Renormalizing a BRST-invariant composite operator
of mass dimension 2 in Yang-Mills theory

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Abstract

We discuss the renormalization of a BRST and anti-BRST invariant composite operator of mass dimension 2 in Yang-Mills theory with the general BRST and anti-BRST invariant gauge fixing term of the Lorentz type. The interest of this study stems from a recent claim that the non-vanishing vacuum condensate of the composite operator in question can be an origin of mass gap and quark confinement in any manifestly covariant gauge, as proposed by one of the authors. First, we obtain the renormalization group flow of the Yang-Mills theory. Next, we show the multiplicative renormalizability of the composite operator and that the BRST and anti-BRST invariance of the bare composite operator is preserved under the renormalization. Third, we perform the operator product expansion of the gluon and ghost propagators and obtain the Wilson coefficient corresponding to the vacuum condensate of mass dimension 2. Finally, we discuss the connection of this work with the previous works and argue the physical implications of the obtained results.

Key words: Renormalization, composite operator, BRST symmetry, Yang-Mills theory, mass gap, quark confinement

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1 Introduction

It is still a challenging and unsolved problem to prove quark confinement in the framework of quantum chromodynamics (QCD). A very beginning question in deriving quark confinement is in what sense quark is confined? A simple criterion of quark confinement which has been widely used so far is the area law decay of the Wilson loop (defined by the vacuum expectation value of the Wilson loop operator). The area law implies the presence of a linear piece $\sigma r$ proportional to the interquark distance $r$ in the static interquark potential $V(r)$. The dual superconductivity of QCD vacuum [1] is one of the most promising mechanisms of quark confinement in compatible with this picture. However, it is well known that this criterion is not so useful in the presence of dynamical matter, since the interquark force is screened by a pair of quark and anti-quark created from the vacuum and the linear piece no longer appears in the potential.

In the previous paper [2], one of the authors (K.-I. K.) has proposed a non-vanishing vacuum condensate $\langle O \rangle$ of mass dimension 2 as the origin of mass gap and quark confinement in Yang-Mills theory. The proposed composite operator of mass dimension 2 is given by

$$O := \frac{1}{\Omega(D)} \int d^D x \, \text{tr} \left[ \frac{1}{2} A_\mu(x) \cdot A_\mu(x) + \alpha i \bar{C}(x) \cdot \bar{C}(x) \right], \quad (1.1)$$

where $A_\mu$ is the gauge field, $C$ ($\bar{C}$) is the ghost (anti-ghost) field and $\Omega(D)$ denotes the volume of the $D$-dimensional spacetime. It has been shown [2] that the composite operator $O$ is invariant under the Becchi-Rouet-Stora-Tyutin (BRST) [3] and anti-BRST [4] transformations in the manifestly Lorentz covariant gauge, especially in the most general Lorentz gauge [11] and the Maximal Abelian (MA) gauge [19–20]. In (1.1), the trace is taken over the broken generators of the Lie algebra $G$ of the original group $G$ when the original gauge group $G$ is broken to $H$ by a local gauge fixing condition chosen, i.e., $G$ itself in the Lorentz gauge and $G/H$ in the MA gauge corresponding to the maximal torus group $H$ of $G$. Especially, in the limit $\alpha \to 0$ (which we call the Landau gauge), the composite operator reduces to $O = (\Omega(D))^{-1} \int d^D x \, \text{tr} \left[ \frac{1}{2} A_\mu(x) \cdot A_\mu(x) \right]$ and hence it becomes gauge invariant, since the contribution from the ghost and anti-ghost disappears. The vacuum condensate includes the ghost condensation proposed in the MA gauge [19,20] and reduces to the gluon condensation proposed recently by several authors [21–24], see also [25,26].

The physical implication of the existence of such a condensate $\langle O \rangle$ has been argued based on the operator product expansion (OPE) of the gluon and ghost propagators (2-point functions) and the vertex function (3-point function) [2,21,24]. However, the actual calculation has been performed within the tree level.

In order for such a proposal to be meaningful, it is very indispensable to show that the whole strategy to derive quark confinement based on the novel vacuum condensate

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1 The precise definition of ‘the most general’ is stated later in the text. Roughly speaking, the most general Lorentz gauge is obtained by imposing both the BRST and anti-BRST invariance for the gauge fixing term which corresponds to the Lorentz gauge $\partial^\mu A_\mu(x) = 0$. The resulting gauge fixing term has two parameters. The conventional Lorentz gauge is obtained as a special choice of the parameters.
survives the renormalization. In view of this, we focus on the renormalization of the composite operator \( \mathcal{O} \). The main purpose of this paper is to examine whether or not the composite operator in the integrand of \( \mathcal{O} \) is renormalizable. In addition, we must clarify the meaning of the BRST and anti-BRST symmetry in the renormalized theory. We examine whether or not the renormalized composite operator \( \mathcal{O}^R \) is invariant under the renormalized BRST and anti-BRST transformation. If this is the case, the proposed composite operator of mass dimension 2 and the corresponding vacuum condensate can have a definite physical meaning. The analysis of this paper is restricted to the most general Lorentz gauge fixing, since the analysis of the MA gauge is more involved and hence the result is to be reported in a separate paper \([27]\).

In the most general Lorentz gauge, the multiplicative renormalizability of the Yang-Mills theory has been worked out by Baulieu and Thierry-Mieg \([8]\) by making use of the Slavnov-Taylor identities characterizing the BRST and anti-BRST invariance of the theory (See textbooks and reviews, e.g., \([28–34]\)). In the course of renormalizing the composite operator, however, there is a subtle problem of the operator mixing. In order to discuss the renormalization of a composite operator, we must take into account all the contributions coming from all the other composite operators of the same mass dimension and the same symmetry property. In the OPE, the Wilson coefficient corresponding to arbitrary vacuum condensate can be calculated by the perturbation theory. In the usual Lorentz gauge, the Wilson coefficient associated with the ghost condensate \( \langle \bar{C} \cdot C \rangle \) in the OPE of the propagator vanishes identically due to a special property of the 3-point gluon-ghost-anti-ghost vertex as pointed out in \([35]\). In the most general Lorentz gauge \([3,4]\), however, we show in this paper that the operator mixing between two composite operators, \( \frac{1}{2} A_\mu \cdot A_\mu \) and \( i \bar{C} \cdot \bar{C} \), of mass dimension 2 does exist in general due to the presence of four-ghost interaction (except for the case which is reduced to the conventional Lorentz gauge). We explicitly calculate the matrix of renormalization factors of the composite operator in the one-loop level.

For the Landau gauge, the vacuum condensate of mass dimension 2 in Yang-Mills theory is nothing but the gluon pair condensation. A possibility of gluon pair condensation was already suggested from the existence of the tachyon pole in the two gluon channel by solving approximately the Bethe-Salpeter equation, see e.g. \([37]\) and \([38]\). A gluon pair can be identified as a Cooper pair, that is a bound state caused by the attractive force. Hence the gluon condensation is regarded as the Bose condensation of the gluon with spin 1. A remarkable point of our treatment different from the previous one is to retain the manifest Lorentz covariance and gauge (or BRST and anti-BRST) invariance. Hence the introduction of ghost field is dispensable in this approach. It is important to clarify how the inclusion of the ghost influences the dynamics of gluon to recover the gauge invariance. This paper is a preliminary work toward the complete understanding of this problem.

Another purpose of this paper is to point out that the composite operator discussed above has the analog in the Abelian gauge theory, especially, quantum electrodynamics (QED). This suggests that a confinement phase can exist even in QED probably in the strong coupling region \([39–42]\). In QED, the vacuum condensate in question is reduced in the Landau gauge to the photon pairing. The photon pairing has also been suggested long ago by solving the Cooper equation, see \([13,14]\). From quite a different
viewpoint, one of the authors [36] discussed the existence of a confinement phase in QED based on the total QED Lagrangian with the BRST and anti-BRST invariant gauge fixing term which is identical to the usual Lagrangian in the Lorentz gauge up to a total derivative term. An advantage of rewriting the gauge fixing part of the Lagrangian into the BRST and anti-BRST exact form is that the hidden supersymmetry becomes manifest and that the gauge-fixing part in four spacetime dimensions is reduced to the $O(2)$ non-linear sigma model in two spacetime dimensions owing to the Parisi-Sourlas dimensional reduction. In view of this, the ghost is indispensable in this approach even for the Abelian gauge theory where the ghost decouples and is usually considered to be unnecessary. In the analysis of quark confinement, it is most important to understand the origin of the scale or the mechanism of mass generation which was not so clear in the previous treatments. The detailed analysis of this issue will be reported in a subsequent paper.

This paper is organized as follows. In section 2, we summarize the BRST and anti-BRST transformations and their properties which are necessary in the following analyses.

In section 3, we examine how the renormalization in QED is performed so as to preserve the BRST and anti-BRST symmetry. This section is a preliminary step for dealing with the non-Abelian gauge theory in the subsequent sections.

In section 4, we consider the most general Lagrangian of the Yang-Mills theory which has manifest Lorentz covariance, global gauge invariance and BRST and anti-BRST symmetry. The gauge fixing term contains two gauge fixing parameters. We give Feynman rules of this theory and calculate the renormalization constants in the one-loop level. Although some materials in this section are a reconfirmation of the results obtained by Baulieu and Thierry-Mieg [8], it is necessary to make this paper self-contained and to give basic ingredients in the subsequent sections.

In section 5, we obtain the renormalization group flow in the parameter space of the theory. To one-loop order, we specify the location of the fixed points and obtain the equation of the lines of connecting the fixed points.

In section 6, we discuss the main subject of this paper: the renormalization of the composite operator $O$ of mass dimension 2. First, we show when the composite operator $O$ is both BRST and anti-BRST invariant. Next, we evaluate the renormalization of $O$ by taking into account the mixing of the operators with the same mass dimension and the same symmetry. To the best of our knowledge, the renormalization of the composite operator of mass dimension 2 has not been fully discussed except for a special case, i.e., the Landau gauge in the conventional Lorentz gauge fixing [2].

In section 7, we perform the operator product expansion of the gluon and ghost propagators and obtain the Wilson coefficient associated with the vacuum condensates in question.

In the final section, we give the conclusion of this paper and discuss the future directions of our research. In Appendix, we give some of the calculations omitted in the text.

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2 This formulation has been applied to QED at finite temperature and a new confining phase is claimed to exist, see [15] and references therein.
2 BRST and anti-BRST transformation

We consider the general non-Abelian gauge theory with a gauge group \( G \). In the following we use the notation:

\[
F : G := F^A G^A, \quad F^2 := F \cdot F, \quad (F \times G)^A := f^{ABC} F^B G^C,
\]

where \( f^{ABC} \) are the structure constants of the Lie algebra \( \mathfrak{g} \) of the gauge group \( G \).

For the non-Abelian gauge theory, we define the BRST transformation by

\[
\begin{align*}
\delta_B A_\mu(x) &= D_\mu[A] \tilde{C}(x) := \partial_\mu \tilde{C}(x) + g(A_\mu(x) \times \tilde{C}(x)), \quad (2.2a) \\
\delta_B C(x) &= -\frac{1}{2} g(\tilde{C}(x) \times \tilde{C}(x)), \quad (2.2b) \\
\delta_B \tilde{C}(x) &= iB(x), \quad (2.2c) \\
\delta_B B(x) &= 0, \quad (2.2d)
\end{align*}
\]

where \( A_\mu, B, C \) and \( \tilde{C} \) are the non-Abelian gauge field, the Nakanishi-Lautrup (NL) auxiliary field, the Faddeev-Popov (FP) ghost and anti-ghost fields respectively. Another BRST transformation, i.e., anti-BRST transformation \([\bar{\delta}] \) is defined by

\[
\begin{align*}
\bar{\delta}_B A_\mu(x) &= D_\mu[A] \tilde{\bar{C}}(x) := \partial_\mu \tilde{\bar{C}}(x) + g(A_\mu(x) \times \tilde{\bar{C}}(x)), \quad (2.3a) \\
\bar{\delta}_B \tilde{C}(x) &= -\frac{1}{2} g(\tilde{\bar{C}}(x) \times \tilde{\bar{C}}(x)), \quad (2.3b) \\
\bar{\delta}_B \bar{C}(x) &= i\bar{B}(x), \quad (2.3c) \\
\bar{\delta}_B \bar{B}(x) &= 0, \quad (2.3d)
\end{align*}
\]

where \( \bar{\delta}_B \) is defined by

\[
\bar{B}(x) = -B(x) + ig(\tilde{C}(x) \times \tilde{\bar{C}}(x)). \quad (2.5)
\]

The BRST and anti-BRST transformations are nilpotent and they anti-commute:

\[
\delta_B \delta_B \equiv 0, \quad \bar{\delta}_B \bar{\delta}_B \equiv 0, \quad \delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B \equiv 0. \quad (2.6)
\]

For the Abelian gauge theory, the BRST transformation reads

\[
\begin{align*}
\delta_B a_\mu(x) &= \partial_\mu C(x), \quad (2.7a) \\
\delta_B C(x) &= 0, \quad (2.7b) \\
\delta_B \bar{C}(x) &= iB(x), \quad (2.7c) \\
\delta_B B(x) &= 0, \quad (2.7d)
\end{align*}
\]

\( ^3 \) The last transformation is equivalent to

\[
\bar{\delta}_B B(x) = -g\tilde{C}(x) \times B(x). \quad (2.4)
\]
where $A_\mu, B, C$ and $\bar{C}$ are the Abelian gauge field, the NL auxiliary field, the FP ghost and anti-ghost fields respectively. The anti-BRST transformation is reduced to

\begin{align}
\tilde{\delta}_B a_\mu(x) &= \partial_\mu C(x), \\
\tilde{\delta}_B \bar{C}(x) &= 0, \\
\tilde{\delta}_B C(x) &= iB(x), \\
\tilde{\delta}_B \bar{B}(x) &= 0,
\end{align}

where $\bar{B}$ is defined by

\[
\bar{B}(x) = -B(x).
\]

3 QED in the Lorentz gauge

As a warming-up problem, we consider the quantum electrodynamics (QED). As is well known, the total Lagrangian of QED is given by

\[
\mathcal{L}^{\text{tot}}_{\text{QED}} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu a_\mu + \mathcal{L}_{\text{GF+FP}},
\]

with a gauge-fixing (GF) plus FP ghost term $\mathcal{L}_{\text{GF+FP}}$. The explicit form of the GF+FP term depends on the gauge chosen. In this paper we adopt the most familiar covariant gauge, i.e., the Lorentz gauge

\[
\partial^\mu a_\mu = 0.
\]

Therefore, the GF+FP term is given by

\[
\mathcal{L}_{\text{GF+FP}} = -i\delta_B \left( \bar{C} \partial^\mu a_\mu + \frac{\alpha}{2} \bar{\bar{C}} B \right) = B \partial^\mu a_\mu + \frac{\alpha}{2} B^2 + i\bar{C} \partial^\mu \partial_\mu C.
\]

Although the ghost and anti-ghost fields are free and decouple from other fields, we have included them to study the relationship with the non-Abelian case which will be discussed in the next section.

As pointed out in [36], the GF+FP term (3.3) is rewritten into the BRST and anti-BRST exact form,

\[
\mathcal{L}_{\text{GF+FP}} = i\delta_B \bar{\delta}_B \left( \frac{1}{2} a_\mu a^\mu + \frac{\alpha}{2} i\bar{C} \bar{C} \right).
\]

In fact, this is cast into the form,

\[
\mathcal{L}_{\text{GF+FP}} = i\delta_B \left( \left( \delta_B a^\mu \right) a_\mu - \frac{\alpha}{2} i\bar{C} \delta_B C \right) \\
= i\delta_B \left( \partial^\mu \bar{C} a_\mu - \frac{\alpha}{2} \bar{C} B \right),
\]

which agrees with (3.3) up to a total-derivative term.
If the NL field $B$ is eliminated by performing the functional integration or by making use of the equation of motion, then we obtain

$$\mathcal{L}'_{GF+FP} = -\frac{1}{2\alpha} (\partial^\mu a_\mu)^2 + i\bar{C} \partial^\mu \partial_\mu C. \quad (3.6)$$

The on-shell BRST transformation is given by

$$\delta_B a_\mu(x) = \partial_\mu C(x), \quad (3.7a)$$
$$\delta_B C(x) = 0, \quad (3.7b)$$
$$\delta_B \bar{C}(x) = -\frac{i}{\alpha} \partial^\mu a_\mu(x), \quad (3.7c)$$

while the on-shell anti-BRST transformation is

$$\bar{\delta}_B a_\mu(x) = \partial_\mu \bar{C}(x), \quad (3.8a)$$
$$\bar{\delta}_B \bar{C}(x) = 0, \quad (3.8b)$$
$$\bar{\delta}_B C(x) = +\frac{i}{\alpha} \partial^\mu a_\mu(x). \quad (3.8c)$$

The GF+FP Lagrangian $\mathcal{L}'_{GF+FP}$ and the total Lagrangian $\mathcal{L}^\text{tot}_{QED}$ with $\mathcal{L}'_{GF+FP}$ are separately invariant under the on-shell BRST and on-shell anti-BRST transformations. The nilpotency of the on-shell BRST and anti-BRST transformation is realized only when the equation of motion for the ghost and anti-ghost field is used, since

$$\delta_B^2 a_\mu(x) = 0, \quad (3.9a)$$
$$\delta_B^2 C(x) = 0, \quad (3.9b)$$
$$\delta_B^2 \bar{C}(x) = -\frac{i}{\alpha} \partial^\mu \partial_\mu C(x), \quad (3.9c)$$

and

$$\bar{\delta}_B^2 a_\mu(x) = 0, \quad (3.10a)$$
$$\bar{\delta}_B^2 C(x) = +\frac{i}{\alpha} \partial^\mu \partial_\mu \bar{C}(x), \quad (3.10b)$$
$$\bar{\delta}_B^2 \bar{C}(x) = 0. \quad (3.10c)$$

Moreover, we obtain the similar result for the anti-commutability:

$$\delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B a_\mu(x) = 0, \quad (3.11a)$$
$$\delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B C(x) = -\frac{i}{\alpha} \partial^\mu \partial_\mu \bar{C}(x), \quad (3.11b)$$
$$\delta_B \bar{\delta}_B + \bar{\delta}_B \delta_B \bar{C}(x) = +\frac{i}{\alpha} \partial^\mu \partial_\mu C(x). \quad (3.11c)$$

Now we define the composite operator $\mathcal{O}$ of mass dimension 2 by

$$\mathcal{O} := \frac{1}{\Omega(B)} \int d^D x \mathcal{Q}(x), \quad \mathcal{Q}(x) := \frac{1}{2} a_\mu(x) a^\mu(x) + \alpha \bar{C}(x) C(x). \quad (3.12)$$
This composite operator is BRST and anti-BRST invariant, since
\[ \delta_B Q(x) = \partial^\mu (a_\mu(x)C(x)), \quad \bar{\delta}_B Q(x) = \partial^\mu (a_\mu(x)\bar{C}(x)). \] (3.13)

We consider the renormalization of the composite operator \( Q \). The Abelian case is very simple due to the trivial renormalization factors \( Z_{a^2}, Z_{CC} \) for the composite field \( \frac{1}{2}a_\mu a_\mu \) and \( i\bar{C}C \). Therefore, we have only to take into account the renormalization factor of the fundamental field, \( a_\mu, C, \bar{C} \) and the gauge fixing parameter \( \alpha \). QED is known to be multiplicatively renormalizable in the sense that the divergences are absorbed by introducing the renormalization factors in the following way.

\[
\psi = Z_2^{1/2}\psi^R, \\
a_\mu = Z_3^{1/2}a_\mu^R, \\
C = Z_C C^R, \quad \bar{C} = Z_{\bar{C}} \bar{C}^R, \\
(B = Z_3^{-1/2}B^R), \\
m = Z_m Z_2^{-1}m^R, \\
\alpha = Z_\alpha \alpha^R, \\
e = Z_1 Z_2^{-1}Z_3^{-1/2}e^R. 
\] (3.14 - 3.20)

The renormalization factors are not independent to each other. In fact, the coupling constant is renormalized as

\[ e = Z_3^{-1/2}e^R, \] (3.21)

as a consequence of the Ward relation:

\[ Z_1 = Z_2. \] (3.22)

Moreover, the Ward-Takahashi identity yields

\[ Z_\alpha = Z_3. \] (3.23)

The result of perturbative renormalization in QED is well known and can be seen in the text books. A result:

\[ Z_C = Z_{\bar{C}} = 1, \] (3.24)

means that both ghost and anti-ghost are free and receive no renormalization in the perturbation theory (This is not the case in the non-Abelian case). Consequently, we arrive at the result that the composite operator is renormalized as

\[ Q = Z_3^RQ, \quad Q^R := \frac{1}{2}a_\mu^R(x)a^\mu(x) + \alpha^R i\bar{C}^R(x)C^R(x). \] (3.25)

Therefore, the BRST invariant combination of two composite operators with mass dimension 2 is preserved under the renormalization.

In view of the above results, the renormalized BRST transformation is defined by

\[ \delta_B^R = Z_3^{1/2}\delta_B, \quad \bar{\delta}_B^R = Z_3^{1/2}\bar{\delta}_B. \] (3.26)
This is shown as follows. The Noether current of the BRST symmetry is obtained as

$$J^\mu_B = B \partial^\mu C - \partial^\mu BC - \partial_\nu (f^{\mu\nu} C). \quad (3.27)$$

The Noether charge, i.e., the BRST charge $Q_B$ as the generator of the BRST transformation

$$[i\lambda Q_B, \Phi(x)] = \lambda \delta_B \Phi(x), \quad (3.28)$$

is given by

$$Q_B = \int d^3 x J^0_B = \int d^3 x [B \partial^0 C - \partial^0 BC]. \quad (3.29)$$

In the similar way, the anti-BRST charge $\bar{Q}_B$ can also be defined as the Noether charge for the anti-BRST transformation. Therefore we can define the renormalized BRST charge $Q^R_B$ as

$$Q^R_B = Z^{1/2}_3 Q_B = \int d^3 x [B^R \partial^0 C^R - \partial^0 B^R C^R]. \quad (3.30)$$

This ensures the renormalization of the BRST transformation (3.26). The renormalized BRST transformation for the renormalized field has the same form as the bare BRST transformation for the bare field. Thus, the composite operator $\mathcal{Q}$ is a BRST invariant and multiplicatively renormalizable operator for arbitrary gauge parameter $\alpha$. The renormalized GF+FP term has the same form as the bare one:

$$\mathcal{L}_{GF+FP} = 2\alpha^R_1 A^R_\mu A^R_\mu + 2\alpha^R_2 C^R \cdot \bar{C}^R \left(2\alpha^R_1 \frac{i}{2} \bar{C}^R \cdot \bar{C}^R \right). \quad (3.31)$$

## 4 Yang-Mills theory in the most general Lorentz gauge

### 4.1 Lagrangian

We consider the most general quantum Lagrangian density that is a local function of the fields, $A^A_\mu$, $B^A$, $\bar{C}^A$, $\bar{C}^A$ and satisfies the conditions: The Lagrangian is (1) of mass dimension 4, (2) Lorentz invariant, (3a) BRST invariant, (3b) anti-BRST invariant, (4) Hermitian, (5) of zero ghost number, (6) global gauge invariant, and the theory with this Lagrangian is (7) (multiplicative) renormalizable. Here it is implicitly assumed that the Lagrangian is written as the polynomial of the fields, and that there are no higher derivative terms, since there is no intrinsic mass scale in the Yang-Mills theory. It should be remarked that we have imposed BRST and anti-BRST invariance instead of gauge invariance (we do not require gauge invariance for the Lagrangian). Such a Lagrangian was given by Baulieu and Thierry-Mieg as

$$\mathcal{L}_{YM}^{\text{tot}} = -\frac{1}{4} \alpha_1 \mathcal{F}_{\mu\nu} \cdot \mathcal{F}^{\mu\nu} + \alpha_2 \epsilon_{\mu\nu\rho\sigma} \mathcal{F}^{\mu\nu} \cdot \mathcal{F}^{\rho\sigma}$$

$$+ i \delta_B \delta_B \left(\alpha_3 A_\mu \cdot A^\mu + \alpha_4 \bar{C} \cdot \bar{C} \right) + \frac{\alpha'}{2} \mathcal{B} \cdot \mathcal{B}, \quad (4.1)$$

4 Yang-Mills theory in the most general Lorentz gauge
where $\alpha_i (i = 1, 2, 3, 4)$ is an arbitrary constant, and $\delta_B$ and $\bar{\delta}_B$ are the BRST and anti-BRST transformations. The first term is the Yang-Mills Lagrangian, the second term is the topological term which is not discussed in this paper and omitted hereafter. The first and the second terms are gauge invariant. On the other hand, the third and the fourth terms are identified with the GF and FP term, since they break the gauge invariance of the Lagrangian. After the rescaling of the parameters and the field redefinitions, we can cast the total Lagrangian of the Yang-Mills theory into the form,

$$L_{YM}^{\text{tot}} = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + L_{\text{GF+FP}}, \quad (4.2)$$

with the GF+FP term [8–10]:

$$L_{\text{GF+FP}} = i \delta_B \bar{\delta}_B \left( \frac{1}{2} A_\mu \cdot A^\mu - \frac{\alpha}{2} \bar{c} \cdot \bar{c} \right) + \frac{\alpha'}{2} B \cdot B \quad (4.3)$$

$$= -i \delta_B \left( -\partial_\mu \bar{c} \cdot A^\mu + \frac{\alpha}{2} \bar{c} \cdot B - \frac{i}{4} \alpha g \bar{c} \cdot (\bar{c} \times c) \right) + \frac{\alpha'}{2} B \cdot B. \quad (4.4)$$

The final term is allowed for the renormalizability of the total Lagrangian and is written in either BRST exact or anti-BRST exact form,

$$B \cdot B = -i \delta_B (\bar{c} \cdot B) = i \bar{\delta}_B (c \cdot B). \quad (4.5)$$

However, the GF+FP term (4.4) is simultaneously BRST and anti-BRST exact, i.e., $\delta_B \bar{\delta}_B (\ast)$, only if $\alpha' = 0$. If we impose one more condition, e.g., the FP ghost conjugation invariance,

$$\bar{c} A \rightarrow \pm \bar{c} A, \quad \bar{c} A \rightarrow \mp \bar{c} A, \quad B A \rightarrow - B A, \quad \bar{B} A \rightarrow - \bar{B} A \quad (A_\mu \rightarrow A_\mu), \quad (4.6)$$

the second term of (4.4) is excluded, namely, only the choice $\alpha' = 0$ is allowed.

By performing the BRST and anti-BRST transformations, we obtain

$$L_{\text{GF+FP}} = \frac{\alpha + \alpha'}{2} B \cdot B - \frac{\alpha}{2} i g (\bar{c} \times \bar{c}) \cdot B + B \cdot \partial_\mu A^\mu$$

$$+ i \bar{c} \cdot \partial_\mu D^\mu [A] \bar{c} + \frac{\alpha}{8} g^2 (\bar{c} \times \bar{c}) \cdot (\bar{c} \times \bar{c})$$

$$= \frac{\alpha + \alpha'}{2} B \cdot B - \frac{\alpha}{2} i g (\bar{c} \times \bar{c}) \cdot B + B \cdot \partial_\mu A^\mu$$

$$+ i \bar{c} \cdot \partial_\mu D^\mu [A] \bar{c} + \frac{\alpha}{4} g^2 (i \bar{c} \times \bar{c}) \cdot (i \bar{c} \times \bar{c}). \quad (4.7)$$

The GF+FP term includes the ghost self-interaction where the strength is proportional to the parameter $\alpha$.

When $\alpha = 0$, this theory reduces to the usual Yang-Mills theory in the Lorentz type gauge fixing with the gauge fixing parameter $\alpha'$:

$$L_{\text{GF+FP}} = \frac{\alpha'}{2} B \cdot B + B \cdot \partial_\mu A^\mu + i \bar{c} \cdot \partial_\mu D^\mu [A] \bar{c}. \quad (4.9)$$

This is consistent with the FP prescription.
When \( \alpha \neq 0 \), there exists a quartic ghost interaction which cannot be implemented by the usual FP prescription. Therefore we must go beyond the FP prescription. The GF+FP term is further rewritten as

\[
L_{\text{GF+FP}} = -\frac{1}{2\lambda} (\partial^\mu A_\mu)^2 + (1 - \xi) i \bar{C} \cdot \partial_\mu D^\mu [A] C + \xi i \bar{C} \cdot D^\mu [A] \partial_\mu C
\]

\[
+ \frac{1}{2} \lambda \xi (1 - \xi) g^2 (i \bar{C} \times \bar{C}) \cdot (i \bar{C} \times \bar{C})
\]

\[
+ \frac{\lambda}{2} \left( B + \lambda^{-1} \partial^\mu A_\mu - \xi i g (C \times \bar{C}) \right)^2
\]

\[
= -\frac{1}{2\lambda} (\partial^\mu A_\mu)^2 + i \bar{C} \cdot \partial_\mu \partial^\mu C - (1 - \xi) g_i A_\mu \cdot (\partial_\mu \bar{C} \times C)
\]

\[
+ \xi g_i A_\mu \cdot (\bar{C} \times \partial_\mu C) + \frac{1}{2} \lambda \xi (1 - \xi) g^2 (i \bar{C} \times \bar{C}) \cdot (i \bar{C} \times \bar{C})
\]

\[
+ \frac{\lambda}{2} \left( B + \lambda^{-1} \partial^\mu A_\mu - \xi i g (C \times \bar{C}) \right)^2,
\]

(4.10)

where we have defined the two parameters\(^4\)

\[
\lambda := \alpha + \alpha', \quad \xi := \frac{\alpha/2}{\alpha + \alpha'} = \frac{\alpha}{2\lambda}.
\]

(4.12)

In this form, it is easy to eliminate the Nakanishi-Lautrup field \( B \). We call the gauge (4.11) the most general Lorentz gauge hereafter.

### 4.2 Feynman rules

We obtain the following Feynman rules for the Yang-Mills theory of the Lagrangian (4.2) with (4.11) where the NL field is eliminated.

#### 4.2.1 Propagators

(a) gluon propagator:

\[
A, \mu \rightarrow \rightarrow \rightarrow \rightarrow B, \nu = i D^{AB}_{\mu\nu} = -\frac{i}{p^2} \left[ g_{\mu\nu} - (1 - \lambda) \frac{p_\mu p_\nu}{p^2} \right] \delta^{AB}.
\]

(4.13)

(b) ghost propagator:

\[
A \rightarrow \rightarrow \rightarrow B = i G^{AB} = \frac{1}{p^2} \delta^{AB}.
\]

(4.14)

#### 4.2.2 Three-point vertices

(c) Three-gluon vertex:

\[
A, \mu \rightarrow \rightarrow \rightarrow \rightarrow B, \rho \rightarrow \rightarrow \rightarrow \rightarrow C, \sigma = g f^{ABC} [(q - r)_\mu g_{\rho\sigma} + (r - p)_\rho g_{\sigma\mu} + (p - q)_\sigma g_{\mu\rho}].
\]

(4.15)

\(^4\) The parameters \( \alpha, \alpha', \lambda, \xi \) in this paper corresponds respectively to the \( \lambda_c, \lambda_b, \lambda, \alpha \) in [1] and \( \alpha, \alpha', \lambda, \alpha/2 \) in [2].
(d) Gluon–ghost–anti-ghost vertex:

\[ p \quad A \quad C, \mu \quad B \quad q \]

\[ = igf^{ABC} [\xi(p - q) - p]^\mu. \quad (4.16) \]

4.2.3 Four-point vertices

(e) Four-gluon vertex:

\[ A, \mu \quad B, \nu \quad C, \rho \quad D, \sigma \]

\[ = -i2g^2 \left( f^{EAB} f^{ECD} I_{\mu\nu,\rho\sigma} + f^{EAC} f^{EBD} I_{\mu\rho,\nu\sigma} + f^{EAD} f^{EBC} I_{\mu\sigma,\nu\rho} \right), \quad (4.17) \]

where \( I_{\mu\nu,\rho\sigma} := \frac{(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})}{2}. \)

(f) Four-ghost vertex:

\[ A \quad B \]

\[ C \quad D \]

\[ = -i\lambda\xi(1 - \xi)g^2 \left( f^{ACE} f^{BDE} - f^{ADE} f^{BCE} \right). \quad (4.18) \]

4.3 Multiplicative renormalization

It has been proved by Baulieu and Thierry-Mieg [8] based on the mathematical induction that the Yang-Mills theory in the most general Lorentz gauge (4.11) is multiplicative renormalizable. We introduce the renormalization constant (or renormalization factor) for the field:

\[ A_\mu = Z_A^{1/2} A^R_\mu, \quad C = Z_C^{1/2} C^R, \quad \bar{C} = Z_C^{1/2} \bar{C}^R, \quad B = Z_B^{1/2} B^R = Z_C Z_A^{-1/2} B^R, \quad (4.19) \]

and for the parameters:

\[ \lambda = Z_\lambda \lambda_R, \quad \xi = Z_\xi \xi_R, \quad g = Z_g g_R. \quad (4.20) \]

By substituting (4.19) and (4.20) into the bare Lagrangian, we obtain the total Lagrangian written in terms of the renormalized fields, renormalized parameters and
the renormalization factors:

\[
L_{YM}^{\text{tot}} = -\frac{1}{4}Z_A(\partial_\mu A_\nu^R - \partial_\nu A_\mu^R + Z_g Z_A^{1/2} g_R A_\mu^R \times A_\nu^R)^2 \\
- \frac{1}{2\lambda_R} Z_A Z_\lambda^{-1}(\partial^\mu \bar{A}_\mu^R)^2 + i Z_C \bar{c}^R \cdot \partial_\mu \bar{c}^R \\
- (1 - Z_\xi \xi_R)Z_g Z_A^{1/2} Z_C g_R i A_\mu^R \cdot (\partial_\mu \bar{c}^R \times \bar{c}^R) \\
+ Z_\xi Z_g Z_A^{1/2} Z_C \xi_R g_R i A_\mu^R \cdot (\bar{c}^R \times \partial_\mu \bar{c}^R) \\
+ \frac{1}{2} Z_\xi Z_\xi_R Z_2^2 \lambda_R \xi_R (1 - Z_\xi \xi_R) g_R^2 (i \bar{c}^R \times \bar{c}^R) \cdot (i \bar{c}^R \times \bar{c}^R) \\
+ \frac{\lambda_R}{2} Z_\lambda (Z_C Z_A^{-1/2} B_R + Z_A^{-1/2} \lambda_R^{-1} \partial^\mu A_\mu^R - Z_\xi Z_g Z_C \xi_R g_R \bar{c}^R \times \bar{c}^R)^2.
\]

(4.21)

The total Lagrangian (4.21) is decomposed into a renormalization-factor independent part \(L_{YM}^{\text{tot}}\) and the remaining part \(L_{YM}^{\text{tot}} - L_{YM}^{\text{tot}}\) as

\[
L_{YM}^{\text{tot}} = L_{YM}^{\text{tot}} - L_{YM}^{\text{tot}},
\]

(4.22a)

\[
L_{YM}^{\text{tot}} := -\frac{1}{4}(\partial_\mu A_\nu^R - \partial_\nu A_\mu^R + g_R A_\mu^R \times A_\nu^R)^2 \\
- \frac{1}{2\lambda_R}(\partial^\mu \bar{A}_\mu^R)^2 + i \bar{c}^R \cdot \partial_\mu \bar{c}^R \\
- (1 - \xi_R)Z_g g_R i A_\mu^R \cdot (\partial_\mu \bar{c}^R \times \bar{c}^R) + \xi_R g_R i A_\mu^R \cdot (\bar{c}^R \times \partial_\mu \bar{c}^R) \\
+ \frac{1}{2} \lambda_R \xi_R (1 - \xi_R) g_R^2 (i \bar{c}^R \times \bar{c}^R) \cdot (i \bar{c}^R \times \bar{c}^R) \\
+ \frac{\lambda_R}{2} (\bar{c}^R + \lambda_R^{-1} \partial^\mu A_\mu^R - \xi_R i g_R \bar{c}^R \times \bar{c}^R)^2,
\]

(4.22b)

\[
L_{YM}^{\text{tot}} := L_{YM}^{\text{tot}} - L_{YM}^{\text{tot}}.
\]

(4.22c)

Here \(L_{YM}^{\text{tot}}\) is obtained by setting all renormalization factors \(Z \equiv 1\) in (4.21) and hence it is written in terms of the renormalized fields and renormalized parameters and has the same form as the bare Lagrangian \(L_{YM}\), while \(L_{YM}^{\text{tot}}\) is the counterterm defined by the difference \(L_{YM}^{\text{tot}} - L_{YM}^{\text{tot}}\).

### 4.3.1 Renormalization of two-point functions

First, we calculate the vacuum polarization function of the gluon. To the order \(g^2\), there are three Feynman diagrams, see (a1), (a2) and (a3) in Fig.1

![Figure 1: Vacuum polarization of the gluon.](image-url)

Figure 1: Vacuum polarization of the gluon.
As a gauge-invariant regularization, we adopt the dimensional regularization. Then we obtain the following result (\(\epsilon := 2 - D/2\)).

\[(a1) = C_2(G)\delta^{AB} \frac{(g\mu^{-\epsilon})^2 i}{(4\pi)^2} \left[ \frac{1}{12} q^2 g_{\mu\nu} - \left\{ \xi(1 - \xi) - \frac{1}{6} \right\} q_{\mu} q_{\nu} \right], \quad (4.23a)\]

\[(a2) = \frac{1}{2} C_2(G)\delta^{AB} \frac{(g\mu^{-\epsilon})^2 i}{(4\pi)^2} \left[ \frac{19}{6} q^2 g_{\mu\nu} - \frac{11}{3} q_{\mu} q_{\nu} + (1 - \lambda)(q^2 g_{\mu\nu} - q_{\mu} q_{\nu}) \right], \quad (4.23b)\]

\[(a3) = 0, \quad (4.23c)\]

where \(C_2 = C_2(G)\) is the quadratic Casimir operator in the adjoint representation of the gauge group \(G\) defined by \(\delta^{AB} C_2(G) = f^{ACD} f^{BCD}\). Hence the counterterms \(\delta_T\) and \(\delta_L\) for the transverse and longitudinal part of the vacuum polarization tensor are determined so as to satisfy the relation,

\[(a1) + (a2) + (a3) - i\delta_T (q^2 g_{\mu\nu} - q_{\mu} q_{\nu}) \delta^{AB} - i\delta_L \chi q_{\mu} q_{\nu} \delta^{AB} \equiv 0, \quad (4.24)\]

which yields the result:

\[\delta_T = \left( \frac{13}{6} - \frac{\lambda}{2} \right) \frac{(g\mu^{-\epsilon})^2 C_2(G)}{(4\pi)^2} \frac{\epsilon}{\xi^2}, \quad \delta_L = -\lambda \xi (1 - \xi) \frac{(g\mu^{-\epsilon})^2 C_2(G)}{(4\pi)^2} \frac{\epsilon}{\xi}. \quad (4.25)\]

On the other hand, the relationship

\[\delta_T = Z_A - 1 = Z_A^{(1)} + \ldots, \quad \delta_L = Z_A Z_{\lambda}^{-1} - 1 = Z_A^{(1)} - Z_{\lambda}^{(1)} + \ldots, \quad (4.26)\]

must hold for the multiplicative renormalizability where we have defined the renormalization factor \(Z\) order by order of the loop expansion, \(Z = 1 + Z^{(1)} + Z^{(2)} + \ldots\). Thus we obtain the renormalization factors:

\[Z_A^{(1)} = \delta_T = \left( \frac{13}{6} - \frac{\lambda}{2} \right) \frac{(g\mu^{-\epsilon})^2 C_2(G)}{(4\pi)^2} \frac{\epsilon}{\xi}, \quad (4.27)\]

and

\[Z_{\lambda}^{(1)} = \delta_T - \delta_L = \left[ \left( \frac{13}{6} - \frac{\lambda}{2} \right) + \lambda \xi (1 - \xi) \right] \frac{(g\mu^{-\epsilon})^2 C_2(G)}{(4\pi)^2} \frac{\epsilon}{\xi}. \quad (4.28)\]

Note that \(\delta_T\) and hence \(Z_A\) is the same as in the FP case where the four ghost interaction does not exist. When \(\xi \neq 0, 1\), however, we find that \(\delta_L \neq 0\) or equivalently \(Z_A \neq Z_{\lambda}\). On the contrary to the FP case, the longitudinal part of the gluon propagator must be renormalized in this case.

Next, the vacuum polarization function of the ghost is calculated in the similar way. To the order \(g^2\), there are two Feynman diagrams, see (b1) and (b2) in Fig.\( \text{2} \). The explicit calculation shows that

\[(b1) = \left( \frac{1}{2} + \frac{1 - \lambda}{4} \right) \frac{(g\mu^{-\epsilon})^2 C_2(G)}{(4\pi)^2} \frac{\epsilon}{p^2} \delta^{AB}, \quad (4.29a)\]

\[(b2) = 0. \quad (4.29b)\]
The counterterm $\delta_C$ is determined from
\[
(b1) + (b2) - p^2 \delta^{AB} \delta_C = 0.
\]
Hence the counterterm $\delta_C = Z_C - 1 = Z_C^{(1)} + \cdots$ is equal to the renormalization constant $Z_C^{(1)}$:
\[
Z_C^{(1)} = \delta_C = \frac{3 - \lambda (g\mu^{-\epsilon})^2 C_2(G)}{4 (4\pi)^2 \epsilon}.
\]
This is again the same as in the FP case.

4.3.2 Renormalization of three-point function

We consider the renormalization of three-point vertex. For example, the Feynman diagrams for the radiative correction of the gluon–ghost–anti-ghost vertex to one-loop order is given in Fig.3.

If we write the counterterm for the gluon–ghost–anti-ghost vertex function as
\[
\begin{align*}
(C, \mu) & = ig_R f^{ABC} \left[ \xi R^{\delta_{ACC}}(p - q) - \delta_{ACC}^2 \right] \mu ,
\end{align*}
\]
we find the renormalization factors are related as
\[
\begin{align*}
\delta_{ACC}^1 & = Z_C Z_A^2 Z_\psi Z_\xi - 1 = Z_C^{(1)} + \frac{1}{2} Z_A^{(1)} + Z_\psi^{(1)} + Z_\xi^{(1)} + \cdots , \\
\delta_{ACC}^2 & = Z_C Z_A^2 Z_\psi - 1 = Z_C^{(1)} + \frac{1}{2} Z_A^{(1)} + Z_\psi^{(1)} + \cdots .
\end{align*}
\]
At $p = q$, the respective diagram is calculated as

\begin{align}
(c1)_{p=q} &= -\frac{1}{2}C_2(G)f^{ABC}g^3\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\frac{1}{4}p^\mu, \\
(c2)_{p=q} &= -\frac{1}{2}C_2(G)f^{ABC}g^3\lambda\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\frac{3}{4}p^\mu, \\
(c3)_{p=q} &= 0. 
\end{align}

(4.35a)

By substituting (4.35a), (4.35b) and (4.35c) into

\begin{align}
(c1) &= p = q + (c2) + (c3) - igf^{ABC}\delta_{ACC}p^\mu \equiv 0, 
\end{align}

it follows that

\begin{align}
\delta_{ACC}^2 &= -\frac{1}{2}\frac{\lambda (g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}.
\end{align}

(4.37)

Hence the renormalization factor is obtained as

\begin{align}
Z^{(1)}_g = \delta_{ACC}^2 - \frac{1}{2}Z^{(1)}_C = -\frac{11}{6}\frac{(g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}.
\end{align}

(4.38)

At $p = 0$, the respective diagram is calculated as

\begin{align}
(c1)_{p=0} &= -\frac{1}{2}C_2(G)f^{ABC}g^3\lambda\xi\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\left[(1 - \xi)\left(\xi - \frac{1}{2}\right) + \frac{1}{4}\right]q^\mu, \\
(c2)_{p=0} &= -\frac{1}{2}C_2(G)f^{ABC}g^3\lambda\xi\frac{i}{(4\pi)^2}\frac{3}{4\epsilon}q^\mu, \\
(c3)_{p=0} &= -\frac{1}{2}C_2(G)f^{ABC}g^3\lambda(1 - \xi)\xi\frac{i}{(4\pi)^2}\frac{1}{\epsilon}\left(\xi - \frac{1}{2}\right)q^\mu.
\end{align}

(4.39a)

By substituting (4.39a), (4.39b) and (4.39c) into

\begin{align}
(c1)_{p=0} + (c2)_{p=0} + (c3)_{p=0} - igf^{ABC}\xi R\delta_{ACC}q^\mu \equiv 0, 
\end{align}

it follows that

\begin{align}
\delta_{ACC}^1 &= \left[-\lambda(1 - \xi)\left(\xi - \frac{1}{2}\right) - \frac{1}{2}\lambda\right]\frac{(g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}.
\end{align}

(4.41)

Then we obtain

\begin{align}
Z^{(1)}_\xi = \delta_{ACC}^1 - \delta_{ACC}^2 \\
= \lambda(1 - \xi)\left(\xi - \frac{1}{2}\right)\frac{(g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}.
\end{align}

(4.42)

Accordingly, the renormalization constants of $\alpha$ and $\alpha'$ are obtained as

\begin{align}
Z^{(1)}_\alpha &= \left(\frac{13}{6} - \frac{\alpha}{4}\right)\frac{(g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}, \\
Z^{(1)}_{\alpha'} &= \left(\frac{13}{6} - \frac{\alpha + \alpha'}{2}\right)\frac{(g\mu^-)^2}{(4\pi)^2}\frac{C_2(G)}{\epsilon}.
\end{align}

(4.43)
5 Renormalization group flow and fixed points

Using the above result, the renormalization group (RG) functions are obtained as follows. The $\beta$-function is obtained as

$$\beta(g_R) := \mu \frac{\partial g_R}{\partial \mu} = -g_R \frac{\partial}{\partial \mu} \ln Z_g \approx -g_R \mu \frac{\partial}{\partial \mu} Z_g^{(1)}. \quad (5.1)$$

It turns out that the $\beta$-function does not depend on the gauge parameters $\lambda$ and $\xi$:

$$\beta(g_R) := \mu \frac{\partial g_R}{\partial \mu} = -\frac{1}{16\pi^2} \frac{11}{3} C_2(G) g_R^3. \quad (5.2)$$

Similarly, we obtain the RG functions:

$$\gamma_\xi := \mu \frac{\partial}{\partial \mu} \xi_R = 2\lambda_R \xi_R(\xi_R - 1) \left(\xi_R - \frac{1}{2}\right) \frac{C_2(G) g_R^2}{(4\pi)^2}, \quad (5.3)$$

and

$$\gamma_\lambda := \mu \frac{\partial}{\partial \mu} \lambda_R = 2\lambda_R \left[\frac{13}{6} - \frac{\lambda_R}{2} + \lambda_R \xi_R(1 - \xi_R)\right] \frac{C_2(G) g_R^2}{(4\pi)^2}. \quad (5.4)$$

The RG flow in three-dimensional parameter space ($\xi$, $\lambda$, $g$) is determined by solving simultaneous differential equations:

$$\mu \frac{\partial \xi}{\partial \mu} = 2\lambda \xi(\xi - 1) \left(\xi - \frac{1}{2}\right) \frac{C_2(G) g^2}{(4\pi)^2}, \quad (5.5a)$$

$$\mu \frac{\partial \lambda}{\partial \mu} = 2\lambda \left[\frac{13}{6} - \frac{\lambda}{2} + \lambda \xi(1 - \xi)\right] \frac{C_2(G) g^2}{(4\pi)^2}, \quad (5.5b)$$

$$\mu \frac{\partial g}{\partial \mu} = -\frac{11}{3} \frac{C_2(G) g^3}{(4\pi)^2}, \quad (5.5c)$$

where we have omitted the subscript R for the renormalized quantity.

As is well known, the equation (5.5c) is solved exactly,

$$g^2(\mu) = \frac{g^2(\mu_0)}{1 + \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} g^2(\mu_0) \ln \frac{\mu}{\mu_0}} = \frac{1}{\frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{QCD}}}, \quad (5.6)$$

where we have used the boundary condition $g(\mu_0) = \infty$ at $\mu_0 = \Lambda_{QCD}$. The remaining two equations (5.5a) and (5.5b) can not be solved exactly.

5.1 Fixed points

First, we obtain the fixed point of the RG. Note that the derivative $\frac{1}{g^2} \mu \frac{\partial}{\partial \mu}$ in (5.5b) is rewritten as

$$\frac{1}{g^2} \mu \frac{\partial}{\partial \mu} = \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{QCD}} \mu \frac{\partial}{\partial \mu} = \frac{22}{3} \frac{C_2(G)}{(4\pi)^2} \ln \frac{\mu}{\Lambda_{QCD}}. \quad (5.7)$$
Then the fixed point (to one-loop order) is obtained by solving the algebraic equation simultaneously:

\[ \lambda \xi (\xi - 1) \left( \xi - \frac{1}{2} \right) = 0, \quad \lambda \left[ \frac{13}{6} - \frac{\lambda}{2} + \lambda \xi (1 - \xi) \right] = 0. \]  

(5.8)

We find one line of fixed points and three isolated fixed points in the \((\xi, \lambda)\) plane or equivalently four isolated fixed points in the \((\alpha, \alpha')\) plane:

A. The line of fixed points: \( \lambda = 0, \xi \in \mathbb{R} \) corresponds to an isolated fixed point \((\alpha, \alpha') = (0, 0)\).

B. \((\xi, \lambda) = \left( \frac{1}{2}, \frac{26}{3} \right)\) corresponds to \((\alpha, \alpha') = (\frac{26}{3}, 0)\).

C. \((\xi, \lambda) = (0, \frac{13}{3})\) corresponds to \((\alpha, \alpha') = (0, \frac{13}{3})\).

D. \((\xi, \lambda) = (1, \frac{13}{3})\) corresponds to \((\alpha, \alpha') = (\frac{26}{3}, -\frac{13}{3})\).

If the two parameters \(\xi, \lambda\) are set equal to one of the fixed points, the theory remains forever on the fixed. If the system starts from other points and the scale \(\mu\) is decreased, it evolves into the infrared (IR) region according to a couple of differential equations (5.5a)–(5.5c).

### 5.2 RG flow in the neighborhood of fixed points

In the neighborhood of the respective fixed point \((X_1^*, X_2^*)\) in the plane \((X_1, X_2) = (\xi, \lambda)\) or \((\alpha, \alpha')\), we can study the behavior of the RG flow analytically. By taking into account only the terms which are linear in the infinitesimal deviation \(\delta X_1 := X_1 - X_1^*, \delta X_2 := X_2 - X_2^*\) from the fixed point, a set of RG equations, (5.5a) and (5.5b), is reduced to the form:

\[
\begin{pmatrix}
\gamma X_1 \\
\gamma X_2
\end{pmatrix}
\sim
A
\begin{pmatrix}
\delta X_1 \\
\delta X_2
\end{pmatrix},
\]

where \(A\) is a two by two matrix.

| Eigenvalue \(\frac{13 g^2 C_2(\alpha)}{(4\pi)^2} \times\) | Eigenvector \((\xi, \lambda) (\alpha, \alpha')\) |
|---|---|
| A | 1 | IR fixed point \((1, a)\) any lines |
| B | -1 | UV fixed point \((1, a) (1, a)\) any lines |
| C | 1 | Saddle point \((13, 3) (1, -2)\) line IV |
| | -1 | \((0, 1) (0, 1)\) line II |
| D | 1 | Saddle point \((-13, 3) (0, 1)\) line V |
| | -1 | \((1, -2) (0, 1)\) line III |

Table 1: Eigenvalues and eigenvectors of the linearized RG equation where the lines, II, III, IV are defined below. At the IR fixed point A and UV fixed point B, two eigenvalues are degenerate.
In \((\xi, \lambda)\) plane, the set of linearized RG equations reads

\[
B \left(\frac{1}{2}, \frac{26}{3}\right) : \begin{pmatrix} \gamma_\xi \\ \gamma_\lambda \end{pmatrix} \sim -\frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \xi \\ \delta \lambda \end{pmatrix}. \tag{5.9a}
\]

\[
C \left(0, \frac{13}{3}\right) : \begin{pmatrix} \gamma_\xi \\ \gamma_\lambda \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ \frac{26}{3} & -1 \end{pmatrix} \begin{pmatrix} \delta \xi \\ \delta \lambda \end{pmatrix}. \tag{5.9b}
\]

\[
D \left(1, \frac{13}{3}\right) : \begin{pmatrix} \gamma_\xi \\ \gamma_\lambda \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ -\frac{26}{3} & -1 \end{pmatrix} \begin{pmatrix} \delta \xi \\ \delta \lambda \end{pmatrix}. \tag{5.9c}
\]

Similarly in \((\alpha, \alpha')\) plane, we obtain

\[
A \left(0, 0\right) : \begin{pmatrix} \gamma_\alpha \\ \gamma_{\alpha'} \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \alpha \\ \delta \alpha' \end{pmatrix}. \tag{5.10a}
\]

\[
B \left(\frac{26}{3}, 0\right) : \begin{pmatrix} \gamma_\alpha \\ \gamma_{\alpha'} \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta \alpha \\ \delta \alpha' \end{pmatrix}. \tag{5.10b}
\]

\[
C \left(0, \frac{13}{3}\right) : \begin{pmatrix} \gamma_\alpha \\ \gamma_{\alpha'} \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \delta \alpha \\ \delta \alpha' \end{pmatrix}. \tag{5.10c}
\]

\[
D \left(-\frac{13}{3}, \frac{26}{3}\right) : \begin{pmatrix} \gamma_\alpha \\ \gamma_{\alpha'} \end{pmatrix} \sim \frac{13 g^2 C_2(G)}{3 \left(4\pi^2\right)} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \delta \alpha \\ \delta \alpha' \end{pmatrix}. \tag{5.10d}
\]

The respective matrix characterizing the behavior of the RG flow in the neighborhood of the respective fixed point has the eigenvalue and the corresponding eigenvector enumerated in Table I. The direction of the flow is determined at the respective fixed point. We will see that these results are consistent with the global flow diagram given in Fig.4 below.

### 5.3 Global behavior of the RG flow

We find that \(\xi \equiv 0\), \(\xi \equiv 1/2\), and \(\xi \equiv 1\) are solutions of the equation (5.5a). This implies that the RG flow starting from the point on one of the three planes, \((0, \lambda, g)\), \((1/2, \lambda, g)\), \((1, \lambda, g)\), is always kept on the respective plane. On the three planes, moreover, the equation (5.5b) can be solved exactly. On the plane \((1/2, \lambda, g)\), the RG flow in the region \(0 < \lambda < \frac{26}{3}\) obeys

\[
\lambda(\mu) = \frac{26}{3} \left\{ 1 + C \left( \ln \frac{\mu}{\Lambda_{\text{QCD}}} \right)^{-\frac{\mu}{\Lambda_{\text{QCD}}} - \frac{1}{2}} \right\}^{-1}, \tag{5.11}
\]

where \(C\) is a positive constant. We see that \(\lambda\) approaches to the ultraviolet (UV) fixed point \(\lambda \uparrow \frac{26}{3}\) in the UV limit \(\mu \uparrow \infty\), while \(\lambda \downarrow 0\) monotonically as \(\mu \downarrow \Lambda_{\text{QCD}}\). On the other hand, the RG flow in the region \(\lambda > \frac{26}{3}\) is described by

\[
\lambda(\mu) = \frac{26}{3} \left\{ 1 - C \left( \ln \frac{\mu}{\Lambda_{\text{QCD}}} \right)^{-\frac{\mu}{\Lambda_{\text{QCD}}} - \frac{1}{2}} \right\}^{-1}, \tag{5.12}
\]
where $\lambda$ approaches to the UV fixed point $\lambda \uparrow \frac{26}{3}$ in the UV limit $\mu \uparrow \infty$, while $\lambda \uparrow \infty$ monotonically as $\mu \downarrow \Lambda_{\text{QCD}}$. By substituting $\ln \frac{\mu}{\Lambda_{\text{QCD}}} = \left\{ \frac{22 C_2(G)}{3 (4\pi)^2 g^2} \right\}^{-1}$ into the above equation, the equation of the RG flow on the plane $(1/2, \lambda, g)$ is obtained

$$
\lambda = \frac{26}{3} \left\{ 1 \pm C \left( \frac{22 C_2(G)}{3 (4\pi)^2 g^2} \right)^{\frac{13}{12}} \right\}^{-1}.
$$

(5.13)

The RG flows on the plane $(0, \lambda, g)$ and $(1, \lambda, g)$ are governed by the same equations which are obtained by replacing $26/3$ with $13/3$.

The global behavior of the RG flow is obtained by solving (5.5c)–(5.5d) numerically. In Fig.4, the RG flow is drawn on the plane $(\xi, \lambda)$ and the plane $(\alpha, \alpha')$. The direction of the arrow denotes the direction towards the IR region and the length of the arrow is proportional to the magnitude of the vector $\mu \frac{d}{d\mu} (\xi, \lambda)/g^2$. In the neighborhood of the respective fixed point, we see that the numerical result agrees with the analytical result given in Table II of the previous subsection.

Among the RG flows, the five RG flows (I, II, III, IV, V) connecting the fixed points $A, B, C, D$ form the watershed (or backbone) in the flow diagram.

I. $\xi = \frac{1}{2}$, $\alpha' = 0$.  
II. $\xi = 0$, $\alpha = 0$.  
III. $\xi = 1$, $\alpha' = -\frac{1}{2} \alpha$.  
IV. $\lambda = \frac{13}{3} - \frac{1}{\xi}$, $\alpha' = -\frac{1}{2} \alpha + \frac{13}{3}$.  
V. $\lambda = \frac{13}{3} \frac{1}{\xi}$, $\alpha = \frac{26}{3}$.

(a)  
(b)  

Figure 4: RG flows in $(\xi, \lambda)$ plane (a) and in $(\alpha, \alpha')$ plane (b).
Since the flow is symmetric for the reflection with respect to the straight line \( \xi = 1/2 \), we focus on the region \( \xi \leq 1/2 \). The flow starting from the initial position below \( IV \) runs towards the line \( A \) of fixed points and eventually arrive at \( A \). If it arrive at a fixed point on \( A \) with a certain value of \( \xi \) depending on the initial position, then it does not move anymore. On the other hand, the flow starting from the initial position above \( IV \) runs away into the infinity, \( \lambda = +\infty \). Here the flow on the line \( I \) and \( II \) is not an exception. However, it should be remarked that the fixed point \( B \) is IR repulsive in both directions, while the fixed point \( C \) is IR attractive on \( IV \) and repulsive on \( II \). In view of these, it turns out that any fixed point on \( A \) is IR stable, while the fixed point \( B \) on \( I \) is a rather special fixed point which is IR unstable (UV stable).

We have shown that the three fixed points \( B, C, D \) for the gauge parameter \( \xi, \lambda \) are located on the line \( I, II, III (\xi = 1/2, 0, 1) \), respectively. On the lines \( I, II, III \), the RG flow is confined in the respective line, the Lagrangian takes the following form.

I. \( \xi = \frac{1}{2} \) (i.e., \( \alpha \in \mathbb{R}, \alpha' = 0 \)): The GF+FP term is invariant under the FP ghost conjugation and the orthosymplectic transformation \( OSp(4|2) \).  

\[
\mathcal{L}_{GF+FP} = i \delta_B \bar{\delta}_B \left( \frac{1}{2} A_\mu \cdot A^\mu - \frac{\alpha}{2} i \bar{C} \cdot \bar{C} \right). \tag{5.15}
\]

There is a four-ghost interaction.

II. \( \xi = 0 \) (i.e., \( \alpha = 0, \alpha' \in \mathbb{R} \)): The GF+FP term is invariant under the global shift of anti-ghost \( \bar{C} \):

\[
\mathcal{L}_{GF+FP} = \frac{\alpha'}{2} B \cdot B + B \cdot \partial_\mu A^\mu + i \bar{C} \cdot \partial_\mu D^\mu[A] \bar{C}. \tag{5.16}
\]

There is no 4-ghost interaction. This Lagrangian is the same as that in the conventional Lorentz gauge.

III. \( \xi = 1 \) (i.e., \( \alpha' = -\frac{1}{2} \alpha \)): The GF+FP term is invariant under the global shift of ghost \( C \):

\[
\mathcal{L}_{GF+FP} = \frac{\lambda}{2} B \cdot B + B \cdot \partial_\mu A^\mu + i \bar{C} \cdot D^\mu[A] \partial_\mu C. \tag{5.17}
\]

There is no four-ghost interaction.

The choice II or III eliminates the four ghost interaction and the Yang-Mills theory reduces to the FP case. Once \( \xi = 0 \) or \( \xi = 1 \) is chosen, \( \xi \) is not renormalized by quantum corrections, since \( \xi = 0 \) and \( \xi = 1 \) are fixed point of the renormalization group. Then the FP Lagrangian is preserved under the renormalization.

In II and III, the role of ghost and anti-ghost is interchanged. The FP ghost conjugation invariance is broken in the usual FP Lagrangian where the ghost and anti-ghost are not treated on equal footing (except for the Landau gauge). In other

---

5 This does not imply that the similar result is obtained also for the MA gauge. For example, \( \alpha = 0 \) is not a fixed point in the MA gauge. See \[27\] for details.
words, the FP ghost conjugation invariance is recovered for \( \alpha' = 0 \) (i.e., \( \xi = 1/2 \) or \( \lambda = \alpha \)) by including the quartic ghost interaction even for \( \alpha = 0 \).

We must keep in mind that these results are obtained to one-loop order. Therefore, the details of the flow diagram may change if we include higher-order corrections. The higher-order result is not known to date and will be given elsewhere. Nevertheless, the existence of the fixed point at \( \lambda = 0 \) remains true to any finite order of perturbation. The existence of the lines, I, II and III are also guaranteed even after the inclusion of higher order terms, since it is protected by the symmetry dictated in the above. This is because the symmetry can not be broken as far as the perturbation series to all orders are not summed up.

6 Renormalizing the composite operator of mass dimension 2

In this section we discuss the renormalization of the composite operator of mass dimension 2 and its BRST and anti-BRST invariance under the renormalization.

6.1 On-shell BRST transformation

By eliminating the Nakanishi-Lautrup field \( B \), the on-shell BRST and anti-BRST transformations are obtained as

\[
\begin{align*}
\delta_B \bar{C}(x) &= i \left[ -\frac{1}{\lambda} \partial^\mu A_\mu(x) + \xi g \bar{C}(x) \times \bar{C}(x) \right], \\
\bar{\delta}_B C(x) &= i \left[ \frac{1}{\lambda} \partial^\mu A_\mu(x) - (\xi - 1) ig \bar{C}(x) \times \bar{C}(x) \right].
\end{align*}
\]

(6.1)

The nilpotency of the on-shell transformation is partially broken\(^6\) by the equation of motion of ghost and anti-ghost:

\[
\begin{align*}
(\delta_B)^2 A_\mu(x) &= 0, \\
(\delta_B)^2 \bar{C}(x) &= 0, \\
(\delta_B)^2 \bar{C}(x) &= \frac{-i \delta L_{YM}^{tot}}{\lambda} \delta \bar{C} = \partial^\mu D_\mu \bar{C} - g \xi (\partial^\mu A_\mu \times \bar{C}) + i g^2 \lambda \xi (\xi - 1)(\bar{C} \times \bar{C}) \times \bar{C},
\end{align*}
\]

(6.3a)

(6.3b)

(6.3c)

and

\[
\begin{align*}
(\bar{\delta}_B)^2 A_\mu(x) &= 0, \\
(\bar{\delta}_B)^2 C(x) &= \frac{-i \delta L_{YM}^{tot}}{\lambda} \delta C = \partial^\mu D_\mu C - g \xi (\partial^\mu A_\mu \times C) - i g^2 \lambda \xi (\xi - 1)(C \times \bar{C}) \times C, \\
(\bar{\delta}_B)^2 \bar{C}(x) &= 0.
\end{align*}
\]

(6.4a)

(6.4b)

(6.4c)

\(^6\) An elegant proof of the unitarity of the gauge theory is given based on the nilpotency of the BRST transformation, see e.g. \cite{30}. The nilpotency is indeed broken in the on-shell BRST transformation which is obtained by eliminating the NL field. However, the nilpotency is not the only way to show the unitarity. Even in this case, it is possible to show the unitarity order by order of the perturbation theory based on the Feynman diagrams without the NL fields.
Moreover, the anti-commutativity is also broken in the similar way:

\[
(\delta_B \tilde{\delta}_B + \tilde{\delta}_B \delta_B) A_\mu(x) = 0, \quad (6.5a)
\]

\[
(\delta_B \tilde{\delta}_B + \tilde{\delta}_B \delta_B) \bar{C}(x) = \frac{1}{\lambda} \frac{\delta L_{\text{YM}}^{\text{tot}}}{\delta \bar{C}}, \quad (6.5b)
\]

\[
(\delta_B \tilde{\delta}_B + \tilde{\delta}_B \delta_B) \tilde{C}(x) = \frac{1}{\lambda} \frac{\delta L_{\text{YM}}^{\text{tot}}}{\delta C}. \quad (6.5c)
\]

### 6.2 Composite operator of mass dimension 2

We define the composite operator \( O \) as a linear combination of two composite operators of mass dimension 2:

\[
O = (\Omega^{(D)})^{-1} \int d^D x \left[ \frac{1}{2} A_\mu(x) \cdot A^\mu(x) + \lambda i \bar{C}(x) \cdot \bar{C}(x) \right]. \quad (6.6)
\]

The on-shell BRST transformation of the operator \( O \) is calculated as

\[
\delta_B O = (\Omega^{(D)})^{-1} \int d^D x \delta_B \left[ \frac{1}{2} A_\mu(x) \cdot A^\mu(x) + \lambda i \bar{C}(x) \cdot \bar{C}(x) \right]
\]

\[
= (\Omega^{(D)})^{-1} \int d^D x \left[ A_\mu(x) \cdot \delta_B A^\mu(x) - \lambda i \bar{C}(x) \cdot \delta_B \bar{C}(x) + \lambda i \delta_B \bar{C}(x) \cdot \bar{C}(x) \right]
\]

\[
= (\Omega^{(D)})^{-1} \int d^D x \left[ \partial^\mu A_\mu(x) \cdot \bar{C}(x) + \lambda i \bar{C}(x) \cdot \frac{g}{2} (\bar{C}(x) \times \bar{C}(x)) \right]
\]

\[
+ \partial^\mu A_\mu(x) \cdot \bar{C}(x) - \lambda \xi g (\bar{C}(x) \times \tilde{C}(x)) \cdot \bar{C}(x) \right]
\]

\[
= (\Omega^{(D)})^{-1} \int d^D x \left\{ \partial^\mu [A_\mu(x) \cdot \bar{C}(x)] + \lambda \left( \frac{1}{2} - \xi \right) i \bar{C}(x) \cdot g(\bar{C}(x) \times \bar{C}(x)) \right\}. \quad (6.7)
\]

In the similar way, the on-shell anti-BRST transformation of the operator \( O \) is calculated as

\[
\tilde{\delta}_B O = (\Omega^{(D)})^{-1} \int d^D x \left\{ \partial^\mu [A_\mu(x) \cdot \tilde{C}(x)] + \lambda \left( \frac{1}{2} - \xi \right) i \tilde{C}(x) \cdot g(\bar{C}(x) \times \bar{C}(x)) \right\}. \quad (6.8)
\]

Therefore, the composite operator \( O \) is invariant under the BRST and anti-BRST transformations when

\[
\xi = \frac{1}{2} \quad \text{or} \quad \lambda = 0, \quad (6.9)
\]

i.e., on the line I and A in the \((\xi, \lambda)\) plane, or on the line I in the \((\alpha, \alpha')\) plane. For \( \xi = 1/2 \), the on-shell BRST and anti-BRST transformation reads

\[
\delta_B \bar{C}(x) = - \frac{i}{\alpha} \partial^\mu A_\mu(x) - \frac{1}{2} g(\bar{C}(x) \times \tilde{C}(x)), \quad (6.10)
\]

\[
\tilde{\delta}_B \bar{C}(x) = \frac{i}{\alpha} \partial^\mu A_\mu(x) - \frac{1}{2} g(\bar{C}(x) \times \bar{C}(x)). \quad (6.11)
\]

\[22\]
The special case, $\lambda = 0$ (and $\alpha = 0$ to have a finite $\xi$) is nothing but the Landau gauge in the conventional Lorentz gauge and the BRST and anti-BRST invariant operator $O$ reduces to a simple form:

$$O' = (\Omega^{(D)})^{-1} \int d^D x \left[ \frac{1}{2} A_{\mu}(x) \cdot A^{\mu}(x) \right].$$  \hspace{1cm} (6.12)

Note that $O'$ is invariant under the gauge transformation as well as the BRST and anti-BRST transformation.

### 6.3 Renormalization of the composite operator

Hereafter, we use the following notation to simplify the expressions.

$$A := A^R, \quad C := C^R, \quad \bar{C} := \bar{C}^R, \quad B := B^R.$$ \hspace{1cm} (6.13)

We consider the Green function of the fundamental fields with an insertion of the composite operator of mass dimension 2. In the following, it is assumed that we have already finished the renormalization for the fundamental field in the perturbative theory. Therefore, we have only to consider the extra renormalization for the divergence coming from the inserted composite operators in the renormalized Green function. In order to take into account the operator mixing among composite operators with the same mass dimension and the same quantum number, we must introduce the matrix of renormalization factors $Z_1, \cdots, Z_4$:

$$\left( \begin{array}{cc} \frac{1}{2} AA_R & \frac{1}{2} AA_R \\ i\bar{C}C_R & i\bar{C}C_R \end{array} \right) = \left( \begin{array}{cc} Z_1 & Z_2 \\ Z_3 & Z_4 \end{array} \right) \left( \begin{array}{cc} \frac{1}{2} AA_R & \frac{1}{2} AA_R \\ i\bar{C}C_R & i\bar{C}C_R \end{array} \right).$$ \hspace{1cm} (6.14)

Then, to the lowest nontrivial order, we find

$$\langle AA \left[ \frac{1}{2} AA \right] \rangle = \quad + \quad + \quad + \cdots, \hspace{1cm} (6.15a)$$

$$\langle AA \left[ i\bar{C}C \right] \rangle = 0 + \quad + \quad + \cdots, \hspace{1cm} (6.15b)$$

$$\langle i\bar{C}C \left[ \frac{1}{2} AA \right] \rangle = 0 + \quad + \cdots, \hspace{1cm} (6.15c)$$

$$\langle i\bar{C}C \left[ i\bar{C}C \right] \rangle = \quad + \quad + \cdots, \hspace{1cm} (6.15d)$$

where we have used the Feynman rule:

$$\begin{array}{c}
\quad = \delta^{AB}, \\
\quad = i\delta^{AB},
\end{array} \hspace{1cm} (6.16a, 6.16b)$$

with the dot denoting the insertion of a composite operator.
We show that the divergences coming from the compositeness are absorbed by taking the four renormalization constants $Z_1, Z_2, Z_3, Z_4$ appropriately. The first example is

$$\langle AA \left[ \frac{1}{2} AA \right]_R \rangle = Z_1 \langle AA \left[ \frac{1}{2} AA \right] \rangle + Z_2 \langle AA [i\bar{C}C] \rangle$$

$$= Z_1 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$+ Z_2 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$\equiv \cdots . \quad (6.17)$$

Hence the lowest value of $Z_1$ is 1:

$$Z_1 = 1 + Z_1^{(1)} + \cdots . \quad (6.18)$$

The second example is

$$\langle i\bar{C}C \left[ \frac{1}{2} AA \right]_R \rangle = Z_1 \langle i\bar{C}C \left[ \frac{1}{2} AA \right] \rangle + Z_2 \langle i\bar{C}C [i\bar{C}C] \rangle$$

$$= Z_1 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$+ Z_2 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$\equiv 0 \text{ (no divergence)}. \quad (6.19)$$

Hence $Z_2$ does not have the tree part and begins with the one-loop order:

$$Z_2 = Z_2^{(1)} + \cdots . \quad (6.20)$$

The third example is

$$\langle i\bar{C}C [i\bar{C}C]_R \rangle = Z_3 \langle i\bar{C}C \left[ \frac{1}{2} AA \right] \rangle + Z_4 \langle i\bar{C}C [i\bar{C}C] \rangle$$

$$= Z_3 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$+ Z_4 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$\equiv \cdots . \quad (6.21)$$

Hence, $Z_4$ has the form:

$$Z_4 = 1 + Z_4^{(1)} + \cdots . \quad (6.22)$$

The fourth example is

$$\langle AA [i\bar{C}C]_R \rangle = Z_3 \langle AA \left[ \frac{1}{2} AA \right] \rangle + Z_4 \langle AA [i\bar{C}C] \rangle$$

$$= Z_3 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$+ Z_4 \left\{ \begin{array}{c}
\cdots + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \cdots
\end{array} \right\}$$

$$\equiv 0 \text{ (no divergence)}. \quad (6.23)$$
Hence, $Z_3$ begins with the one-loop order:

$$Z_3 = Z_3^{(1)} + \cdots.$$  

(6.24)

Therefore, up to one-loop order, the renormalization constants must satisfy the relationship:

$$Z_1^{(1)} + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram1.png} \\ + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram2.png} \\ + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram3.png} \\ = 0, \quad \text{(6.25a)} \end{array} \end{array}$$

$$Z_2^{(1)} + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram4.png} \\ = 0, \quad \text{(6.25b)} \end{array}$$

$$Z_3^{(1)} + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram5.png} \\ = 0, \quad \text{(6.25c)} \end{array}$$

$$Z_4^{(1)} + \begin{array}{c} \includegraphics[width=0.1\textwidth]{diagram6.png} \\ = 0. \quad \text{(6.25d)} \end{array}$$

The explicit calculations lead to the following divergent part.

$$\sim C_2(G)\delta^{AB} \left[ 3 + \frac{3}{4} \lambda(1 + \lambda) \right] g_{\mu\nu} \frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.26)

$$\sim -3C_2(G)\delta^{AB} g_{\mu\nu} \frac{3 + \lambda^2 (g\mu^-\epsilon)^2}{4} \frac{1}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.27)

$$\sim iC_2(G)\delta^{AB} \xi (1 - \xi) \lambda^2 \frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.28)

$$\sim -\frac{1}{4} C_2(G)\delta^{AB} g_{\mu\nu} \frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.29)

$$\sim iC_2(G)\delta^{AB} \xi (1 - \xi) \lambda \frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.30)

$$\sim -iC_2(G)\delta^{AB} \xi (1 - \xi) \lambda \frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon}.$$  

(6.31)

Thus the renormalization constants for the composite operators are obtained as

$$Z_1^{(1)} = -\frac{3}{4} (1 + \lambda)C_2(G)\frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.32a)

$$Z_2^{(1)} = -\lambda^2 \xi (1 - \xi)C_2(G)\frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.32b)

$$Z_3^{(1)} = \frac{1}{2} C_2(G)\frac{(g\mu^-\epsilon)^2}{(4\pi)^2} \frac{1}{\epsilon},$$  

(6.32c)

$$Z_4^{(1)} = 0.$$  

(6.32d)

We pay attention to the renormalization constants of composite operators in light
of the inverted relation of (6.14):

\[
\begin{pmatrix}
\frac{1}{2} AA \\
[i\bar{CC}]
\end{pmatrix}
= 
\begin{pmatrix}
Z_1 & Z_2 \\
Z_3 & Z_4
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{1}{2} AA \\
[i\bar{CC}]
\end{pmatrix}_R
= 
\begin{pmatrix}
1 - Z^{(1)}_1 & -Z^{(1)}_2 \\
-Z^{(1)}_3 & 1 - Z^{(1)}_4
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} AA \\
[i\bar{CC}]
\end{pmatrix}_R.
\]

(6.33)

This relation shows that there is an operator mixing between the gluon and ghost composite operators which are of mass dimension 2 and color singlet, as pointed out in [3]. In the absence of four-ghost interaction \((\xi = 0 \text{ or } \xi = 1)\), (6.28), (6.30) and (6.31) vanish and hence we have \(Z^{(1)}_2 = 0 = Z^{(1)}_4\). In this case, there is no contribution from ghost for the renormalization of the gluon composite operator \([\frac{1}{2} AA]\):

\[
\frac{1}{2} AA = (1 - Z^{(1)}_1) \left[\frac{1}{2} AA\right]_R,
\]

(6.34)

\([i\bar{CC}] = [i\bar{CC}]_R - Z^{(1)}_3 \left[\frac{1}{2} AA\right]_R\).

(6.35)

On the other hand, the ghost composite operator can not be finite without the mixing of the gluon composite operator. In the conventional Lorentz gauge fixing, therefore, we do not have to consider the contribution from ghost in treating the renormalization of the gluon composite operator \([\frac{1}{2} AA]\) (at least in the one-loop level).

### 6.4 Multiplicative renormalizability of the composite operator

Now we examine the multiplicative renormalizability of the composite operator \(\mathcal{O}\). Taking into account the renormalization of the fundamental field and the composite field (6.33), we obtain

\[
Q_0 := \frac{1}{2} A_0 A_0 + \lambda_0 i\bar{C} C_0
\]

\[
= (1 + Z^{(1)}_A) \frac{1}{2} AA + (1 + Z^{(1)}_\lambda) \left(1 + Z^{(1)}_C\right) \lambda i\bar{CC}
\]

\[
= (1 + Z^{(1)}_A) \left\{ \left(1 - Z^{(1)}_1\right) \left[\frac{1}{2} AA\right]_R - Z^{(1)}_2 [i\bar{CC}]_R \right\}
\]

\[
+ \left(1 + Z^{(1)}_\lambda\right) \left(1 + Z^{(1)}_C\right) \lambda \left\{-Z^{(1)}_3 \left[\frac{1}{2} AA\right]_R + (1 - Z^{(1)}_4) [i\bar{CC}]_R \right\}
\]

\[
= \left\{1 + Z^{(1)}_A - Z^{(1)}_1 - \lambda Z^{(1)}_3 \right\} \left[\frac{1}{2} AA\right]_R
\]

\[
+ \left\{-\frac{Z^{(1)}_2}{\lambda} + 1 + Z^{(1)}_\lambda + Z^{(1)}_C - Z^{(1)}_4\right\} \lambda [i\bar{CC}]_R.
\]

(6.36)

The multiplicative renormalizability holds (in the one-loop level) if and only if

\[
Z^{(1)}_Q := Z^{(1)}_A - Z^{(1)}_1 - \lambda Z^{(1)}_3 = -\frac{Z^{(1)}_2}{\lambda} + Z^{(1)}_\lambda + Z^{(1)}_C - Z^{(1)}_4.
\]

(6.37)

This is equivalent to the condition:

\[
\lambda \left(\xi - \frac{1}{2}\right)^2 = 0.
\]

(6.38)
If this condition is satisfied, the composite operator is multiplicatively renormalized as
\[ Q_0 = Z_Q \left( \left[ \frac{1}{2} AA \right]_R + \lambda [i \bar{C} C]_R \right), \]  
(6.39)
\[ Z_Q^{(1)} = \left( \frac{35}{12} - \frac{1}{4} \lambda \right) C_2(G) \frac{(g \mu^{-\epsilon})^2 1}{(4\pi)^2 \epsilon}. \]  
(6.40)

In the case of \( \lambda = 0 \), this result reduces to that of Boucaud et al. [22] without operator mixing.

It should be remarked that the composite operator is not multiplicatively renormalizable, unless the renormalization of the composite operators \( AA \) and \( \bar{C} C \) are taken into account. In fact, the multiplicative renormalizability of
\[ Q_0 := \frac{1}{2} A_0 A_0 + \lambda i \bar{C}_0 C_0 \]

without the renormalization of the composite operator leads to the condition: \( Z_A^{(1)} - Z_C^{(1)} = 0 \), which reads \( \lambda \left[ \xi (\xi - 1) + \frac{1}{4} \right] = \frac{3}{4} \). This curve does not have a definite meaning in the renormalization, since the curve is not along the RG flow.

### 6.5 BRST invariance of the renormalized composite operator

Finally, we show that the renormalized composite operator \( O^R \) is invariant under the renormalized BRST and anti-BRST transformation. By requiring that the renormalized BRST and anti-BRST transformations are nilpotent and anti-commute:
\[ \delta_B^R \delta_B^R \equiv 0, \quad \bar{\delta}_B^R \delta_B^R \equiv 0, \quad \delta_B^R \bar{\delta}_B^R + \bar{\delta}_B^R \delta_B^R \equiv 0, \]  
(6.42)
the renormalized BRST and anti-BRST transformation for the renormalized fields \( A_\mu, C, \bar{C}, B \) are determined (by an appropriate rescaling of \( B \) field) as [8, 9]
\[ \delta_B^R A_\mu(x) = X D_\mu [A]^R C(x) := X [\partial_\mu C(x) + Z_A^{1/2} Z_g g_R (A_\mu(x) \times C(x))], \]  
(6.43a)
\[ \delta_B^R C(x) = -\frac{1}{2} X Z_A^{1/2} Z_g g_R (C(x) \times C(x)), \]  
(6.43b)
\[ \delta_B^R \bar{C}(x) = i X B(x), \]  
(6.43c)
\[ \delta_B^R B(x) = 0, \]  
(6.43d)
and
\[ \bar{\delta}_B^R A_\mu(x) = X D_\mu [A]^R C(x) := X [\partial_\mu \bar{C}(x) + Z_A^{1/2} Z_g g_R (A_\mu(x) \times \bar{C}(x))], \]  
(6.44a)
\[ \bar{\delta}_B^R \bar{C}(x) = -\frac{1}{2} X Z_A^{1/2} Z_g g_R (\bar{C}(x) \times \bar{C}(x)), \]  
(6.44b)
\[ \bar{\delta}_B^R \bar{C}(x) = i X \bar{B}(x), \]  
(6.44c)
\[ \bar{\delta}_B^R B(x) = 0, \]  
(6.44d)
where $X$ and $X^\ast$ are arbitrary real numbers and $\bar{B}$ is defined by

$$\bar{B}(x) = -B(x) + iZ_1^{1/2}Z_2g_R(C(x) \times \bar{C}(x)).$$  \hspace{1cm} (6.45)

The Lagrangian is written by making use of the renormalized BRST and anti-BRST transformation and the renormalized fields as

$$\mathcal{L}_{YM}^{\text{tot}} = -\frac{1}{4} Z_A (\partial_\mu A_\nu - \partial_\nu A_\mu + Z_2 Z_1^{1/2} g_R A_\mu \times A_\nu)^2$$

$$+ \frac{Z_C}{X^\ast} i \delta_B^{\text{BRST}} \delta_B^{\text{anti-BRST}} \left( \frac{1}{2} A_\mu \cdot A_\mu - \frac{Z_2 Z_1^{1/2} g_R}{Z_A} iC \cdot \bar{C} \right) + \frac{Z_2^2 Z_3^2}{Z_A^2} B \cdot B. \hspace{1cm} (6.46)$$

This agrees with (4.21).

We derive the condition for the renormalized composite operator $O_R$ to be invariant under the renormalized BRST transformation defined above. We can write a finite composite operator of mass dimension 2 in the form (up to an overall constant):

$$Q_R = \left[ \frac{1}{2} A_\mu(x) \cdot A_\mu(x) \right] + K_R [i\bar{C}(x) \cdot C(x)]_R, \hspace{1cm} (6.47)$$

where $K_R$ is a finite function of the renormalized parameters, $g_R$, $\xi_R$, $\lambda$. Performing the renormalized BRST transformation (6.43d) after the renormalization factors (6.33) of the composite operator are included, we obtain

$$\delta_B^{\text{BRST}} Q_R = \delta_B^{\text{anti-BRST}} \left\{ (Z_1 + K_R Z_3) (\frac{1}{2} A_\mu \cdot A_\mu) + (Z_2 + K_R Z_4) (i\bar{C} \cdot C) \right\}$$

$$= (Z_1 + K_R Z_3) X \partial_\mu C \cdot A_\mu$$

$$+ (Z_2 + K_R Z_4) \left\{ i\bar{C} \cdot (X Z_1^{1/2} Z_2 g_R C \times C) \right\}$$

$$+ X \left( \frac{Z_A}{Z_2 Z_3} \frac{1}{X^\ast} \partial_\mu A_\mu - iZ_1^{1/2} Z_3^2 \xi R g_R C \times \bar{C} \right) \cdot C. \hspace{1cm} (6.48)$$

For the right-hand-side to be a total derivative, we must require two conditions: 1) the coefficient for the term $C \cdot (\bar{C} \times C)$ vanishes, 2) the remaining terms containing the derivative are combined into a total derivative term. The respective condition reads

$$\frac{Z_1^{1/2} Z_2 g_R}{2} = \frac{Z_1^{1/2} Z_2}{Z_A} Z_3 \xi_R,$$  \hspace{1cm} (6.49)

$$Z_1 + K_R Z_3 = (Z_2 + K_R Z_4) \frac{Z_A}{Z_2 Z_3} \frac{1}{X^\ast} \xi_R. \hspace{1cm} (6.50)$$

The first condition reduces to

$$\xi_0 = Z_3 \xi_R = \frac{1}{2}. \hspace{1cm} (6.51)$$

Since $Z_2, Z_3 \sim O(\hbar/\epsilon)$ and $Z_1, Z_4 \sim 1 + O(\hbar/\epsilon)$, the second condition yields for the $O(1)$ term:

$$K_R = \lambda_R, \hspace{1cm} (6.52)$$
and for the $O(1/\epsilon)$ term:
\[
Z_A^{(1)} - Z_1^{(1)} - \lambda R Z_3^{(1)} + \frac{Z_2^{(1)}}{\lambda R} - Z_4^{(1)} + Z_4^{(1)} = 0.
\] (6.53)

This condition is the same as (6.37). In the Landau gauge $\alpha = \lambda = 0$, especially, the condition (6.53) reduces to $Z_2^{(1)} = 0$. This is automatically satisfied in this case.

7 Operator product expansion and vacuum condensate

We apply the operator product expansion (OPE) or short distance expansion (SDE) to the gluon and ghost propagators. The OPE is originally proposed as an operator relation by Wilson [46]. For example, the product of two scalar field operators defined at different spacetime points is expanded as
\[
\phi(x)\phi(y) \sim \sum_i F[O_i](x-y) [O_i \left(x+y \over 2 \right)],
\]
(7.1)
where the composite operators $\{O_i\}$ form a complete set of renormalized local operators. The famous proof of OPE by Zimmermann [47] was given in the framework of the perturbation theory. Quite recently, the OPE was rigorously proved as an operator relation by Bostelman [48].

According to the method [49, 50], the (Fourier transformed) Wilson coefficient $\tilde{F}[\phi_1\cdots\phi_n](p)$ in the OPE:
\[
\phi(x)\phi(y) \sim \sum_n F[\phi_1\cdots\phi_n](x-y) \left[ \phi_1 \cdots \phi_n \left(x+y \over 2 \right) \right],
\]
(7.2)
can be calculated in perturbation theory by equating a $(2+n)$-point one-particle irreducible (1PI) Green’s function — where two of the external legs have hard momentum $p$ and the remaining $n$ external legs are assigned zero momentum $q = 0$ — with the Wilson coefficient times an $n$-point Green’s function with an insertion of the relevant composite operator at zero momentum.

7.1 The OPE in the tree level

First, we consider the OPE of the inverse gluon propagator:
\[
(D^{-1})^{AB}_{\mu\nu}(p) = C^{[1]}_{\mu\nu}(p) \langle 1 \rangle + C^{[AF]}_{\mu\nu}(p) \left( \frac{1}{2} A_\mu \cdot A_\nu \right) + C^{[CC]}_{\mu\nu}(p) \langle i \bar{C} \cdot C \rangle + \cdots,
\]
(7.3)
where the first Wilson coefficient is nothing but the bare inverse gluon propagator:
\[
C^{[1]}_{\mu\nu}(p) = (D_0^{-1})^{AB}_{\mu\nu}(p) := -p^2 (P^T_{\mu\nu} + \lambda^{-1} P^L_{\mu\nu}) \delta^{AB}
\]
\[
= -p^2 \left( g_{\mu\nu} - \frac{p^\mu p^\nu}{p^2} + \lambda^{-1} \frac{p^\mu p^\nu}{p^2} \right) \delta^{AB}.
\]
(7.4)

7 The authors would like to thank Izumi Ojima for informing this reference.
The other Wilson coefficients are calculated in the perturbation theory from the diagrams:

\[ iC_{\mu\nu}^{[A^2]} = \begin{array}{c}
\begin{array}{c}
\text{blob}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{blob}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{blob}
\end{array}
\end{array}, \quad (7.5) \]

\[ C_{\mu\nu}^{[\bar{C}C]} = \begin{array}{c}
\begin{array}{c}
\text{blob}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{blob}
\end{array}
\end{array}. \quad (7.6) \]

In the diagram, two external legs have hard momentum \( p \) and the \((n = 2)\) lines connected to a blob correspond to the external legs with zero momentum \( q = 0 \).

The explicit calculation in the tree level yields the result (see Appendix for the details of calculations):

\[ C_{\mu\nu}^{[A^2]}(p) = - \frac{N_c g^2}{2(N_c^2 - 1)} (1 + \lambda) P_T^{\mu\nu} \delta^{AB}, \quad (7.7) \]

\[ C_{\mu\nu}^{[\bar{C}C]}(p) = 2 \frac{N_c g^2}{(N_c^2 - 1)} \xi (1 - \xi) P_L^{\mu\nu} \delta^{AB}, \quad (7.8) \]

where we have put \( C_2(G) = N_c \) for simplicity. Defining the vacuum polarization tensor of the gluon by

\[ (D^{-1})_{\mu\nu}^{AB}(p) := (D_0^{-1})_{\mu\nu}^{AB}(p) + \Pi_{\mu\nu}^{AB}(p), \quad (7.9) \]

we obtain the vacuum polarization tensor of the gluon:

\[ \Pi_{\mu\nu}^{AB}(p) = \frac{N_c g^2}{4(N_c^2 - 1)} \delta^{AB} \left\{ -(1 + \lambda) P_T^{\mu\nu} (A_\rho \cdot A^\rho) + 2 D \xi (1 - \xi) P_L^{\mu\nu} \langle i \bar{C} \cdot C \rangle \right\}. \quad (7.10) \]

It turns out that even the inclusion of the quartic ghost interaction does not affect the Wilson coefficient \( C_{\mu\nu}^{[A^2]} \) at least in the tree level. For the Wilson coefficient \( C_{\mu\nu}^{[\bar{C}C]} \), however, there is an extra contribution coming from the quartic ghost interaction, as suggested already in \cite{2}. The non-zero Wilson coefficient \( C_{\mu\nu}^{[\bar{C}C]} \) due to the presence of the quartic ghost interaction \((\xi \neq 0, 1)\) breaks the transversality of the gluon polarization tensor, i.e., \( \Pi_{\mu\nu} \neq P_T^{\mu\nu} \Pi \). This result does not contradict with the Slavnov-Taylor identity \cite{5,8,27}. When \( \xi = 0 \) (resp. \( \xi = 1 \)), the ghost condensate \( \langle i \bar{C} \cdot C \rangle \) can not appear in the OPE, since the gluon–ghost–anti-ghost vertex \((4.16)\) is proportional to the outgoing ghost (resp. anti-ghost) momentum \( p_\mu \) (resp. \( q_\mu \)). The above result \((7.10)\) suggests the existence of the effective gluon mass given by

\[ m_A^2 = - \frac{N_c g^2}{4(N_c^2 - 1)} (1 + \lambda) \langle A_\rho \cdot A^\rho \rangle. \quad (7.11) \]

Therefore, the gluon condensation of mass dimension 2 can be an origin of the gluon mass. The effect of higher orders will be investigated in the next subsection.

Next, we perform the OPE for the inverse ghost propagator:

\[-i(G^{-1})^{AB}(p) = C_{AB}^{[\bar{C}]}(p)\langle 1 \rangle + C_{AB}^{[A^2]}(p)\langle \frac{1}{2} A_\rho \cdot A^\rho \rangle + C_{AB}^{[\bar{C}C]}(p)\langle i \bar{C} \cdot C \rangle + \cdots, \quad (7.12)\]
where the first Wilson coefficient agrees with the bare inverse ghost propagator:

$$C^{[1]}_{AB}(p) = -i(G^{-1}_0)_{AB}(p) = -p^2 \delta^{AB}. \quad (7.13)$$

The other Wilson coefficients are calculated from the diagrams,

$$-C^{[A^2]}_{gh} = \text{diagram} + \text{diagram}, \quad (7.14)$$

$$iC^{[CC]}_g = \text{diagram} + \text{diagram}, \quad (7.15)$$

which yield the result:

$$C^{[A^2]}_{AB}(p) = \frac{Ncg^2}{2(N_c^2 - 1)} \delta^{AB}, \quad (7.16)$$

$$C^{[CC]}_{AB}(p) = 0. \quad (7.17)$$

Here the coefficient $C^{[CC]}_{AB}$ vanishes due to cancellation, see Appendix. Defining the vacuum polarization tensor of the ghost by

$$(G^{-1})^{AB}(p) := (G^{-1}_0)_{AB}(p) + i\Pi^{AB}_{gh}(p), \quad (7.18)$$

the vacuum polarization for the ghost is obtained:

$$\Pi^{AB}_{gh}(p) = \frac{Ncg^2}{4(N_c^2 - 1)} \delta^{AB} \langle A_\rho \cdot A^\rho \rangle. \quad (7.19)$$

We find that the ghost vacuum polarization has no contribution from the ghost-anti-ghost condensation even for $\xi \neq 0, 1$. Thus we obtain the effective ghost mass:

$$m^2_C = \frac{Ncg^2}{4(N_c^2 - 1)} \langle A_\rho \cdot A^\rho \rangle. \quad (7.20)$$

This result shows that the gluon condensation of mass dimension 2 can also be an origin of the ghost mass.

The combination of gluon and ghost condensation appearing in the OPE is not BRST invariant in the sense explained in the previous section. This is reasonable, since even the OPE of gauge invariant operators does not give a gauge invariant combination in the OPE, see e.g. [51].

In the Lorenz gauge, the effective gluon mass and ghost mass are generated by the gluon condensation of mass dimension 2 alone in the tree level. This is not the case if we include the high-order correction as will be shown in the next subsection. In the MA gauge, on the contrary, two condensations from the off-diagonal gluon and off-diagonal ghost contribute to the effective off-diagonal gluon and ghost masses already in the tree level, see [2, 20].
7.2 RG improvement of the OPE

One of the advantages of the OPE is that the momentum dependence of the Wilson coefficient is determined by the renormalization group (RG) equation. More accurately, the change of the Wilson coefficient under the RG transformation can be specified by the renormalization factors $Z$ which are to be calculated before the RG improvement of the OPE calculus. Therefore, we can obtain the higher-order corrections for the momentum dependence of the coefficient without any explicit higher-order computations (at least for the leading logarithmic corrections).

7.2.1 RG equation for Wilson coefficients

We begin with an OPE relation in the momentum representation obtained by extracting composite operators up to mass dimension 2 (we omit all the indices, since they are not essential in the following arguments):

$$-i\tilde{A}_R(p)\tilde{A}_R(-p) = D_{\text{pert}}(p)[1] + F_1^A(p)\left[\frac{1}{2}A(0)A(0)\right]_R + F_2^A(p)[i\tilde{C}(0)C(0)]_R + \cdots,$$

(7.21a)

$$\tilde{C}_R(p)\tilde{C}_R(-p) = -iG_{\text{pert}}(p)[1] + F_1^C(p)\left[\frac{1}{2}A(0)A(0)\right]_R + F_2^C(p)[i\tilde{C}(0)C(0)]_R + \cdots,$$

(7.21b)

where $D_{\text{pert}}(p)$ and $G_{\text{pert}}$ denote respectively the perturbative gluon and ghost propagators in which the perturbative loop corrections are included besides the OPE contribution.

First, we try to rewrite all the field operators in both sides of (7.21a) and (7.21b) in terms of bare quantities. Hereafter it is supposed that the Wilson coefficient and composite operators are defined based on the renormalization scheme depending on a certain parameter $\mu$ (corresponding to the mass scale), which is different from the BPHZ prescription at zero momentum, $q = 0$. In the actual calculations, we adopt the minimal subtraction (MS) scheme, although the resulting expressions can be translated into those of the momentum-space subtraction scheme (MOM).

By making use of the $Z$ factors calculated already in the previous section, two OPE relations above are combined into a matrix form:

$$Z_f^{-1}\begin{pmatrix} -i\tilde{A}_0(p)\tilde{A}_0(-p) \\ \tilde{C}_0(p)\tilde{C}_0(-p) \end{pmatrix} = D_{\text{pert}} + F\tilde{Z}\begin{pmatrix} \frac{1}{2}A_0(0)A_0(0) \\ i\tilde{C}_0(0)C_0(0) \end{pmatrix} + \cdots,$$

(7.22)

where we have defined the two by two matrices,

$$Z_f = \begin{pmatrix} Z_A & 0 \\ 0 & Z_C \end{pmatrix}, \quad F := \begin{pmatrix} F_1^A(p) & F_2^A(p) \\ F_1^C(p) & F_2^C(p) \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}, \quad D_{\text{pert}} := \begin{pmatrix} D_{\text{pert}}(p) \\ -iG_{\text{pert}}(p) \end{pmatrix},$$

(7.23)

and a column vector,

$$\begin{pmatrix} D_{\text{pert}}(p) \\ -iG_{\text{pert}}(p) \end{pmatrix}.$$
Introducing a matrix $F_0$ by

$$F_0 = Z_t F \tilde{Z} := \begin{pmatrix} F_{01}^A(p) & F_{02}^A(p) \\ F_{01}^C(p) & F_{02}^C(p) \end{pmatrix},$$

we obtain an OPE relation among the bare quantities as

$$\left( -i A_0(p) A_0(-p) \right) = Z_t D_{\text{pert}} + F_0 \left( \frac{1}{2} A_0(0) A_0(0) \right) + \cdots. \tag{7.25}$$

Second, we observe that the relation (7.26) should have no dependence on the renormalization point $\mu$. Hence, the first term on the right-hand-side of (7.26) is independent of $\mu$, i.e.,

$$\mu \frac{d}{d\mu} (Z_t D_{\text{pert}}) = 0, \tag{7.27}$$

and the coefficient $F_0$ in the second term is also independent of $\mu$, i.e,

$$\mu \frac{d}{d\mu} F_0 = \mu \frac{d}{d\mu} (Z_t F \tilde{Z}) = 0. \tag{7.28}$$

We multiply (7.28) by $Z_t^{-1}$ from the left and by $\tilde{Z}^{-1}$ from the right to obtain

$$\left[ \mu \frac{\partial}{\partial \mu} + \sum_i \beta_i(\alpha) \frac{\partial}{\partial \alpha_i} \right] F + Z_t^{-1} \left( \mu \frac{d}{d\mu} Z_t \right) F + F \left( \mu \frac{d}{d\mu} \tilde{Z} \right) \tilde{Z}^{-1} = 0, \tag{7.29}$$

where $\alpha_i$ denotes the parameters of the theory $(g_R, \xi_R, \lambda_R)$, and $\beta_i$ denotes the corresponding RG function $\beta_i(\alpha) := \mu \frac{\partial}{\partial \alpha_i} \alpha_i$. Here we have used a fact that $\mu \frac{\partial}{\partial \mu} + \sum_i \beta_i(\alpha(\mu)) \frac{\partial}{\partial \alpha_i}$ is just the ordinary differential operator $\mu \frac{d}{d\mu}$.

Defining the RG function (matrix) $\gamma_t, \tilde{\gamma}$ from $Z_t, \tilde{Z}$ by

$$\mu \frac{d}{d\mu} Z_t := Z_t \gamma_t, \quad \mu \frac{d}{d\mu} \tilde{Z} := \tilde{\gamma} \tilde{Z}, \tag{7.30}$$

we obtain the RG equation for the matrix $F$ of the Wilson coefficients:

$$\left[ \mu \frac{\partial}{\partial \mu} + \sum_i \beta_i(\alpha) \frac{\partial}{\partial \alpha_i} \right] F(p, \alpha, \mu) + \gamma_t F(p, \alpha, \mu) + F(p, \alpha, \mu) \tilde{\gamma} = 0. \tag{7.31}$$

Similarly, we can show that $D_{\text{pert}}$ obeys the RG equation:

$$\left[ \mu \frac{\partial}{\partial \mu} + \sum_i \beta_i(\alpha) \frac{\partial}{\partial \alpha_i} \right] D_{\text{pert}}(p, \alpha, \mu) + \gamma_t D_{\text{pert}}(p, \alpha, \mu) = 0. \tag{7.32}$$

\footnote{Were it not for the renormalization of the composite operator, $F_0$ reduced to $F$.}
7.2.2 Solving the RG equation

Now we proceed to solve the RG equation just obtained. A simple dimensional analysis leads to a relation, $F(\kappa p, \alpha, \kappa \mu) = \kappa^{d_f} F(p, \alpha, \mu)$ which is equivalent to the relation:

$$F(\kappa p, \alpha, \mu) = \kappa^{d_f} F(p, \alpha, \mu)$$

(7.33)

where $d_f$ is the canonical dimension of $F$. Hence, $F$ satisfies

$$\left[ \kappa \frac{\partial}{\partial \kappa} + \mu \frac{\partial}{\partial \mu} - d_f \right] F(\kappa p, \alpha, \mu) = 0.$$  

(7.34)

We use this equation to eliminate $\mu \frac{\partial}{\partial \mu}$ in (7.31) to obtain

$$\left[ \kappa \frac{\partial}{\partial \kappa} - \sum_i \beta_i(\alpha) \frac{\partial}{\partial \alpha_i} - d_f \right] F(\kappa p, \alpha, \mu) - \gamma^f F(\kappa p, \alpha, \mu) - F(\kappa p, \alpha, \mu) \tilde{\gamma} = 0.$$  

(7.35)

This is the homogeneous RG equation of Weinberg-'t Hooft type [52] which is adequate for the mass-independent renormalization method.

By the standard method [30, 32], the general solution of the RG equation (7.35) is given by

$$F(\kappa p, \alpha, \mu) = \kappa^{-4} \exp \left\{ \int_1^\kappa d\kappa' \frac{\gamma^f(\kappa')}{\kappa'} \right\} F(p, \tilde{\alpha}(\kappa), \mu) \exp \left\{ \int_1^\kappa d\kappa' \frac{\tilde{\gamma}(\kappa')}{\kappa'} \right\}.$$  

(7.36)

where we have imposed the boundary condition: $\tilde{\alpha}(\kappa = 1) = \alpha$.

The similar consideration yields the general solution of the RG equation (7.32):

$$D_{pert}(\kappa p, \alpha, \mu) = \kappa^{-2} \exp \left\{ \int_1^\kappa d\kappa' \frac{\gamma^f(\kappa')}{\kappa'} \right\} D_{pert}(p, \alpha, \mu).$$  

(7.37)

Once we know the $Z$ factors of the fundamental field and of the composite operator, it is easy to calculate $\gamma^f$, $\tilde{\gamma}$ up to $O(h)$, since we have known all the $Z$ factors of the fundamental field and of the composite operator up to $O(h)$ in this paper. In the high-energy limit $\kappa \to \infty$, it is expected that the solution can be explicitly obtained in the neighborhood of the non-trivial UV stable fixed point in the parameter space, due to asymptotic freedom of the Yang-Mills theory, i.e., $\tilde{g}(\kappa) \to \tilde{g}_\infty = 0$ as $\kappa \to \infty$. In the three-dimensional parameter space $g_R, \xi_R, \lambda_R$, actually, we have found that all the points are flowing into the UV fixed point B in the UV limit except for some...
lines that have higher symmetry. On the other hand, within the perturbation theory using the dimensional regularization, the $\mu$ dependent loop correction of all the $Z$ factors always appears with a factor of $O(g^2_R)$. Therefore, the RG function $\gamma$ as an element of the matrix $\gamma$ defined by differentiating the $Z$ factor with respect to $\mu$ is accompanied by $g^2_R$ to the $O(\hbar)$, like $\gamma \sim g^2_R f(\xi, \lambda) h + O(h^2)$. If the polynomial function $f(\xi, \lambda)$ in the above expression has a nonvanishing value at the fixed point $(\xi^*, \lambda^*)$, the $\mu$ dependence of $\gamma = g^2 f$ is governed by $g^2$ alone and hence we can replace $f(\xi, \lambda)$ with the constant $f(\xi^*, \lambda^*)$ at the UV fixed point. By substituting the fixed-point values $\lambda^*_R = 26/3$, $\xi^*_R = 1/2$ into $\xi$, $\lambda$, the $Z$ factors become

$$
Z^*_A = 1 - \frac{13 g^2 N_c \mu^{-2\epsilon}}{6 16\pi^2 \epsilon}, \\
Z^*_1 = 1 - \frac{29 g^2 N_c \mu^{-2\epsilon}}{4 16\pi^2 \epsilon}, \\
Z^*_2 = -\frac{1}{4} \left( \frac{26}{3} \right) g^2 N_c \mu^{-2\epsilon}, \\
Z^*_3 = \frac{1}{2} \frac{g^2 N_c \mu^{-2\epsilon}}{16\pi^2 \epsilon}, \\
Z^*_4 = 1,
$$

(7.38)

which yield the matrix of the renormalization group function:

$$
\gamma^*_A(g) = \frac{g^2 N_c}{8\pi^2} \begin{pmatrix} \frac{13}{6} & 0 & 0 \\ 0 & \frac{17}{12} & 0 \\ 0 & 0 & \frac{17}{12} \end{pmatrix}, \quad \tilde{\gamma}^*(g) = \frac{g^2 N_c}{8\pi^2} \begin{pmatrix} \frac{61}{12} & \frac{1}{4} \left( \frac{26}{3} \right)^2 \\ -\frac{1}{2} & -\frac{17}{12} \end{pmatrix}.
$$

(7.39)

Furthermore, we define the coefficient matrix $C_{\gamma^*_A}$ and $C_{\tilde{\gamma}^*}$ in (7.39) by

$$
\gamma^*_A(g) := g^2 C_{\gamma^*_A}, \quad \tilde{\gamma}^*(g) := g^2 C_{\tilde{\gamma}^*}.
$$

(7.40)

By taking into account the RG equation $\mu \frac{d}{d\mu} g = -\frac{b}{8\pi^2} g^3$ ($b = \frac{11}{6} N_c$) and the resulting relation: $\frac{d}{d\mu} \ln g^2 = \frac{2}{g} \frac{d}{d\mu} g = -\frac{2b}{8\pi^2} g^2$, the nontrivial integration of (7.38) can be performed as

$$
\int_1^\kappa dk' \frac{\gamma(g(k'))}{k'} = \int_1^\kappa dk' C_{\gamma} \frac{(g(k'))^2}{k'} = C_{\gamma} \frac{8\pi^2}{2b} \ln \frac{g^2(1)}{g^2(\kappa)}.
$$

(7.41)

Hence the solution becomes

$$
F(\kappa p, \alpha, \mu) = \kappa^{-4} \left( \frac{g^2(1)}{g^2(\kappa)} \right) C_{\gamma} \frac{8\pi^2}{2b} \left( F(p, \bar{\alpha}(\kappa), \mu) \right) \left( \frac{g^2(1)}{g^2(\kappa)} \right) C_{\gamma} \frac{8\pi^2}{2b}.
$$

(7.42)

The $\kappa$ dependence of $g^2$ is obtained by solving its RG equation as $g^2(\kappa) \sim \left[ \frac{2b}{8\pi^2} \ln \kappa \right]^{-1}$ for large $\kappa$. Substituting (7.41) into (7.38), therefore, we determine the $\ln \kappa$ dependence of the solution for large $\kappa$:

$$
F(\kappa p, \alpha, \mu) = \kappa^{-4} \left( \ln \kappa \right) C_{\gamma} \frac{8\pi^2}{2b} \left( F(p, \bar{\alpha}(\kappa), \mu) \right) \left( \ln \kappa \right) C_{\gamma} \frac{8\pi^2}{2b}.
$$

(7.43)

In order to cast the matrix power of $\ln \kappa$ into a more tractable form, we shall diagonalize the matrix $C_{\gamma}$ in such a way that $S$ diagonalizes $C_{\gamma}$ by the similarity
transformation $C_{\gamma} \rightarrow S^{-1}C_{\gamma}S$. Such a matrix $S$ and the diagonalized matrix are given by

$$S = \begin{pmatrix} -\frac{13}{3} & -\frac{26}{3} \\ 1 & 1 \end{pmatrix}, \quad S^{-1}C_{\gamma}S = \frac{N_c}{8\pi^2} \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{35}{12} \end{pmatrix}. \quad (7.44)$$

This diagonalization corresponds to redefining the combination between two composite operators of mass dimension 2, i.e., $\frac{1}{2}A(0)A(0)$ and $i\bar{C}(0)C(0)$, by multiplying $S^{-1}$:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = S^{-1} \begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix} = \frac{3}{13} \begin{pmatrix} \frac{1}{2}A^2 + \frac{26}{13}i\bar{C}C \\ -\frac{1}{2}A^2 - \frac{13}{3}i\bar{C}C \end{pmatrix}. \quad (7.45)$$

Inserting the identity matrix $1 = SS^{-1}$ appropriately, the solution $(7.42)$ is rewritten as

$$F(\kappa p, \alpha, \mu) = \kappa^{-4} \begin{pmatrix} \bar{g}^2(1) \\ g^2(\kappa) \end{pmatrix} C_{\gamma}^{\frac{8N_c}{20}} F(p, \bar{\alpha}, \mu) SS^{-1} \begin{pmatrix} \bar{g}^2(1) \\ g^2(\kappa) \end{pmatrix} C_{\gamma}^{\frac{8N_c}{20}} SS^{-1}. \quad (7.46)$$

Now both $C_{\gamma f}$ and $S^{-1}C_{\gamma}S$ are diagonal. Hence we can write down the power explicitly as

$$F(\kappa p) = \kappa^{-4} \begin{pmatrix} \frac{\bar{g}^2(1)}{g^2(\kappa)} \frac{13}{12}N_c \\ 0 \end{pmatrix} T(p) S \begin{pmatrix} \frac{\bar{g}^2(1)}{g^2(\kappa)} \frac{3}{4N_c} \\ 0 \end{pmatrix} \begin{pmatrix} \bar{g}^2(1) \\ g^2(\kappa) \end{pmatrix} \quad (7.47)$$

Here we impose a condition that $T(p) := F(p, \bar{\alpha}(\kappa), \mu)$ coincides with the Wilson coefficient in the tree level obtained in the previous section in which the coupling constant is replaced with the running coupling constant $\bar{\alpha}(\kappa)$. Note that $F$ is the Wilson coefficient of the Green function (not of the one-particle irreducible (1PI) function).\footnote{Except for the Landau gauge in which any operator mixing does not occur, a linear combination of different powers of $\ln \kappa$ appears in the solution, and its combination coefficients cannot be completely determined by perturbation theory alone. But it is important to note that a fitting of the analytical result with the simulation data (or experimental data) can determine the asymptotic behavior of $F$ completely as discussed in the next subsection.} Hence we put

$$T(p) = \begin{pmatrix} T_1(p) \\ T_2(p) \end{pmatrix} \quad (7.48)$$

We notice that each element $T_1, \ldots, T_4$ of $T(p)$ brings an extra $\ln \kappa$ factor to $F$ through $\bar{g}^2(\kappa) \sim \frac{1}{\ln \kappa}$. Therefore, the OPE correction up to dimension 2 operators reads

$$F(p) \begin{pmatrix} \frac{1}{2}A^2 \\ i\bar{C}C \end{pmatrix} = \begin{pmatrix} (-\frac{13}{3}T_1 + T_2)(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}) \frac{8N_c}{20} \\ -\frac{13}{3}T_3(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}) \frac{8N_c}{20} \end{pmatrix} \begin{pmatrix} \frac{\bar{g}^2(1)}{g^2(\kappa)} \frac{3}{4N_c} \\ 0 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \quad (7.49)$$
where we have used $T_4 = 0$. Here we have used the translation rule from the MS scheme to the MOM scheme:

$$\frac{g^2(1)}{g^2(\kappa)} \rightarrow \frac{\ln p/\Lambda_{\text{QCD}}}{\ln \mu/\Lambda_{\text{QCD}}} \quad (7.50)$$

Among the terms with various powers of $\ln \kappa$, the largest-power term (corresponding to the largest eigenvalue of the matrix $C_{\gamma}$) is dominant in the UV limit ($\kappa \gg 1$). Extracting this $\ln \kappa$ contribution, we can simplify the Wilson coefficient of $1\text{PI}$ function in the UV limit as

$$C^{1\text{PI}} = \begin{pmatrix} C_{gl}^{[A]} & C_{gl}^{[CC]} \\ C_{gh}^{[A]} & C_{gh}^{[CC]} \end{pmatrix} = \begin{pmatrix} (iD_{\text{pert}})^{-2} & 0 \\ 0 & (iG_{\text{pert}})^{-2} \end{pmatrix} F \quad (7.51)$$

$$= \frac{8\pi^2}{2b} \frac{N_c}{(N_c^2 - 1)} \begin{pmatrix} (D_{\text{pert}}/D_0)^{-2} & 0 \\ 0 & (G_{\text{pert}}/G_0)^{-2} \end{pmatrix}$$

$$\times \begin{pmatrix} 13(-1-\lambda)P_0 - 6\xi(1-\xi)P_1 & (\ln \frac{p}{\Lambda_{\text{QCD}}} \frac{N_c}{26})^{-1} \\ (\ln \frac{\mu}{\Lambda_{\text{QCD}}} \frac{N_c}{26})^{-1} & 13(-1-\lambda)P_0 - 6\xi(1-\xi)P_1 \frac{N_c}{26}^{-1} \end{pmatrix}.$$

In the similar way, we obtain

$$D_{\text{pert}}(\kappa p, \alpha, \mu) = \kappa^{-2} \begin{pmatrix} \left(\frac{g^2(1)}{g^2(\kappa)}\right)^{\frac{13}{6} \frac{N_c}{26}} & 0 \\ 0 & \left(\frac{g^2(1)}{g^2(\kappa)}\right)^{\frac{13}{6} \frac{N_c}{26}} \end{pmatrix} D_t(p),$$

where the tree expression is given by

$$D_t(p) = \begin{pmatrix} D_0(p) \\ -iG_0(p) \end{pmatrix} = \begin{pmatrix} -\frac{1}{p^2}(P_T + \lambda P_L) \\ \frac{1}{p^2} \end{pmatrix}. \quad (7.54)$$

### 7.2.4 The solution at the conventional Landau gauge

Finally, we consider the OPE on the line A of the fixed points (corresponding to the conventional Landau gauge), the RG matrices read

$$\gamma^* = g^2 C_{\gamma} = \frac{g^2 N_c}{8\pi^2} \begin{pmatrix} -\frac{13}{6} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix}, \quad \tilde{\gamma}^* = g^2 C_{\tilde{\gamma}} = \frac{g^2 N_c}{8\pi^2} \begin{pmatrix} 35/12 & 0 \\ 0 & -3/4 \end{pmatrix}. \quad (7.55)$$

The diagonalization can be performed as

$$S = \begin{pmatrix} 1 & -\frac{13}{6} \\ 0 & 1 \end{pmatrix}, \quad S^{-1}C_{\tilde{\gamma}}S = \frac{N_c}{8\pi^2} \begin{pmatrix} 3/4 & 0 \\ 0 & 35/12 \end{pmatrix}. \quad (7.56)$$

The eigenvalues of $C_{\tilde{\gamma}}$ are the same as those at the fixed point B. Therefore, we obtain the Wilson coefficient $C_{\mu\nu}^{[A]}$ between $\langle A_\mu(p)A_\nu(-p)\rangle^{-1}$ and $\langle (A(0))^2 \rangle$ and $C^{[CC]}$
between \( (C(p)\tilde{C}(-p))^{-1} \) and \( (A(0))^2 \):

\[
F(\kappa p) = \kappa^{-4} \begin{pmatrix}
T_1(p) \left( \frac{g^2(1)}{g^2(\kappa)} \right)^{\frac{3}{2b}} & 0 \\
T_3(p) \left( \frac{g^2(1)}{g^2(\kappa)} \right)^{\frac{13}{2b}} & 0
\end{pmatrix},
\]

where no mixing between gluon and ghost occurs due to \( T_2 = 0 \) in addition to \( T_4 = 0 \). The coefficient of the 1PI OPE read

\[
\begin{pmatrix}
C^{[A^2]}_gl & C^{[CC]}_gl \\
C^{[A^2]}_{gh} & C^{[CC]}_{gh}
\end{pmatrix} = \frac{8\pi^2}{2b} \frac{N_c}{2(N_c^2 - 1)} \begin{pmatrix}
\left( \frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}} \right)^{-\frac{13}{6}} - \frac{N_c}{2b} & 0 \\
0 & \left( \frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}} \right)^{-\frac{3}{2b}} - \frac{N_c}{2b}
\end{pmatrix} \Delta_t(p),
\]

where

\[
D_{pert}(p) = \begin{pmatrix}
\left( \frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}} \right)^{-\frac{13}{6}} - \frac{N_c}{2b} & 0 \\
0 & \left( \frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}} \right)^{-\frac{3}{2b}} - \frac{N_c}{2b}
\end{pmatrix}.
\]

This result for the ghost part is new, while the gluon part reproduces the recent result of Boucaud et. al. \cite{22} in the MOM scheme (Note that their definition of \( \gamma \) is different from ours by a factor 2 and the coefficient \( \gamma_0 \) differs by the signature). In order to transfer from our renormalization scheme to the MOM scheme, we have used the translation rule \( (7.50) \). In the Landau gauge, therefore, we have confirmed that the ghost condensation does not affect the inverse gluon propagator as in the tree level, even if the leading logarithmic corrections are taken into account in the OPE. In other words, the gluon condensation is decoupled from the ghost condensation within this approximation.

### 7.3 Full propagators: momentum dependence

The vacuum polarization tensor of the gluon is decomposed into the transverse and the longitudinal parts:

\[
\Pi^{AB}(p) = [\Pi^T(p^2)P^T_{\mu\nu} + \Pi^L(p^2)P^L_{\mu\nu}]\delta^{AB},
\]

where \( \Pi^T \) and \( \Pi^L \) are functions of \( p^2 \) alone. Once the vacuum polarization functions \( \Pi^T \) and \( \Pi^L \) of the gluon are obtained from the OPE, the propagator is written as

\[
(D)^{AB}_{\mu\nu}(p) = \delta^{AB} \left[ \frac{1}{-p^2 + \Pi^T(p^2)} P^T_{\mu\nu} + \frac{\lambda}{-p^2 + \lambda \Pi^L(p^2)} P^L_{\mu\nu} \right]
\]

\[
= \delta^{AB} \left[ \frac{Z_{gl}(-p^2)}{-p^2} P^T_{\mu\nu} + \frac{\lambda}{-p^2 + \lambda \Pi^L(p^2)} P^L_{\mu\nu} \right],
\]

where we have defined a function \( Z_{gl}(-p^2) \) by

\[
Z_{gl}(-p^2) = Z_{pert}(-p^2) + Z_{OPE}(-p^2) := \frac{-p^2}{-p^2 + \Pi^T(p^2)}.
\]
Note that $\Pi^L(p^2) \equiv 0$ in the conventional Landau gauge.

On the other hand, if the vacuum polarization function of the ghost $\Pi_{gh}^{AB}(p^2) = \delta^{AB} \Pi_{gh}(p^2)$ is calculated by OPE, the ghost propagator is obtained as

$$G^{AB}(p) = [(G_0)^{-1} + i\Pi_{gh}(p^2)]^{-1}_{AB} = \frac{1}{-ip^2 + i\Pi_{gh}(p^2)} \delta^{AB} = (-i)G_{gh}(-p^2) \delta^{AB},$$

(7.64)

where we have introduced a function $G_{gh}(-p^2)$ by

$$G_{gh}(-p^2) = G_{\text{pert}}(-p^2) + G_{\text{OPE}}(-p^2) := \frac{-p^2}{-p^2 + \Pi_{gh}(p^2)}.$$  

(7.65)

The OPE contribution $\Pi^{\text{OPE}}$ to the vacuum polarization function in the inverse propagators (7.3) and (7.12) is related to the Wilson coefficient $C^{\text{IPI}}$ as

$$\Pi^{\text{OPE}} := \left[\Pi_{gh}^{\text{OPE}}(\Pi_{gh}^{G})\right] = C^{\text{IPI}} \left(\frac{1}{2}A^2 \right) = \left(\frac{iD_{\text{pert}}}{iG_{\text{pert}}}\right)^{-2} \left(\frac{iD_{\text{pert}}}{iG_{\text{pert}}}\right)^{-2} F \left(\frac{1}{2}A^2 \right).$$

(7.66)

Substituting the result (7.41) into (7.66), we obtain a pair of vacuum polarization functions:

$$\Pi^{\text{OPE}}(p) = \left(\begin{array}{cc} (T_2 - \frac{13}{3}T_1)(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}) \frac{35}{12} \frac{N_c}{26} & \frac{1}{(iG_{\text{pert}})^2} \\ -\frac{13}{3}T_3(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}) \frac{1}{26} \frac{N_c}{6} & \left(-\frac{26}{3}T_3(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}) \frac{1}{26} \frac{N_c}{6} \right) \end{array}\right) \left(\begin{array}{c} Q_1 \\ Q_2 \end{array}\right).$$

(7.67)

It turns out that the vacuum polarization functions just obtained reduce to the tree results, i.e., (7.10) and (7.19), at $\kappa = 1$ (or $p = \mu$). Therefore, the ghost condensation $\langle i\bar{C}C \rangle$ does contribute to the gluon and ghost vacuum polarization functions in the leading logarithmic corrections of the OPE.

Thus the OPE contribution to the gluon and ghost vacuum polarization functions are obtained:

$$\Pi_\text{OPE}^{(p^2)} = \frac{2\pi^2}{b} N_c (1 + \lambda) \left\{ \left(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}\right)^{\frac{35}{12} \frac{N_c}{26} - 1} \left(\frac{1}{2}A^2 \right) + \frac{26}{3} \langle i\bar{C}C \rangle \right\} - 2 \left(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}\right)^{\frac{13}{6} \frac{N_c}{26} - 1} \left(\frac{1}{2}A^2 \right) + \frac{26}{3} \langle i\bar{C}C \rangle \right) \left(\frac{D_0(p)}{D_{\text{pert}}(p)}\right)^2,$$

(7.68)

$$\Pi_{gh}^{\text{OPE}}(p^2) = \frac{2\pi^2}{b} N_c (1 + \lambda) \left\{ \left(\frac{\ln p/\Lambda_{QCD}}{\ln \mu/\Lambda_{QCD}}\right)^{\frac{13}{6} \frac{N_c}{26} - 1} \left(\frac{1}{2}A^2 \right) + \frac{26}{3} \langle i\bar{C}C \rangle \right\} \left(\frac{G_0(p)}{G_{\text{pert}}(p)}\right)^2.$$

(7.69)
The effective gluon mass is obtained from the pole of $Z_{\text{gl}}(-p^2)$, i.e., a solution of the equation $p^2 = \Pi_T(p^2)$, while the effective ghost mass is obtained from the pole of $G_{\text{gh}}(-p^2)$, i.e., a solution of the equation $p^2 = -i\Pi_{\text{gh}}(p^2)$. In view of this, the solutions (7.68) and (7.69) would give an improvement of the tree-level result, (7.11) and (7.20). However, a BRST non-invariant combination $Q_2$ of composite operators appears together with the BRST invariant combination $Q_1$ discussed in the previous section. Therefore, these results indicate that we need more endeavor in order to reach the BRST invariant pole position in the IR region.

In the Landau gauge, especially, we have

$$Z_{\text{gl}}(-p^2) = -p^2 D_{\text{pert}}(p) - p^{-2} \left( \frac{\pi^2}{b} \frac{N_c}{(N_c^2 - 1)} \left( \ln \frac{p}{\Lambda_{\text{QCD}}} \right)^\frac{3}{4} \frac{N_c - 1}{4} \langle A^2 \rangle \right),$$  

(7.70)

$$G_{\text{gh}}(-p^2) = -ip^2 G_{\text{pert}}(p) + p^{-2} \left( \frac{\pi^2}{b} \frac{N_c}{(N_c^2 - 1)} \left( \ln \frac{p}{\Lambda_{\text{QCD}}} \right)^\frac{11}{4} \frac{N_c - 1}{26} \langle A^2 \rangle \right).$$  

(7.71)

After the Wick rotation to the Euclidean region $p^2 \rightarrow -p_E^2$, we find that the function $Z_{\text{gl}}(p_E^2)$ is monotonically increasing in $p_E^2$ if $\langle A_E^2 \rangle := -\langle A^2 \rangle > 0$, as in the case of constant $\Pi_T(-p_E^2) = M^2 > 0$. On the other hand, if $\langle A_E^2 \rangle := -\langle A^2 \rangle < 0$, $Z_{\text{gl}}(p_E^2)$ has a Landau pole in the IR region and is monotonically decreasing in $p_E^2$ in the UV region. In the conventional Landau gauge, these results can be compared with those of the Schwinger-Dyson equation (see e.g., [53]) and the numerical simulation on a lattice (see e.g., [22, 54–56]). According to these results, $Z_{\text{gl}}(p_E^2)$ is enhanced at intermediate momenta and has a peak at about 1 GeV. It was argued [56] that the enhancement of the gluonic form factor at IR region is related to quark confinement. However, this region is beyond the reach of our study in this paper. Incidentally, the data in the gauge other than the Landau gauge is not yet available.

8 Conclusion and discussion

In this paper we have discussed the multiplicative renormalizability of the composite operator $O$ in QED and Yang-Mills theory. This research is motivated by clarifying the mechanism of mass generation and a possible connection to quark confinement.

In QED, we have shown that the composite operator is trivially renormalizable and that the renormalized composite operator is BRST and anti-BRST invariant for an arbitrary value of the gauge fixing parameter. There is no subtlety related to the renormalization of the composite operator.

In the Yang-Mills theory, we have adopted the most general Lorentz gauge with two gauge-fixing parameters $\xi, \lambda$ which was derived by the Baulieu and Thierry-Mieg [8]. It was known [2] that the bare composite operator $O$ of mass dimension 2 is invariant under the bare BRST and anti-BRST transformation for the choice of gauge parameters $\lambda = 0$ or $\xi = \frac{1}{2}$ and that it is also invariant under the gauge transformation in the Landau gauge $\lambda = 0$. In this paper the composite operator has been renormalized by taking into account the operator mixing carefully. Here the matrix of renormalization factors has been explicitly calculated. Consequently, we
have found that the BRST and anti-BRST invariance of the renormalized composite operator $O^R$ holds if the renormalized parameters take the same value, $\lambda_R = 0$ or $\zeta_R = \frac{1}{2}$, as the bare one. Moreover, we have obtained the RG flow in the $(\xi, \lambda)$ plane to one-loop order. In the RG flow diagram, the RG flow runs only on the line $\xi_R = \frac{1}{2}$ if the initial position of $\xi$ is located somewhere on the line. The line $\lambda_R = 0$ is a line of fixed points. Therefore, if the system is located on a point in the line $\lambda_R = 0$ initially, it can not move from the initial position. This fact guarantees the BRST invariance of the renormalized composite operator $O^R$.

We have also examined in this paper how the conventional calculations are modified in the presence of the vacuum condensate of mass dimension 2. By performing the OPE of the gluon and ghost propagators, we have shown that the effective masses of gluon and ghost are generated due to the non-vanishing vacuum condensate. Although this phenomenon was already suggested based on the tree level calculation, we have taken into account the leading logarithmic corrections in consistent with the RG flow by making use of the RG equation. We have found that the effective masses are provided from the ghost condensation $\langle i\bar{C} \cdot C \rangle$ as well as the gluon condensation $\langle \frac{1}{2} A_\mu \cdot A^\mu \rangle$ (except for the Landau gauge $\lambda = 0$). This result should be compared with the tree level result where the effective mass has the contribution from the gluon condensate alone.

The next step is to show that the non-vanishing vacuum condensates $\langle O \rangle \neq 0$ is actually realized in the QCD vacuum. An attempt in this direction has already been performed in [20] by calculating the effective potential for the ghost condensation $\langle i\bar{C}C \rangle$ in the $SU(2)$ and $SU(3)$ Yang-Mills theories in the MA gauge. Quite recently, Verschelde et al. [57] have carefully obtained the multiplicatively renormalizable effective potential for the gluon condensate $\langle \frac{1}{2} A_\mu A^\mu \rangle$ in the Landau gauge up to two-loop order in the $SU(N)$ Yang-Mills theory. Both results support that the non-zero vacuum condensate of mass dimension 2 is energetically favoured in the Yang-Mills theory. In these approaches, an auxiliary field $\rho(x)$ corresponding to the composite operator has been introduced to obtain the effective potential $V(\sigma)$ of a constant $\sigma = \rho(x)$. However, this treatment has a number of subtle points which have not been discussed in these papers. This issue will be discussed in a subsequent paper [27] in detail.

In massless QED, photon pairing [43, 44] can occur in the strong coupling phase [39–41] where the chiral symmetry is spontaneously broken. Therefore, it will be possible to discuss the interplay between quark confinement and chiral symmetry breaking on equal footing in a unified treatment. The extension of this viewpoint into the non-Abelian case, i.e., gluon pairing [42] is also an interesting subject to be tackled in the future work.

Finally, we point out that the operator $O$ is essentially a mass term for the gluon and the ghost fields. Although a naive introduction of a mass term for the gluon alone breaks the BRST symmetry, our result indicates that there is a BRST invariant combination of mass terms:

$$L_m := \text{tr} \left[ \frac{1}{2} m^2 A_\mu(x) \cdot A^\mu(x) + m^2 \alpha i \bar{C}(x) \cdot \bar{C}(x) \right]. \quad (8.1)$$

This mass term is very similar to that obtained after the spontaneous breakdown caused by the non-vanishing vacuum expectation value of the Higgs scalar field. In
our case, the mass should be of dynamical origin. It is possible to give a proof of the multiplicative renormalizability of the Yang-Mills theory with a mass term preserving the BRST symmetry to all orders of perturbation theory. However, it is known [58, 59] that the introduction of the mass term (8.1) breaks the nilpotency of the off-shell BRST transformation as well as the on-shell one. Consequently, the unitarity of the theory turns out to be spoiled. In this sense, the mass generation should occur in the dynamical way, i.e., $\langle O \rangle \neq 0$ in the limit $m \to 0$. This viewpoint will be discussed in a subsequent paper.

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A OPE calculations

In order to give the OPE correction for the gluon propagator, we need to calculate the following diagrams.

$$\begin{align*}
= & \frac{1}{(N_c^2 - 1)D} \int g^2 \gamma^\delta \delta^{CD} \left[ f^{EAB} f^{ECD} (g_{\mu\gamma} g_{\nu\delta} - g_{\mu\delta} g_{\nu\gamma})
+ f^{EAC} f^{EBD} (g_{\mu\nu} g_{\gamma\delta} - g_{\mu\delta} g_{\gamma\nu}) + f^{EAD} f^{EBC} (g_{\mu\nu} g_{\gamma\delta} - g_{\mu\gamma} g_{\delta\nu}) \right] \\
= & ig^2 \frac{2}{(N_c^2 - 1)D} D N_c [g_{\mu\nu} (D - 1)] \delta^{AB}. \quad (A.2)
\end{align*}$$

For the correction of the ghost propagator, we need the calculation of the following
\[ \frac{\delta^{CC'} D_{\mu}}{D G_{\mu}} \left( p_\mu \right) \frac{-1}{p^2} \delta^{DD'} \left( p_\nu \right) \]
\[ = - \frac{N_c g^2}{D} \delta^{AB}. \]  \hspace{1cm} (A.4)

\[ = - \frac{\delta^{CD} D_{\mu}}{D} \left( -ig^2 \right) \lambda \left( 1 - \xi \right) \left( f^{EAB} f^{EDC} - f^{EAC} f^{EDB} \right) \]
\[ = i \frac{N_c g^2}{D} \lambda \left( 1 - \xi \right) \delta^{AB}. \]  \hspace{1cm} (A.5)

\[ \frac{\delta^{DD'} D_{\mu}}{D} \left( p_\mu \right) \left( 1 - \xi \right) \frac{-1}{p^2} \left( P_T + \lambda P_L \right) \mu \nu \delta^{CC'} \left( p_\nu \right) \]
\[ = - i \frac{N_c g^2}{D} \lambda \left( 1 - \xi \right) \delta^{AB}. \]  \hspace{1cm} (A.6)

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