Analysis of an Embedded-Hybridizable Discontinuous Galerkin Method for Biot’s Consolidation Model

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Abstract
We present an embedded-hybridizable discontinuous Galerkin finite element method for the total pressure formulation of the quasi-static poroelasticity model. Although the displacement and the Darcy velocity are approximated by discontinuous piece-wise polynomials, $H(\text{div})$-conformity of these unknowns is enforced by Lagrange multipliers. The semi-discrete problem is shown to be stable and the fully discrete problem is shown to be well-posed. Additionally, space-time a priori error estimates are derived, and confirmed by numerical examples, that show that the proposed discretization is free of volumetric locking.

Keywords Biot’s consolidation model · Poroelasticity · Discontinuous Galerkin · Finite element methods · Hybridization

Mathematics Subject Classification 65M12 · 65M15 · 65M60 · 76S99 · 74B99

1 Introduction
Poroelasticity models are systems of partial differential equation that describe the physics of deformable porous media saturated by fluids. They were originally developed for geophysics applications in petroleum engineering but nowadays they are also widely used for biomechanical modeling. The first poroelasticity models were derived by Biot [4, 5]. Since then, mathematical properties and numerical methods for these models have been widely studied. Here we give a brief literature review.
Early studies on linear poroelasticity models include well-posedness analysis and finite element discretizations for quasi-static [46, 47] and dynamic [42, 43, 52] models. For quasi-static models with incompressible elastic grains, Murad et al. [33] observed spurious pressure oscillations of certain finite element discretizations for small time and studied their asymptotic behavior. Phillips and Wheeler [36] connected these pressure oscillations to volumetric locking due to incompressibility of the displacement. They further developed numerical methods in [35, 36] coupling mixed methods and discontinuous Galerkin methods that do not show pressure oscillations. Yi [49–51] proposed numerical methods coupling mixed and nonconforming finite elements that are also free of pressure oscillations. An analysis to address the volumetric locking problem for poroelasticity was first presented in [29] adopting mixed methods for linear elasticity. Various numerical methods avoiding this locking problem have since been studied using nonconforming or stabilized finite elements [6, 24, 30, 41], the total pressure formulation [17, 31, 34], and exactly divergence-free finite element spaces [22, 25].

Discontinuous Galerkin methods are known to be computationally expensive. A remedy for this was provided by Cockburn et al. [12] by introducing the hybridizable discontinuous Galerkin (HDG) framework for elliptic problems. Indeed, element unknowns can be eliminated from the problem resulting in a global problem for facet unknowns only. The number of globally coupled degrees-of-freedom can be reduced even further using the embedded discontinuous Galerkin (EDG) framework [13, 20]; where the HDG method uses a discontinuous trace approximation, the EDG method uses a continuous trace approximation. HDG, and related hybrid high-order (HHO), methods have recently been introduced for the poroelasticity problem [7, 18, 28]. These discretizations consider the primal bilinear form for linear elasticity. In contrast, in this paper we adopt the total pressure formulation [31, 34] and present novel HDG and EDG-HDG methods for the quasi-static poroelasticity models. (It is possible to also consider an EDG method for the poroelasticity model, however, such a discretization is sub-optimal.) The total pressure formulation provides a natural decoupling of the linear elasticity and Darcy equations in the incompressible limit. Indeed, in this limit our discretizations reduce to the exactly divergence-free HDG and EDG-HDG discretizations of [37, 39] for the Stokes problem and the hybridized formulation of [3] for the Darcy problem. We further remark that the total pressure formulation has been applied also in the context of magma/mantle dynamics problems [26, 27] where it was shown to be advantageous in the context of coupled physics problems beyond quasi-static poroelasticity problems.

We present an analysis of the proposed HDG and EDG-HDG methods in which we show that the space-time discretizations are well-posed. We further determine an a priori error estimate for all unknowns that is robust in the incompressible limit and for arbitrarily small specific storage coefficient. We remark that the standard approach of analyzing time-dependent problems is to use discrete Grönewall inequalities. However, this results in error bounds with a coefficient that grows exponentially in time. We present an alternative approach that avoids this exponential term.

The remainder of this paper is organized as follows. We present Biot’s consolidation model in Sect. 2. The HDG and EDG-HDG methods for Biot’s model is presented in Sect. 3 together with a stability proof for the semi-discrete problem. Well-posedness and a priori error estimates for the fully discrete problem are shown in Sect. 4. The analysis is verified by numerical examples in Sect. 5 and conclusions are drawn in Sect. 6.
2 Biot’s Consolidation Model

To introduce Biot’s consolidation model, let us introduce the following notation. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded polygonal domain with a boundary partitioned as $\partial \Omega = \Gamma_p \cup \Gamma_f$ and $\partial \Omega = \Gamma_D \cup \Gamma_T$, where $\Gamma_p \cap \Gamma_f = \emptyset$, $|\Gamma_p| > 0$, $\Gamma_D \cap \Gamma_T = \emptyset$, and $|\Gamma_D| > 0$. We denote the unit outward normal to $\partial \Omega$ by $n$ and we denote by $I = (0, T]$ the time interval of interest.

Let $f : \Omega \times I \rightarrow \mathbb{R}^d$ be a given body force and let $g : \Omega \times I \rightarrow \mathbb{R}$ be a given source/sink term. Furthermore, let $\kappa > 0$ be a scalar constant that represents the permeability of the porous media, $c_0 \geq 0$ the specific storage coefficient, and $0 < \alpha < 1$ the Biot–Willis constant. Denoting Young’s modulus of elasticity by $E > 0$ and Poisson’s ratio by $0 < \nu < 1/2$, in the case of plane strain, the Lamé constants are given by $\lambda = E\nu/(1 + \nu)(1 - 2\nu)$ and $\mu = E/(2(1 + \nu))$.

Biot’s consolidation model describes a system of equations for the displacement of the porous media, $u : \Omega \times I \rightarrow \mathbb{R}^d$, and the pore pressure of the fluid $p : \Omega \times I \rightarrow \mathbb{R}$. Denoting by $\sigma = 2\mu \varepsilon(u) + \lambda \nabla \cdot u I - \alpha p I$ the total Cauchy stress, where $I$ is the $d \times d$-dimensional identity matrix, $\varepsilon(u) := (\nabla u + (\nabla u)^T)/2$ is the symmetric gradient in which $(\nabla u)^T$ is the transpose of $\nabla u$, this model is given by

$$-\nabla \cdot \sigma = f, \quad \partial_t (c_0 p + \alpha \nabla \cdot u) - \nabla \cdot (\kappa \nabla p) = g, \quad \text{in } \Omega \times I. \quad (2.1)$$

Following [31], by introducing the total pressure $p_T := -\lambda \nabla \cdot u + \alpha p$ and the Darcy velocity $z := -\kappa \nabla p$, we may write Biot’s consolidation model as a system of equations for $(u, p_T, z, p)$ such that

$$-\nabla \cdot 2\mu \varepsilon(u) + \nabla p_T = f \text{in } \Omega \times I, \quad (2.2a)$$

$$-\nabla \cdot u - \lambda^{-1}(p_T - \alpha p) = 0 \text{in } \Omega \times I, \quad (2.2b)$$

$$\partial_t (c_0 p + \lambda^{-1}\alpha(p_T - p_T)) + \nabla \cdot z = g \text{in } \Omega \times I, \quad (2.2c)$$

$$\kappa^{-1} z + \nabla p = 0 \text{in } \Omega \times I, \quad (2.2d)$$

which will be the formulation studied in this article. Noting that $\sigma = 2\mu \varepsilon(u) - p_T I$, we close the model by imposing the following boundary and initial conditions:

$$u = 0 \text{ on } \Gamma_D \times I, \quad (2.3a)$$

$$p = 0 \text{ on } \Gamma_p \times I, \quad (2.3b)$$

$$z \cdot n = 0 \text{ on } \Gamma_F \times I, \quad (2.3c)$$

$$\sigma n = 0 \text{ on } \Gamma_T \times I, \quad (2.3d)$$

$$p(x, 0) = p_0(x) \text{ in } \Omega, \quad (2.3e)$$

$$u(x, 0) = u_0(x) \text{ in } \Omega. \quad (2.3f)$$

In the remainder of this article we assume that $c_0, \kappa, \mu$ are constants on $\Omega$. We furthermore assume that there exists a $\nu_s$ such that $0 < \nu_s \leq 0.5$ on $\Omega$. As a consequence, $C_s \mu \leq \lambda$ with $C_s = 2\nu^*/(1 - 2\nu^*)$. 
3 The Embedded-Hybridizable Discontinuous Galerkin Method

3.1 Notation

On a Lipschitz domain $D$ in $\mathbb{R}^d$, we denote by $W^{l,p}(D)$ the usual Sobolev spaces for $l \geq 0$ and $1 \leq p \leq \infty$ (see, for example, [1]). When $p = 2$, we define on $H^l(D) = W^{l,2}(D)$ the norm $\| \cdot \|_{l,D}$ and semi-norm $|\cdot|_{l,D}$. We note that $L^2(D) = H^0(D)$ is the Lebesgue space of square integrable functions with norm $\| \cdot \|_{0,D}$ and inner product $(\cdot, \cdot)_D$. Vector-valued function spaces will be denoted by $[L^2(D)]^d$ and $[H^l(D)]^d$. On a set $S$ of positive $(d - 1)$-dimensional Lebesgue measure, $(\cdot, \cdot)_S$ will denote the $L^2$-inner product with $(d - 1)$-dimensional Lebesgue measure on $S$.

Let $X$ be a Banach space and $I = (0, T]$, $T > 0$ a time interval. We denote by $C^0(I; X)$ the space of continuous functions $f : I \rightarrow X$, which is equipped with the norm $\| f \|_{C^0(T, X)} := \sup_{t \in T} \| f(t) \|_X$. By $C^k(I; X)$, $k \geq 0$, we denote the space of continuous functions $f : I \rightarrow X$ such that $\partial_t^i f \in C^0(I, X)$ for $1 \leq i \leq k$. For $1 \leq p < \infty$, $W^{k,p}(I; X)$ is defined to be the closure of $C^k(I; X)$ with respect to the norm

$$
\| f \|_{W^{k,p}(I; X)} := \left( \int_0^T \sum_{i=0}^k \| \partial_t^i f(t) \|_X^p \right)^{1/p} dt.
$$

We note that for $k = 0$, $W^{k,p}(I; X) = L^p(I; X)$. For simplicity, we will use $\| f_1, f_2 \|_{W^{k,p}(I; X)}$ to denote $\| f_1 \|_{W^{k,p}(I; X)} + \| f_2 \|_{W^{k,p}(I; X)}$.

Let $T_h$ be a family of shape-regular simplicial triangulations of the domain $\Omega$. We will denote the diameter of an element $K \in T_h$ by $h_K$, the meshsize by $h := \max_{K \in T_h} h_K$, and the sets of interior facets and facets that lie on $\Gamma_D$, $\Gamma_F$, and $\Gamma_T$ by, respectively, $\mathcal{F}_h$, $\mathcal{F}_h^D$, $\mathcal{F}_h^F$, and $\mathcal{F}_h^T$. The set of all facets is denoted by $\mathcal{F}_h$ and their union is denoted by $\Gamma_h$. On the boundary of an element $K$, we denote by $n_K$ the outward unit normal vector, although, where no confusion will occur we drop the subscript $K$. On the mesh and skeleton we define the inner products

$$(\phi, \psi)_\Omega := \sum_{K \in T_h} (\phi, \psi)_K, \quad (\phi, \psi)_{\partial \Omega} := \sum_{K \in T_h} (\phi, \psi)_{\partial K}, \quad \text{if } \phi, \psi \text{ are scalar},$$

$$(\phi, \psi)_\Omega := \sum_{i=1}^d (\phi_i, \psi_i)_\Omega, \quad (\phi, \psi)_{\partial \Omega} := \sum_{i=1}^d (\phi_i, \psi_i)_{\partial \Omega}, \quad \text{if } \phi, \psi \text{ are vector-valued}.$$

The norms induced by these inner products are denoted by $\| \cdot \|_\Omega$ and $\| \cdot \|_{\partial \Omega}$, respectively.

Sets of polynomials of degree not larger than $l \geq 0$ defined on, respectively, an element $K \in T_h$ and a facet $F \in \mathcal{F}_h$ will be denoted by $\mathcal{P}_l(K)$ and $\mathcal{P}_l(F)$. As approximation spaces we then use:

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The analysis in this paper holds for both the HDG and EDG-HDG methods. For notational convenience, element and facet function pairs will be denoted by boldface, for example, \( \mathbf{Q} \). Norms on the extended spaces will be denoted by \( \| \cdot \|_Q \) and \( \| \cdot \|_P \), respectively.

The HDG method seeks an approximation in the form:

\[
\psi_h = (\psi_h, \bar{\psi}_h) \in Q_h := Q_h \times \bar{Q}_h,
\]

and it will also be useful to define \( X_h := V_h \times Q_h \times V_h \times Q_h \).

**Remark 1** The HDG method seeks an approximation in \( X_h \) with \( V_h, \bar{V}_h, Q_h, \bar{Q}_h, \) and \( \bar{Q}_h \) defined in Eq. 3.1. If \( \bar{V}_h \) is replaced by \( \bar{V}_h \cap C^0(\Gamma_0) \) then we obtain the EDG-HDG method. The analysis in this paper holds for both the HDG and EDG-HDG methods. For notational purposes, in the analysis, \( X_h \) and \( V_h \) will refer both to the HDG and EDG-HDG spaces.

Denoting by \( \bar{V}, \bar{Q}, \bar{Z}, \) and \( \bar{Q}_0 \) the trace spaces of, respectively, \( V, Q, Z, \) and \( Q_0 \) to the mesh skeleton, we introduce the extended spaces

\[
\begin{align*}
V(h) := V_h + V \times \bar{V}, & \quad Z(h) := V_h + Z, \\
Q(h) := Q_h + Q \times \bar{Q}, & \quad Q_0(h) := Q_h + Q_0 \times \bar{Q}_0.
\end{align*}
\]

Norms on the extended spaces \( Q(h), Q_0(h), \) and \( V(h) \) are defined as:

\[
\begin{align*}
\| q_h \|^2_Q := \| q_h \|^2_{\Omega} + \sum_{K \in T_h} h_K \| \bar{q}_h \|^2_{\partial K} & \quad \forall q_h \in Q(h), \\
\| v \|^2_v := \| \varepsilon(v) \|^2_{\Omega} + \sum_{K \in T_h} h_K^{-1} \| v - \bar{v} \|^2_{\partial K} & \quad \forall v \in V(h), \\
\| v \|^2_{v'} := \| v \|^2_v + \sum_{K \in T_h} h_K^2 \| \nabla v \|^2_{\partial K} & \quad \forall v \in V(h),
\end{align*}
\]

where \( \| \cdot \|_Q \) is the discrete \( L^2 \)-norm on the pressure space and \( \| \cdot \|_v \) is the discrete \( H^1 \)-norm on the velocity space. In our analysis we further require the norm \( \| \cdot \|_{v'} \); this norm is equivalent to \( \| \cdot \|_{v} \) on \( V_h \), see [48], but stronger than \( \| \cdot \|_{v} \) on \( V(h) \). To conclude this section, we remark that \( C > 0 \) will denote a constant independent of \( h \) and the model parameters \( c_0, \mu, \lambda, \kappa \).

### 3.2 The Semi-Discrete Problem

In this section, we present the semi-discrete problem and provide an energy estimate for this discretization. The fully-discrete problem is presented in Sect. 3.3 which is analysed in Sect. 4.

The semi-discrete HDG method for Biot’s consolidation model Eqs. 2.2, 2.3 is given by: Find \( (u_h, z_h) \in C^0(I; V_h \times V_h) \) and \( (p_{Th}, p_h) \in C^1(I; Q_h \times Q_0 h) \) such that for all \( (v_h, q_{Th}, w_h, q_h) \in V_h \times Q_h \times V_h \times Q_0 h \):

\[
a_h(u_h, v_h) + b_h(v_h, p_{Th}) = (f, v_h)_{\Omega},
\]

where

\[
\begin{align*}
a_h(u_h, v_h) &= a_h(u_h, v_h) + b_h(v_h, p_{Th}) = (f, v_h)_{\Omega}, \\
b_h(v_h, p_{Th}) &= b_h(v_h, p_{Th}) = (g, v_h)_{\Omega}, \\
c_h(w_h, q_{Th}) &= c_h(w_h, q_{Th}) = (j, w_h)_{\Omega}.
\end{align*}
\]
\[ b_h(u_h, q_{Th}) - \left( \kappa^{-1}(p_{Th}, \alpha p_h), q_{Th}) \right)_\Omega = 0, \quad (3.2b) \]
\[ \left( \kappa^{-1}z_h, w_h \right)_\Omega + b_h((w_h, 0), p_h) = 0, \quad (3.2c) \]
\[ \left( \vartheta \left( c_0 p_h + \kappa^{-1} \alpha (\alpha p_h - p_{Th}) \right), q_h \right)_\Omega - b_h((z_h, 0), q_h) = (g, q_h)_\Omega, \quad (3.2d) \]

where
\[ a_h(u, v) := (2\mu \varepsilon(u), \varepsilon(v))_\Omega + \left( \frac{2\beta}{h_k} (u - \bar{u}), v - \bar{v} \right)_\partial \Gamma_h \]
\[ - (2\mu \varepsilon(u)n, v - \bar{v})_\partial \Gamma_h - \left( 2\mu \varepsilon(v)n, u - \bar{u} \right)_\partial \Gamma_h, \quad (3.3a) \]
\[ b_h(v, q) := -(q, \nabla \cdot v)_\Omega + \langle \bar{q}, (v - \bar{v}) \cdot n \rangle_\partial \Gamma_h. \quad (3.3b) \]

To analyze the HDG and EDG-HDG methods, let us recall some properties of the bilinear forms \( a_h \) and \( b_h \). It was shown in [37, Lemma 4.2] and [11, Lemma 2] that there exist constants \( C \) and \( \beta_0 > 0 \) such that for \( \beta > \beta_0 \),
\[ a_h(v_h, v_h) \geq C \mu ||v_h||^2_v \quad \forall v_h \in V_h. \quad (3.4) \]

Additionally, \( a_h \) satisfies the following continuity result [11, Lemma 3]:
\[ a_h(u, v) \leq C \mu ||u||^2_v ||v||^2_v \quad \forall u, v \in V(h). \quad (3.5) \]

The bilinear form \( b_h \) satisfies the following stability results:
\[
\inf_{q_h \in Q_h} \sup_{0 \neq w_h \in V_h} \frac{b_h(v_h, q_h)}{||v_h||_v ||q_h||_q} \geq C, \quad (3.6a)
\]
\[ \inf_{q_h \in Q_h} \sup_{0 \neq w_h \in V_h} \frac{b_h((w_h, 0), q_h)}{||w_h||_\Omega ||q_h||_q} \geq C, \quad (3.6b) \]

where the first inequality was shown in [38, Lemma 1] and [39, Lemma 8] and the second is proven in Appendix 1. Continuity of the bilinear form \( b_h \) was established in [37]:
\[ |b_h(v, q)| \leq C ||v||_v ||q||_q \quad \forall v \in V(h), q \in Q(h). \quad (3.7) \]

**Lemma 1** (Consistency) Let \((u, p_T, z, p)\) be a solution to Eqs. 2.2, 2.3 and let \( \bar{u}, \bar{p}_T \), and \( \bar{p} \) be the traces of, respectively, \( u, p_T \), and \( p \) on the mesh skeleton. Then \((u, p_T, z, p)\) satisfies Eq. 3.2.

**Proof** The proof is standard and follows by integration by parts, smoothness of the solution to Eqs. 2.2, 2.3, single-valuedness of \( \bar{v}_h \) and \( \bar{q}_h \) on element boundaries, and using that \( \bar{v}_h = 0 \) on \( \Gamma_D \) and \( \bar{q}_h = 0 \) on \( \Gamma_P \). \( \square \)

The following theorem now shows energy stability of the semi-discrete problem.

**Theorem 1** (Stability) Suppose that \((u_h, p_{Th}, z_h, p_h) \in C^1(I; X_h) \) is a solution to Eq. 3.2 with \( f \in W^{1,1} \) and \( g \in L^2(\Omega) \). Let \( X(t) \geq 0 \) and \( Y(t) \geq 0 \) be defined by:
\[
X(t)^2 = a_h(u_h(t), u_h(t)) + \left( \kappa^{-1}(p_{Th}(t), \alpha p_h(t)), p_{Th}(t) - \alpha p_h(t) \right)_\Omega + (c_0 p_h(t), p_h(t))_\Omega,
\]
\[ Y(t)^2 = (\kappa^{-1}z_h(t), z_h(t))_\Omega. \]

Then, there exists \( C > 0 \), independent of \( t > 0 \), such that
\[ X(t) \leq X(0) + CF(t), \quad (3.8a) \]
\[
\left( \int_0^t Y(s)^2 \, ds \right)^{\frac{1}{2}} \leq CX(0) + CF(t) \tag{3.8b}
\]

with
\[
F(t) := \left[ \mu^{-\frac{1}{2}} \left( \max_{0 \leq s \leq t} \|f(s)\|_{\Omega} + \int_0^t \|\partial_t f(s)\|_{\Omega} \, ds \right) + \left( \int_0^t \|g(s)\|^2_{\Omega} \, ds \right)^{\frac{1}{2}} \right].
\]

**Proof** We first note that by the inf-sup condition Eqs. 3.6b, 3.2c, and the Cauchy–Schwarz inequality,
\[
\|p_h(t)\|_{\Omega} \leq C \sup_{0 \neq w_h \in V_h} \frac{|b_h((w_h, 0), p_h)|}{\|w_h\|_{\Omega}} \leq C \sup_{0 \neq w_h \in V_h} \frac{|(\kappa^{-1} z_h, w_h)|}{\|w_h\|_{\Omega}} \leq CK^{-1} \|z_h(t)\|_{\Omega}.
\tag{3.9}
\]

Now, in Eqs. 3.2a, 3.2c, 3.2d set \((v_h, w_h, q_h) = (\partial_t u_h, z_h, p_h)\). Take the time derivative of Eq. 3.2b and set \(q_{Th} = -p_{Th}\). Adding the resulting equations we find:
\[
\frac{1}{2} \frac{d}{dt} X(t)^2 + Y(t)^2 = (f(t), \partial_t u_h(t))_{\Omega} + (g(t), p_h(t))_{\Omega}.
\tag{3.10}
\]

Integrating Eq. 3.10 in time from 0 to \(t\) results in
\[
\frac{1}{2} (X(t)^2 - X(0)^2) + \int_0^t Y(s)^2 \, ds = \int_0^t [(f(s), \partial_t u_h(s))_{\Omega} + (g(s), p_h(s))_{\Omega}] \, ds.
\]

Integration by parts, Young’s inequality and Eq. 3.9 imply
\[
\frac{1}{2} (X(t)^2 - X(0)^2) + \int_0^t Y(s)^2 \, ds \leq (f(t), u_h(t))_{\Omega} - (f(0), u_h(0))_{\Omega}
- \int_0^t (\partial_t f(s), u_h(s))_{\Omega} \, ds
+ C \int_0^t \|g(s)\|^2_{\Omega} \, ds + \frac{1}{2} \int_0^t Y(s)^2 \, ds.
\]

A discrete Poincaré inequality (cf. [14, Corollary 5.4] or [8]), a discrete Korn’s inequality (cf. [9]), and the coercivity of \(a_h\) Eq. 3.4 imply that
\[
\|u_h(t)\|_{\Omega} \leq C \mu^{-1/2} a_h(u_h, u_h)^{1/2} \leq C \mu^{-1/2} X(t).
\]

Therefore, by the Cauchy-Schwarz inequality, for any \(t \geq 0\),
\[
X(t)^2 + \int_0^t Y(s)^2 \, ds \leq X(0)^2 + C \mu^{-1/2} \left( \|f(t)\|_{\Omega} X(t) + \|f(0)\|_{\Omega} X(0) \right)
+ C \mu^{-1/2} \int_0^t \|\partial_t f(s)\|_{\Omega} X(s) \, ds + C \int_0^t \|g(s)\|^2_{\Omega} \, ds.
\tag{3.11}
\]

To show Eq. 3.8a it suffices to show
\[
X(t) \leq X(0) + CF(t), \quad t \in A := \{ t \in I : \max_{0 \leq s \leq t} X(s) = X(t) > 0 \}.
\tag{3.12}
\]
This is because for \( t \notin \mathcal{A} \) there exists a \( 0 \leq t_M < t, \ t_M \in \mathcal{A} \) such that \( X(t) < X(t_M) \), and therefore, by Eq. 3.12,

\[
X(t) \leq \max_{0 \leq s \leq t} X(s) = \max_{0 \leq s \leq t_M} X(s) = X(t_M) \leq X(0) + CF(t_M) \\
\leq X(0) + CF(t).
\]

We therefore now proceed to prove Eq. 3.8a for \( t \in \mathcal{A} \). From Eqs. 3.11, 3.12, we find

\[
X(t)^2 + \int_0^t Y(s)^2 \, ds \\
\leq \left( X(0) + C \mu^{-1/2} \left( \| f(t) \|_{\Omega} + \| f(0) \|_{\Omega} + \int_0^t \| \partial_t f(s) \|_{\Omega} \, ds \right) \right) X(t) \\
+ C \int_0^t \| g(s) \|^2_{\Omega} \, ds.
\]

Define \( \alpha(t) := \left( C \int_0^t \| g(s) \|^2_{\Omega} \, ds \right)^{1/2} > 0 \). If \( \alpha(t) \leq X(t) \), dividing Eq. 3.13 by \( X(t) \) implies

\[
X(t) \leq X(0) + C \mu^{-1/2} \left( \| f(t) \|_{\Omega} + \| f(0) \|_{\Omega} + \int_0^t \| \partial_t f(s) \|_{\Omega} \, ds \right) + \alpha(t).
\]

Note that this inequality holds trivially if \( X(t) < \alpha(t) \). Proceeding, we find

\[
X(t) \leq X(0) + C \mu^{-1/2} \left( 2 \max_{0 \leq s \leq t} \| f(s) \|_{\Omega} + \int_0^t \| \partial_t f(s) \|_{\Omega} \, ds \right) + \alpha(t) \\
= X(0) + C \mu^{-1/2} \left( 2 \max_{0 \leq s \leq t} \| f(s) \|_{\Omega} + \int_0^t \| \partial_t f(s) \|_{\Omega} \, ds \right) + C \left( \int_0^t \| g(s) \|^2_{\Omega} \, ds \right)^{1/2},
\]

so that Eq. 3.8a follows.

For Eq. 3.8b, if \( t \in \mathcal{A} \), then Eq. 3.8b follows by combining Eqs. 3.8a and 3.13. For general \( t \), there exists \( 0 \leq t_M < t \) such that \( X(t) < X(t_M) = \max_{0 \leq s \leq t} X(s) \). A direct modification of Eq. 3.11 using \( X(t) < X(t_M) \) gives

\[
X(t)^2 + \int_0^t Y(s)^2 \, ds \\
\leq \left( X(0) + C \mu^{-1/2} \left( \| f(t) \|_{\Omega} + \| f(0) \|_{\Omega} + \int_0^t \| \partial_t f(s) \|_{\Omega} \, ds \right) \right) X(t_M) \\
+ C \int_0^t \| g(s) \|^2_{\Omega} \, ds.
\]

Then, Eq. 3.8b follows by applying Eq. 3.8a to \( X(t_M) \). \( \square \)

### 3.3 The Fully Discrete Problem

To define the fully discrete scheme, let \( \{ t_m \}_{0 \leq m \leq N} \) be a uniform partition of \( I \) and let \( \Delta t > 0 \) be the corresponding time step. We will denote the value of a function \( f(t) \) at \( t = t_m \) by \( f^m := f(t_m) \). For a sequence \( \{ f^m \}_{m \geq 1} \), \( d_t f^m := \frac{f^m - f^{m-1}}{\Delta t} \) defines a first order difference
operator. Using the backward Euler time stepping, the fully discrete problem reads: Find $(u_h^{m+1}, p_T^{m+1}, z_h^{m+1}, p_h^{m+1}) \in X_h$, with $m \geq 0$, such that
\begin{equation}
a_h(u_h^{m+1}, v_h) + b_h(v_h, p_T^{m+1}) = (f^{m+1}, v_h)_\Omega, \tag{3.14a}
\end{equation}
\begin{equation}
b_h(u_h^{m+1}, q_T h) + \lambda^{-1}(\alpha p_h^{m+1} - p_T^{m+1}, q_T h)_\Omega = 0, \tag{3.14b}
\end{equation}
\begin{equation}
\left(\kappa^{-1} z_h^{m+1}, w_h\right)_\Omega + b_h((w_h, 0), p_h^{m+1}) = 0, \tag{3.14c}
\end{equation}
\begin{equation}
\frac{1}{\Delta t} \left( (c_0 p_h^m, q_h)_\Omega + \lambda^{-1} (\alpha p_h^m - p_T^m, \alpha q_h)_\Omega \right) - b_h((z_h^{m+1}, 0), q_h)_\Omega,
\end{equation}
for all $(v_h, q_T h, w_h, q_h) \in X_h$. We first show that Eq. 3.14 is well-posed.

**Theorem 2** There exists a unique solution to Eq. 3.14.

**Proof** It is sufficient to show that if the data is equal to zero then the solution is zero. As such, suppose that $f^{m+1} = 0$, $g^{m+1} = 0$, $p_T^{m+1} = 0$, and $p_h^m = 0$. Then, setting $v_h = u_h^{m+1}$, $q_T h = -p_T^{m+1}$, $w_h = \Delta t z_h^{m+1}$, and $q_h = \Delta t p_h^{m+1}$ in Eq. 3.14 and adding the equations, we obtain:
\begin{equation}
a_h(u_h^{m+1}, u_h^{m+1}) + c_0 \|p_h^m\|_{\Omega}^2 + \lambda^{-1} \|\alpha p_h^m - p_T^m\|_{\Omega}^2 + \kappa^{-1} \Delta t \|p_h^{m+1}\|_{\Omega}^2 = 0.
\end{equation}
Coercivity of $a_h$, Eq. 3.4, positivity of $\kappa$ and $\lambda$, and nonnegativity of $c_0$ directly imply that $u_h^{m+1} = 0$ and $z_h^{m+1} = 0$. Substituting $u_h^{m+1} = 0$ in Eq. 3.14a, $p_T^{m+1} = 0$ follows from the inf-sup condition Eq. 3.6a. This then implies $p_h^{m+1} = 0$ since $\alpha, \lambda > 0$.

Using a BDM local lifting of the normal trace [15, Proposition 2.10], there exists $\tilde{w}_h \in V_h$ such that $(\tilde{w}_h \cdot n, p_h^{m+1})_{\partial T_h} = \|p_h^{m+1}\|_{\partial T_h}^2$. Setting $z_h^{m+1} = 0$, $p_h^{m+1} = 0$ and choosing $w_h = \tilde{w}_h$ in Eq. 3.14c, we obtain $p_h^{m+1} = 0$. This completes the proof. \(\square\)

Let us also note that the fully-discrete scheme Eq. 3.14 results in divergence-conforming solutions for the displacement $u_h^m$ and velocity $z_h^m$. To show this, let us first introduce the usual jump operator for a vector function $f \in \mathbb{R}^d$ on a facet $F$. Let $K^+$ and $K^-$ be two neighboring elements in $T_h$ such that $F = K^+ \cap \partial K^-$. Let $n^+$ and $n^-$ be the outward unit normal vectors on the boundaries of, respectively, $K^+$ and $K^-$. The jump operator on $F$ is then defined as $[f \cdot n] = f|_{K^+ \cdot n^+} + f|_{K^- \cdot n^-}$. If $F \subset \partial K$ is a boundary facet, then $[f \cdot n] = f \cdot n$. Now, set $u_h = 0$, $q_h = 0$, $w_h = 0$, and $q_T h = 0$ in Eq. 3.2 and note that since $u_h^m \cdot n \in P_k(F)$ and $u_h^m \cdot n = 0$ on $\Gamma_D$,
\begin{equation}
[u_h^m \cdot n] = 0, \forall x \in F, \quad \forall F \in \mathcal{F}_h \setminus \mathcal{F}_h^T, \tag{3.15a}
\end{equation}
\begin{equation}
u_h^m \cdot n = u_h^m \cdot n, \forall x \in F, \quad \forall F \in \mathcal{F}_h^T. \tag{3.15b}
\end{equation}
Similarly, by setting $v_h = 0$, $q_T h = 0$, $w_h = 0$, and $q_h = 0$ in Eq. 3.2, and noting that $z_h^m \cdot n \in P_k(F)$, we find that
\begin{equation}
z_h^m \cdot n = 0, \quad \forall x \in F, \quad \forall F \in \mathcal{F}_h \setminus \mathcal{F}_h^P. \tag{3.16}
\end{equation}

**4 A Priori Error Estimates**

To facilitate the a priori error analysis, we introduce various interpolation operators. First, let $\Pi_V : [H^1(\Omega)]^d \to V_h$ be the BDM interpolation operator [10, Section III.3], [21, Lemma 7] with the following interpolation estimate:
\[ \| z - \Pi V z \|_K \leq C h^\ell_K \| z \|_{\ell,K}, \quad 1 \leq \ell \leq k + 1. \]  

(4.1)

The elliptic projection operator, \( \Pi_{V}^{\text{ell}} := (\Pi_{V}^{\text{ell}}, \hat{\Pi}_{V}^{\text{ell}}) : [H^1_D(\Omega)]^d \rightarrow V_h \) is defined by:

\[ a_h(\Pi_{V}^{\text{ell}} u, v_h) = a_h((u, u), v_h), \quad \forall v_h \in V_h. \]

Standard a priori error estimate theory for second order elliptic equations imply

\[ a_h(u - \Pi_{V}^{\text{ell}} u, u - \Pi_{V}^{\text{ell}} u)^{1/2} \leq C \mu^{1/2} h^{-1} \| u \|_{\ell; \Omega}, \quad 1 \leq \ell \leq k + 1. \]  

(4.2)

By \( \Pi_{Q}, \hat{\Pi}_{Q}, \) and \( \hat{\Pi}_{Q}^0 \) we denote the \( L^2 \)-projections onto, respectively, \( Q_h \) and the trace spaces \( \hat{Q}_h \) and \( \hat{Q}_h^0 \). Given the interpolation/projection operators, the numerical initial data is set by imposing the interpolation/projection of continuous initial data as follows:

\[ ((u_0^0, \hat{u}_0^0), (\hat{p}_T^0, \hat{\bar{p}}_T^0), z_0^0, (p_0^0, \bar{p}_h^0)) \]

\[ = ((\Pi_{V}^{\text{ell}} u(0), \hat{\Pi}_{V}^{\text{ell}} u(0)), (\Pi_{Q} P T h(0), \hat{\Pi}_{Q} P T h(0)), \Pi_{V} z(0), (\Pi_{Q}^0 p(0), \hat{\Pi}_{Q}^0 p(0))). \]  

(4.3)

In the error analysis it will be convenient to split the error into approximation and interpolation errors:

\[ \omega - \omega_h = e^I - e^h, \quad \omega = u, p_T, z, p. \]  

(4.4a)

\[ \xi \big|_{\Gamma^0} - \bar{\xi}_h = e^I - e^h, \quad \xi = u, p_T, p. \]  

(4.4b)

where

\[ e^I_u = u - \Pi_{V}^{\text{ell}} u, e^h_u = u_h - \Pi_{V}^{\text{ell}} u, e^I_z = z - \Pi_{V} z, e^h_z = z_h - \Pi_{V} z, \]

\[ e^I_p = p - \Pi_{Q} P, e^h_p = p_h - \Pi_{Q} P, e^I_{p_T} = p_T - \Pi_{Q} P T, e^h_{p_T} = p_T h - \Pi_{Q} P T, \]

and where

\[ \bar{e}^I_u = u|_{\Gamma^0} - \hat{\Pi}_{V}^{\text{ell}} u, \bar{e}^h_u = \bar{u}_h - \hat{\Pi}_{V}^{\text{ell}} u, \bar{e}^I_p = p|_{\Gamma^0} - \hat{\Pi}_{Q}^0 p, \bar{e}^h_p = \bar{p}_h - \hat{\Pi}_{Q}^0 p, \]

\[ \bar{e}^I_{p_T} = p_T|_{\Gamma^0} - \hat{\Pi}_{Q} P T, \bar{e}^h_{p_T} = \bar{p}_T h - \hat{\Pi}_{Q} P T. \]

Following the convention introduced earlier in this paper, we use boldface notation for element/facet error pairs, i.e., \( e^I_\xi = (e^I_u, e^I_z) \) and \( e^h_\xi = (e^h_u, \bar{e}^h_z) \) for \( \xi = u, p_T, p. \)

It will also be useful to introduce the following error estimates: let \( \psi \) be a function defined on \([0, T] \times D\) that satisfies \( \| \partial_t \psi \|_{L^1(0,T; L^2(D))} + \| \partial_t \psi \|_{L^1(0,T; H^l(D))} < \infty \) for some domain \( D \subset \mathbb{R}^d \) and \( 1 \leq l \leq k \), then, as a consequence of Taylor’s theorem (see Appendix 2),

\[ \sum_{i=0}^{m} \Delta t \| \partial_t \psi^i - d_t \psi^i \|_{0,D} \leq \Delta t \| \partial_t \psi \|_{L^1(0,T; L^2(D))}, \]  

(4.5a)

\[ \Delta t \sum_{i=0}^{m} \| d_t e^I_{\psi}^i \|_{0,D} \leq C h^l \| \partial_t \psi \|_{L^1(0,T; H^l(D))}, \psi = p, p_T. \]  

(4.5b)

Lemma 2 Let \( \{A_i\}_i, \{B_i\}_i, \{E_i\}_i, \) and \( \{D_i\}_i \) be nonnegative sequences. Suppose these sequences satisfy

\[ A_m^2 + \sum_{i=0}^{m} B_i^2 \leq A_0^2 + \sum_{i=1}^{m} E_i A_i + \sum_{i=0}^{m} D_i, \]  

(4.6)
for all \( m \geq 0 \). Then for any \( m \geq 0 \),
\[
A_m \leq A_0 + \sum_{i=1}^{m} E_i + \left( \sum_{i=0}^{m} D_i \right)^{1/2},
\]
(4.7a)
\[
\left( \sum_{i=0}^{m} B_i^2 \right)^{1/2} \leq C (\| \mathbf{l} \|) A_0 + \sum_{i=1}^{m} E_i + \left( \sum_{i=0}^{m} D_i \right)^{1/2},
\]
(4.7b)
with \( C > 0 \) independent of \( m \).

**Proof** First, note that Eqs. 4.6 and 4.7a directly imply Eq. 4.7b. It is therefore sufficient to prove Eq. 4.7a. As in the proof of Theorem 1, we point out that to prove Eq. 4.7a it suffices to prove
\[
A_m \leq A_0 + \sum_{i=1}^{m} E_i + \left( \sum_{i=0}^{m} D_i \right)^{1/2}, \quad \forall m \in \mathcal{A} := \{ m \geq 0 : A_m = \max_{0 \leq i \leq m} A_i \}. \quad (4.8)
\]
This is because for \( m \notin \mathcal{A} \), there exists \( 0 \leq m_0 < m \), \( m_0 \in \mathcal{A} \) such that \( A_m \leq A_{m_0} \) and therefore, by Eq. 4.8,
\[
A_m \leq A_{m_0} \leq A_0 + \sum_{i=1}^{m_0} E_i + \left( \sum_{i=0}^{m_0} D_i \right)^{1/2} \leq A_0 + \sum_{i=1}^{m} E_i + \left( \sum_{i=0}^{m} D_i \right)^{1/2},
\]
which is Eq. 4.7a for \( m \). Therefore, we now proceed to prove Eq. 4.7a for \( m \) assuming that \( 0 < A_m = \max_{0 \leq i \leq m} A_i \) holds. If \( A_m \leq \left( \sum_{i=0}^{m} D_i \right)^{1/2} \), then Eq. 4.7a is satisfied trivially. On the other hand, if \( A_m > \left( \sum_{i=0}^{m} D_i \right)^{1/2} \), then Eq. 4.6 implies
\[
A_m^2 + \sum_{i=0}^{m} B_i^2 \leq A_0 A_m + A_m \sum_{i=0}^{m} E_i + A_m \left( \sum_{i=0}^{m} D_i \right)^{1/2}.
\]
The result now follows by dividing by \( A_m \).

Before addressing the main result (Theorem 3) of this section, we first determine the error equation.

**Lemma 3** (Error equation) Suppose that \( (u_h^m, p_{T h}^m, z_h^m, p_h^m) \) \( m \geq 1 \) is the solution of (3.14) with the numerical initial data (4.3). The approximation and interpolation errors satisfy
\[
\begin{align*}
& a_h(e_u^{h,m+1}, v_h) + b_h(v_h, e_{p r}^{h,m+1}) \\
& + b_h(d_t e_u^{h,m+1}, q_{T h}) + \lambda^{-1}(d_t (\alpha e_p^{h,m+1} - e_{p r}^{h,m+1}), \alpha q_h + q_{T h})_\Omega \\
& + \kappa^{-1}(e_z^{h,m+1}, w_h)_\Omega - b_h((e_z^{h,m+1}, 0), q_h) \\
& = b_h(d_t e_u^{h,m+1}, q_{T h}) + \kappa^{-1}(e_z^{h,m+1}, w_h)_\Omega + c_0 \partial_t p_{T h}^{m+1} - d_t \Pi Q p_h^{m+1})_\Omega \\
& + \lambda^{-1}(\alpha \partial_t p_{T h}^{m+1} - d_t \Pi Q p_h^{m+1} - d_t \Pi Q p_{T h}^{m+1}, \alpha q_h)_\Omega,
\end{align*}
\]
for any \( (v_h, q_{T h}, w_h, q_h) \in X_h \).
**Theorem 3** Let \( u^{n+1} \), \( p_T^{m+1} \), \( s^{m+1} \), \( p^{m+1} \) be the solution to Eqs. 2.2, 2.3 evaluated at \( t = t^{m+1} \), into Eq. 3.2. Then, subtracting Eq. 3.14, applying \( d_t \) to the second equation of the resulting set of equations, and adding all the equations, we obtain the following:

\[
a_h(e^{l,m+1}_u, v_h) + b_h(v_h, e^{h,m+1}_{p_T}) + b_h(d_t e^{h,m+1}_u, q_T h) + \lambda^{-1}(d_t(\alpha e^{h,m+1}_p - e^{h,m+1}_{p_T}), \alpha q_h + q_T h) = 0
\]

The result follows by noting that: \( a_h(e^{l,m+1}_u, v_h) = 0 \) by definition of \( \Pi v \), \( b_h(v_h, e^{l,m+1}_{p_T}) = 0 \) because \( \Pi_Q \) and \( \Pi_h \) are \( L^2 \) projections into \( Q_h \) and \( \widetilde{Q}_h \), respectively, and \( \nabla \cdot V_h = Q_h; b_h(e^{l,m+1}_v, q_h) = 0 \) by the commuting property of the BDM interpolation operator \( \Pi v \), \( \nabla v = \Pi v \) for \( v \in [H^1(\Omega)]^d \), the \( (H, \text{div}) \)-conformity of \( e^{l,m+1}_v \), and the boundary conditions on \( \Gamma^f \); and \( \lambda^{-1}(d_t(\alpha e^{l,m+1}_p - e^{l,m+1}_{p_T}), \alpha q_h + q_T h) = 0 \) because \( \Pi _Q \) is the \( L^2 \)-projection into \( Q_h \).

We are now ready to prove an a priori error estimate for the HDG and EDG-HDG methods Eq. 3.14.

**Theorem 3** Let \( u, p_T, z, p \) be a solution to Eqs. 2.2, 2.3 on the time interval \( I = (0, T) \) and let \( \bar{u}, \bar{p}_T, \) and \( \bar{p} \) be the traces of, respectively, \( u, p_T, \) and \( p \) on the mesh skeleton. Let \( (u^m, p_T^m, z^m, p^m) \in X_h \) be the solution to Eq. 3.14. Suppose the numerical initial data is imposed according to Eq. 4.3 and that \( u, p_T, z, p \) satisfy

\[
\|u\|_{W^{1,1}(I; H^1(\Omega))} + \|\partial_t u\|_{L^1(I; H^1(\Omega))} < \infty,
\]

\[
\|z\|_{C^0(I; H^1(\Omega))} < \infty,
\]

\[
\|p, p_T\|_{W^{1,1}(I; H^1(\Omega))} + \|p, p_T\|_{W^{2,1}(I; L^2(\Omega))} < \infty,
\]

for \( 1 \leq l \leq k \). Then, the following error estimates hold:

\[
c_0^{1/2} \|p^m - p_T^m\|_{C^0(\Omega)} + \lambda^{-1/2} \|\alpha(p^m - p_T^m) - (p_T^m - p_T h^m)\|_{C^0(\Omega)} + \lambda^{-1/2}\|u^m - u_T^m\|_{H^1(\Omega)} + \kappa^{-1/2}\left( \sum_{i=1}^m \Delta t |k^i - e_{z,h}^i|^2 \right)^{1/2} \leq C_1 \Delta t + C_2 h^l, \quad (4.9a)
\]

\[
\mu^{-1/2}\|p^m - p_T^m\|_q \leq C_1 \Delta t + C_3 h^l, \quad (4.9b)
\]

where

\[
C_1 = C \max\{c_0^{1/2}, \lambda^{-1/2}\} \|p, p_T\|_{W^{2,1}(I; L^2(\Omega))},
\]

\[
C_2 = C\left( \mu^{1/2}\|u\|_{W^{1,1}(I; H^1(\Omega))} + \kappa^{-1/2}T^{1/2}\|z\|_{C^0(I; H^1(\Omega))} \right).
\]
We define the error equation in Lemma 3. Then, we arrive at

\[ C_3 = C(\mu^{1/2} \| \partial_t u \|_{L^1(I; H^1(\Omega))} + \max \{ c_0^{1/2}, \mu^{-1/2} \| p \|_{W^{1,1}(I; H^1(\Omega))} \}) \]

\[ + \kappa^{-1/2} T^{1/2} \| z \|_{C^0(I; H^1(\Omega))}. \]

**Proof** Let us first remark that for a Hilbert space \( X \) with inner product \( \langle \cdot, \cdot \rangle_X \) we have:

\[ \langle a, a - b \rangle_X = \frac{1}{2} \langle (a, a) \rangle_X + \langle b, a - b \rangle_X - \langle b, b \rangle_X \geq \frac{1}{2} \langle (a, a) \rangle_X - \langle b, b \rangle_X. \]

(4.10)

Let us now choose \( v_h = d_1 e_{u}^{h,m+1}, q_{Th} = -e_{pT}^{h,m+1}, w_h = e_{z,m+1} \), and \( q_h = e_{p,m+1} \) in the error equation in Lemma 3. Then,

\[
\begin{align*}
& a_h(e_{u}^{h,m+1}, d_1 e_{u}^{h,m+1}) + \kappa^{-1}(e_{z,m+1}^{h,m+1}, e_{z,m+1}^{h,m+1}) \\
& + (c_0 d_1 e_{p}^{h,m+1}, e_{pT}^{h,m+1}) + \lambda^{-1}((d_1 e_{pT}^{h,m+1} - e_{p,m+1}^{h,m+1}), \alpha e_{p,m+1}^{h,m+1} - e_{pT}^{h,m+1}) \\
& = -b_h(d_1 e_{u}^{l,m+1}, e_{pT}^{h,m+1}) + \kappa^{-1}(e_{z,m+1}^{l,m+1}, e_{z,m+1}^{h,m+1}) \\
& + (c_0 (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}), e_{p,m+1}^{h,m+1}) \\
& + \lambda^{-1}(\alpha (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}) - (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}), \alpha e_{p,m+1}^{h,m+1}).
\end{align*}
\]

Multiplying the above equation by \( \Delta t \), and using Eq. 4.10 with

\[
\begin{align*}
& X = L^2(\Omega), \quad \langle \cdot, \cdot \rangle_X = c_0(\cdot, \cdot)_{\Omega}, \quad a = e_{p,m+1}^{h,m+1}, b = e_{p,m+1}^{h,m}, \\
& X = L^2(\Omega), \quad \langle \cdot, \cdot \rangle_X = \lambda^{-1}(\cdot, \cdot)_{\Omega}, \quad a = \alpha e_{p,m+1}^{h,m+1} - e_{pT}^{h,m+1}, b = \alpha e_{p,m}^{h,m} - e_{pT}^{h,m}, \\
& X = V_h, \quad \langle \cdot, \cdot \rangle_X = a_h(\cdot, \cdot), \quad a = e_{h,m+1}^{h,m+1}, b = e_{h,m}^{h,m}.
\end{align*}
\]

we arrive at

\[
\begin{align*}
& \frac{c_0}{2} \left( \| e_{p}^{h,m+1} \|_2^2 - \| e_{p}^{h,m} \|_2^2 \right) + \frac{\lambda}{2} \left( \| \alpha e_{p}^{h,m+1} - e_{pT}^{h,m+1} \|_2^2 - \| \alpha e_{p}^{h,m} - e_{pT}^{h,m} \|_2^2 \right) \\
& + \frac{1}{2} (a_h(e_{u}^{h,m+1}, e_{u}^{h,m+1}) - a_h(e_{u}^{h,m}, e_{u}^{h,m})) + \kappa^{-1} \Delta t \| e_{z,m+1}^{h,m+1} \|_2^2 \\
& \leq -\Delta t b_h(d_1 e_{u}^{l,m+1}, e_{pT}^{h,m+1}) + \Delta t \kappa^{-1}(e_{z,m+1}^{l,m+1}, e_{z,m+1}^{h,m+1}) \\
& + \Delta t (c_0 (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}), e_{p,m+1}^{h,m+1}) \\
& + \lambda^{-1} \Delta t (\alpha (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}) - (\partial_t p_{m+1} - d_1 \Pi Q p_{m+1}), \alpha e_{p,m+1}^{h,m+1}) \\
& =: I_{m+1} + I_{2,m+1} + I_{3,m+1} + I_{4,m+1}.
\end{align*}
\]

We define

\[
\begin{align*}
A_i^2 := & \frac{c_0}{2} \| e_{p}^{h,i} \|_2^2 + \frac{\lambda}{2} \| \alpha e_{p}^{h,i} - e_{pT}^{h,i} \|_2^2 + \frac{1}{2} a_h(e_{u}^{h,i}, e_{u}^{h,i}), \\
B_i^2 := & \frac{\kappa}{2} \Delta t \| e_{z}^{h,i} \|_2^2,
\end{align*}
\]

so that Eq. 4.11 can be written as

\[
A_{m+1}^2 + 2B_{m+1}^2 \leq A_m^2 + I_1^{m+1} + I_2^{m+1} + I_3^{m+1} + I_4^{m+1}.
\]

(4.12)

We proceed by bounding \( I_1^i, I_2^i, I_3^i, \) and \( I_4^i \), starting with \( I_1^i \).
Restricting the error equation in Lemma 3 for $v_h$ with general index $i$, we find the error equation

$$a_h(e^{h,i}_u, v_h) + b_h(v_h, e^{h,i}_{p_T}) = 0 \quad \forall v_h \in V_h.$$ 

By Eq. 3.6a, the above equality, Eq. 3.5, the equivalence between $|||\cdot|||$ and $|||\cdot|||_v$ [48, eq. (5.5)], and Eq. 3.4,

\[
C|||e^{h,i}_{p_T}|||_q \leq \sup_{0 \neq v_h \in V_h} \frac{b_h(v_h, e^{h,i}_{p_T})}{|||v_h|||_v} = \sup_{0 \neq v_h \in V_h} \frac{-a_h(e^{h,i}_u, v_h)}{|||v_h|||_v} \leq C\mu|||e^{h,i}_u|||_v \leq C\mu^{1/2} \left( a_h(e^{h,i}_u, e^{h,i}_u) \right)^{1/2},
\]

(4.13)

implying that

$$\mu^{-1/2}|||e^{h,i}_{p_T}|||_q \leq CA_i.$$  

(4.14)

We may now bound $I_1^i$ using Eqs. 3.7 and 4.14:

$$I_1^i \leq \Delta t |b_h(d_i e^{i}_u, e^{i}_{p_T})| \leq C\Delta t |||d_i e^{i}_u|||_v |||e^{i}_{p_T}|||_q \leq C\Delta t \mu^{1/2} |||d_i e^{i}_u|||_v A_i.$$ (4.15)

A bound for $I_2^i$ follows from the Cauchy–Schwarz and Young’s inequalities:

$$I_2^i \leq \kappa^{-1} \Delta t |||e^{i}_z||| \|e^{h,i}_z\| \leq \frac{\kappa^{-1}}{2} \Delta t |||e^{i}_z|||_2^2 + B_i^2.$$  

Using the Cauchy–Schwarz and triangle inequalities, we bound $I_3^i$ as follows:

$$I_3^i \leq c_0 \Delta t \|\partial_t p^i - d_i \Pi_Q p^i \|_2 |||e^{h,i}_p|||_2 \leq c_0 \Delta t \left( \|\partial_t p^i \|_2 + |||d_i e^{i}_p|||_2 \right) |||e^{h,i}_p|||_2 \leq (2c_0)^{1/2} \Delta t \left( \|\partial_t p^i \|_2 + |||d_i e^{i}_p|||_2 \right) A_i.$$  

To estimate $I_4^i$, we first derive an auxiliary result. By the assumption that $C_\mu \leq \lambda$ (see Sect. 2),

$$\lambda^{-1} \alpha^2 |||e^{h,i}_p|||_2 \leq 2\lambda^{-1} \left( \|\alpha e^{h,i}_p - e^{h,i}_{p_T} \|_2^2 \right) + 2\lambda^{-1} |||e^{h,i}_{p_T}|||_2^2 \leq 2\lambda^{-1} \left( \|\alpha e^{h,i}_p - e^{h,i}_{p_T} \|_2^2 \right) + 2(C_\mu)^{-1} |||e^{h,i}_{p_T}|||_2^2.$$  

Combining this estimate with Eq. 4.14 we obtain:

$$\lambda^{-1/2} \alpha |||e^{h,i}_p|||_2 \leq CA_i.$$  

(4.16)

The Cauchy–Schwarz and triangle inequalities, together with Eq. 4.16 now imply

$$I_4^i \leq \lambda^{-1} \alpha \Delta t \|\alpha (\partial_t p^i - d_i \Pi_Q p^i) - (\partial_t p^i - d_i \Pi_Q p^i) \|_2 |||e^{h,i}_p|||_2 \leq C\lambda^{-1/2} \Delta t \left( \|\alpha \partial_t p^i - d_i p^i \|_2 + |||d_i e^{i}_p|||_2 \right) A_i.$$  

If we define

$$E_i = C\mu^{1/2} \Delta t |||d_i e^{i}_u|||_v + (2c_0)^{1/2} \Delta t \left( \|\partial_t p^i \|_2 + |||d_i e^{i}_p|||_2 \right) + C\lambda^{-1/2} \Delta t \left( \|\alpha \partial_t p^i - d_i p^i \|_2 + |||d_i e^{i}_p|||_2 \right) A_i,$$

$$D_i = \frac{\kappa^{-1}}{2} \Delta t |||e^{i}_z|||_2^2,$$

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we find, using Eq. 4.12, that

\[(A_i^{2} - A_i^{2}) + B_i^{2} \leq E_i A_i + D_i.\]  

(4.17)

Summing now for \(i = 0\) to \(i = m - 1\), and using \(A_0 = 0\), we obtain, after shifting indices and using \(D_0 \geq 0\) and \(B_0 = 0\),

\[A_m^2 + \sum_{j=1}^{m} B_j^2 \leq \sum_{j=1}^{m} E_j A_j + \sum_{j=1}^{m} D_j.\]  

(4.18)

Then, by Lemma 2 we obtain

\[A_m + \left(\sum_{i=1}^{m} B_i^2\right)^{1/2} \leq C \left(\sum_{i=1}^{m} E_i + \left(\sum_{i=1}^{m} D_i\right)^{1/2}\right).\]  

(4.19)

To prove Eq. 4.9a, we therefore need to estimate \(\sum_{i=1}^{m} E_i\) and \((\sum_{i=1}^{m} D_i)^{1/2}\).

By Eqs. 3.4, 4.2, we note that

\[\mu^{1/2} \Delta t \sum_{i=1}^{m} ||d_i e_{\omega}^{l,i}||_v \leq C \Delta t \sum_{i=1}^{m} (a_i (d_i e_{\omega}^{l,i}, d_i e_{\omega}^{l,i}))^{1/2} \leq C \Delta t \mu^{1/2} \sum_{i=1}^{m} h^l ||d_i u||_{L^1(I; H^{l+1}(\Omega))},\]

where we used \(\Delta t d_i u = \int_{\mu_{i-1}}^{\mu_i} \partial_t u(s) ds\) for the last inequality. Using this estimate, together with Eqs. 4.5a, 4.5b, we find:

\[\sum_{i=1}^{m} E_i = C \sum_{i=1}^{m} \left[\mu^{1/2} \Delta t ||d_i e_{\omega}^{l,i}||_v + (2c_0)^{1/2} \Delta t (||\partial_t p^i - d_i p^i||_\omega + ||d_i e_{\omega}^{l,i}||_v) \right.\]

\[\left. + \lambda^{-1/2} \Delta t \left(\alpha ||d_i p^i||_\omega + \alpha ||d_i e_{\omega}^{l,i}||_v + ||\partial_t p_T - d_i p_T||_\omega + ||d_i e_{\omega}^{l,i}||_v\right)\right].\]

\[\leq C \mu^{1/2} h^l ||\partial_t u||_{L^1(I; H^{l+1}(\Omega))}\]

\[+ C \max\{c_0^{1/2}, \lambda^{-1/2}\} \left(\Delta t ||\partial_T p, \partial_T p_T||_{L^1(I; L^2(\Omega))} + h^l ||\partial_T p, \partial_T p_T||_{L^1(I; H^l(\Omega))}\right).\]  

(4.20)

Next, by Eq. 4.1,

\[\sum_{i=1}^{m} D_i = \sum_{i=1}^{m} \frac{k^{-1}}{2} \Delta t ||e_{\omega}^{l,i}||_v^2 \leq C k^{-1} (\Delta t) h^2 \sum_{i=1}^{m} ||z(\iota^i)||_{H^l(\Omega)}^2 \leq C k^{-1} T h^2 ||z||_{C^0(I; H^l(\Omega))}^2.\]

(4.21)
where in the last inequality $m \Delta t \leq T$ is used. Combining Eq. 4.19 with Eqs. 4.20, 4.21 and the coercivity of $a_h$ Eq. 3.4, we find:

\[
c_0^{1/2} \|e_p^m\|_{\Omega} + \lambda^{-1/2} \|ae_p^h,m - e_p^h,m\|_{\Omega} + \lambda^{1/2} \|e_p^m\|_{\Omega} + \kappa^{-1/2} \left( \sum_{i=1}^{m} \Delta t \|e_z^i\|^2_{\Omega} \right)^{1/2} \leq C \mu^{-1/2} h^l \|\partial_t u\|_{L^1(I; H^{m+1}(\Omega))} + C \max\{c_0^{1/2}, \lambda^{-1/2}\} \left( \|\partial_t p\|_{L^1(I; L^2(\Omega))} + T\|\partial_t p\|_{L^1(I; L^2(\Omega))} + h^l \|\partial_t p\|_{L^1(I; H^l(\Omega))} \right) + C \kappa^{-1/2} T^{1/2} h^l \|z\|_{C^0(I; H^l(\Omega))} \leq c_1 \Delta t + c_2 h^l.
\]

where

\[
c_1 = C \max\{c_0^{1/2}, \lambda^{-1/2}\} \|\partial_t p\|_{L^1(I; L^2(\Omega))} + C \max\{c_0^{1/2}, \lambda^{-1/2}\} \|p\|_{W^{1,1}(I; L^2(\Omega))},
\]

\[
c_2 = C \mu^{1/2} \|\partial_t u\|_{L^1(I; H^{m+1}(\Omega))} + \max\{c_0^{1/2}, \lambda^{-1/2}\} \|\partial_t p\|_{L^1(I; H^l(\Omega))} + \kappa^{-1/2} T^{1/2} h^l \|z\|_{C^0(I; H^l(\Omega))}.
\]

Next, by Eq. 4.4, the triangle inequality, and the approximation error Eq. 4.22,

\[
c_0^{1/2} \|p^m - p_h^m\|_{\Omega} + \lambda^{-1/2} \|e_p(p^m - p_h^m)\|_{\Omega} + \mu^{1/2} \|u^m - u_h^m\|_{\Omega} + \lambda^{-1/2} \left( \sum_{i=1}^{m} \Delta t \|z^i\|_{\Omega} \right)^{1/2} \leq c_0^{1/2} \|e_p^m\|_{\Omega} + \lambda^{-1/2} \|e_p^m - e_p^h\|_{\Omega} + \mu^{1/2} \|e_p^m\|_{\Omega} + \left( \sum_{i=1}^{m} D_i \right)^{1/2} \leq c_1 \Delta t + c_2 h^l.
\]

Note that

\[
c_0^{1/2} \|e_p^m\|_{\Omega} \leq C c_0^{1/2} h^l \|p^m\|_{l, \Omega},
\]

\[
\lambda^{-1/2} \|e_p^m - e_p^h\|_{\Omega} \leq C \lambda^{-1/2} h^l (\|p^m\|_{l, \Omega} + \|p_h^m\|_{l, \Omega}),
\]

\[
\mu^{1/2} \|e_p^m\|_{\Omega} \leq C \mu^{1/2} h^l \|u^m\|_{l+1, \Omega}.
\]

When combined with Eqs. 4.21, 4.23,

\[
c_0^{1/2} \|p^m - p_h^m\|_{\Omega} + \lambda^{-1/2} \|e_p(p^m - p_h^m)\|_{\Omega} + \mu^{1/2} \|u^m - u_h^m\|_{\Omega} + \kappa^{-1/2} \left( \sum_{i=1}^{m} \Delta t \|z^i_h\|^2_{\Omega} \right)^{1/2} \leq c_1 \Delta t + (c_2 + c_3) h^l.
\]

where

\[
c_3 = C \left( c_0^{1/2} \|p^m\|_{l, \Omega} + \lambda^{-1/2} (\|p^m\|_{l, \Omega} + \|p_h^m\|_{l, \Omega}) + \mu^{1/2} \|u^m\|_{l+1, \Omega} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right) = C \left( c_0^{1/2} + \lambda^{-1/2} \|p^m\|_{l, \Omega} + \lambda^{-1/2} \|p_h^m\|_{l, \Omega} + \mu^{1/2} \|u^m\|_{l+1, \Omega} + \kappa^{-1/2} T^{1/2} \|z\|_{C^0(I; H^l(\Omega))} \right).
\]
Let us have a closer look at $c_2 + c_3$. Using the Sobolev embedding $W^{s,q}(I; H^l(\Omega)) \hookrightarrow C^0(I; H^l(\Omega))$ for $(s, q) = (1, 1)$ and $(s, q) = (2, 1)$,

$$c_2 + c_3 \leq C \left( \mu^{1/2} \{ \| \partial_t u \|_{L^1(I; H^1(\Omega))} + \| \mu_m \|_{L^1(\Omega)} \} + \kappa^{-1/2} T^{1/2} \| z \|_{C^0(I; H^l(\Omega))} \right)$$

$$+ \max \{ c_0^{1/2}, \lambda^{-1/2} \} \{ \| \partial_t p \|_{L^1(I; H^1(\Omega))} + \| \mu \|_{L^1(\Omega)} \}$$

$$\leq C \left( \mu^{1/2} \{ \| \partial_t u \|_{L^1(I; H^1(\Omega))} + \| \mu \|_{C^0(I; H^1(\Omega))} \} + \kappa^{-1/2} T^{1/2} \| z \|_{C^0(I; H^l(\Omega))} \right)$$

$$+ \max \{ c_0^{1/2}, \lambda^{-1/2} \} \{ \| \partial_t p \|_{L^1(I; H^1(\Omega))} + \| \mu \|_{C^0(I; H^l(\Omega))} \}$$

proving Eq. 4.9a.

Finally, Eq. 4.9b follows from Eq. 4.4 and the triangle inequality, Eqs. 4.14, 4.22, and usual interpolation estimates for the $L^2$-projection:

$$\mu^{-1/2} \| p_T^m - p_{Th}^m \|_q \leq \mu^{-1/2} \| e_{pT}^{I,m} \|_q + C A_{m}$$

$$\leq c_1 \Delta t + \left( C \mu^{-1/2} \| p_T^m \|_{I,\Omega} + c_2 \right) h^l \quad (4.26)$$

Let us have a closer look at the constant in front of the second term:

$$C \mu^{-1/2} \| p_T \|_{C^0(I; H^l(\Omega))} + c_2$$

$$\leq C \left( \mu^{1/2} \| \partial_t u \|_{L^1(I; H^1(\Omega))} + \max \{ c_0^{1/2}, \lambda^{-1/2} \} \| \partial_t p \|_{L^1(I; H^l(\Omega))} \right)$$

$$+ \kappa^{-1/2} T^{1/2} \| z \|_{C^0(I; H^l(\Omega))} + C \mu^{-1/2} \| p_T \|_{C^0(I; H^l(\Omega))}$$

$$\leq C \left( \mu^{1/2} \| \partial_t u \|_{L^1(I; H^1(\Omega))} + \max \{ c_0^{1/2}, \mu^{-1/2} \} \| p_T \|_{W^{1,1}(I; H^l(\Omega))} \right)$$

$$+ \kappa^{-1/2} T^{1/2} \| z \|_{C^0(I; H^l(\Omega))}$$

where in the last step we used that $C \lambda^{-1/2} \leq \mu^{1/2}$. This proves Eq. 4.9b.

We end this section by noting that the estimates in Theorem 3 for the displacement, Darcy velocity, and total pressure are unconditionally robust in the incompressible limit $c_0 \to 0$ and $\lambda \to \infty$.

## 5 Numerical Examples

We now validate our theoretical analysis. As stated previously in Remark 1, the analysis in this paper holds both for HDG and EDG-HDG. As such, both methods are implemented using the Netgen/NGSolve finite element library [44, 45]. Numerical results are compared to analytical solutions and some benchmark problems.

### 5.1 Rates of Convergence for a Manufactured Solution

We start by considering a manufactured solution for Eq. 2.2 on the unit square. We divide the boundary of our domain into

$$\Gamma_1 = \{(x_1, x_2) \in \partial \Omega : x_2 = 0\}, \quad \Gamma_2 = \{(x_1, x_2) \in \partial \Omega : x_1 = 1\},$$
and set $\Gamma_D = \Gamma_1 \cup \Gamma_3 \cup \Gamma_4$, $\Gamma_P = \Gamma_1 \cup \Gamma_2$, $\Gamma_T = \Gamma_2$, and $\Gamma_F = \Gamma_3 \cup \Gamma_4$. As exact solution we take
\[
\begin{align*}
\Gamma_3 &= \{(x_1,x_2) \in \partial \Omega : x_2 = 1\}, \\
\Gamma_4 &= \{(x_1,x_2) \in \partial \Omega : x_1 = 0\},
\end{align*}
\] and set body force terms, source/sink terms, initial and boundary conditions accordingly. As parameters we set $E = 10^4$, $\kappa = 10^{-2}$, $\alpha = 0.1$, $c_0 = 0.1$, and we consider two values for $\nu$, namely, $\nu = 0.2$ and the quasi-incompressibility case of $\nu = 0.49999$. We consider the solution over the time interval $I = (0, 0.1]$.

To verify Theorem 3, let us introduce the following notation:
\[
\begin{align*}
\|e_h^m\|_E &= e_0^{1/2} \|p^m - p_h^m\|_\Omega + \lambda^{-1/2} \|\kappa (p^m - p_h^m) - (p_T^m - p_{T_h}^m)\|_\Omega \\
&
\end{align*}
\]

5.2 The Footing Problem

The two-dimensional footing problem has been proposed in the literature to study the locking-free properties of numerical methods for the Biot equations [19, 34]. We follow here the setup of [2] and consider the domain $\Omega = (-50, 50) \times (0, 75)$ and model parameters $\kappa = 10^{-4}$, $c_0 = 10^{-3}$, $\alpha = 0.1$, $E = 3 \cdot 10^4$, and $\nu = 0.4995$ (so that $\lambda \approx 10^7$). We define the boundaries $\Gamma_1 = \{(x_1, x_2) \in \partial \Omega, |x_1| \leq 50/3, x_2 = 75\}$, $\Gamma_2 = \{(x_1, x_2) \in \partial \Omega, |x_1| > 50/3, x_2 = 75\}$, and $\Gamma_3 = \partial \Omega \setminus (\Gamma_1 \cup \Gamma_2)$ and impose the following boundary conditions:

\[\sigma n = (0, -\sigma_0)^T \text{ on } \Gamma_1, \quad \sigma n = 0 \text{ on } \Gamma_2, \quad u = 0 \text{ on } \Gamma_3, \quad p = 0 \text{ on } \partial \Omega,\]

where $\sigma_0 = 10^4$. As initial conditions we impose $u(x,0) = 0$ and $p(x,0) = 0$. We solve this problem with EDG-HDG until $T = 50$ using backward Euler time stepping. We choose a time step of size $\Delta t = 0.25$, take $k = 2$ in our finite element spaces, and compute the solution on an unstructured mesh consisting of 169984 simplices.

We show the solution to this problem at time $T = 50$ in Fig. 1. In this incompressible limit we observe that the discretization results in pressure and displacement solutions are free of, respectively, spurious oscillations and locking effects.
Table 1  Spatial rates of convergence for HDG and EDG-HDG for the test case described in Sect. 5.1 for \( k = 1 \) and \( k = 2 \). Here \( \epsilon_{\omega} = \omega - \omega_{h} \) and \( \| \epsilon_{\omega} \|_{L_{\infty}} \Omega = \| \epsilon_{\omega} \|_{L_{\infty}} \Omega \). For \( \omega = u, p_{T}, \zeta, p \), dofs are the total number of degrees-of-freedom and \( R \) is the rate of convergence.

| Dofs | \( \| \epsilon_{u} \|_{L_{\infty}} \Omega \) | \( R \) | \( \| \epsilon_{pT} \|_{L_{\infty}} \Omega \) | \( R \) | \( \| \epsilon_{\zeta} \|_{L_{\infty}} \Omega \) | \( R \) | \( \| \epsilon_{p} \|_{L_{\infty}} \Omega \) | \( R \) | \( \| e^{\text{int}} \|_{L_{\infty}} \Omega \) | \( R \) |
|------|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|-------------------------------|-----|
| HDG (\( \nu = 0.2 \)) \( k = 1 \) | 948 5.9e-02 1.2 2.6e-01 0.6 1.4e-01 1.9 2.8e-01 1.4 3.3e+01 0.6 | 3716 1.2e-02 2.3 1.4e-01 0.9 4.1e-02 1.8 1.3e-01 1.1 1.8e+01 0.9 | 16064 2.2e-03 2.5 5.9e-02 1.3 9.2e-03 2.2 5.9e-02 1.1 7.4e+00 1.3 | 62444 5.5e-04 2.0 3.0e-02 1.0 2.4e-03 2.0 2.9e-02 1.0 3.8e+00 1.0 | 248336 1.3e-04 2.0 1.5e-02 1.0 5.9e-04 2.0 1.4e-02 1.0 1.9e+00 1.0 |
| HDG (\( \nu = 0.49999 \)) \( k = 1 \) | 1728 7.0e-03 4.4 3.8e-02 3.0 1.5e-02 3.7 3.9e-02 2.2 4.6e+00 3.0 | 6816 7.1e-04 3.3 8.3e-03 2.2 1.4e-03 3.5 7.8e-03 2.3 1.0e+00 2.2 | 29568 7.0e-05 3.4 1.8e-03 2.2 1.5e-04 3.2 1.8e-03 2.1 2.2e-01 2.2 | 115104 8.5e-06 3.0 4.4e-04 2.0 1.9e-05 3.0 4.3e-04 2.1 5.3e-02 2.0 | 458112 1.0e-06 3.1 1.1e-04 2.0 2.3e-06 3.0 1.1e-04 2.0 1.3e-02 2.0 |
| HDG-HDG (\( \nu = 0.2 \)) \( k = 1 \) | 948 9.1e-02 4.6 2.5e-01 0.5 6.4e-02 2.7 2.7e-01 1.4 3.1e+03 0.6 | 3716 1.8e-02 2.4 1.2e-01 1.0 1.3e-02 2.3 1.3e-01 1.1 1.5e+03 1.0 | 16064 3.8e-03 2.2 5.6e-02 1.1 3.0e-03 2.2 5.9e-02 1.1 7.1e+02 1.1 | 62444 9.6e-04 2.0 2.8e-02 1.0 7.3e-04 2.0 2.9e-02 1.0 3.6e+02 1.0 | 248388 2.4e-04 2.0 1.4e-02 1.0 1.8e-04 2.0 1.4e-02 1.0 1.8e+02 1.0 |
| EDG-HDG (\( \nu = 0.2 \)) \( k = 1 \) | 1728 7.3e-03 4.4 3.6e-02 3.1 1.2e-02 3.9 3.9e-02 2.2 4.6e+02 3.1 | 6816 7.3e-04 3.3 8.1e-03 2.2 1.1e-03 3.5 7.8e-03 2.3 1.0e+02 2.2 | 29568 6.9e-05 3.4 1.7e-03 2.2 1.2e-04 3.1 1.8e-03 2.1 2.2e+01 2.2 | 115104 8.3e-06 3.1 4.3e-04 2.0 1.5e-05 3.0 4.3e-04 2.1 5.4e+00 2.0 | 458688 9.8e-07 3.1 1.1e-04 2.0 1.8e-06 3.0 1.1e-04 2.0 1.3e+00 2.0 |
| HDG-HDG (\( \nu = 0.2 \)) \( k = 2 \) | 764 8.2e-02 1.0 2.7e-01 0.8 1.7e-01 1.9 2.8e-01 1.4 3.3e+01 0.7 | 2996 1.7e-02 2.3 1.4e-01 1.0 4.7e-02 1.8 1.3e-01 1.1 1.6e+01 1.0 | 12962 3.5e-03 2.3 6.5e-02 1.1 1.1e-02 2.1 5.9e-02 1.1 7.6e+00 1.1 | 50408 8.8e-04 2.0 3.2e-02 1.0 2.9e-03 2.0 2.9e-02 1.0 3.8e+00 1.0 | 200522 2.2e-04 2.0 1.6e-02 1.0 7.2e-04 2.0 1.4e-02 1.0 1.9e+00 1.0 |
| HDG-HDG (\( \nu = 0.2 \)) \( k = 2 \) | 1544 7.6e-03 4.3 3.9e-02 3.0 1.8e-02 3.5 4.0e-02 2.2 4.8e+00 3.0 | 6096 7.7e-04 3.3 8.6e-03 2.2 1.8e-03 3.3 7.8e-03 2.3 1.0e+00 2.2 |
Table 1 continued

\[ \begin{array}{ccccccc}
\text{\( k = 2 \)} & & & & & & \\
26466 & 7.5e-05 & 3.4 & 1.9e-03 & 2.2 & 2.0e-04 & 3.2 & 1.8e-03 & 2.1 & 2.2e-01 & 2.2 \\
103068 & 9.1e-06 & 3.0 & 4.6e-04 & 2.0 & 2.4e-05 & 3.0 & 4.3e-04 & 2.1 & 5.5e-02 & 2.0 \\
410298 & 1.1e-06 & 3.1 & 1.1e-04 & 2.0 & 3.0e-06 & 3.0 & 1.1e-04 & 2.0 & 1.4e-02 & 2.0 \\
\end{array} \]

EDG-HDG (\( \nu = 0.49999 \))

\[ \begin{array}{ccccccc}
\text{\( k = 1 \)} & & & & & & \\
764 & 9.0e-02 & 3.8 & 2.5e-01 & 0.5 & 6.4e-02 & 2.7 & 2.7e-01 & 1.4 & 3.1e+03 & 0.6 \\
2996 & 1.8e-02 & 2.3 & 1.2e-01 & 1.0 & 1.3e-02 & 2.3 & 1.3e-01 & 1.1 & 1.5e+03 & 1.0 \\
12962 & 4.0e-03 & 2.2 & 5.6e-02 & 1.1 & 3.0e-03 & 2.2 & 5.9e-02 & 1.1 & 7.1e+02 & 1.1 \\
50408 & 1.0e-03 & 2.0 & 2.8e-02 & 1.0 & 7.3e-04 & 2.0 & 2.9e-02 & 1.0 & 3.6e+02 & 1.0 \\
200564 & 2.5e-04 & 2.0 & 1.4e-02 & 1.0 & 1.8e-04 & 2.0 & 1.4e-02 & 1.0 & 1.8e+02 & 1.0 \\
\end{array} \]

\[ \begin{array}{ccccccc}
\text{\( k = 2 \)} & & & & & & \\
1544 & 8.5e-03 & 4.2 & 3.6e-02 & 3.1 & 1.2e-02 & 3.9 & 3.9e-02 & 2.2 & 4.6e+02 & 3.1 \\
6096 & 8.4e-04 & 3.3 & 8.1e-03 & 2.2 & 1.1e-03 & 3.5 & 7.8e-03 & 2.3 & 1.0e+02 & 2.2 \\
26466 & 7.9e-05 & 3.4 & 1.7e-03 & 2.2 & 1.2e-04 & 3.1 & 1.8e-03 & 2.1 & 2.2e+01 & 2.2 \\
103068 & 9.1e-06 & 3.1 & 4.3e-04 & 2.0 & 1.5e-05 & 3.0 & 4.3e-04 & 2.1 & 5.4e+00 & 2.0 \\
410298 & 1.1e-06 & 3.1 & 1.1e-04 & 2.0 & 1.8e-06 & 3.0 & 1.1e-04 & 2.0 & 1.3e+00 & 2.0 \\
\end{array} \]

Table 2 Temporal rates of convergence for HDG and EDG-HDG for the test case described in Sect. 5.1. Here \( \nu = 0.49999 \) and \( R = \frac{\|\mathbf{v}\|_{\Omega}}{\|\mathbf{v}\|_{\Omega}} \) for \( \mathbf{v} = u, p, z, p \). The time step is denoted by \( \Delta t \), and \( R \) is the rate of convergence.

\[ \begin{array}{ccccccccc}
\Delta t & \|\mathbf{u}\|_{R, \Omega} & R & \|p_T\|_{R, \Omega} & R & \|z\|_{R, \Omega} & R & \|p\|_{R, \Omega} & R & \|\mathbf{e}^m\|_{\mathbf{E}} \\\n\hline
1/4 & 1.7e-05 & 0.9 & 2.0e-05 & 0.9 & 1.5e+00 & 0.9 & 9.6e-01 & 0.9 & 6.2e-01 & 1.0 \\
1/8 & 9.0e-06 & 0.9 & 1.1e-05 & 0.9 & 7.6e-01 & 0.9 & 5.0e-01 & 1.0 & 3.1e-01 & 1.0 \\
1/16 & 4.6e-06 & 1.0 & 5.4e-06 & 1.0 & 3.9e-01 & 1.0 & 2.5e-01 & 1.0 & 1.6e-01 & 1.0 \\
1/32 & 2.3e-06 & 1.0 & 2.7e-06 & 1.0 & 2.0e-01 & 1.0 & 1.3e-01 & 1.0 & 7.8e-02 & 1.0 \\
1/64 & 1.2e-06 & 1.0 & 1.4e-06 & 1.0 & 9.9e-02 & 1.0 & 6.4e-02 & 1.0 & 3.9e-02 & 1.0 \\
\hline
\text{HDG (\( \nu = 0.2 \))} & & & & & & \\
1/4 & 2.7e-09 & 0.9 & 4.6e-14 & 0.9 & 1.5e+00 & 0.9 & 9.6e-01 & 0.9 & 6.1e-01 & 1.0 \\
1/8 & 1.4e-09 & 0.9 & 2.4e-14 & 0.9 & 7.6e-01 & 0.9 & 5.0e-01 & 1.0 & 3.1e-01 & 1.0 \\
1/16 & 7.2e-10 & 1.0 & 1.2e-14 & 1.0 & 3.9e-01 & 1.0 & 2.5e-01 & 1.0 & 1.6e-01 & 1.0 \\
1/32 & 3.7e-10 & 1.0 & 6.2e-15 & 1.0 & 2.0e-01 & 1.0 & 1.3e-01 & 1.0 & 7.8e-02 & 1.0 \\
1/64 & 1.8e-10 & 1.0 & 3.1e-15 & 1.0 & 9.9e-02 & 1.0 & 6.4e-02 & 1.0 & 3.9e-02 & 1.0 \\
\hline
\text{EDG-HDG (\( \nu = 0.2 \))} & & & & & & \\
1/4 & 1.7e-05 & 0.9 & 2.0e-05 & 0.9 & 1.5e+00 & 0.9 & 9.6e-01 & 0.9 & 6.2e-01 & 1.0 \\
1/8 & 9.0e-06 & 0.9 & 1.1e-05 & 0.9 & 7.6e-01 & 0.9 & 5.0e-01 & 1.0 & 3.1e-01 & 1.0 \\
\end{array} \]
Table 2 continued

|                | 1/16 | 1/32 | 1/64 |
|----------------|------|------|------|
| EDG-HDG ($\nu = 0.2$) |      |      |      |
| $1/16$         | $4.6e^{-06}$ | $2.3e^{-06}$ | $1.2e^{-06}$ |
|                | $1.0$ | $5.4e^{-06}$ | $1.4e^{-06}$ |
|                | $1.0$ | $3.9e^{-01}$ | $9.9e^{-02}$ |
|                | $1.0$ | $2.5e^{-01}$ | $6.4e^{-02}$ |
|                | $1.0$ | $1.6e^{-01}$ | $3.9e^{-02}$ |

|                | 1/32 | 1/64 |
|----------------|------|------|
| EDG-HDG ($\nu = 0.49999$) |      |      |
| $1/4$          | $2.7e^{-09}$ | $1.4e^{-09}$ |
|                | $0.9$ | $2.4e^{-14}$ |
| $1/8$          | $7.2e^{-10}$ | $3.7e^{-10}$ |
|                | $1.0$ | $1.2e^{-14}$ |
| $1/16$         | $0.9$ | $3.9e^{-01}$ |
|                | $1.0$ | $2.5e^{-01}$ |
| $1/32$         | $1.0$ | $6.2e^{-15}$ |
|                | $1.0$ | $2.0e^{-01}$ |
| $1/64$         | $1.0$ | $3.1e^{-15}$ |
|                | $1.0$ | $9.9e^{-02}$ |
|                | $1.0$ | $6.4e^{-02}$ |
|                | $1.0$ | $3.9e^{-02}$ |

Fig. 1 The solution to the footing problem of Sect. 5.2 in the deformed domain at $t = 50$

5.3 The Cantilever Bracket Problem

The cantilever bracket problem was used in [2, 32, 36] to study locking phenomena at low permeability and when the storage coefficient is zero. Consider the domain $\Omega = (0, 1)^2$ and define

$$
\Gamma_1 = \{(x_1, x_2) \in \partial \Omega, \ x_2 = 0\}, \quad \Gamma_2 = \{(x_1, x_2) \in \partial \Omega, \ x_1 = 1\},
$$
The solution to the cantilever bracket problem Sect. 5.3 using EDG-HDG. Left: the pressure solution at $t = 0.001$. Right: the pressure solution along different $x$-lines at time $t = 0.005$

$$\Gamma_3 = \{(x_1, x_2) \in \partial \Omega, \ x_2 = 1\}, \quad \Gamma_4 = \{(x_1, x_2) \in \partial \Omega, \ x_1 = 0\}.$$  

We impose the boundary conditions

$$z \cdot n = 0 \text{ on } \partial \Omega, \quad \sigma n = (0, -1)^T \text{ on } \Gamma_3, \quad \sigma n = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad u = 0 \text{ on } \Gamma_4.$$  

At $t = 0$ we set $u = 0$ and $p = 0$. The model parameters are chosen as $E = 10^5$, $\nu = 0.4$, $\alpha = 0.94$, $c_0 = 0$, and $\kappa = 10^{-7}$ [36]. As shown in [36], with these parameters continuous Galerkin numerical methods show spurious oscillations in the pressure on a very short time interval. Therefore, we consider here the time interval $I = [0, 0.005]$. In our discretization we combine the EDG-HDG discretization with backward Euler time stepping, choose a time step of $\Delta t = 0.00025$, set $k = 2$ in our finite element spaces, and compute the solution on a mesh consisting of 128 simplices.

We plot the solution in Fig. 2. In Fig. 2a we observe that the pressure field at $t = 0.001$ is free from spurious oscillations, similar to the discontinuous Galerkin solutions obtained in [36]. We further show in Fig. 2b that the pressure solution along the lines $x = 0.26$, $x = 0.33$, $x = 0.4$, and $x = 0.46$ at $t = 0.005$ is free of oscillations, agreeing with other stable finite element methods for this problem [2, 32, 36].

6 Conclusions

An HDG and an EDG-HDG method have been presented and analyzed for the total pressure formulation of the quasi-static poroelasticity model. Both discretization methods are shown to be well-posed and space-time a priori error estimates show robustness of the proposed methods when $\lambda \to \infty$ and $c_0 \to 0$; both methods are free of volumetric locking. Numerical examples confirm our theory and further show optimal spatial rates of convergence in the $L^2$-norm.

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The Inf-Sup Condition for $b_h((w_h, 0), q_h)$

By definition of $b_h$, Eq. 3.3b,

$$ b_h((w_h, 0), q_h) := -(q_h, \nabla \cdot w_h)_\Omega + (\bar{q}_h, w_h \cdot n)_{\partial \Omega} \quad \forall w_h \in V_h, \forall q_h \in Q_h. \quad (A.1) $$

Let $q_h \in Q_h$. It is known (see, for example, [16, Section 4.1.4] or [31, Remark 3.3]) that there exists $w \in [H^1(\Omega)]^d$ such that

$$ - (\nabla \cdot w, q_h)_\Omega = \|q_h\|_{\Omega}^2, \quad \|w\|_{1,\Omega} \leq C \|q_h\|_{\Omega}. \quad (A.2) $$

for some positive constant $C$ that only depends on $\Omega$. Let $\Pi_V : [H^1(\Omega)]^d \rightarrow V_h$ be the BDM interpolation operator [10, Section III.3] and observe that by the single-valuedness of $\bar{q}_h$, continuity of $\Pi_V w \cdot n$ across interior faces, and since $\bar{q}_h = 0$ on $\Gamma_P$ and $w = 0$ on $\Gamma_F$,

$$ b_h^2(\Pi_V w, \bar{q}_h) = (\bar{q}_h, \Pi_V w \cdot n)_{\Gamma_F} = 0, $$

i.e., $\Pi_V w \in \text{Ker } b_2 := \{ w_h \in V_h : b_2(w_h, \bar{q}_h) = 0 \ \forall \bar{q}_h \in Q_h \}$. Recall also that $(q_h, \nabla \cdot \Pi_V w)_\Omega = (q_h, \nabla \cdot w)_\Omega$ and $\|\Pi_V w\|_{\Omega} \leq C \|w\|_{1,\Omega}$. Then, by Eq. A.2,

$$ \sup_{0 \neq w_h \in \text{Ker } b_2} \frac{b_2^1(w_h, q_h)}{\|w_h\|_{\Omega}} \geq \frac{-(q_h, \nabla \cdot \Pi_V w)_\Omega}{\|\Pi_V w\|_{\Omega}} \geq \frac{\|q_h\|_{\Omega}^2}{C \|q_h\|_{\Omega}} = C \|q_h\|_{\Omega}. $$

Next, let $w_h := L\tilde{q}_h \in P_k(K)^d$ where $L$ is the local BDM interpolation operator [10] such that

$$ (L\tilde{q}_h) \cdot n = \bar{q}_h, \quad \|L\tilde{q}_h\|_K \leq Ch_{1/2} \|\tilde{q}_h\|_{\partial K}, \quad K \in T. \quad (A.3) $$

Then

$$ \sup_{0 \neq w_h \in V_h} \frac{b_h^2(w_h, \tilde{q}_h)}{\|w_h\|_{\Omega}} \geq \frac{\|\tilde{q}_h\|_{\partial T_h}^2}{\|w_h\|_{\Omega}} \geq \frac{\|\tilde{q}_h\|_{\partial T_h}^2}{C \sum_{K \in T} h_{K}^{-1} \|w_h\|_{\Omega}} \geq C h_{\min} h_{\max}^{-1} \left( \sum_{K \in T} h_{K} \|\tilde{q}_h\|_{\partial K}^2 \right)^{1/2}. $$

Therefore, by [23, Theorem 3.1],

$$ \sup_{0 \neq w_h \in V_h} \frac{b_h((w_h, 0), q_h)}{\|w_h\|_{\Omega}} = \sup_{0 \neq w_h \in V_h} \frac{b_h^1(w_h, q_h) + b_h^2(w_h, \tilde{q}_h)}{\|w_h\|_{\Omega}} \geq C \left( \|q_h\|_{\Omega} + h_{\min} h_{\max}^{-1} \left( \sum_{K \in T} h_{K} \|\tilde{q}_h\|_{\partial K}^2 \right) \right) \geq C \|q_h\|_{q}. $$

Error Estimates Following from Taylor’s Theorem

We prove here Eqs. 4.5a, 4.5b. Let $D \subset \mathbb{R}^d$. Then for $\psi$ a regular enough function defined on $[0, T] \times D$, using Taylor’s theorem,

$$ \Delta t \|\partial_t \psi^i - d_t \psi^i\|_{0,D} = \|\Delta t \partial_t \psi^i - (\psi^i - \psi^{i-1})\|_{0,D} $$

Data Availability Statement Enquiries about data availability should be directed to the authors.
\[
\begin{align*}
&= \| \int_{t_{i-1}}^{t_i} \partial_t \psi(t) \left( t - t_{i-1} \right) \, dt \|_{0,D} \\
&\leq \Delta t \int_{t_{i-1}}^{t_i} \| \partial_t \psi(t) \|_{0,D} \, dt \\
&= \Delta t \| \partial_t \psi \|_{L^1(t_{i-1},t_i;L^2(D))},
\end{align*}
\]
from which Eqs. 4.5a follows. Next, to show Eq. 4.5b we use the identity

\[
\Delta t d_t e^{L,i}_\psi = \int_{t_{i-1}}^{t_i} \left( \partial_t \psi(s) - \Pi_Q \partial_t \psi(s) \right) ds,
\]
and Eq. 4.1. Then, by the approximation property of the $L^2$-projection,

\[
\Delta t \sum_{i=1}^{m} \| d_t e^{L,i}_\psi \|_{0,D} \leq \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \| \partial_t \psi(s) - \Pi_Q \partial_t \psi(s) \|_{0,D} \, ds \\
\leq Ch^l \| \partial_t \psi \|_{L^1(t;H^l(D))}.
\]

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