K-CLASSES OF BRILL-NOETHER LOCI AND A DETERMINANTAL FORMULA

DAVE ANDERSON, LINDA CHEN, AND NICOLA TARASCA

Abstract. We prove a determinantal formula for the K-theory class of certain degeneracy loci, and apply it to compute the Euler characteristic of the structure sheaf of the Brill-Noether locus of linear series with special vanishing at marked points. When the Brill-Noether number \( \rho \) is zero, we recover the Castelnuovo formula for the number of special linear series on a general curve; when \( \rho = 1 \), we recover the formulas of Eisenbud-Harris, Pirola, and Chan-Lópe-Pflueger-Teixidor for the arithmetic genus of a Brill-Noether curve of special divisors. Our degeneracy locus formula also specializes to new determinantal expressions for the double Grothendieck polynomials corresponding to 321-avoiding permutations, and gives double versions of the flagged skew Grothendieck polynomials recently introduced by Matsumura. Our result extends the formula of Billey-Jockusch-Stanley expressing Schubert polynomials for 321-avoiding permutations as generating functions for skew tableaux.

Consider a smooth projective curve \( C \) of genus \( g \) over an algebraically closed field. The classical Brill-Noether theorem describes the locus of special divisors,

\[
W^r_d(C) = \{ L \in \text{Pic}^d(C) \mid h^0(C, L) \geq r + 1 \}.
\]

A parameter count — reviewed at the end of this introduction — estimates the dimension of \( W^r_d(C) \) as \( \rho = \rho(g, r, d) = g - (r + 1)(g - d + r) \), and the theorem states that if \( C \) has general moduli, \( W^r_d(C) \) is in fact nonempty of dimension \( \rho \) whenever \( \rho \geq 0 \). A connection with degeneracy loci for maps of vector bundles was implicit in the original work by Brill and Noether, and was brought into focus by Kleiman and Laksov in one of several modern proofs of the theorem given in the 1970s.

We prove two main theorems in this article. The first gives a formula for the (arithmetic) genus of the Brill-Noether locus — and in fact, for the generalized Brill-Noether loci parametrizing linear series having specific vanishing profiles at one or two points. Our results extend the classical computation by Castelnuovo, who studied the zero-dimensional case \( \rho = 0 \); Eisenbud-Harris [13] and Pirola [31], who studied the case \( \rho = 1 \); and Chan-Lópe-Pflueger-Teixidor [9], whose remarkable computation uses the combinatorics

\[1\]

MSC2010. 14H51, 14M15 (primary), 19E20, 05E05 (secondary).

Key words and phrases. Brill-Noether loci, K-theory classes, determinantal formulas, Young diagrams, double Schubert and Grothendieck polynomials.

DA was partially supported by NSF Grant DMS-1502201.
of tableaux and the geometry of limit linear series to treat the case when
the two-pointed locus is one-dimensional.

Our genus formulas are deduced from the second main theorem of the
article: a new determinantal formula for the K-theory class of a certain
type of degeneracy locus. These loci arise naturally not only from the Brill-
Noether problem, but also in combinatorics — they are built from a special
class of permutations called 321-avoiding permutations. As a corollary, we
find new formulas for families of polynomials occurring in algebraic combi-
natorics, the double Schubert and double Grothendieck polynomials. These
results extend recent work of Matsumura [28], Hudson-Matsumura [24], and
Hudson-Ikeda-Matsumura-Naruse [23].

Another goal of this work is to highlight the connection between recent
developments in Schubert calculus and the geometry of curves. The results
of this paper expand on the fruitful interactions which led to the growth of
both subjects, as discussed extensively in [4]. On one hand, an approach
to linear series via degeneracy loci unifies (and perhaps simplifies) several
results in Brill-Noether theory. On the other hand, constructions arising
in the study of linear series led us to the geometric proof of the general
determinantal formula presented in §1. It seems natural to expect further
progress can be made in both subjects by exploiting this bridge.

We now turn to more precise statements of the main results. The locus
$W^r_d(C)$ of special divisors on $C$ has a canonical desingularization by the
variety of linear series $G^r_d(C)$, which parametrizes pairs $\ell = (L, V)$ with
$L \in \text{Pic}^d(C)$ and $V \subseteq H^0(C, L)$ an $(r + 1)$-dimensional subspace. For a
given linear series $\ell$ and a point $P \in C$, the vanishing sequence of $\ell$ at $P$ is
the sequence

$$a^\ell(P) = \left(0 \leq a^\ell_0(P) < a^\ell_1(P) < \cdots < a^\ell_r(P) \leq d\right)$$

of distinct orders of vanishing of sections in $V$ at $P$.

The two-pointed Brill-Noether locus is defined as follows. Fix two points
$P$ and $Q$ on a smooth curve $C$ of genus $g$. Given sequences of integers

$$a = (0 \leq a_0 < a_1 < \cdots < a_r \leq d) \quad \text{and} \quad b = (0 \leq b_0 < b_1 < \cdots < b_r \leq d),$$

we wish to parametrize linear series $\ell$ of projective dimension $r$ and degree
d on $C$ with $a^\ell(P)$ dominating $a$, and $a^\ell(Q)$ dominating $b$. That is,

$$G^{a,b}_d(C, P, Q) := \left\{ \ell \in G^r_d(C) \mid a^\ell_i(P) \geq a_i \quad \text{and} \quad a^\ell_i(Q) \geq b_i \quad \text{for all} \quad 0 \leq i \leq r \right\}.$$
has dimension equal to the \textit{two-pointed Brill-Noether number}:

\[
\rho := \rho(g, r, d, a, b) = g - \sum_{i=0}^{r} (g - d + a_i + b_{r-i}).
\]

This was first proved by Eisenbud and Harris using limit linear series and a construction on a singular curve [13, §1]. More recently, explicit examples of smooth two-pointed curves satisfying the two-pointed Brill-Noether theorem (in any genus) have been constructed, by studying curves on decomposable elliptic ruled surfaces [16, §2]. In contrast to the situation with \( G_{d}^{r}(C) \), the condition \( \rho \geq 0 \) is not sufficient to guarantee that the pointed locus \( G_{d}^{a,b}(C, P, Q) \) is nonempty. A numerical criterion for nonemptiness was given by Osserman [29], and also follows from our results; see Proposition 4.2.

Our first main theorem computes the holomorphic (sheaf) Euler characteristic of \( G_{d}^{a,b}(C, P, Q) \) when it has expected dimension \( \rho \). To state it, we need some more notation. Given sequences \( a \) and \( b \) as above, we define two partitions, \( \lambda \) and \( \mu \), by setting

\[
\lambda_i = n + a_{r+1-i} - (r + 1 - i), \quad \text{and} \quad \mu_i = n - b_{r+1-i} + i - 1 - g + d - r
\]

for \( 1 \leq i \leq r + 1 \), where \( n \) is a fixed, sufficiently large nonnegative integer.

Partitions are commonly represented as \textit{Young diagrams}, so \( \lambda \) is a collection of boxes with \( \lambda_i \) boxes in the \( i \)-th row. When \( \mu_i \leq \lambda_i \) for all \( i \), one has \( \mu \subseteq \lambda \), and one represents the sequence \( \lambda_i - \mu_i \) as a \textit{skew Young diagram} \( \lambda/\mu \) (the complement of \( \mu \) in \( \lambda \)). Borrowing this notation, we will write \[|l/m| = \sum_{i=1}^{r+1} (l_i - m_i)\] for any sequences of integers \( l \) and \( m \) of length \( r + 1 \), regardless of whether \( l_i - m_i \geq 0 \).

**Theorem A.** Let \((C, P, Q)\) be any smooth two-pointed curve of genus \( g \), and fix \( d, a, b \). If \( G = G_{d}^{a,b}(C, P, Q) \) has dimension equal to \( \rho \), then the Euler characteristic is

\[
\chi(O_G) = \sum_{l,m} \left( \prod_{i=1}^{r+1} \left( \frac{\mu_i}{\mu_i - m_i} \right) \left( \frac{-\lambda_i}{l_i - \lambda_i} \right) \right) \frac{1}{(l_i - m_j + j - i)!} |l/m| \quad 1 \leq i, j \leq r+1
\]

the sum being taken over all nonnegative integer sequences \( l \) and \( m \) such that \( m_i \leq \mu_i \) and \( l_i \geq \lambda_i \) for all \( i \), and such that \(|l/m| = |\lambda/\mu| + \rho \).

The proof is given in §3. In this theorem, the sequences \( l \) and \( m \) need not be partitions, and even when they are, \( l/m \) need not be a skew Young diagram. However, with a more detailed combinatorial analysis of the formula, one can rewrite it so that terms where \( l/m \) is a skew Young diagram are the only ones which contribute to the sum — see Theorem C.

We now turn to the degeneracy locus formulas. Recently, Hudson-Ikeda-Matsumura-Naruse gave a determinantal formula for the K-theory class of the structure sheaf of a Schubert variety in a Grassmann bundle [23], and
this was refined in [2] and [24]. The loci considered by these authors are not quite sufficient to compute the class of a two-pointed Brill-Noether variety — so we require a new determinantal formula. A special case of the result we obtain is related algebraically to a formula of Matsumura [28].

Here is the general setup. Given decreasing sequences of integers \( p \) and \( q \), consider vector bundles
\[
E_{p_i} \hookrightarrow \cdots \hookrightarrow E_{p_1} \twoheadrightarrow F_q \rightarrow \cdots \rightarrow F_{q_i},
\]
on a nonsingular variety \( X \), with the ranks indicated by subscripts. The degeneracy locus is
\[
W_{p,q} = \{ x \in X \mid \dim \ker(E_{p_j} \to F_{q_i}) \geq 1 + i - j \text{ for all } i, j \}.
\]
From the data \( p, q \), we define partitions \( \lambda \) and \( \mu \) by
\[
\lambda_i = q_i - t + i, \quad \mu_j = p_j - (t + 1 - j).
\]
(These partitions are related to the ones associated to the Brill-Noether loci; see the discussion at the end of this introduction for a special case, and §2 for more detail.)

In order for the rank conditions defining \( W_{p,q} \) to be feasible and nontrivial, one should require \( \lambda_i \geq \mu_i \), so that \( \lambda/\mu \) forms a skew Young diagram. The locus \( W_{p,q} \) has expected codimension equal to \( |\lambda/\mu| \).

We compute the class of \( W_{p,q} \) as a variation of a skew Schur determinant. Given partitions \( \lambda = (\lambda_1 \geq \cdots \geq \lambda_t) \) and \( \mu = (\mu_1 \geq \cdots \geq \mu_t) \), and indexed variables \( c(i, j) \) for \( 1 \leq i, j \leq t \), let us define the determinant
\[
\Delta_{\lambda/\mu}(c; \beta) = \sum_{k \geq 0} \left( \begin{array}{c} \lambda_i - \mu_j + k - 1 \\ k \end{array} \right) c_{\lambda_i-\mu_j+j-i+k}(i, j) \bigg|_{1 \leq i, j \leq t}. \]
The notation for the entries of this determinant can be condensed by using the operator \( T \) which raises the index of \( c(i, j) \), so \( T^k \cdot c_{m}(i, j) = c_{m+k}(i, j) \). Then
\[
\Delta_{\lambda/\mu}(c; \beta) = \left| (1 - \beta T)^{-\lambda_i+\mu_j} c_{\lambda_i-\mu_j+j-i}(i, j) \right|_{1 \leq i, j \leq t}.
\]
When \( \beta = 0 \) and
\[
c(i, j) = \prod_{k=1}^{t} (1 - x_k)^{-1}
\]
for all \( i, j \), this is the classical Jacobi-Trudi formula for the skew Schur function \( s_{\lambda/\mu}(x) \).

**Theorem B.** Assume that \( \lambda_i - \mu_i \geq 0 \) for all \( i \), and that \( W = W_{p,q} \) has codimension \( |\lambda/\mu| \). The class of \( W \) in \( K^o(X) \) is
\[
[\mathcal{O}_W] = \Delta_{\lambda/\mu}(c; -1),
\]
where \( c \) is the \( \mathcal{O}_W \) class for the K-theoretic Chern class.

This is proved in §1, as part (ii) of Theorem 1.1. Part (i) of Theorem 1.1 provides a more general statement needed for the proof of Theorem A. In fact, all the formulas we prove take place in the connective K-theory of \( X \), a
module over \( \mathbb{Z}[\beta] \) which interpolates K-theory (at \( \beta = -1 \)) and Chow groups (at \( \beta = 0 \)). So our formulas also specialize directly to cohomology.

There is a general correspondence between degeneracy loci and permutations, as explained in [17], for example. Our loci \( W_{p,q} \) are exactly those corresponding to 321-avoiding permutations, i.e., permutations with no decreasing subsequence of length three. Under this correspondence, the formulas for general degeneracy loci are related to the (double) Schubert polynomials and Grothendieck polynomials of Lascoux and Schützenberger. Our K-theoretic results therefore give new determinantal formulas for the double Grothendieck polynomials of 321-avoiding permutations, extending work by Matsumura [28]. Specializing to cohomology, we recover formulas of Billey-Jockusch-Stanley [6] and Chen-Li-Louck [10], giving new proofs via geometry. The details, including the correspondence between \((p, q)\) and 321-avoiding permutations, are described in §5.

In §4, we explain how our results can be phrased in terms of the combinatorics of tableaux. A row semi-standard Young tableau on a skew diagram \( \lambda/\mu \) is a filling of the boxes of \( \lambda/\mu \) whose entries are strictly increasing along rows and weakly decreasing down columns. A strict Young tableau is a filling whose entries are strictly increasing across each row and down each column. A standard Young tableau is a strict Young tableau using the numbers \( 1, \ldots, |\lambda/\mu| \).

The number of standard Young tableaux on a skew shape \( \lambda/\mu \) is denoted by \( f_{\lambda/\mu} \). We will use \( \alpha_{\lambda/\mu} \) to denote the number of row semi-standard Young tableaux on \( \lambda/\mu \) whose entries in row \( i \) are in \( \{1, \ldots, \lambda_i\} \); and \( \zeta_{\lambda/\mu} \) for the number of strict Young tableaux whose entries in row \( i \) are in \( \{1, \ldots, \lambda_i - 1\} \).

**Theorem C.** If \( \dim G_{a,b}^d(C, P, Q) = \rho \), then the Euler characteristic is

\[
\chi \left( \mathcal{O}_{G_{a,b}^d(C, P, Q)} \right) = \sum_{\lambda^+, \mu^-} \left( -1 \right)^{\lambda^+/\lambda^-} \cdot \alpha_{\mu^-/\mu^+} \cdot \zeta_{\lambda^+/\lambda^-} \cdot f_{\lambda^+/\mu^-}
\]

where the sum is over partitions \( \mu^- \subseteq \mu \) and \( \lambda^+ \supseteq \lambda \) of length \( r + 1 \) such that \( |\lambda^+/\mu^-| = |\lambda/\mu| + \rho \).

Special cases of this Theorem C include Castelnuovo’s formula and the Eisenbud-Harris-Pirola formula. Its proof is given in §4, along with a discussion of other special cases and further connections to the combinatorics of tableaux. We also establish the one-pointed case of a conjecture of Chan and Pflueger, expressing \( \chi(\mathcal{O}_{G_{a,b}^d(C, P)}) \) as an enumeration of set-valued tableaux.

To conclude this introduction, we briefly sketch the argument for the classical case of our main theorem, describing the Euler characteristic of the locus \( \mathcal{W}_d^r(C) \subseteq \text{Pic}^d(C) \). The construction is standard; see [25], [4, §VII], or [18, (14.4.5)].

Fix a point \( P \in C \), and let \( \mathcal{L} \) be a Poincaré bundle on \( C \times \text{Pic}^d(C) \), normalized so that \( \mathcal{L}_{\{P\} \times \text{Pic}^d(C)} \) is trivial. Choose a nonnegative integer \( n \) large enough so that all divisors of degree \( n + d \) are non-special; any
nonnegative $n \geq 2g - 1 - d$ will do. Writing $\pi_1$ and $\pi_2$ for the projections from $C \times \text{Pic}^d(C)$ to $C$ and $\text{Pic}^d(C)$, respectively, let $E = \pi_2^*(L \otimes \pi_1^*\mathcal{O}_C(nP))$ and $F = \pi_2^*(L \otimes \pi_1^*\mathcal{O}_{nP})$. Then the exact sequence on $C$

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(nP) \to \mathcal{O}_{nP} \to 0$$

transforms via $\pi_1^*$ and $\pi_2^*$ into an exact sequence

$$0 \to \pi_2^*L \to E \xrightarrow{\varphi} F$$

on $\text{Pic}^d(C)$. The Brill-Noether variety $W^r_d(C)$ is thereby identified with the locus where $\dim \ker(\varphi) \geq r + 1$.

Since $L(nP)$ is non-special for all $L$ in $\text{Pic}^d(C)$, Riemann-Roch shows that the sheaf $E$ is locally free of rank equal to $h^0(C, L(nP))$; that is,

$$\text{rk}(E) = n + d - g + 1.$$ 

The sheaf $F$ is also locally free, of rank

$$\text{rk}(F) = n,$$

and in fact, the normalization of $L$ shows that $F$ has a filtration with trivial subquotients. (Apply $\pi_1^*$ and $\pi_2^*$ to the exact sequence

$$0 \to \mathcal{O}_P \to \mathcal{O}_{nP} \to \mathcal{O}_{(n-1)P} \to 0.$$)

This means the Chern classes of $F$ are trivial, so $c(F - E) = c(-E)$.

The Brill-Noether dimension estimate comes from a basic fact about matrices: the locus of $q \times p$ matrices having kernel of dimension at least $t$ has codimension $t(q-p+t)$ inside the affine space of all matrices. (Take $t = r+1$, $p = n + d - g + 1$, and $q = n$ to get the Brill-Noether number.) Furthermore, applying the K-theoretic Giambelli formula of [2] yields

$$[\mathcal{O}_{W^r_d(C)}] = \sum_{k \geq 0} \binom{g - d + r + k - 1}{k} (-1)^k e_{g-d+r-j-i+k}(-E)_{1 \leq i,j \leq r+1}$$

whenever $\dim W^r_d(C) = \rho(g, r, d)$. The Euler characteristic formula is then deduced from Hirzebruch-Riemann-Roch and some linear algebra (see §4).

Acknowledgements. Our initial motivation for this project came from studying the papers [9] and [23], and we thank these authors for their inspiring work. This collaboration began at the Fields Institute Thematic Program on Combinatorial Algebraic Geometry, and we are grateful to the organizers and the Institute for providing a stimulating working environment.

1. A Determinantal Formula

Let us return to the setup of Theorem B. We have a sequence of vector bundles

$$E_{p_t} \hookrightarrow \cdots \hookrightarrow E_{p_1} = E \xrightarrow{\varphi} F = F_{q_t} \to \cdots \to F_{q_t}$$
on a (now possibly singular) variety $X$, where subscripts indicate rank, so that

$$0 < p_t < \cdots < p_1 \quad \text{and} \quad q_1 > \cdots > q_t > 0.$$ 

Let $V = E \oplus F$. This includes isomorphic copies of the subbundles $E_\bullet$ via the graph of $\varphi$:

$$E_{p_t} \subseteq \cdots \subseteq E_{p_1} = E_\varphi \subseteq V,$$

and also comes with natural projections $V \to F_{q_i}$ for all $i$.

The degeneracy loci we will consider lie in $X$, and in a Grassmann bundle over $X$:

$$\Omega \hookrightarrow \text{Gr} = \text{Gr}(t, V)$$

$$\downarrow \quad \quad \quad \quad \downarrow \pi$$

$$W \hookrightarrow X$$

The locus $W = W_{p,q} \subseteq X$ is defined by the conditions

$$\dim \ker(E_{p_j} \to F_{q_i}) \geq 1 + i - j,$$

for all $i, j$, and here we assume

$$(*) \quad q_i \geq p_i - 1 \quad \text{for all } i,$$

to avoid trivially satisfied conditions. (Evidently, it also suffices to require conditions only for $j \leq i$.)

To define $\Omega$, let $S \subseteq V$ be the tautological rank $t$ subbundle on $\text{Gr}$. (Here $V$ should be understood as $\pi^*V$ — following a common abuse, we omit notation for such pullbacks.) Using the inclusions and projections $E_{p_j} \hookrightarrow V \to F_{q_i}$ described above, $\Omega = \Omega_{p,q} \subseteq \text{Gr}$ is defined by the conditions

$$\dim(S \cap E_{p_j}) \geq t + 1 - j \quad \text{and} \quad \dim \ker(S \to F_{q_i}) \geq i$$

for all $1 \leq i, j \leq t$. No restrictions on $p$ and $q$ are needed here.

As in the introduction, we define partitions $\lambda$ and $\mu$ by

$$\lambda_i = q_i - t + i \quad \text{and} \quad \mu_j = p_j - (t + 1 - j).$$

The condition $(*)$ is equivalent to requiring that $\lambda_i \geq \mu_i$ for all $i$, i.e., that $\lambda/\mu$ is a skew shape.

Our main theorem about degeneracy loci gives formulas for $\pi_*[\Omega]$ and $[W]$, in the connective K-homology of $X$. Foundational facts about this theory can be found in [8, 12], and briefer digests are in [23] and [2, Appendix A]. The main features we will require are the following:

(a) Connective K-homology $CK_\ast(X)$ is a graded module over $\mathbb{Z}[\beta]$, with $\beta$ having degree 1.

(b) There are Chern classes operators for vector bundles; if $\alpha \in CK_\ast(X)$, then $c_k(E) \cdot \alpha \in CK_{\ast-k}(X)$. 
(c) Specializing $\beta = 0$ and $\beta = -1$ induces natural isomorphisms

$$CK_*(X)/(\beta = 0) \cong A_*(X) \quad \text{and} \quad CK_*(X)/(\beta = -1) \cong K_c(X)$$

with Chow homology and the Grothendieck group of coherent sheaves, respectively.

(d) There are fundamental classes $[Z] \in CK_*(X)$ for closed subvarieties $Z \subseteq X$, specializing to $[Z] \in A_*(X)$ and $[O_Z] \in K_c(X)$.

Given indexed variables $c(i, j)$ for $1 \leq i, j \leq t$, we define the determinant

$$\Delta_{\lambda/\mu}(c; \beta) = \left| (1 - \beta T)^{-\lambda_i - \mu_j} c_{\lambda_i - \mu_j + j - i}(i, j) \right|_{1 \leq i, j \leq t}$$

as in the introduction. Now we can state the theorem.

**Theorem 1.1.** Assume $X$ is an irreducible Cohen-Macaulay variety, and let $c(i, j) = c(F_{pq} - E_{pq})$ be the K-theoretic Chern classes.

(i) If $\Omega = \Omega_{p,q} \subseteq \text{Gr}(s, V)$ has codim $\Omega \leq |\lambda/\mu| + t(p_1 + q_1 - t)$. If equality holds, then $\Omega$ is Cohen-Macaulay, and

$$\pi_*[\Omega] = \Delta_{\lambda/\mu}(c; \beta) \cdot [X]$$

in $CK_*(X)$.

(ii) Assume $(*), i.e., \lambda_i \geq \mu_i$ for all $i$. Then $W \subseteq X$ has codim $W \leq |\lambda/\mu|$. If equality holds, then $W$ is Cohen-Macaulay, and

$$[W] = \Delta_{\lambda/\mu}(c; \beta) \cdot [X]$$

in $CK_*(X)$.

The statement in (ii) specializes to Theorem B from the introduction.

The proof of the theorem will occupy the rest of this section. We begin by reviewing the meaning of these degeneracy classes. As usual, a degeneracy locus inherits its scheme structure by pullback from a universal case; the classes are also pulled back. The key statement is this (cf. [12, Theorem 7.4]):

**Lemma 1.2.** Let $f$ be a morphism from a pure-dimensional Cohen-Macaulay scheme $X$ to a nonsingular variety $Y$. Suppose $Z \subseteq Y$ is a Cohen-Macaulay subscheme of pure codimension $d$. Then $W = f^{-1}Z$ has codimension $\leq d$. If $W$ has pure codimension $d$ in $X$, then it is Cohen-Macaulay and $[W] = f^*[Z]$ in $CK_*(X)$.

**Proof.** Everything except the last statement is contained in [19, Lemma, p. 108], and the equality $[W] = f^*[Z]$ is also proved there for cohomology (or Chow) classes. So it suffices to prove this equality for K-theory, which we do by a slight refinement of the standard argument for cohomology. Let
$\Gamma_f \subseteq X \times Y$ be the graph, so $W$ is identified with $\Gamma_f \cap X \times Z$ via the first projection.

If $\dim Y = m$, then the graph $\Gamma_f \subseteq X \times Y$ is locally cut out by a regular sequence $y_1, \ldots, y_m$; that is, the Koszul complex $K_\bullet(y)$ is exact and resolves $O_{\Gamma_f}$. Indeed, there is an exact sequence

$$0 \leftarrow O_{\Gamma_f} \leftarrow T \leftarrow \Lambda^2 T \leftarrow \cdots \leftarrow \Lambda^m T \leftarrow 0,$$

where $T = \text{pr}_2^* T_Y^*$ is the cotangent bundle of $Y$, pulled back to $X \times Y$.

Since $X \times Z$ is Cohen-Macaulay and $W \cong \Gamma_f \cap X \times Z$ has codimension $d + m$ in $X \times Y$, the restrictions $\mathcal{F}_1, \ldots, \mathcal{F}_m$ to $X \times Z$ also form a regular sequence. This means the Koszul complex $K_\bullet(\mathcal{F}) = K_\bullet(y) \otimes O_{X \times Z}$ is also exact, so by restricting the above resolution to $X$ via the graph morphism, we obtain an exact sequence

$$0 \leftarrow O_W \leftarrow T \otimes O_X \leftarrow \Lambda^2 T \otimes O_X \leftarrow \cdots \leftarrow \Lambda^m T \otimes O_X \leftarrow 0.$$

Recalling that $f^*[O_Z]$ is defined to be $\sum (-1)^i [\text{Tor}^Y_i(O_X, O_Z)]$, we see that $f^*[O_Z] = [O_X \otimes O_Y, O_Z] = [O_W]$, since exactness of the above sequence shows that the higher Tor terms vanish. \hfill $\Box$

In the present setting, the flag $E_\bullet$ of subbundles of $V$ defines a section of a partial flag bundle,

$$\sigma: X \to \text{Fl} = \text{Fl}(p, V),$$

so that $\sigma^* E_\bullet = E_\bullet$, where $E_\bullet$ is the tautological flag bundle on $\text{Fl}$. In this flag bundle, there is the universal locus $W = W_{p,q}$ defined by the analogous conditions,

$$\dim \ker(E_{p_j} \to F_{q_i}) \geq 1 + i - j,$$

for all $i, j$, again assuming $(\ast)$. By construction, $W = \sigma^{-1} W$.

The section $\sigma$ lifts to a section

$$\tilde{\sigma}: \text{Gr} \to \text{Gr} \times_X \text{Fl},$$

and the universal locus $\Omega = \Omega_{p,q}$ in $\text{Gr} \times_X \text{Fl}$ is defined similarly, by the conditions

$$\dim(S \cap E_{p_j}) \geq t + 1 - j \quad \text{and} \quad \dim \ker(S \to F_{q_i}) \geq i$$

(with no restrictions on $p, q$).

The situation (and notation) is summarized in the diagram:
The relationship between the loci $W$ and $\Omega$ is described by the following proposition.

**Proposition 1.3.** Given $p, q$, there is a $q'$ so that $p, q'$ satisfies ($*$) and 
\[ \tilde{\pi}(\Omega_{p,q}) = W_{p,q}'. \]
When $q' = q$, the projection $\tilde{\pi}: \Omega_{p,q} \to W_{p,q}$ is birational, and moreover, 
\[ \tilde{\pi}_*(\Omega_{p,q}) = [W_{p,q}]. \]

Together with Lemma 1.2, the proposition implies that corresponding statements also hold for the pulled back loci $\pi: \Omega_{p,q} \to W_{p,q}$. In fact, this proves all of the statements of Theorem 1.1, except for the determinantal formulas.

Since the statements of Proposition 1.3 are local, in proving them we may replace $X$ by the flag variety $Fl = Fl(p, V)$, and consider the map $\pi: Fl \times Gr \to Fl$, where $Gr = Gr(t, V)$ is the Grassmannian.

The locus $W$ is a Schubert variety in $Fl$, so it corresponds to some permutation $w \in S_n$, where $n = p_1 + q_1$ is the dimension of $V$. The first task is to determine this permutation.

For any $p, q$, define an associated permutation $w$ by setting
\[ w(p_i) = \max\{q_i + 1, p_i\} \quad \text{for } 1 \leq i \leq t, \]
and then filling in the remaining entries minimally with unused numbers in increasing order. For example, if $p = (5, 4, 1)$ and $q = (5, 2, 1)$ then 
\[ w = 2 \begin{array}{cccc} {1} & {3} & {4} & {6} & {5} \end{array}. \]
Given $p, q$, let us define $q'$ by
\[ q_i' = \max\{q_i, p_i - 1\}. \]
Evidently the associated permutation $w$ is unchanged, and the new pair $p, q'$ satisfies ($*$), so the corresponding partitions $\lambda', \mu$ form a skew diagram $\lambda' / \mu$.

**Lemma 1.4.** The permutation $w$ is the unique one of minimal length such that
\[ \#\{p \leq p_j \mid w(p) > q_i\} \geq 1 + i - j \]
for all $i, j$. Its length is equal to $|\lambda' / \mu| = \sum_{i=1}^{t} (q_i' - p_i + 1)$.
(The length of $w$ is defined to be $\#\{a < b \mid w(a) > w(b)\}$. It is the codimension of the corresponding Schubert variety in $Fl$.) The lemma can be proved using Fulton’s essential set [17].

**Proof of Proposition 1.3.** Let $W = W_{p,q'}$ and $\Omega = \Omega_{p,q}$. The conditions defining $\Omega$ imply that after forgetting the $t$-dimensional subspace $S \subseteq V$, one has
\[ \dim \ker(E_{p_j} \to F_{q_i}) \geq 1 + i - j \]
for all $i, j$. Lemma 1.4 implies that these same conditions suffice to define the Schubert variety for $w$. (See [17].) In particular, the conditions corresponding to $p, q$ and $p, q'$ define the same locus $W \subseteq Fl$. It follows that $\pi(\Omega_{p,q}) \subseteq W_{p,q'}$. 

On the other hand, the projection \( \pi : \Omega \to W \) is \( B \)-equivariant for the standard action of \( B \subseteq GL(V) \) (a Borel subgroup fixing the flag \( F \)) on \( Fl \times Gr \). To show that \( \pi \) is surjective, it suffices to show that the fiber over \( E \in W \) is nonempty, for any choice of general \( E \), i.e., one such that

\[
K_{j,i} := \ker(E_{p_j} \to F_{q_i})
\]

has dimension equal to \( 1 + i - j \), for all \( j \leq i + 1 \). This is straightforward: from the assumption on \( E \), one sees

\[
dim K_{i,i} = \dim \ker(E_{p_i} \to F_{q_i}) = 1
\]

and these lines are independent, since \( \dim \ker(E_{p_{i+1}} \to F_{q_i}) = 0 \) means \( K_{i,i} \cap E_{p_{i+1}} = 0 \). So

\[
S = K_{1,1} \oplus K_{2,2} \oplus \cdots \oplus K_{t,t} \subseteq E_{p_1}
\]

has dimension \( t \), and \( S \cap E_{p_j} = K_{j,j} \oplus \cdots \oplus K_{t,t} \) has dimension \( t + 1 - j \). Since \( F_{q_i} \to F_{q_i} \), one has \( K_{j,i} \subseteq \ker(E_{p_j} \to F_{q_i}) \) for all \( i, j \), so

\[
\ker(S \to F_{q_i}) \supseteq K_{1,1} \oplus \cdots \oplus K_{i,i}
\]

has dimension at least \( i \).

We conclude that \( \pi(\Omega) = W \). If \( q' = q \), the construction in the previous paragraph determines \( S \) uniquely, so \( \Omega \to W \) is birational in this case.

Being a Schubert variety, \( W = W_{p,q} \) has rational singularities. We will show that \( \Omega \) also has rational singularities. Let \( \pi_2 : Fl \times Gr \to Gr \) be the second projection. Then \( \Omega = \Omega' \cap \pi_2^{-1} \Omega \), where

\[
\Omega' = \{ S \mid \dim \ker(S \to F_{q_i}) \geq i \ \text{for all} \ i \} \subseteq Gr
\]

and

\[
\Omega = \{ (E, S) \mid \dim (S \cap E_{p_j}) \geq t + 1 - j \ \text{for all} \ j \} \subseteq Fl \times Gr.
\]

Restricting the projections to \( \Omega' \) produces flat morphisms \( \pi : \Omega' \to Fl \) and \( \pi_2 : \Omega' \to Gr \), whose fibers can be described explicitly.

The fiber of the first projection \( \Omega' \to Fl \) over a flag \( E \) is a Schubert variety \( \Omega_\nu(E) \subseteq Gr \), where \( \nu = \mu^\vee \) is the complementary partition to \( \mu \) inside the \( t \times (p_1 + q_1 - t) \) rectangle. (Specifically, \( \nu_j = p_1 + q_1 - t - \mu_{t+1-j} = p_1 + q_1 - t - p_{t+1-j} + j \).) It follows that \( \dim \Omega' = \dim Fl + |\mu| \).

The fiber of the second projection \( \Omega' \to Gr \) over \( S \) is a Schubert variety in \( Fl \). (Its associated permutation is the inverse of the Grassmannian permutation for the partition \( \nu \).) In particular, after intersecting with \( \pi_2^{-1} \Omega_\nu \), the morphism

\[
\pi_2 : \Omega \to \Omega_\nu
\]

is again flat, and both the base and fibers have rational singularities. By [14], we conclude that \( \Omega \) has rational singularities. It follows that \( \pi_*[\Omega] = [W] \) when \( q' = q \).

We now return to the general setup of Theorem 1.1 and complete the proof.
Proof of Theorem 1.1. The argument for Proposition 1.3 shows that the universal loci have expected codimensions:

$$\text{codim } \Omega_{p,q} = |\lambda/\mu| + t(p_1 + q_1 - t)$$

and (when $p,q$ satisfy (⋆))

$$\text{codim } W_{p,q} = |\lambda/\mu|$$

in $\text{Fl}_X \times \text{Gr}$ and $\text{Fl}_t$, respectively. The inequalities for the pulled back loci $\Omega$ and $W$ therefore follow from Lemma 1.2.

Now let us assume that equality holds for the codimension of $\Omega$. We will prove the determinantal formula for $\pi_*[\Omega]$ in part (i) of Theorem 1.1; the formula for $[W]$ follows using the second statement of Proposition 1.3.

As in the proof of Proposition 1.3, write $\Omega = \Omega' \cap \pi_2^{-1}\Omega_\lambda$ as an intersection of degeneracy loci, where $\Omega'$ is defined by $\dim(S \cap E_{p_j}) \geq t + 1 - j$ for all $j$, and $\Omega_\lambda$ is defined by $\dim \ker(S \to F_{q_i}) \geq i$ for all $i$. Using Lemma 1.2 again, we have $[\Omega] = [\Omega'] \cdot \pi_2^*\Omega_\lambda$.

The locus $\Omega'$ can be resolved in the standard way, via a sequence of projective bundles:

$$\tilde{\Omega}' = \mathbb{P}(E_{p_1}/S_{t-1}) \xrightarrow{\pi^{(1)}} \cdots \xrightarrow{\pi^{(3)}} \mathbb{P}(E_{p_{t-1}}/S_1) \xrightarrow{\pi^{(2)}} \mathbb{P}(E_{p_t}) \xrightarrow{\pi^{(1)}} X,$$

where $S_{j+1}/S_j \subset E_{p_{t-j}}/S_j$ is the tautological line bundle on $\mathbb{P}(E_{p_{t-j}}/S_j)$ (suppressing notation for pullbacks of bundles, as usual). The rank $t$ bundle $S_t \subset E_{p_1} = E$ on $\tilde{\Omega}'$ defines a map

$$f: \tilde{\Omega}' \to \Omega' \subset \text{Gr}(t, E),$$

which is a desingularization. Write $\pi': \tilde{\Omega}' \to X$ for the composition.

Since $\Omega'$ has rational singularities, $f_*[\Omega'] = [\Omega']$. By the projection formula, we obtain

$$f_*f^*\pi_2^*[\Omega_\lambda] = [\Omega'] \cdot \pi_2^*[\Omega_\lambda] = [\Omega].$$

Now we can compute the pushforward as

$$\pi_*[\Omega] = \pi_*f_*f^*\pi_2^*[\Omega_\lambda] = \pi_*f^*\pi_2^*[\Omega_\lambda].$$

The class $[\Omega_\lambda]$ is computed by [2, Theorem 1], and therefore so is its pullback:

$$f^*\pi_2^*[\Omega_\lambda] = \det \left((1 - \beta T)^{-\lambda_i c_{\lambda_i + j - i}(F_{q_i} - S_i)}\right)_{1 \leq i,j \leq t}.$$

A formal determinantal identity used in the “general case” of the proof of [2, Theorem 1] shows that

$$\left|(1 - \beta T)^{-\lambda_i c_{\lambda_i + j - i}(F_{q_i} - S)}\right| = \left|(1 - \beta T)^{-\lambda_i c_{\lambda_i + j - i}(F_{q_i} - S_{t+1-j})}\right|_{1 \leq i,j \leq t}.$$

Finally, for all $i,j$, one has

$$\pi_*^{(i)} \left((1 - \beta T)^{-m c_m(F_{q_i} - S_{t+1-j})}\right) = (1 - \beta T)^{-m + p_j - (t+1-j)c_{m-p_j+t+1-j}(F_{q_i} - E_{p_j})}. $$

(This is Eq. (5) in [2], and an explanation is given there.)
Applying this to the entries of the determinant gives
\[
\pi_\ast[\Omega] = \pi'_\ast \left( \left| (1 - \beta T)^{-\lambda_i} c_{\lambda_i + j - i} (F_{q_i} - E_{p_j}) \right|_{1 \leq i, j \leq t} \right)
\]
= \left| (1 - \beta T)^{-\lambda_i + p_j - (t + 1 - j)} c_{\lambda_i - p_j + (t + 1 - j) + j - i} (F_{q_i} - E_{p_j}) \right|_{1 \leq i, j \leq t}
\]
= \left| (1 - \beta T)^{-\lambda_i + \mu_i} c_{\lambda_i - \mu_j + j - i} (F_{q_i} - E_{p_j}) \right|_{1 \leq i, j \leq t},
\]
so the theorem is proved. □

In the rest of the paper, we will discuss some basic applications of the degeneracy locus formula. One of them is a direct generalization of Kleiman and Laksov’s proof of the existence theorem: by standard intersection theory, one can deduce a criterion for non-emptiness of a degeneracy locus from a formula for its class.

**Corollary 1.5.** Let \( p, q, \) and \( q' \) be as above, so \( p, q' \) satisfies \((*)\), and let \( \lambda' / \mu \) be the skew diagram corresponding to \( p, q' \). If \( \Delta_{\lambda' / \mu}(c; 0) \cdot [X] \neq 0 \) in \( A_\ast(X) \), then \( W_{p, q'} \) is nonempty, and so \( \Omega_{p, q} \) is also nonempty.

The converse holds when \( X \) is projective: If \( \text{codim} W_{p, q'} \geq |\lambda' / \mu| \) and \( W_{p, q'} \) (or equivalently, \( \Omega_{p, q} \)) is nonempty, then \( \text{codim} W_{p, q} = |\lambda' / \mu| \) and \( \Delta_{\lambda' / \mu}(c; 0) \cdot [X] \) is nonzero.

**Remark 1.6.** In [23], formulas are proved in connective K-cohomology \( CK^\ast(X) \), under the hypothesis that \( X \) is smooth. The relationship with our more general setup is best described in the framework of the operational cohomology theory associated to a (generalized oriented Borel-Moore) homology theory \([3, 21]\). One can define \( CK^\ast \) to be the operational cohomology ring associated to the homology theory \( CK_\ast \), so that \( CK^\ast(X) \) is defined for any scheme. This is a graded algebra over \( \mathbb{Z}[\beta] \), where now \( \beta \) has degree \(-1\), and \( CK^\ast(X) \) is a module for \( CK^\ast(X) \), with \( c \in CK^i(X) \) acting as a homomorphism \( CK^\ast(X) \to CK^\ast-i(X) \).

Specializing at \( \beta = 0 \) and \( \beta = -1 \) produces natural isomorphisms
\[
CK^\ast(X)/(\beta = 0) \cong A^\ast(X) \quad \text{and} \quad CK^\ast(X)/(\beta = -1) \cong \text{op}K^\circ(X),
\]
where \( A^\ast(X) \) is the Fulton-MacPherson operational Chow ring, and \( \text{op}K^\circ(X) \) is the operational K-theory developed in [3]. When \( X \) is smooth, Poincaré isomorphisms show that the operational \( CK^\ast(X) \) agrees with the connective K-cohomology used in [23], and that \( CK^\ast(X) \cong CK_{\dim X-\ast}(X) \).

## 2. Varieties of linear series as degeneracy loci

Let \( C \) be a smooth curve of genus \( g \). Given a linear series \( \ell = (L, V) \) in \( G_d^\ell(C) \) and a point \( P \in C \), the definition of the vanishing sequence
\[
a^\ell(P) = \left( 0 \leq a_0^\ell(P) < \cdots < a_r^\ell(P) \leq d \right)
\]
from the introduction can be equivalently phrased by the condition
\[ V \cap H^0(C, L(-a_{r+1-i}P)) \geq i \]
for \( 1 \leq i \leq r+1 \). Fixing two points \( P \) and \( Q \), and two sequences \( a \) and \( b \), the variety of linear series \( G^{a,b}_d(C, P, Q) \) is therefore defined by the conditions
\[
\dim(V \cap H^0(C, L(-a_{r+1-i}P))) \geq i \quad \text{and} \\
\dim(V \cap H^0(C, L(-b_{r+1-i}Q))) \geq i
\]
for all \( 1 \leq i \leq r+1 \). We will construct \( G^{a,b}_d(C, P, Q) \) as a degeneracy locus \( \Omega_{p,q} \) inside a certain Grassmann bundle \( \pi: \text{Gr} \to \text{Pic}^d(C) \), where \( p \) and \( q \) are indices to be specified below.

The construction generalizes the description of \( W^r_d(C) \) reviewed in the introduction. As before, choose \( n \geq 0 \) large enough so that line bundles of degree \( d+n-b_r \) are non-special, that is, \( n \geq 2g-1-d+b_r \). Fix a Poincaré line bundle \( L \) on \( C \times \text{Pic}^d(C) \), normalized so that \( L|_{(P)} \times \text{Pic}^d(C) \) is trivial. Let \( \pi_1 \) and \( \pi_2 \) be the projections from \( C \times \text{Pic}^d(C) \) to \( C \) and \( \text{Pic}^d(C) \), and set
\[
\mathcal{E}_j = (\pi_2)_*(L \otimes \pi_1^* O_C(nP - b_j Q)) \quad \text{and} \\
\mathcal{F}_i = (\pi_2)_*(L \otimes \pi_1^* O_{(n+a_{r+1-i})P}),
\]
for \( 1 \leq i, j \leq r+1 \). The sheaf \( \mathcal{E}_j \) is a vector bundle of rank
\[ p_j := \text{rk}(\mathcal{E}_j) = n + d - b_{j+1} + 1 - g, \]
and \( \mathcal{F}_i \) is a vector bundle of rank
\[ q_i := \text{rk}(\mathcal{F}_i) = n + a_{r+1-i}. \]

One obtains a sequence
\[ \mathcal{E}_{r+1} \hookrightarrow \mathcal{E}_r \hookrightarrow \cdots \hookrightarrow \mathcal{E}_1 \xrightarrow{\varphi} \mathcal{E} \xrightarrow{\pi} \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \cdots \rightarrow \mathcal{F}_{r+1} \]
of vector bundles over \( \text{Pic}^d(C) \).

Define \( V := \mathcal{E} \oplus \mathcal{F} \), with the natural maps \( V \to \mathcal{F}_i \), and with \( \mathcal{E}_j \hookrightarrow V \) included via the graph of \( \varphi \). Consider the Grassmann bundle
\[ \pi: \text{Gr}(r+1, V) \to \text{Pic}^d(C), \]
and let \( S \subseteq V \) be the tautological rank \( r+1 \) subbundle on \( \text{Gr}(r+1, V) \). The locus in \( \text{Gr}(r+1, V) \) defined by the conditions
\[ \dim \ker(S \to \mathcal{F}_i) \geq i \quad \text{for all } i, \text{ and } \ S \subseteq \mathcal{E} \]
coincides with the locus of linear series \( (L, V) \in G^r_d(C) \) such that
\[
\dim(V \cap H^0(C, L(-a_{r+1-i}P))) \geq i \quad \text{for all } i, \text{ and} \\
\dim(V \cap H^0(C, L(-b_0 Q))) \geq r+1.
\]
Imposing the additional conditions \( \dim(S \cap \mathcal{E}_j) \geq t + 1 - j \) for all \( j \), one recovers the locus of linear series satisfying (2). Thus \( G^{a,b}_d(C, P, Q) \) can be identified with the degeneracy locus \( \Omega_{p,q} \subseteq \text{Gr}(r+1, V) \).
We now study the image of $G^a_b(C, P, Q)$ in $\text{Pic}^d(C)$ via the map $\pi$. Let $W^a_b(C, P, Q)$ be the degeneracy locus $W_{p, q}$ in $\text{Pic}^d(C)$ as in §1, that is,

$$W^a_b(C, P, Q) = \left\{ L \in \text{Pic}^d(C) \mid \dim \ker(\mathcal{E}_j \rightarrow \mathcal{F}_i) \geq 1 + i - j \right\}.$$ 

Equivalently, $W^a_b(C, P, Q)$ is the locus of line bundles $L \in \text{Pic}^d(C)$ such that

$$h^0(C, L(-a_{r+1-i}P - b_{j-1}Q)) \geq 1 + i - j$$

for all $i, j$. Recall the definition of the two partitions $\lambda$ and $\mu$ associated to the data $g, d, a, b :$

$$\lambda_i = n + a_{r+1-i} - (r + 1 - i), \quad \mu_i = n - b_{i-1} + i - 1 - g + d - r$$

for $1 \leq i \leq r + 1$. When the diagram $\lambda/\mu$ is a skew shape, one has $\pi(G^a_b(C, P, Q)) = W^a_b(C, P, Q),$ and in this case $\pi : G^a_b(C, P, Q) \rightarrow W^a_b(C, P, Q)$ is birational. In general, let $a'$ be the sequence defined as

$$a'_i := a_i + \max\{0, d - g - a_i - b_{r-i}\}.$$

The diagram $\lambda'/\mu$ is a skew shape by construction, where

$$\lambda'_i = n + a'_{r+1-i} - (r + 1 - i)$$

for all $i$. (If $\lambda/\mu$ is already a skew shape, then $a' = a$ and $\lambda' = \lambda$.)

We have the following diagram

$$
\begin{array}{ccc}
G^a_b(C, P, Q) & \xrightarrow{} & \text{Gr} (r + 1, V) \\
\downarrow & & \downarrow \pi \\
W^a_b(C, P, Q) & \xrightarrow{} & \text{Pic}^d(C)
\end{array}
$$

fitting into the framework studied in §1, so we can apply Theorem 1.1. Recall that the expected dimension for pointed Brill-Noether loci is

$$\rho(g, r, d, a, b) = g - \sum_{i=0}^{r} (g - d + a_i + b_{r-i})$$

$$= g - |\lambda/\mu|.$$ 

Assume that $G^a_b(C, P, Q)$ has dimension equal to $\rho$. Then it is Cohen-Macaulay, and the determinantal formula in §1 gives

(3) \[ \pi_* \left[ G^a_b(C, P, Q) \right] = \left[ (1 - \beta T)^{-\lambda_i + \mu_j} c^K_{\lambda_i - \mu_j + j - i}(-\mathcal{E}_j) \right]_{1 \leq i, j \leq r+1} \]

$$= \sum_{k \geq 0} \binom{\lambda_i - \mu_j + k - 1}{k} \beta^k c^K_{\lambda_i - \mu_j + j - i + k}(-\mathcal{E}_j) \left|_{1 \leq i, j \leq r+1} \right.$$ 

in $CK^*(\text{Pic}^d(C))$, where $c^K$ is the K-theoretic Chern class.
Similarly, assume $\lambda/\mu$ is a skew shape. If $W_d^{a,b}(C, P, Q)$ has dimension equal to $\rho$, then it is Cohen-Macaulay, and has class given by (3) in $C K^*(\text{Pic}^d(C))$.

When both $G_d^{a,b}(C, P, Q)$ and $W_d^{a',b}(C, P, Q)$ have the expected codimension,

$$
\pi_* \left[ G_d^{a,b}(C, P, Q) \right] = \beta^{a'|-a|} \pi_* \left[ G_d^{a',b}(C, P, Q) \right] = \beta^{a'|-a|} \left[ W_d^{a',b}(C, P, Q) \right].
$$

The first equality follows from (3) and by linearity, since for partitions $\lambda$ and $\mu$, the determinant $|c_{\lambda_i-\mu_j+i}^K|$ vanishes unless $\lambda/\mu$ is a skew shape. (If $\lambda_k < \mu_k$ for some $k$, then the matrix is singular, since it has 0’s in positions $(i, j)$ for all $i \geq k \geq j$.)

## 3. Euler characteristics

In this section we give a formula for the Euler characteristic of the two-pointed Brill-Noether loci $G_d^{a,b}(C, P, Q)$. In order to simplify the determinant formula (3) for the $K$-classes of varieties of linear series, we prove some general lemmas on K-theoretic Chern classes and apply them to the bundles $E_i$ in §2.

**Lemma 3.1.** Suppose a rank-$e$ vector bundle $E$ has $ch(E)_i = 0$ for $i > 1$. Then $ch(c^K_i(E)) = c_i(E)$.

That is, if the Chern character of $E$ is $ch(E) = e + c_1(E)$, then K-theory Chern classes agree with cohomology Chern classes under the Chern character isomorphism.

**Proof.** First recall that the Chern class of a line bundle has Chern character $ch(c^K_1(L)) = 1 - e^{-c_1(L)}$. Now let $L_1, \ldots, L_e$ be K-theoretic Chern roots of $E$, i.e., line bundle classes so that $c^K(E) = c^K(L_1) \cdots c^K(L_e)$. Then

$$
ch(c^K_i(E)) = ch \left( e_i \left( c^K_1(L_1), \ldots, c^K_1(L_e) \right) \right) = e_i \left( ch \left( c^K_1(L_1) \right), \ldots, ch \left( c^K_1(L_e) \right) \right) = e_i(1 - e^{-c_1(L_1)}, \ldots, 1 - e^{-c_1(L_e)}),
$$

where $e_i(x_1, \ldots, x_e)$ is the elementary symmetric polynomial. The lowest degree term in the last line is equal to $c_i(E)$. The higher-degree terms vanish, thanks to the hypothesis $ch(E)_i = 0$ for $i > 1$, and Lemma 3.2.  

**Lemma 3.2.** Let $e_i$ be the elementary symmetric polynomial, let $e^x$ the formal power series $\sum_{k \geq 0} \frac{x^k}{k!}$, and $p_i(x_1, \ldots, x_n) := x_1^{i} + \cdots + x_n^{i}$ the power sum symmetric function. Then

$$
e_i(1 - e^{-x_1}, \ldots, 1 - e^{-x_n}) \equiv e_i(x_1, \ldots, x_n) \mod \text{the ideal } (p_2, \ldots, p_n) + (x_1, \ldots, x_n)^{n+1}.
$$
Proof. In the ring $\mathbb{Q}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{n+1}$, one has
\[ p_i(1 - e^{-x_1}, \ldots, 1 - e^{-x_n}) \equiv p_i(x_1, \ldots, x_n) =: p_i \]
for all $i \geq 1$, modulo the ideal $(p_2, \ldots, p_n)$. From Newton’s identities, it follows that
\[ e_i(1 - e^{-x_1}, \ldots, 1 - e^{-x_n}) \equiv \frac{1}{i} e_{i-1}(1 - e^{-x_1}, \ldots, 1 - e^{-x_n}) \cdot p_1, \]
which is equivalent to $e_i(x_1, \ldots, x_n)$ modulo $(p_2, \ldots, p_n)$. □

Let $(C, P, Q)$ be a two-pointed curve of genus $g$, and consider the vector bundles $E_i$ from §2. Lemma 3.1 applies to these bundles. Indeed, modulo numerical (or homological) equivalence, the Chern classes of $-E_i$ are
\[ c_j(-E_i) = \frac{\theta^j}{j!}, \]
where $\theta$ is the cohomology class of the theta divisor. (The proof given in [4, §VII] is for singular cohomology, but it works as well in numerical or homological equivalence.) Equivalently, $\text{ch}(-E_i) = \text{rank}(-E_i) + \theta$. We therefore have
\[ \text{ch}\left(c^K_j(-E_i)\right) = \frac{\theta^j}{j!}. \]

Now we can compute the Euler characteristic of the loci $G_{a,b}^d(C, P, Q)$ via Hirzebruch-Riemann-Roch. The Todd class of $\text{Pic}^d(C)$ is trivial, so
\[ \chi\left(\mathcal{O}_{G_{a,b}^d(C, P, Q)}\right) = \int_{\text{Pic}^d(C)} \text{ch}\left(\pi_*\left[\mathcal{O}_{G_{a,b}^d(C, P, Q)}\right]\right). \]
Combining (4) with the specialization of (3) at $\beta = -1$, the Euler characteristic $\chi\left(\mathcal{O}_{G_{a,b}^d(C, P, Q)}\right)$ is equal to
\[ \int_{\text{Pic}^d(C)} \left(1 + T\right)^{-g+d-a_{r+1-i}-b_{j-1}+j-i} c_{g-d+a_{r+1-i}+b_{j-1}} \left|_{1 \leq i,j \leq r+1} \right. \]
From the Poincaré formula $\int \theta^g = g!$, it follows that the Euler characteristic is $g!$ times the coefficient of $\theta^g$ in the expansion of the determinant. The next step is to analyze this expansion.

Let $\rho := \rho(g, r, d, a, b)$, and recall that
\[ \lambda_i - \mu_j + j - i = g - d + a_{r+1-i} + b_{j-1}. \]
If we expand the operators $(1 + T)^{-\lambda_i+\mu_j}$ in powers of $T$, the constant term is the cohomology class
\[ |c_{\lambda_i-\mu_j+j-i}|_{1 \leq i,j \leq r+1}, \]
and is a multiple of $\theta^{g-\rho}$ (possibly zero). The determinant in (5) is obtained by applying the operator $(1 + T)^{\mu j}$ to the $j$-th column of the matrix in (6), and the operator $(1 + T)^{-\lambda i}$ to its $i$-th row. By linearity, (5) equals

$$\sum_{|l/m|=|\lambda/\mu|+\rho} \left| \prod_{i=1}^{r+1} \left( \frac{\mu_i}{\mu_i - m_i} \right) \left( \frac{-\lambda_i}{l_i - \lambda_i} \right) \right| T^{l_i - \lambda_i + \mu_j - m_j} c_{\lambda_i - \mu_j + j - i} |1 \leq i, j \leq r+1|,$$

the sum over all sequences $l$ and $m$, with $l_i \geq \lambda_i$ and $m_i \leq \mu_i$. (Here $l$ and $m$ are not required to be partitions, but we still use the notation $|l/m| = \sum(l_i - m_i).$)

This proves Theorem A. More precisely, we have proved:

**Theorem 3.3.** Let $(C, P, Q)$ be any smooth two-pointed curve of genus $g$. If $G_{d}^{a,b}(C, P, Q)$ has dimension equal to $\rho$, then the Euler characteristic $\chi\left( O_{G_{d}^{a,b}(C, P, Q)} \right)$ equals

$$\sum_{|l/m|=|\lambda/\mu|+\rho} g! \left| \prod_{i=1}^{r+1} \left( \frac{\mu_i}{\mu_i - m_i} \right) \left( \frac{-\lambda_i}{l_i - \lambda_i} \right) \right| \left| \frac{1}{(l_i - m_j + j - i)!} \right| |1 \leq i, j \leq r+1|.$$

Assume furthermore that $\lambda_i \geq \mu_i$ for all $i$. If $W_{d}^{a,b}(C, P, Q)$ has dimension equal to $\rho$, then $\chi\left( O_{W_{d}^{a,b}(C, P, Q)} \right) = \chi\left( O_{G_{d}^{a,b}(C, P, Q)} \right)$.

4. **Determinantal and tableau formulas**

In this section, we will give a simplified expression for the Euler characteristic of the loci $G_{d}^{a,b}(C, P, Q)$, expressing it as a weighted enumeration of standard Young tableaux, by performing a combinatorial analysis of the sum. Along the way, we find a nonemptiness criterion for these loci, stated in Proposition 4.2. Then we examine several special cases of particular interest.

Recall that the notation $\lambda/\mu$ usually denotes a skew Young diagram — that is, $\lambda$ and $\mu$ are both partitions, and $\lambda_i \geq \mu_i$ for all $i$; the shape $\lambda/\mu$ is represented as the complement of $\mu$ in $\lambda$. A standard Young tableau on a skew diagram $\lambda/\mu$ is a filling of the boxes of $\lambda/\mu$ by numbers $1, \ldots, |\lambda/\mu|$ such that the entries in each row and in each column are strictly increasing. The number of standard Young tableaux on $\lambda/\mu$ is commonly denoted by $f_{\lambda/\mu}$, and is given by the determinantal formula

$$f_{\lambda/\mu} = |\lambda/\mu|! \left| \frac{1}{(\lambda_i - \mu_j + j - i)!} \right| |1 \leq i, j \leq r+1|,$$

(see [1]).

We will extend the above notation to arbitrary sequences $l = (l_1, \ldots, l_{r+1})$ and $m = (m_1, \ldots, m_{r+1})$ of nonnegative integers, writing $l/m$ for a “generalized skew diagram” — note that we allow the differences $l_i - m_i$ to be
negative. Extending the notation for skew shapes, we will write

$$|l/m| := \sum_{i=1}^{r+1} (l_i - m_i)$$

and

$$f^{l/m} := |l/m|! \left| \frac{1}{(l_i - m_j + j - i)!} \right|_{1 \leq i,j \leq r+1}. \tag{8}$$

There are two basic facts underpinning our arguments in this section:

**Fact 1:** Suppose $\lambda$ and $\mu$ are partitions of length $r+1$. Then

$$f^{\lambda/\mu} = |\lambda/\mu|! \left| \frac{1}{(\lambda_i - \mu_j + j - i)!} \right|_{1 \leq i,j \leq r+1}$$

is nonzero if and only if $\lambda_i \geq \mu_i$ for all $i$. (Here one should read reciprocals of factorials of negative integers as 0.)

**Fact 2:** Suppose $\lambda = (\lambda_1, \ldots, \lambda_{r+1})$ is a partition, and $l = (l_1, \ldots, l_{r+1})$ is any sequence of nonnegative integers such that $l_i \geq \lambda_i$ for all $i$. If the sequence $(l_1 - 1, \ldots, l_{r+1} - (r+1))$ consists of distinct integers, and $w$ is the permutation which sorts them into decreasing order, then the sequence $\lambda_i^+ = l_{w(i)} - w(i) + i$ is a partition with $\lambda_i^+ \geq \lambda_i$ for all $i$.

**Proof of Theorem C.** We can rewrite the formula of Theorem 3.3 as

$$\chi \left( \mathcal{O}_{G^a(C,P,Q)} \right) = \sum_{|l/m|=|\lambda/\mu|+\rho} \left( \prod_{i=1}^{r+1} \left( \frac{\mu_i}{\mu_i - m_i} \right) \left( -\lambda_i \right) \left( l_i - \lambda_i \right) \right) f^{l/m} \tag{9}$$

since $|\lambda/\mu| + \rho = g$. (Recall that the sums are over sequences $l$ and $m$ such that $l_i \geq \lambda_i$ and $m_i \leq \mu_i$ for all $i$.)

When the determinant

$$\frac{1}{|l/m|!} f^{l/m} = \left| \frac{1}{(l_i - m_j + j - i)!} \right|_{1 \leq i,j \leq r+1}$$

is nonzero, there is a unique permutation $w \in S_{r+1}$ acting on the columns of the matrix which sorts the entries across rows into decreasing order; equivalently,

$$\mu_j^- := m_{w(j)} + j - w(j) \tag{10}$$
defines a partition \( \mu^- \subseteq \mu \), using a variation of Fact 2. Then \( f^{l/m} = (-1)^{\text{sgn}(w)} f^{l/\mu^-} \), and collecting terms gives

\[
\chi \left( \mathcal{O}_{G_d}^{b,(C,P,Q)} \right) = \sum_{\mu^- \subseteq \mu} \alpha^{\mu/\mu^-} \left( \prod_{i=1}^{r+1} \left( \frac{l_i - 1}{l_i - \lambda_i} \right) \right) (-1)^{|l/\lambda|} f^{l/\mu^-},
\]

where the sum is over partitions \( \mu^- \subseteq \mu \) and sequences \( l = (l_1, \ldots, l_{r+1}) \) of nonnegative integers such that \( l_i \geq \lambda_i \) for all \( i \), \( |\mu/\mu^-| + |l/\lambda| = \rho \), and

\[
\alpha^{\mu/\mu^-} := \sum_{w \in S_{r+1}} (-1)^{\text{sgn}(w)} \left( \prod_{j=1}^{r+1} \left( \frac{\mu_{w(j)}}{\mu_j + j - w(j)} \right) \right)
\]

\[
\left| \left( \frac{\mu_i}{\mu_i - \mu_j + j - i} \right) \right|_{1 \leq i, j \leq r+1}.
\]

Similarly, using Fact 2 again, when the determinant

\[
\frac{1}{|l/\mu^-|!} f^{l/\mu^-} = \left| \frac{1}{(l_i - \mu_j + j - i)!} \right|_{1 \leq i, j \leq r+1}
\]

is nonzero, there is a unique permutation \( w \in S_{r+1} \) acting on the rows which sorts the entries into decreasing order down columns. Equivalently,

\[
\lambda_i^+ := l_{w(i)} - w(i) + i \quad \text{for} \quad 1 \leq i \leq r + 1
\]

defines a partition \( \lambda^+ \supseteq \lambda \). Then \( f^{l/\mu^-} = (-1)^{\text{sgn}(w)} f^{\lambda^+/\mu^-} \). Collecting terms gives

\[
\left( \prod_{i=1}^{r+1} \left( \frac{l_i - 1}{l_i - \lambda_i} \right) \right) f^{l/\mu^-} = \sum_{\lambda^+ \supseteq \lambda} \zeta^{\lambda^+/\lambda} \cdot f^{\lambda^+/\mu^-},
\]

where the sum is over partitions \( \lambda^+ \) of length \( r + 1 \) (so \( \lambda^+/\mu^- \) is a skew diagram) such that \( |\lambda^+/\lambda| = |l| \), and

\[
\zeta^{\lambda^+/\lambda} := \sum_{w \in S_{r+1}} (-1)^{\text{sgn}(w)} \left( \prod_{i=1}^{r+1} \left( \frac{\lambda_i^+ + w(i) - i - 1}{\lambda_i^+ - \lambda_{w(i)} + w(i) - i} \right) \right)
\]

\[
\left| \left( \frac{\lambda_i^+ + j - i - 1}{\lambda_i^+ - \lambda_j + j - i} \right) \right|_{1 \leq i, j \leq r+1}.
\]

The binomial determinants \( \alpha^{\mu/\mu^-} \) and \( \zeta^{\lambda^+/\lambda} \) enumerate tableaux, by the method of Gessel-Viennot. A (column) semi-standard Young tableau on a given shape is a filling of the boxes by positive integers such that the entries are weakly increasing across each row and strictly increasing down each column. A filling is a row semi-standard Young tableau if the transpose condition holds: the entries are strictly increasing across each row and weakly increasing down each column. A strict Young tableau is a filling whose entries are strictly increasing across each row and down each column.
By [20, Theorem 14], the determinant $\alpha^\mu/\mu^-$ is equal to the number of row semi-standard Young tableaux on $\mu/\mu^-$ whose entries in row $i$ are between 1 and $\mu_i$, inclusive, and the determinant $\zeta^{\lambda^+}/\lambda^+$ is equal to the number of semi-standard Young tableaux on $\lambda^+/\lambda$ whose entries in row $i$ are between $-\lambda_i$ and $-1$, inclusive. Such tableaux are in bijection with strict Young tableaux on $\lambda^+/\lambda$ whose entries in row $i$ are between 1 and $\lambda^+_i - 1$: given a semi-standard tableau on $\lambda^+/\lambda$ with $i$-th row entries in $\{-\lambda_i, \ldots, -1\}$, add to each entry the index of its column to obtain a strict tableau with $i$-th row entries in $\{1, \ldots, \lambda^+_i - 1\}$. Combining equations (11) and (14) concludes the proof of Theorem C.

Next we will prove a nonemptiness criterion for the variety $G_d^{a,b}(C, P, Q)$, using Corollary 1.5. By setting $\beta = 0$ in (3), and passing to numerical equivalence, we obtain a variation on the formula for the cohomology class of $W^a_d(C)$:

**Proposition 4.1.** Let $(C, P, Q)$ be a smooth two-pointed curve of genus $g$. If $\lambda_i \geq \mu_i$ for all $i$ and $W^{a,b}_d(C, P, Q)$ has dimension equal to $\rho$, then its numerical class is

$$\left[ W^{a,b}_d(C, P, Q) \right] = \frac{1}{(a_{r-i} + b_j + g - d)!} |_{0 \leq i,j \leq r} g^{g-\rho}. $$

If $G^{a,b}_d(C, P, Q)$ has dimension equal to $\rho$, then $\pi_*[G^{a,b}_d(C, P, Q)]$ equals $[W^{a,b}_d(C, P, Q)]$ when $\lambda/\mu$ is a skew diagram, and vanishes otherwise.

(In comparing with (3), note the shift of indexing of the matrix, and recall that the definitions of $\lambda$ and $\mu$ imply $\lambda_i - \mu_j + j - i = a_{r-i+1} + b_{j-1} + g - d$. The vanishing statement follows algebraically from Fact 1, or geometrically from the fact that $\dim(G^{a,b}_d(C, P, Q)) < \dim(G^{a,b}_d(C, P, Q))$ unless $\lambda/\mu$ is a skew diagram.)

Now we can state the nonemptiness criterion.

**Proposition 4.2.** Let $(C, P, Q)$ be a smooth two-pointed curve of genus $g$. If

$$\rho' := g - \sum_{i=0}^r \max\{0, a_i + b_{r-i} + g - d\} \geq 0, $$

then the locus of special linear series $G^{a,b}_d(C, P, Q)$ is non-empty.

This was first proved by Osserman, using degeneration techniques [29]. When $b = (0, 1, \ldots, r)$, it recovers the statement for the one-pointed case in [13, Proposition 1.2].

**Proof.** The nonemptiness of $G^{a,b}_d(C, P, Q)$ is equivalent to the nonemptiness of its image $W = W^{a',b}_d(C, P, Q)$ in $\text{Pic}^d(C)$. By Corollary 1.5, $W$ is
nonempty when the class $\Delta_{\lambda/\mu}(c; 0)$ is nonzero. By Proposition 4.1, this class is numerically equivalent to

$$\frac{1}{(a'_{r-i} + b_j + g - d)!} g^{g-\rho'},$$

where, as before, $a'$ is the sequence defined by

$$a'_i := a_i + \max\{0, d - g - a_i - b_{r-i}\}.$$ 

This means $\rho' = \rho(g, r, d, a', b)$.

Associating partitions $\lambda'$ and $\mu$ to the data $g, d, a', b$ as usual, the definition of $a'$ guarantees that $\lambda'_i \geq \mu_i$ for all $i$. It follows that the determinantal coefficient is nonzero (see Fact 1), so the expression (17) is nonzero if and only if $\rho' \geq 0$. This is equivalent to the condition in (16). \qed

Now we turn to some special cases.

4.1. **The curve case.** Let us write $\lambda + \epsilon_i$ for the diagram obtained by adding one box to the right of the $i$-th row of $\lambda$, and $\mu - \epsilon_i$ for the diagram obtained by subtracting one box from the $i$-th row of $\mu$. (This means the diagram $\lambda/(\mu - \epsilon_i)$ is obtained by *adding* one box to the left of the $i$-th row of $\lambda/\mu$.)

Now assume $\rho(g, r, d, a, b) = 1$. The reformulation in (9) of Theorem A reduces to

$$\chi\left(\mathcal{O}_{G^a_d(C, P, Q)}\right) = \sum_{i=1}^{r+1} \mu_i f^{\lambda/(\mu-\epsilon_i)} - \sum_{i=1}^{r+1} \lambda_i f^{(\lambda+\epsilon_i)/\mu}.$$ 

By Fact 1, $f^{\lambda/(\mu-\epsilon_i)}$ vanishes when $\lambda/(\mu - \epsilon_i)$ is not a skew diagram, and $f^{(\lambda+\epsilon_i)/\mu}$ vanishes when $(\lambda + \epsilon_i)/\mu$ is not a skew diagram. Using the identity

$$(r+1)(|\lambda/\mu|+1)f^{\lambda/\mu} = \sum_{i=1}^{r+1} (\lambda_i + r + 2 - i) f^{(\lambda+\epsilon_i)/\mu} - \sum_{i=1}^{r+1} (\mu_i + r + 1 - i) f^{\lambda/(\mu-\epsilon_i)},$$

for the number of standard skew Young tableaux, we recover [9, Theorem 1.2].

4.2. **The one-pointed case.** When $b = (0, \ldots, r)$, the locus $G^a_b(C, P, Q)$ is identical to the one-pointed locus

$$G^a_d(C, P) := \left\{ \ell \in G^a_d(C) \mid a^\ell(P) \geq a \right\}.$$ 

On the other hand, when the points $P$ and $Q$ collide together on the curve $C$, the locus $G^a_b(C, P, Q)$ specializes to the locus of linear series $(L, V) \in G^a_d(C)$ such that

$$V \cap h^0(L(-\ell) + P) \geq 1 + j - i.$$ 

Fix $l$ such that $l_i \geq \lambda_i$ and $|l/\lambda| = \rho$. By an application of the Vandermonde identity,

$$\left| \frac{g_{l_i+j-i}}{(l_i + j - i)!} \right|_{1 \leq i, j \leq r+1} = g^{l/\lambda} \left| \frac{\Pi_{1 \leq i < j \leq r+1} (l_i - l_j + j - i)}{\Pi_{i=1}^{r+1} (l_i + r + 1 - i)!} \right|,$$
so Theorem A reduces to

\[ \chi \left( \mathcal{O}_{G_a^d}(C, P) \right) = \sum_{|\nu|/\lambda = \rho} \left( -\lambda_1 \right) \cdots \left( -\lambda_{r+1} \right) \]

\[ \times g! \prod_{1 \leq i < j \leq r+1} \left( l_i - l_j + j - i \right) \prod_{i=1}^{r+1} \left( l_i + r + 1 - i \right)! \]  

When in addition \( \rho(g, r, d, a, b) = 1 \), this sum becomes

\[ \chi \left( \mathcal{O}_{G_a^d}(C, P) \right) = -g! \sum_{k=0}^{r} \left( g - d + r + a_k - k \right) \prod_{0 \leq i < j \leq r} (a_j - a_i + \delta^k - \delta^k_i) \prod_{i=0}^{r} (g - d + r + a_i + \delta^k)! \]

where \( \delta \) is the Kronecker delta.

### 4.3. Set-valued tableaux and the one-pointed case.

In the one-pointed case, we can re-write the Euler characteristic in terms of set-valued tableaux. Given a partition \( \lambda \) and a nonnegative integer \( \rho \), a \( \rho \)-standard set-valued tableau on \( \lambda \) is a filling of the boxes of \( \nu \) by non-empty subsets of \( \{1, \ldots, |\lambda|+\rho\} \) such that the entries in each row and in each column are strictly increasing, with each of \( 1, \ldots, |\lambda|+\rho \) appearing exactly once. (See §5 for more about set-valued tableaux and the connection with Grothendieck polynomials.)

Chan and Pflueger conjectured a formula expressing the Euler characteristic of a two-pointed Brill-Noether locus via set-valued tableaux on a skew shape. The following establishes the one-pointed version of their conjecture.

**Corollary 4.3.** Suppose \( \dim G_a^d(C, P) = \rho \), and let \( \lambda \) be the partition corresponding to \( a \). Then

\[ \chi \left( \mathcal{O}_{G_a^d}(C, P) \right) = (-1)^\rho \cdot \# \{ \rho \text{-standard set-valued tableaux on } \lambda \}. \]

This is zero if and only if \( W_a^d(C, P) = \text{Pic}^d(C) \).

**Proof.** In the one-pointed case, Theorem C becomes

\[ \chi \left( \mathcal{O}_{G_a^d}(C, P) \right) = (-1)^\rho \sum_{|\lambda^+/\lambda| = \rho} \zeta_{\lambda^+/\lambda} \cdot f_{\lambda^+}, \]

so we must identify the sum on the RHS with the number of \( \rho \)-standard set-valued tableaux on \( \lambda \).

For any partition \( \nu \), it follows from a theorem of Lenart [27, Theorem 2.2] that

\[ \# \{ \rho \text{-standard set-valued tableaux on } \nu \} = \sum_{|\nu^+|/\nu = \rho} g^{\nu^+/\nu} f^{\nu^+}, \]

where \( g^{\nu^+/\nu} \) is the number of strict Young tableaux on \( \nu^+ / \nu \) whose entries in row \( i \) are between 1 and \( i - 1 \), inclusive, and \( f^{\nu^+} \) is the number of standard Young tableaux on \( \nu^+ \). (To deduce this from Lenart’s theorem, which writes the Grothendieck polynomial for \( \nu \) as a sum of Schur polynomials, compare the coefficient of the monomial \( x_1 \cdots x_{|\nu|+\rho} \) on each side of his formula.)
Our claim follows by taking \( \nu \) to be the conjugate partition \( \lambda' \), i.e., the diagram obtained by reflecting across the diagonal, so that rows and columns are interchanged. It is easy to see that \( \rho \)-standard set valued tableaux on \( \lambda \) and \( \nu \) are in bijection. Similarly, standard Young tableaux on \( \lambda^+ \) and \( \nu^+ = (\lambda^+)' \) are also in bijection, so \( f^{\lambda^+} = f^{\nu^+} \). Finally, \( \zeta^{\lambda^+/\lambda} = g^{\nu^+/\nu} \), for \( \nu = \lambda' \) and \( \nu^+ = (\lambda^+)' \), because sending a tableau \( T \) to its conjugate \( T' \) defines a bijection from strict tableaux on \( \lambda^+/\lambda \) with \( i \)-th row entries in \( \{1, \ldots, \lambda_i^+ - 1\} \) to strict tableaux on \( \nu^+/\nu \) with \( i \)-th row entries in \( \{1, \ldots, i-1\} \). \( \square \)

4.4. The classical case. Here there are no point conditions, and in the formulas one can take \( \mu = \emptyset \), and let \( \lambda = (g-d+r)^{r+1} \) be the rectangular shape. Any partition \( \lambda^+ \supseteq \lambda \) of length \( r+1 \) can be written as \( \lambda + \gamma \), for some partition \( \gamma \) of length \( r+1 \). The determinant \( \zeta^{\lambda^+/\lambda} \) can therefore be written as

\[
\zeta^{\lambda^+/\lambda} = \frac{(g-d+r+\gamma_i+j-i-1)_{\gamma_i+j-i}}{(\gamma_i+j-i)!} \bigg|_{1 \leq i,j \leq r+1},
\]

where \( (n)_k = n(n-1) \cdots (n+1-k) \) is the falling factorial. Manipulating the matrix leads to a factorization of this determinant as

\[
\left| \frac{1}{(\gamma_i+j-i)!} \right|_{1 \leq i,j \leq r+1} \cdot \prod_{i=1}^{r+1} (g-d+r+\gamma_i-i)_{\gamma_i}.
\]

Applying this simplification to Theorem C, we obtain:

**Corollary 4.4.** If \( \dim G^r_d(C) = \rho(g,r,d) \leq g \), then

\[
\chi\left(\mathcal{O}_{G^r_d(C)}\right) = \chi\left(\mathcal{O}_{W^r_d(C)}\right) = \frac{(-1)^\rho}{\rho!} \sum_{|\gamma| = \rho} f^\gamma \cdot \prod_{i=1}^{r+1} (g-d+r+\gamma_i-i)_{\gamma_i} \cdot f^{\lambda^+\gamma}
\]

where the sum is over partitions \( \gamma = (\gamma_1 \geq \cdots \geq \gamma_{r+1} \geq 0) \), and \( \lambda + \gamma \) is the partition \( (g-d+r+\gamma_1, \ldots, g-d+r+\gamma_{r+1}) \).

Finally, we include some examples for the classical case.

4.5. The classical case with \( \rho \leq 3 \). In low dimensions, the Euler characteristic can be written in a fairly simple closed form.

When \( \rho(g,r,d) = 0 \), the formula in Corollary 4.4 for \( \chi(\mathcal{O}_{G^r_d(C)}) \) has a single summand, namely \( \gamma = \emptyset \), and it recovers Castelnuovo’s count for the number of line bundles of degree \( d \) with \( r+1 \) sections:

\[
N^r_{g,d} := \chi\left(\mathcal{O}_{G^r_d(C)}\right) = \frac{g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}}{g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}}.
\]
When \( \rho(g, r, d) = 1 \), the only non-zero contribution in the formula for \( \chi(\mathcal{O}_{G_d}(C)) \) is the summand with \( \gamma_1 = 1 \), and we recover [13, Theorem 4]:

\[
\chi(\mathcal{O}_{G_d}(C)) = -g! \frac{(g - d + r)(r + 1)}{g - d + 2r + 1} \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}.
\]

When \( \rho(g, r, d) = 2 \), there are two non-zero contributions in the formula for \( \chi(\mathcal{O}_{G_d}(C)) \), corresponding to terms with \( \gamma = (2, 0) \) and \( \gamma = (1, 1) \). This gives

\[
\chi(\mathcal{O}_{G_d}(C)) = \left(1 + 2r\right) \frac{(r + 1)^2 (g - d + r)^2}{2(g - d + 2r)(g - d + 2r + 2)} \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!}.
\]

(One can verify that this formula for \( \chi(\mathcal{O}_{G_d}(C)) = \chi(W_d(C)) \) satisfies Noether’s relation 12 \( \chi = c_1^2 + c_2 \) for surfaces, using computations of the classes \( c_1^2(W_d(C)) \) and \( c_2(W_d(C)) \) from [22] and [30].)

Finally, when \( \rho(g, r, d) = 3 \), there are three non-zero contributions in the formula for \( \chi(\mathcal{O}_{G_d}(C)) \). We have

\[
\chi(\mathcal{O}_{G_d}(C)) = \frac{(r + 1)^2 s^2 \left[\left(\frac{(r + s + 1)^2}{s + r + 1}\right)^2 - 2\right]}{6(s + r - 1)(s + r)(s + r + 1)(s + r + 2)(s + r + 3)} N_{g,d}^r,
\]

where \( s := g - d + r \).

4.6. **The classical case with** \( g - d + r = 1 \). In this case, one has \( \rho = g - r - 1 \). From Corollary 4.4, the contributions of all terms with \( \gamma_1 < \rho \) vanish; indeed, one has \( (\gamma_i - i + 1) \gamma_i = 0 \) for \( i > 1 \). The Euler characteristic of \( \mathcal{O}_{G_d}(C) \) is therefore given by a single binomial coefficient:

\[
\chi(\mathcal{O}_{G_d}(C)) = (-1)^\rho \binom{r + \rho}{\rho} = \binom{-r - 1}{\rho}.
\]

5. **Schubert and Grothendieck polynomials**

As another application, we deduce determinantal formulas for \((double) Schubert and Grothendieck polynomials\) for 321-avoiding permutations from the degeneracy locus formula of Theorem B. The latter will identify our K-theory formulas with double versions of the \emph{flagged skew Grothendieck polynomials} recently introduced by Matsumura [28].

Given decreasing sequences \( p = (p_1, \ldots, p_t) \) and \( q = (q_1, \ldots, q_t) \), we defined partitions \( \lambda \) and \( \mu \) by

\[
\lambda_i = q_i - t + i, \\
\mu_j = p_j - (t + 1 - j).
\]

When \( p, q \) satisfy

\[
(*) \quad q_i \geq p_i - 1 \quad \text{for all } i,
\]
the partitions form a skew diagram $\lambda/\mu$, and we defined an associated permutation $w$ by setting
\[ w(p_i) = q_i + 1, \]
for $1 \leq i \leq t$, and then filling in the remaining entries with the unused numbers in increasing order. This is a 321-avoiding permutation, and all 321-avoiding permutations arise this way, for some $p,q$, since any such permutation is a shuffle of two increasing subsequences (see e.g., [15]).

**Remark 5.1.** The above is equivalent to the bijection of 321-avoiding permutations with labeled skew tableaux of Billey-Jockusch-Stanley ([6]), which can be re-formulated as follows. For a 321-avoiding permutation $w$, the skew shape $\sigma(w)$ considered in [6] is a 180 degree rotation of our skew shape $\lambda/\mu$, that is, $\sigma(w) = \eta/\tau$ where $\eta_i = \lambda_1 - \mu_{t+1-i}$ and $\tau_i = \lambda_1 - \lambda_{t+1-i}$. Let $f_w = (f_1, f_2, \ldots, f_t)$ be the increasing sequence of indices $j$ such that $w(j) > j$, and let $e_i = w(f_i) - 1$. Then the labeling $\omega(w)$ of the skew shape $\sigma(w)$ is obtained by placing the entries $f_i, f_i + 1, \ldots, e_i$ in the $i$-th row of $\sigma(w)$ such that the entries increase by one in each column, and decrease by one in each row. In our setup, the labeling $\omega(w)$ is determined by $f_i = p_{t+1-i}$ and $e_i = q_{t+1-i}$.

For any permutation $w$, the **double Schubert polynomial** of Lascoux and Schützenberger is a canonical representative $S_w(x; y)$ for the corresponding Schubert class (see [17]). These polynomials are defined inductively, but for special types of permutations, one can give direct formulas. We do this here for 321-avoiding permutations.

Given sets of variables $x$ and $y$, let
\[ c(i, j) = \frac{\prod_{a=1}^{p_j}(1 - uy_a)}{\prod_{b=1}^{q_i}(1 - ux_b)}, \]
and define $c_k(i, j)$ by collecting the coefficient of $u^k$ in the expansion of this rational function (in positive powers of $x$ and $y$). For example, if $y = 0$, then $c_k(i, j)$ is the complete homogeneous symmetric polynomial $h_k(x_1, \ldots, x_{p_j})$ (for any $i$), and if $x = 0$, then $c_k(i, j)$ is the elementary symmetric polynomial $(-1)^k e_k(y_1, \ldots, y_{q_i})$ (for any $j$).

**Corollary 5.2.** Let $w$ be a 321-avoiding permutation, associated to tuples $p,q$ (satisfying *), and let $\lambda/\mu$ be the corresponding skew Young diagram. The double Schubert polynomial for $w$ has the following determinantal expression:
\[ \mathcal{S}_w(x; y) = \Delta_{\lambda/\mu}(c; 0) = |c_{\lambda_i-\mu_j+i-j}(i, j)|_{1 \leq i,j \leq t}. \]

This recovers a formula of Lascoux and Chen-Yan-Yang (see [11]), which in turn generalized a formula of Billey-Jockusch-Stanley [6] for the single Schubert polynomials of 321-avoiding permutations — that is, the case $y = 0$. More precisely, the matrices computing these formulas in [11] are obtained by reflecting about the anti-diagonal the matrices computing the determinants in Corollary 5.2. The right-hand side is a flagged double skew Schur function,
a variant of the flagged double Schur function introduced by Chen-Li-Louck [10]. (“Flagging” refers to the nested sets of variables appearing along rows and columns of the determinant: the \( i \)-th row uses \( \{ y_1, \ldots, y_{q_i} \} \), and the \( j \)-th column uses \( \{ x_1, \ldots, x_{p_j} \} \).)

**Example 5.3.** An example of a 321-avoiding permutation which is not also vexillary (another class having determinantal expressions, thanks to an older theorem of Wachs) is \( w = 3 \ 1 \ 2 \ 5 \ 4 \). Here \( p = (4, 1) \) and \( q = (4, 2) \), so \( \lambda = (3, 2) \) and \( \mu = (2, 0) \), and the formula says

\[
\mathfrak{S}_{31254} = \begin{vmatrix} c_1(1, 1) & c_4(1, 2) \\ 0 & c_2(2, 2) \end{vmatrix} = c_1(1, 1) \cdot c_2(2, 2) \\
= (x_1 + x_2 + x_3 + x_4 - y_1 - y_2 - y_3 - y_4) \cdot (x_1^2 - x_1y_1 - x_1y_2 + y_1y_2).
\]

Comparing with [6], and using their notation, the labeled skew diagram \((\sigma(w) = \eta/\tau, \omega(w))\) associated to \( w = 3 \ 1 \ 2 \ 5 \ 4 \) is given by:

\[
\begin{array}{ccc}
2 & 1 \\
& 4
\end{array}
\]

where \( \eta = (3, 1), \tau = (1, 0), f = (1, 4), \) and \( e = (2, 4) \). The matrix computing the determinant \( \mathfrak{S}_{31254} \) in [6] is obtained by reflecting the above matrix about the anti-diagonal.

The proof of Corollary 5.2 is immediate from Theorem 1.1(ii). One can take the base \( X \) to be a product of two flag varieties, and the vector bundles \( E_\bullet \) and \( F_\bullet \) tautological from each factor. The \( x \) variables are then Chern roots of \( E_{p_j}^* \), and the \( y \) variables are Chern roots of \( F_{q_i}^* \).

More generally, we obtain a similar result for **double Grothendieck polynomials** \( \mathfrak{G}_w(x; y) \), where \( w \) is a 321-avoiding permutation. Grothendieck polynomials were introduced by Lascoux and Schützenberger as representatives of the K-class of the structure sheaf of the corresponding Schubert variety. Here the variables should be specialized as follows. Let

\[
c(E_{p_j}) = \prod_{b=1}^{p_j} \frac{1 + \beta x_b - ux_b}{1 + \beta x_b}, \quad c(F_{q_i}) = \prod_{a=1}^{q_i} \frac{1 + \beta y_a - uy_a}{1 + \beta y_a},
\]

and set

\[
c(i, j) = \frac{c(F_{q_i})}{c(E_{p_j})} = \prod_{a=1}^{q_i} \prod_{b=1}^{p_j} \frac{(1 + \beta y_a - uy_a)(1 + \beta x_b)}{(1 + \beta y_a)(1 + \beta x_b - ux_b)}.
\]

The term \( c_k(i, j) \) is obtained as before, by expanding and collecting the coefficient of \( u^k \).
Corollary 5.4. Let \( w \) be a 321-avoiding permutation, associated to tuples \( p, q \) (satisfying \((\ast)\)), and let \( \lambda/\mu \) be the corresponding skew Young diagram. The double Grothendieck polynomial for \( w \) has the following determinantal expression:

\[
G_w(x; y) = \Delta_{\lambda/\mu}(c; \beta) = \left| \sum_{k \geq 0} \left( \lambda_i - \mu_j + k - 1 \right) \beta^k c_{\lambda_i - \mu_j - i + j + k} \right|_{1 \leq i, j \leq t}.
\]

The proof is the same as for the previous Corollary, with the \( x \) variables defined to be Chern roots of \( E^*_p \), and the \( y \) variables defined to be the Chern roots of \( F^*_q \), in \( CK^*(X) \).

In [28, \( \S \) 4], Matsumura defines flagged skew Grothendieck polynomials as generating functions of flagged set-valued tableaux and proves that they have determinantal expressions. A set-valued tableau of skew shape \( \lambda/\mu \) is a labelling of the boxes of \( \lambda/\mu \) by finite non-empty subsets of \( \mathbb{N} \) such that the maximum element of the label of any box \((i, j)\) is at most the minimum element of the label at \((i, j + 1)\), and smaller than the minimum element of the label at \((i + 1, j)\) (see [7]). Given a skew shape \( \lambda/\mu \) and a flagging \( f = (f_1, \ldots, f_t) \), a flagged skew set-valued tableau of skew shape \( \lambda/\mu \) with flagging \( f \) is a set-valued tableau on \( \lambda/\mu \) such that every entry in the \( i \)-th row is a subset of \( \{1, 2, \ldots, f_i\} \). Let \( FSVT(\lambda/\mu, f) \) denote the set of all such flagged skew tableaux.

Corollary 5.5. Let \( w \) be a 321-avoiding permutation and let \( \sigma(w) = \eta/\tau \) be the skew Young diagram with flagging \( f_w \) corresponding to \( w \) via the Billey-Jockusch-Stanley bijection, as in Remark 5.1. The Grothendieck polynomial \( G_w(x) = G_w(x; 0) \) is equal to

\[
\sum_{T \in FSVT(\sigma(w), f_w)} \beta^{|T| - |\sigma(w)|} x^T,
\]

where \( x^T := \prod_{k \in T} x_k \).

Proof. Corollary 5.4 also holds after the matrix is reflected about the anti-diagonal — by replacing the \((i, j)\) entry with the \((t + 1 - j, t + 1 - i)\) entry — since the determinant is unchanged by this operation. The entries of this reflected matrix are equal to those in the determinantal formulas of [28, \( \S \) 4], as explained in [2, Remark 1.1]. \( \square \)

In particular, we recover the tableau formula [6, Theorem 2.2] for Schubert polynomials of 321-avoiding permutations.

Remark 5.6. There is a bijection between flagged set-valued tableaux and pipe dreams, extending the bijection between (ordinary) skew tableaux and reduced pipe dreams of 321-avoiding permutations used to prove [6, Theorem 2.2]. A pipe dream \( P \) is a tiling of the fourth quadrant of the plane by crosses
and elbows that uses only finitely many crosses. When no two pipes of $P$ cross each other more than once, $P$ is called a reduced pipe dream. The permutation of a reduced pipe dream is the permutation $w$ such that the pipe that starts at the top of the $w_i$-th column exits the $i$-th row on the left side of the quadrant (numbered by the absolute value of the $y$-coordinate).

Following the notation of Remark 5.1, to $T \in \text{FSVT}(\sigma(w), f_w)$ we can associate a pipe dream $P = \Omega(T)$ as follows:

For each entry $i$ in a box $b$ of $\sigma(w)$, place a crossing pipe in position $(i, \omega(b) - i + 1)$.

Given a pipe dream $P$, let $x^P := \prod_{(i,j)} x_i$, where the product is over crosses $(i,j)$ in $P$. If $T \in \text{FSVT}(\sigma(w), f_w)$ is a flagged set-valued skew tableau, let $\mathcal{T}$ be the flagged skew tableau obtained by taking the smallest element of each box of $T$. We obtain a reduced pipe dream $P = \Omega(\mathcal{T})$ for which we know $x^P = x^\mathcal{T}$ — this is exactly the bijection in [6] and [5].

It remains to describe how the extra entries in the filling of the set-valued tableau $T$ correspond to extra crossings for a non-reduced pipe dream for $w$. Let $b$ be a box in the labelled skew diagram $\sigma(w)$. The pipe dream $\Omega(T)$ maps an entry $i$ of box $b$ to $(i, \omega(b) - i + 1) \in \Omega(T)$. Let $i_0$ be the smallest entry of $b$. Then for $i > i_0$, the additional entries occur in the same antidiagonal $\{(i,j) \mid i + j = \omega(b) - 1\}$, southwest of the entry $(i_0, \omega(b) - i_0 + 1)$, and conversely.

We observe that this bijection satisfies $x^T = x^P$ for $P = \Omega(T)$ and $|\sigma(w)| = |T| = |\Omega(T)| = |\mathcal{T}| = l(w)$. This shows that the generating function formulas for Grothendieck polynomials in terms of pipe dreams and for skew flagged Grothendieck polynomials agree term by term.

This bijection gives an alternative proof of Corollary 5.5. It also extends the bijection between set-valued tableaux and pipe dreams for Grassmann permutations [26, Proposition 5.3].

References

[1] A. C. Aitken. The monomial expansion of determinantal symmetric functions. Proc. Roy. Soc. Edinburgh, Sect. A., 61:300–310, 1943. ↑18
[2] D. Anderson. K-theoretic Chern class formulas for vexillary degeneracy loci. Preprint, arXiv:1701.00126, 2017. ↑14, ↑7, ↑12, ↑28
[3] D. Anderson and S. Payne. Operational K-theory. Doc. Math., 20:357–399, 2015. ↑13
[4] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of algebraic curves. Vol. I, volume 267 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, New York, 1985. ↑2, ↑5, ↑17
[5] N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. Experiment. Math., 2(4):257–269, 1993. ↑29
[6] S. C. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. J. Algebraic Combin., 2(4):345–374, 1993. ↑15, ↑26, ↑27, ↑28, ↑29
[7] A. S. Buch. A Littlewood-Richardson rule for the K-theory of Grassmannians. Acta Math., 189(1):37–78, 2002. ↑28
[8] S. Cai. Algebraic connective K-theory and the niveau filtration. J. Pure Appl. Algebra, 212(7):1695–1715, 2008. ↑
[9] M. Chan, A. L. Martín, N. Pflueger, and M. Teixidor i Bigas. Genera of Brill-Noether curves and staircase paths in Young tableaux. To appear in Transactions of the AMS, 2017. ↑1, ↑6, ↑22
[10] W. Y. C. Chen, B. Li, and J. D. Louck. The flagged double Schur function. J. Algebraic Combin., 15(1):7–26, 2002. ↑5, ↑27
[11] W. Y. C. Chen, G.-G. Yan, and A. L. B. Yang. The skew Schubert polynomials. European J. Combin., 25(8):1181–1196, 2004. ↑26
[12] S. Dai and M. Levine. Connective algebraic K-theory. J. K-Theory, 13(1):9–56, 2014. ↑17, ↑18
[13] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. Invent. Math., 90(2):359–387, 1987. ↑1, ↑3, ↑21, ↑25
[14] R. Elkik. Singularités rationnelles et déformations. Invent. Math., 47(2):139–147, 1978. ↑11
[15] K. Eriksson and S. Linusson. The size of Fulton’s essential set. Electron. J. Combin., 2(Research Paper 6):18pp., 1995. ↑26
[16] G. Farkas and N. Tarasca. Du Val curves and the pointed Brill-Noether Theorem. Selecta Mathematica, to appear, 2017. ↑3
[17] W. Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. Duke Math. J., 65(3):381–420, 1992. ↑5, ↑10, ↑26
[18] W. Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998. ↑5
[19] W. Fulton and P. Pragacz. Schubert varieties and degeneracy loci, volume 1689 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1998. Appendix J by the authors in collaboration with I. Ciocan-Fontanine. ↑8
[20] I. M. Gessel and X. G. Viennot. Determinants, paths, and plane partitions. Preprint, 1989. ↑21
[21] J. L. González and K. Karu. Bivariant algebraic cobordism. Algebra Number Theory, 9(6):1293–1336, 2015. ↑13
[22] J. Harris and L. Tu. Chern numbers of kernel and cokernel bundles. Invent. Math., 75(3):467–475, 1984. ↑25
[23] T. Hudson, T. Ikeda, T. Matsumura, and H. Naruse. Degeneracy loci classes in K-theory — Determinantal and Pfaffian formula. Preprint, arXiv:1504.02828, 2016. ↑2, ↑12, ↑13
[24] T. Hudson and T. Matsumura. Vexillary degeneracy loci classes in K-theory and algebraic cobordism. preprint, arXiv:1701.00204, 2017. ↑2, ↑4
[25] S. L. Kleiman and D. Laksov. Another proof of the existence of special divisors. Acta Math., 132:163–176, 1974. ↑5
[26] A. Knutson, E. Miller, and A. Yong. Gröbner geometry of vertex decompositions and of flagged tableaux. J. Reine Angew. Math., 630:1–31, 2009. ↑29
[27] C. Lenart. Combinatorial aspects of the K-theory of Grassmannians. Ann. Comb., 4(1):67–82, 2000. ↑23
[28] T. Matsumura. Flagged Grothendieck polynomials. Preprint, arXiv:1701.03561, 2017. ↑12, ↑14, ↑25, ↑28
[29] B. Osserman. A simple characteristic-free proof of the Brill-Noether theorem. Bull. Braz. Math. Soc. (N.S.), 45(4):807–818, 2014. ↑3, ↑21
[30] A. Parusiński and P. Pragacz. Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors. J. Amer. Math. Soc., 8(4):793–817, 1995. ↑25
[31] G. P. Pirola. Chern character of degeneracy loci and curves of special divisors. Ann. Mat. Pura Appl. (4), 142:77–90 (1986), 1985. ↑1
