Linearisation of Simple Pendulum

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Abstract

The motion of a pendulum is described as Simple Harmonic Motion (SHM) in case the initial displacement given is small. If we relax this condition then we observe the deviation from the SHM. The equation of motion is non-linear and thus difficult to explain to under-graduate students. This manuscript tries to simplify things.

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One of the basic experiments which a physics student will try out is that of the pendulum. A pendulum consists of a massive bob suspended from a massless string, in actual terms, the string does have a mass but it is negligible as compared to the mass of the bob. Also, for making the assumption that the bob is a point mass, the length of the string is made far greater than the radius of the bob.

The general equation of motion (EOM) of pendulum was derived in our previous work \[1, 2\]. The equation \[1\] was:

\[
\cos \left( \frac{\Theta}{2} \right) \frac{d^2\Theta}{dt^2} - \frac{1}{2} \sin \left( \frac{\Theta}{2} \right) \left( \frac{d\Theta}{dt} \right)^2 = - \omega^2 \sin \Theta
\]  

In above eq.(1) $\Theta$ is the angular displacement and $\omega = \sqrt{\frac{g}{l}}$, where the terms have their usual meaning. This is a non-linear differential equation and is not solvable analytically. We adopted numerical methods to get the solutions of this equation under various conditions. If the initial displacement is small, \textit{i.e.} under small oscillation approximation we can write $\sin\theta \sim \theta$ and $\cos\theta \sim 1$. Using this, eq.(1) goes to

\[
\frac{d^2\Theta}{dt^2} - \frac{1}{4} \Theta \left( \frac{d\Theta}{dt} \right)^2 = - \omega^2 \Theta
\]  

the second term on the left side of above eqn. is a second order term in $\Theta$, so we can neglect this term being the higher order term (in $\Theta$). So the eqn becomes

\[
\frac{d^2\Theta}{dt^2} = - \omega^2 \Theta
\]  

this is the equation of SHM. So under small oscillation approximation the EOM of non-linear pendulum turns to a linear second order differential equation in time, more well known as the simple harmonic motion.

Let’s now discuss the solutions of both linear (eq.(3)) and non-linear (eq.(1)) equations. As we discussed earlier work [2] that the eq.(1) can’t be solved analytically, so we tried out numerical solution for the non-linear equation. But the solutions of linear equation(3) are well known and can be written as:

\[
\Theta(t) = A\cos(\omega t) + B\sin(\omega t)
\]
where A and B are the constants (we expects two constants to be there in the general solution as the equation which we are trying to solve is a second order equation in time), and $\omega_i = \sqrt{\frac{g}{x_t}}$, where the i subscript indicates that the EOM is SHM i.e. a linear second order equation. The solution (equation [4]) is a harmonic function of time. One can determine the unknown coefficients A and B by using the initial conditions.

![Figure 1: The plot of the displacement at various times (t = 5, 10, 15, 20 seconds) with the initial displacement, $\omega = 4$](image)

For a simple pendulum the total time taken for completing N oscillations can be written as:

$$ t = T_1 + T_2 + \ldots + T_N $$

(5)

where $T_1, T_2, \ldots T_N$ are time periods of first, second, \ldots and $N^{th}$ oscillation respectively. For a simple pendulum undergoing SHM all the time periods are same i.e.

$$ T_1 = T_2 = \ldots = T_N = T_i $$

(6)
leading to

\[ t_i = NT_i \tag{7} \]

where the subscript \( i \) indicates the SHM. However in the case of pendulum obeying the non-linear EOM (eq. (1)), as we have shown in previous work \[2\], the time period of oscillation won’t be the same as in the case of SHM (in fact the displacement given by non-linear EOM lags behind in phase to that of SHM). Let’s parameterise this lag in phase by saying that a constant phase difference (say \( \alpha \)) is introduced to the time period after each successive oscillation \( i.e. \)

\[ T = T_i - \alpha \tag{8} \]

where \( T \) is the time period of the pendulum using non-linear EOM. So after \( N \) oscillations the difference between the two times would be

\[ t = t_i - N\alpha \tag{9} \]

where \( t \) and \( t_i \) are the time taken to complete \( N \) oscillations by the non-linear and linear oscillators respectively, while \( \alpha \) is the small variation introduced with each successive oscillations. As time increases, the pendulum whose motion is described by eq(1) completes more oscillations as compared to the simple pendulum, for a given time. Thus, the only correction called for seems to be a correction factor in the angular frequency. As we have also shown earlier \[2\] that the difference between the two time periods increases as the initial amplitude of the oscillation increases. So we can say that the correction factor is dependent of the initial displacement. Also, for the initial displacement tending to be small, the correction factor should approach zero and hence the non-linear oscillator should approach the SHM. So effectively we can say that the motion of non-linear oscillator can still the thought of simple harmonic, with the difference that the frequency is now dependent on the initial amplitude. So we can write solution of actual pendulum of type :

\[ \Theta(t) = a \cos(\omega_i t + f(a)t) \tag{10} \]

where the constant \( A \) has been replaced with initial amplitude \( a \). The function \( f(a) \) is the function of initial displacement as argued above.
In figure (1) we have plotted the displacement at various times with the initial displacement. In figure (2) we have plotted the phase difference introduced between the two displacements which we get by solving the linear (SHM) eqn. (3) and the non-linear eq. (1) after the first oscillation w.r.t. the initial displacement. As stated, $\alpha$ is the constant delay introduced in successive oscillations, hence it is sufficient to plot between initial displacement and $\alpha$ after the first oscillation. As can be seen $f(\alpha)$ is essentially a function of the initial displacement. We have also tried to fit polynomial function of $f(a)$ to the numerical results which we get by solving eq. (1), the result is:

$$f(a) = a + bx + cx^2$$  \hspace{1cm} (11)

with

$$a = -0.0016674238 \ , \ b = 0.0026202282 \ , \ c = 0.024899586$$
Conclusion

In summary we have shown the difference in the results when we use the linear EOM (eq.(3)) and non-linear EOM (eq.(4)) for describing a pendulum. The time taken to complete an oscillation decreases with successive oscillations for a pendulum whose initial displacement is large. In short it seems that under this condition the oscillator will oscillate more rapidly. Thus, in case of a pendulum oscillating under the non-linear EOM condition, a modified value of $\omega$ has to be considered which is a function of the initial displacement. Apart from this modification the solution of non-linear EOM can also be taken to be harmonic.

References

[1] Peter V. O’Neil, “Advanced Engineering Mathematics”, 3ed edition, PWS publishing company, Boston, 1993.

[2] P.Arun & Naveen Gaur, physics/0106097.