Limits of Jensen polynomials for partitions and other sequences

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Abstract

It was discovered in [GORZ19] that the Jensen polynomials associated to many sequences have Hermite polynomial limits. We develop this theory in detail, based on the log-polynomial property which is a refinement of log-concavity and log-convexity. Applications to various partition sequences are given. An application to the sequence of factorials leads naturally to evaluating limits of generalized Laguerre polynomials.

1 Introduction

A sequence of real numbers $\alpha(0), \alpha(1), \alpha(2), \ldots$ is log-concave if

$$\alpha(n+1)^2 \geq \alpha(n) \cdot \alpha(n+2)$$

(1.1)

holds for $n \geq 0$. If the sequence is positive then being log-concave is equivalent to the sequence $\beta(n) = \log \alpha(n)$ being concave:

$$2\beta(n+1) \geq \beta(n) + \beta(n+2).$$

(1.2)

Many combinatorial and number theoretic sequences are log-concave, such as the binomial coefficients $\binom{n}{m}$ as $n$ varies, for example. Reversing the inequalities in (1.1) and (1.2) defines log-convex and convex sequences, respectively. Strictly log-concave means (1.1) holds with a strict inequality and similarly in the other cases. If the inequalities hold only for $n$ large enough then we say the sequence has the property asymptotically.

The following result mimics the second derivative test and is an easy exercise.

**Proposition 1.1.** Let $\alpha(n)$ be a sequence of positive real numbers. Suppose there exist $\delta(n) > 0$, $A(n)$ and $\kappa = \pm 1$ so that

$$\log \left( \frac{\alpha(n+j)}{\alpha(n)} \right) = A(n)j + \kappa \cdot \delta(n)^2 j^2 + o(\delta(n)^2)$$

(1.3)

as $n \to \infty$ for $j = 1, 2$. Then the sequence $\alpha$ is asymptotically strictly log-concave if $\kappa = -1$ and asymptotically strictly log-convex if $\kappa = 1$.

For another approach to log-concavity, note that (1.1) is equivalent to the polynomial

$$J_{2,n}^{\alpha}(X) := \binom{2}{0} \alpha(n) + \binom{2}{1} \alpha(n+1)X + \binom{2}{2} \alpha(n+2)X^2$$

having real roots. Define the $d$th Jensen polynomial associated to the sequence $\alpha$, and with shift $n$, as

$$J_{\alpha}^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j)X^j.$$
For any \( d \in \mathbb{Z}_{\geq 2} \) then \( J_{d,n}^\alpha(X) \) having all real zeros implies (1.1) by a result of Newton; see [HLP52, p. 53]. We are also interested in the reciprocal polynomials

\[
K_{\alpha}^{d,n}(X) := X^d J_{\alpha}^{d,n}(1/X) = \sum_{j=0}^{d} \binom{d}{j} \alpha(n + j) X^{d-j}.
\] (1.5)

In fact it was the \( K_{\alpha}^{d,0}(X) \) polynomials that Jensen used in [Jen13] to give criteria for an entire function to have only real zeros. Shifting by \( n \) corresponds to working with the \( n \)th derivative of the function; see for example [O'S21, Sect. 3].

**Proposition 1.2.** Let \( \alpha(n) \) be a sequence of positive real numbers and let \( d \) be a positive integer. Suppose there exists \( A(n) \) so that

\[
\log \left( \frac{\alpha(n + j)}{\alpha(n)} \right) = A(n)j + o(1)
\]

as \( n \to \infty \) for \( j = 1, 2, \ldots, d \). Then, as \( n \to \infty \),

\[
\frac{1}{\alpha(n)} J_{\alpha}^{d,n} \left( \frac{X - 1}{\exp(A(n))} \right) \to X^d,
\] (1.6)

\[
\exp(A(n))^{-d} K_{\alpha}^{d,n} \left( \frac{X - 1}{\exp(-A(n))} \right) \to X^d.
\] (1.7)

**Proof.** This is straightforward using the definitions (1.4), (1.5) and that \( \exp(o(1)) = 1 + o(1) \).

Throughout this paper, limits such as (1.6), (1.7), and (1.12), (1.14) below, mean that the coefficients of the polynomial on the left tend to the corresponding coefficients on the right. The following definition, based on [GORZ19, Eq. (15)], describes a situation when we have more information about the sequence \( \alpha \) than Propositions 1.1, 1.2 and log \( \alpha(n + j) \) can be well approximated by a polynomial in \( j \) for large \( n \).

**Definition 1.3.** A sequence of positive real numbers \( \alpha(n) \) is log-polynomial of degree \( m \geq 2 \), with data \( \{A(n), \kappa, \delta(n)\} \) for \( \kappa = \pm 1 \), if it satisfies the following conditions. There exist sequences \( A(n), \delta(n) \) and \( g_k(n) \) for \( k = 3, 4, \ldots, m \) so that

\[
\log \left( \frac{\alpha(n + j)}{\alpha(n)} \right) = A(n)j + \kappa \cdot \delta(n)^2 j^2 + \sum_{k=3}^{m} g_k(n) j^k + o(\delta(n)^{m+1})
\] (1.8)

as \( n \to \infty \), for \( j = 1, 2, \ldots, m + 1 \). We also require \( \delta(n) > 0, \delta(n) \to 0 \) and \( g_k(n) = o(\delta(n)^k) \) as \( n \to \infty \).

Some basic properties of log-polynomial sequences are contained in the next result.

**Proposition 1.4.** Suppose \( \alpha(n) \) is a log-polynomial sequence of degree \( m \) with data \( \{A(n), \kappa, \delta(n)\} \).

(i) Then \( \alpha(n) \) is log-polynomial for all degrees \( 2, 3, \ldots, m \).

(ii) The reciprocal sequence \( 1/\alpha(n) \) is log-polynomial of degree \( m \) with data \( \{-A(n), -\kappa, \delta(n)\} \).

(iii) If \( \alpha(n) \) is also log-polynomial of degree \( m \) with different data \( \{A^*(n), \kappa^*, \delta^*(n)\} \) then \( \kappa^* = \kappa \) and

\[
A^*(n) = A(n) + o(\delta(n)^{m+1}), \quad \delta^*(n)^2 = \delta(n)^2 + o(\delta(n)^{m+1}) \quad \text{as} \quad n \to \infty.
\] (1.9)

Let \( H_d(X) \) be the \( d \)th Hermite polynomial. As reviewed in section 3, we have

\[
H_d(X/2) = \sum_{k=0}^{d} \frac{d!}{(d-k)!} \binom{d-k}{k} (-1)^k X^{d-2k},
\] (1.10)

\[
i^{-d} H_d(iX/2) = \sum_{k=0}^{d} \frac{d!}{(d-k)!} \binom{d-k}{k} X^{d-2k}.
\] (1.11)
Griffin, Ono, Rolen and Zagier in [GORZ19] discovered a refinement of Proposition [12] showing that in some cases the \(d\)th Jensen polynomial of a sequence has limit \(H_d(X)\) as the shift \(n\) increases. Precisely, Theorem 3 of [GORZ19] says that if \(\alpha(n)\) is a log-polynomial sequence of degree \(d\) with data \(\{A(n), -1, \delta(n)\}\) then
\[
\frac{\delta(n)^{-d}}{\alpha(n)} J_{d,n}^\alpha \left( \frac{\delta(n)X - 1}{\exp(A(n))} \right) \rightarrow H_d(X/2) \quad \text{as} \quad n \rightarrow \infty.
\]
(1.12)

(Unfortunately [GORZ19] Thm. 3 is slightly misstated, omitting the sum over \(k\) in (1.8); it is given correctly in [GORZ19] Eq. (15)].) Replacing \(X\) by \(X/\delta(n)\) in (1.12) implies that
\[
\frac{1}{\alpha(n)} J_{d,n}^\alpha \left( \frac{X - 1}{\exp(A(n))} \right) \approx X^d - 2 \left( \frac{d}{2} \right) \delta(n)^{2} X^{d-2} + 12 \left( \frac{d}{4} \right) \delta(n)^{4} X^{d-4} + \cdots,
\]
where (1.13) means that given any \(\varepsilon > 0\) the coefficients of \(X^{d-j}\) on both sides agree to within \(\varepsilon \cdot \delta(n)^{j}\) for \(n\) sufficiently large.

The next definition allows us to include the natural extensions of (1.12) to the reciprocal Jensen polynomials \(K_{d,n}^\alpha\) and also the \(\kappa = 1\) case.

**Definition 1.5.** A sequence of positive real numbers \(\alpha(n)\) is **Hermite-Jensen of degree \(d\) and data \(\{A(n), \kappa, \delta(n)\}\)** if it satisfies the following conditions. As \(n \rightarrow \infty\),
\[
\frac{\delta(n)^{-d}}{\alpha(n)} J_{d,n}^\alpha \left( \frac{\delta(n)X - 1}{\exp(A(n))} \right) \rightarrow \begin{cases} H_d(X/2) & \text{if} \ \kappa = -1, \\ i^{-d} H_d(-iX/2) & \text{if} \ \kappa = 1. \end{cases}
\]
(1.14)

As (1.14) is always true for \(d = 0\) we restrict our attention to degrees \(d \geq 1\). Wagner in [Wag20, Def. 1.1] uses the term ‘Hermite-Jensen’ to refer to what we are calling log-polynomial sequences, though with the sum over \(k\) in (1.8) missing.

**Theorem 1.6.** Suppose \(\alpha(n)\) is a log-polynomial sequence of degree \(m\) with data \(\{A(n), \kappa, \delta(n)\}\). Then this sequence is Hermite-Jensen of degree \(d\) for \(1 \leq d \leq m + 1\) and the same data.

It is perhaps surprising that in (1.14) we also find Hermite polynomials as the limit of the reciprocal polynomials \(K_{d,n}^\alpha\). The Hermite polynomials themselves do not seem to satisfy any reciprocal relations.

As the orthogonal polynomials \(H_d(x)\) have distinct real roots, we will see that it follows from (1.14) that \(J_{d,n}^\alpha(X)\) must also have distinct real roots for \(n\) large enough when \(\kappa = -1\). In this way, Theorem 1.6 may be used to show that the Jensen polynomials associated to many sequences must have distinct real roots for \(n\) large. This idea was introduced in [GORZ19]. An application to the partitions sequence \(p(n)\) in Theorem 5 of [GORZ19] proved the real roots Conjecture 1.5 of [CJW19]. [GORZ19] Thm. 7] generalized this to sequences of coefficients of weakly holomorphic modular forms and a further application to the Taylor coefficients of the Riemann zeta \(\zeta(s)\) at \(s = 1/2\) was given in [GORZ19] Thm. 2).

The precise properties of Jensen polynomials associated to log-polynomial sequences may be stated as follows:

**Theorem 1.7.** Assume the sequence \(\alpha(n)\) is log-polynomial for some degree \(m \geq 2\).

(i) If \(\kappa = -1\) then the sequence is asymptotically strictly log-concave. If \(\kappa = 1\) then the sequence is asymptotically strictly log-convex.

(ii) Fix \(d\) with \(1 \leq d \leq m + 1\). For \(n\) sufficiently large, the zeros of \(J_{d,n}^\alpha(X)\) and \(K_{d,n}^\alpha(X)\) can be described as follows. Firstly, they are all distinct. If \(\kappa = -1\) they are all real. If \(\kappa = 1\) they are all not real, except for a single real zero when \(d\) is odd.

In just a few pages, the authors of [GORZ19] introduce many striking results and techniques. Our aim in this article is to expand and clarify these ideas, supply some omitted details, and to give further applications.
2 Applications to partitions

In sections 4,5 we provide detailed proofs of Theorems 1.6 and 1.7. It is mentioned in [GORZ19, Sect. 6] that (1.12) is still valid if $A(n)$ and $\delta(n)$ are replaced by approximations. The precise conditions needed are given next.

**Theorem 2.1.** Let $\alpha(n)$ be a log-polynomial sequence of degree $m$ with data \{ $A(n)$, $\kappa$, $\delta(n)$ \}. Suppose $A^*(n)$ and $\delta^*(n)$ satisfy

$$
A^*(n) = A(n) + o(\delta(n)), \quad \delta^*(n) = \delta(n)(1 + o(1)) \quad \text{as} \quad n \to \infty.
$$

Then $\alpha(n)$ is Hermite-Jensen of degree $d$ for $1 \leq d \leq m + 1$ with data \{ $A^*(n)$, $\kappa$, $\delta^*(n)$ \}.

Comparing (1.9) and (2.1) shows that the data required for a sequence to be Hermite-Jensen can be much less precise than the data required for it to be log-polynomial.

Theorems 1.6 and 2.1 are illustrated next with applications to various partition functions. A partition of $n$ is a non increasing sequence of positive integers that sum to $n$. The partition function $p(n)$ counts all the partitions of $n$. The number of overpartitions of $n$ is $\overline{p}(n)$ and these are partitions where the first appearance of a part of each size may or may not be overlined. For $k \in \mathbb{Z}_{\geq 2}$, let $b_k(n)$ denote the number of partitions of $n$ where no part sizes are multiples of $k$. These are called $k$-regular partitions. Glaisher’s theorem of 1883 implies that $b_k(n)$ also counts the number of partitions of $n$ where parts appear at most $k - 1$ times. Hence $b_2(n)$ gives the number of partitions of $n$ into distinct parts.

The well-known generating functions for $p(n)$, $\overline{p}(n)$ and $b_k(n)$ are

$$
\sum_{n=0}^{\infty} p(n)q^n = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}, \quad \sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{j=1}^{\infty} \frac{1 + q^j}{1 - q^j}, \quad \sum_{n=0}^{\infty} b_k(n)q^n = \prod_{j=1}^{\infty} \frac{1 - q^{kj}}{1 - q^j}.
$$

Multiplying the first product in (2.2) by the third when $k = 2$ gives the second. This implies the relation

$$
\overline{p}(n) = \sum_{r=0}^{n} p(r) \cdot b_2(n - r).
$$

Precise asymptotics are available for these partition functions. As $n \to \infty$,

$$
p(n) = \frac{1}{4\sqrt{3\pi}n^{3/2}} \left(1 - \frac{1}{\sqrt[4]{\gamma n}}\right) e^{\sqrt{n/\gamma}} \left(1 + O(e^{-\sqrt{n/\gamma}})\right),
$$

$$
\overline{p}(n) = \frac{1}{8\pi} \left(1 - \frac{1}{\pi \sqrt{n}}\right) e^{\pi \sqrt{n}} \left(1 + O(e^{-2\pi \sqrt{n}})\right),
$$

$$
b_k(n) = \frac{\sqrt{k'}}{2\sqrt{k' \cdot n''}} I_1 \left(\sqrt{k' \cdot n''} \right) \left(1 + O\left(e^{-\varepsilon_k \sqrt{n}}\right)\right),
$$

for certain $\varepsilon_k > 0$. We are using the notation

$$
\gamma := \frac{2\pi^2}{3}, \quad k' := \frac{2\pi^2}{3} \left(1 - \frac{1}{k}\right), \quad n' := n - \frac{1}{24}, \quad n'' := n + \frac{k - 1}{24}.
$$

Also $I_1$ is a modified Bessel function of the first kind [AAR99, Eq. (4.12.2)]:

$$
I_\alpha(z) := z^\alpha \sum_{j=0}^{\infty} \frac{z^{2j}}{\Gamma(\alpha + j + 1)j!}.
$$

The estimate (2.3) follows from the work of Hardy and Ramanujan in [HR18] and is given in this form by Rademacher in [Rad73, p. 278]. Then (2.4) follows similarly, is mentioned in [HR18], and generalized in [Si10]. The estimate (2.5) is Corollary 4.1 of [Hag71].
Theorem 2.2. With \( S := \pi(2/3)^{1/2}(n - 1/24)^{1/2} \), set
\[
A(n) := \frac{\pi^2}{3} \left( \frac{1}{S} - \frac{3}{S^2} \right), \quad \delta(n) := \frac{\pi}{3} \left( \frac{1}{2S(S - 1)} - \frac{3}{S^2} \right)^{1/2}.
\] (2.8)
Then for every degree \( m \geq 2 \) the partition sequence \( p(n) \) is log-polynomial with data \( \{A(n), -1, \delta(n)\} \). Hence \( p(n) \) is also Hermite-Jensen for all positive integer degrees and the same data or, more simply, \( \{A^*(n), -1, \delta^*(n)\} \) for
\[
A^*(n) := \frac{\pi}{6^{1/2}n^{1/2}}, \quad \delta^*(n) := \frac{\sqrt{\pi}}{2 \cdot 6^{1/4}n^{3/4}}.
\] (2.9)

Theorem 2.2 is proved in section 6. It agrees with the results for \( p(n) \) in sections 3 and 6 of [GORZ19], giving a more precise statement. See also [LW19 Eq. (1.1)], though this is stated incorrectly. Theorem 7.2 gives the corresponding result for overpartitions. For \( k \)-regular partitions we have

Theorem 2.3. Fix an integer \( k \geq 2 \) and let \( G(z) \) be the Bessel ratio \( I_0(z)/I_1(z) \). With (2.6), set
\[
A(n) := \frac{1}{2} \sqrt{\frac{k' n''}{n''}} \left( \sqrt{k'' n''} - \frac{1}{n''} \right), \quad \delta(n) := \frac{1}{2n''} \left( \frac{k'' n''}{2} \left( G\left( \sqrt{k'' n''} \right)^2 - 1 \right) - 2 \right)^{1/2}.
\] (2.10)
Then for every degree \( m \geq 2 \) the \( k \)-regular partition sequence \( b_k(n) \) is log-polynomial with data \( \{A(n), -1, \delta(n)\} \). Hence \( b_k(n) \) is also Hermite-Jensen for all positive integer degrees and the same data or, more simply, \( \{A^*(n), -1, \delta^*(n)\} \) for
\[
A^*(n) := \frac{\pi(1 - 1/k)^{1/2}}{6^{1/2}n^{1/2}}, \quad \delta^*(n) := \frac{\sqrt{\pi}(1 - 1/k)^{1/4}}{2 \cdot 6^{1/4}n^{3/4}}.
\] (2.11)

Theorem 2.3 extends and clarifies the main result of [CP21] as discussed in section 8. Theorem 1.7 then implies

Corollary 2.4. The sequences \( p(n), \overline{p}(n) \) and \( b_k(n) \) are asymptotically strictly log-concave. For fixed \( d \), the Jensen polynomials \( J_{\overline{p},d,n}(X) \), \( J_{p,d,n}(X) \) and \( J_{b_k,d,n}(X) \) each have distinct real roots for all \( n \) sufficiently large.

The asymptotic strict log-concavity of these sequences may also be shown more directly from (2.3), (2.4) and (2.5), reducing to the fact that \( \sqrt{n} \) is concave. By using more precise asymptotic expansions, it is proved in [Nic78] and [DP15] that \( p(n) \) is log-concave (i.e. (1.1) holds) exactly for \( n \geq 25 \), and shown in [Eng17] that \( \overline{p}(n) \) is log-concave for all \( n \). The precise \( n \)s for which \( J_{p,d,n}(X) \) has real roots are given in [CIW19] for \( d = 3 \) and also in [LW19] for \( d = 3, 4, 5 \).

![Figure 1: Renormalized Jensen polynomials for \( b_2(n) \) approaching \( H_7(X/2) \)](image)

Figure 1 illustrates Theorem 2.3 for \( k = 2 \) and the sequence of partitions into distinct parts. The graphs of the renormalized Jensen polynomials on the left of (1.14) are displayed for \( d = 7 \) and \( n = 10^7 \) with \( A(n) \) and \( \delta(n) \) given by (2.11). They appear close to the graph of \( H_d(X/2) \), which is in the middle.

In sections 7 and 9 we give further general families of log-polynomial sequences and it is seen that the Hermite-Jensen property is a very general phenomenon. See also [Far] where this ‘Hermite universality’ is discussed for the Taylor coefficients of entire functions.
3 Some polynomial families

The De Moivre polynomials $A_{n,k}(a_1, a_2, a_3, \ldots)$ appear when taking powers of power series:

$$\left(a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)^k = \sum_{n=k}^{\infty} A_{n,k}(a_1, a_2, a_3, \ldots) x^n \quad (k \in \mathbb{Z}_{\geq 0}).$$  \hfill (3.1)

Expanding the left side of (3.1) with multinomial coefficients then shows

$$A_{n,k}(a_1, a_2, a_3, \ldots) = \sum_{j_1+j_2+\cdots+j_m=n, j_1+j_2+\cdots+j_m=k} \binom{k}{j_1, j_2, \ldots, j_m} a_1^{j_1} a_2^{j_2} \cdots a_m^{j_m}$$  \hfill (3.2)

where $m = n - k + 1$ and the sum is over all possible $j_1, j_2, \ldots, j_m \in \mathbb{Z}_{\geq 0}$. Therefore $A_{n,k}(a_1, a_2, a_3, \ldots)$ is a polynomial in $a_1, a_2, \ldots, a_{n-k+1}$ of homogeneous degree $k$ with positive integer coefficients. They were introduced by De Moivre in 1697 and are also known in the literature as a type of Bell polynomial. See [Com74, Sect. 3.3], [O'S] for more information. We will only need the following easily established properties:

$$A_{n,k}(ca_1, c^2 a_2, c^3 a_3, \ldots) = c^n A_{n,k}(a_1, a_2, a_3, \ldots),$$  \hfill (3.3)

$$A_{n,k}(x, y, 0, 0, \ldots) = \binom{k}{n-k} x^{2k-n} y^{n-k} \quad (n \geq k).$$  \hfill (3.4)

If we set

$$h_r(y) := \begin{cases} 0, & \text{if } r \text{ is odd;} \\ y^{r/2}/(r/2)!, & \text{if } r \text{ is even}, \end{cases}$$  \hfill (3.5)

then it is clearly true that

$$e^{2Xt+yt^2} = e^{2Xt} e^{yt^2} = \sum_{d=0}^{\infty} \left[ d! \sum_{r=0}^{d} h_{d-r}(y) \frac{(2X)^r}{r!} \right] \frac{t^d}{d!}.$$  \hfill (3.6)

With $y = -1$ in (3.6) we have the generating function of the Hermite polynomials $H_d(X)$:

$$H_d(X) = d! \sum_{r=0}^{d} h_{d-r}(-1) \frac{(2X)^r}{r!}.$$  \hfill (3.7)

We will need the $y = 1$ case as well. It can be related to $H_d(X)$ by replacing $X$ by $iX$ and $t$ by $t/i$ in (3.6):

$$d! \sum_{r=0}^{d} h_{d-r}(1) \frac{(2X)^r}{r!} = i^{-d} H_d(iX).$$  \hfill (3.8)

Expanding the left of (3.6) we may also write

$$d! \sum_{r=0}^{d} h_{d-r}(y) \frac{(2X)^r}{r!} = \sum_{k=0}^{d} \frac{d!}{k!} A_{d,k}(2X, y, 0, 0, \ldots).$$  \hfill (3.9)

Hence (3.4) and (3.9) imply the simple formulas (1.10) and (1.11).

The generalized Laguerre polynomials will also be required in section 9:

$$L_d^{(r)}(X) := \frac{\Gamma(d + r + 1)}{d!} \sum_{j=0}^{d} \binom{d}{j} \frac{(-X)^j}{\Gamma(j + r + 1)}.$$  \hfill (3.10)

The Hermite polynomials may be expressed in terms of the Laguerre polynomials in (3.10) with $r = \pm 1/2$; see [AAR99, p. 284].

Proposition 1.4 is proved next, giving some first properties of log-polynomial sequences. We need a lemma.
Lemma 3.1. Fix \( m \geq 0 \). Suppose that \( P_n(j) = o(\Phi(n)) \) as \( n \to \infty \) when \( j = 0, 1, 2, \ldots, m \) for the polynomial
\[
P_n(x) := \phi_0(n) + \phi_1(n)x + \phi_2(n)x^2 + \cdots + \phi_m(n)x^m.
\]
Then \( \phi_k(n) = o(\Phi(n)) \) for \( k = 0, 1, 2, \ldots, m \).

Proof. Use induction on \( m \) with the case \( m = 0 \) clearly true. The forward difference operator \( \Delta \) acts on functions by \( \Delta f(x) := f(x + 1) - f(x) \), and repeated application gives
\[
\Delta^r f(x) = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r+j} f(x + j).
\] (3.11)

Also, see [Com74, Sect. 1.6],
\[
\Delta^r x^k \big|_{x=0} = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r+j} j^k = r! \left\{ \frac{k}{r} \right\}
\] (3.12)
where the Stirling subset numbers \( \left\{ \frac{k}{r} \right\} \) count the number of ways to partition \( k \) elements into \( r \) nonempty subsets with \( \left\{ \frac{k}{1} \right\} = 1 \) and \( \left\{ \frac{k}{r} \right\} = 0 \) for \( r > k \). Therefore
\[
\sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} P_n(j) = \Delta^m P_n(x) \big|_{x=0} = \sum_{k=0}^{m} \phi_k(n) m! \left\{ \frac{k}{m} \right\} = m! \phi_m(n).
\]

It follows that \( \phi_m(n) = o(\Phi(n)) \). We can then include this highest degree term in the error and assume \( P_n(x) \) is of degree \( m - 1 \), allowing the induction to proceed. \( \square \)

Proof of Proposition 1.4. Parts (i) and (ii) are clear. For part (iii), assume that (1.8) and its associated conditions are true and also a starred version: there exist sequences \( A^*(n), \delta^*(n) \) and \( g_k^*(n) \) for \( k = 3, 4, \ldots, m \) so that
\[
\log \left( \frac{\alpha(n+j)}{\alpha(n)} \right) = A^*(n) j + \delta^*(n)^2 j^2 + \sum_{k=3}^{m} g_k^*(n) j^k + o(\delta^*(n)^{m+1})
\] (3.13)
as \( n \to \infty \), for \( j = 1, 2, \ldots, m + 1 \). Here \( \delta^*(n) > 0, \delta^*(n) \to 0 \) and \( g_k^*(n) = o(\delta^*(n)^k) \) as \( n \to \infty \).

By Proposition 1.1 we must have \( \kappa^* = \kappa \). Then the difference of (1.8) and (3.13) has
\[
(A^*(n) - A(n)) j + (\kappa \cdot \delta^*(n)^2 - \kappa \cdot \delta(n)^2) j^2 + \sum_{k=3}^{m} (g_k^*(n) - g_k(n)) j^k = o(\delta^*(n)^{m+1} + \delta(n)^{m+1})
\]
as \( n \to \infty \), for \( j = 0, 1, \ldots, m + 1 \). Apply Lemma 3.1 to find
\[
A^*(n) - A(n) = o(\delta^*(n)^{m+1} + \delta(n)^{m+1}), \tag{3.14}
\]
\[
\delta^*(n)^2 - \delta(n)^2 = o(\delta^*(n)^{m+1} + \delta(n)^{m+1}). \tag{3.15}
\]
We may simplify the error estimates in (3.14) and (3.15) as follows. Since \( m \geq 2 \), (3.15) implies
\[
\delta^*(n)^2 - \delta(n)^2 = o(\delta^*(n)^3 + \delta(n)^3).
\]
Hence
\[
\delta^*(n) - \delta(n) = o(\delta^*(n)^2 + \delta(n)^2). \tag{3.16}
\]
Let \( h(n) := \delta^*(n)/\delta(n) - 1 \). Then as \( n \to \infty \), (3.16) is equivalent to
\[
\frac{h(n)}{\delta^*(n)h(n) + \delta^*(n) + \delta(n)} \to 0 \iff \frac{\left| \delta^*(n)h(n) + \delta^*(n) + \delta(n) \right|}{|h(n)|} \to \infty \iff \frac{\left| \delta^*(n) + \delta(n) \right|}{h(n)} \to \infty \iff \frac{\delta^*(n) + \delta(n)}{h(n)} \to \infty \iff \frac{h(n)}{\delta^*(n) + \delta(n)} \to 0.
\]
(We may omit the subsequence of \( n \text{s} \) where \( h(n) = 0 \) in the above argument.) It follows that \( \delta^*(n) = \delta(n) (1 + O(\delta^*(n) + \delta(n))) \) and, more simply, \( \delta^*(n) = \delta(n) (1 + o(1)) \). Use this in (3.14) and (3.15) to complete the proof of Proposition 1.4 and we have also shown that

\[
g_k^*(n) = g_k(n) + o(\delta(n)^{n+1}).
\]  

(3.17)

\[\]

4 Jensen limits

The next result is a slight extension of [GORZ19, Thm. 8] and we include its proof for completeness.

**Theorem 4.1.** Let \( \alpha(n) \) be a sequence of positive real numbers and fix a positive integer \( d \). Suppose there exist \( R(n) \), \( \delta(n) \) and \( C_r(n) \) for \( r = 0, 1, \ldots, d \) so that, as \( n \to \infty \),

\[
\frac{\alpha(n+j)}{\alpha(n)R(n)^j} = C_0(n) + \sum_{r=1}^{d} C_r(n) \delta(n)^r j^r + o(\delta(n)^d)
\]

(4.1)

for \( j = 1, 2, \ldots, d \). Suppose also that \( \delta(n) > 0 \), \( \delta(n) \to 0 \) and \( C_r(n) \to c_r \) as \( n \to \infty \). Then as \( n \to \infty \),

\[
\frac{\delta(n)^{-d}}{\alpha(n)} j^d \alpha \left( \frac{\delta(n)X - 1}{R(n)} \right) \to d! \sum_{k=0}^{d} (-1)^{d-k} c_{d-k} \frac{X^k}{k!},
\]

(4.2)

\[
\frac{\delta(n)^{-d}}{\alpha(n)} K_d X \left( \frac{\delta(n)X - 1}{R(n)} \right) \to d! \sum_{k=0}^{d} c_{d-k} \frac{X^k}{k!}.
\]

(4.3)

We necessarily have \( c_0 = 1 \) and the polynomials on the right of (4.2) and (4.3) are of degree \( d \) and monic. The proof of Theorem 4.1 relies on the following key combinatorial lemma.

**Lemma 4.2.** For integers \( d, k, r \) with \( d \geq k \) and \( r \geq 0 \), set

\[
\sigma(d, k; r) := \sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} j^r.
\]

Then \( \sigma(d, k; r) = 0 \) if \( r < d - k \) and \( \sigma(d, k; d-k) = (-1)^{d-k/(d-k)!} \).

**Proof.** Employing generating functions we have

\[
\sum_{r=0}^{\infty} \sigma(d, k; r) \frac{z^r}{r!} = \sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} e^z
\]

\[
= e^{kz} \sum_{j=k}^{d} \binom{d-k}{j-k} (-e^z)^{j-k} = e^{kz} (1 - e^z)^{d-k}
\]

and the result follows as this last expression is

\[
(1 + kz + O(z^2)) (1 - 1 - z + O(z^2))^d = (-1)^{d-k} z^d - k + O(z^{d-k+1}).
\]

For a second proof, as in [GORZ19, Sect. 2], \( \sigma(d, k; r) \) may be recognized with (3.11) as an iterated forward difference of the monomial \( x^r \) with

\[
\sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} j^r = (-1)^{d-k} \left. \Delta^d x^r \right|_{x=k},
\]

(4.4)

Since every application of \( \Delta \) reduces the degree by \( 1 \), (4.4) must equal \( 0 \) for \( r < d - k \). For \( r = d - k \) the degree of \( \Delta^r x^r \) is zero and (4.4) equals \( (-1)^r r! \) by (3.12). \( \square \)
Proof of Theorem 4.1. Write
\[
\frac{\delta(n)}{\alpha(n)} - \alpha(n) \sum_{j=0}^{d} \binom{d}{j} \alpha(n+j) \left( \frac{\delta(n)X - 1}{R(n)} \right)^j
\]
\[
= \sum_{k=0}^{d} \binom{d}{k} \delta(n)^{k-d} X^k \sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} \frac{\alpha(n+j)}{\alpha(n)R(n)^j}.
\]
Insert (4.1) and interchange the order of summation to find that the coefficient of $X^k$ equals
\[
\binom{d}{k} \sum_{r=0}^{d} C_r(n) \delta(n)^{k-d+r} \sum_{j=k}^{d} (-1)^{j-k} \binom{d-k}{j-k} j^r + o\left(\delta(n)^k\right).
\]
(4.5)

By Lemma 4.2, (4.5) is
\[
\frac{d!}{k!} (-1)^{d-k} C_d-k(n) + \binom{d}{k} \sum_{r=d-k+1}^{d} C_r(n) \delta(n)^{k-d+r} \sigma(d, k; r) + o\left(\delta(n)^k\right)
\]
(4.6)
and (4.2) follows. The limit (4.3) is shown similarly with $(-1)^{d-k} \sigma(d - k, 0; r)$ appearing in place of $\sigma(d, k; r)$.

Lemma 4.3. Fix $j \in \mathbb{R}$ and suppose $\delta(n) \to 0$ as $n \to \infty$ and $\rho(n) = o(\delta(n))$. Then for every positive integer $m$ we can write
\[
(1 + \rho(n))^{\delta(n)} = 1 + \sum_{\ell=1}^{m-1} \rho(\ell) n^{\delta(n)} + o\left(\delta(n)^m\right) \quad \text{with} \quad \rho(\ell) = o\left(\delta(n)^\ell\right).
\]

Proof. We have the useful elementary inequality
\[
|\log(1 + z)| \leq 2|z| \quad \text{for} \quad z \in \mathbb{C}, |z| \leq 3/4.
\]
(4.7)
Therefore $j \log(1 + \rho(n))$ is small for large $n$ and so
\[
(1 + \rho(n))^{\delta(n)} = e^{j \log(1 + \rho(n))} = 1 + \sum_{\ell=1}^{m-1} \frac{1}{\ell!} \log^\ell(1 + \rho(n)) \cdot j^\ell + O(|\log(1 + \rho(n))|^{m-j^m}).
\]
The lemma follows as $\log(1 + \rho(n)) = O(|\rho(n)|) = o(\delta(n))$ by (4.7).

Lemma 4.3 is used next to show there is some flexibility in how we choose $R(n)$ and $\delta(n)$ in (4.1).

Proposition 4.4. Let $\alpha(n)$ be a sequence of positive real numbers and fix a positive integer $d$. Suppose there exist $R(n)$, $\delta(n)$ and $C_r(n)$ for $r = 0, 1, \ldots, d$ satisfying the conditions of Theorem 4.1. This means that, as $n \to \infty$,
\[
\alpha(n+j) \int_{\alpha(n)}^{\delta(n)} \sigma(n)^r j^r + o(\delta(n)^d)
\]
for $j = 1, 2, \ldots, d$. Also $\delta(n) > 0$, $\delta(n) \to 0$ and $C_r(n) \to c_r$ as $n \to \infty$. If $R^*(n)$ and $\delta^*(n)$ satisfy
\[
R^*(n) = R(n)(1 + o(\delta(n))), \quad \delta^*(n) = \delta(n)(1 + o(1)) \quad \text{as} \quad n \to \infty
\]
(4.9)
then there exist $C_r^*(n)$ so that (4.8) remains true with $R(n)$, $C_r(n)$ and $\delta(n)$ replaced by their starred versions. Precisely, as $n \to \infty$,
\[
\alpha(n+j) \int_{\alpha(n)R^*(n)}^{\delta^*(n)} \sigma(n)^r j^r + o(\delta^*(n)^d)
\]
for $j = 1, 2, \ldots, d$, where $C_r^*(n) \to c_r$.
Proof. For clarity, write \( R^*(n) = R(n)(1 + \rho(n)) \) and \( \delta^*(n) = \delta(n)(1 + \phi(n)) \) where \( \rho(n) = o(\delta(n)) \) and \( \phi(n) = o(1) \). Then (4.8) implies

\[
\frac{\alpha(n + j)}{\alpha(n) R^*(n)^j} = (1 + \rho(n))^{-j} \left( \sum_{r=0}^{d} C_r(n)(1 + \phi(n))^{-r} \delta^*(n)^r j^r + o(\delta^*(n)^d) \right). \tag{4.11}
\]

With Lemma 4.3, write

\[
(1 + \rho(n))^{-j} = \sum_{\ell=0}^{d} \rho_\ell(n) j^\ell + o\left(\delta^*(n)^{d+1}\right)
\]

for \( \rho_\ell(n) := \frac{(-1)^\ell}{\ell!} \log^\ell(1 + \rho(n)) \) and \( \rho_\ell(n) = o(\delta^*(n)^\ell) \). Hence

\[
\frac{\alpha(n + j)}{\alpha(n) R^*(n)^j} = \sum_{u=0}^{d} \delta^*(n)^u j^u \sum_{\ell=0}^{u} C_{u-\ell}(n)(1 + \phi(n))^{\ell-u} \frac{\rho_\ell(n)}{\delta^*(n)^\ell} + \sum_{d+1}^{2d} \delta^*(n)^u j^u \sum_{\ell=0}^{d} C_{u-\ell}(n)(1 + \phi(n))^{\ell-u} \frac{\rho_\ell(n)}{\delta^*(n)^\ell} + o(\delta^*(n)^d).
\]

The sum with \( u \) between \( d + 1 \) and \( 2d \) is \( o(\delta^*(n)^d) \). Consequently we obtain (4.10) for

\[
C_r^*(n) := C_r(n)(1 + \phi(n))^{-r} + \sum_{\ell=1}^{r} C_{r-\ell}(n)(1 + \phi(n))^{\ell-r} \frac{\rho_\ell(n)}{\delta^*(n)^\ell}.
\]

The bounds for \( \phi(n) \) and \( \rho_\ell(n) \) now ensure that \( C_r^*(n) \to c_r \) as \( n \to \infty \). \( \square \)

5 Main theorems

Proof of Theorem 1.6 We have

\[
\log \left( \frac{\alpha(n + j)}{\alpha(n)} \right) = A(n) j + \kappa \cdot \delta(n)^2 j^2 + \sum_{k=3}^{m} g_k(n) j^k + o(\delta(n)^{m+1}) \tag{5.1}
\]

for \( j = 1, 2, \ldots, m + 1 \). Also \( \delta(n) > 0, \delta(n) \to 0 \) and \( g_k(n) = o(\delta(n)^k) \) as \( n \to \infty \). Let \( Y(n, j) := \kappa \cdot \delta(n)^2 j^2 + \sum_{k=3}^{m} g_k(n) j^k \). Then \( Y(n, j) \ll \delta(n)^2 \) and (5.1) implies

\[
\frac{\alpha(n + j)}{\alpha(n) \exp(A(n))^j} = \left( \sum_{u=0}^{t} \frac{1}{u!} Y(n, j)^u + O\left(|Y(n, j)|^t+1\right) \right) (1 + o(\delta(n)^{m+1}))
\]

\[
= \sum_{u=0}^{t} \frac{1}{u!} Y(n, j)^u + O\left(\delta(n)^{2t+2}\right) + o(\delta(n)^{m+1}). \tag{5.2}
\]

Now, recalling (3.1) and (5.3),

\[
\sum_{u=0}^{t} \frac{1}{u!} Y(n, j)^u = \sum_{u=0}^{t} \frac{1}{u!} \left( \kappa \cdot \delta(n)^2 j^2 + \sum_{k=3}^{m} g_k(n) j^k \right)^u
\]

\[
= \sum_{u=0}^{t} \frac{1}{u!} \left( \sum_{r=u}^{m} A_{r,u} \left( 0, \kappa, \delta(n)^3, \ldots, g_3(n), g_m(n), 0, \ldots \right) j^r \right)
\]

\[
= \sum_{u=0}^{t} \frac{1}{u!} \left( \sum_{r=u}^{m} A_{r,u} \left( 0, \kappa, \frac{g_3(n)}{\delta(n)^3}, \ldots, \frac{g_m(n)}{\delta(n)^m}, 0, \ldots \right) \delta(n)^3 j^r \right)
\]

\[
= \sum_{r=0}^{m} C_r(n) \delta(n)^r j^r
\]
for

\[ C_r(n) = \sum_{u=0}^{t} \frac{1}{u!} A_{r,u}(0, \kappa, g_3(n) \delta(n)^3, \ldots, g_m(n) \delta(n)^m, 0, \ldots). \]

To make the error in (5.2) of size \( o(\delta(n)^d) \) we choose \( d \) with \( 1 \leq d \leq m + 1 \) and then \( t \geq 1 \) so that \( 2t \geq d \). Our assumptions for \( g_k(n) \) mean that \( C_r(n) \ll 1 \) and so

\[
\frac{\alpha(n + j)}{\alpha(n) \exp(A(n)^j)} = C_0(n) + \sum_{r=1}^{d} C_r(n) \delta(n)^r j^r + o(\delta(n)^d). \tag{5.3}
\]

Also

\[
\lim_{n \to \infty} C_r(n) = \sum_{u=0}^{t} \frac{1}{u!} A_{r,u}(0, \kappa, 0, 0, \ldots),
\]

where \( A_{r,u}(0, \kappa, 0, 0, \ldots) \) equals \( \kappa^u \) if \( r = 2u \) and otherwise equals 0. Therefore, for \( r \leq d \leq 2t \),

\[
\lim_{n \to \infty} C_r(n) = c_r = \begin{cases} 0, & \text{if } r \text{ is odd;} \\ \kappa^{r/2}/(r/2)!, & \text{if } r \text{ is even.} \end{cases} \tag{5.4}
\]

Theorem 4.1 now applies to (5.3) with \( R(n) = \exp(A(n)) \). The polynomials on the right of (4.2) and (4.3) are recognized by (3.7) and (3.8) as the desired ones on the right of (1.14). This finishes the proof of Theorem 1.6.

Proof of Theorem 2.1 Using the proof of Theorem 1.6 we obtain (5.3) satisfying the conditions of Theorem 4.1 with \( R(n) = \exp(A(n)) \). For \( A^*(n) \) and \( \delta^*(n) \) satisfying (2.1), set \( R^*(n) = \exp(A^*(n)) \). Then \( R^*(n) = R(n)(1 + o(\delta(n))) \) and Proposition 4.4 implies that (5.3) is also valid with \( A(n), \delta(n) \) and \( C_r(n) \) replaced by \( A^*(n), \delta^*(n) \) and \( C^*_r(n) \), with \( C^*_r(n) \) giving the same limit (5.4). Applying Theorem 4.1 now completes the proof.

The next result lets us prove the properties mentioned in Theorem 1.7 for the zeros of shifted Jensen polynomials associated to Hermite-Jensen sequences.

Theorem 5.1. [US77] Let \( Q_n(X) \) be a sequence of monic polynomials of the same degree in \( \mathbb{C}[x] \) so that

\[
\lim_{n \to \infty} Q_n(X) = Q(X). \tag{5.5}
\]

Let \( r \) be any root of \( Q(X) \) and suppose it has multiplicity \( m \). Then for every \( \epsilon > 0 \) we can find an \( N \) so that \( Q_n(X) \) has \( m \) roots inside the complex ball of radius \( \epsilon \) centered at \( r \) for all \( n \geq N \).

Proof of Theorem 1.7 Part (i) follows from Proposition 1.1. For part (ii), it is clearly enough to prove it for \( J^{d,n}_\alpha(X) \). Theorem 1.6 implies the sequence is Hermite-Jensen of degree \( d \). Hence there exist sequences of positive reals \( A_n, B_n, C_n \) so that

\[
A_nJ^{d,n}_\alpha(B_nX - C_n) \to Q(X) := \begin{cases} H_d(X/2) & \text{if } \kappa = -1, \\ i^{-d}H_d(iX/2) & \text{if } \kappa = 1. \end{cases} \tag{5.6}
\]

The coefficient of \( X^d \) on the left of (5.6) is \( \alpha(n + d)A_nB_n^d \) which is nonzero and tends to 1. Divide by this to get the polynomials

\[ Q_n(X) := J^{d,n}_\alpha(B_nX - C_n)/(\alpha(n + d)B_n^d). \]

Then \( Q_n(X) \) is monic of degree \( d \) with real coefficients and tends to \( Q(X) \) as \( n \to \infty \).

The Hermite polynomials \( H_d(X) \) have distinct real roots with 0 as a root exactly when \( d \) is odd. Choose \( \epsilon \) in Theorem 5.1 small enough so that the balls of this radius around each root of \( Q(x) \) don’t overlap. It follows that there exists an \( N \) so that the roots of \( Q_n(X) \) are distinct for \( n \geq N \). Hence \( J^{d,n}_\alpha(X) \) must also have distinct roots for \( n \) large enough.
If $\kappa = -1$ then the roots of $Q(X)$ are real. The roots of $Q_n(X)$ for $n \geq N$ must also be real as otherwise two conjugate roots would lie within the same ball. Therefore the roots of $J_{n}^{d,n}(X)$ must also be real in this case. If $\kappa = 1$ then the roots of $Q(X)$ are all imaginary (and not real) except for the root 0 when $d$ is odd. Choose a smaller $\epsilon$ if necessary so that the balls around the non-real roots do not intersect the real line. If $d$ is odd then the single root of $Q_n(X)$ in the ball about 0 must be real for $n \geq N$. All other roots of $Q_n(X)$ must be not real. Therefore the roots of $J_{n}^{d,n}(X)$ must also be non-real in this case with the exception of one real root when $d$ is odd.

\section{Partitions}

The general properties of asymptotic expansions such as

$$f(x) = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + O\left(\frac{1}{x^4}\right) \quad \text{when } x \to \infty,$$

are explained in [Olv74, pp. 16 - 22]. These expansions are unique to each order and may be added, multiplied and divided. It is also valid to integrate the expansion of $f(x)$ to obtain the expansion of the integral of $f(x)$. However, differentiating asymptotic expansions is not always valid in the same way. It may be justified in the following situation, as described in [Olv74, p. 21], by employing integration and uniqueness. We give the details here for the case we require.

\textbf{Lemma 6.1.} Let $f(x)$ be a differentiable function on some positive interval $[c, \infty)$ with $f'(x)$ continuous. Suppose $f$ and $f'$ have the asymptotic expansions

$$f(x) = \sum_{j=1}^{R} \frac{a_j}{x^{j/2}} + O\left(\frac{1}{x^{(R+1)/2}}\right), \quad f'(x) = \sum_{j=0}^{S} \frac{b_j}{x^{j/2}} + O\left(\frac{1}{x^{(S+1)/2}}\right)$$

as $x \to \infty$. Then $b_0 = b_1 = b_2 = 0$ and $b_{j+2} = -(j/2)a_j$ for $3 \leq j \leq \min(R, S - 2)$. In other words, we may differentiate the asymptotic expansion of $f(x)$ to get the asymptotic expansion of $f'(x)$ if $f'(x)$ is continuous and known to have an asymptotic expansion.

\textbf{Proof.} The identity

$$f(x) = f(c) + \int_{c}^{x} f'(t) \, dt$$

shows that we must have $b_0 = b_1 = b_2 = 0$ as otherwise $f(x)$ becomes unbounded as $x \to \infty$. Set

$$E_S(x) := f'(x) - \sum_{j=3}^{S} \frac{b_j}{x^{j/2}}.$$ 

Then

$$f(x) = \int_{\infty}^{x} f'(t) \, dt = \int_{\infty}^{x} \sum_{j=3}^{S} \frac{b_j}{x^{j/2}} \, dt + \int_{\infty}^{x} E_S(t) \, dt = \sum_{j=1}^{S-2} \frac{c_j}{x^{j/2}} + O\left(\frac{1}{x^{(S-1)/2}}\right)$$

for $b_{j+2} = -(j/2)c_j$. Hence, for $M = \min(R, S - 2)$,

$$0 = f(x) - f(x) = \sum_{j=1}^{M} \frac{a_j - c_j}{x^{j/2}} + O\left(\frac{1}{x^{(T+1)/2}}\right). \quad (6.1)$$

Multiply both sides of (6.1) by $x^{1/2}$ and take the limit as $x \to \infty$ to see that $a_1 = c_1$. Repeat this procedure to show that $a_j = c_j$ for $j \leq M$. This completes the proof. \qed

We will also use the next result following from basic complex variables; see [Ahl78] pp. 125-126.
Lemma 6.2. Suppose $F(z)$ is holomorphic for $\text{Re}(z) > T_0$ and for $T > 2T_0$ we have the bound

$$|F(T(1 + \lambda))| \leq \beta(T) \quad \text{for} \quad \lambda \in \mathbb{C}, \ |\lambda| \leq 1/2.$$ 

Then for all $T > 2T_0$ and $|\lambda| \leq 1/4$, say, we have

$$F(T(1 + \lambda)) = \sum_{r=0}^{m} \frac{T^r}{r!}F^{(r)}(T) \cdot \lambda^r + O\left(\beta(T) \cdot |\lambda|^{m+1}\right),$$

where the implied constant depends only on $m$.

The method we use next to prove that the sequence $p(n)$ is Hermite-Jensen will also work for overpartitions, $k$-regular partitions and many other examples.

Proof of Theorem 2.2. Put $\gamma := 2\pi^2/3$ and $n' := n - 1/24$ for convenience, and also set

$$T := \gamma n', \quad \lambda := j/n', \quad F(z) := -\log z + \log\left(1 - \frac{1}{\sqrt{z}}\right) + \sqrt{z}.$$ 

Then (2.3) implies

$$\log\left(\frac{p(n + j)}{p(n)}\right) = \log\left(\frac{\gamma}{4\sqrt{3}}\right) + F(T(1 + \lambda)) + O(e^{-\sqrt{m}/2}).$$

Let $m \geq 2$ be a fixed integer. By Lemma 6.2 we have the expansion

$$F(T(1 + \lambda)) = \sum_{r=0}^{m} \frac{T^r}{r!}F^{(r)}(T) \cdot \lambda^r + O\left(n^{1/2}|\lambda|^{m+1}\right)$$

and hence, for $j$ in the range we want, $1 \leq j \leq m + 1$,

$$\log\left(\frac{p(n + j)}{p(n)}\right) = \sum_{r=1}^{m} \frac{\gamma^r}{r!}F^{(r)}(T) \cdot j^r + O\left(\frac{1}{n^{m+1/2}}\right) \quad (6.2)$$

as $n \to \infty$ for an implied constant depending only on $m$. A comparison of (6.2) with (1.8) now gives explicit expressions for $A(n)$, $\kappa \cdot \delta(n)^2$ and $g_k(n)$ in terms of derivatives of $F(z)$.

We wish to apply Lemma 6.1 to these derivatives to obtain their asymptotic expansions. To do this write

$$F'(z) = \frac{1}{2z^{1/2}} - \frac{1}{z} + \frac{1}{2z^{3/2}}G(\sqrt{z}) \quad \text{for} \quad G(z) := \frac{1}{1 - 1/z}. \quad (6.3)$$

Then $G(z)$ has the nice properties $G'(z) = -G(z)^2/z^2$ and

$$G(z) = \sum_{j=0}^{m} \frac{1}{z^j} + O\left(\frac{1}{|z|^{m+1}}\right) \quad (m \in \mathbb{Z}_{\geq 0}, |z| > 1). \quad (6.4)$$

Therefore, when $x > 1$ and working inductively, $F^{(r)}(x)$ may be expressed as a linear combination of terms of the form $G'(\sqrt{x})^a/x^{b/2}$ where $a, b \in \mathbb{Z}_{\geq 0}$. It then follows from (6.4) that $F^{(r)}(x)$ has an asymptotic expansion of the form

$$F^{(r)}(x) = \sum_{j=0}^{S} \frac{c_j}{x^{j/2}} + O\left(\frac{1}{x^{(S+1)/2}}\right),$$

for any $S$, as $x \to \infty$. The first case, by (6.3), has

$$F'(x) = \frac{1}{2x^{1/2}} - \frac{1}{x^{3/2}} + \frac{1}{2x^{5/2}} + \cdots + \frac{1}{2x^{S/2}} + O\left(\frac{1}{x^{(S+1)/2}}\right). \quad (6.5)$$
It is also clear that \( F^{(r)}(x) \) is continuous for \( x > 1 \). Lemma 6.1 now tells us that successive derivatives of both sides of (6.5) agree. From (6.2) we have

\[
A(n) = \frac{\gamma}{1!} F'(T) = \frac{\gamma}{2\sqrt{T}} + O\left(\frac{1}{T}\right),
\]

(6.6)

\[\]

\[-\delta(n)^2 = \frac{\gamma^2}{2!} F''(T) = -\frac{\gamma^2}{8T^{3/2}} + O\left(\frac{1}{T^2}\right),\]

(6.7)

\[\]

\[g_k(n) = \frac{\gamma^k}{k!} F^{(k)}(T) = O\left(\frac{1}{n^{k-1/2}}\right) \quad (k \geq 3).\]

(6.8)

Then in terms of \( n \),

\[
A(n) = \frac{\sqrt{\gamma}}{2\sqrt{n}} \left(1 - \frac{1}{24n}\right)^{-1/2} + O\left(\frac{1}{n}\right) = \frac{\sqrt{\gamma}}{2\sqrt{n}} + O\left(\frac{1}{n}\right),
\]

(6.9)

\[\]

\[\delta(n) = \frac{\gamma^{1/4}}{8^{1/2}n^{3/4}} \left(1 - \frac{1}{24n}\right)^{-3/4} \left(1 + O\left(\frac{1}{n^{1/2}}\right)\right)^{1/2} = \frac{\gamma^{1/4}}{8^{1/2}n^{3/4}} \left(1 + O\left(\frac{1}{n^{1/2}}\right)\right).
\]

(6.10)

With (6.2), (6.6) – (6.10) we have shown that the sequence \( p(n) \) is log-polynomial of degree \( m \) with data \( \{A(n), -1, \delta(n)\} \) where \( A(n) \) and \( \delta(n) \) are given exactly in (6.6), (6.7), simplifying to (2.8). Hence, by Theorem 1.6 \( p(n) \) is also Hermite-Jensen for all positive integer degrees and the same data. Take \( A^*(n), \delta^*(n) \) to be the main terms on the right of (6.9), (6.10). They satisfy the conditions of Theorem 2.1 and so \( p(n) \) is also Hermite-Jensen for the simpler data \( \{A^*(n), -1, \delta^*(n)\} \) in (2.9), as we wanted to show.

An alternative approach supplies a more explicit version of (6.2). Write

\[
F(T(1 + \lambda)) - F(T) = -\log(1 + \lambda) + \sqrt{T} (\sqrt{1 + \lambda} - 1) + \log\left(1 - \frac{1}{\sqrt{T - 1}} [(1 + \lambda)^{-1/2} - 1]\right)
\]

and expanding this as a series in \( \lambda \) using (3.1) produces

\[
F(T(1 + \lambda)) - F(T) = \sum_{r=1}^{\infty} \rho_r \left(\sqrt{T}\right) \lambda^r
\]

for

\[
\rho_r(x) := \frac{(-1)^r}{r} + \left(\frac{1}{2}\right)^r x - \sum_{u=1}^{r} \frac{1}{u(x - 1)^u} A_{r,u} \left(\left(-\frac{1}{2}\right)^{u}, \left(-\frac{1}{2}\right)^r\right), \ldots.
\]

(6.11)

In particular,

\[
\rho_1(x) = \frac{1}{2} \left(\frac{x^2}{x - 1} - 3\right), \quad \rho_2(x) = \frac{1}{8} \left(\frac{-x^3}{(x - 1)^2} + 6\right).
\]

(6.12)

We have proved

**Proposition 6.3.** For \( j = 1, 2, \ldots, m + 1 \) we have

\[
\log\left(\frac{p(n + j)}{p(n)}\right) = \sum_{r=1}^{m} \rho_r \left(\sqrt{\gamma}n^j\right) \lambda^r + O\left(\frac{1}{n^{m+1/2}}\right)
\]

as \( n \to \infty \) with an implied constant depending only on \( m \).

### 7 Overpartitions and other sequences

To help get an understanding of the kinds of sequences that are log-polynomial, consider

\[
\alpha(n) = n^a \exp(c \cdot n^b) \quad (a, b, c \in \mathbb{R})
\]

(7.1)
with \( a, c \) not both 0. Then for \( 1 \leq j \leq n/2 \), say,
\[
\log \left( \frac{\alpha(n + j)}{\alpha(n)} \right) = \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{a}{k} + c \cdot n^b \binom{b}{k} \frac{j^k}{n^k}.
\]

From the \( k = 1 \) and \( k = 2 \) coefficients
\[
A(n) = -\frac{a}{n} + bc \cdot n^{b-1}, \quad \kappa \cdot \delta(n)^2 = -\frac{a}{2n^2} + \frac{b(b-1)c}{2n^2-b}.
\]

The condition \( b < 2 \) is needed to ensure that \( \delta(n) \to 0 \). If \( b(b-1)c = 0 \) or \( b < 0 \) then \( \alpha(n) \) is not log-polynomial since \( g_k(n) \neq o(\delta(n)^k) \). The log-polynomial conditions are met in the remaining cases and we obtain the following:

**Proposition 7.1.** Let \( \alpha(n) \) be given by (7.1). Suppose \( 0 < b < 2, b \neq 1 \) and \( c \neq 0 \). Set \( \kappa = \text{sgn}(b(b-1)c) \) and define \( A(n) \) and \( \delta(n) \) with (7.2). Then, for all degrees \( m \geq 2 \), this sequence is log-polynomial with data \( \{A(n), \kappa, \delta(n)\} \).

Then \( \alpha(n) \) satisfying Proposition 7.1 is Hermite-Jensen and it follows from Theorem 1.7 that if \( b(b-1)c < 0 \) then the zeros of \( J_{d,n}^\alpha(X) \) are all distinct and real for \( n \) sufficiently large. At the ‘boundary’ case \( b = 1 \) we see that
\[
J_{d,n}^\alpha(X) = n\alpha e^{cn} \left( 1 + e^{c}X \right)^d + O \left( \frac{1}{n} \right)
\]
and the roots are all close to \(-1/e^c \) for large \( n \).

For another example, the sequence \( \gamma(n) \) coming from the Taylor coefficients of the Riemann zeta function at \( s = 1/2 \) is studied in [GORZ19]. In their Theorem 1 they show that the zeros of \( J_{g,n}^\alpha(X) \) are all real for sufficiently large \( n \). By [O’S21 Thm. 1.4], the simpler sequence \( \alpha(n) := n^{-n} \) gives a crude approximation to \( \gamma(n) \). Use the method of Theorem 2.2 with \( F(z) = -z \log z \) to show that this \( \alpha(n) \) is log-polynomial with data \( \{-1 - \log n, -1, (2n)^{-1/2}\} \).

Lastly in this section we consider the sequence of overpartitions \( \overline{\gamma}(n) \) from (2.2).

**Theorem 7.2.** With \( S := \pi \sqrt{n} \), set
\[
A(n) := \frac{\pi^2}{2} \left( \frac{1}{S-1} - \frac{3}{S^2} \right), \quad \delta(n) := \frac{\pi^2}{2} \left( \frac{1}{2S(S-1)^2} - \frac{3}{S^4} \right)^{1/2}.
\]

Then for every degree \( m \geq 2 \) the overpartition sequence \( \overline{\gamma}(n) \) is log-polynomial with data \( \{A(n), -1, \delta(n)\} \). Hence \( \overline{\gamma}(n) \) is also Hermite-Jensen for all positive integer degrees and the same data or, more simply, \( \{A^*(n), -1, \delta^*(n)\} \) for
\[
A^*(n) := \frac{\pi}{2n^{1/2}}, \quad \delta^*(n) := \frac{\sqrt{\pi}}{8^{1/2}n^{3/4}}.
\]

**Proof.** The proof is the same as for Theorem 2.2 but with \( \gamma = \pi^2 \) and \( n' = n \). \( \square \)

We can also write a similar expansion to Proposition 6.3 recalling (6.11):

**Proposition 7.3.** For \( j = 1, 2, \ldots, m + 1 \) we have
\[
\log \left( \frac{\overline{\gamma}(n + j)}{\overline{\gamma}(n)} \right) = \sum_{r=1}^{m-1} \rho_r \left( \frac{\pi \sqrt{n}}{n^r} \right) j^r + O \left( \frac{1}{n^{m+1/2}} \right)
\]
as \( n \to \infty \) with an implied constant depending only on \( m \).
8 \textit{k-regular partitions}

We first review from \cite{AAR99} Sect. 4.12 some results we will require for the Bessel functions $I_\alpha(z)$, defined in (2.7). For $\alpha, z \in \mathbb{C}$ they satisfy

$$I_{\alpha-1}(z) - I_{\alpha+1}(z) = 2\alpha I_\alpha(z)/z, \quad 2I'_\alpha(z) = I_{\alpha-1}(z) + I_{\alpha+1}(z),$$

(8.1)

from which we obtain

$$I'_0(z) = I_1(z), \quad I'_1(z) = I_0(z) - I_1(z)/z.$$  \hspace{1cm} (8.2)

As $|z| \to \infty$ with $|\arg z| < \pi/2$ there is the asymptotic expansion

$$I_\alpha(z) = \frac{e^z}{\sqrt{2\pi z}} \left( \sum_{j=0}^{n-1} \binom{j - 1/2 + \alpha}{j} \binom{j - 1/2 - \alpha}{j} \frac{j!}{(2z)^j} + O\left( \frac{1}{|z|^n} \right) \right).$$

(8.3)

**Proof of Theorem 2.3**  Make these definitions:

$n'' := n + \frac{k - 1}{24}, \quad k' := \frac{2\pi^2}{3} \left( 1 - \frac{1}{k} \right), \quad T := k' \cdot n'', \quad F(z) := \log\left( \frac{1}{\sqrt{z}} I_1(\sqrt{z}) \right).$

Then by (2.5), for $\lambda = j/n''$,

$$\log(b_k(n + j)) = \log\left( \frac{\pi^2}{3\sqrt{k}} \left( 1 - \frac{1}{k} \right) \right) + F(T(1 + \lambda)) + O\left( e^{-\epsilon_k \sqrt{n}} \right).$$

For $T$ large and $\lambda \in \mathbb{C}$ small we have $F(T(1 + \lambda)) = O(\sqrt{T}) = O(\sqrt{n})$ by (8.3). Hence we have the Taylor expansion

$$F(T(1 + \lambda)) = \sum_{r=0}^{m} \frac{T^r}{r!} F^{(r)}(T) \cdot \lambda^r + O\left( \frac{1}{n^{m+1/2}} \right),$$

and so, for $j$ with $1 \leq j \leq m + 1$,

$$\log\left( \frac{b_k(n + j)}{b_k(n)} \right) = \sum_{r=1}^{m} \frac{(k')^r}{r!} F^{(r)}(T) \cdot j^r + O\left( \frac{1}{n^{m+1/2}} \right),$$

(8.4)

as $n \to \infty$ for an implied constant depending only on $m$. Next write

$$F'(z) = \frac{1}{2\sqrt{z}} G(\sqrt{z}) - \frac{1}{z} \quad \text{for} \quad G(z) := \frac{I_0(z)}{I_1(z)}.$$  \hspace{1cm} (8.5)

It follows from (8.3) that we have the expansion

$$G(z) = \sum_{j=0}^{m} \frac{\tau_j}{z^j} + O\left( \frac{1}{|z|^{m+1}} \right) \quad (m \in \mathbb{Z}_{\geq 0})$$

(8.6)

for certain coefficients $\tau_j$. A computation shows

$$F'(x) = \frac{1}{2x^{1/2}} - \frac{3}{4x^{3/2}} + \frac{3}{16x^{5/2}} + \frac{3}{16x^{7/2}} + \cdots + \frac{c_R}{x^{R/2}} + O\left( \frac{1}{x^{(R+1)/2}} \right),$$

(8.6)

as $x \to \infty$. Use (8.2) to check that $G'(z) = 1 + G(z)/z - G(z)^2$. Consequently, all the derivatives of $F(z)$ may be expressed using powers of $G(z)$. Lemma 6.1 now implies that the repeated derivatives of both sides of (8.6) agree. Comparing (1.8) and (8.4) shows

$$A(n) = \frac{k'}{2!} F'(T) = \frac{k'}{2\sqrt{T}} + O\left( \frac{1}{T} \right),$$

(8.7)

$$-\delta(n)^2 = \frac{(k')^2}{2!} F''(T) = -\frac{(k')^2}{8T^{3/2}} + O\left( \frac{1}{T^2} \right),$$

(8.8)

g_{r}(n) = \frac{(k')^r}{r!} F^{(r)}(T) = O\left( \frac{1}{n^{r-1/2}} \right) \quad (r \geq 3),$$

(8.9)
In terms of \( n \) we obtain the estimates
\[
A(n) = \frac{\sqrt{k'}}{2\sqrt{n}} + O \left( \frac{1}{n} \right), \quad \delta(n) = \frac{(k')^{1/4}}{8^{1/2} n^{3/4}} \left( 1 + O \left( \frac{1}{n^{1/2}} \right) \right),
\]
(8.10)
\[
g_r(n) = \left( \frac{1}{2} \right)^{1/2} \frac{\sqrt{k'}}{n^{r-1/2}} + O \left( \frac{1}{n^r} \right).
\]
(8.11)

The conditions of Definition 1.3 and Theorems 1.6, 2.1 are seen to hold and the theorem follows.

Theorem 1.1 of [CP21] claims that the sequence \( b_k(n) \) is Hermite-Jensen (for the \( \lambda_{d,n} \) polynomials). There are several problems with the proof in [CP21 Sect. 2], the main one being that they try to use the misstated version of [GORZ19 Thm. 3]. To establish that \( b_k(n) \) is \( n \)-log-polynomial, their version of (8.4) has
\[
\log \left( \frac{b_k(n + j)}{b_k(n)} \right) = A^*(n) j - \delta^*(n)^2 j^2 + \sum_{r=3}^{m} g_r^*(n) j^r + O \left( \frac{1}{n^{m+1/2}} \right)
\]
for \( g_k^*(n), \ldots, g_m^*(n) \) all identically 0. But these \( g_r^*(n) \)s cannot be 0 since Proposition 1.4 and (3.17) demonstrate that \( g_r^*(n) \) and \( g_r(n) \), estimated in (8.11), must agree up to the error \( o(\delta^*(n)^m) \) which becomes arbitrarily small for large \( m \). Nevertheless, Theorem 1.1 of [CP21] is stated correctly because their expressions for \( A^*(n) \) and \( \delta^*(n) \) match ours in (8.7), (8.8), (8.10) to within the error allowed by Theorem 2.1.

9 Gamma sequences

In this final section we see how an application of Theorems 1.6 and 2.1 to the simple sequence \( \alpha(n) := \Gamma(n + \beta) \) of gamma values leads naturally to the classical limit for generalized Laguerre polynomials
\[
\left( \frac{2}{r} \right)^{d/2} L_d^{(r)}(r + \sqrt{2r} X) \to H_d(-X) \frac{H_d(1)}{d!} \quad \text{when } r \to \infty,
\]
(9.1)
as in [AAR99 Eq. (6.2.14)]. Set \( L_d^{(r)}(X) := X^d L_d^{(r)}(1/X) \), and we also obtain a reciprocal version of (9.1) that we have not found in the literature:
\[
(2r)^{d/2} L_d^{(r)} \left( \frac{1}{r} + \sqrt{\frac{2}{r}} \right) \to H_d(X) \frac{H_d(1)}{d!} \quad \text{when } r \to \infty.
\]
(9.2)

Start by putting \( n' := n + \beta \) where we may assume \( 0 \leq \beta \leq 1 \). Also set
\[
T := n', \quad \lambda := j/n', \quad F(z) := \log \Gamma(z).
\]
Then
\[
\log(\alpha(n + j)) = F(T(1 + \lambda)).
\]

Let \( m \geq 2 \) be a fixed integer. Using Stirling’s formula to bound \( F \), and Lemma 6.2 yields the expansion
\[
F(T(1 + \lambda)) = \sum_{r=0}^{m} \frac{T^r}{r!} F^{(r)}(T) \cdot \lambda^r + O(n \log(n) |\lambda|^{m+1})
\]
and hence, for \( j \) in the range we want \( 1 \leq j \leq m + 1 \),
\[
\log \left( \frac{\alpha(n + j)}{\alpha(n)} \right) = \sum_{r=1}^{m} \frac{1}{r!} F^{(r)}(T) \cdot j^r + O \left( \frac{\log n}{n^m} \right)
\]
(9.3)
as \( n \to \infty \) for an implied constant depending on \( m \). The first derivative is the digamma function,
\[
F'(z) = \frac{\Gamma'(z)}{\Gamma(z)} = \psi(z),
\]
with a well-known asymptotic expansion involving Bernoulli numbers \[6.3.18\]:

\[ F'(z) = \psi(z) = \log z - \frac{1}{2z} - \sum_{j=2}^{m} \frac{B_j}{jz^j} + O\left(\frac{1}{|z|^{m+1}}\right). \]

Its derivatives are the polygamma functions and with \[6.4.11\] they have the expansions for \( r \geq 2 \)

\[ F^{(r)}(z) = \psi^{(r-1)}(z) = (-1)^r \left[ \frac{(r-2)!}{z^{r-1}} + \frac{(r-1)!}{2z^r} + \sum_{j=2}^{m} \frac{(j+r-2)!B_j}{j!z^{j+r-1}} + O\left(\frac{1}{|z|^{m+r}}\right) \right]. \]

From (9.3) we obtain

\[ A(n) = \frac{1}{1!} F'(T) = \psi(T) = \log T + O\left(\frac{1}{T}\right), \quad (9.4) \]

\[ \delta(n)^2 = \frac{1}{2!} F''(T) = \frac{\psi^{(1)}(T)}{2} = \frac{1}{2T} + O\left(\frac{1}{T^2}\right), \quad (9.5) \]

\[ g_k(n) = \frac{1}{k!} F^{(k)}(T) = \frac{\psi^{(k-1)}(T)}{k!} = O\left(\frac{1}{n^{k-1}}\right) \quad (k \geq 3). \quad (9.6) \]

Therefore

\[ A(n) = \log(n + \beta) + O\left(\frac{1}{n}\right), \quad \delta(n) = \frac{1}{\sqrt{2(n + \beta)}} \left(1 + O\left(\frac{1}{n}\right)\right), \quad (9.7) \]

Checking the conditions of Definition \[1.3\] and Theorems \[1.6\] \[2.1\] we have proved

**Theorem 9.1.** The sequence \( \alpha(n) = \Gamma(n + \beta) \) is log-polynomial for all degrees \( m \geq 2 \) with data \( \{\psi(n + \beta), 1, \left(\psi^{(1)}(n + \beta)\right)/2\}^{1/2} \). It is Hermite-Jensen for this data as well as the simpler

\( \{\log(n + \beta), 1, (2(n + \beta))^{-1/2}\} \).

The Jensen polynomials for the reciprocal sequence \( 1/\alpha(n) \) are essentially the generalized Laguerre polynomials we saw in (5.10):

\[ J^{d,n}_1/\alpha(X) := \sum_{j=0}^{d} \binom{d}{j} \frac{1}{\Gamma(n + j + \beta)} X^j = \frac{d!}{\Gamma(n + d + \beta)} L^{(n+\beta-1)}_d(-X). \]

This sequence is Hermite-Jensen for the data

\[ \{-\log(n + \beta), -1, (2(n + \beta))^{-1/2}\} \]

by part (ii) of Proposition \[1.4\] and for each \( \beta \) we obtain

\[ \lim_{n \to \infty} \frac{(2(n + \beta))^{d/2}}{d!} \frac{\Gamma(n + \beta)}{\Gamma(n + \beta + d)} L^{(n+\beta-1)}_d \left((n + \beta) \left(1 - \frac{X}{\sqrt{2(n + \beta)}}\right)\right) = \frac{H_d(X/2)}{d!}. \quad (9.8) \]

**Proposition 9.2.** When \( r \to \infty \) we have

\[ (2r)^{d/2} \frac{\Gamma(r)}{\Gamma(r + d)} L^{(r-1)}_d \left(r + \sqrt{2r}X\right) \to \frac{H_d(-X)}{d!}, \quad (9.9) \]

\[ (2r^3)^{d/2} \frac{\Gamma(r)}{\Gamma(r + d)} L^{(r-1)}_d \left(\frac{1 + \sqrt{2}X}{r^{3/2}}\right) \to \frac{H_d(X)}{d!}, \quad (9.10) \]

where \( L^{(r-1)}_d(x) \) is the reciprocal version of \( L^{(r-1)}_d(x) \), defined after (9.1).
Proof. Let \( r = n + \beta \) and replace \( X \) by \(-2X\) on the left side of (9.8) to get the left side of (9.9). It can be easily checked that the coefficient of \( X^{d-j} \) in this expression is \( r^{j/2} \) times a rational function of \( r \). Therefore, if the limit (9.9) holds for \( r \) in the subsequence of integers it must hold for all \( r \to \infty \). Similarly, for the reciprocal Jensen version with \( K_{1,n}^{d,n} \) in (1.14), we obtain (9.10).

It follows from Proposition 9.2 that the zeros of \( L_d^{(r)}(x) \) are real for \( r \) sufficiently large. That is already clear in this case since \( L_d^{(r)}(x) \) is orthogonal for \( r \geq -1 \).

The identity

\[
L_d^{(r-1)}(x) = L_d^{(r)}(x) - L_{d-1}^{(r)}(x)
\]

from [AS64, Eq. (22.7.30)] may be used to show that (9.9) and (9.1) are really equivalent. For this, write

\[
L_d^{(r)}(x + \sqrt{2r}X) = \sum_{j=0}^{d} \ell_{d,j}(r)X^j, \quad \frac{H_d(-X)}{d!} = \sum_{j=0}^{d} h_{d,j}X^j.
\]

Then (9.9) is true if and only if

\[
(2r)^{d/2} \frac{\Gamma(r)}{\Gamma(r + d)} \left[ \ell_{d,r}(r) - \ell_{d-1,r}(r) \right] \to h_{d,j}
\]

for \( j = 0, 1, \ldots, d \) as \( r \to \infty \). Since

\[
\frac{\Gamma(r)}{\Gamma(r + d)} = \frac{1}{(r + d - 1)(r + d - 2) \cdots (r)} = \frac{1}{r^d} \left( 1 + O\left( \frac{1}{r} \right) \right),
\]

we see that (9.11) is equivalent to

\[
\left( \frac{2}{r} \right)^{d/2} \ell_{d,j}(r) - \left( \frac{2}{r} \right)^{1/2} \left[ \left( \frac{2}{r} \right)^{(d-1)/2} \ell_{d-1,j}(r) \right] \to h_{d,j}
\]

for \( j = 0, 1, \ldots, d \). Now, (9.12) being true for every positive integer \( d \) is equivalent to

\[
\left( \frac{2}{r} \right)^{d/2} \ell_{d,j}(r) \to h_{d,j} \quad \text{for} \quad j = 0, 1, \ldots, d
\]

being true for every positive integer \( d \) because the \( d = 0 \) case of (9.13) is just \( 1 \to 1 \). It follows that (9.9) can be used to prove (9.11), and vice versa. In a similar way, (9.10) and (9.2) are equivalent.

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