SURFACES CONTRACTING WITH SPEED $|A|^2$

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ABSTRACT. We show that strictly convex surfaces contracting with normal velocity equal to $|A|^2$ shrink to a point in finite time. After appropriate rescaling, they converge to spheres. We describe our algorithm to find the main test function.

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1. INTRODUCTION

We consider closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$ that contract with normal velocity equal to the square of the norm of the second fundamental form

$$\frac{d}{dt} X = -|A|^2 \nu.$$ (1.1)

This is a parabolic flow equation. We obtain a solution on a maximal time interval $[0, T)$, $0 < T < \infty$. For $t \uparrow T$, the surfaces converge to a point. After appropriate rescaling, they converge to a round sphere. We say that the surfaces $M_t$ converge to a “round point”. The key step in the proof, Theorem 3.3, is to show that

$$\max_{M_t} \left( \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \right)$$ (1.2)

is non-increasing in time.

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Here, we used standard notation as explained in Section 2.

Our main theorem is

**Theorem 1.1.** For any smooth closed strictly convex surface \( M \) in \( \mathbb{R}^3 \), there exists a smooth family of surfaces \( M_t, t \in [0, T) \), solving (1.1) with \( M_0 = M \). For \( t \uparrow T \), \( M_t \) converges to a point \( Q \). The rescaled surfaces \( (M_t - Q) \cdot (6(T - t))^{-1/3} \) converge smoothly to the unit sphere \( S^2 \).

We will also consider other normal velocities for which similar results hold. Therefore, we have to find quantities like (1.2) that are monotone during the flow and vanish precisely for spheres. In general, this is a complicated issue. In order to find these test quantities, we used an algorithm that checks, based on randomized tests, whether possible candidates fulfill certain inequalities. These inequalities guarantee especially that we can apply the maximum principle to prove monotonicity. We used that algorithm only to propose useful quantities. The presented proofs do not depend on it. So far, all candidates turned out to be appropriate for proving convergence to a round point. Our algorithm yields also candidates for many other normal velocities. We have only included a discussion of some interesting normal velocities. Moreover, for a fixed normal velocity, there are mostly several candidates for monotone quantities. In these cases, we have picked those involving not too complicated polynomials of low homogeneity.

In Table 1, we have collected some normal velocities \( F \) (1st column) and quantities \( w \) (2nd column) such that \( \max M_t w \) is non-increasing in time for surfaces contracting with normal velocity \( F \). In each case, we obtain convergence to round points for smooth closed strictly convex initial surfaces \( M_0 \).

In [5–7, 21], Gerhard Huisken and Ben Andrews proved that convex hypersurfaces contracting with certain normal velocities homogeneous of degree one converge to “round points”, i.e., they converge to a point and, after appropriate rescaling, to round spheres. For homogeneities larger than one, this was shown by Ben Andrews and Felix Schulze [5, 31], if the initial hypersurfaces are pinched appropriately. Kaising Tso proved that Gauß curvature flow shrinks strictly convex hypersurfaces to points [36]. If the homogeneity is less than one, there are examples by Koichi Anada, Masayoshi Tsutsumi, and Ben Andrews, where hypersurfaces do not become spherical [2, 3, 13]. Expanding flows were studied by Claus Gerhardt, John Urbas, Bennett Chow, Dong-Ho Tsai, Nina Ivochkina, Thomas Nehring, Friedrich Tomi, Knut Smoczyk, Gerhard Huisken, and Tom Ilmanen [15, 17, 23, 25, 32, 33, 37, 38]. Similar problems were also studied in manifolds (e.g. [8, 9, 22]) and for anisotropic flow equations (e.g. [11]). It is often required that the normal velocity is a concave function of the second fundamental form. There are many papers, concerned with contracting curves, e.g. by Michael Gage, Richard Hamilton, Matthew Grayson, and Steven Altschuler [1, 16, 18].

In [12], Ben Andrews shows that convex surfaces moving by Gauß curvature converge to round points. This normal velocity is homogeneous of degree two in the principal curvatures. He does not require any pinching condition for the initial surface. Our paper extends this result to other flow equations. We consider also normal velocities of degree larger than one and do not have to impose any pinching condition on the initial surface. Any smooth strictly convex surface converges to a round point.

The rest of this paper is organized as follows. In Section 2, we explain our notation. Section 3 concerns the proof for the normal velocity \( |A|^2 \). We describe our
algorithm to find test quantities in Section 4. In the remaining sections, we prove convergence for some other normal velocities and discuss the expected convergence rate.

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2. Notation

We use $X = X(x, t)$ to denote the embedding vector of a manifold $M_t$ into $\mathbb{R}^3$ and $\frac{\partial}{\partial t}X = \dot{X}$ for its total time derivative. It is convenient to identify $M_t$ and its

| Table 1. Monotone quantities |
|--------------------------------|
| $|A|^2$                       | $\frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ |
| $K$ [12]                     | $\lambda_1 - \lambda_2$ |
| $H^2$                        | $\frac{(\lambda_1 + \lambda_2)^3(\lambda_1 - \lambda_2)^2}{(\lambda_1^2 + \lambda_2^2) \lambda_1 \lambda_2}$ |
| $H^3$                        | $\frac{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) (\lambda_1 + \lambda_2)^2(\lambda_1 - \lambda_2)^2}{(\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2) \lambda_1 \lambda_2}$ |
| $H^4$                        | $\frac{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) (\lambda_1 + \lambda_2)^2(\lambda_1 - \lambda_2)^2}{\lambda_1^2 \lambda_2^2}$ |
| $|A|^2 + \beta H^2, 0 \leq \beta \leq 5$ | $\frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ |
| $\text{tr } A^3$            | $\frac{(3\lambda_1^2 + 2\lambda_1 \lambda_2 + 3\lambda_2^2) (\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ |
| $\text{tr } A^\alpha, \alpha = 2, 4, 5, 6$ | $\frac{(\lambda_1^{\alpha-2} + \lambda_2^{\alpha-2}) (\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ |
| $H|A|^2$                     | $\frac{(\lambda_1 + \lambda_2)^2(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}$ |
| $|A|^4$                      | $\frac{(\lambda_1^4 + 2\lambda_1^2 \lambda_2 + 4\lambda_1^2 \lambda_2 + 2\lambda_1 \lambda_2^2 + \lambda_2^4) (\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)\lambda_1 \lambda_2}$ |
embedding in \( \mathbb{R}^3 \). We choose \( \nu \) to be the outer unit normal vector to \( M_t \). The embedding induces a metric \((g_{ij})\) and a second fundamental form \((h_{ij})\). We use the Einstein summation convention. Indices are raised and lowered with respect to the metric or its inverse \((g^{ij})\). The inverse of the second fundamental form is denoted by \((\tilde{h}_{ij})\). The principal curvatures \( \lambda_1, \lambda_2 \) are the eigenvalues of the second fundamental form with respect to the induced metric. A surface is called strictly convex, if all principal curvatures are strictly positive. We will assume this throughout the paper.

Symmetric functions of the principal curvatures are well-defined, we will use the mean curvature \( H = \lambda_1 + \lambda_2 \), the square of the norm of the second fundamental form \( |A|^2 = \lambda_1^2 + \lambda_2^2 \), \( \text{tr} A^k = \lambda_1^k + \lambda_2^k \), and the Gauß curvature \( K = \lambda_1 \lambda_2 \). We write indices, preceded by semi-colons, e.g. \( h_{ij;k} \), to indicate covariant differentiation with respect to the induced metric.

Whenever we use this notation, we will also assume that we have fixed such a coordinate system. We will only use Euclidean coordinate systems for \( \mathbb{R}^3 \) so that \( h_{ij;k} \) is symmetric according to the Codazzi equations.

A normal velocity \( F \) can be considered as a function of \((\lambda_1, \lambda_2)\) or \((h_{ij}, g_{ij})\). We set \( F^{ij} = \frac{\partial F}{\partial h_{ij}} \), \( F^{ij,kl} = \frac{\partial^2 F}{\partial h_{ij} \partial h_{kl}} \). Note that in coordinate systems with diagonal \( h_{ij} \) and \( g_{ij} = \delta_{ij} \) as mentioned above, \( F^{ij} \) is diagonal. For \( F = |A|^2 \), we have \( F^{ij} = 2h^{ij} = 2\lambda_i g^{ij} \).

Recall, see e.g. [21, 28, 29], that for a hypersurface moving according to \( \frac{d}{dt} X = -F \nu \), we have

\[
\frac{d}{dt} g_{ij} = -2Fh_{ij},
\]
\[
\frac{d}{dt} h_{ij} = F_{ij} - Fh^k_i h^j_k,
\]
\[
\frac{d}{dt} g^{\alpha\beta} = g^{ij} F_{i;j} X^\alpha_{;j},
\]

where Greek indices refer to components in the ambient space \( \mathbb{R}^3 \). In order to compute evolution equations, we use the Gauß equation and the Ricci identity for the second fundamental form

\[
R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk},
\]
\[
h_{ik;lj} = h_{ik;jl} + h^a_i R_{a;lj} + h^a_l R_{a;ik}.
\]

We will also employ the Gauß formula and the Weingarten equation

\[
X^\alpha_{;ij} = -h_{ij} \nu^\alpha \quad \text{and} \quad \nu^\alpha = h^k_i X^\alpha_{;k}.
\]

For tensors \( A \) and \( B \), \( A_{ij} \geq B_{ij} \) means that \((A_{ij} - B_{ij})\) is positive definite. Finally, we use \( c \) to denote universal, estimated constants.
3. Surfaces Contracting With Speed $|A|^2$

3.1. Convergence to a Point. It is known, that (1.1) is a parabolic evolution equation for strictly convex initial data and that is has a solution on a maximal time interval.

We show that $M_t$ stays uniformly strictly convex. The following lemma is similar to results in [7].

**Lemma 3.1.** For a smooth closed strictly convex surface $M$ in $\mathbb{R}^3$, flowing according to $X = -|A|^2\nu$, the minimum of the principal curvatures is non-decreasing.

**Proof.** Consider $M_{ij} = h_{ij} - \varepsilon g_{ij}$ with $\varepsilon > 0$ so small that $M_{ij}$ is positive semi-definite for some time $t_0$. We wish to show that $M_{ij}$ is positive semi-definite for $t > t_0$. Combine (2.2), (2.4), and (2.5) to obtain

$$\frac{d}{dt} h_{ij} - F^{kl} h_{ij;kl} = 2 \text{tr} A^2 h_{ij} - 3|A|^2 h^k_k h_{kj} + 2g^{kr} g^{js} h_{kl;js;ij}.$$ 

In the evolution equation for $M_{ij}$, we drop the positive definite terms involving derivatives of the second fundamental form

$$\frac{d}{dt} M_{ij} - F^{kl} M_{ij;kl} \geq 2 \text{tr} A^2 h_{ij} - 3|A|^2 h^k_k h_{kj} + 2\varepsilon|A|^2 h_{ij}.$$ 

Let $\xi$ be a zero eigenvalue of $M_{ij}$ with $|\xi| = 1$, $M_{ij}\xi^j = h_{ij}\xi^j - \varepsilon g_{ij}\xi^j = 0$. So we obtain in a point with $M_{ij} \geq 0$

$$(2 \text{tr} A^2 h_{ij} - 3|A|^2 h^k_k h_{kj} + 2\varepsilon|A|^2 h_{ij}) \xi^j \xi^j = 2\varepsilon \text{tr} A^3 - 3\varepsilon^2|A|^2 + 2\varepsilon^2|A|^2$$

$$= 2\varepsilon \text{tr} A^3 - \varepsilon^2|A|^2$$

$$\geq 2\varepsilon^2|A|^2 - \varepsilon^2|A|^2 > 0$$

and the maximum principle for tensors [14, 19, 20] gives the result. \qed

The next result shows that $|A|^2$ stays uniformly bounded as long as $M_t$ encloses a ball of fixed positive radius. A similar estimate is used in [36].

**Lemma 3.2.** For a strictly convex solution of (1.1), $|A|^2$ is uniformly bounded in terms of the radius $R$ of an enclosed sphere $B_R(x_0)$, $\max_{M_0} \frac{|A|^2}{(X-x, \nu)-\frac{R}{2}R^2}$, and $\max_{M_0} |X-x_0|$. More precisely, we have

$$\sup_{t} \max_{M_t} |A|^2 \leq \max \left\{ \max_{M_0} |X-x_0| \cdot \max_{M_0} \frac{|A|^2}{(X-x, \nu)-\frac{R}{2}R^2}, \frac{18}{R^2} \right\}.$$ 

**Proof.** We may assume that $x_0 = 0$. Let $\alpha = \frac{1}{2}R$. Then $\alpha$ is a positive lower bound for $\langle X, \nu \rangle - \alpha$. Standard computations [21, 28, 29] yield the evolution equations

$$\frac{d}{dt} X^\beta - F^{ij} X^\beta_{;ij} = |A|^2 \nu^\beta,$$

$$\frac{d}{dt} \nu^\beta - F^{ij} \nu^\beta_{;ij} = 2 \text{tr} A^3 \nu^\beta,$$

$$\frac{d}{dt} \langle X, \nu \rangle - F^{ij} \langle X, \nu \rangle_{;ij} = -3|A|^2 + 2 \text{tr} A^3 \langle X, \nu \rangle,$$

$$\frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} = 2|A|^2 \text{tr} A^3.$$
In a critical point of \( \frac{|A|^2}{\langle X, \nu \rangle - \alpha} \), we obtain
\[
\frac{d}{dt} \log \frac{|A|^2}{\langle X, \nu \rangle - \alpha} = F^{ij} \left( \log \frac{|A|^2}{\langle X, \nu \rangle - \alpha} \right)_{;ij} = \frac{1}{\langle X, \nu \rangle - \alpha} (3|A|^2 - 2 \text{tr} A^3 \alpha).
\]

Note that \( \langle X, \nu \rangle - \alpha \leq \max_{M_t} |X| \) as a sphere of radius \( \max_{M_t} |X| \), centered at the origin, will enclose any \( M_t \). We only have to prove that we preserve the bound in Equation (3.1), when \( \max_{M_t} \frac{|A|^2}{\langle X, \nu \rangle - \alpha} \) increases. Then we have \( 0 \leq 3|A|^2 - 2 \text{tr} A^3 \alpha \) at a point, where \( \max_{M_t} \frac{|A|^2}{\langle X, \nu \rangle - \alpha} \) is attained. This inequality and elementary calculations for convex surfaces give
\[
|A|^2 \leq 2^{1/3} \cdot (\text{tr} A^3)^{2/3} \leq 2 \left( \frac{\text{tr} A^3}{|A|^2} \right)^{2/3} \leq \frac{9}{2\alpha^2}
\]
at such a maximum point and the Lemma follows.

We obtain that the second fundamental form of the surface stays bounded as long as \( M_t \) encloses some ball. The estimates of Krylov, Safonov, Evans (see also [4]), and Schauder imply that the solution stays smooth. Then, similarly as in [36], the positive lower bound on the minimum principal curvature implies that the surfaces converge to a point in finite time.

### 3.2. A Monotone Quantity.

**Theorem 3.3.** For a family of smooth closed strictly convex surfaces \( M_t \) in \( \mathbb{R}^3 \) flowing according to \( \dot{X} = -\frac{|A|^2}{X} \nu \),
\[
\max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)}{2\lambda_1 \lambda_2} = \max_{M_t} \frac{H \cdot (2|A|^2 - H^2)}{H^2 - |A|^2} \equiv \max_{M_t} w
\]
is non-increasing in time.

An immediate consequence of this theorem is

**Corollary 3.4.** The only homothetically shrinking smooth closed strictly convex surfaces \( M_t \), solving the flow equation \( \dot{X} = -\frac{|A|^2}{X} \nu \) in \( \mathbb{R}^3 \), are spheres.

**Proof.** The quantity \( \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2} \) is positive homogeneous of degree one in the principal curvatures and non-negative. If \( M \) is homothetically shrinking, Theorem 3.3 implies that \( (\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 = 0 \) everywhere. Thus \( M_t \) is umbilic and [34, Lemma 7.1] implies that \( M_t \) is a sphere.

**Proof of Theorem 3.3.** We combine (2.1), (2.2), (2.4), and (2.5) in order to get the following general evolution equation
\[
\frac{d}{dt} \text{tr} A^\alpha - F^{kl} (\text{tr} A^\alpha)_{;kl} = \alpha \sum_i F^{ii} \lambda_i^3 \text{tr} A^\alpha + \alpha \left( F - \sum_i F^{ii} \lambda_i \right) \text{tr} A^{\alpha + 1} - \alpha \sum_{r=0}^{\alpha - 2} \sum_{i,j,k} F^{kk} \lambda_i^{\alpha - 2 - r} \lambda_j^r h^2_{ij;k} + \alpha \sum_{k,l,r,s,i} F^{kl} r^s h_{kl;rs} i h_{rs;i} \lambda_i^{\alpha - 1}.
\]
(3.3)
for solutions to the flow equation
\[ \frac{d}{dt} \mathbf{X} = -F\nu. \]

Using (3.3) for \( F = |A|^2 \) yields
\[ \frac{d}{dt} H - F^{ij} H_{;ij} = -\left( |A|^2 \right)^2 + 2H \text{ tr } A^3 + 2 \sum h_{ij;k}^2 \]

and
\[ \frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} = 2|A|^2 \text{ tr } A^3. \]

For the reader’s convenience, we include the details to obtain (3.4).
\[ H = g^{ij} h_{ij}, \]
\[ F = |A|^2 = g^{ij} h_{jk} g^{kl} h_{li}, \]
\[ F^{ij} = 2h^{ij}, \]
\[ \frac{d}{dt} H = -g^{ia} g^{bj} h_{ij} \frac{d}{dt} g_{ab} + g^{ij} \frac{d}{dt} h_{ij} \]
\[ = -g^{ia} g^{bj} h_{ij} (-2|A|^2 h_{ab}) \]
\[ + g^{ij} \left( (|A|^2)_{;ij} - |A|^2 h_{kj} \right) \]
\[ = (|A|^2)^2 + g^{ij} \left( 2h^{kl} h_{kl;i,j} + 2g^{kr} g^{ls} h_{kl;i,j} \right) \]
\[ = (|A|^2)^2 + 2g^{ij} h_{kl} (h_{ij;kl} + h_{kl}^{il} R_{ailj} + h_{kl}^{il} R_{aklj}) \]
\[ + 2 \sum h_{ij;k}^2 \]
\[ = F^{kl} (g^{ij} h_{ij})_{;kl} + (|A|^2)^2 \]
\[ + 2g^{ij} h_{kl} h_{al} h_{ij} - 2g^{ij} h_{kl} h_{aj} h_{il} \]
\[ + 2g^{ij} h_{kl} h_{al} h_{kj} - 2g^{ij} h_{kl} h_{aj} h_{kl} + 2 \sum h_{ij;k}^2 \]
\[ = F^{ij} H_{;ij} - (|A|^2)^2 + 2H \text{ tr } A^3 + 2 \sum h_{ij;k}^2. \]

For the rest of the proof, we consider a critical point of \( w|_{M_t} \), for some \( t > 0 \), where \( w > 0 \). It suffices to show that \( \tilde{w} := \log w \) is non-increasing in such a point. Then our theorem follows.

We rewrite \( \tilde{w} \)
\[ \tilde{w} = \log H + \log \left( 2|A|^2 - H^2 \right) - \log(H^2 - |A|^2) \]
\[ \equiv \log A + \log B - \log C. \]

In a critical point of \( \tilde{w} \), we obtain
\[ \frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{;ij} = \frac{1}{A} \left( \frac{d}{dt} A - F^{ij} A_{;ij} \right) + \frac{1}{B} \left( \frac{d}{dt} B - F^{ij} B_{;ij} \right) \]
\[ - \frac{1}{C} \left( \frac{d}{dt} C - F^{ij} C_{;ij} \right) - \frac{1}{AB} F^{ij} (A_{;i} B_{;j} + A_{;j} B_{;i}). \]
and

\[
0 = \frac{1}{H} H_{;k} + \frac{1}{2|A|^2 - H^2} (2|A|^2 - H^2) ;_{;k} - \frac{1}{H^2 - |A|^2} (H^2 - |A|^2) ;_{;k}
\]

\[
= \frac{2\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 (\lambda_1^2 - \lambda_2^2)} h_{11;1} + \frac{2\lambda_2^2 + \lambda_1 \lambda_2 + \lambda_1^2}{\lambda_2 (\lambda_2^2 - \lambda_1^2)} h_{22;2}.
\]

So we deduce that

\[
h_{22;1} = \frac{\lambda_2}{\lambda_1} 2\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 = a_1 h_{11;1}
\]

and a similar formula holds for \(h_{11;2} = a_2 \cdot h_{22;2}\). We now combine all these results and obtain in a straightforward calculation

\[
\frac{d}{dt} \dot{w} - F^{ij} \dot{w}_{;ij} = \left( \frac{1}{H} \frac{2H}{2|A|^2 - H^2} \right) \cdot \left( \frac{d}{dt} H - F^{ij} H_{;ij} \right) \\
+ \left( \frac{2}{2|A|^2 - H^2} + \frac{1}{H^2 - |A|^2} \right) \cdot \left( \frac{d}{dt} |A|^2 - F^{ij} (|A|^2) ;_{;ij} \right) \\
+ \left( \frac{6}{2|A|^2 - H^2} + \frac{2}{H^2 - |A|^2} \right) F^{ij} H_{;i} H_{;j} \\
- \frac{2}{H}(2|A|^2 - H^2) F^{ij} (|A|^2) ;_{;i} H_{;j} + (|A|^2) ;_{;j} H_{;i}
\]

\[
= \left( \frac{1}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 \lambda_1^2 \lambda_2} \right) \cdot \left( - (|A|^2)^2 + 2H \text{tr} A^3 \right) \\
+ \frac{(\lambda_1 + \lambda_2)^2}{2(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2} \cdot 2|A|^2 \text{tr} A^3 \\
- 2\lambda_1^4 + \lambda_1^3 \lambda_2 + 4\lambda_1^2 \lambda_2^2 + \lambda_1 \lambda_2^3 + \lambda_2^4 \\
(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \sum h_{ij;1} \\
+ 2\lambda_2^2 + 4\lambda_1 \lambda_2 + \lambda_1^2 \\
(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \sum \lambda_k h_{ii;1} h_{jj;1} \\
- \frac{8}{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^3} \sum \lambda_k (\lambda_1 + \lambda_2) h_{ii;1} h_{jj;1} \\
\lambda_1^6 + 3\lambda_1^5 \lambda_2 + 4\lambda_1^4 \lambda_2^2 + 9\lambda_1^3 \lambda_2^3 - 2\lambda_1^2 \lambda_2^4 \\
(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \\
- 9\lambda_1^3 \lambda_2^3 + 4\lambda_1^2 \lambda_2^2 + 2\lambda_1 \lambda_2^3 + \lambda_2^4 \\
(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \\
+ \lambda_1^7 + 2\lambda_1^6 \lambda_2 + 2\lambda_1^5 \lambda_2^2 + 3\lambda_1^4 \lambda_2^3 + 3\lambda_1^3 \lambda_2^4 + 2\lambda_1^2 \lambda_2^5 + 2\lambda_1 \lambda_2^6 + \lambda_2^7 \\
(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \\
- 2\lambda_2^4 + 4\lambda_1 \lambda_2 + \lambda_1^2 \\
(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \cdot (1 + a_1^2) \cdot h_{11;1} + (1 + 3a_2^2) \cdot h_{22;2} \right) \\
+ 2\lambda_2^2 + 4\lambda_1 \lambda_2 + \lambda_1^2 \\
(\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2 \cdot (1 + a_1^2) \cdot h_{11;1} + \lambda_2 (1 + a_2^2) \cdot h_{22;2}
Corollary 3.7. For a smooth closed strictly convex surface $c \cdot (\lambda_1 + \lambda_2 a_1)(1 + a_1) \cdot h_{11}^2 \cdot (\lambda_1 + \lambda_2)(1 + a_2) \cdot h_{22}^2$)

$$= -\frac{16}{(\lambda_1 - \lambda_2)^2(\lambda_1 + \lambda_2)} \cdot (\lambda_1(\lambda_1 + \lambda_2 a_1)(1 + a_1) \cdot h_{11}^2 \cdot (\lambda_1 + \lambda_2)(1 + a_2) \cdot h_{22}^2)$$

$$= -\frac{4K^2}{H}$$

$$= \frac{5\lambda_1^4 - 4\lambda_1^2 \lambda_2 + 46\lambda_1^6 \lambda_2^2 + 48\lambda_1^2 \lambda_2^3 + 72\lambda_1^4 \lambda_2^2}{\lambda_1 + \lambda_2} (\lambda_1 - \lambda_2)^2 (\lambda_1^2 + \lambda_1 \lambda_2 + 2\lambda_2^2) \lambda_2^2 h_{11}^2$$

$$- \frac{5\lambda_1^4 - 4\lambda_1^2 \lambda_2 + 46\lambda_1^6 \lambda_2^2 + 48\lambda_1^2 \lambda_2^3 + 72\lambda_1^4 \lambda_2^2}{\lambda_1 + \lambda_2} (\lambda_1 - \lambda_2)^2 (\lambda_1^2 + \lambda_1 \lambda_2 + 2\lambda_2^2) \lambda_2^2 h_{22}^2$$

$$\leq 0.$$  

We finally apply the maximum principle and our theorem follows. \qed

3.3. Direct Consequences. We obtain a pinching estimate

**Corollary 3.5.** For a smooth closed strictly convex surface $M_t$ in $\mathbb{R}^3$, flowing according to $\dot{X} = -|A|^2 \nu$, there exists $c = c(M_0)$ such that $0 < \frac{\lambda_1}{\lambda_2} \leq \frac{\lambda_1}{\lambda_2} \leq c$.

**Proof.** Choose $\varepsilon > 0$ such that $\lambda_1, \lambda_2 > \varepsilon$ at $t = 0$. Theorem 3.3 and Lemma 3.1 imply that

$$2\varepsilon \left( \frac{\lambda_1}{\lambda_2} - 1 \right)^2 = 2\varepsilon \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \leq \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \leq c.$$  

We obtain the upper bound on $\frac{\lambda_1}{\lambda_2}$ claimed above. Similarly, we obtain an upper bound on $\frac{\lambda_1}{\lambda_2}$. \qed

Let $\rho_+$ be the minimal radius of enclosing spheres and $\rho_-$ the maximal radius of enclosed spheres. The quotient of these radii can be estimated as follows

**Corollary 3.6.** Under the assumptions of Corollary 3.5, $\rho_+ / \rho_-$ is bounded above by a constant depending only on the constant $c(M_0)$ in Corollary 3.5.

**Proof.** Combine Corollary 3.5, [7, Theorem 5.1], and [7, Lemma 5.4]. \qed

We also obtain a bound for $|\lambda_1 - \lambda_2|

**Corollary 3.7.** For a smooth closed strictly convex surface $M_t$ in $\mathbb{R}^3$, flowing according to $\dot{X} = -|A|^2 \nu$, there exists a constant $c = c(M_0)$ such that $|\lambda_1 - \lambda_2| \leq c \cdot |A|^2 \nu \leq c \cdot \sqrt{H}$.

**Proof.** This is a direct consequence of Theorem 3.3 and Corollary 3.5. \qed

As in [12], this estimate on $|\lambda_1 - \lambda_2|$ is “better” than scaling invariant. It is crucial for the rest of the proof of Theorem 1.1.

Let us recall a form of the maximum principle for evolving hypersurfaces.
Lemma 3.8. Let $M_t$ and $\tilde{M}_t$ be two smooth closed strictly convex solutions to (1.1) on some time interval $[0, T^*)$. If $M_0$ encloses $\tilde{M}_0$, then $M_t$ encloses $\tilde{M}_t$ for any $t \in [0, T^*)$.

Proof. This is a standard consequence of the maximum principle. □

The next result describes the evolution of spheres.

Lemma 3.9. Spheres $\partial B_{r(t)}(x_0)$ solve (1.1) for $t \in [0, T)$ with $r(t) = (6(T-t))^{1/3}$ and $T = \frac{1}{6}r^3(0)$.

Proof. The evolution equation for the radius of a sphere is

$$\dot{r}(t) = -\frac{2}{r^2(t)}.$$ □

As a consequence, we can estimate the life span of a solution in terms of inner and outer radii.

Lemma 3.10. Let $\rho_+(t)$ and $\rho_-(t)$ be the inner and outer radii of $M_t$, respectively. Assume that $M_t$ is a smooth closed strictly convex solution of (1.1) on a maximal time interval $[0, T)$. Then we have for $t \in [0, T)$

$$\frac{1}{6}\rho_+^3(t) \leq T - t \leq \frac{1}{6}\rho_-^3(t).$$

Proof. As $M_t$ contracts to a point, we deduce from Lemma 3.8 that $T - t$ is bounded below by the life span of $\partial B_{\rho_-(t)}$ evolving according to (1.1). So the lower bound follows from Lemma 3.9. The upper bound is obtained similarly. □

3.4. Convergence to a Round Point. We closely follow the corresponding part of [12].

Proposition 3.11. Define $q(t) := \frac{1}{4\pi} \int_{M_t} KX$. Then

$$\left| \langle X - q, \nu \rangle - \frac{1}{8\pi} \int_{M_t} H \right| \leq \frac{1}{4\pi} \cdot \sup_{M_t} |\lambda_1 - \lambda_2| \cdot \mathcal{H}^2(M_t),$$

where $\mathcal{H}^2(M_t)$ denotes the area of $M_t$.

Proof. This is [12, Proposition 4]. □

We define $r_+(t)$ to be the minimal radius of a sphere, centered at $q(t)$, that encloses $M_t$. Similarly, we define $r_-(t)$ to be the maximal radius of a sphere, centered at $q(t)$, that is enclosed by $M_t$.

Lemma 3.12. Under the assumptions of Theorem 1.1, for $T-t$ sufficiently small, $r_+$ and $r_-$ are estimated as follows

$$r_+(t) \leq (6(T-t))^{1/3} \cdot \left(1 + c \cdot (T-t)^{1/6}\right),$$

$$r_-(t) \geq (6(T-t))^{1/3} \cdot \left(1 - c \cdot (T-t)^{1/6}\right),$$

and

$$1 \leq \frac{r_+}{r_-} \leq 1 + c \cdot (T-t)^{1/6}.$$
Proof. Denote the bounded component of $\mathbb{R}^3 \setminus M_t$ by $E_t$. The transformation formula for integrals implies that
\[
\frac{1}{4\pi} \int_{M_t} KX = \frac{1}{4\pi} \int_{\mathbb{S}^2} X(\nu^{-1}(\cdot)).
\]
So we see that $q(t) \in E_t$. We have
\[
\begin{align*}
    r_+ &= \max_{M_t} \langle X - q(t), \nu \rangle, \\
    \rho_+ &= \min_{p \in \mathbb{R}^3} \max_{M_t} \langle X - p, \nu \rangle, \\
    r_- &= \min_{M_t} \langle X - q(t), \nu \rangle, \\
    \rho_- &= \max_{p \in E_t} \min_{M_t} \langle X - p, \nu \rangle.
\end{align*}
\]
Recall the first variation formula for a vector field $Y$ along $M_t$ [24]
\[
\int_{M_t} H(Y, \nu) = \int_{M_t} \text{div}_{M_t} Y
\]
and get for $p \in E_t$ such that $\rho_+ = \max_{M_t} \langle X - p, \nu \rangle$
\[
\int_{M_t} H \geq \frac{1}{\rho_+} \int_{M_t} H \cdot \langle X - p, \nu \rangle = \frac{1}{\rho_+} \int_{M_t} \text{div}_{M_t} X = \frac{1}{\rho_+} \int_{M_t} 2 = \frac{2}{\rho_+} \mathcal{H}^2(M_t).
\]
We employ Proposition 3.11 and deduce that
\[
\begin{align*}
    r_- &\geq \frac{1}{8\pi} \int_{M_t} H \cdot \left(1 - 2 - \left(\int_{M_t} H\right)^{-1} \cdot \sup_{M_t} |\lambda_1 - \lambda_2| \cdot \mathcal{H}^2(M_t) \right) \\
    &\geq \frac{1}{8\pi} \int_{M_t} H \cdot \left(1 - \rho_+ \cdot \sup_{M_t} |\lambda_1 - \lambda_2| \right).
\end{align*}
\]
We estimate as follows
\[
\begin{align*}
    \rho_+ \cdot \sup_{M_t} |\lambda_1 - \lambda_2| &\leq c \cdot (|A|^2)^{1/4} \quad \text{by Corollary 3.7} \\
    &\leq c \cdot \rho_+ \cdot \left(c + \frac{c}{\rho_+^2}\right)^{1/4} \quad \text{by Lemma 3.2} \\
    &\leq c \cdot (T - t)^{1/6}
\end{align*}
\]
by Corollary 3.6 and Lemma 3.10 for $(T - t)$ small. So we obtain
\[
(3.6) \quad r_-(t) \geq \frac{1}{8\pi} \int_{M_t} H \cdot \left(1 - c \cdot (T - t)^{1/6}\right).
\]
Similar calculations yield
\[
(3.7) \quad r_+(t) \leq \frac{1}{8\pi} \int_{M_t} H \cdot \left(1 + c \cdot (T - t)^{1/6}\right).
\]
We employ Lemma 3.10
\[
r_- \leq \rho_- \leq (6(T - t))^{1/3} \leq \rho_+ \leq r_+
\]
and obtain for $(T - t)$ small
\[
(6(T - t))^{1/3} \cdot \left(1 - c \cdot (T - t)^{1/6}\right) \leq \frac{1}{8\pi} \int_{M_t} H \leq (6(T - t))^{1/3} \cdot \left(1 + c \cdot (T - t)^{1/6}\right).
\]
Proof. We define for Lemma 3.14. The dual function to concave. The function $\Phi$ is called the dual function to $F$. Therefore, we get $$\Phi = \Phi(\lambda) = \Phi(h_{ij}, g_{ij}) = -\left(1 - \frac{\lambda_1}{\lambda_2}\right) = \frac{2K - H^2}{K^2} = \frac{2 \det h^i_j - (\text{tr} h^i_j)^2}{(\det h^i_j)^2}$$ and obtain $$\Phi^{ij} = \frac{\partial \Phi}{\partial h_{ij}} = \frac{1}{K^2} \left\{ \frac{2H^2 - 2K}{h^{ij}} - 2H g^{ij} \right\},$$ $$\Phi^{ij, kl} = \frac{1}{K^2} \left\{ \frac{2K - 4H^2}{h^{ij}} \tilde{h}^{ij} \tilde{h}^{kl} + \frac{[2K - 2H^2]}{h^{ij} h^{kl}} \right\}.$$ According to [10, (5.4)], it suffices to show that $$\Phi^{ij, kl} \eta_{ij} \eta_{kl} \leq \frac{\alpha - 1}{\alpha \Phi} \Phi^{ij} \eta_{ij} \Phi^{kl} \eta_{kl}$$ for some $\alpha > 0$. If $\Phi$ is $\alpha$-concave, if $\Phi = \text{sgn} \cdot B^\alpha$ for some $B$, where $B$ is positive and concave. The function $\Phi$ is called the dual function to $F$.

**Corollary 3.13.** Under the assumptions of Theorem 1.1, we have the estimate

$$|q(t) - Q| \leq c \cdot (T - t)^{1/3 + 1/6}.$$ Therefore, we obtain the same estimates as in Lemma 3.12, if we define $r_+$ and $r_-$ using $Q$ instead of $q(t)$.

**Proof.** Fix $t_0 \in [0, T)$. A sphere of radius $(6(T - t_0))^{1/3} (1 + c(T - t_0)^{1/6}$, centered at $q(t_0)$ as defined in Proposition 3.11, will enclose $M_0$ for all $t_0 \leq t < T$. Thus its radius at time $t = T$ is an upper bound for $|q(t_0) - Q|$. According to Lemma 3.9, the radius of that sphere evolves as follows

$$r(t) = \left(6 \left(\frac{1}{6} r_3(0) - (t - t_0)\right)\right)^{1/3}.$$ Therefore, we get

$$|q(t_0) - Q| \leq r(T)$$

$$= \left(6 \left(\frac{1}{6} (T - t_0) \left(1 + c(T - t_0)^{1/6}\right)^3 - (T - t_0)\right)\right)^{1/3}$$

$$= (6(T - t_0))^{1/3} \cdot \left(\left(1 + c(T - t_0)^{1/6}\right)^3 - 1\right)^{1/3}$$

$$\leq (6(T - t_0))^{1/3} \cdot c \cdot (T - t_0)^{1/6}.$$ □

Next, we want to check that we can apply a Harnack inequality [10, Theorem 5.17]. For $F = F(\lambda_i)$, $\lambda_i > 0$, we define

$$\Phi(\kappa_i) := -F(\kappa_i^{-1}).$$ We say that $\Phi$ is $\alpha$-concave, if $\Phi = \text{sgn} \cdot B^\alpha$ for some $B$, where $B$ is positive and concave. The function $\Phi$ is called the dual function to $F$.

**Lemma 3.14.** The dual function to $F = |A|^2 = \lambda_1^2 + \lambda_2^2$ is $\alpha$-concave for $\alpha \leq -2$.

**Proof.** We define for $\lambda_i > 0$
for symmetric matrices $(\eta_{ij})$. Terms involving $\eta_{12}$ clearly have the right sign. The remaining terms are a quadratic form in $(\eta_{11}, \eta_{22})$. Thus (3.8) is fulfilled, if

$$|A|^2 \begin{cases} [2K - 4H^2] \left( \frac{1}{\lambda_1} \quad \frac{1}{\lambda_2} \right) + [2K - 2H^2] \left( \frac{1}{\lambda_1^2} \quad 0 \quad \frac{1}{\lambda_2^2} \right) \\ +4H \left( \alpha \frac{2}{\lambda_1} + \frac{1}{\lambda_2} \right) - 2 \left( 1 \quad 1 \right) \right) \leq \\
\leq -\frac{\alpha - 1}{\alpha} \left\{ [2H^2 - 2K] \left( \frac{1}{\lambda_1^2} \right) - 2H \left( 1 \right) \right\} \otimes \left\{ [2H^2 - 2K] \left( \frac{1}{\lambda_2^2} \right) - 2H \left( 1 \right) \right\} \\
or equivalently \\
-6 (\lambda_1^2 + \lambda_2^2) \begin{pmatrix} \frac{1}{\lambda_1} \quad 0 \quad \frac{1}{\lambda_2} \end{pmatrix} \leq -4 \frac{\alpha - 1}{\alpha} \begin{pmatrix} \frac{\lambda_1^4}{\lambda_1^2} \quad \lambda_1 \lambda_2 \quad \frac{\lambda_2^4}{\lambda_2^2} \end{pmatrix}.
$$

As $\frac{\alpha - 1}{\alpha} \leq \frac{2}{3}$ for $\alpha \leq -2$, we obtain that $\Phi$ is $\alpha$-concave. \hfill \Box

We are now able to improve our velocity bounds.

**Lemma 3.15.** Under the assumptions of Theorem 1.1, we obtain

$$2(6(\tau^3 - 6\tau)^{1/3}) \leq |A|^2 \leq 2(6(T - \tau))^{-2/3} \cdot \left( 1 + c \cdot (T - \tau)^{1/2} \right)$$

everywhere on $M_t$ for $(T - \tau)$ sufficiently small.

**Proof.** We may assume that $T - \tau > 0$ is so small that we can use the results obtained before. Parameterize $M_t$ by $S^2$ such that the normal image of $M_t$ at $X(z, t)$ equals $z \in S^2$. Let us define the support function $s$ of $M_t$ as

$$s(z, t) := \langle X(z, t), z \rangle.$$ 

Its evolution equation, see e.g. [10], is

$$\frac{d}{dt} s(z, t) = -|A|^2(z, t).$$

The $\alpha$-concavity proved in Lemma 3.14 allows us to use [10, Theorem 5.17]. We obtain for $0 < t_1 < t_2 < T$ and $z \in S^2$, for two points $(z, t_1)$ and $(z, t_2)$ with the same normal,

$$\frac{|A|^2(z, t_2)}{|A|^2(z, t_1)} \geq \left( \frac{t_1}{t_2} \right)^{2/3}.$$ 

Let us assume that $q(t)$ is the origin for some fixed time $t$. As $M_t$ lies between $\partial B_{r_+(t)}(0)$ and $\partial B_{r_-(t)}(0)$, $M_t$ lies outside $B_{(r_+(t) - 6\tau)^{1/3}}(0)$ for any $0 < \tau < T - t$, so

$$r_-(t) \leq s(\cdot, t) \leq r_+(t) \quad \text{and} \quad (r_+^3 - 6\tau)^{1/3} \leq s(\cdot, t + \tau).$$

Set $\tau = r_+^{5/2}(t) \cdot (r_+(t) - r_-(t))^{1/2}$ and observe that $t + \tau < T$, if $(r_+ - r_-)^{1/2} \leq \frac{1}{6} t^{1/2}$ (by Lemma 3.10), or, if $T - t$ is sufficiently small (by Lemma 3.12). We estimate

$$|A|^2(z, t) \leq \inf_{0 \leq \tilde{\tau} \leq \tau} \left\{ \left( \frac{t + \tilde{\tau}}{t} \right)^{2/3} \cdot |A|^2(z, t + \tilde{\tau}) \right\} \quad \text{by (3.10)}$$
Lemma 3.16. Under the assumptions of Theorem 1.1, we obtain 
\[(6(T-t))^{-1/3} \cdot (1 - c \cdot (T - t)^{1/12}) \leq \lambda_1, \lambda_2 \leq (6(T-t))^{-1/3} \cdot (1 + c \cdot (T - t)^{1/12})\] 
on M_t for small \(T - t\).

Proof. As \(H^2 = 2|A|^2 - (\lambda_1 - \lambda_2)^2\), we obtain
\[
\lambda_1 = \frac{1}{2} (\lambda_1 + \lambda_2) + \frac{1}{2} (\lambda_1 - \lambda_2) \\
= \frac{1}{2} \sqrt{2|A|^2 - (\lambda_1 - \lambda_2)^2} + \frac{1}{2} (\lambda_1 - \lambda_2).
\] 
Combining Lemmata 3.7 and 3.15, we get \(|\lambda_1 - \lambda_2| \leq c \cdot (T - t)^{-1/6}\). We use Lemma 3.15 and (3.12). The claimed inequality follows. \(\square\)
Proof of Theorem 1.1: Lemma 3.16 implies, that, everywhere on $M_t$, the quotient $\lambda_1/\lambda_2$ tends to 1 as $t \uparrow T$. Then we can apply known results, see e.g. [5, Theorem 2], to conclude that the rescaled surfaces converge smoothly to the unit sphere $S^2 \subset \mathbb{R}^3$. □

A standard way of rescaling [7] is to consider the embeddings $\tilde{X}(z, t)$, $\tilde{X}(z, t) := \frac{(6(T-t))^{-1/3}}{3}(X(z, t) - Q)$ with $Q$ as in Theorem 1.1. Define the time function $\tau(t) := \frac{1}{6} \log T - \frac{1}{6} \log(T-t)$. Then we have, using suggestive notation, the following evolution equation

$$d \frac{d}{d\tau} \tilde{X} = -|\tilde{A}|^2 \tilde{\nu} + 2 \tilde{X}$$

and our a priori estimates imply, that, for $\tau \to \infty$, $\tilde{M}_t$ converges exponentially to $S^2$.

4. Finding Monotone Quantities

4.1. The Algorithm. We use a sieve algorithm and start with symmetric rational functions of the principal curvatures as candidates for test functions, e.g.

$$w = \frac{p_1(\lambda_1, \lambda_2)}{p_2(\lambda_1, \lambda_2)} = \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}.$$  

Here, $p_1 \neq 0$ and $p_2 \neq 0$ are homogeneous polynomials.

In the end, we want to find functions $w$ such that $W := \sup_{M_t} w$ is monotone and ensures convergence to round spheres.

We check, whether these test functions $w$ fulfill the following conditions.

1. (a) $p_1(\lambda_1, \lambda_2), p_2(\lambda_1, \lambda_2) \geq 0$ for $0 < \lambda_1, \lambda_2$,
   (b) $p_1(\lambda_1, \lambda_2) = 0$ for $\lambda_1 = \lambda_2 > 0$.
2. $\deg p_1 > \deg p_2$.
3. $\frac{\partial w(1, \lambda_2)}{\partial \lambda_2} < 0$ for $0 < \lambda_2 < 1$ and $\frac{\partial w(1, \lambda_2)}{\partial \lambda_2} > 0$ for $\lambda_2 > 1$.
4. $d \frac{d}{dt} w - F_{ij} w_{,ij} \leq 0$
   (a) for terms without derivatives of $(h_{ij})$,
   (b) for terms involving derivatives of $(h_{ij})$, if $w_{,i} = 0$ for $i = 1, 2$.

4.2. Motivation and Randomized Tests. We restrict our attention to non-negative polynomials $p_1$. For all flow equations considered, spheres contract to points and stay spherical. So we can only find monotone quantities, if $\deg p_1 \leq \deg p_2$ or $p_1(\lambda, \lambda) = 0$.

If $\deg p_1 < \deg p_2$, we obtain that $W$ is decreasing on any self-similarly shrinking surface. So this does not imply convergence to a sphere. The counterexamples in [5] show for normal velocities of homogeneity larger than 1, that the pinching ratio $\sup_{M_t} \lambda_2/\lambda_1$ (for $\lambda_2 > \lambda_1$) will increase during the flow for appropriate initial surfaces. Therefore, we require in step (2), that $\deg p_1 > \deg p_2$.

Condition (3) ensures that the quantity decreases, if the eigenvalues approach each other. This excludes especially local zeroes of $w(1, \lambda_2)$ for $\lambda_2 \neq 1$.

In all these steps, inequalities are tested by evaluating both sides at random numbers. If an inequality is not violated for sufficiently many tuples of random numbers, we move to the next step and keep the candidate, otherwise we start with another candidate.
The hard part is to test, whether \( \frac{d}{dt} w - F_{ij} w_{;ij} \leq 0 \) holds at a point, where \( w_{;i} = 0 \).

We assume in step (4a), that \( h_{ij;k} = 0 \). Recall the algebraic fact, that all products of \( H \) and \(|A|^2\) of given homogeneity form a basis for the symmetric homogeneous polynomials of that degree. We represent the polynomials \( p_i \) in this basis. At random values for \( \lambda_1, \lambda_2 \), we compute \( \frac{d}{dt} H - F_{ij} H_{;ij} \) and \( \frac{d}{dt} |A|^2 - F_{ij} (|A|^2)_{;ij} \). This can be combined according to the rules of differentiation and yields \( \frac{d}{dt} w - F_{ij} w_{;ij} \), evaluated at these numbers.

If not all components of \( h_{ij;k} \) vanish (step (4b)), we also have to choose random numbers for \( h_{ij;k} \) that fulfill the extremal condition \( w_{;i} = 0 \). As above, we evaluate \( \frac{d}{dt} w - F_{ij} w_{;ij} \) at the random numbers chosen. Here we can ignore all terms that do not contain derivatives of the second fundamental form. The evaluation of the remaining terms is more involved than in the last step, but can be done similarly according to the various rules of differentiation.

Now we iterate steps (4a) and (4b). If all tests yield \( \frac{d}{dt} w - F_{ij} w_{;ij} \leq 0 \) at critical points of \( w \), it seems likely that we have found an appropriate test quantity. Indeed, if this inequality is fulfilled, the maximum principle implies that \( W = \max_{M_t} w \) is non-increasing in time.

We implemented this algorithm in a C-program and used it to find all the new test functions of this paper.

Obviously, the computing time depends on the number of tests performed. The following computing times are measured for the quantity (3.2) on an Intel Pentium 4, 2.4 GHz, running Linux 2.4.24 and GNU C-compiler 2.95.4. The number of tests per second for step (4a) is \( 1.6 \cdot 10^5 \) and \( 5.8 \cdot 10^3 \) for step (4b). The other steps are comparable to step (4a). In steps (1) to (3), the calculations do not depend on the normal velocity.

It seems worth noting, that, after testing with enough random numbers in an appropriate range, every candidate for a monotone quantity that we checked, turned out to be a useful test quantity. In that sense, algorithm and program seem to be correct.

We are convinced that it is possible to implement this algorithm for surfaces without using random numbers. Evolution equations can be computed algebraically and Sturm’s algorithm can be used to test for non-negativity.

We expect that similar algorithms will be used to find (monotone) test functions for other (geometric) problems.

5. \( H^3 \)-Flow

In this and the following sections, we will consider strictly convex surfaces contracting according to \( \frac{d}{dt} X = -F\nu \) for several normal velocities \( F \). We will not repeat parts of the argument that are very similar to the respective parts in the proof for \( F = |A|^2 \).

As the theorems for these flow equations agree essentially with Theorem 1.1, we will state them in concise form as follows.

**Theorem 5.1.** A smooth closed strictly convex surface in \( \mathbb{R}^3 \), contracting with normal velocity \( H^3 \), converges to a round point in finite time.

5.1. A Monotone Quantity.
Theorem 5.2. For a family of smooth closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$, flowing according to $\frac{d}{dt} X = -H^3 \nu$,

$$\max_{M_t} \frac{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) (\lambda_1 + \lambda_2)^2 (\lambda_1 - \lambda_2)^2}{2 (\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2) \lambda_1 \lambda_2}$$

is non-increasing in time.

Proof. We compute

$$F = H^3,$$

$$F_{ij} = 3H^2 g^{ij},$$

$$\frac{d}{dt} H - F_{ij} H_{;ij} = H^3 |A|^2 + 6H \sum h_{ii; k} h_{jj; k},$$

$$\frac{d}{dt} |A|^2 - F_{ij} (|A|^2)_{;ij} = 6H^2 (|A|^2)^2 - 4H^3 \text{tr} A^3$$

$$\dot{\tilde{w}} = \log \left( \frac{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) (\lambda_1 + \lambda_2)^2 (\lambda_1 - \lambda_2)^2}{2 (\lambda_1^2 - \lambda_1 \lambda_2 + \lambda_2^2) \lambda_1 \lambda_2} \right)$$

$$= \log \left( \frac{-H^6 + 3H^2 |A|^4 - 4 |A|^2 H^2 - 3 |A|^2}{-H^4 + 4 |A|^2 H^2 - 3 |A|^2} \right).$$

In a critical point of $\tilde{w}$, we obtain

$$h_{22;1} = \frac{\lambda_2}{\lambda_1} \cdot \frac{3\lambda_1^6 - 2\lambda_1^5 \lambda_2 + 4\lambda_2^4 \lambda_2^2 + 4\lambda_2^3 \lambda_4^2 + 4\lambda_1^2 \lambda_4^2 - 2\lambda_1 \lambda_2^2 + \lambda_4^6}{3\lambda_1^5 - 2\lambda_1^4 \lambda_2 + 4\lambda_2^3 \lambda_4^2 + 4\lambda_1^2 \lambda_4^2 - 2\lambda_1 \lambda_2^2 + \lambda_4^6} \cdot h_{11;1},$$

$$\frac{d}{dt} \tilde{w} - F_{ij} \tilde{w}_{;ij} = \left( \frac{-6H^2 + 8 |A|^2 H^2 - 8 |A|^2 H^2}{-H^4 + 4 |A|^2 H^2 - 3 |A|^2} \right) \cdot \left( \frac{-6H^2 + 8 |A|^2 H^2 - 8 |A|^2 H^2}{-H^4 + 4 |A|^2 H^2 - 3 |A|^2} \right).$$
Proof. We applied Sturm’s algorithm [35] to obtain the last inequality. Here and in the rest of the paper, we have sometimes used a computer algebra program for the calculations involving longer polynomials. Moreover, we use \((\ldots)\) in \((\ldots)\cdot h_{22;2}^2\) to abbreviate terms that are, up to interchanging \(\lambda_1\) and \(\lambda_2\), equal to the respective factors in front of \(h_{11;1}^2\).

We have applied the following two identities in order to rewrite terms involving derivatives of the second fundamental form

\[
\sum \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma h_{i;j;:k}^2 = \left( \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma a_1^2 + \lambda_2^\alpha \lambda_3^\beta \lambda_4^\gamma a_1^2 + \lambda_3^\alpha \lambda_4^\beta \lambda_2^\gamma a_1^2 \right) \cdot h_{11;1}^2 + (\ldots) \cdot h_{22;2}^2,
\]

\[
\sum \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma h_{i;j;:k} h_{j;j;:k} = (\lambda_1^\alpha + \lambda_2^\alpha a_1) \left( \lambda_1^\beta + \lambda_2^\beta a_1 \right) \lambda_3^\gamma h_{11;1}^2 + (\ldots) \cdot h_{22;2}^2.
\]

They hold for all \(\alpha, \beta, \gamma \in \mathbb{R}\). \(\square\)

5.2. Velocity Bounds. The following lemma is known, see [30]. Nevertheless, we include it, as we will use some of the calculations of its proof later on.

**Lemma 5.3.** For a family of smooth closed strictly convex surfaces \(M_t \subset \mathbb{R}^3\), \(0 \leq t < T\), flowing according to \(\frac{d}{dt} X = -F\nu\) with \(F = H^\alpha\), \(\alpha > 1\), a positive lower bound on the principal curvatures, \(\lambda_1, \lambda_2 \geq \varepsilon > 0\), is preserved during the evolution.

**Proof.** Combining (2.2), (2.4), and (2.5) yields

\[
\frac{d}{dt} h_{ij} - F^{kl} h_{ij;kl} = F^{kl} h^a_k h_{al} \cdot h_{ij} - F^{kl} h_{kl} \cdot h^a_i h_{aj} - F h^k_i h_{kj} + F^{kl} h_{kl} h^a_i h_{aj} - F h^k_i h_{kj}.
\]

We wish to apply the maximum principle for tensors [14,19,20]. So we define

\[
M_{ij} = h_{ij} - \varepsilon g_{ij},
\]

use (2.1) and obtain

\[
\frac{d}{dt} M_{ij} - F^{kl} M_{ij;kl} = F^{kl} h^a_k h_{al} \cdot h_{ij} - F^{kl} h_{kl} h^a_i h_{aj} - F h^k_i h_{kj} + F^{kl} h_{kl} h^a_i h_{aj} - F h^k_i h_{kj} + 2\varepsilon F h_{ij}.
\]
We now specialize to the normal velocity $F = H^\alpha$. It is easy to see that

$$F^{kl,rs}h_{kl;rs;i} \geq 0.$$  

We have to test the zero eigenvalue condition. Assume that $\xi$ is a zero eigenvalue of $(M_{ij})$, $h_{ij}\xi^j = \varepsilon g_{ij}\xi^j$, with $g_{ij}\xi^i\xi^j = 1$. We may assume, that in our coordinate system, we have $\xi = (1, 0)$ and $(h_{ij}) = (\varepsilon 0\lambda^00\varepsilon)$. Direct calculations yield

$$\left( \frac{d}{dt}M_{ij} - F^{kl}M_{ij;kl} \right) \xi^i \xi^j \geq \alpha H^{\alpha-1}|A|^2 \varepsilon - \alpha H^\alpha \varepsilon^2 - H^\alpha \varepsilon^2 + 2\varepsilon H^\alpha \varepsilon$$

$$= H^{\alpha-1} \left( \alpha \varepsilon \lambda (\lambda - \varepsilon) + \varepsilon^3 + \varepsilon^2 \lambda \right) > 0.$$  

The lemma follows from the maximum principle. □

The next lemma appears also in [30]. Once again, we will use the following calculations later on.

**Lemma 5.4.** For a family of closed smooth strictly convex surfaces $M_t \subset \mathbb{R}^3$, $0 \leq t < T$, flowing according to $\frac{d}{dt}X = -F\nu$ with $F = H^\beta$, $\beta > 0$, the velocity $F$ is bounded as in Lemma 3.2, in terms of $\beta$, the initial data, and the radius $R$ of an enclosed sphere.

**Proof.** We proceed as in Lemma 3.2, use $\alpha = \frac{1}{2}R$ and obtain, see e.g. [27, Lemma 5.4], [29],

$$\frac{\partial F}{\partial g_{kl}} = -F^{il}h_{kl}^i,$$

$$\frac{d}{dt}F - F^{ij}F_{;ij} = FF_{ij}h_{kl}^i,$$

$$\frac{d}{dt}X^\alpha - F^{ij}X_{;ij}^\alpha = (-F + F_{ij}h_{ij})\nu^\alpha,$$

$$\frac{d}{dt}\nu^\alpha - F^{ij}\nu_{;ij}^\alpha = F^{ij}h_{ij}^k\nu^\alpha,$$

$$\frac{d}{dt}(X, \nu) - F^{ij}(X, \nu)_{;ij} = -F - F^{ij}h_{ij} + F^{ij}h_{kl}^i,$$

$$\frac{d}{dt}\log \left( \frac{F}{(X, \nu) - \alpha} \right) = \frac{1}{(X, \nu) - \alpha} (F + F^{ij}h_{ij} - \alpha F^{ij}h_{kl}^i).$$

So far, we did not use the fact, that $F = H^\beta$. In an increasing maximum of $\frac{F}{(X, \nu) - \alpha}$, we get the inequality $F + F^{ij}h_{ij} - \alpha F^{ij}h_{kl}^i \geq 0$ and deduce there

$$\frac{1 + \beta}{\alpha} \geq \frac{|A|^2}{H} \geq \frac{1}{2}H.$$  

Our Lemma follows. □

### 5.3. Concavity

**Lemma 5.5.** The dual function to $F = H^\alpha$, $\alpha > 0$, defined before Lemma 3.14, is $-\alpha$-concave.
Proof. We use the same notation as before and obtain
\[
\Phi = -H^\alpha K^{-\alpha},
\]
\[
\Phi_{ij} = -\alpha H^\alpha K^{-\alpha} g^{ij} + \alpha H^\alpha K^{-\alpha} \tilde{h}^{ij},
\]
\[
\Phi_{ij,kl} = -\alpha \left( \alpha - 1 \right) H^\alpha K^{-\alpha} g^{ij} g^{kl}
+ \alpha^2 H^\alpha K^{-\alpha} g^{ij} \tilde{h}^{kl} + \alpha^2 H^\alpha K^{-\alpha} \tilde{h}^{ij} \tilde{h}^{kl}
- \alpha^2 H^\alpha K^{-\alpha} \tilde{h}^{ij} \tilde{h}^{kl} - \alpha H^\alpha K^{-\alpha} \tilde{h}^{ik} \tilde{h}^{jl}.
\]
We have to prove that
\[
\Phi_{ij,kl} \eta_{ij} \eta_{kl} \leq \frac{\alpha + 1}{\alpha} \Phi_{ij} \eta_{ij} \Phi_{kl} \eta_{kl}
\]
or
\[
0 \leq - \left( \alpha + 1 \right) \left( \begin{array}{cc} \frac{\lambda^2}{\lambda_1} & 1 \\ 1 & \frac{\lambda^2}{\lambda_2} \end{array} \right) + \left( \alpha - 1 \right) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)
- \alpha H \left( \frac{2}{\lambda_1} + \frac{1}{\lambda_2} \frac{1 + \frac{1}{\lambda_2}}{\lambda_1^2} \right) + \alpha H^2 \left( \begin{array}{cc} \frac{1}{\lambda_1 \lambda_2} & \frac{1}{\lambda_1 \lambda_2} \\ \frac{1}{\lambda_1 \lambda_2} & \frac{1}{\lambda_1 \lambda_2} \end{array} \right)
+ H^2 \left( \begin{array}{cc} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{array} \right)
= \left( \begin{array}{cc} 2 \frac{\lambda^2}{\lambda_1} & -2 \\ -2 & 2 \frac{\lambda^2}{\lambda_2} \end{array} \right).
\]

5.4. Some Constants. We wish to obtain precise bounds on the normal velocity \( F \) near \( t = T \). This proof is almost identical for all our test functions in Table 1 with a factor \( \lambda_1 \lambda_2 \) in the denominator. So it seems appropriate to state this proof only once with appropriate constants depending on the normal velocity and the test function. These constants are

- \( c_h \): the homogeneity of \( F \) in terms of the principal curvatures.
- \( c_1 \): the value of \( F \) at \( \lambda_1 = \lambda_2 = 1 \).
- \( c_\alpha \): a positive constant, such that the dual function to \( F \), defined before Lemma 3.14, is a \(-c_\alpha\)-concave function. It turns out, that, for all flows considered here, we can choose \( c_\alpha = c_h \).
- \( c_d \): a constant depending on the difference of the degrees of the numerator \( d_n \) and the denominator \( d_d \) of the test function \( w \), \( c_d := \frac{1}{2} \left( 2 - d_n + d_d \right) \).

This constant is defined such that
\[
\left| \lambda_1 - \lambda_2 \right| \leq c \cdot H^c_d
\]
for a pinched surface for which \( \max M_t \) \( w \) is non-increasing.

For the flow equations considered here, these constants are as in Table 2. We assume there that \( \alpha \geq 2 \) and \( \beta \geq 0 \).

It is important for us that \( c_d < 1 \) as it implies that Inequality (5.1) is not scaling invariant.
5.5. **Pinching.** We show that our surfaces are pinched during the evolution, i.e., that there exists a constant $c > 0$, depending on our test quantity, especially on the upper bound for it, and on the positive lower bound for the principal curvatures, $\varepsilon$, such that

$$0 < \frac{1}{c} \leq \frac{\lambda_i}{\lambda_j} \leq c$$

for $i, j \in \{1, 2\}$.

The following proof does not apply directly to the case $F = K$ considered in [12], but the result is also true in that case.

By direct inspection, we see that all our test quantities $w$ are such that $w \cdot (\lambda_1 - \lambda_2)$ is bounded below by a positive constant, depending especially on $\varepsilon$. So we see that

$$\left(\frac{\lambda_1 - \lambda_2}{\lambda_1 \lambda_2}\right)^2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \leq c.$$  

Thus $\frac{\lambda_1}{\lambda_2}$ is bounded above and the surface is pinched.

5.6. **Evolution of Spheres.** The radius of contracting spheres fulfills the ordinary differential equation

$$\dot{r}(t) = -\frac{c_1}{r(t)^c_1}.$$  

A solution is given by

$$r(t) = (c_1 \cdot (1 + c_h) \cdot (T - t))^{\frac{1}{1+c_h}},$$  

so that inner and outer radii are related to the life span $T - t$ of $M_t$ as follows

$$\rho_- \leq (c_1 \cdot (1 + c_h) \cdot (T - t))^{\frac{1}{1+c_h}} \leq \rho_+(t).$$

5.7. **Bounds for Radii.** In order to prove bounds for the radii $\rho_-$ and $\rho_+$, we proceed as in the proof of Lemma 3.12 and use $|\lambda_1 - \lambda_2| \leq c \cdot H^{c_d}$

$$r_- \geq \frac{1}{8\pi} \int_{M_t} H \cdot \{1 - c \cdot \rho_+ \cdot H^{c_d}\}.$$  

For each $F$ in Table 2, there exists $c_F > 0$ such that

$$0 < \frac{1}{c_F} H \leq F^{1/c_F} \leq c_F H,$$
so we can apply the variants of Lemma 3.2 for other curvature functions. For \( t \) close to \( T \), we get
\[
    r_+ \geq \frac{1}{8\pi} \int_{M_t} H \cdot \left\{ 1 + c \cdot (T - t) \frac{1-c_h}{1+c_h} \right\}
\]
and similarly, we obtain
\[
    r_- \leq \frac{1}{8\pi} \int_{M_t} H \cdot \left\{ 1 - c \cdot (T - t) \frac{1-c_h}{1+c_h} \right\}
\]
Then we get for small \( T - t \)
\[
    r_+ \leq (c_1 \cdot (1 + c_h) \cdot (T - t))^{1 + c_h} \cdot \left\{ 1 - c \cdot (T - t) \frac{1-c_h}{1+c_h} \right\}
\]
and
\[
    r_- \geq (c_1 \cdot (1 + c_h) \cdot (T - t))^{1 + c_h} \cdot \left\{ 1 - c \cdot (T - t) \frac{1-c_h}{1+c_h} \right\}
\]
and
\[
    1 \leq \frac{r_+}{r_-} \leq 1 + c \cdot (T - t) \frac{1-c_h}{1+c_h}.
\]

5.8. Precise Velocity Bounds. We use the notation of Lemma 3.15. The Harnack inequality [10, Theorem 5.17] implies for \( t_2 > t_1 > 0 \)
\[
    F(z, t_2) \leq \left( \frac{t_1}{t_2} \right)^{\frac{c_h}{1+c_h}}.
\]
Note that spheres evolve such that we get
\[
    (r_-(t))^{1+c_h} - c_1 (1 + c_h) \tau \frac{1}{1+c_h} \leq s(\cdot, t + \tau).
\]
We set
\[
    \tau := r_-(t)^{1+c_h} \cdot \left( \frac{r_+(t)}{r_-(t)} - 1 \right)^{1/2}
\]
and get
\[
    F(z, t) \leq \frac{1 + c \cdot (T - t)}{r_-(t)^{c_h} \left( \frac{r_+(t)}{r_-(t)} - 1 \right)^{1/2}} \left( \frac{r_+(t)}{r_-(t)} - \left( 1 - c_1 \cdot (1 + c_h) \cdot \left( \frac{r_+(t)}{r_-(t)} - 1 \right)^{1/2} \right)^{1/2} \frac{1-c_h}{1+c_h} \right).
\]
Use for \( 0 \leq x \leq c(c_h) \)
\[
    - (1 - x)^{1/c_h} \leq -1 + \frac{1}{1+c_h} x + \frac{1}{1+c_h} x^2.
\]
We get

\[ F(z, t) \leq \frac{1 + c \cdot (T - t)}{r_+(t)^{c_k}} \cdot \left( c_1 + c \left( \frac{r_+}{r_-} - 1 \right)^{1/2} \right) \]

\[ \leq c_1 \cdot (c_1 \cdot (1 + c_k) \cdot (T - t))^{-\frac{c_k}{1+c_k}} \cdot \left( 1 + c \cdot (T - t)^{\frac{1}{2}} \cdot \frac{1}{1+c_k} \right) \]

and a similar lower bound follows.

5.9. **Convergence of Principal Curvatures.** We consider \( F = H^\alpha \) and obtain

\[ \lambda_1 = \frac{1}{2} (\lambda_1 + \lambda_2) + \frac{1}{2} (\lambda_1 - \lambda_2) \]

\[ = \frac{1}{2} F^{1/\alpha} + \frac{1}{2} (\lambda_1 - \lambda_2) \]

\[ \leq (2^\alpha \cdot (1 + \alpha) \cdot (T - t))^{-\frac{1}{1+\alpha}} \cdot \left( 1 + c \cdot (T - t)^{\frac{1}{2}} \cdot \frac{1}{1+c_k} \right) \cdot \left( 1 + c \cdot (T - t)^{\frac{1}{2}} \cdot \frac{1}{1+c_k} \right). \]

A similar lower bound is proved analogously. Theorem 5.1 follows.

6. **\( H^2 \)-Flow**

**Theorem 6.1.** A smooth closed strictly convex surface in \( \mathbb{R}^3 \), contracting with normal velocity \( H^2 \), converges to a round point in finite time.

**Theorem 6.2.** For a family of smooth closed strictly convex surfaces \( M_t \) in \( \mathbb{R}^3 \), flowing according to \( \frac{d}{dt} X = -H^2 \nu \),

\[ \max_{M_t} \frac{(\lambda_1 + \lambda_2)^3 (\lambda_1 - \lambda_2)^2}{2 (\lambda_1^2 + \lambda_2^2) \lambda_1 \lambda_2} \]

is non-increasing in time.

**Proof.** We set

\[ w(\lambda_1 + \lambda_2)^3 (\lambda_1 - \lambda_2)^2 = H^3 \frac{(2|A|^2 - H^2)}{|A|^2 (H^2 - |A|^2)}. \]

In a critical point of \( \dot{w} \), we get, based on computer algebra calculations,

\[ \frac{d}{dt} w - F^{ij} w_{,ij} = -2 \frac{(\lambda_1 + \lambda_2)^4 (\lambda_1 - \lambda_2)^2 \lambda_1 \lambda_2}{(\lambda_1^2 + \lambda_2^2)^2} \]

\[ - \frac{(\lambda_1 + \lambda_2)^4}{(\lambda_1^2 + \lambda_2^2)^2} \left( \lambda_1^4 - \lambda_1^2 \lambda_2 + 7 \lambda_1^2 \lambda_2^2 - \lambda_1 \lambda_2^2 + 2 \lambda_2^4 \right) \lambda_1 \]

\[ \cdot \left( 5 \lambda_1^2 - 24 \lambda_1^{11} \lambda_2 + 112 \lambda_1^{10} \lambda_2^2 - 164 \lambda_1^9 \lambda_2^3 + 529 \lambda_1^8 \lambda_2^4 \right) \]

\[ - 448 \lambda_1^7 \lambda_2^5 + 952 \lambda_1^6 \lambda_2^6 - 312 \lambda_1^5 \lambda_2^7 + 391 \lambda_1^4 \lambda_2^8 - 72 \lambda_1^3 \lambda_2^9 \]

\[ + 56 \lambda_1^2 \lambda_2^{10} - 4 \lambda_1 \lambda_2^{11} + 3 \lambda_2^{12}) \cdot h_{11,1} + (\ldots) \cdot h_{22,2} \leq 0. \]

We apply the maximum principle. Our claim follows. \hfill \Box

Theorem 6.1 follows.
7. $H^4$-Flow

**Theorem 7.1.** A smooth closed strictly convex surface in $\mathbb{R}^3$, contracting with normal velocity $H^4$, converges to a round point in finite time.

**Theorem 7.2.** For a family of smooth closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$, flowing according to $\frac{d}{dt} X = -H^4 \nu$,

$$\max_{M_t} \frac{(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2) (\lambda_1 + \lambda_2)^6 (\lambda_1 - \lambda_2)^2}{2 \lambda_1^2 \lambda_2^2}$$

is non-increasing in time.

**Proof.** We proceed as above.

$$F = H^4,$$

$$F^{ij} = 4H^3 g^{ij},$$

$$F^{ij, kl} = 12H^2 g^{ij} g^{kl},$$

$$\tilde{w} = \log\left(\frac{-H^{10} + H^8 |A|^2 + 2H^6 (|A|^2)^2}{H^4 - 2H^2 |A|^2 + (|A|^2)^2}\right)\cdot \frac{2 \lambda_1^2 \lambda_2^2}{(\lambda_1 + \lambda_2)^6 (\lambda_1 - \lambda_2)^2}.$$

In a critical point of $\tilde{w}$, we get

$$h_{22; 1} = a_1 h_{11; 1},$$

$$\frac{d}{dt} H - F^{ij} H_{, ij} = H^4 |A|^2 + 12H^2 (1 + a_1)^2 \cdot h_{11; 1}^2 + \ldots \cdot h_{22; 2}^2,$$

$$\frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{, ij} = 8H^3 (|A|^2)^2 - 6H^4 \text{tr} A^3 - 8H^3 (1 + 3a_1^2) \cdot h_{11; 1}^2 + 24H^2 \lambda_1 (1 + a_1)^2 \cdot h_{11; 1}^2 + \ldots \cdot h_{22; 2}^2.$$

$$-F^{ij} H_{, i} H_{, j} = -4H^3 (1 + a_1)^2 \cdot h_{11; 1}^2 + \ldots \cdot h_{22; 2}^2,$$

$$-F^{ij} (|A|^2)_{, i} H_{, j} + (|A|^2)_{, j} H_{, i} = -16H^3 (\lambda_1 + \lambda_2 a_1) (1 + a_1) \cdot h_{11; 1}^2 + \ldots \cdot h_{22; 2}^2,$$

$$-F^{ij} (|A|^2)_{, i} (|A|^2)_{, j} = -16H^3 (\lambda_1 + \lambda_2 a_1)^2 \cdot h_{11; 1}^2 + \ldots \cdot h_{22; 2}^2.$$
Here we used once more a computer algebra system and Sturm’s theorem to obtain the last inequality.

\[ \leq 0. \]

Theorem 7.1 follows.
normal velocity flowing according to Theorem 8.1.

Theorem 8.1. A smooth closed strictly convex surface in $\mathbb{R}^3$, contracting with normal velocity $|A|^2 + \beta H^2$, $0 \leq \beta \leq 5$, converges to a round point in finite time.

Theorem 8.2. For a family of smooth closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$, flowing according to $\dot{X} = -(|A|^2 + \beta H^2)\nu$, $0 \leq \beta \leq 5$,

$$\max_{M_t} \frac{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{2\lambda_1 \lambda_2}$$

is non-increasing in time.

Proof. Similarly as above, we obtain for $F = |A|^2 + \beta H^2$

$$\frac{d}{dt} |A|^2 - F^{ij} (|A|^2);_{ij} = 2|A|^2 \operatorname{tr} A^3 + \beta \left(4 |A|^2 - 2H^2 \operatorname{tr} A^3\right) + \beta \left(-4H h_{ij, k}^2 + 4 \sum h_{ii; k} h_{jj; k} h_{kk}^2\right).$$

As in the proof of Theorem 3.3, we set

$$\tilde{w} = \log H + \log (2|A|^2 - H^2) - \log (H^2 - |A|^2)$$

and obtain in a critical point of $\tilde{w}$, where $h_{22; 1} = a_1 h_{11; 1}$

$$-F^{ij} H_{;i} H_{;j} = -2(\lambda_1 + \beta H)(1 + a_1) \cdot h_{11; 1} + (\ldots) \cdot h_{22; 2},$$

$$-F^{ij} \left((|A|^2)_{;i} H_{;j} + (|A|^2)_{;j} H_{;i}\right) = -8(\lambda_1 + \beta H)(\lambda_1 + \lambda_2 a_1)(1 + a_1) \cdot h_{11; 1} + (\ldots) \cdot h_{22; 2},$$

$$h_{22; 1} = \frac{\lambda_2}{\lambda_1} 2 \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 \lambda_1 h_{11; 1}.$$

In a critical point of $\tilde{w}$, we obtain the evolution equation

$$\frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{;ij} = \left(1 - \frac{2H}{|A|^2 - H^2} - \frac{2H}{H^2 - |A|^2}\right) \cdot \left(\frac{d}{dt} H - F^{ij} H_{;ij}\right) + \left(\frac{2}{|A|^2 - H^2} + \frac{1}{H^2 - |A|^2}\right) \cdot \left(\frac{d}{dt} |A|^2 - F^{ij} (|A|^2);_{ij}\right) + \left(\frac{2}{|A|^2 - H^2} + \frac{6}{H^2 - |A|^2}\right) F^{ij} H_{i} H_{j} - \frac{2}{H \cdot (2|A|^2 - H^2)} F^{ij} \left((|A|^2)_{;i} H_{;j} + (|A|^2)_{;j} H_{;i}\right) = -4 \frac{K^2}{H} - 2\beta HK - 2(5\lambda_1^8 - 4\lambda_1^7 \lambda_2 + 46\lambda_1^6 \lambda_2^2 + 48\lambda_1^5 \lambda_2^3 + 72\lambda_1^4 \lambda_2^4) \lambda_2 h_{11; 1}.$$
We wish to show for $\alpha$ derivatives of $F$. Proof.

The dual function to $\lambda \leq -\lambda$ we observe that the terms without derivatives of the second fundamental form yields

$$
- \frac{2(44\lambda_1^3\lambda_2^2 + 34\lambda_1^2\lambda_2^2 + 8\lambda_1\lambda_2^3 + 3\lambda_2^3)\lambda_2}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 (\lambda_2^2 + \lambda_1\lambda_2 + 2\lambda_2^2)^2}\lambda_1^2 h_{11;1}^2
+ 2\beta \frac{(\lambda_1 - 16\lambda_2^2\lambda_2 - 6\lambda_1\lambda_2^2 - 8\lambda_1\lambda_2 - 3\lambda_2^3)(\lambda_1 + \lambda_2)^3\lambda_2}{(\lambda_1 - \lambda_2)^2 (\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2)^2}\lambda_1^2 h_{11;1}^2
+ (\ldots) \cdot h_{22;2}^2.
$$

For $\beta = 5$, the factor in front of $h_{11;1}^2$ equals

$$
- \frac{4\lambda_2^2}{(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 (\lambda_1^2 + \lambda_1\lambda_2 + 2\lambda_2^2)^2}\lambda_1^3
\cdot (28\lambda_1^7 + 183\lambda_1^5\lambda_2 + 334\lambda_1^3\lambda_2^2 + 371\lambda_1\lambda_2^3 + 272\lambda_1\lambda_2^3 + 157\lambda_2^5 + 54\lambda_1\lambda_2^3 + 9\lambda_2^2).
$$

Our claim follows.

In order to see that the range for $\beta$ is sharp for applying the maximum principle, we observe that the terms without derivatives of the second fundamental form require that $\beta \geq 0$. For $\lambda_2 = 1$ and $\lambda_1 \rightarrow \infty$, the factor in front of $h_{11;1}^2$ behaves like $-112\lambda_1^{-3}$ for $\beta = 5$ and like $-10\lambda_1^{-2}$ for $\beta = 0$, so we need the upper bound $\beta \leq 5$. □

Lemma 8.3. For a family of smooth closed strictly convex surfaces $M_t \subset \mathbb{R}^3$, $0 \leq t < T$, flowing according to $\frac{d}{dt}X = -F$ with $F = |A|^2 + \beta H^2$, $\beta \geq 0$, a positive lower bound on the principal curvatures, $\lambda_1, \lambda_2 \geq \varepsilon > 0$, is preserved during the evolution.

Proof. We proceed similarly as in Lemma 5.3. Dropping the term involving second derivatives of $F$ yields

$$
\left( \frac{d}{dt} M_{ij} - F^{kl} M_{ij;kl} \right) \xi^i \xi^j \geq 2 \left( \text{tr} A^2 + \beta H |A|^2 \right) \varepsilon - (|A|^2 + \beta H^2) \varepsilon^2
= \varepsilon^4 + \varepsilon \lambda^3 + \varepsilon^2 (\lambda - \varepsilon) + \beta \left( \varepsilon^4 + \varepsilon^2 \lambda^2 + 2\varepsilon \lambda^3 \right) > 0.
$$

□

Similar calculations as before give an upper bound on the velocity for $F = |A|^2 + \beta H^2$, $\beta \geq 0$.

Lemma 8.4. The dual function to $F = |A|^2 + \beta H^2$, $\beta \geq 0$, is $\alpha$-concave for $\alpha \leq -2$.

Proof. We have $\Phi = \frac{2K - (1 + \beta)H^2}{K^2}$,

$$
\Phi^{ij} = \frac{1}{K^2} \left( (-2K + 2(1 + \beta)H^2) \tilde{h}^{ij} - 2(1 + \beta)H g^{ij} \right),
\Phi^{ijkl} = \frac{1}{K^2} \left( (2K - 4(1 + \beta)H^2) \tilde{h}^{ijkl} - (-2K + 2(1 + \beta)H^2) \tilde{h}^{ik} \tilde{h}^{jl}
+ 4(1 + \beta)H (g^{ij} \tilde{h}^{kl} + g^{kl} \tilde{h}^{ij}) - 2(1 + \beta)g^{ij} g^{kl} \right).
$$

We wish to show for $\alpha \leq -2$ and for symmetric matrices $(\eta_{ij})$, that

$$
\Phi^{ij;kl} \eta_{ij} \eta_{kl} \leq \frac{\alpha - 1}{\alpha \Phi} \Phi^{ij} \eta_{ij} \Phi^{kl} \eta_{kl}.
$$
Terms involving $\eta_{i,j}^2$ have the right sign.

Consider $\alpha = -2$. Then it suffices to prove the inequality

$$
\left( \left( 1 + \beta \right) H^2 - 2K \right) \left( \frac{6 \lambda_1^2}{\lambda_1} + \frac{6 \beta \lambda_1^2}{\lambda_1} + \frac{4 \beta \lambda_2^2}{\lambda_1} \right) \geq
\left( \frac{2 \beta}{2 \lambda_2} + \frac{6 \lambda_1^2}{\lambda_1} + \frac{4 \beta \lambda_2^2}{\lambda_1} \right).
$$

\begin{equation}
\begin{aligned}
\geq & \frac{2}{2 \lambda_2} \left( \frac{2 \beta}{2 \lambda_1} + \frac{6 \beta \lambda_1^2}{\lambda_1} + 2 \beta \lambda_2 \right) \otimes \left( \frac{2 \beta}{2 \lambda_1} + \frac{6 \beta \lambda_1^2}{\lambda_1} + 2 \beta \lambda_1 \right)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\geq & \frac{3}{2 \lambda_2} \left( \frac{(3 \beta^2 + 2 \beta \lambda_2)}{2 \lambda_1 (2 \beta + \lambda_2)} \right) \left( \frac{(3 \beta^2 + 2 \beta \lambda_1)}{2 \lambda_1 (2 \beta + \lambda_1)} \right) \left( \frac{(3 \beta^2 + 2 \beta \lambda_2)}{2 \lambda_1 (2 \beta + \lambda_2)} \right)
\end{aligned}
\end{equation}

in order to obtain $\alpha$-concavity for all $\alpha \leq -2$. This inequality is fulfilled, if

$$
\left\{ 6 \lambda_1 \lambda_2 + \beta \left( 4 \lambda_1^2 + 12 \lambda_1 \lambda_2 + 4 \lambda_2^2 \right) + \beta^2 \left( 4 \lambda_1^2 + 8 \lambda_1 \lambda_2 + 4 \lambda_2^2 \right) \right\} \left( \frac{\lambda_1 A - A}{\lambda_1} \right) \left( \frac{-A}{\lambda_2} \right)
$$

is positive semi-definite.

We want to derive precise bounds on the principal curvatures. To this end, we use (5.1) and (5.2)

$$
(2 + 4\beta) \left( (2 + 4\beta) 3 (T - t) \right)^{-2/3} \cdot \left( 1 - c \cdot (T - t)^{1/12} \right)
\leq F = |A|^2 + \beta H^2
\leq \lambda_1^2 + (\lambda_1 + |\lambda_1 - \lambda_2|)^2 + \beta (\lambda_1 + (\lambda_1 + |\lambda_1 - \lambda_2|))^2
\leq (2 + 4\beta) \lambda_1^2 + c \cdot F^{1/2} \cdot F^{1/4} + c \cdot F^{1/2}
\leq (2 + 4\beta) \lambda_1^2 + c \cdot (T - t)^{-2/3} \cdot (T - t)^{1/6} + (T - t)^{1/3}
\cdot (T - t)^{1/12}.
$$

We get

$$
\lambda_1 \geq ((2 + 4\beta) 3 (T - t))^{-1/3} \cdot \left( 1 + c \cdot (T - t)^{1/12} \right)
$$

and a similar upper bound follows analogously.

We obtain Theorem 8.1.

9. tr $A^3$-Flow

**Theorem 9.1.** A smooth closed strictly convex surface in $\mathbb{R}^3$, contracting with normal velocity $\text{tr} \ A^3$, converges to a round point in finite time.

**Theorem 9.2.** For a family of smooth closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$, flowing according to $\dot{X} = - \text{tr} \ A^3 \nu$,

$$
\max_{M_t} \frac{(3 \lambda_1^2 + 2 \lambda_1 \lambda_2 + 3 \lambda_2^2) (\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2}
$$

is non-increasing in time.

**Proof.** Calculations as above yield

$$
\dot{\omega} = \log \left( \frac{(3 \lambda_1^2 + 2 \lambda_1 \lambda_2 + 3 \lambda_2^2) (\lambda_1 - \lambda_2)^2}{2 \lambda_1 \lambda_2} \right)
$$
\[
\begin{align*}
\frac{d}{dt} \log \left( \frac{-H^4 + 4 (|A|^2)^2}{H^2 - |A|^2} \right), \\
\frac{d}{dt} H - F^{ij} H_{;ij} &= 3 \text{tr} A^4 H - 2 \text{tr} A^3 |A|^2 + 3 \sum (\lambda_i + \lambda_j) h_{ij;k}^2 \nonumber \\
&= 3 \text{tr} A^4 H - 2 \text{tr} A^3 |A|^2 + 3 \left( 2 \lambda_1 + 2 \lambda_1 \lambda_2 + 4 \lambda_2^2 \right) \cdot h_{11;1}^2 \\
&\quad + (\ldots) \cdot h_{22;2}^2, \\
\frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} &= 6 \text{tr} A^4 |A|^2 - 4 (\text{tr} A^3)^2 \\
&= \sum 6 \lambda_i^2 h_{ij;k}^2 + 6 \sum \lambda_k (\lambda_i + \lambda_j) h_{ij;k}^2 \nonumber \\
&= 6 \text{tr} A^4 |A|^2 - 4 (\text{tr} A^3)^2 - 6 (\lambda_1^2 + \lambda_2^2 \lambda_2 + 2 \lambda_2^2 \lambda_1^2) \cdot h_{11;1}^2 \\
&\quad + 6 (2 \lambda_1^2 + 4 \lambda_1 \lambda_2 \lambda_2 + 2 \lambda_2^2 \lambda_1^2) \cdot h_{11;1}^2 + (\ldots) \cdot h_{22;2}^2, \\
-F^{ij} H_{;ij} &= -3 \lambda_1^2 (1 + a_1^2) \cdot h_{11;1}^2 + (\ldots) \cdot h_{22;2}^2, \\
-F^{ij} (|A|^2)_{;i} (|A|^2)_{;j} &= -12 \lambda_1^2 (1 + a_1 a_2) \cdot h_{11;1}^2 + (\ldots) \cdot h_{22;2}^2, \\
h_{22;1} &= \frac{\lambda_1 9 \lambda_2^2 + \lambda_2^2 \lambda_2 + 3 \lambda_1 \lambda_2^2 + 3 \lambda_2^2}{\lambda_1 9 \lambda_2^2 + \lambda_2^2 \lambda_1 + 3 \lambda_2 \lambda_2^2 + 3 \lambda_1^2} \cdot h_{11;1}, \\
\frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{;ij} &= \left( \frac{-4 H^3}{-H^4 + 4 (|A|^2)^2} - \frac{2 H}{H^2 - |A|^2} \right) \left( \frac{d}{dt} H - F^{ij} H_{;ij} \right) \\
&\quad + \left( \frac{8 |A|^2}{-H^4 + 4 (|A|^2)^2} + \frac{1}{\lambda_1 9 \lambda_2^2 + \lambda_2^2 \lambda_2 + 3 \lambda_1 \lambda_2^2 + 3 \lambda_2^2} \right) \cdot \left( \frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} \right) \\
&\quad + \left( \frac{-12 H^2}{-H^4 + 4 (|A|^2)^2} - \frac{2}{H^2 - |A|^2} \right) \cdot (-F^{ij} H_{;ij}) \\
&\quad + \left( \frac{8}{-H^4 + 4 (|A|^2)^2} \right) \left( -F^{ij} (|A|^2)_{;i} (|A|^2)_{;j} \right) \\
&= \frac{2 (\lambda_1^4 - 2 \lambda_1^3 \lambda_2 + 18 \lambda_1^2 \lambda_2^2 - 2 \lambda_1 \lambda_2^3 + \lambda_2^4) \lambda_1 \lambda_2}{3 \lambda_1^2 + 2 \lambda_1 \lambda_2 + 3 \lambda_2^2} \\
&\quad - \frac{(3 \lambda_1^2 + 2 \lambda_1 \lambda_2 + 3 \lambda_2^2) (\lambda_1 - \lambda_2)^2 \lambda_2^2}{6 \lambda_2} \\
&\quad - \frac{1}{(3 \lambda_1^2 + 3 \lambda_2^2 \lambda_2 + \lambda_1 \lambda_2^2 + 9 \lambda_2^3)^2} \\
&\quad \cdot (63 \lambda_1^4 + 138 \lambda_1^3 \lambda_2 + 2883 \lambda_1^2 \lambda_2^2 + 2883 \lambda_1 \lambda_2^3 + 36 \lambda_2^4 \lambda_2^2 - 1218 \lambda_1^2 \lambda_2^2 + 2294 \lambda_2^4 \lambda_2 + 582 \lambda_2^3 \lambda_2^2 + 855 \lambda_1 \lambda_2^4} \\
&\quad \cdot (945 \lambda_1^4 \lambda_2^2 + 135 \lambda_1^3 \lambda_2^3 + 135 \lambda_1^2 \lambda_2^4 + 54 \lambda_1^2 \lambda_2^4 + 54 \lambda_2^4) \cdot h_{11;1}^2 \\
&\quad + (\ldots) \cdot h_{22;2}^2 \leq 0.
\end{align*}
\]
Lemma 9.3. For a family of smooth closed strictly convex surfaces $M_t \subset \mathbb{R}^3$, $0 \leq t < T$, flowing according to $\frac{d}{dt} X = -\nabla F$ with $F = \text{tr} A^\alpha$, $\alpha \geq 2$, a positive lower bound on the principal curvatures, $\lambda_1, \lambda_2 \geq \varepsilon > 0$, is preserved during the evolution.

Proof. We proceed similarly as in Lemma 5.3. Once again, the term involving second derivatives of $F$ is non-negative. As before, we obtain

$$\left( \frac{d}{dt} M_{ij} - F^k l M_{ij, kl} \right) \xi^i \xi^j \geq \alpha \text{tr} A^{\alpha+1} \varepsilon - \alpha \text{tr} A^\alpha \varepsilon^2 - \alpha \varepsilon \text{tr} A^\alpha \varepsilon$$

$$= \varepsilon \left( \lambda^{\alpha+1} - \varepsilon \lambda^\alpha \right) + \varepsilon^{\alpha+1} + \varepsilon \lambda^\alpha > 0.$$

$\square$

Similar calculations as in Lemma 5.4, using $F + F^{ij} h_{ij} - \alpha F^{ij} h^k l h_{kj} \geq 0$ for $F = \text{tr} A^\beta$, $\beta \geq 2$, yield for some $c_\beta > 0$

$$\frac{1}{c_\beta} (\text{tr} A^\beta)^{1/\beta} \leq \frac{1}{c_\beta} H \leq \frac{\text{tr} A^{\beta+1}}{\text{tr} A^\beta} \leq \frac{\beta + 1}{\beta}$$

and an estimate as in Lemma 5.4 follows.

Lemma 9.4. For $\alpha > 0$, the dual function to $F = \text{tr} A^\alpha = \lambda_1^\alpha + \lambda_2^\alpha$ is $-\alpha$-concave.

Proof. We set $\Phi = -\text{tr} A^\alpha \cdot K^{-\alpha}$ and have to show that

$$\Phi^{ij, kl} \eta_{ij} \eta_{kl} \leq \frac{-\alpha - 1}{-\alpha \Phi} \Phi^{ij} \eta_{ij} \Phi^{kl} \eta_{kl}$$

for symmetric matrices $(\eta_{ij})$. Direct computations yield that this inequality is equivalent to

$$- \alpha (\alpha - 1) K^{-\alpha} \left( \frac{\eta_{11}}{\eta_{22}} \right)^{\text{tr}} \left( \begin{array}{cc} \lambda_1^{\alpha-2} & 0 \\ 0 & \lambda_2^{\alpha-2} \end{array} \right) \left( \begin{array}{c} \eta_{11} \\ \eta_{22} \end{array} \right) - 2 \alpha K^{-\alpha} \sum_{r=0}^{\alpha-2} \eta_{11} \lambda_1^{\alpha-2-r} \eta_{12}^2$$

$$+ \alpha^2 K^{-\alpha} \left( \frac{\eta_{11}}{\eta_{22}} \right)^{\text{tr}} \left( \frac{2 \lambda_1^{\alpha-2}}{\lambda_1^x} \text{tr} A^\alpha \frac{1}{\lambda_2^{\alpha-2}} \eta_{11} \frac{1}{\lambda_1^x} \eta_{22} \right)$$

$$- \alpha^2 \text{tr} A^\alpha K^{-\alpha} \left( \frac{\eta_{11}}{\eta_{22}} \right)^{\text{tr}} \left( \frac{1}{\lambda_1^x} \frac{1}{\lambda_2^{\alpha-2}} \eta_{11} \frac{1}{\lambda_1^x} \eta_{22} \right)$$

$$- \alpha \text{tr} A^\alpha K^{-\alpha} \left( \frac{\eta_{11}}{\eta_{22}} \right)^{\text{tr}} \left( \frac{1}{\lambda_1^x} \frac{1}{\lambda_2^{\alpha-2}} \eta_{11} \frac{1}{\lambda_1^x} \eta_{22} \right) - 2 \alpha \text{tr} A^\alpha K^{-\alpha} \frac{1}{\lambda_1^x} \frac{1}{\lambda_2^{\alpha-2}} \eta_{12}^2$$

$$\leq - \frac{\alpha (\alpha + 1) K^{-\alpha}}{\text{tr} A^\alpha} \left( \frac{\eta_{11}}{\eta_{22}} \right)^{\text{tr}} \left( \begin{array}{cc} \lambda_1^{\alpha} & \lambda_1^{\alpha-1} \lambda_2^{\alpha-1} \\ \lambda_1^{\alpha-1} \lambda_2^{\alpha-1} & \lambda_2^{\alpha} \end{array} \right) \left( \begin{array}{c} \eta_{11} \\ \eta_{22} \end{array} \right).$$

Further computations show that this is fulfilled, if

$$0 \leq \left( \begin{array}{cc} \lambda_1^{\alpha-2} \lambda_2^\alpha & -\lambda_1^{\alpha-1} \lambda_2^{\alpha-1} \\ -\lambda_1^{\alpha-1} \lambda_2^{\alpha-1} & \lambda_1^{\alpha-2} \lambda_2^\alpha \end{array} \right).$$

$\square$

Then we proceed as before. Similar calculations as for $F = |A|^2 + \beta H^2$ give for $\alpha = 3$

$$\lambda_1, \lambda_2 \leq (2(1 + \alpha) \cdot (T - t))^{-\frac{1}{1+\alpha}} \cdot \left( 1 + c \cdot (T - t)^{\frac{1}{3} \cdot \alpha + 1} \right)$$
and a corresponding lower bound holds. This estimate holds also for \( F = \text{tr} A^\alpha \), \( \alpha = 2, 3, 4, 5, 6 \).

This finishes the proof of Theorem 9.1.

10. \textit{tr} \( A^\alpha \)-Flow

**Theorem 10.1.** A smooth closed strictly convex surface in \( \mathbb{R}^3 \), contracting with normal velocity \( \text{tr} A^4, \text{tr} A^5, \) or \( \text{tr} A^6 \), converges to a round point in finite time.

**Theorem 10.2.** For a family of smooth closed strictly convex surfaces \( M_t \) in \( \mathbb{R}^3 \), flowing according to

\[
\dot{X} = - \text{tr} A^\alpha + 2 \nu, \quad \alpha = 2, 3, 4, \quad \max_{M_t} (\lambda_{ij}^\alpha + \lambda_{ij}^\alpha)(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2 / \lambda_1 \lambda_2 \]

is non-increasing in time.

**Proof.** One might conjecture, that this quantity is also monotone for other values of \( \alpha \). For \( \alpha = 0 \), corresponding to \( F = |A|^2 \), we have already checked that in Theorem 3.3. Further computations for \( \alpha = 1, 5, 6, 7 \) suggest, however, that this quantity is not monotone for these values of \( \alpha \).

We obtain

\[
\frac{d}{dt} H - F_{ij} H_{ij} = (\alpha + 2) \text{tr} A^{\alpha + 3} H - (\alpha + 1) \text{tr} A^{\alpha + 2} |A|^2
\]

\[
+ (\alpha + 2) \sum_{r=0}^{\alpha} \sum_{i,j,k=1}^2 \lambda_i^r \lambda_j^{\alpha - r} h_{ij}^2,
\]

\[
\frac{d}{dt} |A|^2 - F_{ij} (|A|^2)_{ij} = 2(\alpha + 2) \text{tr} A^{\alpha + 3} |A|^2 - 2(\alpha + 1) \text{tr} A^{\alpha + 2} \text{tr} A^3
\]

\[
- 2(\alpha + 2) \sum_{i,j,k=1}^2 \lambda_i^{\alpha + 1} h_{ij}^2,
\]

\[
+ 2(\alpha + 2) \sum_{r=0}^{\alpha} \sum_{i,j,k=1}^2 \lambda_i^r \lambda_j^{\alpha - r} h_{ij}^2,
\]

\[
\frac{d}{dt} \text{tr} A^\alpha - F_{ij} (\text{tr} A^\alpha)_{ij} = \alpha(\alpha + 2) \text{tr} A^{\alpha + 3} \text{tr} A^\alpha - \alpha(\alpha + 1) \text{tr} A^{\alpha + 2} \text{tr} A^{\alpha + 1}
\]

\[
- \alpha(\alpha + 2) \sum_{r=0}^{\alpha - 2} \lambda_i^{\alpha + 1} \lambda_j^{\alpha - 2 - r} \lambda_j^{\alpha - 2 - r} h_{ij}^2
\]

\[
+ \alpha(\alpha + 2) \sum_{r=0}^{\alpha} \lambda_i^r \lambda_j^{\alpha - r} \lambda_k^{\alpha - 1} h_{ij}^2,
\]

\[
\ddot{\bar{w}} = \log \left( \frac{(\lambda_1^2 + \lambda_2^2)(\lambda_1 + \lambda_2)(\lambda_1 - \lambda_2)^2}{\lambda_1 \lambda_2} \right)
\]

\[
\equiv \log A + \log B + \log C - \log D,
\]

\[
\frac{d}{dt} \ddot{\bar{w}} - F_{ij} \ddot{\bar{w}}_{ij} = \frac{1}{A} \left( \frac{d}{dt} A - F_{ij} A_{ij} \right) + \frac{1}{B} \left( \frac{d}{dt} B - F_{ij} B_{ij} \right)
\]

\[
+ \frac{1}{C} \left( \frac{d}{dt} C - F_{ij} C_{ij} \right) - \frac{1}{D} \left( \frac{d}{dt} D - F_{ij} D_{ij} \right)
\]
\[
\begin{align*}
&\frac{2}{B^2} F^{ij} B_{i; j} + \frac{2}{C^2} F^{ij} C_{i; j} + \frac{1}{BC} F^{ij} (B_{i; C, j} + B_{j; C, i}) \\
&- \frac{1}{DD} F^{ij} (B_{i; D, j} + B_{j; D, i}) - \frac{1}{CD} F^{ij} (C_{i; D, j} + C_{j; D, i}) \\
&= \frac{1}{\text{tr} A^\alpha} \cdot \left( \frac{d}{dt} \text{tr} A^\alpha - F^{ij} (\text{tr} A^\alpha)_{; ij} \right) \\
&+ \left( \frac{1}{H - \frac{2|A|^2 - H^2}{H^2 - |A|^2}} - \frac{2H}{H^2 - |A|^2} \right) \left( \frac{d}{dt} H - F^{ij} H_{; ij} \right) \\
&+ \left( \frac{2}{2|A|^2 - H^2 + \frac{1}{H^2 - |A|^2}} \right) \left( \frac{d}{dt} |A|^2 - F^{ij} |A|^2_{; ij} \right) \\
&+ \left( \frac{2}{2|A|^2 - H^2 + \frac{2}{H^2 - |A|^2} + \frac{8H^2}{(2|A|^2 - H^2)^2}} \right) \\
&- \frac{4H}{H (2|A|^2 - H^2)} - \frac{4H}{H (H^2 - |A|^2)} \\
&- \frac{8H^2}{(2|A|^2 - H^2) (H^2 - |A|^2)} \cdot F^{ij} H_{i; j} H_{; j} \\
&+ \left( \frac{8}{(2|A|^2 - H^2)^2} + \frac{4}{(2|A|^2 - H^2) (H^2 - |A|^2)} \right) \cdot F^{ij} (|A|^2)_{; ij} H_{; j} \\
&+ \left( \frac{1}{H (H^2 - |A|^2)} - \frac{8H}{(2|A|^2 - H^2)^2} + \frac{2}{H (2|A|^2 - H^2)} \right) \\
&- \frac{6H}{(2|A|^2 - H^2) (H^2 - |A|^2)} \right) \cdot F^{ij} \left( H_{; i} (|A|^2)_{; j} + H_{; j} (|A|^2)_{; i} \right).
\end{align*}
\]

Plugging this into a computer algebra program yields for \( \alpha = 2 \), corresponding to \( F = \text{tr} A^4 \),

\[
\begin{align*}
h_{22; 1} &= \frac{\lambda_2 4\lambda_1^4 + \lambda_2^3 \lambda_1^2 + \lambda_2^2 \lambda_1^3 + \lambda_2 \lambda_1^4 + \lambda_2^4}{\lambda_1 4\lambda_2^4 + \lambda_2^3 \lambda_1^2 + \lambda_2^2 \lambda_1^3 + \lambda_2 \lambda_1^4 + \lambda_1^4 h_{11; 1}}, \\
\frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{; ij} &= \frac{-24\lambda_1^2 \lambda_2^2}{(\lambda_1 - \lambda_2) (\lambda_1^2 + \lambda_2^2)} \\
&+ \frac{-4\lambda_2}{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2) \lambda_1^2 (\lambda_1^2 + \lambda_2^2)} \\
&\cdot \left( \frac{1}{(11\lambda_1^6 + 24\lambda_1^5 \lambda_2 + 39\lambda_1^4 \lambda_2^2 - 328\lambda_1^3 \lambda_2^3 + 482\lambda_1^2 \lambda_2^4 + 192\lambda_1^1 \lambda_2^5 + 215\lambda_1^1 \lambda_2^4 + 236\lambda_1^0 \lambda_2^3 + 432\lambda_1^0 \lambda_2^2 + 200\lambda_1^0 \lambda_2^1 + 6 \lambda_1^6 \lambda_2^0 + 173\lambda_1^5 \lambda_2^1 + 144\lambda_1^4 \lambda_2^2 + 32\lambda_1^3 \lambda_2^3 + 21\lambda_1^2 \lambda_2^4 + 12\lambda_1^1 \lambda_2^5 + 5\lambda_1^0 \lambda_2^6)}{h_{11; 1}} \right) \\
&+ \ldots \right) \cdot h_{22; 2}^2.
\end{align*}
\]
for $\alpha = 3$ \( (F = \text{tr} A^5) \)

\[
\begin{align*}
\mathcal{h}_{22;1} &= \frac{\lambda_2 5 \lambda_1^4 - 4 \lambda_1^4 \lambda_2 + 2 \lambda_1^2 \lambda_2^2 + \lambda_2^4}{\lambda_1 5 \lambda_1^2 - 4 \lambda_1^2 \lambda_2 + 2 \lambda_1 \lambda_2^2 + \lambda_1^3} h_{11;1}, \\
\frac{d}{dt} \dot{w} - F^{ij} \dot{w}_{;ij} &= -4 \left( 2 \lambda_1^4 - 7 \lambda_1^3 \lambda_2 + 12 \lambda_1^2 \lambda_2^2 - 7 \lambda_1 \lambda_2^3 + 2 \lambda_2^4 \right) \lambda_2^2 \lambda_2^2 \\
&+ \frac{\lambda_2^3 (\lambda_1 - \lambda_2)^2 (\lambda_1^4 + 2 \lambda_1^2 \lambda_2^2 - 4 \lambda_1 \lambda_2^3 + 5 \lambda_2^4)}{\lambda_1^4 + 2 \lambda_1^2 \lambda_2^2 - 4 \lambda_1 \lambda_2^3 + 5 \lambda_2^4} \left( 7 \lambda_1^6 - 65 \lambda_1^{15} \lambda_2 + 397 \lambda_1^4 \lambda_2^2 - 1295 \lambda_1^{13} \lambda_2^3 + 2464 \lambda_1^{12} \lambda_2^4 \\
&- 2981 \lambda_1^{11} \lambda_2^5 + 2645 \lambda_1^{10} \lambda_2^6 - 2007 \lambda_1^9 \lambda_2^7 + 1510 \lambda_1^8 \lambda_2^8 \\
&- 1011 \lambda_1^7 \lambda_2^9 + 583 \lambda_1^6 \lambda_2^{10} - 309 \lambda_1^5 \lambda_2^{11} + 176 \lambda_1^4 \lambda_2^{12} - 71 \lambda_1^3 \lambda_2^{13} \\
&+ 23 \lambda_1^2 \lambda_2^{14} - 5 \lambda_1 \lambda_2^{15} + 3 \lambda_2^{16} \right) \cdot h_{11;1} \\
&+ (\ldots) \cdot h_{22;2}^2,
\end{align*}
\]

and for $\alpha = 4$ \( (F = \text{tr} A^6) \)

\[
\begin{align*}
\mathcal{h}_{22;1} &= \frac{\lambda_2 6 \lambda_1^5 + \lambda_2^5 \lambda_2 - 3 \lambda_1^4 \lambda_2^2 + 2 \lambda_1^3 \lambda_2^3 + \lambda_1^2 \lambda_2^4 + \lambda_2^6}{\lambda_1 6 \lambda_1^4 + \lambda_2^5 \lambda_1 + 3 \lambda_1^3 \lambda_2 + 2 \lambda_1^2 \lambda_2^2 + \lambda_1 \lambda_2^3 + \lambda_1^2} h_{11;1}, \\
\frac{d}{dt} \dot{w} - F^{ij} \dot{w}_{;ij} &= -10 \left( \lambda_1^5 - 6 \lambda_1^4 \lambda_2 + 6 \lambda_1^3 \lambda_2^2 - 3 \lambda_1^2 \lambda_2^3 + 3 \lambda_1 \lambda_2^4 + \lambda_2^5 \right) \lambda_2^2 \\
&\cdot \left( \lambda_1^4 + \lambda_2^4 \right) \\
&+ \frac{\lambda_2^3 (\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2) (\lambda_1^4 + \lambda_2^4)}{\lambda_1^4 + \lambda_2^4} \\
&\cdot \frac{1}{\lambda_1^4 + \lambda_2^4} \\
&\cdot \left( 17 \lambda_1^{24} - 124 \lambda_1^{23} \lambda_2 + 218 \lambda_1^{22} \lambda_2^2 + 646 \lambda_1^{21} \lambda_2^3 - 642 \lambda_1^{20} \lambda_2^4 \\
&- 2586 \lambda_1^{19} \lambda_2^5 + 2536 \lambda_1^{18} \lambda_2^6 + 3576 \lambda_1^{17} \lambda_2^7 - 2411 \lambda_1^{16} \lambda_2^8 \\
&- 2928 \lambda_1^{15} \lambda_2^9 + 1524 \lambda_1^{14} \lambda_2^{10} + 1724 \lambda_1^{13} \lambda_2^{11} + 548 \lambda_1^{12} \lambda_2^{12} \\
&- 276 \lambda_1^{11} \lambda_2^{13} - 696 \lambda_1^{10} \lambda_2^{14} - 8 \lambda_1^9 \lambda_2^{15} + 499 \lambda_1^8 \lambda_2^{16} + 236 \lambda_1^7 \lambda_2^{17} \\
&+ 146 \lambda_1^6 \lambda_2^{18} - 66 \lambda_1^5 \lambda_2^{19} - 2 \lambda_1^4 \lambda_2^{20} + 46 \lambda_1^3 \lambda_2^{21} + 48 \lambda_1^2 \lambda_2^{22} \\
&+ 16 \lambda_1 \lambda_2^{23} + 7 \lambda_2^{24} \right) \cdot h_{11;1} \\
&+ (\ldots) \cdot h_{22;2}^2.
\end{align*}
\]

In each case, Sturm’s algorithm yields, that the right-hand side is non-positive. \(\square\)

Theorem 10.1 follows.

11. \(H|A|^2\)-Flow

**Theorem 11.1.** A smooth closed strictly convex surface in \(\mathbb{R}^3\), contracting with normal velocity \(H|A|^2\), converges to a round point in finite time.

**Theorem 11.2.** For a family of smooth closed strictly convex surfaces \(M_t\) in \(\mathbb{R}^3\), flowing according to \(\frac{d}{dt} X = -H|A|^2 \nu\),

\[
\max_{M_t} \frac{(\lambda_1 + \lambda_2)^2(\lambda_1 - \lambda_2)^2}{2\lambda_1 \lambda_2}
\]

would
is non-increasing in time.

Proof. We proceed as above.

\[ w = \log \left( \frac{(\lambda_1 + \lambda_2)^2(\lambda_1 - \lambda_2)^2}{2\lambda_1\lambda_2} \right) \]
\[ = \log \left( \frac{-H^4 + 2|A|^2H^2}{H^2 - |A|^2} \right), \]
\[ h_{22;1} = \frac{3\lambda^2 + \lambda^2_2}{3\lambda^2 + \lambda^2_1} h_{11;1}. \]

\[ \frac{d}{dt} H - F^{ij} H_{;ij} = 2H^2 tr A^3 - H \left( |A|^2 \right)^2 \]
\[ + 2 \sum (\lambda_i + \lambda_j) h_{ii; k} h_{jj; k} + 2H \sum h^2_{ij; k} \]
\[ = 2H^2 tr A^3 - H \left( |A|^2 \right)^2 \]
\[ + 4(\lambda_1 + \lambda_2a_1)(1 + a_1) \cdot h^2_{11;1} + 2H(1 + 3a^2_1) \cdot h^2_{11;1} \]
\[ + (\ldots) h^2_{22;2}, \]

\[ \frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} = 2 \left( |A|^2 \right)^3 - 2|A|^2 \sum h^2_{ij; k} + 4 \sum (\lambda_i + \lambda_j) \lambda_k h_{ii; k} h_{jj; k} \]
\[ = 2 \left( |A|^2 \right)^3 - 2|A|^2 (1 + 3a^2_1) \cdot h^2_{11;1} \]
\[ + 8\lambda_1(\lambda_1 + \lambda_2a_1)(1 + a_1) \cdot h^2_{11;1} \]
\[ + (\ldots) h^2_{22;2}, \]

\[ -F^{ij} H_{;i} H_{;j} = - \sum (|A|^2 + 2H \lambda_k) h_{ii; k} h_{jj; k} \]
\[ = - |A|^2 (1 + a_1)^2 \cdot h^2_{11;1} - 2H\lambda_1(1 + a_1)^2 \cdot h^2_{11;1} \]
\[ + (\ldots) h^2_{22;2}, \]

\[ -F^{ij} \left( (|A|^2)_{;i} H_{;j} + (|A|^2)_{;j} H_{;i} \right) = - 2 \sum (|A|^2 + 2H \lambda_k) (\lambda_i + \lambda_j) \lambda_k h_{ii; k} h_{jj; k} \]
\[ = - 4|A|^2 (\lambda_1 + \lambda_2a_1)(1 + a_1) \cdot h^2_{11;1} \]
\[ - 8\lambda_1(\lambda_1 + \lambda_2a_1)(1 + a_1) \lambda_1 h^2_{11;1} \]
\[ + (\ldots) h^2_{22;2}, \]

\[ \frac{d}{dt} w - F^{ij} w_{;ij} = \left( \frac{-4H^3 + 4|A|^2 H}{H^4 + 2|A|^2 H^2} - \frac{2H}{H^2 - |A|^2} \right) \left( \frac{d}{dt} H - F^{ij} H_{;ij} \right) \]
\[ + \left( \frac{2H^2}{H^4 + 2|A|^2 H^2} + \frac{1}{H^2 - |A|^2} \right) \left( \frac{d}{dt} |A|^2 - F^{ij} (|A|^2)_{;ij} \right) \]
\[ + \left( \frac{-12H^2 + 4|A|^2}{H^4 + 2|A|^2 H^2} - \frac{2}{H^2 - |A|^2} \right) (-F^{ij} H_{;i} H_{;j}) \]
\[ + \frac{4H}{-H^4 + 2|A|^2 H^2} (-F^{ij} \left( (|A|^2)_{;i} H_{;j} + (|A|^2)_{;j} H_{;i} \right)) \]
Lemma 11.3. For a family of smooth closed strictly convex surfaces $M_t \subset \mathbb{R}^3$, $0 \leq t < T$, flowing according to $\frac{d}{dt} X = -Fv$ with $F = H|A|^2$, a positive lower bound on the principal curvatures, $\lambda_1, \lambda_2 \geq \varepsilon > 0$, is preserved during the evolution.

Proof. We proceed similarly as in Lemma 5.3 and compute

\[ F^{kl,rs} h_{kl;1} h_{rs;1} = (6\lambda_1 + 2\lambda_2) \cdot h_{11;1}^2 + 4(\lambda_1 + \lambda_2) \cdot h_{11;1} h_{22;1} + 4(\lambda_1 + \lambda_2) \cdot h_{11;2}^2 + (2\lambda_1 + 6\lambda_2) \cdot h_{22;1}^2 \geq 0, \]

\[
\left( \frac{d}{dt} M_{ij} - F^{kl} M_{ij;kl} \right) \xi^i \xi^j \geq \left( (|A|^2)^2 + 2H \text{tr} A^3 \right) \cdot h_{ij} \xi^i \xi^j - 4H |A|^2 h^k_{ij} h_{kj} \xi^i \xi^j + 2\varepsilon H |A|^2 h_{ij} \xi^i \xi^j \]
\[= \varepsilon^5 + 3\varepsilon \lambda^4 > 0. \]

Similar calculations as in Lemma 5.4 using $F + F^{ij} h_{ij} - \alpha F^{ij} h^k_{ij} h_{kj} \geq 0$ for $F = H|A|^2$ give

\[ \alpha \left( (|A|^2)^2 + 2H \text{tr} A^3 \right) \leq 4H |A|^2, \]
\[ \frac{1}{\varepsilon} \left( H|A|^2 \right)^{1/3} \leq \frac{4}{H} \leq \frac{4}{\alpha} \]
and an estimate as in Lemma 5.4 follows.

Lemma 11.4. The dual function to $F = H|A|^2$ is $-3$-concave.

Proof. We set $\Phi = -H|A|^2 K^{-3}$ and want to prove that

\[ \Phi^{ij, kl} \eta_{ij} \eta_{kl} \leq \frac{4}{3\Phi} \Phi^{ij} \eta_{ij} \Phi^{kl} \eta_{kl}. \]

We compute

\[ \Phi^{ij} = -|A|^2 K^{-3} g^{ij} - 2H K^{-3} h^{ij} + 3H |A|^2 K^{-3} \tilde{h}^{ij}, \]
\[ \Phi^{kl, ij} = -2K^{-3} \left( g^{ij} h^{kl} + h^{ij} g^{kl} \right) + 3|A|^2 K^{-3} \left( g^{ij} \tilde{h}^{kl} + \tilde{h}^{ij} g^{kl} \right) + 6H K^{-3} \left( \tilde{h}^{ij} h^{kl} + \tilde{h}^{ij} h^{kl} \right) \]
\[ -2H K^{-3} g^{ik} g^{il} - 9H |A|^2 K^{-3} \tilde{h}^{ij} \tilde{h}^{kl} - 3H |A|^2 K^{-3} \tilde{h}^{ik} \tilde{h}^{il}. \]
We have to check that

$$-2H|A|^2 \left( \frac{2\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1 + \lambda_2}{2\lambda_2} \right) + 3H (|A|^2)^2 \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_1 + \lambda_2}{2\lambda_2} \right)$$

$$+ 6H^2|A|^2 \left( \frac{2}{\lambda_1} + \lambda_2 \right) - 2H^2|A|^2 \left( 1 \quad 0 \right) \left( 1 \quad 0 \right)$$

$$- 9H^2 (|A|^2)^2 \left( \frac{1}{\lambda_1} \quad \frac{1}{\lambda_2} \right) - 3H^2 (|A|^2)^2 \left( \frac{1}{\lambda_1} \quad 0 \right) \left( \frac{1}{\lambda_2} \right).$$

This is equivalent to

$$0 \leq \left( 2\lambda_1^4 + \frac{24}{3} \lambda_1^3 \lambda_2 + \frac{24}{3} \lambda_1^2 \lambda_2^2 + \frac{24}{3} \lambda_1 \lambda_2^3 + 2\lambda_2^4 \right) \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{1}{\lambda_2} \right).$$

Calculations as before show that

$$(16(T-t))^{-1/4} \left( 1 - c \cdot (T-t)^{1/4} \right) \leq \lambda_1, \lambda_2 \leq (16(T-t))^{-1/4} \left( 1 + c \cdot (T-t)^{1/4} \right).$$

This finishes the proof of Theorem 11.1.

12. $|A|^4$-Flow

**Theorem 12.1.** A smooth closed strictly convex surface in $\mathbb{R}^3$, contracting with normal velocity $|A|^4$, converges to a round point in finite time.

**Theorem 12.2.** For a family of smooth closed strictly convex surfaces $M_t$ in $\mathbb{R}^3$, flowing according to $\frac{d}{dt} X = -|A|^4 \nu = -(|A|^2)^2 \nu$,

$$\max_{M_t} \frac{(\lambda_1^4 + 2\lambda_1^3 \lambda_2 + 4\lambda_1^2 \lambda_2^2 + 2\lambda_1 \lambda_2^3 + \lambda_2^4) (\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}$$

is non-increasing in time.

**Proof.** We calculate

$$\tilde{w} = \log \left( \frac{(\lambda_1^4 + 2\lambda_1^3 \lambda_2 + 4\lambda_1^2 \lambda_2^2 + 2\lambda_1 \lambda_2^3 + \lambda_2^4) (\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} \right)$$

$$= \log \left( \frac{-H^6 + 2|A|^2 H^4 - (|A|^2)^2 H^2 + 2(|A|^2)^3}{H^3 - |A|^2 H} \right),$$

$$\frac{d}{dt} H = F^{ij} H_{ij} = 4H|A|^2 \text{tr} A^3 - 3(|A|^2)^3$$

$$+ 8 \sum \lambda_i \lambda_j h_{ij} k h_{jj} k + 4|A|^2 \sum h_{ij}^2 k,$$

$$\frac{d}{dt}|A|^2 = F^{ij} (|A|^2)_{ij} = 2(|A|^2)^2 \text{tr} A^3 + 16 \sum \lambda_i \lambda_j \lambda k h_{ij} k h_{jj} k.$$
\[
\frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{;ij} = \left( \begin{array}{c}
-6H^5 + 8|A|^2 H^3 - 2 (|A|^2)^2 H \\
-6H + 2|A|^2 H^4 - (|A|^2)^3 H^2 + 2 (|A|^2)^2 H^2
\end{array} \right) - \left( \begin{array}{c}
3H^2 - |A|^2 \\
H^3 - |A|^2 H
\end{array} \right) \cdot \left( \frac{d}{dt} H - F^{ij} H_{;ij} \right)
\]

\[
+ \left( \frac{2H^4}{-6H^6 + 2|A|^2 H^4 - (|A|^2)^2 H^2 + 2 (|A|^2)^3} - \frac{6H}{H^3 - |A|^2 H} \right) \cdot \left( -F^{ij} H_{;i} H_{;j} \right)
\]

\[
+ \left( \frac{-30H^4 + 24|A|^2 H^2 - 2 (|A|^2)^2}{-6H^6 + 2|A|^2 H^4 - (|A|^2)^2 H^2 + 2 (|A|^2)^3} - \frac{6H}{H^3 - |A|^2 H} \right) \cdot \left( -F^{ij} (|A|^2)_{;i} (|A|^2)_{;j} \right)
\]

\[
+ \left( \frac{8H^3 - 4|A|^2 H}{-6H^6 + 2|A|^2 H^4 - (|A|^2)^2 H^2 + 2 (|A|^2)^3} - \frac{-1}{H^3 - |A|^2 H} \right) \cdot \left( -F^{ij} (H_{;i} (|A|^2)_{;j} + H_{;j} (|A|^2)_{;i}) \right).
\]

We use a computer algebra program and obtain

\[
h_{22;1} = \frac{\lambda_2}{\lambda_1 + \lambda_2} \left( \frac{\lambda_1 + 2 \lambda_1^2 + 4 \lambda_2^2 + 2 \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 + \lambda_2} \right) \left( \frac{\lambda_1 + 2 \lambda_1^2 + 4 \lambda_2^2 + 2 \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 + \lambda_2} \right)
\]

\[
\frac{d}{dt} \tilde{w} - F^{ij} \tilde{w}_{;ij} = -4\lambda_2 \frac{(\lambda_1 + 2 \lambda_1^2 + 4 \lambda_2^2 + 2 \lambda_1 \lambda_2 + \lambda_2^2)}{\lambda_1 + \lambda_2}
\]

\[
\cdot \left( \frac{\lambda_1 + 2 \lambda_1^2 + 4 \lambda_2^2 + 2 \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 + \lambda_2} \right) \left( \frac{\lambda_1 + 2 \lambda_1^2 + 4 \lambda_2^2 + 2 \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 + \lambda_2} \right)
\]

\[
\cdot \left( 1 + \frac{1}{\lambda_1 + \lambda_2} \right) \cdot h_{22;1}
\]

We apply Sturm’s algorithm and obtain that the right-hand side is non-positive. \qed

**Lemma 12.3.** For a family of smooth closed strictly convex surfaces \( M_t \subset \mathbb{R}^3 \), \( 0 \leq t < T \), flowing according to \( \frac{d}{dt} X = -F v \) with \( F = |A|^2 \), a positive lower bound on the principal curvatures, \( \lambda_1, \lambda_2 \geq \varepsilon > 0 \), is preserved during the evolution.
Proof. The term involving second derivatives of $F$ is non-negative, so we have
\[
\left(\frac{d}{dt}M_{ij} - F^{kl}M_{ij;kl}\right)\xi^i\xi^j \geq |A|^2\varepsilon \left(4\text{ tr } A^3 - 3\varepsilon |A|^2\right)
\]
\[
= |A|^2\varepsilon (\lambda^3 + 3\lambda^2(\lambda - \varepsilon) + \varepsilon^3) > 0.
\]

\[\square\]

Similar calculations as in Lemma 5.4 using $F + F^{ij}h_{ij} - \alpha F^{ij}h_{ij} \geq 0$ for $F = |A|^4$ give
\[
5|A|^2 - 4\alpha \text{ tr } A^3 \geq 0,
\]
\[
\frac{1}{\varepsilon} \left(|A|^4\right)^{1/4} \leq \frac{\text{ tr } A^3}{|A|^2} \leq \frac{5}{4\alpha}
\]
and an estimate as in Lemma 5.4 follows.

The following lemma implies that the dual function to $F = |A|^4$ is $-4$-concave.

Lemma 12.4. If the dual function to $F$ is $\alpha$-concave for some $\alpha < 0$, then the dual function to $F^\beta$ is $\alpha \cdot \beta$-concave for $\beta > 0$.

Proof. We use similar notation as before. Assume that
\[
-G^{ij;kl}\eta_{ij}\eta_{kl} \leq \frac{\alpha - 1}{\alpha(-G)} G^{ij}\eta_{ij} G^{kl}\eta_{kl}.
\]
We have to show for $\Phi = -G^\beta$
\[
\Phi^{ij;kl}\eta_{ij}\eta_{kl} \leq \frac{\alpha\beta - 1}{\alpha\beta\Phi} \Phi^{ij}\eta_{ij} \Phi^{kl}\eta_{kl}.
\]
Direct calculations yield
\[
\Phi^{ij} = -\beta G^{\beta-1}G^{ij},
\]
\[
\Phi^{ij,kl}\eta_{ij}\eta_{kl} = -\beta G^{\beta-1}G^{ij,kl} - \beta(\beta - 1)G^{\beta-2}G^{ij} G^{kl},
\]
\[
\Phi^{ij;kl}\eta_{ij}\eta_{kl} = -\beta G^{\beta-1}G^{ij;kl}\eta_{ij}\eta_{kl} - \beta(\beta - 1)G^{\beta-2} \left(G^{ij}\eta_{ij}\right)^2
\]
\[
\leq -\beta \frac{\alpha - 1}{\alpha} G^{\beta-2} \left(G^{ij}\eta_{ij}\right)^2 - \beta(\beta - 1)G^{\beta-2} \left(G^{ij}\eta_{ij}\right)^2
\]
\[
= -\frac{\alpha\beta - 1}{\alpha\beta G^{\beta}} G^{2\beta-2} \left(G^{ij}\eta_{ij}\right)^2
\]
\[
= \frac{\alpha\beta - 1}{\alpha\beta\Phi} \left(\Phi^{ij}\eta_{ij}\right)^2.
\]

\[\square\]

Calculations as before show that
\[
\lambda_1, \lambda_2 \leq (20(T - t))^{-4/5} \cdot \left(1 + c \cdot (T - t)^{3/20}\right).
\]
A corresponding lower estimate is also true. Theorem 12.1 follows.
13. Convergence Rate

In order to find out what the optimal convergence rate might be, we consider the evolution equation

\[ \frac{d}{dt} X = -|A|^2 \nu + 2X, \]

that appropriately rescaled solutions of \( \frac{d}{dt} X = -|A|^2 \nu \) fulfill. As in [29, Appendix], we represent the surfaces \( M_t \) as graphs over the sphere with embeddings

\[ S^2 \ni x \mapsto x \cdot u(x), \]

where \( u : S^2 \to \mathbb{R}_+ \). Let \( (\sigma_{ij}) \) be the standard metric on the sphere and \( (\sigma^{ij}) \) its inverse. Then we get as in [29],

\[ g_{ij} = u^2 (\sigma_{ij} + \varphi_i \varphi_j), \quad \text{where } \varphi = \log u, \]
\[ h_{ij} = \frac{1}{uw} g_{ij} - \frac{u}{w} \varphi_{ij}, \quad \text{where } w = \sqrt{1 + \varphi_i \sigma^{ij} \varphi_j}, \]
\[ \frac{\partial u}{\partial t} = -|A|^2 w + 2u. \]

We linearize our equation around the stationary solution \( u = 1 \) and take \( u = 1 + \varepsilon v \). Then

\[ \frac{d}{d\varepsilon} w \bigg|_{\varepsilon=0} = 0, \]
\[ \frac{d}{d\varepsilon} g_{ij} \bigg|_{\varepsilon=0} = 2v \sigma_{ij}, \]
\[ \frac{d}{d\varepsilon} h_{ij} \bigg|_{\varepsilon=0} = v \sigma_{ij} - v_{ij}. \]

So the linearized equation becomes

\[ \frac{\partial v}{\partial t} = 2\Delta v + 6v. \]

There are spherical harmonics \( u \) solving \( \Delta u = -l(l+1)u \) for \( l \in \mathbb{N} \) [26]. We do not need to consider \( l = 0 \) as a corresponding eigenfunction is positive (or negative) everywhere. Thus the resulting surface does not contract to a point for \( t \uparrow T \) in the unrescaled setting. Similarly, we can exclude \( l = 1 \) as the respective eigenfunctions correspond to translations and translated surfaces converge to infinity in the rescaled setting.

Using the ansatz eigenfunction multiplied with \( e^{-\lambda t} \), we get \( \lambda = 2l(l+1) - 6 \). This is positive for \( l = 2, \lambda = 6 \). So we should not expect a convergence rate better than \( ||X| - 1| \leq c \cdot e^{-6t} \) after rescaling, corresponding to estimates like

\[ r_+ \leq (6(T-t))^{1/3} \cdot (1 + c \cdot (T-t)). \]

Here, we have assumed that \( q(t) = 0 \).

We can still improve our convergence rate by considering

\[ \max_{M_t} \frac{(\lambda_1^2 + \lambda_2^2) (\lambda_1 - \lambda_2)^2}{4\lambda_1^2 \lambda_2^2} = \max_{M_t} \frac{2|A|^2 - H^2}{(H^2 - |A|^2)^2} \equiv \max_{M_t} w. \]
Under the flow equation $\frac{d}{dt}X = -|A|^2 \nu$, we obtain, using calculations as above, that $w$ fulfills in a critical point of $w$

$$\frac{d}{dt}w - F^{ij}w_{;ij} = -\left(\frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^3 - \lambda_1 \lambda_2^2 + 2 \lambda_2^3} \cdot \frac{\lambda_1}{\lambda_2^5}\right) \cdot \left(5\lambda_1^9 - 3\lambda_1^8 \lambda_2 - 6\lambda_1^7 \lambda_2^2 + 26\lambda_1^6 \lambda_2^3 - 20\lambda_1^5 \lambda_2^4 + 8\lambda_1^4 \lambda_2^5 + 6\lambda_1^3 \lambda_2^6 - 2\lambda_1^2 \lambda_2^7 - \lambda_1 \lambda_2^8 + 3\lambda_2^9\right) \cdot h_{11;1}^2 \cdot h_{22;2}^2 \leq 0.$$

This implies convergence rates like

$$r_+ \leq (6(T-t))^{1/3} \cdot \left(1 + c \cdot (T-t)^{1/3}\right)$$

similar to [12], where the author also has a scaling invariant upper bound for $|\lambda_1 - \lambda_2|$. As there are no negative constant terms left for this choice of $w$, it might be, that there is no monotone quantity as studied in this paper, that allows to improve this convergence rate.

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