ADAPTATIVE DECOMPOSITION: THE CASE OF THE DRURY-ARVESON SPACE

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Abstract. The maximum selection principle allows to give expansions, in an adaptive way, of functions in the Hardy space $H_2$ of the disk in terms of Blaschke products. The expansion is specific to the given function. Blaschke factors and products have counterparts in the unit ball of $\mathbb{C}^N$, and this fact allows us to extend in the present paper the maximum selection principle to the case of functions in the Drury-Arveson space of functions analytic in the unit ball of $\mathbb{C}^N$. This will give rise to an algorithm which is a variation in this higher dimensional case of the greedy algorithm. We also introduce infinite Blaschke products in this setting and study their convergence.

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1. Introduction

In [15] the authors introduced an algorithm based on the maximum selection principle, to decompose a given function of the Hardy space $H_2(\mathbb{D})$ of the unit disk into intrinsic components which correspond to modified Blaschke products

\begin{equation}
B_n(z) = \frac{\sqrt{1 - |a_n|^2}}{1 - z a_n} \prod_{k=1}^{n-1} \frac{z - a_k}{1 - z a_k}, \quad n = 1, 2, \ldots
\end{equation}

where the points $a_n \in \mathbb{D}$ are adaptively chosen according to the given function. These points $a_n$ do not necessarily satisfy the so-called hyperbolic non-separability condition

\begin{equation}
\sum_{n=1}^{\infty} (1 - |a_n|) = \infty,
\end{equation}

and so the functions $B_n(z)$ do not necessarily form a complete system in $H_2(\mathbb{D})$. This decomposition may be obtained in an adaptive way, see [14], making the algorithm more...
efficient than the greedy algorithm of which it is a variation.

In [1] the above algorithm is extended to the matrix-valued case and the choice of a point and of a projection is based at each step on the maximal selection principle. The extension is possible because of the existence of matrix-valued Blaschke factors and is based on the existence of solutions of interpolation problems in the matrix-valued Hardy space of the disk.

When leaving the realm of one complex variable, a number of possibilities occur, and in particular the unit ball \( B_N \) of \( \mathbb{C}^N \) and the polydisk. The polydisk case will be studied in a future publication. In this paper we focus on the case of the unit ball. For the present purposes, it is more convenient to consider the Drury-Arveson space rather than the Hardy space of the ball, and we extend some of the results of [13] and [1] to the setting of the Drury-Arveson space, denoted here \( H(B_N) \). This is the space with reproducing kernel

\[
\frac{1}{1 - \langle z, w \rangle}, \quad z, w \in B_N,
\]

with

\[
\langle z, w \rangle = \sum_{u=1}^{N} z_u \overline{w_u} = zw^*,
\]

where \( z = (z_1, \ldots, z_N) \) and \( w = (w_1, \ldots, w_N) \) belong to \( B_N \). This space has a long history (see for instance [8, 11, 2, 10, 12, 17]) and is used in the proof of a von Neumann inequality for row contractions. Interpolation inside the space \( H(B_N) \) was done in [7]. A key tool in [7] was the existence in the ball of the counterpart of a Blaschke factor (appearing in [16]; see (2.5) below). The existence of these Blaschke factors and the fact that one can solve interpolation problems in \( H(B_N) \) allow us to develop the asserted extension.

The approach in [7] is based on the solution of Gleason’s problem. For completeness we recall that given a space, say \( F \), of functions analytic in \( \Omega \subset \mathbb{C}^N \), Gleason’s problem consists in finding for every \( f \in F \) and \( a = (a_1, \ldots, a_N) \in \Omega \), functions \( g_1(z, a), \ldots, g_N(z, a) \in F \) and such that

\[
f(z) - f(a) = \sum_{u=1}^{N} (z_u - a_u)g_u(z, a), \quad z \in \Omega.
\]

Using power series, one sees that there always exist analytic functions satisfying (1.3). The requirement is that one can choose them in \( F \).

The paper consists of five sections besides the introduction. In Sections 2 and 3 we review some basic facts on the Drury-Arveson space, and on the interpolation in it. The latter will be necessary to prove the maximum selection principle. This principle is proved in Section 4. In Section 5 we prove the convergence of the algorithm. In the last section, which is of independent interest, we consider infinite Blaschke products. When \( N > 1 \) the \( a_n \) in (1.2) are vectors in \( B_N \) and condition (1.2) is replaced by the requirement

\[
\sum_{n=1}^{\infty} \sqrt{1 - a_n a_n^*} < \infty.
\]
We note that most of the analysis presented here still holds for general complete Nevanlinna-Pick kernels, that is kernels of the form

\[
\frac{1}{c(z) c(w) - \langle d(z), d(w) \rangle_H},
\]

where \( c \) is scalar and \( d \) is \( H \)-valued where \( H \) is some Hilbert space or more generally, in some reproducing kernel Hilbert spaces in which Gleason’s problem is solvable with bounded operators; see [5] for the latter.

\section{The Drury-Arveson space}

We use the multi-index notations

\[ z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}, \quad \text{and} \quad \alpha! = \alpha_1! \cdots \alpha_N!, \]

with \( z = (z_1, \ldots, z_N) \in \mathbb{C}^N \) and \( \alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N \). For \( z, w \in \mathbb{B}_N \) we have

\[
\frac{1}{1 - z w^*} = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha!}{z^\alpha} w^\alpha.
\]

The function (2.1) is thus positive definite in \( \mathbb{B}_N \). The associated reproducing kernel Hilbert space, which we denote by \( H(\mathbb{B}_N) \), is called the Drury-Arveson space, and can be characterized as

\[
(2.2) \quad H(\mathbb{B}_N) = \left\{ f(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha f_\alpha : \| f \|_{H(\mathbb{B}_N)}^2 = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\alpha!}{|\alpha|!} |f_\alpha|^2 < \infty \right\}.
\]

For \( N > 1 \) the Drury-Arveson space is contractively included in, but different from, the Hardy space of the ball. The latter has reproducing kernel

\[
\frac{1}{(1 - \langle z, w \rangle)^N}, \quad z, w \in \mathbb{B}_N.
\]

See [6] for an expression for the inner product (not in terms of a surface integral).

We define \((H(\mathbb{B}_N))^{n \times m}\) as in (2.2), but with now \( f_\alpha, g_\alpha \in \mathbb{C}^{n \times m} \) and define for \( f, g \in (H(\mathbb{B}_N))^{n \times m} \), with \( g(z) = \sum_{\alpha \in \mathbb{N}_0^N} z^\alpha g_\alpha \),

\[
(2.3) \quad [f, g]_{(H(\mathbb{B}_N))^{n \times m}} = \sum_{\alpha \in \mathbb{N}_0^N} g_\alpha^* f_\alpha, \quad \text{and}
\]

\[
(2.4) \quad \langle f, g \rangle_{(H(\mathbb{B}_N))^{n \times m}} = \text{Tr} [f, g]_{(H(\mathbb{B}_N))^{n \times m}}.
\]

In the sequel we will not write anymore explicitly the space in these forms. We will write sometimes \( \mathbb{C}^N \) instead of \( \mathbb{C}^{1 \times N} \).

For \( a \in \mathbb{B}_N \) we will use the notations \( e_a \) and \( b_a \) for the normalized Cauchy kernel and the \( \mathbb{C}^N \)-valued Blaschke factor at the point \( a \) respectively, that is:

\[
(2.5) \quad e_a(z) = \frac{\sqrt{1 - \|a\|^2}}{1 - \langle z, a \rangle} \quad \text{and} \quad b_a(z) = \frac{(1 - \|a\|^2)^{1/2}}{1 - \langle z, a \rangle} (z - a)(I_N - a^* a)^{-1/2}.
\]
Let \( w \in \mathbb{B}_N \). Then (see [16]; another more analytic and maybe easier proof can be found in [7]):

\[
\frac{1 - b_a(z)b_a(w)^*}{1 - zw^*} = \frac{1 - aa^*}{(1 - za^*)(1 - w^*)}, \quad z, w \in \mathbb{B}_N.
\]

Gleason’s problem is solvable in the Drury-Arveson space and in the Hardy space; see [5].

For \( a = 0 \) and by setting \( g_u(z, 0) = g_u(z) \), a solution is given by

\[
g_u(z, 0) = \int_0^1 \frac{\partial}{\partial z_u} f(tz) dt = \sum_{\alpha \in N^0_N} \frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u},
\]

where \( \epsilon_u \) is the \( N \)-index with all the other entries equal to 0, but the \( u \)-th one equal to 1, and with the understanding that

\[
\frac{\alpha_u}{|\alpha|} z^{\alpha - \epsilon_u} = 0
\]

if \( \alpha_u = 0 \). We set \((R_uf)(z) = f_0^1 \frac{\partial}{\partial z_u} f(tz) dt\). We thus have

\[
f(z) - f(0) = \sum_{u=1}^N z_u (R_uf)(z).
\]

When \( N = 1 \), then \( R_1 \) reduces to the classical backward-shift operator which to \( f \) associates the function \( \frac{f(z) - f(0)}{z} \) for \( z \neq 0 \) and \( f'(0) \) for \( z = 0 \).

3. Interpolation in the Drury-Arveson space

This section is based on [7] and reviews the tools necessary to develop the maximum selection principle and the convergence result in the next section. We provide the proofs for completeness.

**Proposition 3.1.** Let \( 0 \neq c \in \mathbb{C}^{n \times 1} \), \( a \in \mathbb{B}_N \), and let \( f \in \mathcal{H}(\mathbb{B}_N)^{n \times 1} \). Then

\[
c^* f(a) = 0 \iff f(z) = B(z)g(z),
\]

where \( B \) is given by

\[
B(z) = U \begin{pmatrix} b_a(z) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix},
\]

where \( b_a(z) \in \mathbb{C}^{1 \times N} \), \( U \in \mathbb{C}^{n \times n} \) is a unitary matrix with the first column equal to \( \frac{c^*c}{c^*c} \), and \( g \) is an arbitrary element of \( \mathcal{H}(\mathbb{B}_N)^{(N+n-1) \times 1} \).

**Proof.** We recall the proof of the proposition; see [7 Proposition 4.5, p. 15]. We note that

\[
c^* U = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix},
\]

and hence

\[
c^* B(a) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0_{1 \times N} & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{pmatrix} = 0_{n \times (N+(n-1))}.
\]
and so every function of the form $Bg$ with $g \in H(\mathbb{B}_N)^{(N+n-1)\times 1}$ is a solution of the interpolation problem. To prove the converse statement, we first remark that

$$\frac{I_n - B(z)B(w)^*}{1 - zw^*} = \frac{cc^*}{c^*c/(1 - za^*)} - \frac{1 - aa^*}{1 - wz^*}.$$  

It follows that the one dimensional subspace $\mathcal{H}_1$ of $H(\mathbb{B}_N)^n$ spanned by the vector $\frac{c}{1 - za^*}$ has reproducing kernel $I_n - B(z)B(w)^*/(1 - zw^*)$. Thus the decomposition of kernels

$$\frac{I_n}{1 - zw^*} = \frac{I_n - B(z)B(w)^*}{1 - zw^*} + \frac{B(z)B(w)^*}{1 - zw^*}$$

leads to an orthogonal decomposition of the space $H(\mathbb{B}_N)^n$ as

$$H(\mathbb{B}_N)^n = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp,$$

where $\mathcal{H}_1^\perp$ is the subspace of $H(\mathbb{B}_N)^n$ consisting of functions $g$ such that $c^*g(a) = 0$. Since the reproducing kernel of $\mathcal{H}_1^\perp$ is $\frac{B(z)B(w)^*}{1 - zw^*}$ we have

$$\mathcal{H}_1^\perp = \{Bg; g \in H(\mathbb{B}_N)^{(N+n-1)}\},$$

with norm

$$\|Bg\|_{H(\mathbb{B}_N)^n} = \inf_{g \in H(\mathbb{B}_N)^{(N+n-1)}} \|g\|_{H(\mathbb{B}_N)^{(N+n-1)}}.$$  

We note that we do not write the dependence of $B$ on $a$ and $c$.

**Definition 3.2.** The $\mathbb{C}^{n \times (N+n-1)}$-valued function $B$ is an elementary Blaschke factor. A (possibly infinite) Blaschke product is a product of terms of the form (3.1) of compatible (growing) sizes.

**Remark 3.3.** Let $B$ be a $\mathbb{C}^{n \times m}$-valued Blaschke product (or taking values operators from $\mathbb{C}^n$ into $\ell_2$ if $m = \infty$). Then $B$ is a Schur multiplier, meaning that the kernel $I_n - B(z)B(w)^*/(1(z,w))$ is positive definite in $\mathbb{B}_N$. When $N > 1$, the family of Schur multipliers is strictly included in the family of functions analytic and contractive in the unit ball. For the realization theory of Schur multipliers, see for instance [9, 10].

More generally than (3.1) we have (see [7, Theorem 5.2, p. 17]):

**Theorem 3.4.** Given $a_1, \ldots, a_M \in \mathbb{B}_N$ and vectors $c_1, \ldots, c_M \in \mathbb{C}^{n\times 1}$ different from $0_{n\times 1}$, a function $f \in H(\mathbb{B}_N)^{n\times 1}$ satisfies

$$c^*_j f(a_j) = 0, \quad j = 1, \ldots, M$$

if and only if it is of the form $f(z) = B(z)u(z)$, where $B(z)$ is a rational $\mathbb{C}^{n\times(n+k(N-1))}$-valued function, for some integer $k \leq M$, taking coisometric values on the boundary of $\mathbb{B}_N$, and $u$ is an arbitrary element in $H(\mathbb{B}_N)^{(n+k(N-1))\times 1}$.

**Proof.** Indeed, starting with $j = 1$ we have that $f = B_1g_1$, where $B_1$ is given by (3.1) with $a = a_1$ (and an appropriately constructed matrix $U$) and $g_1 \in H(\mathbb{B}_N)^{(N+n-1)\times 1}$. The interpolation condition $c^*_j f(a_j) = 0$ becomes

$$c^*_2 B_1(a_2)g_1(a_2) = 0.$$
If $c^2 B_1(a_2) = 0_{1 \times (n+n-1)}$, any $g_1$ will be a solution. Otherwise, we solve (3.2) using Proposition 3.1 and get

$$g_1(z) = B_2(z) g_2(z),$$

where $B_2$ is $\mathbb{C}^{(n+(N-1)) \times (n+2(N-1))}$-valued and obtained from (3.1) with $a = a_2$ and an appropriately constructed matrix $U$. Iterating this procedure we obtain the result. The fact that $k$ may be strictly smaller than $M$ comes from the possibility that conditions as (3.2) occur. This will not happen when $N = 1$ and when all the $a_j$ chosen are different. □

4. The maximum selection principle

The proof is similar to the one in the original paper [15] and in [4], but one relevant difference is the use of orthogonal projections in $\mathbb{C}^{n \times n}$ of fixed rank. The fact that the set of such projections is compact in $\mathbb{C}^{n \times n}$ ensures the existence of a maximum. Besides the use of the normalized Cauchy kernel, the possibility of approximating by polynomials is a key tool in the proof.

Proposition 4.1. Let $B$ be a $\mathbb{C}^{n \times n}$-valued rational function of the variables $z_1, \ldots, z_N$, analytic in a neighborhood of the closed unit ball $\mathbb{B}_N$, and taking co-isometric values on the unit sphere, let $r_0 \in \{1, \ldots, n\}$, and let $F \in H(\mathbb{B}_N)^{n \times m}$. There exists $w_0 \in \mathbb{B}_N$ and a $\mathbb{C}^{n \times n}$-valued orthogonal projection $P$ of rank $r_0$ such that

$$(1 - \|w_0\|^2) (\text{Tr} [B(w_0) P F(w_0), B(w_0) P F(w_0)])$$

is maximum.

Proof. We first recall that for $f \in H(\mathbb{B}_N)$ (that is, $n = m = 1$), with power series $f(z) = \sum_{\alpha \in \mathbb{N}_0^n} f_\alpha z^\alpha$, and for $w \in \mathbb{B}_N$, we have

$$(1 - \|w\|^2)|f(w)| = |\langle f, e_w \rangle| \leq \|f\|.$$ (4.1)

Let $F = (f_{ij}) \in H(\mathbb{B}_N)^{n \times m}$, where the entries $f_{ij} \in H(\mathbb{B}_N)$ ($i = 1, \ldots, n$ and $j = 1, \ldots, m$), and let $P$ denote a projection of rank $r_0$. Then:

$$\text{Tr} F(w)^* P B(w)^* B(w) P F(w) \leq \text{Tr} F(w)^* F(w) \quad \text{(since } B(w) \text{ is contractive inside the sphere)}$$

$$= \sum_{i=1}^n \sum_{j=1}^m |f_{ij}(w)|^2.$$ (4.2)

Hence, using (4.1) for every $f_{ij}$, we obtain

$$(1 - \|w\|^2) (\text{Tr} [B(w) P F(w), B(w) P F(w)]) \leq \sum_{i=1}^n \sum_{j=1}^m \|f_{ij}\|^2 = \|F\|^2.$$ (4.3)

Let $\epsilon > 0$. In view of the power series expansion characterization (2.2) of the elements of the Drury–Arveson space, there exists a $\mathbb{C}^{n \times m}$-valued polynomial $p$ in $z_1, \ldots, z_N$ such
that \( \|F - p\| \leq \epsilon \). We have
\[
(1 - \|w\|^2) \left( \text{Tr} [B(w)PF(w), B(w)PF(w)] \right)
\leq (1 - \|w\|^2) (\text{Tr} [F(w), F(w)])
= (1 - \|w\|^2) \| (F - p)(w) + p(w) \|^2
\leq 2(1 - \|w\|^2) \| (F - p)(w) \| + \| p(w) \|\|^2
\leq 2(1 - \|w\|^2) \| (F - p)(w) \|^2 + 2(1 - \|w\|^2) \| p(w) \|^2
\leq 2\|F - p\|^2 + 2(1 - \|w\|^2) \| p(w) \|^2 \quad \text{(where we have used (4.1))}
\leq 2\epsilon^2 + 2(1 - \|w\|^2) \| p(w) \|^2.
\]
Since \( (1 - \|w\|^2) \| p(w) \|^2 \) tends to 0 as \( w \) approaches the unit sphere, the expression
\( (1 - \|w\|^2) \left( \text{Tr} [B(w)PF(w), B(w)PF(w)] \right) \) can be made arbitrary small, uniformly with
respect to \( P \), as \( w \) approaches the unit sphere. Thus,
\[
(1 - \|w\|^2) \left( \text{Tr} [B(w)PF(w), B(w)PF(w)] \right)
\]
is uniformly bounded as \( w \in \mathbb{B}_N \) and \( P \) runs through the projections of rank \( r_0 \), and goes
to 0 as \( w \) tends to the boundary. It has therefore a finite supremum, which is in fact a
maximum and is in \( \mathbb{B}_N \) (and not on the boundary), as is seen by taking a subsequence
tending to this supremum, and this ends the proof. \( \square \)

Let us rewrite \( F(z) \) as
\[
F(z) = P_0F(w_0)e_{w_0}(z)\sqrt{1 - \|w_0\|^2} + F(z) - P_0F(w_0)e_{w_0}(z)\sqrt{1 - \|w_0\|^2}.
\]
We now show that (4.3) gives an orthogonal decomposition of \( F \), which is the first step
in the expansion of \( F \) that we are looking for (see (5.2) for a more precise way of writing
the decomposition) and for the algorithm that will arise repeating this construction.

**Lemma 4.2.** Let
\[
H(z) = F(z) - P_0F(w_0)e_{w_0}(z)\sqrt{1 - \|w_0\|^2}
\]
and
\[
H_0(z) = P_0F(w_0)e_{w_0}(z)\sqrt{1 - \|w_0\|^2},
\]
where \( w_0, P_0 \) are as in Proposition (4.1). It holds that
\[
P_0H(w_0) = 0
\]
and
\[
[F, F] = [H_0, H_0] + [H, H].
\]

**Proof.** First we have (4.4) since
\[
P_0H(w_0) = P_0F(w_0) - P_0F(w_0)e_{w_0}(w_0)\sqrt{1 - \|w_0\|^2} = 0.
\]
Using (4.4) we have
\[
[H, P_0F(w_0)e_{w_0}(z)\sqrt{1 - \|w_0\|^2}] = F(w_0)^*P_0H(w_0)(1 - \|w_0\|^2) = 0.
\]
So, \([H, H_0] = 0\) and
\[
[F, F] = [H_0 + H, H_0 + H] = [H_0, H_0] + [H, H].
\]

\( \square \)
5. The algorithm

To proceed and take care of the condition (4.4) (that is, in the scalar case, to divide by a Blaschke factor) we use a factor of the form (3.1). Then, we use Theorem 3.4 to find a \( C^{n \times (n + r'_0(N - 1))} \)-valued rational function \( B_{w_0, P_0} \) with \( r'_0 \leq r_0 \) and such that

\[
\text{ran } P_0 e_{w_0} = H(B_N)^{n \times m} \oplus B_{w_0, P_0}(H(B_N))^{(n + r'_0(N - 1)) \times m},
\]

and so

\[
H(B_N)^{n \times m} = (H(B_N)^{n \times m} \oplus B_{w_0, P_0}(H(B_N))^{(n + r'_0(N - 1)) \times m}) \oplus B_{w_0, P_0}(H(B_N))^{(n + r'_0(N - 1)) \times m}.
\]

Let \( F \in (H(B_N))^{n \times m} \). We choose \( w_0 \in B_N \) and \( r_0 \in \{1, \ldots, n\} \). Using the maximum selection principle with \( B(z) = I_n \) we get a decomposition of the form (5.1). We rewrite (4.3) as

\[
F(z) = P_0 F(w_0 e_{w_0}(z)) \sqrt{1 - \|w_0\|^2} + B_{w_0, P_0}(z) F_1(z),
\]

where \( F_1 \in (H(B_N))^{(n + r'_0(N - 1)) \times m} \) (which, as \( F_1 \) is uniquely defined when \( N > 1 \)). We now select \( w_1 \in B_N \) and \( r_1 \in \{1, \ldots, n + r'_0(N - 1)\} \), and apply the maximum selection principle to the pair \( (B_{w_0, P_0}(z), F_1(z)) \). We have then

\[
F_1(z) = P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} + B_{w_1, P_1}(z) F_2(z),
\]

where \( F_2 \in (H(B_N))^{(n + r'_0 + r'_1(N - 1)) \times m} \) (with \( r'_1 \leq r_1 \)) is not uniquely defined when \( N > 1 \). So

\[
F(z) = P_0 F(w_0 e_{w_0}(z)) \sqrt{1 - \|w_0\|^2} + B_{w_0, P_0}(z) P_1 F_1(w_1) e_{w_1}(z) \sqrt{1 - \|w_1\|^2} + B_{w_0, P_0}(z) B_{w_1, P_1}(z) F_2(z).
\]

We iterate the procedure with the pair \( (B_{w_0, P_0}(z) B_{w_1, P_1}(z), F_2(z)) \) and observe the appearance of the Blaschke product

\[
B_k(z) = B_{w_0, P_0} B_{w_1, P_1} B_{w_2, P_2} \cdots B_{w_{k-1}, P_{k-1}}, \quad \text{for } k \geq 1,
\]

which will be \( C^{n \times (1 + s_k(N - 1))} \)-valued for some \( s_k \leq \sum_{j=0}^{k-1} r_j \). We set

\[
M_k = F_k(w_k) \in C^{s_k \times m},
\]

and

\[
\mathcal{B}_k(z) = \begin{cases} 
\sqrt{1 - \|w_0\|^2} e_{w_0}(z) & \text{for } k = 0, \\
\sqrt{1 - \|w_k\|^2} e_{w_k}(z) B_{w_0, P_0}(z) B_{w_1, P_1}(z) B_{w_2, P_2}(z) \cdots B_{w_{k-1}, P_{k-1}}(z) & \text{for } k \geq 1.
\end{cases}
\]

Note that

\[
\mathcal{B}_k(w_k) = B_k(w_k), \quad k \geq 1.
\]

We have

\[
F(z) = \sum_{k=0}^{u} \mathcal{B}_k(z) M_k + \mathcal{B}_{u+1}(z) F_{u+1}(z).
\]

Moreover,

\[
\langle \mathcal{B}_k M_k, \mathcal{B}_\ell M_\ell \rangle_{H(B_N)} = 0 \quad \text{for } k \neq \ell
\]
and we have by the orthogonality of the decomposition that

\[(5.8) \quad \|F\|^2_{H(\mathcal{B}_N)} = \sum_{k=0}^{u} \|\mathcal{B}_k M_k\|^2_{H(\mathcal{B}_N)} + \|\mathcal{B}_{u+1}(z) F_{u+1}\|^2_{H(\mathcal{B}_N)}.\]

This recursive procedure gives, at the \(k\)-th step, the best approximation. However we have to ensure that when \(k\) tends to infinity the algorithm converges. This is guaranteed by virtue of the next result.

**Theorem 5.1.** Suppose that in (5.6) at each step one selects \(w_k\) and \(P_k\) according to the maximum selection principle applied to \((\mathcal{B}_{w_k}(z), F_k(z))\). Then the algorithm converges, meaning that

\[F(z) = \sum_{k=0}^{\infty} \mathcal{B}_k(z) M_k\]

in the norm of the Drury-Arveson space.

**Proof.** We follow the arguments of [15] and [4]. We set

\[(5.9) \quad R_u(z) = F(z) - \sum_{k=0}^{u} \mathcal{B}_k(z) M_k = \mathcal{B}_{w_{u+1}}(z) F_{u+1}(z)\]

(where \(F_{u+1}\) is not uniquely defined when \(N > 1\)) and

\[S_u(z) = \sum_{k=u+1}^{\infty} \mathcal{B}_k(z) M_k.\]

In view of (5.7)-(5.8) the sum \(\sum_{k=0}^{\infty} \mathcal{B}_k(z) M_k\) converges in the Drury-Arveson space. Let \(G\) be its limit, and assume that \(G \neq F\). Thus there exists \(w \in \mathcal{B}_N\) such that \(G(w) \neq F(w)\). We now proceed in a number of steps to obtain a contradiction.

**STEP 1:** There exists \(u_0 \in \mathbb{N}\) such that for \(u \geq u_0\)

\[(5.10) \quad \sqrt{1 - \|w\|^2} \cdot \|R_u(w)\| > \sup_{c \in \mathbb{C}^n, \|c\|=1, \text{ any } d \in \mathbb{C}^m, \|d\|=1, \langle (F - G) d, ce_w \rangle_{(H(\mathcal{B}_N))^n} \geq \frac{1}{2}.\]

Indeed, \(S_u\) tends to 0 in norm in \((H(\mathcal{B}_N))^{n \times m}\). Since in a reproducing kernel Hilbert space convergence in norm implies pointwise convergence, we have \(\lim_{u \to \infty} S_u(w) = 0_{n \times m}\) in the norm of \(\mathbb{C}^{n \times m}\), and there exists \(u_0 \in \mathbb{N}\) such that

\[u \geq u_0 \implies \|S_u(w)\| < \frac{\|F(w) - G(w)\|}{2}.\]

Thus

\[\|R_u(w)\| + \frac{\|F(w) - G(w)\|}{2} > \|R_u(w)\| \geq \|F(w) - G(w)\|,
\]

and so

\[\|R_u(w)\| > \frac{\|F(w) - G(w)\|}{2},\]

which can be rewritten as (5.10).
STEP 2: It holds that

\[ (5.11) \lim_{k \to \infty} (1 - \|w_k\|^2)\|B_k(w_k)M_k\|^2 = 0 \]

Indeed, from the convergence of \( \sum_{k=0}^{\infty} B_k M_k \) we have

\[ \lim_{k \to \infty} \|B_k M_k\|_{(H(B_N))^n \times m} = 0. \]

Thus, with \( c \in \mathbb{C}^m \) and \( d \in \mathbb{C}^n \), we have:

\[ |\langle B_k M_k c, d \rangle_{(H(B_N))^n}| = \|B_k M_k c\|_{(H(B_N))^n} \cdot \frac{\|d\|}{\sqrt{1 - \|w_k\|^2}} \leq \|B_k M_k\|_{(H(B_N))^n \times m} \cdot \|c\| \cdot \frac{\|d\|}{\sqrt{1 - \|w_k\|^2}}, \]

where we have used the Cauchy-Schwarz inequality. So, after taking supremum on \( c \) and \( d \),

\[ \|\sqrt{1 - \|w_k\|^2} B_k(w_k)M_k\| \leq \|B_k M_k\|_{(H(B_N))^n \times m} \to 0 \text{ as } n \to \infty, \]

and so (5.11) holds in view of (5.5).

STEP 3: We conclude the proof.

Let \( u \geq u_0 \), where \( u_0 \) is as in Step 1. Since \( R_u(z) = B_{w_{u+1}}(z)F_{u+1}(z) \) and since \( w \) is such that \( F(w) \neq G(w) \) we have

\[ (5.12) \sqrt{1 - \|w\|^2} \cdot \|B_{w_{u+1}}(w)F_{u+1}(w)\| > \sup_{c \in \mathbb{C}^n, \|c\|=1, \ d \in \mathbb{C}^m, \|d\|=1} \frac{|\langle (F - G)d, ce_w \rangle_{(H(B_N))^n}|}{2} \]

By definition of \( w_{u+1} \) we have

\[ \sqrt{1 - \|w_{u+1}\|^2} \cdot \|B_{w_{u+1}}(w_{u+1})F_{u+1}(w_{u+1})\| < \sup_{c \in \mathbb{C}^n, \|c\|=1, \ d \in \mathbb{C}^m, \|d\|=1} \frac{|\langle (F - G)d, ce_w \rangle_{(H(B_N))^n}|}{2}, \]

and using (5.5) we contradict (5.11).

\[ \Box \]

6. INFINITE BLASCHKE PRODUCTS

In the previous sections appeared the counterpart of finite Blaschke products in the setting of the ball. We now consider the case of infinite products.

Let \( a \in B_N \), and let \( b_a(z) \) be a \( \mathbb{C}^{1 \times N} \)-valued Blaschke factor. We use the formula

\[ (6.1) b_a(z) = \frac{a - za^*}{aa^*}a - \sqrt{1 - aa^*} \left( z - \frac{za^*}{aa^*}a \right) \]

from [16] (2), p. 25] rather than the formula in (2.5). See [7, Lemma 4.2, p. 13] for the equality between the two expressions.
We first prove a technical lemma useful in the proof of the convergence of an infinite Blaschke product.

**Lemma 6.1.** Let \( \alpha = \frac{-a}{\sqrt{aa^*}} \in \partial \mathbb{B}_N \). Then,
\[
\begin{align*}
  b_a(z) - b_a(\alpha) &= \frac{(z - \alpha) \left( a^*a \left( \frac{1 - \sqrt{1 - aa^*}}{aa^*} \right) - I_N \right) + z(\alpha a^*) - \alpha (za^*)}{(1 - za^*)(1 + \sqrt{aa^*})} \cdot \sqrt{1 - aa^*} \\
  \|b_a(z) - b_a(\alpha)\| &\leq \frac{4\sqrt{1 - aa^*}}{1 - \|z\|}.
\end{align*}
\]

**Proof.** We write \( b_a(z) - b_a(\alpha) = \frac{\Delta}{(1 - za^*)(1 - \alpha a^*)} \), where the numerator
\[
\begin{align*}
  \Delta &= \left( a - \frac{za^*}{aa^*}a - \sqrt{1 - aa^*} \left( z - \frac{za^*}{aa^*}a \right) \right) (1 - \alpha a^*) - \\
  &\quad - \left( a - \frac{\alpha a^*}{aa^*}a - \sqrt{1 - aa^*} \left( \alpha - \frac{\alpha a^*}{aa^*}a \right) \right) (1 - za^*)
\end{align*}
\]
has 16 terms. Out of there, \( a \) and \( -a \) cancel each other, and
\[
\frac{za^*}{aa^*}a(\alpha a^*) = \frac{\alpha a^*}{aa^*}a(za^*)
\]
and
\[
\sqrt{1 - aa^*} \frac{za^*}{aa^*}a(\alpha a^*) = \sqrt{1 - aa^*} \frac{\alpha a^*}{aa^*}a(za^*).
\]
We are thus left with 10 terms, which can be rewritten as:
\[
\Delta = (z - \alpha) \left( \left( -\frac{a^*a}{aa^*} - \sqrt{1 - aa^*}I_N + \sqrt{1 - aa^*} \frac{a^*a}{aa^*} + a^*a \right) + \sqrt{1 - aa^*} (z(\alpha a^*) - \alpha (za^*)) \right) .
\]
Note that \( z(\alpha a^*) - \alpha (za^*) \) does not vanish when \( N > 1 \). Therefore
\[
\begin{align*}
  b_a(z) - b_a(\alpha) &= \frac{(z - \alpha) \left( -\frac{a^*a}{aa^*} - \sqrt{1 - aa^*}I_N + \sqrt{1 - aa^*} \frac{a^*a}{aa^*} + a^*a \right) + \sqrt{1 - aa^*} (z(\alpha a^*) - \alpha (za^*))}{(1 - za^*)(1 + \sqrt{aa^*})}.
\end{align*}
\]

\( \square \)

**Remark 6.2.** We note that
\[
\|a^*a \left( \frac{1 - \sqrt{1 - aa^*}}{aa^*} \right) - I_N\| = \sqrt{1 - aa^*},
\]
as can be seen by computing the eigenvalues of the matrix in the left hand side.

We now consider a term of the form \( (3.1) \) and write (where \( \alpha = -\frac{a}{\sqrt{aa^*}} \) and \( W \) is a unitary matrix to be determined)
\[
\begin{align*}
  B_a(z) &= B(z)W \\
  &= \left( U \left( \begin{array}{cc}
  b_a(\alpha) & 0_{1 \times (n-1)} \\
  0_{(n-1) \times N} & I_{n-1}
\end{array} \right) + U \left( \begin{array}{cc}
  (b_a(z) - b_a(\alpha)) & 0_{1 \times (n-1)} \\
  0_{(n-1) \times N} & I_{n-1}
\end{array} \right) \right) W,
\end{align*}
\]

where \( \alpha = -\frac{a}{\sqrt{aa^*}} \) and \( W \) is a unitary matrix to be determined.
where we do not stress the dependence on the matrices $U$ and $W$. Since $b_\alpha(\alpha)$ is a unit vector, the matrix
\[
U \left( \begin{array}{cc} b_\alpha(\alpha) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{array} \right)
\]
is coisometric, and we can complete the columns of its adjoint to a unitary matrix $W$. Then we have
\[
(6.6) \quad U \left( \begin{array}{cc} b_\alpha(\alpha) & 0_{1 \times (n-1)} \\ 0_{(n-1) \times N} & I_{n-1} \end{array} \right) W = (I \ 0)
\]
and show that the corresponding infinite product will converge when $\sum_{n=1}^{\infty} \sqrt{1 - a_n a_n^*}$ converges.

In Theorem 6.3 below we imbed $\mathbb{C}^m$ inside $\ell_2$ via the formula:
\[
(6.7) \quad i_m(z_1, \ldots, z_m) = (z_1, \ldots, z_m, 0, 0, \ldots).
\]
We also need some notation and introduce the matrices
\[
E_k = \left( \begin{array}{cc} 1 & 0_{1 \times k(N-1)} \\ \end{array} \right) \quad (= 1 \text{ when } N = 1),
\]
\[
F_k = \left( I_{1+(k-1)(N-1)} \ 0_{(1+(k-1)(N-1)) \times (N+(k-1)(N-1))} \right) \in \mathbb{C}^{1+(k-1)(N-1) \times (N+(k-1)(N-1))},
\]
and note that $E_1 = F_1$ and
\[
(6.8) \quad E_k = E_1 F_2 \cdots F_k \quad \text{and} \quad E_{k+1} = E_k F_{k+1}.
\]
We also note that multiplication by $F_k$ on the right imbeds $\mathbb{C}^{1+(k-1)(N-1)}$ into $\mathbb{C}^{1+k(N-1)}$. It will be useful to use the notation
\[
(6.9) \quad F_{m2}^{m1} = \prod_{k=m+1}^{m2} E_k.
\]

**Theorem 6.3.** The infinite product $b_{w_0}(z) B_{w_1}(z) B_{w_2}(z) \cdots B_{w_{k-1}}(z) \cdots$ where the factors are normalized as in (6.6) converges pointwise for $z \in \mathbb{B}_N$ to a non-identically vanishing $\ell_2$-valued function analytic in $\mathbb{B}_N$ if
\[
(6.10) \quad \sum_{k=0}^{\infty} \sqrt{1 - a_k a_k^*} < \infty.
\]

**Proof.** The idea is to follow the proof for the scalar case appearing in sources such as [11, 13] and reproduced in [3, pp. 104-105]. We consider the product
\[
\prod_{k=1}^{m} (F_k + A_k(z))
\]
with
\[
A_k(z) = U_k \left( \begin{array}{cc} b_{a_k}(z) - b_{a_k}(\alpha) & 0_{1 \times (k-1)(N-1)} \\ 0_{(k-1)(N-1) \times N} & I_{(k-1)(N-1)} \end{array} \right) W_k \in \mathbb{C}^{1+(k-1)(N-1) \times (N+(k-1)(N-1))},
\]
and note that, in view of (6.3),
\[
(6.11) \quad \|A_k(z)\| \leq \frac{4 \sqrt{1 - a_k a_k^*}}{1 - \|z\|}.
\]
Following the classical proof we now prove the convergence in a number of steps and use [3, pp. 104-105] as a source.

Note that, to ease the notation, in Steps 1-3 we do not stress the dependence of $A_k$ on the variable $z$.

**STEP 1:** It holds that

\[
\| \overset{m}{\prod_{k=1}} (F_k + A_k) - E_m \| \leq \prod_{k=1}^m (1 + \|A_k\|) - 1, \quad m \in \mathbb{N}.
\]

We proceed by induction, the case $m = 1$ being trivial since $E_1 = F_1$. We have

\[
\begin{split}
\| \overset{m+1}{\prod_{k=1}} (F_k + A_k) - E_{m+1} \| &= \| \left( \overset{m}{\prod_{k=1}} (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_{m+1} \| \\
&= \| \left( \overset{m}{\prod_{k=1}} (F_k + A_k) \right) (F_{m+1} + A_{m+1}) - E_{m}F_{m+1} \| \\
&\leq \| \left( \overset{m}{\prod_{k=1}} (F_k + A_k) \right) - E_{m} \| F_{m+1} \| + \\
&\quad + \|A_{m+1}\| \left( \overset{m}{\prod_{k=1}} (1 + \|A_k\|) \right) \\
&\leq \left( \left( \overset{m}{\prod_{k=1}} (1 + \|A_k\|) \right) - 1 \right) + \|A_{m+1}\| \left( \overset{m}{\prod_{k=1}} (1 + \|A_k\|) \right) \\
&= \left( \overset{m+1}{\prod_{k=1}} (1 + \|A_k\|) \right) - 1,
\end{split}
\]

where we have used the induction hypothesis to go from the third to the fourth line.

Replacing $A_k$ by $A_{k+m_1}$ we have for $m_2 > m_1$:

\[
\| \left( \overset{m_2}{\prod_{k=m_1+1}} (E_k + A_k) \right) - \overset{m_2}{\prod_{k=m_1+1}} E_{m_2} \| \leq \left( \overset{m_2}{\prod_{k=m_1+1}} (1 + \|A_k\|) \right) - 1.
\]

**STEP 2:** Let $Z_m \equiv \overset{m}{\prod_{k=1}} (F_k + A_k)$. Then,

\[
\|Z_m\| \leq e^{\sum_{k=1}^m \|A_k\|} < \infty.
\]
Indeed,

\[ \|Z_m\| \leq \prod_{k=1}^{m} \|F_k + A_k\| \]
\[ \leq \prod_{k=1}^{m} (1 + \|A_k\|) \]
\[ \leq \prod_{k=1}^{m} e^{\|A_k\|} \leq e^{\sum_{k=1}^{\infty} \|A_k\|} < \infty, \]

in view of \((6.10)\) and \((6.11)\).

**STEP 3:** Let \(i_m\) be defined by \((6.7)\). Then, \((i_m(Z_m))_{m \in \mathbb{N}}\) is a Cauchy sequence in \(\ell_2\).

For \(m_2 > m_1\) and using \((6.13)\), we have

\[ \|i_{m_2}(Z_{m_2}) - i_{m_1}(Z_{m_1})\|_{\ell_2} = \|Z_{m_2} - i_{m_2} \cdots i_{m_1+1}(Z_{m_1})\|_{\ell_2} \]
\[ = \left( \prod_{k=1}^{m_1} (F_k + A_k) \right) \cdot \left( \prod_{k=m_1+1}^{m_2} (F_k + A_k) - F_{m_1+1}^{m_2} \right) \]
\[ \leq \left( \prod_{k=1}^{m_1} (1 + \|A_k\|) \right) \cdot \left( \prod_{k=m_1+1}^{m_2} (F_k + A_k) - F_{m_1+1}^{m_2} \right) \]
\[ \leq e^K \left\{ \left( \prod_{k=m_1+1}^{m_2} (1 + \|A_k\|) \right) - 1 \right\} \]
\[ \leq e^K \left\{ \left( \prod_{k=m_1+1}^{m_2} e^{\|A_k\|} \right) - 1 \right\} \]
\[ \leq \left( \sum_{k=m_1+1}^{m_2} \|A_k\| \right) e^{2K}, \]

with \(K = \sum_{k=1}^{\infty} \|A_k\|\) (which is finite, thanks to \((6.10)\) and \((6.11)\)), and using inequality
\[ e^x \leq 1 + xe^x, \quad x \geq 0, \]

with \(x = \sum_{k=m_1+1}^{m_2} \|A_k\|\).

**STEP 4:** The \(\ell_2\)-valued function \(Z(z) = \lim_{m \to \infty} Z_m(z)\) does not vanish identically in \(\mathbb{B}_N\).

We first assume that \(\sum_{k=1}^{\infty} \|A_k(z)\| < \frac{1}{2}\) and prove by induction that

\[ \|Z_m(z)\| \geq 1 - \sum_{k=1}^{m} \|A_k(z)\|. \]

\[ (6.15) \]
The claim $Z \neq 0$ will then follow by letting $m \to \infty$. For $m = 1$ the claim is trivial. Assume that (6.13) holds for $m$. We then have:

\[
\|Z_{m+1}(z)\| = \|Z_m(z)(F_{m+1} + A_{m+1}(z))\| \\
\geq \|Z_m(z)F_{m+1}\| - \|Z_m(z)A_{m+1}(z)\| \quad \text{(since } \|Z_m(z)F_{m+1}\| = \|Z_m(z)\|) \\
\geq \|Z_m(z)\| - \|Z_m(z)\|\|A_{m+1}(z)\| \quad \text{(since } \|Z_m(z)A_{m+1}(z)\| \leq \|Z_m(z)\|\|A_{m+1}(z)\|) \\
= \|Z_m(z)\| \cdot (1 - \|A_{m+1}(z)\|) \\
\geq (1 - \sum_{k=1}^{m} \|A_k(z)\|)(1 - \|A_{m+1}(z)\|) \\
\geq (1 - \sum_{k=1}^{m+1} \|A_k(z)\|).
\]

Let $M \in \mathbb{N}$ (depending on $z$) be such that $\sum_{k=M}^{\infty} \|A_k(z)\| < \frac{1}{2}$. Then the same inequality holds in an open neighborhood $V$ of $z$ in view of (6.11), and so the same $M$ can be taken for $z \in V$. Let

\[
Z_{M-1}(z) = \prod_{u=1}^{M-1} (F_u + A_u(z)) \in \mathbb{C}^{1 \times (1 + (M-2)(N-1))},
\]

where

\[
\widetilde{Z}_M(z) = \prod_{u=M}^{\infty} (F_u + A_u(z)).
\]

We can patch together all the $Z_{M-1}(z)\widetilde{Z}_M(z)$ to a common function defined in $\mathbb{B}_N$. Assume that $Z_{M-1}(z)\widetilde{Z}_M(z) \equiv 0$ in one of the neighborhoods $V$. Then the infinite product vanishes identically in $\mathbb{B}_N$. Letting $z$ go to the boundary we get a contradiction since

\[
Z_{M-1}(z)\widetilde{Z}_M(z)
\]

takes coisometric values on $\partial \mathbb{B}_N$.

STEP 5: Using (6.12) and (6.14), we obtain the bound:

\[
(6.16) \quad \| \prod_{k=1}^{m} (F_k + A_k(z)) - Z \| \leq e^{2K} \left( \sum_{k=m+1}^{\infty} \|A_k(z)\| \right).
\]

It is worthwhile to note that the above theorem allows to further extending the results of [7] to the case of an infinite number of points.

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