Counterexamples to $C^\infty$ well posedness for some hyperbolic operators with triple characteristics

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Abstract: In this paper we prove a well posed and an ill posed result in the Gevrey category for a simple model hyperbolic operator with triple characteristics, when the principal symbol cannot be smoothly factorized, and whose propagation cone is not transversal to the characteristic manifold, thus confirming the conjecture that the Ivrii-Petkov condition is not sufficient for the $C^\infty$ well posedness unless the propagation cone is transversal to the characteristic manifold, albeit for a limited class of operators. Moreover we are able not only to disprove $C^\infty$ well posedness, but we can actually estimate the precise Gevrey threshold where well posedness will cease to hold.

Key words: Cauchy problem; well-posedness; Gevrey class; triple characteristics.

1. Introduction. Hyperbolic operators with double characteristics have been thoroughly investigated in the past years, and at least in the case when there is no transition between different types on the set where the principal symbol vanishes of order 2, essentially everything is known, see e.g. [8] and [1] for a general survey and [5] and [3] for classical introductions. When $C$, the propagation cone, see [8] for a definition, is transversal to the manifold of multiple points, we are again, in a way, effectively hyperbolic. When this happens and the lower order terms satisfy a generic Ivrii-Petkov vanishing condition, it is known that we have well posedness in $C^\infty$. See [6] for a very complete analysis of this situation. Here we prove a well posedness result in the Gevrey category for a simple model hyperbolic operator with triple characteristics and whose propagation cone is not transversal to the triple manifold. Also we are able not only to disprove $C^\infty$ well posedness, but we can actually estimate the precise Gevrey threshold, by exhibiting a special class of solutions, through which we can violate weak necessary solvability conditions. This threshold appears at $s = 2$, thus beyond the canonical value of $s = 3/2$ dictated by the classical result of Bronstein [2]. The choice of the lower order terms will be the easiest possible, i.e. zero. We consider the operator

\begin{equation}
P(x,D) = D_0^3 - (D_1^2 + x_1^2 D_2^2)D_0 - b_0 x_1^2 D_0^3.
\end{equation}

Here $x = (x_0, x') \in \mathbb{R}^{n+1}$ with $x' = (x_1, x', x_n)$ and the local estimates below will be proven in a neighborhood of $x = 0$. Clearly hyperbolicity is equivalent to $b_0^2 \leq 4/27$. We will also assume that the principal symbol vanishes exactly of order 3 on the triple manifold $\Sigma_3$, thus we will require $b_0^2 < 4/27$, i.e. outside $\Sigma_3$, $P$ is strictly hyperbolic. We assume familiarity with the definition of $\gamma^{(s)}(\mathbb{R}^n)$, the Gevrey s class and with the notion of locally solvable in $\gamma^{(s)}$ Cauchy problem (see [1]). In this note we say that the Cauchy problem for $P$ is well posed in the Gevrey s class if for any $\phi_j(x') \in \gamma^{(s)}(\mathbb{R}^n)$, $j = 0, 1, 2$ one can find a neighborhood $\omega$ of the origin such that there is a $u \in C^3(\omega)$ verifying $Pu = 0$ in $\omega$ and $D_0^j u(0, x') = \phi_j(x')$ in $\omega \cap \{x_0 = 0\}$ for $j = 0, 1, 2$.

The main results in this paper are then precisely stated:

Theorem 1.1. Assume that $b_0^2 < 4/27$. Then the Cauchy problem for $P$ is well posed in the Gevrey 2 class.

That this is actually the best one can hope for is proven in

Theorem 1.2. If $s > 2$, it is possible to choose $b_0 \in [0, \frac{2}{3\sqrt{3}})$ such that the Cauchy problem for $P$ is not locally solvable at the origin in the Gevrey s class.

2. Estimates in Gevrey classes. Since the coefficients of $P$ are independent of $x_n$, we first
make the Fourier transform with respect to \( x_n \) and regard \( \xi_n \), the dual variable of \( x_n \), as a parameter. We define

\[
(u, v) = \int_{\mathbb{R}^{n-1}} \hat{u}(x_0, x_1, \ldots, x_{n-1}, \xi_n) \overline{\hat{v}}(x_0, x_1, \ldots, x_{n-1}, \xi_n) dx_1 \ldots dx_{n-1}
\]

with \( \hat{u} \) denoting the partial Fourier transform with respect to \( x_n \). In a similar way we have for the \( L^2 \) norm \( ||u||^2 = \int_{\mathbb{R}^n} ||\hat{u}(x_0, x_1, \xi_n)||^2 dx_1 dx_n \). Before dealing with the operator (1.1) itself, we need a preliminary result on the multiplier operator \( M \). Let

\[
E_j(u) = [\{D_0^2u\}]^j + [\{D_1^2u\}]^j + [\{(x_1\xi_n)^j\}]^j u^2
\]

where \( E_0(u) = ||u||^2 \). Let \( 0 < \theta < 1/2 \) and we start by proving the following

**Lemma 2.1.** Let \( M = D_0^2 - \theta \Omega \) with \( \Omega = D_1^2 + x_1^2\xi_n^2 \). Then for any \( s \geq 1 \), \( s \in \mathbb{R} \) and any \( \tau \) large we have for any \( u \in \mathcal{C}_0^\infty (\mathbb{R}^n) \)

\[
(2.2) \quad \theta^{-1} \int_0^\infty W[|\mathcal{M}u|^2] dx_0 \geq W(0) \sum_{j=0}^\infty \frac{\tau^{3j-3}}{3!} W_j(u(0, \cdot)) + \int_0^\infty W(\tau^{3j} \langle \xi_n \rangle^3 \xi_n) E_j(u) dx_0,
\]

where \( W = \exp(2\tau \langle \xi_n \rangle^3 (x_0 - a)) \) with \( a > 0 \) and \( \langle \xi_n \rangle = \sqrt{1 + \xi_n^2} \).

**Proof.** We compute

\[
-2\mathcal{M}(\mathcal{M}u, D_0 u) = -2\mathcal{M}(D_0^2 u, D_0 u) + 2\mathcal{M}(\Omega u, D_0 u) = \partial_{\xi_n} [||D_0 u||^2 + \theta(\Omega u, u)].
\]

Since \( (\partial_{\xi_n} u) = ||D_0 u||^2 + ||x_1\xi_n||^2 \) and \( -W \partial_{\xi_n} = -\partial_{\xi_n} W + 2\tau \langle \xi_n \rangle^2 W \) using Cauchy-Schwarz inequality we see

\[
(2.3) \quad \int_0^\infty W[|\mathcal{M}u|^2] dx_0 \geq \theta \tau^{3/2} W(0) E_1(u(0, \cdot)) + \tau^{2/3} \int_0^\infty W(\xi_n^3) [||D_0 u||^2 + \theta ||D_1 u||^2] dx_0 + \theta ||x_1\xi_n||^2 dx_0.
\]

Repeating similar arguments we have

\[
\int_0^\infty W[|D_0 u|^2] dx_0 \geq \tau \langle \xi_n \rangle^3 W(0) ||u(0, \cdot)||^2
\]

and replacing \( (1 - \theta) \int_0^\infty W[|D_0 u|^2] dx_0 \) in (2.3) by the above estimate the right-hand side of (2.3) is bounded from below by

\[
\theta W(0) \sum_{j=0}^\infty \frac{\tau^{3j-2} \langle \xi_n \rangle^3 \xi_n E_j(u(0, \cdot))}{j!} + \theta \tau^2 \int_0^\infty W(\xi_n^3) \frac{1}{j!} \sum_{j=0}^\infty \tau^{j-2} \langle \xi_n \rangle^3 \xi_n E_j(u) dx_0.
\]

It is now easy to see that (2.2) holds.

**Proof.** We now move to the proof of Theorem 1.1. If \( b_0 = 0 \) we do have \( C^\infty \) well posedness which is an easy consequence of the double characteristics theory. So we will assume \( b_0 \neq 0 \) in the following.

**Proof.** We will make use of standard energy estimates. We choose \( \theta = 1/3 \) and with \( M(x, D) = D_0^2 - \Omega/3 \) compute \( 2\mathcal{M}(\mathcal{P}u, \mathcal{M}u) \) which is, with \( B = b_0 x_1^2 \xi_n^3 \),

\[
(2.4) \quad 2\mathcal{M}(\mathcal{P}u, \mathcal{M}u) = -\frac{\partial}{\partial x_0} [||\mathcal{M}u||^2 + 2\mathcal{M}(\Omega D_0 u, D_0 u)]
\]

\[
= -2\mathcal{M}(D_0 u, D_0 u) + 2\mathcal{M}(\Omega D_0 u, D_0 u) + 2\mathcal{M}(\Omega D_0 u, \Omega/3) + 2\mathcal{M}(\Omega D_0 u, D_0^2 u) + 2\mathcal{M}(\Omega D_0 u, -b_0 x_1^2 \xi_n u, D_0^2 u) + 2\mathcal{M}(\Omega D_0 u, -b_0 x_1^2 \xi_n u, -\Omega/3 u).
\]

From (2.4) we get

\[
2\mathcal{M}(\mathcal{P}u, \mathcal{M}u) = -\frac{\partial}{\partial x_0} \mathcal{E}(u) + \mathcal{R}(u),
\]

where \( \mathcal{R}(u) = b_0 \mathcal{M}(D_0^2, x_1^2 \xi_n^3 u) u/3 \) and

\[
(2.5) \quad \mathcal{E}(u) = ||M u||^2 + \frac{2}{3} [\Omega D_0 u, D_0 u] + \frac{2}{9} [||\Omega u||^2 + 2b_0 \text{Re}(x_1^2 \xi_n^3 u, D_0 u)].
\]

From (2.5) we have

\[
(2.6) \quad \mathcal{E}(u) = ||M u||^2 + 2b_0 \text{Re}(x_1^2 \xi_n^3 u, x_1 \xi_n D_0 u) + \frac{2}{3} [||D_0 u||^2 + ||x_1 \xi_n D_0 u||^2] + \frac{2}{9} [||D_1 u||^2 + ||x_1^2 \xi_n^3 u||^2 + 2\text{Re}(D_1^2 u, x_1^2 \xi_n^3 u)].
\]

We write (2.6) like this:

\[
(2.7) \quad \mathcal{E}(u) = ||M u||^2 + \frac{2}{3} ||D_1 D_0 u||^2 + \frac{2}{3} \left[ \left( \frac{2}{3} x_1 \xi_n D_0 u + b_0 \left( \frac{2}{3} x_1^2 \xi_n^3 u \right) \right)^2 + \frac{2}{9} ||D_1^2 u||^2 \right]
\]

\[
+ \frac{2}{9} \left( 1 - \frac{27}{4} b_0^2 \right) [x_1^2 \xi_n^3 u]^2 + \frac{4}{9} \text{Re}(D_1^2 u, x_1^2 \xi_n^3 u).
\]
Noticing that $\text{Re}(D^2_t u, x^2_1 u) = [(x_1 D_1 u)^2 - \|u\|^2]$, we get from (2.7) that

$$E(u) = \|Mu\|^2 + \frac{2}{3} \|D_1 D_0 u\|^2$$

$$+ \left[ \left( \frac{2}{3} x_1 \xi_0 D_0 u + b_0 \sqrt{\frac{\beta}{2}} \xi_0 u \right)^2 + \frac{2}{3} \|D^2_t u\|^2 \right]$$

$$+ \frac{2}{9} \left( 1 - \frac{27}{4} b_0^2 \right) [(x_1^2 \xi_0^2 u)^2 + \frac{4}{9} (x_1 \xi_0 D_1 u)^2]$$

$$- \frac{4}{9} \xi_0^2 \|u\|^2.$$

Multiplying by $W$ and integrating from 0 to $\infty$ we have

$$W \int_0^\infty \text{Im} \langle Pu, Mu \rangle dx_0 = W(0) \langle E(u, 0, \cdot) \rangle$$

$$+ 2\tau (\xi_0)^\frac{1}{3} \int_0^\infty W \left\{ \|Mu\|^2 + \frac{2}{3} \|D_1 D_0 u\|^2 \right\}$$

$$+ \left[ \left( \frac{2}{3} x_1 \xi_0 D_0 u + b_0 \sqrt{\frac{\beta}{2}} \xi_0 u \right)^2 + \frac{2}{3} \|D^2_t u\|^2 \right]$$

$$+ \frac{2}{9} \left( 1 - \frac{27}{4} b_0^2 \right) [(x_1^2 \xi_0^2 u)^2 + \frac{4}{9} (x_1 \xi_0 D_1 u)^2]$$

$$- \frac{4}{9} \xi_0^2 \|u\|^2 \right\} dx_0$$

$$- 2 b_0 (\xi_0)^{\frac{1}{3}} \int_0^\infty W \text{Re} \langle x^2_1 u, D_1 u \rangle dx_0.$$

Recalling (2.2) from Lemma 2.1 now with $1 \leq s \leq 2$ we can dispose of the negative contribution in (2.8) $-4 \xi_0^2 \|u\|^2/9$, choosing $\tau$ large because $\langle \xi_0 \rangle^2 \geq \xi_0^2$.

Let us now deal with the remainder term

$$\int_0^\infty W \text{Re} \langle x^2_1 u, D_1 u \rangle dx_0.$$

Applying Cauchy-Schwarz inequality we get

$$\langle \xi_0 \rangle^\frac{1}{2} (\|x^2_1 \xi_0^2 u\|^2 + \langle \xi_0 \rangle^2 \|D_1 u\|^2)$$

$$\leq \langle \xi_0 \rangle^\frac{1}{2} (\|x^2_1 \xi_0^2 u\|^2 + \langle \xi_0 \rangle^4 \|u\|^2 + \|D^2_t u\|^2).$$

It is clear that $\|D^2_t u\|^2 \leq 4 (\|Mu\|^2 + \|x_1 \xi_0^2 u\|^2 + \|D^2_t u\|^2).$ Using (2.8) and $1 + 2/s \geq 2 \geq 4 - 4/s$ we obtain for any $u \in C^\infty_0 (\mathbb{R}^n)$

$$\int_0^\infty W \|Pu\|^2 dx_0 \geq CW(0) \sum_{j=0}^2 2^{j-3/2}$$

$$\times \langle \xi_0 \rangle^{\frac{5-2j}{3}} \int_0^\infty W \|E_j(u(0, \cdot)) + C \sum_{j=0}^2 \|u^{6-2j}\| \int_0^\infty W \|E_j(u(0, \cdot)) dx_0$$

if $\tau$ is large enough and $1 \leq s \leq 2$. Let $s = 2$ and

$$E_j(u) = \int \text{Im} \langle Pu, Mu \rangle dx_0$$

$$= \int_0^\infty \|D^2_t u\|^2 + |D^2_t u|^2 + |(x_1 D_1 u)^2| dx'.

Then for any $u \in C^\infty_0 (\mathbb{R}^{n+1})$ vanishing in $x_0 \geq a$ we integrate (2.2) with respect to $\xi_0$ we get

$$\int_0^a \|\mathbb{E}^{\xi_0} (D^2_t (x_0-a) Pu)\|^2 dx_0$$

$$\geq C \sum_{j=0}^2 \int_0^a \mathbb{E}_j \|\mathbb{E}^{\xi_0} (D^2_t (x_0-a) Pu)\|^2 (\xi_0, 0) \right\rangle dx_0.$$
\[ D_j^k u \in L^2(I; H'^{-3/2-j}(\mathbb{R}^n)) \]
for \( j = 0, 1, 2, 3 \). Thus we get a smooth solution in \( I \times \mathbb{R}^n \) provided (2.9) is verified and choosing \( \ell \) large.

3. Optimality of the Gevrey index.

3.1. Sibuya’s results. The differential equation

\[(3.10) \quad w''(y) = (y^3 + \zeta y)w(y)\]
will play a very important role in the construction of the family of solutions leading to the optimality of the Gevrey index \( s = 2 \). Therefore we recap briefly, in this special setting, the general theory of subdominant solutions and Stokes coefficients for the equation (3.10), following the presentation found, for example, in the book of Sibuya [9].

Theorem 6.1 in [9] states that the differential equation (3.10) has a unique solution

\[ w(y; \zeta) = Y(y; \zeta) \]
such that

(i) \( Y(y; \zeta) \) is an entire function of \( (y, \zeta) \).
(ii) \( Y(y; \zeta) \) and its derivative \( Y'(y; \zeta) \) admit asymptotic representations

\[ y^{-3/4} \left[ 1 + \sum_{N=1}^{\infty} B_N y^{-N/2} \right] e^{-E(y, \zeta)} = Y(y, \zeta), \]
\[ y^{3/4} \left[ -1 + \sum_{N=1}^{\infty} C_N y^{-N/2} \right] e^{-E(y, \zeta)}, \]
uniformly on each compact set in the \( \zeta \) space as \( y \) goes to infinity in any closed subsector of the open sector \( |\arg y| < 3\pi/5 \) more over

\[ E(y, \zeta) = \frac{2}{5} y^{5/2} + \zeta y^{1/2} \]
and \( B_N, C_N \) are polynomials in \( \zeta \).

We note that if we set \( \omega = \exp(2\pi i/5) \) and

\[ Y_k(y; \zeta) = Y(\omega^{-k} y; \omega^{-2k} \zeta) \]
where \( k = 0, 1, 2, 3, 4 \) then \( Y_k(y; \zeta) \) solve (3.10). In particular \( Y_0(y; \zeta) = Y(y; \zeta) \). Then we have

(i) \( Y_k(y; \zeta) \) is an entire function of \( (y, \zeta) \).
(ii) \( Y_k(y; \zeta) \sim Y(\omega^{-k} y; \omega^{-2k} \zeta) \) uniformly on each compact set in the \( \zeta \) space as \( y \) goes to infinity in any closed subsector of the open sector

\[ |\arg y - 2k\pi/5| < 3\pi/5. \]

Let \( S_k \) denote the open sector defined by

\[ |\arg y - 2k\pi/5| < 3\pi/5. \]

Since

\[(3.11) \quad \text{Re}[y^{5/2}] > 0 \quad \text{for} \quad y \in S_0 \]
and \( \text{Re}[y^{5/2}] < 0 \quad \text{for} \quad y \in S_{-1} = S_1 \) and for \( S_1 \) the solution \( Y_0(y; \zeta) \) is subdominant in \( S_0 \) and dominant in \( S_1 \) and \( S_0 \). Similarly \( Y_k(y; \zeta) \) is subdominant in \( S_k \) and dominant in \( S_{k-1} \) and \( S_{k+1} \). It is clear that \( Y_{k+1} \) and \( Y_{k+2} \) are linearly independent. Therefore \( Y_k \) is a linear combination of those two:

\[ Y_k(y; \zeta) = C_k(\zeta)Y_{k+1}(y; \zeta) + \tilde{C}_k(\zeta)Y_{k+2}(y; \zeta) \]
where \( C_k, \tilde{C}_k \) are called the Stokes coefficients for \( Y_k(y; \zeta) \). We summarize in the following statements some of the known and useful facts about the Stokes coefficients for our particular equation (3.10). Proofs can be found in Chapter 5 of [9].

**Proposition 3.1.** The following results hold.

(i) \( \tilde{C}_k(\zeta) = -\omega^j \) for any \( k \) and \( \zeta \).
(ii) \( C_k(\zeta) = C_0(\omega^{-2k} \zeta) \) for any \( k, \zeta \) and \( C_0(\zeta) \) is an entire function of \( \zeta \).
(iii) \( C_0(0) = 1 + \omega \) for any \( k \).
(iv) \( \partial_\zeta C_0(\zeta)|_{\zeta = 0} \neq 0 \).

**Proposition 3.2.** We have

\[ C_0(\zeta) + \omega^2 C_0(\omega \zeta) C_0(\omega^4 \zeta) = 0 \]
for any \( \zeta \in \mathbb{C} \).

**3.2. Localization of zeros.** We now state a key lemma which is proved in [1].

**Lemma 3.1.** The Stokes coefficient \( C_0(\zeta) \) vanishes in at least one (non zero) \( \zeta_0 \).

Now that we know that \( C_0(\zeta) \) vanishes somewhere, we would like to find out where exactly this happens. We begin with a symmetry result, recalling Proposition 3 in [10]:

**Lemma 3.2.** The Stokes coefficient \( C_0(\zeta) \) verifies the equivalence

\[ C_0(\zeta) = 0 \iff C_0(\omega \zeta) = 0. \]

The following is a very important step in the construction of the null solutions, and is a sharp result on the location of the zeros of the entire function \( C_0(\zeta) \).

**Lemma 3.3.** There exists \( \zeta_0 \in W = \{ z \in \mathbb{C} | \pi < \arg z \leq 19\pi/15 \} \) where \( C_0(\zeta_0) = 0 \).

**Proof.** We recall from Proposition 3.1 in [7] that \( C_0(\zeta) = 0 \) implies either \( \zeta \in W_1 = \{ \pi \leq \arg \zeta \leq 19\pi/15 \} \) or \( \zeta \in W_2 = \{ \pi/3 \leq \arg \zeta \leq 3\pi/5 \} \). But \( W_1 \) and \( W_2 \) are symmetric under the mapping \( \zeta \to \bar{\zeta} \). We just have to show the \( \arg \zeta \neq \pi \).

Proposition 3.2 and Lemma 3.2 above together
imply that \( C_0(\zeta) \neq 0 \) if \( \zeta \) is real.

### 3.3. Proof of Theorem 1.2

Consider again the operator:

\[
P(x, D) = D_0^2 - (D_1^2 + x_2^2 D_2^2)D_0 - b_0 x_1^2 D_1^2.
\]

Let \( \lambda > 0 \) be a positive large parameter, \( R > 0, \theta \in [0, \pi[ \) to be chosen later and consider:

\[
U(x, \lambda, R, \theta) = E(x_0, x_n, \lambda)w(Ax + B),
\]

with \( E(x_0, x_n, \lambda) = e^{i\pi x_0^3}e^{i\pi x_n^3} \) and \( A, B \) to be chosen together with \( w \). Sometimes the \( x_0 \) components of \( x \) will be omitted to enhance readability. It is easy to see

\[
 PU = \left( \frac{1}{2} R^2 e^{i\pi} - \lambda^{1/2} Re^{-\theta} x_2^2 - b_0 \lambda^2 x_1^2 x_2^2 + \lambda^{1/2} Re^{-\theta} \frac{w}{w'} (Ax + B) \right) U.
\]

Thus setting \( y = Ax + B \) we have from (3.13) and the request that \( PU = 0 \),

\[
w''(y) = \lambda^{1/2} R^{-1} e^{-i\theta} A^{-2} \left[ \frac{b_0 \lambda^3}{A^2} y^3 + \left( -3 \frac{b_0 \lambda B}{A^3} + \lambda^{1/2} Re^{-\theta} \right) y \right] + \left( \frac{3b_0 \lambda B^2}{A^3} - 2 \lambda^{1/2} Re^{-\theta} \right) y - \frac{b_0 \lambda^2 B}{A^2} + \lambda^{1/2} Re^{-\theta} \frac{B^2}{A^2} - \lambda^{1/2} e^{i\theta} \frac{R^2}{A^3} w(y).
\]

The following choices are then made:

\[
\lambda^{1/2} R^{-1} e^{-i\theta} \frac{b_0 \lambda^3}{A^2} = 1, \quad -3 \frac{b_0 \lambda B}{A^3} + \lambda^{1/2} Re^{-\theta} = 0.
\]

Then (3.15) yields

\[
A = \lambda^{1/2} b_0^{1/5} R^{-1/5} e^{-i\theta/5}, \quad B = \frac{R^{1/5} b_0^{-4/5} e^{i\theta/5}}{3}.
\]

Using these values we have from (3.14)

\[
w''(y) = (y^3 + \zeta y + \mu)w(y),
\]

with

\[
\zeta = -\frac{b_0^{3/5} e^{i\theta/5} R^{1/5}}{3}, \quad \mu = \lambda R^2 e^{i\theta} A^{-2} \left( \frac{2}{27 b_0^{2/5}} - 1 \right).
\]

It is now clear that choosing \( b_0 = \frac{\sqrt{2}}{3} \in ]0, \frac{2}{3}] \) will give us equation (3.10).

We now choose \( w(y; \zeta) = y \theta(y; \zeta) \) with \( \theta \) found in Lemma 3.3 and from (3.16) we take \( y = \frac{1}{b_0} \frac{1}{R^2} R^{-1/5} R^{1/5} e^{-i\theta} x_1 + \frac{1}{b_0} \frac{1}{R^2} R^{1/5} e^{i\theta} x_1 \).

We have that \( b_0^{3/5} R^2 e^{i\theta} \pi/15 = 3|\zeta| e^{i\theta} \pi |\zeta| = \pi < \arg \zeta \leq 19\pi/15 \). This clearly leaves us with \( 0 < \theta = \arg \zeta < \pi/6 \), while the number \( R \), still at our disposal, is chosen to fix the absolute values, thus \( R = R_0 > 0 \), depending on \( b_0 \) and \( |\zeta| \).

Recall that \( y \theta(y; \zeta) = y \theta(\omega^{-1} y, \omega^{-1} \zeta) \) and that

\[
y \theta(y; \zeta) = -\omega y \theta(y; \zeta) \quad \text{since} \quad C_0(\zeta) = 0.
\]

We notice that when \( x_1 > 0 \) and \( \lambda \) is large \( \arg(y) \in [-\pi/3, 0] \) clearly well inside the subdomain sector \( \mathcal{S}(R) \) and moreover \( w(y; \zeta) \) is bounded on \( R \) uniformly in \( \lambda \). Let

\[
U_\lambda = e^{i(T-x_0)} e^{i\theta/5} x_2^5 w(y; \zeta)
\]

then \( PU_\lambda = 0 \) because \( P(x_1, D_1, D_2, -D_0) = -P(x_1, D_0, D_1, D_2) \). Suppose now that there exist a neighborhood \( \omega \) of the origin and \( u \in C^3(\omega) \) satisfying

\[
\begin{cases}
PU = 0 \quad \text{in} \omega \\
u(0, x') = 0, \quad D_0 u(0, x') = 0 \quad \text{on} \omega' \\
D_0^2 u(0, x') = \phi_1(x_1) \psi_2(x'' \phi_3(x_n)) \quad \text{on} \omega' \end{cases}
\]

where \( \omega' = \omega \cap \{ x_0 = 0 \} \) and \( \phi_1(x_1) \in \gamma_0^{(c)}(\mathbf{R}), \phi_2(x'') \in \gamma_0^{(c)}(\mathbf{R}^{n-2}), \phi_3(x_n) \in \gamma_0^{(c)}(\mathbf{R}) \).

From the Holmgren uniqueness theorem we can assume that \( u(x) = 0 \) if \( 0 < x_0 \leq T \), \( |x'| \geq r \) for small \( T > 0 \) and \( r > 0 \). Then from \( 0 = \int_0^T (PU_\lambda, u)dx_0 - \int_0^T (U_\lambda, Pu)dx_0 \) we have

\[
(U_\lambda(0), D_0^2 u(0)) = \sum_{j=0}^2 (D_0^2 U_\lambda(T), D_0^2 u(T)).
\]

The right-hand side is \( O(\lambda^2) \) because \( w(y; \zeta) \)

\[
\lambda^{-1/2} D_0 w(y; \zeta) \] are bounded uniformly in \( \lambda \). On the other hand the left-hand side is

\[
\psi(\lambda) e^{i\lambda^{1/2} T R e^{C \lambda}} \int w(y; \zeta) \phi_1(x_1) \phi_2(x'')dx_1dx''
\]
We choose $\phi_2$ so that $\int \phi_2 (x''') dx'' \neq 0$. Recall that $\psi \in \gamma_0^\infty (\mathbb{R})$ if and only if $|\psi(\xi)| \leq C e^{-L|\xi|^p}$ with some $L > 0$, $C > 0$. Thus if we take $\psi \notin \gamma_0^\infty (\mathbb{R})$ which is even then $\lambda^{-N} \psi(\lambda) e^{2i\lambda^2 \frac{R_n}{n} e^{\theta_i \eta_i}}$ is not bounded as $\lambda \to \infty$ for any $N \in \mathbb{N}$. Checking that

$$\lambda^n \int w(y; \zeta_n) \phi_1 (x_1) dx_1 \to c \neq 0$$

with a suitable choice of $\phi$ and $\kappa = \zeta$ in $\mathbb{R}$ we could get a contradiction proving non local solvability of (3.19).

Let $
abla = b_0^{1/5} R_0^{-1/5} e^{-i\theta_i/5}$, $\beta = b_0^{-1/5} R_0^{1/5} e^{i\theta_i/5}/3$; and note that it is enough to show $\int w(\alpha x_1 + \beta; \zeta_0) x_1 dx_1 \neq 0$ for at least one $k = 0, 1, 2$. Put

$$v(\xi) = \int e^{-i\xi x} w(\alpha x + \beta; \zeta_0) dx$$

then $v(\xi)$ satisfies the equation

$$(i\alpha \frac{d}{d\xi} + \beta)^3 v(\xi) + \zeta_0 (i\alpha \frac{d}{d\xi} + \beta) v(\xi) + \alpha^{-2} \xi^2 v(\xi) = 0$$

and

$$v^{(k)}(0) = (-i)^k \int w(\alpha x + \beta; \zeta_0) x^k dx.$$ 

So if $v^{(k)}(0) = 0$ for $k = 0, 1, 2$ then we would have $v(\xi) = 0$ so that $w(\alpha x + \beta; \zeta_0) = 0$ which is a contradiction.

4. Cones and factorization. Here we briefly verify that the propagation cone is not transversal to the triple manifold. Let $p(x, \xi) = \xi_1 + (\xi_1^2 + x_2^2) \zeta_0 = b_0 x_2^2 \xi_1^3$ be the symbol of the operator (1.1). $p$ vanishes exactly of order $3$ on $\Sigma_3 = \{ x_1 = \zeta_0 = \xi_1 = 0 \}$ near $(0; 0, \ldots, 0; 0)$. Fix $z \in \Sigma_3$ and take $\delta v = (-1, 0, \ldots, 0; 0).$ Clearly $\delta v \in T_z \Sigma_3$ and, since $\sigma(\delta v, (\delta y, \delta y)) = -\delta \eta_0 \leq 0$ if $(\delta y, \delta y) \in \Gamma_2$, we have that $\Gamma_2 \cap T_z \Sigma_3 \neq 0$. On the other hand $\Gamma_2$ cannot be completely contained in $T_z \Sigma_3$, because otherwise $T_z \Sigma_3 \subset C_2$ and this would imply that $\langle H_0, \bar{H}_0, H_0 \rangle \subset C_2$, which is false. Therefore $\Gamma_2$ is neither disjoint from nor totally inside $T_z \Sigma_3$. For the next item we change slightly the notations in order to simplify the treatment of a third degree equation naturally associated with the problem.

Let us show that for our model no root is $C^\infty$. Let $p = x^3 - 3(x^2 + \xi^2)^2 x - 2bx^3$, with $0 < |b| < 1$. If $p$ could be written like $p = (\tau - L(x, \xi))(\tau^2 + A(x, \xi) \tau + B(x, \xi))$ with $C^\infty$ functions $L, A, B, one then would get $A = L, \ L^2 - B = 3(x^2 + \xi^2)$ and $LB = 2bx^3$. This shows that we have $L = xL_1(x, \xi)$ or $B = xB_1(x, \xi)$ with $C^\infty$ smooth $L_1, B_1$. If $B = xB_1$ then $L(0, \xi)^2 = 3\xi^2$ so that $L(0, \xi) = \pm \sqrt{3}\xi$ or $L(0, \xi) = -\sqrt{3}\xi$ and hence $L = \pm \sqrt{3}\xi + xL_1$. From $LB = x(\pm \sqrt{3}\xi + xL_1)B_1 = -2bx^3$, we would have $B_1 = x^2 B_2$ so that $B = x^2 B_1$ which is incompatible with $LB = -2bx^3$. If $L = xL_1$ then from $L^2 - B = x^2 L_1^2 - B = 3(x^2 + \xi^2)$ we would have $B = -3\xi^2 + x^2 B_1$. Then from $LB = x(-3\xi^2 + x^2 B_1)B_1 = -2bx^3$ we would have $L_1 = x^2 L_2$ which is incompatible with $LB = -2bx^3$. This contradiction proves that $p$ cannot be smoothly factorized.

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