WHITTAKER MODULES FOR THE SUPER-VIRASORO ALGEBRAS

DONG LIU, YUFENG PEI, AND LIMENG XIA

Abstract. In this paper, we define and study Whittaker modules for the super-Viraoro algebras, including the Neveu-Schwarz algebra and the Ramond algebra. We classify the simple Whittaker modules and obtain necessary and sufficient conditions for irreducibility of these modules.

1. Introduction

Whittaker vectors and Whittaker modules play a critical role in the representation theory of finite-dimensional simple Lie algebras (cf. [2, 17]). Recently Whittaker modules have been intensively studied for many infinite dimensional Lie algebras such as the Virasoro algebra [11, 12, 22, 23], Heisenberg algebras [8], and affine Kac-Moody algebras [11]. Analogous results in similar settings have been worked out for many Lie algebras with triangular decompositions (cf. [7, 18, 28, 27, 30] and references therein). A general categorial framework for Whittaker modules was proposed in [3, 19]. Degenerate Whittaker vectors of the Virasoro algebra naturally appear in the AGT conjecture in physics [10]. An explicit formula for degenerate Whittaker vectors has been obtained in terms of the Jack symmetric functions in [29].

Whittaker modules and Whittaker categories have been generalized to finite-dimensional simple Lie superalgebras based on some representations of nilpotent finite-dimensional Lie superalgebras in [5]. In this paper, we define and study Whittaker modules for the super-Viraoro algebras, which are Lie superalgebras and known in literature under the name the N=1 superconformal algebras [6, 20, 24]. It is known that they can be viewed as certain supersymmetric extensions of the Virasoro algebra and arise as the covariant constraints in the classical formulation of the RNS model. Representations for the super-Viraoro algebras have been extensively investigated (cf. [13, 14, 15, 16, 26]). We classify all finite-dimensional simple modules over certain subalgebra over the super-Virasoro algebra. Furthermore we classify the simple Whittaker modules and obtain necessary and sufficient conditions for irreducibility of these modules. It is worth remarking that degenerate Whittaker vectors of the super-Virasoro algebras have been investigated in [9].

The aforementioned results demonstrate that the Whittaker modules defined in present paper satisfy some properties that their non-super analogues do. However, there are several differences and some features that are new in the super case. It has been observed in [5], Mathematics Subject Classification: 17B65; 17B66; 17B70.
simple finite-dimensional modules for a finite-dimensional nilpotent Lie superalgebra are not always one-dimensional [25]. This leads to an additional challenge for generalizing Lie algebra results in the Lie superalgebra setting. In our settings, finite-dimensional simple modules over the positive parts of the super-Virasoro algebras are proved to be two-dimensional, in contrast to the situation for the Virasoro algebra. For this reason, our definition of Whittaker modules is closely related to certain smaller subalgebras.

The paper is arranged as follows. In Section 2, we recall some notations and collect known facts about the super-Viraoro algebras. In Section 3, we classify all finite-dimensional simple modules over certain subalgebra of the super-Viraoro algebras. In Section 4, we classify all simple Whittaker modules and obtain necessary and sufficient conditions for irreducibility of these modules.

Throughout this paper, we shall use \( \mathbb{C}, \mathbb{C}^*, \mathbb{N}, \mathbb{Z}_+ \) and \( \mathbb{Z} \) to denote the sets of complex numbers, nonzero complex numbers, non-negative integers, positive integers and integers respectively. For convenience, all elements in superalgebras and modules are homogenous unless specified.

2. Preliminaries

In this section, we introduce the notation and conventions that will be used throughout the paper.

Let \( V = V_0 \oplus V_1 \) be any \( \mathbb{Z}_2 \)-graded vector space. Then any element \( u \in V_0 \) (resp. \( u \in V_1 \)) is said to be even (resp. odd). We define \( |u| = 0 \) if \( u \) is even and \( |u| = 1 \) if \( u \) is odd. Elements in \( V_0 \) or \( V_1 \) are called homogeneous. For convenience all elements in superalgebras and modules are homogenous unless specified throughout this paper.

Let \( \mathfrak{g} \) be a Lie superalgebra, a \( \mathfrak{g} \)-module is a \( \mathbb{Z}_2 \)-graded vector space \( V \) together with a bilinear map \( \mathfrak{g} \times V \to V \), denoted \( (x, v) \mapsto xv \) such that

\[
x(yv) - (-1)^{|x||y|}y(xv) = [x, y]v
\]

and

\[
\mathfrak{g}_i V_j \subseteq V_{i+j}
\]

for all \( x, y \in \mathfrak{g}, v \in V \). Thus there is a parity-change functor \( \Pi \) on the category of \( \mathfrak{g} \)-modules, which interchanges the \( \mathbb{Z}_2 \)-grading of a module. We use \( U(\mathfrak{g}) \) to denote the universal enveloping algebra.

All modules for Lie superalgebras considered in this paper are \( \mathbb{Z}_2 \)-graded and all simple modules are nontrivial.

**Definition 2.1.** The super-Virasoro algebras are the Lie superalgebras

\[
\text{SVir}_\epsilon = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{r \in \mathbb{Z}+\epsilon} \mathbb{C}G_r \oplus \mathbb{C}\epsilon
\]
which satisfies the following commutation relations:

\[ [L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} C, \]
\[ [L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r}, \]
\[ [G_r, G_s] = 2L_{r+s} + \frac{1}{3} \delta_{r+s,0} \left( r^2 - \frac{1}{4} \right) C, \]
\[ [SVir_\epsilon, C] = 0, \]

for all \( m, n \in \mathbb{Z}, \epsilon = \frac{1}{2}, 0, r, s \in \mathbb{Z} + \epsilon \). SVir\(_0\) is called the Ramond algebra and SVir\(_{\frac{1}{2}}\) is called the Neveu-Schwarz algebra.

By definition, we have the following decompositions:

\[ SVir_\epsilon = SVir^0_\epsilon \oplus SVir^1_\epsilon, \]

where

\[ SVir^0_\epsilon = \bigoplus_{n \in \mathbb{Z}} C L_n \oplus C C, \quad SVir^1_\epsilon = \bigoplus_{r \in \mathbb{Z} + \epsilon} C G_r. \]

It is clear that SVir\(^0_\epsilon\) is isomorphic to the well-known Virasoro algebra Vir.

The Neveu-Schwarz algebra SVir\(_\epsilon\) has a \((1 - \epsilon)\mathbb{Z}\)-grading by the eigenvalues of the adjoint action of \( L_0 \). It follows that SVir\(_\epsilon\) possesses the following triangular decomposition:

\[ SVir_\epsilon = SVir^{+}_\epsilon \oplus SVir^{0}_\epsilon \oplus SVir^{-}_\epsilon \]

where

\[ SVir^{\pm}_\epsilon = \bigoplus_{n \in \mathbb{N}} C L_{\pm n} \oplus \bigoplus_{r \in \mathbb{N} + \epsilon} C G_{\pm r}, \quad SVir^{0}_\epsilon = C L_0 \oplus C \delta_{\epsilon,0} G_0 \oplus C C. \]

Set

\[ p_\epsilon = \bigoplus_{n \geq 1} C L_n \oplus \bigoplus_{n \geq 2} C G_{n-\epsilon}. \]

It is clear that \( p_\epsilon = p^{0}_\epsilon \oplus p^{1}_\epsilon \) is a subalgebra of SVir\(^+\), where

\[ p^{0}_\epsilon = \bigoplus_{n \geq 1} C L_n, \quad p^{1}_\epsilon = \bigoplus_{n \geq 2} C G_{n-\epsilon}. \]

**Definition 2.2.** For \( c \in \mathbb{C} \), let \( \psi : p_\epsilon \to \mathbb{C} \) be a Lie superalgebra homomorphism. A SVir\(_\epsilon\)-module \( M_\epsilon \) is called a Whittaker module of type \((\psi, c)\) if

(i) \( M_\epsilon \) is generated by a homogeneous vector \( w \);

(ii) \( xw = \psi(x)w \) for any \( x \in p_\epsilon \);

(iii) \( Cw = cw \),

where \( w \) is called a Whittaker vector of \( M \).
Let $\psi : p_e \to \mathbb{C}$ be a Lie superalgebra homomorphism. Let $Cw_\psi$ be one-dimensional $p_e$-module with $xw_\psi = \psi(x)w_\psi$ for any $x \in p_e$ and $Cw_\psi = cw_\psi$ for some $c \in \mathbb{C}$. Define induced module

$$W_\epsilon(\psi, c) =: \text{Ind}_{p_e \otimes C}^{\mathfrak{svir}} Cw_\psi = U(\mathfrak{svir}_\epsilon) \otimes_{U(p_e \otimes C)} Cw_\psi.$$  

Then $W_\epsilon(\psi, c)$ is a Whittaker module of level $c$ for $\mathfrak{svir}_\epsilon$.

Clearly $W_\epsilon(\psi, c)$ contains a unique maximal submodule, and its irreducible quotient is denoted by $L_\epsilon(\psi, c)$.

**Remark 2.3.** When $\psi$ is trivial ($\psi = 0$), the Whittaker module $W_\epsilon(0, c)$ and any $h \in \mathbb{C}$, $L_0w - hw$ is a Whittaker vector. Moreover $W_\epsilon(0, c)/ < L_0 - hw >$ is the standard Verma module over the super Virasoro algebra $\mathfrak{svir}_\epsilon$, which were studied in [13], [14], etc..

We define a **pseudopartition** $\lambda$ to be a non-decreasing sequence of non-negative integers

$$\lambda = (0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m). \tag{2.1}$$

Denote by $\mathcal{P}$ the set of pseudopartitions. Similarly, we define a **strict pseudopartition** $\lambda$ to be a strict increasing sequence of non-negative integers

$$\lambda = (0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_m). \tag{2.2}$$

Denote by $\mathcal{Q}$ the set of strict pseudopartitions. We also introduce an alternative notation for pseudopartitions. For $\lambda \in \mathcal{P}$ or $\mathcal{Q}$, write

$$\lambda = (0^{\lambda(0)}, 1^{\lambda(1)}, 2^{\lambda(2)}, \ldots), \tag{2.3}$$

where $\lambda(k)$ is the number of times $k$ appearing in the pseudopartition and $\lambda(k) = 0$ for $k$ sufficiently large. Clearly if $\mu \in \mathcal{Q}$, then $\mu(k) \in \{0, 1\}$.

For $\lambda \in \mathcal{P}$, $\mu \in \mathcal{Q}$, define elements $L_{-\lambda}, G_{\pm \epsilon - \mu} \in U(\mathfrak{svir}_\epsilon)$ by

$$L_{-\lambda} = L_{-\lambda_s} \cdots L_{-\lambda_1} L_{-\lambda_1} = \cdots L_{-1}^{\mu(1)} L_0^{\lambda(0)},$$

$$G_{\pm \epsilon - \mu} = G_{\pm \epsilon - \mu_s} \cdots G_{\pm \epsilon - \mu_1} G_{\pm \epsilon - \mu_1} = \cdots G_{\pm \epsilon - 2} G_{\pm \epsilon - 1} G_{\pm \epsilon}^{\mu(1)} G_{\pm \epsilon}^{\mu(0)}.$$  

Define

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_s = \sum_{i \geq 0} i\lambda(i) \quad (\text{the size of } \lambda),$$

$$|\mu \pm \epsilon| = (\mu_1 \pm \epsilon) + (\mu_2 \pm \epsilon) + \cdots + (\mu_r \pm \epsilon).$$

Define $\mathcal{Q} = (0^0, 1^0, 2^0, \ldots)$, and write $L_{\mathcal{Q}} = 1 \in U(\mathfrak{svir}_\epsilon)$. We will consider $\mathcal{Q}$ to be an element of $\mathcal{P}$. For any $\lambda \in \mathcal{P}$ and $\mu \in \mathcal{Q}$, $L_{-\lambda} G_{\pm \epsilon - \mu} \in U(\mathfrak{svir}_\epsilon)_{-|\lambda|-|\mu|+\epsilon}$, where $U(\mathfrak{svir}_\epsilon)_{-|\lambda|-|\mu|+\epsilon}$ is the $-|\lambda| - |\mu| + \epsilon$-weight space of $U(\mathfrak{svir}_\epsilon)$ under the adjoint action. In particular, if $\lambda \in \mathcal{P}, \mu \in \mathcal{Q}$, then $L_{-\lambda} G_{\pm \epsilon - \mu} \in U(\mathfrak{svir}_\epsilon)_{-|\lambda|-|\mu|+\epsilon}$. 

3. Simple modules over SVir$^+_\epsilon$

In this section we classify all finite dimensional simple modules over the subalgebras SVir$^+_\epsilon$ and $p_\epsilon$.

**Lemma 3.1.** Let $\psi : p_\epsilon \to \mathbb{C}$ be a Lie superalgebra homomorphism. Then

$$\psi(L_n) = 0, \quad \psi(G_r) = 0$$

for $n \in \mathbb{Z}, n \geq 3$ and $r \in \mathbb{Z} + \epsilon, r \geq 2 - \epsilon$

**Proof.** It follows immediately from Definition 2.1. \qed

**Lemma 3.2.** Let $\psi : p_\epsilon \to \mathbb{C}$ be a Lie superalgebra homomorphism and $A_\epsilon(\psi) = \mathbb{C}w_\psi \oplus \mathbb{C}u_\psi$ with be a two-dimensional vector space with

$$xw_\psi = \psi(x)w_\psi, \quad G_{1-\epsilon}w_\psi = u_\psi, \quad \forall x \in p_\epsilon.$$

Then

1. $A_\epsilon(\psi)$ is a $(1|1)$-dimensional SVir$^+_\epsilon$-module.
2. $A_{\epsilon}\mathbb{Z}(\psi)$ is simple if and only if $\psi$ is non-trivial.
3. $A_0(\psi)$ is simple if and only if $\psi(L_2) \neq 0$.
4. $A_0(\psi)$ with $\psi(L_2) = 0$ and $\psi(L_1) \neq 0$ has a nontrivial submodule spanned by the vector $u_\psi$.
5. $A_\epsilon(0)$ has a trivial submodule spanned by the vector $u_\psi$.

**Proof.** It is straightforward to verify. \qed

**Proposition 3.3.** Assume $V$ is a finite-dimensional simple module over SVir$^+_\epsilon$. Then $V$ is isomorphic to $A_\epsilon(\psi)$ or its simple sub(or quotient)-module, up to parity-change, where $\psi : p_\epsilon \to \mathbb{C}$ is a non-trivial Lie superalgebra homomorphism.

**Proof.** Suppose that $V$ is a finite-dimensional simple SVir$^+_\epsilon$-module. $V$ can be regarded as a finite-dimensional Vir$^+$-module. Let $W$ be a simple Vir$^+$-submodule of $V$. It follows from Proposition 12 in [19] that $W$ has to be one-dimensional. Then $W = \mathbb{C}w_\psi$ and $L_nw_\psi = c_nw_\psi$ for all $n \in \mathbb{Z}_+$, where $c_1, c_2 \in \mathbb{C}$ and $c_i = 0$ for all $i \geq 3$.

Next we always assume that $\epsilon = \frac{1}{2}$ since the proof of $\epsilon = 0$ case can be managed by essentially the same way.

If $G_{\frac{1}{2}}w_\psi = 0$, $V$ becomes a trivial SVir$^+_\epsilon$-module. Assume that $G_{\frac{1}{2}}w_\psi \neq 0$. Let $U$ be a vector space spanned by $u_i = G_{i+\frac{1}{2}}w_\psi$ for $i \in \mathbb{N}$.

**Claim 1.** $U$ is a finite-dimensional Vir$^+$-module.

It follows from

$$L_nu_i = \psi(L_n)u_i + \left(\frac{n}{2} - i - \frac{1}{2}\right)u_{n+i} \in U, \quad n \in \mathbb{Z}_+, i \in \mathbb{N}.$$
Corollary 3.5. Let \( U \) be the vector space spanned by \( u_0, \ldots, u_i \). If there \( m \in \mathbb{Z}_+ \) such that \( u_m \in U_{m-1}, u_{m+k} \in U_{m-1} \) for \( k \in \mathbb{Z}_+ \).

In fact, if

\[
    u_m = a_0 u_0 + a_1 u_1 + \cdots + a_{m-1} u_{m-1}
\]

for \( a_i \in \mathbb{C} \), for \( k = 1 \), we get

\[
    u_{m+1} = -\frac{1}{m} \left( \sum_{i=0}^{m-1} a_i L_1 u_i \right) + \psi(L_1) u_m \in U_{m-1}.
\]

By induction on \( k \), we have \( u_{m+k} \in U_{m-1} \) for \( k \in \mathbb{Z}_+ \).

Claim 2. Let \( U_i \) be the vector space spanned by \( u_0, \ldots, u_i \). If there \( m \in \mathbb{Z}_+ \) such that

\[
    u_m \in U_{m-1}, \quad u_{m+k} \in U_{m-1} \quad \text{for} \quad k \in \mathbb{Z}_+.
\]

By Claim 3, there exists a positive integer \( m \in \mathbb{Z}_+ \) such that \( u_m = 0 \).

Since \( U \) is finite-dimensional, we can choose minimal \( p \in \mathbb{Z}_+ \) such that

\[
    a_{i_1} u_{i_1} + a_{i_2} u_{i_2} + \cdots + a_{i_p} u_{i_p} = 0,
\]

for \( a_{i_1}, a_{i_2}, \ldots, a_{i_p} \in \mathbb{C}^* \), where \( i_1 < i_2 < \cdots < i_p \). Then by action of \( L_{2i_1+1} \) on both sides of

(3.2) we get

\[
    (i_1 - i_2) a_{i_2} u_{2i_1+1+i_2} + (i_1 - i_3) a_{i_3} u_{2i_1+1+i_3} + \cdots + (i_1 - i_p) a_{i_p} u_{2i_1+1+i_p} = 0,
\]

which implies \( p = 1 \). It gets the Claim 3.

By Claim 3, there exists \( n_0 \in \mathbb{Z}_+ \) such that \( G_{n_0+\frac{1}{2}} w_{\psi} = 0 \). Furthermore we get \( G_{n+\frac{1}{2}} w_{\psi} = 0 \)

for all \( n \geq n_0 \) by actions of \( L_1 \). Choose minimal \( m \in \mathbb{Z}_+ \) such that \( G_{m-\frac{1}{2}} w_{\psi} \neq 0 \) and \( G_{m+\frac{1}{2}} w_{\psi} = 0 \) for all \( n \geq m \). In this case if we set \( w' = G_{m-\frac{1}{2}} w_{\psi} \), then \( G_{m-\frac{1}{2}} w' = 0 \) and \( G_{m+\frac{1}{2}} w' = 0 \) for all \( n \geq m \), which means \( m = 1 \). Then

\[
    V = \mathbb{C} w_{\psi} \oplus \mathbb{C} u_{\psi} \cong A_1(\psi).
\]

\[\square\]

**Proposition 3.4.** Any finite-dimensional simple module over \( p \) is one-dimensional.

**Proof.** For \( \epsilon = \frac{1}{2} \), suppose that \( V \) is a finite-dimensional simple \( p \)-module. Then there exists \( w \in V \) such that \( L_n w = c_n w \) for all \( n \in \mathbb{Z}_+ \), where \( c_1, c_2 \in \mathbb{C} \) and \( c_i = 0 \) for all \( i \geq 3 \). With the same proof in Proposition 3.3 we have

\[
    G_r w = 0, \quad r \geq \frac{3}{2}.
\]

Then \( V = \mathbb{C}w \). Since the proof for \( \epsilon = 0 \) case is similar, we omit the details. \[\square\]

**Corollary 3.5.** Let \( V \) be finite-dimensional simple module over \( SVir_{\epsilon}^+ \) with \( C \) acts as a scalar \( c \in \mathbb{C} \). Define induced \( SVir_{\epsilon} \)-module

\[
    M_{\epsilon}(V,c) = U(SVir_{\epsilon}) \otimes_{U(SVir_{\epsilon}^+ \oplus \mathbb{C}C)} V.
\]
Then there exists a Lie superalgebra homomorphism \( \psi : p_\epsilon \rightarrow \mathbb{C} \) such that
\[
M(V, c) \cong W_\epsilon(\psi, c)
\]
as modules over \( SVir_\epsilon \).

4. WHITTAKER MODULES

In this section, we classify all Whittaker vectors of \( W_\epsilon(\psi, c) \) and obtain a criterion of simplicity of \( W_\epsilon(\psi, c) \).

**Lemma 4.1.** \( \{L_{-\lambda}G_{\epsilon-\mu}w_\psi \mid \lambda \in P, \mu \in Q\} \) forms a basis of the Whittaker module \( W_\epsilon(\psi, c) \).

*Proof.* It is a consequence of the PBW theorem. \( \square \)

**Lemma 4.2.** Let \( w_\psi \in W_\epsilon(\psi, c) \) be the Whittaker vector. For \( v = uw_\psi \in W_\epsilon(\psi, c) \), we have
\[
(x - \psi(x))v = [x, u]w_\psi, \quad \forall x \in p_\epsilon.
\]

*Proof.* For \( x \in p_\epsilon \), by Lemma 3.1 we have
\[
[x, u]w_\psi = xuw_\psi \pm uxw_\psi = xuw_\psi \pm \psi(x)uw_\psi = (x - \psi(x))uw_\psi = (x - \psi(x))v.
\]

For
\[
v = \sum_{(\lambda, \mu) \in P \times Q} p_{\lambda, \mu}L_{-\lambda}G_{\epsilon-\mu}w_\psi \in W_\epsilon(\psi, c),
\]
we define
\[
\text{maxdeg}(v) = \max\{|\lambda| + |\mu - \epsilon| \mid p_{\lambda, \mu} \neq 0\}
\]

**Lemma 4.3.** For any \( (\lambda, \mu) \in P \times Q \), \( k \in \mathbb{Z}_+ \), we have

(i) \( \text{maxdeg}([G_{k+2-\epsilon}, L_{-\lambda}G_{\epsilon-\mu}]w_\psi) \leq |\lambda| + |\mu - \epsilon| - k + \epsilon; \)

(ii) If \( \mu(i) = 0 \) for all \( 0 \leq i \leq k \), then
\[
\text{maxdeg}([G_{k+2-\epsilon}, L_{-\lambda}G_{\epsilon-\mu}]w_\psi) \leq |\lambda| + |\mu - \epsilon| - k + 1 + \epsilon;
\]

(iii) If \( \mu(i) = 0 \) for all \( i \leq k \) while \( \mu(k) = 1 \), then
\[
[G_{k+2-\epsilon}, L_{-\lambda}G_{\epsilon-\mu}]w_\psi = v + 2\mu(k)\psi(L_2)L_{-\lambda}G_{\epsilon-\mu}w_\psi,
\]
\[
\text{maxdeg}(v) \leq |\lambda + \mu| - k - 1 + \epsilon,
\]
and \( \mu' \) satisfies that \( \mu'(i) = \mu(i) \) for all \( i \neq k \) and \( \mu'(k) = 0 \) and then
\[
\text{maxdeg}(L_{-\lambda}G_{\frac{1}{2} - \mu'}w_\psi) = |\lambda| + |\mu - \epsilon| - k + \epsilon.
\]
Proof. Assume that $\epsilon = \frac{1}{2}$.

To prove (i), we write $[G_{\frac{3}{2}+k}, L_{-\lambda}G_{\frac{1}{2}-\mu}]$ as a linear combination of the basis of $U(SVir_{\frac{3}{2}})$:

$$[G_{\frac{3}{2}+k}, L_{-\lambda}G_{\frac{1}{2}-\mu}]w_\psi = [G_{\frac{3}{2}+k}, L_{-\lambda}]G_{\frac{1}{2}-\mu}w_\psi.$$

For the second summand, due to $[G_{\frac{3}{2}+k}, G_{\frac{1}{2}-\mu}]w_\psi = 2\psi(L_2)w_\psi$, then the maxdeg of second summand maybe become $|\lambda| + |\mu - \frac{1}{2}| - k + \frac{3}{2}$. (ii) follows from the calculations as the second summand in (i). Combining with (i) and (ii), we get (iii).

Since the proof for $\epsilon = 0$ case is similar, we omit the details. \hfill $\square$

**Proposition 4.4.** Suppose that $\psi$ is nontrivial and $w_\psi \in W_\epsilon(\psi,c)$ be the Whittaker vector.

(i) If $\psi(L_2) \neq 0$, then $w' \in W_\frac{3}{2}(\psi,c)$ is a Whittaker vector if and only if $w' \in \text{span}_C\{w_\psi\}$.

(ii) If $\psi(L_2) = 0$ and $\psi(L_1) \neq 0$, then $w' \in W_\frac{3}{2}(\psi,c)$ is a Whittaker vector if and only if $w' \in \text{span}_C\{w_\psi, G_{\frac{1}{2}}w_\psi\}$.

(iii) $v \in W_0(\psi,c)$ is a Whittaker vector if and only if $w' \in \text{span}_C\{w_\psi, G_{\frac{1}{2}}w_\psi\}$.

Proof. Let $w' = uw \in W_\frac{3}{2}(\psi,c)$ be a Whittaker vector, we can write $w'$ a linear combination of the basis of $W_\frac{3}{2}(\psi,c)$:

$$w' = \sum_{(\lambda,\mu) \in \mathcal{P} \times \mathcal{Q}} p_{\lambda,\mu}L_{-\lambda}G_{\frac{1}{2}-\mu}w,$$

where $p_{\lambda,\mu} \in \mathbb{C}$. Set

$$N := \max\{|\lambda| + |\mu - \frac{1}{2}| \mid p_{\lambda,\mu} \neq 0\},$$

$$\Lambda_N := \{(\lambda,\mu) \in \mathcal{P} \times \mathcal{Q} \mid p_{\lambda,\mu} \neq 0, |\lambda| + |\mu - \frac{1}{2}| = N\}.$$

Let $\psi' : p_{\frac{1}{2}} \to \mathbb{C}$ be a Lie superalgebra homomorphism and $\psi' \neq \psi$. Then there exists at least one of $\{L_1, L_2\}$, denoted by $E$, such that $\psi(E) \neq \psi'(E)$. Assume that $w'$ is a Whittaker vector of $W_\epsilon(\psi',c)$. On the one hand,

$$Ew' = v + \sum_{(\lambda,\mu) \in \Lambda_N} p_{\lambda,\mu}L_{-\lambda}G_{\frac{1}{2}-\mu}\psi(E)w,$$

where $\maxdeg(v) < N$. On the other hand, $Ew' = \psi'(E)w'$. Thus $\psi'(E) = \psi(E)$, which is contrary to our assumption that $\psi'(E) \neq \psi(E)$.

Next, for $w'$, we will show that if there is $(\bar{0},\bar{0}) \neq (\lambda,\mu) \in \mathcal{P} \times \mathcal{Q}$ such that $p_{\lambda,\mu} \neq 0$, then there is $E_n \in \{L_n, G_{n+\frac{1}{2}} \mid n \geq 1\}$ such that $(E_n - \psi(E_n))w' \neq 0$. In this case $w'$ is not a Whittaker vector, which proves the necessity.

Assume that $p_{\lambda,\mu} \neq 0$ for some $(\lambda,\mu) \neq (\bar{0},\bar{0})$, then, by Lemma 1.2 we have

$$(E_n - \psi(E_n))w' = \sum_{\lambda,\mu} p_{\lambda,\mu}[E_n, L_{-\lambda}G_{\frac{1}{2}-\mu}]w.$$

**Case (i):** $\psi(L_2) \neq 0$. 

Suppose that there exists \((\lambda, \mu) \in \Lambda_N\) with \(\mu(n) = 1\) for some \(n \in \mathbb{N}\). Now we can set \(k := \min\{n \in \mathbb{N} | \mu(n) = 1\} \) for some \((\lambda, \mu) \in \Lambda_N\), then
\[
G_{k+\frac{3}{2}} w' = \sum_{(\lambda, \mu) \notin \Lambda_N} p_{\lambda, \mu} [G_{k+\frac{3}{2}}, L-\lambda G_{\frac{1}{2}-\mu}] w \\
+ \sum_{(\lambda, \mu) \in \Lambda_N} p_{\lambda, \mu} [G_{k+\frac{3}{2}}, L-\lambda G_{\frac{1}{2}-\mu}] w \\
+ \sum_{(\lambda, \mu) \in \Lambda_N} p_{\lambda, \mu} [G_{k+\frac{3}{2}}, L-\lambda G_{\frac{1}{2}-\mu}] w.
\]

By using Lemma 4.3 (i) to the first summand, we know that the maxdeg of it is strictly smaller than \(N - k + \frac{1}{2}\). As for the second summand, note that \(\mu(i) = 0\) for \(0 \leq i \leq k\), the maxdeg of \([G_{k+\frac{3}{2}}, L-\lambda G_{\frac{1}{2}-\mu}]\) is also strictly smaller than \(N - k\) by Lemma 4.3 (ii). Now using Lemma 4.3 (iii) to the third summand, we know it has form:
\[
v + 2 \sum_{(\lambda, \mu) \in \Lambda_N} \mu(k) \psi(L_2) p_{\lambda, \mu} L_{-\lambda} G_{\frac{1}{2}-\mu} w,
\]
where \(\maxdeg(v) \leq N - k - \frac{1}{2}\), and \(\mu'\) satisfies \(\mu'(k) = 0\), \(\mu'(i) = \mu(i)\) for all \(i > k\). So the maxdeg of the third summand is \(N - k + \frac{1}{2}\). This implies that \(G_{k+\frac{3}{2}} w' \neq 0\).

Now we can suppose that \(\mu(i) = 0\) for all \(i \geq 1\) if \((\lambda, \mu) \in \Lambda_N\). Then we can set \(l := \min\{n \in \mathbb{N} | \lambda(n) \neq 0\} \) for some \((\lambda, \mu) \in \Lambda_N\). Then
\[
(L_{l+2} - \psi(L_{l+2})) w' = \sum_{(\lambda, \mu) \notin \Lambda_N} p_{\lambda, \mu} [L_{l+2}, L-\lambda G_{\frac{1}{2}-\mu}] w \\
+ \sum_{(\lambda, \mu) \in \Lambda_N} p_{\lambda, \mu} [L_{l+2}, L-\lambda] w \\
+ \sum_{(\lambda, \mu) \in \Lambda_N} p_{\lambda, \mu} [L_{l+2}, L-\lambda] w.
\]
Since the maxdeg’s of the first summand and second summand are strictly smaller than \(N - l\), and the maxdeg’s of the third summand is \(N - l\), we have \((L_{l+2} - \psi(L_{l+2})) w' \neq 0\) and get the proposition.

**Case (ii).** \(\psi(L_2) = 0\) and \(\psi(L_1) \neq 0\).

In this case, we can also get \(\mu(i) = 0\) for all \(i \geq 1\) if \((\lambda, \mu) \in \Lambda_N\) by action of \(G_{k+\frac{3}{2}}\) as in Case I.
Set \( l := \min\{n \in \mathbb{N} \mid \lambda(n) \neq 0 \text{ for some } (\lambda, \mu) \in \Lambda_N \} \). Then we have
\[
(L_{l+1} - \psi(L_{l+1}))w' = \sum_{(\lambda, \mu) \notin \Lambda_N} p_{\lambda, \mu}[L_{l+1}, L_{-\lambda}G_{\frac{1}{2} - \mu}]w + \sum_{(\lambda, \mu) \in \Lambda_N : \lambda(l) = 0} p_{\lambda, \mu}[L_{l+1}, L_{-\lambda}]G_{\frac{i}{2}}w + \sum_{(\lambda, \mu) \in \Lambda_N : \lambda(l) \neq 0} p_{\lambda, \mu}[L_{l+2}, L_{-\lambda}]G_{\frac{j}{2}}w,
\]
where \( i, j = 0, 1 \). Since the maxdeg’s of the first summand and second summand are strictly smaller than \( N - l \), and the maxdeg’s of the third summand is \( N - l \), we have \((L_{l+1} - \psi(L_{l+1}))w' \neq 0\).

Since the proof for \( \epsilon = 0 \) case (iii) is similar, we omit the details.

\[\square\]

**Lemma 4.5.** Let \( \lambda \in \mathcal{P}, \mu \in \mathcal{Q} \) and \( E_n = L_n \) or \( G_{n+1-\epsilon} \).

(i) For all \( n > 1 \), \( E_n(L_{-\lambda}G_{\epsilon-\mu}w) \in \text{span}_C \{L_{-\lambda'}G_{\epsilon-\mu'}w' \mid |\mu' - \epsilon| + |\lambda'| + \lambda'(0) \leq |\lambda| + |\mu - \epsilon| + \lambda(0)\} \).

(ii) If \( n > |\lambda| + |\mu - \epsilon| + 2 \), then \( E_n(L_{-\lambda}G_{\epsilon-\mu}w) = 0 \).

**Proof.** By direct calculation as Lemma 4.3. \( \square \)

**Lemma 4.6.** Let \( V \) be a Whittaker module, and \( v \in V \). Regarding \( V \) as a \( p_\epsilon \)-module, \( U(p_\epsilon)v \) is a finite-dimensional submodule of \( V \).

**Proof.** It follows from Lemma 4.5. \( \square \)

**Lemma 4.7.** Let \( U \) be a submodule of \( W_\epsilon(\psi, c) \). Then there is a nonzero Whittaker vector \( u \in U \).

**Proof.** It follows from Lemma 4.6 and Proposition 3.4. \( \square \)

Our main result can be stated as follows:

**Theorem 4.8.** Suppose that \( \psi : p_\epsilon \to \mathbb{C} \) is a nontrivial Lie superalgebra homomorphism and \( c \in \mathbb{C} \). Then

(i) \( W_{\frac{1}{2}}(\psi, c) \) is simple.

(ii) \( W_{0}(\psi, c) \) is simple if and only if \( \psi(L_2) \neq 0 \).

**Proof.** For \( \epsilon = \frac{1}{2} \), if \( \psi(L_2) = 0 \) and \( \psi(L_1) \neq 0 \), then \( G_{\frac{1}{2}}w_\psi \) is also a Whittaker vector. However the Whittaker module generated by \( G_{\frac{1}{2}}w_\psi \) is same as \( W_{\frac{1}{2}}(\psi, c) \) since \( G_{\frac{1}{2}}G_{\frac{1}{2}}w_\psi = \psi(L_1)w_\psi \). So (i) and the sufficient condition (ii) follow from Proposition 4.4 and Lemma 4.7.
We only need to prove the necessary condition for (ii). If \( \psi(L_2) = 0 \) and \( \psi(L_1) \neq 0 \), then \( G_1w_\psi \) is also a Whittaker vector of \( W_0(\psi, c) \). In this case the Whittaker module generated by \( G_1w_\psi \) is a proper submodule of \( W_0(\psi, c) \) since \( w_\psi \notin \langle G_1w_\psi \rangle \). □

Remark 4.9. For \( \epsilon = 0 \), if \( \psi(L_2) = 0 \) and \( \psi(L_1) \neq 0 \), the vector \( G_1w_\psi \) is called the degenerate Whittaker vector [9]. Moreover \( W_0(\psi, c)/\langle G_1w_\psi \rangle \) is a Whittaker module of \( (\phi, c) \), where \( \phi : SVir_0^+ \to \mathbb{C} \) such that \( \phi(L_2) = 0 \) and \( \phi(L_1) \neq 0 \) (in this case \( \phi(G_1) = 0 \) since \( \phi(L_2) = \phi(G_1)^2 \)). Similar to the proof of Proposition 4.4 we can show that \( W_0(\psi, c)/\langle G_1w_\psi \rangle \) is simple. Then in this case \( L_0(\psi, c) = W_0(\psi, c)/\langle G_1w_\psi \rangle \).

Corollary 4.10. Assume \( \psi : p_\epsilon \to \mathbb{C} \) is a nontrivial Lie superalgebra homomorphism and \( c \in \mathbb{C} \). Let \( M_\epsilon \) be a simple Whittaker module of type \( (\psi, c) \) for \( SVir_\epsilon \). Then \( M_\epsilon \cong L_\epsilon(\psi, c) \) or \( \Pi L_\epsilon(\psi, c) \).

Proof. Let \( w \in M_\epsilon \) be a Whittaker vector corresponding to \( \psi \). If \( |w| = |w_\psi| \) (the \( \mathbb{Z}_2 \) degree’s), then by the universal property of \( W_\epsilon(\psi, c) \), there exists a module homomorphism \( \varphi : W_\epsilon(\psi, c) \to M_\epsilon \) with \( uw_\psi \mapsto uw \). This map is surjective since \( w \) generates \( M_\epsilon \). Then \( M_\epsilon \cong L_\epsilon(\psi, c) \) since \( M_\epsilon \) is simple. Similarly, if \( |w| \neq |w_\psi| \), then \( M_\epsilon \cong \Pi L_\epsilon(\psi, c) \). As a corollary of Theorem 4.8 the proof is complete. □

Corollary 4.11. Let \( V \) be finite-dimensional module over \( SVir_\epsilon^+ \) with \( C \) acts as a scalar \( c \in \mathbb{C} \). Then \( M_\epsilon(V, c) \) is a simple module for \( SVir_\epsilon \) if and only if \( V \) is simple module for \( SVir_\epsilon^+ \).

ACKNOWLEDGMENTS

We gratefully acknowledge the partial financial support from the NNSF (No.11871249, No.11771142), the ZJNSF (No.LZ14A010001), the Shanghai Natural Science Foundation (No.16ZR1425000) and the Jiangsu Natural Science Foundation (No.BK20171294). Part of this work was done while the authors were visiting the Chern Institute of Mathematics, Tianjin, China. The authors would like to thank the institute and Prof. Chengming Bai for their warm hospitality and support. We also thank the referee for his/her helpful suggestions.

REFERENCES

[1] D. Adamović, R. Lu, K. Zhao, Whittaker modules for the affine Lie algebra \( A_1^{(1)} \), Adv. Math. 289 (2016), 438–479.
[2] D. Arnal and G. Pinczon, On algebraically irreducible representations of the Lie algebra \( \mathfrak{sl}_2 \), J. Math. Phys. 15 (1974), 350–359.
[3] P. Batra and V. Mazorchuk, Blocks and modules for Whittaker pairs, J. Pure Appl. Algebra, 215 (2011), 1552–1568.
[4] E. Backelin, Representation of the category in Whittaker categories, Internat. Math. Res. Notices (1997), 4, 153–172.

[5] I. Bagci, K. Christodoulopoulou, and E. Wiesner, Whittaker categories and strongly typical Whittaker modules for Lie superalgebras, Comm. Algebra 42 (2014), 11, 4932–4947.

[6] O. Blondeau-Fournier, P. Mathieu, D. Ridout, and S. Wood, Superconformal minimal models and admissible Jack polynomials, Adv. Math. 314, (2017), 71–123.

[7] Y. Cai, R. Shen, and J. Zhang, Whittaker modules and quasi-Whittaker modules for the Euclidean Lie algebra e(3), J. Pure and Applied Algebra. 220 (2016), 1419–1433.

[8] K. Christodoupoulou, Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine lie algebras, J. Algebra. 320 (2008), 2871-2890.

[9] P. Desrosiers, L. Lapointe, and P. Mathieu, Superconformal field theory and Jack superpolynomials, JHEP 09 (2012) 037.

[10] D. Gaiotto, Asymptotically free N = 2 theories and irregular conformal blocks, Journal of Physics: Conference Series, Volume 462, (2013).

[11] R. Lu, X. Guo and K. Zhao, Irreducible modules over the Virasoro algebra, Documenta Math. 16 (2011) 709–721

[12] E. Felinska, Z. Jaskolski, and M. Kosztolowicz, Whittaker pairs for the Virasoro algebra and the Gaiotto-Bonelli-Maruyoshi-Tanzini states, J. Math. Phys. 53 (2012), 033504.

[13] K. Iohara, Y. Koga, Representation theory of Neveu-Schwarz and Ramond algebras I: Verma modules, Adv. Math. 177 (2003), 61-69.

[14] K. Iohara, Y. Koga, Representation theory of Neveu-Schwarz and Ramond algebras II: Fock modules, Ann. Inst. Fourier 53 (2003), 1755-1818.

[15] V. Kac and J. van de Leur. On classification of superconformal algebras, Strings 88, World Sci. (1989), 77–106.

[16] V. Kac, M. Wakimoto, Unitarizable highest weight representation of the Virasoro, Neveu-Schwarz and Ramond algebras, Lecture Notes in Physics, Vol. 261, Springer, Berlin, 1986.

[17] B. Kostant, On Whittaker vectors and representation theory, Invent. Math. 48 (1978), 101–184.

[18] D. Liu Y. Wu, and L. Zhu, Whittaker modules over the twisted Heisenberg-Virasoro algebra, J. Math. Phys. 51 (2010), 023524.

[19] V. Mazorchuk and K. Zhao, Simple Virasoro modules which are locally finite over a positive part, Selecta Math. New Ser., 20 (2014), 839–854.

[20] A. Neveu and J. Schwarz, Factorizable dual model of poins, Nucl. Phys. B, 31 (1971), 86–112.

[21] M. Ondrus, Whittaker modules for \( U_q(sl_2) \), J.Algebra, 289 (2005), 192–213.

[22] M. Ondrus and E. Wiesner, Whittaker modules for the Virasoro algebra, J. Algebra Appl. 8 (2009), 363–377

[23] M. Ondrus, E. Wiesner, Whittaker categories for the Virasoro algebra, Comm. Algebra, 41,(2013) 3910–3930.

[24] P. Ramond. Dual theory for free fermions, Phys. Rev. D, 3 (1971), 2415–2418.

[25] A. Sergeev, Irreducible representations of solvable Lie superalgebras, Represent. Theory, 3 (1999), 435–443.

[26] Y. Su. Classification of Harish-Chandra modules over the super-Virasoro algebras, Comm. Algebra, 23 (1995), 3653–3675.

[27] S. Tan, Q. Wang and C. Xu, On Whittaker modules for a Lie algebra arising from the 2-dimensional torus, Pacific J. Math. 273 (2015), 147–167.
[28] B. Wang, Whittaker modules for graded Lie algebras, Algebra Represent. Theory, 14 (2011), 1–12.
[29] S. Yanagida, Whittaker vectors of the Virasoro algebra in terms of Jack symmetric polynomial, J. Algebra. 333 (2011) 273-294.
[30] X. Zhang, S. Tan, and H. Lian, Whittaker modules for the Schrödinger-Witt algebra. J. Math. Phys. 51 (2010), 083524.

Department of Mathematics, Huzhou University, Zhejiang Huzhou, 313000, China
E-mail address: liudong@zjhu.edu.cn

Department of Mathematics, Shanghai Normal University, Shanghai, 200234, China
E-mail address: pei@shnu.edu.cn

Institute of Applied System Analysis, Jiangsu University, Jiangsu Zhenjiang, 212013, China
E-mail address: xialimeng@ujs.edu.cn