LOCAL COHOMOLOGY—AN INVITATION

ULI WALThER AND WENLIANG ZHANG

ABSTRACT. This article is part introduction and part survey to the mathematical area centered around local cohomology.

CONTENTS

Acknowledgments 3
1. Introduction 3
   1.1. Koszul cohomology 4
   1.2. The Čech complex 4
   1.3. Limits of Ext-modules 5
   1.4. Local duality 6
2. Finiteness and vanishing 7
   2.1. Finiteness properties. 7
   2.2. Vanishing 11
   2.3. Annihilation of local cohomology 16
3. $D$- and $F$-structure 20
   3.1. $D$-modules 20
      3.1.1. Characteristic 0 21
      3.1.2. $D$-modules and group actions 27
      3.1.3. Coefficient fields of arbitrary characteristic 30
   3.2. $F$-modules 32
      3.2.1. $F$-modules 34
      3.2.2. $A\{f\}$-modules: action of Frobenius 35
      3.2.3. The Lyubeznik functor $H_{R,A}$ 40
   3.3. Interaction between $D$-modules and $F$-modules 43
4. Local cohomology and topology 44
   4.1. Arithmetic rank 45
      4.1.1. Some examples and conjectures 45
      4.1.2. Endomorphisms of local cohomology 48
   4.2. Relation with de Rham and étale cohomology 50
      4.2.1. The Čech–de Rham complex 50
      4.2.2. Algebraic de Rham cohomology 53
      4.2.3. Lefschetz and Barth Theorems 54

UW acknowledges support through Simons Foundation Collaboration Grant for Mathematicians #580839, and through NSF Grant DMS-2100288. WZ acknowledges the support by the NSF through DMS-1752081.
This article is a mixture of an introduction to local cohomology, and a survey of the recent advances in the area, with a view towards relations to other parts of mathematics. It thus proceeds at times rather carefully, with definitions and examples, and sometimes is more cursory, aiming to give the reader an impression about certain parts of the mathematical landscape. As such, it is more than a reference list but less than a monograph. One possible use we envision is as a guide for a novice, such as a beginning graduate student, to get an idea what the general thrust of local cohomology is, and where one can read more about certain topics.

While the article is rather much longer than originally anticipated, several active areas that interact with local cohomology have been left out. For instance, we refer the reader to [Har67, Sch82b, Sch98] for connections with dualizing complexes which are not discussed in this article. What we have put into the article is driven by personal preferences and lack of expertise; we apologize to those offended by our choices.

Over time, several excellent survey articles on local cohomology and related themes have been written, and we strongly recommend the reader study the following ones. One should name [Lyu02] on the state of the art 20 years since, the article [NuBWZ16] specifically geared at Lyubeznik numbers, and the survey [Hoc19].

In the more expository direction, we and many others have been fortunate to be able to study Hochster’s unpublished notes (available on his website) and Huneke’s point of view in [Hun07]. These notes come with our highest recommendations and have strongly influenced us and this article. For a treatment de-emphasizing Noetherianness we point at [SS18].

We close this thread of thoughts with mentioning the books concerned with local cohomology as main subject: the original account of Grothendieck as recorded by Hartshorne [Har67], the classic [BS98] by Brodmann and Sharp, and the outgrowth [ILL+07] of a summer school on local cohomology.

Some words on the prerequisites for reading this article are in order. Inasmuch as pure commutative algebra is concerned, we imagine the reader be familiar with the contents of the book by Atiyah and Macdonald [AM69] or an appropriate subset of the book by Eisenbud [Eis95]. For homological
algebra one should know about injective and projective resolutions, Ext and Tor and the principles of derived functors, and perhaps a bit about spectral sequences at the level of Rotman [Rot09]. Hartshorne’s opus [Har77] covers all that is needed on varieties, schemes and sheaves in chapters 1-3.

Acknowledgments

We are grateful to Josep Alvarez, Robin Hartshorne, Jack Jeffries, Peter Schenzel, Craig Huneke, Karl Schwede, Kazuma Shimomoto, and Matteo Varbaro for telling us about corrections and suggestions in the manuscript.

Our main intellectual debt and gratitude is due to Gennady Lyubeznik, our both advisor. We also happily acknowledge the impact our many teachers and collaborators have had on our understanding of the subjects discussed in this article. Special thanks go to Mel Hochster and Anurag K. Singh for patience, insights, and friendship.

1. Introduction

Notation 1.1. Throughout, $A$ will denote a commutative Noetherian ring. On occasion, $A$ will be assumed to be local; then its maximal ideal is denoted by $m$ and the residue field by $k$.

We reserve the symbol $R$ for the case that $A$ is regular, while $M$ will generally denote a module over $A$.

Definition 1.2. For an ideal $I \subseteq A$ the (left-exact) section functor with support in $I$ (also called the $I$-torsion functor) $\Gamma_I(\cdot)$ and the local cohomology functors $H_I^j(\cdot)$ with support in $I$ are

$$\Gamma_I: M \sim \{m \in M | \exists \ell \in \mathbb{N}, I^\ell m = 0\}$$

and its right derived functors $H_I^j(\cdot)$. Since $\Gamma_I(\cdot)$ is left exact, $\Gamma_I(\cdot)$ agrees with $H_I^0(\cdot)$.

Local cohomology was invented by Grothendieck, at least in part, for the purpose of proving Lefschetz and Barth type theorems (comparisons between a smooth ambient variety and a possibly singular subvariety). The idea rests on the fact, already exploited by Serre in [Ser55], that the geometry of projective varieties is encoded in the algebra of its coordinate ring. Grothendieck makes it clear in his Harvard seminar that, for this purpose, studying general properties of the concept of local cohomological dimension is of great importance [Har67, p. 79].

Definition 1.3. The local cohomological dimension $\text{lcd}_A(I)$ of the $A$-ideal $I$ is

$$\text{lcd}_A(I) = \max\{k \in \mathbb{N} | H_I^k(A) \neq 0\}.$$ 

One can show, using long exact sequences and direct limits, that $H_I^{\text{lcd}_A(I)}(M)$ vanishes for every $A$-module $M$. 
It is an essential feature of the theory of local cohomology and its applications that there are several different ways of calculating $H^k_I(M)$ for any $A$-module $M$, all compatible with natural functors. We review briefly three other approaches; for a more complete account we refer to [ILL+07].

1.1. **Koszul cohomology.** Let $x \in A$ be a single element and consider the multiplication map $A \xrightarrow{x} A$ by $x$, also referred to as the *cohomological Koszul complex* $K^\bullet(A; x)$, so the displayed map is a morphism from position 0 to position 1 in the complex. We write $H^i_p(A; x_q)$ for the cohomology modules of this complex.

Replacing $x$ with its own powers, one arrives at a tower of commutative diagrams

\[
\begin{array}{ccc}
A & \xrightarrow{x} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{x^2} & A \\
\downarrow & & \downarrow \\
A & \xrightarrow{x^3} & A \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}
\]

which induces maps on the cohomology level, $x: H^i(A; x^\ell) \to H^i(A; x^{\ell+1})$ and hence a direct system of cohomology modules over the index set $\mathbb{N}$. It is an instructive exercise (using the fact that $\mathbb{N}$ is an index set that satisfies: for all $n, n' \in \mathbb{N}$ there is $N \in \mathbb{N}$ exceeding both $n, n'$) to check that the direct limit $\lim_{\ell} H^k(A; x^\ell)$ agrees with the local cohomology module $H^k_p(I; x_q)$. In particular, it is independent of the chosen generating set for $I$.

If $M$ is an $A$-module and the ideal $I$ is generated by $x_1, \ldots, x_m$ then to each such generating set there is a *cohomological Koszul complex*

\[K^\bullet(M; x_1, \ldots, x_m) := M \otimes_A \bigotimes_{i=1}^m K^\bullet(A; x_i)\]

whose cohomology modules are denoted $H^\bullet(M; x_1, \ldots, x_m)$. Again, one can verify that replacing each $x_i$ by powers of themselves leads to a tower of complexes whose direct limit has a cohomology that functorially equals the local cohomology $H^\bullet_p(M)$. In particular, it is independent of the chosen generating set for $I$.

1.2. **The Čech complex.** Inspection shows that the direct limit of the tower $A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots$ is functorially equal to the localization $A[x^{-1}]$ which we also write as $A_x$. Thus, the limit complex to the tower (1.1.0.1) is the localization complex $A \to A_x$. In greater generality, the module that appears in the limit complex $C^\bullet(M; x_1, \ldots, x_m)$ of the tower

$K^\bullet(M; x_1, \ldots, x_m) \to K^\bullet(M; x_1^2, \ldots, x_m^2) \to K^\bullet(M; x_1^3, \ldots, x_m^3) \to \cdots$
in cohomological degree $k$ is the direct sum of all localizations of $M$ at $k$ of the $m$ elements $x_1, \ldots, x_m$. Hence,

$$
\hat{C}^*(M; x_1, \ldots, x_m) = \lim_{\ell} K^*(M; x_1^\ell, \ldots, x_m^\ell)
$$

and a corresponding statement links the cohomology modules on both sides.

The point of view of the Čech complex provides a useful link to projective geometry. Indeed, suppose $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ is the homogeneous ideal defining the projective variety $X$ in $\mathbb{P}_R^n$. Then the cohomological dimension $\text{cd}(U)$ of $U = \mathbb{P}_R^n \setminus X$, the largest integer $k$ for which $H^k(U, -)$ is not the zero functor on the category of quasi-coherent sheaves on $U$, equals $\text{lcd}_R(I) - 1$. This follows from the exact sequence

$$
0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow \bigoplus_{k \in \mathbb{Z}} \Gamma(\mathbb{P}_R^n \setminus X, \tilde{M}(k)) \longrightarrow H^1_I(M) \longrightarrow 0
$$

and the isomorphisms $\bigoplus_{k \in \mathbb{Z}} H^i(\mathbb{P}_R^n \setminus X, \tilde{M}(k)) = H^{i+1}_I(M)$ for any $R$-module $M$ with associated quasi-coherent sheaf $\tilde{M}$.

1.3. Limits of Ext-modules. Again, let $I = (x_1, \ldots, x_m)$ be an ideal of $A$. The natural projections $A/I^{\ell+1} \longrightarrow A/I^\ell$ lead to a natural tower of morphisms $\text{Ext}^k_A(A/I^\ell, M) \longrightarrow \text{Ext}^k_A(A/I^\ell, M) \longrightarrow \text{Ext}^k_A(A/I^\ell, M) \longrightarrow \cdots$. An exercise involving $\delta$-functors (also known as connected sequences of functors) shows that the direct limit of this system functorially agrees with $H^k_I(M)$.

We have thus the functorial isomorphisms

$$
H^k_I(M) \simeq \lim_{\ell} H^k(M; x_1^\ell, \ldots, x_m^\ell) \simeq H^k \hat{C}^*(M; x_1, \ldots, x_m) \simeq \lim_{\ell} \text{Ext}^k_A(A/I^\ell, M)
$$

for all choices of generating sets $x_1, \ldots, x_m$ for $I$.

Remark 1.4. (1) The derived functor version of local cohomology shows that $H^*_I(-)$ and $H^*_J(-)$ are the same functor whenever $I$ and $J$ have the same radical.

(2) It follows easily from the Čech complex interpretation that local cohomology satisfies a local-to-global principle: for any multiplicatively closed subset $S$ of $A$ one has $S^{-1} \cdot H^1_I(M) = H^1_I(S^{-1}A)(S^{-1}M)$, and in particular $H^1_I(M) = 0$ if and only if $H^1_{IA_p}(M_p) = 0$ for all $p \in \text{Spec } A$.

(3) If $I$ is (up to radical) a complete intersection in the localized ring $A_p$, then $H^k_I(A) \otimes_A A_p$ is zero unless $k = \text{ht}(IA_p)$. If $R$ is a regular local ring and $I$ reduced then $I$ is a complete intersection in every smooth point. It follows that for equidimensional $I$ the support of $H^k_I(R)$ with $k > \text{ht}(I)$ only contains primes $p$ contained in the singular locus of $I$.

(4) It is in general a difficult question to predict how the natural maps $\text{Ext}^k_A(A/I^\ell, M) \longrightarrow H^k_I(M)$ and $H^k(M; x_1^\ell, \ldots, x_m^\ell) \longrightarrow H^k_I(M)$ behave; some information can be found in [EMS00, SW07, BBL+19, MSW21].

(5) If $\phi: A' \longrightarrow A$ is a ring morphism, and if $M$ is an $A$-module and $I'$ an ideal of $A'$, then there is a functorial isomorphism between $H^k_{A'}(\phi_\ast M)$ and $\phi_\ast(H^k_I(M))$, where $\phi_\ast$ denotes restriction of scalars from $A$ to $A'$. The easiest way to see this is by comparison of the two Čech complexes involved.
Remark 1.5. Let $I$ be an ideal of a Noetherian commutative ring $A$. A sequence of ideals $\{I_k\}$ is called cofinal with the sequence of powers $\{I^k\}$ if, for all $k \in \mathbb{N}$, there are $\ell, \ell' \in \mathbb{N}$ such that both $I_\ell \subseteq I^k$ and $I_{\ell'} \subseteq I_k$.

Sequences $\{I_k\}$ cofinal with $\{I^k\}$ are of interest in the study of local cohomology since

$$\lim_{k \to \infty} \Ext^i_R(R/I_k, M) \cong \lim_{k \to \infty} \Ext^i_R(R/I^k, M) = H^i_I(M).$$

This provides one with the flexibility of using sequences of ideals other than $\{I^n\}$. In characteristic $p > 0$, the sequence of ideals defined next plays an extraordinary part in the story.

Let $A$ be a Noetherian commutative ring of prime characteristic $p$ and $I$ be an ideal of $A$. The $e$-th Frobenius power of $I$, denoted by $I^{[p^e]}$, is defined to be the ideal generated by the $p^e$-th powers of all elements of $I$.

Since the Frobenius endomorphism $A \xrightarrow{a \mapsto a^p} A$ is a ring homomorphism, $I^{[p^e]} = (f_1^{p^e}, \ldots, f_t^{p^e})$ for every set of generators $\{f_1, \ldots, f_t\}$ of $I$.

It is straightforward to check that $\{I^{[p^e]}\}$ is cofinal with $\{I^k\}$ since $A$ is Noetherian and thus $I$ is finitely generated.

1.4. Local duality. Matlis duality over a complete local ring $(A, m, k)$ provides a one-to-one correspondence between the Artinian and the Noetherian modules over $A$; in both directions it is given by the functor

$$D(M) := \Hom_A(M, E_A(k))$$

of homomorphisms into the injective hull of the residue field. Of course, one can in principle apply $D(-)$ to any module, but the property $D(D(M)) = M$ is likely to fail when $M$ does not enjoy any finiteness condition.

A natural question is what the result of applying $D(-)$ to $H^i_m(A)$ should be or, more generally, how to describe $D(H^i_m(M))$ for Noetherian $A$-modules $M$. It turns out that when $A$ “lends itself to duality”, then this question has a pleasing answer:

Theorem 1.6. Suppose $(A, m, k)$ is a local Gorenstein ring. Then

$$D(H^i_m(M)) \cong \Ext^{\dim(A) - i}_A(M, A)$$

for every finitely generated $A$-module $M$.

The original version is due to Grothendieck [Har67], and then expanded in Hartshorne’s opus [Har66]. As it turns out, there are extensions of local duality to Cohen–Macaulay rings with a dualizing module, and yet more generally to rings with a dualizing complex. Duality on formal or non-Noetherian schemes and other generalizations are discussed in [ATJLL99].

\footnote{Strictly speaking, one should write $D_A(-)$, but in all cases the underlying ring will be understood from the context.}
In particular, Chapter 4 of [Har66] contains a discussion on Cousin complexes and their connection to local cohomology, that we do not have the space to give justice to. Further accounts in this direction can be found in [Sha69, Sha77, Sch82b, Lip02].

2. Finiteness and vanishing

2.1. Finiteness properties. In general, local cohomology modules are not finitely generated. For instance, the Grothendieck nonvanishing theorem says:

**Theorem 2.1.** Let \((A, m)\) be a Noetherian local ring and \(M\) be a nonzero finitely generated \(A\)-module. Then \(H^\dim(M)_m(M) \neq 0\).

Moreover, if \(\dim(M) > 0\), then \(H^\dim(M)_m(M)\) is not finitely generated.

Finiteness is more unusual yet than this theorem indicates. For example, over a ring \(R\) of polynomials over \(\mathbb{C}\), a local cohomology module \(H^k_I R_q\) is a finite \(R\)-module precisely if \(I \neq 0\) and \(k = 0\), or if \(H^k_I(R) = 0\). This lack of finite generation prompted people to look at other types of finiteness properties, and in this section we survey various fruitful avenues of research that pertain to finiteness.

In [Gro68, exposé 13, 1.2] Grothendieck conjectured that, if \(I\) is an ideal in a Noetherian local ring \(A\), then \(\text{Hom}_A(A/I, H^1_I(A))\) is finitely generated. Hartshorne refined this finiteness of \(\text{Hom}_A(A/I, H^1_I(A))\) and introduced the notion of cofinite modules in [Har70].

**Definition 2.2.** Let \(A\) be a Noetherian commutative ring and \(I \subseteq A\) an ideal. An \(A\)-module \(M\) is called \(I\)-cofinite if \(\text{Supp}_A(M) \subseteq V(I)\) and \(\text{Ext}^i_A(A/I, M)\) is finitely generated for all \(i\).

In [Har70] Hartshorne constructed the following example which answered Grothendieck’s conjecture on finiteness of \(\text{Hom}_A(A/I, H^1_I(M))\) in the negative.

**Example 2.3.** Let \(k\) be a field and put \(A = \frac{k[x,y,u,v]}{(xu-ye)}\). Set \(\mathfrak{a} = (x, y)\) and \(\mathfrak{m} = (x, y, u, v)\). Then \(\text{Hom}_A(A/\mathfrak{m}, H^2_{\mathfrak{a}}(A))\) is not finitely generated and hence neither is \(\text{Hom}_A(A/\mathfrak{a}, H^2_{\mathfrak{a}}(A))\).

We note in passing, that while the socle dimension of \(H^2_{\mathfrak{a}}(A)\) is infinite, it is nonetheless a finitely generated module over the ring of \(k\)-linear differential operators on \(A\), [Hsi12].

The ring \(A\) in Hartshorne’s example is not regular; one may ask whether local cohomology modules \(H^1_I(R)\) of a Noetherian regular ring \(R\) are \(I\)-cofinite. Huneke and Koh showed in [HK91] that this is not the case even for a polynomial ring over a field.
Example 2.4. Let $k$ be a field of characteristic 0 and let $R = k[x_{1,1}, \ldots, x_{2,3}]$ be the polynomial ring over $k$ in 6 variables. Set $I$ to be the ideal generated by the $2 \times 2$ minors of the matrix $(x_{ij})$.

The geometric origins and connections of this example, including a discussion of the interaction of the relevant local cohomology groups with de Rham cohomology and $D$-modules, can be found in Examples 2.14, 4.8 and Remark 4.9 below. In particular, Example 4.8 discusses that $H^3_I(R)$ is isomorphic to the injective hull of $k$ over $R$, which means that $\text{Hom}_R(R/I, H^3_I(R))$ is the injective hull of $k$ over $R/I$ and thus surely not finitely generated.

Huneke and Koh further proved in [HK91] that:

Theorem 2.5. Let $R$ be a regular local ring and $I$ be an ideal in $R$. Set $b$ to be the biggest height of any minimal prime of $I$ and set $c = \text{lcd}_R(I)$, compare Definition 1.3.

1. If $R$ contains a field of characteristic $p > 0$ and if $j > b$ is an integer such that $\text{Hom}_R(R/I, H^j_I(R))$ is finitely generated, then $H^j_I(R) = 0$.
2. If $R$ contains $\mathbb{Q}$ then $\text{Hom}_R(R/I, H^j_I(R))$ is not finitely generated.

In Example 2.4, it turns out that the socle $\text{Hom}_R(R/m, H^3_I(R))$ of $H^3_I(R)$ is finitely generated. It is natural to ask whether the socle of local cohomology of a Noetherian regular ring is always finitely generated; as a matter of fact this was precisely [Hun92a, Conjecture 4.3].

In [Hun92a], Huneke proposed a number of problems on local cohomology which guided the study of local cohomology modules for decades.

Problem 2.6 (Huneke’s List).

1. When is $H^j_I(M) = 0$?
2. When is $H^j_I(M)$ finitely generated?
3. When is $H^j_I(M)$ Artinian?
4. If $M$ is finitely generated, is the number of associated primes of $H^j_I(M)$ always finite?

Huneke remarked that “all of these problems are connected with another question:

5. What annihilates the local cohomology module $H^j_I(M)$?

More concretely, Huneke conjectured:

Conjecture 2.7 (Conjectures 4.4 and 5.2 in [Hun92a]). Let $R$ be a regular local ring and $I$ be an ideal. Then

1. the Bass numbers $\text{Ext}^i_{R_p}(\kappa(p), H^j_I(R_p))$ are finite for all $i$, $j$, and prime ideals $p$, and
2. the number of associated primes of $H^j_I(R)$ is finite for all $j$.
Later in [Lyu93b] Lyubeznik conjectured further that the finiteness of associated primes holds for local cohomology of all Noetherian regular rings. Substantial progress has been made on these conjectures. If the regular ring has prime characteristic $p > 0$, then these conjectures were completely settled by Huneke and Sharp in [HS93]; in equi-characteristic 0, Lyubeznik proved these conjectures for two large classes of regular rings in [Lyu93b]; for complete unramified regular local rings of mixed characteristic, these conjectures were first settled by Lyubeznik in [Lyu00b] (different proofs can also be found in [NnB13] and [BBL+14]). The finiteness of associated primes of local cohomology was also proved in [BBL+14] for smooth $\mathbb{Z}$-algebras. We summarize these results as follows.

**Theorem 2.8.** Assume that $R$ is

1. a Noetherian regular ring of characteristic $p > 0$, or
2. a complete regular local ring containing a field of characteristic 0, or
3. regular of finite type over a field of characteristic 0, or
4. an unramified regular local ring of mixed characteristic, or
5. a smooth $\mathbb{Z}$-algebra.

Then the Bass numbers and the number of associated primes of $H^j_I(R)$ are finite for every ideal $I$ of $R$ and every integer $j$.

**Remark 2.9.** When $R$ is a smooth $\mathbb{Z}$-algebra, then finiteness of Bass numbers was not addressed in [BBL+14]. However, one can conclude readily from the unramified case in [Lyu00b] as follows. The Zariski-local structure theorem for smooth morphism says that $\mathbb{Z} \to R$ factors as a composition of a polynomial extension and a finite etale morphism, which implies that locally $R$ is an unramified regular local ring of mixed characteristic. Since the finiteness of Bass numbers is a local problem, the desired conclusion follows from the results in [Lyu00b].

Conjecture 2.7 is still open when $R$ is a ramified regular local ring of mixed characteristic. Theorem 2.8(1) was proved in [HS93] using properties of the Frobenius endomorphism; this approach was later conceptualized by Lyubeznik to his theory of $F$-modules in [Lyu97]. The proof of Theorem 2.8(2)-(5) uses $D$-modules (i.e. modules over the ring of differential operators). Both $F$-modules and $D$-modules will be discussed in the sequel.

For a non-regular Noetherian ring $A$, if $\dim(A) \leq 3$ ([Mar01]), or if $A$ is a 4-dimensional excellent normal local domain ([HKM09]), then the number of associated primes of $H^j_I(M)$ is finite for every finitely generated $A$-module $M$, for every ideal $I$ and for all integers $j$. Once the restriction on $\dim(A)$ is removed, then the number of associated primes of local cohomology modules can be infinite; such examples have been discovered in [Sin00, Kat02, SS04]. Note that all these examples are hypersurfaces; the hypersurface in [SS04] has rational singularities.

As local cohomology modules may have infinitely many associated primes in general, one may ask a weaker question ([HKM09, p. 3195]):
**Question 2.10.** Let $A$ be a Noetherian ring, $I$ be an ideal of $A$ and $M$ be a finitely generated $A$-module. Does $H^j_I(M)$ have only finitely many minimal associated primes? Or equivalently, is the support of $H^j_I(M)$ Zariski-closed?

It is stated in [HKM09] that “this question is of central importance in the study of cohomological dimension and understanding the local-global properties of local cohomology”.

When $\dim(A) \leq 4$, then Question 2.10 has a positive answer due to [HKM09]. If $\mu(I)$ denotes the number of generators of $I$ and $A$ has prime characteristic $p$, it was proved and attributed to Lyubeznik in [Kat06] that $H^\mu_I(A)$ has a Zariski-closed support. When $A = R/(f)$ where $R$ is a Noetherian ring of prime characteristic $p$ with isolated singular closed points, it was proved independently in [KZ18] and in [HNnB17] that $H^j_I(A)$ has a Zariski-closed support for every ideal $I$ and integer $j$.

A classical result in commutative algebra (e.g. [BH93, Theorem 3.1.17]) says that if $A$ is a Noetherian local ring and $M$ is a finitely generated $A$-module $M$ that has finite injective dimension, then

$$\dim(M) \leq \text{injdim}_A(M) = \text{depth}(A)$$

where $\text{injdim}_A(M)$ denotes the injective dimension of $M$ over $A$. Interestingly, for local cohomology modules over regular rings, the inequality seems to be reversed. More precisely, the following was proved in [HS93] and [Lyu93b]

**Theorem 2.11.** Assume that $R$ is

1. a Noetherian regular ring of characteristic $p > 0$, or
2. a complete regular local ring of characteristic $0$, or
3. regular of finite type over a field of characteristic $0$.

Then

$$\text{injdim}_R(H^j_I(R)) \leq \dim(\text{Supp}_R(H^j_I(R)))$$

for every ideal $I$ and integer $j$.

In [Put14], Puthenpurakal showed that if $R = k[x_1, \ldots, x_n]$ where $k$ is a field of characteristic 0 then $\text{injdim}_R(H^j_I(R)) = \dim(\text{Supp}_R(H^j_I(R)))$ for every ideal $I$. Later this was strengthened in [Zha17, Theorem 1.2] as follows: assume that either $R$ is a regular ring of finite type over an infinite field of prime characteristic $p$ and $M$ is an $F$-finite $F$-module, or $R = k[x_1, \ldots, x_n]$ where $k$ is a field of characteristic 0 and $M$ is a holonomic $^2$ $D$-module. Then

$$\text{injdim}_R(M) = \dim(\text{Supp}_R(M)).$$

--2The notions of holonomic $D$-modules and $F$-finite $F$-modules will be explained in the sequel; local cohomology modules with argument $R$ and $R$ as discussed here are primary examples of those.
Subsequently [SZ19] proved that, if $M$ is either a holonomic $D$-module over a formal power series ring $R$ with coefficients in a field of characteristic 0, or an $F$-finite $F$-module over a Noetherian regular ring $R$ of prime characteristic $p$, then

$$\dim(\text{Supp}_R(M)) - 1 \leq \text{injdim}_R(M) \leq \dim(\text{Supp}_R(M)).$$

When the regular ring $R$ does not contain a field, the bounds on injective dimension of local cohomology modules of $R$ are different. In [Zho98], Zhou proved that, if $(R, \mathfrak{m})$ is an unramified regular local ring of mixed characteristic and $I$ is an ideal of $R$, then $\text{injdim}_R(H^j_I(R)) \leq \dim(\text{Supp}_R(H^j_I(R))) + 1$ and $\text{injdim}_R(H^1_\mathfrak{m}H^2_I(R)) \leq 1$. Moreover, it may be the case that $\text{injdim}_R(H^1_\mathfrak{m}H^2_I(R)) = 1$, as shown in [HNnBPW19, DSZ19].

2.2. Vanishing. Problem 1 in Huneke’s list of problems in [Hun92a] asks: when is $H^j_I(M) = 0$? Vanishing results on local cohomology modules have a long and rich history. Note that $H^j_I(M) = 0$ for all $j > t$ and all $A$-modules $M$ if and only if $H^j_A = 0$ for $j > t$. Recall the notion of local cohomological dimension from Definition 1.3. For a Noetherian local ring $A$, we set

$$\text{mdim}(A) = \min\{\dim(A/Q) \mid Q \text{ is a minimal prime of } A\}$$

and we write $\text{embdim}(A)$ for the embedding dimension (the number of generators of the maximal ideal) of a local ring $A$. For an ideal $I$ of $A$, we set

$$c_A(I) = \text{embdim}(A) - \text{mdim}(A/I).$$

Note that if $A$ is regular then $c_A(I)$ is called the big height, i.e. the biggest height of any minimal prime ideal of $I$.

We now summarize the most versatile vanishing theorems on local cohomology.

- (Grothendieck Vanishing) Let $A$ be a Noetherian ring and $M$ be a finitely generated $R$-module. Then $H^j_I(M) = 0$ for all integers $j > \dim(M)$ and ideals $I$. In particular, this implies that $\text{lcd}_A(I) \leq \dim(A)$ for all ideals $I$.

- (Hartshorne–Lichtenbaum Vanishing) Let $(A, \mathfrak{m})$ be a Noetherian local ring and $I$ be an ideal of $R$. Then $\text{lcd}_A(I) \leq \dim(A) - 1$ if and only if $\dim(\hat{A}/(I\hat{A} + P)) > 0$ for every minimal prime $P$ of $\hat{A}$ such that $\dim(\hat{A}/P) = \dim(A)$, where $\hat{A}$ denotes the completion of $A$. In particular, this implies that if $A$ is a complete local domain and $\sqrt{I} \neq \mathfrak{m}$ then $\text{lcd}_A(I) \leq \dim(A) - 1$. cf. [Har68].

- (Faltings Vanishing) Let $R$ be a complete equi-characteristic regular local ring with a separably closed residue field. Then

$$\text{lcd}_R(I) \leq \dim(R) - \left\lfloor \frac{\dim(R) - 1}{c_R(I)} \right\rfloor,$$
This bound is sharp according to [Lyu85].

(Second Vanishing Theorem) Let $R$ be a complete regular local ring that contains a separably closed coefficient field and $I$ be an ideal. Then $\text{lcd}_{R}(I) \leq \dim(R) - 2$ if and only if $\dim(R/I) \geq 2$ and the punctured spectrum of $R/I$ is connected.

A version of this vanishing theorem for projective varieties was first obtained by Hartshorne in [Har68, Theorem 7.5] who coined the name ‘Second Vanishing Theorem’. The local version stated here was left as a problem by Hartshorne in [Har68, p. 445]. Subsequently, this theorem was proved in prime characteristic in [PS73], in equi-characteristic 0 in [Ogu73] (a unified proof for equi-characteristic regular local rings can be found in [HL90]), and for unramified regular local rings in mixed characteristic in [Zha21b].

(Peskine–Szpiro Vanishing) Let $(R, \mathfrak{m})$ be a Noetherian regular local ring of prime characteristic $p$ and $I$ be an ideal. Then $\text{lcd}_{R}(I) \leq \dim(R) - \text{depth}(R/I)$, cf. [PS73].

(Vanishing via action of Frobenius) Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic $p$ and $I$ be an ideal. Set $d = \dim(R)$. Then $H^j_{\mathfrak{m}}(R) = 0$ if and only if the Frobenius endomorphism on $H^{d-j}_{\mathfrak{m}}(R/I)$ is nilpotent. cf. [Lyu06b].

There have been various extensions of the vanishing theorems mentioned above. Most notably, [HL90] initiated an investigation on finding bounds of local cohomological dimension under topological and/or geometric assumptions. For instance, [HL90, Theorem 3.8] asserts that if $A$ is a complete local ring containing a field and $I$ is a formally geometrically irreducible ideal such that $0 < c_A(I) < \dim(A)$ then

\begin{equation}
\text{lcd}_{A}(I) \leq \dim(A) - 1 - \left\lfloor \frac{\dim(A) - 2}{c_A(I)} \right\rfloor.
\end{equation}

Furthermore, if $A/I$ is normal then

\[ \text{lcd}_{A}(I) \leq \dim(A) - \left\lfloor \frac{\dim(A) + 1}{c_A(I) + 1} \right\rfloor - \left\lfloor \frac{\dim(A)}{c_A(I) + 1} \right\rfloor. \]

The bound on cohomological dimension in (2.2.0.1) was later extended to reducible ideals in [Lyu07] as follows.

**Theorem 2.12.** Let $(A, \mathfrak{m}, \mathbb{k})$ be a $d$-dimensional local ring containing $\mathbb{k}$. Assume $d > 1$. Let $c$ be a positive integer, let $t = \lfloor (d-2)/c \rfloor$ and $\nu = d - 1 - \lfloor (d-2)/c \rfloor$. Let $I$ be an ideal of $A$ with $c(IA) \leq c$. Let $B$ be the completion of the strict Henselization of the completion of $A$. Let $I_1, \ldots, I_n$ be the minimal primes of $IB$ and let $P_1, \ldots, P_m$ be the primes of $B$ such that $\dim(B/P_i) = d$. Let $\Delta_i$ be the simplicial complex on $n$ vertices $\{1, 2, \ldots, n\}$ such that a simplex $\{j_0, \ldots, j_s\}$ belongs to $\Delta_i$ if and only if $I_{j_0} + \cdots + I_{j_s} + P_i$ is not $\mathfrak{m}B$-primary. Let $\check{H}_{t-1}(\Delta_i; \mathbb{k})$ be the $(t-1)st$ singular homology group

\footnote{This is the floor function $\lfloor x \rfloor = \max\{k \in \mathbb{Z}, k \leq x\}$.}
of $\Delta_i$ with coefficients in $k$. Then $\text{lcd}_A(I) \leq v$ if and only if $\tilde{H}_{t-1}(\Delta_i; k) = 0$ for every $i$.

We ought to point out that the simplicial complex introduced in Theorem 2.12 has spurred a line of research on connectedness dimension, cf. [KLZ16], [DT16], [NnBSW19], [Var19].

Also, [DT16] shows that the same bound as in (2.2.0.1) holds when $A$ is a complete regular local ring containing a field such that $A/I$ has positive dimension and satisfies Serre’s condition $(S_2)$. This result is in the spirit of a question raised by Huneke in [Hun92a].

**Question 2.13** (Huneke). Let $R$ be a complete regular local ring with separably closed residue field and $I$ be an ideal of $R$. Assume that $R/I$ satisfies Serre’s conditions $(S_i)$ and $(R_j)$. What is the maximal possible cohomological dimension for such an ideal?

In the same spirit as Huneke’s Question 2.13, one may ask about the possibility of an implication

(2.2.0.2) $\quad [\text{depth}_R(R/I) \geq t] \quad ? \quad [\text{lcd}_R(I) \leq \text{dim}(R) - t].$

In prime characteristic $p$, such implication holds due to Peskine–Szpiro Vanishing. On the other hand, Peskine–Szpiro Vanishing can fail in characteristic 0.

**Example 2.14.** Let $R$ be the polynomial ring in the variables $x_{1,1}, \ldots, x_{2,3}$ over the field $K$, localized at $x = (x_{1,1}, \ldots, x_{2,3})$. Let $I$ be the ideal of maximal minors of the matrix $(x_{i,j})$. Then $I$ is the radical ideal associated to the 4-dimensional locus $V$ of the $2 \times 3$ matrices of rank one, which agrees with the image of the map $K^2 \times K^3 \rightarrow K^{2 \times 3}$ that sends $((s,t), (x,y,z))$ to $(xs, ys, zs, xt, yt, zt)$. In particular, $I$ is the prime ideal associated to the image of the Segre embedding of $\mathbb{P}^1_k \times \mathbb{P}^2_k \hookrightarrow \mathbb{P}^5_k$.

Thus $I$ is 3-generated of height $2 = 6 - 4$, and in fact $R/I$ is Cohen–Macaulay of depth 4. Since the origin is the only singular point of $V$, the local cohomology groups $H^k_I(R)$ are supported at the origin for $k \neq 2$ and zero for $k \neq \{2,3\}$. Cohen–Macaulayness of $R/I$ forces the vanishing of $H^k_I(R)$ for $k \neq 2$ in prime characteristic, but if the characteristic of $k$ is zero then $H^2_I(R)$ is actually nonzero. For a computational discussion involving $D$-modules see [Oak97, Wal99, OT01, Wal02]. We will return to this situation in Example 4.8.

In Example 2.14, $\text{depth}(R/I) = 4$ but $H^3_I(R) \neq 0$. This shows that the implication (2.2.0.2) can fail in characteristic 0 when $t \geq 4$. When $t \leq 2$, the implication (2.2.0.2) holds due the Second Vanishing Theorem and the Hartshorne–Lichtenbaum Theorem. The case $t = 3$ is not completely settled, but there have been positive results. In [Var13], continuing his work on the number of defining equations in [Var12], Varbaro proved that if a homogeneous ideal $I$ in a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$ over a field $\mathbb{k}$ satisfies $\text{depth}(R/I) \geq 3$ then $\text{lcd}_R(I) \leq n - 3$. He also conjectured:
Conjecture 2.15 (Varbaro). Let $R$ be a regular local ring containing a field and $I$ be an ideal of $R$. If depth$(R/I) \geq 3$, then $\text{lcd}_R(I) \leq \dim(R) - 3$.  

The fact that Implication (2.2.0.2) can fail for complex projective threefolds (Example 2.14) raises the question what exact features are responsible for failure when $t = 3$. Clearly, more knowledge about the singularity is required than just depth$_R(R/I)$.

Dao and Takagi prove Conjecture 2.15 in [DT16] when $R$ is essentially of finite type over a field. More specifically, they show the following facts about the inequality $\text{lcd}_R(I) \leq \dim_R - 3$. Suppose $R$ is a regular local ring essentially of finite type over its algebraically closed residue field of characteristic zero. Take an ideal $I$ such that $R/I$ has depth 2 or more and $H^2_m(R/I)$ is a $\mathbb{K}$-vector space (i.e., it is killed by $m$). Then $\text{lcd}_R(I) \leq \dim R - 3$ if and only if the torsion group of Pic$(\text{Spec}(R/I))$ is finitely generated on the punctured completed spectrum. In case that the depth of $R/I$ is at least 4, one even has $\text{lcd}_R(I) \leq \dim(R) - 4$ if and only if the Picard group is torsion on the punctured completed spectrum of $R/I$. In Example 2.14, the depth of $R/I$ is four, but the Picard group on the punctured spectrum is not torsion but $\mathbb{Z}$. Conjecture 2.15 remains open in general.

Both the Hartshorne–Lichtenbaum Vanishing Theorem and the Second Vanishing Theorem may viewed as topological criteria for vanishing and have applications to topology of algebraic varieties (cf. [Lyu93a] and [Hun07]). It would be desirable to have an analogue of the Second Vanishing Theorem for non-regular rings. In [Lyu02, p. 144] Lyubeznik asked the following questions.

Question 2.16. Let $(A, m)$ be a complete local domain with a separably closed residue field.

(1) Find necessary and sufficient conditions on $I$ such that $\text{lcd}_A(I) \leq \dim(A) - 2$.

(2) Let $I$ be a prime ideal. Is it true that $\text{lcd}_A(I) \leq \dim(A) - 2$ if and only if $(P + I)$ is not primary to the maximal ideal for any prime ideal $P$ of height 1?

Question 2.16(1) remains open. It turns out that Question 2.16(2) has a negative answer due to [HZ18, Proposition 7.7]:

Example 2.17. Let $A = \mathbb{C}[[x,y,z,u,v]]/(x^3 + y^3 + z^2 - ux - vy)$ and $I = (x, y, z)$. Then

(1) $\dim(A) = 3$ and ht$(I) = 1$;
(2) $I + P$ is not primary to the maximal ideal for every height-1 prime ideal $P$;
(3) $H^2_I(A) \neq 0$.  

Example 2.17.
Given the connections between local cohomology and sheaf cohomology (cf. (1.2.0.1)), vanishing of sheaf cohomology can be interpreted in terms of local cohomology. The classical Kodaira Vanishing Theorem asserts that: If $X$ is smooth projective variety over a field $\mathbb{K}$ of characteristic 0, then $H^i(X, \mathcal{O}(j)) = 0$ for $i < \dim(X)$ and all $j < 0$. This result has an equivalent formulation in terms of local cohomology: If $R$ is a standard graded domain over a field $\mathbb{K}$ of characteristic 0 such that $\text{Proj}(R)$ is smooth, then $H^j_\mathfrak{m}(R)_{<0} = 0$ for all $j < \dim(R)$, where $\mathfrak{m}$ is the homogeneous maximal ideal of $R$. For ideal-theoretic interpretations and connections with tight closure and Frobenius, we refer the interested reader to [HS97, Smi97b].

The Kodaira Vanishing Theorem fails for singular varieties in characteristic 0 and also fails for smooth varieties in characteristic $p$. It is proved in [BBL+19] that, if one focuses on the range $i < \text{codim}(\text{Sing}(X))$, then the Kodaira Vanishing Theorem can be extended to thickenings of local complete intersections. More precisely:

**Theorem 2.18.** Let $X$ be a closed local complete intersection subvariety of $\mathbb{P}^n_\mathbb{K}$ over a field $\mathbb{K}$ of characteristic 0 and let $I$ be its defining ideal. Let $X_t$ denote the scheme defined by $I_t$. Then

$$H^i(X_t, \mathcal{O}_{X_t}(j)) = 0$$

for all $i < \text{codim}(\text{Sing}(X))$, all $t \geq 1$, and all $j < 0$.

Or, equivalently, let $S = \mathbb{K}[x_0, \ldots, x_n]$ and $I$ be as above. Then

$$H^\ell_\mathfrak{m}(S/I^t)_{<0} = 0$$

for $\ell < \text{codim}(\text{Sing}(X)) + 1$ and all $t \geq 1$.

A natural question is whether the restriction on $\text{codim}(\text{Sing}(X))$ can be relaxed or even removed. The following example from [BBL+21] shows that this is not the case.

**Example 2.19.** Let $R = \mathbb{K}[x, y, u, v, w]$ where $\mathbb{K}$ is a field of characteristic 0. Fix an integer $c \geq 2$ and set $I := (uy - vx, vy - wx) + (u, v, w)^c$. Then one can check that

1. $X = \text{Proj}(R/I)$ is local complete intersection in $\mathbb{P}^4_\mathbb{K}$;
2. $H^2_\mathfrak{m}(R/I)_{-ct+1} \neq 0$;
3. $H^2_\mathfrak{m}(R/I)_{\leq -ct} = 0$.

Example 2.19 indicates that, if one removes the restriction on the homological degree by $\text{codim}(\text{Sing}(X))$, the best vanishing result one can hope for is an asymptotic vanishing bounded by a linear function of $t$. Such an asymptotic vanishing turns out to be true, as shown in [BBL+21].

---

A standard graded algebra over a field $\mathbb{K}$ is a graded quotient of a polynomial ring over $\mathbb{K}$ with the standard grading.
Theorem 2.20. Let $X$ be a closed local complete intersection subscheme of $\mathbb{P}^n$ over a field of arbitrary characteristic. Then there exists an integer $c \geq 0$ such that for each $t \geq 1$ and $i < \dim(X)$, one has

$$H^i(X_t, \mathcal{O}_{X_t}(j)) = 0, \ \forall \ j < -ct.$$ 

When $\text{Proj}(R/I)$ is a local complete intersection (here $R = \mathbb{K}[x_0, \ldots, x_n]$ and $I$ is a homogeneous ideal of $R$), the local cohomology modules $H^j_m(R/I)$ have finite length for $j < \dim(R/I)$ and consequently $H^j_m(R/I)_{\ell \leq 0} = 0$. This is one of the underlying reasons for the vanishing in Theorems 2.18 and 2.20. Once the local complete intersection assumption is dropped, $H^j_m(R/I)$ may not have finite length and hence the vanishing may fail.

However, since $H^j_m(R/I)$ are Artinian (even when $j \geq \dim(R/I)$), the socles $\text{Hom}_R(R/m, H^j_m(R/I))$ are finite dimensional and vanish in all sufficiently negative degrees. Therefore, one can ask:

Question 2.21. Let $R = \mathbb{K}[x_0, \ldots, x_n]$ and $I$ be a homogeneous ideal of $R$. For each $j \geq 0$, does there exist an integer $c$ such that

$$\text{Hom}_R(R/m, H^j_m(R/I))_{\ell} = 0$$

for all $t \geq 1$ and all $\ell < -ct$? 

For related questions and applications, we refer the interested reader to [Zha21a].

2.3. Annihilation of local cohomology. We now turn to the question: what annihilates the local cohomology module $H^j_I(M)$?

If $R$ is a Noetherian regular ring of prime characteristic $p$, then Huneke and Koh proved in [HK91] that $\text{ann}_R(H^j_I(R)) \neq 0$ if and only if $H^j_I(R) = 0$. The same conclusion for Noetherian regular rings of characteristic 0 was established implicitly in [Lyu93b]. The aforementioned result due to Huneke–Koh was later generalized to strongly $F$-regular domains in [BE18]. Inspired by the results due to Huneke–Koh and Lyubeznik, Lynch [Lyn12] conjectured that $\dim(A/\text{ann}_A(H^j_I(A))) = \dim(A/H^0_I(A))$ for every Noetherian local ring $A$, where $\delta = \text{lcd}_A(I)$. This conjecture turns out to be false in general, cf. [Bah17] and [SW20]. Note that the rings in the counterexamples in [Bah17] and [SW20] are not equidimensional. In [Hoc19, Question 6] Hochster asks the following.

Question 2.22. If $A$ is a Noetherian local domain and $I$ is an ideal of cohomological dimension $c$, is $H^j_I(A)$ a faithful $A$-module? 

In [HJ20], Hochster and Jeffries answer this question in the affirmative in the following cases:

- $\text{ch}(A) = p > 0$ and $c$ equals the arithmetic rank of $I$, see Subsection 4.1 below;
- $A$ is a pure subring of a regular ring containing a field.
In [DSZ19], Datta, Switala and Zhang answer Question 2.22 in the negative by the following (equidimensional) example.

**Example 2.23.** Let \( R = \mathbb{Z}_2[x_0, \ldots, x_5] \) and let \( I \) be the ideal of \( R \) generated by the 10 monomials
\[
\{x_0x_1x_2, x_0x_1x_3, x_0x_2x_4, x_0x_3x_5, x_0x_4x_5, x_1x_2x_5, x_1x_3x_4, x_2x_3x_4, x_2x_3x_5\}.
\]
Then \( \text{cd}(I) = 4 \), but \( \text{ann}_R(H^4_4(R)) \) is the ideal generated by 2 in \( R \). \( \diamond \)

When \((A, \mathfrak{m})\) is a local ring, the annihilation of \( H^j_\mathfrak{m}(A) \) is particularly interesting for \( j < \dim(A) \), and has a wide range of applications. We recall that an element \( x \in A^\circ \) is called a uniform local cohomology annihilator of \( A \) if \( xH^j_\mathfrak{m}(A) = 0 \) for \( j < \dim(A) \), where \( A^\circ = A \setminus \bigcup_{\mathfrak{p} \in \text{min}(A)} \mathfrak{p} \). Since \( H^j_\mathfrak{m}(A) \) may not be finitely generated, it is not clear whether such a uniform annihilator should exist. Surprisingly, in [Zho07] Zhou proved that if \( A \) is an excellent local ring then \( A \) admits a uniform local cohomology annihilator if and only if \( A \) is equidimensional. If \( x \) is a uniform local cohomology annihilator then \( Ax \) is Cohen–Macaulay (cf. [HH90], [Zho06]); in fact, there is a deep connection between the existence of uniform local cohomology annihilators and the Cohen–Macaulay locus. To explain this connection, we need to recall some definitions from [Hun92b]. For a Noetherian ring \( A \), a finite complex of finitely generated free \( A \)-modules
\[
G_\bullet : 0 \longrightarrow G_n \xrightarrow{f_n} G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{f_1} G_0
\]
is said
- to satisfy the standard condition on rank if \( \text{rank}(f_i) + \text{rank}(f_{i-1}) = \text{rank}(G_{i-1}) \) for \( 1 \leq i \leq n \) and \( \text{rank}(f_n) = \text{rank}(G_n) \), where the rank of a map is the determinantal rank;
- to satisfy the standard condition on height if \( \text{ht}(I(f_i)) \geq i \) for all \( i \), where \( I(f_j) \) is the ideal generated by the rank-size minors of \( f_j \) which is viewed as a matrix.

For a Noetherian ring \( A \), we denote by \( \text{CM}(A) \) the set of elements \( x \in A \) such that for all finite complexes \( G_\bullet \) of finitely generated free \( A \)-modules satisfying the standard conditions on rank and height, \( xH_i(G_\bullet) = 0 \) for \( i \geq 1 \). Huneke conjectured in [Hun92b] that if \( A \) is an equidimensional excellent Noetherian ring then \( \text{CM}(A) \) is not contained in any minimal prime of \( A \). Zhou proved this conjecture in [Zho07] by showing the following theorem.

**Theorem 2.24.** Let \( A \) be an excellent local ring \( A \). Then \( A \) admits a uniform local cohomology annihilator if and only if \( \text{CM}(A) \) is not contained in any minimal prime of \( A \).

One may consider the uniform annihilation of local cohomology in a different direction.
Question 2.25. Let \((A, \mathfrak{m})\) be a Noetherian local ring of characteristic \(p\) and \(I\) be an ideal of \(A\). Does there exist a constant \(B\) such that
\[
\mathfrak{m}^{Bp^e} H_0^\mathfrak{m}(A/I[p^e]) = 0
\]
for all \(e \geq 1\)?

The special case of Question 2.25 where \(I\) is primary to a prime ideal of height \(\dim(A) - 1\) was explicitly asked by Hochster and Huneke in [HH90]; a positive answer to this special case would have significant consequences in tight closure theory, especially to the notion of \(F\)-regularity. Question 2.25 is wide open to the best of our knowledge. The graded analog, when \(A\) is a standard graded ring over a field of characteristic \(p\) and \(I\) is homogeneous, has also attracted attention. When \(\dim(A/I) = 1\), the graded version was settled independently in [Hun00] and [Vra00]. Let \(A\) be a standard graded ring over a field and let \(M\) be a finitely generated graded \(A\)-module. We set
\[
a_j(M) := \max \{\ell \mid H_\ell^\mathfrak{m}(M) \neq 0\}
\]
for each integer \(j\). Since \(H_\ell^\mathfrak{m}(M)\) is Artinian, \(a_j(M) < \infty\) for each \(j\). Hence for a homogeneous ideal \(J\), if \(a_0(A/J) \leq t\), then \(\mathfrak{m}^{t+1} H_0^\mathfrak{m}(A/J) = 0\). Consequently, if there is an integer \(B\) such that \(a_0(A/I[p^e]) \leq Bp^e\) for all \(e\), then \(\mathfrak{m}^{Bp^e+1} H_0^\mathfrak{m}(A/I[p^e]) = 0\) for all \(e\). In general, it is an open question whether there exists an integer \(B\) (independent of \(e\)) such that \(a_0(A/I[p^e]) \leq Bp^e\) for all \(e\). On the other hand, \(a_0(M)\) may be considered as a partial Castelnuovo–Mumford regularity since the regularity of \(M\) is defined as
\[
\reg(M) := \max \{a_j(M) + j \mid 0 \leq j \leq \dim(M)\},
\]
so that \(a_0(M) \leq \reg(M)\). Therefore, one may ask for a stronger conclusion on the linear growth of \(\reg(A/I[p^e])\) with respect to \(p^e\). Indeed, the following was asked in [Kat98, p. 212].

Question 2.26. Let \(A\) be a standard graded ring over a field of characteristic \(p\) and \(I\) be a homogeneous ideal. Does there exist a constant \(C\) such that
\[
\reg(A/I[p^e]) \leq Cp^e
\]
for all \(e\)?

Some progress has been made: for cases of small singular locus see [Cha04], [Bre05] and [Zha15]; for rings of finite Frobenius representation type, see [KSSZ14].

At the crux of the homological conjectures stands the existence of big Cohen–Macaulay algebras: the assertion that each Noetherian complete local domain \((A, \mathfrak{m})\) admits an algebra (not necessarily Noetherian) in which every system of parameters of \(A\) becomes a regular sequence. A beautiful result of Hochster–Huneke in [HH92] says that if \(A\) is an excellent Noetherian local domain of characteristic \(p\) then its absolute integer closure\(^5\) \(A^+\) is a

\(^5\)The absolute integral closure of an integral domain \(A\) is defined to be the integral closure of \(A\) in the algebraic closure of the field of fractions of \(A\).
big Cohen–Macaulay \(A\)-algebra. In [HL07], Huneke and Lyubeznik gave a much simpler proof using annihilation of local cohomology.

The following result, proved in [HL07, Lemma 2.2], has been referred to as ‘the equational lemma’.

**Theorem 2.27.** Let \(A\) be a commutative Noetherian domain that contains a field of characteristic \(p\), let \(K\) be its field of fractions and \(\overline{K}\) be the algebraic closure of \(K\). Let \(I\) be an ideal of \(A\) and let \(\alpha\) be an element in \(H^i_{I^p}(R)\) such that the elements \(\alpha, \alpha^p, \ldots, \alpha^{p^t}, \ldots\) belong to a finitely generated submodule of \(H^i_{I^p}(A)\). Then there is a module-finite extension \(A'\) of \(A\) inside \(\overline{K}\) such that the natural map \(H^i_{I^p}(A) \longrightarrow H^i_{I^p}(A')\) induced by \(A \longrightarrow A'\) sends \(\alpha\) to 0.

Since the module-finite extension \(A'\) is constructed using the equations satisfied by \(\alpha\), Theorem 2.27 is referred in the literature as an “equational lemma”. Using Theorem 2.27, Huneke and Lyubeznik proved the following in [HL07, Theorem 2.1, Corollary 2.3]:

**Theorem 2.28.** Let \((A, m)\) be a commutative Noetherian domain that contains a field of characteristic \(p\), let \(K\) be its field of fractions and \(\overline{K}\) be the algebraic closure of \(K\). Assume, furthermore, that \(A\) is a homomorphic image of a Gorenstein local ring. For every module-finite extension \(A'\) of \(A\) inside \(\overline{K}\), there exists module-finite extension \(A' \subseteq A''\) inside \(\overline{K}\) such that the natural maps

\[
H^i_m(A') \longrightarrow H^i_m(A'')
\]

are the zero map for each \(i < \dim(A)\).

In particular,

1. \(H^i_m(A^+) = 0\) for \(i < \dim(A)\);
2. every system of parameter of \(A\) is a regular sequence on \(A^+\).

The Huneke–Lyubeznik equational lemma (or equivalently, the technique of annihilating local cohomology with finite extensions) in characteristic \(p\) has found many applications, for instance [Bha12] and [BST15]. In equi-characteristic 0, such annihilation of local cohomology is not possible once the dimension is at least 3: every module-finite extension of a normal domain must split in equi-characteristic 0. The situation in mixed characteristic has long been a mystery. However, in a very surprising turn of events, Bhatt proved in [Bha20, Theorem 5.1] the following:

**Theorem 2.29.** Let \((A, m)\) be an excellent Noetherian local domain with mixed characteristic \((0, p)\) and let \(A^+\) be an absolute integral closure of \(A\). Then

1. \(H^i_m(A^+/pA^+) = 0\) for \(i < \dim(A/pA)\) and \(H^i_m(A^+) = 0\) for \(i < \dim(A)\).

---

6Here \(\alpha^p\) denotes \(f(\alpha)\) where \(f\) is the natural action of Frobenius on \(H^1_I(A)\) induced by the Frobenius endomorphism on \(A\).
Every system of parameters of \( A \) is a Koszul regular sequence\(^7\) on \( A^+ \).

(3) If \( A \) admits a dualizing complex, then there exists a module-finite extension \( A \rightarrow B \) with \( H^i_m(A/pA) \rightarrow H^i_m(B/pB) \) being the 0 map for all \( i < \dim(A/pA) \).

For other connections between annihilators of local cohomology modules and homological conjectures, we refer the reader to [Rob76, Sch82a].

3. \( D- \) and \( F- \) structure

In this section we discuss some special structures that local cohomology have. In positive characteristic the Frobenius endomorphism is the main tool, while in any case they have a structure over the ring of differential operators.

3.1. \( D- \) modules. Following Grothendieck’s approach in [Gro67], we reproduce the definition of differential operators as follows. Let \( A \) be a commutative ring. The differential operators \( D^j_A \) on \( A \) are classified by their order \( j \) (a natural number), and defined inductively as follows. The differential operators \( D^0_A \) of order zero are precisely the multiplication maps \( \tilde{a}: A \rightarrow A \) where \( a \in A \); for each positive integer \( j \), the differential operators \( D^j_A \) of order less than or equal to \( j \) are those additive maps \( \tilde{P}: A \rightarrow A \) for which the commutator \( [\tilde{a}, \tilde{P}] = \tilde{a} \circ \tilde{P} - \tilde{P} \circ \tilde{a} \) is a differential operator on \( A \) of order less than or equal to \( j - 1 \). If \( P' \) and \( P'' \) are differential operators of orders at most \( j' \) and \( j'' \) respectively, then \( P' \circ P'' \) is again a differential operator and its order is at most \( j' + j'' \). Thus, the differential operators on \( R \) form an \( \mathbb{N} \)-filtered subring \( D(R) \) of \( \text{End}_\mathbb{Z}(R) \), and the order filtration is (by definition) increasing and exhaustive.

When \( A \) is an algebra over the central subring \( \mathbb{k} \), we define \( D(A, \mathbb{k}) \) to be the subring of \( D(A) \) consisting of those elements of \( D(A) \) that are \( \mathbb{k} \)-linear. Thus, \( D(A, \mathbb{Z}) = D(A) \) and \( D(A, \mathbb{k}) = D(A) \cap \text{End}_\mathbb{k}(A) \). It turns out that if \( A \) is an algebra over a perfect field \( F \) of prime characteristic, then \( D(A, F) = D(A) \), see, for example, [Lyu97, Example 5.1 (c)].

By a \( D(A, \mathbb{k}) \)-module, we mean a left \( D(A, \mathbb{k}) \)-module, unless we expressly indicate a right module. The standard example of a \( D(A, \mathbb{k}) \)-module is \( A \) itself. Using the quotient rule, localizations \( A' \) of \( A \) also carry a natural \( D(A, \mathbb{k}) \)-structure and the formal quotient rule induces a natural map

---

\( ^7 \)A sequence of elements \( z_1, \ldots, z_t \) in a commutative ring \( C \) is a Koszul regular sequence if \( H_i(K_\bullet(C; z_1, \ldots, z_t)) = 0 \) for \( i > 0 \) where \( K_\bullet(C; z_1, \ldots, z_t) \) is the Koszul complex of \( C \) on \( z_1, \ldots, z_t \).
\( \mathcal{D}(A, k) \rightarrow \mathcal{D}(A', k) \). Suppose \( a \) is an ideal of \( A \). The Čech complex on a generating set for \( a \) is a complex of \( \mathcal{D}(A, k) \)-modules; it then follows that each local cohomology module \( H^k_a(A) \) is a \( \mathcal{D}(A, k) \)-module.

More generally, if \( M \) is a \( \mathcal{D}(A, k) \)-module, then each local cohomology module \( H^k_a(M) \) is also a \( \mathcal{D}(A, k) \)-module. This was used by Kashiwara as early as 1970 as an inductive tool in algebraic analysis via reduction of dimension \[\text{[Kas95]}\] and was introduced to commutative algebra in \[\text{[Lyu93b, Examples 2.1 (iv)]}\].

If \( R \) is a polynomial or formal power series ring in the variables \( x_1, \ldots, x_n \) over a commutative ring \( k \), then \( \frac{1}{t!} \partial_{x_i}^{t} \) can be viewed as a differential operator on \( R \) even if the integer \( t_i! \) is not invertible. In these cases, \( \mathcal{D}(R, k) \) is the free \( R \)-module with basis elements

\[
\frac{1}{t_1! \partial_{x_1}^{t_1}} \ldots \frac{1}{t_n! \partial_{x_n}^{t_n}} \quad \text{for } (t_1, \ldots, t_n) \in \mathbb{N}^n,
\]

see \[\text{[Gro67, Théorème 16.11.2]}\]. When \( R \) is a polynomial ring or formal power series ring over a field \( k \) of characteristic 0, then the ring of differential operators

\[
\mathcal{D}(R, k) = R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}),
\]

is known as the Weyl algebra, a simple ring in the sense that it has no non-trivial two-sided ideals.

If \( k \) is a field and if \( A \) is a singular \( k \)-algebra then the structure of \( \mathcal{D}(R, A) \) can be very complicated, even in characteristic zero. For example, the ring of differential operators on the cone over an elliptic curve is not Noetherian and also not generated by homogeneous operators of bounded finite degree, \[\text{[BGfG72, Hsi15]}\]. In most cases, differential operators on singular spaces are completely mysterious, except for toric varieties, Stanley-Reisner rings and hyperplane arrangements, see \[\text{[Mus87, Jon94, MSTW01, Tra06, Tri97, Hol04]}\].

3.1.1. **Characteristic 0.** As references for background reading in this section we recommend \[\text{[Bjö79, Kas95, Kas03, KS90, HTT08]}\].

Let \( k \) denote a field of characteristic 0 and fix \( n \in \mathbb{N} \). Let \( R \) denote either \( k[x_1, \ldots, x_n] \) or \( k[[x_1, \ldots, x_n]] \), and let \( \mathcal{D} \) denote \( \mathcal{D}(R, k) \), unless specified otherwise. The partial differential operator \( \frac{\partial}{\partial x_i} \) is denoted by \( \partial_i \) for each variable \( x_i \).

Note that here the order of \( r \partial_1^{e_1} \cdots \partial_n^{e_n} \quad (r \in R) \) equals simply \( \sum_i e_i \). The order (i.e., the filtration level) of an element \( \sum_{\alpha, \beta \neq 0} c_{\alpha, \beta} x^\alpha \partial^\beta \in \mathcal{D} \) is the maximum of the orders \( |\beta| \) of its terms \( x^\alpha \partial^\beta \). Then we have \( \mathcal{D}_j = \{ r \partial_1^{e_1} \cdots \partial_n^{e_n} \mid r \in R, \sum_i e_i \leq j \} \), an increasing and exhaustive filtration of \( \mathcal{D} \), called the order filtration of \( \mathcal{D} \).
Using the order filtration \( \{D_j\} \), one can form the associated graded ring,

\[
\text{gr}(D) := D_0 \oplus \frac{D_1}{D_0} \oplus \cdots.
\]

Since the only nonzero commutators of pairs of generators in \( D \) are the \([\partial_i, x_i] = 1 \in D_0\), it follows that \( \text{gr}(D) \) is isomorphic to a (commutative) ring of polynomials \( R[\xi_1, \ldots, \xi_n] \) where \( \xi_i \) is the image of \( \partial_i \) in \( D_1/D_0 \). Note that \( \text{gr}(D) \) is naturally the coordinate ring on the cotangent space of \( k^n \), if \( R \) is a ring of polynomials. We use this to construct varieties from \( D \)-modules as follows.

**Definition 3.1.** Let \( M \) be a \( D \)-module. A filtration of \( M \) with respect to the order filtration \( \{D_j\} \) is a sequence of \( R \)-submodules \( \{F_i M\} \) such that

1. \( F_0 M \subseteq F_1 M \subseteq \cdots \subseteq F_i M \subseteq F_{i+1} M \subseteq \cdots \);
2. \( \bigcup_i F_i M = M \);
3. \( D_j \cdot F_i M \subseteq F_{i+j} M \).

Such filtration is called a good filtration if the associated graded module \( \text{gr}^F(M) := F_0 M \oplus \frac{F_1 M}{F_0 M} \oplus \cdots \) is finitely generated over \( \text{gr}(D) \).

Every finitely generated \( D \)-module admits a good filtration \( \{F_i M\} \); for instance, if \( M \) can be generated by \( m_1, \ldots, m_d \), then setting \( F_i M := \sum_j D_j m_j \) produces a good filtration of \( M \). Set \( J \) to be the radical of \( \text{ann}_{\text{gr}(D)}(\text{gr}^F(M)) \). This ideal \( J \) is independent of the good filtration \( \{F_i M\} \) (cf. [Bjö79, 1.3.4], [Cou95, 11.1]), and is called the characteristic ideal of \( M \). The characteristic ideal of \( M \) induces the notion of dimension of \( M \) (as a \( D \)-module) and characteristic variety of \( M \).

**Definition 3.2.** Let \( M \) be a \( D \)-module with good filtration and let \( J \) be its characteristic ideal. The dimension of \( M \) is defined as

\[
d(M) := \dim(\text{gr}(D)/J).
\]

The characteristic variety \( \text{Ch}(M) \) of \( M \) is defined as the subvariety of \( \text{Spec}(\text{gr}(D)) \) defined by \( J \). The set of the irreducible components of \( \text{Ch}(M) \), paired with their multiplicities in \( \text{gr}(M) \) is called the characteristic cycle of \( M \).

It turns out that dimensions cannot be small:

**Theorem 3.3** (Bernstein Inequality). Let \( M \) be a nonzero finitely generated \( D \)-module. Then

\[
n \leq d(M) \leq 2n.
\]

The nonzero modules of minimal dimension form a category with many good features.

**Definition 3.4.** A finitely generated \( D \)-module \( M \) is called holonomic if \( d(M) = n \) or \( M = 0 \).
Example 3.5. (1) Set $F_iR = R$ for all $i \in \mathbb{N}$. Then one can check that \{F_i\} is a good filtration on $R$ and $\text{gr}^F(R) \cong R$. Hence

$$J = \sqrt{\text{ann}_{\mathcal{D}}(\text{gr}^F(R))} = (\xi_1, \ldots, \xi_n).$$

This shows that $d(R) = n$. Therefore, $R$ is a holonomic $\mathcal{D}$-module.

(2) Denote $H^n_m(R)$ by $E$ and set $\eta = \left[ \frac{1}{x_1 \cdots x_n} \right]$, the class of the given fraction inside $E$. Set $F_iE = D_i \cdot \eta$. Then one can check that \{F_i\} is a good filtration of $E$ and $\text{gr}^F(E) \cong k[\xi_1, \ldots, \xi_n]$ where $\xi_i$ denotes the image of $\partial_i$ in $D_1/D_0$. Hence

$$J = \sqrt{\text{ann}_{\mathcal{D}}(\text{gr}^F(E))} = (x_1, \ldots, x_n).$$

This shows that $d(E) = n$. Therefore, $E = H^n_m(R)$ is a holonomic $\mathcal{D}$-module.

We collect next some of the basic properties of holonomic $\mathcal{D}$-modules.

Theorem 3.6. (1) Holonomic $\mathcal{D}$-modules form an Abelian subcategory of the category of $\mathcal{D}$-modules that is closed under the formation of submodules, quotient modules and extensions ([Bjö79, 1.5.2]).

(2) If $M$ is holonomic, then so is the localization $M_f$ for every $f \in R$ ([Bjö79, 3.4.1]). Consequently, each local cohomology module $H^i_f(M)$ of $M$ is holonomic.

(3) Each holonomic $\mathcal{D}$-module admits a finite filtration in the category of $\mathcal{D}$-modules in which each composition factor is a simple $\mathcal{D}$-module ([Bjö79, 2.7.13]).

(4) A simple holonomic $\mathcal{D}$-module has only one associated prime ([Bjö79, 3.3.16]).

Certain finiteness properties of $H^i_f(R)$ are enjoyed by arbitrary holonomic $\mathcal{D}$-modules. In the following list, the first is a special case of Kashiwara equivalence; the latter were established in [Lyu93b, Theorem 2.4].

Theorem 3.7. Let $R = k[[x_1, \ldots, x_n]]$ and let $m$ denote the maximal ideal. Let $M$ be a finitely generated $\mathcal{D}$-module.

(1) If $\dim(Supp_R(M)) = 0$, then $M$ is a direct sum of copies of $\mathcal{D}/\mathcal{D}m$.

(2) $\text{injdim}_R(M) \leq \dim(Supp_R(M))$.

(3) If $M$ is finitely generated (as a $\mathcal{D}$-module), then $M$ has finitely many associated primes (as an $R$-module).

(4) If $M$ is holonomic, then the Bass numbers of $M$ are finite.

Similar statements hold when $R = k[x_1, \ldots, x_n]$.

Remark 3.8. Let $S = k[y_1, \ldots, y_{2n}]$ be the polynomial ring over $k$ in $2n$ variables. When $R = k[x_1, \ldots, x_n]$, we have seen that $\text{gr}(\mathcal{D}) \cong S$. The Poisson bracket on $S$ is defined as follows:

$$\{f, g\} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial y_{n+i}} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial y_{n+i}} \frac{\partial f}{\partial y_i} \right).$$
An ideal $a$ of $S$ is said to be closed under the Poisson bracket if $\{f, g\} \in a$ whenever $f, g \in a$.

The Poisson bracket is closely related to symplectic structures on $\mathbb{C}^{2n}$ and involutive subvarieties of $\mathbb{C}^{2n}$. A symplectic structure $\omega$ on $\mathbb{C}^{2n}$ is a non-degenerate skew-symmetric form; the standard one is given by

$$\begin{bmatrix}
0 & -I_n \\
I_n & 0
\end{bmatrix}$$

where $I_n$ is the $n \times n$ identity matrix. Fix a symplectic structure $\omega$ on $\mathbb{C}^{2n}$.

Given any subspace $W$ of $\mathbb{C}^{2n}$, its skew-orthogonal complement is defined as $W^\perp := \{ \bar{v} \in \mathbb{C}^{2n} \mid \omega(\bar{w}, \bar{v}) = 0 \ \forall \ \bar{w} \in W \}$.

A subspace $W$ is called involutive if $W^\perp \subseteq W$. A subvariety $X$ of $\mathbb{C}^{2n}$ is called involutive if the tangent space $T_xX \subseteq \mathbb{C}^{2n}$ is an involutive subspace for every smooth point $x \in X$. One can show that an affine variety $X \subseteq \mathbb{C}^{2n}$ is involutive with respect to the standard symplectic structure on $\mathbb{C}^{2n}$ if and only if its (radical) defining ideal $I(X)$ is closed under the Poisson bracket.

The following was conjectured in [GQS70] by Guillemin–Quillen–Sternberg and proved in [SKK73] for sheaves of differential operators with holomorphic coefficients on a complex analytic manifold by Kashiwara–Kawai–Sato. The first algebraic proof was discovered by Gabber in [Gab81].

**Theorem 3.9.** Let $R = k[x_1, \ldots, x_n]$ and $M$ be a holonomic $\mathcal{D}$-module. Then the characteristic ideal $J$ of $M$ is closed under the Poisson bracket on $gr(\mathcal{D})$.

Again, let $R$ be either $k[x_1, \ldots, x_n]$ or $k[[x_1, \ldots, x_n]]$. Then each $\mathcal{D}$-module $M$ admits a (global) de Rham complex. This is a complex of length $n$, denoted $\Omega^*_R \otimes M$ (or simply $\Omega^*_R$ in the case $M = R$), whose objects are $R$-modules but whose differentials are merely $k$-linear. It is defined as follows [Bjo79, §1.6]: for $0 \leq i \leq n$, $\Omega^i_R \otimes M$ is a direct sum of $K_i^n$ copies of $M$, indexed by $i$-tuples $1 \leq j_1 < \cdots < j_i \leq n$. The summand corresponding to such an $i$-tuple will be written $M \, dx_{j_1} \wedge \cdots \wedge \, dx_{j_i}$. The $k$-linear differentials $d^i : \Omega^i_R \otimes M \to \Omega^{i+1}_R \otimes M$ are defined by

$$d^i(m \, dx_{j_1} \wedge \cdots \wedge \, dx_{j_i}) = \sum_{s=1}^n \partial_s(m) \, dx_s \wedge \, dx_{j_1} \wedge \cdots \wedge \, dx_{j_i},$$

with the usual exterior algebra conventions for rearranging the wedge terms, and extended by linearity to the direct sum. We remark that in the polynomial case we are simply using the usual Kähler differentials to build this complex, whereas in the formal power series case, we are using the $m$-adically continuous differentials (since in this case the usual module $\Omega^1_{R/k}$ of Kähler differentials is not finitely generated over $R$). An alternative way is to view
$\Omega^*_R \otimes M$ as a representative of $\omega_R \otimes_\mathcal{D}^R M$, where $\omega_R$ is the right $\mathcal{D}$-module $\mathcal{D}/(c_1, \ldots, c_n) \mathcal{D}$ which is as $R$-module simply $R$.

The cohomology objects $H^i(M \otimes \Omega^*_R)$ are $k$-spaces and called the de Rham cohomology spaces of the left $\mathcal{D}$-module $M$, and are denoted $H^i_{\text{dR}}(M)$. The simplest de Rham cohomology spaces (the 0th and $n$th) of $M$ take the form

$$H^0_{\text{dR}}(M) = \{ m \in M \mid \hat{c}_1(m) = \cdots = \hat{c}_n(m) = 0 \} \subseteq M$$

$$H^n_{\text{dR}}(M) = M/(\hat{c}_1 \cdot (M) + \cdots + \hat{c}_n \cdot (M)).$$

The de Rham cohomology spaces are not finite dimensional in general, even for finitely generated $M$. The following theorem is (for the Weyl algebra) a special case of fact that the $\mathcal{D}$-module theoretic direct image functor preserves holonomicity, [HTT08, Section 3.2]. It can be found in [Bjö79, 1.6.1]) for the polynomial case and in [vdE85, Prop. 2.2] for the formal power series case.

**Theorem 3.10.** Let $M$ be a holonomic $\mathcal{D}$-module. The de Rham cohomology spaces $H^i_{\text{dR}}(M)$ are finite-dimensional over $k$ for all $i$.

Let $E$ denote $H^n_m(R)$. If $R = \mathbb{k}[[x_1, \ldots, x_n]]$, then we use $D(-)$ to denote $\text{Hom}_R(-, E)$ (this is the Matlis dual; it should not be confused with the holonomic duality functor $D$ which is quite different). If $R = \mathbb{k}[x_1, \ldots, x_n]$, we consider the following “natural” grading on $R$ and on $\mathcal{D}$:

$$\text{deg}(x_i) = 1, \text{deg}(\hat{c}_i) = -1, \quad i = 1, \ldots, n.$$ 

Note that this is really a grading on $\mathcal{D}$ since the relations $[\hat{c}_i, x_i] = 1$ are homogeneous of degree zero. Then $E$ inherits a grading from setting $\text{deg}(\frac{1}{x_1 \cdots x_n}) = -n$. In this graded setting, we use $^*\text{Hom}_R$ to denote the graded $\text{Hom}$ and use $D^*(-)$ to denote $^*\text{Hom}_R(-, E)$ (the graded Matlis dual).

It turns out that $D(-)$ is a functor on the category of $\mathcal{D}$-modules, that is compatible with de Rham cohomology. The following theorem is a combination of [Swi17a, Theorem 5.1] and [SZ18, Theorem A].

**Theorem 3.11.**

1. Let $R = \mathbb{k}[[x_1, \ldots, x_n]]$ and $M$ be a holonomic $\mathcal{D}$-module. Then $$H^i_{\text{dR}}(M)^\vee \cong H^{n-i}_{\text{dR}}(D(M)), \quad i = 1, \ldots, n,$$ where $(-)^\vee$ denotes the $k$-dual of a $k$-vector space.

2. Let $R = \mathbb{k}[x_1, \ldots, x_n]$ and $M$ be a graded $\mathcal{D}$-module. Assume that $\dim_k(H^1_{\text{dR}}(M)) < \infty$. Then $$H^i_{\text{dR}}(M)^\vee \cong H^{n-i}_{\text{dR}}(D^*(M)).$$

As shown in [SZ18, Example 3.14], $D(M)$ may not be holonomic even if $M$ is. The duality statements in Theorem 3.11 show that the (graded) Matlis duals of holonomic $\mathcal{D}$-modules still have finite dimensional de Rham cohomology.
Remark 3.12. The idea of applying Matlis duality to local cohomology modules already appears in the work of Ogus and Hartshorne. For example, Proposition 2.2 in [Ogu73] states that in a local Gorenstein ring $A$ with dualizing functor $D(\cdot)$, the dual $D(H^i_I(A))$ of the local cohomology module $H^i_I(A)$ is equal to the local cohomology module $H^{\dim(A)-i}_\mathfrak{p}(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ where $\mathfrak{X}$ is the completion of $\text{Spec}(A)$ along $I$, and $\mathfrak{p}$ its closed point.

In much greater generality, Greenlees–May duality [GM92] states that (the derived functor of sections with support in $I$) $R\Gamma_I(\cdot)$ and (the derived functor of completion along $I$) $L\Lambda^I(\cdot)$ are adjoint functors. See also [ATJLL99, Lip02].

We briefly discuss algorithmic aspects. The Weyl algebra is both left and right Noetherian and has a Poincaré–Birkhoff–Witt basis of a polynomial ring in $2n$ variables; this makes it possible to extend the usual Gröbner basis techniques to $\mathcal{D}$-modules, see for example [Gal85].

When $R$ is a polynomial ring over the rational numbers, algorithms have been formulated that compute:

1. the local cohomology modules $H^j_I(R)$ in [Wal99], but see also [Oak97, OT01, BL10];
2. the characteristic cycles and Bass numbers of $H^j_I(R)$ when $I$ is a monomial ideal in [AM00, AM04];
3. an algorithm to compute the support of local cohomology modules in [AML06].

In a nutshell, the algorithms are based on the fact that the modules that appear in a Čech complex $\check{C}^*(R; f_1, \ldots, f_m)$ are holonomic and sums of modules generated by fractions of the form $(f_{i_1} \cdots f_{i_t})^e$ for sufficiently small $e \in \mathbb{Z}$. In general, $e = -n$ is sufficient by [Sai09], but in the spirit of computability, it is desirable to know the largest $e$ that may be used. This number turns out to be the smallest integer root of the Bernstein–Sato polynomial $b_{f}(s)$ of the polynomial $f$ in question. Indeed, as was shown by Bernstein in [Ber72], for every polynomial $f \in R$ there is a linear differential operator $P$ depending polynomially on the additional variable $s$ such that

$$P(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, s) \cdot f^{s+1} = b_{P,f}(s) \cdot f^s,$$

where $0 \neq b_{P,f}(s) \in k[s]$ with $k$ a field of definition for $f$. Since $k[s]$ is a PID, Bernstein’s theorem implies there is a monic generator for the ideal of all $b_{P,f}(s)$ that arise this way; this then is called the Bernstein–Sato polynomial $b_f(s)$. It was shown to factor over the rational numbers in [Mal75, Kas77] and is a fascinating invariant of $f$ as it relates to monodromy of the Milnor fiber, multiplier ideals, (Igusa, topological, motivic) zeta functions, the log-canonical threshold and various other geometric notions with differential background. See [Kol97, Wal15] for more details and [AMHNnB17] for a generalization of Bernstein-Sato polynomials to direct summands of polynomial rings.
The polynomial $b_{f}(s)$ can be computed as the intersection of a left ideal (derived from $f_{1}, \ldots, f_{k}$) inside a Weyl algebra with one more variable $t$, with a “diagonal subring” $\mathbb{Q}[t \hat{c}]$. The idea of how to compute this intersection, and then to give a presentation for the corresponding localization $R_{f}$, is due to Oaku. In [Wal99] it was realized how to read off the $D$-structure of the resulting local cohomology $H_{I}^{1}(R)$ and the process was scaled up to non-principal ideals. The algorithm in [OT01] is different in nature and exploits the fact that local cohomology can be seen as certain Tor-modules along the geometric diagonal in $2n$-space. It is, however, still based on the computation of certain $b$-functions that generalize the notion of a Bernstein–Sato polynomial. To understand conceptually how exactly the singularity structure of $I$ influences the structure of the $D$-module $H_{I}^{k}(R)$ remains a question of great interest.

3.1.2. $D$-modules and group actions. We start with discussing the ideal determining the space of matrices of bounded rank, and then outline more recent developments that consider more general actions by Lie groups.

Let for now $\mathbb{K}$ be a field, choose natural numbers $m \leq n$ and set $R = \mathbb{K}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. Let $I_{m,n,t}$ be the ideal generated by the $t$-minors of the matrix $(x_{ij})$. Then $R/I$ is Cohen–Macaulay and $I$ has height $(m - t + 1)(n - t + 1)$, compare [Bru89, BS90].

Thus, in characteristic $p > 0$ one has vanishing $H_{I_{m,n,t}}^{k}(R)$ for any $k \neq (m - t + 1)(n - t + 1)$, because of the Frobenius (via the Peskine–Szpiro vanishing result in Subsection 2.2). In characteristic zero, by [BS90], $\text{lcd}_{R}(I) = mn - t^{2} + 1$. Therefore, $\text{lcd}_{R}(I) - \text{depth}(I, R) = (m + n - 2t)(t - 1) > 0$, unless $m = n = t$ or $t = 1$.

Bruns and Schwänzl also proved in all characteristics that a determinantal variety is cut out set-theoretically by $mn - t^{2} + 1$ equations, and no fewer. In fact, these equations can be chosen to be homogeneous; their methods rest on results involving étale cohomology. In particular, $I_{m,n,t}$ is a set-theoretic complete intersection if and only if $n = m = t$. The same questions for the case of symmetric and skew-symmetric matrices were answered completely in [Bar95] by Barile. In many but not all cases the number of defining equations agree with the local cohomological dimension.

Consider now the integral version of $I_{m,n,p}$ inside $R_{\mathbb{Z}} = \mathbb{Z}[x_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$. By [LSW16], $H_{I_{m,n,t}}^{k}(R_{\mathbb{Z}})$ is a vector space over $\mathbb{Q}$ when $k$ exceeds the height of $I_{m,n,t}$. Similar results are shown for the case of generic matrices that are symmetric or anti-symmetric. As a corollary, $H_{a}^{mn-t^{2}+1}(A)$ vanishes for every commutative ring $A$ of dimension less than $mn$ where $a$ is the ideal of $t$-minors of any $m \times n$ matrix over $A$. The initial version of this result ($m = 2 = n - 1 = t$) appeared in [HKM09].

If $\mathbb{K}$ is algebraically closed, Barile and Macchia study in [BM19] the number of elements needed to generate the ideal of $t$-minors of a matrix $X$ up to radical, if the entries of $X$ outside some fixed $t \times t$-submatrix are algebraically dependent over $\mathbb{K}$. They prove that this number drops at least by
one with respect to the generic case; under suitable assumptions, it drops at least by \( k \) if \( X \) has \( k \) zero entries.

**Notation 3.13.** We now specialize the base field to \( \mathbb{C} \) and let \( G \) be a connected linear algebraic group acting on a smooth connected complex algebraic variety \( X \).

Suppose \( R = \mathbb{C}[x_1, \ldots, x_n] \) and \( G \) is an algebraic Lie group acting algebraically on \( X = \mathbb{C}^n \). There is a natural map

\[
\psi: g \rightarrow \text{Der}(\mathbb{C}^n)
\]

from the Lie algebra to the global vector fields on \( \mathbb{C}^n \), i.e., the derivations inside the Weyl algebra \( D = D(R, \mathbb{C}) \).

The induced action \( \star \) of \( G \) on \( R \) can be extended to an action on \( D \) that we also denote by \( \star \). If \( M \) is a \( D \)-module with a \( G \)-action, it is **equivariant** if the actions of \( G \) on \( D \) and \( M \) are compatible:

\[
(g \cdot P) \cdot (g \cdot m) = g \cdot (P \cdot m)
\]

for all \( g \in G, P \in D, m \in M \).

Differentiating the \( G \)-action on \( M \) one obtains an action of \( g \) on \( M \). One can now ask whether the Lie algebra element \( \gamma \) acts on \( M \) via differentiation of the \( G \)-action the same way that \( \psi(\gamma) \) acts on \( M \) as element of \( D \). This is not necessarily the case.

**Example 3.14.** Let \( G = \mathbb{C}^* \) act on \( \mathbb{C} \) by standard multiplication. The Lie algebra \( \text{Lie}(G) \) has an equivariant generator \( \gamma \) that via \( \psi \) becomes \( x \partial_x \in D \).

Let \( M = D/(x \partial_x - \lambda) \), with \( G \)-action inherited from the standard \( G \)-action on \( D \): \( g \cdot x = g^{-1}x \), \( g \cdot \partial_x = g \partial_x \). Since \( x \partial_x - \lambda \) is \( g \)-invariant, this is indeed a \( G \)-action on \( M \). Since \( 1 \in D \) is \( G \)-invariant, the effect of \( \gamma \) on \( 1 \in M \) should be zero. On the other hand, \( \psi(\gamma) \cdot 1 = \lambda \). Thus, the two actions agree if and only if \( \lambda = 0 \).

Now note that there are other ways to act with \( G \) on \( M \). Indeed, a \( \mathbb{C}^* \)-action is the same as the choice of a \( \mathbb{Z} \)-grading on \( M \). Our choice above was \( \deg(1) = 0 \); we now consider the choice \( \deg(1) = k \in \mathbb{Z} \). This corresponds to \( g \cdot 1 = g^k 1 \), so that \( \gamma \) must act on \( 1 \) as multiplication by \( k \). We conclude that the two actions of \( \gamma \) agree if and only if \( \lambda \) is an integer and the degree of \( 1 \) is \( \lambda \).

**Definition 3.15.** The \( D \)-module \( M \) is **strongly equivariant** if the differential action of \( G \) on \( M \) agrees with the effect of \( \psi \) on \( M \). In other words, \( \gamma \cdot m = \psi(\gamma)m \) for all \( \gamma \in g, m \in M \).

**Remark 3.16.** Strong \( G \)-equivariance of a group acting on a variety \( X \) can be also phrased as follows, see [HTT08, Dfn. 11.5.2]: let \( \pi \) and \( \mu \) be the projection and multiplication maps

\[
\pi: G \times X \rightarrow X, \\
\mu: G \times X \rightarrow X,
\]
respectively. Then $M$ is strongly equivariant if there is a $D_{G \times X}$-isomorphism 

$$\tau : \pi^* M \to \mu^* M$$

that satisfies the usual compatibility conditions on $G \times G \times X$, see [vdB, Prop. 2.6]. If such $\tau$ exists, it is unique. 

Strongly $G$-equivariant $D_X$-modules are rather special $D$-modules. A $G$-equivariant morphism of smooth varieties with $G$-action automatically preserves $G$-equivariance under direct and inverse images (since $G$ is connected, see [vdB99, before Prop. 3.1.2]). If $G$ has finitely many orbits on $X$, strong equivariance implies that the underlying $D$-module is regular holonomic; this is a growth condition of the solution sheaf of the module and a critical component of the Riemann–Hilbert correspondence. In this case, the simple and strongly equivariant $D_X$-modules are labeled by pairs consisting of a $G$-orbit $G/H$ and a finite-dimensional irreducible representation of the component group of $H$ (in other words, a simple $G$-equivariant local system on the orbit), [HTT08, Prop. 11.6.1]. For example, if $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$, these simple modules are the modules $H^j_{I_S}(R)$ where $R = \mathbb{C}[x_1, \ldots, x_n]$, $S \in 2^n$ and $I_S = \{(x_s | s \in S)\}$.

If $I$ is an ideal of $R$ and $Y$ the corresponding variety, then $I$ is $G$-stable if and only if $Y$ is. In this case, the localization of a strongly equivariant module $M$ at an equivariant $g \in R$ is also strongly equivariant. It follows that all local cohomology modules $H^j_I(M)$ are as well. In particular, this holds when $M$ is $R$ or a local cohomology module of $R$ obtained in this way.

In [RWW14], the authors initiated the study of the $GL$-equivariant decomposition of the local cohomology modules of determinantal ideals in characteristic zero. The main result of the paper is a complete and explicit description of the character of this representation. An important consequence is a complete and explicit description of exactly which local cohomology modules $H^j_I(M)$ vanish and which do not, in the case $t = n$. This was then refined and extended to Pfaffians in [RWW14]. The restriction $t = n$ was removed in [RW14]. Generalizations to symmetric and skew-symmetric matrices were published in [RW16, Nan12].

In [Rai16], Raicu obtains results on the structure of the $G$-invariant simple $D$-modules and their characters for rank-preserving actions on matrices, extending work of Nang [Nan08, Nan12]. Remarkably, for the case of symmetric matrices, this provides a correction to a conjecture of Levasseur. Raicu’s methods produce composition factors for certain local cohomology modules. In [LR20] then this was taken the furthest, to give character formulae for iterated local cohomology modules.

A more general approach was used in [LR20, LW19] in order to study decompositions and categories of equivariant modules in the category of $D$-modules, specifically with regards to quivers. These arise when when $G$ acts on $X$ with finitely many orbits and more particularly when $X$ is a spherical vector space and $G$ is reductive and connected. This leads to the study of
the “representation type” of the underlying quiver (shown to be finite or
tame) and the quivers are described explicitly for all irreducible $G$-spherical
vector spaces of connected reductive groups using the classification of Kac.
An early paper on this regarding the determinantal case was [Nan04]. More
recently, cases of exceptional representations and their quivers have been
studied: [Per20, LRW19].

**Remark 3.17.** Invariant theory has also recently been aimed at singularity
invariants such as multiplier and test ideals [HV16], and $F$-pure thresholds
[MSV14].

### 3.1.3. Coefficient fields of arbitrary characteristic.

Let here $k$ be a field and set $R = \mathbb{k}[x_1, \ldots, x_n]$ or $R = \mathbb{k}[[x_1, \ldots, x_n]]$. We have seen that $\mathcal{D} = \mathcal{D}(R, \mathbb{k})$ is the free $R$-module with basis

$$
\frac{1}{t_1!} \partial x_1^{t_1} \ldots \frac{1}{t_n!} \partial x_n^{t_n}
$$

for $(t_1, \ldots, t_n) \in \mathbb{N}^n$

When $\text{char}(k) = p > 0$, the ring $\mathcal{D}$ is no longer left or right Noetherian. However, some desirable properties of $\mathcal{D}$-modules in characteristic 0 extend
to the finite characteristic case.

**Theorem 3.18.** Let $R = \mathbb{k}[x_1, \ldots, x_n]$ or $R = \mathbb{k}[[x_1, \ldots, x_n]]$, where $\mathbb{k}$ is a
field. Then

1. $\text{injdim}_R(M) \leq \dim(\text{Supp}_R(M))$ for every $\mathcal{D}$-module $M$ ([Lyu00c]).
2. $Rf$ has finite length in the category of $\mathcal{D}$-modules for each $f \in R$
   ([Lyu00a], [Lyu11]).

   Consequently, local cohomology modules $H_f^1(R)$ have finite length
   in the category of $\mathcal{D}$-modules.

In general, it is a difficult problem to calculate the length $\ell_{\mathcal{D}}(H_f^1(R))$, or
even just $\ell_{\mathcal{D}}(Rf)$. Some results are in [Tor09] and [Bit20]. The following
upper bounds were obtained in [KMSZ18].

**Theorem 3.19.** Let $\mathbb{k}$ be a field and $R = \mathbb{k}[x_1, \ldots, x_n]$.

1. For each $f \in R$,
   $$
   \ell_{\mathcal{D}}(Rf) \leq (\deg(f) + 1)^n.
   $$

2. Assume an ideal $I$ can be generated by $f_1, \ldots, f_t$. Then
   $$
   \ell_{\mathcal{D}}(H_I^1(R)) \leq \sum_{1 \leq i_1 < \cdots < i_j \leq t} (\deg(f_{i_1}) + \cdots + \deg(f_{i_j}) + 1)^n - 1.
   $$

**Example 3.20.** Let $R = \mathbb{k}[x_1, x_2, x_3]$ and $f = x_1^3 + x_2^3 + x_3^3$. Then

$$
\ell_{\mathcal{D}}(H_f^1(R)) = \begin{cases}
1 & \text{if } \text{char}(k) \equiv 2 \pmod{3}; \\
1 & \text{if } \text{char}(k) = 3; \\
2 & \text{if } \text{char}(k) \equiv 1 \pmod{3}; \\
2 & \text{if } \text{char}(k) = 0.
\end{cases}
$$
If \( \text{ch}(k) = 0 \) and \( R = \mathbb{k}[x_1, \ldots, x_n] \) or \( R = \mathbb{k}[[x_1, \ldots, x_n]] \), then \( R_f \) can always be generated by \( 1/f^n \) as a \( \mathcal{D} \)-module, but may not be generated by \( 1/f \). For instance, let \( f, R \) be as in Example 3.20, then \( 1/f \) generates a proper \( \mathcal{D} \)-submodule of \( R_f \) in characteristic 0. On the other hand, in characteristic \( p \), the situation is quite different as shown in [AMBL05], and generalized to rings of \( F \)-finite representation type in [TT08].

**Theorem 3.21.** Let \( \mathbb{k} \) be a field of characteristic \( p > 0 \) and let \( R = \mathbb{k}[x_1, \ldots, x_n] \) or \( R = \mathbb{k}[[x_1, \ldots, x_n]] \). Then \( R_f \) can be generated by \( 1/f \) as a \( \mathcal{D} \)-module for every \( f \in R \).

We have seen that, when \( \mathbb{k} \) is a field, \( R = \mathbb{k}[x_1, \ldots, x_n] \), and \( M \) is a \( \mathcal{D} \)-module, then \( \text{injdim}_R(M) \leq \dim(\text{Supp}_R(M)) \). Thus, if \( \dim(\text{Supp}_R(M)) = 0 \), then \( M \) must be an injective \( R \)-module. Let \( I \) be a homogeneous ideal of \( R \) and assume that \( \text{Supp}_R(H^1_I(R)) = \{m\} \) where \( m = (x_1, \ldots, x_n) \). Then \( H^1_I(R) \cong \bigoplus H^n_{m^j}(R) \), a direct sum of finitely many copies of \( H^n_{m}(R) \). Since both \( H^1_I(R) \) and \( H^n_{m}(R) \) are graded, a natural question is whether this isomorphism is degree-preserving. To answer this question, the notion of Eulerian graded \( \mathcal{D} \)-modules was introduced in [MZ14].

Recall that \( R = \mathbb{k}[x_1, \ldots, x_n] \) and \( \mathcal{D} = \mathcal{D}(R, \mathbb{k}) \) are naturally graded via:

\[
\text{deg}(x_i) = 1, \quad \text{deg}(\partial_i) = -1.
\]

**Definition 3.22.** Denote the operator \( \frac{1}{t!} \frac{\partial^t_i}{\partial x_i^t} \) by \( \partial_i^{[t]} \).

The \( t \)-th Euler operator \( E_t \) is defined as

\[
E_t := \sum_{\substack{t_1 + t_2 + \cdots + t_n = t \\ t_1 \geq 0, \ldots, t_n \geq 0}} x_1^{t_1} \cdots x_n^{t_n} \partial_1^{[t_1]} \cdots \partial_n^{[t_n]}.
\]

In particular \( E_1 \) is the usual Euler operator \( \sum_{i=1}^n x_i \partial_i \).

A graded \( \mathcal{D} \)-module \( M \) is called **Eulerian**, if each homogeneous element \( z \in M \) satisfies

\[
E_t \cdot z = \binom{\text{deg}(z)}{t} \cdot z
\]

for every \( t \geq 1 \).

We collect some basic properties of Eulerian graded \( \mathcal{D} \)-modules as follows.

**Theorem 3.23.** Let \( M \) be an Eulerian graded \( \mathcal{D} \)-module. Then

1. Graded \( \mathcal{D} \)-submodules of \( M \) and graded \( \mathcal{D} \)-quotients of \( M \) are Eulerian.
2. If \( S \) is a homogeneous multiplicative system in \( R \), then \( S^{-1}M \) is Eulerian. In particular, \( M_g \) is Eulerian for every homogeneous \( g \in R \).
3. The local cohomology modules \( H^j_I(M) \) are Eulerian for every homogeneous ideal \( I \).
4. The degree-shift \( M(\ell) \) is Eulerian if and only if \( \ell = 0 \).
It follows from Theorem 3.23 that, if \( \text{Supp}_R(H^j_I(R)) = \{m\} \) for a homogeneous ideal \( I \), then \( H^j_I(R) \cong \oplus H^a_m(R)^{\mu_j} \) is a degree-preserving isomorphism. Consequently,

\[
H^j_I(R) \cong \oplus H^a_m(R)^{\mu_j} \cong \mu_j = 0
\]

This turns out to be a source of vanishing results for sheaf cohomology. For example, (3.1.3.1) is one of the ingredients in [BBL+19] to prove Theorem 2.18 which is an extension of Kodaira vanishing to a non-reduced setting.

Extensions of Eulerian \( \mathcal{D} \)-modules may not be Eulerian as shown in [MZ14, Remark 3.6]. In [Put15] the notion of generalized Eulerian \( \mathcal{D} \)-module in characteristic 0 was introduced as follows. Fix integers \( w_1, \ldots, w_n \) and set

\[
\deg(x_i) = w_i \quad \deg(\partial_i) = -w_i
\]

A graded \( \mathcal{D} \)-module \( M \) is called generalized Eulerian if, for every homogeneous element \( m \in M \), there is an integer \( a \) (which may depend on \( m \)) such that

\[
(E_1 - \deg(m))^a \cdot m = 0.
\]

It was shown that the category of generalized Eulerian \( \mathcal{D} \)-modules is closed under extension. This notion of generalized Eulerian \( \mathcal{D} \)-modules turns out to be useful in calculating de Rham cohomology of local cohomology modules in characteristic 0 (cf. [Put15], [PS19], [RWZ21]).

In characteristic \( p \), the fact that \( H^j_I(R) \cong \oplus H^a_m(R)^{\mu_j} \) is a degree-preserving isomorphism when \( \text{Supp}_R(H^j_I(R)) = \{m\} \) was also established in [Zha12] using \( F \)-modules, a technique that we discuss next.

3.2. \( F \)-modules. Let \( A \) be a Noetherian commutative ring of characteristic \( p \). Then \( A \) is equipped with the Frobenius endomorphism

\[
F : A \xrightarrow{a \mapsto a^p} A.
\]

The Frobenius endomorphism plays a very important role in the study of rings of characteristic \( p \). For instance, in [Kun69], regularity of \( A \) is characterized by the flatness of the Frobenius endomorphism.

**Definition 3.24** (Peskine–Szpiro functor). Let \( A \) be a Noetherian commutative ring of characteristic \( p \). For each \( A \)-module \( M \), denote by \( F_*M \) the \( A \)-bimodule whose underlying Abelian group is the same as \( M \), whose left \( A \)-module structure is the usual one: \( a \cdot z = az \) for each \( z \in F_*M \), and whose right \( A \)-module structure is given via the Frobenius \( F \): \( z \cdot a := a^p z \) for each \( z \in F_*M \).

The Peskine–Szpiro functor \( F_A(-) \) from the category of left \( A \)-modules to itself is defined via

\[
F_A(M) := F_*A \otimes_A M
\]

for each \( A \)-module \( M \), where the tensor product uses the right \( A \)-structure on \( F_*A \).
Geometrically, consider the morphism of spectra induced by the Frobenius $F: A \rightarrow A$. Then the right $A$-module structure of $F_\ast(M)$ is obtained via restriction of scalars along $F$, and hence agrees with the pushforward of $M$. On the other hand, $F_A(M)$ is the pullback of a module under the Frobenius.

If $A$ is regular, then it follows from [Kun69] that $F_\ast A$ is a flat $A$-module and hence $F_A(-)$ is an exact functor.

**Remark 3.25.** Let $R$ be a Noetherian regular ring of characteristic $p$ and $I$ be an ideal of $R$.

1. We have
   \[
   F_R(R^m) \cong R^m, \quad F_R(R/I) \cong R/I^{[p]},
   \]
   Here $I^{[p]}$ is the Frobenius power from Remark 1.5

2. Moreover,
   \[
   F_R(\text{Ext}^j_R(R/I, R)) \cong \text{Ext}^j_R(F_R(R/I), F_R(R)) \cong \text{Ext}^j_R(R/I^{[p]}, R).
   \]
   The natural surjection $R/I^{[p]} \twoheadrightarrow R/I$ induces
   \[
   \beta: \text{Ext}^j_R(R/I, R) \twoheadrightarrow \text{Ext}^j_R(R/I^{[p]}, R)
   \]
   and by iteration produces a directed system
   \[
   \text{Ext}^j_R(R/I, R) \xrightarrow{\beta} \text{Ext}^j_R(R/I^{[p]}, R) \xrightarrow{F_R(\beta)} \text{Ext}^j_R(R/I^{[p^2]}, R) \cdots
   \]
   which agrees with
   \[
   \text{Ext}^j_R(R/I, R) \xrightarrow{\beta} F_R(\text{Ext}^j_R(R/I, R)) \xrightarrow{F_R(\beta)} F_R^2(\text{Ext}^j_R(R/I, R)) \cdots
   \]
   Since \{I^{[p^n]}\}_{n \geq 0} and \{I^j\}_{j \geq 0} are cofinal (that is, the two families of ideals define the same topology on the ring), the direct limit of this direct system is $H^j_I(R)$.

3. The previous items suggest that $H^j_I(R)$ may be built from the finitely generated $R$-module $\text{Ext}^j_R(R/I, R)$ using Frobenius, and hence it is natural to expect some properties of $H^j_I(R)$ to be reflected in $\text{Ext}^j_R(R/I, R)$. Indeed, it was proved in [HS93] that
   \[
   \text{Ass}_R(H^j_I(R)) \subseteq \text{Ass}_R(\text{Ext}^j_R(R/I, R)), \quad \mu_p(M)(H^j_I(R)) \leq \mu_p(\text{Ext}^j_R(R/I, R))
   \]
   for every prime ideal $p$, where $\mu_p(M)$ denotes the $i$-th Bass number of an $R$-module $M$ with respect to $p$. (This was generalized to rings of $F$-finite representation type in [TT08]).

Based on the idea of building $H^j_I(R)$ using $\text{Ext}^j_R(R/I, R)$, [KZ18] describes a practical algorithm to calculate the support of $H^j_I(R)$; this algorithm has been implemented in Macaulay2 [GS].
### 3.2.1. $F$-modules

In order to conceptualize the approach in [HS93], Lyubeznik introduced the theory of $F$-modules in [Lyu97]. Throughout 3.2.1, $R$ is a regular (not necessarily local) Noetherian ring of characteristic $p > 0$, and $I$ is an ideal of $R$.

**Definition 3.26.** An $F$-module over $R$ (or $F_R$-module) is a pair $(M, \theta_M)$ where $M$ is an $R$-module and $\theta_M : M \to F_R(M)$ is an $R$-module isomorphism, called the structure morphism. (When the underlying ring is understood, we sometimes refer simply to $M$ as an “$F$-module”. ) The category of $F_R$-modules will be denoted by $\mathcal{F}_R$ (or $\mathcal{F}$ when $R$ is clear from the context).

If $R$ is graded, a graded $F$-module is an $F$-module $M$ such that $M$ is graded and the structure isomorphism $M \to F_R(M)$ is degree-preserving.

**Example 3.27.** One can check that $F_*R \otimes_R R \xrightarrow{r' \otimes_{p} \to r'p} R$ is an $R$-linear isomorphism. Hence $R$ is an $F$-module; consequently so are all free $R$-modules.

Given any $g \in R$, one can check that $F_*R \otimes_R R_g \xrightarrow{r' \otimes_{p} \to r'p} R_g$ is an $R$-linear isomorphism. Hence $R_g$ is an $F$-module.

When $R = k[x_1, \ldots, x_n]$ with standard grading, then for each graded $R$-module $M$ we define a grading on $F_R(M) = F_*R \otimes_R M$ via

$$\deg(r' \otimes m) = \deg(r') + p \deg(m)$$

for all homogeneous $r' \in R$ and $m \in M$.

In this setting, $F_*R \otimes_R R \xrightarrow{r' \otimes_{p} \to r'p} R$ is a degree-preserving $R$-linear isomorphism and so $R$ is a graded $F$-module. Likewise, if $g \in R$ is homogeneous, then $F_*R \otimes_R R_g \xrightarrow{r' \otimes_{p} \to r'p} R_g$ is a degree-preserving $R$-linear isomorphism and hence $R_g$ is a graded $F$-module.

**Definition 3.28.** Let $(M, \theta_M)$ be an $F$-module. We say that $M$ is $F$-finite if there exists a finitely generated $R$-module $M'$ and an $R$-linear map $\beta : M' \to F_R(M')$ such that

$$(3.2.1.1) \quad \lim_{\ell}(M' \xrightarrow{\beta} F_R(M') \xrightarrow{F_R(\beta)} F_R^2(M') \to \cdots) \cong M,$$

and the structure morphism $\theta_M$ is induced by taking the direct limit over $\ell$ of $F_R^\ell(\beta) : F^\ell_R(M') \to F^{\ell+1}_R(M')$. In this case we call $M'$ a generator of $M$ and $\beta$ a generating morphism. A generator $M'$ of an $F$-finite $F$-module $M$ is called a root if the generating morphism $\beta : M' \to F_R(M')$ is injective.

A graded $F$-finite $F$-module is defined to be an $F$-finite $F$-module for which the modules and morphisms in (3.2.1.1) can be chosen to be homogeneous.

**Example 3.29.** From Remark 3.25, one can see that every local cohomology module $H^j_I(R)$ is an $F$-finite $F$-module since it is the direct limit of

$$\Ext^j_R(R/I, R) \xrightarrow{\beta} F_R(\Ext^j_R(R/I, R)) \xrightarrow{F_R(\beta)} F_R^2(\Ext^j_R(R/I, R)) \to \cdots$$
and $\text{Ext}^j_R(R/I,R)$ is finitely generated.

When $R = \mathbb{k}[x_1, \ldots, x_n]$ and $I$ is a homogeneous ideal of $R$, the local cohomology modules $H^j_I(R)$ are graded $F$-finite $F$-modules.

There is a fruitful analogy between $(F$-finite) $F$-modules and (holonomic) $\mathcal{D}$-modules. We collect some basic properties of $F$-modules, which are parallel to those of $\mathcal{D}$-modules, as follows.

**Theorem 3.30.** Let $R$ be a Noetherian regular ring of characteristic $p > 0$.

1. If $M$ is an $F$-module, then $\text{injdim}_R(M) \leq \dim(\text{Supp}_R(M))$, [Lyu97, 1.4].
2. $F$-finite $F$-modules form a full Abelian subcategory of the category of $R$-modules that is closed under the formation of submodules, quotient modules, and extensions, [Lyu97, 2.8].
3. If $M$ is an $F$-finite $F$-module, then so is the localization $M_g$ for each $g \in R$, [Lyu97, 2.9].
4. A simple $F$-module has a unique associated prime, [Lyu97, 2.12].
5. $F$-finite $F$-modules have finite length in the category of $F$-modules, [Lyu97, 3.2].

**Remark 3.31.** The theory of $F$-modules plays a crucial role in the extension of the Riemann–Hilbert correspondence to characteristic $p$ by Emerton and Kisin [EK04], which is beyond the scope of this survey.

### 3.2.2. $A\{f\}$-modules: action of Frobenius

Let $A$ be a Noetherian commutative ring of characteristic $p$. We will use $A\{f\}$ to denote the associative $A$-algebra with one generator $f$ and relations $fa = a^p f$ for all $a \in A$.

**Remark 3.32.** Let $M$ be an $A$-module $M$. The following are equivalent.

1. $M$ is an $A\{f\}$-module.
2. $M$ admits an additive map $f : M \rightarrow M$ such that $f(am) = a^p f(m)$ for every $a \in A$ and $m \in M$; this $f$ is called a Frobenius action on $M$.
3. $M$ admits an $A$-linear map $M \rightarrow F_\ast M$ where $F : A \rightarrow A$ is the Frobenius endomorphism on $A$.
4. $M$ admits an $A$-linear map $F_\ast A \otimes_A M \rightarrow M$ where $F : A \rightarrow A$ is the Frobenius endomorphism on $A$.

In (2), we still use $f$ to denote the Frobenius action since multiplication on the left by $f$ on $M$ is indeed a Frobenius action for each $A\{f\}$-module $M$.

Of course, the standard example of a Frobenius action is $A$ with the $p$-th power map. Note that the image $f(M)$ is in general just a group, but acquires the structure of a $\mathbb{k}$-space when $\mathbb{k}$ is perfect.

The Frobenius on $A$ induces a natural Frobenius action on each $H^j_\alpha(A)$ for every ideal $\alpha$; hence $H^j_\alpha(A)$ is an $A\{f\}$-module. In this paper, we always consider $H^j_\alpha(A)$ as an $A\{f\}$-module with the Frobenius action $f$ induced by the Frobenius endomorphism on $A$. For this reason, some authors denote by
F (instead of f) the Frobenius action on $H^i_\mathfrak{a}(A)$ induced by the Frobenius endomorphism on $A$.

**Definition 3.33.** Given an $A\{f\}$-module $M$ with Frobenius action $f : M \to M$, the intersection

$$M_{\text{st}} := \bigcap_{t \geq 1} f^t(M)$$

is called the $f$-stable part of $M$.

An element $z \in M$ is called $f$-nilpotent if $f^t(z) = 0$ for some integer $t$.

An $A\{f\}$-module $M$ is called $f$-torsion if every element in $M$ is in the kernel of some iterate of $f$, and it is called $f$-nilpotent if there is an integer $t$ such that $f^t(M) = 0$.

**Remark 3.34.** When $M = H^i_\mathfrak{a}(A)$ is a local cohomology module of $A$, the notions of $f$-torsion and $f$-nilpotent are also denoted by $F$-torsion and $F$-nilpotent, respectively, since the Frobenius action $f$ is induced by the Frobenius endomorphism on $A$.

Assume $(A, \mathfrak{m}, k)$ is a local ring and $x_1, \ldots, x_d$ is a full system of parameters. Then the Frobenius action $f$ on $H^d_\mathfrak{m}(A)$ can be described as follows. Let $\eta = \frac{a}{x_1^{1p} \cdots x_d^{up}}$ be an element in $H^d_\mathfrak{m}(A)$, then

$$f(\eta) = \frac{a^p}{x_1^{np} \cdots x_d^{mp}}.$$

An $A\{f\}$-module that is also an Artinian $A$-module is called a cofinite $A\{f\}$-module. Cofinite $A\{f\}$-modules enjoy an amazing property.

**Theorem 3.35.** Let $A$ be a local ring of characteristic $p > 0$. Assume that $M$ is an $f$-torsion cofinite $A\{f\}$-module. Then $M$ must be $f$-nilpotent.

Theorem 3.35 was first proved by Hartshorne and Speiser in [HS77]. There, Hartshorne and Speiser created a version of some of Ogus’ results from [Ogu73] in characteristic $p > 0$. Their motivating question was to determine when the cohomology of every coherent sheaf on the complement of a projective variety be a finite dimensional vector space. Hartshorne and Speiser use the Frobenius endomorphism on $\mathcal{O}_X$ to supply the information given by the connection used by Ogus in characteristic zero, and $\mathbb{Z}/p$-étale cohomology turns up in place of de Rham cohomology. Theorem 3.35 was later generalized by Lyubeznik in [Lyu97] (using the $\mathcal{H}_{R,A}$-functor discussed in the sequel). It has found applications in [KLZ09, BSTZ10, BB11] in the study of singularities and invariants defined by Frobenius.

Theorem 4.6 in [Lyu06b] reads as follows: if $k$ is an algebraically closed field of positive characteristic, and if $(A, \mathfrak{m}, k)$ is a complete local ring with connected punctured spectrum and $k \subseteq A$, then $H^d_\mathfrak{m}(A)$ is $f$-torsion. Lyubeznik derives this via a comparison with local cohomology in a complete regular local ring that surjects onto $A$. In [SW08b], this result is sharpened.
to a numerical statement over an algebraically closed coefficient field: the number of connected components of the punctured spectrum of $A$ is one more than the dimension of the $f$-stable part of $H^1_m(A)$.

A general study of Frobenius operators started with [LS01] and later was carried out by various authors: aside from Sharp’s article [Sha07a] we should point at [Sha07b] by the same author, [BB11] who develop the notion of Cartier modules (which are approximately modules with a Frobenius action), and [Gab04]. The article [KSSZ14] contains positive results on finiteness dual to [Sch11b] as well as examples of failure.

**Definition 3.36.** Let $(A, m, k)$ be a local ring of characteristic $p > 0$. Given a cofinite $A\{f\}$-module $W$, a prime ideal $p$ is called a special prime of $W$ if it is the annihilator of an $A\{f\}$-submodule of $W$.

It is proved in [Sha07a, Corollary 3.7] and [EH08, Theorem 3.6] that if the Frobenius action $f: M \rightarrow M$ on the $A\{f\}$-module $M$ is injective then $M$ admits only finitely many special primes. This will be useful when we discuss the $F$-module length of local cohomology modules in the sequel.

**Definition 3.37 ([EH08]).** Let $(A, m)$ be a Noetherian local ring of characteristic $p$. Let $f : H^j_m(A) \rightarrow H^j_d(A)$ denote the Frobenius action induced by the Frobenius on $A$.

A submodule $N$ of $H^j_m(A)$ is called $F$-stable if $f(N) \subseteq N$.

The ring $A$ is called FH-finite if $H^j_m(A)$ admits only finitely many $F$-stable submodules for each $0 \leq j \leq \dim(A)$.

Also, $A$ is called $F$-injective if the natural Frobenius action $f : H^j_m(A) \rightarrow H^j_m(A)$ is injective for each integer $j \leq \dim(A)$.

The Frobenius action on local cohomology modules connects with a very important type of singularities, that of $F$-rationality, which we recall next.

**Definition 3.38.** Let $A$ be a Noetherian ring of characteristic $p$, let $A^o$ denote the complement of the union of minimal primes in $A$ and let $a$ be an ideal of $A$. An element $a \in A$ is in the tight closure of $a$ if there is a $c \in A^o$ such that $ca^e \in a[b^e]$ for all $e > 0$. Let $a^*$ denote the set of elements $a \in A$ that are in the tight closure of $a$; it is an ideal of $A$. An ideal $a$ is called tightly closed if $a = a^*$.

A local ring $A$ is called $F$-rational if $a = a^*$ for every parameter ideal $a$.

In her work to relate $F$-rationality (an algebraic notion) to rational singularity (a geometric notion), Smith [Smi97a] proves the following characterization of $F$-rationality using a Frobenius action.

**Theorem 3.39.** Let $(A, m)$ be a $d$-dimensional excellent local domain of characteristic $p$. Then $A$ is $F$-rational if and only if $A$ is normal, Cohen–Macaulay, and $H^d_m(A)$ contains no non-trivial $F$-stable submodules.
In independent work of Smith, Mehta–Srinivas, and Hara, F-rationality was shown to be the algebraic counterpart to the notion of rational singularities [MS97, Smi97a, Har98]. The purpose of these studies was to establish a parallelism between the concept of a rational singularity in characteristic zero, and invariants based on the Frobenius for its models in finite (large) characteristic. The development of such connections has a fascinating and distinguished history, and we recommend the recent and excellent survey article [TW18] by two experts in the field.

A related construction goes back to [Ene03]. For an element \( x \in A \) and a parameter ideal \( I \) of \( A \) let \( I[x] \) be the ideal of elements \( c \in A \) that multiply \( x^p \) into \( I[x^p] \) for all large \( e \) (cf. Definition 3.38). Enescu shows in [Ene03] that if \( A \) is F-injective and Cohen–Macaulay, then the set of maximal elements in \( \{ I(x) : x \notin I \} \) does not depend on \( I \), is finite and consists only of prime ideals. These are called F-stable primes, and the collection of them is denoted by \( \text{FS}(R) \). Enescu shows further that for an F-injective Cohen–Macaulay complete local ring \( A \), the F-stable primes can be expressed in terms of F-unstability, introduced by Fedder and Watanabe. Enescu and Sharp continued the study of properties of F-stable primes in [Sha07a, Ene09].

Along with FH-finiteness goes another property of rings that will come back to us later:

**Definition 3.40.** \( A \) is called F-pure if \( (A \xrightarrow{a^p} A) \otimes_A M \) is injective for all \( A \)-modules \( M \). ☐

**Remark 3.41.** For background to this remark we refer to the excellent article [TW18].

A standard question on “deformation” in commutative algebra is to ask “If a quotient \( A/(x) \) of \( A \) by a regular element has a nice property, is \( A \) forced to share it?”.

It turns out that F-purity does not deform in this sense, [Fed83, Sin99a]. The reader familiar with the concepts of F-regularity and F-rationality may know that F-rationality deforms [HH94] while F-regularity does not [Sin99b] although it does so for \( \mathbb{Q} \)-Gorenstein rings [HH94, AKM98]. Very recently, Polstra and Simpson proved in [PS20] that F-purity deforms in \( \mathbb{Q} \)-Gorenstein rings.

It is still an open question whether F-injectivity deforms, but some progress has been made. Fedder showed in [Fed83] that F-injectivity deforms when the ring is Cohen–Macaulay. In [HMS14], it was proved that if \( R/xR \) is F-injective and \( H^j_m(A/(x^\ell)) \longrightarrow H^j_m(A/(x)) \) is surjective for all \( \ell > 1 \) and \( j \) then \( A \) is F-injective. Ma and de Stefani established deformation when the local cohomology modules \( H^*_m(A) \) have secondary decompositions that are preserved by the Frobenius [DSM20]. ☐

In [EH08] it is proved that face rings of finite simplicial complexes are FH-finite. They showed further that an F-pure and quasi-Gorenstein local ring is
FH-finite, and raised the question whether all F-pure and Cohen–Macaulay local rings are FH-finite. Ma answered this question in the affirmative by proving the following result in [Ma14].

**Theorem 3.42.** Let \((A, \mathfrak{m})\) be a Noetherian local ring of characteristic \(p\). If \(A\) is F-pure, then \(A\) and all power series rings over \(A\) are FH-finite.

In the paper he also proved that if \(A\) is F-pure (even just on the punctured spectrum) then \(H^p_{\mathfrak{m}}(A)\) is a finite length \(A\{f\}\)-module, and he also established that the finite length property is stable under localization. With Quy, he introduced more recently in [MQ18] the notions F-full (when the Frobenius action is surjective) and F-anti-nilpotent (when the action is injective on every \(A\{f\}\)-subquotient of local cohomology). They established that F-anti-nilpotence implies F-fullness and equals FH-finiteness of [EH08]. Inspired by ideas from [HMS14], they prove the interesting fact that both F-anti-nilpotence and F-fullness do deform.

The action of the Frobenius also ties in naturally with the action of the Frobenius on the cohomology of projective varieties via the identification (1.2.0.1). For example, the Segre product of a smooth elliptic curve \(E\) with \(\mathbb{P}_K^1\) has F-injective coordinate ring (recall Definition 3.40) if and only if the curve is ordinary (the group \(H^1(E; \mathcal{O}_E)\) is the degree zero part of \(H^2_{\mathfrak{m}}(A)\) and the Frobenius action is the induced one; here \(A\) is the coordinate ring of \(E\)). Compare Example 4.2.

Hartshorne and Speiser in [HS77], and Fedder and Watanabe in [FW89] studied F-actions on local cohomology with regards to vanishing of cohomology on projective varieties, and with regards to singularity types of local rings respectively.

According to [ST17], a local ring \((A, \mathfrak{m})\) is F-nilpotent if the Frobenius action is nilpotent on \(H^{<\dim(A)}_{\mathfrak{m}}(A)\) and \(0^{\ast}_{H^{\dim(A)}_{\mathfrak{m}}(A)}\) (the tight closure of the zero submodule of \(H^{\dim(A)}_{\mathfrak{m}}(A)\)), and Srinivas and Takagi show that \(A\) is F-injective and F-nilpotent if and only if it is F-rational. In [PQ19], Polstra and Quy characterize F-nilpotence as (under mild hypotheses) being equivalent to the equality of tight and Frobenius closure for all parameter ideals. This work extends the finite length case discussed in [Ma15] and is somewhat surprising since the complementary notion of F-injectivity is not equivalent to the Frobenius-closedness of all parameter ideals, [QS17], but only implied by it.

Ma also shows in [Ma15], in his setting of finite length lower local cohomology, that F-injectivity implies the ring being Buchsbaum (a generalization of Cohen–Macaulay, [SV86]), and that the analogous statement in characteristic zero is true in the sense that, if \(A\) is a normal standard graded \(\mathbb{K}\)-algebra with \(\mathbb{K} \supseteq \mathbb{Q}\) that is Du Bois and has finite length lower local cohomology, then \(A\) is Buchsbaum. (A singularity \(X\) embedded inside a smooth scheme over the complex numbers is du Bois, following Schwede’s paper [Sch07], if and only if an embedded resolution \(\pi: Y \longrightarrow X\) of \(X = \text{Spec}(A)\) with
reduced total transform $E$ leads to an isomorphism $O_X = R\pi_*(O_E)$. Initially, Du Bois singularities arose from Hodge-theoretic filtrations of the de Rham complex in [DB81]; they include normal crossings and quotient singularities. Du Bois singularities are closely related to (and conjecturally equivalent to) singularities of dense $F$-injective type. Recall that, a finite $\mathbb{Z}$-algebra $A_\mathbb{Z}$ is of dense $F$-injective type if its reductions $A_p$ modulo $p$ are $F$-injective for infinitely many primes $p \in \mathbb{Z}$. Schwede proved in [Sch09c] that if a finite $\mathbb{Z}$-algebra $A_\mathbb{Z}$ is of dense $F$-injective type then the complex model $A_\mathbb{C} = A_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ is Du Bois. The other implication remains an open problem and was proved to be equivalent to the Weak Ordinariness Conjecture (see [BST17] for details).

We close this section with a brief discussion on the very interesting topic of the interaction of the Frobenius with Hodge theory, crossing characteristics. Suppose $A$ is a finitely generated graded $\mathbb{C}$-algebra, and set $X := \text{Proj}(A)$. It is known that certain aspects of the Hodge theory of $X$ are encoded in the combinatorics of the resolution of singularities of $X$, [Del71, Del74, ABW13]. In this context, Srinivas and Takagi proposed and studied in [ST17] the following local conjecture.

**Conjecture 3.43.** If $x$ is a normal isolated singularity on the $n$-dimensional $\mathbb{C}$-scheme $X$ then the local ring at $x$ is of $F$-nilpotent type if and only if for all $i < \dim(X)$, the zeroth graded piece $\text{Gr}_F^0(H^i_F(X^\text{an}, \mathbb{C}))$ of the Hodge filtration is zero.

How much is still unknown in this fascinating area between characteristics can be seen from the fact that the following conjectural statement is still open: let $V$ be an $(n-1)$-dimensional projective simple normal crossings variety in characteristic zero; then the Frobenius action on $H^i(V_p, O_{V_p})$ is not nilpotent for an infinite set of reductions $V_p$ modulo $p$ of $V$. Srinivas and Takagi [ST17] prove the case $n-1 = 2$ of this and derive from it the case $n = 3$ for the conjecture above.

**3.2.3. The Lyubeznik functor $\mathcal{H}_{R,A}$.** Assume that $A$ is a homomorphic image of a Noetherian regular ring $R$. The approach of building $H^i_F(R)$ using a finitely generated $R$-module results in a very useful functor $\mathcal{H}_{R,A}$ from the category of cofinite $A$-modules to the category of $F$-finite $F$-modules.

**Remark 3.44.** Let $R = \mathbb{k}[[x_1, \ldots, x_n]]$ and $E = H^n_{(x_1,\ldots,x_n)}(R)$. Denote as before the Matlis dual functor $\text{Hom}_R(-, E)$ by $D(-)$. Then there is a functorial $R$-module isomorphism

$$\tau : D(F_R(M)) \cong F_R(D(M))$$

for all Artinian $R$-modules $M$.

Let $A$ be a homomorphic image of $R$. Let $M$ be an $A\{f\}$-module. One can check that

$$\alpha : F_R(M) \xrightarrow{r \otimes m \mapsto rf(m)} M$$

(3.2.3.1)
is an $R$-module homomorphism. Now, assume that $M$ is a cofinite $A\{f\}$-module. Taking the Matlis dual of $\alpha$, we have an $R$-module homomorphism

$$\beta = \tau \circ D(\alpha) : D(M) \to FR(D(M)),$$

and hence we have a direct system of Noetherian $R$-modules:

$$D(M) \xrightarrow{\beta} FR(D(M)) \xrightarrow{FR(\beta)} FR^2(D(M)) \to \cdots$$

Analogously, let $R = \mathbb{k}[x_1, \ldots, x_n]$ and denote the graded Matlis dual functor $^\ast \text{Hom}_R(\cdot, E)$ by $D^\ast(\cdot)$. There is a functorial graded $R$-module isomorphism

$$\tau : D^\ast(FR(M)) \cong FR(D^\ast(M))$$

for all Artinian graded $R$-modules $M$.

Assume $R = \mathbb{k}[x_1, \ldots, x_n]$ and $A$ is a graded homomorphic image of $R$. By a graded $A\{f\}$-module, we mean a graded $A$-module $M$ such that $f : M_\ell \to M_{p\ell}$ for all integers $\ell$. One can check that then (3.2.3.1) is a graded $R$-module homomorphism. Now, assume that $M$ is a cofinite graded $A\{f\}$-module. Taking the graded Matlis dual of $\alpha$, we have a graded $R$-module homomorphism

$$\beta = \tau \circ D^\ast(\alpha) : D^\ast(M) \to FR(D^\ast(M)),$$

and hence we have a direct system of graded Noetherian $R$-modules:

$$D^\ast(M) \xrightarrow{\beta} FR(D^\ast(M)) \xrightarrow{FR(\beta)} FR^2(D^\ast(M)) \to \cdots$$

\(\triangleleft\)

**Definition 3.45.** Let $R$ be a complete regular local ring $R$ of characteristic $p$ and let $A$ be a homomorphic image of $R$. For each cofinite $A\{f\}$-module $M$, we define

$$\mathcal{H}_{R,A}(M) := \lim\limits_{\to}(D(M) \xrightarrow{\beta} FR(D(M)) \xrightarrow{FR(\beta)} FR^2(D(M)) \to \cdots)$$

The graded version $\mathcal{H}^\ast_{R,A}$ is defined analogously on homogeneous input. \(\triangleleft\)

**Example 3.46.** Let $R = \mathbb{k}[x_1, \ldots, x_n]$ (or $R = \mathbb{k}[x_1, \ldots, x_n]$ respectively) and let $I$ be an ideal of $R$ (homogeneous, if $R = \mathbb{k}[x_1, \ldots, x_n]$). Set $A = R/I$. Hence $H^I_m(A)$ is an $A\{f\}$-module according to Remark 3.32. Since $H^I_m(A)$ is Artinian, it is a cofinite $A\{f\}$-module. Then one can check that

$$\mathcal{H}_{R,A}(H^I_m(A)) \cong H^{n-j}_I(R)$$

(which reads

$$\mathcal{H}^\ast_{R,A}(H^I_m(A)) \cong H^{n-j}_I(R)$$

when $R = \mathbb{k}[x_1, \ldots, x_n]$). \(\triangleleft\)
Remark 3.47. The functor $\mathcal{H}_{R,A}$ (resp. $\mathcal{H}^*_{R,A}$) from the category of cofinite (graded) $A\{f\}$-modules to the category of (graded) $F$-finite $F$-modules is contravariant, additive, and exact.

Given a cofinite (graded) $A\{f\}$-module $M$, $\mathcal{H}_{R,A}(M) = 0$ (or $\mathcal{H}^*_{R,A}(M) = 0$ respectively) if and only if the additive map $\varphi : M \rightarrow M$ in Remark 3.32 is nilpotent.

Now Lyubeznik’s vanishing theorem in characteristic $p$ follows from Example 3.46: $H^{n-\delta}_{I}(R) = 0$ if and only if the natural Frobenius (induced by the Frobenius on $R$) on $H^d_m(A)$ is nilpotent.

The nilpotence of the action of Frobenius on $H^d_m(A)$ prompts the following definition (cf. [Lyu06b, Definition 4.1]).

Definition 3.48. Let $(A, m)$ be a local ring of characteristic $p$. The $F$-depth of $A$ is the smallest $i$ such that $H^i_m(A)$ is not $f$-nilpotent, where $f$ is the natural action of Frobenius on $H^i_m(A)$ induced by the Frobenius endomorphism on $A$.

Remark 3.49. One can show that (cf. [Lyu06b, §4])

(1) $\text{depth}(A) \leq F\text{-depth}(A) \leq \dim(A)$,
(2) $F\text{-depth}(A) = F\text{-depth}(A)$,
(3) $F\text{-depth}(A) = F\text{-depth}(A_{\text{red}})$ where $A_{\text{red}} = A/\sqrt{(0)}$.

In terms of $F$-depth, the vanishing theorem via Frobenius in characteristic $p$ can be restated as follows: let $(R, m)$ be a regular local ring of characteristic $p$ and $I$ be an ideal. Then

$$\text{lcm}_R(I) = \dim(R) - F\text{-depth}(R/I).$$

(Compare also the corresponding statement in characteristic zero, Theorem 4.12).

In general, $F\text{-depth}(A)$ can be different from $\text{depth}(A)$ as shown in the following example (cf. [Lyu06b, §5]).

Example 3.50. Let $k$ be a perfect field of characteristic $p$ and let $C \subseteq \mathbb{P}_k^2$ denote the Fermat curve defined by $x^3 + y^3 + z^3$. Let $R = k[x_0, \ldots, x_5]$ and $I \subseteq R$ be the defining ideal of $C \times \mathbb{P}_k^1 \subseteq \mathbb{P}_k^5$. Set $A = (R/I)_m$ where $m = (x_0, \ldots, x_5)$.

If $3 \mid (p - 2)$, then

$F\text{-depth}(A) = 3 > 2 = \text{depth}(A)$.

See also Example 4.2.

Since $F$-finite $F$-modules have finite length in the category of $F$-modules, it is natural to ask whether one can compute the length, especially for local cohomology modules. It turns out that $F$-module length of local cohomology modules is closely related to singularities defined by the Frobenius, and Lyubeznik’s functor $\mathcal{H}_{R,A}$ is a useful tool for studying this length. To illustrate this, let $R$ be a regular local ring of characteristic $p$. That $\mathcal{H}_{R,A}$ sets
up a link between the length of $H^m_I(R)$ and the singularities of $A = R/I$ was first discovered in [Bli04]; this was later extended and strengthened in [KMSZ18] as follows, see also [Bit20]

**Theorem 3.51.** Let $R = \mathbb{k}[[x_1, \ldots, x_n]]$ (or $R = \mathbb{k}[x_1, \ldots, x_n]$), and set $m = (x_1, \ldots, x_n)$. Let $A = R/I$ be a reduced and equidimensional (and graded, if $R = \mathbb{k}[x_1, \ldots, x_n]$) ring of dimension $d \geq 1$. Let $c$ denote the number of minimal primes of $A$.

1. If $A$ has an isolated non-$F$-rational point at $m$ and $\mathbb{k}$ is separably closed, then
   $$\ell_{\mathcal{F}_R}(H^d_I R) = \dim_\mathbb{k}(H^d_m(A)_{\text{st}}) + c.$$

2. If the non-$F$-rational locus of $A$ has dimension $\leq 1$ and $\mathbb{k}$ is separably closed, then
   $$\ell_{\mathcal{F}_R}(H^d_I R) \leq \sum_{\dim(A/p) = 1} \dim_\mathbb{k}(H^{d-1}_{p^\alpha_p}(A_p)_{\text{st}}) + \dim_\mathbb{k}(H^d_m(A)_{\text{st}}) + c,$$

3. If $A$ is $F$-pure, then $\ell_{\mathcal{F}_R}(H^d_I R)$ is at least the number of special primes of $H^d_m(A)$. Moreover, if $A$ is $F$-pure and quasi-Gorenstein, then $\ell_{\mathcal{F}_R}(H^d_I R)$ is precisely the number of special primes of $H^d_m(A)$.

It remains an open problem whether one can extend Theorem 3.51 to the case of a higher dimensional non-$F$-rational locus.

Recently, in [AMBZ20], Álvarez Montaner, Boix and Zarzuela computed $\ell_{\mathcal{F}_R}(H^1_I R)$ and $\ell_{\mathcal{D}_R}(H^1_I R)$ when $R$ is a polynomial ring over a field and $I$ is generated by square-free monomials and pure binomials (i.e., $I$ is a toric face ideal).

### 3.3. Interaction between $D$-modules and $F$-modules

In characteristic $p$, the theories of $D$-modules and $F$-modules are entwined; it has been fruitful to consider local cohomology modules from both perspectives.

**Remark 3.52.** Let $\mathbb{k}$ be a field of characteristic $p$ and let $R = \mathbb{k}[x_1, \ldots, x_n]$ or $R = \mathbb{k}[[x_1, \ldots, x_n]]$. It is clear from the definition that, if $(M, \theta)$ is an $F$-module, the map

$$\alpha_\epsilon: M \to F_R(M) \xrightarrow{F_R(\theta)} F^2_R(M) \xrightarrow{F^2_R(\theta)} \cdots \xrightarrow{F^e_R(\theta)} F^e_R(M)$$

induced by $\theta$ is also an isomorphism.

This induces a $\mathcal{D} = \mathcal{D}(R, \mathbb{k})$-module structure on $M$. To specify the induced $\mathcal{D}$-module structure, it suffices to specify how $\mathcal{c}^{[i_1]}_1 \cdots \mathcal{c}^{[i_n]}_n$ acts on $M$. Choose $e$ such that $p^e \geq (i_1 + \cdots + i_n) + 1$. Given $z \in M$, we consider $\alpha_\epsilon(z)$ and we will write it as $\sum r_j \otimes z_j$ with $r_j \in F^e_* R$ and $z_j \in M$. Then define

$$\mathcal{c}^{[i_1]}_1 \cdots \mathcal{c}^{[i_n]}_n \cdot z := \alpha_\epsilon^{-1}(\sum c^{[i_1]}_1 \cdots \mathcal{c}^{[i_n]}_n r_j \otimes z_j),$$

that this is legal is due to a simplification in the product rule in characteristic $p$: $(x^p g)' = x^p (g)'$. 


Therefore, every $F$-module is also a $D$-module.

When $R = k[x_1, \ldots, x_n]$ with its standard grading, the $D$-module structure on each graded $F$-module as in Remark 3.52 is also graded. Moreover, [MZ14] proves the following:

**Theorem 3.53.** Let $R = k[x_1, \ldots, x_n]$. Every graded $F$-module is an Eulerian graded $D$-module.

Since every $F$-module is a $D$-module, given an $F$-finite $F$-module $M$, one may compare $\ell_F(M)$ and $\ell_D(M)$. A quick observation is that, since each filtration of $M$ in $F$ is also a filtration in $D$, one always has

$$\ell_F(M) \leq \ell_D(M).$$

It turns out that this inequality can be strict.

**Theorem 3.54** (Proposition 7.5 in [KMSZ18]). Let $p$ be a prime number such that $7 \mid (p - 4)$. Let $R = \mathbb{Z}_p[x, y, z, t]$ and $f = tx^7 + ty^7 + z^7$. Then

$$\ell_F(H^1_{(f)}(R)) = 3 < 7 = \ell_D(H^1_{(f)}(R)).$$

On the other hand, the equality holds when hypotheses are added:

**Theorem 3.55.** Let $R, I, A$ be as in Theorem 3.51. If $A$ has an isolated non-$F$-rational point at $m$ and $k$ is separably closed, then

$$\ell_F(H_{I^p}^d(R)) = \ell_D(H_{I^p}^d(R)).$$

$F$-modules and $D$-modules are deeply connected via a generating property. The following is a special case of [AMBL05, Corollary 4.4].

**Theorem 3.56.** Let $k$ be a field of characteristic $p$ such that $[k : k^p] < \infty$ and let $R = k[x_1, \ldots, x_n]$ or $R = k[[x_1, \ldots, x_n]]$. Let $M$ be an $F$-finite $F$-module. If $z_1, \ldots, z_t \in M$ generate a root of $M$, then $z_1, \ldots, z_t \in M$ generate $M$ as a $D$-module.

Theorem 3.56 plays a crucial role in proving that $1/g$ generates $R_g$ as a $D$-module in [AMBL05], and also in proving the finiteness of associated primes of local cohomology of smooth $\mathbb{Z}$-algebras in [BBL+14].

4. **Local cohomology and topology**

In this section we discuss the interaction of local cohomology with various themes of topological flavor. The interactions can typically be seen as a failure of flatness in some family witnessed by specific elements of certain local cohomology.

We start with a classical discussion of the number of defining equations for a variety, then elaborate on the more recent developments that originate from this basic question. We survey interactions with topology in characteristic zero, and with the Frobenius map in positive characteristic. We discuss a collection of applications of local cohomology to various areas: hypergeometric functions, the theory of Milnor fibers, the Bockstein morphism from
4.1. **Arithmetic rank.** The main object of interest here is described in our first definition.

**Definition 4.1.** The *arithmetic rank* \( \text{ara}_A(I) \) of the \( A \)-ideal \( I \) is the minimum number of generators for an ideal with the same radical as \( I \):

\[
\text{ara}_A(I) = \min\{\ell \in \mathbb{N} | \exists x_1, \ldots, x_\ell \in A, \sqrt{I} = \sqrt{(x_1, \ldots, x_\ell)}\}.
\]

Here, \( \sqrt{\cdot} \) denotes the radical of the given ideal.

The arithmetic rank of an ideal has been of interest to algebraists for as long as they have looked at ideals. In a polynomial ring over an algebraically closed field it answers the question by how many hypersurfaces the affine variety defined by \( I \) is cut out. The problem of finding this number has a long history that is detailed excellently in [Lyu89, Lyu02]. Some ground-breaking contributions before the turn of the millennium included [Har68, EE73, Ogu73, PS73, Har74, HS77, Spe78, BS90, Lyu92], and [Kun85] contains a gentle introduction to the problem.

4.1.1. **Some examples and conjectures.** Local cohomology is sensitive to arithmetic rank and relative dimension. Indeed, it follows from the Čech complex point of view that

\[
\max\{k \in \mathbb{N} | H^k_I(A) \neq 0\} = \text{lcd}_A(I) \leq \text{ara}_A(I),
\]

while a standard theorem in local cohomology asserts that

\[
\min\{k \in \mathbb{N} | H^k_I(A) \neq 0\} = \text{depth}_A(I, A),
\]

where \( \text{depth}_A(I, M) \) is the length of the longest \( M \)-regular sequence in \( I \). If \( A \) is a Cohen–Macaulay ring, \( \text{depth}_A(I, A) \) is the height of the ideal.

There are examples where the arithmetic rank exceeds the local cohomological dimension, but it is often not easy to verify this since the determination of \( \text{lcd}_A(I) \) and \( \text{ara}_A(I) \) is tricky.

**Example 4.2.** (1) Let \( E \) be an elliptic curve over any field of characteristic \( p > 0 \), and consider the Segre embedding \( E \times \mathbb{P}^1_K \hookrightarrow \mathbb{P}^5_K \). The curve \( E \) is *supersingular* if the Frobenius acts as zero on the one-dimensional space \( H^1(E, \mathcal{O}_E) \). It is known that if \( E \) is defined over the integers then there are infinitely many \( p \) for which the reduction \( E_p \) is supersingular [Elk87], and infinitely many primes for which it is ordinary. For example, for \( E = \text{Var}(x^3 + y^3 + z^3) \), supersingularity is equivalent to \( p - 2 \) being a multiple of 3. By [HS77, Ex. 3], the local cohomological dimension of the ideal defining \( E \times \mathbb{P}^1_K \) in \( \mathbb{P}^5_K \) equals three if and only if \( E \) is supersingular (and it is 4 otherwise). However, by [SW05], the arithmetic rank is always four, independently of supersingularity (and even in characteristic zero).
(2) Let \( I \subseteq R = \mathbb{C}[x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}] \) be the ideal describing the image of the second Veronese map from \( \mathbb{P}^2 \) to \( \mathbb{P}^5 \) over the complex numbers. Then \( \text{lcd}_R(I) = 3 = \text{ht}(I) \). On the other hand, as will be discussed in Example 4.10, the arithmetic rank of \( I \) is 4, not 3. The underlying method, de Rham cohomology, is the topic of Subsection 4.2. Replacing de Rham arguments with étale cohomology, similar results hold in prime characteristic, Example 4.14. This is an example where the étale cohomological dimension of the projective complement \( U \) surpasses the sum of the dimension and the cohomological dimension, \( \text{ecd}(U) = 6 > 2 + 3 = \text{cd}(U) + \dim(U) \), compare also the discussion around Conjecture 4.13.

Finding the arithmetic rank in concrete cases can be extremely difficult; some of the long-standing open problems in this area include general questions about “large” ambient spaces, but also about concrete curves:

- Hartshorne’s conjecture ([Har74]: If \( Y = \text{Proj}(R/I) \) is a smooth \( s \)-dimensional subvariety of \( \mathbb{P}^n_{\mathbb{C}} \), and \( s > \frac{2n}{3} \), is then \( Y \) a global complete intersection (i.e., is \( Y \) the zero set of \( \text{codim}(Y) \) many projective hypersurfaces that, at each point of \( Y \), are smooth and meet transversally)?
- Is the Macaulay curve in \( \mathbb{P}^3_\mathbb{K} \), parameterized as \( \{(s^4, s^3t, st^3, t^4)\}_{s,t \in \mathbb{P}^1_\mathbb{K}} \), a set-theoretic complete intersection (i.e., does the defining ideal have arithmetic rank 2, realized by homogeneous generators)?

This question is specific to characteristic zero, as in prime characteristic \( p \), Hartshorne proved in [Har79] that the Macaulay curve is a set-theoretic complete intersection for each \( p \).

The (degree 5) Plücker embedding of the (6-dimensional) Grassmann variety \( \text{Gr}_{C}(2, 5) \) of affine \( C \)-planes in \( \mathbb{C}^5 \) into \( \mathbb{P}_{\mathbb{K}}^9 \) is not contained in a hyperplane, so Bezout’s Theorem indicates that we are not looking at a complete intersection. Thus, the factor 2/3 in the Hartshorne Conjecture is, in a weak sense, optimal. Asymptotically, the coefficient must be at least 1/2, but Hartshorne writes in [Har74]: “I do not know any infinite sequences of examples of noncomplete intersections which would justify the fraction of the conjecture as \( n \to \infty \)”. On the other hand, even less is known when \( \dim(Y) \) is small. For example, scores of articles have been devoted to the study of monomial curves in \( \mathbb{P}^3_{\mathbb{C}} \); in larger ambient spaces [Vog71] contains a criterion for estimating arithmetic rank in terms of ideal transforms, the functors \( \lim_{\rightarrow} \text{Ext}_A^*(I^k, -) \).

Some of the major vanishing theorems in local cohomology came out of an unsuccessful attempt to use local cohomology in order to show that certain curves in \( \mathbb{P}^3_{\mathbb{K}} \) cannot be defined set-theoretically by two equations. For example, let \( I \subseteq R = \mathbb{K}[x_1, \ldots, x_4] \) define an irreducible projective curve. In order for the arithmetic rank of \( I \) to be 2, \( H^2_I(R) \) and \( H^2_I(R) \) should both be zero. That \( H^1_I(R) \) vanishes follows from the Hartshorne–Lichtenbaum theorem. That \( H^3_I(R) \) is also always zero is the Second Vanishing Theorem...
discussed in Subsection 2.2, in its incarnations due to Ogus, Peskine–Szpiro, Hartshorne, and Huneke–Lyubeznik. In particular, the desired obstruction to ara_A(I) = 2 cannot not materialize, but the attempt led to the discovery of the Second Vanishing theorem.

On the positive side, Moh proved in [Moh85] that in positive characteristic every monomial curve in \( \mathbb{P}_K^n \) is defined set-theoretically by two binomials; compare also [CN78, Fer79, Har79, BSR81, RS79]. The construction of the two binomials uses heavily the Frobenius and, as one might expect, the equations that work in one characteristic do not work in another [BM98]. In characteristic zero, Kneser proved that a curve in \( \mathbb{P}_K^n \) is cut out by three equations if it has a \( K \)-rational point, and monomial space curves are cut out by three binomials by [BM98], but nothing better is known at this point.

There is recent progress on arithmetic rank and local cohomological dimension in toric and monomial situations.

In [Var12], Varbaro shows that if \( X \) is a general smooth hypersurface of projective \( n \)-space of degree less than \( 2n \) then the arithmetic rank of the natural embedding of the Segre product of \( X \) with a projective line is at most \( 2n \). This generalizes an observation that appeared in [SW05] where \( X \) is an elliptic curve. Moreover, Varbaro continues, if \( X \) is a smooth conic in the projective plane then its Segre product with projective \( m \)-space has arithmetic rank exactly \( 3m \), as long as the characteristic is not 2.

Toric varieties, by which we mean here the spectra of semigroup rings \( K[S] \) where \( S \subseteq \mathbb{Z}^d \) is a finitely generated semigroup, provide a standard testing ground for theories and conjectures. Note that, for example, the Macaulay curve falls into this category.

Barile and her coauthors have studied the question whether a toric variety is a complete or almost complete intersection in [Bar06a, Bar06b, BMT00, BMT02]. Building on this, [Bar07] shows that certain toric ideals of codimension two are not complete intersections, and that their arithmetic rank is equal to 3. The combinatorial condition with arithmetic flavor of being \( p \)-glued has been shown to be pertinent here. A semigroup can be \( p \)-glued for exactly one prime \( p \), [BL05]. That such examples might be possible is explained in part by the fact that the depth of the semigroup ring may depend on the chosen characteristic: Hochster’s theorem from [Hoc72] indicates for example how Cohen–Macaulayness can toggle with \( p \).

Monomial ideals and their local cohomology have been studied by Álvarez-Montaner and his collaborators, see [Mon13] for notes to a lecture series. At the heart of this work stands the Galligo–Granger–Maisonobe correspondence between perverse sheaves and hypercubes detailed in [GGM85]. Morally, this is similar to the quiver encoding from Subsection 3.1.2 and will receive a second look in Section 4.4; compare specifically [AM05] on the category of regular holonomic \( D \)-modules with support on a normal crossing divisor and variation zero, and [AMGLZA03].
In [SV79] a technique is given how to find generators (up to radical) for ideals that are intersections of ideals with given generators. Application to monomial ideals relates to systematic search for the arithmetic rank of certain intersections, compare [Bar04]. Goresky and MacPherson noted in [GM88, Jew94] a formula on the singular cohomology of the complement of a complex subspace arrangement. The article [Yan00] generalizes the formula to subspace arrangements over any separably closed field using étale cohomology and sheaf theory. These results are then applied to determine the arithmetic rank of monomial ideals. In [Yan99], Yan studies a question of Lyubeznik on the arithmetic rank of certain resultant systems and again uses étale cohomology to get some lower bounds. More recently, Kimura and her collaborators have produced a wealth of new information on arithmetic rank of monomial ideals, cf. [KM17] and its bibliography tree.

4.1.2. Endomorphisms of local cohomology. As always, \((A, \mathfrak{m}, k)\) is a Noetherian local ring and \(a\) an ideal of \(A\). In this subsection we discuss some challenges that have arisen in the last two decades, connecting the question of finding the arithmetic rank to problems about \(\mathcal{D}\)-modules, with the focus on the question of determining whether a given ideal be a complete intersection.

We recall that the local cohomological dimension \(\text{lcd}_{A}(a)\) is a lower bound for the arithmetic rank \(\text{ara}_{A}(a)\) and that the two invariants may not be equal, Example 4.2. Nonetheless, as work primarily of Hellus and Stückrad shows, local cohomology modules contain information that can lead to the determination of arithmetic rank. However, decoding it successfully is at this point a serious challenge.

The story starts with a result of Hellus from [Hel05]. Denote \(E = E_A(k)\) the injective hull of the residue field. Suppose \(f_1, \ldots, f_c\) are elements of \(a\), and write for simplicity \(b_i = \text{the A-ideal generated by} f_1, \ldots, f_i\). Assuming that \(\text{lcd}_{A}(a) = c\), Hellus showed that these elements generate \(a\) up to radical if and only if \(f_i\) operates surjectively on \(H_{c+1-i}(A/b_{i-1})\) for \(1 \leq i \leq c\). This has the following corollary pertaining to set-theoretic complete intersections: if \(f_1, \ldots, f_c\) is an \(A\)-regular sequence (in our situation this means that \(H_i^{c+1}(A) = 0\) unless \(i = c\)), then the sequence generates \(a\) up to radical if and only if they form a regular sequence on \(D(H_{c}^c(A))\) where, as before,

\[
D(M) := \text{Hom}_A(M, E)
\]

is the Matlis dual. This is discussed from a new angle in [HP21a].

This motivates (when only one \(H_{c}^c(A)\) is nonzero) the study of the multiplication operators \(f_i : D(H_{c}^c(A)) \rightarrow D(H_{c-1}^{c-1}(A))\), and in particular the associated primes of \(D(H_{c}^c(A))\). In fact, Hellus offers the following conjecture: if \((A, \mathfrak{m}, k)\) is local Noetherian,

\[
(4.1.2.1) \text{ Is } \text{Ass}_A(D(H_{b_i}^1(A))) = \{p \in \text{Spec } A \mid H_{b_i}^1(A/p) \neq 0\} ?
\]

(One always has the inclusion \(\subseteq\) above, and in the equi-characteristic case, the set \(\{p \in \text{Spec}(A) \mid f_1, \ldots, f_i\text{ is part of an s.o.p. for } A/p\}\) is contained in
Ass\(_A(D(H^i_{b_1}(A)))\)—but this may not be an equality. In mixed characteristic, a similar statement can be made. Hellus proceeds to show that this conjecture is equivalent to Ass\(_A(D(H^i_{b_1}(A)))\) being stable under generalization, and also gives the following reformulation:

**Problem 4.3.** For all Noetherian local domains \((A,\mathfrak{m},k)\) and for all \(f_1,\ldots,f_c \in A\), show that the nonvanishing of \(H^i_{(f_1,\ldots,f_i)}(A)\) implies that the zero ideal is associated to \(D(H^i_{(f_1,\ldots,f_i)}(A))\).

**Remark 4.4.** A significant part of Problem 4.3 was resolved positively in [LY18]. Namely, if \(R\) is a regular Noetherian local ring of prime characteristic, then Ass\(_R(D(H^1_I(R)))\) contains \(\{0\}\), as long as \(H^1_I(R)\) is nonzero. In fact, it is shown for all \(F\)-finite \(F\)-modules \(M\) that \(\{0\}\) has to be associated to at least one of \(M,D(M)\). The proof is an explicit construction of an element that is not torsion.

Motivated by their work in prime characteristic, they conjectured in [LY18, Conjecture 1] that, if \((R,\mathfrak{m})\) is a regular local ring and \(I\) is an ideal such that \(H^1_I(R) \neq 0\), then \((0) \in \text{Ass}_R(D(H^1_I(R)))\).

**Remark 4.5.** Let \(R = \mathbb{Z}_2[[x_0,\ldots,x_5]]\) and let \(I\) be the monomial ideal as in Example 2.23. It follows from [DSZ19, Remark 5.3] that the arithmetic rank of \(I\) is 4; equivalently there are \(f_1,\ldots,f_4 \in R\) such that \(H^4_I(R) = H^4_{(f_1,\ldots,f_4)}(R)\). By [DSZ19, Proposition 5.5], \(H^4_I(R) \cong E_R(\bar{R}/\mathfrak{m})\), where \(\bar{R} = R/(2)\) and \(\mathfrak{m} = (2,x_0,\ldots,x_5)\). Hence \(\text{alg dim} I = \text{dim} \bar{R}\). Consequently the zero ideal is not associated to \(D(H^4_{(f_1,\ldots,f_4)}(R))\). This answers Hellus’ question in Problem 4.3 in the negative for unramified regular local rings of mixed characteristic, and provides a counterexample to the conjecture of Lyubeznik and Yildirim in mixed characteristic.

In [Hel07b], an example is given where arithmetic rank and local cohomological dimension differ. What is special here is that \(\text{lcm}_A(a) = 1\); Hellus gives a criterion for the arithmetic rank to be one, based on the prime avoidance property of Ass\(_A(D(H^1_{b_1}(A)))\). In the same year and journal [Hel07a], he shows for Cohen–Macaulay rings the curious identity \(H^2_{\mathfrak{a}}(D(H^2_{\mathfrak{a}}(R))) = D(R)\), provided that \(c = \text{lcm}_A(a)\) is also the grade of \(a\). This was subsequently generalized in [Kha07].

In [HS08c], Hellus and St"uckrad continue their study of associated primes of, and regular sequences on, \(D(H^2_{\mathfrak{a}}(A))\). They show that \(H^m_{(f_1,\ldots,f_m)}(A)\) always surjects onto \(H^{m+n}_{(f_1,\ldots,f_m,g_1,\ldots,g_n)}(A)\) for \(m > 0\) and derive from this some insights about the inclusion \((4.1.2.1)\), and about Problem 4.3 when \(A\) is a complete domain and \(\mathfrak{a}\) a 1-dimensional prime. In [HS08a] the same authors show that in a complete local ring, when \(\mathfrak{a}\) has the local cohomological behavior of a complete intersection \(i.e., H^1_{\mathfrak{a}}(A) = 0\) unless \(i = c\), then the natural map \(A \longrightarrow \text{End}_A(H^c_{\mathfrak{a}}(A))\) is an isomorphism. (In general, this map
is not surjective and has a kernel. In particular, no element of $A$ annihilates $H^d_A(A)$. By results mentioned above, this means that if $a$ behaves local cohomologically like a complete intersection and if $f_1, \ldots, f_c$ is an $A$-regular sequence in $a$, then $D(H^d_a(D(H^c_A(A))))$ is an ideal of $A$ which, if computable, predicts whether $a$ is a complete intersection. For more on $\text{End}_A(H^d_a(A))$, see [Sch09b, Kha10, Sch10, Sch11a].

In [HS08b] it is investigated which ideals behave like a complete intersection from the point of local cohomology, by establishing relations to iterated local cohomology functors which then lead to Lyubeznik numbers (see Section 4.4). For example, if $a = (f_1, \ldots, f_c)$ is an ideal of dimension $d$ in a local Gorenstein ring, and if $a$ is a complete intersection outside the maximal ideal, then $[H^d(A) = 0$ unless $i = c$] precisely when $[H^d_m(H^c(a)) = E_A(k)$ and $H^i_m(H^c(a)) = 0$ for $i \neq d]$. In particular, the complete intersection property of $a$ is then completely detectable from $H^d_m(a)$ alone. A new version of some of these ideas is given in a recent work of Hartshorne and Polini, who introduce and investigate coregular sequences and codelength in [HP21a].

4.2. Relation with de Rham and étale cohomology.

4.2.1. The Čech–de Rham complex. Suppose $I \subseteq R_K := \mathbb{K}[x_1, \ldots, x_n]$ is generated by $f_1, \ldots, f_m$ and assume that $\mathbb{K}$ is a field containing $\mathbb{Q}$. The finitely many coefficients of $f_1, \ldots, f_m$ all lie in some finite extension field $\mathbb{K}$ of $\mathbb{Q}$, and because of flatness one has $H^i_I(\mathbb{K}[x_1, \ldots, x_n]) = H^i_{I_{\mathbb{K}}}(\mathbb{K}[x_1, \ldots, x_n]) \otimes_{\mathbb{K}} \mathbb{K}$, where $I_{\mathbb{K}} = (f_1, \ldots, f_m)R_{\mathbb{K}} = I \cap R_{\mathbb{K}}$ with $R_{\mathbb{K}} = \mathbb{K}[x_1, \ldots, x_n]$. The finite extension $\mathbb{K}$ can be embedded into $\mathbb{C}$ and then, by flatness again, $H^i_I(\mathbb{C}[x_1, \ldots, x_n]) = H^i_{I_{\mathbb{K}}}((\mathbb{K}[x_1, \ldots, x_n]) \otimes_{\mathbb{K}} \mathbb{C}$. It follows that most aspects of the behavior of local cohomology in characteristic zero can be studied over the complex numbers.

Convention 4.6. In this subsection, $\mathbb{K} = \mathbb{C}$ and $I$ is an ideal of $R = \mathbb{C}[x_1, \ldots, x_n]$. The advantage of working over $\mathbb{C}$ is that one has access to topological notions and tools.

The arithmetic rank of the ideal $I$ is the smallest number of principal open affine sets $U_f$ that cover the complement $U_I = \mathbb{C}^n \setminus \text{Var}(I)$. Any $U_f$ arises also as the closed affine set defined by $f \cdot x_0 = 1$ inside $\mathbb{C}^n \times \mathbb{C}\mathbb{P}_Q$.

Complex affine space as well as all its Zariski closed subsets are Stein spaces. This is a complex analytic condition that includes separatedness by holomorphic functions, and a convexity condition about compact sets under holomorphic functions. It implies, among other things, that a Stein space of complex dimension $n$ has the homotopy type of an $n$-dimensional CW-complex. In particular, a Stein space $S$ of complex dimension $n$ cannot have singular cohomology $H^n_{\text{Sing}}(S; -)$ beyond degree $n$. That complex affine varieties have this latter property is the Andreotti–Frankel Theorem. (For example, a Riemann surface is Stein exactly when it is not compact). In the “spirit of GAGA”, [Ser55, Ser56], Stein spaces are the notion that corresponds to affine varieties.
Now consider the complement $U_I = U_{f_1} \cup \cdots \cup U_{f_m}$ of the variety $\text{Var}(I)$. It follows from the Mayer–Vietoris principle that $H^i_{\text{Sing}}(U; -) = 0$ for all $i > n + m - 1$ and all coefficients. Being Stein is not a local property:

**Example 4.7.** Let $I = (x, y) \subseteq \mathbb{C}[x, y]$. Then $U_I$ is homotopy equivalent to the 3-sphere and in particular cannot be Stein.

\[
\text{Čech complex on a set of generators for } I \text{ is always a complex in the category of } \mathcal{D}\text{-modules.}
\]

Let $X \rightarrow Y$ be a morphism of smooth algebraic varieties. We refer to [HTT08] for background and details on the following continuation of the discussion on functors on $\mathcal{D}$-modules in Section 3.

There are (both regular and exceptional) direct and inverse image functors between the categories of bounded complexes of $\mathcal{D}$-modules on $X$ and $Y$. These functors preserve the categories of complexes with holonomic cohomology. In particular, one can apply them to the structure sheaf, or to local cohomology modules and Čech complexes.

If $\iota: U \hookrightarrow X$ is an open embedding and $M$ a $\mathcal{D}_U$-module, then the direct image of $M$ under $\iota$ as $\mathcal{D}$-module agrees with the direct image as $\mathcal{O}$-module.

For example, in both categories there is an exact triangle
\[
\text{R} \Gamma_X \cdot \iota(-) \rightarrow \text{id} \rightarrow \iota_*((-)\vert_U)^{+1}.
\]

Let $X = \mathbb{C}^n$ and choose $\varphi: X \rightarrow Y$ be the projection to a point $Y$. Write $\omega_X = \mathcal{D}_X/(\partial_1, \ldots, \partial_n) \cdot \mathcal{D}_X$; this gives the canonical sheaf of the manifold $X$ a right $\mathcal{D}_X$-structure in a functorial way. Then under $\iota: U \hookrightarrow X$, $\mathcal{O}_U$ turns into a complex of sheaves that is represented on global sections by the Čech complex on generators of the ideal $I = (f_1, \ldots, f_m)$ describing $X \setminus U$. The $\mathcal{D}$-module direct image under $\varphi$ corresponds to the functor $\omega_X \otimes_{\mathcal{D}_X} (-)$ whose output is a complex of vector spaces. Applying this functor to the Čech complex for $I$ invites the inspection of a Čech–de Rham spectral sequence starting with $\text{Tor}^D_X(\omega_X, H^j_\mathcal{O}(\mathcal{O}_X))$. With $R = \Gamma(X, \mathcal{O}_X)$, $\omega_R = \Gamma(X, \omega_X)$, and $D = \Gamma(X, \mathcal{D}_X)$, the Grothendieck Comparison Theorem [Gro66] asserts that on global sections, the abutment of the spectral sequence is the reduced de Rham cohomology of $U$,

\[
E_2^{i,j} = \text{Tor}^D_{n-j}(\omega_R, H^j(R)) \Rightarrow \tilde{H}^{i+j-1}_{\text{dR}}(U; \mathbb{C}).
\]

We note in passing that there are algorithmic methods that can compute the pages of this spectral sequence as vector spaces over $\mathbb{C}$, see [OT99, Wal00, Wal01a, OT01]. In the sequence (4.2.1.1), the Tor-groups involved vanish for the index exceeding $\dim X$, and so the spectral sequence operates clearly inside the rectangle $0 \leq i \leq \text{lcd}_R(I), 0 \leq j \leq n$. However, it is actually limited to a much smaller, triangular region, compare [RWZ21].

This now opens the door to direct comparisons between local cohomology groups of high index and singular cohomology groups of high index; the de Rham type arguments in the following example are written down in
[LSW16, HP21b], but are folklore and were known to the authors of [Ogu73] and [Har75]. For example, Theorem 2.8 in [Ogu73] shows that in a regular local ring \( R \) over \( \mathbb{Q} \) with closed point \( p \), the vanishing of local cohomology \( H^j_I(R) \) for all \( j > r \) implies the vanishing of the local de Rham cohomology groups \( H^i_p(\text{Spec}(R/I)) \) for all \( i < \dim(R) - r \) (and is in fact equivalent to it if one already knows that the support of \( H^j_I(R) \) is inside \( p \) for \( j > r \)).

**Example 4.8.** We continue Example 2.14 with \( K = \mathbb{C} \). The open set \( U = \mathbb{C}^6 \setminus \text{Var}(I) \) consists of the set of 2 \( \times \) 3 complex matrices of rank two. The closed set \( V = \text{Var}(I) \) is smooth outside the origin, as one sees from the \( GL(2, \mathbb{C}) \)-action. Since \( \dim(R/I) = 4 \), the height of \( I \) is 2 and so \( H_i^3(R) \) must be, if nonzero, supported at the origin only, by Remark 1.4.

Since \( H_3^3(R) \) is also a holonomic \( D \)-module, \( D \) being the ring of \( \mathbb{C} \)-linear differential operators on \( R \), Kashiwara equivalence ([HT08, §1.1.6]) asserts that \( H_3^3(R) \) is a finite direct sum of \( \lambda \) copies of \( E \), the \( R \)-injective hull of the residue field at the origin. The number \( \lambda \) can be evaluated as follows.

Since \( I \) is 3-generated, \( H_2^3 \lambda(R) = 0 \) and the \( \check{\text{C}} \)ech–de Rham spectral sequence shows that \( H^i(U; \mathbb{C}) \) vanishes for \( i > 6 + 3 - 1 = 8 \). Moreover, an easy exercise shows that \( \text{Tor}^D_{n-j}(\omega_R, E) = 0 \) unless \( j = 0 \), and in that case returns one copy of \( \mathbb{C} \) so that the only possibly nonzero \( E_2 \)-entry in the spectral sequence \((4.2.1.1)\) in column 3 is the entry \( E_2^{3,6} = \mathbb{C}^\lambda \). The workings of the spectral sequence make it clear that all differentials into and out of position \((3,6)\) on all pages numbered 2 and up vanish. So, \( \mathbb{C}^\lambda = E_2^{3,6} = E_\infty^{3,6} = H^8(U; \mathbb{C}) \). We now compute this group explicitly via the following argument taken from Mel Hochster’s unpublished notes on local cohomology.

Let \( A \) be a point of \( U \), representing a rank two \( 2 \times 3 \) matrix. Consider the deformation that scales the top row to length 1, followed by the deformation (based on gradual row reduction) that makes the bottom row perpendicular to the top row and then scales it to length 1 as well. Then the top row varies in the 5-sphere, and for each fixed top row the bottom row varies in a 3-sphere. Let \( M \) be this retract of \( U \) and note that, projecting to the top row, it is the total space of an \( S^3 \)-bundle over \( S^5 \). Both base and fiber are orientable, and the base is simply connected. Thus, \( M \) is an orientable compact manifold of dimension 8 which forces \( 1 = \dim \mathbb{C} H^8(M; \mathbb{C}) = \dim \mathbb{C} H^8(U; \mathbb{C}) = \lambda \).

**Remark 4.9.** Already Ogus proved in [Ogu73] results that relate the local cohomology module \( H_3^1(R) \) of Example 4.8 to topological information. We discuss this in and after Theorem 4.12 below. In brief, the non-vanishing of \( H_1^1(R) \) is “to be blamed” on the failure of the restriction map \( H^2_{\text{dr}}(\mathbb{P}_\mathbb{C}^2) \to H^2_{\text{dr}}(Y) \) to be surjective. Here, \( Y \) is the image of the Segre map and \( \dim \mathbb{C}(H^2_{\text{dr}}(Y)) = \dim \mathbb{C}(H^2_{\text{dr}}(\mathbb{P}_\mathbb{C}^1 \times \mathbb{P}_\mathbb{C}^2)) = 2 \) by the K"unneth theorem.

**Example 4.10** (Compare [Ogu73, Exa. 4.6]). Let \( \nu : \mathbb{P}_\mathbb{C}^2 \to \mathbb{P}_\mathbb{C}^2 \) be the second Veronese morphism, denote the target by \( X \), the image by \( Z \) and write
There is a long exact sequence of singular (local) cohomology
\[ H^p_Z(X; -) \to H^p(U; -) \to H^p(U; -) \xrightarrow{+1} \]
and a natural identification \( H^p_Z(X; -) \cong (H^p_{2 \dim X-p}(Z; -))^\vee \) with compactly supported cohomology, for any coefficient field, compare [Ive86, §6.6]. Via Poincaré duality, this allows to identify the map \( H^p_Z(X; -) \to H^p(U; -) \) as the dual to \( H^{2 \dim X-p}(Z; -) \to H^{2 \dim X-p}(X; -) \). Now take \( \mathbb{Z}/2\mathbb{Z} \) as coefficients. Then, since \( \iota^* \) sends the generator of \( H^2(Z; \mathbb{Z}) \) to twice the generator of \( H^2(X; \mathbb{Z}) \), the long exact sequence shows that \( H^8(U; \mathbb{Z}/2\mathbb{Z}) \) is nonzero. Thus, \( U \) cannot be covered by three affine sets and \( \text{ara}_A(I) \geq 4 \).

4.2.2. Algebraic de Rham cohomology. In [Har75], Hartshorne defines and develops for (possibly singular) schemes over a field of characteristic zero a purely algebraic (co)homology theory that he connects to singular cohomology via comparison theorems. In a nutshell, the de Rham cohomology \( H^q_{dR}(Y) \) of \( Y \) embedded into a smooth scheme \( X \) is the \( q \)-th hypercohomology on \( X \) of the de Rham complex on \( X \), completed along \( Y \). Similarly, the de Rham homology \( H_q(Y) \) of \( Y \) is the \( (\dim X - q) \)-th local hypercohomology group with support in \( Y \) of the de Rham complex on \( X \). (We add here a pointer to Remark 3.12). Hartshorne develops many tools of singular (co)homology: Mayer–Vietoris sequences, Thom–Gysin sequences, Poincaré duality, and a local (relative) version. With it, he shows foundational finiteness as well as Lefschetz type theorems.

One of the most remarkable applications of his theory as it relates to local cohomology is worked out in the thesis of Ogus, and based on the following definition.

**Definition 4.11** ([Ogu73, Dfn. 2.12]). Let \( Y \) be a scheme over a field of characteristic zero. The de Rham depth \( dR-depth(Y) \) of \( Y \) is the greatest integer \( d \) such that for every point \( y \in Y \) (closed or not) one has
\[ H^i_y(Y) = 0 \quad \text{for } i < d - \dim(y). \]

This number never exceeds the dimension of \( Y \) as one sees by looking at a closed point \( y \). Ogus uses it in the following fundamental result; we point here at Remark 3.49 for the corresponding result in positive characteristic and note the formal similarities both of de Rham and \( F \)-depth, and the corresponding results on local cohomological dimension.

**Theorem 4.12** ([Ogu73, Thm. 2.13]). If \( Y \) is a closed subset of a smooth Noetherian scheme \( X \) of dimension \( n \) over a field \( k \) of characteristic zero, then for each \( d \in \mathbb{N} \) one has
\[ \text{lcd}(X, Y) \leq n - d \iff \text{dR-depth}(Y) \geq d. \]

In particular, if \( Y = \text{Spec}(R/I) \) for some regular \( k \)-algebra \( R \) then \( n - \text{lcd}_R(I) = \text{dR-depth}(Y) \) is intrinsic to \( Y \) and does not depend on \( X \).
Now let $Y$ be a projective variety over the field $k$ of characteristic zero, embedded into $\mathbb{P}_k^n$. Let $R$ be the coordinate ring of $\mathbb{P}_k^n$ and $I$ the defining ideal of $Y$; of course, these are not determined by $Y$. Then Ogus obtains in [Ogu73, Thm. 4.1] the equivalences

$$\text{lcd}(\mathbb{P}_k^n, Y) \leq r \iff [\text{Supp}_R(H^i(R)) \subseteq m \text{ for } i > r]$$

$$\iff [\text{dR-depth}(Y) \geq n - r].$$

In particular, for any such embedding, the smallest integer $r$ such that $H^{\dim(Y)}_i(R)$ is Artinian is intrinsic to $Y$.

One might wonder whether a similar result holds for $\text{lcd}(\mathbb{P}_k^n, I)$ itself. With the same notations as in the previous theorem, Ogus proves in [Ogu73, Thm. 4.4]:

$$\text{cd}(\mathbb{P}_k^n \setminus Y) < r$$

(that is, $\text{lcd}(R, I) \leq r$) is equivalent to

$$[\text{dR-depth}(Y) \geq n - r \text{ and } H^i_{\text{dR}}(\mathbb{P}_k^n) \to H^i_{\text{dR}}(Y) \text{ for } i < n - r].$$

Note that these restriction maps are always injective, and surjectivity is preserved under Veronese maps.

### 4.2.3. Lefschetz and Barth Theorems

Let $X \subseteq \mathbb{P}^n_C$ be a projective variety and $H \subseteq \mathbb{P}^n_C$ a hyperplane. Setting $Y = X \cap H$, the Lefschetz hyperplane theorem states that under suitable hypotheses the natural restriction map

$$\rho^{i}_{X,Y}: H^i(X; \mathbb{C}) \longrightarrow H^i(Y; \mathbb{C})$$

is an isomorphism for $i < \dim(Y)$ and injective for $i = \dim(Y)$. In the original formulation by Lefschetz, $X$ is supposed to be smooth and $H$ should be generic (which then entails $Y$ being smooth). Inspection showed that the relevant condition is that the affine scheme $X \setminus Y$ be smooth, since then the relative groups $H^i_{\text{Sing}}(X, Y; \mathbb{C})$ are zero in the required range.

It is clear that one can iterate this procedure and derive similar connections between the cohomology of $X$ and the cohomology of complete intersections on $X$ that are well-positioned with respect to the singularities of $X$. (Recall that any hypersurface section can be cast as a hyperplane section via a suitable Veronese embedding of $X$).

A rather more difficult problem is to establish connections when $Y$ is not a complete intersection. At the heart of the problem is the issue that in general $X \setminus Y$ will not be affine and thus might allow more complicated cohomology.

In [Bar70], Barth developed theorems that connect, for $Y \subseteq \mathbb{P}^n_C$ smooth (and of small codimension), the surjectivity of $\rho^i_{\mathbb{P}^n_C, Y}$ to the surjectivity of corresponding restrictions $\rho^i_{\mathbb{P}^n_C, Y}(\mathcal{F})$ of coherent sheaves $\mathcal{F}$ and hence to the cohomological dimension of $\mathbb{P}^n_C \setminus Y$ and the arithmetic rank of the defining ideal of $Y$. More precisely, he proved that surjectivity of $\rho^i_{\mathbb{P}^n_C, Y}(\mathcal{F})$ occurs for $i \leq 2 \dim(Y) - n$ and proved for $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_C}$ that surjectivity of $\rho^i_{\mathbb{P}^n_C, Y}(\mathcal{F})$ is
equivalent to surjectivity of $\rho_{P^n, Y}$ in the sense of Equation (4.2.3.1) above. As a corollary, he obtained a more general form of the Lefschetz Hyperplane Theorem: if $P^n_Y \supseteq X, Y$ are smooth with $\dim(X) = a, \dim(Y) = b$ then
$$\rho_{Y, X \cap Y}^i : H^i(Y; \mathbb{C}) \longrightarrow H^i(X \cap Y; \mathbb{C})$$
is an isomorphism for $i \leq \min(2b - n, a + b - n - 1)$. It is worth looking at the special case when $X$ is the ambient projective space. For $i = 0$ the theorem then generalizes the fundamental fact that a smooth subvariety of pure dimension $a$ is connected whenever $2a \geq n$. But it also gives obstructions for embedding varieties into projective spaces of given dimension, since it forces the singular cohomology groups $H^i(Y; \mathbb{C})$ to agree with those of $P^n_Y$ in the range $i \leq 2\dim(Y) - n$. For example, an Abelian variety $Y$ of dimension $b$ cannot be embedded in $Y = P^{2b-1}_C$ since with such embedding the map $H^1(P^{2b-1}; \mathbb{C}) \longrightarrow H^1(Y; \mathbb{C})$ should be surjective.

Barth uses the special unitary group action on $P^n_C$ to “spread” the classes on $Y$ to classes on $P^n_C$ near $Y$. In order to glue them, he then needs a suitable cohomological triviality of the complement of $Y$. In [Ogu73], Ogus gives an algebraic version of Barth’s transplanting technique, and succeeds (in his section 4) in proving various statements that connect the isomorphy of the restriction maps of de Rham cohomology of two schemes $X \subseteq Y \subseteq P^n_C$ to the de Rham depths of $X, Y$ and $X \setminus Y$.

In [Spe78], Speiser studies in varying characteristics the cohomological dimension of the complement $C_Y$ of the diagonal in $Y \times Y$. As a stepping stone he studies $C_{P^n}$ for arbitrary fields. In any characteristic, the diagonal scheme is the set-theoretic intersection of $2n - 1$ very ample divisors. However, a big difference appears for cohomological dimension: $cd(C_{P^n}) = 2n - 2$ when $Q \subseteq K$, but $cd(C_{P^n}) = n - 1$ in positive characteristic. The discrepancy is due to the Peskine–Szpiro Vanishing since the diagonal comes with a Cohen–Macaulay coordinate ring.

In characteristic zero, Speiser’s results imply that the diagonal of projective space is cut out set-theoretically by $2n - 1$ and no fewer hypersurfaces. More generally, for Cohen–Macaulay $Y$, he shows in [Spe78, Thm. 3.3.1] a similar vanishing result about $C_Y$ in positive characteristic over algebraically closed fields: the cohomological dimension of $Y \times Y \setminus \Delta$ is bounded by $2n - 2$ whenever $Y \subseteq P^n$ is a Cohen–Macaulay scheme of dimension $s \geq (n + 1)/2$.

4.2.4. Results via étale cohomology. Suppose $U$ is an open subset of affine space $X = \mathbb{C}^n$ whose closed complement $V = X \setminus U$ is defined by an ideal $I$ in the appropriate polynomial ring $R$. We have seen in (4.2.1.1) that the local cohomological dimension $\text{lcd}_R(I)$ is related to the de Rham cohomology via the vanishing
$$(4.2.4.1) \quad [H^i_{dR}(U; \mathbb{C}) = 0] \text{ whenever } [i \geq \text{lcd}(I) + n - 1 = \text{cd}(U) + n].$$
We mention here a variant of this in arbitrary characteristic, involving étale cohomology. This is a cohomology theory that interweaves topological data
with arithmetic information. We refer to \cite{Mil80, Mil21} for guidance on \'{e}tale cohomology.

One significant difference to the de Rham case is that the basic version of \'{e}tale cohomology involves coefficients that are torsion (\textit{i.e.}, sheaves with stalk \(\mathbb{Z}/(\ell \mathbb{Z})\)) of order not divisible by \(p = \text{ch}(k)\).

In many aspects, over a separably closed field \(k\), \'{e}tale cohomology behaves quite similar to de Rham or singular cohomology over the complex numbers. For example, on non-singular projective varieties there is a version of Poincaré duality, there is a Künneth theorem, and if a variety is defined over \(\mathbb{Z}\) then its model over \(\mathbb{C}\) has singular cohomology group ranks equal to the corresponding \'{e}tale cohomology ranks of the reductions modulo \(p\) for most primes \(p\).

The \'{e}tale cohomology groups on a scheme \(X\) vanish beyond \(2 \dim X\), and even beyond \(\dim(X)\) if \(X\) is affine, similar to the Andreotti–Frankel Theorem. So, it makes sense to talk of \'{e}tale cohomological dimension \(\text{ecd}(\_\_\_\_)\), the largest index of a non-vanishing \'{e}tale cohomology group. The Mayer–Vietoris principle implies that if \(V\) is a variety inside affine \(n\)-space \(X \neq V\) over the algebraically closed field \(k\), cut out by the ideal \(I\), then with \(U = X \setminus V\) one has

\[
\text{ecd}(U) \leq n + \text{ara}_{A}(I) - 1.
\]

Note that \(\text{ara}_{A}(I) \geq \text{lcd}_{R}(I) = \text{cd}(U) + 1\).

In \cite{Lyu02}, Lyubeznik formulates the following conjecture.

**Conjecture 4.13.** Over a separably closed field \(k\),

\[
\text{ecd}(U) \geq \dim(U) + \text{cd}(U).
\]

In this conjecture, \(U\) need not be the complement of an affine variety or even smooth. Comparison with (4.2.4.1) shows that (for complements of varieties in affine or projective spaces) the conjecture can be interpreted to say that \'{e}tale cohomology always provides a better lower bound for arithmetic rank than local cohomological dimension does. At present, this conjecture seems wide open. Varbaro shows in \cite{Var12} that it holds over \(\mathbb{C}\) in the case that \(U\) is the complement in projective space \(\mathbb{P}^n_{\mathbb{C}} \setminus V\) of a smooth variety \(V\) with \(\text{cd}(\mathbb{P}^n_{\mathbb{C}} \setminus V) > \text{codim}_{\mathbb{P}^n_{\mathbb{C}}}(V) - 1\).

**Example 4.14.** We continue Example 4.10. For \(K = \mathbb{C}\) and all other field coefficients of characteristic not equal to 2, one has \(H^2(U; K) = 0\). Thus, we cannot conclude that \(\text{lcd}_R(I) \geq 4\) in the way we concluded in Example 4.8. In fact, as Ogus \cite[Exa. 4.6]{Ogu73} proved, \(\text{cd}(U) = 2\) (and so \(\text{lcd}_R(I) = 3\)) and, in particular, \(\text{ecd}(U) > \dim(U) + \text{cd}(U)\).

In finite characteristic different from 2, if one replaces “singular” by “\'{e}tale”, the same formal arguments as in Example 4.10 show that the arithmetic rank of the defining ideal of the Plücker embedding is 4 while (since the coordinate ring is Cohen–Macaulay) \(\text{lcd}_R(I) = 3\).
In characteristic 2, the arithmetic rank drops to 3 and the ideal is generated up to radical by \( \{ t \_xx t \_yy - t \_xy t \_xx, t \_xx t \_zz, t \_yy t \_zz - t \_yz t \_xz \} \) since, for example, \( t \_xx t \_zz (t \_xx t \_yy - t \_xy t \_xx) + t \_xy t \_yy (t \_xx t \_zz - t \_xz t \_xx) + t \_xx t \_xx (t \_yy t \_zz - t \_yz t \_xz) = (t \_xx t \_yz - t \_xy t \_xz)^2 \) in characteristic 2.

Remark 4.15. In [Var12, Rmk. 2.13], Varbaro points out that Example 4.14 shows that the étale cohomological dimension of the complement of an embedding of \( \mathbb{P}^2_\mathbb{K} \) into \( \mathbb{P}^5_\mathbb{K} \) depends on the embedding: for a subspace embedding it is at most 3 + 4 since the subspace is covered by three affine spaces of dimension 5, but for the Veronese it is 8 (compare also [Bar95] for arithmetic rank consequences that highlight variable behavior in varying characteristic). This contrasts with his Theorem 2.4, which states that the quasi-coherent cohomological dimension is independent of the embedding (intrinsic to the given smooth projective subvariety).

Ogus proved in [Ogu73, Ex. 4.6] for any Veronese map of a projective space in characteristic zero that the local cohomological dimension agrees with the height of the defining ideal. In positive characteristic, the same follows from Peskine–Szpiro [PS73, Prop. III.4.1]. In [Pan21], Pandey shows that this is even true over the integers, and by extension then over every commutative Noetherian ring.

Now, recall Speiser’s result from Subsection 4.2.3, on the arithmetic rank \( 2n - 1 \) of the diagonal of \( \mathbb{P}^n_\mathbb{K} \times \mathbb{P}^n_\mathbb{K} \). In [Var12] Varbaro shows that it remains true in every characteristic as long as \( \mathbb{K} \) is separably closed; note, however, that the cohomological dimension of the complement is much smaller in finite characteristic, always equal to \( n - 1 \). The main ingredient comes from Künneth theorems on étale cohomology.

There are Lefschetz and Barth type results for étale cohomology. For example, in [Lyu93a, Prop 9.1], Lyubeznik proves the following: assume \( \mathbb{K} \) to be separably closed, of any characteristic, and pick two varieties \( Y \subseteq X \) with \( X \setminus Y \) smooth. If \( \text{cd}(U) < 2 \dim(X) - r \) then \( H^i_{\text{ét}}(X, \mathbb{Z}/(\ell \mathbb{Z})) \to H^i_{\text{ét}}(Y, \mathbb{Z}/(\ell \mathbb{Z})) \) is an isomorphism for \( i < r \) and injective for \( i = r \).

In the [Var12], Varbaro also investigates the interaction of étale cohomological dimension with intersections: let \( \mathbb{K} \) be an algebraic closed field of arbitrary characteristic and let \( X \) and \( Y \) be two smooth projective varieties of dimension at least 1. Set \( Z = X \times Y \subseteq \mathbb{P}_\mathbb{K}^n \) (any embedding) and \( U = \mathbb{P}^N \setminus Z \). Then \( \text{cd}(U) \geq 2N - 3 \). In particular, if \( \dim Z \geq 3 \) then \( Z \) cannot be a set-theoretic complete intersection by (4.2.4.2).

4.3. Other applications of local cohomology to geometry.

4.3.1. Bockstein morphisms. In this subsection we discuss a construction that originates (to our knowledge) in topology but can, in principle, be used as a tool to study any linear functor in prime characteristic.

For this we need the following concept. A collection of functors \( \{ F^\bullet \} \) is a covariant \( \delta \)-functor (in the sense of Grothendieck) if for each short exact
sequence of $A$-modules $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ one obtains a functorial long exact sequence

$$\cdots \rightarrow F^1(M') \rightarrow F^1(M) \rightarrow F^1(M'') \rightarrow F^{i+1}(M') \rightarrow \cdots$$

Now suppose that for some $A$-module $M$, multiplication by $f \in A$ induces an injection $0 \rightarrow M \xrightarrow{f} M \xrightarrow{\pi} M/fM \rightarrow 0$. If $F^\bullet$ is a covariant $\delta$-functor that is $A$-linear (i.e., each $F^i$ is additive, and $F^i(M \xrightarrow{a,h} N) = F^i(M) \xrightarrow{aF^i(h)} F^i(N)$ for all $a \in A$ and all $h \in \text{Hom}_A(M, N)$) then there is an induced long exact sequence

$$\cdots \rightarrow F^i(M/fM) \xrightarrow{\delta^F_{i,i}} F^{i+1}(M) \xrightarrow{f} F^{i+1}(M) \xrightarrow{\pi^F_{i,i+1}} F^{i+1}(M/fM) \xrightarrow{\delta^F_{i+1,j}} \cdots$$

Now one can define a sequence of Bockstein morphisms

$$\beta^F_{i,i} : F^i(M/fM) \rightarrow F^{i+1}(M/fM)$$

as the composition

$$\beta^F_{i,i} = \pi^F_{i,i+1} \circ \delta^F_{i,i}.$$

**Remark 4.16.** (1) Clearly, $f, i, F$ and the $A/fA$-module $M/fM$ are ingredients of a Bockstein morphism. However, while the notation does not indicate this, is also depends on $A$ and the avatar $M \xrightarrow{f} M$ for $M/fM$ (or at least an infinitesimal avatar $0 \rightarrow M/fM \rightarrow M/f^2M \rightarrow M/fM \rightarrow 0$).

Bocksteins are not intrinsic but arise from a specialization.

(2) It is possible to modify the constructions to include contravariant functors, or $A$-modules $N$ on which $f$ acts surjectively.

The original version of a Bockstein morphism appeared in topology, where $A = \mathbb{Z}$, $f$ is a prime number, $M$ is an Abelian group without $p$-torsion, and $F^\bullet$ is singular homology (or cohomology) with coefficients $M$ on a fixed space $X$. Generally, in this context there is a Bockstein spectral sequence that arises from the short exact sequence of singular chains on $X$ with coefficients in $M, M$ and $M/pM$ respectively. It starts with $E^1_{i,j} = H_{i+j}(X; M/pM)$, the differential on the $E^1$-page is the Bockstein morphism, and it converges to the tensor product of $\mathbb{Z}/p\mathbb{Z}$ with the free part of $H_{i+j}(X; M)$.

In [SW11], Bockstein maps were introduced and studied in local cohomology. So, $A$ is a Noetherian $\mathbb{Z}$-algebra, $I = (g = g_1, \ldots, g_m) \subseteq A$ is an ideal, and $M$ is a $p$-torsion free $A$-module. In this setup there are several $\delta$-functors $F^\bullet$ that arise naturally: the local cohomology functor $F^i = H^i_I(-)$ with support in $I$, the extension functors $F^i = \text{Ext}^i_A(A/I^\ell, -)$, the Koszul cohomology functors $F^i = H^i(-; g)$. It is shown in [SW11] that in the same way that these three $\delta$-functors allow natural transformations, the three families of Bockstein morphisms are compatible. Several examples
are given, based (for example) on the arithmetic of elliptic curves and on subspace arrangements.

One result of [SW11] states that when $A$ is a polynomial ring over $\mathbb{Z}$ containing the ideal $I$, then the Bockstein on $H^*(R/pR)$ is zero except for a finite set of primes $p$. On a more topological note, the same article investigates the interplay between Bocksteins on local cohomology and those on singular homology in the context of Stanley–Reisner rings. More precisely, let $R = \mathbb{Z}[x]$ be the $\mathbb{Z}^n$-graded polynomial ring on the vertices of the simplicial complex $\Delta$ on $n$ vertices, and let $m = (x)$ be the graded maximal ideal. Hochster linked the multi-graded components of the local cohomology $H^m_\mathfrak{m}(M \otimes_{\mathbb{Z}} R/I)$ with the singular cohomology with coefficients in $M$ of a certain simplicial subcomplex of $\Delta$ determined by the chosen multi-degree, [Hoc77]. Then [SW11] shows that the topological Bocksteins on these links are compatible with the local cohomology Bocksteins via Hochster’s identification, and that it behaves well with respect to local duality.

It follows easily from the definitions that the composition of Bocksteins $\beta^F_{i+1} \circ \beta^F_i$ is zero; this is the origin of the Bockstein spectral sequence mentioned above. Its ingredients are the Bockstein cohomology modules $\ker(\beta^F_{i+1})/\operatorname{im}(\beta^F_i)$. In [Put19], this notion is used to study the extended Rees ring $A[It, t^{-1}]$ of an $m$-primary ideal in the local ring $(A, \mathfrak{m})$ as $M$, using $t$ for $f$ and $F$ is the local cohomology with support in $\mathfrak{m}$. The accomplishment consists in vanishing theorems for local cohomology of the associated graded ring $\text{gr}_f(A)$, extending earlier such results of Narita, and Huckaba–Huneke [Nar63, HH99].

4.3.2. Variation of Hodge structures and GKZ-systems. Here we give a brief motivation of $A$-hypergeometric systems and explain how local cohomology of toric varieties enters the picture. We recommend [SST00, Sti98, RSSW21] for more detailed information and literature sources.

**Notation 4.17.** Let $\{a_1, \ldots, a_n\} = A \subseteq \mathbb{Z}^d \times n$ satisfy the following properties:

(1) the cone $C_A := \mathbb{R}_{\geq 0}A$ spanned by the columns of $A$ inside $\mathbb{R}^d$ is $d$-dimensional and its lineality (the dimension of the largest real vector space that it contains) is zero;

(2) there exists a $\mathbb{Z}$-linear functional $h: \mathbb{Z}^d \rightarrow \mathbb{Z}$ such that $h(a_j) = 1$ for $1 \leq j \leq n$;

(3) the semigroup $\mathbb{N}A := \sum_{j=1}^n \mathbb{N}a_j$ agrees with the intersection $\mathbb{Z}^d \cap C_A$.

The graded (via $h$) semigroup ring $S_A := \mathbb{C}[\mathbb{N}A]$
Here, \( \tilde{h} \) does not necessarily gives rise to a projective toric variety \( Y_A \subseteq \mathbb{P}^{n-1}_\mathbb{C} \) of dimension \( d-1 \) and its cone \( X_A = \text{Spec}(S_A) \subseteq \mathbb{C}^n \). They can be viewed as (partial) compactifications of the \( (d-1) \)-torus

\[
\mathbb{T} := \frac{\text{Hom}_\mathbb{Z}(\mathbb{Z}^d, \mathbb{C}^*)}{\text{Hom}_\mathbb{Z}(\mathbb{Z} a_0, \mathbb{C}^*)},
\]

and \( \mathbb{\tilde{T}} \) respectively, where \( a_0 \) is a suitable element of \( \mathbb{Z}^d \cap C_A \) that induces \( h \) in the sense that \( h(\mathfrak{a}_j) \) is the dot product \( \langle \mathfrak{a}_0, \mathfrak{a}_j \rangle \).

A global section \( F_{A,\nu} \in \Gamma(Y_A, \mathcal{O}_{Y_A}(1)) \) is an element \( \sum \mathfrak{a}_j t^{\mathfrak{a}_j} \) of the Laurent polynomial ring \( \mathbb{C}[t_1^\pm, \ldots, t_d^\pm] \) that is equivariant under the action of \( \text{Hom}_\mathbb{Z}(\mathbb{Z} a_0, \mathbb{C}^*) \). Its vanishing defines a hypersurface \( Z_\nu \) inside \( \mathbb{T} \) with complement \( U_\nu = \mathbb{T} \setminus Z_\nu \). Batyrev initiated the study of the Hodge theory of these objects in his search for mirror symmetry on toric varieties and their hypersurfaces \cite{Bat93}. As is explained in Stienstra's article \cite{Sti98}, for understanding the weight filtration on the cohomology of \( Z_\nu \) it is useful to study Hodge aspects of the cohomology \( H^*(\mathbb{T}, \mathbb{\tilde{Z}}_\nu ; \mathbb{C}) \) relative to the affine cone

\[
\mathbb{\tilde{Z}}_\nu := \mathbb{\tilde{T}} \cap \text{Var}(F_{A,\nu} - 1).
\]

A powerful tool in this endeavor is the idea of letting the section vary and studying these cohomology groups as a family, viewing the coefficients of the Laurent polynomial as parameters. For this, read the parameter \( \nu \) of \( F_{A,\nu} \) as a point in \( \mathbb{C}^n \). For any face \( \tau \) of the cone \( C_A \) let \( F_{A,\tau}^\tau \) be the subsum of \( F_{A,\nu} \) of terms with support on \( \tau \). Then \( \nu \) is non-degenerate if the singular locus of \( F_{A,\nu}^\tau \) does not meet \( \mathbb{\tilde{T}} \) for any \( \tau \), including the case \( \tau = A \).

For non-degenerate \( \nu \), \( H^i(\mathbb{T}, \mathbb{\tilde{Z}}_\nu ; \mathbb{C}) \) is nonzero only when \( i = d \) and there is a natural identification of \( H^d(\mathbb{T}, \mathbb{\tilde{Z}}_\nu ; \mathbb{C}) \) with the stalk of the solutions of a certain natural \( D \)-module that we describe next.

For what is to follow, we assume that \( A \) satisfies condition 4.17(1) but not necessarily 4.17(2) and 4.17(3), unless indicated expressly.

Let \( D_A \) be the Weyl algebra \( \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \) with subring \( O_A = \mathbb{C}[x_1, \ldots, x_n] \), and let \( L_A \) be the \( \mathbb{Z} \)-kernel of \( A \). Define two types of operators

\[
E_i := \sum_{j=1}^n a_{i,j} x_j \partial_j \quad \text{(Euler operators)};
\]

\[
\square_u := \partial^{u+} - \partial^{u-} \quad \text{(box operators)}.
\]

Here, \( 1 \leq i \leq d \) and \( u \in L_A \) with \( (u_+)_i = \max\{u_i, 0\} \) and \( (u_-)_i = \max\{-u_i, 0\} \). Then choose a parameter vector \( \beta \in \mathbb{C}^d \) and define the hypergeometric ideal

\[
H_A(\beta) = D_A \cdot \{E_i - \beta_i\}_{i=1}^d + D_A \cdot \{\square_u\}_{u \in L_A}
\]

and the hypergeometric module

\[
M_A(\beta) := D_A / H_A(\beta)
\]
to $A$ and $\beta$. These modules were defined by Gelfand, Graev, Kapranov and Zelevinsky in a string of articles including [GGZ87, GZK89] during their investigations of Aomoto type integrals. The modules are always holonomic [Ado94], and they are regular holonomic if and only if $A$ satisfies Condition 4.17.(2), [SW08a]. We refer to [SST00] for extensive background on hypergeometric functions, their associated differential equations, and how they relate to hypergeometric modules $M_A(\beta)$ via a dehomogenization technique investigated in [BMW19]. The article [RSSW21] is a gentle introduction to hypergeometric $D$-modules, combined with a survey on recent applications to Hodge theory.

Let $R_A = \mathbb{C}[\partial_1, \ldots, \partial_n]$; while this is a subring of operators of $D_A$, one can also view it as a polynomial ring in its own right. The ideal

$$I_A := R_A \cdot \{a \in L_A\}$$

that forms part of the defining equations for $H_A(\beta)$ is called the toric ideal; its variety in $\hat{\mathbb{C}}^n = \text{Spec} \ R_A$ is the toric variety $X_A$. We use here the "hat" to distinguish the copy of complex $n$-space that arises as $\text{Spec} \ R_A$ from that which arises as $\text{Spec} \ O_A$. The two are domain and target of the Fourier–Laplace transform $\text{FL}(\cdot)$ which, on elements of $D_A$, amounts to $x_j \mapsto \partial_j, \partial_j \mapsto -x_j$.

Local cohomology arises in two ways in the study of $M_A(\beta)$: in connection with the dimension of the space of solutions, and in the limitations of a functorial description of $M_A(\beta)$ via a $D$-module theoretic pushforward.

For any holonomic $D_A$-module $M$ there is a Zariski open set of $\mathbb{C}^n$ on which $M$ is a connection; we call the rank of this connection the rank of $M$. For $M_A(\beta)$, this open set is determined by the non-vanishing of the $A$-discriminant, a generalization of the discriminant of a polynomial. In particular, it does not depend on $\beta$; we denote it $U_A$. If $A$ satisfies Condition 4.17.(3) then the connection on $U_A$ has rank equal to the simplicial volume in $\mathbb{R}^d$ of the convex hull of the origin and the columns of $A$, [GGZ87, GZK89, Ado94]. Indeed, the hypothesis implies that the semigroup ring $S_A$ is Cohen–Macaulay by Hochster’s theorem [Hoc72], and this allows a certain spectral sequence to degenerate, which determines the rank. In fact, one can even produce the solutions often in explicit forms, by writing down suitable hypergeometric series and proving convergence [GGZ87, SST00].

In the absence of Condition 4.17.(3), the situation can be more interesting since then there may be choices of $\beta$ with the effect of changing the rank [MMW05]. That the possibility of changing rank exists at all was discovered in [ST98]. A certain Koszul-like complex based on the Euler operators $E_i$ that appeared in [MMW05] can be used to substitute for the (now not degenerating) spectral sequence.

A natural question is which parameters $\beta$ will show a change in rank. Because of basic principles, the rank at special $\beta$ can only go up [MMW05]. Since $S_A$ is $A$-graded via $\deg_A(\partial_j) = a_j \in \mathbb{Z}A$, so are its local cohomology
modules $H^i_\beta(S_A)$ supported at the homogeneous maximal ideal. Set
\[
\mathcal{E}_A := \bigcup_{i=0}^{d-1} \deg_A(H^i_\beta(S_A))
\]
the Zariski closure of the union of all $A$-degrees of nonzero elements in a local cohomology module with $i < d$. Note that the union of these degrees can be seen as witnesses to the failure of $S_A$ being Cohen–Macaulay: the union is empty if and only if $S_A$ has full depth. In generalization of the implication of equal rank for all $\beta$ in the Cohen–Macaulay case, it is shown in [MMW05] that
\[
[rk(M_A(\beta) > vol(A)) \iff [\beta \in \mathcal{E}_A].
\]
Consider now the monomial map
\[
\varphi = \varphi_A: \tilde{T} \longrightarrow \mathbb{C}^n,
\]
\[
t \mapsto (t^{a_1}, \ldots, t^{a_n})
\]
induced by $A$. The map is an isomorphism onto the image by Condition 4.17(1), and its closure is the toric variety $X_A$. On $\tilde{T}$ one has for each $\beta$ the (regular) connection $L_\beta = D_T/D_T \cdot \{t_\ast \partial_{t_i} + \beta_t\}$. In [GKZ90], Gelfand, Kapranov and Zelevinsky proved that if $\beta$ is sufficiently generic then the Fourier–Laplace transform $FL(M_A(\beta))$ agrees with the $D$-module direct image $\varphi_+(L_\beta)$, where the set of “good” $\beta$ forms the complement of a countably infinite and locally finite hyperplane arrangement called the resonant parameters, and given by all $L_A$-shifts of the bounding hyperplanes of the cone $C_A$. In [SW09] this result was refined and completed to an equivalence
\[
[M_A(\beta) = \varphi_+(L_\beta)] \iff [\beta \text{ is not strongly resonant}].
\]
Here, following [SW09], $\beta \in \mathbb{C}^d$ is strongly resonant if and only if there is a finitely generated $R_A$-submodule of $\bigoplus_{j=1}^n H^j_\beta(S_A)$ containing $\beta$ in the Zariski closure of its $A$-degrees. (Since the local cohomology modules here are not coherent, being strongly resonant is more special than being in the Zariski closure of the $A$-degrees of the direct sum). Some further improvements have been made in [Ste19b, Ste19a].

Remark 4.18. As it turns out, when Conditions 4.17 are in force in full strength, then certain $M_A(\beta)$, including the case $\beta = 0$, are not just a regular $D_A$-module but in fact carry a mixed Hodge module structure in the sense of Saito, [Sai90]. The Hodge and weight filtrations of hypergeometric systems have been studied in [RS20, Rei14, RW], showing connections to intersection homology of toric varieties. See [RSSW21] for a survey. 

4.3.3. Milnor fibers and torsion in the Jacobian ring. Let $f$ be a non-unit in $R = \mathbb{C}[x_1, \ldots, x_n]$ and put $X := \mathbb{C}^n = \text{Spec}(R)$. By the ideal $J_f$ we mean the ideal generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$; this ideal varies
with the choice of coordinate system in which we calculate. In contrast, the Jacobian ideal \( \text{Jac}(f) = J_f + (f) \) is independent.

If \( \xi \in \text{Var}(f) \), let \( B(\xi, \varepsilon) \) denote the \( \varepsilon \)-ball around \( \xi \in \text{Var}(f) \subseteq \mathbb{C}^n \). Milnor [Mil68] proved that the diffeomorphism type of the open real manifold

\[
M_{f,\xi,t,\varepsilon} = B(\xi, \varepsilon) \cap \text{Var}(f - t)
\]

is independent of \( \varepsilon, t \) as long as \( 0 < |t| \ll \varepsilon \ll 1 \). Abusing language, for \( 0 < t \ll \varepsilon \ll 1 \) denote by \( M_{f,\xi} \) the fiber of the bundle

\[
B(\xi, \varepsilon) \cap \{ \eta \in \mathbb{C}^n \mid 0 < |f(\eta)| < t \} \rightarrow f(\eta).
\]

If \( f \) has an isolated singularity at \( \xi \) then the Milnor fiber \( M_{f,\xi} \) is a bouquet of \((n-1)\)-spheres, and \( H^{n-1}(M_{f,\xi}; \mathbb{C}) \) can be identified non-canonically with the Jacobian ring \( R \) of \( p \) in that case. Suppose from now on that \( f \) is homogeneous, and the eigenvalues of the action of \( s \) are actually true for general isolated singularities, not just in the presence of homogeneity, and the eigenvalues of the action of \( s \) turn out to be the non-trivial roots of the local Bernstein–Sato polynomial of \( f \) at \( \xi \). The \( s \)-action comes then from the Gauß–Manin connection. Compare also [Sai09, Sai16a, Mal83, Kas83, Wal15]. Compare [Ste87] for details on the Hodge structure on the cohomology of the Milnor fiber.

For non-isolated singularities, most of this must break down, since \( R/\text{Jac}(f) \) is not Artinian in that case. Suppose from now on that \( f \) is homogeneous, and that \( \xi \) is the origin. Note that now \( \text{Jac}(f) = J_f \); we abbreviate \( M_{f,\xi} \) to \( M_f = \text{Var}(f - 1) \). The Jacobian module

\[
H^0_m(R/J_f) = \{ g + J_f \mid \exists k \in \mathbb{N}, \forall i, x_i^k g \in J_f \}
\]

has been studied in [Pel88, vSW15] for various symmetry properties and connections with geometry. Note that this finite length module agrees with the Jacobian ring in the case of an isolated singularity, it can hence be considered a generalization of it in more general settings.

If

\[
\eta = \sum_i x_i dx_1 \wedge \cdots \wedge d\tilde{x}_i \wedge \cdots \wedge dx_n
\]

denotes the canonical \((n-1)\)-form on \( X \), then (via residues) every class in \( H^{n-1}(M_f; \mathbb{C}) \) is of the form \( g\eta \) for suitable \( g \in R \), and if \( g \in R \) is the smallest degree homogeneous polynomial such that \( g\eta \) represents a chosen class in \( H^{n-1}(M_f; \mathbb{C}) \) then \( -\deg(g\eta)/\deg(f) \) is a root of the Bernstein–Sato polynomial of \( f \) [Wal05]. Suppose the singular locus of \( f \) is (at most) 1-dimensional. Then by [Sai16b, Wal17, Sai17], with \( 1 \leq k \leq d \) and \( \lambda = \exp(2\pi \sqrt{-1}k/d) \), the following holds:

\[
\dim \mathbb{C}[H^0_m(R_n/Jac(f))]_{d-n+k} \leq \dim \mathbb{C} \mathfrak{gr}^{\text{Hodge}}_{n-2}(H^{n-1}(M_f; \mathbb{C})_{\lambda}),
\]
where the right hand side indicates the $\lambda$-eigenspace of the associated graded object to the Hodge filtration on $H^{n-1}(M;\mathbb{C})$. Dimca and Sticlaru have used this inequality to study nearly free divisors and pole order filtrations, [DS19a, DS19b]. It would be interesting to find more general inequalities of this type. The above estimate is based on local cohomology of logarithmic forms introduced in [Sai80]; such modules have been calculated in [DSS+13] for generic hyperplane arrangements. See [Wal15] for more connections to monodromy and zeta-functions.

4.4. Lyubeznik numbers. Let $(R, \mathfrak{m}, k)$ be a commutative regular local Noetherian ring of dimension $n$ that contains its residue field. For any ideal $I$ of $R$, Lyubeznik proved in [Lyu93b] that the $k$-dimension

$$\lambda_{i,j}(R, I) := \dim_k(\Ext^i_R(k, H^{n-j}_I(R)))$$

is for each $i, j \in \mathbb{N}$ only a function of $R/I$ and so does not depend on the presentation of $R/I$ as a quotient of a regular local ring.

In his seminal paper, Lyubeznik also showed that $\lambda_{i,j}(R, I)$ agrees with the socle dimension in $H^i_\mathfrak{m}(H^{n-j}_I(R))$, and hence with the $i$-th Bass number of $H^{n-j}_I(R)$ with respect to $\mathfrak{m}$. In fact, $H^i_\mathfrak{m}(H^{n-j}_I(R))$ is the direct sum of $\lambda_{i,j}(R/I)$ many copies of $E_R(k)$, the injective hull of $k$ when viewed as $R$-module.

It follows from the local cohomology interpretation that $\lambda_{i,j}(R, I) = \lambda_{i,j}(\hat{R}, I\hat{R})$ is invariant under completion. By the Cohen structure theorems, every complete local Noetherian ring containing its residue field is the quotient of a complete regular Noetherian local ring containing its residue field. One can thus define for every local Noetherian ring $A$ the $(i, j)$-Lyubeznik number

$$\lambda_{i,j}(A) := \lambda_{i,j}(R, I)$$

via any surjection $R \to R/I = \hat{A}$ from a complete regular ring $R$ onto the completion of $A$.

**Notation 4.19.** Throughout this subsection, $(R, \mathfrak{m}, k)$ is a regular local ring containing its residue field, $\hat{R}$ its completion along $\mathfrak{m}$, and $I$ an ideal of $R$ such that $A = R/I$. Set $d := \dim(A)$. Field extensions $R \to \mathbb{K} \otimes_k \hat{R}$ have no impact on the Lyubeznik numbers, so that one can always assume $k$ to be algebraically or separably closed if necessary. Moreover, since $\Gamma_I(M) = \Gamma_{\sqrt{I}}(M)$, one may assume that $A$ is reduced.

By Grothendieck’s vanishing theorem, $\lambda_{i,j}(A)$ is zero if $j < 0$, and by the depth sensitivity of local cohomology, $\lambda_{i,j}(A) = 0$ if $j > \dim(A)$, [ILL+07]. By construction, the dimension of the support of $H^{n-j}_I(R)$ is contained in the variety of $I$, so that $\lambda_{i,j}(A) = 0$ for all $i > d$. 
We can thus write $\Lambda(A)$ for the Lyubeznik table

$$\Lambda(A) := \begin{pmatrix} 
\lambda_{0,0}(A) & \ldots & \lambda_{0,d}(A) \\
\vdots & \ddots & \vdots \\
\lambda_{d,0}(A) & \ldots & \lambda_{d,d}(A)
\end{pmatrix}$$

It has been shown in [HS93] in the case $\text{char}(R) > 0$, and then in [Lyu93b] when $Q \subseteq k$ that the injective dimension of $H^k_I(R)$ is always bounded above by the dimension of its support. However, it is standard that the support of $H^{n-j}_I(R)$ is contained in a variety of dimension at most $j$. This implies that the nonzero entries of $\Lambda(A)$ are on or above the main diagonal of $\Lambda(A)$.

There is a Grothendieck spectral sequence

$$(4.4.0.1) \quad H^i_m(H^j_I(R)) \Longrightarrow H^{i+j}_m(R).$$

It follows directly from this spectral sequence that

- the alternating sum $\sum_{i,j} (-1)^{i+j} \lambda_{i,j}(A)$ equals 1;
- $\lambda_{0,d}(A) = \lambda_1(A) = 0$ for all $A$ unless $\dim(A) \leq 1$;
- if $R/I$ is a complete intersection, then $\lambda_{i,j}(A)$ vanishes unless $i = j = d$. (We say that the Lyubeznik table is trivial).
- Moreover, following [AMY18] let $\rho_j(A) := -\delta_{0,j} + \sum_{i=0}^{d-j} \lambda_{i,i+j}(A)$ be the reduced sum along the $j$-th super-diagonal in $\Lambda(A)$, where $\delta$ denotes the Kronecker-$\delta$. Then $\rho_d(A)$ is always zero, and non-vanishing of $\rho_j(A)$ implies that of either $\rho_{j-1}(A)$ or $\rho_{j+1}(A)$, compare [NnBSW19].

In characteristic $p > 0$, the (iterated) Frobenius functor sends a free resolution of the ideal $I$ to a free resolution of the Frobenius power $I^{p^r}$. As the Frobenius powers of $I$ are cofinal with the usual powers, $H^k_I(R) = 0$ whenever $k$ exceeds the projective dimension of $R/I$. In particular, if $I$ is perfect (i.e., $R/I$ is Cohen–Macaulay), the Lyubeznik table of $R/I$ is trivial in positive characteristic. In characteristic zero, this is not so; for example, the Lyubeznik table for the (perfect) ideal of the $2 \times 2$ minors of a $2 \times 3$ matrix of indeterminates over $k \supseteq Q$ is

$$\begin{pmatrix}
0 & 0 & 1 & 0 \\
\cdot & 0 & 0 & 0 \\
\cdot & 0 & 0 & 1 \\
\cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & 1
\end{pmatrix}$$

as one sees from the fact that $I$ is 3-generated and $H^2_I(R) = E_R(k)$, compare Example 2.14, or the computations in [Wal99]. (Single dots indicate a zero entry.)

**Definition 4.20.** The highest Lyubeznik number of $A$ is $\lambda_{d,d}(A)$. ◦

It follows directly from the spectral sequence that for $d \leq 1$, only $\lambda_{d,d}(A)$ is nonzero (and thus equal to 1).
Lyubeznik proved in [Lyu93b] that \( \lambda_{d,d}(A) \) is always positive. For 2-dimensional complete local rings, with separably closed residue field, it was shown in [Wal01b, Kaw00] that the Lyubeznik table is independent of the 1-dimensional components of \( I \). Indeed, one has:

\[
\Lambda(A) = \begin{pmatrix}
0 & t - 1 & 0 \\
\cdot & 0 & 0 \\
\cdot & \cdot & t
\end{pmatrix}
\]

where \( t \) is the number of components of the punctured spectrum of \( A \). In any dimension \( d \), the number \( \lambda_{d,d}(A) \) is 1 if \( A \) is analytically normal [Lyu93b] or has Serre’s condition \( S_2 \) [Kaw02]. On the other hand, \( \lambda_{d,d}(A) \) can be 1 without \( A \) being Cohen–Macaulay or even \( S_2 \), [Kaw02].

More generally, consider the Hochster–Huneke graph of \( A \): the vertices of \( \text{HH}(A) \) are the \( d \)-dimensional primes of \( A \) and an edge links two such primes if the height of their sum is 1. Then Zhang, generalizing the case \( d \leq 2 \) from [Wal01b] and the case \( \text{char}(k) > 0 \) from [Lyu06a], proved (in a characteristic-independent way) in [Zha07] that \( \lambda_{d,d}(A) \) agrees with the number of connected components of \( \text{HH}(A) \). The main result in [Zha07] has been extended to mixed characteristic in [Zha21b]. See also [MS12, Sch09a] for more on the relationship between connectedness and the structure of local cohomology.

4.4.1. Combinatorial cases and topology. If \( I \) is a monomial ideal, then Alvarez, Vahidi and Yanagawa [Yan03, AMV14, AM15, AMY18] have obtained the following results:

- Lyubeznik numbers of monomial ideals relate to linear strands of the minimal free resolution of their Alexander duals;
- If \( A \) is sequentially Cohen–Macaulay (i.e., every \( \text{Ext}_R^i(A, R) \) is zero or Cohen–Macaulay of dimension \( i \)) then both in characteristic \( p > 0 \) and also if \( I \) is monomial then the Lyubeznik table is trivial.
- there are Thom–Sebastiani type results for Lyubeznik tables of monomial ideals in disjoint sets of variables.
- Lyubeznik numbers of Stanley–Reisner rings are topological invariants attached to the underlying simplicial complex.

In a different direction, consider the case when \( I_{r,s,t} \) is the ideal generated by the \( (t + 1) \times (t + 1) \) minors of an \( r \times s \) matrix of indeterminates over the field \( k \). In positive characteristic, the Cohen–Macaulayness of \( R/I \) implies triviality of the Lyubeznik table. In characteristic zero, however, these numbers carry interesting combinatorial information related to representations of the general linear group. Lőrincz and Raicu proved in [LR20] the following. Write the Lyubeznik numbers into a bivariate generating function

\[
L_{r,s,t}(q, w) := \sum_{i,j \geq 0} \lambda_{i,j}(A_{r,s,t}) \cdot q^i \cdot w^j
\]
with \( A_{r,s,t} = \mathbb{C}[\{x_{i,j}|1 \leq i \leq r, 1 \leq j \leq s\}]/I_{r,s,t} \), with \( r > s > t \). Then

\[
L_{r,s,t}(q, w) = \sum_{i=0}^{t} q^{i^2+i(r-s)} \binom{s}{i} . w^{r^2+2t+i(r+s-2t-2)} . \binom{s-1-i}{t-i} . w^2.
\]

Here, the subscripts to the binomial coefficient indicate the Gaussian q-binomial expression

\[
\binom{p}{q}_{c} = \frac{(1-c^{p})(1-c^{p-1})... (1-c^{p-b+1})}{(1-c)(1-c^{b-1})... (1-c^{q})}.
\]

There is a similar formula for the case \( r = s > t \).

We now turn to topological interpretations on Lyubeznik tables. The earliest such results were formulated by García López and Sabbah. Suppose \( A \) has an isolated singularity at \( \mathfrak{m} \). Then \( H_{I}^{n-j}(R) \) is \( \mathfrak{m} \)-torsion for \( n - j \neq d \). Hence, by the spectral sequence (4.4.0.1), \( \Lambda(A) \) is concentrated in the top row and the rightmost column, and there are equalities \( \lambda_{0,j} + \delta_{j+1,d} = \lambda_{j+1,d} \), using again Kronecker notation. It is shown in [GLS98] that, if the coefficient field is \( \mathbb{C} \), then \( \lambda_{0,j} \) equals the \( \mathbb{C} \)-dimension of the topological local cohomology group of the analytic space \( \text{Spec}(V) \) with support in the vertex \( \mathfrak{m} \).

This result was then generalized by Blickle and Bondu as follows. Suppose (over \( \mathbb{C} \)) that the constant sheaf on the spectrum of \( A \) is self-dual in the sense of Verdier outside the vertex \( \mathfrak{m} \). This is the case when \( \mathfrak{m} \) is an isolated singularity, but it also occurs in more general cases. For example, on a hypersurface \( f = 0 \) this condition is equivalent to the Bernstein–Sato polynomial of \( f \) having no other integral root but \(-1\), and \(-1\) occurring with multiplicity one, [Tor09]. Blickle and Bondu prove in [BB05] that in this situation the same interpretation of \( \Lambda(A) \) can be made as in the article by García López and Sabbah. In parallel, they also show that if the field has finite characteristic, a corresponding interpretation can be made in terms of local étale cohomology with supports at the vertex.

Lyubeznik numbers also contain information on connectedness of algebraic varieties. For example, as mentioned before, for \( \dim(A) = 2 \) over a separably closed field, the Lyubeznik table is entirely characterized by the number of connected components of the 2-dimensional part of the punctured spectrum.

Suppose \( A \) is equidimensional, with separably closed coefficient field \( \mathbb{K} \). Denote by \( \kappa(A) \) the connectedness dimension of \( A \), the smallest dimension \( t \) of a subvariety \( Y \) in \( \text{Spec}(A) \) whose removal leads to a disconnection. Núñez-Betancourt, Spiroff and Witt discuss in [NnBSW19] the relationship between the number \( \kappa(A) \) and the vanishing of certain Lyubeznik numbers. Their results generalize a consequence of the Second Vanishing Theorem that can be phrased as: \( H_{I}^{n-1}(R) = 0 \) if and only if \( \kappa(A) \neq 0 \). To be precise, they show for an equidimensional ring \( A \):

- \( \kappa(A) \geq 1 \) if and only if \( \lambda_{0,1}(A) = 0 \);
- \( \kappa(A) \geq 2 \) if and only if \( \lambda_{0,1}(A) = \lambda_{1,2}(A) = 0 \);
- for \( i < \dim(A) \), \( \kappa(A) \geq i \) if and only if \( \lambda_{0,1}(A) = \cdots = \lambda_{i-1,i}(A) = 0 \).
Earlier, Dao and Takagi, inspired by remarks of Varbaro, showed that over any field, Serre’s condition $S_3$ implies that $\lambda_{d-1,d} = 0$, [DT16], while in increasing generality it was shown in [Wal01b, Lyu06a, Zha07] that $[\kappa(A) \geq \dim(A) - 1] \iff [\lambda_{d,d}(A) = 1]$. In [RWZ21] are some other results on the effect of Serre’s conditions $(S_t)$ on $\Lambda(A)$.

4.4.2. Projective Lyubeznik numbers. Suppose $X = X_k$ is a projective variety of dimension $d - 1$, with embedding $\iota: X \hookrightarrow \mathbb{P} := \mathbb{P}^{n-1}_k$ via sections of the line bundle $\mathcal{L} = \iota^*(\mathcal{O}_\mathbb{P}(1))$. With this embedding comes a global coordinate ring $\Gamma_*(\mathbb{P})$ of $\mathbb{P}$ and a homogeneous ideal defining the cone $C(X)$ over $X$ in the corresponding affine space. Let $R$ be the localization of $\Gamma_*(\mathbb{P})$ at the vertex, and let $I$ be the ideal defining the germ of $C(X)$ in $R$. A natural question is to ask:

**Problem 4.21.** To what extent are the Lyubeznik numbers of $R/I$ dependent on the embedding $\iota$?

Certainly, if two such cones $(R, I)$ and $(R', I')$ arise from one another by an automorphism of $\mathbb{P}$, then the attached Lyubeznik tables are equal. It is less clear from the definitions whether two embeddings that produce the same sheaf $\mathcal{L}$ on $X$, or at least the same element in the Picard group, should give the same Lyubeznik tables. And even more difficult is the question whether $\iota, \iota'$ should give rise to equal Lyubeznik tables when $\mathcal{L}_\iota \not\cong \mathcal{L}_{\iota'}$ in the Picard group.

We say that $\Lambda(X)$ (or just $\lambda_{i,j}(X)$) is projective if each cone derived from a projective embedding of $X$ produces the same $\Lambda$-table (or at least the same $\lambda_{i,j}$). Positive known results include the following:

- If $\dim(X) \leq 1$ then $\Lambda(X)$ is projective by [Wal01b], since then each cone ring is at most 2-dimensional, and connectedness of the punctured $d$-dimensional spectrum of $R/I$ is equivalent to connectedness of the $(d - 1)$-dimensional part of $X$.
- If $X$ is smooth and $k = \mathbb{C}$, then each cone has an isolated singularity, so that the Lyubeznik numbers can be expressed in terms of topological local cohomology as in [GLS98]. Switala proves in [Swi15] that these data are actually intrinsic to $X$, appearing as cokernels of the cup product with the Chern class of the embedding on singular cohomology of $X$. By independence of Lyubeznik numbers under field extensions, this also works when just $\mathbb{Q} \subseteq k$.
- Since $\lambda_{0,1}(A) = 0$ is equivalent to $H^{n-1}_i(R) = 0$, which in turn is equivalent to connectedness of the punctured spectrum of $A$, $\lambda_{0,1}(X)$ is projective.
- Similarly, the simultaneous vanishing of $\lambda_{0,1}, \lambda_{1,2}, \ldots, \lambda_{i-1,i}$ is projective since it measures by [NnBSW19] the connectedness dimension of the cone, which corresponds to connectedness dimension of $X$ itself.
Consider the module $E_{i,j}(\iota) := \text{Ext}^{n-i}_R(\text{Ext}^{n-j}_R(R/I, \Omega_R), \Omega_R)$ where $\Omega_R$ is the canonical module of $R$. In [Zha11], Zhang proves that in finite characteristic, the degree zero part of $E_{i,j}(\iota)$ supports a natural action of Frobenius, whose stable part is independent of $\iota$ and has $k$-dimension $\lambda_{i,j}(R/I)$. In particular, $\Lambda$ is projective in positive characteristic.

In characteristic zero, after base change to $\mathbb{C}$, the modules $H^m_i(R/I)$ have a natural structure as mixed Hodge modules. This has been exploited in [RSW21] to prove that in this setting, on the level of constructible sheaves via the Riemann–Hilbert correspondence,

$$\lambda_{i,j}(R/I) = \dim_{\mathbb{C}} H^j \tau^! p\mathcal{H}^{-j}(\mathbb{D}\mathbb{Q}_C).$$

Here, $\mathbb{Q}_C$ is the constant sheaf on the cone $C = C(X)$ under any embedding of $X$, $\mathbb{D}$ is Verdier duality (corresponding to holonomic duality), $p\mathcal{H}$ is taking perverse cohomology (corresponding to usual cohomology for $D$-modules via the Riemann–Hilbert correspondence) and $\tau^!$ is the exceptional inverse image for constructible sheaves under the embedding $\tau$ of the vertex into the cone. One can then recast this as the dimension of the cohomology of a certain related sheaf on the punctured cone, and this cohomology is the middle term in an exact sequence whose other terms are kernels and cokernels of the Chern class of $L_i$ on certain sheaves on $X$. These sheaves are relatives of, but not always equal to, intersection cohomology of $X$. This difference is then exploited to construct examples of (reducible) varieties whose Lyubeznik numbers are not projective. In [Wan20], the construction was modified to yield irreducible ones with non-projective $\Lambda$-table.

The construction of [RSW21] starts with a variety whose Picard number is greater than one, and from it constructs a suitable $X$. In [RWZ21] it is shown that if the rational Picard group of $X$ is $\mathbb{Q}$ then almost all Lyubeznik numbers of $X$ are projective. In particular, this applies to determinantal varieties so that the Lörincz–Raicu computation in [LR20] determines the vast majority of the entries of the Lyubeznik tables for such varieties under all embeddings.

**Remark 4.22.** A similar set (to Lyubeznik numbers) of invariants is introduced in [Bri20] (but see also [Swi17b]). It is shown that if $I$ is an ideal in a polynomial ring over the complex numbers then the Čech-to-de Rham spectral sequence whose abutment is the reduced singular cohomology of the complement of the variety of $I$ has terms on page two that do not depend on the embedding of the variety of $I$ into an affine space, at least when suitably re-indexed. Using algebraic de Rham cohomology, this is actually shown over all fields of characteristic zero. These Čech–de Rham numbers are further investigated in [RWZ21] from the viewpoint of projectivity since, if $I$ is homogeneous, one can ask to what extent these numbers are defined by the associated projective variety (rather than the affine cone). [RWZ21] studies their behavior under Veronese maps, and the degeneration of the spectral sequence. ☐
References

[ABW13] Donu Arapura, Parsa Bakhtary, and Jaroslaw Wlodarczyk, *Weights on cohomology, invariants of singularities, and dual complexes*, Math. Ann. 357 (2013), no. 2, 513–550. MR 3096516 3.2.2

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. 73 (1994), no. 2, 269–290. 4.3.2

[AKM08] Ian Aberbach, Mordechai Katzman, and Brian MacCrady, *Weak F-regularity deforms in $\mathbb{Q}$-Gorenstein rings*, J. Algebra 204 (1998), no. 1, 281–285. MR 1623973 3.41

[AM00] Josep Alvarez Montaner, *Characteristic cycles of local cohomology modules of monomial ideals*, J. Pure Appl. Algebra 150 (2000), no. 1, 1–25. MR 1762917 2

[AM04] ———, *Characteristic cycles of local cohomology modules of monomial ideals. II*, J. Pure Appl. Algebra 192 (2004), no. 1-3, 1–20. MR 2067186 2

[AM05] ———, *Operations with regular holonomic $\mathcal{D}$-modules with support a normal crossing*, J. Symbolic Comput. 40 (2005), no. 2, 999–1012. MR 2167680 4.1.1

[AM15] ———, *Lyubeznik table of sequentially Cohen-Macaulay rings*, Comm. Algebra 43 (2015), no. 9, 3695–3704. MR 3360843 4.4.1

[AMBL05] Josep Alvarez-Montaner, Manuel Blickle, and Gennady Lyubeznik, *Generators of $\mathcal{D}$-modules in positive characteristic*, Math. Res. Lett. 12 (2005), no. 4, 459–473. MR 2155224 3.1.3, 3.3, 3.3

[AMBZ20] Josep Álvarez Montaner, Alberto F. Boix, and Santiago Zarzuela, *On some local cohomology spectral sequences*, Int. Math. Res. Not. IMRN (2020), no. 19, 6197–6293. MR 4165477 3.2.3

[AMGLZA03] Josep Álvarez Montaner, Ricardo García López, and Santiago Zarzuela Armengou, *Local cohomology, arrangements of subspaces and monomial ideals*, Adv. Math. 174 (2003), no. 1, 35–56. MR 1959890 4.1.1

[AMHNnB17] Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes*, Contemporary Mathematics, vol. 244, American Mathematical Society, Providence, RI, 1999. MR 1716706 1.4, 3.12

[AML06] Josep Álvarez Montaner and Anton Leykin, *Computing the support of local cohomology modules*, J. Symbolic Comput. 41 (2006), no. 12, 1328–1344. MR 2271328 3

[AMV14] Josep Álvarez Montaner and Alireza Vahidi, *Lyubeznik numbers of monomial ideals*, Trans. Amer. Math. Soc. 366 (2014), no. 4, 1829–1855. MR 3152714 4.4.1

[AMY18] Josep Álvarez Montaner and Kohji Yanagawa, *Lyubeznik numbers of local rings and linear strands of graded ideals*, Nagoya Math. J. 231 (2018), 23–54. MR 3845587 4.4, 4.4.1

[ATJLL99] Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Studies in duality on Noetherian formal schemes and non-Noetherian ordinary schemes*, Contemporary Mathematics, vol. 244, American Mathematical Society, Providence, RI, 1999. MR 1716706 1.4, 3.12

[Bah17] Kamal Bahmanpour, *A note on Lynch’s conjecture*, Comm. Algebra 45 (2017), no. 6, 2738–2745. MR 3594553 2.3

[Bar70] W. Barth, *Transplanting cohomology classes in complex-projective space*, Amer. J. Math. 92 (1970), 951–967. MR 287032 4.2.3
Margherita Barile, Arithmetical ranks of ideals associated to symmetric and alternating matrices, J. Algebra 176 (1995), no. 1, 59–82. MR 1345294 3.1.2, 4.15

Margherita Barile, On the computation of arithmetical ranks, Int. J. Pure Appl. Math. 17 (2004), no. 2, 143–161. MR 2104196 4.1.1

Margherita Barile, On toric varieties of high arithmetical rank, Yokohama Math. J. 52 (2006), no. 2, 125–130. MR 2222158 4.1.1

Margherita Barile, On toric varieties which are almost set-theoretic complete intersections, J. Pure Appl. Algebra 207 (2006), no. 1, 109–118. MR 2244384 4.1.1

Victor V. Batyrev, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, Duke Math. J. 69 (1993), no. 2, 349–409. 4.3.2

Manuel Blickle and Raphael Bongu, Local cohomology multiplicities in terms of étale cohomology, Ann. Inst. Fourier (Grenoble) 55 (2005), no. 7, 2239–2256. MR 2207383 4.4.1

Manuel Blickle and Gebhard Böckle, Cartier modules: finiteness results, J. Reine Angew. Math. 661 (2011), 85–123. MR 2863904 3.2.2

Bhargav Bhatt, Manuel Blickle, Gennady Lyubeznik, Anurag K. Singh, and Wenliang Zhang, Local cohomology modules of a smooth $\mathbb{Z}$-algebra have finitely many associated primes, Invent. Math. 197 (2014), no. 3, 509–519. MR 3251828 2.1, 2.9, 3.3

Bhargav Bhatt, Manuel Blickle, Gennady Lyubeznik, Anurag K. Singh, and Wenliang Zhang, Stabilization of the cohomology of thickenings, Amer. J. Math. 141 (2019), no. 2, 531–561. MR 3928045 4, 2.2, 3.1.3

Bhargav Bhatt, Manuel Blickle, Gennady Lyubeznik, Anurag K. Singh, and Wenliang Zhang, An asymptotic vanishing theorem for the cohomology of thickenings, Math. Ann. 380 (2021), no. 1-2, 161–173. MR 4263681 2.2, 2.2

Alberto F. Boix and Majid Eghbali, Annihilators of local cohomology modules and simplicity of rings of differential operators, Beitr. Algebra Geom. 59 (2018), no. 4, 665–684. MR 3871100 2.3

I. N. Bernšteĭn, Analytic continuation of generalized functions with respect to a parameter, Funkcional. Anal. i Priložen. 6 (1972), no. 4, 26–40. MR 0320735 3.1.1

I. N. Bernšteĭn, I. M. Gel’fand, and S. I. Gel’fand, Differential operators on a cubic cone, Uspehi Mat. Nauk 27 (1972), no. 1(163), 185–190. MR 0385159 3.1

Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR 1251956 2.1

Bhargav Bhatt, Annihilating the cohomology of group schemes, Algebra Number Theory 6 (2012), no. 7, 1561–1577. MR 3007159 2.3

Bhargav Bhatt, Cohen-Macaulayness of absolute integral closures, arXiv:2008.08070, 2020. 2.3

Thomas Bitoun, Length of local cohomology in positive characteristic and ordinarity, Int. Math. Res. Not. IMRN (2020), no. 7, 1921–1932. MR 4089437 3.1.3, 3.2.3

J.-E. Björk, Rings of differential operators, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam-New York, 1979. MR 549189 3.1.1, 3.1.1, 1, 2, 3, 4, 3.1.1

Margherita Barile and Gennady Lyubeznik, Set-theoretic complete intersections in characteristic $p$, Proc. Amer. Math. Soc. 133 (2005), no. 11, 3199–3209. MR 2160181 4.1.1
[CN78] R. C. Cowsik and M. V. Nori, Affine curves in characteristic $p$ are set theoretic complete intersections, Invent. Math. 45 (1978), no. 2, 111–114. MR 472835 4.1.1

[Con95] S. C. Coutinho, A primer of algebraic $D$-modules, London Mathematical Society Student Texts, vol. 33, Cambridge University Press, Cambridge, 1995. MR 1356713 3.1.1

[DB81] Philippe Du Bois, Complexe de de Rham filtré d’une variété singulière, Bull. Soc. Math. France 109 (1981), no. 1, 41–81. MR 613848 3.2.2

[Del71] Pierre Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 498551 3.2.2

[Del74] ———, Théorie de Hodge. III, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR 498552 3.2.2

[DS19a] Alexandru Dimca and Gabriel Sticlaru, Computing the monodromy and pole order filtration on Milnor fiber cohomology of plane curves, J. Symbolic Comput. 91 (2019), 98–115. MR 3860886 4.3.3

[DS19b] ———, Line and rational curve arrangements, and Walther’s inequality, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 30 (2019), no. 3, 615–633. MR 4002214 4.3.3

[DSM20] Alessandro De Stefani and Linquan Ma, F-stable secondary representations and deformation of $F$-injectivity, Preprint arXiv:2009.09638, 2020. 3.41

[DSS+13] G. Denham, H. Schenck, M. Schulze, M. Wakefield, and U. Walther, Local cohomology of logarithmic forms, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 3, 1177–1203. MR 3137483 4.3.3

[DSZ19] Rankeya Datta, Nicholas Switala, and Wenliang Zhang, Annihilators of $D$-modules in mixed characteristic, arXiv:1907.09948, Math. Res. Lett., to appear. 2.1, 2.3, 4.5

[DT16] Hailong Dao and Shunsuke Takagi, On the relationship between depth and cohomological dimension, Compos. Math. 152 (2016), no. 4, 876–888. MR 3484116 2.2, 2.2, 4.4.1

[EE73] David Eisenbud and E. Graham Evans, Jr., Every algebraic set in $n$-space is the intersection of $n$ hypersurfaces, Invent. Math. 19 (1973), 107–112. MR 327783 4.1

[EH08] Florian Enescu and Melvin Hochster, The Frobenius structure of local cohomology, Algebra Number Theory 2 (2008), no. 7, 721–754. MR 2460693 3.2.2, 3.37, 3.2.2, 3.2.2

[Eis95] David Eisenbud, Commutative algebra, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (document)

[EK04] Matthew Emerton and Mark Kisin, The Riemann-Hilbert correspondence for unit $F$-crystals, Astérisque (2004), no. 293, vi+257. MR 2071510 3.31

[Elk87] Noam D. Elkies, The existence of infinitely many supersingular primes for every elliptic curve over $Q$, Invent. Math. 89 (1987), no. 3, 561–567. MR 903384 4

[EMS00] David Eisenbud, Mircea Mustaţă, and Mike Stillman, Cohomology on toric varieties and local cohomology with monomial supports, J. Symbolic Comput. 29 (2000), no. 4-5, 583–600, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). MR 1769656 4

[Ene03] Florian Enescu, F-injective rings and F-stable primes, Proc. Amer. Math. Soc. 131 (2003), no. 11, 3379–3386. MR 1990626 3.2.2

[Ene09] ———, Local cohomology and F-stability, J. Algebra 322 (2009), no. 9, 3063–3077. MR 2567410 3.2.2

[Fal80] Gerd Faltings, Über lokale Kohomologiegruppen hoher Ordnung, J. Reine Angew. Math. 313 (1980), 43–51. MR 552461 2.2
[Fed83] Richard Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480. MR 701505 3.41

[Fer79] Daniel Ferrand, *Set-theoretical complete intersections in characteristic $p > 0$*, Algebraic geometry (Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978), Lecture Notes in Math., vol. 732, Springer, Berlin, 1979, pp. 82–89. MR 556992 4.1.1

[FW89] Richard Fedder and Keiichi Watanabe, *A characterization of $F$-regularity in terms of $F$-purity*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 227–245. MR 1015520 3.2.2

[Gab81] Ofer Gabber, *The integrability of the characteristic variety*, Amer. J. Math. 103 (1981), no. 3, 445–468. MR 618321 3.1.1

[Gab04] *Notes on some t-structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 711–734. MR 2099084 3.2.2

[Gal85] Andrè Galligo, *Some algorithmic questions on ideals of differential operators*, EUROCAL ’85, Vol. 2 (Linz, 1985), Lecture Notes in Comput. Sci., vol. 204, Springer, Berlin, 1985, pp. 413–421. MR 826576 3.1.1

[GGM85] A. Galligo, M. Granger, and Ph. Maisonobe, *$\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier (Grenoble) 35 (1985), no. 1, 1–48. MR 781776 4.1.1

[GGZ87] I. M. Gel’fand, M. I. Graev, and A. V. Zelevinsky, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR 295 (1987), no. 1, 14–19. MR 902936 4.3.2

[GKZ90] Israel M. Gel’fand, Mikhail M. Kapranov, and Andrei V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. 84 (1990), no. 2, 255–271. 4.3.2

[GLS98] R. García López and C. Sabbah, *Topological computation of local cohomology multiplicities*, Collect. Math. 49 (1998), no. 2-3, 317–324, Dedicated to the memory of Fernando Serrano. MR 1677136 4.4.1, 4.4.2

[GM88] Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR 932724 4.1.1

[GM92] J. P. C. Greenlees and J. P. May, *Derived functors of $I$-adic completion and local homology*, J. Algebra 149 (1992), no. 2, 438–453. MR 1172439 3.12

[GQS70] Victor W. Guillemin, Daniel Quillen, and Shlomo Sternberg, *The integrability of characteristics*, Comm. Pure Appl. Math. 23 (1970), no. 1, 39–77. MR 461597 3.1.1

[Gro66] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 95–103. MR 199194 4.2.1

[Gro67] *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. (1967), no. 32, 361. MR 238860 3.1

[Gro68] Alexander Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*, North-Holland Publishing Co., Amsterdam; Masson & Cie, Éditeur, Paris, 1968, Augmenté d’un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2. MR 0476737 2.1

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. 3.25
I. M. Gel’fand, A. V. Zelevinsky, and M. M. Kapranov, Hypergeometric functions and toric varieties, Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 12–26. MR 1011353

Robin Hartshorne, Residues and duality, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin- New York, 1966. MR 0222093

Robin Hartshorne, Local cohomology, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin-New York, 1967. MR 0224620 (document)

Robin Hartshorne, Varieties of small codimension in projective space, Bull. Amer. Math. Soc. 80 (1974), 1017–1032. MR 384816

Robin Hartshorne, Affine duality and cofiniteness, Invent. Math. 9 (1969/70), 145–164. MR 257096

M. Hellus, On the associated primes of Matlis duals of top local cohomology modules, Comm. Algebra 33 (2005), no. 11, 3997–4009. MR 2183976

M. Hellus, Finiteness properties of duals of local cohomology modules, Comm. Algebra 35 (2007), no. 11, 3590–3602. MR 2362672

Michael Hellus, Matlis duals of top local cohomology modules and the arithmetic rank of an ideal, Comm. Algebra 35 (2007), no. 4, 1421–1432. MR 2313677

Melvin Hochster and Craig Huneke, Tight closure, invariant theory, and the Briançon-Skoda theorem, J. Amer. Math. Soc. 3 (1990), no. 1, 31–116. MR 1017784

Melvin Hochster and Craig Huneke, Normal ideals in regular rings, J. Reine Angew. Math. 510 (1999), 63–82. MR 1696091

Melvin Hochster and Jack Jeffries, Faithfulness of top local cohomology modules in domains, Math. Res. Lett. 27 (2020), no. 6, 1755–1765. MR 4216603

Craig Huneke and Jee Koh, Cofiniteness and vanishing of local cohomology modules, Math. Proc. Cambridge Philos. Soc. 110 (1991), no. 3, 421–429. MR 1120477

Craig Huneke, Daniel Katz, and Thomas Marley, On the support of local cohomology, J. Algebra 322 (2009), no. 9, 3194–3211. MR 2567416

C. Huneke and G. Lyubeznik, On the vanishing of local cohomology modules, Invent. Math. 102 (1990), no. 1, 73–93. MR 1069240
Craig Huneke and Gennady Lyubeznik, Absolute integral closure in positive characteristic, Adv. Math. 210 (2007), no. 2, 498–504. MR 2303230.

Jun Horiuchi, Lance Edward Miller, and Kazuma Shimomoto, Deformation of F-injectivity and local cohomology, Indiana Univ. Math. J. 63 (2014), no. 4, 1139–1157, With an appendix by Karl Schwede and Anurag K. Singh. MR 3269325.

Melvin Hochster and Luis Núñez Betancourt, Support of local cohomology modules over hypersurfaces and rings with FFRT, Math. Res. Lett. 24 (2017), no. 2, 401–420. MR 3685277.

Daniel J. Hernández, Luis Núñez Betancourt, Felipe Pérez, and Emily E. Witt, Lyubeznik numbers and injective dimension in mixed characteristic, Trans. Amer. Math. Soc. 371 (2019), no. 11, 7533–7557. MR 3955527.

M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. of Math. (2) 96 (1972), 318–337. MR 304376.

Melvin Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), 1977, pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. MR 0441987.

Robin Hartshorne and Claudia Polini, Quasi-cyclic modules and coregular sequences, Math. Z. 299 (2021), no. 1-2, 123–138. MR 4311598.

Michael Hellus and Peter Schenzel, On cohomologically complete intersections, J. Amer. Math. Soc. 21 (2008), no. 1, 45–79. MR 2390499.

Michael Hellus and Jürgen Stückrad, Matlis duals of top local cohomology modules, Proc. Amer. Math. Soc. 136 (2008), no. 2, 489–498. MR 2358488.

Jen-Chieh Hsiao, D-module structure of local cohomology modules of toric algebras, Trans. Amer. Math. Soc. 364 (2012), no. 5, 2461–2478. MR 2888215.

Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, D-modules, perverse sheaves, and representation theory, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008, Translated from the 1995
Japanese edition by Takeuchi. MR 2357361 3.1.1, 3.1.1, 3.1.6, 3.1.2, 4.2.1, 4.8

[Hun92a] Craig Huneke, Problems on local cohomology, Free resolutions in commutative algebra and algebraic geometry (Sundance, UT, 1990), Res. Notes Math., vol. 2, Jones and Bartlett, Boston, MA, 1992, pp. 93–108. MR 1165320 2.1, 2.7, 2.2, 2.2

[Hun92b] ———, Uniform bounds in Noetherian rings, Invent. Math. 107 (1992), no. 1, 203–223. MR 1135470 2.3

[Hun00] ———, Lectures on local cohomology, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, Appendix 1 by Amelia Taylor, pp. 51–99. MR 2355770 (document) 2.2

[Hun07] ———, Test, multiplier and invariant ideals, Adv. Math. 287 (2016), 704–732. MR 3422690 3.17

[HZ18] Melvin Hochster and Wenliang Zhang, Content of local cohomology, parameter ideals, and robust algebras, Trans. Amer. Math. Soc. 370 (2018), no. 11, 7789–7814. MR 3852449 2.2

[ILL+07] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, Twenty-four hours of local cohomology, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR 2355715 4.10

[Ive86] Birger Iversen, Cohomology of sheaves, Universitext, Springer-Verlag, Berlin, 1986. MR 842190 2.1

[Jew94] Ken Jewell, Complements of sphere and subspace arrangements, Topology Appl. 56 (1994), no. 3, 199–214. MR 1269311 4.1.1

[Jon94] A. G. Jones, Rings of differential operators on toric varieties, Proc. Edinburgh Math. Soc. (2) 37 (1994), no. 1, 143–160. MR 1258039 3.1

[Kas83] M. Kashiwara, Vanishing cycle sheaves and holonomic systems of differential equations, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 134–142. MR 726425 (85e:58137) 4.3.3

[Kas95] Masaki Kashiwara, Algebraic study of systems of partial differential equations, Mém. Soc. Math. France (N.S.) (1995), no. 63, xiv+72. MR 1384226 3.1, 3.1.1

[Kas03] ———, D-modules and microlocal calculus, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003, Translated from the 2000 Japanese original by Mutsumi Saito, Iwanami Series in Modern Mathematics. MR 1943036 3.1.1

[Kas77] ———, B-functions and holonomic systems. Rationality of roots of B-functions, Invent. Math. 38 (1976/77), no. 1, 33–53. MR 430304 3.1.1

[Kat98] Mordechai Katzman, The complexity of Frobenius powers of ideals, J. Algebra 203 (1998), no. 1, 211–225. MR 1620654 2.3

[Kat02] ———, An example of an infinite set of associated primes of a local cohomology module, J. Algebra 252 (2002), no. 1, 161–166. MR 1922391 2.1

[Kat06] ———, The support of top graded local cohomology modules, Commutative algebra, Lect. Notes Pure Appl. Math., vol. 244, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 165–174. MR 2184796 2.1
Ken-ichiroh Kawasaki, *On the Lyubeznik number of local cohomology modules*, Bull. Nara Univ. Ed. Natur. Sci. 49 (2000), no. 2, 5–7. MR 1814657

Kazem Khashyarmanesh, *On the Matlis duals of local cohomology modules*, Arch. Math. (Basel) 88 (2007), no. 5, 413–418. MR 2316886

Mordechai Katzman, Gennady Lyubeznik, and Wenliang Zhang, *On the discreteness and rationality of $F$-jumping coefficients*, J. Algebra 322 (2009), no. 9, 3238–3247. MR 2567418

Mordechai Katzman, Linquan Ma, Ilya Smirnov, and Wenliang Zhang, *Discrete and $F$-module length of local cohomology modules*, Trans. Amer. Math. Soc. 370 (2018), no. 12, 8551–8580. MR 3864387

János Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287. MR 1492525 (99m:14033)

Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1990, Translated from the French by Christian Houzel. MR 1074006

Mordechai Katzman, Karl Schwede, Anurag K. Singh, and Wenliang Zhang, *Rings of Frobenius operators*, Math. Proc. Cambridge Philos. Soc. 157 (2014), no. 1, 151–167. MR 3211813

Mordechai Katzman and Wenliang Zhang, *The support of local cohomology modules*, Int. Math. Res. Not. IMRN (2018), no. 23, 7137–7155. MR 3920344

Joseph Lipman, *Lectures on local cohomology and duality*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 39–89. MR 1888195

András C. Lőrincz and Claudiu Raicu, *Iterated local cohomology groups and Lyubeznik numbers for determinantal rings*, Algebra Number Theory 14 (2020), no. 9, 2533–2569. MR 4172715

András C. Lőrincz, Claudiu Raicu, and Jerzy Weyman, *Equivariant $D$-modules on binary cubic forms*, Comm. Algebra 47 (2019), no. 6, 2457–2487. MR 3957110

Gennady Lyubeznik and Karen E. Smith, *On the commutation of the test ideal with localization and completion*, Trans. Amer. Math. Soc. 353 (2001), no. 8, 3149–3180. MR 1826062

Gennady Lyubeznik, Anurag K. Singh, and Uli Walther, *Local cohomology modules supported at determinantal ideals*, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 11, 2545–2578. MR 3562351
[LW19] András C. Lőrincz and Uli Walther, *On categories of equivariant $D$-modules*, Adv. Math. 351 (2019), 429–478. MR 3952575 3.1.2

[Lyu18] Gennady Lyubeznik and Tuğba Yıldırım, *On the Matlis duals of local cohomology modules*, Proc. Amer. Math. Soc. 146 (2018), no. 9, 3715–3720. MR 3825827 4.4

[Lyn12] Laura R. Lynch, *Annihilators of top local cohomology*, Comm. Algebra 40 (2012), no. 2, 542–551. MR 2889480 2.3

[Lyu85] Gennady Lyubeznik, *Some algebraic sets of high local cohomological dimension in projective space*, Proc. Amer. Math. Soc. 95 (1985), no. 1, 9–10. MR 796437 2.2

[Lyu89], *A survey of problems and results on the number of defining equations*, Commutative algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 375–390. MR 1015529 4.1

[Lyu92], *The number of defining equations of affine algebraic sets*, Amer. J. Math. 114 (1992), no. 2, 413–463. MR 1156572 4.1

[Lyu93a], *´Etale cohomological dimension and the topology of algebraic varieties*, Ann. of Math. (2) 137 (1993), no. 1, 71–128. MR 1200077 2.2

[Lyu93b], *Finiteness properties of local cohomology modules (an application of $D$-modules to commutative algebra)*, Invent. Math. 113 (1993), no. 1, 41–55. MR 1223223 2.1, 2.1, 2.3, 3.1, 3.1.1, 4.4, 4.4, 4.4

[Lyu97], *F-modules: applications to local cohomology and $D$-modules in characteristic $p > 0$*, J. Reine Angew. Math. 491 (1997), 65–130. MR 1476089 2.1, 3.1, 3.2.1, 1, 2, 3, 4, 5, 3.2.2

[Lyu00a], *Finiteness properties of local cohomology modules: a characteristic-free approach*, J. Pure Appl. Algebra 151 (2000), no. 1, 43–50. MR 1770642 2

[Lyu00b], *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case*, Comm. Algebra 28 (2000), no. 12, 5867–5882, Special issue in honor of Robin Hartshorne. MR 1808608 2.1, 2.9

[Lyu00c], *Injective dimension of $D$-modules: a characteristic-free approach*, J. Pure Appl. Algebra 149 (2000), no. 2, 205–212. MR 1757731 1

[Lyu02], *A partial survey of local cohomology*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 121–154. MR 1888197 (document), 2.2, 4.1, 4.2.4

[Lyu06a], *On some local cohomology invariants of local rings*, Math. Z. 254 (2006), no. 3, 627–640. MR 2244370 4.4, 4.4.1

[Lyu06b], *On the vanishing of local cohomology in characteristic $p > 0$*, Compos. Math. 142 (2006), no. 1, 207–221. MR 2197409 2.2, 3.2.2, 3.2.3, 3.49, 3.2.3

[Lyu07], *On some local cohomology modules*, Adv. Math. 213 (2007), no. 2, 621–643. MR 2326204 2.2

[Lyu11], *A characteristic-free proof of a basic result on $D$-modules*, J. Pure Appl. Algebra 215 (2011), no. 8, 2019–2023. MR 2776441 2

[Ma14] Linquan Ma, *Finiteness properties of local cohomology for $F$-pure local rings*, Int. Math. Res. Not. IMRN (2014), no. 20, 5489–5509. MR 3271179 3.2.2

[Ma15], *F-injectivity and Buchsbaum singularities*, Math. Ann. 362 (2015), no. 1-2, 25–42. MR 3343868 3.2.2

[Mal75] B. Malgrange, *Le polynôme de Bernstein d’une singularité isolée*, Fourier integral operators and partial differential equations (Colloq. Internat.,
Univ. Nice, Nice, 1974), Springer, Berlin, 1975, pp. 98–119. Lecture Notes in Math., Vol. 459. MR 0419827 (54 #7845) 3.1.1, 4.3.3

[Mal83] ______, Polynômes de Bernstein-Sato et cohomologie évanescente, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 243–267. MR 737934 (86f:58148) 4.3.3

[Mar01] Thomas Marley, The associated primes of local cohomology modules over rings of small dimension, Manuscripta Math. 104 (2001), no. 4, 519–525. MR 1836111 2.1

[Mil21] J.S Milne, Lectures on étale cohomology, Version 2.21, https://www.jmilne.org/math/CourseNotes/lec.html 4.2.4

[Mil68] John Milnor, Singular points of complex hypersurfaces, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968. MR 0239612 (39 #969) 4.3.3

[Mil80] James S. Milne, Étale cohomology, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980. MR 559531 4.2.4

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, Homological methods for hypergeometric families, J. Amer. Math. Soc. 18 (2005), no. 4, 919–941 (electronic). 4.3.2

[Moh85] T. T. Moh, Set-theoretic complete intersections, Proc. Amer. Math. Soc. 94 (1985), no. 2, 217–220. MR 784166 4.1.1

[Mon13] Josep Álvarez Montaner, Local cohomology modules supported on monomial ideals, Monomial ideals, computations and applications, Lecture Notes in Mathematics, vol. 2083, Springer, Heidelberg, 2013, pp. 109–178. MR 3184122 4.1.1

[MQ18] Linquan Ma and Pham Hung Quy, Frobenius actions on local cohomology modules and deformation, Nagoya Math. J. 232 (2018), 55–75. MR 3866500 3.2.2

[MS97] V. B. Mehta and V. Srinivas, A characterization of rational singularities, Asian J. Math. 1 (1997), no. 2, 249–271. MR 1491985 3.2.2

[MS12] Waqas Mahmood and Peter Schenzel, On invariants and endomorphism rings of certain local cohomology modules, J. Algebra 372 (2012), 56–67. MR 2990000 4.4

[MSTW01] Mircea Mustaţă, Gregory G. Smith, Harrison Tsai, and Uli Walther, D-modules on smooth toric varieties, J. Algebra 240 (2001), no. 2, 744–770. MR 1841355 3.1

[MSV14] Lance Edward Miller, Anurag K. Singh, and Matteo Varbaro, The F-pure threshold of a determinantal ideal, Bull. Braz. Math. Soc. (N.S.) 45 (2014), no. 4, 767–775. MR 3296192 3.17

[MSW21] Linquan Ma, Anurag K. Singh, and Uli Walther, Koszul and local cohomology, and a question of Dutta, Math. Z. 298 (2021), no. 1-2, 697–711. MR 4257105 4

[Mus87] Ian M. Musson, Rings of differential operators on invariant rings of tori, Trans. Amer. Math. Soc. 303 (1987), no. 2, 805–827. MR 902799 3.1

[MZ14] Linquan Ma and Wenliang Zhang, Eulerian graded D-modules, Math. Res. Lett. 21 (2014), no. 1, 149–167. MR 3247047 3.1.3, 3.1.3, 3.3

[Nan04] Philibert Nang, D-modules associated to the determinantal singularities, Proc. Japan Acad. Ser. A Math. Sci. 80 (2004), no. 5, 74–78. MR 2062805 3.1.2

[Nan08] ______, On a class of holonomic D-modules on \( M_n(\mathbb{C}) \) related to the action of \( GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \), Adv. Math. 218 (2008), no. 3, 635–648. MR 2414315 3.1.2
On the classification of regular holonomic $D$-modules on skew-symmetric matrices, J. Algebra 356 (2012), 115–132. MR 2891125

Masao Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, Proc. Cambridge Philos. Soc. 59 (1963), 269–275. MR 146212

Luis Núñez Betancourt, Local cohomology modules of polynomial or power series rings over rings of small dimension, Illinois J. Math. 57 (2013), no. 1, 279–294. MR 3224571

Masao Narita, A note on the coefficients of Hilbert characteristic functions in semi-regular local rings, Proc. Cambridge Philos. Soc. 59 (1963), 269–275. MR 146212

Luis Núñez Betancourt, Sandra Spiroff, and Emily E. Witt, Connectedness and Lyubeznik numbers, Int. Math. Res. Not. IMRN (2019), no. 13, 4233–4259. MR 3978438

Luis Núñez Betancourt, Sandra Spiroff, and Emily E. Witt, Connectedness and Lyubeznik numbers, Int. Math. Res. Not. IMRN (2019), no. 13, 4233–4259. MR 3978438

Luis Núñez Betancourt, Emily E. Witt, and Wenliang Zhang, A survey on the Lyubeznik numbers, Mexican mathematicians abroad: recent contributions, Contemp. Math., vol. 657, Amer. Math. Soc., Providence, RI, 2016, pp. 137–163. MR 3466449

Toshinori Oaku, An algorithm of computing $b$-functions, Duke Math. J. 87 (1997), no. 1, 115–132. MR 1440065

Arthur Ogus, Local cohomological dimension of algebraic varieties, Ann. of Math. (2) 98 (1973), 327–365. MR 506248

Toshinori Oaku and Nobuki Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via $D$-module computation, J. Pure Appl. Algebra 139 (1999), no. 1-3, 201–233, Effective methods in algebraic geometry (Saint-Malo, 1998). MR 1700544

Toshinori Oaku and Nobuki Takayama, An algorithm for de Rham cohomology groups of the complement of an affine variety via $D$-module computation, J. Pure Appl. Algebra 139 (1999), no. 1-3, 201–233, Effective methods in algebraic geometry (Saint-Malo, 1998). MR 1700544

Vaibhav Pandey, Cohomological dimension of ideals defining Veronese subrings, Proc. Amer. Math. Soc. 149 (2021), no. 4, 1387–1393. MR 4242298

Ruud Pellikaan, Projective resolutions of the quotient of two ideals, Nederl. Akad. Wetensch. Indag. Math. 50 (1988), no. 1, 65–84. MR 934475

Michael Perlman, Equivariant $D$-modules on $2 \times 2 \times 2$ hypermatrices, J. Algebra 544 (2020), 391–416. MR 4027737

Thomas Polstra and Pham Hung Quy, Nilpotence of Frobenius actions on local cohomology and Frobenius closure of ideals, J. Algebra 529 (2019), 196–225. MR 3938852

C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119. MR 374130

Tony J. Puthenpurakal and Jyoti Singh, On derived functors of graded local cohomology modules, Math. Proc. Cambridge Philos. Soc. 167 (2019), no. 3, 549–565. MR 4015650

Thomas Polstra and Austyn Simpson, $F$-purity deforms in $Q$-gorenstein rings, arXiv:2009.13444, 2020.

Tony J. Puthenpurakal, On injective resolutions of local cohomology modules, Illinois J. Math. 58 (2014), no. 3, 709–718. MR 3395959

Tony J. Puthenpurakal, On injective resolutions of local cohomology modules, Illinois J. Math. 58 (2014), no. 3, 709–718. MR 3395959

Thomas Polstra and Austyn Simpson, $F$-purity deforms in $Q$-gorenstein rings, arXiv:2009.13444, 2020.

Tony J. Puthenpurakal, On injective resolutions of local cohomology modules, Illinois J. Math. 58 (2014), no. 3, 709–718. MR 3395959
[QS17] Pham Hung Quy and Kazuma Shimomoto, *F-injectivity and Frobenius closure of ideals in Noetherian rings of characteristic* $p > 0$, Adv. Math. **313** (2017), 127–166. MR 3649223

[Rai16] Claudiu Raicu, *Characters of equivariant D-modules on spaces of matrices*, Compos. Math. **152** (2016), no. 9, 1935–1965. MR 3568944

[RW14] Claudiu Raicu and Jerzy Weyman, *Local cohomology with support in generic determinantal ideals*, Algebra Number Theory **8** (2014), no. 5, 1231–1257. MR 3263142

[RW16] __________, *Local cohomology with support in ideals of symmetric minors and Pfaffians*, J. Lond. Math. Soc. (2) **94** (2016), no. 3, 709–725. MR 3614925

[RWW14] Claudiu Raicu, Jerzy Weyman, and Emily E. Witt, *Local cohomology with support in ideals of maximal minors and sub-maximal Pfaffians*, Adv. Math. **250** (2014), 596–610. MR 3122178

[Rei14] Thomas Reichelt, *Laurent Polynomials, GKZ-hypergeometric Systems and Mixed Hodge Modules*, Compositio Mathematica (150) (2014), 911–941.

[RS20] Thomas Reichelt and Christian Sevenheck, *Hypergeometric Hodge modules*, Algebr. Geom. **7** (2020), no. 3, 263–345.

[RSW21] Thomas Reichelt, Morihiko Saito, and Uli Walther, *Dependence of Lyubeznik numbers of cones of projective schemes on projective embeddings*, Selecta Math. (N.S.) **27** (2021), no. 1, Paper No. 6, 22. MR 4202748

[RSSW21] Thomas Reichelt, Mathias Schulze, Christian Sevenheck, and Uli Walther, *Algebraic aspects of hypergeometric differential equations*, Beitr. Algebra Geom. **62** (2021), no. 1, 137–203. MR 4249859

[RW] Thomas Reichelt and Uli Walther, *Weight filtrations on GKZ-systems*, Preprint arXiv:1809.04247.

[RWZ21] Thomas Reichelt, Uli Walther, and Wenliang Zhang, *On Lyubeznik type invariants*, arXiv:2106.04457, Topology Appl., to appear.

[Rob76] Paul Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), no. 1, 103–106. MR 399075

[RS79] Hartmut Roloff and Jürgen Stückrad, *Bemerkungen über Zusammenhangseigenschaften und mengentheoretische Darstellung projektiver algebraischer Mannigfaltigkeiten*, Wiss. Beitr. Martin-Luther-Univ. Halle-Wittenberg M **12** (1979), 125–131, Beiträge zur Algebra und Geometrie, 8. MR 571359

[Rot09] Joseph J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009. MR 2455920 (document)

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR 586450

[Sai80a] Morihiko Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333.

[Sai90] __________, *On b-function, spectrum and multiplier ideals*, Algebraic analysis and around, Adv. Stud. Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, pp. 355–379. MR 2499561

[Sai80b] __________, *Bernstein–Sato polynomials for projective hypersurfaces with weighted homogeneous isolated singularities*, Preprint arXiv:1609.04801, 2016.

[Sai16a] __________, *Bernstein-Sato polynomials of hyperplane arrangements*, Selecta Math. (N.S.) **22** (2016), no. 4, 2017–2057. MR 3573952
[Sai17] ______, Roots of Bernstein–Sato polynomials of certain homogeneous polynomials with two-dimensional singular loci, Pure Appl. Math. Q. 16 (2020), no. 4, 1219–1280. MR 4180246 4.3.3

[Sch82a] Peter Schenzel, Cohomological annihilators, Math. Proc. Cambridge Philos. Soc. 91 (1982), no. 3, 345–350. MR 654081 2.3

[Sch82b] ______, Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe, Lecture Notes in Mathematics, vol. 907, Springer-Verlag, Berlin-New York, 1982. With an English summary. MR 654151 (document) 1.4

[Sch98] ______, On the use of local cohomology in algebra and geometry, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 241–292. MR 1648667 (document) 1.4

[Sch07] Karl Schwede, A simple characterization of Du Bois singularities, Compos. Math. 143 (2007), no. 4, 813–828. MR 2339829 3.2.2

[Sch09a] Peter Schenzel, On connectedness and indecomposibility of local cohomology modules, Manuscripta Math. 128 (2009), no. 3, 315–327. MR 2481047 4.4

[Sch09b] ______, On endomorphism rings and dimensions of local cohomology modules, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1315–1322. MR 2465654 4.1.2

[Sch10] Peter Schenzel, Matlis duals of local cohomology modules and their endomorphism rings, Arch. Math. (Basel) 95 (2010), no. 2, 115–123. MR 2674247 4.1.2

[Sch11a] ______, On the structure of the endomorphism ring of a certain local cohomology module, J. Algebra 344 (2011), 229–245. MR 2831938 4.1.2

[Sch11b] Karl Schwede, Test ideals in non-$\mathbb{Q}$-Gorenstein rings, Trans. Amer. Math. Soc. 363 (2011), no. 11, 5925–5941. MR 2817415 3.2.2

[Ser55] Jean-Pierre Serre, Faisceaux algébriques cohérents, Ann. of Math. (2) 61 (1955), 197–278. MR 68874 1, 4.2.1

[Ser56] ______, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier (Grenoble) 6 (1955/56), 1–42. MR 82175 4.2.1

[Sha69] Rodney Y. Sharp, The Cousin complex for a module over a commutative Noetherian ring, Math. Z. 112 (1969), 340–356. MR 263800 1.4

[Sha77] ______, Local cohomology and the Cousin complex for a commutative Noetherian ring, Math. Z. 153 (1977), no. 1, 19–22. MR 442062 1.4

[Sha07a] ______, Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure, Trans. Amer. Math. Soc. 359 (2007), no. 9, 4237–4258. MR 2309183 3.2.2, 3.2.2, 3.2.2

[Sha07b] ______, On the Hartshorne-Speiser-Lyubeznik theorem about Artinian modules with a Frobenius action, Proc. Amer. Math. Soc. 135 (2007), no. 3, 665–670. MR 2262861 3.2.2

[Sin99a] Anurag K. Singh, Deformation of $F$-purity and $F$-regularity, J. Pure Appl. Algebra 140 (1999), no. 2, 137–148. MR 1693967 3.41

[Sin99b] ______, $F$-regularity does not deform, Amer. J. Math. 121 (1999), no. 4, 919–929. MR 1704481 3.41

[Sin00] ______, p-torsion elements in local cohomology modules, Math. Res. Lett. 7 (2000), no. 2-3, 165–176. MR 1764314 2.1

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, Microfunctions and pseudo-differential equations, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), 1973, pp. 265–529. Lecture Notes in Math., Vol. 287. MR 0420735 3.1.1
Karen E. Smith, *F*-rational rings have rational singularities*, Amer. J. Math. 119 (1997), no. 1, 159–180. MR 1428062

Robert Speiser, *Projective varieties of low codimension in characteristic p > 0*, Trans. Amer. Math. Soc. 240 (1978), 329–343. MR 491703

Anurag K. Singh and Irena Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. (2004), no. 33, 1703–1733. MR 2058025

Peter Schenzel and Anne-Marie Simon, *Completion, Čech and local homology and cohomology*, Springer Monographs in Mathematics, Springer, Cham, 2018, Interactions between them. MR 3838396

Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. MR 1734566

Bernd Sturmfels and Nobuki Takayama, *Gröbner bases and hypergeometric functions*, Gröbner bases and applications (Linz, 1998), London Math. Soc. Lecture Note Ser., vol. 251, Cambridge Univ. Press, Cambridge, 1998, pp. 246–258. MR 1708882

Jan Stienstra, *Resonant hypergeometric systems and mirror symmetry*, Integrable systems and algebraic geometry (Kobe/Kyoto, 1997 ), World Sci. Publ., River Edge, NJ, 1998, pp. 412–452. MR 1734570

Thomas Schmitt and Wolfgang Vogel, *Note on set-theoretic intersections of subvarieties of projective space*, Math. Ann. 245 (1979), no. 3, 247–253. MR 533433

Jürgen Stuckrad and Wolfgang Vogel, *Buchsbaum rings and applications*, Springer-Verlag, Berlin, 1986, An interaction between algebra, geometry and topology. MR 881220

Anurag K. Singh and Uli Walther, *On the arithmetic rank of certain Segre products*, Commutative algebra and algebraic geometry, Contemp. Math., vol. 390, Amer. Math. Soc., Providence, RI, 2005, pp. 147–155. MR 2187332

Mathias Schulze and Uli Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*, Duke Math. J. 142 (2008), no. 3, 465–509. MR 2378800
85

[SW09] Mathias Schulze and Uli Walther, *Hypergeometric D-modules and twisted Gauf-Manin systems*, J. Algebra 322 (2009), no. 9, 3392–3409.

[SW11] Amurag K. Singh and Uli Walther, *Bockstein homomorphisms in local cohomology*, J. Reine Angew. Math. 655 (2011), 147–164. MR 2806109

[SW20] *On a conjecture of Lynch*, Comm. Algebra 48 (2020), no. 6, 2681–2682. MR 4107600

[Swi15] Nicholas Switala, *Lyubeznik numbers for nonsingular projective varieties*, Bull. Lond. Math. Soc. 47 (2015), no. 1, 1–6. MR 3312957

[Swi17a] *On the de Rham homology and cohomology of a complete local ring in equicharacteristic zero*, Compos. Math. 153 (2017), no. 10, 2075–2146. MR 3705285

[Swi17b] *On the de Rham homology and cohomology of a complete local ring in equicharacteristic zero*, Compos. Math. 153 (2017), no. 10, 2075–2146. MR 3705285

[SZ18] Nicholas Switala and Wenliang Zhang, *Duality and de Rham cohomology for graded D-modules*, Adv. Math. 340 (2018), 1141–1165. MR 3886190

[SZ19] *A dichotomy for the injective dimension of F-finite F-modules and holonomic D-modules*, Comm. Algebra 47 (2019), no. 6, 2525–2539. MR 3957114

[Tor09] Tristan Torrelli, *Intersection homology D-module and Bernstein polynomials associated with a complete intersection*, Publ. Res. Inst. Math. Sci. 45 (2009), no. 2, 645–660. MR 2510514

[Tra06] William N. Traves, *Differential operators on orbifolds*, J. Symbolic Comput. 41 (2006), no. 12, 1295–1308. MR 2271326

[Tri97] J. R. Tripp, *Differential operators on Stanley-Reisner rings*, Trans. Amer. Math. Soc. 349 (1997), no. 6, 2507–2523. MR 1376559

[TT08] Shunsuke Takagi and Ryo Takahashi, *D-modules over rings with finite F-representation type*, Math. Res. Lett. 15 (2008), no. 3, 563–581. MR 2407232

[TT18] Shunsuke Takagi and Kei-Ichi Watanabe, *F-singularities: applications of characteristic p methods to singularity theory* [translation of MR3135334], Sugaku Expositions 31 (2018), no. 1, 1–42. MR 3784697

[Var12] Matteo Varbaro, *On the arithmetical rank of certain Segre embeddings*, Trans. Amer. Math. Soc. 364 (2012), no. 10, 5091–5109. MR 2931323

[Var13] *Cohomological and projective dimensions*, Compos. Math. 149 (2013), no. 7, 1203–1210. MR 3078644

[Var19] *Connectivity of hyperplane sections of domains*, Comm. Algebra 47 (2019), no. 6, 2540–2547. MR 3957115

[vdB] Michel van den Bergh, *Some generalities on quasi-coherent $\mathcal{O}_X$ and $\mathcal{D}_X$-modules*, Preprint, https://hardy.uhasselt.be/personal/vdbergh/Publications/Geq.ps.

[vdB99] *Local cohomology of modules of covariants*, Adv. Math. 144 (1999), no. 2, 161–220. MR 1695237

[vdB85] A. van den Essen, *The cokernel of the operator $\frac{\partial}{\partial x_i}$ acting on a $\mathcal{D}_n$-module*, II, Compos. Math. 56 (1985), no. 2, 259–269. MR 786497

[Vog71] W. Vogel, *Eine Bemerkung über die Anzahl von Hyperflächen zur Darstellung algebraischer Varietäten*, Monatsb. Deutsch. Akad. Wiss. Berlin 13 (1971), 629–633. MR 325630

[Vra00] Adela Vraciu, *Local cohomology of Frobenius images over graded affine algebras*, J. Algebra 228 (2000), no. 1, 347–356. MR 1760968
Duco van Straten and Thorsten Warnt, *Gorenstein-duality for one-dimensional almost complete intersections—with an application to non-isolated real singularities*, Math. Proc. Cambridge Philos. Soc. **158** (2015), no. 2, 249–268. MR 3310244

Uli Walther, *Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties*, J. Pure Appl. Algebra **139** (1999), no. 1-3, 303–321, Effective methods in algebraic geometry (Saint-Malo, 1998). MR 1700548

Uli Walther, *Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties*, J. Pure Appl. Algebra **139** (1999), no. 1-3, 303–321, Effective methods in algebraic geometry (Saint-Malo, 1998). MR 1700548

Uli Walther, *Algorithmic computation of de Rham cohomology of complements of complex affine varieties*, J. Symbolic Comput. **29** (2000), no. 4-5, 795–839, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). MR 1769667

Uli Walther, *Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements*, Compos. Math. **141** (2005), no. 1, 121–145. MR 2099772 (2005k:32030)

Uli Walther, *Survey on the D-module fs*, Commutative algebra and noncommutative algebraic geometry. Vol. I, Math. Sci. Res. Inst. Publ., vol. 67, Cambridge Univ. Press, New York, 2015, With an appendix by Anton Leykin, pp. 391–430. MR 3525478

Uli Walther, *The Jacobian module, the Milnor fiber, and the D-module generated by fs*, Invent. Math. **207** (2017), no. 3, 1239–1287. MR 3608290

Botong Wang, *Lyubeznik numbers of irreducible projective varieties depend on the embedding*, Proc. Amer. Math. Soc. **148** (2020), no. 5, 2091–2096. MR 4078092

Zhao Yan, *Minimal resultant systems*, J. Algebra **216** (1999), no. 1, 105–123. MR 1694582

Zhao Yan, *An étale analog of the Goresky-MacPherson formula for subspace arrangements*, J. Pure Appl. Algebra **146** (2000), no. 3, 305–318. MR 1742346

Kohji Yanagawa, *Stanley-Reisner rings, sheaves, and Poincaré-Veirdard duality*, Math. Res. Lett. **10** (2003), no. 5-6, 635–650. MR 2024721

Wenliang Zhang, *On the highest Lyubeznik number of a local ring*, Compos. Math. **143** (2007), no. 1, 82–88. MR 2295196

Wenliang Zhang, *A note on the growth of regularity with respect to Frobenius*, arXiv:1512.00049, 2015.

Wenliang Zhang, *On injective dimension of F-finite F-modules and holonomic D-modules*, Bull. Lond. Math. Soc. **49** (2017), no. 4, 593–603. MR 3725482
[Zha21a] On asymptotic socle degrees of local cohomology modules, J. Pure Appl. Algebra 225 (2021), no. 12, 106789. MR 4260035

[Zha21b] The second vanishing theorem for local cohomology modules, arXiv:2102.12545, submitted.

[Zho98] Caijun Zhou, Higher derivations and local cohomology modules, J. Algebra 201 (1998), no. 2, 363–372. MR 1612378

[Zho06] Uniform annihilators of local cohomology, J. Algebra 305 (2006), no. 1, 585–602. MR 2264146

[Zho07] Uniform annihilators of local cohomology of excellent rings, J. Algebra 315 (2007), no. 1, 286–300. MR 2344347

U. Walther: Purdue University, Dept. of Mathematics, 150 N. University St., West Lafayette, IN 47907, USA

Email address: walther@math.purdue.edu

W. Zhang: Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL 60607, USA

Email address: wlzhang@uic.edu