SASAKI MANIFOLDS, KÄHLER CONE MANIFOLDS
AND BIHARMONIC SUBMANIFOLDS

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Abstract. For a Legendrian submanifold $M$ of a Sasaki manifold $N$, we study harmonicity and biharmonicity of the corresponding Lagrangian cone submanifold $C(M)$ of a Kähler manifold $C(N)$. We show that, if $C(M)$ is biharmonic in $C(N)$, then it is harmonic; and $M$ is proper biharmonic in $N$ if and only if $C(M)$ has a non-zero eigen-section of the Jacobi operator with the eigenvalue $m = \dim M$.

1. Introduction

Harmonic maps play a central role in geometry; they are critical points of the energy functional $E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g$ for smooth maps $\varphi$ of $(M, g)$ into $(N, h)$. The Euler-Lagrange equations are given by the vanishing of the tension filed $\tau(\varphi)$. In 1983, J. Eells and L. Lemaire [9] extended the notion of harmonic map to biharmonic map, which are, by definition, critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g.$$ (1.1)

After G.Y. Jiang [16] studied the first and second variation formulas of $E_2$, extensive studies in this area have been done (for instance, see [5], [19], [22], [29], [30], [11], [12], [15], etc.). Notice that harmonic maps are always biharmonic by definition. We say, for a smooth map $\varphi : (M, g) \to (N, h)$ to be proper biharmonic if it is biharmonic, but not harmonic. B.Y. Chen raised ([7]) so called B.Y. Chen’s conjecture and later, R. Caddeo, S. Montaldo, P. Piu and C. Oniciuc raised ([5]) the generalized B.Y. Chen’s conjecture.

B.Y. Chen’s conjecture:
Every biharmonic submanifold of the Euclidean space $\mathbb{R}^n$ must be harmonic (minimal).

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The generalized B.Y. Chen’s conjecture:
Every biharmonic submanifold of a Riemannian manifold of non-positive curvature must be harmonic (minimal).

For the generalized Chen’s conjecture, Ou and Tang gave ([28], [29]) a counter example in a Riemannian manifold of negative curvature. For the Chen’s conjecture, affirmative answers were known for the case of surfaces in the three dimensional Euclidean space ([7]), and the case of hypersurfaces of the four dimensional Euclidean space ([10], [8]). Furthermore, Akutagawa and Maeta gave ([1]) recently a final supporting evidence to the Chen’s conjecture:

**Theorem 1.1.** Any complete regular biharmonic submanifold of the Euclidean space \(\mathbb{R}^n\) is harmonic (minimal).

To the generalized Chen’s conjecture, we showed ([26]) that

**Theorem 1.2.** Let \((M, g)\) be a complete Riemannian manifold, and the curvature of \((N, h)\), non-positive. Then,
1. every biharmonic map \(\varphi : (M, g) \rightarrow (N, h)\) with finite energy and finite bienergy must be harmonic.
2. In the case \(\text{Vol}(M, g) = \infty\), under the same assumption, every biharmonic map \(\varphi : (M, g) \rightarrow (N, h)\) with finite bienergy is harmonic.

We also obtained (cf. [24], [25], [26])

**Theorem 1.3.** Assume that \((M, g)\) is a complete Riemannian manifold, \(\varphi : (M, g) \rightarrow (N, h)\) is an isometric immersion, and the sectional curvature of \((N, h)\) is non-positive. If \(\varphi : (M, g) \rightarrow (N, h)\) is biharmonic and \(\int_M |H|^2 v_g < \infty\), then it is minimal. Here, \(H\) is the mean curvature normal vector field of the isometric immersion \(\varphi\).

Theorem 1.3 gives an affirmative answer to the generalized B.Y. Chen’s conjecture under the \(L^2\)-condition and completeness of \((M, g)\).

In this paper, for every Legendrian submanifold \(\varphi : (M^m, g) \rightarrow (N^{2m+1}, h)\) of a Sasaki manifold \((N^{2m+1}, h)\), and the Lagrangian cone submanifold \(\mathcal{C} : (C(M), \mathcal{J}) \rightarrow (C(N), \mathcal{J})\) of a Kähler cone manifold \((C(N), \mathcal{J})\), we show (Theorems 3.3 and 4.4) that (1) \(\mathcal{C} : (C(M), \mathcal{J}) \rightarrow (C(N), \mathcal{J})\) is biharmonic if and only if it is harmonic, which is equivalent to that \(\varphi : (M, g) \rightarrow (N, h)\) is harmonic. (2) \(\varphi : (M, g) \rightarrow (N, h)\) is proper biharmonic if and only if \(\tau(\mathcal{J})\) is a non-zero eigen-section of the Jacobi operator \(J_{\mathcal{J}}\) with the eigenvalue \(m = \dim M\). The assertion
(2) can be regarded as a biharmonic map version of T. Takahashi’s theorem (cf. Theorem 4.5) which claims that each coordinate function of the isometric immersion of \((M^m, g)\) into the unit sphere \(S^n \hookrightarrow \mathbb{R}^{n+1}\) is the eigenfunction of the Laplacian of \((M, g)\) with the eigenvalue \(m = \dim M\).

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2. Preliminaries

We first prepare the materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map \(\varphi : (M, g) \to (N, h)\), of a compact Riemannian manifold \((M, g)\) into another Riemannian manifold \((N, h)\), which is an extremal of the energy functional defined by

\[
E(\varphi) = \int_M e(\varphi) \, v_g,
\]

where \(e(\varphi) := \frac{1}{2}|d\varphi|^2\) is called the energy density of \(\varphi\). That is, for any variation \(\{\varphi_t\}\) of \(\varphi\) with \(\varphi_0 = \varphi\),

\[
\frac{d}{dt} \bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \tag{2.1}
\]

where \(V \in \Gamma(\varphi^{-1}TN)\) is a variation vector field along \(\varphi\) which is given by \(V(x) = \frac{d}{dt} \bigg|_{t=0} \varphi_t(x) \in T_{\varphi(x)}N, \, (x \in M)\), and the tension field is given by \(\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)\), where \(\{e_i\}_{i=1}^m\) is a locally defined orthonormal frame field on \((M, g)\), and \(B(\varphi)\) is the second fundamental form of \(\varphi\) defined by

\[
B(\varphi)(X, Y) = (\tilde{\nabla}d\varphi)(X, Y) = (\tilde{\nabla}_X d\varphi)(Y) = \nabla_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \tag{2.2}
\]

for all vector fields \(X, Y \in \mathfrak{X}(M)\). Here, \(\nabla\), and \(\nabla^N\), are Levi-Civita connections on \(TM, TN\) of \((M, g), (N, h)\), respectively, and \(\tilde{\nabla}\), and \(\tilde{\nabla}\)
are the induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By (2.1), $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that $\varphi$ is harmonic. Then,

$$
\frac{d^2}{dt^2} \bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V)v_g, \quad (2.3)
$$

where $J$ is an elliptic differential operator, called the Jacobi operator acting on $\Gamma(\varphi^{-1}TN)$ given by

$$
J(V) = \overline{\Delta} V - \mathcal{R}(V), \quad (2.4)
$$

where $\overline{\Delta} V = \nabla^* \nabla V = -\sum_{i=1}^m \{ \nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V \}$ is the rough Laplacian and $\mathcal{R}$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $\mathcal{R}(V) = \sum_{i=1}^m R^N(U, V, d\varphi(e_i))d\varphi(e_i)$, and $R^N$ is the curvature tensor of $(N, h)$ given by $R^N(U, V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla^N_{[U, V]}$ for $U, V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire [9] proposed polyharmonic ($k$-harmonic) maps and Jiang [16] studied the first and second variation formulas of biharmonic maps. Let us consider the bienergy functional defined by

$$
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g, \quad (2.5)
$$

where $|V|^2 = h(V, V)$, $V \in \Gamma(\varphi^{-1}TN)$.

The first variation formula of the bienergy functional is given by

$$
\frac{d}{dt} \bigg|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V)v_g, \quad (2.6)
$$

Here,

$$
\tau_2(\varphi) := J(\tau(\varphi)) = \overline{\Delta}(\tau(\varphi)) - \mathcal{R}(\tau(\varphi)), \quad (2.7)
$$

which is called the bitension field of $\varphi$, and $J$ is given in (2.4).

A smooth map $\varphi$ of $(M, g)$ into $(N, h)$ is said to be biharmonic if $\tau_2(\varphi) = 0$. By definition, every harmonic map is biharmonic. We say, for an immersion $\varphi : (M, g) \to (N, h)$ to be proper biharmonic if it is biharmonic but not harmonic (minimal).

3. LEGENDRIAN SUBMANIFOLDS AND LAGRANGIAN SUBMANIFOLDS

In this section, we first show a correspondence between the set of all Legendrian submanifolds of a Sasakian manifold and the one of all Lagrangian submanifolds of a Kähler cone manifold.

An $n(= 2m + 1)$ dimensional contact Riemannian manifold $(N, h)$ with a contact form $\eta$ is said to be a contact metric manifold if there...
exist a smooth \((1, 1)\) tensor field \(J\) and a smooth vector field \(\xi\) on \(N\), called a basic vector field, satisfying that
\[
J^2 = -\mathbb{I} + \eta \otimes \xi, \tag{3.1}
\]
\[
\eta(\xi) = 1, \tag{3.2}
\]
\[
J \xi = 0, \tag{3.3}
\]
\[
\eta \circ J = 0, \tag{3.4}
\]
\[
h(JX, JY) = h(X, Y) - \eta(X) \eta(Y), \tag{3.5}
\]
\[
\eta(X) = h(X, \xi), \tag{3.6}
\]
\[
d\eta(X, Y) = h(X, JY), \tag{3.7}
\]
for all smooth vector fields \(X, Y\) on \(N\). Here, \(\mathbb{I}\) is the identity transformation of \(T_x N\) \((x \in N)\). A contact metric manifold \((N, h, J, \xi, \eta)\) is Sasakian if \((C(N), \overline{h}, I)\) is a Kähler manifold. Here, a cone manifold \(C(N) := N \times \mathbb{R}^+\) where \(\mathbb{R}^+ := \{ r \in \mathbb{R} | r > 0 \}\), \(\overline{h}\) is a cone metric on \(C(N)\), \(\overline{h} := dr^2 + r^2 h\), which is a Hermitian metric with respect to an almost complex structure \(I\) on \(C(N)\) given by
\[
\begin{align*}
    IY &:= JY + \eta(Y) \Psi, \quad (Y \in \mathfrak{X}(N)), \\
    I\Psi &:= -\xi,
\end{align*}
\tag{3.8}
\]
where \(\Psi := r \frac{\partial}{\partial r}\) is called the Liouville vector field on \(C(N)\). We denote by \(\mathfrak{X}(N)\), the set of all smooth vector fields on \(N\). A contact metric manifold \((N, h, J, \xi, \eta)\) is Sasakian if and only if
\[
(\nabla^N_X J)(Y) = h(X, Y) \xi - \eta(Y) X \quad (X, Y \in \mathfrak{X}(N)). \tag{3.9}
\]

Let us recall the definition

**Definition 3.1.** Let \(M^m\) be an \(m\)-dimensional manifold, an immersion \(\varphi : M^m \to N^{2m+1}\). \(M^m\) is called to be a Legendrian submanifold of an \((2m + 1)\)-dimensional Sasakian manifold \((N, h, J, \xi, \eta)\) if \(\varphi^* \eta \equiv 0\) which is equivalent to that
\[
\varphi^* x(x_x) \in \text{Ker}(\eta_{\varphi(x)}) \tag{3.10}
\]
for all \(X_x \in T_x M\) \((x \in M)\).

A Legendrian submanifold \(M^m\) satisfies the following two conditions:

1. \(\varphi^*(T_x M)\) is orthogonal \(J(\varphi^*(T_x M))\) with respect to \(h\) for all \(x \in M\). This is equivalent to that the normal bundle \(T^\perp M\) of \(\varphi : M \to N\) has the following splitting:
\[
T_x M^\perp = \mathbb{R} x_{\varphi(x)} \oplus J \varphi^* T_x M \quad (x \in M).
\]
(2) The second fundamental form $B$ of $\varphi(M) \subset N$ has its value at $\text{Ker}(\eta)$, that is,

$$B(\varphi_*X, \varphi_*Y) = \nabla^N_X \varphi_*Y - \varphi_*(\nabla_X Y) \in \varphi_*(T_x M) \perp,$$

where $T_x M \perp$ is $\varphi_*(T_x M) \perp$, which is

$$\{W_{\varphi(x)} \in T_{\varphi(x)} N \mid h(W_{\varphi(x)}, \varphi_* x X_x) = 0 \ (\forall \ X_x \in T_x M)\}.$$

Here, $\nabla$, $\nabla^N$ are Levi-Civita connections of $(M, g)$, $(N, h)$ where $g$ is the induced metric on $M$ by $g := \varphi^* h$.

In the following, we identify $\varphi(M)$ with $M$, itself. The following theorem is well known, but essentially important for us.

**Theorem 3.2.** Let $M^m$ be an $m$-dimensional submanifold of a Sasakian manifold $(N^{2m+1}, h, J, \xi, \eta)$. Then, $M$ is a Legendrian submanifold of a Sasaki manifold $N$ if and only if $C(M) \subset C(N)$ is a Lagrangian submanifold of a Kähler cone manifold $(C(N), \overline{h}, \overline{I})$.

**Proof** We have the equivalence that $M \subset N$ is Legendrian if and only if

$$\begin{cases}
\xi_x = T_x M \perp + JT_x M, \\
h(T_x M, JT_x M) = \{0\}
\end{cases}$$

for all $x \in M$. That is, $h(\xi, X) = 0$ and $h(X, JY) = 0$ for all $X, Y \in \mathfrak{X}(M)$. Then, (3.11) is equivalent to that

$$\Omega(f_1 \Phi + X, f_2 \Phi + Y) = r^2 \{f_1 h(\xi, Y) - f_2 h(\xi, X) + h(X, JY)\}$$

$$= 0$$

(3.12)

for all smooth functions $f_1, f_2$ on $C(M)$ and $X, Y \in \mathfrak{X}(M)$. Here, $\Omega$ is the Kähler form of $C(N)$ which is given by $\Omega = 2 r dr \wedge \eta + r^2 d\eta$.

Finally, (3.12) is equivalent to that $C(M) \subset C(N)$ is Lagrangian. □

Now our main theorem is as follows:

**Theorem 3.3.** Let $\varphi : (M, g) \to (N, h)$ be a Legendrian submanifold of a Sasakian manifold $(N^n, h, J, \xi, \eta)$ $(n = 2m + 1)$ and $\overline{\varphi} : (C(M), \overline{g}) \ni (r, x) \mapsto (r, \varphi(x)) \in (C(N), \overline{h}, \overline{I})$, a Lagrangian submanifold of a Kähler cone manifold. Here $C(M) := M \times \mathbb{R}^+ \subset C(N) := N \times \mathbb{R}^+$, $\overline{g} = dr^2 + r^2 g$, and $\overline{h} = dr^2 + r^2 h$. Then,

(1) it holds that

$$\tau(\overline{\varphi}) = \frac{1}{r^2} \tau(\varphi).$$

(3.13)

Thus, we have the equivalence that $\varphi : (M, g) \to (N, h)$ is harmonic if and only if $\overline{\varphi}(C(M), \overline{g}) \to (C(N), \overline{h})$ is also harmonic.
(2) Secondly, it holds that
\[ \tau_2(\varphi) = \frac{1}{r^4} \tau_2(\varphi) + \frac{m}{r^2} \tau(\varphi). \]  
Then, we have the equivalence that \( \varphi : (M, g) \to (N, h) \) is proper biharmonic if and only if for \( \varphi : (C(M), \bar{g}) \to (C(N), \bar{h}) \), the tension field \( \tau(\varphi) \) is a non-zero eigen-section of the Jacobi operator \( J_\varphi \) with the eigenvalue \( m = \dim M \). And we have the equivalence that \( \varphi : (M, g) \to (N, h) \) is biharmonic if and only if it is harmonic, which is equivalent to that \( \varphi : (M, g) \to (N, h) \) is harmonic.

(3) Thirdly, it holds that
\[ \tau_2(\varphi) \perp = \frac{1}{r^4} \tau_2(\varphi) \perp + \frac{m}{r^2} \tau(\varphi). \]  
Then, we have the equivalence that \( \varphi : (M, g) \to (N, h) \) is minimal if and only if \( \varphi : (C(M), \bar{g}) \to (C(N), \bar{h}) \) is bi-minimal.

(4) Finally, it holds that
\[ \text{div}_{\bar{g}}(I \tau(\varphi)) = \frac{1}{r^2} \text{div}_g(J \tau(\varphi)). \]  
Then, we have also the equivalence that \( \varphi : (M, g) \to (N, h, J, \xi, \eta) \) is Legendrian minimal if and only if \( \varphi : (C(M), \bar{g}) \to (C(N), \bar{h}, I) \) is also Lagrangian minimal.

To prove Theorem 3.3, we need the following lemma.

**Lemma 3.4.** The Levi-Civita connection \( \nabla^{C(M)} \) of the cone manifold \((C(M), \bar{g})\) of a Riemannian manifold \((M, g)\), where the cone metric \( \bar{g} = dr^2 + r^2 g \), is given as follows:

\[
\begin{align*}
\nabla^{C(M)}_X Y &= \nabla_X Y - g(X, Y) \frac{\partial}{\partial r}, \\
\nabla^{C(M)}_X \frac{\partial}{\partial r} &= \frac{1}{r} X, \\
\nabla^{C(M)}_\varphi Y &= \frac{1}{r} Y, \\
\nabla^{C(M)}_\varphi \frac{\partial}{\partial r} &= 0.
\end{align*}
\]

Here, \( X, Y \in \mathfrak{X}(M) \), and \( \nabla \) is the Levi-Civita connection of \((M, g)\).

The proof of Lemma 3.4 is a direct computation which is omitted.

To proceed to give a proof of Theorem 3.3, we first take a locally defined orthonormal frame field \( \{e_i\}_{i=1}^m \) on \((M, g)\). Define \( \bar{e}_i := \frac{1}{r} e_i \).
(i = 1, . . . , m), and e_{m+1} := \frac{\partial}{\partial r}. Then, \{\mathbf{e}_i\}_{i=1}^{m+1} is a locally defined orthonormal frame field on the cone manifold (C(M), \overline{g}).

Let \varphi : (M^m, g) \rightarrow (N^n, h) (n = 2m + 1) be a Legendrian submanifold of a Sasakian manifold, and \overline{\varphi} : (C(M), \overline{g}) \rightarrow (C(N), \overline{h}) be the corresponding cone submanifold of a Kähler cone (C(N), \overline{h}). We should see a relation between the induced bundles \varphi^{-1}T_N and \overline{\varphi}^{-1}TC(N). We denote by \Gamma(E), the space of all smooth sections of the vector bundle E. Then, every smooth section W of the induced bundle \overline{\varphi}^{-1}TC(N) can be written as

\[ W = V + B \frac{\partial}{\partial r} \]  \hspace{1cm} (3.18)

where V is a smooth section of the induced bundle \varphi^{-1}T_N and B is a smooth function on \mathbb{C}(M) = M \times \mathbb{R}^+. Because, for every point (x, r) \in C(M) = M \times \mathbb{R}^+, \overline{\varphi}(x, r) = (\varphi(x), r), and \overline{W}_{(x,r)}C(N) = T_{(\varphi(x),r)}(N \times \mathbb{R}^+) = T_{\varphi(x)}N \oplus T_r\mathbb{R}^+, so we can write as \overline{W}_{(x,r)} = V_x + B(x,r) \frac{\partial}{\partial r}, where V_x \in T_{\varphi(x)}N and B(x,r) \in \mathbb{R}.

Then, if we denote by \nabla, and \overline{\nabla}, the induced connections of the induced bundles \varphi^{-1}T_N and \overline{\varphi}^{-1}TC(N) from the connections \nabla^N, \nabla^{C(N)} of (N, h) and (C(N), \overline{h}), respectively, then we have for every \overline{W} \in \Gamma(\overline{\varphi}^{-1}TC(N)), with \overline{W} = V + B \frac{\partial}{\partial r} and V \in \Gamma(\varphi^{-1}T_N) and B \in \mathcal{C}^\infty(M \times \mathbb{R}^+),

\[
\begin{cases}
\nabla_X \overline{W} = \nabla_X V + \frac{B}{r} X + (XB) \frac{\partial}{\partial r}, & (X \in \mathfrak{X}(M)), \\
\nabla_{\frac{\partial}{\partial r}} \overline{W} = \frac{\partial B}{\partial r} \frac{\partial}{\partial r}. & \hspace{1cm} (3.19)
\end{cases}
\]

Proof of Theorem 3.3.

(1) We have, for i = 1, . . . , m, (m = \text{dim } M),

\[
\overline{\varphi}_* \nabla_{\mathbf{e}_i}^{C(M)} \mathbf{e}_i = \overline{\varphi}_* \left( \frac{1}{r^2} \nabla_{\mathbf{e}_i}^{C(M)} \mathbf{e}_i \right) \\
= \frac{1}{r^2} \overline{\varphi}_* \left( \nabla_{\mathbf{e}_i} \mathbf{e}_i - r g(\mathbf{e}_i, \mathbf{e}_i) \frac{\partial}{\partial r} \right) \hspace{1cm} \text{(by Lemma 3.4 (3.17))} \\
= \frac{1}{r^2} \left( \nabla_{\mathbf{e}_i} \mathbf{e}_i - r \frac{\partial}{\partial r} \right) \hspace{1cm} (3.20)
\]

since \overline{\varphi} is the inclusion map of C(M) into C(N). For i = m + 1, we have

\[ \overline{\varphi}_* \left( \nabla_{\mathbf{e}_{m+1}}^{C(M)} \mathbf{e}_{m+1} \right) = \overline{\varphi}_* \left( \nabla_{\frac{\partial}{\partial r}}^{C(M)} \frac{\partial}{\partial r} \right) = 0. \hspace{1cm} (3.21) \]
Furthermore, we have, for $i = 1, \ldots, m$,
\[
\nabla e^* \phi^* e_i = \nabla^C(N) \frac{1}{r} e_i \\
= \frac{1}{r^2} \left\{ \nabla^N e_i - r h(e_i, e_i) \frac{\partial}{\partial r} \right\} \\
= \frac{1}{r^2} \left\{ \nabla^N e_i - r \frac{\partial}{\partial r} \right\}
\]
(3.22)
since $\phi^* \tilde{h} = \tilde{g}$ and $\phi^* h = g$. For $i = m + 1$, we have also
\[
\nabla e^{m+1} \phi^* e^{m+1} = \nabla^C(N) \frac{\partial}{\partial r} = 0.
\]
(3.23)
Thus, we have
\[
\tau(\phi) = \sum_{i=1}^{m+1} \left\{ \nabla e_i \varphi^* e_i - \varphi^* \left( \nabla^C(M) e_i \right) \right\} \\
= \frac{1}{r^2} \sum_{i=1}^{m} \left\{ \nabla^N e_i - \nabla e_i \right\} \\
= \frac{1}{r^2} \tau(\phi),
\]
(3.24)
which is (3.13).

For (2), we have to see relations between
\[
J_\phi(V) = \varphi^* (\Delta \phi V) - \sum_{i=1}^{m} R^N(V, \varphi^* e_i) \varphi^* e_i, \quad (V \in \Gamma(\phi^{-1}TN)),
\]
(3.25)
\[
J_\varphi(W) = \varphi^* (\Delta \varphi W) - \sum_{i=1}^{m+1} R^C(N)(W, \varphi^* e_i) \varphi^* e_i, \quad (W \in \Gamma(\varphi^{-1}TC(N)).
\]
(3.26)
where
\[
\Delta \phi V := - \sum_{i=1}^{m} \{ \nabla e_i (\nabla e_i V) - \nabla_{\nabla e_i e_i} V \};
\]
(3.27)
\[
\Delta \varphi W := - \sum_{i=1}^{m+1} \{ \nabla e_i (\nabla e_i W) - \nabla_{\nabla e_i e_i} W \}.
\]
(3.28)

Here, $\nabla$, and $\varphi^*$ are the induced connections of $\phi^{-1}TN$ and $\varphi^{-1}TC(N)$ from the Levi-Civita connections $\nabla^N$ and $\nabla^C(N)$ of $(N, h)$ and $(C(N), \tilde{h})$ with $\tilde{h} = dr^2 + r^2 h$, respectively.
By (3.19), we have

\[
\begin{aligned}
\nabla_X(\nabla_Y W) &= \nabla_X(\nabla_Y V) + \frac{B}{r} \nabla_X Y + \frac{X B}{r} Y + \frac{Y B}{r} X \\
&\quad + X(Y B) \frac{\partial}{\partial r}, \quad (X, Y \in \mathfrak{X}(M)),
\end{aligned}
\]

(3.29)

where we used that \(\nabla_X(\nabla_Y V) = \nabla_X(\nabla_Y V), \nabla_X Y = \nabla_X Y = \nabla_X Y\) and \(\nabla_X \frac{\partial}{\partial r} = \frac{1}{r} X\) for every \(X, Y \in \mathfrak{X}(M)\).

(3.30)

Therefore, we obtain, for \(W = V + B \frac{\partial}{\partial r} \in \Gamma(\varphi^{-1}TC(N))\) with \(V \in \Gamma(\varphi^{-1}TN)\) and \(B \in C^\infty(M \times \mathbb{R}^+)\),

\[
\Delta_\varphi W = \frac{1}{r^2} \Delta V - B \frac{\tau}{r^3} \frac{\partial}{\partial r} - \frac{m}{r^2} \text{grad}_M B
\]

\[
+ \left( \frac{1}{r^2} \Delta M - \frac{\partial^2 B}{\partial r^2} - \frac{m}{r} \partial B \right) \frac{\partial}{\partial r},
\]

(3.32)

where let us recall

\[
\Delta_\varphi V = - \sum_{i=1}^m \{\nabla_{e_i}(\nabla_{e_i} V) - \nabla_{\nabla_{e_i} e_i} V\} \quad (V \in \Gamma(\varphi^{-1}TN)),
\]

\[
\tau(\varphi) = \sum_{i=1}^m (\nabla_{e_i e_i} - \nabla_{e_i} e_i), \quad \text{grad}_M B = \sum_{i=1}^m (e_i B) e_i,
\]

\[
\Delta_M B = - \sum_{i=1}^m \{e_i(e_i B) - \nabla_{e_i e_i} B\} \quad (B \in C^\infty(M \times \mathbb{R}^+)).
\]

(3.31)

\[
R^C(N)(X, Y) Z = R^N(X, Y) Z - h(Y, Z) X + h(X, Z) Y,
\]

\[
R^C(N)\left(X, \frac{\partial}{\partial r}\right) \frac{\partial}{\partial r} = 0,
\]

\[
R^C(N)\left(\frac{\partial}{\partial r}, Y\right) Z = 0,
\]

(3.32)

for every \(X, Y, Z \in \mathfrak{X}(M)\). Therefore, we obtain

\[
\sum_{i=1}^m R^C(N)(W, \varphi_* e_i, \varphi_* e_i) = \frac{1}{r^2} \sum_{i=1}^m R^N(V, \varphi_* e_i, \varphi_* e_i) - \frac{m}{r^2} V + \frac{1}{r^2} V^T,
\]

(3.32)

for \(W = V + B \frac{\partial}{\partial r} \in \Gamma(\varphi^{-1}TC(N))\).
Therefore, we have

\[ J_\varphi(W) = \nabla_\varphi W - \sum_{i=1}^{m} R^{C(N)}(W, \varphi_* e_i)\varphi_* e_i \]
\[ = \frac{1}{r^2} \left( \Delta \varphi V - \sum_{i=1}^{m} R^N(V, \varphi_* e_i)\varphi_* e_i \right) + \frac{m}{r^2} V - \frac{1}{r^2} V^T \]
\[ - \frac{B}{r^3} \tau(\varphi) - \frac{2}{r^3} \text{grad}_M B \]
\[ + \left( \frac{1}{r^2} \Delta_M B - \frac{\partial^2 B}{\partial r^2} - \frac{m}{r} \frac{\partial B}{\partial r} \right) \frac{\partial}{\partial r}. \quad (3.33) \]

Here, we have already \( \tau(\varphi) = \frac{1}{r^2} \tau(\varphi) \) in Theorem 3.3 (1) (3.13). For this \( W := \tau(\varphi) \), we have \( V = \frac{1}{r^2} \tau(\phi), B = 0 \) and \( V^T = 0 \), and we have

\[ J_\varphi(\tau(\varphi)) = \frac{1}{r^4} \left( \nabla_\varphi(\tau(\varphi)) - \sum_{i=1}^{m} R^N(\tau(\varphi), \varphi_* e_i)\varphi_* e_i \right) + \frac{m}{r^2} \tau(\varphi) \]
\[ = \frac{1}{r^4} J_\varphi(\tau(\varphi)) + \frac{m}{r^2} \tau(\varphi). \quad (3.34) \]

We have (3.14) in (2). By (3.34), we have the equivalence between the bi-harmonicity of \( \varphi \) and that \( \tau(\varphi) \) is a non-zero eigen-section of the Jacobi operator \( J_\varphi \) with eigenvalue \( m = \text{dim} M \). Furthermore, \( \tau_2(\varphi) = 0 \) if and only if \( \tau_2(\varphi) + mr^2 \tau(\varphi) = 0 \) for all \( r > 0 \), which is equivalent to that \( \tau(\varphi) = 0 \).

For (3) in Theorem 3.3, we only observe the following orthogonal decompositions:

\[ T_x N = T_x M \oplus T_x M^\perp, \quad T_x M^\perp = J T_x M \oplus \mathbb{R} \xi_x, \quad (3.35) \]
\[ T_{(x,r)} C(N) = T_x N \oplus T_r \mathbb{R}^+ \]
\[ = T_x M \oplus J T_x M \oplus \mathbb{R} \xi_x \oplus T_r \mathbb{R}^+ \]
\[ = T_{(x,r)} C(M) \oplus J T_x M \oplus \mathbb{R} \xi_x \]
\[ = T_{(x,r)} C(M) \oplus T_x M^\perp, \quad (3.36) \]

for every \( x \in M \subset N \). So let us decompose \( \tau_2(\varphi) = \frac{1}{r^2} \tau_2(\varphi) \) following (3.35) and (3.36). Then, we have

\[ \tau_2(\varphi) = \tau_2(\varphi)^T + \tau_2(\varphi)^\perp \quad (3.37) \]

where \( \tau_2(\varphi)^T \in T_{(x,r)} C(M) \) and \( \tau_2(\varphi)^\perp \in T_x M^\perp \), and also we have

\[ \frac{1}{r^4} \tau_2(\varphi) + \frac{m}{r^2} \tau(\varphi) = \frac{1}{r^4} \tau_2(\varphi)^T + \frac{1}{r^4} \tau_2(\varphi)^\perp + \frac{m}{r^2} \tau(\varphi), \quad (3.38) \]
where $\tau_2(\varphi)^T \in T_x M$ and $\tau_2(\varphi)^\perp \in T_x M^\perp$. But, since we have $T_x M \subset T_{(x,r)} C(M)$, we have

$$
\begin{cases}
\tau_2(\varphi)^T = \frac{1}{r^4} \tau_2(\varphi)^T, \\
\tau_2(\varphi)^\perp = \frac{1}{r^4} \tau_2(\varphi)^\perp + \frac{m}{r^2} \tau(\varphi).
\end{cases}
$$

Then, we have $\tau_2(\varphi)^\perp = 0$ if and only if $\tau_2(\varphi)^\perp + mr^2 \tau(\varphi) = 0$ for all $r > 0$, which is equivalent to that $\tau(\varphi) = 0$.

For (4), we first show that

$$
I \tau(\varphi) = J \tau(\varphi) + \eta(\tau(\varphi)) \Psi
$$

$$
= \frac{1}{r^2} J \tau(\varphi) + \frac{1}{r^2} \eta(\tau(\varphi)) \Psi
$$

$$
= \frac{1}{r^2} J \tau(\varphi)
$$

(3.40)

Because for a Legendrian submanifold of a Sasaki manifold, the second fundamental form $B$ takes its value in $\text{Ker}(\eta)$, so $\tau(\varphi) = \text{Trace}(B) \subset \text{Ker}(\eta)$, that is,

$$
\eta(\tau(\varphi)) = 0.
$$

(3.41)

Then, we have

$$
\text{div}_g(I \tau(\varphi)) = \sum_{i=1}^{m+1} g(e_i, \nabla_{e_i}^{C(M)}(I \tau(\varphi)))
$$

$$
= \frac{1}{r^4} \sum_{i=1}^{m} g(e_i, \nabla_{e_i}^{C(M)}(J \tau(\varphi)))
$$

$$
+ \frac{1}{r^2} g \left( \frac{\partial}{\partial r}, \nabla_{\frac{\partial}{\partial r}}^{C(M)}(J \tau(\varphi)) \right).
$$

(3.42)

But, the first term of the right hand side of (3.42) coincides with

$$
\frac{1}{r^4} \sum_{i=1}^{m} g(e_i, \nabla_{e_i}(J \tau(\varphi)) - rg(e_i, J \tau(\varphi)) \frac{\partial}{\partial r})
$$

$$
= \frac{1}{r^2} \sum_{i=1}^{m} g(e_i, \nabla_{e_i}(J \tau(\varphi)))
$$

$$
= \frac{1}{r^2} \text{div}_g(J \tau(\varphi)).
$$

(3.43)
On the other hand, the second term of the right hand side of (3.42) coincides with
\[
\frac{1}{r^2} g\left( \frac{\partial}{\partial r}, \nabla^C_{\frac{\partial}{\partial r}}(J \tau(\varphi)) \right) = \frac{1}{r^3} g\left( \frac{\partial}{\partial r}, J \tau(\varphi) \right) = 0
\]
(3.44) because \( J \tau(\varphi) \) is tangential to \( T_xM \) for the Legendrian immersion \( \varphi : (M,g) \to (N,h,J) \). Therefore, we obtain the desired formula:
\[
\text{div}_{\varphi}(I \tau(\varphi)) = \frac{1}{r^2} \text{div}_g(J \tau(\varphi)).
\]
We obtain Theorem 3.3.

Remark 3.5. The assertion (4) in Theorem 3.3 was given by I. Castro, H.Z. Li and F. Urbano ([6]), and H. Iriyeh ([14]), independently in a different manner from ours.

4. Biharmonic Legendrian submanifolds of Sasakian manifolds

By Theorem 3.3, we turn to review studies of a proper biharmonic Legendrian submanifold of a Sasaki manifold \((N^n, h, J, \xi, \eta)\) and give Takahashi-type theorem (cf. Theorem 4.4). First let us recall the equations of biharmonicity of an isometric immersions (cf. [21]).

Lemma 4.1. Let \( \varphi : (M^m, g) \to (N^n, h) \) be an isometric immersion. Then, for \( \varphi \) to be biharmonic if and only if
\[
\begin{align*}
\sum_{i=1}^{m} (\nabla_{e_i} A_H)(e_i) + \sum_{i=1}^{m} A_{\nabla^\perp e_i H}(e_i) - \sum_{i=1}^{m} \left( R^N_H(e_i) e_i \right)^\perp &= 0, \\
\Delta^\perp H + \sum_{i=1}^{m} B(A_H(e_i), e_i) - \sum_{i=1}^{m} \left( R^N_H(e_i) e_i \right)^\perp &= 0,
\end{align*}
\]
(4.1)
where \( H = \frac{1}{m} \sum_{i=1}^{m} B(e_i, e_i) \) the mean curvature vector field along \( \varphi \), \( B \) is the second fundamental form, and \( A \) is the shape operator for the isometric immersion \( \varphi : (M,g) \to (N,h) \).

For an isometric immersion of a Legendrian submanifold into a Sasakian manifold, we have

Theorem 4.2. Let \( \varphi : (M^m, g) \to (N^n, h, J, \xi, \eta) \) \((n = 2m + 1)\) be an isometric immersion of a Legendrian submanifold of a Sasakian
manifold. Then, for $\varphi$ to be biharmonic if and only if

$$
\sum_{i=1}^{m}(\nabla_{e_i}A_H)(e_i) + \sum_{i=1}^{m}A_{\nabla_{e_i}H}(e_i)
- \sum_{i,j=1}^{m}h((\nabla_{e_j}B)(e_i, e_i) - (\nabla_{e_i}B)(e_j, H)) e_j
= 0,
$$

(4.2)

$$
\Delta^\perp H + \sum_{i=1}^{m}B(A_H(e_i), e_i)
+ \sum_{j=1}^{m}\text{Ric}^N(JH, e_j)Je_j - \sum_{j=1}^{m}\text{Ric}^M(JH, e_j)Je_j
- \sum_{i=1}^{m}JA_B(JH, e_i)(e_i) + mJA_H(JH) + H
= 0.
$$

(4.3)

In the case that $(N^{2m+1}, h, J, \xi, \eta)$ is a Sasaki space form $N^{2m+1}(\epsilon)$ of constant $J$-sectional curvature $\epsilon$ whose curvature tensor $R^N$ is given by

$$
R^N(X, Y)Z = \frac{\epsilon + 3}{4}\{h(Y, Z) X - h(Z, X) Y\}
+ \frac{\epsilon - 1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + h(X, Z)\eta(Y)\xi - h(Y, Z)\eta(X)\xi
+ h(Z, JY) JX - h(Z, JX) JY + 2h(X, JY) JZ\},
$$

(4.4)

for all $X, Y, Z \in \mathfrak{X}(N)$, we have ([13], [32])

**Theorem 4.3.** Let $\varphi : (M^m, g) \to N^{2m+1}(\epsilon)$ be a Legendrian submanifold of a Sasaki space form of constant $J$-sectional curvature $\epsilon$. Then, for $\varphi$, to be biharmonic if and only if

$$
\Delta_{\varphi}H = \frac{\epsilon(m + 3) + 3(m - 1)}{4} H
$$

(4.5)

which is equivalent to

$$
\begin{cases}
\sum_{i=1}^{m}(\nabla_{e_i}A_H)(e_i) + \sum_{i=1}^{m}A_{\nabla_{e_i}H}(e_i) = 0, \\
\Delta^\perp H + \sum_{i=1}^{m}B(A_H(e_i), e_i) - \frac{\epsilon(m + 3) + 3(m - 1)}{4} H = 0.
\end{cases}
$$

(4.6)
Now, let us consider a Legendrian submanifold $M^m$ of the $(2m+1)$-dimensional unit sphere $S^{2m+1}(1)$ with the standard metric $ds^2_{\text{std}}$ of constant sectional curvature 1. Then, we have, due to Theorem 3.3, and $\Delta = \Delta$ which follows from that $R^C(N) = 0$ because of $(C(N), \overline{h}) = (C^{m+1}, ds^2)$:

**Theorem 4.4.** Let $\varphi : (M^m, g) \rightarrow (S^{2m+1}(1), ds^2_{\text{std}})$ be a Legendrian submanifold of $(S^{2m+1}(1), ds^2_{\text{std}})$, and $\overline{\varphi} : (C(M), \overline{g}) \rightarrow (C^{m+1}, ds^2)$, the corresponding Lagrangian cone submanifold of the standard complex space $(C^{m+1}, ds^2)$. Then, it holds that $\varphi : (M^m, g) \rightarrow (S^{2m+1}(1), ds^2_{\text{std}})$ is proper biharmonic if and only if $\tau(\overline{\varphi}) = \frac{1}{r^2} \tau(\varphi) = \frac{1}{r^2} H_m$ is a non-zero eigen-section of the rough Laplacian $\overline{\Delta}_\varphi$ acting on $\Gamma(\overline{\varphi}^{-1}T\overline{C}^{m+1})$ with the eigenvalue $m = \dim M$: $\overline{\Delta}_\varphi \tau(\overline{\varphi}) = m \tau(\overline{\varphi})$.

This Theorem 4.4 could be regarded as a biharmonic map version of the following T. Takahashi’s theorem ([33]). Our theorem is a different type from Theorem 4.3. For Takahashi-type theorem for harmonic maps into Grassmannian manifolds, see pp. 42 and 46 in [23]:

**Theorem 4.5.** (T. Takahashi) Let $(M^m, g)$ be a compact Riemannian manifold, and let $\varphi : (M^m, g) \rightarrow (S^n, ds^2_{\text{std}})$ be an isometric immersion. We write $\varphi = (\varphi_1, \cdots, \varphi_{n+1})$ where $\varphi_i \in C^\infty(M)$ ($1 \leq i \leq n+1$) via the canonical embedding $S^n \hookrightarrow \mathbb{R}^{n+1}$. Then, $\varphi : (M, g) \rightarrow (S^n, ds^2)$ is minimal if and only if $\Delta_g \varphi_i = m \varphi_i$, ($1 \leq i \leq n+1$). Here, $\Delta_g$ is the positive Laplacian acting on $C^\infty(M)$.

Certain classification theorems about proper biharmonic Legendrian immersions into the unit sphere $(S^{2m+1}(1), ds^2_{\text{std}})$ were obtained by T. Sasahara ([30], [31], [32]).

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