Concurrent Composition Theorems for all Standard Variants of Differential Privacy

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Abstract

We study the concurrent composition properties of interactive differentially private mechanisms, whereby an adversary can arbitrarily interleave its queries to the different mechanisms. We prove that all composition theorems for non-interactive differentially private mechanisms extend to the concurrent composition of interactive differentially private mechanisms for all standard variants of differential privacy including $(\epsilon, \delta)$-DP with $\delta > 0$, Rényi DP, and $f$-DP, thus answering the open question by [VW21]. For $f$-DP, which captures $(\epsilon, \delta)$-DP as a special case, we prove the concurrent composition theorems by showing that every interactive $f$-DP mechanism can be simulated by interactive post-processing of a non-interactive $f$-DP mechanism. For Rényi DP, we use a different approach by showing a triangle-like inequality of Rényi divergence in the special setting of concurrent interaction.

1 Introduction

1.1 Differential Privacy

Differential privacy is a statistical notion of database privacy, which ensures that the output of an algorithm will still have approximately the same distribution if a single data entry were to be changed. Differential privacy can be defined in terms of a general database space $\mathcal{X}$, and a binary neighboring relation on $\mathcal{X}$, which we think of as capturing whether “two datasets” differ on one individual’s data. For example, if databases are real-valued and contain a fixed number $n$ of entries, then $\mathcal{X} = \mathbb{R}^n$, and two datasets $x, x' \in \mathbb{R}^n$ are said to be neighboring if they differ in at most one coordinate.

Definition 1.1 (Differential Privacy [DMNS06]). A randomized algorithm $\mathcal{M} : \mathcal{X} \to \mathcal{R}$ is $(\epsilon, \delta)$-differentially private if for every pair of neighboring datasets $x, x' \in \mathcal{X}$, and for every subset of possible outputs $S \subseteq \mathcal{R}$,

$$\Pr[\mathcal{M}(x) \in S] \leq \exp(\epsilon) \cdot \Pr[\mathcal{M}(x') \in S] + \delta.$$ 

Thus, differential privacy requires that for all neighboring datasets $x, x'$, $\mathcal{M}(x)$ and $\mathcal{M}(x')$ are close as probability distributions (as measured by the parameters $\epsilon$ and $\delta$). A number of variants of differential privacy have been defined based on other ways of measuring closeness, leading to Concentrated differential privacy (CDP) [DR16, BS16] and Rényi differential privacy (RDP) [Mir17] and $f$-differential privacy ($f$-DP) [DRS19].

1.2 Interactive Differential Privacy

Definition 1.1 considers only non-interactive mechanisms $\mathcal{M}$ that release query answers in one shot, but data analysts often interact with a database in an adaptive fashion. In fact, many useful primitives in differential
privacy such as the Sparse Vector Technique \[\text{DNR}^{+}09, \text{DNPR}10, \text{DR}14\], and the Private Multiplicative Weights \[\text{HR}10\] allow analysts to ask an adaptive sequence of queries about a dataset. It motivates the study of interactive mechanisms to capture full-featured privacy-preserving data analytics. Here, we view the mechanism $\mathcal{M}$ as a party in an interactive protocol, interacting with a (possibly adversarial) analyst.

**Definition 1.2 (Interactive protocols).** An interactive protocol $(A, B)$ is any pair of functions on tuples of binary strings. The interaction between $A$ with input $x_A$ and $B$ with input $x_B$ is the following random process (denoted $(A(x_A), B(x_B))$):

1. Uniformly choose random coins $r_A$ and $r_B$ for $A$ and $B$, respectively.
2. Repeat the following for $i = 0, 1, \ldots$.
   a) If $i$ is even, let $m_i = A(x_A, m_1, m_3, \ldots, m_{i-1}; r_A)$.
   b) If $i$ is odd, let $m_i = B(x_B, m_0, m_2, \ldots, m_{i-1}; r_B)$.
   c) If $m_i = \text{halt}$, then exit loop.

The view of a party in an interactive protocol captures everything the party “sees” during the execution.

**Definition 1.3 (View of a party in an interactive protocol).** Let $(A, B)$ be an interactive protocol. Let $r_A$ and $r_B$ be the random coins for $A$ and $B$, respectively. $A$’s view of $(A(x_A; r_A), B(x_B; r_B))$ is the tuple $\text{View}_A(A(x_A; r_A) \leftrightarrow B(x_B; r_B)) = (r_A, x_A, m_1, m_3, \ldots)$ consisting of all the messages received by $A$ in the execution of the protocol together with the private input $x_A$ and random coins $r_A$. $B$’s view of $(A(x_A; r_A), B(x_B; r_B))$ is defined symmetrically.

In the setting of differentially private mechanisms, party $A$ is the adversary that does not have an input $x_A$. Party $B$ is the mechanism, where the input $x_B$ is the dataset. Since we only care about the view of the adversary, we will drop the subscript and denote the view of the adversary as $\text{View}(A \leftrightarrow \mathcal{M}(x))$. With this notation, interactive differential privacy is defined by asking for the views of an adversary on any pair of neighboring datasets $\text{View}(A \leftrightarrow \mathcal{M}(x))$ and $\text{View}(A \leftrightarrow \mathcal{M}(x'))$ satisfying the same $(\epsilon, \delta)$-closeness notion as in non-interactive differential privacy.

**Definition 1.4.** A randomized algorithm $\mathcal{M}$ is an $(\epsilon, \delta)$-differentially private interactive mechanism if for every pair of neighboring datasets $X, X' \in \mathcal{X}$, every adversary algorithm $A \in \mathcal{A}$, and every subset of possible views $\mathcal{S} \subseteq \text{Range}(\text{View})$, we have

$$\Pr[\text{View}(A \leftrightarrow \mathcal{M}(x)) \in \mathcal{S}] \leq \exp(\epsilon) \cdot \Pr[\text{View}(A \leftrightarrow \mathcal{M}(x')) \in \mathcal{S}] + \delta.$$ 

### 1.3 Concurrent Composition

A fundamental problem in differential privacy is studying how the privacy degrades under composition as more computations are performed on the same database. The composition property is particularly useful when we want to ask interactive queries on the same database, and it also allows us to design a complex differentially private algorithm by combining several building blocks. Formally, we define the composition of a sequence of non-interactive $k$ mechanisms $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ as the non-interactive mechanism $\mathcal{M} = \text{Comp}(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k)$ defined as

$$\mathcal{M}(x) := (\mathcal{M}_1(x), \mathcal{M}_2(x), \ldots, \mathcal{M}_k(x)), \tag{1}$$

where each mechanism $\mathcal{M}$ is executed using independent random coins.

The composition of non-interactive mechanisms has been studied extensive in the literature. The basic composition theorem \[\text{DKM}^{+}06\] states that the privacy parameters add up linearly when composing private mechanisms. The advanced composition theorem \[\text{DRV}10\] provides a tighter bound where the privacy parameter grows sublinearly under $k$-fold adaptive composition. Later, the optimal composition theorem \[\text{KOV}15, \text{MV}16\] gives an exact characterization of the privacy guarantee under $k$-fold composition. The relaxations of differential privacy including zero-concentrated differential privacy (zCDP) \[\text{DR}16, \text{BST}16\], Renyi
differential privacy (RDP) [Mir17], and \( f \)-differential privacy (\( f \)-DP) [DRS19] allows for tighter reasoning about composition. In the abovementioned work, some of them [DRV10, MV16] are framed in a way that the adversary can adaptively choose the mechanisms \( M_1, M_2, \ldots, M_k \), and thus the adaptive composition can be viewed as an interactive mechanism.

![Figure 1: Concurrent composition of interactive mechanisms](image)

In many cases, analysts may wish to perform multiple interactive analyses on the same dataset, which raises the question of concurrent composition, first studied for differential privacy in [VW21]. In this setting (illustrated in Figure 1), an adversary can arbitrarily interleave its queries to several differentially private mechanisms, and those queries might be correlated and depends on the answer received in other mechanisms. As a motivating example, several organizations might set up multiple DP query systems on datasets that may refer to the same set of individuals. Each query system has its own privacy budget \( \epsilon \). Suppose an adversary can concurrently access those systems, and a query sent to one system might depends on all the previous messages that received from other systems. For example, when we run two Sparse Vector mechanisms \( M_1 \) and \( M_2 \) concurrently, the queries for \( M_1 \) depends on the all previous answers from \( M_1 \) and \( M_2 \). We only know the overall privacy guarantees for \( M_1 \) and \( M_2 \) when they are executed independently. Directly using sequential composition technique is not applicable for analyzing the concurrent composition. It is not clear if the adversary can run any concurrent attack to break the privacy guarantees, and therefore, we wish to provide a provable guarantee to account for the total privacy loss in such systems. Formally, the concurrent composition of interactive mechanisms is defined as follows.

**Definition 1.5** (Concurrent composition of interactive mechanisms [VW21]). Let \( M_1, \ldots, M_k \) be interactive mechanisms. \( M = \text{ConComp}(M_1, \ldots, M_k) \) is the concurrent composition of mechanisms \( M_1, \ldots, M_k \) defined as follows:

1. Random sample \( r = (r_1, \ldots, r_k) \) where \( r_j \) are random coin tosses for \( M_j \).
2. Inputs for \( M \) consists of \( x = (x_1, \ldots, x_k) \) where \( x_j \) is a private dataset for \( M_j \).
3. \( M(x, m_0, \ldots, m_{i-1}; r) \) is defined as follows:
   
   (a) Parse \( m_{i-1} \) as \((j, q)\) where \( j = 1, \ldots, k \) and \( q \) is a query to \( M_j \). If \( m_{i-1} \) cannot be parsed correctly, output \text{halt}.
   
   (b) Extract history \((m_0^j, \ldots, m_{t-1}^j)\) from \((m_0, \ldots, m_{i-1})\) where \( m_i^j \) are all of the queries to mechanism \( M_j \).
   
   (c) Output \( M_j(x_j, m_0^j, \ldots, m_{t-1}^j; r_j) \).
For an adversary $A$, we will use the notation $\text{View}(A \leftrightarrow (M_1, \ldots, M_k))$ as shorthand for $\text{View}(A \leftrightarrow \text{ConComp}(M_1, \ldots, M_k))$

Vadhan and Wang [WV21] showed that the advanced and optimal composition theorems extend to the concurrent composition of interactive pure DP mechanisms.

**Theorem 1.1 ([WV21]).** Suppose that for all non-interactive mechanisms $M_1, \ldots, M_k$ such that $M_i$ is $(\epsilon_i, \delta_i)$-differentially private for $\delta_1 = \delta_2 = \ldots = \delta_k = 0$, their composition $\text{Comp}(M_1, \ldots, M_k)$ is $(\epsilon, \delta)$-differentially private. Then for all interactive mechanisms $M_1, \ldots, M_k$ such that $M_i$ is $(\epsilon_i, \delta_i)$-differentially private for $\delta_1 = \delta_2 = \ldots = \delta_k = 0$, the concurrent composition $\text{ConComp}(M_1, \ldots, M_k)$ of interactive mechanisms $M_1, \ldots, M_k$ is $(\epsilon, \delta)$-differentially private.

They prove this by reducing the analysis of interactive pure DP mechanism to that of analyzing the Randomized Response mechanism [War65, DMNS06].

**Theorem 1.2 ([WV21]).** Suppose that $M$ is an interactive $(\epsilon, 0)$-differentially private mechanism. Then for every pair of neighboring datasets $x, x'$, there exists an interactive post-processing function $T$ such that for every adversary $A \in A$, we have

$$\text{View}(A \leftrightarrow (x)) \equiv \text{View}(A \leftrightarrow T(\text{RR}_e(0))) \quad \text{View}(A \leftrightarrow (x')) \equiv \text{View}(A \leftrightarrow T(\text{RR}_e(1))).$$

Here $T$ is an interactive post-processing function that depends on $M$ and a fixed pair of neighboring datasets $x, x'$. It receives a single bit as an output of $\text{RR}_e(0)$ or $\text{RR}_e(1)$, and then interacts with the adversary $A$.

Note that Theorem 1.1 and Theorem 1.2 do not apply to the case where the composed mechanisms $M_i$ are $(\epsilon_i, \delta_i)$-DP for $\delta_i > 0$. In this case, [WV21] only show a bound that is similar to the “group privacy” property of $(\epsilon, \delta)$-DP. In particular, if $\epsilon_1 = \epsilon_2 = \ldots = \epsilon_k = \epsilon$ and $\delta_1 = \delta_2 = \ldots = \delta_k = \delta$, they show that the concurrent composition $\text{ConComp}(M_1, M_2, \ldots, M_k)$ is $(k\epsilon, exp(k\epsilon-1)\delta)$-differentially private. This is suboptimal even compared to basic composition. It left as an open problem that if any composition theorems for non-interactive mechanisms can extend to all variants of DP interactive mechanisms.

**Open Question.** Does Theorem 1.1 extend to other variants of DP (such as $(\epsilon_i, \delta_i)$-DP with $\delta_i > 0$, Rényi DP, $f$-DP)?

### 1.4 Our Results on Concurrent Composition

In this paper, we close this gap and show that any composition theorems of non-interactive mechanisms also extend to the concurrent composition of interactive DP mechanisms for all standard variants of DP. In particular, we show that Theorem 1.1 extends to the case that $\delta_i > 0$:

**Theorem 1.3** (Concurrent composition for $(\epsilon, \delta)$-DP interactive mechanisms). Suppose that for all non-interactive mechanisms $M_1, \ldots, M_k$ such that $M_i$ is $(\epsilon_i, \delta_i)$-differentially private for $i = 1, 2, \ldots, k$, their composition $\text{Comp}(M_1, \ldots, M_k)$ is $(\epsilon, \delta)$-differentially private. Then for all interactive mechanisms $M_1, \ldots, M_k$ with finite communication such that $M_i$ is $(\epsilon_i, \delta_i)$-differentially private for $i = 1, 2, \ldots, k$, the concurrent composition $\text{ConComp}(M_1, \ldots, M_k)$ of interactive mechanisms $M_1, \ldots, M_k$ is $(\epsilon, \delta)$-differentially private.

We also handle general $f$-DP and Rényi DP as defined and discussed in the sections below.

**Theorem 1.4** (Concurrent composition for $f$-DP interactive mechanisms). The analogue of Theorem 1.3 holds for $f$-DP.

**Theorem 1.5** (Concurrent composition for RDP interactive mechanisms). The analogue of Theorem 1.3 holds for Rényi DP of any fixed order.

**Theorem** 1.3 follows directly from Theorem 1.4 because $f$-DP defined below captures $(\epsilon, \delta)$-DP as a special case [WZ10, DRS19]. Interestingly, the generalization to $f$-DP is important for the proof, even if we only want to prove Theorem 1.3. The proof technique for Rényi DP (Theorem 1.5) is very different. We explain the detailed proof techniques for these results in the sections below.
1.5 Proof for $f$-DP

$f$-differential privacy (f-DP) \cite{DRS19} is a generalization of $(\epsilon, \delta)$-differential privacy based on the hypothesis testing interpretation of differential privacy. Differential privacy attempts to measure the difficulty of distinguishing two neighboring databases based on the output of a mechanism. Specifically, an adversary considers the following hypothesis testing problem:

$$H_0 : \text{the dataset is } x \quad \text{versus} \quad H_1 : \text{the dataset is } x'.$$

Denote by $Y$ and $Y'$ the output distributions of $M$ on the two neighboring datasets, namely $M(x)$ and $M(x')$. For a given rejection rule $\phi$, the type I error $\alpha_\phi = E[\phi(Y)]$ is the probability of rejecting $H_0$ when $H_0$ is true, while the type II error $\beta_\phi = 1 - E[\phi(Y')]$ is the probability of failing to reject $H_0$ when $H_1$ is true. A trade-off function serves as the optimal boundary of the achievable and unachievable regions of these errors.

**Definition 1.6 (Trade-off function \cite{DRS19}).** For any two probability distributions $Y$ and $Y'$ on the same space, define the trade-off function $T(Y, Y') : [0, 1] \to [0, 1]$ as

$$T(Y, Y')(\alpha) = \inf \{ \beta_\phi : \alpha_\phi \leq \alpha \}, \quad (3)$$

where the infimum is taken over all (measurable) rejection rules $\phi$.

Proposition 1.7 gives the necessary and sufficient condition for $f$ to be a trade-off function.

**Proposition 1.7 (Class of trade-off functions \cite{DRS19}).** A function $f : [0, 1] \to [0, 1]$ is a trade-off function if and only if $f$ is convex, continuous, non-increasing, and $f(x) \leq 1 - x$ for $x \in [0, 1]$.

$f$-DP allows the full trade-off between type I and type II errors in the simple hypothesis testing problem to be governed by a trade-off function $f$. A larger trade-off functions implies stronger privacy guarantees.

**Definition 1.8 ($f$-differential privacy \cite{DRS19}).** Let $f$ be a trade-off function. A mechanism $M : X \to R$ is $f$-differentially private if for every pair of neighboring datasets $x, x' \in X$, we have

$$T(M(x), M(x')) \geq f.$$

$(\epsilon, \delta)$-DP is a special case of $f$-DP, taking $f = f_{\epsilon, \delta}$, where $f_{\epsilon, \delta} = \max\{0, 1 - \delta - \exp(\epsilon)\alpha, \exp(-\epsilon)(1 - \delta - \alpha)\}$ \cite{WZ10, DRS19}.

To prove Theorem 1.4 (and hence Theorem 1.3), we prove the following analogue of Theorem 1.2 showing that every interactive $f$-DP mechanism can be simulated by an interactive post-processing of a non-interactive mechanism.

**Theorem 1.6.** For every trade-off function $f$, every interactive $f$-DP mechanism $M$ with finite communication, and every pair of neighboring datasets $x, x'$, there exists a non-interactive $f$-DP mechanism $N$ and an randomized interactive post-processing mechanism $T$ such that for every adversary $A \in A$, we have

$$\text{View}(A \leftrightarrow M(x)) \equiv \text{View}(A \leftrightarrow T(N(x))) \quad \text{and} \quad \text{View}(A \leftrightarrow M(x')) \equiv \text{View}(A \leftrightarrow T(N(x'))). \quad (4)$$

Similarly to Theorem 1.2 in the case of $(\epsilon, \delta)$-DP, one can take the non-interactive mechanism $N$ as the $(\epsilon, \delta)$-Randomized Response mechanism of \cite{KO15}. Indeed, \cite{KO15} show that every non-interactive $(\epsilon, \delta)$-DP mechanism can be simulated as a post-processing of $(\epsilon, \delta)$-Randomized Response.

Theorem 1.4 follows from Theorem 1.6 in the same way as Theorem 1.1 follows from Theorem 1.2. Indeed, Theorem 1.4 implies that to analyze the concurrent composition of interactive mechanisms $M_i$, it suffices to consider the composition of the non-interactive mechanisms $N_i$. As a result, composition theorems for non-interactive mechanisms extend to the concurrent composition of interactive $f$-DP mechanisms.

Theorem 1.6 is an interesting statement about statistical hypothesis testing even without the application to differential privacy. Normally, hypothesis testing is the presented as the task of distinguishing between two distributions or sets of distributions. This is a noninteractive task: a sample from the distribution is
generated and given to the hypothesis tester, which then tries to decide whether the distribution is in $H_0$ or $H_1$. However, suppose instead we consider the task of distinguishing between two interactive mechanisms $M_0$ and $M_1$, each of which responds to queries in a randomized and stateful manner. Since the mechanisms are stateful, the hypothesis tester may never learn everything there is to know about the mechanism; in particular it cannot find out how the mechanism would have answered if different queries had been asked in the past. This is in contrast to ordinary hypothesis testing, where the full sample from the distribution is given to the hypothesis tester. Nevertheless, by viewing $M_0$ as $M(x)$ and $M_1$ as $M(x')$, Theorem 1.6 implies that the two interactive mechanisms $M_0$ and $M_1$ can be simulated perfectly by noninteractive random variables $N_0 = N(x)$ and $N_1 = N(x')$ such that even if we give $N_0$ or $N_1$ to a hypothesis tester in its entirety (thereby revealing how $M_0$ or $M_1$ would answer all questions), it cannot distinguish them any better than it could distinguish $M_0$ and $M_1$. The trick, of course, is that the simulation is “perfect” only when executing a single interaction with $M_0$ or $M_1$ (with no rewinding to explore multiple paths in the interaction tree).

The proof of Theorem 1.6 relies on the following two lemmas.

**Lemma 1.7 (Coupling property of f-DP).** Let $f$ be a trade-off function, and suppose we have random variables $X_0$, $X_1$ and $X'_0$, $X'_1$ such that

$$T(X_0, X'_0) \geq f \text{ and } T(X_1, X'_1) \geq f.$$ 

Then there exists couplings $(X_0, X_1)$ and $(X'_0, X'_1)$ such that

$$T((X_0, X_1)||(X'_0, X'_1)) \geq f.$$ 

A coupling of random variables $X_0$ and $X_1$ is any random vector $(Y_0, Y_1)$ such that the marginal distributions are identically distributed to $X_0$ and $X_1$ respectively, i.e., $Y_0 \equiv X_0$ and $Y_1 \equiv X_1$. Subject to this constraint on the marginal, $Y_0$ and $Y_1$ can be arbitrarily correlated. Allowing correlations is critical to Lemma 1.7. For example, for the case of $(\varepsilon, \delta)$-DP, if we keep $X_0$, $X_1$ and $X'_0$, $X'_1$ independent, then we would just get the “group privacy” like bound.

**Lemma 1.8 (Chain rule of f-DP).** For every pair of random variables $X, X'$ with finite support, there exists a function $\text{ChainRule}_{X, X'}$ such that for every random variable $Y$ jointly distributed with $X$, and every random variable $Y'$ jointly distributed with $X'$, we have

$$T((X, Y), (X', Y')) = \text{ChainRule}_{X, X'}((T(Y|X = x, Y'|X' = x'))_{x \in \text{supp}(X) \cap \text{supp}(X')}).$$

Moreover, $\text{ChainRule}$ is a “continuous monotone function” on the partially ordered set of trade-off functions (see Section 2 for formal definition).

Lemma 1.8 says that the trade-off function between $(X, Y)$ and $(X', Y')$ can be determined by a collection of trade-off functions between $Y$ and $Y'$ conditioned on $X = X' = x$ for every $x \in \text{supp}(X) \cap \text{supp}(X')$ through a ChainRule function. The terminology “chain rule” is by analogy with the standard chain rule for KL divergence, which says

$$\text{KL}(X, Y) (| (X', Y')) = \text{KL}(X || X') + E_{x \sim X} \text{KL}(Y|X = x || Y'|X' = x).$$

(5)

So fixing $X$ and $X'$, we can calculate the KL divergence for arbitrary $Y$ and $Y'$ as a function of the KL divergences $\text{KL}(Y|X = x || Y'|X' = x)$. $(\varepsilon, \delta)$-DP does not admit the chain rule property, because no pair of $(\varepsilon, \delta)$ can exactly capture the “closeness” of $(X, Y)$ and $(X', Y')$ given a collection of $\{\varepsilon_j, \delta_j\}_j$ that characterize the “closeness” of $Y|X = x$ and $Y'|X' = x'$. Working with the general $f$-DP allows us to capture a complete characterization of “privacy”.

To prove Theorem 1.6 using Lemmas 1.8 and 1.8, our strategy is to apply induction on the number of messages exchanged (which we can do since $\mathcal{M}$ has finite communication by assumptions). To reduce $k$ rounds of interactions to $k - 1$ rounds, we consider the subsequent interaction conditioned on the first message. Depending on whether the first message sent from the mechanism $\mathcal{M}$ or the adversary $A$, we consider the following two cases.
Case 1. The adversary $B$ sends the first query $q_1$ to the mechanism $M$. Fix a pair of neighboring datasets $x, x'$. Fixing $q_1$, we denote the the subsequent interactive mechanism by $M_{q_1}$. By induction, $M_{q_1}$ can be simulated by a post-processing of a non-interactive $f$-DP mechanism $N_{q_1}$. Then we are coupling all possible pairs of $N_{q_1}$ on $x, x'$ on all the values of $q_1$, which is finite by our assumption of finite communication. We note that the coupling lemma 1.7 extends to finite number of random variables $X_0, X_1, \ldots, X_k$ and $X'_0, X'_1, \ldots, X'_k$ by induction on $k$. Following the coupling lemma, we have $T(((N_{q_1}(x))_{q_1}), ((N_{q_1}(x'))_{q_1})) \geq f$. Since the coupling $((N_{q_1})_{q_1})$ are obtained by post-processing of the non-interactive mechanisms $N_{q_1}$ for all $q_1$, we have Theorem 1.6 holds for $k$ rounds of interactions.

Case 2. Mechanism $M$ sends the first message $a_1$ to the adversary $B$. Fix a pair of neighboring datasets $x, x'$. We denote the mechanism conditioned on every $a_1$ by $M_{a_1}$, and let $f_{a_1}$ be the trade-off function of $M_{a_1}$ (maximized over all adversaries). We denote the random variable of the first message as $A_1, A'_1$ on datasets $x, x'$, respectively. By induction, $M_{a_1}$ can be simulated by a post-processing of a non-interactive $f_{a_1}$-DP mechanism $N_{a_1}$. Thus, $M$ can be simulated by a post-processing of the non-interactive mechanism $N$ where $N(x) \equiv (A_1, N_{A_1}(x))$ and $N(x') \equiv (A'_1, N_{A'_1}(x'))$. We conclude that Theorem 1.6 holds for $k$ rounds of interactions.

1.6 Proof for Rényi DP

Rényi differential privacy (RDP) [M17] is based on Rényi divergence between the output distributions of $M$ on the two neighboring datasets.

Definition 1.9 (Rényi differential privacy (RDP) [M17]). A randomized algorithm $M : \mathcal{X} \rightarrow \mathcal{R}$ is $(\alpha, \epsilon)$-RDP if for every pair of neighboring datasets $x, x' \in \mathcal{X}$,

$$D_\alpha(M(x)||M(x')) \leq \epsilon.$$

Rényi divergence has the chain rule property, similar to KL divergence in Equation 5 (which is the case $\alpha = 1$). However, it does not satisfy the coupling property, and that Theorem 1.6 does not hold for Rényi DP. Thus, proving Theorem 1.6 requires a different approach.

First, we reformulate Theorem 1.5 as follows:

Theorem 1.9. For all $\alpha > 1$, $k \in \mathbb{N}$, $\epsilon_1, \ldots, \epsilon_k > 0$, and all interactive mechanisms $M_1, \ldots, M_k$ such that $M_i$ is $(\alpha, \epsilon_i)$-RDP for $i = 1, 2, \ldots, k$, the concurrent composition $\text{ConComp}(M_1, \ldots, M_k)$ of interactive mechanisms $M_1, \ldots, M_k$ is $(\alpha, \sum_{i=1}^k \epsilon_i)$-RDP.

The reason that Theorem 1.9 is equivalent to Theorem 1.5 is that when the order $\alpha$ of the Rényi divergence is fixed, then summing the $\epsilon_i$'s is the optimal composition bound for RDP [M17].

To prove a summation bound like the above, it is tempting to use a hybrid argument and argue via a triangle inequality that:

$$D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x))))||\text{View}(B \leftrightarrow (M_1(x'), \ldots, M_k(x')))) \leq \sum_{i=1}^k D_\alpha(\text{View}(B \leftrightarrow (M_i(x'), \ldots, M_{i-1}(x'), M_i(x), M_{i+1}(x), \ldots, M_k(x))))$$

That is, we change the inputs to the mechanisms from $x$ to $x'$ one at a time. The hybrids in the $i$'th term differ only in the input to mechanism $i$, so the Rényi divergence can be bounded by $\epsilon_i$. Unfortunately, Rényi divergence does not satisfy a triangle inequality (except when $\alpha = \infty$). Nevertheless, we prove the following similar statement for the setting of concurrent interactions that suffices:
Lemma 1.10. For every adversary $B$ concurrently interacting with interactive RDP mechanisms $M_1, M_2, \ldots, M_k$ operating on two neighboring datasets $x, x'$, we have

$$D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x)))) \mid \text{View}(B \leftrightarrow (M_1(x'), \ldots, M_k(x')))) = \sum_{i=1}^k D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x)))) \mid \text{View}(B \leftrightarrow (M_1(x), \ldots, M_{i-1}(x), M_i(x'), M_{i+1}(x), \ldots, M_k(x)))),$$

where $\text{View}(B \leftrightarrow (M_1(x'), \ldots, M_{i-1}(x'), M_i(x'), M_{i+1}(x), \ldots, M_k(x)))$ is defined similarly to $\text{View}(B \leftrightarrow \text{ConComp}(M_1, \ldots, M_k(x)))$, except that we give mechanism $M_j$ input $x'$ rather than input $x$. In contrast to the hybrid argument approach, here we always compare the case that all of the inputs are $x$ to the case where one input is $x'$. Still, the fact that in the $i$'th term, the inputs only differ in the input to mechanism $M_i$ allows us to still obtain a bound of $\sum_i \epsilon_i$.

### 2 Generalized Definitions of DP Mechanisms

To prove our results and discuss the several variants of differential privacy, it is convenient to introduce a more general abstraction, where distances between probability distribution can be in an arbitrary partially ordered set.

**Definition 2.1 (Generalized distance).** A generalized probability distance measure is a tuple $(D, \preceq, D)$ such that

1. $(D, \preceq)$ is a partially ordered set (poset).
2. $D$ is a mapping that takes any two random variables $X, X'$ taking values in the same set to an element $D(X, X')$ of $D$.
3. (Post-processing.) The generalized distance mapping $D$ is closed under post-processing, meaning that for every function $g$, $D(g(X), g(X')) \preceq D(X, X')$.
4. (Joint Convexity.) Suppose we have a collection of random variables $(X_i, X'_i)_{i \in I}$ and a random variable $I$ distributed on $I$. If $D(X_i, X'_i) \preceq d$ for all $i \in I$, then $D(X_i, X'_i) \preceq d$.

For the generalized notion $d$-DP, the difficulty of distinguishing two neighboring datasets is measured by the generalized distance between the distributions of an adversary’s views. The partially ordered set allows us to compare the level of privacy guarantees of mechanisms.

**Definition 2.2 (d-DP).** Let $(D, \preceq, D)$ be a generalized probability distance. For $d \in D$, we call an interactive mechanism $M$ $d$-DP if for every adversary $A \in A$ and every pair of neighboring datasets $x, x'$, we have

$$D(\text{View}(A \leftrightarrow M(x)), \text{View}(A \leftrightarrow M(x'))) \preceq d.$$

Let us instantiate the standard pure DP and its variants using the definition above by specifying the generalized distances.

**Example: pure DP.** For pure DP, a smaller $\epsilon$ provides stronger privacy guarantee, so the partially ordered set $D$ is defined as $((\mathbb{R}^{\geq 0}) \cup \{\infty\}, \leq)$. The distance mapping is the max-divergence $D_\infty$. For two probability distributions $P$ and $Q$, the max-divergence is

$$D_\infty(P||Q) := \sup_{T \subseteq \text{supp}(P)} \log \left( \frac{\Pr(P(x) \in T)}{\Pr(Q(x) \in T)} \right).$$

Max-divergence is closed under post-processing due to the data-processing inequality. Max-divergence satisfies joint convexity due to the following lemma.

**Lemma 2.1 ([YEH14]).** For every two pairs of probability distributions $(P_0, Q_0)$ and $P_1, Q_1$, and every $\lambda \in (0, 1)$,

$$D_\infty((1 - \lambda)P_0 + \lambda P_1 || (1 - \lambda)Q_0 + Q_1) \leq \max\{D_\infty(P_0 || Q_0), D_\infty(P_1 || Q_1)\}.$$
Example: Rènyi DP. For Rènyi DP of order $\alpha$, the partially ordered set $\mathcal{D}$ is also $((\mathbb{R}^{\geq 0}) \cup \{\infty\}, \leq)$. The distance mapping is $\alpha$-Rènyi divergence for $\alpha \in (1, \infty)$. The Rènyi divergence is defined as follows.

**Definition 2.3** (Rènyi divergence [Rén61]). For two probability distribution $P$ and $Q$, the Rènyi divergence of order $\alpha > 1$ is

$$D\alpha(P||Q) = \frac{1}{\alpha - 1} \log \left( E_{x \sim Q} \left[ \left( \frac{P(x)}{Q(x)} \right)^\alpha \right] \right).$$

Rènyi divergence is also closed under post-processing due to the data-processing inequality, and it satisfies the joint convexity because an analogue of Lemma 2.1 also holds for Rènyi divergence:

**Lemma 2.2** ([VEH14]). For every order $\alpha > 1$, every two pairs of probability distributions $(P_0, Q_0)$ and $P_1, Q_1$, and every $\lambda \in (0, 1)$,

$$D\alpha((1 - \lambda)P_0 + \lambda P_1||Q_0 + (1 - \lambda)Q_1) \leq \max\{D\alpha(P_0||Q_0), D\infty(P_1||Q_1)\}.$$

Example: $f$-DP. For $f$-DP, the partially ordered set $\mathcal{D}$ is defined as $(\mathcal{F}, \preceq)$, where $\mathcal{F}$ is the set of all trade-off functions that satisfies the conditions in Proposition 1.7. The partial ordering is defined as $f_1 \preceq f_2$ if $f_1(\alpha) \geq f_2(\alpha)$ holds for all $\alpha \in [0, 1]$. Note that the direction of the inequalities is reversed, corresponding to the fact that a larger trade-off function means less privacy loss. The distance mapping is the trade-off function $T$ in Definition 1.6.

$f$-DP also satisfies the two properties. First, $f$-DP is preserved under post-processing. We will only need to show the joint convexity of $f$-DP.

**Lemma 2.3.** Suppose we have a collection of random variables $(X_i, X'_i)_{i \in \mathcal{I}}$ and a random variable $I$ distributed on $\mathcal{I}$. If $T(X_i, X'_i) \geq f$ for all $i \in \mathcal{I}$, then $T(X_I, X'_I) \geq f$.

**Proof.** For any random variable $I$ distributed on $\mathcal{I}$, we have

$$T(X_I, X'_I)(\alpha) = \inf_{\phi} \{ E[1 - \phi(X'_I)] : E[\phi(X'_I)] \leq \alpha \}$$

$$= \inf_{\phi} \{ E_{i \sim I}[1 - \phi(X'_i)] : E_{i \sim I}[\phi(X_i)] \leq \alpha \}$$

$$\geq \inf_{\phi} \{ E_{i \sim I}[f(\phi(X_i))] : E_{i \sim I}[\phi(X_i)] \leq \alpha \}$$

$$\geq \inf_{\phi} \{ f(E_{i \sim I}[\phi(X_i)]) : E_{i \sim I}[\phi(X_i)] \leq \alpha \}$$

$$= f(\alpha).$$

Therefore, we have $T(X_I, X'_I) \geq f$.

It is useful to work with distance posets that are complete:

**Definition 2.4** (Complete poset). A partially ordered set (poset) $(\mathcal{D}, \preceq)$ is complete if for every nonempty subset $S \subseteq \mathcal{D}$ has a supremum $\sup(S)$, where $s \preceq \sup(S)$ for every $s \in S$, and $\sup(S) \preceq t$ for every $t$ satisfying $s \preceq t$ for every $s \in S$.

We note that $\sup(S)$ is always unique. The poset $((\mathbb{R}^{\geq 0}) \cup \{\infty\}, \leq)$ used in pure DP and Rènyi DP is complete by the usual completeness of the real numbers. For the poset $(\mathcal{F}, \preceq)$ used in $f$-DP, we prove it below. Note that if $(\mathcal{D}, \preceq)$ is complete then in Definition 2.2 we can take $d = \sup_{A} D(\text{View}(A \leftrightarrow \mathcal{M}(x)), \text{View}(A \leftrightarrow \mathcal{M}(x'))) as the optimal privacy loss for a given interactive mechanism $\mathcal{M}$.

**Lemma 2.4.** The partially ordered set $(\mathcal{F}, \preceq)$, where $\mathcal{F}$ consists of all trade-off functions satisfying the conditions in Proposition 1.7, is complete. Specifically, for $S \subseteq \mathcal{F}$, $\sup S$ is the trade-off function $h$ defined as follows.

$$h(\alpha) = \inf_{F: \supp(F) \subseteq S} \{ E[F(A(F))] : E[A(F)] \leq \alpha \},$$

where $F$ is a random variable that takes value in $S$ and $A : S \rightarrow [0, 1]$ is a function.
Proof. We first show that $h$ is the least upper bound for $S$. We shall show that for any tradeoff function $h’$ such that $f \preceq h’$ for every $f \in S$, we have $h \preceq h’$. Let $F$ be a random variable such that $\text{supp}(F) \subseteq S$, and let $A : S \to [0,1]$ be a function such that $\mathbb{E}[A(F)] \leq \alpha$. As stated in Proposition 1.7, a trade-off function is convex and non-increasing, so by Jensen’s inequality, we have

$$h’(\alpha) \leq h’(\mathbb{E}[A(F)]) \leq \mathbb{E}[h’(A(F))].$$

By the definition of the partial ordering, we have $h’(\alpha) \leq f(\alpha)$ for every $\alpha \in [0,1]$ and every $f \in S$, so $\mathbb{E}[h’(A(F))] \leq \mathbb{E}[F(A(F))]$. Therefore, we have

$$h’(\alpha) \leq \mathbb{E}[F(A(F))],$$

Taking the infimum over $F$ and $A$ on both sides, we get $h’(\alpha) \leq h(\alpha)$, and therefore, $h \preceq h’$.

Next, we shall show that $h$ is a trade-off function. Following the proposition 1.7, it suffices to check the four properties for $h$. We begin with proving the convexity of $h$. For every $a, b \in [0,1]$, and every $\lambda \in [0,1]$, we have

$$h(\lambda a + (1 - \lambda)b) = \inf_{F,A} \{\mathbb{E}[F(A(F))] : \mathbb{E}[A(F)] \leq \lambda a + (1 - \lambda)b\}$$

$$\leq \lambda \inf_{F,A} \{\mathbb{E}[F(A(F))] : \mathbb{E}[A(F)] \leq a\} + (1 - \lambda) \inf_{F,A} \{\mathbb{E}[F(A(F))] : \mathbb{E}[A(F)] \leq b\}$$

(7)

$$\leq \lambda h(a) + (1 - \lambda) h(b).$$

where inequality (7) is because that for every $A_a, F_a$ that satisfies $\mathbb{E}[A(F)] \leq a$ and every $A_b, F_b$ that satisfies $\mathbb{E}[A(F)] \leq b$, the linear combination $A(F) = \lambda A_a(F_a) + (1 - \lambda)A_b(F_b)$ satisfies $\mathbb{E}[A(F)] \leq \lambda a + (1 - \lambda)b$. Thus, $h$ is convex. $h$ is non-increasing and continuous on $[0,1]$ due to the monotonicity and continuity of $f \in F$ (Proposition 1.7). Finally, since $f(x) \leq 1 - x$ for every $f \in F$, we have

$$h(\alpha) \leq \mathbb{E}[F(A(F))] \leq \mathbb{E}[1 - A(F)] \leq 1 - \alpha.$$  

Therefore, $h$ is a trade-off function, and $\text{sup} S$ exists.

A convenient consequence of joint convexity is that it suffices to consider deterministic adversaries.

Lemma 2.5. An interactive mechanism $M$ is $d$-DP, if and only if for every pair of neighboring datasets $x, x'$, for every deterministic adversary algorithm $A$, we have $D(\text{View}(A, M(x)), \text{View}(A, M(x'))) \leq d$.

Proof. The necessity is immediately implied by Definition 2.2. We shall prove the sufficiency. Let $A$ be a randomized adversary. If we fix the coin tosses of $A$ to a value $r$, we obtain a deterministic adversary $A_r$. By hypothesis, we have $D(\text{View}(A_r, M(x)), \text{View}(A_r, M(x'))) \leq d$. Now let random variable $R$ be uniformly distributed over the coins of $A$. Then the view of the randomized adversary $A$ when interacting with $M$ consists of the coins $R$ and the view of the deterministic adversary $A_R$. That is,

$$\text{View}(A \leftrightarrow M(x)) = (R, \text{View}(A_R \leftrightarrow M(x))),$$

and similarly for $x'$. By joint convexity, we deduce:

$$\text{View}(A \leftrightarrow M(x)) \leq d.$$
3 Coupling and Chain Rule Properties of $f$-DP

In this section, we prove that $f$-DP has the coupling and chain rule properties that we use to prove Theorems 1.4 and 1.6.

**Definition 3.1 (Coupling property).** We say that a generalized distance $D$ has the coupling property if for any two pairs of random variables $X, X'$ and $Y, Y'$, we have $D(X, X') \preceq d$ and $D(Y, Y') \preceq d$, then there exists a coupling of $X$ and $Y$ (denoted as $(X, Y)$), and a coupling of $X'$ and $Y'$ (denoted as $(X', Y')$), such that $D((X, Y), (X', Y')) \preceq d$.

**Lemma 3.1 (Lemma 1.7 restated).** $(F, \preceq, T)$ has the coupling property: Suppose $f$ is a trade-off function and we have random variables $X_0$, $X_1$ and $X_0'$, $X_1'$ such that
\[ T(X_0, X_0') \geq f \quad \text{and} \quad T(X_1, X_1') \geq f. \]

Then there exists couplings $(X_0, X_1)$ and $(X_0', X_1')$ such that
\[ T((X_0, X_1)||((X_0', X_1')) \geq f. \]

We prove this lemma using the following result:

**Theorem 3.2 (Blackwell Theorem [DRS19] also see [Bla53, KOV15]).** Let $P, Q$ be probability distributions on $X$ and $P', Q'$ be probability distributions on $Y$. The following two statements are equivalent:

1. $T(P, Q) \leq T(P', Q')$.
2. There exists a randomized algorithm $\text{Proc} : X \rightarrow Y$ such that $\text{Proc}(P) = P'$ and $\text{Proc}(Q) = Q'$.

**Proof of Lemma 3.1.** Since a function is called a trade-off function if it is equal to $T(P, Q)$ for some distribution $P$ and $P'$, for a given trade-off function $f$, there exists a pair of random variables $P, P'$ such that $T(P, P') = f$. By the Blackwell Theorem, since $T(P, P') = f \leq T(X_0, X_0')$, there exists a randomized algorithm $\text{Proc}_0$ such that $\text{Proc}_0(P)$ and $\text{Proc}_0(P')$ are identically distributed to $X_0$ and $X_0'$, respectively. Similarly, since $T(P, P') = f \leq T(X_1, X_1')$, there exists a randomized algorithm $\text{Proc}_2$ such that $\text{Proc}_2(P)$ and $\text{Proc}_2(P')$ are identically distributed to $X_1$ and $X_1'$, respectively. We construct a coupling of $X_0$ and $X_0'$ as $(\text{Proc}_0(P), \text{Proc}_0(P'))$, and a coupling of $X_1$ and $X_1'$ as $(\text{Proc}_2(P), \text{Proc}_2(P'))$. Then the trade-off function between the two couplings satisfies the following inequality.

\[
T((\text{Proc}_0(P), \text{Proc}_1(P)), (\text{Proc}_0(P'), \text{Proc}_1(P'))) \\
\quad \geq T(P, P') \\
\quad = f,
\]

where Equation (8) follows from Lemma 2.9 in [DRS19]. Let $Y_0 = \text{Proc}_0(P)$, $Y_1 = \text{Proc}_1(P)$, $Y_0' = \text{Proc}_0(P')$ and $Y_1' = \text{Proc}_1(P')$, completing the proof.

To formally state the chain rule property, we need a couple of definitions.

**Definition 3.2 (Continuous function).** Let $(A, \preceq)$ and $(B, \preceq)$ be complete posets. A function $f : A \rightarrow B$ is continuous if $f(\sup(S)) = \sup(f(S))$ for every set $S$.

Observe that every continuous function is monotone: if $a \preceq a'$ are elements of $A$, then $f(a') = f(\sup(a, a')) = \sup(f(a), f(a')) \preceq f(a)$.

**Definition 3.3.** If $(D, \preceq)$ is a poset, and $S$ is a finite set, then $(D^S, \preceq)$ is the poset defined by $((a)_s\in S) \preceq ((b)_s\in S)$ if $a_s \preceq b_s$ for every $s \in S$. 

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**Lemma 3.3 (Lemma [1,8] restated).** For every pair of random variables \(X, X'\) with finite support, there exists a function \(\text{ChainRule}_{X,X'}\) such that for every random variable \(Y\) jointly distributed with \(X\), and every random variable \(Y'\) jointly distributed with \(X'\), we have
\[
T((X,Y),(X',Y')) = \text{ChainRule}_{X,X'}((T(Y|X = x, Y'|X' = x'))_{x \in \text{supp}(X) \cap \text{supp}(X')}).
\]
where \(T\) is a trade-off function.

**Proof.**

**Claim 3.4.** The \(\text{ChainRule}\) function for \(f\text{-DP}\) is given as follows.
\[
\text{ChainRule}_{X,X'}(f_x)_{x \in \text{supp}(X) \cap \text{supp}(X')} = \inf_{\alpha_x \in [0,1]} \{E_{x \sim X'}[f_x(\alpha_x)] : E_{x \sim X}[\alpha_x] \leq \alpha\}.
\]
We first prove this claim. Suppose \(Y\) is jointly distributed with \(X\), and \(Y'\) is jointly distributed with \(X'\). We consider hypothesis tests distinguishing \((X,Y)\) and \((X',Y')\). Let \(\phi\) be any decision rule for this testing, \(\alpha(\phi)\) and \(\beta(\phi)\) be the corresponding Type I error and Type II error, respectively. For a given instance \(x \in \text{supp}(X) \cap \text{supp}(X')\), let \(\phi_x(y) := \phi(x, y)\). Additionally, let \(f_x\) be the trade-off function conditioned on \(x\), i.e.,
\[
f_x(\alpha) := T(Y|X = x, Y'|X' = x)(\alpha).
\]
The type I error \(\alpha_\phi\) and type II error \(\beta_\phi\) are given as
\[
\alpha_\phi = E[\phi(x,y)] = E_{x \sim X}E_{y \sim Y}[\phi_x(y)],
\]
and
\[
\beta_\phi = 1 - E[\phi(x',y')] = 1 - E_{x \sim X}E_{y' \sim Y'}[\phi_{x'}(y')].
\]
For every fixed \(x \in \text{supp}(X) \cap \text{supp}(X')\) and every decision rule \(\phi\) such that \(E_{y \sim Y}[\phi_x(y)] = \alpha_x\), by the definition of \(f_x\) in (11), we have
\[
1 - E_{y' \sim Y'}[\phi_{x'}(y')] \geq f_x(\alpha_x).
\]
Therefore, the trade-off function between \((X,Y)\) and \((X',Y')\) satisfies the following inequality:
\[
T((X,Y),(X',Y'))(\alpha) = \inf \{\beta_\phi : \alpha_\phi \leq \alpha\} = \inf \{1 - E_{x \sim X}E_{y' \sim Y'}[\phi_{x'}(y')] : E_{x \sim X}E_{y \sim Y}[\phi_x(y)] \leq \alpha\} \geq \inf \{E_{x \sim X}[f_x(\alpha_x)] : E_{x \sim X}[\alpha_x] \leq \alpha\}.
\]
On the other hand, by the definition of \(f_x\), for every \(0 < \alpha_x < 1\) and \(\delta > 0\), there exists a decision rule \(\phi_\delta\) such that \(1 - E_{y' \sim Y'}[\phi_{x'}^\delta(y')] \leq f_x(\alpha_x) + \delta\) and \(E_{y' \sim Y'}[\phi_{x'}^\delta(y')] \leq \alpha_x\). Then we have
\[
\inf \{E_{x \sim X}[f_x(\alpha_x)] : E_{x \sim X}[\alpha_x] \leq \alpha\} \leq \inf \{E_{x \sim X}[1 - E_{y' \sim Y'}[\phi_{x'}^\delta(y')]] - \delta : E_{x \sim X}[\alpha_x] \leq \alpha\}.
\]
Let $\delta$ go to 0, the decision function $\phi^\delta$ will converge to the decision rule $\phi^*$ such that the infimum in (11) is reached over all the decision rules such that $E_{y \sim Y}[\phi^*_y(y)] \leq \alpha_x$, and thus, we get
\[
\inf \{ E_{x \sim X}[f_x(\alpha_x)] : E_{x \sim X}[\alpha_x] \leq \alpha \} \geq \inf \{ E_{x \sim X} [1 - E_{y \sim Y}[\phi^*_y(y)]] : E_{x \sim X} [\alpha_x] \leq \alpha \}
\]
\[= T((X, Y), (X', Y'))(\alpha). \tag{13} \]
Combining Equation (12) and Equation (13), we have
\[T((X, Y), (X', Y'))(\alpha) = \inf \{ E_{x \sim X}[f_x(\alpha_x)] : E_{x \sim X}[\alpha_x] \leq \alpha \}. \tag{14} \]
completing the proof for this claim.

Next, we shall show that the ChainRule function defined in (10) is continuous. By assumption, $\text{supp}(X) \cap \text{supp}(X')$ has finite elements, and we have $X = x_j$ with probability $p_j$ and $X' = x_j$ with probability $p'_j$. We can write the ChainRule function as
\[\text{ChainRule}_{X, X'} \left( \left( f_x \right)_{x \in \text{supp}(X) \cap \text{supp}(X')} \right) = \inf_{\alpha_x \in [0, 1]} \left\{ \sum_j p'_j f_{x_j}(\alpha_{x_j}) : \sum_j p_j \alpha_{x_j} \leq \alpha \right\}. \tag{15} \]
Our goal is to show that $\text{ChainRule}_{X, X'}(\text{sup}(S)) = \sup(\text{ChainRule}_{X, X'}(S))$ for every finite set $S$. By induction, it suffices to consider sets of size 2, so let the set $S = \{(f_{x_j})_{x_j \in \text{supp}(X) \cap \text{supp}(X')}, (g_{x_j})_{x_j \in \text{supp}(X) \cap \text{supp}(X')}\}$.
We have
\[\sup \text{sup}(\text{ChainRule}_{X, X'}(S))\]
\[= \sup \left( \text{ChainRule} \left( \left( f_{x_j} \right)_{x_j} \right) \text{ChainRule} \left( \left( g_{x_j} \right)_{x_j} \right) \right) \]
\[= \sup \left( \inf_{\alpha_{x_j} \in [0, 1]} \left\{ \sum_j p'_j f_{x_j}(\alpha_{x_j}) : \sum_j p_j \alpha_{x_j} \leq \alpha \right\} \right) \text{ by (15)} \]
\[= \sup_{p \alpha_1 + (1-p) \alpha_2 = \alpha} \left\{ p \sum_j p_{j \alpha_{x_j} \leq \alpha_1} \left\{ \sum_j p'_j f_{x_j}(\alpha_{x_j}) \right\} + (1-p) \sum_j p_{j \alpha_{x_j} \leq \alpha_2} \left\{ \sum_j p'_j g_{x_j}(\alpha'_{x_j}) \right\} \right\} \text{ by Lemma 2.4} \]
On the other hand,
\[\text{ChainRule}_{X, X'}(\text{sup}(S)) \]
\[= \inf_{\alpha_x \in [0, 1]} \left\{ \sum_j p'_j \sup(f_{x_j}, g_{x_j})(\alpha_{x_j}) : \sum_j p_j \alpha_{x_j} \leq \alpha \right\} \text{ by (15)} \]
\[= \inf_{\sum_j p_j \alpha_{x_j} \leq \alpha} \left\{ \sum_j p'_j \left( \inf_{\sum_j p_j \alpha_{x_j} + (1-p) \alpha_{x_j} = \alpha_j} \left\{ p f_{x_j}(\alpha_{x_j}) + (1-p) g_{x_j}(\alpha'_{x_j}) \right\} \right) \right\} \text{ by Lemma 2.4} \]
\[= \inf_{\sum_j p_{j \alpha_{x_j} = \alpha_1} \sum_j p_{j \alpha'_{x_j} = \alpha_2} \sum_j p_{j \alpha_{x_j} + (1-p) \alpha_{x_j} = \alpha}} \left\{ p \sum_j p_j f_{x_j}(\alpha_{x_j}) + (1-p) \sum_j p_j g_{x_j}(\alpha'_{x_j}) \right\} \]
Therefore, ChainRule$_{X, X'}(\text{sup}(S)) = \sup(\text{ChainRule}_{X, X'}(S))$. \[\square\]
4 Concurrent Composition of d-D DP

Theorem 4.1 shows that if the generalized distance satisfies the coupling property in Definition 3.1 and the chain rule in Definition 3.4, then every interactive d-D DP mechanism can be simulated by an interactive post-processing of a non-interactive d-D DP mechanism. This is a generalized statement of Theorem 1.6 as f-DP is an example of d-D DP.

**Theorem 4.1 (Theorem 1.6 generalized).** Assume that the generalized probability distance measure \((D, \preceq, d)\) satisfies

1. \((D, \preceq)\) is complete.
2. \(D\) satisfies the chain rule.
3. every \(d \in D\) satisfies the coupling property.

Then for every \(d \in D\) and every interactive d-D DP mechanism \(M\) with finite communication complexity, and every pair of two neighboring datasets \(x\) and \(x'\), there exists a pair of random variables \(Y, Y'\) and an randomized interactive post-processing mechanism \(T\) such that \(D(Y, Y') \preceq d\), and for every adversary \(A \in A\), we have

\[
\text{View}(A, M(x)) \equiv \text{View}(A, T(Y))
\]

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\[
\text{View}(A, M(x')) \equiv \text{View}(A, T(Y')).
\]

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Note that the theorem is stated for mechanisms with finite communication, which is formally defined as follows.

**Definition 4.1.** Let \((A, B)\) be an interactive protocol (as in Definition 1.2). We say that \(A\) has finite communication if for every \(x_A\) there is a constant \(c\), such that for all \(r_A, m_1, \ldots, m_{i-1}\), we have

1. If \(\max\{i, |m_1|, \ldots, |m_{i-1}|\} > c\), then \(A(x_A, m_1, m_3, \ldots, m_{i-1}; r_A) = \text{halt}\).
2. If \(\max\{i, |m_1|, \ldots, |m_{i-1}|\} \leq c\), then \(\sum_{j=0}^{i-1} |A(x_A, m_1, m_3, \ldots, m_j; r_A)| \leq c\).

Here \(|y|\) denotes the bit length of string \(y\). \(B\) having finite communication is defined symmetrically.

**Proof.** Our strategy is to apply the induction argument by the number of rounds of interactions. Fix a pair of neighboring datasets \(x, x'\). We consider two cases depending on whether the first message sent from the mechanism \(M\) or the adversary \(A\).

**Case 1.** The adversary \(A\) sends the first query \(q_1\) to the mechanism \(M\). Fixing \(q_1\), the subsequent interactive mechanism \(M_{q_1}\) is defined by

\[
M_{q_1}(x, q_2, \ldots, q_m, r) = M(x, q_1, \ldots, q_m, r).
\]

We claim that View(\(A \leftrightarrow M_{q_1}\)) consists of \(m - 1\) messages, and \(M_{q_1}\) satisfies d-D DP on the two neighboring datasets \(x\) and \(x'\). By induction, there exists a randomized interactive post-processing \(T_{q_1}\) and a pair of random variables \(Y_{q_1}, Y'_{q_1}\) such that

\[
D(Y_{q_1}, Y'_{q_1}) \preceq d,
\]

and

\[
T_{q_1}(Y_{q_1}) \equiv M_{q_1}(x) \quad T_{q_1}(Y'_{q_1}) \equiv M_{q_1}(x').
\]

By coupling property, there exists a pair of random variables \(Y, Y'\) and a randomized post-processing function \(S_{q_1}\) such that

\[
S_{q_1}(Y) = Y_{q_1}.
\]
and

\[ S_{q_i}(y') = y'_i. \]

Then the interactive post-processing \( T \) is defined by \( T_{q_i} \circ S_{q_i}, \) i.e.,

\[ T(y, q_1, q_2, \ldots, q_{m-1}; r_S, r_T) = T_{q_i}(S_{q_i}(y; r_S), q_2, \ldots, q_{m}; r_T)). \]

**Case 2.** The mechanism \( \mathcal{M} \) sends the first message \( a_1 \) to the adversary \( A \). Let \( q_1, \ldots, q_{m-1} \) be the queries from the adversary, and \( A_1, \ldots, A_m \) be messages from the mechanism. Conditioned on \( A_1 = a_1 \), the subsequent interactive mechanism \( \mathcal{M}_{a_1} \) is defined by

\[ \mathcal{M}_{a_1}(x, q_1, \ldots, q_{m-1}; g_x(r)) = \mathcal{M}(x, q_1, \ldots, q_{m-1}; r). \]

\( \mathcal{M}_{a_1} \) uses its randomness to choose uniformly from randomness of \( \mathcal{M} \) conditioned on \( \mathcal{M}(x) = a_1 \). Specifically, let \( g_x \) be a random transformation such that if \( R \) is uniform random for \( \mathcal{M}_{a_1} \), then for all \( x \), \( g_x(R) \) is uniform on the randomness of \( \mathcal{M} \) conditioned on \( \mathcal{M}(x) = a_1 \).

We define the subsequent adversary \( A_{a_1}(A_2, \ldots, A_m) = A(a_1, A_2, \ldots, A_m) \). We know that for all adversary strategy \( A \in \mathcal{A} \), we have \( D(\text{View}(A \leftrightarrow \mathcal{M}(x)), \text{View}(A \leftrightarrow \mathcal{M}(x'))) \leq d \), so we have

\[ \sup_A D(\text{View}(A \leftrightarrow \mathcal{M}(x)), \text{View}(A \leftrightarrow \mathcal{M}(x'))) \leq d. \]

We have

\[ \sup_A D(\text{View}(A \leftrightarrow \mathcal{M}(x)), \text{View}(A \leftrightarrow \mathcal{M}(x'))) = \sup_A \text{ChainRule}_{A_1, A'_1}(\text{View}(A_{a_1} \leftrightarrow \mathcal{M}_{a_1}(x)), \text{View}(A_{a_1} \leftrightarrow \mathcal{M}_{a_1}(x')))_{a_1 \in \text{supp}(A_1) \cap \text{supp}(A'_1))} \]

\[ \text{ChainRule}_{A_1, A'_1}(\sup_A D(\text{View}(A_{a_1} \leftrightarrow \mathcal{M}_{a_1}(x)), \text{View}(A_{a_1} \leftrightarrow \mathcal{M}_{a_1}(x')))_{a_1 \in \text{supp}(A_1) \cap \text{supp}(A'_1))}, \]

where (18) follows from the chain rule, and (19) is because that the ChainRule function is continuous. By induction, for every \( a \in \text{supp}(A_1) \cap \text{supp}(A'_1) \), suppose \( \mathcal{M}_a \) is \( d_a \)-DP, then there exists a pair of random variables \( Y_a, Y'_a \) and a post-processing \( T_a \) such that

\[ D(Y_a, Y'_a) \leq d_a, \]

and

\[ T_a(Y_a) \equiv \mathcal{M}_a(x) \text{ and } T_a(Y'_a) \equiv \mathcal{M}_a(x'). \]

Let \( Y_A \) be the random variable that defined as \( \Pr[Y_A = b] = \Pr[Y = b| A = a] \) and \( A \) denote the random variable of the first message \( a_1 \). Thus, by the chain rule, we have

\[ D((A_1, Y_A), (A'_1, Y'_A)) \leq d. \]

Let \( Y = (A_1, Y_A) \) and \( Y' = (A'_1, Y'_A) \), we define the post-processing \( T \) as

\[ T(y, q_1, \ldots, q_{m-1}; r_T) = (a_1, T_{a_1}(y, q_1, \ldots, q_{m-1}; r_{T_a})). \]

We now use Theorem 4.1 to prove that the concurrent composition of interactive mechanisms can be reduced to the composition of the non-interactive mechanisms.

**Theorem 4.2** (Theorem 4.1 generalized). Suppose that the generalized probability distance \( (\mathcal{D}, \preceq, \mathcal{D}) \) satisfies the chain rule, every \( d \in \mathcal{D} \) satisfies the coupling property, and \( (\mathcal{D}, \preceq) \) is complete. Suppose for all non-interactive mechanism \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) such that \( \mathcal{M}_i \) is \( d_i \)-DP for \( i = 1, 2, \ldots, k \), their composition \( \text{Comp}({\mathcal{M}_1, \ldots, \mathcal{M}_k}) \) is \( d \)-DP, then the concurrent composition \( \text{ConComp}({\mathcal{M}_1, \ldots, \mathcal{M}_k}) \) of interactive mechanisms \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) such that \( \mathcal{M}_i \) is \( d_i \)-DP is also \( d \)-DP.
Proof of Theorem 4.2. Following Theorem 1.1 for every interactive \( d_j \)-DP mechanism \( \mathcal{M}_j \), \( j = 1, \ldots, k \), and every pair of neighboring datasets \( x, x' \), there exists a pair of of random variables \( Y_j, Y_j' \) and an interactive post-processing \( T_j \) such that \( D(Y_j, Y_j') \leq d_j \), and for every adversary \( A \in \mathcal{A} \), \( \text{View}(\mathcal{A}, \mathcal{M}_j(x)) \) (resp., \( \text{View}(\mathcal{A}, \mathcal{M}_j(x')) \)) is identically distributed as \( \text{View}(A, T_j(Y_j)) \) (resp., \( \text{View}(A, T_j(Y_j')) \)). Since \( Y_j, Y_j', j = 1, \ldots, k \), are noninteractive random variables, which can be viewed as the output distributions of a noninteractive mechanism \( \mathcal{N}_j \) on \( x, x' \). Suppose \( \text{Comp}(\mathcal{N}_1, \ldots, \mathcal{N}_k) \) is \( d \)-DP. By the post-processing property, we know that \( \text{Comp}((T_1(\mathcal{N}_1), \ldots, T_k(\mathcal{N}_k)) \) is also \( d \)-DP. Therefore, we have \( \text{ConComp}(\mathcal{M}_1, \ldots, \mathcal{M}_k) \) is also \( d \)-DP.

\[ \square \]

5 Concurrent Composition of Rényi DP

In this section, we show that privacy adds up for the concurrent composition of Rényi DP, which is the same as in the sequential or non-interactive composition.

Theorem 5.1 (Theorem 1.5 restated). For all \( \alpha > 1 \), \( k \in \mathbb{N}, \epsilon_1, \ldots, \epsilon_k > 0 \), and all interactive mechanisms \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) such that \( \mathcal{M}_i \) is \((\alpha, \epsilon_i)\)-RDP for \( i = 1, 2, \ldots, k \), the concurrent composition \( \text{ConComp}(\mathcal{M}_1, \ldots, \mathcal{M}_k) \) of interactive mechanisms \( \mathcal{M}_1, \ldots, \mathcal{M}_k \) is \((\alpha, \sum_{i=1}^k \epsilon_i)\)-RDP.

Our proof relies Lemma 5.2 which describes a property of Rényi divergence in the concurrent interactive protocol. The Rényi divergence is defined as follows.

Definition 5.1 (Rényi divergence [Rény01]). For two probability distribution \( P \) and \( Q \), the Rényi divergence of order \( \alpha > 1 \) is

\[
D_\alpha (P || Q) = \frac{1}{\alpha - 1} \log E_{x \sim P} \left( \frac{P(x)}{Q(x)} \right)^\alpha.
\]

Lemma 5.2 (Lemma 1.10 restated). For any adversary \( B \) concurrently interacting with RDP mechanisms \( \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k \) operating on two neighboring datasets \( x, x' \), it is equivalent to consider a set of hybrid experiments where we compare \( (\mathcal{M}_1(x), \ldots, \mathcal{M}_{j-1}(x), \mathcal{M}_j(x), \mathcal{M}_{j+1}(x), \ldots, \mathcal{M}_k(x)) \) to \( (\mathcal{M}_1(x), \ldots, \mathcal{M}_{j-1}(x), \mathcal{M}_j(x'), \mathcal{M}_{j+1}(x), \ldots, \mathcal{M}_k(x)) \), that is

\[
D_\alpha (\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x)))) || \text{View}(B \leftrightarrow (\mathcal{M}_1(x'), \ldots, \mathcal{M}_k(x')))) = \sum_{j=1}^k D_\alpha (\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x)))) || \text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_{j-1}(x), \mathcal{M}_j(x'), \mathcal{M}_{j+1}(x), \ldots, \mathcal{M}_k(x))).
\]

The proof of the lemma will use the chain rule for Rényi divergence:

Proposition 5.2 (Chain rule for Rényi divergence). Let \( X, X' \) be a pair of random variables and \( Y, Y' \) are random variables jointly distributed with \( X, X' \), respectively. Then,

\[
D_\alpha ((X, Y) || (X', Y')) = D_\alpha (X || X') + D_\alpha (Y || Y' || X')
\]

where the conditional Rényi divergence is defined as

\[
D_\alpha (Y || X' | X' = x) = E_{x \sim X} D_\alpha (Y | X = x || Y' | X' = x).
\]

Proof of Lemma 5.2. By Lemma 2.5 we will only need to consider deterministic adversaries. Let \( R_1, \ldots, R_k \) be the randomness of mechanism \( \mathcal{M}_1, \ldots, \mathcal{M}_k \), respectively. The random coins \( R_1, \ldots, R_k \) are mutually independent. Let \( R \) be the randomness of the adversary \( B \).

We define \( \text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x))) = (R, A_1, \ldots, A_r) \). Let \( Q_1, \ldots, Q_r \) be the queries from round 1 to \( r \). We also define \( J_1, \ldots, J_r \) as \( J_i = j \) if \( i \)th query is sent to the \( j \)th mechanism. The \( i \)th query is uniquely determined by the adversary \( (J_i, Q_i) = B(A_1, \ldots, A_{i-1}; R) \), and the message \( A_i \) is determined by
the mechanism given the history of all queries, i.e., \( A_i = \mathcal{M}_{J_i}(x,(Q_t)_{t\leq i},R_{J_i}) \). Similarly, we define \( \text{View}(B \leftrightarrow (\mathcal{M}_1(x'),\ldots,\mathcal{M}_k(x'))) = (R',A'_1,\ldots,A'_k) \).

We also define a set of experiments that the adversary \( B_j \) interacts with the \( k \) mechanism with only the \( j \)th mechanism operating on dataset \( x' \), i.e., \( \mathcal{M}_1(x),\ldots,\mathcal{M}_{j-1}(x),\mathcal{M}_j(x'),\mathcal{M}_{j+1}(x),\ldots,\mathcal{M}_k(x) \). Let \( \text{View}(B_j \leftrightarrow (\mathcal{M}_1(x),\ldots,\mathcal{M}_{j-1}(x),\mathcal{M}_j(x'),\mathcal{M}_{j+1}(x),\ldots,\mathcal{M}_k(x))) = (R^{(j)},A_1^{(j)},\ldots,A_r^{(j)}) \). In this experiment, we denote the random coins of the \( k \) mechanisms as \( R_1^{(j)},\ldots,R_k^{(j)} \), the queries as \( Q_1^{(j)},\ldots,Q_k^{(j)} \) and the indices as \( J_1^{(j)},\ldots,J_k^{(j)} \). Then the message \( A_i^{(j)} \) is defined as follows.

\[
A_i^{(j)} = \begin{cases} \mathcal{M}_{J_i}(x',(Q_t)_{t\leq i},R_{J_i}), & \text{if } J_i = j, \\ \mathcal{M}_{J_i}(x,(Q_t)_{t\leq i},R_{J_i}), & \text{otherwise.} \end{cases}
\]

We first analyze the Rînyi divergence of the \( i \)th message conditioned on the history up to round \( i \). We have the following claim.

**Claim 5.3.**

\[
D_\alpha(A_i|(R, A_{<i})||A'_i|(R', A'_{<i})) = \sum_{j=1}^k D_\alpha(A_i|(R, A_{<i})||A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i}))
\]

(20)

To prove this claim, we apply the definition of the conditional Rînyi divergence, and we have

\[
D_\alpha(A_i|(R, A_{<i})||A'_i|(R', A'_{<i})) = E_{(r,a_{<i}) \sim (R,A_{<i})} [D_\alpha(A_i|(R, A_{<i}) = (r,a_{<i})||A'_i|(R', A'_{<i}) = (r,a_{<i}))]
\]

(21)

\[
= E_{(r,a_{<i}) \sim (R,A_{<i})} [D_\alpha(A_i|(R, A_{<i}) = (r,a_{<i})||A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i}) = (r,a_{<i}))]
\]

(22)

where Equation (22) follows from the fact that

\[
A'_i|(R', A'_{<i}) = (r,a_{<i}) \equiv A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i}) = (r,a_{<i}).
\]

This is because the randomness of \( A_i \) and \( A_i^{(j)} \) only comes from the mechanism being queried at round \( i \), i.e., \( M_{J_i} \) given the history \( (r,a_{<i}) \), which fully determines the queries \( q_1,\ldots,q_i \) and the corresponding mechanisms \( J_1,\ldots,J_i \).

We can rewrite Equation (22) as follows. This is because that

\[
E_{(r,a_{<i}) \sim (R,A_{<i})} [D_\alpha(A_i|(R, A_{<i}) = (r,a_{<i})||A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i}) = (r,a_{<i}))] = 0
\]

when \( j \neq j_i \).

\[
= \sum_{j=1}^k E_{(r,a_{<i}) \sim (R,A_{<i})} [D_\alpha(A_i|(R, A_{<i}) = (r,a_{<i})||A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i}) = (r,a_{<i}))]
\]

(22)

\[
= \sum_{j=1}^k D_\alpha(A_i|(R, A_{<i})||A_i^{(j)}|(R^{(j)}, A^{(j)}_{<i})).
\]

We complete the proof for Claim 5.3. Then using Claim 5.3 we have
\[D_\alpha(\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x)))) || \text{View}(B \leftrightarrow (\mathcal{M}_1(x'), \ldots, \mathcal{M}_k(x'))))
\]
\[= D_\alpha((R, A_1, \ldots, A_r) || (R', A'_1, \ldots, A'_r))
\]
\[= D_\alpha((R, A_1) || (R', A'_1)) + \sum_{i=2}^{r} D_\alpha(A_i | (R, A_{<i}) || A'_i | (R', A'_{<i}))
\]
\[= \sum_{j=1}^{k} D_\alpha((R, A_1, \ldots, A_r) || (R^{(j)}, A_1^{(j)}, \ldots, A_r^{(j)}))
\]
\[= \sum_{j=1}^{k} D_\alpha(\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x)))) || \text{View}(B_j \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_{j-1}(x), \mathcal{M}_j(x'), \mathcal{M}_{j+1}(x), \ldots, \mathcal{M}_k(x))))
\]

\[\square\]

Now we provide the proof of Theorem 5.1.

**Proof of Theorem 5.1.** Our proof relies on the following claim.

**Claim 5.4.** For any adversary \(B\) concurrently interacting with mechanisms \(M_1, \ldots, M_k\), there exists a non-concurrent adversary \(C_j\) only interacting with \(M_j\) which simulates \(M_i(x)\) \((i \neq j)\) on its own (always on dataset \(x\), not \(x'\)), such that

\[D_\alpha(\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x)))) || \text{View}(B_j \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_{j-1}(x), \mathcal{M}_j(x'), \mathcal{M}_{j+1}(x), \ldots, \mathcal{M}_k(x))))
\]
\[\leq D_\alpha(\text{View}(C \leftrightarrow M_j(x))) || \text{View}(C \leftrightarrow M_j(x'))
\]
(23)

Let us first prove Claim 5.4. We define a non-concurrent adversary \(C_j\) only interacting with \(M_j\) which simulates \(M_i(x)\) \((i \neq j)\) on its own (always on dataset \(x\), not \(x'\)). Let the randomness for \(C_j\) consist of \(R_C = (R_B, R_1, \ldots, R_{j-1}, R_{j+1}, R_k)\), where \(R_B\) is the randomness of the adversary \(B\) and \(R_1, \ldots, R_{j-1}, R_{j+1}, R_k\) is the randomness of the mechanisms except \(M_j\). We define the adversary’s strategy \(C_j(x)(a_1^{(j)}, a_2^{(j)}, \ldots, a_{j-1}^{(j)}; r)\) as follows.

1. Let \(a_{s<t,j_s=j} = a_{<i}^{(j)}\).
2. Compute \(q_t, j_t = B(a_{<t}, r_B)\).
3. If \(j_t \neq j\), then \(a_t = M_j(x, q_{<t}, r_{j_t})\).
4. Let \(t = t + 1\) and repeat step 1-3 until \(j_t = j\).
5. Output \(q_t = q_t\).

We can recover \(\text{View}(B \leftrightarrow (\mathcal{M}_1(x), \ldots, \mathcal{M}_k(x))))\) given a transcript of \(\text{View}(C_j \leftrightarrow M_j(x))\) through the following post-processing function \(\text{Post}(R_B, R_1, \ldots, R_{j-1}, R_{j+1}, \ldots, R_k, a_1^{(j)}, \ldots, a_r^{(j)})\).

1. Let \(t = 1\).
2. For \(t_j = 1, 2, \ldots, r\), compute \((q_t, j_t) = B(a_{<t}, r_B)\).
3. If \(j_t \neq j\), then \(a_t = M_j(x, q_{<t}, r_{j_t})\).
4. Else, let \(a_t = a_{t_j}\) and \(t = t + 1\).
5. Output \((R_B, a_1, \ldots, a_t)\).

Therefore, we have
\[
\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x))) = \text{Post}(\text{View}(C_j \leftrightarrow M_j(x))).
\]

Similarly, we have
\[
\text{View}(B \leftrightarrow (M_1(x), \ldots, M_{j-1}(x), M_j(x), M_{j+1}(x), \ldots, M_k(x))) = \text{Post}(\text{View}(C_j \leftrightarrow M_j(x'))),
\]
for the same post-processing function \(\text{Post}(R_B, R_1, \ldots, R_{j-1}, R_{j+1}, \ldots, R_k, a_1^{(j)}, \ldots, a_r^{(j)})\). Thus, we have
\[
D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x)))) || \text{View}(B \leftrightarrow (M_1(x), \ldots, M_{j-1}(x), M_j(x'), M_{j+1}(x), \ldots, M_k(x))) \\
= D_\alpha(\text{Post}(\text{View}(C_j \leftrightarrow M_j(x)))) || \text{Post}(\text{View}(C_j \leftrightarrow M_j(x')))) \\
\leq D_\alpha(\text{View}(C_j \leftrightarrow M_j(x))) || \text{View}(C_j \leftrightarrow M_j(x'))), \tag{24}
\]
where inequality (24) follows from the data-processing inequality.

Combining Lemma 5.2 and Claim 5.4 we arrive at
\[
D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x)))) || \text{View}(B \leftrightarrow (M_1(x'), \ldots, M_k(x')))) \\
= \sum_{j=1}^k D_\alpha(\text{View}(B \leftrightarrow (M_1(x), \ldots, M_k(x)))) || \text{View}(B \leftrightarrow (M_1(x), \ldots, M_{j-1}(x), M_j(x'), M_{j+1}(x), \ldots, M_k(x)))) \\
\leq \sum_{j=1}^k D_\alpha(\text{View}(C_j \leftrightarrow M_j(x))) || \text{View}(C_j \leftrightarrow M_j(x'))) \\
\leq \sum_{j=1}^k \epsilon_j.
\]

\[\square\]

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