Research Article
Uncertainty Principles for Heisenberg Motion Group

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Abstract and Applied Analysis

This paper is organized as follows. In Section 2, we present the group G and the Fourier transform on G, and we will cite some of its fundamental properties. Section 3 is devoted to expressing this uncertainty principle. The proof of the theorem was given by Hörmander [3], and it states that if \( f \in L^2(\mathbb{R}^n) \) satisfying

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)| |\hat{f}(y)| e^{j|x|y} dx dy < \infty,
\]

then, \( f = 0 \) almost everywhere.

The above theorem of Hörmander was further generalized by Bonami, Demange, and Jaming [4], as follows:

**Theorem 1.** Let \( N \geq 0 \) and let \( f \in L^2(\mathbb{R}^n) \) satisfying

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)| |\hat{f}(y)|}{(1 + |x| + |y|)^N} e^{j|x|y} dx dy < \infty.
\]

Then, \( f = 0 \) almost everywhere whenever \( N \leq n \), and if \( N > n \), then \( f(x) = P(x)e^{-a|x|^2} \), where \( a \) is a positive real number and \( P \) is a polynomial on \( \mathbb{R}^n \) of degree \( (N - n)/2 \).

This last theorem admits another modified version proved by Parui and Sarkar [5]. It is of the following form.

**Theorem 2.** Let \( \delta \geq 0 \) and \( f \in L^2(\mathbb{R}^n) \) be such that

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)| |\hat{f}(y)|}{(1 + |x| + |y|)^N} e^{j|x|y} |Q(y)|^\delta dx dy < \infty,
\]

where \( Q \) is a polynomial of degree \( m \). Then, \( f(x) = P(x)e^{-a|x|^2} \), where \( a \) is a positive real number and \( P \) is a polynomial with deg \((P) < (N - n - m\delta)/2\).

Beurling’s theorem has been extended to different settings. Huang and Liu established an analogue of Beurling’s theorem on the Heisenberg group [6]. An analogue of Beurling’s theorem for Euclidean motion groups was also formulated by Sarkar and Thangavelu [7].

In [8], Baklouti and Thangavelu gave an analogue of Hardy’s theorem for the Heisenberg motion group by means of the heat kernel and also proved an analogue of Miyachi’s theorem and Cowling-Price uncertainty principle. In my paper, we would like to establish other uncertainty principles such as Beurling’s theorem and Gelfand-Shilov and prove Hardy’s theorem as a consequence of Beurling’s theorem.

This paper is organized as follows. In Section 2, we present the group G and the Fourier transform on G, and we will cite some of its fundamental properties. Section 3 is devoted...
to formulate and prove an analogue of Beurling’s theorem associated to the group Fourier transform on the Heisenberg motion group and prove a modified version of this principle. Finally, we derive some other versions of uncertainty principles such as Hardy uncertainty principle and Gelfand-Shilov.

2. Heisenberg Motion Group

Let \( \mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \) be the Heisenberg group with the group law

\[
(z, t), (w, s) = (z + w, t + s + \frac{1}{2} \text{Im} (z \bar{w})).
\]

where \( z, w \in \mathbb{C}^n \), \( t, s \in \mathbb{R} \).

Let \( K \) be the unitary group \( U(n) \), we define the Heisenberg motion group \( G \) to be the semidirect product of \( \mathbb{H}^n \) and \( K \), with the group law

\[
(z, t, k)(w, s, h) = ((z, t), (kw, s), kh),
\]

where \( (z, t), (w, s) \in \mathbb{H}^n, k, h \in K \).

The Haar measure on \( G \) is given by \( dg = dzdtdk \), where \( dzdt \) and \( dk \) are the normalized Haar measures on \( \mathbb{H}^n \) and \( K \), respectively.

For \( \lambda \in \mathbb{R} \setminus \{0\} \), we define the Schrödinger representation of \( \mathbb{H}^n \) on \( L^2(\mathbb{R}^n) \) by

\[
\pi_\lambda(z, t) \varphi(x) = e^{i\lambda t} e^{i\lambda (x \cdot (\bar{z}, \bar{t} \cdot 2))} \varphi(x + y),
\]

where \( z = x + iy \) and \( \varphi \in L^2(\mathbb{R}^n) \).

\( \sigma \) be any irreducible, unitary representation of \( K \). For each \( \lambda \neq 0 \), we consider the representations \( \rho_\lambda \) of \( G \) on the tensor product space \( L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma \) defined by

\[
\rho_\lambda(z, t, k) = (\pi_\lambda(z, t) \mu_\lambda(k)) \otimes \sigma(k),
\]

where \( \mu_\lambda \) are the metaplectic representations \([9]\), satisfying

\[
\pi_\lambda(kz, t) = \mu_\lambda(k) \pi_\lambda(z, t) \mu_\lambda(k)^*, \text{ for all } (z, t, k) \in G.
\]

Proposition 1 \([9]\). Each \( \rho_\lambda \) is unitary and irreducible.

For \( f \in L^1 \cap L^2(G) \), consider the group Fourier transform

\[
\hat{f}(\lambda, \sigma) = \int_{\mathbb{H}^n} \int_\mathbb{R} f(z, t, k) \rho_\lambda(z, t, k) dzdtdk
\]

For \( f \in L^1 \cap L^2(G) \), we have

\[
\int_{\mathbb{H}^n} \int_\mathbb{R} \left| f^\lambda(z, k) \right|^2 dzdk = \left( 2\pi \right)^{-n} \left| \lambda \right|^n \sum_{\sigma \in \mathcal{K}} d_\sigma \left\| \hat{f}(\lambda, \sigma) \right\|_{HS}^2,
\]

and the Plancherel formula for the Fourier transform on \( G \) reads as

\[
\int_{\mathbb{H}^n} \int_\mathbb{R} \left| f(z, t, k) \right|^2 dzdtdk = \sum_{\sigma \in \mathcal{K}} d_\sigma \int_{\mathcal{H}_\sigma} \left\| \hat{f}(\lambda, \sigma) \right\|_{HS}^2 d\tau(\lambda),
\]

where \( d\tau(\lambda) = (2\pi)^{-n-1} |\lambda|^n d\lambda \) is the measure defined on \( \mathbb{R} \setminus \{0\} \), \( d_\sigma \) is the dimension of the space \( \mathcal{H}_\sigma \), and \( \left\| \hat{f}(\lambda, \sigma) \right\|_{HS} \) denote the Hilbert-Schmidt norm of \( \hat{f}(\lambda, \sigma) \) \([9]\).

At the end of this paragraph, we introduce an orthonormal basis for \( L^2(\mathbb{C}^n \times K) \) \([10]\). Let \( H_k(t) \) be the Hermite polynomials defined by

\[
H_k(t) = (-1)^k e^{-t^2} \frac{d^k}{dt^k} \left( e^{t^2} \right).
\]

The normalized Hermite functions are defined by

\[
\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j),
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \).

It is well known that \( \{ \Phi_\alpha, \alpha \in \mathbb{N}^n \} \) form an orthonormal basis for \( L^2(\mathbb{R}^n) \) \([2]\). Then, an orthonormal basis for \( L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma \) is given by \( B_\sigma = \{ \Phi_\alpha \otimes \epsilon_{\alpha}^i : \alpha \in \mathbb{N}^n, 1 \leq i \leq d_\sigma \} \), where \( \{ \epsilon_{\alpha}^i : 1 \leq i \leq d_\sigma \} \) is an orthonormal basis for \( \mathcal{H}_\sigma \) and \( d_\sigma = \dim \mathcal{H}_\sigma \).

Define the Fourier-Wigner transform \( V^\sigma_{fj}(z, k) \) of \( f, g \in L^2(\mathbb{R}^n) \otimes \mathcal{H}_\sigma, 1 \leq f, g \leq 2 \), the following identity holds.

\[
\int_{\mathbb{H}^n} \int_\mathbb{R} V^\sigma_{f_1}(z, k) V^\sigma_{g_1}(z, k) dzdk = \langle f_1, f_2 \rangle \langle g_1, g_2 \rangle.
\]
Set $\Psi_{a,j} = \Phi_a \otimes e_j^\sigma$, then, by Lemma 1, we infer that the set

$$V_B = \left\{ V_{\psi_{a,i}}^{\sigma}, \psi_{a,i} \in B_{\sigma} \right\}$$

(18)

is an orthonormal basis for $V_{\sigma} = \text{span}(V_f : f, g \in L^2(\mathbb{R}^n) \otimes \mathcal{H})$.

**Proposition 2** [10]. The family $\mathcal{B} = \{ V_{\psi_{a,i}} : \sigma \in \mathcal{K} \}$ is an orthonormal basis for $L^2(C^n \times K)$.

**Lemma 2.** The function $(z, k) \mapsto V_{\psi_{a,i}}^{\sigma}(z, k)$ is a bounded function.

**Proof.** Let $(z, k) \in C^n \times K$ and $\Psi_{a,i}^{\sigma}, \Psi_{B,i}^{\sigma} \in B_{\sigma}$, we have

$$V_{\psi_{a,i}}^{\sigma}(z, k) = \left(2\pi\right)^{-n/2} \left\langle p_i(z) \phi_\alpha \otimes \phi_{\beta} \otimes e^\sigma_j \right\rangle$$

$$= \left(2\pi\right)^{-n/2} \left\langle \pi_i(z) \phi_\alpha \otimes \phi_{\beta} \otimes e^\sigma_j \right\rangle$$

$$= \left(2\pi\right)^{-n/2} \left\langle \pi_i(z) \phi_\alpha \otimes \phi_{\beta} \otimes e^\sigma_j \right\rangle$$

$$= \left(2\pi\right)^{-n/2} \left\langle \pi_i(z) \phi_\alpha \otimes \phi_{\beta} \otimes e^\sigma_j \right\rangle.$$

(19)

We know that $\mu_i(k) \Phi_\alpha = \sum_{|y|=|\alpha|} \langle \mu_i(k) \Phi_\alpha, \phi_y \rangle \phi_y$ (see [9] p.21); then

$$\left| V_{\psi_{a,i}}^{\sigma}(z, k) \right| \leq \left(2\pi\right)^{-n/2} \sum_{|y|=|\alpha|} \left| \langle \mu_i(k) \Phi_\alpha, \phi_y \rangle \right| \left| \langle \pi_i(z) \phi_y, \phi_{\beta} \rangle \right| \left| \langle \sigma(k) e^\sigma_j \rangle \right|.$$  

(20)

Since $\pi_1, \mu_1, \sigma$ are unitary representations, then

$$\left| V_{\psi_{a,i}}^{\sigma}(z, k) \right| \leq \left(2\pi\right)^{-n/2} \frac{(m+n-1)!}{m!(n-1)!},$$

(21)

where $m = |\alpha|$. □

**3. An Analogue of Beurling’s Theorem**

In this section, we prove an analogue of Beurling’s theorem on the Heisenberg motion group $G = \mathbb{H}^n \times K$ whose statement is as follows:

**Theorem 3.** Let $f \in L^2(G)$ and $d \geq 0$. Suppose that

$$\int_G \int_R |f(z, t, k)| \left| \langle \sum_{\delta \in \mathcal{K}} d_{\delta} f(\lambda, \sigma) \rangle_{L^2(\mathbb{H}^n)} \right|^{1/2} e^{i|t||\lambda|} dt \, d\lambda < \infty.$$  

(22)

Then,

$$f(z, t, k) = e^{-at} \left( \sum_{j=0}^{m} \phi_j(z, k)t^j \right),$$

(23)

where $a > 0$, $\phi_j \in L^1(\mathbb{R})$ and $m < (d - n/2 - 1)/2$.

**Lemma 3.** If $f$ satisfies the hypotheses of Theorem 3, then $f \in L^1(G)$.

**Proof.** As $f$ is not identically zero, then, there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$\sum_{\sigma \in \mathcal{K}} \left| d_\sigma \right| \left| \langle f(\lambda, \sigma) \rangle_{L^2(\mathbb{H}^n)} \right|^{1/2} \neq 0.$$  

(24)

By (22), we have

$$\int_G \int_R \left| f(z, t, k) \right| e^{i|t||\lambda|} dt \, d\lambda < \infty.$$  

(25)

On the other hand, the function $(t, \lambda) \mapsto (1+|t|+|\lambda|)^d e^{i|t||\lambda|}$ is bounded, so there exists a constant $C$ such that

$$\frac{(1+|t|+|\lambda|)^d}{e^{i|t||\lambda|}} \leq C.$$  

(26)

From where

$$\int_G \int_R |f(z, t, k)| \, dz \, dt \, dk \leq C \int_G \left| \langle f(z, t, k) \rangle_{L^2(\mathbb{H}^n)} \right| (1+|t|+|\lambda|)^d e^{i|t||\lambda|} \, dz \, dt \, dk$$

and using (25), we have $f \in L^1(G)$.

**Proof of Theorem 3.** For any $\varphi \in \mathcal{S}(C^n \times K)$, the Schwartz space of $C^n \times K$, consider the function

$$F_\varphi(t) = \int_{C^n \times K} f(z, t, k) \varphi(z, k) \, dz \, dk.$$  

(27)

Since $f \in L^1(G)$, then, $F_\varphi$ is integrable on $\mathbb{R}$, and for any $\lambda \in \mathbb{R} \setminus \{0\}$, the Fourier transform of $F_\varphi$ is given by

$$\widehat{F_\varphi}(\lambda) = \left(2\pi\right)^{-1/2} \int_{C^n \times K} F_\varphi(t) e^{-i\lambda t} \, dt$$

$$= \left(2\pi\right)^{-1/2} \int_{C^n \times K} f(z, t, k) \varphi(z, k) e^{-i\lambda t} \, dz \, dt \, dk$$

(28)
then by (11)

$$\left| \tilde{F}_\varphi(\lambda) \right| \leq (2\pi)^{-1/2} \| \varphi \|_2 \left( \int_{C^n \times K} \left| f^{(-)}(z, k) \right|^2 \, dz \, dk \right)^{1/2} \leq (2\pi)^{-n/2} \| \varphi \|_2 \left| \lambda \right|^{n/2} \left( \sum_{\sigma \in \Lambda} d_\sigma \| f(\lambda, \sigma) \|_{HS} \right)^{1/2} \cdot \left. \right|_{\varphi}$$. \hspace{1cm} (29)

As a result,

$$I = \int_{R^2} \left| F_\varphi(t) \right| \left| \tilde{F}_\varphi(\lambda) \right| \frac{e^{i|\lambda|t}}{(1 + |t| + |\lambda|^2)^d} \frac{d\lambda}{\| \varphi \|_2}$$

$$\cdot \int_{R^2} \left| f(z, t, k) \right| \left( \sum_{\sigma \in \Lambda} d_\sigma \| f(\lambda, \sigma) \|_{HS} \right)^{1/2} \frac{e^{i|\lambda|t}}{(1 + |t| + |\lambda|^2)^d} \, dz \, dt \, dk < \infty.$$ \hspace{1cm} (30)

In particular,

$$\int_{R^2} \left| F_\varphi(t) \right| \left| \tilde{F}_\varphi(\lambda) \right| \frac{e^{i|\lambda|t}}{(1 + |t| + |\lambda|^2)^d} \frac{d\lambda}{\| \varphi \|_2}$$

$$< \infty.$$ \hspace{1cm} (31)

Note that the previous calculations are generalized to a bounded function $\varphi \in L^2(C^n \times K)$, in particular for the bounded functions $\varphi$ in the basis $B$ of $L^2(C^n \times K)$ defined in (18).

According to Beurling’s theorem in the Euclidean case, modified version (Theorem 2), for every function $\varphi \in B$, there exists a polynomial function $P_\varphi$ with deg ($P_\varphi$) < $(d - (n/2) - 1)/2$ and a real $a_\varphi > 0$ such that

$$\int_{C^n \times K} f(z, t, k) \varphi(z, k) \, dz \, dk = P_\varphi(t) e^{-a_\varphi|t|^2},$$ \hspace{1cm} (32)

from where

$$f(z, t, k) = \sum_{\varphi \in B} P_\varphi(t) e^{-a_\varphi|t|^2} \varphi(z, k).$$ \hspace{1cm} (33)

Let $\varphi, \psi \in B$, since

$$\int_{R^2} \int_{R^2} \left| F_\varphi(t) \right| \left| \tilde{F}_\psi(\lambda) \right| \left| \frac{e^{i|\lambda|t}}{(1 + |t| + |\lambda|^2)^d} \right| \frac{d\lambda}{\| \varphi \|_2}$$

$$< \infty.$$ \hspace{1cm} (34)

then by Lemma 2.2 in [5], we obtain that $a_\varphi = a_\psi = a$ are independent of $\varphi$.

Let $P_\varphi(t) = \sum_{j=0}^m c_{j\varphi} t^j$, then

$$f(z, t, k) = e^{-a|t|^2} \sum_{j=0}^m c_{j\varphi} \varphi(z, k) \cdot t^j = e^{-a|t|^2} \sum_{j=0}^m \xi_j(z, k) \cdot t^j,$$ \hspace{1cm} (35)

where $\xi_j(z, k) = \sum_{j=0}^m c_{j\varphi} \varphi(z, k) \in L^2(C^n \times K)$.

The proof of Theorem 3 is completed. \hspace{1cm} \Box

We will finish this section with a modified version of previous Theorem 3 as follows:

**Proposition 3.3.** Let $f \in L^2(G)$ and $p, \delta \geq 0$. Suppose that

$$\int_{G} \int_{R^2} \left| \sum_{\sigma \in \Lambda} d_\sigma \| f(\lambda, \sigma) \|_{HS} \right|^{1/2} \frac{d\lambda}{(1 + |\lambda|^2)^{d/2}} < \infty.$$ \hspace{1cm} (36)

Then,

$$f(z, t, k) = e^{-a|t^2|} \left( 1 + \| z \| \right)^p \left( \sum_{j=0}^m \varphi_j(z, k) t^j \right),$$ \hspace{1cm} (37)

where $a > 0$, $\varphi_j \in L^1 \cap L^2(C^n \times K)$ and $m < (d - (n/2) - 1 - \delta)/2$.

**Proof.** By replacing $f(z, t, k)$ by $f(z, t, k)/(1 + |z|)^p$ and proceeding as in the proof of Theorem 3, one can apply Theorem 3 to get the result. \hspace{1cm} \Box

4. Applications to Other Uncertainty Principles

Let us first state and prove the following analogue of Hardy’s theorem for $G$.

**Theorem 5** (Hardy type). Suppose $f$ is a measurable function on $G$ satisfying

(i) $|f(z, t, k)| \leq g(z, k) e^{-a|t|^2}$, where $g \in L^1 \cap L^2(C^n \times K)$ and $a > 0$.

(ii) $\| \lambda \|^{n/2} |f(\lambda, \sigma)|_{HS} \leq c_\beta e^{-c_{beta}|\lambda|}$, for some $\beta > 0$ and $c_\beta > 0$ such that $\sum_{\sigma \in \Lambda} d_\sigma c_{\beta} < \infty$.

Then,

(1) If $a\beta > 1/4$, $f = 0$ almost everywhere on $G$.

(2) If $a\beta = 1/4$, $f(z, t, k) = e^{-a|t^2|} f(z, 0, k)$, for all $(z, t, k) \in C^n \times R \times K$.

**Proof.** From (i) and (ii), we have

$$\int_{G} \int_{R^2} \frac{\left| \sum_{\sigma \in \Lambda} d_\sigma \| f(\lambda, \sigma) \|_{HS} \right|^{1/2} \frac{d\lambda}{(1 + |\lambda|^2)^{d/2}}}$$

$$\leq \int_{R^2} \int_{R^2} \left| F_\varphi(t) \right| \left| \tilde{F}_\varphi(\lambda) \right| \frac{e^{i|\lambda|t}}{(1 + |t| + |\lambda|^2)^d} \frac{d\lambda}{\| \varphi \|_2}$$

$$\left| \lambda \|^{n/2} e^{-a|t|^2} (1 + |z|)^p \left( \sum_{j=0}^m \varphi_j(z, k) t^j \right) \right| \leq C \times M,$$ \hspace{1cm} (38)

where

$$M = \int_{R} \frac{\left| \lambda \|^{n/2} \left( e^{-a|t|^2} - (1/2\pi) \right) \right|^{1/2}}{(1 + |t| + |\lambda|^2)^{d/2}} \, dt \, d\lambda.$$ \hspace{1cm} (39)
Then, if \( \alpha \beta \geq 1/4 \), the expression above for \( d > (n/2) + 2 \) is finite.

According to Theorem 3, there exist \( a > 0 \) and \( \varphi_j \in L^1 \cap L^2(\mathbb{C}^n \times K) \) such that

\[
    f(z, t, k) = e^{-at^2} \left( \sum_{j=0}^{m} \varphi_j(z, k)t^j \right), \tag{40}
\]

where \( K \) is a positive constant.

From condition (i), we have

\[
    \int_{C^* \times K} \left| f(z, 0, k) \right| dzdk \leq K \times e^{(a-\alpha)t^2}, \tag{42}
\]

where \( K \) is a positive constant.

We have

\[
    f^1(z, k) = \int_{R} f(z, t, k)e^{i\lambda t} dt = \sqrt{\frac{\pi}{a}} e^{-\lambda/4a} f(z, 0, k). \tag{43}
\]

Then,

\[
    \int_{C^* \times K} \left| f^1(z, k) \right|^2 dzdk = \frac{\pi}{a} e^{-\lambda/2a} \int_{C^* \times K} \left| f(z, 0, k) \right|^2 dzdk. \tag{44}
\]

From (11), we have

\[
    (2\pi)^{-n}|\lambda|^n \int_{\sigma \in K} \left\| \mathcal{F}(\lambda, \sigma) \right\|_{H^S}^2 = \frac{\pi}{a} e^{-\lambda/2a} \int_{C^* \times K} \left| f(z, 0, k) \right|^2 dzdk. \tag{45}
\]

From condition (ii), we obtain

\[
    \int_{C^* \times K} \left| f(z, 0, k) \right|^2 dzdk \leq C_0 (1/2a - 2\beta) \lambda^2, \tag{46}
\]

where \( C \) is a positive constant.

(1) Case \( \alpha \beta > (1/4) \):

(i) Suppose that \( a < \alpha \); we have \( (1/2a) < 2\beta \) and \( \lim_{t \to \infty} e^{(1/2a - 2\beta)\lambda^2} = 0. \) From (46), we have \( \int_{C^* \times K} |f(z, 0, k)|^2 dzdk = 0 \), and finally \( f = 0 \).

(ii) Suppose that \( a \geq \alpha \); we have \( (1/2a) > 2\beta \) and \( \lim_{t \to \infty} e^{(1/2a - 2\beta)\lambda^2} = 0. \) From (46), we have \( \int_{C^* \times K} |f(z, 0, k)|^2 dzdk = 0 \), and finally \( f = 0 \).

(2) Case \( \alpha \beta = 1/4 \):

(i) Suppose that \( f \neq 0 \); then (42) and (46) hold if and only if \( a = \alpha \). We conclude that \( f(z, t, k) = e^{-at^2} f(z, 0, k) \).

\[ \square \]

**Theorem 6** (Gelfand-Shilov type). Let \( f \in L^2(G) \) and \( d \geq 0 \) satisfy

\[
    \int_{G} \left| f(z, t, k) \right| e^{-at^2} dt dzdk < +\infty, \tag{47}
\]

\[
    \int_{R} \left( \sum_{\sigma \in K} d_{\sigma} \left\| \mathcal{F}(\lambda, \sigma) \right\|_{H^S}^2 \right)^{1/2} e^{\beta \lambda^2} d\tau(\lambda) < +\infty,
\]

for some positive constants \( \alpha \) and \( \beta \). Then, \( \alpha \beta > 1/4 \) implies \( f = 0 \).

**Proof.** Suppose that \( \alpha \beta > 1/4 \), and consider the following numerical sequences \( \{\alpha_p\}_{p \in \mathbb{N}^*} \) and \( \{\beta_p\}_{p \in \mathbb{N}^*} \) defined by

\[
    \alpha_p = \alpha - \frac{1}{p}, \quad \beta_p = \beta - \frac{1}{p}. \tag{48}
\]

(i) \( \lim_{p \to \infty} \alpha_p = \alpha > 0 \); therefore, there exists \( \alpha_p \in \mathbb{N}^* \) such that \( \alpha_p > 0 \) for \( p > N_{\alpha_p} \).

(ii) \( \lim_{p \to \infty} \beta_p = \beta > 0 \); therefore, there exists \( \beta_p \in \mathbb{N}^* \) such that \( \beta_p > 0 \) for \( p > N_{\beta_p} \).

(iii) \( \lim_{p \to \infty} \alpha_p \beta_p = \alpha \beta > 1/4 \); therefore, there exists \( N_{\alpha_p \beta_p} \in \mathbb{N}^* \) such that \( \alpha_p \beta_p > 1/4 \) for \( p > N_{\alpha_p \beta_p} \).

Let \( N = \max(N_{\alpha_p}, N_{\beta_p}, N_{\alpha_p \beta_p}) \), then, for all \( p > N \).

\[
    \alpha_p > 0, \quad \beta_p > 0, \quad \alpha_p \beta_p > \frac{1}{4}, \tag{49}
\]

\[
    \left| f(\lambda) \right| \leq \alpha_p t^2 + \beta_p \lambda^2.
\]
We have

$$\frac{|f(z, t, k)| \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2}}{(1 + |t| + |\lambda|)^d} e^{\rho t^2} \leq \frac{|f(z, t, k)| e^{\rho t^2} \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2} e^{\rho t^2}}{(1 + |t| + |\lambda|)^d} h(t, \lambda),$$

(50)

where $h(t, \lambda) = (1 + |t|)^d(1 + |\lambda|)^d/(1 + |t| + |\lambda|)^d e^{-(1/p)t^2} e^{-(1/p)|\lambda|^2}$. The function $h(t, \lambda)$ is a bounded function, then there exists a positive constant $K$ such that

$$\frac{|f(z, t, k)| \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2}}{(1 + |t| + |\lambda|)^d} e^{\rho t^2} \leq K \frac{|f(z, t, k)| e^{\rho t^2} \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2} e^{\rho t^2}}{(1 + |t| + |\lambda|)^d}.$$  

(51)

According to the hypotheses of the theorem, the integral

$$\int_G \int_R \frac{|f(z, t, k)| \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2}}{(1 + |t| + |\lambda|)^d} e^{\rho t^2} dt d\lambda dzdk$$

(52)

is finite. According to Theorem 3, there exist $a > 0$ and $\psi_j \in L^2(\mathbb{C}^n \times \mathbb{K})$ such that

$$f(z, t, k) = e^{-\beta t^2} \left(\sum_{j=0}^m \psi_j(z, k)t^j\right).$$

(53)

where $m < (d - (n/2 + 1))/2$.

We have

$$f^\lambda(z, k) = \sum_{j=0}^m \psi_j(z, k) \int_R t^j e^{-\beta t^2} \lambda^t dt = \sum_{j=0}^m \psi_j(z, k) P_j(\lambda) e^{-((1/4a)|\lambda|^2)/(1 + |t|)^d}$$

(54)

\(\text{(where } P_j(\lambda) \text{ is a polynomial function)}\)

$$= \left(\sum_{j=0}^m \psi_j(z, k) P_j(\lambda)\right) e^{-((1/4a)|\lambda|^2)}.$$

Then $\|f^\lambda\|_2 = h(\lambda) e^{-(1/4a)|\lambda|^2}$, where $h(\lambda)$ is a polynomial function, and from (11), we have

$$\left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2} = \frac{(2\pi)^{n/2}}{|\lambda|^{n/2}} h(\lambda) e^{-(1/4a)|\lambda|^2}.$$

(55)

From the hypotheses of the theorem, we obtain

$$\int_R \frac{|\lambda|^{n/2}}{(1 + |\lambda|)^d} h(\lambda) e^{((\beta - 1/4a)|\lambda|^2)} d\lambda < +\infty.$$  

(56)

(i) Suppose that $a \geq \alpha$; then $\beta - (1/4a) > 0$, so $h(\lambda) = 0$ implies $\hat{f}(\lambda, \sigma) = 0$

By Plancherel formula (12), we conclude that $f = 0$.

(ii) Suppose that $a < \alpha$, according to the hypotheses of the theorem, we have

$$\int_R \frac{e^{(a-\alpha) t^2}}{(1 + |t|)^d} \left(\int_{\mathbb{C}^n \times \mathbb{K}} \left|\sum_{j=0}^m \psi_j(z, k)t^j\right| dzdk\right) dt < +\infty.$$  

(57)

Since $\alpha - a > 0$ and $\sum_{j=0}^m \psi_j(z, k)t^j \in L^1(\mathbb{C}^n \times \mathbb{K})$ for almost every $t \in \mathbb{R}$, then, $\psi_j = 0$ for all $j = 0, \cdots, m$ and $f = 0.$

\[\square\]

Corollary 1 (Another version of Hardy). Suppose $f \in L^1 \cap L^2(G)$ satisfy

(i) $|f(z, t, k)| \leq g(z, k) e^{-\beta t^2}$, where $g \in L^1 \cap L^2(\mathbb{C}^n \times \mathbb{K})$ and $\alpha > 0$.

(ii) $\|\hat{f}(\lambda, \sigma)||_{HS} \leq c_\sigma e^{-\beta |\lambda|^2}$, for some $\beta > 0$ and $c_\sigma > 0$ such that $\sum_{k \in \mathbb{K}} |d_A| e^{c_\sigma} < +\infty$.

Then:

(1) If $\alpha \beta > 1/4$, $f = 0$ almost everywhere on $G$.

(2) If $\alpha \beta < 1/4$, there exist an infinite number of linearly independent functions meeting hypotheses (i) and (ii).

Proof.

(1) Case $\alpha \beta > 1/4$: if $\alpha \beta > 1/4$ and $d > n + 1$, then, from the hypothesis of the theorem, we have

$$\int_G |f(z, t, k)| e^{\rho t^2} dzdk \leq C \int_R \frac{1}{(1 + |t|)^d} dt < +\infty,$$

$$\int_R \left(\sum_{k \in \mathbb{K}} |d_A||f \wedge (\lambda, \sigma)||_{HS}^2\right)^{1/2} e^{\rho t^2} d\lambda \leq K \int_R \frac{|\lambda|^n}{(1 + |\lambda|)^d} d\lambda < +\infty,$$

(58)

where $C, K > 0$. Since $\alpha \beta > 1/4$, then, the Gelfand-Shilov theorem implies that $f = 0$.

(2) Case $\alpha \beta < 1/4$: If $\alpha \beta < 1/4$, then, any function of the form $g(z, k)h_k(t)$ where $h_k$ are the one-dimensional
Hermite functions satisfy the hypotheses of the theorem.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflict of interest.

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