Hopf bifurcation of the stochastic model on business cycle

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Abstract: A stochastic model on business cycle was presented in this paper. Simplifying the model through the quasi Hamiltonian theory, the Ito diffusion process was obtained. According to Oseledec multiplicative ergodic theory and singular boundary theory, the conditions of local and global stability were acquired. Solving the stationary FPK equation and analyzing the stationary probability density, the stochastic Hopf bifurcation was explained. The result indicated that the change of parameter $a$ was the key factor to the appearance of the stochastic Hopf bifurcation.

1. Introduction
There have been presented numerous interesting contributions to business cycle modeling during the last decades, using more complicated relations. It is, however, interesting to see how complex the scenarios can become with even the most simplified assumptions, to which complications of the model can only add even more complexity. The dynamical characters like bifurcation and chaos had been discussed, but most of the models are discrete with no stochastic factors [1-5].

Recently, nonlinear theory has been applied in many areas, such as atom motions, HAB, flexible rotor, microbial continuous culture model with time delay, viscoelastic moving belt and neuron models [6-11]. More and more economists have realized that there exists many stochastic factors impacting on economics, so the economical system presents randomicity and uncertainty. In this cases, certain economical models cannot truly reflect the action of economical system, it is necessary to do research on stochastic theory of economics, more and more research have been done on stochastic factors impacting.

2. Stochastic dynamical model
Considering the influence of stochastic factors, a stochastic dynamical model has been set up:

$$\ddot{q} + (1-a)\dot{q} + (1+a)q^3 + bq = \beta qW(t)$$

(1)

Where $q$ presents output, $a = v - s$ , ( $a > 0$ ), $v$ is the “accelerator”, ( $v > 1$), $s$ presents the complementary proportion saved, ( $0 \leq s \leq 1$), $b = (1-\varepsilon)s$ , ( $0 \leq \varepsilon \leq 1$), $0 \leq b \leq 1$, $\beta$ are parameters, $W(t)$ is in possession of zero mean value Gauss white noise and its intensity is 2D, $\beta qW(t)$ presents the influence of inner stochastic factors, the system (1) is defined as a weakly damp and weakly excitation quasi Hamiltonian system.

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The system (1) can be changed as follows:

\[
\begin{align*}
\dot{q} &= p \\
\dot{p} &= -(a + 1)p^3 + (a - 1)p + bq + \beta q W(t)
\end{align*}
\]  

(2)

The Hamiltonian function is:

\[
H(p, q) = p^2 / 2 + bq^2 / 2
\]  

(3)

According to the quasi Hamiltonian theory, the system (2) converges to the one-dimensional Ito diffusion process:

\[dH = m(H)dt + \sigma(H)dB(t)\]  

(4)

Where: B(t) is a standard Weiner process, \(m(H)\) and \(\sigma(H)\) present drift coefficient and diffusion coefficient, they can be calculated by quasi non-integrable Hamiltonian stochastic average method as follows:

\[
m(H) = \left(\frac{D\beta^2}{b} + a - 1\right)H - \frac{3}{2}(a + 1)H^2
\]

\[
\sigma^2(H) = \frac{D\beta^2H^2}{b}
\]

3. Stochastic stability

3.1. Local stochastic stability

We get the linearized Ito stochastic differential equation, its max Lyapunov exponent is as follows:

\[
\lambda = \lim_{t \to \infty} \frac{1}{t} \ln H^{1/2}(t) = \frac{1}{2} \left[ \frac{D\beta^2}{2b} + a - 1 \right]
\]  

(5)

According to Oseledec multiplicative ergodic theory, the equilibrium point of system (4) is asymptotic stable when \(\lambda < 0\) or unstable when \(\lambda > 0\); if \(\lambda = 0\), it means the stochastic bifurcation may occur. So we obtain the asymptotic stable condition of the equilibrium point is that:

\[
a < 1 - \frac{D\beta^2}{2b}
\]  

(6)

3.2. Global stochastic stability

The max Lyapunov exponent based on Oseledec multiplicative ergodic theory can only be used to judge the local stability, here we judge the global stability by the singular boundary theory.

\(H = 0\) is the first kind of singular boundary of system (4) when \(\sigma^2(H) = 0\); when \(H = +\infty\), we find \(m(H) = \infty\), so \(H = +\infty\) is the second kind of singular boundary of system (4).

According to the singular boundary theory, we calculate the diffusion exponent, drifting exponent and characteristic value of boundary \(H = 0\) and the results are as follows:

\[
\alpha_1 = 2, \beta_1 = 1
\]
\[ c_i = 2 + \frac{2(a - 1)b}{D\beta^2} \]  

(7)

If \( c_i > 1 \), the boundary \( H = 0 \) is exclusively natural boundary.
If \( c_i < 1 \), the boundary \( H = 0 \) is attractively natural boundary.
If \( c_i = 1 \), the boundary \( H = 0 \) is strictly natural boundary.

We can also calculate the diffusion exponent, drifting exponent and characteristic value of boundary \( H = +\infty \) and the results are as follows:

\[ \alpha_r = 2, \quad \beta_r = 2 \]
\[ c_r = -\frac{3(a + 1)b}{D\beta^2} \]

So, the boundary \( H = +\infty \) is entrance boundary.
If the singular boundary \( H = 0 \) is attractively natural boundary and \( H = +\infty \) is entrance boundary, this situation is all the solve curves enter the inner system from the right boundary \( H = +\infty \) and is attracted by the left boundary \( H = 0 \), the equilibrium point is global stable.

From the analysis above, we can draw a conclusion that the equilibrium point is global stable when the singular boundary \( H = 0 \) is attractively natural boundary and \( H = +\infty \) is entrance boundary.
Combine the condition of local stability, the equilibrium point \( H = 0 \) is stable when

\[ 0 < a < 1 - \frac{D\beta^2}{2b} \]  

(8)

3.3. Conclusion of local and global stabilities
From the stochastic stability analysis above, we can draw a conclusion that: the equilibrium point of system (4) is asymptotic stable under the condition (8); The results indicate that the system’s stability is influenced by the inner multiplicative excitations.

4. Stochastic Hopf bifurcation
It is the only strong solution of system (4) initialized by \( H \). Here \( m(0) = 0, \sigma(0) = 0 \), so zero is fixed point. Solve the stationary FPK equation, we obtain:

\[ \frac{\partial p}{\partial t} = -\frac{\partial}{\partial H}[m(H)p] + \frac{1}{2} \frac{\partial^2 [\sigma^2(H)p]}{\partial H^2} \]  

(11)

Solve (11), the stationary probability density has been obtained:

\[ p(H) = CH^\eta \exp\left[-\frac{3(a + 1)b}{D\beta^2}H\right] \]  

(12)

where: \( \eta = \frac{2(a - 1)b}{D\beta^2} \).

Near the equilibrium point, drift coefficient \( m(H) \) and diffusion coefficient \( \sigma(H) \) can be denoted as:
\[ m(H) = O(H^\beta), \quad \beta_i = 1, \quad H \to 0 \]  

(13)

\[ \sigma^2(H) = O(H^{\alpha_i}), \quad \alpha_i = 2, \quad H \to 0 \]  

(14)

\[ c_i = \lim_{H \to 0} \frac{2m(H)H^{\alpha_i-\beta_i}}{\sigma^2(H)} \]  

(15)

So, the stationary probability density can be rewritten as:

\[ P(H) = O\left[H^{-\alpha_i} \exp\left[c_i \int_0^H x^{(\beta_i-\alpha_i)} dx\right]\right], \quad H \to 0 \]  

(16)

When \( \beta_i - \alpha_i = -1 \) and \( H \to 0 \), \( P(H) = O(H^\eta) \), where \( \eta = c_i - \alpha_i = \frac{2(a-1)b}{D\beta^2} \).

When \( \eta < -1 \), \( P(H) \) is function \( \delta \); when \(-1 < \eta < 0 \), \( P(H) \) get its maximum value at \( H = 0 \), the first bifurcation occurs; when \( \eta > 0 \), \( P(H) \) still has its maximum value but the position is away from \( H = 0 \), the second bifurcation occurs. So we can say that the bifurcation occurs at \( \eta = -1 \) and \( \eta = 0 \), that is, \( a = 1 - \frac{D\beta^2}{2b} \) and \( a = 1 \).

Here we mainly discuss the stochastic Hopf bifurcation at \( a = 1 \), fix up the parameters \( b, \beta, D \), the stationary probability density \( P(H) \) and joint probability density \( P(p,q) \) can be seen in Fig 1-Fig 8 (\( b = 0.5, \beta = 0.8, D = 0.4 \)).

![Figures](image-url)
From Fig1-Fig8, we can see that the figures of the stationary probability density and the joint probability density changes, especially the joint probability density has turned into crater. The appearance of crater means stochastic Hopf bifurcation occurs.

5. Conclusion
According to the analysis above, we can conclusion that the parameter \( a \) is the bifurcation parameter, the stochastic Hopf bifurcation may occur at \( a = 1 \). The stochastic Hopf bifurcation can cause shock in the economy system which may do a great harm to the society, so we should do our best to avoid it. The detailed measure is to expand the investment, that is, enlarge the parameter \( a \), to make it larger far from 1. In developed countries, parameter \( b \) is small, the effective way to avoid bifurcation is to enlarge the parameter \( a \), but in developing countries, the parameter \( a \) is small (sometimes less than 1) and the parameter \( b \) is larger than developed countries, this situation is dangerous with the increase of the economy when \( a = 1 \), at that time, the government should take effective measure to increase investment as quickly as possible so as to reduce the damage of stochastic Hopf bifurcation.

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