ON THE ESTIMATION OF $Z_2(s)$

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ABSTRACT

Estimates for $Z_2(s) = \int_1^{\infty} |\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx$ ($\Re s > 1$) are discussed, both pointwise and in the mean square. It is shown how these estimates can be used to bound $E_2(T)$, the error term in the asymptotic formula for $\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt$.

1. Introduction

The function $Z_2(s)$ is the analytic continuation of the function

$$Z_2(s) = \int_1^{\infty} |\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx$$

and represents the (modified) Mellin transform of $|\zeta(\frac{1}{2} + ix)|^4$. It was introduced by Y. Motohashi [15] (see also [7], [10], [11] and [16]), who showed that it has meromorphic continuation over $\mathbb{C}$. In the half-plane $\sigma = \Re s > 0$ it has the following singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j$ ($\kappa_j = \sqrt{\lambda_j - \frac{1}{4}}$) and poles at $s = \frac{1}{2} \rho$, where $\rho$ denotes complex zeros of $\zeta(s)$. Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ is the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2, \mathbb{Z})$-automorphic forms (see [16, Chapters 1–3] for a comprehensive account of spectral theory and the Hecke $L$-functions).

The aim of this note is to study the estimation $Z_2(s)$, both pointwise and in mean square. This research was begun in [11], and continued in [7]. It was shown there that we have

$$\int_0^T |Z_2(\sigma + it)|^2 \, dt \ll_{\varepsilon} T^\varepsilon \left( T + T^{\frac{2-\sigma}{1-\sigma}} \right) \quad (\frac{1}{2} < \sigma < 1),$$

and we also have unconditionally

$$\int_0^T |Z_2(\sigma + it)|^2 \, dt \ll T^{\frac{2-\sigma}{1-\sigma}} \log C T \quad (\frac{1}{2} < \sigma < 1, C > 0).$$

Here and later $\varepsilon$ denotes arbitrarily small, positive constants, which are not necessarily the same ones at each occurrence, while $\sigma$ is assumed to be fixed. The constant $c$ appearing in (1.1) is defined by

$$E_2(T) \ll_{\varepsilon} T^{c+\varepsilon},$$

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where the function $E_2(T)$ denotes the error term in the asymptotic formula for the mean fourth power of $|\zeta(\frac{1}{2} + it)|$. It is customarily defined by the relation

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = TP_4(\log T) + E_2(T),
\end{equation}

with

\begin{equation}
P_4(x) = \sum_{j=0}^4 a_j x^j, \quad a_4 = \frac{1}{2\pi^2}.
\end{equation}

For the explicit evaluation of the $a_j$'s in (1.5), see [3].

Mean value estimates for $Z_2(s)$ are a natural tool to investigate the eighth power moment of $|\zeta(\frac{1}{2} + it)|$. Indeed, one has (see [7, (4.7)])

\begin{equation}
\int_T^{2T} |\zeta(\frac{1}{2} + it)|^8 \, dt \ll \varepsilon T^{2\sigma-1} \int_0^{T^{1+\varepsilon}} |Z_2(\sigma + it)|^2 \, dt \quad (\frac{1}{2} < \sigma < 1).
\end{equation}

We shall prove here the pointwise estimate for $Z_2(s)$ given by

**THEOREM 1.** For $\frac{1}{2} < \sigma \leq 1$ fixed and $t \geq t_0 > 0$ we have

\begin{equation}
Z_2(\sigma + it) \ll \varepsilon t^{\frac{1}{4}(1-\sigma) + \varepsilon}.
\end{equation}

Theorem 1 corrects an oversight in the proof of Theorem 3 of [7], where the better exponent $1 - \sigma$ was claimed, since unfortunately the condition $1/3 \leq \xi \leq 1/2$ (see [11, (4.22)]) has to be observed, and our argument needs $\xi = \varepsilon$ to hold. It improves the exponent $2(1 - \sigma)$ that was obtained in [11]. Probably the exponent $1 - \sigma$ could be reached with further elaboration. In any case this is much weaker than the bound conjectured in [7] by the author, namely that for any given $\varepsilon > 0$ and fixed $\sigma$ satisfying $\frac{1}{2} < \sigma < 1$, one has

\begin{equation}
Z_2(\sigma + it) \ll \varepsilon t^{\frac{1}{4}(1-\sigma)} \quad (t \geq t_0 > 0).
\end{equation}

Both pointwise and mean square estimates for $Z_2(s)$ may be used to estimate $E_2(T)$. This connection is furnished by

**THEOREM 2.** Suppose that for some $\rho \geq 0$ and $r \geq 0$ we have

\begin{equation}
Z_2(\sigma + it) \ll \varepsilon t^{\rho + \varepsilon}, \quad \int_1^T |Z_2(\sigma + it)|^2 \, dt \ll \varepsilon T^{1+2r+\varepsilon} \quad (\frac{1}{2} < \sigma \leq 1).
\end{equation}

Then we have

\begin{equation}
E_2(T) \ll \varepsilon T^{\frac{2\rho+1}{2\rho+2} + \varepsilon}, \quad E_2(T) \ll \varepsilon T^{\frac{2r+1}{2r+2} + \varepsilon},
\end{equation}

and

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^8 \, dt \ll \varepsilon T^{\frac{2\rho+1}{2\rho+2} + \varepsilon}.
\end{equation}

Note that from (1.2) with $\sigma = \frac{1}{2} + \varepsilon$ one can take in (1.8) $r = \frac{1}{2}$, hence (1.9) gives

\begin{equation}
E_2(T) \ll \varepsilon T^{\frac{4}{3} + \varepsilon}.
\end{equation}
The bound (1.11) is, up to “ε”, currently the best known one (see [10] and [15], where \(E_2(T) \ll T^{2/3} \log^8 T\) is proved). Thus any improvement of the existing mean square bound for \(Z_2(s)\) at \(\sigma = \frac{1}{2} + \varepsilon\) would result in (1.11) with the exponent strictly less than 2/3, which would be important. Of course, if the first bound in (1.8) holds with some \(\rho\), then trivially the second bound will hold with \(r = \rho\). Observe that the known value \(r = \frac{1}{2}\) and (1.10) yield
\[
\int_0^T |\zeta(\frac{1}{2} + it)|^8 dt \ll \varepsilon T^{3/2+\varepsilon},
\]
which is, up to “ε”, currently the best known bound for the eighth moment (see [1, Chapter 8]), and any value \(r < \frac{1}{2}\) would reduce the exponent 3/2 in the above bound.

2. The necessary lemmas

This section contains the lemmas needed for the proof of Theorem 1. Let, as usual, \(\alpha_j = |\rho_j(1)|^2 (\cosh \pi \kappa_j)^{-1}\), where \(\rho_j(1)\) is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue \(\lambda_j\) to which the Hecke \(L\)-function \(H_j(s)\) is attached.

**LEMMA 1.** We have
\[
\sum_{K-G \leq \kappa_j \leq K+G} \alpha_j H_j^3(\frac{1}{2}) \ll \varepsilon GK^{1+\varepsilon} (K^\varepsilon \leq G \leq K).
\]

This result is proved by the author in [6]. Note that M. Jutila [12] obtained
\[
\sum_{K-K^{1/3} \leq \kappa_j \leq K+K^{1/3}} \alpha_j H_j^4(\frac{1}{2}) \ll \varepsilon K^{4/3+\varepsilon},
\]
but this result and (2.1) do not seem to apply each other. Both, however, imply the hitherto sharpest bound for \(H_j(\frac{1}{2})\), namely
\[
H_j(\frac{1}{2}) \ll \varepsilon \kappa_j^{1/3+\varepsilon}.
\]

This bound is still quite far away from the conjectural bound
\[
H_j(\frac{1}{2} + it) \ll \varepsilon (\kappa_j + |t|)^\varepsilon,
\]
which may be thought of as the analogue of the classical Lindelöf hypothesis \((\zeta(\frac{1}{2} + it) \ll \varepsilon |t|^\varepsilon)\) for the Hecke series.

**LEMMA 2.** Let \(\xi \in (0, 1)\) be a constant, and set
\[
\psi(T) = \frac{1}{\sqrt{\pi T^\xi}} \int_{-\infty}^\infty |\zeta(\frac{1}{2} + i(T + t))|^4 \exp(-(t/T^\xi)^2) dt.
\]

Then we have
\[
\psi(T) = I_{2,r}(T, T^\xi) + I_{2,h}(T, T^\xi) + I_{2,c}(T, T^\xi) + I_{2,d}(T, T^\xi).
\]
Here \(I_{2,r}\) is an explicit main term, the contribution of \(I_{2,h}\) is small,
\[
I_{2,c}(T, T^\xi) = \pi^{-1} \int_{-\infty}^\infty \frac{|\zeta(\frac{1}{2} + ir)|^6}{|\zeta(1 + 2ir)|^2} \Lambda(r; T, T^\xi) dr,
\]
\[
I_{2,d}(T, T^\xi) = \sum_{j=1}^\infty \alpha_j H_j^3(\frac{1}{2}) \Lambda(\kappa_j; T, T^\xi),
\]
where

\[ \Lambda(r; T, T^\xi) = \frac{1}{2} \text{Re} \left\{ \left( 1 + \frac{i}{\sinh \pi r} \right) \Xi(ir; T, T^\xi) + \left( 1 - \frac{i}{\sinh \pi r} \right) \Xi(-ir; T, T^\xi) \right\} \quad (r \in \mathbb{R}) \]

(2.5)

with

\[ \Xi(ir; T, T^\xi) = \frac{1}{2} \frac{\Gamma^2\left( \frac{1}{2} + ir \right)}{\Gamma(1 + 2ir)} \int_0^\infty (1 + y)^{-\frac{1}{4} + it} y^{-\frac{1}{4} + ir} \times \exp \left( -\frac{1}{4} T^{2\xi} \log^2(1 + y) \right) F\left( \frac{1}{2} + ir, \frac{1}{2} + ir; 1 + 2ir; -y \right) dy, \]

(2.6)

and \( F \) is the hypergeometric function.

This fundamental result is the spectral decomposition formula of \( Y \). Motohashi (see [16, Section 5.1] with \( G = T^\xi \)). The holomorphic part \( I_{2, h}(T, T^\xi) \) is “small”, namely by [16, Lemma 4.1] it is \( \ll T^{-2} \). Motohashi (op. cit.) gives an explicit evaluation of the main term \( I_{2, r}(T, T^\xi) \) (\( \ll \log^4 T \)), which can be used to show that the contribution from this function to the relevant expression in Section 4 will be indeed absorbed by the other terms. The structure of the continuous part \( I_{2, c}(T, T^\xi) \) is similar in nature to (2.4), only the presence of integration instead of summation over \( \kappa_j \) makes this term less difficult to deal with than (2.4).

**Lemma 3.** The hypergeometric function, defined for \(|z| < 1\) by

\[ F(\alpha, \beta; \gamma; z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k \quad ((\alpha)_k = \alpha (\alpha + 1) \ldots (\alpha + k - 1)), \]

satisfies

\[ F(\alpha, \alpha; 2\alpha; z) = \left( 1 + \frac{\sqrt{1 - z}}{2} \right)^{-2\alpha} F\left( \alpha, \frac{1}{2}, \alpha + \frac{1}{2}; \left( \frac{1 - \sqrt{1 - z}}{1 + \sqrt{1 - z}} \right)^2 \right). \]

(2.7)

This is a special case of the classical quadratic transformation formula for the hypergeometric function (see e.g., [12, (9.6.12)]).

**Lemma 4.** If \( \Xi(ir; T, T^\xi) \) is defined by (2.6), then

\[ \Xi(ir; x, x^\xi) \ll \begin{cases} \frac{1}{|r|^C} & |r| \leq x \log^2 x, \\ 1 & |r| > x \log^2 x, \end{cases} \]

(2.8)

which holds uniformly for any fixed \( \xi \in (0, 1) \) and any fixed, large \( C > 0 \). We also have, for \( r \geq 2x^{1-\xi} \log^5 x \) and any fixed, large \( D > 0 \),

\[ \Xi(-ir; x, x^\xi) \ll (rx)^{-D}. \]

(2.9)

The first bound in (2.8) (see [11, eq. (4.20)]) follows trivially from the defining relation (2.6) if the hypergeometric function is represented by the Gaussian integral formula (see e.g., [16, (6.17)-(6.18)]). The other bound in (2.8) and the one in (2.9) are contained in [16, Lemma 4.2].
3. The estimation of \( Z_2(s) \)

We are ready now to proceed with the estimation of \( Z_2(s) \). We suppose that \( \frac{1}{2} < \sigma_0 \leq \sigma \leq 1 \), where \( \sigma_0 \) is fixed and let for some \( C > 1 \)

\[
T \leq t \leq 2T, \quad s = \sigma + it, \quad \frac{1}{2} < \sigma_0 \leq \sigma \leq 1, \quad T^{1+\varepsilon} \leq X \leq T^C.
\]

The reason for introducing \( T \) is for potential applications of our method to mean square estimates of \( Z_2(s) \).

We start from the decomposition

\[
Z_2(s) = \int_1^{2X} \rho(x)|\zeta(\frac{1}{2} + ix)|^4x^{-s} \, dx + \int_X^{2Y} \sigma(x)(\zeta(\frac{1}{2} + ix)|^4 - \psi(x))x^{-s} \, dx + \int_X^{2Y} \sigma(x)\psi(x)x^{-s} \, dx + \int_x^{\infty} \omega(x)|\zeta(\frac{1}{2} + ix)|^4x^{-s} \, dx
\]

\[
= Z_{12}(s) + Z_{22}(s) + Z_{32}(s) + Z_{42}(s),
\]

say. It is by introducing \( \psi(T) \), given by (2.2), that we are able to exploit the spectral decomposition furnished by (2.3). We suppose that \( 0 < \xi \leq \frac{1}{2} \), but eventually we shall take \( \xi = \varepsilon \), namely arbitrarily small. This will follow, in the course of the estimation of \( Z_{32}(s) \), by an analysis similar to the one made in [7]. We suppose that \( Y = Y(T, \sigma_0) (\ll T^C) \) is a large parameter such that \( Y > X \). The function \( \rho(x) (\geq 0) \) is a smooth function supported in \([1, 2X]\) such that \( \rho(x) = 1 \) for \( 1 \leq x \leq X \), and \( \rho(x) \) monotonically decreases from 1 to 0 in \([X, 2X]\). The function \( \sigma(x) \) is a smooth non-negative function supported in \([X, 2Y]\). We set \( \sigma(x) = 1 - \rho(x) \) for \( X \leq x \leq Y \), and let \( \sigma(x) \) monotonically decrease from 1 to 0 in \([Y, 2Y]\). Thus \( \sigma(x) \) is supported in \([X, 2Y]\), \( \sigma^{(\ell)}(x) = \sigma^{(\ell)}(2Y) = 0 \) for \( \ell = 0, 1, 2, \ldots \), and for \( \ell \in \mathbb{N} \) we have

\[
\sigma^{(\ell)}(x) = \begin{cases} O_{\ell}(X^{-\ell}) & \text{if } X \leq x \leq 2X, \\ 0 & \text{if } 2X \leq x \leq Y, \\ O_{\ell}(Y^{-\ell}) & \text{if } Y \leq x \leq 2Y. \\ \end{cases}
\]

For \( x \geq Y \) we set \( \omega(x) = 1 - \sigma(x) \). Then we have \( \omega^{(\ell)}(Y) = 0 \) and \( \omega^{(\ell)}(x) \ll x^{-\ell} \) for \( \ell = 0, 1, 2, \ldots \), and \( \omega'(x) = 0 \) for \( x \geq 2Y \). This decomposition of \( Z_2(s) \) differs from the one that was made in [11]. Namely we have introduced here the parameters \( X, Y \) and the smoothing functions \( \rho, \sigma \) and \( \omega \).

Clearly the functions \( Z_{12}(s), Z_{22}(s), Z_{32}(s) \) are entire functions for \( s \) belonging to the region defined by (1.1). The function \( Z_{42}(s) \) is initially defined for \( \sigma > 1 \), but we shall presently see that it admits analytic continuation to the region \( \frac{1}{2} < \sigma_0 \leq \sigma \leq 1 \), and moreover its contribution (for \( Y \) sufficiently large) will be negligible. To see this write (see (1.4))

\[
|\zeta(\frac{1}{2} + ix)|^4 = Q_4(log x) + E'_2(x), \quad Q_4(z) := P_4(z) + P'_4(z).
\]

Then

\[
Z_{42}(s) = \int_Y^{\infty} \omega(x)(Q_4(log x) + E'_2(x))x^{-s} \, dx \quad (\sigma > 1).
\]

Integrating by parts we obtain

\[
Z_{42}(s) = \frac{1}{s-1} \int_Y^{\infty} x^{1-s}(\omega(x)Q_4(log x))' \, dx - \int_Y^{\infty} E_2(x)(\omega'(x)x^{-s} - s\omega(x)x^{-s-1}) \, dx = Z_{52}(s) + Z_{62}(s),
\]
say. Since \( \omega'(x) \ll 1/x \), it follows from the mean square bound for \( E_2(T) \) (see e.g., [9]), by the Cauchy-Schwarz inequality for integrals, that \( Z_{62}(s) \) is regular for \( \sigma > \frac{1}{2} \) and that

\[
Z_{62}(s) \ll TY^{\frac{1}{2} - \sigma} \log^C Y
\]

holds for \( s \) satisfying (3.1). Now choose

\[
Y = T^{\frac{3}{10}}.
\]

Then for \( s \) satisfying (3.1) we have

\[
Z_{62}(s) \ll TY^{\frac{1}{2} - \sigma_0} \log^C Y \ll T^{-\frac{1}{2}} \log^C T,
\]

hence the contribution of \( Z_{62}(s) \) will be negligible. Repeated integration by parts gives, since \( \omega'(x) \) is supported in \([Y, 2Y]\),

\[
Z_{52}(s) = \sum_{j=1}^{5} \frac{1}{(s-1)^j} \int_{Y}^{2Y} x^{1-s} \omega'(x) Q_4^{(j-1)}(\log x) \, dx
\]

\[
+ \frac{1}{(s-1)^4} \int_{Y}^{\infty} x^{-s} \omega(x) Q_4^{(4)}(\log x) \, dx.
\]

Note that \( Q_4^{(4)}(\log x) = C \), a constant, since \( Q_4(z) \) is a polynomial of degree four in \( z \). Thus the last integral above becomes, for \( \ell \geq 2 \),

\[
\frac{C}{(s-1)^4} \int_{Y}^{\infty} x^{-s} \omega(x) \, dx = \frac{C}{(s-1)^5(s-2) \ldots (s-\ell)} \int_{Y}^{2Y} x^{\ell-s} \omega(\ell)(x) \, dx
\]

\[
\ll Y^{1-\sigma} T^{-4-\ell} \leq Y^\frac{1}{2} T^{-4-\ell} \ll T^{-\frac{1}{2}}
\]

on taking \( \ell = \ell(\sigma_0) \) sufficiently large. The remaining integrals with \( Q_4^{(j-1)}(\log x) \) are treated in an analogous way. Integration by parts is applied a large number of times, until each summand by trivial estimation is estimated as \( O(T^{-\frac{1}{2}}) \). Therefore the total contribution of \( Z_{62}(s) \) will be negligible, as asserted.

Now we trivially estimate \( Z_{12}(s) \) by the fourth moment of \( |\zeta(\frac{1}{2} + it)| \) as

\[Z_{12}(s) = \int_{1}^{2X} \rho(x)|\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx \ll X^{1-\sigma} \log^4 X, \tag{3.6}\]

for \( s = \sigma + it \) and \( \frac{1}{2} < \sigma_0 \leq \sigma \leq 1 \).

Next, the change of variable \( t = ax^\xi \log x \) in (2.2) gives

\[
Z_{22}(s) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{X}^{2Y} (|\zeta(\frac{1}{2} + i(x + ax^\xi \log x))|^4 - |\zeta(\frac{1}{2} + ix)|^4)
\]

\[
\times \sigma(x) \exp(-a^2 \log^2 x) x^{-s} \log x \, dx \, da.
\]

The integral over \( \alpha \) may be truncated with a negligible error at \( |\alpha| = b \), with \( b \) a small, positive constant. The relevant portion of \( Z_{22}(s) \) will be a multiple of

\[
Z_{22}^*(s) = \int_{-b}^{b} \int_{X}^{\infty} (E_2'(x + ax^\xi \log x) - E_2'(x))
\]

\[
\times \sigma(x) \exp(-a^2 \log^2 x) x^{-s} \log x \, dx \, da,
\]

where (3.4) is used. Namely, for \( \xi = \varepsilon \), the portion of \( Z_{22}(s) \) containing \( Q_4 \) makes a total contribution that does not exceed the one in (3.6).
In the $x$-integral in (3.7) we make the change of variable $\tau = \tau(x, \alpha) = x + \alpha x^\xi \log x$. If $b$ is sufficiently small, then $\tau(x, \alpha)$ is monotonically increasing as a function of $x$ for $x \geq 1$, and there is a monotonic inverse function $x = x(\tau, \alpha)$. For $x = x(\tau, \alpha)$, we have $x - \tau \ll (\log \tau)\tau^\xi$, hence $x \asymp \tau$, and the implicit equation for $x$ shows that

$$x(\tau, \alpha) = \tau - \alpha \tau^\xi \log \tau + O(\tau^{2\xi\log 2} \tau).$$

Also we have

$$\frac{\partial x(\tau, \alpha)}{\partial \tau} = 1 - \alpha \tau^{\xi-1}(1 + \xi \log \tau) + O(\tau^{2\xi\log 2} \tau).$$

For given positive $\alpha$, we combine the contributions of $\alpha$ and $-\alpha$ in (3.7). In the respective integrals we put $\tau = \tau(x, \pm \alpha)$, in the integral involving $E_2^s(x)$ we simply change the notation $x$ to $\tau$, and we set

$$G(u) = \exp(-\alpha^2 \log^2 u) \cdot (1 + \alpha u^{\xi-1}(1 + \xi \log u))^{-1} \sigma(u) \log u.$$

Then in view of (3.9) it is seen that (3.7) becomes

$$Z_{22}(s) = \int_0^b \int_0^\infty E_2^s(\tau)G(x(\tau, \alpha))(x(\tau, \alpha))^{-s} \, d\tau \, d\alpha$$

$$+ \int_0^b \int_0^\infty E_2^s(\tau)G(x(\tau, -\alpha))(x(\tau, -\alpha))^{-s} \, d\tau \, d\alpha$$

$$- 2 \int_0^b \int_X E_2^s(\tau)\sigma(\tau)\tau^{-s} \exp(-\alpha^2 \log^2 \tau) \log \tau \, d\tau \, d\alpha.$$

In the integrals with $G$ we replace the lower bounds of integration in the $\tau$-integrals by $X$. Changing the variable back to $x = x(\tau, \pm \alpha)$, using Lemma 1 and (1.14) it follows that the total error made in this process will be ($\xi = \varepsilon$)

$$\ll_\varepsilon X^{1-\sigma+\varepsilon}.$$

Now we write

$$G(x(\tau, \pm \alpha)) = \exp(-\alpha^2 \log^2 \tau)\sigma(\tau) \log \tau + H(x(\tau, \pm \alpha)),$$

say, where $H(x(\tau, \pm \alpha))$ is independent of $s$. Taking into account (3.8), it follows by using the mean value theorem that

$$H(x(\tau, \pm \alpha)) \ll \tau^{\xi-1} \log^3 \tau.$$

The portion of the integrals in (3.10) with $H(x(\tau, \pm \alpha))$, on changing the variable again to $x = x(\tau, \pm \alpha)$, is estimated by Lemma 1. Its contribution does not exceed (3.11), hence we are left with

$$Z_{22}^{**}(s) := \int_0^b \int_X E_2^s(\tau)\{x(\tau, \alpha)^{-s}$$

$$+ (x(\tau, -\alpha))^{-s} - 2\tau^{-s}\} \exp(-\alpha^2 \log^2 \tau)\sigma(\tau) \log \tau \, d\tau \, d\alpha.$$

The expression in curly braces in (3.12) is expanded by Taylor’s formula at $\tau$. It becomes

$$(x(\tau, \alpha) + x(\tau, -\alpha) - 2\tau)(-s\tau^{-s-1})$$

$$+ \frac{1}{2!} (x(\tau, \alpha) - \tau)^2 + (x(\tau, -\alpha) - \tau)^2 \frac{\partial}{\partial \tau}(-s\tau^{-s-1}) + \ldots.$$
Note that \(|s|^\tau^{-1} \ll TX^{-1} = T^{-\varepsilon}\) by (3.1), so that each time \(\partial^\tau\) is taken in forming a new derivative in (3.13), its order will decrease by a factor of \(|s|^\tau^{-1} (\ll T^{-\varepsilon})\). We shall take sufficiently many terms in (3.13) in such a way that the error will make a negligible contribution (i.e., absorbed by (3.11)), since (3.8) holds and \(\xi = \varepsilon\). The expression in (3.13) will be

\[
(3.14) \quad \ll \varepsilon T^2 \tau \varepsilon - \sigma - 2.
\]

Since (3.1) holds we obtain, by using the fourth moment for \(|\zeta(\frac{1}{2} + it)|\),

\[
(3.15) \quad Z_{22}(s) \ll \varepsilon T^2 X^{-1 - \sigma} \ll \varepsilon X^{\varepsilon + 1 - \sigma}.
\]

We pass now to the contribution of \(Z_{22}(s)\). While the estimation of \(Z_{12}(s)\) and \(Z_{22}(s)\) was essentially elementary, it is the function \(Z_{32}(s)\) that is the most delicate one in (3.2) and its treatment requires the application of spectral theory, namely Lemma 2.

As discussed in [11] and in Section 2, after Lemma 2, the main contribution to \(Z_{32}(s)\) will come from \(I_{2,d}\) in (2.4), namely from the discrete spectrum. Thus only this contribution will be treated in detail. It equals

\[
(3.16) \quad \int_X^{2Y} \sigma(x) I_{2,d}(x, x^\xi) x^{-s} \, dx = \sum_{j=1}^{\infty} \alpha_j \Lambda_j^3(\frac{1}{2}) \int_X^{2Y} \sigma(x) \Lambda(x; x, x^\xi) x^{-s} \, dx,
\]

where (3.1) is assumed. The interchange of integration and summation follows from (2.8) of Lemma 4, which ensures absolute convergence on the right-hand side. Note that in place of the integral on the right-hand side of (3.16) we can consider

\[
(3.17) \quad X_r(s) := \int_X^{2Y} \sigma(x) \Theta(ir; x, x^\xi) x^{-s} \, dx \quad (r = \kappa_j),
\]

since the term with \(\Theta(ir; x, x^\xi)\) (see (2.5)) has no saddle-point, and its estimation is less difficult. Next, note that in view of (2.9) of Lemma 4 the sum in (3.16) can be restricted to \(\kappa_j \leq T^{C_1}\) with a suitable constant \(C_1 = C_1(\sigma_0, \xi)\), since the tails of the series will make a negligible contribution.

We make the change of variable \(y = z/x\) in the \(\Theta\)-integral (see (2.6)) in (3.17). This is done to regulate the location of the corresponding saddle point, similarly as in [7] and [11]. After the change of variable the integral \(X_r(s)\) becomes

\[
(3.18) \quad \frac{\Gamma^2(\frac{1}{2} - ir)}{\Gamma(1 - 2ir)} \int_X^{2Y} \sigma(x) x^{-\frac{1}{2} - ir} L^\ast(r; x) \, dx,
\]

where

\[
(3.19) \quad L^\ast(r; x) := \int_0^\infty z^{-\frac{1}{2} - ir} \left(1 + \frac{z}{x}\right)^{-\frac{1}{2} + ir} \times \exp \left(-\frac{1}{4} x^2 s \log^2 \left(1 + \frac{z}{x}\right)\right) F \left(\frac{1}{2} - ir, \frac{1}{2} - ir; 1 - 2ir; -\frac{z^2}{x}\right) \, dz.
\]

In (3.19) we consider separately the ranges \(z/x \leq x^{-\delta}\) and \(z/x > x^{-\delta}\) for a sufficiently small, fixed \(\delta > 0\). In the latter range, the exponential factor is \(\ll x^{-A}\) for any fixed positive \(A\) provided that \(\xi > \delta\), which we may assume, and thus the total contribution of the range \(z/x > x^{-\delta}\) in (3.19) is negligible. Therefore so far we have reduced the problem to the estimation of a finite sum over \(\kappa_j\) in (3.16) and a finite \(z\)-integral in (3.19).
In the range \( z/x \leq x^{-\delta} \) in (3.19) we transform the hypergeometric function by (2.7) of Lemma 3, noting that the new hypergeometric series converges rapidly. Namely in the series expansion for the hypergeometric function (see Lemma 3) we can take a finite number of terms so that the tails of the series will make a negligible contribution. Each term, since in our case \( \alpha = \frac{1}{2} - ir, \beta = \frac{1}{2}, \gamma = 1 - ir \), will yield similar expressions, and each contribution will be smaller than the one coming from the preceding term. Therefore the most significant term in the above series expansion will be simply the leading term 1, so it suffices to consider its contribution. Then the essential part of \( L^*(r; x) \), say \( L(r; x) \), takes the form

\[
L(r; x) := 2^{1-2ir} \int_0^{z_0} \frac{z^{1-\delta}}{z} \left(1 + \frac{z}{x}\right)^{-\frac{1}{2}} e^{i\varphi(z)} \, dz,
\]

where

\[
\varphi(z) = \varphi(r, x; z) := -r \log z + x \log \left(1 + \frac{z}{x}\right) + 2r \log \left(1 + \frac{1 + \frac{z}{x}}{2}\right).
\]

The integral in (3.20) can be approximately evaluated by the saddle point method (see e.g., [2, Chapter 2]). The main contribution to the integral comes from the saddle point \( z_0 \) satisfying \( \varphi'(z_0) = 0 \). We have

\[
\varphi'(z) = \frac{\partial \varphi}{\partial z} = -\frac{r}{z} + \frac{x}{z} + \frac{r}{x} \left(\frac{1}{1 + \frac{z}{x}} + 1 + \frac{z}{x}\right),
\]

and

\[
\varphi''(z) = \frac{r}{z^2} - \frac{x}{(x+z)^2} - \frac{r}{x^2} - \frac{1}{2} \frac{1}{\left(1 + \frac{z}{x} + 1 + \frac{z}{x}\right)^2}.
\]

This gives

\[
z_0 = r \left(1 + \frac{r}{2x} + \frac{r^2}{8x^2} + O \left(\frac{r^3}{x^3}\right)\right),
\]

and the error term in (3.24) admits an asymptotic expansion in powers of \( r/x \), since the relevant range is \( r/x \ll T^{-\epsilon} \), in view of (3.1) and (3.32). Similarly we find that

\[
\left(\frac{z_0}{x}\right)' = \frac{\partial}{\partial x} \left(\frac{z_0}{x}\right) = -\frac{r}{x^2} - \frac{r^2}{x^3} + O \left(\frac{r^3}{x^4}\right),
\]

\[
z'_0 = r \left(-\frac{r}{2x^2} - \frac{r^2}{8x^3} + O \left(\frac{r^3}{x^4}\right)\right), \quad \varphi''(z_0) = \frac{1}{r} \left(1 + O \left(\frac{r}{x}\right)\right),
\]

where again the error terms admit an asymptotic expansion in powers of \( r/x \). A calculation then shows that we obtain

\[
\frac{\partial \varphi(z_0)}{\partial x} = -\frac{r^3}{24x^3} + O \left(\frac{r^4}{x^4}\right).
\]

As already asserted the main contribution to the integral in (3.20) comes from the saddle point \( z_0 \), and equals a multiple of

\[
C(z_0) \varphi''(z_0) e^{i\varphi(z_0)},
\]
where

\[
C(z) := C(\xi, x; z) = z^{-\frac{1}{2}} \left(1 + \frac{z}{x}\right)^{-\frac{1}{2}} \times \exp \left( -\frac{1}{2} x^{2 \xi} \log^2 \left(1 + \frac{z}{x}\right) \right) \left(1 + \sqrt{1 + \frac{z}{x}}\right)^{-1}.
\]

From (3.24) and (3.26) we have \( (z_0 \phi_0''(z_0))^{-1/2} \sim 1 \). Hence inserting (3.28)-(3.29) in (3.18) it is seen that the main contribution will be (by using Stirling’s formula to simplify the gamma-factors) a multiple of

\[
\begin{align*}
&\int_{X}^{2Y} \sigma(x) x^{-\frac{1}{2} + ir - s} \left(z_0 \phi''(z_0)\right)^{-1/2} \left(1 + \frac{z_0}{x}\right)^{-1/2} \times \\
&\times \exp \left( -\frac{1}{4} x^{2 \xi} \log^2 \left(1 + \frac{z_0}{x}\right) \right) \left(1 + \sqrt{1 + \frac{z_0}{x}}\right)^{-1} e^{i \sigma(z_0)} d x
\end{align*}
\]

say, where in view of (3.27) we have

\[
\frac{\partial^\ell h(r, x)}{\partial x^\ell} \ll_\ell \frac{r^4}{x^{3+\ell}}, \quad \frac{d^\ell g(x)}{dx^\ell} \ll_\ell x^{-\ell} \quad (\ell = 0, 1, 2, \ldots).
\]

From the term \( \exp \left( -\frac{1}{2} x^{2 \xi} \log^2 \left(1 + \frac{z_0}{x}\right) \right) \) it transpires that the range \( r \geq x^{1-\xi} \log x \) will make a negligible contribution. Similarly if

\[
|r - t| > r^\alpha > \frac{r^{3\epsilon}}{x^3},
\]

repeated integration by parts shows that the contribution is negligible. But as \( r \leq x^{1-\xi} \log x \), (3.31) will hold for \((3 - \alpha)(1 - \xi) < 2\), namely for

\[
\alpha > 3 - \frac{2}{1 - \xi}.
\]

If \( 0 < \xi < 1/3 \), as we assume, then it follows that \( 0 < \alpha < 1 \), hence only the range \( |r - t| < r^\alpha \) is relevant, which gives

\[
C_1 T \leq r = \kappa_j \leq C_2 T \quad (0 < C_1 < C_2)
\]

for suitable constants \( C_1, C_2 \).

Repeated integration by parts in (3.30) shows then that, after each integration, the order of the integrand is lowered by a factor of

\[
\frac{x}{1 + |t - r|} \left(1 + \frac{r^{3\epsilon}}{x^3}\right).
\]

It follows that, if \( x \geq r^{3/2} \approx T^{3/2} \), then the above expression is \( \ll T^{-\epsilon} \) for \(|r - t| > T^\epsilon\), and hence its total contribution is negligible. If \(|r - t| \leq T^\epsilon\), then by trivial estimation and Lemma 1 the total contribution will be

\[
\ll t^{-1/2} \sum_{|r - t| \leq T^\epsilon} \alpha_j H_j^3(\ell) X^{1/2 - \sigma} \ll_\epsilon T^{1/2 + \epsilon} X^{1/2 - \sigma} \ll_\epsilon t^{1 - \sigma + \epsilon},
\]
hence absorbed by the right-hand side of (1.7). In view of (3.33) this shows that the relevant range for \( x \) and \( r \) is

\[
T^{1+\varepsilon} \leq x \leq T^{3/2}, \quad T^\varepsilon < |r - t| < T^{3+\varepsilon} X^{-2}.
\]

The main contribution in (3.30) will, as in the previous case, come also from the saddle point. If we write the integral as

\[
r^{-1/2} \int_X^{2Y} \sigma(x) g(x) x^{-\frac{1}{2} - \sigma} e^{iH(r,x)} \, dx,
\]

where

\[
H(r, x) := (r - t) \log x + \frac{r^3}{48x^2} + h(r, x),
\]

then in this case the saddle point \( x_0 \) is the solution (in \( x \)) of \( H'(r, x) = 0 \), so that

\[
x_0 = \sqrt[3]{\frac{r^3}{24(r - t)}} \left(1 + O \left( \frac{r}{x_0} \right) \right),
\]

where the \( O \)-term admits an asymptotic expansion. We have also

\[
H''(r, x_0) \sim 48(r - t)^2r^{-3},
\]

and we suppose \( G < r - t \leq 2G, \ G = 2^k T^s, \ k \in \mathbb{N} \). Then it is seen that the contribution coming from the ranges

\[
x \geq C_1 T^{3/2} G^{-1/2}, \quad x \leq C_2 T^{3/2} G^{-1/2}
\]

for sufficiently large \( C_1 \) and sufficiently small \( C_2 > 0 \) will be negligible. There remains a multiple of \((S(x) := \sigma(x) g(x) x^{-1/2-\sigma})\)

\[
\sum_{t+G < \kappa_j \leq t+2G} \alpha_j H_j^3(1/2) \kappa_j^{-1/2} S(x_0)(H''(\kappa_j, x_0))^{-1/2} e^{iH(\kappa_j, x_0)} \ll T^{-2} \sum_{t+G < \kappa_j \leq t+2G} \alpha_j H_j^3(1/2) x_0^{-1/2-\sigma} x_0^2 \kappa_j^{-3/2} \ll T^{-2} \sum_{t+G < \kappa_j \leq t+2G} \alpha_j H_j^3(1/2)(T/2G)^{-1/2} 3^{2/3} \ll e^{T^{-1} GT^{9/4-3\sigma/2} G^\sigma/2-3/4 = T^{5/4-3\sigma/2} 4^{-\varepsilon} G^{1/4+\sigma/2}}
\]

where Lemma 1 was used. From (3.34) it follows that \( G \ll T^{3+\varepsilon} X^{-2} \), hence we obtain

\[
\sum \ll_T T^{2+\varepsilon} X^{-1/2-\sigma},
\]

and the same bound will hold for \( Z_{32}(s) \). Hence finally, in view of (3.6) and (3.15), we obtain

\[
Z_2(\sigma + it) \ll_\varepsilon T^\varepsilon (X^{1-\sigma} + t^2 X^{-1/2-\sigma}) \ll_\varepsilon T^{\varepsilon(1-\sigma)+\varepsilon} \quad (\frac{1}{2} < \sigma \leq 1)
\]

for \( X = t^{4/3} \). This completes the proof of Theorem 1.

### 4. Proof of Theorem 2

The proof follows the analysis outlined in [7], where the first bound in (1.9) was mentioned. From the defining relation (1.4) it is not difficult to obtain (see e.g., [4, (5.3)]) that \((C_1, C_2 > 0, 1 \ll H \leq \frac{1}{4} T)\)

\[
E_2(T) \leq C_1 H^{-1} \int_T^{T+H} E_2(x) f(x) \, dx + C_2 H \log^4 T,
\]

(4.1)
where \( f(x) (> 0) \) is a smooth function supported in \([T, T+H]\), such that \( f(x) = 1 \) for \( T + \frac{1}{2}H \leq x \leq T + \frac{3}{2}H \).

Taking account that \( Z_2(s) \) is the (modified) Mellin transform of \( |\zeta(\frac{1}{2} + ix)|^4 \), it follows by the Mellin inversion formula that (see [7, (3.27)])

\[
|\zeta(\frac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \log x \quad (x > 1),
\]

where we have set, as in (3.4), \( Q_4(x) = P_4(x) + P'_4(x) \) and \( L \) denotes the line \( \Re s = 1 + \varepsilon \) with a small indentation to the left of \( s = 1 \). If we integrate (4.2) from \( x = 1 \) to \( x = T \) and take into account the defining relation of \( E_2(T) \), we shall obtain

\[
E_2(T) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \log x \quad (T > 1).
\]

Then from (4.1) and (4.3) we have by Cauchy’s theorem \((\frac{1}{2} < c < 1, T > 1)\)

\[
E_2(T) \leq \frac{C_1}{2\pi i H} \int_{c-iT}^{c+iT} \frac{Z_2(s)}{s} \int_{T}^{T+H} f(x) x^s dx ds + O(1) \quad (T > 1).
\]

and we also have an analogous lower bound for \( E_2(T) \). Since \( f^{(r)}(x) \ll H^{-r} \) it follows that the \( s \)-integral in (4.4) can be truncated at \( |3m| = T^{1+\varepsilon}H^{-1} \) with a negligible error, for any \( c \) satisfying \( \frac{1}{2} < c < 1 \). We take \( c = \frac{1}{2} + \varepsilon \) and use the first bound in (1.8) to obtain

\[
E_2(T) \ll \varepsilon \int_{1}^{T^{1+\varepsilon}H^{-1}} t^{\varepsilon+1} t^{\frac{1}{4}+\varepsilon} dt + HT^\varepsilon
\]

\[
\ll \varepsilon T^{\varepsilon} (T^{\frac{1}{4}+\varepsilon}H^{-\varepsilon} + H) \ll \varepsilon T^{\frac{1}{4}+\varepsilon}
\]

with the choice \( H = T^{\frac{1}{2r+1}} \). This proves the first part of Theorem 2. To prove the second, we proceed similarly, but use the Cauchy-Schwarz inequality and the second bound in (1.8). We have

\[
E_2(T) \ll \varepsilon \int_{1}^{T^{1+\varepsilon}H^{-1}} |Z_2(\frac{1}{2} + \varepsilon + it)| t^{-1} t^{\frac{1}{4}+\varepsilon} dt + HT^\varepsilon
\]

\[
\ll \varepsilon T^{\frac{1}{4}+\varepsilon} \left( \int_{1}^{T^{1+\varepsilon}H^{-1}} |Z_2(\frac{1}{2} + \varepsilon + it)|^2 t^{-1} dt \right)^{1/2} + HT^\varepsilon
\]

\[
\ll \varepsilon T^{\varepsilon} (T^{\frac{1}{4}+\varepsilon}H^{-\varepsilon} + H) \ll \varepsilon T^{\frac{1}{4}+\varepsilon}
\]

with \( H = T^{\frac{2r+1}{2r+2}} \).

Finally to prove (1.10), note that by [8, eq. (4.9)] we have \((c_2(\frac{1}{2} + \varepsilon) = 1 + 2r \) in this notation\)

\[
\int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{4+4A} dt \ll \varepsilon T^{A+\varepsilon} \quad (1 \leq A \leq 2).
\]

Let \( 0 \leq C \leq 8 \) be a constant. Then, for \( p > 0, q > 0, 1/p + 1/q = 1 \), Hölder’s inequality for integrals gives

\[
\int_{0}^{T} |\zeta(\frac{1}{2} + it)|^8 dt = \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^C |\zeta(\frac{1}{2} + it)|^{8-C} dt
\]

\[
\leq \left( \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{Cp} dt \right)^{1/p} \left( \int_{0}^{T} |\zeta(\frac{1}{2} + it)|^{(8-C)q} dt \right)^{1/q}.
\]
Now choose $p, q$ and $C$ so that

$$Cp = 4, \quad (8 - C)q = 4 + 4A, \quad \frac{1}{p} + \frac{1}{q} = 1.$$  

Using the fourth moment and (4.5) the above inequality gives then

$$\int_0^T |\zeta(\frac{1}{2} + it)|^8 \, dt \ll_{\varepsilon} T^{2A^{-1}+\varepsilon} = T^{\frac{4r+1}{3r+1}+\varepsilon},$$

since $A = 1 + 2r$. This completes the proof of Theorem 2.

In concluding let us remark that $r = 0$ in (1.8) was conjectured by the author in [7]. This is a very strong conjecture since it gives, by (1.6) and (1.9),

$$(4.5) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^8 \, dt \ll_{\varepsilon} T^{1+\varepsilon}, \quad E_2(T) \ll_{\varepsilon} T^{\frac{1}{2}+\varepsilon},$$

and both of these bounds are, up to "$\varepsilon$", best possible. Namely one has (see e.g., [1] and [16])

$$\int_0^T |\zeta(\frac{1}{2} + it)|^8 \, dt \gg T \log 16 T, \quad E_2(T) = \Omega_{\pm} (\sqrt{T}).$$

So far it is not known whether either of the bounds in (4.5) implies the other one. However, it is known that either of them implies (see [2, Theorem 1.2 and Lemma 4.1]) the hitherto unproved bound

$$\zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^\frac{1}{8} + \varepsilon.$$

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