On Quantum Markov Chains on Cayley tree III: Ising model

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Abstract

In this paper, we consider the classical Ising model on the Cayley tree of order \( k \) \((k \geq 2)\), and show the existence of the phase transition in the following sense: there exists two quantum Markov states which are not quasi-equivalent. It turns out that the found critical temperature coincides with usual critical temperature.

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1 Introduction

The present paper is a continuation of our previous works \([6, 7]\). In \([6]\) we have introduced forward type of quantum Markov chains (QMC) defined on the Cayley tree was studied\(^1\). It was provided a construction of such kind of chains, in which a QMC is defined as a weak limit of finite volume states with boundary conditions\(^2\). By means of the provided construction we proved uniqueness of QMC associated with XY-model on a Cayley tree of order two. Furthermore, in \([7]\) we have defined a notion of phase transition in QMC scheme. Namely, such a notion is based on the quasi-equivalence of QMC. Therefore, such a phase transition is purely noncommutative. We point out that phase transitions in a quantum setting play an important role to understand quantum spin systems (see for example \([9, 11, 14]\)). Furthermore, in \([7]\) it was established the existence of the phase transition for XY-model on the Cayley tree of order three. From the provided definition of the phase transition, it naturally appears a

\(^1\)We remark that backward quantum Markov chains on lattices and trees have been investigated in \([2, 8]\).

\(^2\)Note that similar kind of constructions of QMC on integer lattice were known in the literature (see for example \([1]-[5]\).
question: would this definition be compatible with well-known definitions of phase transitions for lattice models (see [10] [15] [19]).

In this paper, we consider the classical Ising model on the Cayley tree of order \( k \) \((k \geq 2)\), and show the existence of the phase transition in sense of [7]. It turns out that the found critical temperature coincides with usual one (see [15] [19]). This means that our definition of the phase transition is compatible with known ones. Note that very recently, in [16] other new kind of phase transitions have been observed, for the classical Ising model on Caley tree.

We stress that noncommutative approach to the phase transition for the Ising model was studied in [9]. But our way to define the phase transition is bit different from the mentioned paper. In general, phase transition for quantum systems is defined as the existence of two distinct KMS-state corresponding to the model (see [13] for review). In our approach we follow the same definition but we require the existence of two non-quasi equivalent QMC associated with a model. We point out that, in general, two distinct QMC may generate in GNS-representation the same type of von Neumann algebras [17].

The paper is organized as follows. In section 2 we give necessary definitions and construction of QMC. In section 3 we consider Ising model and formulate the main result of the paper. In section 4 we derive a dynamical system related to our model. In section 5 asymptotic behavior of the dynamical system will be studied. In section 6 we prove the diagonalizability of the QMC. In section 7 the first part of the main theorem will be proved. In the final section 8, it will be established the existence of the phase transition.

2 Construction of QMC on the Cayley tree

In this section we recall needed definitions which will be used in the paper (see [6] [7] for more information).

Let \( \Gamma^k = (L, E) \) be a semi-infinite Cayley tree of order \( k \geq 1 \) with the root \( x^0 \) (i.e. each vertex of \( \Gamma^k \) has exactly \( k + 1 \) edges, except for the root \( x^0 \), which has \( k \) edges). Here \( L \) is the set of vertices and \( E \) is the set of edges. If the vertices \( x \) and \( y \) are connected by an edge, then they are called nearest neighbors and denoted by \( < x, y > \). A collection of the pairs \( < x, x_1 >, \ldots, < x, x_d, y > \) is called a path from the point \( x \) to the point \( y \). The distance \( d(x, y) \), \( x, y \in V \), on the Cayley tree, is the length of the shortest path from \( x \) to \( y \). One can define a coordinate structure in \( \Gamma^k \) as follows: every vertex \( x \) (except for \( x^0 \)) of \( \Gamma^k \) has coordinates \((i_1, \ldots, i_n)\), here \( i_m \in \{1, \ldots, k\} \), \( 1 \leq m \leq n \) and for the vertex \( x^0 \) we put \((0)\). Namely, the symbol \((0)\) constitutes level 0, and the sites \((i_1, \ldots, i_n)\) form level \( n \) (i.e. \( d(x^0, x) = n \)) of the lattice (see Fig. 1).

Let us set

\[ W_n = \{ x \in L : d(x, x_0) = n \}, \quad \Lambda_n = \bigcup_{k=0}^{n} W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^{m} W_k, \quad (n < m) \]

\[ E_n = \{ < x, y > \in E : x, y \in \Lambda_n \}, \quad \Lambda_n^c = \bigcup_{k=n}^{\infty} W_k \]

Now we rewrite the elements of \( W_n \) in the following order, i.e.

\[ \overline{W}_n := \left( x_{W_n}^{(1)} , x_{W_n}^{(2)} , \ldots , x_{W_n}^{(|W_n|)} \right), \quad \overline{W}_n := \left( x_{W_n}^{(|W_n|-1)} , x_{W_n}^{(|W_n|-2)} , \ldots , x_{W_n}^{(1)} \right). \]
Note that $|W_n| = k^n$. Vertices $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \ldots, x_{W_n}^{(|W_n|)}$ of $W_n$ can be represented in terms of the coordinate system as follows

$$
x_{W_n}^{(1)} = (1, 1, \ldots, 1, 1), \quad x_{W_n}^{(2)} = (1, 1, \ldots, 1, 2), \quad \ldots \quad x_{W_n}^{(k)} = (1, 1, \ldots, 1, k),
$$

$$
x_{W_n}^{(k+1)} = (1, 1, \ldots, 2, 1), \quad x_{W_n}^{(2)} = (1, 1, \ldots, 2, 2), \quad \ldots \quad x_{W_n}^{(2k)} = (1, 1, \ldots, 2, k),
$$

$$
\vdots
$$

$$
x_{W_n}^{(|W_n|-k+1)} = (k, k, \ldots, k, 1), \quad x_{W_n}^{(|W_n|-k+2)} = (k, k, \ldots, k, 2), \quad \ldots \quad x_{W_n}^{(|W_n|)} = (k, k, \ldots, k, k).
$$

For a given vertex $x$, we shall use the following notation for the set of direct successors of $x$:

$$
\overline{S(x)} := ((x, 1), (x, 2), \ldots, (x, k)), \quad \overline{S'(x)} := ((x, k), (x, k - 1), \ldots, (x, 1)).
$$

In what follows, for the sake of simplicity, we will use notation $i \in \overline{S(x)}$ (resp. $i \in \overline{S'(x)}$) instead of $(x, i) \in S(x)$ (resp. $(x, i) \in S'(x)$).

The algebra of observables $B_x$ for any single site $x \in L$ will be taken as the algebra $M_d$ of the complex $d \times d$ matrices. The algebra of observables localized in the finite volume $\Lambda \subseteq L$ is then given by $B_\Lambda = \bigotimes_{x \in \Lambda} B_x$. As usual if $\Lambda^1 \subset \Lambda^2 \subset L$, then $B_{\Lambda^1}$ is identified as a subalgebra of $B_{\Lambda^2}$ by tensoring with unit matrices on the sites $x \in \Lambda^2 \setminus \Lambda^1$. Note that, in the sequel, by $B_{\Lambda,+}$ we denote the positive part of $B_\Lambda$. The full algebra $B_L$ of the tree is obtained in the usual manner by an inductive limit

$$
B_L = \bigcup_{\Lambda_n} B_{\Lambda_n}.
$$

Assume that for each edge $< x, y > \in E$ of the tree an operator $K_{<x,y>} \in B_{\{x,y\}}$ is assigned. We would like to define a state on $B_{\Lambda_n}$ with boundary conditions $w_0 \in B_{\langle 0 \rangle,+}$ and $h = \{h_x \in B_{x,+} \mid x \in L\}$.

Let us denote

$$
K_{[m-1,m]} := \prod_{x \in \overline{W}_{m-1}} \prod_{y \in S(x)} K_{<x,y>}, \tag{2.1}
$$

$$
h_n^{1/2} := \prod_{x \in \overline{W}_n} h_x^{1/2}, \quad h_n := h_n^{1/2}(h_n^{1/2})^*, \tag{2.2}
$$

$$
K_n := w_0^{1/2} K_{[0,1]} K_{[1,2]} \cdots K_{[n-1,n]} h_n^{1/2}, \tag{2.3}
$$

$$
W_{[n]} := K_n K_n^*. \tag{2.4}
$$

It is clear that $W_{[n]}$ is positive.

In what follows, by $\text{Tr}_\Lambda : B_L \to B_\Lambda$ we mean normalized partial trace (i.e. $\text{Tr}_\Lambda(\mathbf{1}_L) = \mathbf{1}_\Lambda$, here $\mathbf{1}_\Lambda = \bigotimes_{y \in \Lambda} \mathbf{1}$, for any $\Lambda \subseteq \text{fin} L$. For the sake of shortness we put $\text{Tr}_{[n]} := \text{Tr}_{\Lambda_n}$.

Let us define a positive functional $\varphi_{w_n,h}$ on $B_{\Lambda_n}$ by

$$
\varphi_{w_n,h}^{(n,f)}(a) = \text{Tr}(W_{n+1}(a \otimes \mathbf{1}_{W_{n+1}})), \tag{2.5}
$$
for every \(a \in \mathcal{B}_\Lambda\). Note that here, Tr is a normalized trace on \(\mathcal{B}_L\) (i.e. \(\text{Tr}(\mathbf{1}_L) = 1\)).

To get an infinite-volume state \(\varphi^{(f)}\) on \(\mathcal{B}_L\) such that \(\varphi^{(f)}|_{\mathcal{B}_\Lambda} = \varphi^{(n,f)}_{w_0,h}\), we need to impose some constraining to the boundary conditions \([w_0,h]\) so that the functionals \(\{\varphi^{(n,f)}_{w_0,h}\}\) satisfy the compatibility condition, i.e.

\[
\varphi^{(n+1,f)}_{w_0,h}|_{\mathcal{B}_\Lambda} = \varphi^{(n,f)}_{w_0,h}. \tag{2.6}
\]

**Theorem 2.1** ([6]). Let the boundary conditions \(w_0 \in \mathcal{B}_{(0),+}\) and \(h = \{h_x \in \mathcal{B}_{x,+}\}_{x \in L}\) satisfy the following conditions:

\[
\text{Tr}(w_0 h_0) = 1 \tag{2.7}
\]

\[
\text{Tr}_{x^*} \left[ \prod_{y \in S(x)} K_{x,y}^> \prod_{y \in S(x)} h_y \prod_{y \in S(x)} K_{x,y}^< \right] = h(x) \text{ for every } x \in L. \tag{2.8}
\]

Then the functionals \(\{\varphi^{(n,f)}_{w_0,h}\}\) satisfy the compatibility condition \(2.6\). Moreover, there is a unique forward quantum d-Markov chain \(\varphi^{(b)}_{w_0,h}\) on \(\mathcal{B}_L\) such that \(\varphi^{(f)}_{w_0,h} = w - \lim_{n \to \infty} \varphi^{(n,f)}_{w_0,h}\).

From direct calculation we can derive the following

**Proposition 2.2.** If \(2.7\) and \(2.8\) are satisfied then one has \(\varphi^{(n,f)}_{w_0,h}(a) = \text{Tr}(W_n(a))\) for any \(a \in \mathcal{B}_\Lambda\).

In [7] we have introduced a notion of the phase transition quantum Markov chains associated with the given family \(\{K_{x,y}\}\) of operators. Heuristically, such a phase transition means the existence of two distinct QMC for the given \(\{K_{x,y}\}\). Let us provide a more exact definition.

**Definition 2.3.** We say that there exists a phase transition for a family of operators \(\{K_{x,y}\}\) if \(2.7\), \(2.8\) have at least two \((w_0, \{h_x\}_{x \in L})\) and \((v_0, \{s_x\}_{x \in L})\) solutions such that the corresponding quantum d-Markov chains \(\varphi_{w_0,h}\) and \(\varphi_{v_0,s}\) are not quasi equivalent. Otherwise, we say there is no phase transition.

In [7] we have established the existence of the phase transition for XY-model on the Cayley tree of order three.

### 3 QMC associated with Ising model and main results

In this section, we consider the Ising model and formulate the main results of the paper. In what follows, we consider a semi-infinite Cayley tree \(\Gamma^k = (L, E)\) of order \(k\). Our starting \(C^*\)-algebra is the same \(\mathcal{B}_L\) but with \(\mathcal{B}_x = M_2(\mathbb{C})\) for \(x \in L\). By \(\sigma_x^{(u)}, \sigma_y^{(u)}, \sigma_z^{(u)}\) we denote the Pauli spin operators at site \(u \in L\). Here are they

\[
\mathbf{1}^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x^{(u)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y^{(u)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.1}
\]

\(^3\text{For the definition of quantum Markov chain we refer } [8].\)
For every edge \( <u,v> \in E \) put
\[
K_{<u,v>} = \exp\{\beta H_{<u,v>}\}, \quad \beta > 0,
\]
where
\[
H_{<u,v>} = \frac{1}{2} (I(u)I(v) + \sigma_z(u)\sigma_z(v)).
\]

Such kind of Hamiltonian is called *Ising model* per edge \( <x,y> \).

Now taking into account the following equalities
\[
H_m^{<u,v>} = H_{<u,v>} = \frac{1}{2} (I(u)I(v) + \sigma_z(u)\sigma_z(v)),
\]
one finds
\[
K_{<u,v>} = I(u)I(v) + (\exp{\beta} - 1)H_{<u,v>}.
\]

It follows from (3.3) that
\[
K_{<u,v>} = K_0 I(u)I(v) + K_3 \sigma_z(u)\sigma_z(v).
\]

where, \( K_0 = \frac{\exp{\beta} + 1}{2} \) and \( K_3 = \frac{\exp{\beta} - 1}{2} \).

The main result of the present paper concerns the existence of the phase transition for the model (3.2). Namely, we have the following result.

**Theorem 3.1.** Let \( \{K_{<x,y>}\} \) be given by (3.2) on the Cayley tree of order \( k \geq 2 \) and \( \theta = \exp\{2\beta\}, \beta > 0 \). Then the following assertions hold:

(i) If \( \theta \leq \frac{k+1}{k} \) then there is a unique forward quantum \( d \)-Markov chain.

(ii) If \( \theta > \frac{k+1}{k} \) then there is a phase transition for a given model, i.e. there are two distinct forward quantum \( d \)-Markov chains.

**Remark 3.2.** We point out that in the classical case, for the Ising model on the Cayley tree, the phase transition occurs if \( \theta > \frac{k+1}{k} \) (see for example [13, 19]). This means that our definition of the phase transition is compatible with well-known definitions.

The rest of the paper will be devoted to the proof of this theorem. To do it, we shall use a dynamical system approach, which is associated with the equations (2.7), (2.8).

## 4 A dynamical system related to boundary conditions

In this section we shall reduce equations (2.8) to some dynamical system. Our goal is to describe all solutions \( h = \{h_x\} \) of that equation.

Furthermore, we shall assume that \( h_u = h_v \) for every \( u, v \in W_n, n \in \mathbb{N} \). Hence, we denote \( h_{x}^{(n)} := h_{(x)} \) and \( h_{y}^{(n+1)} := h_{(y)} \) if \( x \in W_n \) and \( y \in W_{n+1} \). Now from (3.2), (3.3) one can see that \( K_{<u,u>} = K_{<x,y>}^{(n)} \), therefore, equation (2.8) can be rewritten as follows
\[
\text{Tr}_x \left[ \prod_{y \in S(x)} K_{<x,y>} \prod_{y \in S(x)} h_y^{(n+1)} \prod_{y \in S(x)} K_{<x,y>} \right] = h_{x}^{(n)} \quad \text{for every} \quad x \in L.
\]
Assume that
\[
h^{(n)}_x = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix}, \quad h^{(n+1)}_y = \begin{pmatrix} a_{11}^{(n+1)} & a_{12}^{(n+1)} \\ a_{21}^{(n+1)} & a_{22}^{(n+1)} \end{pmatrix}.
\]
Then after simple calculations, one can reduce (4.1) to
\[
\left( \frac{a_{11}^{(n+1)} \exp 2\beta + a_{22}^{(n+1)}}{2} \right)^k = a_{11}^{(n)}
\]
\[
0 = a_{12}^{(n)}
\]
\[
0 = a_{21}^{(n)}
\]
\[
\left( \frac{a_{11}^{(n+1)} + a_{22}^{(n+1)} \exp 2\beta}{2} \right)^k = a_{22}^{(n)}
\]
From the last system of equations we obtain
\[
a_{12}^{(n)} = a_{21}^{(n)} = 0, \quad \forall n \in \mathbb{N},
\]
and
\[
a_{11}^{(n+1)} = \frac{2 \exp 2\beta \sqrt{a_{11}^{(n)}} - 2 \sqrt{a_{22}^{(n)}}}{\exp 4\beta - 1}
\]
\[
a_{22}^{(n+1)} = \frac{-2 \sqrt{a_{11}^{(n)}} + 2 \exp 2\beta \sqrt{a_{22}^{(n)}}}{\exp 4\beta - 1}
\]
The last equalities allow us to consider a dynamical system \( f : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) given by
\[
\begin{cases}
  x' = \frac{2 \sqrt{x} - 2 \sqrt{y}}{\theta^2 - 1} \\
  y' = \frac{-2 \sqrt{x} + 2 \theta \sqrt{y}}{\theta^2 - 1}
\end{cases}
\] (4.3)
where \( \beta > 0, \theta = \exp\{2\beta\} \), and \((x', y') = f(x, y)\). Then one has that \( \theta > 1 \).

**Remark 4.1.** The dynamical system \( f : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) given by (4.3) is well-defined if and only if \( x, y \geq 0 \) and \( \frac{1}{\theta^2} y \leq x \leq \theta^k y \). In what follows, we will only consider the case in which \( x > 0 \) and \( y > 0 \).

## 5 Asymptotical behavior of the dynamical system.

In this section we shall find fixed points of (4.3) and prove the absence of periodic points. Moreover, we investigate an asymptotical behavior it.

It is clear from (4.3) that
\[
\frac{x'}{y'} = \frac{2 \sqrt{x} - 2 \sqrt{y}}{-2 \sqrt{x} + 2 \theta \sqrt{y}} = \frac{\theta \sqrt{\frac{x}{y}} - 1}{\theta - \sqrt{\frac{x}{y}}}.
\]
Therefor, let us define the function $g_\theta : \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$g_\theta(t) = \frac{\theta \sqrt{t} - 1}{\theta - \sqrt{t}}. \quad (5.1)$$

One can see that the domain of $g_\theta$ is $\Delta := [0, \theta^k) \cup (\theta^k, \infty)$.

Let us study an asymptotical behavior of the function $g_\theta : \Delta \rightarrow \mathbb{R}$.

**Proposition 5.1.** The function $g_\theta : \Delta \rightarrow \mathbb{R}$ given by (5.1) is strictly increasing.

**Proof.** Let us calculate the derivative of the function $g_\theta$.

$$g'_\theta(t) = \frac{\frac{\theta}{2 \sqrt{t}}(\theta - \sqrt{t}) + \frac{1}{2 \sqrt{t}}(\theta \sqrt{t} - 1)}{(\theta - \sqrt{t})^2} = \frac{\theta^2 - 1}{2 \sqrt{t}(\theta - \sqrt{t})^2}.$$ 

Since $\theta > 1$, hence $g'_\theta(t) > 0$ for all $t \in \Delta$. This means that the function $g_\theta : \Delta \rightarrow \mathbb{R}$ given by (5.1) is increasing in its domain $\Delta$. \hfill \square

**Remark 5.2.** Let $g_\theta : \Delta \rightarrow \mathbb{R}$ be a given function by (5.1). Then one can easily check that:

(i) The function $g_\theta$ does not have any $m$ periodic point in its domain $\Delta$, where $m > 1$;

(ii) The function $g_\theta$ is positive if and only if $t \in \left(\frac{1}{\theta^k}, \theta^k\right)$.

**Proposition 5.3.** Let $g_\theta : D \rightarrow \mathbb{R}$ be a function given by (5.1). Then the following assertions hold:

(i) If $\theta > \frac{k+1}{k-1}$ then it has three fixed points which are $t_1, t_2, t_3$ such that

$$\frac{1}{\theta^k} < t_2 < t_1 = 1 < t_3 < \theta^k;$$

(ii) If $1 < \theta \leq \frac{k+1}{k-1}$ then it has a unique fixed point which is $t_1 = 1$.

**Proof.** In order to find all fixed points of $g_\theta$ we should solve the following equation

$$\frac{\theta \sqrt{t} - 1}{\theta - \sqrt{t}} = t. \quad (5.2)$$

Let us denote

$$x = \theta \frac{\sqrt{t} - 1}{\theta - \sqrt{t}}.$$ 

After some algebraic manipulations the equation (5.2) takes the following form

$$\frac{x}{\theta^k} = \left(\frac{x + 1}{x + \theta^2}\right)^k. \quad (5.3)$$

In this case, we should find all positive solutions of (5.3). It was shown in [18] that if $\theta > \frac{k+1}{k-1}$ then (5.3) has three positive solutions which are $x_1, x_2, x_3$ such that $0 < x_2 < x_1 = \theta < x_3$ and if $1 < \theta \leq \frac{k+1}{k-1}$ then the equation (5.3) has a unique solution which is $x_1 = \theta$. Then the corresponding solutions of (5.2) are $t_1, t_2, t_3$. This completes the proof. \hfill \square
Proposition 5.4. Let $g_\theta : \Delta \to \mathbb{R}$ be a function given by (5.1). Then the following assertions hold:

(i) Let $\theta > \frac{k+1}{k-1}$. Then one has

(a1) $g_\theta(t) < t$ for any $t \in (\sqrt[\theta]{t}, t_2) \cup (t_1, t_3)$ and $g_\theta(t) > t$ for any $t \in (t_2, t_1) \cup (t_3, \theta^k)$;

(b1) If $t_0 \in (t_2, t_3)$ then the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$, starting from the point $t_0$, converges to the fixed point $t_1$ which is equal to one;

(c1) If $t_0 \in (\sqrt[\theta]{t}, t_2) \cup (t_3, \theta^k)$ then the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$, starting from the point $t_0$, is finite.

(ii) Let $1 < \theta \leq \frac{k+1}{k-1}$. Then one has

(a2) $g_\theta(t) < t$ for any $t \in (\sqrt[\theta]{t}, t_1)$ and $g_\theta(t) > t$ for any $t \in (t_1, \theta^k)$.

(b2) for any initial point $t_0 \in (\sqrt[\theta]{t}, \theta^k)$, the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$ starting from the point $t_0$ is finite.

Proof. (i) Assume that $\theta > \frac{k+1}{k-1}$. Let us prove (a1). One can see that

$$g_\theta(t) - t = \left(\sqrt[\theta]{t} - \sqrt[\theta]{t_1}\right)\left(\sqrt[\theta]{t} - \sqrt[\theta]{t_2}\right)\left(\sqrt[\theta]{t} - \sqrt[\theta]{t_3}\right)\phi\left(\sqrt[\theta]{t}\right) \quad (5.4)$$

where $\phi\left(\sqrt[\theta]{t}\right)$ is a polynomial of the argument $\sqrt[\theta]{t}$ and $\phi\left(\sqrt[\theta]{t}\right) > 0$ for any $t \in (\sqrt[\theta]{t}, \theta^k)$.

It is clear from (5.4) that if $t \in (\sqrt[\theta]{t}, t_2) \cup (t_1, t_3)$ then $g_\theta(t) < t$ and if $t \in (t_1, t_2) \cup (t_3, \theta^k)$ then $g_\theta(t) > t$.

(b1) Assume that $t_0 \in (t_2, t_3)$. Since the function $g_\theta$ is strictly increasing and $t_1, t_2, t_3$ such that $\frac{1}{\theta^{k}} < t_2 < t_1 < t_3 < \theta^k$ are its fixed points, the segments $(t_2, t_1), (t_1, t_3), (t_2, t_3)$ are invariant w.r.t. the function $g_\theta$ and $g_\theta(t) > 0$ for any $t \in (t_2, t_3)$. Therefore, the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$, starting from the point $t_0$, is well-defined.

Without loss any generality, we assume that $t_0 \in (t_2, t_1)$. According to (a1) we have $g_\theta(t_0) > t_0$. Since the function $g_\theta$ is strictly increasing, one finds

$$t_2 < t_0 < g_(t_0) < g_\theta^{(2)}(t_0) < \cdots < g_\theta^{(n)}(t_0) < \cdots < t_1.$$ 

Then $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$ is a convergent sequence and its limiting point should be a fixed point which is equal to $t_1 = 1$.

Analogously, one can show that if $t_0 \in (t_1, t_3)$ then the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$, starting from the point $t_0$, is a monotone decreasing sequence on $(t_1, t_3)$ and its limiting point is a fixed point $t_1 = 1$.

(c1) Now assume that $t_0 \in (\sqrt[\theta]{t}, t_2) \cup (t_3, \theta^k)$.

Without loss of any generality we suppose that $t_0 \in (\sqrt[\theta]{t}, t_2)$. Let us assume that the trajectory $\{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty}$ is well-defined and it has an infinite number of distinct terms. According to Remark 5.2(ii) all terms of the trajectory should be inside of the interval $(\sqrt[\theta]{t}, \theta^k)$, i.e.,

$$g_\theta^{(n)}(t_0) > \frac{1}{\theta^k}, \quad (5.5)$$
for any \( n \in \mathbb{N} \).

On the other hand, since \( g_\theta : \Delta \to \mathbb{R} \) is increasing, due to (a1) one gets

\[
t_2 > t_0 > g_\theta(t_0) > g_\theta^{(2)}(t_0) > \cdots > g_\theta^{(n)}(t_0) > \cdots > \frac{1}{\theta^k}.
\]  

(5.6)

It follows from (5.6) that the sequence \( \{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty} \) converges and its limiting point should be a fixed point which is less than \( t_2 \). However, the function \( g_\theta \) does not have any fixed points except \( t_1, t_2, t_3 \). This contradiction shows that the trajectory \( \{g_\theta^{(n)}(t_0)\}_{n=1}^{\infty} \), starting from the point \( t_0 \), is finite.

(ii) Let \( 1 < \theta \leq \frac{k+1}{k-1} \). Then (a2) is evident. Using the same argument as (b1) one can prove (b2).

This completes the proof.

Let us study an asymptotical behavior of the dynamical system \( f : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) given by (4.3)

**Proposition 5.5.** Let \( f : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) be a dynamical system given by (4.3). Then the following assertions hold:

(i) If \( \theta > \frac{k+1}{k-1} \) then it has three fixed points which are equal to \( (A_i, B_i) \), where

\[
A_i = \frac{2\theta \sqrt{t_i} - 2}{\theta^2 - 1}, \quad B_i = \frac{2\theta - 2 \sqrt{t_i}}{\theta^2 - 1}, \quad i = 1, 2, 3,
\]

(ii) If \( 1 < \theta \leq \frac{k+1}{k-1} \) then it has one fixed point which is equal to \( (A_1, B_1) \).

Proof. From (4.3) we immediately find

\[
\frac{x'}{y'} = \frac{\theta \sqrt{x} - \sqrt{y}}{-\sqrt{x} + \theta \sqrt{y}} = \frac{\theta \sqrt{\frac{x}{y}} - 1}{\theta - \sqrt{\frac{x}{y}}}.
\]  

(5.7)

Hence, fixed points of \( f : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \) should satisfy the following equation

\[
\frac{x}{y} = \frac{\theta \sqrt{\frac{x}{y}} - 1}{\theta - \sqrt{\frac{x}{y}}}
\]  

(5.8)

Denote \( t = \frac{x}{y} \). Then (5.8) takes the form \( g_\theta(t) = t \). Consequently, Proposition 5.3 implies that if \( \theta > \frac{k+1}{k-1} \) then the function \( g_\theta \) has three fixed points which are equal to \( t_1, t_2, t_3 \) and if \( 1 < \theta \leq \frac{k+1}{k-1} \) then the function \( g_\theta \) has one fixed point which is equal to \( t_1 \). Therefore, we have that if \( \theta > \frac{k+1}{k-1} \) then

\[
\frac{x}{y} = t_i, \quad i = 1, 2, 3
\]

and if \( 1 < \theta \leq \frac{k+1}{k-1} \) then

\[
\frac{x}{y} = t_1.
\]
After elemental calculation one can see that if $\theta > \frac{k+1}{k-1}$ then the dynamical system $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ has three fixed points which are equal to $(A_i^k \sqrt[2k]{B_i}; B_i^k \sqrt[2k]{B_i})$, where

$$A_i = \frac{20 \sqrt[2k]{T_i} - 2}{\theta^2 - 1}, \quad B_i = \frac{2 \theta - 2 \sqrt[2k]{T_i}}{\theta^2 - 1}, \quad i = 1, 2, 3,$$

and if $1 < \theta \leq \frac{k+1}{k-1}$ then it has one fixed point which is equal to $(A_1^k \sqrt[2k]{B_1}; B_1^k \sqrt[2k]{B_1})$.

**Proposition 5.6.** Let $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ be a dynamical system given by (1.3). Then the following assertions hold true:

(i) If $\theta > \frac{k+1}{k-1}$ then it has three invariant semi-lines $l_i$ which are defined by $y = \frac{1}{t_i} x$, $i = 1, 2, 3$

(ii) If $1 < \theta \leq \frac{k+1}{k-1}$ then it has one invariant semi-line $l_1$ which is defined by $y = \frac{1}{t_1} x$.

**Proof.** It follows from (5.7) that if $t^*$ is a fixed point of the function $g_\theta$, then $\frac{x}{y} = t^*$ yields $\frac{x'}{y'} = t^*$. Therefore, if $\theta > \frac{k+1}{k-1}$ then the dynamical system $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ has three invariant semi-lines given by $y = \frac{1}{t_i} x$, $i = 1, 2, 3$ and if $1 < \theta \leq \frac{k+1}{k-1}$ then it has one invariant semi-line defined by $y = \frac{1}{t_1} x$.

**Theorem 5.7.** Let $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ a dynamical system be given by (1.3) and $\theta > \frac{k+1}{k-1}$. Then the following assertions hold true:

(i) If an initial point $(x^0, y^0)$ belongs to an invariant semi-line $l_i$ (where $i = 1, 2, 3$) of the dynamical system $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$ then the trajectory $\{f^{(n)}(x^0, y^0)\}_{n=1}^\infty$ starting from the point $(x^0, y^0)$, converges to a fixed point $(A_i^k \sqrt[2k]{B_i}; B_i^k \sqrt[2k]{B_i})$ which belongs to an invariant line $l_i$.

(ii) If an initial point $(x^0, y^0)$ satisfies the following condition

$$\frac{x^0}{y^0} \in (t_2, t_1) \cup (t_1, t_3)$$

then the trajectory $\{f^{(n)}(x^0, y^0)\}_{n=1}^\infty$ starting from the point $(x^0, y^0)$, converges to a fixed point $(A_1^k \sqrt[2k]{B_1}; B_1^k \sqrt[2k]{B_1})$ which belongs to an invariant semi-line $l_1$.

(iii) If an initial point $(x^0, y^0)$ satisfies the following condition

$$\frac{x^0}{y^0} \in \left(\frac{1}{\theta^k}, t_2\right) \cup (t_3, \theta^k)$$

then the trajectory $\{f^{(n)}(x^0, y^0)\}_{n=1}^\infty$ starting from the point $(x^0, y^0)$, is finite.

**Proof.** Assume that $\theta > \frac{k+1}{k-1}$.

(i). Now we suppose that $(x^0, y^0)$ belongs to an invariant semi-line $l_i$ ($i = 1, 2, 3$) of the dynamical system $f : \mathbb{R}^2_+ \rightarrow \mathbb{R}^2_+$. Then one has

$$\frac{x^{(n)}}{y^{(n)}} = t_i,$$
for any \( n \in \mathbb{N} \), where \((x^{(n)}, y^{(n)}) \equiv f^{(n)}(x^0, y^0)\). Hence, one finds

\[
\begin{cases}
  x^{(n)} = A_i \sqrt[k]{y^{(n-1)}} \\
  y^{(n)} = B_i \sqrt[k]{y^{(n-1)}}
\end{cases}
\]

Therefore

\[
\begin{cases}
  x^{(n)} = A_i \sqrt[k]{B_i \sqrt[k]{B_i \cdots \sqrt[k]{B_i}}} \sqrt[k]{y^0} = A_i B_i^{1 – \frac{1}{k}} \sqrt[k]{y^0} \\
  y^{(n)} = B_i \sqrt[k]{B_i \sqrt[k]{B_i \cdots \sqrt[k]{B_i}}} \sqrt[k]{y^0} = B_i^{1 – \frac{1}{k}} \sqrt[k]{y^0}.
\end{cases}
\]

If we take into account \( \frac{x_0}{A_i^{k-\sqrt[k]{B_i}}} = \frac{y_0}{B_i^{k-\sqrt[k]{B_i}}} \) then

\[
\begin{cases}
  x^{(n)} = A_i^{k-\sqrt[k]{B_i}} \sqrt[k]{\left( \frac{x^0}{A_i^{k-\sqrt[k]{B_i}}} \right)} \\
  y^{(n)} = B_i^{k-\sqrt[k]{B_i}} \sqrt[k]{\left( \frac{y^0}{B_i^{k-\sqrt[k]{B_i}}} \right)}
\end{cases}
\]

(5.9)

It is clear that the sequence \((x^{(n)}, y^{(n)})\) converges to \((A_i^{k-\sqrt[k]{B_i}}; B_i^{k-\sqrt[k]{B_i}})\) which is a fixed point belonging to \( l_i \).

(ii). Assume that an initial point \((x^0, y^0)\) satisfies

\[
\frac{x^0}{y^0} \in (t_2, t_1) \cup (t_1, t_3).
\]

It follows from (5.7) that

\[
\frac{x^{(n)}}{y^{(n)}} = g \left( \frac{x^{(n-1)}}{y^{(n-1)}} \right).
\]

According to Proposition 5.4 the sequence \( \left\{ \frac{x^{(n)}}{y^{(n)}} \right\}_{n=1}^{\infty} \) converges to the fixed point \( t_1 \) of the function \( g_{\theta} : \Delta \to \mathbb{R} \).

Taking \( \frac{x^{(n)}}{y^{(n)}} = c_n \), then one gets that \( g_{\theta}(c_n) = c_{n+1} \) and

\[
\begin{cases}
  x^{(n+1)} = a_n \sqrt[k]{y^{(n)}} \\
  y^{(n+1)} = b_n \sqrt[k]{y^{(n)}}
\end{cases}
\]

where

\[
a_n = \frac{2\theta \sqrt[k]{c_n} - 2}{\theta^2 - 1}, \quad b_n = \frac{2\theta - 2\sqrt[k]{c_n}}{\theta^2 - 1}.
\]
So, we find that

\[
\begin{align*}
    x^{(n+1)} &= a_n \sqrt[n+1]{b_{n-1} \sqrt[n-2]{b_{n-2}} \cdots \sqrt{b_0}} \\
    y^{(n+1)} &= b_n \sqrt[n+1]{b_{n-1} \sqrt[n-2]{b_{n-2}} \cdots \sqrt{b_0}}
\end{align*}
\]

The following lemma is useful to calculate the limiting point of the sequence \(\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}\).

**Lemma 5.8.** If a sequence \(\{b_n\}_{n=0}^{\infty}\), with positive terms, converges to \(\beta_0 > 0\) then the sequence

\[
\beta_n = b_n \sqrt[n+1]{b_{n-1} \sqrt[n-2]{b_{n-2}} \cdots \sqrt{b_0}}
\]

converges to \(\beta_0^{k \sqrt[n+1]{b_0}}\).

We know that

\[
e_n \to t_1, \quad b_n \to B_1, \quad a_n \to A_1.
\]

Then, according to Lemma 5.8, the sequence \((x^{(n)}, y^{(n)})\) converges to \((A_1^{k \sqrt[n+1]{b_1}}, B_1^{k \sqrt[n+1]{b_1}})\) which belongs to \(l_1\).

(iii). Now assume that an initial point \((x^0, y^0)\) satisfies

\[
\frac{x^0}{y^0} \in \left(\frac{1}{\theta^k}, t_2\right) \cup (t_3, \theta^k).
\]

It follows from (5.7) that

\[
\frac{x^{(n+1)}}{y^{(n+1)}} = g\left(\frac{x^{(n)}}{y^{(n)}}\right),
\]

for any \(n \in \mathbb{N}\). According to Proposition 5.4 the sequence \(\left\{\frac{x^{(n)}}{y^{(n)}}\right\}_{n=1}^{\infty}\) has a finite number of terms. Therefore the sequence \(\{(x^{(n+1)}, y^{(n+1)})\}_{n=0}^{\infty}\) is finite. \(\Box\)

Analogously, one can prove the following

**Theorem 5.9.** Let \(f : \mathbb{R}_+^2 \to \mathbb{R}_+^2\) be a dynamical system given by (4.3) and \(1 < \theta \leq \frac{k+1}{k-1}\).

(i) If an initial point \((x^0, y^0)\) belongs to an invariant semi-line \(l_1\) of the dynamical system \(f : \mathbb{R}_+^2 \to \mathbb{R}_+^2\) then the trajectory \(\{f^{(n)}(x^0, y^0)\}_{n=1}^{\infty}\), starting from the point \((x^0, y^0)\), converges to a fixed point \((A_1^{k \sqrt[n+1]{b_1}}, B_1^{k \sqrt[n+1]{b_1}})\) which belongs to an invariant semi-line \(l_1\).

(ii) If an initial point \((x^0, y^0)\) satisfies the following condition

\[
\frac{x^0}{y^0} \in \left(\frac{1}{\theta^k}, t_1\right) \cup (t_1, \theta^k)
\]

then the trajectory \(\{f^{(n)}(x^0, y^0)\}_{n=1}^{\infty}\), starting from the point \((x^0, y^0)\), is finite.
Remark 5.10. One can easily check that if an initial point \((x^0, y^0)\) belongs to an invariant line \(l_i\) of the dynamical system (4.3) then the trajectory \((x^{(n)}, y^{(n)})\) starting from \((x^0, y^0)\) has the following form

\[
\begin{align*}
    x^{(n)} &= A_i^{-k-1}B_i^{k} x^{(n)} \
    y^{(n)} &= B_i^{-k-1}B_i^{k} y^{(n)}
\end{align*}
\]  

(5.10)

In the case \(\theta > \frac{k+1}{k-1}\) the formula (5.10) was already shown in (5.9) where \(i = 1, 2, 3\). In the case \(\theta \leq \frac{k+1}{k-1}\) the dynamical system (4.3) has unique an invariant line \(l_i\) at \(i = 1\) in the formula (5.10).

6 Diagonalizability of forward QMC

In previous section we have found fixed points of the dynamical system (4.3) and prove the absence of periodic points for any \(\theta > 1\). Moreover, we investigated an asymptotical behavior of (4.3). It is clear that every fixed point of (4.3) defines boundary conditions which are solutions of (2.7), (2.8). Namely, we have that if \(\theta \leq \frac{k+1}{k-1}\) then there are boundary conditions \((w_0(\alpha_0), \{h_0(\alpha_0)\})\) of the model (3.4)

\[
w_0(\alpha_0) = \frac{1}{\alpha_0} \sigma_0, \quad h_0(\alpha_0) = \alpha_0 \sigma_0^{(x)}
\]

(6.1)

and if \(\theta > \frac{k+1}{k-1}\) then apart the previous one, there are two extra boundary conditions \((w_0(\beta), \{h_0(\beta)\})\) and \((w_0(\gamma), \{h_0(\gamma)\})\) of the model (3.4)

\[
w_0(\beta) = \frac{1}{\beta_0} \sigma_0, \quad h_0(\beta) = \beta_0 \sigma_0^{(x)} + \beta_3 \sigma_3^{(x)}
\]

(6.2)

\[
w_0(\gamma) = \frac{1}{\gamma_0} \sigma_0, \quad h_0(\gamma) = \gamma_0 \sigma_0^{(x)} + \gamma_3 \sigma_3^{(x)}
\]

(6.3)

where \(\alpha_0 = k^{-\sqrt{\theta_0}}, \beta = (\beta_0, \beta_3), \gamma = (\gamma_0, \gamma_3)\) such that

\[
\begin{align*}
    \beta_0 &= \frac{A_2 k^{-\sqrt{B_2}} + B_2 k^{-\sqrt{B_2}}}{2} \
    \beta_3 &= \frac{A_2 k^{-\sqrt{B_2}} - B_2 k^{-\sqrt{B_2}}}{2} \
    \gamma_0 &= \frac{A_3 k^{-\sqrt{B_3}} + B_3 k^{-\sqrt{B_3}}}{2} \
    \gamma_3 &= \frac{A_3 k^{-\sqrt{B_3}} - B_3 k^{-\sqrt{B_3}}}{2}
\end{align*}
\]

Note that these boundary conditions (6.1), (6.2), (6.3) due to Theorem 2.1 define the forward QMC. Hence, the existence of the boundary conditions imply the existence of forward QMC for the model (3.4) for any \(\theta > 1\).

We are going to prove diagonalizability of the forward QMC corresponding to any boundary conditions which are solutions of (2.7) and (2.8).

Recall that the diagonal subalgebra \(M^d_2(\mathbb{C})\) of the algebra \(M_2(\mathbb{C})\) is defined as follows

\[
M^d_2(\mathbb{C}) = \{ a \in M_2(\mathbb{C}) : a = a_0 1 + a_3 \sigma_3 \}.
\]
Since the elements $\mathbb{1}, \sigma_x, \sigma_y, \sigma_z$ are basis in $M_2(\mathbb{C})$ then every element $a \in M_2(\mathbb{C})$ can be written in the following form $a = a_0 \mathbb{1} + a_1 \sigma_x + a_2 \sigma_y + a_3 \sigma_z$. Then for any $a \in M_2(\mathbb{C})$ an element $a_d = a_0 \mathbb{1} + a_3 \sigma_z$ is called its diagonal part and an element $a_{xy} = a_1 \sigma_x + a_2 \sigma_y$ is called its $xy$-part. It is clear that a linear span of the elements $\mathbb{1}, \sigma_z$ is a commutative diagonal subalgebra $M_2^d(\mathbb{C})$. In these notions, any element $a \in M_2(\mathbb{C})$ can be written as $a = a_d + a_{xy}$.

Let us consider a conditional expectation $E : M_2(\mathbb{C}) \to M_2^d(\mathbb{C})$ defined by

$$E(a) = e_{11}a_{11} + e_{22}a_{22},$$

where $e_{11}, e_{22}$ are two minimal projectors of the algebra $M_2(\mathbb{C})$. It is clear that $E(a) = a_d$.

The diagonal subalgebra $B_L^d$ of the full algebra $B_L$ is defined by an inductive limit

$$B_L^d = \bigcup_{\Lambda_n} B_{\Lambda_n}^d, \quad B_{\Lambda_n}^d = \bigotimes_{u \in \Lambda_n} B_u^d, \quad B_u^d = M_2^d(\mathbb{C}), \quad \forall u \in \Lambda.$$

We will define a conditional expectation $E : B_L \to B_L^d$ on the algebra $B_L$ as follows: the value of this expectation at any linear generator $a_{\Lambda_n} = \bigotimes_{x \in \Lambda_n} a_x$ of the algebra $B_{\Lambda_n}$ is defined by

$$E(a_{\Lambda_n}) = \bigotimes_{x \in \Lambda_n} E(a_x) \quad (6.4)$$

and the linear extension of (6.4) defines it on the whole algebra $B_{\Lambda_n}$. By inductive limit we define the expectation $E$ on the full algebra $B_L$. For any $a_{\Lambda} \in B_{\Lambda}$ the value $E(a_{\Lambda})$ is called a diagonal part of $a_{\Lambda}$ and denote by $a_{\Lambda}^d = E(a_{\Lambda})$.

**Theorem 6.1.** Let $\varphi^{(f)}_{w_0, h}$ be a forward QMC of the model (3.4) with boundary conditions which are solutions of (2.7) and (2.8). Let $E : B_L \to B_L^d$ be a conditional expectation given (6.4) and $\theta > 1$. Then for any $a \in B_L$ one has

$$\varphi^{(f)}_{w_0, h}(a) = \varphi^{(f)}_{w_0, h}(E(a)).$$

**Proof.** Since $B_\Lambda$ is a quasi-local algebra it is enough to show that

$$\varphi^{(f)}_{w_0, h}(a_{\Lambda_n}) = \varphi^{(f)}_{w_0, h}(E(a_{\Lambda_n})) \quad (6.5)$$

for any $a_{\Lambda_n} \in B_{\Lambda_n}$. It follows from the definition of the QMC that

$$\varphi^{(f)}_{w_0, h}(a_{\Lambda_n}) = w - \lim_{m \to \infty} \varphi^{(m,f)}_{w_0, h}(a_{\Lambda_n}) = \varphi^{(n,f)}_{w_0, h}(a_{\Lambda_n}).$$

Analogously, one has

$$\varphi^{(f)}_{w_0, h}(E(a_{\Lambda_n})) = \varphi^{(n,f)}_{w_0, h}(E(a_{\Lambda_n})).$$

It follows from (4.2) that any solutions $\{h_x\}_{x \in \Lambda}$ of the equation (2.8) lie in the diagonal algebra $M_2^d(\mathbb{C})$. Let us choose $w_0 = \frac{1}{\text{Tr}(h_0)} \mathbb{1}$. Since an element $K_{<u,v>}$ given by (3.5) lies in the diagonal algebra $M_2^d(\mathbb{C}) \otimes M_2^d(\mathbb{C})$ then an element $W_n$ given by (2.4) lies in the diagonal algebra $B_L^d$. We then get

$$\varphi^{(n,f)}_{w_0, h}(E(a_{\Lambda_n})) = \text{Tr}(W_n(E(a_{\Lambda_n})) = \text{Tr}(E(W_n a_{\Lambda_n})) = \text{Tr}(W_n a_{\Lambda_n}) = \varphi^{(n,f)}_{w_0, h}(a_{\Lambda_n}).$$

This completes the proof. \qed
7 Uniqueness of forward QMC: regime \( \theta \leq \frac{k+1}{k-1} \)

In this section we prove the first part of the main theorem (see Theorem 3.1), i.e. we show the uniqueness of the forward quantum \( d \)-Markov chain in the regime \( 1 < \theta \leq \frac{k+1}{k-1} \).

We assume that \( \theta \leq \frac{k+1}{k-1} \). Since \( t_1 = 1 \) we have \( \Theta := A_1 = B_1 = \frac{2}{\theta + 1} \). Then it follows from Theorem 5.9 and Remark 5.10 that the equation (2.8) does not have any solution except the following parametrical solutions \( \{ h_\alpha(x) \} \) given by

\[
h^{(n)}_x(\alpha) = \begin{pmatrix}
\frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}} & 0 \\
0 & \frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}}
\end{pmatrix},
\]

for every \( x \in W_n \), here \( \alpha \) is any positive real number and \( n \in \mathbb{N} \setminus \{0\} \). One of the solutions of the equation (2.7) has the following form

\[
w_0(\alpha) = \begin{pmatrix}
\frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}} & 0 \\
0 & \frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}}
\end{pmatrix}
\]

The boundary conditions corresponding to the fixed point \( (\frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}}, \frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}}) \) of the dynamical system (7.3) are the following

\[
w_0(\alpha_0) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

\[
h^{(n)}_x(\alpha_0) = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

which correspond to the value of \( \alpha_0 = \frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}} \) in (7.1), (7.2).

Let us consider the states \( \varphi^{(n,f)}_{w_0(\alpha),h(\alpha)} \) corresponding to the solutions \( \{ w_0(\alpha), \{ h^{(n)}_x(\alpha) \} \} \). By definition we have

\[
\varphi^{(n,f)}_{w_0(\alpha),h(\alpha)}(x) = \text{Tr} \left( w_0^{1/2}(\alpha) \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{x \in W_n} h^{(n)}_x(\alpha) \prod_{i=0}^{n-1} K^*_{[n-1-i,n-i]} w_0^{1/2}(\alpha) x \right)
\]

\[
= \frac{(k-\sqrt{\Theta^k})^n}{\alpha} \left( \frac{k-\sqrt{\Theta^k}}{k+\sqrt{\Theta^k}} \right)^n \text{Tr} \left( \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{i=0}^{n-1} K^*_{[n-1-i,n-i]} x \right)
\]

\[
= \frac{\alpha_0^n}{\alpha_0} \left( \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{i=0}^{n-1} K^*_{[n-1-i,n-i]} x \right)
\]

\[
= \text{Tr} \left( w_0^{1/2}(\alpha) \prod_{i=0}^{n-1} K_{[i,i+1]} \prod_{x \in W_n} h^{(n)}_x(\alpha) \prod_{i=0}^{n-1} K^*_{[n-1-i,n-i]} w_0^{1/2}(\alpha) x \right)
\]

\[
= \varphi^{(n,f)}_{w_0(\alpha_0),h(\alpha_0)}(x),
\]

for any \( \alpha \). Hence, from the definition of forward QMC one finds that \( \varphi^{(f)}_{w_0(\alpha),h(\alpha)} = \varphi^{(f)}_{w_0(\alpha_0),h(\alpha_0)} \), which yields that the uniqueness of QMC associated with the model (3.2). Hence, Theorem 3.1 (i) is proved.
8 Existence of phase transition: regime $\theta > \frac{k+1}{k-1}$

This section is devoted to the proof of part (ii) of Theorem 3.1. In the sequel we suppose that $\theta > \frac{k+1}{k-1}$.

In this section, for the sake of simplicity of formulas, we will use the following notations, for the Pauli matrices:

$$\sigma_0 := \mathbf{1}, \quad \sigma_1 := \sigma_x, \quad \sigma_2 := \sigma_y, \quad \sigma_3 := \sigma_z$$

According to Proposition 5.5 there are three fixed points of the dynamical system (4.3) in the considering regime. Then the corresponding solutions of equations (2.7), (2.8) can be written as follows: $(w_0(\alpha_0), \{h_x(\alpha_0)\}), (w_0(\beta), \{h_x(\beta)\})$ and $(w_0(\gamma), \{h_x(\gamma)\})$, where

$$w_0(\alpha_0) = \frac{1}{\alpha_0} \sigma_0, \quad h_x(\alpha_0) = \alpha_0 \sigma^{(x)}_0$$
$$w_0(\beta) = \frac{1}{\beta_0} \sigma_0, \quad h_x(\beta) = \beta_0 \sigma^{(x)}_0 + \beta_3 \sigma^{(x)}_3$$
$$w_0(\gamma) = \frac{1}{\gamma_0} \sigma_0, \quad h_x(\gamma) = \gamma_0 \sigma^{(x)}_0 + \gamma_3 \sigma^{(x)}_3$$

here $\alpha_0 = \sqrt[3]{k} \frac{\Theta}{\gamma}$, and $\beta = (\beta_0, \beta_3)$, $\gamma = (\gamma_0, \gamma_3)$ are vectors with

$$\beta_0 = \frac{A_2 \sqrt{k} \sqrt{B_2} + B_2 \sqrt{k} \sqrt{B_2}}{2}, \quad \beta_3 = \frac{A_2 \sqrt{k} \sqrt{B_2} - B_2 \sqrt{k} \sqrt{B_2}}{2},$$
$$\gamma_0 = \frac{A_3 \sqrt{k} \sqrt{B_3} + B_3 \sqrt{k} \sqrt{B_3}}{2}, \quad \gamma_3 = \frac{A_3 \sqrt{k} \sqrt{B_3} - B_3 \sqrt{k} \sqrt{B_3}}{2}.$$  \hspace{1cm} (8.1, 8.2)

By $\varphi^{(f)}_{\alpha_0(\alpha_0), \{h_0(\alpha_0)\}}, \varphi^{(f)}_{\alpha_0(\beta), \{h(\beta)\}}$, and $\varphi^{(f)}_{\alpha_0(\gamma), \{h(\gamma)\}}$ we denote the corresponding forward quantum Markov chains. To prove the existence of the phase transition, we need to show that there are two states which are not quasi-equivalent. We will show that two states $\varphi^{(f)}_{\alpha_0(\alpha_0), \{h(\alpha_0)\}}$, $\varphi^{(f)}_{\alpha_0(\gamma), \{h(\gamma)\}}$ are not quasi-equivalent. To do so, we will need some auxiliary facts and results.

First of all, we recall some properties of $2 \times 2$ special matrices which are not required their proof.

Let $M, N$ be matrices given as follows

$$M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad N = \begin{pmatrix} c & d \\ d & c \end{pmatrix}.$$  \hspace{1cm} (8.3)

Then, these matrices commute each other, i.e., $MN = NM$. For any $n \in \mathbb{N}$ one has

$$M^n = \frac{1}{2} \begin{pmatrix} (a+b)^n + (a-b)^n & (a+b)^n - (a-b)^n \\ (a+b)^n - (a-b)^n & (a+b)^n + (a-b)^n \end{pmatrix}.$$  \hspace{1cm} (8.3)

For the sake of simplicity we will use the following notations

$$\Theta_+ = \frac{\gamma_0 + \frac{\Theta - 1}{2}}{2}, \quad \Theta_- = \frac{\gamma_0 - \frac{\Theta - 1}{2}}{2}.$$  \hspace{1cm} (8.4)
where \( \gamma_0, \gamma_3 \) are given by (8.2). Let us denote by

\[
A = \frac{1}{2} \begin{pmatrix}
\frac{\theta + 1}{2} \left( \Theta_1^{k-1} + \Theta_2^{k-1} \right) & \frac{\theta - 1}{2} \left( \Theta_1^{k-1} - \Theta_2^{k-1} \right) \\
\frac{\theta + 1}{2} \left( \Theta_3^{k-1} - \Theta_4^{k-1} \right) & \frac{\theta - 1}{2} \left( \Theta_3^{k-1} + \Theta_4^{k-1} \right)
\end{pmatrix}.
\]

(8.5)

Let us study some properties of this matrix \( A \). The next proposition deals with eigenvalues of the matrix \( A \).

**Proposition 8.1.** Let \( A \) be the matrix given by (8.5). Then the following assertions hold true:

(i) The matrix \( A \) has the following form

\[
A = \frac{1}{(\theta + t_3)(\theta t_3 + 1)} \begin{pmatrix}
\frac{\theta + 1}{2}(t_3^2 + 2\theta t_3 + 1) & \frac{\theta - 1}{2}(t_3^2 - 1) \\
\frac{\theta + 1}{2}(t_3^2 - 1) & \frac{\theta - 1}{2}(t_3^2 + 2\theta t_3 + 1)
\end{pmatrix}
\]

(8.6)

where \( t_3 \) is a fixed point of the function given by (5.1);

(ii) The numbers \( \lambda_1 = 1, \lambda_2 = \det(A) \in (0, 1) \) are eigenvalues of the matrix \( A \);

(iii) The vectors

\[
(x_1, y_1) = (t_3 + 1, t_3 - 1), \quad \lambda_1 = 1
\]

(8.7)

\[
(x_2, y_2) = (-(\theta - 1)(t_3 - 1), (\theta + 1)(t_3 + 1)), \quad \lambda_2 = \det(A)
\]

(8.8)

are eigenvectors of the matrix \( A \) corresponding to the eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = \det(A) \), respectively;

(iv) If the matrix \( P \) has the following form

\[
P = \begin{pmatrix}
t_3 + 1 & -(\theta - 1)(t_3 - 1) \\
t_3 - 1 & (\theta + 1)(t_3 + 1)
\end{pmatrix},
\]

then

\[
P^{-1}AP = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix};
\]

(8.9)

(v) For any \( n \in \mathbb{N} \) one has

\[
A^n = \begin{pmatrix}
\frac{(\theta + 1)x_1^2 + (\theta - 1)y_1^2 \lambda_2}{x_1 y_1(\theta + 1)(1 - \lambda_2)} & \frac{x_1 y_1(\theta - 1)(1 - \lambda_2)}{(\theta + 1)x_1^2 + (\theta - 1)y_1^2} \\
\frac{(\theta + 1)x_1^2 + (\theta - 1)y_1^2}{x_1 y_1(\theta + 1)(1 - \lambda_2)} & \frac{(\theta + 1)x_1^2 + (\theta - 1)y_1^2}{(\theta + 1)x_1^2 + (\theta - 1)y_1^2}
\end{pmatrix},
\]

(8.10)

where \( (x_1, y_1) \) is an eigenvector of the matrix \( A \).

**Proof.** (i). We know that \( t_3 \) is a fixed point of (5.1), i.e.,

\[
\frac{\theta^{k-1}t_3 - 1}{\theta - \sqrt{k}t_3} = t_3.
\]

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It follows from the last identity that
\[ t_3 = \left( \frac{\theta t_3 + 1}{\theta + t_3} \right)^k . \]  

(8.11)

By means (8.2), (8.4), (8.11) one can easily get that the matrix \( A \) has the form (8.6).

(ii). We know that the following equation
\[ \lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \]
is a characteristic equation of the matrix \( A \) given by (8.6). Without forcing by detail we can make sure
\[ \text{Tr}(A) - \det(A) = 1, \]
this means that \( \lambda_1 = 1 \) and \( \lambda_2 = \det(A) = \frac{(\theta^2-1)t_3}{(\theta + t_3)(\theta t_3 + 1)} \) are eigenvalues of the matrix \( A \).

(iii). The eigenvector \((x_1, y_1)\) of the matrix \( A \), corresponding to \( \lambda_1 = 1 \) satisfies the following equation
\[ (t_3 - 1)x_1 = (t_3 + 1)y_1. \]

Then, one finds
\[
\begin{cases}
  x_1 = t_3 + 1 \\
  y_1 = t_3 - 1.
\end{cases}
\]

Analogously, one can show that the eigenvector \((x_2, y_2)\) of the matrix \( A \), corresponding to \( \lambda_2 = \det(A) \), is equal to
\[
\begin{cases}
  x_2 = -(\theta - 1)(t_3 - 1) \\
  y_2 = (\theta + 1)(t_3 + 1).
\end{cases}
\]

It is worth noting that \((x_2, y_2) = (-\theta - 1)y_1, (\theta + 1)x_1)\).

(iv). It is clear that
\[ P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \]
where the vectors \((x_1, y_1)\) and \((x_2, y_2)\) are defined by (8.7), (8.8). We then get
\[
P^{-1}A P = \frac{1}{\det(P)} \begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix} \begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},
\]
where \(\det(P) = (\theta + 1)x_1^2 + (\theta - 1)y_1^2 > 0.\)

(v). From (8.9) it follows that
\[ A = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}. \]
Therefore, for any \( n \in \mathbb{N} \) we obtain

\[
A^n = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} y_2 \lambda_1^n & -x_2 \lambda_1^n \\ -y_1 \lambda_2^n & x_1 \lambda_2^n \end{pmatrix}
\]

\[
= \frac{1}{\det(P)} \begin{pmatrix} x_1 y_2 \lambda_1^n - x_2 y_1 \lambda_2^n & x_1 x_2 (\lambda_2^n - \lambda_1^n) \\ y_1 y_2 (\lambda_1^n - \lambda_2^n) & x_1 y_2 \lambda_2^n - x_2 y_1 \lambda_1^n \end{pmatrix}
\]

\[
= \begin{pmatrix} (\theta + 1)x_1^2 + (\theta - 1)y_1^2 \lambda_2^n & x_1 y_1 (\theta - 1)(1 - \lambda_2^n) \\ x_1 y_1 (\theta + 1)(1 - \lambda_1^n) & (\theta + 1)x_1^2 + (\theta - 1)y_1^2 \end{pmatrix}.
\]

This completes the proof. \( \Box \)

In what follows, for the sake of simplicity, let us denote

\[
K_0 = \sqrt{\theta} + \frac{1}{2}, \quad K_3 = \sqrt{\theta} - \frac{1}{2},
\]

here as before \( \theta = \exp\{2\beta\} \). In these notations, the operator \( K_{<u,v>} \) given by (8.12) can be written as follows

\[
K_{<u,v>} = K_0 \sigma_0^{(u)} \otimes \sigma_0^{(v)} + K_3 \sigma_3^{(u)} \otimes \sigma_3^{(v)}.
\]

**Remark 8.2.** In the sequel, we will frequently use the following identities for the numbers \( K_0, K_3 \) given by (8.12):

(i) \( K_0^2 + K_3^2 = \frac{\theta + 1}{2} \);

(ii) \( 2K_0 K_3 = \frac{\theta - 1}{2} \).

**Proposition 8.3.** Let \( K_{<u,v>} \) be given by (8.13), \( \overrightarrow{S(x)} = \{1, 2, \ldots, k\} \), and \( h^{(i)} = h_0^{(i)} \sigma_0^{(i)} + h_3^{(i)} \sigma_3^{(i)} \), where \( i \in \overrightarrow{S(x)} \). Then we have

\[
\text{Tr}_x \left[ \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \prod_{i \in \overrightarrow{S(x)}} h^{(i)}_0 \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \right] = g_0^{(x)} \sigma_0^{(x)} + g_3^{(x)} \sigma_3^{(x)}
\]

where

\[
\begin{pmatrix} g_0^{(x)} \\ g_3^{(x)} \end{pmatrix} = A_{h^{(1)}} A_{h^{(2)}} \cdots A_{h^{(k)}} e_1, \quad e_1 = (1, 0),
\]

\[
A_{h^{(i)}} = \begin{pmatrix} (K_0^2 + K_3^2) h_0^{(i)} \\ 2K_0 K_3 h_3^{(i)} \end{pmatrix}, \quad i \in \overrightarrow{S(x)}.
\]

**Proof.** It is clear that

\[
\text{Tr}_x \left[ \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \prod_{i \in \overrightarrow{S(x)}} h^{(i)}_0 \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \right] = \text{Tr}_x \left[ K_{<x,1>} h^{(1)} \text{Tr}_x \left[ K_{<x,2>} h^{(2)} \cdots \text{Tr}_x \left[ K_{<x,k>} h^{(k)} K_{<x,k>} \cdots K_{<x,2>} \right] K_{<x,1>} \right] \right]
\]

\[
= \text{Tr}_x \left[ K_{<x,1>} h^{(1)} \right] \text{Tr}_x \left[ K_{<x,2>} h^{(2)} \right] \cdots \text{Tr}_x \left[ K_{<x,k>} h^{(k)} K_{<x,k>} \cdots K_{<x,2>} \right] K_{<x,1>}
\]

\[
= \text{Tr}_x \left[ K_{<x,1>} h^{(1)} K_{<x,2>} h^{(2)} \cdots K_{<x,k>} h^{(k)} K_{<x,k>} \cdots K_{<x,2>} \right] K_{<x,1>}
\]

\[
= \text{Tr}_x \left[ K_{<x,1>} h^{(1)} \right] \text{Tr}_x \left[ K_{<x,2>} h^{(2)} \right] \cdots \text{Tr}_x \left[ K_{<x,k>} h^{(k)} K_{<x,k>} \cdots K_{<x,2>} \right] K_{<x,1>}
\]
Let us first evaluate \( g_k^{(x)} := \text{Tr}_x \left[ K_{<x,k>} h^{(k)} K_{<x,k>} \right] \). From (8.13) it follows that

\[
K_{<x,k>} h^{(k)} K_{<x,k>} = K_0^2 \sigma_0^{(x)} \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)}) + K_0 K_3 \sigma_3^{(x)} \otimes (h^{(k)}_{0} \sigma_3^{(k)} + h^{(k)}_{3} \sigma_0^{(k)}) + K_0 K_3 \sigma_3^{(x)} \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)}) + K_3^2 \sigma_0^{(x)} \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)})
\]

Therefore, one gets

\[
g_k^{(x)} = g_{k,0}^{(x)} + g_{k,3}^{(x)} \tag{8.17}
\]

where

\[
(g_{k,0}^{(x)}, g_{k,3}^{(x)}) = A_{h(k)} e_1, \quad e_1 = (1, 0). \tag{8.18}
\]

Now, evaluate \( g_{k-1}^{(x)} := \text{Tr}_x \left[ K_{<x,k-1>} h^{(k-1)} g_k^{(x)} K_{<x,k-1>} \right] \). Using (8.13) and (8.17) we find

\[
K_{<x,k-1>} h^{(k-1)} g_k^{(x)} K_{<x,k-1>} = K_0^2 \left( g_{k,0}^{(x)} + g_{k,3}^{(x)} \right) \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)}) + K_0 K_3 \left( g_{k,0}^{(x)} + g_{k,3}^{(x)} \right) \otimes (h^{(k)}_{0} \sigma_3^{(k)} + h^{(k)}_{3} \sigma_0^{(k)}) + K_0 K_3 \left( g_{k,0}^{(x)} + g_{k,3}^{(x)} \right) \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)}) + K_3^2 \left( g_{k,0}^{(x)} + g_{k,3}^{(x)} \right) \otimes (h^{(k)}_{0} \sigma_0^{(k)} + h^{(k)}_{3} \sigma_3^{(k)})
\]

Hence, one has

\[
g_{k-1}^{(x)} = g_{k-0}^{(x)} + g_{k-3}^{(x)} \tag{8.19}
\]

where

\[
(g_{k-0}^{(x)}, g_{k-3}^{(x)}) = A_{h(k-1)} \left( g_{k,0}^{(x)}, g_{k,3}^{(x)} \right) = A_{h(k-1)} A_{h(k)} e_1 \tag{8.20}
\]

Similarly, one can evaluate

\[
g_{i}^{(x)} := \text{Tr}_x \left[ K_{<x,i>} h^{(i)} g_{i+1}^{(x)} K_{<x,i>} \right] = g_{i,0}^{(x)} + g_{i,3}^{(x)} \tag{8.21}
\]

where

\[
(g_{i,0}^{(x)}, g_{i,3}^{(x)}) = A_{h(i)} \left( g_{i+1,0}^{(x)}, g_{i+1,3}^{(x)} \right) = \cdots = A_{h(i)} A_{h(i+1)} \cdots A_{h(k-1)} A_{h(k)} e_1 \tag{8.22}
\]

Consequently, we have

\[
\text{Tr}_x \left[ \prod_{i \in S(x)} K_{<x,i>} \prod_{i \in S(x)} h^{(i)} \prod_{i \in S(x)} K_{<x,i>} \right] = g_{0}^{(x)} \sigma_0^{(x)} + g_{3}^{(x)} \sigma_3^{(x)},
\]

where

\[
(g_{0}^{(x)}, g_{3}^{(x)}) = A_{h(1)} A_{h(2)} \cdots A_{h(k-1)} A_{h(k)} e_1.
\]

This completes the proof. \( \square \)
Remark 8.4. One can easily check that for any permutation \( \pi \) of the set \( \overrightarrow{S(x)} \) we have

\[
A_{h^{(1)}}A_{h^{(2)}} \cdots A_{h^{(k-1)}}A_{h^{(k)}} = A_{h^{(\pi(1))}}A_{h^{(\pi(2))}} \cdots A_{h^{(\pi(k-1))}}A_{h^{(\pi(k))}},
\]
in other words the matrices \( A_{h^{(i)}} \), \( i \in \overrightarrow{S(x)} \) commute each other.

Corollary 8.5. Let \( K_{<u,v>} \) be given by \( (8.13) \), \( \overrightarrow{S(x)} = (1, 2, \ldots, k) \), and

\[
\mathbf{h}^{(1)} = h_3 \sigma_3^{(1)}, \quad \mathbf{h}^{(i)} = \alpha_i \sigma_0^{(i)}, \quad i = 2, k,
\]
where \( \alpha_0 = \frac{k-1}{\sqrt{\Theta}}, \Theta = \frac{2}{\theta + 1} \), and \( h_3 \) is some positive number. Then we have

\[
\text{Tr}_x \left[ \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \prod_{i \in \overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \right] = h_3 \frac{\theta - 1}{\theta + 1} \sigma_3^{(x)}
\]

(8.23)

Proof. Let us calculate the matrices \( A_{h^{(i)}} \), \( i \in \overrightarrow{S(x)} \) which are given by \( (8.16) \). It is clear that

\[
A_{h^{(1)}} = \begin{pmatrix} 0 & 2K_0K_3h_3 \\ 2K_0K_3h_3 & 0 \end{pmatrix}, \quad A_{h^{(i)}} = \begin{pmatrix} (K_0^2 + K_3^2) \alpha_0 & 0 \\ 0 & (K_0^2 + K_3^2) \alpha_0 \end{pmatrix},
\]
where \( i = 2, k \). We then have

\[
A_{h^{(1)}}A_{h^{(2)}} \cdots A_{h^{(k)}} = \begin{pmatrix} 0 & 2\alpha_0^{-1}(K_0^2 + K_3^2)k^{-1}K_0K_3h_3 \\ 2\alpha_0^{-1}(K_0^2 + K_3^2)k^{-1}K_0K_3h_3 & 0 \end{pmatrix}.
\]

Therefore, it follows from Remark 8.2 and Proposition 8.3 that

\[
\text{Tr}_x \left[ \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \prod_{i \in \overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \right] = h_3 \alpha_0^{-k-1} \left( \frac{\theta - 1}{2} \right) \left( \frac{\theta + 1}{2} \right) k^{-1} \sigma_3^{(x)}
\]

\[
= h_3 \frac{\theta - 1}{\theta + 1} \sigma_3^{(x)}.
\]

\[
\square
\]

Corollary 8.6. Let \( K_{<u,v>} \) be given by \( (8.13) \), \( \overrightarrow{S(x)} = (1, 2, \ldots, k) \), and

\[
\mathbf{h}^{(1)} = h_0 \sigma_0^{(1)} + h_3 \sigma_3^{(1)}, \quad \mathbf{h}^{(i)} = \gamma_0 \sigma_0^{(i)} + \gamma_3 \sigma_3^{(i)}, \quad i = 2, k,
\]
where the numbers \( \gamma_0, \gamma_3 \) are given by \( (8.2) \) and \( h_0, h_3 \) are some positive numbers. Then we have

\[
\text{Tr}_x \left[ \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \prod_{i \in \overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i \in \overrightarrow{S(x)}} K_{<x,i>} \right] = h_0^{(x)} \sigma_0^{(x)} + h_3^{(x)} \sigma_3^{(x)},
\]

(8.24)

where \( h^{(x)} = \mathbf{h}h^{(x)} \), \( \mathbf{h}^{(x)} = (h_0^{(x)}, h_3^{(x)}) \), \( h = (h_0, h_3) \) are vectors.
Proposition 8.7. Let \( \omega \) with boundary conditions \( \omega \) where the matrix \( A \) Let us calculate the matrices \( A_{h^{(i)}} \), \( i \in S(x) \) which are given by (8.16). It follows from Remark 8.2 that

\[
A_{h^{(1)}} = \begin{pmatrix} h_0 \frac{\theta + 1}{2} & h_3 \frac{\theta - 1}{2} \\ h_3 \frac{\theta - 1}{2} & h_0 \frac{\theta + 1}{2} \end{pmatrix}, \quad A_{h^{(k)}} = \begin{pmatrix} \gamma_0 \frac{\theta + 1}{2} & \gamma_3 \frac{\theta - 1}{2} \\ \gamma_3 \frac{\theta - 1}{2} & \gamma_0 \frac{\theta + 1}{2} \end{pmatrix},
\]

where \( i = 2, k \). By means (8.3) we then get

\[
A_{h^{(2)}} \cdots A_{h^{(k)}} = A_{h^{(k)}}^{-1} \left( \begin{array}{cc} \Theta^{k-1} & \Theta^{k-1} \\ \Theta^{k-1} & \Theta^{k-1} \end{array} \right).
\]

here as before \( \Theta_+, \Theta_- \) are given by (8.4). After simple algebra, it follows from Proposition 8.3 that

\[
h(x) = A_{h^{(1)}} A_{h^{(2)}} \cdots A_{h^{(k)}} e_1 = A h
\]

where the matrix \( A \) is given by (8.5) and \( h(x) = (h_0, h_3) \), \( h = (h_0, h_3) \) are vectors.

Let us consider the following elements:

\[
\sigma_0^\Lambda := \bigotimes_{x \in \Lambda} \sigma_0^{(x)} \in B_\Lambda, \Lambda \subset \Lambda_n, \quad \sigma_3^{S(x),1} := \sigma_3^{(1)} \otimes \sigma_0^{(2)} \otimes \cdots \otimes \sigma_0^{(k)} \in B_{S(x)}, \quad (8.25)
\]

\[
\sigma_3^{W_{n+1},1} := \sigma_3^{S(x^{W_{n+1}}),1} \otimes \sigma_0^{S(x^{W_{n+1}})} \in B_{W_{n+1}}, \quad (8.26)
\]

\[
a_{\sigma_3}^{\Lambda_{N+1}} := \bigotimes_{i=0}^n \sigma_3^{W_{i}} \otimes \sigma_3^{W_{n+1},1} \in B_{\Lambda_{n+1}}. \quad (8.27)
\]

Proposition 8.7. Let \( \varphi_{w_0(a_0), h(a_0)} \) be a forward QMC corresponding to the model (8.13) with boundary conditions \( w_0(a_0) = \frac{1}{\alpha_0} \sigma_0 \) and \( h(x) = \alpha_0 \sigma_0^{(x)} \) for all \( x \in L \), where \( \alpha_0 = k^{-1} \sqrt{\theta} \), \( \theta = \frac{2}{\kappa} \). Let \( a_{\sigma_3}^{\Lambda_{N+1}} \) be an element given by (8.27) and \( \theta > \frac{k+1}{k-1} \). Then one has

\[
\varphi_{w_0(a_0), h(a_0)} (a_{\sigma_3}^{\Lambda_{N+1}}) = 0, \text{ for any } N \in \mathbb{N}.
\]

Proof. Due to (2.8) (see Theorem 2.1) the compatibility condition holds \( \varphi_{w_0(a_0), h(a_0)} [B_{\Lambda_n} = \varphi_{w_0(a_0), h(a_0)}] \). Therefore,

\[
\varphi_{w_0(a_0), h(a_0)} (a_{\sigma_3}^{\Lambda_{N+1}}) = w - \lim_{n \to \infty} \varphi_{w_0(a_0), h(a_0)} (a_{\sigma_3}^{\Lambda_{N+1}}) = \varphi_{w_0(a_0), h(a_0)} (a_{\sigma_3}^{\Lambda_{N+1}}). \quad (8.28)
\]

Taking into account \( w_0(a_0) = \frac{1}{\alpha_0} \sigma_0 \) and due to Proposition 2.2 it is enough to evaluate the following

\[
\varphi_{w_0(a_0), h(a_0)} (a_{\sigma_3}^{\Lambda_{N+1}}) = \text{Tr} \left( W_{N+1} (a_{\sigma_3}^{\Lambda_{N+1}}) \right)
\]

\[
= \frac{1}{\alpha_0} \text{Tr} \left[ K_{[0,1]} \cdots K_{[N,N+1]} h_{N+1} K_{[N,N+1]}^* \cdots K_{[0,1]}^* a_{\sigma_3}^{\Lambda_{N+1}} \right]
\]

\[
= \frac{1}{\alpha_0} \text{Tr} \left[ K_{[0,1]} \cdots K_{[N-1,N]} \right] \text{Tr} \left[ K_{[N,N+1]} h_{N+1} K_{[N,N+1]}^* \cdots K_{[0,1]}^* \right].
\]

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Now let us calculate \( \tilde{h}_N := \text{Tr}_N \left[ K_{[N,N+1]} h_{N+1}^* K_{[N,N+1]}^* \sigma_3 \right] \). Since \( K_{<u,v>} \) is a self-adjoint, we then get

\[
\tilde{h}_N = \text{Tr}_{\tilde{x}_{\tilde{W}_N}} \left[ \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} K_{x_{\tilde{W}_N}^{(1)},y}^{(1)} \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} h^{(y)} \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} K_{x_{\tilde{W}_N}^{(1)},y}^{(1)} \sigma_3 \sigma_3 \right] \boxtimes \text{Tr}_x \left[ \prod_{y \in S(x)} K_{x,y}^{(x)} \prod_{y \in S(x)} h^{(y)} \prod_{y \in S(x)} K_{x,y}^{(x)} \right].
\]

We know that

\[
\text{Tr}_x \left[ \prod_{y \in S(x)} K_{x,y}^{(x)} \prod_{y \in S(x)} h^{(y)} \prod_{y \in S(x)} K_{x,y}^{(x)} \right] = h^{(x)}, \quad (8.29)
\]

for every \( x \in \tilde{W}_N \setminus x_{\tilde{W}_N}^{(1)} \). On the other hand, since operators \( K_{<u,v>} \) and \( \sigma_3^{(x)} \) commute each other for any \( u, v, x \in \tilde{L} \) it follows from Corollary \( 8.5 \) that

\[
\text{Tr}_{\tilde{x}_{\tilde{W}_N}} \left[ \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} K_{x_{\tilde{W}_N}^{(1)},y}^{(1)} \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} h^{(y)} \prod_{y \in S(x_{\tilde{W}_N}^{(1)})} K_{x_{\tilde{W}_N}^{(1)},y}^{(1)} \sigma_3 \sigma_3 \right] = \tilde{h}_{\tilde{x}_{\tilde{W}_N}}^{(1)}, \quad (8.30)
\]

where

\[
\tilde{h}_{\tilde{x}_{\tilde{W}_N}}^{(1)} = \frac{\theta - 1}{\theta + 1} \sigma_3^{(x_{\tilde{W}_N}^{(1)})}.
\]

Hence, we obtain

\[
\tilde{h}_N = \tilde{h}_{\tilde{x}_{\tilde{W}_N}^{(1)}} \boxtimes h^{(x)}.
\]

Therefore, one finds

\[
\varphi_{\omega_0,h(\alpha_0)}^{(N+1,f)} \left( d^{A_{N+1}} \sigma_3 \right) = \frac{1}{\alpha_0} \text{Tr} \left[ K_{[0,1]} \cdots K_{[N-2,N-1]} \right] \text{Tr}_{N-1} \left[ K_{[N-1,N]} \tilde{h}_N K_{[N-1,N]}^* K_{[N-2,N-1]}^* \cdots K_{[0,1]}^* \right].
\]

So, after \( N \) times applying Corollary \( 8.5 \), we get

\[
\varphi_{\omega_0,h(\alpha_0)}^{(N+1,f)} \left( d^{A_{N+1}} \sigma_3 \right) = \alpha_0^{N-1} \left( \frac{\theta - 1}{\theta + 1} \right)^{2N} \text{Tr}(\sigma_3^{(0)}) = 0.
\]

This completes the proof. \( \Box \)
Proposition 8.8. Let $\varphi^{(f)}_{u_0, h(\gamma)}$ be a forward QMC corresponding to the model (8.13) with boundary conditions $\omega_0(\gamma) = \frac{1}{\gamma_0(\gamma)}$ and $h(x) = \gamma_0(\gamma)_0 + \gamma_3(\gamma)_3$ for all $x \in L$, where $\gamma_0, \gamma_3$ are given by (8.2). Let $a_{\sigma_3}^\Lambda_{N+1}$ be an element given by (8.27) and $\theta > \frac{h+1}{\kappa-1}$. Then one has

$$\varphi^{(f)}_{u_0, h(\gamma)}(a_{\sigma_3}^\Lambda_{N+1}) = \frac{1}{\gamma_0} \left( A_{N+1} h_{\gamma_0, \gamma_3}, e \right) \quad \forall N \in \mathbb{N},$$

where $A$ is a matrix given by (8.6), $\langle \cdot, \cdot \rangle$ is an inner product of vectors and $e = (1, 0)$, $h_{\gamma_0, \gamma_3} = (\gamma_3, \gamma_0)$ are vectors.

Proof. Again the compatibility condition yields that

$$\varphi^{(f)}_{u_0, h(\gamma)}(a_{\sigma_3}^\Lambda_{N+1}) = w - \lim_{n \to \infty} \varphi^{(n,f)}_{u_0, h(\gamma)}(a_{\sigma_3}^\Lambda_{N+1}) = \varphi^{(N+1,f)}_{u_0, h(\gamma)}(a_{\sigma_3}^\Lambda_{N+1}).$$

Due to Proposition 2.2 it is enough to evaluate the following

$$\varphi^{(N+1,f)}_{u_0, h(\gamma)}(a_{\sigma_3}^\Lambda_{N+1}) = \frac{1}{\gamma_0} \left[ \text{Tr} \left[ K_{[0,1]} \cdots K_{[N-1,N]} \right] + \text{Tr} \left[ K_{[N,N+1]} \gamma_{N+1} \right] K_{[N-1,N]} \cdots K_{[0,1]} \right].$$

Let us calculate $\tilde{h}_N := \text{Tr} \left[ K_{[N,N+1]} \gamma_{N+1} \right] K_{[N-1,N]} \cdots K_{[0,1]}$. Self-adjointness of $K_{\langle u,v \rangle}$ implies that

$$\tilde{h}_N = \text{Tr}_{x^{(1)}_{W_N}} \left[ \prod_{y \in S(x^{(1)}_{W_N})} K_{\langle x^{(1)}_{W_N}, y \rangle} \prod_{y \in S(x^{(1)}_{W_N})} h^{(y)} \prod_{y \in S(x^{(1)}_{W_N})} K_{\langle x^{(1)}_{W_N}, y \rangle} S_{x^{(1)}_{W_N}, 1} \right] \ni \text{Tr}_y \left[ \prod_{y \in S(x)} K_{\langle x,y \rangle} \prod_{y \in S(x)} h^{(y)} \prod_{y \in S(x)} K_{\langle x,y \rangle} \right].$$

We know that

$$\text{Tr}_y \left[ \prod_{y \in S(x)} K_{\langle x,y \rangle} \prod_{y \in S(x)} h^{(y)} \prod_{y \in S(x)} K_{\langle x,y \rangle} \right] = h(x),$$

for every $x \in \tilde{W}_N \setminus x^{(1)}_{W_N}$. On the other hand, since operators $K_{\langle u,v \rangle}$ and $\sigma_3^{(x)}$ commute each other for any $u, v, x \in L$ it follows from Corollary 8.6 that

$$\text{Tr}_{x^{(1)}_{W_N}} \left[ \prod_{y \in S(x^{(1)}_{W_N})} K_{\langle x^{(1)}_{W_N}, y \rangle} \prod_{y \in S(x^{(1)}_{W_N})} h^{(y)} \prod_{y \in S(x^{(1)}_{W_N})} K_{\langle x^{(1)}_{W_N}, y \rangle} S_{x^{(1)}_{W_N}, 1} \right] = \tilde{h}^{(1)}_{x^{(1)}_{W_N}},$$

where

$$\tilde{h}^{(1)}_{x^{(1)}_{W_N}} = h_0 \sigma_0^{(x)}_{x^{(1)}_{W_N}} + h_3 \sigma_3^{(x)}_{x^{(1)}_{W_N}}, \quad (h_0, h_3) = A_h_{\gamma_0, \gamma_3}, \quad h_{\gamma_0, \gamma_3} = (\gamma_3, \gamma_0).$$
Thus we obtain
\[ \overline{h}_N = \overline{h}^{(x_{W_N})} \bigotimes_{x \in \overline{W}_N \setminus x^{(1)}_{W_N}} h^{(x)}. \]

Therefore, one gets
\[ \varphi_{w_0, h(\gamma)}^{(N+1,f)} \left( a_{\sigma_3}^{\Lambda_{N+1}} \right) = \frac{1}{\gamma_0} \text{Tr} \left[ K_{[0,1]} \cdots K_{[N-2,N-1]} \right] \text{Tr}_{N-1} \left[ K_{[N-1,N]} h_N K_{[N-1,N]}^* K_{[N-2,N-1]}^* \cdots K_{[0,1]}^* \right]. \]

Again applying \( N \) times Corollary 8.6 one finds
\[ \varphi_{w_0, h(\gamma)}^{(N+1,f)} \left( a_{\sigma_3}^{\Lambda_{N+1}} \right) = \frac{1}{\gamma_0} \langle h^{N+1} h_{\gamma_0, \gamma_3}, e \rangle. \]

This completes the proof.

To prove our main result we are going to use the following theorem (see [12], Corollary 2.6.11).

**Theorem 8.9.** Let \( \varphi_1, \varphi_2 \) be two states on a quasi-local algebra \( \mathfrak{A} = \mathfrak{A}_\Lambda \). The states \( \varphi_1, \varphi_2 \) are quasi-equivalent if and only if for any given \( \varepsilon > 0 \) there exists a finite volume \( \Lambda \subset L \) such that \( \| \varphi_1(a) - \varphi_2(a) \| < \varepsilon \| a \| \) for all \( a \in B_{\Lambda'} \) with \( \Lambda' \cap \Lambda = \emptyset \).

Now by means of the Theorem 8.9 we will show that the states \( \varphi_{w_0(\alpha_0), h(\alpha_0)}^{(f)} \) and \( \varphi_{w_0(\gamma), h(\gamma)}^{(f)} \) are not quasi-equivalent. Namely, we have the following

**Theorem 8.10.** Let \( \theta > \frac{k+1}{k-1} \) and \( \varphi_{w_0(\alpha_0), h(\alpha_0)}^{(f)}, \varphi_{w_0(\gamma), h(\gamma)}^{(f)} \) be two forward QMC corresponding to the model \( \Lambda_3 \) with two boundary conditions \( \omega_0(\alpha_0) = \frac{1}{\alpha_0} \sigma_0, h(x) = \alpha_0 \sigma_0(x), \forall x \in L \) and \( \omega_0(\gamma) = \frac{1}{\gamma_0} \sigma_0, h(x) = \gamma_0 \sigma_0(x) + \gamma_3 \sigma_3(x), \forall x \in L \), respectively, here as before \( \alpha_0 = \frac{k-1}{k+1} \sqrt{\Theta^2}, \gamma_0 := (1,0), \gamma_3 \) are given by (8.2). Then \( \varphi_{w_0(\alpha_0), h(\alpha_0)}^{(f)} \) and \( \varphi_{w_0(\gamma), h(\gamma)}^{(f)} \) are not quasi-equivalent.

**Proof.** Let \( a_{\sigma_3}^{\Lambda_{N+1}} \) be an element given by (8.27). It is clear that \( \| a_{\sigma_3}^{\Lambda_{N+1}} \| = 1 \), for all \( N \in \mathbb{N} \).

If \( \theta > \frac{k+1}{k-1} \), then according to Propositions 8.7 and 8.8 we have
\[ \varphi_{w_0(\alpha_0), h(\alpha_0)}^{(f)} \left( a_{\sigma_3}^{\Lambda_{N+1}} \right) = 0, \]
\[ \varphi_{w_0(\gamma), h(\gamma)}^{(f)} \left( a_{\sigma_3}^{\Lambda_{N+1}} \right) = \frac{1}{\gamma_0} \langle h^{N+1} h_{\gamma_0, \gamma_3}, e \rangle \]
for all \( N \in \mathbb{N} \), here as before \( e = (1,0), h_{\gamma_0, \gamma_3} = (\gamma_3, \gamma_0) \) and \( \Lambda \) is given by (8.6). Then from (8.33) with Proposition 8.1 one finds
\[ \varphi_{w_0(\gamma), h(\gamma)}^{(f)} \left( a_{\sigma_3}^{\Lambda_{N+1}} \right) = \frac{(\theta + 1)x_1^2 y_1^2 + (\theta - 1)x_1 y_1 \gamma_0}{\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)} + \frac{(\theta - 1)(y_1^2 \gamma_3 - x_1 y_1 \gamma_0)}{\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)} \Lambda^{N+1}_{\sigma_3}. \]
where $\lambda_2$ is an eigenvalue of $\Lambda$ and $(x_1, y_1)$ is an eigenvector of the matrix $\Lambda$ corresponding to the eigenvalue $\lambda_1 = 1$ (see Proposition 3.1). Since $0 < \lambda_2 < 1$ then there exists $N_0 \in \mathbb{N}$ such that

$$\left| \frac{(\theta + 1)x_1^2\gamma_3 + (\theta - 1)x_1y_1\gamma_0}{\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)} + \frac{(\theta - 1)(y_1^2\gamma_3 - x_1y_1\gamma_0)}{\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)} \Lambda^{N+1}_2 \right| \geq \frac{(\theta + 1)x_1^2\gamma_3 + (\theta - 1)x_1y_1\gamma_0}{2\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)} \quad (8.36)$$

for all $N > N_0$.

Now putting $\varepsilon_0 = \frac{(\theta + 1)x_1^2\gamma_3 + (\theta - 1)x_1y_1\gamma_0}{2\gamma_0((\theta + 1)x_1^2 + (\theta - 1)y_1^2)}$ and using (8.33), (8.35), (8.36) we obtain

$$\left| \varphi^{(f)}_{\omega(0), h(0)}(a^{\Lambda_N}_{a_0}) - \varphi^{(f)}_{\omega(\gamma), h(\gamma)}(a^{\Lambda_N}_{a_3}) \right| \geq \varepsilon_0 \| a^{\Lambda_{N+1}}_{a_3} \|,$$

for all $N > N_0$, which means $\varphi^{(f)}_{\omega(0), h(0)}$ and $\varphi^{(f)}_{\omega(\gamma), h(\gamma)}$ are not quasi-equivalent. This completes the proof. \hfill \Box

Analogously, one can prove the following result.

**Theorem 8.11.** Let $\theta > \frac{k + 1}{k - 1}$ and $\varphi^{(f)}_{\omega(0), h(0)}$, $\varphi^{(f)}_{\omega(\beta), h(\beta)}$ be two forward quantum $d-$Markov chains corresponding to the model (8.13) with two boundary conditions $\omega_0(\alpha_0) = \frac{1}{a_0} \sigma_0$, $h(\alpha) = a_0 \sigma_0(\alpha)$, $\forall \alpha \in L$ and $\omega_0(\beta) = \frac{1}{a_0} \sigma_0$, $h(\alpha) = \beta_0 \sigma_0(\alpha) + \beta_3 \sigma_3(\alpha)$, $\forall x \in L$, respectively, here as before $\alpha_0 = \frac{k - 1}{k + 1}$, $\beta_0$ and $\beta_3$ are given by (8.11). Then $\varphi^{(f)}_{\omega(0), h(0)}$ and $\varphi^{(f)}_{\omega(\beta), h(\beta)}$ are not quasi-equivalent.

From Theorem 8.10 we immediately get the occurrence of the phase transition for the model (8.13) on the Cayley tree of order $k$ in the regime $\theta > \frac{k + 1}{k - 1}$. This completely proves our main Theorem 3.1.

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