{H}$-graph measurable random set contained in $\Gamma$ a.s., then $\tilde{\Gamma}(\omega) \subseteq m(\Gamma|\mathcal{H})(\omega)$ $P$-almost all $\omega \in \Omega$. The following result is proved in [4] and is an extension of the primal result of [3]. Recall that, if $\Gamma$ is a random set, $L^0(\Gamma, \mathcal{H}) \subseteq L^0(\mathcal{X}, \mathcal{H})$ is the set of all $\mathcal{H}$-measurable random variables $\gamma$ with values $\gamma(\omega) \in \Gamma(\omega)$ a.s.

**Proposition 2.2** Suppose that $\Gamma$ is a closed $F$-graph measurable set that may be empty. Then, the conditional core $m(\Gamma|\mathcal{H})$ exists and we have

$$L^0(m(\Gamma|\mathcal{H}), \mathcal{H}) = L^0(\Gamma, \mathcal{H}).$$

If $\gamma \in L^0(\mathcal{X}, \mathcal{H})$ and $r \in L^0([0, \infty), \mathcal{F})$, $m(\widetilde{B}(\gamma, r)|\mathcal{H}) = \widetilde{B}(\gamma, \text{ess inf}_\mathcal{H} r)$. The conditional core plays a role in mathematical finance as it naturally appears when considering the dynamics of a self-financing discrete-time portfolio process $(V_t)_{t=0}^T$ of the form $V_{t-1} \in V_t + G_t$, $t \geq 1$, where $G_t$ is the solvency set, see [6], i.e., $V_{t-1} \in m(V_t + G_t|\mathcal{F}_{t-1})$, see [5]. When the $\sigma$-algebra is trivial, the conditional core becomes the set of fixed points of $X$; it is also related to the essential intersection considered in [9]. Notice that the $\mathcal{H}$-measurable conditional core is mainly independent of $F$, i.e., instead of completing the underlying measurable space with respect to a single (or a dominated family of) probability measure(s), one may pass to the universal completion, and hence, avoid postulating the existence of any reference measure.

We now introduce an open version of the conditional core:

**Definition 2.2** The $\mathcal{H}$-graph measurable interior $o(\Gamma|\mathcal{H})$ of a set-valued mapping $\Gamma$ is the largest $\mathcal{H}$-graph measurable random open set $\Gamma'$ such that $\Gamma'(\omega) \subseteq \Gamma(\omega)$ a.s.

Notice that such a conditional interior is necessarily unique by definition. We first show the existence of the conditional interior, when $\Gamma$ is open-valued.

**Theorem 2.2** Let us consider an $F$-graph measurable open random set $O$. Then, there exists a unique $\mathcal{H}$-graph measurable interior of $O$.

**Proof** By Proposition 2.1, $\text{cl} O$ and $\partial O := \text{cl} O \setminus O$ are closed $\mathcal{F}$-graph measurable random sets, see Corollary 2.1. When $\partial O(\omega) = \emptyset$, we define $d(x, \partial O(\omega)) = \infty$ for all $x \in \mathcal{X}$. Otherwise, $\partial O$ admits a Castaing representation on $[\partial O \neq \emptyset]$ by Proposition 2.1. We deduce that the random mapping $(\omega, x) \mapsto d(x, \partial O(\omega))$ is $\mathcal{F} \otimes B(\mathcal{X})$-measurable. Therefore, the random sets

$$F^n := \{ x : d(x, \partial O) \geq 1/n \} \cap \text{cl} O, \quad n \geq 1,$$

are closed $\mathcal{F}$-graph measurable random subsets of $O$ and we have $O = \bigcup_n F^n$. Let us define the $\mathcal{H}$-graph measurable open random set (see Corollary 2.1 and Proposition 2.2):

$$o(O|\mathcal{H}) = \text{int} \left( \bigcup_n m(F^n|\mathcal{H}) \right) \subseteq O.$$
Let us show that this is the largest $\mathcal{H}$-graph measurable open random subset of $\mathcal{O}$. To do so, let $\mathcal{O}_\mathcal{H} \subseteq \mathcal{O}$ be a $\mathcal{H}$-graph measurable open subset of $\mathcal{O}$. As for $\mathcal{O}$, we may write $\mathcal{O}_\mathcal{H} = \bigcup_n H^n$ where $H^n := \{ x : d(x, \partial \mathcal{O}_\mathcal{H}) \geq 1/n \} \cap \text{cl} \mathcal{O}_\mathcal{H}$ are $\mathcal{H}$-graph measurable closed subsets of $\mathcal{O}_\mathcal{H}$.

We claim that $H^n \subseteq F^n$. Indeed, if $x \in H^n$, it suffices to show that $d(x, \partial \mathcal{O}_\mathcal{H}) \leq d(x, \partial \mathcal{O})$ a.s. In the contrary case, on a non-null set, there exists $o \in \partial \mathcal{O}$ such that $o \in B(x, d(x, \partial \mathcal{O}_\mathcal{H}))$. By Lemma 2.1, this implies that $o \in \mathcal{O}_\mathcal{H} \subseteq \mathcal{O}$ which yields a contradiction.

As $H^n$ is $\mathcal{H}$-graph measurable and closed, $H^n \subseteq \mathcal{m}(F^n \mid \mathcal{H})$ for all $n$. We deduce that $\mathcal{O}_\mathcal{H} \subseteq \bigcup_n \mathcal{m}(F^n \mid \mathcal{H})$, hence, $\mathcal{O}_\mathcal{H} \subseteq \mathcal{O}(\mathcal{O} \mid \mathcal{H})$. \hfill \square

We then deduce the general case:

**Theorem 2.3** For any $\mathcal{F}$-graph measurable random set $\Gamma$, the $\mathcal{H}$-graph measurable interior of $\Gamma$ exists and $\mathcal{O}(\Gamma \mid \mathcal{H}) = \mathcal{O}(\text{int} \Gamma \mid \mathcal{H})$ is unique.

The following lemma is recalled for the sake of completeness. It is used in the proof above.

**Lemma 2.1** Let $\mathcal{O}$ be an open set in a normed space. For every $x \in \mathcal{O}$, $B(x, d(x, \partial \mathcal{O})) \subseteq \mathcal{O}$.

**Proof** Let us consider $r^* = \sup R$ where $R$ is the non-empty set of all $r > 0$ such that $B(x, r) \subseteq \mathcal{O}$. It is trivial that $B(x, r^*) \subseteq \mathcal{O}$. So, for all $o \in \partial \mathcal{O}$, $d(x, o) \geq r^*$ hence $r^* \leq d(x, \partial \mathcal{O})$. Moreover, by definition of $r^*$, for all $n$ there exists $z^n \in B(x, r^* + n^{-1})$ such that $z^n \notin \mathcal{O}$ hence $r^* \leq \|z^n - x\| \leq r^* + n^{-1}$. As the sequence $(z^n)_n$ is bounded, we deduce by a compactness argument that for a subsequence $z_n \rightarrow z$ as $n \rightarrow \infty$. Then, $\|z - x\| = r^*$ hence $z \in \text{cl} \mathcal{O}$. In the case where $z \in \mathcal{O}$, $z^n \in \mathcal{O}$ for $n$ large enough since $\mathcal{O}$ is open, which yields a contradiction. Therefore, $z \in \partial \mathcal{O}$. This implies that $r^* = d(x, z) \geq d(x, \partial \mathcal{O})$ and finally $r^* = d(x, \partial \mathcal{O})$. \hfill \square

## 3 Conditional Closure

We now introduce the concept of conditional closure. The existence is proved in the following theorem which is the second main contribution of this paper. As previously, $\mathcal{F}$ and $\mathcal{H}$ are supposed to be complete with respect to some probability measure $\mathcal{P}$.

**Definition 3.1** The $\mathcal{H}$-graph measurable conditional closure $\text{cl} (\Gamma \mid \mathcal{H})$ of a set-valued mapping $\Gamma$ is the smallest $\mathcal{H}$-graph measurable random closed set $\Gamma'$ such that $\Gamma(\omega) \subseteq \Gamma'(\omega)$ a.s.

**Theorem 3.1** For any $\mathcal{F}$-graph measurable random set $\Gamma$, the $\mathcal{H}$-graph measurable conditional closure $\text{cl} (\Gamma \mid \mathcal{H})$ of $\Gamma$ exists and is unique. We have

$$\text{cl} (\Gamma \mid \mathcal{H}) = \mathcal{X} \setminus \mathcal{O}(\mathcal{X} \setminus \Gamma \mid \mathcal{H}).$$

Moreover, for all measurable selection $\gamma$ of $\text{cl} (\Gamma \mid \mathcal{H})$ and $\varepsilon \in \mathcal{L}^0(0, \infty, \mathcal{H})$, for all $H \in \mathcal{H}$ such that $\mathcal{P}(H) > 0$, we have $\mathcal{P}((\Gamma \cap B(\gamma, \varepsilon) \neq \emptyset) \cap H) > 0$. 

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Proof The first part is a direct consequence of Corollary 2.3. Indeed, we first observe that \( \Gamma \subseteq \mathcal{X} \setminus \partial(\mathcal{X} \setminus \Gamma | \mathcal{H}) \). Moreover, if \( \Gamma_{\mathcal{H}} \) is a closed-valued \( \mathcal{H} \)-graph measurable set containing \( \Gamma \), then \( \mathcal{X} \setminus \Gamma_{\mathcal{H}} \subseteq \mathcal{X} \setminus \Gamma \). We deduce that \( \mathcal{X} \setminus \Gamma_{\mathcal{H}} \subseteq \partial(\mathcal{X} \setminus \Gamma | \mathcal{H}) \) and, finally, \( \mathcal{X} \setminus \partial(\mathcal{X} \setminus \Gamma | \mathcal{H}) \subseteq \Gamma_{\mathcal{H}} \). Suppose that \( \mathcal{P}(\{ \Gamma \cap B(\gamma, \varepsilon) \neq \emptyset \} \cap \mathcal{H}) = 0 \) for some \( \mathcal{H} \in \mathcal{H} \) and \( \varepsilon \in \mathcal{L}^0((0, \infty), \mathcal{H}) \). Therefore, by definition of the conditional closure as a smallest set, we have

\[
\text{cl}(\Gamma | \mathcal{H}) = \text{cl}(\Gamma | \mathcal{H}) 1_{\Omega \setminus \mathcal{H}} + \text{cl}(\Gamma | \mathcal{H}) \cap (\mathcal{X} \setminus B(\gamma, \varepsilon)) 1_H.
\]

Indeed, the \( \mathcal{H} \)-graph measurable set in the r.h.s. above contains \( \Gamma \) by assumption; hence, it contains \( \text{cl}(\Gamma | \mathcal{H}) \). We get a contradiction since \( \gamma \in \mathcal{L}^0(\text{cl}(\Gamma | \mathcal{H}), \mathcal{H}) \) a.s. by assumption. Uniqueness is clear as the conditional closure of \( \Gamma \) is the smallest closed set, up to a negligible set, in the sense that it is included in any other \( \mathcal{H} \)-measurable set containing \( \Gamma \).

\[\square\]

Remark 3.1 We may deduce the conditional support of a random variable \( X \in \mathcal{L}^0(\mathcal{X}, \mathcal{F}) \). Precisely, there exists a smallest \( \mathcal{H} \)-graph measurable random closed set denoted by \( \text{supp}_{\mathcal{H}}(X) \) such that \( \mathcal{P}(X \in \text{supp}_{\mathcal{H}}(X)) = 1 \). It is given by \( \text{supp}_{\mathcal{H}}(X) = \text{cl}(\{X\} | \mathcal{H}) \) and is called the \( \mathcal{H} \)-conditional support of \( X \). Moreover, for all \( \gamma \in \mathcal{L}^0(\text{supp}_{\mathcal{H}}(X), \mathcal{H}) \) and \( \varepsilon \in \mathcal{L}^0([0, \infty], \mathcal{H}) \), for all \( \mathcal{H} \in \mathcal{H} \) such that \( \mathcal{P}(H) > 0 \), we have \( \mathcal{P}(\{X \in B(\gamma, \varepsilon)\} \cap \mathcal{H}) > 0 \).

Notice that the conditional support is necessarily non-empty a.s. hence, it admits a measurable selection. In the following, we adopt the notation \( kA = \{ka : a \in A\} \) for any subset \( A \subseteq \mathcal{X} \) and \( k \in A \).

Lemma 3.1 Let \( \mathcal{H} \in \mathcal{H} \). Then, \( \text{cl}(\Gamma 1_H | \mathcal{H}) = 1_H \text{cl}(\Gamma | \mathcal{H}) \), for every \( \mathcal{F} \)-graph measurable set \( \Gamma \).

Proof First, observe that \( \mathcal{P}(\Gamma 1_H \subseteq 1_H \text{cl}(\Gamma | \mathcal{H})) = 1 \) as \( 1_H \text{cl}(\Gamma | \mathcal{H}) = \{0\} \) on \( \Omega \setminus \mathcal{H} \). We deduce that \( \text{cl}(\Gamma 1_H | \mathcal{H}) \subseteq 1_H \text{cl}(\Gamma | \mathcal{H}) \) a.s. and \( \text{cl}(\Gamma 1_H | \mathcal{H}) = \{0\} \) on \( \Omega \setminus \mathcal{H} \). Let us define \( Z = \text{cl}(\Gamma 1_H | \mathcal{H}) 1_H + \text{cl}(\Gamma | \mathcal{H}) 1_{\Omega \setminus \mathcal{H}} \). As \( \Gamma \subseteq 1_H \) on \( \mathcal{H} \), we deduce that \( \Gamma \subseteq Z \) a.s. hence \( \text{cl}(\Gamma | \mathcal{H}) \subseteq Z \) a.s. Therefore, we deduce that \( \text{cl}(\Gamma | \mathcal{H}) 1_H \subseteq Z 1_H \) and finally \( \text{cl}(\Gamma | \mathcal{H}) 1_H \subseteq \text{cl}(\Gamma 1_H | \mathcal{H}) \) as the latter set is \( \{0\} \) on \( \Omega \setminus \mathcal{H} \).

Example 3.1 Let \( \mathcal{X} = \mathbb{R} \) and \( \gamma_1, \gamma_2 \in \mathcal{L}^0(\mathcal{X}, \mathcal{F}) \) be such that \( \gamma_1 \leq \gamma_2 \) a.s. Consider \( \Gamma = [\gamma_1, \gamma_2] \). Then, \( \text{cl}(\Gamma | \mathcal{H}) = [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R} \). Indeed, first observe that \( \Gamma \subseteq [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R} \) a.s. Consider a \( \mathcal{H} \)-measurable closed set \( \Gamma_{\mathcal{H}} \) containing \( \Gamma \) and let us show that

\[ [\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R} \subseteq \Gamma_{\mathcal{H}}. \]

To do so, consider \( \xi \in \mathcal{L}^0([\text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R}, \mathcal{H}) \) and suppose that \( \mathcal{P}(\xi \notin \Gamma_{\mathcal{H}}) > 0 \). We suppose w.l.o.g. that \( \mathcal{P}(\xi \notin \Gamma_{\mathcal{H}}) = 1 \). As \( \Gamma \subseteq \Gamma_{\mathcal{H}} \), we necessarily have \( \xi \in [\text{ess inf}_{\mathcal{H}} \gamma_1, \gamma_1] \cap \mathbb{R} \) or \( \xi \in [\gamma_2, \text{ess sup}_{\mathcal{H}} \gamma_2] \cap \mathbb{R} \). Let us consider the case where \( \xi \in [\text{ess inf}_{\mathcal{H}} \gamma_1, \gamma_1] \cap \mathbb{R} \), the other case being similar. If \( \text{ess inf}_{\mathcal{H}} \gamma_1 = \gamma_1 \), then \( \xi = \gamma_1 \in \Gamma_{\mathcal{H}} \), i.e., a contradiction. Therefore, \( \text{ess inf}_{\mathcal{H}} \gamma_1 < \gamma_1 \) and, since \( \xi \)

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Lemma 4.1 Let \( h \) be a non-empty closed \( \mathcal{H} \)-programming, see our example below. We recall that an integrand \( \xi \) is \( \mathcal{H} \)-measurable and \( \xi \leq \gamma_1 \) a.s., we deduce that \( \xi \leq \text{ess inf}_{\mathcal{H}} \gamma_1 \). It follows that \( \xi = \text{ess inf}_{\mathcal{H}} \gamma_1 \in \mathbb{R} \). By definition of the essential supremum, there is a non-null set on which \( \text{ess inf}_{\mathcal{H}} \gamma_1 + \frac{1}{2} d(\xi, \Gamma_\mathcal{H}) > \gamma_1 \) where \( d(\xi, \Gamma_\mathcal{H}) > 0 \) as \( \xi \notin \Gamma_\mathcal{H} \). We may suppose that \( \xi + \frac{1}{2} d(\xi, \Gamma_\mathcal{H}) \in [\gamma_1, \gamma_2] \) even if we have to change \( d(\xi, \Gamma_\mathcal{H}) \) by a smaller \( \mathcal{F} \)-measurable term. Therefore, \( \xi + \frac{1}{2} d(\xi, \Gamma_\mathcal{H}) \in \Gamma_\mathcal{H} \) and \( d(\xi, \Gamma_\mathcal{H}) \leq d(\xi + \frac{1}{2} d(\xi, \Gamma_\mathcal{H}), \Gamma_\mathcal{H}) \leq \frac{1}{2} d(\xi, \Gamma_\mathcal{H}) \). We get a contradiction and, finally, \( \text{cl} (\Gamma|\mathcal{H}) = \text{ess inf}_{\mathcal{H}} \gamma_1, \text{ess sup}_{\mathcal{H}} \gamma_2 \) \( \cap \mathbb{R} \).

**Example 3.2** Let \( \gamma \in \mathcal{L}^0(\mathcal{X}, \mathcal{H}) \) and \( r \in \mathcal{L}^0((0, \infty), \mathcal{F}) \). Consider \( \Gamma \) the random closed ball \( \overline{B}(\gamma, r) \) of center \( \gamma \) and radius \( r \). We shall prove that

\[ \text{cl} (\Gamma|\mathcal{H}) = \overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r). \]

First observe that \( \Gamma \subseteq \overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r) \) a.s. Moreover, consider \( \Gamma_{\mathcal{H}} \) a \( \mathcal{H} \)-graph measurable random closed set containing \( \Gamma \) a.s. Suppose that there is \( \xi \in \mathcal{L}^0(\overline{B}(\gamma, \text{ess sup}_{\mathcal{H}} r), \mathcal{H}) \) such that \( \mathbb{P}(\xi \notin \Gamma_{\mathcal{H}}) > 0 \). We first consider the case where \( \text{ess sup}_{\mathcal{H}} r < \infty \). On the set \( A_{\mathcal{H}} = \{ \xi \notin \Gamma_{\mathcal{H}} \} \subseteq \mathcal{H} \), \( d(\xi, \Gamma_{\mathcal{H}}) > 0 \). Note that \( d(\xi, \Gamma_{\mathcal{H}}) < \infty \). There is a non-null \( \mathcal{F} \)-measurable subset of \( A_{\mathcal{H}} \) on which \( \text{ess sup}_{\mathcal{H}} r - d(\xi, \Gamma_{\mathcal{H}}) < r \) by definition of the essential supremum. Let us define \( \tilde{\xi} = \gamma + \alpha(\xi - \gamma) \) where \( \alpha = r/(r + d(\xi, \Gamma_{\mathcal{H}})) \). As we have \( \|\tilde{\xi} - \gamma\| < r + d(\xi, \Gamma_{\mathcal{H}}) \) when \( \text{ess sup}_{\mathcal{H}} r - d(\xi, \Gamma_{\mathcal{H}}) < r \), we deduce that \( \tilde{\xi} \in \Gamma \) and \( \tilde{\xi} \) is only \( \mathcal{F} \)-measurable. Therefore, \( \xi \in \Gamma_{\mathcal{H}} \) and \( d(\xi, \Gamma_{\mathcal{H}}) \leq \|\tilde{\xi} - \xi\| \) where we may show that \( \|\tilde{\xi} - \xi\| < d(\xi, \Gamma_{\mathcal{H}}) \) hence a contradiction. Consider now the case where \( \text{ess sup}_{\mathcal{H}} r = \infty \). We need to show that \( \Gamma_{\mathcal{H}} = \mathcal{X} \). In the contrary case, on a non-null set, we may construct a \( \mathcal{H} \)-measurable selection \( \zeta \) of \( \mathcal{X} \setminus \Gamma_{\mathcal{H}} \). Moreover, on a smaller non-null set, \( r \geq \|\zeta - \gamma\| \) as \( \text{ess sup}_{\mathcal{H}} r = \infty \). It follows that \( \zeta \in \Gamma \) hence \( \zeta \in \Gamma_{\mathcal{H}} \), i.e., a contradiction.

### 4 Applications

#### 4.1 Conditional Optimization

The second main contribution is the following. It allows one to compute numerically an essential supremum as a pointwise supremum on the conditional closure. This result is a generalization of [10, Proposition 2.9] and is useful in robust finance dynamic programming, see our example below. We recall that an integrand \( h(\omega, x) \), \( (\omega, x) \in \Omega \times \mathcal{X} \), is a jointly measurable function, which is lower semi-continuous in \( x \) and takes values in the extended real line \( \mathbb{R} \cup \{+\infty\} \), see [11, Corollary 14.34].

**Lemma 4.1** Let \( h(\omega, x) \), \( (\omega, x) \in \Omega \times \mathcal{X} \), be an \( \mathcal{H} \otimes \mathcal{B}(\mathcal{X}) \)-measurable integrand. Let \( \Gamma \) be a non-empty closed \( \mathcal{F} \)-graph measurable set of \( \mathcal{X} \). Then, \( \sup_{x \in \text{cl} (\Gamma|\mathcal{H})(\omega)} h(x) = \sup_n h(\gamma_n) \) where \( (\gamma_n)_{n \in \mathbb{N}} \) is a Castaing representation of \( \text{cl} (\Gamma|\mathcal{H}) \).

**Proof** As \( \text{cl} (\Gamma|\mathcal{H}) \) is \( \mathcal{H} \)-graph measurable and closed-valued, it admits a Castaing representation \( \text{cl} (\Gamma|\mathcal{H})(\omega) = \text{cl}\{\gamma_n(\omega) : n \in \mathbb{N}\} \) a.s. where, for all \( n \),
\(\gamma_n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\), by Proposition 2.1. Notice that we may adjust the values on a set of measure zero, and therefore assume that the equality holds everywhere on \(\Omega\), see the proof of [6, Proposition 5.4.4].

As \((\gamma_n)_n \subset \text{cl}\{\gamma_n : n \in \mathbb{N}\} = \text{cl}(\Gamma|\mathcal{H}), h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)\). Therefore, \(\sup_n h(\gamma_n) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)\). Moreover, if \(x \in \text{cl}(\Gamma|\mathcal{H})(\omega)\), we may write almost surely \(x = \lim_n \gamma_n(\omega)\) by the Castaing representation \((\gamma_n)_n\) of \(\text{cl}(\Gamma|\mathcal{H})\). By the lower semicontinuity of \(h\), we get that \(h(x) \leq \lim \inf_n h(\gamma_n)\). Thus, \(h(x) \leq \sup_n h(\gamma_n)\) and \(\sup_{x \in \text{cl}(\Gamma|\mathcal{H})(\omega)} h(x) \leq \sup_n h(\gamma_n)\) on \(\Omega\). The equality is then deduced. \(\square\)

**Theorem 4.1** Let \(h(\omega, x), (\omega, x) \in \Omega \times \mathcal{X}\), be an \(\mathcal{H} \otimes \mathcal{B}(\mathcal{X})\)-measurable integrand. Let \(\Gamma\) be a non-empty closed \(\mathcal{F}\)-graph measurable set of \(\mathcal{X}\). Then, with the notation \(h(\gamma)(\omega) = h(\omega, \gamma(\omega))\), we have

\[
\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})(\omega)} h(x), \text{ a.s.}
\]

**Proof** As \(\text{cl}(\Gamma|\mathcal{H})\) is \(\mathcal{H}\)-graph measurable and closed-valued, it admits a Castaing representation \(\text{cl}(\Gamma|\mathcal{H})(\omega) = \text{cl}\{\gamma_n(\omega) : n \in \mathbb{N}\}\) where, for all \(n\), \(\gamma_n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\), by Proposition 2.1, see also the lemma above.

**Step 1.** We show that

\[
\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.
\]

To see it, notice that by Lemma 4.1,

\[
\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n) \leq \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\}.
\]

If \(\gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\), then \(\gamma \in \text{cl}\{\gamma_n : n \in \mathbb{N}\}\), i.e., \(\gamma = \lim_n \gamma_n\) for a subsequence and, by lower semi-continuity,

\[
h(\gamma) = \lim \inf_n h(\gamma_n) \leq \sup_n h(\gamma_n) = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x).
\]

Since the family \(\{h(\gamma), \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}))\}\) is directed upward, we also deduce that \(\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \lim_n \uparrow h(\gamma^n)\) where \(\gamma^n \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H})\).

**Step 2.** We show that \(\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} = \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)\) a.s. First notice that \(\sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) = \sup_n h(\gamma_n)\) is \(\mathcal{H}\)-measurable, since \(h\) is \(\mathcal{H} \otimes \mathcal{B}(\mathcal{X})\)-measurable. Moreover, as \(\Gamma(\omega) \subset \text{cl}(\Gamma|\mathcal{H})(\omega)\) a.s., we deduce that \(h(\gamma) \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x)\), for every \(\gamma \in L^0(\Gamma, \mathcal{F})\). We deduce that

\[
\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F})\} \leq \sup_{x \in \text{cl}(\Gamma|\mathcal{H})} h(x) \text{ a.s.}
\]
To show the reverse inequality, consider any $\mathcal{H}$-measurable selection $\gamma$ of $\text{cl}(\Gamma|\mathcal{H})$ and a deterministic sequence $\varepsilon_n > 0$ with $\lim_n \varepsilon_n = 0$. Let us define

$$\Lambda_n = \{ \Gamma \cap \hat{B}(\gamma, \varepsilon_n) \neq \emptyset \} \in \mathcal{F}.$$ 

By Corollary 3.1, $\mathbb{P}(\Lambda_n|\mathcal{H}) > 0$ a.s. Indeed, in the contrary case, on a non-null $\mathcal{H}$-measurable set $\Lambda_n$, we have $\mathbb{P}(\Lambda_n \cap \Lambda_n|\mathcal{H}) = 0$ hence $\mathbb{P}(\Lambda_n \cap \Lambda_n) = 0$, i.e., a contradiction with Theorem 3.1.

By a measurable selection argument, we consider $\hat{\gamma}_n \in L^0(\Gamma, \mathcal{F})$ such that $\hat{\gamma}_n \in \hat{B}(\gamma, \varepsilon_n)$ on $\Lambda_n$. We define $\hat{\Lambda}_n = \{ \hat{\gamma}_n \in \hat{B}(\gamma, \varepsilon_n) \}$. Since $\Lambda_n \subseteq \hat{\Lambda}_n$, we have $\mathbb{P}(\Lambda_n|\mathcal{H}) > 0$ a.s. Moreover, using the conditional expectation, we have

$$\text{ess sup}_{\mathcal{H}} h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F}) \mid \mathbb{P}(\Lambda_n|\mathcal{H}) \geq \mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}), \text{ a.s.}$$

Moreover, we have

$$\mathbb{E}(h(\hat{\gamma}_n)1_{\hat{\Lambda}_n}|\mathcal{H}) \geq \mathbb{E}(\inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z)1_{\hat{\Lambda}_n}|\mathcal{H}).$$

Notice that $\inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z) = \inf_{z \in \chi} \tilde{h}(z)$ where $\tilde{h} = h$ on $\hat{B}(\gamma, \varepsilon_n)$ and $\tilde{h} = +\infty$ otherwise. As $\tilde{h}$ is also an integrand, we deduce by [11, Theorem 14.37] that $\inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z)$ is $\mathcal{H}$-measurable. Therefore,

$$\mathbb{E}(\inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z)1_{\hat{\Lambda}_n}|\mathcal{H}) = \inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z)\mathbb{P}(\Lambda_n|\mathcal{H}).$$

As $\mathbb{P}(\hat{\Lambda}_n|\mathcal{H}) > 0$ a.s., we deduce by (1) that

$$\text{ess sup}_{\mathcal{H}} h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F}) \geq \inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z), \text{ a.s.}$$

for every $n \geq 1$. Therefore,

$$\text{ess sup}_{\mathcal{H}} h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F}) \geq \lim_n \inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z), \text{ a.s.}$$

Since $\hat{B}(\gamma, \varepsilon_n)$ is a.s. compact and $h$ is a.s. lower semi-continuous, we deduce that $\inf_{z \in \hat{B}(\gamma, \varepsilon_n)} h(z) = h(z_n)$ where $z_n \in \hat{B}(\gamma, \varepsilon_n)$ converges pointwise to $\gamma$ as $n \to \infty$. We finally deduce by lower semi-continuity that

$$\text{ess sup}_{\mathcal{H}} h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F}) \geq \lim \inf_n h(z_n) \geq h(\gamma).$$

This inequality holds for any selection $\gamma$ of $\text{cl}(\Gamma|\mathcal{H})$. Therefore, we get that

$$\text{ess sup}_{\mathcal{H}} h(\gamma) : \gamma \in L^0(\Gamma, \mathcal{F}) \geq \text{ess sup}_{\mathcal{H}} h(\gamma), \gamma \in L^0(\text{cl}(\Gamma|\mathcal{H}), \mathcal{H}).$$
The conclusion of the lemma follows. □

Recall that a set $\Lambda$ of measurable random variables is said $\mathcal{F}$-decomposable if for any finite partition $(F_i)_{i=1}^n \subseteq \mathcal{F}$ of $\Omega$, and for every family $(\gamma_i)_{i=1}^n$ of $\Lambda$, we have $\sum_{i=1}^n \gamma_i 1_{F_i} \in \Lambda$. Decomposability was initially introduced by Rockafellar: see also [12]. In the following, we denote by $\Sigma(\Lambda)$ the $\mathcal{F}$-decomposable envelope of $\Lambda$, i.e., the smallest $\mathcal{F}$-decomposable family containing $\Lambda$. Notice that

$$\Sigma(\Lambda) = \left\{ \sum_{i=1}^n \gamma_i 1_{F_i} : n \geq 1, (\gamma_i)_{i=1}^n \subseteq \Lambda, (F_i)_{i=1}^n \subseteq \mathcal{F} \text{ s.t. } \sum_{i=1}^n F_i = \Omega \right\}.$$ 

The closure $\overline{\Sigma}(\Lambda)$ in probability of $\Sigma(\Lambda)$ is decomposable even if $\Lambda$ is not decomposable. By [6, Proposition 5.4.3], there exists an $\mathcal{F}$-graph measurable closed random set $\sigma(\Lambda)$ such that $\overline{\Sigma}(\Lambda)$ coincides with $L^0(\sigma(\Lambda), \mathcal{F})$, the set of all measurable selectors of $\sigma(\Lambda)$.

**Theorem 4.2** Let $h(\omega, x), x \in \mathcal{X}$, be an $\mathcal{H} \otimes \mathcal{B}(\mathcal{X})$-measurable integrand. Let us consider a family $\Lambda$ of measurable random variables so that

$$\overline{\Sigma}(\Lambda) = L^0(\sigma(\Lambda), \mathcal{F})$$

is the set of all measurable selectors of some $\mathcal{F}$-graph measurable random closed set $\sigma(\Lambda)$. Then,

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \Lambda\} = \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \overline{\Sigma}(\Lambda)\} = \sup_{x \in \text{cl}(\sigma(\Lambda) | \mathcal{H})} h(x).$$

**Proof** Notice that for any finite partition $(F_i)_{i=1}^n \subseteq \mathcal{F}$ of $\Omega$, $n \geq 1$, and for every family $(\gamma_i)_{i=1}^n$ of $\Lambda$, we have

$$h \left( \sum_{i=1}^n \gamma_i 1_{F_i} \right) = \sum_{i=1}^n h(\gamma_i) 1_{F_i}.$$ 

Therefore, as $\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Lambda$, we deduce that $\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \Lambda\} \geq h(\gamma)$ a.s. for any $\gamma \in \Sigma(\Lambda)$. Since $h$ is l.s.c. and any $\gamma \in \overline{\Sigma}(\Lambda)$ is a limit of elements of $\Sigma(\Lambda)$, we get that the inequality also holds for any $\gamma \in \overline{\Sigma}(\Lambda)$. Taking the essential supremum overall $\gamma \in \overline{\Sigma}(\Lambda)$, we deduce that

$$\text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \Lambda\} \geq \text{ess sup}_{\mathcal{H}} \{h(\gamma) : \gamma \in \overline{\Sigma}(\Lambda)\}$$

and, finally, the equality holds since $\Lambda \subseteq \overline{\Sigma}(\Lambda)$. The last equality of the corollary is deduced from Theorem 4.1. □
4.2 Application in Finance: Robust Super-Hedging of an European or Asian Option

We consider a financial market in discrete time defined by a complete stochastic basis $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. We suppose that there is a non-risky asset whose price is $S_0 = 1$, without loss of generality. The (discounted) prices are modeled by a vector-valued stochastic process $(S_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$ with values in $\mathbb{R}^d$, $d \geq 1$.

We consider the one step super-hedging problem between two dates $t - 1$ and $t$ with $t \geq 1$. We suppose that after time $t - 1$ but strictly before time $t$ the portfolio manager observes the price $S_{t-1}$, as a consequence of her/his order. More precisely, the portfolio manager knows $(S_u)_{u \leq t-2}$ at time $t - 1$ and sends an order at time $t - 1$ which is executed with a delay so that the executed price $S_{t-1}$ is only observed strictly after $t - 1$.

Let us consider, for each $t \leq T$, $\Lambda_t \subseteq L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ an $\mathcal{F}_t$-measurable random set representing the possible prices for the risky assets at time $t$. We suppose that, at time $t$, the set $\Lambda_t$ may depend on the observed prices before time $t - 1$, i.e., to each vector of prices $(S_u)_{u \leq t-1}$, we associate a set $\Lambda_t = \Lambda_t((S_u)_{u \leq t-1})$ representing the possible next prices at time $t$ given that we have observed the executed prices $(S_u)_{u \leq t-1}$. Therefore:

**Definition 4.1** A price process is an $(\mathcal{F}_t)_{t=0}^T$-adapted non-negative process $(S_t)_{t=0}^T$ such that $S_t \in \Lambda_t((S_u)_{u \leq t-1})$ for all $t = 1, \cdots, T$ and $S_{-1} \in \mathbb{R}^d$ is given.

Note that $S_t$ represents the prices $(S_1^1, \cdots, S_d^d)$ of the risky assets proposed by the market to the portfolio manager when selling or buying. A typical case could be $\Lambda_t = L^0(I_t, \mathcal{F}_t)$ with

$$I_t = \Pi_{j=1}^d [S_t^{bj}, S_t^{aj}],$$

where $(S_t^{bj})_{j=1}^d$ and $(S_t^{aj})_{j=1}^d$ are, respectively, the bid and the ask price processes observed in the market at time $t$ that may depend on $(S_u)_{u \leq t-1}$. They are not necessary the best bid/ask prices as, in practice, the real transaction price may be a convex combination of bid and ask prices. Indeed, a transaction is the result of an agreement between sellers and buyers, but it also depends on the traded volume. Clearly, the portfolio manager does not benefit from the last price observed in the market when sending an order. On the contrary, he should face an uncertain price $S_t$ which depends on the type of order (which may be not executed), but it also depends on some random events he does not control, e.g., slippage. A simple way to model this phenomenon is to suppose that the executed prices obtained by the manager belong to random intervals.

Another interesting case could be when $\Lambda_t$ coincides with a parametrized family $\{S_t^\theta : \theta \in \Theta\}$ of random variables. For instance, consider fixed processes $(\xi_u)_{u \leq T}$ and $(m_u)_{u \leq T}$ adapted to $(\mathcal{F}_t)_{t=0}^T$ and independent of $\mathcal{F}_{t-1}$. Let $C$ be a compact set and suppose that $S_{-1}$ is given. We define recursively

$$\Lambda_t((S_u)_{u \leq t-1}) = \{S_{t-1} \exp(\sigma \xi_t + m_t) : S_{t-1} \in \Lambda_{t-1}, \sigma \in C\}, \quad t \leq T.$$
In this model, there is an uncertainty on prices because of the unknown parameter (e.g., volatility) $\sigma$.

In the following, we consider the $\sigma$-algebra $\mathcal{F}_{t-1} = \sigma(S_u : u \leq t-1)$ for all $t \geq 1$. Let us consider a random function $g_t$ defined on $\mathbb{R}^t$, $t \geq 1$. We assume that the mapping $(\omega, z) \mapsto g_t(S_0(\omega), \cdots, S_{t-1}(\omega), z)$ is $\mathcal{F}_{t-1} \times \mathcal{B}(\mathbb{R})$-measurable and $z \mapsto g_t(S_0, S_1, \cdots, S_{t-1}, z)$ is lower-semicontinuous (l.s.c.) almost surely whatever the price process $(S_u)_{u \leq t-1}$. Our goal is to characterize the set $\mathcal{P}_{t-1}$ of all $V_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})$ such that

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_1, \cdots, S_t), \text{ a.s. for all } S_t \in \Lambda_t((S_u)_{u \leq t-1}),$$

for some $\theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. We observe that, by lower-semicontinuity, (2) holds if and only if

$$V_{t-1} + \theta_{t-1} \Delta S_t \geq g_t(S_1, \cdots, S_t), \text{ for all } S_t \in \overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})).$$

Recall that $\overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1}))$ is defined in the previous section. This means that we may suppose w.l.o.g. that $\overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})) = \Lambda_t((S_u)_{u \leq t-1})$. In the following, we denote by $I_t((S_u)_{u \leq t-1})$ the $\mathcal{F}_t$-measurable closed random set such that $\overline{\Sigma}(\Lambda_t((S_u)_{u \leq t-1})) = L^0(I_t((S_u)_{u \leq t-1}), \mathcal{F}_t)$: see [4, Theorem 2.4].

By Theorem 4.1, we deduce that (3) is equivalent to $V_{t-1} \geq p_{t-1}$ where $p_{t-1} = p_{t-1}(S_u)_{u \leq t-1}, \theta_{t-1}$ is given by

$$p_{t-1} = \theta_{t-1} S_{t-1} + \sup_{z \in \text{cl}(I_t((S_u)_{u \leq t-1}), \mathcal{F}_{t-1})} (g_t(S_1, \cdots, S_{t-1}, z) - \theta_{t-1} z),$$

$$= \theta_{t-1} S_{t-1} + f_{t-1}^*( -\theta_{t-1}).$$

In the formula above, $f_{t-1}^*(y) = \sup_{z \in \mathbb{R}^d} (yz - f_{t-1}(z))$ is the Fenchel–Legendre conjugate function of $f_{t-1}$ defined as

$$f_{t-1}(z) := -g_t(S_1, \cdots, S_{t-1}, z) + \delta_{\text{cl}(I_t((S_u)_{u \leq t-1}), \mathcal{F}_{t-1})}(z),$$

where $\delta_{\text{cl}(I_t((S_u)_{u \leq t-1}), \mathcal{F}_{t-1})} \in [0, \infty]$ is infinite on the complimentary of $\text{cl}(I_t((S_u)_{u \leq t-1}), \mathcal{F}_{t-1})$ and 0 otherwise. Notice that $f_{t-1}^*$ is convex and l.s.c. as a supremum (on $\text{cl}(I_t((S_u)_{u \leq t-1}), \mathcal{F}_{t-1})$) of convex and l.s.c. functions. Moreover, by Theorem 4.1, $(\omega, y) \mapsto f_{t-1}^*(\omega, y)$ is $\mathcal{F}_{t-1} \times \mathcal{B}(\mathbb{R}^d)$-measurable. Therefore, Dom $f_{t-1}^* := \{ y : f_{t-1}^*(\omega, y) < \infty \}$ is an $\mathcal{F}_{t-1}$-measurable random set. We deduce that the $\mathcal{F}_{t-1}$-measurable prices at time $t - 1$ are given by

$$\mathcal{P}_{t-1}((S_u)_{u \leq t-1}) = \left\{ \theta_{t-1} S_{t-1} + f_{t-1}^*(-\theta_{t-1}) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\} \cup L^0(\mathbb{R}^+, \mathcal{F}_{t-1}).$$

The second step is to determine the infimum super-hedging price as

$$p_{t-1}((S_u)_{u \leq t-1}) = \text{ess inf}_{\mathcal{F}_{t-1}} \mathcal{P}_{t-1}((S_u)_{u \leq t-1}).$$
To do so, we use the arguments of [10, Theorem 2.8] and we obtain that:

\[
p_{t-1}(\{S_u \mid u \leq t-1\}) = \ess inf_{\mathcal{F}_{t-1}} \left\{ \theta_{t-1} S_{t-1} + f^*_{t-1}(\cdot) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\},
\]

\[
= \ess inf_{\mathcal{F}_{t-1}} \left\{ -\theta_{t-1} S_{t-1} + f^*_{t-1}(\cdot) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\},
\]

\[
= -\ess sup_{\mathcal{F}_{t-1}} \left\{ \theta_{t-1} S_{t-1} - f^*_{t-1}(\cdot) : \theta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\},
\]

\[
= -\ess sup_{z \in \text{Dom} f^*_t} (z S_{t-1} - f^*_t(z)),
\]

\[
= -f^*_{t-1}(S_{t-1}).
\]

In the following, we suppose that, for all price process \((S_u)_{u \leq t-1}\), there exists \(\alpha_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})\) and \(\beta_{t-1} \in L^0(\mathbb{R}, \mathcal{F}_{t-1})\) such that

\[
g_t(S_0, S_1, \ldots, S_{t-1}, x) \leq \alpha_{t-1} x + \beta_{t-1}, \quad \forall x \in \text{cl}(I_t((S_u)_{u \leq t-1})|\mathcal{F}_{t-1}).
\]

This is the case for instance for Asian options whose payoff is of the form \(k(S_0 + S_1 + \cdots + S_t - K)^+\), \(k > 0\). By [10, Theorem 2.8], we then deduce that

\[
p_{t-1}(\{S_u \mid u \leq t-1\})
\]

\[
= \inf \left\{ \alpha S_{t-1} + \beta : \alpha x + \beta \geq g_t(S_0, \ldots, S_{t-1}, x), \forall x \in \text{cl}(I_t((S_u)_{u \leq t-1})|\mathcal{F}_{t-1}) \right\}.
\]

In this section, we have solved the super-hedging problem without any no-arbitrage condition, contrarily to what it is usual to do.

5 Conclusions

In Sect. 4.2, we have solved the general super-hedging problem in one step for an Asian option. The next step is to repeat the whole procedure to deduce backwardly the infimum prices and the associated super-hedging strategy from the maturity date \(T\) to the starting date \(t = 0\). This is a current project, that shows the relevance of the conditional closure.

The conditional closure could be also useful more generally for robust finance dynamic programming as in the paper [13]. We conjecture that it is possible to solve a discrete-time stochastic control problem through random set conditioning.

At last, some interesting problems leave open in the direction of conditional topologies: see [14]. A deeper study of the basic properties of the conditional closure and interior of random sets may be interesting with a comparison to the classical results of topology but also with the paper by Truffert [3]. This also allows to consider new types of martingales, see [12], and, in continuous time, new problems should arise.
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