"Distance distribution in random graphs
and application to network exploration."

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ABSTRACT

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Distance distribution in random graphs and application to networks exploration

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We consider the problem of determining the proportion of edges that are discovered in an Erdős-Rényi graph when one constructs all shortest paths from a given source node to all other nodes. This problem is equivalent to the one of determining the proportion of edges connecting nodes that are at identical distance from the source node. The evolution of this quantity with the probability of existence of the edges exhibits intriguing oscillatory behavior. In order to perform our analysis, we introduce a new way of computing the distribution of distances between nodes. Our method outperforms previous similar analyses and leads to estimates that coincide remarkably well with numerical simulations. It allows us to characterize the phase transitions appearing when the connectivity probability varies.

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I. INTRODUCTION

The small-world phenomenon has attracted increasing attention over the last few years \cite{2,9}. In a small-world network, the average distance between two nodes is small as compared to the total number of nodes. In many natural networks, it is typically of the order of $\log(n)$ ($n$ is the total number of nodes) and several models have been proposed to explain this phenomenon (see, e.g. \cite{2,3,10}). In some applications though, one is interested not only in this so-called “average inter-vertex distance”, but in the whole inter-vertex distance distribution.

Even though this distribution is of much interest, it has not been studied very much in the literature. A theoretical method for the computation of the distances in uncorrelated random networks of infinite size has been proposed by Dorogovtsev et al. in 2003 \cite{4}. In 2004, Fronczak et al. have analyzed the distance between nodes for a wide class of random networks of finite size that generalizes the Erdős-Rényi graphs, the so-called uncorrelated random networks with hidden variables \cite{5}. They propose an approximation of the distribution of the distance between nodes that performs well for a certain range of the parameter values. Their formula has the advantage of being simple and analytical, but the approximations done in the calculations lead to significative differences with the numerical evidence for some ranges of the parameters.

Our work is motivated by the analysis of algorithms that have been recently developed for analysing networks, such as the internet. A typical way of doing that is to use the freeware traceroute, that provides the user a short path from his computer to any other one in the internet. In the ASP model (All Shortest Paths), introduced to model this strategy, one chooses a particular node $s$ of the network, and then constructs all shortest paths from $s$ to all other nodes of the network \cite{6}. Some edges of the network may not belong to any of these shortest paths and so they are left undiscovered. The problem considered in \cite{6} is that of determining the proportion of edges of the network that are discovered. Thus the question is: “what is the proportion of edges that are on at least one shortest path starting from the source?”. As pointed out in \cite{6}, the edges that are not discovered are exactly those connecting nodes that are at identical distances from the source. Indeed, if an edge connects two equidistant nodes, it cannot be on a shortest path from the source, since any path using this edge (say going from $v_1$ to $v_2$) can be shortened by going directly to $v_2$ via the shortest path to $v_2$. Conversely, if an edge links two nodes that are not at the same distance, then it links a node $v_1$ at a certain distance $d$ to a node $v_2$ at a distance $d+1$, and at least one shortest path to $v_2$ passes through this edge. We are therefore interested in computing the number of edges connecting nodes that are at the same distance from the source. Other models exist for representing network analysis strategies. For instance, \cite{6} introduces the USP model (for Unique Shortest Path). In the USP model one chooses only one shortest path from
the source to each node in the graph and so there are possibly more edges that are left undiscovered. Our work is also relevant to the analysis of this model, as it counts the proportion of edges that are never found by any single or multiple USP searches.

In [6], massive numerical simulations have been performed to analyze the proportion of edges that are on shortest paths in Erdős-Rényi graphs. In such random graphs, edges are all equally likely to be present and the probability of presence is given by some fixed probability $p$. We do not consider self loops nor multiple edges. So, for constructing an Erdős-Rényi graph, one needs to fix two parameters: the number of nodes $n$ and the probability of existence for every edge $p$. As shown in FIG. 1, the proportion of edges that are discovered in the ASP model presents an interesting dependence in the parameter $p$. One can directly explain some characteristics of this curve. When $p$ is very small the graph is highly disconnected and consists in small connected components. Most edges do therefore not belong to any path starting from the source, and the proportion of observed edges is close to zero. Conversely, if $p$ is very high, the graph is almost complete, and every shortest path has length one. So $n-1$ edges are found, while there are almost $\frac{1}{2}n(n-1)$ edges in the graph, and thus the proportion also vanishes.

The aims of this paper are first to introduce a new simple model of inter-vertex distances in Erdős-Rényi graphs that can be used to compute the curve of FIG. 1 without any numerical experiment, and second to analyze the oscillating behavior of this curve and explain the phase transitions appearing with variations of the graph connectivity. Note that similar oscillating behaviors in random graphs have recently been observed [8], and that these phenomena seem to open challenging questions in random graphs theory. This paper proposes a precise analysis of such an oscillating behavior in the simple theoretical framework of Erdős-Rényi graphs. One could imagine exploiting these oscillations to optimize the design of a network or to develop method for its analysis, although this is beyond the scope of this paper. Besides, such applications of the concepts developed here would probably require some further analysis and extension of our results, because real networks often exhibit non-trivial correlations between nodes that do not occur in Erdős-Rényi graphs. These extensions would however most likely not lead to the derivation of simple analytical solutions providing an intuitive understanding of the phenomena as it is done here.

The remainder of the paper is organized as follows. In Section II we introduce a recurrence equation allowing to evaluate the inter-vertex distance distribution for Erdős-Rényi graphs, and compare to previously published results [5]. From this function we derive a theoretical expression for values shown on FIG. 1. In Section III we analyze this curve, we characterize the phase transitions, and give analytical expressions in different phases (proved in Appendix A). In Section IV we conclude and make some remarks on practical applications of the phenomena studied in the paper.

II. APPROXIMATION OF INTERVERTEX DISTANCE DISTRIBUTION

In this section we propose an approximation for inter-vertex distance distribution in Erdős-Rényi graphs. We compare our results to those obtained by Frączak et al. [5] in a more generic situation, and show how our results outperform theirs in the particular case of Erdős-Rényi graphs. We also analyze the accuracy of our model and
its dependence on the graph connectedness. We then use our inter-vertex distance distribution to estimate the proportion of equidistant pairs of nodes.

In the sequel, we consider the distance between a randomly selected node and a fixed but initially randomly selected “source node”. Since this source is randomly selected, all results obtained for the distance probability can also be applied to the distance between two randomly selected vertices. Let $F_d$ be the probability for a randomly selected node to be at a distance larger than $d$ from the source, that is, the probability that there is no path of length smaller than or equal to $d$ from the source to this node. The probability $f_d$ for the node to be at a distance exactly $d$ of the source is then given by $f_d = F_{d-1} - F_d$. Obviously, $F_0 = 1 - 1/n$. We now derive a recurrence relation allowing the computation of $F_d$ for higher values of $d$. A node is at a distance larger than $d$ from the source if it is not the source itself, which happens with probability $1 - 1/n$, and if it is connected to no node at distance less than $d$ from the source, which happens with probability $(1 - p)^{n_d}$, $n_d$ being the number of nodes at distance less than $d$ from the source. We have therefore the following simple relation:

$$F_d = \left(1 - \frac{1}{n}\right) \sum_{k=1}^{n_d-1} P[n_d = k](1 - p)^k,$$

(1)

where $P[n_d = k]$ denotes the probability that $n_d = k$. In order to express the probability $F_d$, we should thus know the distribution of $n_d$. We approximate this quantity to be always exactly equal to its expectation $\langle n_d \rangle = (1 - F_{d-1})n$. Introducing this approximation in (1) we obtain a recurrence relation for $F$.

$$F_d = \left(1 - \frac{1}{n}\right)(1 - p)^{(1-F_{d-1})n},$$

(2)

which allows us to compute $f_d$ for any $d$. This formula is different, but provably equivalent to Equation (6) in [1] that has been derived independently for other purposes.

In [5], Fronczak et al. propose an expression for the intervertex distance distribution of any “random graph with hidden variables”, that are generalizations of Erdős-Rényi graphs. In these graphs, two nodes $i$ and $j$ are connected with a probability $p_{i,j} = h_i h_j / \beta$, where each node $v$ has its own “hidden variable” $h_v$, and $\beta = \langle h \rangle n$. So, in a large graph, the hidden variables represent the expected degree of the vertices. In the particular case of Erdős-Rényi graphs, that is when $h_v = np$ for all $v \in V$, their expression for the function $F$ of inter-vertex distance distribution reduces to:

$$F_d = e^{-\frac{1}{\beta}(np)^d}.$$  \[3\]

This result has a straightforward interpretation as the solution of an other recurrence equation on $d$, although it is not obtained in that way in [5]. A vertex $i$ is at a distance larger than $d$ from the source node if all its neighbors are at distance larger than $d - 1$ from the source. Approximating the number of neighbors by its expectation $np$ and neglecting the dependence effects, one obtains the recurrence $F_d = F_{d-1}^{np}$. The relation (3) is then re-obtained by taking $F_0 = e^{-\frac{1}{n}} \approx 1 - \frac{1}{n}$ as initial condition. Numerical experiments confirm indeed that taking $e^{-\frac{1}{n}}$ or $1 - \frac{1}{n}$ as initial condition has no influence on the results if $n$ is sufficiently large.

In FIG. 2 we compare the predictions from the two models, with numerical results. One can see that both models perform very well when the average degree $np$ is significantly larger than 1 and if $p$ is not too big, as in FIG. 2(c). For an average degree $np < 1$, that is below the emergence of the giant connected component (see [2]), our results match approximately the experimental observations while Fronczak et al.’s model is not valid as it gives an increasing curve (see FIG. 2(a)). For values of $np$ larger than but close to 1, both models present significant errors but ours is closer to the experimental observations (see FIG. 2(b)). Finally, for a large $p$, one can see in FIG. 2(d) that our results match the experimental data very well while those obtained with the model of [5] are significantly different.

The fact that the model derived in [5] behaves very differently from our model for a certain range of values of $p$ may seem surprising. Our derivation presents indeed various similarities with the interpretation of Fronczak et al.’s model as a solution of a recursive equation. Three reasons can however explain why a model based on this interpretation gives less accurate results than ours. First, for $np << 1$, the possibility for the randomly selected node to be the source could not be neglected, as very few nodes are in the connected component of the source. When $np$ is larger than but close to 1, the approximation that a node has exactly $np$ neighbors leads to proportionally more important errors. This problem could be solved by considering a binomial distribution for the number of neighbors in
FIG. 2: Evolution of $F_d$, the probability for a random node to be at a distance larger than $d$ from the source node, for $n = 1000$ nodes and for (a) $np = 0.5$, (b) $np = 2$, (c) $np = 10$, (d) $np = 900$. The three curves represent the experimental observations (averaged on 500 graphs), our model, and the model of Fronczak et al given in [5].

our interpretation of Fronczak et al.’s model. Finally, for large values of $p$, the number of neighbors of the randomly selected node is large, so that some independence problems are not negligible. Indeed, the probabilities for two neighbors of the randomly selected node to be the source are not independent, as there is exactly one source in the graph.

The errors of our model, observed for values of $np$ larger than but close to 1 are due to the approximation mentioned above: To obtain the recurrence equation (2) from (1), we suppose that the number $n_d$ of vertices at a distance smaller than $d$ from the source is exactly equal to its expectancy $n(1 - F_{d-1})$ instead of considering its probability distribution. In this range of parameters, the distribution is far from being centered because of the existence of a peak around 0 (see FIG. 3(a)). For these values indeed, the graph is not totally connected. If the source happens not to be in the giant connected component, almost all nodes are at an infinite distance of it, so that $n_d$ is close to 0 for any $d$. The weight of the peak represents thus the probability for a randomly selected source not to be in the giant connected component. It is known that when $np$ grows, this probability tends exponentially to 0 independently of $n$ [7, Theorem 5.4]. This problem does therefore only appear when the average degree $np$ is very small (but larger than 1), independently of the size $n$ of the graph. FIG. 3(b) shows that the problem is already almost negligible when $np = 4$ (for these values, the giant connected component already contains more than 98% of the vertices). Note that for $np < 1$ the graph is highly disconnected so that almost no nodes are at a finite distance from the source. The distribution $P[n_d = k]$ consists thus only in one peak around 0 and is therefore centered.

We close this section by explaining how the distance distribution can be used to compute the proportion $P_s$ of edges belonging to shortest paths starting at the source node. As explained in the introduction, the edges that do not belong to any shortest path are those connecting nodes that are at the same distance from the source, in addition to all edges that are not in the same connected component as the source. Since the expectation of the number of nodes at distance $d$ from the source is equal to $nf_d$, the expected number of edges connecting these nodes is roughly equal to $\frac{1}{2}p(nf_d)^2$. Taking $\frac{1}{2}nm^2$ as the total number of edges, we obtain the following expression for the proportion of edges that lie on a shortest path in an Erdős-Rényi graph, which we denote by $P_s(n, p)$ in the sequel:

$$P_s(n, p) = 1 - \frac{\sum_{d=1}^{n} p(nf_d)^2}{n^2p} = 1 - \sum_{d=1}^{n} f_d^2.$$  (4)
reached for np not all edges lie on shortest paths anymore. As a result of the two conflicting phenomena a (global) optimum is with the apparition of a giant component, there also appears a non negligible number of cycles in the graphs, so that the edges are also in this component, the proportion of discovered edges increases rapidly with a giant component emerges very quickly, and the source is in this component with a large probability. Since most of behavior. In particular, the successive values of the maxim a seem to tend to indicate that in this range of parameters been able to prove this, nor to express analytically the values of np less accurate. Moreover, the range of values of is larger than but close to one, which is the range of parameters for which our model has already been shown to be a large peak around 0, while the peak in (b) is much smaller. Our approximation of nd by its average value n(1 – Fd–1) leads thus to larger errors for np = 2 (a) than for np = 4 (b).

Note that this expression implicitly handles the edges that are not in the same connected component as the source if we take fn = Fn−1 ≈ Fn. Indeed, this quantity represents those nodes that are not connected to the source, as they are at a distance larger than n − 1. The evolutions with p of Ps using the two models presented above are represented in FIG.4 for n = 1000 and n = 10000. One can see that our results match the experiments very well except when np is larger than but close to one, which is the range of parameters for which our model has already been shown to be less accurate. Moreover, the range of values of np for which our model is less accurate appears not to grow with n.

III. ANALYSIS OF THE CURVE

In this section we analyze the function Ps(n, p) generated with our model and (4). We show the appearance of a sort of phase transition: for some particular values, a weak variation of the probability p may cause abrupt changes in the proportion of discovered edges with the ASP model, and affect dramatically the properties of the graph. We give analytical formulas for the asymptotic behavior in several phases.

We begin by analyzing the first transition, starting from small values of p. It is well known [2] that in an Erdős-Rényi graph, a giant component emerges when p becomes larger than 1/n. If the average degree np is sufficiently small, the graph is not connected and the only edges that the observer can see are in the (small) connected component of the source. This quantity is negligible in view of the total number of edges, and so the function is approximately zero. Note however that such graphs do not contain many cycles, so that most paths starting from the source are shortest paths. Therefore, the observer discovers approximately all edges in its connected component. When p grows the size of the connected components increases, so that more and more edges are discovered. Now when np ≈ 1, the giant component emerges very quickly, and the source is in this component with a large probability. Since most of the edges are also in this component, the proportion of discovered edges increases rapidly with np. Simultaneously with the apparition of a giant component, there also appears a non negligible number of cycles in the graphs, so that not all edges lie on shortest paths anymore. As a result of these two conflicting phenomena a (global) optimum is reached for np ≈ 2. Experimentally our model gives an optimum that seems to lie exactly at np = 2, but we have not been able to prove this, nor to express analytically the values of Ps(n, p) around np ≈ 2. However, experiments seem to indicate that in this range of parameters Ps(n, p) only depends on np. All this can be seen in FIG.5 for different values of n. When np becomes larger, one can see that Ps does not only depend on np, and presents an oscillatory behavior. In particular, the successive values of the maxima seem to tend to $\frac{1}{2}$. We explain this phenomenon in the sequel.
As can be observed in FIG. 4 the shape of $P_s(p)$ tends to the parabola $2p(1-p)$ on any interval $[\epsilon, 1]$ when $n$ increases (note that the x-axis is in linear scale). This fact can be proved theoretically, based on our model of evolution of $F_d$. In the sequel, for the sake of clarity in our analysis, we modify (2), and study the slightly different one:

$$F_d(n, p) = (1 - p)\frac{n}{1 - F_d(n, p)}.$$  

This new approximation is justified by the fact that we will consider asymptotic behaviors for $n \to \infty$. Moreover, the results that we derive can be obtained without making this approximation. Observe that $F_1(n, p) = 1 - p$, so that $F_2(n, p) = (1 - p)^{np}$. When $n$ grows $F_0 = 1 - \frac{1}{n} \to 1$, and if $p$ is bounded from below by an arbitrary positive constant $\epsilon$, $F_2(n, p) = (1 - p)^{np}$ tends uniformly to 0. As a consequence, the probability $f(d)$ for a node to be at a distance $d$ from the source tends uniformly to 0 for all $d$ except for $d = 1, 2$, for which $f_1 = F_0(n, p) - F_1(n, p) \to 0$ and $f_2 = F_1(n, p) - F_2(n, p) \to 1 - p$. It follows then from (3) that

$$P_s(n, p) \to 1 - p^2 - (1 - p)^2 = 2p(1 - p),$$

so that asymptotically, the last maximum of $P_s$ is $\frac{1}{2}$ and is reached at $p = \frac{1}{2}$. The asymptotic parabolic character of $P_s$ is thus here due to the fact that almost all nodes tend to be at a distance either 1 or 2 from the source when $n$ grows and $p$ is sufficiently large, as can for example be observed in FIG. 2(d).

We now analyze the oscillating behavior between the first and last maximum. One can see in FIG. 4 that around the

FIG. 4: Comparison of the evolution of $P_s(n, p)$ with $n = 1000$ (a) and $n = 10000$ (b) according to numerical experiments, to our model, and to Fronczak et al.’s model.
FIG. 5: Evolution of $P_s(n, p)$ with $np$ for different values of $n$. All curves present a sharp increase between $np = 1$ and $np = 2$, and a global maximum in $np \approx 2$. For larger values, the curves present several oscillations, with local maxima tending to 0.5. (b) is a zoomed-in linear-scale version of (a).

FIG. 6: Evolution of $P_s(n, p)$ with $p$ for different values of $n$. On any interval $[\varepsilon, 1]$, $P_s$ tends to the parabola $2p(1 - p)$ when $n$ increases.
second rightmost maximum, $P_s$ only depends on $n^{\frac{1}{2}}p$, and that $P_s$ asymptotically behaves as

$$P_s \simeq 2e^{-(n^{\frac{1}{2}}p)^2} \left(1 - e^{-(n^{\frac{1}{2}}p)^2}\right)$$

(6)

around this maximum. The maximum therefore tends to $\frac{1}{2}$ when $n \to \infty$ and is attained for $(n^{\frac{1}{2}}p)^2 = \log 2$. To explain (6), we show in the appendix that similarly as above, all nodes are asymptotically at distance either 2 or 3 when $n \to \infty$ with $\epsilon < np^2 < R$, where $\epsilon, R$ are arbitrarily positive constants. As in the case of the parabola, this together with (4) implies that $P_s$ then asymptotically behaves as $2(1 - F_2)F_2$. We also show that $F_2(n, p)$ tends to $e^{-np^2}$, which implies (6).

Actually the previous relations can be generalized inductively: we prove in the appendix that when $n \to \infty$ with $\epsilon < n^{d-1}p^d < R$, $F_d$ converges uniformly to $e^{-n^{d-1}p^d}$, while all $F_{d'}$ with $d' < d$ converge uniformly to 1 and all others to 0. This means that in this range of parameters, and when $n$ tends to infinity, almost all nodes are at distance $d$ or
Asymptotically, local maxima $\frac{1}{2}$ appear for $n^{\frac{2}{3}}p = \sqrt{\log 2}$ and $n^{\frac{3}{4}}p = \sqrt{\log 2}$. The "*" represent the theoretical asymptotic behavior.

$d + 1$ from the source. It follows then from (4) that

$$\lim_{\epsilon < n^{d-1}p^d < R} P_s(n, p) = 2 \left(1 - e^{-n^{d-1}p^d}\right) e^{-n^{d-1}p^d},$$

which, as for $d = 1, 2$, is a parabolic curve with respect to $F_d$. This parabolic curve attains its maximum $\frac{1}{2}$ when $e^{-n^{d-1}p^d} = \frac{1}{2}$. So, when $n \to \infty$, $P_s$ contains an unbounded number of oscillations and local maxima with asymptotic values $\frac{1}{2}$, and these maxima are attained when $n^{d-1}p^d = \log 2$ for each $d > 1$ as can be seen on some additional examples in FIG. 8. Experimentally, all local maxima but the first global one can be explained in that way. Between two maxima, there is a zone where asymptotically $F_d \simeq 1$ and $F_{d+1} \simeq 0$, so that almost all nodes are at distance $d + 1$ from the source, and $P_s \simeq 1 - \frac{1}{2} = 0$. Such behavior is obtained when $n \to \infty$ with either large values of $n^{d-1}p^d$ but still $\epsilon < n^{d-1}p^d < R$, or small values of $n^d p^{d+1}$ but still $\epsilon < n^d p^{d+1} < R$. One can indeed see in FIG. 7 and 8 for example that the values of the local minima decrease significantly when $n$ increases. Let us mention that an explanation of the oscillatory behavior based on the fact that almost all nodes are at distance $d$ or $d + 1$ from the source had been suggested without proof in [6].
IV. CONCLUSIONS AND FUTURE WORK

The goal of this paper was twofold: First, we have proposed a simple model for the computation of the inter-vertex distance distribution in a random graph, via a recurrence equation for the probability for two randomly chosen nodes to be at distance more than \( d \). Contrary to the model of Fronczak et al., our recurrence equation is not explicitly solvable, but it is more accurate. It has to be noted that for the range of parameters corresponding to the oscillating behavior analyzed at the end of Section III, the two models are equally valid, and that the analysis that we have made for such values could also be made using Fronczak et al.’s model. Let us add that the ideas behind the derivation of the formula remain valid for more general graphs such as random graphs with hidden variables. In the particular case of Erdős-Rényi graphs, these ideas lead to a simple recurrence equation, allowing to compute explicit values numerically, and to prove the asymptotic behavior of the curve experimentally obtained in [6]. Nevertheless, a further analysis for more general graphs could be interesting.

Second, following numerical simulations in previous works motivated by practical graph exploration questions [6], we have analyzed the proportion of edges connecting nodes that are equidistant from a certain source node in random graphs. The evolution of this quantity with the parameter \( p \) exhibits an intriguing oscillating behavior, which we have been able to explain and reproduce with a great accuracy using our model. We have also characterized precisely the (infinite number of) transitions for this quantity, and the analytical evolution with \( p \) in the different phases.

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APPENDIX A: EXPRESSION OF THE ASYMPTOTIC BEHAVIOR

In this appendix, we provide an analytical expression for \( F_d \) when \( n \) tends to infinity with \( \epsilon < n^{d-1}p^d < R \), and we show that in this range of parameters, almost all nodes are at distance \( d \) or \( d + 1 \). Suppose first that \( n \to \infty \) with \( 0 < np^2 < R \) for an arbitrary \( R \). Then \( p \to 0 \) so that \( F_1 = (1-p) \to 1 \) uniformly with \( np^2 \). From our recurrence formula [5], we have

\[
F_2(n, p) = (1-p)^{np} = \left( (1-p)^{\frac{1}{p}} \right)^{np^2},
\]

which, together with the classical relation \( \lim_{p \to 0} (1-p)^{\frac{1}{p}} = e^{-1} \), implies that

\[
\lim_{np^2 < R} F_2(n, p) = e^{-np^2}
\]

holds uniformly for \( 0 < np^2 < R \). We now show that

\[
\lim_{\epsilon < np^2 < R} F_3(n, p) = \lim_{\epsilon < np^2 < R} (1-p)^{n(1-F_2)} = 0
\]
for any two arbitrary constants \( \epsilon \) and \( R \). This implies that almost all nodes are at distance 2 or 3 from the source. It follows from (A1) that \( 1 - F_d \) is uniformly bounded from below by a positive constant when \( n \to \infty \) with \( \epsilon < np^2 < R \), so that we just need to prove the uniform decay of \((1 - p)^n\). The latter expression can be rewritten as

\[
(1 - p)^n = \left( 1 - \frac{np^2}{(np)} \right)^{n(p)} \quad (A2)
\]

Since \( np \to \infty \) when \( n \to \infty \) with \( \epsilon < np^2 < R \), there uniformly holds

\[
e^{-R} \leq \lim_{\epsilon < np^2 < R} \left( 1 - \frac{np^2}{(np)} \right)^{np} \leq e^{-\epsilon}.
\]

And since \( \frac{1}{p} \to \infty \), it follows from (A2) that

\[
\lim_{\epsilon < np^2 < R} (1 - p)^n = 0,
\]

which implies the desired result.

There remains to prove our assertions about the asymptotic behavior of \( F_d \) for any \( d > 2 \). We first prove by induction that the two following relations hold uniformly for \( n^{d-1}p^d < R \) where \( R \) is any arbitrary positive constant.

\[
\lim_{n^{d-1}p^d < R} F_{d-1}(n, p) = 1, \quad (A3)
\]

\[
\lim_{n^{d-1}p^d < R} F_d(n, p) = e^{-n^{d-1}p^d}. \quad (A4)
\]

These relations hold for \( d = 2 \) as shown above. Let us now assume that they hold for a certain \( d - 1 \) and prove that they then hold for \( d \). Observe first that when \( n \to \infty \) with \( n^{d-1}p^d < R \), \( n^{d-2}p^{d-1} \) tends uniformly to 0 and is bounded. It follows then from the induction hypothesis that

\[
F_{d-1}(n, p) \to e^{-n^{d-2}p^{d-1}} \to 1 - n^{d-2}p^{d-1}
\]

uniformly when \( n \to \infty \), \( n^{d-1}p^d < R \). So Equation (A3) is proved. By our recurrence relation (5), \( F_d(n, p) = (1 - p)^{n(1-F_{d-1})} \). Therefore, there holds

\[
\lim_{n^{d-1}p^d < R} F_d(n, p) = (1 - p)^{n(p)^{d-1}} = \left( 1 - \frac{1}{p} \right)^{n^{d-1}p^d}.
\]

Since \( n^{d-1}p^d \) is bounded, and since \( p \) tends thus uniformly to 0 when \( n \to \infty \), the last equation becomes

\[
\lim_{n^{d-1}p^d < R} F_d(n, p) = e^{-n^{d-1}p^d}
\]

uniformly for \( n^{d-1}p^d \in (0, R) \), which proves (A4).

Using the results above we now prove that for any \( d > 2 \), the following holds uniformly

\[
\lim_{\epsilon < n^{d-1}p^d < R} F_{d+1}(n, p) = 0, \quad (A5)
\]

where \( \epsilon \) and \( R \) are two arbitrary positive constants. By (5), we have

\[
F_{d+1}(n, p) = (1 - p)^{n(1-F_d)}.
\]

It follows from the results above that \( 1 - F_d \) is uniformly bounded from below by a positive constant when \( n \to \infty \) with \( \epsilon < n^{d-1}p^d < R \), so that we just need to prove the uniform decay of \((1 - p)^n\). The latter expression can be rewritten as

\[
(1 - p)^n = \left( 1 - \frac{n^{d-1}p^d}{(np)^{d-1}} \right)^{n^{d-1}p^d}. \quad (A6)
\]
Since \((np)^{d-1} \to \infty\) when \(n \to \infty\) with \(\epsilon < n^{d-1}p^d < R\), there uniformly holds
\[
e^{-R} \leq \lim_{\epsilon < n^{d-1}p^d < R} \left(1 - \frac{n^{d-1}p^d}{(np)^{d-1}}\right)^{(np)^{d-1}} \leq e^{-\epsilon}.
\]
And since \(n^{d-2}p^{d-1} \to 0\) when \(n \to \infty\) with \(\epsilon < n^{d-1}p^d < R\), it follows from (A6) that
\[
\lim_{\epsilon < n^{d-1}p^d < R} (1 - p)^n = 0,
\]
which implies the desired result (A5).