THE AUSLANDER-GORENSTEIN PROPERTY FOR Z-ALGEBRAS

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Abstract. We provide a framework for part of the homological theory of Z-algebras and their generalizations, directed towards analogues of the Auslander-Gorenstein condition and the associated double Ext spectral sequence that are useful for enveloping algebras of Lie algebras and related rings. As an application, we prove the equidimensionality of the characteristic variety of an irreducible representation of the Z-algebra, and for related representations over quantum symplectic resolutions. In the special case of Cherednik algebras of type A, this answers a question raised by the authors.

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1. Introduction

1.1. Throughout the paper, all rings will be algebras over a fixed base field k. An unadorned tensor product ⊗ will denote a tensor product over k. A lower triangular Z-algebra is a Z-bigraded associative algebra

\[ S = \bigoplus_{0 \leq q \leq p \in \mathbb{Z}} S_{p,q} \]

with matrix-style multiplication and where each non-zero \( S_{q,q} \) is a noetherian, unital algebra and the \( S_{p,q} \) are finitely generated modules over both \( S_{p,p} \) and \( S_{q,q} \).

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These algebras are important in noncommutative algebraic geometry – see, for example, [SV] or [VdB]. They are also useful in geometric representation theory, including in the study of rational Cherednik algebras [GS1, GS2], deformed preprojective algebras [Bo, Mu], finite W-algebras [Gi] and deformations of conical symplectic singularities [BPW]. In applications, the \( \mathbb{Z} \)-algebra provides an effective way to relate the representation theory of the given algebra to the geometry of a resolution of singularities for its associated graded ring: indeed, the \( \mathbb{Z} \)-algebra can be regarded as a quantization of that resolution. In the above examples, the \( \mathbb{Z} \)-algebras quantize Hilbert schemes of points on the plane, minimal resolutions of Kleinian singularities, resolutions of Slodowy slices and, most generally, symplectic resolutions of conical symplectic singularities, respectively.

What is missing, and what is provided in this paper, is a suitable homological machine to relate the commutative and noncommutative theories. Our aim is show how the Auslander-Gorenstein condition and the related double Ext spectral sequence, that are so useful for enveloping algebras and related rings, can be generalised to work for \( \mathbb{Z} \)-algebras. As an application we generalize Gabber’s equidimensionality result to \( \mathbb{Z} \)-algebras.

1.2. To explain our results in more detail, we need some notation. Write \( S_{0,0}\text{-}\text{mod} \) for the category of noetherian left \( S_{0,0} \)-modules, \( S\text{-grmod} \) for the category of noetherian left \( S \)-modules and \( S\text{-qgr} \) for the quotient category of \( S\text{-grmod} \) modulo the bounded modules. Let \( \pi_S : S\text{-grmod} \to S\text{-qgr} \) be the natural projection. If the \( S_{p,q} \) are \( (S_{p,p}, S_{q,q}) \)-progenerators, we say that \( S \) is a Morita \( \mathbb{Z} \)-algebra. In this case, for any \( q \geq 0 \), there is an equivalence of categories \( S_{q,q}\text{-}\text{mod} \sim S\text{-qgr} \) given by \( \Phi : M \mapsto \pi_S(S_{*,q} \otimes_{S_{q,q}} M) \), where \( S_{*,q} = \bigoplus_{p \geq q} S_{p,q} \).

We assume that there is a filtration \( F \) on \( S \) such that the associated graded algebra \( \text{gr}_F S = \bigoplus \text{gr}_F S_{p,q} = \Delta(R) \). Here, \( \Delta(R) = \bigoplus \Delta(R)_{p,q} \) is the \( \mathbb{Z} \)-algebra associated to some finitely generated commutative graded algebra \( R = \bigoplus R_n \) by defining \( \Delta(R)_{p,q} = R_{p-q} \) for all \( p \geq q \). There are natural equivalences \( \Delta(R)\text{-qgr} \simeq R\text{-qgr} \simeq \text{Coh}(X) \), for \( X = \text{Proj}(R) \). If \( M \in S_{0,0}\text{-}\text{mod} \) has a good filtration \( F \), then \( M = \Phi(M) \) is naturally filtered and has associated graded sheaf \( \text{gr}_F M \in \text{Coh}(X) \). The characteristic variety \( \text{Char}(M) \subseteq X \) of \( M \) (or of \( \mathcal{M} \)) is then the support of \( \text{gr}_F \mathcal{M} \).

1.3. If, for example, \( S_{0,0} = U_{\lambda} \) is the spherical subalgebra of a rational Cherednik algebra of type \( A \) – see Subsection 9.8 for the definitions – then \( S \) is defined by setting \( S_{p,p} = U_{\lambda+p} \), with a canonical choice of \( (U_{\lambda+p}, U_{\lambda+q}) \)-bimodules \( S_{p,q} \). In this case \( S \) is a Morita \( \mathbb{Z} \)-algebra whenever \( \Re(\lambda) \geq \frac{1}{2} \). Moreover, \( X = \text{Hilb}^n(\mathbb{C}^2) \) is the Hilbert scheme of \( n \) points on the place and the natural map \( X \to Y = \text{Spec}(R_0) \) is a resolution of singularities of the quotient variety \( (\mathbb{C}^n \times \mathbb{C}^n)/S_n \), see Section 9 or [GS1] for more details. The structure of \( \text{Char}(M) \) is complex: when \( M \) is an irreducible \( U_{\lambda} \)-representation from category \( \mathcal{O} \) it has irreducible components parametrized by partitions of \( n \), see [GS2, Section 6].
1.5. As we will describe in 1.7 there are a number of important examples of Theorem. Suppose that of the paper culminate in the following theorem, which summarises Theorems 7.5 and 8.3. This uses the notion of a filtered Auslander-Gorenstein algebra $U$ and the convergent spectral sequence

$$\text{Ext}^p_U(\text{Ext}^q_U(M, U), U) \Rightarrow H^{p-q}(M) = \begin{cases} M & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

In order to generalise this spectral sequence to $\mathbb{Z}$-algebras, we need extra conditions. Suppose now that $S = \bigoplus_{p \geq q \geq 0} S_{p,q}$ and $T = \bigoplus_{p \geq q \geq 0} T_{p,q}$ are two filtered $\mathbb{Z}$-algebras with $S_{0,0} = T_{0,0}$. We say that $q_S$ is a good parameter for $S$ if $S_{\geq \lambda} = \bigoplus_{p \geq 2q \geq \lambda} S_{p,q}$ is a Morita $\mathbb{Z}$-algebra. We consider the following hypotheses.

(H1) There exists a good parameter $q_S$ for $S$.

(H2) $\text{gr} S \cong \Delta(R)$ where $R = \bigoplus_{n \geq 0} R_n$ is a finitely generated commutative domain satisfying $R_m R_n = R_{m+n}$ for all $m, n \geq 0$.

(H3) Conditions (H1) and (H2) hold for $T$ with $q_T = q_T$. Furthermore, under the tensor product filtration defined in Notation 3.1, $\text{gr}(S_{p,0} \otimes_{S,0} T_{q,0}) \cong R_{p+q}$ for all $p, q \geq q_S$.

(H4) $X = \text{Proj}(R)$ is Gorenstein, $\text{Spec}(R_0)$ is normal, and the canonical morphism $X \to \text{Spec}(R_0)$ is birational.

1.6. Let us explain some of the undefined terms in this theorem, and the significance of the hypotheses (H1–H4). First, $\mathcal{E}\mathcal{X}\mathcal{T}$ is the analogue for $S$-agr of sheaf Ext and is defined in Definition 3.2. Part (2) of the theorem is then the natural analogue of the Auslander-Gorenstein condition for $\mathbb{Z}$-algebras. Given the earlier discussions, conditions (H1), (H2) and (H4) are natural and so it is only the requirement of an auxiliary algebra $T$ and the description of $X$ that require explanation.

The problem comes in generalizing $\text{Ext}^p(U, M)$. One choice is to consider $\mathcal{A} = \mathcal{E}\mathcal{X}\mathcal{T}_S(M, S) = \bigoplus_n \text{Ext}^p_S(M, S_{s,n})$. There are two problems here. First, one really wants to take the image of this module in a quotient category $agr$. However, finitely generated right $S$-modules are bounded and so $agr-S = 0$ (see 2.7)! Thus one does need an auxiliary algebra $T$. For example, $\mathcal{A}$ is

Theorem. Suppose that $S$ and $T$ are $\mathbb{Z}$-algebras that satisfy Hypotheses (H1–H4).

1. There exists a module $X \in (S \otimes T)-agr$ such that, for $M \in S-agr$, there is a convergent spectral sequence

$$\mathcal{E}\mathcal{X}\mathcal{T}^p_T(\mathcal{E}\mathcal{X}\mathcal{T}^q_S(M, X), X) \Rightarrow H^{p-q}(M) = \begin{cases} M & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

2. If $M \in S-agr$ and $N$ is a $T$-submodule of $\mathcal{E}\mathcal{X}\mathcal{T}^q_S(M, X)$, then $\mathcal{E}\mathcal{X}\mathcal{T}^p_T(M, X) = 0$ for $p < q$.
3. If $\Phi(M) \in S-Qgr$ is irreducible for some $M \in S_{q,q}-\text{mod}$, then $\text{Char}(M)$ is equidimensional.
naturally a right module over the upper triangular \( \mathbb{Z} \)-algebra \( S^* = \bigoplus_{0 \leq q \leq p} S^*_{p,q} \) where \( S^*_{p,q} = \text{Hom}_{S_{q,q}}(S_{p,q}, S_{q,q}) \). For more technical reasons \( T = S^* \) does not work; basically because one would then need \( \text{gr}(S^*_{p,q}) \otimes \text{gr}(S_{p,q}) = \text{gr}(S_{q,q}) \) for \( p \geq q \), and this essentially never holds. Fortunately, in all the standard examples there is a natural candidate for \( T \). For example given the Cherednik algebra as in [1.3], where \( T = S^* \) does not work; basically because one would then need \( \text{gr}(S^*_{p,q}) \otimes \text{gr}(S_{p,q}) = \text{gr}(S_{q,q}) \) for \( p \geq q \), and this essentially never holds. Fortunately, in all the standard examples there is a natural candidate for \( T \). For example given the Cherednik algebra as in 1.3, where \( S \) is defined in terms of the sequence of algebras \( S_{p,p} = U_{\lambda+p} \), the algebra \( T \) is defined by moving in the “opposite” direction: \( T_{p,p} = U_{\lambda-p}^{op} \). The module \( X \) is now the image in \( (S \otimes T)\text{-qgr} \) of the natural \( S \otimes T \)-bimodule \( S^*_{0,0} \otimes S_{0,0} \).

1.7. The hypotheses (H1–H4) hold for the example of rational Cherednik algebras and Hilbert schemes, thanks mostly to [GGS]. In Proposition [9.6] we show that they also hold for the \( \mathbb{Z} \)-algebras constructed in [BPW]. These algebras are attached to \( \mathbb{C}^* \)-equivariant symplectic resolutions of singularities \( X \to Y \), where \( Y \) is an irreducible affine symplectic singularity with a \( \mathbb{C}^* \)-action that both attracts to a unique fixed point and scales the given symplectic form \( \omega \) by \( t \cdot \omega = t^m \omega \) for all \( t \in \mathbb{C}^* \) and some \( m > 0 \). Examples include:

- The minimal resolution \( \mathcal{M} \to \mathcal{M}_0 = \mathbb{C}^2 / \Gamma \) for a finite subgroup \( \Gamma \subseteq SL(2, \mathbb{C}) \) of type \( \mathcal{A} \);
- \( X = \text{Hilb}^n(\mathcal{M}) \) and \( Y = \text{Sym}^n(\mathcal{M}_0) \) with \( \mathcal{M} \) and \( \mathcal{M}_0 \) as above;
- The Springer resolution \( T^*(G/B) \to \mathcal{N} \). More generally, one takes \( X = T^*(G/P) \), where \( P \) is a parabolic subgroup of the reductive algebraic group \( G \) and \( Y \) is the affinization of \( X \);
- Various Nakajima quiver varieties and their affinizations.

In these examples, the \( \mathbb{Z} \)-algebra is constructed from a \( \mathbb{C}^* \)-equivariant deformation quantization of \( X \) depending on \( \lambda \in H^2(X, \mathbb{C}) \) and a very ample line bundle \( L \) on \( X \). The characteristic variety can be interpreted, in this case, in terms of the deformation quantization.

1.8. It may be worth noting that we do not require \( X \to \text{Spec}(R_0) \) in (H4) to be a resolution of singularities. So it makes sense to try to apply Theorem [1.5] in the case which arises from the minimal model programme, namely when the morphism \( X \to Y \) is crepant with \( X \) \( \mathbb{Q} \)-factorial and terminal.

1.9. Outline of the paper. The natural generality for the results of this paper are for algebras indexed by \( \mathbb{Z}^n \times \mathbb{Z}^n \) for \( n \geq 1 \). Following [Gi] we call them directed algebras; this generality has appeared for [Gi] and [Mu] and has potential applications for other, more general symplectic reflection algebras. The essentials of directed algebras, combined with basic results about their Ext groups, filtrations and associated graded modules are given in Sections 2, 3 and 4. The analogue of dualizing and the spectral sequence relating the cohomology of a module to that of its associated graded module appears in Section 5. This is used in Section 7 to prove parts (1) and (2) of Theorem [1.5]. The analogue of Gabber’s theorem, Theorem [1.5(3)], is then proved for general directed algebras in Section 8. The application of the theorem to a number of different quantizations of symplectic singularities, including Cherednik algebras, is given in Section 9.
2. Directed algebras.

2.1. As remarked in the introduction, the results in this paper work for more than just \( \mathbb{Z} \)-algebras, and so in this section we provide the relevant definitions and basic results.

Let \( \Lambda \) be a submonoid of a finitely generated free additive abelian group \( G(\Lambda) \) such that \( \Lambda \) generates \( G(\Lambda) \) and \( \Lambda \cap -\Lambda = \{0\} \). For \( \lambda, \mu \in G(\Lambda) \) we write \( \lambda \geq \mu \) if \( \lambda - \mu \in \Lambda \). Note that if \( \mu_1, \ldots, \mu_n \in G(\Lambda) \) then there exists \( \lambda \in \Lambda \) such that \( \lambda \geq \mu_i \) for each \( i \), namely \( \lambda = \sum \mu_i \).

**Definition.** A \( \Lambda \)-directed algebra is a \( \Lambda \)-bigraded \( k \)-algebra \( S = \bigoplus_{\lambda, \mu \in \Lambda} S_{\lambda, \mu} \) such that each \( S_{\lambda, \lambda} \) is a \( k \)-algebra and multiplication is defined matrix style: \( S_{\lambda, \mu} S_{\mu, \tau} \subseteq S_{\lambda, \tau} \) and \( S_{\lambda, \mu} S_{\nu, \tau} = 0 \) for \( \mu \neq \nu \). We further assume that each \( S_{\lambda, \lambda} \) is unital, with unit \( 1_\lambda \), and that each \( S_{\lambda, \mu} \) is finitely generated and unital as a left module over \( S_{\lambda, \lambda} \) and as a right module over \( S_{\mu, \mu} \).

The \( \Lambda \)-directed algebra \( S \) is called a lower \( \Lambda \)-directed algebra if \( S_{\lambda, \mu} \neq 0 \) only if \( \lambda \geq \mu \), respectively an upper \( \Lambda \)-directed algebra if \( S_{\lambda, \mu} \neq 0 \) only if \( \lambda \leq \mu \).

2.2. **Examples.** There are three examples of \( \Lambda \)-directed algebras that will interest us.

1. Let \( R = \bigoplus_{\lambda \in \Lambda} R_\lambda \) be a \( \Lambda \)-graded \( k \)-algebra. Set \( R_\lambda = 0 \) for \( \lambda \in G(\Lambda) \setminus \Lambda \). Define a lower \( \Lambda \)-directed algebra \( \Delta(R) \) by \( \Delta(R) = \bigoplus_{\lambda, \mu \in \Lambda} \Delta(R)_{\lambda, \mu} \) where \( \Delta(R)_{\lambda, \mu} = R_{\lambda - \mu} \).

2. If \( S \) is an upper \( \Lambda \)-directed algebra, then its “transpose” \( S^{tr} \) is a lower \( \Lambda \)-directed algebra. Formally \( S^{tr} = \bigoplus (S^{tr})_{\lambda, \mu} \) where \( (S^{tr})_{\lambda, \mu} = S_{\mu, \lambda} \), with the opposite multiplication.

3. If \( S \) is lower \( \Lambda \)-directed and \( T \) is lower \( \Gamma \)-directed, then \( S \otimes T \) is lower \( (\Lambda \times \Gamma) \)-directed.

In this paper we will only consider upper and lower directed algebras. By (2) it suffices to consider only the latter case.

For the rest of this section we assume that \( S \) is a lower \( \Lambda \)-directed algebra.

2.3. **Definition.** A graded (left) \( S \)-module is a left \( S \)-module \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \) with matrix style multiplication \( S_{\lambda, \mu} M_\nu = 0 \) for \( \nu \neq \mu \) and \( S_{\lambda, \mu} M_\mu \subseteq M_\lambda \). Each \( M_\lambda \) is assumed to be a unital left \( S_{\lambda, \lambda} \)-module.

The category of all such modules will be denoted \( S \text{-Grmod} \), where the morphisms are the homogeneous \( S \)-homomorphisms. The category of graded right modules is denoted \( \text{Grmod}-S \).

If \( S = \Delta(R) \) as in Example 2.2(1), then \( \Delta(R) \text{-Grmod} \) is the category of \( \Lambda \)-graded \( R \)-modules.

2.4. The category \( S \text{-Grmod} \) admits direct limits and these direct limits preserve exactness. Moreover the category has a set of distinguished objects parametrised by \( \Lambda \),

\[
S_{*, \lambda} := \bigoplus_{\tau \in \Lambda} S_{\tau, \lambda} = S \cdot 1_\lambda.
\]

The set \( \{ S_{*, \lambda} : \lambda \in \Lambda \} \) generates the category \( S \text{-Grmod} \): if \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \in S \text{-Grmod} \) and \( m \in M_\lambda \), then \( 1_\lambda \mapsto m \) induces a unique homomorphism \( S_{*, \lambda} \rightarrow M \). It follows from [Ga] Chapitre II, §6, Théorème 2 that every object of \( S \text{-Grmod} \) has an injective hull.
Definition. $S$ is called locally left noetherian if $S_{*,\lambda}$ is noetherian in $S\text{-Grmod}$ for all $\lambda \in \Lambda$. The full subcategory of noetherian objects in $S\text{-Grmod}$ will be denoted by $S\text{-grmod}$.

If $R$ is a $\Lambda$-graded algebra, then $\Delta(R)_{*,\lambda} = R[-\lambda]$ where $R[-\lambda]_\tau = R_{\tau-\lambda}$ for all $\tau$. So $\Delta(R)$ is locally left noetherian if and only if $R$ is left noetherian. In general, when $S$ is locally left noetherian, $S\text{-grmod}$ is the subcategory of finitely generated modules.

2.5. As for unital graded algebras, it is natural to consider the quotient category of graded noetherian modules modulo the Serre subcategory of torsion modules.

Definition. An object $M \in S\text{-grmod}$ is called torsion if there exists $\lambda \in \Lambda$ such that $M_\tau = 0$ for all $\tau \leq \lambda$. An object $M$ of $S\text{-Grmod}$ is torsion if it is the direct limit of noetherian torsion objects.

We denote the corresponding full subcategories by $S\text{-tors}$ and $S\text{-Tors}$ respectively.

The category $S\text{-tors}$ is a Serre subcategory of $S\text{-grmod}$ and so we can form the quotient category $S\text{-qgr} = S\text{-grmod}/S\text{-tors}$. If $S$ is locally left noetherian (which is the main case of interest in this paper) then $S\text{-Tors}$ is a localising subcategory of $S\text{-Grmod}$, [Ga, Chapitre III, §3, Corollaire 1] and so we can form the quotient category $S\text{-Qgr} = S\text{-Grmod}/S\text{-Tors}$. By [Ga, Chapitre III, §1&2], $S\text{-Qgr}$ has an exact quotient functor $\pi_S : S\text{-Grmod} \rightarrow S\text{-Qgr}$ whose right adjoint is the section functor $\sigma_S : S\text{-Qgr} \rightarrow S\text{-Grmod}$.

2.6. Vorsicht! There is a substantial lack of symmetry in concepts from the last two subsections. For example, if $T$ is lower $\mathbb{N}$-directed then every graded noetherian right $T$-module is torsion.

2.7. Shifting. Let $S$ be lower $\Lambda$-directed. For $\lambda \in \Lambda$ we define a new lower $\Lambda$-directed algebra by

$$S_{\geq \lambda} := \bigoplus_{\mu,\tau \in \Lambda} S_{\mu+\lambda,\tau+\lambda}.$$

We have shift functors

$$[\lambda] : S\text{-Grmod} \rightarrow S_{\geq \lambda}\text{-Grmod}, \quad [-\lambda] : S_{\geq \lambda}\text{-Grmod} \rightarrow S\text{-Grmod}$$

defined as follows. Given $M \in S\text{-Grmod}$, set $M[\lambda] = \bigoplus_{\tau \in \Lambda} M[\lambda]_\tau$ where $M[\lambda]_\tau = M_{\lambda+\tau}$, while if $N \in S_{\geq \lambda}\text{-Grmod}$ set $N[-\lambda] = \bigoplus_{\tau \in \Lambda} N[-\lambda]_\tau$ where $N[-\lambda]_\tau = N_{\tau-\lambda}$ if $\tau \geq \lambda$ and is zero otherwise. It is immediate that $N[-\lambda][\lambda] = N$, whilst there is a short exact sequence

$$0 \rightarrow M[\lambda][-\lambda] \rightarrow M \rightarrow C \rightarrow 0$$

where $C_\tau = 0$ for all $\tau \geq \lambda$ and so $C \in S\text{-Tors}$. This has the following useful consequences.

Lemma. Let $S$ be a locally left noetherian lower $\Lambda$-directed algebra.

(i) The functors $[\lambda]$ and $[-\lambda]$ induce equivalences of categories between $S\text{-Qgr}$ and $S_{\geq \lambda}\text{-Qgr}$ and hence between the noetherian subcategories $S\text{-qgr}$ and $S_{\geq \lambda}\text{-qgr}$.

(ii) Suppose that there exists $\lambda \in \Lambda$ such that $S_{\geq \lambda} = \Delta(R)$ for some $\Lambda$-graded commutative algebra $R$. Then $S\text{-Qgr}$ and $R\text{-Qgr}$ are equivalent categories. \qed
2.8. **Morita directed algebras.** Let $S$ be a lower $\Lambda$-directed algebra.

**Definition.** We call $\lambda \in \Lambda$ a good parameter for $S$ and $S_{\geq \lambda}$ a Morita $\Lambda$-directed algebra provided:

1. $S_{\lambda, \lambda}$ is a noetherian ring;
2. for all $\mu \geq \lambda$, tensoring with the $(S_{\mu, \mu}, S_{\lambda, \lambda})$-bimodule $S_{\mu, \lambda}$ induces a Morita equivalence between $S_{\lambda, \lambda} \text{-Mod}$ and $S_{\mu, \mu} \text{-Mod}$;
3. for any $\mu \geq \tau \geq \lambda$ the multiplication map $S_{\mu, \tau} \otimes_{S_{\tau, \tau}} S_{\tau, \lambda} \rightarrow S_{\mu, \lambda}$ is an isomorphism.

A routine application of Morita theory shows that if $\lambda$ is good then so is any $\mu \geq \lambda$. If $\lambda$ is a good parameter and $\mu \geq \lambda$ we write $S^{\ast}_{\mu, \lambda} = \text{Hom}_{S_{\lambda, \lambda}}(S_{\mu, \lambda}, S_{\lambda, \lambda})$ for the dual of $S_{\mu, \lambda}$. By hypothesis it is an $(S_{\lambda, \lambda}, S_{\mu, \mu})$-bimodule that is projective on both sides.

2.9. A mild generalisation of [GS2, Lemma 5.5] or [Bo, Theorem 12] gives:

**Proposition.** Suppose that $S$ is a locally left noetherian lower $\Lambda$-directed algebra and suppose that $\lambda$ is good. Then there exists an equivalence of categories $\Psi : S_{\lambda, \lambda} \text{-Mod} \sim \rightarrow S_{\geq \lambda} \text{-Qgr}$ given by $M \mapsto \pi_{S} \left( \bigoplus_{\tau} S_{\tau + \lambda, \lambda} \otimes_{S_{\lambda, \lambda}} M \right)[-\lambda]$. It restricts to an equivalence between $S_{\lambda, \lambda} \text{-mod}$ and $S \text{-qgr}$.

**Proof.** We first claim that the functor $\Psi' : M \mapsto \bigoplus_{\tau} S_{\tau + \lambda, \lambda} \otimes_{S_{\lambda, \lambda}} M$ provides an equivalence $S_{\lambda, \lambda} \text{-Mod} \sim \rightarrow S_{\geq \lambda} \text{-Qgr}$. This follows from the cited references, after the following minor modifications. First, they only deal with the $\mathbb{N}$-directed case, but the proof extends trivially. Second, those papers only prove the result at the level of noetherian objects. This implies the general case since the functor is defined on all modules and, by the Morita equivalence, it commutes with direct limits.

The equivalence $S_{\geq \lambda} \text{-Qgr} \sim \rightarrow S \text{-Qgr}$ and hence the proposition now follow from Lemma 2.7. □

2.10. **Remark.** The functor $\Phi$ inverse to $\Psi$, is easy to describe for noetherian objects. Specifically, if $M = \pi_{S} \left( \bigoplus_{\mu \in \Lambda} M_{\mu} \right) \in S \text{-agr}$ then $\Phi(M) = S^{\ast}_{\mu, \lambda} \otimes M_{\mu} \in S_{\lambda, \lambda} \text{-mod}$ for any $\mu \gg \lambda$. The details can be found in [GS2, Lemma 5.5].

3. **Homological notions.**

In this section we assume that $S$, $T$ and $S \otimes T$ are, respectively, $\Lambda$-directed, $\Gamma$-directed and $(\Lambda \times \Gamma)$-directed. Each is assumed to be locally left noetherian.

3.1. Here, we introduce several homological concepts which are to be used throughout the paper. In particular, we will find the appropriate analogues for directed algebras of a locally free sheaf and of the dual functor $\text{Hom}_{A}(-, A)$ over a unital ring $A$.

3.2. **Definition.** (i) Let $N \in (S \otimes T) \text{-Grmod}$ and $\gamma \in \Gamma$. Mimicking (2.4.1), set

$$N_{* \gamma} = \bigoplus_{\lambda \in \Lambda} N_{\lambda, \gamma} \in S \text{-Grmod}.$$
(ii) For \( i \geq 0 \), define a functor \( \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(-, N) : S\text{-Qgr} \to T\text{-Qgr} \) by
\[
(3.2.1) \quad \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, N) = \pi_T \left( \bigoplus_{\gamma \in \Gamma} \text{Ext}^i_{S,\text{-Qgr}}(\mathcal{M}, \pi_S(N_{*,\gamma})) \right) \quad \text{for } \mathcal{M} \in S\text{-Qgr}.
\]

To see that part (ii) of this definition makes sense, use [2.1] to pick an injective resolution \( I^\bullet \) of \( N_{*,\gamma} \). By \([Ga, Corollaire 2, p.375] \) \( \pi_S(I^\bullet) \) is an injective resolution of \( \pi_S(N_{*,\gamma}) \) and so this can be used to calculate the Ext-groups in question. Moreover for any \( t \in T_{\gamma_1,\gamma_2} \), left multiplication by \( t \) induces a morphism \( t \cdot : N_{*,\gamma_2} \to N_{*,\gamma_1} \) in \( S\text{-Grmod} \) and hence in \( S\text{-Qgr} \). This morphism provides the direct sum in (3.2.1) with the required structure of a graded left \( T \)-module.

3.3. We can modify the above definition, replacing \( N \in S \otimes T\text{-Grmod} \) with \( N' \in S \otimes T\text{-Qgr} \), by setting
\[
(3.3.1) \quad \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, N') = \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, \sigma_{S \otimes T}(N')) \quad \text{for } \mathcal{M} \in S\text{-Qgr}.
\]
In this definition note that the section functor \( \sigma_{S \otimes T} : S \otimes T\text{-Qgr} \to S \otimes T\text{-Grmod} \) is not necessarily right exact, just as proper pushforward for quasi-coherent sheaves is not.

We need to check that this definition coincides with (3.2.1) when \( N' = \pi_{S \otimes T}(N) \), and \( \mathcal{M} \) is noetherian.

**Lemma.** Assume that \( \mathcal{M} \in S\text{-Qgr} \). Then for all \( i \geq 0 \) and \( N \in S \otimes T\text{-Grmod} \) there exists a natural isomorphism in \( N \)
\[
\mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, N) \cong \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, \pi_{S \otimes T}(N))
\]

**Proof.** Let \( Z \in S \otimes T\text{-Grmod} \) be torsion; thus \( Z = \lim \rightarrow Z(j) \) where each \( Z(j) \in S \otimes T\text{-grmod} \) has the property that there exists \( (\lambda_j, \gamma_j) \in \Lambda \times \Gamma \) such that \( Z(j)_{\alpha,\beta} = 0 \) for all \( \alpha \geq \lambda_j \) and \( \beta \geq \gamma_j \). The first step of the proof will be to show that
\[
(3.3.2) \quad \mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, Z) = 0 \quad \text{for any } i \geq 0.
\]

Well,
\[
\mathcal{E}X \mathcal{T}^i_{S,\text{-Qgr}}(\mathcal{M}, Z) \cong \pi_T \left( \bigoplus_{\gamma \in \Gamma} \text{Ext}^i_{S,\text{-Qgr}}(\mathcal{M}, \pi_S(Z_{*,\gamma})) \right)
\]
\[
\cong \pi_T \left( \bigoplus_{\gamma \in \Gamma} \text{Ext}^i_{S,\text{-Qgr}}(\mathcal{M}, \lim \rightarrow \pi_S(Z(j)_{*,\gamma})) \right)
\]
\[
\cong \pi_T \left( \bigoplus_{\gamma \in \Gamma} \lim \rightarrow \pi_T \left( \bigoplus_{\gamma \in \Gamma} \text{Ext}^i_{S,\text{-Qgr}}(\mathcal{M}, \pi_S(Z(j)_{*,\gamma})) \right) \right)
\]

Here the second and the last isomorphism hold since \( \pi_S \) and \( \pi_T \) commute with direct limits, \([Ga, p.378–9] \), while the third isomorphism holds because \( \mathcal{M} \) is noetherian. But for any \( \gamma \geq \gamma_j \), the object \( Z(j)_{*,\gamma} \) is \( S \)-torsion since \( (Z(j)_{*,\gamma})_{\lambda} = Z(j)_{\lambda,\gamma} = 0 \) for all \( \lambda \geq \lambda_j \). Hence \( \pi_S(Z(j)_{*,\gamma}) = 0 \)
and the $T$-module $\bigoplus_{\gamma \in T} \text{Ext}^i_{S,\text{qgr}}(M, \pi_S(Z(j)_{*\gamma}))$ is torsion. Thus $\pi_T$ kills each such module and (3.3.2) is proven.

Set $N' = \sigma_{S \otimes T}(\pi_{S \otimes T}(N))$. By [Ga] Proposition 3(2), p.371] there are graded $S \otimes T$-modules $M \subseteq N$ and $M' \subseteq N$ such that (i) $M$ and $N'/M'$ are torsion, and (ii) there exists an isomorphism $\psi : N/M \rightarrow M'$. Now apply the functor $\mathcal{E}X^T_{S,\text{qgr}}(M, -)$ to the short exact sequences

$$0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow M' \rightarrow N' \rightarrow N'/M' \rightarrow 0.$$ 

Using the fact that $\pi_S$ and $\pi_T$ are exact, we get long exact sequences

$$\cdots \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, M) \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, N) \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, N/M) \rightarrow \cdots$$

and

$$\cdots \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, M') \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, N') \rightarrow \mathcal{E}X^T_{S,\text{qgr}}(M, N'/M') \rightarrow \cdots.$$ 

By (3.3.2), it follows that

$$\mathcal{E}X^T_{S,\text{qgr}}(M, N) \cong \mathcal{E}X^T_{S,\text{qgr}}(M, N/M) \quad \text{and} \quad \mathcal{E}X^T_{S,\text{qgr}}(M, M') \cong \mathcal{E}X^T_{S,\text{qgr}}(M, N').$$

But $\psi$ induces an isomorphism $\mathcal{E}X^T_{S,\text{qgr}}(M, N/M) \cong \mathcal{E}X^T_{S,\text{qgr}}(M, M')$. Combining these three isomorphisms gives the desired result. \hfill $\square$

3.4. Acyclic sheaves. Assume that $S_{00} = T_{00}$. We define

$$(3.4.1) \quad \mathcal{X}_{S \otimes T} = \pi_{S \otimes T}(S_{*0} \otimes S_{00} T_{*0}) = \pi_{S \otimes T}\left(\bigoplus_{\lambda \in \Lambda, \gamma \in \Gamma} S_{\lambda,0} \otimes S_{00} T_{\gamma,0}\right) \in S \otimes T\text{qgr}$$

and, for $\gamma \in \Gamma$, set

$$(3.4.2) \quad \mathcal{X}_\gamma = \mathcal{X}_{*\gamma} = \pi_S\left(\bigoplus_{\nu} S_{\nu,0} \otimes S_{0,0} T_{\gamma,0}\right) \in S\text{qgr}.$$ 

We will usually drop the subscript from $\mathcal{X}_{S \otimes T}$ as it will be clear from the context.

**Definition.** A module $M \in S\text{qgr}$ is called an acyclic sheaf, or more strictly an $(S, T)$-acyclic sheaf, if $\mathcal{E}X^T_{S,\text{qgr}}(M, \mathcal{X}_{S \otimes T}) = 0$ for all $i > 0$.

The significance of this definition is that the dual object $\text{Hom}_{S,\text{qgr}}(M, \pi_S(S))$ does not behave well over a lower $\Lambda$-directed algebra $S$. Indeed, [2,6] implies that $S$ is a torsion right $S$-module and so $\pi_S(S) = 0$. This makes $\text{Hom}_{S,\text{qgr}}(M, \pi_S(S))$ useless in applications. As will be seen, an appropriate replacement is the $T$-module $\mathcal{H}OM_{S,\text{qgr}}(M, \mathcal{X}_{S \otimes T})$, for a suitable directed algebra $T$.

Now suppose that $S = T = \Delta(R)$ for a commutative $\Lambda$-graded algebra $R$ for which $X = \text{Proj}(R)$ is smooth. If we identify $S\text{qgr} = \text{Coh}(X)$, then $\mathcal{X}$ is just $\iota_*\mathcal{O}_X$ under the diagonal embedding $\iota : X \rightarrow X \times X$. Since $\mathcal{E}xt^i_X(M, \mathcal{O}_X)$ can be calculated on affine patches, it follows that in this case an acyclic sheaf is just a vector bundle on $X$. 

9
3.5. **Proposition.** Assume that \( S_{00} = T_{00}^{\text{op}} \). Let \( \lambda \in \Lambda \) be a good parameter for \( S \) and \( \gamma \in \Gamma \) a good parameter for \( T \).

1. \( S_\lambda = \pi_S(S_{\lambda, \lambda}) \) is a projective object in \( S \cdot \text{Qgr} \). In particular, for all \( N \in S \cdot \text{Qgr} \) and \( i > 0 \) we have \( \text{Ext}^i_{S \cdot \text{Qgr}}(S_\lambda, N) = 0 \) and so \( S_\lambda \) is an acyclic sheaf.
2. \( X_\gamma \in S \cdot \text{qgr} \) for any \( \gamma \in \Lambda \). Consequently, \( X \) is noetherian as an object in \( (S \otimes T) \cdot \text{Qgr} \).

**Proof.** (1) Under the equivalence of categories \( S_{\lambda, \lambda} \cdot \text{Mod} \xrightarrow{\sim} S_{\geq \lambda} \cdot \text{Qgr} \) defined in the proof of Proposition 2.9, the projective object \( S_{\lambda, \lambda} \) is sent to \( \bigoplus \mu \nu \cdot S_{\tau + \lambda, \lambda} \). On shifting by \((- \lambda)\) we then get the object whose \( \mu \)th component is \( S_{\mu, \lambda} \) if \( \mu \geq \lambda \) and is zero otherwise. But on applying \( \pi_S \) this is \( S_\lambda \).

(2) By Proposition 2.9 again, the noetherianity of \( X_\gamma \) is equivalent to the noetherianity of the \( S_{\lambda, \lambda} \)-module \( S_{\lambda, 0} \otimes_{S_{0, 0}} T_{\gamma, 0} \). This is noetherian since \( T_{\gamma, 0} \) is a finitely generated module over \( S_{0, 0} = T_{0, 0}^{\text{op}} \) and \( S_{\lambda, 0} \) is a finitely generated \( \Lambda \)-module.

In order to prove that \( X \in (S \otimes T) \cdot \text{qgr} \), recall that \( T_{\mu, \gamma} \) is a progenerator for all \( \mu \geq \gamma \) and hence \( X_\mu = \pi_S(\bigoplus \nu S_{0, 0} \otimes_{S_{0, 0}} T_{\mu, 0}) = \pi_S(T_{\mu, \gamma} \otimes_{T_{\gamma, \gamma}} X_\gamma) \) for all such \( \mu \). Therefore,

\[
X = \pi_{S \otimes T}(\bigoplus \nu X_\nu) = \pi_{S \otimes T}(\bigoplus_{\mu \geq \gamma} X_\mu) = \pi_{S \otimes T}(T_{\geq \gamma} X_\gamma).
\]

By the above paragraph this is a noetherian object in \( S \otimes T \cdot \text{Qgr} \).

Further acyclic sheaves are provided by Theorem 7.4.

4. **Filtrations.**

4.1. In this section we discuss the basic properties of filtrations on directed algebras.

**Notation.** An \( \mathbb{N} \)-filtration \( F^\bullet S \) on a lower \( \Lambda \)-directed algebra \( S \) will always be assumed to respect the graded structure of \( S \). Thus \( F^m(S) = \bigoplus_{\nu \geq \lambda} F^m S_{\nu, \lambda} \) for all \( m \geq 0 \). This ensures that the associated graded object \( \text{gr}_F S \) has an induced lower \( \Lambda \)-directed algebra structure.

Suppose that \( A = \bigcup F^i A \) is a filtered right module and that \( B = \bigcup F^i B \) is a filtered left module over some algebra \( U \). Then the **tensor product filtration** on the vector space \( A \otimes_U B \) is defined by \( F^m(A \otimes_U B) = \sum_{j} F^j A \otimes_U F^{m-j} B \) for \( m \in \mathbb{N} \). Note that for a filtered algebra \( U = \bigcup F^\bullet U \), the multiplication map \( \mu : U \otimes U \to U \) is automatically filtration preserving in the sense that \( \mu(F^m(U \otimes U)) \subseteq F^m U \) when the left hand side is given the tensor product filtration.

4.2. **Hypotheses.** Given a lower \( \Lambda \)-directed algebra \( S \) with an \( \mathbb{N} \)-filtration \( F^\bullet S \), we assume

- (H1) There exists a good parameter \( \lambda_S \) for \( S \);
- (H2) As \( \Lambda \)-directed algebras, \( \text{gr} S \cong \Delta(R) \), where \( R = \bigoplus_{\lambda \in \Lambda} R_\lambda \) is a finitely generated commutative \( \Lambda \)-graded domain satisfying \( R_\nu \cdot R_{\nu'} = R_{\nu + \nu'} \) for all \( \nu, \nu' \in \Lambda \).

Given a second \( \Lambda \)-directed algebra \( T \) satisfying \( T_{00}^{\text{op}} = S_{00} \), we also assume

- (H3) Hypotheses (H1) and (H2) also hold for \( T \) with \( \lambda_S = \lambda_T \). Furthermore, under the tensor product filtration, \( \text{gr}(S_{\tau, 0} \otimes_{S_{0, 0}} T_{\gamma, 0}) = R_{\tau + \gamma} \) for each \( \tau, \gamma \geq \lambda_S \).
Similarly, the first part of Hypothesis (H3) ensures that \( T \) and \( S \otimes T \) are locally left noetherian. Therefore the underlying assumptions of Section 3 follow from these hypotheses.

The final part of Hypothesis (H2) guarantees that the \( n \)-th graded piece \( \text{gr}_n(S_{\nu,\tau}) \) of \( S_{\nu,\tau} \) satisfies

\[
\text{gr}_n(S_{\nu,\tau}) = \sum_{j \leq n} \text{gr}_j(S_{\nu,\rho}) \text{gr}_{n-j}(S_{\rho,\tau}) \quad \text{for each } n \text{ and } \nu \geq \rho \geq \tau \in \Lambda.
\]

Thus (H2) implies that the multiplication map \( \mu \) is filtered-surjective in the sense that under the tensor product filtration

\[
\mu(F^m(S_{\nu,\rho} \otimes S_{\rho,\tau})) = F^m(S_{\nu,\tau}) \quad \text{for each } m \text{ and } \nu \geq \rho \geq \tau \in \Lambda.
\]

4.3. Here are our conventions concerning filtrations on objects from \( \text{Qgr} \).

**Definition.** A filtration on \( M \in \text{S-Qgr} \) is a pair \((M, F^\bullet)\) consisting of choice of a lift \( M \in \text{S-Grmod} \) of \( M \) and a \( \mathbb{Z} \)-filtration \( F^\bullet \) on \( M \) that is separated and exhaustive. We typically abuse notation and denote such data by \( F^\bullet M \). A filtration on \( M \) gives rise to the quasi-coherent sheaf \( \text{gr}_F M = \pi_{\Delta(R)}(\text{gr} M) \) on \( X \). A filtration preserving morphism between \( F^\bullet M \) and \( F^\bullet N \) is just a filtration preserving morphism in \( \text{S-Grmod} \) between the given lifts \( M \) and \( N \). It induces a morphism \( \text{gr}_F M \to \text{gr}_F N \) in \( R\text{-Qgr} \). A good filtration on \( M \in \text{S-Qgr} \) is a filtration such that \( \text{gr}_F M \in R\text{-qgr} \).

Note that if \( G^\bullet M \) is another filtration of \( M \) that agrees with \( F^\bullet M \) in high degree, then \( \text{gr}_F M = \text{gr}_G M \). The existence of a good filtration implies that \( M \in \text{S-qgr} \); conversely, any object of \( \text{S-qgr} \) can be given a good filtration, thanks to the combination of Proposition 2.9 and \([\text{GS2}, \text{Lemma 2.5}]\).

4.4. Assume that \( S \) satisfies Hypotheses (H1) and (H2) and write \( X = \text{Proj } R \). For \( \lambda \geq \lambda_S \), recall the module \( S_\lambda = \pi_S(S_{s,\lambda}) \) from Corollary 3.3. Then the given filtration \( F^\bullet \) on \( S \) provides induced filtrations, again written \( F^\bullet \), on \( S_{s,\lambda} \) and \( S_\lambda \). We have \( \text{gr}_F S_\lambda \cong \pi_{R(R)}(R[-\lambda]) = \mathcal{O}_X(-\lambda) \) as objects in \( \Delta(R)\text{-qgr} \).

**Definition.** Let \( M \in \text{S-Qgr} \) have a good filtration \( F^\bullet M \). We will say that an exact sequence \( P^\bullet \to M \to 0 \) is a filtered projective resolution of \( M \) in \( \text{S-Qgr} \) if the following hold.

1. For some \( \tau \geq \lambda_S \) each \( P^\tau \) can be written \( P^\tau = \bigoplus_{1 \leq j \leq n_\tau} S_\tau \epsilon_{jr} \cong S_{(n_\tau)} \) for some basis \( \{\epsilon_{jr}\} \).
2. Give each \( S_{s,\tau} \) the filtration induced from \( S \), and give \( S_{s,\tau} \epsilon_{jr} \) the induced filtration \( G^\bullet \) by assigning \( \epsilon_{jr} \in G_{k_{jr}} \) for some \( k_{jr} \). Then the resolution \( P^\bullet \to M \) is filtered and the associated graded complex \( \text{gr}_G P^\bullet \to \text{gr}_F M \to 0 \) is exact.

Note that in the second part of the definition \( \text{gr}_G P^\tau = \bigoplus_j \mathcal{O}_X(-\tau) \epsilon_{jr} \).

4.5. We can form filtered projective resolutions in \( \text{S-qgr} \).

**Lemma.** Let \( S \) satisfy Hypotheses (H1) and (H2) and suppose that \( M \in \text{S-Qgr} \) has a good filtration \((M, F^\bullet)\). Then \( M \) has a filtered projective resolution.

**Proof.** We will reduce the problem to a case where we can apply \([\text{GS2}, \text{Lemma 2.5(i)}]\). It does no harm to replace \( M \) by \( M_{\geq \tau} \), for any \( \tau \geq \lambda_S \), so we do. Now, by Proposition 2.9, \( \text{S-Qgr} \) is \( \text{S}_{\tau,\tau}\text{-Mod} \) and \( S_{\geq \tau} \) is a Morita \( \Lambda \)-algebra.
Let $\gr_F M$ be generated by $\sum_{i=1}^n \gr_F M_{i\ell}$ and pick $\tau \geq \lambda_S, \tau_1, \ldots, \tau_n$. By (H2) $\gr_F M_{\geq \tau}$ is generated by $\gr M_{\tau}$, so replacing $M$ by $M_{\geq \tau}$ means that $M$ is now generated by $N = M_{\tau}$. The multiplication map $\mu_\nu: S_{\nu,\tau} \otimes S_{\nu,\tau} N \rightarrow M_\nu$ is therefore surjective for all $\nu \geq \tau$. If it is not an isomorphism for some $\nu$ then $\Ker \mu_\nu \neq 0$. But as each $S_{\nu,\nu}$ is a progenerator, this would imply that $\Ker \mu_\alpha \neq 0$ for all $\alpha \geq \nu$, contradicting Proposition 2.9. Hence each $\mu_\nu$ is an isomorphism.

Let $F^\bullet N$ denote the induced filtration on $N$. We claim that each $\mu_\nu$ is a filtered isomorphism, where the domain is given the tensor product filtration $T^\bullet$ induced from the filtrations on $S_{\nu,\tau}$ and $N$ while the range is given the filtration $F^\bullet$ induced from that on $M$. To see this, note that $\mu_\nu$ is filtered by construction and so it is a filtered injection. That it is a filtered surjection for all $\nu$ is equivalent to the fact that $\gr_F M$ is generated by $\gr_F N$. This proves the claim. To summarise, replacing $M$ by some $M_{\geq \tau}$, we can assume that its filtration is induced from that on $N = M_{\tau}$.

Set $U = S_{\tau,\tau}$ and apply the proof of [GS2, Lemma 2.5(i)]. This constructs a free $U$-module $Q = \bigoplus U \epsilon_i$, with a filtration $G^\bullet$ defined by giving each $\epsilon_i$ some degree $k_i \geq 0$, for which there exists a filtered surjection $\phi: Q \rightarrow N$. Set $P(\nu) = S_{\nu,\tau} \otimes_U Q$ for $\nu \geq \tau$ and $P = S_{\star,\tau} \otimes_U Q = \bigoplus_{\nu \geq \tau} P(\nu)$ with the tensor product filtration $T^\bullet$. As in [loc.cit] it follows that $\gr_T P \cong \bigoplus R \epsilon_i$ is a free $R$-module such that the induced map $\gr \phi: \gr_T P \rightarrow \gr_F M$ is also surjective.

Now repeat this procedure for $N' = \Ker(\phi)$, with the filtration induced from that of $Q$, and iteratively obtain a filtered resolution $Q^\bullet \rightarrow N \rightarrow 0$ that satisfies all the usual properties of filtered free resolutions of modules. Note that, as we are working with $S_{\tau,\tau}$-modules here, the choice of $\tau$ stays fixed throughout this procedure. Finally, using Proposition 2.9 one sees that $S_{\star,\tau} \otimes_{S_{\tau,\tau}} Q^\bullet \rightarrow S_{\star,\tau} \otimes_{S_{\tau,\tau}} M_{\tau} \rightarrow 0$ is then the desired filtered projective resolution of $M$.

4.6. **Corollary.** Let $S$ and $T$ satisfy Hypotheses (H1–H3) and pick $M \in S\text{-qgr}$ and $N \in S\text{-Qgr}$. Then

1. The groups $\Ext^\star_{S\text{-Qgr}}(M, N)$ can be computed as the cohomology groups $\RHom_{S\text{-Qgr}}(P^\bullet, N)$ for any filtered projective resolution $P^\bullet \rightarrow M \rightarrow 0$.

2. Define $X = X_{S \otimes T}$ by (3.4.1). Then

$$\mathcal{E}X T^\bullet_{S\text{-Qgr}}(M, X) = \pi_T \left( \bigoplus_{\lambda \in A} \RHom_{S\text{-Qgr}}(P^\bullet, \pi_S(S_{\star,0} \otimes S_{0,\lambda}, T_{\lambda,0})) \right).$$

**Proof.** (1) Since everything can be computed in $S_{\tau,\tau}$-Mod for $\tau \gg 0$, this follows from Lemma 4.5.

(2) As the members of the projective resolution of $M$ are noetherian, this follows from (1) combined with Lemma 3.3.

5. **Dualising and associated graded modules.**

Assume throughout this section that $S$ and $T$ are filtered, locally left noetherian $A$-directed algebras that satisfy (H1–H3) from 4.2. Fix objects $M, N \in S\text{-qgr}$ with good filtrations $(M, F^\bullet)$, respectively $(N, G^\bullet)$, where $M, N \in S\text{-grmod}$. Set $X = \text{Proj}(R)$.

5.1. In this section results on filtrations and associated graded modules for unital algebras are generalised to directed algebras. This will culminate in a spectral sequence for the associated graded modules of Ext groups; see Proposition 4.9 for the details.
5.2. Recall from [Ga, p.365] that $\text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N}) = \lim \text{Hom}_{S_{\text{Grmod}}}(M', N/N')$, where $M' \subseteq M$ and $N' \subseteq N$ are submodules for which $M/M'$ and $N'$ are torsion. Since $M$ and $N$ are both noetherian, $(N/N)_\nu = N_\nu$ and $M'_\nu = M_\nu$ for $\nu > 0$ and so any $\theta \in \text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N})$ belongs to $\text{Hom}_{S_{\text{Grmod}}}(M_{\geq \nu}, N_{\geq \nu})$ for $\nu > 0$. We therefore obtain a well-defined filtration $\Psi^\bullet$ on $\text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N})$ by

$$
\Psi^m = \Psi^m \text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N}) = \{ \theta \in \text{Hom}_{S_{\text{Grmod}}}(M_{\geq \nu}, N_{\geq \nu}) \text{ such that } \theta(F^t M_{\geq \nu}) \subseteq G^{t+m} N_{\geq \nu} \text{ for all } t \in \mathbb{Z} \text{ and } \nu > 0 \}.
$$

(5.2.1)

5.3. **Lemma.** The filtration (5.2.1) is separated and exhaustive.

**Proof.** For $\beta > \alpha \in \Lambda$ and $\theta \in \text{Hom}(M_{\geq \alpha}, N_{\geq \alpha})$, we will denote the image of $\theta$ in $\text{Hom}(M_{\geq \beta}, N_{\geq \beta})$ by $\theta$ too. Given $\alpha \in \Lambda$, define the usual exhaustive filtration $\Psi^\bullet_\alpha$ on $\text{Hom}_{S_{\text{Grmod}}}(M_{\geq \alpha}, N_{\geq \alpha})$ by

$$
\Psi^m_\alpha = \{ \theta \in \text{Hom}(M_{\geq \alpha}, N_{\geq \alpha}) : \theta(F^t M_{\geq \alpha}) \subseteq G^{t+m} N_{\geq \alpha} \text{ for all } \beta \geq \alpha \text{ and } t \in \mathbb{Z} \}.
$$

Since $\Psi^m_\alpha \subseteq \Psi^m_{\beta} \subseteq \Psi^m$, for all $m$ and $\beta \geq \alpha$, it follows that $\Psi$ is exhaustive. Thus it remains to prove separability.

For $\theta \in \text{Hom}(M_{\geq \alpha}, N_{\geq \alpha})$ write $m_\alpha(\theta) = \min \{ m : \theta \in \Psi^m_\alpha \}$. Since $G^\bullet$ is good, there exists $\alpha \in \Lambda$ such that $\text{gr}_G(N)_{\geq \alpha} = \text{gr}_G(N_{\geq \alpha})$ has zero torsion submodule; $\text{Tors}(\text{gr}_G(N_{\geq \alpha})) = 0$. Clearly $m_\beta(\theta) \geq m_\gamma(\theta)$ for any $\gamma > \beta \geq \alpha$ and $\theta \in \text{Hom}(M_{\geq \beta}, N_{\geq \beta})$, so the result will follow if $m_\beta(\theta) = m_\gamma(\theta)$ always holds.

Suppose that $r = m_\beta(\theta) > m_\gamma(\theta)$ for some such $\gamma, \beta$ and $\theta$. Then we can find some $\beta' \geq \beta$ and $\alpha \in F^{u} M_{\beta'}$ such that $\theta(\alpha) \in G^{u+r} N_{\beta'} \setminus G^{u+r-1} N_{\beta'}$. Replace $\beta$ by $\beta'$ and $\gamma$ by $\gamma' = \max\{ \beta', \gamma \}$; noting that we still have $r = m_\beta(\theta) > m_\gamma(\theta)$. Clearly $x \alpha \in F^{u+p} M_{\delta}$ for any $x \in F^{p} S_{\delta \beta} \cap F^{p-1} S_{\delta \beta}$ with $\delta \geq \gamma$ and $p \in \mathbb{Z}$. Since $m_\delta(\theta) \leq m_\gamma(\theta) \leq r - 1$, it follows that $x \theta(\alpha) = \theta(x \alpha) \in G^{u+p+r-1} N_{\delta}$. But this implies that the principal symbol $\sigma(\theta(\alpha))$ satisfies $\sigma(x) \sigma(\theta(\alpha)) = 0$. In other words, we have $\Delta(R_{\delta \beta}) \cdot \sigma(\theta(\alpha)) = 0$ for any $\delta \geq \gamma$. Thus $\sigma(\theta(\alpha)) \in \text{Tors}(\text{gr}_G(N_{\geq \alpha})) = 0$, contradicting the choice of $\alpha$. This means that $m_\beta(\theta) = m_\gamma(\theta)$ for all $\gamma > \beta \geq \alpha$ and hence that $\Psi$ is separated. \hfill \Box

5.4. Any $\theta \in \Psi^m \text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N})$ induces a mapping from $F^t M_{\nu} / F^{t+1} M_{\nu}$ to $F^{i+m} N_{\nu} / F^{i+1+m} N_{\nu}$ for all large enough $\nu$ and hence defines a homomorphism

$$(5.4.1) \quad \Theta_{\mathcal{M}, \mathcal{N}} : \text{gr}_\Psi \text{Hom}_{S_{\text{Qgr}}}(\mathcal{M}, \mathcal{N}) \longrightarrow \text{Hom}_{\Delta(R)_{\text{Qgr}}}(\text{gr}_F \mathcal{M}, \text{gr}_F \mathcal{N}) = \text{Hom}_X(\text{gr}_F \mathcal{M}, \text{gr}_F \mathcal{N}).$$

It follows from the definition of the filtration $\Psi$ that $\Theta_{\mathcal{M}, \mathcal{N}}$ is injective. It is natural in both entries in the category of filtered objects.

**Lemma.** Give $S_\nu = \pi_S(S_{\ast \nu})$ the filtration induced from $S$. For large enough $\nu$, depending on $\mathcal{N}$ and its filtration, the map

$$
\Theta = \Theta_{S_\nu, \mathcal{N}} : \text{gr}_\Psi \text{Hom}_{S_{\text{Qgr}}}(S_\nu, \mathcal{N}) \longrightarrow \text{Hom}_{\Delta(R)_{\text{Qgr}}}(\text{gr}_F S_\nu, \text{gr}_F \mathcal{N}) = H^0(\text{gr}_F \mathcal{N} \otimes \mathcal{O}_X(\nu))
$$

is an isomorphism.
Lemma 3.3 provides an explicit lift of $M$ Definition. after (5.4.1), it is an isomorphism.

This induces a filtration, again written $\Psi F$ whose composition is the identity on $gr\nu$ for all $\nu \gg 0$. Given $x \in F\mathcal{N}_\nu$ the map $\chi : 1_\nu \mapsto x$ defines an element of $\Psi^1\operatorname{Hom}_{\mathcal{S}-\operatorname{Qgr}}(S_\nu, \mathcal{N})$. Therefore, for $\nu \gg 0$ we have induced graded homomorphisms

$$\operatorname{gr} F\mathcal{N}_\nu \xrightarrow{\operatorname{gr} \chi} \operatorname{gr} \operatorname{Hom}_{\mathcal{S}-\operatorname{Qgr}}(S_\nu, \mathcal{N}) \xrightarrow{\Theta} \operatorname{Hom}_{\Delta(R)-\operatorname{Qgr}}(\operatorname{gr} F S_\nu, \operatorname{gr} F \mathcal{N}) \xrightarrow{\cong} \operatorname{gr} F\mathcal{N}_\nu$$

whose composition is the identity on $\operatorname{gr} F\mathcal{N}_\nu$. It follows that $\Theta$ is surjective and so, by the comments after (5.4.1), it is an isomorphism.

5.5. Definition. Let $\mathcal{X} = \mathcal{X}S \otimes T$ be as defined in (3.4.1) and recall $\lambda_S = \lambda_T$ from (H3). Define

$$\mathcal{M}^\tau = \mathcal{E}\mathcal{X}T^0_{\mathcal{S}-\operatorname{Qgr}}(\mathcal{M}, \mathcal{X}) = \pi_T\left( \bigoplus_{\gamma \in \Lambda} \operatorname{Hom}_{\mathcal{S}-\operatorname{Qgr}}(\mathcal{M}, \mathcal{X}_\gamma) \right) \in T\operatorname{-Qgr}. $$

Lemma 3.3 provides an explicit lift of $\mathcal{M}^\tau$ to $T\operatorname{-Grmod}$: $\mathcal{M}^\tau = \pi_T\mathcal{M}^\tau$ where

$$\mathcal{M}^\tau = \bigoplus_{\gamma \geq \lambda_T} \operatorname{Hom}_{\mathcal{S}-\operatorname{Qgr}}(\mathcal{M}, \pi_S\left( \bigoplus_{\nu} S_{\nu,0} \otimes S_{0,0} T_{\gamma,0} \right)) \in T\operatorname{-Grmod}. $$

Symmetric definitions apply to $\mathcal{M} \in T\operatorname{-qgr}$.

We will always give both $\mathcal{X}$ and $\mathcal{X}_\nu$ the filtration induced from the tensor product filtration on the summands $S_{\nu,0} \otimes S_{0,0} T_{\gamma,0}$. Then (5.2.1) induces a filtration $\Psi^\tau$ on $\mathcal{M}^\tau$ by

$$(\theta_\gamma) \in \mathcal{M}^\tau : \theta_\gamma(F^t M_\nu) \subseteq F^{t+m}(S_{\nu,0} \otimes S_{0,0} T_{\gamma,0}) : $$

for all $\gamma \geq \lambda_T, t \in \mathbb{Z}$ and $\nu \gg 0$.

This induces a filtration, again written $\Psi^\tau$, on $\mathcal{M}^\tau$. We observe that, although $\mathcal{X} \not\in S\operatorname{-qgr}$, any $(\theta_\gamma) \in \mathcal{M}^\tau$ only has finitely many non-zero entries and is therefore contained in $\operatorname{Hom}(\mathcal{M}, \mathcal{Y})$ for some $\mathcal{Y} \in S\operatorname{-qgr}$. Therefore, Lemma 5.3 can be applied to give

**Lemma.** The filtration (5.5.1) is separated and exhaustive.

5.6. The assumption in (H1) that some large enough $\lambda_S$ is good, rather than all $\lambda \in \Lambda$ are good, is necessary in applications. However, it does have the disadvantage that that we have less control over terms like $S_{\lambda_S,0}$. The next lemma will allow us to avoid these problems. For $\nu \geq \lambda_S$, recall the definition of $S_{\nu, \lambda_S}$ from (2.8).

**Lemma.**

1. $S_{\mu,0} = S_{\mu,0} S_{\nu,0} \cong S_{\mu,0} \otimes S_{\nu,0}$ for all $\mu \geq \nu \geq \lambda_S$.

2. For all $\nu \geq \tau \geq \lambda_S$ we have $S_{\tau,0} = S_{\nu,\tau} \otimes S_{0,0}$.

**Proof.** The fact that $S_{\mu,0} = S_{\mu,0} S_{\nu,0}$ follows from (4.2.1), while the isomorphism follows from the fact that $S_{\mu,\nu}$ is a progenerator over $S_{\nu,\nu}$.

(2) As $S_{\nu,\nu}$ is a progenerator the assertion is equivalent to (1).
5.7. As we show next, the hom filtration (5.5.1) on \(S^\nabla_\tau\) has a natural interpretation in terms of the tensor product filtration.

**Lemma.** Fix \(\tau \geq \lambda_S = \lambda_T\).

1. There is an isomorphism of noetherian \(T\)-graded modules

(5.7.1) \[ \Theta : S^\nabla_\tau \xrightarrow{\cong} \bigoplus_{\gamma \geq \lambda_T} S_{\tau,0} \otimes_{S_{0,0}} T_{\gamma,0}. \]

This isomorphism is a filtered isomorphism provided we give the left-hand side the filtration \((5.5.1)\) on \(S_\tau\) and the right-hand side the tensor product filtration.

2. For \(\phi \geq \lambda_T\) we have

\[(S^\nabla_\tau)^\nabla \cong \bigoplus_{\mu \geq \lambda_S} \text{Hom}_{T_\phi,\phi}(S_{\tau,0} \otimes_{S_{0,0}} T_{\phi,0}, \ S_{\mu,0} \otimes_{S_{0,0}} T_{\phi,0}).\]

**Proof.** (1) Let \(\Phi_\tau : S_{-\text{qgr}} \to S_{\tau,\tau} \text{-Mod}\) be the equivalence of categories given by Proposition 2.10 and Remark 2.10. Then, as right \(T\)-modules, we have

\[S^\nabla_\tau = \bigoplus_{\gamma \geq \lambda_T} \text{Hom}_{S_{-\text{qgr}}}(S_{\tau,0} \pi_S(\mathcal{X}_\gamma), S_{\tau,0} \pi_S(\mathcal{X}_\gamma)) \cong \bigoplus_{\gamma \geq \lambda_T} \text{Hom}_{S_{\tau,\tau} \text{-Mod}}(S_{\tau,\tau}, \Phi_\tau(\pi_S(\mathcal{X}_\gamma))).\]

Remark 2.10 and Lemma 5.6 (2) imply that, for any \(\mu \gg 0\) and \(\gamma \geq \lambda_T\) we have

\[\Phi_\tau(\pi_S(\bigoplus_{\nu \in A_{\gamma}} S_{\nu,0} \otimes_{S_{0,0}} T_{\gamma,0})) = S^\ast_{\mu,\tau} \otimes_{S_{\mu,\mu}} (S_{\mu,0} \otimes_{S_{0,0}} T_{\gamma,0}) \cong S_{\tau,0} \otimes_{S_{0,0}} T_{\gamma,0}.\]

This proves (5.7.1). The argument of Corollary 3.5 (2) ensures that \(\bigoplus_{\gamma \geq \lambda_T} S_{\tau,0} \otimes_{S_{0,0}} T_{\gamma,0} \in T_{\text{qgr}}\).

It remains to examine the filtered structure of \(\Theta\). Let

\[x = \sum_{\gamma} \sum_{i_\gamma} x_\gamma^{i_\gamma} \otimes x_\gamma^{i_\gamma} \in Y = \bigoplus_{\gamma \geq \lambda_T} S_{\tau,0} \otimes_{S_{0,0}} T_{\gamma,0}.\]

Then, for any \(\nu \geq \tau\), the computations of the last paragraph show that \(\Theta^{-1}(x) \in S^\nabla_\tau\) is the homomorphism that maps

\[f \in S_{\nu,\tau} \mapsto \sum_{\gamma \geq \lambda_T} \sum_{i_\gamma} f \otimes x_\gamma^{i_\gamma} \otimes x_\gamma^{i_\gamma} \mapsto \sum_{\gamma \geq \lambda_T} \sum_{i_\gamma} f \cdot x_\gamma^{i_\gamma} \otimes x_\gamma^{i_\gamma} \in W_\nu = \sum_{\gamma \geq \lambda_T} S_{\nu,\tau} \otimes_{S_{\tau,\tau}} S_{\tau,0} \otimes_{S_{0,0}} T_{\gamma,0} \cong Z_\nu = \sum_{\gamma \geq \lambda_T} S_{\nu,0} \otimes_{S_{0,0}} T_{\gamma,0}.\]

By (4.2.1), the tensor product filtration on \(S_{\nu,\tau} \otimes_{S_{\tau,\tau}} S_{\tau,0}\) agrees under multiplication with the given filtration on \(S_{\nu,0} = S_{\nu,\tau} S_{\tau,0}\). Hence the treble tensor product filtration on \(W_\nu\) agrees with the tensor product filtration on \(Z_\nu\). Thus, by the last displayed equation, if \(x \in F^\tau(Y) \setminus F^{\tau-1}(Y)\), then \(\Theta^{-1}(x)\) maps \(F^m(S_{\nu,\tau})\) to \(F^{m+r}(Z_\nu)\). In order to complete the proof of the lemma, we need to prove that \(\Theta^{-1}(x) \in F^\tau(S^\nabla_\tau) \setminus F^{\tau-1}(S^\nabla_\tau)\) or, equivalently, that there exists \(\nu \gg 0\) such that \(xF^m(S_{\nu,\tau}) \not\subset F^{m+r-1}(Z_\nu)\) for some \(m \geq 0\).
So, suppose that \( xF^m(S_{\nu,\tau}) \subseteq F^{m+r-1}(Z_{\nu}) \) for all \( \nu \geq \nu_0 \gg 0 \) and all \( m \geq 0 \). Then Hypotheses (H2) and (H3) imply that, inside \( R \), one has \( \sigma(x) \cdot \text{gr}(\bigoplus_{\nu \geq \nu_0} S_{\nu,\tau}) = 0 \) and hence that \( \sigma(x) \cdot R_{(\nu_0, \tau)} = 0 \). This contradicts the fact that, by Hypothesis (H2), \( R \) is a domain.

(2) By (1)

\[
(\mathcal{S}^\vee) = \bigoplus_{\mu \geq \lambda_5} \text{Hom}_{T,Qgr}(\pi_T(\bigoplus_{\gamma \geq \lambda_T} S_{\gamma,0} \otimes_{S_{0,0}} T_{\gamma,0}), \pi_T(\bigoplus_{\nu} S_{\nu,0} \otimes_{S_{0,0}} T_{\nu,0})).
\]

Now, as in (1), apply the equivalence of categories \( \Phi : T,Qgr \rightarrow T_{\lambda_5},\text{-Mod} \) defined by Proposition 2.9 and Remark 2.10 for the algebra \( T \).

\[ \square \]

5.8. Combining Lemma 5.7(1) with Hypothesis (H3) gives:

**Corollary.** Pick \( \tau \geq \lambda_5 \), and give \( \mathcal{S}^\vee \), the filtration \( F^* \) of Lemma 5.7(1). Then

\[
\text{gr}_F \mathcal{S}^\vee = \bigoplus_{\gamma \geq \lambda_T} \text{gr}_F(S_{\gamma,0} \otimes_{S_{0,0}} T_{\gamma,0}) = \bigoplus_{\gamma \geq \lambda_T} R_{\tau + \gamma}
\]

as objects in \( \Delta(R)\cdot\text{grmod} \).

In particular \( F^* \) is a good filtration on \( \mathcal{S}^\vee \) and \( \text{gr} \mathcal{S}^\vee = \pi_R(R[\tau]) = \mathcal{O}_X(\tau) \). \( \square \)

5.9. We next want to filter the \( T \)-module \( \mathcal{E} \mathcal{X}\mathcal{T}_S^p(M, X) \), and to relate its associated graded module to the sheaf Ext group \( \mathcal{E}xt^p_{\mathcal{X}}(M, \mathcal{O}_X) \subseteq \mathcal{Qcoh}(X) \). For this we introduce the following spectral sequence.

**Proposition.** There is a convergent spectral sequence in \( \Delta(R)\cdot\text{Qgr} = \mathcal{Qcoh}(X) \)

\[
(5.9.1) \quad \delta : E_1^p = \mathcal{E}xt^p_{\mathcal{X}}(M, \mathcal{O}_X) \Rightarrow \text{gr}_F \mathcal{E} \mathcal{X}\mathcal{T}_S^p(M, X),
\]

where the good filtration \( F^* \) of \( \mathcal{E} \mathcal{X}\mathcal{T}_S^p(M, X) \) is defined in the proof.

**Proof.** The proof will follow the argument in [Bj1, Chapter 2, Section 4]. We remark that Bjork’s result is phrased at the level of filtered abelian groups, and this is general enough to encompass much of the present proof.

Using Lemma 4.5 pick a filtered projective resolution \( P^* \rightarrow M \rightarrow 0 \), where each \( P^* = \bigoplus_j S_{\nu} \epsilon_j \) for some basis elements \( \epsilon_j \) and some \( \nu \geq \lambda_5 \). Here, the summands \( S_{\nu} \epsilon_j \) are filtered by giving \( \epsilon_j \) some degree \( k_{jr} \). Write

\[
(K^*, d) = (P^*) = \bigoplus_{\gamma \geq \lambda_T} \text{Hom}_{S,Qgr}(P^*, \pi_S(\bigoplus_{\nu \in \Lambda} S_{\nu,0} \otimes_{S_{0,0}} T_{\gamma,0}))
\]

for the induced complex with differential \( d \). By Lemma 5.5 this is a filtered complex in \( T\cdot\text{Grmod} \).

Let \( (K^*, d) \in T\cdot\text{Qgr} \) denote the image of \( (K^*, d) \) under \( \pi_T \). By Lemma 5.7(1), for each \( r \)

\[
(5.9.2) \quad K^r = \bigoplus_j S^\vee \epsilon^\vee_{jr} = \bigoplus_j \left( \bigoplus_{\gamma \geq \lambda_T} S_{\nu,0} \otimes_{S_{0,0}} T_{\gamma,0} \right) \epsilon^\vee_{jr}
\]

where \( \deg \epsilon^\vee_{jr} = -\deg \epsilon_{jr} \) and then the right hand side of (5.9.2) is given the (good) tensor product filtration \( F^* \).
In the notation of [Bj1] pp.48–51 our complex $K^* = \bigoplus K^p$ with filtration $F^j$ is Bjork’s abelian group $(A, d)$ with filtration $\Gamma_j$. For each $j \in \mathbb{N}$, set $Z^\infty_j = \text{Ker}(d) \cap F^j$ and notice that the group $H := \text{Ker}(d) / \text{Im}(d)$ is filtered by the $F^j(H) = Z^\infty_j + \text{Im}(d) / \text{Im}(d)$ with associated graded group $\text{gr} H = \bigoplus_j \left( Z^\infty_j + \text{Im}(d) \right) / \left( Z^\infty_{j-1} + \text{Im}(d) \right)$. By Corollary [4.3]

\begin{equation}
(5.9.3)
H = H^*(K^*) = \bigoplus_{p} \bigoplus_{\gamma \geq \lambda T} \Ext^p_{S \cdot \text{Qgr}}(M, \pi_S \left( \bigoplus_{\nu} S_{\nu,0} \otimes S_{0,0} T_{\gamma,0} \right)).
\end{equation}

Now consider the graded group $\text{gr} F K^* = \bigoplus F^j / F^{j-1}$ with the induced action $\text{gr} d$ of the differential $d$ and its cohomology group $H = \text{Ker}(\text{gr} d) / \text{Im}(\text{gr} d) = \bigoplus_p H^p(\text{gr} F K^*)$. By [Bj1] Chapter 2, Theorem 4.3 and (5.9.3), for any $p \geq 0$, we have a spectral sequence of graded $\Delta(R)$-modules:

$$S': E_1 = H^p(\text{gr} F K^*) \Rightarrow \text{gr} F \left( \bigoplus_{\gamma \geq \lambda T} \Ext^p_{S \cdot \text{Qgr}}(M, \pi_S \left( \bigoplus_{\nu} S_{\nu,0} \otimes S_{0,0} T_{\gamma,0} \right)) \right)$$

As we noted, $F^*$ is a good filtration on each $K^r$ and so, for each $p$, $F^q K^p = 0$ for $q \ll 0$. Therefore, by the proof of [Se2] Theorem, p.II.15, this sequence converges if for all $p$ there exists $s(p) \geq 0$ such that

\begin{equation}
(5.9.4)
F^q K^{p+1} \cap d(K^p) \subseteq d(F^{q+s(p)} K^p) \quad \text{for all } q \in \mathbb{Z}.
\end{equation}

Since $F^*$ is a good filtration on each $K^r$, both $\{G_j = F^j K^{p+1} \cap d(K^p)\}$ and $\{G'_j = d(F^j K^p)\}$ are good filtrations on $d(K^p)$. By [MR] Proposition 8.6.13 there therefore exists $s(p) \geq 0$ such that $G_q \subseteq G'_{q+s(p)}$ for all $q$. Thus (5.9.4) holds and $S'$ converges.

Applying the exact functor $\pi_{\Delta(R)}$ to $S'$ gives a necessarily convergent spectral sequence:

$$S: E_1^p = H^p(\pi_{\Delta(R)}(\text{gr} F K^*)) \Rightarrow \pi_{\Delta(R)} \left( \text{gr} F \left( \bigoplus_{\gamma \geq \lambda T} \Ext^p_{S \cdot \text{Qgr}}(M, \pi_S \left( \bigoplus_{\nu} S_{\nu,0} \otimes S_{0,0} T_{\gamma,0} \right)) \right) \right).$$

By definition, the right hand side of $S$ is $\text{gr} F(\mathcal{E}X^p_{\Delta(R)} \cdot S \cdot \text{Qgr}(M, \mathcal{X}))$ and so it remains to identify the left hand side. However, by construction, the filtered resolution $\mathcal{P}^* \to \text{gr} F \mathcal{M} \to 0$ of $\text{gr} F \mathcal{M}$ in $\Delta(R) \cdot \text{Qgr}$. Also, Hypothesis (II3) ensures that $\text{gr} F \mathcal{X}_\gamma$ equals $\mathcal{O}_X(\gamma)$ for $\gamma \gg 0$. Thus

$$\pi_{\Delta(R)}(\text{gr} F K^*) \cong \pi_{\Delta(R)} \left( \bigoplus_{\gamma \geq \lambda T} \text{gr} \psi \text{Hom}_{S \cdot \text{Qgr}}(\mathcal{P}^*, \mathcal{X}_\gamma) \right)$$

$$\cong \pi_{\Delta(R)} \left( \bigoplus_{\gamma \geq \lambda T} \text{Hom}_{\Delta(R) \cdot \text{Qgr}}(\text{gr} F \mathcal{P}^*, \text{gr} F \mathcal{X}_\gamma) \right) \quad \text{by Lemma 5.4}$$

$$\cong \pi_{\Delta(R)} \left( \bigoplus_{\gamma \geq \lambda T} \text{Hom}_{\Delta(R) \cdot \text{Qgr}}(\text{gr} F \mathcal{P}^*, \mathcal{O}_X(\gamma)) \right)$$

$$\cong \text{Hom}_X(\text{gr} F \mathcal{P}^*, \mathcal{O}_X).$$

Therefore $H^p(\pi_{\Delta(R)}(\text{gr} F K^*)) = \mathcal{E}xt^p_X(\text{gr} F \mathcal{M}, \mathcal{O}_X)$, as required. $\Box$
5.10. The following standard consequence of the spectral sequence \([5.9.1]\) will be particularly useful.

**Corollary.**

(1) For each \(p \geq 0\) and under the induced grading, the sheaf \(\text{gr} \, \mathcal{E} \mathcal{X} \mathcal{T}^p_{S\text{-}qgr}(\mathcal{M}, \mathcal{X}) \in \Delta(R)\text{-}qgr\) is a subquotient of \(\mathcal{E} \mathcal{X} \mathcal{T}^p_{X}(\text{gr}_F \mathcal{M}, \mathcal{O}_X)\).

(2) Let \(\mathcal{M}' \in \mathcal{T}\text{-}qgr\) with a good filtration \(F\). Then for each \(p \geq 0\) and under the induced grading, \(\text{gr} \, \mathcal{E} \mathcal{X} \mathcal{T}^p_{T\text{-}qgr}(\mathcal{M}', \mathcal{X}) \in \Delta(R)\text{-}qgr\) is a subquotient of \(\mathcal{E} \mathcal{X} \mathcal{T}^p_{X}(\text{gr}_F \mathcal{M}', \mathcal{O}_X)\). \(\square\)

**Proof.** Part (1) follows from the proposition combined with the observation that each term \(E^p_{r+1}\) is a subquotient of \(E^p_r\). As our hypotheses are symmetric in \(S\) and \(T\), this also proves (2). \(\square\)

5.11. **Remark.** It is well-known that filtrations that are not good have bad properties. This is especially true when filtering objects in \(S\text{-}qgr\), or even \(U\text{-}qgr\) for a graded unitary ring \(U\). For an extreme example, suppose that \(F^0 S = \bigoplus_{\tau} S_{\tau}\) and that \(0 \neq \mathcal{M} = \pi_S(M) \in S\text{-}qgr\). Now filter \(\mathcal{M}\) by \(F^0(M) = \mathcal{M}\). Then \(\text{gr}_F \mathcal{M}\) is killed by \(\text{gr}_{\geq 1} \Delta(R) \supseteq R_{\geq 1}\) and so \(\text{gr}_F \mathcal{M}\) is torsion. Thus, in the notation of \([2.3]\) \(\text{gr}_F \mathcal{M} = \pi_{\Delta(R)}(\text{gr}_F(M)) = 0\).

Fortunately for good filtrations this problem does not arise as we have

**Lemma.** Let \(\mathcal{M} \in S\text{-}qgr\) have a good filtration \((\mathcal{M}, F^*)\) and suppose that \(\text{gr}_F \mathcal{M} = 0\). Then \(\mathcal{M} = 0\).

**Proof.** Since \(F^*\) is a good filtration, \(\text{gr}_F(\mathcal{M})\) is finitely generated. Since \(\pi_{\Delta(R)} = 0\) this means that \(\text{gr}_F(\mathcal{M})\) is torsion and so \(\mathcal{M}\) is also torsion. Hence \(\mathcal{M} = 0\). \(\square\)

6. **Commutative theory.**

| Fix a finitely generated commutative \(\Lambda\)-graded algebra \(R = \bigoplus_{\lambda \in \Lambda} R_{\lambda}\) and lower \(\Lambda\)-directed algebras \(S\) and \(T\) that satisfy Hypotheses (H1–H3) from \([4.2]\). Set \(X = \text{Proj}(R)\). |

6.1. We prove results on the support of \(R\)-modules and more specifically the support of \(\text{gr} \mathcal{M}\) for \(\mathcal{M} \in S\text{-}qgr\). These provide analogues for projective varieties of results proven in \([L]\) for affine varieties and our proofs closely mimic Levasseur’s.

6.2. **Notation.** We will always identify \(X = \text{Proj}(R)\) with the set of graded prime ideals of \(R\) that contain no irrelevant ideal. Given \(p \in \text{Proj}(R)\), write \(\mathcal{L}_p\) for the homogeneous elements in \(R \setminus p\), with \(G(\Lambda)\)-graded localization \(R_p = R_{p}^{-1}\) and set \(\mathcal{L}_p\) to be the degree zero component \((R_p)_0\). Thus \(\mathcal{L}_p\) is the local ring of \(X\) corresponding to the complement of the subscheme \(V(p) \subset X\).

Similarly we write \(\mathcal{N}_p = (\mathcal{N} \otimes R_p)_0\) for \(\mathcal{N} \in R\text{-}Gmod\). By \([R]\) Propostions 8.2.2 and 9.1.2] there is an equivalence \(R_p\text{-}\mathcal{M} \overset{\sim}{\longrightarrow} R_p\text{-}Gmod\) given by \(M \mapsto R_p \otimes_{R_p} M\), with inverse \(L \mapsto L_0\).

Given \(\mathcal{L} \in \Delta(R)\text{-}qgr = \text{Coh}(X)\), its support \(\text{Supp} \, \mathcal{L}\) is defined to be its support in \(X\):

\[
\text{Supp} \, \mathcal{L} = \{ p \in \text{Proj}(R) : \mathcal{L}_p \neq 0 \}.
\]

By the multihomogeneous analogue of \([H]\) Chapter II, Proposition 5.11(a), if \(\mathcal{L} = \pi_R L\) for \(L \in R\text{-}gmod\), then \(\mathcal{L}_p = L_{(p)}\). Thus \(\text{Supp} \, \mathcal{L} = \text{Supp} \, L \cap \text{Proj}(R)\), where \(\text{Supp} \, L\) denotes the module-theoretic support of \(L\).
Combining these observations with the multi-graded analogue of [Hr] Proposition III.6.8 gives the following description of the sheaf Ext groups of $\mathcal{O}_X$-modules, denoted by $\mathcal{E}xt_X$.

**Lemma.** Let $p \in \text{Proj } R$ and $N \in R$-$\text{grmod}$ with corresponding coherent sheaf $\mathcal{N} = \pi_R N \in R$-$\text{qgr}$. Then $\text{Ext}^t_{R(p)}(N(p), R(p)) \cong \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)_p$ for any $t \geq 0$. 

6.3. By [Ba, Theorem (f)], and Lemma 6.2, $X$ is Gorenstein if and only if $\mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X) = 0$ for all $t > \dim X$ and $N \in \text{Coh}(X) = R$-$\text{qgr}$. Given a module $M$ over a ring $A$, we write $\ell_A(M)$ for the length of $M$.

**Lemma.** Assume that $X$ is Gorenstein. Let $N \in R$-$\text{grmod}$ with corresponding coherent sheaf $\mathcal{N} = \pi_R N$, and let $t \in \mathbb{N}$.

1. If $p \in \text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)$, then $\text{ht}(p) = \dim R(p) \geq t$.
2. Let $p$ be a minimal prime in $\text{Supp } N$. Then $\text{Ext}^t_{R(p)}(N(p), R(p)) = 0$ if $t \neq \dim R(p)$. If $t = \dim R(p)$ then $\text{Ext}^t_{R(p)}(N(p), R(p))$ has finite length, equal to $\ell_{R(p)}(N(p))$.
3. $\text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X) = \bigcup_{p \in \mathcal{N}} \mathcal{V}(p) \cup \mathcal{D}$, where the $\mathcal{V}(p)$ run through the irreducible components of $\text{Supp } \mathcal{N}$ of codimension $t$ in $X$ and $\mathcal{D}$ consists of a union of irreducible components of codimension different from $t$.

**Proof.** (1) If $p \in \text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)$, then $\text{Ext}^t_{R(p)}(N(p), R(p)) \neq 0$ by Lemma 6.2. Since $R(p)$ is Gorenstein, [Ba, Theorem] implies that $\text{ht}(p) = \text{injdim } R(p) = \dim R(p) \geq t$.

(2) If $\ell_{R(p)}(N(p)) = \infty$, then $N(q) \neq 0$ for some $q \subset p$, contradicting the minimality of $p$ in $\text{Supp } \mathcal{N}$. Hence $\ell_{R(p)}(N(p)) < \infty$. Set $k_p = R(p)/p(p)$. Then, as $R(p)$ is Gorenstein, [Ba, Proposition 2.9] implies that

$$\text{Ext}^t_{R(p)}(k_p, R(p)) = \begin{cases} 0 & \text{if } t \neq \text{ht}(p) \\ k_p & \text{if } t = \text{ht}(p). \end{cases}$$

Let $L$ be an $R(p)$-module of finite length. By induction on the length of $L$, it follows that $\text{Ext}^t_{R(p)}(L, R(p)) = 0$ if $t \neq \text{ht}(p)$, while $\ell_{R(p)}(\text{Ext}^t_{R(p)}(L, R(p))) = \ell_{R(p)}(L(p)) < \infty$. So (2) holds.

(3) If $\mathcal{V}(p)$ is an irreducible component of $\text{Supp } \mathcal{N}$, then $p$ is minimal in $\text{Supp } \mathcal{N}$. Thus if $t = \text{codim } \mathcal{V}(p) = \text{injdim } R(p)$, then part (2) and Lemma 6.2 show that $p \in \text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)$ and hence that $\mathcal{V}(p) \subseteq \text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)$. Conversely, if $q \in \text{Supp } \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)$, then $\text{ht}(q) \geq t$ by part (1). By Lemma 6.2, $0 \neq \mathcal{E}xt^t_X(\mathcal{N}, \mathcal{O}_X)_q \cong \text{Ext}^t_{R(q)}(N(q), R(q))$ and so $q \in \text{Supp } \mathcal{N}$. 

6.4. **Definition.** Let $M \in S$-$\text{grmod}$ and pick a good filtration $F^\bullet$ for $M$. Following [GS2, Definition 2.6], we define the characteristic variety of $M$ to be $\text{Char } M = \text{Supp}(\text{gr}_F M) \subseteq X$. The characteristic dimension of $M$ is defined to be $\text{chdim}(M) = \dim \text{Char } M$. If $M \in S_{X, \lambda}$-$\text{mod}$, for a good parameter $\lambda$, we write $\text{Char } (M) = \text{Char } (\Psi(M))$, in the notation of Proposition 2.9.

By the proof of [GS2, Lemma 2.5(3)], $\text{Char } M$ is independent of the choice of $F^\bullet$. Moreover, $\text{Char } N = \emptyset$ for any torsion module $N$ and so $\text{Char } M = \text{Char } M'$ whenever $\pi_S(M) = \pi_S(M')$. Thus $\text{Char } M$ and $\text{chdim}(M)$ are also well-defined for $M \in S$-$\text{qgr}$, or indeed for $M \in R$-$\text{qgr}$. 

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6.5. Recall the cohomology groups $\mathcal{E}_t X$ defined in (3.3.1) for objects in $S$-$\text{Qgr}$.

**Proposition.** Assume that $X$ is Gorenstein. Suppose that $M \in S$-$\text{Qgr}$ has a good filtration $F$ and give $\mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})$ the induced filtration $F$, as defined in Proposition 5.9. If $\mathcal{V}(p)$ is an irreducible component of $\text{Char} M$ of codimension $t$ in $X$, then

$$
\left(\text{gr}_F \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})\right)_p \cong \text{Ext}_X^i (\text{gr}_F (M), O_X)_p.
$$

**Proof.** We use the spectral sequence $S$ from Proposition 5.9. Here $E_1^t = \text{Ext}_X^t (\text{gr}_F (M), O_X)_p$, and by definition $E_1^t = H^t (E^*_t, d)$ for $t \geq 0$. We have $E_1^m = E_1^\infty = \text{Ext}_F S_{-\text{Qgr}}(M, \mathcal{X})$ for all $m \gg 0$. Clearly, $(E_1^t)_p = H^t ((E_1^*)_p)$ and so, by Lemma 6.2

$$(E_1^t)_p = \text{Ext}_X^t (\text{gr}_F (M), O_X)_p \cong \text{Ext}_{R(p)}^t ((\text{gr}_F (M))_p, R(p)).$$

Since $p$ is a minimal prime ideal in $\text{Char} M = \text{Supp}(\text{gr}_F (M))$, Lemma 6.3(2) implies that the only nonzero term in $(E_1^*)_p$ is $\text{Ext}_X^t ((\text{gr}_F (M))_p, R(p))$. Therefore, by recurrence, $(E_1^m)_p = (E_1^1)_p$ for all $m \geq 1$. By Proposition 5.9 this implies that

$$
\left(\text{gr}_F \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})\right)_p = (E_1^*)_p = (E_1^1)_p = \text{Ext}_X^t (\text{gr}_F (M), O_X)_p,
$$

as required. \qed

6.6. The main result of this section is the following analogue of [Lc Corollary 3.3.4].

**Corollary.** Assume that $X$ is Gorenstein. Let $M \in S$-$\text{Qgr}$ with a good filtration $F$ and pick $t \in \mathbb{N}$. Then

$$
\text{Char} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X}) = \bigcup_{i=1}^p \mathcal{V}(p_i) \cup \mathcal{D},
$$

where the $\mathcal{V}(p_i)$ are the irreducible components of $\text{Char} M$ of codimension $t$ in $X$ and $\mathcal{D}$ consists of a union of irreducible components of codimension $\geq t$ in $X$, each of which is contained in the union of the irreducible components of $\text{Char} M$ of codimension different from $t$.

**Proof.** By Corollary 5.10 and under the induced filtrations, we know that $\text{gr} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})$ is a subquotient of $\mathcal{E}_t^1 X (\text{gr}_F (M), O_X)_p$. Therefore,

$$
\text{Char} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X}) \subseteq \text{Supp} \mathcal{E}_t^1 X (\text{gr}_F (M), O_X)_p.
$$

Lemma 6.3(3) then implies that $\text{Char} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X}) \subseteq \bigcup_{i=1}^p \mathcal{V}(p_i) \cup \mathcal{D}$.

It remains to show that each irreducible component $\mathcal{V}(q)$ of $\text{Char} M$ of codimension $t$ actually appears in $\text{Char} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})$. However, by Lemma 6.3(3), such a $\mathcal{V}(q)$ is also an irreducible component of $\text{Supp} \mathcal{E}_t^1 X (\text{gr}_F (M), O_X)_p$. Therefore, by Proposition 6.5 $(\text{gr} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X}))_q \neq 0$, and so $\mathcal{V}(q)$ does indeed appear in $\text{Char} \mathcal{E}_t^i S_{-\text{Qgr}}(M, \mathcal{X})$. \qed
7. The double Ext spectral sequence.

Fix lower $\Lambda$-directed algebras $S$ and $T$ and a finitely generated commutative $\Lambda$-graded algebra $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$ that satisfy (H1–H3) from [4.2]. Set $X = \text{Proj}(R)$.

7.1. One of the most useful tools for applying homological techniques to enveloping algebras and other filtered algebras is the notion of an Auslander-Gorenstein ring and the related double Ext spectral sequence [Bj1, Chapter 2, Theorem 4.15]. As we prove in Theorem 7.5, these concepts have direct analogues for directed algebras.

7.2. Hypothesis. The following hypothesis will be in force for the rest of the section.

(H4) $X$ is Gorenstein, $\text{Spec } R_0$ is normal, and the canonical morphism $X \to \text{Spec } R_0$ is birational.

7.3. Lemma. Pick $\nu \in \Lambda$ such that $R_\nu \neq 0$. Then $R_0 = \text{End}_{R_0}(R_\nu)$.

Proof. By (H2), $R$ is a domain and so the field of rational functions $\mathbb{k}(X)$ equals $R[C^{-1}_R]_0$, where $C_R$ denotes the set of nonzero homogeneous elements of $R$. By (H4), $\mathbb{k}(X) = Q(R_0)$, the field of fractions of $R_0$. Therefore $R_\nu x^{-1} \subseteq \mathbb{k}(X)$ for any $0 \neq x \in R_\nu$ and hence $R_\nu x^{-1} y \subseteq R_0$ for some $0 \neq y \in R_0$.

Set $E = \text{End}_{R_0}(R_\nu)$. By the last paragraph, we may identify $E = \{ \theta \in \mathbb{k}(X) : \theta R_\nu \subseteq R_\nu \} \supseteq R_0$. For any $\omega \geq \nu$, (H2) implies that $R_\omega = R_\nu R_{\omega-\nu}$ and hence that $ER_\omega \subseteq R_\omega$. Therefore, if $I = R_{\geq \nu}$ then $E \subseteq F = \text{End}_R(I)$. Since $R$ is a $\Lambda$-graded domain, so is $F$. Moreover, $F$ is a finitely generated $R$-module since $F \cong Fx \subseteq R$ for any $0 \neq x \in R_\nu$. By the same observation, $R$ and $F$ have the same graded quotient ring $R[C^{-1}_R] = F[C^{-1}_F]$, and hence $F[C^{-1}_F]_0 = \mathbb{k}(X) = Q(R_0)$.

By restriction to degree zero, it follows that $F_0$ and hence $E$ are finitely generated as $R_0$-modules. Moreover,

$$Q(R_0) \subseteq Q(E) \subseteq Q(F_0) \subseteq F[C^{-1}_F]_0 = Q(R_0).$$

As $R_0$ is integrally closed by (H4), this implies that $R_0 = E$. \qed

7.4. The next result is fundamental to our approach, since it implies that $\mathcal{E}\mathcal{T}^*(-, \mathcal{X})$ for directed algebras plays the rôle of $\text{Ext}^*(-, R)$ for a unital algebra $R$ and it is this that allows us to mimic the homological approach of Gabber [Le]. Recall the definition of acyclic objects in $S\text{-qgr}$ from Definition 3.4.

Theorem. Let $\nu \geq \lambda_S$. Then

1. $S^\nu_\nu$ is an acyclic sheaf in $T\text{-qgr}$, and
2. $S^\nu_\nu \cong S_\nu$ as objects in $S\text{-qgr}$.

Proof. (1) Fix $j > 0$. By Corollary 5.8, $\text{gr}_F S^\nu_\nu = O_X(\nu)$ is a vector bundle on $X$ and so certainly $\mathcal{E}\text{xt}_X^j(\text{gr}_F S^\nu_\nu, O_X) = 0$. Thus, by Corollary 5.10(2), $\text{gr}_F \mathcal{E}\mathcal{T}^j_T(\text{qgr})(S^\nu_\nu, \mathcal{X}) = 0$, where $F$ is the good filtration defined by Proposition 5.9. Therefore $\mathcal{E}\mathcal{T}^j_T(\text{qgr})(S^\nu_\nu, \mathcal{X}) = 0$ by Lemma 5.11.
(2) By Lemma 5.4(2),
\[(S^\gamma_\phi)^{\vee} \cong \bigoplus_{\gamma \geq \lambda_S} \text{Hom}_{T_{\phi,0}}(S_{\gamma,0} \otimes S_{0,0}, T_{\phi,0}, S_{\gamma,0} \otimes S_{0,0}, T_{\phi,0})\]
for any \(\phi \geq \lambda_T = \lambda_S\). Thus for \(\gamma \geq \nu\), left multiplication by \(S_{\gamma,\nu}\) induces a homomorphism
\[(7.4.1) \quad (S_\nu)_\gamma = S_{\gamma,\nu} \rightarrow \text{Hom}_{T_{\phi,0}}(S_{\nu,0} \otimes S_{0,0}, T_{\phi,0}, S_{\gamma,0} \otimes S_{0,0}, T_{\phi,0}) = (S^\gamma_\nu)^{\vee}.
\]
This produces a homomorphism \(\Theta : S_\nu \rightarrow \pi_S((S^\gamma_\nu)^{\vee}) = S^\vee_\nu\) in \(S\cdot\text{Qgr}\). In order to prove that \(\Theta\) is an isomorphism, it suffices to prove the same for \((7.4.1)\).

Set \(Z = S_{\nu,0} \otimes S_{0,0}, T_{\phi,0}\). Then Definition 2.8(3) implies that \((7.4.1)\) is the multiplication map
\[(7.4.2) \quad S_{\gamma,\nu} \rightarrow \text{Hom}_{T_{\phi,0}}(Z, S_{\gamma,\nu} \otimes S_{0,0}, T_{\phi,0}, Z).
\]
As \(S_{\gamma,\nu}\) is a projective right \(S_{\nu,\nu}\)-module, we can write \(S_{\gamma,\nu} \otimes P \cong S_{\nu,\nu}^{(m)}\) and so, in order to prove that \((7.4.2)\) is bijective it suffices to prove that the multiplication map \(S_{\nu,\nu}^{(m)} \rightarrow \text{Hom}_{T_{\phi,0}}(Z, S_{\nu,\nu}^{(m)} \otimes S_{\nu,\nu}, Z)\) is bijective. Equivalently, it suffices to prove that the action map \(\theta : S_{\nu,\nu} \rightarrow \text{Hom}_{T_{\phi,0}}(Z, Z)\) is bijective. Since multiplication by \(S_{\nu,\nu}\) preserves the tensor product filtration on \(Z = S_{\nu,0} \otimes S_{0,0}, T_{\phi,0}\), the map \(\theta\) is filtered and hence induces an associated graded morphism
\[(7.4.3) \quad \chi : \text{gr} S_{\nu,\nu} \xrightarrow{\text{gr} \theta} \text{gr} \text{Hom}_{T_{\phi,0}}(Z, Z) \hookrightarrow \text{Hom}_{\text{gr} T_{\phi,0}}(\text{gr}(Z), \text{gr}(Z)),\]
where the final inclusion follows from the observations in Subsection 5.4. Here \(\chi\) is again the natural multiplication map.

Finally, Hypothesis (H3) implies that \(\text{gr}(Z) = R_{\gamma+\phi}\) and so, by Lemma 7.3, \(\chi\) is an isomorphism for \(\gamma \gg 0\). It follows that \(\theta\) and hence \((7.4.1)\) are also isomorphisms, as required. \(\square\)

7.5. We are now ready to state and prove the second spectral sequence; in essence it shows that the Auslander-Gorenstein condition and related double Ext spectral sequence, that are so useful for unitary algebras, have a natural analogue for directed algebras. We remark that, as in [BJ2, Section I.1], we prove the result for rings of finite injective dimension. However, it will often be more convenient to mimic the argument of [BJ1 Chapter 2, Theorem 4.15] which unnecessarily assumes finite global dimension.

For this result, define
\[\mathcal{E}^{p,q}(\mathcal{M}) = \mathcal{E}X_T^{p,q}(\mathcal{E}X_T^{q,p}(\mathcal{M}, \mathcal{X}), \mathcal{X}), \quad \text{for all } p, q \geq 0.\]

**Theorem.** Let \(\mathcal{M} \in S\cdot\text{qgr}\) and set \(d = \dim X\). Then

1. For all \(v < j\), and all subobjects \(\mathcal{N} \subseteq \mathcal{E}X_T^{j,p}(\mathcal{M}, \mathcal{X})\) one has \(\mathcal{E}X_T^{j,p}(\mathcal{N}, \mathcal{X}) = 0\).
2. If \(\mathcal{P} \subseteq U\cdot\text{qgr}\), for \(U = S\) or \(T\), then \(\mathcal{E}X_T^{j,p}(\mathcal{P}, \mathcal{X}) = 0\) for \(j > \dim X\).
3. There is a spectral sequence
\[E_2^{p,q} = \mathcal{E}^{p,q}(\mathcal{M}) \implies \mathbb{H}^{p-q}(\mathcal{M}),\]
where \(\mathbb{H}^{p-q}(\mathcal{M}) = \mathcal{M}\) if \(p = q\) and \(\mathbb{H}^{p-q}(\mathcal{M}) = 0\) otherwise.
(4) There exists a filtration $0 = \mathcal{R}_{-1}(\mathcal{M}) \subseteq \mathcal{R}_0(\mathcal{M}) \subseteq \mathcal{R}_1(\mathcal{M}) \subseteq \cdots \subseteq \mathcal{R}_d(\mathcal{M}) = \mathcal{M}$ whose sections $\mathcal{M}_v = \mathcal{R}_v(\mathcal{M})/\mathcal{R}_{v-1}(\mathcal{M})$ appear in exact sequences

$$0 \to \mathcal{M}_v \to \mathcal{E}^{d-v,d-v}(\mathcal{M}) \to \mathcal{W}_v \to 0.$$ 

The cokernel $\mathcal{W}_v$ is isomorphic to a subfactor of the direct sum of double Ext groups of the form $\mathcal{E}^{a,b}(\mathcal{M})$ for $(a, b) = (d - v + j + 2, d - v + j + 1)$ and $0 \leq j \leq v - 2$.

Proof. We need two preliminary observations. Let $\mathcal{P} \in S$-qgr. First, by Corollary 6.6 and in the notation of Definition 6.4

$$\text{chdim} \left( \mathcal{E}^{\mathcal{X}\mathcal{T}^i_{S\text{-qgr}}}(\mathcal{P}, \mathcal{X}) \right) \leq d - j \quad \text{for all } 0 \leq j \leq d. \tag{7.5.1}$$

Next, Lemma 6.2 and [Ba, Proposition 2.9] implies that, for any $p \in \text{Proj}(R)$,

$$\mathcal{E}t^{d}_{\mathcal{X}}((\mathfrak{g} \mathfrak{f} \mathfrak{F} \mathcal{P}, \mathcal{O}_\mathcal{X}) = 0 \quad \text{if } v < d - \text{chdim}(\mathcal{P}).$$

This implies that $\mathcal{E}t^{d}_{\mathcal{X}}((\mathfrak{g} \mathfrak{f} \mathfrak{F} \mathcal{P}, \mathcal{O}_\mathcal{X}) = 0$ for $v < d - \text{chdim}(\mathcal{P})$. Thus, by Corollary 5.10 and Lemma 5.11

$$\mathcal{E}^{\mathcal{X}\mathcal{T}^v_{S\text{-qgr}}}(\mathcal{P}, \mathcal{X}) = 0 \quad \text{if } v < d - \text{chdim}(\mathcal{P}). \tag{7.5.2}$$

By symmetry, these two equations also hold for $\mathcal{P} \in T$-qgr. We now turn to the proof of the theorem.

(1) Applying (7.5.1) to $\mathcal{P} = \mathcal{M}$ shows that $\alpha = \text{chdim} N \leq \text{chdim} \mathcal{E}^{\mathcal{X}\mathcal{T}^j_{S\text{-qgr}}}(\mathcal{M}, \mathcal{X}) \leq d - j$. Thus if $\mathcal{E}^{\mathcal{X}\mathcal{T}^v_{S\text{-qgr}}}(\mathcal{N}, \mathcal{X}) \neq 0$ then the analogue of (7.5.2) for $\mathcal{N} \in T$-qgr shows that $v \geq d - \alpha \geq j$, as required.

(2) Since injdim $R = d$ by (H4), this is immediate from Corollary 5.10

(3) By Lemma 6.5 and the comments in Subsection 4.3 we may form a finitely generated projective resolution $\cdots \to \mathcal{P}_d \to \cdots \to \mathcal{P}_0 \to \mathcal{M} \to 0$ in $S$-qgr. Now form the complex

$$\mathcal{P}^\gamma: \quad 0 \to \mathcal{P}_0^\gamma \to \mathcal{P}_1^\gamma \to \cdots \to \mathcal{P}_d^\gamma \to \cdots,$$

where $\mathcal{P}_r^\gamma = \mathcal{HOM}_{S\text{-qgr}}(\mathcal{P}_r, \mathcal{X})$.

By construction, there exists some fixed $\nu \gg 0$ such that the $\mathcal{P}_r$ are direct sums of shifts of $\mathcal{S}_\nu$ and so, by Theorem 7.3(1), each $\mathcal{P}_r$ is an acyclic sheaf in $T$-qgr. Moreover, by Lemma 5.7(1), each $\mathcal{P}_r^\gamma \in T$-qgr. We claim that we can form a double complex

$$\begin{array}{ccccccc}
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\end{array}$$

$$\begin{array}{ccccccc}
\mathcal{Q}_0 \to & \mathcal{Q}_1 \to & \cdots & \to & \mathcal{Q}_d \to & \cdots \\
\downarrow & & & & & & \downarrow \\
\mathcal{P}_0^\gamma \to & \mathcal{P}_1^\gamma \to & \cdots & \to & \mathcal{P}_d^\gamma \to & \cdots \\
\downarrow & & & & & & \downarrow \\
0 & & & & & & 0 \\
\end{array} \tag{7.5.3}$$

in $T$-qgr satisfying the following conditions:
• Each $Q_{ij}$ is a projective object in $T$-$qgr$ and each column is exact;
• the cohomology groups $\{H_{ij}\}$ of the row complexes $0 \to Q_{i0} \to Q_{i1} \to \cdots \to Q_{id} \to \cdots$ are projective;
• for each $j$, the complex $\cdots H_{2j} \to H_{1j} \to H_{0j} \to EAXT_{S,qgr}^j(M, X)$ is a projective resolution in $T$-$qgr$ of $EAXT_{S,qgr}^j(M, X)$.

The proof of this assertion is almost immediate. As is proved in [Bj1, pp.58-9], starting with any complex of finitely generated modules over a noetherian ring in place of $P^\bullet$, then such a double complex exists. So it exists in $T_\gamma,\gamma$-$mod$ for $\gamma \gg 0$. But $T_\gamma,\gamma$-$mod \simeq T$-$qgr$ by Proposition 2.9 applied to $T$. So use this equivalence to translate the given complex $P^\bullet$ in $T_\gamma,\gamma$-$mod$ to a complex $C^\bullet$ in $T$-$qgr$ and apply the above construction.

Now consider the double complex $\{Q_{jv}^\vee = \text{HOM}_{T$-$qgr}(Q_{jv}, X)\}$ of modules in $S$-$Qgr$. We remark that the rest of the proof closely follows that of Bjork ([Bj1, pp.60–62] or [Bj2, pp.62–64]). We will cite both books since, although the former assumes finite global dimension, the arguments given there are closer to the ones we need.

**Sublemma.** The hypercohomology groups in the double complex $\{Q_{jv}^\vee\}$ vanish everywhere except on the $d$th diagonal, where the cohomology group is $H^d \cong M$ as objects in $S$-$Qgr$.

**Proof.** Consider (7.5.3). By Theorem 7.4 and construction, the $P_{jv}^\vee$ and $Q_{uv}$ are acyclic sheaves, while the columns of (7.5.3) are exact. Thus the dual complex $0 \to P_{jv}^\vee \to Q_{jv}^\vee \to Q_{1j}^\vee \to \cdots$ will be exact in $S$-$Qgr$. Moreover, by Theorem 7.4 (2), $P_{jv}^\vee \cong P_j$. Up to a change of notation, the proof of [Bj2, Proposition 1.2] may now be used without further change to prove the sublemma. □

The remainder of proof of [Bj1, Chapter 2, Theorem 4.15], from the sublemma on page 60 through to its conclusion on page 61, may now be used to prove the rest of parts (2) and (3) of the present theorem; the only changes being to substitute projective module by acyclic sheaf, $(\cdots)^*\vee$ by $(-)^\vee$, and hence Ext$(\cdots, A)$ by $EAXT(\cdots, X)$. The proof only uses the fact that $A$ has finite injective dimension rather than finite global dimension, and the relevant analogue of that assertion is given by part (2) of this theorem. □

7.6. By combining Theorem 7.5 (3) with (7.5.1) we obtain

**Corollary.** For $M \in S$-$qgr$ and $0 \leq v \leq d$, define $M_v$, $R_v(M)$ and $W_v$ as in Theorem 7.5. Then

1. $\text{chdim}(M_v) \leq v$.
2. $\text{chdim}(W_v) \leq d - (d - v + 2) = v - 2$.
3. $\text{chdim}(R_v(M)) \leq v$ for all $0 \leq v \leq d$. □
8. Equidimensionality.

Fix lower Λ-directed algebras $S$ and $T$ with $T_{00}^{\text{op}} = S_{00}$, and a finitely generated commutative Λ-graded algebra $R = \bigoplus_{\lambda \in \Lambda} R_{\lambda}$. We assume that these satisfy hypotheses (H1–H4) from 4.2 and 7.2. Write $X = \text{Proj}(R)$, and set $\dim X = d$.

8.1. We are now ready to prove the main result of the paper: the characteristic variety $\text{Char} \mathcal{M}$ of a simple object $\mathcal{M} \in S_{-\text{qgr}}$ is equidimensional. The proof also works for the following more general modules.

An object $\mathcal{M} \in S_{-\text{qgr}}$ will be called $s$-homogeneous if each nonzero subobject $N \subseteq \mathcal{M}$ satisfies $\text{chdim}(N) = s$, in the sense of Definition 6.4. It is not clear how to characterise $s$-homogeneous objects, but at least if $\mathcal{M}$ is simple then $\mathcal{M}$ is $s$-homogeneous for some $s$. Here is another example, the proof of which is left to the reader: if $P$ is a prime ideal of $S_{\lambda, \lambda}$ for a good parameter $\lambda$, then the image $\Psi(S_{\lambda, \lambda}/P) \in S_{-\text{qgr}}$ is $s$-homogeneous for some $s$.

8.2. Definition. If $\mathcal{M} \in S_{-\text{qgr}}$, define the grade $j(\mathcal{M})$ of $\mathcal{M}$ to be

$$j(\mathcal{M}) = \min\{j : \mathcal{E} \mathcal{A}^j T_{S_{-\text{qgr}}}(\mathcal{M}, \mathcal{X}) \neq 0\}.$$ 

Recall the notation $\mathcal{E}^{p,q}(\mathcal{M}) = \mathcal{E} \mathcal{A}^p T_{-\text{qgr}}(\mathcal{E} \mathcal{A}^q S_{-\text{qgr}}(\mathcal{M}, \mathcal{X}'), \mathcal{X}')$ from 7.5.

**Proposition.** Let $\mathcal{M} \in S_{-\text{qgr}}$.

(1) $j(\mathcal{M}) + \text{chdim}(\mathcal{M}) = d$.

(2) Let $p \in \mathbb{N}$. Then $\mathcal{E}^{p,p}(\mathcal{M})$ is either zero or $(d-p)$-homogeneous.

(3) If $\text{chdim}(\mathcal{M}) \leq d - v$ for some $v \in \mathbb{N}$, then $\text{chdim}(\mathcal{E}^{j,v}(\mathcal{M})) \leq d - j - 2$ for all $j > v$.

**Proof.** (1) By (7.5.2) we know that $j(\mathcal{M}) \geq d - \text{chdim}(\mathcal{M})$, so we only need to prove the opposite inequality. Suppose that $v > d - j(\mathcal{M})$. Then $d - v < j(\mathcal{M})$ and $\mathcal{E} \mathcal{A}^j T_{S_{-\text{qgr}}}(\mathcal{M}, \mathcal{X}) = 0$. Therefore, $0 = \mathcal{E}^{d,v,d-v}(\mathcal{M}) = \mathcal{M}_v$. This implies that $\mathcal{M} = R_{d-j(\mathcal{M})}(\mathcal{M})$ and so Corollary 7.6 gives $\text{chdim}(\mathcal{M}) \leq d - j(\mathcal{M})$.

(2) This is a formal consequence of the spectral sequence Theorem 7.5.2). Modulo replacing $\text{Ext}^i(-, A)$ by $\mathcal{E} \mathcal{A}^i T^i(-, \mathcal{X})$, the proof of [Bj2] Proposition 1.18 and Corollary 1.20, or indeed of [Bj1] Chapter 2, Theorem 7.10, can be used mutatis mutandis.

(3) This result, which will not be needed in this paper, is again proved by investigating the spectral sequence from Theorem 7.5.2. Modulo replacing $\text{Ext}^i(-, A)$ by $\mathcal{E} \mathcal{A}^i T^i(-, \mathcal{X})$, the proof of [Bj1] Chapter 2, Lemma 7.11] can be used verbatim. □

8.3. At last, we reach our destination. The proof of this result follows that of [Le Théorème 3.3.2].

**Theorem.** Suppose that $\mathcal{M} \in S_{-\text{qgr}}$ is $s$-homogeneous for some $s \in \mathbb{N}$. Then each irreducible component of $\text{Char} \mathcal{M}$ has dimension $s$. 25
Proof. Keep the notation from Theorem 7.5 and let \( p \in \mathbb{Z} \). If \( \mathcal{E}^{p,p}(\mathcal{M}) = 0 \), then \( \mathcal{M}_{d-p} = 0 \) by Theorem 7.5(3). Conversely, if \( \mathcal{M}_{d-p} = 0 \) then \( \mathcal{E}^{p,p}(\mathcal{M}) = W_{d-p} \) and so, by Corollary 7.6(2),
\[
\text{chdim}(\mathcal{E}^{p,p}(\mathcal{M})) = \text{chdim}(W_{d-p}) \leq d - p - 2.
\]
By Proposition 8.2(2), this implies that \( \mathcal{E}^{p,p}(\mathcal{M}) = 0 \).

Suppose that there exists an irreducible component \( \mathcal{V}(\mathcal{p}) \) of \( \mathbf{Char} \mathcal{M} \) with \( \dim \mathcal{V}(\mathcal{p}) < s \). Let \( \mathcal{E} = \mathcal{E}X^{p}T_{\mathcal{S} \text{-qgr}}(\mathcal{M}, \mathcal{X}) \in T\text{-qgr} \). By Corollary 6.6 \( \mathcal{V}(\mathcal{p}) \) is an irreducible component of \( \mathbf{Char} \mathcal{E} \). Applying Corollary 6.6 again, but with \( \mathcal{M} \) replaced by \( \mathcal{E} \), shows that \( \mathcal{V}(\mathcal{p}) \) is also an irreducible component of \( \mathcal{E}X^{p}T_{\mathcal{S} \text{-qgr}}(\mathcal{E}, \mathcal{X}) = \mathcal{E}^{p,p}(\mathcal{M}) \). In particular, \( \mathcal{E}^{p,p}(\mathcal{M}) \neq 0 \). By the first paragraph of this proof, it follows that \( \mathcal{M}_{d-t} \neq 0 \) and hence that \( \mathcal{R}_{d-t}(\mathcal{M}) \neq 0 \). By Corollary 7.6 \( \text{chdim} \mathcal{R}_{d-t}(\mathcal{M}) \leq d - t < s \). This contradicts the fact that \( \mathcal{M} \) is \( s \)-homogeneous. Hence each component of \( \mathbf{Char} \mathcal{M} \) has dimension at least \( s \).

Conversely since \( \text{chdim}(\mathcal{M}) = s \) each component of \( \mathbf{Char} \mathcal{M} \) has dimension at most \( s \). This completes the theorem. \( \square \)

9. APPLICATIONS TO QUANTIZATIONS OF SYMPLECTIC SINGULARITIES

9.1. We end by showing that many important examples of \( \mathbb{Z} \)-algebras arising from geometric representation theory satisfy Hypotheses (H1–H4). As an application, we answer a question from [GS2] on rational Cherednik algebras.

9.2. We start by recalling results presented in [BPW]. Let \( Y \) be an affine irreducible symplectic singularity with a \( \mathbb{C}^* \)-action and \( \pi : X \to Y \) a \( \mathbb{C}^* \)-equivariant symplectic resolution of singularities. We assume first that \( \mathbb{C}^* \) acts on \( \mathbb{C}[Y] \) with non-negative weights and such that \( \mathbb{C}[Y]^{\mathbb{C}^*} = \mathbb{C} \), and second that there exists \( 0 < m \in \mathbb{N} \) such that the \( \mathbb{C}^* \)-action scales the given symplectic form \( \omega \) by \( t \cdot \omega = t^m \omega \) for all \( t \in \mathbb{C}^* \). This is called a conical symplectic resolution. As we saw in the introduction, there are many interesting examples of these varieties.

9.3. Given \( \lambda \in H^2(X, \mathbb{C}) \) there is a unique sheaf of \( \mathbb{C}^* \)-equivariant complete \( \mathbb{C}[[x]] \)-algebras \( Q^\lambda_X \) on \( X \) with the following properties:

1. \( Q^\lambda_X/h Q^\lambda_X \cong \mathcal{O}_X \);
2. the Poisson bracket on \( X \) induced by \( \omega \) equals the Poisson bracket defined by \( \{ f_1, f_2 \} \equiv h^{-1} [\hat{f}_1, \hat{f}_2] \mod h \), where \( \hat{f}_1, \hat{f}_2 \) are lifts to \( Q^\lambda_X \) of \( f_1, f_2 \in \mathcal{O}_X \);
3. \( t \cdot h = t^{-m} h \).

Let \( D^\lambda_X = Q^\lambda_X[[h^{-1/m}]] \), a sheaf of \( \mathbb{C}((h^{1/m})) \)-algebras on \( X \). Write \( D^\lambda_X \text{-mod} \) for the full subcategory of \( \mathbb{C}^* \)-equivariant \( D^\lambda_X \)-modules whose objects \( \mathcal{M} \) admit a \( \mathbb{C}^* \)-equivariant \( Q^\lambda_X \)-lattice \( \mathcal{M}(0) \) such that \( \mathcal{M}(0)/h \mathcal{M}(0) \) is a coherent \( \mathcal{O}_X \)-module. [BPW, § 4]. Setting \( U_\lambda = (D^\lambda_X(X))^{\mathbb{C}^*} \), there is a functor \( \Gamma^{\mathbb{C}^*} : D^\lambda_X \text{-mod} \to U_\lambda \text{-mod} \) which takes \( \mathcal{M} \) to the \( \mathbb{C}^* \)-invariant global sections \( \Gamma(X, \mathcal{M})^{\mathbb{C}^*} \). This has a left adjoint \( \text{Loc} : U_\lambda \text{-mod} \to D^\lambda_X \text{-mod} \) which sends \( N \) to \( D^\lambda_X \otimes_{U_\lambda} N \). The pair \( (X, \lambda) \) is called localizing if \( (\Gamma^{\mathbb{C}^*}, \text{Loc}) \) are mutually inverse equivalences.
9.4. There is an \( \mathbb{N} \)-algebra attached to \((X, \lambda)\) and the choice of a very ample line bundle \( \mathcal{L} \) on \( X \). To construct it, let \( \eta \in H^2(X, \mathbb{C}) \) be the first Chern class of \( \mathcal{L} \). Then, for any pair of integers \( k, m \), [BPW] Proposition 5.2 shows that there is a unique sheaf of \( \mathbb{C}^* \)-equivariant \( (\mathcal{Q}_X^{\lambda+k\eta}, \mathcal{Q}_X^{\lambda+m\eta}) \)-bimodules \( k\eta \mathcal{B}_{m\eta}(\lambda) \) which quantizes the line bundle \( \mathcal{L}^{k-m} \). Set
\[
 k\eta \mathcal{B}_{m\eta}(\lambda) = \Gamma^{\mathbb{C}^*}(k\eta \mathcal{B}_{m\eta}(\lambda)[h^{-1/m}]) \in (U_{\lambda+k\eta}, U_{\lambda+m\eta})\text{-bimod}.
\]
Then the desired \( \mathbb{N} \)-algebra is
\[
 S(\lambda, \eta) = \bigoplus_{k \geq m \geq 0} k\eta \mathcal{B}_{m\eta}(\lambda),
\]
with multiplication induced from tensor products (see [BPW] Definition 5.5).

9.5. For each \( p \in \mathbb{N} \), the following new \( \mathbb{N} \)-algebras can be constructed from \( S \):
\[
 S^{(p)}(\lambda, \eta) = \bigoplus_{k \geq m \geq 0} p k\eta \mathcal{B}_{m\eta}(\lambda) \quad \text{and} \quad S_{\geq r}(\lambda, \eta) = S(\lambda, \eta)_{\geq r} = \bigoplus_{k \geq m \geq r} k\eta \mathcal{B}_{m\eta}(\lambda).
\]
The former of these algebras is called the \( p^{\text{th}} \) Veronese ring of \( S(\lambda, \eta) \).

**Proposition.** ([BPW] Propositions 5.10 and 5.14) Let \( \lambda \in H^2(X, \mathbb{C}) \) and let \( \eta \in H^2(X, \mathbb{C}) \) be the first Chern class of a relatively very ample line bundle on \( X \).

1. There exists \( q \gg 0 \) such that \( S_{\geq r}(\lambda, \eta) \) is Morita for all \( r \geq q \).
2. \((X, \lambda)\) is a good parameter if and only if \( S^{(p)}(\lambda, \eta) \) is Morita for \( p \gg 0 \).

9.6. The algebras constructed by Proposition 9.5 satisfy the Hypotheses (H1–H4) from Subsections 4.2 and 7.2.

**Proposition.** Let \( \lambda \in H^2(X, \mathbb{C}) \) and let \( \eta \in H^2(X, \mathbb{C}) \) be the first Chern class of a relatively very ample line bundle on \( X \). Assume that \((X, \lambda)\) is localizing. Then Hypotheses (H1–H4) hold for a pair of directed algebras \((S^{(p)}(\lambda, \eta), T)\), where the integer \( p \) and algebra \( T \) are constructed within the proof.

**Proof.** By Proposition 9.5 we can find \( p \gg 0 \) such that \( S^{(p)}(\lambda, \eta) \) is Morita. Now consider \( (\mathcal{Q}_X^{\lambda})^{op} \), another \( \mathbb{C}^* \)-equivariant quantization of \( \mathcal{O}_X \). By [BPW] Proposition 3.2 it is isomorphic to \( \mathcal{Q}_X^{-\lambda} \). Therefore, \( U_{\lambda}^{op} \cong U_{-\lambda} \), and so repeating the above construction, with the same line bundle \( \mathcal{L} \), we produce another \( \mathbb{Z} \)-algebra \( S(-\lambda, \eta) \). It may be that \((X, -\lambda)\) is not localizing. By Proposition 9.5 however, we do know that for \( q \gg 0 \) the \( \mathbb{N} \)-algebra
\[
 S_{\geq q}(-\lambda, \eta) = \bigoplus_{k \geq m \geq q} k\eta \mathcal{B}_{m\eta}(-\lambda)
\]
is Morita.

So we will consider the following algebras, where \( r \) is the least common multiple of \( p \) and \( q \).
\[
 S = \bigoplus_{a \geq b \geq 0} ra\eta \mathcal{B}_{rb\eta}(\lambda, \eta), \quad T = \bigoplus_{a \geq b \geq 0} ra\eta \mathcal{B}_{rb\eta}(-\lambda, \eta).
\]

(H1) By construction, \( S \) is Morita and so all values are good for it, while \( T_{\geq 1} \) is Morita.
(H2) As observed in [BPW] Section 5.2, \( \text{gr} S = \text{gr} T = \Delta(R) \) where \( R_a = H^0(X, L^{\otimes a}) \). Now it may be that \( R_m R_n \neq R_{m+n} \). But \( X \) is smooth, hence normal, and so this property will hold if we replace \( R \) by some further Veronese subring. Also, since \( X \) is reduced and irreducible, certainly \( R \) is a domain. So, after passing to the appropriate Veronese ring, (H2) holds.

(H3) We must prove that \( \text{gr}(S_{m0} \otimes S_{n0} T_{n0}) \cong R_{m+n} \) for all \( m, n \). By Lemma 7.3 each \( R_m \) is a torsion-free rank one \( R_0 \)-module. Since \( r_m S_0(\lambda) = \text{gr} r_m S_0(-\lambda) = R_m \), it follows that both \( r_m S_0(\lambda) \) and \( r_m S_0(-\lambda) \) are torsion-free of rank one on both sides. Since \( S \) is Morita, certainly \( S_{m0} = r_m S_0(\lambda) \) is a projective \( 0 S_0(\lambda) \)-module, so the tensor product \( r_m S_0(\lambda) \otimes_{0 S_0(\lambda)} r_m S_0(-\lambda) \) is therefore torsion-free of rank one.

As observed earlier, \( Q_X^N \cong (Q_X^N)^{op} \). Thus the \( (Q_X^N)^{-1+r_m N}, Q_X^N \)-bimodule \( r_m B_0(-\lambda) \) may be considered as a \( (Q_X^N, Q_X^N)^{-r_m N}) \)-bimodule quantizing \( L^{\otimes r_m N} \), and as such must be isomorphic to \( 0 B_0(-r_m N) \) by the uniqueness of quantization property mentioned in 9.4. Hence, by [BPW] Proposition 5.2, multiplication produces an isomorphism of \( (Q_X^N, Q_X^N)^{-r_m N}) \)-bimodules

\[
\text{gr} S_{m0} \otimes_{S_{n0}} T_{n0} = r_m S_0(\lambda) \otimes_{0 S_0(\lambda)} 0 B_0(-r_m N) \xrightarrow{\sim} r_m S_0(-\lambda).
\]

Taking invariant global sections then produces a non-zero multiplication map

\[
\mu : S_{m0} \otimes_{S_{n0}} T_{n0} \to r_m S_0(-\lambda).
\]

By torsion-freeness, \( \mu \) must be injective. But \( \mu \) is also filtered surjective because

\[
\text{gr} S_{m0} \cdot \text{gr} T_{n0} = R_m \cdot R_n = R_{m+n} = \text{gr} r_m S_0(-r_m N).
\]

Thus \( \text{gr}(S_{m0} \otimes S_{n0} T_{n0}) = R_{m+n} \), as required.

(H4) Symplectic singularities are automatically normal and so (H4) follows from the fact that \( X \to Y \) is a resolution of singularities.

9.7. Given a \( U_\lambda \)-module \( N \), set \( N = \text{Loc}(N) \). We define \( \text{Char}_{\text{loc}}(N) \) to be the support on \( X \) of the coherent sheaf \( Y := N(0)/h^{1/m} N(0) \) where \( N(0) \) is a lattice for \( N \). This is well-defined, independent of the choice of lattice.

We claim that \( \text{Char}_{\text{loc}}(N) = \text{Char}(\Psi(N)) \), where \( \text{Char}(\Psi(N)) \) is defined using the \( \mathbb{Z} \)-algebra in Proposition 9.6 as in Definition 6.4. Indeed the discussion preceding Theorem 5.6 in [BPW] together with the first two paragraphs of the proof of [BPW] Proposition 5.9 show that there is an isomorphism

\[
\bigoplus_{a \geq 0} \Gamma^C(r_m B_0(\lambda)[h^{-1/m}] \otimes_{U_\lambda} N) \xrightarrow{\sim} \bigoplus_{a \geq 0} r_m S_0(\lambda) \otimes_{U_\lambda} N = \Psi(N)
\]

and furthermore that a lattice \( N(0) \) for \( N \) induces a good filtration on \( \Psi(N) \) such that

\[
\text{gr}(\Psi(N)) = \bigoplus_{a \geq 0} \Gamma(X, Y \otimes L^{\otimes ra})
\]

This shows two things: first that, by its goodness, this filtration on \( \Psi(N) \) may be used to calculate \( \text{Char}(\Psi(N)) \); second that this equals the support of \( Y \). In other words \( \text{Char}_{\text{loc}}(N) \).

It follows that \( \text{Char}(N) \) is independent of the choice of \( \eta \) and of Veronese algebra in Proposition 9.6. So combining this with Theorem 8.3 gives:
Corollary. Let $\lambda \in H^2(X, \mathbb{C})$ and suppose that $(X, \lambda)$ is localizing. If $M$ is either a simple $U_\lambda$-module or a prime factor of $U_\lambda$, then the characteristic variety $\text{Char}(M)$ is equidimensional.

9.8. Rational Cherednik algebras. We apply the above analysis to the symplectic resolution

$$\pi : X = \text{Hilb}^n \mathbb{C}^2 \rightarrow Y = \text{Sym}^n \mathbb{C}^2,$$

where $\pi$ is the Hilbert-Chow morphism. We take the $\mathbb{C}^*$-action which is induced from dilation on $\mathbb{C}^2$. For $\lambda \in H^2(\text{Hilb}^n \mathbb{C}^2, \mathbb{C}) = \mathbb{C}$ there is an explicit realization of the algebra $U_\lambda = (D_X^N(X))^{\mathbb{C}^*}$ as the spherical rational Cherednik algebra of type $A$, defined as follows.

Let $W = S_n$ be the symmetric group and write $e = \frac{1}{|W|} \sum_{w \in W} w \in CW$ for the trivial idempotent. Let $\mathfrak{h} = \mathbb{C}^n$ denote the permutation representation of $W$ and

$$\mathfrak{h}^{\text{reg}} = \{(z_1, \ldots, z_n) \in \mathfrak{h} : z_i \neq z_j \text{ for all } 1 \leq i < j \leq n\}
$$

the subset of $\mathfrak{h}$ on which $W$ acts freely.

Define $H_\lambda$ to be the subalgebra of $D(\mathfrak{h}^{\text{reg}}) \rtimes W$ generated by $W$, the vector space $\mathfrak{h}^* = \sum x_i \mathbb{C} \subset \mathbb{C}[\mathfrak{h}]$ of linear functions, and the Dunkl operators

$$D_\lambda(y_i) = \frac{\partial}{\partial x_i} - \frac{1}{2} \sum_{j \neq k} \left(\lambda - \frac{1}{2}\right) \frac{\langle y_i, x_j - x_k \rangle}{x_j - x_k} (1 - s_{jk})$$

(9.8.1)

where $\{y_1, \ldots, y_n\} \subset \mathfrak{h}$ is the dual basis to $\{x_1, \ldots, x_n\}$ and the $s_{jk} \in W$ are simple transpositions. The spherical subalgebra is defined to be $eH_\lambda e$: thanks to [GGS, Theorem 2.8] it is isomorphic to $U_\lambda$. One can also define $H_\lambda$ in terms of the reflection representation of $S_n$ instead of $\mathbb{C}^n$. However, as noted in [GS1], the two theories are exactly parallel and the same results hold in the two cases.

Set $C = \{z + \frac{1}{d} : z = \frac{m}{d} \text{ where } m, d \in \mathbb{Z} \text{ with } 2 \leq d \leq n, \text{ and } z \notin \mathbb{Z}\}$. By [GS1, Theorem 1.6] and Proposition 9.5 $(X, \lambda)$ is localizing provided $\lambda \notin C \cap (-1/2, 1/2)$. (We remark that [GS1] has the additional restriction that $\lambda \notin \mathbb{Z}$, but this is removed using [BE, Theorem 4.3] and [GS1, Remark 3.14(1)].)

9.9. Combining this discussion with Corollary 9.7 gives the following result.

Corollary. Assume that $\lambda \notin C \cap (-1/2, 1/2)$. Let $M$ be a simple $U_\lambda$-module or a prime factor ring of $U_\lambda$. Then the characteristic variety $\text{Char} M \subseteq \text{Hilb}^n \mathbb{C}^2$ is equidimensional. \hfill \square

This answers [GS2, Question 4.9] and [Rq1, Problem 8]; by [GGS, Theorem 7.6], the definition of a characteristic variety given there agrees with the definition given here.

9.10. We end with an amusing consequence of the above corollary for which we need to assume that $\lambda \notin \mathbb{Z}$. As in [GS2 (2.7.1)], write $\text{ch}(M)$ for the characteristic cycle of a module $M \in \mathcal{O}_\lambda(S_n)$, the category $\mathcal{O}$ for the rational Cherednik algebra. By [GS2, Theorem 6.7], the cycles are fully supported on the subvarieties $Z_\mu \subset \text{Hilb}^n \mathbb{C}^2$ defined in [GS2, Section 6.4] where $\mu$ is a partition of $n$. Write $\text{ch}_\mu(M)$ for the multiplicity of $\text{ch}(M)$ along $Z_\mu$.

\footnote{In much of the literature the rational Cherednik algebra with parameter $\lambda$ is what we call $H_{\lambda+\frac{1}{2}}$.}
As in [GS2, 6.10 and 6.11], there is an isomorphism \( \chi : K_0(\mathcal{O}_\lambda(S_n)) \rightarrow \Lambda_n \) where \( \Lambda_n \) is the degree \( n \) part of the ring of symmetric functions. It is defined by

\[
\chi(M) = \sum_{\mu} \text{ch}_{\mu}(M)m_{\mu}
\]

where \( m_{\mu} \) is the monomial symmetric function associated to \( \mu \). Under this isomorphism it was shown in [loc.cit] that \( \chi(\Delta_\lambda(\mu)) = s_\mu \), the Schur function associated to \( \mu \). The same is true for the co-standard modules \( \nabla_\lambda(\mu) \) since they have the same image in \( K_0(\mathcal{O}_\lambda(S_n)) \) as \( \Delta_\lambda(\mu) \).

Since \( \lambda \notin \mathbb{Z} \) we may apply [Rq2, Theorem 6.13] which shows that \( S_q(n,n) \text{-mod} \) and \( \mathcal{O}_\lambda(S_n) \) are equivalent. Here \( S_q(n,n) \) is the \( q \)-Schur algebra at \( q = -\exp(2\pi \sqrt{-1}\lambda) \), which is the equivalent to the category of representations of the quantum group \( G_q(n) \) of degree \( n \). By taking characters we have a mapping

\[
\chi_q : S_q(n,n) \text{-mod} \rightarrow \Lambda_n.
\]

The category \( S_q(n,n) \text{-mod} \) has Weyl modules, written \( \nabla_q(\mu) \): the Weyl character formula shows that \( \chi_q(\nabla_q(\mu)) = s_\mu \).

Putting these observations together gives an equivalence

\[
\Theta : S_q(n,n) \text{-mod} \rightarrow \mathcal{O}_\lambda(S_n)
\]

which sends Weyl modules to co-standard modules, so intertwines \( \chi_q \) with \( \chi \). Now, in \( \chi_q(V) \), the coefficient \( m_{\mu} \) is by definition the multiplicity of the weight space \( V_\mu \). We therefore deduce:

**Proposition.** Assume \( \lambda \notin \mathbb{C} \cap (-1/2, 1/2) \) and that \( \lambda \notin \mathbb{Z} \). Then, in the above notation, \( \text{ch}_{\mu}(M) = \dim \Theta^{-1}(M)_\mu \) for any partition \( \mu \) of \( n \).

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### References

[Ba] H. Bass, On the ubiquity of Gorenstein rings, *Math. Z.*, 82 (1963), 8–28.

[BE] R. Bezrukavnikov and P. Etingof, Parabolic induction and restriction functors for rational Cherednik algebras, *Selecta Math. (N. S.)* 14 (2009), 397–425.
