THE SINGULAR LIMIT OF A CHEMOTAXIS-GROWTH SYSTEM WITH GENERAL INITIAL DATA

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Abstract
We study the singular limit of a system of partial differential equations which is a model for an aggregation of amoebae subjected to three effects: diffusion, growth and chemotaxis. The limit problem involves motion by mean curvature together with a nonlocal drift term. We consider rather general initial data. We prove a generation of interface property and study the motion of interface. We also obtain an optimal estimate of the thickness and the location of the transition layer that develops.

1 Introduction
Let us start by a short description of life-cycles of the cellular slime molds (amoebae). The cells feed and divide until exhaustion of food supply. Then, the amoebae aggregate to form a multicellular assembly called a slug. It migrates to a new location, then forms into a fruiting body, consisting of a stalk formed from dead amoebae and spores on the top (fruiting bodies that are visible to the naked eye are often referred to as mushrooms). Under suitable conditions of moisture, temperature, spores release new amoebae. The cycle then repeats itself.

It is known that the aggregation stage is mediated by chemotaxis, i.e. the tendency of biological individuals to direct their movements according to certain chemicals in their environment. The chemotactant (acrasin) is produced by the amoebae themselves and degraded by an extracellular enzyme (acrasinase). For more details on the biological background, we refer to [15], [20] or [10].

So the amoebae have a random motion analogous to diffusion coupled with an oriented chemotactic motion in the direction of a positive gradient.

1 AMS Subject Classifications: 35K57, 35B50, 35R35, 92C17.
of acrasin. In 1970, Keller and Segel [15] proposed the following system as a model to describe such movements leading to slime mold aggregation:

\[
(\text{KS}) \quad \begin{cases}
  u_t &= \nabla \cdot (D_2 \nabla u) - \nabla \cdot (D_1 \nabla v), \\
  v_t &= D_v \Delta v + f(v)u - k(v)v,
\end{cases}
\]

inside a closed region \( \Omega \). Here, \( u \), respectively \( v \), denotes the concentration of amoebae, respectively of acrasin; \( f(v) \) is the production rate of acrasin, and \( k(v) \) the degradation rate of acrasin (due to acrasinase); \( D_2 = D_2(u, v) \), respectively \( D_1 = D_1(u, v) \), measures the vigor of the random motion of the amoebae, respectively the strength of the influence of the acrasin gradient on the flow of amoebae; \( D_v \) is a positive and constant diffusion coefficient. The problem is completed by initial data \( u_0 \) and \( v_0 \) and, assuming that there is no flow of the amoebae or the acrasin across the boundary \( \partial \Omega \), by homogeneous Neumann boundary conditions

\[
\nabla u \cdot \nu = \nabla v \cdot \nu = 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty),
\]

\( \nu \) being the unit outward normal to \( \partial \Omega \).

An often used simplified model is obtained as follows. By some receptor mechanism, cells do not measure the gradient of \( v \) but of some \( \chi(v) \), with a sensitive function \( \chi \) satisfying \( \chi' > 0 \), so that \( D_1(u, v) = u \chi'(v) \). By taking \( D_2 \), \( f \) and \( k \) as constant functions and using some rescaling arguments, the system reduces to

\[
(\text{KS}') \quad \begin{cases}
  u_t &= d_u \Delta u - \nabla \cdot (u \nabla \chi(v)), \\
  \tau v_t &= d_v \Delta v + u - \gamma v,
\end{cases}
\]

with \( d_u \), \( d_v \), \( \tau \) and \( \gamma \) some positive constants.

Many analyses of the Keller-Segel model for the aggregation process were proposed. Chemotaxis having some features of “negative diffusion”, Nanjundiah [20] suggests that the whole population concentrates in a single point; we refer to this phenomenon as the chemotactic collapse. In mathematical terms, this amounts to blow up in finite time. As a matter of fact, it turns out that the possibility of collapse depends upon the space dimension. In particular it never happens in the one-dimensional case whereas in two space dimensions, assuming radially symmetric situations, it only occurs if the total amoebae number is sufficiently large. The problem of global existence and blow up of solutions has been intensively studied; we refer in particular to [9], [22], [16], [13], [19], [11], [12].

In a different framework, Mimura and Tsujikawa [17], consider aggregating pattern-dynamics arising in the following chemotaxis model with growth:

\[
(\text{MT}^\varepsilon) \quad \begin{cases}
  u_t &= \varepsilon^2 \Delta u - \varepsilon \nabla \cdot (u \nabla \chi(v)) + f(u), \\
  \tau v_t &= \Delta v + u - \gamma v,
\end{cases}
\]
where $\varepsilon > 0$ is a small parameter. The function $f$ is cubic, 0 and 1 being its stable zeros, and satisfies $\int_0^1 f > 0$. In this model, the population is subjected to three effects: diffusion, growth and chemotaxis. The diffusion rate and the chemotactic rate are both very small compared with the growth rate. They observe that, in a first stage, internal layers — which describe the boundaries of aggregating regions — develop; in a second stage, the motion of the aggregating regions — which can be described by that of internal layers — takes place. The balance of the three effects (diffusion, growth and chemotaxis) makes the aggregation mechanism possible. Taking the limit $\varepsilon \to 0$, they formally derive the equation for the motion of the limit interface and study the stability of radially symmetric equilibrium solutions.

The purpose of this paper is to extend some of the results obtained by Bonami, Hilhorst, Logak and Mimura [4] about the singular limit of a variant of system $(MT^\varepsilon)$, where the second equation is elliptic ($\tau = 0$):

\begin{equation}
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u \nabla \chi(v)) + \frac{1}{\varepsilon^2} f_\varepsilon(u) & \text{in } \Omega \times (0, +\infty), \\
  0 = \Delta v + u - \gamma v & \text{in } \Omega \times (0, +\infty), \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \partial \Omega \times (0, +\infty), \\
  u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\end{equation}

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $\nu$ is the Euclidian unit normal vector exterior to $\partial \Omega$. We assume that $\gamma$ is a positive constant and that the nonlinearity $f_\varepsilon$ is given by

\begin{equation}
  f_\varepsilon(u) = u(1-u)(u-\frac{1}{2}) + \varepsilon \alpha u(1-u) =: f(u) + \varepsilon g(u),
\end{equation}

with $\alpha > 0$. The role of the function $g$ is to break the balance of the two stable zeros slightly. The sensitive function $\chi$ is smooth and satisfies $\chi'(v) > 0$ for $v > 0$.

We also assume that the initial datum satisfies $u_0 \in C^2(\overline{\Omega})$ and $u_0 \geq 0$. Throughout the present paper, we fix a constant $C_0 > 1$ that satisfies

\begin{equation}
  \|u_0\|_{C^0(\overline{\Omega})} + \|\nabla u_0\|_{C^0(\overline{\Omega})} + \|\Delta u_0\|_{C^0(\overline{\Omega})} \leq C_0.
\end{equation}

Furthermore we define the "initial interface" $\Gamma_0$ by

\begin{equation}
  \Gamma_0 := \{ x \in \Omega, \quad u_0(x) = 1/2 \}.
\end{equation}

We suppose that $\Gamma_0$ is a $C^{2+\vartheta}$ hypersurface without boundary, for a $\vartheta \in (0, 1)$, such that, $n$ being the Euclidian unit normal vector exterior to $\Gamma_0$,

\begin{equation}
  \Gamma_0 \subset \subset \Omega \quad \text{and} \quad \nabla u_0(x) \cdot n(x) \neq 0 \quad \text{if } x \in \Gamma_0,
\end{equation}

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\[ u_0 > 1/2 \quad \text{in} \quad \Omega_0^{(1)}, \quad u_0 < 1/2 \quad \text{in} \quad \Omega_0^{(0)}, \quad (1.4) \]

where \( \Omega_0^{(1)} \) denotes the region enclosed by \( \Gamma_0 \) and \( \Omega_0^{(0)} \) the region enclosed between \( \partial \Omega \) and \( \Gamma_0 \).

The existence of a unique smooth solution to Problem \((P^\varepsilon)\) is proved in [4], Lemma 4.2:

**Lemma 1.1.** There exists \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique solution \((u^\varepsilon, v^\varepsilon)\) to Problem \((P^\varepsilon)\) on \( \Omega \times [0, +\infty) \), with \( 0 \leq u^\varepsilon \leq C_0 \) on \( Q_T \).

To study the interfacial behavior associated with this model, it is useful to consider a formal asymptotic limit of Problem \((P^\varepsilon)\) as \( \varepsilon \to 0 \). Then the limit solution \( u^0(x, t) \) will be a step function taking the value 1 on one side of the interface, and 0 on the other side. This sharp interface, which we will denote by \( \Gamma_t \), obeys a law of motion, which can be obtained by formal analysis (see Section 2):

\[
\begin{cases}
V_n = -(N - 1)\kappa + \frac{\partial \chi(v^0)}{\partial n} + \sqrt{2}\alpha & \text{on } \Gamma_t, \\
\Gamma_t|_{t=0} = \Gamma_0 \\
-\Delta v^0 + \gamma v^0 = u^0 & \text{in } \Omega \times (0, T], \\
\frac{\partial v^0}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T],
\end{cases}
\]

where \( V_n \) is the normal velocity of \( \Gamma_t \) in the exterior direction, \( \kappa \) the mean curvature at each point of \( \Gamma_t \). We set \( Q_T := \Omega \times [0, T] \) and for each \( t \in [0, T] \), we define \( \Omega_t^{(1)} \) as the region enclosed by the hypersurface \( \Gamma_t \) and \( \Omega_t^{(0)} \) as the region enclosed between \( \partial \Omega \) and \( \Gamma_t \). The step function \( u^0 \) is determined straightforwardly from \( \Gamma_t \) by

\[
u^0(x, t) = \begin{cases}
1 & \text{in } \Omega_t^{(1)} \\
0 & \text{in } \Omega_t^{(0)}
\end{cases} \quad \text{for } t \in [0, T].
\]

By a contraction fixed-point argument in suitable Hölder spaces, the well-posedness, locally in time, of the free boundary Problem \((P^0)\) is proved in [4], Theorem 2.1:

**Lemma 1.2.** There exists a time \( T > 0 \) such that \((P^0)\) has a unique solution \((v^0, \Gamma)\) on [0, T], with

\[
\Gamma = \bigcup_{0 \leq t \leq T} (\Gamma_t \times \{t\}) \in C^{2+\theta, \frac{2+\theta}{2}},
\]

and \( v^0|_{\Gamma} \in C^{2+\theta, \frac{2+\theta}{2}}. \)
Bonami, Hilhorst, Logak and Mimura [4] have proved a motion of interface property; more precisely, for some prepared initial data, they show that \((u^\varepsilon, v^\varepsilon)\) converges to \((u^0, v^0)\) as \(\varepsilon \to 0\), on the interval \((0, T)\). So the evolution of \(\Gamma_t\) determines the aggregating patterns of the individuals. Here we consider the case of arbitrary initial data. Our first main result, Theorem 1.3, describes the profile of the solution after a very short initial period. It asserts that, given a virtually arbitrary initial datum \(u_0\), the solution \(u^\varepsilon\) quickly becomes close to 1 or 0, except in a small neighborhood of the initial interface \(\Gamma_0\), creating a steep transition layer around \(\Gamma_0\) (generation of interface). The time needed to develop such a transition layer, which we will denote by \(t^\varepsilon\), is of order \(\varepsilon^2 |\ln \varepsilon|\). The theorem then states that the solution \(u^\varepsilon\) remains close to the step function \(u_0\) on the time interval \([t^\varepsilon, T]\) (motion of interface). Moreover, as is clear from the estimates in the theorem, the “thickness” of the transition layer is of order \(\varepsilon\).

**Theorem 1.3 (Generation and motion of interface).** Let \(\eta \in (0, 1/4)\) be arbitrary and set
\[
\mu = f'(1/2) = 1/4.
\]
Then there exist positive constants \(\varepsilon_0\) and \(C\) such that, for all \(\varepsilon \in (0, \varepsilon_0)\), all \(t^\varepsilon \leq t \leq T\), where \(t^\varepsilon = \mu^{-1} \varepsilon^2 |\ln \varepsilon|\), we have
\[
u^\varepsilon(x, t) \in \begin{cases} [-\eta, 1 + \eta] & \text{if } x \in \mathcal{N}^\varepsilon_\epsilon(\Gamma_t) \\ [-\eta, \eta] & \text{if } x \in \Omega_t^0 \setminus \mathcal{N}^\varepsilon_\epsilon(\Gamma_t) \\ [1 - \eta, 1 + \eta] & \text{if } x \in \Omega_t^1 \setminus \mathcal{N}^\varepsilon_\epsilon(\Gamma_t), \end{cases}
\] (1.6)
where \(\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, \text{dist}(x, \Gamma_t) < r\}\) denotes the \(r\)-neighborhood of \(\Gamma_t\).

**Corollary 1.4 (Convergence).** As \(\varepsilon \to 0\), the solution \((u^\varepsilon, v^\varepsilon)\) converges to \((u^0, v^0)\) everywhere in \(\bigcup_{0 < t \leq T} (\Omega_t^0 \cup 1) \times \{t\}\).

The next theorem deals with the relation between the set \(\Gamma^\varepsilon_t := \{x \in \Omega, u^\varepsilon(x, t) = 1/2\}\) and the solution \(\Gamma_t\) of Problem \((P^0)\).

**Theorem 1.5 (Error estimate).** There exists \(C > 0\) such that
\[
\Gamma^\varepsilon_t \subset \mathcal{N}^\varepsilon_\epsilon(\Gamma_t) \quad \text{for } 0 \leq t \leq T.
\] (1.7)

**Corollary 1.6 (Convergence of interface).** There exists \(C > 0\) such that
\[
d_H(\Gamma^\varepsilon_t, \Gamma_t) \leq C\varepsilon \quad \text{for } 0 \leq t \leq T,
\] (1.8)
where
\[
d_H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}
\]
denotes the Hausdorff distance between two compact sets \(A\) and \(B\). Consequently, \(\Gamma^\varepsilon_t \to \Gamma_t\) as \(\varepsilon \to 0\), uniformly in \(0 \leq t \leq T\), in the sense of the Hausdorff distance.
As far as we know, the best thickness estimate in the literature was of order $\varepsilon |\ln \varepsilon|$ (see [5], [6]). We refer to a forthcoming article [14], respectively [1], in which an order $\varepsilon$ estimate is established for a Lotka-Volterra competition-diffusion system, respectively for the FitzHugh-Nagumo system.

The organization of this paper is as follows. Section 2 is devoted to preliminaries: we recall the method of asymptotic expansions to derive the equation of the interface motion; we also recall a relaxed comparison principle used in [4]. In Section 3 we prove a generation of interface property. The corresponding sub- and super-solutions are constructed by modifying the solution of the ordinary differential equation $u_t = \varepsilon^{-2} f(u)$, obtained by neglecting diffusion and chemotaxis. In Section 4 in order to study the motion of interface, we construct a pair of sub- and super-solutions that rely on a related one-dimensional stationary problem. Finally, in Section 5 by fitting the two pairs of sub- and super-solutions into each other, we prove Theorem 1.3, Theorem 1.5 and theirs corollaries.

2 Some preliminaries

2.1 Formal derivation

A formal derivation of the equation of interface motion was given in [3]. Nevertheless we briefly present it in a slightly different way: we use arguments similar to those in [21] where the first two terms of the asymptotic expansion determine the interface equation. The observations we make here will help the rigorous analysis in later sections, in particular for the construction of sub- and super-solutions for the study of the motion of interface in Section 4.

Let $(u^\varepsilon, v^\varepsilon)$ be the solution of Problem $(P^\varepsilon)$. We recall that $\Gamma^\varepsilon_t := \{x \in \Omega, u^\varepsilon(x, t) = 1/2\}$ is the interface at time $t$ and call $\Gamma^\varepsilon := \bigcup_{t \geq 0} (\Gamma^\varepsilon_t \times \{t\})$ the interface. Let $\Gamma = \bigcup_{0 \leq t \leq T} (\Gamma_t \times \{t\})$ be the solution of the limit geometric motion problem and let $\tilde{d}$ be the signed distance function to $\Gamma$ defined by:

$$\tilde{d}(x, t) = \begin{cases} 
\text{dist}(x, \Gamma_t) & \text{for } x \in \Gamma^{(0)}_t \\
-\text{dist}(x, \Gamma_t) & \text{for } x \in \Gamma^{(1)}_t, \end{cases}$$

where $\text{dist}(x, \Gamma_t)$ is the distance from $x$ to the hypersurface $\Gamma_t$ in $\Omega$. We remark that $\tilde{d} = 0$ on $\Gamma$ and that $|\nabla \tilde{d}| = 1$ in a neighborhood of $\Gamma$. We then define

$$Q^{(1)}_T = \bigcup_{0 < t \leq T} (\Omega^{(1)}_t \times \{t\}), \quad Q^{(0)}_T = \bigcup_{0 < t \leq T} (\Omega^{(0)}_t \times \{t\}).$$

We assume that the solution $u^\varepsilon$ has the expansions

$$u^\varepsilon(x, t) = \{0 \text{ or } 1\} + \varepsilon u_1(x, t) + \cdots$$

(2.2)
away from the interface \( \Gamma \) (the outer expansion) and
\[
u^\varepsilon(x,t) = U_0(x,t, \frac{d(x,t)}{\varepsilon}) + \varepsilon U_1(x,t, \frac{d(x,t)}{\varepsilon}) + \cdots \tag{2.3}
\]

near \( \Gamma \) (the inner expansion). Here, the functions \( U_k(x,t,z) \), \( k = 0,1,\cdots \), are defined for \( x \in \Omega \), \( t \geq 0 \), \( z \in \mathbb{R} \). The stretched space variable \( \xi := \frac{\tilde{d}(x,t)}{\varepsilon} \) gives exactly the right spatial scaling to describe the rapid transition between the regions \( \{ u^\varepsilon \approx 1 \} \) and \( \{ u^\varepsilon \approx 0 \} \). We use the normalization conditions
\[
U_0(x,t,0) = 1/2, \quad U_k(x,t,0) = 0,
\]
for all \( k \geq 1 \). The matching conditions between the outer and the inner expansion are given by
\[
U_0(x,t,\pm \infty) = 0, \quad U_k(x,t,\pm \infty) = 0,
\]
for all \( k \geq 1 \). We also assume that the solution \( v^\varepsilon \) has the expansion
\[
v^\varepsilon(x,t) = v_0(x,t) + \varepsilon v_1(x,t) + \cdots \tag{2.5}
\]
in \( \Omega \times (0,T) \).

We now substitute the inner expansion (2.3) and the expansion (2.5) into the parabolic equation of \((P^\varepsilon)\) and collect the \( \varepsilon^{-2} \) terms. We omit the calculations and, using \( |\nabla \tilde{d}| = 1 \) near \( \Gamma \), the normalization and matching conditions, we deduce that \( U_0(x,t,z) = U_0(z) \) is the unique solution of the stationary problem
\[
\left\{ \begin{array}{l}
U_0'' + f(U_0) = 0 \\
U_0(-\infty) = 1, \quad U_0(0) = 1/2, \quad U_0(+\infty) = 0.
\end{array} \right. \tag{2.6}
\]
This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates. Recalling that the nonlinearity is given by \( f(u) = u(1-u)(u-1/2) \), we have
\[
U_0(z) = \frac{1}{2}(1 - \tanh \frac{z}{2\sqrt{2}}) = \frac{e^{-z/\sqrt{2}}}{1 + e^{-z/\sqrt{2}}} \tag{2.7}
\]

We claim that \( U_0 \) has the following properties.

**Lemma 2.1.** There exist positive constants \( C \) and \( \lambda \) such that the following estimates hold.
\[
0 < U_0(z) \leq Ce^{-\lambda |z|} \quad \text{for } z \geq 0,
\]
\[
0 < 1 - U_0(z) \leq Ce^{-\lambda |z|} \quad \text{for } z \leq 0.
\]

In addition, \( U_0 \) is a strictly decreasing function and
\[
|U_0'(z)| + |U_0''(z)| \leq Ce^{-\lambda |z|} \quad \text{for } z \in \mathbb{R}.
\]
Next we collect the $\varepsilon^{-1}$ terms. Since $U_0$ depends only on the variable $z$, we have $\nabla U_{0z} = 0$ which, combined with the fact that $|\nabla \tilde{d}| = 1$ near $\Gamma_t$, yields

$$U_{1zz} + f'(U_0)U_1 = U_0'(\tilde{d}_t - \Delta \tilde{d} + \nabla \tilde{d} \cdot \nabla \chi(v^0)) - g(U_0),$$  \hspace{1cm} (2.8)

a linearized problem corresponding to (2.6). The solvability condition for the above equation, which can be seen as a variant of the Fredholm alternative, plays the key role for deriving the equation of interface motion. It is given by

$$\int_{\mathbb{R}} \left[ U_0'^2(z)(\tilde{d}_t - \Delta \tilde{d} + \nabla \tilde{d} \cdot \nabla \chi(v^0))(x, t) - g(U_0(z))U_0'(z) \right] dz = 0,$$

for all $(x, t) \in Q_T$. By the definition of $g$ in (1.1), we compute

$$\int_{\mathbb{R}} g(U_0(z))U_0'(z)dz = -\int_0^1 g(u)du = -\alpha/6,$$

whereas the equality (2.7) yields

$$\int_{\mathbb{R}} U_0'^2(z)dz = 1/\sqrt{2} \int_0^{+\infty} u/\left(1 + u^2\right)du = 1/6\sqrt{2}.$$

Combining the above expressions, we obtain

$$\left(\tilde{d}_t - \Delta \tilde{d} + \nabla \tilde{d} \cdot \nabla \chi(v^0)\right)(x, t) = -\sqrt{2}\alpha.$$  \hspace{1cm} (2.9)

Since $\nabla \tilde{d} (= \nabla_x \tilde{d}(x, t))$ coincides with the outward normal unit vector to the hypersurface $\Gamma_t$, we have $\tilde{d}_t(x, t) = -V_n$, where $V_n$ is the normal velocity of the interface $\Gamma_t$. It is also known that the mean curvature $\kappa$ of the interface is equal to $\Delta \tilde{d}/(N - 1)$. Thus the above equation reads as

$$V_n = -(N - 1)\kappa + \frac{\partial \chi(v^0)}{\partial n} + \sqrt{2}\alpha \quad \text{on} \quad \Gamma_t, \hspace{1cm} (2.10)$$

that is the equation of interface motion in $(P^0)$. Summarizing, under the assumption that the solution $u^\varepsilon$ of Problem $(P^\varepsilon)$ satisfies

$$u^\varepsilon \rightarrow \begin{cases} 1 & \text{in } Q_T^{(1)} \\ 0 & \text{in } Q_T^{(0)} \end{cases} \quad \text{as } \varepsilon \to 0,$$

we have formally showed that the boundary $\Gamma_t$ between $\Omega_t^{(0)}$ and $\Omega_t^{(1)}$ moves according to the law (2.10).

One can note that, using the equality (2.7), we clearly have $\sqrt{2}\alpha U_0' + g(U_0) \equiv 0$ so that, substituting (2.9) into (2.8) yields $U_1 \equiv 0$. 

8
2.2 A comparison principle

The definition of sub- and super-solutions is the one proposed in [4].

**Definition 2.2.** Let \((u_\varepsilon^-, u_\varepsilon^+\rangle\) be two smooth functions with \(u_\varepsilon^- \leq u_\varepsilon^+\) on \(Q_T\) and
\[
\frac{\partial u_\varepsilon^-}{\partial \nu} \leq 0 \leq \frac{\partial u_\varepsilon^+}{\partial \nu} \quad \text{on} \quad \partial \Omega \times (0, T).
\]
By definition, \((u_\varepsilon^-, u_\varepsilon^+\rangle\) is a pair of sub- and super-solutions if, for any \(v_\varepsilon\) which satisfies
\[
\begin{cases}
  u_\varepsilon^- \leq -\Delta v_\varepsilon + \gamma v_\varepsilon \leq u_\varepsilon^+ & \text{on} \ Q_T,
  \\
  \frac{\partial v_\varepsilon}{\partial \nu} = 0 & \text{on} \ \partial \Omega \times (0, T),
\end{cases}
\]
we have
\[
L_{v_\varepsilon}[u_\varepsilon^-] \leq 0 \leq L_{v_\varepsilon}[u_\varepsilon^+],
\]
where the operator \(L_{v_\varepsilon}\) is defined by
\[
L_{v_\varepsilon}[\phi] = \phi_t - \Delta \phi + \nabla \cdot (\phi \nabla \chi(v_\varepsilon)) - \frac{1}{\varepsilon^2} f_\varepsilon(\phi).
\]

As proved in [4], the following comparison principle holds.

**Proposition 2.3.** Let a pair of sub- and super-solutions be given. Assume that, for all \(x \in \Omega\),
\[
u_\varepsilon^-(x, 0) \leq u_0(x) \leq u_\varepsilon^+(x, 0).
\]
Then, if we denote by \((u_\varepsilon, v_\varepsilon)\) the solution of Problem \((P_\varepsilon)\), the function \(u_\varepsilon\) satisfies, for all \((x, t) \in Q_T\),
\[
u_\varepsilon^-(x, t) \leq u_\varepsilon^-(x, t) \leq u_\varepsilon^+(x, t).
\]

3 Generation of interface

In this section we study the rapid formation of internal layers in a neighborhood of \(\Gamma_0 = \{x \in \Omega, u_0(x) = 1/2\}\) within a very short time interval of order \(\varepsilon^2 |\ln \varepsilon|\). In the sequel, we shall always assume that \(0 < \eta < 1/4\). The main result of this section is the following.

**Theorem 3.1.** Let \(\eta\) be arbitrary and define \(\mu\) as the derivative of \(f(u)\) at the unstable equilibrium \(u = 1/2\), that is
\[
\mu = f'(1/2) = 1/4.
\]
Then there exist positive constants \(\varepsilon_0\) and \(M_0\) such that, for all \(\varepsilon \in (0, \varepsilon_0)\),
for all \( x \in \Omega \),
\[
-\eta \leq u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq 1 + \eta,
\]
(3.2)

- for all \( x \in \Omega \) such that \( |u_0(x) - \frac{1}{2}| \geq M_0 \varepsilon \), we have that
  \[
  \begin{align*}
  &\text{if } u_0(x) \geq 1/2 + M_0 \varepsilon \text{ then } u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \geq 1 - \eta, \\
  &\text{if } u_0(x) \leq 1/2 - M_0 \varepsilon \text{ then } u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq \eta.
  \end{align*}
  \]
(3.3)

The above theorem will be proved by constructing a suitable pair of sub and super-solutions.

### 3.1 The perturbed bistable ordinary differential equation

We first consider a slightly perturbed nonlinearity:
\[
f_\delta(u) = f(u) + \delta,
\]
where \( \delta \) is any constant. For \( |\delta| \) small enough, this function is still cubic and bistable; more precisely, we claim that it has the following properties.

**Lemma 3.2.** Let \( \delta_0 > 0 \) be small enough. Then, for all \( \delta \in (-\delta_0, \delta_0) \),
- \( f_\delta \) has exactly three zeros, namely \( \alpha_- (\delta) < a(\delta) < \alpha_+ (\delta) \). More precisely,
  \[
  f_\delta(u) = (u - \alpha_- (\delta))(\alpha_+ (\delta) - u)(u - a(\delta)),
  \]
  (3.5)

  and there exists a positive constant \( C \) such that
  \[
  |\alpha_- (\delta)| + |a(\delta) - 1/2| + |\alpha_+ (\delta) - 1| \leq C|\delta|.
  \]
(3.6)

- We have that
  \[
  \begin{align*}
  &f_\delta \text{ is strictly positive in } (-\infty, \alpha_- (\delta)) \cup (a(\delta), \alpha_+ (\delta)), \\
  &f_\delta \text{ is strictly negative in } (\alpha_- (\delta), a(\delta)) \cup (\alpha_+ (\delta), \infty).
  \end{align*}
  \]
(3.7)

- Set
  \[
  \mu(\delta) := f_\delta'(a(\delta)) = f'(a(\delta)),
  \]

  then there exists a positive constant, which we denote again by \( C \), such that
  \[
  |\mu(\delta) - \mu| \leq C|\delta|.
  \]
(3.8)

In order to construct a pair of sub and super-solutions for Problem \((P^\varepsilon)\) we define \( Y(\tau, \xi; \delta) \) as the solution of the ordinary differential equation
\[
\begin{cases}
  Y_\tau(\tau, \xi; \delta) = f_\delta(Y(\tau, \xi; \delta)) \quad \text{for } \tau > 0, \\
  Y(0, \xi; \delta) = \xi,
\end{cases}
\]
(3.9)

for \( \delta \in (-\delta_0, \delta_0) \) and \( \xi \in (-2C_0, 2C_0) \), where \( C_0 \) has been chosen in (1.2). We present below basic properties of \( Y \).
Lemma 3.3. We have $Y_\xi > 0$, for all $\xi \in (-2C_0, 2C_0) \backslash \{\alpha_- (\delta), a(\delta), \alpha_+ (\delta)\}$, all $\delta \in (-\delta_0, \delta_0)$ and all $\tau > 0$. Furthermore,

$$Y_\xi (\tau, \xi; \delta) = \frac{f_\delta (Y(\tau, \xi; \delta))}{f_\delta (\xi)}.$$  

Proof. We differentiate (3.9) with respect to $\xi$ to obtain

$$\begin{cases} Y_{\xi \tau} = Y_\xi f'(Y), \\ Y_\xi (0, \xi; \delta) = 1, \end{cases}$$

which is integrated as follows:

$$Y_\xi (\tau, \xi; \delta) = \exp \left[ \int_0^\tau f'(Y(s, \xi; \delta)) ds \right] > 0. \quad (3.10)$$

Then differentiating (3.9) with respect to $\tau$, we obtain

$$\begin{cases} Y_{\tau \tau} = Y_\tau f'(Y), \\ Y_\tau (0, \xi; \delta) = f_\delta (\xi), \end{cases}$$

which in turn implies

$$Y_\tau (\tau, \xi; \delta) = f_\delta (\xi) \exp \left[ \int_0^\tau f'(Y(s, \xi; \delta)) ds \right],$$

which enables to conclude. \hfill \Box

We define a function $A(\tau, \xi; \delta)$ by

$$A(\tau, \xi; \delta) = \frac{f'(Y(\tau, \xi; \delta)) - f'(\xi)}{f_\delta (\xi)}. \quad (3.11)$$

Lemma 3.4. We have, for all $\xi \in (-2C_0, 2C_0) \backslash \{\alpha_- (\delta), a(\delta), \alpha_+ (\delta)\}$, all $\delta \in (-\delta_0, \delta_0)$ and all $\tau > 0$,

$$A(\tau, \xi; \delta) = \int_0^\tau f'''(Y(s, \xi; \delta)) Y_\xi (s, \xi; \delta) ds.$$  

Proof. We differentiate the equality of Lemma 3.3 with respect to $\xi$ to obtain

$$Y_{\xi \xi}(\tau, \xi; \delta) = A(\tau, \xi; \delta) Y_\xi (\tau, \xi; \delta). \quad (3.12)$$

Then differentiating (3.10) with respect to $\xi$ yields

$$Y_{\xi \xi} = Y_\xi \int_0^\tau f'''(Y(s, \xi; \delta)) Y_\xi (s, \xi; \delta) ds.$$  

These two last results complete the proof of Lemma 3.4. \hfill \Box

Next we prove estimates on the growth of $Y$, $A$ and theirs derivatives. We first consider the case where the initial value $\xi$ is far from the stable equilibria, more precisely when it lies between $\eta$ and $1 - \eta$.  

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Lemma 3.5. Let $\eta$ be arbitrary. Then there exist positive constants $\delta_0 = \delta_0(\eta)$, $C_1 = \tilde{C}_1(\eta)$, $C_2 = \tilde{C}_2(\eta)$ and $C_3 = C_3(\eta)$ such that, for all $\delta \in (-\delta_0, \delta_0)$, for all $\tau > 0$,

- if $\xi \in (a(\delta), 1 - \eta)$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(a(\delta), 1 - \eta)$, we have
  \[ \bar{C}_1 e^{\mu(\delta)\tau} \leq Y_\xi(\tau, \xi; \delta) \leq \bar{C}_2 e^{\mu(\delta)\tau}, \]  \tag{3.13}
  and
  \[ |A(\tau, \xi; \delta)| \leq C_3(e^{\mu(\delta)\tau} - 1); \]  \tag{3.14}

- if $\xi \in (\eta, a(\delta))$ then, for every $\tau > 0$ such that $Y(\tau, \xi; \delta)$ remains in the interval $(\eta, a(\delta))$, (3.13) and (3.14) hold as well.

Proof. We take $\xi \in (a(\delta), 1 - \eta)$ and suppose that for $s \in (0, \tau)$, $Y(s, \xi; \delta)$ remains in the interval $(a(\delta), 1 - \eta)$. Integrating the equality

\[ \frac{Y_\xi(s, \xi; \delta)}{f_\delta(Y(s, \xi; \delta))} = 1 \]

from 0 to $\tau$ and using the change of variable $q = Y(s, \xi; \delta)$ leads to

\[ \int_\xi^{Y(\tau, \xi; \delta)} \frac{dq}{f_\delta(q)} = \tau. \]  \tag{3.15}

Moreover, the equality in Lemma 3.3 enables to write

\[ \ln Y_\xi(\tau, \xi; \delta) = \int_\xi^{Y(\tau, \xi; \delta)} \frac{f'(q)}{f_\delta(q)} dq \]
\[ = \int_\xi^{Y(\tau, \xi; \delta)} \left[ \frac{f'(a(\delta))}{f_\delta(q)} + \frac{f'(q) - f'(a(\delta))}{f_\delta(q)} \right] dq \]  \tag{3.16}
\[ = \mu(\delta)\tau + \int_\xi^{Y(\tau, \xi; \delta)} h_\delta(q) dq, \]

where

\[ h_\delta(q) = \frac{f'(q) - f'(a(\delta))}{f_\delta(q)}. \]

In view of (3.8), respectively (3.6), we can choose $\delta_0 = \delta_0(\eta) > 0$ small enough so that, for all $\delta \in [-\delta_0, \delta_0]$, we have $\mu(\delta) \geq \mu / 2 > 0$, respectively $(a(\delta), 1 - \eta] \subset (a(\delta), \sigma_+ (\delta))$. Since

\[ h_\delta(q) \rightarrow \frac{f''(a(\delta))}{f'_\delta(a(\delta))} = \frac{f''(a(\delta))}{f'(a(\delta))} \quad \text{as} \quad q \rightarrow a(\delta), \]

we see that the function $(q, \delta) \mapsto h_\delta(q)$ is continuous in the compact region \{$|\delta| \leq \delta_0, \ a(\delta) \leq q \leq 1 - \eta \}$. It follows that $|h_\delta(q)|$ is bounded by a constant.
\( H = H(\eta) \) as \((q, \delta)\) varies in this region. Since \(|Y(\tau, \xi; \delta) - \xi|\) takes its values in the interval \([0, 1 - \eta - a(\delta)]\) \(\subset [0, 1]\), it follows from (3.16) that

\[
\mu(\delta) \tau - H \leq \ln Y(\tau, \xi; \delta) \leq \mu(\delta) \tau + H,
\]

which, in turn, proves (3.13). Next Lemma 3.4 and (3.13) yield

\[
|A(\tau, \xi; \delta)| \leq \|f''\|_{L^\infty(0, 1)} \int_0^\tau \tilde{C}_2 e^{\mu(\delta)s} ds
\]

\[
\leq \frac{\|f''\|_{L^\infty(0, 1)} \tilde{C}_2}{\mu(\delta)} (e^{\mu(\delta)\tau} - 1)
\]

\[
\leq \frac{2}{\mu} \|f''\|_{L^\infty(0, 1)} \tilde{C}_2 (e^{\mu(\delta)\tau} - 1),
\]

which completes the proof of (3.14). The case where \(\xi\) and \(Y(\tau, \xi; \delta)\) are in \((\eta, a(\delta))\) is similar and omitted. \(\square\)

**Corollary 3.6.** Let \(\eta\) be arbitrary. Then there exist positive constants \(\delta_0 = \delta_0(\eta), C_1 = C_1(\eta)\) and \(C_2 = C_2(\eta)\) such that, for all \(\delta \in (-\delta_0, \delta_0)\), for all \(\tau > 0\),

- if \(\xi \in (a(\delta), 1 - \eta)\) then, for every \(\tau > 0\) such that \(Y(\tau, \xi; \delta)\) remains in the interval \((a(\delta), 1 - \eta)\), we have
  \[
  C_1 e^{\mu(\delta)\tau} (\xi - a(\delta)) \leq Y(\tau, \xi; \delta) - a(\delta) \leq C_2 e^{\mu(\delta)\tau} (\xi - a(\delta)), \quad (3.17)
  \]

- if \(\xi \in (\eta, a(\delta))\) then, for every \(\tau > 0\) such that \(Y(\tau, \xi; \delta)\) remains in the interval \((\eta, a(\delta))\), we have
  \[
  C_2 e^{\mu(\delta)\tau} (\xi - a(\delta)) \leq Y(\tau, \xi; \delta) - a(\delta) \leq C_1 e^{\mu(\delta)\tau} (\xi - a(\delta)). \quad (3.18)
  \]

**Proof.** In view of (3.3), respectively (3.6), we can choose \(\delta_0 = \delta_0(\eta) > 0\) small enough so that, for all \(\delta \in [-\delta_0, \delta_0]\), we have \(\mu(\delta) \geq \mu/2 > 0\), respectively \((a(\delta), 1 - \eta] \subset (a(\delta), \alpha_+(\delta))\). Since

\[
\frac{f_\delta(q)}{q - a(\delta)} \to \mu(\delta) \quad \text{as} \quad q \to a(\delta),
\]

it follows that \((q, \delta) \mapsto f_\delta(q)/(q - a(\delta))\) is a strictly positive and continuous function in the compact region \(\{ \delta \leq \delta_0, a(\delta) \leq q \leq 1 - \eta \}\), which insures the existence of constants \(B_1 = B_1(\eta) > 0\) and \(B_2 = B_2(\eta) > 0\) such that, for all \(q \in (a(\delta), 1 - \eta)\), all \(\delta \in (-\delta_0, \delta_0)\),

\[
B_1 (q - a(\delta)) \leq f_\delta(q) \leq B_2 (q - a(\delta)). \quad (3.19)
\]
We write the inequalities (3.19) for \( q = Y(\tau, \xi; \delta) \in (a(\delta), 1 - \eta) \) and then for \( q = \xi \in (a(\delta), 1 - \eta) \), which, together with Lemma 3.3 implies that

\[
\frac{B_1}{B_2} (Y(\tau, \xi; \delta) - a(\delta)) \leq (\xi - a(\delta)) Y_\xi(\tau, \xi; \delta) \leq \frac{B_2}{B_1} (Y(\tau, \xi; \delta) - a(\delta)).
\]

In view of (3.13), this completes the proof of inequalities (3.17). The proof of (3.18) is similar and omitted. \( \square \)

We now present estimates in the case that the initial value \( \xi \) is smaller than \( \eta \) or larger than \( 1 - \eta \).

**Lemma 3.7.** Let \( \eta \) and \( M > 0 \) be arbitrary. Then there exist positive constants \( \delta_0 = \delta_0(\eta, M) \) and \( C_4 = C_4(M) \) such that, for all \( \delta \in (-\delta_0, \delta_0) \),

- if \( \xi \in [1 - \eta, 1 + M] \), then, for all \( \tau > 0 \), \( Y(\tau, \xi; \delta) \) remains in the interval \([1 - \eta, 1 + M]\) and
  \[
  |A(\tau, \xi; \delta)| \leq C_4 \tau \quad \text{for} \quad \tau > 0; \quad (3.20)
  \]
- if \( \xi \in [-M, \eta] \), then, for all \( \tau > 0 \), \( Y(\tau, \xi; \delta) \) remains in the interval \([-M, \eta]\) and (3.20) holds as well.

**Proof.** Since the two statements can be treated in the same way, we will only prove the former. The fact that \( Y(\tau, \xi; \delta) \), the solution of the ordinary differential equation (3.9), remains in the interval \([1 - \eta, 1 + M]\) directly follows from the bistable properties of \( f_\delta \), or, more precisely, from the sign conditions \( f_\delta(1 - \eta) > 0 \), \( f_\delta(1 + M) < 0 \) valid if \( \delta_0 = \delta_0(\eta, M) \) is small enough.

To prove (3.20), suppose first that \( \xi \in [\alpha_+(\delta), 1 + M] \). By the above arguments, \( Y(\tau, \xi; \delta) \) remains in this interval. Moreover \( f' \) is negative in this interval. Hence, it follows from (3.14) that \( Y_\xi(\tau, \xi; \delta) \leq 1 \). We then use Lemma 3.4 to deduce that

\[
|A(\tau, \xi; \delta)| \leq \|f''\|_{L^\infty(-M, 1+M)} \tau =: C_4 \tau.
\]

The case \( \xi \in [1 - \eta, \alpha_+(\delta)] \) being similar, this completes the proof of the lemma. \( \square \)

Now we choose the constant \( M \) in the above lemma sufficiently large so that \([-2C_0, 2C_0] \subset [-M, 1 + M] \), and fix \( M \) hereafter. Therefore the constant \( C_4 \) is fixed as well. Using the fact that \( \tau \mapsto \tau(e^{\mu(\delta) \tau} - 1)^{-1} \) is uniformly bounded for \( \delta \in (-\delta_0, \delta_0) \), with \( \delta_0 \) small enough (see (3.8)), and for \( \tau > 0 \), one can easily deduce from (3.14) and (3.20) the following general estimate.

**Lemma 3.8.** Let \( \eta \) be arbitrary and let \( C_0 \) be the constant defined in (1.2). Then there exist positive constants \( \delta_0 = \delta_0(\eta) \), \( C_5 = C_5(\eta) \) such that, for all \( \delta \in (-\delta_0, \delta_0) \), all \( \xi \in (-2C_0, 2C_0) \) and all \( \tau > 0 \),

\[
|A(\tau, \xi; \delta)| \leq C_5(e^{\mu(\delta) \tau} - 1).
\]
3.2 Construction of sub and super-solutions

We now use $Y$ to construct a pair of sub- and super-solutions for the proof of the generation of interface theorem. We set

$$w_\varepsilon^\pm(x,t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 \text{r}(\pm \varepsilon G, \frac{t}{\varepsilon^2}); \pm \varepsilon G\right), \quad (3.21)$$

where the constant $G$ is defined by

$$G = \sup_{u \in [-2C_0, 2C_0]} |g(u)|,$$

and the function $r(\delta, \tau)$ is given by

$$r(\delta, \tau) = C_6 \left(e^{\mu(\delta)\tau} - 1\right).$$

For simplicity, we make the following additional assumption:

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega. \quad (3.22)$$

In the general case where $(3.22)$ does not necessary hold, we have to slightly modify $w_\varepsilon^\pm$ near the boundary $\partial \Omega$. This will be discussed in the next remark.

**Lemma 3.9.** There exist positive constants $\varepsilon_0$ and $C_{15}$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the functions $w_\varepsilon^-$ and $w_\varepsilon^+$ are respectively sub- and super-solutions for Problem $(P_\varepsilon)$, in the domain

$$\{(x,t) \in Q_T, \ x \in \Omega, \ 0 \leq t \leq \mu^{-1} \varepsilon^2 \ln \varepsilon\}.$$

**Proof.** First, $(3.22)$ implies the homogeneous Neumann boundary condition

$$\frac{\partial w_\varepsilon^\pm}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0, +\infty).$$

Let $v_\varepsilon$ be such that

$$\begin{cases}
    w_\varepsilon^- \leq -\Delta v_\varepsilon + \gamma v_\varepsilon \leq w_\varepsilon^+ \\
    \frac{\partial v_\varepsilon}{\partial \nu} = 0.
\end{cases} \quad (3.23)$$

According to Definition 2.2, what we have to show is

$$L_{v_\varepsilon}[w_\varepsilon^+] := (w_\varepsilon^+)_t - \Delta w_\varepsilon^+ + \nabla \cdot (w_\varepsilon^+ \nabla (v_\varepsilon^+)) - \frac{1}{\varepsilon^2} f_\varepsilon(w_\varepsilon^+) \geq 0.$$

Let $C_{15}$ be a positive constant which does not depend on $\varepsilon$. If $\varepsilon_0$ is sufficiently small, we note that $\pm \varepsilon G \in (-\delta_0, \delta_0)$ and that, in the range $0 \leq t \leq \mu^{-1} \varepsilon^2 \ln \varepsilon$,

$$|\varepsilon^2 C_{15}(\mu(\pm \varepsilon G) t/\varepsilon^2 - 1)| \leq \varepsilon^2 C_{15}(\varepsilon^{-\mu(\pm \varepsilon G)}/\mu - 1) \leq C_0,$$
which implies that

\[ u_0(x) \pm \epsilon^2 r(\pm \epsilon G, t/\epsilon^2) \in (-2C_0, 2C_0). \]

These observations allow us to use the results of the previous subsection with \( \tau = t/\epsilon^2 \), \( \xi = u_0(x) + \epsilon^2 r(\epsilon G, t/\epsilon^2) \) and \( \delta = \epsilon G \). In particular, setting \( F_1 := \| f' \|_{L^\infty(-2C_0, 2C_0)} \), this implies, using (3.10), that

\[ e^{-F_1 T} \leq Y_\xi \leq e^{F_1 T}. \]

Straightforward computations yield

\[
L_{\epsilon^2}[w^+_{\xi}] = \frac{1}{\epsilon^2} Y_\tau + C_0 \mu(\epsilon G)e^{\mu(\epsilon G)t/\epsilon^2} Y_\xi - \Delta u_0 Y_\xi - |\nabla u_0|^2 Y_{\xi\xi} \\
+ Y_\xi \nabla u_0 \cdot \nabla \chi(\epsilon^2) + Y \Delta \chi(\epsilon^2) - \frac{1}{\epsilon^2} f(Y) - \frac{1}{\epsilon} g(Y),
\]

and then, in view of the ordinary differential equation (3.9), \( \epsilon G \) playing the role of \( \delta \),

\[
L_{\epsilon^2}[w^+_{\xi}] = \frac{1}{\epsilon}[G - g(Y)] + Y_\xi \left[C_0 \mu(\epsilon G)e^{\mu(\epsilon G)t/\epsilon^2} - \Delta u_0 - \frac{Y_\xi}{Y_{\xi}} |\nabla u_0|^2 + Y \Delta \chi(\epsilon^2) \right].
\]

By the definition of \( G \) the first term above is positive. Now, using the choice of \( C_0 \) in (1.2), the fact that \( Y_{\xi\xi}/Y_\xi = A \) and Lemma 3.8, we obtain, for a \( C_5 \) independent of \( \epsilon \),

\[
L_{\epsilon^2}[w^+_{\xi}] \geq Y_\xi \left[C_0 \mu(\epsilon G)e^{\mu(\epsilon G)t/\epsilon^2} - C_0 - C_5 \delta \mu(\epsilon G)t/\epsilon^2 - 1)C_0^2 \\
- C_0 |\Delta \chi(\epsilon^2)| - 2C_0 e^{F_1 T} |\Delta \chi(\epsilon^2)| \right].
\]

Moreover, the inequalities in (3.23) can be written as \(-\Delta v^\epsilon + \gamma v^\epsilon = h^\epsilon \), with \(-2C_0 \leq h^\epsilon \leq 2C_0 \), so that the standard theory of elliptic equations gives a uniform bound \( M \) for \( |v^\epsilon|, |\nabla v^\epsilon| \) and \( |\Delta v^\epsilon| \). Hence, using the smoothness of \( \chi \), we have a uniform bound \( M' \) for \( |\nabla \chi(\epsilon^2)| \) and \( |\Delta \chi(\epsilon^2)| \). It follows that

\[
L_{\epsilon^2}[w^+_{\xi}] \geq Y_\xi \left[(C_0 \mu(\epsilon G) - C_5 C_0^2)\epsilon^{\mu(\epsilon G)t/\epsilon^2} - C_0 + C_5 C_0^2 - C_0 M' - 2C_0 e^{F_1 T} M' \right].
\]

Hence, in view of (3.8), we have, for \( \epsilon_0 \) small enough (recall that \( Y_\xi > 0 \)),

\[
L w^+_{\xi} \geq Y_\xi \left[(C_0 \frac{1}{2} \mu - C_5 C_0^2) - C_0 - C_0 M' - 2C_0 e^{F_1 T} M' \right] \geq 0,
\]

for \( C_0 \) large enough, so that \( w^+_{\xi} \) is a super-solution for Problem \((P^\epsilon)\). We omit the proof that \( w^-_{\xi} \) is a sub-solution. \( \square \)
Now, since $w^\pm_\varepsilon(x,0) = Y(0,u_0(x);\pm\varepsilon G) = u_0(x)$, the comparison principle set in Proposition 2.3 asserts that, for all $x \in \Omega$, for all $0 \leq t \leq \mu^{-1}\varepsilon^2\ln\varepsilon$,
\[ w^\pm_\varepsilon(x,t) \leq u^\varepsilon(x,t) \leq w^\pm_\varepsilon(x,t). \tag{3.25} \]

Remark 3.10. In the more general case where (3.22) is not valid, one can proceed in the following way: in view of (1.3) and (1.4) there exist positive constants $d_1$ and $\rho$ such that $u_0(x) \leq 1/2 - \rho$ if $d(x,\partial\Omega) \leq d_1$. Let $\chi$ be a smooth cut-off function defined on $[0, +\infty)$ such that $0 \leq \chi \leq 1$, $\chi(0) = \chi'(0) = 0$ and $\chi(z) = 1$ for $z \geq d_1$. Then define
\[ u^+_0(x) = \chi(d(x,\partial\Omega))u_0(x) + (1 - \chi(d(x,\partial\Omega))(1/2 - \rho), \]
\[ u^-_0(x) = \chi(d(x,\partial\Omega))u_0(x) + (1 - \chi(d(x,\partial\Omega))\min_{x \in \Omega} u_0(x). \]

Clearly, $u^-_0 \leq u_0 \leq u^+_0$, and both $u^\pm_0$ satisfy (3.22). Now we set
\[ \tilde{w}^\pm_\varepsilon(x,t) = Y\left(\frac{t}{\varepsilon^2}, u^\pm_0(x) \pm \varepsilon^2\eta(\pm\varepsilon G, \frac{t}{\varepsilon^2}); \pm\varepsilon G\right). \]

Then the same argument as in Lemma 3.9 shows that $(\tilde{w}^-_\varepsilon, \tilde{w}^+_\varepsilon)$ is a pair of sub and super-solutions for Problem $(P^\varepsilon)$. Furthermore, since $\tilde{w}^-_\varepsilon(x,0) = u^-_0(x) \leq u_0(x) \leq u^+_0(x) = \tilde{w}^+_\varepsilon(x,0)$, Proposition 2.3 asserts that, for all $x \in \Omega$, for all $0 \leq t \leq \mu^{-1}\varepsilon^2\ln\varepsilon$; we have $\tilde{w}^-_\varepsilon(x,t) \leq u^\varepsilon(x,t) \leq \tilde{w}^+_\varepsilon(x,t)$. □

### 3.3 Proof of Theorem 3.1

In order to prove Theorem 3.1, we first present a key estimate on the function $Y$ after a time of order $\tau \sim |\ln\varepsilon|$.

**Lemma 3.11.** Let $\eta$ be arbitrary; there exist positive constants $\varepsilon_0 = \varepsilon_0(\eta)$ and $C_7 = C_7(\eta)$ such that, for all $\varepsilon \in (0,\varepsilon_0)$,
\[ \bullet \text{ for all } \xi \in (-2C_0,2C_0), \]
\[ -\eta \leq Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \leq 1 + \eta, \tag{3.26} \]
\[ \bullet \text{ for all } \xi \in (-2C_0,2C_0) \text{ such that } |\xi - \frac{\eta}{2}| \geq C_7\varepsilon, \text{ we have that} \]
\[ \begin{align*}
\text{if } \xi \geq 1/2 + C_7\varepsilon & \text{ then } Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \geq 1 - \eta, \tag{3.27} \\
\text{if } \xi \leq 1/2 - C_7\varepsilon & \text{ then } Y(\mu^{-1}|\ln\varepsilon|, \xi; \pm\varepsilon G) \leq \eta. \tag{3.28}
\end{align*} \]

**Proof.** We first prove (3.27). In view of (3.6), we have, for $C_7$ large enough, $1/2 + C_7\varepsilon \geq a(\pm\varepsilon G) + \frac{1}{2}C_7\varepsilon$, for all $\varepsilon \in (0,\varepsilon_0)$, with $\varepsilon_0$ small enough. Hence
for $\xi \geq 1/2 + C_7 \varepsilon$, as long as $Y(\tau, \xi; \pm \varepsilon G)$ has not reached $1 - \eta$, we can use (3.17) to deduce that

$$Y(\tau, \xi; \pm \varepsilon G) \geq a(\pm \varepsilon G) + C_1 e^{\mu(\pm \varepsilon G)\tau}(\xi - a(\pm \varepsilon G))$$

$$\geq a(\pm \varepsilon G) + \frac{1}{2} C_1 C_7 \varepsilon e^{\mu(\pm \varepsilon G)\tau}$$

$$\geq \frac{1}{2} - \varepsilon CG + \frac{1}{2} C_1 C_7 \varepsilon e^{\mu(\pm \varepsilon G)\tau}$$

$$\geq 1 - \eta$$

provided that

$$\tau \geq \tau^e := \frac{1}{\mu(\pm \varepsilon G)^2} \ln \frac{1/2 - \eta + CG\varepsilon}{C_1 C_7 \varepsilon / 2}.$$ 

To complete the proof of (3.27) we must choose $C_7$ so that $\mu - 1 \lvert \ln \varepsilon \rvert - \tau^e \geq 0$.

A simple computation shows that

$$\mu - 1 \lvert \ln \varepsilon \rvert - \tau^e = \frac{\mu(\pm \varepsilon G) - \mu}{\mu(\pm \varepsilon G)^2} \lvert \ln \varepsilon \rvert - \frac{1}{\mu(\pm \varepsilon G)} \ln \frac{1/2 - \eta + CG\varepsilon}{C_1 / 2}$$

$$+ \frac{1}{\mu(\pm \varepsilon G)} \ln C_7.$$ 

Thanks to (3.8), as $\varepsilon \to 0$, the first term above is of order $\varepsilon \lvert \ln \varepsilon \rvert$ and the second one of order 1. Hence, for $C_7$ large enough, the quantity $\mu - 1 \lvert \ln \varepsilon \rvert - \tau^e$ is positive, for all $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0$ small enough. The proof of (3.28) is similar and omitted.

Next we prove (3.26). First note that, by taking $\varepsilon_0$ small enough, the stable equilibria of $f_{\pm \varepsilon G}$, namely $\alpha_-(\pm \varepsilon G)$ and $\alpha_+(\pm \varepsilon G)$, are in $[-\eta, 1 + \eta]$. Hence, $f_{\pm \varepsilon G}$ being a bistable function, if we leave from a $\xi \in [-\eta, 1 + \eta]$ then $Y(\tau, \xi; \pm \varepsilon G)$ will remain in the interval $[-\eta, 1 + \eta]$. Now suppose that

$$1 + \eta \leq \xi \leq 2C_0 \quad \text{(note that this work is useless if } 2C_0 < 1 + \eta).$$

We check below that $Y(\mu - 1 \lvert \ln \varepsilon \rvert, \xi; \pm \varepsilon G) \leq 1 + \eta$. As long as $1 + \eta \leq Y \leq 2C_0$, (3.9) leads to the inequality $Y \leq f(1 + \eta) + \varepsilon G \leq f(1 + \eta) < 0$, for $\varepsilon_0 = \varepsilon_0(\eta)$ small enough. By integration from 0 to $\tau$, it follows that

$$Y(\tau, \xi; \pm \varepsilon G) \leq \xi + \frac{1}{2} f(1 + \eta)\tau$$

$$\leq 2C_0 + \frac{1}{2} f(1 + \eta)\tau$$

$$\leq 1 + \eta,$$

provided that

$$\tau \geq \frac{2C_0 - 1 - \eta}{-f(1 + \eta)/2},$$

and a fortiori for $\tau = \mu - 1 \lvert \ln \varepsilon \rvert$, which completes the proof of (3.26). \qed
We are now ready to prove Theorem 3.1. By setting \( t = \mu^{-1} \varepsilon^2 |\ln \varepsilon| \) in (3.25), we obtain
\[
Y \left( \mu^{-1} |\ln \varepsilon|, u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1} |\ln \varepsilon|); -\varepsilon G \right)
\leq u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq Y \left( \mu^{-1} |\ln \varepsilon|, u_0(x) + \varepsilon^2 r(\varepsilon G, \mu^{-1} |\ln \varepsilon|); +\varepsilon G \right).
\]
(3.29)

In view of (3.8),
\[
\lim_{\varepsilon \to 0} \frac{\mu - \mu(\pm \varepsilon G)}{\mu} \ln \varepsilon = 0,
\]
(3.30)
so that, for \( \varepsilon_0 \) small enough, we have
\[
\varepsilon^2 r(\pm \varepsilon G, \mu^{-1} |\ln \varepsilon|) = C_6 \varepsilon \left( \frac{\mu - \mu(\pm \varepsilon G)}{\mu} \right) \varepsilon \in \left( \frac{1}{2} C_6 \varepsilon, \frac{3}{2} C_6 \varepsilon \right).
\]

It follows that \( u_0(x) \pm \varepsilon^2 r(\pm \varepsilon G, \mu^{-1} |\ln \varepsilon|) \in (-2C_0, 2C_0) \). Hence the result (3.2) of Theorem 3.1 is a direct consequence of (3.26) and (3.29).

Next we prove (3.3). We take \( x \in \Omega \) such that \( u_0(x) \geq 1/2 + M_0 \varepsilon \) so that
\[
u_0(x) - \varepsilon^2 r(-\varepsilon G, \mu^{-1} |\ln \varepsilon|) \geq 1/2 + M_0 \varepsilon - \frac{3}{2} C_6 \varepsilon
\geq 1/2 + C_7 \varepsilon,
\]
if we choose \( M_0 \) large enough. Using (3.29) and (3.27) we obtain (3.3), which completes the proof of Theorem 3.1.

4 Motion of interface

We have seen in Section 3 that, after a very short time, the solution \( u^\varepsilon \) develops a clear transition layer. In the present section, we show that it persists and that its law of motion is well approximated by the interface equation in \( (P^0) \) obtained by formal asymptotic expansions in subsection 2.1.

More precisely, take the first term of the formal asymptotic expansion (2.3) as a formal expansion of the solution:
\[
u^\varepsilon(x, t) \approx \tilde{u}^\varepsilon(x, t) := U_0 \left( \frac{\tilde{d}(x, t)}{\varepsilon} \right),
\]
(4.1)
where \( U_0 \) is defined in (2.6). The right-hand side is a function having a well-developed transition layer, and its interface lies exactly on \( \Gamma_t \). We show that this function is a good approximation of the solution; more precisely:

*If \( u^\varepsilon \) becomes very close to \( \tilde{u}^\varepsilon \) at some time moment \( t = t_0 \), then it stays close to \( \tilde{u}^\varepsilon \) for the rest of time. Consequently, \( \Gamma^\varepsilon_t \) evolves roughly like \( \Gamma_t \).*
To that purpose, we will construct a pair of sub- and super-solutions \( u^- \) and \( u^+ \) for Problem \( (P^\varepsilon) \) by slightly modifying \( \tilde{u}^\varepsilon \). It then follows that, if the solution \( u^\varepsilon \) satisfies
\[
u^-(x,t_0) \leq u^\varepsilon(x,t_0) \leq u^+(x,t_0),
\]
for some \( t_0 \geq 0 \), then
\[
u^-(x,t) \leq u^\varepsilon(x,t) \leq u^+(x,t),
\]
for \( t_0 \leq t \leq T \). As a result, since both \( u^+, u^- \) stay close to \( \tilde{u}^\varepsilon \), the solution \( u^\varepsilon \) also stays close to \( \tilde{u}^\varepsilon \) for \( t_0 \leq t \leq T \).

### 4.1 Construction of sub- and super-solutions

To begin with we present mathematical tools which are essential for the construction of sub and super-solutions.

**A modified signed distance function.** Rather than working with the usual signed distance function \( \tilde{d} \), defined in (2.1), we define a “cut-off signed distance function” \( d \) as follows. Choose \( d_0 > 0 \) small enough so that \( \tilde{d}(\cdot,\cdot) \) is smooth in the tubular neighborhood of \( \Gamma \)
\[
\{(x,t) \in \overline{Q_T}, |\tilde{d}(x,t)| < 3d_0\},
\]
and such that
\[
dist(\Gamma_t, \partial \Omega) > 4d_0 \quad \text{for all } t \in [0,T].
\] (4.2)

Next let \( \zeta(s) \) be a smooth increasing function on \( \mathbb{R} \) such that
\[
\zeta(s) = \begin{cases} 
  s & \text{if } |s| \leq 2d_0 \\
  -3d_0 & \text{if } s \leq -3d_0 \\
  3d_0 & \text{if } s \geq 3d_0.
\end{cases}
\]

We define the cut-off signed distance function \( d \) by
\[
d(x,t) = \zeta(\tilde{d}(x,t)).
\] (4.3)

Note that \(|\nabla d| = 1\) in the region \( \{(x,t) \in \overline{Q_T}, |d(x,t)| < 2d_0\} \) and that, in view of the above definition, \( \nabla d = 0 \) in a neighborhood of \( \partial \Omega \). Note also that the equation of motion interface in \( (P^0) \), which is equivalent to (2.9), is now written as
\[
d_t = \Delta d - \nabla d \cdot \nabla \chi(v^0) - \sqrt{2} \alpha \quad \text{on } \Gamma_t.
\] (4.4)

**Construction.** We look for a pair of sub- and super-solutions \( u^\pm \) for Problem \( (P^\varepsilon) \) of the form
\[
u^\pm(x,t) = U_0\left(\frac{d(x,t) \mp \varepsilon p(t)}{\varepsilon}\right) \pm q(t),
\] (4.5)
where $U_0$ is the solution of (2.6), and where
\[
\begin{align*}
p(t) &= -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \\
q(t) &= \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 Le^Lt).
\end{align*}
\]
Note that $q = \sigma \varepsilon^2 p_t$. Let us remark that the construction (4.5) is more precise than the procedure of only taking a zeroth order term of the form $U_0$, since we have shown in the formal derivation that the first order term $U_1$ in (2.3) vanishes. It is clear from the definition of $u^\pm_\varepsilon$ that
\[
\lim_{\varepsilon \to 0} u^\pm_\varepsilon(x, t) = \begin{cases} 
1 & \text{for all } (x, t) \in Q^1_T \\
0 & \text{for all } (x, t) \in Q^0_T.
\end{cases}
\]

The main result of this section is the following.

**Lemma 4.1.** There exist positive constants $\beta, \sigma$ with the following properties. For any $K > 1$, we can find positive constants $\varepsilon_0$ and $L$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, the functions $u^-_\varepsilon$ and $u^+_\varepsilon$ are respectively sub- and supersolutions for Problem $(P^\varepsilon)$ in the range $x \in \Omega, 0 \leq t \leq T$.

### 4.2 Proof of Lemma 4.1

First, since $\nabla d = 0$ in a neighborhood of $\partial \Omega$, we have the homogeneous Neumann boundary condition
\[
\frac{\partial u^\pm_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial \Omega \times [0, T].
\]
Let $v^\varepsilon$ be such that (2.11) holds. We have to show that
\[
L_{v^\varepsilon}[u^+_\varepsilon] := (u^+_\varepsilon)_t - \Delta u^+_\varepsilon + \nabla u^+_\varepsilon \cdot \nabla \chi(v^\varepsilon) + u^+_\varepsilon \Delta \chi(v^\varepsilon) - \frac{1}{\varepsilon^2} f'_{\varepsilon}(u^+_\varepsilon) \geq 0,
\]
the proof of inequality $L_{v^\varepsilon}[u^-_\varepsilon] \leq 0$ following by the same arguments.

**Computation of $L_{v^\varepsilon}[u^+_\varepsilon]$.** By straightforward computations we obtain the following terms:
\[
(u^+_\varepsilon)_t = U_0'(\frac{dt}{\varepsilon} - p_t) + q_t,
\]
\[
\nabla u^+_\varepsilon = U_0' \frac{\nabla d}{\varepsilon},
\]
\[
\Delta u^+_\varepsilon = U_0'' \frac{(|\nabla d|^2)}{\varepsilon^2} + U_0 \frac{\Delta d}{\varepsilon},
\]
where the function $U_0$, as well as its derivatives, is taken at the point $(d(x, t) - \varepsilon p(t))/\varepsilon$. We also use expansions of the reaction terms:
\[
f(u^+_\varepsilon) = f(U_0) + qf'(U_0) + \frac{1}{2} q^2 f''(\theta),
\]
\[
g(u^+_\varepsilon) = g(U_0) + qg'(\omega),
\]
where \( \theta(x,t) \) and \( \omega(x,t) \) are some functions satisfying \( U_0 < \theta < u^+_\varepsilon \), \( U_0 < \omega < u^+_\varepsilon \). Combining the above expressions with equation (2.6) and the fact that \( \sqrt{2\alpha U'_0} + g(U_0) \equiv 0 \), we obtain

\[
L_{\nu^e}[u^+_\varepsilon] = E_1 + \cdots + E_5,
\]

where:

\[
E_1 = -\frac{1}{\varepsilon^2} q[f'(U_0) + \frac{1}{2}qf''(\theta)] - U'_0 p_t + q_t,
\]

\[
E_2 = \frac{U''_0}{\varepsilon^2} (1 - |\nabla d|^2),
\]

\[
E_3 = \frac{U'_0}{\varepsilon} (d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) + \sqrt{2\alpha}),
\]

\[
E_4 = -\frac{1}{\varepsilon} g'(\omega),
\]

\[
E_5 = \frac{U'_0}{\varepsilon} \nabla d \cdot \nabla (\chi(v^\varepsilon) - \chi(v^0)) + u^+_\varepsilon \Delta \chi(v^\varepsilon).
\]

In order to estimate the terms above, we first present some useful inequalities. As \( f'(0) = f'(1) = -1/2 \), we can find strictly positive constants \( b \) and \( m \) such that

\[
\text{if } U_0(z) \in [0, b] \cup [1 - b, 1] \text{ then } f'(U_0(z)) \leq -m. \tag{4.8}
\]

On the other hand, since the region \( \{ z \in \mathbb{R} \mid U_0(z) \in [b, 1 - b] \} \) is compact and since \( U'_0 < 0 \) on \( \mathbb{R} \), there exists a constant \( a_1 > 0 \) such that

\[
\text{if } U_0(z) \in [b, 1 - b] \text{ then } U'_0(z) \leq -a_1. \tag{4.9}
\]

We then define

\[
F = \sup_{-1 \leq z \leq 2} |f(z)| + |f'(z)| + |f''(z)|, \tag{4.10}
\]

\[
\beta = \frac{m}{4}, \tag{4.11}
\]

and choose \( \sigma \) that satisfies

\[
0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2), \tag{4.12}
\]

where

\[
\sigma_0 := \frac{a_1}{m + F}, \quad \sigma_1 := \frac{1}{\beta + 1}, \quad \sigma_2 := \frac{4\beta}{F(\beta + 1)}.
\]

Hence, combining (4.8) and (4.9), we obtain, using that \( \sigma \leq \sigma_0 \),

\[
-U'_0(z) - \sigma f'(U_0(z)) \geq 4\sigma \beta \quad \text{for } -\infty < z < \infty. \tag{4.13}
\]
Now let $K > 1$ be arbitrary. In what follows we will show that $L e^\varepsilon[u_\varepsilon^+] \geq 0$ provided that the constants $\varepsilon_0$ and $L$ are appropriately chosen. From now on, we suppose that the following inequality is satisfied:

$$\varepsilon_0^2 L e^{LT} \leq 1.$$  \hspace{1cm} (4.14)

Then, given any $\varepsilon \in (0, \varepsilon_0)$, since $\sigma \leq \sigma_1$, we have $0 \leq q(t) \leq 1$, hence, recalling that $0 < U_0 < 1$,

$$-1 \leq u_\varepsilon^+(x,t) \leq 2.$$  \hspace{1cm} (4.15)

**We first estimate the term $E_1$.** A direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2} (I - \sigma \beta) + L e^{LT} (I + \varepsilon^2 \sigma L),$$

where

$$I = -U_0' - \sigma f'(U_0) - \frac{\sigma^2}{2} f''(\theta) (\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{LT}).$$

In virtue of (4.13) and (4.15), we obtain

$$I \geq 4\sigma \beta - \frac{\sigma^2}{2} F(\beta + \varepsilon^2 L e^{LT}).$$

Then, in view of (4.14), using that $\sigma \leq \sigma_2$, we have

$$I \geq 2\sigma \beta.$$  

Consequently, the following inequality holds.

$$E_1 \geq \frac{\sigma \beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma \beta L e^{LT} =: \frac{C_1}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + C_1' L e^{LT}.$$  

The term $E_2$. First, in the points where where $|d(x,t)| < d_0$, we have that $|\nabla d| = 1$ so that $E_2 = 0$. Next we consider the points where $|d(x,t)| \geq d_0$. We deduce from Lemma 2.1 that:

$$|E_2| \leq \frac{C}{\varepsilon^2} (1 + \|d\|_{\infty}^2) e^{-\lambda |d + \varepsilon p|/\varepsilon} \leq \frac{C}{\varepsilon^2} (1 + \|d\|_{\infty}^2) e^{-\lambda (d_0/\varepsilon - |p|)}.$$  

In view of the definition of $p$ in (4.6), we have that $0 < K - 1 \leq p \leq e^{LT} + K$, and suppose from now that the following assumption holds:

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}.$$  \hspace{1cm} (4.16)}
Then \( \frac{d_0}{\varepsilon} - |p| \geq \frac{d_0}{2\varepsilon} \), so that

\[
|E_2| \leq \frac{C}{\varepsilon^2} (1 + \|\nabla d\|_\infty^2) e^{-\lambda d_0/(2\varepsilon)} \\
\leq C_2 := \frac{16C}{(e\lambda d_0)^2} (1 + \|\nabla d\|_\infty^2).
\]

Next we consider the term \( E_3 \). We recall that

\[
d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) + \sqrt{2} \alpha = 0 \quad \text{on} \quad \Gamma_t = \{ x \in \Omega, d(x, t) = 0 \}.
\]

Since \( v^0 \) is of class \( C^{1+\vartheta', \frac{1+\vartheta'}{2}} \), for any \( \vartheta' \in (0, 1) \), and since the interface \( \Gamma_t \) is of class \( C^{2+\vartheta, \frac{2+\vartheta}{2}} \), the functions \( \nabla d, \Delta d, d_t \) and \( \nabla \chi(v^0) \) are Lipschitz continuous near \( \Gamma_t \). It then follows, from the mean value theorem applied separately on both sides of \( \Gamma_t \), that there exists \( N_0 > 0 \) such that:

\[
|(d_t - \Delta d + \nabla d \cdot \nabla \chi(v^0) + \sqrt{2} \alpha)(x, t)| \leq N_0 |d(x, t)| \quad \text{for all} \quad (x, t) \in Q_T.
\]

Applying Lemma 2.1 we deduce that

\[
|E_3| \leq N_0 C \left( \frac{|d(x, t)|}{\varepsilon} e^{-\lambda |d(x, t)|/\varepsilon} |p(t)| \right) \\
\leq N_0 C \max_{y \in \mathbb{R}} |y| e^{-\lambda |y| + p(t)} \\
\leq N_0 C \left( |p(t)|, \frac{1}{\lambda} \right) \\
\leq N_0 C \left( |p(t)| + \frac{1}{\lambda} \right).
\]

Taking the expression of \( p \) into account, we see that \( |p(t)| \leq e^{Lt} + K \), which implies

\[
|E_3| \leq C_3 (e^{Lt} + K) + C_3',
\]

where \( C_3 := N_0 C \) and \( C_3' := N_0 C/\lambda \).

**The term \( E_4 \).** We set \( G_1 := \|g'\|_{L^\infty(-1, 2)} \) and, substituting the expression for \( q \), obtain that

\[
|E_4| \leq \sigma G_1 \left( \frac{\beta}{\varepsilon} e^{-\beta t/\varepsilon} + \varepsilon Le^{Lt} \right) \\
\leq \frac{C_4}{\varepsilon} e^{-\beta t/\varepsilon^2} + C_4' \varepsilon Le^{Lt}.
\]

**We continue with the term \( E_5 \).** This term requires a more delicate analysis. We need a precise estimate of \( v^\varepsilon - v^0 \). We recall that \( v^0 \) satisfies \(-\Delta v^0 + \gamma v^0 = u^0 \), with \( u^0 \) a step function discontinuous when crossing the interface.
Lemma 4.2. There exists a positive constant $C_G$ such that, for all $(x,t) \in Q_T$,
\begin{equation}
\left( |v^\varepsilon| + |\nabla v^\varepsilon| + |\Delta v^\varepsilon| \right)(x,t) \leq C_G,
\end{equation}
\begin{equation}
\left( |v^\varepsilon - v^0| + |\nabla d \cdot \nabla (v^\varepsilon - v^0)| \right)(x,t) \leq C_G(\varepsilon p(t) + q(t)).
\end{equation}
We postpone the proof of this lemma and pursue the proof of Lemma 4.1. Using the smoothness of $\chi$ and (4.17), we obtain a uniform bound $C_G'$ for $\Delta \chi(v^\varepsilon)$. Moreover, we write
\begin{equation}
\nabla d \cdot \nabla \left( \chi(v^\varepsilon) - \chi(v^0) \right) = \chi'(v^\varepsilon) \nabla d \cdot \nabla (v^\varepsilon - v^0) + \left( \chi'(v^\varepsilon) - \chi'(v^0) \right) \nabla d \cdot \nabla v^0.
\end{equation}
Since $v^0$ is of class $C^{1+\vartheta', \frac{1+\vartheta'}{2}}$, for any $\vartheta' \in (0,1)$, there exists a constant, which we denote again by $C_G'$, such that
\begin{equation}
\|v^0\|_{L^\infty(Q_T)} + \|\nabla v^0\|_{L^\infty(Q_T)} \leq C_G.
\end{equation}
which, combined with (4.19), yields
\begin{equation}
|\nabla d \cdot \nabla \left( \chi(v^\varepsilon) - \chi(v^0) \right)(x,t)| \leq C_G''(\varepsilon p(t) + q(t)).
\end{equation}
Hence, using (4.15) and the above estimates, we obtain,
\begin{equation}
|E_5| \leq C \frac{C_G'' \varepsilon}{\varepsilon} (\varepsilon p(t) + q(t)) + 2C_G'.
\end{equation}
Then, substituting the expressions for $p$ and $q$, we easily obtain positive constants $C_5, C_5'$ and $C_5''$ such that
\begin{equation}
|E_5| \leq C_5 + \frac{C_5'}{\varepsilon} e^{-\beta t/\varepsilon^2} + C_5''(1 + \varepsilon L)e^{Lt}.
\end{equation}
Completion of the proof. Collecting the above estimates of $E_1$—$E_5$ yields
\begin{equation}
L_{u^\varepsilon}[u^\varepsilon] \geq \frac{C_1 - \varepsilon C_4}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + \left( L(C_1' - \varepsilon C_4' - \varepsilon C_5'') - C_3 - C_5'' \right) e^{Lt} - C_7,
\end{equation}
where $C_7 := C_2 + KC_3 + C_3' + C_5$. Now, we set
\begin{equation}
L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0},
\end{equation}
25
which, for $\varepsilon_0$ small enough, validates assumptions (4.14) and (4.16). If $\varepsilon_0$ is chosen sufficiently small (i.e. $L$ large enough), $C_1/\varepsilon^2 - (C_4 + C_5')/\varepsilon$ is positive, $C_1 - \varepsilon C_4' - \varepsilon C_5'' \geq \frac{1}{2} C_1'$, and

$$L_{uv}[u^+_{\varepsilon}] \geq \left[\frac{1}{2} L C_1' - C_3 - C_5''\right] e^{Lt} - C_7 \geq \frac{1}{2} L C_1' - C_7 \geq 0.$$ 

The proof of Lemma 4.1 is now completed, with the choice of the constants $\beta, \sigma$ as in (4.11), (4.12).

### 4.3 Proof of Lemma 4.2

Lemma 4.2 is inspired by Lemma 4.9 in [4]. Since our pair of sub- and super-solutions is different from the one in [4], we need to perform some minor changes. First we give a useful estimate on “shifted $U_0$”.

**Lemma 4.3.** For all $a \in \mathbb{R}$, all $z \in \mathbb{R}$, we have

$$|U_0(z + a) - \chi_{[-\infty,0]}(z)| \leq Ce^{-\lambda |z+a|} + \chi_{[-a,a]}(z).$$

**Proof.** Let us give the proof for $a > 0$. We distinguish three cases and use the estimates of Lemma 2.1. For $z \leq -a$, we have $|U_0(z + a) - 1| \leq Ce^{-\lambda |z+a|}$. For $-a < z \leq 0$, we have $|U_0(z + a) - 1| \leq |U_0(z + a)| + 1 \leq Ce^{-\lambda |z+a|} + 1$. For $z > 0$, we have $|U_0(z + a)| \leq Ce^{-\lambda |z+a|}$. We proceed in the same way for $a < 0$. \(\square\)

We turn to the proof of Lemma 4.2. First, we recall that $v^\varepsilon$ is such that (2.11) holds; hence, in view of (4.15), the estimate (4.17) is a direct consequence of the standard theory of elliptic equations. Next we prove (4.18). The function $w = w^\varepsilon := v^\varepsilon - u^0$ is solution of

$$\begin{cases}
-\Delta w + \gamma w = h & \text{on } Q_T, \\
\frac{\partial w}{\partial \nu} = 0 & \text{on } \partial \Omega \times (0, T),
\end{cases} \tag{4.20}$$

with $u^-_0 - u^0 \leq h = h^\varepsilon \leq u^+_0 - u^0$, where $u^0$ is the step function defined by $u^0(x,t) = \chi\{d(x,t) \leq 0\}$. The key idea of the proof is the fact that $h$ is exponentially small with respect to $\varepsilon$, except possibly in a thin neighborhood of $\Gamma_t$ of width of order $\varepsilon p(t)$. More precisely, from the definitions of $u^\pm_0$ in (4.5) and from the above lemma for $z = d(x,t)/\varepsilon$ and $a = \pm p(t)$, we deduce that

$$|h(x,t)| \leq C(e^{-\lambda |d(x,t)/\varepsilon + p(t)|} + e^{-\lambda |d(x,t)/\varepsilon - p(t)|}) + \chi\{|d(x,t)| \leq \varepsilon p(t)\} + q(t). \tag{4.21}$$
By linearity, we successively consider equation (4.20) with the various terms appearing in the right-hand side of (4.21). By the standard elliptic estimates, the solution $w$ of (4.20) satisfies
\[ |w(x, t)| + |\nabla w(x, t)| \leq C' \sup_{y \in \Omega} |h(y, t)|, \tag{4.22} \]
which gives the term $C_G q(t)$ that appears in the right-hand side of inequality (4.18) for $h(y, t) = q(t)$. We now suppose that the function $h$ satisfies one of the three following assumptions:

\begin{align*}
(H_1) & \quad |h(y, t)| \leq \chi_{\{|d(y, t)| \leq d_0\}} \chi_{\{|d(y, t)| \leq \varepsilon p(t)\}} \\
(H_2^\pm) & \quad |h(y, t)| \leq \exp \left( -\lambda \frac{d(y, t)}{\varepsilon} \pm p(t) \right),
\end{align*}

and write
\[ h(y, t) = h(y, t) \chi_{\{|d(y, t)| \leq d_0\}} + h(y, t) \chi_{\{|d(y, t)| > d_0\}}. \]

We first consider the term $h(y, t) \chi_{\{|d(y, t)| > d_0\}}$. In virtue of (4.16), we have
\[ 0 < K - 1 \leq p(t) \leq d_0/2\varepsilon. \tag{4.23} \]
Under assumption $(H_1)$, it follows that $h$ is supported in $\{|d(y, t)| \leq d_0/2\}$, which implies $h(y, t) \chi_{\{|d(y, t)| > d_0\}} = 0$. Moreover, under assumption $(H_2^\pm)$, using again (4.23),
\[ |h(y, t)| \chi_{\{|d(y, t)| > d_0\}} \leq \exp \left[ -\lambda (d_0/\varepsilon - p(t)) \right] \leq \exp(-\lambda d_0/2\varepsilon) \leq \frac{2}{\lambda d_0 e} \frac{1}{\varepsilon} \leq \frac{2}{\lambda d_0 e} K - 1 \varepsilon p(t). \]
Thus, under either of the assumptions $(H_1)$ or $(H_2^\pm)$, the estimate (4.18) — for the term $h(y, t) \chi_{\{|d(y, t)| > d_0\}}$ — directly follows from inequality (4.22).

From now on, we assume that $h$ is supported in $\{|d(y, t)| \leq d_0\}$. We have that
\[ w(x, t) = \int_{|d(y, t)| \leq d_0} G(x, y) h(y, t) dy, \]
and
\[ \nabla d(x, t) \cdot \nabla w(x, t) = \int_{|d(y, t)| \leq d_0} (\nabla_x G(x, y) \cdot \nabla d(x, t)) h(y, t) dy, \]
where $G$ is the Green’s function associated to the homogeneous Neumann boundary value problem on $\Omega$ for the operator $-\Delta + \gamma$. More precisely,
\( G(x, y) = g_\gamma(|x - y|) + H_\gamma(x, y) \), where \( g_\gamma(|x - y|) \) is the Green’s function associated to the operator \(-\Delta + \gamma \) on \( \mathbb{R}^N \) and where \( H_\gamma(x, y) \) is smooth for \( x \) and \( y \) far away from \( \partial \Omega \). It is known that \( g_\gamma \) is the Bessel function defined by
\[
g_\gamma(r) = c_N \int_0^{+\infty} e^{-\frac{r^2}{4s}} e^{-\gamma \frac{r^2}{2s}} \frac{ds}{s},
\]
with \( c_N > 0 \) a normalization constant. We use the following estimates (see [4]):
\[
|G(x, y)| \leq \begin{cases} 
C & \text{for } N \geq 3 \\
C \ln |y - x| & \text{for } N = 2,
\end{cases}
\]
(4.24)
\[
|\nabla_x G(x, y) \cdot \nabla d(x, t)| \leq \frac{C |d(y, t) - d(x, t)|}{|y - x|^N} + \frac{C}{|y - x|^{N-2}} \quad \text{for } N \geq 2.
\]
(4.25)
This last inequality follows from
\[
|\nabla_x G(x, y) \cdot \nabla d(x, t)| \leq \frac{C |\nabla d(x, t) \cdot (y - x)|}{|y - x|^N},
\]
and from \( d(y, t) - d(x, t) = \nabla d(x, t) \cdot (y - x) + O(|y - x|^2) \). Now, under respectively assumptions \((H_1)\), \((H_2^\pm)\), we define a function \( \tilde{h} = \tilde{h}^\varepsilon \) on \( \mathbb{R} \times [0, T] \), respectively by
\[
\tilde{h}(r, t) := \begin{cases} 
\chi_{\{|r| \leq \varepsilon p(t)\}} \\
\exp \left( -\lambda \frac{r}{\varepsilon} \pm p(t) \right).
\end{cases}
\]
(4.26)
Note that \( |h(y, t)| \leq \tilde{h}(d(y, t), t) \). Moreover, using (4.23), straightforward computations show that, under either of the assumptions \((H_1)\) or \((H_2^\pm)\), there exists \( \tilde{C} > 0 \) such that
\[
0 \leq \int_{-d_0}^{d_0} \tilde{h}(r, t) dr \leq \tilde{C} \varepsilon p(t),
\]
(4.27)
which is an analogue of (4.20) in [4]. The end of the proof is now identical to that of Lemma 4.10 in [4]. We omit the details and refer to this article. \( \Box \)

5 Proof of the main results

In this section, we prove our main results by fitting the two pairs of sub- and super-solutions, constructed for the study of the generation and the motion of interface, into each other.
5.1 Proof of Theorem 1.3

Let \( \eta \in (0, 1/4) \) be arbitrary. Choose \( \beta \) and \( \sigma \) that satisfy (4.11), (4.12) and
\[
\sigma \beta \leq \frac{\eta}{3}. \tag{5.1}
\]

By the generation of interface Theorem 3.1, there exist positive constants \( \varepsilon_0 \) and \( M_0 \) such that (3.2), (3.3) and (3.4) hold with the constant \( \eta \) replaced by \( \sigma \beta / 2 \). Since \( \nabla u_0 \cdot n \neq 0 \) everywhere on the initial interface \( \Gamma_0 = \{ x \in \Omega, u_0(x) = 1/2 \} \) and since \( \Gamma_0 \) is a compact hypersurface, we can find a positive constant \( M_1 \) such that
\[
\begin{align*}
\text{if } d_0(x) &\geq M_1 \varepsilon \text{ then } u_0(x) \leq 1/2 - M_0 \varepsilon, \\
\text{if } d_0(x) &\leq -M_1 \varepsilon \text{ then } u_0(x) \geq 1/2 + M_0 \varepsilon. \tag{5.2}
\end{align*}
\]

Here \( d_0(x) := d(x, 0) \) denotes the cut-off signed distance function associated with the hypersurface \( \Gamma_0 \). Now we define functions \( H^+(x), H^-(x) \) by
\[
H^+(x) = \begin{cases} 
1 + \frac{1}{2} \sigma \beta & \text{if } d_0(x) < M_1 \varepsilon \\
\frac{1}{2} \sigma \beta & \text{if } d_0(x) \geq M_1 \varepsilon,
\end{cases}
\]
\[
H^-(x) = \begin{cases} 
1 - \frac{1}{2} \sigma \beta & \text{if } d_0(x) \leq -M_1 \varepsilon \\
-\frac{1}{2} \sigma \beta & \text{if } d_0(x) > -M_1 \varepsilon.
\end{cases}
\]

Then from the above observation we see that
\[
H^-(x) \leq u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq H^+(x) \quad \text{for } x \in \Omega. \tag{5.3}
\]

Next we fix a sufficiently large constant \( K > 1 \) such that
\[
U_0(M_1 - K) \geq 1 - \frac{\sigma \beta}{3} \quad \text{and} \quad U_0(-M_1 + K) \leq \frac{\sigma \beta}{3}. \tag{5.4}
\]

For this \( K \), we choose \( \varepsilon_0 \) and \( L \) as in Lemma 4.1. We claim that
\[
u^\varepsilon_0(x, 0) \leq H^-(x), \quad H^+(x) \leq u^\varepsilon_0(x, 0) \quad \text{for } x \in \Omega. \tag{5.5}
\]

We only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that
\[
u^\varepsilon(x, 0) = U_0\left(\frac{d_0(x)}{\varepsilon} + K\right) - \sigma(\beta + \varepsilon^2 L) \leq H^-(x). \tag{5.6}
\]

In the range where \( d_0(x) > -M_1 \varepsilon \), the second inequality in (5.4) and the fact that \( U_0 \) is a decreasing function imply
\[
U_0\left(\frac{d_0(x)}{\varepsilon} + K\right) - \sigma(\beta + \varepsilon^2 L) \leq \frac{1}{3} \sigma \beta - \sigma \beta \leq H^-(x).
\]
On the other hand, in the range where \( d_0(x) \leq -M_1 \varepsilon \), we have
\[
U_0(\frac{d_0(x)}{\varepsilon} + K) - \sigma(\beta + \varepsilon^2 L) \leq 1 - \sigma \beta \leq H^-(x).
\]

This proves (5.6), hence (5.5) is established.

Combining (5.3) and (5.5), we obtain
\[
u^-_\varepsilon(x,0) \leq u^-_\varepsilon(x,\mu^{-1}e^{2\mid\ln \varepsilon\mid}) \leq u^+_\varepsilon(x,0).
\]

Since \( u^-_\varepsilon \) and \( u^+_\varepsilon \) are sub- and super-solutions for Problem \((P_\varepsilon)\) thanks to Lemma 4.1, the comparison principle yields
\[
u^-_\varepsilon(x,t) \leq u^-_\varepsilon(x,t+t^\varepsilon) \leq u^+_\varepsilon(x,t) \quad \text{for} \quad 0 \leq t \leq T-t^\varepsilon,
\]
where \( t^\varepsilon = \mu^{-1}e^{2\mid\ln \varepsilon\mid} \). Note that, in view of (4.7), this is enough to prove Corollary 1.4. Now let \( C \) be a positive constant such that
\[
U_0(-C + e^{LT} + K) \geq 1 - \frac{\eta}{2} \quad \text{and} \quad U_0(C - e^{LT} - K) \leq \frac{\eta}{2}.
\]

One then easily checks, using successively (5.7), (4.5), (5.8) and (5.1), that, for \( \varepsilon_0 \) small enough, for \( 0 \leq t \leq T-t^\varepsilon \), we have
\[
\text{if} \quad d(x,t) \geq C\varepsilon \quad \text{then} \quad u^-_\varepsilon(x,t+t^\varepsilon) \leq \eta,
\]
\[
\text{if} \quad d(x,t) \leq -C\varepsilon \quad \text{then} \quad u^-_\varepsilon(x,t+t^\varepsilon) \geq 1 - \eta,
\]
and
\[
u^-_\varepsilon(x,t+t^\varepsilon) \in [-\eta, 1+\eta],
\]
which completes the proof of Theorem 1.3.

\[\Box\]

5.2 Proof of Theorem 1.5

In the case where \( \mu^{-1}e^{2\mid\ln \varepsilon\mid} \leq t \leq T \), the assertion of the theorem is a direct consequence of Theorem 1.3. All we have to consider is the case where \( 0 \leq t \leq \mu^{-1}e^{2\mid\ln \varepsilon\mid} \). We shall use the sub- and super-solutions constructed for the study of the generation of interface in Section 3. To that purpose, we first prove the following lemma concerning \( Y(\tau,\xi;\delta) \), the solution of the ordinary differential equation (3.9), in the initial time interval.

Lemma 5.1. There exist constants \( C_8 > 0 \) and \( \varepsilon_0 > 0 \) such that, for all \( \varepsilon \in (0,\varepsilon_0) \),
\[
\text{if} \quad \xi \geq 1/2 + C_8 \varepsilon \quad \text{then} \quad Y(\tau,\xi;\pm\varepsilon G) > 1/2 \quad \text{for} \quad 0 \leq \tau \leq \mu^{-1}\mid\ln \varepsilon\mid,
\]
\[
\text{if} \quad \xi \leq 1/2 - C_8 \varepsilon \quad \text{then} \quad Y(\tau,\xi;\pm\varepsilon G) < 1/2 \quad \text{for} \quad 0 \leq \tau \leq \mu^{-1}\mid\ln \varepsilon\mid.
\]

(5.10)
Proof. We only prove the first inequality. Assume $\xi \geq 1/2 + C_8\varepsilon$. By (3.6), for $C_8 \geq CG$, we have that $\xi \geq 1/2 + C_8\varepsilon \geq a(\pm \varepsilon G)$. It then follows from (3.17) that

$$Y(\tau, \xi; \pm \varepsilon G) \geq a(\pm \varepsilon G) + C_1 e^{\mu(\pm \varepsilon G)\tau}(1/2 + C_8\varepsilon - a(\pm \varepsilon G))$$
$$\geq 1/2 - CG\varepsilon + C_1 (-CG\varepsilon + C_8\varepsilon)$$
$$\geq 1/2 + \varepsilon(C_1 C_8 - CG(C_1 + 1))$$
$$> 1/2,$$

provided that $C_8$ is sufficiently large.

Now we turn to the proof of Theorem 1.5. We first claim that there exists a positive constant $M_2$ such that for all $t \in [0, \mu^{-1}\varepsilon^2|\ln \varepsilon|]$,

$$\Gamma_t^\varepsilon \subset N_{M_2\varepsilon}(\Gamma_0).$$

(5.11)

To see this, we choose $M_0'$ large enough, so that $M_0' \geq C_8 + 2C_6$, where $C_6$ is as in Lemma 3.9. As is done for (5.2), there is a positive constant $M_2$ such that

if $d_0(x) \geq M_2\varepsilon$ then $u_0(x) \leq 1/2 - M_0'\varepsilon$,

if $d_0(x) \leq -M_2\varepsilon$ then $u_0(x) \geq 1/2 + M_0'\varepsilon$.

(5.12)

In view of this last condition, we see that, if $\varepsilon_0$ is small enough, if $d_0(x) \geq M_2\varepsilon$, then for all $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln \varepsilon|$,

$$u_0(x) + \varepsilon^2 r(\varepsilon G, \frac{t}{\varepsilon^2}) \leq 1/2 - M_0'\varepsilon + \varepsilon^2 C_6 \left[ e^{\mu(\varepsilon G)|\ln \varepsilon|/\mu} - 1 \right]$$
$$\leq 1/2 + \varepsilon \left[ -M_0' + C_6\varepsilon^{(\mu-\mu(\varepsilon G))/\mu} - \varepsilon C_6 \right]$$
$$\leq 1/2 + \varepsilon (-M_0' + 2C_6) \quad \text{thanks to (3.30)}$$
$$\leq 1/2 - C_8\varepsilon.$$

This inequality and Lemma 5.1 imply $w_\varepsilon^+(x, t) < 1/2$, where $w_\varepsilon^+$ is the sub-solution defined in (3.21). Consequently, by (3.25),

$$u_\varepsilon(x, t) < 1/2 \quad \text{if } d_0(x) \geq M_2\varepsilon.$$

In the case where $d_0(x) \leq -M_2\varepsilon$, similar arguments lead to $u_\varepsilon(x, t) > 1/2$. This completes the proof of (5.11). Note that we have proved that, for all $0 \leq t \leq \mu^{-1}\varepsilon^2|\ln \varepsilon|$,

$$u_\varepsilon(x, t) > 1/2 \quad \text{if } x \in \Omega_0^{(1)} \setminus N_{M_2\varepsilon}(\Gamma_0),$$

$$u_\varepsilon(x, t) < 1/2 \quad \text{if } x \in \Omega_0^{(0)} \setminus N_{M_2\varepsilon}(\Gamma_0).$$

(5.13)
Next, since $\Gamma_t$ depends on $t$ smoothly, there is a constant $\tilde{C} > 0$ such that, for all $t \in [0, \mu^{-1}\varepsilon^2|\ln \varepsilon|]$,
\[ \Gamma_0 \subset \mathcal{N}_{\tilde{C}\varepsilon^2|\ln \varepsilon|}(\Gamma_t), \]  
\tag{5.14}

and
\[ \Omega_t^{(1)} \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^{(1)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0), \]  
\[ \Omega_t^{(0)} \setminus \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma_t) \subset \Omega_0^{(0)} \setminus \mathcal{N}_{M_2\varepsilon}(\Gamma_0). \]  
\tag{5.15}

As a consequence of (5.11) and (5.14) we get
\[ \Gamma^\varepsilon_t \subset \mathcal{N}_{M_2\varepsilon + \tilde{C}\varepsilon^2|\ln \varepsilon|}(\Gamma_t) \subset \mathcal{N}_{C\varepsilon}(\Gamma_t), \]  
which completes the proof of Theorem 1.5.

\[ \square \]

**Proof of Corollary 1.6.** In view of Theorem 1.5 and the definition of the Hausdorff distance, to prove this corollary we only need to show the reverse inclusion, that is
\[ \Gamma_t \subset \mathcal{N}_{C^\varepsilon}(\Gamma_t^\varepsilon) \quad \text{for} \quad 0 \leq t \leq T, \]  
\tag{5.16}

for some constant $C' > 0$. To that purpose let $C'$ be a constant satisfying $C' > \max(C, \tilde{C})$, where $C$ is as in Theorem 1.3 and $\tilde{C}$ as in (5.15). Choose $t \in [0, T]$, $x_0 \in \Gamma_t$ arbitrarily and, $n$ being the Euclidian normal vector exterior to $\Gamma_t$ at point $x_0$, define a pair of points:
\[ x^{(0)} := x_0 + C'\varepsilon n \quad \text{and} \quad x^{(1)} := x_0 - C'\varepsilon n. \]

Since $C' > C$ and since the curvature of $\Gamma_t$ is uniformly bounded as $t$ varies over $[0, T]$, we see that, if $\varepsilon_0$ is sufficiently small,
\[ x^{(0)} \in \Omega_t^{(0)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t) \quad \text{and} \quad x^{(1)} \in \Omega_t^{(1)} \setminus \mathcal{N}_{C\varepsilon}(\Gamma_t). \]

Therefore, if $t \in [\mu^{-1}\varepsilon^2|\ln \varepsilon|, T]$, then, by Theorem 1.3, we have
\[ u^\varepsilon(x^{(0)}, t) < 1/2 < u^\varepsilon(x^{(1)}, t). \]  
\tag{5.17}

On the other hand, if $t \in [0, \mu^{-1}\varepsilon^2|\ln \varepsilon|]$, then from (5.13), (5.15) and the fact that $C' > \tilde{C}$, we again obtain (5.17). Thus (5.17) holds for all $t \in [0, T]$. Now, by the mean value theorem, we see that, for each $t \in [0, T]$, there exists a point $\bar{x}$ such that
\[ \bar{x} \in [x^{(0)}, x^{(1)}] \quad \text{and} \quad u^\varepsilon(\bar{x}, t) = 1/2. \]

This implies $\bar{x} \in \Gamma_t^\varepsilon$. Furthermore we have $|x_0 - \bar{x}| \leq C'\varepsilon$, since $\bar{x}$ lies on the line segment $[x^{(0)}, x^{(1)}]$. This proves (5.16).  

\[ \square \]
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