Well Posedness of the Problem of Estimation
Fractional Derivative for a Distribution Function.

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\textbf{Abstract.}

We study the problem of nonparametric estimation of the fractional derivative of unknown distribution function and of spectral function and show that these problems are well posed when the order of derivative is less than 0.5.

We prove also the unbiaseness and asymptotical normality of offered estimates with optimal speed of convergence.

For the construction of the confidence region in some functional norm we establish the Central Limit Theorem in correspondent Lebesgue-Riesz space for offered estimates, and deduce also the non-asymptotical deviation of our estimates in these spaces.

\textit{Key words and phrases:} Fractional derivatives and integrals of a Riemann-Liouville type, empirical and exact function of distribution, reliability function, loss functional, indicator function, density, spectral function and density, sample, estimate, confidence region, periodogram, asymptotical normality, bias and unbiased estimate, Gaussian random process, Kolmogorov's theorem, Central Limit Theorem in Banach space, Lebesgue-Riesz and Grand Lebesgue spaces (GLS), measurable set, random variable (r.v.) and random process (r.p.), measurable function.

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1 Notations. Statement of problem.

"Fractional derivatives have been around for centuries but recently they have found new applications in physics, hydrology and finance", see [31].

Another applications: in the theory of Differential Equations are described in [32]; in statistics see in [1], [3], see also [13], [8]; in the theory of integral equations etc. see in the classical monograph [36].
We consider here the problem of the nonparametric estimation of the fractional derivative for a distribution function based on the sample of a "great" volume, and analogously estimation of the fractional derivative of the spectral function of Gaussian stationary sequence.

We will prove that if the order of the fractional derivative $\alpha$ is less than $1/2$, then these problems are well posed. In particular, the speed of convergence of offered unbiased estimate is $1/\sqrt{n}$, as in the case of estimation of ordinary distribution function $F(x)$; they are asymptotical normal still in some rearrangement invariant norm.

Our results improve ones in the articles [4], [13], [8], [23] etc., but does not contradict to the known results.

More detail description. Let $\xi_1, \xi_2, \ldots, \xi_n$ be a sample of a volume $n$, i.e. independent identical distributed numerical random variable with common distribution function $F=F(x)$. In what follows we restrict ourselves by consideration of the following class $K$ of distributions:

$\forall F \in K \Rightarrow \exists (a, b) \in \mathbb{R}^2, 0 \leq a < b \leq \infty, F(a+0) = 0, F(b-0) = 1,$

such that on the interval $(a, b)$ the function $F(x)$ is continuous and strictly increasing.

We can and will suppose further without loss of generality $a = 0$.

Let $\alpha = \text{const} \in (0,1)$; and let $g = g(x), x \in \mathbb{R}$ be measurable numerical function. The fractional derivative of a Riemann-Liouville type of order $\alpha$: $D^\alpha[g](x) = g^{(\alpha)}(x)$ is defined as follows:

$$
\Gamma(1-\alpha) D^\alpha[g](x) = \Gamma(1-\alpha) D_x^\alpha[g](x) \overset{\text{def}}{=} \frac{d}{dx} \int_0^x \frac{g(t) \, dt}{(x-t)^\alpha}.
$$

(1.1)

see, e.g. the classical monograph of S.G.Samko, A.A.Kilbas and O.I.Marichev [36], pp. 33-38; see also [32].

Hereafter $\Gamma(\cdot)$ denotes the ordinary $\Gamma$ function.

We agree to take $D^\alpha[g](x_0) = 0$, if at the point $x_0$ the expression $D^\alpha[g](x_0)$ does not exists.

Notice that the operator of the fractional derivative is non-local, if $\alpha$ is not integer non-negative number.

Recall also that the fractional integral $I^{(\alpha)}[\phi](x) = I^\alpha[\phi](x)$ of a Riemann-Liouville type of an order $\alpha, 0 < \alpha < 1$ is defined as follows:

$$
I^{(\alpha)}[\phi](x) \overset{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_0^x \frac{\phi(t) \, dt}{(x-t)^{1-\alpha}}, x, t > 0.
$$

(1.1a)

It is known (theorem of Abel, see [36], chapter 2, section 2.1) that the operator $I^{(\alpha)}[\cdot]$ is inverse to the fractional derivative operator $D^{(\alpha)}[\cdot]$, at least in the class of absolutely continuous functions.

Note that for the considered further functions this fractional derivative there exists almost everywhere.
Let us consider the following important example. Define the function

\[ g_h(x) = I(h < x), \quad x > 0, \quad h = \text{const} > 0. \]

We conclude after simple calculations taking into account our agreement

\[ g_h^{(\alpha)}(x) = \frac{1}{\Gamma(1 - \alpha)} \cdot I(h < x) \cdot (x - h)^{-\alpha}, \quad \alpha = \text{const} \in (0, 1). \]

Let us calculate for the verification the fractional integral \( I^\alpha \) of order \( \alpha \) from the function \( x \to g_h^{(\alpha)}(\cdot) \). We have

\[ \Gamma(\alpha) \Gamma(1 - \alpha) I^\alpha [g_h^{(\alpha)}] (x) = 0, \quad x \leq h, \]

and in the case \( x > h \)

\[ \Gamma(\alpha) \Gamma(1 - \alpha) I^\alpha [g_h^{(\alpha)}] (x) = \int_0^x \frac{I(t > h) (t - h)^{-\alpha} \, dt}{(x - t)^{1-\alpha}} = \int_h^x (t - h)^{-\alpha} (x - t)^{\alpha-1} \, dt. \]

We make the substitution

\[ t = h + y(x - h); \quad dt = (x - h) dy : \]

\[ \Gamma(\alpha) \Gamma(1 - \alpha) I^\alpha [g_h^{(\alpha)}] (x) = \int_0^1 y^{-\alpha} (1 - y)^{\alpha-1} \, dy = B(1 - \alpha, \alpha) = \Gamma(\alpha) \Gamma(1 - \alpha)/\Gamma(1) = \Gamma(\alpha) \Gamma(1 - \alpha), \]

where \( B(\cdot, \cdot) \) denotes the usually Beta function.

Thus,

\[ I^\alpha [g_h^{(\alpha)}] (x) = I(h < x) = g_h(x). \]

Note that since the function \( x \to g_h(x) \) is not absolutely continuous, this result can not be obtained from the results of chapter 2 from the monograph [36].

Further, we define as the capacity of a loss function the following \( L_q(R, dF) \) functional

\[ \hat{W}_{n,q}[D^\alpha[F](\cdot), \hat{F}_{a,n}(\cdot)] \overset{def}{=} \sqrt{n} \times \left[ \mathbb{E} \int_R |D^\alpha[F](x) - \hat{F}_{a,n}(x)|^q \, dF(x) \right]^{1/q}, \quad (1.2) \]

where \( q = \text{const} \geq 1 \), \( \hat{F}_{a,n}(x) \) is arbitrary estimation of \( D^\alpha[F](\cdot) \) based on our sample.

But it is more convenient sometimes to consider the equivalent problem of estimation of the fractional derivative of so-called “reliability” function \( D^\alpha[G](x) \), where

\[ G(x) = P(\xi_i \geq x) = 1 - F(x), \quad (1.3) \]
and to take \( G^{(\alpha)}(x) = D^\alpha G(x) \),

\[
W_{n,q}[D^\alpha[G](\cdot), \tilde{G}_{\alpha,n}(\cdot)] \overset{\text{def}}{=} \sqrt{n} \times [E \int_{\mathbb{R}} |D^\alpha[G](x) - \tilde{G}_{\alpha,n}(x)|^q dF(x)]^{1/q}, \tag{1.4}
\]

where in turn \( \tilde{G}_{\alpha,n}(x) \) is arbitrary estimation of \( D^\alpha[G](\cdot) \) based on our sample.

Note that

\[
D^\alpha[1](x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \neq 0,
\]

so that

\[
D^\alpha[G](x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)} - D^\alpha[F](x) \neq -D^\alpha[F](x). \tag{1.4a}
\]

For the practical using the expression (1.4) may be consistent approximate as \( n \to \infty \) as follows

\[
W_q[D^\alpha[G](\cdot), \tilde{G}_{\alpha,n}(\cdot)] \approx \sqrt{n} \times [E \int_{\mathbb{R}} |D^\alpha[G](x) - \tilde{G}_{\alpha,n}(x)|^q dF_n(x)]^{1/q},
\]

where \( F_n(x) \) is ordinary empirical function of distribution.

We can define analogously the following estimate of the function \( G(x) \) :

\[
G_n(x) := n^{-1} \sum_{i=1}^{n} I(\xi_i \geq x), \tag{1.5}
\]

empirical reliability function. Here \( I(\xi_i \geq x) \) is the usually indicator function:

\[
I(\xi_i \geq x) = 1 \iff \xi_i \geq x; \quad I(\xi_i \geq x) = 0 \iff \xi_i < x.
\]

Evidently, (Kolmogorov’s theorem), the problem of distribution function estimation (\( \alpha = 0 \)) is well posed. V.D.Konakov in [18] proved in contradiction that the problem of density estimation, i.e. when \( \alpha = 1 \), is ill posed.

Roughly speaking, the result of V.D.Konakov may be reformulated as follows. Certain problem of statistical estimation is well posed iff there exists an estimate (more exactly, a sequence of estimates) such that the speed of convergence is equal (or less than) \( 1/\sqrt{n} \). As a rule these estimations are asymptotically normal.

\section{Point estimate.}

0. \textit{We suppose in what follows in this section that} \( x > 0 \) \textit{and} \( 0 < \alpha < 1/2 \), \textit{so that} \( a = 0, \ 0 < b \leq \infty \).

1. Let us consider the following function

\[
x \to f_h(x) = I(x < h), \ h = \text{const} > 0. \tag{2.1}
\]
It is easy to calculate that

\[ f_{\alpha,h}(x) := \Gamma(1 - \alpha) \, D^\alpha[f_h](x) = x^{-\alpha} - (x - h)^{-\alpha} \cdot I(x > h), \quad x > 0. \quad (2.2) \]

2. It is reasonable to offer as a capacity of the estimate \( G_{\alpha,n}(x) \) at the fixed point \( x, \ x > 0 \) of the fractional derivative \( D^\alpha[G](x) \) the following statistic:

\[
\Gamma(1 - \alpha) \cdot G_{\alpha,n}(x) \overset{\text{def}}{=} n^{-1} \sum_{i=1}^{n} f_{\alpha,\xi_i}(x). \quad (2.3)
\]

Denote also

\[
\Sigma^2_\alpha(x) := 2x^{-\alpha} \Gamma(1 - \alpha) G^{(\alpha)}(x) - \Gamma(1 - 2\alpha) \, G^{(2\alpha)}(x) - \left( \Gamma(1 - \alpha) \, G^{(\alpha)}(x) \right)^2. \quad (2.4)
\]

3. Theorem 2.1.

A. Suppose that at the fixed positive point \( x \) the fractional derivative \( G^{(\alpha)}(x) \) there exists. Then the estimate \( G_{\alpha,n}(x) \) is not biased:

\[
\mathbb{E}G_{\alpha,n}(x) = G^{(\alpha)}(x). \quad (2.5)
\]

B. Suppose in addition that at the fixed positive point \( x \) the fractional derivative \( G^{(2\alpha)}(x) \) there exists. Then the estimate \( G_{\alpha,n}(x) \) is asymptotically as \( n \to \infty \) normal with the variance

\[
\text{Var}[G_{\alpha,n}(x)] = \frac{\Sigma^2_\alpha(x)}{n \, \Gamma^2(1 - \alpha)}:\]

\[
\text{Law}\{G_{\alpha,n}(x)\} \sim N \left( G^{(\alpha)}(x), \frac{\Sigma^2_\alpha(x)}{n \, \Gamma^2(1 - \alpha)} \right). \quad (2.6)
\]

Proof. It is sufficient to prove the equality (2.5) only for the value \( n = 1 \). We deduce by direct computation using the source definition (1.1)

\[
\Gamma(1 - \alpha) \cdot D^\alpha I(\xi \geq x) = x^{-\alpha} - (x - \xi)^{-\alpha} \cdot I(\xi < x) =
\]

\[
f_{\alpha,\xi}(x), \quad \xi = \xi_1. \quad (2.8)
\]

It remains to take the expectation \( \mathbb{E} \) from both the sides of the relationship (2.8) to establish the unbiaseness.

Let us calculate the variance; we consider of course the case \( n = 1 \).

\[
\text{Var} := \text{Var} \left[ x^{-\alpha} - (x - \xi)^{-\alpha} I(\xi < x) \right] =
\]

\[
\text{Var} \left[ (x - \xi)^{-\alpha} I(\xi < x) \right] = S_2 - S_1^2, \quad (2.9)
\]

where
\[ S_1 = \mathbf{E}(x - \xi)^{-\alpha} I(\xi < x) \]

and we know that

\[ S_1 = x^{-\alpha} - \Gamma(1 - \alpha) G^{(\alpha)}(x). \quad (2.10) \]

Further, we will use the formula (2.24), section 2, pp. 35-37 from the book [36]:

\[ \Gamma(1 - \alpha) D_{\alpha}^2[F] = \frac{F(0)}{x^{\alpha}} + \int_0^x \frac{dF(t)}{(x-t)^\alpha} = \int_0^x \frac{dF(t)}{(x-t)^\alpha}, \]

since \( F(0) = F(0+) = 0 \). Therefore

\[ S_2 = \mathbf{E}(x - \xi)^{-2\alpha} I(\xi < x) = \int_0^x \frac{dF(t)}{(x-t)^{2\alpha}} = \Gamma(1 - 2\alpha) D_{2\alpha}^2[F](x). \quad (2.11) \]

It remains to substitute into equality (2.9), taking into account the relation (1.4a).

The asymptotical normality our estimate follows now from the classical one-dimensional CLT.

**Remark 2.1.** As follows from the relation (2.8), under condition \( |G^{(\alpha)}(x)| < \infty \) the variable \( f_{\alpha,\xi}(x) \) has a finite absolute expectation:

\[ \mathbf{E}|f_{\alpha,\xi}(x)| < \infty. \]

Therefore, on the basis of the Law of Large Numbers, the estimate \( G_{\alpha,n}(x) \) is consistent with probability one only under the condition \( |G^{(\alpha)}(x)| < \infty \).

**Remark 2.2.** We are not sure that offered in this report estimate \( G_{\alpha,n}(x) \) of the value \( G^{(\alpha)}(x) \) is optimal, in the contradiction to the Kolmogorov’s estimate of the ordinary distribution function.

**Remark 2.3.** Emerging in the theorem 2.1 the variable \( G^{(2\alpha)}(x) \), which may be used by the practical application, may be consistent estimated as follows:

\[ G^{(2\alpha)}(x) \approx G_{2\alpha,n}(x), \]

as long as \( \alpha < 1/2 \).

**Remark 2.4.** Non-asymptotical approach.

Let \( x \) be fixed positive number; we consider a non-asymptotical deviation

\[ P_{\alpha,x}(y) \overset{df}{=} \sup_n \mathbf{P}\left(\sqrt{n} \cdot \left| G_{\alpha,n}(x) - G^{(\alpha)}(x) \right| > y \right), \quad y \geq 3. \quad (2.12) \]

Note that the summand r.v. \( f_{\alpha,\xi} = x^{-\alpha} - (x - \xi)^{-\alpha} \cdot I(x > \xi) \) has a heavy tail. Namely, if

\[ F(x) - F(0) = F(x) \sim C_1 x^\Delta, \quad x \to 0+, \quad \Delta = \text{const} \in (0, 1], \quad (2.13) \]

then

\[ \mathbf{P}\left(\left| G_{\alpha,1}(x) - G^{(\alpha)}(x) \right| > y \right) \sim C_2(\alpha, x) \cdot y^{-\Delta/\alpha}, \quad y \to \infty. \quad (2.14) \]
The non-asymptotical bounds for \(\sqrt{n}\) normed deviations of sums of these variables are obtained, e.g. the articles [1], [5], [6], [26]. We deduce the upper bound for considered probability using these results:

\[
P_{\alpha,x}(y) \leq C_3(\alpha, x) \cdot y^{-\Delta/\alpha} \ln y, \ y > 3.
\] (2.15)

The lower estimate for this probability is trivial: as \(y \to \infty\)

\[
P_{\alpha,x}(y) \geq \mathbb{P} \left( |G_{\alpha,1}(x) - G^{(\alpha)}(x)| > y \right) = C_4(\alpha, x) \cdot y^{-\Delta/\alpha}(1 + o(1)).\] (2.16)

The ultimate value of the degree of the value \(\ln y\) in (2.15) is now unknown.

3 Main result: error estimation in Lebesgue-Riesz norm.

As long as the function \(D^{\alpha}[G](\cdot)\) and correspondingly its estimate \(G_{\alpha,n}(\cdot)\) both are discontinuous and all the more so are unbounded, we can not do the error estimation in the uniform norm, and still can not apply the CLT in the Prokhorov-Skorokhod space, in contradiction to the classical Kolmogorov’s theorem.

We intent to investigate the \(L_q(dF)\) deviation of empirical derivative reliability function \(G_{\alpha,n}\) from its true value \(D^{\alpha}G\) : \(W_{q,n} :=\)

\[
W_{q,n}[D^{\alpha}[G](\cdot), G_{\alpha,n}(\cdot)] \overset{\text{def}}{=} \sqrt{n} \cdot \left[ \mathbb{E} \int_R |D^{\alpha}[G](x) - G_{\alpha,n}(x)|^q \, dF(x) \right]^{1/q},
\] (3.1)

As usually, in order to evaluate the variable \(L_q\), we need to establish the Central Limit Theorem in the Lebesgue-Riesz space \(L_q(dF)\).

Note first of all that the expression for \(W_q\) in (3.1) does not depend on the function \(F(\cdot)\) inside the set \(K = \{F\}\), as in the Kolmogorov’s theorem; therefore we can and will suppose \(F(x) = x, 0 \leq x \leq 1\), i.e. \(a = 0, b = 1\) and the r.v. \(\{\xi_i\}\) have the uniform distribution on the set \([0, 1]\).

We introduce some new notations. \(x \cap y := \min(x, y), 1 \leq q = \text{const} < 1/\alpha,\)

\[
R_\alpha(x, y) \overset{\text{def}}{=} \frac{(x \cap y)^{1-2\alpha}}{1-2\alpha} - \frac{(x y)^{1-\alpha}}{(1-\alpha)^2}, \ 0 \leq x, y < 1, \ 0 < \alpha < 1/2,
\] (3.2)

\[
\sigma^2_\alpha(x) := R_\alpha(x, x) = \frac{x^{1-2\alpha}}{1-2\alpha} - \frac{x^{2-2\alpha}}{(1-\alpha)^2},
\]

\[
g_\alpha(x) = x^{-\alpha} - \frac{x^{1-\alpha}}{1-\alpha},
\]

\[
\Gamma(1-\alpha) \zeta^{(\alpha)}_n(x) = \Gamma(1-\alpha) \zeta_n(x) :=
\]
\[ n^{-1/2} \sum_{i=1}^{n} \left\{ f_{\alpha,\xi_i}(x) - \left[ x^{-\alpha} - \frac{x^{1-\alpha}}{1-\alpha} \right] \right\} = \]

\[ n^{-1/2} \sum_{i=1}^{n} \{ f_{\alpha,\xi_i}(x) - g_{\alpha}(x) \}, \]  \hspace{1cm} (3.3)

\[ \zeta_{\alpha}^{(n)}(x) = \zeta_{\alpha}^{(n)}(x) = \zeta(x), \ 0 \leq x \leq 1 \] be a separable (moreover, continuous with probability one) centered Gaussian random process with covariation function \( R_{\alpha}(x, y) : \)

\[ \text{Cov}(\zeta_{\alpha}(x), \zeta_{\alpha}(y)) = E \zeta_{\alpha}(x) \cdot \zeta_{\alpha}(y) = R_{\alpha}(x, y). \]  \hspace{1cm} (3.4)

The ordinary Lebesgue-Riesz space \( L_q(dF) = L_q(R_+, dF) \) consists by definition on all the measurable functions \( f : R_+ \to R \) with finite norms

\[ ||f||_q = ||f||_{L_q(dF)} \overset{\text{def}}{=} \left[ \int_0^\infty |f(x)|^q \ dF(x) \right]^{1/q}, \ q = \text{const} \geq 1. \]

**Theorem 3.1.** Let \( F(\cdot) \in K \) and let \( 1 \leq q < 1/\alpha \). Our statement: the sequence of distributions generated in the space \( L_q(dF) \) by the random processes \( \zeta_{\alpha}^{(n)}(\cdot) \) converges weakly as \( n \to \infty \) to the random process \( \zeta_{\alpha}^{(\infty)}(x) \) (the CLT in the space \( L_q(R_+, dF) \)).

**Proof.** Note first of all that here

\[ G^{(\alpha)}(x) = D^{\alpha}[G](x) = \frac{1}{\Gamma(1-\alpha)} \cdot \left[ x^{-\alpha} - \frac{x^{1-\alpha}}{1-\alpha} \right], \]

as long as in this section \( G(x) = 1 - x, \ x \in (0, 1) \). Therefore, all the processes \( \zeta_n(x) \) are centered.

As before, it is sufficient to consider the centered random process \( \zeta_1^{(\alpha)}(x), \ x \in (0, 1) \). It is easy to calculate its covariation function; it coincides with \( R_{\alpha}(x, y) \).

It remains to establish the CLT in the Lebesgue-Riesz space \( L_q(0, 1) \) for the sequence \( \zeta_n^{(\alpha)}(\cdot) \). The using for us version of CLT in this spaces is obtained, for example, in the fundamental monograph [20], pp. 308-319. Namely, the sufficient condition

\[ E||\zeta_1||^q < \infty \]

is here satisfied, as long as \( q < 1/\alpha \).

In detail, let us denote

\[ K(\alpha, q) := \frac{2^{1-1/q}}{\Gamma(1-\alpha)} \cdot \left[ (1 - \alpha)^{-q} + (1 - \alpha q)^{-1} \right]^{1/q} < \infty, \]

then

\[ \Gamma(1 - \alpha) |\zeta_1(x)| \leq \frac{x^{1-\alpha}}{1-\alpha} + (|(x - \xi)| I(x > \xi))^{-\alpha}; \]

\[ \Gamma^q(1 - \alpha) |\zeta_1(x)|^q \leq 2^{q-1} \left[ (1 - \alpha)^{-q} + x^{-aq} \right]; \]
\[ \Gamma^q(1 - \alpha) \| |\zeta_1(\cdot)\| \|^q = \Gamma^q(1 - \alpha) \int_0^1 |\zeta_1(x)|^q \, dx \leq 2^{q-1} \left[ (1 - \alpha)^{-q} + (1 - \alpha q)^{-1} \right]; \]

so

\[ \| |\zeta_1(\cdot)\| \| \leq \frac{2^{1-1/q}}{\Gamma(1 - \alpha)} \cdot \left[ (1 - \alpha)^{-q} + (1 - \alpha q)^{-1} \right]^{1/q} = K(\alpha, q) < \infty, \quad (3.7a) \]

since \(0 < \alpha < 1, \ 1 \leq q < 1/\alpha.\)

This completes the proof of theorem 3.1.

**Remark 3.1.** Note that the obtained estimate (3.7a) is deterministic, i.e. is true still without the expectation \(E.\)

**Remark 3.2.** Let us denote

\[ Q^{(q)}(\alpha)(u) = P (\| |\zeta_1(\cdot)\|L_q(0, 1) > u), \quad u = \text{const} > 0. \quad (3.8) \]

We deduce as a consequence of the theorem 3.1 for the values \(q = \text{const} \in [1, 1/\alpha), \ 0 \in (0, 1/2)\) and \(u > 0\)

\[ \lim_{n \to \infty} P \left( \| |G_{\alpha,n}(\cdot) - G^{(\alpha)}\|L_q(dF) > \frac{u}{\sqrt{n \Gamma(1 - \alpha)}} \right) = Q^{(q)}(\alpha)(u), \quad (3.9) \]

therefore for sufficiently greatest values \(n\)

\[ P \left( \| |G_{\alpha,n}(\cdot) - G^{(\alpha)}\|L_q(dF) > \frac{u}{\sqrt{n \Gamma(1 - \alpha)}} \right) \approx Q^{(q)}(\alpha)(u). \quad (3.10) \]

The asymptotical behavior of the probability \(Q_\alpha(u)\) as \(u \to \infty\) is known, see [33],[34]. Briefly, let us denote \(q' = q/(q - 1), \ q > 1,\) and introduce the variable

\[ \beta := \sup_{h: \|h\|L_q=1} \left\{ \int_0^1 \int_0^1 R_\alpha(x, y)h(x)h(y)dxdy \right\}, \]

which may be computed in turn through solving of some non-linear integral equation; then

\[ \ln Q^{(q)}(\alpha)(u) \sim \frac{-u^2}{2\beta^2}, \ u \to \infty. \]

The non-asymptotical estimates of this probability is obtained in [24], chapter 4, section 4.8; see also [25], chapter 3.

It is clear that the equality (3.10) may be used by construction of confidence region for the unknown function \(G^{(\alpha)}(\cdot)\) in the Lebesgue-Riesz norm \(L_q(dF)\) and for the testing of non-parametrical hypotheses.

**Remark 3.3.** Verification. It is interest to note that on the case \(\alpha = 0,\) more exactly when \(\alpha \to 0+\), the obtained before results coincide with the classical belonging to Kolmogorov, Mises etc.
4 Non-asymptotical error estimation in the Lebesgue-Riesz norm.

We intent to obtain in this section the non-asymptotical upper estimate for the supremum of loss function

\[ W_q \overset{\text{def}}{=} \sup_n W_{q,n}[D^\alpha[G](\cdot), G_{\alpha,n}(\cdot)] \] (4.0)

and as a consequence by means of Tchebychev’s inequality the probability

\[ \sup_n P \left( \left| G_{\alpha,n}(\cdot) - G^{(\alpha)}(\cdot) \right| L_q(dF) > \frac{u}{\sqrt{n} \Gamma(1-\alpha)} \right) =: \overline{c}_\alpha(u). \] (4.1)

We retain in this section all the notations and restrictions of third section; for instance, \( F(\cdot) \in K \); therefore we can and will suppose that the r.v. \( \{\xi(i)\} \) are independent and uniformly distributed on the set \([0, 1]\).

The case \( \alpha = 0 \), i.e. when we consider the classical problem of estimation of ordinary distribution function \( F(x) \) or equally the reliability function \( G(x) = 1 - F(x) \) by means of empirical distribution function \( F_n(x) \), is investigated, and at once in the multidimensional case, even in the uniform norm, i.e. when formally \( q = \infty \), in the work of J.Kiefer [17]; more exact estimate see in the article [11].

Indeed,

\[ P \left( \sqrt{n} \sup_x |G_n(x) - G(x)| > u \right) \leq 2e^{-2u^2}, \ u \geq 1, \] (4.2)
i.e. the exponential bound for \( \sqrt{n} \) normed uniform deviation \( \sup_x |G_n(x) - G(x)| \).

Note first of all that the \( \sqrt{n} \) exponential tail distributed confidence region for \( G^{(\alpha)}(\cdot) \) based on our estimate \( G_{\alpha,n}(\cdot) \) in the \( L_q \) norm is impossible when \( p \geq 1/\alpha \), in contradiction to the classical ordinary case \( \alpha = 0 \). Namely, we can deduce the following simple lower bound for \( \overline{W}_q \)

\[ \overline{W}_q \geq W_{q,1}[D^\alpha[G](\cdot), G_{\alpha,1}(\cdot)], \]

and it is easily to calculate analogously to the relations (3.7) - (3.7a) that

\[ \Gamma(1-\alpha) \cdot E \left[ ||D^\alpha[G](\cdot) - G_{\alpha,n}(\cdot)||_q \right] \geq \frac{C_1(\alpha)}{(1-\alpha q)^{1/q}}, \ q < 1/\alpha, \] (4.3)

and

\[ \Gamma(1-\alpha) \cdot E \left[ ||D^\alpha[G](\cdot) - G_{\alpha,n}(\cdot)||_q \right] = \infty, \ q \geq 1/\alpha, \] (4.4)

We are going now to the obtaining of upper estimates for the value \( \overline{W}_q \). We suppose in the sequel \( 2 \leq q < 1/\alpha \).

**Theorem 4.1.** We propose under formulated above conditions \( 0 < \alpha < 1/2, \ 2 \leq q < 1/\alpha \) etc.

\[ \Gamma(1-\alpha) \cdot \sup_n W_{q,n}[D^\alpha[G](\cdot), G_{\alpha,n}(\cdot)] \leq \frac{C_2(\alpha)}{(1-\alpha q)^{1/q}}, \] (4.5)
where $C_2(\alpha)$ is continuous positive function on the closed segment $\alpha \in [0, 1/2]$.

**Proof.** Denote

$$\tau_i(x) = f_{\alpha, \xi(i)}(x) - E f_{\alpha, \xi(i)}(x),$$  \hspace{1cm} (4.6)

$$S_n(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tau_i(x),$$ \hspace{1cm} (4.7)

then $\tau_i(x)$ is a sequence of independent identical distributed centered random fields which are proportional with coefficient $1/\Gamma(1-\alpha)$ to the considered before r.f. $\zeta_i(x)$.

Let us consider the sequence of random variables

$$V_{\alpha,q}(n) = ||S_n(\cdot)||_q^q = \int_0^1 |S_n(x)|^q \, dx,$$

then we have using Fubini-Tonelli theorem under our condition $2 \leq q < 1/\alpha$

$$EV_{\alpha,q}(n) = \int_0^1 E|S_n(x)|^q \, dx = \int_0^1 E \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tau_i(x) \right|^q \, dx.$$ \hspace{1cm} (4.8)

We intent to exploit the famous Rosenthal’s inequality, see [35], [27]. Namely, for arbitrary sequence $\{\zeta_k\}$ of independent centered random variables

$$\left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \zeta_k \right|_q \leq K_R \cdot \frac{q}{\ln q} \cdot \sqrt{\sum_{k=1}^{n} |\zeta_k|^2/n}, \quad q \geq 2,$$

where the ”Rosenthal’s” constant $K_R$ is less than 0.6535, see [27].

If the r.v. $\{\zeta_k\}$ are in addition identically distributed, then

$$\left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \zeta_k \right|_q \leq K_R \cdot \frac{q}{\ln q} \cdot \zeta_1,$$

As long as in this section $2 \leq q < 1/\alpha$,

$$\frac{q}{\ln q} \leq \max \left\{ \frac{2}{\ln 2}, \frac{1/\alpha}{|\ln \alpha|} \right\},$$

and if we denote

$$K_{R,\alpha} = K_R \cdot \max \left\{ \frac{2}{\ln 2}, \frac{1/\alpha}{|\ln \alpha|} \right\},$$ \hspace{1cm} (4.11)

then there holds the following inequality (under our conditions)

$$\left| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \zeta_k \right|_q \leq K_{R,\alpha} \cdot |\zeta_1|_q.$$ \hspace{1cm} (4.12)

We conclude after substitution into (4.8)

$$EV_{\alpha,q}(n) \leq K(\alpha, q)^q \cdot K_{R,\alpha}^q.$$ \hspace{1cm} (4.13)
It remains to extract the root of degree $q$ from last inequality to obtain the estimate

$$
\sup_n W_{q,n}[D^\alpha[G](\cdot), G_{\alpha,n}(\cdot)] \leq K(\alpha, q) \cdot K_{R,\alpha},
$$

which is equivalent to the assertion if theorem 4.1 with explicit evaluate of constant.

5 Estimation of fractional derivatives of spectral function.

Let us consider in this section the classical problem of spectral density estimation.

Let $\eta_k, \, k = 1, 2, \ldots, n$ be real valued, centered: $E\eta(k) = 0$ Gaussian distributed stationary random sequence (process) with (unknown) even covariance function $r = r(m)$, spectral function $F(\lambda)$, $\lambda \in [0, 2\pi]$, $F(0^+) = F(0) = 0$, and with spectral density $f(\lambda)$, (if there exists):

$$
r(m) = \text{Cov}(\eta(j + m), \eta(j)) = \mathbb{E}\eta(j + m) \cdot \eta(j) =
\int_{[0,2\pi]} \cos(\lambda m) dF(\lambda) = \int_{[0,2\pi]} \cos(\lambda m) f(\lambda) \, d\lambda,
$$

so that

$$
F(\lambda) = \int_0^\lambda f(t) dt = I^{(1)}[f](\lambda).
$$

The periodogram of this sequence will be denoted by $J_n(\lambda), \, 0 \leq \lambda \leq 2\pi$:

$$
J_n(\lambda) := (2\pi n)^{-1} \left| \sum_{k=1}^n e^{i k\lambda} \eta(k) \right|^2 ; \, i^2 = -1.
$$

We intent here to estimate the fractional derivative $F^{(\alpha)}(\lambda)$ of the spectral function $F(\lambda)$.

Recall that the problem of $F(\cdot)$ estimation is well posed, in contradiction to the problem of spectral density $f(\cdot) = F^{(1)}(\cdot)$ estimation.

We assume as before $0 < \alpha < 1/2$, and denote $\beta = 1 - \alpha; \, \beta \in (1/2, 1)$.

Heuristic arguments. We have using the group properties of the fractional derivative-integral operators

$$
F^{(\alpha)} = D^\alpha[F] = D^\alpha I_1[f] = D^\alpha D^{-1}[f] = D^{\alpha - 1}[f] =
I^{1-\alpha}[f] = I^\beta[f] \approx I^\beta[J_n].
$$

Thus, we can offer as an estimation of $F^{(\alpha)}$ the following statistics

$$
F_{\alpha,n}(\lambda) := I^\beta[J_n](\lambda) = \frac{1}{\Gamma(\beta)} \int_0^\lambda \frac{J_n(t)}{(t - \lambda)^{1-\beta}} \, dt = \frac{1}{\Gamma(1 - \alpha)} \int_0^\lambda \frac{J_n(t)}{(t - \lambda)^{\alpha}}.
$$
Theorem 5.1. Suppose as before $0 < \alpha < 1/2$ and that the spectral density $f(\lambda)$ there exists and is continuous on the circle $[0, 2\pi]$, in particular

\[ f(0) = f(0+) = f(2\pi - 0) = f(2\pi). \]

Then the estimate $F_{\alpha,n}(\lambda)$ of the fractional derivative $F(\alpha)(\lambda)$ is asymptotically unbiased:

\[ \mathbb{E}F_{\alpha,n}(\lambda) = F(\alpha)(\lambda) + O(1/n), \quad (5.5) \]

and

\[ \lim_{n \to \infty} n \cdot \text{Var}[F_{\alpha,n}(\lambda)] = \frac{4\pi}{\Gamma^2(1 - \alpha)} \cdot \int_0^\lambda \frac{f^2(\nu)d\nu}{(\lambda - \nu)^{2\alpha}} = \frac{4\pi\Gamma(1 - 2\alpha)}{\Gamma^2(1 - \alpha)} \cdot J_{2\alpha}[f^2](\lambda) < \infty. \quad (5.6) \]

Note that the last integral is finite since the function $f$ is bounded and $\alpha < 1/2$.

More generally,

\[ \lim_{n \to \infty} n \cdot \text{Cov}\{F_{\alpha,n}(\lambda), F_{\alpha,n}(\mu)\} = \frac{4\pi\Gamma(1 - 2\alpha)}{\Gamma^2(1 - \alpha)} \cdot D_{\alpha}[D_{\mu}^{-\alpha}[f^2]] = \frac{4\pi}{\Gamma^2(1 - \alpha)} \cdot \int_0^{\lambda \cap \mu} \frac{f^2(\nu)d\nu}{(\lambda - \nu)^{\alpha}(\mu - \nu)^{\alpha}} =: \Theta_{\alpha}(\lambda, \mu). \quad (5.7) \]

Proof. Our assertion follows immediately from the following proposition, see the fundamental monograph of T.W. Anderson [2], chapter 5, page 564-572, theorem 9.3.1: if $w(\lambda, \nu)$ is non-negative integrable function, then

\[ \lim_{n \to \infty} \int_0^{2\pi} w(\lambda, \nu) J_n(\nu)d\nu = \int_0^{2\pi} w(\lambda, \nu) f(\nu)d\nu, \quad (5.8a) \]

\[ \lim_{n \to \infty} n \cdot \text{Cov}\left( \int_0^{2\pi} w(\lambda_1, \nu) J_n(\nu)d\nu, \int_0^{2\pi} w(\lambda_2, \nu) J_n(\nu)d\nu \right) = 4\pi \int_0^{2\pi} w(\lambda_1, \nu) w(\lambda_2, \nu) f^2(\lambda) d\lambda, \quad (5.8b) \]

with remainder terms. We choose $w(\lambda, \nu) = |\lambda - \nu|^{-\alpha}$; it is easy to verify that all the conditions of the mentioned result are satisfied.

Recall also that the considered stationary sequence $\{\eta(k)\}$ is Gaussian, i.e. without cumulant function.

Remark 5.1. Emerging in the equality (5.6) the variable

\[ J_{2\alpha}[f^2](\lambda) = \frac{1}{\Gamma(2\alpha)} \int_0^\lambda \frac{f^2(\nu)d\nu}{(\lambda - \nu)^{1-2\alpha}} \]

may be $n^{-1/2}$ — consistent estimated as follows:
\[ I^{2\alpha}[f^2](\lambda) \approx \frac{1}{\Gamma(2\alpha)} \int_0^\lambda \frac{J_n^2(\nu)d\nu}{(\lambda - \nu)^{1-2\alpha}}. \] (5.9)

**Remark 5.2.** I.A.Ibragimov in [14] proved the asymptotical normality of the random process \( \sqrt{n} \{ I^1[J_n](\lambda) - I^1[F](\lambda) \} \) as \( n \to \infty \) in the space \( C(0,2\pi) \) of continuous functions. See also [7], [21]. A fortiori, the sequence of random processes

\[ \zeta_n(\lambda) = \sqrt{n} \cdot \{ F_{\alpha,n}(\lambda) - F^{(\alpha)}(\lambda) \} \]

converges weakly in the space \( C(0,2\pi) \) as \( n \to \infty \) to the centered separable Gaussian process \( \zeta_\infty \) with covariation function \( \Theta_\alpha(\lambda,\mu) \). Therefore

\[ P \left( \sqrt{n} \cdot \max_\lambda \left| \{ F_{\alpha,n}(\lambda) - F^{(\alpha)}(\lambda) \} \right| > u \right) \approx P(\max_\lambda |\zeta_\infty(\lambda)| > u), \quad u = \text{const} > 0. \] (5.10)

The asymptotical as \( u \to \infty \) behavior of the last probability is fundamental investigated in the monograph [34], see also [33]:

\[ P(\max_\lambda |\zeta_\infty(\lambda)| > u) \sim H(\alpha) u^{\kappa-1} \exp \left( -u^2/\sigma^2 \right), \] (5.11)

\[ H(\alpha), \quad \kappa = \text{const}, \quad \sigma^2 = \sigma^2(\alpha) = \max_{\lambda \in (0,2\pi)} \Theta_\alpha(\lambda,\lambda). \quad (5.11a) \]

The last equalities may be used by construction of confidence region for \( F^{(\alpha)}(\cdot) \) in the uniform norm. Indeed, let \( 1 - \delta \) be the reliability of confidence region, for example, 0.95 or 0.99 etc. Let \( u_0 = u_0(\delta) \) be a maximal root of the equation

\[ H(\alpha) u_0^{\kappa-1} \exp \left( -u_0^2/\sigma^2 \right) = \delta, \]

then with probability \( \approx 1 - \delta \)

\[ \sup_{\lambda \in (0,2\pi)} \left| F_{\alpha,n}(\lambda) - F^{(\alpha)}(\lambda) \right| \leq \frac{u_0(\delta)}{\sqrt{n}}. \] (5.12)

### 6 Multidimensional case.

We consider in this section the problem of statistical estimates of fractional derivative for multidimensional distribution function. We restrict ourselves for simplicity only two-dimensional case \( d = 2 \).

In detail, let \( \{(\xi(i),\eta(i))\}, i = 1,2,\ldots, n \) be a two dimensional non-negative sample with common distribution function. We define the reliability function \( G = G(x,y) \) as follows:

\[ G(x,y) = P(\xi(i) \geq x, \eta(i) > y), \quad x,y \geq 0. \] (6.1)

Let \( \alpha, \beta = \text{const} \) be two numbers such that \( 0 < \alpha, \beta < 1 \); (we will suppose further that \( 0 < \alpha, \beta < 1/2 \).) The partial mixed fractional derivative \( D_{x,y}^{\alpha,\beta}[G](x,y) \) again
of Riemann-Liouville type of order \((\alpha, \beta)\) of a function \(G(\cdot, \cdot)\) at the positive points \((x, y)\) is defined as follows:

\[
G^{(\alpha, \beta)}(x, y) = D^{\alpha, \beta}_{x,y}[G](x, y) \overset{\text{def}}{=} D^{\alpha}_{x}D^{\beta}_{y}[G] = \frac{1}{\Gamma(1 - \alpha)} \frac{1}{\Gamma(1 - \beta)} \times \\
\frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y G(t, s) \, dt \, ds \cdot (x - t)^{\alpha}(y - s)^{\beta},
\]

(6.2)

see, e.g. [36], chapter 24. We put as before \(D^{\alpha, \beta}_{x,y}[G](x, y) = 0\) if at the point \((x, y)\) the expression (4.2) for \(D^{\alpha, \beta}_{x,y}[G](x, y)\) does not exists.

Note that in general case \(D^{\alpha}_{x}D^{\beta}_{y}[H] \neq D^{\beta}_{y}D^{\alpha}_{x}[H]\), but if the function \(G = G(x, y)\) is factorable: \(H(x, y) = g_1(x)g_2(y)\) and both the functions \(g_1(\cdot)\) and \(g_2(\cdot)\) are "differentiable" at the points \(x\) and \(y\) correspondingly:

\[
\exists D^{\alpha}[g_1](x), \quad \exists D^{\beta}[g_2](y),
\]

then really

\[
D^{\alpha}_{x}D^{\beta}_{y}[H] = D^{\beta}_{y}D^{\alpha}_{x}[H] = D^{\alpha}_{x}[g_1](x) \cdot D^{\beta}_{y}[g_2](y).
\]

(6.3)

Introduce as a capacity of the function \(H\) the following:

\[
H(x, y) := I(\xi \geq x, \eta \geq y), \quad \xi = \xi(1), \eta = \eta(1).
\]

As long as

\[
I(\xi \geq x, \eta \geq y) = I(\xi \geq x) \cdot I(\eta \geq y),
\]

the function \(H = H(x, y)\) is factorable and therefore (see (6.3))

\[
\Gamma(1 - \alpha)\Gamma(1 - \beta)D^{\alpha}_{x}D^{\beta}_{y}[H] = f_{\alpha, \xi}(x) \cdot f_{\beta, \eta}(y).
\]

(6.4)

The consistent with probability one in each fixed point \((x, y)\) estimate of the function \(G^{(\alpha, \beta)}(x, y)\) is follows:

\[
\Gamma(1 - \alpha)\Gamma(1 - \beta)G_{\alpha, \beta, n}(x, y) \overset{\text{def}}{=} n^{-1} \sum_{i=1}^{n} f_{\alpha, \xi(i)}(x) \cdot f_{\beta, \eta(i)}(y).
\]

(6.5)

It is easily to verify that the estimate \(G_{\alpha, \beta, n}(x, y)\) obeys at the same properties as its one-dimensional predecessor \(G_{\alpha, n}(x)\), for example, is unbiased, satisfies LLN and CLT.

Note that despite the function

\[
h_{\alpha, \beta}(x, y) = h_{\alpha, \beta}(\xi, \eta; x, y) := f_{\alpha, \xi}(x) \cdot f_{\beta, \eta}(y), \quad \xi = \xi(1), \eta = \eta(1)
\]

(6.6)

is also factorable, we do not suppose the independence of the r.v. \((\xi, \eta)\).

Define as before the following sequence of random fields

\[
S_{n}(x, y) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( h_{\alpha, \beta}(\xi(i), \eta(i); x, y) - \Gamma(1 - \alpha)\Gamma(1 - \beta)G^{(\alpha, \beta)}(x, y) \right),
\]

(6.7)
so that
\[
\Gamma(1 - \alpha)\Gamma(1 - \beta) \left[ G_{(\alpha,\beta,n)}(x, y) - G^{(\alpha,\beta)}(x, y) \right] = n^{-1/2} S_n(x, y). \tag{6.8}
\]

Further, it is clear that
\[
||h_{\alpha,\beta}(\cdot, \cdot)||_{q,x;r,y} \asymp (1 - \alpha q)^{-1/q} (1 - \beta r)^{-1/r}, \quad 1 \leq q < 1/\alpha, 1 \leq r < 1/\beta. \tag{6.9}
\]

Let return to the source problem and let us consider only the non-mixed case \( r = q \); then
\[
||h(\cdot, \cdot)||_{q,q} = ||h(\cdot, \cdot)||_{q,F} = \left[ \int_X \int_Y |h(x, y)|^q F(dx, dy) \right]^{1/q}. \tag{6.10}
\]

Assume also \( q \geq 2 \), so \( 2 \leq q = r < \min(1/\alpha, 1/\beta) \).

We will distinguish two essentially different variants: \( V_1 : \beta < \alpha \) and \( V_2 : \beta = \alpha \).

The case \( \beta > \alpha \) may be considered analogously.

**First possibility:** \( \beta < \alpha \).

We find after simple calculations as in fourth section using Rosenthal’s inequality:
\[
\sup_n E ||S_n(\cdot, \cdot)||_{q,q,F} \asymp C_1(\alpha, \beta, q) \cdot (1 - \alpha q)^{-1}, \quad 1 \leq q < 1/\alpha, \tag{6.11}
\]
and
\[
\sup_n E ||S_n(\cdot, \cdot)||_{q,q,F} = \infty, \quad q \geq 1/\alpha. \tag{6.11a}
\]

**Second possibility:** \( \beta = \alpha \).

We have analogously
\[
\sup_n E ||S_n(\cdot, \cdot)||_{q,q,F} \asymp C_2(\alpha, q) \cdot (1 - \alpha q)^{-2}, \quad 1 \leq q < 1/\alpha, \tag{6.12}
\]
and
\[
\sup_n E ||S_n(\cdot, \cdot)||_{q,q,F} = \infty, \quad q \geq 1/\alpha. \tag{6.12a}
\]

As a consequence: the sequence of r.f. \( S_n(\cdot, \cdot) \) in both the considered cases satisfies the CLT in the space \( L_q(F(dx, dy)) \) iff \( 1 \leq q < 1/\alpha \).

Thus, there is a possibility to built the asymptotical and non-asymptotically confidence region for estimated mixed fractional derivative \( F^{(\alpha,\beta)}(\cdot, \cdot), \alpha, \beta < 1/2 \) still in the multivariate case in the \( L_q(dF) \) norm as well as in the fixed point \( (x_0,y_0) \).
7 Estimation of fractional derivative in Grand Lebesgue Space norm.

Let \((X, M, \mu)\) be a probability space with non-trivial probability measure \(\mu\), and let also \(\psi = \psi(q), 1 \leq q < s, \ s = \text{const} \in (1, \infty)\) be continuous on the open interval \((1, s)\) bounded from below function. By definition, a Grand Lebesgue Space (GLS) \(G \psi\) over our triplet \((X, M, \mu)\) consists on all the measurable functions \(f : X \to R\) with finite norm

\[
||f||_{G \psi} \overset{\text{def}}{=} \sup_{q \in (1, s)} \left[ \frac{|f|_q}{\psi(q)} \right].
\] (7.1)

Hereafter

\[
|f|_q = \left[ \int_X |f(x)|^q \mu(dx) \right]^{1/q}
\]

and we will denote \(s = \text{supp} \psi\).

The detail investigation of these spaces see, e.g. in [9], [10], [15], [16], [19], [24], [30], [22].

We choose supposing without loss of generality \(a = 0, \ b = 1\), so that \(\dim \xi = 1\) and \(F(0+) = 0, \ F(1-) = 1, \ F \in K, \ \alpha \in (0, 1/2)\),

\[
X = [0, 1] \otimes \Omega,
\]

so that the measure \(\mu\) is direct product of ordinary Lebesgue measure \(dx\) and probability measure \(P\):

\[
\mu(A \otimes B) = \int_A dx \cdot P(B), \ A \subset [0, 1], \ B \subset \Omega,
\] (7.2)

where \(\Omega = \{\omega\}\) is source probability space, i.e. in which is defined our sample \(\{\xi(i)\}\).

Proposition 7.1.

\[
\sup_n \mu\{ (x, \omega) : |S_n(x)| > u \} \leq C_3(\alpha) \ u^{-\alpha} \ \ln u, \ u \geq e.
\] (7.3)

Proof. Put

\[
\psi_\alpha(q) = (1 - \alpha q)^{-1/q}, \ 1 \leq q < 1/\alpha.
\]

Note that

\[
\psi_\alpha(q) \asymp (1 - \alpha q)^{-\alpha}, \ 1 \leq q < 1/\alpha.
\]

The assertion (4.5) may be rewritten as follows.

\[
\sup_n |S_n(\cdot, \cdot)|_{q, \mu} \leq C_4(\alpha) \ \psi_\alpha(q),\] (7.4)

or equally on the language of the Grand Lebesgue Spaces.
The tail estimate (7.3) follows immediately from one of results of the article [30]; see also [22].

**Remark 7.1.** It is not hard to generalize this result into the multidimensional case described in the 6th section. Namely, if in the notations of the 6th section $0 < \beta < \alpha < 1/2$, then

$$\sup_n \mu \{|S_n(x,y)| > u\} \leq C_5(\alpha, \beta) u^{-\alpha} \ln u, \ u \geq e; \quad (7.6)$$

if $0 < \beta = \alpha < 1/2$, then

$$\sup_n \mu \{|S_n(x,y)| > u\} \leq C_6(\alpha) u^{-\alpha} \ln^2 u, \ u \geq e. \quad (7.7)$$

### 8 Concluding remarks.

**A. Weight case.**

Perhaps, it is interest to investigate the error of the approximation of a form

$$V(x) \cdot (W \cdot F)^{(\alpha)}(x) \approx V(x) \cdot (W \cdot F)_{n,\alpha}(x),$$

or analogously

$$V(x) \cdot (W \ast F)^{(\alpha)}(x) \approx V(x) \cdot (W \ast F)_{n,\alpha}(x),$$

or analogously

$$V(x) \ast (W \ast F)^{(\alpha)}(x) \approx V(x) \ast (W \ast F)_{n,\alpha}(x),$$

where $V(x), W(x)$ are two weight functions, for instance, $V(x) = |x|^\gamma$, $W(x) = |x|^\Delta$, $\gamma, \Delta = \text{const.}$

**B. Semi-parametric case.**

Let $\xi(i), \ i = 1, 2, \ldots, n$ be a sample of a volume $n$ with parametric family of regular distribution of a form

$$\mathbb{P}(\xi(i) < x) = F(x; \theta),$$

where $\theta \in \Theta \subset R^k$, $k < \infty$ is (multidimensional, in general case) unknown numerical parameter.

Denote by $\hat{\theta}_n$ the maximum likelihood estimate of the parameter $\theta$ builded on our sample. The asymptotical tail behavior of distribution for the following statistic

$$\tau_n := \sqrt{n} \cdot ||F(\cdot; \hat{\theta}_n) - F_n(\cdot)||L,$$
where $|| \cdot ||_L$ is some Banach functional norm in the space $x \in R$, is in detail investigated in [33], [34], chapter 5.

By our opinion, it is interest to obtain also the asymptotical tail behavior of the following statistic

$$
\tau_{n,\alpha} := \sqrt{n} \cdot ||F^{(\alpha)}(\cdot, \hat{\theta}_n) - F_{n,\alpha}(\cdot)||_L,
$$

$\alpha = \text{const} \in (0, 1/2)$.

C. Applications (possible) in statistics.

The asymptotical tail behavior of the statistic $\tau_{n,\alpha}$ may be used perhaps in turn in statistics, for instance, for the verification of semi-parametrical hypotheses and detection of distortion times etc.

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