On Symmetry Properties of Frobenius Manifolds and Related Lie-Algebraic Structures

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Abstract: The aim of this paper is to develop an algebraically feasible approach to solutions of the oriented associativity equations. Our approach was based on a modification of the Adler–Kostant–Symes integrability scheme and applied to the co-adjoint orbits of the diffeomorphism loop group of the circle. A new two-parametric hierarchy of commuting to each other Monge type Hamiltonian vector fields is constructed. This hierarchy, jointly with a specially constructed reciprocal transformation, produces a Frobenius manifold potential function in terms of solutions of these Monge type Hamiltonian systems.

Keywords: Witten–Dijkgraaf–Verlinde-Verlinde associativity equations; oriented associativity equations; loop lie algebras; Frobenius manifold potential function; Adler–Kostant–Symes scheme; Lie-algebraic analysis; compatible Hamiltonian flows; reciprocal transformation

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1. The Introductory Setting

Let us start with an interesting mathematical structure, suggested in [1–5], on the space of smooth functions: consider a real-valued $C^\infty$-smooth differentiable Frobenius manifold potential function $F \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and denote their partial derivatives as

$$F_{ij}(t) := \frac{\partial^2 F(t)}{\partial t_i \partial t_j}, F_{ijk}(t) := \frac{\partial^3 F(t)}{\partial t_i \partial t_j \partial t_k}$$

(1.1)

for $i, j,$ and $k = \overline{1,n}, n \in \mathbb{N}$. These partial derivatives are symmetrical, with respect to permutations of their indices. Let us assume additionally that the symmetric matrix $\eta := \{\eta_{ij}(t) := F_{ij}(t) : i, j = \overline{1,n}\}$ is non-degenerate, and call it an induced metric on the $\mathbb{R}^n$. In addition,

$$F_{ijk}(t) = \sum_{s \in \overline{1,n}} \eta_{is}(t) C^{is}_{jk}(t)$$

(1.2)

where, by definition,

$$C^{is}_{jk}(t) := \sum_{k \in \overline{1,n}} F_{ijk}(t) \eta^{ks}(t), \quad \sum_{k \in \overline{1,n}} \eta^{sk}(t) \eta_{kj}(t) = \delta^s_j$$

(1.3)

for all $i, j,$ and $s \in \mathbb{N}$. Assume now that the set $\mathbb{R}^n$ represents a local coordinate frame [6,7] of an a finite-dimensional manifold $M$. Then its tangent space $T_t(M)$ at a point $t \in M$ is described by means of the local vector field system $\{\partial/\partial t_i \in T_t(M) : i = \overline{1,n}\}$, which a priori commute to each other: $[\partial/\partial t_i, \partial/\partial t_j] = 0$ for all $i, j = \overline{1,n}$. Let us now assume that the manifold $M$ is a Frobenius manifold [8–10], i.e., its tangent space $T_t(M)$ at any point $t \in M$ forms an associative Frobenius algebra $F_M$ with respect to some multiplication “$\circ$” on $F_M$.
\[
\frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} := \sum_{s=1}^{\infty} C_{ij}^s(t) \frac{\partial}{\partial t_s}, \quad \left( \frac{\partial}{\partial t_i} \circ \frac{\partial}{\partial t_j} \right) \circ \frac{\partial}{\partial t_s} = \frac{\partial}{\partial t_i} \circ \left( \frac{\partial}{\partial t_j} \circ \frac{\partial}{\partial t_s} \right) \tag{1.4}
\]

for any \(i, j, s = 1, n\) with the structure constants defined by the expression (1.3). Define now a set of matrices \(C_i(t) := \{ C_{ij}^k(t) = C_{ij}^k(t) : j, k \in \mathbb{N}, i = 1, n \}\). Then, as it easily follows from (1.4), the structure constants (1.3) should satisfy the following additional constraints:

\[
[C_i(t), C_j(t)] = 0, \quad \partial C_i(t) / \partial t = \partial C_j(t) / \partial t_i \tag{1.5}
\]

for any \(t \in M\) and all \(i, j = 1, n\). (1.5) are called the Witten–Dijkgraaf–Verlinde–Verlinde, or oriented associativity WDVV equations. These equations were first investigated in [11–13] for problems related with topological and string quantum field theory of elementary particles. A nice introduction into the topic can be found in B. Dubrovin Lecture Notes [2]. Lie-algebraic aspects of these equations and related integrability properties can be found in recent works [14,15].

The notion of a Frobenius manifold was first axiomatized and thoroughly studied by B. Dubrovin [2–5] in the early nineties, and plays a central role in mirror field theory symmetry [16–18], theory of unfolding spaces of singularities [19], quantization theory [20,21], quantum cohomology [8], and integrability theory [1,19,22–31] of dispersion-less many-dimensional systems.

A full Frobenius structure on \(M\) consists of the data \((\circ, e, \eta, E)\). Here \(\circ : T(M) \otimes S T(M) \to T(M)\) is an associative and commutative multiplication on the tangent sheaf, so that \(T(M)\) becomes a sheaf of commutative algebras over the ring \(\mathbb{R}\{t\}\) of convergent series with identity \(e \in T(M), \eta\) is a metric on \(M\) (non-degenerate quadratic form \(T(M) \otimes S T(M)\)), and \(E\) is a so called Euler vector field. These structures are connected by various constraints and compatibility conditions, and are presented in [2,3] and [32,33]. For example, the metric \(\eta\) must be flat and \(\eta\subset\subset–\)-invariant, i.e., \(\langle a \circ b | c \rangle_{\eta} = \langle a \circ b | c \rangle_{\eta}\) for the metric \((\cdot | \cdot)_{\eta}\) on \(M\) and any \(a, b, c \in T(M)\). Various weaker versions of the Frobenius structure are interesting in themselves and also appear in [19–21] in different contexts.

Let us also mention an additional notion of a unital Frobenius manifold \(F_M\), introduced in [10] and further studied in [9]. This structure consists of an associative and commutative multiplication \(\circ\subset\subset–\) on the tangent sheaf as above, satisfying the following properties: \(1^0\) a flat structure \(T(M)\) on \(M\) subject to a flat connection \(d_{\omega} : \Gamma(A(M) \otimes T(M)) \to \Gamma(A(M) \otimes T(M))\), \(d_{\omega}d_{\omega} = 0\), is compatible with a multiplication \(\circ\subset\subset–\), if in a neighborhood of any point there exists a vector field \(C \in \Gamma(T(M))\), such that for arbitrary local flat vector fields \(X, Y \in \Gamma(T(M))\) one has

\[
X \circ Y = [X, [Y, C]], \quad \tag{1.6}
\]

where \(C \in \Gamma(T(M))\) is called a local vector potential for \(\circ; 2^0\) \(T(M)\) is called compatible with \((\circ, e), e \in \Gamma(T(M))\) is an identity element, if \(1^0\) holds and moreover, the identity element \(e := \partial / \partial t_1\) is flat, that is the corresponding covariant derivative \(\nabla_{X} e = 0\) for any \(X \in \Gamma(T(M))\). From (1.6) one easily ensues the relationships (1.5), where

\[
C_{ij}^k(t) = \partial / \partial t_i \partial / \partial t_j \mathcal{C}^k(t), \quad \partial / \partial t_i \circ \partial / \partial t_j = \partial / \partial t_i, \tag{1.7}
\]

for any \(i, j, k = 1, n\) and \(t \in M\).

As a very interesting example of the above construction can be obtained for the special case \(n = 3\). We can take into account a reduction of the commuting matrices \(C_j \in \text{End } \mathbb{R}^4, j = 1, 3\), presented in [1–3]. Namely, assume that a smooth Frobenius manifold potential function \(F \in C^\infty(\mathbb{R}^4, \mathbb{R})\) is representable as

\[
F(t) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + f(t_1, t_2, t_3), \quad \tag{1.8}
\]
where a smooth mapping $f : \mathbb{R}^3 \to \mathbb{R}$ satisfies, following from (1.4) in the form $(\partial/\partial t_2 \circ \partial/\partial t_2) \circ \partial/\partial t_3 = \partial/\partial t_2 \circ (\partial/\partial t_2 \circ \partial/\partial t_3)$, $\partial/\partial t_1 \circ \partial/\partial t_1 = \partial/\partial t_1, j = 1, 3$, such a partial differential equation:

$$f_{t_2 t_3}^2 - f_{t_3 t_3}^2 - f_{t_2 t_2} f_{t_2 t_3} = 0$$  \hspace{1cm} (1.9)

for any $(t_1, t_2, t_3) \in \mathbb{R}^3$. In particular, as it was shown by B. Dubrovin and Y. Manin \cite{2,3,32,33}, the Equation (1.9) allows the following system of compatible (for any parameter $p \in C\backslash\{0\}$) linear differential equations:

$$\frac{\partial x}{\partial t_1} = \frac{1}{p} C_1 x, \quad \frac{\partial x}{\partial t_2} = \frac{1}{p} C_2 x, \quad \frac{\partial x}{\partial t_3} = \frac{1}{p} C_3 x$$  \hspace{1cm} (1.10)
on vectors $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, determined by matrices

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & b & c \\ 1 & a & b \\ 0 & 1 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & c & b^2 - ac \\ 0 & b & c \\ 1 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (1.11)

where $a := f_{t_2 t_2}, b := f_{t_2 t_3}, c := f_{t_2 t_3}$ and generating the corresponding loop $Diff(R^3)$-group diffeomorphisms. It is easy also to check that matrices (1.11) satisfy the matrix Equation (1.5), that is

$$[C_2, C_3] = 0 = [C_1, C_j], \quad \frac{\partial C_3}{\partial t_2} = \frac{\partial C_2}{\partial t_3}, [C_2, C_3] = 0 = \frac{\partial C_j}{\partial t_1},$$  \hspace{1cm} (1.12)

for $t \in M, j = 1, 3$. An effective Lie-algebraic analysis of the Dubrovin–Manin linear system (1.10) was recently presented in \cite{14,15}.

In the present work, based on a modification of the Adler–Kostant–Symes integrability scheme, applied to the co-adjoint orbits of the loop diffeomorphism group of circle, a new two-parametric hierarchy of commuting to each other Monge type Hamiltonian vector fields

$$u_{t_1} = u_x, \quad v_{t_1} = v_x, \quad u_{t_2} = -(u^2 + 2v)x, \quad v_{t_2} = (v^2 - 2uv)x, \quad u_{t_3} = (3/2) u^2 - 6uv - u^3) + v) = (-3v^3 - 3uv^2 + 3u^2v - 3v^2)x,...$$  \hspace{1cm} (1.13)

and

$$u_{t_3} = (3/2) u^2 - 6uv - u^3) + v) = (-3v^3 - 3uv^2 + 3u^2v - 3v^2)x,...$$  \hspace{1cm} (1.14)
on a pair of smooth functions $(u, v) \in C^\infty(M; \mathbb{R}^2)$ is constructed. Making use of a suitably constructed reciprocal transformation, applied to this hierarchy, one gives rise to constructing a Frobenius manifold potential function in terms of solutions to these Hamiltonian systems. In particular, we succeeded in describing a class of Frobenius manifold structures, generated by the non-linear Monge type evolution systems (1.13) and (1.14).

Proposition 1. Let a function $F : M \to \mathbb{R}$ be defined by the following differential relationships

$$\frac{\partial^2 F(t_1, t_2, t_3)}{\partial t_1 \partial t_2} = v, \quad \frac{\partial^2 F(t_1, t_2, t_3)}{\partial t_1 \partial t_3} = v(2u - v),$$  \hspace{1cm} (1.15)

where the pair of functions $(u, v) \in C^\infty(M; \mathbb{R}^2)$ satisfies the evolution flows (1.13) and (1.14).

Then this function $F : M \to \mathbb{R}$ is a potential function of the Frobenius manifold $M$, describing the related Frobenius manifold algebraic structures.
2. Frobenius Manifolds, the Related Compatible Co-Adjoint Loop Lie Algebra and Integrability

Consider now the functional Lie algebra \( \mathcal{G} \simeq (C^\infty(T^*(\mathbb{S}^1)); \mathbb{R}; \{\cdot,\cdot\}) \), generated by special Hamiltonian vector fields on the cotangent space \( T^*(\mathbb{S}^1) \) to the circle \( \mathbb{S}^1 \) and endowed with the canonical Lie commutator

\[
\{a, b\}(x; p) := \frac{\partial}{\partial p} a(x; p) \frac{\partial}{\partial x} b(x; p) - \frac{\partial}{\partial p} b(x; p) \frac{\partial}{\partial x} a(x; \lambda)
\]  

(2.1)

for any \( a, b \in \mathcal{G} \) at point \((x, p) \in T^*(\mathbb{S}^1)\). This algebra possesses the following symmetric and non-degenerate bi-linear form:

\[
(a|b) := \int_{\mathbb{R}} dp \int_{\mathbb{S}^1} dxa(x; p)b(x; p)dx,
\]

(2.2)

with respect to which \( \mathcal{G}^* \simeq \mathcal{G} \). Moreover, the Lie algebra is metrized with respect to the bilinear form (2.2) as it is \( ad \)-invariant: \( (a|b|c) = (a,b|c) \) for any \( a, b, c \in \mathcal{G} \).

Below, we will consider the case when the Lie algebra \( \mathcal{G} \) allows splitting into the direct sum of two sub-algebras: \( \mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_- \), where

\[
\mathcal{G}_+ := \{a(x; p) = \sum_{j \in \mathbb{N}} a_j(x)p^j \in \mathcal{G}\}
\]  

(2.3)

and

\[
\mathcal{G}_- := \{b(x; p) = \sum_{0 \leq j < \infty} b_j(x)p^{-j} \in \mathcal{G}\},
\]

(2.4)

as \( p \to \infty \), for which the following dual isomorphisms \( \mathcal{G}_+^* \simeq \mathcal{G}_-^*, \mathcal{G}_-^* \simeq \mathcal{G}_+^* \) hold.

Proceed now to describing via the classical Adler–Kostant–Symes scheme [34–39] commuting co-adjoint orbits of the Lie algebra \( \mathcal{G} \) on the adjoint space \( \mathcal{G}^* \simeq \mathcal{G} \), generated by smooth Casimir functionals \( h \in I(\mathcal{G}^*) \) with respect to the classical Lie-Poisson bracket on \( \mathcal{G}^* \simeq \mathcal{G} \):

\[
\{h(l), (l|a)\} := (l|[\nabla h(l), a]) = 0
\]  

(2.5)

for \( l \in \mathcal{G}^* \) and arbitrary \( a \in \mathcal{G} \), where, by definition, \( \frac{d}{dt} h(l + eb)|_{t=0} := (\nabla h(l)|b) \) for any \( b \in \mathcal{G} \). Namely, the following Hamiltonian flows on \( \mathcal{G}^* \)

\[
\frac{d}{dt} l = -ad^*_{\nabla h^{(k)}(l)} l = [l, \nabla h^{(k)}(l)],
\]

(2.6)

where, by definition, \( \nabla h^{(k)}(l) := \nabla h^{(k)}(l)|_{\mathcal{G}^*} \), are commuting to each other subject to the corresponding evolution parameters \( t_k \in \mathbb{R}, k \in \mathbb{Z}_+ \), for arbitrary infinite hierarchy of smooth functionally independent Casimir functionals \( h^{(k)} \in I(\mathcal{G}^*), k \in \mathbb{Z}_+ \). The latter is, evidently, equivalent to the following Lax-Sato type vector field representations:

\[
[\partial/\partial t_k + \nabla h^{(k)}(l), \partial/\partial t_m + \nabla h^{(m)}(l)] = 0
\]  

(2.7)

for all \( k, m \in \mathbb{Z}_+ \), where, by definition, any element \( a \in \mathcal{G} \) via the expression \( \bar{a}(x; p) := \frac{\partial}{\partial x} p - \frac{\partial}{\partial p} \in \Gamma(T(x,p)(T^*(\mathbb{S}^1))) \) generates a canonical Hamiltonian vector field on \( T^*(\mathbb{S}^1) \) at point \((x; p) \in T^*(\mathbb{S}^1)\).

Take now an analytic at the momentum \( p \in \mathbb{R} \) element \( l \in \mathcal{G}^* \simeq \mathcal{G} \) in the following asymptotic form as \( p \to \infty \):

\[
l(x; p) = p + u(x) + \sum_{j \in \mathbb{N}} l_j(x)p^{-j}
\]  

(2.8)
where the element \( p \in G^* \) is considered here as an infinitesimal Lie algebra \( G \) character, satisfying the conditions \([G_z, p] \in G_z\), that can be easily checked by direct computations. The flows (2.6) are equivalent to the following co-adjoint action
\[
\partial l_\pm / \partial t_k = -ad^*_{\partial l_\pm / \partial x} l_\pm = [\nabla h_k^{(k)}(l), l_\pm]_-
\] (2.9)
on \( G^* \cong G \) with respect to the evolution parameters \( t_k \in \mathbb{R} \) for all \( k \in \mathbb{Z}_+ \).

It is worthy to observe now that in the case of the Casimir functionals \( h^{(k)} := \frac{1}{k+1}(l^k|l), k \in \mathbb{Z}_+ \), the flows (2.9) can be equivalently rewritten as the Hamiltonian systems
\[
\partial \tilde{l}(x; p)/\partial t_k = [\tilde{l}_k(x; p), \tilde{l}(x; p)]
\] (2.10)on \( G^* \) for all \( k \in \mathbb{Z}_+ \), where, by definition, \( \tilde{l}(x; p) := \frac{\partial l}{\partial p} - \frac{\partial l}{\partial x} \in \Gamma(T(x;p)(T^*(\mathbb{S}^1))) \) at point \((x;p) \in T^*(\mathbb{S}^1)\). Using the Lie bracket (2.1), the equations (2.10) can be rewritten as the Hamiltonian flows on the cotangent space \( T^*(\mathbb{S}^1) \)
\[
\partial \tilde{l}(x; p)/\partial t_k = \{H_k(x, p), l(x; p)\},
\] (2.11)where, by definitions, \( H_k(x, p) = l_k(x; p) \) for any \( k \in \mathbb{N}, (x; p) \in T^*(\mathbb{S}^1) \).

**Remark 1.** It is worth also to remark here that we can pose the following vector field iso-spectral problem
\[
\tilde{l}(x; p)\psi(x; p|z) = z \psi(x; p|z),
\] (2.12)where \( \psi(;z) \in C^\infty(T^*(\mathbb{S}^1);\mathbb{C}) \) is the eigenfunction corresponding to an eigenvalue \( z \in \mathbb{C} \), which is a priori invariant with respect to all vector fields (2.10). The latter naturally allows to apply to (2.12) the modified inverse scattering transform technique developed in [40] and describe many classes of symbols \( l \in G \), generating important dispersion-less heavenly type [41] dynamical systems, important for applications in modern mathematical physics.

As the point variables \((x; p) \in T^*(\mathbb{S}^1)\) are constant parameters for the evolution flows (2.10) on analytic at \( p = \infty \) element \( l \in G^* \), one can put, by definition, \( l(x; p) = z \in \mathbb{C} \) and resolve the functional equation \( l(x; p) = z \) with respect to the symbol parameter \( p \in \mathbb{R} \), obtaining the following expression:
\[
p := \xi(x; z) = z - u - \sum_{j \in \mathbb{N}} \xi_j(x)z^{-j}
\] (2.13)with coefficients \( \xi_j \in C^\infty(\mathbb{S}^1;\mathbb{R}), j \in \mathbb{N} \), characterized by the following lemma.

**Lemma 1.** The element \( \xi \in C^\infty(\mathbb{S}^1 \times \mathbb{R};\mathbb{C}) \) satisfies the following hierarchy of compatible evolution equations
\[
\frac{\partial}{\partial t_k} \xi(x; z) = \frac{\partial H_k(x; z)}{\partial x},
\] (2.14)where the elements \( H_k(x; z) := l_k(x; \xi(x; z)), k \in \mathbb{N} \), are determined, using the following simple algebraic expressions:
\[
H_k(x; z) := H_k(x; \xi(x; z)),
\] (2.15)which hold jointly with compatibility relationships
\[
\frac{\partial H_k(x; z)}{\partial t_k} = \frac{\partial H_k(x; z)}{\partial t_s}
\] (2.16)for all \( k, s \in \mathbb{N} \).
Proof. Making use of the Equation (2.10), one can easily calculate for any $k \in \mathbb{N}$ the evolution equations

$$\frac{\partial}{\partial t_k} \left( \frac{1}{\xi(x;z) - p} \right) := \left\{ H_k(x;p), \frac{1}{\xi(x;z) - p} \right\},$$

giving rise to the following expressions

$$\frac{\partial \xi(x;z)}{\partial t_k} = \frac{\partial H_k(x;p)}{\partial x} + \frac{\partial H_k(x;p)}{\partial p} \bigg|_{p=\xi(x;z)} \frac{\partial \xi(x;z)}{\partial x} = \frac{dH_k(x;\xi(x;z))}{dx} := \frac{\partial H_k(x;z)}{\partial x},$$

(2.17)

which hold for all $k \in \mathbb{N}$ and all $z \in \mathbb{R}$. The compatibility relationships are obvious, following from the commuting to each other flows (2.14).

Consider now the functional identity

$$\frac{1}{\xi(x;z) - p} = \sum_{k \in \mathbb{N}} \frac{z^{-k}}{k} \frac{\partial}{\partial p} H_k(x;p),$$

(2.18)

which is satisfied as $z \to \infty$, owing to the following residuum calculation:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{z^{-k-1}dz}{\xi(x;z) - p} = \frac{1}{2\pi i} \oint_{|z|=R} \frac{z^{-k-1}dz}{\xi(x;z) - p} = (x;p)^{k-1} l(x;p) / \partial p \bigg|_{z=\infty} = \frac{1}{k} \frac{\partial}{\partial p} H_k(x;p),$$

(2.19)

which holds for any $k \in \mathbb{N}$. Consider now Hamiltonian functions $H_k : T^*(S^1) \to \mathbb{R}, k \in \mathbb{N}$, and consider the related canonical Hamiltonian vector fields on the cotangent space $T^*(\mathbb{R})$:

$$\frac{\partial x}{\partial t_k} = \frac{\partial H_k(x;p)}{\partial p}, \quad \frac{\partial p}{\partial t_k} = - \frac{\partial H_k(x;p)}{\partial x},$$

(2.20)

with respect to a point $(x,p) \in T^*(S^1)$ subject to the evolution parameter $t_k \in \mathbb{R}, k \in \mathbb{N}$. Taking into account the evolution flows (2.20) and the fact that $\partial / \partial t_1 = \partial / \partial x$, the identity (2.18) can be rewritten as

$$\frac{1}{\xi(x;z) - p} = \sum_{k \in \mathbb{N}} \frac{z^{-k}}{k} \frac{\partial x}{\partial t_k} = D(z)x(t),$$

from which and the relationships (2.16) one ensues the functional representation

$$\xi(x;z) = z - \frac{\partial \mathcal{F}(t)}{\partial x} - D(z) \frac{\partial \mathcal{F}(t)}{\partial x}$$

(2.21)

for some smooth function $\mathcal{F} : M \to \mathbb{R}$. Based now on Lemma 1 and relationships (2.18), (2.19) one can state now the following proposition.

**Proposition 2.** Let $F : M \to \mathbb{R}$ be a potential function on the Frobenius manifold $M$, defined by means of the set of asymptotic relationship

$$D(y)F(t) + D(y)D(z)F(t) = - \ln(1 - z/y) - \sum_{k \in \mathbb{N}} \frac{y^{-k}}{k} H_k(x;z)$$

(2.22)

where, by definition, the operator $D(\alpha) = \sum_{k \in \mathbb{N}} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}, \alpha \in \mathbb{R},$ is the well known vertex operator. Then the element (2.13) satisfies the asymptotic representation (2.21) for all $x \in S^1$ as $z \to \infty$. 
The functional identity (2.22) easily reduces to the set of asymptotic expressions

$$\mathcal{H}_k(x;z) = z^k - \partial F/\partial t_k - D(z)\partial F/\partial t_k$$

(2.23)

for all $k \in \mathbb{N}$ as $z \to \infty$. Simultaneously one can observe that the expression (2.14) and (2.15) reduce to the representation (2.21), proving the proposition. □

This proposition is useful for constructing Frobenius manifolds, naturally related with some generating function $\mathcal{F} : M \to \mathbb{R}$, satisfying the relationship (2.21). As an example, we suggest the following element

$$l(x;p) = p + u(x) + \ln\left(1 + \frac{v(x)}{p}\right) \in \mathcal{G}^*,$$

(2.24)

where $u, v \in C^\infty(S^1; \mathbb{R})$ are some functional parameters. The corresponding Casimir functions $h^{(t_1)} := (l^1)/2, h^{(t_2)} := (l^2)/3$ and $h^{(t_3)} := (l^3)/4, h^{(t_4)} := (l^4)/5$, etc., generate the following Hamiltonian flows on $\mathcal{G}^* \simeq \mathcal{G}$:

$$\partial l/\partial x = [l_+, l_1], \partial l/\partial y = [l_1, l], \quad \partial l/\partial t = [l_3, l], \partial l/\partial s = [l_4, l]$$

(2.25)

with respect to the evolution parameters $x = t_1 \in \mathbb{R}, t_2, t_3 \in \mathbb{R}$, etc., where, for instance,

$$l_+^2 : = H_2(x;p) = p^2 + 2pu \in \mathcal{G}_+, \quad l_3 : = H_3(x;p) = p^3 + 3p^2u + 3pu^2 + 3v \in \mathcal{G}_+$$

(2.26)

and so on. The above commutator expressions with respect to the evolution parameters $t_1, t_2$ and $t_3 \in \mathbb{R}$ reduce to the next commuting to each other non-linear Monge type evolution systems

$$u_{t_1} = u_x, \quad v_{t_1} = v_x, \quad u_{t_2} = -(u^2 + 2v)_x, \quad v_{t_2} = (u^2 - 2uv)_x$$

(2.27)

and

$$u_{t_3} = (\frac{3}{2}v^2 - 6uv - u^3)_x, \quad v_{t_3} = (-3v^3 - 3uv^2 + 3u^2v - 3v^2)_x$$

(2.28)

being also compatible dispersion-less Hamiltonian flows on the corresponding functional phase. Moreover, the evolution systems (2.27) and (2.28) are equivalent to the Lax-Sato vector field commutator representation (2.7), where

$$\nabla h^{(l)}_+ (\bar{l}) = (p + u) \frac{\partial}{\partial x} - u_s p \frac{\partial}{\partial p}, \quad \nabla h^{(l)}_+ (\bar{l}) = (p^2 + 2up + 2v + u^2) \frac{\partial}{\partial x} - (u_s p^2 + v_s p + 2uv p) \frac{\partial}{\partial p}.$$  

(2.29)

The vector fields (2.29), being considered as elements of the Lie algebra $\mathcal{G} \simeq \text{diff}(S^1 \times \mathbb{C})$ of holomorphic with respect to the variable $p \in \mathbb{C}$ vector fields on $S^1 \times \mathbb{C}$, naturally splits into the direct sum of two sub-algebras $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$, holomorphic in the parameter $p \in \mathbb{C}$ inside $\mathbb{D}^2_+(0)$ of the unit circle $\mathbb{D}^2_+(0) \subset \mathbb{C}$ and outside $\mathbb{D}^2_-(0)$ of this disk, respectively, appear to be generated by the corresponding Casimir functionals on the adjoint space $\mathcal{G}^* \simeq \Omega^1(S^1 \times \mathbb{C})$ at some root element $\bar{l} \in \mathcal{G}^*$ subject to the following canonical non-degenerate bi-linear form on $\mathcal{G}^* \times \mathcal{G}$:

$$(\bar{l}|\bar{a}) := \int_0^{2\pi} \text{res}_p (\bar{l}|p) dx, \quad (2.30)$$
where we put, by definition, \( \tilde{I} := \langle l | dx \rangle, \tilde{a} := \langle a | \partial / \partial x \rangle, x := (p; x) \in \mathbb{C} \times \mathbb{S}^1 \). Based on the definition of Casimir functionals, one easily enough obtains that this root element equals

\[
\tilde{I} = (u_x p^2 + (v + u^2)_x) dx + (p^2 + 2up + v + u^2) dp =
\]

\[
= d\left( \frac{1}{3} p^3 + up^2 + (v + u^2)p \right),
\]

being a complete derivative of the scalar element \( \tilde{\eta} = \frac{1}{2} p^3 + up^2 + (v + u^2)p \in \Omega^0(\mathbb{S}^1 \times \mathbb{C}), \tilde{I} = d\tilde{\eta}, \) for all \( (p; x) \in \mathbb{C} \times \mathbb{S}^1 \). Moreover, the system of evolution equations (2.27) and (2.28) becomes equivalent to the following co-adjoint flows

\[
\partial \tilde{I} / \partial y = -ad^*_{\tilde{\nu}^{(2)}(\tilde{I})}, \partial \tilde{I} / \partial t = -ad^*_{\tilde{\nu}^{(3)}(\tilde{I})},
\]

on the adjoint space \( \tilde{G}^* \), generated by the corresponding Casimir functionals \( \tilde{h}^{(2)}, \tilde{h}^{(3)} \in I(\tilde{G}^*) \) and satisfying the determining relationships \( ad^*_{\tilde{\nu}^{(2)}(\tilde{I})} \tilde{I} = 0, ad^*_{\tilde{\nu}^{(3)}(\tilde{I})} \tilde{I} = 0 \). As now the basic Lie algebra \( \tilde{G} \simeq diff(\mathbb{S}^1 \times \mathbb{C}) \) of holomorphic vector fields on \( \mathbb{S}^1 \times \mathbb{C} \) is not, evidently, metrized, the flows (2.32) on \( \tilde{G}^* \) do not possess the standard Lax type commutator representation.

Taking into account the expressions (2.21) and (2.24), one can formulate the following proposition.

**Proposition 3.** Let a function \( F : M \rightarrow \mathbb{R} \) be defined by the following differential relationships

\[
\frac{\partial^2 F(t_1, t_2, l_3)}{\partial t_1 \partial t_2} = v, \quad \frac{\partial^2 F(t_1, t_2, l_3)}{\partial t_1 \partial t_3} = v(2u - v),
\]

\[
\frac{\partial^2 F(t_1, t_2, l_3)}{\partial t_2 \partial t_3} = 2v[v^2 + 3v - 3u(u - v)],
\]

where the pair of functions \( (u, v) \in \mathbb{C}^\infty(M; \mathbb{R}^2) \) satisfies the evolution flows (2.27) and (2.28). Then it is a potential function of the Frobenius manifold \( M \), describing the related Frobenius manifold algebraic structures.

This result makes it possible to describe a wide variety of Frobenius manifold potential functions in terms of solutions to these Monge type Hamiltonian systems (2.27) and (2.28).

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