ANTI-FROBENIUS ALGEBRAS AND ASSOCIATIVE YANG-BAXTER EQUATION

A. I. ZOBNIN

Abstract. Associative Yang-Baxter equation arises in different areas of algebra, e.g., when studying double quadratic Poisson brackets, non-abelian quadratic Poisson brackets, or associative algebras with cyclic 2-cocycle (anti-Frobenius algebras). Precisely, faithful representations of anti-Frobenius algebras (up to isomorphism) are in one-to-one correspondence with skew-symmetric solutions of associative Yang-Baxter equation (up to equivalence). Following the work of Odesskii, Rubtsov and Sokolov and using computer algebra system Sage, we found some constant skew-symmetric solutions of associative Yang-Baxter equation and construct corresponded non-abelian quadratic Poisson brackets.

MSC 16T25, 14H70

1. Introduction

Let $V$ be a finite dimensional vector space and $r$ be a linear operator on $V \otimes V$. We consider skew-symmetric solutions of associative Yang-Baxter equation \[ r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13} = 0, \]

where $r_{ij}$ denotes an operator $r$ acting on $i$th and $j$th component of $V \otimes V \otimes V$. Fix a basis $e_\alpha$ in $V$ and let

$r(e_\alpha \otimes e_\beta) = r_{\alpha\beta}^{\gamma\xi} e_\gamma e_\xi$.

Then (1) can be rewritten:

\[ r_{\alpha\beta}^{\gamma\xi} = -r_{\beta\alpha}^{\xi\gamma}, \]

\[ r_{\alpha\beta}^{\lambda\sigma} r_{\mu\nu}^{\tau\sigma} + r_{\mu\sigma}^{\lambda\tau} r_{\nu\alpha}^{\tau\beta} + r_{\nu\sigma}^{\lambda\mu} r_{\lambda\beta}^{\sigma\alpha} = 0. \]

Such solutions appear in different areas of algebra, e.g., in describing double quadratic Poisson brackets \[2, 6\], non-abelian quadratic Poisson brackets \[4\] and anti-Frobenius algebras \[1, 5\]. Using the latter correspondence we construct some solutions of (1).

In [5] Odesskii, Rubtsov and Sokolov considered a special class of non-abelian linear and quadratic Poisson brackets related to ODE systems of the form

\[ \frac{dx_\alpha}{dt} = F_\alpha(x_1, \ldots, x_N), \]

where $x_i$ are $m \times m$-matrices in independent variables and $F_\alpha$ are non-commutative polynomials. They generalized $m$-dimensional Manakov top (which is itself a generalization of $m$-dimensional Euler top) to the case of arbitrary $N$. They used bi-Hamiltonian approach, i.e., they constructed a pair of compatible (in some sense) Poisson brackets. Poisson brackets in question

- are $GL_m$-adjoint invariant;
- send traces of any two matrix polynomials to the trace of some other matrix polynomial.

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Such brackets form an important class, since the corresponding Hamilton operator can be expressed in terms of left and right multiplication operators given by polynomials in matrices \(x_1, \ldots, x_N\). They were called non-abelian Poisson brackets in [5].

Quadratic non-abelian Poisson brackets are of the form

\[
\{ x^{j_1}_{i_1, \alpha}, x^{j_2}_{i_2, \beta} \} = r^{\gamma \epsilon}_{\alpha \beta} x^{j_2}_{i_1, \gamma} x^{j_1}_{i_2, \epsilon} + a^{\gamma \epsilon}_{\alpha \beta} x^{j_2}_{i_1, \gamma} x^{j_1}_{i_2, \epsilon} - a^{\gamma \epsilon}_{\beta \alpha} x^{j_2}_{i_2, \gamma} x^{j_1}_{i_1, \epsilon}.
\]

We consider the case when \(a^{\gamma \epsilon}_{\alpha \beta} = 0\). Then the constraints on coefficients \(r^{\gamma \epsilon}_{\alpha \beta}\) are precisely (2) and (3), i.e., tensor \(r\) is a constant skew-symmetric solution of the associative Yang-Baxter equation for an \(N\)-dimensional vector space \(V\).

The classification of non-abelian quadratic Poisson brackets (even with \(a^{\gamma \epsilon}_{\alpha \beta} = 0\)) is an open question. The complete solution for zero \(a\) is known in the case \(N = 2\) (Aguiar [1] and Odesskii, Rubtsov and Sokolov [6]) and \(N = 3\) (Sokolov [8]). For example, all solutions (up to equivalence) in the case \(N = 2\) are either of the form

\[
r^{21}_{22} = -r^{12}_{22} = \lambda,
\]
or

\[
r^{22}_{21} = -r^{21}_{12} = \lambda.
\]

(Here we presented only non-zero components of tensor \(r\).)

2. Anti-Frobenius algebras

It would be tempting to find an appropriate algebraic structure for \(r\). Odesskii, Rubtsov and Sokolov proved ([5], see also [1, Proposition 2.7]) that the solutions of these equations (up to equivalence, i.e., change of basis) are in one-to-one correspondence with faithful representations of anti-Frobenius algebras (up to isomorphism). An associative algebra \(A\) is called anti-Frobenius if it is equipped with a non-degenerate anti-symmetric bilinear form \((\cdot, \cdot)\) such that

\[
(x, yz) + (y, zx) + (z, xy) = 0
\]

for all \(x, y, z \in A\). Such form is a cyclic 2-cocycle in the sense of Connes [3]. This correspondence is constructive, i.e., it is possible to obtain explicitly the components of \(r\) from an anti-Frobenius algebra and vice versa. Precisely, let \(\varphi: A \to \text{Mat}_N\) be a faithful representation, \(\{e_a\}\) be a basis of \(\varphi(A)\) and \(G\) be the matrix of the bilinear form \((x, y)\). Then

\[
r^{ab}_{cd} = g^{a, \beta} e^a_{c, \alpha} e^b_{d, \beta},
\]

where \((g^{a, \beta}) = G^{-1}\).

Odesskii, Rubtsov and Sokolov [3] considered the following example of an anti-Frobenius algebra: \(A\) consists of \(N \times N\)-matrices with zero \(N\)th row, and \((x, y) = l([x, y])\) for a generic element \(l \in A^*\). (Similar construction for Lie algebras was considered in [4].) They found the corresponding solution of an associative Yang-Baxter equation, which is equivalent to the following one:

\[
r^{\alpha \beta}_{\alpha \beta} = r^{\beta \alpha}_{\alpha \beta} = r^{\alpha \alpha}_{\alpha \beta} = -r^{\alpha \alpha}_{\alpha \beta} = \frac{1}{\lambda_{\alpha} - \lambda_{\beta}}, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, \ldots, N
\]

(here \(\lambda_1, \ldots, \lambda_N\) are arbitrary pairwise distinct parameters, and the other components of \(r\) are zero). We found that this solution can be directly obtained from the anti-Frobenius algebra

\[
A_{N,1} = \{ A \in \text{Mat}_N \mid \sum_i a_{ij} = 0 \quad \forall j = 1, \ldots, N \}
\]
equipped with bilinear form

\[
(x, y) = \text{tr}([x, y] \cdot \text{diag}(\lambda_1, \ldots, \lambda_N)),
\]

which is isomorphic to \(A\).
3. Main result

Let’s generalize the construction of $\mathcal{A}_{N,1}$. Let $M$ be a proper divisor of $N$. We consider $N(N-M)$-dimensional algebra

$$\mathcal{A}_{N,M} = \{ A \in \text{Mat}_N \mid \sum_{i=r \pmod{M}} a_{ij} = 0 \quad \forall r = 1, \ldots, M, \; \forall j = 1, \ldots, N \}$$

equipped with bilinear form (7). It is easy to check that this skew-symmetric form satisfies (1).

**Theorem.** Let $\lambda_i$ be equal to $\lambda_j$ iff $\left[ \frac{i}{M} \right] = \left[ \frac{j}{M} \right]$. Then the form (4) is non-degenerate, i.e., the algebra $\mathcal{A}_{N,M}$ is anti-Frobenius.

**Proof.** Let’s consider the following basis in $\mathcal{A}_{N,M}$:

$$B = \left\{ e_{i,j} \colon= E_{i,j} - E_{\bar{i},j} \mid \bar{i} \neq i \right\}.$$ 

Here $\bar{i}_j$ denotes the integer that has the same remainder modulo $M$ as $i$ and the same quotient modulo $M$ as $j$, i.e., if $i = q_i M + r_i$ and $j = q_j M + r_j$ then $\bar{i}_j = Mq_j + r_i$. There are $N(N-M)$ elements in $B$. Note that

$$e_{i,j} \in B \iff q_i \neq q_j \iff e_{j,i} \in B$$

and $e_{i,i} \notin B$. By assumption we have $\lambda_j = \lambda_{q_j}$ and hence $\lambda_j = \lambda_{\bar{i}_j}$ for all $j$. Let’s divide all elements in $B$ into pairs $\{e_{i,j}, e_{j,i}\}$. We have

$$(e_{i,j}, e_{j,i}) = \text{tr} \left( [E_{i,j} - E_{\bar{i},j}, E_{j,i} - E_{\bar{i},i}] \cdot \text{diag}(\lambda_1, \ldots, \lambda_N) \right) = \lambda_i - \lambda_j$$

and, if $p \neq i$ and $q \neq j$,

$$(e_{i,j}, e_{q,p}) = \text{tr} \left( [E_{i,j} - E_{\bar{i},j}, E_{q,p} - E_{\bar{q},\bar{p}}] \cdot \text{diag}(\lambda_1, \ldots, \lambda_N) \right) = \delta_{i,p}\delta_{j,q}(\lambda_i - \lambda_j) - \delta_{i,p}\delta_{j,\bar{q}}(\lambda_p - \lambda_{\bar{q}}) - \delta_{i,j}\delta_{\bar{q},\bar{p}}(\lambda_{\bar{i}} - \lambda_j) = 0.$$ 

Thus, $(x, y)$ has a canonical block diagonal form in the basis $B$ with nonzero blocks and is non-degenerate. \hfill $\square$

Using (5), we obtain the following components for the corresponding tensor $r$:

$$r^{ab}_{cd} = \begin{cases}
\sum_{i,j} \frac{1}{\lambda_j - \lambda_i} \left( E_{i,j} - E_{\bar{i},j} \right)^a \left( E_{j,i} - E_{\bar{j},i} \right)^b, & \text{if } c \neq d, \\
0, & \text{otherwise}.
\end{cases}$$

In particular, $r^{ab}_{cd} = 0$ when either $a \neq c \pmod{M}$ or $b \neq d \pmod{M}$.

Note that for $M = 1$ this formula becomes

$$r^{ab}_{cd} = \begin{cases}
\frac{(\delta^a_d - \delta^a_{\bar{c}})(\delta^b_c - \delta^b_{d})}{\lambda_{\bar{c}} - \lambda_d}, & \text{if } c \neq d, \\
0, & \text{otherwise},
\end{cases}$$

which coincides with (6). As we have seen, the special case $M = 1$ is equivalent to the construction of Odesskii, Rubtsov and Sokolov.

Consider now the case of arbitrary pairwise distinct $\lambda_i$. One can check that in this case the form $(x, y)$ is non-degenerate too. Using computer algebra system Sage, we obtained
for small $N$ and generalized for all $N$ the following formula for components of tensor $r$ in this case:

$$r_{cd}^{ab} = 0,$$

$$r_{ca}^{ab} = -r_{ac}^{ba} = \frac{1}{\lambda_a - \lambda_c},$$

$$r_{cd}^{ab} = 0,$$

$$r_{ba}^{ab} = \frac{1}{\lambda_a - \lambda_b} \left( \prod_{b' \equiv b, b' \neq b} (\lambda_a - \lambda_{b'}) \prod_{a' \equiv a, a' \neq a} (\lambda_{a'} - \lambda_{a'}) \prod_{b' \equiv b, b' \neq b} (\lambda_{a'} - \lambda_{a'}) \right),$$

$$r_{cd}^{ab} = \frac{1}{\lambda_a - \lambda_b} \left( \prod_{c' \equiv c, c' \neq c} (\lambda_a - \lambda_{c'}) \prod_{d' \equiv d, d' \neq d} (\lambda_{a'} - \lambda_{d'}) \prod_{b' \equiv b, b' \neq b} (\lambda_{a'} - \lambda_{a'}) \right),$$

if $a \neq b$,

otherwise.

Here $x \equiv y$ means that $x \equiv y \pmod{M}$.

With these formulae one can construct corresponding quadratic Poisson brackets. For example, in the case $N = 2M$ and $n = 1$ the corresponding scalar Poisson bracket has the form

$$\{x_\alpha, x_\beta\} = \frac{(x_\alpha - x_\alpha')(x_\beta - x_\beta')(\lambda_{\alpha'} - \lambda_{\beta'})}{(\lambda_a - \lambda_{\alpha'})(\lambda_{\beta'} - \lambda_{\beta'})},$$

where $\gamma'$ relates to $\gamma$ as $|\gamma' - \gamma| = M$.

Finally, we note that for the similar algebra

$$A_{N,M}^{T} = \{A^T \mid A \in A_{N,M}\}$$

the corresponding solution $\hat{r}$ can be computed as follows:

$$r_{cd}^{ab} = r_{cd}^{ab}.$$

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