CONSTRUCTION OF DISCRETE ANALOGUE OF THE DIFFERENTIAL OPERATOR $\frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1$ AND ITS PROPERTIES

A.R. HAYOTOV

Abstract. In the present paper the discrete analogue of the differential operator $\frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1$ is constructed and its some properties are proved.

The optimization problem of approximate integration formulas in the modern sense appears as the problem of finding the minimum of the norm of a error functional $\ell$ given on some set of functions.

The minimization problem of the norm of the error functional by coefficients was reduced in [1] to the system of difference equations of Wiener-Hopf type in the space $L^2(m)$, where $L^2(m)$ is the space of functions with square integrable $m-$th generalized derivative. Existence and uniqueness of the solution of this system was proved by S.L. Sobolev. In the work [1] the description of some analytic algorithm for finding the coefficients of optimal formulas is given. For this S.L. Sobolev defined and investigated the discrete analogue of the differential operator of the polyharmonic operator $\Delta^m$. The problem of construction of the discrete operator $D_{hH}^{(m)}(\beta)$ for $n-$dimensional case was very hard. In one dimensional case the discrete analogue $D_{h}^{(m)}\{\beta\}$ of the differential operator $\frac{d^{2m}}{dx^{2m}}$ was constructed by Z.Zh. Zhamalov and Kh.M. Shadimetov [2,3].

Further, in the work [4] the discrete analogue of the differential $\frac{d^{2m-2}}{dx^{2m-2}}$ was constructed. The constructed discrete analogue of the operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ was applied for finding the coefficients of optimal quadrature formulas in the space $W_2^{(m,m-1)}(0,1)$ (see [5]).

Here, we mainly use a concept of functions of a discrete argument and the corresponding operations (see [1]). For completeness we give some of definitions.

Let $[\beta] = h\beta, \beta \in \mathbb{Z}, h = \frac{1}{N}, N = 1,2,...$. Assume that $\varphi(x)$ and $\psi(x)$ are real-valued functions defined on the real line $\mathbb{R}$.

Definition 1. The function $\varphi[\beta]$ is a function of discrete argument if it is given on some set of integer values of $\beta$.

1991 Mathematics Subject Classification. 65D32.

Key words and phrases. Discrete argument function, discrete analogue of the differential operator, Fourier transformation.
Definition 2. The inner product of two discrete argument functions $\varphi[\beta]$ and $\psi[\beta]$ is given by

$$[\varphi, \psi] = \sum_{\nu=-\infty}^{\infty} \varphi[\beta] \cdot \psi[\beta],$$

if the series on the right hand side converges absolutely.

Definition 3. The convolution of two functions $\varphi[\beta]$ and $\psi[\beta]$ is the inner product

$$\varphi[\beta] * \psi[\beta] = [\varphi[\gamma], \psi[\beta - \gamma]] = \sum_{\gamma=-\infty}^{\infty} \varphi[\gamma] \cdot \psi[\beta - \gamma].$$

In the present paper we consider the problem of construction of the discrete function $D[\beta]$ which satisfies the following equation

$$D[\beta] * \psi[\beta] = \delta[\beta], \quad (1)$$

where

$$G[\beta] = \frac{\text{sign}[\beta]}{4} (\sin[\beta] - [\beta] \cdot \cos[\beta]), \quad (2)$$

$$\delta[\beta] = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0 \end{cases}$$

is the discrete delta function.

The discrete function $D[\beta]$ has important role in calculation of the coefficients of the optimal quadrature formulas in the Hilbert space

$$K_2(P_2) = \left\{ \varphi : [0, 1] \to \mathbb{R} \mid \varphi' \text{ is absolutely continuous and } \varphi'' \in L_2(0, 1) \right\},$$

equipped with the norm

$$\|\varphi\| = \left\{ \int_0^1 (P_2(d/dx) \varphi(x))^2 \, dx \right\}^{\frac{1}{2}},$$

where $P_2(d/dx) = d^2/dx^2 + 1$.

The equation (1) is the discrete analogue of the following equation

$$\left( \frac{d^4}{dx^4} + 2 \frac{d^2}{dx^2} + 1 \right) G(x) = \delta(x), \quad (3)$$

where $G(x) = \frac{\text{sign}(x)}{4} (\sin x - \cos x)$, $\delta(x)$ is Dirac’s delta function. Moreover the discrete function $D[\beta]$ has similar properties as the differential operator

$$\frac{d^4}{dx^4} + 2 \frac{d^2}{dx^2} + 1,$$

i.e. the zeros of the discrete operator $D[\beta]$ are the discrete functions corresponding to the zeros of the operator

$$\frac{d^4}{dx^4} + 2 \frac{d^2}{dx^2} + 1.$$
Theorem 1. The discrete analogue of the differential operator \( \frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1 \) satisfying equation (1) has the form

\[
D[\beta] = \frac{2}{\sin h - h \cdot \cos h} \begin{cases} 
A_1 \cdot \lambda_1^{[\beta]-1}, & |\beta| \geq 2, \\
1 + A_1, & |\beta| = 1, \\
\frac{2h \cos 2h - \sin 2h}{\sin h \cdot h \cos h} + \frac{A_1}{\lambda_1}, & \beta = 0,
\end{cases}
\]

(4)

where

\[ A_1 = \frac{4h^2 \sin^4 h \cdot \lambda_1^2}{(\lambda_1^2 - 1)(\sin h - h \cos h)^2}, \lambda_1 = \frac{2h - \sin 2h - 2 \sin h \cdot \sqrt{h^2 - \sin^2 h}}{2(h \cos h - \sin h)}, \]

|\lambda_1| < 1, h is a small parameter.

Theorem 2. The discrete analogue \( D[\beta] \) of the differential operator \( \frac{d^4}{dx^4} + 2\frac{d^2}{dx^2} + 1 \) satisfies the following equalities

1) \( D[\beta] \ast \sin[\beta] = 0, \)

2) \( D[\beta] \ast \cos[\beta] = 0, \)

3) \( D[\beta] \ast [\beta] \sin[\beta] = 0, \)

4) \( D[\beta] \ast [\beta] \cos[\beta] = 0, \)

5) \( D[\beta] \ast \psi[\beta] = \delta[\beta]. \)

Here \( G[\beta] \) is defined by (2), and \( \delta[\beta] \) is discrete delta function.

In the proofs of these theorems we need the following well known formulas from the theory of generalized (distribution) functions and Fourier transformations (see, for instance, [1])

\[
F[\varphi(x)] = \int_{-\infty}^{\infty} \varphi(x)e^{2\pi ipx} \, dx, \quad F^{-1}[\varphi(p)] = \int_{-\infty}^{\infty} \varphi(p)e^{-2\pi ipx} \, dp,
\]

(5)

\[
F[\varphi \ast \psi] = F[\varphi] \cdot F[\psi],
\]

(6)

\[
F[\varphi \cdot \psi] = F[\varphi] * F[\psi],
\]

(7)

\[
F[\delta^{(\alpha)}(x)] = (-2\pi ip)^\alpha, \quad F[\delta(x)] = 1,
\]

(8)

\[
\delta(hx) = h^{-1}\delta(x),
\]

(9)

\[
\delta(x - a) \cdot f(x) = \delta(x - a) \cdot f(a),
\]

(10)

\[
\delta^{(\alpha)}(x) \ast f(x) = f^{(\alpha)}(x),
\]

(11)

\[
\phi_0(x) = \sum_{\beta = -\infty}^{\infty} \delta(x - \beta), \quad \sum_{\beta} e^{2\pi ix\beta} = \sum_{\beta} \delta(x - \beta).
\]

(12)

Proof of Theorem 1.

According to the theory of periodic generalized functions and Fourier transformations instead of the function \( D[\beta] \) it is convenient to search the harrow-shaped function (see [1])

\[
\overline{D}(x) = \sum_{\beta = -\infty}^{\infty} D[\beta] \delta(x - \beta).
\]
In the class of harrow-shaped functions equation (1) will be in the following form
\[ \hat{D} (x) \ast \hat{G} (x) = \delta(x), \]  
(13)

where \( \hat{G} (x) = \sum \limits_{\beta=-\infty}^{\infty} G[\beta] \delta(x - h\beta) \) is the harrow-shaped function corresponding to the function \( G[\beta] \).

Applying the Fourier transformation to both sides of equation (13) and taking into account (6), (8) we have
\[ F[\hat{D} (x)] = 1 / F[\hat{G} (x)]. \]  
(14)

Now we calculate the Fourier transformation \( F[\hat{G} (x)] \). Using equalities (10) and (12), we get
\[ \hat{G} (x) = h^{-1} G(x) \cdot \phi_0(h^{-1}x). \]  
(15)

Further, use of equation (9) gives
\[ F[\phi_0(h^{-1}x)] = h\phi_0(hp). \]  
(16)

Then, taking account of (15), (16) and (7), we obtain
\[ F[\hat{G} (x)] = F[G(x)] \ast \phi_0(hp). \]  
(17)

To calculate the Fourier transformation \( F[G(x)] \) we use equation (3). Taking into account (11), we rewrite equation (3) in the following form
\[ (\delta^{(4)}(x) + 2\delta^{(2)}(x) + \delta(x)) \ast G(x) = \delta(x). \]

Hence, keeping in mind (6), (8), we have
\[ F[G(x)] = \frac{1}{(2\pi p - 1)^2(2\pi p + 1)^2}. \]  
(18)

Taking into account (18) from (17) we get
\[ F[\hat{G} (x)] = \frac{h^3}{(2\pi)^4} \sum \limits_{\beta=-\infty}^{\infty} \frac{1}{(\beta - h(p + \frac{1}{2\pi}))^2(\beta - h(p - \frac{1}{2\pi}))^2}. \]

Then from (14) we have
\[ F[\hat{D}](p) = \frac{(2\pi)^4}{h^3} \left[ \sum \limits_{\beta=-\infty}^{\infty} \frac{1}{(\beta - h(p + \frac{1}{2\pi}))^2(\beta - h(p - \frac{1}{2\pi}))^2} \right]^{-1}. \]  
(19)

Suppose the Fourier series of the function \( F[\hat{D}](p) \) has the following form
\[ F[\hat{D}](p) = \sum \limits_{\beta=-\infty}^{\infty} \hat{D}[\beta] e^{2\pi iph\beta}, \]  
(20)
where \( \hat{D}[\beta] \) is the Fourier coefficients of the function \( F[\hat{D}](p) \), i.e.

\[
\hat{D}[\beta] = \int_0^{h^{-1}} F[\hat{D}](p) \ e^{-2\pi i ph \beta} dp.
\] (21)

Applying the inverse Fourier transformation to both sides of (20) we obtain the following harrow-shaped function

\[
\hat{D}(x) = \sum_{\beta=-\infty}^{\infty} \hat{D}[\beta]\delta(x - h\beta).
\]

Then according to the definition of harrow-shaped functions the discrete function \( \hat{D}[\beta] \) is searching function of discrete argument \( D[\beta] \). Here for finding of the function \( \hat{D}[\beta] \) we will not use the formula (21). We will find it by the following way.

To calculate the series (19) we use the following well known formula from the residual theory (see [6], p.296)

\[
\sum_{\beta=-\infty}^{\infty} f(\beta) = - \sum_{z_1,z_2,\ldots,z_n} \text{res}(\pi\text{ctg}(\pi z) \cdot f(z))
\] (22)

where \( z_1, z_2, \ldots, z_n \) are poles of the function \( f(z) \).

We denote \( f(z) = \frac{1}{(z-h(p + \frac{1}{2\pi}))^2(z-h(p - \frac{1}{2\pi}))^2} \). Here \( z_1 = h(p + \frac{1}{2\pi}) \) and \( z_2 = h(p - \frac{1}{2\pi}) \) are the poles of order 2. Then taking into account (22) from (19) we have

\[
F[\hat{D}](p) = -\frac{(2\pi)^4}{h^3} \left[ \sum_{z_1,z_2} \text{res}(\pi\text{ctg}(\pi z) \cdot f(z)) \right]^{-1}.
\] (23)

Since

\[
\text{res}_{z=z_1} (\pi\text{ctg}(\pi z) \cdot f(z)) = -\frac{2\pi^4}{h^3} \left( \frac{h}{1 - \cos(2\pi hp + h)} + \frac{\sin(2\pi hp + h)}{1 - \cos(2\pi hp + h)} \right); \\
\text{res}_{z=z_2} (\pi\text{ctg}(\pi z) \cdot f(z)) = -\frac{2\pi^4}{h^3} \left( \frac{h}{1 - \cos(2\pi hp - h)} - \frac{\sin(2\pi hp - h)}{1 - \cos(2\pi hp - h)} \right).
\]

Denoting \( \lambda = e^{2\pi i p h} \), using the last two equalities and taking into account the following well known formulas

\[
\cos z = \frac{e^z + e^{-z}}{2}, \quad \sin z = \frac{e^z - e^{-z}}{2i}
\]

after some simplifications from (23) for \( F[\hat{D}](p) \) we get

\[
F[\hat{D}](p) = \frac{2}{\sin h - h \cos h} \left( \lambda^4 - 4 \cos h \lambda^3 + (2 \cos(2h) + 4) \lambda^2 - 4 \cos h \lambda + 1 \right) \left( \lambda^2 + \frac{2h - \sin(2h)}{\sin h - h \cos h} \right).
\] (24)
To find the explicit form of the discrete operator $D[\beta]$ the equality (24) we expand to the sum of elementary fractions. Since the polynomial $Q_2(\lambda) = \lambda^2 + \frac{2h - \sin(2h)}{\sinh h} \lambda + 1$ has two real roots

$$
\lambda_1 = \frac{2h - \sin(2h) - 2 \sin h \cdot \sqrt{h^2 - \sin^2 h}}{2(h \cos h - \sin h)},
$$

$$
\lambda_2 = \frac{2h - \sin(2h) + 2 \sin h \cdot \sqrt{h^2 - \sin^2 h}}{2(h \cos h - \sin h)},
$$

and $\lambda_1 \cdot \lambda_2 = 1$, $|\lambda_1| < 1$.

Then from (24) we have

$$
\frac{2}{\sin h - h \cos h} \cdot \frac{\lambda^4 - 4 \cos h \lambda^3 + (2 \cos(2h) + 4) \lambda^2 - 4 \cos h \lambda + 1}{\lambda \left( \lambda^2 + \frac{2h - \sin(2h)}{\sinh h} \lambda + 1 \right)} = \frac{\lambda \left( \lambda^2 + \frac{2h - \sin(2h)}{\sinh h} \lambda + 1 \right)}{\lambda^3 - 1}.
$$

For finding unknown coefficients $A$, $A_1$ and $B_1$ in the equation (25) we put $\lambda = 0$, $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

For $\lambda = 0$

$$
A = 1, \quad (26)
$$

for $\lambda = \lambda_1$

$$
A_1 = \frac{\lambda_1^4 - 4 \cos h \lambda_1^3 + (2 \cos(2h) + 4) \lambda_1^2 - 4 \cos h \lambda_1 + 1}{\lambda_1^3 - 1}, \quad (27)
$$

for $\lambda = \lambda_2$

$$
B_1 = \frac{\lambda_2^4 - 4 \cos h \lambda_2^3 + (2 \cos(2h) + 4) \lambda_2^2 - 4 \cos h \lambda_2 + 1}{\lambda_2^3 - 1}.
$$

Hence, taking into account $\lambda_1 \cdot \lambda_2 = 1$, we have

$$
B_1 = -\frac{1}{\lambda_1^2} \cdot A_1. \quad (28)
$$

Finally, taking account of (26), (27), (28) and $|\lambda| = 1$, $|\lambda_1| < 1$, using the formula for geometric progression from (25) consequently we get

$$
F[D](p) = \frac{2}{\sin h - h \cos h} \cdot \left( \frac{\lambda + \frac{2h \cos(2h) - \sin(2h)}{\sin h - h \cos h}}{\lambda^3 - 1} + \frac{A_1}{\lambda - \lambda_1} - \frac{B_1}{\lambda_2 - \lambda_2^2} \right) = \frac{2}{\sin h - h \cos h} \cdot \left( \frac{\lambda + \frac{2h \cos(2h) - \sin(2h)}{\sin h - h \cos h}}{\lambda^3 - 1} + \frac{B_1}{\lambda_2 - \lambda_2^2} \right)
$$

$$
+ \frac{A_1}{\lambda} \sum_{\gamma=0}^{\infty} \left( \frac{\lambda_1}{\lambda} \right)^\gamma - B_1 \lambda_1 \sum_{\gamma=0}^{\infty} (\lambda_1 \lambda)^\gamma \right) =
$$
DISCRETE ANALOGUE OF THE OPERATOR

\[
\lambda(1 + A_1) + \frac{2h \cos(2h) - \sin(2h)}{\sin h - h \cos h} + \frac{A_1}{\lambda_1} + \\
(1 + A_1)\frac{1}{\lambda} + A_1 \sum_{\gamma = -2}^{\infty} \lambda_1^{-\gamma - 1} \lambda^\gamma + A_1 \sum_{\gamma = 2}^{\infty} \lambda_1^{-\gamma - 1} \lambda^\gamma = \sum_{\gamma = -\infty}^{\infty} D[\gamma] \lambda^\gamma.
\]

Hence keeping in mind \( \lambda = e^{2\pi i \phi} \) we obtain the explicit form (4) of the discrete function \( D[\beta] \).

Theorem 1 is proved.

Theorem 2 is proved using Definition 3 and by direct calculation of the left hand sides of 1)-5).

**Remark.** From (4) we note that \( D[\beta] \) is the even function, i.e. \( D[\beta] = D[-\beta] \) and since \( |\lambda_1| < 1 \), then the function \( D[\beta] \) is exponentially decreased as \( \beta \to \infty \).

**REFERENCES**

[1] Sobolev S.L. Introduction to the theory of cubature formulas. Nauka, Moscow, 1974.
[2] Zhamalov Z.Zh. A difference analogue of the operator \( d^{2m}/dx^{2m} \). Direct and inverse problems for partial differential equations and their applications, pp. 97-108, 186, "Fan", Tashkent, 1978.
[3] Shadimetov Kh.M. The discrete analogue of the differential operator \( d^{2m}/dx^{2m} \) and its construction. Questions of Computations and Applied Mathematics. Tashkent, (1985) 22-35. [arXiv:1001.0556v1 [math.NA] Jan. 2010.
[4] Shadimetov Kh.M, Hayotov A.R. Construction of the discrete analogue of the differential operator \( d^{2m}/dx^{2m} - d^{2m-2}/dx^{2m-2} \). Uzbek Math. Zh., 2004, no 2, pp. 85-95.
[5] Shadimetov Kh.M, Hayotov A.R. Computation of coefficients of optimal quadrature formulas in the space \( W^2_{2}(m,m-1) \). Uzbek Math. Zh. 2004, no 3, pp.67-82.
[6] Maqsudov Sh., Salokhidinov M.S., Sirojiddinov S.H. The theory of complex variable functions. -Tashkent, 1976.

Institute of Mathematics, National University of Uzbekistan, Do'rmon yo'li str. 29, Tashkent-100125, Uzbekistan

E-mail address: hayotov@mail.ru