ON TORIC IND-VARIETY AND PRO-AFFINE SEMIGROUPS

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ABSTRACT. An ind-variety is an inductive limit of closed embeddings of algebraic varieties and an ind-group is a group object in the category of ind-varieties. These notions were first introduced by Shafarevich in the study of the automorphism group of affine spaces and have been studied by many authors afterwards. An ind-torus is an ind-group obtained as an inductive limit of closed embeddings of algebraic tori that are also algebraic group homomorphisms. In this paper, we introduce the natural definition of toric ind-varieties as ind-varieties having an ind-torus as an open set and such that the action of the ind-torus on itself by translations extends to a regular action on the whole ind-variety. We also introduce the notion of pro-affine semigroup that turn out to be unital semigroups isomorphic to closed subsemigroups of the group of arbitrary integer sequences with the product topology such that their projection to the first \( i \)-th coordinates is finitely generated for all positive integers \( i \). Our main result is a duality between the categories of affine toric ind-varieties and the the category of pro-affine semigroups.

INTRODUCTION

Shafarevich first introduced in [12, 13] the notion of infinite-dimensional algebraic varieties and infinite-dimensional algebraic groups, the so called ind-varieties and ind-groups, respectively. These notions were later expanded and revisited by several authors, see for instance [8, 7, 15] and the recent preprint [6] that includes a detailed exposition of generalities on ind-varieties and ind-groups.

We work over the field of complex numbers \( \mathbb{C} \). An ind-variety is a set \( V \) together with a filtration \( V_1 \hookrightarrow V_2 \hookrightarrow \ldots \) such that \( V = \bigcup V_i \), where each \( V_i \) is a finite-dimensional algebraic variety and the inclusions \( \varphi_i : V_i \hookrightarrow V_{i+1} \) are a closed embeddings. Morphisms in the category of ind-varieties are defined in the natural way, see Section 1.3 for details. An ind-group is a group object in the category of ind-varieties, i.e., it is an ind-variety endowed with a group structure such that the inversion and multiplication maps are morphisms of ind-varieties. Any algebraic variety or algebraic group is an example of an ind-variety and ind-group, respectively when taken with the trivial filtration. Furthermore, the set

\[
(\mathbb{C}^*)^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}
\]

with the canonical structure of ind-variety given by the filtration \( \mathbb{C}^* \xrightarrow{\varphi_1} (\mathbb{C}^*)^2 \xrightarrow{\varphi_2} (\mathbb{C}^*)^3 \xrightarrow{\varphi_3} \ldots \), where \( \varphi_i (a_1, \ldots, a_i) = (a_1, \ldots, a_i, 1) \) for all integer \( i > 0 \), has a natural structure of ind-group where the group law is given by component-wise multiplication. An algebraic torus \( T \) is an algebraic group isomorphic to \((\mathbb{C}^*)^k \) for some integer \( k \geq 0 \). An ind-torus \( \mathcal{T} \) is an ind-group isomorphic to either an algebraic torus or \((\mathbb{C}^*)^\infty \).

A toric variety \( V \) is an irreducible algebraic variety having an algebraic torus \( T \) as an open set and such that the action of \( T \) on itself by translations extends to a regular action on \( V \). Toric varieties can be classified by certain combinatorial devices, see [10, 4, 2]. This classification allows to translate many algebro-geometric properties of a toric variety in combinatorial terms that may then be computed algorithmically. Hence, toric varieties represent a fertile testing ground for theories in algebraic geometry. Toric morphisms between toric varieties are characterized by the property that they restrict to a morphism
of algebraic groups between the corresponding algebraic tori. For affine toric varieties their combinatorial nature is represented by the fact that the category of affine toric varieties is dual to the category of affine semigroups, i.e., finitely generated semigroups that can be embedded in \( \mathbb{Z}^k \) for some integer \( k \geq 0 \). By convention, all our semigroups will be commutative and unital.

In this paper we introduce the natural notion of toric ind-variety. A toric ind-variety \( V \) is an ind-variety having an ind-torus \( T \) as an open set and such that the action of \( T \) on itself by translations extends to a regular action on \( V \), see Definition \[ \ref{def:toric-ind-variety} \]. Furthermore, toric morphisms between toric ind-varieties are morphisms that restrict to morphisms of ind-groups between the corresponding ind-tori, see Definition \[ \ref{def:toric-morphism} \]. Our first result in this paper, contained in Theorem \[ \ref{thm:ind-variety-duality} \], shows that every toric ind-variety can be obtained as an inductive limit of toric varieties. This result allows us to investigate toric ind-varieties applying usual methods from toric geometry.

In Section \[ 3 \] we introduce the natural dual objects to affine toric ind-varieties that we call pro-affine semigroups. Let \( S \) be a commutative unital semigroup. In analogy with the case of topological algebras \[ \[ \ref{top-alg} \] Section 9.2], the natural way to endow the semigroup \( S \) with a topology is with a descending filtration \( R_1 \supset R_2 \supset \cdots \) of \( S \times S \) of equivalence relations on \( S \) that satisfy certain compatibility condition with respect to the semigroup operation allowing to define a semigroup operation in the set of equivalence classes \( S / R_i \), see Section \[ 5 \] for details. We call a semigroup \( S \) endowed with such a filtration a filtered semigroup. A pro-affine semigroup \( S \) is a filtered semigroup with filtration \( R_1 \supset R_2 \supset \cdots \) of compatible equivalence relations on \( S \) that is complete and such that \( S / R_i \) is an affine semigroup, for all integer \( i > 0 \). Our main result concerning pro-affine semigroups is contained in Corollary \[ \ref{cor:pro-affine-semigroup} \] and is a classification of pro-affine semigroups as semigroups isomorphic to subsemigroups \( S \times \mathbb{Z}^\omega \) of arbitrary sequences of integers, that are closed in the product topology and such that \( \pi_i(S) \) is finitely generated for all integer \( i > 0 \), where \( \pi_i : \mathbb{Z}^\omega \to \mathbb{Z}^i \) is the projection to the first \( i \)-th coordinates.

Finally, our main result in this paper is Theorem \[ \ref{thm:main} \] where we show that the category of affine toric ind-varieties with toric morphisms is dual to the category of pro-affine semigroups with homomorphisms of semigroups.

The contents of the paper is as follows. In Section \[ 1 \] we collect the preliminary notions of toric varieties, inductive and projective limits and ind-varieties required in this paper. In Section \[ 2 \] we introduce toric ind-varieties. In Section \[ 3 \] we define pro-affine semigroups. Finally, in Section \[ 4 \] we prove the duality of categories that is our main result.

**Acknowledgements.** Part of this work was done during a stay of both authors at IMPAN in Warsaw. We would like to thank IMPAN and the organizers of the Simons semester “Varieties: Arithmetic and Transformations” for the hospitality.

1. **Preliminaries**

In this section we recall the notions of toric geometry, injective and projective limits and ind-varieties needed for this paper.

1.1. **Toric varieties.** To fix notation we recall the basics of toric geometry. For details, see \[ \[ \ref{toric} \] 4 \] 2. An algebraic torus \( T \) is a linear algebraic group isomorphic to \( (\mathbb{C}^*)^k \) for some integer \( k \geq 0 \). A toric variety on \( \mathbb{C} \) is an irreducible algebraic variety \( V \) having an algebraic torus as a dense open set such that the action of \( T \) on itself by translations extends to a regular action of \( T \) on \( V \). Similarly to \[ \[ \ref{toric} \] \], we will not assume that a toric variety is necessarily normal. It is well known that affine toric varieties are in correspondence with affine semigroups \( S \), i.e., with finitely generated semigroups that admit an embedding in \( \mathbb{Z}^k \) for some integer \( k \geq 0 \). By convention, all our semigroups are commutative and unital.

Indeed, given an affine semigroup \( S \), the corresponding affine toric variety is given by \( V(S) = \text{Spec} \ \mathbb{C}[S] \), where \( \mathbb{C}[S] \) is the semigroup algebra given by \( \mathbb{C}[S] = \bigoplus_{m \in S} \mathbb{C} \cdot \chi^m \). Here, \( \chi^m \) are new symbols and the multiplication rule is defined by \( \chi^0 = 1 \) and \( \chi^m \cdot \chi^{m'} = \chi^{m+m'} \). On the other, the character lattice \( M \) of the torus \( T \) is a finitely generated free abelian group \( M \simeq \mathbb{Z}^k \) of rank \( k = \dim T \). Let \( V \) be an affine toric variety with acting torus \( T \). We define the semigroup \( S(V) \) of the toric variety \( V \) as the semigroup of characters of \( T \) in \( M \) that extend to regular functions on \( V \).
A toric morphism between toric varieties is a regular map that restricts to a morphism of algebraic groups between the corresponding algebraic tori acting on each toric variety. It is well known that the assignments \(V(\cdot)\) and \(S(\cdot)\) extend to functors from the category of affine varieties with toric morphisms to the category of affine semigroups and vice versa, respectively. Furthermore, the functors \(V(\cdot)\) and \(S(\cdot)\) together form a duality between the categories of affine toric varieties with toric morphisms and affine semigroups with homomorphisms of semigroups.

1.2. \textbf{Inductive and projective limits.} In this paper we will require several instances of inductive and projective limits of algebraic and geometric objects. We give here a brief account to fix notation, for details, see any reference on category theory such as \([9, \text{Chapter III}]\). All the systems of morphism required in this paper will be indexed by the positive integers with the usual order. Hence we restrict the exposition to this setting.

An inductive system indexed by the positive integers in a category \(C\) is a sequence

\[
X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \ldots ,
\]

where \(X_i\) are objects in \(C\) and \(\varphi_i: X_i \to X_{i+1}\) are morphisms in \(C\). We denote such an inductive system by \((X_i, \varphi_i)\). For every \(i, j > 0\) with \(i \leq j\), we define \(\varphi_{ij}: X_i \to X_j\) as \(\varphi_{ij} = \varphi_j \circ \varphi_{j-1} \circ \cdots \circ \varphi_i\), where by definition \(\varphi_{ii} = \text{id}: X_i \to X_i\). The inductive limit of an inductive system \((X_i, \varphi_i)\) is an object \(\varinjlim X_i\) in \(C\) and morphisms \(\psi_i: X_i \to \varinjlim X_i\) verifying \(\psi_i \circ \varphi_{ij} = \psi_j\circ \varphi_{ij}\) and satisfying the following universal property: if there exist another object \(Y\) and morphisms \(\psi_i': X_i \to Y\) verifying \(\psi'_i = \psi'_j \circ \varphi_{ij}\), then there exist a unique morphism \(u: \varinjlim X_i \to Y\) such that \(\psi'_i = u \circ \psi_i\) for all \(i > 0\).

The dual notion of inductive limits is defined as follows. A projective system indexed by the positive integers in a category \(C\) is a sequence

\[
X_1 \xleftarrow{\varphi_1} X_2 \xleftarrow{\varphi_2} X_3 \xleftarrow{\varphi_3} \ldots ,
\]

where \(X_i\) are objects in \(C\) and \(\varphi_i: X_{i+1} \to X_i\) are morphisms in \(C\). We denote such a projective system by \((X_i, \varphi_i)\). For every \(i, j > 0\) with \(i \leq j\), we define \(\varphi_{ij}: X_j \to X_i\) as \(\varphi_{ij} = \varphi_i \circ \varphi_{i+1} \circ \cdots \circ \varphi_j\), where by definition \(\varphi_{ij} = \text{id}: X_i \to X_i\). The projective limit of a projective system \((X_i, \varphi_i)\) is an object \(\varprojlim X_i\) in \(C\) and morphisms \(\pi_i: \varprojlim X_i \to X_i\) verifying \(\pi_i \circ \varphi_{ij} = \pi_j\circ \varphi_{ij}\) and satisfying the following universal property: if there exist another object \(Y\) and morphisms \(\pi'_i: Y \to X_i\) verifying \(\pi'_i = \pi'_j \circ \varphi_{ij}\), then there exist a unique morphism \(u: Y \to \varprojlim X_i\) such that \(\pi'_i = \pi_i \circ u\) for all \(i > 0\).

Both limits may not exist in arbitrary categories but in the categories of our interest (sets, groups, rings, algebras, semigroups, topological space) both limits can be realized by explicit constructions. Indeed, the inductive limit \(\varinjlim\) of an inductive system \((X_i, \varphi_i)\) can be constructed as \(\varinjlim X_i = \bigsqcup_{i \geq 0} X_i / \sim\), where \(\sim\) is the equivalence relation given by \(x_i \sim x_j\), where \(x_i \in X_i\) and \(x_j \in X_j\), if there exist \(k\) verifying \(i \leq k\) and \(j \leq k\) such that \(\varphi_{ik}(x_i) = \varphi_{jk}(x_j)\). The morphisms \(\psi_i: X_i \to \varinjlim X_i\) are induced by the natural injections \(X_i \to \bigsqcup_{i \geq 0} X_i\). Furthermore, if the morphisms \(\varphi_i\) are injective, then we can naturally regard each \(X_i\) as a subobject of the inductive limit \(\varinjlim X_i\). On the other hand, the projective limit \(\varprojlim\) of the projective system \((X_i, \varphi_i)\) can be constructed as

\[
\varprojlim X_i = \left\{ (x_1, x_2, \ldots ) \in \bigsqcup_{i>0} X_i \mid x_i \in X_i \text{ and } \varphi_{ij}(x_j) = x_i \right\},
\]

and the morphisms \(\pi_i: \varprojlim X_i \to X_i\) are induced by the natural projections \(\prod_{i>0} X_i \to X_i\). Furthermore, if the morphisms \(\varphi_i\) are surjective, then we can naturally regard each \(X_i\) as a quotient of the projective limit \(\varprojlim X_i\). Finally, in the case where \(X_i\) are topological spaces, the topology on the projective limit \(\varprojlim X_i\) coincides with the subspace topology on \(\prod_{i>0} X_i\) with the product topology.

\textbf{Example 1.1.} Two particular instances of the above construction will appear very often in this paper. Recall that \(\mathbb{Z}^\omega\) is the group of arbitrary sequences of integer numbers. This group is also called the Baer-Specker group. A sequence in \(a \in \mathbb{Z}^\omega\) is denoted by \(a = (a_1, a_2, \ldots )\). Equivalently, \(\mathbb{Z}^\omega\) is the projective limit of the system \(\mathbb{Z}^1 \leftarrow \mathbb{Z}^2 \leftarrow \ldots \), where the morphisms \(\varphi_i: \mathbb{Z}^{i+1} \to \mathbb{Z}^i\) are the projections forgetting the last coordinate. Furthermore, the subgroup of \(\mathbb{Z}^\omega\) of eventually zero sequences is denoted by \(\mathbb{Z}^\infty\), so...
Example 1.3. $Z^\omega$ is isomorphic to that $Z$ system admits a section. It is a straightforward computation to show that for any split projective system $Z^{n_1} \leftarrow Z^{n_2} \leftarrow \ldots$, with a strictly increasing sequence $n_1 < n_2 < \ldots$ of positive integers, the limit is isomorphic to $Z^\omega$. Similarly, for any split inductive system $Z^{n_1} \rightarrow Z^{n_2} \rightarrow \ldots$, with a strictly increasing sequence $n_1 < n_2 < \ldots$ of positive integers, the limit is isomorphic to $Z^\omega$.

In the sequel we will need the following lemma showing that $Z^\omega$ and $Z^\infty$ are mutually dual. Showing that $\text{Hom}(Z^\infty, Z) \simeq Z^\omega$ is a straightforward exercise, but showing $\text{Hom}(Z^\omega, Z) \simeq Z^\infty$ is more involved, see [14] for the original proof or [3, Example 3.22] for a modern proof.

**Lemma 1.2.** The groups $Z^\omega$ and $Z^\infty$ are mutually dual and this duality is realized by usual dot product

$$\langle \cdot, \cdot \rangle : Z^\omega \times Z^\infty \rightarrow Z, \quad (m, p) \mapsto \sum_{i > 0} (m_i \cdot p_i).$$

### 1.3. General ind-varieties.

In this section we introduce the necessary notions and results regarding ind-varieties. The definitions are borrowed from [8], [7] and [6].

Recall that an ind-variety is a set $\mathcal{V}$ together with a filtration $V_1 \leftarrow V_2 \leftarrow \ldots$ such that $\mathcal{V} = \varinjlim V_i := \bigcup V_i$, each $V_i$ is a finite-dimensional variety over $\mathbb{C}$, and the inclusion $\varphi_i : V_i \hookrightarrow V_{i+1}$ is a closed embedding. An ind-variety $\mathcal{V}$ is affine if each $V_i$ is affine. We also define the ind-topology on an ind-variety $\mathcal{V}$ as the topology where a set $U \subset \mathcal{V}$ is open if and only if $U \cap V_i$ is open in $V_i$ for all $i > 0$. In particular, the filtration $V_1 \leftarrow V_2 \leftarrow \ldots$ is an inductive system and the set $\mathcal{V}$ is the inductive limit. The topology defined on $\mathcal{V}$ corresponds to the inductive topology given by this inductive system.

A morphism between ind-varieties $\mathcal{V}$ and $\mathcal{V}'$ with filtrations $V_i$ and $V'_j$ respectively, is a map $\varphi : \mathcal{V} \rightarrow \mathcal{V}'$ satisfying that for every $i > 0$ there exist a positive integer $j > 0$ such that $\varphi(V_i) \subset V'_j$ and $\varphi|_{V_j} : V_j \rightarrow V'_j$ is a morphism of varieties. A morphism $\varphi$ of ind-varieties is an isomorphism if $\varphi$ is bijective and $\varphi^{-1}$ is a morphism of ind-varieties. Furthermore, two filtrations $V_1 \leftarrow V_2 \leftarrow \ldots$ and $W_1 \rightarrow W_2 \rightarrow \ldots$ on the same underlying set $\mathcal{V}$ are equivalent if the identity map is a isomorphism of ind-varieties. In analogy with similar Example [6.1] if we take any subfiltration of the filtration $V_1 \leftarrow V_2 \leftarrow \ldots$, the ind-varieties obtained by both filtrations are isomorphic. An ind-group is an ind-variety $\mathcal{G}$ endowed with a group structure such that the inversion and multiplication maps are morphisms of ind-varieties.

Recall that a set in a topological space is locally closed if it is the intersection of an open set and a closed set. Let $\mathcal{V} = \varprojlim V_i$ be an ind-variety. A subset $A \subset \mathcal{V}$ is called algebraic if it is locally closed and contained in $V_i$ for some $i > 0$, so $A$ has a natural structure of an algebraic variety. A morphism $\alpha : \mathcal{V} \rightarrow \mathcal{V}'$ is called an embedding if the image $\alpha(\mathcal{V}) \subset \mathcal{V}'$ is locally closed and induces an isomorphism of ind-varieties between $\mathcal{V}$ and $\alpha(\mathcal{V})$. An embedding is called a closed embedding (resp. an open embedding) if $\alpha(\mathcal{V}) \subset \mathcal{V}'$ is closed (resp. open). Finally, recall that a constructible set is a finite union of locally closed subsets.

**Example 1.3.**

1. The infinite-dimensional vector space

$$\mathbb{C}^\infty := \{(a_1, \ldots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 0 \text{ for finitely many } i\}$$

has a canonical structure of ind-variety given by the filtration $\mathbb{C} \xrightarrow{\varphi_1} \mathbb{C}^2 \xrightarrow{\varphi_2} \mathbb{C}^3 \xrightarrow{\varphi_3} \cdots$ where $\varphi_i(a_1, \ldots, a_i) = (a_1, \ldots, a_i, 0)$, for all $i > 0$. This ind-variety is called the infinite-dimensional affine space. Remark that we can change the complex number $0$ in the filtration definition of $\mathbb{C}^\infty$ and in $(i + 1)$-th coordinate of $\varphi_i$ by any other number. The ind-variety obtained this
way is easily seen to be isomorphic to \( \mathbb{C}^\infty \). For instance, we denote by \( \mathbb{C}_1^\infty \) the ind-variety isomorphic to the infinite-dimensional affine space given by \( \mathbb{C}_1^\infty := \{(a_1, \ldots) \mid a_i \in \mathbb{C} \text{ and } a_i \neq 1 \text{ for finitely many } i\} \).

(2) The set
\[
(\mathbb{C}^*)^\infty = \{(a_1, a_2, \ldots) \mid a_i \in \mathbb{C}^* \text{ and } a_i \neq 1 \text{ for finitely many } i\}
\]
has a canonical structure of ind-variety given by the filtration \( \mathbb{C}^* \overset{\varphi_i}{\twoheadrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_i}{\twoheadrightarrow} (\mathbb{C}^*)^3 \overset{\varphi_i}{\twoheadrightarrow} \cdots \)
where \( \varphi_i(a_1, \ldots, a_i) = (a_1, \ldots, a_i, 1) \) for all \( i > 0 \). This ind-variety is an open set in the infinite-dimensional affine space. This follows straightforward from the isomorphism \( \mathbb{C}^\infty \simeq \mathbb{C}_1^\infty \) above. Remark that \( (\mathbb{C}^*)^\infty \) has a natural structure of ind-group given by component-wise multiplication.

A commutative topological \( \mathbb{C} \)-algebra \( \mathcal{A} \) is pro-affine if it is Hausdorff, complete and admits a base \( \{I_i\}_{i>0} \) of open neighborhoods of 0, where \( I_i \subset \mathcal{A} \) is an ideal for all \( i \). Furthermore, we can assume that \( I_i \) form a descending filtration \( I_1 \supset I_2 \supset \cdots \) of ideals of \( \mathcal{A} \). Recall that Hausdorff property is equivalent to \( \bigcap I_i = \{0\} \) and completeness is equivalent to \( \mathcal{A} = \varprojlim A_i \) where the algebra \( A_i := \mathcal{A}/I_i \) is taken with the discrete topology, see [11, Section 9.2] for details. A pro-affine algebra \( \mathcal{A} \) is algebraic if \( A_i \) is finitely generated over \( \mathbb{C} \) for all \( i > 0 \). Every finitely generated algebra over \( \mathbb{C} \) is a pro-affine algebra with \( I_i = \{0\} \) for all \( i > 0 \). In the sequel all pro-affine algebras are algebraic, so we will drop algebraic from the notation.

For an ind-variety \( \mathcal{V} \) with filtration \( V_1 \overset{\varphi_1}{\twoheadrightarrow} V_2 \overset{\varphi_2}{\twoheadrightarrow} \cdots \) the ring of regular functions \( \mathbb{C}[\mathcal{V}] \) is defined as \( \varinjlim \mathbb{C}[V_i] \) where each \( \mathbb{C}[V_i] \) is taken with the discrete topology and \( \varinjlim \mathbb{C}[V_i] \) has the projective limit topology i.e.,
\[
\mathbb{C}[\mathcal{V}] = \varinjlim \mathbb{C}[V_i] = \left\{ (f_1, f_2, \ldots) \mid f_i \in \mathbb{C}[V_i] \text{ and } \varphi_i(f_{i+1}) = f_i \right\} \subset \prod_{i>0} \mathbb{C}[V_i],
\]
with subspace topology. The projective limit comes equipped with natural projections \( \pi_i : \mathbb{C}[\mathcal{V}] \to \mathbb{C}[V_i] \).

Let \( \alpha : \mathcal{V} \to \mathcal{V}' \) be a morphism of ind-varieties, then for every \( i > 0 \) there exists \( j > 0 \) such that \( \alpha \) induces an homomorphism \( \mathbb{C}[\mathcal{V}'] \to \mathbb{C}[\mathcal{V}] \) and so \( \alpha \) induces a continuous pro-affine algebras homomorphism \( \alpha^* : \mathbb{C}[\mathcal{V}'] \to \mathbb{C}[\mathcal{V}] \). Conversely, every continuous homomorphism \( \beta : \mathbb{C}[\mathcal{V}'] \to \mathbb{C}[\mathcal{V}] \) of pro-affine algebras induces for every \( i > 0 \) a homomorphism \( \mathbb{C}[\mathcal{V}'] \to \mathbb{C}[V_i] \) for some \( j > 0 \) and so it induces a morphism \( V_i \to V_j \) which in turns gives a morphism \( \beta^* : \mathcal{V} \to \mathcal{V}' \) [8, 7]. This yields an equivalence of categories between pro-affine algebras and affine ind-varieties.

2. TORIC IND-VARIETY

An algebraic torus \( T \) is an algebraic group isomorphic to \( (\mathbb{C}^*)^i \) for some \( i \geq 0 \). An ind-torus \( T \) is an ind-group isomorphic to either an algebraic torus or \( (\mathbb{C}^*)^\infty \). A regular action of an ind-torus \( T \) on an ind-variety \( \mathcal{V} \) is a group action \( \alpha : T \times \mathcal{V} \to \mathcal{V} \) by automorphisms of \( \mathcal{V} \) such that \( \alpha \) is also a morphism of ind-varieties.

**Definition 2.1.** A toric ind-variety is an irreducible ind-variety \( \mathcal{V} \) having an ind-torus \( T \) as an open subset such that the action of \( T \) on itself by translations extends to a regular action of \( T \) on \( \mathcal{V} \).

If \( \mathcal{V} \) is finite dimensional, then this definition coincides with the usual notion of toric variety, see for instance [2, Definition 1.1.3]. Remark that similarly to [2] and unlike other references [10, 4], we do not require toric varieties to be normal.

**Example 2.2.** Recall that \( \mathbb{Z}^\infty \) is defined as the inductive limit of the inductive system \( \mathbb{Z} \to \mathbb{Z}^2 \to \cdots \) where the maps are the injections setting the last coordinate to 0. Taking tensor product of this system with \( \mathbb{C}^* \) we obtain the inductive system defining \( (\mathbb{C}^*)^\infty \). In analogy with the finite-dimensional case, we denote this by \( (\mathbb{C}^*)^\infty = \mathbb{Z}^\infty \otimes_{\mathbb{Z}} \mathbb{C}^* \). Now, it follows directly from Example [9, 1] that for every sequence \( \mathbb{C}^* \overset{\varphi_1}{\twoheadrightarrow} (\mathbb{C}^*)^2 \overset{\varphi_3}{\twoheadrightarrow} (\mathbb{C}^*)^3 \overset{\varphi_5}{\twoheadrightarrow} \cdots \) with \( \varphi_i \) injective homomorphisms of algebraic groups, the corresponding ind-variety is an ind-group isomorphic to \( (\mathbb{C}^*)^\infty \).
In the next theorem we show that for every toric ind-variety, we can find an equivalent filtration composed of toric varieties and toric morphisms.

**Theorem 2.3.** Let \( \mathcal{V} = \varinjlim V_j \) be an affine ind-variety endowed with a regular action of the ind-torus \( T \). Then \( \mathcal{V} \) is an affine toric ind-variety with respect to \( T \) if and only if \( \mathcal{V} \cong \varprojlim W_j \) where \( W_j \) are toric varieties with acting torus \( T_j \), the closed embedding \( \varphi_j : W_j \hookrightarrow W_{j+1} \) are toric morphisms and the ind-torus \( T \) is the inductive limit \( \varprojlim T_j \).

**Proof.** The finite dimensional case is trivial since we can take \( W_j = \mathcal{V} \) and \( T_j = T \), for all \( j > 0 \). Hence, we only deal with the case where \( \mathcal{V} \) is a closed set in \( \mathcal{V} \). Then \( \mathcal{V} \) is an ind-variety with acting torus \( T \), the closed embedding \( \varphi_j : V_j \hookrightarrow V_{j+1} \) are toric embeddings and the \( j \)-th action on \( T_j \) by translations extends to a \( T_j \)-action on \( W_j \) for every \( j \). Furthermore, the \( j \)-th action on \( T_j \) is continuous and so by [6, Lemma 1.1.5], there exist \( i > 0 \) such that \( W_{j+i} \subseteq V_i \). Moreover, the inclusion \( (\mathbb{C}^*)^j \hookrightarrow (\mathbb{C}^*)^j+1 \) induces an inclusion \( \varphi_j : W_j \hookrightarrow W_{j+1} \). Since \( V_j \) is a closed set in \( \mathcal{V} \) we have that \( W_j \) and \( W_{j+1} \) are closed in \( V_i \) and so \( \varphi_i \) is a closed embedding.

We claim that the varieties \( W_j \) are toric with respect to the algebraic tori \( T_j = (\mathbb{C}^*)^j \) and the morphisms \( \varphi_j : W_j \hookrightarrow W_{j+1} \) are toric. Indeed, since \( (\mathbb{C}^*)^j \) is irreducible, \( W_j \) is also irreducible, for all \( j \). Furthermore, the \( j \)-th action on \( T_j \) by translations extends to a \( T_j \)-action on \( W_j \) for every \( t \in T_j \). We have \( t W_j = W_j \cdot t \) equals the closure of \( t(\mathbb{C}^*)^j \hookrightarrow (\mathbb{C}^*)^j \) and so \( W_j \) is stabilized by \( T_j \). Finally, by [11 Proposition 1.11], the \( j \)-th orbit \( (\mathbb{C}^*)^j \) is locally closed in \( W_j \) and so we conclude that \( (\mathbb{C}^*)^j \) is an open set in \( W_j \). Hence \( W_j \) is a toric variety. Furthermore, the morphism \( \varphi_j : W_j \hookrightarrow W_{j+1} \) is toric since its restriction to the acting torus its a group homomorphism by definition.

Finally, we prove that \( \mathcal{V} \cong \varprojlim W_j \) by proving that the filtrations given by \( V_i \) and \( W_j \), respectively are equivalent. We already proved above that for every \( j > 0 \) there exists \( i > 0 \) such that \( W_j \subseteq V_i \) is a closed embedding. To prove the other direction, observe that the set \( X = V_i \cap (\mathbb{C}^*)^j \) is an algebraic subset of \( \mathcal{V} \). Furthermore, since \( (\mathbb{C}^*)^j \subseteq \bigcup_{j=0}^\infty W_j \) and \( X \subseteq (\mathbb{C}^*)^j \) we have \( X = \bigcup_{j=0}^\infty X \cap W_j \). By [6, Lemma 1.3.1], there exists a positive integer \( k \) such that \( X = X \cap W_k \) and so \( X \subseteq W_k \). Moreover, the closure of \( X \) in \( \mathcal{V} \) is \( V_i \). Since \( W_k \) is closed, we conclude that \( V_i \subseteq W_k \) is a closed embedding. This concludes the proof of the “only if” part of the theorem.

We now prove the “if” direction of the theorem. The irreducibility of \( \mathcal{V} \cong \varprojlim W_j \) is a direct consequence of [15, Proposition 8]. Furthermore, by Example 2.2 the limit \( T = \varprojlim T_j \) is an ind-torus. Moreover, \( T \) is an open set in \( \varprojlim W_j \) by the definition of the ind-topology. Moreover, the action of \( T \) on itself by multiplication extends to \( \varprojlim W_j \) since the same holds in all the strata for \( T_j \) acting on \( W_j \). This concludes the proof. \( \square \)

Let \( \mathcal{V} = \varinjlim V_j \) be a toric ind-variety. We say that \( V_1 \hookrightarrow V_2 \hookrightarrow \ldots \) is a toric filtration if for every \( i > 0 \) the variety \( V_i \) is toric with acting torus \( T_i \), the closed embedding \( \varphi_i : V_i \hookrightarrow V_{i+1} \) is a toric morphism and the acting ind-torus \( T_i \) is the inductive limit \( \varprojlim T_i \). Theorem 2.3 above ensures the every toric ind-variety admits a toric filtration.

We define toric morphisms in direct analogy with the case of classical toric varieties.

**Definition 2.4.** Let \( \mathcal{T}_\mathcal{V} \) and \( \mathcal{T}_\mathcal{V}' \) be ind-tori acting on toric ind-varieties \( \mathcal{V} = \varinjlim V_i \) and \( \mathcal{V}' = \varinjlim V'_i \), respectively. A morphism \( \alpha : \mathcal{V} \rightarrow \mathcal{V}' \) of ind-varieties is toric if the image of \( \mathcal{T}_\mathcal{V} \) by \( \alpha \) is contained in \( \mathcal{T}_\mathcal{V}' \), and \( \alpha|_{T_i} : T_i \rightarrow T'_i \) is a morphism of ind-group.
coming from the toric filtration of $V$ and $V'$, respectively. By the definition of toric morphism, we have $\alpha(T_i) \subset T_V$ and so $\alpha(T_i) \subset H_j = V_j' \cap T_V$. Since $\alpha: T_V \to T_V'$ is a group homomorphism, the same holds for $\alpha|_{T_i}: T_i \to H_j$. This proves this direction of the proposition.

To prove the “if” part, we let $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$ and $V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots$ be toric filtrations of $V$ and $V'$, respectively. We further assume that for every $i > 0$, there exist an integer $j > 0$ such that $\alpha|_{V_i}: V_i \to V_j'$ is a toric morphism. Furthermore, replacing the toric filtration of $V'$ by a subfiltration we may and will assume $\alpha|_{V_i}: V_i \to V_j'$ is a toric morphism. It follows that $\alpha(T_i) \subset H_j$, where $T_V = \varprojlim T_i$ and $T_V' = \varprojlim H_j$ be the acting tori with the filtration coming from the toric filtration of $V$ and $V'$, respectively. Hence, we conclude $\alpha(T_V) \subset T_V'$. Similarly, the fact that $\alpha|_{T_i}: T_i \to H_j$ is a homomorphism of groups implies that $\alpha|_{T_V}: T_V \to T_V'$ is a homomorphism of ind-groups proving the proposition.

**Remark 2.6.** It is straightforward to show that a toric morphism $\alpha: V \to V'$ of toric ind-varieties is equivariant, i.e., $\alpha(t,x) = \alpha(t) . \alpha(x)$, for all $t \in T_V$ and all $x \in V$.

A character of an ind-torus $T$ is a morphism $\chi: T \to \mathbb{C}^*$ of ind-varieties that is also group homomorphism. The set of characters of $T$ forms group denoted by $\mathcal{M}$. If $\dim T < \infty$ it is well known that $\mathcal{M}$ is a finitely generated free abelian group of rank $\dim T$. Similarly, a one-parameter subgroup of $T$ is a morphism $\lambda: \mathbb{C}^* \to T$ of ind-varieties that is also a group homomorphism. The set of one-parameter subgroups of $T$ forms group denoted by $\mathcal{N}$. If $\dim T < \infty$ it is well known that $\mathcal{N}$ is also a finitely generated free abelian group of rank $\dim T$. Furthermore, if $\dim T < \infty$, then the groups $\mathcal{M}$ and $\mathcal{N}$ are dual with duality $\mathcal{M} \times \mathcal{N} \to \mathbb{Z}$ given by $\langle \chi, \lambda \rangle$ is the only integer $k$ such that $\chi \circ \lambda: \mathbb{C} \to \mathbb{C}$ maps $t$ to $t^k$.

We now compute the groups of characters and one-parameter subgroups of the ind-torus and prove the analogous duality result. Let $T$ be the infinite-dimensional ind-torus with toric filtration $T_1 \hookrightarrow T_2 \hookrightarrow \ldots$. Letting $M_i$ and $N_i$ be the character lattice and the one-parameter subgroup lattice of $T_i$, respectively, the filtration induces naturally a projective system $M_1 \leftarrow M_2 \leftarrow \ldots$ and an inductive system $N_1 \to N_2 \to \ldots$.

**Proposition 2.7.** Let $T$ be the infinite-dimensional ind-torus with toric filtration $T_1 \hookrightarrow T_2 \hookrightarrow \ldots$ Then

1. The group of characters $\mathcal{M}$ of $T$ is $\varprojlim M_i$ and is isomorphic to $\mathbb{Z}^\omega$.
2. The group of one-parameter subgroups $\mathcal{N}$ of $T$ is $\varinjlim N_i$ and is isomorphic to $\mathbb{Z}^\omega$.
3. The groups $\mathcal{M}$ and $\mathcal{N}$ are natural dual to each other and the duality is realized by the pairing $\langle \chi, \lambda \rangle: \mathcal{M} \times \mathcal{N} \to \mathbb{Z}$ given by $\langle \chi, \lambda \rangle = k$, where $\lambda \circ \chi: \mathbb{C}^* \to \mathbb{C}^*$ maps $t \mapsto t^k$ making $\mathcal{M}$ and $\mathcal{N}$ dual groups.

**Proof.** To prove (1), we let $\chi: T \to \mathbb{C}^*$ be a character of $T$. By the definition of morphism of ind-varieties, we have that $\chi|_{T_i}: T_i \to \mathbb{C}^*$ is a character of $T_i$ for all $i > 0$. This produces homomorphisms $\pi_i: M \to M_i$ satisfying $\pi_i = \varphi_i^* \circ \pi_{i+1}$, where $\varphi_i^*: M_{i+1} \to M_i$ is the map induced by $\varphi: T_i \to T_{i+1}$. By the universal property of the projective limit we have a homomorphism $M \to \varprojlim M_i$. On the other hand, we define the inverse homomorphism $\varprojlim M_i \to M$ in the following way. Let $(\chi_1, \chi_2, \ldots)$ be an element in the projective limit $\varprojlim M_i$. We associate a character $\chi \in \mathcal{M}$ given by $\chi: T \to \mathbb{C}^*$ via $t \mapsto \chi_k(t)$ for any $k > 0$ such that $t \in T_k$. By the definition of projective limit this map is well defined. It is a straightforward verification that it is a homomorphism. This proves that $\mathcal{M}$ is the projective limit $\varprojlim M_i$. Finally, $\mathcal{M}$ is isomorphic to $\mathbb{Z}^\omega$ by Example [Ex].

To prove (2), we let $\lambda_i: \mathbb{C}^* \to T_i$ be a one-parameter subgroup in $N_i$. Composing with the injection $T_i \hookrightarrow T$ we obtain a one-parameter subgroup $\lambda: \mathbb{C}^* \to T$ of the ind-torus. This yields homomorphisms $\psi_i: N_i \to \mathcal{N}$. By the universal property of the inductive limit we have a homomorphism $\varinjlim N_i \to \mathcal{N}$. On the other hand, we define the inverse homomorphism in the following way. Let $\lambda: \mathbb{C}^* \to T$ be a one-parameter subgroup of $T$. By the definition of morphism of ind-varieties, we have that there exists $k > 0$ such that the one-parameter subgroup $\lambda$ restricts to $\lambda_k: \mathbb{C}^* \to T_k$ is a one-parameter subgroup of $T_k$. Hence, $\lambda_k \in N_k$ and composing with $\psi_k: N_k \to \varprojlim N_i$ we obtain a homomorphism $\mathcal{N} \to \varinjlim N_i$.
By the definition of inductive limit this map is well defined. It is a straightforward verification that it is a homomorphism. Finally, $N$ is isomorphic to $\mathbb{Z}^\infty$ by Example 1.1.

To prove (3), a routine computation shows that $\langle , \rangle$ is bilinear and under the isomorphisms in (1) and (2) corresponds to the usual dot product defined in Lemma 1.2. This proves the proposition.

In the proof of our main result, we will need the following lemma whose proof is straightforward.

**Lemma 2.8.** Let $\mathcal{T}$ and $\mathcal{T}'$ be ind-tori and let $\alpha: \mathcal{T} \to \mathcal{T}'$ be an ind-group homomorphism with character group $M_\mathcal{T}$ and $M_\mathcal{T}'$ and one-parameter subgroup group $N_\mathcal{T}$ and $N_\mathcal{T}'$. Then $\alpha$ induces homomorphisms $\alpha^+: M_\mathcal{T}' \to M_\mathcal{T}$ and $\alpha_*: N_\mathcal{T} \to N_\mathcal{T}'$.

### 3. Pro-affine semigroup

A semigroup is a set $(S, +)$ with an associative binary operation. All our semigroups will be commutative and unital. A semigroup $S$ is called affine if it is finitely generated and can be embedded in a $\mathbb{Z}^k$ for some $k \geq 0$. It is well known that the category of affine toric varieties with toric morphisms is dual to the category of affine semigroups with homomorphisms of semigroups. The main result of this paper is a generalization of this result to the case of affine toric ind-variety. In this section, we define and study the semigroups $S$ that will appear as the semigroup of an affine toric ind-variety $V$.

Recall that the ring of regular functions $\mathbb{C}[V]$ of an ind-variety is a pro-affine algebra and so it is endowed with a topology holding the information of the filtration of $V$. We will first transport the notion of pro-affine algebra into the context of semigroups. A pro-affine algebra $A$ is defined using a filtration of ideals on $A$ and the projective limit topology induced by the quotients of $A$ by the ideals in this filtration. In the case of semigroups, there exists an analog notion of ideal, but there is no bijection between ideals and quotient semigroups. For this reason, in the context of semigroups, we need the more general notion of compatible equivalence relations to keep track of all the possible quotients.

An equivalence relation on a set $S$ is a subset $R \subset S \times S$ satisfying the usual properties of being reflexive, symmetric and transitive. An equivalence relation on a semigroup $S$ is called compatible if for every $(m, n)$ and $(m', n')$ in $R$ we have that $(m + m', n + n')$ also belongs to $R$. In this case, the set of equivalence classes $S/R$ inherits a natural structure of semigroup with binary operation given by $[m] + [m'] = [m + m']$, where $[m]$ denotes the class of $m$ in $S/R$.

A filtered semigroup is a couple $(S, F)$, where $S$ is a semigroup and $F$ is a descending filtration $R_1 \supset R_2 \supset \ldots$ of $S \times S$ of compatible equivalence relations on $S$. We denote a filtered semigroup simply by $S$ if $F$ is clear from the context. In close analogy with Section 9.2, the filtration of compatible equivalence relations on $S$ defines a topology on $S$ having basis $\{E_{m,k} \mid m \in S, k > 0\}$, where $E_{m,k} = \{m' \in S \mid (m, m') \in R_k\}$ is the equivalence class of $m$ under the equivalence relation $R_k$. It is straightforward to verify that this topology coincides with the finest topology making all the quotient morphisms $S \to S/R_k$ continuous where $S/R_k$ is taken with the discrete topology. The trivial equivalence relation on $S$ corresponds to the diagonal in $S \times S$. The trivial filtration on a semigroup $S$ is given by setting each equivalence relation $R_i$ to be trivial. In this case the induced topology on $S$ is the discrete topology.

Let $S$ be filtered semigroup with filtration $R_1 \supset R_2 \supset \ldots$ of compatible equivalence relations in $S$. It is straightforward to verify that the topology on $S$ is Hausdorff if and only if $\bigcap_{k>0} R_k$ equals the diagonal in $S \times S$. Additionally, we can generalize the notion of Cauchy sequence to this context of semigroups. Indeed, a sequence $\{a^{(i)}\}_{i > 0} \subset S$ in the semigroup is say to be Cauchy sequence if given any $k > 0$ there exists an integer $N$ such that $(a^{(i)}, a^{(j)}) \in R_k$ for all $i, j > N$. A direct computation shows that a convergent sequence is always Cauchy. We say that a filtered semigroup $S$ is complete if every Cauchy sequence converges. Given a projective system $S_1 \leftarrow S_2 \leftarrow \ldots$ of semigroups we define a filtration $R_1 \supset R_2 \supset \ldots$ of compatible equivalence relations on the projective limit $S = \varinjlim S_i$ by $R_i = \{(m, m') \in S \times S \mid \pi_i(m) = \pi_i(m')\}$. The topology induced on $S$ by this filtration coincides with the projective limit topology.
Proposition 3.1. Let \( S_1 \leftarrow S_2 \leftarrow \ldots \) be a projective system of semigroups where each \( S_i \) carries the discrete topology. Then the projective limit semigroup \( S = \lim_{{i \to \infty}} S_i \) is Hausdorff and complete.

Proof. A couple \((m, m') \subseteq S \times S\) belongs to \( R_k\) if and only if \( m_i = m'_i \) for all \( i \leq k \). Hence, the couple \((m, m')\) belongs to \( \bigcap_{{k \to 0}} R_k\) if and only if \( m = m'\). We conclude that \( \bigcap_{{k \to 0}} R_k\) equals the diagonal of \( S \times S \) and so \( S \) is Hausdorff. To prove that \( S \) is complete, let \( \{m^{(i)}\}_{i \to 0} \subseteq S \) be a Cauchy sequence in \( S \). Recall that, by the definition of projective limit, each \( m^{(i)}\) equals \((m_1^{(i)}, m_2^{(i)}, \ldots) \in \prod_{{i \to 0}} S_i\). For every \( k > 0 \) there exist \( N \) such that \((m^{(i)}, m^{(i+1)}) \in R_k\) for all \( i > N \). Hence, for every \( k > 0 \) there exist \( N \) such that \( m_k^{(i)} = m_k^{(i+1)} = m_k^{(i+2)} = \ldots\) when \( i > N \). Letting \( m_k = m_k^{(i)} \in S_k\) for any \( i > N \), we let \( m = (m_1, m_2, \ldots) \in S \). Now, for every \( k > 0 \) there exist \( N \) such that \((m, m^{(i)}) \in R_k\) for all \( i > N \) and so the Cauchy sequence \( \{m^{(i)}\}_{i \to 0} \subseteq S \) converges to \( m \).

Remark 3.2. If a filtered semigroup \( S \) with filtration \( R_1 \supset R_2 \supset \ldots \) of compatible equivalence relations in \( S \) is Hausdorff and complete, then \( \lim_{{i \to 0}} S_i \), where \( S_i = S/R_i \) with the morphism induced from \( R_i \supset R_{i+1}\), is canonically isomorphic to \( S \). Indeed, the canonical map \( S \to \lim_{{i \to 0}} S_i \) into the projective limit given by \( m \mapsto (\pi_1(m), \pi_2(m), \ldots)\) has inverse given by \((\{m_1\}, \{m_2\}, \ldots) \mapsto \lim_{{i \to 0}} m_i\), where \( \{m_i\}_{i \to 0} \) is the Cauchy sequence given in \( S \) by \( \{m_1, m_2, \ldots\} \).

We now define the natural notion of morphism of filtered semigroups.

Definition 3.3. Let \( S \) and \( S' \) be filtered semigroup with filtrations \( R_1 \supset R_2 \supset \ldots \) and \( R'_1 \supset R'_2 \supset \ldots \), respectively. A map \( \beta : S \to S' \) is called a morphism of filtered semigroups if \( \beta \) is a semigroup homomorphism and for every \( i > 0 \) there exists \( j > 0 \) such that \( (\beta \times \beta)(R_j) \subseteq R'_j\). In particular, every morphism \( \beta : S \to S' \) of filtered semigroups is continuous since the condition \( (\beta \times \beta)(R_j) \subseteq R'_j \) implies point-wise continuity at every \( m \in S \). As usual, an isomorphism \( \beta : S \to S' \) of filtered semigroups is a bijective morphism whose inverse is also a morphism. We also say that two filtrations \( R_1 \supset R_2 \supset \ldots \) and \( R'_1 \supset R'_2 \supset \ldots \) on the same semigroup \( S \) are equivalent if the identity map is an isomorphism of filtered semigroups.

Lemma 3.4. With the notation in Definition 3.3, the morphism \( \beta : S \to S' \) of filtered semigroups induces a natural homomorphism of semigroup \( \beta_{ij} : S_j \to S'_i \) where \( S_j = S/R_j \) and \( S'_i = S'/R'_i \) such that the following diagram commutes.

\[
\begin{array}{ccc}
S & \xrightarrow[\pi_j]{} & S'_i \\
\beta \downarrow & & \beta_{ij} \downarrow \\
S_j & \xrightarrow[\pi_j]{} & S'_i
\end{array}
\]

Proof. The map \( \beta_{ij} : S_j \to S_i \) defined naturally by \( [m] \mapsto [\beta(m)] \) is well defined due to the condition \( (\varphi \times \varphi)(R_j) \subseteq R'_i \). The rest of the proof is straightforward.

We now define pro-affine semigroups that are the generalization of the affine semigroups that are the objects dual to classical affine toric varieties.

Definition 3.5.

1. A pro-affine semigroup \( S \) is a filtered semigroup with filtration \( R_1 \supset R_2 \supset \ldots \) of compatible equivalence relations in \( S \) that is complete and such that every \( S/R_i \) is an affine semigroup.

2. Let \( S \) be a filtered semigroup with filtration \( R_1 \supset R_2 \supset \ldots \). A filtered subsemigroup is an affine semigroup \( S' \subseteq S \) endowed with the filtration of compatible equivalence relations \( R_i \cap (S' \times S') \) on \( S' \).

Example 3.6.

1. We define the canonical filtration \( \widetilde{R}_1 \supset \widetilde{R}_2 \supset \ldots \) of equivalence relations on the semigroup \( \mathbb{Z}^\omega \) by \( \widetilde{R}_k = \{(m, m') \in \mathbb{Z}^\omega \times \mathbb{Z}^\omega \mid m_i = m'_i, \text{ for all } i \leq k \} \). By Proposition 3.1 we conclude
that \( \mathbb{Z}^\omega \) is complete. Furthermore, \( \mathbb{Z}^\omega / R_1 \) is naturally isomorphic to \( \mathbb{Z}^j \) with quotient morphism \( \pi_i : \mathbb{Z}^\omega \to \mathbb{Z}^i \) the projection to the first \( i \)-th coordinates. Hence, \( \mathbb{Z}^\omega / R_i \) is an affine semigroup and so the filtered semigroup \( \mathbb{Z}^\omega \) is a pro-affine semigroup.

(2) The filtered subsemigroups \( \mathbb{Z}^\omega_{\geq 0} \) of \( \mathbb{Z}^\omega \) of arbitrary sequences of non-negative integers is also pro-affine with a similar argument as in (1).

(3) Any affine semigroup \( S \subset \mathbb{Z}^i \) with the constant filtration given by the trivial equivalence relation is pro-affine.

(4) Let \( e_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{Z}^\omega \), where the non-zero coefficient is located in the position \( i > 0 \). The subsemigroup \( S = \mathbb{Z}^\omega_{\geq 0} \setminus \{e_i\} \) of \( \mathbb{Z}^\omega \) is not complete and so is not pro-affine. Indeed, the sequence \( \{a_i = \epsilon_1 + \epsilon_i\}_{i > 0} \) is Cauchy but not convergent in \( S \).

**Theorem 3.7.** Let \( S \) be a pro-affine semigroup, then \( S \) is isomorphic to a filtered subsemigroup of \( \mathbb{Z}^\omega \). Moreover, we can assume that \( S \) is embedded in \( \mathcal{M} \) with \( \mathbb{Z} S = M \), where \( \mathcal{M} \simeq \mathbb{Z}^\omega \) or \( \mathcal{M} \simeq \mathbb{Z}^k \) for some \( k > 0 \).

**Proof.** Letting \( R_1 \supset R_2 \supset \ldots \) be the filtration of compatible equivalence relations in \( S \) we let \( S_i = S / R_i \) and \( \varphi_i : S_{i+1} \to S_i \) be the homomorphisms given by the inclusion \( R_i \supset R_{i+1} \). Hence, we have a commutative diagram

\[
\begin{array}{c}
S_1 & \longrightarrow & S_2 & \longrightarrow & S_3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
Z S_1 & \longrightarrow & Z S_2 & \longrightarrow & Z S_3 & \longrightarrow & \cdots
\end{array}
\]

where \( Z S_i \) is group generated by \( S_i \) for any embedding \( S_i \hookrightarrow \mathbb{Z}^k \) and the homomorphisms \( Z S_i \to Z S_{i+1} \) are induced by \( S_i \to S_{i+1} \), for all \( i > 0 \). Since the homomorphisms in the upper system are surjective, the same holds for the lower system. Hence, the lower projective system is split. If the homomorphisms in the lower system become also injective for \( i \) large enough, then the projective limit of the lower system is isomorphic to \( \mathbb{Z}^k \) for some \( k > 0 \). Furthermore, since \( \mathbb{Z}^k \) is embedded in \( \mathbb{Z}^\omega \) the first statement follows in this case. Assume now that there is no integer \( i > 0 \) such that \( \mathbb{Z} S_i \) is a subgroup of \( Z S_i \) and this is an embedding of filtered semigroups by Remark 5.2. The second statement follows directly from the construction above in this proof. \( \square \)

In the following example we show the surprising consequence of Specker Theorem (Lemma 1.2) that every group homomorphism \( \beta : \mathbb{Z}^\omega \to \mathbb{Z}^\omega \) is a morphism of filtered semigroups for the canonical filtration \( R_i \).

**Example 3.8.**

(1) Every homomorphism \( \beta : \mathbb{Z}^\omega \to \mathbb{Z}^\omega \) is a morphism of filtered semigroups with respect to the canonical filtration. Indeed, since \( \mathbb{Z}^\omega \) is a group, we have that \( E_{0,k} \) is a subgroup of \( \mathbb{Z}^\omega \) and

\[
\tilde{R}_k = \bigcup_{m \in \mathbb{Z}^\omega} (m + E_{0,k}) \times (m + E_{0,k})
\]

Hence, it is enough to show that for every \( i > 0 \) there exists \( j > 0 \) such that \( \beta(E_{0,j}) \subset E_{0,i} \). By Lemma 1.2, the composition \( \pi_i \circ \beta : \mathbb{Z}^\omega \to \mathbb{Z}^i \) corresponds to an element in \( (p_1, \ldots, p_i) \in (\mathbb{Z}^\omega)^i \) under the isomorphism \( \text{Hom}(\mathbb{Z}^\omega, \mathbb{Z}) \simeq \mathbb{Z}^\omega \) given by the duality map, see also [5] Theorem 94.3 and Corollary 94.5. By definition of inductive limit, each \( p_i \in \mathbb{Z}^j \) for some \( j_i > 0 \). Taking \( j \) to be the maximum of \( \{j_1, \ldots, j_i\} \) we obtain that \( \beta(E_{0,j}) \subset E_{0,i} \).

(2) A similar argument shows that every homomorphism \( \beta : \mathbb{Z}^\omega \to \mathbb{Z}^k \) is a morphism of filtered semigroups with respect to the canonical filtration in \( \mathbb{Z}^\omega \) and trivial filtration in \( \mathbb{Z}^k \), for every \( k \geq 0 \).

The above example allows us to prove that every homomorphism between pro-affine semigroups is a morphism.
Proposition 3.9. Let $S$ and $S'$ be algebraic pro-affine semigroups. If $\beta : S \to S'$ is any homomorphism of semigroup then $\beta$ is a morphism of filtered semigroups.

Proof. By Theorem 3.7, we can assume that $S$ is a subsemigroup of $M = \mathbb{Z}^\omega$ or $M = \mathbb{Z}^k$ for some $k \geq 0$ with $ZS = M$. Similarly, we can assume that $S'$ is a subsemigroup of $M' = \mathbb{Z}^\omega$ or $M' = \mathbb{Z}^\ell$ for some $\ell \geq 0$ with $ZS' = M'$. The homomorphism $\beta$ can be extended to a homomorphism $\tilde{\beta} : M \to M'$ via $m \mapsto \beta(m) - \beta(m')$. If $M = \mathbb{Z}^k$, then $\tilde{\beta}$ is trivially a morphism of filtered semigroups since the filtration by equivalence relation on $\mathbb{Z}^k$ is trivial. Furthermore, if $M = \mathbb{Z}^\omega$ the homomorphism $\beta$ is also a morphism of filtered semigroups by Example 3.8. Now, the proposition follows since $S$ and $S'$ are filtered subsemigroups of $M$ and $M'$, respectively. 

Remark 3.10. It follows from Proposition 3.9 above that two different filtrations $R_1 \supset R_2 \supset \ldots$ and $R'_1 \supset R'_2 \supset \ldots$ of compatible equivalence relations in a pro-affine semigroup $S$ are always equivalent since the identity is an isomorphism of semigroups and so it is also isomorphism of filtered semigroups.

It is straightforward to prove, mimicking the classical argument for metric spaces, that a subsemigroup in a complete filtered semigroup is complete if and only if it is closed. This allows us to derive the following corollary that acts as alternative definition of pro-affine semigroups. Recall that $\mathbb{Z}^\omega/R_1$ is naturally isomorphic $\mathbb{Z}^\omega$ with quotient morphism $\pi_i : \mathbb{Z}^\omega \to \mathbb{Z}^i$ the projection to the first $i$-th coordinates.

Corollary 3.11. An abstract semigroup $S$ admits a filtration by compatible equivalence relations on $S$ making $S$ a pro-affine semigroup if and only if there exists an embedding $\iota : S \hookrightarrow \mathbb{Z}^\omega$ where $\iota(S)$ is closed and $(\pi_i \circ \iota)(S)$ is finitely generated for every $i > 0$. Moreover, if such a filtration exits, then it is unique.

Proof. If $S$ admits a structure of pro-affine semigroup, then the corollary follows from Theorem 3.7. On the other hand, if $S$ is embedded in $\mathbb{Z}^\omega$, then it inherits a filtration $R_1 \supset R_2 \supset \ldots$ from this embedding. By definition $S/R_1 \simeq (\pi_i \circ \iota)(S)$ which is assume to be finitely generated. Furthermore, $S$ is complete with the induced filtration since $\iota(S)$ is closed in $\mathbb{Z}^\omega$. This yields that $S$ is a pro-affine semigroup with this filtration. Finally, the uniqueness statement follows from Proposition 3.9 and Remark 3.10. 

4. AFFINE TORIC IND-VARIETIES AND PRO-AFFINE SEMIGROUPS

In this section we prove that the category of affine toric ind-varieties with toric morphisms is dual to the category of pro-affine semigroups with homomorphisms of semigroups.

Given an affine toric ind-variety $V$ with toric filtration $V_1 \hookrightarrow V_2 \hookrightarrow \ldots$, applying the functor $S(\cdot)$ defined in Section 3.11 we obtain a projective system

$$
\begin{array}{cccc}
V_1 & \xrightarrow{\varphi_1} & V_2 & \xrightarrow{\varphi_2} & V_3 & \xrightarrow{\varphi_3} & \ldots \\
S_1 & \xrightarrow{S(\varphi_1)} & S_2 & \xrightarrow{S(\varphi_2)} & S_3 & \xrightarrow{S(\varphi_3)} & \ldots
\end{array}
$$

where each semigroup $S_i = S(V_i)$ is the affine semigroup associated to the toric variety $V_i$, i.e, $\mathbb{C}[V_i] = \mathbb{C}[S_i]$ and $S(\varphi_i) : S_{i+1} \to S_i$ is the semigroup homomorphism corresponding to the toric morphism $\varphi_i : V_i \to V_{i+1}$ [2 Proposition 1.3.14]. We define the semigroup $S(V)$ associated to $V$ as the projective limit $\varprojlim S_i$ of this projective system. By Proposition 3.11 and the paragraph preceding it, we have that $S(V)$ is a pro-affine semigroup.

On the other hand, given a pro-affine semigroup $S$ with filtration $R_1 \supset R_2 \supset \ldots$ and $R'_1 \supset R'_2 \supset \ldots$ of compatible equivalence relations on $S$, we let $S_1 \leftarrow S_2 \leftarrow \ldots$ be the associated projective system of semigroups where each $S_i = S/R_i$ is an affine semigroup and the homomorphisms $\varphi_i : S_{i+1} \to S_i$ are given by $[m]_{i+1} \mapsto [m]_i$, where $[m]_i$ is the class of $m \in S$ inside the quotient $S_i$. The homomorphisms $\varphi_i$ are surjective. Hence, applying the functor $V(\cdot)$ defined in Section 3.11 for toric varieties, we obtain an
inductive system of closed embeddings

\[
\begin{align*}
S_1 & \xrightarrow{\varphi_1} S_2 \xrightarrow{\varphi_2} S_3 \xrightarrow{\varphi_3} \ldots \\
V_1 & \xrightarrow{\mathcal{V}(\varphi_1)} V_2 \xrightarrow{\mathcal{V}(\varphi_2)} V_3 \xrightarrow{\mathcal{V}(\varphi_3)} \ldots
\end{align*}
\]

where each \( V_i = \mathcal{V}(S_i) \) is the toric variety associated to the semigroup \( S_i \) and \( \mathcal{V}(\varphi_i) : V_i \to V_{i+1} \) is the toric morphism corresponding to the semigroup homomorphism \( \varphi_i : S_{i+1} \to S_i \). The corresponding inductive limit \( \lim_i V_i \) of this system is an affine toric ind-variety by Theorem 2.3 that we denote by \( \mathcal{V}(S) \). The ind-torus acting on \( \mathcal{V}(S) \) is \( T = \lim_i T_i \), where \( T_i \) is the algebraic torus acting on \( V_i \). It is clear that these constructions provide a bijection between affine toric varieties and pro-affine semigroups up to isomorphisms.

Let now \( V \) be an affine toric ind-variety and let \( S = \mathcal{S}(V) \). In general, projective limits do not commute with direct sums, hence we cannot expect to have, as in the classical case, an isomorphism between the ring of regular functions \( \mathbb{C}[V] \) on \( V \) and the semigroup algebra \( \mathbb{C}[S] \). Nevertheless, the semigroup algebra carries a natural descending filtration of ideals \( I_1 \supset I_2 \supset \ldots \), where \( I_i = \ker \pi_i \) and \( \pi_i \) is the natural projection \( \pi_i : \mathbb{C}[S] \to \mathbb{C}[V_i] \), for all \( i > 0 \) induced by the projections \( \pi_i : S \to S_i \) coming from the projective limit. It follows directly from [11, Chapter 9, Theorem 10] that the algebra \( \mathbb{C}[V] \) is the completion of \( \mathbb{C}[S] \).

In the following proposition, we summarize the considerations above.

**Proposition 4.1.** The assignments \( V \mapsto \mathcal{S}(V) \) for every affine toric ind-variety and \( S \mapsto \mathcal{V}(S) \) for every pro-affine semigroup are inverses up to isomorphism, i.e., \( \mathcal{V}(\mathcal{S}(V)) \) is isomorphic to \( V \) for every affine toric ind-variety and \( \mathcal{S}(\mathcal{V}(S)) \) is isomorphic to \( S \) for every pro-affine semigroup \( S \). Furthermore, for every affine toric ind-variety \( V \), the ring of regular functions \( \mathbb{C}[V] \) is isomorphic as filtered algebra to the completion of \( \mathbb{C}[S] \).

We will also need the following lemma generalizing the usual equivalent statement in the classical case.

**Lemma 4.2.** Let \( V \) be an affine toric ind-variety with acting ind-torus \( T \) whose character lattice is \( M \). Then \( \mathcal{S}(V) \) is naturally embedded in \( M \) with \( \mathbb{Z}\mathcal{S}(V) = M \). On the other hand, let \( S \) be a pro-affine semigroup embedded in \( M \simeq \mathbb{Z}^\omega \) or \( M \simeq \mathbb{Z}^k \) for some \( k \geq 0 \) as filtered semigroup with \( \mathbb{Z}\mathcal{S} = M \). Then the character lattice of the ind-torus \( T \) acting in the affine toric ind-variety is naturally isomorphic to \( M \).

**Proof.** The case where \( M \simeq \mathbb{Z}^k \) corresponds to the classical case of affine toric varieties. Hence, we will only deal with the case where \( M \simeq \mathbb{Z}^\omega \). Assume first that \( V \) is an affine toric ind-variety. With the above notation, by the classical case we have that each \( S_i \) is naturally embedded in the character lattice \( M_i \) of the algebraic torus \( T_i \) acting on \( V_i \) with \( M_i = \mathbb{Z}S_i \). By Theorem 2.3, we have that \( T \) equals the inductive limit \( \lim_i T_i \). Furthermore, by Proposition 2.7, we have that \( M \) equals \( \lim_i M_i \). The first assertion now follows. On the other hand, given \( S \) embedded in \( M \simeq \mathbb{Z}^\omega \), we let \( M_i \) be the character lattice of the torus \( T_i \) acting on \( V_i \). By the classical finite dimensional case of the lemma, we have \( \mathbb{Z}S_i = M_i \). The result now follows again from Proposition 2.7. \qed

We come now to morphisms in both categories. Let first \( S \) and \( S' \) be pro-affine semigroups and let \( \beta : S \to S' \) be a semigroup homomorphism. By Proposition 3.9, the pro-affine semigroups \( S \) and \( S' \) admit equivalent filtrations of equivalence relations \( R_1 \supset R_2 \supset \ldots \) and \( R'_1 \supset R'_2 \supset \ldots \), respectively, such that \( \beta \) is a morphism of filtered semigroups with respect to these filtrations. We let \( V = \mathcal{V}(S) \) and \( V' = \mathcal{V}(S') \). The corresponding affine toric ind-varieties defined above with the toric filtrations \( V_1 \hookrightarrow V_2 \hookrightarrow \ldots \) and \( V'_1 \hookrightarrow V'_2 \hookrightarrow \ldots \), respectively, where \( V_i = \mathcal{V}(S_i) \) and \( V'_i = \mathcal{V}(S'_i) \) and the closed embeddings are \( \mathcal{V}(\varphi_i) \) and \( \mathcal{V}(\varphi'_i) \), respectively. We define a homomorphism \( \mathbb{C}[S] \to \mathbb{C}[S'] \) of semigroup algebras by \( \chi^m \mapsto \chi^{\beta(m)} \), for all \( m \in S \). By abuse of notation, we denote this map also by \( \beta : \mathbb{C}[S] \to \mathbb{C}[S'] \).
Lemma 4.3. The homomorphism \( \beta : \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}'] \) is a continuous homomorphism of topological algebras and so we can extend \( \beta \) to an unique continuous homomorphism \( \mathcal{V}(\beta)^* : \mathbb{C}[\mathcal{V}] \to \mathbb{C}[\mathcal{V}'] \) whose comorphism defines a toric morphism of affine toric ind-varieties \( \mathcal{V}(\beta) : \mathcal{V}' \to \mathcal{V} \).

Proof. To prove that \( \beta : \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}'] \) is continuous we have to prove that for all \( i > 0 \) there exists \( j > 0 \) such that \( \beta(I_j) \subset I_i' \). Here \( I_j = \ker \pi_j \) and \( \pi_j \) is the projection \( \pi_j : \mathcal{C}[\mathcal{S}] \to \mathcal{C}[\mathcal{S}_j] \) induced by \( \mathcal{S} \to \mathcal{S}_j \), for all \( j > 0 \) and similarly \( I_i' = \ker \pi_i' \) and \( \pi_i' \) is the projection \( \pi_i' : \mathcal{C}[\mathcal{S}'] \to \mathcal{C}[\mathcal{S}_i'] \) induced by \( \mathcal{S}' \to \mathcal{S}_i' \), for all \( i > 0 \).

Let \( i > 0 \) be an integer. By the definition of morphism of filtered semigroup, there exists \( j > 0 \) such that \( \beta(\mathcal{S}(R_j')) \subset \mathcal{S}_j \). Let \( f = \sum a_m \chi^m \) be an element in \( I_j \) where the sum is finite. Belonging to \( I_j \) is equivalent to \( \pi_j(f) = \sum a_m \chi_j^m \pi_j(m) = 0 \). On the other hand, \( \pi_j'(\beta(f)) = \sum a_m \chi_{j_0}^m \pi_j'(m) \). By Lemma 4.4, the homomorphism \( \beta \) induces a homomorphism \( \beta_{ij} : \mathcal{S}_j \to \mathcal{S}_i' \) and we have \( \pi_j'(\beta(f)) = \beta_{ij}(\sum a_m \chi_{j_0}^m \pi_j'(m)) = 0 \). We conclude that \( \beta(I_j) \subset I_i' \) and so \( \beta : \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}'] \) is continuous.

Finally, the algebra \( \mathbb{C}[\mathcal{S}] \) is dense in \( \mathbb{C}[\mathcal{V}] \) by the second statement of Proposition 4.1. Hence, the homomorphism \( \beta \) can be extended to a continuous homomorphism \( \mathcal{V}(\beta)^* : \mathbb{C}[\mathcal{V}] \to \mathbb{C}[\mathcal{V}'] \) as required, see [11, Ch.9, Th. 5]. Moreover, by Proposition 2.5, the morphism \( \mathcal{V}(\beta) : \mathcal{V}' \to \mathcal{V} \) is toric.

Let now \( \alpha : \mathcal{V} \to \mathcal{V}' \) be a toric morphism of affine toric ind-varieties and let \( \mathcal{S} = \mathcal{S}(\mathcal{V}) \) and \( \mathcal{S}' = \mathcal{S}(\mathcal{V}') \) be the corresponding pro-affine semigroups. By Lemma 4.2 we have that \( \mathcal{S} \) and \( \mathcal{S}' \) are naturally embedded in \( \mathcal{M} \) and \( \mathcal{M}' \), respectively. In particular, by Lemma 2.8 we have that \( \alpha|_{\mathcal{M}_v} : \mathcal{M}_v \to \mathcal{M}_v' \) and so the homomorphism \( (\alpha|_{\mathcal{M}_v})^* \) induces a semigroup homomorphism \( \alpha^* : \mathcal{M}' \to \mathcal{M} \) on the character lattices via \( (\alpha|_{\mathcal{M}_v})^*(\chi^m) = \chi^m \alpha^*(m) \). Furthermore, given \( m \in \mathcal{S}' \), the regular function \( \chi^m \in \mathbb{C}[\mathcal{V}'] \) is mapped to the regular function \( \chi^{\alpha^*(m)} \in \mathbb{C}[\mathcal{V}] \). This yields \( \alpha^*(m) \in \mathcal{S} \), for all \( m \in \mathcal{S}' \). Hence \( \alpha^* \) restricts to a homomorphism \( \mathcal{S}' \to \mathcal{S} \). We denote this homomorphism by \( \mathcal{S}(\alpha) \).

The above constructions provide for every pair of pro-affine semigroups \( \mathcal{S} \) and \( \mathcal{S}' \) a bijection between semigroup homomorphisms \( \mathcal{S} \to \mathcal{S}' \) and toric morphisms \( \mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S}) \). In the following proposition, we summarize the considerations above.

Proposition 4.4. Let \( \mathcal{S} \) and \( \mathcal{S}' \) be pro-affine semigroups. Then, for every homomorphism \( \beta : \mathcal{S} \to \mathcal{S}' \) the map \( \mathcal{V}(\beta) : \mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S}) \) is a toric morphism of affine toric ind-varieties. Moreover, for every toric morphism \( \alpha : \mathcal{V}(\mathcal{S}') \to \mathcal{V}(\mathcal{S}) \) there exists a unique \( \beta : \mathcal{S} \to \mathcal{S}' \) such that \( \alpha = \mathcal{V}(\beta) \).

The assignment \( \mathcal{V}(\mathcal{S}) \) is a contravariant functor, i.e., \( \mathcal{V}(\text{id}) = \text{id} \) and \( \mathcal{V}(\beta' \circ \beta) = \mathcal{V}(\beta) \circ \mathcal{V}(\beta') \), for every pair of semigroup homomorphisms \( \beta : \mathcal{S} \to \mathcal{S}' \) and \( \beta' : \mathcal{S}' \to \mathcal{S}'' \), where \( \mathcal{S}, \mathcal{S}', \) and \( \mathcal{S}'' \) are pro-affine semigroups. This follows directly from the definition of \( \mathcal{V}(\beta) \) as the comorphism of the unique extension of the morphism \( \mathbb{C}[\mathcal{S}] \to \mathbb{C}[\mathcal{S}'] \) given by \( \chi^m \mapsto \chi^{\beta(m)} \).

On the other hand, the assignment \( \mathcal{S}(\mathcal{S}) \) is also a contravariant functor. Indeed, let \( \alpha' : \mathcal{V}' \to \mathcal{V} \) and \( \alpha : \mathcal{V} \to \mathcal{V}' \) be morphisms of affine toric ind-varieties \( \mathcal{V}, \mathcal{V}' \) and \( \mathcal{V}' \). By Proposition 4.4 and Proposition 4.4 there exists pro-affine semigroups such that \( \mathcal{V}(\mathcal{S}) = \mathcal{V}(\mathcal{S}') = \mathcal{V}(\mathcal{S}'') \) with morphisms \( \beta : \mathcal{S} \to \mathcal{S}' \) and \( \beta' : \mathcal{S}' \to \mathcal{S}'' \) such that \( \beta = \mathcal{S}(\alpha) \) and \( \beta' = \mathcal{S}(\alpha') \). By Proposition 4.4 we have \( \mathcal{V}(\beta' \circ \beta) = \alpha' \circ \alpha \) or, equivalently, \( \beta' \circ \beta = \mathcal{S}(\alpha \circ \alpha') \) so that \( \mathcal{S}(\alpha') \circ \mathcal{S}(\alpha) = \mathcal{S}(\alpha \circ \alpha') \).

In the following theorem, that is our main result, we summarize the results in this section.

Theorem 4.5.

1. The assignment \( \mathcal{V}(\mathcal{S}) \) is a contravariant functor from the category of pro-affine semigroups with homomorphisms of semigroups to the category of affine toric ind-varieties with toric morphisms.
2. The assignment \( \mathcal{S}(\mathcal{S}) \) is a contravariant functor from the category of affine toric ind-varieties with toric morphisms to the category of pro-affine semigroups with homomorphisms of semigroups.
3. The pair \( (\mathcal{V}(\mathcal{S}), \mathcal{S}(\mathcal{S})) \) is a duality between the categories of affine toric ind-varieties and pro-affine semigroups.

A well-known feature of the classical duality between affine toric varieties and affine semigroups is the correspondence between points on the toric variety and semigroup homomorphism to \((\mathbb{C}, \cdot)\). In the
following proposition, we generalize this result to the case of affine toric ind-varieties. Let $(\mathbb{C}, \cdot)$ be the semigroup of complex numbers under multiplication. This semigroup is not pro-affine since it is not cancelative and all pro-affine semigroup inherit the cancellation property from the embedding in $\mathbb{Z}^\omega$ shown in Corollary 3.11. We endow $(\mathbb{C}, \cdot)$ with the trivial descending filtration $R_1^1 \supset R_2^1 \supset \ldots$ of compatible equivalence relations $R^j = \{(t, t) \in \mathbb{C} \times \mathbb{C} \mid t \in \mathbb{C}\}$ so that $C/R^j \simeq \mathbb{C}$. Unlike the case of pro-affine semigroups, not every semigroup homomorphism $\mathcal{S} \to (\mathbb{C}, \cdot)$ is a filtered morphism. For instance, see [5] page 159] and apply the fact that $(\mathbb{C}, \cdot)$ contains an isomorphic copy $Q$ of the additive group of the rational numbers. For instance we can take $Q = \{a^q \mid q \in \mathbb{Q}\}$ for any $a \in \mathbb{C}^\omega$.

Proposition 4.6. Let $\mathcal{V}$ an affine toric ind-variety and let $\mathcal{S} = \mathcal{S}(\mathcal{V})$. Then there are bijective correspondence between the following:

1. Points $v$ in $\mathcal{V}$.
2. Closed maximal ideals $m$ in $\mathbb{C}[\mathcal{V}]$ that is equal to the completion of $\mathbb{C}[\mathcal{S}]$.
3. Morphisms of filtered semigroups $\Lambda: \mathcal{S} \to (\mathbb{C}, \cdot)$.

Proof. The equivalence of (1) and (2) is general for ind-varieties and was first proven in [2]. Let $R_1 \supset R_2 \supset \ldots$ be the filtration of compatible equivalence relation in $\mathcal{S}$ and let $\Lambda: \mathcal{S} \to \mathbb{C}$ be a filtered semigroup morphism. By the definition of filtered semigroup, there exists $j > 0$ such that $(\Lambda \times \Lambda)(R^j_j)$ is contained in the diagonal in $\mathbb{C} \times \mathbb{C}$ defining the trivial equivalence relation in $\mathbb{C}$. By Lemma 3.4 the morphism $\Lambda$ induces a semigroup homomorphism $\Lambda_j: S_j \to \mathbb{C}$, where $S_j = \mathcal{S}/R^j_j$. The homomorphism $\Lambda_j: S_j \to \mathbb{C}$ induced a surjective $\mathbb{C}$-algebra homomorphism $\Lambda_j\mathcal{S}[S_j] \to \mathbb{C}$ given by $\chi^m \mapsto \Lambda_j(m)$. Since $\mathbb{C}$ is a field, we have $\mathfrak{m} = \ker \Lambda_j$ is a maximal ideal. The preimage $m$ of $\mathfrak{m}$ by $\bar{\pi}_j: \mathbb{C}[\mathcal{V}] \to \mathbb{C}[S_j]$ the morphism coming from the projective system $\mathbb{C}[S_j] \leftarrow \mathbb{C}[S_2] \leftarrow \ldots$ is also maximal. By [2] Proposition 1.2.2 (i) we have that $m$ is closed since $\bar{I}_j = \ker \bar{\pi}_j$ is subset of $m$.

On the other hand, let $m$ be a closed maximal ideal in $\mathbb{C}[\mathcal{V}]$. By [2] Proposition 1.2.2 (iv), there exist $j > 0$ and $\mathfrak{m}$ a maximal ideal of $\mathbb{C}[S_j]$ such that $m$ is the preimage of $\mathfrak{m}$ by $\bar{\pi}_j$. This maximal ideal $\mathfrak{m}$ defines an algebra homomorphisms $\Lambda_j: \mathbb{C}[S_j] \to \mathbb{C} \simeq \mathbb{C}[S_j]/\mathfrak{m}$. By [2] proposition 1.3.1, this algebra homomorphism defines a semigroup homomorphism $\Lambda_j: S_j \to \mathbb{C}$ given by $\Lambda_j(m) = \chi^m_j$. We define $\Lambda: \mathcal{S} \to \mathbb{C}$ by $\Lambda = \Lambda_j \circ \pi_j$, where $\pi_j: \mathcal{S} \to S_j$ is the quotient morphism. The semigroup homomorphism $\Lambda$ is a filtered semigroup morphism since $(\Lambda \times \Lambda)(R^j_j)$ is contained in the diagonal in $\mathbb{C} \times \mathbb{C}$ defining the trivial equivalence relation in $\mathbb{C}$. It is a straightforward verification that both this constructions provide the required bijection. □

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