HEIGHT BOUND AND PREPERIODIC POINTS
FOR JOINTLY REGULAR FAMILIES OF RATIONAL MAPS

CHONG GYU LEE

Abstract. Silverman [14] proved a height inequality for jointly regular family of rational maps and the author [10] improved it for jointly regular pairs. In this paper, we provide the same improvement for jointly regular family; let $h : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}$ be the logarithmic absolute height on the projective space, let $r(f)$ be the $D$-ratio of a rational function $f$ which is defined in [10] and let \{f_1, \ldots, f_k\} be a finite set of rational maps which is defined over a number field $K$. If the intersection of all indeterminacy loci of $f_i$ is empty, then

\[ \sum_{i=1}^k \deg f_i h(f_i(P)) > \left(1 + \frac{1}{r}\right) f(P) - C \]

where $r = \max_i r(f_i)$.

1. Introduction

Let $K$ be a number field and $h : \mathbb{P}^n(K) \to \mathbb{R}$ be the logarithmic absolute height on the projective space. If $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ is a morphism defined on $K$, then we can make a good estimate of the height $h(P)$ with $h(f(P))$. We can define the degree of given morphism algebraically;

**Definition 1.1.** Let $g : V(K) \to W(K)$ be a rational map. Then, we define the degree of $f$ to be \[ \deg g := [\mathcal{C}(V(K)) : g^*\mathcal{C}(W(K))] \]

where $\mathcal{C}(V(K)), \mathcal{C}(W(K))$ is the function field on $V(K)$ and $W(K)$ respectively.

If $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ is a morphism on a projective space, we can find the degree from geometric information;

\[ f^*H = \deg f \cdot H \text{ in } \text{Pic}(\mathbb{P}^n). \]

Then, the functorial property of the Weil height machine will prove the Northcott’s theorem. The author refer [15, Theorem B.3.2] to the reader for the details of the Weil height machine.

**Theorem 1.2** (Northcott [12]). If $f : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{P}^n(\mathbb{Q})$ is a morphism defined over a number field $K$, then there are two constants $C_1$ and $C_2$, which are independent of point $P$, such that

\[ \frac{1}{\deg f} h(f(P)) + C_1 > h(P) > \frac{1}{\deg f} h(f(P)) - C_2 \]

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for all \( P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \).

If \( f \) is not a morphism but a rational map, then the functoriality breaks down; two height functions \( h_f \cdot H(P) \) and \( h_H(f(P)) \) are not equivalent. Hence, Northcott’s Theorem is not valid for rational maps. (However, we still have \( h(P) > \frac{1}{\deg f} h(f(P)) + C_2 \) by the triangular inequality. See [15, Proposition B.7.1].)

Silverman [14] suggested a way of studying height for rational maps using jointly regular family.

**Definition 1.3.** Let \( S = \{ f_1, \ldots, f_k \mid f_i : \mathbb{P}^n(\overline{\mathbb{K}}) \rightarrow \mathbb{P}^n(\overline{\mathbb{K}}) \} \) and \( Z(f) \) be the indeterminacy locus of \( f \). We say \( S \) is jointly regular when
\[
\bigcap_{i=1}^k Z(f_i) = \emptyset.
\]
We also say that a finite set of affine morphisms \( S' = \{ g_1, \ldots, g_k \mid g_i : \mathbb{A}^n(\mathbb{K}) \rightarrow \mathbb{A}^n(\mathbb{K}) \} \) is jointly regular if corresponding set of rational maps \( S = \{ f_i \mid f_i \text{ is the meromorphic extension of } g_i \in S' \} \) is jointly regular.

Then, a jointly regular set will bring an upper bound of \( h(P) \);

**Theorem 1.4** (Silverman, 2006). Let \( \{ f_1, \ldots, f_k \mid f_i : \mathbb{A}^n(\mathbb{K}) \rightarrow \mathbb{A}^n(\mathbb{K}) \} \) be a jointly regular family of rational maps defined over \( K \). Then, there is a constant \( C \) satisfying
\[
\sum_{l=1}^k \frac{1}{\deg f_l} h(f_l(P)) > h(P) - C
\]
for all \( P \in \mathbb{A}^n(\overline{\mathbb{K}}) \).

In this paper, we will improve Theorem 1.4.

**Theorem 1.5.** Let \( H \) be a hyperplane of \( \mathbb{P}^n(\overline{\mathbb{K}}) \), let \( S = \{ f_1, \ldots, f_k \mid f_i : \mathbb{A}^n(\mathbb{K}) \rightarrow \mathbb{A}^n(\mathbb{K}) \} \) be a jointly regular family of affine automorphisms defined over a number field \( K \) and let \( r(f) \) be \( D \)-ratio of \( f \). Suppose that \( S \) has at least two elements and \( r = \max_{i} r(f_i) \). Then, there is a constant \( C \) satisfying
\[
\sum_{l=1}^k \frac{1}{\deg f_l} h(f(P)) > \left( 1 + \frac{1}{r} \right) h(P) - C
\]
for all \( P \in \mathbb{A}^n(\overline{\mathbb{K}}) \).

Thus, Silverman’s result for preperiodic points [14, Theorem 4] is also improved;

**Theorem 1.6.** Let \( S = \{ f_1, \ldots, f_k \mid f_i : \mathbb{A}^n(\mathbb{K}) \rightarrow \mathbb{A}^n(\mathbb{K}) \} \) be jointly regular and let \( \Phi \) be the monoid of rational maps generated by \( S \). Define
\[
\delta_S := \left( \frac{1}{1 + 1/r} \right) \sum_{l=1}^k \frac{1}{\deg f_l}
\]
where $r = \max\{r(f_i)\}$.

If $\delta_S < 1$, then,

$$\text{Preper}(\Phi) := \bigcap_{f \in \Phi} \text{Preper}(f)$$

is a set of bounded height.

From now on, we will let $K$ be a number field, let $H$ be an infinity hyperplane of $\mathbb{A}^n$ in the projective space $\mathbb{P}^n(K)$ and let $f$ be an affine automorphism unless stated otherwise.

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2. Preliminaries

We need two main ingredients of this paper, the theory of resolution of indeterminacy and the $D$-ratio of rational maps. For details, the author refers readers to [11] and [3, II.7] for the resolution of indeterminacy and blowups, and [10] for the $D$-ratio.

2.1. Blowup and resolution of indeterminacy.

**Theorem 2.1** (Resolution of Indeterminacy). Let $f : V \to W$ be a rational map between proper varieties such that $V$ is nonsingular. Then there is a proper nonsingular variety $\tilde{V}$ with a birational morphism $\pi : \tilde{V} \to V$ which satisfy that $\phi = f \circ \pi : \tilde{V} \to W$ is a morphism.

For notational convenience, we will define the followings;

**Definition 2.2.** Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map and let $V$ be a blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$. We say that a pair $(V, \pi)$ is a resolution of indeterminacy of $f$ if $f \circ \pi : V \to \mathbb{P}^n$ is extended to a morphism $\phi$. And we call the extended morphism $\phi := f \circ \pi$ a resolved morphism of $f$.

**Definition 2.3.** Let $\pi : W \to V$ be a birational morphism. We say $\pi$ is a monoidal transformation if its center scheme is a smooth irreducible subvariety.

**Theorem 2.4** (Hironaka). Let $f : X \to Y$ be a rational map between proper varieties such that $V$ is nonsingular. Then, there is a sequence of proper varieties $X_0, \ldots, X_m$ such that

1. $X_0 = X$.
2. $\rho_i : X_i \to X_{i-1}$ is a monoidal transformation.
3. If $T_i$ is the center of blowup of $X_i$, then $\rho_0 \circ \cdots \circ \rho_i(T_i) \subset Z(f)$ on $X$.
4. $f$ is extended to a morphism $\tilde{f} : X_m \to Y$ on $X_m$. 
(5) Consider the composition of all monoidal transformation: \( \rho : X_m \to X \). Then, the underlying set of the center of blowup for \( X_m \) is exactly \( Z(f) \) on \( X \).

Proof. See [4, Question (E) and Main Theorem II]. \( \square \)

**Definition 2.5.** Let \( \pi : V \to \mathbb{P}^n \) be a birational morphism. Then, we define \( \mathfrak{I} \) is the center scheme of \( \pi \) if its corresponding ideal sheaf \( \mathcal{S} \) generates \( V \):

\[
V = \text{Proj} \bigoplus_{d \geq 0} \mathcal{S}^d.
\]

**Definition 2.6.** Let \( \pi : \tilde{V} \to V \) be a birational morphism with center scheme \( \mathfrak{I} \) and let \( D \) be an irreducible divisor of \( V \). We define the proper transformation of \( D \) by \( \pi \) to be

\[
\pi^# D = \pi^{-1}(D \cap U)
\]

where \( U = V - Z(\mathfrak{I}) \) and \( Z(\mathfrak{I}) \) is the underlying subvariety made by the zero set of the ideal \( \mathfrak{I} \).

2.2. \( \mathbb{A}^n \)-effectiveness and the \( D \)-ratio.

**Proposition 2.7.** Let \( \pi : V \to \mathbb{P}^n \) be a birational morphism which is a composition of monoidal transformation. Then, \( \text{Pic}(V) \) is a free \( \mathbb{Z} \)-module. Furthermore, let \( H \) be a hyperplane on \( \mathbb{P}^n \) and let \( E_i \) be the proper transformation of the exceptional divisor of \( i \)-th blowup. Then,

\[
\{ H_V = \pi^# H, E_1, \cdots, E_r \}
\]

is a linearly independent generator of \( \text{Pic}(V) \).

Proof. [3, Exer.II.7.9] shows that

\[
\text{Pic}(\tilde{X}) \simeq \text{Pic}(X) \oplus \mathbb{Z}
\]

if \( \pi : \tilde{X} \to X \) is a monoidal transformation. More precisely,

\[
\text{Pic}(\tilde{X}) = \{ \pi^# D + nE \mid D \in \text{Pic}(X) \}.
\]

Now suppose that \( X_0 = \mathbb{P}^n \), \( \rho_i : X_i \to X_{i-1} \) is a monoidal transformation. Then, we get the desired result. \( \square \)

**Definition 2.8.** Let \( V \) be a blowup of \( \mathbb{P}^n \), \( H \) be a fixed hyperplane of \( \mathbb{P}^n \) and

\[
\text{Pic}(V) = \mathbb{Z}H_V \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_r.
\]

Then, we define the \( \mathbb{A}^n \)-effective cone

\[
\text{AFE}(V) = \mathbb{Z}^{\geq 0}H_V \oplus \mathbb{Z}^{\geq 0}E_1 \oplus \cdots \oplus \mathbb{Z}^{\geq 0}E_r
\]

where \( \mathbb{Z}^{\geq 0} \) is the set of nonnegative integers. We say a divisor \( D \) of \( V \) is \( \mathbb{A}^n \)-effective if \( D \in \text{AFE}(V) \) and denote it by

\[
D \succ 0.
\]

Moreover, we will say

\[
D_1 \succ D_2
\]

if \( D_1 - D_2 \) is \( \mathbb{A}^n \)-effective.
Proposition 2.9. Let $V$ be a blowup of $\mathbb{P}^n$ with birational morphism $\pi : V \to \mathbb{P}^n$ and $D, D_i \in \text{Pic}(V)$.

1. (Effectiveness) If $D$ is $\mathbb{A}^n$-effective, then $D$ is effective.
2. (Boundedness) If $D$ is $\mathbb{A}^n$-effective, then $h_D(P)$ is bounded below on $V \setminus (H_V \cup (\bigcup_{i=1}^r E_i))$.
3. (Transitivity) If $D_1 \triangleright D_2$ and $D_2 \triangleright D_3$, then $D_1 \triangleright D_3$.
4. (Functoriality) If $W$ is a blowup of $V$, a map $\rho : W \to V$ is a birational morphism and $D_1 \triangleright D_2$, then $\rho^*D_1 \triangleright \rho^*D_2$.

Proof. See [10, Proposition 3.3]

Definition 2.10. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map with $Z(f) \subset H$, let $(V, \pi_V)$ be a resolution of indeterminacy of $f$ and let $\phi_V$ be a resolved morphism.

\[
\begin{array}{c}
V \\
\pi_V \\
\mathbb{P}^n \xrightarrow{f} \mathbb{P}^n \\
\phi_V
\end{array}
\]

Suppose that

\[
\pi_V^*H = a_0H_V + \sum_{i=1}^r a_iE_i \quad \text{and} \quad \phi_V^*H = b_0H_V + \sum_{i=1}^r b_iE_i
\]

where $a_i, b_i$ are nonnegative integers. If all $b_i$ are nonzero for all $i$ satisfying $a_i \neq 0$, we define the $D$-ratio of $\phi_V$,

\[
r(\phi_V) = \deg \phi_V \cdot \max_i \left( \frac{a_i}{b_i} \right).
\]

Otherwise; if there is an $i$ satisfying $a_0 \neq 0$ and $b_i = 0$, define

\[
r(\phi_V) = \infty.
\]

Lemma 2.11. Let $(V, \pi_V)$ and $(W, \pi_V)$ be resolutions of indeterminacy with resolved morphisms $\phi_V = f \circ \pi_V$ and $\phi_W = f \circ \pi_W$ respectively.

\[
\begin{array}{c}
W \\
\pi_W \\
\mathbb{P}^n \xrightarrow{f} \mathbb{P}^n \\
\phi_W \quad \phi_V
\end{array} \quad \begin{array}{c}
V \\
\pi_V \\
\mathbb{P}^n \xrightarrow{f} \mathbb{P}^n
\end{array}
\]

Then,

\[
r(\phi_V) = r(\phi_W).
\]

Proof. See [10, Lemma 4.3]

Definition 2.12. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map with $Z(f) \subset H$. Then, we define the $D$-ratio of $f$,

\[
r(f) = r(\phi_V)
\]

for any resolution of indeterminacy $(V, \pi_V)$ of $f$ with resolved morphism $\phi_V$. 
Proposition 2.13. Let \( f, g : \mathbb{P}^n \rightarrow \mathbb{P}^n \) be rational maps with \( Z(f), Z(g) \subset H \). Then,

1. \( r(f) = 1 \) if \( f \) is a morphism.
2. \( r(f) \in [1, \infty] \).
3. \( \frac{r(f)}{\deg f} \cdot \frac{r(g)}{\deg g} \geq \frac{r(g \circ f)}{\deg(g \circ f)} \).
4. If \( g \) is a morphism and \( f \) is a rational map on \( \mathbb{P}^n \), then \( r(g \circ f) = r(f) \).

Example 2.14. Let \( f : \mathbb{A}^n \rightarrow \mathbb{A}^n \) be an affine automorphism with the inverse map \( f^{-1} : \mathbb{A}^n \rightarrow \mathbb{A}^n \). Then, \( r(f) = \deg f \times \deg f^{-1} \). (For details, see [9].) For example, a Hénon map

\[
\begin{align*}
  f_H(x, y, z) &= (z, x + z^2, y + x^2) \\
  f_H^{-1}(x, y, z) &= (y - x^2, z - (y - x^2)^2, x).
\end{align*}
\]

Thus,

\[
r(f_H) = r(f_H^{-1}) = \deg f_H \times \deg f_H^{-1} = 2 \times 4 = 8.
\]

Example 2.15. Let \( f[x, y, z] = [x^2, yz, z^2] \). Then, the indeterminacy locus is \( P = [0, 1, 0] \). Then, the blowup \( V \) along closed scheme corresponding ideal sheaf \( (yz, x^2) \) will resolves indeterminacy, which is a successive blowup along \( P \) and \( H^\# \cap E_1 \).

\[
\begin{align*}
  f_1[x, y, z][x_1, z_1] &= [x_1x, z_1y, z_1z] \\
  \phi &= f_2[x, y, z][x_1, z_1][x_2, z_2] = [x_2, z_2y, x_2z_1^2]
\end{align*}
\]

Let \( E_1, E_2 \) be the exceptional divisors on each step.

Then, the intersection number \( E_2^2 = -1, E_1^2 = -2, (H^\#)^2 = -1, H^\# \cdot E_1 = 0 \) and \( H^\# \cdot E_2 = E_1 \cdot E_2 = 1 \)

Furthermore,

\[
\begin{align*}
  H^\# \cdot \phi^*H &= \phi_*H^\# \cdot H = 0, \\
  E_1 \cdot \phi^*H &= \phi_*E_1 \cdot H = 0, \\
  E_2 \cdot \phi^*H &= \phi_*E_2 \cdot H = 1.
\end{align*}
\]

Since \( \text{Pic}(V) = \langle H^\#, E_1, E_2 \rangle \), we may assume that

\[
\phi^*H = aH^\# + bE_1 + cE_2
\]
Then, by previous facts,
\[ \phi^* H \cdot H^# = -a + c = 0, \quad \phi^* H \cdot E_1 = a - 2b = 0. \]

Therefore,
\[ \phi^* H = 2H^# + E_1 + 2E_2, \quad \pi^* H = H^# + E_1 + 2E_2 \]
and hence
\[ r(f) = 2 \times 1 = 2 \]

3. Jointly Regular Families of Rational maps

Proof of Theorem 1.5. For notational convenience, let
- \( d_l = \deg f_l \)
- \( r_l = r(f_l) \)
- \((V_l, \pi_l)\) be a resolution of indeterminacy of \( f_l \) constructed by Theorem 2.4; assume \( \pi_l \) is a composition of monoidal transformation and \( \{ \pi_l^# H = H_{V_l}, E_{l1}, \ldots, E_{ls_l} \} \) is the generator of \( \text{Pic}(V_l) \) given by Proposition 2.7
- \( \phi_l \) be the resolved morphism of \( f_l \) on \( V_l \).

\[ \pi_l^* H = a_0 H_{V_l} + \sum_{i=1}^{s_l} a_{li} E_{li} \quad \text{and} \quad \phi_l^* H = b_0 H_{V_l} + \sum_{i=1}^{s_l} b_{li} E_{li} \]
in \( \text{Pic}(V_l) = \mathbb{Z}\pi_l^# H \oplus \mathbb{Z}E_{l1} \oplus \cdots \oplus \mathbb{Z}E_{ls_l} \).

We can easily check that \( a_0 = 1 \) and \( b_0 = d_l \) from \( \pi_{ls_l} \phi_l^* H = H \) and \( \pi_{ls_l} \phi_l^* H = \text{deg} \phi_l \cdot H \) For details, see [10, Proposition 4.5.(2)].

Let \( T_l \) be the center scheme of blowup for \( V_l \) and \( W \) is the blowup of \( \mathbb{P}^n \) whose center scheme is \( \sum T_l \). Then, \( W \) is a blowup of \( V_l \) for all \( l \). Furthermore, since the underlying set of \( T_l \) is exactly \( Z(f_l) \), the underlying set of \( \sum T_l = \cup Z(f_l) \). Let \( \rho_l : W \to V_l, \pi_W \) be the monoidal transformations defined by construction of \( W \):

Then, still \( W \) is a blowup of \( \mathbb{P}^n \) and hence \( \text{Pic}(W) \) is generated by \( \pi_W \) and the irreducible components of the exceptional divisor:

\[ \text{Pic}(W) = \mathbb{Z}\pi_W^# H \oplus \mathbb{Z}F_1 \oplus \cdots \oplus \mathbb{Z}F_s \]
where \( F_j \) are irreducible components of exceptional divisor of \( W \). Thus, we can represent \( \pi_w^*H \) as follows:

\[
\pi_w^*H = \pi_w^#H + \sum_{j=1}^{s} \alpha_j F_j.
\]

To describe \( \phi_i^*H \) precisely, let’s define sets of indices

\[
\mathcal{I}_l = \{1 \leq j \leq s \mid \pi_W(F_j) \subset Z(f_l)\} \quad \text{and} \quad \mathcal{I}_l^c = \{1 \leq j \leq s \mid \pi_W(F_j) \not\subset Z(f_l)\}.
\]

By definition, it is clear that

\[
\mathcal{I}_l \cup \mathcal{I}_l^c = \{1, \cdots, s\} \quad \text{and} \quad \mathcal{I}_l \cap \mathcal{I}_l^c = \emptyset.
\]

Thus, we can say

\[
\tilde{\phi}_l^*H = d_l \pi_w^#H + \sum_{j=1}^{s} \beta_{ij} F_j = d_l \pi_w^#H + \sum_{j \in \mathcal{I}_l^c} \beta_{ij} F_j + \sum_{j \in \mathcal{I}_l} \beta_{ij} F_j.
\]

Moreover, we have the following lemmas;

**Lemma 3.1.**

\[
\bigcup_{l=1}^{k} \mathcal{I}_l = \bigcup_{l=1}^{k} \mathcal{I}_l^c = \{1, \cdots, s\}.
\]

**Proof.** \( \bigcup_{l=1}^{k} \mathcal{I}_l = \{1, \cdots, s\} \) is clear; because the underlying set of the center scheme of \( W \) is \( \bigcup Z(f_l) \), \( \bigcup \pi_W(F_j) = \pi_W(\bigcup F_j) = \bigcup Z(f_l) \).

Suppose \( \bigcup_{l} \mathcal{I}_l^c \subseteq \{1, \cdots, s\} \). Then, there is an index \( k_0 \) satisfying \( \pi_W(F_{k_0}) \subset Z(f_l) \) for all \( l \).

This implies \( \pi_W(F_{k_0}) \subset Z(f_l) \) for all \( l \) and hence \( \emptyset \neq \pi_W(F_{k_0}) \subset \bigcap_l Z(f_l) \) which contradicts to that \( S \) is jointly regular. \( \square \)

**Lemma 3.2.** Let \( \alpha_j \) and \( \beta_{ij} \) be the coefficients of \( F_j \) in \( \pi_i^*H \) and \( \tilde{\phi}_l^*H \) respectively. Then,

\[
d_l \frac{\alpha_j}{\beta_{ij}} \leq r_l.
\]

Especially, if \( j \in \mathcal{I}_l^c \), then

\[
d_l \alpha_j = \beta_{ij}.
\]

**Proof.** By definition of the D-ratio, the first inequality is clear:

\[
 r_l = d_l \cdot \max_i \left( \frac{\alpha_i}{\beta_{li}} \right) \geq d_l \cdot \frac{\alpha_j}{\beta_{ij}}.
\]

Now, suppose that

\[
\rho_{l}^* \pi_i^# H = \gamma_{l00} \pi_w^# H + \sum_{j=1}^{s} \gamma_{l0j} F_j = \gamma_{l00} \pi_w^# H + \sum_{j \in \mathcal{I}_l^c} \gamma_{l0j} F_j + \sum_{j \in \mathcal{I}_l} \gamma_{l0j} F_j
\]

\[
\rho_{l}^* E_{li} = \gamma_{li0} \pi_w^# H + \sum_{j=1}^{s} \gamma_{lij} F_j = \gamma_{li0} \pi_w^# H + \sum_{j \in \mathcal{I}_l^c} \gamma_{lij} F_j + \sum_{j \in \mathcal{I}_l} \gamma_{lij} F_j.
\]
First of all, $\gamma_{l00} = 1$ and $\gamma_{li0} = 0$ for all $i \neq 0$; if $i \neq 0$, $\pi_{W^*}(\rho_i^* E_i) = 0$ because $\pi_W(\rho_i^* E_i) \subset \cup Z(f_l)$. On the other hand,

$$
\pi_W^* \left( \gamma_{l00} \pi_W^# H + \sum_{j=1}^{s_l} \gamma_{lij} F_j \right) = \gamma_{l00} H.
$$

Hence, $\gamma_{li0} = 0$. For $\gamma_{l00}$, we have

$$
\pi_W^* (\pi_W^# H) = H
$$

because $\pi_W$ is one-to-one outside of the center of blowup of $W$. Therefore,

$$
\pi_W^* (\rho_i^* \pi_i^# H) = \pi_W^* \left( \pi_W^# H - \sum_{j=1}^{s_l} a_{li} \rho_i^* E_{li} \right) = H
$$

and hence $\gamma_{l00} = 1$.

Moreover, because $\pi_i(E_{li}) \subset Z(f_l)$ and $\pi_W(F_j) \not\subset Z(f_l)$ for any $j \in I^c_l$, the multiplicity of $\rho_i(F_j)$ on $E_l$ is zero and hence $\gamma_{lij} = 0$. Thus, we can say

$$
\rho_i^* E_{li} = \sum_{j \in I_l} \gamma_{lij} F_j.
$$

Since $\tilde{\phi}_l = \rho_l \circ \phi_l$ and $\pi_W = \rho_l \circ \pi_l$, we have

$$
\tilde{\phi}_l^* H = \rho_i^* \phi_i^* H = \rho_i^* \left( d_l \pi_i^# H + \sum_{i=1}^{s_l} b_{li} E_{li} \right) = d_l \rho_i^* \pi_i^# H + \sum_{i=1}^{s_l} b_{li} \rho_i^* E_{li}.
$$

Thus,

$$
\pi_{W^*}^* H = \rho_i^* \pi_i^# H
$$

$$
= \rho_i^* \left( \pi_i^# H + \sum_{i=1}^{s_l} a_{li} E_{li} \right)
$$

$$
= \left( \pi_{W^*}^# H + \sum_{j \in I_l} \gamma_{li0j} F_j + \sum_{j \in I^c_l} \gamma_{li0j} F_j \right) + \sum_{i=1}^{s_l} a_{li} \left( \sum_{j \in I_l} \gamma_{lij} F_j \right)
$$

$$
= \pi_{W^*}^# H + \sum_{j \in I^c_l} \gamma_{li0j} F_j + \sum_{j \in I_l} \left( \sum_{i=0}^{s_l} a_{li} \gamma_{lij} \right) F_j.
$$
\begin{align*}
\tilde{\phi}_l^* H &= \rho_1^* \phi_l^* H \\
&= \rho_1^* \left( \pi_i^* H + \sum_{i=1}^{s_l} b_{li} E_{li} \right) \\
&= d_l \left( \pi_i^* H + \sum_{j \in I_i} \gamma_{i0j} F_j + \sum_{j \in I_i^c} \gamma_{i0j} F_j \right) + \sum_{i=1}^{s_l} b_{li} \left( \sum_{j \in I_i} \gamma_{ij} F_j \right) \\
&= d_l \pi_i^* H + \sum_{j \in I_i^c} d_l \gamma_{i0j} F_j + \sum_{j \in I_i} \left( \sum_{i=1}^{s_l} b_{li} \gamma_{ij} \right) F_j.
\end{align*}

Therefore,
\[ d_l \alpha_j = d_l \sum_{j \in I_i^c} \gamma_{i0j} = \beta_j \quad \text{for all } j \in I_i^c. \]

\[ \square \]

We now complete the proof of Theorem 1.5. Let \( r = \max r_i \). Note that

\[ p_0 \pi_i^* H + \sum_{j=1}^{s} p_j F_j \succ q_0 \pi_i^* H + \sum_{j=1}^{s} q_j F_j \]

if \( p_j \geq q_j \) for all \( j = 0, \cdots, s \). Then, we have

\[ \sum_{l=1}^{k} \frac{1}{d_l} \tilde{\phi}_l^* H \]

\[ > \sum_{l=1}^{k} \pi_i^* H + \sum_{l=1}^{k} \sum_{j \in I_i^c} \alpha_j F_j + \sum_{l=1}^{k} \left( \sum_{j \in I_i} \frac{\alpha_j}{r} F_j \right) \quad (\because \text{Lemma 3.2}) \]

\[ > k \pi_i^* H + \sum_{l=1}^{k} \sum_{j \in I_i^c} \alpha_j F_j + \sum_{l=1}^{k} \left( \sum_{j \in I_i} \frac{\alpha_j}{r} F_j \right) \quad (\because r \geq r_l) \]

\[ > k \pi_i^* H + \sum_{j=1}^{s} \alpha_j F_j + \frac{1}{r} \sum_{j=1}^{s} \alpha_j F_j \quad (\because \text{Lemma 3.1}) \]

\[ > \left( 1 + \sum_{j=1}^{s} \frac{1}{r} \right) \pi_i^* H \]

and hence

\[ D = \sum_{l=1}^{k} \frac{1}{d_l} \tilde{\phi}_l^* H - \left( 1 + \sum \min \frac{1}{r_l} \right) \pi_i^* H \]

is an \( \mathbb{A}^n \)-effective divisor.
Thus, by Proposition 2.9, \( h_D \) is bounded below on \( \pi_W^{-1} A^n \). Therefore, there is a constant \( C \) such that

\[
\begin{align*}
  h_D(Q) &= \sum_{l=1}^{k} \frac{1}{d_l} \sum_{r_l} h(\tilde{\varphi}_l(Q)) - \left( 1 + \sum \min \left\{ \frac{1}{r_l} \right\} h_H(Q) \right) \\
  &= k \sum_{l=1}^{k} \frac{1}{d_l} h_H(\tilde{\varphi}_l(Q)) - \left( 1 + \sum \min \left\{ \frac{1}{r_l} \right\} h_H(Q) \right) \\
  &> C
\end{align*}
\]

for all \( Q \in \pi_W^{-1}(A^n)(\overline{K}) \). Finally, for \( P = \pi(Q), \tilde{\varphi}_l(Q) = f(P) \) and \( \pi_W(Q) = P \) and hence

\[
\begin{align*}
  \sum_{l=1}^{k} \frac{1}{d_l} h_H(P) - \left( 1 + \sum \min \left\{ \frac{1}{r_l} \right\} h_H(P) \right) > C.
\end{align*}
\]

\[ \qed \]

**Example 3.3.** Let

\[
f_1 = (z, y + z^2, x + (y + z^2)^2), \quad f_2 = (x, y^2, z), \quad f_3 = (x^2 + y, y, z^3).
\]

Then, the \( r(f_1) = 8, r(f_2) = 2 \) and \( r(f_3) = 3/2 \). (For details of the D-ratio calculation, see [10].) Therefore,

\[
\begin{align*}
  h((z, y + z^2, x + (y + z^2)^2)) + h((x, y^2, z)) + h((x^2 + y, y, z^3)) &\geq \left( 1 + \frac{1}{8} \right) h((x, y, z)) - C
\end{align*}
\]

for some constant \( C \).

**Corollary 3.4.** Let \( S \) be a jointly regular set of affine morphisms. Then,

\[
\kappa(S) := \liminf_{P \in A^n(K)} \frac{1}{h(P) \to \infty} \sum_{f \in S} \frac{1}{\deg f} h(f(P)) \geq h(P) + \frac{1}{r}
\]

where \( r = \max_{f \in S} r(f) \).

**Remark 3.5.** Corollary 3.4 may not be the exact limit infimum value. For example, If there is a subset \( S' \subset S \) such that \( S' \) is still jointly regular and \( \max_{f \in S'} r(f) < \max_{f \in S} r(f) \), then

\[
\kappa(S) > \kappa(S') \geq \frac{1}{r'} > 1 + \frac{1}{r'}.
\]

**Example 3.6.** We have some examples for \( \kappa(S) = 1 + \min_{f \in S} \left( \frac{1}{\deg f} \right) \cdot \frac{1}{\deg f} \):

1. \( S = \{ f, g \} \) where \( f, g \) are morphisms. If \( f, g \) are morphism, then \( r(f) = r(g) = 1 \). Therefore,

\[
\frac{1}{\deg f} h(f(P)) + \frac{1}{\deg g} h(g(P)) = h(P) + h(P) + O(1).
\]

2. \( S = \{ f, f^{-1} \} \) where \( f \) is a regular affine automorphism and \( f^{-1} \) is the inverse of \( f \). It is proved by Kawaguchi. See [6].
4. AN APPLICATION TO ARITHMETIC DYNAMICS

This result is a generalization of [14, Section 4]. The only difference is that we have an improved inequality for jointly regular family. The proof is almost the same.

Fix an integer \( m \geq 1 \) and let \( S = \{ f_1, \cdots, f_k \} \subset \text{Rat}^n(H) \) be a jointly regular family. For each \( m \geq 0 \), let \( W_m \) be the collection of ordered \( m \)-tuples chosen from \( \{ 1, \cdots, k \} \),

\[
W_m = \{ (i_1, \cdots, i_m) \mid i_j \in \{ 1, \cdots, k \} \} = \{ 1, \cdots, k \}^m,
\]

and let

\[
W_* = \bigcup_{m \geq 0} W_m.
\]

Thus \( W_* \) is the collection of words on \( r \) symbols.

For any \( I = (i_1, \cdots, i_m) \in W_m \), let \( f_I \) denote the corresponding composition of the rational maps \( f_1, \cdots, f_k \),

\[
f_I = f_{i_1} \circ \cdots \circ f_{i_m}.
\]

**Definition 4.1.** We denote the monoid of rational maps generated by \( f_1, \cdots, f_k \) under composition by

\[
\Phi = \{ \phi = f_I \mid I \in W_* \}.
\]

Let \( P \in \mathbb{A}^n \). The \( \Phi \)-orbit of \( P \) is

\[
\Phi(P) = \{ \phi(P) \mid \phi \in \Phi \}.
\]

The set of (strongly) \( \Phi \)-preperiodic points is the set

\[
\text{Preper}(\Phi) = \{ P \in \mathbb{A}^n \mid \Phi(P) \text{ is finite} \}.
\]

**Proof of Theorem 1.6.** By Theorem 1.5, we have a constant \( C \) such that

\[
0 \leq \left( \frac{1}{1 + \frac{r}{r+1}} \right) \sum_{l=1}^k \frac{1}{d_l} h(f_l(Q)) - h(Q) + C \quad \text{for all } Q \in \mathbb{A}^n.
\]

(1)

Note that if \( r = \infty \), then \( \left( \frac{1}{1 + \frac{r}{r+1}} \right) = 1 \) and theorem holds because of [14, Section 4]. Thus, we may assume that \( r \) is finite.

We define a map \( \mu : W_* \to \mathbb{Q} \) by the following rule:

\[
\mu_I = \mu_{(i_1, \cdots, i_m)} = \prod_{l=1}^k \mathcal{P}_{l,l}^{p_{l,l}}
\]

where \( p_{l,l} = -|\{ t \mid i_t = l \}| \). Then, by definition of \( \delta \) and \( \mu_I \), the following is true:

\[
\delta^m = \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \frac{1}{\deg f_{i_1} \cdots \deg f_{i_m}} = \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \mu_I.
\]
Let $P \in \mathbb{A}^n(\overline{\mathbb{Q}})$. Then, (1) holds for $f_I(P)$ for all $I \in W_m$:

$$0 \leq \left( \frac{r}{r+1} \right)^k \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) - h(f_I(P)) + C.$$  

and hence

$$0 \leq \sum_{m=0}^{M} \sum_{I \in W_m} \mu_I \left( \frac{r}{r+1} \right)^m \left[ \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) - \left( 1 + \frac{1}{r} \right) h(f_I(P)) + C \right].$$  

(2)

The main difficulty of the inequality is to figure out the constant term. From the definition of $\delta$, we have

$$\sum_{m=0}^{M-1} \left( \frac{r}{r+1} \right)^m \sum_{I \in W_m} \mu_I = \sum_{m=1}^{M} \delta^m \leq \frac{1}{1-\delta}.$$  

Now, do the telescoping sum and most terms in (2) will be canceled;

$$\left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) \right) - \left( \sum_{m=0}^{M} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I h(f_I(P)) \right)$$

$$= \left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \left( \frac{r}{r+1} \right)^m \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) \right) - \left( \sum_{m=0}^{M-1} \sum_{I \in W_m} \sum_{l=1}^{k} \left( \frac{r}{r+1} \right)^m \frac{\mu_I}{d_l} h(f_i(f_I(P))) \right)$$

$$= 0$$

Therefore, the remaining terms in (2) are

$$0 \leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) \right] - h(P) + \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I C$$

$$\leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} h(f_i(f_I(P))) \right] - h(P) + \frac{1}{1-\delta} C,$$

$$\sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} = \left( \frac{r}{r+1} \right)^M \sum_{I \in W_{M+1}} \mu_I = \left( 1 + \frac{1}{r} \right) \delta^{M+1}.$$  

Define the height of the images of $P$ by the monoid $\Phi$:

$$h(\Phi(P)) = \sup_{R \in \Phi(P)} h(R).$$

Then, if $P \in \text{Preper}(\Phi)$, $h(\Phi(P))$ is finite and hence we have an upper bound for $h(P)$:

$$h(P) \leq \left[ \sum_{I \in W_M} \left( \frac{r}{r+1} \right)^M \mu_I \sum_{l=1}^{k} \frac{1}{d_l} \right] h(\Phi(P)) + \frac{1}{1-\delta} C$$

$$\leq \left( 1 + \frac{1}{r} \right) \delta^{M+1} h(\Phi(P)) + \frac{1}{1-\delta} C.$$
By assumption, \( \delta < 1 \) and \( h(\Phi(P)) \) is finite, so letting \( M \to \infty \) shows that \( h(P) \) is bounded by a constant that depends only on \( S \).

\[ \square \]

References

[1] Cutkosky, Steven Dale. *Resolution of singularities*, Graduate Studies in Mathematics, Vol 63, American Mathematics Society, 2004
[2] Fulton, W., *Intersection theory*, Second edition, Springer-Verlag, Berlin, 1998
[3] Hartshorne, R., *Algebraic geometry*, Springer, 1977
[4] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero. I*, Ann. of Math. (2) 79 (1964), 109-203
[5] Kawaguchi, S., *Canonical height functions for affine plane automorphisms*, Math. Ann. 335 2006, no. 2, 285–310
[6] Kawaguchi, S., *Local and global canonical height functions for affine space regular automorphisms*, preprint, [arXiv:0909.3573], 2009.
[7] Lang, S., *Fundamentals of diophantine geometry*, Berlin Heidelberg New York: Springer 1983
[8] Lazarsfeld, R., *Positivity in Algebraic Geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Bd. 48 (Springer, New York, 2004
[9] Lee, C., *The upper bound of height and regular affine automorphisms on \( \mathbb{A}^n \)*, submitted, [arXiv:0909.3107], 2009
[10] Lee, C., *The maximal ratio of coefficients of divisors and an upper bound for height for rational maps*, submitted, [arXiv:1002.3357], 2010
[11] Marcello, S., *Sur la dynamique arithmetique des automorphismes de l’espace affine*, Bull. Soc. Math. France, 131, 229-257, 2003
[12] Northcott, D. G., *Periodic points on an algebraic variety*, Ann. of Math. (2), 51, 167-177, 1950
[13] Shafarevich, I. *Basic Algebraic Geometry*, Springer, 1994
[14] Silverman, J. H. *Height bounds and preperiodic points for families of jointly regular affine maps*, Pure Appl. Math. Q. 2, 2006, no. 1, part 1, 135–145.
[15] Silverman, J. H., Hindry, M. *Diophantine Geometry, An introduction*, Springer 2000
[16] Silverman, J. H. *The arithmetic of Dynamical systems*, Springer, 2007
[17] Weil, A., *Arithmetic on algebraic varieties*, Ann. of Math. (2) 53, 412-444, 1951

Department of Mathematics, Brown University, Providence RI 02912, US

E-mail address: phiel@math.brown.edu