Stabilization of heterodimensional cycles

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Abstract

We consider diffeomorphisms $f$ with heteroclinic cycles associated with saddles $P$ and $Q$ of different indices. We say that a cycle of this type can be stabilized if there are diffeomorphisms close to $f$ with a robust cycle associated with hyperbolic sets containing the continuations of $P$ and $Q$. We focus on the case where the indices of these two saddles differ by one. We prove that, excluding one particular case (so-called twisted cycles that additionally satisfy some geometrical restrictions), all such cycles can be stabilized.

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1. Introduction

In [17] Palis proposed a program whose main goal is a geometrical description for the behaviour of most dynamical systems. This program pays special attention to the generation of non-hyperbolic dynamics and to robust dynamical properties (i.e. properties that hold for open sets of dynamical systems). An important part of this program is the Density Conjecture (of cycles versus cycles): the two main sources of non-hyperbolic dynamics are heterodimensional cycles and homoclinic tangencies (shortly, cycles), see [17, conjecture 1]. The goal of this paper is to study the generation of robust heterodimensional cycles (see definition 1.1).

In addition to Palis’ program, we have the following two motivations for this paper:

Motivation I ([16, 21, 22]). Every $C^2$-diffeomorphism having a homoclinic tangency associated with a saddle $P$ is in the $C^2$-closure of the set of diffeomorphisms having $C^2$-robust homoclinic tangencies. Moreover, these robust homoclinic tangencies can be taken associated with hyperbolic sets containing the continuations of the saddle $P$.

This conjecture was proved by Pujals–Sambarino for $C^1$-surface diffeomorphisms in [18] (since heterodimensional cycles only can occur in manifolds of dimension $n \geq 3$, for surface diffeomorphisms it is enough to consider homoclinic tangencies).
Using the terminology that will be introduced in this paper this means that homoclinic
tangencies of $C^2$-diffeomorphisms can be stabilized, see definition 1.1. On the other hand, for
$C^1$-diffeomorphisms of surfaces homoclinic tangencies cannot be stabilized, see in [14]. This
leads to the following motivation.

**Motivation II ([8]).** Every diffeomorphism with a heterodimensional cycle associated with
a pair of hyperbolic saddles $P$ and $Q$ with $\dim E^s(P) = \dim E^s(Q) \pm 1$ belongs to the
$C^1$-closure of the set of diffeomorphisms having $C^1$-robust heterodimensional cycles. Here
$E^s$ denotes the stable bundle of a saddle.

One may think of the result in motivation II as a version of the results in motivation I
for heterodimensional cycles in the $C^1$-setting. However, the results in [8] do not provide
information about the relation between the hyperbolic sets involved in the robust cycles and
the saddles in the initial one. Thus, one aims for an extension of [8] giving some information
about the hyperbolic sets displaying the robust cycles, see [8, question 1.9].

In this paper we prove that, with the exception of a special type of heterodimensional
cycles (so-called twisted cycles, see definition 4.6), the hyperbolic sets exhibiting the robust
cycles can be taken containing the continuations of the saddles in the initial cycle. In fact,
by [10] our results cannot be improved: there are twisted cycles that cannot be stabilized, that
is, the hyperbolic sets with robust cycles cannot be taken containing the continuations of the
saddles in the initial cycle.

To state precisely our results we need to introduce some definitions. Recall that if $\Lambda$ is
a hyperbolic basic set of a diffeomorphism $f: M \to M$ then there are a neighbourhood $U_f$
of $f$ in the space of $C^1$-diffeomorphisms and a continuous map $U_f \to M; g \mapsto \Lambda_g$, such
that $\Lambda_f = \Lambda, \Lambda_g$ is a hyperbolic basic set, and the dynamics of $f|_\Lambda$ and $g|_{\Lambda_g}$ are conjugate.
The set $\Lambda_g$ is called the continuation of $\Lambda$ for $g$. Note that these continuations are uniquely
defined.

**Definition 1.1 (Heterodimensional cycles).**

- The $s$-index ($u$-index) of a transitive hyperbolic set is the dimension of its stable (unstable)
bundle.

- A diffeomorphism $f$ has a heterodimensional cycle associated with transitive hyperbolic
basic sets $\Lambda$ and $\Sigma$ of $f$ if these sets have different $s$-indices and their invariant manifolds
meet cyclically, that is, if $W^s(\Lambda, f) \cap W^u(\Sigma, f) \neq \emptyset$ and $W^u(\Lambda, f) \cap W^s(\Sigma, f) \neq \emptyset$.

- The heterodimensional cycle has coindex $k$ if $s(\Lambda) = s(\Sigma) \pm k$. In such a
case we just write coindex $k$ cycle.

- A diffeomorphism $f$ has a $C^1$-robust heterodimensional cycle associated with its
hyperbolic basic sets $\Lambda$ and $\Sigma$ if there is a $C^1$-neighbourhood $U$ of $f$ such that every
diffeomorphism $g \in U$ has a heterodimensional cycle associated with the continuations
$\Lambda_g$ and $\Sigma_g$ of $\Lambda$ and $\Sigma$, respectively.

- Consider a diffeomorphism $f$ with a heterodimensional cycle associated with a pair
of saddles $P$ and $Q$. This cycle can be $C^1$-stabilized if every $C^1$-neighbourhood $U$ of $f$
contains a diffeomorphism $g$ with hyperbolic basic sets $\Lambda_g \ni P_g$ and $\Sigma_g \ni Q_g$ having a
robust heterodimensional cycle. Otherwise the cycle is said to be $C^1$-fragile.

Note that, by the Kupka–Smale genericity theorem (invariant manifolds of hyperbolic
periodic points of generic diffeomorphisms are in general position), at least one of the
hyperbolic sets involved in a robust cycle is necessarily non-trivial, that is, not a periodic orbit.
Definition 1.2 (Homoclinic class). The homoclinic class of a saddle \( P \) is the closure of the transverse intersections of the stable and unstable manifolds \( W^s(P, f) \) and \( W^u(P, f) \) of the orbit of \( P \). We denote this class by \( H(P, f) \). A homoclinic class is non-trivial if it contains at least two different orbits.

A homoclinic class can be also defined as the closure of the set of saddles that are homoclinically related to \( P \). Here we say that a saddle \( Q \) is homoclinically related to \( P \) if the invariant manifolds of the orbits of \( P \) and \( Q \) meet cyclically and transversely, that is, \( W^s(P, f) \cap W^u(Q, f) \neq \emptyset \) and \( W^u(P, f) \cap W^s(Q, f) \neq \emptyset \).

The following is a consequence of our results (see theorem 2).

Theorem 1. Let \( f \) be a \( C^1 \)-diffeomorphism with a coindex one cycle associated with saddles \( P \) and \( Q \). Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then the heterodimensional cycle of \( f \) associated with \( P \) and \( Q \) can be \( C^1 \)-stabilized.

A simple consequence of this result is the following:

Corollary 1. Let \( f \) be a \( C^1 \)-diffeomorphism with a heterodimensional cycle associated with saddles \( P \) and \( Q \) such that \( s\text{-index}(P) = s\text{-index}(Q) + 1 \). Suppose that the intersection \( W^u(P, f) \cap W^u(Q, f) \) contains at least two different orbits. Then the cycle can be \( C^1 \)-stabilized.

The question of the stabilization of cycles is relevant for describing the global dynamics of diffeomorphisms (indeed this is another motivation for this paper). We explain this point succinctly. Following [1, 12, 15], this global dynamics is structured by means of homoclinic or/and chain recurrence classes. The goal is to describe the dynamics of these classes and their relating cycles. In general, homoclinic classes are (properly) contained in chain recurrence classes. For \( C^1 \)-generic diffeomorphisms and for hyperbolic periodic points, these two kinds of classes coincide [4]. However, there are non-generic situations where two different homoclinic classes are ‘joined’ by a cycle. In this case these classes are contained in one common chain recurrence class which hence is strictly larger. We would like to know under which conditions after small perturbations these two homoclinic classes explode and fall into the very same homoclinic class \( C^1 \)-robustly. Indeed this occurs if the cycle can be \( C^1 \)-stabilized. Examples where this stabilization is used for describing global dynamics can be found in [5, 23, 24]. See [11, chapter 10.3-4] and [3] for a broader discussion of these questions.

To prove our results we analyse the dynamics associated with different types of coindex one cycles. This analysis essentially depends on two factors: the central multipliers of the cycle and its unfolding map. We will now discuss this point briefly, for further details we refer to section 4.

1.1. Multipliers and unfolding map of a cycle

Let \( f \) be a diffeomorphism with a coindex one cycle associated with saddles \( P \) and \( Q \). In what follows we will assume that \( s\text{-index}(P) = s\text{-index}(Q) + 1 \). Denote by \( \pi(R) \) the period of a periodic point \( R \).

We say that the cycle is partially hyperbolic if there are heteroclinic points \( X \in W^s(P, f) \cap W^u(Q, f) \) and \( Y \in W^u(P, f) \cap W^s(Q, f) \) such that the closed set formed by the orbits of \( P, Q, X \) and \( Y \) has a partially hyperbolic splitting of the form \( E^u \oplus E^c \oplus E^s \), where \( E^c \) is one-dimensional, \( E^s \) is uniformly contracting and \( E^u \) is uniformly expanding. We call \( E^c \) the central bundle. Note that, in particular, this implies that \( X \) is a transverse intersection and \( Y \) is a quasi-transverse intersection of the invariant manifolds. Also observe
that the bundle $E^c$ is necessarily non-hyperbolic. Bearing in mind this property we introduce the following definition.

**Definition 1.3 (Central multipliers).** The cycle has real central multipliers if there are a contracting real eigenvalue $\lambda$ of $Df^{\pi(P)}(P)$ and an expanding real eigenvalue $\beta$ of $Df^{\pi(Q)}(Q)$ such that: (i) $\lambda$ and $\beta$ have multiplicity one, (ii) $|\lambda| > |\sigma|$ for every contracting eigenvalue $\sigma$ of $Df^{\pi(P)}(P)$ and (iii) $|\beta| < |\eta|$ for every expanding eigenvalue $\eta$ of $Df^{\pi(Q)}(Q)$. In this case, we say that $\lambda$ and $\beta$ are the real central multipliers of the cycle.

Similarly, the cycle has non-real central multipliers if either (i) there are a pair of non-real (conjugate) contracting eigenvalues $\lambda$ and $\bar{\lambda}$ of $Df^{\pi(P)}(P)$ such that $|\lambda| = |\bar{\lambda}| \geq |\sigma|$ for every contracting eigenvalue $\sigma$ of $Df^{\pi(P)}(P)$, or (ii) there are a pair of non-real (conjugate) expanding eigenvalues $\beta$ and $\bar{\beta}$ of $Df^{\pi(Q)}(Q)$ such that $|\beta| = |\bar{\beta}| \leq |\eta|$ for every expanding eigenvalue $\eta$ of $Df^{\pi(Q)}(Q)$.

We note that cycles with central real multipliers can be perturbed to obtain partially hyperbolic ones (associated with the continuations of the saddles in the initial one).

In the case of cycles with real central multipliers we will distinguish so-called twisted and non-twisted cycles, see definition 4.6. An intuitive explanation of these two sorts of cycles goes as follows, see figure 1.

In order to study the dynamics of the cycle we select heteroclinic points $X \in W^s(P, f) \cap W^u(Q, f)$ and $Y \in W^u(P, f) \cap W^s(Q, f)$. Typically, $X$ is a transverse intersection point and $Y$ is a quasi-transverse intersection point (due to dimension deficiency). The next step is to consider a neighbourhood of the cycle, that is, an open set $V$ containing the orbits of $P, Q, X$ and $Y$, and study the dynamics of perturbations of $f$ in such a neighbourhood. If the neighbourhood $V$ is small enough, possibly after a perturbation of $f$, the dynamics of $f$ in $V$ is partially hyperbolic with a splitting of the form $E^{\text{ss}} \oplus E^c \oplus E^{\text{uu}}$ (recall the definition above).

Replacing $Y$ by some backward iterate, we can assume that the heteroclinic point $Y$ is close to $P$. We pick some large number $k$ such that $f^k(Y)$ is nearby $Q$ and consider the map $\Sigma_1 = f^k$ defined in a small neighbourhood of $Y$. This map is called the unfolding map. If it is possible to pick $k$ in such a way that $Df^k$ preserves the orientation of the central bundle then we say that the cycle is non-twisted. Otherwise, the cycle is twisted. Note that in the previous discussion the choice of the heteroclinic point $X$ does not play any relevant role.

More precisely, the dynamics of the unfolding of the cycle mostly depends on the signs of the central eigenvalues $\lambda$ (associated with $P$) and $\beta$ (associated with $Q$) and on the restriction...
of $\Sigma_1$ to the central bundle. We associate with the cycle the signs, sign$(Q)$, sign$(P)$ and sign$(\Sigma_1)$ in $\{+,-\}$ determined by the following rules:

- sign$(Q) = +$ if $\beta > 0$ and sign$(Q) = -$ if $\beta < 0$;
- sign$(P) = +$ if $\lambda > 0$ and sign$(P) = -$ if $\lambda < 0$ and
- sign$(\Sigma_1) = +$ if $\Sigma_1$ preserves the orientation in the central direction and sign$(\Sigma_1) = -$ if the orientation is reversed.

A cycle is twisted if sign$(Q) = +$, sign$(P) = +$ and sign$(\Sigma_1) = -$. Otherwise the cycle is non-twisted. For details see definition 4.6.

We observe that the discussion above is reminiscent of the one in [19, section 2] about bifurcations of homoclinic tangencies of surface diffeomorphisms. It involves similar ingredients to the ones above: the signs of the eigenvalues of the derivatives, the sides of the tangencies and the connections (homoclinic and heteroclinic intersections).

To state our main result we need to introduce the bi-accumulation property. Given a periodic point $R$ of $f$, let $\lambda_1(R), \ldots, \lambda_n(R)$ be the eigenvalues of $Df^{\pi(R)}(R)$ ordered in increasing modulus and counted with multiplicity. If $R$ is hyperbolic, has s-index $k$, and $|\lambda_{k-1}(R)| < |\lambda_k(R)|$ then there is a unique invariant manifold $W^s(R, f)$ (the strong stable manifold of $R$) tangent to the eigenspace associated with $\lambda_1(R), \ldots, \lambda_{k-1}(R)$ (the strong stable bundle). The manifold $W^s_{\text{loc}}(R, f)$ has codimension one in $W^s_{\text{loc}}(R, f)$ and $W^s_{\text{loc}}(R, f)$ splits each component of $W^s_{\text{loc}}(R, f)$ into two parts.

Definition 1.4 (Bi-accumulation property). A saddle $R$ of s-index $k$ such that $|\lambda_{k-1}(R)| < |\lambda_k(R)|$ is s-bi-accumulated (by homoclinic points) if every component of $W^s_{\text{loc}}(R, f) \setminus W^s_{\text{loc}}(R, f)$ contains transverse homoclinic points of $R$.

A heterodimensional cycle associated with saddles $P$ and $Q$ with s-index$(P) = s$-index$(Q) + 1$ is bi-accumulated if either $P$ is s-bi-accumulated for $f$ or $Q$ is s-bi-accumulated for $f^{-1}$.

We are now ready to state our main result.

Theorem 2. Consider a diffeomorphism $f$ having a coindex one cycle associated with saddles $P$ and $Q$. Suppose that one of the conditions below holds

(A) the cycle has a non-real central multiplier,
(B) the cycle has real multipliers and is non-twisted,
(C) the cycle has real multipliers, is twisted, and satisfies the bi-accumulation property.

Then the cycle of $f$ associated with $P$ and $Q$ can be $C^1$-stabilized.

Our results can be summarized as follows:

Corollary 2. Consider a diffeomorphism $f$ with a fragile cycle associated with saddles $P$ and $Q$ with s-index$(P) = s$-index$(Q) + 1$. Then

- the cycle has positive central real multipliers,
- the cycle is persistently twisted (i.e. the cycle cannot be perturbed to obtain a non-twisted cycle associated with $P$ and $Q$),
- the intersection $W^s(P, f) \cap W^s(Q, f)$ consists of exactly one orbit and
- the homoclinic classes of $P$ and $Q$ are both trivial.

We observe that theorem 2 cannot be improved. Indeed examples of fragile cycles satisfying the four properties in corollary 2 can be found in [10].
2. Ingredients of the proofs

In this section, we review some tools of our constructions.

2.1. Reduction to the case of cycles with real multipliers

A first step is to see that to prove our results it is enough to consider cycles with real central multipliers. Our main technical result, involving these sorts of cycles, is the following:

**Theorem 2.1.**

(A) Every non-twisted cycle can be $C^1$-stabilized.

(B) Every twisted cycle with the bi-accumulation property can be $C^1$-stabilized.

The first ingredient of our constructions is a result we borrow from [8].

**Theorem 2.2 (Theorem 2.1 in [8]).** Let $f$ be a diffeomorphism with a coindex one cycle associated with saddles $P$ and $Q$. Then there are diffeomorphisms $g$ arbitrarily close to $f$ with a coindex one cycle with real central multipliers associated with saddles $P_g$ and $Q_g$, which are homoclinically related to the continuations $P_g$ and $Q_g$ of $P$ and $Q$. In this result one may have $P = P_g$ and/or $Q = Q_g$.

Note that the previous theorem means the following.

**Remark 2.3.** Assume that the saddle $P$ in theorem 2.2 has non-real central multipliers. Then the homoclinic class of $P_g$ is non-trivial and contains $P$.

There is also the following simple fact:

**Lemma 2.4.** Consider a diffeomorphism $f$ with a heterodimensional cycle associated with $P$ and $Q$. Suppose that there are saddles $P'_g$ and $Q'_g$ homoclinically related to $P_g$ and $Q_g$, respectively, with a heterodimensional cycle that can be $C^1$-stabilized. Then the initial cycle can also be $C^1$-stabilized.

**Proof.** The stabilization of the cycle associated with $P'_g$ and $Q'_g$ means that there is $h$ arbitrarily close to $g$ having a pair of basic hyperbolic sets $\Lambda'_h \ni P'_h$ and $\Sigma'_h \ni Q'_h$ with a robust cycle. Since the saddles $P_h$ and $P'_h$ are homoclinically related there is a basic set $\Lambda_h$ containing $\Lambda'_h$ and $P_h$. Similarly, there is a basic set $\Sigma_h$ containing $\Sigma'_h$ and $Q_h$. Since $W^s(\Lambda_h, h) \supset W^u(\Lambda'_h, h)$ and $W^s(\Sigma_h, h) \supset W^u(\Sigma'_h, h)$, it is immediate that there is a robust cycle associated with $\Lambda_h \ni P_h$ and $\Sigma_h \ni Q_h$. □

**Remark 2.5.** Theorem 2.2 and lemma 2.4 mean that to prove theorems 1 and 2 it is enough to stabilize cycles with real central multipliers. Thus, in what follows we will focus on these types of cycles.

2.2. Strong homoclinic intersections and blenders

A key ingredient for obtaining robust heterodimensional cycles in [8] is the notion of a blender. A blender is a hyperbolic set with some additional geometrical intersection properties that guarantee some robust intersections, see section 3.1 and definition 3.1. The key step in [8] to obtain robust cycles is that coindex one cycles yield periodic points of saddle-node/flip type with strong homoclinic intersections: the strong stable manifold of the saddle-node/flip intersects its strong unstable manifold, see definition 3.3. These strong homoclinic intersections generate blenders yielding robust cycles, see proposition 3.4.
In [8] the generation of blenders is not controlled and in general the saddle-node/flip has 'nothing to do' with the saddles in the initial cycle. This is why in [8] the hyperbolic sets with robust cycles are not related (in general) to the saddles in the initial cycle. Here we control the 'generation' of the saddle-node/flip with strong homoclinic intersections, obtaining blenders that contain the continuation of a saddle in the initial cycle and intersecting the invariant manifolds of the other saddle in the cycle. This configuration provides robust cycles associated with hyperbolic sets containing the continuation of both initial saddles, see theorem 3.5.

We next explain the 'generation' of saddle-node/flip points with strong homoclinic intersections.

2.3. Simple cycles and iterated function systems (IFSs)

To analyse the dynamics of cycles with real multipliers we borrow some constructions and the notion of a simple cycle from [8], see section 4.

In very rough terms, if a diffeomorphism has a simple cycle then its dynamics in a neighbourhood of the cycle is affine and preserves a partially hyperbolic splitting \( E^{ss} \oplus E^c \oplus E^{uu} \), where \( E^{ss} \) is uniformly contracting, \( E^{uu} \) is uniformly expanding and \( E^c \) is one-dimensional and non-hyperbolic, see proposition 4.1. Following [8], to prove our results it is enough to consider simple cycles and their (suitable) unfoldings.

We consider one-parameter families of diffeomorphisms \((f_t)\), unfolding a simple cycle at \( t = 0 \) and preserving the affine structure associated with the splitting \( E^{ss} \oplus E^c \oplus E^{uu} \). In particular, the foliation of hyperplanes parallel to \( E^{ss} \oplus E^{uu} \) is preserved. Considering the central dynamics given by the quotient of the dynamics of the diffeomorphism \( f_t \) by these hyperplanes one gets a one-parameter family of iterated function systems (IFSs). Some properties of these IFSs are translated to properties of the diffeomorphisms \( f_t \), see proposition 4.9. This IFS provides relevant information about the dynamics of the diffeomorphisms \( f_t \) such as, for example, the existence of saddle-nodes with strong homoclinic intersections. Such IFSs play a role similar to the one of the quadratic family in the setting of homoclinic bifurcations, compare [20, chapter 6.3].

2.4. Organization of the paper

The discussion above corresponds to the contents in sections 3 and 4. The key step is to analyse the dynamics of the IFSs associated with simple cycles. Using these IFSs, in section 5 we analyse non-twisted cycles (which is the principal case) and explain how they yield saddle-nodes/flips with strong homoclinic intersections as well as further intersection properties, see proposition 5.3. We study (twisted and non-twisted) cycles with the bi-accumulation property in section 5.3. In section 6 we prove theorem 2.1, which is the main technical step in the paper. Finally, in section 7 we see how theorems 1 and 2 can be easily derived from theorem 2.1.

3. Robust cycles and blenders

In this section, we recall the definition and main properties of blenders. We also state the tools to obtain the stabilization of heterodimensional cycles, see proposition 3.4 and theorem 3.5.

3.1. Blenders

We recall the definition of a cu-blender in [9]. See also the examples in [6] and the discussion in [11, chapter 6]:
Definition 3.1 (cu-blender, definition 3.1 in [9]). Let \( f : M \to M \) be a diffeomorphism. A transitive hyperbolic compact (non-trivial) set \( \Gamma \) of \( f \) with \( \mu\)-index(\( \Gamma \)) = \( k \), \( k \geq 2 \), is a cu-blender if there are a \( C^1 \)-neighbourhood \( \mathcal{U} \) of \( f \) and a \( C^1 \)-open set \( \mathcal{D} \) of embeddings of \((k - 1)\)-dimensional discs \( D \) into \( M \) such that for every \( g \in \mathcal{U} \) and every disc \( D \in \mathcal{D} \) the local stable manifold \( W_{\text{loc}}(\Gamma g) \) of \( \Gamma g \) intersects \( D \). The set \( D \) is called the superposition region of the blender.

Remark 3.2. Let \( \Gamma \) be a blender of \( f \). Then for every \( g \) close enough to \( f \) the continuation \( \Gamma g \) of \( \Gamma \) is a blender of \( g \).

In fact, the cu-blenders considered in [8] to obtain robust cycles are a special class of blenders, called blender-horseshoes, see [9, definition 3.8]. In this definition, the blender-horseshoe \( \Gamma \) is the maximal invariant set in a ‘cube’ \( C \) and has a hyperbolic splitting with three non-trivial bundles \( T_{\Gamma}M = E^s \oplus E^c \oplus E^u \), such that the unstable bundle of \( \Gamma \) is \( E^u = E^{cs} \oplus E^{mu} \) and \( E^{cs} \) is one-dimensional. Moreover, the set \( \Gamma \) is conjugate to the complete shift of two symbols. Thus it has exactly two fixed points, say \( A \) and \( B \), called distinguished points of the blender, and that play a special role in the definition of a blender-horseshoe.

The definition of a blender-horseshoe involves a \( Df \)-invariant strong unstable cone-field \( C^u \) corresponding to the strong unstable direction \( E^u \), the local stable manifolds \( W_{\text{loc}}^s(A, f) \) and \( W_{\text{loc}}^s(B, f) \) of the distinguished saddles \( A \) and \( B \) (defined as the connected component of \( W^s(R, f) \cap C \) containing \( R \), \( R = A, B \)), and the local strong unstable manifolds \( W_{\text{loc}}^u(A, f) \) and \( W_{\text{loc}}^u(B, f) \) of \( A \) (the component of \( W^u(R, f) \cap C \) containing \( R \)). Recall that the strong unstable manifold of \( R \) is the only invariant manifold of dimension \( \dim(E^u) \) that is tangent to \( E^u \) at \( R \).

Let \( \dim(E^u) = u \). One considers vertical discs through the blender, that is, discs \( \Delta \) of dimension \( u \) tangent to the cone-field \( C^u \) joining the ‘top’ and the ‘bottom’ of the cube \( C \). Then there are two isotopy classes of vertical discs that do not intersect \( W_{\text{loc}}^s(A, f) \) (respectively \( W_{\text{loc}}^s(B, f) \)), called discs at the right and at the left of \( W_{\text{loc}}^s(A, f) \) (resp. \( W_{\text{loc}}^s(B, f) \)). For instance, \( W_{\text{loc}}^u(B, f) \) (that is a vertical disc) is at the right of \( W_{\text{loc}}^s(A, f) \). Similarly, \( W_{\text{loc}}^u(A, f) \) is at the left of \( W_{\text{loc}}^s(B, f) \). The superposition region \( \mathcal{D} \) of the blender-horseshoe consists of the vertical discs in between \( W_{\text{loc}}^s(A, f) \) and \( W_{\text{loc}}^s(B, f) \) (i.e. at the right of \( W_{\text{loc}}^s(A, f) \) and at the left of \( W_{\text{loc}}^s(B, f) \)). See figure 2.

3.2. Generation of blenders and robust cycles

To state a criterion for the existence of robust cycles we need some definitions.

Definition 3.3. Let \( S \) be a periodic point of a diffeomorphism \( f \).
We say that $S$ is a partially hyperbolic saddle-node (respectively flip) of $f$ if the derivative of $D_f^{\pi(S)}(S)$ has exactly one eigenvalue $\sigma$ of modulus 1, the eigenvalue $\sigma$ is equal to 1 (respectively, $-1$), and there are eigenvalues $\lambda$ and $\beta$ of $D_f^{\pi(S)}(S)$ with $|\lambda| < 1 < |\beta|$.

Consider the strong unstable (respectively stable) invariant direction $E_{uu}$ (respectively $E_{ss}$) corresponding to the eigenvalues $\kappa$ of $D_f^{\pi(S)}(S)$ with $|\kappa| > 1$ (resp. $|\kappa| < 1$). The strong unstable manifold $W_{uu}(S, f)$ of $S$ is the unique $f$-invariant manifold tangent to $E_{uu}$ of the same dimension as $E_{uu}$. The strong stable manifold $W_{ss}(S, f)$ of $S$ is defined similarly considering $E_{ss}$.

We say that $S$ has a strong homoclinic intersection if $W_{uu}(S, f) \cap W_{ss}(S, f)$ contains points which do not belong to the orbit of $S$.

**Proposition 3.4 (Criterion for robust cycles. Theorem 2.4 in [8]).** Let $f$ be a diffeomorphism having a partially hyperbolic saddle-node/flip $S$ with a strong homoclinic intersection. Then there is a diffeomorphism $h$ arbitrarily $C^1$-close to $f$ with a robust heterodimensional cycle.

Note that this result does not provide information about the sets involved in the robust cycle.

We state in theorem 3.5 a version of this proposition providing some information about these sets. Before proving this theorem we explain the main steps of the proof of proposition 3.4, for further details see [8].

**Sketch of the proof of proposition 3.4.** For simplicity, we assume that $S$ is a saddle-node of $f$ of period one. After a perturbation, we can suppose that the saddle-node $S$ splits into two hyperbolic fixed points $S^-_g$ (contracting in the central direction) and $S^+_g$ (expanding in the central direction), here $g$ is a diffeomorphism obtained by a small perturbation of $f$. The saddles $S^+_g$ and $S^-_g$ have different indices and the manifolds $W^s(S^-_g)$ and $W^u(S^+_g)$ have a transverse intersection that contains the interior of a ‘central’ curve joining $S^-_g$ and $S^+_g$. Note that this intersection property is $C^1$-robust. The proof has three steps (see figure 3):

(I) There is a blender-horseshoe $\Gamma_g$ having $S^-_g$ as a distinguished fixed point.

(II) The unstable manifold of $S^-_g$ contains a vertical disc $\Delta$ in the superposition region $D$ of the blender-horseshoe $\Gamma_g$. Thus, by the definition of blender-horseshoe, $W^u(\Gamma_g, g)$ intersects $W^s(S^-_g, g)$. Hence, as $S^-_g \in \Gamma_g$ and $W^u(S^-_g, g) \cap W^s(S^-_g, g) \neq \emptyset$, there is a heterodimensional cycle associated with $\Gamma_g$ and $S^-_g$.

(III) The following properties are open ones: (i) the continuation of the hyperbolic set $\Gamma_g$ to be a blender (the elements in the definition of a blender depend continuously on $g$, see remark 3.2), (ii) $W^u(S^-_g, g)$ to contain a vertical disc in the superposition region $D$ of the blender and (iii) $W^s(S^-_g, g) \cap W^u(S^-_g, g) \neq \emptyset$.

Therefore, every diffeomorphism $h$ that is $C^1$-close to $g$ has a heterodimensional cycle associated with $S^-_g$ and $\Gamma_g$. Since $g$ can be taken arbitrarily close to $f$ this concludes the proof.

The next result is just a reformulation of the construction above that allows us to obtain robust cycles associated with sets that contain the continuations of a given saddle. This theorem will be the main tool for stabilizing cycles.

**Theorem 3.5.** Let $f$ be a diffeomorphism, $P$ a saddle of $f$, and $S$ a partially hyperbolic saddle-node/flip of $f$ such that:

1. $s$-index $(P) = \dim(W^u(S)) + 1 = s + 1$,
2. $S$ has a strong homoclinic intersection,
(3) $W^u(P, f) \cap W^s(S, f) \neq \emptyset$ and
(4) $W^s(P, f) \cap W^{uu}(S, f) \neq \emptyset$.

Then there is a diffeomorphism $h$ arbitrarily $C^1$-close to $f$ with a robust heterodimensional cycle associated with the continuation $P_h$ of $P$ and a transitive hyperbolic set $\Gamma_h$ containing a hyperbolic continuation $S_h^+$ of $S$ of s-index $s$.

Proof. One proceeds as in the proof of proposition 3.4, considering a perturbation $h$ of $g$ with saddles $S^+_h$ satisfying conditions (I) and (II) above and such that the saddles $P_h$ and $S^+_h$ having the same index are homoclinically related,

$W^u(P_h, h) \cap W^s(S_h^-, h) \neq \emptyset$ and $W^s(P_h, h) \cap W^{uu}(S_h^-, h) \neq \emptyset$.

Since $W^u(S_h^+, h) \cap W^s(S_h^-, h) \neq \emptyset$, the inclination lemma implies that $W^u(S_h^+, h)$ accumulates to $W^u(S_h^-, h)$ and thus

$W^u(P_h, h) \cap W^u(S_h^+, h) \neq \emptyset$, \hspace{1cm} (3.1)

see figure 4.

Recall that $W^u(S_h^-, h)$ contains a vertical disc in the superposition region of the blender $\Gamma_h$. Since $W^u(P_h, h) \cap W^u(S_h^-, h) \neq \emptyset$, the inclination lemma implies that the same holds for $W^u(P_h, h)$. Thus we can repeat the construction in proposition 3.4 replacing $S_h^-$ by $P_h$. Hence $W^u(P_h, \varphi)$ intersects $W^s(\Gamma_{\varphi}, \varphi)$ for any diffeomorphism $\varphi$ close to $h$. Since $W^u(P_h, \varphi) \cap W^u(S_h^+, \varphi) \neq \emptyset$ (recall (3.1)) and $S_h^+ \in \Gamma_{\varphi}$ for every $\varphi$ close to $h$, there is a robust heterodimensional cycle associated with $P_h$ and $\Gamma_{\varphi}$, ending the proof of the theorem. □
4. Simple cycles and systems of iterated functions

In this section, following [8], we introduce simple cycles (section 4.1) and their associated one-dimensional dynamics (section 4.3). We see that given any diffeomorphism \( f \) with a coindex one cycle with real central multipliers (associated with saddles \( P \) and \( Q \)) there is a diffeomorphism \( g \) arbitrarily \( C^1 \)-close to \( f \) with a cycle associated with \( P \) and \( Q \) whose dynamics in a neighbourhood of the cycle is affine, see proposition 4.1. In such a case we say that this cycle of \( g \) is simple.

In fact, for a diffeomorphism \( g \) with a simple cycle there is a one-parameter family of diffeomorphisms \( (g_t) \), \( g_0 = g \), preserving a (semi-local) partially hyperbolic splitting \( E^s \oplus E^c \oplus E^u \) such that the bundles \( E^s \) and \( E^u \) are non-trivial and hyperbolic (uniformly contracting and uniformly expanding, respectively) and the bundle \( E^c \) is not hyperbolic and one-dimensional. We consider the quotient dynamics by the hyperplanes \( E^s \oplus E^u \), obtaining a one-parameter family of one-dimensional iteration function systems (IFSs) which describe the central dynamics of the maps \( g_t \). Properties of these IFSs are translated to properties of the diffeomorphisms \( g_t \), see proposition 4.9.

In section 5 we will write intersection properties implying the existence of robust cycles (similar to the ones in theorem 3.5) in terms of properties of the IFSs associated with simple cycles. We now discuss simple cycles and their IFSs.

4.1. Simple cycles

The next proposition summarizes the results in [8] about simple cycles and their unfoldings. This proposition means that if \( (f_t) \) is a ‘model arc’ unfolding a simple cycle then the dynamics of the maps \( f_t \) in a neighbourhood of the cycle is given by suitable compositions of two linear maps (the dynamics nearby the saddles in the cycle) and two affine maps (iterations corresponding to the ‘transition’ and the ‘unfolding maps’). The ingredients in the next proposition are depicted in figure 5.

**Proposition 4.1 (Proposition 3.5 and section 3.2 in [8]).** Let \( f \) be a diffeomorphism having a coindex one cycle with real central multipliers associated with saddles \( P \) and \( Q \) such that

\[
s\text{-index}(Q) + 1 = s\text{-index}(P).
\]

Then there is a one-parameter family of diffeomorphisms \( (g_t)_{t \in [-\epsilon, \epsilon]} \), \( \epsilon > 0 \), such that it satisfies properties (C1)–(C3) below and \( g_0 \) is arbitrarily close to \( f \).

Let \( s \) and \( u \) be the dimensions of \( W^s(Q, f) \) and of \( W^u(P, f) \), respectively. There are linear maps

- \( \phi_\lambda, \psi_\beta : \mathbb{R} \rightarrow \mathbb{R}, \phi_\lambda(x) = \lambda x \) and \( \psi_\beta(x) = \beta(x) \),
- \( A^s, B^s, T^s_1, T^s_2 : \mathbb{R}^s \rightarrow \mathbb{R}^s \), which are contractions (i.e. their norms are strictly less than one),
- \( A^u, B^u, T^u_1, T^u_2 : \mathbb{R}^u \rightarrow \mathbb{R}^u \), which are expansions (i.e. their inverse maps are contractions),

such that:

**(C1) There are local charts \( U_P \) and \( U_Q \) centred at \( P \) and \( Q \) such that in these coordinates we have, for all \( t \),

\[
f_t^{\pi(P)}(x^s, x^c, x^u) = (A^s(x^s), \phi_\lambda(x^c), A^u(x^u)),
\]

\[
f_t^{\pi(Q)}(x^s, x^c, x^u) = (B^s(x^s), \psi_\beta(x^c), B^u(x^u)),
\]

where \( \pi(P) \) and \( \pi(Q) \) are the projections of \( f_t \) to the stable and unstable manifolds at \( P \) and \( Q \), respectively.

In this representation, \( f_t \) is the identity for \( t = 0 \) and \( f_t \) is close to \( f \).
Figure 5. A simple cycle and its unfolding.

where $|\lambda| \in (0, 1)$ and $|\beta| > 1$, $x^s \in \mathbb{R}^s$, $x^c \in \mathbb{R}$ and $x^u \in \mathbb{R}^u$, and $\pi(P)$ and $\pi(Q)$ are the periods of $P$ and $Q$, respectively.

(C2) There is a quasi-transverse heteroclinic point $Y_P \in W^s(Q, f_0) \cap W^u(P, f_0)$ in $U_P$ such that, in the coordinates in the chart $U_P$, it holds:

1. For every $t$, $Y_P = (0^s, 0, a^u) \in W^u(P, f_t)$, $a^u \in \mathbb{R}^u$.
2. There is a neighbourhood $C_{Y_P} \subset W^s(Q, f_0) \cap U_P$ of the form $(-1, 1)^s \times \{0, a^u\}$.
3. There is $\tau_{p,q} \in \mathbb{N}$ such that for all $t$ $Y_Q, t = (a^s, t, 0^u) = f^{\tau_{p,q}}(Y_P) \in U_Q \cap W^u(P, f_t)$, $a^s \in \mathbb{R}^s$ and $Y_Q, t \in J = f^{\tau_{p,q}}(I) = \{0^s\} \times [-1 - \delta, -1 + \delta] \times \{0^u\} \subset U_P$.

(C3) For every $t$, there is a point $X_Q \in U_Q \subset W^u(Q, f_t) \cap W^s(P, f_t)$ (independent of $t$) such that, in the coordinates in the chart $U_Q$, it holds:

1. $X_Q = (0^s, 1, 0^u)$ and there is $\delta > 0$ such that $X_Q \subset \mathbb{I} = [0^s] \times [1 - \delta, 1 + \delta] \times \{0^u\} \subset W^u(Q, f_t) \cap W^s(P, f_t)$.
2. There is $\tau_{q,p} \in \mathbb{N}$ such that $X_P = f^{\tau_{q,p}}(X_Q) = (0, -1, 0) \in U_P$ and $X_P \in \mathbb{I} = f^{\tau_{q,p}}(I) = \{0^s\} \times [-1 - \delta, -1 + \delta] \times \{0^u\} \subset U_P$.
3. There is a neighbourhood $U_{X_Q} \subset U_Q$ such that $\tau_{2, t} = \tau_2 = f^{\tau_{q,p}}(X_Q) \in U_P$ is an affine map of the form

$$\tau_2(x^s, x^c, x^u) = (T_2^s(x^s), \pm (x^c - 1), T_2^u(x^u)) + (0^s, -1, 0^u)$$

According to [8, sections 3.1-2], we give the following definition.
to emphasize the unfolding and the transition times \( \tau_{p,q} \).

In the local coordinates \( U_Q \), if it satisfies conditions (C1)–(C3) in proposition 4.1 and

\[ \tau_{p,q} \quad \text{and} \quad \tau_{q,p} \quad \text{are the unfolding and the transition times}, \]

\[ \lambda \quad \text{and} \quad \beta \quad \text{are the central multipliers} \]

\[ \phi_t(x) = \lambda x \quad \text{and} \quad \psi_\beta(x) = \beta x \quad \text{are the linear central maps of the cycle}. \]

**Remark 4.3.** Since we are only interested in the dynamics in the central direction of the simple cycle, we denote the simple cycle and its unfolding model by \( sc( f, Q, P, \beta, \lambda, \pm, \pm) \).

We now state some generalizations of the simple cycles above.

### 4.1.1. Simple cycles with homoclinic intersections and semi-simple cycles

In our constructions we will consider cycles associated with saddles with non-trivial homoclinic classes. We want to find that some of these homoclinic intersections associated with this saddle were ‘detected’ by the cycle and ‘well posed’ in relation to it. This leads to the next definition.

**Definition 4.4 (Simple cycles with adapted homoclinic intersections).** Consider a simple cycle \( sc(f, Q, P, \beta, \lambda, \pm, \pm) \). Write \( f = f_0 \) and let \( (f_t)_{t \in [\varepsilon, \varepsilon]} \) be a model unfolding family of \( f_0 \). The family \( (f_t)_{t \in [\varepsilon, \varepsilon]} \) has **adapted homoclinic intersections** (associated with \( P \)) if it satisfies conditions (C1)–(C3) in proposition 4.1 and

(C4) In the local coordinates in \( U_Q \), there is \( \tilde{a}^t \in (-1, 1)^s \) such that

\[ \Delta_0 = \{(\tilde{a}^t, 1)\} \times [-1, 1]^s \subset W^u(P, f_t), \quad \text{for every } t \text{ close to } 0. \]

This implies that \( (\tilde{a}^t, 1, 0) \) is a transverse homoclinic point of \( P \) of \( f_t \) for all \( t \) close to 0.

The family \( (f_t)_{t \in [\varepsilon, \varepsilon]} \) has a **sequence of adapted homoclinic intersections** (associated with \( P \)) if it satisfies conditions (C1)–(C4) and

(C5) In the local coordinates in \( U_Q \), for every \( t \) close to 0 there are sequences

\[ \tilde{a}^t_i \to \tilde{a}^t \quad \text{and} \quad x_i \to 1, \quad \tilde{a}^t_i \in (-1, 1)^s \quad \text{and} \quad x_i \in (1 - \delta, 1 + \delta), \]

such that

\[ \Delta_i = \{(\tilde{a}^t_i, x_i)\} \times [-1, 1]^s \subset W^u(P, f_t) \quad \text{for every } t \text{ close to } 0. \]

Moreover, the orbits by \( f_t \) of the discs \( \Delta_i, i \geq 0 \), are pairwise disjoint.

As above, this implies that \( (\tilde{a}^t_i, x_i, 0) \) is a transverse homoclinic point of \( P \) of \( f_t \).

In these cases, we say that \( f_0 \) has a **simple cycle with an adapted (sequence of) homoclinic intersection(s)**.

Since we will consider perturbations of simple cycles, in some cases we will need to consider diffeomorphisms with ‘simple cycles’ such that the maps \( \psi_\beta \) and \( \phi_t \) in proposition 4.1 are not linear.

**Definition 4.5 (Semi-simple cycles).** A diffeomorphism \( f \) has a **semi-simple cycle** associated with saddles \( P \) and \( Q \) if it satisfies the properties of the diffeomorphism \( f_0 \) in proposition 4.1, where the linear central maps \( \phi_t \) and \( \psi_\beta \) in (C1) are replaced by maps \( \bar{\phi}_t, \bar{\psi}_\beta : \mathbb{R} \to \mathbb{R} \) with

\[ \bar{\phi}(0) = \bar{\psi}(0) = 0, \quad \bar{\phi}'(0) = \lambda, \quad \bar{\psi}'(0) = \beta. \]

For such a semi-simple cycle we use the notation \( ssc(f, Q, \psi_\beta, \phi_t, \pm, \pm) \).
4.2. Twisted and non-twisted cycles

To a simple cycle \( \text{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2) \) we associate \( \text{sign}(Q), \text{sign}(P) \) and \( \text{sign}(\Xi_1) \) in \([+, -]\) by the following rules:

- \( \text{sign}(Q) = + \) if \( \beta > 0 \) and \( \text{sign}(Q) = - \) if \( \beta < 0 \),
- \( \text{sign}(P) = + \) if \( \lambda > 0 \) and \( \text{sign}(P) = - \) if \( \lambda < 0 \) and
- \( \text{sign}(\Xi_1) = + \) if \( \pm_1 = + \) (i.e. \( \theta_{1,0}(x^\prime) = x^\prime \)) and \( \text{sign}(\Xi_1) = - \) if \( \pm_1 = - \) (i.e. \( \theta_{1,0}(x^\prime) = -x^\prime \)).

**Definition 4.6 (Twisted and non-twisted cycles).** We say that a simple cycle is twisted if \((\Xi_2)\) in proposition 4.1. By construction, the central component of such a cycle is non-twisted.

A diffeomorphism \( f \) with a coincex one cycle with real central multipliers (associated with \( P \) and \( Q \)) is twisted (respectively non-twisted) if there is a diffeomorphism \( h \) arbitrarily \( C^1 \)-close to \( f \) with a twisted (respectively non-twisted) simple cycle associated with \( P \) and \( Q \).

Next lemma means that after a perturbation non-twisted cycles can be chosen satisfying \((\text{sign}(Q), \text{sign}(P), \text{sign}(\Xi_1)) = (\pm, \pm, +)\) (i.e. the case \((-, -, -)\) can be discarded).

**Lemma 4.7.** Consider a non-twisted simple cycle \( \text{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2) \). Then there is a diffeomorphism \( g \) arbitrarily close to \( f \) with a simple cycle associated with \( P \) and \( Q \) of such that

\[
(\text{sign}(Q), \text{sign}(P), \text{sign}(\Xi_1^g)) = (\pm, \pm, +).
\]

This notation emphasizes that \( \Xi_1^g \) is the unfolding map of the cycle associated with \( g \).

**Proof.** If \( \text{sign}(\Xi_1) = + \) we are done. If \( \text{sign}(\Xi_1) = - \) then the definition of non-twisted cycle implies that at least one of the central multipliers \( \lambda \) and \( \beta \) of the cycle is negative. To prove the lemma we fix a constant \( K > 0 \) (with \( K > |\beta|^2 \) and \( K^{-1} < |\lambda|^2 \)) and replace the unfolding map \( \Xi_1 \) by a composition of the form

\[
(Df^{\pi(Q)})^m \circ \Xi_{1,0} \circ (Df^{\pi(P)})^n,
\]

where \( n \) and \( m \) are arbitrarily large and

\[
\lambda^n \beta^m < 0 \quad \text{and} \quad K^{-1} < |\lambda^n \beta^m| < K.
\]

In this way, we obtain a new ‘unfolding map’ \( \tilde{\Xi}_{1,0} = f^m \circ \Xi_{1,0} \circ f^n \), defined on a small neighbourhood of \( f^{-n}(Y_P) \), where \( Y_P \in W^u(Q, f) \cap W^s(P, f) \) is the heteroclinic point in \((\text{C}2)\) in proposition 4.1. By construction, the central component \( \tilde{\theta}_{1,0} \) of \( \tilde{\Xi}_{1,0} \) satisfies

\[
\tilde{\theta}_{1,0}(x') = -\lambda^n \beta^m x' = |\lambda^n \beta^m| x'.
\]

Consider now the segment of orbit

\[
\{ f^{-n}(Y_P), \ldots, Y_P, \ldots, f^{\tau_1}(Y_P), \ldots, f^{\tau_1+\gamma}(Y_P) \}.
\]

Since \( n \) and \( m \) are arbitrarily big and \( K^{-1} < |\lambda^n \beta^m| < K \), we can modify the map \( f \) along this segment of orbit to obtain \( \tilde{\theta}_{1,0}(x') = x' \). This perturbation can be taken arbitrarily small if \( n \) and \( m \) are arbitrarily large. Therefore the new simple cycle is of type \((\pm, \pm, +)\). This completes the sketch of the proof of the lemma. For further details see [8, proposition 3.5]. □
4.3. Quotient dynamics. Families of iterated function systems

In what follows, \((f_t)_{t \in [-\epsilon, \epsilon]}\) is a model unfolding family associated with a diffeomorphism \(f = f_0\) with a semi-simple cycle. We use the notation in proposition 4.1. Next remark allows us to consider (in a neighbourhood of a semi-simple cycle) the quotient dynamics by the strong stable/unstable hyperplanes.

**Remark 4.8.** Consider a semi-simple cycle \(ssc(f, Q, P, \psi_\beta, \phi_\lambda, \pm 1, \pm 2)\) and its model unfolding map \((f_t)_{t \in [-\epsilon, \epsilon]}\), where \(f_0 = f\). Consider the partially hyperbolic splitting \(E^{ss} \oplus E^c \oplus E^{uu}\), defined over the orbits of \(P\) and \(Q\), that in the local charts \(U_P\) and \(U_Q\) is of the form

\[
E^{ss} = \mathbb{R}^s \times \{(0', 0)\}, \quad E^c = \{0\} \times \mathbb{R} \times \{0\}, \quad E^{uu} = \{(0', 0)\} \times \mathbb{R}^u.
\]

This splitting is extended to \(U_P \cup U_Q\) as constant bundles. Proposition 4.1 implies that the maps \(T_1, t\) and \(T_2, t\) are affine maps preserving \(E^{ss} \oplus E^c \oplus E^{uu}\).

The open set \(V\) defined by

\[
V = U_P \cup U_Q \cup \left( \bigcup_{j=0}^{\tau_{Q, P}} f_0^j(U_{X_Q}) \right) \cup \left( \bigcup_{j=0}^{\tau_{P, Q}} f_0^j(U_{Y_P}) \right)
\]

is the neighbourhood associated with the cycle. For small \(t\), we consider the maximal invariant set \(\Lambda t(V)\) of \(f_t\) in \(V\),

\[
\Lambda t(V) = \bigcap_{i \in \mathbb{Z}} f_i^t(V).
\]

By construction, for \(f_t\), there is a partially hyperbolic extension of the splitting \(E^{ss} \oplus E^c \oplus E^{uu}\) over the set \(\Lambda t(V)\). With a slight abuse of notation, we also denote this extension by \(E^{ss} \oplus E^c \oplus E^{uu}\).

This remark implies that the returns of points \(X \in U_{X_Q} \cap \Lambda t(V)\) to \(U_{X_Q}\),

\[
X \in U_{X_Q} \cap \Lambda t(V) \mapsto f_i^t(X) \in U_{X_Q},
\]

preserve the codimension one foliation \(\mathbb{R}^s \times \{x^c\} \times \mathbb{R}^u\) tangent to \(E^{ss} \oplus E^{uu}\). We consider the ‘quotient dynamics’ by these hyperplanes, obtaining a one-parameter family of IFSs defined on the interval \(I = [1 - \delta, 1 + \delta]\) (see item (1) in (C3) in proposition 4.1). This family describes the ‘central’ dynamics of these returns. We will provide in proposition 4.9 a ‘dictionary’ translating properties of this IFS to properties of the diffeomorphisms \(f_t\). These properties are about the existence of periodic orbits, homoclinic and heteroclinic intersections and cycles.

4.3.1. Families of IFSs induced by the quotient dynamics

Consider a semi-simple cycle \(ssc(f, Q, P, \psi_\beta, \phi_\lambda, \pm 1, \pm 2)\) and its model unfolding family \((f_t)_{t \in [-\epsilon, \epsilon]}\), here \(f = f_0\). Consider the segment \(I\) in condition (C3)(1) in proposition 4.1. For each pair \((k, n)\) of large natural numbers and small \(t\), define the map

\[
\Gamma^{k,n}_t : [l_{k,n}^t, [l_{k,n}^t] \to I, \quad \Gamma^{k,n}_t(x) = (\psi_\beta \circ \theta_1 \circ \phi_\lambda \circ \theta_2)(x),
\]

where \([l_{k,n}^t\) is the maximal subinterval of \(I\) where the map \(\Gamma^{k,n}_t\) is defined. Note that there are choices of \(k, n, t\) such that the set \([l_{k,n}^t\) is empty.

The one-parameter family \((\Gamma^{k,n}_t)_{t \in [-\epsilon, \epsilon]}\) is the IFS associated with \((f_t)_{t \in [-\epsilon, \epsilon]}\).
**4.3.2. Dictionary IFS—Global dynamics** Using the invariance of the splitting $E^{as} \oplus E^c \oplus E^{uu}$ above one gets the following extension of [8, proposition 3.8]:

**Proposition 4.9 (Quotient dynamics—Global dynamics).** Consider a semi-simple cycle $ssc(f, Q, P, \psi_\beta, \phi_\lambda, \tau_{p,q}, \tau_{q,p})$, its model unfolding family $(f_t)_{t \in \{-\epsilon, \epsilon\}}$, here $f = f_0$, and its associated IFS $(\Gamma_{t,n,m})_{t \in \{-\epsilon, \epsilon\}}$. Suppose that the saddles $P$ and $Q$ have $s$-indices $(s+1)$ and $s$, respectively.

**(A) Periodic points:** Suppose that there is $r \in \mathbb{R}^n$ such that

$$\Gamma_t^k(r) = r.$$  

Then there are $r^s \in \mathbb{R}^s$ and $r^u \in \mathbb{R}^u$ such that

$$R = (r^s, r, r^u) \in U_Q \cap \Lambda_t(V)$$

is a periodic point of $f_t$ of period

$$\pi(R) = k \pi(Q) + n \pi(P) + \tau_{p,q} + \tau_{q,p}.$$  

The eigenvalue of $Df^{\pi(R)}(r)$ corresponding to central direction $\{0^s\} \times \mathbb{R} \times \{0^u\}$ is

$$\left(\Gamma_{t,n}^l(r)\right) (r) = \left(\psi^{\beta}_t\right)^{(0_1, (\phi_\lambda^0(\theta_2(r))))} \left(\phi_\lambda^0\right)^{(\theta_2(r))}.$$  

In particular, if $\left|\left(\Gamma_{t,n}^l(r)\right) (r)\right| > 1$ (resp. $< 1$) the periodic point $R$ has s-index $s$ (resp. $s$-index $s+1$).

Moreover, the periodic point $R$ also satisfies

$$W^{ss}(R, f_t) \cap W^{u}(R, f_t) \neq \emptyset \quad \text{and} \quad W^{uu}(R, f_t) \cap W^{s}(P, f_t) \neq \emptyset. \quad (4.3)$$

In what follows, let $r$, $R$ and $(k, n)$ be as in item (A).

**(B) Strong homoclinic intersections:** Suppose that there is a pair $(\bar{k}, \bar{n}) \neq (k, n)$ such that

$$\Gamma_{t,n}^{\bar{k},\bar{n}}(r) = r.$$  

Then $W^{ss}(R, f_t) \cap W^{uu}(R, f_t)$ contains points that do not belong to the orbit of $R$.

**(C) Heterodimensional cycles:** Suppose that there are $d \in \mathbb{I}$ and $d^s \in \mathbb{R}^s$ such that (in the coordinates in $U_Q$)

$$\Upsilon = \Upsilon(d^s, d) = \{(d^s, d) \times [-1, 1]^u \subset W^{ss}(P, f_t)\}.$$  

If there is $i \in \mathbb{N}$ such that

$$\theta_{1,i} \circ \phi_{k_i}^i \circ \theta_2(d) = 0$$  

then

$$W^{ss}(P, f_t) \cap W^{u}(P, f_t) \neq \emptyset.$$  

Thus, as $W^{ss}(P, f_t) \cap W^{uu}(Q, f_t) \neq \emptyset$, the diffeomorphism $f_t$ has a heterodimensional cycle associated with $P$ and $Q$.

In particular, if there are $i, h \in \mathbb{N}$ such that

$$\theta_{1,i} \circ \phi_{k_i}^i \circ \theta_2 \circ \psi^{\beta}_h(t) = \theta_{1,i} \circ \phi_{k_i}^i \circ \theta_2 \circ \psi^{\beta}_h \circ \theta_{1,i}(0) = 0$$  

then $f_t$ has a heterodimensional cycle associated with $P$ and $Q$.  

**(D) Heteroclinic intersections (I):** Suppose that there are $i, \tilde{k}, \tilde{n} \in \mathbb{N}$ such that

$$\theta_{1,i} \circ \phi_{k_i}^i \circ \theta_2 \circ \Gamma_{t,n}^{\tilde{k},\tilde{n}}(r) = 0.$$
Then
\[ W^{uu}(R, f_i) \cap W^s(Q, f_i) \neq \emptyset. \]
If \((\bar{k}, \bar{n}) = (0, 0)\) the previous identity just means \(\theta_{1,i} \circ \phi^i_1 \circ \theta_2(r) = 0.\)

(E) Heteroclinic intersections (II): Let \((d^+, d^-)\) be as in item (C) (i.e. \(\Upsilon(d^+, d^-) \subseteq W^u(P, f_i)).\)
If there are \(i, j \in \mathbb{N}\) such that
\[ \Gamma^i_{j,i}(d) = r \]
then
\[ W^u(P, f_i) \cap W^s(R, f_i) \neq \emptyset. \]
In particular, if
1. either \(r = d\) and \((i, j) = (0, 0),\)
2. or there is \(i\) such that \(\psi^i_\beta \circ \theta_{1,i}(0) = \psi^i_\beta(t) = r\)
then
\[ W^u(P, f_i) \cap W^s(R, f_i) \neq \emptyset. \]

(F) Homoclinic points: Suppose that there is \(i\) such that
\[ \psi^i_\beta \circ \theta_{1,i}(0) = \psi^i_\beta(t) = \bar{h} \in [1 - \delta, 1 + \delta]. \]
Then there is \(\bar{h}^+ \in (-1, 1)^t\) such that \(\bar{h} = (\bar{h}^+, \bar{h}, 0^\rho) \in U_Q\) is a transverse homoclinic point of \(P\) for \(f_i\) and
\[ \{(\bar{h}^+, \bar{h})\} \times [-1, 1]^u \subset W^u(P, f_i). \]

**Proof.** For notational simplicity, we assume that \(P\) and \(Q\) are fixed points.

Items (A) and (B) are stated in [8, proposition 3.8]. To prove item (A) it is enough to observe that the definition of the pair \((k, n)\) and the product structure provide a pair of cubes \(\Delta^u \subseteq [-1, 1]^u\) and \(\Delta^s \subseteq [-1, 1]^s\) such that
\[ f^\ell_{s}([-1, 1]^s \times \{r\} \times \Delta^u) = \Delta^s \times \{r\} \times [-1, 1]^u, \quad \ell = k + n + \tau_{p,q} + \tau_{q,p}, \]
if \(k\) and \(n\) are large enough (note that \(k, n \to \infty\) as \(l \to 0\)). Note that \(Df^\ell_{s}\) uniformly contracts vectors parallel to \(\mathbb{R}^s \times \{0, 0^\rho\}\) and uniformly expands vectors parallel to \([0^\rho] \times \mathbb{R}^u\).
This gives the periodic point \(R = (r^\ell, r, r^u)\) of period \(\ell\). Note that our arguments also imply that
\[ W^{uu}(R, f_i) \supset \{(r^\ell, r)\} \times [-1, 1]^u, \quad W^u(R, f_i) \supset [-1, 1]^s \times \{(r, r^u)\}. \tag{4.4} \]
Note also that from (C3)(1) in proposition 4.1, in the coordinates in \(U_Q\), one has that
\[ \{0^\rho\} \times [1 - \delta, 1 + \delta] \times [-1, 1]^u \subset W^u(Q, f_i) \quad \text{and} \quad [-1, 1]^s \times [1 - \delta, 1 + \delta] \times \{0^\rho\} \subset W^s(P, f_i). \tag{4.5} \]
The intersection properties between the invariant manifolds of \(R, P\) and \(Q\) in item (A) follow immediately from equations (4.4) and (4.5) and \(r \in [1 - \delta, 1 + \delta].\)

To prove item (B) one argues exactly as in item (A). Note that the choice of \((\bar{k}, \bar{n})\) (large \(\bar{k}, \bar{n})\) provides a cube \(\bar{\Delta}^u \subseteq [-1, 1]^u\) and a point \(\bar{r}^\ell \in [-1, 1]^t\) such that
\[ f^m_{s}((\bar{r}^\ell, \bar{r}) \times \bar{\Delta}^u) = (\bar{r}^\ell, \bar{r}) \times [-1, 1]^u, \quad m = \bar{k} + \bar{n} + \tau_{p,q} + \tau_{q,p}. \]
Since \((\bar{r}^\ell, \bar{r}) \times \bar{\Delta}^u \subset W^{uu}(R, f_i)\) and \((\bar{r}^\ell, \bar{r}, r^u) \in W^u(R, f_i)\) there is a strong homoclinic intersection associated with \(R\).
To prove the first part of item (C) note that if \( t \) is small then \( i \) is large and thus
\[
f_{t}^{\tau_{p,q}^{*}+i_{p,q}^{*}}(\Upsilon) \cap U_{Q} = f_{t}^{\tau_{p,q}^{*}+i_{p,q}^{*}}((d^{*}, d)) \times [-1, 1]^{w} \cap U_{Q} \\
\supset ((d^{*}, \theta_{1,i} \circ \phi_{k}^{i} \circ \theta_{2}(d))) \times [-1, 1]^{w} \\
= ((d^{*}, 0)) \times [-1, 1]^{w},
\]
for some \( d^{*} \in (-1, 1)^{w} \). Since \([-1, 1]^{w} \times \{(0, 0^{w})\} \subset W^{u}(Q, f_{i}) \) and \( \Upsilon \subset W^{u}(P, f_{i}) \) we obtain \( W^{u}(P, f_{i}) \cap W^{u}(Q, f_{i}) \neq \emptyset \).

To prove the second part of item (C) consider \( a^{r} \in \mathbb{R}^{r} \) and the linear map \( B^{r} \) as in (C2)(3) and (C1) in proposition 4.1, respectively. Note that
\[
((B^{r})^{h}(a^{r}), \psi_{p}^{h}(t), 0^{w}) = (d^{*}, \tilde{d}, 0^{w}), \quad \bar{d} = \psi_{p}^{h}(t) = \psi_{p}^{h} \circ \theta_{1,i}(0) \in [1 - \delta, 1 + \delta]
\]
is a transverse homoclinic point of \( P \) such that
\[
\tilde{\Upsilon} = ((\tilde{d}^{*}, \tilde{d})) \times [-1, 1]^{w} \subset W^{u}(P, f_{i}) \cap U_{Q}.
\]
The intersection between \( W^{u}(P, f_{i}) \) and \( W^{u}(Q, f_{i}) \) now follows applying the first part of item (C) to the disc \( \tilde{\Upsilon} \): just note that by hypothesis and the definition of \( \bar{d} = \psi_{p}^{h} \circ \theta_{1,i}(0) \) one has \( \theta_{1,i} \circ \phi_{k}^{i} \circ \theta_{2}(\bar{d}) = 0 \).

Item (D) follows similarly. Let (in the coordinates in \( U_{Q} \))
\[
\Delta = \{(r^{*}, r) \times [-1, 1]^{w} \subset W^{uu}(R, f_{i})
\]
In the coordinates in \( U_{Q} \), we have
\[
f_{t}^{\tau_{p,q}^{*}+i_{p,q}^{*}+\tilde{w}+i_{p,q}^{*}+\tilde{k}+\tilde{t}+s}(\Delta) \supset ((\tilde{r}^{*}, \theta_{1,i} \circ \phi_{k}^{i} \circ \theta_{2}^{h}(r)), (\tilde{r}^{*}, 0)) \times [-1, 1]^{w}
\]
for some \( \tilde{r}^{*} \). As \([-1, 1]^{w} \times \{(0, 0^{w})\} \subset W^{u}(Q, f_{i}) \) we obtain \( W^{uu}(R, f_{i}) \cap W^{u}(Q, f_{i}) \neq \emptyset \).

The remainder assertions (E) and (F) in the proposition follow analogously, so we omit their proofs.

\[\square\]

5. Simple non-twisted cycles

In this section we first consider non-twisted cycles and explain how these cycles yield partially hyperbolic saddle-node/flip points with strong homoclinic intersections as well as further intersection properties, see proposition 5.3. Using proposition 4.9 we will write these properties in terms of the IFSs associated with the cycle. We also see how these intersections are realized by perturbations (model families) of the initial cycle. These intersection properties are the main ingredient for the stabilization of cycles. Finally, in section 5.3 we consider cycles involving a saddle with a non-trivial homoclinic class and introduce the bi-accumulation property.

5.1. Non-twisted simple cycles with adapted homoclinic intersections

The first step is to see that non-twisted simple cycles yield simple cycles with adapted homoclinic intersections.

**Lemma 5.1.** Consider a non-twisted cycle \( sc(f, Q, P, \beta, \lambda, \pm_{1}, \pm_{2}) \). There is \( g \) arbitrarily \( C^{1} \)-close to \( f \) having a non-twisted simple cycle (associated with \( Q \) and \( P \)) with a sequence of adapted homoclinic intersections (associated with \( P \)).
Proof. Note that by lemma 4.7 we can assume that \( \theta_{1,i}(x) = x + t_i \). The proof has two steps. We first perturb the cycle to obtain a cycle with one adapted homoclinic intersection. In the second step we perturb this new cycle with an adapted homoclinic intersection to obtain a cycle with a sequence of adapted homoclinic intersections.

A cycle with one adapted homoclinic intersection. Observe that, after an arbitrarily small perturbation, we can assume that the central multipliers of the cycle satisfy \( \lambda_k^2 = \beta^{-m} > 0 \) for some arbitrarily large \( k \) and \( m \). We fix small \( t_k > 0 \) such that

\[
t_k = \lambda_k^2 = \beta^{-m}.
\]  

(5.1)

This choice gives

\[
\psi^m_P(\theta_{1,i}(0)) = \psi^m_P(t_k) = 1.
\]

Therefore, by (F) in proposition 4.9, the point \( H = (h^t, 1, 0) \in U_Q \) is a transverse homoclinic point of \( P \) such that

\[
\{(h^t, 1)\} \times [-1, 1]^u \subset W^u(P, f_{t_k}).
\]

The point \( H \) will provide the adapted homoclinic point in definition 4.4.

To see that \( f_{t_k} \) has a cycle associated with \( P \) and \( Q \) just note that

\[
\theta_{1,i} \circ \phi_{k}^l \circ \theta_2 \circ \psi^m_P \circ \theta_{1,i}(0) = \theta_{1,i} \circ \phi_{k}^l \circ \theta_2(1) = -\lambda_k^2 + t_k = 0.
\]

(5.2)

Item (C) in proposition 4.9 implies that \( W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k}) \neq \emptyset \).

Let \( \hat{Y}_P \) be the heteroclinic point in \( W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k}) \) corresponding to the condition in (5.2). This implies that \( f_{t_k} \) has a cycle associated with \( P \) and \( Q \) and that the points \( X_Q \in W^s(P, f_{t_k}) \cap W^u(Q, f_{t_k}) \) (in condition (C3)(1)) and \( \hat{Y}_P \in W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k}) \) are heteroclinic points associated with this cycle. Using the transverse homoclinic point \( H \) of \( P \) and arguing as in lemma 4.7, we will obtain a cycle with an adapted homoclinic intersection.

Indeed, repeating the previous argument we can assume that the cycle has two ‘adapted homoclinic points’. The additional one is of the form \( V = (v', 1 + v, v^u) \), where \( 1 + v \in [1 - \delta, 1 + \delta] \) (in principle \( v \neq 0 \)) and \( \Delta_V = [(v', 1 + v) \times [-1, 1]^u \subset W^u(P, f_{t_k}) \). Also have that the discs \( \Delta_V \) and \( \Delta_H = [(h^t, 1)] \times [-1, 1]^u \subset W^u(P, f_{t_k}) \) have disjoint orbits. We use the disc \( \Delta_V \) to obtain the sequence of adapted homoclinic intersections.

A cycle with a sequence of adapted homoclinic intersections. To obtain a cycle with a sequence of adapted homoclinic intersections we argue as above, but now starting with a cycle with ‘two adapted homoclinic intersections’, say \( H \) and \( V \) as above. We assume that \( \theta_2(1 + x) = (-1 + x) \). The case \( \theta_2(1 + x) = (1 + x) \) is analogous. As above we can assume that equation (5.1) holds for infinitely many \( m \) and \( k \).

To obtain a sequence of homoclinic points \( H_i \) accumulating to \( H \) write

\[
\delta_i = \beta^{-m} \lambda^{2i} (1 - v) > 0
\]

and consider the sequence

\[
\psi_P^m \circ \theta_{1,i} \circ \phi_{k}^l \circ \theta_2(1 + v) = \psi_P^m(t_k - \lambda_k^2 (1 - v)) = 1 - \beta^{-m} \lambda^{2i} (1 - v) = 1 - \delta_i.
\]

Item (F) in proposition 4.9 implies that for each \( i \) there is \( h_{1,i} \) such that

\[
H_i = (h_{1,i}, 1 - \delta_i, 0) \in U_Q, \quad \delta_i > 0,
\]

(5.3)

is a transverse homoclinic point of \( P \) and

\[
\Delta_i = [(h_{1,i}, 1 - \delta_i)] \times [-1, 1]^u \subset W^u(P, f_{t_k}).
\]

This sequence accumulates to \( \Delta_H \) and the discs \( \Delta_i \) and \( \Delta_H \) have disjoint orbits by construction.
Finally, arguing exactly as above we have that $f_0$ has a heterodimensional cycle associated with $P$ and $Q$.

Write $\tilde{f} = f_0$. We perturb $\tilde{f}$ to obtain a simple cycle with a sequence of adapted homoclinic intersections. Note that $\tilde{f}$ preserves the partially hyperbolic splitting $E^{ss} \oplus E^{c} \oplus E^{uu}$ in the neighbourhood $V$ of the initial simple cycle (recall (4.1)). For this new cycle we have ‘transition maps’ say $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ (in principle, these maps do not satisfy all the properties of ‘true’ transitions). These new ‘transitions’ $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ are obtained considering compositions of the maps $T_1$, $T_2$, $Df^{\pi(P)}(P)$, and $Df^{\pi(Q)}(Q)$ defined for the initial cycle and replacing the heteroclinic points $X_Q$ and $Y_P$ by some backward iterates of them. Note that the central maps $\tilde{T}_1$, $\tilde{T}_2$, and $\tilde{T}_3$ associated with the ‘new transitions’ may fail to be isometries.

Now, exactly as in the proof of lemma 4.7, we consider an arbitrarily small perturbation of $\tilde{f}$ obtained taking multiplications (in the central direction) by numbers close to one throughout long segments of the orbits of $X_Q$ and $Y_P$. This is possible since $\theta$ can be taken arbitrarily small and $k$ and $m$ arbitrarily big. The resulting diffeomorphism has a simple cycle with a sequence of adapted homoclinic intersections associated with $P$ (obtained considering appropriate iterations of the points $H_i$ and $H$). This completes the proof of the lemma. \hfill $\square$

Remark 5.2. Using equation (5.3), we can assume that in the coordinates in $U_Q$, the adapted transverse homoclinic points of $P$ are such that
\[ H = (h', 1, 0^u) \quad \text{and} \quad \{(h', 1)\} \times [-1, 1]^u \subset W^u(P, f), \]
\[ H_i = (h_i', \zeta_i, 0^u) \quad \text{and} \quad \{(h_i', \zeta_i)\} \times [-1, 1]^u \subset W^u(P, f), \]
where ($\zeta$) is an increasing sequence converging to 1.

5.2. Dynamics generated by non-twisted cycles

Consider a diffeomorphism $f$ with a simple cycle and its associated neighbourhood $V$ in (4.1). For $g$ close to $f$ let $\Lambda_g(V) = \cap g(V)$ be the maximal invariant set of $g$ in $V$. Note that the set $\Lambda_g(V)$ has a partially hyperbolic splitting of the form $E^{ss}_g \oplus E^c_g \oplus E^{uu}_g$, where $E^{ss}_g$ is one-dimensional and $E^c_g$ and $E^{uu}_g$ uniformly contracting and expanding, respectively.

Proposition 5.3. Consider a non-twisted cycle $sc(f, Q, P, \beta, \lambda, +, \pm, \pm)$ with a sequence of adapted homoclinic intersections (associated with $P$). Then there is a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ with a partially hyperbolic saddle-node/flip $S_g \in \Lambda_g(V)$ of arbitrarily large period satisfying the following properties:

1. $W^s(S_g, g) \cap W^u(Q, g) \neq \emptyset$,
2. $W^u(S_g, g) \cap W^s(P, g) \neq \emptyset$,
3. $W^u(P, g) \cap W^s(S_g, g) \neq \emptyset$,
4. $W^u(P, g) \cap W^s(Q, g) \neq \emptyset$ and this intersection is quasi-transverse and
5. the homoclinic class of $P$ for $g$ is non-trivial.

Remark 5.4. Indeed, the proof of this proposition will imply that the strong unstable manifold of $S_g$ transversely intersects the disc $[-1, 1]^u \times \mathbb{I} \times \{0^u\}$ contained in $W^u(P, g)$ in (C3)–(1) in proposition 4.1. Now item (3) in proposition 5.3 implies that $W^u(P, g)$ accumulates to $W^u(S_g, g)$ (may be after a perturbation). Thus after a perturbation we can assume that $W^u(P, g) \cap \{(\{-1, 1\}^u \times \mathbb{I} \times \{0^u\}) \neq \emptyset$. 
5.2.1. Proof of proposition 5.3  The main step in the proof of the proposition is the next lemma about the IFS associated with a simple cycle.

**Lemma 5.5.** Consider a non-twisted cycle \( \text{sc}(f, Q, P, \beta, \lambda, +, \pm 2) \) with an increasing sequence of adapted homoclinic intersections \( (h_i, \zeta_i, 0^u) \) as in remark 5.2.

Then there are sequences of parameters \( (t_i), t_i \to 0 \), and of perturbations \( \psi_{\beta,i} \) of \( \psi_{\beta}(x) = \beta x \), \( \psi_{\beta,i} \to \psi_{\beta} \), such that the IFS \( \tilde{\Gamma}_{\beta,i}^{n,k} \) associated with \( \phi_{\lambda}, \psi_{\beta,i}, \theta_1, \theta_2 \) in equation (4.2) satisfies the following properties:

1. There is a sequence of pairs \( (v_i, w_i) \), \( v_i, w_i \to \infty \), such that \( \tilde{\Gamma}_{\beta,i}^{v_i,w_i}(1) = 1 \), \( \lambda^2 \left| \frac{1}{2(1 - |\lambda|^2)} - \lambda_{c}(S_g) \right| < \lambda_{c}(S_g) < \frac{2 |\lambda|}{1 - |\lambda|} \).
2. There are large \( j \) and \( \ell \in \mathbb{N} \) such that \( \theta_1 \circ \phi_{\ell}^{j} \circ \theta_2(\zeta_j) = 0 \).
3. There are \( j_0 \in \{j - 1, j + 1\} \) (as in item (2)) and \( \bar{n}, \bar{\ell} \in \mathbb{N} \) such that \( \Gamma_{\beta,i}^{\bar{n},\bar{\ell}}(\zeta_{j_0}) = 1 \).

We postpone the proof of this lemma to the next subsection.

**Proof of proposition 5.3.** Note that for each \( t_i \) there is a perturbation \( f_i \) of \( f \), \( f_i \to f \) as \( i \to \infty \), having a semi-simple cycle \( \text{ssc}(f_i, Q, P, \psi_{\beta,i}, \lambda, +, \pm 2) \) 'close' to the initial cycle \( \text{ssc}(f, Q, P, \beta, \lambda, +, \pm 2) \) (i.e. we replace the linear map \( \psi_{\beta} \) by its perturbation \( \psi_{\beta,i} \), while preserving the cycle configuration).

For large \( i \), write \( g = f_i \) and select the pair \( (v_i, w_i) \) in item (1) of lemma 5.5. Let \( S_g = (s^r, 1, s^u) \) be the saddle associated with this pair and the central coordinate '1' given by (A) in proposition 4.9. By construction, the eigenvalue \( \lambda_{c}(S_g) \) of \( Dg_{\pi}(S_g)(S_g) \) corresponding to the central direction \( E_{c}^{s} \) satisfies

\[
\left| \frac{\lambda^2}{2(1 - |\lambda|^2)} \right| < |\lambda_{c}(S_g)| < \frac{2 |\lambda|}{1 - |\lambda|}.
\]

We claim that \( S_g \) also satisfies the intersection properties in the proposition (note that in principle \( S_g \) is not yet a saddle-node/flip).

- Items (1) and (2) in the proposition follow from equation (4.3) in item (A) of proposition 4.9.
- Item (3) in the proposition follows from (3) in lemma 5.5 and (E) in proposition 4.9, where \( d = \zeta_{j+1} \) corresponds to adapted homoclinic points (recall also remark 5.2). Note that using these points we also obtain that \( W^{u}(P, g) \) transversely intersects \( [-1, 1]^r \times 1 \times \{0^u\} \), proving remark 5.4.
- Item (4) in the proposition follows from (2) in lemma 5.5 and (C) in proposition 4.9, where \( d = \xi_j \) corresponds to an adapted homoclinic point.
- Since transverse homoclinic intersections persist and the saddle \( P \) has transverse homoclinic points for the diffeomorphism \( f \), we obtain (5) in the proposition.

It remains to see that we can take \( S_g \) with \( \lambda_{c}(S_g) = \pm 1 \). Observe that the period \( \pi(S_g) \) of \( S_g \) can be taken arbitrarily large and \( |\lambda_{c}(S_g)| \) is uniformly bounded (independent of the period). Arguing as in lemma 4.7, we perturb \( g \) along the orbit of \( S_g \) in order to transform this point into a saddle-node (if \( \lambda_{c}(S_g) > 0 \)) or a flip (if \( \lambda_{c}(S_g) < 0 \)). In this way one gets a partially hyperbolic saddle-node/flip. This perturbation can be done preserving the intersection properties in the proposition. This concludes the proof of the proposition. □
5.2.2. Proof of lemma 5.5 We first consider the case $\beta > 0$ and $\lambda > 0$.

**Positive central multipliers.** As above, after an arbitrarily small perturbation of the central multipliers of cycle, we can assume that there are arbitrarily large $m$ and $k$ with

$$\beta_m = \lambda^k (1 - \lambda).$$  \hfill (5.4)

Consider the parameter $t_k = \lambda^k$. This choice gives

$$\Gamma_{t_k}^{m, k+1}(1) = \psi_{\beta}^m \circ \theta_{1, t_k} \circ \phi_{\lambda}^{k+1} \circ \theta_2(1) = \psi_{\beta}^m \circ \theta_{1, t_k} (-\lambda^{k+1}) = \beta_m^m \lambda^k (1 - \lambda) = 1.$$

Take $(v_k, w_k) = (m, k + 1)$ and note that

$$(\Gamma_{t_k}^{v_k, w_k})'(1) = \pm \beta_m^m \lambda^k = \pm \frac{\lambda}{1 - \lambda}.$$

This gives (1) in the lemma. To obtain the other conditions we consider perturbations $\tilde{\psi}_\beta$ of $\psi_\beta$ preserving the condition $\Gamma_{t_k}^{v_k, w_k}(1) = 1$. From now on we fix the parameter $t_k$. We first consider the case where $\theta_2$ has derivative +1.

**Case $\theta_2(1 + x) = -1 + x$: For every small enough $\mu$, define $\beta(\mu)$ by**

$$\beta(\mu) = \frac{\lambda - \lambda^k}{1 - \lambda}.$$

and consider its associated linear map $\psi_{\beta(\mu)}(x) = \beta(\mu) x$. Write $\phi_\lambda(x) = \lambda x$. Note that the IFS $\tilde{\Gamma}_{t_k, \mu}^{i, j}$ associated with $\phi_\lambda, \psi_{\beta(\mu)}, \theta_{1, t_k} \mu$ and $\theta_2$ satisfies

$$\tilde{\Gamma}_{t_k, \mu}^{i, j}(1) = \psi_{\beta(\mu)}^m \circ \theta_{1, t_k} \mu \circ \phi_{\lambda}^{k+1} \circ \theta_2(1) = 1,$$

for all small $\mu$. \hfill (5.5)

Thus, for $(v_k, w_k) = (m, k + 1)$,

$$(\tilde{\Gamma}_{t_k, \mu}^{v_k, w_k})'(1) = \beta(\mu) \lambda^k = \frac{\lambda}{1 - \lambda + \mu}.$$

Thus, for small $\mu$, these derivatives also satisfy (1).

Consider $\zeta_i$ as in remark 5.2, that is $\zeta_i = 1 - \delta_i, \delta_i \to 0^+$ and $\delta_i > \delta_{i+1}$. For large $i$ define

$$\omega_i(\mu) = \theta_{1, t_k + \mu} \circ \phi_{\lambda}^k \circ \theta_2(\zeta_i) = \theta_{1, t_k + \mu} (-\lambda^k - \lambda^k \delta_i) = \mu - \lambda^k \delta_i.$$

(5.7)

Note that

$$\omega_{i+1}(\mu) - \omega_i(\mu) = \lambda^k (\delta_{i+1}).$$

(5.8)

Define small $\mu_j > 0$ by the condition

$$\omega_j(\mu_j) = 0, \quad \mu_j = \lambda^k \delta_j, \quad \lim_{j \to \infty} \mu_j \to 0.$$

By the choice of $\mu_j$ and (5.8) one has

$$\omega_{j+1}(\mu_j) = \lambda^k (\delta_j - \delta_{j+1}), \quad \lim_{j \to \infty} \omega_{j+1}(\mu_j) \to 0^+.$$

In particular, $\omega_{j+1}(\mu_j)$ can be taken arbitrarily small in comparison with $\beta(\mu_j)^{-m} = \lambda^k (1 - \lambda) + \mu_j$. This immediately implies the following:

**Fact 5.6.** Given any $N > 0$ there is large $j$ such that $[\omega_{j+1}(\mu_j), \beta(\mu_j)^{-m}]$ contains at least $N$ consecutive fundamental domains of $\psi_{\beta(\mu)}$.

Using this fact, we obtain that for every large $j$ there is a small perturbation $\tilde{\psi}_{\beta(\mu)}$ of the linear map $\psi_{\beta(\mu)}$ such that:

- $\tilde{\psi}_{\beta(\mu)}(x) = \psi_{\beta(\mu)}(x)$ if $x \in [\beta(\mu)^{-m-1}, 1]$. 

Remark 5.7. Note that the first two conditions above imply that
\[ \tilde{\psi}_{\beta(\mu_j)}^{n_j/m}(\omega_j+1(\mu_j)) = 1. \] (5.9)
Also important, note that this perturbation can be done (and we do) in such a way previous conditions (5.5) (5.6) and (5.7) are preserved.

The previous construction can be summarized as follows. Fix large \( k \) and the sequence of parameters \( t_k, j = t_k + \mu_j \). For each large \( j \), consider the perturbation \( \tilde{\psi}_{\beta(\mu_j)} \) of \( \psi_{\beta(\mu_j)} \) and the IFS \( \hat{f}_{t_k,j}^{\mu_k} \) corresponding to \( \tilde{\psi}_{\beta(\mu_j)}, \phi_{1,k}, \theta_{1,k,j} \) and \( \theta_2 \). Then

(i) \( \hat{f}_{t_k,j}^{\mu_k}X_1(1) = 1 \) (recall (5.5)),
(ii) \( \hat{f}_{t_k,j}^{\mu_k}X(1) = \frac{\lambda}{\lambda^{1+\delta}} \) (recall (5.6)),
(iii) \( \theta_{1,k,j} \circ \phi_{1,k}^{j-1} \circ \theta_{2,j}(\xi_j) = \omega_j(\mu_j) > 0 \) (recall the choice of \( \mu_j \) and (5.7)) and
(iv) \( \hat{f}_{t_k,j}^{\mu_k} X_{j+1}(1) = \tilde{\psi}_{\beta(\mu_j)}^{n_j/m}(\omega_j+1(\mu_j)) = 1, \) (recall (5.9)).

To conclude the proof the lemma in this first case (positive multipliers and (5.5), (5.6) and (5.7)) just note that (i)–(ii) correspond to (1) in the lemma (iii) to (2) in the lemma and (iv) to (3) in the lemma.

Case \( \theta_2(1 + x) = -1 - x. \) We proceed as in the previous case and define the sequence \( \omega_i(\mu) \) similarly. In this case, instead of equation (5.7) we obtain

\[ \omega_i(\mu_j) = \theta_{1,k,j} \circ \phi_{1,k}^{i} \circ \theta_{2}(\xi_j) = \theta_{1,k,j}(\xi_j) = \omega_j(\mu_j) = \mu + \lambda^k \delta_i. \]

The proof now follows as above.

Non-positive central multipliers. In this case, after an arbitrarily small perturbation of the central multipliers of cycle, we can assume that there are arbitrarily large \( m \) and \( k \) with

\[ \beta^{-2m} = \lambda^{2k}(1 - \lambda^2). \] (5.10)

We consider the parameter \( t_k = \lambda^{2k}. \) The proof now follows exactly as in the case where the multipliers are both positive considering the sequences

\[ \omega_i(\mu_j) = \theta_{1,k,j} \circ \phi_{1,k}^{i} \circ \theta_{2}(\xi_j) = \theta_{1,k,j}(\xi_j) = \mu \pm \lambda^k \delta_i. \]

This completes the proof of lemma 5.5. \( \square \)

5.3. Cycles associated with bi-accumulated saddles

Given a periodic point \( R \) of \( f \), consider the eigenvalues \( \lambda_1(R), \ldots, \lambda_n(R) \) of \( Df(R)(R) \) ordered in increasing modulus and counted with multiplicity. Denote by \( \text{Per}^m(f) \) the set of (hyperbolic) saddles \( R \) of \( f \) of s-index \( k \) satisfying \( |\lambda_{k-1}(R)| < |\lambda_k(R)| < 1 \). Given such a saddle \( R \in \text{Per}^m(f) \), its local strong stable manifold \( W^s_{\text{loc}}(R, f) \) is well defined (recall that \( W^s(R, f) \) is the unique invariant manifold tangent to the eigenspace associated with \( \lambda_1(R), \ldots, \lambda_{k-1}(R) \)). Moreover, \( W^s_{\text{loc}}(R, f) \) has codimension one in \( W^s(R, f) \) and \( W^s_{\text{loc}}(R, f) \setminus W^s_{\text{loc}}(R, f) \) has \( 2 \pi(R) \) connected components (indeed \( W^s_{\text{loc}}(R, f) \) splits each component of \( W^s_{\text{loc}}(R, f) \) into two parts).

Given a saddle \( P \) of s-index \( s + 1 \), we consider the following subsets of \( H(P, f) \):

- \( \text{Per}_h(H(P, f)) \) is the subset of \( H(P, f) \) of hyperbolic periodic points \( R \) which are homoclinically related to \( P \) (thus \( R \) also has index \( s + 1 \)).
\begin{itemize}
\item $\text{Per}^{s+1}(H(P, f)) = \text{Per}_h^{s+1}(H(P, f)) \cap \text{Per}^{s+1}(f)$.
\end{itemize}

**Definition 5.8 (Bi-accumulation property).** A saddle $R \in \text{Per}^{s+1}(f)$ is $s$-bi-accumulated (by homoclinic points) if every component of $(W_{\text{loc}}^s(R, f) \setminus W_{\text{loc}}^u(R, f))$ contains transverse homoclinic points of $R$.

We have the following result.

**Lemma 5.9.** Let $f$ be a diffeomorphism with a coindex one cycle associated with $P$ and $Q$ such that $H(P, f)$ is non-trivial. Let $s$-index$(P) = s + 1$. Then there is $g$ arbitrarily $C^1$-close to $f$ such that

- there is a saddle $\bar{P}_g \in \text{Per}^{s+1}_h(H(P_g, g))$ that is $s$-bi-accumulated and
- the diffeomorphism $g$ has a cycle associated with $\bar{P}_g$ and $Q_g$.

**Proof.** The lemma follows from [2, 13]. From [2, proposition 2.3], if $H(P, f)$ is non-trivial then there is $g$ arbitrarily $C^1$-close to $f$ with a cycle associated with $P_g$ and $Q_g$ such that $\text{Per}^{s+1}_h(H(P_g, g))$ is infinite.

By [13, lemma 3.4], if the set $\text{Per}^{s+1}_h(H(P, f))$ is infinite then there is a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ with a cycle associated with $P_g$ and $Q_g$ and such that $\text{Per}^{s+1}_h(H(P_g, g))$ contains infinitely many $s$-bi-accumulated saddles. Pick one of these saddles $\bar{P}_g$ and note that to be bi-accumulated is a property that persists under perturbations. We can now perturb $g$ to obtain $h$ with a cycle associated with $\bar{P}_h$ and $Q_h$, ending the proof of the lemma. \qed

6. Stabilization of cycles. Proof of theorem 2.1

6.1. Stabilization of non-twisted cycles

The next proposition is the main step to prove the stabilization of non-twisted cycles.

**Proposition 6.1.** Let $f$ be a diffeomorphism with a non-twisted cycle associated with saddles $P$ and $Q$ such that $s$-index$(P) = s$-index$(Q) + 1$. Then there is a diffeomorphism $g$ arbitrarily $C^1$-close to $f$ with a partially hyperbolic saddle-node/flip $S_g$ such that:

1. $W^s(S_g, g) \cap W^u(Q_g, g) \neq \emptyset$,
2. $W^u(S_g, g) \cap W^s(P_g, g) \neq \emptyset$,
3. $W^u(S_g, g) \cap W^s(S_g, g)$ contains a point that is not in the orbit of $S_g$ (strong homoclinic intersection),
4. $W^u(S_g, g) \cap W^u(P_g, g) \neq \emptyset$ and
5. $W^u(S_g, g) \cap W^s(Q_g, g) \neq \emptyset$.

The dynamical configuration in the proposition is depicted in figure 6.

We postpone the proof of this proposition to section 6.1.1. We now prove (A) in theorem 2.1.

6.1.1. Proposition 6.1 implies (A) in theorem 2.1

Note that the transverse intersection conditions immediately imply that $s$-index$(P) = \dim(W^s(S)) + 1 = s + 1$ (condition (1) in theorem 3.5). Moreover, conditions (2)–(4) in proposition 6.1 imply that $S$ and $P$ satisfy (2)–(4) in theorem 3.5. Thus the diffeomorphism $g$ satisfies all conditions in theorem 3.5 and hence there is $h$ arbitrarily $C^1$-close to $g$ having a robust heterodimensional cycle associated with $P_h$ and a (transitive) hyperbolic set $\Gamma_h$ containing a continuation $S_h^s$ of $s$-index $s$ of $S_g$. 

Observe that items (1) and (5) in proposition 6.1 imply that the saddle $S^+_h$ of $h$ can be chosen such that

$$W^s(S^+_h, h) \cap W^u(Q_h, h) \neq \emptyset \quad \text{and} \quad W^u(S^+_h, h) \cap W^s(Q_h, h) \neq \emptyset.$$ 

Thus the saddles $S^+_h$ and $Q_h$ are homoclinically related and then there is a transitive hyperbolic set $\Sigma_h$ containing $Q_h$ and $\Gamma_h$. In particular, for every diffeomorphism $\varphi$ close to $h$ it holds $W^{su}(\Gamma_h, \varphi) \subset W^{su}(\Sigma_h, \varphi)$. Thus, by the first step of the proof, the diffeomorphism $h$ has a robust cycle associated with $\Sigma_h$ and $\Phi_h$, ending the proof of (A) in theorem 2.1.

\[\square\]

6.1.2. Proof of proposition 6.1

This proposition follows from proposition 5.3. First note that by lemma 5.1, after a small perturbation, we can assume that the cycle (associated with $P$ and $Q$) has a sequence of adapted homoclinic intersections associated with the saddle $P$. Thus applying proposition 5.3 we obtain $g$ close to $f$ with a partially hyperbolic saddle-node/flip satisfying conditions (1), (2) and (4) in proposition 6.1. It remains to obtain conditions (3) ($W^{uu}(S, g) \cap W^{ss}(S, g)$ contains a point that is not in the orbit of $S$) and (5) ($W^{uu}(S, g) \cap W^{s}(P, g) \neq \emptyset$) in proposition 6.1. To obtain these two properties we use arguments analogous to the ones in lemmas 5.1 and 5.5.

Since in what follows we do not modify the orbits of $P$, $Q$ and $S$ we omit the dependence on $g$. Note that since $W^{uu}(S, g) \cap W^{s}(P, g)$ (condition (2) in proposition 5.3) we have that $W^{uu}(S, g)$ accumulate to $W^{u}(P, g)$. Since by condition (4) in proposition 5.3 we have that $W^{u}(P, g) \cap W^{s}(Q, g) \neq \emptyset$, thus $W^{uu}(S, g)$ also accumulates to $W^{s}(Q, g)$. In particular there are segments of $W^{uu}(S, g)$ (with disjoint orbits) arbitrarily close to $W^{uu}(S, g)$. Use one of these segments to get $W^{uu}(S, h) \cap W^{s}(Q, h) \neq \emptyset$ for some $h$ close to $g$ (condition (5) in proposition 6.1).

Moreover, the previous perturbation can be done in such a way that there are segments of $W^{uu}(S, h)$ close to $W^{s}(Q, h)$ in the ‘same side’ of $W^{s}(Q, h)$ as $W^{uu}(S, h)$. See figures 7 and 8. We see that modifying the derivative of $Q$ in the central direction we obtain that $W^{uu}(S, h)$ intersects $W^{s}(S, h)$ (condition (3) in proposition 6.1). We briefly explain this last step. We can modify the derivative of $Q$ in the central direction such a way that $\Delta^uu$ intersects $W^{s}(S, h)$ (condition (3) in proposition 6.1). We briefly explain this last step. We can modify the derivative of $Q$ in the central direction such a way that $\Delta^uu$ intersects $W^{s}(S, h)$ (condition (3) in proposition 6.1).
Figure 7. Accumulation of $W^{uu}(S)$ to $W^{\text{loc}}_\text{inc}(Q)$.

Figure 8. Accumulation of $W^{uu}(S)$ to $W^{\text{loc}}_\text{inc}(Q)$.

intersects $\Delta^{uu}$ and therefore $W^{uu}(S, h) \cap W^{ss}(S, h) \neq \emptyset$. Note that these perturbations can be done preserving the saddle-node/flip $S$ and the intersection properties (1), (2) and (4) in proposition 6.1.

6.2. Stabilization of bi-accumulated twisted cycles

In this section we prove item (B) in theorem 2.1.

**Proposition 6.2 (Generation of non-twisted cycles).** Let $f$ be a diffeomorphism with a twisted cycle associated with saddles $P$ and $Q$ with $s$-index($P$) = $s$-index($Q$) + 1. Assume that $P$ is $s$-bi-accumulated. Then there is $g$ arbitrarily $C^1$-close to $f$ with a non-twisted cycle associated with $Q_g$ and a saddle $R_g$ that is homoclinically related to $P_g$.

Item (A) in theorem 2.1 implies that the cycle associated with $R_g$ and $Q_g$ can be stabilized. Since $R_g$ is homoclinically related to $P_g$, lemma 2.4 implies that the cycle associated with $P_f$ and $Q_f$ can also be stabilized. Thus proposition 6.2 implies (B) in theorem 2.1.

6.2.1. Proof of proposition 6.2 The proposition is an immediate consequence of the following two lemmas:

**Lemma 6.3.** Under the hypotheses of proposition 6.2, there is $g$ arbitrarily $C^1$-close to $f$ with a twisted simple cycle associated with $P$ and $Q$ and with an adapted homoclinic point of $P$. 
Lemma 6.4. Consider a twisted cycle \( sc(f, Q, P, \beta, \lambda, -\), \( \pm 2\), \( \lambda, \beta > 0\), with an adapted homoclinic intersection (associated with \( P\)). Then there is \( g \) arbitrarily \( C^1\)-close to \( f \) with a saddle \( R_g \) such that

- \( R_g \) is homoclinically related to \( P_g \) and
- \( g \) has a non-twisted cycle associated with \( R_g \) and \( Q_g \).

6.2.2. Proof of lemma 6.3 We claim that (in the coordinates in \( U_Q \) in proposition 4.1) there are sequences of points \( (x_i) \) and \( (\alpha' \)i \), \( x_i \in \mathbb{R} \) and \( \alpha' \)i \( \in \mathbb{R}^2 \), and of discs \( \Delta_i \) of dimension \( u \) such that

- \( (\alpha' \)i \), \( x_i, 0 \) \( \in \Delta_i \) where \( x_i \rightarrow 0^+ \) and \( \alpha' \)i \( \rightarrow \alpha \) and
- \( \Delta_i \rightarrow \{(\alpha', 0)\} \times [-1, 1]^u \) and \( \Delta_i \subset W^u(P, f) \).

here \( (\alpha', 0, 0^u) \) is the heteroclinic intersection between \( W^u(P, f) \) and \( W^s(Q, f) \) in (C2) in proposition 4.1.

To see why this assertion is so just note that, by the bi-accumulation property, there is a sequence of unstable discs \( \tilde{\Delta}_i \subset W^u(P, f) \) of dimension \( u \) approaching \( W^u(P, f) \) from the ‘negative side’, see figure 9. Since the cycle is twisted the map \( T^1, 0 \) reverses the ordering in the central direction. Thus these discs are mapped by \( \Sigma_{1,0} \) into discs \( \Delta_i \) that approaches \( (\alpha', 0, 0^u) \) from the ‘positive side’. See figure 9. We need to perform a perturbation in order to put these discs in ‘vertical’ position.

Arguing exactly as in the proof of lemma 5.5, after an arbitrarily small perturbation we can assume that \( \beta \) is such that \( \psi_{\beta}(xi) = 1 \) for some arbitrarily large \( i \) and \( ki \). This provides a transverse homoclinic point of \( P \) of the form \( (h^i, 1, 0) \). This follows from (F) in proposition 4.9. Note that this perturbation can be done preserving the cycle between \( P \) and \( Q \).

Finally, using this transverse homoclinic point and after an arbitrarily small perturbation, we obtain the simple cycle with an adapted homoclinic intersection associated with \( P \) and \( Q \) (the argument is similar to the one in lemma 4.7.) \( \square \)

6.3. Proof of lemma 6.4

The lemma follows arguing as in [8, lemma 3.13] and using proposition 4.9. Note that we can assume (after a small modification of \( \beta \) and \( \lambda \)) that \( \beta^{-m} = \lambda^k \). Noting that the cycle is twisted (i.e. \( \theta_{t, i}(x) = t - x \)) we have that this equality implies that

\[ \Gamma_m(1) = \psi_m \circ \theta_{1,0} \circ \phi_{\lambda^k} \circ \theta_2(1) = \psi_m \circ (- \phi_{\lambda^k}(-1)) = 1. \]
In this case we also have
\[
(G^{m,k}_{\beta})'(1) = (\psi^m_{\beta})'(-\phi^k_{\beta}(-1)) (\phi^k_{\beta})'(1) = \pm \beta^m \lambda^k = \pm 1.
\]
Thus modifying the central derivatives at \(P\) and \(Q\), we can assume that the cycle is semi-simple with central maps \(\psi_{\beta}\) and \(\phi_{\beta}\) such that there are large \(m, k\) and \(\ell\), with \(\ell >> k\), satisfying
\[
\tilde{\psi}^m_{\beta}(-\phi^k_{\beta}(-1) + \phi^\ell_{\beta}(-1)) = 1
\]
and
\[
|[\tilde{\psi}^m_{\beta}(-\phi^k_{\beta}(-1) + \phi^\ell_{\beta}(-1)) (\phi^\ell_{\beta})'(1)| < 1.
\]
For that note that \(\tilde{\phi}^k_{\beta}(-1)\) is arbitrarily small in comparison with \(\tilde{\phi}^\ell_{\beta}(-1)\).

Let \(R = (r^1, 1, r^m) \in U_Q\) be the saddle of \(f_t\) associated with 1 and the itinerary \((m, k)\) given by (A) in proposition 4.9. Note that \(W^u(R, f_t) = W^u(R, f_t)\).

By equation (6.3) in proposition 4.9 we have that
\[
W^s(R, f_t) \cap W^u(Q, f_t) \neq \emptyset \quad \text{and} \quad W^u(R, f_t) \cap W^s(P, f_t) \neq \emptyset.
\]
From the existence of an adapted homoclinic intersection and item (E)(1) in proposition 4.9:

- \(H = (h', 1, 0)\) is a transverse homoclinic point of \(P\),
- \([(h', 1)] \times [-1, 1)^m \subset W^u(P, f_t) \cap U_Q\) and
- \([-1, 1]^m \times (1, r^m) \subset W^s(R, f_t)\).

This implies that \(W^u(P, f_t) \cap W^s(R, f_t) \neq \emptyset\). Thus, by the second part of (6.3), the saddles \(P\) and \(R\) are homoclinically related for \(f_t\).

To obtain a cycle associated with \(R\) and \(Q\) note that the choice of \(t\) implies that
\[
\theta_{t,1} \circ \phi^k_{\beta} \circ \theta_2(1) = -\phi^k_{\beta}(-1) + t = 0.
\]
Since \(R = (r^1, 1, r^m)\), condition (D) in proposition 4.9 implies that \(W^u(R, f_t) \cap W^s(Q, f_t) \neq \emptyset\). Thus by the first part of (6.3) the diffeomorphism \(f_t\) has a cycle associated with \(R\) and \(Q\).

We claim that this cycle is non-twisted. If \(\theta_2\) reverses the orientation then the central multiplier of \(R\) is negative and the cycle is non-twisted. Otherwise, we have a cycle whose central ‘unfolding map’ is obtained considering the composition \(\theta_{t,1} \circ \phi_{\beta} \circ \theta_2\). This map preserves the central orientation: just note that \(\theta_{t,1}\) and \(\theta_2\) both reverse the orientation and \(\phi_{\beta}\) preserves this orientation (recall that \(\lambda > 0\)). This completes the proof of the lemma. \(\square\)
7. Proof of theorems 1 and 2

7.1. Proof of theorem 2.

Note that (B) and (C) in theorem 2 are immediate consequences of (A) and (B) in theorem 2.1, respectively.

To prove item (A) in theorem 2 we assume that, for instance, the saddle \( P \) has non-real central multipliers. By theorem 2.2 (see also remark 2.3) there is \( g \) close to \( f \) having saddles \( P'_g \) and \( Q'_g \) such that

- there is a cycle with real central multipliers associated with \( P'_g \) and \( Q'_g \),
- \( P'_g \) and \( Q'_g \) are homoclinically related to \( P_g \) and \( Q_g \),
- the homoclinic class of \( P'_g \) is non-trivial (note that we may have \( Q'_g = Q_g \) and a trivial homoclinic class \( H(Q_g,g) \)).

By lemma 2.4 it is enough to prove that this new cycle can be stabilized.

If the cycle associated with \( P'_g \) and \( Q'_g \) is non-twisted the stabilization follows from (A) in theorem 2.1. Otherwise, if the cycle is twisted, by lemma 5.9 there is a diffeomorphism \( h \) close to \( g \) having a saddle \( \bar{P}_h \) such that

- \( \bar{P}_h \) is homoclinically related to \( P'_h \) and has the bi-accumulation property,
- there is a cycle associated with \( Q'_h \) and \( \bar{P}_h \). Note that this cycle has real central multipliers.

As above, it is enough to prove that this cycle can be stabilized. The stabilization of this cycle follows from theorem 2.1. This ends the proof of the theorem.

7.2. Proof of theorem 1

By theorem 2.2 and lemma 2.4 we can assume that the cycle associated with the saddles \( P \) and \( Q \) has real central multipliers and that, for instance, the homoclinic class of \( P \) is non-trivial. If the cycle is non-twisted the result follows from (A) in theorem 2.1.

Otherwise, if the cycle is twisted, arguing as in the proof of theorem 2, there is a diffeomorphism \( g \) close to \( f \) having a cycle associated with \( Q_g \) and to a saddle \( \bar{P}_g \) that is homoclinically related to \( P_g \) and satisfies the s-bi-accumulation property. By (B) in theorem 2.1 this cycle can be stabilized. Since \( \bar{P}_g \) is homoclinically related to \( P_g \) the initial cycle also can be stabilized, ending the proof of the theorem.

7.3. Proof of corollary 1

This result follows immediately from theorem 1 considering the following perturbation of the initial cycle. First, we preserve one of the heteroclinic orbits in \( W^u(P, f) \cap W^s(Q, f) \). We can also assume that \( W^u(P, f) \) transversely intersects \( W^u(Q, f) \) and thus accumulates to \( W^s(Q, f) \). We can now use the second heteroclinic orbit in \( W^u(P, f) \cap W^s(Q, f) \) to obtain a transverse homoclinic point of \( P \). In this way we obtain a cycle satisfying theorem 1.

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