Compactness and existence results for the $p$-Laplace equation

Marino Badiale$^{a,b}$ - Michela Guida$^{a,c}$ - Sergio Rolando$^{d,c}$

Abstract

Given $1 < p < N$ and two measurable functions $V(r) \geq 0$ and $K(r) > 0$, $r > 0$, we define the weighted spaces

$$W = \left\{ u \in D^{1,p}(\mathbb{R}^N): \int_{\mathbb{R}^N} V(|x|)|u|^p \, dx < \infty \right\}, \quad L^q_K = L^q(\mathbb{R}^N, K(|x|) \, dx)$$

and study the compact embeddings of the radial subspace of $W$ into $L^{q_1}_K + L^{q_2}_K$, and thus into $L^q_K (= L^q_K + L^q_K)$ as a particular case. We consider exponents $q_1, q_2, q$ that can be greater or smaller than $p$. Our results do not require any compatibility between how the potentials $V$ and $K$ behave at the origin and at infinity, and essentially rely on power type estimates of their relative growth, not of the potentials separately. We then apply these results to the investigation of existence and multiplicity of finite energy solutions to nonlinear $p$-Laplace equations of the form

$$- \Delta_p u + V(|x|)|u|^{p-1} u = g(|x|, u) \quad \text{in } \mathbb{R}^N, 1 < p < N,$$

where $V$ and $g(|\cdot|, u)$ with $u$ fixed may be vanishing or unbounded at zero or at infinity. Both the cases of $g$ super and sub $p$-linear in $u$ are studied and, in the sub $p$-linear case, nonlinearities with $g(|\cdot|, 0) \neq 0$ are also considered.

MSC (2010): Primary 35J92; Secondary 35J20, 46E35, 46E30

Keywords: Quasilinear elliptic equations with $p$-Laplacian, unbounded or decaying potentials, weighted Sobolev spaces, compact embeddings

1 Introduction

In this paper we pursue the work we made in papers [3,4,9], where we studied embedding and compactness results for weighted Sobolev spaces in order to get existence and multiplicity results for semilinear elliptic equations in $\mathbb{R}^N$, by variational methods.

In the present paper, we face nonlinear elliptic $p$-Laplace equations with radial potentials, whose prototype is

$$- \Delta_p u + V(|x|)|u|^{p-1} u = K(|x|) f(u) \quad \text{in } \mathbb{R}^N$$

\footnote{\text{Dipartimento di Matematica “Giuseppe Peano”, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino, Italy. e-mails: marino.badiale@unito.it, michela.guida@unito.it}}

\footnote{\text{Partially supported by the PRIN2012 grant “Aspetti variazionali e perturbativi nei problemi di.renziali nonlineari”.}}

\footnote{\text{Member of the Gruppo Nazionale di Alta Matematica (INdAM).}}

\footnote{\text{Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Via Roberto Cozzi 53, 20125 Milano, Italy. e-mail: sergio.rolando@unito.it}}
(more general nonlinear terms will be actually considered in the following). Here $1 < p < N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying $f(0) = 0$ and $V \geq 0$, $K > 0$ are given potentials, which may be vanishing or unbounded at the origin or at infinity.

To study this problem we introduce the weighted Sobolev space

$$ W := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|) |u|^p \, dx < \infty \right\} $$

equipped with the standard norm

$$ \|u\|^p := \int_{\mathbb{R}^N} \left| \nabla u \right|^p + V(|x|) |u|^p \, dx, $$

and we say that $u \in W$ is a weak solution to (1) if

$$ \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx + \int_{\mathbb{R}^N} V(|x|) |u|^{p-2} uh \, dx = \int_{\mathbb{R}^N} K(|x|) f(u) h \, dx \quad \text{for all } h \in W. $$

The natural approach in studying weak solutions to equation (1) is variational, since these solutions are (at least formally) critical points of the Euler functional

$$ J(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} K(|x|) F(u) \, dx, $$

where $F(t) := \int_0^t f(s) \, ds$. Then the problem of existence is easily solved if $V$ does not vanish at infinity and $K$ is bounded, because standard embeddings theorems of $W$ and its radial subspace $W_r$ into the weighted Lebesgue space $L^q_K := L^q(\mathbb{R}^N, K(|x|) \, dx)$ are available (for suitable $q$’s). As we let $V$ and $K$ to vanish, or to go to infinity, as $|x| \to 0$ or $|x| \to +\infty$, the usual embeddings theorems for Sobolev spaces are not available anymore, and new embedding theorems need to be proved. This has been done in several papers: see e.g. the references in [3,4,9] for a bibliography concerning the usual Laplace equation, and [1,5–8,10,13–15,17–19] for equations involving the $p$-laplacian.

The main novelty of our approach (in [3,4] and in the present paper) is two-fold. First, we look for embeddings of $W_r$ not into a single Lebesgue space $L^q_K$, but into a sum of Lebesgue spaces $L^{q_1}_K + L^{q_2}_K$. This allows to study separately the behaviour of the potentials $V$ and $K$ at 0 and $\infty$, and to assume different set of hypotheses about these behaviours. Second, we assume hypotheses not on $V$ and $K$ separately but on their ratio, so allowing asymptotic behaviours of general kind for the two potentials.

Thanks to these novelties, our embedding results yield existence of solutions for (1) in cases which are not covered by the previous literature. Moreover, one can check that our embeddings are also new in some of the cases already treated in previous papers (see e.g. Example 3.5), thus giving existence results which improve some well-known theorems in the literature.

The proofs of our embedding theorems for the space $W_r$ are generalizations of those presented in [4] for the Hilbertian case $p = 2$. The generalizations to the case $1 < p < N$ are not difficult but boring and lengthy, because one needs to repeat a lot of detailed computations, the basic ideas remaining the same. In view of this, in the present paper we limit ourselves to state our embedding results and to present in detail some examples, leading to new existence results for equation (1). For all the proofs, with full details, we refer the reader to the specific document [2], which is essentially a longer version of Section 2 below.
This paper is organized as follows. In Section 2 we state our main results: a general result concerning the embedding properties of \( W_p \) into \( L^{q_1}_K + L^{q_2}_K \) (Theorem 2.1) and some explicit conditions ensuring that the embedding is compact (Theorems 2.2 and 2.3). In Section 3 we apply our compactness results to some examples, with a view to both illustrate how to use them in concrete cases and to compare them with the related literature. In Section 4 we present existence and multiplicity results for equations like (1), but with more general nonlinearities, whose proofs are given in Section 5.

Notations. We end this introductory section by collecting some notations used in the paper.

- We denote \( \| \cdot \|_X \) and let \( X' \) denote the norm and the dual space of a Banach space \( X \), in which \( \to \) and \( \rightharpoonup \) mean strong and weak convergence respectively.
- \( \to \) denotes continuous embeddings.
- \( C^\infty_c(\Omega) \) is the space of the infinitely differentiable real functions with compact support in \( \Omega \subseteq \mathbb{R}^d \) open.
- For any measurable set \( A \subseteq \mathbb{R}^d \), \( L^p(A) \) and \( L^q(A) \) are the usual real Lebesgue spaces and, if \( \rho : A \to \mathbb{R}_+ \) is a measurable function, \( L^p(A, \rho(z)dz) = \) the real Lebesgue space with respect to the measure \( \rho(z)dz \) (\( dz \) stands for the Lebesgue measure on \( \mathbb{R}^d \)). In particular, if \( K : \mathbb{R}_+ \to \mathbb{R}_+ \) is measurable, we denote \( L^q_K(E) := L^q(E, K(|x|) \, dx) \) for any measurable set \( E \subseteq \mathbb{R}^N \).
- For \( 1 < p < N \), \( p^* := \frac{pN}{N-p} \) is the Sobolev critical exponent and \( D^{1,p}(\mathbb{R}^N) = \{ u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N) \} \) is the usual Sobolev space, which identifies with the completion of \( C^\infty_c(\mathbb{R}^N) \) with respect to the norm of the gradient. \( D^{1,p}_{\text{rad}}(\mathbb{R}^N) \) is the radial subspace of \( D^{1,p}(\mathbb{R}^N) \).

2 Compactness results

Assume \( 1 < p < N \) and consider two functions \( V, K \) such that:

(V) \( V : \mathbb{R}_+ \to [0, +\infty] \) is a measurable function such that \( V \in L^1((r_1, r_2)) \) for some \( r_2 > r_1 > 0 \);

(K) \( K : \mathbb{R}_+ \to \mathbb{R}_+ \) is a measurable function such that \( K \in L^{s}_{\text{loc}}(\mathbb{R}_+) \) for some \( s > 1 \).

Define the function spaces

\[
W := D^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V(|x|)dx), \quad W_r := D^{1,p}_{\text{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V(|x|)dx)
\]

and let \( \| u \| \) be the standard norm of (3) in \( W \) (and \( W_r \)). Assumption (V) implies that the spaces \( W \) and \( W_r \) are nontrivial, while hypothesis (K) ensures that \( W_r \) is compactly embedded into the weighted Lebesgue space \( L^{q_i}_r(B_R \setminus B_r) \) for every \( 1 < q < \infty \) and \( R > r > 0 \) (see \( \mathbf{2} \) Lemma 3.1). In what follows, the summability assumptions in (V) and (K) will not play any other role than this.

Given \( V \) and \( K \), we define the following functions of \( R > 0 \) and \( q > 1 \):

\[
S_0(q,R) := \sup_{u \in W_r, \| u \|=1} \int_{B_R} K(|x|) |u|^q \, dx,
\]

\[
S_\infty(q,R) := \sup_{u \in W_r, \| u \|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|) |u|^q \, dx.
\]

Clearly \( S_0(q, \cdot) \) is nondecreasing, \( S_\infty(q, \cdot) \) is nonincreasing and both of them can be infinite at some \( R \).

Our first result concerns the embedding properties of \( W_r \) into the sum space

\[
L^{q_1}_K + L^{q_2}_K := \{ u_1 + u_2 : u_1 \in L^{q_1}_K(\mathbb{R}^N), u_2 \in L^{q_2}_K(\mathbb{R}^N) \}, \quad 1 < q_i < \infty.
\]
We recall from [5] that such a space can be characterized as the set of measurable mappings \( u : \mathbb{R}^N \to \mathbb{R} \) for which there exists a measurable set \( E \subseteq \mathbb{R}^N \) such that \( u \in L^p_K (E) \cap L^q_K (E^c) \). It is a Banach space with respect to the norm

\[
\| u \|_{L^p_K + L^q_K} := \inf_{u_1 + u_2 = u} \max \left\{ \| u_1 \|_{L^p_K (\mathbb{R}^N)} , \| u_2 \|_{L^q_K (\mathbb{R}^N)} \right\}
\]

and the continuous embedding \( L^q_K \to L^p_K + L^q_K \) holds for all \( q \in \left[ \min \{ q_1 , q_2 \} , \max \{ q_1 , q_2 \} \right] \). The assumptions of our result are quite general, sometimes also sharp (see claim (iii)), but not so easy to check, so that the next results will be devoted to provide more handy conditions ensuring such general assumptions.

**Theorem 2.1.** Let \( 1 < p < N \), let \( V, K \) be as in (V), (K) and let \( q_1 , q_2 > 1 \).

(i) If \( S_0 ( q_1 , R_1 ) < \infty \) and \( S_\infty ( q_2 , R_2 ) < \infty \) for some \( R_1 , R_2 > 0 \), \( (S'_{q_1 , q_2}) \)

then \( W_r \) is continuously embedded into \( L^p_K (\mathbb{R}^N) + L^q_K (\mathbb{R}^N) \).

(ii) If

\[
\lim_{R \to 0^+} S_0 ( q_1 , R ) = \lim_{R \to +\infty} S_\infty ( q_2 , R ) = 0,
\]

\( (S''_{q_1 , q_2}) \)

then \( W_r \) is compactly embedded into \( L^p_K (\mathbb{R}^N) + L^q_K (\mathbb{R}^N) \).

(iii) If \( K (|\cdot|) \in L^1 (B_1) \) and \( q_1 \leq q_2 \), then conditions \( (S'_{q_1 , q_2}) \) and \( (S''_{q_1 , q_2}) \) are also necessary to the above embeddings.

Observe that, of course, \((S''_{q_1 , q_2})\) implies \((S'_{q_1 , q_2})\). Moreover, these assumptions can hold with \( q_1 = q_2 = q \) and therefore Theorem 2.1 also concerns the embedding properties of \( W_r \) into \( L^q_K \), \( 1 < q < \infty \).

We now look for explicit conditions on \( V \) and \( K \) implying \((S''_{q_1 , q_2})\) for some \( q_1 \) and \( q_2 \). More precisely, we will ensure \((S''_{q_1 , q_2})\) through a more stringent condition involving the following functions of \( R > 0 \) and \( q > 1 \):

\[
\mathcal{R}_0 ( q , R ) := \sup_{u \in W_r , \ h \in W , \ \| u \| = \| h \| = 1} \int_{B_R} K (|x|) |u|^{q-1} |h| \, dx,
\]

\[
\mathcal{R}_\infty ( q , R ) := \sup_{u \in W_r , \ h \in W , \ \| u \| = \| h \| = 1} \int_{R^N \setminus B_R} K (|x|) |u|^{q-1} |h| \, dx.
\]

Note that \( \mathcal{R}_0 ( q , \cdot ) \) is nondecreasing, \( \mathcal{R}_\infty ( q , \cdot ) \) is nonincreasing and both can be infinite at some \( R \). Moreover, for every \( (q , R) \) one has \( S_0 ( q , R ) \leq \mathcal{R}_0 ( q , R ) \) and \( S_\infty ( q , R ) \leq \mathcal{R}_\infty ( q , R ) \), so that \((S''_{q_1 , q_2})\) is a consequence of the following, stronger condition:

\[
\lim_{R \to 0^+} \mathcal{R}_0 ( q_1 , R ) = \lim_{R \to +\infty} \mathcal{R}_\infty ( q_2 , R ) = 0,
\]

\( (R''_{q_1 , q_2}) \).

In Theorems 2.2 and 2.7 we will find ranges of exponents \( q_1 \) such that \( \lim_{R \to 0^+} \mathcal{R}_0 ( q_1 , R ) = 0 \). In Theorems 2.3 and 2.4 we will do the same for exponents \( q_2 \) such that \( \lim_{R \to +\infty} \mathcal{R}_\infty ( q_2 , R ) = 0 \). Condition \((R''_{q_1 , q_2})\) then follows by joining Theorem 2.2 or 2.7 with Theorem 2.3 or 2.4.

For \( \alpha \in \mathbb{R} \) and \( \beta \in [0, 1] \), define two functions \( \alpha^* ( \beta ) \) and \( q^* ( \alpha , \beta ) \) by setting

\[
\alpha^* ( \beta ) := \max \left\{ p \beta - 1 - \frac{p - 1}{p} N , - (1 - \beta) N \right\} = \begin{cases} p \beta - 1 - \frac{p - 1}{p} N & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ - (1 - \beta) N & \text{if } \frac{1}{p} \leq \beta \leq 1 \end{cases}
\]

4
and
\[
q^* (\alpha, \beta) := p \frac{\alpha - p\beta + N}{N - p}.
\]

Note that \( \alpha^* (\beta) \leq 0 \) and \( \alpha^* (\beta) = 0 \) if and only if \( \beta = 1 \).

The first two Theorems \( 2.2 \) and \( 2.3 \) only rely on a power type estimate of the relative growth of the potentials and do not require any other separate assumption on \( V \) and \( K \) than \( (V) \) and \( (K) \), including the case \( V = 0 \) (see Remark \( 2.4 \)).

**Theorem 2.2.** Let \( 1 < p < N \) and let \( V, K \) be as in \( (V), (K) \). Assume that there exists \( R_1 > 0 \) such that \( V (r) < +\infty \) almost everywhere in \( (0, R_1) \) and
\[
\text{ess sup}_{r \in (0, R_1)} \frac{K(r)}{r^\alpha V(r)^{\beta_0}} < +\infty \quad \text{for some } 0 \leq \beta_0 \leq 1 \text{ and } \alpha_0 > \alpha^* (\beta_0).
\]
Then \( \lim_{R \to 0^+} R_0 (q_1, R) = 0 \) for every \( q_1 \in \mathbb{R} \) such that
\[
\max \{1, p\beta_0\} < q_1 < q^* (\alpha_0, \beta_0).
\]

**Theorem 2.3.** Let \( 1 < p < N \) and let \( V, K \) be as in \( (V), (K) \). Assume that there exists \( R_2 > 0 \) such that \( V (r) < +\infty \) for almost every \( r > R_2 \) and
\[
\text{ess sup}_{r > R_2} \frac{K(r)}{r^\alpha V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \leq \beta_\infty \leq 1 \text{ and } \alpha_\infty \in \mathbb{R}.
\]
Then \( \lim_{R \to +\infty} R_\infty (q_2, R) = 0 \) for every \( q_2 \in \mathbb{R} \) such that
\[
q_2 > \max \{1, p\beta_\infty, q^* (\alpha_\infty, \beta_\infty)\}.
\]

We observe explicitly that for every \( (\alpha, \beta) \in \mathbb{R} \times [0, 1] \) one has
\[
\max \{1, p\beta, q^* (\alpha, \beta)\} = \begin{cases} q^* (\alpha, \beta) & \text{if } \alpha \geq \alpha^* (\beta) \\ \max \{1, p\beta\} & \text{if } \alpha \leq \alpha^* (\beta) \end{cases}.
\]

**Remark 2.4.**

1. We mean \( V (r)^0 = 1 \) for every \( r \) (even if \( V (r) = 0 \)). In particular, if \( V (r) = 0 \) for almost every \( r > R_2 \), then Theorem \( 2.3 \) can be applied with \( \beta_\infty = 0 \) and assumption \( 10 \) means
\[
\text{ess sup}_{r > R_2} \frac{K(r)}{r^{\alpha_\infty}} < +\infty \quad \text{for some } \alpha_\infty \in \mathbb{R}.
\]

Similarly for Theorem \( 2.2 \) and assumption \( 8 \), if \( V (r) = 0 \) for almost every \( r \in (0, R_1) \).

2. The inequality \( \max \{1, p\beta_0\} < q^* (\alpha_0, \beta_0) \) is equivalent to \( \alpha_0 > \alpha^* (\beta_0) \). Then, in \( 9 \), such inequality is automatically true and does not ask for further conditions on \( \alpha_0 \) and \( \beta_0 \).
3. The assumptions of Theorems 2.2 and 2.3 may hold for different pairs \((\alpha_0, \beta_0), (\alpha_\infty, \beta_\infty)\). In this case, one chooses them in order to get the ranges for \(q_1, q_2\) as large as possible. For instance, if \(V\) is essentially bounded in a neighbourhood of 0 and condition (8) holds true for a pair \((\alpha_0, \beta_0)\), then (8) also holds for all pairs \((\alpha_0', \beta_0')\) such that \(\alpha_0' < \alpha_0\) and \(\beta_0' < \beta_0\). Therefore, since max \(\{1, p\beta\}\) is nondecreasing in \(\beta\) and \(q^*(\alpha, \beta)\) is increasing in \(\alpha\) and decreasing in \(\beta\), it is convenient to choose \(\beta_0 = 0\) and the best interval where one can take \(q_1\) is \(1 < q_1 < q^*= (\varpi, 0)\) with \(\varpi := \sup\{\alpha_0 : \text{ess sup}_{r \in (0, R)} K(r)/r^{\alpha_0} < +\infty\}\) (we mean \(q^* = +\infty, 0) = +\infty\).

For any \(\alpha \in \mathbb{R}, \beta \leq 1\) and \(\gamma \in \mathbb{R}\), define
\[
q_*(\alpha, \beta, \gamma) := p\frac{\alpha - \gamma \beta + N}{N - \gamma} \quad \text{and} \quad q^{**}(\alpha, \beta, \gamma) := p\frac{\alpha + (1 - p\beta)\gamma + p(N - 1)}{p(N - 1) - \gamma(p - 1)}.
\]

(12)

Of course \(q_*\) and \(q^{**}\) are undefined if \(\gamma = N\) and \(\gamma = \frac{p\beta}{\beta-1} (N - 1)\), respectively.

The next Theorems 2.5 and 2.7 improve the results of Theorems 2.2 and 2.3 by exploiting further informations on the growth of \(V\) (see Remarks 2.6.2 and 2.8.3).

**Theorem 2.5.** Let \(1 < p < N\) and let \(V, K\) be as in (V), (K). Assume that there exists \(R_2 > 0\) such that \(V(r) < +\infty\) for almost every \(r \geq R_2\) and
\[
\text{ess sup}_{r > R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty \quad \text{for some} \ 0 \leq \beta_\infty \leq 1 \text{ and } \alpha_\infty \in \mathbb{R}
\]
(13)

and
\[
\text{ess inf}_{r > R_2} r^{\gamma_\infty} V(r) > 0 \quad \text{for some} \ \gamma_\infty \leq p.
\]
(14)

Then \(\lim_{R \to +\infty} \mathcal{R}_{\infty}(q_2, R) = 0\) for every \(q_2 \in \mathbb{R}\) such that
\[
q_2 > \max\{1, p\beta_\infty, q_*, q^{**}\},
\]
(15)

where \(q_* = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)\) and \(q^{**} = q^{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)\).

For future convenience, we define three functions \(\alpha_1 := (\alpha, \gamma), \alpha_2 := \alpha_2(\beta)\) and \(\alpha_3 := \alpha_3(\beta, \gamma)\) by setting
\[
\alpha_1 := -\frac{(1 - \beta)\gamma}{N}, \quad \alpha_2 := -(1 - \beta) N, \quad \alpha_3 := -\frac{(p - 1)N + (1 - p\beta)\gamma}{p}.
\]
(16)

Then an explicit description of max \(\{1, p\beta, q_*, q^{**}\}\) is the following: for every \((\alpha, \beta, \gamma) \in \mathbb{R} \times (-\infty, 1] \times (-\infty, N)\) we have
\[
\max\{1, p\beta, q_*, q^{**}\} = \begin{cases} 
q^{**}(\alpha, \beta, \gamma) & \text{if } \alpha \geq \alpha_1 \\
q_*(\alpha, \beta, \gamma) & \text{if } \max\{\alpha_2, \alpha_3\} \leq \alpha \leq \alpha_1 \\
\max\{1, p\beta\} & \text{if } \alpha < \min\{\alpha_2, \alpha_3\} 
\end{cases}
\]
(17)

where max \(\{\alpha_2, \alpha_3\} < \alpha_1\) for every \(\beta < 1\) and max \(\{\alpha_2, \alpha_3\} = \alpha_1 = 0\) if \(\beta = 1\).

**Remark 2.6.**

1. The proof of Theorem 2.5 does not require \(\beta_\infty > 0\), but this condition is not a restriction of generality in stating the theorem. Indeed, under assumption (14), if (13) holds with \(\beta_\infty < 0\), then it also holds with \(\alpha_\infty - \beta_\infty \gamma_\infty\) replaced by \(\alpha_\infty - \beta_\infty \gamma_\infty\), and \(\beta_\infty\) replaced by \(\alpha_\infty - \beta_\infty \gamma_\infty\) and 0 respectively, and this does not change the thesis (15), because \(q_*(\alpha_\infty - \beta_\infty \gamma_\infty, 0, \gamma_\infty) = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)\) and \(q^{**}(\alpha_\infty - \beta_\infty \gamma_\infty, 0, \gamma_\infty) = q^{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)\).
2. Denote \( q^* = q^* (\alpha_\infty, \beta_\infty) \) for brevity. If \( \gamma_\infty < p \), then one has

\[
\max \left\{ 1, p\beta_\infty, q^* \right\} = \begin{cases} 
\max\{1, p\beta_\infty\} = \max\{1, p\beta_\infty, q_*, q_{**}\} & \text{if } \alpha_\infty \leq \alpha^*(\beta_\infty) \\
q^* > \max\{1, p\beta_\infty, q_*, q_{**}\} & \text{if } \alpha_\infty > \alpha^*(\beta_\infty)
\end{cases}
\]

so that, under assumption 14, Theorem 2.5 improves Theorem 2.2. Otherwise, if \( \gamma_\infty = p \), we have \( q_* = q_{**} = q^* \) and Theorems 2.3 and 2.5 give the same result. This is not surprising, since, by Hardy inequality, the space \( W \) coincides with \( D^{1,p}(\mathbb{R}^N) \) if \( V(r) = r^{-p} \) and thus, for \( \gamma_\infty = p \), we cannot expect a better result than the one of Theorem 2.3 which covers the case of \( V = 0 \), i.e., of \( D^{1,p}(\mathbb{R}^N) \).

3. Description 17 shows that \( q_* \) and \( q_{**} \) are not relevant in inequality 15 if \( \alpha_\infty \leq \alpha_2 (\beta_\infty) \). On the other hand, if \( \alpha_\infty > \alpha_2 (\beta_\infty) \), both \( q_* \) and \( q_{**} \) turn out to be increasing in \( \gamma \) and hence it is convenient to apply Theorem 2.5 with the smallest \( \gamma_\infty \) for which 14 holds. This is consistent with the fact that, if 14 holds with \( \gamma_\infty \), then it also holds with every \( \gamma'_\infty \) such that \( \gamma_\infty \leq \gamma'_\infty \leq p \).

In order to state our last result, we introduce, by the following definitions, an open region \( A_{\beta, \gamma} \) of the \( \alpha \)-plane, depending on \( \beta \in [0, 1] \) and \( \gamma \geq p \). Recall the definitions 12 of the functions \( q_* = q_* (\alpha, \beta, \gamma) \) and \( q_{**} = q_{**} (\alpha, \beta, \gamma) \). We set

\[
A_{\beta, \gamma} := \{(\alpha, q) : \max\{1, p\beta\} < q < \min\{q_*, q_{**}\}\}
\text{ if } p \leq \gamma < N,
\]

\[
A_{\beta, \gamma} := \{(\alpha, q) : \max\{1, p\beta\} < q < q_{**}, \alpha > -(1 - \beta) N\} \text{ if } \gamma = N,
\]

\[
A_{\beta, \gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*\} < q < q_{**}\} \text{ if } N < \gamma < \frac{p}{p - 1} (N - 1),
\]

\[
A_{\beta, \gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*\} < q, \alpha > -(1 - \beta) \gamma\} \text{ if } \gamma = \frac{p}{p - 1} (N - 1),
\]

\[
A_{\beta, \gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*\} < q\} \text{ if } \gamma > \frac{p}{p - 1} (N - 1).
\]

Notice that \( \frac{p}{p - 1} (N - 1) > N \) because \( p < N \). For more clarity, \( A_{\beta, \gamma} \) is sketched in the following five pictures, according to the five cases above. Recall the definitions 16 of the functions \( \alpha_1 = \alpha_1 (\beta, \gamma) \), \( \alpha_2 = \alpha_2 (\beta) \) and \( \alpha_3 = \alpha_3 (\beta, \gamma) \).

**Fig.1:** \( A_{\beta, \gamma} \) for \( p \leq \gamma < N \).

- If \( \gamma = p \), the two straight lines above are the same.
- If \( \beta < 1 \) we have

\[
\max\{\alpha_2, \alpha_3\} < \alpha_1 < 0.
\]

If \( \beta = 1 \) we have

\[
\alpha_3 < \alpha_2 = \alpha_1 = 0
\]

and \( A_{1, \gamma} \) reduces to the angle \( p < q < q_{**} \).
Fig. 2: $A_{\beta, \gamma} \gamma = N$.
- If $\beta < 1$ we have $\alpha_1 = \alpha_2 = \alpha_3 < 0$.
- If $\beta = 1$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $A_{1, \gamma}$ reduces to the angle $p < q < q_{**}$. 

Fig. 3: $A_{\beta, \gamma}$ for $N < \gamma < \frac{p}{p-1}(N - 1)$.
- If $\beta < 1$ we have $\alpha_1 < \min \{\alpha_2, \alpha_3\} < 0$.
- If $\beta = 1$ we have $0 = \alpha_1 = \alpha_2 < \alpha_3$ and $A_{1, \gamma}$ reduces to the angle $p < q < q_{**}$. 

Fig. 4: $A_{\beta, \gamma}$ for $\gamma = \frac{p}{p-1}(N - 1)$.
- If $\beta < 1$ we have $\alpha_1 < \min \{\alpha_2, \alpha_3\} < 0$.
- If $\beta = 1$ we have $0 = \alpha_1 = \alpha_2 < \alpha_3$ and $A_{1, \gamma}$ reduces to the angle $\alpha > 0$, $q > p$. 
Fig. 5: $A_{\beta,\gamma}$ for $\gamma > \frac{p}{p - 1}(N - 1)$.

- If $\beta < 1$ we have $\alpha_1 < \min\{\alpha_2, \alpha_3\} < 0$.
- If $\beta = 1$ we have $0 = \alpha_1 = \alpha_2 < \alpha_3$ and $A_{1,\gamma}$ reduces to the angle $q > \max\{p, q_+\}$.

**Theorem 2.7.** Let $N \geq 3$ and let $V, K$ be as in (V), (K). Assume that there exists $R_1 > 0$ such that $V(r) < +\infty$ almost everywhere in $(0, R_1)$ and

$$\text{ess sup}_{r \in (0, R_1)} \frac{K(r)}{r^{\beta_0} V(r)^{\beta_0}} < +\infty \text{ for some } 0 \leq \beta_0 \leq 1 \text{ and } \alpha_0 \in \mathbb{R}$$

(19)

and

$$\text{ess inf}_{r \in (0, R_1)} r^{\gamma_0} V(r) > 0 \text{ for some } \gamma_0 \geq p.$$  

(20)

Then $\lim_{R \to 0^+} R_0(q_1, R) = 0$ for every $q_1 \in \mathbb{R}$ such that

$$(\alpha_0, q_1) \in A_{\beta_0,\gamma_0}.$$  

(21)

**Remark 2.8.**

1. Condition (21) also asks for a lower bound on $\alpha_0$, except for the case $\gamma_0 > \frac{p}{p - 1}(N - 1)$, as it is clear from Figures 1-5.

2. The proof of Theorem 2.7 does not require $\beta_0 \geq 0$, but this is not a restriction of generality in stating the theorem (cf. Remark 2.6.1). Indeed, under assumption (20) if (19) holds with $\beta_0 < 0$, then it also holds with $\alpha_0$ and $\beta_0$ replaced by $\alpha_0 - \beta_0 \gamma_0$ and 0 respectively, and one has that $(\alpha_0, q_1) \in A_{\beta_0,\gamma_0}$ if and only if $(\alpha_0 - \beta_0 \gamma_0, q_1) \in A_{0,\gamma_0}$.

3. If (20) holds with $\gamma_0 > p$, then Theorem 2.7 improves Theorem 2.2. Otherwise, if $\gamma_0 = p$, then one has $\max\{\alpha_2, \alpha_3\} = \alpha^*(\beta_0)$ and $(\alpha_0, q_1) \in A_{\beta_0,\gamma_0}$ is equivalent to $\max\{1, p \beta_0\} < q_1 < q^*(\alpha_0, \beta_0)$, i.e., Theorems 2.7 and 2.2 give the same result, which is consistent with Hardy inequality (cf. Remark 2.6.2).

4. Given $\beta \leq 1$, one can check that $A_{\beta,\gamma_1} \subseteq A_{\beta,\gamma_2}$ for every $p \leq \gamma_1 < \gamma_2$, so that, in applying Theorem 2.7 it is convenient to choose the largest $\gamma_0$ for which (20) holds. This is consistent with the fact that, if (20) holds with $\gamma_0$, then it also holds with every $\gamma'_0$ such that $p \leq \gamma'_0 \leq \gamma_0$.

**Remark 2.9.** If $p = 2$, the above compactness theorems exactly reduces to the ones of [4], except for the fact that there we required assumption (K) with $s > 2N/(N + 2)$ instead of $s > 1$. In this respect, the result we present here are improvements of the ones of [4] also for $p = 2$. 

9
3 Examples

In this section we give some examples of application of our compactness results, which might clarify how to use them in concrete cases. We also compare them with the most recent and general related results [14,15], which unify and extend the previous literature. Essentially, the spirit of the results of [14,15] is the following: assuming that $V, K$ are continuous and satisfy power type estimates of the form:

$$
\liminf_{r \to 0^+} \frac{V(r)}{r^a} > 0, \quad \liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0, \quad \limsup_{r \to 0^+} \frac{K(r)}{r^b} < \infty, \quad \limsup_{r \to +\infty} \frac{K(r)}{r^b} < \infty, \quad (22)
$$

the authors find two limit exponents $\underline{q} = q(a,b)$ and $\overline{q} = q(a_0, b_0)$ such that the embedding $W_r \hookrightarrow L_K^q$ is compact if $q < \underline{q} < \overline{q}$. The case with $q > p$ is studied in [15], the one with $q < p$ in [14]. The exponent $\underline{q}$ is always defined, while $\overline{q}$ exists provided that suitable compatibility conditions between $a_0$ and $b_0$ occur. Moreover, the condition $\underline{q} < q < \overline{q}$ also asks for $\underline{q} < \overline{q}$, which is a further assumption of compatibility between the behaviours of the potentials at zero and at infinity.

In the following it will be always understood that $1 < p < N$.

Example 3.1. Consider the potentials

$$V(r) = \frac{1}{r^a}, \quad K(r) = \frac{1}{r^{a-1}}, \quad a \leq p.$$  

Since $V$ satisfies (14) with $\gamma_\infty = a$ (cf. Remark 2.6.3 for the best choice of $\gamma_\infty$), we apply Theorems 2.2 and 2.3 where we choose $\beta_0 = \beta_\infty = 0$ and $\alpha_0 = \alpha_\infty = 1 - a$. Note that $a \leq p$ implies $\alpha_0 = 1 - a > \alpha^*(0)$ and $q_* (\alpha_\infty, 0, a) \leq q_* (\alpha_\infty, 0, a)$. Hence we get that $(R_{\alpha_0, \beta_0}^\alpha)$ holds for every exponents $q_1, q_2$ such that

$$1 < q_1 < q^* = p \frac{N - a + 1}{N - p}, \quad q_2 > q_{**} = p \frac{pN - a (p - 1)}{p (N - 1) - a(p - 1)}. \quad (23)$$

If $a < p$, then one has $q_* < q^*$ and therefore Theorem 2.7 gives the compact embedding

$$W_r \hookrightarrow L_K^q \quad \text{for} \quad q_* < q < q^*. \quad (24)$$

If $a = p$, then $q_* = q^*$ and we get the compact embedding

$$W_r \hookrightarrow L_K^{q_1} + L_K^{q_2} \quad \text{for} \quad 1 < q_1 < p + \frac{p}{N - p} < q_2.$$  

Since $V$ and $K$ are power potentials, one can also apply the results of [15], which give two suitable limit exponents $\underline{q}$ and $\overline{q}$ such that the embedding $W_r \hookrightarrow L_K^q$ is compact if $q < \underline{q} < \overline{q}$. These exponents $\underline{q}$ and $\overline{q}$ are exactly exponents $q_*$ and $q^*$ of (22) respectively, so that one obtains (24) again provided that $a < p$ (which implies $q < \overline{q}$). If $a = p$, instead, one gets $q = \overline{q}$ and no result is available in [15]. The results of [14] do not apply to $V$ and $K$, since the top and bottom exponents of $[14]$ turn out to be equal to one another for every $a \leq p$.

The next Examples 3.2, 3.3 and 3.4 concern potentials for which no result is available in [14,15], since they do not satisfy (22).

Example 3.2. Taking $V = 0$, $K$ as in (K) and $\beta_0 = \beta_\infty = 0$, from Theorems 2.2 and 2.3 (see also Remark 2.4.1) we get that $(R_{\alpha_0, \beta_0}^\alpha)$ holds for

$$1 < q_1 < p \frac{\alpha_0 + N}{N - p} \quad \text{and} \quad q_2 > \max \left\{ 1, p \frac{\alpha_\infty + N}{N - p} \right\}, \quad (25)$$

provided that \( \exists R_1, R_2 > 0 \) such that
\[
\text{ess sup}_{r > R_2} K(r) < +\infty \quad \text{and} \quad \text{ess sup}_{r \in (0,R_1)} K(r) < +\infty \quad \text{with} \quad \alpha_0 > -1 - \frac{p-1}{p} N.
\]

Correspondingly, Theorem 2.7 gives a compact embedding of \( D^{1,p}_\mathrm{rad}(\mathbb{R}^N) \) into \( L^q_{\alpha} + L^q_{\beta} \), which was already proved in [3] Theorem 4.1] assuming \( K \in L^\infty_\mathrm{loc}(\mathbb{R}_+) \). Of course, according to [3], in (26) it is convenient to choose \( \alpha_0 \) as large as possible and \( \alpha_\infty \) as small as possible. For instance, if \( K(r) = r^d \) with \( d > -1 - N(p-1)/p \), we choose \( \alpha_0 = \alpha_\infty = d \) and obtain the compact embedding
\[
D^{1,2}_\mathrm{rad}(\mathbb{R}^N) \hookrightarrow L^{q_1}_K + L^{q_2}_K, \quad \text{for} \quad 1 < q_1 < p\frac{d+N}{N-p} < q_2.
\]

Observe that, if (26) holds for some \( \alpha_0 > \alpha_\infty \), then we can take \( q_1 = q_2 \) in (26) and get the compact embedding
\[
D^{1,2}_\mathrm{rad}(\mathbb{R}^N) \hookrightarrow L^q_K, \quad \text{for} \quad \max \left\{ 1, \frac{\alpha_\infty + N}{N-p} \right\} < q < p\frac{\alpha_0 + N}{N-p}.
\]

Example 3.3. Essentially the same result of Example 3.2 holds if \( V \) is not singular at the origin and, roughly speaking, decays at infinity much faster than \( K \) (or is compactly supported). The result becomes different (and better) if \( K \) decays at infinity similarly to \( V \), or much faster. For example, consider the potentials
\[
V(r) = e^{-ar}, \quad K_1(r) = r^d, \quad K_2(r) = r^d e^{-br}, \quad a, b > 0, \quad d > -1 - \frac{p-1}{p} N.
\]

Since \( V \) does not satisfy (26) or (24), we use Theorems 2.2 and 2.8. According to Remark 2.4.3, both for \( K = K_1 \) and \( K = K_2 \), Theorem 2.2 leads to take \( 1 < q_1 < p(d+N)/(N-p) \). If \( K = K_1 \), the ratio in (10) is bounded only if \( \beta_\infty = 0 \) and the best \( \alpha_\infty \) we can take is \( \alpha_\infty = d \), which yields \( q_2 > p(d+N)/(N-p) \). Then, via condition (\( R''_{q_1,q_2} \)), Theorem 2.7 gives the compact embedding
\[
W_r \hookrightarrow L^{q_1}_K + L^{q_2}_K, \quad \text{for} \quad 1 < q_1 < p\frac{d+N}{N-p} < q_2.
\]

If \( K = K_2 \), instead, assumption (10) holds with \( \beta_\infty = 0 \) and \( \alpha_\infty \in \mathbb{R} \) arbitrary, so that we can take \( q_2 > 1 \) arbitrary. Then, via condition (\( R''_{q_1,q_2} \)) with \( q_2 = q_1 \), Theorem 2.7 gives the compact embedding
\[
W_r \hookrightarrow L^{q}_K, \quad \text{for} \quad 1 < q < p\frac{d+N}{N-p}.
\]

Example 3.4. Consider the potentials
\[
V(r) = e^{\frac{a}{r^2}}, \quad K(r) = e^{\frac{b}{r}}, \quad 0 < b \leq 1.
\]

Since \( V \) satisfies (20), we apply Theorem 2.7 together with Theorems 2.2 and 2.8. Assumption (10) holds for \( \alpha_\infty \geq 0 \) and \( 0 \leq \beta_\infty \leq 1 \), so that the best choice for \( \alpha_\infty \), which is \( \alpha_\infty = 0 \), gives
\[
\max \{ 1, p\beta_\infty, q^*(0,\beta_\infty) \} = p\frac{N-p\beta_\infty}{N-p}.
\]
Then we take $\beta_\infty = 1$, so that Theorem 2.3 gives $q_2 > p$. As to Theorem 2.7 hypothesis (20) holds with $\gamma_0 \geq p$ arbitrary and therefore the most convenient choice is to assume $\gamma_0 > (N - 1)p/(p - 1)$ (see Remark 2.8.4). On the other hand, we have
\[
K (r) = e^{\frac{b \cdot b_0}{r^{\alpha_0}}} \quad \text{and thus hypothesis (19) holds for some } \alpha_0 \in \mathbb{R}
\]
for $0 \leq b \leq b_0 \leq 1$. We now distinguish two cases. If $b < 1$, we can take $b_0 > b$ and thus (19) holds for every $\alpha_0 \in \mathbb{R}$, so that Theorem 2.7 gives $q_1 > \max \{1, p \beta_0\}$ (see Fig.5), i.e., $q_1 > \max \{1, 2b\}$. If $b = 1$, then we need to take $b_0 = 1$ and thus (19) holds for $\alpha_0 = 0$. Since $\gamma_0 > (N - 1)p/(p - 1)$ implies
\[
A_{1, \gamma_0} = \left\{ (\alpha, q) \in \mathbb{R}^2 : q > \max \left\{ p, p - \frac{\alpha p^2}{\gamma_0 (p - 1) - p (N - 1)} \right\} \right\}
\]
the best choice for $\alpha_0 = 0$ is $\alpha_0 = 0$ and we get that $(0, q_1) \in A_{1, \gamma_0}$ if and only if $q_1 > p$. Hence Theorem 2.7 gives $q_1 > \max \{1, 2b\}$ again. As a conclusion, observing that $0 < b \leq 1$ implies $\max \{1, pb\} \leq p$, we obtain condition $(\mathcal{R}''_{q,q})$ and the compact embedding $W_r \hookrightarrow L^q_K$ for $q > p$.

If we now modify $V$ by taking a compactly supported potential $V_1$ such that $V_1 (r) \sim V (r)$ as $r \to 0^+$, everything works as above in applying Theorem 2.7 but now we need to take $\beta_\infty = 0$ and $\alpha_\infty \geq 0$ in Theorem 2.3. This gives
\[
\max \{1, p \beta_\infty, q^* (\alpha_\infty, \beta_\infty)\} = p \frac{\alpha_\infty + N}{N - p}
\]
and thus, choosing $\alpha_\infty = 0$, we get $(\mathcal{R}''_{q,q})$ and the compact embedding $W_r \hookrightarrow L^q_K$ for $q > p^*$.

Similarly, if we modify $V$ by taking a potential $V_2$ such that $V_2 (r) \sim V (r)$ as $r \to 0^+$ and $V_2 (r) \sim r^N$ as $r \to +\infty$, Theorem 2.7 yields $q_1 > \max \{1, pb\}$ as above and Theorem 2.3 gives $q_2 > 1$ (apply it for instance with $\alpha_\infty = -N/2$ and $\beta_\infty = 1/2$), so that we get $(\mathcal{R}''_{q,q})$ and the compact embedding $W_r \hookrightarrow L^q_K$ for $q > \max \{1, pb\}$.

The last example shows that our results also extend the ones of [14, 15] for power-type potentials.

**Example 3.5.** Consider the potential
\[
V (r) = r^a, \quad -\frac{p}{p - 1} (N - 1) < a < -N,
\]
and let $K$ be as in (K) and such that
\[
K (r) = O (r^{b_0}), \quad b_0 > a, \quad b \in \mathbb{R}.
\]
Since $V$ satisfies (20) with $\gamma_0 = -a$ (cf. Remark 2.8.4 for the best choice of $\gamma_0$), we apply Theorem 2.7 together with Theorems 2.3 and 2.7 where assumptions (16) and (19) hold for $\alpha_\infty \geq b - a \beta_\infty$ and $\alpha_0 \leq b_0 - a \beta_0$ with $0 \leq \beta_\infty \leq 1$ and $\beta_0 \leq 1$ arbitrary. Note that $N < \gamma_0 < (N - 1)p/(p - 1)$. According to (11) and (21) (see in particular Fig.3), it is convenient to choose $\alpha_\infty$ as small as possible and $\alpha_0$ as large as possible, so we take
\[
\alpha_\infty = b - a \beta_\infty, \quad \alpha_0 = b_0 - a \beta_0.
\]
Then $q^* = q^* (\alpha_\infty, \beta_\infty), q_+ = q_+ (\alpha_0, \beta_0, -a)$ and $q^{**} = q^{**} (\alpha_0, \beta_0, -a)$ are given by
\[
q^* = p \frac{N + b - (a + p) \beta_\infty}{N - p}, \quad q_+ = p \frac{N + b_0}{N + a} \quad \text{and} \quad q^{**} = p \frac{p (N - 1) + p b_0 - a}{p (N - 1) + a (p - 1)}.
\]
Since \( a + p < p - N < 0 \), the exponent \( q^* \) is increasing in \( \beta_0 \) and thus, according to (11) again, the best choice for \( \beta_\infty \) is \( \beta_\infty = 0 \). This yields

\[
q_2 > \max \left\{ 1, p \frac{N + b}{N - p} \right\}.
\]  

(30)

As to Theorem 2.2 we observe, thanks to the choice of \( \alpha_0 \), the exponents \( q_* \) and \( q_{**} \) are independent of \( \beta_0 \), so that we can choose \( \beta_0 = 0 \) in order to get the region \( A_{\beta_0, -\alpha} \) as large as possible (cf. Fig.3 or the third definition in (13)). Then we get \( \alpha_0 = b_0 > a = \alpha_1 \) (recall (29) and the definition (16) of \( \alpha_1 \)), so that \( (\alpha_0, q_1) \in A_{\beta_0, -\alpha} \) if and only if

\[
\max \left\{ 1, p \frac{N + b_0}{N + a} \right\} < q_1 < p \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)}.
\]  

(31)

As a conclusion, via condition \((R'_{q_1, q_2})\), we obtain the compact embedding

\[
W_r \hookrightarrow L^q_K \quad \text{for every } q_1, q_2 \text{ satisfying (30) and (31)}.
\]

If furthermore \( a, b, b_0 \) are such that

\[
\frac{N + b}{N - p} < p \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)},
\]  

(32)

then we can take \( q_1 = q_2 \) and we get the compact embedding

\[
W_r \hookrightarrow L^q_K \quad \text{for } \max \left\{ 1, p \frac{N + b_0}{N + a}, p \frac{N + b}{N - p} \right\} < q < p \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)}.
\]  

(33)

Observe that the potentials \( V \) and \( K \) behave as a power and thus they fall into the classes considered in (14)-(15). In particular, the results of (15) provide the compact embedding

\[
W_r \hookrightarrow L^q_K \quad \text{for } \max \left\{ p, p \frac{N + b}{N - p} \right\} =: q < q := p \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)}.
\]  

(34)

This requires condition (32), which amounts to \( q < \overline{q} \), and no compact embedding is found in (15) if (32) fails. Moreover, our result improves (32) even if (32) holds. Indeed, \( b_0 > a \) and \( N + a < 0 \) imply \( \frac{N + b_0}{N + a} < 1 \) and thus one has

\[
q = \begin{cases} 
\frac{pN + b}{N - p} = \max \left\{ 1, p \frac{N + b_0}{N + a}, p \frac{N + b}{N - p} \right\} & \text{if } b \geq -p \\
p > \max \left\{ 1, p \frac{N + b_0}{N + a}, p \frac{N + b}{N - p} \right\} & \text{if } b < -p
\end{cases}
\]

so that (35) is exactly (34) if \( b \geq -p \) and it is better if \( b < -p \). This last case actually concerns exponents less than \( p \), so it should be also compared with the results of (14), where, setting

\[
\begin{align*}
b_1 &:= \frac{p(N - 1) + a(p - 1)}{p^2} - N, \quad b_2 := \frac{p(N - 1) + a(p - 1)}{p} - N, \quad b_3 := \frac{N - p}{p} - N \\
\end{align*}
\]

(notice that \( -N < b_1 < b_2 < b_3 < -p \) for \( a \) as in (27) and)

\[
q' := \begin{cases} 
\frac{pN + b}{p^2(N + b)} & \text{if } b \in [b_3, -p) \\
\frac{pN + b}{p^2(N + b)} & \text{if } b \in [b_1, b_2) \\
\end{cases}, \quad \overline{q}' := \begin{cases} 
\frac{pN + b}{p^2(N + b)} & \text{if } b_0 \in (b_3, -p] \\
\frac{pN + b}{p^2(N + b)} & \text{if } b_0 \in (b_1, b_2].
\end{cases}
\]  

(35)
the authors find the compact embedding

\[ W_r \hookrightarrow L^q_K \quad \text{for} \quad q' < q < q'' \].

Our result (33)-(32) extends (36) in three directions. First, (36) requires that \( q' \) and \( q'' \) are defined, i.e., \( b \) and \( b_0 \)
lie in the intervals considered in (35), while (33) and (32) do not need such a restriction, also covering cases of \( b \in (-\infty, b_1) \cup [b_2, b_3) \) or \( b_0 \in (a, b_1) \cup (b_2, b_3) \) (take for instance \( b_0 > a \) arbitrary and \( b \) small enough to satisfy (32)). Moreover, (36) asks for the further condition \( q' < q'' \), which can be false even if \( q' \) and \( q'' \) are defined (take for instance \( b = b_0 \in (b_3, -p) \) or \( b = b_0 \in (b_1, b_2) \), which give \( q' = q'' \)), while condition (32) does not. Actually, as soon as \( q' \) and \( q'' \) are defined, one has \( b < -p \) and \( b_0 > -N \), which imply

\[
\frac{N + b}{N - p} - \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)} < 1 + \frac{p + a}{p(N - 1) + a(p - 1)} = \frac{N + a}{p(N - 1) + a(p - 1)} < 0.
\]

Finally, setting for brevity

\[ q' := \max \left\{ 1, \frac{N + b_0}{N + a}, \frac{N + b}{N - p} \right\}, \]

some computations (which we leave to the reader) show that, whenever \( q' \) and \( q'' \) are defined, one has

\[
q' = \begin{cases} p \frac{N + b}{N - p} = q'' & \text{if } b \in [b_3, -p) \\ \frac{p^2(N + b)}{p(N - 1) + a(p - 1)} = 1 = q'' & \text{if } b = b_1 \\ \frac{p^2(N + b)}{p(N - 1) + a(p - 1)} > 1 = q'' & \text{if } b \in (b_1, b_2) \end{cases} \quad \text{and} \quad q'' < \frac{p(N - 1) + pb_0 - a}{p(N - 1) + a(p - 1)}.
\]

This shows that (32) always gives a wider range of exponents \( q \) than (36).

4 Existence and multiplicity results

Let \( 1 < p < N \). In this section we state our existence and multiplicity results about radial weak solutions to the equation

\[
- \Delta_p u + V(|x|) |u|^{p-2} u = g(|x|, u) \quad \text{in } \mathbb{R}^N,
\]

i.e., functions \( u \in W_r \) such that

\[
\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla h \, dx + \int_{\mathbb{R}^N} V(|x|) |u|^{p-2} u h \, dx = \int_{\mathbb{R}^N} g(|x|, u) h \, dx \quad \text{for all } h \in W_r,
\]

where \( V \) is a potential satisfying (V) and \( W_r \) are the Banach spaces defined in (3), equipped with the uniformly convex standard norm given by (2). As concerns the nonlinearity, we assume that \( g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that

\( (g_0) \) the linear operator \( h \mapsto \int_{\mathbb{R}^N} g(|x|, 0) h \, dx \) is continuous on \( W_r \)

and that there exist \( f \in C(\mathbb{R}; \mathbb{R}) \) and a function \( K \) satisfying (K) such that:

\( (g) \ |g(r, t) - g(r, 0)| \leq K(r) |f(t)| \) for almost every \( r > 0 \) and all \( t \in \mathbb{R} \).
The model cases of \( g \) we have in mind are, of course, \( g(r,t) = K(r)f(t) \) or \( g(r,t) = K(r)f(t) + Q(r) \) with \( Q = g(\cdot,0) \) such that \( (g_0) \) holds (see Remark 4.1). On the function \( f \) we will also require the following condition (see also Remarks 4.4.2 and 4.9.2), where \( q_1, q_2 \) will be specified each time:

\[
(f_{q_1,q_2}) \exists M > 0 \text{ such that } |f(t)| \leq M \min \{|t|^{q_1-1}, |t|^{q_2-1}\} \text{ for all } t \in \mathbb{R}.
\]

Observe that, if \( q_1 \neq q_2 \), the double-power growth condition \((f_{q_1,q_2})\) is more stringent than the more usual single-power one, since it implies \(\sup_{t > 0} |f(t)| / t^{q-1} < +\infty \) for \( q = q_1, q = q_2 \) and every \( q \) in between. On the other hand, we will never require \( q_1 \neq q_2 \) in \((f_{q_1,q_2})\), so that our results will also concern single-power nonlinearities as long as we can take \( q_1 = q_2 \) (see Example 4.1 below).

**Remark 4.1.** Of course assumption \((g_0)\) will be relevant only if \( g(\cdot,0) \neq 0 \) (meaning that \( g(\cdot,0) \) does not vanish almost everywhere). In this case, the radial estimates satisfied by the \( W_r \) mappings (see Lemmas 4.2 and 4.3 of [2]) provide simple explicit conditions ensuring assumption \((g_0)\), which turns out to be fulfilled if \( g(\cdot,0) \) belongs to \( L^p_\infty(\mathbb{R}^N \setminus \{0\}) \) and satisfies suitable decay (or growth) conditions at zero and at infinity. On the other hand, it is not difficult to find explicit conditions on \( g(\cdot,0) \) ensuring \((g_0)\) even on the whole space \( W_r \), for example \( g(\cdot,0) \in L^{p/(p-1)}(\mathbb{R}^+, |x|^{N-1/(p-1)} \, dx) \). Indeed, this means \( g(\cdot,0) \in L^{p/(p-1)}(\mathbb{R}^N, |x|^{p/(p-1)} \, dx) \) and thus it implies

\[
\int_{\mathbb{R}^N} g(|x|,0) \, h \, dx \leq \left( \int_{\mathbb{R}^N} g(|x|,0) |x|^{p-2} |x|^{p/2} \, dx \right)^{p-1} \left( \int_{\mathbb{R}^N} |h|^p |x|^p \, dx \right)^{1/p} \leq (\text{const.}) \|h\|
\]

for all \( h \in W_r \rightarrow D^{1,p}(\mathbb{R}^N) \), by Hölder and Hardy inequalities. Other conditions ensuring the same result are \( g(\cdot,0) \in L^{p/N/(p-N) + p}(\mathbb{R}^+, r^{N-1} \, dr) \) or \( V^{-1/p} g(\cdot,0) \in L^{p/(p-1)}(\mathbb{R}^+, r^{N-1} \, dr) \).

Set \( G(r,t) := \int_0^r g(r,u) \, ds \) and

\[
I(u) := \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} G(|x|,u) \, dx.
\]

From the continuous embedding result of Theorem 2.1 and the results of [5] about Nemytskii operators on the sum of Lebesgue spaces, we have that \([39]\) defines a \( C^1 \) functional on \( W_r \) provided that there exist \( q_1, q_2 > 1 \) such that \((f_{q_1,q_2})\) and \((S'_{q_1,q_2})\) hold. In this case, the Fréchet derivative of \( I \) at any \( u \in W_r \) is given by

\[
I'(u) h = \int_{\mathbb{R}^N} (\nabla u|^p - 2u \cdot \nabla h + V(|x|) |u|^{p-2} u h) \, dx - \int_{\mathbb{R}^N} g(|x|,u) h \, dx, \quad \forall h \in W_r
\]

and therefore the critical points of \( I : W_r \rightarrow \mathbb{R} \) satisfy \([38]\) for all \( h \in W_r \). Our first result shows that such critical points are actually weak solutions to equation \([37]\), provided that the following slightly stronger version of condition \((S'_{q_1,q_2})\) holds:

\[
\mathcal{R}_0(q_1,R_1) < \infty \quad \text{and} \quad \mathcal{R}_\infty(q_2,R_2) < \infty \quad \text{for some} \quad R_1, R_2 > 0.
\]

Observe that the classical Palais’ Principle of Symmetric Criticality \([11]\) does not apply in this case, because we do not know whether or not \( I \) is differentiable, not even well defined, on the whole space \( W \).

**Proposition 4.2.** Assume \( s > \frac{Np}{N(p-1)+p} \) in condition \((K)\) and assume that \((g_0)\) holds on the whole space \( W \) (cf. Remark 4.7). Assume furthermore that there exist \( q_1, q_2 > 1 \) such that \((f_{q_1,q_2})\) and \((S'_{q_1,q_2})\) hold. Then every critical point of \( I : W_r \rightarrow \mathbb{R} \) is a weak solution to equation \([37]\).
By Proposition 4.2, the problem of radial weak solutions to (37) reduces to the study of the critical points of $I : W_r \to \mathbb{R}$, which is the aim of our next results.

Concerning the case of super $p$-linear nonlinearities, we will prove the following existence and multiplicity theorems.

**Theorem 4.3.** Assume $g(\cdot, 0) = 0$ and assume that there exist $q_1, q_2 > p$ such that $(f_{q_1, q_2})$ and $(S''_{q_1, q_2})$ hold. Assume furthermore that $g$ satisfies:

1. $\exists \theta > p$ such that $0 \leq \theta G(r, t) \leq g(r, t) t$ for almost every $r > 0$ and all $t \geq 0$;
2. $\exists \theta > p$ such that $0 < \theta G(r, t) \leq g(r, t) t$ for almost every $r > 0$.

If $K(\cdot|\cdot) \in L^1(\mathbb{R}^N)$, we can replace assumptions (g1)-(g2) with:

1. $\exists \theta > p$ and $\exists \theta > 0$ such that $0 < \theta G(r, t) \leq g(r, t) t$ for almost every $r > 0$ and all $t \geq t_0$.

Then the functional $I : W_r \to \mathbb{R}$ has a nonnegative critical point $u \neq 0$.

**Remark 4.4.**

1. Assumptions (g1) and (g2) imply (g3), so that, in Theorem 4.3, the information $K(\cdot|\cdot) \in L^1(\mathbb{R}^N)$ actually allows weaker hypotheses on the nonlinearity.
2. In Theorem 4.3 assumptions $(g)$ and $(f_{q_1, q_2})$ need only to hold for $t \geq 0$. Indeed, all the hypotheses of the theorem still hold true if we replace $g(r, t)$ with $\chi_{\mathbb{R}_+}(t) g(r, t)$ ($\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$) and this can be done without restriction since the theorem concerns nonnegative critical points.

**Theorem 4.5.** Assume that there exist $q_1, q_2 > p$ such that $(f_{q_1, q_2})$ and $(S''_{q_1, q_2})$ hold. Assume furthermore that:

1. $\exists m > 0$ such that $G(r, t) \geq m K(r) \min \{t^{q_1}, t^{q_2}\}$ for almost every $r > 0$ and all $t \geq 0$;
2. $g(r, t) = -g(r, -t)$ for almost every $r > 0$ and all $t \geq 0$.

Finally, assume that $g$ satisfies (g3), or that $K(\cdot|\cdot) \in L^1(\mathbb{R}^N)$ and $g$ satisfies (g3). Then the functional $I : W_r \to \mathbb{R}$ has a sequence of critical points $\{u_n\}$ such that $I(u_n) \to +\infty$.

**Remark 4.6.** The condition $g(\cdot, 0) = 0$ is implicit in Theorem 4.3 (and in Theorem 4.10 below), as it follows from assumption (g5).

As to sub $p$-linear nonlinearities, we will prove the following results, where we also consider the case $g(\cdot, 0) \neq 0$.

**Theorem 4.7.** Assume that there exist $q_1, q_2 \in (1, p)$ such that $(f_{q_1, q_2})$ and $(S''_{q_1, q_2})$ hold. Assume furthermore that $g$ satisfies (g0) and at least one of the following conditions:

1. $\exists \theta < p$ and $\exists \theta > m > 0$ such that $G(r, t) \geq m K(r) t^\theta$ for almost every $r > 0$ and all $0 \leq t \leq t_0$;
2. $g(\cdot, 0)$ does not vanish almost everywhere in $(r_1, r_2)$.

If (g7) holds, we also allow the case $\max \{q_1, q_2\} = p > \min \{q_1, q_2\} > 1$. Then there exists $u \neq 0$ such that

$I(u) = \min_{v \in W_r} I(v)$.

16
If \( g(\cdot, t) \geq 0 \) almost everywhere for all \( t < 0 \), the minimizer \( u \) of Theorem 4.7 is nonnegative, since a standard argument shows that all the critical points of \( I \) are nonnegative (test \( I'(u) \) with the negative part \( u_\cdot \) and get \( I'(u) u_\cdot = -\|u_\cdot\|^p = 0 \). The next corollary gives a nonnegative critical point just requiring \( g(\cdot, 0) \geq 0 \).

**Corollary 4.8.** Assume the same hypotheses of Theorem 4.7. If \( g(\cdot, 0) \geq 0 \) almost everywhere, then \( I : W_r \to \mathbb{R} \) has a nonnegative critical point \( \bar{u} \neq 0 \) satisfying

\[
I(\bar{u}) = \min_{u \in W_r, u \geq 0} I(u).
\]

**Remark 4.9.**

1. In Theorem 4.7 and Corollary 4.8, the case \( \max \{q_1, q_2\} = p > \min \{q_1, q_2\} > 1 \) cannot be considered under assumption (g6), since (g6) and \( (f_{q_1, q_2}) \) imply \( \max \{q_1, q_2\} \leq \theta < p \).

2. Checking the proof, one sees that Corollary 4.8 actually requires that assumptions (g) and \( (f_{q_1, q_2}) \) hold only for \( t \geq 0 \), which is consistent with the concern of the result about nonnegative critical points.

**Theorem 4.10.** Assume that there exist \( q_1, q_2 \in (1, p) \) such that \( (f_{q_1, q_2}) \) and \( (S''_{q_1, q_2}) \) hold. Assume furthermore that \( g \) satisfies \( (g_5) \) and \( (g_6) \). Then the functional \( I : W_r \to \mathbb{R} \) has a sequence of critical points \( \{u_n\} \) such that \( I(u_n) < 0 \) and \( I(u_n) \to 0 \).

All the above existence and multiplicity results rely on assumption \( (S''_{q_1, q_2}) \), which is quite abstract but can be granted in concrete cases through Theorems 2.2 and 2.7 and 2.8, which ensure the stronger condition \( (R''_{q_1, q_2}) \) for suitable ranges of exponents \( q_1 \) and \( q_2 \) by explicit conditions on the potentials. This has been already discussed in Section 2 and exemplified in Section 3. Moreover, explicit conditions on a forcing term \( g(\cdot, 0) \neq 0 \) in order that \( (g_6) \) holds has been examined in Remark 4.1, so here we limit ourselves to give some basic examples of nonlinearities of the form \( g(r, t) = K(r) f(t) \) satisfying the assumptions of our results and to apply them to a sample equation.

**Example 4.11.** Let \( g(r, t) = K(r) f(t) \) with \( K \) satisfying (K). The simplest \( f \in C(\mathbb{R}; \mathbb{R}) \) such that \( (f_{q_1, q_2}) \) holds is \( f(t) = \min \left\{ |t|^{q_1 - 2} t, |t|^{q_2 - 2} t \right\} \), which also ensures \( (g_1) \) if \( q_1, q_2 > p \) (with \( \theta = \min \{q_1, q_2\} \)), and \( (g_6) \) if \( q_1, q_2 < p \) (with \( \theta = \max \{q_1, q_2\} \)). Another model example is

\[
f(t) = \frac{|t|^{q_2 - 2} t}{1 + |t|^{q_2 - q_1}}, \quad \text{with} \quad 1 < q_1 \leq q_2,\]

which ensures \( (g_1) \) if \( q_1 > p \) (with \( \theta = q_1 \)) and \( (g_6) \) if \( q_2 < p \) (with \( \theta = q_2 \)). Note that, in both these cases, also \( (g_2), (g_4) \) and \( (g_5) \) hold true. Moreover, both of these functions \( f \) become \( f(t) = |t|^{q_2 - 2} t \) if \( q_1 = q_2 = q \). Other examples of nonlinearities \( f \) ensuring \( (f_{q_1, q_2}) \) are

\[
f(t) = \frac{|t|^{q_1 + q - 1} - |t|^{q_2 - 1}}{1 + |t|^q}, \quad f(t) = \frac{|t|^{q_2 - 1} + \varepsilon}{1 + |t|^{q_2 - q_1 + 2\varepsilon}} \ln |t|,
\]

(the latter extended at 0 by continuity) with \( 1 < q_1 \leq q_2 < q_1 + \varepsilon > 0 \), for which \( (g_1) \) and \( (g_6) \) do not hold, but \( (g_3) \) is granted if \( q_1 > p \) and \( \varepsilon \) is small enough.
Example 4.12. Let $1 < p < N$ and $\alpha > 0$, and consider the equation

$$-\Delta_p u + \frac{e^{-\alpha |x|}}{|x|^N} |u|^{p-1} u = K (|x|) |u|^{q-1} u \quad \text{in } \mathbb{R}^N$$

(42)

where $K : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $K (r) = O(r^b_0), r \to 0^+$ and $K (r) = O(r^b) r \to +\infty$ with $b_0 > -N$ and $b \in \mathbb{R}$. By Theorems 2.2 and 2.7 (applied with $\gamma_0 = N, \alpha_0 = b_0, \alpha_\infty = b, \beta_0 = \beta_\infty = 0$), condition $(S''_{p, q, 2})$ hold if

$$1 < q_1 < \overline{q} := p \left(1 + \frac{N + b_0}{N - p} \right) \quad \text{and} \quad q_2 > q := \max \left\{1, p \frac{N + b}{N - p} \right\}.$$  

(43)

Note that $\overline{q} > p$, since $b_0 > -N$. Then, by Theorem 4.3, Corollary 4.8 and Proposition 4.2 the equation has a nonnegative radial weak solution in the following cases:

- $b < p (N + b_0 - 1)$ and $\max \left\{p, p \frac{N + b}{N - p} \right\} < q < \overline{q}$;
- $b < -p$ and $\overline{q} < q < p$.

If $b \geq p (N + b_0 - 1)$, instead, we cannot pick $q_1 = q_2 > p$ in (43) (since $\overline{q} \leq q$), nor $q_1 = q_2 < p$ (since $q > p$).

In this case our results do not apply to equation (42), but they apply to the equation

$$-\Delta_p u + \frac{e^{-\alpha |x|}}{|x|^N} |u|^{p-1} u = K (|x|) f (u) \quad \text{in } \mathbb{R}^N$$

where $f$ is any nonlinearity ensuring $(g_1)$ and $(g_2)$ and satisfying $(f_{q_1, q_2})$ (for instance one of the first two functions considered in Example 4.11), for which Theorem 4.3 and Proposition 4.2 provide a nonnegative radial weak solution if $p < q_1 < \overline{q}$ and $q_2 > q$.

We end this section by observing that from the above results one can also derive existence and multiplicity results for equation (1) with Dirichlet boundary conditions in bounded balls or exterior radial domains, where a single-power growth condition on the nonlinearity is sufficient and, respectively, only assumptions on $V$ and $K$, near the origin or at infinity are needed. This can be done by suitably modifying the potentials $V$ and $K$, in order to reduce the Dirichlet problem to the problem in $\mathbb{R}^N$ (see [3, Section 5]). We leave the details to the interested reader, as well as the precise statements of the results.

5 Proofs

This section is devoted to proof of the results of Section 4, so we keep the notation and assumptions of that section. We begin by recalling the following lemma from [2].

Lemma 5.1 ([2, Lemma 3.1]). Let $R > r > 0$ and $1 < q < \infty$. Then there exist $\tilde{C} = \tilde{C} (N, p, r, R, q, s) > 0$ and $l = l (p, q, s) > 0$ such that $\forall u \in W_r$ one has

$$\int_{B_R \setminus B_r} K (|x|) |u|^q \, dx \leq \tilde{C} \|K (|\cdot|)\|_{L^r (B_R \setminus B_r)} \|u\|^q \|u\|^{q-lp} \left(\int_{B_R \setminus B_r} |u|^p \, dx\right)^l.$$
Moreover, if \( s > \frac{Np}{N(p-1)+p} \) in assumption (K), then there exists \( \tilde{C}_1 = \tilde{C}_1 (N, p, r, R, q, s) > 0 \) such that \( \forall u \in W_r \) and \( \forall h \in W \) one has

\[
\frac{\int_{B_R \setminus B_{R_2}} K (|x|) |u|^{q_1 - 1} |h| \, dx}{C_1 \| K (|\cdot|) \|_{L^r (B_R \setminus B_{R_2})}} \leq \begin{cases} 
\left( \int_{B_{R_2} \setminus B_{R_1}} |u|^p \, dx \right)^{\frac{q_1 - 1}{p}} \| h \| & \text{if } q \leq \tilde{q} \\
\left( \int_{B_{R_2} \setminus B_{R_1}} |u|^p \, dx \right)^{\frac{q_1 - 1}{p}} \| u \|^{q_1 - \tilde{q}} \| h \| & \text{if } q > \tilde{q}
\end{cases}
\]

where \( \tilde{q} := p \left( 1 + \frac{1}{N} - \frac{1}{r} \right) \) (note that \( s > \frac{Np}{N(p-1)+p} \) implies \( \tilde{q} > 1 \)).

**Proof of Proposition 4.2.** Let \( u \in W_r \). By the monotonicity of \( R_0 \) and \( R_\infty \), it is not restrictive to assume \( R_1 < R_2 \) in hypothesis (\( R_{q_1, q_2} \)). So, by Lemma 5.1 there exists a constant \( C > 0 \) (dependent on \( u \)) such that for all \( h \in W \) we have

\[
\int_{B_{R_2} \setminus B_{R_1}} K (|x|) |u|^{q_1 - 1} |h| \, dx \leq C \| h \|
\]

and therefore, by (g) and (\( f_{q_1, q_2} \)),

\[
\int_{\mathbb{R}^N} |g (|x|, u) - g (|x|, 0)| |h| \, dx \leq M \int_{\mathbb{R}^N} K (|x|) \min \{|u|^{q_1 - 1}, |u|^{q_2 - 1}\} |h| \, dx
\]

\[
\leq M \left( \int_{B_{R_1}} K (|x|) |u|^{q_1 - 1} |h| \, dx + \int_{B_{R_2}} K (|x|) |u|^{q_2 - 1} |h| \, dx + C \| h \| \right)
\]

\[
\leq M \left( \| u \|^{q_1 - 1} R_0 (q_1, R_1) + \| u \|^{q_2 - 1} R_\infty (q_2, R_2) + C \right) \| h \|.
\]

Together with the assumption on the continuity on \( W \) of the operator \( h \mapsto \int_{\mathbb{R}^N} g (|x|, 0) \, h \, dx \), this gives that the linear operator

\[
T (u) h := \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \cdot \nabla h + V (|x|) |u|^{p-2} u h) \, dx - \int_{\mathbb{R}^N} g (|x|, u) \, h \, dx
\]

is well defined and continuous on \( W \). Hence, by uniform convexity, there exists a unique \( \tilde{u} \in W \) such that

\[
T (u) \tilde{u} = \| \tilde{u} \|^2 = ||T (u)||^2_{W'}.
\]

Denoting by \( O (N) \) the orthogonal group of \( \mathbb{R}^N \), by means of obvious changes of variables it is easy to see that for every \( h \in W \) one has

\[
T (u) h (S \cdot) = T (u) h \quad \text{and} \quad ||h (S \cdot)|| = ||h|| \quad \text{for all } S \in O (N),
\]

whence, applying with \( h = \tilde{u} \), one deduces \( \tilde{u} (S \cdot) = \tilde{u} \) by uniqueness. This means \( \tilde{u} \in W_r \), so that, if \( T (u) h = 0 \) for all \( h \in W_r \), one gets \( T (u) \tilde{u} = 0 \) and hence \( ||T (u)||_{W'} = 0 \).

For future reference, we observe here that, by assumption (g), if (\( f_{q_1, q_2} \)) holds then there exists \( \tilde{M} > 0 \) such that for almost every \( r > 0 \) and all \( t \in \mathbb{R} \) one has

\[
|G (r, t) - g (r, 0, t)| \leq \tilde{M} K (r) \min \{|t|^{q_1}, |t|^{q_2}|. \quad (44)
\]

**Lemma 5.2.** Assume (\( g_0 \)) and let \( L_0 \) be the norm of the operator therein. If there exist \( q_1, q_2 > 1 \) such that (\( f_{q_1, q_2} \)) and (\( S'_{q_1, q_2} \)) hold, then there exist two constants \( c_1, c_2 > 0 \) such that

\[
I (u) \geq \frac{1}{p} ||u||^p - c_1 \| u \|^{q_1} - c_2 \| u \|^{q_2} - L_0 \| u \| \quad \text{for all } u \in W_r. \quad (45)
\]
If \( (S''_{R_1,q_2}) \) also holds, then \( \forall \varepsilon > 0 \) there exist two constants \( c_1 (\varepsilon), c_2 (\varepsilon) > 0 \) such that \( (45) \) holds both with \( c_1 = \varepsilon, c_2 = c_2 (\varepsilon) \) and with \( c_1 = c_1 (\varepsilon), c_2 = \varepsilon \).

**Proof.** Let \( i \in \{1, 2\} \). By the monotonicity of \( S_0 \) and \( S_\infty \), it is not restrictive to assume \( R_1 < R_2 \) in hypothesis \( (S'_{q_1,q_2}) \). Then, by Lemma 5.1 and the continuous embedding \( W \hookrightarrow L^p_{loc} (\mathbb{R}^N) \), there exists a constant \( c^{(i)}_{R_1,R_2} > 0 \) such that for all \( u \in W_r \) we have

\[
\int_{B_{R_2} \setminus B_{R_1}} K (|x|) |u|^q \, dx \leq c^{(i)}_{R_1,R_2} \|u\|^q.
\]

Therefore, by \( (44) \) and the definitions of \( S_0 \) and \( S_\infty \), we obtain

\[
\left| \int_{\mathbb{R}^N} G (|x|, u) \, dx \right| \leq \int_{\mathbb{R}^N} |G (|x|, u) - g (|x|, 0) u| \, dx + \int_{\mathbb{R}^N} g (|x|, 0) u \, dx
\]

\[
\leq \tilde{M} \int_{\mathbb{R}^N} K (|x|) \min \{|u|^{q_1}, |u|^{q_2}\} \, dx + L_0 \|u\|
\]

\[
\leq \tilde{M} \left( \int_{B_{R_1}} K (|x|) |u|^{q_1} \, dx + \int_{B_{R_2} \setminus B_{R_1}} K (|x|) |u|^{q_2} \, dx + \int_{B_{R_2} \setminus B_{R_1}} K (|x|) |u|^{q_1} \, dx \right) + L_0 \|u\|
\]

\[
\leq \tilde{M} \left( |u|^{q_1} s_0 (q_1, R_1) + |u|^{q_2} s_\infty (q_2, R_2) + c^{(i)}_{R_1,R_2} \|u\|^{q_1} \right) + L_0 \|u\|.
\]

(46)

with obvious definition of the constants \( c_1 \) and \( c_2 \), independent of \( u \). This yields \( (45) \). If \( (S''_{q_1,q_2}) \) also holds, then \( \forall \varepsilon > 0 \) we can fix \( R_{1,\varepsilon} < R_{2,\varepsilon} \) such that \( \tilde{M} s_0 (q_1, R_{1,\varepsilon}) < \varepsilon \) and \( \tilde{M} s_\infty (q_2, R_{2,\varepsilon}) < \varepsilon \), so that inequality \( (46) \) becomes

\[
\left| \int_{\mathbb{R}^N} G (|x|, u) \, dx \right| \leq \varepsilon \|u\|^{q_1} + \varepsilon \|u\|^{q_2} + c^{(i)}_{R_{1,\varepsilon},R_{2,\varepsilon}} \|u\|^{q_1} + L_0 \|u\|.
\]

The result then ensues by taking \( i = 2 \) and \( c_2 (\varepsilon) = \varepsilon + c^{(2)}_{R_{1,\varepsilon},R_{2,\varepsilon}} \), or \( i = 1 \) and \( c_1 (\varepsilon) = \varepsilon + c^{(1)}_{R_{1,\varepsilon},R_{2,\varepsilon}} \).

Henceforth, we will assume that the hypotheses of Theorem 4.3 also include the following condition:

\[
g (r, t) = 0 \quad \text{for all } r > 0 \text{ and } t < 0.
\]

(47)

This can be done without restriction, since the theorem concerns nonnegative critical points and all its assumptions still hold true if we replace \( g (r, t) \) with \( g (r, t) \chi_{\mathbb{R}^+} (t) \) (\( \chi_{\mathbb{R}^+} \) is the characteristic function of \( \mathbb{R}^+ \)).

**Lemma 5.3.** Under the assumptions of each of Theorems 4.3 (including (47) and (45)), the functional \( I : W_r \to \mathbb{R} \) satisfies the Palais-Smale condition.

**Proof.** By \( (47) \) and \( (g_5) \) respectively, under the assumptions of each of Theorems 4.3 and 4.5 we have that either \( g \) satisfies \( (g_1) \) for all \( t \in \mathbb{R} \), or \( K (|\cdot|) \in L^1 (\mathbb{R}^N) \) and \( g \) satisfies

\[
\theta G (r, t) \leq g (r, t) t \quad \text{for almost every } r > 0 \text{ and all } |t| \geq t_0.
\]

(48)

Let \( \{u_n\} \) be a sequence in \( W_r \) such that \( \{I (u_n)\} \) is bounded and \( I' (u_n) \to 0 \) in \( W_r' \). Hence

\[
\frac{1}{p} \|u_n\|^p - \int_{\mathbb{R}^N} G (|x|, u_n) \, dx = O (1) \quad \text{and} \quad \|u_n\|^p - \int_{\mathbb{R}^N} g (|x|, u_n) u_n \, dx = o (1) \|u_n\|.
\]

20
If $g$ satisfies $(g_1)$, then we get
\[
\frac{1}{p} \|u_n\|^p + O(1) = \int_{\mathbb{R}^N} G(|x|, u_n) \, dx \leq \frac{1}{\theta} \int_{\mathbb{R}^N} g(|x|, u_n) \, u_n \, dx = \frac{1}{\theta} \|u_n\|^p + o(1) \|u_n\|,
\]
which implies that $\{|\|u_n\||\}$ is bounded since $\theta > p$. If $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and $g$ satisfies $(\mathbf{48})$, then we slightly modify the argument: we have
\[
\int_{\{|u_n| \geq t_0\}} g(|x|, u_n) \, u_n \, dx \leq \int_{\mathbb{R}^N} g(|x|, u_n) \, u_n \, dx + \int_{\{|u_n| < t_0\}} |g(|x|, u_n)\| u_n \, dx
\]
where (thanks to $(g)$ and $(f_{q_1, q_2})$)
\[
\int_{\{|u_n| < t_0\}} |g(|x|, u_n)\| u_n \, dx \leq M \int_{\{|u_n| < t_0\}} K(|x|) \min\{\|u_n\|^{q_1}, \|u_n\|^{q_2}\} \, dx \leq M \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)}
\]
so that, by $(\mathbf{44})$, we obtain
\[
\frac{1}{p} \|u_n\|^p + O(1) = \int_{\mathbb{R}^N} G(|x|, u_n) \, dx \leq \int_{\{|u_n| < t_0\}} G(|x|, u_n) \, dx + \int_{\{|u_n| \geq t_0\}} G(|x|, u_n) \, dx
\]
\[
\leq \hat{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta} \int_{\mathbb{R}^N} g(|x|, u_n) \, u_n \, dx + \frac{M}{\theta} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)}
\]
\[
= \left(\hat{M} + \frac{M}{\theta}\right) \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta} \|u_n\|^p + o(1) \|u_n\|.
\]
This yields again that $\{|\|u_n\||\}$ is bounded. Now, thanks to assumption $(\mathbf{S}_{q_1, q_2}''')$, we apply Theorem $5.1$ to deduce the existence of $u \in W_r$ such that (up to a subsequence) $u_n \rightharpoonup u$ in $W_r$ and $u_n \to u$ in $L^{q_1}_{K} + L^{q_2}_{K}$. Setting
\[
I_1 (u) := \frac{1}{p} \|u\|^p \quad \text{and} \quad I_2 (u) := I_1 (u) - I (u)
\]
for brevity, we have that $I_2$ is of class $C^1$ on $L^{q_1}_{K} + L^{q_2}_{K}$ by $(\mathbf{5})$ Proposition $3.8$ and therefore we get $\|u_n\|^p = I' (u_n) u_n + I_2' (u_n) u_n = I_2' (u_n) u_n + o(1)$. Hence $\lim_{n \to \infty} \|u_n\|$ exists and one has $\|u\|^p \leq \lim_{n \to \infty} \|u_n\|^p$ by weak lower semicontinuity. Moreover, the convexity of $I_1 : W_r \to \mathbb{R}$ implies
\[
I_1 (u) - I_1 (u_n) \geq I_1 (u_n) (u - u_n) = I' (u_n) (u - u_n) + I_2 (u_n) (u - u_n) + o (1)
\]
and thus
\[
\frac{1}{p} \|u\|^p = I_1 (u) \geq \lim_{n \to \infty} I_1 (u_n) = \frac{1}{p} \lim_{n \to \infty} \|u_n\|^p.
\]
So $\|u_n\| \to \|u\|$ and one concludes that $u_n \to u$ in $W_r$ by the uniform convexity of the norm.

\[\Box\]

\textbf{Proof of Theorem 4.3} We want to apply the Mountain-Pass Theorem. To this end, from $(45)$ of Lemma $5.2$ we deduce that, since $L_0 = 0$ and $q_1, q_2 > p$, there exists $\rho > 0$ such that
\[
\inf_{u \in W_r, \|u\| = \rho} I (u) > 0 = I (0).
\]
Therefore, taking into account Lemma 5.3, we need only to check that \( \exists \tilde{u} \in W_r \) such that \( \| \tilde{u} \| > \rho \) and \( I(\tilde{u}) < 0 \). To this end, from assumption \((g_3)\) (which holds in any case, since \((g_1)\) and \((g_2)\) imply \((g_3)\)), we infer that
\[
G(r, t) \geq \frac{G(r, t_0)}{t_0^\theta} t^\theta \quad \text{for almost every } r > 0 \text{ and all } t \geq t_0.
\]

Then, by assumption \( (V) \), we fix a nonnegative function \( u_0 \in C_c^\infty(B_{r_2} \setminus \overline{B_{r_1}}) \cap W_r \) such that the set \( \{ x \in \mathbb{R}^N : u_0(x) \geq t_0 \} \) has positive Lebesgue measure. We now distinguish the case of assumptions \((g_1)\) and \((g_2)\) from the case of \( K (|\cdot|) \in L^1(\mathbb{R}^N) \). In the first one, \((g_1)\) and \((g_2)\) ensure that \( G \geq 0 \) and \( G (\cdot, t_0) > 0 \) almost everywhere, so that for every \( \lambda > 1 \) we get
\[
\int_{\mathbb{R}^N} G(|x|, \lambda u_0) \, dx \geq \int_{\{ \lambda u_0 \geq t_0 \}} G(|x|, \lambda u_0) \, dx \geq \frac{\lambda^\theta}{t_0^\theta} \int_{\{ \lambda u_0 \geq t_0 \}} G(|x|, t_0) \, u_0^\theta dx \\
\geq \frac{\lambda^\theta}{t_0^\theta} \int_{\{ u_0 \geq t_0 \}} G(|x|, t_0) \, u_0^\theta dx \geq \lambda^\theta \int_{\{ u_0 \geq t_0 \}} G(|x|, t_0) \, dx > 0.
\]

Since \( \theta > p \), this gives
\[
\lim_{\lambda \to +\infty} I(\lambda u_0) \leq \lim_{\lambda \to +\infty} \left( \frac{\lambda^p}{p} \| u_0 \|^p - \lambda^\theta \int_{\{ u_0 \geq t_0 \}} G(|x|, t_0) \, dx \right) = -\infty.
\]

If \( K (|\cdot|) \in L^1(\mathbb{R}^N) \), assumption \((g_3)\) still gives \( G (\cdot, t_0) > 0 \) almost everywhere and from (44) we infer that
\[
G(r, t) \geq -\tilde{M} K(r) \min \{ t_0^\varphi_1, t_0^\varphi_2 \} \quad \text{for almost every } r > 0 \text{ and all } 0 \leq t \leq t_0.
\]

Therefore, arguing as before about the integral over \( \{ \lambda u_0 \geq t_0 \} \), for every \( \lambda > 1 \) we obtain
\[
\int_{\mathbb{R}^N} G(|x|, \lambda u_0) \, dx = \int_{\{ \lambda u_0 < t_0 \}} G(|x|, \lambda u_0) \, dx + \int_{\{ \lambda u_0 \geq t_0 \}} G(|x|, \lambda u_0) \, dx \\
\geq -\tilde{M} \min \{ t_0^\varphi_1, t_0^\varphi_2 \} \int_{\{ \lambda u_0 < t_0 \}} K(|x|) \, dx + \lambda^\theta \int_{\{ u_0 \geq t_0 \}} G(|x|, t_0) \, dx,
\]
which implies
\[
\lim_{\lambda \to +\infty} I(\lambda u_0) \leq \lim_{\lambda \to +\infty} \left( \frac{\lambda^p}{p} \| u_0 \|^p + \tilde{M} \min \{ t_0^\varphi_1, t_0^\varphi_2 \} \| K \|_{L^1(\mathbb{R}^N)} - \lambda^\theta \int_{\{ u_0 \geq t_0 \}} G(|x|, t_0) \, dx \right) = -\infty.
\]

So, in any case, we can take \( \tilde{u} = \lambda u_0 \) with \( \lambda \) sufficiently large and the Mountain-Pass Theorem provides the existence of a nonzero critical point \( u \in W_r \) for \( I \). Since (44) implies \( I'(u) u_- = -\| u_- \|^p \) (where \( u_- \in W_r \) is the negative part of \( u \)), one concludes that \( u_- = 0 \), i.e., \( u \) is nonnegative.

In order to conclude the proof of Theorem 4.5 we recall from [5 Corollary 2.19] that for every \( p_1, p_2 \in (1, +\infty) \) and \( u \in L^{p_1}_K + L^{p_2}_K \) one has
\[
\| u \|_{L^{p_1}_K + L^{p_2}_K} \leq \| u \|_{L^{\min(p_1, p_2)}_K (\mathcal{A}_u)} + \| u \|_{L^{\max(p_1, p_2)}_K (\mathcal{A}_u)} \quad \text{where} \quad \mathcal{A}_u := \{ x \in \mathbb{R}^N : |u(x)| > 1 \}.
\]

(50)
Proof of Theorem 4.5. By the oddness assumption \((g_5)\), one has \(I(u) = I(-u)\) for all \(u \in W_r\) and thus we can apply the Symmetric Mountain-Pass Theorem (see e.g. [12, Chapter 1]). To this end, we deduce \([49]\) as in the proof of Theorem 4.3 and therefore, thanks to Lemma 5.3, we need only to show that we can apply the Symmetric Mountain-Pass Theorem (see e.g. [12, Chapter 1]). To this end, we deduce (49) as in (50) one has

\[
\|u_n\|_{L^p_{K}(\lambda u_n)} + \|u_n\|_{L^p_{K}(\Lambda u_n)} \geq \|u_n\|_{L^p_{K} + L^p_{K}} \geq m_1 \|u_n\| \to +\infty \tag{51}
\]

for some constant \(m_1 > 0\), where \(p_1 := \min\{q_1, q_2\}\) and \(p_2 := \max\{q_1, q_2\}\). Hence, up to a subsequence, at least one of the sequences \(\{\|u_n\|_{L^p_{K}(\lambda u_n)}\}\), \(\{\|u_n\|_{L^p_{K}(\Lambda u_n)}\}\) diverges. We now use assumptions \((g_4)\) and \((g_5)\) to deduce that

\[
G(r, t) \geq mK(r) \min\{|t|^{p_2}, |t|^{p_1}\} \quad \text{for almost every } r > 0 \text{ and all } t \in \mathbb{R},
\]

which implies

\[
\int_{\mathbb{R}^N} G(|x|, u_n) \, dx \geq m \int_{\Lambda u_n} K(|x|) |u_n|^{p_1} \, dx + m \int_{\lambda u_n} K(|x|) |u_n|^{p_2} \, dx.
\]

Hence, using inequalities (51), there exists a constant \(m_2 > 0\) such that

\[
I(u_n) \leq m_2 \left( \|u_n\|_{L^p_{K}(\lambda u_n)} + \|u_n\|_{L^p_{K}(\Lambda u_n)} \right) - m \left( \|u_n\|_{L^p_{K}(\lambda u_n)} + \|u_n\|_{L^p_{K}(\Lambda u_n)} \right),
\]

so that \(I(u_n) \to -\infty\) since \(p_1, p_2 > p\). The Symmetric Mountain-Pass Theorem thus implies the existence of an unbounded sequence of critical values for \(I\) and this completes the proof.

Lemma 5.4. Under the assumptions of each of Theorems 4.7 and 4.10, the functional \(I : W_r \to \mathbb{R}\) is bounded from below and coercive. In particular, if \(g\) satisfies \((g_6)\), then

\[
\inf_{v \in W_r} I(v) < 0. \tag{52}
\]

Proof. The fact that \(I\) is bounded below and coercive on \(W_r\) is a consequence of Lemma 5.2. Indeed, the result readily follows from (45) if \(q_1, q_2 \in (1, p)\), while, if max \(\{q_1, q_2\} = p > \min\{q_1, q_2\} > 1\), we fix \(\varepsilon < 1/p\) and use the second part of the lemma in order to get

\[
I(u) \geq \left( \frac{1}{p} - \varepsilon \right) \|u\|^p - c(\varepsilon) \|u\|^{\min\{q_1, q_2\}} - L_0 \|u\| \quad \text{for all } u \in W_r,
\]

which yields again the conclusion. In order to prove (52) under assumption \((g_6)\), we use assumption (V) to fix a function \(u_0 \in C_0^\infty(B_{r_2} \setminus \overline{B_{r_1}}) \cap W_r\) such that \(0 \leq u_0 \leq t_0, u_0 \not\equiv 0\). Then, by \((g_6)\), for every \(0 < \lambda < 1\) we get that \(\lambda u_0 \in W_r\) satisfies

\[
I(\lambda u_0) = \frac{1}{p} \|\lambda u_0\|^p - \int_{\mathbb{R}^N} G(|x|, \lambda u_0) \, dx \leq \frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta m \int_{\mathbb{R}^N} K(|x|) u_0^p \, dx.
\]

Since \(\theta < p\), this implies \(I(\lambda u_0) < 0\) for \(\lambda\) sufficiently small and therefore (52) ensues. \(\square\)
Proof of Theorem 4.7. Let 

\[ \mu := \inf_{v \in W_r} I(v) \]

and take any minimizing sequence \( \{v_n\} \) for \( \mu \). From Lemma 5.4 we have that the functional \( I : W_r \to \mathbb{R} \) is bounded from below and coercive, so that \( \mu \in \mathbb{R} \) and \( \{v_n\} \) is bounded in \( W_r \). Thanks to Theorem 2.1 and assumption \((S'_{q_1, q_2})\), the embedding \( W_r \hookrightarrow L^q_{K_r} + L^{q_2}_{K_r} \) is compact and thus we can assume that there exists \( u \in W_r \) such that, up to a subsequence, one has \( v_n \rightharpoonup^* u \) in \( W_r \) and \( v_n \to u \) in \( L^q_{K_r} + L^{q_2}_{K_r} \). Then, thanks to \((g_0)\) and the continuity of the functional \( v \mapsto \int_{\mathbb{R}^N} (G(|x|, v) - g(|x|, 0) v) \, dx \) on \( L^q_{K_r} + L^{q_2}_{K_r} \) (which follows from \((g)\), \((f_{q_1, q_2})\) and [5] Proposition 3.8), \( u \) satisfies

\[
\int_{\mathbb{R}^N} G(|x|, v_n) \, dx = \int_{\mathbb{R}^N} (G(|x|, v_n) - g(|x|, 0) v_n) \, dx + \int_{\mathbb{R}^N} g(|x|, 0) v_n \, dx \to \int_{\mathbb{R}^N} G(|x|, u) \, dx.
\]

By the weak lower semi-continuity of the norm, this implies

\[
I(u) = \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} G(|x|, u) \, dx \leq \lim_{n \to \infty} \left( \frac{1}{p} \|v_n\|^p - \int_{\mathbb{R}^N} G(|x|, v_n) \, dx \right) = \mu
\]

and thus we conclude \( I(u) = \mu \). It remains to show that \( u \neq 0 \). This is obvious if \( g \) satisfies \((g_0)\), since \( \mu < 0 \) by Lemma 5.4. If \((g_7)\) holds, assume by contradiction that \( u = 0 \). Since \( u \) is a critical point of \( I \in C^1(W_r; \mathbb{R}) \), from (40) we get

\[
\int_{\mathbb{R}^N} g(|x|, 0) \, h \, dx = 0, \quad \forall h \in C^\infty_{c, rad}(B_{r_2} \setminus \overline{B_{r_1}}) \subset W_r.
\]

This implies \( g(., 0) = 0 \) almost everywhere in \((r_1, r_2)\), which is a contradiction.

Proof of Corollary 4.8. Setting

\[
\tilde{g}(r, t) := \begin{cases} 
  g(r, t) & \text{if } t \geq 0 \\
  2g(r, 0) - g(r, |t|) & \text{if } t < 0
\end{cases}
\]

and

\[
\tilde{G}(r, t) := \int_0^t \tilde{g}(r, s) \, ds = \begin{cases} 
  G(r, t) & \text{if } t \geq 0 \\
  2g(r, 0) t + G(r, |t|) & \text{if } t < 0
\end{cases}
\]

it is easy to check that the function \( \tilde{g} \) still satisfies all the assumptions of Theorem 4.7. Then there exists \( \overline{u} \neq 0 \) such that

\[
\overline{I}(\overline{u}) = \min_{u \in W_r} \overline{I}(u), \quad \text{where} \quad \overline{I}(u) := \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} \tilde{G}(|x|, u) \, dx.
\]

For every \( u \in W_r \) one has

\[
\overline{I}(u) = \frac{1}{p} \|u\|^p - \int_{\{u \geq 0\}} G(|x|, u) \, dx - 2 \int_{\{u < 0\}} g(|x|, 0) u \, dx - \int_{\{u < 0\}} G(|x|, |u|) \, dx
\]

\[
= \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} G(|x|, |u|) \, dx + 2 \int_{\mathbb{R}^N} g(|x|, 0) u_- \, dx
\]

\[
= I(|u|) + 2 \int_{\mathbb{R}^N} g(|x|, 0) u_- \, dx,
\]

24
Proof of Theorem 4.10.

Then, for any \( \forall \) bounded in the Palais-Smale condition, is even, bounded from below and such that \( J \) in order to check (54), we first deduce from and the geometric condition (54). By coercivity (see Lemma 5.4 again), every Palais-Smale sequence for \( \bar{u} \) is still a minimizer for \( \bar{I} \), so that we can assume \( \bar{u} \geq 0 \). Finally, \( \bar{u} \) is a critical point for \( \bar{I} \) since \( \bar{u} \) is a critical point of \( \bar{I} \) and \( \bar{g} (r, t) = g (r, t) \) for every \( t \geq 0 \).

In proving Theorem 4.10 we will use a well known abstract result, which we recall it here in a version of [16].

**Theorem 5.5 (16 Lemma 2.4).** Let \( X \) be a real Banach space and let \( J \in C^1 (X; \mathbb{R}) \). Assume that \( J \) satisfies the Palais-Smale condition, is even, bounded from below and such that \( J (0) = 0 \). Assume furthermore that \( \forall k \in \mathbb{N} \setminus \{0\} \) there exist \( \rho_k > 0 \) and a \( k \)-dimensional subspace \( X_k \) of \( X \) such that

\[
\sup_{u \in X_k, \|u\|_X = \rho_k} J (u) < 0.
\]

Then \( J \) has a sequence of critical values \( c_k < 0 \) such that \( \lim_{k \to \infty} c_k = 0 \).

**Proof of Theorem 4.10.** Since \( I : W_r \to \mathbb{R} \) is even by assumption (5.5) and bounded below by Lemma 5.4 for applying Theorem 5.5 (with \( X = W_r \) and \( J = I \)) we need only to show that \( I \) satisfies the Palais-Smale condition and the geometric condition 5.4. By coercivity (see Lemma 5.4 again), every Palais-Smale sequence for \( I \) is bounded in \( W_r \) and one obtains the existence of a strongly convergent subsequence as in the proof of Lemma 5.3. In order to check (54), we first deduce from (5.5) and (5.6) that

\[
G (r, t) \geq mK (r) \|t\|^\theta \quad \text{for almost every } r > 0 \text{ and all } \|t\| \leq t_0.
\]

Then, for any \( k \in \mathbb{N} \setminus \{0\} \), we take \( k \) linearly independent functions \( \phi_1, ..., \phi_k \in C^\infty_c (B_{r_2} \setminus \overline{B_{r_1}}) \) such that

\[
0 \leq \phi_i \leq t_0 \quad \text{for every } i = 1, ..., k \quad \text{and set}
\]

\[
X_k := \text{span} \{ \phi_1, ..., \phi_k \} \quad \text{and} \quad \| \lambda_1 \phi_1 + ... + \lambda_k \phi_k \|_{X_k} := \max_{1 \leq i \leq k} |\lambda_i|.
\]

This defines a subspace of \( W_r \) by assumption (V) and all the norms are equivalent on \( X_k \), so that there exist \( m_k, l_k > 0 \) such that for all \( u \in X_k \) one has

\[
\|u\|_{X_k} \leq m_k \|u\| \quad \text{and} \quad \|u\|^\theta \|u\|^\theta \geq t_0 \|u\|^\theta.
\]

Fix \( \rho_k > 0 \) small enough that \( km_k \rho_k < 1 \) and \( \rho_k^\theta / p - m_k \rho_k^\theta < 0 \) (which is possible since \( \theta < p \)) and take any \( u = \lambda_1 \phi_1 + ... + \lambda_k \phi_k \in X_k \) such that \( \|u\| = \rho_k \). Then by (56) we have \( |\lambda_i| \leq \rho_k \|X_k \| \leq m_k \rho_k < 1 / k \) for every \( i = 1, ..., k \) and therefore

\[
|u (x)| \leq \sum_{i=1}^k |\lambda_i| \phi_i (x) \leq t_0 \sum_{i=1}^k |\lambda_i| < t_0 \quad \text{for all } x \in \mathbb{R}^N.
\]

By (55) and (56), this implies

\[
\int_{\mathbb{R}^N} G (|x|, u) \, dx \geq m \int_{\mathbb{R}^N} K (|x|) |u|^\theta \, dx \geq m \|u\|^\theta
\]

and hence we get \( I (u) \leq \|u\|^p / p - m_k \|u\|^\theta = \rho_k^\theta / p - m_k \rho_k^\theta < 0 \). This proves (54) and the conclusion thus follows from Theorem 5.5.
References

[1] Anoop, T. V ., Drábek, P., Sasi, S.: Weighted quasilinear eigenvalue problems in exterior domains. Calc. Var. Partial Differential Equations 53, 961-975 (2015)

[2] Badiale, M., Guida, M., Rolando, S.: Compactness results for the $p$-Laplace equation. [arXiv:1510.03879 [math.AP]]

[3] Badiale, M., Guida, M., Rolando, S.: Compactness and existence results in weighted Sobolev spaces of radial functions. Part II: Existence. NoDEA, Nonlinear Differ. Equ. Appl., to appear [arXiv:1506.00056 [math.AP]]

[4] Badiale, M., Guida, M., Rolando, S.: Compactness and existence results in weighted Sobolev spaces of radial functions. Part I: Compactness. Calc. Var. Partial Differential Equations 54, 1061-1090 (2015)

[5] Badiale, M., Pisani, L., Rolando, S.: Sum of weighted Lebesgue spaces and nonlinear elliptic equations. NoDEA, Nonlinear Differ. Equ. Appl. 18, 369-405 (2011)

[6] Bartolo, R., Candela, A.M., Salvatore, A.: Multiplicity results for a class of asymptotically $p$-linear equations on $\mathbb{R}^N$. Commun. Contemp. Math. 18, 1550031, 24 pages (2016).

[7] Cai, H., Su, J., Sun, Y.: Sobolev type embeddings and an inhomogeneous quasilinear elliptic equation on $\mathbb{R}^N$ with singular weights. Nonlinear Anal. 96, 59-67 (2014)

[8] Chen, S., Wang, Z.-Q.: Existence and multiple solutions for a critical quasilinear equation with singular potentials. NoDEA, Nonlinear Differ. Equ. Appl. 22, 699-719 (2015).

[9] Guida, M., Rolando, S.: Nonlinear Schrödinger equations without compatibility conditions on the potentials. J. Math. Anal. Appl. 439, 201-219 (2007)

[10] Li, A., Cai, H., Su, J.: Quasilinear elliptic equations with singular potentials and bounded discontinuous nonlinearities. Topol. Methods Nonlinear Anal. 43, 439-450 (2014).

[11] Palais R.S.: The principle of symmetric criticality. Commun. Math. Phys. 69, 19-30 (1979).

[12] Rabinowitz P.H.: Minimax methods in critical point theory with applications to differential equations. CBMS Regional Conference Series in Mathematics, no. 65, Providence, 1986.

[13] Su, J.: Quasilinear elliptic equations on $\mathbb{R}^N$ with singular potentials and bounded nonlinearity. Z. Angew. Math. Phys. 63, 51-62 (2012)

[14] Su, J., Tian, R.: Weighted Sobolev type embeddings and coercive quasilinear elliptic equations on $\mathbb{R}^N$. Proc. Amer. Math. Soc. 140, 891-903 (2012)

[15] Su, J., Wang, Z.-Q., Willem, M.: Weighted Sobolev embedding with unbounded and decaying radial potentials. J. Differential Equations 238, 201-219 (2007)

[16] Wang Z.Q.: Nonlinear boundary value problems with concave nonlinearities near the origin. NoDEA, Nonlinear Differ. Equ. Appl. 8, 15-33 (2001).

[17] Yang, Y., Zhang, J.: A note on the existence of solutions for a class of quasilinear elliptic equations: an Orlicz-Sobolev space setting. Bound. Value Probl. 2012, 2012:136, 7 pages
[18] Zhang, G.: Weighted Sobolev spaces and ground state solutions for quasilinear elliptic problems with unbounded and decaying potentials. Bound. Value Probl. 2013, 2013:189, 15 pages

[19] Zhao, L., Su, J., Wang, C.: On the existence of solutions for quasilinear elliptic problems with radial potentials on exterior ball. Math. Nachr. 289, 501-514 (2016).