Square Difference Labeling of Cycle, Path and Tree Related Graphs

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Abstract

In this paper, we prove that the duplication of each vertex of a path by an edge, the duplication of a pendant vertex by an edge of CnʘK₁, 2-tuple graph of K₁,n, the switching of a pendant vertex in path admit square difference labeling (SDL).

Keywords: Square difference graph, duplication, switching of vertex, 2-tuple graph

I. Introduction

For our study, we consider a simple, undirected, finite graph and we follow [I, II, III] for all terminology and notations. The square difference labeling is introduced by Shiama [IV]. Shiama investigated that Path, Fan and Gear graphs are Square difference [V]. A. Sugumaran, and Mohan proved that some special graphs are Difference Cordial labeling [VI]. P.L.Vihol and Shah verified 2-tuple graphs of some graphs are Difference Cordial Labeling [VII]. In this paper, we prove some special graphs are Square difference.

II. Definitions

Definition 2.1. [IV].
A graph G is said to be a square difference graph if it admits a bijective function g : V → {0, 1, 2, … p−1} such that the induced function g* : E(G) → N given by g*(xy) = |[g(x)]² − [g(y)]²| ∀xy ∈ E(G) are all distinct.

Definition 2.2. [VI]
A vertex switching of a graph G is a graph acquired by switching the pendant vertex v of G, detaching all the edges incident with v and attaching edges joining v to other vertices which are not adjacent to v in G.
Definition 2.3. [VI]
Duplication of a vertex \( v_i \) by a new edge \( e = v'_i v''_i \) in a graph \( G \) produces a new

graph \( G' \) such that \( N(v'_i) = N(v_i v''_i) \) and \( N(v''_i) = N(v_i v'_i) \).

Definition 2.4. [VII].
The 2 - tuple graph of \( G \) vertex \( v_i \) of \( G \) is attained by joining each vertex \( v \) of \( G \) to the corresponding

vertex \( v' \) of \( G' \) (copy of \( G \)) by an edge and is denoted by \( T^2(G) \)

III. Main Results

Theorem 3.1
The switching of a pendant vertex in path is Square difference.

Proof:
Consider the graph \( G(P_n) \) with the vertices \( u_1, u_2, \ldots, u_n \) denotes the vertex

switching of \( P \) with respect to the pendant vertex \( u_1 \).

Clearly \( |V(G(P_n))| = n \) and

\( |E(G(P_n))| = 2n - 4 \)

We explore a vertex valued function \( f \) as:

\( f(u_i) = i-1 \)

Then the induced function \( f^* \) yields edge labels as,

\[ f^*(u_i u_{i+1}) = 2i - 1, \quad 2 \leq i \leq n - 1 \]

\[ f^*(u_i u_1) = [i - 1]^2, \quad 3 \leq i \leq n - 1 \]

Thus, all the edge labeling are distinct i.e., \( f^*(e_i) \neq f^*(e_j), \forall e_i, e_j \in (G(P_n)) \).

Hence the theorem.

Example 1

![Diagram](image)

Fig 1. The switching of \( u_0 \) in path is SD graph

Theorem 3.2
The duplication of each vertex of path by an edge is Square difference Graph.

Proof:
Let the graph $G$ be obtained by duplicating each vertex of the path $P_n$ by an edge with the vertices
$$V(G) = \{v_j, v'_j, v''_j / 1 \leq j \leq n\}$$
and the edges
$$E(G) = \{v_j v_{j+1} \cup \{v_j v'_j\} \cup \{v_j v''_j\}\}$$

Consider the vertex valued function $f: V(G) \rightarrow \{0, 1, \ldots, 2n\}$ as follows:

For $1 \leq i \leq n$,
$$f(v_i) = j - 1,$$
$$f(v'_i) = n + 2j - 2,$$
$$f(v''_i) = n + 2j - 1$$

and the induced function $f^*: E(G) \rightarrow \mathbb{N}$ given by $f^*(uv) = |f(u)|^2 - |f(v)|^2|$, $\forall uv \in E(G)$.

For the above labeling pattern, we attain the edge labels $f^*$ as,

For $1 \leq i \leq n$,
$$f^*(v_i, v_{i+1}) = 2j - 1,$$
$$f^*(v'_i, v''_i) = 2n + 4j - 3,$$
$$f^*(v_i, v'_i) = n^2,$$
$$f^*(v_i, v''_i) = (n+1)^2$$

and also $f^*(v'_j, v''_j)$ for $j = 1, 2, \ldots, n$ receives edge labels as an increasing sequence of odd integers when it has label with one end vertex of odd integer and the other end vertex of even integer. Similarly, it receives labels of even integers when it have label with both end vertices are either odd (or) even. Thus, all the edge labeling are distinct and strictly increasing sequence. Hence the graph $G$ admits SDL.

**Theorem 3.3**
The duplication of a pendant vertex by an edge of $Cn \Theta K_1$ is SDG.

**Proof:**
Consider the graph $G$ with
$$V(G) = \{u_i, v_i, v_{ij} / 1 \leq i \leq n, 1 \leq j \leq 2\},$$
and
$$E(G) = \{u_i u_{i+1} / 1 \leq i \leq n - 1\} \cup \{u_iv_i, v_{i1} v_{i2}, v_{i,j}, i = 1 \text{ to } n, 1 \leq j \leq 2\} \cup \{u_n u_1\}$$

Clearly, $|V| = 4n$, $|E| = 5n$. Define a bijective function $g$ as
$$g(u_i) = i - 1,$$
$$g(v_i) = n + 3(i-1),$$
$$g(v_{ij}) = n + 3(i-1) + j, \text{ for } i = 1, 2, \ldots, n, j = 1, 2$$

Then the induced edge function $g^*$ yields the edge labeling as follows.
$$g^*(u_i u_{i+1}) = 2i - 1, 1 \leq i \leq n - 1$$

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\[ g^*(u_1u_2) = (n-I)^2 \]
\[ g^*(u_1v_1) = \begin{cases} 1 \mod 2, & n \text{ is odd} \\ 0 \mod 2, & n \text{ is even} \end{cases} \]
\[ g^*(v_1v_{i,2}) = 2n + 6i - 3 \]
\[ g^*(v_1v_{1,1}) = 2n + 6i - 5 \]
\[ g^*(v_1v_{2,1}) = 4n + 12j - 8 \]

Thus, the entire 5n edges are all distinct. Hence the theorem is verified.

Example 3:

Fig 3. SDL for Duplication of all the vertex by an edge of \( C_3 \otimes K_1 \)

Theorem 3.4

The 2-tuple graph of \( K_{1,n} \) admits Square difference Labeling.

Proof:

Let \( T^2(\overline{K}_{1,n}) \) be the graph with the apexes \( x \) and \( x' \) and the vertices \( x_1, x_2, \ldots, x_n \) and \( x_1', x_2', \ldots, x_n' \).

Clearly, the edge set \( E = \{xx'\} \cup \{xx_i\} \cup \{x'x_i'\} \cup \{x_1x_i\}. \)

The number of vertices and edges are 2n and 3n+1 resp.

Let the vertex valued function \( g: V \rightarrow \{0, 1, 2, \ldots, 2n-1\} \) be defined as follows

\[ g(x) = 0 \]
\[ g(x') = 1 \]
\[ g(x_i) = 2i \]
\[ g(x_i') = 2i + 1, \text{ for } 1 \leq i \leq n. \]

For the above labeling pattern, \( g^* \) satisfies the condition of SD Labeling. Thus, the edges of G receive labels as,

\[ g^*(xx') = 1 \]
\[ g^*(xx_i) = (2i)^2 \]
Thus, all the edge labeling are distinct. i.e., $g^*(e_i) \neq g^*(e_j), \forall e_i \neq e_j \in E(G)$. Hence, the graph $T^2(K_{1,n})$ admits SDL.

**Example 4**

![Figure 3. Square Difference Labeling for $T^2(K_{1,4})$](image)

**IV. Conclusion**

In this work, we proved some special graphs of cycle, path and tree admit Square difference labeling.

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