Quantum Probability Aspects to Lexicographic and Strong Products of Graphs

Nobuaki OBATA*

Graduate School of Information Sciences, Tohoku University, Sendai 980-8579, Japan

The adjacency matrix of the lexicographic product of graphs is decomposed into a sum of monotone independent random variables in a certain product state. The adjacency matrix of the strong product of graphs admits an expression in terms of commutative independent random variables in a product state. Their spectral distributions are obtained by using the monotone, classical and Mellin convolutions of probability distributions.

KEYWORDS: adjacency matrix, convolution of probability distributions, lexicographic product, spectral distribution, strong product

1. Products of Graphs

A graph \( G = (V, E) \) is a pair, where \( V \) is a non-empty set of vertices and \( E \) a set of edges, i.e., a subset of unordered pairs of distinct vertices. If \( (x, y) \in E \), we say that \( x \) and \( y \) are adjacent and write \( x \sim y \). We deal with both finite and infinite graphs, but always assume that a graph is locally finite, i.e., \( \deg(x) < \infty \) for all vertices \( x \in V \). The adjacency matrix of \( G \), denoted by \( A = A[G] \), is a matrix with index set \( V \times V \) defined by

\[
(A)_{xy} = \begin{cases} 
1, & \text{if } x \sim y, \\
0, & \text{otherwise}.
\end{cases}
\]

Given two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) there is a large variety of forming their product to obtain a larger graph, see e.g., [4] and references cited therein. From the quantum probability viewpoint we have so far studied the Cartesian, star, comb and free products of graphs [1,6,9,10]. In this paper, being based on a similar spirit, we will discuss the lexicographic and strong products of graphs, and derive their spectral distributions using certain concepts of independence in quantum probability.

Definition 1.1. The lexicographic product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \triangleright_L G_2 \), is the graph on \( V = V_1 \times V_2 \), where two distinct vertices \( (x_1, y_1) \) and \( (x_2, y_2) \) are adjacent whenever (i) \( x_1 \sim x_2 \); or (ii) \( x_1 = x_2 \) and \( y_1 \sim y_2 \).

Lemma 1.2. Let \( G_1 \) and \( G_2 \) be graphs with adjacency matrices \( A_1 \) and \( A_2 \), respectively. Then the adjacency matrix of the lexicographic product \( G_1 \triangleright_L G_2 \) satisfies

\[
A[G_1 \triangleright_L G_2] = A_1 \otimes J_2 + I_1 \otimes A_2,
\]

where \( J_2 \) is the matrix with index set \( V_2 \times V_2 \) whose entries are all one, and \( I_1 \) is the identity matrix with index set \( V_1 \times V_1 \). In particular, the graph operation \( \triangleright_L \) is associative: \( (G_1 \triangleright_L G_2) \triangleright_L G_3 \equiv G_1 \triangleright_L (G_2 \triangleright_L G_3) \), but it is not commutative.

The proof is straightforward by definition and is omitted. The Cartesian product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \times_C G_2 \), is the graph on \( V = V_1 \times V_2 \), where two distinct vertices \( (x_1, y_1) \) and \( (x_2, y_2) \) are adjacent whenever (i) \( x_1 = x_2 \) and \( y_1 \sim y_2 \); or (ii) \( x_1 \sim x_2 \) and \( y_1 = y_2 \). The adjacency matrix of \( G_1 \times_C G_2 \) is given by

\[
A[G_1 \times_C G_2] = A_1 \otimes I_2 + I_1 \otimes A_2.
\]

The Cartesian product is associative and commutative. By definition, \( G_1 \times_C G_2 \) is a subgraph of \( G_1 \triangleright_L G_2 \), which is viewed also from the adjacency matrices (1.1) and (1.2).

Definition 1.3. The strong product of \( G_1 \) and \( G_2 \), denoted by \( G_1 \times_S G_2 \), is the graph on \( V = V_1 \times V_2 \), where two distinct vertices \( (x_1, y_1) \) and \( (x_2, y_2) \) are adjacent whenever (i) \( x_1 = x_2 \) or \( x_1 \sim x_2 \); and (ii) \( y_1 = y_2 \) or \( y_1 \sim y_2 \).
Lemma 1.4. Let $G_1$ and $G_2$ be graphs with adjacency matrices $A_1$ and $A_2$, respectively. Then the adjacency matrix of the strong product $G_1 \times_s G_2$ satisfies

$$A[G_1 \times_s G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2.$$  

(1.3)

The proof is obvious. In the recent paper [7] we studied the spectral distribution of the Kronecker product $G_1 \times_K G_2$, which is the graph on $V = V_1 \times V_2$, where two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ are adjacent whenever $x_1 \sim x_2$ and $y_1 \sim y_2$. Then, the adjacency matrix is given by

$$A[G_1 \times_K G_2] = A_1 \otimes A_2.$$  

(1.4)

It is noted that the Kronecker product is a subgraph of the distance-2 graph of the Cartesian product $G_1 \times C G_2$.

There are quite a few concepts of "graph product" and the terminologies have not been unified in literatures. Our definitions are mostly in accordance with those in the handbook [4]. The Kronecker product is called conjunction in [2], the cardinal product in [3], the direct product in [4], and the strong product in [8].

2. Adjacency Matrices As Algebraic Random Variables

Let $G$ be a (locally finite) graph and $A$ the adjacency matrix. The adjacency algebra of $G$ is the $*$-algebra generated by $A$ and $I = A^0$ (identity matrix), and is denoted by $\mathcal{A}(G)$. Equipped with a state, $\mathcal{A}(G)$ becomes an algebraic probability space and the adjacency matrix $A$ is regarded as a real algebraic random variable, where a state means a linear function $\varphi : \mathcal{A}(G) \to \mathbb{C}$ satisfying $\varphi(a^*a) \geq 0$ and $\varphi(I) = 1$. Then it is well known (see e.g., [6, 10]) that there exists a probability distribution $\mu$, called the spectral distribution of $A$ in the state $\varphi$, such that

$$\varphi(A^m) = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m \geq 0.$$  

(2.1)

The left-hand side is called the $m$th moment of $A$ in the state $\varphi$, while the right-hand side is the usual $m$th moment of a probability distribution $\mu$. Note that the spectral distribution $\mu$ is not necessarily determined uniquely by (2.1) due to the famous indeterminate moment problem, but it is unique if $\sup\{|\deg(x)|; x \in V| < \infty$.

Let $C(V)$ be the space of all $\mathbb{C}$-valued functions on $V$. A matrix $T$ with index set $V \times V$ acts on $C(V)$ by means of usual matrix multiplication:

$$T f(x) = \sum_{y \in V} (T)_{xy} f(y),$$

whenever the right-hand side converges absolutely. It is convenient to define the “inner product” of $f, g \in C(V)$ by

$$\langle f, g \rangle = \sum_{x \in V} f(x) g(x),$$

whenever the right-hand side converges absolutely. With each $x \in V$ we define $e_x \in C(V)$ by $e_x(y) = \delta_{xy}$. Then we have $\langle e_x, e_y \rangle = \delta_{xy}$ and $(T)_{xy} = \langle e_x, Te_y \rangle$.

A state $\varphi$ on $\mathcal{A}(G)$ is called a vector state if it is of the form $\varphi(a) = \langle \xi, a\xi \rangle$, where $\xi \in C(V)$ with $\langle \xi, \xi \rangle = 1$. In particular, the vacuum state at a vertex $o \in V$ is defined by

$$\langle a \rangle_o = \langle e_o, ae_o \rangle = \langle a \rangle_{oo}, \quad a \in \mathcal{A}(G).$$

If a graph $G$ is finite, the normalized trace is defined by

$$\varphi_\mu(a) = \frac{1}{|V|} \text{Tr}(a) = \frac{1}{|V|} \sum_{x \in V} (a)_{xx} = \frac{1}{|V|} \sum_{x \in V} \langle e_x, ae_x \rangle, \quad a \in \mathcal{A}(G).$$

We are also interested in the vector state with state vector given by

$$\psi = \frac{1}{|V|} \sum_{x \in V} e_x.$$  

In fact, slightly abusing symbols, we see easily that

$$\varphi(a) = \langle \psi, a\psi \rangle = \frac{1}{|V|} \sum_{x \in V} \langle e_x, ae_x \rangle = \frac{1}{|V|} \sum_{x \in V} \langle a \rangle_{xx}, \quad a \in \mathcal{A}(G).$$  

(2.2)

It is noteworthy that the moments $\varphi(A^m)$ are related to counting walks in the graph $G$. Let $W_m(x, y; G)$ denote the number of $m$-step walks from a vertex $x$ to another $y$ in a graph $G$. As is easily verified by definition, we have

$$W_m(x, y; G) = \varphi(A^m)_{xy} = \langle e_x, A^m e_y \rangle, \quad m \geq 0.$$  

Therefore, $\langle A^m \rangle_o$ coincides with the number of $m$-step walks from a fixed vertex $o \in V$ to itself. Moreover, $\varphi_\mu(A^m)$ is the average number of $m$-step walks from a vertex to itself (the average is taken over all vertices). Let $W_m(x, *; G)$
denote the number of $m$-step walks starting from $x$, and $\bar{W}_m$ the average of $W_m(x, \ast; G)$ over all vertices $x \in V$. Then we have

$$\bar{W}_m = \frac{1}{|V|} \sum_{x \in V} W_m(x, \ast; G) = \frac{1}{|V|} \sum_{x, y \in V} W_m(x, y; G) = \frac{1}{|V|} \sum_{x, y \in V} \langle e_x, A^m e_y \rangle = \langle \psi, A^m \psi \rangle = \psi(A^m).$$

### 3. Lexicographic Products

**Theorem 3.1.** Let $G = G_1 \triangleright_L G_2$ be the lexicographic product of two graphs $G_1$ and $G_2$, where the latter is assumed to be finite. Then the adjacency matrix $A = A(G_1 \triangleright_L G_2)$ is expressed as in (1.1) and the right-hand side is a sum of monotone independent random variables in the product state $\varphi \otimes \psi$, where $\varphi$ is an arbitrary state on $A(G_1)$ and $\psi$ is the vector state on $A(G_2)$ defined as in (2.2).

**Proof.** Set $T_1 = A_1 \otimes 1$ and $T_2 = 1 \otimes A_2$. It is sufficient to show the following factorization property:

$$\varphi \otimes \psi(T_1^x T_2^y \cdot \cdot \cdot) = \varphi \otimes \psi(T_2^y \cdot \cdot \cdot T_1^x \cdot \cdot \cdot), \quad x, y \geq 1.$$

The verification is straightforward from definition. In fact, since $T_2$ is a constant multiple of a rank-one projection such that $T_2 \psi = |V_2| \psi$, the argument is similar to the case of comb products [1, 6]. \(\square\)

**Corollary 3.2.** Notations and assumptions being as in Theorem 3.1, let $\mu_1$ and $\mu_2$ be the spectral distributions of $A_1$ in $\varphi$ and that of $A_2$ in $\psi$, respectively. Let $\mu$ be the spectral distribution of $A = A(G_1 \triangleright_L G_2)$ in $\varphi \otimes \psi$. Then $\mu = (D\mu_1) \triangleright \mu_2$, where $D\mu_1$ is the dilation defined by $D\mu_1(dx) = \mu_1(|V_2|^{-1} dx)$ and $\triangleright$ is the monotone convolution.

**Proof.** Since $J_0^n = |V_2|^{m-1} J_2$ we have

$$\varphi \otimes \psi(A_1 \otimes J_0^n) = \varphi(A_1^n) \varphi(J_0^n) = \varphi(A_1^n)|V_2|^{m-1} \varphi(J_2) = \varphi(A_1^n)|V_2|^{m-1} \cdot |V_2| = |V_2|^m \varphi(A_1^n), \quad m \geq 0.$$

Hence the spectral distribution of $A_1 \otimes J_2$ in the product state $\varphi \otimes \psi$ is $D\mu_1$. On the other hand, the spectral distribution of $I_1 \otimes A_2$ in the product state $\varphi \otimes \psi$ is $\mu_2$. Since $A = A_1 \otimes J_2 + I_1 \otimes A_2$ is a sum of monotone independent random variables, the spectral distribution of $A$ is given by the monotone convolution of $D\mu_1$ and $\mu_2$. \(\square\)

For explicit calculation of the monotone convolution $\mu = \mu_1 \triangleright \mu_2$ we may employ Muraki’s formula (see e.g., [5]). For a probability distribution $\mu$ on $\mathbb{R}$ the *Stieltjes transform* and the reciprocal *Stieltjes transform* are defined by

$$G_{\mu}(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{z-x}, \quad H_{\mu}(z) = \frac{1}{G_{\mu}(z)}, \quad \text{respectively. Then, for } \mu = \mu_1 \triangleright \mu_2 \text{ we have }$$

$$H_{\mu}(z) = H_{\mu_1}(H_{\mu_2}(z)), \quad \text{Im} z > 0.$$

As a simple consequence, for the point masses we have $\delta_a \triangleright \delta_b = \delta_{a+b}$ for $a, b \in \mathbb{R}$. We should remind that the monotone convolution is not commutative, namely, $\mu_1 \triangleright \mu_2$ does not coincide with $\mu_2 \triangleright \mu_1$ in general.

**Example 3.3.** Let $G = K_n$ be the complete graph on $n$ vertices and $A$ the adjacency matrix. The spectral distribution of $A$ in the state $\psi$ defined as in (2.2) is the point mass $\delta_{n-1}$, since we have $\psi(A^m) = \bar{W}_m = (n-1)^m$ for $m \geq 0$. Now let $G_1 = K_m$ and $G_2 = K_n$. It follows from Corollary 3.2 that the spectral distribution of $A = A(K_m \triangleright_L K_n)$ in the product state $\varphi \otimes \psi$ is given by the monotone convolution:

$$(D\delta_{m-1}) \triangleright \delta_{n-1} = \delta_{m(n-1)} \triangleright \delta_{n-1} = \delta_{m(n-1) + (n-1)} = \delta_{mn-1}. \quad (3.1)$$

On the other hand, we see easily that $K_m \triangleright_L K_n \cong K_{mn}$ (hence we meet an exceptional case where $K_m \triangleright_L K_n \cong K_n \triangleright_L K_m$). Moreover, the product state $\varphi \otimes \psi_2$ coincides with the state $\psi$ similarly defined for $K_{mn}$. Hence the spectral distribution of $A = A(K_m n)$ in $\psi$ is the point mass $\delta_{mn-1}$, which, of course, coincides with (3.1).

### 4. Strong Products

For two probability distributions $\mu_1$ and $\mu_2$ on $\mathbb{R}$, the (classical) convolution $\mu_1 * \mu_2$ is defined by

$$\int_{\mathbb{R}} f(z) \mu_1 * \mu_2(dz) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x + y) \mu_1(dx) \mu_2(dy), \quad f \in C_c(\mathbb{R}),$$

where $C_c(\mathbb{R})$ stands for the space of continuous functions on $\mathbb{R}$ with compact supports. It is apparent that $\delta_a * \delta_b = \delta_{a+b}$ for $a, b \in \mathbb{R}$. Similarly, the Mellin convolution $\mu_1 *_{M} \mu_2$ is defined by
The above definition is a natural extension of the standard one for the Mellin convolution of probability distributions supported by the half line \([0, \infty)\). We see immediately that \(\delta_a \ast_M \delta_b = \delta_{ab}\) for \(a, b \in \mathbb{R}\).

**Theorem 4.1.** Let \(G = G_1 \times S G_2\) be the strong product of two graphs \(G_1\) and \(G_2\). Let \(\mu_i\) be the spectral distribution of the adjacency matrix \(A_i\) of \(G_i\) in \(\varphi_i\) for \(i = 1, 2\). Then, the spectral distribution \(\mu\) of \(A = A[G_1 \times S G_2]\) in \(\varphi_1 \otimes \varphi_2\) is given by \(\mu = S^{-1}(S\mu_1 \ast_M S\mu_2)\), where \(S\) is the shift defined by \(S\mu(dx) = \mu(dx - 1)\).

**Proof.** Since \(A + I_1 \otimes I_2 = (A_1 + I_1) \otimes (A_2 + I_2)\), we see that \(S\mu\) is the Mellin convolution of \(S\mu_1\) and \(S\mu_2\). \(\square\)

**Example 4.2.** We keep the same notations and assumptions as in Example 3.3. We apply Theorem 4.1 to obtain

\[
S^{-1}(S\delta_{m-1} \ast_M S\delta_{n-1}) = S^{-1}(\delta_m \ast_M \delta_n) = S^{-1}\delta_{mn} = \delta_{mn-1}.
\]

On the other hand, since \(K_m \times K_n \cong K_{mn}\), which is easily verified by definition, the spectral distribution of \(A = A[K_{mn}]\) in \(\varphi\) is the point mass \(\delta_{mn-1}\), which coincides with (4.1).

**Acknowledgements**

The notion of lexicographic product was brought to the author by Professor Edy Tri Baskoro through his work [11] and that of strong product by Professor Akihiro Munemasa. The author thanks them for interesting conversation and for their instructing references.

**REFERENCES**

[1] L. Accardi, A. Ben Ghorbal and N. Obata: *Monotone independence, comb graphs and Bose-Einstein condensation*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 7: 419–435 (2004).

[2] A. E. Brouwer and W. H. Haemers: *Spectra of Graphs*, Springer, New York, 2012.

[3] K. Čulik: *Zur Theorie der Graphen*, Časopis Pro Pěstování Matematiky 83: 133–155 (1958).

[4] R. Hammack, W. Imrich and S. Klavžar: ‘Handbook of Product Graphs, (2nd Ed.),’ CRC Press, Boca Raton, FL, 2011.

[5] T. Hasebe: *Monotone convolution and monotone infinite divisibility from complex analytic viewpoint*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 13: 111–131 (2010).

[6] A. Hora and N. Obata: “Quantum Probability and Spectral Analysis of Graphs,” Springer, Berlin, 2007.

[7] H. H. Lee and N. Obata: *Kronecker product graphs and counting walks in restricted lattices*, arXiv:1607.06808, 2016.

[8] L. Lovász: *On the Shannon capacity of a graph*, IEEE Transactions on Information Theory, IT-25: 1–7 (1979).

[9] N. Obata: Quantum probabilistic approach to spectral analysis of star graphs, Interdiscip. Inform. Sci. 10: 41–52 (2004).

[10] N. Obata: Notions of independence in quantum probability and spectral analysis of graphs, in “Selected papers on analysis and related topics,” pp. 115–136, Amer. Math. Soc. Transl. Ser. 2, 223, Amer. Math. Soc., Providence, RI, 2008.

[11] S. W. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. T. Baskoro, A. N. M. Salman and M. Bača: The metric dimension of the lexicographic product of graphs, Discrete Math. 313: 1045–1051 (2013).

*Note added in proof:* The author is grateful to Professor Takahiro Hasebe who pointed out an error in Section 4.