Quasi-invariant states on $W^*$–algebras

Luigi Accardi
Volterra Center, University of Roma Tor Vergata
Via Columbia 2, 00133 Roma, Italy
accardi@volterra.uniroma2.it

Ameur Dhahri
Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, I-20133 Milano, Italy
ameur.dhahri@polimi.it

Abstract

We define the notion of a quasi–invariant (resp. strongly quasi–invariant) state under the action of a group $G$ of normal $*$–automorphisms of a von Neumann algebra $\mathcal{A}$. We prove that these states are naturally associated to left–$G$–1–cocycles. If $G$ is compact, the structure of strongly $G$–quasi–invariant states is determined. The case of unimodular locally compact group is also discussed showing that many of the structure properties of strongly $G$–quasi–invariant states continue to hold in this case. For any $G$–quasi–invariant state $\varphi$, we construct a unitary representation associated to the triple $(\mathcal{A}, G, \varphi)$. We prove that any quantum Markov chain, with commuting, invertible and hermitean conditional density amplitudes, on a countable tensor product of type I factors is strongly quasi–invariant with respect to the natural action of the group $S_\infty$ of local permutations and we give the explicit form of the associated cocycle. This provides a family of non–trivial examples of strongly quasi–invariant states for locally compact groups obtained as inductive limit of an increasing sequence of compact groups whose structure is formally identical to the one deduced in the compact case. However, if also for groups in this larger class this structure characterizes all possible strongly quasi–invariant states, is an open problem.
## Contents

1 Introduction ................................................................. 2

2 Quasi–invariant states under a group of normal $*$-automorphisms 3

2.1 The structure of strongly quasi–invariant states: $G$ compact . 10
2.2 The case of a unimodular locally compact group . . . . . . . . 13
2.3 Inductive limits of compact groups . . . . . . . . . . . . . . . 17

3 Unitary representations associated to strongly quasi–invariant states 18

4 Examples of quasi–invariant states under permutations 21

4.1 Product states . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
4.2 Markov Chains with commuting conditional density ampli-
tudes . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

5 On the quasi–invariant states under the group of permutations $S_\infty$ 28

## 1 Introduction

Given a normal faithful state on von Neumann algebra $\mathcal{A}$ and a group $G$ of normal $*$-automorphisms of $\mathcal{A}$, we say that $\varphi$ is $G$-invariant if

$$\varphi(g(a)) = \varphi(a), \quad \forall g \in G, \ a \in \mathcal{A} \quad (1.1)$$

This class of states is of paramount importance in many branches of mathematics and physics and it has been studied in different contexts (cf [1], [3], [11]...). In this paper, motivated by the analogy with the classical case, we extend definition (1.1) as follows: we assume that for every $g \in G$ there exists $x_g \in \mathcal{A}$ such that

$$\varphi(g(x)) = \varphi(x_g a) \quad (1.2)$$

and we call $G$-quasi–invariant a state satisfying (1.2). Moreover, if in addition $x_g = x_g^*$ for every $g \in G$, the state $\varphi$ is said to be $G$–strongly quasi–invariant. In this case we prove that the Radon–Nikodym derivatives $x_g$’s are strictly positive operators and satisfy the normalized left–$G$–1–cocycle property

$$x_{g_2 g_1} = x_{g_1} (g_1^{-1} (x_{g_2})), \quad x_e = 1$$
Moreover, we show that the $C^*$-algebra generated by the Radon–Nikodym derivatives $x_g$’s is commutative. We further study the case of a unimodular locally compact group $G$. In particular, we give the structure of a $G$–strongly quasi–invariant state $\varphi$, when $G$ is a compact group, and we prove that it has the following form

$$\varphi(a) = \varphi_G(\kappa^{-1}_G a), \quad \forall a \in \mathcal{A},$$

for some bounded strictly positive operator $\kappa_G$ in the centraliser of $\varphi$ whose structure is explicitly given (see Theorem 1). Conversely, if $\varphi_G$ is a $G$–invariant state and $\kappa_G$ is an invertible positive operator in the centraliser of $\varphi$, then the state $\varphi$, defined by the right hand side of (1.3), is a $G$–quasi–invariant state with cocycle

$$x_g = \kappa_G g^{-1}(\kappa^{-1}_G), \quad \forall g \in G$$

We study the inductive limit associated to a uni–modular group $G$ which is the union of an increasing sequence of compact groups. For any $G$–strongly quasi–invariant state $\varphi$, the unitary representation of the group $G$ is also given. We finally give some examples and we investigate the structure of a $S_\infty$–quasi–invariant state $\varphi$, where $S_\infty = \cup_{N=1}^\infty S_N$ is the group of local permutations on an infinite tensor product of a von Neumann algebra ($S_N$ is the group of permutations on $\{1, 2, \ldots, N\}$). If the sequence $\kappa^{-1}_N$ converges in norm to a positive bounded operator $\rho$, we prove that the state $\varphi$ is of the form

$$\varphi(a) = \varphi_\infty(\rho a) \quad \forall a \in \mathcal{A}$$

where $\varphi_\infty$ is a $S_\infty$–invariant state and, if $\rho$ is invertible, the associated cocycle is given by (1.3) with $\kappa^{-1}_G$ replaced by $\rho$.

## 2 Quasi–invariant states under a group of normal *-automorphisms

In the following $\mathcal{A}$ will denote a von Neumann algebra, $\varphi$ a faithful normal state on $\mathcal{A}$ and $G \subseteq \text{Aut}(\mathcal{A})$ a group of normal *–automorphisms of $\mathcal{A}$.

**Proposition 1** Suppose that, for any $g \in G$, there exists a $x_g \in \mathcal{A}$ such that:

$$\varphi(g(a)) = \varphi(x_g a) ; \quad \forall a \in \mathcal{A}$$

(2.1)
Then the map \( g \in G \mapsto x_g \in A \) is a normalized left \( G^{-1} \)-co-cycle, i.e. it satisfies the identities

\[
x_e = 1 \quad (2.2)
\]

\[
x_{g_2g_1} = x_{g_1}(g_1^{-1}(x_{g_2})) \quad \forall g_2, g_1 \in G \quad (2.3)
\]

In particular each \( x_g \) is invertible and its inverse is

\[
x_g^{-1} = g^{-1}(x_g) \quad (\iff x_g^{-1} = g(x^{-1}_g)) \quad (2.4)
\]

**Proof.** One has

\[
\varphi(a) = \varphi(e(a)) = \varphi(x_ea) \quad \forall a \in A
\]

Since \( \varphi \) is faithful, (2.2) follows. Now for any \( g_2, g_1 \in G \), one has

\[
\varphi(g_2g_1(a)) = \varphi(x_{g_2g_1}(a)) = \varphi(g_1(g_1^{-1}(x_{g_2})a)) = \\
\varphi(x_{g_1}(g_1^{-1}(x_{g_2})a)) = \varphi((x_{g_1}g_1^{-1}(x_{g_2})a)
\]

This is equivalent to

\[
0 = \varphi(x_{g_2g_1}a) - \varphi(x_{g_1}(g_1^{-1}(x_{g_2})a) = \\
\varphi((x_{g_2g_1} - x_{g_1}(g_1^{-1}(x_{g_2}))a)
\]

Since \( a \) is arbitrary, one can choose \( a = (x_{g_2g_1} - x_{g_1}(g_1^{-1}(x_{g_2})))^* \). This gives

\[
0 = \varphi((x_{g_2g_1} - x_{g_1}(g_1^{-1}(x_{g_2})))^*^2)
\]

Since \( \varphi \) is faithful, (2.3) follows. Taking \( g_2 := g_1^{-1} \) in (2.3), one finds, using \( x_e = 1, \)

\[
1 = x_e = x_{g_1}(g_1^{-1}(x_{g^{-1}_1}))
\]

This implies that \( x_{g_1} \) is invertible and its inverse is

\[
x_{g_1}^{-1} = g_1^{-1}(x_{g_1}^{-1})
\]

since \( g_1 \) is arbitrary, the first identity in (2.4) follows. Multiplying both sides of it by \( g_1 \), one obtains

\[
g_1(x_{g_1}^{-1}) = x_{g_1}^{-1}
\]

which is equivalent to the second identity in (2.4). \( \square \)

The following result is well known, we include it simple proof for completeness.
Lemma 1 Suppose that
\[ x = g(y) \]
with \( x \) and \( y \) are selfadjoint, invertible in \( \mathcal{A} \). Then
\[ x^s = g(y^s) \quad \forall s \in \mathbb{R} \tag{2.5} \]

Proof.
\[ 1 = g(yy^{-1}) = g(y)g(y^{-1}) \]
Therefore \( g(y) \) is invertible and
\[ g(y)^{-1} = g(y^{-1}) = x^{-1} \tag{2.6} \]
Thus
\[ x^s = g(y^s) \quad \forall s \in \mathbb{Z} \]
For \( n \in \mathbb{N} \) one has
\[ g(y^{\frac{1}{n}})^n = g((y^{\frac{1}{n}})^n) = g(y) = x \]
Therefore
\[ x^{\frac{1}{n}} = g(y^{\frac{1}{n}}) \quad \forall n \in \mathbb{N} \]
Consequently
\[ x^{\frac{m}{n}} = (x^{\frac{1}{n}})^m = g(y^{\frac{1}{n}})^m = g(y^{\frac{m}{n}}) \]
and, because of (2.6), \( x^s = g(y^s) \) for all \( s \in \mathbb{Q} \). By continuity
\[ x^s = g(y^s) \quad \forall s \in \mathbb{R} \]
□

Corollary 1 If \( x_g = x_g^s \), then for any \( s \in \mathbb{R} \),
\[ x_g^{-s} = g^{-1}(x_g^{s-1}) \tag{2.7} \]

Proof. Replacing in (2.5) \( g \) by \( g^{-1}, x \) by \( x_g^{-1} \) and \( y \) by \( x_{g-1} \), one finds
\[ x_g^{-s} = g^{-1}(x_g^{s-1}) \]
which is (2.7). □

Definition 1 If the pair \((G, x)\) satisfies condition (2.1) of Proposition 1, the state \( \phi \) is \((G, x)\)-quasi-invariant, or quasi-invariant with cocycle \( x \).
Lemma 2 If $\varphi$ is $(G,x)$–quasi–invariant, then for all $g \in G$

\[ a \in \mathcal{A}, \ a \geq 0 \Rightarrow \varphi(x_g a) \geq 0 , \ 1 = \varphi(x_g) \quad (2.8) \]

\[ \varphi(x_g a) = \varphi(ax_g^*) ; \quad \forall a \in \mathcal{A} \quad (2.9) \]

**Proof.** If $a \in \mathcal{A}, \ a \geq 0$ and $g \in G$, one has $0 \leq \varphi(g(a)) = \varphi(x_g a)$ and this proves the inequality in (2.8). The equality follows from

\[ 1 = \varphi(1) = \varphi(g(1)) = \varphi(x_g) ; \quad \forall g \in G \]

Moreover one has

\[ \varphi(g(a^*)) = \varphi(x_g a^*) ; \quad \forall a \in \mathcal{A} \]

On the other hand

\[ \varphi(g(a^*)) = \varphi((g(a))^*) = \overline{\varphi(g(a))} = \overline{\varphi(x_g a)} = \varphi((x_g a)^*) = \varphi(a^* x_g^*) \]

It follows that

\[ \varphi(x_g a^*) = \varphi(a^* x_g^*) ; \quad \forall a \in \mathcal{A} \]

and, since $a \in \mathcal{A}$ is arbitrary, this is equivalent to (2.9). \hfill \square

**Remark 1** Recall that the **centralizer** of $\varphi$, hereinafter denoted $\text{Centr}(\varphi)$, is characterized by the property

\[ c \in \text{Centr}(\varphi) \iff \varphi(ac) = \varphi(ca) ; \quad \forall a \in \mathcal{A} \quad (2.10) \]

**Lemma 3** If $\varphi$ is $(G,x)$–quasi–invariant, for any $g \in G$ the following are equivalent:

(i) $x_g$ is a hermitean element of $\mathcal{A}$

(ii) $x_g$ is in the centralizer of $\varphi$.

In both cases $x_g$ is strictly positive, i.e. its spectrum, $\text{Spec}(x_g) \subset (0, \infty)$. 

6
Proof. (i) $\Rightarrow$ (ii). If $x_g = x_g^*$, (ii) follows from (2.9).

(ii) $\Rightarrow$ (i). If $x_g$ is in the centralizer of $\varphi$ then the same is true for $x_g^*$. Therefore (2.9) implies that for all $a \in \mathcal{A}$,

$$\varphi(x_g a) = \varphi(ax_g) = \varphi(x_g^* a) \iff \varphi((x_g - x_g^*) a) = 0 \overset{a = (x_g - x_g^*)^*}{\Rightarrow} \varphi(|x_g - x_g^*|^2) = 0$$

and (i) follows because $\varphi$ is faithful. If either (i) or (ii) holds, then $x_g = x_g^* = x_{g,+} - x_{g,-}$, with $x_{g,\pm}$ are positive such $x_{g,+}x_{g,-} = 0$. Therefore, choosing in (2.9) $a = x_{g,-}$, one finds

$$0 \leq \varphi(x_g x_{g,-}) = \varphi((x_{g,+} - x_{g,-}) x_{g,-}) = -\varphi(x_{g,-}^2) \iff 0 = \varphi(x_{g,-}^2) \iff 0 = x_{g,-}$$

Thus $x_g = x_g^* = x_{g,+} \geq 0$. Since $x_g$ is invertible,

$$\text{Spec}(x_g) \subset (0, \infty)$$

□

Definition 2 A $(G,x)$–quasi–invariant state $\varphi$ is called $(G,x)$–strongly quasi–invariant if, for any $g \in G$, $x_g$ is hermitean.

Note that if $\varphi$ is $G$ strongly quasi–invariant, then $\text{Spec}(x_g) \subset (0, \infty)$ and

$$\varphi(g(a)) = \varphi(x_g a) = \varphi(x_g^{1/2} ax_g^{1/2}) \quad \forall a \in \mathcal{A}$$

The following Proposition shows that, if $\mathcal{A} = \mathcal{B}(\mathcal{H})$, there exist many $G$–strongly quasi–invariant states.

Proposition 2 If $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\varphi = \text{Tr}(W \cdot)$ where $W$ is a density operator, then $x_g$ is a solution of equation (2.9) if and only if

$$W x_g = x_g^* W$$

and the solutions of equation (2.11) are given by the set

$$S_W := \{W^{-1} z: z \in \mathcal{B}(\mathcal{H}), \quad z = z^*\}$$

(2.12)

Proof. In the assumptions of the Proposition, equation (2.9), i.e.

$$\varphi(x_g a) = \varphi(ax_g^*) \quad \forall a \in \mathcal{A}$$

(2.13)
becomes equivalent to
\[ \text{Tr}(Wx_\gamma a) = \text{Tr}(Wax_\gamma^*) = \text{Tr}(x_\gamma^* Wa) \quad \forall a \]
\[ \iff Wx_\gamma = x_\gamma^* W \]
which is (2.11). If \( x \) is a solution of equation (2.11), defining \( z := Wx \)
one has
\[ z^* = (Wx)^* = x^* W \quad \text{equation (2.11)} \]
thus \( x = W^{-1} z \in S_W \). Conversely, if \( x \in S_W \), then \( x = W^{-1} z \) for some \( z = z^* \).
Therefore
\[ Wx = W(W^{-1}z) = z = zW^{-1}W = (W^{-1}z)^*W = x^*W \]
i.e. \( x \) is a solution of (2.11). \( \square \)

**Remark 2**

1) If \( z = z^* \in \mathcal{B}(\mathcal{H}) \) commutes with \( W \), then it commutes with \( W^{-1} \) and therefore, if \( x := W^{-1}z \), one has
\[ x^* = zW^{-1} = W^{-1}z = x \]
In other words, equation (2.11) always admits hermitean solutions when \( \mathcal{A} = \mathcal{B}(\mathcal{H}) \).

2) From Proposition 2, one also deduces that the family of strong quasi-invariant states is strictly smaller than the family of quasi-invariant states.

In fact, let
\[ \mathcal{G} = \{ g_0^n : n \in \mathbb{N} \} ; \quad g_0^0 := e \neq g_0 \]
be a group of \(*\)-automorphism of \( \mathcal{B}(\mathcal{H}) \) with a single generator \( g_0 \neq e \).

Given \( W \), fix \( z_{g_0} = z_{g_0}^* \) subject to the only condition
\[ W^{-1}[z_{g_0}, W] \neq 0 \quad \text{(2.14)} \]
(clearly, for any \( W \), there are many \( z_{g_0} \) satisfying (2.14)) and define
\[ x_{g_0} := W^{-1}z_{g_0} \quad \text{(2.15)} \]
The identity
\[ x_{g_0}^n = x_{g_0} g_0^{-1}(x_{g_0}) \cdots g_0^{-(n-1)}(x_{g_0}) \quad n \in \mathbb{N} \]
easily verified by induction on the cocycle equation, shows that (2.15) uniquely determines \( x_{g_0}^n \) for any \( n \in \mathbb{N} \), i.e. the cocycle \( x : G \to A \). On the other hand condition (2.14) means that \( W^{-1} z_{g_0} \) is not in the centralizer of \( \varphi = \text{Tr}(W \cdot) \). Hence it is not hermitian by Lemma 3.

**Lemma 4** If \( \varphi \) is \((G, x)\)–strongly quasi–invariant, \( x \) satisfies
\[ x_{g_1} (g_1^{-1}(x_{g_2})) = (g_1^{-1}(x_{g_2})) x_{g_1} \quad \forall g_2, g_1 \in G \]which is equivalent to
\[ x_{g_2} x_{g_1} = x_{g_1} x_{g_2} \quad \forall g_2, g_1 \in G \]

**Proof.** From the cocycle identity (2.3) and the hermiteanity of the \( x_g \), one has, for all \( g_2, g_1 \in G \),
\[ x_{g_1} (g_1^{-1}(x_{g_2})) = x_{g_2 g_1} = x_{g_2} g_1 = (g_1^{-1}(x_{g_2})) x_{g_1} \]
which is (2.16). (2.16) is equivalent to
\[ (g_1^{-1}(x_{g_2})) x_{g_1}^{-1} = x_{g_1}^{-1} (g_1^{-1}(x_{g_2})) \]
\[ \iff g_1^{-1}(x_{g_2} x_{g_1}) = g_1^{-1}(g_1^{-1}(x_{g_2})) g_1^{-1}(x_{g_1}) = g_1^{-1}(x_{g_1} g_1^{-1}(x_{g_2})) \]
\[ \iff g_1^{-1}(x_{g_2} x_{g_1}) = g_1^{-1}(x_{g_1} g_2) \iff x_{g_2} x_{g_1} = x_{g_1} x_{g_2} \]
which is (2.17). \( \square \)

**Remark 3** Denote \( C \) the abelian \( C^* \)–algebra generated by the \( x_g \) (\( g \in G \)). Then \( C \) contains the identity of \( A \) (\( 1_A = x_e \)). Therefore, by Gelfand Theorem ((see [3], section 1.1.9) \( C \) can be identified to the algebra \( C_C(S) \) of continuous complex valued functions on a compact space \( S \). Since \( S \) is compact, in this identification, the \( x_g \) become **strictly positive functions**, i.e.
\[ x_g(S) = \text{Spec}(x_g) \subset (0, \infty) \quad \forall g \in G \] \( \quad \) (2.18)

**Lemma 5** If \( \varphi \) is \((G, x)\)–strongly quasi–invariant and \( x \in \text{Centr}(\varphi) \), then \( \forall g \in G \) and \( a \in A \)
\[ \varphi(g(x)a) = \varphi(a g(x x_g^{-1})) \] \( \quad \) (2.19)
In particular,
\[ g(C) \subseteq \text{Centr}(\varphi) \quad , \quad \forall g \in G \] \( \quad \) (2.20)
Proof. In the above notations, one has
\[ \varphi(g(x)a) = \varphi(g(xg^{-1}(a))) = \varphi(xg xg^{-1}(a)) = \varphi(g^{-1}(ag(xg)x)) = \varphi(xg^{-1}ag(xg)g(x)) = \varphi(ag(xg)g(x)g(xg^{-1})) \]
which is (2.19). (2.20) follows from (2.19) because, \( C \subseteq \text{Centr}(\varphi) \) and, if \( x \in C \), then it is known that \( x^{-1} \in C \) (in fact, if \( x \in \text{Centr}(\varphi) \) is invertible, \( \varphi(x^{-1}xa) = \varphi(ax^{-1}x) = \varphi(xax^{-1}) \) and, since the map \( a \in A \mapsto xa \) is invertible, it follows that, if \( a \in A \) is arbitrary, so it is \( xa \)). Therefore \( \varphi(axg xg^{-1}) = \varphi(xg xg^{-1}a) \) for any \( g \in G \). □

2.1 The structure of strongly quasi–invariant states: \( G \) compact

In this section we shall assume that the action of the group \( G, g \in G \mapsto x_g \in C \), is continuous and that \( G \) is a compact group of normal \( * \)-automorphisms of \( \text{Aut}(A) \) with normalized Haar measure \( dg \).

Theorem 1 In the above assumptions the element
\[ \kappa := \int_G x_g dg \in C \subset A, \quad (2.21) \]
\( \kappa \) is invertible (hence \( \kappa^{-1} \) belongs to \( C \)).

Proof. Using the identification in Remark 3 we realize the \( x_g' \)s as continuous strictly positive functions on the compact Hausdorff space \( S \). Then each function \( s \in S \mapsto x_g(s) \) is bounded away from zero, i.e. \( x_g(S) = \text{Spec}(x_g) \subset (0, \infty) \). Then, for each \( s \in S \),
\[ \kappa(s) = \int_G x_g(s) dg > 0, \quad \forall s \in S \]
The continuity of the map \( g \in G \mapsto x_g \in C \), then implies that \( \kappa \) is continuous and bounded away from zero. Hence \( \kappa^{-1} \) is bounded strictly positive and \( \kappa^{-1} \) belongs to \( C \). □
Theorem 2 Define
\[ E_G = \int_G g \, dg \] (2.22)

Then
(i) \( E_G \) is a \ast\-map from \( \mathcal{A} \) to \( \mathcal{A} \).

(ii) For any \( g \in G \), \( E_G \) satisfies the following identity
\[ gE_G = E_G g = E_G \] (2.23)
\[ E_G(g(a)) = gE_G(a) = E_G(a) \] (2.24)

In particular
\[ \text{Range}(E_G) = \text{Fix}(G) := \{ a \in \mathcal{A} : g(a) = a \} \] (2.25)
and \( E_G(\mathcal{A}) \) is a \( W^* \)-sub-algebra of \( \mathcal{A} \).

(iii) \( E_G \) is a Umegaki conditional expectation from \( \mathcal{A} \) to \( E_G(\mathcal{A}) \).

Proof. For each \( a \in \mathcal{A} \)
\[ \mathcal{A} \ni (E_G(a))^* = \left( \int_G g(a) \, dg \right)^* = \int_G g(a)^* \, dg = \int_G g(a^*) \, dg = E_G(a^*) \]

Therefore \( E_G(\mathcal{A}) \) is a \ast\-sub-space of \( \mathcal{A} \). This proves (i).

Left-invariance of the Haar measure implies that, for each \( a \in \mathcal{A} \) and \( g \in G \),
\[ gE_G(a) = \int_G gh(a) \, dh = \int_G gh(a) \, d(gh) = \int_G g'(a) \, dg' = E_G(a) \]

Therefore the second identity in (2.24) holds and it implies (2.25). Similarly, right-invariance of the Haar measure implies that, for each \( a \in \mathcal{A} \) and \( g \in G \),
\[ E_G(g(a)) = \int_G h(g(a)) \, dh = \int_G h(g(a)) \, dhg = \int_G g'(a) \, dg' = E_G(a) \]

Therefore the first identity in (2.24) holds. This proves (ii).

To prove (iii) notice that \( E_G \) is completely positive being a convex combination of automorphisms and identity preserving because \( g(1_{\mathcal{A}}) = 1_{\mathcal{A}} \) for any \( g \in G \). Moreover, for any \( a \in \mathcal{A} \), one has
\[ E_G^2(a) = \int_G dg \, g(E_G(a)) \overset{(2.24)}{=} \int_G dg \, E_G(a) = E_G(a) \]
due to the normalization of the Haar measure. Thus $E_G$ is a completely positive norm–1 projector on $A$ and, by Tomijama’s theorem [12], $E_G(A)$ is a sub–$W^*$–algebra of $A$ and $E_G$ is a Umegaki conditional expectation from $A$ to $E_G(A)$. This proves (iii). □

The structure of a strongly quasi–invariant state is described by the following theorem.

**Theorem 3** Let $G$ be a compact group of normal $*$–automorphisms of $A$ and let $\varphi$ be a $G$–strongly quasi–invariant faithful normal state on $A$. Denote $x : g \in G \to x_g \in A$ the (continuous, hermitean) left–$G$–1–cocycle associated to $\varphi$ and

$$
\kappa := \int_G x_g dg \in \mathcal{C}
$$

Then $\kappa$ is invertible with inverse in $\mathcal{C}$ and $\varphi$ can be written in the form

$$
\varphi(a) = (\varphi \circ E_G)(\kappa^{-1}a) =: \varphi_G(\kappa^{-1}a) \quad \forall a \in A \quad (2.26)
$$

In particular $\varphi_G$ is a $G$–invariant state.

Conversely, if $\varphi_G$ is a $G$–invariant state on $A$ and $\kappa$ is any element of $A$, invertible with inverse in $A$, then

$$
\varphi(\cdot) := \varphi_G(\kappa^{-1} \cdot) \quad (2.27)
$$

is a $(G, x)$–quasi-invariant state with cocycle

$$
x_g := \kappa g^{-1}(\kappa^{-1}) \quad (2.28)
$$

$\varphi_G$ is $(G, x)$–strongly quasi-invariant if and only if $x_g$ is hermitean.

**Proof.** If $\varphi$ is a $(G, x)$ strongly–quasi–invariant state, define $\varphi_G := \varphi \circ E_G$.

Since $E_G g^{-1} E_G = E_G$ for every $g \in G$, the state $\varphi_G$ is $G$–invariant. From Theorem [11] one knows that $\kappa$ is an invertible element of $A$ with inverse in $A$. Moreover, for any $a \in A$ one has:

$$
\varphi_G(\kappa^{-1}a) = \varphi(E_G(\kappa^{-1}a)) = \int_G dg \varphi(g(\kappa^{-1}a)) = \int_G dg \varphi(x_g \kappa^{-1}a) = \varphi(\kappa \kappa^{-1}a) = \varphi(a)
$$
which is (2.26). Conversely, let $\varphi_G$ be $G$–invariant and let $\varphi$ be defined by (2.27) with $\kappa$ as in the statement. Then for any $a \in A$ and $g \in G$, one has

$$\varphi(g(a)) = \varphi_G(\kappa^{-1}g(a)) = \varphi_G(g[g^{-1}(\kappa^{-1})a]) = \varphi_G([g^{-1}(\kappa^{-1})a])$$

$$= \varphi_G(\kappa^{-1}[g^{-1}(\kappa^{-1})a]) = \varphi(\kappa^{-1}[\kappa g^{-1}(\kappa^{-1})a]) = \varphi(x_g a)$$

where $x_g$ is given by (2.28), which is a left–$G$–1–cocycle (see Lemma 7). Therefore $\varphi$ is $(G, x)$–quasi–invariant. Finally, from Definition 2 $\varphi$ is $(G, x)$–strongly quasi–invariant iff each $x_g$ is hermitean. □

**Lemma 6** We have

$$E_G(\kappa^{-1}) = 1$$

**Proof.** Note that the Haar measure on a unimodular group (in particular on a compact group) is invariant by inversion. Therefore, one gets

$$E_G(\kappa^{-1}) = \int_G g(\kappa^{-1})dg \overset{(2.29)}{=} \int_G \kappa^{-1}x_g^{-1}dg = \kappa^{-1}\int_G x_g^{-1}dg = \kappa^{-1}\kappa = 1$$

□

2.2 The case of a unimodular locally compact group

In the following, we will assume that $G$ is unimodular locally compact group of normal $*$–automorphisms of $A$ with Haar measure denoted $\mu_G$ or simply $dg$ and we will assume that $\varphi$ is $(G, x)$–strongly quasi–invariant. Note that compact and discrete groups are particular cases of unimodular locally compact groups.

If the Haar measure of $G$ is not finite (i.e. $G$ is not compact), the integral

$$E_G = \int_G g dg$$

with domain

$$\text{Dom}(E_G) := \left\{ a \in A : \int_G g(a)dg \in A \right\}$$

(2.30)
is not a Umegaki conditional expectation, but an **operator valued weight** in the sense of Haagerup [6], [7] with values in $\text{Fix}(G)_+$. This means that, $E_G$ satisfies the following conditions:

$$
E_G(\lambda x) = \lambda E_G(x) \quad , \quad \lambda \geq 0 , \ x \in \mathcal{A}_+
$$

$$
E_G(x + y) = E_G(x) + E_G(y) \quad , \quad x, y \in \mathcal{A}_+
$$

$$
E_G(a^*xa) = a^*E_G(x)a \quad , \quad x \in \mathcal{A}_+ , \ a \in \text{Range}(E_G) \quad (2.31)
$$

It is known (see [4], Lemma 1.10) that, denoting

$$
\text{Dom}^+(E_G) := \{ x \in \mathcal{A}_+: \|E_G(x)\| < +\infty \}
$$

$$
\text{Dom}^{1/2}(E_G) := \{ x \in \mathcal{A}: \|E_G(x^*x)\| < +\infty \}
$$

$$
= \left\{ \sum_{i=1}^{n} x_i^*y_i : x_i, y_i \in \text{Dom}^{1/2}(E_G) , \ n \in \mathbb{N} \right\}
$$

$$
\text{Dom}(E_G) := \left( \text{Dom}^{1/2}(E_G) \right)^* \text{Dom}^{1/2}(E_G)
$$

(i) $\text{Dom}(E_G)$ is linearly spanned by $\text{Dom}^+(E_G)$.

(ii) $\text{Dom}(E_G)$ and $\text{Dom}^{1/2}(E_G)$ are two sided modules over $\text{Range}(E_G)$.

(iii) $E_G$ has a unique linear extension, still denoted $E_G: \text{Dom}(E_G) \to \text{Range}(E_G)$ and it is a module map, i.e.

$$
E_G(axb) = aE_G(x)b \quad , \quad a, b \in \text{Range}(E_G) , \ x \in \text{Dom}(E_G) \quad (2.32)
$$

**Lemma 7** Let $y : g \in G \to y_g \in \mathcal{A}$ be a left–$G$–1–cocycle such that

$$
\kappa := \int_G y_g dg \in \mathcal{A} \quad (2.33)
$$

is an invertible element of $\mathcal{A}$. Then

$$
y_g = \kappa g^{-1}(\kappa^{-1}) \quad , \quad \forall \ g \in G
$$

$$
E_G(\kappa^{-1}) = 1
$$

Conversely, if $\kappa$ is any invertible element of $\mathcal{A}$, then the map

$$
g \in G \mapsto y_g := \kappa g^{-1}(\kappa^{-1}) \in \mathcal{A} \quad (2.34)
$$

is a left–$G$–1–cocycle, integrable if and only if $\kappa^{-1} \in \text{Dom}(E_G)$ and hermitean if $\kappa$ is hermitean and

$$
\kappa g^{-1}(\kappa^{-1}) = g^{-1}(\kappa^{-1})\kappa
$$

(2.35)
Proof. If \( y \) is an integrable left–\( G \)–1–cocycle with values in \( A \) such that \( \kappa \), defined by \( (2.33) \) is invertible then, since the Haar measure on \( G \) is invariant under translation, for any \( g' \in G \) one has

\[
\kappa = \int_G y_g dg = \int_G y_g' dg = \int_G y_g'(g')^{-1}(y_g) dg = y_g'(g')^{-1} \left( \int_G y_g dg \right)
\]

By assumption \( \kappa \) is invertible and by \( (2.4) \) \( y_g' \) is invertible. Therefore, since \((g')^{-1}\) is an automorphism, \((2.36)\) is equivalent to

\[
y_g^{-1} = (g')^{-1}(\kappa)^{-1} \iff (g')^{-1}(y_{g'}^{-1}) = (g')^{-1}(\kappa)^{-1} \]

Since \( g' \in G \) is arbitrary one can replace \( g' \) by \( g^{-1} \) and, with this replacement, \( (2.36) \) becomes \((2.34)\). Finally, \((2.34)\) is equivalent to

\[
\kappa^{-1} = g^{-1}(\kappa^{-1})y_g^{-1} \iff g(\kappa^{-1}) = \kappa^{-1}g(y_g^{-1}) \iff \kappa^{-1}y_{g^{-1}}, \quad \forall g \in G
\]

Now, since \( G \) is unimodular locally compact group, then the Haar measure on \( G \) is invariant by inversion. Therefore we can prove, in the same way as in Lemma \( \Box \) that \( E_G(\kappa^{-1}) = 1 \).

Conversely, if \( \kappa \in A \) is invertible then \( y_g \), defined by \((2.34)\), satisfies

\[
y_{g_1g_2} = \kappa(g_2g_1)^{-1}(\kappa^{-1}) = \kappa g_1^{-1}g_2^{-1}(\kappa^{-1}) = \kappa g_1^{-1}(\kappa^{-1}g_2^{-1}(\kappa^{-1}))
\]

\[
= \kappa g_1^{-1}(\kappa^{-1})g_2^{-1}(\kappa g_2^{-1}(\kappa^{-1})) = y_{g_1}g_1^{-1}(y_{g_2})
\]

i.e. the map \( g \in G \mapsto y_g := \kappa g^{-1}(\kappa^{-1}) \in A \) is a left–\( G \)–1–cocycle. Finally, if \( \kappa \) is hermitean and \((2.34)\) holds, then

\[
y_g^* = (\kappa g^{-1}(\kappa^{-1}))^* = g^{-1}(\kappa^{-1})^* = \kappa g^{-1}(\kappa^{-1}) = y_g
\]

\( \Box \)
Remark 4 From the results of the paper [4] it follows that $E_G$ can be further extended to the case in which, for a large class of positive operators $a \in \mathcal{A}$, $E_G(a)$ is an unbounded positive operator affiliated to $\mathcal{A}$.

This extension should allow to weaken the assumption of bounded invertibility in Lemma [7].

Many of the results of Theorem [2] remain true if restricted from $\mathcal{A}$ to Dom$(E_G)$.

Theorem 4  
(i) $E_G$ is a $*$-map in the sense that, if $a \in \text{Dom}(E_G)$, then $a^* \in \text{Dom}(E_G)$ and $E_G(a)^* = E_G(a^*)$. In particular Dom$(E_G)$ is a $*$-sub-space of $\mathcal{A}$.

(ii) The identities (2.23), (2.24) must be understood in the sense that these identities hold for any $a \in \text{Dom}(E_G)$ such that $g \in G$ such that $g(a) \in \text{Dom}(E_G)$. Moreover, Fix$(G)$ is defined by

$$\text{Fix}(G) := \{a \in \text{Dom}(E_G) : \forall g \in G, g(a) \in \text{Dom}(E_G) \text{ and } (2.24) \text{ holds}\}$$

(iii) The identity (2.32) must be understood in the sense that $E_G(a)bE_G(c) \in \text{Dom}(E_G)$ and the equality holds.

(iv) For all $a, b \in \text{Dom}(E_G)$ such that $E_G(a)b \in \text{Dom}(E_G)$,

$$E_G(a)E_G(b) = E_G(E_G(a)b)$$

In particular, for any $x \in \text{Fix}(G)$,

$$xb \in \text{Dom}(E_G) \iff b \in \text{Dom}(E_G)$$

Proof. The proofs of (i), (ii) and (iii) are similar as in Theorem [2]. Now in order to prove (iv), let $a, b \in \text{Dom}(E_G)$ such that $E_G(a)b \in \text{Dom}(E_G)$. Then, one has

$$E_G(a)E_G(b) = \int_G g(a)dg \int_G h(b)dh = \int_G \int_G dg dh g(a)h(b)$$

(2.37)

$$= \int_G \int_G dg dh h(h^{-1}g(a)b) = \int_G dh h \left( h^{-1} \int_G dg g(a)b \right)$$

16
\[ \int_G \, dh \, (h^{-1}E_G(a)b) = \int_G \, dh \, (E_G(a)b) = E_G(E_G(a)b) \in \text{Range}(E_G) \]

It follows that, for any \( x \in \text{Fix}(G) \),

\[ xb \in \text{Dom}(E_G) \iff b \in \text{Dom}(E_G) \]

Remark 5: Theorem 3 completely determines the structure of the \((G, x)\)-strongly-quasi-invariant states in the case when the group \( G \) is compact. As such it is not directly applicable to the countable symmetric group, which is only a unimodular locally compact. In the following sections we shall develop the tools needed to achieve this extension.

2.3 Inductive limits of compact groups

Let \((G_N)_{N \in \mathbb{N}}\) be an increasing sequence of compact groups of normal automorphisms of a von Neumann algebra \( \mathcal{A} \) and assume that the group

\[ G := \bigcup_{N \in \mathbb{N}} G_N \quad (2.38) \]

is unimodular and locally compact. Denote

\[ E_{G_N} = \int_{G_N} gd_Ng \quad (2.39) \]

where \( d_Ng \) is the Haar measure on \( G_N \).

**Proposition 3** The family \((E_{G_N})_{N \in \mathbb{N}}\) is a projective family of conditional expectations, i.e.

\[ E_{G_{N+1}}E_{G_N} = E_{G_{N+1}} , \quad \forall N \in \mathbb{N} \quad (2.40) \]

**Proof.** Since \( G_N \subseteq G_{N+1} \), one has

\[ \text{Range}(E_{G_{N+1}}) = \text{Fix}(G_{N+1}) \subseteq \text{Fix}(G_N) = \text{Range}(E_{G_N}) \quad (2.41) \]

Therefore projectivity is equivalent to \((2.40)\). Moreover

\[ E_{G_{N+1}}E_{G_N} = E_{G_{N+1}} \int_{G_N} gd_Ng = \int_{G_N} d_NgE_{G_{N+1}}g = \int_{G_N} d_NgE_{G_{N+1}} = E_{G_{N+1}} \]

\[ \Box \]
Lemma 8 A state $\varphi$ on $A$ is $G$–quasi–invariant if and only if it is $G_N$–quasi–invariant for each $N \in \mathbb{N}$. If this is the case, denoting for each $g \in G$

$$N_g := \min\{N \in \mathbb{N} : g \in G_N\} \quad (2.42)$$

$$x_g := \text{the } G\text{–cocycle} \quad ; \quad x_{G_N} := \text{the } G_N\text{–cocycle} \quad (N \in \mathbb{N}) \quad (2.43)$$

one has

$$x_g = x_{G_N}g \quad , \quad \forall g \in G_N \quad (2.44)$$

In particular, $\varphi$ is $G$–strongly quasi–invariant if and only if it is $G_N$–strongly quasi–invariant for each $N \in \mathbb{N}$.

Proof. Since any $g \in G$ belongs to some $G_N$, $N_g$ is well defined by (2.42). Since, for each $N \in \mathbb{N}$, $G_N \subseteq G$, it is clear that, if $\varphi$ is $G$–quasi–invariant, it is also $G_N$–quasi–invariant. To prove that (2.44) holds notice that, for any $N \in \mathbb{N}$ such that $g \in G_N$,

$$\varphi(x_g a) = \varphi(g(a)) = \varphi(x_{G_N}g a)$$

The faithfulness of $\varphi$ then implies that $x_g = x_{G_N}g$ for any $N \in \mathbb{N}$ such that $g \in G_N$. Thus in particular (2.44) holds.

Conversely, suppose that $\varphi$ is $G_N$–quasi–invariant for each $N \in \mathbb{N}$ and notice that, since any $g \in G$ belongs to some $G_N$, Then one has, for any $N \geq N_g$,

$$\varphi(x_{G_N}g a) = \varphi(g(a)) = \varphi(x_g a)$$

The faithfulness of $\varphi$ then implies that, defining $x_g$ by the right hand side of (2.44), one has

$$\varphi(g(a)) = \varphi(x_g a) \quad , \quad \forall g \in G$$

i.e. that $\varphi$ is $G$–quasi–invariant with cocycle $g \in G \mapsto x_g$. This proves the first statement of the theorem. Given this and (2.44), the second statement is clear. □

3 Unitary representations associated to strongly quasi–invariant states

Let $G \subseteq \text{Aut}(A)$ be a group of normal $*$–automorphisms of $A$. In the following the cyclic representation of $\{A, \varphi\}$ shall be denoted $\{A_\varphi, \pi, \Phi\}$.
Theorem 5 If \( \varphi \) is \((G,x)\)-strongly quasi-invariant, there exists a unique unitary representation \( U \) of \( G \) on \( \mathcal{A}_\varphi \) characterized by the property

\[
U_g \pi(a) \Phi = \pi(g(a) x_{g^{-1}}^{1/2}) \Phi \quad \forall \ a \in \mathcal{A}
\]

where \( x_{g^{-1}}^{1/2} \) is the positive square root of \( x_{g^{-1}} \). Moreover

\[
U_g^* \pi(a) U_g = \pi(g^{-1}(a)) \quad \forall \ g \in G \quad \forall \ a \in \mathcal{A}
\]

Proof. In the above notations, for any \( a, b \in \mathcal{A} \) and \( g \in G \), one has

\[
\langle U_g \pi(a) \Phi, U_g \pi(b) \Phi \rangle \overset{\text{def}}{=} \langle \pi(g(a) x_{g^{-1}}^{1/2}) \Phi, \pi(g(b) x_{g^{-1}}^{1/2}) \Phi \rangle
= \langle \Phi, \pi(x_{g^{-1}}^{1/2} g(a^* b) x_{g^{-1}}^{1/2}) \Phi \rangle
= \varphi(x_{g^{-1}}^{1/2} g(a^* b) x_{g^{-1}}^{1/2})
\]

\[\overset{\text{Lemma 3}}{=} \varphi(g(a^* b) x_{g^{-1}})
= \varphi \circ g(a^* b \cdot g^{-1}(x_{g^{-1}}))
\overset{\text{Lemma 3}}{=} \varphi(x_g a^* b g^{-1}(x_{g^{-1}}))
\]

\[
\overset{\text{Lemma 3}}{=} \varphi(a^* b g^{-1}(x_{g^{-1}}) x_g)
\overset{\text{3.1}}{=} \varphi(a^* b) = \langle \pi(a) \Phi, \pi(b) \Phi \rangle
\]

Thus \( U_g \) is isometric. Now, if \( g, g' \in G \), then

\[
U_g U_{g'} \pi(a) \Phi = U_g \pi(g'(a) x_{g'^{-1}}^{1/2}) \Phi \quad \overset{\text{3.3}}{=}
= \pi(g'(a) x_{g'^{-1}}^{1/2}) x_{g'^{-1}}^{1/2}) \Phi
= \pi(g g'(a) \cdot g(x_{g'^{-1}}^{1/2}) x_{g'^{-1}}^{1/2}) \Phi
\]

One has

\[
U_g^* = U_{g^{-1}}
\]

19
In fact, for any \(a, b \in \mathcal{A}\),
\[
\langle \pi(a) \Phi, U^*_g \pi(b) \Phi \rangle = \langle U_g \pi(a) \Phi, \pi(b) \Phi \rangle = \langle \pi(g(a)) x^{1/2} \Phi, \pi(b) \Phi \rangle = \varphi(x^{1/2} g(a^*) b) = \varphi(g(g^{-1}(x^{1/2} a^* g^{-1}(b)))) = \varphi(x g^{-1}(x^{1/2}) a^* g^{-1}(b)) \tag{2.7}
\]
\[
\text{Lemma 3} = \varphi(a^* g^{-1}(b)x^{1/2}) = \varphi^*(a^* g^{-1}(b)x^{1/2}_{g^{-1}}) = \langle \pi(a) \Phi, U_{g^{-1}} \pi(b) \Phi \rangle
\]
which is equivalent to (3.4). Since \(x^g\) and \(g^{-1}(x_g)\) are hermitean and commute because of (2.16), the cocycle property implies that
\[
g(x^{1/2}_g x^{1/2}_{g^{-1}}) = (x^{1/2}_{g^{-1}} g(x^{1/2}_g)) \tag{2.3}
\]
Therefore the right hand side of (3.3) becomes
\[
\pi(g g'(a) \cdot x^{1/2}_{(g g')^{-1}}) \Phi = U_{g g'} \pi(a) \Phi
\]
Thus \(U\) is a representation hence, by isometry and (3.4), a unitary representation. Finally, for \(a, b, c \in \mathcal{A}\) and \(g \in G\), one has
\[
U^*_g \pi(a) U_g \pi(b) \Phi = U_{g^{-1}} \pi(a \cdot g(b) \cdot x^{1/2}_{g^{-1}}) \Phi = \pi(g^{-1}[a g(b) x^{1/2}_{g^{-1}}] \cdot x^{1/2}_{g^{-1}}) \Phi \tag{3.5}
\]
\[
= \pi(g^{-1}(a) b \cdot [g^{-1}(x^{1/2}_{g^{-1}}) x^{1/2}_{g}]) \Phi
\]
and, by (2.7) with \(s = 1/2\), the right hand side of (3.5) is equal to
\[
\pi(g^{-1}(a) b) \Phi = \pi(g^{-1}(a)) \pi(b) \Phi
\]
which implies (3.2) by the cyclicity of \(\Phi\). \(\square\)

If \(G\) is the union of an increasing sequence of compact groups, we have the following.
Proposition 4 Let $G$ be given by (2.38) If $\varphi$ is $G$–strongly quasi–invariant, there exists a unique unitary representation $U$ of $G$ on $\mathcal{A}_\varphi$ characterized by the property

$$U_g \pi(a) \Phi = \pi(g(a)x_{g^{-1}}^{1/2})) \Phi \quad \forall a \in \mathcal{A}$$

(3.6)

where $x_{g^{-1}}^{1/2}$ is the positive square root of $x_{g^{-1}}$ and $x_g$ is given by (2.44). Moreover

$$u_g(\pi(a)) := U_g^* \pi(a) U_g = \pi(g^{-1}(a)) \quad \forall g \in G ; \quad \forall a \in \mathcal{A}$$

(3.7)

Proof. Each step of the proof of Theorem 5 involves only one or two elements of $G$. Therefore one can always suppose that these elements belong to some $G_N$. From Lemma 8 one knows that the associated cocycles do not depend on $N$. Therefore the proof of Theorem 5 holds unaltered for $G$. □

Corollary 2 The family $(u_g)_{g \in G}$ of $\ast$–automorphisms of $\pi(\mathcal{A})$ extends uniquely to a representation of $G$ into the normal $\ast$–automorphisms of $\pi(\mathcal{A})''$, still denoted with the same symbol. Denoting, for each $N \in \mathbb{N}$,

$$\hat{E}_{G_N} := \int_{G_N} u_g dN_g$$

(3.8)

the family $(\hat{E}_{G_N})$ is a projective family of normal Umegaki conditional expectations. Moreover, for each $N \in \mathbb{N}$,

$$\text{Range}(\hat{E}_{G_N}) = \text{Fix}(u_G)$$

(3.9)

Proof. From (3.7) it follows that each $u_g$ is a strongly continuous $\ast$–automorphism of $\pi(\mathcal{A})$. Hence it extends uniquely to a representation of $G$ into the normal $\ast$–automorphisms of $\pi(\mathcal{A})''$. Compactness of the $G_N$ implies that each $\hat{E}_{G_N}$ is normal. The proof of Proposition 3 holds for any representation of $G$ into automorphisms of any $C^*$–algebra. In particular (3.9) holds. □

4 Examples of quasi–invariant states under permutations

Let $\mathcal{B} := \mathcal{B}(\mathcal{H})$ for some (separable) Hilbert space $\mathcal{H}$. Denote

$$\mathcal{A} = \bigotimes_N \mathcal{B}$$

(4.1)
and let $j_n$ be the natural embedding from $B$ onto the $n$-th factor of $A_n$ (intuitively $j_n(B) =: A_n \equiv B \otimes \bigotimes_{n \in \mathbb{N} \setminus \{1\}} 1_B$). Denote by $S_N$ the permutation group on $\{0, 1, \ldots, N\}$ and define $S_\infty = \cup_{N \in \mathbb{N}} S_N$. $S_\infty$ has a natural action on $A$ by $*$-automorphisms given by $\pi \circ j_n := j_{\pi n}$ where we use the same symbol for the actions of $S_\infty$ on $\mathbb{N}$ and on $A$. With this identification, $S_\infty$ is called the group of local permutations on $A$.

### 4.1 Product states

The example considered in Proposition 5 below is in fact a particular case of the one discussed in section 4.2. We include it because the idea is best understood in the simplest case.

**Proposition 5** Any product state on $A$

$$\varphi = \bigotimes_{n \in \mathbb{N}} \varphi_n = \bigotimes_{n \in \mathbb{N}} Tr(W_n \cdot) =: Tr_N \left( \bigotimes_{n \in \mathbb{N}} W_n \cdot \right) = Tr_N \left( \prod_{n \in \mathbb{N}} j_n(W_n) \cdot \right)$$

with $W_n$ invertible for each $n \in \mathbb{N}$, is $S_\infty$-strongly quasi-invariant with cocycle

$$x_g := \prod_{n \in \Lambda_g} j_n(W_{n}^{-1})j_{\pi n}(W_n) , \quad \forall g \in S_\infty$$

where $\Lambda_g$ is the support of $g$. 

---

22
Proof. Let \( g \in \mathcal{S}_\infty \) with support \( \Lambda_g \). Note that \( g(\Lambda_g) = \Lambda_g \) and for any \( n \in \Lambda_g, g^{-1}n \neq n \). Then for any \( a \in \mathcal{A} \) one has

\[
\varphi(g(a)) = Tr_N \left( \bigotimes_{n \in \mathbb{N}} W_n g(a) \right)
\]

\[
= Tr_N \left( g \left( \bigotimes_{n \in \mathbb{N}} W_n \right) a \right)
\]

\[
= Tr_N \left( g^{-1} \left( \bigotimes_{n \in \mathbb{N}} W_n \right) a \right)
\]

\[
= Tr_N \left( \prod_{n \in \mathbb{N}} j_{g^{-1}n}(W_n)a \right)
\]

\[
= Tr_N \left( \left( \prod_{n \in \mathbb{N}} j_n(W_n) \right) \left( \prod_{n \in \Lambda_g} j_{g^{-1}n}(W_n) \right) a \right)
\]

\[
= Tr_N \left( \left( \prod_{n \in \mathbb{N}} j_n(W_n) \right) \left( \prod_{n \in \Lambda_g} j_n(W_n^{-1}) \right) \left( \prod_{n \in \Lambda_g} j_{g^{-1}n}(W_n) \right) a \right)
\]

\[
= \varphi \left( \left( \prod_{n \in \Lambda_g} j_n(W_n^{-1})j_{g^{-1}n}(W_n) \right) a \right)
\]

Thus, if for each \( g \in \mathcal{S}_\infty \), \( x_g \) is defined by (4.3), one obtains

\[
\varphi(g(a)) = \varphi(x_g a)
\]

i.e. \( \varphi \) is \( \mathcal{S}_\infty \)-quasi-invariant with cocycle given by (4.3). Finally it is clear that \( x_g \) is hermitean and therefore \( \varphi \) is \( \mathcal{S}_\infty \)-strongly quasi-invariant.

\[\square\]

Proposition 6 Any product state on \( \mathcal{A} \) of the form (4.2) can be written in the form

\[
\varphi = \psi(x_\varphi \cdot )
\]

where

\[
\psi := \bigotimes_{N} \psi_0 = \bigotimes_{N} Tr(V_0 \cdot )
\]
ψ₀ = Tr(V₀·) ∈ ℳ(B) is an arbitrary faithful state and
\[ x_n := V₀⁻¹W_n, \quad ∀ n ∈ ℕ; \quad x_φ := ∏_{νₙ} V₀⁻¹W_n \]  \hspace{1cm} (4.6)
x_φ is hermitean iff \( V₀ \) commutes with all the \( W_n \). Under this assumption, \( φ \) is \( S_∞ \)-strongly quasi–invariant and the cocycle associated to the decomposition (4.4) is given by
\[ x_g := \prod_{n ∈ Λ_g} j_n(x_n⁻¹)j_{g⁻¹}n(x_n), \]  \hspace{1cm} (4.7)
where Λ_g is the support of \( g \).

**Proof.** \( ψ \) is exchangeable by construction and one has
\[ \varphi = ∏_{N} Tr(V₀V₀⁻¹W_n ·) = ψ \left( ∏_{N} V₀⁻¹W_n · \right) \]

Defining
\[ \varphi = ψ \left( ∏_{n ∈ N} x_n · \right) = ψ \left( ∏_{n ∈ N} j_n(x_n) · \right) = ψ(x_φ ·) \]
which is (4.4).

Notice that
\[ x_n = x_n* ⇐⇒ V₀⁻¹W_n = W_nV₀⁻¹ \]  \hspace{1cm} (4.8)
i.e. if and only if \( V₀⁻¹ \) and \( W_n \) commute. If this is the case \( x_φ = x_φ* \) is in the centralizer of \( ψ \). Notice that \( x_n \) is invertible and suppose that (4.8) holds,
then for any $g \in S_\infty$ one has
\[
\varphi(g(a)) = \psi(x_\varphi g(a)) \\
= \psi\left(\bigotimes_{n \in \mathbb{N}} x_n g(a)\right) \\
= \psi\left(g \left( g^{-1} \left( \bigotimes_{n \in \mathbb{N}} x_n \right) a \right) \right) \\
= \psi\left(g^{-1} \left( \bigotimes_{n \in \mathbb{N}} x_n \right) a \right) \\
= \psi\left(\prod_{n \in \mathbb{N}} j_{g^{-1}n}(x_n)\right) \\
= \psi\left(\prod_{n \in \Lambda_g} j_{g^{-1}n}(x_n) \left( \prod_{n \in \Lambda_g} j_n(x_n) \right) a \right) \\
= \psi\left(\prod_{n \in \mathbb{N}} j_n(x_n) \left( \prod_{n \in \Lambda_g} j_n(x_n^{-1}) j_{g^{-1}n}(x_n) \right) a \right) \\
= \varphi\left(\prod_{n \in \Lambda_g} j_n(x_n^{-1}) j_{g^{-1}n}(x_n) \right) a
\]

\[\square\]

4.2 Markov Chains with commuting conditional density amplitudes

Recall that, by the $C^*$-version of de Finetti’s Theorem, all localized sub–algebras of $A = \bigotimes_{n \in \mathbb{N}} B$ are $\psi$–expected for any exchangeable state $\psi$ on $A$. For $I \subseteq \mathbb{N}$, the sub–algebra of $A$ localized in $I$ is denoted $A_I$. We fix an exchangeable state $\psi$ on $A$, we denote $E_I^\psi$ the (unique) Umegaki conditional expectation from $A$ onto $A_I$ satisfying
\[
\psi \circ E_I^\psi = \psi \quad (A_I \ \psi\text{–expected })
\]

Let $j_n$ be the natural embedding from $A$ to $A_{[n,n+1]} =: A_n \otimes A_{n+1}$. Recall that an $E_I^\psi$–conditionally density amplitude (CDA) localized in $A_{[n,n+1]}$
\((n \in \mathbb{N})\) is an operator \(K_{n,n+1} \in \mathcal{A}_{[n,n+1]}\) satisfying
\[
E_{\{n\}}^\psi(j_n(K_{n,n+1}^*K_{n,n+1})) = \left(E_{\{n\}}^\psi(j_n(K_{n,n+1}^*K_{n,n+1}))\right) = 1_{\mathcal{A}_{\{n\}}}
\]
and that the family \((K_{n,n+1})_{n \in \mathbb{N}}\) of CDA is called \textit{commutative} if the \(C^*\)-algebra generated by it is commutative.

**Proposition 7** In the above notations, the state \(\varphi\) on \(\mathcal{A}\) defined by
\[
\varphi(a) = \psi\left(\left(\prod_{n \in \mathbb{N}} j_n(K_{n,n+1}^*)\right)a\left(\prod_{n \in \mathbb{N}} j_n(K_{n,n+1})\right)\right) \tag{4.9}
\]
satisfies
\[
\varphi(g(a)) = \varphi\left(\left(\prod_{n \in \Lambda_g} j_n((K_{n,n+1}^*)^{-1})j_g^{-1}n(K_{n,n+1})\right)a\left(\prod_{n \in \Lambda_g} j_n((K_{n,n+1})^{-1})j_g^{-1}n(K_{n,n+1})\right)\right) \tag{4.10}
\]
where \(\Lambda_g\) is the support of \(g\). In particular, if each \(K_{n,n+1}\) is in the centralizer of \(\psi\), then \(\varphi\) is a \((\mathcal{S}_\infty, x)\)-strongly quasi–invariant state with
\[
x_g = \prod_{n \in \Lambda_g} j_n(|K_{n,n+1}|^2)^{-1})j_g^{-1}n(|K_{n,n+1}|^2), \quad \forall g \in \mathcal{S}_\infty \text{ with support } \Lambda_g \tag{4.11}
\]

**Remark 6** The right hand side of \((4.9)\) is a formal expression. The precise meaning of the right hand side of \((4.9)\) is given by the general theory of quantum Markov chains (see the proof below for this meaning).
**Proof.** Let \( g \in S_\infty \), with support \( \Lambda_g \). Then, one has

\[
\varphi(g(a)) = \psi\left( \left( \prod_{n \in \mathbb{N}} j_n(K_{n,n+1}^{*}) \right) g(a) \left( \prod_{n \in \mathbb{N}} j_n(K_{n,n+1}) \right) \right)
\]

\[
= \psi\left( g \left( \left( \prod_{n \in \mathbb{N}} g^{-1}(j_n(K_{n,n+1}^{*})) \right) a \left( \prod_{n \in \mathbb{N}} g^{-1}(j_n(K_{n,n+1})) \right) \right) \right)
\]

\[
= \psi\left( \left( \prod_{n \in \mathbb{N}} j_{g^{-1}n}(K_{n,n+1}^{*}) \right) a \left( \prod_{n \in \mathbb{N}} j_{g^{-1}n}(K_{n,n+1}) \right) \right)
\]

\[
= \psi\left( \left( \prod_{n \in \Lambda_g^c} j_n(K_{n,n+1}^{*}) \right) \left( \prod_{n \in \Lambda_g} j_{g^{-1}n}(K_{n,n+1}^{*}) \right) a \left( \prod_{n \in \Lambda_g^c} j_{g^{-1}n}(K_{n,n+1}) \right) \left( \prod_{n \in \Lambda_g} j_n(K_{n,n+1}) \right) \right)
\]

\[
= \psi\left( \left( \prod_{n \in \Lambda_g} j_n(K_{n,n+1}) \right) \left( \prod_{n \in \Lambda_g} j_{g^{-1}n}(K_{n,n+1}) \right) a \left( \prod_{n \in \Lambda_g} j_{g^{-1}n}(K_{n,n+1}) \right) \left( \prod_{n \in \Lambda_g} j_n(K_{n,n+1}) \right) \right)
\]

In the last equality we used the fact that for any \( n \in \Lambda_g \), \( g^{-1}n \neq n \) and therefore the terms in the products commute.
Now, if each $K_{n,n+1}$ is in the centralizer of $\psi$, then (4.10) becomes

\[
\varphi(g(a)) = \varphi\left(\prod_{n \in \Lambda_g} j_n((K_{n,n+1})^{-1}(K_{n,n+1}^*)^{-1})j_g^{-1}n(K_{n,n+1}K_{n,n+1}^*)a\right)
\]

\[
= \varphi\left(\prod_{n \in \Lambda_g} j_n(|K_{n,n+1}|^{-1})j_g^{-1}n(|K_{n,n+1}|^2)a\right)
\]

\[
= \varphi(x_g a)
\]

Therefore $\varphi$ is $(S_\infty, x)$–strongly quasi–invariant state with cocycle given by (4.11).

\[\square\]

**Remark.** Notice that all the examples of $S_\infty$–strongly quasi–invariant states constructed in this section, have the form $\varphi = \psi(\rho \cdot)$ where $\psi$ is a $S_\infty$–invariant state and $\rho$ is a positive element of $\mathcal{A}$ satisfying $\psi(\rho) = 1$. Conversely, if $\varphi$ has this form and $\rho$ is invertible, then $\varphi$ is $S_\infty$–strongly quasi–invariant with cocycle $x_g = \rho^{-1}g^{-1}(\rho)$, which is the same structure as in (1.4) with $\kappa_G^{-1}$ replaced by $\rho$.

### 5 On the quasi–invariant states under the group of permutations $S_\infty$

In the notations of section 4 let $\varphi$ be a $S_\infty$–strongly quasi–invariant state with cyclic and separating vector $\Phi$. Then, for each $N \in \mathbb{N}^*$, $\varphi$ is a $S_N$–strongly quasi–invariant state with

\[
\kappa_N = \frac{1}{N!} \sum_{\pi \in S_N} x_\pi, \quad E_N := E_{S_N}
\]

Let $\rho_N = \kappa_N^{-1}$ and define the state

\[
\varphi_N := \varphi \circ E_N
\]

Denote $\mathcal{A}_N = \text{Range}(E_N)$. Since, from (2.41), $E_N : \mathcal{A} \to \mathcal{A}_N$ is a conditional expectation with $\{\mathcal{A}_N\}_N$ is a decreasing sequence of sub von Neumann algebras of $\mathcal{A}$, then from [10], there exists a unique conditional expectation
$E_\infty : \mathcal{A} \to \cap_{n \in \mathbb{N}} \mathcal{A}_n$. Moreover, in [2] Proposition 3, it is shown that for all $a \in \mathcal{A}$, the sequence $E_N(a)$ converges ultrastrongly (i.e. $\sigma$-strongly) to $E_\infty(a)$ as $N \to \infty$.

**Lemma 9** If $\rho_N$ converges in norm to a positive operator $\rho_\infty \in \mathcal{A}$, then for every $a \in \mathcal{A}$

$$\lim_{N} \varphi_N(\rho_N a) = \lim_{N} \varphi_N(\rho_\infty a)$$

**Proof.** Notice that we have

$$|\varphi_N(\rho_N a) - \varphi_N(\rho a)| \leq ||\rho_N - \rho_\infty|| |a|,$$

which converges to 0 when $\rho_N$ converges in norm to $\rho_\infty$.

□

**Lemma 10** One has

$$\lim_{N} \varphi_N(\rho a) = \lim_{N} \varphi_\infty(\rho_\infty a)$$

where $\varphi_\infty = \varphi \circ E_\infty$ is a $S_\infty$-invariant state.

**Proof.** Remember that for all $a \in \mathcal{A}$, $\varphi(a) = \langle \Phi, a\Phi \rangle$. Then, we have

$$|\varphi_N(\rho_\infty a) - \varphi_\infty(a)| = |\langle \Phi, (E_N(a) - E_\infty(a))\Phi \rangle|$$

which converges to 0 as $N \to \infty$ (because from Proposition 3 in [2], $E_N(a)$ converges ultrastrongly to $E_\infty(a)$, and in particular it converges weakly to $E_\infty(a)$). This is proves the first statement of the lemma. Now, for $g \in S_\infty$, there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$, $g$ is an element of $S_N$. Then, from (2.23), one has

$$\langle \Phi, E_N(g(a))\Phi \rangle = \langle \Phi, E_N(\rho a)\Phi \rangle \quad \forall N \geq N_0, \quad \forall a \in \mathcal{A} \quad (5.1)$$

By taking the limit in (5.1) as $N \to \infty$, one gets

$$\varphi(E_\infty(g(a))) = \varphi(E_\infty(a)) \quad \forall a \in \mathcal{A}$$

and therefore $\varphi_\infty$ is a $S_\infty$-invariant state. □

Using together Lemmas 1 and 2, we have proved the following.
Theorem 6 If \( \varphi \) is a \( S_\infty \)-strongly quasi-invariant state and if \( \rho_N = \sigma_N^{-1} \) converges in norm to a positive operator \( \rho_\infty \in \mathcal{A} \), then \( \varphi \) is of the form

\[
\varphi(a) = \varphi_\infty(\rho_\infty a), \quad \forall a \in \mathcal{A}
\]

with cocycle \( x_g = \rho_\infty^{-1}g^{-1}(\rho_\infty) \), where \( \varphi_\infty \) is a \( S_\infty \)-invariant state.

Remark 7 The result of Theorem 6 remains true if \( \varphi \) is \( G \)-strongly quasi-invariant on a von Neumann algebra \( \mathcal{A} \), where \( G \) is given by \( (2.38) \).

References

[1] Bratteli O., Robinson D. W.: Operator Algebras and Quantum Statistical Mechanics I, Springer-Verlag Berlin Heidelberg 1979, Second Edition 1987.

[2] Dang-Ngoc N.: Pointwise convergence of martingales in von Neumann algebras, Israel Journal of Mathematics 34 (4) (1979) 273–280.

[3] Diaconis P., Freedman D.: De Finetti’s Theorem for Markov Chains, The Annals of Probability 8 (1) (1980) 115–130.

[4] Falcone T., Takesaki M.: Operator valued weights without structure theory, Trans. Amer. Math. Soc. 351 (1) (1999) 323–341.

[5] Fannes M., Lewis J.T., Verbeure A.: Symmetric states of composite systems, Lett. Math. Phys. 15 (1988) 255–260.

[6] Haagerup U.: Operator valued weights in von Neumann algebras I, J. Funct. Anal. 32 (1979) 175–206.

[7] Haagerup U.: Operator valued weights in von Neumann algebras II, J. Funct. Anal. 33 (1979) 339–361.

[8] Herman R. H., Takesaki M.: States and Automorphism Groups of Operator Algebras, Comm. Math. Phys. 19 (1970) 142–160.

[9] Sakai S.: \( C^* \)-Algebras and \( W^* \)-algebras, Springer (1971).

[10] Takesaki M.: Conditional Expectations in von Neumann Algebras, J. Funct. Anal. 9 (1972) 306–321.
[11] Størmer E.: States and Invariant Maps of Operator Algebras, J. Funct. Anal. 5 (1970), 44–65.

[12] J. Tomijama: On the projection of norm one in $W^*$–algebras, Proc. Japan Acad. 33 (1957) 608–612.