The construction of two-dimensional optimal systems for the invariant solutions

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Abstract

To search for inequivalent group invariant solutions, a general and systematic approach is established to construct two-dimensional optimal systems, which is based on commutator relations, adjoint matrix and the invariants. The details of computing all the invariants for two-dimensional subalgebras is presented and the optimality of two-dimensional optimal systems is shown clearly under different values of invariants, with no further proof. Applying the algorithm to \((1+1)\)-dimensional heat equation and \((2+1)\)-dimensional Navier-Stokes (NS) equation, their two-dimensional optimal systems are obtained, respectively. For the heat equation, eleven two-parameter elements in the optimal system are found one by one, which are discovered more comprehensive. The two-dimensional optimal system of NS equations is used to generate intrinsically different reduced ordinary differential equations and some interesting explicit solutions are provided.

Keywords: adjoint matrix; invariants; optimal system; Heat equation; Navier-Stokes equation

1. Introduction

The study of group-invariant solutions of differential equations plays an important role in mathematics and physics. The machinery of Lie group theory provides the systematic method to search for these special group invariant solutions. For a \(n\)-dimensional differential system, any of its \(m\)-dimensional \((m < n)\) symmetry subgroup can transform it into a \((n - m)\)-dimensional differential system, which is generally easier to solve than the original system. By solving these reduced equations, rich group-invariant solutions are found. For two group-invariant solutions, one may connect them with some group transformation and in this case, one calls them equivalent. Naturally, it is a significant job to find these inequivalent branches of group-invariant solutions, which leads to the concept of the optimal systems. For the classification of group-invariant solutions, it is more convenient to work in the space of the Lie algebra and this problem reduces to the problem of finding an optimal system of subalgebras under the adjoint representation.

The adjoint representation of a Lie group on its Lie algebra was known to Lie. The construction of the one-dimensional optimal system of Lie algebra is demonstrated by Ovsiannikov \cite{1}, using a global matrix for the adjoint transformation. This is also the technique used by Galas \cite{2} and Ibragimov \cite{3}. Then Olver \cite{4} uses a slightly different and elegant technique for one-dimensional optimal system, which is based on commutator table and adjoint table, and presents detailed instructions on the KdV equation and the heat equation. For the two-dimensional optimal systems, Ovsiamnikov sketches the construction by showing a simple example. Galas refines Ovsiamnikov’s method by removing equivalent subalgebras for the solvable algebra, and he also discusses the problem of a nonsolvable algebra, which is generally harder. In Ref. \cite{4}, the details of the construction of the two-dimensional optimal systems are shown for the three-dimensional, one-temperature hydrodynamic equations. In a fundamental series of papers, Patera et al. \cite{5,6,7,8} have developed a different and powerful method to classify subalgebras and many optimal systems of important Lie algebras arising in mathematical physics are obtained.

For the construction of one-dimensional optimal system, Olver has pointed out that the Killing form of the Lie algebra as an “invariant” for the adjoint representation is very important since it places restrictions on how far one can
expect to simplify the Lie algebra. Recently, Chou et al. [13][12] develop this idea by introducing more numerical invariants (which are different from common invariants such as the Casimir operator, harmonics and rational invariants) to demonstrate the inequivalence among the elements in the optimal system. However, to the best of our knowledge, in spite of the importance of the common invariants for the Lie algebra, there are few literatures to use more invariants except the killing form in the process of constructing optimal systems.

In this paper, we continue to introduce a systematic method for two-dimensional optimal system, taking full advantage of the invariants of two-dimensional subalgebras. In almost all of literatures, one-dimensional optimal direct and valid method for providing all the general invariants which are not numerical invariants and then make the best of them with the adjoint matrix to construct one-dimensional optimal system. On the basis of all the invariants, the new method can both guarantee the comprehensiveness and the inequivalence of the one-dimensional optimal system. In this paper, we continue to introduce a systematic method for two-dimensional optimal system, taking full advantage of the invariants of two-dimensional subalgebras. In almost all of literatures, one-dimensional optimal system is required for the calculation of two-dimensional optimal system. Here our new algorithm will start from Lie algebra directly, without the prior construction of one-dimensional optimal system. We shall demonstrate the new technique by treating a couple of illustrative examples, the \((1+1)\)-dimensional heat equation and \((2+1)\)-dimensional Navier-Stokes (NS) equation. Due to the two-dimensional optimal system of Lie algebra, the number of independent variables of any differential equations would be reduced by two.

The layout of this paper is as follows. In section 2, a systematic algorithm of two-dimensional optimal systems for the general symmetry algebra is proposed. Since the realization of our new algorithm builds on different invariants and the adjoint matrix, a valid method for computing all the invariants of the two-dimensional subalgebras is also given in this section. In section 3, we apply the new algorithm to a six-dimensional Lie algebra of \((1+1)\)-dimensional heat equation, and construct its two-dimensional optimal system step by step. In section 4, the two-dimensional optimal system of \((2+1)\)-dimensional Navier-Stokes equation is presented and all the corresponding reduced equations with some interesting exact group invariant solutions are obtained. Finally, a brief conclusion is given in section 5.

2. A general algorithm for constructing two-dimensional optimal system

Consider the \(n\)-dimensional symmetry algebra \(G\) of a differential system, which is generated by the vector fields \([v_1, v_2, \ldots, v_n]\). The corresponding symmetry group of \(G\) is denoted as \(G\). A family of \(r\)-dimensional subalgebras \([[v_\alpha]]_{\alpha \in \mathbb{N}}\) is an \(r\)-parameter optimal system if (1) any \(r\)-dimensional subalgebra is equivalent to some \(\tilde{v}_\alpha\) and (2) \(\tilde{g}_\alpha\) and \(g_\alpha\) are inequivalent for distinct \(\alpha\) and \(\beta\).

Let
\[
\begin{align*}
    w_1 & = \sum_{i=1}^{n} a_i v_i, \\
    w_2 & = \sum_{j=1}^{n} b_j v_j, \\
    w'_1 & = \sum_{j=1}^{n} a'_j v_j, \\
    w'_2 & = \sum_{j=1}^{n} b'_j v_j.
\end{align*}
\] (1)

For two-dimensional optimal system denoted by \(\Theta_2\), we call its two elements \(g_\alpha = [w_1, w_2]\) and \(g_\beta' = [w'_1, w'_2]\) are equivalent if one can find some transformation \(g \in G\) and some constants \([k_1, k_2, k_3, k_4]\) so that
\[
\begin{align*}
    w'_1 & = k_1 Ad_g(w_1) + k_2 Ad_g(w_2), \\
    w'_2 & = k_3 Ad_g(w_1) + k_4 Ad_g(w_2)
\end{align*}
\] (2)
and the inverse is also true. For the invertibility of (2), it requires \(k_1 k_4 - k_2 k_3 \neq 0\).

Here \(Ad_g(w_i)(i = 1, 2)\) is the adjoint representation action of the group \(g\) on the algebra \(w_i\) with \(Ad_g(w_i) = g^{-1} w_i g\).

2.1. Construction of the refined two-dimensional algebra

For constructing two-dimensional optimal system \(\Theta_2\) of \(n\)-dimensional Lie algebra \(G\), what we should do is to form a list of two-dimensional algebras \(G(w_1, w_2)\) that will later be separated into equivalence classes for different \(a_i\) and \(b_j\) under the adjoint transformation. For an arbitrary element \(g_\alpha = [w_1, w_2] \in \Theta_2\), firstly it requires \(w_1\) and \(w_2\) form a two-dimensional subalgebra, i.e. \([w_1, w_2] = \lambda w_1 + \mu w_2\) with \(\lambda\) and \(\mu\) being constants. Here \([,]\) represents the commutator relation. Then Galas refined this selection by showing that \(w_2\) must be an element from the normalizer of \(w_1\). That is to say one can select \(w_2\) for
\[
[w_1, w_2] = \lambda w_1.
\] (3)

Similarly, if \(g_\beta' = [w'_1, w'_2]\) is equivalent to \(g_\alpha\), it is necessary that
\[
[w'_1, w'_2] = \lambda' w'_1.
\] (4)
Substituting (11) into (3) and collecting all the coefficients of $v_j$, it will give some relations among $a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\lambda$, which are called the determined equations by us. What we need do is to find all the representative elements in $\Theta_2$ under the conditions of the determined equations.

In fact, we can split the determined equations into two inequivalent classes: $\lambda = 0$ and $\lambda \neq 0$. For this inequivalence, we give the following remark.

**Remark 1**: If $g_0 = [w_1, w_2]$ and $g_0' = [w_1', w_2']$ are equivalent and they satisfy (3) and (4), we have:

1) when $\lambda = 0$, it is natural that $\lambda' = 0$;
2) when $\lambda \neq 0$, there must be $\lambda' \neq 0$ with $k_2 = 0$ in (2).

**Proof**: Since the equivalence of $[w_1, w_2]$ and $[w_1', w_2']$, Eqs. (2) hold for some $g = e^v$ with $v \in G$. We have

$$\begin{bmatrix} w_1', w_2' \end{bmatrix} = [k_1Ad_g(w_1) + k_2Ad_g(w_2), k_3Ad_g(w_1) + k_4Ad_g(w_2)]$$

$$= (k_1k_4 - k_2k_3)[Ad_g(w_1), Ad_g(w_2)]$$

$$= (k_1k_4 - k_2k_3)[e^{-v}w_1e^v, e^{-v}w_2e^v]$$

$$= (k_1k_4 - k_2k_3)e^{-v}[w_1, w_2]e^v$$

$$= (k_1k_4 - k_2k_3)\lambda Ad_g(w_1)$$

If there is $\lambda = 0$, we obtain $[w_1', w_2'] = 0$, i.e. $\lambda' = 0$. If there is $\lambda \neq 0$, in the condition of $[w_1', w_2'] = \lambda'w_1' = \lambda'(k_1Ad_g(w_1) + k_2Ad_g(w_2)),$

there must be $\lambda' \neq 0$ and $k_2 = 0$ for $k_1k_4 - k_2k_3 \neq 0$.

### 2.2. Calculation of the adjoint transformation matrix

For $w_1 = \sum_{i=1}^n a_i v_i$, its general adjoint transformation matrix $A$ is the product of the matrices of the separate adjoint actions $A_1, A_2, \ldots, A_n$, each corresponding to $Ad_{exp(ev_i)}(w_1), i = 1 \ldots n$.

For example, applying the adjoint action of $v_1$ to $w_1 = \sum_{i=1}^n a_i v_i$ and with the help of adjoint representation table, one has

$$Ad_{exp(ev_1)}(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$$

$$= a_1Ad_{exp(ev_1)}v_1 + a_2Ad_{exp(ev_1)}v_2 + \cdots + a_nAd_{exp(ev_1)}v_n$$

$$= R_1v_1 + R_2v_2 + \cdots + R_nv_n,$$

with $R_i \equiv R_i(a_1, a_2, \ldots, a_n, e_i), i = 1 \cdots n$. To be intuitive, the formula (7) can be rewritten into the following matrix form:

$$v \overset{\text{Ad}_{exp(ev_1)}}{\longrightarrow} (R_1, R_2, \ldots, R_n) = (a_1, a_2, \ldots, a_n)A.$$

Similarly, the matrices $A_2, A_3, \ldots, A_n$ of the separate adjoint actions of $v_2, v_3, \ldots, v_n$ can be constructed, respectively. Then the general adjoint transformation matrix $A$ is the product of $A_1, \ldots, A_n$ taken in any order

$$A \equiv A(e_1, e_2, \ldots, e_n) = A_1A_2 \cdots A_n.$$

That is to say, applying the most general adjoint action $Ad_g = Ad_{exp(ev_1)} \cdots Ad_{exp(ev_n)}$ to $w_1$, we have

$$w_1 \overset{\text{Ad}_{exp(ev_1)}}{\longrightarrow} Ad_g(w_1) \overset{\text{Ad}_{exp(ev_1)}}{\longrightarrow} (a_1, a_2, \ldots, a_n)A.$$

Likewise, there is

$$w_2 \overset{\text{Ad}_{exp(ev_1)}}{\longrightarrow} Ad_g(w_2) \overset{\text{Ad}_{exp(ev_1)}}{\longrightarrow} (b_1, b_2, \ldots, b_n)A.$$

When $g_0' = [w_1', w_2']$ is equivalent to $[w_1, w_2]$, we can rewrite (2) as

$$\begin{align*}
\{a_1', a_2', \ldots, a_n'\} &= k_1(a_1, a_2, \ldots, a_n)A + k_2(b_1, b_2, \ldots, b_n)A, \\
\{b_1', b_2', \ldots, b_n'\} &= k_3(a_1, a_2, \ldots, a_n)A + k_4(b_1, b_2, \ldots, b_n)A.
\end{align*}$$
Remark 2: In fact, Eqs. (11) can be regarded as $2n$ algebraic equations with respect to $\epsilon_1, \ldots, \epsilon_n$ and $k_1, k_2, k_3, k_4$, which will be taken to judge whether any two given two-dimensional algebras $\{w_1, w_2\}$ and $\{w'_1, w'_2\}$ are equivalent. If Eqs. (11) have the solution, it means that $\{\sum_{i=1}^n a_i v_i, \sum_{j=1}^n b_j v_j\}$ is equivalent to $\{\sum_{i=1}^n a'_i v_i, \sum_{j=1}^n b'_j v_j\}$; If Eqs. (11) are incompatible, it shows that they are inequivalent.

2.3. Calculation of the equations about the invariants

For the two-dimensional subalgebra, a real function $\phi$ is called an invariant if $\phi(a_{11}A_{d}(w_1) + a_{12}A_{d}(w_2), a_{21}A_{d}(w_1) + a_{22}A_{d}(w_2)) = \phi(w_1, w_2)$ for any two-dimensional subalgebra $\{w_1, w_2\}$ and all $g \in G$ with $a_{11}, a_{12}, a_{21}, a_{22}$ being arbitrary constants. For a general two-dimensional subalgebra $\{\sum_{i=1}^n a_i v_i, \sum_{j=1}^n b_j v_j\}$ of $G$, the corresponding invariant is a $2n$-dimensional function of $(a_1, \cdots, a_n, b_1, \cdots, b_n)$. Now we will propose a valid method to compute all the invariants of two-dimensional subalgebras and further make the best use of them to construct two-dimensional optimal system.

Let $v = \sum_{k=1}^n c_k v_k$ be a general element of $G$. In conjunction with the commutator table, we have

$$Ad_v(w_1) = Ad_{\exp(v)}(w_1)$$

$$= w_1 - \epsilon [v, w_1] + \frac{1}{2!} \epsilon^2 [v, [v, w_1]] - \cdots$$

$$= (a_1 v_1 + \cdots + a_n v_n) - \epsilon (c_1 v_1 + \cdots + c_n v_n) + a_1 v_1 + \cdots + a_n v_n + o(\epsilon)$$

(12)

where $\Theta_i \equiv \Theta_i(a_1, \cdots, a_n, c_1, \cdots, c_n)$ is obtained directly by replacing $a_i$ with $b_i$ in $\Theta_i(i = 1 \cdots n)$.

More intuitively, we denote

$$w_1 = (a_1, a_2, \cdots, a_n), \quad w_2 = (b_1, b_2, \cdots, b_n),$$

$$Ad_v(w_1) = (a_1 - \Theta_1', a_2 - \Theta_2', \cdots, a_n - \Theta_n') + o(\epsilon),$$

$$Ad_v(w_2) = (b_1 - \Theta_1', b_2 - \Theta_2', \cdots, b_n - \Theta_n') + o(\epsilon).$$

(14)

For the two-dimensional subalgebra $\{w_1, w_2\}$, according to the definition of the invariant we have

$$\phi(a_{11}A_{d}(w_1) + a_{12}A_{d}(w_2), a_{21}A_{d}(w_1) + a_{22}A_{d}(w_2)) = \phi(w_1, w_2).$$

(15)

Further, to guarantee $a_{11}A_{d}(w_1) + a_{12}A_{d}(w_2) = w_1$ and $a_{21}A_{d}(w_1) + a_{22}A_{d}(w_2) = w_2$ after the substitution of $\epsilon = 0$, we require

$$a_{11} \equiv 1 + \epsilon a_{11}, \quad a_{12} \equiv a_{12}, \quad a_{21} \equiv a_{21}, \quad a_{22} \equiv 1 + \epsilon a_{22}.$$

(16)

Remark 3: Following “Remark 1”, we need consider two cases to determine the invariants $\phi$.

(a) When $[w_1, w_2] = 0$, i.e. $\lambda = 0$, substituting (14) and (16) into Eq. (15), then taking the derivative of Eq. (15) with respect to $\epsilon$ and setting $\epsilon = 0$, extracting all the coefficients of $c_i, a_{11}, a_{12}, a_{21}, a_{22}$, some linear differential equations of $\phi$ are obtained. By solving these equations, all the invariants $\phi$ can be found.

(b) When $[w_1, w_2] \neq 0$, i.e. $\lambda \neq 0$, there must be $a_{12} = 0$ in Eq. (15). Then one does the same steps just as case (a) to obtain linear differential equations of $\phi$. 

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2.4. the construction of two-dimensional optimal system

(1) **First step:** give the commutator table and the adjoint representation table of the generators \( \{v_i\}_{i=1}^n \) which are firstly used by Olver [4]. Then following 2.3 and 2.4, compute the corresponding adjoint transformation matrix \( A \) and determine the equations about the invariants \( \phi \).

(2) **Second step:** in terms of \( [w_1, w_2] = \lambda w_1 \), present “the determined equations” about \( a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \) and \( \lambda \). For these equations, two cases including \( \lambda = 0 \) and \( \lambda \neq 0 \) need to be discussed, respectively. The aim is to find all the inequivalent elements \( \{g_n\} \) which form the two-dimensional optimal system.

(3) **Third step:** for the general element \( \{w_1, w_2\} \), in terms of every restricted condition shown in the determined equations, compute their respective invariants and select the corresponding eligible representative elements \( \{w'_1, w'_2\} \).

For ease of calculations, we rewrite (2) as

\[
\begin{cases}
\text{Ad}_p(w_1) = k'_1w'_1 + k'_2w'_2, \\
\text{Ad}_q(w_2) = k'_3w'_1 + k'_4w'_2.
\end{cases}
\]

That is to say

\[
\begin{cases}
(a_1, a_2, \ldots, a_n)A = k'_1(a'_1, a'_2, \ldots, a'_n) + k'_2(b'_1, b'_2, \ldots, b'_n), \\
(b_1, b_2, \ldots, b_n)A = k'_3(a'_1, a'_2, \ldots, a'_n) + k'_4(b'_1, b'_2, \ldots, b'_n).
\end{cases}
\]

Following “Remark 2”, if Eqs. (13) have the solution with respect to \( \varepsilon_1, \varepsilon_2, k'_1, k'_2, k'_3, k'_4 \), it signifies that the selected representative element is right; if Eqs. (13) have no solution, another new representative element \( \{w'_1, w'_2\} \) should be reselected. Repeat the process until all the cases are finished in the “the determined equations”.

3. the new approach for the (1+1)-dimensional heat equation

The equation for the conduction of heat in a one-dimensional road is written as

\[ u_t = u_{xx}. \]

The Lie algebra of infinitesimal symmetries for this equation is spanned by six vector fields

\[
\begin{align*}
v_1 &= \partial_x, & v_2 &= \partial_t, & v_3 &= u\partial_u, & v_4 &= x\partial_x + 2t\partial_t, \\
v_5 &= 2t\partial_x - xu\partial_u, & v_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u,
\end{align*}
\]

and the infinitesimal subalgebra

\[
v_6 = h(x, t)\partial_x,
\]

where \( h(x, t) \) is an arbitrary solution of the heat equation. Since the infinite-dimensional subalgebra \( \langle v_6 \rangle \) does not lead to group invariant solutions, it will not be considered in the classification problem.

The commutator table and actions of the adjoint representation, which are taken from [6], are in table 1 and table 2, respectively.

| \( v_i \) | \( v_1 \) | \( v_2 \) | \( v_3 \) | \( v_4 \) | \( v_5 \) | \( v_6 \) |
|---|---|---|---|---|---|---|
| \( v_1 \) | 0 | 0 | 0 | \( -v_3 \) | \( 2v_5 \) | |
| \( v_2 \) | 0 | 0 | 0 | \( 2v_2 \) | \( 2v_1 \) | \( 4v_4 - 2v_3 \) |
| \( v_3 \) | 0 | 0 | 0 | 0 | 0 | |
| \( v_4 \) | \( -v_1 \) | \( -2v_2 \) | 0 | 0 | \( v_5 \) | \( 2v_6 \) |
| \( v_5 \) | \( v_3 \) | \( -2v_1 \) | 0 | \( -v_5 \) | 0 | 0 |
| \( v_6 \) | \( -2v_5 \) | 2v_3 - 4v_4 | 0 | \( -2v_6 \) | 0 | 0 |
3.1. Calculation of the adjoint transformation matrix

For \( w_1 = \sum_{i=1}^{6} a_i v_i \), its general adjoint transformation matrix \( A \) is the product of the matrices of the separate adjoint actions \( A_1, A_2, \cdots, A_6 \), each corresponding to \( Ad_{exp(v_i)}(w_1) \), \( i = 1 \cdots 6 \).

For example, under the adjoint action of \( v_1 \) and with the help of Table 2, \( w_1 \) can be transformed into

\[
Ad_{exp(v_1)}(a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 + a_5 v_5 + a_6 v_6) = (a_1 - a_2 e_1 v_1 + a_2 v_2 + (a_3 - a_4 e_1 - a_6 e_2 v_3 + a_4 v_4 + (a_5 - 2a_1 a_6) v_5 + a_6 v_6.
\]

One can rewrite above formula (21) into the following matrix form:

\[
w_1 \equiv (a_1, a_2, \cdots, a_6) \xrightarrow{Ad_{exp(v_1)}} (a_1, a_2, \cdots, a_6) A_1
\]

where

\[
A_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-\epsilon_1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \epsilon_1 & 0 & 1 & 0 \\
0 & 0 & \epsilon_1 & 0 & -2\epsilon_1 & 1
\end{pmatrix}.
\]  

(22)

Similarly, the rest matrices of the separate adjoint actions of \( v_2, \cdots, v_6 \) are found to be:

\[
A_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -2\epsilon_2 & 0 & 1 & 0 & 0 \\
-2\epsilon_2 & 0 & 0 & 0 & 1 & 0 \\
0 & 4\epsilon_2^2 & 2\epsilon_2 & -4\epsilon_2 & 0 & 1
\end{pmatrix}, \\
A_4 = \begin{pmatrix}
e^{\epsilon_2} & 0 & 0 & 0 & 0 & 0 \\
e^{\epsilon_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & e^{-\epsilon_2} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{2\epsilon_2} & 0
\end{pmatrix},
\]  

(23)

\[
A_5 = \begin{pmatrix}
1 & 0 & -\epsilon_3 & 0 & 0 & 0 \\
2\epsilon_5 & 1 & -\epsilon_3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \\
A_6 = \begin{pmatrix}
1 & 0 & 0 & 0 & 2\epsilon_6 & 0 \\
0 & 1 & -2\epsilon_6 & 0 & 4\epsilon_6^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2\epsilon_6 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(24)

with \( A_3 = E \) being the identity matrix. Then the general adjoint transformation matrix \( A \) is the product of \( A_1, \cdots, A_6 \) which can be taken in any order:

\[
A \equiv (a_1)_{6 \times 6} = A_1 A_2 A_3 A_4 A_5 A_6.
\]

(25)
Similarly, applying the adjoint action with

\[ a_{11} = e^{i\alpha}, \quad a_{12} = 0, \quad a_{13} = -\epsilon e^{i\alpha}, \quad a_{14} = 0, \quad a_{15} = 2\epsilon e^{i\alpha}, \quad a_{16} = 0, \quad a_{21} = 2\epsilon e^{2i\alpha}, \quad a_{22} = e^{2i\alpha}, \]

\[ a_{23} = -(\epsilon^2 + 2\epsilon)e^{i\alpha}, \quad a_{24} = 4\epsilon e^{2i\alpha}, \quad a_{25} = 4\epsilon e^{6e^{2i\alpha}}, \quad a_{26} = 4\epsilon^2 e^{2i\alpha}, \quad a_{31} = a_{32} = a_{34} = a_{35} = a_{36} = 0, \]

\[ a_{33} = 1, \quad a_{41} = -e^{i\alpha}(\epsilon^1 + 4\epsilon e^{3i\alpha}), \quad a_{42} = -2\epsilon e^{2i\alpha}, \quad a_{43} = 4\epsilon e^{2e^{2i\alpha}} + \epsilon e^{i\alpha}(\epsilon^1 + 2\epsilon e^{3i\alpha}), \]

\[ a_{44} = 1 - 8\epsilon e^{2e^{2i\alpha}}, \quad a_{45} = e^{-2\epsilon e^{2i\alpha}}(\epsilon^1 + 4\epsilon e^{3i\alpha}), \quad a_{46} = 2\epsilon e^{2e^{2i\alpha}} - \epsilon e^{i\alpha}(\epsilon^1 + 2\epsilon e^{3i\alpha}), \quad a_{51} = -2\epsilon e^{i\alpha}, \quad a_{52} = 0, \]

\[ a_{53} = e^{i\alpha} + 2\epsilon e^{2i\alpha}, \quad a_{54} = 0, \quad a_{55} = e^{-i\alpha}(1 - 4\epsilon e^{2i\alpha}), \quad a_{56} = 0, \quad a_{61} = 4\epsilon e^{2i\alpha}(\epsilon^1 + 2\epsilon e^{3i\alpha}), \]

\[ a_{62} = 4\epsilon^2 e^{2i\alpha}, \quad a_{63} = -(-e^{i\alpha}(1 - 4\epsilon e^{2i\alpha})) + 2\epsilon(1 - 4\epsilon e^{2i\alpha}), \quad a_{64} = -4\epsilon(1 - 4\epsilon e^{2i\alpha}), \]

\[ a_{65} = 8\epsilon e^{2i\alpha}(\epsilon^1 + 2\epsilon e^{3i\alpha}) - 4\epsilon e^{i\alpha} - 4\epsilon e^{2i\alpha}, \quad a_{66} = -e^{-i\alpha}(1 - 4\epsilon e^{2i\alpha})^2. \]

3.2. Calculation of the invariants

For the heat equation, take

\[ w_1 = 6 \sum_{i=1} a_{1i}v_i, \quad w_2 = 6 \sum_{j=1} b_{1j}v_j. \]  

(27)

For a general two-dimensional subalgebra \([w_1, w_2]\) of the heat equation, the corresponding invariant \(\phi\) is a twelve-dimensional function of \((a_1, \cdots, a_6, b_1, \cdots, b_6)\). Let \(v = 6 \sum_{k=1} c_kv_k\) be a general element of \(G\), then in conjunction with the commutator table 1, we have

\[ Ad_{\phi}(w_1) = Ad_{exp(\phi)}(w_1) \]

\[ = w_1 - \epsilon[v, w_1] + \frac{1}{2!}[v, [v, w_1]] - \cdots \]

\[ = (a_{11}v_1 + \cdots + a_{16}v_6) - \epsilon(c_{11}v_1 + \cdots + c_{66}v_6, a_{11}v_1 + \cdots + a_{66}v_6) + o(\epsilon) \]

\[ = (a_{11}v_1 + \cdots + a_{16}v_6) - \epsilon(\Theta^1_1v_1 + \cdots + \Theta^6_6v_6) + o(\epsilon) \]

\[ = (a_1 - \epsilon\Theta^1_1)v_1 + (a_2 - \epsilon\Theta^2_2)v_2 + \cdots + (a_6 - \epsilon\Theta^6_6)v_6 + o(\epsilon) \]

(28)

with

\[ \Theta^1_1 = -c_{a1} - 2c_{b2} + c_{1a} + 2c_{2a5}, \quad \Theta^2_2 = -2c_{d2} + 2c_{2a4}, \quad \Theta^3_3 = c_{3a1} + 2c_{6a2} - c_{1a} - 2c_{2a6} \]

\[ \Theta^4_4 = -4c_{6a2a4} + 4c_{2a6}, \quad \Theta^5_5 = -2c_{6a1} - c_{3a} + c_{5a} + 2c_{3a6}, \quad \Theta^6_6 = -2c_{6a4} + 2c_{3a6}. \]

(29)

Similarly, applying the same adjoint action \(v = 6 \sum_{k=1} c_kv_k\) to \(w_2\), we get

\[ Ad_{\phi}(w_2) = Ad_{exp(\phi)}(w_2) \]

\[ = (b_1 - \epsilon\Theta^{\beta}_{\beta})v_1 + (b_2 - \epsilon\Theta^{\beta}_{\beta})v_2 + \cdots + (b_6 - \epsilon\Theta^{\beta}_{\beta})v_6 + o(\epsilon), \]

(30)

with

\[ \Theta^1_1 = -c_{a1} - 2c_{b2} + c_{1a} + 2c_{2a5}, \quad \Theta^2_2 = -2c_{d2} + 2c_{2a4}, \quad \Theta^3_3 = c_{3a1} + 2c_{6a2} - c_{1a} - 2c_{2a6} \]

\[ \Theta^4_4 = -4c_{6a2a4} + 4c_{2a6}, \quad \Theta^5_5 = -2c_{6a1} - c_{3a} + c_{5a} + 2c_{3a6}, \quad \Theta^6_6 = -2c_{6a4} + 2c_{3a6}. \]

(31)

For the two-dimensional subalgebra \([w_1, w_2]\), according to the definition of the invariant, we have

\[ \phi((1 + \epsilon a_{11})Ad_{\phi}(w_1) + \epsilon a_{12}Ad_{\phi}(w_2), \epsilon a_{21}Ad_{\phi}(w_1) + (1 + \epsilon a_{22})Ad_{\phi}(w_2)) = \phi(w_1, w_2). \]

(32)

Following “Remark 1” and “Remark 3”, Eq. (32) can be separated into two cases.

(a) When \(\lambda = 0\), all the \(c_i(i = 1 \cdots 6), a_{11}, a_{12}, a_{21}, a_{22}\) in Eq. (32) are arbitrary. Now taking the derivative of Eq. (32) with respect to \(\epsilon\) and then setting \(\epsilon = 0\), extracting the coefficients of all \(c_i, a_{11}, a_{12}, a_{21}, a_{22}\), one can directly
obtain nine differential equations about $\phi \equiv \phi(a_1, \ldots, a_6, b_1, \ldots, b_6)$:

$$
\begin{align*}
& a_1\phi_{a_1} + a_2\phi_{a_2} + a_3\phi_{a_3} + a_4\phi_{a_4} + a_5\phi_{a_5} + a_6\phi_{a_6} = 0, \\
& a_1\phi_{b_1} + a_2\phi_{b_2} + a_3\phi_{b_3} + a_4\phi_{b_4} + a_5\phi_{b_5} + a_6\phi_{b_6} = 0, \\
& 2a_2\phi_{b_2} + 2a_2\phi_{b_2} - a_1\phi_{a_1} - b_1\phi_{b_1} + a_2\phi_{a_2} + b_2\phi_{b_2} = 0, \\
& -a_3\phi_{a_3} - a_4\phi_{a_4} + b_3\phi_{b_3} + a_5\phi_{a_5} + b_5\phi_{b_5} - 2a_6\phi_{a_6} - 2b_6\phi_{b_6} = 0, \\
& a_1\phi_{a_1} + a_2\phi_{a_2} + 2a_2\phi_{a_2} - 2a_2\phi_{a_2} - a_3\phi_{a_3} - b_3\phi_{b_3} = 0, \\
& -a_4\phi_{a_4} - a_5\phi_{a_5} + b_4\phi_{b_4} + a_6\phi_{a_6} + b_6\phi_{b_6} - 2a_6\phi_{a_6} - 2b_6\phi_{b_6} = 0, \\
& -a_3\phi_{a_3} - b_3\phi_{b_3} - a_4\phi_{a_4} - b_4\phi_{b_4} + a_5\phi_{a_5} + b_5\phi_{b_5} - 2a_6\phi_{a_6} - 2b_6\phi_{b_6} = 0, \\
& -a_4\phi_{a_4} - b_4\phi_{b_4} + 2a_2\phi_{a_2} + 2b_2\phi_{b_2} + a_3\phi_{a_3} + b_3\phi_{b_3} + a_5\phi_{a_5} + b_5\phi_{b_5} = 0,
\end{align*}
$$

(33)

and

$$
\begin{align*}
& b_1\phi_{a_1} + b_2\phi_{a_2} + b_3\phi_{a_3} + b_4\phi_{a_4} + b_5\phi_{a_5} + b_6\phi_{a_6} = 0.
\end{align*}
$$

(34)

Here the subscripts indicate partial derivatives.

(b) When $\lambda \neq 0$, we have $a_{12} = 0$ and all the $c(i = 1 \cdots 6)$, $a_{11}, a_{21}, a_{22}$ in Eq. (32) are arbitrary. Plugging $a_{12} = 0$ into Eq. (32) and making the same process to case (a), we get eight equations about $\phi$ which are just Eqs. (33).

3.3. the two-dimensional optimal system for the heat equation

Due to the refined Lie algebra $[w_1, w_2] = \lambda w_1$, we find its determined equations are

$$
\begin{align*}
& a_1b_3 + 2a_2b_5 - a_1b_1 - 2a_3b_2 = \lambda a_1, \\
& 2a_2b_4 - 2a_4b_2 = \lambda a_2, \\
& -a_1b_3 - 2a_2b_5 - a_3b_1 + a_1b_1 = \lambda a_3, \\
& 2a_1b_6 + a_2b_5 - a_3b_4 - 2a_4b_2 = \lambda a_5, \\
& 4a_3b_6 - 4a_4b_2 = \lambda a_4, \\
& 2a_4b_6 - 2a_6b_4 = \lambda a_6.
\end{align*}
$$

(35)

Then Eqs. (35) are separated into two inequivalent classes: $\lambda = 0$ and $\lambda \neq 0$.

3.3.1. the case of $\lambda = 0$ in the determined equations

Substituting $\lambda = 0$ into Eqs. (35), two cases need to be considered.

Case 1: Not all $a_2, a_4, a_6, b_2, b_4$ and $b_6$ are zeroes. Without loss of generality, we adopt $a_6 \neq 0$. In fact, when only one of $a_2, a_4, a_6, b_2, b_4$ and $b_6$ is not zero, one can choose appropriate adjoint transformation to transform it into the case $a_6 \neq 0$. By solving Eqs. (35) with $\lambda = 0$ and $a_6 \neq 0$, we have three kinds of solutions:

(i) $a_3, a_4, a_5, a_6, b_3$ and $b_6$ are independent with

$$
\begin{align*}
& a_1 = \frac{1}{2} \frac{a a a_5}{a_6}, \\
& a_2 = \frac{1}{4} \frac{a^2}{a_6}, \\
& b_1 = \frac{1}{2} \frac{a a a_5 b_6}{a_6^2}, \\
& b_2 = \frac{1}{4} \frac{a^2 b_6}{a_6^2}, \\
& b_3 = \frac{a a b_6}{a_6}, \\
& b_4 = \frac{a b_6}{a_6}, \\
& b_5 = \frac{a b_6}{a_6}.
\end{align*}
$$

(36)

(ii) $a_3, a_4, a_5, a_6, b_3, b_5$ and $b_6$ are arbitrary but with

$$
\begin{align*}
& a_1 = \frac{1}{2} \frac{a a a_5}{a_6}, \\
& a_2 = \frac{1}{4} \frac{a^2}{a_6}, \\
& b_1 = \frac{1}{2} \frac{a a b_5}{a_6}, \\
& b_2 = \frac{1}{4} \frac{a^2 b_5}{a_6^2}, \\
& b_3 = \frac{a a b_6}{a_6}, \\
& b_4 = \frac{a b_6}{a_6}, \\
& b_5 \neq \frac{a b_6}{a_6}.
\end{align*}
$$

(37)

(iii) $a_1, a_2, a_3, a_4, a_5, a_6, b_3$ and $b_6$ are arbitrary but with

$$
\begin{align*}
& a_1 = \frac{1}{2} \frac{a a a_5}{a_6} \text{ or } a_2 = \frac{1}{4} \frac{a^2}{a_6}, \\
& b_1 = \frac{1}{2} \frac{a a b_6}{a_6}, \\
& b_2 = \frac{a a b_6}{a_6}, \\
& b_3 = \frac{a a b_6}{a_6}, \\
& b_4 = \frac{a b_6}{a_6}, \\
& b_5 = \frac{a b_6}{a_6}.
\end{align*}
$$

(38)

Substituting the above three conditions into Eqs. (32) and Eq. (34), we find that $\phi \equiv \text{constant}$, i.e. there is no invariant. Then for each case, select the corresponding representative element $(w'_1, w'_2)$ and verify whether Eqs. (18) have the solution.

For case (i), select a representative element $(w'_1, w'_2) = (v_6, v_3)$. Substituting (36) and $w'_1 = v_6, w'_2 = v_3$ into Eqs. (19), we obtain the solution

$$
\begin{align*}
& k'_1 = a_6 e^{-2a_6}, \\
& k'_2 = k'_5 = \frac{a_3 a_6 + 2a_2 a_6 + a_4^2}{4a_6}, \\
& k'_3 = \frac{4b_2 a_6^2 + b_3 a_6^2 + 2a_3 a_6 b_6}{4a_6^2}, \\
& \epsilon_1 = \frac{a_6}{2a_6}, \\
& \epsilon_2 = \frac{a_4}{4a_6}.
\end{align*}
$$

(39)
Hence case (i) is equivalent to \([v_6, v_3]\).

For case (ii), there exist three circumstances in terms of the following expression

\[
\Lambda_1 \equiv 2a_0[a_4(\alpha_4 s - b_3 a_6)^2 - 2a_0(\alpha_4 s b_6 - b_3 a_6)^2 - 2a_0(\alpha_4 s b_6 - b_3 a_6)(\alpha_3 b_5 - b_3 a_5)].
\]

\[
(40)
\]

(iia) When \(\Lambda_1 > 0\), case (ii) is equivalent to \([v_3, v_6, v_5]\). After the substitution of (37) with \(w_1' = v_3 + v_6, w_2' = v_5\), Eqs. (18) hold for

\[
k_1' = -\frac{\Lambda_1}{4a_0(\alpha_4 s b_6 - b_3 a_6)^2}, \quad k_2' = \frac{a_4 s(\alpha_4 s b_6 - b_3 a_6) + 2a_0(\alpha_4 s b_6 - b_3 a_6)}{2a_0(\alpha_4 s b_6 - b_3 a_6)^2} \sqrt{\Lambda_1}, \quad k_3' = \frac{b_6}{a_0}, \quad e_2 = \frac{a_4}{4a_0},
\]

\[
(41)
\]

(iiia) When \(\Lambda_1 = 0\), case (ii) is equivalent to \([v_6, v_3]\). By solving Eqs. (18), we obtain

\[
k_1' = a_6 e^{-2\alpha}, \quad k_2' = a_4 s(\alpha_4 s - b_3 a_6) + 2a_0(\alpha_4 s b_6 - b_3 a_6), \quad k_3' = a_4(\alpha_4 s b_6 - b_3 a_6) + 2a_0(\alpha_4 s b_6 - b_3 a_6),
\]

\[
e_1 = \frac{a_2 b_6 - b_3 a_6}{a_6 s b_6 - b_3 a_6}, \quad e_2 = \frac{a_6}{2a_0(\alpha_4 s b_6 - b_3 a_6)^2}.
\]

(iii) For case (iii), it can be divided into the following several types.

(iii) \(4a_2a_6 - a_3^2 > 0\). Select a representative element \([v_2 + v_6, v_3]\). After substituting (38) into Eqs. (18), we have

\[
k_1' = (a_2 - 2a_4 s + 4a_6 s^2)e^{2\alpha}, \quad k_2' = a_3 + \frac{a_4^2 s^2 - a_1 a_4 s + a_2 a_4^2}{4a_4 s^2 - a_1^2 a_4}, \quad k_3' = \frac{b_6(a_2 - 2a_4 s + 4a_6 s^2)}{a_0} e^{2\alpha},
\]

\[
(44)
\]

(iiv) \(4a_2a_6 - a_3^2 < 0\). Choose a representative element \([v_2 + v_6, v_3]\). Now Eqs. (18) have the solution

\[
k_1' = -(a_2 - 2a_4 s + 4a_6 s^2)e^{2\alpha}, \quad k_2' = a_3 + \frac{a_4^2 s^2 - a_1 a_4 s + a_2 a_4^2}{4a_4 s^2 - a_1^2 a_4}, \quad k_3' = \frac{b_6(a_2 - 2a_4 s + 4a_6 s^2)}{a_0} e^{2\alpha},
\]

\[
(45)
\]

(iii) \(4a_2a_6 - a_3^2 = 0\). In this case, two conditions should be considered.
When $2a_1a_6 - a_4a_5 > 0$, adopt the representative element $\{v_1 + v_6, v_3\}$. Then the solution for Eqs. (18) is

$$
k_1' = \frac{a_6}{Z'}, \quad k_2' = -\frac{(e_2^2 + e_5)(2a_1a_6 - a_4a_5)}{2a_6}Z + a_3 + \frac{a_4}{2} + \frac{a_5}{4a_6}, \quad k_3' = \frac{a_6(2a_1a_6 - a_4a_5)}{2a_6}Z, \quad e_2 = \frac{a_4}{4a_6},
$$

with $Z = (\frac{2a_4}{2a_6 a_4 a_5})^{1/3}$.

When $2a_1a_6 - a_4a_5 < 0$, adopt the representative element $\{-v_1 + v_6, v_3\}$. By solving Eqs. (18), we find

$$
k_1' = \frac{a_6}{Z'}, \quad k_2' = -\frac{(e_2^2 - e_5)(2a_1a_6 - a_4a_5)}{2a_6}Z' + a_3 + \frac{a_4}{2} + \frac{a_5}{4a_6}, \quad k_3' = \frac{a_6(2a_1a_6 - a_4a_5)}{2a_6}Z', \quad e_2 = \frac{a_4}{4a_6},
$$

with $Z' = (-\frac{2a_4}{2a_6 a_4 a_5})^{1/3}$.

**Case 2**: $a_2 = a_4 = a_6 = b_2 = b_4 = b_6 = 0$. Now the determined equations (35) become

$$-a_1b_5 + a_5b_1 = 0. \quad (48)$$

Here we just need consider not all $a_1, a_5, b_1$ and $b_5$ are zeroes. Without loss of generality, let $a_5 \neq 0$. Similarly, if one of $a_1, a_5, b_1, b_5$ is not zero, one can choose appropriate adjoint transformation to transform it into the case $a_5 \neq 0$. By solving Eq. (48), we obtain

$$b_1 = \frac{a_1b_5}{a_5}. \quad (49)$$

Now adopt a representative element $\{v_5, v_3\}$. Then one can verify that all the $\{a_1v_1 + a_3v_3, a_3v_3, b_1v_1 + b_3v_3 + b_3v_3\}$ with the condition (49) are equivalent to $\{v_5, v_3\}$ since the solution for Eqs. (18) is

$$k_1' = a_5 e^{-a_5}, \quad k_2' = a_3 + a_5 e^{a_5}, \quad k_3' = b_5 e^{-a_5}, \quad k_4' = b_3 + b_5 e^{a_5}, \quad e_2 = \frac{a_1}{2a_5}. \quad (50)$$

### 3.3.2. the case of $\lambda \neq 0$ in the determined equations

**Case 3**: Not all $a_2, a_4$ and $a_6$ are zeroes. Without loss of generality, we also adopt $a_6 \neq 0$.

For $\lambda = 0$ and $a_6 \neq 0$, by solving Eqs. (35), we find the relations

$$a_1 = \frac{a_4a_5}{2a_6}, \quad a_2 = \frac{a_7}{4a_6}, \quad a_3 = \frac{a_2^2 + 2a_2a_6}{4a_6}, \quad b_1 = \frac{a_4 a_6 b_5 - a_5(a_4b_6 - b_3a_6)}{2a_6^2}, \quad b_2 = \frac{a_1(2a_4b_4 - a_4b_6)}{4a_6^2}. \quad (51)$$

Substituting (51) into Eqs. (33), it leads to an invariant for $\{w_1, w_2\}$:

$$\phi = \Delta_1 = \frac{2(b_4 + 2b_5)e_2^2 - a_5(a_4b_6 - 2b_3a_6)}{a_6(a_4b_6 - a_5b_3)}. \quad (52)$$

In condition of (51) and $\Delta_1 = e$, choose the corresponding representative element $\{v_6, (-\frac{1}{b} - \frac{e}{b})v_3 + v_4\}$ and Eqs. (18) have the solution

$$k_1' = a_6 e^{-a_4}, \quad k_2' = 0, \quad k_3' = 2e_6 b_4 + b_6 e^{-a_4} - \frac{2e_6 a_1 b_6}{a_5} e^{a_4}, \quad k_4' = \frac{a_6 b_4 - b_3 a_4}{a_5}, \quad e_1 = \frac{a_5}{2a_6}, \quad e_2 = \frac{a_4}{4a_6}, \quad e_3 = \frac{a_5b_4 - a_5b_3}{a_1b_6 - a_4b_4} e^{-a_4}. \quad (53)$$

**Case 4**: $a_2 = a_4 = a_6 = 0$. 


For the same reason, we just think about \( a_5 \neq 0 \). Taking \( a_2 = a_4 = a_6 = 0 \) with \( a_5 \neq 0 \) and \( \lambda \neq 0 \) into (55), we have

\[
b_1 = \frac{a_1(2a_1b_6 + a_6b_2) - a_3a_5b_4}{a_5}, \quad b_2 = \frac{a_1(a_4b_4 - a_1b_6)}{a_5}, \quad \lambda = \frac{2a_1b_6 - a_5b_4}{a_5} \neq 0.
\]

Substituting the relations (54) into Eqs. (33), we obtain an invariant for \( \{w_1, w_2\} \), i.e.

\[
\phi = \Delta_2 = \frac{4a_5(a_1b_5 + a_5b_3) - 2a_2^2(b_4 + 2b_1)}{a_5(2a_1b_6 - a_5b_4)}.
\]

In this case, choose a representative element \( \{v_5, (\frac{1}{4} - \frac{1}{2}v_3 + v_4)\} \) for \( \Delta_2 = c \). Then solving Eqs. (18), one get

\[
k'_1 = a_5e^{-\epsilon_1}, \quad k'_2 = 0, \quad k'_3 = \frac{(a_3b_5 + 2a_5b_3)e^{-\epsilon_1} + \epsilon_3(a_5b_4 - 2a_1b_6)}{a_5}, \quad k'_4 = \frac{a_3b_5 - 2a_1b_6}{a_5},
\]

\[
\epsilon_1 = -\frac{a_3}{a_5}, \quad \epsilon_2 \equiv \frac{a_1}{2a_5}, \quad \epsilon_6 = \frac{a_3b_6}{2(2a_1b_6 - a_5b_4)}e^{-2\epsilon_1}.
\]

In summary, we have completed the construction of the two-dimensional optimal system \( \Theta_2 \):

\[
g_1 = \{v_6, v_3\}, \quad g_2 = \{v_3 + v_6, v_5\}, \quad g_3 = \{-v_3 + v_6, v_3\}, \quad g_4 = \{v_6, v_3\}, \quad g_5 = \{v_5 + v_6, v_3\}, \quad g_6 = \{-v_3 + v_6, v_3\}, \quad g_7 = \{v_1 + v_6, v_3\}, \quad g_8 = \{-v_1 + v_6, v_3\}, \quad g_9 = \{v_1, v_3\}, \quad g_{10} = \{v_6, v_4 + \beta v_3\}, \quad g_{11} = \{v_5, v_4 + \beta v_3\}. \quad (\beta \in \mathbb{R})
\]

**Remark 4:** The process of the construction ensures that all \( g_i(i = 1 \cdots 11) \) are mutually inequivalent since each case is closed. One can also easily find this inequivalence from the incompatibility of Eqs. (18). In [10], Chou et al. gave a two-parameter optimal system \( \{M_i\}_{i=1}^{10} \) for the same Lie algebra (20) of the heat equation and showed their inequivalences by sufficient numerical invariants. One can see that \( \{M_i\}_{i=1}^{10} \) in [10] are equivalent to our \( \{g_1, g_2, g_6, g_8, g_9, g_{10}, g_{11}\} \), respectively. Furthermore, we realize that \( g_7 \) is inequivalent to any of the elements in \( \{M_i\}_{i=1}^{10} \). Hence, here the two-dimensional optimal system \( \Theta_2 \) given by (57) is complete and really optimal.

4. the new approach for the the (2+1)-dimensional Navier-Stokes equation

One of the most important open problems in fluid is the existence and smoothness problem of the Navier-Stokes (NS) equation, which has been recognized as the basic equation and the very starting point of all problems in fluid physics [13][14]. In Ref. [15], by means of the classical Lie symmetry method, we investigate the (2+1)-dimensional Navier-Stokes equations:

\[
\omega = \psi_{xx} + \psi_{yy},
\]

\[
\omega_t + \psi_x\omega_y - \psi_y\omega_x - \gamma(\omega_{xx} + \omega_{yy}) = 0.
\]

One can rewrite Eqs. (58) into

\[
\psi_{xx} + \psi_{yy} + \psi \psi_{xy} - \psi_y \psi_{yy} - \psi_x \psi_{xy} - \gamma(\psi_{xxxx} + 2\psi_{xxyy} + \psi_{yyyy}) = 0.
\]

The associated vector fields for the one-parameter Lie group of the NS equation (59) are given by

\[
v_1 = \frac{\partial}{\partial x} + \frac{\gamma}{2} \frac{\partial}{\partial t}, \quad v_2 = \partial_t, \quad v_3 = -yt \partial_x + xt \partial_y + \frac{x^2 + y^2}{2} \partial_{\phi}, \quad v_4 = -y \partial_x + x \partial_y,
\]

\[
v_5 = f(t) \partial_x - f'(t) y \partial_{\phi}, \quad v_6 = g(t) \partial_y + g'(t) x \partial_{\phi}, \quad v_7 = h(t) \partial_{\phi}.
\]

Here, ignoring the discussion of the infinite dimensional subalgebra, we apply the new approach to construct the two-dimensional optimal system and the corresponding invariant solutions for the four-dimensional Lie algebra spanned by \( v_1, v_2, v_3, v_4 \) in (60).

The commutator table and the adjoint representation table for \( \{v_1, v_2, v_3, v_4\} \) are given in table 3 and table 4, respectively.
For the general two-dimensional subalgebra $\mathfrak{g} = \mathfrak{Ad}_2 = \mathfrak{v}_1 \oplus \mathfrak{v}_2$, then in conjunction with the commutator table 3, we have

\begin{align*}
Ad_{\exp(e_1)}(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4) &= a_1v_1 + a_2e^{e_1}v_2 + a_3e^{e_1}v_3 + a_4v_4.
\end{align*}

Hence the corresponding adjoint transformation matrix $A_1$ is saying

\begin{align*}
A_1 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{e_1} & 0 & 0 \\
0 & 0 & e^{-e_1} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}

Similarly, one can get

\begin{align*}
A_2 &= \begin{pmatrix}
1 & -e_2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -e_2 \\
0 & 0 & 0 & 1
\end{pmatrix}, &
A_3 &= \begin{pmatrix}
1 & 0 & e_3 & 0 \\
0 & 1 & 0 & e_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, &
A_4 &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}

Then the most general adjoint matrix $A$ can be taken as

\begin{align*}
A &= A_1A_2A_3A_4 = \begin{pmatrix}
1 & -e_2 & -e_2e_3 & 0 \\
0 & e^{e_1} & 0 & e^{e_1} \\
0 & 0 & e^{-e_1} & -e^{-e_1}e_2 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\end{align*}

Let

\begin{align*}
w_1 &= \sum_{i=1}^4 a_i v_i, & w_2 &= \sum_{j=1}^4 b_j v_j,
\end{align*}

For the general two-dimensional subalgebra $\{w_1, w_2\}$, the corresponding invariant $\phi$ is some eight-dimensional function of $(a_1, \ldots, a_4, b_1, \ldots, b_4)$. Let $v = \sum_{k=1}^4 c_kv_k$ be a general element of $\mathcal{G}$, then in conjunction with the commutator table 3, we have

\begin{align*}
Ad_{\phi}(w_1) &= Ad_{\exp(e_1)}(w_1) \\
&= w_1 - \epsilon [v, w_1] + \frac{1}{2!} \epsilon^2[v, [v, w_1]] - \cdots \\
&= (a_1v_1 + \cdots + a_4v_4) - \epsilon(c_1v_1 + \cdots + c_4v_4 + a_1v_1 + \cdots + a_4v_4) + o(\epsilon) \\
&= a_1v_1 + (a_2 - \epsilon(c_2a_1 - c_1a_2))v_2 + (a_3 - \epsilon(c_3a_1 - c_1a_3))v_3 + (a_4 - \epsilon(c_4a_1 - c_1a_4))v_4 + o(\epsilon).
\end{align*}
Similarly, applying the same adjoint action \( \nu = \sum_{k=1}^{4} c_k v_k \) to \( w_2 \), we get

\[
Ad_\nu(w_2) = b_1 v_1 + (a_2 - \epsilon(c_2 b_1 - c_1 b_2)) v_2 + (b_3 - \epsilon(c_1 b_1 - c_3 b_1)) v_3 + (b_4 - \epsilon(c_2 b_1 - c_3 b_2)) v_4 + o(\epsilon). 
\]

(67)

Substituting (66) and (67) into Eq. (62), we treat two cases in the following.

(a) When \( \lambda = 0 \), the invariant function \( \phi = \phi(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \) is determined by

\[
a_1 \phi_{a_1} + a_2 \phi_{a_2} + a_3 \phi_{a_3} + a_4 \phi_{a_4} = 0, \quad a_1 \phi_{b_1} + a_2 \phi_{b_2} + a_3 \phi_{b_3} + a_4 \phi_{b_4} = 0,
\]

\[
b_1 \phi_{b_1} + b_2 \phi_{b_2} + b_3 \phi_{b_3} + b_4 \phi_{b_4} = 0, \quad a_2 a_1 - a_3 b_1 + a_4 b_2 - b_3 b_1 = 0, 
\]

and

\[
b_1 \phi_{a_1} + b_2 \phi_{a_2} + b_3 \phi_{a_3} + b_4 \phi_{a_4} = 0. 
\]

(68)

(b) When \( \lambda \neq 0 \), the invariant function \( \phi = \phi(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) \) only needs to meet Eqs. (68) in terms of the condition \( a_2 = 0 \).

Now we set out to construct the two-dimensional optimal system for \([v_1, v_2, v_3, v_4]\).

4.2. the construction of two-dimensional optimal system for the heat equation

Substituting (65) into \([w_1, w_2] = \lambda w_1\), the determined equations are found as follows

\[
\lambda a_1 = 0, \quad a_2 b_1 - a_1 b_2 = \lambda a_2, \quad a_1 b_3 - a_3 b_1 = \lambda a_3, \quad a_2 b_3 - a_3 b_2 = \lambda a_4. 
\]

(70)

4.2.1. the case of \( \lambda = 0 \) in the determined equations (67)

Case 1: Not all \( a_1 \) and \( b_1 \) are zeroes. Without loss of generality, we adopt \( a_1 \neq 0 \). Solving (70), we get

\[
b_2 = \frac{a_2 b_1}{a_1}, \quad b_3 = \frac{a_3 b_1}{a_1}, 
\]

(71)

with \( a_1, a_2, a_3, a_4, b_1 \) and \( b_3 \) being arbitrary.

Substituting the condition (71) into Eqs. (68) and Eq. (69), we find that \( \phi = constant \). Hence according to (71), select the corresponding representative element \([v_1, v_4]\). Since Eqs. (18) have the solution

\[
k'_1 = a_1, \quad k'_2 = \frac{a_4 a_1 - a_2 a_3}{a_1}, \quad k'_3 = b_1, \quad k'_4 = \frac{b_4 a_1^2 - a_2 a_3 b_1}{a_1^2}, \quad \epsilon_2 = \frac{\epsilon_1^a a_2}{a_1}, \quad \epsilon_3 = -\frac{a_3}{a_1 a_1^e}, 
\]

(72)

case (1) is equivalent to \([v_1, v_4]\).

Case 2: \( a_1 = b_1 = 0 \). Now the determined equations (70) become

\[
a_2 b_3 - a_3 b_2 = 0. 
\]

(73)

Case 2.1: Not all \( a_2 \) and \( b_2 \) are zeroes and we let \( a_2 \neq 0 \). From Eq. (73), we get \( b_1 = \frac{a_1 b_2}{a_2} \). Then by solving Eqs. (68) and Eq. (69), we find \( \phi = constant \). In this case, there exist three circumstances in terms of the sign of \( a_2 a_3 \).

(i). When \( a_3 = 0 \), choose the representative element \([v_2, v_4]\) and Eqs. (18) have the solution

\[
k'_1 = e^{\epsilon_1} a_2, \quad k'_2 = e^{\epsilon_1} e^{\epsilon_2} a_2 + a_4, \quad k'_3 = e^{\epsilon_2} b_2, \quad k'_4 = e^{\epsilon_2} e^{\epsilon_3} b_2 + b_4. 
\]

(74)

(ii). For \( a_2 a_3 > 0 \), we select \([v_2 + v_3, v_4]\) as a representative element. Eqs. (18) hold for

\[
k'_1 = \sqrt{\frac{a_2}{a_2}} a_2, \quad k'_2 = \sqrt{\frac{a_2}{a_2}} (e_3 - e_2) a_2 + a_4, \quad k'_3 = \sqrt{\frac{a_2}{a_2}} b_2, \quad k'_4 = \sqrt{\frac{a_2}{a_2}} (e_3 - e_2) b_2 + b_4. 
\]

(75)

(iii). For \( a_2 a_3 < 0 \), we select \([v_2 - v_3, v_4]\) as a representative element. and Eqs. (18) have the solution

\[
k'_1 = \sqrt{-\frac{a_2}{a_2}} a_2, \quad k'_2 = \sqrt{-\frac{a_2}{a_2}} (e_3 + e_2) a_2 + a_4, \quad k'_3 = \sqrt{-\frac{a_2}{a_2}} b_2, \quad k'_4 = \sqrt{-\frac{a_2}{a_2}} (e_3 + e_2) b_2 + b_4. 
\]

(76)
Case 2.2: For $a_2 = b_2 = 0$, Eqs. (18) always stand up and the general two-dimensional Lie algebra becomes 
$\{a_3 \partial_3, b_3 \partial_3, a_4 \partial_4, b_4 \partial_4, \partial_3, \partial_4\}$. Then if not all $a_3$ and $b_3$ are zeroes (and let $a_3 \neq 0$), it will equivalent to $\{v_3, v_4\}$ since that Eqs. (18) have the solution

$$
k'_1 = e^{-\epsilon_1} a_3, \quad k'_2 = -e^{\epsilon_1} e_2 a_3 + a_4, \quad k'_3 = e^{\epsilon_1} b_3, \quad k'_4 = -e^{\epsilon_1} e_2 b_3 + b_4. \tag{77}$$

For the case of $a_3 = b_3 = 0$, the general two-dimensional Lie algebra $\{a_4 \partial_4, b_4 \partial_4\}$ is trivial.

4.2.2. the case of $\lambda \neq 0$ in the determined equations [70]

Substituting $\lambda \neq 0$ into Eqs. (70), there must be $a_1 = 0$.

Case 3: $a_2 \neq 0$. Now, Eqs. (70) require

$$
\lambda = b_1(\neq 0), \quad a_3 = 0, \quad a_4 = \frac{a_2 b_1}{b_1}.
$$

(78)

Substituting (78) into Eqs. (68), it leads to an invariant for $\{w_1, w_2\}$:

$$
\phi = \Delta_3 \equiv \frac{b_1 b_4 - b_2 b_3}{b_1^2}.
$$

(79)

In condition of (78) and $\Delta_3 = c$, choose the corresponding representative element $\{v_2, v_1 + c v_4\}$ and Eqs. (18) have the solution

$$
k'_1 = a_2 e^{\epsilon_1}, \quad k'_2 = 0, \quad k'_3 = -\epsilon_2 b_1 + e^{\epsilon_1} b_2, \quad k'_4 = b_1, \quad \epsilon_3 = \frac{b_3}{b_1} e^{-\epsilon_1}.
$$

(80)

Case 4: $a_2 = 0$. By solving Eqs. (70), we get

$$
\lambda = -b_1(\neq 0), \quad b_2 = \frac{b_1 a_4}{a_3}.
$$

(81)

Substituting (81) with $a_1 = a_2 = 0$ into Eqs. (68), one can obtain an invariant as follows:

$$
\phi = \Delta_4 \equiv \frac{b_1 b_4 - b_2 b_3}{b_1^2}.
$$

(82)

In condition of (81) and $\Delta_4 = c$, select a representative element $\{v_3, v_1 + c v_4\}$ and Eqs. (18) have the solution

$$
k'_1 = a_3 e^{-\epsilon_1}, \quad k'_2 = 0, \quad k'_3 = \epsilon_3 b_1 + e^{-\epsilon_1} b_3, \quad k'_4 = b_1, \quad \epsilon_4 = \frac{a_4}{a_3} e^{\epsilon_1}.
$$

(83)

In summary, a two-dimensional optimal system $\Theta_2$ for the four-dimensional Lie algebra spanned by $v_1, v_2, v_3, v_4$ in (80) is shown as follows:

$$
g'_1 = [v_1, v_4], \quad g'_2 = [v_2, v_4], \quad g'_3 = [v_2 + v_3, v_4], \quad g'_4 = [v_2 - v_3, v_4],
$$

$$
g'_5 = [v_1, v_4], \quad g'_6 = [v_2, v_1 + c v_4], \quad g'_7 = [v_3, v_1 + c v_4]. \quad (c \in \mathbb{R})
$$

(84)

4.3. two-dimensional reductions for the NS equation [59]

Using a two-dimensional Lie algebra, one can reduce the $(2+1)$-dimensional NS equation to the ordinary equation and further get the invariant solutions. For each two-parameter element in the two-dimensional optimal system (84), there will be a corresponding class of group invariant solutions which will be determined from a reduced ordinary differential equation. For the case of $g'_1 = [v_1, v_4]$ and $g'_2 = [v_2, v_4]$, one can refer to [15]. The case of $g'_3$ leads to no group invariant solutions. Then we just consider the rest elements in (84).

(a) $g'_3 = [v_2 + v_3, v_4]$ and $g'_4 = [v_2 - v_3, v_4]$. By solving $(v_2 \pm v_3)(\psi) = 0$ and $v_4(\psi) = 0$, we have $\psi = F(x^2 + y^2) \pm \frac{1}{2} t(x^2 + y^2)$. Substituting it into the NS equation (59), one can get

$$
8 \gamma [\xi^2 F^{(4)}(\xi) + 4 \xi F''(\xi) + 2 F''(\xi)] \equiv 1 = 0
$$

(85)
with \( \xi = x^2 + y^2 \). By solving Eqs. (85), we find \( g'_3 \) and \( g'_4 \) lead to the same group invariant solution

\[ \psi = c_1 + c_2(x^2 + y^2) + c_3 \ln(x^2 + y^2) + c_4(x^2 + y^2)[\ln(x^2 + y^2) - 1] + \frac{1}{32y}(x^2 + y^2) + \frac{1}{2}(x^2 + y^2). \]  

(86)

(b) \( g'_5 = \{v_2, v_1 + cv_4\} \). From \( v_2(\psi) = 0 \) and \( (v_1 + cv_4)(\psi) = 0 \), one can get \( \psi = F(\arctan(\frac{y}{x}) - c \ln(x^2 + y^2)) \). Substituting it into (59) and integrating the reduced equation once, we have

\[ \gamma[(4c^2 + 1)G''(\xi) + 8cG'(\xi) + 4G(\xi)] - G^2(\xi) = 0, \quad (G(\xi) = F'(\xi)) \]  

with \( \xi = \arctan(\frac{y}{x}) - c \ln(x^2 + y^2) \). Specially, when \( c = 0 \) in Eq. (87), there is a solution

\[ G(\xi) = -6\text{sech}^2(\xi + c_0) + 4\gamma. \]  

Then it leads to the solution of the NS equation

\[ \psi = -6\gamma \tanh(\arctan(\frac{y}{x}) + c_0) + 4\gamma \arctan(\frac{y}{x}) + c_1. \]  

(89)

(c) \( g'_1 = \{v_3, v_1 + cv_4\} \). In this case, we have \( \psi = \arctan(\frac{y}{x}) + c \ln(t) + F(\frac{x^2 + y^2}{t}) \). The reduced equation for Eqe (59) is

\[ 4\gamma Z^2 G''(Z) + Z(Z + 8\gamma - 2)G'(Z) + (Z - 2)G(Z) = 0, \quad (F'(Z) = G(Z)) \]  

with \( Z = \frac{x^2+y^2}{t} \). In particular, for \( \gamma = \frac{1}{4} \), we obtain a solution

\[ \psi = \arctan(\frac{y}{x}) + c \ln(t) + c_1 + c_2 \ln(Z) + c_3(2 \ln(Z) + 3e^{-Z} + Ze^{-Z} + 2Ei(1, Z) + \frac{4}{3}). \]  

(91)

for \( \gamma = 1 \), there is

\[ \psi = \arctan(\frac{y}{x}) + c \ln(t) + c_1 + c_2 \ln(Z) + \frac{105}{8} c_3 \left( -\sqrt{\text{erf}(\frac{\sqrt{Z}}{2})} + \sqrt{\text{hypergeom}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{Z}{4})} \right). \]  

(92)

5. Summary and discussion

Since many important equations arising from physics are of low dimensions, only the determination of small parameter optimal systems can reduce them to ODEs which often lead to inequivalent group invariant solutions. In this paper, we give an elementary algorithm for constructing two-dimensional optimal system which only depends on fragments of the theory of Lie algebras. The intrinsic idea of our method is that every element in the optimal system corresponds to different values of invariants, the definition of which have been refined in this paper. Thanks to these invariants which are often overlooked except the Killing form in the almost existing methods, all the elements in the two-dimensional optimal system are found one by one and their inequivalences are evident with no further proof. Moreover, the construction of two-dimensional optimal system by us just starts from subalgebras, and does not require the prior one-dimensional optimal system as usual.

Before manipulating the given algorithm to construct two-dimensional optimal systems, one should compute two-dimensional subalgebras with “the determined equations”, the general adjoint transformation matrix and the invariants equations, which seem much complicated but can all be carried out in mechanization with the computer software "Maple". A new method is shown to provide all the invariants for the two-dimensional subalgebras, which is based on the idea of “invariant” under the meaning of adjoint transformation and combination act. Applying the algorithm to (1+1)-dimensional heat equation and (2+1)-dimensional NS equation, we obtain their two-dimensional optimal systems respectively. For the heat equation, the obtained two-dimensional optimal system contains eleven elements, which are discovered more comprehensive than that in Ref. [10] after a detailed comparison. For the (2+1)-dimensional NS equation, all the reduced ordinary equations and some exact group invariant solutions which come from the obtained two-dimensional optimal system are found. Since group invariant solutions are connected with the specific differential systems, inequivalent subalgebras may lead to the same solutions.

The algorithm considered in this paper is elementary and practical, without too much algebraic knowledge. Due to the programmatic process, to give a Maple package on the computer for two-dimensional optimal system is necessary and under our consideration. How to apply all the invariants to construct r-parameter \((r > 2)\) optimal systems is also an interesting job.
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