Quantum correlations in connected multipartite Bell experiments

Armin Tavakoli\textsuperscript{1,2,3}

\textsuperscript{1}Department of Physics, Stockholm University, SE-10691 Stockholm, Sweden
\textsuperscript{2}ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, E-08860 Castelldefels (Barcelona), Spain
\textsuperscript{3}Computer Science Division, University of California, Berkeley, CA 94720, USA

E-mail: armin.tavakoli@hotmail.com

Received 5 September 2015, revised 28 January 2016
Accepted for publication 1 February 2016
Published 23 February 2016

Abstract
Bell experiments measure correlations between outcomes of a number of observers measuring on a shared physical state emitted from a single source. Quantum correlations arising in such Bell experiments have been intensively studied over the last decades. Much less is known about the nature of quantum correlations arising in network structures beyond Bell experiments. Such networks can involve many independent sources emitting states to observers in accordance with the network configuration. Here, we will study classical and quantum correlations in a family of networks which can be regarded as compositions of several independent multipartite Bell experiments connected together through a central node. For such networks we present tight Bell-type inequalities which are satisfied by all classical correlations. We study properties of the violations of our inequalities by probability distributions arising in quantum theory.

Keywords: quantum correlations, Bell inequality, quantum network

(Some figures may appear in colour only in the online journal)

1. Introduction
Statistical correlations between outcomes obtained in different measurement events can provide insight to the physical causes of the statistics. One example is the celebrated theorem of John Bell which shows that statistical correlations arising in quantum theory cannot be explained by any theory that respects the principles of locality and realism [1]. The bipartite Bell experiment, see figure 1(a), considers two observers Alice and Bob each performing measurements $x$ and $y$ respectively, that are randomly chosen from some set of possible
measurement setting, in space-like separated measurement events on a shared state. The measurements return outcomes \(a\) and \(b\) for Alice and Bob respectively. If the resulting probability distribution \(P_{ab}(x, y)\) respects the principle of locality, then no influence can propagate fast enough between the measurement events in order for the outcomes to directly influence each other i.e. all correlations between the outcomes of Alice and Bob must be due to some cause \(\lambda\) originating from the common past of the two particles. Furthermore, if \(P_{ab}(x, y)\) also respects realism, the physical properties of the system are well-defined before a measurement takes place. Correlations satisfying such a description are deemed classical and can be written on the form

\[
P(a, b|x, y) = \int dq(\lambda)P(a|x, \lambda)P(b|y, \lambda),
\]

where \(q\) is some probability density function. However, if \(P(\cdot)\) is given by quantum theory then it may not admit the above form and is thus deemed intrinsically quantum. The properties of quantum correlations in such bipartite Bell experiments have been thoroughly studied [2].

A straightforward extension of the bipartite Bell experiment is the multipartite Bell experiment in which a source emits a many-particle state and each particle is measured at a different measurement event, see figure 1(b). An interesting property of the arising multipartite quantum correlations is that they can exponentially outperform the classical bound on Bell inequalities, as was first demonstrated for Mermin’s inequality [3].

However, the properties of quantum and classical correlations in more sophisticated network configurations than the standard Bell experiments are to great extent unknown. Such networks can involve multiple independent sources each distributing a state to some set of observers performing randomly chosen measurements. Despite the initial independence of the involved sources, suitably chosen measurements can give rise to strong quantum correlations spanning the whole network.

\[4\] The notation \(P(\cdot)\) will be used as an abbreviation for a probability distribution when the argument is clear from context. e.g. in this case \(P(\cdot) = P(a, b|x, y)\).
Quantifying correlations in network structures was first considered for the network consisting of three observers in a chain configuration with two independent sources [4, 5], see figure 1(c). Bell-type inequalities, and their quantum violations, for such chain networks involving arbitrary many observers have been studied in [6]. Another class of networks with star-shaped configuration i.e. configurations involving many bipartite sources connected by a center node, were studied in [7]. Recently, it was shown that given knowledge of a Bell-type inequality for a network, one can recursively derive another Bell-type inequality for the same network but with one added source connected to one observer [8]. Advances on a method for deriving Bell-type inequalities for networks has been made in [9]. Furthermore, correlations in network structures have been more broadly studied through the lens of causal inference [10–19].

There are several motivations for studying quantum correlations in networks beyond the Bell experiments. Firstly, the notion of classical correlations in a network naturally leads to constraints stronger than the assumption (1) associated to classical correlations in Bell experiments. This is realized from the fact that in Bell experiments the observers always share a common local random variable $\lambda$ which allows any two observers to be directly correlated with each other whereas in networks this need not to be the case. To what extent networks limit the strength of classical and quantum correlations is interesting for foundational studies. Secondly, most implementations of quantum information protocols exploiting quantum correlations e.g. [20], involve a small number of observers connected by a single source. However, large-scale multiuser quantum communication networks are arguably one of the main objectives of applied quantum information. Therefore, it is relevant to study the ability of networks to support quantum correlations. Examples in which network structures are important include entanglement swapping experiments [21], entanglement percolation [22] and quantum repeaters protocols [23, 24]. Thirdly, it is known that the ability of a quantum state to violate a Bell inequality can be activated by considering many copies of the same state distributed in a network structure [25]. Thus, quantum correlations on networks can yield advantages over those in Bell experiments. However, this result was shown without invoking the stronger constraints associated to networks i.e. the independence of the sources. It is of evident interest to search for such advantages when imposing this stronger constraint.

Nevertheless, it is not obvious (i) which networks are the most interesting to study, and (ii) how to derive Bell-type inequalities for general networks. Therefore, we have restricted ourselves to a particular class of networks.

In this work, we will explore classical and quantum correlations on a class of networks consisting of many, initially independent, multipartite Bell experiments that are all connected through a central node, see figure 1(d). These networks can also be viewed as generalizations of the networks considered in [4, 5, 7] to scenarios involving many multipartite sources. We will derive Bell-type inequalities for such networks and study the properties of their quantum violations. In the light of the above motivations for studying quantum correlations on networks, we are in particular interested in searching for advantages over standard Bell experiments.

2. Classical correlations in connected Bell experiments

The networks we will investigate are composed of $n$ sources, each distributing a state of $L + 1$ particles in such a way that for each source $L$ distinct observers receive one particle each and the final particle is sent to a particular observer (called Bob) that acts as the center node connecting the $n$ Bell experiments. Thus, Bob will have $n$ particles at his disposal while
the other $n \times L$ observers each hold one particle. We can index the described network by the pair $(n, L)$ and we abbreviate the network configuration by $\mathcal{N}_n^L$. For example, figure 1(d) represents the network $\mathcal{N}_2^2$ since there are two independent sources and each distribute a 2 + 1 particle state.

Each observer (except Bob) will randomly choose one of two local measurements. The measurement choice of the $k$th observer associated to the $j$th source is denoted $x_0, 1_{j,k}$ and the associated outcome is denoted $a_0, 1_{j,k}$. Bob will randomly select a measurement $y_0, 1, 2_{L}$ from which he will output $b_0, 1$. One can also consider variations in which Bob performs a measurement which returns more than two possible outcomes.

2.1. Defining classical correlations

Let us begin with defining the notion of classical correlations in $\mathcal{N}_n^L$. The natural extension of the classical probability distribution in the standard bipartite Bell experiment (1) is as follows: a probability distribution in $\mathcal{N}_n^L$ is classical if it is a mixture of local outcomes that depend only on the local measurement setting and the relevant physical causes rendering the associated outcome deterministic i.e.

$$P(\pi, b|\pi, y) = \int \prod_{j=1}^{n} \left( \sum_{\lambda_j} q_j(\lambda_j) \prod_{k=1}^{L} P(a_j^k|x_j^k, \lambda_j) \right) P(b|\bar{\lambda}, y),$$

(2)

where we have by $q_j$ denoted the probability distribution function associated to the physical cause $\lambda_j$ associated to the $j$th source. Also, we will frequently make use of the bar-notation to denote a collection of associated variables e.g. $\bar{\pi} = (a_1, \ldots, a_L)$ and similarly for $\pi$ and $\lambda$.

2.2. Bell-type inequalities for $\mathcal{N}_n^L$

We will now derive a family of Bell-type inequalities for the network $\mathcal{N}_n^L$.

Introduce a set of quantities $K_{X}$ indexed by $X$ which are linear combinations of conditional probabilities $P(\pi, b|\pi, y_x)$ arising in $\mathcal{N}_n^L$. The index $X$ runs over the power set (the set of all subsets), $\mathcal{P}$, of the set $\mathcal{N}_L = \{1, \ldots, L\}$, i.e. every subset of $\mathcal{N}_L$ corresponds to a value of $X$ to which we associate a quantity $K_X$. For every element of $\mathcal{P}(\mathcal{N}_L)$, we define

$$K_X = \frac{1}{2^nL} \sum_{\pi,b} g(X) \sum_{\pi,b} (-1)^{b+\sum_{x \in X} x} P(\pi, b|\pi, y_x),$$

(3)

where in the expression $P(\pi, b|\pi, y_x)$ we use the index $X$ in $y_X$ to identify the particular measurement of Bob (chosen from the $2^L$ possible settings) associated to the quantity $K_X$, and where the function $g(X)$ is defined as

$$g(X) = \prod_{j=1}^{n} (-1)^{\sum_{x \in X} x_j}.$$

(4)

We will now state and prove the following theorem:

**Theorem 1.** If given a probability distribution $P$ in $\mathcal{N}_n^L$ that admits a classical model, then the following inequality holds

$$\sum_{X \in \mathcal{P}(\mathcal{N}_L)} |K_X|^n \leq 1.$$

(5)
Proof. Implementing the classical model (2) in $A^L_n$ with the quantities $K_X$ yields

$$K_X = \frac{1}{2nL} \sum_X g(X) \int \prod_{j=1}^n \left[ \prod_{k=1}^L \left( \sum_{a^k_j} (-1)^{a^k_j} P(a^k_j | x^k_j, \lambda_j) \right) \right] \left( -1 \right)^{b_1 P(b | \lambda_X, \gamma_X)} \right].$$

(6)

Introducing the following notations

$$(B_{x})_X = \sum_b (-1)^b P(b | \lambda_X, \gamma_X),$$

(7)

$$(A_{x}^{i,j})_\lambda = \sum_{a^i_j} (-1)^{a^i_j} P(a^i_j | x^i_j, \lambda_j)$$

(8)

and using the fact that $|\langle B_{x} \rangle_X| \leq 1$, the quantity $|K_X|$ can be bounded from above by

$$|K_X| \leq \frac{n}{2nL} \int \prod_{j=1}^n d\lambda_j q_j(\lambda_j) \prod_{k=1}^L \left( \prod_{s^k_j} (-1)^{s^k_j A_{x}^{i,j}} (A_{x}^{i,j})_\lambda \right).$$

(9)

where we have introduced the binary function $\delta^k_X = 1$ if $k \in X$ and $\delta^k_X = 0$ otherwise.

At this point, we need to introduce the following lemma: let $c^l_i$ be non-negative real numbers and $m, n$ be positive integers, then it holds that

$$\prod_{l=1}^m \left( \prod_{j=1}^n c^l_i \right)^{1/n} \leq \prod_{j=1}^n \left( \prod_{l=1}^m c^l_i \right)^{1/n}.$$  

(10)

This was proven in [7]: for every $l$ use that the arithmetic mean of the sequence $\{c^l_i\}_{i=1}^n$ is always larger than or equal to the geometric mean. Establishing such a relation for every $l$ and then summing the right- and left-hand sides over $l$ will prove the lemma.

Applying the relation (10) to the inequalities (9) with $c^l_i$ corresponding to each factor of the product series over $j$ in (9), we can construct the following inequality

$$\sum_{X \in \mathbb{P}(\mathbb{N}_j)} |K_X|^{1/n} \leq \prod_{j=1}^n \left[ \frac{1}{2nL} \int \prod_{j=1}^n d\lambda_j q_j(\lambda_j) \sum_{X \in \mathbb{P}(\mathbb{N}_j)} \prod_{k=1}^L \left( \prod_{s^k_j} (-1)^{s^k_j A_{x}^{i,j}} (A_{x}^{i,j})_\lambda \right) \right]^{1/n}.$$  

(11)

To find an upper bound on the sum over the product series in the integrand, we note that for a given $j$ and $X$ we can write the absolute value in the integrand as $r^{j,k}_x \equiv \langle A_{0}^{j,k} \rangle_{\lambda} + (-1)^{\delta_x} \langle A_{1}^{j,k} \rangle_{\lambda}$ where the $\pm$ index refers to the sign inside the modulus determined by $(-1)^{\delta_x}$. The integrand consists of products of $L$ such factors over which a sum is taken so that all possible arrangements of the sign inside the absolute value of each factor in the product occurs. Therefore, we can factor the integrand into a product of $L$ factors as follows: $(r^{j,k}_x + r^{j,k}_x)(r^{j,k}_x + r^{j,k}_x) \ldots (r^{j,k}_x + r^{j,k}_x)$. Since all $\langle A_{x}^{i,j} \rangle_{\lambda}$ are real and their modulus is bounded by 1, it must be that $r^{j,k}_x + r^{j,k}_x \leq 2$ for any given $j, k$. Therefore, we conclude that for all $j$:
Implementing this upper bound with (11), and using that \( \int q_j(\lambda_j) d\lambda_j = 1 \) for any \( j \), we obtain that

\[
\sum_{X \in \mathcal{P}(\mathbb{N}_L)} |K_X|^{\ell/n} \leq 1. \tag{13}
\]

Notice that the inequality (5) admits various special cases that have been derived in earlier work: when \((n, L) = (2, 1)\) the inequality reduces to that found in [5], and the Bell-type inequality for the star-network found in [7] is recovered by considering only bipartite sources \((L = 1)\).

2.3. Tightness of the inequality

We will now state and prove the following theorem:

**Theorem 2.** The inequality (5) is tight in the sense that whenever a conditional probability distribution \( P(\cdot) \) on \( \Lambda_n \) satisfies (5), then \( P(\cdot) \) necessarily admits a classical model.

**Proof.** We show tightness of (5) by explicitly constructing a classical model that continuously saturates the classical bound i.e. we find a family of classical strategies parametrized by a set of continuous variables that saturate the classical bound for given \((n, L)\).

Let \( \pi \) be a string of \( n \times L \) random bits i.e. for \( j = 1, \ldots, n \) and \( k = 1, \ldots, L \), there is an \( \alpha_j^k \in \{0, 1\} \), subject to the probability distribution \( P(\alpha_j^k) = p_j^k \). Introduce a family of classical strategies \( D(\{p_j^k\}) \) which depend on the probability distribution of all the bits in \( \pi \),

\[
D : \ a_j^k = \lambda_j \oplus \alpha_j^k b = \begin{cases} \bigoplus \lambda_j & \text{if } L \text{ is odd} \\ 0 & \text{if } L \text{ is even} \end{cases}
\]

with the distribution of \( \lambda_j \in \{0, 1\} \) being \( q(\lambda_j) = \frac{1}{2} \).

For any \( X \in \mathcal{P}(\mathbb{N}_L) \) the strategy \( D(\{p_j^k\}) \) yields

\[
K_X = \prod_{j=1}^{n} \left( \prod_{k \in X} (1 - p_j^k) \prod_{k \notin X} p_j^k \right). \tag{15}
\]

Let us denote the expression inside the bracket by \( c_j^X \). We can then bound the left-hand side of (5) by

\[
\sum_{X \in \mathcal{P}(\mathbb{N}_L)} |K_X|^{\ell/n} = \sum_{X \in \mathcal{P}(\mathbb{N}_L)} \left( \prod_{j=1}^{n} c_j^X \right)^{\ell/n} \leq \prod_{j=1}^{n} \left( \sum_{X \in \mathcal{P}(\mathbb{N}_L)} c_j^X \right)^{\ell/n} = 1, \tag{16}
\]

where in the second step we have used the lemma (10) and in the last step used that \( \sum_{X} c_j^X = \prod_{k=1}^{L} (p_j^k + (1 - p_j^k))^{\ell/n} = 1 \) for every \( j \). Importantly, the inequality in the second step becomes an equality if and only if we impose that \( c_j^X = c_1^X = \ldots = c_n^X \) corresponding to constraining the probability distribution by \( p_j^0 = p_0^1 = p_1^1 = \ldots = p_n^0 \). Thus, the strategy \( D(p_0^1, \ldots, p_n^1) \) achieves the bound of (5) for every given value of \( \{p_j^k\} \). We did only consider the case in which all \( K_X \)'s are positive. However, since the inequality (5) only involves the
absolute values of $K_X$ it directly follows that all $2^L$ sign combinations in \{\pm K_0, \ldots, \pm K_{1,\ldots,L}\} lead to the bound (16). Therefore, we conclude that the inequality is tight in the sense that whenever a conditional probability distribution on $\mathcal{V}_n$ satisfies (5) then it necessarily admits a classical model.

Since we know that the inequality (5) is tight we can identify an important property for the set of classical correlations that satisfies the derived inequality: the set of points in the space $(K_0, \ldots, K_{1,\ldots,L})$ which satisfies the inequality (5) is non-convex. We illustrate the non-convexity in figure 2 (left sub-figure) in which we have set $(n, L) = (2, 2)$ and plotted the subset of the region that satisfies (5) in $(K_0, K_{1,1}, K_{1,2}, K_{1,3})$-space associated to $K_{1,2} = 1/16$. To illustrate that the non-convexity of the classical set associated to our inequality arises from the network structure with multiple sources, we have also plotted a subset of the classical set arising in the standard multipartite Bell experiment corresponding to $(n, L) = (1, 2)$ which is illustrated in figure 2 (right sub-figure) for $K_{1,2} = 1/4$.

3. Quantum correlations in connected Bell experiments

Quantum theory may predict violations of the inequality (5). A quantum model for the probability distribution $P(\hat{a}, b|\bar{x}, y)$ takes the form

$$P(a_1^\dagger a_n^\dagger b|x_1^\dagger \ldots x_n^\dagger y) = \text{tr}((M_{a_1|x_1^\dagger} \otimes \ldots \otimes M_{a_L|x_L^\dagger} \otimes M_{b|y})(\rho_1 \otimes \ldots \otimes \rho_n)),$$

where $M_{a_j|x_j^\dagger}$ is the measurement operator associated to outcome $a_j^\dagger$ when the measurement choice is $x_j$, and $\rho_j$ is the state of $L + 1$ qubits emitted by the $j$th source. The tensors are computed over the relevant Hilbert spaces (and the subspaces should therefore be rearranged accordingly).
Let Bob perform the measurements

$$M_{b|y} = \sum_{b_1 \oplus \ldots \oplus b_n = b} \Pi_{b_1}^y \otimes \ldots \otimes \Pi_{b_n}^y$$

for $y \in \{0, 1\}$ where we define $\Pi_{b_i}^0 = \frac{1}{2}(1 + (-1)^{b_i}X)$ and $\Pi_{b_i}^1 = \frac{1}{2}(1 + (-1)^{b_i}Y)$ corresponding to the projectors onto the positive and negative subspaces of the Pauli operators $X$ and $Y$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (19)$$

The action of Bob at the center node can be regarded as performing $n$ measurements, either all $X$ or all $Y$, with outcomes $b_1, \ldots, b_n$ and then process these $n$ outcomes into $b = b_1 \oplus \ldots \oplus b_n$ which is announced as the final output. Hence, Bob’s measurement is separable and the probability distribution (17) can be written

$$P(a_1^1, \ldots, a_n^L, b|x_1^1, \ldots, x_n^L, y_X)$$

\[= \sum_{b_1 \oplus \ldots \oplus b_n = b} \prod_{j=1}^n \text{tr}\left( \left( \bigotimes_{k=1}^L M_{a_j^k}\right) \Pi_{b_j}^y \right) \]

\[= \sum_{b_1 \oplus \ldots \oplus b_n = b} \prod_{j=1}^n P(a_j^1, \ldots, a_j^L, b|x_j^1, \ldots, x_j^L, y_X) \]

\[= \sum_{b_1 \ldots b_n} \prod_{j=1}^n P(a_j^1, \ldots, a_j^L, b_j|x_j^1, \ldots, x_j^L, y_X) \times \]

\[\times P(a_{n-1}^1, \ldots, a_{n-1}^L, b_{n-1}|x_{n-1}^1, \ldots, x_{n-1}^L, y_X) \times \]

\[\times P(a_n^1, \ldots, a_n^L, b \oplus b_1 \oplus \ldots \oplus b_{n-1}|x_n^1, \ldots, x_n^L, y_X). \quad (20)\]

In the first equality we have used the linearity of the trace operation and that $\text{tr}(O_1 \otimes O_2) = \text{tr}(O_1)\text{tr}(O_2)$. In the final equality we have expanded the product and rewritten the domain of the sum to go over $b_1, \ldots, b_{n-1}$ by setting $b_n = b \oplus b_1 \oplus \ldots \oplus b_{n-1}$.

Let all the $n$ sources distribute the $(L + 1)$-qubit Greenberger–Horne–Zeilinger (GHZ) state defined as $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0^\otimes L+1\rangle + |1^\otimes L+1\rangle)$ i.e. $\rho_j = |\text{GHZ}\rangle\langle \text{GHZ}|$ for every $j$. If we let the measurements associated to observers acting on a single qubit be either $X$ (labeled 0) or $Y$ (labeled 1) the probability distribution obtained in every separate multipartite Bell experiment takes the form [26]

$$P(a_j^1, \ldots, a_j^L, b|x_j^1, \ldots, x_j^L, y_X) = \frac{1}{2^{L+1}} \left[ 1 + (-1)^b \sum_{i=1}^L a_j^i \cos \left( \frac{\pi}{2} \sum_{k=1}^L x_j^k + y_X \right) \right]. \quad (21)$$

Substituting this into (20) we obtain the joint probability distribution

$$P(\vec{a}, b|\vec{x}, y_X) = \frac{1}{2^{n(L+1)}} \sum_{b_1 \ldots b_n} \left[ 1 + \ldots + (-1)^b \sum_{j=1}^n a_j^i \prod_{k=1}^n \cos \left( \frac{\pi}{2} \sum_{k=1}^L x_j^k + y_X \right) \right]. \quad (22)$$

The sum is taken over $2^n$ terms We observe that all the terms not explicitly written out contain a factor on the form $(-1)^b$ and that this will cause all terms not explicitly written to be cancelled when executing the sum. In conclusion we are left with
\[ P(\pi, b|\pi, y_X) = \frac{1}{2^{nL+1}} \left( 1 + (-1)^{b+\sum_{k=1}^{nL} \cos \left( \frac{\pi}{2} \sum_{k=1}^{nL} x_k^j + y_X \right) \right). \] (23)

Let us now compute the value of the left-hand side of (5) under the above calculated probability distribution. By direct insertion of (23) into the definition (3) of \( K_X \), we find that

\[ K_X = \frac{1}{2^{nL}} \sum_{\ell} \prod_{j=1}^{nL} (-1)^{\sum_{k=1}^{nL} \cos \left( \frac{\pi}{2} \sum_{k=1}^{nL} x_k^j + y_X \right) \right) . \] (24)

First, we need to define how we assign values to \( y_X \). We introduce the convention \( y_X = \frac{1}{2}(1 + (-1)^{L+R(L)}) \) where we define \( R(L) = 1 \) when \( L \equiv 0 \mod 4 \), and \( R(L) = 0 \) otherwise.

Let us begin with computing \( |K_\varnothing| \) for which all the factors of the form \((-1)^i\) in (24) are simply 1. Observe that each cosine-factor in the product series over \( j \) only can attain three different values, \( \pm 1 \) and 0. Therefore, for given \( x \), we can only have a non-zero contribution to \( K_\varnothing \) when all the cosine-factors in the product series take values \( \pm 1 \). Thus, there will be a contribution to \( K_\varnothing \) only from half of the allowed strings of measurement settings per constituent Bell experiment. Let \( C \) of the \( 2^{L-1} \) contributing strings (per Bell experiment) be the total number of strings that lead to a positive contribution to \( K_\varnothing \). Then, the number of strings making a negative contribution is \( 2^{L-1} - C \). We can therefore write \( |K_\varnothing| \) as

\[ |K_\varnothing| = \frac{1}{2^{nL}} \times (2 \max \{ C, 2^{L-1} - C \} - 2^{L-1} \), \] (25)

where we have used that the sum and products in (24) can be interchanged when \( X = \varnothing \).

Let us distinguish between when \( R(L) = 1 \), in which case we label \( C \rightarrow C_1 \) and when \( R(L) = 0 \) in which case we label \( C \rightarrow C_0 \). It is straightforward to compute \( C_0 \) and \( C_1 \). When \( R(L) = 1 \), we have a positive contribution to \( K_\varnothing \) from every string \( x_1^j ... x_L^j \) in which the entry 1 appears a multiple of four number of times:

\[ C_1 = \sum_{j=0}^{\lfloor L/4 \rfloor} \binom{L}{4j} = 2^{L/2-1} \left( 2^{L/2} + \cos \left( \frac{\pi L}{4} \right) + (-1)^L \cos \left( \frac{3\pi L}{4} \right) \right) . \] (26)

Similarly, when \( R(L) = 0 \), we have a positive contribution to \( K_\varnothing \) whenever the entry 1 appears in \( x_1^j ... x_L^j \) a number of times which takes the form \( 4j + 3 \) for some non-negative integer \( j \):

\[ C_0 = \sum_{j=0}^{\lfloor (L-3)/4 \rfloor} \binom{L}{4j+3} = 2^{L/2-1} \left( 2^{L/2} - \sin \left( \frac{\pi L}{4} \right) + (-1)^L \sin \left( \frac{3\pi L}{4} \right) \right) . \] (27)

However, we can simplify the above by noting that if \( L = 2 \mod 4 \), that is \( L = 4k + 2 \) for some non-negative integer \( k \), then \( C_0 \) can be written \( C_0 = 16^k - 4^k (-1)^k \). Similarly, using \( L = 0 \mod 4 \), that is \( L = 4k \) for some positive integer \( k \), we find \( C_1 = 4^{k-1}(4^k + 2(-1)^k) \).

Inserting either of these expressions into (25), we find that \( |K_\varnothing| = 1/\sqrt{2^{2nL}} \) which is thus true for any even \( L \). Performing the analog calculation for odd values of \( L \), one will find that \( |K_\varnothing| = 1/\sqrt{2^{2n(L+1)}} \).

In order to determine the value of all other \( |K_X| \) for \( X = \varnothing \), it is sufficient to note that the symmetries of (24) together with the conventions introduced for \( y_X \), will lead to \( |K_X| = |K_\varnothing| \) for all \( X \). Therefore, we end up with
A peculiarity is that our choice of measurements has led to a weaker violation for odd values of \( L \) (and in fact no violation for \( L = 1 \)). However, the violation for odd \( L \) can be improved by a modification of the measurements of the non-Bob observers. For such purpose, we keep Bob’s measurements as \((18)\) but change the measurements of all remaining observers to \((X + Y)/\sqrt{2} \) (labeled 0) and \((X - Y)/\sqrt{2} \) (labeled 1). For example, if we consider quantum correlations in \( \mathcal{A}_3 \) we find the probability distribution associated to a constituent Bell experiment, i.e. the analog of \((21)\) takes the form

\[
P(a^1 \ldots a^3, b|x^1 \ldots x^3, y_x) = \frac{1}{16} \left[ 1 + \frac{1}{\sqrt{2}} (-1)^{b + \sum_{i=1,2,3} (a^i + y_x x^i)} \right].
\]

Substituting this into \((20)\) and computing \(|K_X|\) in analogy with the above, one will find the violation \(\sum |K_X|^{1/n} = \sqrt{2} \not\leq 1\) which improves the previous result for odd \( L \). It is straightforward to generalize this to arbitrary \((n, L)\) which will lead to the violation \(\sum |K_X|^{1/n} = \sqrt{2^L}\) holding true for arbitrary \( L \).  

\[3.1.\text{Noise tolerance of quantum correlations}\]

Let us briefly study the possibility to violate the inequality \((5)\) by the above procedure in the presence of environments with white noise. That is, with some probability \( p \) a source will emit the GHZ state while with the probability \( 1 - p \) the emitted state is a random noise signal modeled by the fully mixed state \(1_{2L+1}\). Thus, the state emitted by the \( j \)th source is

\[
|\phi_j\rangle = p_j |\text{GHZ}\rangle \langle \text{GHZ}| + (1 - p_j) \frac{1}{2^{L+1}}.
\]

The visibility of the system is the product over the visibility of each source i.e. \( V = p_1 \ldots p_n \) \([5, 7, 8]\). Let us find the critical value of \( V \), by which we mean the largest number \( V \) such that we can no longer violate the inequality \((5)\) by the above method.

Since all the quantities \( K_X\) are linear combinations of conditional probabilities and \( K_X = 0 \) for a maximally mixed state, the \( V \)-dependent value of \( K_X \) scales linearly with \( V \) : \( K_X(V) = K_X \times V \). Thus, the critical value of \( V \) is found from solving \(2^L \times (V/\sqrt{2^L})^{1/n} = 1\) which returns \( V_{\text{crit}} = 2^{-1/4L} \). The noise tolerance of the quantum correlations thus scales exponentially with the number of observers and sources in the network.

This result can be compared to the critical visibility in the analog standard Bell scenario in which we remove the center node Bob and the \( n \) sources and instead introduce only a single source emitting an \( nL \)-particle state shared between the remaining observers. For such a scenario with all observers performing one of two two-outcome measurements the most popular Bell inequality is due to Mermin [3]. Analyzing quantum violations of Mermin’s inequality in the presence of imperfect visibilities yields a critical visibility identical to what we have obtained on our network \( \mathcal{A}_n \). However, the two scenarios are far from equivalent despite their common critical visibilities. Our assumption \((2)\) is a significantly stronger constraint than the assumption of local causality used in standard Bell inequalities. Evidently, not only does this stronger constraints affect the strength of classical correlations, but it also translates into a stronger constraint on the strength of quantum correlations on \( \mathcal{A}_n \) rendering the critical visibility the same as in Bell experiments.
3.2. Entanglement swapping in the center node

We have seen that there is no advantage in terms of critical visibility for the quantum correlations we have found over those obtained in Bell experiments. It is then reasonable to ask if such stronger quantum correlations can be obtained by letting Bob perform a more general measurement. Such a measurement could be to jointly measure the \( n \) qubits at Bob’s disposal in a basis of entangled states. This would cause the global state of the system, which initially is a tensor over independent sources, to become entangled i.e. the entanglement in the network has been swapped. The basis of entangled states can be taken as a set of \( 2^n \) GHZ-like states, indexed by a bit-string \( b^1 \ldots b^n \), obtained from

\[
\left| \psi_{b^1 \ldots b^n} \right> = Z b^1 \otimes X^{b^2} \otimes \ldots \otimes X^{b^n} \text{[GHZ]},
\]

where \( Z = |0\>\langle 0| - |1\>\langle 1| \).

Thus, such a complete entanglement swapping measurement would return one of \( 2^n \) possible outputs \( b^1 \ldots b^n \in \{0, 1\}^n \). However, the current form of our inequality (5) involves quantities \( K_X \) for each of which we may associate a different measurement of Bob (labeled by \( y_X \)) and which returns one of two outcomes. Therefore, we need to modify our inequality so that all \( K_X \) are associated to the single entanglement swapping measurement, and are conditioned on its output. Instead of conditioning \( K_X \) on the measurement \( y_X \) in (3), introduce a condition on some suitably chosen bit \( \hat{b}_X \) obtained from manipulations of Bob’s outcome string \( b^1 \ldots b^n \) by some function \( q_{\hat{b}} : \{0, 1\}^n \rightarrow \{0, 1\} \). That is, each quantity \( K_X \) is characterized by some mapping of the \( n \) output bits of Bob into a single bit \( \hat{b}_X = q_{\hat{b}}(b^1 \ldots b^n) \). We write

\[
K_X = \frac{1}{2^n} \sum_g \mathcal{G}(X) \sum_{x, b^1 \ldots b^n} (-1)^{\hat{b}_X + \sum_{x} a_{x} P(x, b^1 \ldots b^n | x)},
\]

for which the inequality (5) still holds. The function \( b_X = q_{\hat{b}}(b^1 \ldots b^n) \) can be freely chosen, preferably such that the violation of the inequality is maximized.

We have performed case studies for analyzing the strength of correlations arising from such entanglement swapping strategies. In particular, we have considered \( A \frac{\sqrt{2}}{2} \) for which we have modified the inequality (5) such that we let the outcome of Bob associated to \( K_{(1,2)} \) be \( b^1 \) and similarly \( b^2 \) for \( K_{(1)} \) and \( K_{(2)} \) i.e. \( q_{b_1} = q_{(1,2)} = b^1 \) and \( q_{b_2} = q_{(1,2)} = b^2 \). Our method is analogous to that outlined in [7]: we modify the problem so that it can be treated as a semi-definite program (SDP). To this end, we have to overcome the problem of the classical set of correlations being non-convex. We therefore restrict to the convex subset obtained from enforcing the restriction \( K_{a} = \ldots = K_{(1,2)} \). Also, we fix the measurements of all observers except Bob to \( (X + Y) / \sqrt{2} \) and \( (X - Y) / \sqrt{2} \), which makes the optimization linear and thus suitable for techniques relying on SDPs. An SDP has been run optimizing the left-hand side of our inequality over the measurement of Bob. Unsurprisingly such an optimization returns a measurement of Bob projecting the two qubits in a basis of Bell states of the type (31). However, the violation is the same as when using the separable measurement for Bob as in the above analysis, i.e. we find \( \sum_{x} \left| K_x \right| / 2 = 2 \). In addition several variations of the above have been tried: the condition \( K_{a} = \ldots = K_{b1} \) has been varied i.e. optimization has been performed along different convex subsets in combination with changing the fixed measurements of the non-Bob observers. Furthermore, numerical methods not based on SDPs have been used in brute-force optimizations in which we fix the measurement of Bob to a projection onto a basis of Bell states and optimize over the measurements of the non-Bob observers. Since this optimization problem is nonlinear it can easily return a local maxima. Therefore, the optimization was performed many times with different initial conditions.
Nevertheless, no improvement over the quantum violations from the previous subsection have been found. Our results fall in line with the work of \[7\] for the star-network (corresponding to \(L = 1\)) for which no gain was found over separable measurements by introducing a complete joint measurement on \(n\) qubits.

3.3. Transforming the inequality for \(\mathcal{N}_n^L\) into an inequality for a Bell experiment

The quantities \(K_X\) can be written on a more compact form by introducing correlators over the outcomes of all observers, \(R_{j,k}^L\), and Bob in \(\mathcal{N}_n^L\) defined as 
\[
\langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle \equiv \sum_{a,b} (-1)^{b+\sum_{k} d_k} P(a, b|\beta_k, y_k).
\]
We can therefore re-write the definition (3) compactly as 
\[
K_X = \frac{1}{2} \sum_{k} g(X) \langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle.
\]

Now, notice that the measurement chosen for Bob in the above analysis, namely (18), is a tensor product over the Hilbert spaces associated to respective qubit in his possession. Therefore, Bob’s observable factors

\[
B_{\lambda_1} = M_{0(\lambda_1)} - M_{1(\lambda_1)} = (\Pi_{j=1}^n - \Pi_{j=1}^n \geq 0) = \bigotimes_{j=1}^n B_j.
\]

This leads to a factorization 
\[
\langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle = \langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle \langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle.
\]
Thus, we re-write \(K_X\) as

\[
K_X = \frac{1}{2^n L} \sum_{j=1}^n \sum_{x_1 \ldots x_n} (-1)^{x_1 \ldots x_n} \langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle.
\]

In the above product over the \(n\) sources, we can restrict the correlators to being independent of the choice of \(j\). This makes the index \(j\) superfluous. Our inequality (5) takes the form

\[
\sum_{X \in \mathcal{P}(\mathcal{N}_X)} |K_X|^{1/n} = \sum_{X \in \mathcal{P}(\mathcal{N}_X)} \left| \frac{1}{2^n L} \sum_{x_1 \ldots x_n} (-1)^{x_1 \ldots x_n} \langle B_{\lambda_1} R_{j_1}^{l_1} \ldots R_{j_n}^{n_{l_n}} \rangle \right| \leq 1,
\]

which only considers the measurements and outcomes of a single Bell experiment (associated to a single multipartite source). Observe that for \(L = 1\) this is equivalent to the CHSH inequality [27]. This can be seen from the fact that the domain of the sum reduces to \(\mathcal{P}(\mathcal{N}_X) = \{\emptyset, \{1\}\}\). Then

\[
\sum_{X = \emptyset, \{1\}} \left| \frac{1}{2} \sum_{x_1 = 0, 1} (-1)^{x_1} \langle B_{\lambda_1} R_{j_1}^{l_1} \rangle \right| = \frac{1}{2} \left| \langle B_{\lambda_1} R_{j_1}^{l_1} \rangle + \langle B_{\lambda_1} R_{j_1}^{l_1} \rangle \right| + \frac{1}{2} \left| \langle B_{\lambda_1} R_{j_1}^{l_1} \rangle - \langle B_{\lambda_1} R_{j_1}^{l_1} \rangle \right| \leq 1.
\]

Identifying the labels \(y_0\) and \(y_{11}\) with ‘0’ and ‘1’ respectively, we see that we have derived the CHSH inequality.

By a similar procedure we can also derive Mermin’s inequality for three observers \((L = 2)\) as a special case of (35) by removing the absolute value with a plus sign for \(X = \emptyset, \{1\}, \{2\}\) and with a minus sign for \(X = \{1, 2\}\, and reduce Bob’s four measurements to only two by setting \(y_{12} = y_{11}\) and \(y_{1|2} = y_0\). We will then find
Figure 3. The network $\mathcal{N}^{1,2,3}_3$.

\[
K_{\emptyset}^{1/n} + K_1^{1/n} + K_2^{1/n} - K_{\{1,2\}}^{1/n} = \frac{1}{2} \left( \langle B_1R_1^1R_2^2 \rangle - \langle B_2R_0^1R_1^2 \rangle + \langle B_2R_0^1R_2^3 \rangle + \langle B_2R_0^1R_2^3 \rangle - \langle B_1R_1^1R_2^2 \rangle \right) \leq 1. \tag{37}
\]

Identifying the labels $y_\emptyset$ and $y_1$ with ‘0’ and ‘1’ returns Mermin’s inequality.

4. Generalization to a broader class of networks

Let us briefly show how the technique for deriving Bell-type inequalities on $\mathcal{N}^L_n$ can be generalized to include also other networks. So far, we have let every source in the network emit a system of $L + 1$ qubits so that one qubit is sent to Bob and each of the remaining $L$ qubits to a distinct observer. We will now relax this condition and let the $j$th source emit an arbitrary number, $L_j + 1$, of qubits such that one qubit is sent to Bob and the remaining $L_j$ sent to distinct observers. This new family of networks can be intuitively understood as connecting $n$ Bell experiments through a central node such that the $j$th Bell experiment involves $L_j + 1$ observers. We can characterize the configuration of the network by the numbers $nL_{L_1\ldots L_n}$ and we abbreviate the network as $\mathcal{N}^{L_1\ldots L_n}_n$. In Figure 3 we exemplify this by illustrating the network $\mathcal{N}^{1,2,3}_3$.

To extend our inequalities (5) to account for correlations on $\mathcal{N}^{L_1\ldots L_n}_n$, we will define quantities analogous of those in (3): for every subset $X$ of $\mathbb{N}_L = \{1,\ldots,L\}$ with $L \equiv \max\{L_1,\ldots,L_n\}$ we define

\[
Q_X = \frac{1}{2^{\sum_{j=1}^n L_j}} \sum_X h(X) \langle B_{x_1}R_{x_1}^{1,1}\ldots R_{x_j}^{1,L_j}\ldots R_{x_n}^{n,L_n} \rangle, \tag{38}
\]

where the $h(X) = \prod_{i=1}^n (-1)^{\sum_{j=0}^{L_j-1}(-1)^j}$ and $X_j = \{s \in X | s \leq L_j\}$. Note that when $L_1 = \ldots = L_n$, $Q_X$ reduces to $K_X$.

To bound $|Q_X|$, we can use a direct analogy of the technique used to bound the quantities $|K_X|$ in (9). Using (10), we will then be led to the Bell-type inequality

\[
\sum_{X \in \mathbb{P}(\mathbb{N}_L)} |Q_X|^{1/n} \leq 2^L \times 2^{\sum_{j=1}^n L_j}, \tag{39}
\]
where the classical bound arises from the fact that for every $X_j$ there exists $2^{L-L_j}$ different sets $X \subset \mathbb{N}_L$ that contain $X_j$. Again, note that when $L_1 = \ldots = L_n$, the classical bound reduces to that of inequality (5), namely one.

The analysis of the quantum violations of inequality (39) is again a straightforward modification of our previous analysis in section 3. We will need to impose measurements of Bob that are somewhat different from (18). Let us note that there can be at most $L$ different elements appearing in the set $\{L_1, \ldots, L_n\}$, and let us denote these elements by $r_1, \ldots, r_L$ for some $L \leq L$. Also, let $y_1, \ldots, y_L$ be an $l$-bit string. Bob’s measurements are defined as

$$M_{\theta | y_1, \ldots, y_L} = \sum_{b_1 \oplus \ldots \oplus b_N = b} \pi^{y_1}_{b_1} \otimes \ldots \otimes \pi^{y_L}_{b_L} \otimes \ldots \otimes \pi^{y_{N-l}}_{b_{N-l}}.$$  

(40)

In the case of $l = 1$, we recover the two measurements (18). More generally, there will be $2^l$ different measurements of Bob.

Since the measurements (40) are separable, the resulting joint probability distribution in $\mathcal{A}^{L_{L_1}, L_{L_2}}$ will be subjected to a decomposition analogous to that in (20) and thus the analysis reduces to considering the probability distribution associated to each source separately. If we distribute the state $\frac{1}{\sqrt{2}}(|0\rangle^{\otimes L_j} + |1\rangle^{\otimes L_j})$ in the $j$th source, and let the non-Bob parties perform measurements $(X + Y)/\sqrt{2}$ or $(X - Y)/\sqrt{2}$, we will be lead to strong quantum violations analogous to those obtained in section 3: $|Q_\mathcal{X}| = 2^{-\frac{L}{2}} \sum_{j=1}^{L_j}$ leading to the violation of the inequality

$$\sum_{X \in \mathcal{P}(\mathbb{N}_L)} |Q_\mathcal{X}|^{1/n} = 2L \times 2^{L - \frac{1}{2} \sum_{j=1}^{L_j}}$$  

(41)

which is clearly a violation of (39) for all $\{L_1, \ldots, L_n\}$.

5. Conclusion

We have studied correlations in a class of network configurations which can be understood as connecting many independent multipartite Bell experiments through a central node. We have derived tight Bell-type inequalities for such networks. By investigating quantum models of the arising probabilities, we found strong violations of our inequalities scaling exponentially with the number of observers connected to a source. However, the violation was found to be independent of the number of sources in the network. Somewhat surprisingly, we did not manage to increase the violation of our inequalities by performing an entanglement swapping measurement at the center node, which intuition suggests should offer a stronger result than performing a separable measurement. If entanglement swapping would have yielded an enhancement, then that would have constituted the sought for advantage over Bell experiments. We note that also previous works have studied network correlations in unsuccessful attempts to find an advantage over Bell experiments [5, 7]. In this sense, also our networks with multiparticle sources failed in finding the sort of advantage anticipated from [25]: the quantum correlations found on the studied networks were not stronger (more resistant to noise) than the analog quantum correlations in Bell experiments. In the original work finding the advantage of networks [25], a particular scenario is considered similar to the networks studied here in which the advantage over a Bell experiment on a single copy of a state appears in a network with seven sources in which a center node performing entanglement swapping. Therefore, it may be the case that an advantage over Bell experiments can be found for the networks analyzed here in the case of Bob performing an entanglement swapping measurement if there are sufficiently many sources in the network. However, due to limited available
computation power, this could not be efficiently investigated with numerics for networks of somewhat large size.

Our negative result also suggests that it may be necessary to look for advantages over Bell experiments in other types of correlation experiments. It is interesting to study correlations in networks in which all observers perform measurements with more than two outcomes. For such purposes, novel techniques going beyond those presented here will most likely be necessary.

Acknowledgments

The author thanks Paul Skrzypczyk, Daniel Cavalcanti and Antonio Acín for discussions.

References

[1] Bell J S 1964 On the Einstein–Podolsky–Rosen paradox Physics 1 195
[2] Brunner N, Cavalcanti D, Pironio S, Scarani V and Wehner S 2014 Bell nonlocality Rev. Mod. Phys. 86 419478
[3] Mermin N D 1990 Extreme quantum entanglement in a superposition of macroscopically distinct states Phys. Rev. Lett. 65 1838
[4] Branciard C, Gisin N and Pironio S 2010 Characterizing the nonlocal correlations created via entanglement swapping Phys. Rev. Lett. 104 170401
[5] Branciard C, Rosset D, Gisin N and Pironio S 2012 Bilocal versus non-bilocal correlations in entanglement swapping experiments Phys. Rev. A 85 032119
[6] Mukherjee K, Paul B and Sarkar D 2015 Correlations in n-local scenario Quantum Inf. Process. 14 2025–42
[7] Tavakoli A, Skrzypczyk P, Cavalcanti D and Acín A 2014 Nonlocal correlations in the star-network configuration Phys. Rev. A 90 062109
[8] Rosset D, Branciard C, Barnea T J, Pütz G, Brunner N and Gisin N 2015 Nonlinear Bell inequalities tailored for quantum networks Phys. Rev. Lett. 116 010403
[9] Chaves R 2015 Polynomial Bell inequalities Phys. Rev. Lett. 116 010402
[10] Chiribella G, D’Ariano G M and Perinotti P 2010 Probabilistic theories with purification Phys. Rev. A 81 062348
[11] Fritz T 2012 Beyond Bell’s theorem: correlations scenarios New J. Phys. 14 103001
[12] Fritz T 2014 Beyond: II. Bell’s theorem scenarios with arbitrary causal structure Comm. Math. Phys. 341 391–434
[13] Leifer M S and Spekkens R W 2013 Towards a formulation of quantum theory as a causally neutral theory of Bayesian inference Phys. Rev. A 88 052130
[14] Chaves R, Klyachko A, Maciel T O, Gross D, Janzing D and Schölkopf B 2014 Inferring latent structures via information inequalities Proc. of the 30th Conf. on Uncertainty in Artificial Intelligence (UAI 2014) (AUAI Press) pp 112–21
[15] Henson J, Lal R and Pusey M F 2014 Theory-independent limits on correlations from generalized Bayesian networks New J. Phys. 16 113043
[16] Lee C M and Spekkens R W 2015 Causal inference via algebraic geometry: necessary and sufficient conditions for the feasibility of discrete causal models arXiv:1506.03880
[17] Chaves R, Kueng R, Brask J B and Gross D 2015 Unifying framework for relaxations of the causal assumptions in Bells theorem Phys. Rev. Lett. 114 140403
[18] Wood C J and Spekkens R W 2015 The lesson of causal discovery algorithms for quantum correlations: causal explanations of Bell-inequality violations require fine-tuning New J. Phys. 17 073020
[19] Chaves R, Majenz C and Gross D 2015 Information theoretic implications of quantum causal structures Nat. Commun. 6 5766
[20] Pironio S et al 2010 Random numbers certified by Bells theorem Nature 464 1021
[21] Zukowski M, Zeilinger A, Horne M A and Ekert A K 1993 Event-ready-detectors Bell experiment via entanglement swapping Phys. Rev. Lett. 71 4287
[22] Acín A, Cirac I and Lewenstein M 2007 Entanglement percolation in quantum networks Nat. Phys. 3 256
[23] Sangouard N, Simon C, de Riedmatten H and Gisin N 2011 Quantum repeaters based on atomic ensembles and linear optics Rev. Mod. Phys. 83 33
[24] Sen(De) A, Sen U, Brukner C, Buzek V and Zukowski M 2005 Entanglement swapping of noisy states: a kind of superadditivity in nonclassicality Phys. Rev. A 72 042310
[25] Cavalcanti D, Almeida M L, Scarani V and Acín A 2010 Quantum networks reveal quantum nonlocality Nat. Commun. 2 184
[26] Liang Y-C, Curchod F J, Bowles J and Gisin N 2014 Anonymous quantum nonlocality Phys. Rev. Lett. 113 130401
[27] Clauser J F, Horne M A, Shimony A and Holt R A 1969 Proposed experiment to test local hidden-variable theories Phys. Rev. Lett. 23 880