Counting BPS Blackholes in Toroidal Type II String Theory

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We derive a $U$-duality invariant formula for the degeneracies of BPS multiplets in a D1-D5 system for toroidal compactification of the type II string. The elliptic genus for this system vanishes, but it is found that BPS states can nevertheless be counted using a certain topological partition function involving two insertions of the fermion number operator. This is possible due to four extra toroidal U(1) symmetries arising from a Wigner contraction of a large $\mathcal{N} = 4$ algebra $A_{\kappa,\kappa'}$ for $\kappa' \to \infty$. We also compare the answer with a counting formula derived from supergravity on $AdS_3 \times S^3 \times T^4$ and find agreement within the expected range of validity.
1. Introduction

Supersymmetric indices have proven to be invaluable in the program of accounting for black-hole entropy using D-branes [1]. In particular, in those cases where the computation of BPS black holes can be related to counting functions in a conformal field theory, the elliptic genus has been of particular use. Nevertheless, there are examples, notably toroidal compactification of type II string, where the relevant elliptic genus vanishes, thus giving little indication about the D-brane BPS state degeneracies. Perhaps surprisingly, the degeneracies are therefore more subtle for compactification on $T^4$ than for $K3$. These degeneracies were first seriously investigated in [2][3][4]. In this paper we study these degeneracies further in the case of the three charge system of [1] consisting of $Q_1 D1$-branes, $Q_5$, $D5$-branes and momentum $N$. Using a function closely related to the elliptic genus we derive $E_6,6(\mathbb{Z}) U$-dual expressions for the case of primitive charges, i.e., charges such that $gcd(Q_1, Q_5, N) = 1$. The formula is given in equations (6.2), (6.3) below and is easily derived from our central result, the counting formula for $1/8$ BPS states given in equation (5.9) below, valid for $gcd(Q_1, Q_5) = 1$.

Our approach to the problem is to define an “index” in the same spirit as the “new supersymmetric index” of [5]. These authors investigated the traces in supersymmetric quantum mechanics defined by:

$$E_\ell = \text{Tr}_H (-1)^F F^\ell e^{-\beta H}, \quad (1.1)$$

where $H$ is the Hamiltonian and $F$ is a fermion number operator. For $\mathcal{N} = 2$ supersymmetric theories one can take $F$ to be the generator of a $U(1)$ invariance and the “index” with $\ell = 1$ is invariant under perturbations of $D$-terms (but not $F$-terms). Moreover, in general $E_\ell$ has no special invariances for $\ell \geq 2$. In this paper we consider the case $\ell = 2$ in the context of certain conformal field theories. In the problem of interest we have some extra symmetry, namely the four $U(1)$ translation symmetries of the torus. The full symmetry is a Wigner contraction of the large $\mathcal{N} = 4$ supersymmetry algebra $\mathcal{A}_{\kappa, \kappa'}$ [6]. We show that the presence of this large $\mathcal{N} = 4$ algebra leads to invariance of $E_{\ell=2}$ under a class of perturbations discussed below. From the point of view of the five dimensional theory these indices are particular cases of supertrace formulas [4], which are invariant under deformations of the theory.
2. Setting the Stage

We will consider black strings in 6D compactification of IIB theory on $T^4$ and the black holes in 5D compactifications on $T^5$ obtained by wrapping these strings. In this section we summarize some standard facts about $U$-duality. See [8] for background.

The low energy theory of IIB on $T^5$ is given by the 32-supercharge supergravity supermultiplet. This has 27 gauge fields and 42 scalars. The scalar moduli space is $E_{6,6}(\mathbb{R})/USp(8)$. We will work in a regime of moduli space where $T^5 = S^1 \times T^4$ is metrically a product with a large radius for the $S^1$. Moreover, we assume there are no Wilson lines (of 6D gauge fields) along the $S^1$. This submanifold of moduli space is described by the moduli of 6D compactification

$$[O(5, 5; \mathbb{R})/(O(5) \times O(5))] \times \mathbb{R}^+ \tag{2.1}$$

where the last factor is the radius of the large $S^1$. A subgroup of the $U$-duality group preserving this submanifold is $O(5, 5; \mathbb{Z})$ (not to be confused with the Narain duality group in 5D).

In 5D there are particles charged under the 27 gauge fields. Their charges form the $\mathbb{Z}^{27}$ representation of $E_{6,6}(\mathbb{Z})$. Since the $U$-duality symmetry is broken to $O(5, 5; \mathbb{Z})$ along (2.1) the 5D particle charges accordingly decompose as the representation:

$$\mathbb{Z}^{27} \to II^{5,5} \oplus \mathbb{Z}^{16} \oplus \mathbb{Z} \tag{2.2}$$

of $O(5, 5; \mathbb{Z})$. These representations have the following interpretations. The lattice $II^{5,5}$ is the electric/magnetic charge lattice of 6D strings. The representation $\mathbb{Z}^{16}$ corresponds to the 6D charges of particles. Finally, the singlet $\mathbb{Z}$ is the momentum along the large circle.

We will denote a 5d charge vector in this decomposition as $\gamma = (S; P; N)$.

We are interested in charged black holes arising from wrappings of 6D strings on the large $S^1$, and in their BPS excitations. In the following sections we will count these BPS excitations using a mapping to instanton moduli space sigma models. We will then verify that this counting is invariant under a certain subgroup of the $U$-duality group $E_{6,6}(\mathbb{Z})$. To explain this subgroup we need to understand the physics of the three summands in (2.2).

The first summand is the charge lattice of 6D strings (general considerations show it is a lattice, i.e., has a symmetric nondegenerate bilinear form [8]). We can write $II^{5,5} \cong H_{\text{even}}(T^4) \oplus II^{1,1}$. Corresponding to the decomposition in terms of D-branes
and (fundamental strings, wrapped NS5 branes), respectively. We can further decompose
\[ H_{\text{even}}(T^4) = (H_0 \oplus H_4) \oplus H_2 \cong I I^{1,1} \oplus I I^{3,3} \]
corresponding to a natural basis of D1 strings parallel to the large \( S^1 \), wrapped D5 branes, and wrapped D3-branes, respectively.

The particle charges \( P \) in 6D form the spinor representation \( \mathbb{Z}^{16} \) of \( O(5, 5; \mathbb{Z}) \). Writing
the decomposition under the \( O(4, 4; \mathbb{Z}) \) Narain subgroup this decomposes as \( \mathbb{Z}^{16} = I I^{4,4} \oplus H_{\text{odd}}(T^4; \mathbb{Z}) \), corresponding to momentum, fundamental string winding, and wrapping of D1, D3 branes. In this paper we often take \( P = 0 \).

Now let us consider \( U \)-duality. Let us first assume the string charge \( S \in I I^{5,5} \) is a primitive vector. It is then a standard result of lattice theory (see, e.g. \[10\], Theorem 1.1.2 or Theorem 1.14.4) that all primitive vectors \( S \in I I^{5,5} \) of a given length are equivalent under \( O(5, 5; \mathbb{Z}) \). Since \[10\] uses some heavy machinery it is worth giving the following elementary example of this phenomenon. We may identify the lattice \( I I^{2,2} \) with the set of integral \( 2 \times 2 \) matrices. The signature \( (2, 2) \) quadratic form is simply the determinant. The \( O(2, 2; \mathbb{Z}) \) automorphism group acts by left- and right-multiplication by \( SL(2, \mathbb{Z}) \):

\[
M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A_L \begin{pmatrix} a & b \\ c & d \end{pmatrix} A_R
\]

Now, using the standard fact that if \( \gcd(a, b) = 1 \) then there exist \( p, q \) with \( ap + bq = 1 \), it is easy to show that \( M \) can be bidiagonalized over \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) to Smith normal form:

\[
M \cong \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix}
\]

Thus, if \( M \) is primitive then the only invariant is the determinant, i.e., the norm-square. In a similar way, if \( S \in I I^{5,5} \) is primitive we can, without loss of generality, put it in the form
\[
S = (Q_1, Q_5) \oplus \bar{0} \oplus (0, 0)
\]
with \( \gcd(Q_1, Q_5) = 1 \) (These are the cases for which there is a sigma model description). In other words, we can map any general string charge into a D1-D5 system. We then simply write \( S = (Q_1, Q_5) \) and henceforth consider the charge vectors

\[
\gamma = (Q_1, Q_5; P; N). \tag{2.5}
\]

Charge vectors of the form (2.5) are special because states with these charges can be described using an instanton sigma model as in the original discussion of \[1\]. It follows that invariance of physical quantities under \( U \)-duality transformations which preserve the form (2.5) can lead to nontrivial predictions for the instanton sigma model. For simplicity
we will henceforth consider only those charges $\gamma$ which can be mapped to the standard 3-charge system $\gamma = (Q_1, Q_5; \vec{0}; N)$ of $\mathbb{Z}$. 

The $U$-duality transformations preserving the 3-charge system $\gamma = (Q_1, Q_5; \vec{0}; N)$ form a subgroup

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \ltimes S_3 \subset E_{6,6}(\mathbb{Z}).$$

This group is generated by 3 transformations:

$$\mathcal{R} : (Q_1, Q_5; \vec{0}; N) \to (-Q_1, Q_5; \vec{0}; -N)$$

$$\mathcal{T} : (Q_1, Q_5; \vec{0}; N) \to (Q_5, Q_1; \vec{0}; N)$$

$$\mathcal{T'} : (Q_1, Q_5; \vec{0}; N) \to (N, Q_5; \vec{0}; Q_1)$$

The transformation $\mathcal{R}$ is simply a rotation by $\pi$ and is certainly an invariance of the sigma-model. Also, $\mathcal{T}$ is an order two element of the Narain duality group $O(4,4; \mathbb{Z})$ corresponding to $T$-duality in all four directions. This is supposed to be a symmetry of the conformal field theory on the instanton moduli space. However $\mathcal{T'}$ is not an invariance of the instanton sigma model. This is an “STS” type transformation in 5D which is not in $O(5,5; \mathbb{Z})$. Thus, the nontrivial predictions of $E_{6,6}(\mathbb{Z})$ $U$-duality for the instanton sigma model are reduced to checking invariance under (2.9). This is what we will check below for degeneracies of BPS states, when $Q_1, Q_5$ are relatively prime.

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1 Whether an arbitrary charge $\gamma$ can be so mapped is a subtle arithmetic question, but the answer is probably that every charge vector is equivalent to a 3-charge system, at least if the cubic invariant is nonzero. The strategy for showing this is the following (we have not carried out all the details). Using the description of [11] this is equivalent to diagonalizability of $3 \times 3$ Hermitian matrices over the integral split octonions $\tilde{O}_{\mathbb{Z}}$ using $E_{6,6}(\mathbb{Z})$ transformations. It is straightforward to show that for any $\gamma$ there is in fact a 3-charge system $\gamma'$ such that $\gamma \cong \gamma'$ $p$-adically for all $p$. Using some facts about the topology of $F_{4,4}$ and a result from number theory called the “strong approximation theorem,” the necessary Hasse-Minkowski local \rightarrow global principle can be justified, so the matrix can in fact be diagonalized over $\mathbb{Z}$. We thank B. Gross for very helpful comments on this problem.
3. The Instanton Sigma Model and its Superconformal Symmetry

Now we consider the standard $D1D5$ system as an effective string in the $05$ direction. For the present purposes we will approximate the CFT for the low energy excitations of the theory on the string by a supersymmetric sigma model [3]:

$$\sigma[\mathbb{R}^4 \times T^4] \times \sigma[\text{Hilb}^k(T^4)].$$

(3.1)

Here $\sigma(X)$ denotes a supersymmetric sigma model with target space $X$, and $k = Q_1 Q_5$. The factor $\sigma[\mathbb{R}^4 \times T^4]$ is the free sigma model from the diagonal $U(1)$ factor in the $U(Q_1) \times U(Q_5)$ gauge symmetry. The other degrees of freedom come from the hypermultiplets of interacting $D1D5$ degrees of freedom. Their target space is approximated by $\text{Hilb}^k(T^4)$, the Hilbert scheme of $k$ points on $T^4$. This is a smooth resolution of the singular orbifold $\text{Sym}^k(T^4)$, and is endowed with a smooth hyperkähler metric.

It is important to realize that the innocent-looking (3.1) has several subtleties. First of all, there should be an orbifold by certain translation symmetries. Because of a restriction to a charge zero sector, described below, this can be ignored. Furthermore, we will be working at a point in moduli space where the D1 branes cannot leave the fivebranes. At some special points in moduli space, for example when all B-fields are zero, the D1 branes can leave the system and the CFT becomes singular.

The symmetries of the CFT can be deduced from standard Dbrane technology. We assume the D1 string is in the $05$ direction and the D5 wraps the $T^4$ and is in the $056789$ direction. The spinors, which initially transform in the $16_+$ of the ten dimensional Lorentz group now transform under

$$\text{Spin}(1,1)_{05} \times [SU(2)^+ \times SU(2)^-]_{1234} \times [SU(2)^+ \times SU(2)^-]_{6789}.$$  

(3.2)

Note that the last factor is not really a full symmetry of the CFT since we are on $T^4$, but it is useful to classify spinors. The ten-dimensional supersymmetries are in the $16_+$, but only those invariant under $SU(2)_{6789}$ survive, i.e. only the ones with positive chirality in the $051234$ directions. Thus the unbroken supersymmetry is in the representation

$$\left(\frac{1}{2}; 1, 1, 1\right) \oplus \left(-\frac{1}{2}; 1, 2, 1\right)$$

(3.3)

In 1+1 dimensions $\pm \frac{1}{2}$ chiralities correspond to left and right movers, so we see that we get (4,4) supersymmetry. We also see that spacetime rotations in the directions 1234 act
as R-symmetries of this conformal field theory. Since we have 8 supersymmetries we can denote the two possible multiplets as vectors and hypers. From the center-of-mass (COM) CFT $\sigma(\mathbb{R}^4 \times T^4)$ we get a vector and a hyper. The vector describes motion in $\mathbb{R}^4$ and the hyper describes motion in $T^4$.

The left-moving part of the vector multiplet transforms as:

$$X \in (0; 2, 2; 1, 1)$$
$$\lambda \in (+\frac{1}{2}; 1, 2; 2, 1) \quad (3.4)$$

and similarly for the right-moving part exchanging the $SU(2)_{1234}$ factors from $[3.2]$.

The left-moving part of the hypermultiplet describing motion on $T^4$ transforms as

$$X \in (0; 1, 1; 2, 2)$$
$$\lambda \in (+\frac{1}{2}; 2, 1; 1, 2) \quad (3.5)$$

The D1D5 strings give hypermultiplets $(h, \psi)$ transforming as

$$h \in (0; 1, 1; 2, 1)$$
$$\psi \in (+\frac{1}{2}; 2, 1; 1, 1) \quad (3.6)$$

The full CFT $[3.1]$ has a global $SU(2)^{+}_{1234} \times SU(2)^{-}_{1234}$ symmetry corresponding to space-time rotations. This is the massive little group of particles in 5D and will be used below to enumerate BPS representations. The quantum numbers of the fields under this symmetry follow from $[3.4][3.5](3.6)$. Note that for the $\mathbb{R}^4$ factor the bosons transform under the global symmetry. Note also that all hypermultiplets $[3.5][3.6]$ transform in the same way under $SO(1, 1) \times SU(2)^{+}_{1234} \times SU(2)^{-}_{1234}$ and in a different way from the vector multiplets $[3.4]$. This difference is what distinguishes a vectormultiplet from a hypermultiplet in 1+1 dimensions.

For the $T^4$ and $\text{Sym}^k(T^4)$ factors the $SU(2)^{+}_{1234} \times SU(2)^{-}_{1234}$ are zeromodes of left and right-moving $SU(2)$ current algebras of level $k$ which are part of the left- and right-moving $\mathcal{N} = 4$ superconformal algebra. In fact, in the example of toroidal compactification there is a larger superconformal algebra. This arises because there is a $U(1)^4$ current algebra which commutes with the $SU(2)_k$, and can be understood as follows. The unsymmetrized product of $k$ copies of $T^4$ has four currents which generate simultaneous translation along the four axes of all $k$ copies of $T^4$. These four currents are permutation invariant and
therefore descend to four \( U(1) \) currents in the orbifold theory on \( \text{Sym}^k(T^4) \). The resolved conformal field theory on \( \text{Hilb}^k(T^4) \) is determined by twenty parameters (= 4\( h_{1,1} \)) which determine the complex structure, Kahler class and \( B \)-fields [12]. Sixteen of these are essentially associated to each \( T^4 \) and the last four are involved in blowing up the orbifold points. The values of these 20 parameters are invariant under the \( U(1)^4 \) action. Therefore, the \( U(1)^4 \) current algebra survives the resolution of \( \text{Sym}^k(T^4) \) to \( \text{Hilb}^k(T^4) \).

Put more geometrically, the resolution \( p : \text{Hilb}^k(T^4) \rightarrow \text{Sym}^k(T^4) \) only depends on local data (such as the direction along which points approach each other at the orbifold loci) so the obvious translation symmetry of \( \text{Sym}^k(T^4) \) lifts to an action of \( U(1)^4 \) on \( \text{Hilb}^k(T^4) \).

\( U(1)^4 \) can be regarded as the \( \kappa' \rightarrow \infty \) limit of \( SU(2)_{\kappa'} \times U(1) \). Since the large \( \mathcal{N} = 4 \) current algebra is \( SU(2)_{\kappa} \times SU(2)_{\kappa'} \times U(1) \), we conclude that \( \text{Hilb}^k(T^4) \) conformal field theory has a degenerate large \( \mathcal{N} = 4 \) algebra, \( \mathcal{A}_{\kappa,\infty} \). (In the following we will sometimes abuse language and refer to \( \mathcal{A}_{\kappa,\infty} \) as a large \( \mathcal{N} = 4 \) algebra.)

In the study of 5D black holes in \( S^1 \times K3 \) compactifications a key role was played by the elliptic genus for \( \mathcal{N} = 2 \) conformal field theories defined by [13]

\[
\mathcal{E} := \text{Tr}[(-1)^{2J^3_0 - 2\tilde{J}^3_0} q^{L_0} \bar{q}^{\bar{L}_0} y^{2J^3_0}],
\]

(3.7)

where \( J^3_0 \) and \( \tilde{J}^3_0 \) are the half-integral left and right \( U(1) \) charges. Here and henceforth we normalize \( L_0 \) so that the Ramond ground states have \( L_0 = 0 \). The elliptic genus \( \mathcal{E} \) is a useful object because it is invariant under all smooth deformations of the theory. The trace is taken in the RR sector of the conformal field theory. Of course, it can also be defined for \( \mathcal{N} = 4 \) theories by embedding the \( U(1) \) charges in \( SU(2) \). But in theories having large \( \mathcal{N} = 4 \) symmetry it is not useful because it always vanishes. We will now show that the modified partition function

\[
\mathcal{E}_2 := \text{Tr}[(-1)^{2J^3_0 - 2\tilde{J}^3_0} (2J^3_0)^2 q^{L_0} \bar{q}^{\bar{L}_0} y^{2J^3_0}],
\]

(3.8)

is an analogous topological invariant for theories with the large \( \mathcal{N} = 4 \) symmetry. (Note that \( \mathcal{E}_1 = 0 \), and indeed, \( \text{Tr}(J^3_0)^n = 0 \) in any \( SU(2) \) representation, for \( n \) odd.) This amounts to showing that the massive representations of this degenerate large \( \mathcal{N} = 4 \) algebra do not contribute to \( \mathcal{E}_2 \). Consider the subalgebra generated by the Ramond-sector zero mode
generators $G_0^{±}, Q_0^{±}, J_0^3$ and $L_0$. Since $L_0$ commutes with the rest of the generators we can just think of it as a c-number. The relevant commutation relations are
\[
\begin{align*}
\{G_0^{++}, G_0^{--}\} &= 2L_0, \quad \{G_0^{+-}, G_0^{-+}\} = 2L_0 \\
\{Q_0^{++}, Q_0^{--}\} &= 1, \quad \{Q_0^{+-}, Q_0^{-+}\} = 1, \\
[J_0^3, G_0^{±−}] &= ±\frac{1}{2}G_0^{±−}, \quad [J_0^3, G_0^{±+}] = ±\frac{1}{2}G_0^{±+} \\
[J_0^3, Q_0^{±−}] &= ±\frac{1}{2}Q_0^{±−}, \quad [J_0^3, Q_0^{±+}] = ±\frac{1}{2}Q_0^{±+}.
\end{align*}
\] (3.9)

The rest of the commutators, including those of $G$’s with $Q$’s, vanish if we consider states neutral with respect to $U(1)^4$, i.e. with no momentum or winding on $T^4$. The general case will be discussed momentarily.

For a massive representation, by definition $L_0 > 0$. This implies that the commutation relations of the $G$’s and $Q$’s are those of fermionic creation and annihilation operators. We have four creation operators $b_i^\dagger$ which we choose to have $J_0^3 = 1/2$. The annihilation operators then have $J_0^3 = −1/2$. Let $|0, j\rangle$ denote the state that is annihilated by all the annihilation operators and obeys $J_0^3|0, j\rangle = j|0, j\rangle$ for some $j$. Acting with the creation operators we get four states with $J_0^3 = j + 1/2$, six states with $J_0^3 = j + 1$, four with $J_0^3 = j + 3/2$ and one with $J_0^3 = j + 2$. The fermion numbers of these states alternate. It is easy to check that the traces over this zero mode representation $\text{Tr}_j(−1)^F = \text{Tr}_j(−1)^{2J_0^3}$ as well as $\text{Tr}_j(−1)^{2J_0^3}J_0^3$ vanish. One also finds by direct computation
\[
\text{Tr}_j(−1)^{2J_0^3}(J_0^3)^2 \propto j^2 − 4(j + \frac{1}{2})^2 + 6(j + 1)^2 − 4(j + \frac{3}{2})^2 + (j + 2)^2 = 0.
\] (3.10)

We conclude the massive representations do not contribute to $\mathcal{E}_2$.

If we now relax the assumption that the $U(1)^4$ charges vanish, then the anti commutation relations of the $G$’s and $Q$’s (denoted collectively as $b_i, b_i^\dagger, i = 1, \cdots, 4$) are of the form
\[
\{b_i, b_j\} = 0, \quad \{b_i^\dagger, b_j^\dagger\} = 0, \quad \{b_i, b_j^\dagger\} = M_{ij}
\] (3.11)

where $M_{ij}$ is an Hermitian matrix which depends on $L_0$ and the four $U(1)$ charges. We can diagonalize $M$ by a unitary transformation. If the eigenvalues of $M$ are all non-zero, then

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2 Our notation is as follows. $G_0^{±±}$ are the supercharges, and the first ± superscript indicates the charge under $J_0^3$ of $SU(2)_k$. $Q_0^{±±}$ are the fermionic partners of the $U(1)^4$ current algebra. $(G_0^{±±})^\dagger = G_0^{±±}$ and $(Q_0^{±±})^\dagger = Q_0^{±±}$.
the $b$’s are usual creation and annihilation operators and the trace of $(-1)^{2J_3} J_3^2$ vanishes. This is the case when $L_0 > \sum_{i=1}^{4} u_i^2$ where $u_i$ are the eigenvalues of the four U(1) charges (appropriately normalized). If $M$ has zero eigenvalues, this is no longer the case. This happens when $L_0 = \sum_{i=1}^{4} u_i^2$. It would be very interesting to understand these BPS states carrying additional charges. In this paper, however, we concentrate on the case where all these charges are zero.

For non-degenerate large $\mathcal{N} = 4$ algebras $A_{\kappa, \kappa'}$ with finite $SU(2)$ levels $\kappa$ and $\kappa'$, the commutators of $G_0$ and $Q_0$ have $SU(2)_\kappa \times SU(2)_{\kappa'}$ current algebra zero modes on the right hand side. This complicates the preceding argument. However in this case one may conclude from direct examination of formulae in [14] that the massive characters do not contribute to $E_2$. This implies that the index (3.8) will be useful to analyze the conformal field theory related to $AdS_3 \times S^3 \times S^3 \times S^1$ [13]. In fact it would be very interesting to compute the supergravity result since it could teach us something about the dual conformal field theory.

As we shall see shortly, the massless characters with $L_0 = 0$ do contribute to $E_2$. This contribution is independent of the continuous parameters describing the resolution of $\text{Sym}^k(T^4)$ to $\text{Hilb}^k(T^4)$. Hence we can compute $E_2$ for all cases from the limiting case of $\text{Sym}^k(T^4)$.

4. Counting spacetime BPS multiplets

In this section we explain the spacetime interpretation of (3.8). The D1D5 system on $S^1 \times T^4$ and its excitations describe particles in 5 dimensions. These all transform in representations of the 5d Poincaré supersymmetry algebra with 32 real supercharges. The different representations can be characterized by the $Spin(4)_{1234}$ characters

$$\chi(y, \tilde{y}) := \text{Tr}_{H_{\text{little}}} (-1)^{F_{\text{spacetime}}} y^{2J_0^3} \tilde{y}^{2\tilde{J}_0^3}$$

of the representation of the little superalgebra. The long representations built with 32 active (i.e. broken) supercharges have character

$$\chi_{0/32}(y, \tilde{y}) = \chi_{jL}(-y) \chi_{jR}(-\tilde{y}) (y^{1/2} - y^{-1/2})^8 (\tilde{y}^{1/2} - \tilde{y}^{-1/2})^8.$$  

(4.2)

Here the subscript indicates the number of preserved supercharges, $(j_L, j_R)$ are arbitrary half-integral spins, and

$$\chi_{j}(y) = y^{-2j} + y^{-2j+2} + \cdots + y^{2j} = \frac{y^{2j+1} - y^{-2j-1}}{y - y^{-1}}.$$  

(4.3)
The BPS states we will encounter in the D1D5 system come in three kinds of short representations:

A. \( M = Z_1 \), \( M \neq |Z_i|, i > 1 \), where \( Z_i \) are the skew eigenvalues of the central charge matrix. The characters are

\[
\chi_{4/32}^+ = x_{jL}(-y)x_{jR}(-\tilde{y})(y^{1/2} - y^{-1/2})^8(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^6
\]  

(4.4)

B. If instead \( M = -Z_1 \) we get

\[
\chi_{4/32}^- = x_{jL}(-y)x_{jR}(-\tilde{y})(y^{1/2} - y^{-1/2})^6(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^8
\]  

(4.5)

C. Finally, a shorter representation has character

\[
\chi_{8/32} = x_{jL}(-y)x_{jR}(-\tilde{y})(y^{1/2} - y^{-1/2})^6(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^6.
\]  

(4.6)

U-duals of massive Dabholkar-Harvey states turn out to be in representations of type C. There are also other BPS states in other representations for example 1/2 BPS states, etc.

We now discuss how these characters show up in CFT partition functions. In general for the CFT \( \sigma(X) \) we denote

\[
Z(\sigma(q, y, \bar{q}, \tilde{y})) = \text{Tr}_{RR}(-1)^{2j_0^3 - 2\tilde{j}_0^3}q^{J_0}q^{\tilde{J}_0}y^{2j_0^3}\tilde{y}^{2\tilde{j}_0^3}
\]  

(4.7)

For the conformal field theory (3.1) this trace is a product of two factors: One for the COM degrees of freedom and one for the CFT \( \sigma(\text{Hilb}^k(T^4)) \). The first factor can be computed straightforwardly in terms of oscillators using the quantum numbers (3.4)(3.5)(3.6). We will discuss the second factor in section five.

The trace (4.7) for the CFT (3.1) can be decomposed in terms of the characters of the massive little superalgebra:

\[
Z_{\sigma}(q, y, \bar{q}, \tilde{y}) = \sum_{jL, jR} \chi_{8/32}(y, \tilde{y})D_{8/32}(Q_1, Q_5; jL, jR)
\]

\[
+ \sum_{N=1}^{\infty} \sum_{jL, jR} q^N \chi_{4/32}^+(y, \tilde{y})D_{4/32}^+(Q_1, Q_5, N; jL, jR)
\]

\[
+ \sum_{\tilde{N}=1}^{\infty} \sum_{jL, jR} q^{\tilde{N}} \chi_{4/32}^-(y, \tilde{y})D_{4/32}^-(Q_1, Q_5, \tilde{N}; jL, jR)
\]

\[
+ \sum_{\Delta, \tilde{\Delta} > 0} \sum_{jL, jR} q^{\Delta}q^{\tilde{\Delta}} \chi_{0/32}(y, \tilde{y})D_{0/32}(Q_1, Q_5, \Delta, \tilde{\Delta}; jL, jR)
\]  

(4.8)

\[\text{Again, we ignore a discrete translation orbifold action.}\]
Comparing with (4.8) we finally obtain the desired counting formula for representations:

\[
\text{where } (\Delta, \bar{\Delta}) \text{ run over the massive spectrum of the CFT (3.1).}
\]

In (4.8) the \( D \)'s measure the degeneracies of various types of representations of the spacetime \( D = 5, N = 4 \) superalgebra. In particular, \( D_{8/32}(Q_1, Q_5; j_L, j_R) \) is the number of BPS multiplets of charge \( (Q_1, Q_5, N = 0) \) in the representation (4.6). \( D^+_{4/32}(Q_1, Q_5, N; j_L, j_R) \) is the number of BPS multiplets of charge \( (Q_1, Q_5, N > 0) \) in the representation (4.4). These are macroscopically black holes with positive horizon area, etc.

Note that part of the structure of (4.8) as a function of \( y, \tilde{y} \) follows from the representation theory of the algebra \( A_{\kappa, \infty} \). From the COM sigma model we have an overall factor of \( (y^{1/2} - y^{-1/2})^4(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^4 \). Then, in the \( \text{Sym}^k(T^4) \) sigma model we have \( \text{Tr}(-1)^F \ell = 0 \) for \( \ell = 0, 1 \) and therefore there is an extra factor of \( (y^{1/2} - y^{-1/2})^2(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^2 \) coming from this piece. For massive reps \( \Delta > 0 \) of \( A_{\kappa, \infty} \) we showed in section three that in fact \( \text{Tr}(-1)^F \ell = 0 \) for \( \ell = 0, 1, 2, 3 \) and therefore reps with \( \Delta > 0, \bar{\Delta} = 0 \) give a factor of \( (y^{1/2} - y^{-1/2})^2(\tilde{y}^{1/2} - \tilde{y}^{-1/2})^2 \), etc.

In order to give a counting formula for BPS multiplets we should take a derivative of (4.8) by \( \frac{1}{6!}(\frac{d}{dy})^6 \) at \( \tilde{y} = 1 \). From the CFT of the sigma model \( \sigma(\mathbb{R}^4 \times T^4) \times \sigma(\text{Sym}^k(T^4)) \) we need 4 derivatives to act on the COM part of the sigma model and 2 derivatives to act on the \( S^k(T^4) \) part. There is a surprising cancellation of the COM contributions from \( \mathbb{R}^4 \) and \( T^4 \) after setting \( \tilde{y} = 1 \) and the \( D1D5 \) CFT gives simply:

\[
(y^{1/2} - y^{-1/2})^4 \frac{1}{2} (\frac{d}{dy})^2 \bigg|_{\bar{y}=1} \mathcal{Z}(S^k(T^4)). \tag{4.9}
\]

Comparing with (4.8) we finally obtain the desired counting formula for representations:

\[
\frac{1}{2} (\frac{d}{dy})^2 \bigg|_{\bar{y}=1} \mathcal{Z}(S^k(T^4)) =
\]

\[
(y^{1/2} - y^{-1/2})^2 \sum_{j_L, j_R} \chi_{j_L}(\bar{y})(-1)^{2j_R}(2j_R + 1)D_{8/32}(Q_1, Q_5; j_L, j_R)
\]

\[
+(y^{1/2} - y^{-1/2})^4 \sum_{N=1}^{\infty} \sum_{j_L, j_R} q^N \chi_{j_L}(\bar{y})(-1)^{2j_R}(2j_R + 1)D^+_{4/32}(Q_1, Q_5, N; j_L, j_R) \tag{4.10}
\]

5. Computation of \( E_{\ell=2} \)

In this section we evaluate \( E_2 \) more explicitly. In [16] a general formula was derived relating the partition function for a conformal field theory with target \( X \) to that of a
conformal field theory whose target is the orbifold Sym$^k(X)$. The partition function for a single copy of $X$ defines the degeneracies $c(\Delta, \bar{\Delta}, \ell, \bar{\ell})$ via:

$$Z(X) = \sum_{\Delta, \bar{\Delta}, \ell, \bar{\ell}} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) q^\Delta \bar{q}^{\bar{\Delta}} y^\ell \bar{y}^{\bar{\ell}}. \quad (5.1)$$

Here the trace is in the RR sector. The spectrum of $U(1)$ charges $\ell, \bar{\ell}$ is integer or half-integer, according to the parity of the complex dimension of $X$ and $\Delta, \bar{\Delta}$ runs over the spectrum of $L_0, \bar{L}_0$. The values of $\Delta, \bar{\Delta}$ are in general arbitrary nonnegative real numbers, although the difference $\Delta - \bar{\Delta}$ is integral.

In terms of $c$, the partition function over Sym$^k(X)$ may be derived using a small modification of the discussion in \[16\], and is:

$$Z(p, q, \bar{q}, y, \bar{y}) := \sum_{k=0}^{\infty} p^k Z(\text{Sym}^k(X)) = \prod_{n=1}^{\infty} \prod_{\Delta, \bar{\Delta}, \ell, \bar{\ell}}' \frac{1}{(1 - p^n q^n \bar{q}^n y^n \bar{y}^n)^{c(\Delta, \bar{\Delta}, \ell, \bar{\ell})}} \quad (5.2)$$

where the prime on the product indicates that $\Delta, \bar{\Delta}$ are restricted so that $\frac{\Delta - \bar{\Delta}}{n}$ is an integer.

We now specialize to a target space such that $Z(X)|_{\bar{y}=1} = Z(X)'|_{\bar{y}=1} = 0$ (as, for example, in the case $X = T^4$ due to fermion zero modes.). Thus we have

$$\sum_{\bar{\ell}} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) = 0$$
$$\sum_{\bar{\ell}} \bar{\ell} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) = 0 \quad (5.3)$$

Moreover, we assume that the conformal field theory for $X$ has a realization of the superconformal algebra $\mathcal{A}_{\kappa, \kappa'}$ or its $\kappa' \to \infty$ contraction. In this case we may use the results of the previous section to obtain the identity

$$\sum_{\bar{\ell}} \bar{\ell}^2 c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) = 0 \quad \text{for} \quad \bar{\Delta} > 0 \quad (5.4)$$

(Recall that we are taking the $U(1)$ charges to be zero.)

Let us now compute $Z''$. It follows from (5.3) that $Z$ in (5.2) is equal to one for $\bar{y} = 1$.

Differentiating with respect to $\bar{y}$ gives

$$\partial_{\bar{y}} Z = \left[ \sum_{n, \Delta, \bar{\Delta}, \ell, \bar{\ell}} \bar{\ell} c(\Delta, \bar{\Delta}, \ell, \bar{\ell}) p^n q^n \bar{q}^n y^n \bar{y}^{n-1} \right] Z \quad (5.5)$$
if we set \( \tilde{y} = 1 \) we get zero by \((5.3)\). Next we compute the second derivative of \((5.2)\) with respect to \( \tilde{y} \) and set \( \tilde{y} = 1 \). If the second derivative acts on the \( Z \) factor in \((5.5)\) the result vanishes when we set \( \tilde{y} = 1 \). So the second derivative must act on the sum in \((5.5)\). After setting \( \tilde{y} = 1 \) one finds that the sum over \( \bar{\Delta} \) drops out, since only the \( \bar{\Delta} = 0 \) term contributes by \((5.4)\). This implies that \( \Delta \) is integral and divisible by \( n \): \( \Delta = nm \) with \( m = 0, 1, 2, \ldots \). So we get

\[
\frac{1}{2} \partial_2^2 \tilde{y} Z|_{\tilde{y}=1} = \sum_{n \geq 1} \sum_{m \geq 0} \sum_{\ell \in \mathbb{Z}} \hat{c}(nm, \ell) p^n q^m y^\ell \left( 1 - p^n q^m y^\ell \right)^2
\]

(5.6)

Here we have defined

\[
\hat{c}(\Delta, \ell) := \frac{1}{2} \sum_{\tilde{\ell}} \tilde{\ell}^2 c(\Delta, 0, \tilde{\ell})
\]

(5.7)

Expanding \((5.6)\) yields

\[
\frac{1}{2} \partial_2^2 \tilde{y} Z|_{\tilde{y}=1} = \sum_{s,n,m,\ell} s \left( p^n q^m y^\ell \right)^s \hat{c}(nm, \ell)
\]

(5.8)

where \( s, n \geq 1, m \geq 0, \ell \in \mathbb{Z} \). Collecting powers of \( p, q, y \) we finally obtain our counting formula:

\[
(y^{1/2} - y^{-1/2})^4 \sum_{j_L, j_R} \chi_{j_L}(-y)(-1)^{2j_R}(2j_R + 1) D_{4/32}^+(Q_1, Q_5; N; j_L, j_R)
\]

\[
= \sum_{\ell} y^\ell \sum_{s|Q_1, Q_5; s|N, s|\ell} s \hat{c}\left( \frac{Q_1 Q_5 N}{s^2}, \ell \right)
\]

(5.9)

We stress that this formula is only applicable for \( \gcd(Q_1, Q_5) = 1 \) (i.e., for a primitive vector in the string charge lattice) because otherwise the possibility of bound states at threshold obscures the relationship between the sigma model and the \( D1D5 \) system.

Let us now specialize to the particular example of \( X = T^4 \). The partition function \((5.1)\) becomes

\[
\left( \sum_{\ell^4} q^{1/2} p^{1/2} q^{1/2} p_R^{1/2} \right) \left( \frac{\vartheta_1(z|\tau)}{\eta} \right)^2 \frac{1}{\eta^4} \left( \frac{\vartheta_1(z|\tau)}{\eta} \right)^2 \frac{1}{\eta^4}
\]

(5.10)
Here $\Gamma^{4,4}$ is a lattice of zeromodes. We will be interested in states with zero $U(1)$ charges and as we discussed above this implies that only states with $\tilde{L}_0 = 0$ will contribute to the index we will be computing. This implies that $p_R = 0$ for each copy of the symmetric product of $T^4$'s. For generic values of the $T^4$ moduli this implies that also $p_L = 0$. If we go to the particular values where we have additional values of $p_L$ allowed we see that they should appear in pairs so that their contribution to the index cancels. We will therefore drop the lattice sum in (5.10).

Taking explicit derivatives and using the product formula:

$$\vartheta_1(z|\tau) = i(y^{1/2} - y^{-1/2})q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^n)(1 - y^{-1}q^n)$$  \hspace{1cm} (5.11)

with $y = e^{2\pi iz}$ gives:

$$-\left(\frac{\vartheta_1(z|\tau)}{\eta}\right)^2 \frac{1}{\eta^4} = \sum \hat{c}(n, \ell) q^n y^\ell$$  \hspace{1cm} (5.12)

The left hand side is a weak Jacobi form of weight $-2$ and index 1. Therefore, the coefficients $\hat{c}(n, \ell)$ are actually functions of only one variable $\hat{c}(n, \ell) = \hat{c}(4n - \ell^2)$ \[^8\]. (This can also be seen by bosonizing the $U(1)$ current.) Using the sum formula

$$\vartheta_1(z|\tau) = i q^{1/8}(y^{1/2} - y^{-1/2}) \sum_{n=0}^{\infty} q^{n(n+1)/2}(-1)^n \chi_{j=n}(y^{1/2})$$  \hspace{1cm} (5.13)

and $\chi_{j=n}(y^{1/2}) = \chi_{j=n/2}(y) + \chi_{j=(n-1)/2}(y)$ (valid for $n > 0$ and integral) and expanding one easily derives explicit formulae for the expansion coefficients:

$$\sum \hat{c}(n, \ell) q^n y^\ell = (y^{1/2} - y^{-1/2})^2 \left[ 1 - 2(y^{1/2} - y^{-1/2})^2 q - (y^{1/2} - y^{-1/2})^2(8\chi_0 - \chi_{1/2})q^2 - (y^{1/2} - y^{-1/2})^2(24\chi_0 - 8\chi_{1/2})q^3 + \cdots \right]$$  \hspace{1cm} (5.14)

Note that the positive powers of $q$ have an extra factor of $(y^{1/2} - y^{-1/2})^2$, and that $(y^{1/2} - y^{-1/2})^2 = \chi_{1/2} - 2\chi_0$ allowing a decomposition into $SU(2)$ characters.

Finally, let us close with two remarks.

1. First, the expression $Z''$ in (5.6) can be interpreted more geometrically. Recall that there is an action of $T^4$ on Hilb$^k(T^4)$ lifting the action by translation on Sym$^k(T^4)$.  

14
The quotient space \( \widetilde{\text{Hilb}}^k(T^4) := \text{Hilb}^k(T^4)/T^4 \) is a simply connected irreducible hyperkähler manifold. Working in the charge zero sector the partition function factorizes:

\[
Z(\text{Hilb}^k(T^4)) = Z(T^4)Z(\widetilde{\text{Hilb}}^k(T^4))
\]

for \( k \geq 1 \). Therefore,

\[
\left( \sum_{k=1}^{\infty} p^k Z(\widetilde{\text{Hilb}}^k(T^4)) \right) |_{\tilde{y}=1} = \lim_{\tilde{y} \to 1} \frac{Z(p,q,\tilde{q},y,\tilde{y}) - 1}{Z(T^4)} = - \frac{1}{2} \eta_1^6 \frac{d^2}{d\tilde{y}^2} Z''
\]

so \( Z'' \) is essentially just the generating function for elliptic genera of the hyperkähler spaces \( \text{Hilb}^k(T^4) \).

2. Second, similar multiplet counting formulae apply to compactifications on \( S^1 \times K3 \).

In this case, \( K3 \) breaks half of the supersymmetries, the BPS multiplets are smaller, the sigma model is now \( \sigma(\mathbb{R}^4) \times \sigma(\text{Sym}^k(K3)) \) and the analog of (5.9) is obtained by taking \( \frac{1}{2} \frac{d^2}{d\tilde{y}^2} |_{\tilde{y}=1} \) to get:

\[
(y^{1/2} - y^{-1/2})^2 \sum_{jL,jR} x_{jL}(-y)x_{jR}(-1)D_{8/16}(Q_1, Q_5; jL, jR)
\]

\[
+ (y^{1/2} - y^{-1/2})^4 \sum_{N>0} \sum_{jL,jR} q^N x_{jL}(-y)x_{jR}(-1)D_{4/16}^+(Q_1, Q_5, N; jL, jR)
\]

\[
= (y^{1/2} - y^{-1/2})^2 \prod_{n=1}^{\infty} \frac{1}{(1 - yq^n)^2(1 - y^{-1}q^n)^2} \text{Tr}_{S^k(K3)}[(-1)^F q^{L_0} q^{\bar{L}_0} y^{2J_3^0}]
\]

\[
= -(y^{1/2} - y^{-1/2})^4 \frac{\eta_1^6(\tau)}{\vartheta_1(z|\tau)^2} \text{Tr}_{S^k(K3)}[(-1)^F q^{L_0} q^{\bar{L}_0} y^2 J_3^0]
\]

We see that in this case the center of mass sigma model contributes to the index for spacetime BPS states.

6. \textit{U-duality and the Long String Interpretation}

\textit{U-duality} has interesting implications in connection with the long-string picture of \cite{20}. The six dimensional \( O(5,5; \mathbb{Z}) \) \textit{U-duality} group does not transform \( N \), but, as mentioned above can be used to put the string charge \( S \) in a canonical form, which we take

\[\text{See, e.g. \cite{19}, and references therein. We thank N. Seiberg and E. Witten for discussions about this space.}\]
to be $S = (Q_1 Q_5, 1)$. By a permutation like (2.9) we can then map to $S = (Q_1 Q_5, N)$. Then, if $N$ and $Q_1 Q_5$ are relatively prime we can again use U-duality to map to a charge vector of the form $\gamma = (1, 1; \tilde{\psi}; Q_1 Q_5 N)$. This state is just a single D1 and a single D5 with momentum $N' = Q_1 Q_5 N$, and its degeneracy is the same as that of a single long string. This implies that if we think in terms of strings in the fivebrane $[4]$, only the long string contributes and all other contributions cancel. It can be seen from (5.9) that indeed in this case only the term with $s = 1$ contributes to (5.9).

This description in terms of a long string applies when we take $N$ to be coprime with $Q_1 Q_5$. However, given $k = Q_1 Q_5$ we should consider all possible values of $N$ and the structure of the Hilbert space is of the form:

$$\mathcal{H}(\text{Sym}^k X) = \bigoplus_{\{k_r\}} \otimes_{r > 0} \text{Sym}^{k_r} (\mathcal{H}_r(X))$$

(6.1)

where $\sum r k_r = k$, and $\mathcal{H}_r(X)$ is the single string Hilbert space for a string of length $r$. The sectors which contribute to $Z''$ are of the form $\bigoplus_{r | k} \text{Sym}^{k/r} (\mathcal{H}_r(X)) \oplus \cdots$ and correspond to collections of strings of a single length $k/r$.

6.1. A formula for all primitive vectors

In this section we extend the counting formula from the case $\gcd(Q_1, Q_5) = 1$ to all primitive vectors equivalent to the three charge system.

$U$-duality under the transformation $\mathcal{T}$ (2.8) is obvious. To check $U$-duality under $\mathcal{T}'$ we should remember that our formula is valid only if $\gcd(Q_1, Q_5) = 1$, therefore we can compute the right hand side of (2.9) only if $\gcd(N, Q_5) = 1$ as well. In that case it is easy to see that the sum over $s$ is such that the two results agree. If one drops the restriction $\gcd(Q_1, Q_5) = 1$ then (5.9) is not $U$-duality invariant. As a simple example, let $p_1, p_2$ be two distinct primes. Then $\gamma = (p_1, p_1; 0; p_2)$ on the RHS of (5.9) gives $\sum_\ell y^{\ell} \hat{c}(p_1^2 p_2, \ell)$ while $\gamma = (p_2, p_1; 0; p_1)$ gives $\sum_\ell y^{\ell} \hat{c}(p_2^2 p_1, \ell) + \sum_\ell p_1 y^{\ell} p_2 \hat{c}(p_1 p_2, \ell)$.

To cure this problem we begin by noting that, in close analogy to the remark of [16], the expression on the RHS of (5.9) is just a transform by a Hecke operator $V_{Q_1 Q_5}$ applied to a Jacobi form [18]. Since $V_{Q_1 Q_5} = V_{Q_1 Q_5}$ for $Q_1, Q_5$ relatively prime one might wonder

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7 The power of $s$ in (5.9) is nonstandard. A weight $= k$ form would have a power of $s^{k-1}$. In our case, $\hat{c}(n, \ell)$ are coefficients of a weight $k = -2$ form, but the power of $s$ in (5.3) corresponds to a weight $k = +2$ form. We do not understand this peculiarity very well, but it does not affect the following argument.

16
if the general formula is given by \(V_Q V_{Q_5}\). Indeed, this guess has some very attractive features. Following [18], pp. 44-45 one can write (for any \(Q_1, Q_5\), not necessarily relatively prime):

\[
(V_Q V_{Q_5}) (N, \ell) = \sum_s N(s) s \hat{c}(\frac{NQ_1Q_5}{s^2}, \frac{\ell}{s})
\]  

(6.2)

where \(N(s)\) is the number of integral divisors \(\delta\) of

\[
N, Q_1, Q_5, s, \frac{NQ_1}{s}, \frac{NQ_5}{s}, \frac{Q_1Q_5}{s^2}, \frac{NQ_1Q_5}{s^2}
\]

(6.3)

It follows that \(V_Q V_{Q_5} = V_{Q_5} V_Q\), and, more importantly, that (6.2) is completely symmetric in \(Q_1, Q_5, N\). Thus, this is a natural \(U\)-duality invariant ansatz for the general case. Indeed, it is the unique \(U\)-duality invariant extension to primitive 3-charge systems.

**7. Comparison to Supergravity**

In this section we compute \(\mathcal{E}_2\) by summing over multiparticle supergravity excitations of \(AdS_3 \times S^3 \times T^4\) and using the AdS/CFT correspondence [21]. A similar comparison with the elliptic genus for the \(K3\) case was made by de Boer [22].

The supergravity computation is in the NS sector while the CFT partition function is normally calculated in the R sector. Under spectral flow between the sectors a state with weight \(h_R\) and (half-integer) \(U(1)\) charge \(j_R\) is mapped into a state with weights

\[
h_{NS} = \frac{k}{4} + h_R + j_R \quad j_{NS} = j_R + \frac{k}{2}
\]

(7.1)

where \(k = c/6\) \((c = 6Q_1Q_5\) is the central charge of the CFT). We use a convention such that \(h_{NS} = -k/4\) for the NS vacuum. The partition function in the NS sector can be obtained from the partition function in the R sector by the following replacements

\[
p \rightarrow pq^{1/4} y, \quad q \rightarrow q, \quad y \rightarrow yq^{1/2}
\]

(7.2)

In principle one expects agreement with supergravity only for small conformal weights, not much bigger than the NS vacuum \(h_{NS} = -k/4\). When conformal weights are of order \(k\) the stringy exclusion principle [23] is relevant and supergravity breaks down. We shall in fact find agreement for all negative values of \(h_{NS}, -k/4 \leq h_{NS} < 0\).
For the CFT we start from (5.8) in the RR sector and we find the NS-NS partition function

\[ \frac{1}{2} Z''_{\text{NS}} = \sum_{n,m,l,s} \hat{c}(4nm - l^2)s \left( p^n q^{n/4 + m + l/2} y^{n + l} \right)^s. \] (7.3)

Now we concentrate on the terms in this expansion with negative powers of \( q \), relevant for the comparison to supergravity. The only possibility is \( n = 1, m = 0, l = -1 \), since \( \hat{c}(r) = 0 \) for \( r < -1 \). Using \( \hat{c}(-1) = 1, \hat{c}(0) = -2 \) this gives

\[ \frac{1}{2} Z''_{\text{NS}} = \sum_s s(pq^{-1/4})^s + \ldots \] (7.4)

where the dots involve non-negative powers of \( q \).

Now we consider the supergravity calculation. We need to define an appropriate notion of a “supergravity elliptic genus” \( Z_{\text{sugra}}(p,q,y) \). We will follow the proposal of de Boer [22]. The single particle supergravity Hilbert space can be derived by group theory and Kaluza-Klein reduction. It decomposes as a representation of \( SU(2|1,1) \times SU(2|1,1) \):

\[ \mathcal{H}_{\text{single particle}} = \bigoplus_{j,\tilde{j} \geq 0} N_{j,\tilde{j}} (j) \otimes (\tilde{j}) \] (7.5)

Short \( SU(2|1,1) \) reps are labelled by the maximal spin, i.e., a nonnegative half-integer \( j \). The highest weight has \( h = j \). Label it by \( (j) \). It turns out that single particle states are always products of short representations. There is no analog of the long \( \otimes \) short of CFT. (These latter come from multiparticle supergravity states.) It turns out that the degeneracies in (7.3) can be read off from the identity of [17]

\[ \sum_{k \geq 0} \sum_{r,\tilde{r}} p^k h^{r,\tilde{r}}(S^k X) y^r \tilde{y}^{\tilde{r}} = \prod_{n=1}^{\infty} \prod_{r,\tilde{r}} (1 - (-1)^{r+\tilde{r}} y^r \tilde{y}^\tilde{r} + (-1)^{r+\tilde{r}} h^{r,\tilde{r}}(X)) \] (7.6)

where \( h(X) = \sum_{r,\tilde{r}} (-1)^{r+\tilde{r}} h^{r,\tilde{r}} y^r \tilde{y}^\tilde{r} \) is the Hodge polynomial. The generating function (7.6) counts \( (c,c) \) primaries. Each \( (c,c) \) primary in turn corresponds to a short \( SU(2|1,1) \times SU(2|1,1) \) representation. De Boer [22] proposes to associate a new quantum number to the supergravity states, the degree, in order to take into account the exclusion principle. The degree is the power of \( p \) multiplying the various factors in (7.6). Thus, representations are now labelled by \( (r,\tilde{r};d) \) where \( d \) is the degree. Notice that this assignment of degree breaks the \( SO(4,5) \) continuous U-duality symmetry of supergravity on \( AdS_3 \times S^3 \times T^4 \).

With this innovation the single-particle Hilbert space is:

\[ \mathcal{H}_{\text{single particle}} = \bigoplus_{n \geq 0, r,\tilde{r}} h^{r,\tilde{r}}(X) \left( \frac{1}{2}(n + r), \frac{1}{2}(n + \tilde{r}); n + 1 \right) \] (7.7)
where \( h^{r,\tilde{r}}(X) \) are the Hodge numbers of \( X = K3, T4 \). For the torus the Hodge polynomial factorizes as \( (1 - y)^2(1 - \tilde{y})^2 \) so we can introduce the useful device for the torus Hilbert space:

\[
\mathcal{H}_{\text{single particle}} = \bigoplus_{n \geq 0} \bigoplus_{r, \tilde{r} = 0, 1, 2} d(r)d(\tilde{r})\left(\frac{1}{2}(n + r), \frac{1}{2}(n + \tilde{r}); n + 1\right) \tag{7.8}
\]

Here \( d(0) = d(2) = 1, d(1) = -2 \) (the sign is for a fermionic representation). Notice that we are including the identity.

We now define the “supergravity elliptic genus” as the free field theory partition function for the Fock space built up from \( \mathcal{H}_{\text{single particle}} \):

\[
\mathcal{Z}_{\text{sugra}}(p, q, \bar{q}, y, \tilde{y}) := \prod_{\mathcal{H}_{\text{single particle}}} (1 - p^d q^h \tilde{q}^\tilde{h} y^\ell \tilde{y}^{\tilde{\ell}})^{-(-1)^{\ell+\tilde{\ell}}} \tag{7.9}
\]

(here it is more convenient to use \( \ell = 2j \) which is integral). Since we will eventually set \( \tilde{y} = 1 \) and expect only holomorphic quantities from left chiral primaries we will temporarily suppress \( \bar{q} \). This is not totally innocent, and we will return to the \( \bar{q} \)-dependence at the end of this section. Suppressing \( \bar{q}, \tilde{y}, \) we can rewrite (7.9) as a product of factors \( (1 - p^n q^h y^\ell)^{-c_s(n, h, \ell)} \) where \( c_s(n, h, \ell) \) is the number of single particle states with \( L_0 = h, U(1) \) charge = \( \ell \) and ‘degree’ \( n \). (As usual \( c < 0 \) for fermions). Here we are measuring \( L_0 \) relative to the NS vacuum, as is conventional in \( AdS \) discussions. So in order to compare with the above formulae we need to replace \( p \to pq^{-1/4} \). The effects of the exclusion principle are approximated by truncating the supergravity spectrum to states with total degree \( k = Q_1 Q_5 \).

The full exclusion-principle-modified supergravity partition function is thus

\[
\mathcal{Z}_{\text{sugra}} = \prod_{n, h, \ell} \frac{1}{(1 - p^n q^h - n/4 y^\ell)^{c_s(n, h, \ell)}} \tag{7.10}
\]

Of course, as written \( \mathcal{Z}_{\text{sugra}} = 1 \) for \( T^4 \) at \( \tilde{y} = 1 \). We therefore need to put back \( \tilde{y} \) and take derivatives to get a nontrivial quantity. The \( \tilde{y} \) that appears in the R partition function differs from the one appearing in the NS partition function by a factor of \( \bar{q}^{1/2} \) arising in the spectral flow. So after differentiating twice we set \( \tilde{y} = \bar{q}^{1/2} \). This selects the chiral primaries. Manipulations similar to those in section five then lead to

\[
\mathcal{Z}''_{\text{sugra}}|_{\tilde{y} = \bar{q}^{1/4}} = \sum_{s, h, n, \ell} \hat{c}_s(n, h, \ell)s(p^n q^h - n/4 y^\ell)^s \tag{7.11}
\]
where \( \hat{c}_s(n, h, \ell) = \sum \ell^2 \tilde{c}_s(n, h, \ell, \tilde{\ell}) \) counts the number of right chiral primaries with the given properties and the sum over \( \tilde{\ell} \) runs over all the chiral primaries of degree \( n \).

Next we need a good way to enumerate chiral primaries in this theory. Using (7.8) above we can perform the sum over \( \ell^2 \) over chiral primaries of given degree. It is easy to see that \( \sum_r d(\tilde{r}) = 0, \sum d(\tilde{r})(n + \tilde{r}) = 0 \) and \( \sum_r d(\tilde{r})(n + \tilde{r})^2 = 2 \). This final sum is independent of \( n \) and just gives an overall factor, as in the CFT result. We now need to compute \( c(n, h, \ell) \) just for the left-moving piece. Ignoring for a moment the sum over \( s \) we see that we have:

\[
\sum c(n, h, \ell) p^n q^{h-n/4} y^\ell = \sum_{n,k,r,t} d(r)d(t)p^{n+1}q^{\frac{n+2r+2t-1}{4}+k} \sum_{\ell=-n+r-t}^{n+r-t} y^\ell \tag{7.12}
\]

where \( r = 0, 1, 2 \) as above and \( t = 0, 1, 2 \) takes into account the descendants of the form \( G_{-1/2} \), etc. The sum over \( k \) takes into account the descendants of the form \( L_{k,1}^k, k = 0, 1, 2 \). The sum over \( \ell \) is in steps of 2. We have replaced \( p \to pq^{-1/4} \) to take into account the ground state energy so that we can compare to (7.4). The sum over \( s \) is taken into account by replacing \( (p, q, y) \to (p^s, q^s, y^s) \) multiplying by \( s \) and then summing over \( s \). We are interested in terms with negative powers of \( q \). This requires:

\[
\frac{n-1}{4} + \frac{r+t}{2} + k < 0 \tag{7.13}
\]

The only possibility is \( k = r = t = n = 0 \), and this reproduces (7.4). Hence the supergravity and CFT calculations of \( Z'' \) agree exactly for all negative powers of \( q \). Notice that basically only the ground state is contributing to (7.4). So the agreement boils down to the statement that all the gravity contributions cancel at low enough energies.

It is not hard to see that the agreement does not persist for nonnegative powers of \( q \) (indeed, there is a discrepancy at order \( q^0 \)). This is not surprising because supergravity becomes strongly coupled before this point. Indeed a black hole which is a left-chiral primary appears at this level. This black hole is an extremal rotating black hole with angular momentum on \( S^3 \).

Finally, let us return to the issue of the \( \bar{q} \) dependence of \( Z''_{\text{sugra}} \). In fact if \( \bar{q} \) is reinstated, one finds at these excited levels dependence on positive powers of \( \bar{q} \). This might seem to be a contradiction because we argued in section 3 that the large \( \mathcal{N} = 4 \) algebra forbids \( \bar{q} \)-dependence of \( Z'' \). What happens is that the implementation of the exclusion principle as a cutoff on supergravity states breaks the large \( \mathcal{N} = 4 \), which for example maps single particle
states below the cutoff to multi-particle states above the cutoff. Hence this implementation, while very successful at low energies, is too naive to describe the Hilbert space at high energies. Indeed, the $\bar{q}$ dependence at order $p^N$ first shows up at order $\bar{q}^{N/4}$. Thus, in the large $N$ limit the action of the large $\mathcal{N} = 4$ algebra is restored, in accord with the AdS/CFT correspondence.

8. Open Questions

As we have stressed, (5.9) is false when $Q_1, Q_5$ have common factors, i.e., when the Mukai vector of the instanton moduli space is not primitive. This is not terribly surprising since it is known that the moduli space is singular under such circumstances, and there are even resolutions of the space not equivalent to the Hilbert scheme of points [24]. Physically, nonprimitive vectors are associated with the possibility of boundstates at threshold so we expect subtleties in counting BPS states. Very similar subtleties were found already in the work of Vafa and Witten in [23]. In view of this one should be cautious about the existing formulae for BPS states in $S^1 \times K3$ compactification for nonprimitive Mukai vectors. Unfortunately, $U$-duality is not a useful tool for probing this question.

In (6.2)(6.3) we extended (5.9) to all primitive 3-charge systems, but this leaves open the question of what the degeneracies really are when $\gamma$ is not primitive. Because of bound-states at threshold this question requires careful definition. One way to approach this question is to use the trick in [2], compactifying on another circle and turning on a charge to remove the boundstates at threshold.

The results of this paper raise some interesting open problems. It is natural to expect that the full set of BPS states for toroidally compactified type II string is counted by some interesting automorphic functions transforming nontrivially under $E_{d,d}(\mathbb{Z})$. One might hope that such forms might appear in quantum corrections along the lines of the BPS counting formulae appearing in quantum corrections. (See [8], p.10 for a list of references.) At present such automorphic forms remain part of the Great Unknown.

Finally it would be interesting to compute this index for supergravity on $AdS_3 \times S^3 \times S^3 \times S^1$ [15] in order to see what we can learn about the conformal field theory.

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