Boundary conditions for conformally coupled scalar in AdS$_4$

Jae-Hyuk Oh

Department of Physics, Hanyang University, Seoul 133-791, Korea

Abstract

We consider conformally coupled scalar with $\phi^4$ coupling in AdS$_4$ and study its various boundary conditions on AdS boundary. We have obtained perturbative solutions of equation of motion of the conformally coupled scalar with power expansion order by order in $\phi^4$ coupling $\lambda$ up to $\lambda^2$ order. In its dual CFT, we get 2, 4 and 6 point functions by using this solution with Dirichlet and Neumann boundary conditions via AdS/CFT dictionary. We also consider marginal deformation on AdS boundary and get its on-shell and boundary effective actions.

1e-mail:jack.jaehyuk.oh@gmail.com
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1 Introduction

Alternative quantization is studied by Breitenlohner and Freedman[1] in gauged supergravity context and Klevanov and Witten et. al. [2, 3, 4] discussed this in AdS/CFT which gives interesting conformal field theories as duals of the gravity theories defined in AdS space by imposing various possible boundary conditions on AdS boundary.

For scalar field theories in AdS space[5, 6], the dual conformal field theory is known to be unitary when its mass is in the region of $-\frac{d^2}{4} \leq m^2 \leq -\frac{d^2}{4} + 1$, where $d$ is dimensionality of AdS boundary and $m$ is mass of the scalar field. In this mass window, there are two possible quantization schemes in the dual CFT, which are $\Delta^+$ and $\Delta^-$ theories respectively, where $\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$, the conformal dimensions of the boundary operator in each dual CFT.

Since the unitarity bound of the scalar operators in CFT is $\Delta \geq \frac{d}{2} - 1$, two point correlation functions in both CFTs are positive definite[2].

The bulk scalar field shows near AdS boundary expansion as $\phi(r, x_i) = \phi^{(0)}(x_i) r^{d-\Delta^+} + \phi^{(1)}(x_i) r^{\Delta^+}$ as the AdS radial coordinate $r \to 0$, where $\phi^{(0)}(x_i)$ becomes a source term in the dual CFT whereas $\phi^{(1)}(x_i)$ is certain boundary composite operator according to standard AdS/CFT dictionary. This is called $\Delta^+$ theory and achieved by imposing Dirichlet boundary condition $\delta \phi^{(0)} = 0$. This gives standard quantization scheme to the dual CFT. Even in the case that the scalar field mass in not in the mass range, $\Delta^+$ satisfies unitarity bound and so Dirichlet boundary condition is always possible boundary condition.

Alternative quantization can be achieved by imposing Neumann boundary condition as $\delta \phi^{(1)} = 0$. In this quantization scheme, the role of the source and composite operator in the dual CFT is switched, so $\phi^{(0)}(x_i)$ becomes the boundary composite operator and $\phi^{(1)}(x_i)$ is the source term. The corresponding CFT is called $\Delta^-$ theory.

A way of imposing boundary conditions is to add boundary term on AdS boundary and variational principle provides the boundary condition upto bulk equation of motion. Therefore, once we define on-shell action as $I_{os} = S_{bulk} + S_{bdy}$ and then the boundary condition is $\delta I_{os} = 0$, where $S_{bulk}$ is boundary contribution of the bulk scalar field action upto the equation of motion and $S_{bdy}$ is boundary action. Dirichlet boundary condition is achieved by $S_{bdy} = 0$. Without adding any boundary term, $\delta I_{os} = \int \phi^{(1)} \delta \phi^{(0)} = 0$, so we can request $\delta \phi^{(0)} = 0$. Neumann boundary condition is obtained by adding $S_{bdy} = -\int \phi^{(0)} \phi^{(1)}$ to the bulk action. Then $\delta I_{os} = -\int \phi^{(0)} \delta \phi^{(1)}$, so the boundary condition becomes $\delta \phi^{(1)} = 0$.

There are many discussions about alternative quantization schemes in Dirac field[7], Rarita-Schwinger field[8], $U(1)$ gauge field[9] and $SU(2)$ Yang-Mills field[10].
In this paper, we study conformally coupled scalar in AdS$_4$. Mass of the conformally coupled scalar is $m^2 = -\frac{d^2-1}{4}$. Therefore in 4-dimension, $m^2 = -2$ so $\Delta_+ = 2$ and $\Delta_- = 1$ which satisfy unitarity bound. Definitely two possible quantization schemes can be applied to the conformally coupled scalar and there are various boundary CFTs obtained by imposing many kinds of boundary conditions. One interesting property of this conformally coupled scalar is that a suitable field redefinition transform this theory into a massless scalar field theory in half of flat space, $\mathbb{R}_+$. In our study, we mostly work in this frame.

Conformally coupled scalar in AdS and their boundary CFTs are studied in several literatures. In [11], the author have considered conformally coupled scalar in AdS space and their boundary effective action with two derivative kinetic term by using derivative expansion(by ignoring higher derivative terms). In [5, 6, 11], the authors have considered conformally coupled scalar and its instanton solution. By looking at their solutions, they suggested that the dual CFT theory shows $\phi^6$ interaction with standard kinetic term.

However, we study this theory in different regime and obtain its boundary CFTs. In section 2, we solve conformally coupled scalar field equation of motion with power expansion order by order in $\lambda$, which is $\phi^4$ coupling in the theory. We obtain the solutions upto $\lambda^2$ order. Therefore, we study small coupling regime of this theory. In section 3, by utilizing the bulk solutions, we compute on-shell and boundary effective actions by imposing several boundary conditions. For Dirichlet boundary condition, we obtained 2, 4 and 6-point functions in dual CFTs. For example, 4-point function is given by
\[
\langle O_p O_q O_s O_u \rangle = -\frac{3!\lambda}{(2\pi)^3} \frac{\delta^3(u + q + s + p)}{|u| + |q| + |s| + |p|},
\] (1)
where $u$, $q$, $s$ and $p$ are 3-momenta along boundary direction and $O$ indicate the dual CFT operator. This is exotic boundary momentum dependent correlation function. For Neumann boundary condition, the 4-point function of boundary composite operators becomes
\[
\frac{\langle O_p O_q O_s O_u \rangle}{|u||q||s||p|}.
\] (2)

We also consider another type of boundary condition, called marginal deformation[5] and obtained its boundary effective action too.

## 2 Conformally coupled scalar in AdS$_4$ and its perturbative solutions

In this section, we solve the equation of motion of the conformally coupled scalar with $\phi^4$ interaction term with power expansion order by order in its coupling $\lambda$. We start with the action
\[
S = \int dr d^3 x \sqrt{g} L(\phi) + \int d^3 x L_{c.t.}(\phi) + S_{bdy},
\] (3)
where $L$ is an action of conformally coupled scalar in AdS$_4$, which is given by
\[
L = \frac{1}{2} g^\mu^\nu \partial_\mu \phi \partial_\nu \phi - \phi^2 + \frac{\lambda}{4} \phi^4,
\] (4)
\( \mathcal{L}_{c.t.} \) is counter term Lagrangian and \( S_{bdy} = \int d^3x \mathcal{L}_b \) is boundary action. (Euclidean) AdS\(_4\) metric is given by

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{dr^2 + \sum_{i=1}^3 dx^i dx^i}{r^2},
\]

where \( r = x^0 \) is the radial coordinate of AdS space (\( r=0 \) is AdS boundary and \( r = \infty \) is Poincar horizon), \( x^i \) is boundary directional coordinate and the spacetime indices \( i, j \)... run from 1 to 3. The action of conformally coupled scalar enjoys an interesting property as follows. Once we define a new field \( f(r, x) = \frac{\phi(r, x)}{r} \), then by using the explicit form of the background metric (5), the action (3) is transformed into

\[
S = \int dr d^3x \left( \frac{1}{2} \partial_r f \partial_r f + \frac{1}{2} \delta^{ij} \partial_i f \partial_j f + \frac{\lambda}{4} f^4 \right) + \int d^3x \left( \mathcal{L}_{c.t.}(\phi) + \frac{f^2}{2r} \right) + \int d^3x \mathcal{L}_b(\phi),
\]

where the \( \frac{f^2}{2r} \) term may divergent as it approach AdS boundary. Once we take \( \mathcal{L}_{c.t.} = -\sqrt{\gamma} \frac{\phi^2}{2} \), then this term is canceled with \( \mathcal{L}_{c.t.} \), where \( \gamma \) is determinant of an induced metric \( \gamma_{ij} = \frac{\partial x^\mu}{\partial x^i} \frac{\partial x^\nu}{\partial x^j} g_{\mu\nu} \).

Equation of motion is obtained by variation of the action (6), which is given by

\[
0 = \partial_r^2 f + \delta^{ij} \partial_i \partial_j f - \lambda f^3.
\]

By using Fourier transform,

\[
f(x) = \frac{1}{(2\pi)^3} \int e^{-ip\cdot x} f_p(r) d^3p
\]

one can get this equation in momentum space:

\[
0 = \partial_r^2 f_p(r) - p^2 f_p(r) - \frac{\lambda}{(2\pi)^3} \int d^3[q, s, t] \delta^3(q + s + t - p)f_q(r)f_s(r)f_t(r),
\]

where \( d^3[q, s, ..., t] \equiv d^3q d^3s ... d^3t \).

We solve the equation perturbatively order by order in \( \lambda \) upto \( \lambda^2 \) order. Namely, we try the following ansatz:

\[
f_p(r) = \bar{f}_p(r) + \lambda \tilde{f}_p(r) + \lambda^2 \check{f}_p(r) + O(\lambda^3)
\]

In the zeroth order in \( \lambda \), the equation becomes

\[
0 = \partial_r^2 \bar{f}_p(r) - p^2 \bar{f}_p(r),
\]

and its solution is given by

\[
\bar{f}_p(r) = \bar{f}_{0,p} \cosh(|p|r) + \frac{\tilde{f}_{1,p}}{|p|} \sinh(|p|r),
\]

We will use \( p, q, s, t, u, v, w \) to indicate 3-momenta along boundary directions.
where \( \bar{f}_{0,p} \) and \( \bar{f}_{l,p} \) are boundary momenta, \( p_i \) dependent functions and \( |p| = \sqrt{p_1^2 + p_2^2 + p_3^2} \), which are absolute value of momentum along boundary direction. This solution should be regular everywhere, and for this we request that

\[
\bar{f}_{0,p} + \frac{\bar{f}_{l,p}}{|p|} = 0,
\]

(14)

Thus, the regular solution is given by

\[
\bar{f}_p(r) = \bar{f}_{0,p} e^{-|p|r}.
\]

(15)

In the first order in \( \lambda \), the equation is given by

\[
0 = (\partial_r^2 - p^2) \bar{f}_p(r) - \frac{1}{(2\pi)^3} \int d^3[q, s, t] \delta^3(q + s + t - p) \bar{f}_{0,q} \bar{f}_{0,s} \bar{f}_{0,t} \bar{f}_{0} e^{-(|q| + |s| + |t|)r},
\]

(16)

where the last term is a source term from the zeroth order solution. The first order solution is given by

\[
\tilde{f}_p(r) = \bar{f}_{0,p} e^{-|p|r} + \frac{1}{(2\pi)^3} \int d^3[q, s] \bar{f}_{0,p-q-s} \bar{f}_{0,q} \bar{f}_{0,s} \frac{e^{-(|q-s|+|q|+|s|)r}}{(p - q - s + |q| + |s|)^2 - p^2},
\]

(17)

where the first term is homogeneous solution and the last term is inhomogeneous one.

Finally, we will obtain the second order solution in \( \lambda \). The equation of motion is given by

\[
0 = (\partial_r^2 - p^2) \tilde{f}_p(r) - \frac{3}{(2\pi)^6} \int d^3[t, q, s, v, u] \delta^3(t + q + s - p) \times f_{0,t} f_{0,q} f_{0,s} f_{0,v} f_{0,u} f_{0,w} \frac{e^{-(|t|+|q|+|u|+|v|+|s-u-v|)r}}{(|s-u-v| + |u| + |v|)^2 - s^2},
\]

(18)

and its solution becomes

\[
\hat{f}_p(r) = \tilde{f}_{0,p} e^{-|p|r} + \frac{3}{(2\pi)^6} \int d^3[t, q, s, u, v, w] f_{0,t} f_{0,q} f_{0,u} f_{0,v} f_{0,w} \frac{\delta^3(t + q + s - p) \delta^3(w + u + v - s)}{[|w| + |v| + |u|)^2 - s^2][(|w| + |t| + |q| + |v| + |u|)^2 - p^2]} e^{-(|w|+|t|+|q|+|v|+|u|)r},
\]

(19)

where again the first term in the solution is homogeneous part and the last term is inhomogeneous one.

The homogeneous solutions in the first and the second order in \( \lambda \) can be absorbed in the zeroth order solution. Therefore, we set \( \bar{f}_{0,p} = \tilde{f}_{0,p} = 0 \). Moreover, we define a few complex expressions as

\[
\alpha_p(u, q, s) \equiv \frac{\delta^3(u + q + s - p)}{(|u| + |q| + |s|)^2 - p^2},
\]

(20)

\[
\beta_p(t, q; v, s, u) \equiv \frac{\delta^3(q + u + s + t + v - p)}{[|v| + |s| + |u|)^2 - (v + s + u)^2]} \times \frac{1}{[|v| + |t| + |q| + |s| + |u|^2 - p^2]}
\]

(21)

then, the form of the solution is much more simplified.
Near boundary expansion

Near conformal boundary \( r = 0 \), our solution is expanded as

\[
f_p(r) = f_{0,p} + \frac{\lambda}{(2\pi)^3} \int d^3[q, s, u] f_{0,q} f_{0,s} f_{0,u} \alpha_p(q, s, u)
\]

\[+ \frac{3\lambda^2}{(2\pi)^6} \int d^3[t, q, s, u, v] f_{0,q} f_{0,s} f_{0,u} f_{0,v} \beta_p(t, q; v, s, u)
\]

\[+ r \left[ -|p| f_{0,p} - \frac{\lambda}{(2\pi)^3} \int d^3[q, s, u] f_{0,q} f_{0,s} f_{0,u} (|q| + |s| + |u|) \alpha_p(q, s, u)
\]

\[= \left. -\frac{3\lambda^2}{(2\pi)^6} \int d^3[t, q, s, u, v] f_{0,q} f_{0,s} f_{0,u} f_{0,v} (|v| + |t| + |q| + |s| + |u|) \beta_p(t, q; v, s, u) \right] + O(r^2).
\]

The first two lines are boundary value of the field \( f_p(r) \) while the third and fourth lines are the boundary value of \( \partial_r f_p(r) \). We define this boundary value of \( f_p(r) \) to be \( f_p^{(0)} \) and the boundary value of \( \partial_r f_p(r) \equiv f_p^{(1)} \). One can rewrite \( f_p^{(1)} \) in terms of the boundary value \( f_p^{(0)} \), which is given by

\[
f_p^{(1)} = -|p| f_p^{(0)} - \frac{\lambda}{(2\pi)^3} \int d^3[q, s, u] f_u^{(0)} f_q^{(0)} f_s^{(0)} \alpha_p(u, q, s)(|u| + |q| + |s| - |p|)
\]

\[+ \frac{3\lambda^2}{(2\pi)^6} \int d^3[q, u, t, v, w] f_u^{(0)} f_q^{(0)} f_t^{(0)} f_v^{(0)} f_w^{(0)} \alpha_p^{(2)}(u, q; t, v, w)(|u| + |q| + |t + v + w| - |p|)
\]

\[= \beta_p(t, q; v, w, u)(|v| + |t| + |q| + |w| + |u| - |p|),
\]

where

\[
\alpha_p^{(2)}(u, q; t, v, w) \equiv \int d^3 s \alpha_p(u, q, s) \alpha_s(t, v, w)
\]

\[= \frac{\delta^3(t + v + w + q + u - p)}{((|t| + |v| + |w|)^2 - (t + v + w)^2)[(|t + v + w| + |q| + |u|)^2 - p^2}.
\]

3 Boundary conditions and effective actions

In this section, we discuss various boundary conditions and on-shell and boundary effective actions followed by those boundary conditions. One can evaluate on-shell action by using equation of motion from (6). This is given by

\[
I_{os} = S_{bulk} + S_{bdy} = \int_{r=0} d^3 x \frac{1}{2} f(x, r) \partial_r f(x, r) - \int dr d^3 x \frac{\lambda}{4} f^4(x, r) + \int d^3 x L_b(f).
\]

By using Fourier transform defined in (9), one can write this in momentum space as

\[
S_{os} = \frac{1}{2} \int_{r=0} d^3 p f_p(r) \partial_r f_p(r) - \frac{\lambda}{4(2\pi)^3} \int_{r=0} d^3[p, q, s, t] d r f_p(r) f_q(r) f_s(r) f_t(r) \delta^3(p + q + s + t)
\]

\[+ \int d^3 p L_b(f_p).
\]
We define the boundary value of the bulk canonical momentum, $\partial_r f(r)$ as

$$\Pi_p = \frac{\delta S_{\text{bulk}}}{\delta f_p^{(0)}}.$$  

(27)

With this canonical momentum, the on-shell action is now functional of the boundary value of $f_p(r)$, $f_p^{(0)}$ and its canonical momentum $\Pi_p$. In AdS/CFT context, the bulk on-shell action becomes generating functional of the dual CFT as

$$Z[J] = e^{-W[J(f_p^{(0)}, \Pi_p)]} = \int D[f_p^{(0)}, \Pi_p] \exp \left[ -S_{\text{bulk}}(f_p^{(0)}) - S_{\text{bdy}}(f_p^{(0)}, \Pi_p) \right],$$  

(28)

where $J$ is source which couples to certain boundary composite operator and $W$ is the generating functional with the source term $J$. This generating functional is identified with the on-shell action as $W[J] = I_{\text{os}}[f_p^{(0)}, \Pi_p]$. In standard quantization ($\Delta_+$ theory), $J = f_p^{(0)}$ and the boundary composite operator becomes $\Pi_p$. In general, however, the source $J$ is generic function of $f_p^{(0)}$ and $\Pi_p$.

The boundary condition is obtained by looking at saddle point of the on-shell action as

$$\frac{\delta I_{\text{os}}[f_p^{(0)}, \Pi_p]}{\delta f_p^{(0)}} = 0, \quad \text{and} \quad \frac{\delta I_{\text{os}}[f_p^{(0)}, \Pi_p]}{\delta \Pi_p} = 0,$$  

(29)

which gives the relation between $f_p^{(0)}$ and $\Pi_p$ and with this, one can rewrite the on-shell action in terms of $f_p^{(0)}$ only, which also gives the correct boundary condition in its saddle point.

The boundary effective action is given by Legendre transform of the generating functional, which if given by

$$\Gamma[\sigma] = -\int J \sigma + W[J],$$  

(30)

where $\sigma$ is vacuum expectation value of the boundary composite operator. Followed by (30), $\sigma$ and $J$ satisfy the following relations:

$$\sigma = \frac{\delta W[J]}{\delta J} \quad \text{and} \quad J = -\frac{\delta \Gamma[\sigma]}{\delta \sigma}.$$  

(31)

Suppose that $W$ and $\Gamma$ are the generating functional and boundary effective action without any boundary deformation. Let us deform this with boundary term $S_{\text{bdy}}$, and assume that $\sigma$ does not change by the deformation. The deformed boundary effective action may have a form of

$$\Gamma_d[\sigma] = \Gamma[\sigma] + \int g(\sigma),$$  

(32)

where $\Gamma_d$ is deformed boundary effective action and $g$ is a function of $\sigma$. Deformed source $J_d$ will be given by

$$J_d \equiv \frac{\delta \Gamma_d[\sigma]}{\delta \sigma} = J - \frac{dg(\sigma)}{d\sigma}.$$  

(33)

Therefore, deformed generating functional $W_d[J_d] = \Gamma_d[\sigma] + \int J_d \sigma$ is given by

$$W_d[J_d] = W[J] + \int (g(\sigma) - \sigma g'(\sigma)).$$  

(34)
In sum, the boundary deformation term is given by

$$S_{bdy} = \int (g(\sigma) - \sigma g'(\sigma)) \delta(\sigma - \frac{\delta W[J]}{\delta \sigma}).$$

(35)

**Dirichlet boundary condition** Dirichlet boundary condition is achieved without adding any deformation term. Then, the boundary condition is obtained by finding saddle point of the bulk action as

$$\delta I_{os} = \delta S_{bulk} = \int \Pi_{-p} \delta f_p^{(0)},$$

(36)

Then the boundary condition is $\delta f_p^{(0)} = 0$. By plugging the on-shell solution into (26), one can evaluate this on-shell action in terms of the boundary value of the field $f(r)$ as

$$I_{os}^D = W^D[f_p^{(0)}] = \frac{1}{2} \int d^3p d^3q f_p^{(0)} f_q^{(0)} \langle O_p O_q \rangle + \frac{1}{4!} \int d^3[p, q, s, u] f_p^{(0)} f_q^{(0)} f_s^{(0)} f_u^{(0)} \langle O_p O_q O_s O_u \rangle\) (37)

$$+ \frac{1}{6!} \int d^3[p, q, t, v, w, u] f_p^{(0)} f_q^{(0)} f_t^{(0)} f_v^{(0)} f_w^{(0)} f_u^{(0)} \langle O_p O_q O_t O_v O_w O_u \rangle$$

where

$$\langle O_p O_q \rangle = -|p| \delta^3(p + q),$$

(38)

$$\langle O_p O_q O_s O_u \rangle = - \frac{3! \lambda}{2} \delta^3(u + q + s + p)$$

(39)

and

$$\langle O_p O_q O_w O_u O_t O_v \rangle = \frac{9 \cdot 5! \lambda^2}{(2\pi)^6} \left( \frac{\delta^3(t + v + w + q + u + p)}{[(|t| + |v| + |w|)^2 - (t + v + w)^2][|t| + v + w + |u| + |q| + |p|]} \right. (40)

$$- \frac{2}{3} \delta^3(t + v + w + q + u + p)

$$- \left. \frac{\delta^3(t + v + w + q + u + p)}{2[(|t| + |v| + |w| + |u| + |q| + |p|)[|t| + |v| + |w| + |t + v + w| + |u| + |q| + |p|]}\right).$$

This provides boundary momentum dependent 2,4, and 6-point functions in dual CFTs.

From the definition (27), we get canonical momentum of $f^{(0)}$:

$$\Pi_{-p} = \int d^3s d^3q \delta^3(s - p) f_q^{(0)} \langle O_s O_q \rangle + \frac{1}{3!} \int d^3[t, q, s, u] \delta^3(t - p) f_q^{(0)} f_s^{(0)} f_u^{(0)} \langle O_t O_q O_s O_u \rangle$$

(41)

$$+ \frac{1}{5!} \int d^3[p, q, t, v, w, u] f_p^{(0)} f_q^{(0)} f_t^{(0)} f_v^{(0)} f_w^{(0)} f_u^{(0)} \langle O_p O_q O_t O_v O_w O_u \rangle$$

By Legendre transform (30), Boundary effective action is given by

$$\Gamma^D(\Pi) = -\frac{1}{2} \int d^3[p, q] \frac{\langle O_p O_q \rangle}{|p||q|} \Pi_{-p} \Pi_{-q} + \frac{1}{4!} \int d^3[p, q, s, t] \frac{\langle O_p O_q O_s O_t \rangle}{|p||q||s||t|} \Pi_{-p} \Pi_{-q} \Pi_{-s} \Pi_{-t}$$

(42)

$$+ \frac{1}{3!} \int d^3[p, q, s, t, u, v] \Pi_{-p} \Pi_{-q} \Pi_{-s} \Pi_{-t} \Pi_{-u} \Pi_{-v} \left( \frac{1}{2} \int d^3w \frac{\langle O_p O_q O_s O_w \rangle}{|p||q||s||w|} |w| \frac{\langle O_t O_s O_v \rangle}{|w||t||u||v|} + \frac{1}{20} \frac{\langle O_p O_q O_s O_t O_v \rangle}{|p||q||s||t||u||v|} \right),$$

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Neumann boundary condition Neumann boundary condition is achieved by adding boundary deformation to the bulk action as

\[ S_{\text{bdy}}^N = - \int d^3 p f_p^{(0)} \Pi_{-p}, \]  

then the boundary condition is achieved by variation of the on-shell action

\[ \delta I_{\text{os}} = \delta S_{\text{bulk}} + \delta S_{\text{bdy}} = - \int d^3 p f_p^{(0)} \delta \Pi_{-p}. \]  

Therefore, one can request \[ \delta \Pi_{-p} = 0, \] which is Neumann boundary condition. Since adding such boundary deformation provides effective Legendre transform from bulk action, then the on-shell action has the same form with the boundary effective action in Dirichlet boundary condition case. Moreover, its boundary effective action will be the form of the on-shell action in Dirichlet boundary condition case too. In sum,

\[ I_{\text{os}}^N = W^N[\Pi_{-p}] = \Gamma^D[\Pi_{-p}], \quad \text{and} \quad I_{\text{os}}^D = W^D[f_p^{(0)}] = \Gamma^N[f_p^{(0)}]. \]  

Marginal deformation Marginal deformation is achieved by adding the following boundary deformation term:

\[ S_{\text{bdy}}^M = - \frac{\alpha}{3} \int \frac{d^3[q, t]}{(2\pi)^{3/2}} f_q^{(0)} f_t^{(0)} \delta^3(p + q + t), \]  

where \( \alpha \) is a free real parameter. Then, followed by this, the boundary condition is given by

\[ 0 = \frac{\delta S}{\delta f_p^{(0)}} = \Pi_{-p} - \frac{\alpha}{3} \int \frac{d^3[q, t]}{(2\pi)^{3/2}} f_q^{(0)} f_t^{(0)} \delta^3(p + q + t) \]  

On-shell action in marginal deformation case is

\[ I_{\text{os}}^M = I_{\text{os}}^D[f_p^{(0)}] + S_{\text{bdy}}^M[f_p^{(0)}] \]  

By using the procedure introduced in the beginning of this section to derive boundary effective action from on-shell action and the source \( J \), we get those as

\[ \Gamma^M(f_p^{(0)}) = - \frac{1}{2} \int d^3 p d^3 q f_p^{(0)} f_q^{(0)} \langle O_p O_q \rangle + \frac{1}{3} \int d^3[q, s, t] f_q^{(0)} f_s^{(0)} f_t^{(0)} \langle O_q O_s O_t \rangle - \frac{1}{3 \cdot 4!} \int d^3[p, q, s, u] f_p^{(0)} f_q^{(0)} f_s^{(0)} f_u^{(0)} \langle O_p O_q O_s O_u \rangle \]

\[ - \frac{1}{5 \cdot 6!} \int d^3[p, q, t, v, w, u] f_p^{(0)} f_q^{(0)} f_t^{(0)} f_v^{(0)} f_w^{(0)} f_u^{(0)} \langle O_p O_q O_t O_v O_w O_u \rangle, \]

\[ \text{Even in this case, Dirichlet boundary condition is possible to be imposed. Dirichlet boundary condition is always possible boundary condition.} \]
\[ J^M_{\rho} = \int d^3q f^{(0)}_q \langle O_p O_q \rangle - \int d^3[s, t] f^{(0)}_s f^{(0)}_t \langle O_p O_s O_t \rangle \]
\[ + \frac{1}{3 \cdot 3!} \int d^3[q, s, u] f^{(0)}_q f^{(0)}_s f^{(0)}_u \langle O_p O_q O_s O_u \rangle \]
\[ + \frac{1}{5 \cdot 5!} \int d^3[q, t, v, w, u] f^{(0)}_q f^{(0)}_t f^{(0)}_v f^{(0)}_w f^{(0)}_u \langle O_p O_q O_s O_t O_v O_w O_u \rangle, \]

where
\[ \langle O_q O_s O_t \rangle = \frac{\alpha}{2 (2\pi)^{3/2}} \delta^3(q + s + t). \] (51)

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