ENERGY STABILITY FOR THERMO-VISCOUS FLUIDS WITH A FADING MEMORY HEAT FLUX

Giovambattista Amendola
Dipartimento di Matematica
Largo Bruno Pontecorvo 5
Pisa, 56127, Italy

Mauro Fabrizio
Dipartimento di Matematica
Piazza di Porta S. Donato 5
Bologna, 40127, Italy

John Murrough Golden
School of Mathematical Sciences, Dublin Institute of Technology
Kevin Street
Dublin 8, Ireland

Adele Manes
Dipartimento di Matematica
Largo Bruno Pontecorvo 5
Pisa, 56127, Italy

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Abstract. In this work we consider the thermal convection problem in arbitrary bounded domains of a three-dimensional space for incompressible viscous fluids, with a fading memory constitutive equation for the heat flux. With the help of a recently proposed free energy, expressed in terms of a minimal state functional for such a system, we prove an existence and uniqueness theorem for the linearized problem. Then, assuming some restrictions on the Rayleigh number, we also prove exponential decay of solutions.

1. Introduction. In a previous paper [2], two of the authors studied the stability of a viscous fluid with a fading memory extra-stress. Similar problems were studied for viscous fluids without memory by Preziosi and Rionero [13], Lozinsky and Owens [11] and Doering et al [8]. A comprehensive review of the older results may be found in [15].

In [2], the Bénard problem was considered for a viscoelastic fluid with fading memory, using the Boussinesq approximation in the temperature. Assuming incompressibility for the fluid, a theorem on existence and uniqueness of solutions for the linearized system of equations was proved subject to a restriction on the Rayleigh number.

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In this work we study a more general problem involving a thermo-viscous fluid. We extend Slemrod’s stability results [14] by assuming memory effects in the constitutive equation for the heat flux; in [2], such effects were considered for the stress tensor. Also, by applying semigroup theory [5], exponential stability is established for a large domain of initial conditions.

For this problem we need to introduce a new free energy, recently introduced in [7] for viscoelastic solids and in [1] for viscoelastic fluids. This is defined in a very large domain of minimal states (see, for example, [3]).

The layout of the paper is as follows. In Section 2, we introduce the constitutive equations for a thermo-viscous fluid and the new free energy. In Section 3, the basic equations describing thermal convection in the fluid are given, together with their linear approximation. Finally, in Section 4, we prove an existence and uniqueness theorem for the linearized thermal problem and derive conditions for exponential stability.

2. A viscous incompressible fluid with a fading memory heat flux. We consider a simple incompressible viscous fluid characterized by the following constitutive equations

\[
T(x,t) = -p(x,t)I + 2\mu D(x,t), \quad D = \frac{1}{2} (\nabla v + \nabla v^T), \tag{1}
\]

\[
\nabla \cdot v = 0, \tag{2}
\]

where \(T\) denotes the stress tensor, \(p\) is the pressure, \(I\) is the unit tensor, \(D\) is the strain rate tensor and \(v\) denotes the velocity, which must satisfy the equation (2) for an incompressible fluid. The quantity \(\mu\) is the viscosity.

Let the quantity \(\theta > 0\) be the absolute temperature and \(\theta_0\) the average temperature of the fluid. We put

\[
\theta(t) = \theta_0 + \vartheta(t), \tag{3}
\]

where \(\vartheta(t)\) will be used to denote the temperature of the body. We assume for the heat flux \(q\) a constitutive equation with memory effects of the type

\[
q(x,t) = -\int_0^{+\infty} k(s) \nabla \vartheta^t(x,s) ds, \tag{4}
\]

where

\[
\vartheta^t(x,s) = \vartheta(x,t-s) \quad \forall s \in \mathbb{R}^+, \tag{5}
\]

denotes the history up to time \(t\) of \(\vartheta\) at any point \(x\) of the fluid and \(k \in H^2(\mathbb{R}^+, L^\infty(\Omega))\) is the relaxation function, which has the fading memory property

\[
\lim_{s \to +\infty} k(s) = 0. \tag{6}
\]

It is useful to introduce the extra stress \(T_E\), defined by

\[
T_E(x,t) = T(x,t) + p(x,t)I = 2\mu D(x,t). \tag{7}
\]

In the following, it is sometimes convenient to use the temperature integrated history

\[
\tilde{\vartheta}(x,t) = \int_{-\infty}^{t} \vartheta(x,\tau) d\tau, \tag{8}
\]

and to introduce the history of such a quantity relative to its value at time \(t\), defined by

\[
\tilde{\vartheta}_t^t(x,s) = \tilde{\vartheta}(x,s) - \tilde{\vartheta}(x,t) \quad \forall s \in \mathbb{R}^+ = [0, +\infty), \tag{9}
\]
where
\[ \tilde{\vartheta}^t(x, s) = \tilde{\vartheta}(x, t-s). \] (10)
Thus, we obtain, in particular,
\[ \int_{t-s}^t \nabla \vartheta(x, \tau) d\tau = - \left[ \nabla \tilde{\vartheta}(x, t-s) - \nabla \tilde{\vartheta}(x, t) \right] := - \nabla \tilde{\vartheta}^t(x, s). \] (11)
Also,
\[ \frac{\partial}{\partial s} \left[ - \nabla \tilde{\vartheta}^t(x, s) \right] = \frac{\partial}{\partial s} \int_{t-s}^t \nabla \vartheta(x, \tau) d\tau = - \nabla \tilde{\vartheta}^t(x, s). \] (12)
and
\[ \frac{\partial}{\partial t} \left[ \nabla \tilde{\vartheta}^t(x, s) \right] = \nabla \tilde{\vartheta}^t(x, s). \] (13)
Integrating by parts in (4), taking account of (6) and substituting (11)
2, we find
\[ q(x, t) = \int_0^{+\infty} k'(s) \left[ \int_{t-s}^t \nabla \vartheta(x, \tau) d\tau \right] ds = - \int_0^{+\infty} k'(s) \nabla \tilde{\vartheta}^t(x, s) ds. \] (14)
The dependence on \( x \) will be often omitted in what follows. Also, the inequalities
\[ k(s) > 0, \quad k'(s) < 0, \quad k''(s) \geq 0 \quad \forall s \in \mathbb{R}^+ \] (15)
and
\[ 0 \neq |k'(0)| < +\infty \] (16)
will be assumed.
We shall consider the free energy which was studied in [7] for viscoelastic solids
and denoted by \( \psi_F \). Analogous functionals can be defined for other materials [3].
For the constitutive equation (4), this free energy functional has the form
\[ \psi_F(t) = -\frac{1}{2} \int_0^{+\infty} \frac{1}{k'(\tau)} Q^t(\tau) \cdot Q^t(\tau) d\tau, \] (17)
where \( Q^t(\tau) \) denotes the derivative with respect to \( \tau \) of the function \( Q^t(\tau) \), defined by
\[ Q^t(\tau) = \int_0^{+\infty} k'(\tau + s) \nabla \tilde{\vartheta}^t(s) ds \quad \forall \tau \in \mathbb{R}^+. \] (18)
So, \( Q^t(\tau) \) is given by
\[ Q^t(\tau) := \frac{\partial}{\partial \tau} Q^t(\tau) = \int_0^{+\infty} k''(\tau + s) \nabla \tilde{\vartheta}^t(s) ds. \] (19)
Since the functional (17) must give a non-negative quantity, the assumed relation
(15)2 must hold.
The domain of definition of the functional \( \psi_F \) is the following space
\[ H^t_F(\mathbb{R}^+) = \left\{ Q^t; \left| \int_0^{+\infty} \frac{1}{k'(\tau)} Q^t(\tau) \cdot Q^t(\tau) d\tau \right| < +\infty \right\}, \] (20)
which, as already observed in several works and in particular in [3], yields a much
larger space than applies for many other free energies, in particular that corresponding
to the Graffi-Volterra free energy \( \psi_G \).
From (18), taking into account (14), we have
\[ Q_t(0) = \int_0^{+\infty} k'(s) \nabla \tilde{\vartheta}_t(s) ds \equiv -q(t); \] (21)
moreover, from (19), by integrating by parts and using (12), we have
\[ Q_t(0) = \int_0^{+\infty} k''(s) \nabla \tilde{\vartheta}_t(s) ds = \int_0^{+\infty} k'(s) \nabla \vartheta_t(s) ds. \] (22)

The quantity \( \psi_F(t) \) plays an important role in later developments (see (49) below). Also, the Graffi-Volterra free energy (see [3], for example) for the temperature history, introduced in (50), will be useful.

3. **Basic equations.** In this section we study thermal convection in an incompressible viscous fluid, with memory effects in the heat flux, which occupies an arbitrary bounded domain \( \Omega \subset \mathbb{R}^3 \). For this purpose, we adopt the Boussinesq approximation for the density, which is assumed to be constant except in the body force due to gravity.

With respect to a steady state solution we consider a perturbation denoted by \((v(x, t), \vartheta(x, t), p(x, t))\). The system of differential equations, which must be satisfied by such a perturbation, has been studied in the general case and considered also in its linearized form in [15]. This form, with a few change due to the presence of memory effects, was derived in [2] (see also [4]) and put in a non-dimensional form. Let us redefine \( \vartheta(x, t) \) as a dimensionless quantity by writing (3) as
\[ \theta(t) = 1 + \vartheta(t), \quad \vartheta(t) = \frac{\theta(t) - \theta_0}{\theta_0}. \] (23)

Taking account of the constitutive equations (1), (4) and the condition (2), which yields \( \nabla \cdot D = \frac{1}{2} \left[ \nabla^2 v + \nabla (\nabla \cdot v) \right] = \frac{1}{2} \nabla^2 v \), such a system assumes the form
\[ v_t = -\nabla p + \mu \nabla^2 v + R \vartheta \mathbf{k} + b, \] (24)
\[ \mbox{Pr} \vartheta_t = \int_0^t k'(s) \nabla^2 \vartheta_r(t-s) ds + R v \cdot \mathbf{k} + f, \] (25)
\[ f(x, t) = r(x, t) + \int_0^{+\infty} k'(\eta + t) \nabla^2 \vartheta_r(-\eta) d\eta, \] (26)

where \( b \) and \( r \) are the perturbations to the corresponding sources in the steady state solution. Also, \( R \) is the Rayleigh number, \( \mbox{Pr} \) is the Prandtl number and \( \mathbf{k} = (0, 0, 1) \). Moreover, we put
\[ k(0) = 1. \] (27)

It is necessary to associate boundary and initial conditions with these equations. We assume that on the boundary \( \partial \Omega \) of the domain \( \Omega \)
\[ v(x, t) = 0, \quad \vartheta(x, t) = 0 \quad \forall x \in \partial \Omega, \] (28)
for any time \( t \neq 0 \), while for the initial conditions we remember that, by means of a suitable modification of the sources, it is always possible to change the most general initial conditions, given by
\[ v(x, 0) = v_0(x), \quad \vartheta(x, 0) = \vartheta_0(x), \quad \vartheta(x, -s) = \vartheta_0(x, -s) \quad \forall s \in \mathbb{R}^+, \] (29)
to initial conditions of the form
\[ v(x, 0) = 0, \quad \vartheta(x, 0) = 0, \quad \vartheta^0(x, s) = \vartheta_0(x, s) \quad \forall s \in \mathbb{R}^+. \] (30)
Now we define the function spaces
\[
\mathcal{H}_\varphi(\mathbb{R}^+, \Omega) = \left\{ v \in H^{1/2}(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H^1_0(\Omega)) \right\},
\]
\[
\mathcal{H}_\varphi(\mathbb{R}^+, \Omega) = \left\{ \vartheta \in H^{1/2}(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H^1_0(\Omega)) \right\}
\]
\[
\int_{-\infty}^{+\infty} \int_{\Omega} \hat{k}_c(\omega)|\nabla \hat{\vartheta}_+(x, \omega)|^2 dxd\omega < +\infty.
\]

The quantity \( \hat{k}_c(\omega) \) is the Fourier cosine transform of \( k(s) \) and \( \hat{\vartheta}_+(x, \omega) \) denotes the Fourier transform of \( \vartheta(x, s) \) where \( \vartheta(x, s) = 0, \ \forall \ s \in \mathbb{R}^- \). Taking account of (2) in \( \mathcal{H}_\varphi(\mathbb{R}^+, \Omega) \) we have introduced the function space
\[
H^1_0(\Omega) = \left\{ v \in H^1(\Omega); \nabla \cdot v = 0 \right\}.
\]

We also define
\[
\mathcal{M}_f(\mathbb{R}^+, \Omega) = \left\{ f \in L^2(\mathbb{R}^+; L^2(\Omega)); \right\}
\]
\[
\int_{-\infty}^{+\infty} \int_{\Omega} \frac{1 + |\omega|}{\hat{k}_c(\omega)} |\hat{f}_+(x, \omega)|^2 dxd\omega < +\infty
\]

where \( \hat{f}_+(x, \omega) \) is the Fourier transform of \( f(x, s) \) where \( f(x, s) = 0, \ \forall \ s \in \mathbb{R}^- \).

**Definition 3.1.** A pair \((v, \vartheta) \in \mathcal{H}_\varphi(\mathbb{R}^+, \Omega) \times \mathcal{H}_\varphi(\mathbb{R}^+, \Omega)\) is said to be a weak solution of the problem (24)-(25), with boundary conditions (28), and initial data (30), \( f \in \mathcal{M}_f(\mathbb{R}^+, \Omega) \) and \( b \in L^2(\mathbb{R}^+; L^2(\Omega)) \), if satisfies the equations
\[
\int_0^{+\infty} \int_{\Omega} (v \cdot w_t - \mu \nabla v \cdot \nabla w + R\vartheta k \cdot w) \ dx dt = -\int_0^{+\infty} \int_{\Omega} b \cdot w \ dx dt,
\]
(34)
\[
\int_0^{+\infty} \int_{\Omega} \left[ \text{Pr} \vartheta \varphi_t - \int_0^t k'(s) \nabla \hat{\vartheta}_+(t-s) ds \cdot \nabla \varphi + Rv \cdot k \varphi \right] \ dx dt = -\int_0^{+\infty} \int_{\Omega} f \varphi \ dx dt
\]
(35)

for all \( w \in \mathcal{H}_\varphi(\mathbb{R}^+, \Omega) \) and \( \varphi \in \mathcal{H}_\varphi(\mathbb{R}^+, \Omega) \).

By applying the Parseval-Plancherel theorem to (34)-(35), the following theorem was proved in [2].

**Theorem 3.2.** If the kernel \( k \in H^1(\mathbb{R}^+; L^\infty(\Omega)) \) satisfies the condition (15) and, moreover, \( f \in \mathcal{M}_f(\mathbb{R}^+, \Omega) \) and \( b \in L^2(\mathbb{R}^+; L^2(\Omega)) \), then, for sufficiently small values of \( R \), there exists a unique weak solution \((v, \vartheta) \in \mathcal{H}_\varphi(\mathbb{R}^+, \Omega) \times \mathcal{H}_\varphi(\mathbb{R}^+, \Omega)\) as specified in Definition 3.1.

4. Existence, uniqueness and exponential stability of the solution. In this section we shall consider the linearized system of equations in the non-dimensional form (24)-(25) in order to prove an existence and uniqueness theorem for its solution and to study the exponential stability of the solution.
We assume that the relaxation function \( k \), such that \( k(s) \in H^2(\mathbb{R}^+) \), satisfies (15)-(16) together with the following conditions
\[
 k'(s) + \xi k(s) \leq 0, \quad k''(s) + \xi k'(s) \geq 0 \quad \xi > 0,
\]
where \( \xi \) is a positive constant.

Using the constitutive equation for the extra-stress (7) and for the heat flux, given by (14), the system assumes the form
\[
\frac{\partial \nu}{\partial t} = -\nabla p + \nabla \cdot T_E + R\partial k, \quad (37)
\]
\[
\frac{\partial \vartheta}{\partial t} = \frac{1}{Pr} (\nabla \cdot q + R \nu \cdot k), \quad (38)
\]
\[
\nabla \cdot \nu = 0, \quad (39)
\]
\[
\frac{\partial q}{\partial t} = -\int_0^{+\infty} k'(s) \nabla \vartheta(t,s) ds, \quad (40)
\]
where the supplies \( b \) and \( f \) have been taken to be zero. We associate the initial conditions (29) and the boundary conditions (28) with these equations. Note that (40) follows from (13) and (14). Also, we have
\[
\frac{\partial Q_{(1)}^t(\tau)}{\partial t} = \frac{\partial Q_{(1)}^t(\tau)}{\partial \tau} + k'(\tau) \nabla \vartheta(t). \quad (41)
\]
This equation derives from (19) since, by using (8), (9) and (10) together with an integration by parts, we find that
\[
\frac{\partial Q_{(1)}(\tau)}{\partial t} = \int_0^{+\infty} k''(\tau + s) \frac{\partial}{\partial t} \left[ \nabla \tilde{\vartheta}(s) - \nabla \tilde{\vartheta}(t) \right] ds
\]
\[
= -\int_0^{+\infty} k''(\tau + s) \frac{\partial}{\partial s} \nabla \tilde{\vartheta}(s) ds - \int_0^{+\infty} k''(\tau + s) ds \frac{\partial}{\partial t} \nabla \tilde{\vartheta}(t)
\]
\[
= \frac{\partial}{\partial \tau} \int_0^{+\infty} k''(\tau + s) \nabla \tilde{\vartheta}(s) ds + k'(\tau) \nabla \vartheta(t)
\]
\[
= \frac{\partial Q_{(1)}(\tau)}{\partial \tau} + k'(\tau) \nabla \vartheta(t).
\]
Finally, from (5) and (9), it follows that
\[
\frac{\partial \vartheta_r^t(s)}{\partial t} = -\frac{\partial \vartheta_r^t(s)}{\partial s} - \frac{\partial \vartheta(t)}{\partial t}. \quad (42)
\]
We now introduce the quantity
\[
\chi = (\nu, \vartheta, q, Q_{(1)}^t, \vartheta_r^t) \quad (43)
\]
and suppose that
\[
\chi \in \mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{B}(\Omega, \mathbb{R}^+) \times \mathcal{D}(\Omega, \mathbb{R}^+), \quad (44)
\]
where
\[
\mathcal{B}(\Omega, \mathbb{R}^+) = \left\{ Q_{(1)}^t(x, \cdot) : \mathbb{R}^+ \to V^3; \right. \\
- \int_\Omega \int_0^{+\infty} \frac{1}{k'(\tau)} \left[ Q_{(1)}^t(x, \tau) \right]^2 d\tau dx < +\infty \right\}. \quad (45)
\]
and

\[ \mathcal{D}(\Omega, \mathbb{R}^+) = \left\{ \vartheta^t(x, \cdot) : \mathbb{R}^+ \to \mathbb{R}; -\int_0^{+\infty} k'(s) \left[ \vartheta^t(x, s) \right]^2 ds dx < +\infty \right\} \]  

(46)

are Hilbert’s spaces with the inner products defined by

\[ \langle Q_{(1)_1}, Q_{(1)_2} \rangle_B = -\int_{\Omega} \int_0^{+\infty} \frac{1}{k'(\tau)} Q_{(1)_1}^t(\tau) \cdot Q_{(1)_2}^t(\tau) d\tau dx, \]

\[ \langle \vartheta_{(1)_1}^t, \vartheta_{(2)_2}^t \rangle_D = -Pr \int_0^{+\infty} k'(s) \vartheta_{(1)_1}^t(s) \vartheta_{(2)_2}^t(s) ds dx, \]

respectively.

The important key to this work is the free energy \( \psi_F(t) \) expressed by (17), that is

\[ \psi_F(Q^t) = -\frac{1}{2} \int_0^{+\infty} \frac{1}{k'(\tau)} \left[ Q_{(1)_1}^t(\tau) \right]^2 d\tau \]

(49)

with (18)-(22), together with a pseudo-energy \( \psi_q(t) \) defined by

\[ \psi_q(\vartheta_{(2)_2}^t) = -\frac{Pr}{2} \int_0^{+\infty} k'(s) \left[ \vartheta_{(2)_2}^t(s) \right]^2 ds, \]

(50)

which, as noted earlier in [4], has the Graffi-Volterra form.

To prove the next theorem, we shall consider an inequality, analogous to that of Slemrod, proved in [14], where the history of the relative velocity is involved.

If \( \vartheta \in L^2(\mathbb{R}; H^1_0(\Omega)) \) while \( k \in H^2(\mathbb{R}^+) \) is a function which satisfies the conditions (15)-(16), together with (36)1, then, for any \( t \in \mathbb{R}^+ \), we have the useful inequality [14]

\[ \int_\Omega |\vartheta(x, t)|^2 dx \leq c \left[ \int_\Omega |\vartheta(x, 0)|^2 dx - Pr \int_0^{+\infty} k'(s) \int_\Omega |\vartheta_r^t(x, s)|^2 dx ds \right], \]

(51)

where \( c \) is a suitable positive constant depending on \( \Omega \).

The notations

\[ \|v(t)\| = \int_\Omega |v(t)|^2 dx, \quad \|\vartheta(t)\|^2 = \int_\Omega \vartheta^2(t) dx, \]

\[ \|q(t)\|^2 = \int_\Omega |q(t)|^2 dx. \]

(52)

will be used later on.

We can now prove the following theorem.

**Theorem 4.1.** The problem (37)-(42), under the hypothesis of the Theorem 3.2, for any initial conditions \( \chi_0 \in \mathcal{H} \), if the relaxation function \( k \) satisfies the hypotheses (15)-(16) and (36), admits a unique solution \( \chi(t) \in \mathcal{H} \) such that

\[ \frac{1}{2} \left[ \|v(t)\|^2 + Pr \|\vartheta(t)\|^2 + \|q(t)\|^2 \right] + \int_\Omega \psi_F(Q^t) dx + \int_\Omega \psi_q(\vartheta_{(2)_2}^t) dx \]

\[ \leq Me^{-\varepsilon t} \left\{ \frac{1}{2} \left[ \|v(0)\|^2 + Pr \|\vartheta(0)\|^2 \right] + \|q(0)\|^2 + \int_\Omega \psi_F(Q^{t=0}) dx + \int_\Omega \psi_q(\vartheta_{(2)_2}^{t=0}) dx \right\}, \]

(53)

where \( M \) and \( \varepsilon \) are suitable positive constants.
Proof. Consider the following functional

$$E(t) = \frac{1}{2} \left[ \|\psi(t)\|^2 + Pr \|\vartheta(t)\|^2 + \|q(t)\|^2 \right]$$

$$+ \int \varphi F(Q^t) dx + \int \varphi q(t^r) dx,$$

(54)

which is the total energy of the system. We seek to evaluate its derivative with respect to time $t$. To do this, we firstly differentiate the two functionals $\varphi F(Q^t)$ and $\varphi q(t^r)$. From (49), by using (41), integrating by parts and taking account of (21)$_2$, we obtain

$$\int \frac{\partial \varphi F(Q^t)}{\partial t} dx = - \int \frac{1}{k^t(\tau)} \left[ \frac{\partial Q^t(\tau)}{\partial \tau} + k^t(\tau) \nabla \vartheta(t) \right]$$

$$\cdot Q^t(\tau) d\tau dx = - \int \frac{1}{k^t(\tau)} \int_0^{+\infty} \frac{1}{2} \frac{\partial}{\partial \tau} \left[ Q^t(\tau) \right]^2 d\tau dx$$

$$- \int \frac{1}{k^t(\tau)} \int_0^{+\infty} Q^t(\tau) \cdot \nabla \vartheta(t) dx = - \frac{1}{2} \int \left\{ - \int_0^{+\infty} \frac{d}{d\tau} \left[ \frac{1}{\mu^t(\tau)} \right] \right\} dx$$

$$\times \left[ Q^t(\tau) \right]^2 d\tau - \frac{1}{k^t(0)} \int_0^{+\infty} \left[ Q^t(0) \right]^2 d\tau dx$$

$$+ \frac{1}{2k^t(0)} \int \left[ Q^t(0) \right]^2 dx - \int q(t) \cdot \nabla \vartheta(t) dx.$$  (55)

We observe that the first two terms in (55)$_4$ are non-positive quantities by virtue of the hypotheses (15)-(16). These terms are of course the integrals over the fluid of the negative of the rate of dissipation associated with the free energy functional $\varphi F(t)$ ([3], for example).

From (50), taking account of (42) and integrating by parts, we obtain

$$\int \frac{\partial \varphi q(t^r)}{\partial t} dx = - \frac{Pr}{2} \int \left\{ \int_0^{+\infty} 2k'(s) \frac{\partial q^r(t^r)}{\partial t} \vartheta^r(t) \right\} dx$$

$$= Pr \int \left\{ \int_0^{+\infty} k'(s) \left[ \frac{\partial q^r(t^r)}{\partial t} \vartheta^r(t) + \frac{\partial \vartheta(t)}{\partial t} \right] \right\} dx$$

$$= - \frac{Pr}{2} \int_0^{+\infty} k'(s) \left[ \vartheta^r(t) \right]^2 ds dx$$

$$+ Pr \int \left[ \int_0^{+\infty} k'(s) \vartheta^r(t) ds \right] \frac{\partial \vartheta(t)}{\partial t} dx,$$  (56)

where the first term of (56)$_3$ is non-positive, and once again, is the integral over the fluid of the negative of the rate of dissipation associated with the Graffi-Volterra free energy.

Then, we consider (40), whence, by integrating by parts and using (28)$_2$, we evaluate the following quantity

$$\frac{1}{2} \int \frac{\partial}{\partial t} |q(t)|^2 dx = - \int \left[ \int_0^{+\infty} k'(s) \nabla \vartheta^r(s) ds \right] \cdot q(t) dx$$
obtaining we firstly take the scalar product of (37) and \(v_1\) account of (39) together with the boundary condition (28) (57). Thus, we have

\[
\int_\Omega \frac{\partial}{\partial t} \left[ \psi_q (\vartheta^0_t) + \frac{1}{2} |q(t)|^2 \right] \, dx = - \frac{Pr}{2} \int_\Omega \int_0^{+\infty} k''(s) \left[ \vartheta^t_r(s) \right]^2 \, ds \, dx \\
+ R \int_\Omega \int_0^{+\infty} k'(s) \vartheta^t_r(s) \, ds \, \vartheta(t) \cdot k \, dx. \quad (58)
\]

We now evaluate the time derivative of \(\frac{1}{2}||v(t)||^2\) and \(\frac{\partial}{\partial t} ||\vartheta(t)||^2\). Using (52), we firstly take the scalar product of (37) and \(v(t)\). Integrating by parts and taking account of (39) together with the boundary condition (28), we have

\[
\frac{1}{2} \frac{d}{dt} ||v(t)||^2 = \int_\Omega \frac{\partial v(t)}{\partial t} \cdot v(t) \, dx = \int_\Omega [-\nabla p + \nabla \cdot T_E] \\
+ R (\partial \cdot k) \cdot v(t) \, dx = R \int_\Omega \vartheta(t) \cdot k \cdot v(t) \, dx - \int_\Omega T_E \cdot \nabla v \, dx \\
=: R (\partial, k \cdot v) - (T_E, \nabla v). \quad (59)
\]

From (7) and the symmetry of \(T\), it follows that

\[
T_E \cdot \nabla v = \frac{1}{2\mu} T_E(t) \cdot T_E(t) = \frac{1}{2\mu} |T_E(t)|^2. \quad (60)
\]

Let us now consider (38) multiplied by \(\vartheta(t)\). Using (52), we integrate by parts, obtaining

\[
\frac{1}{2} \frac{Pr}{\partial t} ||\vartheta(t)||^2 = Pr \int_\Omega \frac{\partial \vartheta(t)}{\partial t} \vartheta(t) \, dx \\
= \int_\Omega (-\nabla \cdot q \vartheta + R v \cdot k \vartheta) \, dx = \int_\Omega q(t) \cdot \nabla \vartheta(t) \, dx \\
+ R \int_\Omega \vartheta(t) \cdot k \cdot v(t) \, dx = \int_\Omega q(t) \cdot \nabla \vartheta(t) \, dx + R (\vartheta, k \cdot v), \quad (61)
\]

by virtue of (28). Therefore, the time derivative of (54) can be written as

\[
\frac{d}{dt} E(t) = -\frac{1}{2} \int_\Omega \int_0^{+\infty} k''(\tau) \left[ \frac{Q_t^{(1)}(\tau)}{[k'(\tau)]^2} \right]^2 \, d\tau \, dx \\
+ \frac{1}{2k'(0)} \int_\Omega \left[ Q_t^{(1)}(0) \right]^2 \, dx - \frac{1}{2\mu} \int_\Omega |T_E(t)|^2 \, dx \\
- \frac{Pr}{2} \int_\Omega \int_0^{+\infty} k''(s) \left[ \vartheta^t_r(s) \right]^2 \, ds \, dx + 2R (\vartheta, k \cdot v) \\
+ R \int_\Omega \int_0^{+\infty} k'(s) \vartheta^t_r(s) \, ds \, \vartheta(t) \cdot k \, dx, \quad (62)
\]
using (55), (58), (59), (60) and (61). In this expression, only the last two terms may lead to instability. The other terms are the negative of the integrated rates of dissipation related to the free energy $\psi_F(Q')$ and the pseudo-energy $\psi_q(\theta^0_\tau)$ noted in the contexts of (55) and (56), together with the third term, all of which are negative.

Therefore, we must estimate the effects of the last two terms. Firstly, we consider

$$2R(\vartheta, k \cdot v) = 2R \int_\Omega \vartheta(t) k \cdot v(t) \, dx \leq R \int_\Omega 2 \left( \frac{|\vartheta(t)|}{\sqrt{\alpha}} \right) \left( |k \cdot v(t)| \sqrt{\alpha} \right) \, dx$$

$$\leq \frac{R}{\alpha} \|\vartheta\|^2 + \alpha R \|v\|^2 \leq \frac{R}{\alpha} \|\vartheta\|^2 + R\alpha \|v\|^2,$$

(63)

where we have introduced an arbitrary positive number $\alpha$.

Then, for the last term, using (27) and again the arithmetic-geometric mean inequality with another arbitrary number $\beta > 0$ [9], we find that

$$R \int_\Omega \left[ \int_{t_0}^{t_\infty} k'(s) \vartheta^0_\tau(s) \, ds \right] |v(t) \cdot k| \, dx$$

$$\leq R \int_\Omega \left[ \int_{t_0}^{t_\infty} \sqrt{-k'(s)} \sqrt{-k'(s)} |\vartheta^0_\tau(s)|^2 \, ds \right] |v(t)| \, dx$$

$$\leq R \int_\Omega \frac{|v(t)|}{\sqrt{\beta}} \sqrt{-k'(s)} \sqrt{-k'(s)} \sqrt{\int_0^{t_\infty} -k'(s) |\vartheta^0_\tau(s)|^2 \, ds} \, dx$$

$$\leq R \int_\Omega \frac{1}{\sqrt{\beta}} \|v\|^2 - \beta \int_\Omega \int_{t_0}^{t_\infty} k'(s) |\vartheta^0_\tau(s)|^2 \, ds \, dx.$$  

(64)

Substituting these two results into (62), we have the inequality

$$\frac{d}{dt} E(t) \leq -\frac{1}{2} \int_\Omega \int_0^{t_\infty} \frac{k''(\tau)}{k'^2(\tau)} \left[ Q_{(1)}^2(\tau) \right] \, d\tau \, dx$$

$$+ \frac{1}{2k'(0)} \int_\Omega \left[ Q_{(1)}^2(0) \right] \, dx + \frac{R}{\alpha} \|\vartheta\|^2$$

$$- \frac{Pr}{2} \int_\Omega \int_0^{t_\infty} \left[ k''(s) + R \frac{\beta}{Pr} k'(s) \right] |\vartheta^0_\tau(s)|^2 \, ds \, dx$$

$$- \frac{1}{2\mu} \int_\Omega |T_E(t)|^2 \, dx - R \left( \alpha + \frac{R}{2\beta} \right) \|v\|^2.$$

(65)

which holds because $k'(s) < 0$ and for sufficiently small values of $R$. By virtue of the Poincarè inequality there exists a constant $c_1(\Omega)$, which depends on the domain $\Omega$, such that

$$\int_\Omega v^2(t) \, dx \leq c_1(\Omega) \int_\Omega |T_E(t)|^2 \, dx.$$

(66)

Thus, using (36) and (51), we obtain

$$\frac{d}{dt} E(t) \leq -\frac{1}{2} \int_\Omega \int_0^{t_\infty} \frac{k''(\tau)}{k'^2(\tau)} \left[ Q_{(1)}^2(\tau) \right] \, d\tau \, dx$$
which follows by comparing the last two inequalities of (71) and eliminating

ties where again, we have used (27). Thus, from (21), it follows that

\[ - \frac{1}{2\mu} - R \left( \alpha + \frac{1}{2\beta} \right) c_1(\Omega) \int_\Omega |T_E(t)|^2 \, dx. \]  

We now observe that the heat flux \( q \) is bounded by the free energy \( \psi_F \); therefore, recalling (21)-(22), we have

\[ |Q'(0)| \leq \int_0^{+\infty} |Q_{(1)}(\tau)| \, d\tau \leq \int_0^{+\infty} \sqrt{-k'(\tau)} \sqrt{\frac{1}{-k'(\tau)}} |Q_{(1)}'(\tau)|^2 \, d\tau \]

\[ \leq \left[ \int_0^{+\infty} -k'(\tau) \, d\tau \right] \left[ \int_0^{+\infty} \frac{1}{-k'(\tau)} |Q_{(1)}'(\tau)|^2 \, d\tau \right]^{\frac{1}{2}} \]

\[ \leq \left[ \int_0^{+\infty} \frac{1}{-k'(\tau)} |Q_{(1)}'(\tau)|^2 \, d\tau \right]^{\frac{1}{2}}, \]  

where again, we have used (27). Thus, from (21), it follows that

\[ |q(t)|^2 = |Q'(0)| \leq - \int_0^{+\infty} \frac{1}{k'(\tau)} |Q_{(1)}'(\tau)|^2 \, d\tau = 2\psi_F(t) \]  

and hence, by integrating (69) over \( \Omega \), the inequality (67) can be written as

\[ \frac{d}{dt} E(t) \leq - \frac{1}{2} \int_\Omega \int_0^{+\infty} \left\{ \frac{k''(\tau)}{|k'(\tau)|^2} + \frac{R}{\alpha} \frac{1}{k'(\tau)} \right\} |Q_{(1)}'(\tau)|^2 \, d\tau \, dx \]

\[ + \frac{Pr}{2} \left( \xi - R \left( \frac{\beta}{Pr} + \frac{2c}{\alpha} \right) \right) \int_0^{+\infty} k'(s) |\varphi'(s)|^2 \, ds \, dx \]

\[ - \left[ \frac{1}{2\mu} - R \left( \alpha + \frac{1}{2\beta} \right) c_1(\Omega) \right] \int_\Omega |T_E(t)|^2 \, dx \]

\[ + \frac{1}{2k'(0)} \int_\Omega \left[ Q_{(1)}'(0) \right]^2 \, dx. \]  

We now note that, taking into account the hypothesis (36)\(_2\), we have the inequalities

\[ k''(s) \geq -\xi k'(s) \geq -R\frac{2c}{\alpha} k'(s) > 0, \]  

the second of which is satisfied if the coefficient \( R \) is such that

\[ R \leq \frac{\alpha \xi}{2c}, \]  

which follows by comparing the last two inequalities of (71) and eliminating \(-k'(s) > 0\).

Thus, with such a constraint for \( R \), it follows that the quantity given by the first integral in the right-hand side of (70) is non-positive.

If we examine the quantities in square brackets, which multiply the second and third integrals respectively, we can also choose the values of \( R \) which satisfy both
of the inequalities
\[ R \leq \frac{\alpha \xi \Pr}{\alpha \beta + 2e \Pr}, \quad (73) \]
\[ R \leq \frac{\beta}{\mu (1 + 2\alpha \beta) c_1(\Omega)}. \quad (74) \]

These ensure that the second and third terms on the right-hand side of (70) are also non-positive. Furthermore, the final term has the same property since \( k'(0) < 0 \) by virtue of (15)_2.

Thus, the stability of the system is assured if \( R \) satisfies the three inequalities (72)-(74) simultaneously, a property which can be expressed by
\[ R \leq \min \left\{ \frac{\alpha \beta}{2e}, \frac{\alpha \xi \Pr}{\alpha \beta + 2e \Pr}, \frac{\beta}{\mu (1 + 2\alpha \beta) c_1(\Omega)} \right\}. \quad (75) \]

Therefore, the inequality (70) becomes
\[
\frac{d}{dt} E(t) \leq \frac{1}{2} \left( \xi - \frac{2eR}{\alpha} \right) \int_{\Omega} \int_{0}^{\infty} \frac{1}{k'(\tau)} \left[ Q^t_{(1)}(\tau) \right]^2 d\tau dx \\
+ \frac{1}{2k'(0)} \int_{\Omega} \left[ Q^t_{(1)}(0) \right]^2 dx \\
+ \frac{\Pr}{2} \left[ \xi - R \left( \frac{\beta}{\Pr} + \frac{2e}{\alpha} \right) \right] \int_{\Omega} \int_{0}^{\infty} k'(s) \left| \theta^t_r(s) \right|^2 ds dx \\
- \left[ \frac{1}{2\mu} - R \left( \alpha + \frac{1}{2\beta} \right) c_1(\Omega) \right] \int_{\Omega} \left| T_E(t) \right|^2 dx =: -\rho(t) \leq 0, \quad (76) \]

on using (71)_1. An alternative and interesting form of this inequality can be given as follows
\[
\frac{d}{dt} E(t) \leq - \left( \xi - \frac{2eR}{\alpha} \right) \int_{\Omega} \psi_F dx \\
- \left[ \xi - R \left( \frac{\beta}{\Pr} + \frac{2e}{\alpha} \right) \right] \int_{\Omega} \psi_q dx + \frac{1}{2k'(0)} \int_{\Omega} \left[ Q^t_{(1)}(0) \right]^2 dx \\
- \left[ \frac{1}{2\mu} - R \left( \alpha + \frac{1}{2\beta} \right) c_1(\Omega) \right] \int_{\Omega} \left| T_E(t) \right|^2 dx, \quad (77) \]

by using the definitions (49) of \( \psi_F \) and (50) of \( \psi_q \).

Integrating (76)_2 on \((0, t)\), we obtain
\[ E(t) - E(0) \leq - \int_{0}^{t} \rho(\xi) d\xi, \quad (78) \]
giving
\[ 0 \leq E(t) \leq E(0) - \int_{0}^{t} \rho(\xi) d\xi \leq E(0), \quad (79) \]
or
\[ E(0) \geq E(t) + \int_{0}^{t} \rho(\xi) d\xi \geq \int_{0}^{t} \rho(\xi) d\xi \quad \forall t \in \mathbb{R}^+. \quad (80) \]
Letting \( t \to +\infty \), we obtain
\[
E(0) \geq \int_0^{+\infty} \rho(x)dx \geq \int_0^{+\infty} \left\{ \left( x - \frac{2cR}{\alpha} \right) \right\} \int_\Omega \psi_F dx
\]
\[
+ \left[ \xi - R \left( \frac{\beta}{Pr} + 2c \frac{\alpha}{\alpha} \right) \right] \int_\Omega \psi_a dx - \frac{1}{2k(0)} \int_\Omega \left[ Q_{(1)}(0) \right]^2 dx,
\]
\[
+ \left[ \frac{1}{2\mu} - R \left( \alpha + \frac{1}{2\alpha} \right) c_1(\Omega) \right] \int_\Omega \left[ T_E(t) \right]^2 dt.
\] (81)

Finally, by the Theorem 3.2 and the inequalities (69) and (81), we have by the definition (54) of \( E \)
\[
\int_0^{+\infty} E(t)dt = \int_0^{+\infty} \left\{ \frac{1}{2} \left[ \|v(t)\|^2 + Pr \|\partial(t)\|^2 + \|q(t)\|^2 \right] + \right\}
\]
\[
+ \int_\Omega \psi_F (Q^i) dx + \int_\Omega \psi_a (\theta^i) dx \right\} dt < +\infty.
\] (82)

We can now write the system (37)-(42) as follows
\[
\dot{\chi} = A\chi, \quad \chi(0) = \chi_0,
\] (83)
where we have denoted by \( A \) the operator given by the right-hand sides of (37)-(38) and (40)-(42) and defined on the following domain
\[
\mathcal{D}(A) = \left\{ \chi = (v, \theta, q, Q_{(1)}, \theta^t) \in \mathcal{H}; \, \, v \in H^1_0(\Omega) \cap H^2(\Omega), \theta \in H^1_0(\Omega) \cap H^2(\Omega), q \in L^2(\Omega), Q_{(1)} \in B(\Omega, \mathbb{R}^+), v^t \in D(\Omega, \mathbb{R}^+) \right\}.
\] (84)

Then, it is easy to verify that
\[
\langle \dot{\chi}, \chi \rangle \equiv \frac{dE(t)}{dt},
\] (85)
where \( \frac{dE(t)}{dt} \) and \( \dot{\chi} \) are given by (62) and (83). In fact, recalling the inner products (47)-(48), we define
\[
\langle \frac{\partial v}{\partial t}, v \rangle = \int_\Omega \frac{\partial v}{\partial t} \cdot v dx, \quad \langle \dot{\theta}, \theta \rangle = \int_\Omega Pr \frac{\partial \theta}{\partial t} \theta dx,
\]
\[
\langle \frac{\partial q}{\partial t}, q \rangle = \int_\Omega \frac{\partial q}{\partial t} \cdot q dx,
\]
\[
\langle \frac{\partial Q_{(1)}}{\partial t}, Q_{(1)} \rangle_B = -\int_0^{+\infty} \frac{1}{k(\tau)} \frac{\partial Q_{(1)}(\tau)}{\partial t} \cdot Q_{(1)}(\tau) d\tau dx,
\]
\[
\langle \frac{\partial \theta^t}{\partial t}, \theta^t \rangle_D = -Pr \int_0^{+\infty} k(s) \frac{\partial \theta^t(s)}{\partial t} \theta^t(s) ds dx.
\]

Thus, we can write
\[
\langle \dot{\chi}, \chi \rangle \equiv \langle A\chi, \chi \rangle = \int_\Omega \left\{ \frac{\partial v}{\partial t} \cdot v + Pr \frac{\partial \theta}{\partial t} \theta + \frac{\partial q}{\partial t} \cdot q \right\}
\]
where the derivatives with respect to time of $v$, $\vartheta$, $q$, $Q_{\infty}^t$ and $\vartheta^t_r$ are given by the right-hand sides of (37)-(38) and (40)-(42), respectively. By substituting these expressions into (86), then, with some integrations by parts, we obtain the quantities on the right-hand side of (62), so that (85) follows. Therefore, from (76),
\[
\langle A\chi, \chi \rangle \leq \int_0^{+\infty} \frac{Q_{\infty}^t(\tau)}{k^t(\tau)} \cdot Q_{\infty}^t(\tau) d\tau
\]
\[
- \Pr \int_0^{+\infty} k^t(s) \frac{\partial \vartheta^t_r(s)}{\partial t} \vartheta^t_r(s) ds \}
dx,
\]
where
\[
\begin{align*}
- \int_0^{+\infty} \frac{1}{k^t(\tau)} \frac{\partial Q_{\infty}^t(\tau)}{\partial t} \cdot Q_{\infty}^t(\tau) d\tau \\
- \Pr \int_0^{+\infty} k^t(s) \frac{\partial \vartheta^t_r(s)}{\partial t} \vartheta^t_r(s) ds \}
dx,
\end{align*}
\]
where $H$ is a Hilbert space.

Thus, with a proof similar to that given in [9], we can show that the range of $(A - I)$ is $H$ and consequently that $A : D(A) \to H$ is a maximal dissipative operator on $H$ [5, 12]. Moreover, from the Lummer-Phillips theorem, the operator $A$ generates a strongly continuous semigroup of linear contractions $S(t)$ on $H$, so that the solutions of the system (37)-(42) have the form
\[
\chi(t) = S(t)\chi_0.
\]
Furthermore, from (82) it follows that
\[
\int_0^{+\infty} E(t) dt = \frac{1}{2} \int_0^{+\infty} \langle \chi(t), \chi(t) \rangle dt
\]
\[
= \frac{1}{2} \int_0^{+\infty} \langle S(t)\chi_0, S(t)\chi_0 \rangle dt < +\infty
\]
for any $\chi_0 \in H$.

Hence, we use the following lemma due to Dakto [6].

**Lemma 4.2.** Given a strongly continuous semigroup of linear contractions $S(t)$ on a Hilbert space $K$, then, with two suitable constants $M$ and $\varepsilon$, we have
\[
\langle S(t)y_0, S(t)y_0 \rangle = Me^{-\varepsilon t} \langle y_0, y_0 \rangle \quad \forall y_0 \in K
\]
if and only if $\int_0^{+\infty} \langle S(t)y_0, S(t)y_0 \rangle dt < +\infty$.

In our case, a necessary and sufficient condition that a strongly continuous semigroup of linear operators $S(t)$ satisfy the inequality
\[
\langle S(t)\chi_0, S(t)\chi_0 \rangle \leq M \exp(-\varepsilon t) \langle \chi_0, \chi_0 \rangle \quad \forall \chi_0 \in H,
\]
where $M$ and $\varepsilon$ are two suitable positive constants, is that the integral
\[
\int_0^{+\infty} \langle S(t)\chi_0, S(t)\chi_0 \rangle dt < +\infty, \quad \text{with} \ \chi_0 \in H.
\]
Therefore, from (89) and by Dakto’s Lemma, we obtain the inequality (53).
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E-mail address: amendola@dma.unipi.it
E-mail address: mauro.fabrizio@unibo.it
E-mail address: murrough.golden@dit.ie
E-mail address: manes@dm.unipi.it