Off-diagonal Bethe ansatz and exact solution of a topological spin ring

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Integrable models play important roles in statistical physics, quantum field theory and condensed matter physics, because those models provide some benchmarks for understanding the corresponding universal classes. Since Yang and Baxter’s pioneering works [1,2], the Yang-Baxter relation has become a cornerstone for constructing and solving the integrable models. Especially, the $T-Q$ relation method [3,4] and the algebraic Bethe ansatz method [5,6] developed from the Yang-Baxter equation have become two very popular methods for dealing with the exact solutions of the known integrable models. Generally speaking, there are two classes of integrable models. One possesses $U(1)$ symmetry and the other does not. Three well known examples without $U(1)$ symmetry are the XYZ spin chain [5,7], the XXZ spin chain with antiperiodic boundary condition [8,9] and the ones with unparallel boundary fields [13,17]. It has been demonstrated that the algebraic Bethe ansatz and $T-Q$ relation can successfully diagonalize the integrable models with $U(1)$ symmetry. However, for those without $U(1)$ symmetry, only some very special cases such as the XYZ spin chain with even site number [3,4] and the XXZ spin chain with constrained unparallel boundary fields [12,17] can be dealt with because of the existence of a proper “local vacuum state” in these special cases. The main obstacle applying the algebraic Bethe ansatz and Baxter’s method to general integrable models is the inhomogeneous $XXZ$ topological spin ring model, as it is tightly related to the recent study on the topological states of matter. In fact, the topological boundary problem in many body systems has been rarely touched. With the inhomogeneous $XXZ$ topological spin ring model, we elucidate how our method works to derive the exact spectrum and the BAEs by constructing and solving a recursive functional equations. Particular attention is focused on the elementary excitations of the homogeneous $XX$ spin ring with antiperiodic boundary condition, as it is the simplest quantum realization of the Möbius stripe. Our exact solution shows that the elementary excitations of this simple model indeed exhibit a nontrivial topological nature.

We start from the following model Hamiltonian

$$H = -\sum_{j=1}^{N} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cosh \eta \sigma_j^z \sigma_{j+1}^z \right], \quad (1)$$

with the anti-periodic boundary conditions $\sigma_{N+1}^\alpha = \sigma_1^\alpha \sigma_2^\alpha \sigma_3^\alpha$. $N$ is the site number of the system and $\sigma_j^\alpha$ ($\alpha = x,y,z$) is the Pauli matrix on the site $j$ along the $\alpha$ direction. With such a topological boundary condition, the spin on the $N$-th site connects with that on the first site after rotating $\pi$-angle along the $x$-direction (a kink on the $(N,1)$ bond) and forms a torus in the spin space. With an unitary transformation $U_n H U_n^{-1}$, $U_n = \prod_{j=1}^{n} \sigma_j^x$, the kink can be shifted to the $(n, n+1)$ bond without changing the spectrum of the Hamiltonian. Notice here the braiding is in the quantum space rather than in the real space and therefore the present model describes a quantum Möbius stripe. We define a $Z_2$ operator $U_N = \prod_{j=1}^{N} \sigma_j^x$. It can be easily checked that $U_N^{-2} = 1$ and $[H, U_N] = 0$. Therefore, the present model possesses a global $Z_2$ invariance, indicating the double degeneracy of the eigenstates.
The integrability of the present model is associated with the following Lax operator
\[
L_{0j}(\lambda) = \begin{pmatrix}
\sinh(\lambda_j + \frac{\eta}{2} (1 + \sigma_j^-)) & \sinh(\eta \sigma_j^+ - \lambda_j + \frac{\eta}{2} (1 - \sigma_j^-)) \\
\sinh(\eta \sigma_j^+ - \lambda_j + \frac{\eta}{2} (1 - \sigma_j^-)) & \sinh(\lambda_j + \frac{\eta}{2} (1 + \sigma_j^-))
\end{pmatrix},
\]
and the monodromy matrix
\[
T_0(\lambda) = L_{01}(\lambda) \cdots L_{0N}(\lambda) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix},
\]
where \(\lambda_j = \lambda - \theta_j\), \(\lambda\) is the spectral parameter and \(\theta_j\) are the site inhomogeneous constants; \(\eta\) is the crossing parameter as usual; the index 0 indicates the auxiliary space and \(j\) indicates the quantum space. Both the Lax operator and the monodromy matrix satisfy the Yang-Baxter relation
\[
\begin{align*}
R_{12}(\lambda_1 - \lambda_2) L_{1j}(\lambda_1) L_{2j}(\lambda_2) &= L_{2j}(\lambda_2) L_{1j}(\lambda_1) R_{12}(\lambda_1 - \lambda_2), \\
R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) T_2(\lambda_2) &= T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_1 - \lambda_2),
\end{align*}
\]
(2)
with \(R_{12}(\lambda) = L_{12}(\lambda \theta_j = 0)\). The transfer matrix of the system is defined as
\[
\tau(\lambda) = tr_0 \sigma_3^z T_0(\lambda) = B(\lambda) + C(\lambda),
\]
(3)
where \(tr_0\) means tracing the auxiliary space. From Eq.(2), one can prove that the transfer matrices with different spectral parameters are mutually commutative, i.e., \([\tau(\lambda), \tau(\mu)] = 0\). Therefore, \(\tau(\lambda)\) serves as the generating functional of the conserved quantities of the corresponding system. The first order derivative of logarithm of the transfer matrix gives the Hamiltonian
\[
H = -2 \sinh(\eta) \frac{\partial \ln \tau(\lambda)}{\partial \lambda}|_{\lambda=0, \theta_j = 0} + N \cosh \eta.
\]
(4)
Define the state \(|0\rangle = \otimes |\uparrow\rangle_j\). From the definition of the Lax operator we obtain
\[
\begin{align*}
C(\lambda)|0\rangle = 0, & \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle, \\
D(\lambda)|0\rangle = d(\lambda)|0\rangle,
\end{align*}
\]
(5)
where \(a(\lambda) = \prod_{n=1}^N \sinh(\lambda - \theta_n + \eta)\) and \(d(\lambda) = \prod_{n=1}^N \sinh(\lambda - \theta_n)\). Before going further, we introduce the following useful formula
\[
C(\lambda)|B(\mu)|0\rangle = \sum_{l=1}^N M^l(\lambda, \{\mu_j\}) B^{l_{n-1}}|0\rangle
\]
\[
+ \sum_{k=1}^N \tilde{M}^{kl}(\lambda, \{\mu_j\}) B^{kl_{n-1}}|0\rangle,
\]
(6)
which can be obtained from the commutation relations derived from the Yang-Baxter relation [2], where
\[
M^l_{n}(\lambda, \{\mu_j\}) + g(\mu_i, \lambda)a(\mu_i)d(\lambda) \prod_{j \neq l} f(\mu_j, \lambda)f(\mu_j, \mu_i)
\]
\[
\tilde{M}^{kl}_{n}(\lambda, \{\mu_j\}) = g(\lambda, \mu_k)g(\mu_i, \lambda)f(\mu_i, \mu_k)a(\mu_i)d(\mu_k)
\]
\[
\times \prod_{j \neq k,l} f(\mu_j, \mu_k)f(\mu_j, \mu_i) + g(\mu_i, \mu_k)g(\mu_i, \lambda)f(\mu_i, \mu_k)a(\mu_i)d(\mu_i)
\]
\[
\times \prod_{j \neq k,l} f(\mu_j, \mu_k)f(\mu_j, \mu_i),
\]
(7)
(8)
We adopt the procedure introduced in [10]. Suppose \(|\Psi\rangle\) is an eigenstate of \(\tau(\lambda)\) and independent of \(\lambda\). We have \(\tau(\lambda)|\Psi\rangle = \Lambda(\lambda)|\Psi\rangle\). In addition, we define
\[
F_n(\{\mu_j\}) = \langle \Psi | \prod_{j=1}^n B(\mu_j)|0\rangle
\]
and put \(F_0(\{\mu_j\}) = 1\). Consider the quantity \(\langle \Psi | \tau(\lambda) \prod_{j=1}^n B(\mu_j)|0\rangle\). By acting \(\tau(\lambda)\) right alternatively, we have the following functional relations
\[
\Lambda(\lambda) F_n = \sum \lambda^l_{n} F^{l}_{n-1} + \sum \tilde{M}_{n}^{kl} F_{n-1}^{kl} + F_{n+1},
\]
(9)
\[
F_1(\lambda) = \Lambda(\lambda),
\]
\[
F_{N+1} = 0,
\]
where \(F_n = F_n(\{\mu_j\})\), \(F_{n-1}^{l} = F_{n-1}(\{\mu_j\})_{j \neq l}\), \(F_{N+1} = F_{N+1}(\{\mu_j\}_{j \neq k, l})\) and \(\{\mu_j\}\) indicating the parameter set \(\{\mu_1, \cdots, \mu_n\}\) for \(n = 1, \cdots, N\). Notice that we have \(N + 2\) equations and \(N + 2\) unknown functions \(\Lambda(\lambda)\) and \(F_n\). The function \(F_n(\{\mu_j\})\) is symmetric by exchanging the variables \(\mu_j\) because of \([B(\mu_j), B(\mu_i)] = 0\) and is a degree \(N - 1\) trigonometrical polynomial. The eigenvalue \(\Lambda(\lambda)\) therefore can be parameterized as
\[
\Lambda(\lambda) = \Lambda_0 \prod_{j=1}^{N-1} e^{z_j} \sinh(\lambda - z_j),
\]
(10)
where \(\Lambda_0\) is a constant and \(\{z_1, \cdots, z_{N-1}\}\) is a set of roots of \(\Lambda(\lambda)\) with \(\Lambda(z_j) = 0\). The recursion equations [9] determine the eigenvalue \(\Lambda(\lambda)\). From Eq.(11) we easily derive the eigenvalue of the Hamiltonian as
\[
E = -2 \sinh(\eta) \frac{\partial \ln \Lambda(\lambda)}{\partial \lambda}|_{\lambda=0, \theta_j = 0} + N \cosh \eta
\]
\[
= -2 \sinh(\eta) \sum_{j=1}^{N-1} \coth z_j + N \cosh \eta.
\]
(11)
Since \(d(\theta_j) = 0\), all the functions \(M_{n}^l\) and \(\tilde{M}_{n}^{kl}\) are zero as long as their variables belong to the parameter set \(\theta_1, \cdots, \theta_N\) and \(\theta_j \neq \theta_k \neq \theta_l \pm \eta\). Therefore, the
following relations hold
\[
F_n(\theta_1, \ldots, \theta_n) = \prod_{j=1}^{n} \Lambda(\theta_j).
\] (12)

From the \( n = N \) case of Eq. (11) we obtain
\[
\Lambda(\lambda) = \sum_{j=1}^{N} \frac{a(\theta_j)d(\lambda)}{\Lambda(\theta_j)} g(\theta_j, \lambda) \prod_{l \neq j} f(\theta_j, \theta_l) f(\theta_l, \lambda)
\]
\[
= -\sum_{j=1}^{N} \frac{a(\theta_j)d(\theta_j - \eta)}{\Lambda(\theta_j)d(\theta_j - \eta)} a(\lambda),
\] (13)

with \( d_j(\theta_j) = \prod_{l \neq j}^{N} \sinh(\theta_j - \theta_l) \). This equation gives the closed recursive solution of \( \Lambda(\lambda) \). Putting \( \lambda \rightarrow \theta_j - \eta \), we readily have
\[
\Lambda(\theta_j)\Lambda(\theta_j - \eta) = \Delta_\eta(\theta_j), \quad j = 1, \ldots, N,
\] (14)

where \( \Delta_\eta(\theta_j) = -a(\theta_j)d(\theta_j - \eta) \) is the quantum determinant \( |\theta_j| \). Similar relations were also derived in \cite{13, 14, 13} with the separation of variables method. The above equations determine the \( N - 1 \) roots \( \{z_j\} \) and \( \Lambda_0 \) in Eq. (10). In fact, the operator identity \( B(\theta_j)B(\theta_j - \eta) = 0 \) can be demonstrated with the definition of the monodromy matrix. With this operator identity and considering the quantity \( \langle \Psi|\tau(\theta_j)\tau(\theta_j - \eta)|0\rangle \), one can easily deduce Eq. (14). Taking the limit of Eq. (14) with \( \theta_j \rightarrow 0 \) leads to the following equations which completely determine the spectrum \( \Lambda(\lambda) \) of the homogeneous model
\[
\frac{\partial^l}{\partial u^l} \ln(-\sinh^N(u + \eta)\sinh^N(u - \eta))|_{u=0} = \frac{\partial^l}{\partial u^l} \ln(\Lambda(u)\Lambda(u-\eta))|_{u=0}, \quad l = 0, \ldots, N-1.
\] (15)

However, these relations are quite hard to be used to study the physical properties, especially in the thermodynamic limit. Thus a proper set of BAEs in the usual form is still crucial. As \( \Lambda(\lambda) \) is a trigonometrical polynomial of degree \( N-1 \) with the very periodicity \( \Lambda(\lambda + i\pi) = (-1)^N \Lambda(\lambda) \), we conjecture the following modified \( T - Q \) relation \cite{2, 3}
\[
\Lambda(\lambda) = e^{\phi_1}(\lambda) \frac{Q_1(\lambda - \eta)}{Q_2(\lambda)} - e^{\phi_2}(\lambda) \frac{Q_2(\lambda + \eta)}{Q_1(\lambda)} - b(\lambda) \frac{a(\lambda)d(\lambda)}{Q_1(\lambda)Q_2(\lambda)},
\] (16)

where
\[
Q_1(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \mu_j),
\]
\[
Q_2(\lambda) = \prod_{j=1}^{M} \sinh(\lambda - \nu_j),
\] (17)

and \( b(\lambda) \) is an adjust function. For \( N \) even, \( M = \frac{N}{2} \),
\[
b(\lambda) = e^{i\phi_1+\lambda} - e^{i\phi_2-\lambda-\eta},
\] (18)

with
\[
i\phi_1 = \sum_{j=1}^{N} \theta_j - M\eta - 2\sum_{j=1}^{M} \mu_j,
\]
\[
i\phi_2 = \sum_{j=1}^{N} \theta_j - M\eta - 2\sum_{j=1}^{N} \nu_j,
\] (19)

to cancel the leading terms in Eq. (16) when \( \lambda \rightarrow \pm \infty \). Obviously, the conjectured \( \Lambda(\lambda) \) satisfies Eq. (14) automatically. The BAEs determined by the regularity of \( \Lambda(\lambda) \) (which ensures \( \Lambda(\lambda) \) to be a trigonometrical polynomial of degree \( N - 1 \)) read
\[
d(\nu_j) = \frac{e^{\nu_j}}{b(\nu_j)} Q_1(\nu_j - \eta)Q_1(\nu_j),
\]
\[
a(\mu_j) = - \frac{e^{-\mu_j - \eta}}{b(\mu_j)} Q_2(\mu_j + \eta)Q_2(\mu_j),
\] (20)

\[ j = 1, \ldots, N/2. \]

The BAEs for the homogeneous model are exactly the above equations by putting all \( \theta_j = 0 \). The eigenvalues of Hamiltonian (11) take the following form
\[
E(\{\mu_j, \nu_j\}) = 2\sinh \eta \sum_{j=1}^{M} \left\{ \frac{\cosh(\mu_j + \eta)}{\sinh(\mu_j + \eta)} - \frac{\cosh(\nu_j)}{\sinh(\nu_j)} \right\} + N(\cosh \eta - 2) - 2\sinh \eta.
\] (21)

For odd \( N \), we put \( M = (N + 1)/2 \) and
\[
b(\lambda) = \frac{1}{2} [e^{i\phi_1+2\lambda} - e^{i\phi_2-2\lambda-2\eta}],
\] (22)

where \( \phi_1 \) and \( \phi_2 \) take the same form as Eq. (19) with \( M = (N + 1)/2 \) and \( \theta_j = 0 \). In this case, the BAEs and the eigenvalue of the Hamiltonian are still given by Eq. (20) and Eq. (21), respectively.

We note the BAEs about the roots \( \{z_j\} \) can also be derived from the recursive equations Eq. (11). To show the physical effect of the topological boundary clearly, let us focus on the \( \eta = i\pi/2 \) case, i.e., the topological XX spin ring or equivalently the topological free fermion ring (via a Jordan-Wigner transformation). In the homogeneous case, the roots \( \{z_j\} \) satisfy the equation \( a^2(z_j) = d^2(z_j) \) (see below). This equation has \( 2(N - 1) \) solutions. However, we need only \( N - 1 \) roots to form a solution set. That means a selection rule is needed. From the \( n = N \) case of Eq. (10) we obtain that
\[
F_N(z; \{\mu_i\}) = \frac{1}{A_0} \sum_{j=1,2}^{N-1} M^{ij}_N F_{N-1},
\] (23)
From Eqs. (23) and (24), we have

\[ \text{singular point. Notice the fact that } F_{N-1}(z, \{\mu_i\}) \text{ proportional to } F_2(z, z \pm i\pi/2) \text{ when one of } \mu_i \text{ is equal to } z \pm i\pi/2, \text{ which must be zero according to the above analysis. This gives the constrain condition of the root } z \text{ as } a(z)d(z+i\pi/2) = d(z)a(z+i\pi/2). \]

Therefore, the roots \( z_n \) of \( \Lambda(\lambda) \) satisfy the following Bethe ansatz equation

\[ \cosh^{2N}(z_n) = 1, \quad z_j \neq z_k \pm \frac{\pi i}{2}. \quad (26) \]

Equivalently, we have

\[ \cosh(z_n) = e^{\frac{\pi n}{N}} \equiv e^{i\kappa n}, \quad n = \pm 1, \cdots, \pm (N-1). \quad (27) \]

The \( N-1 \) pair solutions \( \{z_j, z_j + \frac{\pi}{2}\} \mod(i\pi) \) are located on two lines with imaginary part \( \pm i\pi/4 \). The root sets are formed by choosing one and only one in each pair. This selection rule comes from that the poles of the right-hand-side of Eq. (25) do not enter into the set of roots \( \{z_j\} \) because the poles and the zeros satisfy the same equation (26). Therefore, there are \( 2^{N-1} \) possible choices to form a solution of \( \Lambda(\lambda) \). With the \( Z_2 \) symmetry of the system, we demonstrate that the solutions are complete.

The ground state is formed by filling all roots along the \( -i\pi/4 \) line (as shown in Fig.1(a)) and the ground state energy reads \( E_g = -2 \cot \frac{\pi}{2N} \) which is slightly different from that of the periodic boundary condition case. The elementary excitations of the system can be constructed by digging some holes in the lower solution line and putting the same number of particles in the upper solution line (as shown in Fig.1(b)). However, the positions of the holes and the particles are not arbitrary but obey the selection rules of \( z_j \neq z_k \pm \frac{i\pi}{2} \). That means if there is a hole at \( -k \), there must be a particle at \( k \) (as shown in Fig.2). The energy of a particle-hole excitation is thus \( \epsilon(k) = 4 \sin |k| \). Such an excitation character is quite unlike to that in the usual Luttinger liquids, where both the forward scattering and backward scattering are allowed and there is no constrain for the particle-hole excitations besides the Pauli principle in the charge neutral sector. In the present topological boundary case, each particle with momentum \( k \) must lock a hole with momentum \( -k \) to form a virtual bound state, indicating the topological nature of the excitations. Whether the topological boundary modifies Haldane’s Luttinger liquid relation is an interesting issue to be studied.

In conclusion, we developed a general method for diagonalizing the integrable models without \( U(1) \) symmetry. As an example, we constructed the exact solution of the \( XXZ \) spin-ring with topological boundary condition. We remark that the present method could be applied to other integrable models without \( U(1) \) symmetry. For those models, some off-diagonal elements of the monodromy matrix enter into the transfer matrix expression \( \tau(\lambda) \). With the commutation relations derived from the corresponding Yang-Baxter equation, similar relation of \( \Lambda(\theta_j)\Lambda(\theta_j - \eta) \sim \Delta_1(\theta_j) \) can be constructed based on...
some operator identities, with which a modified $T - Q$ relation as well as the usual BAEs can be constructed. Details will be given elsewhere.

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