Controllability and Observability Imply
Exponential Decay of Sensitivity in
Dynamic Optimization

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Abstract: We study a property of dynamic optimization (DO) problems that is known as exponential decay of sensitivity (EDS). This property indicates that the sensitivity of the solution at stage \(i\) against a data perturbation at stage \(j\) decays exponentially with \(|i - j|\). Building upon our previous results, we show that EDS holds under uniform boundedness of the Lagrangian Hessian, a uniform second order sufficiency condition (uSOSC), and a uniform linear independence constraint qualification (uLICQ). Furthermore, we prove that uSOSC and uLICQ can be obtained under uniform controllability and observability. Hence, we have that uniform controllability and observability imply EDS. These results provide insights into how perturbations propagate along the horizon and enable the development of approximation and solution schemes. We illustrate the developments with numerical examples.

Keywords: sensitivity analysis, nonlinear, model predictive control, moving horizon estimation

1. INTRODUCTION

This work studies the discrete-time, dynamic optimization (DO) formulation:

\[
\min_{x_{0:N-1}} \sum_{i=0}^{N-1} \ell_i(x_i, u_i; d_i) + \ell_N(x_N; d_N)
\]

s.t. \(T x_0 = d_0 \quad \|\lambda_1\|, \quad i \in [0,N-1] \quad \|\lambda_i\|.

Here, \(N \in \mathbb{N}_{>0}\) is the horizon length; for each stage \(i\), \(x_i \in \mathbb{R}^{n_u}\) are the states, \(u_i \in \mathbb{R}^{n_u}\) are the controls, \(d_i \in \mathbb{R}^{n_d}\) are the data (parameters), \(\lambda_i \in \mathbb{R}^{n_{\lambda}}\) are the dual variables, \(\ell_i : \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}\) are the stage cost functions, \(f_i : \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}\) are the dynamic mapping functions, \(\ell_N : \mathbb{R}^{n_u} \times \mathbb{R}^{n_d} \to \mathbb{R}\) is the final cost function, and the symbol \(|\cdot|\) is used to denote the associated primal-dual solution trajectory \(w = [w_0; \ldots; w_N]\).

The DO problem (1) is a parametric nonlinear program that we denote as \(P_{0:N}(d_{-1:N})\). We assume that all functions \(\ell_i, f_i, \ell_N\) are twice continuously differentiable and potentially nonconvex. Typical MPC problems are formulated with \(T = I\) and typical MHE problems are formulated with an empty matrix \(T \in \mathbb{R}^{0 \times n_u}\) (i.e., the initial constraint is not enforced). State-output mappings encountered in MHE problems are assumed to be embedded within the stage cost functions.

In this paper we study a property of DO problems that is known as exponential decay of sensitivity (EDS) (Na and Anitescu, 2020a; Shin et al., 2021). The property indicates that the sensitivity of the solution at stage \(i\) against a data perturbation at stage \(j\) decays exponentially with \(|i - j|\). This property helps understand how different data perturbations (e.g., disturbances or changes in set-points, initial conditions, and terminal penalties) propagate along the horizon. Moreover, EDS has been shown to be essential in constructing efficient discretization schemes for continuous-time DO formulations (Shin and Zavala, 2020; Grüne et al., 2020b) and in establishing convergence of algorithms (Na et al., 2020; Na and Anitescu, 2020b).

Building upon our previous results (Shin et al., 2021) we show that, under uniform boundedness of the Lagrangian Hessian (uBLH), a uniform second order sufficiency condition (uSOSC), and a uniform linear independence constraint qualification (uLICQ), the primal-dual solution of the DO problem at a given stage decays exponentially with the distance to the stage at which a data perturbation is introduced. In particular, given base data \(d_{-1:N}\) and associated primal-dual solution trajectory \(w_{-1:N}^*\) (at which uBLH, uSOSC, and uLICQ are satisfied), there exist uniform constants \(T > 0\) and \(\rho \in (0,1)\) and a neighborhood \(D_{-1:N}^*\) of \(d_{-1:N}\) such that the following holds for any \(d_{-1:N} - d_{-1:N}^* \in D_{-1:N}^*\):

\[
\|w_i^0(d_{-1:N}) - w_i^0(d_{-1:N}^*)\| \leq \sum_{j=-1}^{N} \Upsilon \rho^{i-j}\|d_j - d_j^*\|,
\]

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where \( w_i(\cdot) \) is the primal-dual solution mapping at stage \( i \). That is, the sensitivity \( \Phi^{(i−j)} \) of the solution at stage \( j \) against a data perturbation at stage \( j \) decays exponentially with respect to the distance \( |i−j| \). Here, it is important that \((\Phi, \rho)\) are uniform constants (independent of horizon length \( N \)); this allows us to maintain \((\Phi, \rho)\) unchanged even if the horizon length becomes indefinitely long (e.g., when approaching an infinite horizon). Our key finding is that \( uBLH, uLICQ, \) and \( uSOSC \) can be obtained under uniform controllability/observability and under uniformly bounded system matrices (standard assumption). This result thus allows to establish EDS directly from fundamental system-theoretic properties.

In summary, our main contribution is showing that controllability and observability provide sufficient conditions for EDS. This result sheds light on how system-theoretic properties influence the propagation of perturbations along the solution trajectory. EDS for continuous-time, linear-quadratic MPC has been established under stabilizability and detectability in Grüne et al. (2019, 2020b,a). EDS has been established for discrete-time, nonlinear MPC and MHE problems.

**Basic Notation:** The set of real numbers and the set of time, nonlinear MPC and MHE problems.

**Definition 1.** (Uniform Bounds). A set \( \{A_i\}_{i \in \mathbb{A}} \) of uniformly bounded quantities. \((uBLH, uSOSC, \) and \( uLICQ)\). We begin by defining the first formally define sufficient conditions for EDS to hold \((uBLH, uLICQ, \) and \( uSOSC)\). This general setting allows us to establish EDS for discrete-time, nonlinear MPC and MHE problems.

**Definition 2.** (uBLH). Given \( d^*_{−1,N} \) and the solution \( w^*_{−1,N} \) of \( P_{0,N}(d^*_{−1,N}) \), \( L\) holds if:

\[
\|\nabla w^*_{−1,N}−\nabla w^*_{−1,N}\| \leq L \tag{3}
\]

with uniform constant \( L < \infty \).

The primal Hessian \( H_{0,N}(\Phi^*_{0,N}) \) and the constraint Jacobian \( J_{0,N} \) are:

\[
H_{0,N} := \nabla^2 z_{0,N} w_{0,N}(w^*_{−1,N}; d^*_{−1,N}) \tag{4a}
\]

\[
J_{0,N} := \nabla z_{0,N} c−1,N−1(z_{0,N}; d^*_{−1,N}) \tag{4b}
\]

where \( c−1,N−1(\cdot) \) is the constraint function for \( P_0(\gamma) \); that is,

\[
c_{−1,N−1}(z_0; d_{−1,N}) := \begin{bmatrix}
T x_0 − d_{−1,N} \\
x_1 − f_1(z_0; d_0) \\
\vdots \\
x_N − f_N−1(\lambda_{N−1}; d_{N−1})
\end{bmatrix}.
\]

**Definition 3.** (uSOSC). Given \( d^*_{−1,N} \) and the solution \( w^*_{−1,N} \) of \( P_{0,N}(d^*_{−1,N}) \), \( \gamma\) has uniform boundedness hold if:

\[
ReH_{0,N}(J_{0,N}) \geq \gamma I, \tag{5}
\]

with uniform constant \( \gamma > 0 \).

Here, \( ReH_{0,N}(J_{0,N}) := Z^T H_{0,N} Z \) is the reduced Hessian and \( Z \) is a null-space matrix of \( J_{0,N} \).

**Definition 4.** (uLICQ). Given \( d^*_{−1,N} \) and the primal-dual solution \( w^*_{−1,N} \) of \( P_{0,N}(d^*_{−1,N}) \), \( \beta\) has uniform boundedness hold if:

\[
J_{0,N} J_{0,N} \geq \beta I. \tag{6}
\]

Note that \( uSOSC \) assumes that the smallest eigenvalue of the reduced Hessian is uniformly bounded below by \( \gamma \), while \( uLICQ \) assumes that the smallest non-trivial singular value of the Jacobian is uniformly bounded below by \( \beta^{1/2} \). Thus, these are strengthened versions of SOSC and LICQ. We require \( uSOSC \) and \( uLICQ \) because, under SOSC and LICQ, the smallest eigenvalue of reduced Hessian or the smallest non-trivial singular value of the Jacobian may become arbitrarily close to 0 as the horizon length \( N \) is extended (e.g., see Shin et al. (2021, Example 4.18)). Under \( uSOSC \) and \( uLICQ \), on the other hand, the lower bounds are independent of \( N \).

**Assumption 5.** Given twice continuously differentiable functions \( \{u_i(\cdot)\}_{i} \), \( \{f_i(\cdot)\}_{i} \), and base data \( d^*_{−1,N} \), there exists a primal-dual solution \( w^*_{−1,N} \) of \( P_{0,N}(d^*_{−1,N}) \) at which \( L\) holds, \( \gamma\) has uniform boundedness, and \( \beta\) is satisfied.

The following lemma is a well-known characterization of solution mappings of parametric nonlinear programs (NLPs) (Robinson, 1980; Dontchev and Rockafellar, 2009).

**Lemma 6.** Under Assumption 5, there exist neighborhoods \( D^*_{−1,N} \) of \( d^*_{−1,N} \) and \( W^*_{−1,N} \) of \( w^*_{−1,N} \) and continuous
w^{\dagger}_{d-1:N} : D^*_{-1:N} \rightarrow \mathbb{R}^{*}_{-1:N} \text{ such that for any } d-1:N \in D^*_{-1:N}, w^{\dagger}_{d-1:N}(d-1:N) \text{ is a local solution of } P_{d/N}(d-1:N).

**Proof.** From Shin et al. (2021, Lemma 3.3).

We can thus see that there exists a well-defined solution mapping \( w^{\dagger}_{d-1:N}(\cdot) \) around the neighborhood of \( d^*_{-1:N} \). We now study stage-wise solution stability by characterizing the dependence of \( w^{\dagger}_{d}(\cdot) \) on the data \( d_{-1:N} \).

**Theorem 7.** Under Assumption 5, there exist uniform constants \( \Upsilon > 0 \) and \( \rho \in (0,1) \) (functions of \( L, \gamma, \beta \)) and neighborhoods \( D^*_{-1:N} \) of \( d^*_{-1:N} \) and \( \mathbb{R}^{*}_{-1:N} \) of \( w^{\dagger}_{d-1:N} \) such that (2) holds for any \( d_{-1:N}, d^*_{-1:N} \in D^*_{-1:N} \) and \( i \in [i_{-1:N}] \).

**Proof.** We observe that \( P_{d/N}(\cdot) \) is graph-structured (induced by \( G = (\mathcal{V}_N, \mathcal{E}_N) \), where \( \mathcal{V}_N = \{ -1, 0, \ldots, N \} \) and \( \mathcal{E}_N = \{ -1, 0 \}, \{ 0, 1 \}, \ldots, \{ N-1, N \} \), and the maximum graph degree \( D = 2 \). From uBLH, uLICQ, and uSOSC, one can see that assumptions in Shin et al. (2021, Theorem 4.9) are satisfied. This implies that the singular values of \( \nabla_{w_{d-1:N}}^2 \mathcal{L}_0(N)(w^{\dagger}_{d-1:N}; d^*_{-1:N}) \) are uniformly upper and lower bounded and those of \( \nabla_{w_{d-1:N}}^2 \mathcal{L}_0(N)(w^{\dagger}_{d-1:N}; d^*_{-1:N}) \) are uniformly upper bounded (uniform constants given by functions of \( L, \gamma, \beta \); see Shin et al. (2021, Equation (4.15))). We then apply Shin et al. (2021, Theorem 3.5) to obtain \( \Upsilon > 0 \) and \( \rho \in (0,1) \) as functions of the upper and lower bounds of the singular values (see Shin et al. (2021, Equation (3.17))). This allows expressing \( \Upsilon, \rho \) as functions of \( L, \gamma, \beta \).

Theorem 7 establishes EDS under the regularity conditions of Assumption 5. It is important that \( \Upsilon, \rho \) can be determined solely in terms of \( L, \gamma, \beta \) (and do not depend on the horizon length \( N \)). Practical DO problems typically have additional equality/inequality constraints that are not considered in (1). Thus, Theorem 7 may not be directly applicable to those problems. However, the results in Shin et al. (2021) are applicable to such problems as long as the DO problem is a graph-structured NLP. Specifically, under uniformly strong SOSC and uLICQ, we can establish EDS using Shin et al. (2021, Theorem 3.5, 4.9). The graph structure breaks when there exist globally coupled variables; typical MPC and MHE problems do not have such variables, but parameter estimation problems may have such variables. Specifically, in the presence of globally coupled variables, the graph distance between any pair of stages is not greater than two.

### 2.2 Regularity from System-Theoretic Properties

Although uSOSC and uLICQ are standard notions of NLP solution regularity, they are not intuitive notions from a system-theoretic perspective. However, we now show that uSOSC and uLICQ can be obtained from uniform controllability and observability. We begin by defining:

\[
Q_i := \nabla_{x_i}^2 \mathcal{L}_i(z^*_i, \lambda^*_{i-1}; d^*_i), \quad R_i := \nabla^2_{u_i u_i} \mathcal{L}_i(z^*_i, \lambda^*_{i-1}; d^*_i), \quad S_i := \nabla_{x_i u_i}^2 \mathcal{L}_i(z^*_i, \lambda^*_{i-1}; d^*_i), \quad E_i := \nabla_{z_i d_i}^2 \mathcal{L}_i(z^*_i, \lambda^*_{i-1}; d^*_i), \quad F_i := \nabla_{u_i d_i}^2 \mathcal{L}_i(z^*_i, \lambda^*_{i-1}; d^*_i), \quad A_i := \nabla_{x_i f_i}(z^*_i; d^*_i), \quad B_i := \nabla_{u_i f_i}(z^*_i; d^*_i), \quad C_i := \nabla_{d_i f_i}(z^*_i; d^*_i).
\]

**Definition 8.** \( \{(A_i)_{1=N}, (B_i)_{1=N} \} \) is \( (N_c, \beta_c) \)-uniformly controllable with \( N_c \in \mathbb{Z}_{\geq 0} \) and \( \beta_c > 0 \) (independent of \( N \)) if, for any \( i, j \in [0, N-1] \) with \( |i-j| \geq N_c \), \( C_{i,j} C_{i,j}^T \geq \beta_c I \) holds,

\[
C_{i,j} := [A_{i+1} \cdots A_j B_j] = \begin{bmatrix} Q_i A_{i-1}^{\dagger} & \ldots & Q_i A_{i-1}^{\dagger} \\ Q_i A_{i-1}^{\dagger} & \ldots & Q_i A_{i-1}^{\dagger} \end{bmatrix}.
\]

**Proof.** The Jacobian \( J_{0:N} \) has the following form:

\[
J_{0:N} = \begin{bmatrix} T & -A_0 - B_0 & I \\ -A_{N-2} - B_{N-2} & I \\ -A_{N-1} - B_{N-1} & I \\ \vdots & \vdots & \vdots \\ -B_{N-1} & I \\ -A_{j-1} - B_{j-1} & I \\ -A_j - B_j \end{bmatrix}.
\]

By inspecting the block structure of \( J_{0:N} \) and Shin et al. (2021, Lemma 4.15), one can see that it suffices to show that the smallest non-trivial singular value of

\[
S = \begin{bmatrix} A - B_I & I \\ \vdots & \vdots \\ -A_{j-1} - B_{j-1} & I \\ -A_j - B_j \end{bmatrix}.
\]
is $\beta^{1/2}$-uniformly bounded below for $S = T$ or $I$ and for any $i, j \in \mathbb{I}_{[0,N-1]}$ with $N_c \leq |i - j| \leq 2N_c$, where $0 < \beta \leq 1$ is a function of $K, \delta, N_c, \beta_c$. This follows from the observation that one can always partition $I_{[0,N-1]}$ into a family of blocks with size between $N_c$ and $2N_c$. For now, we assume $S = I$. By applying a set of suitable block row and column operations (in particular, first apply block row operations to eliminate $A_1, \ldots, A_j$, and then apply block column operations to eliminate $-B_1, \ldots, -A_{j+1}B_{j+2}$) and permutations, one can obtain the following:

$$
\begin{bmatrix}
I & & & -A_{j+1}B_i & \cdots & -A_jB_{j-1} & -B_j
\end{bmatrix}.
$$

(8)

The lower-right blocks constitute the controllability matrix $C_{ij}$; from uniform controllability, the smallest non-trivial singular value of the matrix in (8) is uniformly lower bounded by $\min(1, \beta^{\delta/2}c)$. Here, we have applied block-row and block-column operations as the ones that appear in Lemma 11 (each multiplied uniformly bounded above due to $K$-uniform boundedness of $\{A_i\}_{i=0}^{N-1}$ and $\{B_i\}_{i=0}^{N-1}$). Also, we have applied such operations only uniformly bounded many times (the number of operations is independent of $N$ since the number of blocks in the matrix in (7) is bounded by $4(2N_c+1)(N_c+1)$, which is uniformly bounded above). We thus have that the smallest non-trivial singular value of the block structure of $A_i$ (uniform boundedness of $\{A_i\}_{i=0}^{N-1}$ and $\{B_i\}_{i=0}^{N-1}$) is uniformly lower bounded by $(K_N)^{\delta/2}c$ and $\delta > 0$ is given by a function of $K, N_c, \beta, \beta_0$. Note that, as we are considering $S = \mathbb{I}$, we can observe that the smallest non-trivial singular value of the matrix in (7) with $S = T$ is lower bounded by that with $S = [T; T]$ (here, $T^\top$ is a null space matrix of $T$); and again, it is lower bounded by $\delta^{1/2}$ times that with $S = I$. We thus have that the smallest non-trivial singular value of the matrix in (7) with $S = T$ is uniformly lower bounded by $\beta_0^{1/2}\beta^{1/2}c$. Therefore, the smallest non-trivial singular values of the matrices in (7) with $S = I$ or $T$ are $\beta^{1/2}$-uniformly lower bounded for any $i, j \in \mathbb{I}_{[0,N-1]}$ with $N_c \leq |i - j| \leq 2N_c$, where $\beta := \min(\beta_0, \delta, \beta_0, 1)$. Thus, by Shin et al. (2021, Lemma 4.15) we have (6) if $T \in \mathbb{R}^{0 \times n_1}$, the assumption $TT^\top \geq \delta I$ for uniformly lower bounded $\delta > 0$ holds for an arbitrary $\delta > 0$ due to the convention introduced in the Notation in Section 1.

We now show that uniform observability implies uSOSC.

Lemma 13. $K$-uniform boundedness of $\{A_i\}_{i=0}^{N-1}$, $\{B_i\}_{i=0}^{N-1}$, and $\{Q_i\}_{i=0}^{N-1}$, $Q_i \geq 0$, $S_i = 0$, $R_i \geq \gamma I$ ($r > 0$ is independent of $N$), and $(N_o, \gamma_o)$-uniform observability of $\{A_i\}_{i=0}^{N-1}$, $\{Q_i\}_{i=0}^{N-1}$ implies (5), where $\gamma > 0$ is a function of $K, N_o, \gamma_o, r$.

**Proof.** The primal Hessian $H_{0:N}$ has the following form:

$$
H_{0:N} = \begin{bmatrix}
[Q_0 & R_0 & \cdots & Q_{N-1} & R_{N-1} & Q_N]
\end{bmatrix}.
$$

By inspecting the block structure of $H_{0:N}$ and $J_{0:N}$ and Shin et al. (2021, Lemma 4.14), one can observe that it suffices to show that: first,
Finally, we show that uniform boundedness of system matrices implies uBLH.

**Lemma 14.** If \( \{Q_i\}_{i=0}^{N-1}, \{R_i\}_{i=0}^{N-1}, \{S_{i,j}\}_{i,j=0}^{N-1}, \{A_i\}_{i=0}^{N-1}, \{B_{i,j}\}_{i,j=0}^{N-1}, \{E_{i,j}\}_{i,j=0}^{N-1}, \{G_{i,j}\}_{i,j=0}^{N-1}, \) and \( T \) are \( K \)-uniformly bounded above, (3) holds, where \( L < \infty \) is a function of \( K \).

**Proof.** Uniform boundedness of the system matrices implies that for any \( i,j \in [0,N-1], \| \nabla_{u,i} L_0 N (w^*, d^*) \| \leq 1 \) for \( i \neq j \) (there is only one identity block). Thus, \( \| \nabla_{u,i} L_0 N (w^*, d^*) \| \leq \max (4K,1) \). By noting that the maximum graph degree \( D = 2 \) and applying Shin et al. (2021, Lemma 4.5), we have that \( \nabla_{w-1,i} L_0 N (w^*, d^*) \) is \( 4 \max (4K,1) \)-uniformly bounded above. We set \( L := 4 \max (4K,1) \).

We now state EDS in terms of uniformly bounded system matrices and uniform controllability/observability.

**Assumption 15.** Given twice continuously differentiable functions \( f_i(x) \) and \( g_i(x) \), and data \( d^* \), there exists a primal-dual solution \( w^*, d^* \) of \( P_{0,N}(d^*) \) at which the assumptions in Lemma 12, 13, 14 hold.

**Corollary 16.** Under Assumption 15, there exist uniform constants \( \gamma > 0 \) and \( \rho \in (0,1) \) (functions of \( K, \gamma, \beta, N, N_0, \gamma_0, \delta \)) and neighborhoods \( \mathcal{W}^* \) of \( d^* \) and \( \mathcal{W}^* \) of \( w^* \) such that (2) holds for any \( d_{-1,N} \) and \( w_{-1,N} \) in \( \mathcal{W}^* \) and \( i \in [0,N-1] \).

**Proof.** From Theorem 7 and Lemma 12, 13, 14.

### 2.3 Time-Invariant Setting

Assume now that the system is time-invariant and focus on a region around a steady-state. A corollary of Theorem 7 for such a setting is derived. We present this result since this setting has been of particular interest in the MPC literature. Consider a time-invariant system with a stage-cost function \( l(x) \), terminal regularization function \( l_T(x) \), and dynamic mapping \( f(x) \). The DO problem is given by (1) with \( f_i(x) = f(x) \) for \( i \in [0,N-1] \), \( l(x) = l_T(x) \) for \( i \in [0,N-1] \), \( L(x,u) = l(x,u) + l_T(x,u) \), and \( L(x) = l_T(x) \). The steady-state optimization problem is:

\[
\min_{x,u} l(x,u) \text{ s.t. } x = f(x,u) \quad \text{for} \quad \lambda.
\]

For given \( d^* \) and an associated primal-dual solution \( w^* := [x^*; u^*; \lambda^*] \) of (13), we define:

\[
Q := \nabla_x^2 L(w^*; d^*), \quad S := \nabla_u^2 L(w^*; d^*),
\]

\[
R := \nabla_{u,x} L(w^*; d^*), \quad A := \nabla_x f(z; d^*), \quad B := \nabla_u f(z; d^*),
\]

where \( L(w; d) := f(z; d) - \lambda x^2 + \lambda^* f(z; d) \), for the initial and terminal cost functions \( l_0(x) \) and \( l_T(x) \), we define:

\[
\lambda_0 := \nabla_x l_0(x; d^*), \quad Q_0 := \nabla_x^2 l_0(x; d^*),
\]

\[
\lambda_T := \nabla_x l_T(x; d^*), \quad Q_T := \nabla_x^2 l_T(x; d^*).
\]

The quantities defined above (\( Q, R, \) etc.) are independent of \( N \) since \( w^* \) can be determined independently of \( N \).

**Assumption 17.** Given twice continuously differentiable \( l(x), l_T(x), f(x), \) and data \( d^* \), there exists a steady-state solution \( w^* \), at which \( Q_T \geq 0, Q_0 > 0, \) \( S = 0, R > 0, (A,B) \) controllable, \( (A,Q) \) observable, \( TT^T > 0, \lambda_0 + x^* \in \text{Range}(T^T) \) and \( \lambda_T = \lambda^* \) hold.

**Corollary 18.** Under the time invariance setting and Assumption 17, there exist uniform constants \( \gamma > 0 \) and \( \rho \in (0,1) \) such that the following holds: for any \( N \in [0,\infty) \), there exist neighborhoods \( \mathcal{D}_N \) of \( d_N \) such that \( \mathcal{W}_N \) of \( w_N \) satisfies \( \lambda_0 + x^* \in \text{Range}(T^T) \) and \( (A,Q) \) observable, \( TT^T > 0, \lambda_0 + x^* \in \text{Range}(T^T) \) and \( \lambda_T = \lambda^* \) hold.

**Proof.** From the existence (follows from \( \ell_T, \ell_T \in \text{Range}(T^T) \)) and uniqueness (follows from \( TT^T > 0 \)) of the solution \( T^T \lambda_N^* + \lambda^* \) and \( \lambda^* \) and \( \lambda^* \) are independent of \( w_N \) for (13), \( w_{N} \) satisfies the first-order optimality conditions for \( P_{0,N}(d^*) \). Furthermore, all the assumptions in Lemma 12 are satisfied with some uniform constant \( K \) because \( \ell_T, \ell_T \), \( f(x), T, w^* \), and \( d^* \) are independent of \( N \); thus, by Lemma 14, we have (3) for a uniform constant \( L < \infty \). Moreover, \( TT^T > \delta I \) holds for some uniform constant \( \delta > 0 \), and \( R = \delta I \) for \( i \in [0,N-1] \) with some uniform constant \( r > 0 \), since \( (\cdot), w^*(x,d^*) \) are independent of \( N \). Similarly, \( (A,B) \) controllability implies \( (N_c, \beta) \)-uniform controllability of \( (A_i)_{i=0}^{N-1}, (B_i)_{i=0}^{N-1} \) with some uniform constant \( N_c, \beta \), and \( (A,Q) \) observability implies \( (N_o, \gamma_0) \)-uniform observability of \( (A_i)_{i=0}^{N-1}, (Q_i)_{i=0}^{N-1} \) for some uniform constants \( N_o, \gamma_0 \) (for now, we assume that \( Q_0 = 0 \) and \( Q_T = Q \) implies (5) for any \( Q_0 \geq 0 \) and \( Q_T = Q \)): thus, we have (5) with uniform \( \gamma > 0 \) for any \( Q_0, Q_T \). Since the first and second order conditions of optimality and constraint qualifications are satisfied, \( w^* \) is a strict minimizer for \( P_{0,N}(d^*) \). Since we have (3), (5), and (6) with uniform \( L, \gamma, \beta, \gamma \) and \( \lambda^* \), we have uBLH, uLCQ, and uSOSC at \( (w^*, d^*) \). By applying Theorem 7, we can obtain (2). Lastly, since the parameters \( K, r, N_c, \beta, N_o, \gamma_0 \) are independent of \( N \), so do \( \gamma \) and \( \rho \).

Initial and terminal cost functions that satisfy Assumption 17 can be constructed as:

\[
\ell_0(x,d) := -((I-T^T)\lambda^*)^T x
\]

\[
\ell_T(x,d) := (x-x^*)^T Q(x-x^*) + (\lambda^*)^T x,
\]

where \( x^* \) is the pseudo-inverse of the argument. One can observe that \( \ell_0(\cdot) \) can be set to constantly zero if \( T = I \).

### 3. NUMERICAL RESULTS

We illustrate the results of Theorem 7 and of Corollaries 16, 18. In this study, we solve the problem with base data \( d_{-1,N} \) to obtain the base solution \( w_{-1,N}^* \). We then solve a set of problems with perturbed data; in each of these problems, a random perturbation \( \Delta d \) is introduced at a selected time stage \( j \), while the rest of the data do not have perturbation (i.e., \( \Delta d \neq 0 \) for \( i \neq j \)). The obtained solutions \( w_{-1,N}^*(d_{-1,N}^* + \Delta d_{-1,N}) \) for the perturbed problems are visualized along with the base solution \( w_{-1,N}^* \). The scripts can be found on https://github.com/zavalab/JuliaBox/tree/master/SensitivityNSPCC. We consider a quadrotor motion planning problem (Hehn and
D’Andrea, 2011) with the time-invariant setting; the cost
functions are given by:
\[ \ell(z; d) := (x - d)^T Q (x - d) + u^T Ru \]
\[ \ell_f(x; d) := (x - d)^T Q_f (x - d), \quad \ell_b(x; d) = 0, \]
where \( Q := \text{diag}(1, 1, 1, q, q, 1, 1, 1) \), \( R := I \), \( Q_f := I \), and \( T := I \); and the dynamic mapping is obtained from:
\[
\begin{align*}
\frac{d^2 X}{dt^2} &= a (\cos \gamma \sin \beta \cos \alpha + \sin \gamma \sin \alpha) \quad (14a) \\
\frac{d^2 Y}{dt^2} &= a (\cos \gamma \sin \beta \sin \alpha - \sin \gamma \cos \alpha) \quad (14b) \\
\frac{d^2 Z}{dt^2} &= a \cos \gamma \cos \beta - g \quad (14c) \\
\frac{d\gamma}{dt} &= (b\omega X \cos \gamma + \omega Y \sin \gamma) / \cos \beta \quad (14d) \\
\frac{d\beta}{dt} &= -b\omega X \sin \gamma + \omega Y \cos \gamma \quad (14e) \\
\frac{d\alpha}{dt} &= b\omega X \cos \gamma \tan \beta + \omega Y \sin \gamma \tan \beta + \omega Z, \quad (14f)
\end{align*}
\]
where the state and control variables are defined as: \( x := (X, X, Y, Y, Z, Z, \gamma, \beta, \alpha) \) and \( u := (a, \omega_X, \omega_Y, \omega_Z) \). We use \( q \) and \( b \) as parameters that influence controllability and observability. In particular, the system becomes less observable if \( q \) becomes small and the system loses controllability as \( b \) becomes small (the effect of manipulation on \( \omega_X \) becomes weak). We have empirically tested the sensitivity behavior for \( q = b = 1 \) (Case 1) and \( q = b = 0 \) (Case 2). One can see that some of the assumptions (e.g., \( S_i = 0 \) in Corollary 16) may be violated, but one can also see that, qualitatively, the system is more observable and controllable in Case 1 than in Case 2. The results are presented in Figure 1. The base trajectories are shown as dashed lines, the perturbed trajectories are shown as solid gray lines, and the perturbed stages are highlighted using vertical lines. We can see that, for Case 1 \( (q = 1, b = 1) \), the differences between the base and perturbed solutions become small as moving away from the perturbation point (EDS holds). On the other hand, for Case 2 \( (q = 0, b = 0) \) one cannot observe EDS; this confirms that observability and controllability induce EDS.

Fig. 1. Base and perturbed solutions. Left: Case 1 \((q = 1, b = 1)\). Right: Case 2 \((q = 0, b = 0)\).

4. CONCLUSIONS

We have shown that uniform controllability and observability provide sufficient conditions for exponential decay of sensitivity in dynamic optimization. As part of future work, we will aim to establish exponential decay of sensitivity under mesh refinement settings and will aim to establish formal connections with continuous-time results.

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