Analytic progress on exact lattice chiral symmetry

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Theoretical issues of exact chiral symmetry on the lattice are discussed and related recent works are reviewed. For chiral theories, the construction with exact gauge invariance is reconsidered from the point of view of domain wall fermion. The issue in the construction of electroweak theory is also discussed. For vector-like theories, we discuss unitarity (positivity), Hamiltonian approach, and several generalizations of the Ginsparg-Wilson relation (algebraic and odd-dimensional).

1. Introduction

Three years has passed since the re-discovery of the Ginsparg-Wilson relation [1]. Now we know how to construct a gauge-covariant and local lattice Dirac operator which satisfies the Ginsparg-Wilson relation [2–4]. Lattice fermion theories with such a Dirac operator nicely reproduce the properties of massless Dirac fermion: the exact symmetry of fermion action [5], the chiral anomaly from functional measure [5–10], and the novel index theorem on the lattice [11,12]. Moreover, it has opened the possibility to construct chiral gauge theories on the lattice with exact gauge invariance [13–15].

Let me first recall the basic structure of the Ginsparg-Wilson fermions, which may be summarized as in the figure 1. For the exact chiral symmetry based on the Ginsparg-Wilson relation to make sense, the locality of the lattice Dirac operator is crucial. Otherwise the lattice chiral transformation, defined as follows

$$\delta \psi(x) = \gamma_5(1 - aD)\psi(x), \quad \delta \bar{\psi}(x) = \bar{\psi}(x)\gamma_5,$$

(1)

cannot be regarded to be a local transformation. The locality has been proved rigorously for Neuberger’s (overlap) Dirac operator under the so-called admissibility condition [14,15] on the plaquette variable, $U(p)$:

$$\| 1 - U(p) \| \leq \epsilon, \quad \epsilon < \frac{1}{30} \quad (m_0 = 1).$$

(2)

The admissibility condition has important dual roles. On one side, it ensures the locality and the smooth dependence of lattice Dirac operators with respect to gauge fields. On the other hand, it gives rise to the topological structure in the space of lattice gauge fields. Both are crucial for the index theorem to hold on the lattice at a finite lattice spacing.

![Figure 1. Structure of Ginsparg-Wilson fermions](image-url)
tion, and the proof of the global integrability of the functional measure of Weyl fermions. These issues, in particular for non-abelian cases, turn out to be rather tough to address and one need to struggle with the admissibility condition. In lattice QCD, on the other hand, the gauge field action is the type of the Wilson action
\[ S_G = \beta \sum_p \frac{1}{3} \text{Re} \text{Tr} (1 - U(p)) \] (3)
and the admissibility condition is not imposed. Then it is necessary to understand how far one can maintain chiral property, locality and also universality. This is an urgent issue for numerical applications. It will be covered in the talk (article) by Hernández.

In this talk (article), I would rather discuss the issues in theoretical interests (so far). First of all, I would like to discuss more on lattice chiral gauge theories. It is highly desirable to extend Lüscher’s construction of abelian chiral gauge theories with exact gauge invariance to non-abelian cases. Because of the reason I mentioned above, however, little progress is obtained so far. Another interesting question is to seek a practical formulation of chiral gauge theories which can be usable for non-perturbative studies through numerical methods. Although chiral gauge theories are a difficult case for numerical simulations, such a formulation can provide tools to get insights into the dynamics of chiral gauge theories. I will discuss the use of domain wall fermion in this respect and will argue that it provides a hint for a practical implementation. Also I will discuss the issue in the construction of electroweak SU(2) × U(1) gauge theory on the lattice.

Secondly, I would like to discuss more on the basic properties of the Ginsparg-Wilson Dirac fermions. Unitarity is one of the basic properties which is not fully examined yet and I will review recent results on this question. There are several proposals for new types of lattice Dirac operators. I will discuss an algebraic generalization of the Ginsparg-Wilson relation and its consequences. Finally, I will review recent works on the Ginsparg-Wilson relation in odd dimensions.

For notational simplicity the lattice spacing \( a \) is set to unity in the following.

2. More on chiral gauge theories

Once the lattice Dirac operator satisfying the Ginsparg-Wilson relation is obtained, Weyl fermion and its functional measure can be defined through the chiral projectors \( \gamma_5 = \gamma_5(1 - D) \), \( \gamma_5 \) by introducing orthogonal chiral bases \( \{ v_i \} \) and \( \{ \tilde{v}_k \} \) as \( \gamma_5 v_i(x) = +v_i(x), \tilde{v}_k(x)\gamma_5 = -\tilde{v}_k(x) \). Then the partition function (chiral determinant) of the Weyl fermion can be evaluated as

\[ Z_W = \det (\tilde{v}_k, Dv_j) = \det (\tilde{v}_k, v_j). \] (4)

When we adopt Neuberger’s (overlap) Dirac operator,

\[ D = \frac{1}{2} \left( 1 + \gamma_5 \frac{H}{\sqrt{H^2}} \right) \] (5)

where \( H = \gamma_5(D_w - m_0)(0 < m_0 < 2) \), the above partition function reproduces the vacuum overlap formula of chiral determinant.

However, there remains a certain ambiguity in the definition of the functional measure and the partition function. In fact, when we choose a different chiral basis, which should be related to the original one by a unitary transformation, \( v_i(x) \rightarrow \tilde{v}_i(x) = \sum_j v_j(x)Q_{ij} \), the functional measure is changed by the phase factor \( \det Q \).

Lüscher proposed a method to fix the phase of the functional measure by imposing the locality of the field equation, the gauge-invariance, and the smooth dependence with respect to gauge field. He gave a constructive proof that such a measure indeed exists in anomaly-free abelian chiral gauge theories.

2.1. Use of domain wall fermion

Since the vacuum overlap formula was originally derived from domain wall fermion, we may reconsider the above problem from the point of view of domain wall fermion. For simplicity we adopt the vector-like domain wall fermion, which is defined by the five-dimensional Wilson-Dirac fermion with a negative mass \( 0 < m_0 < 2 \) in a finite extent of the fifth-dimension, \( x_5 = a5t \).
\( t \in [-N + 1, N] \):

\[
\delta \sum_{t=-N+1}^{N} \sum_{x} \psi(x, t) (D_{5w} - m_0) \psi(x, t).
\]

(a_5 is the lattice spacing of the fifth dimension.)

We first note that there is another possibility to couple gauge field to the chiral zero modes of the vector-like domain wall fermion: one may introduce a five-dimensional gauge field which is varying in the fifth direction so that the gauge field on the right wall, \( U_\mu(x) \), is interpolated smoothly back to a trivial field, \( U^0_\mu = 1 \), on the left wall, \( U_\mu(x, t) = (U_k(x, t), 1) : U^0_k = 1 \to U_k(x) \).

The dependence of the partition function of domain wall fermion on the five-dimensional gauge field can be examined by considering a smooth variation of the five-dimensional gauge field. We assume that the variation is supported in the finite interval which is fixed in size w.r.t. \( N \): \( \delta U_\mu(x, t) = \eta_\mu(x, t) U_\mu(x, t); t \in [-\Delta + 1, \Delta] \).

A crucial point here is the locality property of the current \( J^a_\mu(x, t) \) in the interpolation region \( t \in [-\Delta + 1, \Delta] \). It is possible to show rigorously that the current is a local functional of the five-dimensional gauge field, provided that the gauge field satisfies the five-dimensional analog of the admissibility condition:

\[
\| 1 - U_{5d}(p) \| \leq \epsilon, \quad \epsilon < \frac{1}{50} (m_0 = 1),
\]

where \( U_{5d}(p) \) is the five-dimensional plaquette variable. The immediate consequence of this fact is that in the limit \( N \to \infty \), the current \( J^a_\mu(x, t) \) in the interpolation region does not actually depend on the specific choice of the boundary condition for the domain wall fermion (Dirichlet boundary condition). We may replace the current by that of the five-dimensional Wilson fermion subject to the anti-periodic boundary condition (AP) in the fifth dimension which is enlarged twice as large.
as the original size (see the figure 4). Because of the periodicity, the five-dimensional gauge field of the above configuration can be regarded to define a closed path, \( c_1 + (-c_0) : U_k^0 \rightarrow U_k^1 \rightarrow U_k^0 \) (or \( c_2 + (-c_0) \)) in the space of four-dimensional gauge fields. By noting the fact that the interpolation path \( c_0 \) can be chosen arbitrarily, and the reflection property of the five-dimensional Wilson fermion under \( P : t \rightarrow -t + 1; U_k(x,t) \rightarrow U_k(x,-t+1)^{-1}; D_{5w} \rightarrow P \gamma_5 D_{5w}^\dagger \gamma_5 P \),

\[
\begin{align*}
&-N + 1 \\
&\downarrow \hspace{1cm} \downarrow \\
&0 \hspace{1cm} N \hspace{1cm} 2N \hspace{1cm} 3N \hspace{1cm} N \hspace{1cm} -N + 1 \\
&U_k^0(x) \hspace{1cm} c_2 \hspace{1cm} U_k^1(x) \hspace{1cm} \cdots \hspace{1cm} c_1 \hspace{1cm} U_k^0(x) \\
&c_1, c_2 \hspace{1cm} -c_0
\end{align*}
\]

Figure 4. 5D gauge field for anti-periodic b.c.

we can derive an identity for a subtracted partition function of domain wall fermion:

\[
\ln \tilde{Z}_{DW}^c - \ln \tilde{Z}_{DW} = iQ_{5w}^{c^2 - c_1}, \tag{10}
\]

\[
\tilde{Z}_{DW}^c \equiv \lim_{N \to \infty} \ln \det (D_{5w} - m_0)|_{AP} \tag{11}
\]

\( Q_{5w}^{c^2 - c_1} \) is the complex phase of the determinant of five-dimensional Wilson fermion subject to anti-periodic b.c. and coupled to the five-dimensional gauge field representing the loop \( c_2 - c_1 \),

\[
Q_{5w}^{c^2 - c_1} = \lim_{N \to \infty} \ln \det (D_{5w} - m_0)|_{AP}^{c^2 - c_1}. \tag{12}
\]

This term is known to reproduce the Chern-Simons term in the continuum limit \([24][25]\). Then we can conclude that the dependence on the five-dimensional interpolation is governed by the lattice counter part of the Chern-Simons term.

**Local cohomology problem in 5+1 dimensions**

\( Q_{5w}^{c^2 - c_1} \) can always be expressed by a local field, if the closed path \( c_2 - c_1 \) is contractible. In fact, by introducing a continuous parameter \( s \in [0,1] \) which parameterizes the deformation of the five-dimensional gauge field \( (U_k^c(x),1) \rightarrow U_{\mu}(x,t) \), one obtains

\[
Q_{5w}^{c^2 - c_1} = \sum_{x,t} \int_0^1 ds \{ \eta_\mu(x,t) \operatorname{Im} J^\mu_\mu(x,t) \}|_{U_{\mu}(x)}. \tag{11}
\]

The local field in the r.h.s. is not gauge-invariant in general, but there is a possibility to satisfy the gauge-invariance by adding a certain total divergence term, which does not affect \( Q_{5w}^{c^2 - c_1} \) itself:

\[
\delta_Q \{ \eta_\mu(x,t) \operatorname{Im} J^\mu_\mu(x,t) - \partial^\mu K_\mu(x,t) |_{U_{\mu},n_\mu} \} = 0. \tag{12}
\]

The above equation defines a local cohomology problem. It can be re-formulated with the topological field in 5+1 dimensions and can be solved non-perturbatively in abelian chiral gauge theories \([26][27]\) and in all orders of lattice perturbation theory in non-abelian chiral gauge theories \([28][29]\).

In these cases, one can show that there is the gauge-invariant and local density of \( Q_{5w}^{c^2 - c_1} \), if the anomaly-free condition is satisfied \([30]\).

**Reduction to four-dimensions and factorization**

If \( Q_{5w}^{c^2 - c_1} \) can be written in terms of a gauge-invariant local density for all possible closed path \( c_2 - c_1 \) in the space of gauge fields, then it is possible to write \( Q_{5w}^{c^2 - c_1} \) as

\[
Q_{5w}^{c^2 - c_1} = c_{5w}^{c_2} - c_{5w}^{c_1}, \tag{13}
\]

where \( c_{5w}^{c_1} \) and \( c_{5w}^{c_2} \) are gauge-invariant and local terms associated with the interpolation paths \( c_1 \) and \( c_2 \), respectively. Then from the identity Eq. (14) it follows that

\[
\tilde{Z}_{DW}^c \cdot e^{ic_{5w}^c} = \tilde{Z}_{DW}^{c_2} \cdot e^{ic_{5w}^{c_2}}, \tag{14}
\]

This result implies that the partition function of the domain wall fermion can be made independent of the interpolation path by adding the local counter term.

We can work out explicitly the subtracted partition function with the local counter term, \( Z_{DW}^c \cdot e^{ic_{5w}^c} \), using the master formula in which the partition functions of the five-dimensional Wilson-Dirac fermions are expressed in terms of the transfer matrices \([24][25]\). It turns out that the partition function factorizes into the product of two chiral determinants in the overlap formula:

\[
\tilde{Z}_{DW}^c \cdot e^{ic_{5w}^c} = \det (\bar{v}_k,D\psi_j) \cdot \det (\bar{v}_k,D\bar{\psi}_j)^*, \tag{15}
\]
where $D$ is Neuberger’s (overlap) Dirac operator with the hermitian operator $H$ defined through the transfer matrix $T = \exp(-a_5 H)$ of the five-dimensional Wilson-Dirac fermion, or more simply $H = \gamma_5 (D_\text{w} - m_0) / (1 + a_5 (D_\text{w} - m_0))$ [22]. \{$v_j^0\}$ and \{$v_j\}$ are the chiral bases with respect to $U_k^0$ on the left wall and $U_k$ on the right wall, respectively. Note, however, that in this case the phase of the basis \{$v_j\}$ is fixed as follows:

$$v_i(x) = \begin{cases} v_i^1(x) e^{i\phi(c_1)} e^{-iQ_{5w}(c_1)} & (i = 1) \\ v_i^1(x) & (i \neq 1) \end{cases},$$

$$e^{i\phi(c_1)} = \frac{\det(v_j^i, \Pi^\Delta_{s=-\Delta+1} T_s)}{\det(v_j^i, \Pi^\Delta_{s=-\Delta+1} T_s)}.$$

Then the chiral determinant $\det(v_k, Dv_j)$ is gauge-invariant by construction! We can check that the above choice of basis would define the functional measure of Weyl fermions with the desired properties in anomaly-free chiral gauge theories. (See also [33] for a recent proposal how to fix the complex phase in the vacuum overlap formula.)

In this manner, domain wall fermion can provide a gauge-invariant partition function of the Weyl fermions in four dimensions, through the integrability condition for domain wall fermion Eq. (10) and the cohomological problem in 5+1 dimensions. The sufficient condition is that there exists a gauge-invariant and local density of $Q_{5w}^{c_2-c_1}$ (lattice Chern-Simons term) for all possible closed path $c_2 - c_1$ in the space of gauge fields.

2.2. A hint for practical implementation

The fact that domain wall fermion (as a five-dimensional Wilson-Dirac fermion) can provide a concrete example of Lüscher’s construction suggests that the continuous interpolation in the space of (admissible) lattice gauge fields can be discretized, without losing the topological properties of the gauge anomaly of the Ginsparg-Wilson Weyl fermions [24,24]. In fact, it is possible to define the topological field on six-dimensional lattice which can capture the global aspects of the gauge anomaly.

To see this, let us consider a two-dimensional surface in the space of lattice gauge fields and introduce lattice coordinates $t$ and $s$ on the surface. For each lattice site $(s,t)$, we associate a four-dimensional lattice gauge field and a chiral basis with this gauge field: \{$v_j(x)^{(s,t)}\}; U_k(x, s, t)$. Then we can define a two-dimensional U(1) gauge field $(G_s, G_t)$ on the lattice as follows:

$$G_s(s, t) = \frac{\det(v_i^{(s+1,t)}, v_j^{(s,t)})}{\det(v_i^{(s+1,t)}, v_j^{(s,t)})}$$

and a similar definition for $G_t(s, t)$. The change of the basis by a unitary transformation at each site induces U(1) lattice gauge transformation.

Then we may consider the plaquette variable of the U(1) gauge field given by

$$U(s, t) = G_s(s, t) G_t(s + 1, t) G_s(s, t + 1)^{-1} G_t(s, t)^{-1}$$

and may define a topological term

$$\sum_{s,t} \frac{1}{2\pi} \ln U(s, t).$$

One can show that this topological term reduces in the continuum limit to Lüscher’s topological field in 4+2 dimensions [24]. Therefore, as long as the two-dimensional lattice is fine enough, the topological term can capture the global aspect of the gauge anomaly of the Weyl fermion on the lattice. We may also consider a Wilson loop of the U(1) gauge field given by $\prod_{\text{loop}} G_s(s, t)$. It turns out to be identical to the lattice Chern-Simons term $\exp(iG_{5w}^{\text{loop}})$ in a certain limit. Thus we observe the structure over four-, five-, and six-dimensional lattices similar to the descent relation.

The topological field on the six-dimensional lattice may be used to solve the cohomology problem and to construct the gauge-invariant and local density of the Chern-Simons term. Since the interpolation is discrete, we may perform explicitly the check of the global integrability condition.

2.3. Electroweak theory

It is highly desirable to extend Lüscher’s construction of abelian chiral gauge theories with exact gauge invariance to non-abelian cases. As a
first step towards this direction, one may consider to extend the gauge-invariant construction to electroweak SU(2)×U(1) gauge theory\textsuperscript{23}.  

Electroweak theory is the chiral gauge theory of left-handed leptons and quarks in SU(2) doublet and right-handed quarks in SU(2) singlet. Taking into account of the color degrees of freedom, there are four doublets in each generation. The space of the admissible SU(2) gauge fields is divided into the topological sectors\textsuperscript{13,22}, each one is the product of a U(1) topological sector and a SU(2) topological sector.

In the vacuum sector of the U(1) gauge fields, where any configuration can be deformed smoothly to the trivial one $U^{(1)}(x,\mu) = 1$, we can turn off the hypercharge gauge coupling. Then the theory can be regarded as vector-like due to the pseudo reality of SU(2). It is indeed possible to make the fermion measure defined globally for all topological sectors of SU(2) by the following choice of the basis for a pair of doublets (a,b)\textsuperscript{13}:

\[
\begin{align*}
    w_j^{(a)}(x) &= u_j(x), \\
    w_j^{(b)}(x) &= (\gamma_5 C^{-1} \otimes i\sigma_2) [u_j(x)]^*,
\end{align*}
\]

where $\gamma_5 u_j(x) = -u_j(x)$. It is nothing but the symplectic basis for real representations considered by Suzuki\textsuperscript{37}. This fact implies the cancelation of Witten’s SU(2) anomaly (cf. \textsuperscript{38,39}).

Given the basis for the SU(2) doublets defined globally, one may try to extend the fermion measure to incorporate the U(1) gauge field following the reconstruction theorem\textsuperscript{14}.

The issues in this approach are the local cohomology problem and the proof of the global integrability condition. Fortunately, the cohomology problem can be solved for the U(1) part by the trick to treat the SU(2) gauge field as a background\textsuperscript{10,13}. As to the global integrability condition, it is proved for “gauge loops” in the space of the U(1) gauge fields with arbitrary SU(2) gauge field in the background. For “non-gauge loops”, however, the proof is given so far only for the classical SU(2) instanton backgrounds\textsuperscript{35}. To proceed, it seems that the information on the topological structure of the admissible SU(2) gauge field is required.

3. More on Ginsparg-Wilson fermions

3.1. Unitarity and Hamiltonian approach

Unitarity is one of the remaining issues about the Ginsparg-Wilson fermions, which is not fully examined yet. Let us consider the unitarity in lattice QCD defined with Neuberger’s (overlap) Dirac operator. For free theory, Lüscher worked out the spectral representation of the fermion propagator:

\[
\begin{align*}
    \langle \psi(x) | \bar{\psi}(y) \rangle_{x_0>y_0} &= \int_{0}^{\infty} dE \int_{-\pi}^{\pi} \frac{d^3 p}{(2\pi)^3} \rho(E, p) \\
    & \times e^{-E(x_0-y_0)+ip(x-y)},
\end{align*}
\]

where $m_0 = 1$, $\rho^2 = \sum_{k=1}^{3} 4 \sin^2 \frac{m k}{2}$

\[
\rho(E, p) = (\gamma_0 \sin E - i\gamma_k \sin p_k)
\]

\[
\times \left\{ \delta(E-\omega_p) \theta(\cosh E - \frac{1}{2} \rho^2) \frac{cosh E - \frac{1}{2} \rho^2}{\sinh E} \right\}^{1/2}
\]

\[
+ \frac{1}{2} \theta(E-E_p) \left\{ \rho^2 (cosh E - cosh E_p) + (cosh E - \frac{1}{2} \rho^2)^2 \right\}^{1/2}
\]

\[
\cosh E_p = \frac{1}{\frac{1}{2} \rho^2} \left\{ 1 + \frac{1}{2} \sum_{k \neq l} \hat{p}_k \hat{p}_l ^* \right\}, \quad \sinh \omega_p = |\sin^2 p_k|.
\]

This result shows that the spectral function is non-negative

\[
dEd^3 p \zeta^\dagger \rho(E, p) \zeta \geq 0
\]

and unitarity is maintained for any value of the lattice spacing $a$\textsuperscript{11}.

With gauge interaction, it is possible to examine the reflection positivity\textsuperscript{12} using the connection to domain wall fermion. For this purpose, we recall the fact that the partition function of the lattice fermion defined with Neuberger’s (overlap) Dirac operator in a truncated approximation\textsuperscript{30} can be expressed by the partition function of domain wall fermion. Namely, we have

\[
\det D_N = \frac{\det (D_{5w} - m_0)}{\det (D_{5w} - m_0)_{AP}},
\]

where $D_N = (1 + \gamma_5 \tanh (a_5 NH))/2$. In the r.h.s. of Eq. (23), the five-dimensional Wilson fermion subject to the anti-periodic (AP)
boundary condition in the fifth-dimension is introduced \[23\]. Moreover, chiral invariant observables in the original fermion system can be expressed in terms of the boundary field variables of the domain wall fermion \[23\]:

\[
\langle \mathcal{O}[U, (1 - 2D_N)\psi, \bar{\psi}] \rangle^{(N)} = \langle \mathcal{O}[U, q, \bar{q}] \rangle^{(N)}_{\text{DW}}
\]

where \( q(x) = \psi_L(x, -N + 1) + \psi_R(x, N) \). This result follows from the identity,

\[
\langle [q(x)\bar{q}(y)] \rangle_F = D_N^{-1} - \delta(x, y).
\]

Since the r.h.s. of Eq. \[23\] may be expressed as

\[
\frac{\det(D_{5w} - m_0) \cdot \det(D_{5w} - m_0)|_{\text{AP}}}{\det \left\{ (D_{5w} - m_0)^\dagger (D_{5w} - m_0) \right\}_{\text{AP}}},
\]

the original fermion system is equivalent to the set of two five-dimensional Wilson-Dirac fermions (with Dirichlet b.c. and anti-periodic b.c. in the fifth dimension, respectively) and one complex four-component boson (with the anti-periodic b.c.).

\[
S_{\text{DW}} = \sum_{x,t} \bar{\psi}(D_{5w} - m_0)\psi(x, t)
+ \sum_{x,t} \bar{\psi}'(D_{5w} - m_0)\psi'(x, t)|_{\text{AP}}
+ \sum_{x,t} \Phi |D_{5w} - m_0|^2\Phi(x, t)|_{\text{AP}}. \tag{26}
\]

Then the question is reduced to the reflection positivity in the system given by the action Eq. \[26\] and for the observables written in terms of \( U_k \), \( q \) and \( \bar{q} \) in the gauge \( U_4 = 1 \). Under the reflection symmetry transformation with respect to \( x_4 = 1/2 : \theta(x, x_4) = (x, -x_4 + 1) \), \( (\bar{\psi} = \psi, \psi') \)

\[
\theta U(x, x + \hat{k}) = U(\theta x, \theta(x + \hat{k}))
\theta \Psi(x, t) = \bar{\Psi}(\theta x, t)\gamma_4, \quad \theta \bar{\Psi}(x, t) = \gamma_4 \Psi(\theta x, t)
\theta \Phi(x, t) = \Phi^\dagger(\theta x, t)\gamma_4\gamma_5\gamma_5 P (P : t \to -t + 1)
\theta (f_{a_1, \ldots, a_n}(U)\phi_{a_1} \cdots \phi_{a_n})
= f_{a_1, \ldots, a_n}(\theta U)\theta\phi_{a_1} \cdots \theta\phi_{a_n},
\]

\( S_{\text{DW}} \) can be written into the form

\[
S_{\text{DW}} = \theta B_+ + \sum_i (\theta A'_i) A'_i + B_+.
\]

where \( A'_i \) and \( B_+ \) are functions of the field variables in the region \( x_0 > 1/2 \). Then we can show the positivity

\[
\langle \theta \mathcal{O}[U, q, \bar{q}] \rangle_{+} \cdot \langle \mathcal{O}[U, q, \bar{q}] \rangle_{+}^{(N)} \geq 0. \tag{28}
\]

The limit \( N \to \infty \) is defined well as long as the admissibility condition is assumed \[15\]. Therefore the above positivity implies the positivity in lattice QCD with Neuberger’s (overlap) Dirac operator. The analysis and discussion in more detail will be given elsewhere \[46\].

The Hamiltonian approach is a possible direction to the formulation of lattice QCD with manifest chiral symmetry and unitarity. Creutz, Horvath and Neuberger has constructed such a Hamiltonian based on the overlap construction \[17\]. It is defined through the three-dimensional Wilson-Dirac operator \( X = D_{3w} - m_0(0 < m_0 < 2) \) as \( H_F = \sum_x \psi^\dagger(x)H_F\psi(x) \)

\[
H_F = \gamma_0 \left( 1 + X/\sqrt{X^\dagger X} \right) \equiv \gamma_0 D. \tag{29}
\]

Because of the property of \( D \) in three dimensions, \( D^\dagger = \gamma_5 D \gamma_5 = \gamma_0 D \gamma_0 \), there are two kinds of the Ginsparg-Wilson relations with \( \gamma_5 \) and \( \gamma_0 \), respectively. The chiral charge can be introduced as \( Q_5 = \sum_x \psi^\dagger(x)Q_5\psi(x) \)

\[
Q_5 = \gamma_5 \left( 1 - \frac{1}{2} \frac{D}{2} \right) \tag{30}
\]

and it commutes with the Hamiltonian \( H_F \). But it does not commute with the electric term in the Hamiltonian of the gauge field, reflecting chiral anomaly. \( H_F \) and \( Q_5 \) have a correlation

\[
Q_5^2 + \left( \frac{H_F}{2} \right)^2 = 1, \tag{31}
\]

which leads to an interesting interpretation of the chiral properties of the low-lying eigenmodes of \( H_F \). This approach should have physical applications in QCD and also in electroweak theory.
3.2. Algebraic generalization

It is an important and challenging issue to seek new lattice Dirac operators with better chiral and locality properties. Fujikawa and Ishibashi has examined this possibility through an algebraic generalization of the Ginsparg-Wilson relation\ref{48,49}. They consider the following relation for a lattice Dirac operator:

\[ \gamma_5 (\gamma_5 D) + (\gamma_5 D) \gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \tag{32} \]

for \( k = 0, 1, \cdots \). The case \( k = 0 \) corresponds to the usual Ginsparg-Wilson relation. This generalization is intended so that the index relation should be maintained:

\[ \text{Tr} \Gamma_5 = n_+ - n_-, \quad \Gamma_5 \equiv \gamma_5 - (\gamma_5 aD)^{2k+1}. \tag{33} \]

The explicit solution of the relation Eq. (32) has been constructed by noting the relation

\[ \gamma_5 H^{2k+1} + H^{2k+1} \gamma_5 = 2H^{2(2k+1)} \tag{34} \]

for \( H = \gamma_5 D \) and applying the overlap construction to \( H^{2k+1} \). The consistency of free theory, chiral anomaly and locality has been checked\ref{51}. The spectrum of the class of the lattice Dirac operators has been examined numerically by Chiu\ref{52,53}. Unfortunately, it turned out that the locality property becomes worse for larger values of \( k \). Therefore this approach cannot be used to improve the locality property, although the algebraic structure is interesting. It is desirable to have the systematic understanding about the conditions which determines the locality properties of a lattice Dirac operator.

3.3. Odd dimensions

Bietenholz and Nishimura has argued that the Ginsparg-Wilson relation is also useful as a condition for lattice Dirac operator of massless fermions in odd dimensions\ref{25}. They adopt the following relation for the three-dimensional Dirac operator:

\[ D + D^\dagger = D^\dagger D, \tag{35} \]

which solution may be written in general as \( D = (1 - V), \ V^\dagger V = 1 \). The fermion action defined with such a Dirac operator is then invariant under the following parity transformation: \( R : x \rightarrow -x, \ U(x, \mu) \rightarrow U(x, \mu)^P = U^\dagger(-x - \mu, \mu) \) \tag{36} and \( \psi(x) \rightarrow iRV\psi(x) ; \ \bar{\psi}(x) \rightarrow i\bar{\psi}(x)R \) \tag{37} An interesting observation made here is that by this discrete transformation, the fermion measure transforms non-trivially,

\[ d\psi d\bar{\psi} \rightarrow (\det V)^{-1} d\psi d\bar{\psi} \tag{38} \]

and the Jacobian can be regarded as the parity anomaly. The explicit solution of the relation Eq. (35) is again provided by the vacuum overlap formalism\ref{24,25}. It has been known that in odd dimensions, it is possible to choose a phase so that the overlap formula is given by a determinant of a gauge-covariant operator, \( D_{ov} \)\ref{23}. The complex phase of the determinant of \( D_{ov} \) is nothing but the complex phase of the determinant of the three-dimensional Wilson-Dirac operator and it reproduces the Chern-Simons term in the continuum limit\ref{24,25}.

Form the above point of view based on the Ginsparg-Wilson relation, this complex phase is just identified as the parity anomaly from the functional measure

\[ \text{Im} \ln \det V = \text{Im} \ln \det (D_{3w} - m_0) \tag{39} \]

and it provides a reasonable definition of the lattice Chern-Simons term\ref{25}. In five-dimensions, this picture fits well to the structure of the descent relation over four-, five-, and six-dimensional lattices discussed in section 2. The universality classes of the three-dimensional Dirac fermion\ref{27} has also been examined.

4. Conclusion

Core problems still remain. In chiral theories, the construction of non-abelian chiral gauge theories is still an open question. We need to know how to treat the space of the admissible lattice gauge fields. In vector-like theories, we should have better understanding of the behavior of the Ginsparg-Wilson fermions without the admissibility condition (cf.\ref{53}). In particular for Neuberger’s (overlap) Dirac fermion, the behavior of
low-lying eigenvalues of hermitian Wilson-Dirac operator should be understood (cf. [55–57].)

For Neuberger’s (overlap) Dirac fermion, the relation to domain wall fermion is quite useful in theoretical investigations. In numerical investigations, on the other hand, it seems relatively simpler to treat the four-dimensional Dirac operator directly. Still the five-dimensional point of view would be useful to get insights into the problem we face (cf. [58,59]).

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