THE FOURTH POWER MEAN OF DIRICHLET $L$-FUNCTIONS IN $\mathbb{F}_q[T]$

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Abstract. We prove results on moments of $L$-functions in the function field setting, where the moment averages are taken over primitive characters of modulus $R$, where $R$ is a polynomial in $\mathbb{F}_q[T]$. We consider the behaviour as $\deg R \to \infty$ and the cardinality of the finite field is fixed. Specifically, we obtain an exact formula for the second moment provided that $R$ is square-full, and an asymptotic formula for the fourth moment for any $R$. The fourth moment result is a function field analogue of Heath-Brown’s result in the number field setting, which was subsequently improved by Soundararajan. Both the second and fourth moment results extend work done by Tamam in the function field setting who focused on the case where $R$ is prime.

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1. Introduction

The study of moments of families $L$-functions is a central theme in analytic number theory. These moments are connected to the famous Lindelöf hypothesis for such $L$-functions and have many applications in analytic number theory. It is a very challenging problem to establish asymptotic formulas for higher moments of families of $L$-functions and until now we only have asymptotic formulas for the first few moments of any given family of $L$-functions. However, we do have precise conjectures for higher moments of families of $L$-functions due to the work of many mathematicians (see for example [CFK+05] and [DGH03]). In this paper the focus is on the moments of Dirichlet $L$-functions associated to primitive Dirichlet characters.

In 1981, Heath-Brown [HB81] proved that

\begin{equation}
\left| \sum_{\chi \pmod{q}}^\ast \left( \frac{1}{2}, \chi \right) \right|^4 = \frac{1}{2\pi^2} \phi^*(q) \prod_{p \mid q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})^4} (\log q)^4 + O(2^{\omega(q)}q(\log q)^3),
\end{equation}

where, for all positive integers $q$, $\sum_{\chi \pmod{q}}^\ast$ represents a summation over all primitive Dirichlet characters of modulus $q$, $\phi^*(q)$ is the number of primitive characters of modulus $q$, and $\omega(q)$ is the number of distinct prime divisors of $q$ and $L(s, \chi)$ is the associated Dirichlet $L$-function.
In the equation above (1), in order to ensure that the error term is of lower order than the main term, we must restrict $q$ to

$$\omega(q) \leq \frac{\log \log q - 7 \log \log \log q}{\log 2}.$$ 

Soundararajan [Sou07] addressed this by proving that

$$\sum_{\chi \pmod{q}} |L\left(\frac{1}{2}, \chi\right)|^4 = \frac{1}{2\pi^2} \phi^*(q) \prod_{p|q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} (\log q)^4 \left(1 + O\left(\frac{\omega(q)}{\log q} \sqrt{\frac{q}{\phi(q)}}\right)\right) + O(q(\log q)^\frac{5}{2}).$$

Here, the error terms are of lower order than the main term without the need to have any restriction on $q$.

In a breakthrough paper, Young [You11] obtained explicit lower order terms for the case where $q$ is an odd prime and was able to establish the full polynomial expansion for the fourth moment of the associated Dirichlet $L$-functions. In other words, he proved that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* |L\left(\frac{1}{2}, \chi\right)|^4 = \sum_{i=0}^4 c_i (\log q)^i + O\left(q^{\frac{5}{12} + \epsilon}\right),$$

where the constants $c_i$ are computable. The error term was subsequently improved by Blomer et al. [BFK+17] who proved that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* |L\left(\frac{1}{2}, \chi\right)|^4 = \sum_{i=0}^4 c_i (\log q)^i + O\left(q^{\frac{1}{6} + \epsilon}\right).$$

In the function field setting Tamam [Tam14] established that

$$\frac{1}{\phi(Q)} \sum_{\chi \pmod{Q}}^* |L\left(\frac{1}{2}, \chi\right)|^2 = \deg Q + \frac{1}{(q^{\frac{1}{2}} - 1)^2} \left(1 - \frac{2}{|Q|^{\frac{1}{2}} + 1}\right)$$

and

$$\frac{1}{\phi(Q)} \sum_{\chi \pmod{Q}}^* \left|L\left(\frac{1}{2}, \chi\right)\right|^4 = \frac{q - 1}{12q} (\deg Q)^4 + O((\deg Q)^3)$$

as $\deg Q \to \infty$. Here, $Q$ is an irreducible, monic polynomial in $\mathbb{F}_q[T]$ with $\mathbb{F}_q$ a finite field with $q$ elements; $\chi_0$ is the trivial character (in this case, of modulus $Q$); and, for non-trivial characters of modulus $Q$,

$$L\left(\frac{1}{2}, \chi\right) = \sum_{\deg A < \deg Q} \frac{\chi(A)}{|A|^{\frac{1}{2}}},$$

where $\mathcal{M}$ is the set of monic polynomials $\mathbb{F}_q[T]$.

In this paper we prove the function field analogue of Soundararajan’s result, which is also an extension of Tamam’s fourth moment result. In order to accomplish this we prove, along the way, a function field analogue of a special case of Shiu’s Brun-Titchmarsh theorem for multiplicative functions [Shi80]. We also obtain an exact formula for the second moment, similar to Tamam’s second moment result; our result holds for square-full moduli, $R$, whereas Tamam’s result holds for irreducible moduli, $Q$. 
2. Statement of Results

Let $q \in \mathbb{N}$ be a prime-power. We denote the finite field of order $q$ by $\mathbb{F}_q$. We denote the ring of polynomials over the finite field $\mathbb{F}_q$ by $\mathcal{A} := \mathbb{F}_q[T]$. Unless otherwise stated, for a subset $\mathcal{S} \subset \mathcal{A}$ we define $\mathcal{S}_n := \{ A \in \mathcal{S} : \deg A = n \}$. We identify $\mathcal{A}_0$ with $\mathbb{F}_q$. Also, if we have some non-negative real number $x$, then range $\deg A \leq x$ is not taken to include the polynomial $A = 0$.

The norm of $A \in \mathcal{A}\{0\}$ is defined by $|A| := q^{\deg A}$, and for the zero polynomial we define $|0| := 0$.

We denote the set of monic polynomials in $\mathcal{A}$ by $\mathcal{M}$. For $a \in (\mathbb{F}_q)^*$ we denote the set of polynomials, with leading coefficient equal to $a$, by $a\mathcal{M}$. Because $\mathcal{A}$ is an integral domain, an element is prime if and only if it is irreducible. We denote the set of prime monic polynomials in $\mathcal{A}$ by $\mathcal{P}$, and all references to primes (or irreducibles) in the function field setting are taken as being monic primes. Also, when indexing, the upper-case letter $P$ always refers to a monic prime. Furthermore, if we range over polynomials $E$ that divide some polynomial $F$, then these $E$ are taken to be the monic divisors only.

**Definition 2.1** (Dirichlet Characters). Let $R \in \mathcal{M}$. A Dirichlet character on $\mathcal{A}$ with modulus $R$ is a function $\chi : \mathcal{A} \rightarrow \mathbb{C}^*$ satisfying the following properties. For all $A, B \in \mathcal{A}$:

1. $\chi(AB) = \chi(A)\chi(B)$;
2. If $A \equiv B \pmod{R}$, then $\chi(A) = \chi(B)$;
3. $\chi(A) = 0$ if and only if $(A, R) \neq 1$.

Due to point 2 we can view a character $\chi$ of modulus $R$ as a function on $\mathcal{A}\{R, A\}$. This makes expressions such as $\chi(A^{-1})$ well-defined for $A \in (\mathcal{A}\{R, A\})^*$.

We can deduce that $\chi(1) = 1$ and $|\chi(A)| = 1$ when $(A, R) = 1$. We say that $\chi$ is the trivial character of modulus $R$ if $\chi(A) = 1$ when $(A, R) = 1$, and this is denoted by $\chi_0$. Otherwise, we say that $\chi$ is non-trivial. Also, there is only one character of modulus 1 and it simply maps all $A \in \mathcal{A}$ to 1.

It can easily be seen that the set of characters of a fixed modulus $R$ forms an abelian group under multiplication. The identity element is $\chi_0$. The inverse of $\chi$ is $\chi^*$, which is defined by $\chi(A) = \overline{\chi(A)}$ for all $A \in \mathcal{A}$. It can be shown that the number of characters of modulus $R$ is $\varphi(R)$.

A character $\chi$ is said to be even if $\chi(a) = 1$ for all $a \in \mathbb{F}_q$. Otherwise, we say that it is odd. The set of even characters of modulus $R$ is a subgroup of the set of all characters of modulus $R$. It can be shown that there are $\frac{\varphi(R)}{2}$ elements in this group.

**Definition 2.2** (Primitive Character). Let $R \in \mathcal{M}$, $S \mid R$ and $\chi$ be a character of modulus $R$. We say that $S$ is an induced modulus of $\chi$ if there exists a character $\chi_1$ of modulus $S$ such that

$$\chi(A) = \begin{cases} \chi_1(A) & \text{if } (A, R) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

$\chi$ is said to be primitive if there is no induced modulus of strictly smaller norm than $R$. Otherwise, $\chi$ is said to be non-primitive. $\varphi^*(R)$ denotes the number of primitive characters of modulus $R$.

We note that all trivial characters of some modulus $R \neq 1$ are non-primitive as they are induced by the character of modulus 1. We also note that if $R$ is prime, then the only non-primitive character of modulus $R$ is the trivial character of modulus $R$. We denote a sum over primitive characters of
modulus \( R \) by the standard notation \( \sum_{\chi \mod R}^* \).

**Definition 2.3** (Dirichlet L-functions). Let \( \chi \) be a Dirichlet character. The associated L-function, \( L(s, \chi) \), is defined for \( \text{Re}(s) > 1 \) by

\[
L(s, \chi) := \sum_{A \in M} \frac{\chi(A)}{|A|^s}.
\]

This has an analytic continuation to either \( \mathbb{C} \) or \( \mathbb{C} \setminus \{1\} \) depending on the character.

In this paper, we will prove the following two main results. The first one is an exact formula for the second moment of Dirichlet L-functions in function fields.

**Theorem 2.4.** Let \( R \) be a square-full polynomial. That is, if \( P \mid R \) then \( P^2 \mid R \). Then,

\[
\sum_{\chi \mod R}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^2 = \frac{\phi(R)^3}{|R|^2} \deg R + \left( \frac{\phi(R)^3}{|R|^2} - \frac{\phi(R)^2}{|R|^2} \right) \sum_{P \mid R} \frac{\deg P}{|P| - 1} + \frac{1}{(q^{1/2} - 1)^2} \left( - \frac{\phi(R)^3}{|R|^2} + 2 \frac{\phi(R)}{|R|^2} \prod_{P \mid R} \left( 1 - \frac{1}{|P|} \right)^2 \right).
\]

The next result, is an asymptotic formula for the fourth moment of Dirichlet L-functions associated to primitive Dirichlet characters.

**Theorem 2.5.** Let \( R \in \mathcal{M} \). Then,

\[
\sum_{\chi \mod R}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \frac{1 - q^{-1}}{12} \phi^*(R) \prod_{P \text{ prime} \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^4 + O \left( \phi^*(R) \prod_{P \text{ prime} \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^7 \sqrt{\omega(R)} \right).
\]

### 3. Function Field Background

We provide some definitions and results relating to function fields that are needed in this paper. Many of these results are well known and so we do not provide a proof. Some proofs can be found in Rosen’s book [Ros02], particularly chapter 4.

**Definition 3.1** (Möbius Function). We define the Möbius function, \( \mu \), multiplicatively by \( \mu(P) = -1 \) and \( \mu(P^e) = 0 \) for all primes \( P \in \mathcal{A} \) and all integers \( e \geq 2 \).

**Definition 3.2** (\( \omega \) Function). For all \( R \in \mathcal{A} \setminus \{0\} \) we define \( \omega(R) \) to be the number of distinct prime factors of \( R \).

**Definition 3.3** (\( \Omega \) Function). For all \( R \in \mathcal{A} \setminus \{0\} \) we define \( \Omega(R) \) to be the total number of prime factors of \( R \) (i.e. counting multiplicity).

**Definition 3.4** (\( \phi \) Function). For \( R \in \mathcal{A} \) with \( \deg R = 0 \) we define \( \phi(R) := 1 \), and for \( R \in \mathcal{A} \) with \( \deg R \geq 1 \) we define

\[
\phi(R) := |\{ A \in \mathcal{A} : \deg A < \deg R, (A, R) = 1 \}|.
\]

It is not hard to show that

\[
\phi(R) = |R| \prod_{P \mid R} (1 - |P|^{-1}).
\]
Definition 3.5. For all $R \in \mathcal{A}$ with $\deg R \geq 1$ we define $p_-(R)$ to be the largest positive integer such that if $P \mid R$ then $\deg P \geq p_-(R)$. Similarly, we define $p_+(R)$ to be the smallest positive integer such that if $P \mid R$ then $\deg P \leq p_+(R)$.

Proposition 3.6 (Orthogonality Relations). Let $R \in \mathcal{M}$. Then,

$$\sum_{\chi \pmod{R}} \chi(A)\overline{\chi}(B) = \begin{cases} \phi(R) & \text{if } (AB, R) = 1 \text{ and } A \equiv B \pmod{R} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{\chi \pmod{R}} \chi(A)\overline{\chi}(B) = \begin{cases} \frac{1}{q-1}\phi(R) & \text{if } (AB, R) = 1 \text{ and } A \equiv aB \pmod{R} \text{ for some } a \in \mathbb{F}_q \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.7. Let $R \in \mathcal{M}$ and let $A, B \in \mathcal{A}$. Then,

$$\sum_{\chi \pmod{R}} \chi(A)\tilde{\chi}(B) = \begin{cases} \sum_{E,F \mid (A-B)} \mu(E)\phi(F) & \text{if } (AB, R) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 3.8. For all $R \in \mathcal{M}$ we have that

$$\phi^*(R) = \sum_{EF=R} \mu(E)\phi(F).$$

Proof. This follows easily from Proposition 3.7 when we take $A, B = 1$. □

For a character $\chi$ we will, on occasion, write the associated $L$-function as

$$L(s, \chi) = \sum_{n=0}^{\infty} \frac{L_n(\chi)}{q^{-ns}},$$

where we define

$$L_n(\chi) := \sum_{A \in \mathcal{M}} \chi(A) \text{ for all non-negative integers } n \text{ and all characters } \chi.$$ 

Suppose $\chi$ is the character of modulus 1 and $\text{Re}(s) > 1$. Then, $L(s, \chi)$ is simply the zeta-function for the ring $\mathcal{A}$. That is,

$$L(s, \chi) = \sum_{A \in \mathcal{M}} \frac{1}{|A|^s} =: \zeta_{\mathcal{A}}(s).$$

We note further that

$$\zeta_{\mathcal{A}}(s) = \sum_{A \in \mathcal{M}} \frac{1}{|A|^s} = \frac{1}{1-q^{1-s}}.$$

The far-RHS provides a meromorphic extension for $\zeta_{\mathcal{A}}$ to $\mathbb{C}$ with a simple pole at 1. The following Euler product formula will also be useful

$$\zeta_{\mathcal{A}}(s) = \prod_{P \in \mathcal{P}} (1 - |P|^{-s})^{-1},$$

where $\mathcal{P}$ is the set of all prime ideals in $\mathcal{A}$.
for $\Re(s) > 1$.

Now suppose that $\chi_0$ is the trivial character of some modulus $R$ and $\Re(s) > 1$. It can be shown that

$$L(s, \chi_0) = \left( \prod_{P \in \mathcal{P}} 1 - |P|^{-s} \right) \zeta_A(s).$$

So, again, the far-RHS provides a meromorphic extension for $L(s, \chi_0)$ to $\mathbb{C}$ with a simple pole at 1.

Finally, suppose that $\chi$ is a non-trivial character of modulus $R$ and $\Re(s) > 1$. It can be shown that

$$L(s, \chi) = \sum_{A \in \mathcal{M}} \frac{\chi(A)}{|A|^s},$$

This is just a finite polynomial in $q^{-s}$, and so it provides a holomorphic extension for $L(s, \chi)$ to $\mathbb{C}$.

**Proposition 3.9** (Functional Equation for $L$-functions of Primitive Characters). Let $\chi$ be a primitive character of some modulus $R \neq 1$. If $\chi$ is even, then $L(s, \chi)$ satisfies the function equation

$$\left( q^{1-s} - 1 \right) L(s, \chi) = W(\chi) q^{\frac{\deg R}{2}} (q^{-s} - 1)(q^{-s})^{\deg R-1} L(1-s, \overline{\chi});$$

and if $\chi$ is odd, then $L(s, \chi)$ satisfies the function equation

$$L(s, \chi) = W(\chi) q^{\frac{\deg R-1}{2}} (q^{-s})^{\deg R-1} L(1-s, \overline{\chi});$$

where $|W(\chi)| = 1$.

A generalisation of the proposition above appears in Rosen’s book [Ros02, Theorem 9.24 A].

**Proposition 3.10.** Let $\chi$ a primitive odd character of modulus $R$. Then,

$$\left| L\left( \frac{1}{2}, \chi \right) \right|^2 = 2 \sum_{A,B \in \mathcal{M}} \frac{\chi(A) \overline{\chi}(B)}{|AB|^\frac{1}{2}} + c_o(\chi),$$

where we define

$$c_o(\chi) := - \sum_{A,B \in \mathcal{M}} \frac{\chi(A) \overline{\chi}(B)}{|AB|^\frac{1}{2}}.$$

**Proof.** The functional equation for odd primitive characters gives us that

$$\sum_{n=0}^{\deg R-1} L_n(\chi) q^{-ns} = W(\chi) q^{\frac{\deg R-1}{2}} (q^{-s} - 1)(q^{-s})^{\deg R-1} \sum_{n=0}^{\deg R-1} L_n(\overline{\chi}) q^{-n(1-s)}$$

$$= W(\chi) q^{\frac{\deg R-1}{2}} \sum_{n=0}^{\deg R-1} L_n(\overline{\chi}) q^{(1-s)(\deg R-1-n)}.$$

Taking the squared modulus of both sides gives us that

$$\sum_{n=0}^{2\deg R-2} \left( \sum_{0 \leq i,j < \deg R \atop i+j=n} L_i(\chi)L_j(\overline{\chi}) \right) q^{-ns} = q^{-\deg R+1} \sum_{n=0}^{2\deg R-2} \left( \sum_{0 \leq i,j < \deg R \atop i+j=n} L_i(\chi)L_j(\overline{\chi}) \right) q^{(1-s)(2\deg R-2-n)}.$$
We note that both sides are equal to $|L(s, \chi)|^2$, and so by the linear independence of powers of $q^{-s}$ we can take the terms $n = 0, 1, \ldots, \deg R - 1$ on the LHS and the terms $n = 0, 1, \ldots, \deg R - 2$ on the RHS to give

$$|L(s, \chi)|^2 = \sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j < \deg R \atop i + j = n} L_i(\chi)L_j(\chi) \right) q^{-ns} + q^{-\deg R+1} \sum_{n=0}^{\deg R-2} \left( \sum_{0 \leq i, j < \deg R \atop i + j = n} L_i(\chi)L_j(\chi) \right) q^{(1-s)(2\deg R-2-n)}.$$

Hence,

$$\left| L \left( \frac{1}{2}, \chi \right) \right|^2 = \sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j < \deg R \atop i + j = n} L_i(\chi)L_j(\chi) \right) q^{-\frac{n}{2}} + \sum_{n=0}^{\deg R-2} \left( \sum_{0 \leq i, j < \deg R \atop i + j = n} L_i(\chi)L_j(\chi) \right) q^{-\frac{n}{2}}$$

$$= 2 \sum_{A, B \in M \atop \deg AB < \deg R} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}} - \sum_{A, B \in M \atop \deg AB = \deg R-1} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}},$$

as required.

**Proposition 3.11.** Let $\chi$ a primitive even character of modulus $R \neq 1$. Then,

$$\left| L \left( \frac{1}{2}, \chi \right) \right|^2 = 2 \sum_{A, B \in M \atop \deg AB < \deg R} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}} + c_{\epsilon}(\chi),$$

where

$$c_{\epsilon}(\chi) := - \frac{q}{(q^{\frac{1}{2}} - 1)^2} \sum_{A, B \in M \atop \deg AB = \deg R-2} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}} - \frac{2q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1} \sum_{A, B \in M \atop \deg AB = \deg R-1} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}}$$

$$+ \frac{1}{(q^{\frac{1}{2}} - 1)^2} \sum_{A, B \in M \atop \deg AB = \deg R} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}}.$$

**Proof.** The functional equation for even primitive characters gives us that

$$(q^{1-s} - 1) \sum_{n=0}^{\deg R-1} L_n(\chi)q^{-ns} = W(\chi)q^{\frac{\deg R}{2}}(q^{s} - 1)(q^{-s})^{\deg R-1} \sum_{n=0}^{\deg R-1} L_n(\chi)q^{-n(1-s)}$$

$$= W(\chi)q^{\frac{\deg R}{2}}(q^{1-s} - q) \sum_{n=0}^{\deg R-1} L_n(\chi)q^{(1-s)(\deg R-1-n)}$$

For any primitive character $\chi_1$ of modulus $R \neq 1$, we define $L_{-1}(\chi_1) := 0$ and recall that $L_{\deg R}(\chi_1) = 0$. If we define

$$M_i(\chi_1) := qL_{i-1}(\chi_1) - L_i(\chi_1)$$

for $i = 0, 1, \ldots, \deg R$, then (2) gives us that
Similarly as in the proof of Proposition 3.10, we take the squared modulus of both sides, and we take the terms \( n = 0, 1, \ldots, \deg R \) from the LHS and the terms \( n = 0, 1, \ldots, \deg R - 1 \) from the RHS to give

\[
(q^{1-s} - 1)^2 |L(s, \chi)|^2 = \sum_{n=0}^{\deg R} \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} M_i(\chi) M_j(\chi) \right) q^{-ns} + q^{-\deg R} \sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} M_i(\chi) M_j(\chi) \right) q^{(1-s)(2 \deg R-n)}.
\]

We now take \( s = \frac{1}{2} \) and simplify to get

\[
(q^{\frac{1}{2}} - 1)^2 |L\left(\frac{1}{2}, \chi\right)|^2 = 2 \sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} M_i(\chi) M_j(\chi) \right) q^{-\frac{n}{2}} + \sum_{0 \leq i, j \leq \deg R \atop i + j = \deg R} M_i(\chi) M_j(\chi) q^{-\frac{\deg R}{2}}.
\]

Now,

\[
\sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} M_i(\chi) M_j(\chi) \right) q^{-\frac{n}{2}} = \sum_{n=0}^{\deg R-1} q^2 \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} L_{i-1}(\chi) L_{j-1}(\chi) \right) q^{-\frac{n}{2}} - \sum_{n=0}^{\deg R-1} q \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} L_{i-1}(\chi) L_j(\chi) \right) q^{-\frac{n}{2}}
\]

\[
- \sum_{n=0}^{\deg R-1} q \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} L_i(\chi) L_{j-1}(\chi) \right) q^{-\frac{n}{2}} + \sum_{n=0}^{\deg R-1} \left( \sum_{0 \leq i, j \leq \deg R \atop i + j = n} L_i(\chi) L_j(\chi) \right) q^{-\frac{n}{2}}
\]

\[
= \sum_{n=0}^{\deg R-2} q^2 \left( \sum_{0 \leq i, j \leq \deg R-1 \atop i + j = n} L_{i-1}(\chi) L_{j-1}(\chi) \right) q^{-\frac{n}{2}} - \sum_{n=0}^{\deg R-2} q \left( \sum_{0 \leq i, j \leq \deg R-1 \atop i + j = n} L_{i-1}(\chi) L_j(\chi) \right) q^{-\frac{n}{2}}
\]

\[
- \sum_{n=0}^{\deg R-2} q^2 \left( \sum_{0 \leq i, j \leq \deg R-1 \atop i + j = n} L_i(\chi) L_{j-1}(\chi) \right) q^{-\frac{n}{2}} + \sum_{n=0}^{\deg R-2} \left( \sum_{0 \leq i, j \leq \deg R-1 \atop i + j = n} L_i(\chi) L_j(\chi) \right) q^{-\frac{n}{2}}
\]

\[
= (q^{\frac{1}{2}} - 1)^2 \sum_{A, B \in M \atop \deg AB < \deg R} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}} - q \sum_{A, B \in M \atop \deg AB = \deg R-2} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + (2q^{\frac{1}{2}} - q) \sum_{A, B \in M \atop \deg AB = \deg R-1} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\]

and similarly,

\[
\sum_{0 \leq i, j \leq \deg R \atop i + j = \deg R} M_i(\chi) M_j(\chi) q^{-\frac{\deg R}{2}}
\]

\[
= q \sum_{A, B \in M \atop \deg AB = \deg R-2} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}} - 2q^{\frac{1}{2}} \sum_{A, B \in M \atop \deg AB = \deg R-1} \frac{\chi(B) \overline{\chi}(A)}{|AB|^{\frac{1}{2}}} + \sum_{A, B \in M \atop \deg AB = \deg R} \frac{\chi(A) \overline{\chi}(B)}{|AB|^{\frac{1}{2}}}.\]
Hence,
\[
\left|L\left(\frac{1}{2}, \chi\right)\right|^2 = 2 \sum_{A, B \in M} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} - \frac{q}{(q^{\frac{1}{2}} - 1)^2} \sum_{A, B \in M} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} - \frac{2q^{\frac{1}{2}}}{q^{\frac{1}{2}} - 1} \sum_{A, B \in M} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}} + \frac{1}{(q^{\frac{1}{2}} - 1)^2} \sum_{A, B \in M} \frac{\chi(A)\overline{\chi}(B)}{|AB|^{\frac{1}{2}}},
\]
as required. \(\square\)

It is convenient to define
\[
c(\chi) := \begin{cases} 
  c_e(\chi) & \text{if } \chi \text{ is even} \\
  c_o(\chi) & \text{if } \chi \text{ is odd}
\end{cases}
\]

4. Multiplicative Functions on \(\mathbb{F}_q[T]\)

In this section we state and prove some results for the functions \(\mu, \phi\) and \(\omega\) that are required for the proofs of the main theorems.

We will need the following well-known theorem.

**Theorem 4.1** (Prime Polynomial Theorem). We have that
\[
\#P_n = \frac{q^n}{n} + O\left(\frac{q^{n}}{n}\right),
\]
where the implied constant is independent of \(q\). We reserve the symbol \(c\) for the implied constant.

We will also need the following two definitions.

**Definition 4.2** (Radical of a Polynomial, Square-free, and Square-full). For all \(R \in \mathcal{A}\) we define the radical of \(R\) to be the product of all distinct monic prime factors that divide \(R\). It is denoted by \(\text{rad}(R)\). If \(R = \text{rad}(R)\), then we say that \(R\) is square-free. If for all \(P \mid R\) we have that \(P^2 \mid R\), then we say that \(R\) is square-full.

**Definition 4.3** (Primorial Polynomials). Let \((S_i)_{i \in \mathbb{Z}_{\geq 0}}\) be a fixed ordering of all the monic irreducibles in \(\mathcal{A}\) such that \(\deg S_i \leq \deg S_{i+1}\) for all \(i \geq 1\) (the order of the irreducibles of a given degree is not of importance in this paper). For all positive integers \(n\) we define
\[
R_n := \prod_{i=1}^{n} S_i.
\]
We will refer to \(R_n\) as the \(n\)-th primorial. For each positive integer \(n\) we have unique non-negative integers \(m_n\) and \(r_n\) such that
\[
R_n = \left(\prod_{\deg P \leq m_n} P\right)\left(\prod_{i=1}^{r_n} Q_i\right),
\]
where the \(Q_i\) are distinct monic irreducibles of degree \(m_n + 1\). This definition of primorial is not standard.

Now, before proceeding to prove results on the growth of the \(\omega\) and \(\phi\) functions, we note that
\[
\sum_{E \mid R} \frac{\mu(E)}{|E|^s} = \prod_{P \mid R} \left(1 - \frac{1}{|P|^s}\right)
\]
and
\[ \sum_{E \mid R} \frac{\mu(E) \deg E}{|E|^s} = - \left( \prod_{P \mid R} \left( 1 - \frac{1}{|P|^s} \right) \right) \left( \sum_{P \mid R} \frac{\deg P}{|P|^s - 1} \right) \]

for all \( R \in \mathcal{A} \setminus \{0\} \). The first equation holds for all \( s \in \mathbb{C} \). The second holds for all \( s \in \mathbb{C} \setminus \{0\} \) and is obtained by differentiating the first with respect to \( s \).

Also, for all square-full \( R \in \mathcal{A} \setminus \{0\} \) we have that

\[ \sum_{EF = R} \frac{\mu(E) \phi(F)}{|F|^s} = \sum_{EF = R} \frac{\mu(E)}{|F|^s} \frac{\phi(F)}{|E|^s} = \frac{\phi(R)}{|R|^s} \sum_{EF = R} \frac{\mu(E)}{|E|^{1-s}} \]

and

\[ \sum_{EF = R} \frac{\mu(E) \phi(F) \deg F}{|F|^s} = \frac{\phi(R)}{|R|^s} \left( \prod_{P \mid R} \left( 1 - \frac{1}{|P|^{1-s}} \right) \right) \left( \deg R + \sum_{P \mid R} \frac{\deg P}{|P|^{1-s} - 1} \right), \]

The first equation holds for all \( s \in \mathbb{C} \). The second holds for all \( s \in \mathbb{C} \setminus \{1\} \) and is obtained by differentiating the first with respect to \( s \).

**Lemma 4.4.** For all positive integers \( n \) we have that

\[ \log_q \log_q |R_n| = m_n + O(1). \]

**Proof.** By (4) and the prime polynomial theorem, we see that

\[ \log_q |R_n| = \deg R_n \leq \left( \sum_{i=1}^{m_n+1} q^i + O \left( q^{\frac{m_n+1}{2}} \right) \right) \ll q^{m_n+1} \]

and

\[ \log_q |R_n| = \deg R_n \geq \left( \sum_{i=1}^{m_n} q^i + O \left( q^{\frac{m_n}{2}} \right) \right) \gg q^{m_n} . \]

By taking logarithms of both equations above, we deduce that

\[ \log_q \log_q |R_n| = m_n + O(1). \]

Using this result we can obtain results about the growth of the \( \omega \) function.

**Lemma 4.5.** For all primorials \( R_n \) with \( n > 1 \) we have that

\[ \omega(R_n) = \frac{\log_q |R_n|}{\log_q \log_q |R_n|} + O \left( \frac{\log_q |R_n|}{(\log_q \log_q |R_n|)^2} \right). \]

**Proof.** The cases when \( m_n = 0 \) are not difficult. We proceed with the cases where \( m_n \geq 1 \). Using the prime polynomial theorem, we have that

\[ \omega(R_n) = \sum_{i=1}^{m_n} \left( \frac{q^i}{i} + O \left( \frac{q^{\frac{i}{2}}}{i} \right) \right) + r_n = \frac{1}{q-1} \left( \sum_{i=1}^{m_n} q^{i+1} - \frac{q^i}{i+1} + \frac{q^{i+1}}{i(i+1)} \right) + O \left( \sum_{i=1}^{m_n} \frac{q^i}{i} \right) + r_n. \]

We see that
\[
\sum_{i=1}^{m_n} q_i^1 \leq \sum_{i=1}^{m_n} q_i^i + \sum_{i=m_n+1}^{m_n} q_i^i \leq \sum_{i=1}^{m_n} q_i^i + \frac{2}{m_n} \sum_{i=m_n+1}^{m_n} q_i^i = O\left(\frac{q^{m_n}}{m_n}\right).
\]

Similarly,
\[
\sum_{i=1}^{m_n} \frac{q_i^i}{i(i+1)} = O\left(\frac{q^{m_n}}{(m_n)^2}\right).
\]

Whereas, for the main term we apply the more precise calculations
\[
\frac{1}{q-1} \sum_{i=1}^{m_n} q_i^{i+1} - \frac{q_i^i}{i} = \frac{q}{q-1} \left(\frac{q^{m_n}}{m_n+1} - 1\right).
\]

Hence, (9) gives
\[
\omega(R_n) = \frac{q}{q-1} \frac{q^{m_n}}{m_n+1} + O\left(\frac{q^{m_n}}{(m_n)^2}\right) + r_n.
\]

We also have that
\[
\log_q|R_n| = \deg R_n = \left(\sum_{i=1}^{m_n} q_i^i + O\left(\frac{q^{m_n}}{m_n}\right)\right) + (m_n+1)r_n = \frac{q}{q-1} q^{m_n} + O\left(\frac{m_n}{(m_n)^2}\right) + (m_n+1)r_n;
\]

and so
\[
\frac{\omega(R_n)}{\log_q|R_n|} = \frac{1}{m_n+1} + O\left(\frac{1}{(m_n+1)^2}\right) = \frac{1}{\log_q \log_q|R_n|} + O\left(\frac{1}{(\log_q \log_q|R_n|)^2}\right),
\]

where we have used Lemma 4.4. The proof follows.

\[\square\]

Lemma 4.6. We have that
\[
\limsup_{\deg R \to \infty} \omega(R) \frac{\log_q |R|}{\log_q R} = 1.
\]

Proof. This lemma follows from Lemma 4.5 if we also prove that
\[
\omega(R) \leq \frac{\log_q |R|}{\log_q \log_q |R|} + O\left(\frac{\log_q |R|}{(\log_q \log_q |R|)^2}\right)
\]

for all \( R \) with \( \deg R \geq 4 \).

To this end, consider the case where \( R \) is a square-free monic with \( \omega(R) \geq 3 \). Suppose \( R \) has \( n \) prime divisors. Then,
\[
\omega(R) = \omega(R_n) = \frac{\log_q |R_n|}{\log_q \log_q |R_n|} + O\left(\frac{\log_q |R_n|}{(\log_q \log_q |R_n|)^2}\right) \leq \frac{\log_q |R|}{\log_q \log_q |R|} + O\left(\frac{\log_q |R|}{(\log_q \log_q |R|)^2}\right).
\]

For the second relation we used Lemma 4.5. For the third relation we used the fact that for all \( q \) the function \( x \frac{\log_q x}{\log_q \log_q x} \) is increasing at \( x > e \) and that \( \log_q |R_n| \geq \omega(R_n) \geq 3 > e \).

Now consider the more general case where \( R \) is monic and \( \omega(R) \geq 3 \). We have that
\[
\omega(R) = \omega(\text{rad}(R)) \leq \frac{\log q |\text{rad}(R)|}{\log_q \log_q |\text{rad}(R)|} + O\left(\frac{\log q |\text{rad}(R)|}{(\log_q \log_q |\text{rad}(R)|)^2}\right)
\]
\[
\leq \frac{\log_q |R|}{\log_q \log_q |R|} + O\left(\frac{\log_q |R|}{(\log_q \log_q |R|)^2}\right),
\]
where the second relation follows from (11), and, again, the third relation follows from the fact that for all \( q \) the function \( \frac{x}{\log x} \) is increasing at \( x > e \).

Finally consider the case where \( R \) is monic with \( \deg R \geq 4 \) and \( \omega(R) \leq 2 \). We have that
\[
\log q |R| = \deg R \log q \deg R \geq \frac{4}{\log_q 4} \geq 2 = \omega(R).
\]
So, we have proved (10), and this completes the proof. □

**Remark 4.7.** We note that there is an analogous result to Lemma 4.6, in the number field setting:
\[
\limsup_{r \to \infty} \omega(r) \frac{\log \log r}{\log r} = 1.
\]
We now work towards obtaining a result on the growth of the \( \phi \) function.

**Lemma 4.8.** For all integers \( n \geq 2 \) we have that
\[
e^{-1-\frac{1}{n}} \leq \left(1 - \frac{1}{n}\right)^n \leq e^{-1}.
\]

**Proof.** On one hand, we have that
\[
\log \left(\left(1 - \frac{1}{n}\right)^n\right) = n \log \left(1 - \frac{1}{n}\right) = -1 - \frac{1}{2n} - \frac{1}{3n^2} - \frac{1}{4n^3} - \ldots \leq -1.
\]
Taking the exponential of both sides gives
\[
\left(1 - \frac{1}{n}\right)^n \leq e^{-1}.
\]
On the other hand, because \( n \geq 2 \), we have that
\[
\frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{4n^3} + \ldots \leq \frac{1}{2n} \sum_{i=0}^{\infty} \frac{1}{n^i} = \frac{1}{2(n-1)} \leq \frac{1}{n},
\]
and so
\[
\log \left(\left(1 - \frac{1}{n}\right)^n\right) = -1 - \frac{1}{2n} - \frac{1}{3n^2} - \frac{1}{4n^3} - \frac{1}{5n^4} \geq -1 - \frac{1}{n}.
\]
Taking exponential of both side gives
\[
\left(1 - \frac{1}{n}\right)^n \geq e^{-1-\frac{1}{n}}.
\]
\[\square\]

**Proposition 4.9.** For all \( R \in A \) with \( \deg R \geq 1 \) we have that
\[
\phi(R) \geq \frac{e^{-\gamma |R|}}{\log_q \log_q |R| + O(1)} e^{-aq^{-\frac{b}{2}}},
\]
\[\text{(12)}\]
and for infinitely many \( R \in \mathcal{A} \) we have that

\[
\phi(R) \leq \frac{e^{-\gamma}|R|}{\log q \log_q |R| + O(1)} e^{bq^{-\frac{1}{2}}},
\]

where \( a \) and \( b \) are positive constants which are independent of \( q \) and \( R \).

**Proof.** For (12) we need only prove the case where \( R \) is square-free (and \( \deg R \geq 1 \)). Assuming this, we have that

\[
\phi(R) = |R| \frac{\phi(R)}{|R|} = |R| \frac{\phi(\text{rad}(R))}{|\text{rad}(R)|} \geq |R| \frac{e^{-\gamma}}{\log q \log_q |\text{rad}(R)| + O(1)} e^{-aq^{-\frac{1}{2}}}
\]

\[
\geq |R| \frac{e^{-\gamma}}{\log_q \log_q |R| + O(1)} e^{-aq^{-\frac{1}{2}}}.
\]

In fact, we need only prove the inequality for the case where \( R \) is a primorial, as the square-free case follows from this. Indeed, if \( R \) is square-free with \( n \geq 1 \) prime factors, then

\[
\phi(R) \geq |R| \frac{\phi(R_n)}{|R_n|} \geq |R| \frac{e^{-\gamma}}{\log_q \log_q |R_n| + O(1)} e^{-aq^{-\frac{1}{2}}} \geq |R| \frac{e^{-\gamma}}{\log_q \log_q |R| + O(1)} e^{-aq^{-\frac{1}{2}}}.
\]

So, it suffices to prove (12) for the primorials. It is clear that it is also sufficient to prove (13) for the primorials as there are infinitely many of them, and this is exactly what we will do. We first proceed with (12).

We have that

\[
\frac{\phi(R_n)}{|R_n|} = \prod_{P|R_n} \left(1 - \frac{1}{|P|}\right) \geq \prod_{\deg P \leq m+1} \left(1 - \frac{1}{|P|}\right) \geq \prod_{n=1}^{m+1} \left(1 - \frac{1}{q^n}\right)^{\frac{q^n}{m+1}}.
\]

By Lemma 4.8 we see that

\[
\prod_{n=1}^{m+1} \left(1 - \frac{1}{q^n}\right) \geq \prod_{n=1}^{m+1} \left(1 - \frac{1}{q^n}\right)^{\frac{q^n}{m+1}} \geq \prod_{n=1}^{m+1} \exp \left(- \frac{1}{n} - \frac{1}{nq^n}\right) = \exp \left(\sum_{n=1}^{m+1} -\frac{1}{n} - \frac{1}{nq^n}\right)
\]

\[
\geq \exp \left(- \log(m+1) - \gamma - \frac{1}{m+1} - \sum_{n=1}^{m+1} \frac{1}{q^n}\right)
\]

\[
\geq \frac{e^{-\gamma}}{m+1} e^{-\frac{1}{m+1} - 2q^{-1}} \geq \frac{e^{-\gamma}}{m+1} + \frac{1}{m+1} e^{-2q^{-1}} = \frac{e^{-\gamma}}{m+1 + 4} e^{-2q^{-1}},
\]

where for the second to last relation we used the fact that \( e^x \leq 1 + 3x \) for \( x \in [0,1] \). Similarly,

\[
\prod_{n=1}^{m+1} \left(1 - \frac{1}{q^n}\right)^{\frac{q^n}{m+1}} \geq \prod_{n=1}^{m+1} \exp \left(- \frac{c}{nq^n} - \frac{c}{nq^{m+1}}\right) \geq \exp \left(\sum_{n=1}^{m} -\frac{2c}{q^n}\right)
\]

\[
\geq \exp \left(- \frac{2c}{q^n} \frac{1}{1 - q^{-\frac{1}{2}}}\right) \geq e^{-\gamma q^{-\frac{1}{2}}}.
\]

Hence, we deduce that

\[
\frac{\phi(R_n)}{|R_n|} \geq \frac{e^{-\gamma}}{m+1} e^{-2q^{-1}},
\]

where \( a = 7c + 2 \). Finally, we apply Lemma 4.4 and rearrange to see that
\[ \phi(R_n) \geq \frac{e^{-\gamma |R_n|}}{\log_q \log_q |R_n| + O(1)} e^{-aq^{-\frac{1}{2}}}. \]

For (13), we proceed in a similar fashion. We have that
\[ \phi(R_n) \geq \prod_{P | R_n} \left( 1 - \frac{1}{|P|} \right) \leq \prod_{\deg P \leq m_n} \left( 1 - \frac{1}{|P|} \right) \leq \prod_{n=1}^{m_n} \left( 1 - \frac{1}{q^n} \right)^{\frac{n}{2}}. \]

By Lemma 4.8 again, we see that
\[ \prod_{n=1}^{m_n} \left( 1 - \frac{1}{q^n} \right)^{\frac{n}{2}} \leq \prod_{n=1}^{m_n} \exp \left( -\frac{1}{n} \right) \leq \exp \left( -\log(m_n) - \gamma \right) = \frac{e^{-\gamma}}{m_n}. \]

Also, by (14), we see that
\[ \prod_{n=1}^{m_n} \left( 1 - \frac{1}{q^n} \right)^{\frac{n}{2}} \leq \prod_{n=1}^{m_n} \exp \left( -\frac{1}{q^n} \right) \leq \exp \left( -\log(m_n) - \gamma \right) = \frac{e^{-\gamma}}{m_n}. \]

Hence,
\[ \frac{\phi(R_n)}{|R_n|} \leq \frac{e^{-\gamma}}{m_n} e^{bq^{-\frac{1}{2}}}, \]

where \( b = 7c \). Finally, we apply Lemma 4.4 and rearrange to see that
\[ \phi(R_n) \leq \frac{e^{-\gamma |R_n|}}{\log_q \log_q |R_n| + O(1)} e^{bq^{-\frac{1}{2}}}. \]

\[ \Box \]

**Proposition 4.10.** For all \( R \in \mathcal{A} \) with \( \deg R \geq 1 \) we have that
\[ \phi^*(R) \geq \frac{e^{-\gamma \phi(R)}}{\log_q \log_q |R| + O(1)} e^{-cq^{-\frac{1}{2}}}, \]

and for infinitely many \( R \in \mathcal{A} \) we have that
\[ \phi^*(R) \leq \frac{e^{-\gamma \phi(R)}}{\log_q \log_q |R| + O(1)} e^{dq^{-\frac{1}{2}}}, \]

where \( c \) and \( d \) are positive constants which are independent of \( q \) and \( R \).

**Proof.** From Corollary 3.8, we have that
\[ \phi^*(R) = \sum_{EF=R} \mu(E) \phi(F) = \sum_{EF=R} \mu(E) \left( \prod_{P | E, P^2 | R} \frac{1}{|P| - 1} \right) \left( \prod_{P | E, P^2 | R} \frac{1}{|P|} \right) \]
\[ = \left( \prod_{P | R, P^2 | R} \frac{1}{|P| - 1} \right) \left( \prod_{P | R, P^2 | R} \frac{1}{|P|} \right) \]

Now, by similar means as in Proposition 4.9 we can obtain that
\[ \frac{\phi^*(R)}{\phi(R)} \geq \prod_{P | R} \frac{1}{|P| - 1} \geq \frac{e^{-\gamma}}{\log_q \log_q |R| + O(1)} e^{-cq^{-\frac{1}{2}}}. \]
for all \( R \in \mathcal{A} \) with \( \deg R \geq 1 \), and that

\[
\frac{\phi^*(R)}{\phi(R)} \leq \prod_{P|R} \frac{1}{1 - \frac{1}{|P|}} \leq \frac{e^{-\gamma}}{\log_q \log_q |R| + O(1)} e^{dq^{-\frac{1}{2}}} 
\]

for infinitely many \( R \in \mathcal{A} \), where \( c \) and \( d \) are independent of \( q \) and \( R \). \( \square \)

**Remark 4.11.** By similar, but simpler, means as in the proof of Proposition 4.9 and 4.10, we can show that

\[
\prod_{P|R} \frac{1}{1 + 2|P|^{-1}} \asymp \prod_{P|R} \left( \frac{1}{1 + |P|^{-1}} \right)^2 
\]

and

\[
\prod_{P|R} \frac{1}{1 + |P|^{-1}} \asymp \prod_{P|R} 1 - |P|^{-1}. 
\]

Finally, we prove the following four lemmas.

**Lemma 4.12.** For all non-negative integers \( k \) we have that

\[
\left( \prod_{P|R} 1 - |P|^{-1} \right)^k \omega(R) \gg k. 
\]

Also,

\[
\left( \prod_{P|R} 1 - |P|^{-1} \right) \omega(R) \gg \log \omega(R). 
\]

**Proof.** Again, it suffices to prove both results for the primorials. From previous results in this section, we can see that

\[
\left( \prod_{P|R} 1 - |P|^{-1} \right)^k \omega(R) \gg q^{\frac{m_n}{(m_n)^{k+1}}} \gg k. 
\]

For the second result, we note that it is sufficient to show that

\[
\left( \prod_{P|R} 1 - |P|^{-1} \right) \omega(R) \gg \sqrt{\omega(R)}, 
\]

which is equivalent to

\[
\left( \prod_{P|R} 1 - |P|^{-1} \right)^2 \omega(R) \gg 1, 
\]

which we have proved. \( \square \)

**Lemma 4.13.** Let \( R \in \mathcal{M} \). We have that

\[
\sum_{P|R} \deg P \frac{1}{|P| - 1} = O(\log \omega(R)). 
\]
Proof. As we have done in Lemma 4.4 and Proposition 4.9, we can reduce the proof to the case where \( R \) is a primorial. That is, we need only prove that

\[
\sum_{P | R_n} \deg P \left| \frac{P}{|P| - 1} \right| - 1 = O(\log n).
\]

To this end, we recall that \( \#P_m = q^m + O\left( \frac{q^m}{m} \right) \). From this, we can deduce that there is a constant \( c \in (0, 1) \), which is independent of \( q \), such that \( \#P \leq m \geq cq^m \) for all positive integers \( m \). In particular, if we take \( m = \lceil \frac{2}{\log q} \log \frac{n}{c} \rceil \), then \( \#P \leq m \geq n \). So,

\[
\sum_{P | R_n} \deg P \left| \frac{P}{|P| - 1} \right| - 1 \leq \sum_{i=1}^{\frac{2}{\log q} \log \frac{n}{c} + 1} \sum_{P \text{ prime } \deg P = i} \deg P \left| \frac{P}{|P| - 1} \right| \ll \sum_{i=1}^{\frac{2}{\log q} \log \frac{n}{c} + 1} i q^i \ll \log n,
\]

where the second relation follows from the prime polynomial theorem.

□

Lemma 4.14. We have that

\[
\sum_{N \in M} \deg N \leq x \phi(N) \ll x.
\]

Proof. For all \( N \in \mathcal{A} \) we have that

\[
\sum_{E|N} \frac{\mu(E)^2}{\phi(E)} = \prod_{E|N} \frac{1}{|E|-1} = \prod_{E|N} \frac{1}{1-|E|-1} = \frac{|N|}{\phi(N)}.
\]

So,

\[
\sum_{N \in M} \frac{1}{\phi(N)} = \sum_{N \in M, \deg N \leq x} \frac{|N|}{\phi(N)} = \sum_{N \in M, \deg N \leq x} \frac{1}{|N|} \sum_{E|N} \frac{\mu(E)^2}{\phi(E)} = \sum_{E \in M, \deg N \leq x} \frac{\mu(E)^2}{\phi(E)} \sum_{N \in M} \frac{1}{|N|} \ll x,
\]

where the last relation uses the fact that \( \phi(N) \gg \frac{|N|}{\log_q \log_q |N|} \) (see Proposition 4.9).

□

Lemma 4.15. We have that

\[
\sum_{N \in M, \deg N \leq x} \frac{\mu^2(N)}{\phi(N)} \geq x.
\]

Proof. For square-free \( N \) we have that

\[
\frac{1}{\phi(N)} = \frac{1}{|N|} \prod_{p|N} \left( 1 - \frac{1}{|P|-1} \right) = \frac{1}{|N|} \prod_{p|N} \left( 1 + \frac{1}{|P|} + \frac{1}{|P|^2} + \ldots \right) = \sum_{M \in \mathcal{M}, \text{rad}(M) = N} \frac{1}{|M|},
\]

and so
\[
\sum_{N \in \mathcal{M}} \frac{\mu(N)^2}{\phi(N)} = \sum_{N \in \mathcal{M}} \frac{1}{|N|} \geq \sum_{M \in \mathcal{M}} \frac{1}{|M|} = z.
\]

5. The Second Moment

We now proceed to prove Theorem 2.4.

Proof of Theorem 2.4. We have that

\[
\sum_{\chi \pmod{R}} |L\left(\frac{1}{2}, \chi\right)|^2 = \sum_{\chi \pmod{R}} \sum_{A, B \in \mathcal{M}} \frac{\chi(A)\chi(B)}{|AB|^\frac{1}{2}}
\]

\[
= \sum_{EF = R} \mu(E)\phi(F) \sum_{A, B \in \mathcal{M}} \frac{1}{|EF|^\frac{1}{2}} \sum_{A, B \in \mathcal{M}} \frac{\mu(G)}{|G|^\frac{1}{2}} \sum_{G \equiv A \pmod{B}} \frac{1}{|B|^\frac{1}{2}}
\]

\[
= \sum_{EF = R} \mu(E)\phi(F) \sum_{G \equiv R \pmod{A}} \frac{1}{|G|^\frac{1}{2}} \sum_{G \equiv A \pmod{B}} \frac{1}{|B|^\frac{1}{2}}
\]

(15) \[
= \sum_{EF = R} \mu(E)\phi(F) \sum_{G \equiv R \pmod{A}} \frac{1}{|G|^\frac{1}{2}} \left( \sum_{K \in \mathcal{A}} \frac{\mu(G)}{|G|^\frac{1}{2}} \right) \left( \sum_{K \in \mathcal{A}} \frac{1}{|G|^\frac{1}{2}} \right)
\]

The last equality follows from the fact that \(F\) and \(R\) have the same prime factors, and so, if \(\mu(G) \neq 0\), then \(G | F\). Hence, if \(G | A\), then \(A \equiv GK \pmod{F}\) for some \(K \in \mathcal{A}\) with \(deg K < deg F - deg G\) or \(k = 0\).

Now, we note that if \(K \in \mathcal{A} \setminus \mathcal{M}\), then

\[
\sum_{A \in \mathcal{M}} \frac{1}{|A|^\frac{1}{2}} = \sum_{L \in \mathcal{M}} \frac{1}{|LF + GK|^\frac{1}{2}} = \sum_{L \in \mathcal{M}} \frac{1}{|LF|^\frac{1}{2}} = \frac{1}{q^2 - 1} \left( \frac{|R|^\frac{1}{2}}{|F|^\frac{1}{2}} - \frac{1}{|F|^\frac{1}{2}} \right).
\]

Whereas, if \(K \in \mathcal{M}\), then
By applying this to (15), and using (5) to (8), we see that

\[
\sum_{\substack{A \in M \\ \deg A < \deg R \\ A \equiv GK \pmod{F}}} \frac{1}{|A|^2} = \frac{1}{|GK|^2} + \sum_{\substack{L \in M \\ \deg L < \deg R - \deg F}} \frac{1}{|LF + GK|^2} = \frac{1}{|GK|^2} + \frac{1}{q^2 - 1} \left( \frac{|R|^{\frac{1}{2}}}{|F|^2} - \frac{1}{|F|^2} \right).
\]

Hence,

\[
\begin{align*}
\sum_{\substack{K \in A \\ \deg K < \deg F - \deg G \text{ or } K = 0}} & \left( \sum_{\substack{A \in M \\ \deg A < \deg R \\ A \equiv GK \pmod{F}}} \frac{1}{|A|^2} \right) \left( \sum_{\substack{B \in M \\ \deg B < \deg R \\ B \equiv GK \pmod{F}}} \frac{1}{|B|^2} \right) \\
& = \frac{1}{(q^2 - 1)^2} \left( \frac{|R|^{\frac{1}{2}}}{|F|^2} - \frac{1}{|F|^2} \right)^2 \sum_{\substack{K \in A \\ \deg K < \deg F - \deg G \text{ or } K = 0}} 1 \\
& \quad + \frac{2}{q^2 - 1} \left( \frac{|R|^{\frac{1}{2}}}{|F|^2} - \frac{1}{|F|^2} \right) \frac{1}{|G|^2} \sum_{\substack{K \in M \\ \deg K < \deg F - \deg G}} 1 \\
& \quad + \frac{1}{|G|} \sum_{\substack{K \in M \setminus A \\ \deg K < \deg F - \deg G}} \frac{1}{|K|} \\
& = \frac{1}{(q^2 - 1)^2} \left( \frac{|R|}{|FG|^2} - 2 \frac{|R|^{\frac{1}{2}}}{|F||G|^2} - \frac{1}{|G|^2} \right) + \frac{2}{q^2 - 1} \left( \frac{|R|^{\frac{1}{2}}}{|F|^2} - \frac{1}{|F|^2} \right) + \frac{\deg F}{|G|} - \frac{\deg G}{|G|}.
\end{align*}
\]

By applying this to (15), and using (5) to (8), we see that

\[
\sum_{\chi \equiv \frac{1}{2} \pmod{R}} |L(\frac{1}{2}, \chi)|^2 = \frac{\phi(R)^3}{|R|^2} \deg R + 2 \frac{\phi(R)^3}{|R|^2} \sum_{\deg P = 1} 1 + \frac{1}{(q^2 - 1)^2} \left( - \frac{\phi(R)^3}{|R|^2} \prod_{P \mid R} \left( 1 - \frac{1}{|P|^2} \right)^2 \right).
\]

\[\square\]

6. The Brun-Titchmarsh Theorem for the Divisor Function in \( \mathbb{F}_q[T] \)

In this section we prove a specific case of the function field analogue of the generalised Brun-Titchmarsh theorem. The generalised Brun-Titchmarsh theorem in the number field setting was proved by Shiu [Shi80]. It gives upper bounds for sums over short intervals and arithmetic progressions of certain multiplicative functions. We will look at the case where the multiplicative function is the divisor function in the function field setting.

The main results in this section are the following two theorems.

**Theorem 6.1.** Suppose \( \alpha, \beta \) are fixed and satisfy \( 0 < \alpha < \frac{1}{2} \) and \( 0 < \beta < \frac{1}{2} \). Let \( X \in \mathcal{M} \) and \( y \) be a positive integer satisfying \( \beta \deg X < y \leq \deg X \). Also, let \( A \in \mathcal{A} \) and \( G \in \mathcal{M} \) satisfy \( (A, G) = 1 \) and \( \deg G < (1 - \alpha)y \). Then, we have that

\[
\sum_{\substack{N \in \mathcal{M} \\ \deg(N - X) < y \\ N \equiv A \pmod{G}}} d(N) \ll_{\alpha, \beta} \frac{q^y \deg X}{\phi(G)}.
\]
Intuitively, this seems to be a good upper bound. Indeed, all $N$ in the sum are of degree equal to $\deg X$, and so this suggests that the average value that the divisor function will take is $\deg X$. Also, there are $q^y \frac{1}{\phi(G)} \approx q^y \frac{1}{\phi(G)}$ possible values for $N$ in the sum.

**Theorem 6.2.** Suppose $\alpha, \beta$ are fixed and satisfy $0 < \alpha < \frac{1}{2}$ and $0 < \beta < \frac{1}{2}$. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $\beta \deg X < y \leq \deg X$. Also, let $A \in \mathcal{A}$ and $G \in \mathcal{M}$ satisfy $(A, G) = 1$ and $\deg G < (1 - \alpha)y$. Finally, let $a \in (\mathbb{F}_q)^*$. Then, we have that

$$\sum_{N \in A} \deg(N - X) = \frac{y}{\phi(G)} \sum_{N \equiv A \pmod{G}} d(N) \ll_{\alpha, \beta} q^y \deg X \phi(G).$$

Our proofs of these two theorems are based on Shiu’s proof of the more general theorem in the number field setting [Shi80]. We begin by proving preliminary results that are needed for the main part of the proofs.

The Selberg sieve gives us the following result. A proof is given in [Web83].

**Theorem 6.3.** Let $S \subseteq \mathcal{A}$ be a finite subset. For a prime $P \in \mathcal{A}$ we define $S_P = S \cap P\mathcal{A} = \{A \in S : P \mid A\}$. We extend this to all square-free $D \in \mathcal{A}$: $S_D = S \cap D\mathcal{A}$. Furthermore, let $Q \subseteq \mathcal{A}$ be a subset of prime elements. For positive integers $z$ we define $Q_z = \prod_{\deg P \leq z} P$. We also define $S_{Q, z} = S \setminus \cup_{P \in Q, z} S_P$.

Suppose there exists a completely multiplicative function $\omega$ and a function $r$ such that for each $D \mid Q_z$ we have $\#S_D = \frac{\omega(D)}{|D|} \#S_D + r(D)$ and $0 < \omega(D) < |D|$. Also, define $\psi$ multiplicatively by $\psi(P) = \frac{|P|}{\omega(P)} - 1$ and $\psi(P^e) = 0$ for $e \geq 2$.

We then have that

$$\#S_{Q, z} := \#(S \setminus \cup_{P \in Q, z} S_P) = \#\{A \in S : (P \mid A \text{ and } P \in Q) \Rightarrow \deg P > z\} \leq \frac{\#S}{\sum_{F \in \mathcal{M}} \mu^2(F) \frac{\psi(F)}{|F|}} + \sum_{D, E \in \mathcal{M}} \sum_{\deg D, \deg E \leq z} |r([D, E])|.$$

**Corollary 6.4.** Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $y \leq \deg X$. Also, let $K \in \mathcal{M}$ and $A \in \mathcal{A}$ satisfy $(A, K) = 1$. Finally, let $z$ be a positive integer such that $\deg K + z \leq y$. Then,

$$\sum_{N \in \mathcal{M}} \frac{d(N - X) = y}{\phi(K)z} \equiv A \pmod{\mathcal{M}}$$

and $1 \leq \frac{q^y}{\phi(K)z} + O\left(q^{2z}\right)$.

**Proof.** Let us define

$$S = \{N \in \mathcal{M} : \deg(N - X) < y, N \equiv A \pmod{\mathcal{M}}\}$$

and

$$Q = \{P \text{ prime} : \deg P \leq z, P \nmid K\}.$$

Then, we have that
\[ \#S_{Q,z} = \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A \pmod{K} \\ p-N \geq z}} 1, \]

which is what we want to bound.

For \( D | Q_z \) with \( \deg D \leq z \) we have that

\[ \#S_D = \# \{ N \in M : \deg(N - X) < y, N \equiv A \pmod{K}, N \equiv 0 \pmod{0} \} = \frac{q^y}{|KD|}. \]

This follows from the fact that \( K \) and \( D \) are coprime and that \( \deg K + \deg D \leq \deg K + z \leq y \). For \( D | Q_z \) with \( \deg D > z \) we have that

\[ \#S_D = \frac{q^y}{|KD|} + c_D \]

where \( |c_D| \leq 1 \). Therefore, we have \( \omega(D) = 1 \) and \( |r(D)| \leq 1 \) for all \( D | Q_z \). We also have that \( \psi(D) = \phi(D) \) for square-free \( D \).

We can now see that

\[ \sum_{\substack{F \in M \\ \deg F \leq z \atop F|Q_z}} \frac{\mu^2(F)}{\psi(F)} = \sum_{\substack{F \in M \\ \deg F \leq z \atop (F,K)=1}} \frac{\mu(F)^2}{\phi(F)}, \]

and we have that

\[ \sum_{\substack{F \in M \\ \deg F \leq z \atop (F,K)=1}} \frac{\mu(F)^2}{\phi(F)} \geq \sum_{\substack{E \in M \\ \deg E \leq z \atop (E,K)=1}} \frac{\mu(E)^2}{\phi(E)}. \]

To this we apply Lemma 4.15 and the fact that

\[ \sum_{E|K} \frac{\mu(E)^2}{\phi(E)} = \prod_{P|K} 1 + \frac{1}{|P| - 1} = \prod_{P|K} \left( 1 - |P|^{-1} \right)^{-1} = \frac{|K|}{\phi(K)}, \]

to obtain

\[ \sum_{\substack{F \in M \\ \deg F \leq z \atop (F,K)=1}} \frac{\mu(F)^2}{\phi(F)} \geq \frac{\phi(K)}{|K|^2}. \]

Also, we have that

\[ \sum_{\substack{D,E \in M \\ \deg D, \deg E \leq z \atop D, E|Q_z}} |r([D], E)| \leq \left( \sum_{\substack{D \in M \\ \deg D \leq z \atop D, E|Q_z}} 1 \right)^2 \ll q^{2z}. \]

The result now follows by applying Theorem 6.3 and the fact that

The proof of the following corollary is almost identical to the proof above.
Corollary 6.5. Let $X \in \mathcal{M}$ and $y$ be a positive integer satisfying $y \leq \deg X$. Also, let $K \in \mathcal{M}$ and $A \in \mathcal{A}$ satisfy $(A, K) = 1$. Finally, let $z$ be a positive integer such that $\deg K + z \leq y$, and let $a \in \mathbb{F}_q^\times$. Then,

$$
\sum_{N \in A, \deg(N-X)=y, (N-X) \in \mathcal{M}, N \equiv a \pmod{K}, p_{+}(N) \leq z} 1 \leq \frac{q^y}{\phi(K)z} + O\left(q^{2z}\right).
$$

Lemma 6.6. Let $w$ be a positive integer. We have that

$$
\sum_{\deg P \leq w} \frac{1}{\deg P} \ll \frac{q^w}{w^2},
$$

where the implied constant can be taken to be independent of $q$ and will be denoted by $\mathfrak{d}$.

Proof. By using the prime polynomial theorem, we have that

$$
\sum_{\deg P \leq w} \frac{1}{\deg P} = \sum_{n=1}^{w} \frac{1}{n} \left(\frac{q^n}{n} + O\left(\frac{q^w}{n}\right)\right).
$$

Now,

$$
\sum_{n=1}^{w} \frac{q^n}{n^2} = \frac{1}{q-1} \left(\sum_{n=1}^{w} \frac{q^{n+1}}{n^2} - \frac{q^n}{n^2}\right)
$$

$$
= \frac{1}{q-1} \left(\sum_{n=1}^{w} \frac{q^{n+1}}{(n+1)^2} - \frac{q^n}{n^2}\right) + \frac{1}{q-1} \left(\sum_{n=1}^{w} \frac{1}{n} \left(\frac{q^n}{n} + O\left(\frac{q^w}{n}\right)\right)\right)
$$

$$
\leq \frac{q}{q-1} \left(\frac{q^w}{(w+1)^2} - 1\right) + \frac{3q}{q-1} \left(\sum_{n=1}^{w} \frac{q^n}{n^3} + \sum_{n=2}^{w} \frac{q^n}{n^3}\right)
$$

$$
\leq \frac{q}{q-1} \left(\frac{q^w}{(w+1)^2} - 1\right) + \frac{3q}{q-1} \left(\sum_{n=1}^{w} \frac{q^n}{n^3} + \frac{8w}{w^3} \sum_{n=1}^{w} \frac{q^n}{n}\right)
$$

$$
\ll \frac{q^w}{w^2},
$$

Also, we can easily see that

$$
\sum_{n=1}^{w} O\left(\frac{q^n}{n^2}\right) \ll q^{w} \ll \frac{q^w}{w^2}
$$

The result now follows. $\square$

Lemma 6.7. Let $0 < \alpha, \beta < \frac{1}{2}$, let $z > q$ be an integer, and let

$$
w(z) := \log_q z.
$$

Then,

$$
\sum_{N \in \mathcal{M}, \deg N \leq z, p_{+}(N) \leq w(z)} 1 \leq q^{\sqrt{w} (\log w)^\beta},
$$

as $z \to \infty$, where $\mathfrak{d}$ is as in Lemma 6.6. In particular, this implies that
\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} 1 \ll q^z
\]

(under the condition that \(z > q\)).

**Proof.** Let \(\delta > 0\). We will optimise on the value of \(\delta\) later. We have that

\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} 1 \leq q^{\delta z} \sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} |N|^{-\delta} \leq q^{\delta z} \sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} |N|^{-\delta} = q^{\delta z} \prod_{\deg P \leq w(z)} \left(1 + |P|^{-\delta} + |P|^{-2\delta} + \ldots\right)
\]

\[
= q^{\delta z} \prod_{\deg P \leq w(z)} \left(1 + \frac{1}{|P|^\delta - 1}\right) \leq q^{\delta z} \prod_{\deg P \leq w(z)} \left(\exp\left(\frac{1}{|P|^\delta - 1}\right)\right)
\]

\[
\leq q^{\delta z} \prod_{\deg P \leq w(z)} \left(\exp\left(\frac{1}{\delta \log |P|}\right)\right),
\]

where the last two relations follow from the Taylor series for the exponential function. Continuing,

\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} 1 \leq \exp\left((\delta \log q)z + \frac{1}{\delta \log q} \sum_{\deg P \leq w(z)} \frac{1}{\deg P}\right) \leq \exp\left((\delta \log q)z + \frac{1}{\delta \log q} w(z)^2\right),
\]

where the last inequality follows from Lemma 6.6. By using the definition of \(w(z)\), we have that

\[
\frac{\partial q^{w(z)}}{w(z)^2} = \frac{\partial z}{\left(\log q z\right)^2},
\]

and if we take

\[
\delta = \frac{\sqrt{\delta}}{\log z},
\]

then

\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \leq z \\
p_+(N) \leq w(z)}} 1 \leq \exp\left(\frac{\sqrt{\delta}(\log q)z}{\log z} + \frac{\sqrt{\delta}(\log z)z}{(\log q)(\log q z)^2}\right) \leq q^{\frac{\sqrt{\delta} z}{(\log z)}}.
\]

\[\square\]

**Lemma 6.8.** Let \(z\) and \(r\) be a positive integers satisfying \(r \log q r \leq z\). Then,

\[
\sum_{\substack{N \in \mathcal{M} \\
\deg N \geq z \\
p_+(N) \leq w(z)}} \frac{d(N)}{|N|} \ll z^2 \exp\left(-\frac{r \log r}{9}\right).
\]

**Proof.** Let \(\frac{1}{2} \leq \delta < 1\). We will optimise on the value of \(\delta\) later. We have that
\[
\sum_{\begin{array}{c}
N \in \mathcal{M} \\
\deg N \geq \frac{\delta}{2}
\end{array}} \frac{d(N)}{|N|} \leq \frac{\log q}{2} \sum_{\begin{array}{c}
N \in \mathcal{M} \\
\deg N \geq \frac{\delta}{2}
\end{array}} \frac{d(N)}{|N|^\delta} \leq \frac{\log q}{2} \sum_{\begin{array}{c}
N \in \mathcal{M} \\
\deg N \leq \frac{\delta}{2}
\end{array}} \frac{d(N)}{|N|^\delta}
\]

(16)

\[
\leq q^{(\delta-1)\frac{\delta}{2}} \prod_{\deg P \leq \frac{\delta}{2}} \left( 1 + \frac{2}{|P|^{\delta}} + \sum_{l=2}^\infty \frac{l+1}{|P|^{\delta}} \right)
\]

\[
\leq \exp \left( (\log q)(\delta - 1) \frac{z}{2} + 2 \sum_{\deg P \leq \frac{\delta}{2}} \frac{1}{|P|^{\delta}} + \sum_{\deg P \leq \frac{\delta}{2}} \sum_{l=2}^\infty \frac{l+1}{|P|^{\delta}} \right)
\]

where the last relation uses the Taylor series for the exponential function.

Note that

(17)

\[
\sum_{\deg P \leq \frac{\delta}{2}} \sum_{l=2}^\infty \frac{l+1}{|P|^{\delta}} \leq \sum_{P \text{ prime}} \frac{1}{|P|^{\delta}} \sum_{l=0}^\infty \frac{l+1}{|P|^{\delta}} = \sum_{P \text{ prime}} \frac{3}{|P|^{2\delta}} \sum_{l=0}^\infty \frac{1}{|P|^{\delta}} = 3 \sum_{P \text{ prime}} \left( \frac{1}{|P|^{\delta} - 1} \right)^2 = O(1),
\]

where the last relation uses the fact that \(\delta \geq \frac{3}{4}\). Also, we can write \(\frac{1}{|P|^{\delta}} = \frac{1}{|P|} + \frac{1}{|P|^\delta} \left( |P|^{1-\delta} - 1 \right)\). We have that

(18)

\[
\sum_{\deg P \leq \frac{\delta}{2}} \frac{1}{|P|} = \sum_{n=1}^{\frac{z}{2}} \frac{1}{q^n} \left( \frac{q^n}{n} + O \left( \frac{q^n}{n} \right) \right) \leq \log z - \log r + O(1) \leq \log(z) + O(1),
\]

and that

(19)

\[
\sum_{\deg P \leq \frac{\delta}{2}} \frac{1}{|P|} \left( |P|^{1-\delta} - 1 \right) = \sum_{\deg P \leq \frac{\delta}{2}} \frac{1}{|P|} \sum_{n=1}^\infty \frac{((1-\delta) \log |P|)^n}{n!}
\]

\[
\leq \sum_{n=1}^\infty \frac{(1-\delta)^n ((\log q) \frac{z}{2})^{n-1}}{n!} \sum_{\deg P \leq \frac{\delta}{2}} \frac{\log q \deg P}{|P|}
\]

\[
\leq (1 + c) \sum_{n=1}^\infty \frac{(1-\delta)^n ((\log q) \frac{z}{2})^n}{n!} = (1 + c) q^{(1-\delta)\frac{z}{2}},
\]

where the second-to-last relation follows from a similar calculation as (18).

We substitute (17), (18), and (19) into (16) to obtain

\[
\sum_{\begin{array}{c}
N \in \mathcal{M} \\
\deg N \geq \frac{\delta}{2} \\
P_+(N) \leq \frac{\delta}{2}
\end{array}} \frac{d(N)}{|N|} \ll z^2 \exp \left( \log q(\delta - 1) \frac{z}{2} + 2(1 + c)q^{(1-\delta)\frac{z}{2}} \right).
\]

We can now take \(\delta = 1 - \frac{r \log z}{4z}\) (by the conditions on \(r\) given in theorem, we have that \(\frac{3}{4} \leq \delta < 1\), as required). Then,

\[
\sum_{\begin{array}{c}
N \in \mathcal{M} \\
\deg N \geq \frac{\delta}{2} \\
P_+(N) \leq \frac{\delta}{2}
\end{array}} \frac{d(N)}{|N|} \ll z^2 \exp \left( - \frac{r \log r}{8} + 2(1 + c)r^{\frac{1}{4}} \right) \ll z^2 \exp \left( - \frac{r \log r}{9} \right).
\]

\(\square\)
Proof of Theorem 6.1. We will need to break the sum into four parts. First, we define \( z := \alpha \frac{n}{10} y \). Now, for any \( N \) in the summation range, we can write

\[
N = P_1^{e_1} \ldots P_j^{e_j} P_{j+1}^{e_{j+1}} \ldots P_n^{e_n}
\]

where \( \deg P_1 \leq \deg P_2 \leq \ldots \leq \deg P_n \) and \( j \geq 0 \) is chosen such that

\[
\deg\left( P_1^{e_1} \ldots P_j^{e_j} \right) \leq z < \deg\left( P_1^{e_1} \ldots P_j^{e_j} P_{j+1}^{e_{j+1}} \right).
\]

For convenience, we write

\[
B_N := P_1^{e_1} \ldots P_j^{e_j},
\]
\[
D_N := P_{j+1}^{e_{j+1}} \ldots P_n^{e_n}.
\]

We will consider the following cases:

1. \( p-(D_N) > \frac{1}{2} z \);
2. \( p-(D_N) \leq \frac{1}{2} z \) and \( \deg B_N \leq \frac{1}{2} z \);
3. \( p-(D_N) < w(z) \) and \( \deg B_N > \frac{1}{2} z \);
4. \( w(z) \leq p-(D_N) \leq \frac{1}{2} z \) and \( \deg B_N > \frac{1}{2} z \);

where

\[
w(z) := \begin{cases} 
1 & \text{if } z \leq q \\
\log_q(z) & \text{if } z > q.
\end{cases}
\]

**Case 1:** We have that

\[
\sum_{\substack{N \in M \\
\deg(N-X) < y \\
N \equiv A \,(\text{mod } G)} \atop p-(D_N) \geq \frac{1}{2} z} d(N) = \sum_{\substack{N \in M \\
\deg(N-X) < y \\
N \equiv A \,(\text{mod } G)} \atop p-(D_N) \geq \frac{1}{2} z} d(B_N) d(D_N) \leq \sum_{\substack{B \in M \\
\deg B \leq z \\
B \equiv A \,(\text{mod } G)} \atop (B,G) = 1} d(B) \sum_{\substack{D \in M \\
\deg(D-X_B) < y-\deg B \\
D \equiv A_B \,(\text{mod } G)} \atop p-(D) \geq \frac{1}{2} z} d(D),
\]

where \( X_B \) is a monic polynomial of degree \( \deg X - \deg B \) such that \( \deg (X - BX_B) < y \), and \( A_B \) is a polynomial satisfying \( A_B B \equiv A \,(\text{mod } G) \).

We note that

\[
\Omega(D) \leq \frac{\deg D}{p-(D)} \leq \frac{y}{\frac{1}{2} z} = \frac{20}{\alpha},
\]

and so \( d(D) \leq 2 \frac{20}{\alpha} \). Hence,

\[
\sum_{\substack{N \in M \\
\deg(N-X) < y \\
N \equiv A \,(\text{mod } G)} \atop p-(D_N) > \frac{1}{2} z} d(N) \ll_{\alpha} \sum_{\substack{B \in M \\
\deg B \leq z \\
(B,G) = 1}} d(B) \sum_{\substack{D \in M \\
\deg(D-X_B) < y-\deg B \\
D \equiv A_B \,(\text{mod } G)} \atop p-(D) > \frac{1}{2} z} 1.
\]

We can now apply Corollary 6.4 to obtain
\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A \pmod{G} \\ p_-(D_N) > \frac{1}{\varphi(y)}}} d(N) \ll_{\alpha, \beta} \frac{q^y}{\phi(G)z} \sum_{\substack{B \in M \\ \deg B \leq z \\ (B,G)=1}} \frac{d(B)}{|B|} + q^z \sum_{\substack{B \in M \\ \deg B \leq z \\ (B,G)=1}} d(B) \leq \left( \frac{2q^y}{\phi(G)z} + q^2z \right) \sum_{\substack{B \in M \\ \deg B \leq z \\ (B,G)=1}} \frac{d(B)}{|B|} \leq \left( \frac{2q^y}{\phi(G)z} + q^2z \right) z^2 \ll \frac{q^y z}{\phi(G)} \leq \frac{q^y \deg X}{\phi(G)},
\]

where the second-to-last relation uses the fact that \( \deg G \leq (1 - \alpha)y \) and \( z = \frac{\alpha}{\theta}y \).

**Case 2:** Suppose \( N \) satisfies case 2. Then, the associated \( P_{j+1} \) (from (20)) satisfies \( P_{j+1} \pmod{e_{j+1}} \mid N \), \( \deg P_{j+1} \leq \frac{1}{E}z \), and \( \deg P_{j+1} > \frac{1}{E}z \). For a general prime \( P \) with \( \deg P \leq \frac{1}{E}z \) we denote \( e_P \geq 2 \) to be the smallest integer such that \( \deg P^e_P > \frac{1}{E}z \). We will need to note for later that

\[
\sum_{\deg P \leq \frac{1}{E}z} \frac{1}{|P|^{e_P}} \ll \sum_{\deg P \leq \frac{1}{E}z} q^{-\frac{1}{E}z} + \sum_{\frac{1}{E}z < \deg P \leq \frac{1}{E}z} \frac{1}{|P|^{e_P}} \ll q^{-\frac{1}{E}z}.
\]

Let us also note that for \( N \) with \( \deg N \leq \deg X \) we have that

\[
d(N) \ll_{\alpha, \beta} |N|^{\frac{\alpha}{\theta}y} \leq |X|^{\frac{\alpha}{\theta}y} \leq q^{\frac{\alpha}{\theta}y} = q^{\frac{1}{E}z}.
\]

So,

\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A \pmod{G} \\ p_-(D_N) \leq \frac{1}{\varphi(y)}}} d(N) \ll_{\alpha, \beta} q^{\frac{1}{E}z} \sum_{\deg P \leq \frac{1}{E}z} \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A \pmod{G} \pmod{P^e_P} \pmod{N \equiv A \pmod{G}} \pmod{N \eq 0 \pmod{P^e_P}}}} d(N) \ll_{\alpha, \beta} q^{\frac{1}{E}z} \sum_{\deg P \leq \frac{1}{E}z} \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A \pmod{G} \pmod{N \eq 0 \pmod{P^e_P}}}} 1
\]

\[
\leq q^{\frac{1}{E}z} \sum_{\deg P \leq \frac{1}{E}z} \left( \frac{q^y}{|G|^{P^e_P}} + O(1) \right) \leq q^y \frac{1}{|G|} q^{\frac{1}{E}z} \sum_{\deg P \leq \frac{1}{E}z} \frac{1}{|P^{e_P}|} + O(q^{\frac{5}{E}z})
\]

\[
= q^y \frac{1}{|G|} q^{-\frac{1}{E}z} + O(q^{\frac{5}{E}z}) \ll q^y \frac{1}{|G|} q^{-\frac{1}{E}z},
\]

where the last relation follows from the fact that \( z = \frac{\alpha}{\theta}y \) and \( \deg G \leq (1 - \alpha)y \).

**Case 3:** Suppose \( N \) satisfies case 3. For the case where \( z \leq q \) we have that \( w(z) = 1 \), meaning that the only possible value \( N \) could take is 1. At most this contributes \( O(1) \).

So, suppose that \( z > q \), and so \( w(z) = \log_q z \). Case 3 tells us that \( \frac{1}{E}z < \deg B_N \leq z \) and

\[
p_+(B_N) \leq p_-(D_N) < w(z).
\]

Hence,
\[
\sum_{N \in M \atop {\deg (N - X) < y \atop N \equiv A \pmod G}} d(N) \ll_{\alpha, \beta} q^{\frac{1}{2}z} \sum_{N \in M \atop {\deg (N - X) < y \atop N \equiv A \pmod G}} 1 \\
\leq q^{\frac{1}{2}z} \sum_{B \in M \atop {\frac{1}{2}z < \deg B \leq z \atop (B,G) = 1 \atop w(z) < p_{-} (D_{N}) \leq \frac{1}{2}z \atop \frac{1}{2}z < \deg B_{N} \leq z}} 1
\]

\[
\leq q^{\frac{1}{2}z} \sum_{B \in M \atop {\frac{1}{2}z < \deg B \leq z \atop p_{+} (B) < w(z)}} \left( \frac{q^{y}}{|GB|} + O(1) \right)
\]

\[
\leq \left( \frac{q^{y}}{|G|} q^{-\frac{3}{2}z} \sum_{B \in M \atop {\frac{1}{2}z < \deg B \leq z \atop p_{+} (B) < w(z)}} \left( \frac{q^{y}}{|GB|} \right) + O(q^{\frac{9}{8}z}) \right)
\]

\[
\ll \left( \frac{q^{y}}{|G|} q^{-\frac{3}{2}z} q^{\frac{1}{8}z} \right) + O(q^{\frac{9}{8}z})
\]

\[
\ll q^{y} \left( \frac{1}{2} q^{\frac{1}{8}z} \right)
\]

as \( z \to \infty \), where the second-to-last relation follows from Lemma 6.7, and the last relation uses the fact that \( \deg G \leq (1 - \alpha) y \) and \( z = \frac{\alpha}{1 + \alpha} y \).

**Case 4:** The case \( z < 1 \) is trivial, and so we proceed under the assumption that \( z \geq 1 \). We have that

\[
\sum_{N \in M \atop {\deg (N - X) < y \atop N \equiv A \pmod G \atop w(z) < p_{-} (D_{N}) \leq \frac{1}{2}z \atop \frac{1}{2}z < \deg B_{N} \leq z}} d(N) = \sum_{B \in M \atop {\frac{1}{2}z < \deg B \leq z \atop (B,G) = 1 \atop w(z) < p_{-} (D_{N}) \leq \frac{1}{2}z \atop p_{+} (B) \leq p_{+} (B_{N})}} d(B) \sum_{N \in M \atop {\deg (N - X) < y \atop N \equiv A \pmod G \atop w(z) < p_{-} (D_{N}) \leq \frac{1}{2}z \atop p_{-} (D_{N}) \geq p_{+} (B_{N})}} d(D_{N}).
\]

We now divide \( p_{-} (D_{N}) \) into the blocks \( \frac{1}{r+1} z < p_{-} (D_{N}) \leq \frac{1}{r} z \) for \( r = 2, 3, \ldots, r_{1} \) where

\[
r_{1} = \left\lfloor \frac{z}{w(z)} \right\rfloor.
\]

For \( D_{N} \) satisfying \( \frac{1}{r+1} z < p_{-} (D_{N}) \leq \frac{1}{r} z \) we have that

\[
\Omega(D_{N}) \leq \frac{\deg X}{p_{-} (D_{N})} \leq \frac{\deg X}{r+1} \leq \frac{10 (r + 1)}{\alpha \beta} \leq \frac{20r}{\alpha \beta},
\]

and so

\[
d(D_{N}) \leq \frac{a^{20r}}{r} = a^{r},
\]

where \( a = 2^{\frac{20r}{\alpha \beta}} \). So, continuing from (24),
\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ v \langle p_-(D_N) \leq \frac{1}{2}z \\ \deg B_N > \frac{1}{2}z}} d(N) \leq \sum_{r=2}^{r_1} \sum_{\substack{B \in M \\ \frac{1}{2}z < \deg B \leq z \\ (B,G)=1 \\ p_+(B) \leq \frac{1}{2}z}} a^r d(B) \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ N \equiv 0(\text{mod } B) \\ \frac{1}{r+1} z < p_-(D_N) \leq \frac{1}{r} z}} 1,
\]

(25)

where \(X_B\) is a monic polynomial of degree \(\deg X - \deg B\) such that \(\deg X - BX_B < y\), and \(A_B\) is a polynomial satisfying \(A_B B \equiv A(\text{mod } G)\).

Corollary 6.4 tells us that

\[
1 \leq \frac{q^y}{\phi(G)|B|} \frac{r+1}{z} \sum_{r=2}^{r_1} a^r \sum_{\substack{B \in M \\ \frac{1}{2}z < \deg B \leq z \\ (B,G)=1 \\ p_+(B) \leq \frac{1}{2}z}} d(B) \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A_B(\text{mod } G) \\ p_-(D) \geq \frac{1}{r+1} z}} 1,
\]

where the last relation follows from the fact that \(\deg B \leq z\), \(z = \frac{\alpha}{10} y\), and \(\deg G \leq (1 - \alpha) y\). Hence, continuing from (25):

\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ v \langle p_-(D_N) \leq \frac{1}{2}z \\ \deg B_N > \frac{1}{2}z}} d(N) \ll \frac{q^y}{\phi(G)} \frac{1}{z} \sum_{r=2}^{r_1} (r+1)a^r \sum_{\substack{B \in M \\ \frac{1}{2}z < \deg B \leq z \\ (B,G)=1 \\ p_+(B) \leq \frac{1}{2}z}} d(B) \frac{|B|}{|B|}.
\]

Finally, we wish to apply Lemma 6.8. This requires that \(r \log_q r \leq z\). Now, when \(1 \leq z \leq q\) we have that \(w(z) = 1\) and \(r_1 = z\). Hence, \(r \log_q r \leq z \log_q q = z\). When \(z > q\) we have that \(w(z) = \log_q z\) and \(r_1 = \left\lfloor \frac{z}{w(z)} \right\rfloor\). Hence, \(r \log_q r \leq \frac{z}{\log_q z} (\log_q z - \log_q \log_q z) \leq z\), since \(z > q\). Hence,

\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ v \langle p_-(D_N) \leq \frac{1}{2}z \\ \deg B_N > \frac{1}{2}z}} d(N) \ll \frac{q^y}{\phi(G)} \frac{1}{z} \sum_{r=2}^{r_1} (r+1)a^r \exp \left( -\frac{r \log r}{9} \right) \ll \frac{q^y}{\phi(G)} \frac{1}{z} \ll \frac{q^y}{\phi(G)} \deg X.
\]

(26)

The proof now follows from (21), (22), (23), and (26). \(\square\)

**Proof of Theorem 6.2.** The proof of this theorem is almost identical to the proof of Theorem 6.1. Where we applied Corollary 6.4, we should instead apply Corollary 6.5. Also, the calculations

\[
\sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ N \equiv 0(\text{mod } P^\alpha p)}} 1 = \frac{q^y}{|GP^\alpha p|} + O(1) \quad \text{and} \quad \sum_{\substack{N \in M \\ \deg(N-X) < y \\ N \equiv A(\text{mod } G) \\ N \equiv 0(\text{mod } B)}} 1 = \frac{q^y}{|GB|} + O(1)
\]

should be replaced by
\[
\sum_{\substack{N \in A \\ \deg(N-X)=y \\ \text{gcd}(N-X,B)\neq1}} \frac{y^s}{s^k} + O(1) \quad \text{and} \quad \sum_{\substack{N \in A \\ \deg(N-X)=y \\ \text{gcd}(N-X,G)\neq1}} \frac{y^s}{s^k} + O(1),
\]
respectively.

7. Further Preliminary Results

Lemma 7.1. Let \( c \) be a positive real number, and let \( k \geq 2 \) be an integer. Then,
\[
\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \begin{cases} 0 & \text{if } 0 \leq y < 1 \\ \frac{2\pi i}{(k-1)!} \log y^{k-1} & \text{if } y \geq 1 \end{cases}
\]

Proof. We will first look at the case when \( y \geq 1 \). Let \( n \) be a positive integer, and define the following curves:

\[
l_1(n) := [c - ni, c + ni], \\
l_2(n) := [c + ni, ni], \\
l_3(n) := \left\{ ne^{it} : t \in [\pi, \frac{3\pi}{2}] \right\} \text{ (orientated anticlockwise)}, \\
l_4(n) := [-ni, c - ni], \\
L(n) := l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n).
\]

We can see that
\[
\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s} ds = \lim_{n \to \infty} \left( \int_{L(n)} \frac{y^s}{s} ds - \int_{l_2(n)} \frac{y^s}{s} ds - \int_{l_3(n)} \frac{y^s}{s} ds - \int_{l_4(n)} \frac{y^s}{s} ds \right).
\]

For the first integral we apply the residue theorem to obtain that
\[
\lim_{n \to \infty} \int_{L(n)} \frac{y^s}{s} ds = \frac{2\pi i}{(k-1)!} \log y^{k-1}.
\]

For \( j \in \{2, 4\} \) we have that
\[
\lim_{n \to \infty} \left| \int_{l_j(n)} \frac{y^s}{s} ds \right| \leq \lim_{n \to \infty} \frac{y^c}{n^k} \int_{l_j(n)} 1 ds = \lim_{n \to \infty} \frac{cy^c}{n^k} = 0.
\]

For the third integral we note that when \( s \in l_3(n) \) we have \( |y^s| \leq 1 \) (since \( \text{Re } s \leq 0 \) and \( y \geq 1 \)). Hence,
\[
\lim_{n \to \infty} \left| \int_{l_3(n)} \frac{y^s}{s^k} ds \right| \leq \lim_{n \to \infty} \frac{1}{n^k} \int_{l_3(n)} 1 ds = \lim_{n \to \infty} \frac{\pi}{n^{k-1}} = 0.
\]

So, for \( y \geq 1 \) we deduce that
\[
\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds = \frac{2\pi i}{(k-1)!} \log y^{k-1}.
\]

Now we will look at the case when \( 0 \leq y < 1 \). Again, let \( n \) be a positive integer, and define the following curves:
By (5), (6), and Lemma 4.13, we see that

\[ l_1(n) := [c - ni, c + ni], \]
\[ l_3(n) := \{c + ne^{it} : t \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\} \text{ (orientated clockwise)}, \]
\[ L(n) := l_1(n) \cup l_2(n). \]

We can see that

\[ \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds = \lim_{n \to \infty} \left( \int_{L(n)} \frac{y^s}{s^k} ds - \int_{l_2(n)} \frac{y^s}{s^k} ds \right). \]

The limit of the first integral is equal to zero by the residue theorem, because there are no poles inside \( L(n) \). The limit of the second integral is also zero, and this can be shown by a method similar to that applied for the curve \( l_3(n) \) in the case \( y \geq 1 \). So, for \( 0 \leq y < 1 \) we deduce that

\[ \int_{c-i\infty}^{c+i\infty} \frac{y^s}{s^k} ds = 0. \]

We now have the following proposition.

**Proposition 7.2.** Let \( R \in \mathcal{M} \) and let \( x \) be a positive integer. Then,

\[ \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \atop (A,R)=1}} \frac{1}{|A|} = \begin{cases} \frac{\phi(R)}{|R|} x + O(\log \omega(R)) & \text{if } x \geq \deg R \\ \frac{\phi(R)}{|R|} x + O(\log \omega(R)) + O\left(\frac{\omega(R)x}{q}\right) & \text{if } x < \deg R \end{cases}. \]

**Proof.** For all positive integers \( x \) we have that

\[ \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \atop (A,R)=1}} \frac{1}{|A|} = \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x}} \frac{1}{|A|} \sum_{E \mid (A,R)} \mu(E) = \sum_{E \mid R} \frac{\mu(E)}{|E|} \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \atop (A,R)}} 1 = \sum_{E \mid R} \frac{\mu(E)}{|E|} \sum_{\substack{A \in \mathcal{M} \\ \deg A \leq x \atop (A,R)}} \frac{1}{|A|} = \sum_{E \mid R} \frac{\mu(E)}{|E|} (x - \deg E) = \sum_{E \mid R} \frac{\mu(E)}{|E|} (x - \deg E) - \sum_{E \mid R} \frac{\mu(E)}{|E|} (x - \deg E). \]

By (5), (6), and Lemma 4.13 we see that

\[ \sum_{E \mid R} \frac{\mu(E)}{|E|} (x - \deg E) = \frac{\phi(R)}{|R|} x + O\left(\frac{\phi(R)}{|R|} \log \omega(R)\right). \]

When \( x \geq \deg R \), it is clear that

\[ \sum_{E \mid R \atop \deg E > x} \frac{\mu(E)}{|E|} (x - \deg E) = 0. \]

Whereas, when \( x < \deg R \), we have that

\[ \sum_{E \mid R \atop \deg E > x} \frac{\mu(E)}{|E|} (x - \deg E) \ll \sum_{E \mid R \atop \deg E > x} \frac{|\mu(E)| \deg E}{|E|} \ll \frac{x}{q^x} \sum_{E \mid R \atop \deg E > x} |\mu(E)| \ll \frac{2\omega(R)x}{q^x}. \]

The proof follows.
Lemma 7.3. For all $R \in A$ and all $s \in \mathbb{C}$ with $\text{Re}(s) > -1$ we define

$$f_R(s) := \prod_{P|R} \frac{1 - |P|^{-s - 1}}{1 + |P|^{-s - 1}}.$$  

Then, for all $R \in A$ and $j = 1, 2, 3, 4$ we have that

$$f_R^{(j)}(0) \ll (\log_q \log_q |R|)^j \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}}.$$ 

Remark 7.4. We must mention that, in the lemma and the proof, the implied constants may depend on $j$, for example; but because there are only finitely many cases of $j$ that we are interested in, we can take the implied constants to be independent.

Proof. First, we note that

$$f_R'(s) = g_R(s)f_R(s),$$

where

$$g_R(s) := \sum_{P|R} 2 \log |P| \left( \frac{1}{|P|^{s+1} + 1} + \frac{1}{|P|^{2s+2} - 1} \right).$$

We note further that

$$f_R''(s) = \left( g_R(s)^2 + g_R'(s) \right) f_R(s),$$

$$f_R'''(s) = \left( g_R(s)^3 + 3 g_R(s) g_R'(s) + g_R''(s) \right) f_R(s),$$

$$f_R''''(s) = \left( g_R(s)^4 + 6 g_R(s)^2 g_R'(s) + 4 g_R(s) g_R''(s) + 3 g_R'(s)^2 + g_R''(s) \right) f_R(s).$$

For all $R \in A$ and $k = 0, 1, 2, 3$ it is not difficult to deduce that

$$g_R^{(k)}(0) \ll \sum_{P|R} \frac{(\log |P|)^{k+1}}{|P| - 1}.$$ 

The function $\frac{(\log x)^{k+1}}{x-1}$ is decreasing at large enough $x$, and the limit as $x \to \infty$ is 0. Therefore, there exist an independent constant $c \geq 1$ such that for $k = 0, 1, 2, 3$ and all $A, B \in A$ with $\deg A \leq \deg B$ we have that

$$c \frac{(\log |A|)^{k+1}}{|A| - 1} \geq \frac{(\log |B|)^{k+1}}{|B| - 1}.$$

Hence, taking $n = \omega(R)$, we see that

$$\sum_{P|R} \frac{(\log |P|)^{k+1}}{|P| - 1} \ll \sum_{P|R_n} \frac{(\log |P|)^{k+1}}{|P| - 1} \ll \sum_{r=1}^{m_n+1} \frac{q^r}{q^r - 1} \ll \sum_{n=1}^{m_n+1} r^k \ll (m_n + 1)^{k+1}.$$ 

where we have used the prime polynomial theorem and Lemma [1.4].

So, by (27)–(30) and the fact that
we deduce that
\[ f_R^{(j)}(0) \ll \left( \log_q |R| \right)^j \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}}. \]

Proposition 7.5. Let \( R \in A \), and define \( z_R' := \deg R - \log_q 9^\omega(R) \). We have that
\[
\sum_{\substack{N \in M \\ \deg N \leq z_R' \atop (N,R)=1}} \frac{2^{\omega(N)}}{|N|^s} (z_R' - \deg N)^2 = \frac{(1 - q^{-1})}{12} \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^4
\]
\[ + O \left( \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \left( (\deg R)^2 \omega(R) + (\deg R)^3 \log \deg R \right) \right). \]

Proof. \textbf{STEP 1}: Let us define the function \( F \) for \( \Re s > 1 \) by
\[
F(s) = \sum_{\substack{N \in M \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|^s}.
\]
We can see that
\[
F(s) = \prod_{P \text{ prime}} \left( 1 + \frac{2}{|P|^s} + \frac{2}{|P|^{2s}} + \frac{2}{|P|^{3s}} + \cdots \right) = \prod_{P \text{ prime}} \left( \frac{2}{1 - |P|^{-s}} - 1 \right)
\]
\[ = \prod_{P \text{ prime}} \left( \frac{1 + |P|^{-s}}{1 - |P|^{-s}} \right) \prod_{P|R} \left( \frac{1 - |P|^{-s}}{1 + |P|^{-s}} \right) = \zeta_A(s) \prod_{P | R} \left( \frac{2}{1 + |P|^{-s}} \right).
\]

Now, let \( c \) be a positive real number, and define \( y_R := q^{z_R'} \). On the one hand, we have that
\[
\frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} F(1+s) \frac{y_R^s}{s^3} ds = \frac{1}{\pi i} \sum_{\substack{N \in M \\ (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \int_{c-i\infty}^{c+i\infty} \frac{y_R^s}{|N|^s s^3} ds = \sum_{\substack{N \in M \\ \deg N \leq z_R' \atop (N,R)=1}} \frac{2^{\omega(N)}}{|N|} \log \left( \frac{y_R}{|N|} \right)^2
\]
\[ = (\log q)^2 \sum_{\substack{N \in M \\ \deg N \leq z_R' \atop (N,R)=1}} \frac{2^{\omega(N)}}{|N|} (z_R' - \deg N)^2,
\]
where the second equality follows from Lemma 7.4. On the other hand, for all positive integers \( n \) define the following curves:
\[ l_1(n) := \left[ c - \frac{(2n + 1)\pi i}{\log q}, c + \frac{(2n + 1)\pi i}{\log q} \right] \]
\[ l_2(n) := \left[ c + \frac{(2n + 1)\pi i}{\log q}, -\frac{1}{4} + \frac{(2n + 1)\pi i}{\log q} \right] \]
\[ l_3(n) := \left[ -\frac{1}{4} + \frac{(2n + 1)\pi i}{\log q}, -\frac{1}{4} - \frac{(2n + 1)\pi i}{\log q} \right] \]
\[ l_4(n) := \left[ -\frac{1}{4} - \frac{(2n + 1)\pi i}{\log q}, c - \frac{(2n + 1)\pi i}{\log q} \right] \]
\[ L(n) := l_1(n) \cup l_2(n) \cup l_3(n) \cup l_4(n). \]

Then, we have that

\[ (32) \quad \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} F(1 + s) \frac{yR^s}{s^3} ds = \lim_{n \to \infty} \frac{1}{\pi i} \left( \int_{L(n)} F(1 + s) \frac{yR^s}{s^3} ds - \int_{l_2(n)} F(1 + s) \frac{yR^s}{s^3} ds - \int_{l_3(n)} F(1 + s) \frac{yR^s}{s^3} ds - \int_{l_4(n)} F(1 + s) \frac{yR^s}{s^3} ds \right). \]

**STEP 2:** For the first integral in \((32)\) we note that \(F(1 + s) \frac{yR^s}{s^3}\) has a fifth-order pole at \(s = 0\) and double poles at \(s = \frac{2n\pi i}{\log q}\) for \(m = \pm 1, \pm 2, \ldots, \pm n\). By applying Cauchy’s residue theorem we see that

\[ (33) \quad \lim_{n \to \infty} \frac{1}{\pi i} \int_{L(n)} F(1 + s) \frac{yR^s}{s^3} ds = 2 \text{Res}_{s=0} F(s + 1) \frac{yR^s}{s^3} + 2 \sum_{m \in \mathbb{Z}, m \neq 0} \text{Res}_{s=m} \frac{2m\pi i}{\log q} F(1 + s) \frac{yR^s}{s^3}. \]

**STEP 2.1:** For the first residue term we have that

\[ (34) \quad \text{Res}_{s=0} F(s + 1) \frac{yR^s}{s^3} = \frac{1}{4!} \lim_{s \to 0} ds^4 \left( \zeta_A(s + 1) 2^s \frac{1}{\zeta_A(2s + 2)} \prod_{p \mid R} \left( \frac{1 - |P|^{-s - 1}}{1 + |P|^{-s - 1}} \right) yR^s \right). \]

If we apply the product rule for differentiation, then one of the terms will be

\[ \frac{1}{4!} \lim_{s \to 0} \left( \zeta_A(s + 1) 2^s \frac{1}{\zeta_A(2s + 2)} \prod_{p \mid R} \left( \frac{1 - |P|^{-s - 1}}{1 + |P|^{-s - 1}} \right) \frac{d^4}{ds^4} yR^s \right) \]
\[ = \frac{(1 - q^{-1})(\log q)^2}{24} \prod_{p \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (yR')^4 \]
\[ = \frac{(1 - q^{-1})(\log q)^2}{24} \prod_{p \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^4 + O \left( \log q \prod_{p \mid R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R) \right). \]

Now we look at the remaining terms that arise from the product rule. By using the fact that \(\zeta_A(1+s) = \frac{1}{1-q^s}\) and the Taylor series for \(q^{-s}\), we have for \(k = 0, 1, 2, 3, 4\) that

\[ (35) \quad \lim_{s \to 0} \frac{1}{(\log q)^{k-1}} ds^k \frac{d^k}{ds^k} \zeta(s + 1)s = O(1), \]

Similarly,

\[ (36) \quad \lim_{s \to 0} ds^k (2s + 2)^{-1} = \lim_{s \to 0} ds^k \left( 1 - q^{-1 - 2s} \right) = O(1), \]

By \((35), (36), \) and Lemma 7.3 we see that the remaining terms are of order
\[(\log q)^2 \prod_{P|R} \left( \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^3 \log \deg R.\]

Hence,

\[2 \text{Res}_{s=0} F(s + 1) \frac{yR^s}{s^3} = \frac{(1 - q^{-1})(\log q)^2}{12} \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^4 + O \left( (\log q)^2 \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) ((\deg R)^3 \omega(R) + (\deg R)^3 \log \deg R) \right).\]

\textbf{STEP 2.2:} Now we look at the remaining residue terms in (33). By similar (but simpler) means as above can show that

\[\text{Res}_{s = \frac{2m\pi i}{\log q}} F(1 + s) \frac{yR^s}{s^3} = O \left( \frac{1}{m^3} (\log q)^2 \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \deg R \right),\]

and so

\[\sum_{m \in \mathbb{Z} \atop m \neq 0} \text{Res}_{s = \frac{2m\pi i}{\log q}} F(1 + s) \frac{yR^s}{s^3} = O \left( (\log q)^2 \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) \deg R \right).\]

\textbf{STEP 2.3:} By (33), (37) and (38), we see that

\[\lim_{n \to \infty} \frac{1}{\pi i} \int_{L(n)} F(1 + s) \frac{yR^s}{s^3} ds = \frac{(1 - q^{-1})(\log q)^2}{12} \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg R)^4 + O \left( (\log q)^2 \left( \prod_{P|R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) ((\deg R)^3 \omega(R) + (\deg R)^3 \log \deg R) \right).\]

\textbf{STEP 3:} We now look at the integrals over \(l_2(n)\) and \(l_4(n)\). For all positive integers \(n\) and all \(s \in l_2(n), l_4(n)\) we have that \(F(s + 1)yR^s = O_{q,R}(1)\). One can now easily deduce for \(i = 2, 4\) that

\[\lim_{n \to \infty} \frac{1}{\pi i} \int_{l_i(n)} F(1 + s) \frac{yR^s}{s^3} ds = 0.\]

\textbf{STEP 4:} We now look at the integral over \(l_3(n)\). For all positive integers \(n\) and all \(s \in l_3(n)\) we have that

\[\frac{\zeta_A(s + 1)^2}{\zeta_A(2s + 2)} = O(1)\]

and
We can now easily deduce that

\[
\lim_{n \to \infty} \frac{1}{\pi i} \int_{l_3(n)} F(1 + s) \frac{yR^s}{s^3} ds = O(1).
\]

\[\text{(41)}\]

**STEP 5:** By (31), (32), (39), (40) and (41), we deduce that

\[
\sum_{N \in M} \frac{2^{\omega(N)}}{|N|} (z_{R'} - \deg N)^2 = \frac{(1 - q^{-1})}{12} \left( \prod_{P|\mathfrak{R}} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg \mathfrak{R})^4
\]

\[
+ O \left( \left( \prod_{P|\mathfrak{R}} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\deg \mathfrak{R})^3 \omega(\mathfrak{R}) + (\deg \mathfrak{R})^3 \log \deg \mathfrak{R} \right).
\]

□

**Lemma 7.6.** We have that

\[
\sum_{N \in M, \deg N \leq x} \frac{2^{\omega(N)}}{|N|} = \frac{q - 1}{2q} x^2 + \frac{3q + 1}{2q} x + 1.
\]

From this we easily deduce that

\[
\sum_{N \in M, \deg N \leq x} \frac{2^{\omega(N)}}{|N|} = O(x^2).
\]

**Proof.** For \(s > 1\) we define

\[
F(s) := \sum_{N \in M} \frac{2^{\omega(N)}}{|N|^{s+1}}.
\]

We can see that

\[
F(s) = \prod_{P \text{ prime}} \left( 1 + \frac{2}{|P|^{s+1}} + \frac{2}{|P|^{2(s+1)}} + \frac{2}{|P|^{3(s+1)}} + \ldots \right) = \prod_{P \text{ prime}} \left( \frac{2}{1 - |P|^{-s-1}} - 1 \right)
\]

\[
= \prod_{P \text{ prime}} \frac{1 - |P|^{-2(s+1)}}{(1 - |P|^{-s-1})^2} = \frac{\zeta(s + 1)^2}{\zeta(2s + 2)} = \left( \sum_{n=0}^{\infty} q^{-ns} \right)^2 \left( 1 - q^{-1-2s} \right).
\]

By comparing the coefficients of powers of \(q^{-s}\), we see that
\[
\sum_{N \in M, \deg N \leq x} \frac{2^\omega(N)}{|N|} = \left( \sum_{n=0}^{x} n + 1 \right) - \frac{1}{q} \left( \sum_{n=2}^{x} n - 1 \right) = \frac{q-1}{2q} x^2 + \frac{3q+1}{2q} x + 1.
\]

Lemma 7.7. Let \( R \in \mathcal{M} \). We have that

\[
\sum_{N \in M, \deg N \leq \deg R, (N,R)=1} \frac{2^\omega(N)}{|N|} \ll \left( \prod_{P|R} \frac{1}{1+2|P|^{-1}} \right) (\deg R)^2 \times \left( \prod_{P|R} \frac{1-|P|^{-1}}{1+|P|^{-1}} \right) (\deg R)^2.
\]

Proof. We have that

\[
\left( \sum_{N \in M, \deg N \leq \deg R, (N,R)=1} \frac{2^\omega(N)}{|N|} \right) \left( \sum_{E|R} \frac{2^\omega(E)}{|E|} \right) \leq \sum_{N \in M, \deg N \leq 2 \deg R} \frac{2^\omega(N)}{|N|} \ll (\deg R)^2.
\]

where the last relations follows from Lemma 7.6. We also note that

\[
\sum_{E|R} \frac{2^\omega(E)}{|E|} \geq \sum_{E|R} \mu(E)^2 2^\omega(E) = \prod_{P|R} \left( 1 + \frac{2}{|P|} \right).
\]

This proves the first relation in the lemma. The second relation follows from Lemma 4.11. \(\Box\)

Lemma 7.8. Let \( F,K \in \mathcal{M}, x \geq 0 \), and \( a \in (\mathbb{F}_q)^\times \). Suppose also that \( \frac{1}{2} x < \deg KF \leq \frac{3}{4} x \). Then,

\[
\sum_{N \in M, \deg N = x - \deg KF, (N,F)=1} d(N) d(KF + aN) \ll q^x x^2 \frac{1}{|KF|} \sum_{H|K \ deg H \leq \deg \frac{KF}{2}} \frac{d(H)}{|H|}.
\]

Proof. We have that,

\[
\sum_{N \in M, \deg N = x - \deg KF, (N,F)=1} d(N) d(KF + aN) \leq 2 \sum_{N \in M, \deg N = x - \deg KF, (N,F)=1} \sum_{G|N} d(KF + aN) \leq \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF} d(KF + aN) = \sum_{H|K \ deg H \leq \deg \frac{K}{2}} \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF} d(KF + aN) \]

\[
= \sum_{H|K \ deg H \leq \deg \frac{K}{2}} \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF - \deg H} d(\frac{NK}{N}) \cdot d(\frac{NF}{N}) = \sum_{H|K \ deg H \leq \deg \frac{K}{2}} \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF - \deg H} d(\frac{NK}{N}) \cdot d(\frac{NF}{N}).
\]

\[
= \sum_{H|K \ deg H \leq \deg \frac{K}{2}} \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF - \deg H} d(\frac{NK}{N}) \cdot d(\frac{NF}{N}) \ll \sum_{H|K \ deg H \leq \deg \frac{K}{2}} \sum_{G \in M, \deg G \leq \frac{1}{2} \deg KF} \sum_{N \in M, \deg N = x - \deg KF - \deg H} d(\frac{NK}{N}) \cdot d(\frac{NF}{N}).
\]
where $N', G', K'$ are defined by $HN' = N, HG' = G, HK' = K$. Continuing, we have that

$$\sum_{N \in \mathcal{M}, \deg N \leq x - \deg KF} d(N)d(KF + aN) \ll q^x \frac{1}{|KF|} \sum_{H | K} \frac{d(H)}{|H|} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2} \deg KF, (G, F) = 1, (G, K) = H} \frac{1}{\phi(G')}$$

Continuing, we have that

$$\sum_{N \in \mathcal{M}, \deg N = x} d(N)d(KF + N) \ll q^x \frac{1}{|KF|} \sum_{H | K} \frac{d(H)}{|H|} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2} \deg KF, (G, F) = 1, (G, K) = H} \frac{1}{\phi(G')}$$

The third relation holds by Theorem 6.2 with $\beta = \frac{1}{6}$ and $\alpha = \frac{1}{4}$ (one may wish to note that $(K', G') = 1$ and that the other conditions of the theorem are satisfied because $\frac{1}{2}x < \deg KF \leq \frac{3}{4}x$). The last relation follows from Lemma 4.14.

**Lemma 7.9.** Let $F, K \in \mathcal{M}$ and $x \geq 0$ satisfy $\deg KF < x$. Then,

$$\sum_{N \in \mathcal{M}, \deg N = x} d(N)d(KF + N) \ll q^x \frac{1}{|KF|} \sum_{H | K} \frac{d(H)}{|H|}$$

**Proof.** The proof is similar to the proof of Lemma 7.8. We have that

$$\sum_{N \in \mathcal{M}, \deg N = x} d(N)d(KF + N) \leq 2 \sum_{N \in \mathcal{M}, \deg N = x} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2}} d(KF + N) \ll \sum_{N \in \mathcal{M}, \deg N = x} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2}} d(KF + N)$$

$$= \sum_{H | K} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2}, (G, F) = 1, (G, K) = H} \sum_{N \in \mathcal{M}, \deg N = x} d(KF + N)$$

$$= \sum_{H | K} \sum_{G \in \mathcal{M}, \deg G \leq \frac{1}{2}, (G, F) = 1, (G, K) = H} \sum_{N' \in \mathcal{M}, \deg N' = x - \deg H} d(HK'F + HN')$$

where $N', G', K'$ are defined by $HN' = N, HG' = G, HK' = K$. Continuing, we have that
\[
\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F) = 1}} d(N)d(KF + N) \ll \sum_{\substack{H | K \\ \deg H \leq \frac{x}{2}}} d(H) \sum_{\substack{G \in \mathcal{M} \\ \deg G \leq \frac{x}{2} \\ (G,F) = 1 \\ (G,K) = H}} \frac{1}{\phi(G)} \ll q^x x^2 \sum_{\substack{H | K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|},
\]

where we define \( X := T^{x - \deg H} \). We can now apply Theorem 6.1 to obtain that

\[
\sum_{\substack{N \in \mathcal{M} \\ \deg N = x \\ (N,F) = 1}} d(N)d(KF + N) \ll q^x x^2 \sum_{\substack{H | K \\ \deg H \leq \frac{x}{2}}} \frac{d(H)}{|H|}. 
\]

\[\square\]

**Lemma 7.10.** Let \( F \in \mathcal{M} \) and \( z_1, z_2 \) be non-negative integers. Then, for all \( \epsilon > 0 \) we have that

\[
\sum_{\substack{A, B, C, D \in \mathcal{M} \\ \deg AB = z_1 \\ \deg CD = z_2 \\ (ABCBD, F) = 1 \\ AC \equiv BD \mod F \\ AC \neq BD}} \frac{1}{|F|^2} \left( q^{z_1} q^{z_2} \right)^{1+\epsilon} \ll q^{z_1} q^{z_2} (z_1 + z_2)^3
\]

if \( z_1 + z_2 \leq \frac{19}{10} \deg F \)

\[
\ll \frac{1}{\phi(F)} q^{z_1} q^{z_2} (z_1 + z_2)^3
\]

if \( z_1 + z_2 > \frac{19}{10} \deg F \).

**Proof.** We can split the sum into the cases \( \deg AC > \deg BD \), \( \deg AC < \deg BD \), and \( \deg AC = \deg BD \) with \( AC \neq BD \). The first two cases are identical by symmetry.

When \( \deg AC > \deg BD \), we have that \( AC = KF + BD \) where \( K \in \mathcal{M} \) and \( \deg KF > \deg BD \). Furthermore,

\[
2 \deg KF = 2 \deg AC > \deg AC + \deg BD = \deg AB + \deg CD = z_1 + z_2,
\]

from which we deduce that \( \frac{z_1 + z_2}{2} < \deg KF \leq z_1 + z_2 \); and

\[
\deg KF + \deg BD = \deg AC + \deg BD = z_1 + z_2,
\]

from which we deduce that \( \deg BD = z_1 + z_2 - \deg KF \).

When \( \deg AC = \deg BD \), we must have that \( \deg AC = \deg BD = \frac{z_1 + z_2}{2} \) (in particular, this case applies only when \( z_1 + z_2 \) is even). Also, we can write \( AC = KF + BD \), where \( \deg KF < \deg BD = \frac{z_1 + z_2}{2} \) and \( K \) need not be monic.

So, writing \( N = BD \), we have that
\[
\sum_{A,B,C,D \in M} 1 \leq \sum_{K \in M} \sum_{N \in M} d(N)d(KF + N) \\
\sum_{z_1 + z_2 - \deg KF \leq z_1 + z_2} d(N)d(KF + N)
\]

\[
\sum_{ \frac{z_1 + z_2}{2} < \deg KF \leq z_1 + z_2} d(N)d(KF + N)
\]

\[
\ll \epsilon \left( q^{z_1} q^{z_2} \right)^{\frac{1 + \epsilon}{2}} \sum_{K \in M} \sum_{N \in M} \frac{1}{|KF|}
\]

\[
\ll \epsilon \left( q^{z_1} q^{z_2} \right)^{1 + \epsilon} \frac{1}{|F|}
\]

As for the sum

\[
\sum_{K \in A} \sum_{N \in M} d(N)d(KF + N),
\]

we note that it does not apply to this case where \(z_1 + z_2 \leq \frac{19}{10} \deg F\) because \(\deg KF \geq \deg F \geq \frac{20}{19} \frac{z_1 + z_2}{2}\), which does not overlap with range \(\deg KF < \frac{z_1 + z_2}{2}\) in the sum.

Hence,

\[
\sum_{A,B,C,D \in M} 1 \ll \epsilon \left( q^{z_1} q^{z_2} \right)^{1 + \epsilon} \frac{1}{|F|}
\]

**STEP 1**: Let us consider the case when \(z_1 + z_2 \leq \frac{19}{10} \deg F\). By using well known bounds on the divisor function, we have that

**STEP 2**: We now consider the case when \(z_1 + z_2 > \frac{19}{10} \deg F\).

**STEP 2.1**: We consider the subcase where \(\frac{z_1 + z_2}{2} < \deg KF \leq \frac{3(z_1 + z_2)}{4}\). This allows us to apply Lemma 7.8 for the first relation below.
\[ \sum_{\substack{K \in M \\ \deg K \leq \frac{3(z_1 + z_2)}{2}}} \sum_{\substack{N \in M \\ \deg N = \frac{3(z_1 + z_2)}{4}}} d(N)d(KF + N) \]
\[ \ll q^{z_1} q^{z_2} \left( z_1 + z_2 \right)^{\frac{2}{2}} \sum_{\substack{K \in M \\ \deg K \leq \frac{3(z_1 + z_2)}{4}}} \sum_{\substack{H \in M \\ \deg H \leq z_1 + z_2}} \frac{1}{|K|} \sum_{\substack{H \mid K}} \frac{d(H)}{|H|} \]
\[ \leq q^{z_1} q^{z_2} \left( z_1 + z_2 \right)^{\frac{2}{2}} \sum_{\substack{H \in M \\ \deg H \leq z_1 + z_2}} \frac{d(H)}{|H|} \sum_{\substack{K \in A \\ H \mid K \deg K \leq z_1 + z_2}} \frac{1}{|K|} \]
\[ \leq q^{z_1} q^{z_2} \left( z_1 + z_2 \right)^{\frac{3}{2}} \sum_{\substack{H \in M \\ \deg H \leq z_1 + z_2}} \frac{d(H)}{|H|^2} \]
\[ \ll q^{z_1} q^{z_2} \left( z_1 + z_2 \right)^{\frac{3}{2}} \frac{1}{|F|}. \]

**STEP 2.2:** Now we consider the subcase where \( \frac{3(z_1 + z_2)}{4} < \deg KF \leq z_1 + z_2 \). We have that

\[ \sum_{\substack{K \in M \\ \deg K \leq z_1 + z_2}} \sum_{\substack{N \in M \\ \deg N = z_1 + z_2 - \deg K}} d(N)d(KF + N) \]
\[ = \sum_{\substack{N \in M \\ \deg N < z_1 + z_2}} \sum_{\substack{K \in M \\ \deg KF = z_1 + z_2 - \deg N}} d(N)d(KF + N) \]
\[ \leq \sum_{\substack{N \in M \\ \deg N < z_1 + z_2}} d(N) \sum_{\substack{M \in M \\ \deg (M - X(N)) < z_1 + z_2 - \deg N \equiv 0 \pmod{F}}} d(M) \]

where we define \( X(N) = T^{z_1 + z_2 - \deg N} \). We can now apply Theorem 6.1. One may wish to note that

\[ y = z_1 + z_2 - \deg N \geq \frac{3}{4}(z_1 + z_2) \geq \frac{3}{10} \deg F \]

and so

\[ \deg F \leq \frac{40}{3} y = (1 - \alpha)y \]

where \( 0 < \alpha < \frac{1}{10} \), as required. Hence, we have that
STEP 2.3: We now look at the sum

$$\sum_{K \in A} \sum_{N \in M\nmid (N,F)=1} d(N)d(KF + N).$$

By Lemma 7.9 we have that

$$\sum_{K \in A} \sum_{N \in M\nmid (N,F)=1} d(N)d(KF + N) \ll q^{z_1 + z_2}(z_1 + z_2)^2 \sum_{K \in A} \sum_{H \mid K} \frac{d(H)}{|H|}$$

$$\leq q^{z_1 + z_2 - 1}(z_1 + z_2)^2 \frac{1}{|F|} \sum_{K \in A} \sum_{H \mid K} \frac{1}{|K|} \sum_{H \mid K} \frac{d(H)}{|H|}$$

$$\leq q^{z_1 + z_2}(z_1 + z_2)^3 \frac{1}{|F|},$$

where the last relation uses a similar calculation as that in Step 2.1.

STEP 2.4: We apply steps 2.1, 2.2, and 2.3 to (42) and we see that

$$\sum_{A,B,C,D \in M} \sum_{\deg AB = z_1} \sum_{\deg CD = z_2} \sum_{(ABCD,F) = 1} \sum_{AC \equiv BD \pmod{F}} 1 \ll q^{z_1 + z_2}(z_1 + z_2)^3 \frac{1}{|F|},$$

for $z_1 + z_2 \geq \frac{40}{10} \deg F$.

In fact, we can prove the following, more general Lemma:

**Lemma 7.11.** Let $F \in M$, $z_1, z_2$ be non-negative integers, and let $a \in (\mathbb{F}_q)^*$. Then, for all $\epsilon > 0$ we have that
Proof. The case where \( a = 1 \) is just Lemma 7.10 The proof of the case where \( a \neq 1 \) is very similar to the proof of Lemma 7.10. In fact it is easier, because the the case where \( \deg AC = \deg BD \) cannot exist: We would require that \( AC \) and \( BD \) are both monic, but also require that at least one of \( AC \) and \( BD \) have leading coefficient equal to \( a \neq 1 \).

Proposition 7.12. Let \( R \in \mathcal{M} \) and define \( z_R := \deg R - \log_q 2^{\omega(R)} \). Also, let \( a \in (\mathbb{F}_q)^* \). Then,

\[
\sum_{EF=R} \frac{\mu(E)\phi(F)}{|ABCD|^2} \ll \frac{1}{|ABCD|^2} \sum_{\substack{A,B,C,D \in \mathcal{M} \\
\deg AB \leq z_R \\
\deg CD \leq z_R \\
(ABCD,R) = 1 \\
AC \equiv aBD(\text{mod } F) \\
AC \neq BD}} \left( q^{z_1}q^{z_2} \right)^{\frac{1}{2} + \epsilon} + \frac{1}{|\phi(F)|} \sum_{\substack{z_1, z_2 \leq z_R \\
\deg F < z_1 + z_2 \leq 2\deg R \cap \frac{19}{10} \deg F < z_1 + z_2 \leq 2\deg R}} q^{z_1}q^{z_2} (z_1 + z_2)^3
\]

where the second-to-last relation uses the following:

\[
\sum_{EF=R} \frac{\mu(E)|\phi(F)|}{|F|^{\frac{1}{10} - 2\epsilon}} \leq \sum_{EF=R} |\mu(E)|\phi(F) = \phi(R) \sum_{EF=R} |\mu(E)| \left( \prod_{P | E} \frac{1}{|P|} \right) \left( \prod_{P | E} \frac{1}{|P| - 1} \right)
\]

\[
\leq \phi(R) \sum_{EF=R} |\mu(E)| \prod_{P | E} \frac{1}{|P| - 1} = \phi(R) \prod_{P | R} \frac{1}{|P| - 1} = \phi(R) \prod_{P | R} 1 + \frac{1}{|P| - 1} = \phi(R) \phi(R) |R| = |R|.
\]
8. The Fourth Moment

We now proceed to prove Theorem 2.5. In the proof we implicitly state that some terms are of lower order than the main term and that is easy to check. We do not give the justification explicitly, although all the results one needs for a rigorous justification are given in Section 4.

Proof of Theorem 2.5. Let \( \chi \) be a Dirichlet character of modulus \( R \). By Propositions 3.10 and 3.11, we have that

\[
\left| L\left(\frac{1}{2}, \chi\right) \right|^2 = 2 \sum_{A, B \in M, \deg AB < \deg R} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}} + c(\chi) = 2a(\chi) + 2b(\chi) + c(\chi),
\]

where

\[
z_R := \deg R - \log_q(2^{\omega(Q)}),
\]

\[
a(\chi) := \sum_{A, B \in M, \deg AB \leq z_R} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}},
\]

\[
b(\chi) := \sum_{A, B \in M, z_R < \deg AB < \deg R} \frac{\chi(A)\overline{\chi}(B)}{|AB|^\frac{1}{2}},
\]

and \( c(\chi) \) is defined as in (3). Then,

\[
\sum_{\chi \mod R}^* \left| L\left(\frac{1}{2}, \chi\right) \right|^4 = \sum_{\chi \mod R}^* \left( 2a(\chi) + 2b(\chi) + c(\chi) \right)^2.
\]

We will show that \( \sum_{\chi \mod R}^* |a(\chi)|^2 \) has an asymptotic main term of higher order than \( \sum_{\chi \mod R}^* |b(\chi)|^2 \) and \( \sum_{\chi \mod R}^* |c(\chi)|^2 \). From this and the Cauchy-Schwarz inequality, we deduce that \( \sum_{\chi \mod R}^* |a(\chi)|^2 \) gives the leading term in the asymptotic formula.

**STEP 1:** We have that

\[
\sum_{\chi \mod R}^* |a(\chi)|^2 = \sum_{\chi \mod R}^* \sum_{A, B, C, D \in M, \deg AB \leq z_R, \deg CD \leq z_R} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^\frac{1}{2}} = \sum_{A, B, C, D \in M, \deg AB \leq z_R, \deg CD \leq z_R} \frac{1}{|ABCD|^\frac{1}{2}} \sum_{\chi \mod R}^* \chi(AC)\overline{\chi}(BD)
\]

\[
= \sum_{A, B, C, D \in M, \deg AB \leq z_R, \deg CD \leq z_R} \frac{1}{|ABCD|^\frac{1}{2}} \sum_{EF = R, F | (AC - BD)} \mu(E)\phi(F),
\]

where the last equality follows from Proposition 3.7. Continuing,
\[
\sum_{\chi \mod R}^* |a(\chi)|^2 = \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, F|\langle AC-BD \rangle} \frac{1}{|ABCD|^2}
\]

\[
= \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, AC=BD} \frac{1}{|ABCD|^2} + \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, F|\langle AC-BD \rangle, AC \neq BD} \frac{1}{|ABCD|^2}
\]

\[
= \left( \sum_{EF=R} \mu(E)\phi(F) \right) \left( \sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, AC=BD} \frac{1}{|ABCD|^2} \right) + \sum_{EF=R} \mu(E)\phi(F) \sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, F|\langle AC-BD \rangle, AC \neq BD} \frac{1}{|ABCD|^2}
\]

**STEP 1.1:** We will look at the first term on the far-RHS. Since \(AC = BD\), we can write \(A = FU, B = FV, C = GV, D = GU\), where \(F, G, U, V\) are monic and \(U, V\) are coprime. Let us write \(N = UV\), and note that there are \(2^{\omega(N)}\) ways of writing \(N = UV\) with \(U, V\) being coprime. Then,

\[
\sum_{A,B,C,D \in M, \deg AB \leq z_R \deg CD \leq z_R, (ABCD,R)=1, AC=BD} \frac{1}{|ABCD|^2}
\]

\[
= \sum_{F,G,U,V \in M, (U,V)=1, \deg F^2UV \leq z_R \deg G^2UV \leq z_R, (FGUV,R)=1} \frac{1}{|FGUV|} = \sum_{N \in M, \deg N \leq z_R, (N,R)=1} 2^{\omega(N)} \left( \sum_{F \in M, \deg F \leq z_R \deg N, (F,R)=1} \frac{1}{|F|} \right)^2
\]

\[
= \sum_{N \in M, \deg N \leq z_R', (N,R)=1} 2^{\omega(N)} \left( \sum_{F \in M, \deg F \leq z_R \deg N, (F,R)=1} \frac{1}{|F|} \right)^2 + \sum_{z_R' < \deg N \leq z_R, (N,R)=1} 2^{\omega(N)} \left( \sum_{F \in M, \deg F \leq z_R \deg N, (F,R)=1} \frac{1}{|F|} \right)^2
\]

where \(z_R' \colonequals \deg R - \log_q 9^{\omega(R)}\).

Let us look at the first term on the far-RHS of (44). We apply Proposition 7.2. When \(x = \frac{z_R - \deg N}{2}\) and \(\deg N \leq z_R'\), we have that \(\frac{2^{\omega(N)}x^2}{q^2} = O(1)\). Hence,
\[
\sum_{N \in M} \frac{2^\omega(N)}{|N|} \left( \sum_{F \in M} \frac{1}{|F|} \right)^2
\]

\[
= \left( \frac{\Phi(R)}{|R|} \right)^2 \sum_{N \in M} \frac{2^\omega(N)}{|N|} \left( z_R' - \deg N + O(\log \omega(R)) \right)^2
\]

\[
= \left( \frac{\Phi(R)}{|R|} \right)^2 \sum_{N \in M} \frac{2^\omega(N)}{|N|} \left( z_R' - \deg N \right)^2 + O\left( \deg R \log \omega(R) \right)
\]

\[
= \frac{1 - q^{-1}}{48} \prod_{P \text{ prime}} \frac{\left( 1 - |P|^{-1} \right)^3}{1 + |P|^{-1}} \left( \deg R \right)^{4}
\]

\[
+ O\left( \prod_{P \text{ prime}} \frac{\left( 1 - |P|^{-1} \right)^3}{1 + |P|^{-1}} \left( \deg R \right)^{3} \right),
\]

where the last equality follows from Proposition 7.5 and Lemma 7.7.

Now we look at the second term on the far-RHS of (44). Because \( z_R' < \deg N \leq z_R \), we have that \( \deg F \leq \log_q \left( \frac{3}{\sqrt{2}} \right) \omega(R) \). Using this and Proposition 7.2, we have that

\[
\sum_{N \in M} \frac{2^\omega(N)}{|N|} \left( \sum_{F \in M} \frac{1}{|F|} \right)^2 \ll \left( \frac{\Phi(R)}{|R|} \right)^2 \omega(R) \deg R.
\]

Also, by similar means as in Lemma 7.6, we can see that

\[
\sum_{N \in M} \frac{2^\omega(N)}{|N|} \leq \sum_{N \in M} \frac{2^\omega(N)}{|N|} \ll \omega(R) \deg R.
\]

Hence,

\[
(46) \quad \sum_{N \in M} \frac{2^\omega(N)}{|N|} \left( \sum_{F \in M} \frac{1}{|F|} \right)^2 \ll \left( \frac{\Phi(R)}{|R|} \right)^2 \omega(R) \deg R
\]

By (44), (45) and (46), we have that
\[
\left( \sum_{E \in R} \mu(E) \phi(F) \right) \left( \sum_{\substack{A,B,C,D \in \mathcal{M} \\
\deg AB \leq z_R \\
\deg CD \leq z_R \\
(ABC,R) = 1 \\
AC = BD}} \frac{1}{|ABCD|^4} \right)
\]

\[= \frac{1 - q^{-1}}{48} \phi^*(R) \prod_{P \text{ prime} \divides R} \left( \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right)^{(\deg R)^4}
\]

\[+ O\left( \phi^*(R) \left( \prod_{P \text{ prime} \divides R} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right)^{(\deg R)^3} \omega(R) + (\deg R)^3 \log \deg R \right) \].

**STEP 1.2:** For the second term on the far-RHS of (43) we simply apply Proposition 7.12. From this, Step 1.1, and (43), we deduce that

\[
\sum_{\chi \mod R}^{*} |a(\chi)|^2
\]

\[= \frac{1 - q^{-1}}{48} \phi^*(R) \prod_{P \text{ prime} \divides R} \left( \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right)^{(\deg R)^4}
\]

\[+ O\left( \phi^*(R) \left( \prod_{P \text{ prime} \divides R} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right)^{(\deg R)^3} \omega(R) + (\deg R)^3 \log \deg R \right) \].

**STEP 2:** We will now look at \(\sum_{\chi \mod R}^{*} |b(\chi)|^2\). We have that

\[
\sum_{\chi \mod R}^{*} |b(\chi)|^2 \leq \sum_{\chi \mod R} |b(\chi)|^2 = \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\
R < \deg AB < \deg R \\
R < \deg CD < \deg R \\
(ABC,R) = 1 \\
AC = BD \mod R}} \frac{1}{|ABCD|^4}
\]

\[= \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\
R < \deg AB < \deg R \\
R < \deg CD < \deg R \\
(ABC,R) = 1 \\
AC = BD}} \frac{1}{|ABCD|^4} + \phi(R) \sum_{\substack{A,B,C,D \in \mathcal{M} \\
R < \deg AB < \deg R \\
R < \deg CD < \deg R \\
(ABC,R) = 1 \\
AC \equiv BD \mod R}} \frac{1}{|ABCD|^4}
\]

**STEP 2.1:** Looking at the first term on the far-RHS, we apply the same technique as in (44) to obtain
\[
\phi(R) \sum_{A,B,C,D \in M} \frac{1}{|ABCD|^2} \]

where \( z_R' := \deg R - \log q \omega(R) \).

We look at the first term on the far-RHS:

\[
= \phi(R) \sum_{\substack{N \text{ monic} \\
\deg N \leq \deg R \\
(N,R) = 1}} \frac{2^\omega(N)}{|N|} \left( \sum_{\substack{F \text{ monic} \\
\deg F < \deg R - \deg N \\
(F,R) = 1}} \frac{1}{|F|} \right)^2
\]

where for the second relation we applied Proposition 7.2 twice. For the use of this proposition one may wish to note that, because \( \deg N \leq z_R' \), we have that \( \deg R - \deg N \geq \frac{z_R - \deg N}{2} \geq \log_q \left( \frac{3}{\sqrt{2}} \right) \omega(R) \), and so when \( x = \frac{\deg R - \deg N}{2} \) or \( x = \frac{z_R - \deg N}{2} \) we have that \( \frac{\omega(R)}{q^x} = O(1) \). For the last relation we applied Lemma 7.7.

Now we look at the second term on the far-RHS of (48). Because \( z_R' < \deg N < \deg R \), we have that \( \frac{\deg R - \deg N}{2} < \log_q \omega(R) \). Hence,
\[ \phi(R) \sum_{N \text{monic} \atop z_R < \text{deg } N < \text{deg } R} \frac{2^{\omega(N)}}{|N|} \left( \sum_{F \text{monic} \atop \deg F < \text{deg } R - \text{deg } N} \frac{1}{|F|} \right)^2 \leq \phi(R) \sum_{N \text{monic} \atop \deg F < \log_q 9 \frac{\omega(R)}{2}} \frac{2^{\omega(N)}}{|N|} \left( \sum_{F \text{monic} \atop \deg F < \text{deg } R} \frac{1}{|F|} \right)^2 \]

\[ \ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R) \left( \prod_{P \mid R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\text{deg } R)^2, \]

where, again, we have used Propositions 7.2 and Lemma 7.4.

Hence,

\[ \phi(R) \sum_{A,B,C,D \in M} \frac{1}{|ABCD|^2} \ll |R| \left( \frac{\phi(R)}{|R|} \right)^3 \omega(R) \left( \prod_{P \mid R} \frac{1 - |P|^{-1}}{1 + |P|^{-1}} \right) (\text{deg } R)^2 \]

\[ \ll \phi^*(R) \left( \prod_{P \text{ prime} \atop P \mid R} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\text{deg } R)^3 \omega(R). \]

**STEP 2.2:** We now look at the second term on the far right-hand-side of (47):

\[ \phi(R) \sum_{A,B,C,D \text{monic} \atop z_R < \text{deg } AB < \text{deg } R} \frac{1}{|ABCD|^2} \]

\[ = \phi(R) \sum_{z_R < z_1, z_2 < \text{deg } R} \frac{1}{q^{z_1 + z_2}^2} \sum_{A,B,C,D \text{monic} \atop AB = z_1 \atop z_R < \text{deg } CD < \text{deg } R} \frac{1}{|ABCD|^2} \]

\[ = \phi(R) \frac{1}{\phi(R)} \sum_{z_R < z_1, z_2 < \text{deg } R} q^{\frac{z_1 + z_2}{2}} (z_1 + z_2)^3. \]

\[ \ll |R| (\text{deg } R)^3 \ll \phi^*(R) \left( \prod_{P \mid R} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\text{deg } R)^3 \omega(R) \]

as \( \text{deg } R \to \infty \). The second relation follows from Lemma 7.10 with \( F := R \). This can be applied because

\[ z_1 + z_2 \geq 2z_R = 2 \text{deg } R - 2 \log_q 2^{\omega(R)} > \frac{19}{10} \text{deg } R \]

for large enough \( \text{deg } R \).

**STEP 2.3:** Hence, we see that

\[ \sum_{\chi \text{ mod } R}^* |b(\chi)|^2 \ll \phi^*(R) \left( \prod_{P \mid R} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\text{deg } R)^3 \omega(R). \]
**STEP 3:** We will now look at $\sum_{\chi \text{ mod } R} |c(\chi)|^2$. We have that

$$\sum_{\chi \text{ mod } R} |c(\chi)|^2 \leq \sum_{\chi \text{ mod } R} |c(\chi)|^2 = \sum_{\chi \text{ mod } R} |\alpha(\chi)|^2 - \sum_{\chi \text{ even}} |c_0(\chi)|^2 + \sum_{\chi \text{ even}} |c_\chi(\chi)|^2.$$ 

Now,

$$\sum_{\chi \text{ mod } R} |c_0(\chi)|^2 = \sum_{\chi \text{ mod } R} \sum_{A,B,C,D \in \mathcal{M}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^2}$$

$$= \phi(R) \sum_{A,B,C,D \in \mathcal{M}} \frac{1}{|ABCD|^2} + \phi(R) \sum_{A,B,C,D \in \mathcal{M}} \frac{1}{|ABCD|^2}.$$ 

For the first term on the far-RHS we have that

$$\sum_{A,B,C,D \in \mathcal{M}} \frac{1}{|ABCD|^2} \leq \sum_{N \in \mathcal{M}} \frac{2\omega(N)}{|N|} \left( \sum_{F \in \mathcal{M}} \frac{1}{|F|^2} \right)$$

$$= \sum_{N \in \mathcal{M}} \frac{2\omega(N)}{|N|} \ll (\deg R)^2.$$ 

For the second term we have that

$$\sum_{A,B,C,D \in \mathcal{M}} \frac{1}{|ABCD|^2} = \frac{q}{|R|} \sum_{A,B,C,D \in \mathcal{M}} 1 \ll \frac{|R|}{\Phi(R)} (\deg R)^3,$$

where we have used Lemma 7.10. Hence,

$$\sum_{\chi \text{ mod } R} |\alpha(\chi)|^2 \ll |R| (\deg R)^3 \ll \phi^*(R) \left( \prod_{P \text{ prime}} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R).$$

Similarly, by using Lemma 7.11 for the even case, we can show, for $a = 0, 1, 2, 3$, that

$$\sum_{\chi \text{ mod } R} \sum_{A,B,C,D \in \mathcal{M}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^2} \ll |R| (\deg R)^3 \ll \phi^*(R) \left( \prod_{P \text{ prime}} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R),$$

$$\sum_{\chi \text{ even}} \sum_{A,B,C,D \in \mathcal{M}} \frac{\chi(AC)\overline{\chi}(BD)}{|ABCD|^2} \ll |R| (\deg R)^3 \ll \phi^*(R) \left( \prod_{P \text{ prime}} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R).$$

Hence, by using the Cauchy-Schwarz inequality, we can deduce that
\[ \sum_{\chi \mod R}^* |c(\chi)|^2 \ll \phi^*(R) \left( \prod_{P \text{ prime}} \frac{(1 - |P|^{-1})^3}{1 + |P|^{-1}} \right) (\deg R)^3 \omega(R). \]

**STEP 4:** From steps 1 to 3, and the use of the Cauchy-Schwarz inequality (as described at the start of the proof), the result follows.  

\[ \square \]

**REFERENCES**

[BFK+17] Valentin Blomer, Étienne Fouvry, Emmanuel Kowalski, Philippe Michel, Djordje Miličević; *On moments of twisted L-functions*; American Journal of Mathematics; vol. 139(3):(2017), 707–768.

[CFK+05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, N. C. Snaith; *Integral Moments of L-functions*; Proceedings of the London Mathematical Society; vol. 91(1):(2005), 33–104.

[DGH03] A. Diaconu, D. Goldfeld, J. Hoffstein; *Multiple Dirichlet series and moments of zeta and L-functions*; Compos. Math.; vol. 139:(2003), 297–360.

[HB81] D. R. Heath-Brown; *The Fourth Power Mean of Dirichlet’s L-Functions*; Analysis 1; vol. 1:(1981), 25–32.

[Ros02] P. Shi; A *Brun-Titchmarsh theorem for multiplicative functions*; Journal für die Riene und Angewandte Mathematik; vol. 313:(1980), 161–170.

[Sou07] K. Soundararajan; *The fourth moment of Dirichlet L-functions*; Clay Mathematics Proceedings; vol. 7:(2007), 239–246.

[Tam14] Nattalie Tamam; *The Fourth Moment of Dirichlet L-Functions for the Rational Function Field*; International Journal of Number Theory; vol. 10(1):(2014), 183–218.

[Web83] W. A. Webb; *Sieve Methods for Polynomial Rings over Finite Fields*; Journal of Number Theory; vol. 16:(1983), 343–355.

[You11] M. P. Young; *The fourth moment of Dirichlet L-functions*; Annals of Mathematics; vol. 173:(2011), 1–50.

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