On $L^p$-viscosity solutions of bilateral obstacle problems with unbounded ingredients

Shigeaki Koike · Shota Tateyama

Received: 1 September 2018 / Revised: 22 April 2019 / Published online: 8 June 2019
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Abstract
The global equi-continuity estimate on $L^p$-viscosity solutions of bilateral obstacle problems with unbounded ingredients is established when obstacles are merely continuous. The existence of $L^p$-viscosity solutions is established via an approximation of given data. The local Hölder continuity estimate on the first derivative of $L^p$-viscosity solutions is shown when the obstacles belong to $C^{1,\beta}$, and $p > n$.

Mathematics Subject Classification 49L25 · 35D40 · 35R35 · 35B65 · 35J15

1 Introduction

In this paper, we consider the following bilateral obstacle problem

$$\min\{\max\{F(x, Du, D^2 u) - f, u - \psi\}, u - \varphi\} = 0 \quad \text{in } \Omega,$$

under the Dirichlet condition $u = g$ on $\partial \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $F$ is at least a measurable function on $\Omega \times \mathbb{R}^n \times S^n$, and $f, \varphi, \psi$ and $g$ are given. We denote by $S^n$ the set of all $n \times n$ real-valued symmetric matrices with the standard order, and set

Communicated by Y. Giga.
In contrast, unilateral obstacle problems are described by Bellman equations

\[
\max\{F(x, Du, D^2u) - f, u - \psi\} = 0 \quad \text{in } \Omega, \quad (1.2)
\]
or

\[
\min\{F(x, Du, D^2u) - f, u - \varphi\} = 0 \quad \text{in } \Omega. \quad (1.3)
\]

In [27], Lions-Stampacchia first introduced unilateral obstacle problems as an example of variational inequalities. Then, in [3,26], regularity of solutions of obstacle problems was studied by Brezis-Stampacchia and Lewy-Stampacchia. Afterwards, there appeared numerous researches on unilateral obstacle problems when \(F\) are partial differential operators of divergence form. We only refer to [16,19,30] and references therein for the existence and regularity of solutions of obstacle problems and applications.

When \(F\) is a linear second-order uniformly elliptic operator with smooth coefficients in (1.2) or (1.3), as a crucial regularity result of solutions (i.e. \(W^{2,\infty}(\Omega)\)) of unilateral obstacle problems, we refer to [18]. We also refer to [25] for \(W^{2,\infty}_{loc}(\Omega)\) regularity of solutions of (1.2) when \(F\) is given by the maximum of a finite number of linear second-order uniformly elliptic operators with smooth coefficients.

We also note that unilateral obstacle problems arise in stochastic optimal stopping time problems. We refer to [15,34] and references therein for this issue.

Going back to bilateral obstacle problems, we refer to [29] and [11], respectively, for a nice review and a pioneering regularity result. As an application, we also refer to [10].

We note that Eq. (1.1) is formally equivalent to the following problem:

\[
\begin{cases}
F(x, Du, D^2u) \leq f(x) & \text{in } \{x \in \Omega \mid u(x) > \varphi(x)\}, \\
F(x, Du, D^2u) \geq f(x) & \text{in } \{x \in \Omega \mid u(x) < \psi(x)\}, \\
\varphi \leq u \leq \psi & \text{in } \Omega.
\end{cases}
\]

Furthermore, we notice that (1.1) can be regarded as the following Isaacs equation

\[
\min_{\alpha \in [0,1]} \max_{\beta \in [0,1]} \{F_{\alpha,\beta}(x, u, Du, D^2u) - f_{\alpha,\beta}\} = 0 \quad \text{in } \Omega,
\]

where for two parameters \(\alpha, \beta \in [0, 1]]\n
\[
F_{\alpha,\beta}(x, r, \xi, X) = \alpha \beta F(x, \xi, X) + \alpha (1 - \beta) r + (1 - \alpha) r
\]

and

\[
f_{\alpha,\beta}(x) = \alpha \beta f(x) + \alpha (1 - \beta) \psi(x) + (1 - \alpha) \varphi(x),
\]
because of the fact that for $A, B, C \in \mathbb{R}$,

$$
A \land (B \lor C) = \min_{\alpha \in [0,1]} \max_{\beta \in [0,1]} \{ \alpha \beta A + \alpha (1 - \beta) B + (1 - \alpha) C \}.
$$

Here and later, we use the notations: for $a, b \in \mathbb{R}$,

$$
a \lor b := \max\{a, b\}, \quad a \land b := \min\{a, b\}, \quad a^+ := a \lor 0 \quad \text{and} \quad a^- := (-a) \lor 0.
$$

On the other hand, we have few results when $F$ has non-divergence structure even for unilateral obstacle problems. Duque in [12] recently showed interior Hölder estimates on viscosity solutions of bilateral obstacle problems for fully nonlinear uniformly elliptic operators with no variable coefficients, no first derivative terms and constant inhomogeneous terms but only assuming that the obstacles are Hölder continuous;

$$
\begin{aligned}
F(x, \xi, X) &= F(X) \quad \text{for } (x, \xi, X) \in \Omega \times \mathbb{R}^n \times S^n, \\
f &\equiv C, \\
\varphi, \psi &\in C^\alpha(\Omega) \quad \text{for some } \alpha \in (0, 1).
\end{aligned}
$$

Assuming the above hypotheses, in [12], we obtain the existence of viscosity solutions of (1.1) under the Dirichlet condition, and interior Hölder estimates on the first derivative of viscosity solutions of (1.1) when obstacles are in $C^{1, \beta}$ for $\beta \in (0, 1)$. The results associated with parabolic problems are also shown in [12]. We refer to [23,24] for very recent related topics, and to [7] for a different approach via Tug-of-War games.

Although a clever use of the weak Harnack inequality was adapted to show those estimates in [12], in order to extend the results to more general $F$ and $f$, it seems difficult to establish the estimates near the free boundary and near $\partial \Omega$.

Our aim in this paper is to extend results in [12] when $F$ is a fully nonlinear uniformly elliptic operator. More precisely, under more general hypotheses than those in [12], we show the equi-continuity of $L^p$-viscosity solutions of (1.1) in $\overline{\Omega}$, the existence of $L^p$-viscosity solutions of (1.1), and their local Hölder continuity of derivatives under additional assumptions.

For the corresponding results of parabolic obstacle problems, we cannot use the argument in the proof of Hölder estimates on the derivative of $L^p$-viscosity solutions because the domain, where the infimum is taken, differs from that of the $L^{q_0}$ (quasi)-norm in the weak Harnack inequality, which arises in Proposition 2.4 for the elliptic case. The second author finds a new argument to avoid this difficulty. We refer to [32] for the parabolic version of this paper.

For any $p > 0$ and $u : \Omega \to \mathbb{R}$, we denote the quasi-norm:

$$
\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}}.
$$

We note that $\| \cdot \|_{L^p(\Omega)}$ satisfies

$$
\|u + v\|_{L^p(\Omega)} \leq C_p \left( \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \right) \quad \text{for some } C_p \geq 1, \quad (1.4)
$$
where $C_p = 1$ provided $p \geq 1$.

This paper is organized as follows: In Sect. 2, we recall the definition of $L^p$-viscosity solutions, basic properties, and exhibit main results. Section 3 is devoted to the weak Harnack inequality both in $\Omega' \subseteq \Omega$ and near $\partial \Omega$, which yields the global equi-continuity of $L^p$-viscosity solutions. In Sect. 4, we establish the existence of $L^p$-viscosity solutions of (1.1) when the obstacles are only continuous under appropriate hypotheses. We obtain Hölder estimates on the first derivative of $L^p$-viscosity solutions in Sect. 5.

2 Preliminaries and main results

For any $x \in \mathbb{R}^n$ and $r > 0$, we set

$$B_r := \{ y \in \mathbb{R}^n : |y| < r \}, \quad \text{and} \quad B_r(x) := x + B_r.$$  

For any measurable set $A \subset \mathbb{R}^n$, we denote by $|A|$ the Lebesgue measure of $A$.

We recall the definition of $L^p$-viscosity solutions of general elliptic partial differential equations (PDE for short) from [6]:

$$G(x, u, Du, D^2u) = 0 \quad \text{in} \ \Omega, \quad (2.1)$$

where $G : \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ is measurable.

**Definition 2.1** We call $u \in C(\Omega)$ an $L^p$-viscosity subsolution (resp., supersolution) of (2.1) if whenever $u - \eta$ attains its local maximum (resp., minimum) at $x_0 \in \Omega$ for $\eta \in W^{2,p}_{loc}(\Omega)$, it follows that

$$\lim_{r \to 0} \essinf_{B_r(x_0)} G(x, u(x), D\eta(x), D^2\eta(x)) \leq 0 \quad (\text{resp.,} \ \lim_{r \to 0} \esssup_{B_r(x_0)} G(x, u(x), D\eta(x), D^2\eta(x)) \geq 0).$$

We also call $u$ an $L^p$-viscosity solution of (2.1) if it is both an $L^p$-viscosity sub- and supersolution of (2.1).

**Remark 2.2** We will call $C$-viscosity subsolutions (resp., supersolutions, solutions) if we replace $W^{2,p}_{loc}(\Omega)$ by $C^2(\Omega)$ in the above when given $G$ is continuous. We refer to [8] for the theory of $C$-viscosity solutions.

In order to present our main results, we shall prepare some notations and hypotheses. Throughout this paper, under the hypothesis

$$p_0 < p \leq q, \quad q > n, \quad (2.2)$$

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where $p_0 \in \left[\frac{n}{2}, n\right]$ is the constant in [13], we suppose
\[ f \in L^p(\Omega). \tag{2.3} \]
Concerning $F$, we suppose that there exist constants $0 < \lambda \leq \Lambda$, and
\[ \mu \in L^q(\Omega) \tag{2.4} \]
such that
\[ P^-_{\lambda, \Lambda}(X - Y) - \mu(x)|\xi - \eta| \leq F(x, \xi, X) - F(x, \eta, Y) \leq P^+_{\lambda, \Lambda}(X - Y) + \mu(x)|\xi - \eta| \tag{2.5} \]
for $x \in \Omega, \xi, \eta \in \mathbb{R}^n, X, Y \in S^n$, where $P^\pm_{\lambda, \Lambda} : S^n \to \mathbb{R}$ are defined by
\[ P^+_{\lambda, \Lambda}(X) := \max\{-\tr(AX) : A \in S^n\}_{\lambda, \Lambda} \quad \text{and} \quad P^-_{\lambda, \Lambda}(X) := -P^+_{\lambda, \Lambda}(-X) \]
for $X \in S^n$. Since we fix $0 < \lambda \leq \Lambda$ in this paper, we shall write $P^\pm := P^\pm_{\lambda, \Lambda}$ for simplicity. We also suppose that
\[ F(x, 0, O) = 0 \quad \text{for } x \in \Omega. \tag{2.6} \]
We notice that (2.5) and (2.6) yield
\[ \mu \geq 0 \quad \text{in } \Omega. \]
For obstacles $\varphi, \psi$ and the Dirichlet datum $g$, as compatibility conditions, we suppose
\[ \varphi \leq \psi \quad \text{in } \Omega, \quad \text{and } \varphi \leq g \leq \psi \quad \text{on } \partial\Omega. \tag{2.7} \]

### 2.1 Basic properties

We first give a direct consequence from the definition, which will be often used.

**Proposition 2.3** Assume (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7). Let $u \in C(\Omega)$ be an $L^p$-viscosity subsolution (resp., supersolution) of (1.1). Assume that $\gamma \in \mathbb{R}$ satisfies $\gamma \geq \varphi$ (resp. $\gamma \leq \psi$) in an open set $\Omega_0 \subset \Omega$. Then, $u \vee \gamma$ (resp. $u \wedge \gamma$) is an $L^p$-viscosity subsolution (resp., supersolution) of
\[ P^-(D^2u) - \mu|Du| - f^+ = 0 \quad \text{(resp., } P^+(D^2u) + \mu|Du| + f^- = 0) \quad \text{in } \Omega_0. \]

**Proof** We only prove the assertion for subsolutions.

For $\xi \in W^{2,p}_{loc}(\Omega_0)$, we suppose that $(u \vee \gamma) - \xi$ attains its local maximum at $x_0 \in \Omega_0$. 
If we assume $u(x_0) > \gamma$, then $u - \xi$ attains its local maximum at $x_0 \in \Omega$, and $u > \varphi$ near $x_0$. Hence, by the definition, we have

$$\lim_{r \to 0} \inf_{B_r(x_0)} \max\{F(x, D\xi(x), D^2\xi(x)) - f(x), (u - \psi)(x)\} \leq 0,$$

which yields the conclusion by (2.5).

When $u(x_0) \leq \gamma$, it is enough to show that any constant is an $L^p$-viscosity subsolution of

$$P^- (D^2 u) = 0 \text{ in } \Omega. \quad (2.8)$$

In fact, by noting that any constant is a $C$-viscosity subsolution of (2.8), in view of Proposition 2.9 in [6], it is also an $L^p$-viscosity subsolution of (2.8). \(\square\)

We shall recall the scaled version of the weak Harnack inequality and the Hölder continuity in [20]. Modifying the result in [20] by an argument of the compactness, we state the next proposition as simple as possible for later use. See [20] for the original version. Here and later, we use the notation

$$\alpha_0 := 2 - \frac{n}{p \wedge n} \in (0, 1].$$

**Proposition 2.4** (cf. Theorem 4.5, 4.7, Corollary 4.8 in [20]) Assume (2.2), (2.3) and (2.4). There exist $\varepsilon_0 > 0$ and $C_0 > 0$ such that if $v \in C(B_{2r})$ is a nonnegative $L^p$-viscosity supersolution of

$$P^+ (D^2 v) + \mu \abs{Dv} - f = 0 \text{ in } B_{2r}, \quad (2.9)$$

then it follows that

$$\|v\|_{L^{r_0}(B_r)} \leq C_0 r^{\frac{n}{\alpha_0}} \left(\inf_{B_r} v + r^{\alpha_0} \|f\|_{L^{p, n}(B_{2r})}\right).$$

Here, $\varepsilon_0$ and $C_0$ depend on $n, \Lambda, p, q$ and $\|\mu\|_{L^q(B_{2r})}$.

In Sect. 5, we will use the following local maximum principle.

**Proposition 2.5** (cf. Theorem 3.1 in [21]) Under hypotheses (2.2), (2.3), (2.4), for any $\varepsilon > 0$, there exists $C_1 = C_1 (n, \Lambda, \varepsilon, p, q, \|\mu\|_{L^q(B_{2r})}) > 0$ such that if $u \in C(B_{2r})$ is a nonnegative $L^p$-viscosity subsolution of

$$P^- (D^2 u) - \mu \abs{Du} - f = 0 \text{ in } B_{2r},$$

then it follows that

$$\sup_{B_{\frac{r}{2}}} u \leq C_1 \left(\frac{r}{\varepsilon} \|u\|_{L^\varepsilon(B_r)} + r^{\alpha_0} \|f^+\|_{L^{p, n}(B_{2r})}\right).$$
Although it is mentioned in Theorem 6.2 of [20] that Proposition 2.4 implies the Hölder continuity of $L^p$-viscosity solutions of

$$F(x, Du, D^2 u) - f = 0 \text{ in } \Omega, \quad (2.10)$$

to show a key idea of this paper, we recall how to derive Hölder estimates on $L^p$-viscosity solutions of (2.10).

**Proposition 2.6** (cf. Theorem 6.2 in [20]) Assume (2.2), (2.3), (2.4), (2.5) and (2.6). Let $\Omega := B_{2R}$ for $R \in (0, 1]$. Then, there exist constants $K_1 > 0$ and $\hat{\alpha} \in (0, \alpha_0]$ such that if $u \in C(B_{2R})$ is an $L^p$-viscosity solution of (2.10), then it follows that

$$|u(x) - u(y)| \leq K_1 \left( \frac{|x - y|}{R} \right)^{\hat{\alpha} \hat{\alpha}} \|u\|_{L^\infty(B_{2R})} + R^{\alpha_0} \|f\|_{L^{p,n}(B_{2R})} \quad \text{for } x, y \in B_R.$$

**Proof** Fix $x \in B_R$. For $0 < s \leq R$, we set

$$M_s := \sup_{B_s(x)} u, \quad \text{and} \quad m_s := \inf_{B_s(x)} u.$$

Now, for $0 < r \leq \frac{R}{2}$, setting

$$U := u - m_{2r} \geq 0, \quad \text{and} \quad V := M_{2r} - u \geq 0 \quad \text{in } B_{2r}(x),$$

we immediately see that $U$ and $V$ are $L^p$-viscosity supersolutions of (2.9) with $f$ replaced by $- f^-$ and $- f^+$, respectively. Hence, in view of Proposition 2.4, we have

$$\|U\|_{L^0(B_r(x))} \leq C_0 r^\frac{\alpha}{\alpha_0} \left( \inf_{B_s(x)} U + r^{\alpha_0} \|f\|_{L^{p,n}(B_{2r}(x))} \right),$$

$$\|V\|_{L^0(B_r(x))} \leq C_0 r^\frac{\alpha}{\alpha_0} \left( \inf_{B_s(x)} V + r^{\alpha_0} \|f\|_{L^{p,n}(B_{2r}(x))} \right),$$

Therefore, in view of Proposition 2.4, we can find $C'_0 > 1$ such that

$$M_{2r} - m_{2r} = |B_r|^{-\frac{1}{\alpha_0}} \|M_{2r} - m_{2r}\|_{L^0(B_r(x))}$$

$$\leq |B_r|^{-\frac{1}{\alpha_0}} C_{\varepsilon_0} \left( \|V\|_{L^0(B_r(x))} + \|U\|_{L^0(B_r(x))} \right)$$

$$\leq C'_0 \left( M_{2r} - M_r + m_r - m_{2r} + r^{\alpha_0} \|f\|_{L^{p,n}(B_{2R})} \right),$$

where $C_{\varepsilon_0} \geq 1$ is from (1.4). Thus, there exists $\theta_0 \in (0, 1)$ such that

$$\omega(r) \leq \theta_0 \omega(2r) + r^{\alpha_0} \|f\|_{L^{p,n}(B_{2R})}.$$
where $\omega(r) = M_r - m_r$. Hence, the standard argument (e.g. Lemma 8.23 in [17]) implies that

$$|u(x) - u(y)| \leq K_1 \left( \frac{|x - y|}{R} \right)^{\hat{\alpha}} \left( \|u\|_{L^\infty(B_{2R})} + R^{\alpha_0} \|f\|_{L^{p,\infty}(B_{2R})} \right)$$

for some $K_1 > 0$ and $\hat{\alpha} \in (0, \alpha_0]$. \Box

**Remark 2.7** One of key ideas of this paper is a different choice of $M_s$ and $m_s$ in the above for the proof of Lemma 3.1.

When $p > n$ as in (2.16) in Sect. 2.2, we recall the following regularity result for fully nonlinear PDE.

**Proposition 2.8** ([4,5,31]) Let $\Omega = B_2$. Under (2.2), (2.3), there exist $\hat{\beta} \in (0, 1)$ and $K_2 > 0$ such that if $u \in C(B_2)$ is an $L^p$-viscosity subsolution and $L^p$-viscosity supersolution, respectively, of

$$\mathcal{P}^-(D^2u) = 0 \quad \text{and} \quad \mathcal{P}^+(D^2u) = 0 \quad \text{in } B_2,$$

then it follows that

$$\|u\|_{C^{1,\hat{\beta}}(\overline{B_1})} \leq K_2 \|u\|_{L^\infty(B_2)}.$$

We finally give a reasonable property of $L^p$-viscosity solutions of (1.1), which will be often used without mentioning it. We present a proof for the reader’s convenience though it seems standard.

**Proposition 2.9** Under (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7), we assume $\varphi, \psi \in C(\Omega)$. If $u \in C(\Omega)$ is an $L^p$-viscosity subsolution (resp., supersolution) of (1.1), then it follows that

$$u \leq \psi \quad (\text{resp., } u \geq \varphi) \quad \text{in } \Omega.$$

**Proof** We give a proof only for $L^p$-viscosity subsolutions since the other case can be shown similarly. Assume that $(u - \psi)(x_0) =: \theta > 0$ for $x_0 \in \Omega$, then we will have a contradiction. For simplicity, we may suppose $x_0 = 0 \in \Omega$ by translation.

For $\varepsilon > 0$, we let $x_\varepsilon \in \overline{\Omega}$ be such that $\max \{u(x) - \frac{1}{2\varepsilon} |x|^2 \mid x \in \overline{\Omega}\} = u(x_\varepsilon) - \frac{1}{2\varepsilon} |x_\varepsilon|^2$. Since it is easy to see that $\lim_{\varepsilon \to 0} x_\varepsilon = 0$, we may suppose $x_\varepsilon \in \Omega$. Moreover, we may suppose $u \geq \psi + \frac{\theta}{2}$ in $B_r \subseteq \Omega$ for some $r > 0$. Thus, by the first inequality in (2.7), we have

$$u \geq \psi + \frac{\theta}{2} \geq \varphi + \frac{\theta}{2} \quad \text{in } B_r. \quad (2.11)$$

However, from the definition, we have

$$0 \geq \lim_{s \to 0} \inf_{B_s(x_\varepsilon)} \min \left\{ \max \left\{ F \left( x, \frac{x}{\varepsilon}, \frac{1}{\varepsilon^d} \right) - f(x), (u - \psi)(x) \right\}, (u - \varphi)(x) \right\},$$

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which yields

\[ 0 \geq \min\{(u - \psi)(x_\varepsilon), (u - \varphi)(x_\varepsilon)\}. \]

This contradicts to (2.11).

\[ \square \]

### 2.2 Main results

For obstacles, we at least assume that

\[ \varphi, \psi \in C(\overline{\Omega}). \tag{2.12} \]

In order to obtain the estimate near \( \partial \Omega \), we suppose the following condition on the shape of \( \Omega \), which was introduced in [2].

\[
\begin{align*}
\text{There exist } R_0 > 0 \text{ and } \Theta_0 > 0 \text{ such that } \\
|B_r(x) \setminus \Omega| \geq \Theta_0 r^n \text{ for } (x, r) \in \partial \Omega \times (0, R_0). 
\end{align*}
\tag{2.13}
\]

We will also suppose

\[ g \in C(\partial \Omega). \tag{2.14} \]

We call a function \( \omega : [0, \infty) \rightarrow [0, \infty) \) a modulus of continuity if \( \omega \) is nondecreasing and continuous in \([0, \infty)\) such that \( \omega(0) = 0 \).

Our first result is the global equi-continuity estimate on \( L^p \)-viscosity solutions.

**Theorem 2.10** Assume (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.12), (2.13) and (2.14). Then, there exists a modulus of continuity \( \omega_0 \) such that if \( u \in C(\overline{\Omega}) \) is an \( L^p \)-viscosity solution of (1.1) satisfying

\[ u = g \text{ on } \partial \Omega, \tag{2.15} \]

then it follows that

\[ |u(x) - u(y)| \leq \omega_0(|x - y|) \text{ for } x, y \in \overline{\Omega}. \]

If we moreover assume that

\[ \varphi, \psi \in C^{\alpha_1}(\overline{\Omega}), \text{ and } g \in C^{\alpha_1}(\partial \Omega) \text{ for } \alpha_1 \in (0, 1), \]

then there exist \( \alpha_2 \in (0, \alpha_0 \wedge \alpha_1] \) and \( C > 0 \), independent of \( u \), such that

\[ |u(x) - u(y)| \leq C|x - y|^{\alpha_2} \text{ for } x, y \in \overline{\Omega}. \]

Thanks to Theorem 2.10, we establish the following existence result.
Theorem 2.11 Under (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.12), and (2.14), we assume the uniform exterior cone condition on $\Omega$. Then, there exists an $L^p$-viscosity solution $u \in C(\overline{\Omega})$ of (1.1) satisfying (2.15).

For further regularity results, assuming
\[ q \geq p > n, \tag{2.16} \]
we define $\beta_0 \in (0, 1)$ by
\[ \beta_0 := 1 - \frac{n}{p}. \]
To show $C^{1,\beta}$ estimates, we will see in Sect. 5 that it is necessary to suppose that
\[ \varphi, \psi \in C^{1,\beta_1}(\Omega) \text{ for some } \beta_1 \in (0, 1). \tag{2.17} \]
We will use the constant $\beta_2$ defined by
\[ \beta_2 := \beta_0 \land \beta_1 \in (0, 1). \]
We also suppose that obstacles do not coincide in $\Omega$;
\[ \text{there is } r_0 > 0 \text{ such that } \psi - \varphi \geq r_0 \text{ in } \Omega. \tag{2.18} \]

In order to state the next theorem, we prepare some notations. For small $r > 0$, we introduce subdomains of $\Omega$:
\[ \Omega_r := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > r \}. \]
For $u \in C(\Omega)$ such that $\varphi \leq u \leq \psi$ in $\Omega$, we set
\[ C^-[u] := \{ x \in \Omega \mid u(x) = \varphi(x) \}, \quad C^+[u] := \{ x \in \Omega \mid u(x) = \psi(x) \}, \]
\[ C^\pm[u] := C^-[u] \cup C^+[u] \subset \Omega, \]
and the non-coincidence set
\[ N[u] := \Omega \setminus C^\pm[u] = \{ x \in \Omega \mid \varphi(x) < u(x) < \psi(x) \}. \]
For small $r > 0$, we define subdomains of $N[u]$
\[ N_r[u] := \{ x \in \Omega_r \mid \text{dist}(x, C^\pm[u]) > r \}. \]
For $F$ in (1.1), we use the following notation:
\[ \theta(x, y) := \sup_{X \in S^n} \frac{|F(x, 0, X) - F(y, 0, X)|}{1 + \|X\|} \text{ for } x, y \in \Omega. \]
Theorem 2.12 Assume (2.16), (2.3), (2.4), (2.5), (2.6), (2.17) and (2.18). For each small \( \varepsilon > 0 \), there exist \( C > 0 \) and \( \delta_0 > 0 \) such that if \( u \in C(\Omega) \) is an \( L^p \)-viscosity solution of (1.1), and if

\[
\frac{1}{r} \| \theta(y, \cdot) \|_{L^q(B_r(y))} \leq \delta_0 \quad \text{for } r \in (0, \varepsilon) \text{ and } y \in N_\varepsilon[u],
\]

then it follows that

\[
|Du(x) - Du(y)| \leq C|x - y|^{\beta_3} \quad \text{for } x, y \in \Omega_\varepsilon,
\]

where

\[
\beta_3 := \begin{cases} 
\beta_2 & \text{provided } \beta_2 < \hat{\beta}, \\
\text{any } \beta \in (0, \hat{\beta}) & \text{provided } \beta_2 \geq \hat{\beta}.
\end{cases}
\]

3 Global equi-continuity estimates

In what follows, assuming (2.12), we denote by \( \sigma_0 \) the modulus of continuity of \( \varphi \) and \( \psi \) in \( \overline{\Omega} \);

\[
\sigma_0(r) := \max\{|\varphi(x) - \varphi(y)| \vee |\psi(x) - \psi(y)| \mid |x - y| \leq r, \ x, y \in \overline{\Omega}\}.
\]

3.1 Local estimates

We first show the local equi-continuity estimate on \( L^p \)-viscosity solutions of (1.1).

Lemma 3.1 Assume (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) and (2.12). For any \( \Omega' \Subset \Omega \), there exists a modulus of continuity \( \omega_0 \) such that if \( u \in C(\Omega) \) is an \( L^p \)-viscosity solution of (1.1), then it follows that

\[
|u(x) - u(y)| \leq \omega_0(|x - y|) \quad \text{for } x, y \in \overline{\Omega'}.
\]

Proof Let \( r \in (0, \frac{\delta_0}{2}) \), where \( \delta_0 := \text{dist}(\Omega', \partial \Omega) \), and \( x_0 \in \overline{\Omega}' \). We may suppose \( x_0 = 0 \) as before. Setting \( \sigma_0 := \sigma_0(2r) \), we define

\[
u_+ := u \vee (\varphi(0) + \sigma_0) \quad \text{and} \quad u_- := u \wedge (\psi(0) - \sigma_0).
\]

By noting \( \varphi(0) + \sigma_0 \geq \varphi \) and \( \psi \geq \psi(0) - \sigma_0 \) in \( B_{2r} \), Proposition 2.3 shows that \( u_+ \) and \( u_- \) are, respectively, an \( L^p \)-viscosity subsolution and supersolution of

\[
P^{-}(D^2u) - \mu |Du| - f^+ = 0 \quad \text{and} \quad P^+(D^2u) + \mu |Du| + f^- = 0 \quad \text{in } B_{2r}.
\]

Now, for \( s \in (0, \frac{\delta_0}{2}) \), setting

\[
M_s := \sup_{B_s} u_+ \quad \text{and} \quad m_s := \inf_{B_s} u_-,
\]

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we define

\[ U := M_2 r - u_+, \quad \text{and} \quad V := u_- - m_2 r \]

for \( r \in (0, \frac{\delta}{2}) \).

It is easy to see that \( U \) and \( V \) are, respectively, nonnegative \( L^p \)-viscosity super solutions of

\[ \mathcal{P}^+(D^2 u) + \mu |Du| + f^\pm = 0 \quad \text{in} \quad B_{2r}. \]

Hence, by Proposition 2.4, we have

\[ \|U\|_{L^0(B_r)} \leq Cr_{\alpha_0}^{\frac{n}{\alpha_0}} \left( \inf_{B_r} U + r^{\alpha_0} \|f^+\|_{L^{p,n}(\Omega)} \right), \tag{3.1} \]

and

\[ \|V\|_{L^0(B_r)} \leq Cr_{\alpha_0}^{\frac{n}{\alpha_0}} \left( \inf_{B_r} V + r^{\alpha_0} \|f^-\|_{L^{p,n}(\Omega)} \right). \tag{3.2} \]

Here and later, \( C \geq 0 \) denotes the various constant depending only on known quantities. Since \( M_{2r} - m_{2r} = U + (u_+ - u) + (u - u_-) + V \leq U + 4\sigma_0 + V \) by Proposition 2.9, we have

\[ M_{2r} - m_{2r} \leq Cr_{\alpha_0}^{\frac{n}{\alpha_0}} \left( \|U\|_{L^0(B_r)} + \sigma_0 r_{\alpha_0}^{\frac{n}{\alpha_0}} + \|V\|_{L^0(B_r)} \right). \]

Combining this with (3.1) and (3.2), we find \( \theta_1 \in (0, 1) \) such that

\[ M_r - m_r \leq \theta_1 (M_{2r} - m_{2r}) + r^{\alpha_0} \|f\|_{L^{p,n}(\Omega)} + \sigma_0(2r). \]

We note here that

\[ u(x) - u(y) \leq u_+(x) - u_-(y) \quad \text{for} \quad x, y \in B_{2r}. \]

Therefore, as for Proposition 2.6 with Lemma 8.23 in [17], it is standard to find a modulus of continuity \( \omega_0 \) in the conclusion. \( \square \)

**Remark 3.2** As noted in Sect. 2.2, if we suppose \( \varphi, \psi \in C^{\alpha_1}(\Omega) \) for \( \alpha_1 \in (0, 1) \), then we can show \( u \in C^{\alpha_2}(\overline{\Omega}) \) for some \( \alpha_2 \in (0, \alpha_0 \wedge \alpha_1] \) because we can choose \( \sigma_0(r) = Cr^{\alpha_1} \) for some \( C > 0 \) in the above.

### 3.2 Equi-continuity near \( \partial \Omega \)

To state equi-continuity near \( \partial \Omega \), we shall use the following notion: for small \( \varepsilon > 0 \),

\[ \Omega_{\varepsilon} := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \varepsilon \}. \]
Lemma 3.3 Assume (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.12), (2.13) and (2.14). For small $\varepsilon > 0$, there exists a modulus of continuity $\omega_0$ such that if an $L^p$-viscosity solution $u \in C(\overline{\Omega})$ of (1.1) satisfies (2.15), then it follows that

$$|u(x) - u(y)| \leq \omega_0(|x - y|) \quad \text{for } x, y \in \overline{\Omega} \setminus \Omega_\varepsilon.$$ 

Proof Let $r \in (0, \frac{\varepsilon}{2})$ and $x_0 \in \partial \Omega$. We may suppose $x_0 = 0 \in \partial \Omega$. As in the proof of Lemma 3.1, we set

$$u_+ := u \vee (\varphi(0) + \sigma_0) \quad \text{and} \quad u_- := u \wedge (\psi(0) - \sigma_0),$$

where $\sigma_0 := \sigma_0(2r)$. In view of Proposition 2.3 again, we see that $u_+$ and $u_-$ are, respectively, an $L^p$-viscosity subsolution and supersolution of

$$\mathcal{P}^- (D^2 u) - \mu |Du| - f^+ = 0 \quad \text{and} \quad \mathcal{P}^+ (D^2 u) + \mu |Du| + f^- = 0 \quad \text{in } B_{2r} \cap \Omega.$$ 

Now, as in [17,20] for instance, setting

$$M_s := \sup_{B_r \cap \Omega} u_+ \quad \text{and} \quad m_s := \inf_{B_r \cap \Omega} u_-,$$

we define

$$U := \begin{cases} (M_{2r} - u_+) \wedge c_+ & \text{in } B_{2r} \cap \Omega, \\ c_+ & \text{in } B_{2r} \setminus \Omega, \end{cases}$$

and

$$V := \begin{cases} (u_- - m_{2r}) \wedge c_- & \text{in } B_{2r} \cap \Omega, \\ c_- & \text{in } B_{2r} \setminus \Omega, \end{cases}$$

where nonnegative constants $c_{\pm}$ are given by

$$c_+ := M_{2r} - \sup_{B_{2r} \cap \partial \Omega} u_+ \quad \text{and} \quad c_- := \inf_{B_{2r} \cap \partial \Omega} u_- - m_{2r}.$$ 

Hence, it is easy to see that $U$ and $V$ are nonnegative $L^p$-viscosity supersolutions of

$$\mathcal{P}^+ (D^2 u) + \hat{\mu} |Du| + |\hat{f}| = 0 \quad \text{in } B_{2r},$$

where $\hat{\mu}$ and $\hat{f}$ are zero extensions of $\mu$ and $f$ outside of $\Omega$, respectively. Hence, by Proposition 2.4, we have

$$\Theta_0 \left( M_{2r} - \sup_{B_{2r} \cap \partial \Omega} u_+ \right) \leq C \left( M_{2r} - \sup_{B_r \cap \Omega} u_+ + r^{\sigma_0} \|f\|_{L^p(\Omega)} \right)$$

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and

\[ \Theta_0^{\frac{1}{0}} \left( \inf_{B_{2r} \cap \Omega_1} u_- - m_{2r} \right) \leq C \left( \inf_{B_{r} \cap \Omega_1} u_- - m_{2r} + r^{\alpha_0} \| f \|_{L^p(\Omega)} \right) \]  

As in the proof of Lemma 3.1, these inequalities imply that there is \( \theta_0 \in (0, 1) \) such that

\[ M_r - m_r \leq \theta_0 (M_{2r} - m_{2r}) + 2r^{\alpha_0} \| f \|_{L^p(\Omega)} + \sigma_0(2r) + \omega_0(2r), \]

where \( \omega_0(r) := \sup \{|g(x) - g(y)| : |x - y| \leq r, x, y \in \partial \Omega\} \). Therefore, noting that \( u(x) - u(y) \leq u_+(x) - u_-(y) \) for \( x, y \in B_{2r} \cap \Omega \), as before, we can find a modulus of continuity \( \omega_0 \) in the assertion. \( \square \)

**Remark 3.4** As in Remark 3.2, if we suppose

\[ \varphi, \psi \in C^{a_1}(\Omega_1 \setminus \Omega_2), \quad \text{and} \quad g \in C^{a_1}(\partial \Omega) \quad \text{for} \quad a_1 \in (0, 1), \]

then \( u \in C^{a_2}(\Omega_1 \setminus \Omega_2) \) holds for some \( a_2 \in (0, a_0 \wedge a_1] \).

**Proof of Theorem 2.10** In view of Lemmas 3.1 and 3.3, we immediately obtain the assertion. \( \square \)

### 4 Existence Results

In this section, we present an existence result of \( L^p \)-viscosity solutions of (1.1) under suitable conditions when obstacles are merely continuous.

Using the standard mollifier by \( \rho \in C^\infty_0(\mathbb{R}^n) \), we introduce smooth approximations of \( f \) and \( F \) by

\[ f_\varepsilon := f * \rho_\varepsilon, \quad \mu_\varepsilon := \mu * \rho_\varepsilon \quad \text{and} \quad F_\varepsilon(x, \xi, X) := \int_{\mathbb{R}^n} \rho_\varepsilon(x - y) F(y, \xi, X) dy \]

for \( (x, \xi, X) \in \mathbb{R}^n \times \mathbb{R}^n \times S^n \), where \( \rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon) \). Here and later, we use the same notion \( f \) and \( F \) for their zero extension outside of \( \Omega \). Under (2.3), (2.4), (2.5) and (2.6), it is easy to observe that for \( (x, \xi, X) \in \mathbb{R}^n \times \mathbb{R}^n \times S^n \),

\[
\begin{align*}
(i) \quad & |\mathcal{P}^-(X) - \mu_\varepsilon(x)\xi| \leq F_\varepsilon(x, \xi, X) \leq \mathcal{P}^+(X) + \mu_\varepsilon(x)\xi, \\
(ii) \quad & \|f_\varepsilon\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\Omega)}, \\
(iii) \quad & \|\mu_\varepsilon\|_{L^q(\mathbb{R}^n)} \leq \|\mu\|_{L^q(\Omega)}.
\end{align*}
\]  

Furthermore, we shall suppose that \( \varphi \) and \( \psi \) are defined in a neighborhood of \( \Omega \) with the same modulus of continuity. More precisely, there is \( \varepsilon_1 > 0 \) such that

\[ \max\{|\varphi(x) - \varphi(y)|, |\psi(x) - \psi(y)|\} \leq \sigma_0(|x - y|) \quad \text{for} \quad x, y \in \mathcal{N}_{\varepsilon_1}, \]

where \( \mathcal{N}_{\varepsilon_1} \) is the neighborhood of \( \Omega \) with radius \( \varepsilon_1 \).
where $N_{\varepsilon_1} := \{ x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \varepsilon_1 \}$ is a neighborhood of $\overline{\Omega}$. Under (4.2), we define $\varphi_\varepsilon$ and $\psi_\varepsilon$ as follows:

$$
\varphi_\varepsilon := \varphi \ast \rho_\varepsilon - \sigma_0(\varepsilon), \quad \psi_\varepsilon := \psi \ast \rho_\varepsilon + \sigma_0(\varepsilon).
$$

It is easy to see that for $\varepsilon \in (0, \varepsilon_1)$,

$$
\varphi_\varepsilon \leq g \leq \psi_\varepsilon \quad \text{on} \quad \partial\Omega,
$$

and

$$
\max\{|\varphi_\varepsilon(x) - \varphi_\varepsilon(y)|, |\psi_\varepsilon(x) - \psi_\varepsilon(y)|\} \leq \sigma_0(|x - y|) \quad \text{for} \quad x, y \in \overline{\Omega}.
$$

We shall consider approximate equations:

$$
F_\varepsilon(x, Du, D^2u) + \frac{1}{\delta}(u - \psi_\varepsilon)^+ - \frac{1}{\delta}(\varphi_\varepsilon - u)^+ = f_\varepsilon \quad \text{in} \quad \Omega. \quad (4.3)
$$

In order to apply an existence result in [9], we shall suppose the uniform exterior cone condition on $\partial\Omega$ in [28], which is stronger than (2.13).

**Proposition 4.1** Under (2.2), (2.3), (2.4), (2.5), (2.6), (2.7), (2.12), (2.14) and (4.2), we assume the uniform exterior cone condition on $\Omega$. Then, there exists a $C$-viscosity solution $u_\delta^\varepsilon \in C(\overline{\Omega})$ of (4.3) satisfying (2.15).

We first show an existence result for (1.1) when $\varphi$, $\psi$ and $F$ are smooth.

**Theorem 4.2** (cf. Theorem 1.1 in [9]) Under the same hypotheses in Proposition 4.1, let $u_\delta^\varepsilon \in C(\overline{\Omega})$ be $C$-viscosity solutions of (4.3) satisfying (2.15). For each small $\varepsilon > 0$, there exist $\delta_\varepsilon > 0$ and $\tilde{C}_\varepsilon > 0$ such that

$$
0 \leq \frac{1}{\delta}(u_\delta^\varepsilon - \psi_\varepsilon)^+ + \frac{1}{\delta}(\varphi_\varepsilon - u_\delta^\varepsilon)^+ \leq \tilde{C}_\varepsilon \quad \text{in} \quad \overline{\Omega} \quad \text{for} \quad \delta \in (0, \delta_\varepsilon). \quad (4.4)
$$

Furthermore, there exist a subsequence $\{\delta_k\}_{k=1}^\infty$ and $u_\varepsilon \in C(\overline{\Omega})$ such that $\delta_k \to 0$ as $k \to \infty$, (2.15) holds for $u_\varepsilon$,

$$
u_\delta^k \to u_\varepsilon \quad \text{uniformly in} \quad \overline{\Omega}, \quad \text{as} \quad k \to \infty, \quad (4.5)
$$

and $u_\varepsilon$ is a (unique) $C$-viscosity solution of

$$
\min\{\max\{F_\varepsilon(x, Du, D^2u) - f_\varepsilon, u - \psi_\varepsilon\}, u - \varphi_\varepsilon\} = 0 \quad \text{in} \quad \Omega. \quad (4.6)
$$

**Proof** To show the estimate on $\frac{1}{\delta}(u_\delta^\varepsilon - \psi_\varepsilon)^+$, independent of $\delta > 0$, we let $x_0 \in \overline{\Omega}$ be such that

$$
\max_{\overline{\Omega}} \frac{1}{\delta}(u_\delta^\varepsilon - \psi_\varepsilon)^+ = \frac{1}{\delta}(u_\delta^\varepsilon - \psi_\varepsilon)^+(x_0) > 0.
$$
Thus, we see that $x_0 \in \Omega$, and $u^\delta - \psi_\varepsilon$ attains its maximum at $x_0 \in \Omega$. Hence, the definition implies

$$0 \leq \frac{1}{\delta} (u^\delta - \psi_\varepsilon)^+ \leq f_\varepsilon - F_\varepsilon(x_0, D\psi_\varepsilon, D^2\psi_\varepsilon) \quad \text{at } x_0$$

because of $\frac{1}{\delta} (\varphi_\varepsilon - u^\delta)^+(x_0) = 0$.

Following the same argument, we obtain the estimate on $\frac{1}{\delta} (\varphi_\varepsilon - u^\delta)^+$. Thus, we conclude the first assertion (4.4). We then obtain the $L^\infty$ bound of $u^\delta$ independent of $\delta \in (0, 1)$ for each $\varepsilon \in (0, 1)$.

By regarding the penalty term as the right hand side with $L^\infty$-estimates, independent of $\delta > 0$, it is standard to show the equi-continuity and uniform boundedness of $\{u^\delta_k\}_{k=1}^{\infty}$ and $u_\varepsilon \in C(\overline{\Omega})$ satisfying (4.5).

We shall show that $u_\varepsilon$ is a $C$-viscosity subsolution of (4.6) by contradiction. Thus, we suppose that $u - \eta$ attains its local strict maximum at $x_0 \in \Omega$ for $\eta \in C^2(\Omega)$, and

$$\min\{\max\{F_\varepsilon(x_0, D\eta, D^2\eta) - f_\varepsilon, u_\varepsilon - \psi_\varepsilon\}, u_\varepsilon - \varphi_\varepsilon\} \geq 2\theta \quad \text{at } x_0 \quad (4.7)$$

for some $\theta > 0$. By the uniform convergence, we may suppose that $u^\delta_k - \eta$ attains its local maximum at $x^\delta_k \in \Omega$, where $x^\delta_k \to x_0$ as $k \to \infty$. In what follows, we shall write $\delta$ for $\delta_k$.

By (4.7), since we may suppose

$$(u^\delta - \varphi_\varepsilon)(x^\delta) \geq \theta,$$

we have $\frac{1}{\delta} (\varphi_\varepsilon - u^\delta)^+ = 0$ at $x^\delta$. Hence, sending $k \to \infty$ in (4.3) with $\delta = \delta_k$, we have

$$F_\varepsilon(x_0, D\eta, D^2\eta) \leq f_\varepsilon \quad \text{at } x_0,$$

which together with (4.7) yields

$$(u^\delta - \psi_\varepsilon)(x^\delta) \geq \theta$$

for small $\delta > 0$. However, this together with (4.4) yields a contradiction for large $k \geq 1$. \hfill \Box

Now, we shall show our existence result.

**Proof of Theorem 2.11** Let $u_\varepsilon \in C(\overline{\Omega})$ be $C$-viscosity solutions of (4.6) satisfying (2.15) constructed in Theorem 4.2. Since $F_\varepsilon$ and $f_\varepsilon$ are continuous, it is known to see that $u_\varepsilon$ is an $L^p$-viscosity solution of (4.6). We refer to [9] for instance. Furthermore, recalling (4.1), thanks to Theorem 2.10, we find a modulus of continuity $\omega_0$ such that

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq \omega_0(|x - y|) \quad \text{for } x, y \in \overline{\Omega}.$$
Hence, by Proposition 2.9, we can find a subsequence \( \varepsilon_k > 0 \) and \( u \in C(\overline{\Omega}) \) such that \( \varepsilon_k \to 0 \), as \( k \to \infty \), and \( u_{\varepsilon_k} \) converges to \( u \) uniformly in \( \overline{\Omega} \). For simplicity, we shall write \( \varepsilon \) for \( \varepsilon_k \).

It remains to show that \( u \) is an \( L^p \)-viscosity solution of (1.1). To this end, we suppose that for some \( \eta \in W^{2,p}_{loc}(\Omega), u - \eta \) attains its local strict maximum at \( x_0 \in \Omega \), and

\[
\min\{\max\{F(x, Du, D^2 u) - f, u - \psi\}, u - \varphi\} \geq 2\theta \quad \text{a.e. in } B_r(x_0) \subseteq \Omega
\]

for some \( \theta, r > 0 \). For the sake of simplicity, we shall suppose \( x_0 = 0 \in \Omega \). Since we may suppose that for small \( \varepsilon > 0 \),

\[
u_{\varepsilon} - \varphi_{\varepsilon} \geq \theta \quad \text{in } B_r,
\]

it is enough to consider the case when \( u_{\varepsilon} \) is an \( L^p \)-viscosity subsolution of

\[
\max\{F_{\varepsilon}(x, Du, D^2 u) - f_{\varepsilon}, u - \psi_{\varepsilon}\} = 0 \quad \text{in } B_r. \tag{4.8}
\]

Thus, Proposition 2.9 implies

\[
u \leq \psi \quad \text{in } B_r.
\]

Hence, \( \eta \in W^{2,p}(B_r) \) satisfies

\[
F(x, D\eta, D^2 \eta) \geq f + \theta \quad \text{a.e. in } B_r. \tag{4.9}
\]

On the other hand, following the argument in the proof of Theorem 4.1 in [9], since \( u_{\varepsilon} \) is an \( L^p \)-viscosity subsolution of (4.8) together with the uniform convergence of \( u_{\varepsilon} \) to \( u \), we obtain that \( u \) is an \( L^p \)-viscosity subsolution of

\[
F(x, Du, D^2 u) - f = 0 \quad \text{in } B_r,
\]

which contradicts (4.9). We only notice that \( \mu D\eta \in L^p(B_r) \) holds true since \( q > n \), and \( \eta \in W^{2,p}(B_r) \) for \( q \geq p \) though \( \mu \) may not be in \( L^\infty \) in (2.5).

\section{5 Local Hölder continuity of derivatives}

It is well-known that we cannot expect solutions of obstacle problems to be in \( C^2 \) even when obstacles are in \( C^2 \). Furthermore, since \( \varphi_0(x) := -|x|^{1+\beta} + 1 \) for \( x \in [-1, 1] \) with \( \beta \in (0, 1) \) is a \( C \)-viscosity solution of

\[
\min\{-u'', u - \varphi_0\} = 0 \quad \text{in } (-1, 1)
\]

under the Dirichlet condition \( g \equiv 0 \), we cannot expect solutions to be in \( W^{2,\infty}_{loc} \) when obstacles only belong to \( C^{1,\beta} \). Notice that since there is no \( C^2 \) function which touches
\( \varphi_0 \) from below at the origin, we do not have to check the definition of \( C \)-viscosity supersolutions at 0.

### 5.1 Estimates in the non-coincidence set \( N[u] \)

We first note that \( L^p \)-viscosity solutions \( u \in C(\Omega) \) of (1.1) are also \( L^p \)-viscosity solutions of

\[
F(x, Du, D^2 u) - f(x) = 0 \quad \text{in } N[u]. \tag{5.1}
\]

For any compact \( K \subseteq N[u] \), where \( u \in C(\Omega) \) is an \( L^p \)-viscosity solution of (1.1), we show that \( Du \in C^\beta(K) \) for some \( \beta \in (0, \hat{\beta}) \), where \( \hat{\beta} \in (0, 1) \) is the constant in Proposition 2.8.

**Proposition 5.1** (cf. Theorem 2.1 in [31]) Assume (2.16), (2.3), (2.4), (2.5), (2.6), (2.7) and (2.12). Then, there are \( \beta \in (0, \hat{\beta}) \), \( C > 0 \), \( \delta_0 > 0 \) and \( r_1 > 0 \), depending on \( n, \frac{\Lambda}{2}, p, q \), such that if \( u \in C(\Omega) \) is an \( L^p \)-viscosity solution of (1.1), and if (2.19) holds for \( \varepsilon = r_1 \), then \( u \in C^{1,\beta}(N_{r_1}[u]) \). More precisely, if \( B_r(x) \subset N_{r_1}[u] \), then it follows that

\[
|Du(y) - Du(z)| \leq \frac{C|y - z|^\beta}{r^{1+\beta}} \left( \|u\|_{L^\infty(\Omega)} + r^{1+\beta_0} \|f\|_{L^p(\Omega)} \right) \quad \text{for } y, z \in B_{r_1}(x).
\]

**Remark 5.2** For further estimates on \( L^p \)-viscosity solutions of (5.1) under some additional assumptions, we refer to Theorem B.1 in [6]. When \( F \) is given by the maximum of finite uniformly elliptic operators with smooth coefficients, we also refer to [14] for \( C^{2,\alpha} \)-estimates. However, when we have \( \mu \in L^q \) and \( f \in L^p \), we could only expect \( u \) to be in \( W^{2,p} \).

We moreover refer to [33] for some precise equi-continuity estimates on \( C \)-viscosity solutions of (5.1) when \( \mu \equiv 0 \) in (2.5) (i.e. \( F \) is independent of \( \xi \in \mathbb{R}^n \)).

Before going to the proof of Proposition 5.1, we first show a lemma corresponding to Lemma 2.3 in [31]. See also [4,5].

For a modulus of continuity \( \rho \) and a constant \( K > 0 \), we introduce

\[
C(\rho, K; \overline{\Omega}) := \left\{ \xi \in C(\overline{\Omega}) \mid \frac{|\xi(x) - \xi(y)|}{|x - y|} \leq \rho(|x - y|) \quad \text{for} \quad x, y \in \partial \Omega, \text{ and } \|\xi\|_{L^\infty(\Omega)} \leq K \right\}.
\]

**Lemma 5.3** (cf. Lemma 2.3 in [31]) Assume (2.16), (2.4), (2.5) and (2.6) with \( \Omega = B_2 \). For given \( G : B_2 \times \mathbb{R}^n \times S^n \to \mathbb{R} \), we let

\[
g^*(x) := \sup\{ |G(x, \xi, X)| \mid \xi \in \mathbb{R}^n, X \in S^n \}.
\]

For a modulus of continuity \( \rho \), and for constants \( K, \varepsilon > 0 \) and \( p' \in (n, p) \), there exists \( \delta_0 = \delta_0(\varepsilon, p', n, \frac{\Lambda}{2}, p, q, \rho, K) \in (0, 1) \) such that if

\[
\|g^*\|_{L^n(B_2)} \vee \|\mu\|_{L^{p'}(B_2)} \vee \sup_{x \in B_2} \|\theta(x, \cdot)\|_{L^n(B_2)} \leq \delta_0,
\]

(5.2)
then for any two $L^p$-viscosity solutions $v$ and $\xi \in C(\rho, K; \overline{B}_2)$ of

$$F(x, Du, D^2u) + G(x, Du, D^2u) = 0 \quad \text{in } B_2$$

and

$$F(0, 0, D^2u) = 0 \quad \text{in } B_2,$$

respectively, satisfying $(v - \xi)|_{\partial B_2} = 0$, it follows that

$$\|v - \xi\|_{L^\infty(B_2)} \leq \varepsilon.$$

**Remark 5.4** We notice that $\|\mu\|_{L^{p'}(B_2)} \leq \delta_0$ in (5.2) for $p' \in (n, p)$ because we do not know if the equi-continuity of $v_k$ holds true in the proof below when $p' = n$.

**Proof** We argue by contradiction. Thus, suppose that there are $\hat{\varepsilon} > 0$, $v_k, \xi_k \in C(\rho, K; \overline{B}_2)$, $\mu_k \in L^q(B_2)$, $G_k : B_2 \times \mathbb{R}^n \times S^n \to \mathbb{R}$, and $F_k : B_2 \times \mathbb{R}^n \times S^n \to \mathbb{R}$ satisfying (2.5) with $\mu_k$;

$$\mathcal{P}^-(X - Y) - \mu_k(x)|\xi - \eta| \leq F_k(x, \xi, X) - F_k(x, \eta, Y) \leq \mathcal{P}^+(X - Y)$$

$$+ \mu_k(x)|\xi - \eta|$$

for $x \in B_2, \xi, \eta \in \mathbb{R}^n, X, Y \in S^n$ (for $k \in \mathbb{N}$) such that

$$\|g_k^+\|_{L^p(B_2)} \lor \|\mu_k\|_{L^{p'}(B_2)} \lor \sup_{x \in B_2} \|\theta_k(x, \cdot)\|_{L^p(B_2)} \leq \frac{1}{k},$$

(5.3)

where $g_k^+(x) = \sup\{|G_k(x, \xi, X)| \mid (\xi, X) \in \mathbb{R}^n \times S^n\}$, and

$$\theta_k(x, y) = \sup_{X \in S^n} \frac{|F_k(x, 0, X) - F_k(y, 0, X)|}{1 + \|X\|},$$

$v_k$ and $\xi_k$ are, respectively, $L^p$-viscosity solutions of

$$F_k(x, Dv_k, D^2v_k) + G_k(x, Dv_k, D^2v_k) = 0 \quad \text{and} \quad F_k(0, 0, D^2\xi_k) = 0 \quad \text{in } B_2,$$

which satisfy that $(v_k - \xi_k)|_{\partial B_2} = 0$, and

$$\|\xi_k - v_k\|_{L^\infty(B_2)} \geq \hat{\varepsilon}.$$  

(5.4)

Since we may suppose that there are $v, \xi \in C(\rho, K; \overline{B}_2)$ such that $v_k$ and $\xi_k$ converges to $v$ and $\xi$ uniformly in $\overline{B}_2$, respectively, and $v = \xi$ on $\partial B_2$. Because the mapping $X \in S^n \to F_k(0, 0, X)$ is bounded by (2.5), we may suppose $F_k(0, 0, X)$ converges to $F_\infty(X)$, which satisfies

$$F_\infty(O) = 0, \quad \text{and} \quad \mathcal{P}^-(X - Y) \leq F_\infty(X) - F_\infty(Y) \leq \mathcal{P}^+(X - Y).$$
We also notice that by (2.5) and our assumption (5.3),
\[
\lim_{k \to \infty} \sup \{ \| F_k(\cdot, \xi, X) - F_\infty(X) \|_{L^n(B_2)} \mid |\xi| \leq R, \|X\| \leq R \} = 0
\]
holds for each \( R > 0 \). Hence, since \( F_\infty \) is continuous, in view of Lemma 1.7 in \([31]\), we verify that \( v \) and \( \xi \) are \( L^n \)-viscosity (thus, \( C \)-viscosity) solutions of
\[
F_\infty(D^2u) = 0 \quad \text{in} \quad B_2.
\]
Therefore, the comparison principle implies that \( v = \xi \) in \( \overline{B}_2 \), which contradicts (5.4).
\[ \Box \]

Although our proof of Proposition 5.1 follows by the same argument as in \([4, 31]\), we give a proof because we need some modification.

**Proof of Proposition 5.1** Recalling \( \beta_0 := 1 - \frac{n}{p} \in (0, 1) \) and \( \hat{\beta} \in (0, 1) \) from Proposition 2.8, we fix \( \beta \in (0, 1) \) and \( p' > n \) such that
\[
0 < \beta < \beta_0 \land \hat{\beta} \quad \text{and} \quad p' := \frac{p + n}{2} \in (n, p).
\]

For small \( s \in (0, 1) \), which will be fixed later, setting
\[
\varepsilon := K_2 s^{1+\hat{\beta}},
\]
we choose \( \delta_0 = \delta_0(\varepsilon, p', n, \frac{N}{k}, p, q, \rho) \in (0, 1) \) in Lemma 5.3, where the modulus of continuity \( \rho \) is given by
\[
\rho(r) = K_1 r^{\hat{\alpha}}.
\]

Now, we set \( \hat{u}(x) := N^{-1} u(\sigma x) \) for \( \sigma \in (0, \frac{1}{2}) \), where
\[
N = N(y) := 1 \vee \left( 2\|u\|_{L^\infty(B_2(y))} + \frac{2^\beta + 1}{\delta_0} \sup_{0 < r \leq 2} \frac{1}{r^{\hat{\beta}}} \|f\|_{L^n(B_r(y))} \right).
\]

We shall suppose \( y = 0 \) for simplicity.

It is immediate to see that \( \hat{u} \) is an \( L^p \)-viscosity subsolution and supersolution, respectively, of
\[
P^-(D^2u) - \hat{\mu}(x)|Du| - \hat{f}(x) = 0 \quad \text{and} \quad P^+(D^2u) + \hat{\mu}(x)|Du| - \hat{f}(x) = 0 \quad \text{in} \quad B_2,
\]
where \( \hat{\mu}(x) := \sigma \mu(\sigma x) \) and \( \hat{f}(x) = \frac{\sigma^2}{N} f(\sigma x) \). Thus, by Proposition 2.6, we have
\[
|\hat{u}(x) - \hat{u}(y)| \leq K_1 |x - y|^{\hat{\alpha}} \left( \|\hat{u}\|_{L^\infty(B_2)} + \frac{1}{N} \|f\|_{L^n(B_2)} \right) \leq K_1 |x - y|^{\hat{\alpha}}. \tag{5.5}
\]
Notice that the last inequality is derived because of our choice of $\delta_0$ and $N$.

For $s \in (0, s_0]$, where $s_0 := 2^{-\frac{1}{\beta}}$, we shall find affine functions $\ell_k(x) = a_k + \langle b_k, x \rangle$ such that

$$
\begin{align*}
(i) & \quad \|\hat{u} - \ell_k\|_{L^\infty(B_{2s_k})} \leq s^{k(1+\beta)}, \\
(ii) & \quad |a_{k-1} - a_k| \vee s^{k-1}|b_{k-1} - b_k| \leq K_2s^{(k-1)(1+\beta)}, \\
(iii) & \quad |(\hat{u} - \ell_k)(s^kx) - (\hat{u} - \ell_k)(s^ky)| \leq K_1s^{k(1+\beta)}|x - y|^\beta 
\end{align*}
$$

(5.6)

for $x, y \in B_1, k \geq 0$, where $\ell_{-1} = \ell_0 \equiv 0$. When $k = 0$, it is trivial to check (i) and (ii) while (iii) holds by (5.5).

By induction, assume that (5.6) holds for $k = j$. Setting

$$
v(x) := s^{-j(1+\beta)}\left(\hat{u}(s^jx) - \ell_{j}(s^jx)\right),
$$

we observe that $v$ is an $L^p$-viscosity solution of

$$
F_j(x, Du, D^2u) + \hat{G}_j(x, Du, D^2u) - f_j(x) = 0 \quad \text{in } B_2,
$$

where

$$
F_j(x, \xi, X) := \frac{\sigma^2s^{j(1-\beta)}}{N} \left(F\left(\sigma s^jx, \frac{Ns^{j\beta}}{\sigma} \xi, \frac{N}{\sigma^2s^{j(1-\beta)}}X\right) - F\left(\sigma s^jx, \frac{Ns^{j\beta}}{\sigma} \xi, \frac{N}{\sigma^2s^{j(1-\beta)}}X\right) + b_j\right),
$$

and

$$
\hat{G}_j(x, \xi, X) := \frac{\sigma^2s^{j(1-\beta)}}{N} \left\{F\left(\sigma s^jx, \frac{Ns^{j\beta}}{\sigma} (s^j\xi + b_j), \frac{N}{\sigma^2s^{j(1-\beta)}}X\right) - F\left(\sigma s^jx, \frac{Ns^{j\beta}}{\sigma} \xi, \frac{N}{\sigma^2s^{j(1-\beta)}}X\right) + b_j\right\}.
$$

We note that for $(\xi, X), (\eta, Y) \in \mathbb{R}^n \times S^n$,

$$
\mathcal{P}^-(X - Y) - \hat{\mu}_j(x)|\xi - \eta| \leq F_j(x, \xi, X) - F_j(x, \eta, Y) \leq \mathcal{P}^+(X - Y) + \hat{\mu}_j(x)|\xi - \eta|,
$$

where $\hat{\mu}_j(x) = \sigma s^j\mu(\sigma s^jx)$. Also, since

$$
|\hat{G}_j(x, \xi, X)| \leq \sigma |b_j|s^{j(1-\beta)}\mu(\sigma s^jx) \quad \text{for } (x, \xi, X) \in \Omega \times \mathbb{R}^n \times S^n,
$$

setting $\hat{g}_j^\ast(x) := \sup\{|\hat{G}_j(x, \xi, X)| \mid \xi \in \mathbb{R}^n, X \in S^n\}$, we have

$$
\|\hat{g}_j^\ast\|_{L^p(B_2)} \leq \frac{|b_j|}{s^{j\beta}}\|\mu\|_{L^p(B_{2s_j})} \leq 2b_0\sigma^{\beta_0}|b_j|\omega_n\|\mu\|_{L^p(B_{2s_j})},
$$

(5.7)
where $\omega_n := |B_1|^{\frac{1}{\beta} - \frac{1}{\beta'}}$. Hence, we immediately verify that $v$ is an $L^p$-viscosity sub-solution and supersolution, respectively, of

$$\mathcal{P}^-(D^2u) - \hat{\mu}_j |Du| - g_j = 0 \text{ and } \mathcal{P}^+(D^2u) + \hat{\mu}_j |Du| + g_j = 0 \text{ in } B_2,$$

where

$$g_j(x) := |f_j(x)| + \hat{g}_j^*(x).$$

In view of the assumption of our induction, we have

$$|b_j| \leq K_2 \sum_{k=0}^{j-1} s^k \beta \leq K_2 \frac{1}{1 - s \beta} \leq 2K_2$$

for $j \geq 1$ because $0 < s \leq s_0 = 2^{\frac{1}{\beta'}}$. Simple calculations together with our choice of $N$ and (5.7) give

\begin{align*}
(1) \quad & \|f_j\|_{L^p(B_2)} \leq \frac{\delta_0}{2}, \\
(2) \quad & \|\hat{\mu}_j\|_{L^{p'}(B_{2\sigma s_j})} \leq \sigma^{1 - \frac{p}{p'}} \|\mu\|_{L^{p'}(B_{2\sigma s_j})}, \\
(3) \quad & \|\hat{g}_j^*\|_{L^n(B_2)} \leq 2^{1+\beta_0} K_2 \omega_n \sigma \beta_0 \|\mu\|_{L^n(B_{2\sigma s_j})}.
\end{align*}

Now, we can choose $\sigma \in (0, 1)$, independent of $j \geq 0$, such that

$$\|\hat{\mu}_j\|_{L^{p'}(B_2)} \vee \|g_j\|_{L^n(B_2)} \leq \delta_0.$$ 

Because $N \geq 1$ and $\sigma \in (0, 1)$, we verify that

$$0 \leq \theta_{F_j}(x, y) \leq \theta(x, y),$$

where $\theta_{F_j}(x, y) := \sup_{X \in S^n} |F_j(x, 0, X) - F_j(y, 0, X)}/(1 + \|X\|)$.

Let $h \in C(B_1)$ be a $C$-viscosity solution of

$$F_j(0, 0, D^2u) = 0 \text{ in } B_1$$

satisfying $h = v$ on $\partial B_1$. Hence, in view of Lemma 5.3, we have

$$\|v - h\|_{L^{\infty}(B_1)} \leq \epsilon = K_2 s^{1+\beta}.$$ (5.8)

We define

$$\ell_{j+1}(x) := \ell_j(x) + s^{j(1+\beta)} \left(h(0) + \langle Dh(0), \frac{x}{s^j} \rangle\right).$$
Since we observe that for $|x| \leq 1$, by (5.8) and the fact $h \in C^{1+\hat{\beta}}(\overline{B}_2)$, we have

$$|\hat{u}(2s^{j+1}x) - \ell_{j+1}(2s^{j+1}x)| \leq s^{(j+1)(1+\beta)}s^{\hat{\beta}-\beta} \left( K_2 + 2^{1+\hat{\beta}}K_2 \right)$$

for $s \in (0, s_1]$, where $s_1 := s_0 \wedge (K_2 + 2^{1+\hat{\beta}}K_2)^{-\frac{1}{\hat{\beta}-\beta}}$ by (5.8), (i) holds for $k = j + 1$.

To show (ii) for $k = j + 1$, by Proposition 2.8, we first verify

$$\|h\|_{C^{1}(\overline{B}_2)} \leq K_2 \|h\|_{L^\infty(B_1)} = K_2 \max_{\partial B_1} |v| \leq K_2.$$  

Thus, noting $|b_{j+1} - b_j| = s^j|Dh(0)|$ and $a_{j+1} - a_j = s^{j(1+\beta)}h(0)$, we obtain (ii) for $k = j + 1$.

In order to see (iii) for $k = j + 1$, setting

$$\hat{v}(x) := v(x) - h(0) - \langle Dh(0), x \rangle,$$

we observe that for $x \in B_1$,

$$|\hat{u}(2sx)| \leq |v(2sx) - h(2sx)| + |h(2sx) - h(0) - \langle Dh(0), 2sx \rangle|$$

$$\leq 2K_2s^{1+\hat{\beta}}$$

$$= 2s^{1+\beta}s^{\hat{\beta}-\beta}K_2.$$  

Thus, for $s \in (0, s_2]$, where $s_2 := 1/(8K_2)^{\frac{1}{\hat{\beta}-\beta}}$, we have

$$\|\hat{v}\|_{L^\infty(B_2)} \leq s^{1+\beta}.$$

We next verify that $\hat{v}$ is an $L^p$-viscosity solution of

$$F_j(x, Du, D^2u) + G_j(x, Du, D^2u) - f_j = 0$$

in $B_2$,

where

$$G_j(x, \xi, X) := \frac{\sigma^2s^{j(1-\beta)}}{N} \left\{ F \left( \sigma s^j x, \frac{N \sigma}{\sigma} (s^j \xi + b_j + s^j \beta Dh(0)), \frac{N}{\sigma^2s^{j(1-\beta)}} X \right) 
- F \left( \sigma s^j x, \frac{N s^j \xi}{\sigma}, \frac{N}{\sigma^2s^{j(1-\beta)}} X \right) \right\}.$$  

Hence, as before we observe that

$$|G_j(x, \xi, X)| \leq \sigma |b_j + s^j \beta Dh(0)|s^{j(1-\beta)}\mu(\sigma s^j x)$$

$$=: h_j(x) \text{ for } (x, \xi, X) \in \Omega \times \mathbb{R}^n \times S^n.$$
Since we have
\[ \|f_j\|_{L^n(B_{2s})} \leq \frac{s^\beta}{2}, \]
and
\[ \|h_j\|_{L^n(B_{2s})} \leq \frac{3K_2}{s^\beta} \|\mu\|_{L^n(B_{2s})} \leq 6K_2 \omega_n s^{(j+1)\beta_0} \|\mu\|_{L^p(B_{2s})} \]
\[ \leq 6K_2 \omega_n s^{\beta_0} \|\mu\|_{L^p(B_{2s})}, \]
we see that \( x, y \in B_s \),
\[ |\hat{v}(x) - \hat{v}(y)| \leq K_1 s^{1+\beta-\hat{\alpha}}|x - y|^{\hat{\alpha}} \left( \frac{3}{4} + 6s^{\beta_0-\beta} K_2 \omega_n \|\mu\|_{L^p(B_{2s})} \right). \]
Hence, we can choose smaller \( s > 0 \), if necessary, to obtain that
\[ |\hat{v}(x) - \hat{v}(y)| \leq K_1 s^{1+\beta-\hat{\alpha}}|x - y|^{\hat{\alpha}} \text{ for } x, y \in B_s. \]

Now, for \( x, y \in B_1 \), we calculate in the following way:
\[ |(\hat{u} - \ell_{j+1})(s^{j+1}x) - (\hat{u} - \ell_{j+1})(s^{j+1}y)| \]
\[ \leq |(\hat{u} - \ell_j)(s^{j+1}x) - (\hat{u} - \ell_j)(s^{j+1}y) - s^{j(1+\beta)}(Dh)(0, s(x - y))| \]
\[ = s^{j(1+\beta)}|\hat{v}(sx) - \hat{v}(sy)| \]
\[ \leq K_1 s^{j+1(1+\beta)}|x - y|^{\hat{\alpha}}. \]

Thanks to (ii) of (5.6), we find \( a_\infty \in \mathbb{R} \) and \( b_\infty \in \mathbb{R}^n \) such that \( (a_k, b_k) \to (a_\infty, b_\infty) \) as \( k \to \infty \). For any \( x \in B_1 \), we choose \( k \in \mathbb{N} \) such that
\[ 2s^{k+1} \leq |x| < 2s^k. \]
Since (i) of (5.6) yields
\[ |\hat{u}(x) - a_k - \langle b_k, x \rangle| \leq s^{k(1+\beta)} \leq \frac{1}{(2s)^{1+\beta}} |x|^{1+\beta}, \]
by sending \( k \to \infty \), it follows
\[ |\hat{u}(x) - a_\infty - \langle b_\infty, x \rangle| \leq \frac{1}{(2s)^{1+\beta}} |x|^{1+\beta}. \]
Therefore, it is standard to establish the H"{o}lder continuity of \( Du \) with its exponent \( \beta \).
See [22] or [1] for instance. \( \square \)
5.2 Estimates near the coincidence set

We next prove that the first derivative of $L^p$-viscosity solutions $u$ of (1.1) is Hölder continuous with exponent $\beta_0 \wedge \beta_1$ near the coincidence set $C^\pm[u]$, where $u$ touches one of the obstacles.

In what follows, for the $L^p$-viscosity solution $u \in C(\Omega)$ of (1.1), we use the notation of $\epsilon$-neighborhood of $C^\pm[u]$ for small $\epsilon > 0$;

$$C^\pm_{\epsilon}[u] := \{x \in \Omega | \text{dist}(x, C^\pm[u]) < \epsilon\}.$$

**Lemma 5.5** Assume (2.16), (2.3), (2.4), (2.5), (2.6), (2.18) and (2.17). Then, for small $\epsilon > 0$, there exists $\hat{C}_0 = \hat{C}_0(\epsilon) > 0$ such that if $u \in C\left(\Omega\right)$ is an $L^p$-viscosity solution of (1.1), and $x_0 \in C^-[u] \cap \Omega_\epsilon$ (resp., $C^+[u] \cap \Omega_\epsilon$), then it follows that

$$|u(x) - u(x_0) - \langle D\phi(x_0), x - x_0 \rangle| \leq \hat{C}_0 r^{1+\beta_2}$$

(resp., $|u(x) - u(x_0) - \langle D\psi(x_0), x - x_0 \rangle| \leq \hat{C}_0 r^{1+\beta_2}$)

for $x \in B_r(x_0)$. In particular, $u$ is differentiable at $x_0$, and

$$Du(x_0) = D\phi(x_0) \quad (\text{resp., } Du(x_0) = D\psi(x_0)).$$

**Proof** We consider the case when $x_0 \in C^-[u] \cap \Omega_\epsilon$; $(u - \varphi)(x_0) = 0$. For simplicity of notations, we shall suppose $x_0 = 0 \in C^-[u] \cap \Omega_\epsilon$.

Because of (2.18), we choose small $r > 0$ such that

$$u(x) < \psi(x) \quad \text{in } B_{4r}.$$

Hence, setting $v(x) := u(x) - \varphi(0) - \langle D\varphi(0), x \rangle + Ar^{1+\beta_1}$ for a large $A > 0$, we observe that $v$ is a nonnegative $L^p$-viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu |Du| + f^- + |D\varphi(0)|\mu = 0 \quad \text{in } B_{4r}.$$

In view of Proposition 2.4, there is $\epsilon_0 > 0$ such that

$$r^{-\frac{n}{\alpha_0}}\|v\|_{L^{\alpha_0}(B_{2r})} \leq C\left(\inf_{B_{2r}} v + r^{1+\beta_0}\|f^- + \mu\|_{L^p(B_{4r})}\right) \leq C\left(r^{1+\beta_1} + r^{1+\beta_0}\right).$$

Thus, from our choice of $\beta_2$, we have

$$r^{-\frac{n}{\alpha_0}}\|v\|_{L^p(B_{2r})} \leq Cr^{1+\beta_2}. \quad (5.9)$$

On the other hand, we claim that $w := v \vee A'r^{1+\beta_1}$, where $A' > A$, is also an $L^p$-viscosity subsolution of

$$\mathcal{P}^-(D^2u) - \mu |Du| - f^+ - |D\varphi(0)|\mu = 0 \quad \text{in } B_{4r}.$$
Indeed, assuming that $w - \xi$ attains its local maximum at $z \in B_{4r}$ for $\xi \in W^{2,p}(B_{4r})$, we shall conclude the claim. In case of $w(z) = v(z)$, noting

$$u > \varphi \text{ near } z$$

for large $A' > A$, we observe that $w$ is an $L^p$-viscosity subsolution of

$$\mathcal{P}^{-}(D^2u) - \mu |Du| - f^+ - |D\varphi(0)|\mu = 0 \quad (5.10)$$

in $B_r(z)$ for some $\hat{r} > 0$ while in case of $w(z) = A'r^{1+\beta_1}$, we immediately see that any constant is an $L^p$-viscosity subsolution of (5.10). Hence, we verify that $w$ is an $L^p$-viscosity subsolution of (5.10) in $B_{4r}$.

In view of Proposition 2.5, with the above $\varepsilon_0 > 0$, we have

$$\sup_{B_r} v \leq C_1 \left(r^{-\frac{n}{\varepsilon_0}} \|w\|_{L^{\varepsilon_0}(B_{2r})} + r^{1+\beta_0} \|f^+ + \mu\|_{L^p(B_{4r})}\right),$$

where $C_1 = C_1(\varepsilon_0)$ is the constant in Proposition 2.5. This together with (5.9) implies

$$-Cr^{1+\beta_1} \leq u(x) - \varphi(0) - \langle D\varphi(0), x \rangle \leq Cr^{1+\beta_2} \quad \text{in } B_r,$$

which concludes the proof. \hfill \Box

Thanks to Lemma 5.5 with Proposition 5.1, we easily obtain Theorem 2.12. We give a brief proof though it seems standard.

**Proof of Theorem 2.12** In view of Proposition 5.1, to complete the assertion, we may suppose $x, y \in C_{2r_1}^+ [u] \cap \Omega_{2r_1}$. Furthermore, by Lemma 5.5, we may suppose that $x, y \in C_{2r_1}^- [u]$, and $0 < \text{dist}(y, C^- [u]) \leq \text{dist}(x, C^- [u])$. Choose $\hat{x}, \hat{y} \in C^- [u]$ such that $|x - \hat{x}| = \text{dist}(x, C^- [u])$ and $|y - \hat{y}| = \text{dist}(y, C^- [u])$. Thus, we have

$$0 < |y - \hat{y}| \leq |x - \hat{x}|.$$

**Case 1:** $|x - y| < \frac{1}{2} |x - \hat{x}|$. In view of Proposition 5.1, for any $\beta \in (0, \beta_0 \wedge \hat{\beta})$, we easily obtain

$$|Du(x) - Du(y)| \leq C |x - y|^\beta.$$

**Case 2:** $|x - y| \geq \frac{1}{2} |x - \hat{x}| \geq \frac{1}{2} |y - \hat{y}|$. In view of Lemma 5.5, we have

$$|Du(x) - Du(y)| \leq |Du(x) - Du(\hat{x})| + |Du(\hat{x}) - Du(\hat{y})| + |Du(\hat{y}) - Du(y)| \leq C (|x - \hat{x}|^{\beta_2} + |y - \hat{y}|^{\beta_2}) + C |\hat{x} - \hat{y}|^{\beta_1},$$

which is estimated by $C |x - y|^{\beta_2}$ in this case.

Therefore, combining these cases, we obtain the desired estimate. \hfill \Box
Acknowledgements  The authors thank the referees for their careful reading, and several valuable comments, which help us to improve the original manuscript.

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