Measure upper bounds of nodal sets of Robin eigenfunctions

Fang Liu · Long Tian · Xiaoping Yang

Received: 13 January 2022 / Accepted: 13 November 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
In this paper, we will establish the upper bounds of the Hausdorff measure of nodal sets of eigenfunctions with the Robin boundary conditions, i.e.,
\[
\begin{aligned}
\Delta u + \lambda u &= 0, \quad \text{in } \Omega, \\
u_v + \mu u &= 0, \quad \text{on } \partial\Omega,
\end{aligned}
\]
where the domain \(\Omega \subseteq \mathbb{R}^n\), \(u_v\) is the derivative of \(u\) along the outer normal direction on \(\partial\Omega\). We will show that, if \(\Omega\) is bounded and analytic, and the corresponding eigenvalue \(\lambda\) is large enough, then the measure upper bounds for the nodal sets of eigenfunctions are \(C\sqrt{\lambda}\), where \(C\) is a positive constant depending only on \(n\) and \(\Omega\) but not on \(\mu\). We also show that, if \(\partial\Omega\) is \(C^\infty\) smooth and \(\partial\Omega \setminus \Gamma\) is piecewise analytic, where \(\Gamma \subseteq \partial\Omega\) is a union of some \(n-2\) dimensional submanifolds of \(\partial\Omega\), \(\mu > 0\), and \(\lambda\) is large enough, then the corresponding measure upper bounds for the nodal sets of \(u\) are \(C\left(\sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha}\right)\) for any \(\alpha \in (0, 1)\), where \(C\) is a positive constant depending on \(\alpha\), \(n\), \(\Omega\) and \(\Gamma\), and \(c\) is a positive constant depending only on \(n\).

Keywords
Nodal set · Doubling index · Iteration procedure

Mathematics Subject Classification 58E10 · 35J30

This work is supported by National Natural Science Foundation of China (Nos. 12071219, 12141104 and 12090023).

Long Tian
tianlong19850812@163.com

Fang Liu
sdqdlf78@126.com

Xiaoping Yang
xpyang@nju.edu.cn

1 School of Mathematics and Statistics, Nanjing University of Science and Technology, Nanjing 210094, China
2 Department of Mathematics, Nanjing University, Nanjing 210093, China
1 Introduction

In this paper, we focus on the following eigenfunctions with Robin boundary condition

\[
\begin{align*}
\triangle u + \lambda u &= 0, \quad \text{in} \quad \Omega, \\
u v + \mu u &= 0, \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

(1.1)

where \( \Omega \subseteq \mathbb{R}^n \) is a \( C^\infty \) bounded domain, \( \mu \) is a constant, \( v \) is the outer unit normal vector to \( \partial \Omega \), \( u_v \) is the derivative of \( u \) along the direction of \( v \), and \( \lambda > 0 \) is the corresponding eigenvalue. The nodal set of \( u \) means the zero level set of \( u \), i.e., the set \( \{ x \in \Omega \mid u(x) = 0 \} \).

In [22], Yau conjectured that measures of nodal sets of the eigenfunctions on compact \( C^\infty \) Riemannian manifolds without boundary are comparable to \( \sqrt{\lambda} \). In [8], Donnelly and Fefferman proved Yau’s conjecture in the real analytic case. For the nonanalytic case, in [7], Dong showed that an upper bound for the measures of nodal sets of eigenfunctions for the two-dimensional case was \( C \lambda^{\frac{3}{4}} \) and proposed an interesting argument to deal with this problem. Logunov [17] improved the upper bound for the measures of nodal sets of eigenfunctions for any dimensional compact Riemannian manifolds to \( C \lambda^{\alpha} \) for some constant \( \alpha > \frac{1}{2} \). In 1991, Lin in [14] investigated the measure estimates of nodal sets of solutions to uniformly linear elliptic equations of second order with analytic coefficients by using the frequency function which was introduced in [1] in 1979 and also gave the measure upper bound for the nodal sets of eigenfunctions for the analytic case.

The lower bound of the conjecture for two-dimensional surfaces was proven by Brüning [4] and by Yau independently. In the past decades, there has been a lot of work concerning this problem. Sogge and Zelditch [19] proved that a lower bound for the measures of nodal sets of eigenfunctions on compact \( C^\infty \) Riemannian manifolds was \( C \lambda^{(7-3n)/8} \). In [6], Colding and Minicozzi showed that such a lower bound could be improved to be \( C \lambda^{(3-n)/4} \). Recently, A. Logunov in [17] proved the Yau’s conjecture of the lower bound. For the related research work, we refer, for example, [3, 5, 12, 18, 20, 21, 23].

In [13], Kukavica studied a class of general elliptic linear operator of order \( 2m \) and proved that \( C \lambda^{1 \over 2m} \) is the upper measure bound for the nodal set of an eigenfunction \( u \) satisfying the boundary condition \( B_j u = 0 \) on \( \partial \Omega \), where \( B_j \) \( (j = 1, 2, ..., m - 1) \) is a linear boundary differential operator, provided that \( \Omega \) is a bounded, analytic domain in \( \mathbb{R}^n \). In fact, the conclusion indicates that a measure upper bound for the nodal set of an eigenfunction \( u \) with the Robin boundary condition is also \( C \lambda^{1 \over 2m} \), but one only knows that the corresponding constant \( C \) depends on \( n, \Omega \) and \( \mu \). In [2], Ariturk gave some lower bounds for measures of nodal sets of eigenfunctions on smooth Riemannian manifolds with Dirichlet or Neumann boundary conditions.

Note that for the eigenvalue problem (1.1) when \( \mu \to 0 \), it tends to be the Neumann boundary condition; and when \( \mu \to \infty \), it tends to be the Dirichlet boundary condition. Thus one may expect that the upper bound for the measure of the nodal set of the eigenfunction \( u \) is also \( C \sqrt{\lambda} \), where \( C \) is independent of \( \mu \), provided that \( \Omega \) is a bounded analytic domain. In this paper, we will first show that this is true. More precisely, we have the following result.

**Theorem 1.1** If \( \Omega \) is a bounded analytic domain in \( \mathbb{R}^n \), then the upper bound for the Hausdorff measure of the nodal set of \( u \), the solution of the problem (1.1), is

\[
\mathcal{H}^{n-1} \left( \{ x \in \Omega \mid u(x) = 0 \} \right) \leq C \sqrt{\lambda},
\]

(1.2)

where \( C \) is a positive constant depending on \( n \) and \( \Omega \), but independent of \( \mu \).

For the non-analytic case, we obtain the following conclusion.
Theorem 1.2 Suppose that $\Omega$ is a bounded $C^\infty$ domain in $\mathbb{R}^n$ and $\partial \Omega \setminus \Gamma$ is piecewise analytic, where $\Gamma \subseteq \partial \Omega$ is a finite union of some $(n-2)$ dimensional manifolds. Also suppose that $u$ is the solution of the problem (1.1). If $\lambda$ is large enough and $\mu > 0$, then for any $\alpha \in (0, 1)$,

$$H^{\alpha-1}(\{x \in \Omega \mid u(x) = 0 \}) \leq C \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right),$$

where $C$ is a positive constant depending only on $n$, $\Omega$, $\Gamma$ and $\alpha$, and $c$ is a positive constant depending only on $n$.

The rest of this paper is organized as follows. In Sect. 2, we consider the case that $\partial \Omega$ is analytic. We first show that the upper bound for the Hausdorff measure of the nodal set of $u$ is $C(\sqrt{\lambda} + |\mu|)$. Then we show that for $|\mu|$ large enough, the corresponding upper bound is $C\sqrt{\lambda}$. We first extend the eigenfunction $u$ into a neighborhood of $\Omega$. Then we establish two different estimates for the extended function $u$ for small and large $|\mu|$ respectively. Finally, we get the upper Hausdorff measure of the nodal set of $u$ in $\Omega$. In Sect. 3, we focus on the case that $\partial \Omega$ is $C^\infty$ and piecewise analytic, but not global analytic. By using the iteration argument developed in [21], we first give the upper bound for the doubling index of $u$ away from the non-analytic part $\Gamma$, and further control the doubling index near the non-analytic part $\Gamma$. We would like to point out that such an upper bound of the doubling index probably goes to infinity when the center of the doubling index tends to $\Gamma$. Fortunately, with the help of the fact the dimension of $\Gamma$ is $n - 2$, we can control the upper bound of the measure of the nodal set and obtain the desired result.

2 The analytic case

In order to get a measure upper bound for the nodal set of $u$, we first need to extend $u$ into a neighborhood of $\Omega$.

Lemma 2.1 Let $u$ be an eigenfunction of (1.1) and $\lambda$ is the corresponding eigenvalue. Suppose that $\partial \Omega$ is analytic and $\lambda$ is large enough. Then $u$ can be analytically extended into $\Omega_\delta$, where $\Omega_\delta$ denotes the $\delta$ neighborhood of $\Omega$ for some $\delta > 0$ depending only on $n$ and $\Omega$, such that

$$\|u\|_{L^\infty(\Omega_\delta)} \leq e^{C\sqrt{\lambda}}\|u\|_{L^\infty(\Omega)},$$

where $C$ is a positive constants depending only on $n$ and $\Omega$.

Proof By the standard elliptic estimates, we first know that

$$\|u\|_{W^{k,2}(\Omega)} \leq C^k \left( \lambda \frac{2}{k} \|u\|_{L^2(\Omega)} + \|u\|_{W^{k,2}(\partial \Omega)} \right).$$

Now we only need to consider the estimate of $\|u\|_{W^{k,2}(\partial \Omega)}$. For any fixed point $x_0 \in \partial \Omega$, we make a suitable transformation, such that $x_0 = 0$, $v$ is the opposite direction of the axis $x_n$, and the hyperplane $x_n = 0$ is the tangent plane of $\partial \Omega$ at $x_0$. For any point $x$ near the origin point $x_0$, one may assume that $u_{x_n} = \nabla_{\tau} u + \beta u_v$, where $\tau$ is the tangent vector field on $\partial \Omega$, $\nabla_{\tau}$ is an $n - 1$ dimensional vector valued function. Because $\nabla_{\tau}$ and $\beta$ both depend only on $\Omega$, they are also analytic functions. So

$$u_{x_n} = \nabla_{\tau} u + \beta \cdot (u_{x_n})_v$$

$$= \nabla_{\tau} \cdot (\nabla_{\tau} \cdot \nabla_{\tau} u) + \nabla_{\tau} \cdot (\nabla_{\tau}^2 u) + \nabla_{\tau} \cdot \nabla_{\tau} \beta u_v + 2\beta \cdot \nabla_{\tau} \cdot \nabla_{\tau} u_v + \beta \cdot \nabla_{\tau} u_v + \beta^2 u_{vv}.$$
It is also obvious that \( \overline{Y}(0) = 0 \) and \( \beta(0) = -1 \). Then from the Robin boundary condition, we have
\[
 u_{x_n,x_n}(0) = \mu^2 u(0) - \mu \beta_n(0)u(0) - \overline{Y}(0)\nabla_x u(0).
\]
From [13], or the standard elliptic estimate, we know that \( u \) is analytic in \( \overline{\Omega} \). Thus from \( u \) satisfies the equation \( \triangle u + \lambda u = 0 \) in \( \Omega \), we have that \( u \) satisfies the same equation on \( \overline{\Omega} \) and thus it holds that \( \triangle u(0) + \lambda u(0) = 0 \). So from the above calculation, we obtain that
\[
 u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_{n-1}x_{n-1}} - \overline{Y} \nabla_x u + (\lambda + \mu^2 - \beta_n \mu) u = 0,
\]
holds on the origin point \( x_0 \). Because \( x_0 \) is an arbitrary point on \( \partial \Omega \), the equation \( \triangle u + \lambda u = 0 \) becomes
\[
 \triangle_{\partial \Omega} u + \langle \vec{b}, \nabla u \rangle > + (\lambda + \mu^2 + c \mu) u = 0, \quad \text{on} \ \partial \Omega. \tag{2.2}
\]
Here \( \vec{b} \) and \( c \) are coefficient functions depending only on \( n \) and \( \Omega \), \( \nabla_{\partial \Omega} \) is the gradient operator on the submanifold \( \partial \Omega \), \( \langle, \rangle \) is the inner product on the submainfold \( \partial \Omega \), and \( \triangle_{\partial \Omega} \) is the Laplacian on the submanifold \( \partial \Omega \). Because \( \partial \Omega \) is an \( n - 1 \) dimensional analytic compact manifold, the coefficient functions \( \vec{b} \) and \( c \) both are analytic on \( \partial \Omega \). Thus by the standard elliptic estimates on Riemannian manifolds, we have
\[
 \| u \|_{W^{k,2}(\partial \Omega)} \leq C^k(\lambda + \mu^2)^{\frac{k}{2}} \| u \|_{L^2(\partial \Omega)}. \tag{2.3}
\]
Thus
\[
 \| u \|_{W^{k,2}(\Omega)} \leq C^k(\sqrt{\lambda} + |\mu|)^k (\| u \|_{L^2(\Omega)} + \| u \|_{L^2(\partial \Omega)}) \leq C^k(\sqrt{\lambda} + |\mu|)^k \| u \|_{L^\infty(\Omega)}. 
\]
By the Sobolev Embedding Theorem, we have that for any multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \),
\[
 \| D^\alpha u \|_{L^\infty(\Omega)} \leq C|\alpha| + \frac{n+2}{2} (\sqrt{\lambda} + |\mu|)^{|\alpha| + \frac{n+2}{2}} \| u \|_{L^\infty(\Omega)}. 
\]
Thus we can extend \( u \) into a neighborhood of \( \Omega \) by the Taylor series, noted by \( \Omega_\delta \), such that for any multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \),
\[
 \| u \|_{L^\infty(\Omega_\delta)} \leq \| u \|_{L^\infty(\Omega)} + \sum_{k=1}^{\infty} \sum_{|\alpha| = k} \frac{\delta^k}{\alpha!} \| D^\alpha u \|_{L^\infty(\Omega)} \leq \| u \|_{L^\infty(\Omega)} \left( 1 + \sum_{k=1}^{\infty} \sum_{|\alpha| = k} \frac{(C\delta)^k (\sqrt{\lambda} + |\mu|)^{k + \frac{n+2}{2}}}{\alpha!} \right) \leq e^{C(\sqrt{\lambda} + |\mu|)} \| u \|_{L^\infty(\Omega)}
\]
for \( \lambda \) large enough.

On the other hand, by the standard elliptic estimate, the Sobolev Embedding Theorem, and the Robin boundary condition, it also holds that for any multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \),
\[
 \| D^\alpha u \|_{L^\infty(\Omega)} \leq C|\alpha| + \frac{n+2}{2} (\lambda + \mu^2) + \| D^\alpha u \|_{L^\infty(\partial \Omega)} \leq C|\alpha| + \frac{n+2}{2} \| u \|_{L^2(\Omega)} + \frac{1}{|\mu|} \| (D^\alpha u)_{v} \|_{L^\infty(\partial \Omega)}.
\]

\( \otimes \) Springer
Thus for any \( x_0 \in \partial \Omega \) and \( x \in \partial B_r(x_0) \) with \( r \leq \delta \), it holds that
\[
\sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{(x-x_0)^\alpha}{\alpha!} |D^\alpha u(x_0)| \leq \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{C_k r^k}{\alpha!} \|D^\alpha u\|_{L^\infty(\Omega)} \\
\leq \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{C_k r^k}{\alpha!} \left( \lambda_{\frac{1}{2} + \frac{n+2}{4}} \|u\|_{L^2(\Omega)} + \frac{1}{|\mu|} \|(D^\alpha u)_\nu\|_{L^\infty(\partial \Omega)} \right) \\
\leq \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{C_k r^k}{\alpha!} \left( \lambda_{\frac{1}{2} + \frac{n+2}{4}} \|u\|_{L^\infty(\Omega)} + \frac{1}{|\mu|} \|(D^\alpha u)_\nu\|_{L^\infty(\Omega)} \right) \\
\leq e^{C \sqrt{r}} \|u\|_{L^\infty(\Omega)} + \frac{C}{|\mu|} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{C_k r^k}{\alpha!} \|D^\alpha u\|_{L^\infty(\Omega)}.
\]

In the above inequalities, \( C \) may differ from line to line.

Noticing that for \( |\mu| \) large enough, the coefficient \( \frac{C}{|\mu|r} \) on the second term of the last inequality can be controlled by \( \frac{1}{2} \), then the following estimate holds for any \( x \in \partial B_r(x_0) \),
\[
|u(x)| \leq \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{C_k r^k}{\alpha!} \|D^\alpha u\|_{L^\infty(\Omega)} \\
\leq e^{C \sqrt{r}} \|u\|_{L^\infty(\Omega)}.
\]

Thus by the standard elliptic estimates, we know that for any \( x \in B_r(x_0) \), there holds
\[
|u(x)| \leq e^{C \sqrt{r}} \|u\|_{L^\infty(\Omega)}.
\]

Put \( r = \delta \), we can get the desired result, provided that \( |\mu| \geq \frac{C}{\delta} \). Then the proof is finished. \( \square \)

Now we adopt the quantity \( N(x_0, r) \) as follows:
\[
N(x_0, r) = \log_2 \frac{\max_{x \in B_r(x_0)} |u(x)|}{\max_{x \in B_{\frac{r}{2}}(x_0)} |u(x)|}.
\]  (2.4)

It is called in [16] the doubling index of \( u \) centered at \( x_0 \) with radius \( r \).

We give the upper bound for the doubling index as follows.

**Lemma 2.2** Let \( u \) be an eigenfunction in \( \Omega \) and \( \lambda \) be the corresponding eigenvalue. Then it holds that
\[
N(x, r) \leq C \sqrt[\lambda],
\]  (2.5)

with \( x \in \Omega \) and \( r \leq \delta \), \( \delta \) is a positive constant as in Lemma 2.1, and \( C \) is a positive constant depending only on \( n \) and \( \Omega \).

**Proof** Let \( \bar{x} \) be the maximum point of \( u \) at \( \overline{\Omega} \). Then for \( r \leq \delta \),
\[
\|u\|_{L^\infty(B_r(\bar{x}))} \leq \|u\|_{L^\infty(\Omega)} \leq e^{C \sqrt{\lambda}} \|u\|_{L^\infty(\Omega)} \\
\leq e^{C \sqrt{\lambda}} |u(\bar{x})| \leq e^{C \sqrt{\lambda}} \|u\|_{L^\infty(B_{r/2}(\bar{x}))}.
\]  (2.6)

Thus by the definition of the doubling index, we have
\[
N(\bar{x}, r) \leq C \sqrt[\lambda],
\]
for any \( r \leq \delta \). Noting that for any \( x_1 \in B_{r/4}(\overline{x}) \),
\[
\|u\|_{L^\infty(B_{r/2}(x_1))} \geq \|u\|_{L^\infty(B_{r/4}(x_1))} \geq \|u\|_{L^\infty(\Omega)},
\]
and
\[
\|u\|_{L^\infty(B_r(\overline{x}))} \leq e^{c\sqrt{r}}\|u\|_{L^\infty(\Omega)},
\]
we have
\[
N(x_1, r) \leq C\sqrt{r},
\]
for any \( r \leq \delta \) and \( x_1 \in B_{r/4}(\overline{x}) \). For any \( x \) in \( \Omega \), taking \( r = \delta \), we may use the above arguments for at most \( k \) times, where \( k \) is a positive constant depending only on \( n \) and \( \Omega \), to get that the upper bound for the doubling index is
\[
N(x, \delta) \leq C\sqrt{\lambda}.
\]
For the radius \( r < \delta \), we can use the almost monotonicity formula, which is stated in [16] to get the desired estimate. \( \square \)

**Remark 2.3** In fact, Lemma 2.2 tells us that, for any \( x \in \Omega \) and \( r \leq \delta \),
\[
N(x, r) \leq C\sqrt{r},
\]
where \( C \) is a positive constant depending only on \( n \) and \( \Omega \).

Now we state the following lemma, which is stated in [8, 16]. For the completeness, we also give a sketch of its proof.

**Lemma 2.4** Let \( u \) be a nontrivial analytic function in \( B_r(x_0) \). Then it holds that
\[
\mathcal{H}^{\alpha-1}(\{x \in B_{r/16}(x_0) | u(x) = 0\}) \leq CNr^{\alpha-1},
\]
where \( N = \max \{N(x, \rho) | x \in B_{r/2}(x_0), \rho \leq r/2\} \), and \( C \) is a positive constant depending only on \( n \).

**Proof** Without loss of generality, assume that \( \|u\|_{L^\infty(B_r(x_0))} = 1 \). Then from the assumption of \( N \), for any \( p \in B_{r/4}(x_0) \), it holds that \( \|u\|_{L^\infty(B_{r/16}(p))} \geq 4^{-cN} \), where \( c \) is a positive constant depending only on \( n \). So there exists some point \( x_p \in B_{r/16}(p) \) such that \( |u(x_p)| \geq 2^{-cN} \). Choose \( p_j \) be the points on \( \partial B_{r/4}(x_0) \) with \( p_j \) on the \( x_j \) axis, \( j = 1, 2, \ldots, n \). Let \( x_{pj} \in B_{r/16}(p_j) \) be the points such point \( |u(x_{pj})| \geq 2^{-cN} \). For each \( j = 1, 2, \ldots, n \) and \( \omega \) on the unit sphere, let \( f_{j,\omega}(t) = u(x_{pj} + tr\omega) \) for \( t \in (-5/8, 5/8) \). Then \( f_{j,\omega}(t) \) is analytic for \( t \in (-5/8, 5/8) \). So we can extend \( f_{j,\omega}(t) \) to an analytic function \( f_{j,\omega}(z) \) for \( z = t + iy \) with \( |t| < 5/8 \) and \( |y| < y_0 \) for some positive number \( y_0 \). Then we have \( |f_{j,\omega}(0)| \geq 2^{-cN} \) and \( |f_{j,\omega}(z)| \leq 1 \). By applying the corresponding conclusion about the complex analysis in [8, 14], we have
\[
\|u(x_{pj} + tr\omega) = 0, |t| < 1/2\| \leq CN.
\]
It means
\[
\sharp \{t | u(x_{pj} + tr\omega) = 0, x_{pj} + tr\omega \in B_{r/16}(x_0)\} \leq CN.
\]
Then from the integral geometric formula, which can be seen in [9, 15], we have
\[
\mathcal{H}^{\alpha-1}(\{x \in B_{r/16}(x_0) | u(x) = 0\}) \leq CNr^{\alpha-1}.
\]
\( \square \)

From Lemma 2.2 and Lemma 2.4, we can get the conclusion of Theorem 1.1 directly.
3 The non-analytic case

In this section, we consider the case that $\Omega$ satisfies the following two assumptions.

1. $\Omega$ is a $C^\infty$ bounded domain;
2. $\Omega \setminus \Gamma$ is piecewise analytic, where $\Gamma \subseteq \partial \Omega$ is a union of some $n - 2$ dimensional sub-manifolds of $\partial \Omega$.

Because the method in the analytic case cannot be used here directly, we adopt the argument developed in [21] to deal with the non-analytic case. In this section, we also assume that $\mu > 0$.

We use $\partial \Omega (r)$ and $\Gamma (r)$ to denote the following two sets respectively,

$$\partial \Omega (r) = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq r \},$$
$$\Gamma (r) = \{ x \in \Omega : \text{dist}(x, \Gamma) \leq r \}.$$

First we need to do some preparation.

**Lemma 3.1** Let $u$ satisfy the equation $\Delta u + \lambda u = 0$ in $B_r (x_0)$. Let $w(x, x_{n+1}) = u(x)e^{\sqrt{\lambda} x_{n+1}}$. Then $w$ satisfies the equation $\Delta w = 0$ on $\Omega \times \mathbb{R}$, i.e., $w$ is a harmonic function on $\Omega \times \mathbb{R}$.

**Proof** Let $\Theta (x_0, r) = \int_{\partial B_r (x_0)} \frac{|\nabla w|^2 dx}{\int_{\partial B_r (x_0)} w^2 d\sigma}$. Such quantity is called the frequency function of $w$. $\Theta (x_0, r)$ is monotonicity to $r$, and has the following doubling conditions for any $t > 1$.

$$t \Theta (x_0, r/t) \leq \left( \frac{\int_{B_r (x_0)} w^2 dx}{\int_{B_r / t (x_0)} w^2 dx} \right)^{1/2} \leq t \Theta (x_0, r).$$

From the standard interior estimate, for any $\epsilon > 0$,

$$\sup_{B_r (x_0)} |w| \leq C(\epsilon) \left( \int_{B_{(1+\epsilon)r} (x_0)} w^2 dx \right)^{1/2},$$

where $C(\epsilon)$ is a positive constant depending only on $\epsilon$. So for any $r > 0$ and any $\epsilon_1 > 1$, we have

$$\tilde{N} (x_0, r) = \log_2 \sup_{B_r (x_0)} |w| \sup_{B_{r/2} (x_0)} |w| \leq \log_2 \frac{C(\epsilon_1) \left( \int_{B_{(1+\epsilon) r} (x_0)} w^2 dx \right)^{1/2}}{\left( \int_{B_{r/2} (x_0)} w^2 dx \right)^{1/2}} \leq \log_2 2 (1 + \epsilon_1) \Theta (x_0, (1 + \epsilon_1) r) + C_1 (\epsilon_1).$$
We also have
\[
\bar{N}(x_0, r) = \log_2 \sup_{B_r(x_0)} |w| \geq \log_2 \left( \frac{\left( \int_{B_r(x_0)} w^2 \, dx \right)^{1/2}}{C(\epsilon_1) \left( \int_{B_{(1+\epsilon_1)/2}(x_0)} w^2 \, dx \right)^{1/2}} \right) 
\]
\[
\geq \log_2 \left( \frac{2}{1 + \epsilon_1} \right) \Theta(x_0, (1 + \epsilon_1) r/2) - C_2(\epsilon_1). 
\]
Then
\[
\bar{N}(x_0, r) \leq \log_2 2(1 + \epsilon_1) \Theta(x_0, (1 + \epsilon_1) r) + C_1(\epsilon) 
\]
\[
\leq 1 + \log_2(1 + \epsilon_1) \frac{\log(\epsilon_1) N(x_0, 2(1 + \log_2(1 + \epsilon_1))(1 + \epsilon_1))}{1 - \log_2(1 + \epsilon_1)} + C_1(\epsilon_1) + C_2(\epsilon_1). 
\]

For any given \( \epsilon > 0 \), choose \( \epsilon_1 > 0 \) small enough such that \( \frac{1+\log_2(1+\epsilon_1)}{1-\log_2(1+\epsilon_1)} \leq (1 + \epsilon) \), we can get the desired result. \( \square \)

**Remark 3.2** From some direct calculations,
\[
N(x_0, r) \leq \bar{N}(x_0, r) + C \sqrt{\lambda} r, 
\]
for any \( x_0 \in \Omega \), where \( C \) is a positive constant depending only on \( n \).

**Lemma 3.3** Let \( u \) be an eigenfunction and \( \lambda \) be the corresponding eigenvalue. Assume that \( \lambda \) is large enough and \( \mu > 0 \). Then, for
\[
r_0 = C \left( \sqrt{\lambda} + \mu \right) \frac{n+4}{2} \sqrt{\frac{\mu}{1 + \sqrt{\mu}}}, 
\]
we have
\[
\|u\|_{L^2(\partial\Omega_{(r_0)})} \leq \frac{1}{2} \|u\|_{L^2(\Omega)}, 
\]
where \( C \) is a positive constant depending only on \( n \) and \( \Omega \).

**Proof** From the standard elliptic estimate and the Sobolev Embedding Theorem, we have
\[
\|u\|_{L^2(\partial\Omega_{(r_0)})} \leq C_0 \|u\|_{L^\infty(\Omega)} \leq C_0 \left( \lambda^{n+2} \|u\|_{L^2(\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \right). 
\]
Because \( \Omega \) is \( C^\infty \) bounded, (2.2) also holds on \( \partial\Omega \). From the estimate (2.3) and the Sobolev Embedding Theorem again, we have
\[
\|u\|_{L^\infty(\partial\Omega)} \leq C \left( \sqrt{\lambda} + \mu \right) \frac{n+1}{2} \|u\|_{L^2(\partial\Omega)}. 
\]
So
\[
\|u\|_{L^2(\partial\Omega_{(r_0)})} \leq C_0 \left( \lambda^{n+2} \|u\|_{L^2(\Omega)} + \left( \sqrt{\lambda} + \mu \right)^{n+1} \|u\|_{L^2(\partial\Omega)} \right) 
\]
\[
\leq C_0 \left( \sqrt{\lambda} + \mu \right)^{n+2} \left( \|u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial\Omega)} \right). 
\]
From the Robin boundary condition, we have
\[
\|u\|_{L^2(\partial \Omega)}^2 = \int_{\partial \Omega} u^2 d\sigma = \frac{1}{-\mu} \int_{\Omega} uu_v d\sigma = -\frac{1}{\mu} \int_{\Omega} div(u\nabla u) dx
\]
\[
= -\frac{1}{\mu} \int_{\Omega} |\nabla u|^2 dx + \frac{\lambda}{\mu} \int_{\Omega} u^2 dx \leq \frac{\lambda}{\mu} \|u\|_{L^2(\Omega)}^2.
\] (3.5)

By (3.4) and (3.5), we have
\[
\|u\|_{L^2(\partial \Omega(\Omega \setminus \{x\}))} \leq C r_0 \left(\sqrt{\lambda} + \frac{n^4}{2} \left(1 + \frac{1}{\sqrt{\mu}}\right) \|u\|_{L^2(\Omega)}\right),
\]
So there exists some positive constant \(C'\) depending only on \(n\) and \(\Omega\) such that for \(r_0 = C'(\sqrt{\lambda} + \mu)^{-\frac{n^4}{2}} \frac{\sqrt{\mu}}{1 + \sqrt{\mu}}\), the desired result holds. \(\square\)

**Lemma 3.4** Let \(u\) be an eigenfunction and \(\lambda\) be the corresponding eigenvalue. Assume that \(\lambda\) is large enough and \(\mu > 0\). Then for any \(x \in \overline{\Omega} \setminus \Gamma\), it holds that
\[
\|u\|_{L^\infty(B_r(x))} \leq e^{C(\sqrt{\lambda} + \mu - \log \mu - \log r)} \|u\|_{L^2(\Omega)},
\] (3.6)
where \(r = \frac{1}{2} \min \{dist(x, \Gamma), \delta\}\), \(\delta\) is the same constant as in Lemma 2.1, \(C\) is a positive constant depending only on \(n\) and \(\Omega\).

**Proof** Without loss of generality, assume that \(dist(x, \Gamma) \leq \delta\). Since \(\partial \Omega \setminus \Gamma\) is piecewise analytic, all the derivation of \(u\) on the whole domain \(\Omega\) can be estimated in the same way as in Lemma 2.1. Thus we have
\[
\|u\|_{L^\infty(B_r(x))} \leq \sum_{k=0}^{\infty} \sum_{|\alpha| = k} \frac{r^k}{\alpha!} |D^\alpha u(x)|
\]
\[
\leq \sum_{k=0}^{\infty} \sum_{|\alpha| = k} r^k \cdot \frac{C^{k + \frac{n^2}{2}} \left(\sqrt{\lambda} + \mu\right)^{k + \frac{n+2}{2}}}{r^{k + \frac{n+2}{2}} \left(\sqrt{\lambda} + \mu\right)} \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial \Omega)}\right)
\]
\[
\leq e^{C(\sqrt{\lambda} + \mu - \log r)} \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial \Omega)}\right)
\]
\[
\leq e^{C(\sqrt{\lambda} + \mu - \log r)} \left(1 + \frac{\sqrt{\lambda}}{\sqrt{\mu}}\right) \|u\|_{L^2(\Omega)}.
\]
In the last inequality, we have used the same arguments as in the proof of Lemma 3.3. Thus we get the desired result since \(\sqrt{\lambda}/\sqrt{\mu} = e^{C(\log \lambda - \log \mu)}\) and we have already required that the eigenvalue \(\lambda\) is large enough. \(\square\)

Now we will consider the upper bound for the doubling index of \(u\) introduced in Sect. 2.

**Lemma 3.5** Let \(u\) be an eigenfunction on \(\Omega\) and \(\lambda\) is the corresponding eigenvalue. Moreover, we also assume that \(\mu > 0\) and \(\lambda\) is large enough. Then there exists a positive number \(R_0\) depending only on \(n\) and \(\Omega\), such that for any \(x \in \Omega \setminus \Gamma(R_0)\) and any \(\alpha \in (0, 1)\),
\[
N(x, R_0) \leq C \left(\sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha}\right),
\] (3.7)
where \(C\) is a positive constant depending on \(n\), \(\Omega\) and \(\alpha\), and \(c\) is a positive constant depending only on \(n\).
Proof Because $\partial \Omega$ is bounded and $C^\infty$ smooth, there exists some positive constant $R_0$ depending only on $n$ and $\Omega$, such that for any point $x \in \partial \Omega (R_0)$, there is one and only one point $x' \in \partial \Omega$ satisfying that $dist(x, \partial \Omega) = dist(x, x')$. Define $w(x, x_{n+1}) = u(x)e^{\sqrt{n+1}}x_{n+1}$. Let $\bar{x}$ be the maximum point of $u$ on $\Omega \setminus \partial \Omega (r_0)$, where $r_0$ is the same as in (3.2) in Lemma 3.3. On the one hand, it is obvious that

$$\|w\|_{L^\infty(B_{r_0}(\bar{x}))} \leq \|u\|_{L^\infty(\Omega)} e^{\sqrt{n+1}}.$$  

On the other hand, by Lemma 3.3 there holds $\|u\|_{L^2(\Omega \setminus \partial \Omega (r_0))} \geq \frac{1}{2} \|u\|_{L^2(\Omega)}$, and thus

$$\|u\|_{L^\infty(B_{r_0/2}(\bar{x}))} \geq \|u(\bar{x})\| = \|u\|_{L^\infty(\Omega \setminus \partial \Omega (r_0))} \geq C \|u\|_{L^2(\Omega)} \leq C \|u\|_{L^2(\Omega)}.$$  

(3.8)

Because

$$\|u\|_{L^\infty(\partial \Omega)} \leq C \left( \sqrt{\lambda} + \mu \right)^{\frac{n+2}{2}} (\|u\|_{L^2(\Omega)} + \|u\|_{L^2(\partial \Omega)}) \leq C \left( \sqrt{\lambda} + \mu \right)^{\frac{n+2}{2}} (1 + \frac{\sqrt{\lambda}}{\sqrt{\mu}}) \|u\|_{L^2(\Omega)} \leq C \left( \sqrt{\lambda} + \mu \right)^{\frac{n+2}{2}} \|u\|_{L^2(\Omega)},$$

it holds that

$$\|u\|_{L^2(\Omega)} \geq C \sqrt{\mu} \left( \sqrt{\lambda} + \mu \right)^{-\frac{n+4}{2}} \|u\|_{L^\infty(\Omega)}.$$  

(3.9)

Then from (3.8) and (3.9), we have

$$\|w\|_{L^\infty(B_{r_0/2}(\bar{x}))} \geq \|u\|_{L^\infty(B_{r_0/2}(\bar{x}))} \geq C \sqrt{\mu} \left( \sqrt{\lambda} + \mu \right)^{-\frac{n+4}{2}} \|u\|_{L^\infty(\Omega)}.$$  

Therefore

$$\tilde{N}(\bar{x}, r_0) = \log_{2} \sup_{B_{r_0/2}(\bar{x}, 0)} \frac{|w|}{\sup_{B_{r_0}(\bar{x}, 0)} |w|} \leq \log_{2} \frac{\|u\|_{L^\infty(\Omega)}}{C \sqrt{\mu} \left( \sqrt{\lambda} + \mu \right)^{-\frac{n+4}{2}} \|u\|_{L^\infty(\Omega)}} + C \sqrt{\lambda} r_0 \leq C (\log(\sqrt{\lambda} + \mu) - \log \mu + \sqrt{\lambda} r_0).$$

From Lemma 3.1 with $\epsilon$ satisfies that $\log \frac{\alpha}{4} (1 + \epsilon) = \frac{\alpha}{\alpha(n+4)}$, it holds that

$$\tilde{N}(\bar{x}, r_0/2) \leq C (1 + \epsilon) (\log(\sqrt{\lambda} + \mu) - \log \mu + \sqrt{\lambda} r_0),$$

provided that $\lambda$ is large enough.

Let $x_1 \in \partial B_{r_0/4}(\bar{x}) \cap (\Omega \setminus \partial \Omega (r_0))$ and let $r_1 = \frac{\sqrt{\lambda}}{4} r_0$. Then by the triangle inequality, there holds $dist(x_1, \partial \Omega) \leq r_1$. Thus

$$\|w\|_{L^\infty(B_{r_1}(x_1))} \leq \|u\|_{L^\infty(\Omega)} e^{\sqrt{\lambda} r_1}.$$
On the other hand, we have
\[
\|w\|_{L^\infty(B_{1/2}(x_1))} \geq \|w\|_{L^\infty(B_{0/2}(\bar{\Omega}))} \\
\geq 2^{-N(\bar{\Omega}, r_0/2)} \|w\|_{L^\infty(B_{0/2}(\bar{\Omega}))} \\
= 2^{-C \left( \log(\sqrt{\lambda} + \mu) - \log \mu + \sqrt{\lambda} r_0 \right) \sqrt{\mu} (\sqrt{\lambda} + \mu)^{-\frac{\mu \alpha}{2} \epsilon}} \|u\|_{L^\infty(\Omega)} \\
= 2^{-C \left( \frac{\mu \alpha}{2} \epsilon \log(\sqrt{\lambda} + \mu) - \frac{\mu \alpha}{2} \epsilon \log \mu + \sqrt{\lambda} r_0 \right) \|u\|_{L^\infty(\Omega)}}.
\]

Thus
\[
\tilde{N}(x_1, r_1) \leq C(1 + \epsilon) \left( \frac{n + 6}{2} \log(\sqrt{\lambda} + \mu) - \frac{3}{2} \log \mu + \sqrt{\lambda} r_0 \right) + C \sqrt{\lambda} r_1,
\]
and then for any \( r \leq r_1, \)
\[
\tilde{N}(x_1, r) \leq C(1 + \epsilon)^2 \left( \frac{n + 6}{2} \log(\sqrt{\lambda} + \mu) - \frac{3}{2} \log \mu + \sqrt{\lambda} r_0 \right) + C(1 + \epsilon) \sqrt{\lambda} r_1.
\]

Using the above arguments for \( k \) times, such that \((5/4)^{k-1} r_0 < 2 R_0 \) and \((5/4)^k r_0 \geq 2 R_0, \)
then \( k = C \log(2 R_0 / r_0) = C(\log(\sqrt{\lambda} + \mu) - \log \mu). \)
Then for some point \( x_k \in \Omega \) with \( \text{dist}(x_k, \partial \Omega) \geq r_k \geq R_0, \)
it holds that
\[
\tilde{N}(x_k, r_k) \leq C(1 + \epsilon)^k \left( \frac{n + 6}{2} \log(\sqrt{\lambda} + \mu) - \frac{3}{2} \log \mu \right) \\
+ C \sqrt{\lambda} \left( (1 + \epsilon)^k r_0 + (1 + \epsilon)^{k-1} r_0 + \cdots + (1 + \epsilon)^0 r_0 \right) \\
\leq C \left( \frac{5}{4} \right)^k \log_{5/4}(1 + \epsilon) \left( \log(\sqrt{\lambda} + \mu) - \log \mu \right) \\
+ C \sqrt{\lambda} \left( (1 + \epsilon)^k r_0 \left( 1 + \frac{5}{4(1 + \epsilon)} + \left( \frac{5}{4(1 + \epsilon)} \right)^2 + \cdots + \left( \frac{5}{4(1 + \epsilon)} \right)^k \right) \right) \\
\leq C r_0^{-\log_{5/4}(1 + \epsilon)} \left( \log(\sqrt{\lambda} + \mu) - \log \mu \right) + C \sqrt{\lambda} (1 + \epsilon)^k r_0 \left( \frac{5}{4(1 + \epsilon)} \right)^k \\
\leq C \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-\alpha} \right),
\]
provided that \( \alpha < 1, \) where \( c \) is a positive constant depending only on \( n. \) In the above inequalities, \( C \) may differ from line to line and depends on \( \alpha. \) So for any point \( \bar{x} \in \Omega \setminus \Gamma(R_0), \)
we have \( N(\bar{x}, 2 R_0) \leq C \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-\alpha} \right). \)

For any point \( x \in \Omega \setminus \Gamma(R_0), \) by using the same argument for \( l \) times, but keeping the radius unchanged, where \( l \) is a positive constant depending only on \( n \) and \( \Omega, \) we can obtain that
\[
N(x, R_0) \leq C \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-\alpha} \right),
\]
which is the desired result. \( \square \)

**Lemma 3.6** Let \( u \) be an eigenfunction on \( \Omega \) and \( \lambda \) is the corresponding eigenvalue. Moreover, we also assume that \( \mu > 0 \) and \( \lambda \) is large enough. Then for any \( x \in \Gamma(R_0) \setminus \Gamma \) and any \( \bar{r} < \text{dist}(x, \Gamma)/2, \) it holds that for any \( \alpha \in (0, 1), \)
\[
N(x, \bar{r}) \leq C \bar{r}^{-\frac{1}{2}} \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-\alpha} \right), \tag{3.10}
\]
\( \square \) Springer
where $R_0$ and $c$ are the same positive constants as in Lemma 3.5, $C$ is a positive constant depending on $n$, $\Omega$ and $\alpha$.

Proof. Also let $w(x, x_{n+1}) = u(x)e^{\sqrt{\lambda} x_{n+1}}$. From Lemma 3.1 and Lemma 3.5, for any $x_0 \in \Omega \setminus \partial \Omega (R_0)$ and $r < R_0$,

$$\tilde{N}(x_0, r) \leq C(1 + \epsilon) \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right).$$

Choose $\epsilon > 0$ satisfies that $\log_{3/4}(1 + \epsilon) = -\frac{1}{2}$.

Because

$$\|w\|_{L^\infty(B_{R_0/16}(x_0))} \geq \|u\|_{L^\infty(B_{R_0/16}(x_0))} \geq 2^{-C(\sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha})} \|u\|_{L^\infty(\Omega)},$$

for any $x_1 \in B_{R_0/4}(x_0) \cap \partial \Omega (R_0)$, it holds that $dist(x_1, \partial \Omega) \leq \frac{3}{4} R_0$. Let $R_1 = \frac{3}{4} R_0$ then

$$\|w\|_{L^\infty(B_{R_1}(x_1))} \leq \|u\|_{L^\infty(\Omega)} e^{\sqrt{\lambda} R_1}.$$

and

$$\|w\|_{L^\infty(B_{R_1/2}(x_1))} \geq \|w\|_{L^\infty(B_{R_0/16}(x_0))} \geq 2^{-C(\sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha})} \|u\|_{L^\infty(\Omega)}.$$

So

$$\tilde{N}(x_1, R_1) \leq C(1 + \epsilon) \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right) + C \sqrt{\lambda} R_1.$$ 

Furthermore from Lemma 3.1, for any $r \leq R_1$, we have

$$\tilde{N}(x_1, R_1/2) \leq C(1 + \epsilon)^2 \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right) + C(1 + \epsilon) \sqrt{\lambda} R_1,$$

and

$$\|w\|_{L^\infty(B_{R_1/2}(x_1))} \geq \|w\|_{L^\infty(B_{R_0/16}(x_0))} \geq 2^{-C(1+\epsilon)^2 \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha}} - C(1+\epsilon) \sqrt{\lambda} R_1 \|u\|_{L^\infty(\Omega)}.$$

By doing this for $k$ times such that $R_k = (3/4)^{k-1} R_0 > 2\tau$ and $R_k = (3/4)^k R_0 \leq 2\tau$, i.e., $k = -C \log \tau$, we have

$$\tilde{N}(x_k, R_k) \leq C(1 + \epsilon)^k \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right) + C \sqrt{\lambda} \left( 1 + \epsilon \right)^k R_0 + (1 + \epsilon)^{k-1} R_1 + \cdots + (1 + \epsilon)^0 R_k$$

$$\leq C \frac{1}{2^k} \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right) + C \sqrt{\lambda} \left( 1 + \epsilon \right)^k R_0 \left( 1 + \frac{3}{4(1 + \epsilon)} + \left( \frac{3}{4(1 + \epsilon)} \right)^2 + \cdots + \left( \frac{3}{4(1 + \epsilon)} \right)^k \right)$$

$$\leq C \frac{1}{2^k} \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right) + C \sqrt{\lambda} R_0 (1 + \epsilon)^k$$

$$\leq C \frac{1}{2^k} \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right).$$

for any $r < \tau$. Then by repeating the same argument for $l$ times, where $l$ depends only on $n$ and $\Omega$, and keeping the radius unchanged, we have that, for any $x \in \Omega \setminus \Gamma (2\tau)$, the following inequality holds:

$$\tilde{N}(x, 2\tau) \leq C \frac{1}{2^k} \left( \sqrt{\lambda} + \mu^\alpha + \mu^{-c\alpha} \right),$$

where $C$ is a positive constant depending only on $n$ and $\Omega$, provided that $\lambda$ is large enough. Then from Remark 3.2, we can get the desired result. \qed
From the above preparation, we can prove Theorem 1.2 as follows.

**Proof of Theorem 1.2:** From Lemmas 2.4, 3.5 and 3.6, we have

\[
I\mathcal{H}^{n-1} (\{x \in \Omega \mid u(x) = 0\}) \leq \mathcal{H}^{n-1} (\{x \in \Omega \setminus \Gamma (R_0) \mid u(x) = 0\}) + \sum_{k=0}^{\infty} \mathcal{H}^{n-1} \left( \left\{ x \in \Gamma \left( \frac{R_0}{2^{k+1}} \right) \mid u(x) = 0 \right\} \right) + \mathcal{H}^{n-1} (\{x \in \Gamma \mid u(x) = 0\}) . \tag{3.11}
\]

From Lemmas 2.4 and 3.5,

\[
\mathcal{H}^{n-1} (\{x \in \Omega \setminus \Gamma (R_0) \mid u(x) = 0\}) \leq C \left( \sqrt{\lambda} + \mu^a + \mu^{-ca} \right) R_0^{n-1} \frac{1}{R_0^n} \leq C \left( \sqrt{\lambda} + \mu^a + \mu^{-ca} \right) . \tag{3.12}
\]

By Lemmas 2.4 and 3.6,

\[
\sum_{k=0}^{\infty} \mathcal{H}^{n-1} \left( \left\{ x \in \Gamma \left( \frac{R_0}{2^{k+1}} \right) \mid u(x) = 0 \right\} \right) \leq \sum_{k=0}^{\infty} C \left( \sqrt{\lambda} + \mu^a + \mu^{-ca} \right) \left( \frac{R_0}{2^k} \right)^{-\frac{1}{2}} \left( \frac{R_0}{2^k} \right)^{n-1} \frac{1}{R_0^n} \left( \frac{R_0}{2^k} \right)^{n-2} \leq C \left( \sqrt{\lambda} + \mu^a + \mu^{-ca} \right) . \tag{3.13}
\]

Because \( \Gamma \) is a union of some \((n-2)\) dimensional submanifolds of \( \partial \Omega \), we have

\[
\mathcal{H}^{n-1} (\{x \in \Gamma \mid u(x) = 0\}) \leq \mathcal{H}^{n-1} (\Gamma) = 0. \tag{3.14}
\]

Substituting (3.12)–(3.14) into (3.11), we have that

\[
\mathcal{H}^{n-1} (\{x \in \Omega \mid u(x) = 0\}) \leq C \left( \sqrt{\lambda} + \mu^a + \mu^{-ca} \right) ,
\]

where \( C \) is a positive constant depending only on \( n, \Omega \) and \( \Gamma \), provided that \( \lambda \) is large enough. That is the desired result. \[\square\]

**References**

1. F. J. Almgren Jr., Dirichlet’s problem for multiple-valued functions and the regularity of mass minimizing integral currents, M. Obata(Ed.), Minimal Submanifolds and Geodesics, North-Holland, Amsterdam, 1-6 (1979)
2. Ariturk, S.: Lower bounds for nodal sets of Dirichlet and Neumann eigenfunctions. Comm. Math. Phys. 317(3), 817–825 (2013)
3. Bellová, K., Lin, F.H.: Nodal sets of Steklov eigenfunctions. Cal. Var. Par. Differ. Equ. 54(2), 2239–2268 (2015)
4. Brüning, J.: Über Knoten von eigenfunktionen des Laplace-Beltrami operators. Math. Z. 158(1), 15–21 (1978)
5. Chang, J.E.: Lower bounds for nodal sets of biharmonic Steklov problems. J. Lond. Math. Soc. 95(3), 763–784 (2017)
6. Colding, T.H., Minicozzi, W.P., II.: Lower bounds for nodal sets of eigenfunctions. Comm. Math. Phys. 306(3), 777–784 (2011)
7. Dong, R.T.: Nodal sets of eigenfunctions on Riemannian surfaces. J. Differ. Geom. 36, 493–506 (1992)
8. Donnelly, H., Fefferman, C.: Nodal sets of eigenfunctions on Riemannian manifolds. Invent. Math. 93, 161–183 (1988)
9. Federer, H.: Geometric Measure Theory. Springer, New York (1969)
10. Garfalo, N., Lin, F.H.: Unique continuation for elliptic operators: a geometric-variational approach. Comm. Pure. Appl. Math. 40, 347–366 (1987)
11. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equation of Second Order. Springer, Berlin (1983)
12. Hardt, R., Simon, L.: Nodal sets for solutions of elliptic equations. J. Differ. Geom. 30, 505–522 (1989)
13. Kukavica, I.: Nodal volumes for eigenfunctions of analytic regular elliptic problems. J. d’Analyse Mathématique 67(1), 269–280 (1995)
14. Lin, F.H.: Nodal sets of solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 44, 287–308 (1991)
15. Lin, F.H., Yang, X.P.: Geometric measure theory-an introduction. Adv. Math. 1, Science Press/International Press, Beijing/Boston (2002)
16. Logunov, A.: Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. Ann. Math. 187, 221–239 (2018)
17. Logunov, A.: Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. Ann. Math. 187, 241–262 (2018)
18. Logunov, A., Malinnikova, E.: Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimension two and three, 50 Years with Hardy Spaces, 261, 333–344 (2018)
19. Sogge, C.D., Zelditch, S.: Lower bounds on the Hausdorff measure of nodal sets. Math. Res. Lett. 18(1), 25–37 (2011)
20. Tian, L., Yang, X.P.: Nodal sets and horizontal singular sets of H-harmonic functions on the Heisenberg group. Commun. Contemp. Math. 16(4), 1350049 (2014)
21. Tian, L., Yang, X. P.: Measure upper bounds for nodal sets of eigenfunctions of the bi-harmonic operator. J. Lond. Math. Soc. (2021) (to appear)
22. Yau, S.T.: Open problems in geometry. Proc. Sympos. Pure Math. 54(1), 1–28 (1993)
23. Zhu, J.Y.: Interior nodal sets of Steklov eigenfunctions on surfaces. Anal. PDE 9(4), 859–880 (2016)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.