Quantal Modifications to the Wheeler DeWitt Equation

R. Jackiw

Center for Theoretical Physics,
Laboratory for Nuclear Science
and Department of Physics,
Massachusetts Institute of Technology,
Cambridge, MA 02139–4307

Abstract

Commutator anomalies obstruct solving the Wheeler-DeWitt constraint equation in Dirac quantization of quantum gravity-matter theory. When the obstruction is removed, there result quantal modifications to the constraints. The same classical theory gives rise to different quantum theories when different procedures for overcoming anomalies are implemented.

In a canonical, Hamiltonian approach to quantizing a theory with local symmetry — a theory that is invariant against transformations whose parameters are arbitrary functions on space-time — there occur constraints, which are imposed on physical states. Typically these constraints correspond to time components of the Euler-Lagrange equations, and familiar examples arise in gauge theories. The time component of the gauge field equation is the Gauss law.

\[ G_a \equiv D \cdot E^a - \rho^a = 0 \] (1)

Here \( E^a \) is the (non-Abelian) electric field, \( \rho^a \) the matter charge density, and \( D \) denotes the gauge-covariant derivative. When expressed in terms of canonical variables, \( G_a \) does not involve time-derivatives — it depends on canonical coordinates and momenta, which we denote collectively by the symbols \( X \) and \( P \) respectively (\( X \) and \( P \) are fields defined at fixed time) : \( G_a = G_a(X, P) \). Thus in a Schrödinger representation for the theory, the Gauss law condition on physical states

\[ G_a(X, P)|\psi\rangle = 0 \] (2)

corresponds to a (functional) differential equation that the state functional \( \Psi(X) \) must satisfy.

\[ G_a \left( X, \frac{1}{i} \frac{\delta}{\delta X} \right) \Psi(X) = 0 \] (3)

In fact, Eq. (3) represents an infinite number of equations, one for each spatial point \( r \), since \( G_a \) is also the generator of the local symmetry: \( G_a = G_a(r) \). Consequently, questions of consistency (integrability) arise, and these may be examined by considering the commutator of two constraints. Precisely because the \( G_a \) generate the symmetry transformation, one expects their commutator to follow the Lie algebra with structure constants \( f_{abc} \).
\[ [G_a(r), G_b(\tilde{r})] = i f_{abc} G_c(r) \delta(r - \tilde{r}) \]  

(4)

If (4) holds, the constraints are consistent — they are first class — and the constraint equations are integrable, at least locally.

However, it is by now well-known that Eq. (4), which does hold classically with Poisson bracketing, may acquire a quantal anomaly. Indeed when the matter charge density is constructed from fermions of a definite chirality, the Gauss law algebra is modified by an extension — a Schwinger term — the constraint equations become second-class and Eq. (3) is inconsistent and cannot be solved. We call such gauge theories “anomalous.”

This does not mean that a quantum theory cannot be constructed from an anomalous gauge theory. One can adopt various strategies for overcoming the obstruction, but these represent modifications of the original model. Moreover, the resulting quantum theory possesses physical content that is very far removed from what one might infer by studying the classical model. All this is explicitly illustrated by the anomalous chiral Schwinger model, whose Gauss law is obstructed, while a successful construction of the quantum theory leads to massive excitations, which cannot be anticipated from the un-quantized equations [1].

With these facts in mind, we turn now to gravity theory, which obviously is invariant against local transformations that redefine coordinates of space-time.

Indeed over the years there have been many attempts to describe gravity in terms of a gauge theory. That program is entirely successful in three- and two-dimensional space-time, where gravitational models are formulated in terms of Einstein–Cartan variables (spin-connection, Vielbein) as non-Abelian gauge theories, based not on the Yang-Mills paradigm, but rather on Chern-Simons and B-F structures.

But even remaining with the conventional metric-based formulation, it is recognized that the time components of Einstein’s equation comprise the constraints.

\[ \frac{1}{8\pi G} \left( R^0_\nu - \frac{1}{2} \delta^0_\nu R \right) - T^0_\nu = 0 \]  

(5)

The gravitational part is the time component of the Einstein tensor \( R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu R \); weighted by Newton’s constant \( G \), this equals the time component of the matter energy-momentum tensor, \( T^\mu_\nu \). In the quantized theory, the collection of canonical operators on the left side in (5) annihilates physical states. The resulting equations may be presented as

\[ \mathcal{E} | \psi \rangle = 0 \]  

(6)

\[ \mathcal{P}_i | \psi \rangle = 0 \]  

(7)

where \( \mathcal{E} \) is the energy constraint

\[ \mathcal{E} = \mathcal{E}^{\text{gravity}} + \mathcal{E}^{\text{matter}} \]

(8)

and \( \mathcal{P}_i \) is the momentum constraint.

\[ \mathcal{P}_i = \mathcal{P}_i^{\text{gravity}} + \mathcal{P}_i^{\text{matter}} \]

(9)

Taking for definiteness matter to be described by a massless, spinless field \( \varphi \), with canonical momentum \( \Pi \), we have
\[ \mathcal{E}_{\text{matter}} = \frac{1}{2} \left( \Pi^2 + \gamma \gamma^{ij} \partial_i \varphi \partial_j \varphi \right) \]  
(10)

\[ \mathcal{P}^i_{\text{matter}} = \partial_i \varphi \Pi \]  
(11)

Here \( \gamma_{ij} \) is the spatial metric tensor; \( \gamma_i \) its determinant; \( \gamma^{ij} \), its inverse.

The momentum constraint in Eq. (7) is easy to unravel. In a Schrödinger representation, it requires that \( \Psi(\gamma_{ij}, \varphi) \) be a functional of the canonical field variables \( \gamma_{ij}, \varphi \) that is invariant against reparameterization of the spatial coordinates and such functionals are easy to construct.

Of course it is (3), the Wheeler-DeWitt equation, that is highly non-trivial and once again one asks about its consistency. If the commutators of \( \mathcal{E} \) with \( \mathcal{P} \) follow their Poisson brackets one would expect that the following algebra holds.

\[ [\mathcal{P}_i(\mathbf{r}), \mathcal{P}_j(\mathbf{\tilde{r}})] = i \mathcal{P}_j(\mathbf{r}) \partial_i \delta(\mathbf{r} - \mathbf{\tilde{r}}) + i \mathcal{P}_i(\mathbf{\tilde{r}}) \partial_j \delta(\mathbf{r} - \mathbf{\tilde{r}}) \]  
(12a)

\[ [\mathcal{E}(\mathbf{r}), \mathcal{E}(\mathbf{\tilde{r}})] = i \left( \mathcal{P}^i(\mathbf{r}) + \mathcal{P}^i(\mathbf{\tilde{r}}) \right) \partial_i \delta(\mathbf{r} - \mathbf{\tilde{r}}) \]  
(12b)

\[ [\mathcal{E}(\mathbf{r}), \mathcal{P}_i(\mathbf{\tilde{r}})] = i \left( \mathcal{E}(\mathbf{r}) + \mathcal{E}(\mathbf{\tilde{r}}) \right) \partial_i \delta(\mathbf{r} - \mathbf{\tilde{r}}) \]  
(12c)

Here \( \mathcal{P}^i \equiv \gamma \gamma^{ij} \mathcal{P}_j \). If true, Eqs. (12) would demonstrate the consistency of the constraints, since they appear first-class. Unfortunately, establishing (12) in the quantized theory is highly problematical. First of all there is the issue of operator ordering in the gravitational portion of \( \mathcal{E} \) and \( \mathcal{P} \). Much has been said about this, and I shall not address that difficulty here.

The problem that I want to call attention to is the very likely occurrence of an extension in the \([\mathcal{E}, \mathcal{P}_i] \) commutator (12d). We know that in flat space, the commutator between the matter energy and momentum densities possesses a triple derivative Schwinger term [2]. There does not appear any known mechanism arising from the gravity variables that would effect a cancellation of this obstruction.

A definite resolution of this question in the full quantum theory is out of reach at the present time. Non-canonical Schwinger terms can be determined only after a clear understanding of the singularities in the quantum field theory and the nature of its Hilbert space are in hand, and this is obviously lacking for four-dimensional quantum gravity.

Faced with the impasse, we turn to a gravitational model in two-dimensional space-time — a lineal gravity theory — where the calculation can be carried to a definite conclusion: an obstruction does exist and the model is anomalous. Various mechanisms are available to overcome the anomaly, but the resulting various quantum theories are inequivalent and bear little resemblance to the classical model.

In two dimensions, Einstein’s equation is vacuous because \( R^m_n = \frac{1}{2} \delta^m_n R \); therefore gravitational dynamics has to be invented afresh. The models that have been studied recently posit local dynamics for the “gravity” sector, which involves as variables the metric tensor and an additional world scalar (“dilaton” or Lagrange multiplier) field. Such “scalar-tensor” theories, introduced a decade ago [3], are obtained by dimensional reduction from higher-dimensional Einstein theory [3,4]. They should be contrasted with models where quantum fluctuations of matter variables induce gravitational dynamics [5], which therefore are non-local and do not appear to offer any insight into the questions posed by the physical, four-dimensional theory.

The model we study is the so-called “string-inspired dilaton gravity” – CGHS theory [6]. The gravitational action involves the metric tensor \( g_{\mu\nu} \), the dilaton field \( \phi \), and a cosmological constant \( \lambda \). The matter action describes the coupling of a massless, spinless field \( \varphi \).
\[ I_{\text{gravity}} = \int d^2 x \sqrt{-g} e^{-2\phi} (R + 4g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda) \] (13)

\[ I_{\text{matter}} = \frac{1}{2} \int d^2 x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \] (14)

The total action is the sum of (13) and (14), weighted by “Newton’s” constant \( G \):

\[ I = \frac{1}{4\pi G} I_{\text{gravity}} + I_{\text{matter}} \] (15)

In fact this theory can be given a gauge-theoretical “B-F” description based on the centrally extended Poincaré group in (1+1) dimensions [7]. This formulation aided us immeasurably in the subsequent analysis/transformations. However, I shall not discuss this here, because in retrospect it proved possible to carry the analysis forward within the metric formulation (13)–(15).

After a remarkable sequence of redefinitions and canonical transformations on the dynamical variables in (13)–(15), one can present \( I \) in terms of a first-order Lagrange density \( \mathcal{L} \) that is a sum of quadratic terms.

\[ \mathcal{L} = \pi_a \dot{r}^a + \Pi \dot{\phi} - \alpha \mathcal{E} - \beta \mathcal{P} \] (16)

\[ \mathcal{E} = -\frac{1}{2} \left( \frac{1}{\Lambda} \pi^a \pi_a + \Lambda \dot{r}^a \dot{r}_a \right) + \frac{1}{2} \left( \Pi^2 + \varphi' \right) \] (17)

\[ \mathcal{P} = -r^{a'} \pi_a - \varphi' \Pi \] (18)

I shall not derive this, but merely explain it. The index “a” runs over flat 2-dimensional \((t, \sigma)\) space, with signature \((1, -1)\). Dot (dash) signify differentiation with respect to time \(t\) (space \(\sigma\)).

The four variables \(\{r^a, \alpha, \beta\}\) correspond to the four gravitational variables \((g_{\mu\nu}, \phi)\), where only \(r^a\) is dynamical with canonically conjugate momentum \(\pi_a\), while \(\alpha\) and \(\beta\) act as Lagrange multipliers. Notice that regardless of the sign \(\Lambda \equiv \lambda / 8\pi G\), the gravitational contribution to \(\mathcal{E}\), is quadratic with indefinite sign.

\[ \mathcal{E}^{\text{gravity}} = -\frac{1}{2} \left( \frac{1}{\Lambda} \pi^a \pi_a + \Lambda \dot{r}^a \dot{r}_a \right) \]

\[ = -\frac{1}{2} \left( \frac{1}{\Lambda} \pi_0^2 - \frac{1}{\Lambda} \pi_1^2 + \Lambda (r^0')^2 - \Lambda (r^1')^2 \right) \]

\[ = -\mathcal{E}_0 + \mathcal{E}_1 \] (19a)

\[ \mathcal{E}_0 = \frac{1}{2} \left( \frac{1}{\Lambda} \pi_0^2 + \Lambda (r^0')^2 \right) \] (19b)

\[ \mathcal{E}_1 = \frac{1}{2} \left( \frac{1}{\Lambda} \pi_1^2 + \Lambda (r^1')^2 \right) \] (19c)

On the other hand, the gravitational contribution to the momentum does not show alteration of sign.

\[ \mathcal{P}^{\text{gravity}} = -r^{a'} \pi_a \]

\[ = -r^{0'} \pi_0 - r^{1'} \pi_1 \]

\[ = \mathcal{P}_0 + \mathcal{P}_1 \] (20a)

\[ \mathcal{P}_0 = -r^{0'} \pi_0 \] (20b)

\[ \mathcal{P}_1 = -r^{1'} \pi_1 \] (20c)

One may understand the relative negative sign between the two gravitational contributors \((a = 0, 1)\) as follows. Pure metric gravity in two space-time dimensions is described by three functions collected in \(g_{\mu\nu}\). Diffeomorphism invariance involves 2 functions, which reduce the number of variables by \(2 \times 2\), \(i.e.\) pure gravity has \(3 - 4 = -1\) degrees of freedom. Adding the dilaton \(\phi\) gives a net number of \(-1 + 1 = 0\), as in our final gravitational Lagrangian.
The matter contribution is the conventional expression for massless and spinless fields:

\[ \mathcal{E}^\text{matter} = \frac{1}{2} (\Pi^2 + \varphi'^2) \]
\[ \mathcal{P}^\text{matter} = -\varphi' \Pi \]

It is with the formulation in Eqs. (16)–(22) of the theory (13)–(15) that we embark upon the various quantization procedures.

The transformed theory appears very simple: there are three independent dynamical fields \( \{r^a, \varphi\} \) and together with the canonical momenta \( \{\pi_a, \Pi\} \) they lead to a quadratic Hamiltonian, which has no interaction terms among the three. Similarly, the momentum comprises non-interacting terms. However, there remains a subtle “correlation interaction” as a consequence of the constraint that \( \mathcal{E} \) and \( \mathcal{P} \) annihilate physical states, as follows from varying the Lagrange multipliers \( \alpha \) and \( \beta \) in (16)

\[ \mathcal{E} | \psi \rangle = 0 \]
\[ \mathcal{P} | \psi \rangle = 0 \]

Thus, even though \( \mathcal{E} \) and \( \mathcal{P} \) each are sums of non-interacting variables, the physical states \( | \psi \rangle \) are not direct products of states for the separate degrees of freedom. Note that Eqs. (23), (24) comprise the entire physical content of the theory. There is no need for any further “gauge fixing” or “ghost” variables — this is the advantage of the Hamiltonian formalism.

As in four dimensions, the momentum constraint (24) enforces invariance of the state functional \( \Psi(r^a, \varphi) \) against spatial coordinate transformations, while the energy constraint (23) — the Wheeler-DeWitt equation in the present lineal gravity context — is highly non-trivial.

Once again one looks to the algebra of the constraints to check consistency. The reduction of (12) to one spatial dimension leaves (after the identification \( \mathcal{P}_i \rightarrow -\gamma, \gamma^{ij} \rightarrow 1 \))

\[ i[\mathcal{P}(\sigma), \mathcal{P}(\tilde{\sigma})] = (\mathcal{P}(\sigma) + \mathcal{P}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) \]
\[ i[\mathcal{E}(\sigma), \mathcal{E}(\tilde{\sigma})] = (\mathcal{P}(\sigma) + \mathcal{P}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) \]
\[ i[\mathcal{E}(\sigma), \mathcal{P}(\tilde{\sigma})] = (\mathcal{E}(\sigma) + \mathcal{E}(\tilde{\sigma})) \delta'(\sigma - \tilde{\sigma}) - \frac{c}{12\pi} \delta''(\sigma - \tilde{\sigma}) \]

where we have allowed for a possible central extension of strength \( c \), and it remains to calculate this quantity.

The gained advantage in two dimensional space-time is that all operators are quadratic, see (19)-(22); the singularity structure may be assessed and \( c \) computed; obviously it is composed of independent contributions.

\[ c = c^\text{gravity} + c^\text{matter} \]
\[ c^\text{gravity} = c_0 + c_1 \]

Surprisingly, however, there is more than one way of handling infinities and more than one answer for \( c \) can be gotten. This reflects the fact, already known to Jordan in the 1930s [8], that an anomalous Schwinger term depends on how the vacuum is defined.

In the present context, there is no argument about \( c^\text{matter} \), the answer is

\[ c^\text{matter} = 1 \]

The same holds for the positively signed gravity variable (assume \( \Lambda > 0 \), so that \( a_1 \) enters positively).
But the negatively signed gravitational variable can be treated in more than one way, giving different answers for $c_0$. The different approaches may be named “Schrödinger representation quantum field theory” and “BRST string/conformal field theory,” and the variety arises owing to the various ways one can quantize a theory with a negative kinetic term, like the $r^0$ gravitational variable. (This variety is analogous to what is seen in Gupta-Bleuler quantization of electrodynamics: the time component potential $A_0$ enters with negative kinetic term.)

In the Schrödinger representation quantum field theory approach one maintains positive norm states in a Hilbert space, and finds $c_0 = -1$, $c^{\text{gravity}} = c_0 + c_1 = 0$, $c = c^{\text{gravity}} + c^{\text{matter}} = 1$. Thus pure gravity has no obstructions, only matter provides the obstruction. Consequently the constraints of pure gravity can be solved, indeed explicit formulas have been gotten by many people [9]. In our formalism, according to (19) and (20) the constraints read

$$\mathcal{E}^{\text{gravity}}|\psi\rangle_{\text{gravity}} \sim \frac{1}{2} \left( \frac{1}{\Lambda} \frac{\delta^2}{\delta r^a \delta r_a'} - \Lambda r^{a'} r_a' \right) \Psi_{\text{gravity}}(r^a) = 0$$

and

$$\mathcal{P}^{\text{gravity}}|\psi\rangle_{\text{gravity}} \sim i r^{a'} \frac{\delta}{\delta r_a} \Psi_{\text{gravity}}(r^a) = 0$$

with two solutions

$$\Psi_{\text{gravity}}(r^a) = \exp \left\{ \pm \frac{i}{2} \int d\sigma \epsilon_{ab} r^a r^b \right\}$$

This may also be presented by an action of a definite operator on the Fock vacuum state $|0\rangle$,

$$\Psi_{\text{gravity}}(r^a) \propto \left[ \exp \pm \int dk \, a^\dagger_0(k) \epsilon(k) a^\dagger_1(-k) \right] |0\rangle.$$

with $a^\dagger_0(k)$ creating field oscillations of definite momentum.

$$a^\dagger_0(k) = \frac{-i}{\sqrt{4\pi \Lambda |k|}} \int d\sigma \, e^{ik\sigma} \pi_0(\sigma) + \frac{\Lambda |k|}{4\pi} \int d\sigma \, e^{ik\sigma} r^0(\sigma)$$

As expected, the state functional is invariant against spatial coordinate redefinition, $\sigma \to \tilde{\sigma}(\sigma)$; this is best seen by recognizing that integrand in the exponent of (31a) is a 1-form: $d\sigma \, \epsilon_{ab} r^a r^b = \epsilon_{ab} r^a d r^b$. [It is important that two fields, $r^0$ and $r^1$, are in play. One cannot construct an invariant functional out of just one field.]

Although this state is here presented for a gravity model in the Schrödinger representation field theory context, it is also of interest to practitioners of conformal field theory and string theory. The algebra (25), especially when written in decoupled form,

$$\Theta_\pm = \frac{1}{2} (\mathcal{E} \mp \mathcal{P})$$

$$[\Theta_\pm(\sigma), \Theta_\pm(\tilde{\sigma})] = \pm i \left( \Theta_\pm(\sigma) + \Theta_\pm(\tilde{\sigma}) \right) \delta'(\sigma - \tilde{\sigma}) \mp \frac{ic}{24\pi} \delta'''(\sigma - \tilde{\sigma})$$

$$[\Theta_\pm(\sigma), \Theta_\pm(\tilde{\sigma})] = 0$$
is recognized as the position-space version of the Virasoro algebra and the Schwinger term is just the Virasoro anomaly. Usually one does not find a field theoretic non-ghost realization without the Virasoro center; yet the CGHS model, without matter provides an explicit example. Usually one does not expect that all the Virasoro generators annihilate a state, but in fact our states (31) enjoy that property.

Once matter is added, a center appears, $c = 1$, and the theory becomes anomalous. In the same Schrödinger representation approach used above, one strategy is the following modification of a method due to Kuchař [7,10]. The Lagrange density (16) is presented in terms of decoupled constraints.

$$\mathcal{L} = \pi_a \dot{r}^a + \Pi \dot{\varphi} - \lambda^+ \Theta_+ - \lambda^- \Theta_-$$

Then the gravity variables $\{\pi_a, r^a\}$ are transformed by a linear canonical transformation to a new set $\{P_\pm, X^\pm\}$, in terms of which (34a) reads

$$\mathcal{L} = P_+ X^+ + P_- X^- + \Pi \dot{\varphi} - \lambda^+ \left( P_+ X^+ + \vartheta_+^{\text{matter}} \right) - \lambda^- \left( - P_- X^- + \vartheta_-^{\text{matter}} \right)$$

The gravity portions of the constraints $\Theta_\pm$ have been transformed to $\pm P_\pm X^\pm$ — expressions that look like momentum densities for fields $X^\pm$, and thus satisfy the $\Theta_\pm$ algebra (33) without center, as do also momentum densities, see (12a).

The entire obstruction in the full gravity plus matter constraints comes from the commutator of the matter contributions $\vartheta_\pm^{\text{matter}}$. In order to remove the obstruction, we modify the theory by adding $\Delta \Theta_\pm$ to the constraint $\Theta_\pm$, such that no center arises in the modified constraints. An expression for $\Delta \Theta_\pm$ that does the job is

$$\Delta \Theta_\pm = \frac{1}{48\pi} \left( \ln X^\pm \right)^{''}$$

Hence $\tilde{\Theta}_\pm \equiv \Theta_\pm + \Delta \Theta_\pm$ possess no obstruction in its algebra, and can annihilate states. Explicitly, the modified constraint equations read in the Schrödinger representation (after dividing by $X^\pm$)

$$\left( \frac{1}{i} \frac{\delta}{\delta X^\pm} \mp \frac{1}{48\pi X^\pm} \left( \ln X^\pm \right)^{''} \pm \frac{1}{X^\pm} \vartheta_\pm^{\text{matter}} \right) \psi(X^\pm, \varphi) = 0$$

It is recognized that the anomaly has been removed by introducing functional $U(1)$ connections in $X^\pm$ space, whose curvature cancels the anomaly. In the modified constraint there still is no mixing between gravitational variables $\{P_\pm, X^\pm\}$ and matter variables $\{\Pi, \varphi\}$. But the modified gravitational contribution is no longer quadratic — indeed it is non-polynomial — and we have no idea how to solve (37). We suspect, however, that just as its matter-free version, Eq. (37) possesses only a few solutions — far fewer than the rich spectrum that emerges upon BRST quantization, which we now examine.

In the BRST quantization method, extensively employed by string and conformal field theory investigators, one adds ghosts, which carry their own anomaly of $c_{\text{ghost}} = -26$. Also one improves $\Theta_\pm$ by the addition of $\Delta \Theta_\pm$ so that $c$ is increased; for example, with
\[ \Delta \Theta_\pm = \frac{Q}{\sqrt{4\pi}} (\Pi \pm \varphi')' \quad (38) \]
\[ c \rightarrow c + 3Q^2 \quad (39) \]

[The modification (38) corresponds to “improving” the energy momentum tensor by \((\partial_\mu \partial_\nu - g_{\mu\nu} \Box)\varphi\). The “background charge” \(Q\) is chosen so that the total anomaly vanishes.

\[ c + 3Q^2 + c_{\text{ghost}} = 0 \quad (40) \]

Moreover, the constraints are relaxed by imposing that physical states are annihilated by the “BRST” charges, rather than by the bosonic constraints. This is roughly equivalent to enforcing “half” the bosonic constraints, the positive frequency portions. In this way one arrives at a rich and well known spectrum.

Within BRST quantization, the negative signed gravitational field \(r^0\) is quantized so that negative norm states arise — just as in Gupta-Bleuler electrodynamics. [Negative norm states cannot arise in a Schrödinger representation, where the inner product is explicitly given by a (functional) integral, leading to positive norm.] One then finds \(c_0 = 1\); the center is insensitive to the signature with which fields enter the action. As a consequence, \(c^{\text{gravity}} = c_0 + c_1 = 2\) so that even pure gravity constraints possess an obstruction.

Evidently, pure gravity with \(c^{\text{gravity}} = 2\) requires \(Q = 2\sqrt{2}\). The rich BRST spectrum is much more plentiful than the two states (31) found in the Schrödinger representation and does not appear to reflect the fact that the classical pure gravity theory is without excitations.

Gravity with matter carries \(c = 3\), and becomes quantizable at \(Q = \sqrt{23}/3\). Once again a rich spectrum emerges, but it shows no apparent relation to a particle spectrum.

**CONCLUSIONS**

Without question, the CGHS model, and other similar two-dimensional gravity models, are afflicted by anomalies in their constraint algebras, which become second-class and frustrate straightforward quantization. While anomalies can be calculated and are finite, their specific value depends on the way singularities of quantum field theory are resolved, and this leads to a variety of procedures for overcoming the problem and to a variety of quantum field theories, with quite different properties.

Two methods were discussed: (i) a Schrödinger representation with Kuchař-type improvement as needed, i.e. when matter is present, and (ii) BRST quantization. (Actually several other approaches are also available [7].) Only in the first method for pure matterless gravity, with positive norm states and vanishing anomaly, does the quantum theory bear any resemblance to the classical theory, in the sense that the classical gravity theory has no propagating degrees of freedom, while the quantum Hilbert space has only the two states in (31), neither of which contains any further degrees of freedom. In other cases, e.g. with matter, the classical picture of physics seems irrelevant to the behavior of the quantum theory.

Presumably, if anomalies were absent, the different quantization procedures (Schrödinger representation, BRST, ...) would produce the same physics. However, the anomalies are present and interfere with equivalence.
Finally, we remark that our investigation has exposed an interesting structure within Virasoro theory: there exists a field theoretic realization of the algebra without the anomaly, in terms of spinless fields and with no ghost fields. Moreover, there are states that are annihilated by all the Virasoro generators.

What does any of this teach us about the physical four-dimensional model? We believe that an extension in the constraint algebra will arise for all physical, propagating degrees of freedom: for matter fields, as is seen already in two dimensions, and also for gravity fields, which in four dimensions (unlike in two) carry physical energy. How to overcome this obstruction to quantization is unclear to us, but we expect that the resulting quantum theory will be far different from its classical counterpart. Especially problematic is the fact that flat-space calculations of anomalous Schwinger terms in four dimensions yield infinite results, essentially for dimensional reasons. Moreover, it should be clear that any announced “solutions” to the constraints that result from formal analysis must be viewed as preliminary: properties of the Hilbert space and of the inner product must be fixed first in order to give an unambiguous determination of any obstructions.

We believe that our two-dimensional investigation, although in a much simpler and unphysical setting, nevertheless contains important clues for realistic theories. Certainly that was the lesson of gauge theories: anomalies and vacuum angle have corresponding roles in the Schwinger model and in QCD!
REFERENCES

[1] See R. Jackiw, in Quantum Mechanics of Fundamental Systems, C. Teitelboim, ed. (Plenum, New York NY, 1988).

[2] D. Boulware and S. Deser, J. Math. Phys 8, 1468 (1967).

[3] R. Jackiw, C. Teitelboim, in Quantum Theory of Gravity, S. Christensen, ed. (A. Hilger, Bristol UK, 1984).

[4] D. Cangemi, Phys. Lett. B297, 261 (1992); A. Achúcarro, Phys. Rev. Lett. 70, 1037 (1993).

[5] A. Polyakov, Mod. Phys. Lett. A2, 893 (1987), Gauge Fields and Strings (Harwood, New York NY, 1987).

[6] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. D 45 1005 (1992); H. Verlinde in Sixth Marcel Grossmann Meeting on General Relativity, M. Sato and T. Nakamura, eds. (World Scientific, Singapore, 1992).

[7] Our most recent paper on this entire subject, with reference to earlier work, is D. Cangemi, R. Jackiw and B. Zwiebach, Ann. Phys. (NY) (in press).

[8] P. Jordan, Z. Phys. 93, 464 (1935). In the context of the energy-momentum tensor anomaly, this is explained in R. Floreanini and R. Jackiw, Phys. Lett. B175, 428 (1986).

[9] D. Cangemi and R. Jackiw, Phys. Lett. B337, 271 (1994), Phys. Rev. D 50, 3913 (1994); D. Amati, S. Elitzur and E. Rabinovici, Nucl. Phys. B418, 45 (1994); D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B321, 193 (1994); E. Benedict, Phys. Lett. B340, 43 (1994); T. Strobl, Phys. Rev. D 50, 7346 (1994).

[10] K. Kuchař, Phys. Rev. D 39, 2263 (1989); K. Kuchař and G. Torre, J. Math. Phys. 30, 1769 (1989).