Bounded Affine Permutations II. Avoidance of Decreasing Patterns

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Abstract. We continue our study of a new boundedness condition for affine permutations, motivated by the fruitful concept of periodic boundary conditions in statistical physics. We focus on bounded affine permutations of size $N$ that avoid the monotone decreasing pattern of fixed size $m$. We prove that the number of such permutations is asymptotically equal to $(m - 1)^2N^{(m-2)/2}$ times an explicit constant as $N \to \infty$. For instance, the number of bounded affine permutations of size $N$ that avoid 321 is asymptotically equal to $4^N(N/4\pi)^{1/2}$. We also prove a permuton-like result for the scaling limit of random permutations from this class, showing that the plot of a typical bounded affine permutation avoiding $m \cdots 1$ looks like $m - 1$ random lines of slope 1 whose $y$ intercepts sum to 0.

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1. Introduction

This paper is a continuation of the research begun in our companion paper [29]. Accordingly, some of the text and figures in this introduction are drawn from [29, Sec. 1].

Pattern-avoiding permutations have been studied actively in the combinatorics literature for the past 4 decades. (See Sect. 1.1 for definitions of terms we use.) Some sources on permutation patterns include: [4] for essential terminology, [7, Ch. 4] for a textbook introduction, and [34] for an in-depth survey of the literature. Pattern-avoiding permutations arise in a variety of mathematical contexts, particularly algebra and the analysis of algorithms. Research such
as [5,16] have extended these investigations by considering affine permutations that avoid one or more (ordinary) permutations as patterns.

**Definition 1.1.** An affine permutation of size $N$ is a bijection $\sigma : \mathbb{Z} \to \mathbb{Z}$ such that:

1. $\sigma(i + N) = \sigma(i) + N$ for all $i \in \mathbb{Z}$, and
2. $\sum_{i=1}^{N} \sigma(i) = \sum_{i=1}^{N} i$.

Condition (ii) can be viewed as a “centering” condition, since any bijection satisfying (i) can be made to satisfy (ii) by adding a constant to the function. The affine permutations of size $N$ form an infinite Coxeter group under composition, with $N$ generators; see Section 8.3 of Björner and Brenti [6] for a detailed look at affine permutations from this perspective.

For any given size $N > 1$, there are infinitely many affine permutations of size $N$; indeed, for some patterns such as $\tau = 321$, there are infinitely many affine permutations of size $N$ that avoid $\tau$. One can view the following definition, which we introduced in our companion paper [29], as a reasonable attempt to make these sets finite, but there are more compelling reasons for considering this definition, as we describe below.

**Definition 1.2.** A bounded affine permutation of size $N$ is an affine permutation $\sigma$ of size $N$ such that $|\sigma(i) - i| < N$ for all $i$.

Figure 1 illustrates an example of a bounded affine permutation.

**Remark 1.3.** Affine permutations with a different boundedness condition were introduced by Knutson et al. [26], who used them to study the totally non-negative Grassmannian and positroids. The bounded affine permutations in our paper are not the same as those.

Let $S_N$ denote the set of permutations of size $N$, and let $\tilde{S}_{N}^{//}$ denote the set of bounded affine permutations of size $N$. We also define

$$S := \bigcup_{N \geq 0} S_N \text{ and } \tilde{S}^{//} := \bigcup_{N \geq 1} \tilde{S}_{N}^{//}.$$ 

In our companion paper [29], we find exact and asymptotic formulas for $|\tilde{S}_{N}^{//}|$, the total number of bounded affine permutations of size $N$. We show that

$$|\tilde{S}_{N}^{//}| = \sum_{m=0}^{N} \binom{N}{m} \sum_{k=0}^{m} \binom{m}{N-k} (-1)^{N-m} a(m,k)$$

(1.1)

where $a(m, k)$ are the Eulerian numbers (the number of permutations of size $m$ with $k$ excedances), and that

$$|\tilde{S}_{N}^{//}| \sim \sqrt{\frac{3}{2\pi eN}} 2^N N! \text{ as } N \to \infty.$$ (1.2)

If we view a permutation $\pi \in S_N$ as a bijection on $[N]$, then we can extend it periodically by Equation (i) of Definition 1.1 to a bijection $\oplus \pi$ on $\mathbb{Z}$; that is,

$$\oplus \pi(i + kN) = \pi(i) + kN \text{ for } i \in [N] \text{ and } k \in \mathbb{Z}.$$
Figure 1. A bounded affine permutation of size 6, whose values on 1, ..., 6 are 2, 7, −2, −1, 9, 6. For the affine permutation to be bounded, its entries must all lie strictly between the dashed lines. (This figure was previously published in [29])

Observe that $\oplus \pi \in \tilde{S}_N^\parallel$ (see Fig. 2). We call $\oplus \pi$ the infinite sum of $\pi$. The map $\pi \mapsto \oplus \pi$ is an injection from $S_N$ into $\tilde{S}_N^\parallel$.

This paper concerns the set of bounded affine permutations that avoid an (ordinary) permutation $\tau$; this set is denoted $\tilde{S}_N^\parallel(\tau)$, and we define pattern avoidance and related notions in Sect. 1.1.

Let $\tau \in S_k$. It is routine to check that, if $\tau_1 > \tau_k$ (or more generally if $\tau$ is sum-indecomposable), then $\sigma \oplus \pi$ avoids $\tau$ whenever $\sigma$ and $\pi$ both avoid $\tau$. Thus the injection $\pi \mapsto \oplus \pi$ mentioned above is also an injection from $S_N(\tau)$ into $\tilde{S}_N^\parallel(\tau)$. This proves that $|S_N(\tau)| \leq |\tilde{S}_N^\parallel(\tau)|$ whenever $\tau$ is sum-indecomposable. It is harder to find a good general upper bound for $|\tilde{S}_N^\parallel(\tau)|$.

We posed the following conjecture in [29].

**Conjecture 1.4.** The proper growth rate $\text{gr}(\tilde{S}_N^\parallel(\tau)) := \lim_{N \to \infty} |\tilde{S}_N^\parallel(\tau)|^{1/N}$ exists and equals the Stanley–Wilf limit $L(\tau) := \lim_{N \to \infty} |S_N(\tau)|^{1/N}$ for every sum-indecomposable pattern $\tau$.

We remark that the indecomposability condition in the conjecture is important; e.g. the only affine permutation that avoids 2143 is the identity permutation. In our companion paper [29], we prove that the conjecture holds for some specific choices of $\tau$—and the results of this paper show that it holds when $\tau$ is a decreasing pattern—but in general, we cannot even prove that the proper growth rate $\text{gr}(\tilde{S}_N^\parallel(\tau))$ exists. We do know that the upper growth rate
Figure 2. Schematic plot of a permutation $\pi \in S_N$ and its periodic extension $\oplus \pi \in \tilde{S}_N//$. For an affine permutation of size $N$ to be bounded, all points of the plot must lie on or between the two diagonal lines. (This figure was previously published in [29])

$\mathfrak{gr}(\tilde{S}_{//}(\tau)) := \limsup_{N \to \infty} |\tilde{S}_{//}(\tau)|^{1/N}$ is always finite: in the companion paper, we show that $\mathfrak{gr}(\tilde{S}_{//}(\tau)) \leq 3L(\tau)$, where $L(\tau)$ is the Stanley–Wilf limit.

In this paper, we focus on the avoidance of monotone decreasing patterns $m(m-1)\cdots321$ in bounded affine permutations. More specifically, our first main result (Theorem 2.2 in Sect. 2) is that for every $m \geq 3$ we have the asymptotic behaviour

$$\left|\tilde{S}_{//}(m(m-1)\cdots321)\right| \sim A_m N^{(m-2)/2}(m-1)^{2N} \quad \text{as } N \to \infty$$

where the constant $A_m$ is given by

$$A_m = \frac{\sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m-1}{j} (m - 2j - 1)^{m-2}}{(4\pi)^{(m-2)/2} (m - 1)^{(m-1)/2} [(m - 2)!]^2}.$$  

(See Remark 2.3 and the subsequent discussion for comments on the form this result.) The key to proving (1.3) is a counting argument based on the decomposition of any member of $\tilde{S}_{//}(m(m-1)\cdots321)$ into $m-1$ increasing (periodic) subsequences (Proposition 2.1). It turns out that unlike the situation for ordinary permutations avoiding monotone patterns, these $m-1$ subsequences are...
Figure 3. Sketch of typical 321-avoiding bounded affine permutation of size $N$. The plot is completely covered by two diagonal strips of width $\alpha N$ where $\alpha$ is a small positive number. For $(m(m-1) \cdots 21)$-avoidance, we would need $m-1$ such strips. In a typical plot, each strip covers an approximately equal number of points (color figure online)

typically well separated in the bounded affine case, as represented schematically in Fig. 3 in the case $m = 3$.

Indeed, in the plot of a random member of $\tilde{S}_N// (m(m-1) \cdots 321)$, it is highly likely that each of the $m-1$ subsequences is confined to a narrow diagonal strip, and that the points are approximately uniformly distributed within that strip in a sense that we shall make precise in Sect. 2.3 (see Fig. 6 for a simulated example). In addition, each subsequence is likely to have approximately $N/(m-1)$ points with first coordinate in $[1, N]$. This all suggests that as we let $N$ tend to infinity, the plot (scaled down by a factor of $N$) looks more and more like $m - 1$ solid lines of slope 1 (Fig. 4).

Such a phenomenon can be conveniently described in the framework of weak convergence of probability measures in the plane, exactly as in the context of permutons [19,21]. Section 3.1 describes our framework and connections to the permuton literature. Our second main result says that for fixed $m$, the scaling limit as $N \to \infty$ of a random element of $\tilde{S}_N// (m(m-1) \cdots 321)$ (viewed as an atomic measure on the plane) is a uniform measure\(^1\) on $m-1$ parallel lines of slope 1 with $y$-intercepts that are randomly chosen from $[-1, 1]$, independently except for the condition that their sum is 0. Theorem 3.7 in Sect. 3 is a precise statement of this result.

\(^1\)Proportional to one-dimensional Lebesgue measure.
Figure 4. Scaling of a random element of $\tilde{S}^{\parallel}_N(321)$ to a permuton-like limit. As $N \to \infty$, we rescale each axis of $\mathbb{R}^2$ by $1/N$. We view the left figure as a discrete probability measure with an atom of mass $1/N$ at each point of the plot. The right figure represents the uniform probability measure on the line segments $y = x + \delta$ and $y = x - \delta$ ($0 \leq x \leq 1$), where $\delta$ is uniformly distributed on $[-1,1]$ (color figure online).

Our motivation for initiating the study of bounded affine permutations is described in [29]. Briefly, it is our attempt to impose an analogue of “periodic boundary conditions” on the plots of random $\tau$-avoiding (ordinary) permutations for patterns such as $\tau = 4321$ or $\tau = 4231$, inspired by Clisby’s work on self-avoiding walks [15]. We anticipate that (a part of) the plot of a random member of $\tilde{S}^{\parallel}_N(\tau)$ in some sense looks like the middle of the plot a random member of $S_M(\tau)$ (for some suitable $M$), far from the “boundary effects” that come into play near the corners of the square $[1,M]^2$ and constrain the plot of an ordinary permutation (see Figs. 5, 6, 7 and 8).

That is just one potential direction of research into the “shape” of pattern-avoiding bounded affine permutations. There are many other possibilities, such as investigating the asymptotic distribution of the number of inversions, descents, fixed points, or excedances in one period of a random bounded affine permutation. For each of these, we can draw our random permutation uniformly from the set of all bounded affine permutations of size $N$, or from a restricted set such as the ones avoiding 321. Many of these questions have been answered in the case of ordinary permutations, which suggests the possibility of finding analogs of those results for bounded affine permutations. For example, Cheng et al. studied the exact distribution of inversions in 321-avoiding (ordinary) permutations [14], from which it is possible to obtain an asymptotic distribution in terms of the area under a Brownian excursion [23]. Elizalde [17] proved that the (exact) joint distribution of fixed points
and excedances is the same in 321-avoiding permutations as in 132-avoiding permutations. Bóna [8] studied the expected number of occurrences of a given pattern in a uniformly random 132-avoiding permutation. Janson showed that, in a uniformly random 132-avoiding permutation [22] or a uniformly random 321-avoiding permutation [23], the number of occurrences of a given pattern has a limiting distribution that can be expressed in terms of Brownian excursions. It would be interesting to see how each of these results relates to the situation for bounded affine permutations—particularly for the ones involving 321-avoiding bounded affine permutations, since our paper analyzes the shape of those.

Another open problem is to find the exact enumeration of 321-avoiding bounded affine permutations. This seems to be rather more complicated than the classical case of ordinary permutations avoiding 321, which are counted by the Catalan numbers.

1.1. Definitions and Notation
For sequences \( \{a_n\} \) and \( \{b_n\} \), we write \( a_n \sim b_n \) to mean \( \lim_{n \to \infty} a_n/b_n = 1 \). For \( n \in \mathbb{N} \), we write \( [n] = \{1, \ldots, n\} \). We denote the Euclidean norm by \( \| \cdot \| \).
Affine permutations and bounded affine permutations were defined above, along with the sets $S_N, \tilde{S}^{\|}_N, S,$ and $\tilde{S}^{\|}_N$. We represent an ordinary permutation $\sigma \in S_N$ either as a function $\sigma : [N] \to [N]$ or as a finite sequence $\sigma_1 \sigma_2 \ldots \sigma_N$ where $\sigma_i = \sigma(i)$. For affine permutations, we only use the function notation.

We begin by introducing concepts that are standard in permutation patterns research. The diagram or plot of a permutation $\pi \in S_N$ is the set of points $\{(i, \pi(i)) : i \in [N]\}$. Given permutations $\pi$ and $\tau$, we say that $\pi$ contains $\tau$ as a pattern, or simply that $\pi$ contains $\tau$, if the diagram of $\tau$ can be obtained by deleting zero or more points from the diagram of $\pi$ (and shrinking corresponding segments of the axes), i.e. if $\pi$ has a subsequence whose entries have the same relative order as the entries of $\tau$. We may also say that two sequences with the same relative order are order isomorphic. We say $\pi$ avoids $\tau$ if $\pi$ does not contain $\tau$. For instance, for $\pi = 493125876$, the subsequence $9356$ is an occurrence of $\tau = 4123$, but on the other hand $\pi$ avoids $3142$. See Fig. 9.

If $\tau$ is a permutation, then $S(\tau)$ denotes the set of all permutations that avoid $\tau$, and $S_N(\tau)$ is the set of such permutations of size $N$, i.e. $S_N(\tau) = S(\tau) \cap S_N$. The upper growth rate of $S(\tau)$ is defined as $\overline{gr}(S(\tau)) := \limsup_{N \to \infty} |S_N(\tau)|^{1/N}$, and the lower growth rate is defined as $\underline{gr}(S(\tau)) := \liminf_{N \to \infty} |S_N(\tau)|^{1/N}$. If the upper and lower growth rates of $S(\tau)$ are equal, i.e. if $\lim_{N \to \infty} |S_N(\tau)|^{1/N}$ exists (or is $\infty$), then this number is called the proper growth rate of $S(\tau)$, written $gr(S(\tau))$. By the Marcus–Tardos Theorem [30] (formerly the Stanley–Wilf Conjecture), $S(\tau)$ has a finite upper growth rate for every $\tau$. It is also known that $S(\tau)$ has a proper growth rate.
Figure 7. A random 4231-avoiding permutation of size 500. This was generated by Gökhan Yıldırım using a Markov chain Monte Carlo algorithm. Our motivation for the present work was the belief that the part of this plot in the strip $200 < x < 300$, say, should look like part of the plot of a 4231-avoiding bounded affine permutation for every $\tau$ (proved by Arratia [2]); this growth rate is often called the Stanley–Wilf limit and denoted $L(\tau)$.

We now introduce the analogous concepts for affine permutations. The diagram or plot of an affine permutation $\omega \in \tilde{S}_N^{/\tau}$ is the set of points $\{(i, \omega(i)) : i \in \mathbb{Z}\}$. Given an affine permutation $\omega$ and an ordinary permutation $\tau$, we say that $\omega$ contains $\tau$ as a pattern, or simply that $\omega$ contains $\tau$, if the diagram of $\tau$ can be obtained by deleting some points from the diagram of $\omega$, i.e. if $\omega$ has a subsequence whose entries have the same relative order as the entries of $\tau$. We say $\omega$ avoids $\tau$ if $\omega$ does not contain $\tau$, and we let $\tilde{S}_N^{/\tau}$ denote the set of all bounded affine permutations that avoid $\tau$. The idea of an affine permutation containing or avoiding a given ordinary permutation was first used by Crites [16].

We can define $\overline{\text{gr}}(\tilde{S}_N^{/\tau})$, $\underline{\text{gr}}(\tilde{S}_N^{/\tau})$, and $\text{gr}(\tilde{S}_N^{/\tau})$ for bounded affine permutations in the same way as for ordinary permutations, though we do not know whether $\text{gr}(\tilde{S}_N^{/\tau})$ exists for every ordinary permutation $\tau$, as it does in the setting of ordinary permutation classes. As we noted above, $\overline{\text{gr}}(\tilde{S}_N^{/\tau})$ is always finite.
Figure 8. A random 4321-avoiding bounded affine permutation of size 500. This was generated by Quynh Vu using a Markov chain Monte Carlo algorithm, under the supervision of the first author.

Figure 9. The permutation 4123 is contained in the permutation 493125876. (This figure was previously published in [29]).

Note that, if $d \mid N$, then every element of $\tilde{S}^{\parallel}_d$ is also an element of $\tilde{S}^{\parallel}_N$. If $\omega$ is an affine permutation of size $N$, then $N$ need not be the smallest possible size of $\omega$. Thus, for enumeration purposes, our count of affine permutations of size $N$ with a given property includes the affine permutations of size $d$ with that property for $d \mid N$. 
2. Avoiding a Decreasing Pattern: Enumeration

It is well known that a permutation avoids the decreasing pattern $m(m-1) \cdots 21$ if and only if it can be partitioned into $m-1$ increasing subsequences. It is also true that an affine permutation avoids $m(m-1) \cdots 21$ if and only if it can be partitioned into $m-1$ periodic increasing subsequences. Since the number of increasing subsequences is more fundamental to our development than is the length of the pattern, we shall write $k$ for $m-1$ in our work, and state our results with $m$ replaced by $k+1$. We denote the decreasing permutation of size $k+1$ by $(k+1) \cdots 1.0$.

Proposition 2.1. Let $\omega$ be an affine permutation of size $N$, and assume that $N \geq k$. Then $\omega$ avoids $(k+1) \cdots 1$ if and only if $[N]$ can be partitioned into $k$ non-empty sets, $[N] = G_1 \cup \cdots \cup G_k$, such that for each $i$ if $G_i = \{ g_{i,1} < g_{i,2} < \cdots < g_{i,n_i} \}$ (where $n_i = |G_i|$) then $\omega(g_{i,1}) < \omega(g_{i,2}) < \cdots < \omega(g_{i,n_i}) < \omega(g_{i,1} + N)$.

Proof. Just as in the case of ordinary permutations, if there exists a partition $[N] = G_1 \cup \cdots \cup G_k$ satisfying the conditions from the proposition statement, then $\omega$ avoids $(k+1) \cdots 1$: indeed, the positions in an occurrence of $(k+1) \cdots 1$ would have to be in $k+1$ different increasing subsequences.

The converse is proved by the same method as in the classical version for ordinary permutations (see [7, Thm. 4.10]). Assume $\omega$ avoids $(k+1) \cdots 1$. For each $a \in \mathbb{Z}$, define the rank of $a$ (in $\omega$) to be the maximum number $r$ such that $a$ is the start of a sequence of $r$ integers $a = a_1 < a_2 < \cdots < a_r$ such that $\omega(a_1) > \omega(a_2) > \cdots > \omega(a_r)$. That is, the rank of $a$ is the maximum length of a decreasing subsequence of $\omega$ that begins with position $a$. Since $\omega$ avoids $(k+1) \cdots 1$, every integer has rank $r$ satisfying $1 \leq r \leq k$. By the definition of affine permutation, $a$ and $a+N$ have the same rank for all $a \in \mathbb{Z}$.

For each $i \in [k]$ define $G_i$ to be the set of integers of rank $i$. Then $\mathbb{Z} = G_1 \cup \cdots \cup G_k$ is a partition of $\mathbb{Z}$, possibly with some blocks empty, with the property that $a \in G_i$ if and only if $a+N \in G_i$. For each $i$ such that $G_i$ is non-empty, $\{ \omega(a) \}_{a \in G_i}$ is a doubly infinite increasing subsequence of $\omega$ (meaning if $a, a' \in G_i$ and $a < a'$ then $\omega(a) < \omega(a')$).

Finally, if we define $G_i = G_i \cap [N]$, then $[N] = G_1 \cup \cdots \cup G_k$ is a partition of $[N]$ satisfying the conditions given in the proposition statement, except that some $G_i$ may be empty. This last detail can be corrected by removing the empty blocks and subdividing the non-empty blocks until there are exactly $k$ of them (this is possible because $N \geq k$).

Here is the first of the two main theorems of this paper. Everything is trivial for $k = 1$, so in the rest of the paper, we shall always assume $k \geq 2$.

Theorem 2.2. Fix $k \geq 1$. As $N \to \infty$,

$$|\tilde{S}_N^{((k+1) \cdots 1)}| \sim k^{2N} \left( \frac{N}{4\pi} \right)^{(k-1)/2} \frac{Z_k^*}{k^{k/2}(k-1)!}$$

(2.1)
where

\[
Z_k^* = \frac{1}{(k-1)!} \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k}{j} (k - 2j)^{k-1} .
\]

\[ (2.2) \]

**Remark 2.3.** We note that \( Z_k^*/2^{k-1} \) is the value of the probability density function of the sum of \( k \) independent uniform random variables on \([0, 1]\) evaluated at its midpoint, \( k/2 \); see “Irwin–Hall distribution” in [24, Sec. 26.9, Eq. (26.48)]. We easily compute \( Z_1^* = 1 \), \( Z_2^* = 2 \), \( Z_3^* = 3 \), \( Z_4^* = 16/3 \), and \( Z_5^* = 115/12 \).

For instance, for \( k = 2 \) this becomes

\[
|\tilde{S}_{N}(321)| \sim \sqrt{\frac{N}{4\pi}} \cdot 4^N ,
\]

and for \( k = 3 \) we obtain

\[
|\tilde{S}_{N}(4321)| \sim \frac{N}{8\pi \sqrt{3}} \cdot 9^N .
\]

For every \( k \), the proper growth rate of \( \tilde{S}_{N}((k+1) \cdots 1) \) is \( k^2 \), the same as for ordinary permutations avoiding \((k+1) \cdots 1\). More precisely, Regev [32] showed that for the latter,

\[
|S_{N}((k+1) \cdots 1)| \sim I_k k^{2N} N^{-(k^2 - 1)/2}
\]

where

\[
I_k = \frac{1}{(2\pi)^k (k+1)^{k+1} (k+1)!} \int_{\mathbb{R}^{k+1}} \prod_{i<j} (x_i - x_j)^2 e^{-(k+1)(x_1^2 + \cdots + x_{k+1}^2)} \, dx_1 \cdots dx_{k+1} .
\]

We remark that \( |\tilde{S}_{N}((k+1) \cdots 1)| / |S_{N}((k+1) \cdots 1)| \) is asymptotically proportional to \( N^{(k^2 + k - 2)/2} \) as \( N \to \infty \).

2.1. The Setup

Here is the setup that we will use to prove Theorem 2.2, relying on the characterization from Proposition 2.1 that a permutation avoids \((k+1) \cdots 1\) if and only if it can be expressed as the union of \( k \) increasing subsequences.

Given positive integers \( n_1, \ldots, n_k \) whose sum is \( N \), let \( \{G_1, \ldots, G_k\} \) and \( \{H_1, \ldots, H_k\} \) be two partitions of \( \{1, \ldots, N\} \) such that \( |G_i| = |H_i| = n_i > 0 \) for each \( i \in [k] \). For each \( i \), write the elements of the sets \( G_i \) and \( H_i \) as

\[
G_i = \{g_{i,1}, g_{i,2}, \ldots, g_{i,n_i}\} \quad \text{where} \quad g_{i,1} < g_{i,2} < \cdots < g_{i,n_i} ,
\]

\[
H_i = \{h_{i,1}, h_{i,2}, \ldots, h_{i,n_i}\} \quad \text{where} \quad h_{i,1} < h_{i,2} < \cdots < h_{i,n_i} .
\]

Finally, let \( \Delta_1, \ldots, \Delta_k \) be integers such that

\[
\Delta_i \in [-n_i, n_i] \quad (i \in [k]) \quad \text{and} \quad \sum_{i=1}^{k} \Delta_i = 0 .
\]
Figure 10. The case $k = 2$: constructing a 321-avoiding ordinary permutation of size 10 from Eq. (2.5). Here $n_1 = 4$ and $n_2 = 6$. Each 1 on the horizontal (respectively, vertical) axis indicates a member of $G_1$ (respectively, $H_1$), while the 2’s indicate members of $G_2$ (and $H_2$). The dots are the points of the plot, as we pair the $j$th element of $G_i$ with the $j$th element of $H_i$. The shaded squares (green for $G_1 \times H_1$ and blue for $G_2 \times H_2$) will be needed when we extend this construction to affine permutations in Fig. 11 (color figure online).

To shorten the notation, we shall write $\langle n \rangle$ to represent the ordered $k$-tuple $(n_1, \ldots, n_k)$, and similarly for $\langle G \rangle$, $\langle H \rangle$, and $\langle \Delta \rangle$. The procedure described in the next several paragraphs will define a function $\Psi$ whose domain $D_0(N)$ is the set of all $(4k)$-tuples $(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle)$ that satisfy the conditions just described, and whose codomain contains $\tilde{S}_N/((k+1) \cdots 1)$. The correspondence $\Psi$ is the key to our main theorem, as we shall outline soon.

At this point, it is useful to pause and observe that we can use $\langle G \rangle$ and $\langle H \rangle$ to define an ordinary permutation $\sigma$ in $\mathcal{S}_N((k+1) \cdots 1)$ by specifying

$$
\sigma(g_{i,j}) = h_{i,j} \text{ for } j = 1, \ldots, n_i \text{ and } i \in [k].
$$

(2.5)

See Fig. 10. Every permutation in $\mathcal{S}_N((k+1) \cdots 1)$ can be created this way, but not uniquely. The most obvious source of non-uniqueness is that we can permute the subscripts of $n_i$, $G_i$, and $H_i$ in $k!$ ways and get the same $\sigma$. This leads to the bound

$$
|\mathcal{S}_N((k+1) \cdots 1)| \leq \frac{1}{k!} \sum_{n_1, \ldots, n_k \geq 1 \atop n_1 + \cdots + n_k = N} \left( \begin{array}{c} N \\ n_1, n_2, \ldots, n_k \end{array} \right)^2.
$$

(2.6)

We note that the permutation of subscripts is not the only reason that the above association is not unique. For example, we can get the identity permutation by taking $H_i = G_i$ for any choice of $\{G_1, \ldots, G_k\}$. Also, notice that if $\sigma(1) = 1$ and $1 \in G_1$, say, then moving the element 1 from $G_1$ to $G_2$ and
moving 1 from $H_1$ to $H_2$ gives a different decomposition of the same $\sigma$ into $k$ increasing parts. For the case $k = 2$, the upper bound of Eq. (2.6) becomes

$$\frac{1}{2} \left\lvert \sum_{k=1}^{N-1} \binom{N}{k}^2 \right\rvert = \frac{1}{2} \binom{2N}{N} - 1,$$

which is an order of $N$ larger than the correct answer, $|S_N(321)| = \binom{2N}{N} / (2N + 1)$ (e.g. Corollary 4.7 of [7]). In contrast, the analogous bound that we shall derive for $|\tS_N/((k+1) \cdots 1)|$ will be asymptotically exact.

Now we describe the procedure that defines the function $\Psi$, which will take a $(4k)$-tuple $(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle)$ in its domain $D_0(N)$ and use it to construct an affine permutation. The asymptotic upper bound of Sect. 2.2 comes from the fact that each permutation in $\tS_N/((k+1) \cdots 1)$ has at least $k!$ preimages in $D_0(N)$ under $\Psi$ (this leads to Eq. (2.11)). The matching asymptotic lower bound of Sect. 2.3 relies on finding a slightly smaller domain $\text{Dom}$ (depending on $N$ as well as some other parameters) that $\Psi$ maps into $\tS_N/((k+1) \cdots 1)$, and on which $\Psi$ is exactly $k!$-to-one. Indeed, plots such as those suggested by Fig. 3 are images of members of $\text{Dom}$.

We first define some notation as well as the domain $D_0(N)$ of $\Psi$.

**Definition 2.4.** (a) For natural numbers $w$ and $N$, let $\text{Seq}(w,N)$ be the set of all $w$-element subsets of $[N]$. We shall typically identify such a set as an increasing subsequence of $1,2,\ldots,N$, as we do in Eq. (2.3).

(b) For $\langle n \rangle \in \mathbb{N}^k$, let

$$D_\Delta(\langle n \rangle) = \left\{ \langle \Delta \rangle \in \mathbb{Z}^k : |\Delta_i| \leq n_i \quad \forall i \in [k], \quad \text{and} \sum_{i=1}^{k} \Delta_i = 0 \right\}.$$

Also, let $Z(n_1, \ldots, n_k) = |D_\Delta(\langle n \rangle)|$.

(c) For $N \in \mathbb{N}$, define the following set of $(4k)$ tuples:

$$D_0(N) = \left\{ (\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle) : \langle n \rangle \in \mathbb{N}^k, \sum_{i=1}^{k} n_i = N, \langle \Delta \rangle \in D_\Delta(\langle n \rangle), \langle G \rangle \text{ and } \langle H \rangle \text{ are partitions of } [N] \text{ such that } |G_i| = |H_i| = n_i \quad \forall i \in [k] \right\}$$

We now explain how to define $\Psi$ on the domain $D_0(N)$. Let $(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle) \in D_0(N)$. For each $i$, extend the definition of $g_{i,j}$ and $h_{i,j}$ from Eq. (2.3) to all integers $j$ periodically, i.e.

$$g_{i,j+tn_i} := g_{i,j} + tN \quad \text{and} \quad h_{i,j+tn_i} := h_{i,j} + tN \quad \text{for } j \in [n_i], t \in \mathbb{Z}. \quad (2.7)$$

Observe that

$$g_{i,j+tn_i}, h_{i,j+tn_i} \in [1 + tN, N + tN] \quad \text{for } j \in [n_i], t \in \mathbb{Z}. \quad (2.8)$$

In particular, for each $i$, we see that $g_{i,j}$ is a strictly increasing function of $j \in \mathbb{Z}$, and that for each $x \in \mathbb{Z}$ there is a unique choice of $i$ and $j$ such that $g_{i,j} = x$ (and similarly for $h_{i,j}$). We define $\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle)$ to be the function $\sigma$ given by

$$\sigma(g_{i,j}) = h_{i,j+\Delta_i} \quad \text{for } j \in \mathbb{Z}, i \in [k]. \quad (2.9)$$

See Fig. 11.
Figure 11. The action of $\Psi$ for $k = 2$ and $N = 10$: the grid on the left is the example from Fig. 10. On the right, a copy of the same grid colouring (without the dots) is placed above and below the original grid, extending the original colouring periodically from $\{1, \ldots, N\}^2$ to $\{1, \ldots, N\} \times \{-N+1, \ldots, 2N\}$. The boundedness condition requires that all points of the plot lie strictly between the two diagonal dashed lines shown. The plotted points on the right illustrate the result of taking $\Delta_1 = 2$ and $\Delta_2 = -2$ in Eq. (2.9). This has the effect that each black (respectively, red) dot moves up two green spaces (respectively, down two blue spaces) from its original position (i.e. in the left grid) (color figure online)
We remark that if $\Delta_i = 0$ for each $i \in [k]$, then $\sigma$ is just the infinite direct sum of the ordinary permutation that we created in Eq. (2.5) above (recall Fig. 10).

**Lemma 2.5.** Let $\tilde{\nu} = ((n)\langle G \rangle, \langle H \rangle, \langle \Delta \rangle) \in D_0(N)$ and let $\sigma = \Psi(\tilde{\nu})$, as defined by Eqs. (2.7) and (2.9) above. Then $\sigma$ is a (not necessarily bounded) affine permutation of size $N$ that avoids $(k+1) \cdots 1$. Moreover, every member of $\tilde{S}_N^//((k+1) \cdots 1)$ can be obtained in this way, i.e. every member of $\tilde{S}_N^//((k+1) \cdots 1)$ is in the image of $\Psi$.

We remark that $\sigma$ in the image of $\Psi$ is not necessarily in $\tilde{S}_N^//$, since there is no guarantee that the constraint $|\sigma(i) - i| < N$ holds for all $i$.

**Proof.** Observe first that Eq. (2.7) actually holds for every integer $j$. By Eq. (2.8) and the subsequent comments, it is apparent that $\sigma$ is a well-defined bijection of $\mathbb{Z}$. Property (i) of Definition 1.1 follows from Eqs. (2.7) and (2.9). For property (ii), let $f_i(r) = \sum_{j=1}^{n_i} h_{i,j} + r$ for $i \in [k]$ and $r \in \mathbb{Z}$. Then

$$f_i(r + 1) - f_i(r) = h_{i,n_i} + r + h_{i,r+1} = N \quad \text{for every } r \in \mathbb{Z}.$$ 

Moreover,

$$\sum_{i=1}^{k} f_i(0) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} h_{i,j} = \sum_{\ell=1}^{N} \ell.$$ 

It follows that

$$\sum_{\ell=1}^{N} \sigma(\ell) = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sigma(g_{i,j}) = \sum_{i=1}^{k} f_i(\Delta_i)$$

$$= \sum_{i=1}^{k} (f_i(0) + \Delta_i N)$$

$$= \sum_{\ell=1}^{N} \ell + \left( \sum_{i=1}^{k} \Delta_i \right) N.$$ 

(2.10)

Thus condition (ii) follows from the fact that $\sum_{i=1}^{k} \Delta_i = 0$. Finally, we know that $\sigma$ avoids $(k+1) \cdots 1$ because $\{\sigma(i)\}_{i \in \mathbb{Z}}$ can be partitioned into $k$ increasing subsequences.

To prove the final statement of the lemma, let $\phi \in \tilde{S}_N^//((k+1) \cdots 1)$. Partition $\phi$ into $k$ nonempty increasing periodic subsequences as in Proposition 2.1, writing $[N] = G_1 \cup \cdots \cup G_k$ with $G_i = \{g_{i,j} : j \in [n_i]\}$ such that

$$1 \leq g_{i,1} < g_{i,2} < \cdots < g_{i,n_i} \leq N \quad \text{and} \quad \phi(g_{i,1}) < \phi(g_{i,2}) < \cdots < \phi(g_{i,n_i}) < \phi(g_{i,1} + N) = \phi(g_{i,1}) + N.$$

Let $H_i$ be the $n_i$-element subset of $[1,N]$ consisting of the elements that are congruent mod $N$ to $\{\phi(g_{i,j}) : 1 \leq j \leq n_i\}$. Write the elements of $H_i$ as in Eq. (2.3). Next, extend the definition of $h_{i,j}$ to all $j \in \mathbb{Z}$ by Eq. (2.7). Then $\phi(g_{i,1}) = h_{i,J}$ for some integer $J = J(i)$. Set $\Delta_i = J(i) - 1$, so that
\[ \phi = \Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle). \] It remains only to show that Eq. (2.4) holds. Since \( \phi \in \bar{S}_N^{\parallel} \), we know that \( |h_{i,j} - g_{i,1}| < N \). In particular, \( h_{i,j} > -N + g_{i,1} \geq -N + 1 \). Thus, by Eq. (2.8) for \( h \), we conclude that \( J \geq 1 - n_i \), i.e. that \( \Delta_i \geq -n_i \). Similarly, since \( \phi(g_{i,0}) = h_{i,J-1} \), we have \( |h_{i,J-1} - g_{i,0}| < N \) and \( h_{i,J-1} < g_{i,0} + N \leq N \), and hence \( J - 1 \leq n_i \), i.e. \( \Delta_i \leq n_i \). Thus \( |\Delta_i| \leq n_i \) for each \( i \). Finally, the equation \( \sum_{i=1}^k \Delta_i = 0 \) follows from Eq. (2.10) and the fact that \( \phi \in \bar{S}_N^{\parallel} \).

### 2.2. The Asymptotic Upper Bound

Recalling Definition 2.4(b), it follows from the final assertion of Lemma 2.5 that

\[
|\bar{S}_N^{\parallel}(\{k+1\} \cdots 1)| \leq \frac{1}{k!} \sum_{n_1,\ldots,n_k \geq 1 \atop n_1 + \cdots + n_k = N} \binom{N}{n_1,n_2,\ldots,n_k}^2 Z(n_1,\ldots,n_k).
\]  

(2.11)

The division by \( k! \) comes from the interchangeability of the the subscripts of \( G_i, H_i, n_i, \) and \( \Delta_i \) (recall that each \( n_i \) is non-zero). The basic idea behind the proof of Theorem 2.2 is to show that this upper bound is asymptotically tight.

The asymptotic behaviour of the sum of Eq. (2.11) without the \( Z \) terms was established in 2009 by Richmond and Shallit [33]:

**Theorem 2.6.** [33] Fix an integer \( k \geq 2 \). Then as \( N \to \infty \),

\[
\sum_{n_1,\ldots,n_k \geq 0 \atop n_1 + \cdots + n_k = N} \binom{N}{n_1,n_2,\ldots,n_k}^2 \sim k^{2N+k/2} (4\pi N)^{(1-k)/2}.
\]

The dominant terms of the sum of Eq. (2.11) are those for which all \( n_i \)'s are approximately equal. This is quantified in the following result, which is a straightforward application of a well-known bound on tail probabilities.

**Lemma 2.7.** Fix an integer \( k \geq 2 \). Fix \( \alpha \in (0,1/k) \). Then for every \( N \) we have

\[
\sum_{n_1,\ldots,n_k \geq 0 \atop n_1 + \cdots + n_k = N} \binom{N}{n_1,n_2,\ldots,n_k}^2 \leq 4 k^{2N+2} e^{-4N\alpha^2}.
\]  

(2.12)

In particular, as \( N \to \infty \),

\[
\sum_{n_1,\ldots,n_k \geq 1 \atop n_1 + \cdots + n_k = N} \binom{N}{n_1,n_2,\ldots,n_k}^2 \sim \sum_{n_1,\ldots,n_k \geq 1 \atop n_1 + \cdots + n_k = N} \binom{N}{n_1,n_2,\ldots,n_k}^2.
\]  

(2.13)

**Proof.** Consider a sequence of \( N \) independent random variables \( X_1,\ldots,X_N \), where each \( X_j \) is chosen uniformly at random from the set \( \{1,\ldots,k\} \). For \( i \in [k] \), let \( Y_i \) be the number of \( X_j \)'s that are equal to \( i \). Then the joint distribution of \( (Y_1,\ldots,Y_k) \) is multinomial with parameters \( N \) and \( p_1=\cdots=p_k=1/k \).
Also, the (marginal) distribution of each \( Y_i \) is binomial with parameters \( N \) and \( p = 1/k \). Thus we have
\[
\sum_{n_1, \ldots, n_k : n_1 + \cdots + n_k = N, \left| n_i - \frac{N}{k} \right| > \alpha N \text{ for some } i} \binom{N}{n_1, \ldots, n_k} k^{-N} = \Pr \left( \left| Y_i - \frac{N}{k} \right| > \alpha N \text{ for some } i \right)
\]
\[
\leq \sum_{i=1}^{k} \Pr \left( \left| Y_i - \frac{1}{k} \right| > \alpha \right)
\]
\[
\leq 2ke^{-2N\alpha^2},
\]
where the last line uses Hoeffding’s Inequality applied to the binomial distribution (Theorem 2 of [20]). The inequality (2.12) follows directly.

The inequality (2.13) follows from (2.12) and Theorem 2.6. □

We shall also need to understand the asymptotics of \( Z(n_1, \ldots, n_k) \) when each \( n_i \) is close to \( N/k \). We remark that \( Z(n_1, \ldots, n_k) \) is the coefficient of \( x^{N} \) (the middle term) in \( \prod_{i=1}^{k} (1 + x + x^2 + \cdots + x^{2n_i}) \). This connection was made in 1876 by Désiré André, when he proved the following result.

Theorem 2.8. For the case \( n_i = n \) for every \( i = 1, \ldots, k \), we have
\[
Z(n_1, \ldots, n_k) = k^{\lfloor kn/(2n+1) \rfloor} \sum_{j=0}^{[k/2]} (-1)^j \binom{k+kn-j(2n+1)-1}{j} j!(k-j)!(kn-j(2n+1))!.
\]

Proof. This result is given in Remarks 60 and 61 of André [1]. In the notation in that paper, \( Z(n_1, \ldots, n_k) = (k,kn)_{2n} \). The same result is presented with a different proof in [12]. □

Corollary 2.9. For the case \( n_i = n \) for every \( i = 1, \ldots, k \), we have
\[
\lim_{n \to \infty} \frac{Z(n_1, \ldots, n_k)}{n^{k-1}} = \frac{1}{(k-1)!} \sum_{j=0}^{[k/2]} (-1)^j \binom{k}{j} (k-2j)^{k-1} = Z^*_k.
\]

Recall that \( Z^*_k \) was defined in Eq. (2.2).

Proof. This follows directly from Theorem 2.8. Notice that when \( k \) is even, the summand for \( j = k/2 \) is 0. □

We are now ready to prove the asymptotic upper bound corresponding to Theorem 2.2.

Proposition 2.10.
\[
\limsup_{N \to \infty} \frac{|\tilde{S}_N^/((k+1) \cdots 1)|}{k^{2N}N^{(k-1)/2}} \leq \frac{Z^*_k}{k^{k/2} (4\pi)^{(k-1)/2} (k-1)!}.
\]

Proof. Let \( \alpha \in (0,1/k) \). The upper bound (2.11) says that
\[
|\tilde{S}_N^/((k+1) \cdots 1)| \leq \frac{1}{k!} \left( \sum_{N,\leq} + \sum_{N,>} \right),
\]
where
\[ \sum_{N, \leq} = \sum_{n_1, \ldots, n_k \geq 1 : n_1 + \cdots + n_k = N, \left| n_i - \frac{N}{k} \right| \leq \alpha N \text{ for all } i} \left( \begin{array}{c} N \\ n_1, \ldots, n_k \end{array} \right)^2 Z(n_1, \ldots, n_k) \]
and
\[ \sum_{N, >} = \sum_{n_1, \ldots, n_k \geq 1 : n_1 + \cdots + n_k = N, \left| n_i - \frac{N}{k} \right| > \alpha N \text{ for some } i} \left( \begin{array}{c} N \\ n_1, \ldots, n_k \end{array} \right)^2 Z(n_1, \ldots, n_k). \]

In view of the bound \( Z(n_1, \ldots, n_k) \leq (2N + 1)^{k-1} \), we see from inequality (2.12) that \( \sum_{N, >} = o(k^{2N}) \) as \( N \to \infty \). We can now turn to the asymptotics of \( \sum_{N, \leq} \).

Let \( u_N = \lfloor \frac{N}{k} + \alpha N \rfloor \). In every term of \( \sum_{N, \leq} \) we know that \( n_i \leq u_N \) for each \( i \), and hence \( Z(n_1, \ldots, n_k) \leq Z(u_N, \ldots, u_N) \) (since \( Z \) is non-decreasing in each \( n_i \)). Thus we have
\[ \sum_{N, \leq} \leq Z(u_N, \ldots, u_N) \sum_{n_1, \ldots, n_k \geq 0 : n_1 + \cdots + n_k = N} \left( \begin{array}{c} N \\ n_1, n_2, \ldots, n_k \end{array} \right)^2. \tag{2.14} \]

By Corollary 2.9, we have \( Z(u_N, \ldots, u_N) \sim (u_N)^{k-1} Z_k^* \) as \( N \to \infty \). Combining this with Theorem 2.6, we obtain
\[ \sum_{N, \leq} \leq N^{k-1} \left( \frac{1}{k} + \alpha \right)^{k-1} Z_k^* \times k^{2N+k/2} (4\pi N)^{(1-k)/2} \times (1 + o(1)). \]

Since \( \sum_{N, >} = o(k^{2N}) \), it follows that
\[ \limsup_{N \to \infty} \frac{\left| \tilde{S}/((k+1)\cdots1) \right|}{k^{2N} N^{(k-1)/2}} \leq \frac{1}{k!} \left( \frac{1}{k} + \alpha \right)^{k-1} Z_k^* k^{k/2} \left( \frac{1}{4\pi} \right)^{(k-1)/2}. \]

Since the positive number \( \alpha \) can be made arbitrarily close to 0, the proposition follows. \( \square \)

2.3. The Asymptotic Lower Bound

We start with some notation. For positive integers \( w \) and \( N \), recall that \( \text{Seq}(w; N) \) is the set of all \( w \)-element subsets of \([N]\). In this section, we shall write a member of \( \text{Seq}(w; N) \) as a \( w \)-element vector with the entries in increasing order: \( \vec{x} = (x(1), x(2), \ldots, x(w)) \), with \( x(1) < \cdots < x(w) \).

In applying the following definition, we shall want \( \alpha \) to be small, and \( A \) and \( B \) to be large.

**Definition 2.11.** Fix \( k \geq 2 \). Let \( w, N \in \mathbb{N} \), and let \( \alpha, A, \) and \( B \) be positive real numbers.

(a) Define
\[ \text{Seq}^* A(w; N) = \left\{ \vec{x} \in \text{Seq}(w; N) : \left| x(\ell) - \ell \frac{N}{w+1} \right| < A \text{ for all } \ell \in [w] \right\} \]
(Roughly speaking, a \( w \)-element subset of \([N]\) is in \( \text{Seq}^* (w; N) \) if its elements are within distance \( A \) of a uniform spacing configuration over the interval \([0, N]\).)

(b) For \( \langle n \rangle \in \mathbb{N}^k \) such that \( n_1 + \cdots + n_k = N \), define

\[
\mathcal{V}^*_N (\langle n \rangle) = \{(G_1, \ldots, G_k) : \{G_1, \ldots, G_k\} \text{ is a partition of } \{1, \ldots, N\}
\text{ with } G_i \in \text{Seq}^*(n_i; N) \text{ for each } i \in [k]\}.
\]

(c) Let \( \mathcal{M}(N, \alpha) \) be the set of \( \langle n \rangle \in \mathbb{N}^k \) such that \( n_1 + \cdots + n_k = N \) and

\[
\left| n_i - \frac{N}{k} \right| \leq \alpha N \text{ for each } i \in [k].
\]  \hfill (2.15)

(d) Let \( \mathcal{D}_1 (N, \alpha, A, B) \) be the set of all \( \langle \langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle \rangle \in \mathcal{D}_0 (N) \) with the additional constraints that \( \langle n \rangle \in \mathcal{M}(N, \alpha) \), \( |\Delta_i| < n_i - B \) for every \( i \in [k] \), and \( \langle G \rangle, \langle H \rangle \in \mathcal{V}^*_N (\langle n \rangle) \).

The following lemma establishes some regularity properties of the (mildly) reduced domain \( \mathcal{D}_1 (N, \alpha, A, B) \) of \( \Psi \).

**Lemma 2.12.** Let \( k \geq 2 \). Let \( N \in \mathbb{N} \), let \( A, B > 0 \), and let \( \alpha \in (0, 1/k) \). Let \( \vec{v} = (\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle) \in \mathcal{D}_1 (N, \alpha, A, B) \). Let \( \sigma = \Psi(\vec{v}) \) be the affine permutation defined as in Eq. (2.9). Then for every \( i \in [k] \) and \( j \in \mathbb{Z} \),

\[
\left| g_{i,j} - j \frac{N}{n_i} \right| < A + \frac{k}{1 - k\alpha},
\]  \hfill (2.16)

\[
\left| h_{i,j} - j \frac{N}{n_i} \right| < A + \frac{k}{1 - k\alpha},
\]  \hfill (2.17)

\[
\left| \sigma(g_{i,j}) - \left( g_{i,j} + \frac{N}{n_i} \Delta_i \right) \right| < 2 \left( A + \frac{k}{1 - k\alpha} \right),
\]  \hfill (2.18)

and

\[
\left| \sigma(g_{i,j}) - g_{i,j} \right| < N - \frac{kB}{1 + k\alpha} + 2A + \frac{2k}{1 - k\alpha}.
\]  \hfill (2.19)

In particular, \( \sigma \) is a bounded affine permutation if \( \frac{kB}{1 + k\alpha} \geq 2A + \frac{2k}{1 - k\alpha} \).

**Proof.** First we observe that for every \( \ell \in [n_i] \),

\[
\left| \ell \frac{N}{n_i} - \frac{\ell N}{n_i(n_i + 1)} \right| = \frac{\ell N}{n_i(n_i + 1)} < \frac{N}{n_i} \leq \frac{N}{\frac{N}{k} - \alpha N} = \frac{k}{1 - k\alpha}.
\]  \hfill (2.20)

This bound and the definition of \( \mathcal{V}^*_N (\langle n \rangle) \) imply Eqs. (2.16) and (2.17) for \( j \in [n_i] \), and the extension to all \( j \in \mathbb{Z} \) follows from Eq. (2.7). Equation (2.18) follows from Eqs. (2.9), (2.16), and (2.17). Equation (2.19) follows from Eq. (2.18) and

\[
\left| \Delta_i \frac{N}{n_i} \right| \leq \frac{(n_i - B)N}{n_i} \leq \left( 1 - \frac{B}{\frac{N}{k} + \alpha N} \right) N = N - \frac{kB}{1 + k\alpha}
\]

(using Definition 2.11(c,d)). The final assertion of the lemma is a consequence of Eq. (2.19). \( \square \)

We shall now define the restricted domain \( \text{Dom} \) on which \( \Psi \) is \( k! \)-to-one.
Definition 2.13. Let \( N \in \mathbb{N} \), let \( A, B > 0 \), and let \( \alpha \in (0, 1/k) \). Let \( \text{Dom} = \text{Dom}(N, \alpha, A, B) \) be the set of \((4k)\)-tuples \((\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle)\) in \( \mathcal{D}_1(N, \alpha, A, B) \) that also satisfy
\[
\left| \frac{\Delta_i N}{n_i} - \frac{\Delta_{i'} N}{n_{i'}} \right| > 4 \left( 2A + \frac{2k}{1-k\alpha} \right) \quad \text{whenever } i, i' \in [k] \text{ and } i \neq i'.
\]

Lemma 2.14. Let \( N \in \mathbb{N} \), \( A, B > 0 \), and let \( \alpha \in (0, 1/k) \). Then the restriction of the function \( \Psi \) to \( \text{Dom} \) is exactly \( k! \)-to-1.

Proof. Let \( \tilde{v} = (\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle) \in \text{Dom} \) and let \( \sigma = \Psi(\tilde{v}) \). We also define the (truncated) plot of \( \sigma \) to be
\[
\text{Plot}[\sigma] := \{ (i, \sigma(i)) : i \in [N] \}.
\]
For each real \( b \), define
\[
\text{Strip}[b] := \left\{ (x, y) \in \mathbb{R}^2 : |y - (x + b)| < 2A + \frac{2k}{1-k\alpha} \right\},
\]
a diagonal strip shifted vertically by \( b \). For each \( i \in [k] \), Eq. (2.18) of Lemma 2.12 tells us that the points \( (g_{i,j}, \sigma(g_{i,j})) \) \((j = 1, \ldots, n_i)\) are all in \( \text{Strip}[\Delta_i N/n_i] \). Hence
\[
\text{Plot}[\sigma] \subseteq \bigcup_{i=1}^k \text{Strip} \left[ \frac{\Delta_i N}{n_i} \right].
\]
The purpose of the condition (2.21) is to ensure that the \( k \) strips \( \text{Strip}[\Delta_i N/n_i] \) are not only disjoint but also are separated by at least the width of a strip.

For any real \( b \), the strip \( \text{Strip}[b] \) cannot intersect more than one of the \( k \) strips \( \text{Strip}[\Delta_i N/n_i] \). Therefore for any choice of \( b_1, \ldots, b_k \) such that \( \text{Plot}[\sigma] \) is contained in \( \bigcup_{i=1}^k \text{Strip}[b_i] \), the partition of the \( N \) points of \( \text{Plot}[\sigma] \) into the \( k \) parts \( \text{Plot}[\sigma] \cap \text{Strip}[b_i] \) \((i \in [k])\) must be the same partition as the one given by \( \text{Plot}[\sigma] \cap \text{Strip}[\Delta_i N/n_i] \) \((i \in [k])\), up to permutation of the \( k \) parts. This partition determines \( G_i \) (the first coordinates of the points in the \( i \)th part) and \( H_i \) (the second coordinates of the points, modulo \( N \)). Finally, \( \Delta_i \) is determined by Eq. (2.9). Thus the lemma is proved. \( \square \)

Corollary 2.15. Let \( N \in \mathbb{N} \), \( A, B > 0 \), and let \( \alpha \in (0, 1/k) \). Assume \( \frac{kB}{1+k\alpha} \geq 2A + \frac{2k}{1-k\alpha} \). Then the function \( \Psi \) maps \( \text{Dom}(N, \alpha, A, B) \) into \( \tilde{S}^{\parallel}_N((k+1) \cdots 1) \), and
\[
|\tilde{S}^{\parallel}_N((k+1) \cdots 1)| \geq \frac{1}{k!} |\text{Dom}(N, \alpha, A, B)|.
\]
Proof. The first assertion follows from Lemma 2.5 and the last sentence in the statement of Lemma 2.12. The second assertion follows from Lemma 2.14. \( \square \)

Our job now is to estimate the size of \( \text{Dom}(N, \alpha, A, B) \).

Lemma 2.16. Fix \( k \geq 2 \). Fix \( A > 0 \), \( B > 0 \), and \( \alpha \in (0, 1/k) \). For natural numbers \( n_1, \ldots, n_k \), let \( \mathcal{W}(\alpha, A, B, \langle n \rangle) \) be the set of ordered \( k \)-tuples \( \langle \Delta_1, \ldots, \Delta_k \rangle \)
of integers whose sum is 0 and which satisfy Eq. (2.21) (with \(N = n_1 + \cdots + n_k\)) as well as \(|\Delta_i| \leq n_i - B\) for each \(i\). Then we have

\[
|\text{Dom}(N, \alpha, A, B)| = \sum_{\langle n \rangle \in \mathfrak{M}(N, \alpha)} |V^{**A}(\langle n \rangle)|^2 |W(\alpha, A, B, \langle n \rangle)|. \tag{2.22}
\]

Moreover, let \(t_N = \lfloor N/\alpha \rfloor - \alpha N - B\) and \(\Theta_N = 8A + 8k/(1 - k\alpha)\) (which is the right-hand side of Eq. (2.21)). Then for every \(\langle n \rangle \in \mathfrak{M}(N, \alpha)\), we have

\[
|W(\alpha, A, B, \langle n \rangle)| \geq Z(t_n, \ldots, t_N) - \frac{k}{2}(2N)^{k-2}(2\Theta_N + 1). \tag{2.23}
\]

**Proof.** Equation (2.22) follows from the definition of Dom.

Let \(W^{-}\) be the set of ordered \(k\)-tuples \(\langle \Delta \rangle\) of integers whose sum is 0 and satisfy \(|\Delta_i| \leq n_i - B\) for each \(i\). By our assumptions, we have \(n_i - B \geq t_N\) for each \(i\), and hence \(|W^{-}| \geq Z(t_n, \ldots, t_N)\) (since \(Z\) is nondecreasing in each argument). Now, for each two-element subset \(\{i, i'\}\) of \([k]\), the number of \(k\)-tuples \(\langle \Delta \rangle\) in \(W^{-}\) that violate Eq. (2.21) is at most \((2n_i)(2\Theta_N + 1)(2N)^{k-3}\) (first choose \(\Delta_i\), then \(\Delta_{i'}\), then \(\Delta_j\) for \(k - 3\) of the remaining indices \(j\) in \([k]\); the final \(\Delta_j\) is determined because \(\sum_j \Delta_j = 0\)). Equation (2.23) follows.

The main task that remains is to get a lower bound on \(|V^{**A}(\langle n \rangle)|\). This is accomplished by the following lemma. It is an adaptation of part of Lemma 21 in [28].

**Lemma 2.17.** Fix \(k \geq 2\). Let \(A > 0\) and let \(\alpha \in (0,1/k)\). Then there exist positive constants \(C(\alpha)\) and \(\tilde{N}(\alpha)\) such that

\[
\frac{V^{**A}(n_1, \ldots, n_k)}{\binom{n_1 + \cdots + n_k}{N}} \geq 1 - \frac{C(\alpha) N^{3/2}}{A^2} \quad \text{whenever } N \geq \tilde{N}(\alpha) \text{ and } \langle n \rangle \in \mathfrak{M}(N, \alpha).
\]

**Proof.** We shall prove the lemma by converting it into a probabilistic statement. Fix \(N\), and choose \(\langle n \rangle \in \mathfrak{M}(N, \alpha)\). Now, choose \((G_1, \ldots, G_k)\) uniformly at random from the collection of all \(\binom{N}{n_1, \ldots, n_k}\) partitions of \(\{1, \ldots, N\}\) for which the \(i\)th part has size \(n_i\). For each \(i\), by symmetry, the random set \(G_i\) is uniformly distributed on the collection of all \(n_i\)-element subsets of \(\{1, \ldots, N\}\). It follows that

\[
1 - \frac{V^{**A}(n_1, \ldots, n_k)}{\binom{n_1, \ldots, n_k}{N}} = \Pr\left(G_i \notin \text{Seq}^{*A}(n_i, N) \text{ for some } i\right)
\leq \sum_{i=1}^{k} \Pr\left(G_i \notin \text{Seq}^{*A}(n_i, N)\right)
= \sum_{i=1}^{k} \left(1 - \frac{|\text{Seq}^{*A}(n_i; N)|}{\binom{N}{n_i}}\right).
\]

We shall complete the proof by deriving an upper bound on \(1 - |\text{Seq}^{*A}(w; N)|/\binom{N}{w}\), assuming that \(\frac{1}{k} - \alpha \leq \frac{w}{N} \leq \frac{1}{k} + \alpha\) (which is satisfied for \(w = n_i\), since \(\langle n \rangle \in \mathfrak{M}(N, \alpha)\)).
Let \( p \in (0, 1) \). Let \( X_1, X_2, \ldots \) be a sequence of independent random variables having the geometric distribution with parameter \( p \); that is, \( \Pr(X_\ell = \ell) = p(1-p)^{\ell-1} \) for \( \ell = 1, 2, \ldots \). Next, let \( T_i = X_1 + X_2 + \cdots + X_i \) for each \( i \). These random variables have negative binomial distributions

\[
\Pr(T_{j+1} = \ell + 1) = \binom{\ell}{j} p^{j+1} (1-p)^{\ell-j} \quad \text{for } \ell \geq j. \tag{2.24}
\]

Moreover, for any \( \vec{x} \in \text{Seq}(w; N) \) (writing \( x(0) = 0 \) and \( x(w+1) = N+1 \)),

\[
\Pr(T_\ell = x(\ell) \text{ for } \ell = 1, \ldots, w \mid T_{w+1} = N+1) = \frac{\prod_{\ell=1}^{w+1} p(1-p)^{x(\ell)-x(\ell-1)-1}}{\binom{N}{w} p^{w+1}(1-p)^{N-w}} = \binom{N}{w}^{-1}. \tag{2.25}
\]

Equation (2.25) says that the conditional distribution of \((T_1, \ldots, T_w)\), given that \( T_{w+1} = N+1 \), is precisely the uniform distribution on \( \text{Seq}(w; N) \). This assertion is true for any \( p \). Let us now fix \( p = (w+1)/N \); we shall soon see why this is a convenient choice.

By Eq. (2.25),

\[
\frac{|\text{Seq}^*_A(w; N)|}{\binom{N}{w}} = \Pr(|T_\ell - \ell/p| < A \text{ for } \ell = 1, \ldots, w \mid T_{w+1} = N+1),
\]

and, therefore,

\[
0 \leq 1 - \frac{|\text{Seq}^*_A(w; N)|}{\binom{N}{w}} \leq \frac{\Pr(\max_{\ell=1,\ldots,w}|T_\ell - \ell/p| \geq A)}{\Pr(T_{w+1} = N+1)} \tag{2.26}
\]

From Stirling’s Formula \( m! \sim \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \), we see that there is a constant \( C_s > 0 \) such that

\[
\frac{1}{C_s} \frac{m^{m+1/2}}{e^m} \leq m! \leq C_s \frac{m^{m+1/2}}{e^m} \quad \text{for every positive integer } m.
\]

It follows from these bounds and Eq. (2.24) that

\[
\Pr(T_{w+1} = N+1) = \frac{N!}{w!(N-w)!} \frac{(w+1)^{w+1}(N-w-1)^{N-w}}{N^{N+1}} \geq \frac{N^{N+1/2}}{C_s^{w+1/2}w^{w+1/2}(N-w)^{N-w+1/2}} \frac{(w+1)^{w+1}(N-w-1)^{N-w}}{N^{N+1}} \geq \frac{\sqrt{w}}{C_s^3 \sqrt{N} \sqrt{N-w}} \left(\frac{w+1}{w}\right)^{w+1} \left(1 - \frac{1}{N-w}\right)^{N-w}. \tag{2.27}
\]

By calculus, one can show that \( (1 - \frac{1}{\ell})^t \geq \frac{1}{4} \) whenever \( t \geq 2 \). Therefore, we conclude from (2.27) that
\[
\Pr(T_{w+1} = N + 1) \geq \frac{1}{C_3^3} \sqrt{\frac{\frac{1}{k} - \alpha}{(1 - \frac{1}{k} + \alpha) N}} \times 1 \times \frac{1}{4}
\]

if \(N - w \geq 2\) and \(\frac{1}{k} - \alpha \leq \frac{w}{N} \leq \frac{1}{k} + \alpha\).

(2.28)

Observe that under the constraints on \(w/N\), the condition \(N \geq \frac{2}{(1 - \frac{1}{k} - \alpha)}\) implies \(N - w \geq 2\).

Since the random variables \(X_i\) have mean \(1/p\) and variance \((1 - p)/p^2\), we also have

\[
E(T_{\ell}) = \frac{\ell}{p} \quad \text{and} \quad \text{Var}(T_{\ell}) = \frac{\ell(1 - p)}{p^2}.
\]

For the numerator of the right-hand side of Eq. (2.26), we use Kolmogorov’s Inequality (see for example section IX.7 of [18]), which may be viewed as a strengthening of Chebychev’s inequality that is applicable to sums of independent random variables.

\[
\Pr\left(\max_{\ell=1,\ldots,w} |T_{\ell} - \ell/p| \geq A\right) \leq \frac{\text{Var}(T_w)}{A^2} = \frac{w(1 - p)/p^2}{A^2} \leq \frac{wN^2}{(w + 1)^2 A^2} < \frac{N^2}{wA^2} \leq \frac{N}{\left(\frac{1}{k} - \alpha\right) A^2}.
\]

(2.29)

Applying Eqs. (2.28) and (2.29) to Eq. (2.26) shows that

\[
1 - \frac{|\text{Seq}^A_w(N; N)|}{\binom{N}{w}} \leq \frac{4C_3^3 N^{3/2}}{(\frac{1}{k} - \alpha) A^2} \sqrt{\frac{1 - \frac{1}{k} + \alpha}{\frac{1}{k} - \alpha}}.
\]

Taking \(\tilde{N}(\alpha) = 2/\left(1 - \frac{1}{k} - \alpha\right)\) and \(C(\alpha) = 4kC_3^3(1 - \frac{1}{k} + \alpha)(\frac{1}{k} - \alpha)^{-3}\), the proof of Lemma 2.17 is now complete.

We can now complete the proof of Theorem 2.2.

**Proposition 2.18.** Fix \(k \geq 2\). Then

\[
\liminf_{N \to \infty} \frac{|\text{Seq}^A_w(N; N)|}{k^{2N} N^{(k-1)/2}} \geq \frac{Z_k^*}{k^{k/2} (4\pi)^{(k-1)/2} (k - 1)!}.
\]

**Proof.** Let \(\alpha \in (0, 1/(3k))\) (we are really interested in the limit as \(\alpha\) decreases to 0). Let \(A = \alpha N\) and \(B = 2\alpha N\). For each \(\langle n \rangle \in \mathcal{F}(N, \alpha)\), the inequality of Eq. (2.23) becomes

\[
|W(\alpha, A, B, \langle n \rangle)| \geq t_N^{k-1} Z_k^*(1 + o(1)) - \binom{k}{2} \left[2(2N)^{k-2}(16\alpha N) + O(N^{k-2})\right]
\]

(2.30)

with the help of Corollary 2.9. Recalling that \(t_N = \lfloor (N/k) - \alpha N - B \rfloor\), we obtain from Eq. (2.22) and Lemma 2.17 that
\[ |\text{Dom}(N, \alpha, A, B)| \geq \sum_{\langle n \rangle \in \mathfrak{n}(N, \alpha)} N^{n_1, \ldots, n_k} \left( 1 - \frac{C(\alpha)}{\alpha^2 \sqrt{N}} \right)^2 \times N^{k-1} \left[ \left( \frac{1}{k} - 3\alpha \right)^{k-1} Z_k^* - \left( \frac{k}{2} \right)^{2k+2} \alpha + o(1) \right]. \] (2.31)

Observe that the assumption of Corollary 2.15 holds for large $N$ because $B = 2A$. Therefore, applying Corollary 2.15, Eq. (2.13), and Theorem 2.6 to Eq. (2.31) gives

\[
\lim_{N \to \infty} \frac{|\tilde{S}_N^k((k+1) \cdots 1)|}{N^{k-1} k^{2N+k/2} (4\pi N)^{(1-k)/2}} \geq \lim_{N \to \infty} \frac{|\text{Dom}(N, \alpha, \alpha N, 2\alpha N)|}{k! N^{k-1} k^{2N+k/2} (4\pi N)^{(1-k)/2}} \geq \frac{1}{k!} \left[ \left( \frac{1}{k} - 3\alpha \right)^{k-1} Z_k^* - \left( \frac{k}{2} \right)^{2k+2} \alpha \right].
\] (2.32)

Since this inequality holds for arbitrarily small positive $\alpha$, the proposition follows. \hfill \Box

Finally, Theorem 2.2 follows immediately from Propositions 2.10 and 2.18.

### 3. Weak Convergence

In this section, we present a convergence result for $\tilde{S}_N^k((k+1) \cdots 1)$ in the spirit of permutons. Section 3.1 describes the measure-theoretic framework that we use, including an introduction to the Wasserstein distance, and presents the formal statement of the main theorem of this section along with the strategy of its proof. Section 3.2 presents some basic properties of Wasserstein distances that we shall need, particularly in the context of mixtures (i.e. convex combinations) of probability measures. Section 3.3 proves the main result, following the strategy described in Sect. 3.1.

#### 3.1. Overview and Statement of the Main Result

We start with some terminology and notation about measures. We denote the set of all probability measures on a set $\chi$ by $\text{PM}(\chi)$. (We should refer to the set of all probability measures on a measurable space, but the $\sigma$-algebra associated with $\chi$ will always be implicit and unambiguous.) For $x \in \chi$, let $\delta_x$ be the measure on subsets of $\chi$ that assign value 1 to every set containing the point $x$ and value 0 to every other set. We call $\delta_x$ the “point mass at $x$.”

For a permutation $\sigma$ of size $N$, the “empirical measure of $\sigma$” is the measure $\mu_\sigma$ on $\mathbb{R}^2$ defined by

\[
\mu_\sigma := \frac{1}{N} \sum_{i=1}^N \delta_{i, \sigma(i)}. \tag{3.1}
\]
Observe that $\mu_\sigma$ has total mass 1, i.e. it is a probability measure. We can think of $\mu_\sigma$ as describing the selection of one point of the plot of $\sigma$ uniformly at random. We also define the scaled empirical measure $\hat{\mu}_\sigma$ by scaling $[0, N]^2$ down to the unit square:

$$
\hat{\mu}_\sigma := \frac{1}{N} \sum_{i=1}^{N} \delta_{(i/N, \sigma(i)/N)}.
$$

(3.2)

A permuton is a probability measure on $[0, 1]^2$ whose marginal distributions are each the uniform measure on $[0, 1]$ (in the sense of Eq. (3.3) below). A fruitful and fascinating direction of recent research is to examine probability measures on interesting classes of permutations that converge weakly to permutons [19, 21, 25, 31] or more generally to random permutons [3, 9–11].

We shall also use Eqs. (3.1) and (3.2) to define $\mu_\sigma$ and $\hat{\mu}_\sigma$ for bounded affine permutations $\sigma$ in $\tilde{S}_N//$. (We only use the point masses $(i, \sigma(i))$ for $i \in [N]$.) We will be interested in weak limits of these measures, but the limits cannot be permutons because they are not restricted to the unit square. Rather, $\hat{\mu}_\sigma$ and the limits will be measures on the parallelogram

$$
\Diamond = \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, |y - x| \leq 1 \right\}.
$$

For $\sigma \in \tilde{S}_N//$, we have $\hat{\mu}_\sigma \in \text{PM}(\Diamond)$.

**Remark 3.1.** Rather than having marginal distributions that are both uniform, our weak limits $\mu$ of scaled empirical measures of bounded affine permutations will have the property that for every Borel subset $B$ of $[0, 1]$, the values of $\mu(B \times \mathbb{R})$ and $\mu \left( [0, 1] \times \bigcup_{j \in \mathbb{Z}} (B + j) \right)$ both equal the Lebesgue measure of $B$. This is because for every affine permutation $\sigma$ of size $N$, the function on $[N]$ defined by $i \mapsto 1 + (\sigma(i) \mod N)$ is an ordinary permutation.

**Remark 3.2.** Another way to represent an ordinary permutation as a probability measure on the unit square is to replace each point mass $\delta_{(i/N, \sigma(i)/N)}$ in Eq. (3.2) by the uniform probability distribution on the square $\left[ \frac{i-1}{N}, \frac{i}{N} \right] \times \left[ \frac{\sigma(i)-1}{N}, \frac{\sigma(i)}{N} \right]$. With this change, the right-hand side of Eq. (3.2) would be a permuton rather than a discrete measure. An analogous approach for affine permutations would use uniform distributions on scaled-down copies of the parallelogram $\Diamond$ instead of on squares of side $\frac{1}{N}$. In the proof strategy of the present paper, however, this approach would be a bit more awkward than working with point masses.

**Remark 3.3.** One useful approach to proving weak convergence in the permuton framework is by proving convergence of all pattern densities [3]. This has been effective where exact calculations are possible for the permutation classes concerned. We pursue a different strategy in this paper, focusing instead on a more global description of the permutation structure. It seems likely that one could use the pattern density approach to prove the results in this section, but there would still remain numerous technical details that would need comparable effort in both approaches.
Our situation is more complicated than the preceding setup, because we need to think in terms of random measures. In particular, we are interested in what $\hat{\mu}_\sigma$ typically looks like for a randomly chosen $\sigma$ in $\tilde{S}_N^\prime((k+1)\cdots 1)$. We formalize this by considering probability measures on the set of probability measures; that is, our random measures will be members of $\text{PM}(\text{PM}(\Diamond))$. As $\delta_x$ is in $\text{PM}(\Diamond)$ when $x \in \Diamond$, so we have that $\delta_x$ is in $\text{PM}(\text{PM}(\Diamond))$ when $x \in \text{PM}(\Diamond)$. For example, the measure $\nu = 0.5\delta_{\hat{\mu}_{\oplus 132}} + 0.5\delta_{\hat{\mu}_{\oplus 321}}$ is the random measure that is equally likely to produce the scaled empirical measure of either $\oplus 132$ or $\oplus 321$. If $A$ is a (nonempty finite) set of bounded affine permutations, we shall write $\hat{\mu}[A]$ to denote the scaled empirical measure $\hat{\mu}_\sigma$ where $\sigma$ is chosen uniformly at random from $A$. That is,

$$\hat{\mu}[A] := \frac{1}{|A|} \sum_{\sigma \in A} \delta_{\hat{\mu}_\sigma}.$$ 

Observe that $\hat{\mu}[A]$ is in $\text{PM}(\text{PM}(\Diamond))$, whereas $|A|^{-1} \sum_{\sigma \in A} \hat{\mu}_\sigma$ is in $\text{PM}(\Diamond)$.

We shall use Wasserstein distance to show weak convergence. Wasserstein distance is a metric on probability measures (on a given metric space) that corresponds to the topology of weak convergence (provided that the underlying metric space is bounded); see for example Theorem 5.6 of [13]. It is also called the transportation distance, based on the following interpretation. Think of a probability measure $\nu$ on a metric space as representing how a unit of mass is distributed over the space; that is, $\nu(A)$ is the amount of mass in the set $A$. If it costs one dollar to transport one unit of mass for one unit of distance, then the Wasserstein distance between two probability measures $\nu_1$ and $\nu_2$ is the lowest possible cost of a scheme that moves the mass from the initial distribution $\nu_1$ and ends up with the distribution $\nu_2$.

The Wasserstein distance is defined as follows. Let $(\chi, \rho)$ be a metric space. (In this paper, we shall need the example that $\chi$ is the parallelogram $\Diamond$ and $\rho$ is Euclidean distance; and we shall also need the example that $\chi$ is $\text{PM}(\Diamond)$, with $\rho$ being the Wasserstein distance on $\text{PM}(\Diamond)$.) Let $\nu_1$ and $\nu_2$ be two probability measures on $\chi$. Let $\text{Joint}(\nu_1, \nu_2)$ be the set of all probability measures $\mathcal{J}$ on $\chi \times \chi$ whose marginal distributions are $\nu_1$ and $\nu_2$, i.e.

$$\nu_1(B) = \int_{B \times \chi} \mathcal{J}(dx, dy) \quad \text{and}$$

$$\nu_2(B) = \int_{\chi \times B} \mathcal{J}(dx, dy) \quad \text{for every Borel set } B \subset \chi. \quad (3.3)$$

Then the Wasserstein distance between $\nu_1$ and $\nu_2$ is defined to be

$$\text{Wass}(\nu_1, \nu_2) = \inf \left\{ \int_{\chi} \int_{\chi} \rho(x, y) \mathcal{J}(dx, dy) : \mathcal{J} \in \text{Joint}(\nu_1, \nu_2) \right\}. \quad (3.4)$$

That is, $\text{Wass}(\nu_1, \nu_2)$ is the infimum of $E(\rho(X_1, X_2))$ over all jointly distributed pairs of random variables $(X_1, X_2)$ on $\chi \times \chi$ where $X_1$ and $X_2$ have distributions $\nu_1$ and $\nu_2$, respectively. It is known that this infimum is always attained by some joint distribution $\mathcal{J}$ (e.g. Lemma 5.2 of [13]). We shall also use the following convention: if $Y$ and $Z$ are two $\chi$-valued random variables with respective
probability distributions $\nu_Y$ and $\nu_Z$, then we may write $\text{Wass}(Y, Z)$ to denote $\text{Wass}(\nu_Y, \nu_Z)$.

We shall use the following abbreviating notation. We shall write $\text{PM}_1$ for $\text{PM}(\diamondsuit)$, and $\text{PM}_2$ for $\text{PM}(\text{PM}(\diamondsuit))$. Correspondingly, we shall write $\text{Wass}_1$ to denote the Wasserstein distance on $\text{PM}_1$ (determined by the Euclidean metric on $\diamondsuit$), and $\text{Wass}_2$ for the Wasserstein distance on $\text{PM}_2$ (determined by the $\text{Wass}_1$ metric on $\text{PM}(\diamondsuit)$).

Remark 3.4. In general, defining weak convergence in $\text{PM}(X)$ requires specifying a topology on $X$, or equivalently specifying which functions on $X$ are continuous. When $X$ is $\text{PM}(Y)$ for some set $Y$, we need to specify the topology of convergence of measures on $Y$. Weak convergence, corresponding to Wasserstein metric on $X = \text{PM}(Y)$, is a standard choice, and this is our choice. But there are other possibilities, such as Total Variation. The choice of topology on $\text{PM}(Y)$ is a separate decision from the choice of topology on $Y$.

Definition 3.5. Let $\text{Unif}(\mathcal{H})$ denote the uniform distribution on a set $\mathcal{H}$, and let $\mathcal{P}_{\text{Unif}(\mathcal{H})}$ denote the corresponding probability measure. The set $\mathcal{H}$ will always be bounded, and it will be of one of two kinds: a discrete set (i.e. a finite set), or a continuous set of dimension $r$ (i.e. a Borel subset of an $r$-dimensional affine subset of $\mathbb{R}^d$ $(0 < r \leq d$) that has non-zero $r$-dimensional Lebesgue measure). If $\mathcal{H}$ is continuous, then the “uniform distribution” on $\mathcal{H}$ refers to the normalized restriction of $r$-dimensional Lebesgue measure to $\mathcal{H}$.

Now we shall define the random measure $\lambda^{Q_0}$, which, as we shall see, is the weak limit of $\hat{\mu}[\tilde{S}_N/((k+1)\cdots 1)]$ as $N \to \infty$.

Definition 3.6. (a) Given $z \in [-1, 1]$, let $\lambda_z \in \text{PM}_1$ be the probability measure on $\diamondsuit$ that is uniformly distributed on the line segment from $(0, z)$ to $(1, 1+z)$. (Observe that the union of all such line segments is $\diamondsuit$.)

(b) Given $\langle z \rangle = (z_1, \ldots, z_k) \in [-1, 1]^k$, define $\lambda(\langle z \rangle) \in \text{PM}_1$ by

$$\lambda(\langle z \rangle) := \frac{1}{k} \sum_{i=1}^k \lambda_{z_i},$$

the probability measure uniformly distributed on the $k$ line segments in $\diamondsuit$ of slope 1 with $y$-intercepts $z_1, \ldots, z_k$.

(c) Given a set $Q \subset [-1, 1]^k$ (discrete or continuous), let $\lambda^Q \in \text{PM}_2$ be the random measure given by $\lambda(\beta)$ where $(\beta_1, \ldots, \beta_k)$ is uniformly distributed on $Q$. That is, in particular,

$$\lambda^Q = \frac{1}{|Q|} \sum_{\langle z \rangle \in Q} \delta_{\lambda(\langle z \rangle)} \quad \text{if } Q \text{ is discrete}.$$

(d) Define $Q_0 := \{ \langle x \rangle \in [-1, 1]^k : \sum_{i=1}^k x_i = 0 \}$.

The main result of this section, Theorem 3.7 below, states that $\hat{\mu}[\tilde{S}_N/((k+1)\cdots 1)]$ converges weakly to $\lambda^{Q_0}$. Intuitively, this result means that the plot of a random element of $\tilde{S}_N/((k+1)\cdots 1)$ (scaled down to the unit
square) looks like the support of $\lambda^{Q_0}$, which consists of $k$ lines of slope 1 with $y$-intercepts chosen randomly from $[-1,1]$ subject to the constraint that their sum is 0. We shall prove this using Wasserstein distances.

**Theorem 3.7.** $\text{Wass}_2(\hat{\mu}[\tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)], \lambda^{Q_0})$ converges to 0 as $N \to \infty$. That is, the sequence of random measures $\hat{\mu}[\tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)]$ converges weakly to $\lambda^{Q_0}$, with respect to the topology of weak convergence on $\text{PM}(\Diamond)$.

**Remark 3.8.** With reference to Remark 3.4, the weak convergence of $\hat{\mu}[\tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)]$ to $\lambda^{Q_0}$ does not hold with respect to the total variation topology on $\text{PM}(\Diamond)$. This is because the set of points in $\Diamond$ with rational coordinates has probability 1 under every $\hat{\mu}_\sigma$ but has probability 0 under every $\lambda_z$, and hence the total variation distance between $\hat{\mu}_\sigma$ and $\lambda(z)$ is always 1.

Here, in brief, are the main parts of the strategy of the proof.

Step 1. First, recall the set $\text{Dom} \equiv \text{Dom}(N,\alpha,A,B)$ from Definition 2.13, and the $k!$-to-one map $\Psi : \text{Dom} \to \tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)$ whose image is most of $\tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)$. We shall show that the random measures $\hat{\mu}[\tilde{S}^N_{\chi}((k+1)\cdot\cdot\cdot 1)]$ and $\hat{\mu}[\Psi(\text{Dom})]$ are close in $\text{Wass}_2$ distance. (See Proposition 3.15.)

Step 2. Fix $\bar{v} = (\langle n \rangle, \langle \Delta \rangle, \langle G \rangle, \langle H \rangle) \in \text{Dom}$ and let $\sigma = \Psi(\bar{v})$ be its associated affine permutation. We shall show that $\hat{\mu}_\sigma$ is close to $\frac{1}{k} \sum_{i=1}^k \lambda(\Delta_i/n_i)$ in $\text{Wass}_1$ distance. (See Proposition 3.12.)

Step 3. Let $\langle n \rangle \in \mathfrak{N}(N,\alpha)$. Let $\mathcal{W} \equiv \mathcal{W}(\alpha,A,B,\langle n \rangle)$ be as defined in the statement of Lemma 2.16, namely the set of $\langle \Delta \rangle$ such that $(\langle n \rangle, \langle \Delta \rangle, \langle G \rangle, \langle H \rangle)$ is in $\text{Dom}$ for some $\langle G \rangle$ and $\langle H \rangle$. Then we show that the (continuous) set $Q_0$ is well approximated by the discrete set

$$\tilde{\mathcal{W}} := \left\{ \left( \frac{\Delta_1}{n_1}, \ldots, \frac{\Delta_k}{n_k} \right) : \langle \Delta \rangle \in \mathcal{W} \right\},$$

which is a scaled down version of $\mathcal{W}$. We shall show that for every $\langle n \rangle \in \mathfrak{N}(N,\alpha)$, $\lambda^{Q_0}$ is close to $\lambda^{\tilde{\mathcal{W}}}$ (and close to $\hat{\mu}[\Psi(\text{Dom})]$, by Step 2) in $\text{Wass}_2$ distance. (See Proposition 3.14.)

### 3.2. Mixtures and Wasserstein Distance

Throughout this subsection, we assume that $\chi$ is a set with metric $\rho$. We define the diameter of $\chi$ to be

$$\text{diam}(\chi) := \sup\{\rho(a,b) : a, b \in \chi\}.$$ 

In the rest of this paper, we assume the diameter of $\chi$ is finite. We shall use the Borel sigma-algebra of $\chi$, with open sets determined by the metric $\rho$.

Given $m \in \mathbb{N}$, let $\nu_1, \ldots, \nu_m$ be probability measures on $\chi$, and let $a_1, \ldots, a_m$ be real numbers in $[0,1]$ whose sum is 1. Let $\nu = \sum_{i=1}^m a_i \nu_i$. Then $\nu$ is also a probability measure. In other words, every convex combination of probability measures (on a given measurable space) is a probability measure.

In statistical terminology, the measure $\nu$ in the preceding paragraph is also called a “mixture” of $\nu_1, \ldots, \nu_m$. It may be interpreted with the following construction.

**Randomized Algorithm MIX**
(a) Let \( \mathbf{X} = (X_1, \ldots, X_m) \) be a random vector (taking values in \( \chi^d \)) such that the component \( X_i \) has distribution \( \nu_i \) for each \( i \in [m] \). (We do not require that the components be independent.)

(b) Let \( J \) be a random variable, independent of \( \mathbf{X} \), such that \( \Pr(J = i) = a_i \) for each \( i \in [m] \).

(c) Let \( Y = X_J \). (That is, we assign \( Y \) to be \( X_1 \) with probability \( a_1 \), to be \( X_2 \) with probability \( a_2 \), and so on.) Then the distribution of \( Y \) is \( \nu \).

To prove the conclusion of (c), let \( D \) be a measurable subset of \( \chi \). Then

\[
\Pr(X_J \in D) = \sum_{i=1}^m \Pr(J = i \text{ and } X_i \in D) = \sum_{i=1}^m \Pr(J = i) \Pr(X_i \in D) = \sum_{i=1}^m a_i \nu_i(D) = \nu(D).
\]

The above construction leads directly to the following lemmas.

**Lemma 3.9.** Let \( \nu_1, \ldots, \nu_m, \omega_1, \ldots, \omega_m \in \text{PM}(\chi) \). Let \( a_1, \ldots, a_m \) be nonnegative real numbers that add up to 1. Then

\[
\text{Wass} \left( \sum_{i=1}^m a_i \nu_i, \sum_{i=1}^m a_i \omega_i \right) \leq \sum_{i=1}^m a_i \text{Wass}(\nu_i, \omega_i).
\]

**Proof.** Let \( \nu = \sum_{i=1}^m a_i \nu_i \) and \( \omega = \sum_{i=1}^m a_i \omega_i \). For each \( i \in [m] \), let \( (X_i, Z_i) \) be a \( \chi \times \chi \)-valued random vector such that \( E(\rho(X_i, Z_i)) = \text{Wass}(\nu_i, \omega_i) \) (we know that such a random vector exists because the infimum in Eq. (3.4) is always attained). Also let the \( m \) random vectors \( (X_i, Z_i) \) \( (i = 1, \ldots, m) \) be independent. Lastly, let \( J \) be a random variable, independent of the \( (X_i, Z_i) \)'s, such that \( \Pr(J = i) = a_i \) for each \( i \). Then the \( \chi \times \chi \)-valued random vector \( (X_J, Z_J) \) has marginal distributions \( \nu \) and \( \omega \). Therefore,

\[
\text{Wass}(\nu, \omega) \leq E(\rho(X_J, Z_J)) = \sum_{i=1}^m \Pr(J = i) E(\rho(X_i, Z_i)) = \sum_{i=1}^m a_i \text{Wass}(\nu_i, \omega_i).
\]

\( \square \)

**Lemma 3.10.** Let \( \nu_1, \ldots, \nu_m \in \text{PM}(\chi) \). Let \( a_1, \ldots, a_m, b_1, \ldots, b_m \) be nonnegative real numbers such that \( \sum_{i=1}^m a_i = 1 = \sum_{i=1}^m b_i \). Then

\[
\text{Wass} \left( \sum_{i=1}^m a_i \nu_i, \sum_{i=1}^m b_i \nu_i \right) \leq \text{diam}(\chi) \sum_{i=1}^m |a_i - b_i|.
\]

**Proof.** We first assert that there exists a random vector \( (J, K) \) such that \( \Pr(J = i) = a_i \) and \( \Pr(K = i) = b_i \) for each \( i \) and \( \Pr(J \neq K) = \sum_{i=1}^m |a_i - b_i| \). This true by Propositions 4.2 and 4.7 and Remark 4.8 of [27].

Next, let \( (X_1, \ldots, X_m) \) be a random vector, independent of \( (J, K) \), such that the component \( X_i \) has distribution \( \nu_i \) for each \( i \). Then we have
**Theorem 3.11.** Let \( A \subset B \subset \chi \). Assume that \( A \) and \( B \) are either both nonempty finite sets, or both continuous sets in the sense of Definition 3.5. Then we have

\[
\text{Wass}(\text{Unif}(A), \text{Unif}(B)) \leq \text{diam}(B) \text{Pr}(B \setminus A) = 1.
\]

**Proof.** Let \( V_A \) and \( V_B \) be independent random variables, with the Unif(\( A \)) and Unif(\( B \)) distributions respectively. Define the random variables \( X \) and \( Y \) by

\[
X := \begin{cases} 
V_B & \text{if } V_B \in A, \\
V_A & \text{if } V_B \notin A.
\end{cases}
\]

Then \( Y \) has the Unif(\( A \)) distribution. Since \( X \) has the Unif(\( B \)) distribution, we have \( \text{Wass}(\text{Unif}(A), \text{Unif}(B)) \leq E(\rho(Y, X)) \). The result now follows from the fact that \( \rho(Y, X) \leq \text{diam}(B) \mathbb{I}(V_B \notin A) \), where \( \mathbb{I}_E \) is the indicator random variable that equals 1 or 0 according to whether the event \( E \) occurs or not. \( \square \)

We remark that this lemma is not useful when \( A \) is discrete and \( B \) is continuous, nor when \( A \) and \( B \) are continuous sets with different dimensions, since in such cases \( P_{\text{Unif}(B)}(B \setminus A) = 1 \).

### 3.3. Proofs of Wasserstein Approximations

In this section, we fix \( k \geq 2 \).

Recall the set \( \text{Dom}(N, \alpha, A, B) \) from Definition 2.13, as well as the function \( \Psi \) defined on this set by Eqs. (2.7) and (2.9), for suitable values of \( N, \alpha, A, \) and \( B \). We shall first handle Step 2 in the proof strategy outlined at the end of Sect. 3.1.

**Proposition 3.12.** Let \( N \in \mathbb{N} \), let \( A \) and \( B \) be positive real numbers, and let \( \alpha \) be a real number in \((0, 1/k)\). Let \( \bar{v} = (\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle) \in \text{Dom}(N, \alpha, A, B) \), and let \( \sigma = \Psi(\bar{v}) \). As in Eq. (3.5), let \( \lambda(\Delta/n) \) be the probability measure on \( \hat{\Delta} \) defined by

\[
\lambda(\Delta/n) := \frac{1}{k} \sum_{i=1}^{k} \lambda_{\Delta_i/n_i}.
\]

Then

\[
\text{Wass}_1(\hat{\mu}_\sigma, \lambda(\Delta/n)) \leq \frac{1}{N} \left( 2A + \frac{4k}{1 - k\alpha} \right) + 4k\alpha.
\] (3.6)

\( \square \)
Proof. According to the definition of the function $\Psi$ preceding Lemma 2.5, we can write
\[
\hat{\mu}_\sigma = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \delta_{(g_{ij}/N, h_i,(j+\Delta_i)/N)}.
\]
Define the probability measure
\[
\Lambda = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \delta_{(j/n_i,(j+\Delta_i)/n_i)}.
\]
By Lemma 3.9 and the general property that $\text{Wass}(\delta_x, \delta_v) = \rho(x, v)$, we have
\[
\text{Wass}_1(\hat{\mu}_\sigma, \Lambda) \leq \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \left( \left| \frac{g_{ij}}{N} - j/n_i \right| + \left| \frac{h_i,(j+\Delta_i)}{N} - j + \Delta_i/n_i \right| \right)
\]
\[
\leq \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{2}{N} \left( A + \frac{k}{1 - k\alpha} \right) \left( \text{by Lemma 2.12} \right)
\]
\[
= \frac{2}{N} \left( A + \frac{k}{1 - k\alpha} \right). \quad (3.7)
\]

Let $U$ be a random variable with uniform distribution on the interval $(0, 1)$. For $i \in [k]$, define the two $\mathbb{R}^2$-valued random vectors
\[
\vec{\gamma} := \left( U, U + \frac{\Delta_i}{n_i} \right) \quad \text{and} \quad \vec{\kappa} := \left( \left\lfloor \frac{n_i U}{n_i} \right\rfloor, \left\lfloor \frac{n_i U}{n_i} \right\rfloor + \frac{\Delta_i}{n_i} \right)
\]
(here, $\lceil \cdot \rceil$ is the ceiling function). Then $\vec{\gamma}$ has distribution $\lambda_{\Delta_i/n_i}$ and $\vec{\kappa}$ has distribution $\Lambda_i$, where
\[
\lambda_i := \frac{1}{n_i} \sum_{j=1}^{n_i} \delta_{(j/n_i,(j+\Delta_i)/n_i)}.
\]
Since $\left| u - \left\lfloor \frac{n_i u}{n_i} \right\rfloor \right| \leq \frac{1}{n_i}$ for every real $u$, we see that $\rho(\vec{\gamma}, \vec{\kappa}) \leq 2/n_i$ with probability 1, and hence $\text{Wass}_1(\lambda_{\Delta_i/n_i}, \Lambda_i) \leq 2/n_i$. Noting that $\Lambda = \sum_{i=1}^{k} \frac{n_i}{N} \Lambda_i$, we deduce from Lemma 3.9 that
\[
\text{Wass}_1\left( \sum_{i=1}^{k} \frac{n_i}{N} \lambda_{\Delta_i/n_i}, \Lambda \right) \leq \sum_{i=1}^{k} \frac{n_i}{N} \frac{2}{n_i} = \frac{2k}{N}. \quad (3.8)
\]

Next, since $\text{diam}(\hat{\gamma}) = \sqrt{10}$, Lemma 3.10 tells us that
\[
\text{Wass}_1\left( \lambda(\Delta/n), \sum_{i=1}^{k} \frac{n_i}{N} \lambda_{\Delta_i/n_i} \right) \leq \sqrt{10} \sum_{i=1}^{k} \left| \frac{1}{k} - \frac{n_i}{N} \right| < 4k\alpha \quad (3.9)
\]
(where the second inequality uses the definition of $\text{Dom}$, specifically Eq. (2.15)). Finally, the proposition follows from the triangle inequality and Eqs. (3.7–3.9).

To prepare us for Step 3, we first prove a lemma about random measures of the form $\lambda^Q$ as defined in Definition 3.6(c).
Lemma 3.13. Let $QA$ and $QB$ be two (discrete or continuous) subsets of $[-1, 1]^k$. Then
\[ \text{Wass}_2\left(\lambda^{QA}, \lambda^{QB}\right) \leq \text{Wass}_1\left(\text{Unif}(QA), \text{Unif}(QB)\right). \]

Proof. Let $U$ be a uniformly distributed random variable on $[0, 1]$, and let $J$ be a uniformly distributed random variable on $\{1, \ldots, k\}$, independent of $U$. For $\langle x \rangle = (x_1, \ldots, x_k) \in [-1, 1]^k$, the random point $(U, U + x_J)$ has distribution $\lambda(x)$. If also $\langle v \rangle \in [-1, 1]^k$, then
\[ \text{Wass}_1\left(\lambda\langle x \rangle, \lambda\langle v \rangle\right) \leq E\left|\langle x \rangle - \langle v \rangle\right|. \]

Let $(\langle \beta^A \rangle, \langle \beta^B \rangle)$ be an $(\mathbb{R}^k \times \mathbb{R}^k)$-valued random vector such that $\langle \beta^A \rangle$ is uniformly distributed on $QA$, $\langle \beta^B \rangle$ is uniformly distributed on $QB$, and
\[ E\left|\langle \beta^A \rangle - \langle \beta^B \rangle\right| = \text{Wass}_1\left(\text{Unif}(QA), \text{Unif}(QB)\right). \]

Since the random measures $\lambda\langle \beta^A \rangle$ and $\lambda\langle \beta^B \rangle$ have distributions $\lambda^{QA}$ and $\lambda^{QB}$ respectively, we obtain
\[ \text{Wass}_2\left(\lambda^{QA}, \lambda^{QB}\right) \leq E\left(\text{Wass}_1\left(\lambda\langle \beta^A \rangle, \lambda\langle \beta^B \rangle\right)\right) \leq E\left|\langle \beta^A \rangle - \langle \beta^B \rangle\right| \quad \text{(by Eq. (3.10))} \]
\[ = \text{Wass}_1\left(\text{Unif}(QA), \text{Unif}(QB)\right). \]

Proposition 3.14. Fix $k \geq 2$. There is a positive constant $C$, depending only on $k$, such that the following holds. Let $N$ be a natural number, let $A$ and $B$ be positive real numbers, and let $\alpha \in (0, 1/4k)$. Let $\langle n \rangle \in \mathfrak{M}(N, \alpha)$ and write $W$ for $W(\alpha, A, B, \langle n \rangle)$. Let
\[ \lambda^W = \frac{1}{|W|} \sum_{\langle \Delta \rangle \in W} \delta_{\lambda\langle \Delta/n \rangle}. \]

Then
\[ \text{Wass}_2\left(\lambda^W, \lambda^{Q_0}\right) \leq C\left(\alpha + \frac{A + B + 1}{N}\right). \]

Proof. We begin by setting some notation. For $\langle x \rangle = (x_1, \ldots, x_k) \in \mathbb{Z}^k$, let $\langle \hat{x} \rangle$ and $\langle x^* \rangle$ be the rescaled vectors
\[ \langle \hat{x} \rangle = \left(\frac{x_1}{n_1}, \ldots, \frac{x_k}{n_k}\right) \quad \text{and} \quad \langle x^* \rangle = \left(\frac{kx_1}{N}, \ldots, \frac{kx_k}{N}\right), \]
and let the corresponding sets of rescaled $\langle \Delta \rangle$ vectors be

$$\hat{W} = \{ \langle \hat{\Delta} \rangle : \langle \Delta \rangle \in W \} \quad \text{and} \quad W^* = \{ \langle \Delta^* \rangle : \langle \Delta \rangle \in W \}.$$ 

With this notation, the definition of $\lambda \hat{W}$ in Eq. (3.11) is consistent with the definition given in Definition 3.6(c).

If $B \geq N/2k$, then the bound (3.12) holds whenever $C \geq 2k \text{diam}(\Diamond)$. Thus, without loss of generality, we can and shall assume $B < N/2k$ in this proof. Similarly, we shall assume that $N > 4k^2$.

By Lemma 3.13, it suffices to prove the desired upper bound for $\text{Wass}_1(\text{Unif}(\hat{W}), \text{Unif}(Q_0))$. To do this, we shall define an intermediate continuous set $Y$ of dimension $k - 1$, and show that $\text{Wass}_1(\text{Unif}(\hat{W}), \text{Unif}(W^*))$, $\text{Wass}_1(\text{Unif}(W^*), \text{Unif}(Y))$, and $\text{Wass}_1(\text{Unif}(Y), \text{Unif}(Q_0))$ are all small. The third term will be handled with Lemma 3.11, while the other two will be treated directly.

First we show that $\text{Unif}(\hat{W})$ is close to $\text{Unif}(W^*)$. For each $\langle \Delta \rangle \in W$, we have

$$\text{Wass}_1(\delta_{\langle \Delta^* \rangle}, \delta_{\langle \hat{\Delta} \rangle}) = ||\langle \Delta^* \rangle - \langle \hat{\Delta} \rangle || \leq \sum_{i=1}^{k} \left| \frac{k\Delta_i}{N} - \frac{\Delta_i}{n_i} \right|$$

$$= \sum_{i=1}^{k} \frac{|\Delta_i|}{n_i} \left| \frac{k}{N} \cdot n_i - \frac{N}{k} \right|$$

$$< \sum_{i=1}^{k} 1 \cdot \frac{k}{N} \cdot \alpha N$$

$$= k^2 \alpha.$$ 

Using this bound together with Lemma 3.9 shows that

$$\text{Wass}_1(\text{Unif}(W^*), \text{Unif}(\hat{W})) \leq k^2 \alpha. \quad (3.13)$$

Next we define a continuous set $Y$ of dimension $k - 1$ that approximates the discrete set $W^*$. Let $P_0$ be the hyperplane

$$P_0 := \{ (x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \cdots + x_k = 0 \}.$$ 

For each $\langle z \rangle \in P_0$, let $\text{Cube}(\langle z \rangle)$ be the intersection of $P_0$ with translation by $\langle z \rangle$ of the “hypercubical tube” $[0, k/N]^{k-1} \times \mathbb{R}$, i.e.

$$\text{Cube}(\langle z \rangle) := \left\{ (x_1, \ldots, x_k) \in P_0 : z_i \leq x_i < z_i + \frac{k}{N}, i = 1, \ldots, k - 1 \right\}. \quad (3.14)$$

Notice that for $x \in \text{Cube}(\langle z \rangle)$, the relations $x_k = -\sum_{i=1}^{k-1} x_i$ and $z_k = -\sum_{i=1}^{k-1} z_i$ imply that

$$z_k - \frac{k(k-1)}{N} < x_k \leq z_k. \quad (3.15)$$
It is important to observe that the collection of sets \( \{ \text{Cube}(z^*): z \in \mathcal{P}_0 \cap \mathbb{Z}^k \} \) is a partition of \( \mathcal{P}_0 \).

Let \( \langle \Delta \rangle \in \mathcal{W} \) and let \( \langle x \rangle \in \text{Cube}(\Delta^*) \). Then \( ||\langle x \rangle - \langle \Delta^* \rangle|| < 2k(k-1)/N \) by Eqs. (3.14) and (3.15). Therefore
\[
\text{Wass}_1 \left( \text{Unif}(\text{Cube}(\Delta^*)), \delta_{\langle \Delta^* \rangle} \right) \leq \frac{2k(k-1)}{N}. \tag{3.16}
\]
We now define the subset \( \mathcal{Y} \) of \( \mathcal{P}_0 \) to be the union of \( \text{Cube}(\langle z \rangle) \) over all \( z \in \mathcal{W}^* \), i.e.
\[
\mathcal{Y} := \bigcup_{\Delta \in \mathcal{W}} \text{Cube}(\Delta^*).
\]
Since the sets \( \text{Cube}(\langle \Delta \rangle) \) are all translates of one another, we see that the uniform distribution on \( \mathcal{Y} \) is the uniform mixture of the uniform distributions on its constituent Cube sets:
\[
P_{\text{Unif}(\mathcal{Y})} = \frac{1}{|\mathcal{W}|} \sum_{\langle \Delta \rangle \in \mathcal{W}} P_{\text{Unif}(\text{Cube}(\langle \Delta \rangle))} \tag{3.17}
\]
By Lemma 3.9 and Eqs. (3.16) and (3.17), we see that
\[
\text{Wass}_1 \left( \text{Unif}(\mathcal{Y}), \text{Unif}(\mathcal{W}^*) \right) \leq \frac{2k(k-1)}{N}. \tag{3.18}
\]
Now we need to show that \( \mathcal{Y} \) is a good approximation of \( Q_0 \). First we claim
\[
\mathcal{Y} \subset Q_1, \quad \text{where we define} \quad Q_1 := \left( 1 + k\alpha + \frac{k^2}{N} \right) Q_0, \tag{3.19}
\]
using the standard notation for homothety: for positive \( t \), \( tQ_0 = \{ t\langle x \rangle : \langle x \rangle \in Q_0 \} \). Let \( \langle x \rangle \in \mathcal{Y} \). Then \( \langle x \rangle \in \mathcal{P}_0 \), and \( \langle x \rangle \in \text{Cube}(\langle \Delta \rangle) \) for some \( \langle \Delta \rangle \in \mathcal{W} \). Thus for each \( i \in [k] \) we have
\[
|\Delta_i^*| = \frac{|k\Delta_i|}{N} \leq \frac{k}{N} \left( \frac{N}{k} + \alpha N \right) = 1 + k\alpha.
\]
Since \( |x_i| \leq (|\Delta_i^*| + (k-1)) \frac{k}{N} \) by Eqs. (3.14) and (3.15), we obtain Eq. (3.19).

Next we shall show that \( Q_1 \setminus \mathcal{Y} \) has small measure compared to \( Q_1 \). Let \( \langle x \rangle \in Q_1 \setminus \mathcal{Y} \), and define the point \( \langle D \rangle \in \mathbb{Z}^k \) by
\[
D_i = \left\lfloor \frac{Nx_i}{k} \right\rfloor \quad (i \in [k-1]) \quad \text{and} \quad D_k = -\sum_{i=1}^{k-1} D_i.
\]
Then \( \langle D \rangle \in \mathcal{P}_0 \) and \( \langle x \rangle \in \text{Cube}(\langle D \rangle) \). Since \( \langle x \rangle \notin \mathcal{Y} \), the point \( \langle D \rangle \) cannot be in \( \mathcal{W} \). This means that one of two inequalities hold: either
\begin{enumerate}
  \item \( |D_i| > n_i - B \) for some \( i \in [k] \), or
  \item \( \left| \frac{D_iN}{n_i} - \frac{D_jN}{n_j} \right| \leq 8A + 8k/(1 - k\alpha) \) for some \( i, j \in [k] \) with \( i \neq j \).
\end{enumerate}
On the one hand, if (I) holds, then
\[
|D_i^*| = \frac{k}{N} |D_i| > \frac{k}{N} \left( \frac{N}{k} - \alpha N - B \right) = 1 - k\alpha - \frac{kB}{N} \quad \text{for all} \quad i \in [k];
\]
hence $|x_i| > 1 - k\alpha - kB/N - k(k-1)/N$ for all $i$ (by Eqs. (3.14) and (3.15)). Therefore,

$$\langle x \rangle \in Q_1 \setminus \left(1 - k\alpha - \frac{kB}{N} - \frac{k^2}{N}\right) Q_0 \quad \text{in case (I)}. \quad (3.20)$$

(Notice that $1 - k\alpha - kB/N - k^2/N > 0$, due to our assumptions that $\alpha < 1/4k$, $B \leq N/2k$, and $N > 4k^2$ from the beginning of the proof.) On the other hand, if (II) holds for given $i$ and $j$, and (I) does not hold, then

$$|D^*_i - D^*_j| = \left|\frac{kD_i}{N} - \frac{kD_j}{N}\right|$$

$$\leq \left|\frac{kD_i}{N} - \frac{D_i}{n_i}\right| + \left|\frac{D_i}{n_i} - \frac{D_j}{n_j}\right| + \left|\frac{D_j}{n_j} - \frac{kD_j}{N}\right|$$

$$\leq k \left|\frac{D_i}{n_i}\right| n_i - \frac{N}{k} + 8N \left(A + \frac{k}{1 - k\alpha}\right) + k \left|\frac{D_j}{n_j}\right| \frac{N}{n_j} - n_j$$

$$\leq R \quad \text{where} \quad R = 2k\alpha + \frac{8N}{k} \left(A + \frac{k}{1 - k\alpha}\right),$$

and hence $|x_i - x_j| \leq R + \frac{k}{N} + \frac{k(k-1)}{N}$ (again, using Eqs. (3.14) and (3.15)).

Summarizing the results of the preceding paragraph, we have shown

$$Q_1 \setminus \mathcal{Y} \subset Q_1 \setminus \left(1 - k\alpha - \frac{kB}{N} - \frac{k^2}{N}\right) Q_0 \cup \bigcup_{1 \leq i < j \leq k} D_{ij} \left(R + \frac{k^2}{N}\right)$$

(3.21)

where we define

$$D_{ij}(r) := \{\langle x \rangle \in \mathcal{P}_0 \cap [-1, 1]^k : |x_i - x_j| \leq r\}.$$

Write $\text{Leb}_{k-1}$ for $(k-1)$-dimensional Lebesgue measure. Then we have

$$\text{Leb}_{k-1}(tQ_0) = t^{k-1} \text{Leb}_{k-1}(Q_0) \quad \text{for any} \ t > 0,$$

(3.22)

from which it follows that

$$\text{Leb}_{k-1} \left(Q_1 \setminus \left(1 - k\alpha - \frac{kB}{N} - \frac{k^2}{N}\right) Q_0\right) = \left[1 - \left(\frac{1 - k\alpha - \frac{kB}{N} - \frac{k^2}{N}}{1 + k\alpha + \frac{k^2}{N}}\right)^{k-1}\right]$$

$$\text{Leb}_{k-1}(Q_1). \quad (3.23)$$

Next, we make three observations for $i, j \in [k]$ with $i \neq j$.

(a) A set of the form $D_{ij}(r)$ lies between two parallel hyperplanes $x_i - x_j = \pm r$, which are distance $\sqrt{2}r$ apart.

(b) The normal vector to any hyperplane $x_i - x_j = \text{Constant}$ is perpendicular to the normal vector of $\mathcal{P}_0$; and

(c) The diameter of $[-1, 1]^k$ is $2\sqrt{k}$.

By observation (b), we can choose an orthonormal basis $\{\langle e(\ell) \rangle : \ell \in [k]\}$ such that $\langle e(1) \rangle$ is orthogonal to hyperplanes $x_i - x_j = \text{Constant}$ and $\langle e(k) \rangle$ is orthogonal to $\mathcal{P}_0$. Let $\mathbf{H}$ be the set of all vectors in $\mathbb{R}^k$ of the form $\sum_{\ell=1}^{k-1} t_\ell \langle e(\ell) \rangle$ such that $|t_\ell| \leq \sqrt{k}$ for every $\ell \in [k-1]$. Then $\mathbf{H}$ is a $(k-1)$-dimensional
hypercube of side length $2\sqrt{k}$ centered at the origin, contained in $P_0$, with two of its faces contained in the two hyperplanes $x_i - x_j = \pm \sqrt{2k}$. By (c), this hypercube $H$ contains $Q_0$. By (a), $\operatorname{Leb}_{k-1}(D_{ij}(r)) \leq \sqrt{2}r \times (2\sqrt{k})^{k-2}$. Inserting this and Eq. (3.23) into Eq. (3.21) yields

$$
\operatorname{Leb}_{k-1}(Q_1 \setminus \mathcal{Y}) \leq \left[ 1 - \left( \frac{1 - k\alpha - \frac{kB}{N} - \frac{k^2}{N}}{1 + k\alpha + \frac{k^2}{N}} \right)^{k-1} \right] \operatorname{Leb}_{k-1}(Q_1)
+ \frac{k}{2} \sqrt{2} \left( 2k\alpha + \frac{1}{N} \left( 8A + k^2 + \frac{8k}{1-k\alpha} \right) \right) (2\sqrt{k})^{k-2}
$$

(3.24)

Now we can put the pieces together.

$$
\operatorname{Wass}_2\left( \lambda \hat{\mu}, \lambda Q_0 \right)
\leq \operatorname{Wass}_1\left( \operatorname{Unif}(\mathcal{W}), \operatorname{Unif}(Q_0) \right) \quad \text{(by Lemma 3.13)}
\leq \operatorname{Wass}_1\left( \operatorname{Unif}(\tilde{\mathcal{W}}), \operatorname{Unif}(\mathcal{W}^*) \right) + \operatorname{Wass}_1\left( \operatorname{Unif}(\mathcal{W}^*), \operatorname{Unif}(\mathcal{Y}) \right)
+ \operatorname{Wass}_1\left( \operatorname{Unif}(\mathcal{Y}), \operatorname{Unif}(Q_1) \right) + \operatorname{Wass}_1\left( \operatorname{Unif}(Q_1), \operatorname{Unif}(Q_0) \right)
\leq k^2\alpha + \frac{2k^2}{N} + \operatorname{diam}(Q_1) \left[ \frac{\operatorname{Leb}_{k-1}(Q_1 \setminus \mathcal{Y})}{\operatorname{Leb}_{k-1}(Q_1)} + \frac{\operatorname{Leb}_{k-1}(Q_1 \setminus Q_0)}{\operatorname{Leb}_{k-1}(Q_1)} \right]
\text{(by Eqs. (3.13) and (3.18), and Lemma 3.11)}
= O \left( \alpha + \frac{A + B + 1}{N} \right)
\text{(by Eqs. (3.24), (3.22), and (3.19)).}
$$

This completes Step 3 of our outline. We now turn to Step 1.

**Proposition 3.15.** Given $\alpha \in (0, 1/(3k))$, let $\operatorname{Im}(N, \alpha)$ be the image of $\operatorname{Dom}(N, \alpha, \alpha N, 2\alpha N)$ under $\Psi$.

(a) For sufficiently large $N$, we have $\operatorname{Im}(N, \alpha) \subset \tilde{S}_N^{ij}((k+1) \cdots 1)$ and the restriction of $\Psi$ to $\operatorname{Dom}(N, \alpha, \alpha N, 2\alpha N)$ is $k!$-to-one.

(b) Moreover, for any $\epsilon > 0$, there exists an $\alpha_\epsilon \in (0, 1/4k)$ such that

$$\limsup_{N \to \infty} \operatorname{Wass}_2\left( \hat{\mu}[\tilde{S}_N^{ij}((k+1) \cdots 1)], \hat{\mu}[\operatorname{Im}(N, \alpha)] \right) < \epsilon \quad \text{whenever } 0 < \alpha < \alpha_\epsilon.
$$

(3.25)

**Proof.** Part (a) follows from Corollary 2.15 and Lemma 2.14. Next, by Lemma 3.11,

$$
\operatorname{Wass}_2\left( \hat{\mu}[\tilde{S}_N^{ij}((k+1) \cdots 1)], \hat{\mu}[\operatorname{Im}(N, \alpha)] \right)
\leq \operatorname{diam}(\hat{\mathcal{Y}}) \left( 1 - \frac{|\operatorname{Im}(N, \alpha)|}{|\tilde{S}_N^{ij}((k+1) \cdots 1)|} \right).
$$

Part (b) follows from part (a), Eq. (2.32), and Theorem 2.2.

$\Box$
Proof of Theorem 3.7. In this proof, we shall let $A = \alpha N$ and $B = 2\alpha N$, where $\alpha$ is a small positive constant in $(0, 1/4k)$.

Let $\langle n \rangle \in \mathcal{H}(N, \alpha)$ and $\langle \Delta \rangle \in \mathcal{W}(\alpha, A, B, \langle n \rangle)$. Then for every $\langle G \rangle, \langle H \rangle \in \mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)$, Proposition 3.12 tells us that

$$\text{Wass} \left( \lambda(\Delta/n), \hat{\mu}_{\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle)) \right) \leq \frac{1}{N} \left( 2A + \frac{4k}{1 - k\alpha} \right) + 4k\alpha,$$

and hence that

$$\text{Wass} \left( \delta_{\lambda(\Delta/n)}, \delta_{\hat{\mu}_{\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle))} \right) \leq \frac{1}{N} \left( 2A + \frac{4k}{1 - k\alpha} \right) + 4k\alpha. \quad (3.26)$$

Next, for such $\langle n \rangle$ and $\langle \Delta \rangle$, define the mixture

$$\mathcal{M}(\langle n \rangle, \langle \Delta \rangle) := \frac{1}{|\mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)|^2} \sum_{\langle G \rangle, \langle H \rangle \in \mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)} \delta_{\hat{\mu}_{\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle))}.$$ 

It follows from Lemma 3.9 and Eq. (3.26) that

$$\text{Wass} \left( \delta_{\lambda(\Delta/n)}, \mathcal{M}(\langle n \rangle, \langle \Delta \rangle) \right) \leq \frac{1}{N} \left( 2A + \frac{4k}{1 - k\alpha} \right) + 4k\alpha. \quad (3.27)$$

By the properties of $\Psi$ described in Proposition 3.15, we obtain (writing $\text{Dom}$ for $\text{Dom}(N, \alpha, \alpha N, 2\alpha N)$ and $\mathcal{W}(\langle n \rangle)$ for $\mathcal{W}(\langle n \rangle, \alpha, \alpha N, 2\alpha N)$)

$$\hat{\mu}[\text{Im}(N, \alpha)] = \frac{1}{|\text{Im}(N, \alpha)|} \sum_{\sigma \in \text{Im}(N, \alpha)} \delta_{\hat{\mu}_{\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle))}$$

$$= \frac{1}{k! |\text{Im}(N, \alpha)|} \sum_{\bar{v} \in \text{Dom}} \delta_{\hat{\mu}_{\Psi(\langle n \rangle, \langle G \rangle, \langle H \rangle, \langle \Delta \rangle))}$$

$$= \frac{1}{|\text{Dom}|} \sum_{\langle n \rangle \in \mathcal{H}(N, \alpha)} \sum_{\langle \Delta \rangle \in \mathcal{W}(\langle n \rangle)} |\mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)|^2 \mathcal{M}(\langle n \rangle, \langle \Delta \rangle). \quad (3.28)$$

Define the random measure

$$\lambda^* = \sum_{\langle n \rangle \in \mathcal{H}(N, \alpha)} \frac{|\mathcal{W}(\langle n \rangle)| |\mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)|^2 \lambda^*(\langle n \rangle)}{|\text{Dom}|} \hat{\lambda}_{\mathcal{W}(\langle n \rangle)}$$

(c.f. Eq. (2.22)). Then we can write

$$\lambda^* = \frac{1}{|\text{Dom}|} \sum_{\langle n \rangle \in \mathcal{H}(N, \alpha)} \sum_{\langle \Delta \rangle \in \mathcal{W}(\langle n \rangle)} |\mathcal{V}_{\mathcal{N}}^{**A}(\langle n \rangle)|^2 \delta_{\lambda(\langle n \rangle, \langle \Delta \rangle))}. \quad (3.29)$$

Then by Eqs. (3.27–3.29) and Lemma 3.9, we obtain

$$\text{Wass} \left( \lambda^*, \hat{\mu}[\text{Im}(N, \alpha)] \right) \leq \frac{1}{N} \left( 2A + \frac{4k}{1 - k\alpha} \right) + 4k\alpha. \quad (3.30)$$

By Proposition 3.14 and Lemma 3.9, we have

$$\text{Wass} \left( \lambda^*, \lambda^0 \right) \leq C \left( \alpha + \frac{A + B + 1}{N} \right). \quad (3.31)$$
The triangle inequality gives
\[
\text{Wass}_2 \left( \hat{\mu}[\tilde{S}_N^{((k+1)\ldots 1)}], \lambda_{Q_0} \right) \\
\leq \text{Wass}_2 \left( \hat{\mu}[\tilde{S}_N^{((k+1)\ldots 1)}], \hat{\mu}[\text{Im}(N, \alpha)] \right) \\
+ \text{Wass}_2 \left( \hat{\mu}[\text{Im}(N, \alpha)], \lambda^* \right) + \text{Wass}_2 \left( \lambda^*, \lambda_{Q_0} \right).
\] (3.32)

Let \( \epsilon > 0 \) and let \( \alpha_\epsilon \in (0, 1/4k) \) be as specified in Eq. (3.25). Let \( \alpha \in (0, \alpha_\epsilon) \).

Applying Eqs. (3.25), (3.30), and (3.31) to Eq. (3.32), we have
\[
\limsup_{N \to \infty} \text{Wass}_2 \left( \hat{\mu}[\tilde{S}_N^{((k+1)\ldots 1)}], \lambda_{Q_0} \right) \leq \epsilon + 2\alpha + 4k\alpha + 4C\alpha.
\]

Since \( \epsilon \) and \( \alpha \) can both be chosen to be arbitrarily small, the above limsup must be zero. This proves the theorem.

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