Multipair contributions to the spin response of nuclear matter

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We analyse the effect of non-central forces on the magnetic susceptibility of degenerate Fermi systems. These include the presence of contributions from transitions to states containing more than one quasiparticle-quasihole pair, which cannot be calculated within the framework of Landau Fermi-liquid theory, and renormalization of the quasiparticle magnetic moment, as well as explicit non-central contributions to the quasiparticle interaction. Consequently, the relationship between the Landau parameters and the magnetic susceptibility for Fermi systems with non-central forces is considerably more complicated than for systems with central forces. We use sum-rule arguments to place a lower bound on the contribution to the static susceptibility coming from transitions to multipair states.

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I. INTRODUCTION

The spin and spin-isospin responses of nuclear matter with various ratios of protons to neutrons are important ingredients in many calculations of importance for astrophysical applications. Among these we may mention neutral-current processes such as scattering of neutrinos, and emission of neutrino-antineutrino pairs, and charged current ones such as the emission of neutrinos and antineutrinos by variants of the Urca process. For a general review of neutrino processes in dense matter, we refer to Ref. [1].

In calculating rates of neutrino emission, absorption and scattering processes in dense matter, one of the quantities of interest is the effective interaction between nucleons, because this is one of the quantities which determines how matrix elements of the weak current are modified by the medium (see, e.g., Refs. [2,3]). One approach to determining these is to calculate effective interactions directly as, for example, in Ref. [4]. Another approach which has been very successful for liquid \textsuperscript{3}He is to make use of results for the long-wavelength thermodynamic properties of the system. To be specific, in a system with central interactions, some information about the effective interaction may be obtained rather directly from the spin susceptibility. For the purpose of illustration, let us consider such a system with a single species, having spin 1/2. The static, long-wavelength spin susceptibility is given by

\[ \chi = \frac{\mu_0^2 N(0)}{1 + G_0}, \]  

where \( \mu_0 \) is the magnetic moment of a particle in free space, \( G_0 \) is the isotropic part of the Landau quasiparticle interaction in the spin-triplet channel, and \( N(0) = m^* p_F/\pi^2 \hbar^3 \) is the density of states per unit volume at the Fermi surface, \( p_F \) being the Fermi momentum and \( m^* \) the effective mass of a quasiparticle. Thus, knowing the magnetic susceptibility and the quasiparticle effective mass, one can deduce the Landau parameter \( G_0 \).

Landau’s theory of normal Fermi liquids provides a framework for describing the long-wavelength response of the system [5]. In the usual account of the theory, a key role is played by conservation laws. For example, for interactions which conserve the number of particles, addition of a single quasiparticle to the system involves the addition of a single particle. The total number of particles in the system is therefore given by the total number of quasiparticles. For interactions which conserve the total spin, the total spin may likewise be calculated in terms of the quasiparticle distribution, since the spin carried by a quasiparticle is the same as that carried by a particle.

The consequences of conservation laws for matrix elements of the density operator have been described by Pines and Nozières [6], and Leggett has exploited conservation laws to provide a general description of the long-wavelength response in terms of contributions due to excitation of single quasiparticle-quasihole pairs, and those due to multipair excitations [7]. A derivation of the results from Landau theory is given in Ref. [8].

The assumption that interparticle interactions conserve the total spin is an excellent approximation for atomic interactions, since the spin-orbit and magnetic dipole–dipole interactions between electronic spins, which do not conserve total spin, are generally small. The situation is quite different for nuclear forces, because of the importance of the tensor interaction due to exchange of pions and, to a lesser extent, of rho mesons, and the significant spin-orbit contributions to inter-nucleon interactions. For recent analyses of the nucleon-nucleon interaction, see Refs. [9].

The purpose of this paper is to investigate the consequences of the lack of conservation of total spin for the spin response of nuclear matter at long wavelengths. One of them is that the spin carried by a quasiparticle is different from that carried by a bare nucleon. Consequently...
the coupling of the quasiparticle spin to an external field is altered. Such effects are well known for nuclear magnetic moments and for matrix elements of the axial current, and they are reviewed in, e. g., Ref. [10]. They are also discussed in the framework of Landau Fermi-liquid theory in Ref. [11]. A related effect is that the quasiparticle magnetic moment is a tensor, not a scalar. A second well-known effect is that the interaction between quasiparticles has non-central contributions, and the most familiar of these is that due to the tensor force. This has been considered in Ref. [12] and more recent works. A third consequence of non-central forces is that at long wavelengths there are contributions to the response from states with more than one quasiparticle-quasihole pair. This latter effect appears not to have been widely appreciated in work on bulk nuclear matter. In this paper we examine the general structure of the spin-density response function and deduce the qualitative behavior of matrix elements at long wavelengths. We also indicate how the standard Landau-theory expression must be modified when non-central forces are present. Finally, we derive an upper bound on the magnitude of the multipair contribution to the long-wavelength susceptibility from sum-rule arguments.

This paper is organized as follows. In Section II we give a brief introduction to response functions and describe quantities relevant for our later considerations. Section III gives a discussion of the long-wavelength behavior of matrix elements, and how this depends on whether or not the corresponding operator obeys a local conservation law. The consequences of non-central forces for Landau’s theory of a normal Fermi liquid is examined in Section IV. In Section V we derive bounds on the contribution from multipair states to the static, long-wavelength spin susceptibility and use microscopic calculations to estimate how large these are. Concluding remarks are given in Section VI.

II. RESPONSE FUNCTIONS AND SUM RULES

If a system initially in its ground state is subjected to a perturbation

$$H' = O_q^\dagger C_q e^{-i\omega t} + \text{Hermitian conjugate},$$

the Fourier transform of the linear response of the expectation value of the operator $O_q$ at frequency $\omega$ is given by

$$<O_q>_{\omega} = \chi(q, \omega) C_q,$$

where the linear response function is defined by

$$\chi(q, \omega) = \sum_j \frac{|O_q^\dagger|^2}{\omega_j 2\omega_j + (\omega + i\eta)^2}. \tag{4}$$

Here $j$ denotes an excited state and 0 the ground state, and $\omega_j = E_j - E_0$, where $E_j$ is the energy of the excited state, and $E_0$ that of the ground state. The wave vector of the perturbation is denoted by $q / \hbar$. From Eq. (4) it follows that the static response function is given by

$$\chi(q, 0) = \sum_j \frac{|O_q^\dagger|^2}{\omega_j}. \tag{5}$$

Another important quantity is the static structure factor

$$S(q) = \frac{1}{N} \sum_j |O_q^\dagger|^2 \omega_j. \tag{6}$$

Here $N$ is the total number of particles. A third quantity we shall find useful, because it can be calculated directly from knowledge of the ground-state wave function, is the frequency-weighted sum

$$W(q) = \frac{1}{N} \sum_j |O_q^\dagger|^2 \omega_j \omega_j = \frac{1}{N} \langle 0 | O_q | H, O_q^\dagger | 0 \rangle, \tag{7}$$

where $H$ is the Hamiltonian in the absence of the perturbation given in Eq. (2). As we shall demonstrate later, the static structure factor and frequency-weighted sum at long-wavelengths are useful diagnostics for contributions from multipair states.

III. CONSERVATION LAWS AND NON-CENTRAL FORCES

We now investigate the effect of non-central components of the nuclear force on the linear response of the system. One of the reasons for the spectacular success of Landau Fermi-liquid theory in providing a framework for the quantitative description of long-wavelength low-frequency properties of liquid $^3$He at low temperatures is that the physical observables of greatest interest experimentally correspond to quantities such as the particle density, the spin density, and the particle current density, which satisfy local conservation laws. To make this point explicit, let us imagine that the operator $O(r)$ satisfies the conservation law

$$\frac{\partial O(r)}{\partial t} + \nabla \cdot j(r) = 0, \tag{8}$$

where $j(r)$ is the associated current density. On taking the matrix element of the Hermitian conjugate of this equation between an excited state $j$ and the ground state and Fourier transforming in space, one finds

$$\omega_j \langle 0 | O_q^\dagger | j \rangle_{\omega_j} = q \cdot \langle j | O_q^\dagger | 0 \rangle_{\omega_j}. \tag{9}$$

This shows that the matrix element of the operator satisfies the condition

$$\langle O_q^\dagger | j \rangle_{\omega_j} = \frac{q \cdot \langle j | O_q^\dagger | 0 \rangle_{\omega_j}}{\omega_j}. \tag{10}$$
Thus for a state for which $\omega_{j0}$ tends to a nonzero value as $q$ tends to zero, the matrix element of the operator tends to zero in this limit, provided only that the matrix element of the current remains finite.

The expression (10) provides the basis for an investigation of contributions to physical quantities coming from single-pair states and multipair ones. As an example, let us consider the case of density fluctuations, that is single-pair states and multipair ones. As an example, let

\begin{equation}
\delta E[n_p] = \sum_{\alpha\beta} (\epsilon_p)_{\alpha\beta} (\delta n_p)_{\beta\alpha}.
\end{equation}

The quasiparticle interaction is a matrix in two pairs of spin indices, and it is given as the second functional derivative of the energy with respect to the distribution function

\begin{equation}
\delta^2 E[n_p] = \frac{1}{2} \sum_{\alpha\beta\alpha'\beta'} (f_{p'p})_{\alpha\beta,\alpha'\beta'} (\delta n_p)_{\beta\alpha} (\delta n_{p'})_{\beta'\alpha'}.
\end{equation}

The magnetic moment of a quasiparticle is related to the dependence of the quasiparticle energy on the magnetic field $\mathcal{H}$. It is convenient to write the quasiparticle energy in the form

\begin{equation}
(\epsilon_p)_{\alpha\beta} = \epsilon_p \delta_{\alpha\beta} + \sum_{i=1}^{3} \epsilon'_p \sigma_i \epsilon_{\alpha\beta},
\end{equation}

where the $\sigma^i$ are the Pauli matrices. When non-central forces are present, application of a magnetic field will tend to produce a polarization of the quasiparticle distribution for a particular momentum in a direction different from that of the applied field. Thus the magnetic moment of a quasiparticle is a tensor, and it is given by...
\[ \mu_{ij}(p) = -\frac{\partial^i p_j}{\partial \mathcal{H}}. \] (14)

If only central forces act, and one neglects excitation of degrees of freedom other than the nucleons, the total spin is conserved, and the magnetic moment is proportional to the component of the spin in the direction of the magnetic field, that the magnetic moment of a quasiparticle in a single-component Fermi system is equal to the moment of a bare particle.

The form of the magnetic moment tensor when non-central forces are present may be written down immediately from the requirement that it be invariant under simultaneous rotations of the directions of the magnetic field and of the momentum of the quasiparticle. This dictates that the expression for the magnetic moment is

\[ \mu_{ij}(p) = \mu \delta_{ij} + \frac{3 \mu_T}{2} \left( \frac{p_i p_j}{p^2} - \frac{\delta_{ij}}{3} \right). \] (15)

Here \( \mu \), which is not generally equal to the bare moment \( \mu_0 \), is the magnetic moment averaged over directions of the quasiparticle momentum and \( \mu_T \) is a coefficient characterizing the strength of the off-diagonal part of the magnetic moment tensor. For the specific case of a one-pion-exchange potential this form was derived long ago by Miyazawa [14]. The general form of the quasiparticle interaction may be determined by symmetry considerations. For purely central interactions, and for an unpolarized system the interaction must be invariant under rotation of the spins of the two quasiparticles, and therefore the interaction must be of the standard exchange form

\[ f_{p\sigma, p'\sigma'} = f_{pp'} + g_{pp'} \sigma \cdot \sigma', \] (16)

where we have used a compact matrix notation as usual. With the matrix indices included, this equation is

\[ (f_{p\sigma, p'\sigma'})_{\alpha\beta, \alpha'\beta'} = f_{pp'} \delta_{\alpha\beta} \delta_{\alpha'\beta'} + g_{pp'} (\sigma)_{\alpha\beta} \cdot (\sigma')_{\alpha'\beta'}. \] (17)

For calculating low-temperature properties, the quasiparticle interaction is needed only for \( p \) and \( p' \) equal to the Fermi momentum \( p_F \), and in this case \( f \) and \( g \) are functions only of the angle between \( p \) and \( p' \), which we denote by \( \theta \).

When non-central forces are present, the interaction is not invariant under rotations of the spins and the momenta separately, but only under simultaneous rotations of spins and momenta. Then there can be additional terms of the form \( \sigma \cdot p \sigma' \cdot p' \), and other terms with one or both of the momenta \( p \) in this expression replaced by \( p' \). For nuclear matter, the most important contribution to the non-central part of the interaction comes from pion exchange, so it is generally assumed that the interaction may be taken to be of the form suggested by the result one finds for one-pion exchange

\[ f_{p\sigma, p'\sigma'} = f_{pp'} + g_{pp'} \sigma \cdot \sigma' + k_{pp'} (3\sigma \cdot \hat{u} \sigma' \cdot \hat{u} - \sigma \cdot \sigma'), \] (18)

where \( \mathbf{u} = p - p' \). As for the contribution from \( g_{pp'} \) the function \( k_{pp'} \) is also a function only of \( \theta \) at low temperatures. In earlier work, e.g., Refs. [12,13], the non-central contribution to the quasiparticle interaction has usually been expressed in terms of a function \( h \) which is related to \( k \) by the equation \( k = h u^2 / p^2 \). As we shall explain elsewhere, the advantage of the parametrization we use is that the expansion of \( k \) in terms of Legendre polynomials converges much more rapidly than that for \( h \) [10]. Therefore, all of these functions can be expanded in terms of Legendre polynomials of \( \cos \theta \). For example, for \( k \) we write

\[ k_{pp'} = \sum_{l=0}^{\infty} k_l P_l(\cos \theta). \] (19)

Now we quote results for the static magnetic susceptibility. Previously, Haensel and Jerzak [17] performed a similar calculation including the tensor contribution to the quasiparticle interaction but neglecting multipair contributions and the renormalization of the magnetic moment. The magnetization of the matter is

\[ M_i = \sum_p \text{Tr} \mu_{ij} \delta n_p + M_M, \] (20)

where \( M_M \) is the contribution to the magnetization coming from multipair excitations. And the spin susceptibility is defined as

\[ \chi = \frac{\partial M_i}{\partial \mathcal{H}_i} \bigg|_{\mathcal{H}=0}. \] (21)

For simplicity, we here give the result obtained if one includes only Landau parameters with \( l < 2 \) for the central part of the interaction, and takes only the \( l = 0 \) term in the tensor part of the interaction \( k \). Using Eq. (15) for the magnetic moment and including the tensor interaction up to second order, we arrive at the following expression for the static magnetic susceptibility:

\[ \chi = \frac{\mu^2 N(0)}{1 + G_0 - K_0^2 / 8} - \frac{\mu_T N(0) K_0}{1 + G_0} + \frac{1}{2} N(0) \mu_T^2 + \chi_M. \] (22)

The corresponding result if renormalization of the magnetic moment and multipair contributions are neglected is \( \chi = \mu_0^2 N(0) / (1 + G_0 - K_0^2 / 8) \). More details about the above calculation and the choice of parametrization will be given in a later paper, [18]. Next we shall obtain a lower bound on the magnitude of the last term in Eq. (22), the contribution from multipair excitations.
V. A BOUND ON MULTIPAIR CONTRIBUTIONS TO THE SPIN SUSCEPTIBILITY

In this section we shall consider the part of the static susceptibility that cannot be calculated within Landau theory, the contributions from multipair excitations. Our approach will be to employ sum-rule arguments to place a lower bound on the quantity. Since the excitation energies of the system are positive, it follows for any choice of the energy \( \Omega \) that

\[
\sum_j |\langle j| Q_{\mathbf{q}} |0 \rangle|^2 \left( \frac{\Omega - \omega_{\mathbf{q}0}}{\omega_{\mathbf{q}0}} \right)^2 \geq 0. \tag{23}
\]

If one takes \( \Omega \) to be the mean excitation energy \( \bar{\omega} = W(q)/S(q) \), this yields the inequality

\[
\chi(q, 0) \geq \frac{2nS(q)}{\bar{\omega}}, \tag{24}
\]

which is valid for all \( q \). Let us now apply this result in the limit \( q \to 0 \). As we argued in Sec. IV, single-pair excitations and collective modes do not contribute to \( \chi \) in this limit, and therefore it follows that

\[
\chi_{M}(q, 0) \geq \frac{2nS(q)}{\bar{\omega}} \quad (q \to 0), \tag{25}
\]

where \( \chi_{M}(q, 0) \) is the multipair part of the static response defined in Eq. 4.

We now apply this result to pure neutrons, using values of the static structure factor and the mean excitation energy in the limit \( q \to 0 \) taken from Ref. [18]. In that reference the authors calculated the spin-spin response function, while in this paper we have worked in terms of the response of the magnetization density. The response functions we need are therefore \( \mu_0^2 \) times those of Ref. [18].

At the saturation density of nuclear matter, \( n = 0.16 \text{ fm}^{-3} \), these are \( S_{M}(q=0) \approx 0.19\mu_0^2 \) and \( \bar{\omega}(q=0) \approx 63 \text{ MeV} \). For the total susceptibility we take the value \( \chi \approx 0.38\chi_{F} \) from the recent calculations of Fantoni et al. [13] using the auxiliary field diffusion Monte Carlo method. Here \( \chi_{F} = 3n\mu_{0}^2/2\epsilon_{F} \) is the susceptibility of a free Fermi gas. The quantity \( \epsilon_{F} \) is the Fermi energy of the non-interacting gas, and at a density \( n = 0.16 \text{ fm}^{-3} \) this is approximately 59 MeV for neutrons. The value for the susceptibility obtained in Ref. [18] is close to the results of earlier calculations which included fewer correlations. (See e.g. Ref. [20]). Thus from Eq. (25) we find \( \chi_{M}(q, 0)/\chi \geq 0.63 \). If this result were to be taken at face value, it would indicate that multipair states contribute more than \( \sim 60\% \) of the total static response function. This would appear to be unrealistically high. The calculations of Ref. [18] were designed primarily to investigate the spin response at finite wavelengths, at which phenomena such as pion condensation would be expected to occur. Estimating long-wavelength response is a difficult problem because it requires a careful investigation of long-range correlations. The difficulty may be illustrated by considering experience with calculations of the static structure factor for density response for liquid \( ^4\text{He} \). On the basis of general arguments of the sort given above, the static structure factor should vanish in the limit \( q \to 0 \). However, early calculations gave structure factors which were nonzero in this limit. Our conclusion is that sum rule arguments of the sort given above can in principle provide valuable information about multipair states, but that more detailed calculations of structure factors are needed before they can provide useful quantitative estimates.

VI. CONCLUSIONS

One of the main conclusions of this paper is that, when non-central forces are present, there are contributions to the static magnetic susceptibility at long wavelengths that cannot be calculated within the framework of Landau Fermi-liquid theory. In addition, non-central forces renormalize the magnetic moment of a quasiparticle, and introduce additional terms in the expression for the effective interaction between quasiparticles. Consequently, it is much less straightforward to obtain information about Landau parameters from the magnetic susceptibility than it is when there are only central forces. This may be seen by comparing Eqs. (1) and (23).

Among quantities which need to be understood better in order to relate quasiparticle interactions to the magnetic susceptibility are the strength of transitions to multipair states and the renormalization of the magnetic moment of a quasiparticle. In order to place more reliable bounds on the strength of transitions to multipair excitations it would be valuable to have improved estimates of the static spin structure factor at long wavelengths. The renormalization of the magnetic moment has been studied recently by Cowell and Pandharipande using the correlated basis approach developed from the variational methods used for quantum liquids [21]. They find that the magnitudes of magnetic moments are reduced by approximately 10\% for nuclear matter with proton fractions ranging from 0.2 - 0.5 and for densities between 0.08 fm\(^{-3}\) and 0.24 fm\(^{-3}\).

In this paper we have focussed mainly on a single-component system of fermions with spin-1/2. Our arguments may be extended straightforwardly to multi-component systems, such as nuclear matter with an arbitrary ratio of neutrons to protons. In addition, similar arguments may be applied for the spin-isospin response.

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