REPRESENTATIONS OF AUT(M)-INARIANT MEASURES

NATHANAEL ACKERMAN

ABSTRACT. We study probability measures on the space of countable models which are invariant under the group of automorphisms of a fixed structure. We show that for a class of structures, which we call free, every such invariant measure has a representation in terms of a simple collection of measurable functions, akin to the representation given in the Aldous-Hoover-Kallenberg theorem. We then use this to give a characterization of those probability measures which are invariant under the automorphism group of an arbitrary structure, not necessarily free, and which have such a representation.

CONTENTS

1. Introduction 2
1.1. Connections To Other Work 4
1.2. Outline 6
1.3. Notation 7
2. Preliminaries 8
2.1. Polish Group Actions 8
2.2. Infinitary Logic 10
2.3. Definable Expansions 14
2.4. Invariant Measures 17
2.5. Aldous-Hoover-Kallenberg 18
2.6. Existence of an \( \mathcal{N} \)-Invariant Measure 23
3. Free Structures 23
3.1. Canonical Structures 23
3.2. Free Completions and Invariant Measures on Free Structures 25
4. Merging of Measures 30
4.1. Inherited Properties Of Merged Measures 34
5. Representations 36
5.1. Aut(\( M \))-Recipes 36
5.2. Representability and Free Structures 38
5.3. Ergodic Representations 40
6. Acknowledgements 41
References 41
1. Introduction

By studying the symmetries of a collection of random variables one can gain great insight into their collective structure. In particular, one is often able to deduce properties of a collection of random variables simply by knowing that its joint distribution is preserved under a certain group of symmetries (and indeed much of ergodic theory can be recast in this light). Over the last century, there have been several representation results for collections of random variables whose joint distributions are preserved under various, sufficiently large, groups. These representation results show that if a collection of random variables has a joint distribution which is preserved under an appropriate group of symmetries (usually the group of permutations of \( N \) or variants of it) then the random variables must have a very simple form.

As an example, we now consider those collections of random variables which are closed under an action of \( \mathfrak{S}_N \), the group of permutations of \( N \).

**Definition 1.1.** For a standard Borel space \( S \), a sequence of \( S \)-valued random variables \( (X_n)_{n \in N} \) is said to be exchangeable if its joint distribution is unchanged by permutations of \( N \), i.e. for all \( \sigma \in \mathfrak{S}_N \)

\[
(X_n)_{n \in N} \overset{d}{=} (X_{\sigma(n)})_{n \in N}.
\]

A natural class of exchangeable sequences consists of those which are of the form

\[
(f(\zeta_{\emptyset}, \zeta_n))_{n \in N}
\]

where \( f : [0, 1]^2 \to S \) is any measurable function and \( \{\zeta_{\emptyset}\} \cup \{\zeta_n\}_{n \in N} \) is a collection of uniform identically distributed and independent (i.i.d.) \([0, 1]\)-valued random variables. An important result of de Finetti and Hewitt–Savage is that all exchangeable sequences have a distribution which is of this form. This result, known as de Finetti’s theorem, is a fundamental result of modern probability theory with applications throughout Bayesian nonparametric statistics, genetics, statistical mechanics, and ergodic theory (see, e.g. [Kin78], [Ald85], [Kal05] Ch. 1.1).

Let \( N^{<\omega} \) be the collection of finite sequences of distinct natural numbers and let \( \mathcal{P}(\cdot) \) and \( \mathcal{P}_{<\omega}(\cdot) \) be the powerset and finite powerset operations respectively.

**Definition 1.2.** For a standard Borel space \( S \), an array of \( S \)-valued random variables \( (X_a)_{a \in N^{<\omega}} \) is said to be (jointly) exchangeable if its distribution is unchanged under permutations of \( N \), i.e. if for all \( \sigma \in \mathfrak{S}_N \)

\[
(X_{\sigma})_{a \in N^{<\omega}} \overset{d}{=} (X_{\sigma(a)})_{a \in N^{<\omega}}.
\]

The following is a natural class of exchangeable arrays. For \( n \in N \) let \( f_n : [0, 1]^{\mathcal{P}(n)} \to S \) and let \( (\zeta_{\pi})_{\pi \in \mathcal{P}_{<\omega}(N)} \) be a collection of uniform i.i.d. \([0, 1]\)-valued random variables. For \( \pi \in N^{<\omega} \) let \( \widehat{\zeta}_{\pi} = (\zeta_{\pi(a)})_{a \in \mathcal{P}(\pi)} \). The following is
then an exchangeable array
\[
(f_{\pi|a}(\hat{\zeta}_a))_{a \in N[\leq \omega]}.
\]

An important result is that every exchangeable array is equivalent in distribution to one of the above form. This theorem was proved by Aldous, Hoover, and Kallenberg in various levels of generality; for a survey see [Kal08], and for a detailed history, see the Historical Notes of [Kal05]. This result can be seen as a higher dimensional version of de Finetti’s theorem, and is known as the \textit{Aldous-Hoover-Kallenberg theorem (for jointly exchangeable arrays)}. This theorem has also been rediscovered in various forms, notably in terms of the notion of a \textit{graphon}, which can be thought of as the limit of a sequence of dense graphs. For more on this connection see [Aus08] or [DJ08].

The focus in this paper will be on probability measures, and hence on the distributions of random variables. In particular we call the distribution of an exchangeable sequence or array \(S_N\)-invariant as it is preserved by all elements of \(S_N\) under the appropriate action. We will be interested in the distribution of arrays which, instead of being preserved under all elements of \(S_N\), only need to be preserved under elements of a closed subgroup of \(S_N\).

It is well known that a subgroup of \(S_N\) is closed if and only if it is the automorphism group of a countable structure with underlying set \(N\). Now if \(M\) is such a structure then there is a natural collection of arrays whose distribution is preserved under all automorphisms of \(M\), i.e. elements of \(\text{Aut}(M)\). Specifically for \(\pi \in M\) let \(p_{\pi}\) be the orbit of \(\pi\) under \(\text{Aut}(M)\) and for any such orbit let \(f_{p_{\pi}} : [0, 1]^{|p(\pi)|} \to S\) be a measurable function. It is easy to check that the array \((f_{p_{\pi}}(\hat{\zeta}_a))_{a \in N[\leq \omega]}\) is \(\text{Aut}(M)\)-invariant. We call the distribution of such an array \textit{representable}.

A key result of this paper is a characterization of the \(\text{Aut}(M)\)-invariant measures which are representable, where \(M\) is an arbitrary structure in a countable language with underlying set \(N\). Specifically, we introduce in Definition 3.5 the notion of a \textit{free} structure and produce explicit \(S_N\)-invariant measures concentrated on free structures. We then give a general method of combining measures which will allow us, when \(M_{fr}\) is free, to combine an \(\text{Aut}(M_{fr})\)-invariant measure with a \(S_N\)-invariant measure we construct which is concentrated on \(M_{fr}\). We will then obtain a measure to which we can apply the Aldous-Hoover-Kallenberg theorem. This will then let us deduce in Theorem 5.6 that whenever \(M_{fr}\) is free, all \(\text{Aut}(M_{fr})\)-invariant measures are representable. We then show that for all countable structures \(M\), there is a minimal free extension \(\mathfrak{F}(M)\) and that an \(\text{Aut}(M)\)-invariant measure is representable if and only if it can be extended to an \(\text{Aut}(\mathfrak{F}(M))\)-invariant measure (in a sense we will make precise).
These results not only generalize the Aldous-Hoover-Kallenberg theorem to a large class of closed subgroups of $\mathcal{G}_N$, but also provides a way to characterize the representable measures even when such an Aldous-Hoover-Kallenberg like representation theorem fails to hold.

1.1. Connections To Other Work. In this subsection we consider how several related results can be viewed as special cases of our representation theorem.

Suppose $\mathcal{M}_\emptyset$ is the unique structure on $N$ in the empty language. Then $\text{Aut}(\mathcal{M}_\emptyset) = S_N$, and the Aldous-Hoover-Kallenberg theorem says that all $\text{Aut}(\mathcal{M}_\emptyset)$-invariant measures are representable. As $\mathcal{M}_\emptyset$ is free (a fact which is obvious from Definition 3.5) the Aldous-Hoover-Kallenberg theorem can be seen as a special case of Theorem 5.6, which says whenever $\mathcal{M}$ is free all $\text{Aut}(\mathcal{M})$-invariant measures are representable.

In addition to considering arrays whose distributions are invariant under a single, simultaneous, permutation of the indices, Aldous and Hoover also considered arrays whose distributions are invariant when the indices are permuted separately.

**Definition 1.3.** An array of $S$-valued random variables $\left( X_{\langle a \rangle} \right)_{a \in \mathbb{N}}$ is said to be *separately exchangeable* if its distribution is unchanged under coordinatewise permutations of $\mathbb{N}$, i.e. if for all $\sigma_0, \ldots, \sigma_{n-1} \in \mathcal{G}_N$

$$\left( X_{\langle a_i \rangle_{i<n}} \right)_{a_i \in \mathbb{N}} \overset{d}{=} \left( X_{\langle \sigma_i(a_i) \rangle_{i<n}} \right)_{a_i \in \mathbb{N}^n}.$$ 

As for jointly exchangeable arrays Aldous and Hoover proved a representation theorem for separately exchangeable arrays of arbitrary (fixed) dimension. However, to simplify the presentation we will restrict ourselves in the following example to the case when $n=2$. Both the Aldous-Hoover theorem for separately exchangeable arrays and the example below generalizes in the obvious way for $n>2$. For more on separately exchangeable arrays see [Kal05].

The Aldous-Hoover theorem for separately exchangeable arrays can be recast in terms of Theorem 5.6 as follows.

**Example 1.4.** Suppose $\left( X_{a_0,a_1} \right)_{(a_0,a_1) \in \mathbb{N}^2}$ is an $S$-valued array. Aldous and Hoover showed that $\left( X_{a_0,a_1} \right)_{(a_0,a_1) \in \mathbb{N}^2}$ is separately exchangeable if and only if there are uniform i.i.d. $[0,1]$-valued random variables $\beta, (\zeta_a)_{a \in \mathbb{N}}, (\eta_a)_{a \in \mathbb{N}}$ and $(\gamma_{i,j})_{i,j \in \mathbb{N}}$ along with a measurable function $f : [0,1]^4 \to S$ such that

$$\left( X_{a_0,a_1} \right)_{a_0,a_1 \in \mathbb{N}} \overset{d}{=} \left( f(\beta, \zeta_{a_0}, \eta_{a_1}, \gamma_{a_0,a_1}) \right)_{a_0,a_1 \in \mathbb{N}}.$$ 

Let $L = \{ U_0, U_1 \}$ be a language with two unary relations and let $\mathcal{M}_{U_2}$ be the structure with underlying set $\mathbb{N}$ such that $U_0^\mathcal{M} = \{ 2n : n \in \mathbb{N} \}$ and $U_1^\mathcal{M} = \{ 2n+1 : n \in \mathbb{N} \}$. Automorphisms of $\mathcal{M}_{U_2}$ can be represented as pairs $(\sigma, \tau) \in \text{Aut}(\mathcal{M})$.
where the pair \((\sigma, \tau)(2n) = 2\sigma(n)\) and \((\sigma, \tau)(2n + 1) = 2\tau(n) + 1\). Next consider the array \((Y_{2a_0,2a_1+1})(2a_0,2a_1+1) \in \mathbb{N}^2\) where \(Y_{2a_0,2a_1+1} := X_{a_0,a_1}\). Then

\[
(X_{a_0,a_1})_{a_0,a_1 \in \mathbb{N}}
\]
is separately exchangeable if and only if

\[
(Y_{2a_0,2a_1+1})(2a_0,2a_1+1) \in \mathbb{N}^2
\]

has an \(\text{Aut}(\mathcal{M}_{U,2})\)-invariant distribution. Notice that all of the indices of the array

\[
(Y_{2a_0,2a_1+1})(2a_0,2a_1+1) \in \mathbb{N}^2
\]

are in the same \(\text{Aut}(\mathcal{M}_{U,2})\)-orbit. Therefore the Aldous-Hoover theorem for separately exchangeable arrays is equivalent to the statement that all arrays indexed by elements in the orbit of the pair \((0,1)\) in \(\mathcal{M}_{U,2}^2\), and which are \(\text{Aut}(\mathcal{M}_{U,2})\)-invariant, are representable. It is easily seen from Definition 3.5 that \(\mathcal{M}_{U,2}\) is free. Therefore the Aldous-Hoover theorem for separately exchangeable arrays can be seen as a special case of Theorem 5.6.

Independently of the present work, Crane and Towsner ([CT18]) have studied the case of \(\text{Aut}(\mathcal{M})\)-invariant measures on the collection of structures in a fixed finite relational language with universe \(\mathbb{N}\), when \(\mathcal{M}\) is free and ultrahomogeneous. Their notation and framework though are slightly different than ours. In particular, the notion of relative exchangeability which they introduce is, for a non-ultrahomogeneous structure \(\mathcal{M}\), more restrictive than \(\text{Aut}(\mathcal{M})\)-invariance. However, in [CT18] they restrict their attention to ultrahomogeneous structures and, as they observe, in this setting the two notions coincide. Further their notion of an ultrahomogeneous structure whose age has \(n\)-DAP for all \(n \geq 1\) is precisely that of a free ultrahomogeneous structure in our sense.

With these translations Theorem 3.2 of [CT18], which is one of their two main representation theorems, is equivalent to a finitary version of Theorem 5.6. Specifically it is, essentially, Theorem 5.6 restricted to the case where the probability measure is invariant under the action of the automorphisms of a structure which is ultrahomogeneous in a finite relational language (i.e. \(\text{Aut}(\mathcal{M})\) where \(\mathcal{L}_\mathcal{M}\) is finite) and is on a space of structures in a finite relational language (i.e. \(\mathcal{L}_\mathcal{M}\) with \(\mathcal{L}\) finite). One of the main advantages of Theorem 5.6 over Theorem 3.2 of [CT18] is that by allowing the language of the free structure \(\mathcal{M}\) to be infinite we are able to apply Theorem 5.6 to arbitrary free structures by first passing to the canonical structure (which is never in a finite language). This in turn allows us to characterize representable measures over any structure, as in Theorem 5.7.
1.2. Outline. We now give an outline of the rest of the paper. First, in Section 1.3 we collect the notation which we will need. In Section 2 we give background and preliminary material which will be important later. In particular in Section 2.1 we introduce the notion of a canonical structure and review basic facts from the study of Polish group actions. In Section 2.2 we recall fundamental ideas from the study of $L_{\omega_1,\omega}$ and introduce the important notions of non-redundant quantifier free types as well as ordered quantifier free types. In Section 2.3 we introduce the notion of definable expansions and show that for every fragment and every language there is a theory (in a different language) which admits quantifier elimination for the fragment, which is non-redundant, and which is interdefinable with the empty theory in the original language. In Section 2.4 we introduce the notion of an invariant measure. These are the main object of study in this paper. In Section 2.5 we recall the Aldous-Hoover-Kallenberg theorem as well as related results. These results are a key building block of the main results of this paper. We then end the preliminaries with Section 2.6 where we recall the necessary and sufficient conditions for a sentence of $L_{\omega_1,\omega}$ to admit an $S_N$-invariant measure.

In Section 3 we prove various results related to free structures. Specifically, in Section 3.1 we continue our discussion of canonical structures and introduce a notion of map between canonical structures which will be important. Then, in Section 3.2, we introduce the notion of a free structure, show that every canonical structure is contained in a minimal free structure, and give an explicit example, for each free structure, of a $S_N$-invariant measure concentrated on its isomorphism class.

In Section 4 we introduce the notion of merging measures. Specifically we show that if $\text{Aut}(\mathcal{M}) \subseteq \text{Aut}(\mathcal{N})$ then we can combine (in a unique way) an $\text{Aut}(\mathcal{N})$-invariant measure on the space of $L_{\mathcal{M}}$-structures and concentrated on the isomorphism class of $\mathcal{M}$ with an $\text{Aut}(\mathcal{M})$-invariant measure on the space of $L$-structures to get an $\text{Aut}(\mathcal{N})$-invariant measure on the space of $L$-structures which is concentrated on the isomorphism class of $\mathcal{M}$. Further every $\text{Aut}(\mathcal{N})$-invariant measure on the space of $L$-structures which is concentrated on the isomorphism class of $\mathcal{M}$ arises as such a combination. We will use this result to take an $\text{Aut}(\mathcal{M}_{fr})$-invariant measure, when $\mathcal{M}_{fr}$ is free, and combine it with the measure we constructed in Section 3.2 to get a $S_N$-invariant measure to which we can apply the Aldous-Hoover-Kallenberg theorem. In Section 4.1 we will consider properties the merged measures inherit from their parts as well as give a characterization, for any free structure $\mathcal{M}_{fr}$ of those sentences of $L_{\omega_1,\omega}$ which admit an $\text{Aut}(\mathcal{M}_{fr})$-invariant measure.

In Section 5 we put everything together to study the representability of $\text{Aut}(\mathcal{M})$-invariant measures. In Section 5.1 we introduce the notion of an $\text{Aut}(\mathcal{M})$-recipe. Representable measures will be exactly those which are the
distribution of an Aut($\mathcal{M}$)-recipe. In Section 5.2 we prove two of the main results of this paper. First we show that if $\mathcal{M}_{fr}$ is free then every Aut($\mathcal{M}$)-invariant measure is representable. Then we show that for any structure $\mathcal{M}$ an Aut($\mathcal{M}$)-invariant measure is representable if and only if it has an extension to an Aut($\mathcal{F}(\mathcal{M})$)-invariant measure where $\mathcal{F}(\mathcal{M})$ is the minimal free structure containing $\mathcal{M}$. In Section 5.3 we give a characterization of when an Aut($\mathcal{M}$)-recipe is gives rise to an ergodic measure.

1.3. Notation. For $n \in \mathbb{N}$ we let $[n] = \{0, \ldots, n-1\}$ and let $[\omega] = \mathbb{N}$. We will use $\mathbb{N}^{[<\omega]}$ to denote the collection of finite sequences of distinct natural numbers and let $\mathbb{N}^{[<d]}$ be the collection of finite sequences of distinct natural numbers of length less than $d$.

We let $F(X)$ denote the collection of all subsets of $X$. For $n \leq \omega$ and $\sqsubset \in \{=, <, \leq\}$ we let $F_{\sqsubset n}(X)$ be the collection of subsets of $X$ which have cardinality $\sqsubset n$. When $\bar{a}$ is a sequence and no confusion can arise we will abuse notation and let $F(\bar{a})$ denote the set of subsequences of $\bar{a}$.

For $k = (k_0, \ldots, k_{d-1}) \in \mathbb{N}^{[<\omega]}$ and $I = \{i_1, \ldots, i_m\} \in F(d)$ with $i_1 < \cdots < i_m$ we let $k \triangleright I = \{k_{i_1}, \ldots, k_{i_m}\}$. Similarly for $K = \{k_0, \ldots, k_{d-1}\} \in F(d)\mathbb{N}$ with $k_0 < \cdots < k_{d-1}$ and $I = \{i_1, \ldots, i_m\} \in F(d)$ with $i_1 < \cdots < i_m$ we let $K \triangleright I = \{k_{i_1}, \ldots, k_{i_m}\}$. If $(E_\bar{a})_{\bar{b} \in \mathbb{N}^{[<\omega]}}$ is an indexed collection of objects and $\bar{b} \in \mathbb{N}^{[<\omega]}$ we let $E_\bar{b} := \langle E_{\bar{a} \triangleright I} \rangle_{\bar{b} \in \mathbb{N}^{[<\omega]}}$ and if $B \in F_{<\omega}(\mathbb{N})$ we let $E_B := \langle E_{\bar{a} \triangleright I} \rangle_{\bar{b} \in \mathbb{N}^{[<\omega]}}$.

If $\equiv$ is an equivalence relation on a set $X$ and $x \in X$ let $[x]_\equiv$ be the $\equiv$-equivalence class of $x$. If $A, B$ are sets we let $A \triangle B$ be the symmetric difference of $A$ and $B$.

All languages will be countable and relational. Note that by a standard interpretation of functions by their graphs, restricting to relational languages yields no loss of generality. Further $L$ and its variants with decorations will always represent languages. If $R$ is a relation we let $ar(R)$ be its arity. We denote by $L^n$ the sub-language of $L$ consisting of those relations of arity exactly $n$ and by $L^{<n}$ be the sub-language of $L$ consisting of those relations of arity at most $n$. We let $L_{\omega, \omega}(L)$ be the collection of first order formulas in the language $L$ and we let $L_{\omega_1, \omega}(L)$ be the collection of infinitary formulas in the language $L$ with at most countable sized conjunctions and disjunctions. All formulas will be in $L_{\omega_1, \omega}(L)$ for some language $L$. For a formula $\varphi$, and $i \in \mathbb{N}$, we let $\neg^i \varphi$ stand for $\neg \varphi$ if $i$ is odd and $\varphi$ if $i$ is even. For a formula $\varphi(\overline{x}, y) \in L_{\omega_1, \omega}(L)$ we let $(\exists^{=1} y) \varphi(\overline{x}, y)$ stand for “there exists a unique $y$ satisfying $\varphi(\overline{x}, y)$”.

The symbol $\mathcal{M}$ and its variants with decorations will always be countable structures for some language and we will use $\mathcal{M}$ for both the structure and the underlying set when no confusion can arise. We will use $L_{\mathcal{M}}$ to denote the
language of $M$. When $\vec{a}$ is a finite tuple of elements from $M$ we will abuse notation and write $\vec{a} \in M$ to denote $\vec{a} \in M$.

Suppose $\vec{x} = \langle x_i \rangle_{i \in [n]}$ where $n \leq \omega$. We define a function $\gamma_{\vec{x}} : [0, 1] \to \vec{x}$ as follows. First $\gamma_{\vec{x}}(1) := x_0$. If $n < \omega$ and $y \in [0, 1)$ then $\gamma_{\vec{x}}(y) := x_i$ if and only if $y \in \left[\frac{i}{n}, \frac{i+1}{n}\right)$. If $n = \omega$ and $y \in [0, 1)$ then $\gamma_{\vec{x}}(y) := x_i$ if and only if $y \in [1 - 2^{-i}, 1 - 2^{-(i+1)})$.

We say $(\zeta_i)_{i \in I}$ is a $U[0, 1]$-array if $(\zeta_i)_{i \in I}$ consists of i.i.d. $[0, 1]$-valued random variables with uniform distribution. We let $\lambda$ be the Lebesgue measure. For random variables $X, Y$ taking values in the same space we let $X \overset{d}{=} Y$ denote the fact that $X$ and $Y$ have the same distribution. We will use “a.s.” to denote the phrase “almost surely”.

All measures in this paper will be probability measures and all spaces will be standard Borel spaces. Further $S$ and its variants with decorations will always be standard Borel spaces and we let $\mathcal{P}_1(S)$ be the collection of probability measures on $S$.

For a set $X$ we denote by $\mathcal{C}_X$ the collection of permutations of $X$. We will consider $\mathcal{S}_\omega$ as a Polish group with the subspace topology inherited from $\mathbb{N}^\mathbb{N}$. If $M$ is an $L$-structure we denote by $\text{Aut}(M)$ the collection of automorphisms of $M$.

For any notions of probability theory not explicitly covered here we refer the reader to [Kal02]. For any notions of model theory not covered here we refer the reader to [Bar75]. For any notions of descriptive set theory we refer the reader to [Kec95] or [BK96].

2. Preliminaries

In this section we recall some important facts and results which will be used later.

2.1. Polish Group Actions. Suppose $G$ is a closed subgroup of $\mathcal{S}_\omega$. For $\vec{a}, \vec{b} \in \mathbb{N}^{<\omega}$ let $\vec{a} \sim_G \vec{b}$ if there is a $g \in G$ such that $\vec{a} = g(\vec{b})$. It is immediate that $\sim_G$ is an equivalence relation on $\mathbb{N}^{<\omega}$ in which $\sim_G$-equivalent tuples have the same length.

Let $L_G := \{ R_{|\vec{a}|\sim_G} : \vec{a} \in \mathbb{N}^{<\omega} \}$ where $\ar(R_{|\vec{a}|\sim_G}) = |\vec{a}|$ for $\vec{a} \in \mathbb{N}^{<\omega}$, We call $L_G$ the canonical language of $G$. Now let $\mathcal{C}_G$ be the $L_G$-structure with underlying set $\mathbb{N}$ such that $\mathcal{C}_G \models R_A(\vec{b})$ if and only if $\vec{b} \in A$. We call $\mathcal{C}_G$ the canonical structure of $G$. The following two lemmas are then immediate.

Lemma 2.1 ([BK96] Sec. 1.5). If $G$ is any closed subgroup of $\mathcal{S}_\omega$ then

- $G = \text{Aut}(\mathcal{C}_G)$.
- $\mathcal{C}_G$ is the canonical structure of $\text{Aut}(\mathcal{C}_G)$.
- $\mathcal{C}_G$ is ultrahomogeneous, i.e. any isomorphism between finite structures extends to an automorphism.
Lemma 2.2. If $\mathcal{M}$ is a structure with underlying set $\mathbb{N}$ then $\text{Aut}(\mathcal{M})$ is a closed subgroup of $\mathfrak{S}_\mathbb{N}$.

In particular for the purposes of studying closed subgroups of $\mathfrak{S}_\mathbb{N}$ it suffices to restrict our attention to groups of the form $\text{Aut}(\mathcal{M})$ where $\mathcal{M}$ is the $\text{Aut}(\mathcal{M})$-canonical structure. This is significant because there is a concrete representation of actions of $G$ for $G$ a closed subgroup of $\mathfrak{S}_\mathbb{N}$ in terms of its canonical structure.

Definition 2.3. Suppose $G$ is a Polish group. A $G$-space is a pair $(\circ_X, X)$ where

- $X$ is a Borel space.
- $\circ: G \times X \to X$ is a Borel map.

A function $i: (\circ_X, X) \to (\circ_Y, Y)$ is a map of $G$-spaces if it is a Borel function such that $(\forall g \in G)(\forall x \in X) i(\circ_X(g, x)) = \circ_Y(g, i(x))$.

If $(\circ_X, X)$ is a $G$-space then we extend the action of $G$ to subsets of $X$ where, for $A \subseteq X$ and $g \in G$, $gA := \{\circ_X(g, a): a \in A\}$.

Definition 2.4. Suppose $\mathcal{M}$ is an $\mathcal{L}_\mathcal{M}$-structure with underlying set $\mathbb{N}$ and suppose $\mathcal{L}$ is disjoint from $\mathcal{L}_\mathcal{M}$. We define $\mathcal{S}_\mathcal{L}(\mathcal{M})$ to be the collection of $\mathcal{L}_\mathcal{M} \cup \mathcal{L}$ structures with underlying set $\mathbb{N}$ such that whenever $\mathcal{N} \in \mathcal{S}_\mathcal{L}(\mathcal{M})$ then $\mathcal{N}|_{\mathcal{L}_\mathcal{M}} = \mathcal{M}$.

For $k \in \mathbb{N}$ (possibly 0), $\varphi(x_0, \ldots, x_{k-1}) \in \mathcal{L}_{\omega_1, \omega}(\mathcal{L}_\mathcal{M} \cup \mathcal{L})$ and $n_0, \ldots, n_{k-1} \in \mathbb{N}$ define $[\varphi(n_0, \ldots, n_{k-1})]_\mathcal{M} := \{\mathcal{N} \in \mathcal{S}_\mathcal{L}(\mathcal{M}): \mathcal{N} \models \varphi(n_0, \ldots, n_{k-1})\}$.

We give $\mathcal{S}_\mathcal{L}(\mathcal{M})$ the topology generated by the clopen subbasis

$$\{[R(n_0, \ldots, n_{k-1})]_\mathcal{M}: R \in \mathcal{L}, \text{ar}(R) = k, n_0, \ldots, n_{k-1} \in \mathbb{N}\}$$

When $\mathcal{M}_\emptyset$ is the unique structure on $\mathbb{N}$ in the empty language we will denote $\mathcal{S}_\mathcal{L}(\mathcal{M}_\emptyset)$ by $\mathcal{S}_\mathcal{L}$. In this case observe that $\text{Aut}(\mathcal{M}_\emptyset) = \mathfrak{S}_\mathbb{N}$.

Now for any $\mathcal{M}$, there is a natural action of $\text{Aut}(\mathcal{M})$ on $\mathcal{S}_\mathcal{L}(\mathcal{M})$.

Definition 2.5. Suppose $\mathcal{M}$ is an $\mathcal{L}_\mathcal{M}$-structure with underlying set $\mathbb{N}$ and $\mathcal{L}$ is a language disjoint from $\mathcal{L}_\mathcal{M}$. We define the action $\circ_\mathcal{M}: \text{Aut}(\mathcal{M}) \times \mathcal{S}_\mathcal{L}(\mathcal{M}) \to \mathcal{S}_\mathcal{L}(\mathcal{M})$ where, for $g \in \text{Aut}(\mathcal{M})$, $\mathcal{N} \in \mathcal{S}_\mathcal{L}(\mathcal{M})$, $\circ_\mathcal{M}(g, \mathcal{N})$ is the structure $g\mathcal{N}$ such for all $R \in \mathcal{L}$ of arity $k$ and $n_0, \ldots, n_{k-1} \in \mathbb{N}$

$$g\mathcal{N} \models R(n_0, \ldots, n_{k-1}) \text{ if and only if } \mathcal{N} \models R(g^{-1}(n_0), \ldots, g^{-1}(n_{k-1}))$$

It is immediate that $(\circ_\mathcal{M}, \mathcal{S}_\mathcal{L}(\mathcal{M}))$ is an $\text{Aut}(\mathcal{M})$-space.

Lemma 2.6. Suppose $\mathcal{M}$ is an $\mathcal{L}_\mathcal{M}$-structure with underlying set $\mathbb{N}$ and $\mathcal{L}$ is a language disjoint from $\mathcal{L}_\mathcal{M}$. Further suppose $\mathcal{L}$ has relations of unbounded arity. Then $(\circ_\mathcal{M}, \mathcal{S}_\mathcal{L}(\mathcal{M}))$ is a universal $\text{Aut}(\mathcal{M})$-space, i.e. an $\text{Aut}(\mathcal{M})$-space which contains an isomorphic copy of every other $\text{Aut}(\mathcal{M})$-space as a subspace.
Proof. [BK96] Thm. 2.7.4. □

In particular Lemma 2.1 and Lemma 2.6 tell us that if $G$ is a closed subgroup of $\mathfrak{S}_\mathcal{N}$ then the study of $G$-invariant measures on $G$-spaces is equivalent to the study of $\text{Aut}(\mathcal{M})$-invariant measures on $\mathcal{S}_\mathcal{L}(\mathcal{M})$. This is significant as it allows us to translate the study of $G$-invariant measures from the realm of descriptive set theory to the realm of model theory.

Definition 2.7. We say a Borel set $A \subseteq \mathcal{S}_\mathcal{L}(\mathcal{M})$ is $\text{Aut}(\mathcal{M})$-invariant if for all $g \in \text{Aut}(\mathcal{M}), gA = A$.

Note this notion is sometimes called *strict invariance* to contrast it with Definition 2.30.

It is immediate that if $\tau \in \mathcal{L}_{\omega_1,\omega}^{\text{L}_\mathcal{M} \cup \text{L}}$ is a sentence then $\llbracket \tau \rrbracket_\mathcal{M}$ is an $\text{Aut}(\mathcal{M})$-invariant Borel subset of $\mathcal{S}_\mathcal{L}(\mathcal{M})$ and hence $\llbracket \tau \rrbracket_\mathcal{M}$ inherits the structure of an $\text{Aut}(\mathcal{M})$-space.

Definition 2.8. We say a sentence $T \in \mathcal{L}_{\omega_1,\omega}^{\text{L}_\mathcal{M} \cup \text{L}}$ is $\text{Aut}(\mathcal{M})$-universal if $\llbracket T \rrbracket_\mathcal{M}$ is a universal $\text{Aut}(\mathcal{M})$-space.

We will often want to assume our models satisfy some basic syntactic properties, e.g. non-redundancy of relations, Morleyized for a fragment, etc. Provided we can find an $\text{Aut}(\mathcal{M})$-universal theory whose models are exactly those with the desired syntactic properties, then there is no loss in generality in assuming our structures satisfy those properties. We will come back to this in Section 2.3.

2.2. Infinitary Logic. In this section we recall some basic facts and definitions from infinitary logic. In particular it will be important in what follows to pin down various notions of quantifier free type. First we recall some basic properties of structures.

Lemma 2.9 ([Bar75] Ch. VII.6). Suppose $\mathcal{M}$ is a countable $\text{L}_\mathcal{M}$-structure. Then there is a sentence $\sigma_\mathcal{M} \in \mathcal{L}_{\omega_1,\omega}^{\text{L}_\mathcal{M}}$, called the Scott sentence of $\mathcal{M}$, such that for each countable $\text{L}_\mathcal{M}$-structure $\mathcal{N}$

$\mathcal{N} \models \sigma_\mathcal{M}$ if and only if $\mathcal{M} \cong \mathcal{N}$.

We will often want to focus on structures $\mathcal{M}$ which are, in some sense, far from being rigid (and hence will have a large automorphism group). One way to express this is by saying the structure has trivial definable closure.

Definition 2.10. For $\pi \in \mathcal{M}$ the (group theoretic) definable closure of $\pi$ is the set

$$\text{dcl}(\pi) := \{ b \in \mathcal{M} : (\forall g \in \text{Aut}(\mathcal{M})) g(\pi) = \pi \rightarrow g(b) = b \}.$$
We say $\mathcal{M}$ has trivial definable closure, or trivial dcl, if

$$(\forall \pi \in \mathcal{M}) \ dcl(\pi) = \pi.$$  

We will often want to restrict attention to subsets of $\mathcal{L}_{\omega_1, \omega}(L)$ with basic closure properties.

**Definition 2.11.** A fragment is a subset of $A \subseteq \mathcal{L}_{\omega_1, \omega}(L)$ which is closed under

- Sub-formulas,
- finite Boolean operations, and
- $(\exists x), (\forall x)$.

An $A$-theory is a collection of sentences of the form $\{ \varphi \in A : \mathcal{M} \models \varphi \}$ for some $L$-structure $\mathcal{M}$.

In addition to having a notion of trivial dcl for a structure, we will also want an analogous notion for a theory in a fragment.

**Definition 2.12.** Suppose $A$ is a fragment and $T$ is an $A$-theory. We say $T$ has trivial definable closure (or trivial dcl) if there does not exist a formula $\varphi(\bar{x}, y) \in A$ such that

$$T \models (\exists \bar{x})(\exists y=1) \varphi(\bar{x}, y).$$

Note that a structure $\mathcal{M}$ has trivial dcl if and only if for any fragment $A$, any $A$-theory containing $\sigma_{\mathcal{M}}$ has trivial dcl. Similarly a structure $\mathcal{M}$ has trivial dcl if and only if there is some fragment $A$ and some $A$-theory containing $\sigma_{\mathcal{M}}$ which has trivial dcl.

**Definition 2.13.** We say a sentence $T \in \mathcal{L}_{\omega_1, \omega}(L)$ is Morleyized for a fragment $A$ if for all $\varphi(\bar{x}) \in A$ there is a relation $R_{\varphi}(\bar{x}) \in L$ such that $T \models (\forall \bar{x}) \varphi(\bar{x}) \leftrightarrow R_{\varphi}(\bar{x})$.

We now introduce some important notions involving quantifier free types.

**Definition 2.14.** A partial quantifier free $L$-type on $(x_0, \ldots, x_{n-1})$ is a collection of formula, $q$, such that whenever $\eta(x_{i_0}, \ldots, x_{i_{k-1}}) \in q$ we have

- $\{x_{i_0}, \ldots, x_{i_{k-1}}\} \subseteq \{x_0, \ldots, x_{n-1}\}$,
- $\eta$ is either an atomic formula or the negation of an atomic formula,
- for all $0 \leq i < j < n$, $x_i \neq x_j \in q$,
- There is an $L$-structure $\mathcal{M}$ and a tuple $(a_0, \ldots, a_{n-1}) \in \mathcal{M}$ such that $\mathcal{M} \models \bigwedge \{\eta(a_{i_0}, \ldots, a_{i_{k-1}}) : \eta(x_{i_0}, \ldots, x_{i_{k-1}}) \in q\}$.

We say a partial quantifier free type has arity $n$ if it is on a set of variables of size $n$. We denote the arity of a partial quantifier free type $q$ by $\text{ar}(q)$. We say a partial quantifier free type is a (complete) quantifier free type if it is maximal under inclusion.
For $\mathcal{M}$ an $L_{\mathcal{M}}$-structure and $\bar{a} \in \mathcal{M}$ we say $\bar{a}$ realizes a (partial) quantifier free type $p(\bar{x})$ if $\mathcal{M} \models \bigwedge_{\eta(\bar{x}) \in p(\bar{x})} \eta(\bar{a})$. We denote the collection of (complete) quantifier free types realized by elements of $\mathcal{M}$ by $\text{qtp}(\mathcal{M})$.

Throughout this paper we will be interested in constructing random structures in stages, first determining the structure of all singletons, then determining, based on the structure of the singletons, the structure of the pairs, etc. When doing this it is important that the complete structure of all $n$-tuples is determined before we determine the structure of the $(n+1)$-tuples. For this reason we will want to restrict our attention to the case where, whenever an $n$-ary relation holds of a tuple, the tuple has distinct elements (as otherwise the instance of the $n$-ary relation would be about a $k$-tuple of distinct elements for some $k < n$ and not about an $n$-tuple of distinct elements). To this end we define an important class of quantifier free types.

**Definition 2.15.** Suppose $\eta(x_0, \ldots, x_{n-1}) \in L_{\omega_1, \omega}(L)$ is an atomic formula. We say $\eta(x_0, \ldots, x_{n-1})$ is *non-redundant* if for all $0 \leq i < j < n$ we have $x_i \neq x_j$. We say a partial quantifier free type is non-redundant if every atomic formula in it, except perhaps those of the form $x_i = x_i$, are non-redundant.

For $\bar{x} = (x_0, \ldots, x_{n-1})$ distinct elements let $\text{ntp}_L(\bar{x})$ be the collection of non-redundant quantifier free types on $\bar{x}$ in $L$. We will omit $L$ when it is clear from context.

Note $\text{ntp}_L(\bar{x})$ has a natural topology generated by clopen sets of the from

$$\{q: \neg \ell R(x_{i_0}, \ldots, x_{i_{k-1}}) \in q\}$$

where $R \in L$, $\ell \in \{0, 1\}$ and $\{i_0, \ldots, i_{k-1}\} \subseteq [n]$. With this topology $\text{ntp}_L(\bar{x})$ is homeomorphic to Cantor space.

**Definition 2.16.** Suppose we have a theory $T \in L_{\omega_1, \omega}(L)$. We say a formula $\varphi(x_0, \ldots, x_{n-1}) \in L_{\omega_1, \omega}(L)$ is non-redundant over $T$ if

$$T \models (\forall x_0, \ldots, x_{n-1}) \left( \varphi(x_0, \ldots, x_{n-1}) \rightarrow \bigwedge_{0 \leq i < j < n} x_i \neq x_j \right).$$

We say $T$ has non-redundant quantifier free types if each non-equality atomic formula with distinct variables is non-redundant over $T$.

We say a structure $\mathcal{M}$ is non-redundant if $\sigma_M$ has non-redundant quantifier free types.

In particular if $T$ has non-redundant quantifier free types, then every quantifier free type realized in a model of $T$ is non-redundant.

**Example 2.17.** Suppose $\mathcal{M}$ is a canonical structure. Then it is immediate from the definition of a canonical structure that $\sigma_M$ has non-redundant quantifier free types, and so $\mathcal{M}$ is non-redundant.
Another important class of partial quantifier free types are those where the ordering of the variables in the formulas is consistent.

**Definition 2.18.** We say a partial quantifier free type $q$ on distinct elements $(x_0, \ldots, x_{n-1})$ is **ordered** if whenever $\eta(x_{i_0}, \ldots, x_{i_k-1}) \in q$ and $0 \leq j^- < j^+ < k$ we have $i_{j^-} < i_{j^+}$.

We define an **ordered quantifier free type** to be a maximal ordered partial quantifier free type under inclusion (among the collection of ordered quantifier free types on the same variables).

For $x = (x_0, \ldots, x_{n-1})$ let $\text{otp}_{L}(x)$ be the collection of ordered quantifier free types in $L$ on $x$. Note $\text{otp}_{L}(x)$ has a natural topology generated by clopen sets of the form $\{q: \neg \ell R(x_{i_0}, \ldots, x_{i_{j-1}}) \in q\}$ where $R \in L$, $\ell \in \{0,1\}$ and $0 \leq i_0 < \ldots, < i_{j-1} < n$. This topology makes $\text{otp}_{L}(x)$ homeomorphic to Cantor space.

An ordered quantifier free type need not itself be a (complete) quantifier free type as we can often find an extension which doesn’t preserve the order of variables. Despite this we have chosen the current name as “ordered partial quantifier free type” does not convey the maximality condition which we need.

The relationship between non-redundant and ordered quantifier free types is given by the following straightforward lemma.

**Lemma 2.19.** For any $q \in \text{ntp}_{L}(\overline{x})$ there is a unique collection $\langle p_{\tau} \rangle_{\tau \in \mathcal{S}_{\overline{x}}}$ such that

- $p_{\tau} \in \text{otp}_{L}(\tau(\overline{x}))$,
- $q = \bigcup_{\tau \in \mathcal{S}_{\overline{x}}} p_{\tau}$.

The relationship between non-redundant and ordered quantifier free types plays an important role in translating between the distribution of an array of random variables which is $\text{Aut}(\mathcal{M})$-invariant and an $\text{Aut}(\mathcal{M})$-invariant measure on $\mathcal{S}(\mathcal{M})$.

In order to do this translation we want to consider arrays $\langle f_p \rangle_{p \in \text{ntp}(\mathcal{M})}$ where $f_p$ takes values in $\text{otp}_{L}(\overline{x})$ where $|\overline{x}| = \text{ar}(p)$ (which is itself a standard Borel space). In this way we will be able in Definition 5.1 to use Lemma 2.19 to recover the (non-redundant) quantifier free type of a tuple $(n_0, \ldots, n_{k-1})$ from the values of $\langle f_{p_{\tau}}(\overline{x}) \rangle_{\tau \in \mathcal{S}_{k}}$ when $\mathcal{M} \models p_{\tau}(\tau(n_0), \ldots, \tau(n_{k-1}))$. We will then be able to recover an element of $\mathcal{S}(\mathcal{M})$ from $\langle f_p(\overline{x}) \rangle_{p \in \text{ntp}(\mathcal{M})}$.

We end this section recalling the notion of deduction in $\mathcal{L}_{\omega_1}(\mathcal{L})$ (see for example [Bar75] Sec. III.4). This will be important when discussing the theory of an ergodic invariant measure.

**Definition 2.20.** We say $\varphi \in \mathcal{L}_{\omega_1}(\mathcal{L})$ is a **tautology** if for all $L$-structures $\mathcal{M}$, we have $\mathcal{M} \models \varphi$. 
Suppose \( T \subseteq \mathcal{L}_{\omega,1}(L) \) is a collection of sentences. We define the **deductive closure** of \( T \), \( \text{dc}(T) \), to be the smallest subset of \( \mathcal{L}_{\omega,1}(L) \) such that

- (Tautologies) \( \text{dc}(T) \) contains all tautologies.
- (Modus Ponens) If \( \varphi \in \text{dc}(T) \) and \( (\varphi \rightarrow \psi) \in \text{dc}(T) \) then \( \psi \in \text{dc}(T) \).
- (Generalization) If \( (\forall v)(\varphi \rightarrow \psi(v)) \in T \) and \( v \) is not free in \( \varphi \), then \( (\varphi \rightarrow (\forall v)\psi(v)) \in T \).
- (Conjunction) If \( \bigwedge \Phi \in \mathcal{L}_{\omega,1}(L) \) and for all \( \varphi \in \Phi \), \( (\psi \rightarrow \varphi) \in \text{dc}(T) \) then \( (\psi \rightarrow \bigwedge \Phi) \in T \).

We say \( T \) is **consistent** if \( \text{dc}(T) \neq \mathcal{L}_{\omega,1}(L) \).

While it is the case that if \( T \) is countable and consistent it must have a model, this is not in general the case for uncountable \( T \). However, if \( A \) is a countable fragment then \( T \) is an \( A \) theory if and only if it is consistent and complete, i.e. for all sentences \( \varphi \in A \) either \( \varphi \in T \) or \( \neg \varphi \in T \).

### 2.3. Definable Expansions

In this section we recall the notion of a definable expansion and show how they can be used to find \( \text{Aut}(\mathcal{M}) \)-universal sentences with desired properties. This will then let us reduce the general problem of finding representations for \( \text{Aut}(\mathcal{M}) \)-invariant measures on \( \mathcal{S}_\mathcal{M}(\mathcal{M}) \) to the problem of finding representations for such measures which concentrate on the collection of models which have our desired properties.

**Definition 2.21.** Suppose \( L_0 \subseteq L_1 \), \( T_0 \in \mathcal{L}_{\omega,1}(L_0) \) and \( T_1 \in \mathcal{L}_{\omega,1}(L_1) \). We say that \( T_1 \) is a **definable expansion** of \( T_0 \) if

- Every \( L_0 \)-structure \( M_0 \) satisfying \( T_0 \) has a unique expansion to an \( L_1 \)-structure \( M_1 \) satisfying \( T_1 \).
- \( T_1 \models T_0 \).
- For every formula \( \varphi_1 \in \mathcal{L}_{\omega,1}(L_1) \) there is a formula \( \varphi_0 \in \mathcal{L}_{\omega,1}(L_0) \) such that

\[
T_1 \models (\forall \overline{x}) \varphi_1(\overline{x}) \leftrightarrow \varphi_0(\overline{x}).
\]

So \( T_1 \) is a definable expansion of \( T_0 \) if all models of \( T_1 \) are also models of \( T_0 \), the restriction relation is a bijection, and further every formula in \( \mathcal{L}_{\omega,1}(L_1) \) is equivalent (over \( T_1 \)) to one in \( \mathcal{L}_{\omega,1}(L_0) \). The following lemma is then straightforward.

**Lemma 2.22.** Suppose

- \( L_0 \subseteq L_1 \),
- \( T_0 \in \mathcal{L}_{\omega,1}(L_0) \) and \( T_1 \in \mathcal{L}_{\omega,1}(L_1) \),
- \( T_1 \) is a definable expansion of \( T_0 \).

Then for any \( T^* \in \mathcal{L}_{\omega,1}(L_0) \), \( [T_0 \land T^*]_\mathcal{M} \) is isomorphic to \( [T_1 \land T^*]_\mathcal{M} \) as \( \text{Aut}(\mathcal{M}) \)-spaces.
Lemma 2.22 tells us that if $T_1$ is a definable expansion of $T_0$ then when considered as $\text{Aut}(\mathcal{M})$-spaces, $[T_0]_\mathcal{M}$ is isomorphic to $[T_1]_\mathcal{M}$. We will in particular be interested in the case when two theories have a common definable expansion.

**Definition 2.23.** We say theories $T_0 \in \mathcal{L}_{\omega_1,\omega}(L_0)$ and $T_1 \in \mathcal{L}_{\omega_1,\omega}(L_1)$ are interdefinable when there is a language $L_2 \supseteq L_0 \cup L_1$ and a theory $T_2$ which is a definable expansion of both $T_0$ and $T_1$.

Two theories are interdefinable when it is possible to define each from the other (possibly in some larger language).

**Example 2.24.** Suppose $\mathcal{M}$ is an $L$-structure. Then $\sigma_\mathcal{M}$ and $\sigma_{\text{Aut}(\mathcal{M})}$ are interdefinable.

Another important class of examples of interdefinable structures are those obtained by simply relabeling the relations.

**Definition 2.25.** A map of languages between $L_0$ and $L_1$ is a function $i: L_0 \to L_1$ such that for any relation $R \in L_0$, $\text{ar}(R) = \text{ar}(i(R))$. A relabeling of a language $L_0$ by $L_1$ is a bijective map of languages.

Note any relabeling extends to a bijection $i: \mathcal{L}_{\omega_1,\omega}(L_0) \to \mathcal{L}_{\omega_1,\omega}(L_1)$.

**Example 2.26.** If $i: L_0 \to L_1$ is a relabeling then the empty theory in $L_0$ is interdefinable with the empty theory in $L_1$.

There is an important example of a theory which is interdefinable with the empty theory in a language.

For a language $L$ let

$$\mathcal{T}_{L} := \bigwedge_{R \in L, \text{ar}(R) = k} (\forall x_0, \ldots, x_{k-1}) \left( R(x_0, \ldots, x_{k-1}) \to \bigwedge_{0 \leq i < j < k} x_i \neq x_j \right).$$

For a language $L$ let $L_{\text{nr}} := \{ R_{P,\equiv} : P \in L, \text{of arity } n, \equiv \text{ is an equivalence relation on } [n] \text{ with } \text{ar}(R_{P,\equiv}) \text{ many equivalence class} \}$ (here $\text{nr}$ stands for “non-redundant”).

Let $\text{Th}_{L_{\text{nr}}}$ be the conjunction of all sentences of the form

$$(\forall x_0, \ldots, x_{n-1}) R_{P,\equiv}(x_{i_0}, \ldots, x_{i_{k-1}}) \leftrightarrow \left[ P(x_0, \ldots, x_{n-1}) \land \bigwedge_{0 \leq j < \ell < k} x_{i_j} \neq x_{i_\ell} \land \bigwedge_{0 \leq j < \ell < n} x_{j} = x_{\ell} \land \bigwedge_{0 \leq j < \ell < n} x_{j} \neq x_{\ell} \right].$$
This theory replaces every relation with a sequence of non-redundant relations, one for every way in which arguments could be duplicate in the original relation.

The following proposition is immediate from the definitions of $\mathcal{R}_{\text{ext}}$ and $\text{Th}^{\text{nr}}_L$.

**Proposition 2.27.** If $L$ is a language then

(a) $\omega \cdot |L_{\text{ext}}| = \omega \cdot |L|$.

(b) $\text{Th}^{\text{nr}}_L \land \mathcal{R}_{\text{ext}}$ is a definable expansion of the empty theory in $L$ and $\mathcal{R}_{\text{ext}}$ in $L_{\text{ext}}$.

(c) $\mathcal{R}_{\text{ext}}$ has non-redundant quantifier free types.

Proposition 2.27 (b) says that $\mathcal{R}_{\text{ext}}$ is interdefinable with the empty theory in $L$. And, as $L$ has unbounded arity if and only if $L_{\text{ext}}$ does, $\mathcal{R}_{\text{ext}}$ is $\text{Aut}(\mathcal{M})$-universal precisely when $\mathcal{S}_L(\mathcal{M})$ is a universal $\text{Aut}(\mathcal{M})$-space. We will prefer in most circumstances to work with $[\mathcal{R}_{\text{ext}}]_{\mathcal{M}}$ rather than with $\mathcal{S}_L(\mathcal{M})$ and as such we define $\mathcal{S}^*_L(\mathcal{M}) := [\mathcal{R}_{\text{ext}}]_{\mathcal{M}_1}$, and $\mathcal{S}^*_L := [\mathcal{R}_{\text{ext}}]_{\mathcal{M}_0}$ where $\mathcal{M}_0$ is the unique structure on $\mathbb{N}$ in the empty language.

We now give a definable expansion which will give us Morleyizations for a fragment.

Given a countable fragment $A$ we let $L_A := \{R_{\varphi(\overline{x})}(\overline{x}) : \varphi(\overline{x}) \in A\}$. We define the sentence $\text{Th}^{qe}_A \in \mathcal{L}_{\omega_1,\omega}(L_A)$ to be the conjunction of the following:

- If $\varphi(\overline{x}) = \neg \psi(\overline{x})$ then $(\forall \overline{x})R_{\varphi(\overline{x})}(\overline{x}) \leftrightarrow \neg R_{\psi(\overline{x})}(\overline{x})$.
- If $\varphi(\overline{x}) = \wedge_{i \in I} \psi_i(\overline{x})$ then $(\forall \overline{x})R_{\varphi(\overline{x})}(\overline{x}) \leftrightarrow \wedge_{i \in I} R_{\psi_i(\overline{x})}(\overline{x})$.
- If $\varphi(\overline{x}) = \vee_{i \in I} \psi_i(\overline{x})$ then $(\forall \overline{x})R_{\varphi(\overline{x})}(\overline{x}) \leftrightarrow \vee_{i \in I} R_{\psi_i(\overline{x})}(\overline{x})$.
- If $\varphi(\overline{x}) = (\exists y)\psi(\overline{x},y)$ then $(\forall \overline{x})R_{\varphi(\overline{x})}(\overline{x}) \leftrightarrow (\exists y)R_{\psi(\overline{x},y)}(\overline{x})$.
- If $\varphi(\overline{x}) = (\forall y)\psi(\overline{x},y)$ then $(\forall \overline{x})R_{\varphi(\overline{x})}(\overline{x}) \leftrightarrow (\forall y)R_{\psi(\overline{x},y)}(\overline{x})$.

Let $\text{Th}^*_A := \bigwedge_{P \in L}(\forall \overline{x})P(\overline{x}) \leftrightarrow R_P(\overline{x})$. We call $\text{Th}^{qe}_A \land \text{Th}^*_A$ the Morleyization of $A$.

The following is immediate.

**Proposition 2.28.** If $L$ is a language and $A$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(L)$ then

(a) $|L_A| = \omega$.

(b) $\text{Th}^{qe}_A \land \text{Th}^*_A$ is a definable expansion of the empty theory in $L$ and $\text{Th}^{qe}_A$ in $L_A$. Hence $\text{Th}_A$ is interdefinable with the empty theory in $L$, and if $L$ has unbounded arity $\text{Th}_A$ is $\text{Aut}(\mathcal{M})$-universal.

(c) $\text{Th}^{qe}_A \land \text{Th}^*_A$ has Morleyization for $A$.

It is worth noting that in general $\text{Th}_A$ will not have non-redundant quantifier free types. However, if we wish to obtain a universal $\text{Aut}(\mathcal{M})$-theory which both is Morleyized for $A$ and has non-redundant quantifier free types, we can first apply the above to get the theory $\text{Th}_A$ and then apply the transformation to get an interdefinable non-redundant theory. This will result in a universal
Aut($\mathcal{M}$)-theory which is Morleyized for $A$ and which also has non-redundant quantifier free types.

2.4. Invariant Measures. We now introduce the main objects of study in this paper, Aut($\mathcal{M}$)-invariant probability measures. In this subsection $G$ will be a Polish group and $(\circ, X)$ will be a $G$-space.

**Definition 2.29.** Suppose $\mu$ is a measure on $X$. We say $\mu$ is **$G$-invariant** if for all Borel sets $B \subseteq X$ and all $g \in G$

$$\mu(B) = \mu(gB).$$

An important class of invariant measures are the ergodic ones.

**Definition 2.30.** Suppose $\mu \in \mathcal{P}_1(X)$. We say a Borel subset $B \subseteq X$ is $\mu$-a.s. **$G$-invariant** if for every $g \in G$, $\mu(B \triangle g^{-1}B) = 0$. We say $\mu$ is **ergodic** if for every $\mu$-a.s. $G$-invariant Borel set $B$, either $\mu(B) = 0$ or $\mu(B) = 1$.

One of the reasons why ergodic $G$-invariant measures are important is that they are also the extreme ones in the simplex of $G$-invariant measures. The following is standard (see for example [Kal05] Lem. A1.2).

**Lemma 2.31.** For a $G$-invariant measure $\mu$ on $X$ the following are equivalent

- $\mu$ is ergodic.
- $\mu$ is extreme, i.e. is not a non-trivial convex combination of $G$-invariant measures.

The property of being extreme is important because of the following lemma.

**Lemma 2.32 ([Kal05] Thm. A1.3).** Every $G$-invariant measure is a mixture of extreme $G$-invariant measures.

In particular this means that every $G$-invariant measure is a mixture of ergodic $G$-invariant measures.

From the model theoretic point of view one of the most important consequences of ergodicity is that to each ergodic Aut($\mathcal{M}$)-invariant measure on $\mathcal{S}_L(\mathcal{M})$ we can associate a complete consistent $\mathcal{L}_{\omega_1, \omega}(L_\mathcal{M} \cup L)$-theory.

**Definition 2.33.** Suppose $\mu \in \mathcal{P}_1(\mathcal{S}_L(\mathcal{M}))$. Define the **almost sure theory** of $\mu$ to be

$$\text{Th}(\mu) := \{ \tau \in \mathcal{L}_{\omega_1, \omega}(L_\mathcal{M} \cup L) : \mu(\llbracket \tau \rrbracket_{\mathcal{M}}) = 1 \}. $$

**Lemma 2.34.** For any measure $\mu \in \mathcal{P}_1(\mathcal{S}_L(\mathcal{M}))$, Th($\mu$) is consistent.

**Proof.** By $\sigma$-additivity of the measure $\mu$ we have Th($\mu$) must be closed under the rules of deduction of $\mathcal{L}_{\omega_1, \omega}(L)$ in Definition 2.20, i.e. we must have Th($\mu$) = dc(Th($\mu$)). However $\mu(\llbracket (\exists x) x \neq x \rrbracket) = \mu(\emptyset) = 0 \neq 1$ and so Th($\mu$)
is consistent. \hfill \Box_{2.34}

For our purposes we are most interested in the theory of a measure when
the measure is ergodic.

**Lemma 2.35.** If \( \mu \in \mathcal{P}_1(\mathcal{S}_L(M)) \) is ergodic and \( \text{Aut}(M) \)-invariant then
\( \text{Th}(\mu) \) is complete and consistent.

**Proof.** For any sentence \( \tau \in \mathcal{L}_{w_1,\omega}(L) \) we have that \( \llbracket \tau \rrbracket_M \) and \( \llbracket \neg \tau \rrbracket_M \) are
invariant and hence \( \mu(\llbracket \tau \rrbracket_M) \in \{0,1\} \) and \( \mu(\llbracket \neg \tau \rrbracket_M) \in \{0,1\} \). Therefore one
of \( \tau \) or \( \neg \tau \) is in \( \text{Th}(\mu) \) and so \( \text{Th}(\mu) \) is complete.

The consistency of \( \text{Th}(\mu) \) follows from Lemma 2.34. \hfill \Box_{2.35}

We will end this section with a simple but important criteria for when a
function can be extended to an \( \text{Aut}(M) \)-invariant measure on
\( \mathcal{S}_L(M) \).

**Definition 2.36.** For any language \( L \) let \( \text{qf}_\pi(L) \) be the collection of formulas
which are finite conjunctions of atomic and negations of atomic formulas with
parameters in \( \mathbb{N} \).

**Lemma 2.37.** Suppose \( \mu^-: \text{qf}_\pi(L_M \cup L) \to [0,1] \) is such that

(a) For every \( \zeta \in \text{qf}_\pi(L_M) \), if \( M \models \eta \) then \( \mu^-(\eta) = 1 \).

(b) For every \( \zeta \in \text{qf}_\pi(L_M \cup L) \) and every atomic \( L_M \cup L \)-formula \( \eta \) with
parameters from \( \mathbb{N} \),

\[
\mu^-(\zeta) = \mu^-(\zeta \land \eta) + \mu^-(\zeta \land \neg \eta).
\]

Then there is a unique probability measure \( \mu \) on \( \mathcal{S}_L(M) \) such that \( \mu(\llbracket \eta \rrbracket_M) = \mu^-(\eta) \) for all \( \eta \in \text{qf}_\pi(L_M \cup L) \).

Further if \( \mu^-(\zeta(\overline{a})) = \mu^-(\zeta(\overline{b})) \) whenever \( \overline{a}, \overline{b} \) are in the same \( \text{Aut}(M) \)-orbit
then \( \mu \) is \( \text{Aut}(M) \)-invariant.

**Proof.** This follows immediately from the Carathéodory extension theorem. \hfill \Box_{2.37}

Finally the following notion will be important.

**Definition 2.38.** Suppose \( L_0 \subseteq L_1 \) and \( \mu \in \mathcal{P}_1(\mathcal{S}_{L_1}(M)) \). We let \( \mu|_{L_0} \) be the
measure on \( \mathcal{S}_{L_0}(M) \) which agrees with \( \mu \). Note \( \mu|_{L_0} \) is \( \text{Aut}(M) \)-invariant if
\( \mu \) is.

2.5. **Aldous-Hoover-Kallenberg.** In this section we recall the Aldous-Hoover-
Kallenberg theorem which gives a representation for \( \mathcal{S}_N \)-invariant measures.
When \( M \) is free, a notion we will make precise in Definition 3.5, we will be
able to combine an \( \text{Aut}(M) \)-invariant measure \( \mu \) with an explicit \( \mathcal{S}_N \)-invariant
measure \( \vartheta_M \) concentrated on \([\sigma_M]\) to get an \( \mathfrak{S}_N \)-invariant measure. We will then use the Aldous-Hoover-Kallenberg theorem to get a representation of this combined measure from which we will be able to extract a representation of \( \mu \).

One of the definitive sources for the Aldous-Hoover-Kallenberg theorem and related facts is Chapter 7 of [Kal05] which we will use as a reference in this section. However there are two edge cases, Corollary 2.46 and Proposition 2.48, that we will need which, while not technically stated as results in [Kal05], follow immediately with minor modifications of various proofs in [Kal05]. Our proofs of these facts will mention which results in [Kal05] are need to be modified and how, but will leave it to the enthusiastic reader to make the routine modifications.

The following theorem is (a variant of) what is often referred to as the Aldous-Hoover-Kallenberg theorem. See for example [Kal05] Ch. 7.5 or [Aus08].

**Theorem 2.39 (Aldous-Hoover-Kallenberg Theorem).** Let \( \mathcal{X} = (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \) be a collection of random variables such that \( X_\pi \) takes values in a Polish space \( S_\pi \). Then the following are equivalent.

- For all \( \tau \in \mathfrak{S}_N \), \( (X_\tau)_{\pi \in \mathbb{N}^{<\omega}} \overset{d}{=} (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \), i.e. \( (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \) is exchangeable.
- There exists a \( U[0,1] \)-array \((\zeta_\pi)_{\pi \in \mathbb{N}^{<\omega}} \) and a collection of measurable functions \( f_n : [0,1]^\mathfrak{P}(n) \to S_n \) such that
  \[
  (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \overset{d}{=} (f_\pi(\zeta_\pi))_{\pi \in \mathbb{N}^{<\omega}} \text{ a.s.}
  \]

We call \( \langle f_n \rangle_{n \in \mathbb{N}} \) a representation of \( (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \)

Often in the Aldous-Hoover-Kallenberg theorem as stated it is assumed that the array takes values in a single \( S \), possibly \([0,1]\). However, as all standard Borel spaces of the same cardinality are isomorphic, Theorem 2.39 is no more general. By allowing \( X_\pi \) to take values in \( \text{otp}(L[\mathfrak{P}]) \) for some language \( L \), we can assume the array \( (X_\pi)_{\pi \in \mathbb{N}^{<\omega}} \) collectively takes values in \( \mathcal{S}_L \). This then gives us the following equivalent formulation of Theorem 2.39. Note the term *recipe* is due to Austin (see [Aus08]).

**Definition 2.40.** Suppose \( L \) is a language where all relations have arity less than \( N \leq \omega \). Define an \( \mathfrak{S}_N \)-**recipe** to consist of a collection of measurable functions \( \overline{f} := \langle f_n \rangle_{n < N} \) where

\[
f_n : [0,1]^\mathfrak{P}(n) \to \text{otp}(L[\pi]) \quad \text{for} \quad n < N
\]

We let \( \mathcal{M}(\overline{f}) : [0,1]^\mathfrak{P}(N) \to \mathcal{S}_L^\ast \) be the function such that
for all relation $R \in L$ of arity $k$,
• for all $\bar{y} = \langle y_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}_{N}(\mathbb{N})} \in [0, 1]^{\mathcal{P}_{N}(\mathbb{N})}$, and
• for all $\bar{\sigma} = (a_0, \ldots, a_{k-1}) \in \mathbb{N}_{N}$ with $k < N$

\[ \mathcal{M}(\bar{f})(\bar{y}) \models R(a_0, \ldots, a_{k-1}) \] if and only if $R(x_0, \ldots, x_{k-1}) \in f_{\bar{m}}(\bar{y}_{\bar{m}})$. We define the distribution of $\bar{f}$ to be the distribution of $\mathcal{M}(\bar{f})$ (where the domain of $\mathcal{M}(\bar{f})$ is given the Lebesgue measure).

**Proposition 2.41.** Suppose $\mu$ is a measure on $\mathcal{S}^*_L$. Then the following are equivalent.

• $\mu$ is $\mathcal{S}_N$-invariant.
• There exists an $\mathcal{S}_N$-recipe $\bar{f}$ whose distribution is $\mu$.

**Proof.** This follows immediately from Theorem 2.39 by letting the space $S_{\omega}$ be $\text{otp}_{L_n}(x_0, \ldots, x_{n-1})$. \(\square\) 2.41

Also often considered part of the Aldous-Hoover-Kallenberg theorem is a characterization of when two exchangeable arrays have the same distribution. Before we can make this precise we need a definition.

**Definition 2.42.** Suppose $g : [0, 1]^{\mathcal{P}(n)} \to [0, 1]$. We say $g$ preserves $\lambda$ in the highest order arguments if for each $\langle x_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}(n) \backslash \{0, \ldots, n-1\}}$ the map $x_{0, \ldots, n-1} \to g(\langle x_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}(n)})$ preserves $\lambda$.

Similarly, suppose $h : [0, 1]^{\mathcal{P}(n)} \times [0, 1]^{\mathcal{P}(n)} \to [0, 1]$. We say $h$ maps $\lambda^2$ to $\lambda$ in the highest order arguments if for each $\langle x_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}(n) \backslash \{0, \ldots, n-1\}}$, the map $(x_{0, \ldots, n-1}, y_{0, \ldots, n-1}) \to h(\langle x_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}(n)}, \langle y_{\bar{m}} \rangle_{\bar{m} \in \mathcal{P}(n)})$ maps $\lambda^2$ to $\lambda$.

**Theorem 2.43 ([Kal05] Thm. 7.28).** Suppose $N \leq \omega$, $\bar{f}^0 = \langle f^0_{\bar{m}} \rangle_{\bar{m} \in \mathcal{N}_{N}}$ and $\bar{f}^1 = \langle f^1_{\bar{m}} \rangle_{\bar{m} \in \mathcal{N}_{N}}$ are $\mathcal{S}_N$-recipes and $(\eta_{\bar{m}})_{\bar{m} \in \mathcal{P}_{N}(\mathbb{N})}$, $(\eta_{\bar{m}})_{\bar{m} \in \mathcal{P}_{N}(\mathbb{N})}$ are $U[0,1]$-arrays. Then the following are equivalent.

• $(f^0_{\bar{m}}(\widehat{G}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}} \equiv (f^1_{\bar{m}}(\widehat{G}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}}$.
• For each $n < N$ there are functions $g^0_n, g^1_n : [0, 1]^{\mathcal{P}(n)} \to [0, 1]$ which preserve $\lambda$ in the highest order arguments and are such that

\[ (f^0_{\bar{m}}(\widehat{G}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}} = (f^1_{\bar{m}}(\widehat{G}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}} \] a.s.

where $(\widehat{G}^i_{\bar{m}})_{\bar{m} \in \mathcal{N}_{N}} = (\eta_{\bar{m}})_{\bar{m} \in \mathcal{N}_{N}}$.

• For each $n < N$ there is a function $h_n : [0, 1]^{\mathcal{P}(n)} \times [0, 1]^{\mathcal{P}(n)} \to [0, 1]$ which maps $\lambda^2$ to $\lambda$ in the highest order arguments and is such that

\[ (f^0_{\bar{m}}(\bar{G}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}} = (f^1_{\bar{m}}(\bar{H}_{\bar{m}}))_{\bar{m} \in \mathcal{N}_{N}} \] a.s.

where $(\bar{H}_{\bar{m}})_{\bar{m} \in \mathcal{N}_{N}} = (h_{\bar{m}})_{\bar{m} \in \mathcal{N}_{N}}$.
Along with the representation theorem for exchangeable arrays, there is also a characterization of when an exchangeable array has an ergodic distribution. This characterization is both in terms of the possible $S_N$-recipes, as well as in terms of the property of being dissociated.

**Definition 2.44.** Let $X = (X_\pi)_{\pi \in [N]}$ be a collection of random variables. We say $X$ is **dissociated**, if for every disjoint pair of tuples $\pi$ and $\tau$ the random variables $\hat{X}_\pi$ and $\hat{X}_\tau$ are independent. We say a measure is dissociated if it is the distribution of a collection of dissociated random variables.

**Theorem 2.45** ([Kal05] Lem. 7.35). Suppose $\mu$ is an $S_N$-invariant measure on $S^*_L$ where the arities of $L$ are bounded by $N < \omega$. Then the following are equivalent.

- $\mu$ is ergodic.
- $\mu$ is dissociated.
- There is an $S_N$-recipe $\mathcal{F} = \langle f_n \rangle_{n < N}$ such that
  - $\mathcal{M}(\mathcal{F})$ has distribution $\mu$
  - For all $n < N$, $f_n$ does not depend on the coordinate with $\emptyset$-index.

The next corollary follows from Theorem 2.45 as well as from an argument which is essentially the same as that used in its proof.

**Corollary 2.46.** Suppose $L$ has unbounded arity and $\mu$ is an $S_N$-invariant measure on $S^*_L$. Then the following are equivalent.

1. $\mu$ is ergodic.
2. $\mu$ is dissociated.
3. There is an $S_N$-recipe $\mathcal{F} = \langle f_n \rangle_{n \in \mathbb{N}}$ such that
   - $\mathcal{M}(\mathcal{F})$ has distribution $\mu$,
   - For all $n \in \mathbb{N}$, $f_n$ does not depend on the coordinate with $\emptyset$-index.

**Proof.** First we show the equivalence of (1) and (3). Suppose (1) holds. Then for each $n \in \mathbb{N}^*$ we have that $\mu|_{L^n}$ is ergodic as well. By Theorem 2.45 we have an $S_N$-recipe $\mathcal{F}^N = \langle f_n^N \rangle_{n < N}$ for each $\mu|_{L^N-1}$ such that each $f_n^N$ does not depend on the variable indexed by $\emptyset$.

In order to combine these $S_N$-recipes into a single $S_N$-recipe for $\mu$ which satisfies (3) we need to proceed as in Lemma 7.21 of [Kal05] but with the modifications suggested in the last paragraph of the proof of Theorem 7.22 on p. 328 of [Kal05] and beginning the induction at $k = 1$.

Next assume (3) holds. Because $S^*_L$ is a Radon space $\mu$ is inner regular. For every measurable set $I$ we can therefore find a measurable set $A_\epsilon$ such that $\mu(I \Delta A_\epsilon) < \epsilon$ and which only depends on $\{0, \ldots, n - 1\}$ (for some $n$). The proof then follows Lemma 7.35 (iii) $\Rightarrow$ (i) of [Kal05].

Now we show the equivalence of (1) and (2). First note that the Borel $\sigma$-algebra on $S^*_L$ is generated by sets of the form $\llbracket \varphi(\pi) \rrbracket$ where $\varphi$ is quantifier-free.
and first order. Therefore, \( \mu \) is dissociated if and only if \( \mu|_{L_0} \) is dissociated for every finite \( L_0 \subseteq L \).

Now suppose (1) does not hold. Then there must be some finite \( L_0 \subseteq L \) such that \( \mu|_{L_0} \) is not ergodic hence, by Theorem 2.45, not dissociated. But this then implies \( \mu \) is not dissociated and so (2) does not hold.

Finally, suppose (1) does hold. Then (3) holds as well. But then for every finite \( L_0 \subseteq L \) there is a \( \mathcal{S}_N \)-recipe representing \( \mu|_{L_0} \) which does not depend on the coordinates with \( \emptyset \)-index. Therefore, once again by Theorem 2.45, for every finite \( L_0 \subseteq L \), \( \mu|_{L_0} \) is dissociated. But this then implies \( \mu \) is dissociated and so (2) holds.

This result motivates the next definition.

**Definition 2.47.** We say an \( \mathcal{S}_N \)-recipe \( f = \langle f_n \rangle_{n < N} \) (for \( N \leq \omega \)) is ergodic if it doesn’t depend on the coordinate indexed by \( \emptyset \).

Note that there may be \( \mathcal{S}_N \)-recipes which are not ergodic but whose distributions are, as not all representations of an ergodic measure satisfy the conditions of Corollary 2.46. However the distribution of any ergodic \( \mathcal{S}_N \)-recipe is ergodic and by Corollary 2.46 every ergodic \( \mathcal{S}_N \)-invariant measure is the distribution of some ergodic \( \mathcal{S}_N \)-recipe.

**Proposition 2.48.** Suppose \( N \leq \omega \), \( \mathcal{F}^0 = \langle f^0_n \rangle_{n < N} \) and \( \mathcal{F}^1 = \langle f^1_n \rangle_{n < N} \) are ergodic \( \mathcal{S}_N \)-recipes and \( (\zeta_n)_{\pi \in \mathcal{P}_N} \), \( (\eta_n)_{\pi \in \mathcal{P}_N} \) are \( U[0, 1] \)-arrays. Then the following are equivalent.

\begin{itemize}
  \item \( (f^0_n(\zeta))_{\pi \in \mathcal{P}_N} \downarrow (f^1_n(\zeta))_{\pi \in \mathcal{P}_N} \).
  \item For each \( n < N \) there are functions \( g^0_n, g^1_n : [0, 1]^{\mathcal{P}(n)} \rightarrow [0, 1] \) which preserve \( \lambda \) in the highest order arguments, which don’t depend on the argument indexed by \( \emptyset \) and are such that
    \[
    (f^0_n(G^0_n(\zeta)))_{\pi \in \mathcal{P}_N} = (f^1_n(G^1_n(\zeta)))_{\pi \in \mathcal{P}_N} \text{ a.s.}
    \]
    where \( (G^i_n)_{\pi \in \mathcal{P}_N} = (g^i_n)_{\pi \in \mathcal{P}_N} \).
  \item For each \( n < N \) there is a function \( h_n : [0, 1]^{\mathcal{P}(n)} \times [0, 1]^{\mathcal{P}(n)} \rightarrow [0, 1] \) which maps \( \lambda^2 \) to \( \lambda \) in the highest order arguments, which don’t depend on the argument indexed by \( \emptyset \) and are such that
    \[
    (f^0_n(\zeta))_{\pi \in \mathcal{P}_N} = (f^1_n(H_n(\zeta)))_{\pi \in \mathcal{P}_N} \text{ a.s.}
    \]
    where \( (H_n)_{\pi \in \mathcal{P}_N} = (h_n)_{\pi \in \mathcal{P}_N} \).
\end{itemize}

**Proof.** This proof follows closely that of Theorem 7.28 of [Kal05]. \( \square \)
2.6. **Existence of an $S_N$-Invariant Measure.** The following is an important component of our classification in Proposition 4.6 of those sentences of $L_{\omega_1,\omega}(L)$ which admit an $\text{Aut}(\mathcal{M})$-invariant measures, i.e. those sentences $\tau$ for which there is an $\text{Aut}(\mathcal{M})$-invariant measure concentrated on $\llbracket \tau \rrbracket_{\mathcal{M}}$.

**Theorem 2.49 ([AFPa]).** For a sentence $\tau \in L_{\omega_1,\omega}(L)$ the following are equivalent

- There is an $S_N$-invariant measure concentrated on $\llbracket \tau \rrbracket$.
- There is an ergodic $S_N$-invariant measure concentrated on $\llbracket \tau \rrbracket$.
- For all countable fragments $A \subseteq L_{\omega_1,\omega}(L)$ there is an $A$-theories $T$ with trivial dcl containing $\tau$.
- There is a countable fragment $A \subseteq L_{\omega_1,\omega}(L)$ and an $A$-theory $T$ with trivial dcl containing $\tau$.

An immediate consequence of Theorem 2.49 is the following.

**Theorem 2.50 ([AFP16] Thm. 1.1).** For an $L$-structure $\mathcal{M}$ the following are equivalent

- There is an $S_N$-invariant measure concentrated on $\llbracket \sigma_{\mathcal{M}} \rrbracket$.
- $\mathcal{M}$ has trivial dcl.

### 3. Free Structures

In this section we introduce the abstract notion of a *canonical structure* and we show that it corresponds, up to relabeling of the language, with being the canonical structure of a closed subgroup of $S_N$. We then introduce the notion of a free canonical structure and show that every canonical structure has a minimal free extension (in a sense we make precise).

#### 3.1. Canonical Structures.

**Definition 3.1.** We say a structure $\mathcal{M}$ is **canonical** (for a language $L_\mathcal{M}$) if

- $\mathcal{M}$ is ultrahomogeneous,
- $\mathcal{M}$ is non-redundant,
- for all relation $R \in L_\mathcal{M}$ there is a tuple $\overline{a} \in \mathcal{M}$ such that $\mathcal{M} \models R(\overline{a})$, and
- for all $n \in \mathbb{N}$ and all $n$-tuples $\overline{a} \in \mathcal{M}$ with distinct entries there is a unique $n$-ary relation $R \in L_\mathcal{M}$ such that $\mathcal{M} \models R(\overline{a})$.

**Lemma 3.2.** The following are equivalent for an $L_\mathcal{M}$-structure $\mathcal{M}$.

- $\mathcal{M}$ is canonical,
- there is a relabeling $i: L_\mathcal{M} \to L_{\text{Aut}(\mathcal{M})}$ such that for any relation $R \in L_\mathcal{M}$ and tuple $\mathbf{n} \in \mathbb{N}$, $\mathcal{M} \models R(\mathbf{n})$ if and only if $\mathcal{C}_{\text{Aut}(\mathcal{M})} \models i(R)(\mathbf{n})$, i.e. $\mathcal{M}$ and $\mathcal{C}_{\text{Aut}(\mathcal{M})}$ are the same structure up to a relabeling of the language.
Proof. It is clear that for any group $G$, $\mathcal{C}_G$ is canonical. In the other direction if $\mathcal{M}$ is canonical, then for any relation $R \in L$ and any tuple $\bar{b}$ such that $\mathcal{M} \models R(\bar{b})$, we have $\{ \bar{a} : (\exists g \in \text{Aut}(\mathcal{M}))g\bar{a} = \bar{b} \} = \{ \bar{a} : \mathcal{M} \models R(\bar{a}) \}$ (as there is a unique relation holding of any tuple with distinct elements and $\mathcal{M}$ is ultrahomogeneous).

When $i_0, \ldots, i_{k-1} < n$ we will use

$$R(x_0, \ldots, x_{n-1})|_{(x_{i_0}, \ldots, x_{i_{k-1}})} = P$$

as shorthand for the statement

$$(\forall x_0, \ldots, x_{n-1}) [R(x_0, \ldots, x_{n-1}) \rightarrow P(x_{i_0}, \ldots, x_{i_{k-1}})].$$

If $\mathcal{M}$ is a canonical $L$-structure, whenever

$$\mathcal{M} \models (\exists x_0, \ldots, x_{n-1}) [R(x_0, \ldots, x_{n-1}) \land P(x_{i_0}, \ldots, x_{i_{k-1}})]$$

we also have

$$\mathcal{M} \models R(x_0, \ldots, x_{n-1})|_{(x_{i_0}, \ldots, x_{i_{k-1}})} = P$$

and hence for all $R(x_0, \ldots, x_{n-1})$ and $i_0, \ldots, i_{k-1} < n$ there is a unique $P \in L$ such that $\mathcal{M} \models R(x_0, \ldots, x_{n-1})|_{(x_{i_0}, \ldots, x_{i_{k-1}})} = P$. We call $P$ the restriction of $R(x_0, \ldots, x_{n-1})$ to $(x_{i_0}, \ldots, x_{i_{k-1}})$.

There is a natural notion of when one canonical structure is contained in another.

**Definition 3.3.** If $\mathcal{M}$ is an ultrahomogeneous $L$-structure let $\text{Age}(\mathcal{M})$ be the class of all finite $L$-structures isomorphic to a finite substructure of $\mathcal{M}$.

**Definition 3.4.** Suppose $\mathcal{M}_0, \mathcal{M}_1$ are canonical structures in languages $L_0, L_1$ respectively. We say $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$ if

(i) $L_0 \subseteq L_1$.

(ii) For all $A_0 \in \text{Age}(\mathcal{M}_0)$ there is an $A_1 \in \text{Age}(\mathcal{M}_1)$ such that $A_0 = A_1|_{L_0}$.

In particular, as $\mathcal{M}_0$ and $\mathcal{M}_1$ are canonical such an $A_1$ is unique.

We say $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$ if there is a map of languages $i : L_0 \rightarrow L_1^*$ such that $i(\mathcal{M}_0) \subseteq_{\text{can}} \mathcal{M}_1$.

We say $\mathcal{M}_0 \preceq_{\text{can}} \mathcal{M}_1$ if there is a relabeling $i : L_0 \rightarrow L_0^*$ such that $i(\mathcal{M}_0) \subseteq_{\text{can}} \mathcal{M}_1$.

Note because our structures are canonical the $A_1$ in condition (ii) is unique and is trivial on $L_1 \setminus L_0$, i.e. $A_1 \models (\forall \bar{a})(\forall \bar{y}) \neg R(\bar{a}, \bar{y})$ for any $R \in L_1 \setminus L_0$.

An important fact about canonical structures is that if $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$ then every $\text{Aut}(\mathcal{M}_1)$-invariant measure on $\mathcal{S}_L^*(\mathcal{M}_1)$ restricts to an $\text{Aut}(\mathcal{M}_0)$-invariant measure on $\mathcal{S}_L^*(\mathcal{M}_0)$. Specifically, suppose $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$ and $\mu \in \mathcal{S}_L^*(\mathcal{M}_1)$. Let $\mu_0$ the map such that whenever
• $P \in L_{\mathcal{M}_0}$,
• $\mathcal{M}_0 \models P(n_0, \ldots, n_{k-1})$,
• $\mathcal{M}_1 \models P(m_0, \ldots, m_{k-1})$, and
• $\eta \in qf(\pi(L))$ with parameters contained in $\{n_0, \ldots, n_{k-1}\}$
then $\mu_0(\eta(n_0, \ldots, n_{k-1})) = \mu(\eta(m_0, \ldots, m_{k-1}))$.

Note that as $\mu$ is $\text{Aut}(\mathcal{M}_1)$ invariant, the value of $\mu_0$ is independent of the specific choice of $(m_0, \ldots, m_{k-1})$, so long as the tuple satisfies $P$.

By Lemma 2.37 there is then a unique $\text{Aut}(\mathcal{M}_0)$-invariant measure extending $\mu_0$, which we call the restriction of $\mu$ to $\mathcal{M}_0$ and denote $\mu|_{\mathcal{M}_0}$.

3.2. Free Completions and Invariant Measures on Free Structures.

For our purposes we will be interested in a very specific type of canonical structure.

**Definition 3.5.** Suppose $\mathcal{M}$ is a canonical structure. For $n \in \mathbb{N}$ let $X := \{x_k\}_{k \in [n+1]}$ be a set of distinct variables and let $(\overline{x}_i)_{i \in [m]}$ be a collection of distinct sequences of elements of $X$ such that every element of $X$ is in at least one sequence. Further let $\overline{x}$ be an enumeration of $X$.

We say a collection $\{R_i(\overline{x}_A)\}_{i \in [m]}$ is compatible with $\mathcal{M}$ if

• whenever $\overline{y}$ is a sequence of distinct elements contained in $\overline{x}_i$ for some $i \in [m]$ there is a $j \in [m]$ such that $\overline{x}_j = \overline{y}$, and
• for all $i, j \in [m]$, if $\overline{y}$ is a sequence of distinct elements contained in both $\overline{x}_i$ and $\overline{x}_j$ then $\mathcal{M} \models R_i(\overline{x}_i)|_{\overline{y}} = R_j(\overline{x}_j)|_{\overline{y}}$.

We say $X$ is the collection of free variables of $\overline{x}$.

We say an atomic formula $R^*(\overline{x})$ of arity $n+1$ is an extension of $\{R_i(\overline{x}_A)\}_{i \in [m]}$ if for all $i \in [m]$, $\mathcal{M} \models R^*(\overline{x})|_{\overline{x}_i} = R_i$. We say $\{R_i(\overline{x}_i)\}_{i \in [m]}$ is total if it contains an extension of itself.

We say that $\mathcal{M}$ is free if all compatible collections have an extension.

If $\mathcal{M}$ is not canonical, we say $\mathcal{M}$ is free if $\mathcal{C}_{\text{Aut}(\mathcal{M})}$ is free. A free structure can be thought of as a structure where any way of amalgamating types is consistent so long as it is locally consistent.

**Example 3.6.** The quintessential example of a free structure is the Rado graph, $\mathcal{R}$. The canonical structure $\mathcal{C}_{\text{Aut}(\mathcal{R})}$ is the structure where for every finite graph $G$ there is a relation $R_G$ which holds exactly when the parameters form a graph isomorphic to $G$.

We also have the following example of a canonical structure which is not free.

**Example 3.7.** Let $\mathcal{I}$ be the triangle free random graph. The canonical structure of $\mathcal{I}$ is the structure where for every finite triangle free graph $G$
there is a relation $R_G$ which holds exactly when the parameters form a graph isomorphic to $G$.

This structure is not free. To see this let $E$ be the graph with two elements and an edge between them. Then $\{E(x_0, x_1), E(x_1, x_2), E(x_2, x_0)\}$ is a compatible collection which is not the restriction of any relation in the canonical structure of $T$. This is a quintessential example of how a canonical structure can fail to be free.

Even though not all canonical structures are free, every canonical structure is contained in a free canonical structure. Further we can find a minimal such free extension.

**Lemma 3.8.** Suppose $\mathcal{M}$ is a canonical $L_\mathcal{M}$-structure. Then there is a canonical structure $\mathfrak{F}(\mathcal{M})$ such that

- $\mathfrak{F}(\mathcal{M})$ is free,
- $\mathcal{M} \subseteq_{can} \mathfrak{F}(\mathcal{M})$, and
- whenever $\mathcal{M} \subseteq_{can} \mathcal{N}$ and $\mathcal{N}$ is free then $\mathfrak{F}(\mathcal{M}) \subseteq_{can} \mathcal{N}$.

We call $\mathfrak{F}(\mathcal{M})$ the **free completion** of $\mathcal{M}$.

**Proof.** For $n \in \omega$ let $Y$ be the set of collections compatible with $\mathcal{M}$ which are not total. Let $L_{\mathfrak{F}(\mathcal{M})} = L_\mathcal{M} \cup \{P_R : R \in Y\}$ where each $P_R$ is a new relation in $L_\mathcal{M}$ of the same arity as $R$. Let $T$ be the theory where

- $R(\overline{x}) |_{\overline{y}} = P \in T$ for all $R \in L_\mathcal{M}$ and $P \in L_\mathcal{M}$ such that $\mathcal{M} \models R(\overline{x}) |_{\overline{y}} = P$, and

for $\overline{R} = \{R_i(\overline{x}_i)\}_{i \in [m]} \in Y$ of arity $[n]$ with $X$ the set free variables,

- $P_{\overline{R}}(\overline{x}) |_{\overline{y}} = R_i \in T$,
- $P_{\overline{R}}(\overline{x}) |_{\overline{y}} = P \in T$ when there is an $i \in [m]$ such that $\overline{y} \subseteq \overline{x}_i$ and $R_i(\overline{x}_i) |_{\overline{y}} = P \in T$,
- if $\sigma : X \to X$ is a bijection and $\sigma(\overline{R}) = \{R_i(\sigma(\overline{x}_i))\}_{i \in [m]}$ then $\overline{R}(\overline{x}) |_{\sigma(\overline{x})} = \sigma\overline{R} \in T$, and
- whenever
  - $Y \subseteq X$ and $\overline{y}$ is the subtuple of $\overline{x}$ containing the elements of $Y$, and
  - $\mathfrak{F}(\overline{y})$ is the collection of those $R_i(\overline{x}_i)$ such that all elements of $\overline{x}_i$ contained in $Y$, then $P_{\overline{R}}(\overline{y}) |_{\overline{y}} = P_{\overline{y}} \in T$.

For notational convenience if $\overline{R}$ is total of arity $[n]$ with $X$ the set free variables, $\overline{x}$ is an enumeration of $X$ and $R(\overline{x}) \in \overline{R}$ then we let $P_{\overline{R}} = R$.

Let $\text{Age}(\mathfrak{F}(\mathcal{M}))$ be the collection of finite $L_{\mathfrak{F}(\mathcal{M})}$-structures $A$ such that $A \models T$. Note it is easy to check that for each $A \in \text{Age}(\mathfrak{F}(\mathcal{M}))$ and tuple $\overline{a} \in A$ of distinct elements there is a unique relation $R \in L_{\mathfrak{F}(\mathcal{M})}$ such that $A \models R(\overline{a})$. 


For $A \in \text{Age}(\mathfrak{F}(\mathcal{M}))$ with enumeration $\pi$ let $\overline{R}_\pi = \{R_i(\overline{x}_i)\}_{i \in [m]}$ be the collection of relations such that $R_i \in L_M$ and $\overline{x}_i \in A$. Note that $A \models P_{\overline{R}_\pi}(\pi)$. Further, if $B \in \text{Age}(\mathfrak{F}(\mathcal{M}))$ with enumeration $\overline{b}$ and $B \models P_{\overline{R}_\pi}(\overline{b})$ then the map $\pi \mapsto \overline{b}$ is an isomorphism of $A$ and $B$.

Claim 3.9. $\text{Age}(\mathfrak{F}(\mathcal{M}))$ has the hereditary property (HP), the joint embedding property (JEP), and the disjoint amalgamation property (DAP).

Proof. It is immediate that $\text{Age}(\mathfrak{F}(\mathcal{M}))$ has the HP and JEP. We now show it has the DAP. Suppose $X, Y \in \text{Age}(\mathfrak{F}(\mathcal{M}))$ are such that $X|_{X \cap Y} = Y|_{X \cap Y}$ and $\overline{\pi}$ is an enumeration of $X \cup Y$. Let $\overline{R}_{X,Y} = \{R_i(\overline{x}_i)\}_{i \in [m]}$ be the collection of relations such that $R_i \in L_M$ and either $\overline{x}_i \in X$ and $X \models R_i(\overline{x}_i)$, or $\overline{x}_i \in Y$ and $Y \models R_i(\overline{x}_i)$. It is easy to check that $\overline{R}_{X,Y}$ is a collection compatible with $\mathcal{M}$. Further, if $Z \in \text{Age}(\mathfrak{F}(\mathcal{M}))$ is the structure with underlying set $X \cup Y$ such that $\overline{R}_\pi = P_{\overline{R}_{X,Y}}$ then $Z$ is an amalgamation of $X$ and $Y$. \hfill \square_{3.9}

As $\text{Age}(\mathfrak{F}(\mathcal{M}))$ satisfies (HP), (JEP), and (DAP) there is a unique (up to isomorphism) countable structure which is its Fraïssé limit. We denote this structure by $\mathfrak{F}(\mathcal{M})$. Note that by construction of $\text{Age}(\mathfrak{F}(\mathcal{M}))$ we have that $\mathfrak{F}(\mathcal{M})$ is free and canonical. All that is left is to show that $\mathfrak{F}(\mathcal{M})$ is $\succeq_{\text{can}}$-minimal among free structures containing $\mathcal{M}$.

Suppose $\mathcal{M} \prec_{\text{can}} \mathcal{N}$ with $\mathcal{N}$ free with the corresponding injection $i^*: L_M \to L_N$. We will define our map $i: L_{\mathfrak{F}(\mathcal{M})} \to L_\mathcal{N}$ by induction. Note $i$ will extend $i^*$.

First, as $L^1_M = L^1_{\mathfrak{F}(\mathcal{M})}$ we let $i(U) = i^*(U)$ for all $U \in L^1_{\mathfrak{F}(\mathcal{M})}$. Suppose $i$ has been defined for all relations of arity at most $n$ and let $Q$ be a relation of arity $n + 1$. If $Q \in L_M$ let $i(Q) = i^*(Q)$.

Suppose $Q \in L_{\mathfrak{F}(\mathcal{M})} \setminus L_M$ and let $\overline{R} = \{R_i(\overline{x}_i)\}_{i \in [m]}$ be such $X$ is the set of free variables, $\overline{\pi}$ is an enumeration $X$ with $Q(\overline{\pi}) = P_{\overline{R}}(\overline{\pi})$. For $y \in X$ let $\overline{R}^y$ be the collection of those $R_i(\overline{x}_i)$ with $\overline{x}_i \subseteq X \setminus \{y\}$ and let $\overline{\pi}_y$ be the corresponding enumeration of $X \setminus \{y\}$. Note that $\{\overline{R}^y(\overline{x}_y)\}_{y \in X}$ is a compatible collection with respect to $\mathfrak{F}(\mathcal{M})$ all of whose relations have arity $< n$. Therefore $i(P_{\overline{R}^y}(\overline{\pi}_y)\}_{y \in X}$ is a compatible collection with respect to $\mathcal{N}$. But $\mathcal{N}$ is free, and so there must be some extension $Q^+$ of $\{i(P_{\overline{R}^y}(\overline{\pi}_y)\}_{y \in X}$. Let $i(Q) = Q^+$. \hfill \square_{3.8}

If $\mathcal{M}$ is not a canonical structure, then we define $\mathfrak{F}(\mathcal{M}) := \mathfrak{F}(\mathcal{C}_{\text{Aut}(\mathcal{M})})$. It is worth noting that if $\mathcal{M} \subseteq_{\text{can}} \mathcal{N}$ then the corresponding map of languages that witnesses $\mathfrak{F}(\mathcal{M}) \subseteq_{\text{can}} \mathcal{N}$ is not in general unique. However for our purposes this will not be important.

Lemma 3.10. If $\mathcal{M}$ is free then $\mathcal{M}$ has trivial dcl.
where for all \( Z \) there are infinitely many elements of \( M \) of \( \text{Arity } n \) of \( \{ R \} \) induced by that of the language. Note that as \( M \) is a natural \( \mathcal{S}_N \)-invariant measure concentrated on its isomorphism class.

**Definition 3.11.** Suppose \( M \) is a free canonical structure with a chosen ordering on \( L^n_M \) for each \( n \in \mathbb{N} \). We define a **uniform representation** on \( M \) to be an \( \mathcal{S}_N \)-recipe of the following form.

**Arity 1:**
Let \( L^1 \) be the increasing enumeration of \( L^1_M \) under the chosen ordering. Define \( c^M_1 : [0, 1]^2 \to \text{otp}_{L^1_M}(x_0) \) where \( c^M_1(y, z) \) is the type containing \( \gamma_{L^1}(z)(x_0) \). Note the first coordinate is not used. This is representation we are defining is ergodic.

Note that as \( M \) has trivial dcl and is canonical, for any unary relation \( U \) there are infinitely many elements of \( M \) which satisfy \( U \). Therefore \( M(c^M_1) \cong M|_{L^1_M} \) a.s.

**Arity \( n+1 \):**
Assume that \( M(\langle c^M_i \rangle_{i \leq n}) \cong M|_{L^{\leq n}_M} \) a.s.

Suppose \( \overline{R} = \langle R_i(\overline{x}_i) \rangle_{i \in [n]} \) is compatible with \( M \) with free variables \( Z \) and where for all \( Z^* \subseteq Z \) there is some \( i \in [n] \) such that \( \overline{x}_i \) enumerates \( Z^* \). Then let \( X_{\overline{R}} \) be the collection of extensions of \( \overline{R} \) in \( M \) and assume \( X_{\overline{R}} \) has the order induced by that of the language. Note \( X_{\overline{R}} \) is non-empty as \( M \) is free.

Given \( \overline{x} := \langle x_i \rangle_{i \in \Psi(n+1)} \in [0, 1]^{\Psi(n+1)} \) let \( \overline{x}_i := \langle x_i \rangle_{i \in \Psi(n+1) \setminus \{i\}} \). Further let \( \overline{R}^* := \langle c^M_n(\overline{x}_i) \rangle_{i \in [n+1]} \). By induction \( \overline{R}^* \) is a.s. a compatible collection (as \( M(\langle c^M_i \rangle_{i \leq n}) \cong M|_{L^{\leq n}_M} \)). Let \( Y_{n+1}(\overline{x}) = X_{\overline{R}^*} \), i.e. the collection of extensions of \( \overline{R}^* \). Let \( c^M_{n+1}(\overline{x}) = \gamma_{Y_{n+1}}(\overline{x})(x_0, \ldots, x_n) \). As \( M \) has trivial dcl it is easy to check that \( M(\langle c^M_i \rangle_{i \leq n+1}) \cong M|_{L^{\leq n+1}_M} \) a.s.
We let $\bar{\mathcal{M}} := (\mathcal{M}_n)_{n \in \mathbb{N}}$ and call it the Erdős-Rényi random $\mathcal{M}$-structure in analog with the Erdős-Rényi random graph.

The random structure $\mathcal{M}(\bar{\mathcal{M}})$ can be thought of as first assigning to each element of $\mathbb{N}$ a unary relation in an i.i.d. manner. Then assigning a binary relation to every pair of elements in a manner which is i.i.d. conditioned on the unary types that were previously assigned. Then assigning to each triple a ternary relation in a manner which is i.i.d. conditioned on the binary types that were defined, etc.

**Lemma 3.12.** The distribution of $\mathcal{M}(\bar{\mathcal{M}})$ is ergodic.

**Proof.** This follows immediately from Corollary 2.46 and the definition of $\bar{\mathcal{M}}$. □

**Lemma 3.13.** The distribution of $\mathcal{M}(\bar{\mathcal{M}})$ concentrates on $[\sigma_\mathcal{M}]$.

**Proof.** First the quantifier free types realized by $\mathcal{M}(\bar{\mathcal{M}})$ are the same as those realized in $\mathcal{M}$ almost surely. Let $\vartheta_\mathcal{M}$ be the distribution of $\mathcal{M}(\bar{\mathcal{M}})$.

Suppose $\mathcal{M} \models R(\overline{x}, y)|_\mathcal{M} = P$. By a back and forth argument it suffices to show that $\mathcal{M}(\bar{\mathcal{M}})$ satisfies $(\forall \overline{x})P(\overline{x}) \rightarrow (\exists y)R(\overline{x}, y)$ a.s. or equivalently that $\vartheta_\mathcal{M}(\llbracket (\forall \overline{x})P(\overline{x}) \rightarrow (\exists y)R(\overline{x}, y) \rrbracket) = 1$.

But as $\vartheta_\mathcal{M}$ is countably additive and $\mathcal{G}_\mathcal{M}$-invariant it therefore suffices to show $\vartheta_\mathcal{M}(\llbracket P(0, \ldots, k - 1) \rightarrow (\exists y)R(0, \ldots, k - 1, y) \rrbracket) = 1$ where $ar(P) = k$.

Once again using countable additivity and $\mathcal{G}_\mathcal{M}$-invariance it suffices to show $\vartheta_\mathcal{M}(\llbracket P(0, \ldots, k - 1) \wedge R(0, \ldots, k - 1, k) \rrbracket) > 0$. But by construction we know not only that $\vartheta_\mathcal{M}(\llbracket R(0, \ldots, k - 1, k) \rightarrow P(0, \ldots, k - 1) \rrbracket) = 1$ but also that $\vartheta_\mathcal{M}(\llbracket R(0, \ldots, k - 1, k) \rrbracket) > 0$, and so we are done. □

The representations $\bar{\mathcal{M}}$ will play an important role in showing that all $\text{Aut}(\mathcal{M})$-invariant measures, for $\mathcal{M}$ free, are representable. First though we will need a little more notation.

**Definition 3.14.** For all $k \geq n$ and $p \in \text{otp}_{\mathcal{M}}(x_0, \ldots, x_{k - 1})$ define

$$S_p := \{ \overline{x} \in [0, 1]^{\mathbb{P}_{\leq n}(k)} : (\exists \overline{y}). \mathcal{M}(\bar{\mathcal{M}})(\overline{x}, \overline{y}) \models p(0, \ldots, k - 1) \}$$

**Arity 1:**

For each $U(x_0) \in \text{otp}_{\mathcal{M}}(x_0)$ let $S^U_1 = (c_1^\mathcal{M})^{-1}(U(x_0))$. As $S_U$ is the product of $[0, 1]$ with an interval, let $\alpha_U : [0, 1] \times [0, 1] \rightarrow S_U$ be a homeomorphism which doesn’t depend on the first argument such that for every set $X \subseteq S_U$, $\lambda(X)/\lambda(S_U) = \lambda(\alpha_U^{-1}(X))$.

Now suppose $p = \bigwedge_{i \in [k]} U_i(x_i)$ is an element of $\text{otp}_{\mathcal{M}}(x_0, \ldots, x_{k - 1})$. We let $\alpha_p : [0, 1] \times [0, 1]^k \rightarrow S_p$ be the map where $\alpha_p(y, \langle x_i \rangle_{i < k}) = \langle \alpha_{U_i(y, x_i)} \rangle_{i < k}$. 
Arity \( n + 1 \):

Assume \( k \geq n \) and for all \( p \in \text{otp}_{\mathcal{M}}(x_0, \ldots, x_{k-1}) \) we have defined the map \( \alpha_p: [0, 1]^{|\mathcal{M}|} \to S_p \) which is a bijection that doesn’t depend on the coordinate indexed by \( \emptyset \).

Let \( p \in \text{otp}_{\mathcal{M}}(x_0, \ldots, x_n) \) and let \( p^- \) be the restriction of \( p \) to the space \( \text{otp}_{\mathcal{M}}(x_0, \ldots, x_n) \). Let \( S_p = (c_{n+1})^{-1}(p(\mathcal{F})) \). Then because of how \( c_{n+1} \) was defined we know that \( S_p = S_p^- \times I \) where \( I \) is a subinterval of \([0, 1]\). Let \( i: [0, 1] \to I \) be an isomorphism such that for every set \( X \subseteq I \), \( \lambda(X)/\lambda(I) = \lambda(i^{-1}(X)) \). Let \( \alpha_p = \alpha_p^- \times i \).

We then extend the definition of \( \alpha_p \) to the case where \( p \in \text{otp}_{\mathcal{M}}(x_0, \ldots, x_{k-1}) \) and \( k > n + 1 \) in the obvious way.

Note that for each \( p \in \text{otp}(\mathcal{M}) \) of arity \( n \) we have \( p \) only depends on \( \mathcal{M} \) and so the following holds

- If \( X \subseteq S_p \) then \( \lambda(X)/\lambda(S_p) = \lambda(\alpha_p^{-1}(X)) \).
- For any \( n_0 < n \) if \( q_{n_0} \) is the restriction of \( p \) to \( \mathcal{M} \) then \( S_p = S_q \times I_{p,q} \) where \( I_{p,q} \) is the product of intervals. Further \( \alpha_p \) restricts to \( \alpha_q \) on \( S_q \).

### 4. Merging of Measures

One of the key observations of this paper is that if \( \text{Aut}(\mathcal{M}) \subseteq \text{Aut}(\mathcal{N}) \) then we can combine any \( \text{Aut}(\mathcal{M}) \)-invariant measure concentrated on \( \llbracket \sigma_{\mathcal{M}} \rrbracket_{\mathcal{N}} \) with any \( \text{Aut}(\mathcal{M}) \)-invariant measure, in a unique way, to get an \( \text{Aut}(\mathcal{N}) \)-invariant measure which agrees with both. Further every \( \text{Aut}(\mathcal{N}) \)-invariant measure concentrated on \( \llbracket \sigma_{\mathcal{M}} \rrbracket_{\mathcal{N}} \) is of this form.

In particular when \( \mathcal{M} \) is a structure with underlying set \( \mathbb{N} \) and trivial dcl we will be able reduce the question of “for which \( \varphi \in \mathcal{L}_{\omega_1, \omega}(\mathcal{L}) \) is there an \( \text{Aut}(\mathcal{M}) \)-invariant measure on \( \mathcal{S}_L(\mathcal{M}) \) which concentrates on \( \llbracket \varphi \rrbracket_{\mathcal{M}} ? \)” to the question of “for which \( \varphi \in \mathcal{L}_{\omega_1, \omega}(\mathcal{L}) \) is there an \( \mathcal{S}_{\mathcal{N}} \)-invariant measure which concentrates on \( \llbracket \varphi \land \sigma_{\mathcal{M}} \rrbracket ? \), a question which has been completely answered by Ackerman, Freer and Patel in \[\text{AFPa}\].

Further we will be able to use this method of combining measures to show that when \( \mathcal{M} \) is free every \( \text{Aut}(\mathcal{M}) \)-invariant measure can be combined with the distribution of the Erdős-Rényi random \( \mathcal{M} \)-structure to get an \( \mathcal{S}_{\mathcal{N}} \)-invariant measure from whose representation we can extract a representation for the \( \text{Aut}(\mathcal{M}) \)-invariant measure we started with.

Suppose

- \( \mathcal{M} \) and \( \mathcal{N} \) are canonical structures with underlying set \( \mathbb{N} \) and \( \text{Aut}(\mathcal{M}) \subseteq \text{Aut}(\mathcal{N}) \).
- \( \nu_M \) is an \( \text{Aut}(\mathcal{N}) \)-invariant measure on \( \mathcal{S}_{L,M}^* (\mathcal{N}) \) which is concentrated on \( \llbracket \sigma_M \rrbracket \).
- \( \mu \) is an \( \text{Aut}(\mathcal{M}) \)-invariant measure on \( \mathcal{S}_L^* (\mathcal{M}) \).

Let \( \mathbb{B}(\mathcal{M}, L) \) be the collection of sets of the form
\[
\llbracket P(n_0, \ldots, n_{k-1}) \land \varphi(n_{i_0}, \ldots, n_{i_{j-1}}) \rrbracket
\]
where
- \( P \in L_M^k \), and
- \( \varphi(n_{i_0}, \ldots, n_{i_{j-1}}) \in \text{qf}_\pi(L) \).

Suppose
- \( P \in L_M^k \), \( Q \in L_N^k \), and \( \{ \pi \in \mathbb{N} \colon \mathcal{M} \models P(\pi) \} \subseteq \{ \pi \in \mathbb{N} \colon \mathcal{N} \models Q(\pi) \} \), and
- \( \mathcal{M} \models P(n_0, \ldots, n_{k-1}) \) and \( \mathcal{N} \models Q(n_0, \ldots, n_{k-1}) \).

The for \( \llbracket P(n_0, \ldots, n_{k-1}) \land \varphi(n_{i_0}, \ldots, n_{i_{j-1}}) \rrbracket \in \mathbb{B}(\mathcal{M}, L) \) we let
\[
\mu \boxplus \nu_M(\llbracket P(n_0, \ldots, n_{k-1}) \land \varphi(n_{i_0}, \ldots, n_{i_{j-1}}) \rrbracket) \colon= \\
\nu_M(\llbracket P(n_0, \ldots, n_{k-1}) \rrbracket_N) \cdot \mu(\llbracket \varphi(n_{i_0}, \ldots, n_{i_{j-1}}) \rrbracket_M).
\]

**Proposition 4.1.** \( \mu \boxplus \nu_M \) extends uniquely to a \( \text{Aut}(\mathcal{N}) \)-invariant measure, \( \mu \boxplus \nu_M \), on \( \mathcal{S}_{L,M}^* (\mathcal{N}) \) such that

(i) \( \mu \boxplus \nu_M(\llbracket P(n) \rrbracket_N) = \nu_M(\llbracket P(n) \rrbracket_N) \) for all \( p \in L_M \) and \( n \in \mathbb{N} \).

(ii) \( \mu \boxplus \nu_M(\llbracket \eta(n) \rrbracket_N) = \mu(\llbracket \eta(n) \rrbracket_M) \) for all \( \eta(n) \in \text{qf}_\pi(L) \) and \( n \in \mathbb{N} \).

**Proof.** Clearly any extension of \( \mu \boxplus \nu_M \) to a \( \text{Aut}(\mathcal{N}) \)-invariant measure will satisfy (i) and (ii) and each \( \text{Aut}(\mathcal{N}) \)-invariant measure satisfying (i) and (ii) extends \( \mu \boxplus \nu_M \). It therefore suffices to show that there is a unique such extension. We will do this by showing that there is a unique extension to a map \( \mu \boxplus \nu_M \) on \( \text{qf}_\pi(L_M \cup L_M \cup L) \) which satisfies the conditions of Lemma 2.37.

Suppose \( \eta \in \text{qf}_\pi(L_M \cup L_M \cup L) \). Then \( \eta = \zeta_0 \land \zeta_1 \land \zeta_2 \) for some \( \zeta_0 \in \text{qf}_\pi(L_M) \), \( \zeta_1 \in \text{qf}_\pi(L) \) and \( \zeta_2 \in \text{qf}_\pi(L_N) \).

If \( \mathcal{N} \models \zeta_2 \) let \( \mu \boxplus \nu_M(\eta) = \mu \boxplus \nu_M(\zeta_0 \land \zeta_1) \). Otherwise let \( \mu \boxplus \nu_M(\eta) = 0 \).

In particular this implies that \( \mu \boxplus \nu_M \) satisfies Lemma 2.37 condition (a) and condition (b) for atomic formulas from \( L_N \).

Now assume \( \mathcal{N} \models \zeta_2 \). Let \( \pi \) be the union of the parameters from \( \zeta_1 \) and \( \zeta_0 \) and let \( q = R(\pi) \) where \( R \in L_N \) is the unique relation which holds of \( \pi \) (under some fixed ordering). Let \( X_{\zeta_0} := \{ P \in L_M \colon (\exists \pi \in \mathbb{N}) \mathcal{N} \models R(\pi) \text{ and } \mathcal{M} \models P(\pi) \land \zeta_0(\pi) \text{, where } \pi \text{ sits in } \pi \text{ as the parameters of } \zeta_0 \text{ sit in } \pi \} \). If \( X_{\zeta_0} \) is empty then let \( \mu \boxplus \nu_M(\eta) = 0 \).

If \( X_{\zeta_0} \neq \emptyset \) then let \( \mu \boxplus \nu_M(\eta) = \sum_{P \in X_{\zeta_0}} \mu \boxplus \nu_M(\llbracket P \land \zeta_1 \rrbracket_N) \).
Note that
\[
\sum_{P \in X_{\alpha_0}} \mu \boxplus \nu_M(\llbracket P \land \zeta_1 \rrbracket_{\mathcal{M}}) = \left( \sum_{P \in X_{\alpha_0}} \nu_M(\llbracket P \rrbracket_{\mathcal{M}}) \right) \cdot \mu(\llbracket \zeta_1 \rrbracket_{\mathcal{M}})
\]

but as \{\llbracket P \rrbracket : P \in X_{\alpha_0}\} are disjoint this sum equals
\[
\nu_M \left( \bigvee_{P \in X_{\alpha_0}} P \right) \cdot \mu(\llbracket \zeta_1 \rrbracket_{\mathcal{M}}).
\]

In particular this implies the range of \(\mu \boxplus \nu_M\) is a subset of \([0, 1]\).

We now must show Lemma 2.37 condition (b) is satisfied for atomic formulas from \(L_\mathcal{M} \cup L\).

Suppose \(R \in L_\mathcal{M}\). For \(P \in X_{\alpha_0}\), if \(\mathcal{M} \models P(\bar{x})|_{\bar{y}} = R\) and \(\bar{b}\) sits in \(\bar{x}\) as \(\bar{y}\) sits in \(\bar{x}\) then
\[
\mu \boxplus \nu_M((P(\bar{x}) \land R(\bar{b})) \land \zeta_1) = \mu \boxplus \nu_M(P(\bar{x}) \land \zeta_1)
\]
and
\[
\mu \boxplus \nu_M((P(\bar{x}) \land \neg R(\bar{b})) \land \zeta_1) = 0.
\]

Similarly if \(\mathcal{M} \models P(\bar{x})|_{\bar{y}} \neq R\) then
\[
\mu \boxplus \nu_M((P(\bar{x}) \land \neg R(\bar{b})) \land \zeta_1) = \mu \boxplus \nu_M(P(\bar{x}) \land \zeta_1)
\]
and
\[
\mu \boxplus \nu_M((P(\bar{x}) \land R(\bar{b})) \land \zeta_1) = 0.
\]

Therefore
\[
\mu \boxplus \nu_M(\zeta_0 \land \zeta_1) = \sum_{P \in X_{\alpha_0}} \mu \boxplus \nu_M(P \land \zeta_1)
\]

\[
= \left( \sum_{P \in X_{\alpha_0}} \mu \boxplus \nu_M(P \land \zeta_1) \right) + \left( \sum_{P \in X_{\alpha_0}} \mu \boxplus \nu_M(P \land \zeta_1) \right)
\]

\[= \mu \boxplus \nu_M((\zeta_0 \land R(\bar{b})) \land \zeta_1) + \mu \boxplus \nu_M((\zeta_0 \land \neg R(\bar{b})) \land \zeta_1).\]

Now suppose \(\beta \in \text{qf}_{\pi}(L)\). We have
\[
\mu \boxplus \nu_M(\zeta_0 \land (\zeta_1 \land \beta)) = \nu_M \left( \bigvee_{P \in X_{\alpha_0}} P \right) \cdot \mu(\llbracket \zeta_1 \land \beta \rrbracket)
\]
and so if \(\beta\) is an atomic formula
\[
\mu \boxplus \nu_M(\zeta_0 \land \zeta_1) = \mu \boxplus \nu_M(\zeta_0 \land (\zeta_1 \land \beta)) + \mu \boxplus \nu_M(\zeta_0 \land (\zeta_1 \land \neg \beta)).
\]
We therefore have, by Lemma 2.37, that there is a unique measure, \( \mu \boxplus \nu_M \), extending \( \mu \boxplus^- \nu_M \). Further, as \( \mu \boxplus^- \nu_M \) is \( \text{Aut}(\mathcal{M}) \)-invariant so is \( \mu \boxplus \nu_M \).

Finally it is also immediate that \( \mu \boxplus^- \nu_M \) is the unique extension of \( \mu \boxplus \nu_M \) which preserves additivity of the measure and hence satisfies the conditions of Lemma 2.37.

The value \( \mu \boxplus \nu_M(\llbracket \eta(n) \rrbracket_N) \) has a particularly nice description when \( \eta \in \mathcal{L}_{\omega, \omega}(L_M \cup L_N \cup L) \). For \( \eta \in \mathcal{L}_{\omega, \omega}(L_M \cup L_N \cup L) \), \( n \in \mathbb{N}, q \in \text{qtp}(\mathcal{N}) \) with \( \mathcal{N} \models q(n) \) and \( p \in \text{qtp}(\mathcal{M}) \) with \( \{ \pi : \mathcal{M} \models p(\pi) \} \subseteq \{ \pi : \mathcal{N} \models q(\pi) \} \) define

\[
\mu^p(\llbracket \eta(n) \rrbracket_N) = \mu(\llbracket \eta(n^*) \rrbracket_M) \text{ for some } n^* \text{ where } \mathcal{M} \models p(n^*). \]

Note as \( \mu \) is \( \text{Aut}(\mathcal{M}) \)-invariant this is well defined.

**Proposition 4.2.** We have

\[
\mu \boxplus \nu_M(\llbracket \eta(n) \rrbracket) = \sum_{\{ \pi : \mathcal{M} \models p(\pi) \} \subseteq \{ \pi : \mathcal{N} \models q(\pi) \}} \nu_M(\llbracket p(n) \rrbracket) \cdot \mu^p(\llbracket \eta(n) \rrbracket_N)
\]

**Proof.** Let \( A \) be a fragment containing \( \eta \). As \( \text{Th}_A \) is interdefinable with the empty theory in \( L_M \), there is a unique measure \( \mu_A \) on \( \llbracket \text{Th}_A \rrbracket \) which agrees with \( \mu \) on \( L \) and a unique measure on \( \llbracket \text{Th}_A \rrbracket \) which agrees with \( \mu \boxplus \nu_M \) on \( L \cup L_M \). But then \( \mu_A \boxplus \nu_M \) agrees with \( \mu \boxplus \nu_M \) on \( L \cup L_M \) and hence it must be that measure. But the proposition holds for \( \mu_A \boxplus \nu_M \) as \( \text{Th}_A \) proves \( \eta \) is equivalent to a quantifier free formula. Therefore the proposition must also hold of \( \mu \boxplus \nu_M \). \( \square \)

We have shown how to uniquely recover an \( \text{Aut}(\mathcal{N}) \)-invariant measure from an \( \text{Aut}(\mathcal{N}) \)-invariant measure concentrated on \( \llbracket \sigma_M \rrbracket_N \) along with an \( \text{Aut}(\mathcal{M}) \)-invariant measure. We next show that every \( \text{Aut}(\mathcal{N}) \)-invariant measure concentrated on \( \llbracket \sigma_M \rrbracket_N \) must have such a (necessarily unique) decomposition.

**Proposition 4.3.** Suppose \( \eta \) is an \( \text{Aut}(\mathcal{N}) \)-invariant measure on \( \mathcal{H}^*_{L,M \cup L}(\mathcal{N}) \) concentrated on \( \llbracket \sigma_M \rrbracket_N \). There exists unique measures \( \mu^\eta \) and \( \nu_M^\eta \) such that

- \( \mu^\eta \) is an \( \text{Aut}(\mathcal{M}) \)-invariant measure on \( \mathcal{H}^*_{L}(\mathcal{M}) \),
- \( \nu_M^\eta \) is an \( \text{Aut}(\mathcal{N}) \)-invariant measure on \( \mathcal{H}^*_{L,M}(\mathcal{N}) \) concentrated on \( \llbracket \sigma_M \rrbracket_N \),
- \( \eta = \mu^\eta \boxplus \nu_M^\eta \).

**Proof.** First notice that if \( \mu^\eta \) and \( \nu_M^\eta \) exist they must agree with \( \eta \) on their respective domains and hence are uniquely determined by \( \eta \).

By Lemma 2.37, to define \( \mu^\eta \) it suffices to define it on \( \text{qf}_\pi(L_M \cup L) \). In particular suppose \( \zeta_0 \in \text{qf}_\pi(L_M) \) and \( \zeta_1 \in \text{qf}_\pi(L) \). We let \( \mu^\eta(\llbracket \zeta_0 \land \zeta_1 \rrbracket_M) = \eta(\llbracket \zeta_0 \rightarrow \zeta_1 \rrbracket_N) \) if \( \mathcal{M} \models \zeta_0 \) and 0 otherwise. It is then easily checked that the conditions of Lemma 2.37 are satisfied so that this partial definition of \( \mu^\eta \) extends uniquely to an \( \text{Aut}(\mathcal{M}) \)-invariant measure.
Similarly by Lemma 2.37 to define \( \nu_M \) it suffices to define it on \( qf_\pi(L_M \cup L_{M'}) \). In particular suppose \( \zeta_0 \in qf_\pi(L_M) \) and \( \zeta_1 \in qf_c(L_N) \). We let \( \nu_M^\pi(\llbracket \zeta_0 \rrbracket_N) = \eta(\llbracket \zeta_0 \rrbracket_N) \) if \( N |\zeta_1 = \zeta_1 \) and 0 otherwise. It is then easily checked that the conditions of Lemma 2.37 are satisfied so that this partial definition of \( \nu_M^\pi \) extends uniquely to an \( \text{Aut}(N) \)-invariant measure.

But we have that \( \eta \) and \( \mu \boxplus \nu_M \) agree on the domains of \( \mu \) and \( \nu_M \) and so by Proposition 4.1 we have \( \eta = \mu \boxplus \nu_M \). \( \square \) 4.3

4.1. Inherited Properties Of Merged Measures. We now put together the results obtained in the previous subsection to get several key results about merged measures which are inherited from their components.

**Proposition 4.4.** Suppose \( \text{Aut}(M) \subseteq \text{Aut}(N) \), \( \nu_M \) is an \( \text{Aut}(N) \)-invariant measure concentrated on \( [\sigma_M]_N \) and \( \mu \) is an \( \text{Aut}(M) \)-invariant measure \( \mu \) on \( \mathcal{F}_1(M) \). Then the following are equivalent.

1. \( \nu_M \) is an ergodic \( \text{Aut}(N) \)-invariant measure and \( \mu \) is an ergodic \( \text{Aut}(M) \)-invariant measure.
2. \( \mu \boxplus \nu_M \) is an ergodic \( \text{Aut}(N) \)-invariant measure.

**Proof.** First we show (2) implies (1). Suppose \( \mu \boxplus \nu_M \) is ergodic. Note that if \( \alpha_0, \alpha_1 \in (0, 1) \), \( \nu_M^0, \nu_M^1 \) are \( \text{Aut}(N) \)-invariant measures and \( \nu_M = \alpha_0 \cdot \nu_M^0 + \alpha_1 \cdot \nu_M^1 \) then \( \mu \boxplus \nu_M = \alpha_0 \cdot \mu \boxplus \nu_M^0 + \alpha_1 \cdot \mu \boxplus \nu_M^1 \) by Proposition 4.2. Further, by Proposition 4.3 we have \( \mu \boxplus \nu_M = \mu \boxplus \nu_M^i \) if and only if \( \nu_M = \nu_M^i \) (for \( i \in \{0, 1\} \)). Therefore as \( \mu \boxplus \nu_M \) is ergodic (by assumption) so is \( \nu_M \).

Similarly if \( \mu^0, \mu^1 \) are \( \text{Aut}(M) \)-invariant measures and \( \mu = \alpha_0 \cdot \mu^0 + \alpha_1 \cdot \mu^1 \) then \( \mu \boxplus \nu_M = \alpha_0 \cdot \mu^0 \boxplus \nu_M + \alpha_1 \cdot \mu^1 \boxplus \nu_M \) by Proposition 4.2. Also by Proposition 4.3 we have \( \mu \boxplus \nu_M = \mu^i \boxplus \nu_M \) if and only if \( \mu = \mu^i \) (for \( i \in \{0, 1\} \)). So \( \mu \) is extreme (and hence ergodic) as well.

Next we show (1) implies (2). Suppose \( \mu, \nu_M \) are ergodic. Further suppose \( \eta_0, \eta_1 \) are \( \text{Aut}(N) \)-invariant measures on \( \mathcal{F}_1(N) \) and \( \mu \boxplus \nu_M = \alpha_0 \cdot \eta_0 + \alpha_1 \cdot \eta_1 \) with \( \alpha_0, \alpha_1 \in (0, 1) \).

By Proposition 4.3 there are unique \( \text{Aut}(M) \)-invariant measures \( \nu_M^0, \nu_M^1 \) concentrated on \( [\sigma_M]_N \) and \( \text{Aut}(N) \)-invariant measures \( \mu^0, \mu^1 \) such that \( \eta_i = \mu^i \boxplus \nu_M^i \) (for \( i \in \{0, 1\} \)).

But, for any \( p \in L_M \) and \( n \in N \) by Proposition 4.1 we have

\[
\nu_M([p(n)]) = \mu \boxplus \nu_M([p(n)]) = \alpha_0 \cdot \mu^0 \boxplus \nu_M^0([p(n)]) + \alpha_1 \cdot \mu^1 \boxplus \nu_M^1([p(n)]) = \alpha_0 \cdot \nu_M^0([p(n)]) + \alpha_1 \cdot \nu_M^1([p(n)]).
\]

Therefore we have \( \nu_M = \alpha_0 \cdot \nu_M^0 + \alpha_1 \cdot \nu_M^1 \) and so, as \( \nu_M \) is ergodic we have \( \nu_M = \nu_M^0 = \nu_M^1 \).
But then we also have
\[ \mu \Box \nu_M = \alpha_0 \cdot \mu^{M_0} \Box \nu_M + \alpha_1 \cdot \mu^{M_1} \Box \nu_M = (\alpha_0 \cdot \mu^{M_0} + \alpha_1 \cdot \mu^{M_1}) \Box \nu_M \]
with the second equality following from Proposition 4.2. Finally, this implies by Proposition 4.3 that \( \mu = \alpha_0 \cdot \mu^{M_0} + \alpha_1 \cdot \mu^{M_1} \). So, as \( \mu \) is ergodic we have \( \mu = \mu^{M_0} = \mu^{M_1} \).

We therefore have \( \mu \Box \nu_M = \eta_0 = \eta_1 \) and so \( \mu \Box \nu_M \) is extreme and hence by Lemma 2.31, \( \mu \Box \nu_M \) is ergodic. We therefore have (1) implies (2). \( \square_4 \)

The following is an immediate consequence of Proposition 4.1.

**Proposition 4.5.** Suppose \( \text{Aut}(M) \subseteq \text{Aut}(N) \), \( \nu_M \) is an \( \text{Aut}(N) \)-invariant measure concentrated on \( [\sigma_M]_N \) and \( \mu \) is an \( \text{Aut}(M) \)-invariant measure on \( S_L(M) \). Then \( \text{Th}(\mu) = \text{Th}(\mu \Box \nu_M) \).

**Proof.** For \( \tau \) a sentence in \( L_{\omega_1,\omega}(L_M \cup L) \) let \( \tau^\tau := \tau \land (x = x) \) be a unary formula. Note \( \tau \in \text{Th}(\mu) \) if and only if \( \mu([\tau^\tau(0)]) = 1 \). Let \( q \in \text{qtp}(N) \) be such that \( N \models q(0) \).

By Proposition 4.2 we have
\[ \mu \Box \nu_M([\tau^\tau(0)]) = \sum_{p \in \text{qtp}(M) \subseteq [\tau^\tau(0)]_N} \nu_M([p(0)]) \cdot \mu^p([\tau^\tau(0)]_N). \]

But
\[ \sum_{p \in \text{qtp}(M) \subseteq [\tau^\tau(0)]_N} \nu_M([p(0)]) = 1 \]
by construction and so
\[ \mu \Box \nu_M([\tau^\tau(0)]) = 1 \]
if and only if
\[ \bigwedge_{p \in \text{qtp}(M) \subseteq [\tau^\tau(0)]_N} [\mu^p([\tau^\tau(0)]_N) = 1]. \]
Hence \( \tau \in \text{Th}(\mu) \) if and only if \( \tau \in \text{Th}(\mu \Box \nu_M) \). \( \square_4 \)

In particular we have the following easy consequence of Theorem 2.49 and Proposition 4.5.

**Proposition 4.6.** Suppose \( M \) has trivial dcl and \( \varphi \in L_{\omega_1,\omega}(L) \). Then the following are equivalent.

(a) There is an \( \text{Aut}(M) \)-invariant measure on \( S_L(M) \) which concentrates on \( [\varphi]_M \).
(b) There is an ergodic Aut($\mathcal{M}$)-invariant measure on $\mathcal{L}(\mathcal{M})$ which concentrates on $[\varphi]_{\mathcal{M}}$.

(c) For all countable fragment $A \subseteq L_{\omega_1,\omega}(L \cup L_M)$ there is an $A$-theory $T$ with trivial dcl which contains $\sigma_M \land \varphi$.

(d) There is a countable fragment $A \subseteq L_{\omega_1,\omega}(L \cup L_M)$ and an $A$-theory $T$ with trivial dcl which contains $\sigma_M \land \varphi$.

Proof. First notice that (b) immediately implies (a) and that (c) and (d) are equivalent by Theorem 2.49.

Now suppose (a) holds. Note that the linear mixture of measures concentrated on $[\neg \varphi]_{\mathcal{M}}$ is itself concentrated on $[\neg \varphi]_{\mathcal{M}}$. Therefore, by Lemma 2.31 and Lemma 2.32 there must exist an ergodic Aut($\mathcal{M}$)-invariant measure $\mu$ concentrated on $[\varphi]_{\mathcal{M}}$. Therefore (b) holds.

Further, as $\mathcal{M}$ has trivial dcl, by Theorem 2.50 there is an ergodic measure $\nu_M$ concentrated on $[\sigma_M]$. Now suppose (b) holds and $\mu$ is an ergodic Aut($\mathcal{M}$)-invariant measure concentrated on $[\varphi]_{\mathcal{M}}$. By Proposition 4.1 $\mu \boxplus \nu_M$ is an $\mathcal{S}_N$-invariant measure, by Proposition 4.4 $\mu \boxplus \nu_M$ is ergodic, and by Proposition 4.5 $\mu \boxplus \nu_M$ concentrates on $[\sigma_M \land \varphi]$. Now let $A$ be any countable fragment containing $\sigma_M \land \varphi$. As $\mu \boxplus \nu_M$ is ergodic we have $\text{Th}(\mu \boxplus \nu_M) \cap A$ is an $A$-theory and so, by Theorem 2.49 has trivial dcl. Therefore (c) holds.

Suppose (c) holds. Let $A$ be a countable fragment and $T$ an $A$-theory which as trivial dcl and contains $\sigma_M \land \varphi$. Then there is an $\mathcal{S}_N$-invariant measure $\beta$ concentrated on $[\sigma_M \land \varphi]$ by [AFP-Complete classification]. Further, by Proposition 4.3 there is an Aut($\mathcal{M}$)-invariant measure $\mu$ such that $\beta = \mu \boxplus \nu_M$ for some $\mathcal{S}_N$-invariant measure $\nu_M$ concentrated on $[\sigma_M]$. But by Proposition 4.5 we have $\text{Th}(\beta) = \text{Th}(\mu)$ and so $\mu$ must concentrate on $[\varphi]_{\mathcal{M}}$ and so (a) holds.

Note that the case special case of Proposition 4.6 where $\varphi$ is the Scott sentence of a structure $\mathcal{N}$ follows immediately from Theorem 3.21 and Theorem 4.1 in [AFP16].

5. Representations

We now finally prove Theorem 5.6 and Theorem 5.7, the main results of this paper, which gives a characterization of those Aut($\mathcal{M}$)-invariant measures which are representable. We then also give several properties of such representable measures.

5.1. Aut($\mathcal{M}$)-Recipes. We now introduce Aut($\mathcal{M}$)-recipes, which are an immediate generalizations of $\mathcal{S}_N$-recipes from Definition 2.40. In the following definition we assume $\mathcal{M}$ is canonical.
Definition 5.1. An Aut($\mathcal{M}$)-recipe on $L$ is a collection of measurable functions $\overline{f} = \langle f_p \rangle_{p \in \text{ntp}(\mathcal{M})}$ where $f_p : [0,1]^{\mathcal{P}(n)} \to \text{otp}_{L^n}(x_0, \ldots, x_{n-1})$ when $\text{ar}(p) = n$.

If $\overline{f} = \langle f_p \rangle_{p \in \text{ntp}(\mathcal{M})}$ is an Aut($\mathcal{M}$)-recipe let $\mathcal{M}(\overline{f}) : [0,1]^{\mathcal{P}(\omega)} \to \mathcal{S}_L(\mathcal{M})$ be the function such that

- for any relation $R \in L$ of arity $k$,
- $\overline{r} = \langle r_b \rangle_{b \in \mathcal{P}_{\omega}(\omega)} \in [0,1]^{\mathcal{P}_{\omega}(\omega)}$, and
- $\overline{a} = (a_0, \ldots, a_{k-1}) \in \mathbb{N}^{[\omega]}$ with $\mathcal{M} \models \mu(\overline{a})$ for $\mu \in \text{ntp}(\mathcal{M})$.

$\mathcal{M}(\overline{f})(\overline{r}) \models R(a_0, \ldots, a_{k-1})$ if and only if $R(x_0, \ldots, x_{k-1}) \in f_{\mu}(\overline{r})$

We define the distribution of $\overline{f}$, $\mu_\overline{f}$, to be the distribution of $\mathcal{M}(\overline{f})$ where the domain of $\mathcal{M}(\overline{f})$ is given the Lebesgue measure.

An Aut($\mathcal{M}$)-recipe is similar to an $\mathcal{S}_N$-recipe except in the construction of the random structure we are allowed to use the type of the elements in $\mathcal{M}$. In particular every Aut($\mathcal{M}$)-recipe gives rise to an Aut($\mathcal{M}$)-invariant measure.

Lemma 5.2. Suppose $\overline{f} = \langle f_p \rangle_{p \in \text{ntp}(\mathcal{M})}$ is an Aut($\mathcal{M}$)-recipe. Then $\mu_\overline{f}$ is Aut($\mathcal{M}$)-invariant.

Proof. This follows from the fact that if $\overline{v}, \overline{w} \in \mathcal{M}$ with $\overline{v}$ and $\overline{w}$ satisfying the same quantifier free type, $p$, and $\zeta \in \text{qf}_\pi(L)$, then $\mu_\overline{f}(\llbracket \zeta(\overline{w}) \rrbracket_{\mathcal{M}}) = \lambda(f_p^{-1}(\zeta(\overline{v}))) = \mu(\llbracket \zeta(\overline{w}) \rrbracket_{\mathcal{M}})$. \hfill $\Box_{5.2}$

We will be interested in distributions of Aut($\mathcal{M}$)-recipes.

Definition 5.3. We say an Aut($\mathcal{M}$)-invariant measure $\mu$ is representable if there is an Aut($\mathcal{M}$)-recipe $\overline{f}$ such that $\mu = \mu_\overline{f}$. In this case we say $\overline{f}$ is a representation of $\mu$.

The following is important as it shows every representable Aut($\mathcal{M}_0$)-invariant measure can be extended (in a not necessarily unique way) to an Aut($\mathcal{M}_1$)-invariant measure whenever $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$.

Proposition 5.4. Suppose $\overline{f}$ is an Aut($\mathcal{M}_0$)-recipe on $L$ and $\mathcal{M}_0 \subseteq_{\text{can}} \mathcal{M}_1$. Then there is an Aut($\mathcal{M}_1$)-recipe on $\overline{f}$ on $L$ such that whenever $p_0 \in \text{ntp}(\mathcal{M}_0)$, $p_1 \in \text{ntp}(\mathcal{M}_1)$ with $p_0 \subseteq p_1$ we have $g_{p_1} = f_{p_0}$.

Proof. For $p \in \text{ntp}(\mathcal{M}_1)$, if $p |_{\mathcal{M}_1} = q \in \text{ntp}(\mathcal{M}_0)$ let $g_p = f_q$. Otherwise let $g_p$ be the constant function which takes the value of the trivial type, i.e. the type $\{\neg R(\overline{r}) : R \in L^{\text{ar}(p)}\}$. \hfill $\Box_{5.4}$

In particular Lemma 3.8 and Proposition 5.4 imply that any representable measure is the restriction of a measure which is invariant under Aut($\mathcal{M}$) where
$\mathcal{M}$ is free. We will next see that the converse holds as well and every measure which is $\text{Aut}(\mathcal{M})$-invariant for $\mathcal{M}$ free has a representation.

5.2. Representability and Free Structures. We now show that whenever $\mathcal{M}$ is free, every $\text{Aut}(\mathcal{M})$-invariant measure is representable. We can assume with out loss of generality that $\mathcal{M}$ is canonical.

Recall $\mathcal{C}_M$ from Definition 3.11 is an $\mathcal{S}_N$-recipe such that $\mathcal{M}(\mathcal{C}_M)$ is concentrated on $[[\sigma_M]]$. Call its distribution $\vartheta_M$. Also recall the definitions of $S_p$ and $\alpha_p$ (for $p \in \text{otp}(\mathcal{M})$) from Definition 3.14. In what follows $\mathcal{C}_M$, $S_p$ and $\alpha_p$ (for $p \in \text{otp}(\mathcal{M})$) will play an important role.

We begin with a technical lemma.

Lemma 5.5. Suppose $\mathcal{M}$ is free and $\mu$ is an $\text{Aut}(\mathcal{M})$-invariant measure. Then there is an $\mathcal{S}_N$-recipe $\mathcal{E} = \langle e_n \rangle_{n \in \mathbb{N}}$ whose distribution is $\mu \oplus \vartheta_M$ and which agrees with $\mathcal{C}_M$. Further if $\mu$ is ergodic $\mathcal{E}$ can be chosen to be ergodic as well.

Proof. By Proposition 4.1 we know that $\mu \oplus \vartheta_M$ is an $\mathcal{S}_N$-invariant measure and so there is an $\mathcal{S}_N$-recipe $\mathcal{F}$ which represents it. Let $\mathcal{F}^*$ be the restriction of $\mathcal{F}$ to the language $L_M$. As $\mu \oplus \vartheta_M$ restricted to $L_M$ has the same distribution as $\mathcal{M}(\mathcal{F})$ has the same distribution as $\mathcal{M}(\mathcal{C}_M)$, i.e. $\vartheta_M$. So, by Theorem 2.43, there are maps $h_n^\mathcal{F}, h_n^\mathcal{F} : [0, 1]^{\mathbb{N}} \rightarrow [0, 1]$ such that

- $h_n^\mathcal{F}, h_n^\mathcal{F}$ preserve $\lambda$ in the highest order arguments, and
- $c_n^\mathcal{M} \circ \widehat{h_n^\mathcal{F}} = g_n^* \circ \widehat{h_n^\mathcal{F}}$ a.s.

Further, by Proposition 2.48 and the fact that $\mathcal{C}_M$ is ergodic, when $\mu$ is also ergodic we can assume the maps $h_n^\mathcal{F}$ and $\widehat{h_n^\mathcal{F}}$ are independent of the coordinate indexed by $\emptyset$.

But, by Theorem 2.43 $(g_n \circ \widehat{h_n^\mathcal{F}})_{n \in \mathbb{N}}$ is also a representation of $\mu \oplus \vartheta_M$. We can therefore assume without loss of generality that $g_n^* = c_n^\mathcal{M} \circ \widehat{h_n^\mathcal{F}}$ a.s.

We now define $e_n$ by induction.

Arity 1:
For each $U \in \text{ntp}(\mathcal{M})$ of arity 1 let $P_U = (h_1^\mathcal{F})^{-1}(S_U)$. By construction of $\vartheta_M$ we know that $P_U = [0, 1] \times I_U$ where $|I_U| = 2^\omega$ and is Borel with $\lambda(I_U) = \lambda(S_U)$. So there is a measure preserving isomorphism $\beta_U : S_U \rightarrow I_U$. Let $e_1 : S_U \rightarrow \text{ntp}_L(x_0)$ be such that $e_1(x, a) = g_1 \circ \beta_U(a)$ whenever $a \in S_U$. Clearly $\mathcal{M}(g_1)$ and $\mathcal{M}(e_1)$ have the same distribution and $e_1$ does not depend on the first coordinate when $g_1$ doesn’t.

Arity $n + 1$:
Assume we have defined $e_n$ and further that $P_p = (h_n^\mathcal{F})^{-1}(S_p)$ for any $p \in$
Further suppose we have defined a measure preserving bijection $\beta_p: S_p \to P_p$ such that $e_n = g_n \circ \beta_p$ on $S_p$.

Now suppose $p \in \text{ntp}_{\leq n}(x_0, \ldots, x_{\text{ar}(p)-1})$. Further suppose we have defined a measure preserving bijection $\beta_p: S_p \to P_p$ such that $e_n = g_n \circ \beta_p$ on $S_p$.

Let $X_p = \{p^\star \in \text{qtp}(M): p^\star \text{ restricts to } p^\star\}$.

Suppose $\overline{\tau} \in [0, 1]^{\Psi(n)\setminus\{0, \ldots, n-1\}}$ and $y \in [0, 1]$. As $g_{n+1}(\overline{\tau}, y) \cap \Lambda = c_{n+1} \circ h_{n+1}(\overline{\tau}, y)$ we must have

$$(\overline{\tau}, y) \in \bigcup_{p^\star \in X_p} P_{p^\star} \iff (h_{n+1}^\star)(\overline{\tau}, y) \in \bigcup_{p^\star \in X_p} S_p$$

$$\iff (h_{n+1}^\star)(\overline{\tau}) \in S_p$$

$$\iff \overline{\tau} \in P_{p^\star}.$$

Therefore $\bigcup_{p^\star \in X_p} P_{p^\star} = P_{p^\star} \times [0, 1]$.

But we also have that a.s. for all $\overline{\tau} \in P_{p^\star}$ and all $p \in X_{p^\star}$ that $\lambda(\{y: (\overline{\tau}, y) \in P_p\}) = \lambda(\{y: \gamma_{X_{p^\star}}(y) = p\})$ and hence does not depend on $\overline{\tau}$. For each $p \in X_{p^\star}$ there is therefore a measure preserving isomorphism $i_p: P_{p^\star} \times \{y: \gamma_{X_{p^\star}}(y) = p\} \to P_p$. We then let $e_{n+1}(\overline{\tau}) = g_{n+1} \circ \beta_p(\overline{\tau})$ whenever $\overline{\tau} \in S_p$.

It is then immediate that $\overline{\tau}$ agrees with $\overline{\tau}^\Lambda$ and that $\overline{\tau}$ is ergodic when $\mu$ is.

\[\square 5.5\]

**Theorem 5.6.** Suppose $\mathcal{M}$ is free and $\mu$ is an $\text{Aut}(\mathcal{M})$-invariant measure. Then $\mu$ is representable.

**Proof.** By Lemma 5.5 there is an $\mathcal{S}_n$-recipe $\overline{\tau}$ with distribution $\mu \boxplus \vartheta_\mathcal{M}$ and which agrees with $\overline{\tau}^\mathcal{M}$. But by construction, for every $p \in \text{qtp}(\mathcal{M})$, $\alpha_p$ is a bijection from $[0, 1]^{|\text{ar}(p)|}$ to $S_p$. Therefore $\overline{f} := \langle e_p \circ \alpha_p \rangle_{p \in \text{qtp}(\mathcal{M})}$ is an $\text{Aut}(\mathcal{M})$-recipe which is a representation of $\mu$. \[\square 5.6\]

**Theorem 5.7.** The following are equivalent for an $\text{Aut}(\mathcal{M})$-invariant measure $\mu$.

(a) $\mu$ is representable.

(b) There is an $\text{Aut}(\mathcal{F}(\mathcal{M}))$-invariant measure $\mathcal{F}(\mu)$ such that $\mu = \mathcal{F}(\mu)|\mathcal{M}$.

**Proof.** This follows immediately from Proposition 5.4 and Theorem 5.6. \[\square 5.7\]
Condition (b) from Theorem 5.7 can be thought of as being an amalgamation condition on the measure, i.e. it says that any locally consistent properties of the measure can be amalgamated into a measure where they are globally consistent.

5.3. Ergodic Representations. We now give a characterization of when an Aut(\mathcal{M})-recipe gives rise to an ergodic Aut(\mathcal{M})-invariant measure. Recall that by Lemma 2.31 these are exactly the extreme measures in the simplex of Aut(\mathcal{M})-invariant measures and by Lemma 2.35, each ergodic invariant measure has an almost sure complete and consistent theory.

**Proposition 5.8.** Suppose \( \mu \) is an Aut(\mathcal{M})-invariant measure on \( \mathcal{S}_L(\mathcal{M}) \). Then the following are equivalent.

1. \( \mu \) is ergodic.
2. \( \mu \) is dissociated.
3. There is an Aut(\mathcal{M})-recipe \( \mathcal{F} = \langle f_p \rangle_{p \in \text{qtp}(\mathcal{M})} \) such that
   - \( \mathcal{M}(\mathcal{F}) \) has distribution \( \mu \)
   - For each \( p \in \text{ntp}(\mathcal{M}) \), \( f_p \) does not depend on the coordinate with \( \emptyset \)-index.

**Proof.** First assume (3) holds. Recall Definition 3.14. We know by Lemma 3.12 that \( \vartheta_\mathcal{M} \) is ergodic and so by Proposition 4.4 it suffices to show that \( \mu \boxplus \vartheta_\mathcal{M} \) is ergodic.

Let \( \mathcal{E} = \langle e_n \rangle_{n \in \mathbb{N}} \) be the \( \mathcal{S}_N \)-recipe where,

- for each \( n \in \mathbb{N} \),
- for each \( p \in \text{qtp}(\mathcal{M}) \) of arity \( n \)
- for each \( \mathcal{E} \in S_p \), \( e_n(\mathcal{E}) = f_p \circ \alpha_p^{-1}(\mathcal{E}) \).

We then have \( \mathcal{M}(\mathcal{E}) \) has the same distribution as \( \mu \boxplus \vartheta_\mathcal{M} \). But for each \( n \in \mathbb{N} \) it is immediate that the value of \( e_n(\mathcal{E}) \) is independent of the value of the coordinate indexed by \( \emptyset \). Therefore by Corollary 2.46 we have that \( \mu \boxplus \vartheta_\mathcal{M} \) is ergodic and (1) holds.

Next assume (1) holds. By Lemma 5.5 there is an ergodic \( \mathcal{S}_N \)-recipe \( \mathcal{E} = \langle e_n \rangle_{n \in \mathbb{N}} \) whose distribution is \( \mu \boxplus \vartheta_\mathcal{M} \) and which agrees with \( \mathcal{E}^{\text{tr}} \). We therefore have \( \mathcal{F} := \langle e_{\text{ar}(p)} \circ \alpha_p \rangle_{p \in \text{ntp}(\mathcal{M})} \) is an Aut(\mathcal{M})-recipe with distribution \( \mu \) and for which all of the functions are independent of the coordinate indexed by \( \emptyset \). Therefore (3) holds.

Finally note by Corollary 2.46, \( \vartheta_\mathcal{M} \) is dissociated. Therefore, because of how \( \mu \boxplus \vartheta_\mathcal{M} \) was defined, \( \mu \) is dissociated if and only if \( \mu \boxplus \vartheta_\mathcal{M} \) is dissociated. But once again by Corollary 2.46, \( \mu \boxplus \vartheta_\mathcal{M} \) is dissociated if and only if \( \mu \boxplus \vartheta_\mathcal{M} \) is ergodic if and only if \( \mu \) is ergodic. Therefore (1) is equivalent to (2). \( \square \)
6. Acknowledgements

I would like to thank Cameron Freer and Rehana Patel for many helpful conversations as well as for comments on an earlier draft. I would like to thank Alex Kruckman for useful discussions. I would also like to thank Todor Tsankov for posing the question which is answered in Proposition 4.6.

This research was facilitated by participation in the London Mathematical Society – EPSRC Durham Symposium on “Permutation Groups and Transformation Semigroups” from July 20-30, 2015 (Durham, England) as well as the “Logic and Random Graphs” workshop at the Lorentz Center in from August 31 - September 4, 2015 (Leiden, Netherlands).

References

[AFP16] N. Ackerman, C. Freer, and R. Patel. “Invariant measures concentrated on countable structures”. In: Forum Math. Sigma 4 (2016), e17, 59.

[AFPa] N. Ackerman, C. Freer, and R. Patel. “Sentences admitting an invariant measure”. In Preparation.

[Ald85] D. J. Aldous. “Exchangeability and related topics”. In: École d’été de probabilités de Saint-Flour, XIII—1983. Vol. 1117. Lecture Notes in Math. Springer, Berlin, 1985, pp. 1–198.

[Aus08] T. Austin. “On exchangeable random variables and the statistics of large graphs and hypergraphs”. In: Probab. Surv. 5 (2008), pp. 80–145.

[Bar75] J. Barwise. Admissible sets and structures. An approach to definability theory, Perspectives in Mathematical Logic. Springer-Verlag, Berlin-New York, 1975, pp. xiii+394.

[BK96] H. Becker and A. S. Kechris. The descriptive set theory of Polish group actions. Vol. 232. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1996, pp. xii+136.

[CT18] H. Crane and H. Towsner. “Relatively exchangeable structures”. In: J. Symb. Log. 83.2 (2018), pp. 416–442.

[DJ08] P. Diaconis and S. Janson. “Graph limits and exchangeable random graphs”. In: Rend. Mat. Appl. (7) 28.1 (2008), pp. 33–61.

[Hod93] W. Hodges. Model theory. Vol. 42. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1993, pp. xiv+772.

[Kal02] O. Kallenberg. Foundations of modern probability. Second. Probability and its Applications (New York). Springer-Verlag, New York, 2002, pp. xx+638.
[Kal05] O. Kallenberg. *Probabilistic symmetries and invariance principles*. Probability and its Applications (New York). Springer, New York, 2005, pp. xii+510.

[Kal08] O. Kallenberg. “Some highlights from the theory of multivariate symmetries”. In: *Rend. Mat. Appl. (7)* 28.1 (2008), pp. 19–32.

[Kec95] A. S. Kechris. *Classical descriptive set theory*. Vol. 156. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995, pp. xviii+402.

[Kin78] J. F. C. Kingman. “Uses of exchangeability”. In: *Ann. Probability* 6.2 (1978), pp. 183–197.

Harvard University, Cambridge, MA 02138, USA

Email address: nate@aleph0.net