Geometric process solving a class of analytic functions using q-convolution differential operator

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ABSTRACT
In current realization, our object is to use the convolution product in terms of the notion quantum calculus to deliver a propagated q-derivative factor taking a more generalized Salagean formula. By joining both the new factor together with the Janowski formula, we designate a special category of analytic factors in domain of unit disk. Finally, we deliver a set of significant inequalities involving these classes. As applications, we seek the q-differential translator to generalize a denomination of differential equations species Briot-Bouquet and formulate its upper analytic solution using the subordination idea. This application can be employed in information theory and thermo dynamical systems.

1. Introduction
The notion of q-calculus theory is a creative technique for calculations and organizations of the q-special functions such as the q-hypergeometric function, Jackson’s q-Bessel function, the q-gamma and q-beta functions (see [1]). The arrangement of q-calculus develops different classes of orthogonal polynomials, operators and special functions, which are achieving the form of their classical accompaniments. The idea of q-calculus essentially achieved recently by Carmichael [2], Jackson [3], Mason [4], and Trjitzinsky [5]. Study effort joins mathematics and real analysis together with this calculus for the early works presented by Ismail et al. [6]. Various integral and derivative factors introduced by the convolution concept; for example, the Salagean derivative [7], Al-Oboudi derivative (generalization of the Salagean derivative) [8], and the symmetric Salagean derivative [9]. It is important to inform that the technique of convolution calculation utilizes in various research, analysis and study of the geometric possessions of categories of smooth factors.

Recently, Naeem et al. [10] presented a study of categories connecting the Salagean q-derivative factor. In this work, we formalize a novel generalized q-derivative factor call it the symmetric Salagean q-derivative factor. Via this factor, we carry some new classes and study the geometric studies of them. Moreover, El-Qadeem and Mamon presented different classes of q-calculus of p-valent Salagean differential operator. These classes modified some recent works [11].

The study here is regarding the usage of the novel factor in some classical categories of univalent functions such as the classical Janowski function to present new categories of smooth functions in the open unit domain. Finally, we consider a set of significant presences and inequalities of these classes. As applications, we generalize a category of differential equations kind Briot-Bouquet and formulate its upper analytic solution utilizing the subordination idea.

2. Precursory
We shall request the following data throughout this paper, which can be located in [12]. A function \( \psi \in A \) is called univalent in \( U \), where \( U := \{ z \in \mathbb{C} : |z| < 1 \} \) represents to the open unit disk, if it’s certainly not receipts the equal value twice; that is, if \( \zeta_1 \neq \zeta_2 \) in \( U \) then \( \psi(\zeta_1) \neq \psi(\zeta_2) \) or equivalently, if \( \psi(\zeta_1) = \psi(\zeta_2) \) then \( \zeta_1 = \zeta_2 \). Devoid of loss of overview, we can use the letter \( \Lambda \) for univalent types of functions, which indicates by \( \triangleleft \) convincing the contraction

\[
\psi(z) = z + \sum_{n=2}^{\infty} \theta_n z^n, \quad z \in U.
\] (1)

Tow functions \( \varphi \) and \( \ell \in \Lambda \), are satisfied the subordinate relation, represented in the inequality \( \varphi \triangleleft \ell \), if there is a Schwarz function \( \nu \) achieves \( \nu(0) = 0 \), \( |\nu(z)| < 1 \) and \( \varphi(z) = \ell(\nu(z)) \) for all \( z \in U \) (see [13]). Apparently \( \varphi(z) \triangleleft \ell(z) \) is analogous to \( \varphi(0) = \ell(0) \) and \( \varphi(U) \subset \ell(U) \). In addition, the subordinate
The method of convolution product has deep investigation and lim
We proceed to define a new q-derivative factor in
\( \psi \in \Lambda \), the Hadamard product or convolution product is defined by

\[
\psi_1(z) \ast \psi_2(z) = \left( z + \sum_{n=2}^{\infty} an z^n \right) \ast \left( z + \sum_{n=2}^{\infty} bn z^n \right) = \left( z + \sum_{n=2}^{\infty} an bn z^n \right), \quad z \in \mathbb{U}.
\]  

The convolution product of two functions is a third function that states how the geometric representation of one is improved by the other. The concept convolution discusses to both the effect function and to the process of calculating it. It is formulated as the integral (or sum) of the product of the two functions when one is inverted and moved. For two functions \( \psi_1, \psi_2 \in \Lambda \), the convolution product or convolution product is defined by

\[
\psi_1(z) \ast \psi_2(z) = \left( z + \sum_{n=2}^{\infty} an z^n \right) \ast \left( z + \sum_{n=2}^{\infty} bn z^n \right)
= \left( z + \sum_{n=2}^{\infty} an bn z^n \right), \quad z \in \mathbb{U}.
\]  

2.1. Q-operator

We proceed to define a new q-derivative factor in \( \mathbb{U} \). Respecting each non-negative integer \( n \), the value of q-integer number, noted by \( [n, q] \), is given by

\[
[n, q] = (1 - q^n)/(1 - q), \quad \text{where } [0, q] = 0, \quad [1, q] = 1 \text{ and } \lim_{q \to 1} [n, q] = n.
\]

For examples, \( [0, 0.5] = 1, \quad [0, 0.5] = 1.5, \quad [3, 0.5] = 1.75, \quad [2, 0.75] = 1.75, \quad [3, 0.5] = 2.312, \quad [2, 0.99] = 1.99, \quad [3, 0.99] = 2.97, \quad [3, 1] = 3 \).

The main contribution of this work, is to present a new q-differential operator in the open unit disk. This operator indicates a generalization of some well-known operators such as Dunkl differential-difference operator and Salagean differential operator. A major extension to study Dunkl operators appears in their application for the analysis of quantum many body systems. These operators designate combined systems in one dimension and have improved substantial helpfulness in mathematical physics, especially in conformal field theory. Mainly, the most application of this work indicates in the conformal mapping technique (see [14,15]).

The q-difference operator of \( \psi \) is written by the formula

\[
\Delta_q \psi(z) = \frac{\psi(qz) - \psi(z)}{az - z}, \quad z \in \mathbb{U}.
\]  

Clearly, we attain \( \Delta_q z^n = [n, q]z^{n-1} \). Consequentially, for \( \psi \in \Lambda \), we obtain

\[
\Delta_q \psi(z) = \sum_{n=1}^{\infty} \theta_n [n, q]z^{n-1}, \quad z \in \mathbb{U}, \quad \theta_1 = 1.
\]  

For \( \psi \in \Lambda \), the Salagean q-derivative factor [16] is formulated as follows:

\[
S_q^0 \psi(z) = \psi(z), \quad S_q^1 \psi(z) = z \Delta_q \psi(z), \quad \ldots, \quad S_q^k \psi(z) = z \Delta_q \left( S_q^{k-1} \psi(z) \right),
\]  

where \( k \) is a positive integer. A computation based on the definition of \( \Delta_q \), implies that \( S_q^0 \psi(z) = \psi(z) \ast \psi_q^k(z) \), where * is the convolution product, \( \psi_q^k(z) = z + \sum_{n=2}^{\infty} [n, q]z^n \) and \( S_q^k \psi(z) = z + \sum_{n=2}^{\infty} [n, q]^{k} \theta_n z^n \).

Obvious,

\[
\lim_{q \to 1} S_q^k \psi(z) = z + \sum_{n=2}^{\infty} r^k \theta_n z^n,
\]

the Salagean derivative factor [7].

For a function \( \psi \in \Lambda \) and a constant \( \kappa \in \mathbb{R} \), we construct the generalized Salagean differential-difference operator (q-SDD) employing the idea of \( \Delta_q \) as follows:

\[
S_q^{\kappa,0} \psi(z) = \psi(z)
\]

\[
S_q^{\kappa,1} \psi(z) = z \Delta_q \psi(z) + \frac{\kappa}{2} \psi(z) - \psi(-z) - 2z
\]

\[
= z + \sum_{n=2}^{\infty} \left( [n, q] + \frac{\kappa}{2} (-1)^{n+1} + 1 \right) \theta_n z^n
\]

\[
S_q^{\kappa,2} \psi(z) = S_q^{\kappa,1} \left[ S_q^{\kappa,1} \psi(z) \right]
\]

\[
= z + \sum_{n=2}^{\infty} \left( [n, q] + \frac{\kappa}{2} (-1)^{n+1} + 1 \right)^2 \theta_n z^n
\]

\[
\vdots
\]

\[
S_q^{\kappa,k} \psi(z) = S_q^{\kappa,1} \left[ S_q^{\kappa,k-1} \psi(z) \right]
\]

\[
= z + \sum_{n=2}^{\infty} \left( [n, q] + \frac{\kappa}{2} (-1)^{n+1} + 1 \right)^k \theta_n z^n.
\]

Obviously, \( \lim_{q \to 1} S_q^{\kappa,k} \psi(z) \Rightarrow [9] \), when \( \kappa = 0 \) \( \lim_{q \to 1} S_q^{\kappa,k} \psi(z) \Rightarrow [7] \) and \( \kappa = 0 \Rightarrow [16] \). In this place, we remark that similar idea of q-operator has been applied in statistical physics (see [17,18]).

2.2. Convolution classes

Based on the definition of (7), we introduce the following classes. Denote by the following functions

\[
\Psi_q^k(z) := z + \sum_{n=2}^{\infty} [n, q]^k z^n;
\]

\[
\Phi_k^\sigma(z) := z + \sum_{n=2}^{\infty} \left( \frac{\kappa}{2} (1 + (-1)^{n+1}) \right)^k z^n.
\]

Thus, in terms of the convolution product, the factor (7) is formulated as follows:

\[
S_q^{\kappa,k} \psi(z) = \Psi_q^k(z) \ast \Phi_k^\sigma(z) \ast \psi(z), \quad \forall \psi \in \Lambda.
\]  

Let \( \psi \) be assumed a function from \( \Lambda \) and \( \sigma(z) \) be a convex univalent type of functions in \( \mathbb{U} \) such that \( \sigma(0) = 1 \). A function \( \psi \in \Lambda \) is said to be in the class \( \Xi_{\kappa,q_{1,q_{2}}}^k(\sigma) \) if
and only if

\[ Z_{q_1, q_2}^k(\sigma) = \left\{ \psi \in \Lambda : \frac{S_{q_1}^k(\psi(z))}{S_{q_2}^k(\psi(z))} \right\}. \]

When \( k = 0 \), we have the interesting subclass

\[ Z_{q_1, q_2}^{0,k}(\sigma) = \left\{ \psi \in \Lambda : \frac{S_{q_1}^{0,k}(\psi(z))}{S_{q_2}^{0,k}(\psi(z))} \right\}. \]

When \( k = 0 \), we have Dziok subclass [19].

Here, we remark that the class \( Z_{q_1, q_2}^k(\sigma) \) is significant to study the inclusion relation. This because the class \( Z_{q_1, q_2}^k(\sigma) \) indicates the main differential operator for two different quantum order \( S_{q_1}^k \) and \( S_{q_2}^k \). In addition, this combination of two fractional quantum numbers yields a comparison between different classes of analytic functions. Physically, this class spreads the additional flux more or less consistently over the unit disk. Therefore, the average field is maximized and the items (functions) at the place of the flux segment no longer have additional energy.

We indicate by \( S^*(\sigma) \) the class of all functions \( \psi \in \Lambda \) with

\[ S^*(\sigma) = \left\{ \psi \in \Lambda : \frac{z(1-
abla z)^{-1} \psi(z)}{z^2} < \sigma(z), \sigma(0) = 1 \right\}. \]

and \( C^*(\sigma) \) the category of all indicators \( \psi \in \Lambda \) such that

\[ C^*(\sigma) = \left\{ \psi \in \Lambda : \frac{z(1-
abla z)^{-1} \psi(z)}{z^2} < \sigma(z), \sigma(0) = 1 \right\}. \]

The following preliminary can be placed in [20,21] respectively.

**Lemma 2.1:** If \( K \) is smooth (analytic) in \( U \), \( f \in C((1 + z)/(1 - z)) \) is convex and \( g \in S^*((1 + z)/(1 - z)) \) is starlike then

\[ \frac{f * (Kg)}{f * g} \subseteq \text{co}(K(U)), \]  

where \( \text{co}(K(U)) \) is the closed convex hull of \( K(U) \).

**Lemma 2.2:** For analytic functions \( h, h \in U \) the subordination \( h < h \) implies that

\[ \int_0^{2\pi} |h(z)|^p \, d\theta \leq \int_0^{2\pi} |h(z)|^p \, d\theta, \]

where \( z = re^{\theta}, 0 < r < 1 \) and \( p \) is a positive number.

Some of the few studies in \( q \)-calculus are realized in the comparison between two different values of calculus. Class \( Z_{q_1, q_2}^k(\sigma) \) shows the relation between the \( q_1 \)-calculus and \( q_2 \)-calculus depending on the operator (7).

3. Results and discussion

In this section, we present our outcomes. The results are devoted into two subsections named inclusions and integral inequalities. Inequalities are very significant in the investigation of information theory, geometry, statistics and physics. There are a number of different settings in which these inequalities perform. They are approaches of communicating the second law of thermodynamics that is utilized in continuum mechanics. This inequality is predominantly beneficial in defining whether the constitutive relative of a material is thermo-dynamically permissible. This type of inequalities is a declaration regarding the irreversibility of ordinary processes, particularly when energy degeneracy is convoluted.

3.1. Inclusions

This section deals with the geometric representations of the class \( Z_{q_1, q_2}^k(\sigma), q_1 \neq q_2 \) and its consequences.

**Theorem 3.1:** Let \( \psi \in \Lambda \) and let the function \( g := \psi_{q_2}^* \psi \in S^*((1 + z)/(1 - z)), \ z \in U \). If \( \psi \in Z_{q_1, q_2}^{0,k}(\sigma) \) and the function \( \phi_{q_2}^k(z) \in C((1 + z)/(1 - z)) \) then \( \psi \in Z_{q_1, q_2}^{0,k}(\sigma), \sigma(0) = 1 \).

**Proof:** Suppose that \( \psi \in Z_{q_1, q_2}^{0,k}(\sigma) \). This implies that occurs a Schwarz function \( v \) with \( u(0) = 0 \) and \( |v(z)| < 1 \) achieving the following relation:

\[ \frac{\psi_{q_1}^k(z) \psi(z)}{\psi_{q_2}^k(z) \psi(z)} = \sigma(u(z)), \quad (z \in U). \]

This leads to

\[ \psi_{q_1}^k(z) \psi(z) = \left( \psi_{q_2}^k(z) \psi(z) \right) \sigma(u(z)) = g(z) \sigma(u(z)). \]

By employing the convolution’s properties, we arrive at

\[ \frac{\psi_{q_1}^k(z) \psi(z)}{\psi_{q_2}^k(z) \psi(z)} = \frac{\phi_{q_2}^k(z) \psi(z)}{\phi_{q_2}^k(z) \psi(z)} \]

where \( \text{co}(K(U)) \) is the closed convex hull of \( K(U) \).
Accordingly, in virtue of Lemma 2.1, we obtain
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \co(\sigma(\nu(\mathbb{U}))) \subset \co(\sigma(\mathbb{U})). \tag{20}
\]
Since \( \sigma(z) \) is a convex univalent function in \( \mathbb{U} \) with \( \sigma(0) = 1 \), then by the concept of subordination, we conclude that
\[
\frac{\psi_{q_1}^k (z) \ast \Phi_{s}^k (z) \ast \psi(z)}{\psi_{q_2}^k (z) \ast \Phi_{s}^k (z) \ast \psi(z)} < \sigma(z), \tag{21}
\]
which means that \( \psi \in \mathbb{S}_{q_1,q_2}^{\kappa}(\sigma) \). This completes the proof.

In this place, we export that the conclusion of Theorem 3.1 yields \( \mathbb{S}_{q_1,q_2}^{\kappa}(\sigma) \subset \mathbb{S}_{q_1,q_2}^{\kappa}(\sigma) \).

**Theorem 3.2:** Let \( \psi \in \Lambda \) and let the function \( G := \psi_{q_2}^k \ast \Phi_{s}^k \ast \psi \in \mathbb{S}((1 + z)/(1 - z)), z \in \mathbb{U} \). If \( \rho_1 := \psi_{q_1}^k \ast \Phi_{s}^k \ast \psi \ast r \rho_2 := \psi_{q_2}^k \ast \Phi_{s}^k \ast \psi \), for some \( r < 1 \) and the function \( \rho_2 \in \mathbb{C}((1 + z)/(1 - z)) \) then
\[
\mathbb{S}_{q_1,q_2}^{\kappa}(\sigma) \subset \mathbb{S}_{q_1,q_2}^{\kappa}(\sigma). \tag{22}
\]
**Proof:** Suppose that \( \psi \in \mathbb{S}_{q_1,q_2}^{\kappa}(\sigma) \). Then there occurs a Schwarz transform \( \omega \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) achieving
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \sigma(\omega(z)), \quad (z \in \mathbb{U}). \tag{23}
\]
This gives the following equality
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \left( \psi_{q_2}^k \ast \Phi_{s}^k \ast \psi \right) (\sigma(\omega(z))) = G(z) \sigma(\omega(z)). \tag{24}
\]
By considering the convolution’s properties, we conform that
\[
\frac{\psi_{q_1}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi}{\psi_{q_2}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi} (z) = \frac{\rho_1(z) \ast (G(z) \sigma(z))}{\rho_2(z) \ast G(z)}. \tag{25}
\]
Since \( \rho_1 \ast \rho_2 \) then by letting \( r \to 1 \), we obtain \( \rho_1(z) = \rho_2(z) \). As a result, by Lemma 2.1, we deduce that
\[
\frac{\psi_{q_1}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi}{\psi_{q_2}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi} (z) = \frac{\rho_2(z) \ast (G(z) \sigma(z))}{\rho_2(z) \ast G(z)} \in \co(\sigma(\omega(\mathbb{U}))) \subset \co(\sigma(\mathbb{U})). \tag{26}
\]
Since \( \sigma(z) \) is a convex univalent function in \( \mathbb{U} \) with \( \sigma(0) = 1 \), then by the definition of subordination, we obtain
\[
\frac{\psi_{q_1}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi}{\psi_{q_2}^{k+1} \ast \Phi_{s}^{k+1} \ast \psi} (z) \subset \sigma(z) \Rightarrow \psi \in \mathbb{S}_{q_1,q_2}^{\kappa+1}(\sigma), \tag{27}
\]
which completes the proof.

We note that if we replace the condition of Theorem 3.2 by if \( \rho_2 \prec \rho_1 \) such that \( \rho_1 \in \mathbb{C}((1 + z)/(1 - z)) \) then we obtain the same conclusion.

**Theorem 3.3:** Let \( \psi \in \Lambda \) and let the function \( H := \psi_{q_2}^k \ast \Phi_{s}^k \ast \psi \in \mathbb{S}((1 + z)/(1 - z)), z \in \mathbb{U} \). If \( \Phi_{s}^k \prec \Phi_{s}^k \ast r \Phi_{s}^k \) for some \( r < 1 \) then
\[
\mathbb{S}_{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}(\sigma) \subset \mathbb{S}_{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi}(\sigma). \tag{28}
\]
**Proof:** Suppose that \( \psi \in \mathbb{S}_{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}(\sigma) \). Consequently, a Schwarz function \( \vartheta \) with \( \vartheta(0) = 0 \) and \( |\vartheta(z)| < 1 \) occurs with the next conclusion
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \sigma(\vartheta(z)), \quad (z \in \mathbb{U}). \tag{29}
\]
This yields that
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \left( \psi_{q_2}^k \ast \Phi_{s}^k \ast \psi \right) (\sigma(\vartheta(z))) = H(z) \sigma(\vartheta(z)). \tag{30}
\]
But the condition \( \Phi_{s}^k \prec \Phi_{s}^k \ast r \Phi_{s}^k \) implies that \( \Phi_{s}^k \ast r \Phi_{s}^k \) (for some \( r \)). It is clear that \( \vartheta(z) = z \in \mathbb{C}((1 + z)/(1 - z)) \), therefore, by the convolution’s properties we attain
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) = \eta(z) \ast H(z) \sigma(\vartheta(z)), \quad (z \in \mathbb{U}). \tag{31}
\]
Thus, in view of Lemma 2.1, we get
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) \in \co(\sigma(\vartheta(\mathbb{U}))) \subset \co(\sigma(\mathbb{U})). \tag{32}
\]
Since \( \sigma(z) \) is a convex univalent function in \( \mathbb{U} \) with \( \sigma(0) = 1 \), then by the definition of subordination, we obtain
\[
\frac{\psi_{q_1}^k \ast \Phi_{s}^k \ast \psi}{\psi_{q_2}^k \ast \Phi_{s}^k \ast \psi} (z) \subset \sigma(z) \Rightarrow \psi \in \mathbb{S}_{\vartheta_{q_1}^k \ast \Phi_{s}^k \ast \psi}(\sigma), \tag{33}
\]
which completes the proof.

We record that if we change the condition of Theorem 3.3 by \( \Phi_{s}^k \prec \Phi_{s}^k \ast r \Phi_{s}^k \), we have
\[
\mathbb{S}_{\vartheta_{q_1}^k \ast \Phi_{s}^k \ast \psi}(\sigma) \subset \mathbb{S}_{\vartheta_{q_1}^k \ast \Phi_{s}^k \ast \psi}(\sigma). \tag{34}
\]

### 3.2. Integral inequalities

The recent section deals with some inequalities containing the operator (7). For two functions \( h(z) = \sum a_n z^n \) and \( h(z) = \sum b_n z^n \), we have \( h \ll h \) if and only if \( |a_n| \leq |b_n|, \forall n \). This inequality is known as the majorization of two analytic functions.
**Theorem 3.4:** Consider the operator $S_q^k \psi(z)$, $\psi \in \Lambda$. If the coefficients of $\psi$ satisfy the inequality $|\vartheta_n| \leq (1/n\kappa)^k$, $\kappa \in (0, \infty)$ then
\[
\int_0^{2\pi} \left| \frac{S_q^k \psi(z)}{z} \right|^p d\theta \leq \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right|^p d\theta, \quad p > 0.
\]

**Proof:** Let
\[
\sigma(z, \delta) = \left( \frac{1 + z}{1 - z} \right)^\delta, \quad z \in \mathbb{U}, \quad \delta \geq 1.
\]
Then, a computation implies that
\[
\sigma(z, 1) = 1 + \sum_{n=1}^{(2n)} z^n
\]
\[
\sigma(z, 2) = 1 + \sum_{n=1}^{(4n)} z^n
\]
\[
= 1 + 4z + 8z^2 + 12z^3 + 16z^4 + 20z^5 + \cdots
\]
\[
\sigma(z, 3) = 1 + \sum_{n=1}^{(2n^2)} z^n
\]
\[
= 1 + 6z + 18z^2 + 38z^3 + \cdots
\]
\[
\sigma(z, 4) = 1 + \sum_{n=1}^{(1/3)} (8n(2 + n^2))^n z^n
\]
\[
= 1 + 8z + 16z^2 + 24z^3 + \cdots
\]
\[\vdots\]
Comparing Equation (37) and the coefficients of $S_q^k \psi(z)$, which are satisfying
\[
\lim_{q \to 1^-} |\vartheta_n|^k \left( 1 + (-1)^{n+1} \right)^k |\vartheta_n| \leq 1,
\]
we conclude that $S_q^k \psi(z)$ is majorized by the function $\sigma(z, \delta)$ for all $\delta \geq 1$. By the properties of majorization [22], we have
\[
S_q^k \psi(z) < \sigma(z, \delta), \quad z \in \mathbb{U}.
\]
Thus, according to Lemma 2.2, we conclude that
\[
\int_0^{2\pi} \left| \frac{S_q^k \psi(z)}{z} \right|^p d\theta \leq \int_0^{2\pi} \left| \frac{1 + z}{1 - z} \right|^p d\theta, \quad p > 0.
\]
\[\blacksquare\]

Moreover, equality in Theorem 3.4 can be studied in the following result

**Theorem 3.6:** Consider the operator $S_q^k \psi(z)$, $\psi \in \Lambda$. If the coefficients of $\psi$ satisfy the inequality $|\vartheta_n| \leq (1/n\kappa)^k$, $\kappa \in (0, \infty)$ then there is a probability measure $\mu$ on $(\partial \mathbb{U})^2$ for all $\delta > 1$.

**Proof:** Let $\epsilon, \delta \in \partial \mathbb{U}$ then we have
\[
\left( \frac{1 + z\epsilon}{1 + \epsilon z} \right)^{\delta} \leq \frac{1}{1 + \epsilon z} \left( \frac{1 + z\epsilon}{1 - z} \right)^{\delta - 1}
\]
\[
= \left( \frac{1 + z}{1 - z} \right)^{\delta}, \quad \delta > 1.
\]
By the virtue of Theorem 1.11 in [23], the functional $((1 + \epsilon z)/(1 + \epsilon z))^{\delta}$ recognizes a probability measure $\mu$ in $(\partial \mathbb{U})^2$ fulfilling
\[
\chi(z) = \int_{(\partial \mathbb{U})^2} \left( \frac{1 + \epsilon z}{1 + \epsilon z} \right)^{\delta} d\mu(\epsilon, \delta), \quad z \in \mathbb{U}.
\]
Then there occurs a constant $c \in (0, 1)$ (diffusion constant) such that
\[
\int_{(\partial \mathbb{U})^2} \left( \frac{1 + \epsilon z}{1 + \epsilon z} \right)^{\delta} d\mu(\epsilon, \delta)
\]
\[
= c \int_{(\partial \mathbb{U})^2} S_q^k \psi(z) d\mu(\epsilon, \delta), \quad z \in \mathbb{U}.
\]
This completes the proof. \[\blacksquare\]

Since, the operator $S_q^k \psi(z)$ is a generalization of the reference [16], when $\kappa = 0$, then we conclude that our method extended the effort [16] for all values of $\kappa$.

### 4. Applications

The field of complex differential equations increased its application in the last decade. It occurs in the study of neural network [24] and in ocean studies [25]. Here, our application based on a special type of complex differential equations. This kind of differential equations is a gathering of equations involving any types of derivative whose results are terminologies of a complex variable (z) (see for basic information [13] and [26] for recent study). Study of q-ODEs in the complex domain suggest the finding of new special functions, which recognized as q-Briot-Bouquet differential equation (q-BBE).

The special form of BBE is as follows:
\[
e f(z) + (1 - \epsilon) \frac{z\psi(z)'}{\psi(z)}
\]
\[
= \mu(z), \quad \mu(0) = \psi(0), \quad \epsilon \in [0, 1].
\]

Different applications of BBE in the geometric function theory presented in [13]. Our aim is to extend BBE by
employing the (7) and deliver its solutions by applying the subordination inequalities. Now in view of the q-differential operator (7), we have the q-BBE

$$\varepsilon \psi(z) + (1 - \varepsilon) \frac{z(S_q^k \psi(z))'}{S_q^k \psi(z)} = \mu(z), \quad z \in \mathbb{U}. \tag{45}$$

A trivial solution of q-BBE (see (45)) can recognize when \( \varepsilon = 1 \). Thus, our investigation based on the case \( \varepsilon = 0 \). Next theorem indicates the attitude of the outcome of (45).

**Theorem 4.1:** For \( \psi \in \Lambda, \kappa \in [0, \infty) \) and \( \mu \) is univalent convex in \( \mathbb{U} \) if

$$\left( \frac{z(S_q^k \psi(z))'}{S_q^k \psi(z)} \right) < \mu(z), \quad z \in \mathbb{U}, \tag{46}$$

then

$$S_q^k \psi(z) < z \exp \left( \int_0^z \frac{\mu(u(z)) - 1}{u(z)} du(z) \right), \tag{47}$$

where \( \nu \) is a Schwarz function in \( \mathbb{U} \). In addition,

$$|z| \exp \left( \int_0^1 \frac{\mu(u(z)) - 1}{u(z)} du(z) \right) \leq \left| S_q^k \psi(z) \right| \leq |z| \exp \left( \int_0^1 \frac{\mu(u(z)) - 1}{u(z)} du(z) \right). \tag{48}$$

**Proof:** The conclusion of subordination concept stats (see (46)) that there occurs a Schwarz transform \( \nu \) satisfying

$$\left( \frac{z(S_q^k \psi(z))'}{S_q^k \psi(z)} \right) = \mu(\nu(z)), \quad z \in \mathbb{U}. \tag{49}$$

A calculation of the above conclusion, we arrive at

$$\left( \frac{(S_q^k \psi(z))'}{S_q^k \psi(z)} \right) = \frac{1}{z} - \frac{\mu(u(z)) - 1}{z}. \tag{50}$$

Operating the integral from 0 to \( z \), we obtain

$$\log \left( \frac{S_q^k \psi(z)}{z} \right) = \int_0^z \frac{\mu(u(z)) - 1}{z} du(z). \tag{51}$$

By utilizing the subordination definition, we get

$$S_q^k \psi(z) < z \exp \left( \int_0^z \frac{\mu(u(z)) - 1}{u(z)} du(z) \right). \tag{52}$$

Moreover, we confirm that \( \mu(z) \) transfers the disk \( 0 < |z| < \tau \leq 1 \) onto a symmetric convex domain corresponding to the real line, which yields the real part of \( \mu \) obligates the following inequality

$$\mu(-\tau |z|) \leq \Re(\mu(\nu(\tau z))) \leq \mu(\tau |z|), \quad \tau \in (0, 1], \quad |z| \neq \tau, \tag{53}$$

which brings the next inequalities:

$$\mu(-\tau) \leq \mu(-\tau |z|), \quad \mu(\tau |z|) \leq \mu(\tau) \tag{54}$$

and

$$\int_0^1 \frac{\mu(u(-\tau |z|)) - 1}{\tau} d\tau \leq \Re \left( \int_0^1 \frac{\nu(u(\tau |z|)) - 1}{\tau} d\tau \right) \leq \int_0^1 \frac{\nu(u(\tau |z|)) - 1}{\tau} d\tau. \tag{55}$$

By consuming the last two inclusions together with Equation (51), we have

$$\int_0^1 \frac{\mu(u(-\tau |z|)) - 1}{\tau} d\tau \leq \log \left| \frac{S_q^k \psi(z)}{z} \right| \leq \int_0^1 \frac{\nu(u(\tau |z|)) - 1}{\tau} d\tau. \tag{56}$$

This likewise to obtain the result

$$\exp \left( \int_0^1 \frac{\mu(u(-\tau |z|)) - 1}{\tau} d\tau \right) \leq \exp \left( \int_0^1 \frac{\nu(u(\tau |z|)) - 1}{\tau} d\tau \right). \tag{57}$$

**Remark:** Theorem 4.1 provides an exponential growth solution, which is called a geometric growth solution.

### 4.1. Numerical example

Consider the following data: \( \psi(z) = z, K(z) = z/(1 - z), \quad \kappa = 2, \quad q = 0.5 \). It is clear that \( \psi(0) = \mu(0) = 0 \) and \( \Delta_{0.5}(z) = 1 \) in the open unit disk. Under the formula of the operator (7), we have \( S_{0.5}^2(z) = z \). Our aim is to apply Theorem 4.1 to recognize the behaviour of the solution of 0.5-BBE of the function \( \psi(z) = z \in \mathbb{U} \). Moreover, the function \( \psi(z) = z \) achieves \( \psi'(0) = 1 \neq 0, \Re(z^2 \psi''(z)/\psi'(z) + 1) = 1 > 0 \) and \( \Re(z^2 \psi''(z)/\psi'(z)) = 1 > 1/2 \) (so the function \( \psi(z) = z \) is star-like of order 1/2) (see [27]). Similarly, we conclude that

$$\Re \left( \frac{z(S_q^k \psi(z))'}{S_q^k \psi(z)} \right) = 1 > 1/2, \tag{58}$$

and hence, it is star-like of order 1/2. In virtue of the Marx-Strohhaicker result [28] the above construction
yields the main condition of Theorem 4.1,
\[
\begin{pmatrix}
  z(S_q^{\kappa q}(z))' \\
  S_q^{\kappa q} \psi(z)
\end{pmatrix} < \frac{zK'(z)}{K(z)} := \mu(z), \quad z \in \mathbb{U}, \quad \kappa = 2, \quad q = 0.5.
\] (59)

Consequently, by employing Theorem 4.1, the solution of 0.5-BBE achieves the geometric growth inequality
\[
S_{0.5}^{2q}(z) \prec z \exp \left( \int_0^z \frac{\mu(\zeta)}{\zeta} d\zeta \right).
\] (60)

The upper bound of \(S_{0.5}^{2q}(z)\) describes the behaviour of exponential phase growth of population. It is a historical categorized by cell doubling. The sum of new population seems in time-2D space (see Figure 1).

5. Conclusion and future works
Attracted in this method, we get a collection of functions, which are univalent under \(q\)-operator. We delivered appropriate important cases of these subcategories. Applications contains integral inequalities and inclusions are given to describe the behaviour of the operator (7). For extra investigation, we inform the investigators to give some results and consequences related to other categories of smooth functions for instant, one can use harmonic mappings, \(p\)-valent or
meromorphic maps. Moreover, El-Qadeem and Mamon [11] can use (7) to generalize their classes by extend the operator into p-valent class of analytic functions or generalize works in [29–36].

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