A STABILIZED $P_1$ IMMERSED FINITE ELEMENT METHOD FOR THE INTERFACE ELASTICITY PROBLEMS

DO Y. KWAK* AND SANGWON JIN†

Abstract. We develop a new finite element method for solving planar elasticity problems having discontinuous Lamé constants with uniform meshes. This method is based on the ‘broken’ $P_1$-nonconforming finite element method for elliptic interface problems [23] and a stabilizing technique of discontinuous Galerkin method [2, 4, 30] suggested in [21]. We allow the interface cut through the elements, instead modify the basis functions so that they satisfy the traction condition along the interface weakly. We prove optimal $H^1$, $L^2$ and divergence norm error estimates. Numerical experiments are carried out to demonstrate that the our method is optimal for various Lamé parameters $\mu$ and $\lambda$.

Keywords. immersed finite element method, Crouzeix-Raviart finite element, elasticity problems, stability term, traction condition

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1. Introduction. Linear elasticity equation plays an important role in solid mechanics. In particular, when an elastic body is occupied by heterogeneous materials having distinct Lamé parameters $\mu$ and $\lambda$, the governing equation on each disjoint domain and certain jump conditions must be satisfied along the interface of two materials [20]. This kind of problems involving composite materials is getting more attentions from both engineers and mathematicians in recent years, but efficient numerical schemes are not fully developed yet. To solve such equations numerically, one usually uses finite element methods with meshes aligned with the interface between two materials. However, such methods involve unstructured grids resulting in algebraic systems which are hard to solve.

Solving linear elasticity equation with finite element methods has been studied extensively and several methods have been developed, see [3, 10, 18] and references therein. For lower order methods, when $P_1$-conforming element method is applied, the so-called ‘locking phenomena’ is observed when the material is nearly incompressible (11 6 11). Brenner and Sung [10] showed that the Crouizex-Raviart (CR) $P_1$-nonconforming element [16] does not lock on pure displacement problem. But one cannot use this element to a traction-boundary problem since it does not satisfy discrete Korn’s inequality. A remedy was recently suggested by Hansbo et al. [21] who exploited the idea of discontinuous Galerkin methods. By introducing a stabilizing term, they proved the convergence of a locking free $P_1$-nonconforming method for problems with traction boundary conditions.

Solving problems with composite materials is more difficult. Since the traction condition is naturally imposed along the interface, these belong to the traction boundary type problems, even if the Dirichlet boundary condition is imposed on the boundary of the whole domain. Thus the CR element cannot be used without the stability term. In all of the methods, meshes have to be aligned with the interface. For some work related to the interface elasticity problems, we refer to [7, 19, 20, 28].

On the other hand, alternative methods which use uniform mesh have been developed recently for diffusion type of elliptic problems. The immersed finite element methods (IFEM) recently developed in [12, 13, 23, 26, 27] have obvious advantages: simple data structure, no necessity of grid generations, and fast solvers, and so on. The idea of IFEM is to use grids independent of the interface, instead modify the finite element shape functions so that they satisfy the jump conditions along the interface. Similar methods using finite difference were treated in [22, 24, 25, 29].

In this paper, we develop a new immersed finite element based on the broken CR element with a stabilizing term for a linear elasticity problem involving smooth interface. We modify the basis functions so that they satisfy the natural traction condition along the interface, and prove optimal error estimates. Numerical results which support our theory are included.

*Korea Advanced Institute of Science and Technology, Daejeon, Korea 305-701. email:kdy@kaist.ac.kr.
†Korea Advanced Institute of Science and Technology, Daejeon, Korea 305-701 email:jinsangwon@kaist.ac.kr
The rest of our paper is organized as follows. In section 2, we introduce notations and the problem with a traction interface condition. In section 3, we review a stabilized CR finite element method. In section 4, we define new vector basis functions based on the $P_1$ nonconforming elements satisfying the traction interface condition weakly, and introduce our discrete problem using the formulation by Hansbo et al. [21]. In section 5, we prove the approximation property of our finite element space and prove $H^1$, $L^2$ and divergence norm error estimates. Finally, numerical experiments are presented in section 6.

2. preliminaries. Let $\Omega$ be a connected, convex polygonal domain in $\mathbb{R}^2$ which is divided into two subdomains $\Omega^+$ and $\Omega^-$ by a $C^2$ interface $\Gamma = \partial\Omega^+ \cap \partial\Omega^-$, see Figure 1. We assume the subdomains $\Omega^+$ and $\Omega^-$ are occupied by two different elastic materials. For a differentiable function $v = (v_1, v_2)$ and a tensor $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$, we let

$$\nabla v = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix}, \quad \text{div} \tau = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} \\ \frac{\partial \tau_{21}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} \end{pmatrix}. $$

Then the displacement $u = (u_1, u_2)$ of the elastic body under an external force satisfies the Navier-Lamé equation as follows.

$$-\text{div} \sigma(u) = f \quad \text{in } \Omega^s, \quad (s = +, -) \quad (2.1)$$

$$[u]_\Gamma = 0, \quad (2.2)$$

$$[\sigma(u) \cdot n]_\Gamma = 0, \quad (2.3)$$

$$u = 0 \text{ on } \partial\Omega, \quad (2.4)$$

where

$$\sigma(u) = 2\mu \varepsilon(u) + \lambda \text{tr}(\varepsilon(u)) \delta, \quad \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T) \quad (2.5)$$

are the stress tensor and the strain tensor respectively, $n$ outward unit normal vector, $\delta$ the identity tensor, and $f \in (L^2(\Omega))^2$ is the external force. Here

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}, \quad \mu = \frac{E}{2(1 + \nu)}$$

are the Lame constants, satisfying $0 < \mu_1 < \mu < \mu_2$ and $0 < \lambda < \infty$, and $E$ is the Young’s modulus and $\nu$ is the Poisson ratio. When the parameter $\lambda \to \infty$, this equation describes the behavior of nearly incompressible material. Since the material properties are different in each region, we set the Lame constants $\mu = \mu^s, \lambda = \lambda^s$ on $\Omega^s$ for $s = +, -. \quad \text{The bracket } [\cdot] \text{ means the jump across the interface}$$

$$[u]_\Gamma := u|_{\Omega^+} - u|_{\Omega^-}.$$ Multiplying $v \in (H^1_0(\Omega))^2$ and applying Green’s identity in each domain $\Omega^s$, we obtain

$$\int_{\Omega^s} 2\mu^s \varepsilon(u) : \varepsilon(v) dx + \int_{\Omega^s} \lambda^s \text{div} u \text{div} v dx - \int_{\partial\Omega^s} \sigma(u) n \cdot v ds = \int_{\Omega^s} f \cdot v dx, \quad (2.6)$$

where

$$\varepsilon(u) : \varepsilon(v) = \sum_{i,j=1}^2 \varepsilon_{ij}(u) \varepsilon_{ij}(v).$$
Summing over $s = +, -$ and applying the traction condition (2.3), we obtain the following weak form

$$a(u, v) = (f, v),$$  

(2.7)

where

$$a(u, v) = \int_{\Omega} 2\mu \varepsilon(u) : \varepsilon(v) \, dx + \int_{\Omega} \lambda \text{div}u \text{div}v \, dx$$  

(2.8)

and

$$(f, v) = \int_{\Omega} f \cdot v \, dx.$$  

(2.9)

Then we have the following result [15], [21].

**Theorem 2.1.** There exists a unique solution $u$ of (2.1) - (2.3) satisfying

$$u \in (H^1_0(\Omega))^2 \cap (H^2(\Omega^+))^2 \cap (H^2(\Omega^-))^2.$$

3. A stabilized Crouzeix-Raviart finite element method for the elasticity equation.

We briefly review the stabilized version of $P_1$-nonconforming finite element method introduced by Hansbo and Larson [21]. Even though this method was introduced for an elasticity equation without interface, it can be also used for an interface problem as long as the grids are aligned with the interface.

Let ${\mathcal{T}_h}$ be a given quasi-uniform triangulations of $\Omega$ by the triangles of maximum diameter $h$ whose grids are aligned with the interface. For each $T \in \mathcal{T}_h$, one constructs local basis functions using the average value along each edge as degrees of freedom. Let

$$v|_{e} = \frac{1}{|e|} \int_{e} v \, ds$$

denote the average of a function $v \in H^1(T)$ along an edge $e$ of $T$. Let $N_h(T)$ denote the linear space spanned by the six Lagrange basis functions

$$\phi_i = (\phi_{i1}, \phi_{i2})^T, \; i = 1, 2, 3, 4, 5, 6$$

satisfying

$$\phi_{i1}|_{e_j} = \delta_{ij}, \quad \phi_{i2}|_{e_j} = \delta_{i-3,j} \text{ (Kronecker)}$$

for $j = 1, 2, 3$. The Crouzeix-Raviart $P_1$-nonconforming space is given by

$$N_h(\Omega) = \left\{ \phi : \phi|_T \in (P_1(T))^2 \; \text{for each } T \in \mathcal{T}_h; \; \text{if } T_1 \text{ and } T_2 \text{ share an edge } e, \right\}$$

then \( \int_e \phi|_{\partial T_1} \, ds = \int_e \phi|_{\partial T_2} \, ds; \) and \( \int_{\partial T \cap \partial \Omega} \phi \, ds = 0 \).

The stabilized $P_1$-nonconforming finite element method for (2.7) is: find $u_h \in N_h(\Omega)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in N_h(\Omega),$$  

(3.1)

where

$$a_h(u_h, v_h) : = \sum_{T \in \mathcal{T}_h} \int_{T} 2\mu \varepsilon(u_h) : \varepsilon(v_h) \, dx + \sum_{T \in \mathcal{T}_h} \int_{T} \lambda \text{div}u_h \text{div}v_h \, dx$$

$$+ 2\mu \sum_{e \in E} \int_e h^{-1} [u_h] [v_h] \, ds.$$
Remark 3.1. Without the stability term, the bilinear form $a_h(\cdot, \cdot)$ does not satisfy the discrete Korn’s inequality. Hence the scheme does not yield an optimal result. For a problem without an interface, Hansbo and Larson [21] proved the following result.

Theorem 3.1. Let $u$ be the solution of (2.1) and $u_h$ be the solution of (3.1). Then

$$
\|u - u_h\|_{a,h} \leq Ch\|f\|_{L^2(\Omega)},
$$

where $\| \cdot \|_{a,h} = a_h(\cdot, \cdot)^{1/2}$.

4. Construction of broken $P_1$ basis functions satisfying traction condition. In this section, we introduce an immersed finite element (IFEM) based on the scheme in the previous section for elasticity problem. This method was first suggested by the author in [23] and proved to be optimal for elliptic problem. For this purpose, we assume a quasi uniform triangulation $T_h$ of $\Omega$ consisting of triangles of maximum diameter $h$, which is not necessarily aligned with the interface. Typically we use a uniform grid.

We call an element $T \in T_h$ an interface element if the interface $\Gamma$ passes through the interior of $T$, otherwise we call it a noninterface element. Let $T^*_h$ be the collection of all interface elements. We assume the followings which are easily satisfied when $h$ is small enough since the interface is smooth:

- the interface intersects the edges of an element at no more than two points
- the interface intersects each edge at most once, except possibly it passes through two vertices.

The main idea of the IFEM is to use two pieces of linear polynomials on an interface element to form a basis which satisfies the jump conditions. The piecewise linear basis function $\phi_i (i = 1, 2, \ldots, 6)$ of
We are ready to define our new discrete problem for (2.7). We introduce the following mesh dependent energy-like norm.

The conditions lead to a square system of linear equations in twelve unknowns for each $i$. In matrix form, we have

$$M c_i = b_i,$$

where $c_i = (a_0^+, b_0^+, c_0^+, a_0^-, b_0^-, c_0^-)^T$ is the vector of the unknowns and $b_i = (\delta_{i1}, \delta_{i2}, 0, 0, 0, 0, 0, 0)^T$.

With a tedious calculation, we can show that this system has a unique solution regardless of the location of the interface. We skip the details.

**Lemma 4.1.** The system (4.3) has a unique solution which determines $\hat{\phi}_i$ satisfying (4.2), regardless of the location of the intersections.

We denote by $\hat{N}_h(T)$ the space of functions generated by $\hat{\phi}_i, i = 1, 2, 3, 4, 5, 6$ constructed above. Using this local finite element space, we define the global immersed finite element space $\hat{N}_h(\Omega)$ by

$$\hat{N}_h(\Omega) = \left\{ \phi \in \hat{N}_h(T) \text{ if } T \in T_h^*, \text{ and } \phi \in N_h(T) \text{ if } T \notin T_h^*; \right. \left\{ \begin{array}{ll} & \text{if } T_1 \text{ and } T_2 \text{ share an edge } e, \text{ then } \\ & \int_e \phi|_{\partial T_1} ds = \int_e \phi|_{\partial T_2} ds; \text{ and } \int_{\partial T \cap \partial \Omega} \phi ds = 0 \end{array} \right. \right\}.$$

We are ready to define our new discrete problem for (2.7).

**Discrete Problem.** Find $u_h \in \hat{N}_h(\Omega)$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in \hat{N}_h(\Omega),$$

where $a_h(\cdot, \cdot)$ is defined as in (3.1).

5. **Error Analysis.** In this section, we prove an optimal order $H^1$ and $L^2$ norm convergence of our scheme. We introduce the following mesh dependent energy-like norm.

$$\|v\|_{a,h}^2 := a_h(v_h, v_h) = \sum_{T \in T_h} \|v\|_{n,T}^2 + \sum_{e \in F} \int_e \frac{2\mu}{h} |v|^2 ds,$$

where

$$\|v\|_{n,T}^2 = \int_T 2\mu \epsilon(v) : \epsilon(v) dx + \int_T \lambda |\text{div} v|^2 dx.$$
It is obvious that the bilinear form $a_h(\cdot, \cdot)$ is continuous and coercive with respect to the norm $\| \cdot \|_{a,h}$.

We introduce some spaces and norms which are required to prove a priori error estimate. For any domain $D$, we let $W^m_p(D)$ $(W^m_2(D) = H^m(D))$ be the usual Sobolev space with (semi)-norms denoted by $\| \cdot \|_{m,p,D}$ and $\| \cdot \|_{m,2,D}$ $(\| \cdot \|_{m,2} = \| \cdot \|_{m,D})$ $H^1_0(\Omega)$ be the subspace of $H^1(\Omega)$ with zero trace on the boundary. For each $T \in \mathcal{T}_h$, let

$$(\hat{W}_p^m(T))^2 := \{ u \in (L^2(T))^2 : u|_{T \cap \Omega^s} \in (W^m_p(T \cap \Omega^s))^2, s = +, - \},$$

for $p \geq 1$, $m \geq 0$ with norms:

$$|u|_{m,p,T}^2 := |u|_{m,p,T \cap \Omega^+}^2 + |u|_{m,p,T \cap \Omega^-}^2,$$

$$\|u\|_{m,p,T}^2 := \|u\|_{m,p,T \cap \Omega^+}^2 + \|u\|_{m,p,T \cap \Omega^-}^2.$$

We define $(\hat{W}_p^m(\Omega))^2$ to be the space of all functions $u \in (L^2(\Omega))^2$ such that $u|_T \in (\hat{W}_p^m(T))^2$ for all $T \in \mathcal{T}_h$ equipped with the broken (semi)-norms $|u|_{\hat{W}_p^m(\Omega)} := (\sum_T |u|_{m,p,T}^2)^{1/2}$ and $\|u\|_{\hat{W}_p^m(\Omega)} := (\sum_T \|u\|_{m,p,T}^2)^{1/2}$. When $p = 2$, we write $(\hat{H}^m(\Omega))^2 = (\hat{W}_p^m(\Omega))^2$ and denote their (semi)-norms by $|u|_{1,h}$ and $\|u\|_{1,h}$. We also need subspaces of $(\hat{H}^2(\Omega))^2$ and $(\hat{H}^2(\Omega))^2$ satisfying the jump conditions:

$$(\hat{H}^2(\Omega))^2 \{ u \in (H^1(T))^2 : u|_{T \cap \Omega^\pm} \in (H^2(T \cap \Omega^\pm))^2, |\sigma(u) \cdot n|_{\Gamma \cap \Omega} = 0 \},$$

$$(\hat{H}^2(\Omega))^2 \{ u \in (H^1_0(\Omega))^2 : u|_T \in (\hat{H}^2(T))^2, \forall T \in \mathcal{T}_h \}.$$ 

Throughout the paper, the constants $C, C_0, C_1$, etc., are generic constants independent of the mesh size $h$ and functions $u, v$ but may depend on the problem data $\mu, \lambda, f$ and $\Omega$, and are not necessarily the same on each occurrence.

5.1. Approximation property of the immersed finite element space $\hat{N}_h(T)$. In this subsection, we study the approximation property of $\hat{N}_h(T)$. The case of $P_1$ nonconforming element for elliptic problem is shown in [23].

One of the obstacles in proving the approximation property is to treat the jump conditions along the curved interface. The difficulty lies in the fact that $\hat{N}_h(T)$ does not belong to $(\hat{H}_h^2(T))^2$, the restriction of $(\hat{H}_h^2(\Omega))^2$ on $T$, since piecewise linear function cannot satisfy the jump condition along the curved interface $\Gamma$. To overcome this difficulty, we introduce a bigger space which contains both of these spaces. For a given interface element $T$, we consider a function space $X(T)$ such that every $u \in X(T)$ satisfies

$$\begin{cases}
  u \in [H^1(T)]^2 \cap [H^2(T^+ \cap \Omega^+)]^2 \cap [H^2(T^- \cap \Omega^-)]^2 \cap [H^2(T^+)]^2 \cap [H^2(T^-)]^2,
  \\
  \int_{\Gamma \cap T} (\sigma(u)^- - \sigma(u)^+) \cdot n^- ds = 0,
\end{cases}$$

where $\sigma(u)^- = 2\mu^- \epsilon(u) + \lambda^- \text{div}u$, $\sigma(u)^+ = 2\mu^+ \epsilon(u) + \lambda^+ \text{div}u$, $T^s = T \cap \Omega^s$, $s = +, -$ are the regions surrounded by $\Gamma$ and $\overline{DE}$ (Fig 3). For any $u \in X(T)$, we define the following quantities: Let
\( \lambda_M = \max(\lambda^+, \lambda^-) \). We define
\[
\|u\|_{2,T}^2 = \|u\|_{1,T}^2 + \lambda_M \|\text{div}u\|_{0,T}^2,
\]
\[
|u|_{2,X(T)}^2 = |u|_{2,T \cap \Gamma^+}^2 + |u|_{2,T \cap \Gamma^-}^2 + |u|_{2,T_r^+}^2 + |u|_{2,T_r^-}^2,
\]
\[
|u|_{X(T)}^2 = |u|_{1,T}^2 + |u|_{X(T)}^2 + \lambda_M \sum_{\pm} |\text{div}u|_{1,T_{\pm}}^2 + \lambda_M \int_T \text{div}u \, dx^2,
\]
\[
+ \int_{\Gamma \cap T} [\sigma(u)n_T] \, ds^2 + \sum_{i=1}^3 |n_{1T}|_{e_i}^2 + \sum_{i=1}^3 |n_{2T}|_{e_i}^2.
\]

Fig. 5.1. The real interface and the approximated interface

**Remark 5.1.** The difference between the spaces \( X(T) \) and \( \tilde{H}^2(\Omega) \) is: if \( u \in \tilde{H}^2(\Omega) \), then \( u \) satisfies the a strong traction continuity along \( \Gamma \) while \( u \in X(T) \) satisfies a weak traction continuity.

**Lemma 5.1.** For an interface triangle \( T \), every continuous, piecewise linear function \( \phi \) satisfies
\[
\int_{\Gamma \cap T} [\sigma(\phi)n_T] \, ds = 0 \quad \text{if and only if} \quad \int_{DE} [\sigma(\phi)n_{DE}] \, ds = 0.
\]  
In other words, a piecewise linear function \( \phi \) satisfies the traction continuity along the line segment if and only if it satisfies the same condition along the (curved) interface.

**Proof.** This can be easily proved by Green’s theorem since \( \phi \) is piecewise linear. \( \square \)

**Lemma 5.2.** \( \| \cdot \|_{2,T} \) is a norm on the space \( \tilde{H}^2(T) \) which is equivalent to \( \| \cdot \|_{X(T)} \) on \( X(T) \).

**Proof.** Since the terms involving \( \text{div}u \) are common in both norms, it suffices to prove the equivalence ignoring those terms.

Clearly, \( \| \cdot \|_{2,T} \) is a semi-norm. To show it is indeed a norm, assume \( u \in (\tilde{H}^2(T))^2 \) satisfies \( \|u\|_{2,T} = 0 \). Then \( |u|_{X(T)} = 0 \). Hence \( u \) is linear on each of the four regions \( T^+ \cap \Omega^+ \), \( T^- \cap \Omega^- \), \( T^+_r \) and \( T^-_r \). Since \( u \in H^1(T) \), \( u \) is continuous on \( T \). Since \( \int_{\Gamma \cap T} \sigma(u)n_T \, ds = 0 \), \( u \) satisfies the flux condition along \( DE \) by Lemma 5.1. Hence \( u \in \tilde{N}_h(T) \) and together with the fact that \( n_{1T} = 0, \ i = 1, 2, 3 \) and \( n_{2T} = 0, \ i = 1, 2, 3 \), we conclude \( u = 0 \), which shows that \( \| \cdot \|_{2,T} \) is a norm on \( X(T) \).

We now show the equivalence of \( \| \cdot \|_{2,T} \) and \( \| \cdot \|_{X(T)} \) on the space \( X(T) \). (cf. [8, p.77]). Since \( u \in X(T) \), \( \int_{\Gamma \cap T} \sigma(u)n_T \, ds = 0 \). By trace theorem, we have
\[
\sum_{i=1}^3 |n_{1T}|_{e_i}^2 + \sum_{i=1}^3 |n_{2T}|_{e_i}^2 \leq C \|u\|_{1,T}.
\]  
(5.4)
Hence we see
\[ \|u\|_{2,T} \leq C\|u\|_{X(T)}. \]  
(5.5)

Now suppose that the converse
\[ \|u\|_{X(T)} \leq C\|u\|_{2,T}, \quad \forall u \in X(T) \]
fails for any \( C > 0 \). Then there exists a sequence \( \{u_k\} \) in \( X(T) \) with
\[ \|u_k\|_{X(T)} = 1, \quad \|u_k\|_{2,T} \leq \frac{1}{k}, \quad k = 1, 2, \ldots. \]  
(5.6)

Let \( s = + \) or \( s = - \). Since \( W^{1}_t(T^s) \), \( t > 2 \) is compactly imbedded in \( H^1(T^s) \) by Kondrasov theorem [14, p. 114], there exists a subsequence of \( \{u_{k_i}\} \) which converges in \( (H^1(T^s))^2 \), respectively. Without loss of generality, we can assume that the sequences themselves converge in \( (H^1(T^s))^2 \). Let \( u^*_s \in (H^1(T^s))^2 \) be their respective limits. We shall show that \( u^*_s \) defined by \( u^*_s = u^*_+ \) on \( T^+ \) and \( u^*_s = u^*_- \) on \( T^- \) belongs \( (H^1(T))^2 \). Let \( u = (u_1, u_2) \) and \( u^* = (u^*_1, u^*_2) \) respectively. For any \( \phi \in C_0^\infty(T) \) we have for \( i = 1, 2 \)
\[ \int_{T^i} \frac{\partial u^*_i}{\partial n} \phi ds = \int_{\Gamma \cap \partial T^i} u^*_i n_1 \phi ds - \int_{T^i} u^*_i \frac{\partial \phi}{\partial x} dx \]
\[ = \int_{\Gamma \cap \partial T^i} u^*_i n_1 \phi ds - \int_{T^i} u^*_i \frac{\partial \phi}{\partial x} dx, \]
since \( \phi|_{\partial T} = 0 \). Here \( n_1 \) is the first component of the outward normal vector \( n \) of \( \partial T^s \). Adding these
\[ \sum_{s=\pm} \int_{T^s} \frac{\partial u^*_s}{\partial n} \phi ds = \sum_{s=\pm} \int_{\Gamma \cap \partial T^s} u^*_s n_1 \phi ds - \sum_{s=\pm} \int_{T^s} u^*_s \frac{\partial \phi}{\partial x} dx \]
\[ = - \sum_{s=\pm} \int_{T} u^*_s \frac{\partial \phi}{\partial x} ds = - \int_{T} u^* \frac{\partial \phi}{\partial x} dx. \]
Thus the relation
\[ \int_{T} \frac{\partial u^*_s}{\partial x} \phi dx = - \int_{T} u^*_s \frac{\partial \phi}{\partial x} dx, \quad \phi \in C_0^\infty(T) \]
defines \( \frac{\partial u^*_s}{\partial x} \in L^2(T) \). The same argument shows that \( \frac{\partial u^*_s}{\partial y} \) is also well defined on \( T \). Hence \( u^* \in (H^1(T))^2 \) and \( \|u_k - u^*\|_{1,T} \to 0 \). Since
\[ \|u_k - u\|_{X(T)}^2 = \|u_k - u^*\|_{1,T}^2 + \|u_k - u^*\|_{X(T)}^2 \]
\[ \leq \|u_k - u^*\|_{1,T}^2 + \|u^* - u_l\|_{1,T}^2 + \|u^*\|_{X(T)} \to 0 \]
as \( k, l \to \infty \), we see that \( \{u_k\} \) is a Cauchy sequence in \( X(T) \). By completeness, it converges the a limit in \( X(T) \) which is \( u^* \) and hence
\[ \|u^*\|_{X(T)} = 1. \]  
(5.7)

Now (5.5), (5.6) gives
\[ \|u^*\|_{2,T} \leq \|u^* - u_k\|_{2,T} + \|u_k\|_{2,T} \leq C\|u^* - u_k\|_{X(T)} + \frac{1}{k} \to 0. \]
But this is a contradiction to (5.7), since $\|u^*\|_{2,T} = 0$ implies $u^* = 0$.

We define an interpolation operator: for any $u \in X(T)$, we define $I_h u \in \hat{N}_h(T)$ using the average of $u$ on each edge of $T$ by

$$\int_{e_i} I_h u \, ds = \int_{e_i} u \, ds, \quad i = 1, 2, 3$$

and call $I_h u$ the interpolant of $u$ in $\hat{N}_h(T)$. We then define $I_h u$ for $u \in X(T)$ by $(I_h u)|_T = I_h (u|_T)$.

To prove the divergence error estimate, we first need the following property.

**Lemma 5.3.** We have for $u \in (H^1(T))^2$

$$\int_T \text{div}(u - I_h u) \, dx = 0. \quad (5.8)$$

**Proof.** We have

$$\int_T \text{div}(I_h u) \, dx = \sum_{s = \pm} \int_{T^s} \text{div}(I_h u)|_{T^s} \, dx$$

$$= \sum_{s = \pm} \int_{\partial T^s} I_h u \cdot n \, ds$$

$$= \int_{\partial T} I_h u \cdot n \, ds$$

$$= \int_{\partial T} u \cdot n \, ds = \int_T \text{div} u \, dx.$$

We need the following variant of Poincaré inequality. (cf. [17])

**Lemma 5.4.** Let $T$ be an interface element and define

$$\|v\|^2_\tau := \sum_{s = \pm} \|v\|_{1,T^s}^2 + \sum_{s = \pm} \left| \int_{T^s} v \, dx \right|^2, \quad (5.9)$$

$$\|v\|^2_\sigma := \sum_{s = \pm} \|v\|_{1,T^s}^2 + \left| \int_T v \, dx \right|^2. \quad (5.10)$$

Then there exists a constant $C$ independent of $v$ such that

$$\|v\|_\tau^2 \leq C \|v\|_\sigma^2, \quad (5.11)$$

for all $v \in H^1(T^+) \cap H^1(T^-)$ that such that $\text{sgn}(\int_{T^+} v \, dx) = \text{sgn}(\int_{T^-} v \, dx)$.

**Proof.** By a generalized Poincaré inequality [17], we have

$$\|v\|_{2(T^s)}^2 \leq C_1 \|v\|_{1,T^s}^2 + C_2 \left| \int_{T^s} v \, dx \right|^2, \quad \text{for } s = +, - \quad (5.12)$$

Clearly,

$$\|v\|_\tau^2 \leq C \|v\|_\tau^2 \quad (5.13)$$
holds. To show the converse, suppose

$$\|v\|_\tau \leq C\|v\|_\sigma$$  \hspace{1cm} (5.14)$$

fails to hold for any $C > 0$. Then for each $s = \pm$, there exists a sequence $\{v_k^s\}$ in $H^1(T^s)$ such that

$$\|v_k^s\|_\tau = 1, \quad \|v_k^s\|_\sigma \leq \frac{1}{k}, \quad k = 1, 2, \ldots .$$  \hspace{1cm} (5.15)$$

Since $H^1(T^s)$ is compactly imbedded in $L^2(T^s)$ (for each $s = \pm$), by Rellich-Kondrasov theorem, there exists a subsequence $\{v_k^s\}$ which converges in $L^2(T^s)$. Without loss of generality, we may assume the common subsequence $\{v_k\}$ converges to $v^*$ in $L^2(T)$. Then

$$\|v_k - v\|_\tau^2 = \sum_{\pm} |v_k - v|_{1,T^\pm}^2 + \sum_{\pm} \left| \int_{T^\pm} (v_k - v) \, dx \right|^2 \leq \|v_k\|_\sigma^2 + \|v\|_\sigma^2 + C \sum_{\pm} \left| \int_{T^\pm} |v_k - v|^2 \, dx \right| \to 0$$

Hence $\{v_k\}$ is a Cauchy sequence with respect to $\| \cdot \|_\tau$ norm, converging to the same limit $v^*$. Hence

$$\|v^*\|_\sigma \leq \|v^* - v_k\|_\sigma + \|v_k\|_\sigma \leq C\|v^* - v_k\|_\tau + \frac{1}{k} \to 0.$$  

and thus $v^*$ is piecewise constant and $\int_{T^s} v^* \, dx = 0$. Since

$$\|v_k - v^*\|_\sigma^2 = \sum_{\pm} |v_k - v^*|_{1,T^\pm}^2 + \left| \int_{T^\pm} (v_k - v^*) \, dx \right|^2 = \sum_{\pm} |v_k - v^*|_{1,T^\pm}^2 + \left| \int_{T^+} v_k + \int_{T^-} v_k - \int_{T} v^* \, dx \right|^2 \to 0$$

Hence

$$\int_{T^+} v_k \, dx + \int_{T^-} v_k \, dx \to 0.$$  

Since $\int_{T^+} v_k \, dx$ and $\int_{T^-} v_k \, dx$ are of the same sign, $\int_{T^+} v_k \, dx$ and $\int_{T^-} v_k \, dx \to 0$. We also have

$$\sum_{\pm} \left| \int_{T^\pm} (v_k - v^*) \, dx \right| \leq C \left( \int_{T} |v_k - v^*|^2 \, dx \right)^{1/2} \to 0.$$  

Thus

$$\int_{T^s} v^* \, dx = 0, \quad s = +, -.$$  

Since $v^*$ is piecewise constant, $v^* \equiv 0$. This is a contradiction since we have $\|v^*\|_\tau = 1$. \hfill \Box

Applying Lemma 5.4 to $\text{div} \, u$, we obtain

**COROLLARY 5.2.**

$$\|\text{div} \, u\|_{L^2(T)}^2 \leq C \sum_{\pm} |\text{div} \, u|_{1,T^\pm}^2 + C \left| \int_{T} \text{div} \, u \, dx \right|^2$$  \hspace{1cm} (5.16)$$
for all $v \in H^1(T^+) \cap H^1(T^-)$ that such that $\text{sgn}(\int_{T^+} v \, dx) = \text{sgn}(\int_{T^-} v \, dx)$.

Now we are ready to prove the following interpolation error estimate.

**Theorem 5.5.** For any $u \in [\tilde{H}_h^2(\Omega)]^2$, there exists a constant $C > 0$ such that

$$\|u - I_h u\|_{1,h} + \lambda_M^{1/2} \|\text{div}(u - I_h u)\|_{L^2(\Omega)} \leq C h (\|u\|_{\tilde{H}_h^2(\Omega)} + \lambda_M^{1/2} \|\text{div}u\|_{\tilde{H}_h^2(\Omega)}).$$

When the condition about the sign change is not satisfied, one has

$$\|u - I_h u\|_{1,h} \leq C h \|u\|_{\tilde{H}_h^2(\Omega)}.$$

**Proof.** First assume the condition of Lemma 5.4 holds. Let $\tilde{T}$ be a reference interface element, $\tilde{T}$ be the corresponding local reference interface. Then for any $\tilde{u} \in X(T)$, (let us denote $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)$ and $I_h \tilde{u} = (\tilde{u}_1, \tilde{u}_2)$)

$$\|\tilde{u} - I_h \tilde{u}\|_{2,\tilde{T}}^2 = \|\tilde{u} - I_h \tilde{u}\|_{X(\tilde{T})}^2 + \lambda_M |\text{div}(\tilde{u} - I_h \tilde{u})|_{1,\tilde{T}^\pm}^2 + \lambda_M \left|\int_{\tilde{T}} \text{div}(\tilde{u} - I_h \tilde{u}) \, dx\right|^2$$

$$+ \left|\int_{\tilde{T} \cap \tilde{T}'} [\sigma(\tilde{u}) - \sigma(I_h \tilde{u})] \cdot n_T \, ds\right|^2 + \frac{3}{\lambda_M} \sum_{i=1}^3 |(\tilde{u}_1 - \tilde{u}_1)|_e^2 + \frac{3}{\lambda_M} \sum_{i=1}^3 |(\tilde{u}_2 - \tilde{u}_2)|_e^2$$

$$= \|\tilde{u} - I_h \tilde{u}\|_{X(\tilde{T})}^2 + \sum_{s=\pm} \lambda_M |\text{div}(\tilde{u} - I_h \tilde{u})|_{1,\tilde{T}^s}^2 = \|\tilde{u}\|_{X(\tilde{T})}^2 + \sum_{s=\pm} \lambda_M |\text{div} \tilde{u}|_{1,\tilde{T}^s}^2,$$

where we used the properties of the interpolation operator $I_h$, Lemma 5.3 and the fact that $H^2$-seminorm of the piecewise linear function $I_h \tilde{u}$ vanishes. By Lemma 5.2 and Corollary 5.2 and scaling argument

$$\|u - I_h u\|_{b,T} \leq C \|\tilde{u} - I_h \tilde{u}\|_{b,\tilde{T}}$$

$$\leq C \|\tilde{u} - I_h \tilde{u}\|_{X(\tilde{T})}$$

$$\leq C \|\tilde{u} - I_h \tilde{u}\|_{2,\tilde{T}}$$

$$= C (|\tilde{u}|_{X(\tilde{T})} + \lambda_M^{1/2} \sum_{s=\pm} |\text{div} \tilde{u}|_{1,\tilde{T}^s})$$

$$\leq C h (|u|_{X(T)} + \lambda_M^{1/2} \sum_{s=\pm} |\text{div} u|_{1,T^s})$$

$$\leq C h(|u|_{\tilde{H}_h^2(T)} + \lambda_M^{1/2} \sum_{s=\pm} |\text{div} u|_{1,T^s}).$$

When the condition of Lemma 5.4 does not holds, one can proceed exactly the same way without the terms involving $\text{div}u$ in the definition of norms $\|\cdot\|_{b,T}$, $\|\cdot\|_{X(T)}$ and $\|\cdot\|_{2,T}$ to obtain the desired estimate. □
5.2. Consistency error estimate. Let \( u \) be the solution of (2.1). Then since \( u \in (H^1_0(\Omega))^2 \), we see
\[
\begin{align*}
a_h(u - u_h, v_h) &= a_h(u, v_h) - f(v_h) \\
&= \sum_{T \in \mathcal{T}_h} \int_T 2\mu \epsilon(u) : \epsilon(v_h) + \sum_{e \in \partial T} \lambda \text{div} u \text{div} v_h + \sum_{e \in \partial T} \frac{2\mu}{h} |u|^2 |v_h| ds \\
&\quad - \left( \sum_{T \in \mathcal{T}_h} \int_T 2\mu \epsilon(u) : \epsilon(v_h) + \sum_{e \in \partial T} \lambda \text{div} u \cdot \text{div} v_h - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \sigma(u) \cdot n \cdot v_h ds \right) \\
&= - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \sigma(u) n \cdot v_h ds.
\end{align*}
\]

\[
\sum_{T \in \mathcal{T}_h} \int_{\partial T} \sigma(u) n \cdot v_h ds = \sum_{T \in \mathcal{T}_h} \sum_{e \in \partial T} \int_{\partial e} (\sigma(u) - \sigma(u) n) \cdot [v_h] ds
\]
\[
= \sum_{e \in \partial T} \int_{e} (\sigma(u) \cdot n - \sigma(u) \cdot n) \cdot [v_h] ds
\]
\[
\leq \sum_{T \in \mathcal{T}_h} Ch \|\sigma(u) \cdot n\|_{1,T} |v_h|_{1,T}
\]
\[
= \sum_{T \in \mathcal{T}_h} Ch \|2\mu \epsilon(u) \cdot n \|_{1,T} + \|\text{div} u \cdot \delta n\|_{1,T} |v_h|_{1,T}
\]
\[
\leq \sum_{T \in \mathcal{T}_h} Ch (2\mu \|u\|_{2,T} + \|\text{div} u\|_{1,T}) |v_h|_{1,T}
\]

Let \( Q(v_h) := v - \frac{1}{v} \int_{\Gamma} v \, dx \). Then using the facts that
\[
|v_h|_{1,T} \leq C \|v_h\|_{a,T} + \|Q(v_h)\|_{L^2(T)} \leq C \|v_h\|_{a,T} + Ch \|v_h\|_{1,T}
\]
for all \( v_h \in \tilde{N}(T) \) (cf. Thm 3.1 [9]), we see
\[
|v_h|_{1,h} \leq C \|v_h\|_{a,h}.
\]
Hence we have
\[
|a_h(u, v_h) - a_h(u_h, v_h)| \leq Ch R(u)|v_h|_{1,h} \leq Ch R(u)|v_h|_{a,h},
\] 5.17

where
\[
R(u) = \|u\|_{\tilde{H}^2(\Omega)} + \lambda M \|\text{div} u\|_{\tilde{H}^1(\Omega)}.
\]

The following type of elliptic regularity estimate is known when the Lamé constants are continuous (cf. [10]). For a problem with interface, however, such estimate is not known to the author’s knowledge. We set the following hypothesis:

\[ H1 \quad 2\mu \|u\|_{\tilde{H}^2(\Omega)} + \|\text{div} u\|_{\tilde{H}^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \]
5.3. Error estimates. Now we are ready to show our main result.

**Theorem 5.6.** Let \( u \) (resp. \( u_h \)) be the solution of (2.1) (resp. (4.4)). Then we have

\[
\| u - u_h \|_{a,h} \leq C h ( \| u \|_{\tilde{H}^2(\Omega)} + \lambda_M \| \text{div} u \|_{\tilde{H}^1(\Omega)})
\]

under the assumption of Lemma 5.4. When the assumption does not hold, one has

\[
\| u - u_h \|_{1,h} \leq C h \| u \|_{\tilde{H}^2(\Omega)}.
\]

If regularity hypothesis (H1) holds, then both estimates can be bounded by \( C h \| f \|_{L^2(\Omega)} \).

**Proof.** By triangular inequality, we have

\[
\| u - u_h \|_{a,h} \leq \| u_h - I_h u \|_{a,h} + \| u - I_h u \|_{a,h}.
\]

From the coercivity and the interpolation error estimate and (5.17), it follows that

\[
c \| u_h - I_h u \|_{a,h}^2 \leq a_h (u_h - I_h u, u_h - I_h u) = a_h (u - I_h u, u_h - I_h u) \leq C_0 \| u_h - I_h u \|_{a,h} \| u - I_h u \|_{a,h} + C_1 h R(u) \| u_h - I_h u \|_{a,h}.
\]

So we have

\[
\| u_h - I_h u \|_{a,h} \leq C \| u - I_h u \|_{a,h} + C h R(u).
\]

Hence

\[
\| u - u_h \|_{a,h} \leq C \| u - I_h u \|_{a,h} + C h R(u).
\]

Combining with Theorem 5.5 we have

\[
\| u - u_h \|_{a,h} \leq C h (2 \mu)^{1/2} \| u \|_{\tilde{H}^2(\Omega)} + \lambda_M \| \text{div} u \|_{\tilde{H}^1(\Omega)}).
\]

If the elliptic regularity estimate (H1) holds, then we have

\[
\| u - u_h \|_{a,h} \leq C h \| f \|_{L^2(\Omega)}.
\]

We now show the \( L^2 \)-norm estimate:

**Theorem 5.7.** Under the hypothesis (H1), we have

\[
\| u - u_h \|_{L^2(\Omega)} \leq C h^2 \| f \|_{L^2(\Omega)}.
\]

**Proof.** For a given \( g \in (L^2(\Omega))^2 \), define \( z \) as the solution of the dual problem:

\[
- \text{div} \sigma(z) = g \quad \text{in} \quad \Omega^s \quad (s = +, -), \quad (5.18)
\]

\[
[z]_F = 0, \quad (5.19)
\]

\[
[\sigma(z) \cdot n]_F = 0, \quad (5.20)
\]

\[
z = 0 \quad \text{on} \quad \partial \Omega. \quad (5.21)
\]

Then we have by (H1)

\[
2 \mu \| z \|_{\tilde{H}^2(\Omega)} + \lambda \| \text{div} z \|_{\tilde{H}^1(\Omega)} \leq C \| g \|_{L^2(\Omega)}. \quad (5.22)
\]
Let \( z_h \) be the corresponding IFEM solution and let \( e = u - u_h \). Then we have
\[
(u - u_h, g) = \sum_{T \in T_h} \int_T 2\mu \epsilon(u - u_h) : \epsilon(z) \, dx + \sum_{T \in T_h} \int_T \lambda \mathrm{div}(u - u_h) \mathrm{div} z \, dx \\
- \sum_{T \in T_h} \int_{\partial T} \sigma(z) \cdot n(u - u_h) \, ds \\
= a_h(u - u_h, z - z_h) + a_h(u - u_h, z_h) - \sum_{T \in T_h} \int_{\partial T} \sigma(z) \cdot n(u - u_h) \, ds \\
- \sum_{e \in e} \int_{e} \frac{2\mu}{h} |u| |v_h| \, ds \\
= a_h(u - u_h, z - z_h) + \sum_{T \in T_h} \int_{\partial T} \sigma(u) \cdot n z_h \, ds - \sum_{T \in T_h} \int_{\partial T} \sigma(z) \cdot n(u - u_h) \, ds \\
= : I + II + III.
\]
By the continuity and Theorem 5.6,
\[
|I| \leq C\|u - u_h\|_{a,h} \|z - z_h\|_{a,h} \leq Ch\|u - u_h\|_{a,h} \|g\|_{L^2(\Omega)} \\
\leq Ch^2\|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.
\]
\[
|II| = \sum_{T \in T_h} \int_{\partial T} \sigma(u) \cdot n(z_h - z) \, ds \\
\leq \sum_{T \in T_h} \int_{\partial T} (\sigma(u) - \sigma(u)) \cdot n(z_h - z) \, ds \\
\leq Ch \sum_{T \in T_h} \|u\|_{H^2(\Omega)} \|z_h - z\|_{1,T} \\
\leq Ch^2\|u\|_{H^2(\Omega)} \|g\|_{L^2(\Omega)}.
\]
\[
|III| \leq \sum_{T \in T_h} \int_{\partial T} (\sigma(z) - \sigma(z)) \cdot n(u - u_h) \, ds \\
\leq \sum_{T \in T_h} Ch\|\sigma(z)\|_{1,T} \|u - u_h\|_{1,T} \\
\leq \sum_{T \in T_h} Ch^2\|g\|_{0,T} \|f\|_{0,T} \\
\leq Ch^2\|\sigma(z)\|_{1,T} \|g\|_{L^2(\Omega)}.
\]
Hence the proof is complete. \( \Box \)

6. Numerical results. In this section we present numerical examples. We let the domain \( \Omega = (-1, 1) \times (-1, 1) \) be partitioned into uniform right triangles having step size \( h \). Let \( \Omega^+ = \Omega \cap \{ (x, y) | L(x, y) > 0 \} \), \( \Omega^- = \Omega \cap \{ (x, y) | L(x, y) < 0 \} \), where \( L(x, y) = x^2 + y^2 - r_0^2 = 0 \) represents the interface. The exact solution is chosen as
\[
u = \left( \frac{1}{\mu} (x^2 + y^2 - r_0^2) x, \frac{1}{\mu} (x^2 + y^2 - r_0^2) y \right)
\]
with various values of $\mu$ and $\lambda$.

**Example 6.1.** We choose $\mu^- = 1$, $\mu^+ = 100, \lambda = 5\mu$ and see the optimal order of convergence in $H^1, L^2$ and divergence norms.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1/h & \|u - u_h\|_0 & \text{order} & \|u - u_h\|_{1,h} & \text{order} & \|\text{div } u - \text{div } u_h\|_0 & \text{order} \\
\hline
8 & 2.27e-3 & 1.371 & 4.464e-2 & 0.911 & 6.873e-1 & 0.678 \\
16 & 8.294e-4 & 1.453 & 2.184e-2 & 1.031 & 4.430e-2 & 0.634 \\
32 & 2.379e-4 & 1.801 & 1.113e-2 & 0.972 & 2.468e-2 & 0.844 \\
64 & 6.396e-5 & 1.895 & 5.667e-3 & 0.975 & 1.283e-2 & 0.944 \\
128 & 1.666e-5 & 1.941 & 2.848e-3 & 0.993 & 6.528e-3 & 0.974 \\
256 & 4.257e-6 & 1.969 & 1.428e-3 & 0.996 & 3.290e-3 & 0.989 \\
\hline
\end{array}
\]

**Table 6.1**

$\mu^- = 1$, $\mu^+ = 100, \lambda = 5\mu$

**Example 6.2.** We test the case of $\mu^- = 1$, $\mu^+ = 10, \lambda = 5\mu$. We observe similar optimal convergence rates for all norms.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1/h & \|u - u_h\|_0 & \text{order} & \|u - u_h\|_{1,h} & \text{order} & \|\text{div } u - \text{div } u_h\|_0 & \text{order} \\
\hline
8 & 5.032e-3 & 1.778 & 5.960e-2 & 1.051 & 1.231e-1 & 0.934 \\
16 & 1.320e-3 & 1.930 & 2.922e-2 & 1.028 & 6.433e-2 & 0.936 \\
32 & 3.425e-4 & 1.947 & 1.468e-2 & 0.994 & 3.345e-2 & 0.944 \\
64 & 8.907e-5 & 1.943 & 7.340e-3 & 0.988 & 1.709e-2 & 0.969 \\
128 & 2.280e-5 & 1.966 & 3.716e-3 & 0.993 & 8.649e-3 & 0.982 \\
256 & 6.123e-6 & 1.897 & 1.870e-3 & 0.991 & 4.385e-3 & 0.980 \\
\hline
\end{array}
\]

**Table 6.2**

$\mu^- = 1$, $\mu^+ = 10, \lambda = 5\mu$

**Example 6.3** (Nearly incompressible case 1). We let $\mu^- = 1$, $\mu^+ = 10, \lambda = 100\mu$ so that the Poisson ratio is $\nu = 0.495$. No locking phenomena occurs.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
1/h & \|u - u_h\|_0 & \text{order} & \|u - u_h\|_{1,h} & \text{order} & \|\text{div } u - \text{div } u_h\|_0 & \text{order} \\
\hline
8 & 1.089e-1 & 1.750 & 6.947e-2 & 1.456 & 2.002e-0 & 0.868 \\
16 & 2.950e-2 & 1.885 & 3.187e-2 & 1.124 & 1.059e-0 & 0.918 \\
32 & 7.720e-3 & 1.934 & 1.540e-2 & 1.049 & 5.553e-1 & 0.937 \\
64 & 1.990e-3 & 1.956 & 7.581e-3 & 1.022 & 2.823e-1 & 0.971 \\
128 & 5.033e-4 & 1.983 & 3.752e-3 & 1.015 & 1.424e-1 & 0.987 \\
256 & 1.264e-4 & 1.993 & 1.869e-3 & 1.006 & 7.147e-2 & 0.995 \\
\hline
\end{array}
\]

**Table 6.3**

$\mu^- = 1$, $\mu^+ = 10, \lambda = 100\mu$

**Example 6.4** (Nearly incompressible case 2). We let $\mu^- = 1$, $\mu^+ = 10, \lambda = 1000\mu$ so that the Poisson ratio is $\nu = 0.4995$. Still, no locking phenomena occurs.

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| $1/h$ | $\|u - u_h\|_0$ | order | $\|u - u_h\|_{1,h}$ | order | $\|\text{div}u - \text{div}u_h\|_0$ | order |
|-------|-----------------|-------|-----------------|-------|-------------------------------|-------|
|      8 | 1.371e-1        | 1.926 | 6.751e-2        | 1.390 | 2.819e-0                      | 0.958 |
|     16 | 3.660e-2        | 1.905 | 3.124e-2        | 1.112 | 1.451e-0                      | 0.983 |
|     32 | 9.512e-3        | 1.944 | 1.536e-2        | 1.024 | 7.340e-1                      | 0.984 |
|     64 | 2.470e-3        | 1.945 | 7.625e-3        | 1.010 | 3.680e-1                      | 0.996 |
|    128 | 6.235e-4        | 1.986 | 3.782e-3        | 1.012 | 1.835e-1                      | 1.004 |
|    256 | 1.575e-4        | 1.995 | 1.886e-3        | 1.004 | 9.155e-2                      | 1.003 |

$\mu^- = 1$, $\mu^+ = 10$, $\lambda = 1000\mu$

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