Grand Lebesgue Spaces norm estimates for eigen functions for Laplace - Beltrami operator defined on the closed compact smooth Riemannian manifolds.

M.R.Formica, E.Ostrovsky and L.Sirota.

Università degli Studi di Napoli Parthenope, via Generale Parisi 13, Palazzo Pacanowsky, 80132, Napoli, Italy.

e-mail: mara.formica@uniparthenope.it

Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

e-mail:eugostrovsky@list.ru
Department of Mathematics and Statistics, Bar-Ilan University, 59200, Ramat Gan, Israel.

e-mail:sirota3@bezeqint.net

Abstract

We derive a sharp Grand Lebesgue Space norm estimations for normalized eigen functions for the Laplace - Beltrami operator defined on the compact smooth Riemann manifold.

These estimates allow us to deduce in particular the exponential decreasing tail of distribution for these eigen functions.

Key words and phrases:

Compact smooth closed Riemann manifold, Laplace - Beltrami operator, eigen values and functions, Lebesgue - Riesz and Grand Lebesgue norms and spaces, tail function, Young - Fenchel transform, Young inequality, fundamental function, sub-gaussian variables, normalized function, non - asymptotic upper and lower estimate, generating function.
Statement of problem. Notations. Previous results.

Let \((M, g)\) be a compact closed smooth Riemannian manifold of dimension \(d \geq 2\), and let \(\Delta_g\) be the associated Laplace–Beltrami operator. We will consider the \(L^2\)–normalized eigenfunctions satisfying the classical relations

\[-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x), \quad ||e_\lambda||^2 = \int_M |e_\lambda(x)|^2 \, V_g(dx) = 1, \quad \lambda > 0, \quad (1)\]

where \(V_g(dx)\) (measure) is element of volume on \(M\) and as ordinary \(||f||_p\) denotes the classical Lebesgue–Riesz norm for the (measurable) function \(f : M \to \mathbb{R}\):

\[||f||_p \overset{def}{=} \left[ \int_M |f(x)|^p \, V_g(dx) \right]^{1/p}, \quad p \geq 2.\]

Introduce the following variables

\[p_c := \frac{2(d + 1)}{d - 1}, \quad d \geq 2;\]

\[\mu(p) := \frac{d - 1}{2} \cdot \left( \frac{1}{2} - \frac{1}{p} \right), \quad 2 < p \leq p_c;\]

\[\mu(p) := d \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2}, \quad p_c \leq p \leq \infty.\]

We will apply the following important estimate

\[||e_\lambda||_p \leq C(M, g) \lambda^{\mu(p)}, \quad p > 2, \quad (2)\]

see [26], [27] and another works of this author [28]–[34]. See also the articles [9], [35]–[36].

We intent to extend the estimate (1) from the classical Lebesgue–Riesz spaces into the more general ones, namely, into the so-called Grand Lebesgue Spaces.

A brief review of the theory of Grand Lebesgue Spaces.

Let \((a, b) = \text{const}, \quad 1 \leq a < b \leq \infty\), and let \(\psi = \psi(p), \quad p \in (a, b)\) be bounded from below: \(\inf_{p \in (a, b)} \psi(p) > 0\) measurable function. The set of all such a functions will be denoted by \(\Psi(a, b)\); put also

\[\Psi := \cup_{(a, b): 1 < a < b < \infty} \Psi(a, b).\]
Definition 1.1. Recall that the so-called Grand Lebesgue Space \( G_\psi \), \( \psi \in \Psi(a,b) \) builted in particular on the set \( M \) equipped as before with the measure \( V_\psi \), consists by definition on all the integrable numerical valued functions having a finite norm

\[
||f||_{G_\psi} = ||f||_{G_\psi}(M) \overset{\text{def}}{=} \sup_{p \in (a,b)} \left\{ \frac{||f||_p}{\psi(p)} \right\}.
\]  

(3)

The function \( \psi = \psi(p), p \in (a,b) \) is said to be as generating function for this space.

These spaces are rearrangement invariant Banach functional spaces. They was investigated in many works, see e.g. [6], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [22] - [25]. In particular, the belonging of the function to certain Grand Lebesgue Space \( G_\psi \) is closely related with its tail behavior and is related with its moment generating function

\[
\nu[f](z) := \int_M \exp(z f(x)) V_\psi(dx).
\]

Notice that if we choose the following extremal function

\[
\psi_r = \psi_r(p) = 1, \quad r = p, \quad \psi_r(p) = \infty, \quad p \neq r, \quad r = \text{const} > 1,
\]

and agree to take \( C/\infty = 0 \), then

\[
||f||_{G_\psi} = ||f||_{r}.
\]

So, the notion of GLS contains as a particular case the classical Lebesgue - Riesz one.

Further, let \( f(\cdot) \in G_\psi \) and (for definiteness) \( ||f||_{G_\psi} = 1 \). Define the following function (Young - Fenchel transform)

\[
h[\psi](u) := \sup_{p \in (a,b)} (pu - p \ln \psi(p)), \quad u \geq e.
\]

Then the tail function \( T[f](u) \) for \( f(\cdot) \) may be estimated as follows

\[
T[f](u) \overset{\text{def}}{=} V_\psi\{x, x \in M, |f(x)| > u \} \leq \exp(-h[\psi](u)), \quad u \geq e,
\]

the exponential decreasing in general case estimate; and inverse conclusion holds true up to finite constant under appropriate natural conditions.

The fundamental function for these spaces \( \phi[G_\psi](\delta) = \phi[G_\psi(a,b)](\delta), \quad \delta > 0 \) has a form

\[
\phi[G_\psi(a,b)](\delta) = \sup_{p \in (a,b)} \left\{ \delta^{1/p} \frac{\psi(p)}{\psi'(p)} \right\}.
\]  

(4)
These functions were investigated in particular in [25]. They used in functional analysis, theory of Fourier series etc. They are also closely and continuously related with generating function $\psi(p)$.

A very important particular subgaussian case: $\psi(p) = \sqrt{p}$, $p \in (1, \infty)$.

2 Main result.

Case A: small values of the parameter.

We consider here at first the case when $2 < p \leq p_c$. Let $2 < a < b \leq p_c$ and let $\psi \in \Psi(a, b)$.

Theorem 2.1.

$$||e_\lambda||_{G\psi} \leq C(M, g) \lambda^{(d-1)/4} \phi[G\psi]\left(\lambda^{(1-d)/2}\right), \lambda > 0. \quad (5)$$

Proof. We have from the source relation (2) taking into account the restrictions $\lambda > 0$, $2 < p \leq p_c$

$$||e_\lambda||_p \leq C(M, g) \lambda^{(d-1)/4} \lambda^{(1-d)/(2p)}$$

and after dividing over $\psi(p)$

$$\frac{||e_\lambda||_p}{\psi(p)} \leq C(M, g) \lambda^{(d-1)/4} \left(\frac{\lambda^{(1-d)/2}}{\psi(p)}\right)^{1/p}.$$ 

It remains to take the supremum over $p$, $p \in (a, b)$ to get (5).

Case B: great values of the parameter.

Let us consider now the case when $p \geq p_c$. Let here $p_c \leq a < b \leq \infty$ and let $\psi \in \Psi(a, b)$.

Theorem 2.2.

$$||e_\lambda||_{G\psi} \leq C(M, g) \lambda^{(d-1)/2} \phi[G\psi]\left(\lambda^{-d}\right), \lambda > 0. \quad (6)$$

Proof is quite alike ones in the foregoing case. Namely, we have for the values $p \geq p_c$

$$\mu(p) = \frac{d - 1}{2} - \frac{d}{p},$$

following in this case
\[\|e_\lambda\|_p \leq C(M, g) \lambda^{(d-1)/2} \lambda^{-d/p},\]
\[\|e_\lambda\|_p \leq C(M, g) \cdot \lambda^{(d-1)/2} \times \left(\frac{\lambda^{-d}}{\psi(p)}\right)^{1/p},\]
which follows in turn to (6) after taking the supremum over \( p \).

**Example 2.1.** Let us choose \( \psi(p) = 1, \ p > p_c; \) and one can take \( p \to \infty; \) then we conclude
\[
\max_{x \in M} |e_\lambda(x)| \leq C(M, g) \cdot \lambda^{(d-1)/2}, \ \lambda > 0.
\]

**Acknowledgement.** The first author has been partially supported by the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project “sostegno alla Ricerca individuale”.

**References**

[1] I. Ahmed, A. Fiorenza, M.R. Formica, A. Gogatishvili, J.M. Rakotoson. Some new results related to Lorentz G-Gamma spaces and interpolation. J. Math. Anal. Appl., 483, 2, (2020), to appear.

[2] G. Anatriello and A. Fiorenza. Fully measurable grand Lebesgue spaces. J. Math. Anal. Appl. 422 (2015), no. 2, 783–797.

[3] G. Anatriello and M. R. Formica. Weighted fully measurable grand Lebesgue spaces and the maximal theorem. Ric. Mat. 65 (2016), no. 1, 221–233.

[4] J. Bourgain. Besicovitch type maximal operators and applications to Fourier analysis. Geom. Funct. Anal. 1 (1991), no. 2, 147187.

[5] J. Bourgain. Geodesic restrictions and \( L(p) \) – estimates for eigenfunctions of Riemannian surfaces. Linear and complex analysis, Amer. Math. Soc. Transl. Ser. 2, vol. 226, Amer. Math. Soc., Providence, RI, 2009, pp. 2735.

[6] V.V. Buldygin V.V., D.I.Mushtary, E.I.Ostrovsky, M.I.Pushalsky. New Trends in Probability Theory and Statistics. Mokslas, (1992), V.1, p. 78 - 92; Amsterdam, Utrecht, New York, Tokyo.

[7] Capone C, Formica M.R, Giova R. Grand Lebesgue spaces with respect to measurable functions. Nonlinear Analysis 2013; 85: 125 - 131.

[8] Capone C, and Fiorenza A. On small Lebesgue spaces. Journal of function spaces and applications. 2005; 3; 73 - 89.

[9] Donnelly, H. Bounds for eigenfunctions of the Laplacian on compact Riemannian manifolds. J. Func. Anal., 187, (2001), pp. 247 - 261.

[10] S. V. Ermakov, and E. I. Ostrovsky. Continuity Conditions, Exponential Estimates, and the Central Limit Theorem for Random Fields. Moscow, VINITY, 1986. (in Russian).
[11] A. Fiorenza. *Duality and reflexivity in grand Lebesgue spaces.* Collect. Math. 51, (2000), 131 - 148.

[12] 4. A. Fiorenza and G.E. Karadzhov. *Grand and small Lebesgue spaces and their analogs.* Consiglio Nazionale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picone, Sezione di Napoli, Rapporto tecnico 272/03, (2005).

[13] A. Fiorenza, M. R. Formica and A. Gogatishvili. *On grand and small Lebesgue and Sobolev spaces and some applications to PDE’s.* Differ. Equ. Appl. 10 (2018), no. 1, 21–46.

[14] A. Fiorenza, M. R. Formica, A. Gogatishvili, T. Kopaliani and J. M. Rakotoson. *Characterization of interpolation between grand, small or classical Lebesgue spaces.* Preprint arXiv:1709.05892, Nonlinear Anal., to appear.

[15] A. Fiorenza, M. R. Formica and J. M. Rakotoson. *Pointwise estimates for $G$-functions and applications.* Differential Integral Equations 30 (2017), no. 11-12, 809–824.

[16] M. R. Formica and R. Giova. *Boyd indices in generalized grand Lebesgue spaces and applications.* Mediterr. J. Math. 12 (2015), no. 3, 987–995.

[17] T.Iwaniec and C.Sbordone. *On the integrability of the Jacobian under minimal hypotheses.* Arch. Rat.Mech. Anal., 119, (1992), 129-143.

[18] Kozachenko Yu. V., Ostrovsky E.I. (1985). The Banach Spaces of random Variables of subgaussian Type. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

[19] Kozachenko Yu.V., Ostrovsky E., Sirota L. *Relations between exponential tails, moments and moment generating functions for random variables and vectors.* arXiv:1701.01901v1 [math.FA] 8 Jan 2017

[20] E.Ostrovsky, L.Sirot, and E.Rogover. *Integral operators in Grand Lebesgue Sraces.* arXiv:0912.2538v1 [math.FA] 13 Dec 2009

[21] E.Ostrovsky, L.Sirot. *Boundedness of operators in bilateral Grand Lebesgue Spaces, with exact and weakly exact constant calculation.* arXiv:1104.2963v1 [math.FA] 15 Apr 2011

[22] Ostrovsky E.I. (1999). *Exponential estimations for Random Fields and its applications,* (in Russian). Moscow - Obninsk, OINPE.

[23] Ostrovsky E. and Sirot L. *Sharp moment estimates for polynomial martingales.* arXiv:1410.0739v1 [math.PR] 3 Oct 2014

[24] Ostrovsky E. and Sirot L. *Entropy and Grand Lebesgue Spaces approach for the problem of Prokhorov - Skorokhod continuity of discontinuous random fields.* arXiv:1512.01909v1 [math.Pr] 7 Dec 2015

[25] Ostrovsky E. and Sirot L. *Fundamental function for Grand Lebesgue Spaces.* arXiv:1509.03644v1 [math.FA] 11 Sep 2015

[26] Christopher D. Sogge. *Localized $L_p$ estimates for eigenfunctions: II.* arXiv:1610.06639v1 [math.AP] 21 Oct 2016

[27] C. D. Sogge. *Oscillatory integrals and spherical harmonics.* Duke Math. J. 53 (1986), 43 - 65.

[28] C.D.Sogge. *Concerning the $L_p$ norm of spectral clusters for second-order elliptic operators on compact manifolds.* J. Funct. Anal. 77 (1988), 123 - 138.

[29] C.D.Sogge. *Fourier integrals in classical analysis.* Cambridge Tracts in Mathematics, vol. 105, Cambridge University Press, Cambridge, 1993.
[30] C.D. Sogge. Kakeya - Nikodym averages and Lp− norms of eigenfunctions. Tohoku Math. J., (2), 63, (2011), 519-538.

[31] C.D. Sogge. Improved critical eigenfunction estimates on manifolds of non - positive curvature, (2016), arXiv:1512.03725.

[32] C.D. Sogge. Localized Lp− estimates of eigenfunctions: A note on an article of Hezari and Rivière. Adv. Math. 289, (2016), 384 - 396.

[33] C. D. Sogge and S. Zelditch. Riemannian manifolds with maximal eigenfunction growth, Duke Math. J., 114, (2002), 387-437.

[34] C.D. Sogge. On eigenfunction restriction estimates and L4− bounds for compact surfaces with nonpositive curvature. Advances in analysis: the legacy of Elias M. Stein, Princeton Math. Ser., vol. 50, Princeton Univ. Press, Princeton, NJ, 2014, pp. 447-461.

[35] B. Xu. Derivative of the spectral function and Sobolev norms of eigenfunctions on a closed Riemannian manifold. Ann. Glob. Anal. Geo., 26, (2004), 231-252.

[36] B. Xu. Derivative of spectral function and Sobolev norms of eigenfunctions on a closed Riemannian manifold. Harmonic Analysis and Nonlinear PDEs, Surikaisekikenkyusho Kokyuroku, 1389 (2004), 60 - 77.