Nonperturbative stochastic dynamics driven by strongly correlated colored noises

Jun Jing\textsuperscript{1,2,3} \textsuperscript{*} Rui Li\textsuperscript{2} \textsuperscript{*} J. Q. You\textsuperscript{2} \textsuperscript{*} and Ting Yu\textsuperscript{1,2} \textsuperscript{†}

\textsuperscript{1}Center for Controlled Quantum Systems and Department of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030, USA
\textsuperscript{2}Beijing Computational Science Research Center, Beijing 100084, China
\textsuperscript{3}Institute of Atomic and Molecule Physics, Jilin University, Changchun 130012, China

(Dated: June 30, 2014)

We establish for the first time an exact stochastic dynamical equation for an open quantum systems coupled to correlated noises. We rigorously investigate the nonequilibrium dynamics of a qubit system under the influence of two strongly correlated non-Markovian baths by exactly solving the stochastic Schrödinger equation of an open quantum system. We show that the quantum dynamics of the qubit system is profoundly modulated by the mutual correlation between baths and the bath memory times through dissipation and fluctuation. We report a new physical effect on generating large values of the inner-correlation and entanglement of the distant two qubits arising from the strong bath correlation for a class of super-radiation states.

PACS numbers: 42.50.Lc, 03.65.Yz, 05.40.-a

Introduction.—Nonequilibrium quantum open system involves understanding the quantum fluctuation and dissipation processes arising from the interaction between the open system and its surrounding environment \cite{1}. When the environment consists of two or more mutually-interacting sub-environments, the correlations between the sub-environments will bring about new physics into the nonequilibrium quantum relaxation and decoherence across wide parameter regions interpolating totally uncorrelated individual environments and a single common environment. A deep understanding of the multi-correlation dynamical processes of open qubit systems holds enormous promise for both fundamental studies of non-Markovian quantum phenomena as well as practical applications to quantum information processing and quantum devices.

There are several approaches to solving quantum open systems coupled to an environment, including, e.g., the perturbative master equations \cite{2,3}, the quantum trajectories \cite{4,5}, the dressed state method for simple open systems \cite{9}, Feynman-Vernon influence functional \cite{10-12}, and the projection operator approach that has been extensively used in statistical physics \cite{13,14}. In a new approach, a stochastic Schrödinger equation called non-Markovian quantum state diffusion (QSD) equation driven by a complex Gaussian noise was developed to describe a quantum open system that is coupled to a generic bosonic environment \cite{15,16,17,18}. However, when the acting environment is comprised of multi-correlated parts, each consisting of finite or infinite degrees of freedom, it is still an extremely challenging task to establish a fully quantized theory to describe non-Markovian dynamics accurately, especially when the system-environment coupling is strong or ultrastrong, and when the acting environment cannot be approximated by a Markov bath \cite{19}. In this Letter, we develop a nonperturbative approach to studying the nonequilibrium quantum dynamics of a general microscopic non-Markovian spin-dissipation model with two uncoupled qubits (e.g., two-level atoms) as the open system. The environment here is formed by two baths that are correlated through interchanging quanta (i.e., photons). The major difficulty of the study arises from the fact that the usual procedure of converting the functional derivative existing in the stochastic Schrödinger equation into an tractable local operator breaks down when the baths become correlated.

Our strategy of solving this difficult issue is to introduce two tunable uncorrelated artificial baths that simulate the two correlated baths. Our approach recovers the known results for an ordinary common bath (see, e.g., \cite{20}) as well as the case for two uncorrelated individual baths (see, e.g., \cite{21,22}) as limiting cases, and reveals a new coherence dynamics in the strong correlation regime. For a single common bath, it is interesting to note that the quantum dynamics of the qubit system may be protected, wholly or partially due to the symmetry created by the balanced system-environment coupling. For the limiting case corresponding to the two individual baths, typical for the distant qubits in quantum communication and teleportation, it is known that the quantum dynamics involving decoherence and disentanglement will be dominated by an irreversible process when the environment has a broadband spectrum \cite{23,24}. The more general correlated dynamics will inevitably involve multiscale quantum dissipation and quantum backaction induced by the strong bath-interaction, and a genuine correlated quantum dynamics theory must take into account a wide range of memory parameters and the bath coupling strengths in a consistent way. Here our exact quantum theory accounts for all of these important phenomena, and in addition accurately recovers the limiting Markov theory of the correlated environments.

The model and the exact QSD equation.—The total Hamiltonian of two central qubits dissipatively coupled to two correlated baths respectively is given by (setting
where $\tilde{g}_k (\tilde{f}_k)$ is the coupling strength between qubit $A$ ($B$) and bath $d$ ($e$). We assume that $\tilde{f}_k = \kappa \tilde{g}_k$, where $\kappa$ represents the anisotropy degree of the coupling parameter for the two qubits. For simplicity, both baths $d$ and $e$ are supposed to be identical with the same correlation function and at zero temperature. Note that $\lambda_k$ is the correlation strength between every pair of modes in two baths. By using Bogoliubov transformation,

$$d_k = (a_k - b_k)/\sqrt{2}, \quad e_k = (a_k + b_k)/\sqrt{2},$$

(2)

the original Hamiltonian can be converted to the following form,

$$H_{\text{tot}} = H_S + H_I + H_R,$$

(3)

$$H_S = \frac{\omega_A}{2} \sigma^+_A \sigma^-_A + \frac{\omega_B}{2} \sigma^+_B \sigma^-_B,$$

$$H_I = \sum_k g_k \left[ (\sigma^+_A + \kappa \sigma^+_B) a^+_k + (\sigma^-_A - \kappa \sigma^-_B) b^+_k \right] + \text{h.c.},$$

$$H_R = \sum_k \omega_k a^+_k a_k + \sum_k \omega_k b^+_k b_k,$$

where $\omega_k^a = \omega_k + \lambda_k$, $\omega_k^b = \omega_k - \lambda_k$, and $g_k = \tilde{g}_k/\sqrt{2}$. Therefore, we have shown that the correlated baths can be mapped into two uncorrelated fictitious baths, each coupled to two qubits simultaneously, a new feature caused by the transformation. The coupling between the two physical bath $d$ and $e$ has been shifted into a modification to the frequencies in the new structured baths $a$ and $b$. It turns out that this formal transformation has paved an important way of controlling the correlated dynamics by the parameter $\lambda_k$. Note that when $\lambda_k = 0$, i.e., there is no interaction between the two baths, then the model simply reduces to the case with two separable baths [22]. However, it should be noted from Eq. (3) that the qubit quantum dynamics is still significantly different from the case where each qubit is only coupled to a local bath. When $|\lambda_k| = \omega_k$ (the resonant condition as well as the ultrastrong correlation regime), the model reduces to another limiting case with just one common bath [21].

In order to clearly and reliably demonstrate the effect of correlated colored noises on the system, we choose model [11] because of its exact solvability. It can be shown that the exact linear QSD equation, a time-local convolutionless equation, describing quantum trajectories by the Hamiltonian (3) can be formally written as [22]

$$\frac{\partial}{\partial t} \psi(t) = \left( -i H_S + L_a z_{at} - L_b z_{bt} - L_a O_a + L_b O_b \right) \psi(t),$$

(4)

where the two coupling operators are $L_a = \sigma^+_A + \kappa \sigma^+_B$, $L_b = \sigma^-_A - \kappa \sigma^-_B$, and $O_x \equiv O_x(t, z_a, z_b) = \int_0^t ds a_x(t, s) O_x(t, s, z_a, z_b)$, with $O_x$ explicitly given by

$$O_x = f_{x1}(t, s) \sigma^+_A + f_{x2}(t, s) \sigma^-_A + f_{x3}(t, s) \sigma^+_A \sigma^-_A + i \int_0^t ds' p_{x2}(s, s') z_{as},$$

$$+ \int_0^t ds' p_{x3}(s, s') z_{bs}, \quad x = a, b.$$  

(5)

The initial conditions for these O-operators are $O_x(s, s, z_a, z_b) = L_x$. Here $x = a, b$ denote the two baths (sources of environmental noise). $z_{xt} = -i \sum_k g_k z_{kx} e^{i \omega_k t}$ describes a time-dependent, complex Gaussian process that statistically satisfies $M[z_{xt}] = M[z_{xt} z_{xs}] = 0$, and $M[z_{kt}^* z_{ks}] = \alpha_x(t, s)$, where $M[\cdot]$ stands for the ensemble average over the noise $z_{kt}$. It is important to note that $\alpha_x(t, s)$ is an arbitrary correlation function for bath $x$. Note that each O-operator explicitly contains the integrals over noises from both baths due to the fact that baths $a$ and $b$ are indirectly connected with each other through coupling to the system, an important feature that is not encountered in the cases with the local baths [26] or the common bath [21]. It is interesting to note that a set of partial differential equations for $f$’s (noise-free terms) and $p$’s (noise-integral terms) as well as their boundary conditions can be obtained by the QSD approach. To be more efficient numerically, the non-linear QSD equation [23, 24] for the normalized states $\tilde{\psi}_t(z) = \frac{\psi_t(z)}{|\psi_t(z)|}$, is employed in the following numerical simulations.

Non-Markovian nonequilibrium dynamics.—To show the crossover behavior of nonequilibrium dynamics from the Markov limit to the non-Markovian regime, the spectral density function of the physical baths $d$ or $e$ is assumed to be of a Lorentz form: $S(\omega) = \frac{1}{\pi} \frac{\Gamma^2}{\omega_0^2 - \omega^2}$, where $\Gamma$ is the coupling strength between the system and its bath, $\gamma$ denotes the bandwidth of the bath, and $1/\gamma$ represents the bath memory time. The correlation function is obtained via Fourier transform, $\alpha(t, s) = \sum_k |g_k|^2 e^{-i \omega k (t-s)} = \frac{\Gamma}{2} e^{-\gamma |t-s|}$. When $\gamma$ approaches zero, the bath enters into a very strong non-Markovian regime, and typically causes a non-Markovian system dynamics; when $\gamma$ becomes large or when time approaches infinity, $\alpha(t, s) \rightarrow \Gamma \delta(t-s)$, recovering the well-known Markov limit. By definition, the correlation functions for the structured baths $a$ and $b$ are

$$\alpha_x(t, s) = \sum_k g_k^2 e^{-\omega_k (t-s)} = \frac{\Gamma}{2} e^{-\gamma |t-s|},$$

with $x = a, b$. For simplicity, it is supposed that $\lambda_k = \lambda \omega_k$, where the dimensionless parameter $\lambda (|\lambda| \leq 1)$ is used to measure the correlation strength between the physical baths. It is easy to check that when $|\lambda| < 1$, $\alpha_x(t, s) = \frac{\Gamma}{2} e^{-\gamma |t-s|}$, where $\Gamma = \bar{\Gamma}/2$, $\gamma_a = \gamma(1 + \lambda)$, and $\gamma_b = \gamma(1 - \lambda)$; when $|\lambda| = 1$, the correlation function of the effective common bath is $\frac{\Gamma}{2} e^{-2 \gamma |t-s|}$. Here we do not consider the case
with $|\lambda| > 1$. Otherwise the system may become unphysical and lose its positivity.

With the nonlinear QSD equation, the correlated Gaussian noises $z_{xt}$ and the time-evolution function for $\hat{O}_x$, we can efficiently simulate the exact dynamics of the central qubit system by solving the exact stochastic Schrödinger equation: $\rho_t = M[|\psi_t\rangle\langle\psi_t|]$. Below we discuss the nonequilibrium quantum dynamics by examining the inner-correlation $\langle \sigma_x \sigma_x \rangle$ and entanglement $C(t)$ measured by concurrence $[28]$ of the two-qubit system. The parameters are chosen as $\kappa = 1$, $\Gamma = \omega_A = \omega_B = \omega$ and the results are obtained by 1000 quantum trajectories, which are sufficient for the QSD equation to attain convergence for our model.

![Diagram](image)

**FIG. 1.** (Color online) Inner-correlation dynamics with different values of $\gamma$ and positive bath-bath correlation strength (blue solid line for $\lambda = 0.2$, red dashed line for $\lambda = 0.6$, and black dot-dashed line for $\lambda = 1.0$). In (a), (b) and (c), $\psi_0 = (1/\sqrt{2})(|11\rangle + |00\rangle)$; in (d), (e) and (f), $\psi_0 = (1/\sqrt{2})(|10\rangle + |01\rangle)$.

In Fig. 1 we observe the behavior of the inner-correlation between qubit-A and qubit-B along the $x$-direction, $C_{xx} \equiv \langle \sigma_x^A \sigma_x^B \rangle$ describing a collective property relevant to the system coherence, with different initial entangled-states, environment memory times and positive correlation strengths ($0 < \lambda \leq 1$) between the baths $d$ and $e$. The system is initially prepared in the two-photon entangled state $(1/\sqrt{2})(|11\rangle + |00\rangle)$ and the single-photon entangled state $(1/\sqrt{2})(|10\rangle + |01\rangle)$, respectively. In a strong non-Markovian regime with $\gamma = 0.2$, the inner-correlation of the system shows stronger oscillations in the two-photon entangled state than that in the single-photon entangled state [compare Figs. 1(a) and 1(d)]. For the two-photon entangled state, a larger $\lambda$ gives rise to a greater oscillation. But for the single-photon entangled state, the dynamical behavior induced by $\lambda$ is opposite. In a moderate parameter range $\gamma = 1.0$, for $\psi_0 = (1/\sqrt{2})(|11\rangle + |00\rangle)$ [see Fig. 1(b)], only small bumps appear in the inner-correlation decay process when $\lambda = 0.6$. The resonant condition $\lambda = 1$ can extend the survival time of the inner-correlation. While for the state $(1/\sqrt{2})(|10\rangle + |01\rangle)$ [see Fig. 1(e)], the dynamics is almost $\lambda$-independent. When $\gamma = 5.0$, both baths are effectively in the Markov regime, it is shown that the inner-correlation quickly decays into zero monotonously, irrespective of the initial states [see Figs. 1(c) and 1(f)]. However, we notice that, as a rather counterintuitive result shown in Fig. 1(f), the survival time of the inner-correlation in the resonant condition ($\lambda = 1$) is longer than those in the non-resonant cases. This is because the two qubits are embedded in an effective common bath at $\lambda = 1$, and the state $(1/\sqrt{2})(|10\rangle + |01\rangle)$ serves as a super-radiant state due to the Hamiltonian in Eq. (29).

To get a better picture of nonequilibrium processes of the qubit system, it is useful to consider the Markov limit of this correlated-baths model with the bath correlation functions $\alpha_x(t, s) = \Gamma_x \delta(t - s)$, where $\Gamma_a = \Gamma/(1 + \lambda)$ and $\Gamma_b = \Gamma/(1 - \lambda)$. Consequently, for $\lambda < 1$, the Lindblad master equation should be written as $\dot{\rho}_t = -i[H_S, \rho_t] + \sum_{x=a,b} \Gamma_x D(L_x)$, where $D(L_x) \equiv L_x \rho_t L_x^\dagger - \frac{1}{2} L_x^\dagger L_x \rho_t - \frac{1}{2} \rho_t L_x^\dagger L_x$, and for $\lambda = 1$, $\dot{\rho}_t = -i[H_S, \rho_t] + \frac{\Gamma_b}{\Gamma_a} D(a)$. Considering the special initial state $(1/\sqrt{2})(|10\rangle + |01\rangle)$ (the eigenstate of $L_0$), one can see that the damping rate of the system correlation is $2\Gamma_a = \Gamma^2/\Gamma_{\lambda} + \Gamma$, which increases with decreasing $\lambda$ and is larger than the damping rate $\Gamma$ for the $\lambda = 1$ case. Therefore, in this situation, the $\lambda$-influence on bath-correlation in Markov limit [see Fig. 1(f)] is opposite to that observed in a highly non-Markovian case [see Fig. 1(d)]. This observation is helpful in understanding the insensitivity of the dynamics to $\lambda$ in the intermediate non-Markovian regime [see Fig. 1(e)], where the crossover pattern reflects the tradeoff between the two competing elements: $\lambda$ and $\gamma$.

![Diagram](image)

**FIG. 2.** (Color online) Concurrence of distant qubits under the modulation of correlated bath with different correlation strength $\lambda$. Here $\psi_0 = |10\rangle$ and $\gamma = 1.0$.

**Entangling distant qubits via bath correlations.**—A prompt application of this study is to dynamically entangle two remote qubits. In Fig. 2 we keep $\gamma$ (the memory parameter of baths) invariant, and tune the bath-bath correlation strength $\lambda$ to display the entangling proce-
dure of the system that is initially prepared in the separable state $\psi_0 = |10\rangle$. For a weak bath correlation $\lambda = 0.2$ (see the blue solid curve in Fig. 2), the generated concurrence of the two-qubit system is small and has weak oscillations. When $\lambda$ increases (e.g., $\lambda = 0.8$), the generated concurrence becomes large and exhibits appreciable oscillations (see the red dashed curve in Fig. 2). Moreover, it can be seen that the concurrence at $\lambda = 0.2$ and 0.8 tends to zero when $t \to \infty$. This is because the reduced density operator of the system approaches $\rho_{\infty} = |00\rangle\langle 00|$ in the long-time limit for any $\lambda < 1.0$. However, when $\lambda = 1.0$, which corresponds to the two qubits coupled to an effective common bath, the generated concurrence increases faster and then maintains at the value of 0.5. The reason for the observed results is that the reduced density operator of the system approaches $\rho_{\infty} = \frac{1}{4}|\phi\rangle\langle \phi| + \frac{1}{4}|00\rangle\langle 00|$ in the long-time limit for such a common-bath case, where $|\phi\rangle = (1/\sqrt{2})(|10\rangle - |01\rangle)$ is an eigenstate of $L_0 = \sigma_A^4 + \sigma_B^{+}$, which is the only effective Lindblad operator in the ultrastrong correlation case.

Next, we focus on the novel behaviors of the quantum entanglement induced by the ultrastrong bath correlation with $\lambda = 1.0$ which has not been nonperturbatively studied before. We choose the generalized Werner-state as the initial state: $\rho_0 = \frac{Q}{4}I_4 + (1-Q)|\psi\rangle\langle \psi|$, where $0 < Q < 1$, $I_4$ is the identity matrix in the Hilbert space of the system, and $|\psi\rangle = (1/\sqrt{2})(|10\rangle + |01\rangle)$. It is known that the initial concurrence is $C(0) = \max\{0, 1 - \frac{1}{2}Q\}$. Apparently, when $Q < \frac{1}{2}$, the two-qubit system is initially an entangled state. In this case, the system suffers a fast entanglement decay in a short time [see Fig. 3(a)]. Moreover, the entanglement lifetime becomes significantly shortened when increasing the bath memory parameter $\gamma$, which means that the destructive effect on system entanglement from the Markov bath is more serious than a comparatively non-Markovian bath. After a period of time, the entanglement suddenly revives due to the accumulation of correlation between the two baths. Also, as in Fig. 3(a), a larger $\gamma$ yields an earlier revival of entanglement. This correlation-induced behavior is in sharp contrast to the usual observation that a non-Markovian bath is helpful in enhancing the re-coherence process of system compared with a Markov bath. When $Q > \frac{2}{5}$, then the two qubits evolve from a separable mixed state. In this case, only the re-generation of the quantum entanglement occurs [see Fig. 3(b)]. In the long-time limit, the reduced density operator of the system approaches $\rho_{\infty} = \frac{Q}{4}|\phi\rangle\langle \phi| + (1 - \frac{1}{4}Q)|00\rangle\langle 00|$, so $C(\infty) = Q/4$. Therefore all of the curves in Fig. 3 will reach this value, implying that the steady state is irrelevant to $\gamma$. It turns out that the generated entanglement degree must be larger than its original value when $Q > \frac{2}{5}$. This provides a method to increase the quantum entanglement of two distant qubits via bath correlation rather than the qubit-qubit coupling.

**Conclusion.**—In summary, we have derived for the first time an exact dynamical equation for a correlated quantum open system without invoking Born-Markov approximation. With the exact equation, we study the effect of the correlated baths on the nonequilibrium quantum dynamics of the open quantum system consisting of two uncoupled qubits. The new physical phenomena such as entangling two uncoupled distant qubit induced by the correlated baths rather than the direct correlation between these two qubits are observed. The crossover of the dynamics measured by system inner-correlation and entanglement between the weak and the strong bath correlation $\lambda$ is investigated. Our research has not only shed new light on the current fundamental studies in the nonequilibrium quantum dynamics, but also has many applications in ongoing quantum information processing realizations. In particular, we show that the bath-bath correlation in a realistic context such as distant atoms in a quantum network can play an important role as modulators for achieving large entanglement between two qubits. The treatment and results presented in this paper could also be applicable and generalized to many interesting quantum devices involving multi-correlated environments where a new type of quantum manipulation through bath-bath correlation may be realized.

We acknowledge grant support from the NSF No. PHY-0925174, the NBRPC No. 2014CB921401, the NSFC Nos. 91121015 and 11175110, and the NSAF No. U1330201.

---

* J.J. and R.L. equally contributed to this work.

† Corresponding authors:  J.Q.Y. jyou@scrc.ac.cn,  T.Y. Ting.Yu@stevens.edu

[1] E. Calzetta and B. L. Hu, *Non-equilibrium Quantum Field Theory*. (Cambridge University Press, Cambridge,
England, 2008).

[2] H. P. Breuer, Phys. Rev. A 75, 022103 (2007); Md. Manirul Ali, P.-W. Chen, and H.-S. Goan, Phys. Rev. A 82, 022103 (2010); B. Hwang and H.-S. Goan, Phys. Rev. A 85, 032321 (2012).

[3] S. A. Gurvitz and Ya. S. Prager, Phys. Rev. B 53, 15932 (1996); S. A. Gurvitz Phys. Rev. Lett. 85, 812 (2000).

[4] H. Carmichael, An Open System Approach to Quantum Optics (Springer, Berlin, 1994).

[5] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. 68, 580 (1992).

[6] N. Gisin and I. C. Percival, J. Phys. A 25, 5677 (1992); J. Phys. A 26, 2233 (1993).

[7] C. W. Gardiner, A. S. Parkins and P. Zoller, Phys. Rev. A 46, 4363 (1992).

[8] M.B. Plenio and P. L. Knight, Rev. Mod. Phys. 70, 101 (1998).

[9] P. Alsing and H. J. Carmichael, Quantum Opt. 3, 13 (1991); B. M. Garraway, Phys. Rev. A 55, 4636 (1997); B. J. Dalton, Stephen M. Barnett, and B. M. Garraway, Phys. Rev. A 64, 053813 (2001).

[10] R. P. Feynman and F. L. Vernon, Ann. Phys. 24, 118 (1963).

[11] A. O. Caldeira and A. J. Leggett, Physica (Amsterdam) A. 121 587 (1983).

[12] B. L. Hu, J. F. Paz, and Y. Zhang, Phys. Rev. D 45, 2843 (1992).

[13] S. Nakajima, Prog. Theor. Phys. 20, 948 (1958).

[14] R. Zwanzig, J. Chem. Phys.33, 1338 (1960).

[15] L. Diósi and W. T. Strunz, Phys. Lett. A 235, 569 (1997).

[16] W. T. Strunz, L. Diósi and N. Gisin, Phys. Rev. Lett. 82, 1801 (1999).

[17] T. Yu, L. Diósi, N. Gisin and W. T. Strunz, Phys. Rev. A 60, 91 (1999).

[18] J. Jing and T. Yu, Phys. Rev. Lett. 105, 240403 (2010).

[19] G. Lindblad, Commun. Math Phys. 48, 119 (1976).

[20] J. P. Paz and A. J. Roncaglia, Phys. Rev. Lett. 100, 220401 (2008).

[21] X. Zhao, J. Jing, B. Corn, and T. Yu, Phys. Rev. A 84, 032101 (2011).

[22] I. Sinayskiy, E. Ferraro, A. Napoli, A. Messina, and F. Petruccione, J. Phys. A 42, 485301 (2009).

[23] C. Simon, and J. Kempe, Phys. Rev. A 65, 052327 (2002).

[24] Z. Ficek, R. Tanás, Phys. Rep. 372, 369 (2002).

[25] See supplementary material.

[26] T. Yu and J. H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).

[27] L. Diósi, N. Gisin and W. T. Strunz, Phys. Rev. A 58, 1699 (1998).

[28] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

[29] S. Maniscalco, F. Francica, R. L. Zaffino, N. LoGullo, and F. Plastina, Phys. Rev. Lett. 100, 090503 (2008).