An index theorem on asymptotically static spacetimes with compact Cauchy surface

Dawei SHEN & Michał WROCHNA

Abstract. We consider the Dirac operator on asymptotically static Lorentzian manifolds with an odd-dimensional compact Cauchy surface. We prove that if Atiyah–Patodi–Singer boundary conditions are imposed at infinite times then the Dirac operator is Fredholm. This generalizes a theorem due to Bär–Ströhmaier [7] in the case of finite times, and we also show that the corresponding index formula extends to the infinite setting. Furthermore, we demonstrate the existence of a Fredholm inverse which is at the same time a Feynman parametrix in the sense of Duistermaat–Hörmander. The proof combines methods from time-dependent scattering theory with a variant of Egorov’s theorem for pseudo-differential hyperbolic systems.

1. Introduction

1.1. Introduction and main result. In the last few years it was realized that hyperbolic operators on Lorentzian manifolds have an interesting Fredholm and spectral theory despite their non-ellipticity and non-hypoellipticity [75, 9, 33, 7, 27, 39, 77]. Furthermore, it turned out that it is possible to show relationships with geometry that have surprisingly many common features with the Riemannian setting [7, 69, 16, 24, 8].

In particular, Bär–Ströhmaier [7] proved an index formula which can be interpreted as the Lorentzian version of the Atiyah–Patodi–Singer theorem for the Dirac operator on compact spacetimes with space-like boundary. A generalisation to spatially non-compact manifolds was then shown by Braverman [16] in the case of a confining Callias potential, and different boundary conditions were studied by Bär–Hannes [5].

On the other hand, relativistic physics has provided strong motivation for considering hyperbolic operators on Lorentzian manifolds that are non-compact in temporal directions. This has included proofs of the Fredholm property or global invertibility of the wave and Klein–Gordon operator by Vasy [75] and Hintz [44] on asymptotically de Sitter and Kerr–de Sitter spacetimes and by Baskin–Vasy–Wunsch [9], Gell-Redman–Haber–Vasy [33], Gérard–Wrochna [39, 40, 37] and Taira [70] for asymptotically Minkowski spacetimes, cf. Dereziński–Siemssen [27, 28], Vasy [77] and Nakamura–Taira [58] for the closely related question of essential self-adjointness. However, as these results do not concern the Dirac operator, they are unlikely to be tied to a rich index theory, and typically the index is zero indeed.

The objective of the present work is to prove an index theorem for the Dirac operator on a class of Lorentzian manifolds that are non-compact in time, generalizing in this way the result of Bär–Ströhmaier, and to analyze microlocal properties of parametrices needed to make the connection with the local geometry.
Namely, we make the following assumption on the geometry. Suppose that \((M, g)\) is an oriented and time-oriented smooth Lorentzian manifold such that \(M = \mathbb{R} \times \Sigma\) and \(\Sigma\) is compact and odd-dimensional.

\[
\text{Hypothesis 1.1. We assume the metric to be of the form}
\]
\[
g = -c^2(t)dt^2 + h(t),
\]
where \(c \in C^\infty(M)\) satisfies \(c > 0\) and \(\mathbb{R} \ni t \mapsto h(t) \in C^\infty(\Sigma; T^*\Sigma \otimes_s T^*\Sigma)\) is a smooth family of Riemannian metrics on \(\Sigma\) such that for some \(\delta > 1\),

1. \(c(t) - c_{\pm} \in S^{-\delta}(\mathbb{R}_\pm, C^\infty(\Sigma))\) for some \(c_{\pm} \in C^\infty(\Sigma)\) s.t. \(c_{\pm} > 0\),
2. \(h(t) - h_{\pm} \in S^{-\delta}(\mathbb{R}_\pm, C^\infty(\Sigma; T^*\Sigma \otimes_s T^*\Sigma))\) for some Riemannian metric \(h_{\pm}\).

This means in particular that \(g\) decays as \(t \to \pm \infty\) to a static (product type) metric at short-range rate \(|t|^{-\delta}\), and differentiating the coefficients \(k\) times in \(t\) yields decay of order \(|t|^{-\delta-k}\). Thus, \(g\) is an asymptotically static metric. Through the change of coordinates \(\tau_{\pm} = e^{\pm t}\) for \(t\) close to \(\pm \infty\) we can compactify \(M\), which results in a manifold \(\overline{M}\) with boundary \(\partial \overline{M} = \{\tau_+ = 0\} \cup \{\tau_- = 0\}\) with two components representing future and past infinity \(t = \pm \infty\).

**Remark 1.1.** Close to the boundary, if we write \(\tau = \tau_{\pm}\), then \(dt^2 = \pm \frac{d\tau^2}{c^{\pm}}\). We remark that in Hypothesis 1.1 we do not assume that \(c_{\partial \overline{M}} = c_{\pm} = 1\). Furthermore, while the conditions (1)–(2) imply conormality of the metric coefficients, they do not imply smoothness in \(\tau\) at the boundary. This is the main difference as compared to (the Lorentzian analogue of) the class of exact b-metrics considered by Melrose [55] in his version of the Atiyah–Patodi–Singer theorem.

We consider the Dirac operator \(D\) acting on positive-chirality sections of a spinor bundle \(SM\) over \(M\), see §2 for details on the terminology. At each time \(t \in \mathbb{R}\), \(D\) induces a Riemannian Dirac operator denoted by \(A(t)\), and there are two boundary Dirac operators \(A_{\pm}\) corresponding to the limits \(t \to \pm \infty\). If \(I \subset \mathbb{R}\) is an interval we denote by \(1_I(A(t))\) the corresponding spectral projection.

We consider \(D\) as an operator \(D : \mathcal{X} \to \mathcal{Y}\), where \(\mathcal{Y}\) is a weighted \(L^2\)-space with a sufficiently decaying weight in \(t\) (the precise choice has no importance for the result), and \(\mathcal{X}\) is the completion of smooth spinors in the corresponding \(L^2\) graph norm, see §6.3. We then define \(D_{\text{APS}}\) as the restriction of \(D\) to the space \(\mathcal{X}_{\text{APS}}\) consisting of all \(u \in \mathcal{X}\) such that

\[
\lim_{t \to +\infty} 1_{]-\infty,0]}(A(t))u(t) = 0, \quad \lim_{t \to -\infty} 1_{[0,\infty]}(A(t))u(t) = 0.
\]

As in the compact setting of [7], (1.3) are interpreted as Atiyah–Patodi–Singer boundary conditions at \(\partial \overline{M}\). Our first result is the following theorem.

**Theorem 1.2** (cf. Theorem 6.6). The operator \(D_{\text{APS}} : \mathcal{X}_{\text{APS}} \to \mathcal{Y}\) is Fredholm of index

\[
\text{ind}(D_{\text{APS}}) = \int_M \widehat{A} + \int_{\partial \overline{M}} T\widehat{A} + \frac{1}{2}(\eta(A_+, A_-) - \dim \ker(A_+) - \dim \ker(A_-)),
\]
where \( \hat{A} \) the Atiyah–Singer integrand (or \( \hat{\Lambda} \)-form) on \((M, g)\), \( T\hat{A} \) is the transgression form, and \( \eta(A_+, A_-) = \eta(A_+) - \eta(A_-) \) is the difference of the eta forms of \( A_+ \) and \( A_- \).

The content of the above index formula is exactly analogous to the result of Bär–Strohmaier (and its generalisation [16], where \( \eta(A_+, A_-) \) is defined as a relative eta invariant [17]), which in turn closely parallels the Atiyah–Patodi–Singer theorem in Riemannian signature, see [2, 43, 41, 55].

On the other hand, a significant difference with the results known in the finite-time case can be seen on the level of local properties of parametrices of \( D_{APS} \).

The essential question is how do parametrices of \( D_{APS} \) fit into the Duistermaat–Hörmander theory of Fourier integral operators [31], or more precisely, does \( D_{APS} \) have parametrices with Feynman wavefront set, a microlocal condition (see Definition 7.1) which characterizes a parametrix uniquely modulo smoothing operators. In the finite-time setting of [7] this property was only shown to hold true for metrics that are exactly static (product-type) in a neighborhood of the boundary, yet the results for wave operators on non-compact spacetimes [33, 78, 39, 77] suggest that this assumption might not necessarily be needed in situations like Hypothesis 1.1. Our second main result states that the Feynman property holds true indeed.

**Theorem 1.3** (cf. Theorem 7.5). \( D_{APS} \) admits a parametrix which has Feynman wavefront set.

The importance of the Feynman property stems primarily from applications in Quantum Field Theory on curved spacetimes, as already outlined in the work of Duistermaat–Hörmander [31] and further developed by many other authors. Very recently, however, it was demonstrated that is also essential in relating the global theory of hyperbolic operators with the local geometry (as it enables the use of the Hadamard parametrix as an intermediary). In particular, it plays a central rôle in the proof of the Lorentzian spectral action expansion and Kastler–Kalau–Walze theorem by Dang–Wrochna [24]. In the Dirac case, the Feynman wavefront set is the key assumption in the approach of Bär–Strohmaier [8] to local index theory in the Lorentzian signature. Thus, as a consequence of Theorem 1.3, the local index theorem [8] applies to the geometric setting of Hypothesis 1.1.

**1.2. Plan of the proofs.** The proof of the index formula in [7] is based on a spectral flow argument and a reduction to Euclidean signature: this turns out to be very robust and can be applied to our setting with only a few necessary adaptations (see also [63, 74] for a more abstract version). On the other hand, it is the proof of the Fredholm property in Theorem 1.2 which requires the introduction of new techniques (see [63, §6] for a related example where the Fredholm property is not true). The method used in [7] (see also [23] for a generalization to Galois coverings) is based on the calculus of Fourier integral operators and does not apply to the present situation. Instead, we use methods from time-dependent scattering theory combined with a long-time variant of Egorov’s theorem for hyperbolic systems. The arguments are then refined to prove the wavefront set estimate stated in Theorem 1.3. More precisely, the proofs are structured as follows.
(1) In §2 (see also Lemma 6.8) we reduce the Dirac operator to the evolutionary form \( \partial_t - iH(t) \), where \( H(t) \) is a differential operator which is elliptic and similar to a self-adjoint operator for each \( t \in \mathbb{R} \).

(2) We consider the Schrödinger propagator \( U(t,s) \) generated by \( H(t) \) and analyze in §4 the large-time Heisenberg evolution of spectral projections. The key result is Proposition 4.7, which states that for suitable smooth functions \( \chi \), the operator
\[
U(t,0)\chi(H(0))U(0,t) - \chi(H(t))
\]
is compact, and it acquires extra smoothing properties in the limit \( t \to \pm \infty \). For all \( t \) it can also be expressed in terms of pseudo-differential operators of order \(-1\), and in this sense Proposition 4.7 is a variant of Egorov’s theorem. The proof uses a time-dependent pseudo-differential calculus developed by Gérard–Wrochna \([38]\) and adapted to the present setting in §3 including a Beals type characterization.

(3) In §5, the Atiyah–Patodi–Singer boundary conditions are interpreted in terms of asymptotic data in the sense of time-dependent scattering theory. More precisely, as comparison dynamics we take \( e^{itH(t)} \), we show in §4.5 that \( W(0,t) := U(0,t)e^{itH(t)} \) is well-behaved as \( t \to \pm \infty \), and we use this in §5.1 to give meaning to asymptotic data of solutions. We remark that this choice of comparison dynamics is unusual from the point of view of scattering theory but serves our purpose particularly well.

(4) The Fredholm property is concluded in §§5–6 from (1)–(3) in combination with abstract Fredholm theory arguments in the spirit of \([7]\), and then the index formula follows in a similar way as in \([7]\).

(5) The proof of the Feynman property in §7 is essentially reduced to showing that if \( u \) satisfies \( Du \in C^\infty \) and
\[
\lim_{t \to +\infty} 1_{[0, +\infty]}(A(t))u(t) = 0,
\]
then its wavefront set \( \text{WF}(u) \) is contained in at most one of the two components \( N^\pm \) of the characteristic set of \( D \). This statement is shown to be a consequence of (2) (this contrasts with \([7]\) where a parametrix gluing construction is used in the finite-time case). The reduction uses in a crucial way positivity properties of Fredholm inverses of \( D_{\text{APS}} \) which are proved in §5.2.

We remark that the sections §§4–3 involving the key technical result, Proposition 4.7, are self-contained and can be of independent interest.

1.3. Bibliographical remarks. A primary application of index formulae is the computation of chiral anomalies in Quantum Field Theory, see Bär–Strohmaier \([6]\), cf. Zahn \([79]\) and Bär–Strohmaier \([8]\) for local versions.

Index formulae for the scattering matrix of the Dirac operator were previously obtained by Matsui \([53, 54]\), Bunke–Hirschman \([19]\) and Pankrashkin–Richard \([59]\) (cf. Finster \([32]\) for the index of a Lorentzian signature operator). We point out that in our setting \( \text{ind}(D_{\text{APS}}) \) equals the index of a scattering matrix component (see Proposition 6.3 and the proof of Theorem 6.6) for a non-standard comparison dynamics, and one can relate it to the usual scattering matrix under some additional assumptions.

Generalisations and variants of Egorov’s theorem for pseudo-differential systems were considered among others by Cordes \([21, 22]\), Jakobson–Strohmaier \([48]\) and Kordyukov
[49], cf. Brummelhuis–Nourrigat [18], Bolte–Glaser [15] and Assal [1] for the semi-classical case; see also Capoferri–Vassiliev [20] for the closely related construction of pseudo-differential projections. Our proof of Proposition 4.7 is most closely related to [21] and to recent works by Gérard–Stoskopf [36, 35] inspired by time-dependent projections in adiabatic theory, see e.g. Sjöstrand [67].

The wavefront set estimate for parametrices of $D_{\text{APS}}$ is closely related to the construction of Hadamard states from asymptotic data in Quantum Field Theory, a problem considered in different settings by authors including Moretti [57], Dappiaggi–Pinamonti–Moretti [25], Gérard–Wrochna [38, 39], Vasy–Wrochna [78] and Gérard–Stoskopf [35]. Positivity properties are crucial in this context, and have also been studied for Feynman parametrices in the already mentioned references [31, 39, 7] and in works by Vasy [76] and Islam–Strohmaier [47]. The time-dependent pseudo-differential calculus and approximate diagonalisation of the evolution used in [39, 35] are particularly relevant to the present setting, though the approach here is ultimately different (furthermore, in contrast to [39, 35] we do not assume $c_{\pm} = 1$, which leads to new complications).

On the side note we remark that if the boundary is assumed to be time-like instead of being space-like, the Lorentzian Dirac operator behaves very differently, see Drago–Große–Murro for the well-posedness of the corresponding Cauchy problem [29].

Finally, as our result assumes compactness of $M$ in spatial directions, one might ask if an index formula could be shown for e.g. perturbations of Minkowski space and other classes of non-compact spacetimes. We hope that the advances including e.g. [9, 33, 39, 17, 16, 8, 24, 68] make it a viable goal for future research. Generalizations in the spirit of the works of Bismut–Cheeger [13, 14] or Melrose–Piazza [56] also remain an open problem.

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2. The Dirac operator in Lorentzian signature

2.1. Geometric setup. Let $(M, g)$ be a $1 + d$-dimensional globally hyperbolic spacetime\(^1\) with compact Cauchy surface $\Sigma$. We assume that the dimension $d$ of $\Sigma$ is odd.

Let us recall the necessary background on spinors and Dirac operators, see e.g. [10, 50, 4, 72] for more detailed accounts.

Suppose $SM \to M$ is a complex spinor bundle. We denote by $C^\infty(M; SM)$ the space of its smooth sections, and use an analogous notation for sections of other vector bundles.

\(^1\)Recall that an oriented and time-oriented smooth Lorentzian manifold $(M, g)$ is called a globally hyperbolic spacetime if it has a Cauchy surface, i.e., a smooth hypersurface which is intersected by every maximally extended, non-spacelike curve exactly once.
Let us recall that the spinor bundle $SM$ is endowed with a linear map
\[ \gamma : C^\infty(M; TM) \to C^\infty(M; \text{End}(SM)) \]
called **Clifford multiplication** and satisfying
\[ \gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = -2(X \cdot gY)1, \quad X, Y \in C^\infty(M; TM). \] (2.4)
Furthermore, one is given a connection $\nabla^{SM}$ on $SM$, called **spin connection**, such that in particular, for all $X, Y \in C^\infty(M; TM)$ and $\psi \in C^\infty(M; SM)$,
\[ \nabla_X^{SM}(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla_X^{SM}\psi, \]
where $\nabla$ is the Levi-Civita connection on $(M, g)$. In the physicist’s terminology, the **massless Dirac operator** is the differential operator $\slashed{D}$ (or $i$ times $\slashed{D}$) given in a time-oriented local frame $(e_0, e_1, \ldots, e_d)$ of $TM$ by
\[ \slashed{D} = g^{\mu\nu} \gamma(e_\mu)\nabla^{SM}_{e_\nu} : C^\infty(M; SM) \to C^\infty(M; SM) \]
using Einstein’s summation convention. The section $\Gamma = i^{d(d+3)/2}\gamma(e_0) \cdots \gamma(e_d) \in C^\infty(M; \text{End}(SM))$ satisfies $\Gamma^2 = 1$. The spinor bundle has therefore a decomposition $SM = S^+M \oplus S^-M$, where $S^\pm M$ is fiberwise the eigenspace of $\Gamma$ for the eigenvalue $\pm 1$. Recall that we have assumed that $d$ is odd, and consequently one deduces from (2.4) that $\Gamma$ anti-commutes with all $\gamma(X)$, hence $\slashed{D}\Gamma = -\Gamma\slashed{D}$. Thus, in terms of the $S^+M \oplus S^-M$ decomposition,
\[ \slashed{D} = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \]
where
\[ D^\pm : C^\infty(M; S^\pm M) \to C^\infty(M; S^\mp M). \]
The differential geometry literature often calls $D^+$ the Dirac operator (at slight risk of confusion with $\slashed{D}$). Note that in the physics literature the less ambiguous name **Weyl operator** is usually used for $iD^+$.

### 2.2. Foliation by Cauchy surfaces.

By global hyperbolicity and the Bernal–Sánchez theorem [11, 12], $M \cong \mathbb{R} \times \Sigma$ and there exists a foliation $\{\Sigma_t\}_{t \in \mathbb{R}}$ by Cauchy hypersurfaces with $\Sigma_t = \{t\} \times \Sigma$. Furthermore, there exists $c \in C^\infty(M)$ strictly positive and a smooth family of Riemannian metrics $\mathbb{R} \ni t \mapsto h(t)$ on $\Sigma$, such that (disregarding the diffeomorphism $M \to \mathbb{R} \times \Sigma$ in the notation)
\[ g = -c^2(t)dt^2 + h_{ij}(t)dy^i dy^j, \] (2.5)
where we sum over repeated indices, and the dependence on the spatial variable $y$ is dropped in the notation. We denote $|h(t)| := |\det h(t)|$.

Let $n \in C^\infty(M; TM)$ be the unique past-directed vector field such that
\[ n(x) \cdot g(x)n(x) = -1 \quad \text{and} \quad n(x) \cdot g(x)v = 0 \]
for all $x = (t, y) \in M$ and $v \in T_x\Sigma_t$. Then, $\beta := \gamma(n) \in C^\infty(M; \text{End}(SM))$ satisfies $\beta^2 = 1$ and
\[ \forall u \in C^\infty(M; SM), \quad \langle u|u \rangle := (u|\beta u)_{SM} \geq 0. \] (2.6)
The associated Hilbert space will be denoted by $L^2(M; SM)$, and $L^2(M; S^+M)$ is defined analogously.
By the results in [4] and using the assumption that $\Sigma$ is odd-dimensional, for any $t \in \mathbb{R}$ the restriction $(S^+M)|_{\Sigma_t}$ can be identified with a spinor bundle $SS_{t}$ on $\Sigma_t$. It is equipped with the Clifford multiplication $\gamma_t \in C^\infty(\Sigma_t; \text{End}(SS_t))$ defined by restricting $i\beta\gamma$ to $\Sigma_t$. For any $t, s \in \mathbb{R}$ let

$$\tau_t^s : (SM|_{\Sigma_t}) \rightarrow SM|_{\Sigma_t}$$

be for each base point $y \in \Sigma$ the parallel transport for the spin connection $\nabla^{SM}$ along the curve $\mathbb{R} \ni t \mapsto (t, y) \in M$. We denote $\tau_t := \tau^t_0$ to simplify the notation. In what follows we will often write $SS_t$ instead of $(S^+M|_{\Sigma_t})$ for the sake of brevity.

For each $t \in \mathbb{R}$, let $L^2(\Sigma_t; S^+M|_{\Sigma_t})$ be the Hilbert space defined similarly as $\langle \cdot | \cdot \rangle$ using $\beta$ and the volume form $d\text{vol}_{h(t)}$. Let $\rho(t)$ be the unique function such that $d\text{vol}_{h(t)} = \rho(t)^2 d\text{vol}_{h(0)}$, or more explicitly, $\rho(t) = |h(0)|^{-\frac{1}{4}} |h(t)|^{\frac{1}{4}}$. Then, the map

$$U(t) := \rho(t)\tau_t : L^2(\Sigma_t; S^+M|_{\Sigma_t}) \rightarrow L^2(\Sigma; SS)$$

is invertible.

### 2.3. Dirac operators on Cauchy surfaces.

For each $t \in \mathbb{R}$, the spinor connection $\nabla^{SM}$ induces a spinor connection on $SS_t$, denoted by $\nabla^{SS}$, and satisfying (see [4, (3.5)])

$$\nabla^{SM}_X = \nabla^{SS}_X - \frac{1}{2} \beta\gamma(\nabla_X n), \quad X \in C^\infty(\Sigma_t; T\Sigma_t).$$

For $t \in \mathbb{R}$, let $A(t)$ be the Dirac operator associated to the spinor connection $\nabla^{SS}$ and the Clifford multiplication $\gamma_t$, i.e.

$$A(t) = h^{ij}(t)\gamma_t(e_i)\nabla^{SS}_{e_j} : C^\infty(\Sigma_t; SS|_{\Sigma_t}) \rightarrow C^\infty(\Sigma_t; SS|_{\Sigma_t}).$$

The Dirac operator $A(t)$ is elliptic and formally self-adjoint in $L^2(\Sigma_t; S^+M|_{\Sigma_t})$. Its closure, also denoted $A(t)$, has discrete spectrum.

The Dirac operator $\tilde{D}$ on $SM$ can be expressed as follows in terms of $A(t)$ (see [4, (3.6)]):

$$\tilde{D} = \beta \left( \begin{array}{cc} -\nabla^{SM}_n - iA(t) - r(t) & 0 \\ 0 & -\nabla^{SM}_n + iA(t) - r(t) \end{array} \right),$$

where $r(t)$ is the multiplication operator by $\frac{D}{2}$ times the mean curvature of $\Sigma_t$. Our main object of interest is the operator

$$D = -\nabla^{SM}_n - iA(t) - r(t) : C^\infty(\Sigma_t; S^+M) \rightarrow C^\infty(\Sigma_t; S^+M).$$

The practical significance of considering $D$ instead of $D^+$ is that the former acts on sections of the same bundle. Note that $D$ and $D^+$ are simply related through the isomorphism $\beta$, so these two choices are equivalent from the point of view of index theory. By abuse of terminology we also occasionally refer to $D$ as the Dirac operator.

The family $\{L^2(\Sigma_t; S^+M|_{\Sigma_t})\}_{t \in \mathbb{R}}$ and more generally, the family $\{H^m(\Sigma_t; S^+M|_{\Sigma_t})\}_{t \in \mathbb{R}}$ of Sobolev spaces for $m \in \mathbb{R}$, can be considered as a bundle of Hilbert spaces over $\mathbb{R}$, trivialized by the parallel transport $\tau_t$. Let $C^0_t L^2_0(M; S^+M)$, and more generally $C^0_t H^m_y(M; S^+M)$ be the space of continuous sections of that bundle. Seminorms of $u \in C^0_t H^m_y(M; S^+M)$ are by definition the $C^0$ seminorms of

$$\mathbb{R} \ni t \mapsto \|u(t)\|_{H^m(\Sigma_t; S^+M|_{\Sigma_t})}.$$
Furthermore, if we set
\[(Uu)(t) := U(t)u(t)\] (2.10)
then
\[U : C^0_t L^2_y(M; S^+M) \to C^0_t(L^2(M; S^+M))\]
is an isomorphism. Next, we define \(L^2_t H^m_y(M; S^+M)\) in a similar way as the space of weighted \(L^2\) sections of the bundle \(\{H^m_\Sigma t; S^+M[t]\}_{t \in \mathbb{R}}\) with norm given by the \(L^2\)-norm of \(\mathbb{R} \ni t \mapsto \|u(t)\|_{H^m_\Sigma t; S^+M[t]}\) associated with the density \(c(t)dt\). Then, \(U\) extends to an isomorphism
\[U : L^2_t H^m_y(M; S^+M) \to L^2(M; H^m_\Sigma S\Sigma).\]
Thanks to the presence of the weight \(c(t)\) in the definition, the space \(L^2_t L^2_y(M; S^+M) = L^2_t L^2_y(\Sigma; S^+M)\) can be identified with \(L^2(M; S^+M)\).

2.4. Reduction to an evolutionary equation. The next lemma (which follows from the computations in [73]), reduces \(D\) to an evolutionary form which is particularly useful for us.

**Lemma 2.1.** The operator \(D\) defined in (2.9) satisfies
\[D = U(t)^{-1}e^{-1}(t)(\partial_t - iH(t))U(t),\] (2.11)
where \(H(t) = c(t)U(t)A(t)U(t)^{-1}\) and the convention (2.10) is used for the action of the operators on the r.h.s.

**Proof.** We use the same arguments as in the proof of [73, Prop. 3.5] (note that the assumption \(\nabla_n n = 0\) made in [73] is not needed these particular steps). We denote by \(\partial_T\) the time derivative in a local coordinate, then \(n = -c(t)\partial_T\). For \(\psi \in C^\infty(M; S^+M)\), we have
\[
(\partial_t \circ U(t) \psi)(t) = \lim_{\epsilon \to 0} \epsilon^{-1}(U(t + \epsilon)\psi|_{\Sigma t+\epsilon} - U(t)\psi|_{\Sigma t})
= \lim_{\epsilon \to 0} \epsilon^{-1}(\rho(t + \epsilon)\tau_{t+\epsilon}\psi|_{\Sigma t+\epsilon} - \rho(t)\tau_t\psi|_{\Sigma t})
= \tau_t(\lim_{\epsilon \to 0} \epsilon^{-1}(\tau_{t+\epsilon}\rho(t + \epsilon)\psi|_{\Sigma t+\epsilon} - \rho(t)\psi|_{\Sigma t}))
= \tau_t((\nabla_\partial_T^M \rho(t)\psi)(t))
= (\rho(t)^{-1}U(t) \circ \nabla_\partial_T^M \circ \rho(t)\psi)(t),
\]
where we used \(\tau_t = \rho(t)^{-1}U(t)\) in the last step. Hence we have
\[
\partial_t = \rho(t)^{-1}U(t) \circ \nabla_\partial_T^M \circ \rho(t)U(t)^{-1}
= -\rho(t)^{-1}U(t) \circ c(t)\nabla_n^S \circ \rho(t)U(t)^{-1},
\]
which implies
\[c(t)U(t)(-\nabla_n^S)U(t)^{-1} = \rho(t) \circ \partial_t \circ \rho(t)^{-1}.\] (2.12)
In the next step we find a convenient expression for \(r(t)\). On the one hand,
\[
div n = \sum_{j=0}^d g(e_j, \nabla e_j n) = g(n, \nabla_n n) + \sum_{j=1}^d g(e_j, \nabla e_j n) = 2r(t).
\]
On the other hand,
\[ \text{div} n|_{\Sigma_t} = -|g|^{-\frac{1}{2}} \partial_t(|g|^\frac{1}{2} c(t))^{-1} = -c(t)^{-1} |h(t)|^{-\frac{1}{2}} \partial_t(|h(t)|^\frac{1}{2}) = -2c(t)^{-1} |h(t)|^{-\frac{1}{2}} \partial_t(|h(t)|^\frac{1}{2}) = -2c(t)^{-1} \rho(t)^{-1} (\partial_t \rho(t)). \]
So we obtain \( r(t) = -c(t)^{-1} \rho(t)^{-1} (\partial_t \rho(t)) \). In consequence, we have
\[ c(t) U(t) (-r(t)) U(t)^{-1} = \rho(t)^{-1} (\partial_t \rho(t)). \]
Inserting (2.12) and (2.13) into the formula (2.9) for \( D \), we obtain (2.11). \( \square \)

3. **Time-dependent pseudo-differential calculus**

### 3.1. Pseudo-differential operators

We now introduce various classes of pseudodifferential operators that will be needed in the proofs.

For \( \xi \in \mathbb{R}^n \) we use the notation \( \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}} \), in particular \( \langle t \rangle = (1 + t^2)^{\frac{1}{2}} \) if \( t \in \mathbb{R} \). Recall that for \( \ell \in \mathbb{R} \), \( S^\ell(\mathbb{R}) \) is the space of all \( a \in C^\infty(\mathbb{R}) \) such that
\[ \forall \gamma \in \mathbb{N}, \exists C_\gamma \geq 0 \text{ s.t. } |\langle t \rangle^{-\ell+\gamma} |\partial_t^\gamma a(t)| \leq C_\gamma \text{ on } \mathbb{R}. \]
If \( U \subset \mathbb{R}^n \) is an open set and \( m \in \mathbb{R} \), the symbol space \( S^m(\mathbb{T}^*U) \) is the space of all \( a \in C^\infty(U; \math{T}^*U) \) such that:
\[ \forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha,\beta} \geq 0 \text{ s.t. } |\langle \xi \rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \text{ on } U \times \mathbb{R}^d. \]
The best constants \( C_{\alpha,\beta} \) define a family of semi-norms that allows to endow the space \( S^m(\math{T}^*U) \) with a Fréchet space topology.

We introduce a space of \( t \)-dependent symbols that behave in \( t \) as a symbol of order \( \ell \in \mathbb{R} \). Namely, we define \( S^\ell(\mathbb{R}, S^m(\math{T}^*U)) \) to be the space of all \( a \in C^\infty(\mathbb{R} \times U; \math{T}^*U) \) such that:
\[ \forall \alpha, \beta \in \mathbb{N}^n, \gamma \in \mathbb{N}, \langle t \rangle^{-\ell+\gamma} |\langle \xi \rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(t, x, \xi)| \text{ is bounded on } \mathbb{R} \times U \times \mathbb{R}^d. \]
This is equivalent to saying that \( a \in C^\infty(\mathbb{R}, S^m(\math{T}^*U)) \) and for all \( \gamma \in \mathbb{N} \), the semi-norms of \( \langle t \rangle^{-\ell+\gamma} |\partial_x^\alpha \partial_\xi^\beta a(t, \cdot, \cdot) | \) in \( S^m(\math{T}^*U) \) are bounded on \( \mathbb{R} \). More generally, we make the following definition, following [34, §15.1].

**Definition 3.1.** Suppose \( \mathcal{F} \) is a Fréchet space, and let \( \| \cdot \|_j, j \in \mathbb{N} \), be the family of seminorms that defines its topology. If \( I \subset \mathbb{R} \) is an open interval and \( \ell \in \mathbb{R} \), we define \( S^\ell(I, \mathcal{F}) \) to be the space of smooth functions \( I \ni t \mapsto a(t) \in \mathcal{F} \) such that
\[ \forall j, \gamma \in \mathbb{N}, \sup_{t \in I} \langle t \rangle^{-\ell+\gamma} \| \partial_x^\alpha a(t) \|_j < \infty. \]

Next, we consider the setting of a \( k \)-dimensional vector bundle \( E \) over a compact manifold \( \Sigma \). If \( \pi : \math{T}^*\Sigma \to \Sigma \) is the bundle projection, let \( \pi^* \text{End}(E) \to \math{T}^*\Sigma \) be the pullback bundle of \( \text{End}(E) \to \Sigma \) by \( \pi \). A trivialisation \( \varphi_U : \text{End}(E)|_U \cong U \times \mathbb{R}^k \) over an open set \( U \subset \Sigma \) induces a trivialisation
\[ \varphi_{\pi, U} : \pi^* \text{End}(E)|_{\math{T}^*U} \cong \math{T}^*U \times \mathbb{R}^k \]
(3.14) of the vector bundle \( \pi^* \text{End}(E) \) over \( \math{T}^*\Sigma \). The symbol space \( S^m(\math{T}^*\Sigma; \pi^* \text{End}(E)) \) is by definition the space of all \( C^\infty(\math{T}^*\Sigma; \pi^* \text{End}(E)) \) which in local coordinates and in a trivialisation of the form (3.14) are elements of \( S^m(\math{T}^*U) \otimes \mathbb{R}^k \). The seminorms of \( S^m(\math{T}^*U) \otimes \mathbb{R}^k \)
$\mathbb{R}^{k^2}$ for different charts define a Fréchet space topology on $S^m(T^*\Sigma; \pi^* \text{End}(E))$. The principal symbol of $a \in S^m(T^*\Sigma; \pi^* \text{End}(E))$ is the equivalence class

$$[a] \in S^m(T^*\Sigma; \pi^* \text{End}(E))/S^{m-1}(T^*\Sigma; \pi^* \text{End}(E)).$$

Let $L^2(\Sigma; E)$ be the $L^2$ space defined using some positive, smooth section of the density bundle tensored with $E \otimes E^*$. Let $\Psi^m(E)$ be the standard class of pseudo-differential operators on $E \rightarrow \Sigma$ and let us fix a quantization map

$$S^m(T^*\Sigma; \pi^* \text{End}(E)) \ni a \mapsto \text{Op}(a) \in \Psi^m(E)$$

with the extra property that $\text{Op}(a^*) = \text{Op}(a)^*$ in the sense of formal adjoint w.r.t. the $L^2(\Sigma; E)$ inner product. The topology of $S^m(T^*\Sigma; \pi^* \text{End}(E))$ can be used to topologize $\Psi^m(E)$.

We can then consider classes of functions in $t$ with values in symbols or pseudo-differential operators using the notation introduced in Definition 3.1. Note that by applying the quantization map Op pointwise to elements of $S^\ell(I,S^m(T^*\Sigma; \pi^* \text{End}(E)))$ we obtain the $t$-dependent pseudo-differential operators $S^\ell(I,\Psi^m(E))$. For the sake of having more shorthand notation we set

$$\Psi_{td}^m,\ell(I; E) := S^\ell(I,\Psi^m(E)),$$

and

$$S^m_{td,\ell}(I; E) := S^\ell(I,S^m(T^*\Sigma; \pi^* \text{End}(E))$$

for the corresponding time-depending symbols, where the subscript ‘td’ stands for ‘time-decay’. This class generalizes the time-depending pseudo-differential operators introduced in the scalar case in [38], though an important simplification for us is that we consider only compact base manifolds (see [35, 68] for the non-compact setting).

As usual in pseudo-differential calculus, we denote

$$\Psi_{td}^{\infty,\ell}(I; E) := \bigcup_{m \in \mathbb{R}} \Psi_{td}^m,\ell(I; E), \quad \Psi_{td}^{-\infty,\ell}(I; E) := \bigcap_{m \in \mathbb{R}} \Psi_{td}^m,\ell(I; E)$$

and similarly for the $t$-independent classes.

**Definition 3.2.** One says that $A \in \Psi^m(E)$ is principally scalar if its principal symbol has a representative which is a $C^\infty(\Sigma; T^*\Sigma)$ multiple of the identity in $\text{End}(E)$. We say that $A(\cdot) \in \Psi_{td}^m,\ell(I; E)$ is principally scalar if $A(t)$ is principally scalar for all $t \in \mathbb{R}$.

We write $A(t)$ instead of $A(\cdot)$ when it is unlikely to cause any confusion. If $A(t) \in \Psi_{td}^{m_1,\ell_1}(I; E)$ and $B(t) \in \Psi_{td}^{m_2,\ell_2}(I; E)$ for some $m_i, \ell_i \in \mathbb{R}$, $i = 1, 2$, then

$$A(t)B(t) \in \Psi_{td}^{m_1+m_2,\ell_1+\ell_2}(I; E).$$

Furthermore, if $A(t)$ or $B(t)$ is principally scalar then

$$[A(t),B(t)] \in \Psi_{td}^{m_1+m_2-1,\ell_1+\ell_2}(I; E).$$
3.2. Beals type commutator criterion. We show that we can characterize operators in \( \Psi_{td}^{m,\ell}(I; E) \) by a Beals type commutator criterion.

For \( s, r \in \mathbb{R} \), the norm in \( B(H^s(\Sigma; E), H^r(\Sigma; E)) \) is denoted \( \| \cdot \|_{B(H^s, H^r)} \).

**Proposition 3.3.** Suppose \( A(t) \in B(H^s(\Sigma; E), H^{s-m}(\Sigma; E)) \) for all \( t \in I, s \in \mathbb{R} \) and some \( m \in \mathbb{R} \). Then \( A(\cdot) \in \Psi_{td}^{m,\ell}(I; E) \) if and only if \( A(\cdot) \) is infinitely differentiable and

\[
\sup_{t \in I} \langle t \rangle^{-\ell+\gamma} \| \text{ad}_{L_1} \cdots \text{ad}_{L_k} \partial_t^\gamma A(t) \|_{B(H^s, H^{s-m+d(k)})} < +\infty \tag{3.15}
\]

for all \( k \in \mathbb{N} \), all principally scalar \( L_1, \ldots, L_k \in \Psi^1(E) \) and all \( \gamma \in \mathbb{N} \), where \( d(k) = \sum_{j=0}^k (1 - \deg L_j) \), \( \deg L_j \) is the order of \( L_j \), and we denoted \( \text{ad}_{L_j} B := [L_j, B] \).

**Proof.** For fixed \( t \in I \), the Beals criterion on compact manifolds (see [64, §5.3]) generalizes to the vector bundle case, namely, it says in our situation that \( A(t) \in \Psi^m(E) \) if and only if

\[
\| \text{ad}_{L_1} \cdots \text{ad}_{L_k} \partial_t^\gamma A(t) \|_{B(H^s, H^{s-m+d(k)})} < +\infty \tag{3.16}
\]

for all \( k \in \mathbb{N} \) and all principally scalar \( L_1, \ldots, L_k \in \Psi^1(E) \). Indeed, in an arbitrarily chosen local trivialization we can consider (3.16) for the subclass of \( L_i \in \Psi^1(E) \) which are scalar in that particular trivialization: this then implies \( A(t) \in \Psi^m(E) \) by the scalar Beals criterion, applied in this trivialization. The reverse implication is straightforward.

Furthermore, it follows from the proof of [64, Thm. 5.3.1] that the seminorms \( \| a(t) \|_i \) of the symbol of \( a(t) \in S^m(T^*\Sigma; \pi^* \text{End } E) \) of \( A(t) \) are bounded by (3.16) uniformly in \( t \in I \), and the analogous property holds true for the seminorms \( \partial_t^\ell a(t) \) of \( \partial_t^\ell A(t) \). Thus, (3.15) implies \( \sup_{t \in I} \langle t \rangle^{-\ell+\gamma} \| \partial_t^\gamma a(t) \|_i < +\infty \), which means that \( A(\cdot) \in \Psi_{td}^{m,\ell}(I; E) \). The reverse implication is straightforward. \( \square \)

**Remark 3.4.** As in the standard Beals commutator criterion, it suffices to check (3.15) for all differential operators \( L_1, \ldots, L_k \) of order \( \leq 1 \).

3.3. Ellipticity. The principal symbol of \( A = \text{Op}(a) \in \Psi^m(E) \) is by definition the principal symbol \( \sigma_m(A) := [a] \) of \( a \), and \( A \) is elliptic if there exists a symbol \( b \in S^{-m}(T^*\Sigma; \pi^* \text{End } (E)) \) such that

\[
ab - 1, ba - 1 \in S^{-1}(T^*\Sigma; \pi^* \text{End } (E)).
\]

We say that \( A(\cdot) \in \Psi_{td}^{m,\ell}(I; E) \) is elliptic if \( A(t) \) is elliptic for all \( t \in \mathbb{R} \).

If \( A \in \Psi^1(E) \) is positive, elliptic and principally scalar, one can define the Sobolev space of order \( s \in \mathbb{R} \) by

\[
H^s(\Sigma; E) := A^{-s}L^2(\Sigma; E)
\]
equipped with the norm \( \| u \|_{H^s} := \| A^s u \|_{L^2} \), where \( \| u \|_{L^2} \) is the \( L^2(\Sigma; E) \) norm of \( u \). Note that choosing a different \( A \) with the properties mentioned above defines in general a different, but equivalent norm.

The operators of interest to us will typically not be principally scalar, but will often satisfy the following condition instead (alongside self-adjointness properties discussed in the sequel).
Definition 3.5. We say that \( H \in \Psi^m(E) \) is of Dirac type if it is elliptic and \( H^2 \) is principally scalar. We say that \( H(\cdot) \in \Psi^{m,\ell}(I; E) \) is of Dirac type if \( H(t) \) is of Dirac type for all \( t \in \mathbb{R} \).

3.4. Families similar to self-adjoint operators. Let \( \delta > 0 \) and let us fix a family of zero order pseudodifferential operators \( T(t) \), boundedly invertible uniformly in \( t \in \mathbb{R} \) and satisfying:

\[
T(t), T^{-1}(t) \in \Psi^{0,0}_{td}(\mathbb{R}; E), \quad \partial_t T(t) \in \Psi^{0,-1-\delta}_{td}(\mathbb{R}; E). \tag{3.17}
\]

Suppose \( H(t) \in \Psi^{1,0}_{td}(\mathbb{R}; E) \) is a family of elliptic operators of the form

\[
H(t) = T(t)H_0(t)T^{-1}(t), \quad \text{with } H_0(t) \in \Psi^{1,0}_{td}(\mathbb{R}; E) \text{ s.t. } \forall t \in \mathbb{R}, \ (H_0(t))^* = H_0(t). \tag{3.18}
\]

Then, \( H_0(t) \) is elliptic as well and as a consequence it is self-adjoint with domain \( H^1(\Sigma; E) \) and has discrete spectrum (see e.g. [66, Thm. 8.3]). Since multiplication by \( T(t), T^{-1}(t) \) preserves \( H^1(\Sigma; E) \), \( H(t) \) with domain \( H^1(\Sigma; E) \) is closed and it is similar to a self-adjoint operator and has therefore discrete, real spectrum. Furthermore, by (3.18) and self-adjointness of \( H_0(t) \) we also conclude that for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and \( N \in \mathbb{N}_{\geq 0} \),

\[
\| (H(t) - \lambda)^{-N} \| = \| T^{-1}(t)(H_0(t) - \lambda)^{-N}T(t) \| \\
\leq C\| (H_0(t) - \lambda)^{-N} \| \\
\leq C |\text{Im} \lambda|^{-N} \tag{3.19}
\]

uniformly in \( t \in \mathbb{R} \). By the Hille–Yosida theorem (see e.g. [60, §1.3]), for each \( t \in \mathbb{R} \), \( iH(t) \) is the generator of a bounded semi-group.

Remark 3.6. An equivalent point of view is to consider the operators \( H(t) \) as self-adjoint operators in the \( t \)-dependent Hilbert space denoted in the sequel \( L^2_t(\Sigma; E) \) and defined using the norm \( \| u \|_{L^2_t} := \| T^{-1}(t)u \|_{L^2} \) for \( t \in \mathbb{R} \).

3.5. Functional calculus. In view of (3.18) and (3.19) it is possible to define a functional calculus for \( H(t) \) with good properties. In what follows we will use the calculus based on the Helffer–Sjöstrand formula as introduced by Davies [26] for an even more general class of operators.

Namely, for \( f \in S^\rho(\mathbb{R}) \) with \( \rho < 0 \), one defines

\[
f(H(t)) = \frac{1}{2\pi i} \int_C \frac{\partial \hat{f}}{\partial z}(z)(z - H(t))^{-1} \, d\bar{z} \wedge dz, \tag{3.20}
\]

where \( \hat{f} \) is an \( N \)-th order almost analytic extension of \( f \) of the form

\[
\hat{f}(x + iy) = \left( \sum_{k=0}^{N} f^{(k)}(x) \frac{(iy)^k}{k!} \right) \psi \left( \frac{y}{|x|} \right)
\]

for some \( \psi \in C^\infty_c(\mathbb{R}, [0, 1]) \) with \( \psi \equiv 1 \) on \( [-1, 1] \) and \( \psi \equiv 0 \) outside of \( [-2, 2] \), and \( N \in \mathbb{N}_{\geq 0} \) is taken sufficiently large. The crucial properties satisfied by \( \hat{f} \) are:

\[
\hat{f}|_{\mathbb{R}} = f, \quad \left| \frac{\partial \hat{f}}{\partial z} \right| \leq C(x)^{\rho - 1 - N} |y|^N,
\]

\[
\text{supp } \hat{f} \subset \{ x + iy \in \mathbb{C} \mid |y| \leq 2(x), \ x \in \text{supp } f \},
\]
which together with the resolvent bound (3.19) ensure that \( (3.20) \) is well-defined. If \( \rho \geq 0 \) then the proof of [42, Prop. 2.5] shows that

\[
 f(H(t))u := \frac{1}{2\pi i} \lim_{R \to +\infty} \int_{|z| = R} \frac{\partial f}{\partial z}(z - H(t))^{-1} u \, d\bar{z} \wedge dz \tag{3.21}
\]

exists for all \( u \) in the domain of \( (H(t))^\rho \). We also remark that for each \( t \in \mathbb{R} \) we can easily switch to the functional calculus for self-adjoint operators and back thanks to the relation \( (H - \lambda)^{-1} = T^{-1} (H_0 - \lambda)^{-1} T \).

In §4 we will need the following two essential results, Theorem 3.7 and Proposition 3.8 which generalize the scalar case given in [38, Thm. 3.7] (this is a variant of Seeley’s theorem [65]), resp. [38, Prop. 3.10]. Using the principally scalar assumption and the fact that multiplying \( T(t), T^{-1}(t) \) preserves the spaces \( \Psi_{td}^{m,\ell}(I; E) \), the proofs are fully analogous to [38].

**Theorem 3.7.** For \( m > 0 \), if \( A(t) \in \Psi_{td}^{m,0}(I; E) \) is principally scalar, elliptic and \( T(t)A(t)T^{-1}(t) \geq C_1 \) for some \( C > 0 \) and all \( t \in I \), then \( A^s(t) \in \Psi_{td}^{m,0}(I; E) \) and \( A^\ast(t) \) is principally scalar and elliptic for all \( s \in \mathbb{R} \).

**Proposition 3.8.** For \( m > 0 \), if \( A_1(t), A_2(t) \in \Psi_{td}^{m,0}(I; E) \) are principally scalar, elliptic, and satisfy \( T(t)A_i(t)T^{-1}(t) \geq C_i \) for some \( C_i > 0 \), \( i = 1,2 \), and \( A_1(t) - A_2(t) \in \Psi_{td}^{m,\ell}(I; E) \) for some \( \ell < 0 \), then \( A_1^s(t) - A_2^s(t) \in \Psi_{td}^{m,\ell}(I; E) \) for all \( s \in \mathbb{R} \).

4. Evolutionary model and scattering

4.1. Model Dirac equation. As in §3, we consider the setting of a Hermitian bundle \( E \) over a compact manifold \( \Sigma \).

We consider an equation of the form

\[
 (\partial_t - iH(t))\phi(t) = 0,
\]

where \( H(t) \in \Psi_{td}^{1,0}(\mathbb{R}; E) \) satisfies the following hypothesis.

**Hypothesis 4.1.** There exists \( \delta > 0 \) and \( H_\pm \in \Psi^1(E) \) of Dirac type such that:

1. \( H(t) \in \Psi_{td}^{1,0}(\mathbb{R}; E) \) is of Dirac type and \( H(t) - H_\pm \in \Psi_{td}^{1,-\delta}(\mathbb{R}; E) \),
2. \( H(t) \) is of the form (3.18) for some \( T(t) \) satisfying (3.17).

Using the functional calculus introduced in §3.5, we define the family

\[
 \mathbb{R} \ni t \mapsto \Lambda(t) := (1 + H^2(t))^{1/2} \tag{4.22}
\]

which will serve as a reference family of elliptic operators. By the assumption that \( H(t) \) is of Dirac type, \( \Lambda(t) \) is principally scalar. More precisely, we show the following statement.

**Lemma 4.1.** The family \( \Lambda(t) \) defined in (4.22) belongs to \( \Psi_{td}^{1,0}(\mathbb{R}; E) \), is principally scalar, and satisfies for all \( s \in \mathbb{R} \):

1. \( \partial_t \Lambda^s(t) \in \Psi_{td}^{s-1,-\delta}(\mathbb{R}; E) \);
2. there exists \( C_{1,s}, C_{2,s} > 0 \) such that \( C_{1,s}\|u\|_{H^s} \leq \|\Lambda^s(t)u\|_{L^2} \leq C_{2,s}\|u\|_{H^s} \) for all \( u \in H^s(\Sigma; E) \) and \( t \in \mathbb{R} \).
Proof. The fact that $\Lambda(t) \in \Psi_{td}^{1,0}(\mathbb{R}; E)$ (and that is is principally scalar) follows from the assumption that $H(t)$ is of Dirac type and from Theorem 3.7. The latter also gives that $\Lambda^s(t) \in \Psi_{td}^{s,0}(\mathbb{R}; E)$ and $\Lambda^s(t)$ is principally scalar, which proves (2).

Let us prove (1). By definition of $\Lambda(t)$,

$$
\Lambda^2(t) - (1 + H_\pm^2) = H^2(t) - H_\pm^2 = (H(t) + H_\pm)(H(t) - H_\pm) - [H_\pm, H(t) - H_\pm].
$$

By (1) of Hypothesis 4.1, this belongs to $\Psi_{td}^{2,-\delta}(\mathbb{R}_\pm; E)$. Therefore, we can apply Proposition 3.8 to the families $\Lambda^2(t)$ and $1 + H_\pm^2$, which gives

$$
\Lambda^s(t) - (1 + H_\pm^2) \in \Psi_{td}^{s,-\delta}(\mathbb{R}_\pm; E).
$$

By differentiating we obtain $\partial_t \Lambda^s(t) \in \Psi_{td}^{s,-1-\delta}(\mathbb{R}_\pm; E)$, $\partial_s \Lambda^s(t) \in \Psi_{td}^{s,-1-\delta}(\mathbb{R}_\pm; E)$, hence $\partial_t \Lambda^s(t) \in \Psi_{td}^{s,-1-\delta}(\mathbb{R}; E)$. □

4.2. Schrödinger propagator of $H(t)$. We will now be interested in the evolution generated by $H(t)$.

Definition 4.2. Given a family of operators $\mathbb{R} \ni t \mapsto A(t) \in B(H^1(\Sigma; E), L^2(\Sigma; E))$, we say that $\mathbb{R} \ni (t, s) \mapsto U(t, s) \in B(L^2(\Sigma; E))$ is a Schrödinger propagator of $A(t)$ if for all $t, t_0, s \in \mathbb{R}$ it satisfies:

1. $U(t, t) = 1$;
2. $U(t, t_0)U(t_0, s) = U(t, s)$;
3. $U(t, s)$ is strongly continuously differentiable in $B(H^1(\Sigma; E), L^2(\Sigma; E))$, it preserves $H^1(\Sigma; E)$, and

$$
\partial_t U(t, s) = iA(t)U(t, s), \quad \partial_s U(t, s) = -U(t, s)iA(s).
$$

(4.23)

Proposition 4.3. The Schrödinger propagator of $H(t)$, denoted by $U(t, s)$, exists and is unique. Furthermore, $\{U(t, s)\}_{t, s \in \mathbb{R}}$ is uniformly bounded in $B(L^2(\Sigma; E))$.

Proof. The family $t \mapsto H(t)$ is bounded in $B(H^1(\Sigma; E), L^2(\Sigma; E))$ and differentiable. Furthermore, for each $t \in \mathbb{R}$, $H(t)$ is a closed operator in the sense of the Hilbert space $L^2(\Sigma; E)$, with $t$-independent domain $H^1(\Sigma; E)$. It is also the generator of a strongly continuous bounded semi-group. In consequence, we can apply [60, §5, Thm. 4.8] (with $M = 1$ and $\omega = 0$ therein) to conclude the assertion. □

Note that the Schrödinger propagator of $H_\pm$ is simply $e^{it\pm H_\pm}$, and the “free” evolution $e^{itH_\pm}$ is the obvious comparison dynamics for the scattering theory at $t \to \pm \infty$. However, for our purpose it will often be better to consider the family $\{e^{itH(t)}\}_{t \in \mathbb{R}}$, which solves

$$
\partial_t e^{itH(t)} = i(H(t) + t\partial_t H(t))e^{itH(t)}.
$$

(4.24)

Because $e^{itH_\pm}$ commutes with $(1 + H_\pm^2)^{\frac{1}{2}}$, it is straightforward to show that the family $\{e^{itH} \}_{t, s \in \mathbb{R}}$ is bounded in $B(H^m(\Sigma; E))$ for all $m \in \mathbb{R}$. Thanks to our decay assumptions and Lemma 4.1 we also have the following result.

Lemma 4.4. For all $m \in \mathbb{R}$, $\{U(t, s)\}_{t, s \in \mathbb{R}}$ and $\{e^{itH(t)}\}_{t \in \mathbb{R}}$ are uniformly bounded in $B(H^m(\Sigma; E))$. 

Proof. In view of Lemma 4.1 and Proposition 4.3, we can repeat the second paragraph of [38, Proof of Prop. 5.6] verbatim (with \( \Lambda(t) \) instead of \( \epsilon(t) \), and with \( H(t) \) and \( U(t, s) \) instead of \( H^{ad}(t) \) and \( U^{ad}(t, s) \)). The same proof gives also the uniform boundedness of \( \{e^{itH(t)}\}_{t \in \mathbb{R}} \). \( \square \)

4.3. Functions of \( H(t) \) as pseudodifferential operators. Using the Beals type criterion formulated in Proposition 3.3, we show that suitable functions of \( H(t) \) are pseudodifferential operators in the sense of our time-dependent classes.

**Lemma 4.5.** For all \( \chi \in S^0(\mathbb{R}) \), \( \chi(H(t)) \in \Psi^{0,0}_{td}(\mathbb{R}; E) \). Furthermore, \( \partial_t \chi(H(t)) \in \Psi^{0,-1-\delta}_{td}(\mathbb{R}; E) \).

**Proof.** Step 1. In the first step we analyse the resolvent \( (H(t) - z)^{-1} \) for \( z \in \mathbb{C} \setminus \mathbb{R} \).

Let \( \mathcal{R}^{\infty}(z) \) be the algebra finitely generated by \( (H(t) - z)^{-1} \) and by \( (z\text{-independent}) \) elements of \( \Psi^{m,0}_{td}(\mathbb{R}; E) \) for all \( m \in \mathbb{R} \) under operator product and sum. Let us also denote

\[
\mathcal{R}^m(z) := \mathcal{R}^{\infty}(z) \cap \left( \bigcup_{s \in \mathbb{R}} B(H^s(\Sigma; E), H^{s-m}(\Sigma; E)) \right).
\]

This defines a filtration of \( \mathcal{R}^{\infty}(z) \) by \( \mathcal{R}^{m_1}(z) \mathcal{R}^{m_2}(z) \subset \mathcal{R}^{m_1+m_2}(z) \). For all principally scalar \( L_i \in \Psi^1(M; E) \), we have

\[
ad_{L_i} \Psi^{m,0}_{td}(\mathbb{R}; E) \subset \Psi^{m-(1-\deg L_i),0}_{td}(\mathbb{R}; E) \subset \mathcal{R}^{m-(1-\deg L_i)}(z)
\]

and

\[
ad_{L_i}(H - z)^{-1} = (H - z)^{-1}[H, L_i](H - z)^{-1} \in (H - z)^{-1} \mathcal{R}^{-(1-\deg L_i)}(z)
\]

\[
\subset \mathcal{R}^{-1-(1-\deg L_i)}(z).
\]

Since \( \text{ad}_{L_i} \) acts as a derivation, in greater generality we have

\[
ad_{L_i} \mathcal{R}^m(z) \subset \mathcal{R}^{m-(1-\deg L_i)}(z),
\]

and

\[
ad_{L_i}(H - z)^{-1} \mathcal{R}^m(z) \subset (\text{ad}_{L_i}(H - z)^{-1}) \mathcal{R}^m(z) + (H - z)^{-1} \text{ad}_{L_i} \mathcal{R}^m(z)
\]

\[
\subset (H - z)^{-1} \mathcal{R}^{m-(1-\deg L_i)}(z) + (H - z)^{-1} \mathcal{R}^{m-(1-\deg L_i)}(z)
\]

\[
\subset (H - z)^{-1} \mathcal{R}^{m-(1-\deg L_i)}(z).
\]

By iterating this \( k \in \mathbb{N} \) times, for all principally scalar \( L_1, \ldots, L_k \in \Psi^1(M; E) \) we obtain:

\[
ad_{L_1} \cdots \text{ad}_{L_k}(H - z)^{-1} \mathcal{R}^m(z) \subset (H - z)^{-1} \mathcal{R}^{m-d(k)}(z)
\]  
(4.25)

where \( d(k) = \sum_{j=0}^k (1 - \deg L_j) \).

On the other hand, as a consequence of the resolvent identity

\[
(H(t + h) - z)^{-1} - (H(t) - z)^{-1} = (H(t) - z)^{-1}(H(t) - H(t + h))(H(t + h) - z)^{-1}
\]

we have

\[
\partial_t(H(t) - z)^{-1} = -(H(t) - z)^{-1}(\partial_t H(t))(H(t) - z)^{-1}.
\]  
(4.26)

Using (4.26) and \( \partial_t H(t) \in \Psi^{1,-1}_{td}(\mathbb{R}; E) \) repeatedly, for all \( \gamma \in \mathbb{N} \) we get

\[
\langle t \rangle^\gamma \partial_t^\gamma (H(t) - z)^{-1} \in (H(t) - z)^{-1} \mathcal{R}^0(z).
\]
In consequence, we can apply (4.25) with $m = 0$, which yields
\[ \langle t \rangle^\gamma \text{ad}_{L_1} \cdots \text{ad}_{L_k} \partial_t \chi (H - z)^{-1} \in (H - z)^{-1} \mathcal{R}^{-d(k)}(z) \subset \mathcal{R}^{-1-d(k)}(z). \]
Considering the definition of $\mathcal{R}^{-1-d(k)}(z)$ and boundedness properties of pseudo-differential operators, this means that
\[ \sup_{t \in \mathbb{R}} \langle t \rangle^\gamma \| \text{ad}_{L_1} \cdots \text{ad}_{L_k} \partial_t \chi (H(t) - z)^{-1} \|_{B(H^s, H^{s+1+d(k)})} < +\infty. \]
for all $s \in \mathbb{R}$. Alternatively, using the uniform estimate
\[ \| (H(t) - z)^{-1} \|_{B(H^s, H^s)} \lesssim |\text{Im } z|^{-1} \]
one time, we obtain in a similar way
\[ \sup_{t \in \mathbb{R}} \langle t \rangle^\gamma \| \text{ad}_{L_1} \cdots \text{ad}_{L_k} \partial_t \chi (H(t) - z)^{-1} \|_{B(H^s, H^{s+d(k)})} \lesssim |\text{Im } z|^{-1}. \]
By the Beals-type commutator criterion stated in Proposition 3.3, we conclude from (4.27) that $(H - z)^{-1} \in \Psi^{-1,0}_{td}(\mathbb{R}; E)$ and from (4.28) that the seminorms of $(H - z)^{-1}$ in $\Psi_{td}^{0,0}(\mathbb{R}; E)$ are $O(|\text{Im } z|)^{-1}$.

**Step 2.** In the second step, using the Helffer–Sjöstrand formula we deduce $\chi(H(t)) \in \Psi_{td}^{0,0}(\mathbb{R}; E)$ from what we already know on the resolvent. Namely we apply (3.21) in the form
\[ \chi (H(t)) = s - \lim_{R \to +\infty} \frac{1}{2\pi i} \int_C \partial_t (\bar{\chi} \psi_R)(z)(z - H(t))^{-1} dz \wedge dz, \]
where $\psi_R(t) := \psi(t/R)$, $\psi \in C_0^\infty([0, 1])$ with $\psi = 1$ near zero, the limit is taken in the strong operator topology and $(\bar{\chi} \psi_R)$ is an almost analytic extension of $\chi \psi_R$ of order $N \geq 2$, in particular
\[ |\partial_t (\bar{\chi} \psi_R)(z)| \leq c |\text{Re } z|^{-1-N} |\text{Im } z|^N. \]
Above, $c$ depends only on the semi-norms
\[ \sup_{t \in \mathbb{R}} |t|^k |(\chi \psi_R)(k)(t)|, \quad k \in \mathbb{N}, \]
see [42, §2] for more details. We analyse the dependence on $R$:
\[ \sum_{t \in \mathbb{R}} |t|^k (\chi \psi_R)(k)(t)| \lesssim \sum_{t \in \mathbb{R}} |t|^k \chi^{(m)}(t)|(|\psi_R)(k-m)| (t)|
\[ \lesssim \sum_{m=0}^k |t|^m \chi^{(m)}(t) \cdot \sup_{t \in \mathbb{R}} |t|^{k-m} R^{m-k} |\psi^{(k-m)}(t/R)| \]
\[ = O(1), \]
where we used the fact that $|\psi^{(k)}| \in C_c^\infty(\mathbb{R})$ and $\chi \in S^0(\mathbb{R})$ in the last step. Therefore, we can use (4.29) and the $O(|\text{Im } z|)^{-1}$ bound on seminorms of $(H - z)^{-1}$ in $\Psi_{td}^{0,0}(\mathbb{R}; E)$ proved in Step 1 to conclude that $\chi(H(t)) \in \Psi_{td}^{0,0}(\mathbb{R}; E)$.

**Step 3.** To show that $\partial_t \chi(H(t)) \in \Psi_{td}^{-1-\delta}(\mathbb{R}; E)$ we repeat Step 1 and Step 2. The extra decay is obtained by taking into account that $\partial_t H(t) \in \Psi_{td}^{1-1-\delta}(\mathbb{R}; E)$ (which follows from Hypothesis 4.1) when using (4.27). □
More generally, for \( s \in \mathbb{R} \), any \( f \in S^s(\mathbb{R}) \) is of the form \( f(x) = \langle x \rangle^s \chi(x) \) for some \( \chi \in S^0(\mathbb{R}) \). Since by Proposition 3.8, \( (H(t))^s \in \Psi_{td}^{s,0}(\mathbb{R}; E) \), we obtain the following immediate corollary.

**Corollary 4.6.** For all \( s \in \mathbb{R} \), if \( f \in S^s(\mathbb{R}) \) then \( f(H(t)) \in \Psi_{td}^{s,0}(\mathbb{R}; E) \).

This implies in particular that if \( f \in S^s(\mathbb{R}) \) with \( s < 0 \) then \( f(H(t)) \) is compact as an operator in \( B(H^m(\Sigma; E), H^m(\Sigma; E)) \) for each \( m \in \mathbb{R} \).

### 4.4 Large-time evolution of spectral projections

In this part we prove the following key result on the large-time Heisenberg evolution of spectral projections.

**Proposition 4.7.** Let \( \chi \in S^0(\mathbb{R}, [0, 1]) \) be such that \( \chi = 1_{[0, +\infty[} \) outside of a bounded neighborhood of 0. Then there exists \( R(t) \in \Psi_{td}^{-1,0}(\mathbb{R}; E) \) such that for all \( t \in \mathbb{R} \),

\[
U(t, 0)(\chi(H(0)) + R(0))U(0, t) = \chi(H(t)) + R(t), \tag{4.31}
\]

and furthermore,

\[
s - \lim_{t \to \pm\infty} U(0, t)R(t)U(t, 0) = \Psi_{td}^{-\infty}(E). \tag{4.32}
\]

Note that in contrast to the scalar case, \( U(t, 0)R(0)U(0, t) \) is not necessarily pseudodifferential if \( R(0) \) is pseudo-differential, so it cannot be simply absorbed into the r.h.s. of (4.31) as in the standard formulation of Egorov’s theorem (see e.g. \([80, \S 11]\)).

Before giving the proof we need a couple of auxiliary results. For the sake of brevity we denote

\[
P_+(t) = \chi(H(t)), \quad P_-(t) = 1 - \chi(H(t)).
\]

**Lemma 4.8.** Let \( m, \ell \in \mathbb{R} \). Suppose \( A(t) \in \Psi_{td}^{m,\ell}(\mathbb{R}; E) \) satisfies

\[
P_+(t)A(t)P_+(t) \in \Psi_{td}^{-\infty,\ell}(\mathbb{R}; E), \quad P_-(t)A(t)P_-(t) \in \Psi_{td}^{-\infty,\ell}(\mathbb{R}; E). \tag{4.33}
\]

Then there exists \( Z(t) \in \Psi_{td}^{m-1,\ell}(\mathbb{R}; E) \) such that

\[
[H, Z](t) = A(t) \mod \Psi_{td}^{-\infty,\ell}(\mathbb{R}; E). \tag{4.34}
\]

**Proof.** Step 1. Let \( k \in \mathbb{R} \) and suppose \( A_k(t) \in \Psi_{td}^{k,\ell}(\mathbb{R}; E) \) satisfies (4.33). We will show that there exists \( Z_{k-1} \in \Psi_{td}^{k-1,\ell}(\mathbb{R}; E) \) such that

\[
[H, Z_{k-1}] = A_k - A_{k-1} \tag{4.35}
\]

for some \( A_{k-1}(t) \in \Psi_{td}^{k-1,\ell}(\mathbb{R}; E) \) satisfying (4.33).

By the hypothesis that \( A_k(t) \in \Psi_{td}^{k,\ell}(\mathbb{R}; E) \) satisfies (4.33) we can write

\[
A_k = P_+A_kP_+ + P_-A_kP_- \mod \Psi_{td}^{-\infty,\ell}(\mathbb{R}; E).
\]

Let us set

\[
Z_{k-1} = \frac{1}{2} P_+ |H|^{(-1)} A_k P_- - \frac{1}{2} P_- |H|^{(-1)} A_k P_+ \in \Psi_{td}^{k-1,\ell}(\mathbb{R}; E),
\]
where $|H|^{(-1)}(t) \in \Psi^{-1,0}_{\text{td}}(\mathbb{R}; E)$ is defined as $|H|^{(-1)}(t) = f(H^2(t))$ for some $f \in S^{-\frac{1}{2}}(\mathbb{R})$ such that $f(x) = |x|^{-\frac{1}{2}}$ outside of a neighborhood of 0 in $\mathbb{R}$. Note that $|H|^{(-1)}(t)$ is principally scalar because $H(t)$ is of Dirac type. Furthermore, using that $x 1_{[0, +\infty]}(x) |x^2|^{-\frac{1}{2}} = 1_{[0, +\infty]}(x)$, we obtain

\begin{align}
HP_+ |H|^{(-1)} &= P_+ \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E), \\
HP_- |H|^{(-1)} &= -P_- \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E),
\end{align}

(4.36)

where we applied Corollary 4.6 to conclude that the error term is in $\Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E)$. It also commutes with $H(t)$ and $P_\pm(t)$. Therefore,

\begin{align}
HZ_{k-1} &= \frac{1}{2} HP_+ |H|^{(-1)} A_k P_- - \frac{1}{2} HP_- |H|^{(-1)} A_k P_+ \\
&= \frac{1}{2} P_+ A_k P_- + \frac{1}{2} P_- A_k P_+ \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E) \\
&= \frac{1}{2} A_k \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E).
\end{align}

Similarly,

\begin{align}
Z_{k-1}H &= \frac{1}{2} P_+ |H|^{(-1)} A_k HP_- - \frac{1}{2} P_- |H|^{(-1)} A_k HP_+ \\
&= \frac{1}{2} P_+ A_k |H|^{(-1)} HP_- - \frac{1}{2} P_- A_k |H|^{(-1)} HP_+ \\
&\quad + \frac{1}{2} P_+ [H|^{(-1)}, A_k] HP_- - \frac{1}{2} P_- [H|^{(-1)}, A_k] HP_+ \\
&= -\frac{1}{2} P_+ A_k P_- - \frac{1}{2} P_- A_k P_+ \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E) \\
&\quad + \frac{1}{2} P_+ [H|^{(-1)}, A_k] HP_- - \frac{1}{2} P_- [H|^{(-1)}, A_k] HP_+ \\
&= -\frac{1}{2} A_k + \frac{1}{2} P_+ [H|^{(-1)}, A_k] HP_- - \frac{1}{2} P_- [H|^{(-1)}, A_k] HP_+ \mod \Psi^{-\infty,0}_{\text{td}}(\mathbb{R}; E).
\end{align}

(4.38)

By subtracting the two identities (4.37) and (4.38) we find that $[H, Z_{k-1}] = A_k - A_{k-1}$ with $A_{k-1}$ satisfying (4.33) as requested, and belonging to $\Psi^{k-1,\ell}_{\text{td}}(\mathbb{R}; E)$ in view of $[[H]|^{(-1)}, A_k] \in \Psi^{k-2,\ell}_{\text{td}}(\mathbb{R}; E)$.

**Step 2.** Let now $A_m = A$, and for $j \in \mathbb{N}$, let $Z_{m-j} \in \Psi^{m-j,\ell}_{\text{td}}(\mathbb{R}; E)$ and $A_{m-j} \in \Psi^{-j,\ell}_{\text{td}}(\mathbb{R}; E)$ be constructed recursively from $A_{m-j+1}$ using *Step 1*. Let $Z \simeq \sum_{j=1}^{\infty} Z_{m-j} \in \Psi^{-1,\ell}_{\text{td}}(\mathbb{R}; E)$ be defined by asymptotic summation. Then $Z$ satisfies (4.34) in view of (4.35). \[ \square \]

**Lemma 4.9.** Let $m, \ell, \ell_0 \in \mathbb{R}$. Suppose $B(t) \in \Psi^{m,\ell}_{\text{td}}(\mathbb{R}; E)$ solves

\begin{equation}
\partial_t B(t) = i[H, B](t) + E_m(t)
\end{equation}

(4.39)

for some $E_m(t) \in \Psi^{m,\ell}_{\text{td}}(\mathbb{R}; E)$ satisfying (4.33). Then there exists $R(t) \in \Psi^{-1,\ell}_{\text{td}}(\mathbb{R}; E)$ such that

\begin{equation}
\partial_t (B(t) + R(t)) = i[H, B + R](t) \mod \Psi^{-\infty,\ell}_{\text{td}}(\mathbb{R}; E).
\end{equation}

(4.40)
**Proof.** Step 1. Let $k \in \mathbb{N}_0$. Suppose for the moment that instead of (4.39), $B \in \Psi_{td}^{m,\ell_0}(\mathbb{R}; E)$ satisfies

$$\partial_t B = i[H, B] + P_+E_{m-k}P_- + P_-E_{m-k}P_+ \mod \Psi_{td}^{m-k-1,\ell}(\mathbb{R}; E)$$

(4.41)

for some $E_{m-k}(t) \in \Psi_{td}^{m-k,\ell}(\mathbb{R}; E)$. We will show that there exist $Z_{m-k-1}(t), S_{m-k-1}(t) \in \Psi_{td}^{m-k-1,\ell}(\mathbb{R}; E)$ such that

$$\partial_t(B + Z_{m-k-1} + S_{m-k-1}) = i[H, B + Z_{m-k-1} + S_{m-k-1}]$$

$$+ P_+E_{m-k-1}P_- + P_-E_{m-k-1}P_+ \mod \Psi_{td}^{m-k-2,\ell}(\mathbb{R}; E)$$

(4.42)

for some $E_{m-k-1}(t) \in \Psi_{td}^{m-k-1,\ell}(\mathbb{R}; E)$.

First, using Lemma 4.8 we can find $Z_{m-k-1}(t) \in \Psi_{td}^{m-k-1,\ell}(\mathbb{R}; E)$ such that

$$i[H, Z_{m-k-1}] = P_+E_{m-k}P_- + P_-E_{m-k}P_+ \mod \Psi_{td}^{\infty,\ell}(\mathbb{R}; E).$$

Inserting this into (4.41) we get

$$\partial_t(B + Z_{m-k-1}) = i[H, B + Z_{m-k-1}] + E_{m-k-1}$$

(4.43)

for some $E_{m-k-1}(t) \in \Psi_{td}^{m-k,\ell}(\mathbb{R}; E)$. Next, we claim that we can find $S_{m-k-1}(t) \in \Psi_{td}^{m-k-1,\ell}(\mathbb{R}; E)$ such that

$$\partial_t S_{m-k-1} - i[H, S_{m-k-1}] = -P_+E_{m-k-1}P_+ - P_-E_{m-k-1}P_- \mod \Psi_{td}^{m-k-2,\ell}(\mathbb{R}; E).$$

(4.44)

Indeed, it suffices to integrate the resulting ODE for the principal symbol of $S_{m-k-1}$, see e.g. [21, (4.17)–(4.18)]. Then, by adding the identities (4.41) and (4.43) we obtain (4.42) as wanted.

Step 2. Now, if $B$ satisfies (4.39), then it satisfies (4.41) with $k = 0$. By iterating Step 1 indefinitely we obtain two sequences of operators $Z_{m-1-j}, S_{m-1-j} \in \Psi_{td}^{m-1-j,\ell}(\mathbb{R}; E)$ for $j \in \mathbb{N}_0$ such that the asymptotic sum $R \simeq \sum_{j=0}^{\infty}(Z_{m-1-j} + S_{m-1-j}) \in \Psi_{td}^{m-1,\ell}(\mathbb{R}; E)$ satisfies (4.40).  

**Proof of Proposition 4.7.** Recall that we have denoted $P_+(t) = \chi(H(t))$ and $P_-(t) = 1 - \chi(H(t))$, and that $\partial_t P_+(t) \in \Psi_{td}^{0,1-\delta}(\mathbb{R}; E)$ by Lemma 4.5. By differentiating the identity $(P_+(t))^2 = P_+(t)$ we obtain

$$P_\pm(t)(\partial_t P_+(t))P_\pm(t) = 0.$$ 

Furthermore, using that $H(t)$ commutes with $P_+(t)$ we can write

$$\partial_t P_+(t) = i[H(t), P_+(t)] + E_0(t)$$

where $E_0(t) = \partial_t P_+(t) \in \Psi_{td}^{0,1-\delta}(\mathbb{R}; E)$. Therefore, the assumption of Lemma 4.9 applied to $B(t) = P_+(t)$ are satisfied, with $m = \ell_0 = 0$ and $\ell = -1 - \delta$. The lemma gives us the existence of $R_{-1}(t) \in \Psi_{td}^{-1,1-\delta}(\mathbb{R}; E)$ and $R_{-\infty}(t) \in \Psi_{td}^{-\infty,-1-\delta}(\mathbb{R}; E)$ such that

$$\partial_t(P_+(t) + R_{-1}(t)) = i[H(t), P_+(t) + R_{-1}(t)] + R_{-\infty}(t).$$
By Duhamel’s principle we deduce that
\[ U(t,0)(P_+(0) + R_{-1}(0))U(0,t) = P_+(t) + R_{-1}(t) \]
\[ - U(t,0) \left( \int_0^t U(0,s)R_{-\infty}(s)U(s,0)ds \right) U(0,t) \]
\[ =: P_+(t) + R(t). \] (4.45)
With this notation, \( R(t) - R_1(t) \) is bounded in \( B(H^{s_1}(\Sigma; E), H^{s_2}(\Sigma; E)) \) for all \( s_1, s_2 \in \mathbb{R} \), uniformly in \( t \) with all derivatives, therefore \( R(t) - R_{-1}(t) \in \Psi_{td}^{-\infty,0}(\mathbb{R}; E) \) and consequently \( R(t) \in \Psi_{td}^{-1,0}(\mathbb{R}; E) \). Furthermore, \( R(0) = R_{-1}(0) \), which in combination with (4.45) shows that
\[ U(t,0)(P_+(0) + R(0))U(0,t) = P_+(t) + R(t). \]
as wanted. Finally, using the definition of \( R(t) \) and the fact that \( R_{-1}(t) \in \Psi_{td}^{-1,1-\delta}(\mathbb{R}; E) \) and \( R_{-\infty}(t) \in \Psi_{td}^{-\infty,1-\delta}(\mathbb{R}; E) \), we find that
\[ s- \lim_{t \to \pm \infty} U(0,t)R(t)U(t,0) = -s- \lim_{t \to \pm \infty} \int_0^t U(0,s)R_{-\infty}(s)U(s,0)ds \in \Psi^{-\infty}(E). \]
as claimed. \( \Box \)

**Remark 4.10.** It is possible to prove a generalization of Proposition 4.7 in which \( \chi(H(0)) \) is replaced by an arbitrary pseudo-differential operator. The necessary modifications can be carried out as in the proof of [21, Thm. 4.1]; this has in fact no bearing on the large-time aspects on which we focus here.

Note that because of the improvement in Sobolev order, the operators \( R(t) \) in (4.31) are compact in \( B(H^m(\Sigma; E), H^m(\Sigma; E)) \) for all \( t \in \mathbb{R} \). Note also that we obtain immediately an analogous statement if \( \chi = 1_{]-\infty,0]} \) (instead of \( 1_{[0, +\infty[} \)) outside of a bounded neighborhood of 0.

### 4.5. Time-dependent scattering theory.
Recall that by Hypothesis 4.1 we have in particular
\[ \partial_t H(t) \in \Psi_{td}^{1,1-\delta}(\mathbb{R}; E), \]
where so far \( \delta > 0 \) was allowed to be close to zero. In what follows, however, we will frequently need the short-range assumption \( \delta > 1 \) which serves to control integrability of terms involving \( t\partial_t H(t) \) for large \( |t| \).

For \( t, s \in \mathbb{R} \) we define the operators
\[ W(t,s) := e^{-itH(t)}U(t,s)e^{isH(s)}. \]
Remark that \( W(t,t) = 1 \) and \( W(t,s)W(s,r) = W(t,r) \) for \( t, s, r \in \mathbb{R} \).

**Proposition 4.11.** If \( \delta > 1 \) then we have the existence of “Møller wave operators” defined as limits in the strong operator topology, namely,
\[ s- \lim_{t \to \pm \infty} W(0,t) \in B(H^m(\Sigma; E)) \]
for all \( m \in \mathbb{R} \), and the same holds true for \( W(t,0) = (W(0,t))^{-1} \).
Proof. The proof is a standard use of Cook’s method. We have
\[ \partial_t W(0, t) = \partial_t (U(0, t)e^{itH(t)}) = iU(0, t)(t\partial_t H(t))e^{itH(t)}. \]
The r.h.s. is \( O((t)^{-\delta}) \) in \( B(H^{m+1}(\Sigma; E), H^m(\Sigma; E)) \) because \( \partial_t H(t) \in \Psi^{-1-\delta}_{td}(\mathbb{R}; E) \) by our assumptions, and the other factors are uniformly bounded by Lemma 4.4. Therefore, for all \( v \in H^{m+1}(\Sigma; E) \),
\[
\lim_{t \to \pm \infty} W(0, t)v = v + \int_0^{\pm \infty} \partial_t W(0, t)v\, dt
\]
exists in \( H^m(\Sigma; E) \). By density of \( H^{m+1}(\Sigma; E) \) in \( H^m(\Sigma; E) \) and uniform boundedness of \( W(0, t) \), we conclude that \( W(0, t) \) strongly converges on the whole space \( H^m(\Sigma; E) \). \( \square \)

For convenience of notation, instead of writing strong operator limits we will consider time-dependent operators as functions of \( \mathbb{R} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\} \). With this convention, \( H(\pm \infty) = H_{\pm} \) in view of Hypothesis 4.1.

We will use the following consequence of Proposition 4.7.

Lemma 4.12. Let \( I, J \subset \mathbb{R} \) be two intervals such that \( I \cap J \) is bounded. If \( \delta > 1 \) then for all \( t, s \in \mathbb{R} \) and \( m \in \mathbb{R} \) the operator
\[
1_I(H(t))W(t, s)1_J(H(s)) \in B(H^m(\Sigma; E), H^m(\Sigma; E))
\]
is compact. Furthermore, in the case \( t = +\infty \) and \( s = -\infty \) it belongs to \( \Psi^{-\infty}(E) \).

Proof. If \( I \) or \( J \) is bounded then the assertion follows from Corollary 4.6. Otherwise, we can find \( \chi_I, \chi_J \in S^0(\mathbb{R}, [0, 1]) \) such that \( \chi_I = 1_{[0, +\infty[} \) and \( \chi_J = 1_{]-\infty, 0]} \) outside of a bounded neighborhood of 0, or the other way around. By Proposition 4.7, for all \( t \in \mathbb{R} \) we have
\[
\chi_I(H(0)) + R(0) = U(0, t)(\chi_I(H(t)) + R(t))U(t, 0),
\]
with \( R(t) \in \Psi^{-1, 0}_{td}(\mathbb{R}; E) \), and since the l.h.s. does not depend on \( t \) we conclude
\[
U(0, t)(\chi_I(H(t)) + R(t))U(t, 0) = U(0, s)(\chi_I(H(s)) + R(s))U(s, 0)
\]
for all \( s \in \mathbb{R} \), hence
\[
\chi_I(H(t))U(t, s) = U(t, s)(\chi_I(H(s)) + R(s)) - R(t)U(t, s). \tag{4.47}
\]
Recalling \( W(t, s) = e^{-itH(t)}U(t, s)e^{isH(s)} \) and using that functions of \( H(t) \) commute we deduce from (4.47) that
\[
\chi_I(H(t))W(t, s) = W(t, s)\chi_I(H(s)) + W(t, s)\bar{R}(s) - \bar{R}(t)W(t, s), \tag{4.48}
\]
where
\[
\bar{R}(t) = e^{-itH(t)}R(t)e^{itH(t)} \in B(H^m(\Sigma; E), H^{m+1}(\Sigma; E))
\]
for all \( t \in \mathbb{R} \). We can also write
\[
\bar{R}(t) = W(t, 0)U(0, t)R(t)U(t, 0)W(0, t).
\]
When \( t \to \pm \infty \), the family \( U(0, t)R(t)U(t, 0) \) converges in the strong operator topology of \( B(H^m(\Sigma; E), H^{m+1}(\Sigma; E)) \) to an operator in \( \Psi^{-\infty}(E) \) by (4.32) of Proposition 4.7, and \( W(t, 0)W(0, t) \) preserve the mapping properties uniformly in \( t \) by Proposition 4.11. Therefore, \( \bar{R}(t) \) is smoothing for infinite \( t \). It follows that the two last summands in (4.48) extended to \( t \in \mathbb{R} \) are compact (and smoothing for infinite \( t \)) and can be disregarded.
By Corollary 4.6 and the fact that $\chi_{I}\chi_{J}$ is compactly supported, $\chi_{I}(H(t))\chi_{J}(H(s))$ is in $\Psi_{-\infty}(E)$ and thus compact, and we conclude that the operator

$$\chi_{I}(H(t))W(t, s)\chi_{J}(H(s))$$

is compact (and smoothing for infinite $t$). From this we can deduce that the operator $1_{I}(H(t))W(t, s)1_{J}(H(s))$ is compact (and smoothing for infinite $t$) by using again Corollary 4.6 to estimate the difference. □

4.6. Generalization for non-selfadjoint perturbations. In this paragraph we briefly comment on generalisations to perturbations by a non-selfadjoint potential. This discussion is auxiliary and is not needed in the next sections.

Specifically, instead of $H(t)$ we can consider operators of the form $H(t) + V(t)\in \Psi_{td}^{0,-\delta}(\mathbb{R}; E)$, where $V(t)$ is not assumed to have any particular self-adjointness properties, but decays to zero as $t \to \pm\infty$.

Then, if we keep the definition of the principally scalar operator $\Lambda(t) = (1 + H(t)^{2})^{\frac{1}{2}}$ unchanged, we have $[H(t) + V(t), \Lambda^{s}(t)]\in \Psi_{td}^{0,-\delta}(\mathbb{R}; E)$ for all $s \in \mathbb{R}$. Furthermore, for each $t \in \mathbb{R}$, $H(t) + V(t)$ is a bounded perturbation of a generator of a strongly continuous bounded semi-group, and therefore it is also the generator of a strongly continuous semi-group (see e.g. [62, Thm. X.50]). In consequence, $H(t) + V(t)$ has a well-defined Schrödinger propagator $U_{V}(t, s)$ which satisfies the uniform boundedness properties from Lemma 4.4.

Then, the proof of Proposition 4.7 still yields a statement of the form

$$U_{V}(t, 0)(\chi(H(0)) + R(0))U_{V}(0, t) = \chi(H(t)) + R(t),$$

with $R(t)$ uniformly bounded in $B(H^{m}(\Sigma; E), H^{m+1}(\Sigma; E))$ for all $m \in \mathbb{R}$, even though it is no longer true in general that $R(t)$ is necessarily a pseudo-differential operator in this situation. Nevertheless, if we set

$$W_{V}(t, s) = e^{-itH(t)}U_{V}(t, s)e^{isH(s)}$$

then the analogues of Proposition 4.11 and Lemma 4.12 for $W_{V}(t, s)$ instead of $W(t, s)$ remain valid.

In consequence, the arguments of the next section will imply the Fredholm property of the operator $\partial_{t} - i(H(t) + V(t))$ with APS boundary conditions defined using spectral projections of $H(t)$. The inclusion of the potential $V(t)$ will however not be needed when considering the geometric Dirac operator $D$.

5. Fredholm inverses in the evolutionary model

5.1. Fredholm property. We continue the analysis of the evolutionary model introduced in Hypothesis 4.1. Let us denote

$$\tilde{D} = \partial_{t} - iH(t).$$

The tildes will be used to stress that we consider the evolutionary model from §4 rather than the geometric Dirac operator $D$ from §2.
For \( m \in \mathbb{R} \) and small \( \epsilon > 0 \) we define:
\[
\tilde{\mathcal{Y}}^m := (t) = \frac{1}{4\pi t} L_2^2(\mathbb{R}, H^m(\Sigma; E)),
\]
\[
\tilde{\mathcal{X}}^m := \{ u \in C^1(\mathbb{R}, H^m(\Sigma; E)) \mid \tilde{D}u \in \tilde{\mathcal{Y}}^m \}.
\]

For \( u \in \tilde{\mathcal{X}}^m \) and \( t \in \mathbb{R} \) we denote
\[
\tilde{\varrho}_\pm u := \lim_{t \to \pm \infty} e^{-itH}u(t) \in H^m(\Sigma; E)
\]
powered that the limit exists. For each \( u, \tilde{\varrho}_+u \) and \( \tilde{\varrho}_-u \) are interpreted as the asymptotic data of \( u \), and the next proposition asserts that a solution of \( \tilde{D}u = f \) can be recovered from \( \tilde{\varrho}_+u \) (as well as from \( \tilde{\varrho}_-u \)).

**Proposition 5.1.** If \( \delta > 1 \) then for all \( m \in \mathbb{R} \) the operator
\[
\tilde{\varrho}_\pm \oplus \tilde{D} : \tilde{\mathcal{X}}^m \to H^m(\Sigma; E) \oplus \tilde{\mathcal{Y}}^m
\]
is well-defined, bounded and boundedly invertible.

**Proof.** For \( u \in \tilde{\mathcal{X}}^m \), the well-posedness of the inhomogenous Cauchy problem, see e.g. [60, §5, Thm. 5.2], can be expressed by the Duhamel formula
\[
u(t) = U(t, 0)u(0) + \int_0^t U(t, s)f(s)ds
\]
for all \( t \in \mathbb{R} \), where \( f = \tilde{D}u \in \tilde{\mathcal{Y}}^m \). Thus,
\[
e^{-itH}u(t) = e^{-itH}U(t, 0)u(0) + \int_0^t e^{-itH}U(t, 0)U(0, s)f(s)ds,
\]
and this converges as \( t \to \pm \infty \) by Proposition 4.11, Lemma 4.4 and dominated convergence. This proves that \( \tilde{\varrho}_\pm \in B(\tilde{\mathcal{X}}^m, H^m(\Sigma; E)) \) is well-defined as a strong operator limit.

Next, for any \( v \in H^m(\Sigma; E) \) and \( f \in \tilde{\mathcal{Y}}^m \), we claim that \( u = (\tilde{\varrho}_\pm \oplus \tilde{D})^{-1}(v, f) \) is given by the formula
\[
u(s) = \lim_{t \to \pm \infty} \left( U(s, t)e^{itH}v + \int_t^s U(s, r)f(r)dr \right)
\]
(5.50)
Again, this converges by Proposition 4.11, Lemma 4.4 and dominated convergence, and we easily find that \( \tilde{\varrho}_\pm u = v \) and \( \tilde{D}u = f \) indeed.

Next, we introduce abstract Atiyah–Patodi–Singer boundary conditions at infinity by defining for each \( m \in \mathbb{R} \) the space
\[
\tilde{\mathcal{X}}^m_{APS} := \{ u \in \tilde{\mathcal{X}}^m \mid \lim_{t \to -\infty} 1_{(-\infty, 0]}(H(t)u(t)) = 0, \lim_{t \to -\infty} 1_{(0, +\infty]}(H(t)u(t)) = 0 \}.
\]

We denote by \( \tilde{D}_{APS} = \tilde{D}|_{\tilde{\mathcal{X}}^m_{APS}} \) the restriction of \( \tilde{D} \) to \( \tilde{\mathcal{X}}^m_{APS} \).
The dependence on \( m \) is not stressed explicitly in the above notation, which is justified by the facts that for \( m_2 \geq m_1 \),
\[
\tilde{\mathcal{X}}^{m_2}_{\text{APS}} \subset \tilde{\mathcal{X}}^{m_1}_{\text{APS}}, \quad \tilde{\mathcal{Y}}^{m_2} \subset \tilde{\mathcal{Y}}^{m_1},
\]
and that the restriction of \( \tilde{D}|_{\tilde{\mathcal{X}}^{m_1}_{\text{APS}}} \) to \( \mathcal{X}^{m_2}_{\text{APS}} \) coincides with \( \tilde{D}|_{\tilde{\mathcal{X}}^{m_2}_{\text{APS}}} \).

With all the ingredients that we already have, the proof of the Fredholm property of \( \tilde{D}_{\text{APS}} \) can now be concluded from abstract Fredholm theory arguments (summarized in Appendix §A.1) similarly as in [7]. Note that in our setting, the definition of \( \tilde{g}_\pm u \) involves a \( e^{-itH(t)} \) factor, which will however not be a complication thanks to fact that it commutes with functions of \( H(t) \).

**Proposition 5.2.** If \( \delta > 1 \) then the operator \( \tilde{D}_{\text{APS}} : \tilde{\mathcal{X}}^{m}_{\text{APS}} \to \tilde{\mathcal{Y}}^{m} \) is Fredholm of index
\[
\text{ind}(\tilde{D}_{\text{APS}}) = \text{ind}(1_{]-\infty,0]}(H_+)W|_{\text{Ran}1_{]-\infty,0]}(H_-)).
\]
where \( W = W(+\infty, -\infty) = s- \lim_{t \to +\infty, s \to -\infty} W(t,s) \).

**Proof.** We apply Proposition A.2 from the appendix, with:
\[
\mathcal{X} = \tilde{\mathcal{X}}^{m}, \quad \mathcal{Y} = \tilde{\mathcal{Y}}^{m}, \quad \mathcal{P} = \tilde{D}_{\text{APS}} : \tilde{\mathcal{X}}^{m} \to \tilde{\mathcal{Y}}^{m},
\]
\[
\varrho_- = \tilde{\varrho}_- : \text{Ker} \tilde{D}_{\text{APS}} \to H^m(\Sigma; E),
\]
\[
\varrho_+ = \tilde{\varrho}_+ : \text{Ker} \tilde{D}_{\text{APS}} \to H^m(\Sigma; E),
\]
\[
\pi^+ = 1_{[0,+\infty]}(H_-), \quad \pi^- = 1_{]-\infty,0]}(H_+).
\]

The operator \( W^{--} \) from the statement of Proposition A.2 equals then
\[
W^{--} = 1_{]-\infty,0]}(H_+)W|_{\text{Ran}1_{]-\infty,0]}(H_-)),
\]
and similarly, \( W^{+-} = 1_{[0,+\infty]}(H_+)W|_{\text{Ran}1_{]-\infty,0]}(H_-)} \). Proposition A.2 says that \( W^{--} \) is Fredholm and
\[
\text{ind}(W^{--}) = \text{ind}(\tilde{D}|_{\text{Ker} \varrho})
\]
where \( \varrho = \pi^+ \varrho_- \oplus \pi^- \varrho_+ \), provided that the following two statements are true:

a) \( W^{+-} \) is compact,

b) \( \varrho_\pm \oplus P : \mathcal{X} \to \mathcal{H}_\pm \oplus \mathcal{Y} \) is boundedly invertible.

In view of (5.52), a) is in the present situation a direct consequence of Lemma 4.12. By (5.51), b) follows directly from Proposition 5.1.

It remains to prove that \( \tilde{D}_{\text{APS}} = \tilde{D}|_{\text{Ker} \varrho} \), which amounts to checking that \( \tilde{\mathcal{X}}^{m}_{\text{APS}} = \text{Ker} \varrho \). We have indeed:
\[
u \in \tilde{\mathcal{X}}^{m}_{\text{APS}} \iff \lim_{t \to +\infty} 1_{]-\infty,0]}(H(t))u(t) = 0 = \lim_{t \to -\infty} 1_{[0,+\infty]}(H(t))u(t)
\]
\[
\iff \lim_{t \to +\infty} 1_{]-\infty,0]}(H(t))e^{-itH(t)}u(t) = 0 = \lim_{t \to -\infty} 1_{[0,+\infty]}(H(t))e^{-itH(t)}u(t)
\]
\[
\iff \pi^+ \varrho_+ u = 0 = \pi^- \varrho_- u
\]
\[
\iff u \in \text{Ker} \varrho,
\]
which concludes the proof. \( \square \)
5.2. Positivity properties of Fredholm inverses. We will now prove several properties of Fredholm inverses of $\widetilde{D}_{\text{APS}}$ that will be useful in analysing microlocal properties of Fredholm inverses of $\widetilde{D}_{\text{APS}}$.

Let us first introduce the retarded/advanced inverse of $\widetilde{D}$, defined by

$$
(\widetilde{D}_+^{-1}f)(t) := \int_{-\infty}^{t} U(t,s)f(s) \, ds, \quad (\widetilde{D}_-^{-1}f)(t) := \int_{t}^{\infty} U(t,s)f(s) \, ds
$$

for all $f \in \widetilde{Y}^m$. Then, $\widetilde{D}_\pm^{-1} : \widetilde{Y}^m \to \widetilde{X}^m$ is bounded $\widetilde{D} \circ \widetilde{D}_\pm^{-1} = 1$, and from (5.50) we can conclude that

$$
\widetilde{D}_\pm^{-1} = (\tilde{g}_\pm \oplus \widetilde{D})^{-1}(0 \oplus 1).
$$

Recall that for $t \in \mathbb{R}$, $H(t)$ is self-adjoint w.r.t. the $t$-dependent $L^2_t(\Sigma; E)$ scalar product with norm $(\cdot,\cdot)_{L^2_t} = (T^{-1}(t) \cdot | T^{-1}(t) \cdot)_{L^2}$. We also denote by $(\cdot,\cdot)_{L^2_\pm}$ the $L^2(\Sigma; E)$-pairing of $H^m(\Sigma; E)$ and $H^{-m}(\Sigma; E)$, and abbreviate it by $(\cdot,\cdot)_{L^2_\pm}$ in the case $t = \pm \infty$.

For square-integrable functions $f_1$ and $f_2$ with values in respectively $H^m(\Sigma; E)$ and $H^{-m}(\Sigma; E)$, we denote

$$(f_1|f_2) = \int_{\mathbb{R}} (f_1(t)|f_2(t))_{L^2_t} \, dt.$$  

**Lemma 5.3.** Let $\tilde{g}_\pm^{-1} : H^m(\Sigma; E) \to \text{Ker} \, \widetilde{D}$ be defined by $\tilde{g}_\pm^{-1} = (\tilde{g}_\pm \oplus \widetilde{D})^{-1}(1 \oplus 0)$. Then $\tilde{g}_\pm^{-1}$ is the inverse of $\tilde{g}_\pm|_{\text{Ker} \, \widetilde{D}}$, and it satisfies

$$(\tilde{g}_\pm \, \widetilde{D}_\pm^{-1} f|h)_{L^2_\pm} = \pm (f|\tilde{g}_\pm^{-1} h)$$

for all $f \in \widetilde{Y}^{-m}$ and $h \in H^m(\Sigma; E)$.

**Proof.** On the one hand, $$(\widetilde{D}_\pm^{-1}f)(t) = \int_{-\infty}^{\infty} U(t,s)f(s) \, ds,$$ and thus

$$
\tilde{g}_\pm \, \widetilde{D}_\pm^{-1} f = \lim_{t \to \pm \infty} \int_{-\infty}^{\infty} W(t,0)U(0,s)f(s) \, ds.
$$

On the other hand, $$(\tilde{g}_\pm^{-1} h)(s) = \lim_{t \to \pm \infty} U(s,0)W(0,t)h.$$ In consequence,

$$
(\tilde{g}_\pm \, \widetilde{D}_\pm^{-1} f|h)_{L^2_\pm} = \lim_{t \to \pm \infty} \int_{-\infty}^{\infty} (W(t,0)U(0,s)f(s)|h)_{L^2_t} \, ds
$$

$$
= \lim_{t \to \pm \infty} \int_{-\infty}^{\infty} (f(s)|U(s,0)W(0,t)h)_{L^2_t} \, ds = \pm (f|\tilde{g}_\pm^{-1} h)
$$

as claimed, where to go from the first line to the second we used the fact that $U(t,s)$ and $W(t,s) : L^2_t(\Sigma; E) \rightarrow L^2_t(\Sigma; E)$ are unitary for all $t, s \in \mathbb{R}$. The latter property of $U(t,s)$ can be shown using a standard positive energy estimate argument, see e.g. [7, Lem. 2.4], and then the case of $W(t,s)$ follows easily. \hfill \Box

Let now $\widetilde{K}^m \subset \widetilde{X}^m$ and $\widetilde{R}^m \subset \widetilde{Y}^m$ be closed subspaces such that

$$
\widetilde{X}^m_{\text{APS}} = \text{Ker} \, \widetilde{D}_{\text{APS}} \oplus \widetilde{K}^m, \quad \widetilde{Y}^m = \text{Ran} \, \widetilde{D}_{\text{APS}} \oplus \widetilde{R}^m,
$$
and \( \dim \tilde{R}^m < +\infty \). Let \( \tilde{D}^{(-1)} \) be the associated Fredholm inverse\(^2\) of \( \tilde{D}\) : \( \tilde{X}^m \rightarrow \tilde{Y}^m \), uniquely defined by the property that

\[
\tilde{D} \circ \tilde{D}^{(-1)} = 1 - R, \quad \tilde{D}^{(-1)} \circ \tilde{D} = 1 - L,
\]

where \( R \) is the projection to \( \tilde{R}^m \) along \( \text{Ran} \tilde{D} \) and \( L \) the projection to \( \text{Ker} \tilde{D} \) along \( \tilde{K}^m \).

For different \( m \in \mathbb{R} \) we can choose the complementary subspaces in a compatible way, in the sense that

\[
\tilde{K}^m_2 \subset \tilde{K}^m_1, \quad \tilde{R}^m_2 \subset \tilde{R}^m_1
\]

(5.54)

for \( m_2 \geq m_1 \). Furthermore, by density of \( C^\infty(M; S^+ M) \) in \( \mathcal{Y}^m \) we can choose the finite dimensional space \( \tilde{R}^m \) in such way that \( \tilde{R}^m \subset C^\infty(M; S^+ M) \).

The \( m \)-independent notation is justified by the compatibility inclusions (5.54).

**Proposition 5.4.** Assume \( \delta > 1 \). If \( \tilde{D}^{(-1)} \) is a Fredholm inverse as above, then there exists a smoothing operator \( K_\pm \in \bigcap_{m,s \in \mathbb{R}} B(\tilde{Y}^m, \tilde{X}^{m+s}) \) such that for all \( f \in \tilde{Y}^0 \),

\[
(f | \tilde{D}^{(-1)}_{\APS} - \tilde{D}^{-1}_{\pm} + K_{\pm}) f \geq 0.
\]

**Proof.** We focus on \( \tilde{D}^{-1}_{-1} \), the other case being analogous. Let \( Q : \tilde{Y}^m \rightarrow \tilde{X}^m \) be given by \( Q = (1 - \varrho^{-1}_{-1} \pi^+_E \varrho_{-1}) \tilde{D}^{-1}_{-1} \) in the notation introduced in (5.51). Then, by Proposition A.3, the difference

\[
K_- := Q - \tilde{D}^{(-1)}_{\APS} : \tilde{Y}^m \rightarrow \tilde{X}^m
\]

is compact. A close inspection of formulae (A.11) and (A.14) in the proof of Proposition A.3 shows that \( K_- = E_1 + E_2 \), where \( E_1 \) is the composition of

\[
W^{-+} = 1_{]-\infty,0]}(H_+)W|_{\text{Ran}1_{[0,\infty]}(H_-)}.
\]

with operators that preserve regularity, and \( E_2 \) satisfies \( \text{Ran} E_2 \subset \text{Ker} \tilde{D}_{\APS} \). By Lemma 4.12, \( W^{-+} \in \Psi^{-\infty}(E) \), therefore \( E_1 \) is smoothing. Furthermore, elements of \( \text{Ker} \tilde{D}_{\APS} \) are smooth by an argument which is postponed for the moment, see Proposition 7.3 and Corollary 7.4; hence, \( E_2 \) is smoothing as well. We conclude that \( K_- \) is smoothing, and consequently it suffices to prove positivity of \( Q - \tilde{D}^{-1}_{-1} \).

We have \( Q - \tilde{D}^{-1}_{-1} = -\varrho^{-1}_{-1} \pi^+_E \varrho_{-1} \tilde{D}^{-1}_{-1} \), and by Lemma 5.3,

\[
-(f | \varrho^{-1}_{-1} \pi^+_E \varrho_{-1} \tilde{D}^{-1}_{-1} f) = (\varrho_{-1} \tilde{D}^{-1}_{-1} f | \pi^+_E \varrho_{-1} \tilde{D}^{-1}_{-1} f)_{L^2_{-1}}
\]

\[
= (\varrho_{-1} \tilde{D}^{-1}_{-1} f | 1_{[0,\infty]}(H_-) \varrho_{-1} \tilde{D}^{-1}_{-1} f)_{L^2_{-1}} \geq 0
\]

for all \( f \in \tilde{Y}^0 \) as asserted. \qed

\(^2\)If \( P : \mathcal{X} \rightarrow \mathcal{Y} \) is a Fredholm operator acting between two Banach spaces \( \mathcal{X}, \mathcal{Y} \), a **Fredholm inverse** of \( P \) is a bounded operator \( P^{-1} : \mathcal{Y} \rightarrow \mathcal{X} \) such that \( P \circ P^{-1} = 1_\mathcal{Y} \) modulo a compact operator and \( P^{-1} \circ P = 1_\mathcal{X} \) modulo a compact operator. We avoid using the term “parametrix” in this context to avoid confusion with parametrices in the sense of smooth regularity.
6. Index theory

6.1. The spectral flow. Once the Fredholm property is shown, we will employ a spectral flow argument largely analogous to the work of Bär–Strohmaier [7] to compute the index.

We first introduce the relevant terminology; see [51, 61, 7, 74, 30] for more details and various applications.

Let $\mathcal{H}$ be a separable Hilbert space and let $\{B(t)\}_{t \in I}$ be a norm-continuous family of self-adjoint Fredholm operators on $\mathcal{H}$.

**Definition 6.1.** For an interval $J \subset \mathbb{R}$, we denote

$$P_J(t) := 1_J(B(t)), \quad \mathcal{H}_J(t) := \text{Ran}(B_J(t)).$$

For $a \in \mathbb{R}$, we will simply write

$$P_{<a}(t) := P_{(-\infty,a]}(t), \quad \mathcal{H}_{<a}(t) := \text{Ran}(P_{<a}(t)),$$

and similarly for $\geq a$.

We recall the following definition of the spectral flow from [61].

**Definition 6.2.** For a compact interval $I = [t_1,t_2]$, a partition

$$t_1 = \tau_0 < \tau_1 < \ldots < \tau_N = t_2$$

together with numbers $a_j \in \mathbb{R}$ for $0 \leq j \leq N$ is called a flow partition (for $B(\cdot)$) if for each $n$ and $t \in [\tau_{n-1}, \tau_n]$ we have $a_n \notin \text{sp}(B(t))$ and $\mathcal{H}_{[0,a_n]}(t)$ is finite dimensional. For such a partition, the spectral flow is defined as

$$\text{sf}_I(B) = \sum_{n=1}^{N} \dim(\mathcal{H}_{[0,a_n]}(\tau_n)) - \dim(\mathcal{H}_{[0,a_n]}(\tau_{n-1})).$$

The spectral flow is well-defined, i.e., a flow partition exists and the spectral flow is independent of the choice of flow partition, see [61] for more details.

Since in our main case of interest we are dealing with families $\{B(t)\}_{t \in I}$ parametrized by $I = \mathbb{R}$, it is convenient to define

$$\text{sf}_\mathbb{R}(B) := \text{sf}_{[-1,1]}(\overline{B})$$

where $\overline{B}(t) = B(\phi(t))$ and $\phi : [-1,1] \to \mathbb{R}$ is an arbitrarily chosen homeomorphism, and we apply the analogous convention for other infinite intervals.

6.2. The index and the spectral flow. Suppose that $H(t) \in \Psi_{\text{id}}^{1,0}(\mathbb{R}; E)$ satisfies Hypothesis 4.1, in particular $H(t) = T(t)H_0(t)T^{-1}(t)$ with $H_0(t)$ elliptic and self-adjoint, and thus Fredholm. We apply the definitions introduced in the previous paragraph to the family $H_0(t)$ and the Hilbert space $\mathcal{H} = L^2(\Sigma; E)$. In particular, the spectral flow $\text{sf}_\mathbb{R}(H_0)$ is well-defined.

Recall that the family of operators $W(t,s)$ was defined for $t, s \in \mathbb{R}$ in §4.5 and then extended to $t, s \in \mathbb{R}$. If we now denote

$$W_0(t,s) := T(t)W(t,s)T^{-1}(s),$$
then $W_0(t, s)$ has properties analogous to $W(t, s)$, in particular

$$W_0(t, r)W_0(r, s) = W_0(t, s) \text{ for all } t, r, s \in \mathbb{R}.$$ 

Furthermore, $W_0(t, s)$ is bounded in $B(H^m(\Sigma; E), H^m(\Sigma; E))$ for all $m \in \mathbb{R}$ uniformly in $t, s \in \mathbb{R}$ by Lemma 4.4 and uniform boundedness of $T(t), T^{-1}(t)$.

Next, we introduce the notation

$$W^{-}_{0}(t, s) := P_{<0}(t) \circ W_0(t, s)|_{\mathcal{H}_{<0}(s)}, \quad W^{+}_{0}(t, s) := P_{\geq 0}(t) \circ W_0(t, s)|_{\mathcal{H}_{<0}(s)},$$

(and similarly for $W^{+}_{0}(t, s)$ and $W^{-}_{0}(t, s)$) for the components of $W_0(t, s)$ relative to the two decompositions $\mathcal{H} = \mathcal{H}_{<0}(s) \oplus \mathcal{H}_{\geq 0}(s)$ and $\mathcal{H} = \mathcal{H}_{<0}(t) \oplus \mathcal{H}_{\geq 0}(t)$.

We show the following result which equates the spectral flow with an index, and which plays the role of the analogue of [7, Thm. 4.1] in our setting.

**Proposition 6.3.** If $\delta > 1$ then for each $t, s \in \mathbb{R}$ with $t \geq s$, $W^{-}_{0}(t, s)$ is Fredholm and satisfies

$$\text{ind}(W^{-}_{0}(t, s)) = \text{sf}_{[s, t]}(H_0). \quad (6.56)$$

**Proof.** *Step 1.* In view of the relation $H(t) = T(t)H_0(t)T^{-1}(t)$, Lemma 4.12 implies that $W^{-}_{0}(t, s)$ is compact for each $t, s \in \mathbb{R}$. This implies that $W^{-}_{0}(t, s)$ is Fredholm by a simple argument recalled in Proposition A.2 in the appendix.

*Step 2.* From this point on, the arguments are fully analogous to the proof of [7, Thm. 4.1] thanks to the properties of $W_0(t, s)$ which we already showed. For the reader’s convenience we repeat these arguments below (see also [74] for an alternative approach). The cases of $t, s$ finite and infinite will be taken care of simultaneously, in accordance with the notation (6.55).

By Definition 6.2, we can choose a partition $s = \tau_0 < \tau_1 < ... < \tau_N = t$ and numbers $a_j$ such that $\pm a_j \notin \text{sp}(H_0(\tau))$ for all $\tau \in [\tau_{j-1}, \tau_j]$.

$$\text{sf}_{[s, t]}(H_0) = \sum_{j=1}^{N} \left( \dim \mathcal{H}_{[0, a_j]}(\tau_j) - \dim \mathcal{H}_{[0, a_j]}(\tau_{j-1}) \right). \quad (6.57)$$

Since $a_j \notin \text{sp}(H_0(\tau))$, the family of projectors $P_{< a_j}(\tau)$ is continuous on $\mathcal{H}$ over $[\tau_{j-1}, \tau_j]$. By the same arguments as in *Step 1* we can show that

$$P_{< a_j}(\tau) \circ W_0(\tau, s) : \mathcal{H}_{<0}(s) \rightarrow \mathcal{H}_{< a_j}(\tau)$$

is a continuous family of Fredholm operators for $\tau \in [\tau_{j-1}, \tau_j]$. By [52, Lem. 3.2], we obtain

$$\text{ind}(P_{< a_j}(\tau_j) \circ W_0(\tau_j, s)) = \text{ind}(P_{< a_j}(\tau_{j-1}) \circ W_0(\tau_{j-1}, s)). \quad (6.58)$$

If we consider both operators $P_{< a_j}(\tau) \circ W_0(\tau, s)$ and $P_{<0}(\tau) \circ W_0(\tau, s)$ as operators $\mathcal{H}_{<0}(s) \rightarrow \mathcal{H}_{< a_j}(\tau)$, then they differ by $P_{[0, a_j]}(\tau) \circ W_0(\tau, s)$, which is a compact operator by Lemma 4.12 (applied to $I = [0, a_j], \ J = ]s, 0]$) combined with the identity $H = TH_0T^{-1}$. Therefore,

$$\text{ind}(P_{< a_j}(\tau) \circ W_0(\tau, s)) = \text{ind}(P_{<0}(\tau) \circ W_0(\tau, s)), \quad (6.59)$$

where both operators are considered as operators from $\mathcal{H}_{<0}(s)$ to $\mathcal{H}_{< a_j}(\tau)$. 

Now, \( W^1_0(\tau, s) \) coincides with \( P_{< 0}(\tau) \circ W_0(\tau, s) \), with the difference that it is considered as an operator to \( \mathcal{H}_{< 0}(\tau) \). Hence

\[
\text{ind}(W^1_0(\tau, s)) = \text{ind}(P_{< 0}(\tau) \circ W_0(\tau, s)) + \dim \mathcal{H}_{[0, a_j]}(\tau), \quad \forall \tau \in [\tau_{j-1}, \tau_j]. \tag{6.60}
\]

We conclude:

\[
\text{ind}(W^1_0(t, s)) = \sum_{j=1}^{N} \left( \text{ind}(W^1_0(\tau_j, s)) - \text{ind}[W^1_0(\tau_{j-1}, s)] \right)
\]

\[
= \sum_{j=1}^{N} \left( \text{ind}(P_{< 0}(\tau_j) \circ W_0(\tau_j, s)) + \dim \mathcal{H}_{[0, a_j]}(\tau_j)
\right.
\]

\[
- \text{ind}(P_{< 0}(\tau_{j-1}) \circ W_0(\tau_{j-1}, s) - \dim \mathcal{H}_{[0, a_j]}(\tau_{j-1}))
\]

\[
= \sum_{j=1}^{N} \left( \text{ind}(P_{< 0}(\tau_j) \circ W_0(\tau_j, s)) - \text{ind}(P_{< 0}(\tau_{j-1}) \circ W_0(\tau_{j-1}, s)) \right)
\]

\[
+ \text{sf}_{[s, t]}(H_0)
\]

\[
= \sum_{j=1}^{N} \left( \text{ind}(P_{< a_j}(\tau_j) \circ W_0(\tau_j, s)) - \text{ind}(P_{< a_j}(\tau_{j-1}) \circ W_0(\tau_{j-1}, s)) \right)
\]

\[
+ \text{sf}_{[s, t]}(H_0)
\]

\[
= \text{sf}_{[s, t]}(H_0),
\]

where we used (6.60), (6.57), (6.59) and (6.58) from the second to last step. \( \square \)

**Remark 6.4.** Alternatively, one could carry out the same analysis for \( H(t) \) and \( W(t, s) \) directly (rather than for \( H_0(t) \) and \( W_0(t, s) \)), at the slight cost of having to work with operators similar to self-adjoint ones, or with \( t \)-dependent Hilbert spaces as in [7].

### 6.3 Index theorem for the Dirac operator

We now consider the setting of the geometric Dirac operator introduced in §§2.1–2.3, and we use the relationship with the evolutionary model to deduce the index theorem.

Recall in particular that our main operator of interest is

\[
D = -\nabla^M_{\alpha} - iA(t) - \tau(t) : C^\infty(M; S^+M) \to C^\infty(M; S^+M). \tag{6.61}
\]

For small \( \epsilon > 0 \), we define the spaces:

\[
\mathcal{Y} := \langle t \rangle^{-\frac{1}{2} - \epsilon} L_t^2 L_y^2(M; S^+M)
\]

\[
\mathcal{X} := \{ u \in C^0_t L_y^2(M; S^+M) \mid Du \in \mathcal{Y} \}, \tag{6.62}
\]

and we introduce boundary condition at infinity as follows.

**Definition 6.5.** For \( \mathcal{X} \) as in (6.62), the subspace of functions satisfying *Atiyah–Patodi–Singer conditions at infinity* is defined as

\[
\mathcal{X}_{\text{APS}} := \{ u \in \mathcal{X} \mid \lim_{t \to +\infty} 1_{-\infty, 0}(A(t)) u(t) = 0, \lim_{t \to -\infty} 1_{0, +\infty}(A(t)) u(t) = 0 \}.
\]

We denote by \( D_{\text{APS}} \) the restriction of \( D \) to \( \mathcal{X}_{\text{APS}} \).
Theorem 6.6. Assume \((M, g)\) is a Lorentzian spacetime equipped with a spin structure, such that \(M = \mathbb{R} \times \Sigma\) for some \(\Sigma\) compact and odd-dimensional, and such that \(\Sigma_t = \{t\} \times \Sigma\) is a Cauchy surface for each \(t \in \mathbb{R}\). Suppose that the metric \(g\) satisfies Hypothesis 1.1 with \(\delta > 1\), and let \(D_{\text{APS}} : X_{\text{APS}} \to \mathcal{Y}\) be defined as above. Then, \(D_{\text{APS}}\) is Fredholm of index

\[
\text{ind}(D_{\text{APS}}) = \int_M \hat{A} + \int_{\partial M} T\hat{A} + \frac{1}{2}(\eta(A_+, A_-) - \dim \ker(A_+) - \dim \ker(A_-)), \tag{6.63}
\]

where \(\eta(A_+, A_-) = \eta(A_+) - \eta(A_-)\) is the difference of the eta forms of \(A_+\) and \(A_-\).

Above, \(\hat{A}\) is the Atiyah–Singer integrand (or \(\hat{A}\)-form), associated with the Levi–Civita connection \(\nabla\) on \((M, g)\) (see e.g. [71, §10.5–§10.6] for an introductory account). The boundary integral involves the transgression form \(T\hat{A}\) of \((M, g)\), which is defined in terms of \(\nabla\) and a reference connection for an auxiliary Riemannian metric, though its pullback to the boundary does not depend on the choice of the latter, see [7, §4].

Remark 6.7. Theorem 6.6 is valid regardless of the precise choice of decaying weight in the definition of the space \(\mathcal{Y}\); one can also choose a weight that depends on spatial variables.

In the proof of Theorem 6.6 we will use the relationship between \(D\) and the evolutionary setting considered in §4. Recall that in the setting of Hypothesis 1.1, the metric \(g\) is assumed to be of the form \(g = -c^2(t)dt^2 + h_{ij}(t)dy^i dy^j\) with smooth \(c > 0\). By Lemma 2.1, the operator \(D\) satisfies

\[
D = U(t)^{-1}c^{-1}(t) \circ \hat{D} \circ U(t) = U(t)^{-1}c^{-1}(t)(\partial_t - iH(t))U(t), \tag{6.64}
\]

where \(H(t) = c(t)U(t)A(t)U(t)^{-1}\), \(A(t)\) is the Dirac operator on \((\Sigma_t, h(t))\) and \(U(t)\) was defined in §2.2. Let us also recall that Hypothesis 1.1 states that:

1. \(c(t) - c_\pm \in S^{-\delta}(\mathbb{R}, C^\infty(\Sigma))\) for some \(c_\pm \in C^\infty(\Sigma)\) s.t. \(c_\pm > 0\),
2. \(h(t) - h_\pm \in S^{-\delta}(\mathbb{R}, C^\infty(T^*\Sigma \otimes_s T^*\Sigma))\) for some Riemannian metric \(h_\pm\).

Lemma 6.8. Under the assumptions of Theorem 6.6, the family of first order differential operators \(H(t)\) satisfies Hypothesis 4.1, in particular \(H(t) \in \Psi_{td}^{1,0}(\mathbb{R}; S\Sigma)\) and \(H(t) - H_\pm \in \Psi_{td}^{1,1-\delta}(\mathbb{R}; S\Sigma)\) for some \(H_\pm \in \Psi^1(S\Sigma)\) of Dirac type.

Proof. For each \(t \in \mathbb{R}\), by polar decomposition and invertibility \(U(t) = |U^*(t)| U_1(t)\) for some unitary operator \(U_1(t)\). From the definition \(H(t) = c(t)U(t)A(t)U(t)^{-1}\) we obtain immediately \(H(t) = T(t)H_0(t)T^{-1}(t)\) where

\[
H_0(t) = U_1(t)c^{\frac{\delta}{2}}(t)A(t)c^{\frac{\delta}{2}}(t)U_1(t)^{-1}, \quad T(t) = c^{\frac{\delta}{2}}(t)|U^*(t)|.
\]

We first show that \(H_0(t) \in \Psi_{td}^{1,0}(\mathbb{R}; S\Sigma)\). This follows from the fact that \(U(t)\) and the coefficients of \(A(t)\) depend on time only through \(h(t)\) and its derivatives, which all behave as decaying symbols by (2). More precisely, recall that \(U(t) = \rho(t)\tau_t = |h(0)|^{-\frac{1}{2}}|h(t)|^{\frac{\delta}{2}}\tau_t\), and the dependence on \(t\) of the parallel transport \(\tau_t\) can be deduced from the corresponding system of ODEs, see [4, §5].
we introduced an isomorphism \( c(t) \) and \( \partial_t T(t) \) by Proposition 5.2 recalled above, \( \partial_t c(t) \) is satisfied by \( \partial_t \in S^{-1,\delta}(\mathbb{R},C^\infty(\Sigma)) \), \( c_0^{-1} \leq c(t) \leq c_0 \) uniformly in \( t \in \mathbb{R} \) for some \( c_0 > 1 \), so analogous properties are satisfied by \( c^\pm(t) \). Furthermore, we can compute \( U^*(t) \) as in [73, §3.2], and we find that

\[
|U^*(t)|^2 = U(t)U(t)^* = (\tau_t^0,\gamma_0^0) \in S^{0}(\mathbb{R},C^\infty(\Sigma;\text{End}(S\Sigma)))
\]

and \( \partial_t |U^*(t)|^2 \in S^{-1,\delta}(\mathbb{R},C^\infty(\Sigma;\text{End}(S\Sigma))) \) using Hypothesis 1.1. This implies \( |U^*(t)| \in \Psi_{\text{td}}(\mathbb{R};E) \) and \( \partial_t |U^*(t)| \in \Psi_{\text{td}}^{0,-1,\delta}(\mathbb{R};E) \), hence the stated properties of \( T(t) \).

Since \( A(t) \) is elliptic, formally self-adjoint and of Dirac type, the same is true of \( H_0(t) \). Finally, the statements on \( H(t) \) are concluded from the properies of \( H_0(t) \) and \( T(t) \) shown above.

**Proof of Theorem 6.6.** The proof is split in two parts: we first show the Fredholm property using the results from §4, and then the index is computed using a straightforward extension of the method from [7].

**Step 1.** Recall that in §2.3 we introduced an isomorphism \( U : C^0_\mathcal{L} \mathcal{L}_2^2(M;S^+M) \rightarrow C^0(\mathcal{L},\mathcal{L}_2^2;S\Sigma) \), and it is easily seen to extend to an isomorphism \( U\mathcal{Y} = \tilde{\mathcal{Y}}^0 \). By Lemma 2.1, \( D = U^{-1}c(t)^{-1}\tilde{D}U \), hence we also have \( U\mathcal{X} = \tilde{\mathcal{X}}^0 \).

It straightforward to check that boundary conditions are mapped consistently by \( U \), i.e. \( U\mathcal{X} = \tilde{\mathcal{X}}^0 \), and consequently

\[
D_{\text{APS}} = U^{-1}c^{-1}\tilde{D}_{\text{APS}}U : \mathcal{X}_{\text{APS}} \rightarrow \mathcal{Y}.
\]

By Proposition 5.2, \( \tilde{D}_{\text{APS}} \) is Fredholm, and therefore so is \( D_{\text{APS}} \), with \( \text{ind}(D_{\text{APS}}) = \text{ind}(\tilde{D}_{\text{APS}}) \). By Proposition 5.2 combined with Proposition 6.3, we have

\[
\text{ind}(D_{\text{APS}}) = \text{ind}\left( (1|_{-\infty,0}H_+)W\big|_{\text{Ran}(1|_{-\infty,0}H_-)} \right) = \text{ind}(W_0^{-})(+\infty,-\infty) - \dim\text{Ker}H_+ = \text{sf}_{\mathbb{R}}(H_0) - \dim\text{Ker}A_+ (6.66)
\]

provided that we justify the last identity. To show this we write \( H_0(t) \) in the form \( H_0(t) = c^\pm(t)U_1(t)A(t)U_1(t)^{-1}c^\pm(t) \), and since unitary transformations preserve the spectral flow (extended in the natural way to the Hilbert space bundle setting), see e.g. [63, Thm. 3.14], it suffices to know that \( c^{-\frac{1}{2}}(t)H_0(t)c^{-\frac{1}{2}}(t) \) and \( H_0(t) \) have equal spectral flow. This in turn can be proved by considering the continuous deformation

\[
[0,1] \ni s \mapsto c^{-\frac{s}{2}}(t)H_0(t)c^{-\frac{s}{2}}(t)
\]

and by using e.g. [61, Prop. 3] to equate the spectral flows for \( s = 0 \) and \( s = 1 \).

**Step 2.** We can find a smooth Riemannian metric \( g^\mathbb{R} \) on \( \mathbb{R} \times \Sigma \) such that \( g^\mathbb{R} = dt^2 + h_+ \) for all \( t > \frac{1}{2} \) and \( g^\mathbb{R} = dt^2 + h_- \) for all \( t < -\frac{1}{2} \). Let \( D^\mathbb{R} \) be the Riemannian Dirac operator associated to \( g^\mathbb{R} \) and let \( A^\mathbb{R}(t) \) be the induced Dirac operator on \( \{t\} \times \Sigma \), in particular \( A^\mathbb{R}(t) = A_\pm(t) \) for \( t \in \mathbb{R} \). Furthermore, let \( T > \frac{1}{2} \) and let \( D^\mathbb{R}_{\text{APS}} \) be the restriction of \( D^\mathbb{R} \) to the space of \( L^2 \) functions on \([-T,T]\) with Atiyah–Patodi–Singer boundary conditions at \(-T\) and \( T \) (defined using respectively \( A^\mathbb{R}(-T) = A_- \) and \( A^\mathbb{R}(T) = A_+ \)).
By the relationship between the index of $D^k_{\text{APS}}$ and the spectral flow, see e.g. [7, (18)], we have
\[ \text{ind } D^k_{\text{APS}} = \text{sf}_{[-1,1]}(\overline{A}) - \dim \ker A_+ \] (6.67)
using that $\overline{A}(\pm 1) = A_\pm$. Recall that by definition $\text{sf}_{\mathbb{R}}(A) = \text{sf}_{[-1,1]}(\overline{A})$, so in view of (6.66) we get
\[ \text{ind } D^k_{\text{APS}} = \text{ind } D_{\text{APS}}. \] (6.68)
We apply the Atiyah–Patodi–Singer theorem [2, Thm. 3.10], which gives
\[ \text{ind } D^k_{\text{APS}} = \int_{[-T,T] \times \Sigma} \hat{A}^k + \frac{1}{2} \left( \eta(A_+, A_-) - \dim \ker(A_+) - \dim \ker(A_-) \right) \] (6.69)
On the other hand,
\[ \int_{[-T,T] \times \Sigma} \hat{A}^k = \int_{[-T,T] \times \Sigma} \hat{A} + \int_{\partial[-T,T] \times \Sigma} \hat{T}A. \] (6.70)
As long as $T > \frac{1}{2}$, the l.h.s. does not depend on $T$ in view of (6.69) and (6.67). Using Hypothesis 1.1, we control the convergence of the boundary term on the r.h.s., as it is the integral of a polynomial in the curvature tensor, the second fundamental form of the boundary and some derivatives of $c(t)$. Therefore, we can take the $T \to +\infty$ limit in (6.70) and (6.69), which combined with (6.68) yields (6.63). \hfill \Box

7. Microlocal properties of Fredholm inverses

7.1. Preliminaries on wavefront sets. In this final section we analyse microlocal properties of Fredholm inverses of $D_{\text{APS}}$.

Let us start by introducing the necessary background on wavefront sets, see e.g. [45, 46] or [34, §7] for more details.

Let $o$ denote the zero section of $T^*M$. Recall that for each $u \in H^m_{\text{loc}}(M; S^+M)$ with $m \in \mathbb{R}$, its wavefront set $\text{WF}(u)$ is the subset of $T^*M \setminus o$ defined as follows: $q \in T^*M \setminus o$ does not belong to $\text{WF}(u)$ if and only if there exists a properly supported pseudo-differential operator $A \in \Psi^0(M; S^+M)$ such that $Au \in C^\infty(M; S^+M)$. If $G : C^\infty_c(M; S^+M) \to C^\infty(M; S^+M)$ is a continuous operator then $\text{WF}(G)$ is by definition the wavefront set of the Schwartz kernel of $G$. One also uses the somewhat more convenient primed wavefront set $\text{WF}'(G)$, defined by
\[ (q_1, q_2) \in \text{WF}'(G) \iff (q_1, -q_2) \in \text{WF}(G), \]
where the notation $-q_2$ refers to multiplication by $-1$ in the fibers.

In the geometric context introduced in §2, the characteristic set of $D$ is the following subset of $T^*M \setminus o$:
\[ \mathcal{N} = \{ q = (x, \xi) \in T^*M \setminus o \mid \xi \cdot g^{-1}(x)\xi = 0 \}, \]
i.e. $\mathcal{N}$ is the zero set of the principal symbol $p(x, \xi) = \xi \cdot g^{-1}(x)\xi$ of the Lorentzian Laplace–Beltrami operator on $(M, g)$. Integral curves in $\mathcal{N}$ of the (forward, backward)
Hamilton flow of $p$ are called (forward, backward) \textbf{null bicharacteristics}. The characteristic set $\mathcal{N}$ has two connected components which can be distinguished one from the other by setting

$$\mathcal{N}^\pm := \mathcal{N} \cap \{(x, \xi) \in T^*M \setminus o \mid \forall \nu \in T_xM \text{ future directed time-like, } \pm \nu \cdot \xi > 0\}.$$ 

**Definition 7.1.** One says that $G : C_c^\infty(M; S^+M) \to C^\infty(M; S^+M)$ has \textbf{Feynman wavefront set} if

$$\text{WF}'(G) \setminus T^*_\omega(M \times M) \subset \{(q_1, q_2) \in \mathcal{N} \times \mathcal{N} \mid q_1 > q_2\} \quad (7.71)$$

where $q_1 > q_2$ means that $q_1$ can be reached from $q_2$ by a forward null bicharacteristic, and $T^*_\omega(M \times M) = \{(q, q) \mid q \in T^*M\}$ is the diagonal in $T^*M \times T^*M$.

**Remark 7.2.** If $q_1 = (x_1, \xi_1) \in \mathcal{N}^+$ (resp. $\mathcal{N}^-$) and $q_2 = (x_2, \xi_2) \in \mathcal{N}^+$ (resp. $\mathcal{N}^-$) are on the same null bicharacteristic then $q_1 > q_2$ if and only if $x_1$ is in the causal future (resp. past) of $x_2$.

### 7.2. Wavefront set of parametrices of $D_{APS}$.

It is useful to generalize the definitions of the spaces $\mathcal{X}, \mathcal{Y}$ and related objects from §6.3 by setting for $m \in \mathbb{R}$,

$$\mathcal{Y}^m := (t)^{-\frac{1}{2}-s} L^2 \mathcal{H}^m_y(M; S^+M),$$

$$\mathcal{X}^m := \{u \in C^0_t \mathcal{H}^m_y(M; S^+M) \mid Du \in \mathcal{Y}^m\},$$

and

$$\mathcal{X}^m_{APS} := \{u \in \mathcal{X}^m \mid \lim_{t \to +\infty} 1_{[-\infty,0]}(A(t))u(t) = 0, \lim_{t \to -\infty} 1_{[0, +\infty]}(A(t))u(t) = 0\}.$$ 

Since $\mathcal{Y}^{m_2} \subset \mathcal{Y}^{m_1}$ and $\mathcal{X}^{m_2}_{APS} \subset \mathcal{X}^{m_1}_{APS}$ for $m_2 \geq m_1$ it makes sense to write $D_{APS} : \mathcal{X}^{m}_{APS} \to \mathcal{Y}^m$ for the restriction of $D$ to $\mathcal{X}^{m}_{APS}$.

The key argument is summarized in the next proposition.

**Proposition 7.3.** Let $m \in \mathbb{R}$, suppose that $u \in \mathcal{X}^m$ satisfies $Du = f$ with $f \in C^\infty(M; S^+M)$ and

$$\lim_{t \to +\infty} 1_{[0, +\infty]}(A(t))u(t) = 0, \quad (7.72)$$

Then $\text{WF}(u) \subset \mathcal{N}^\pm$. Furthermore, the analogue for $t \to -\infty$ holds true.

**Proof.** For the sake of definiteness we focus on the ‘+’ case in (7.72). Let $\bar{u} = U u$. Then, by (7.72), the asymptotic datum $\bar{u} = \lim_{t \to +\infty} e^{-itH(t)}u(t)$ satisfies

$$1_{[0, +\infty]}(H_+) \bar{u} = 0. \quad (7.73)$$

Furthermore, $\bar{u}$ solves $\tilde{D}\bar{u} = \tilde{f}$, where $\tilde{f} = cUf$. Therefore, we can express it in terms of the asymptotic data using formula (5.50), which gives

$$\bar{u}(t) = \lim_{s \to +\infty} U(t, s)e^{isH(s)} \bar{u} - \int_t^{+\infty} U(t, s)\tilde{f}(s)ds. \quad (7.74)$$

The second term is smooth and will not contribute to $\text{WF}(u)$, so we can assume without loss of generality that $\tilde{f} = 0$. Let now $\chi \in S^0(\mathbb{R}, [0, 1])$ be equal 0 on $]-\infty, \frac{1}{2}]$ and 1 on $[\frac{1}{2}, +\infty[$. We now apply Proposition 4.7 in the same way as in the proof of Lemma 4.12. This gives us the existence of $R(t) \in \Psi^{-1,0}_td(\mathbb{R}; S\Sigma)$ such that

$$\langle \chi(H(t)) + R(t) \rangle U(t, s) = U(t, s)(\chi(H(s)) + R(s))$$
and
\[ s- \lim_{s \to \infty} U(0, s)R(s)U(s, 0) \in \Psi^{-\infty}(S\Sigma). \quad (7.75) \]
In consequence, in combination with (7.74) we obtain
\[
(\chi(H(t)) + R(t))\tilde{u}(t) = (\chi(H(t)) + R(t)) \lim_{s \to +\infty} U(t, s) e^{isH(s)} \tilde{u} + U(t, 0) \lim_{s \to +\infty} (U(0, s)R(s)U(s, 0))W(0, s) \tilde{u} \\
= U(t, 0) \lim_{s \to +\infty} (U(0, s)R(s)U(s, 0))W(0, s) \tilde{u} \\
\in C^\infty(\mathbb{R} \times \Sigma; S\Sigma).
\]

Note that if we interpret \( \mathbf{1} \otimes (\chi(H(t)) + R(t)) \) as an operator acting jointly on spatio-temporal variables, it is not pseudo-differential (tensor products of pseudo-differential operators are not necessarily pseudo-differential). However, an argument from [46, 78, Thm. 18.1.35] yields a slight modification which is pseudo-differential. More precisely, for all \( q \in \mathcal{N}^+ \) we can find \( B_0 \in \Psi^0(\mathbb{R} \times \Sigma) \) such that \( B := U^{-1} \circ B_0 \circ (\mathbf{1} \otimes (\chi(H) + R)) \circ U \) is a pseudo-differential operator in \( \Psi^0(M; S^+M) \) and is elliptic at \( q \). Since by (7.76), \( Bu \) is smooth, we conclude that \( q \notin \mathcal{N}^- \) as claimed. \( \square \)

From the definition of \( X_{APS}^m \) and Proposition 7.3 it follows immediately that if \( u \in X_{APS} \) and \( Du = 0 \) then \( WF(u) \subset \mathcal{N}^+ \cap \mathcal{N}^- = \emptyset \).

**Corollary 7.4.** If \( u \in Ker D_{APS} \) then \( u \in C^\infty(M; S^+M) \).

Let now \( D_{APS}^{(-1)} : \mathcal{Y}^m \to X_{APS}^m \) be a Fredholm inverse of \( D_{APS} : X_{APS}^m \to \mathcal{Y}^m \) associated to a complement \( K^m \) of \( Ker D_{APS} \) in \( X_{APS}^m \) and to a complement \( R^m \) of \( Ran D_{APS} \) in \( \mathcal{Y}^m \).

We can choose the complement in a compatible way, in the sense that
\[
K^m_{m_2} \subset K^m_{m_1}, \quad R^m_{m_2} \subset R^m_{m_1}
\]
for \( m_2 \geq m_1 \). Furthermore, by density of \( C^\infty(M; S^+M) \) in \( \mathcal{Y}^m \) we can choose the finite dimensional space \( R^m \) in such way that \( R^m \subset C^\infty(M; S^+M) \).

Then, \( D_{APS}^{(-1)} \) is a parametrix in the sense that
\[
D \circ D_{APS}^{(-1)} = 1 + L, \quad D_{APS}^{(-1)} \circ D = 1 + R \quad (7.78)
\]
acting on, say, \( H^m_c(M; S^+M) \subset \mathcal{Y}^m \cap X_{APS}^m \), where \( L, R : H^m_c(M; S^+M) \to C^\infty(M; S^+M) \) are smoothing and of finite rank. The \( m \)-independent notation is justified by the compatibility inclusions (7.77).

**Theorem 7.5.** Under the assumptions of Theorem 6.6, let \( D_{APS}^{(-1)} : \mathcal{Y}^m \to X_{APS}^m \) be a Fredholm inverse of \( D_{APS} \) satisfying (7.78). Then \( D_{APS}^{(-1)} \) has Feynman wavefront set.

**Proof.** Let \( D^{(-1)} : H^m_c(M; S^+M) \to H^m_{loc}(M; S^+M) \) be the retarded/advanced inverse of \( D \), see e.g. [47, §4] and references therein for microlocal properties of \( D^{(-1)} \). Then,
\[
D(D_{APS}^{(-1)} - D^{(-1)}_+) = L, \quad (D_{APS}^{(-1)} - D^{(-1)}_-)D = R.
\]
Furthermore, for all \( f \in H_0^m(M; S^+M) \), \( u = (D_{\text{APS}}^{(-1)} - D_\pm^{-1}) f \) satisfies
\[
\lim_{t \to \pm \infty} 1_{\pm [0, +\infty)}(A(t)) u(t) = 0
\]
because \( D_{\text{APS}}^{(-1)} f \in \mathcal{X}_m^{\text{APS}} \) and by definition, \( D_\pm^{(-1)} \) is supported in the causal future/past of \( \text{supp} f \). Therefore, if \( B_\pm \in \Psi^0(M; S^+M) \) is constructed as the operator \( B \) in the proof of Proposition 7.3 then by the computation therein we find that \( B_\pm(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) : H_0^m(M; S^+M) \to C^\infty(M; S^+M) \) continuously for all \( m \in \mathbb{R} \). Therefore,
\[
WF'(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) \subset (\mathcal{N}^\mp \cup o) \times T^*M.
\] (7.79)

By the relationships (6.64)–(6.65) between \( D, D_{\text{APS}} \) and the operators \( \bar{D}, \bar{D}_{\text{APS}} \) from §§4–5, there exists a Fredholm inverse \( \bar{D}_{\text{APS}}^{(-1)} \) of \( D_{\text{APS}}^{(-1)} \) such that
\[
D_{\text{APS}}^{(-1)} - D_\pm^{-1} = U^{-1}(\bar{D}_{\text{APS}}^{(-1)} - \bar{D}_\pm^{-1})cU.
\]
By Proposition 5.4, \( \bar{D}_{\text{APS}}^{(-1)} - \bar{D}_\pm^{-1} \) is a positive operator modulo smooth terms. This allows us to employ a standard Cauchy–Schwarz inequality argument, see e.g. [34, Lem. 8.4.6], to conclude that \( WF'(\bar{D}_{\text{APS}}^{(-1)} - \bar{D}_\pm^{-1}) \) is symmetric. Hence \( WF'(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) \) is symmetric, which in combination with (7.79) yields
\[
WF'(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) \subset (\mathcal{N}^\mp \cup o) \times (\mathcal{N}^\mp \cup o).
\]

In particular, the two wavefront sets corresponding to different signs are disjoint. In view of the identity
\[
(D_{\text{APS}}^{(-1)} - D_+^{-1}) - (D_{\text{APS}}^{(-1)} - D_-^{-1}) = D_-^{-1} - D_+^{-1},
\]
this implies
\[
WF'(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) = WF'(D_-^{-1} - D_+^{-1}) \cap (\mathcal{N}^\mp \times \mathcal{N}^\mp).
\] (7.80)

It is well known that
\[
WF'(D_\pm^{-1}) \setminus T^*_\Delta(M \times M) \subset \{(q_1, q_2) \in \mathcal{N}^\pm \times \mathcal{N}^\pm \mid q_1 > q_2\}
\]
\[
\quad \cup \{(q_1, q_2) \in \mathcal{N}^\mp \times \mathcal{N}^\mp \mid q_2 > q_1\}
\]
and \( WF'(D_-^{-1} - D_+^{-1}) \subset \{(q_1, q_2) \in \mathcal{N} \times \mathcal{N} \mid q_1 > q_2 \text{ or } q_2 > q_1\} \), see e.g. [47, §4], cf. Remark 7.2 for a convenient rephrasing of the relation \( q_1 > q_2 \). From the identity
\[
D_{\text{APS}}^{(-1)} = (D_{\text{APS}}^{(-1)} - D_-^{-1}) + D_+^{-1}
\]
combined with (7.80), we can deduce that
\[
WF'(D_{\text{APS}}^{(-1)}) \setminus T^*_\Delta(M \times M) \subset \bigcap_\pm \left( WF'(D_{\text{APS}}^{(-1)} - D_\pm^{-1}) \cup WF'(D_\pm^{-1}) \right) \setminus T^*_\Delta(M \times M)
\]
\[
\subset \bigcap_\pm \left( \{(q_1, q_2) \in \mathcal{N}^\mp \times \mathcal{N}^\pm \mid q_1 > q_2 \text{ or } q_2 > q_1\} \right.
\]
\[
\quad \cup \{(q_1, q_2) \in \mathcal{N}^\pm \times \mathcal{N}^\pm \mid q_1 > q_2\}
\]
\[
\left. \cup \{(q_1, q_2) \in \mathcal{N}^- \times \mathcal{N}^+ \mid q_1 > q_2\} \right)
\]
\[
\subset \{(q_1, q_2) \in \mathcal{N}^- \times \mathcal{N}^+ \mid q_1 > q_2\}
\]
\[
\cup \{(q_1, q_2) \in \mathcal{N}^\mp \mid q_1 > q_2\}
\]
\[
\subset \{(q_1, q_2) \in \mathcal{N} \times \mathcal{N} \mid q_1 > q_2\}
\]
which implies the bound (7.71) on the wavefront set \( WF'(D_{\text{APS}}^{(-1)}) \).
Appendix A.

A.1. Abstract Fredholm theory. Our objective in this subsection is to provide an abstract version of the results in [7, § 2–3] and complement them with a useful formula for Fredholm inverses.

If $A$, $B$ are two operators, we will say that they have equal index either if both are Fredholm and $\text{ind } A = \text{ind } B$, or if neither is Fredholm (in which case they are said to have “infinite index”).

Following [7], we start by recalling the following elementary lemma, which can be found in, e.g., [3, Prop. A.1].

**Lemma A.1.** Let $X$ be a Hilbert space and $E$, $F$ Banach spaces. Let $K : X \to E$, $L : X \to F$ be bounded and assume that $L$ is surjective. Then $K : \text{Ker } L \to E$ and $K \oplus L : X \to E \oplus F$ have equal index.

**Proof.** Since $X$ is Hilbert, there exists a closed subspace $K \subset X$ complementary to $\text{Ker } L$. A direct computation shows that $L \oplus K : \text{Ker } L \to E$ and $K \oplus L : X \to E \oplus F$ have equal index. One concludes by observing that $1 \oplus (K|_{\text{Ker } L})$ and $(K|_{\text{Ker } L})$ have equal index. □

Let $H_+$, $H_-$ be two Banach spaces. Let $\pi_\pm^+ : H_\pm \to X$ be a pair of complementary projections on $H_+$. and $\pi_\pm^- : H_\pm \to H_-$ be a pair of complementary projections on $H_-$. Let us denote

$$H_\pm^+ = \pi_\pm^+ H_+, \quad H_\pm^- = \pi_\pm^- H_-.$$

If $W : H_- \to H_+$, we use the matrix notation

$$W = \begin{pmatrix} W^{++} & W^{+-} \\ W^{-+} & W^{--} \end{pmatrix} : H_\pm^+ \oplus H_\pm^- \to H_\pm^+ \oplus H_\pm^-,$$

where $W^{++} = \pi_\pm^+ W \pi_\pm^+ : H_\pm^+ \to H_\pm^+$ with $\pi_\pm^+ : H_\pm^+ \to H_\pm^+$ the canonical injection, and similarly for the other components. We state below an abstract version of [7, Thm. 3.2] combined with [7, Thm. 3.3]. The proof is largely analogous to [7].

**Proposition A.2.** Let $X$ be a Hilbert space and $Y$ a Banach space. Let $P : V \to Y$ and $\varrho_\pm : X \to H_\pm$ be bounded, and suppose that:

$$\varrho_\pm \oplus P : X \to H_\pm \oplus Y \text{ is boundedly invertible}. \quad (A.3)$$

Let $\varrho = \pi_\pm^+ \varrho_+ \oplus \pi_\pm^- \varrho_-$. Define

$$W = \varrho_+ \varrho_+^{-1} : H_- \to H_+,$$

where $W$ can be expressed by the formula $Wh = \varrho_+ (\varrho_+ \oplus P)^{-1} (h, 0)$ for all $h \in H_-$.

---

3Equivalently, $W$ can be expressed by the formula $Wh = \varrho_+ (\varrho_+ \oplus P)^{-1} (h, 0)$ for all $h \in H_-$. 

---
where \( q^{-1}_- : \mathcal{H}_- \to \ker P \) is the inverse of \( q_-|_{\ker P} \). If \( W^{--} \) is compact then \( W^{--} \) is Fredholm. Moreover, \( W^{--} \) and \( P : \ker q \to \mathcal{Y} \) have equal index.

**Proof.**  
**Step 1.** Observe that \( W \) is invertible of inverse \( W^{-1} = q_- q_+^{-1} \). Therefore, in the matrix notation generalizing (A.2), the identity \( W^{-1} W = 1 \) implies \((W^{-1})-- \circ W^{--} + (W^{-1})+ \circ W^{++} = 1\), so \((W^{-1})-- \circ W^{--}\) equals \(1\) modulo a compact term, hence \(W^{--}\) is Fredholm.

**Step 2.** We claim that the operators \( W^{--} : \mathcal{H}^-_+ \to \mathcal{H}^-_+ \) and

\[
g : \ker P \to \mathcal{H}^+_+ \oplus \mathcal{H}^-_-
\]

have equal index. On \( \ker P \), we have

\[
g = \pi^+_+ q_- \oplus \pi^-_+ q_+ = (\pi^+_+ \oplus \pi^-_+) W q_-,
\]

where \( q_- : \ker P \to \mathcal{H}_- = \mathcal{H}^+_+ \oplus \mathcal{H}^-_- \) is an isomorphism by (A.3). Furthermore,

\[
(\pi^+_+ \oplus \pi^-_- W) : \mathcal{H}^+_+ \oplus \mathcal{H}^-_- \to \mathcal{H}^+_+ \oplus \mathcal{H}^-_+
\]

\[
(u^+, u^-) \mapsto (u^+, W^{-} W^{--} u^+ + W^{--} u^-)
\]

is represented by the matrix

\[
\begin{pmatrix}
1 & 0 \\
W^{--} & W^{--}
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
W^{--} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & W^{--}
\end{pmatrix},
\]

(A.5)

where \( \begin{pmatrix} 1 & 0 \\ W^{--} & 1 \end{pmatrix} \) is an isomorphism with inverse \( \begin{pmatrix} 1 & 0 \\ -W^{-} & 1 \end{pmatrix} \). Observe that \( W^{--} \) and \( \begin{pmatrix} 1 & 0 \\ 0 & W^{--} \end{pmatrix} \) have equal index. Up to composition with isomorphisms, the latter operator coincides with (A.5), and thus with (A.4). Therefore, \( W^{--} \) and (A.4) have equal index as claimed.

Next, by Lemma A.1 applied to \( \mathcal{E} = \text{ran} \ g = \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \), \( \mathcal{F} = \mathcal{Y} \), \( K = \varrho \) and \( L = P \), the operators

\[
g \odot P : \mathcal{X} \to \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \oplus \mathcal{Y}
\]

and (A.4) have equal index. Finally, by Lemma A.1 applied to \( \mathcal{E} = \mathcal{Y} \), \( \mathcal{F} = \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \), \( K = P \) and \( L = \varrho \), the operators (A.6) and \( P : \ker \varrho \to \mathcal{Y} \) have equal index. We conclude that \( W^{--} \) and \( P|_{\ker \varrho} \) have equal index. \( \square \)

**Proposition A.3.** Under the same assumptions as in Proposition A.2, supposing in addition that \( W^{--} \) is compact, let us define \( P^{-1}_+ = (q_+ \oplus P)^{-1} \circ (0 \oplus 1) : \mathcal{Y} \to \mathcal{X} \), and let

\[
Q = (1 - q^{-1}_+ \pi^+_+ q_-) P^{-1}_+ : \mathcal{Y} \to \mathcal{X}.
\]

Then \( Q \) satisfies \( P \circ Q = 1 \) on \( \mathcal{Y} \) and

\[
g \circ Q : \mathcal{Y} \to \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \text{ is compact.}
\]

(A.7)

Furthermore, if \((P|_{\ker \varrho})^{(-1)} \) is a Fredholm inverse of \( P : \ker \varrho \to \mathcal{Y} \), then

\[
(P|_{\ker \varrho})^{(-1)} - Q : \mathcal{Y} \to \mathcal{X} \text{ is compact.}
\]

(A.8)

**Proof.** **Step 1.** The property \( P \circ Q = 1 \) follows from \( P \circ P^{-1}_- = 1 \) and the fact that by definition, \( q^{-1}_- \) maps to \( \ker P \).
Let us show the compactness of $\varrho \circ Q$. On the one hand, using that $\varrho = \pi^+ \varrho_- + \pi^- \varrho_+ = (\pi^+ \oplus \pi^- W) \varrho_-$ on $\text{Ker} P$, we can write

$$\varrho \varrho^{-1} \pi^+ \varrho_- = (\pi^+ \oplus \pi^- W) \pi^+ \varrho_- = \pi^+ \varrho_- = \pi^+ \varrho_- + W^{-1} \pi^+ \varrho_-$$(A.9)
onumber

on $\text{Ker} P$. On the other hand,

$$\varrho = \pi^+ \varrho_- + \pi^- \varrho_+ = \pi^+ \varrho_- + 0 \text{ on } \text{Ker} \varrho_+.$$ (A.10)

Since $P^{-1}_+ \varrho$ maps to $\text{Ker} \varrho_+$, using (A.9) and (A.10) we get

$$\varrho \circ Q = \varrho (1 - \varrho^{-1} \pi^+ \varrho_-) P^{-1} = \varrho P^{-1} - \varrho \varrho^{-1} \pi^+ \varrho_- P^{-1} = \varrho P^{-1} = \varrho \varrho^{-1} \pi^+ \varrho_- (P^{-1} - P^{-1}_+) = \pi^+ \varrho_- (P^{-1} - P^{-1}_+) = \pi^+ \varrho_- (P^{-1} - P^{-1}_+) = -(0 \oplus W^{-1} \pi^+ \varrho_-) (P^{-1} - P^{-1}_+) = K_1,$$

which is compact by compactness of $W^{-1}$.

**Step 2.** We know from the proof of Proposition A.2 that $\varrho \oplus P : \mathcal{X} \to \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \oplus \mathcal{Y}$ is Fredholm. Let $(\varrho \oplus P)^{-1}$ be a Fredholm inverse; then in particular

$$(\varrho \oplus P)^{-1} (\varrho \oplus P) = 1 - \pi_{\text{Ker}(\varrho \oplus P)},$$ (A.12)

where $\pi_{\text{Ker}(\varrho \oplus P)} : \mathcal{X} \to \mathcal{X}$ projects to the finite dimensional space $\text{Ker} (\varrho \oplus P)$. Furthermore, $(\varrho \oplus P)^{-1}$ can be chosen in such way that

$$(\varrho \oplus P)^{-1} \circ (0 \oplus 1) = (P|_{\text{Ker} \varrho})^{-1}.$$ (A.13)

Indeed, this can be arranged by defining $(\varrho \oplus P)^{-1}$ as

$$(\varrho \oplus P)^{-1} = \begin{pmatrix} \varrho^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (P|_{\text{Ker} \varrho})^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -P \varrho^{-1} & 1 \end{pmatrix},$$

where the matrix notation refers to decomposition $\mathcal{X} = \text{Ran}(P|_{\text{Ker} \varrho})^{-1} \oplus \text{Ker} \varrho$, and where $\varrho^{-1} : \mathcal{H}^+_+ \oplus \mathcal{H}^-_+ \to \text{Ran}(P|_{\text{Ker} \varrho})^{-1}$ is well-defined as the inverse of $\varrho$ restricted to $\text{Ran}(P|_{\text{Ker} \varrho})^{-1}$. The fact that this defines a Fredholm inverse can be checked directly or by using (A.1) with $L = \varrho$ and $K = P$.

If $K_1$ is the compact operator defined in (A.11) then using (A.12) and (A.13) we get

$$Q = (\varrho \oplus P)^{-1} (\varrho \oplus P) Q + \pi_{\text{Ker}(\varrho \oplus P)} Q = (\varrho \oplus P)^{-1} \circ (K_1 \oplus 1) + \pi_{\text{Ker}(\varrho \oplus P)} Q = (P|_{\text{Ker} \varrho})^{-1} + (\varrho \oplus P)^{-1} \circ (K_1 \oplus 0) + \pi_{\text{Ker}(\varrho \oplus P)} Q =: (P|_{\text{Ker} \varrho})^{-1} - K,$$

where $K$ is compact by compactness of $K_1$ and $\pi_{\text{Ker}(\varrho \oplus P)}$. \qed
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Laboratoire Jacques-Louis Lions, Sorbonne Université, 75252 Paris, France

Email address: shend@ljll.math.upmc.fr

CY Cergy Paris Université, 2 av. Adolphe Chauvin, 95302 Cergy-Pontoise, France

Email address: michal.wrochna@cyu.fr