Strong polygamy and monogamy relations for multipartite quantum systems

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Abstract
Monogamy and polygamy are the most striking features of the quantum world. We investigate the monogamy and polygamy relations satisfied by all quantum correlation measures for arbitrary multipartite quantum states. By introducing residual quantum correlations, analytical polygamy inequalities are presented, which are shown to be tighter than the existing ones. Then, similar to polygamy relations, we obtain strong monogamy relations that are better than all the existing ones. Typical examples are presented for illustration.

Keywords Polygamy relation · Monogamy relation · Multipartite systems · Residual quantum correlation

1 Introduction
Quantum correlation is one of the most important properties of quantum physics, which has been extensively studied due to its importance in quantum communication and quantum information processing. One significant property of quantum correlation is known as monogamy. For a tripartite system $A$, $B$ and $C$, the usual monogamy of a quantum correlation measure $Q$ implies that the correlation $Q_{A|BC}$ between $A$ and $BC$ satisfies $Q_{A|BC} \geq Q_{AB} + Q_{AC}$. Dually, the polygamy relation is quantitatively displayed as $Q_{A|BC} \leq Q_{AB} + Q_{AC}$. It is shown that while monogamy inequalities provide an upper bound for multipartite sharability of quantum correlations in a multipar-
tite system, the polygamy inequalities give a lower bound. The first monogamy relation was proven for arbitrary three-qubit states based on the squared concurrence. Later, various monogamy inequalities have been established for a number of entanglement measures in multipartite quantum systems [1–8]. Polygamy relations are also generalized to multiqubit systems [9] and arbitrary dimensional multipartite states [4–6].

As is well known, the usual monogamy and polygamy relations are not always satisfied by any correlation measures like entanglement of formation [10] quantifying the amount of entanglement required for preparation of a given bipartite quantum state. It has been shown that the $\alpha$th ($\alpha \geq 2$) power of concurrence and the $\alpha$th ($\alpha \geq \sqrt{2}$) power of entanglement of formation do satisfy the monogamy relations for $N$-qubit states [2,4]. One may ask whether any measures of quantum correlations satisfy a kind of monogamy or polygamy relations. In this paper, we first show that all quantum correlation measures satisfy some kind of polygamy relations for arbitrary multipartite quantum states. Then, we introduce the residual quantum correlations and present tighter polygamy inequalities that are better than all the existing ones. At last, similar to polygamy relations, we present the strong monogamy relations that are also better than the existing ones.

2 Strong polygamy relations for multipartite quantum systems

Let $Q$ be an arbitrary quantum correlation measure of bipartite systems. $Q$ is said to be polygamous for an $N$-partite quantum state $\rho_{AB_1B_2\cdots B_{N-1}}$, if it satisfies the following inequality:

$$Q(\rho_{AB_1}) + Q(\rho_{AB_2}) + \cdots + Q(\rho_{AB_{N-1}}) \geq Q(\rho_{A|B_1B_2\cdots B_{N-1}}),$$

where $\rho_{AB_i}$, $i = 1, \ldots, N - 1$, are the reduced density matrices, $Q(\rho_{A|B_1B_2\cdots B_{N-1}})$ denotes the quantum correlation $Q$ of the state $\rho_{AB_1B_2\cdots B_{N-1}}$ under bipartite partition $A$ and $B_1B_2\cdots B_{N-1}$, which keeps invariant under discarding subsystems only for states satisfying monogamy relations. For simplicity, we denote $Q(\rho_{AB_i})$ by $Q_{AB_i}$, and $Q(\rho_{A|B_1B_2\cdots B_{N-1}})$ by $Q_{A|B_1B_2\cdots B_{N-1}}$. We define the $Q$-polygamy score for the $N$-partite state $\rho_{AB_1B_2\cdots B_{N-1}}$:

$$\delta_Q = \sum_{i=1}^{N-1} Q_{AB_i} - Q_{A|B_1B_2\cdots B_{N-1}}.$$ 

Nonnegativity of $\delta_Q$ for all quantum states implies the polygamy of $Q$. For instance, the square of the concurrence in terms of the concurrence of assistance has been shown to be polygamous for all multiqubit states [9].

Given any quantum correlation measure that is not polygamous for a multipartite quantum state, it is always possible to find a function of the measure which is polygamous for the same state [11]. It has been proved that for any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$
state $\rho_{A_1B_2\cdots B_{N-1}}$, there exists $\beta_{\text{max}}(Q) \in R$ such that for any $0 \leq \gamma \leq \beta_{\text{max}}(Q)$, the quantum correlation measure $Q$ satisfies the following polygamous relation [11]:

$$Q_{A|B_1B_2\cdots B_{N-1}}^\gamma \leq \sum_{i=1}^{N-1} Q_{A|B_i}^\gamma,$$  

(3)

In the following, we denote $\beta = \beta_{\text{max}}(Q)$ the maximal value such that $Q^\beta$ satisfies the above inequality. Similar to the three tangle of concurrence, for tripartite quantum states $\rho_{ABC}$, we define the residual quantum correlation as a function of $\alpha$:

$$Q_{A|B|C}^\alpha = Q_{AB}^\alpha + Q_{AC}^\alpha - Q_{AB|C}^\alpha, \quad 0 \leq \alpha \leq \beta.$$  

(4)

For the class of GHZ states, the equality (4) is valid for $\beta = 0$.

From the original definition in [12,13], the residual quantum correlation is defined to be $Q_{A|B|C} = Q_{A|B} - Q_{AB} - Q_{AC}$ for some quantum correlation measures $Q$ satisfying the monogamy relations $Q_{AB} \geq Q_{AB} + Q_{AC}$. Generally, it is not the quantum correlation measure $Q$ itself, but the $\alpha$th power satisfies the monogamy inequality, for instance, the $\alpha$th ($\alpha \geq 2$) power of concurrence and the $\alpha$th ($\alpha \geq \sqrt{2}$) power of entanglement of formation [2]. It is also the case for polygamy relations. Therefore, here we use the $\alpha$th power of the quantum correlation to define the “residual quantum correlation.”

The residual quantum correlations quantify the degree of entanglement distributions among the subsystems: The smaller the $\alpha$ in (4), the greater the degree of violation of the polygamy inequality. Let us consider the tripartite systems. The residual quantum correlation is defined by $Q_{A|B|C}^\alpha = Q_{AB}^\alpha + Q_{AC}^\alpha - Q_{AB|C}^\alpha (0 \leq \alpha \leq \beta)$. For two states $\rho_{ABC}$ and $\delta_{ABC}$ such that $Q_{A|B|C}(\rho_{ABC}) = Q_{A|B|C}(\delta_{ABC}) = 0$, $\alpha_1 \leq \alpha_2$, we have $|Q(\rho_{AB}) - Q(\rho_{AC})| \leq |Q(\delta_{AB}) - Q(\delta_{AC})|$. The distribution of quantum correlation in $\rho_{ABC}$ is more averaged than that in state $\delta_{ABC}$. For example, consider the state $|\psi\rangle = \psi_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, where $\lambda_i \geq 0$, $i = 0, \ldots, 4$ and $\sum_{i=0}^{4} \lambda_i^2 = 1$. We have the concurrences $C_{A|B|C} = 2\lambda_0\sqrt{\lambda_2^2 + \lambda_3^2 + \lambda_4^2}$, $C_{AB} = 2\lambda_0\lambda_2$, and $C_{AC} = 2\lambda_0\lambda_3$. Taking $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{5}$, we have $\rho_{ABC} = |\psi_1\rangle\langle \psi_1|$, where $|\psi_1\rangle = \frac{\sqrt{5}}{5}|000\rangle + \frac{\sqrt{5}}{5} e^{i\phi}|100\rangle + \frac{\sqrt{5}}{5}|101\rangle + \frac{\sqrt{5}}{5}|110\rangle + \frac{\sqrt{5}}{5}|111\rangle$. One gets $C(\rho_{AB})^\alpha = (\frac{2\sqrt{5}}{5})^\alpha$, $C(\rho_{AC})^\alpha = (\frac{2\sqrt{5}}{5})^\alpha$, and $C(\rho_{AC})^\alpha \approx 1.26185$ from $Q_{A|B|C}(\rho) = 0$. If we take $\lambda_0 = \lambda_2 = \frac{1}{2}$, $\lambda_1 = \lambda_3 = \lambda_4 = \frac{\sqrt{5}}{6}$, then the state becomes $\delta_{ABC} = |\psi_2\rangle\langle \psi_2|$, where $|\psi_2\rangle = \frac{1}{2}|000\rangle + \frac{\sqrt{6}}{6} e^{i\phi}|100\rangle + \frac{1}{2}|101\rangle + \frac{\sqrt{6}}{6}|110\rangle + \frac{\sqrt{6}}{6}|111\rangle$. One has $\alpha_2 \approx 1.33770$ based on $Q_{A|B|C}(\delta_{ABC}) = 0$. From above, one can easily get that the entanglement distribution between the subsystems in $\rho_{ABC}$ is more averaged than that in $\delta_{ABC}$.

Consider a $d \otimes d_1 \otimes d_2 \otimes d_3$ state $\rho_{A_1B_2B_3}$. Define $Q_{A|B_1'|B_2}' = \max\{Q_{A|B_1|B_2}', Q_{A|B_1|B_3}', Q_{A|B_2|B_3}'\}$, where $B_1'$ and $B_2'$ stand for two of $B_1$, $B_2$ and $B_3$ such that $Q_{A|B_1|B_2}' = \max\{Q_{A|B_1|B_2}, Q_{A|B_1|B_3}, Q_{A|B_2|B_3}\}$. 

$\square$ Springer
Theorem 1 For any $d \otimes d_1 \otimes d_2 \otimes d_3$ state $\rho_{A_1B_2B_3}$, we have

$$Q^{\alpha}_{A|B_1B_2B_3} \leq \sum_{i=1}^{3} Q^{\alpha}_{A|B_i} - Q^{\alpha}_{A|B'_1|B'_2},$$ (5)

for $0 \leq \alpha \leq \beta$.

Proof By definition, we have

$$\sum_{i=1}^{3} Q^{\alpha}_{A|B_i} - Q^{\alpha}_{A|B'_1|B'_2} = Q^{\alpha}_{A|B'_3} + Q^{\alpha}_{A|B'_1B'_2} \geq Q^{\alpha}_{A|B_1B_2B_3},$$

where $B'_3$ is the complementary of $B'_1B'_2$ in the subsystem $B_1B_2B_3$; the equality is due to the definition of the residual quantum correlation. From (3), we get the inequality.

$\square$

Concerning the parameter $\beta$ in Theorem 1, let us consider the following 4-qubit state:

$$|\psi\rangle_{A_1B_2B_3} = \cos \theta_0 |0000\rangle + \sin \theta_0 \cos \theta_1 e^{i\phi} |1000\rangle + \frac{1}{2} \sin \theta_0 \sin \theta_1 |1010\rangle + \frac{3}{4} \sin \theta_0 \sin \theta_1 |1100\rangle + \frac{\sqrt{3}}{4} \sin \theta_0 \sin \theta_1 |1110\rangle,$$ (6)

where $\theta_0, \theta_1 \in [0, \frac{\pi}{2}]$. We have $C_{A_1B_2B_3} = 2 \cos \theta_0 \sin \theta_0 \sin \theta_1$, $C_{A_1B_2} = \cos \theta_0 \sin \theta_0 \sin \theta_1$, $C_{A_1B_3} = \frac{3}{2} \cos \theta_0 \sin \theta_0 \sin \theta_1$ and $C_{A_2B_3} = C_{A|B'_1|B'_2} = 0$. From (5), we obtain $(\frac{1}{3})^{\alpha} + (\frac{3}{4})^{\alpha} \geq 1$, namely $\alpha \leq 1.507126$. Therefore, $\beta = 1.507126$ is the largest value saturating the inequality (5) for the state (6).

Inequality (5) presents a tighter polygamy relations for $0 \leq \alpha \leq \beta$. Specially, inequality (5) is satisfied only when $\alpha = 0$ for particular quantum states like the GHZ-class states. Generalizing the conclusion of Theorem 1 to $N$ partite case, we have the following result.

Theorem 2 For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{A_1B_2\cdots B_{N-1}}$, we have

$$Q^{\alpha}_{A|B_1B_2\cdots B_{N-1}} \leq \sum_{i=1}^{N-1} Q^{\alpha}_{A|B_i} - \sum_{k=2}^{N-2} Q^{\alpha}_{A|B'_1B'_2\cdots B'_k},$$ (7)

for $0 \leq \alpha \leq \beta$, where $Q^{\alpha}_{A|B'_1B'_2\cdots B'_k} = \max_{1 \leq l \leq k+1} \{Q^{\alpha}_{A|B_1\cdots \hat{B}_l\cdots B_{k+1}}\}$ (where $\hat{B}_l$ stands for $B_l$ being omitted in the subindices), $Q^{\alpha}_{A|B_1B_2\cdots B_{k+1}} = \sum_{i=1}^{k+1} Q^{\alpha}_{A|B_i} - Q^{\alpha}_{A|B_1B_2\cdots B_{k+1}} - \sum_{i=2}^{k} Q^{\alpha}_{A|B'_1B'_2\cdots B'_l}, 2 \leq k \leq N - 2, 1 \leq l \leq k + 1, N \geq 4$. 

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Proof We prove the theorem by induction. For $N = 4$, it reduces to Theorem 1. Suppose Theorem 2 holds for $N = n$, i.e.,

$$Q_{A|B_1 B_2 \cdots B_{n-1}}^\alpha \leq \sum_{i=1}^{n-1} Q_{A|B_i}^\alpha - Q_{A|B_i'}^\alpha - \cdots - Q_{A|B_i' B_{n-1}'}^\alpha.$$  \hspace{1cm} (8)

Then, for $N = n + 1$, we have

$$\sum_{i=1}^{n} Q_{A|B_i}^\alpha - Q_{A|B_i'}^\alpha - \cdots - Q_{A|B_i' B_{n-1}'}^\alpha \geq Q_{A|B_1' B_2' \cdots B_{n-1}'}^\alpha + Q_{A|B_i'}^\alpha \geq Q_{A|B_1 B_2 \cdots B_n}^\alpha,$$

where $B_i'$ is the complementary of $B_1', B_2', \ldots, B_{n-1}'$ in the subsystem $B_1, B_2, \ldots, B_n$; the first inequality is due to (8). By (3), we get the last inequality. \hfill \Box

Since the last term $\sum_{k=2}^{N-2} Q_{A|B_1' B_2' \cdots B_k'}^\alpha$, $2 \leq k \leq N - 2$, $N \geq 4$ in (7) is non-negative, the inequality (7) is always tighter than (3). Let us consider the following example based on the quantum entanglement measure concurrence. For a bipartite pure state $|\phi\rangle_{AB}$, the concurrence is $C(|\phi\rangle_{AB}) = \sqrt{2[1 - Tr(\rho_A^2)]}$, where $\rho_A$ is the reduced density matrix by tracing over the subsystem $B$, $\rho_A = Tr_B(|\phi\rangle_{AB}\langle\phi|)$. For a mixed state $\rho_{AB} = \sum_{i} p_i |\phi_i\rangle_{AB} \langle\phi_i|$, the concurrence is defined by the convex roof extension, $C(\rho_{AB}) = \min_{\{p_i, |\phi_i|\}} \sum_{i} p_i C(|\phi_i|)$, where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_{i} p_i |\phi_i\rangle \langle\phi_i|$, with $p_i \geq 0$ and $\sum_{i} p_i = 1$. The concurrence of assistance is defined by $C_a(\rho_{AB}) = \max_{\{p_i, |\phi_i|\}} \sum_{i} p_i C(|\phi_i|)$. And the entanglement of assistance $\tau_a$ is given by $\tau_a(\rho_{AB}) = \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} C_a((\rho_{AB})_{mn}) = \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} (\max_{\{p_i, |\phi_i|\}} (L_A^m \otimes L_B^n) |\phi_i\rangle \langle\phi_i|)) [14]$, where $D_1 = d_1(d_1 - 1)/2$, $D_2 = d_2(d_2 - 1)/2$, $L_A^m = P_A^m (-\langle i|, A\langle j| + \langle j|, A\langle i|) P_A^m$, $L_B^n = P_B^n (-\langle k|, B\langle l| + \langle l|, B\langle k|) P_B^n$, and $P_A^m = |i\rangle, A\langle i| + |j\rangle, A\langle j|$, $P_B^n = |k\rangle, B\langle k| + |l\rangle, B\langle l|$. A general polygamy inequality for any multipartite pure state $|\phi\rangle_{A_1 \cdots A_n}$ was established as [9], $\tau_a^2(|\phi\rangle_{A_1 A_2 \cdots A_n}) \leq \sum_{i=2}^{n} \tau_a^2(\rho_{A_1 A_i})$, where $\rho_{A_1 A_k}$ is the reduced density matrix of subsystems $A_1 A_k$ for $k = 2, \ldots, n$. It has been further shown that [11]

$$\tau_a^\alpha(|\phi\rangle_{A_1 A_2 \cdots A_n}) \leq \sum_{i=2}^{n} \tau_a^\alpha(\rho_{A_1 A_i}), \hspace{1cm} (9)$$

where $0 \leq \alpha \leq 2$

Example 1 Let us consider the entanglement of assistance $\tau_a$ of the following 5-qubit pure state:

$$|\psi\rangle_{AB_1 B_2 B_3 B_4} = \frac{1}{\sqrt{5}} ([10000] + [01000] + [00100] + [00010] + [00001]). \hspace{1cm} (10)$$

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Fig. 1 Solid (blue) line is the $\alpha$th power of $\tau_a$ under bipartition $A|B_1 B_2 B_3 B_4$; dashed (red) line is the upper bound in (9); dotted (green) line is the upper bound in (7) (Color figure online)

We have $\beta = 2$, $\tau_a(|\psi\rangle_A|B_1 B_2 B_3 B_4) = \frac{4}{5}$, $\tau_a(\rho_{AB_i}) = \frac{2}{5}$, $i = 1, 2, 3, 4$. $\tau_a|B_i|B_j = 3\left(\frac{1}{2}\right)^\alpha - \left(\frac{\sqrt{3}}{2}\right)^\alpha$, $i \neq j \neq k \in \{1, 2, 3, 4\}$. From the result (9) in [11], we get $\tau^\alpha_a(|\psi\rangle_A|B_1 B_2 B_3 B_4) \leq 4\left(\frac{2}{5}\right)^\alpha$. From our inequality (7) in Theorem 2, we have $\tau^\alpha_a(|\psi\rangle_A|B_1 B_2 B_3 B_4) \leq 4\left(\frac{2}{5}\right)^\alpha - 3\left(\frac{1}{2}\right)^\alpha + \left(\frac{\sqrt{3}}{2}\right)^\alpha$. Obviously, our result (7) is better than that in [11]; see Fig. 1.

In Theorems 1 and 2, we have taken into account the maximum value among $Q^\alpha_{A|B_1 \cdots \hat{B}_i \cdots B_4}$. If instead of the maximum value, one just considers the mean value of $Q^\alpha_{A|B_1 \cdots \hat{B}_i \cdots B_4}$, one may have the following corollary.

**Corollary 1** For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{A|B_1 B_2 \cdots B_{N-1}}$, we have

$$Q^\alpha_{A|B_1 B_2 \cdots B_{N-1}} \leq \sum_{i=1}^{N-1} Q^\alpha_{AB_i} - \sum_{k=3}^{N-1} \left( \frac{1}{k} \sum_{l=1}^{k} Q^\alpha_{A|B_1 \cdots \hat{B}_i \cdots B_k} \right), \quad (11)$$

for all $0 \leq \alpha \leq \beta$, $N \geq 4$, where

$$Q^\alpha_{A|B_1 B_2 \cdots B_{j}} = \sum_{i=1}^{j} Q^\alpha_{AB_i} - Q^\alpha_{A|B_1 B_2 \cdots B_{j}} - \sum_{k=3}^{j} \left( \frac{1}{k} \sum_{l=1}^{k} Q^\alpha_{A|B_1 \cdots \hat{B}_i \cdots B_k} \right), \quad (12)$$

$3 \leq j \leq N - 1$, $3 \leq k \leq N - 1$ and $1 \leq l \leq k$.

Next, we adopt an approach used in Ref. [15] to improve further the above results on polygamy relations for multipartite quantum correlation measures. First, we give a lemma.
Lemma 1 For any $d_1 \otimes d_2 \otimes d_3$ mixed state $\rho_{ABC}$, if $Q_{AB} \geq Q_{AC}$, we have

$$Q_{A|BC}^\alpha \leq Q_{AB}^\alpha + L Q_{AC}^\alpha,$$

for all $0 \leq \alpha \leq \beta$, where $L = (2^{\frac{\alpha}{\beta}} - 1)$.

Proof For arbitrary $d_1 \otimes d_2 \otimes d_3$ tripartite state $\rho_{ABC}$. If $Q_{AB} \geq Q_{AC}$, we have

$$Q_{A|BC}^\alpha \leq (Q_{AB}^\beta + Q_{AC}^\beta)^{\frac{\alpha}{\beta}} = Q_{AB}^\alpha \left(1 + \frac{Q_{AC}^\beta}{Q_{AB}^\beta}\right)^{\frac{\alpha}{\beta}}$$

$$\leq Q_{AB}^\alpha \left[1 + (2^{\frac{\alpha}{\beta}} - 1) \left(\frac{Q_{AC}^\beta}{Q_{AB}^\beta}\right)^{\frac{\alpha}{\beta}}\right]$$

$$= Q_{AB}^\alpha + (2^{\frac{\alpha}{\beta}} - 1) Q_{AC}^\alpha,$$

where the first inequality is due to (3) and the second inequality is due to the inequality $(1+t)^x \leq 1 + (2^x - 1)t^x$ for $0 \leq x \leq 1$, $0 \leq t \leq 1$.

In the above lemma, without loss of generality, we have assumed that $Q_{AB} \geq Q_{AC}$, as the subsystems $A$ and $B$ are equivalent. Moreover, in the proof of Lemma 1 we have assumed $Q_{AB} > 0$. If $Q_{AB} = 0$ and $Q_{AB} \geq Q_{AC}$, then $Q_{AB} = Q_{AC} = 0$. The upper bound is trivially zero. Generalizing Lemma 1 to multipartite quantum systems, we have the following theorem.

Theorem 3 For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1\cdots B_{N-1}}$, if $Q_{AB_i} \geq Q_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $Q_{AB_j} \leq Q_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, we have

$$Q_{A|B_1B_2\cdots B_{N-1}}^\alpha \leq Q_{AB_1}^\alpha + L Q_{AB_2}^\alpha + \cdots + L^{m-1} Q_{AB_m}^\alpha$$

$$+ L^m Q_{AB_{m+1}}^\alpha + \cdots + Q_{AB_{N-1}}^\alpha,$$

for all $0 \leq \alpha \leq \beta$, where $L = (2^{\frac{\alpha}{\beta}} - 1)$.

Proof By using Lemma 1 repeatedly, one gets

$$Q_{A|B_1B_2\cdots B_{N-1}}^\alpha \leq Q_{AB_1}^\alpha + L Q_{AB_2}^\alpha + \cdots + L^{m-1} Q_{AB_m}^\alpha + L^m Q_{AB_{m+1}}^\alpha + \cdots + Q_{AB_{N-1}}^\alpha.$$

As $Q_{AB_j} \leq Q_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m+1, \ldots, N-2$, by (13) we get

$$Q_{A|B_{m+1}\cdots B_{N-1}}^\alpha \leq L (Q_{AB_{m+1}}^\alpha + Q_{AB_{m+2}}^\alpha + \cdots + Q_{AB_{N-2}}^\alpha) + Q_{AB_{N-1}}^\alpha.$$

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Combining (15) and (16), we have Theorem 3.

Similar to Theorem 2, (14) can be improved by adding a term for residual quantum correlation. By a similar derivation to Theorem 2, we have

Theorem 4 For any \(d \otimes d_1 \otimes \cdots \otimes d_{N-1}\) state \(\rho_{AB_1\cdots B_{N-1}}\), if \(Q_{AB_i} \geq Q_{A|B_{i+1}\cdots B_{N-1}}\) for \(i = 1, 2, \ldots, m\), and \(Q_{AB_j} \leq Q_{A|B_{j+1}\cdots B_{N-1}}\) for \(j = m + 1, \ldots, N - 2\), \(\forall 1 \leq m \leq N - 3, N \geq 4\), we have

\[
Q_{A|B_1B_2\cdots B_{N-1}}^\alpha \leq \sum_{i=1}^{N-1} \hat{Q}_{AB_i} - \sum_{k=2}^{N-2} \hat{Q}_{A|B_i'\cdots B_k'}, \tag{17}
\]

for all \(0 \leq \alpha \leq \beta\), where \(\hat{Q}_{AB_1} = Q_{AB_1}, \hat{Q}_{AB_2} = LQ_{AB_2}, \ldots, \hat{Q}_{AB_m} = L^{m-1}Q_{AB_m}, \hat{Q}_{AB_{m+1}} = L^{m+1}Q_{AB_{m+1}}, \ldots, \hat{Q}_{AB_{N-2}} = L^{m+1}Q_{AB_{N-2}}, \hat{Q}_{AB_{N-1}} = L^{m}Q_{AB_{N-1}}, L = (2^{\frac{\beta}{2}} - 1).\) The residual quantum correlation term \(\hat{Q}_{A|B_i'\cdots B_k'}^\alpha = \max_{1 \leq l \leq k} \{\hat{Q}_{A|B_1\cdots B_l|B_{k-l}}\}, \hat{Q}_{A|B_1\cdots B_k'} = \sum_{i=1}^{k} \hat{Q}_{AB_i} - Q_{A|B_1B_2\cdots B_k} - \sum_{i=2}^{k-1} \hat{Q}_{A|B_i'\cdots B_k'}\)

As an example, let us consider again the concurrence of the state (10). From our inequality (7) in Theorem 2, we have \(\tau_a^\alpha (|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 4\left(\frac{3}{2}\right)^{\alpha} - 3\left(\frac{1}{2}\right)^{\alpha} + (\sqrt{3})^{\alpha}\). From the inequality (17) in Theorem 4, we have \(\tau_a^\alpha (|\psi\rangle_{A|B_1B_2B_3B_4}) \leq 3(2^{\frac{\alpha}{2}})^{\alpha} - 2\left(\frac{3}{2}\right)^{\alpha} - 2\left(\frac{1}{2}\right)^{\alpha} + (\sqrt{3})^{\alpha}.\) Obviously, the inequality (17) is better than the inequality in [11]. We see in Fig. 2 that the bound (7) is improved.
3 Strong monogamy relations for multipartite quantum systems

We now study the monogamy relations for multipartite states. The monogamy relations limit the distributions of quantum correlations among the multipartite systems and play an important role in secure quantum cryptography [16] and in condensed matter physics such as the $n$-representability problem for fermions [17].

Monogamy and polygamy of entanglement can restrict the possible correlations between the authorized users and the eavesdroppers, thus tightening the security bounds in quantum cryptography. The optimized monogamy and polygamy relations give rise to finer characterizations of the entanglement distributions. Furthermore, to optimize the efficiency of entanglement used in quantum cryptography, finer characterizations of the entanglement distributions are preferred in some physical systems for stronger security in quantum key distribution [18].

Monogamy relations of entanglement for multiqubit some higher-dimensional quantum systems have been investigated in terms of various entanglement measures [2,4,5,12,19]. However, there are other measures such as quantum discord, quantum deficit and entanglement of formation, which do not satisfy the monogamy relations for pure three-qubit states [20,21]. In [22], the authors find a monotonically increasing function of quantum measures, from which a quantum correlation can always be made to be monogamous for given state. It has been proved that for arbitrary dimensional tripartite states, there exists $x_{\text{min}}(Q) \in \mathbb{R}$ such that for any $y \geq x_{\text{min}}(Q)$, a quantum correlation measure $Q$ satisfies the following monogamy relation [22]:

$$Q^y_{A|BC} \geq Q^y_{AB} + Q^y_{AC}. \quad (18)$$

In the following, we denote $x = x_{\text{min}}(Q)$ the minimal value such that $Q^x$ satisfies the above inequality. Inequality (18) has been generalized to the $N$-partite case for all measures of quantum correlations [23]:

$$Q^y_{A|B_1B_2\ldots B_{N-1}} \geq \sum_{i=1}^{N-1} Q^y_{AB_i}, \quad (19)$$

for $y \geq x$, $N \geq 3$. (19) has been further improved such that for $y \geq x$, if $Q_{AB_i} \geq Q_{A|B_{i+1}\ldots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $Q_{AB_j} \leq Q_{A|B_{j+1}\ldots B_{N-1}}$ for $j = m+1, \ldots, N-2$, $\forall 1 \leq m \leq N-3$, $N \geq 4$, then [23],

$$Q^y_{A|B_1B_2\ldots B_{N-1}} \geq \sum_{i=1}^{N-1} \hat{Q}^y_{AB_i} + \sum_{k=2}^{N-2} \hat{Q}^y_{A|B'_1|B'_2|\ldots|B'_k}, \quad (20)$$

for all $y \geq x$, $\hat{Q}^y_{AB_1} = Q^y_{AB_1}$, $\hat{Q}^y_{AB_2} = K Q^y_{AB_2}$, $\ldots$, $\hat{Q}^y_{AB_m} = K^{m-1} Q^y_{AB_m}$, $\hat{Q}^y_{AB_{m+1}} = K^{m+1} Q^y_{AB_{m+1}}$, $\ldots$, $\hat{Q}^y_{AB_{N-2}} = K^{m+1} Q^y_{AB_{N-2}}$, $\hat{Q}^y_{AB_{N-1}} = K^m Q^y_{AB_{N-1}}$ and $K = \frac{y}{x}$.

The residual quantum correlation term

$$\hat{Q}^y_{A|B'_1|B'_2|\ldots|B'_{k-1}} = \max_{1 \leq l \leq k} \{\hat{Q}^y_{A|B_1|B'_l|\ldots|B'_k}\}$$

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(where $\hat{B}_l$ stands for $B_l$ being omitted in the subindices), $\hat{Q}_{A|B_1|B_2|\cdots|B_k}^y = \hat{Q}_{A|B_1B_2\cdots B_k}^y - \sum_{i=1}^k \hat{Q}_{A|B_i}^y - \sum_{i=2}^{k-1} \hat{Q}_{A|B_i'B_i''|\cdots|B_k'}^y$, $2 \leq k \leq N - 2$, $1 \leq l \leq k$.

In fact, as a kind of characterization of the quantum correlation distribution among the subsystems, the monogamy inequalities satisfied by the quantum correlations can be further refined and become tighter.

**Lemma 2** For any $d_1 \otimes d_2 \otimes d_3$ mixed state $\rho_{ABC}$, if $Q_{AB} \geq Q_{AC}$, we have

$$Q_{A|BC}^y \geq Q_{AB}^y + LQ_{AC}^y,$$

(21)

for all $y \geq x$, where $L = (2^\frac{y}{x} - 1)$.

**Proof** For arbitrary $d_1 \otimes d_2 \otimes d_3$ tripartite state $\rho_{ABC}$. If $Q_{AB} \geq Q_{AC}$, we have

$$Q_{A|BC}^y \geq (Q_{AB}^x + Q_{AC}^x)^\frac{y}{x} = Q_{AB}^y \left(1 + \frac{Q_{AC}^x}{Q_{AB}^x}\right)^\frac{y}{x},$$

$$\geq Q_{AB}^y \left[1 + L \left(\frac{Q_{AC}^x}{Q_{AB}^x}\right)^\frac{y}{x}\right],$$

$$= Q_{AB}^y + LQ_{AC}^y,$$

where the first inequality is due to (18) and the second inequality is due to the inequality $(1 + t)^x \geq 1 + (2^x - 1)t^x$ for $x \geq 1$, $0 \leq t \leq 1$ [5].

**Theorem 5** For any $d \otimes d_1 \otimes \cdots \otimes d_{N-1}$ state $\rho_{AB_1\cdots B_{N-1}}$, if $Q_{AB_i} \geq Q_{A|B_{i+1}\cdots B_{N-1}}$ for $i = 1, 2, \ldots, m$, and $Q_{AB_j} \leq Q_{A|B_j+1\cdots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, $\forall 1 \leq m \leq N - 3$, $N \geq 4$, we have

$$Q_{A|B_1B_2\cdots B_{N-1}}^y \geq \sum_{i=1}^{N-1} \sum_{k=2}^{N-2} Q_{A|B_iB_i'|\cdots|B_k'}^y,$$

(22)

for all $y \geq x$, where $Q_{AB_1}^y = Q_{AB_1}^y$, $Q_{AB_2}^y = LQ_{AB_2}^{y'}$, $\ldots$, $Q_{AB_m}^y = L^{m-1}Q_{AB_m}^{y''}$, $Q_{AB_{m+1}}^{y'} = L^{m+1}Q_{AB_{m+1}}^{y''} \cdots$, $Q_{AB_{N-2}}^{y''} = L^{m+1}Q_{AB_{N-2}}^{y'''}$, $Q_{AB_{N-1}}^{y'''} = L^m Q_{AB_{N-1}}^{y'''}$, $L = (2^\frac{y}{x} - 1)$. The residual quantum correlation term $\tilde{Q}_{A|B_1'B_1''\cdots B_k'}^y \geq \max_{1 \leq l \leq k} \{ \tilde{Q}_{A|B_1'B_1''\cdots B_k'}^y \}$, $\tilde{Q}_{A|B_1B_2\cdots B_k}^y = Q_{A|B_1B_2\cdots B_k}^y - \sum_{i=1}^k \tilde{Q}_{A|B_i}^y - \sum_{i=2}^{k-1} \tilde{Q}_{A|B_i'B_i''|\cdots|B_k'}^y$, $2 \leq k \leq N - 2$, $1 \leq l \leq k$.

**Proof** By using Lemma 2 repeatedly, one gets

$$Q_{A|B_1B_2\cdots B_{N-1}}^y \geq Q_{AB_1}^y + LQ_{AB_2}^{y''} + \cdots + L^m Q_{AB_m}^{y'''},$$

(23)
As $Q_{AB_j} \leq Q_{A|B_{j+1}\cdots B_{N-1}}$ for $j = m + 1, \ldots, N - 2$, by (15) we get

$$Q_y^{A|B_{m+1}\cdots B_{N-1}} \geq L Q_y^{A|B_{m+1}} + Q_y^{A|B_{m+2}\cdots B_{N-1}} \geq L( Q_y^{A|B_{m+1}} + \cdots + Q_y^{A|B_{N-2}}) + Q_y^{A|B_{N-1}}.$$  \hfill (24)

Combining (23) and (24), we have

$$Q_y^{A|B_1 B_2 \cdots B_{N-1}} \geq \sum_{i=1}^{N-1} \tilde{Q}_y^{A|B_i}.$$  \hfill (25)

Suppose that Theorem 5 holds for $N = n$, i.e.,

$$Q_y^{A|B_1 B_2 \cdots B_{n-1}} \geq \sum_{i=1}^{n-1} \tilde{Q}_y^{A|B_{i+1}} + \tilde{Q}_y^{A|B_i|B_1' B_2'} + \cdots + \tilde{Q}_y^{A|B_{i+1}|B_1' B_2' \cdots B_{n-2}'}.$$  \hfill (26)

Then, for $N = n + 1$, we have

$$\sum_{i=1}^{n} \tilde{Q}_y^{A|B_{i+1}} + \tilde{Q}_y^{A|B_i|B_1' B_2'} + \cdots + \tilde{Q}_y^{A|B_{i+1}|B_1' B_2' \cdots B_{n-1}'}$$

$$\leq \tilde{Q}_y^{A|B_1' B_2' \cdots B_{n-1}'} + \tilde{Q}_y^{A|B_n'} \leq Q_y^{A|B_1 B_2 \cdots B_n},$$

where $B_n'$ is the complementary of $B_1' B_2', \ldots, B_{n-1}'$ in the subsystem $B_1 B_2, \ldots, B_n$. The first inequality is due to (26). By (25), we get the last inequality.
Example 2 For the concurrence of the $W$ state,

\[
|W\rangle_{A|B_1B_2B_3} = \frac{1}{2}(|1000⟩ + |0100⟩ + |0010⟩ + |0001⟩),
\]

we have $x = 2$, $C_{AB_i} = \frac{1}{2}$, $i = 1, 2, 3$, and $C_{A|B_1B_2} = C_{A|B_3} = C_{A|B_2B_3} = \frac{\sqrt{2}}{2}$.

From the inequality (20), one has $\hat{C}_y^{A|B_1B_2} = \hat{C}_y^{A|B_1B_3} = \hat{C}_y^{A|B_2B_3} = (\frac{\sqrt{2}}{2})^y - (1 + \frac{y}{2})(\frac{1}{2})^y$. Hence, the lower bound of $C_y^{A|B_1B_2B_3}$ is $\sum_{i=1}^3 \hat{C}_y^{A|B_i} = \hat{C}_y^{A|B_1B_2} = (\frac{\sqrt{2}}{2})^y + \frac{y}{2}(\frac{1}{2})^y$. From the inequality (22) in Theorem 5, we have $\tilde{C}_y^{A|B_1B_2} = \tilde{C}_y^{A|B_1B_3} = \tilde{C}_y^{A|B_2B_3} = (\frac{\sqrt{2}}{2})^y - (\frac{1}{2})^y$. The lower bound of $C_y^{A|B_1B_2B_3}$ is $\sum_{i=1}^3 \tilde{C}_y^{A|B_i} = \tilde{C}_y^{A|B_1B_2} = (\frac{\sqrt{2}}{2})^y + (2^y - 1)(\frac{1}{2})^y$. One can see that our result is better than (20) in [23]; see Fig. 3.

4 Conclusion

Monogamy and polygamy inequalities are the key features of multipartite entanglement, which distinguish the quantum from the classical correlations. We have investigated the monogamy and polygamy relations satisfied by arbitrary quantum correlation measures for arbitrary multipartite quantum states. Similar to the three tangle of concurrence, we have introduced the $\alpha$th ($0 \leq \alpha \leq \beta$) power of the residual quantum correlation. In terms of the residual quantum correlations, analytical polygamy inequalities have been presented, which are shown to be tighter than the existing ones. Similarly, we have obtained the strong monogamy relations that are also better than all the existing ones. Detailed examples have been given for illustration. The novel residual quantum correlation we introduced may also contribute to improve other relations satisfied by quantum correlation measures.

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References

1. Bai, Y.K., Ye, M.Y., Wang, Z.D.: Entanglement monogamy and entanglement evolution in multipartite systems. Phys. Rev. A 80, 044301 (2009)
2. Zhu, X.N., Fei, S.M.: Entanglement monogamy relations of qubit systems. Phys. Rev. A 90, 024304 (2014)
3. Bai, Y.K., Xu, Y.F., Wang, Z.D.: General monogamy relation for the entanglement of formation in multiqubit systems. Phys. Rev. Lett. 113, 100503 (2014)
4. Jin, Z.X., Fei, S.M.: Tighter entanglement monogamy relations of qubit systems. Quantum Inf. Process. 16, 77 (2017)
5. Jin, Z.X., Li, J., Li, T., Fei, S.M.: Tighter monogamy relations in multiqubit systems. Phys. Rev. A 97, 032336 (2018)
6. Kim, J.S.: Negativity and tight constraints of multiqubit entanglement. Phys. Rev. A 97, 012334 (2018)
7. Kim, J.S.: Weighted polygamy inequalities of multiparty entanglement in arbitrary-dimensional quantum systems. Phys. Rev. A 97, 042332 (2018)
8. Gour, G., Guo, Y.: Monogamy of entanglement without inequalities. Quantum 2, 81 (2018)
9. Gour, G., Bandyopadhyay, S., Sanders, B.C.: Dual monogamy inequality for entanglement. J. Math. Phys. 48, 012108 (2007)
10. Bennett, C.H., Bernstein, H.J., Popescu, S., Schumacher, B.: Concentrating partial entanglement by local operations. Phys. Rev. A 53, 2046 (1996)
11. Jin, Z.X., Fei, S.M.: Superactivation of monogamy relations for nonadditive quantum correlation measures. Phys. Rev. A 99, 032343 (2019)
12. Ekert, A.K.: Quantum cryptography based on Bell’s theorem. Phys. Rev. Lett. 67, 661 (1991)
13. Coffman, V., Kundu, J., Wootters, W.K.: Distributed entanglement. Phys. Rev. A 61, 052306 (2000)
14. Kim, J.S.: Polygamy of entanglement in multipartite quantum systems. Phys. Rev. A 80, 022302 (2009)
15. Jin, Z.X., Fei, S.M.: Finer distribution of quantum correlations among multiqubit systems. Quantum Inf Process 18, 21 (2019)
16. Pawlowski, M.: Security proof for cryptographic protocols based only on the monogamy of Bell’s inequality violations. Phys. Rev. A 82, 032313 (2010)
17. Coleman, A.J., Yukalov, V.I.: Reduced Density Matrices: Coulson’s Challenge. Lecture Notes in Chemistry, vol. 72. Springer, Berlin (2000)
18. Groblacher, S., Jennewein, T., Vaziri, A., Weihs, G., Zeilinger, A.: Experimental quantum cryptography with qutrits. New J. Phys. 8, 75 (2006)
19. Adesso, G., Serafini, A., Illuminati, F.: Multipartite entanglement in three-mode Gaussian states of continuous-variable systems: quantification, sharing structure, and decoherence. Phys. Rev. A 73, 032345 (2006)
20. Giorgi, G.L.: Monogamy properties of quantum and classical correlations. Phys. Rev. A 84, 054301 (2011)
21. Prabh, R., Pati, A.K., Sen, A., Sen, U.: Conditions for monogamy of quantum correlations: Greenberger–Horne–Zeilinger versus W states. Phys. Rev. A 85, 040102(R) (2012)
22. Salini, K., Prabhu, R., Sen, A., Sen, U.: Monotonically increasing functions of any quantum correlation can make all multiparty states monogamous. Ann. Phys. 348, 297–305 (2014)
23. Jin, Z.X., Fei, S.M.: Monogamy relations of all quantum correlation measures for multipartite quantum systems. Optics Commun. 446, 39–43 (2019)

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