THE FOURTH MOMENT OF THE ZETA-FUNCTION

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ABSTRACT. An overview of results and problems concerning the asymptotic formula for $\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt$ is given, together with a discussion of modern methods from spectral theory used in recent work on this subject.

1. INTRODUCTION

A central place in analytic number theory is occupied by the Riemann zeta-function $\zeta(s)$, defined for $\Re s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1},$$

and otherwise by analytic continuation. It admits meromorphic continuation to the whole complex plane, its only singularity being the simple pole $s = 1$ with residue 1. From the functional equation (see [9], [32])

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s),$$

which is valid for any complex $s$, it follows that $\zeta(s)$ has zeros at $s = -2, -4, \ldots$. These zeros are traditionally called the “trivial” zeros of $\zeta(s)$, to distinguish them from the complex zeros of $\zeta(s)$, of which the smallest ones (in absolute value) are $\frac{1}{2} \pm 14.134725\ldots i$. It is well-known that all complex zeros of $\zeta(s)$ lie in the so-called “critical strip” $0 < \sigma = \Re s < 1$, and the Riemann Hypothesis (RH for short) is the conjecture, stated in 1859 by B. Riemann [31], that very likely all complex zeros of $\zeta(s)$ have real parts equal to $\frac{1}{2}$. For this reason the line $\sigma = \frac{1}{2}$ is called the “critical line” in the theory of $\zeta(s)$. The RH is extensively discussed by many authors, and recently by E. Bombieri [3].

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The RH is now probably the most celebrated and difficult open problem in whole Mathematics. Its proof (or disproof) would have very important consequences in multiplicative number theory, especially in problems involving the distribution of primes.

2. Mean values of $|\zeta(\frac{1}{2} + it)|$

Mean values of $|\zeta(\frac{1}{2} + it)|$ are fundamental in the theory and applications of $\zeta(s)$ (see [11] for an extensive account). For $k \geq 1$ a fixed integer let

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt = T P_{k^2}(\log T) + E_k(T),$$

where for some suitable coefficients $a_{j,k}$ one has

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_{j,k} y^j,$$

and in particular

$$P_1(y) = y + 2\gamma - 1 - \log(2\pi),$$

where $\gamma = -\Gamma'(1) = 0.577\ldots$ is Euler’s constant. One hopes that

$$E_k(T) = o(T) \quad (T \to \infty)$$

will hold for every fixed integer $k \geq 1$, but so far this is known to be true only in the cases $k = 1$ and $k = 2$, when $E_k(T)$ is a true error term in (2.1). When $k \geq 3$ it is not even known what should be the values of the coefficients $a_{j,k}$ in (2.2). The connection between $E_k(T)$ and the RH is indirect, namely there is a connection with the Lindelöf hypothesis (LH for short). The LH is also a famous unsettled problem, and it states that ($f \ll g$ means $|f(x)| < Cg(x)$ for some $C > 0$ and $x \geq x_0)$

$$\zeta(\frac{1}{2} + it) \ll \varepsilon t^\varepsilon$$

for any given $\varepsilon > 0$ and $t \geq t_0 > 0$ (since $\zeta(\frac{1}{2} + it) = \overline{\zeta(\frac{1}{2} - it)}$, $t$ may be assumed to be positive). The RH implies LH, in fact it even gives (see [32]) an estimate stronger than (2.3), namely

$$\zeta(\frac{1}{2} + it) \ll \exp\left(\frac{A \log t}{\log \log t}\right) \quad (A > 0, t \geq t_0).$$
It is yet unknown whether the LH implies the RH. The best current unconditional bound for the order of $\zeta(\frac{1}{2} + it)$, which is far from the LH, is due to M.N. Huxley [7]. This is

$$\zeta(\frac{1}{2} + it) \ll \varepsilon t^{c + \varepsilon}, \quad c = \frac{32}{205} = 0.156098 \ldots.$$ 

Huxley [7] also proved

$$E_1(T) \ll \varepsilon T^{\frac{137}{432} + \varepsilon}, \quad \frac{137}{432} = 0.31713 \ldots.$$ 

The LH is equivalent to the bound

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} \, dt \ll_{k, \varepsilon} T^{1 + \varepsilon}$$

for every $k \geq 1$ and any $\varepsilon > 0$, which in turn is the same as

$$E_k(T) \ll_{k, \varepsilon} T^{1 + \varepsilon} \quad (k \in \mathbb{N}).$$

We have $\Omega$-results in the case $k = 1, 2$, which show that $E_1(T)$ and $E_2(T)$ cannot be always small. J.L. Hafner and the author [4] proved that

$$E_1(T) = \Omega_+ \left( (T \log T)^{\frac{1}{4}} (\log \log T)^{\frac{3 + \log 4}{4}} e^{-C \sqrt{\log \log \log T}} \right)$$

and

$$E_1(T) = \Omega_- \left( T^{\frac{1}{4}} \exp \left( \frac{D (\log \log T)^{\frac{1}{4}}}{(\log \log \log T)^{\frac{1}{4}}} \right) \right)$$

for some absolute constants $C, D > 0$. Moreover, the author [10] proved that there exist constants $A, B > 0$ such that, for $T \geq T_0$, every interval $[T, T + B\sqrt{T}]$ contains points $t_1, t_2$ for which

$$E_1(t_1) > At_1^{\frac{1}{2}}, \quad E_1(t_2) < -At_2^{\frac{1}{2}}.$$ 

($f = \Omega_+(g)$ means that $\limsup f/g > 0$, $f = \Omega_-(g)$ means that $\liminf f/g < 0$).

3. **The fourth moment of $|\zeta(\frac{1}{2} + it)|$**

The asymptotic formula for the fourth moment of $\zeta(s)$ on the critical line is customarily written as (this is (2.1) and (2.2) when $k = 2$

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = TP_4(\log T) + E_2(T), \quad P_4(x) = \sum_{j=0}^4 a_j x^j.$$
A classical result of A.E. Ingham [8] from 1926 is that \( a_4 = 1/(2\pi^2) \) and that the error term \( E_2(T) \) satisfies the bound \( E_2(T) \ll T\log^3 T \). In 1979 D.R. Heath-Brown [6] made significant progress in this problem by proving that

\[
E_2(T) \ll \varepsilon T^{\frac{7}{8} + \varepsilon}.
\]

He also calculated

\[
a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}
\]

and produced more complicated expressions for \( a_0, a_1 \) and \( a_2 \). The author [12] made an explicit evaluation of \( a_0, a_1, a_2 \). For \( \Re s > 1 \)

\[
\zeta^2(s) = \sum_{n=1}^{\infty} d(n)n^{-s},
\]

where \( d(n) \) is the sum of positive divisors of \( n \). Note that \( \zeta^4 = \zeta^2 \cdot \zeta^2 \) and that \( \zeta^2 \) can be approximated by a finite sum \( \sum_n d(n)n^{-s} \). Therefore by integrating

\[
|\zeta(\frac{1}{2} + it)|^4 = \zeta^2(\frac{1}{2} + it)\zeta^2(\frac{1}{2} - it)
\]

we are led to the asymptotic evaluation of the sum

\[
\sum_{n \leq x} d(n)d(n + f),
\]

where \( 1 \leq f \leq x \) is not fixed. This is the so-called binary additive divisor problem (see [20], [22] and [27]). Modern approaches both to the study of \( E_2(T) \) and the binary additive divisor problem involve the use of spectral theory.

4. Spectral theory

For a competent account on spectral theory and its applications to \( \zeta(s) \) the reader is referred to Y. Motohashi's monograph [30]. Here we shall only briefly present some basic facts. On the upper complex half-plane \( \mathbb{H} \) the modular group

\[
\Gamma = SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d \in \mathbb{Z}) \land (ad - bc = 1) \right\}
\]

acts by \( \gamma z = (az + b)/(cz + d) \) if \( \gamma \in \Gamma \). The non-Euclidean Laplace operator

\[
L = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]
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is invariant under $\Gamma$, as is the measure $d\mu(z) = y^{-2}dx
dy (z = x + iy)$. Non-holomorphic cusp forms (so-called Maass wave forms) are eigenfunctions $\Psi(z)$ of the discrete spectrum of $L$, which has the form

$$\{\lambda_j\}_{j=1}^{\infty}, \quad \lambda_j = \kappa_j^2 + \frac{1}{4} \quad (\kappa_j > 0).$$

They satisfy $L\Psi(z) = \lambda \Psi(z)$, $\Psi(\gamma z) = \Psi(z)$ for $\gamma \in \Gamma$, and the finiteness condition

$$\int_{\mathcal{D}} |\Psi(z)|^2 \, d\mu(z) = \int_{\mathcal{D}} |\Psi(x + iy)|^2 y^{-2} \, dx \, dy < \infty.$$

Here $\mathcal{D}$ is the fundamental domain of $\Gamma$, namely

$$\mathcal{D} = \{ z : y > 0, |z| > 1, -\frac{1}{2} \leq x \leq \frac{1}{2} \} \cup \{ z : |z| = 1, -\frac{1}{2} \leq x \leq 0 \}.$$

Let henceforth $\varphi_j$ be the Maass wave form attached to $\kappa_j$ so that $\{\varphi_j\}_{j=1}^{\infty}$ forms an orthonormal basis with respect to the Petersson inner product

$$(f_1, f_2) := \int_{\mathcal{D}} f_1 \bar{f_2} \, d\mu(z),$$

and $\varphi_j$ is an eigenfunction of every Hecke operator. The Hecke operator $T_n$ acts on $\mathbb{H}$, for given $n \in \mathbb{N}$, by the relation

$$(T_n f)(z) = n^{-1/2} \sum_{ad-n, d>0} \sum_{b(\text{mod} \, d)} f \left( \frac{az + b}{d} \right).$$

We have the Fourier expansion

$$\varphi_j(z) = \sum_{n \neq 0} \rho_j(n) e^{2\pi ix} \sqrt{y} K_{i\kappa_j}(2\pi |n|y)$$

for $z = x + iy$, $\rho_j(n) = \rho_j(-n)$, where $K$ is the Bessel function (the Mcdonald function)

$$K_s(z) = \frac{1}{2} \int_0^\infty t^{s-1} \exp \left( -\frac{1}{2} z(t + \frac{1}{t}) \right) \, dt \quad (\Re z > 0).$$

For $n \in \mathbb{N}$, $t_j(n)$ is the eigenvalue corresponding to $\varphi_j$ with respect to $T_n$,

$$T_n \varphi_j(z) = t_j(n) \varphi_j(z).$$

Then

$$\rho_j(1) t_j(n) = \rho_j(n).$$
The Hecke series attached to $\varphi_j(z)$ is
\[
H_j(s) := \sum_{n=1}^{\infty} t_j(n) n^{-s} = \prod_{p} (1 - t_j(p)p^{-s} + p^{-2s})^{-1} \quad (\Re s > 1).
\]

It is known that $H_j(s)$ continues analytically to an entire function over $\mathbb{C}$, and that for any $s$ satisfies the functional equation
\[
H_j(s) = \pi^{-1}(2\pi)^{2s-1}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \left\{-\cos(\pi s) + \varepsilon_j \cosh(\pi \kappa_j)\right\} H_j(1-s)
\]
with $\varepsilon_j = \pm 1$.

5. Motohashi’s explicit formula

Recent progress on the fourth moment of $|\zeta(\frac{1}{2}+it)|$ is primarily due to the fundamental explicit formula of Y. Motohashi (see [11], [25], [26], [30]) of 1989 for
\[
I(T, \Delta) := \frac{1}{\Delta \sqrt{\pi}} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it)|^4 e^{-t^2 \Delta^{-2}} \, dt,
\]
under the condition
\[
0 < \Delta \leq \frac{T}{\log T}.
\]

The presence of the Gaussian smoothing factor $e^{-x^2}$ enabled Motohashi to deal with various convergence problems occurring in the proof. In simplified form, the formula is
\[
I(T, \Delta) = \pi 2^{-1/2} T^{-1/2} \sum_{j=1}^{\infty} \alpha_j \kappa_j^{-1/2} H_j^3(\frac{1}{2}) \sin \left( \kappa_j \log \left( \frac{K_j}{4eT} \right) \right) e^{-(\Delta \kappa_j/2T)^2} + O(\log^{C} T).
\]

Here
\[
\alpha_j = \frac{|\rho_j(1)|^2}{\cosh(\pi \kappa_j)},
\]
and, for arbitrary fixed $A > 0$, $C = C(A) > 0$ with $\Delta$ satisfying
\[
\frac{\sqrt{T}}{\log^A T} \leq \Delta \leq T \exp(-\sqrt{\log T}).
\]

The proof of the formula for $I(T, \Delta)$ depends heavily on the use of spectral theory, especially the so-called Kuznetsov trace formulas, which relate sums of Kloosterman sums (see (6.4)) to certain sums involving quantities from spectral theory.
6. New results on the fourth moment

At present the bound
\[ \int_0^T |\zeta(\frac{1}{2} + it)|^k \, dt \ll \varepsilon T^{1+\varepsilon} \]
is not known to hold for any constant \( k > 4 \). Therefore the function \( E_2(T) \) is particularly important in the theory of mean values of \( \zeta(s) \). In recent years much advance has been made in connection with \( E_2(T) \) and related problems. One of the main problems is to get rid of the Gaussian smoothing factor \( e^{-x^2} \) in Motohashi’s formula (5.1) and apply it to obtain results on \( E_2(T) \) itself.

Motohashi and the author in four papers [17]–[20] obtained several results on \( E_2(T) \) and some related problems. It is known now that \( (f = \Omega(g) \text{ means that } \limsup |f|/g > 0) \)

\[ E_2(T) = O(T^{2/3} \log C_1 T), \quad E_2(T) = \Omega(T^{1/2}), \]

\[ \int_0^T E_2(t) \, dt = O(T^{3/2}), \quad \int_0^T E_2^2(t) \, dt = O(T^2 \log C_2 T), \]

with effective constants \( C_1, C_2 > 0 \) (the values \( C_1 = 8, C_2 = 22 \) are admissible).

Y. Motohashi [28] improved the omega-result to \( E_2(T) = \Omega_\pm(T^{1/2}) \).

Finally the author [15] made further progress in this problem by proving the following quantitative omega-result: there exist two constants \( A > 0, B > 1 \) such that for \( T \geq T_0 > 0 \) every interval \([T, BT]\) contains points \( T_1, T_2 \) for which

\[ E_2(T_1) > AT_1^{1/2}, \quad E_2(T_2) < -AT_2^{1/2}. \]

This follows from the asymptotic formula

\[ \int_0^\infty E_2(t)e^{-t/T} \, dt = 2T^{1/2} \Re \left\{ \sum_{j=1}^\infty \alpha_j H_j^3(\frac{1}{2}) R(\kappa_j) \Gamma(\frac{1}{2} - i\kappa_j) T^{-\kappa_j} \right\} \]

\[ + O\left(T^{3/2} \exp\left(-\frac{\log T}{\log \log T}\right) \right), \]

with \( \alpha_j \) given by (5.2) and

\[ R(y) := \sqrt{\frac{\pi}{2}} \left( \frac{2^{iy} \Gamma(\frac{1}{4} + \frac{i}{2}y)}{\Gamma(\frac{1}{4} - \frac{i}{2}y)} \right)^3 \Gamma(-2iy) \cosh(\pi y). \]
Y. Motohashi and the author [19] proved that

\[
\sum_{r=1}^{R} \int_{t_r}^{t_r+\Delta} |\zeta(\frac{1}{2}+it)|^4 \, dt \ll R\Delta \log^4 T + TR^{1/2} \Delta^{-1/2} \log^C T
\]

for some \( C > 0 \), where \( \log T \ll \Delta \ll \log T \) and

\[
T \leq t_1 < t_2 < \ldots < t_R \leq 2T, \quad t_{r+1} - t_r \geq \Delta \quad (r = 1, \ldots, R-1).
\]

From this result it is not difficult to obtain as a corollary the bound

\[
\int_{T}^{\infty} |\zeta(\frac{1}{2}+it)|^{12} \, dt = O(T^2 \log^B T),
\]

proved first (with \( B = 17 \)) by Heath-Brown [5] in 1978.

The asymptotic formula (6.1) for \( \int_0^\infty E_2(t)e^{-t/T} \, dt \) is a Laplace transform formula. Integral transforms such as the Mellin transform (see [13], [21]) and the Laplace transform (see [16], [23]) play an important rôle in analytic number theory, in particular in the theory of \( \zeta(s) \). One can consider the general function

\[
L^k(s) := \int_{0}^{\infty} |\zeta(\frac{1}{2}+ix)|^{2k}e^{-sx} \, dx \quad (k \in \mathbb{N}, \Re s > 0).
\]

A classical result of H. Kober [24] from 1936 says that, as \( \sigma \to 0^+ \),

\[
L_1(2\sigma) = \frac{\gamma - \log(4\pi\sigma)}{2\sin\sigma} + \sum_{n=0}^{N} c_n \sigma^n + O(\sigma^{N+1})
\]

for any given integer \( N \geq 1 \), where the \( c_n \)'s are effectively computable constants.

F.V. Atkinson [2] obtained in 1941 the asymptotic formula

(6.3) \[
L_2(\sigma) = \frac{1}{\sigma} \left( A \log^4 \frac{1}{\sigma} + B \log^3 \frac{1}{\sigma} + C \log^2 \frac{1}{\sigma} + D \log \frac{1}{\sigma} + E \right) + \lambda_2(\sigma),
\]

where \( \sigma \to 0^+ \),

\[
A = \frac{1}{2\pi^2}, \quad B = \pi^2(2\log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2}), \quad \lambda_2(\sigma) \ll \varepsilon \left( \frac{1}{\sigma} \right)^{\frac{13}{14} + \varepsilon}.
\]

He also indicated how, by the use of estimates for Kloosterman sums

(6.4) \[
S(m, n; c) := \sum_{1 \leq d < c, (d,c)=1, dd' \equiv 1 \pmod{c}} e \left( \frac{md + nd'}{c} \right) \quad (e(z) = e^{2\pi iz}),
\]
one can improve the exponent $\frac{13}{14}$ to $\frac{8}{9}$. This is important historically, in view of contemporary importance of Kloosterman sums which stems from the Kuznetsov trace formulas and other important applications.

The author [12] gave explicit, albeit complicated expressions for the remaining coefficients $C, D$ and $E$ in (6.3) and improved Atkinson’s bound for $\lambda_2(\sigma)$ to

$$\lambda_2(\sigma) \ll \sigma^{-1/2} \quad (\sigma \to 0+).$$

Recently in [16] he proved a result which generalizes and sharpens (6.3): Let $0 \leq \phi < \frac{\pi}{2}$ be given. Then for $0 < |s| \leq 1$ and $|\arg s| \leq \phi$ we have

$$L_2(s) = \frac{1}{s}(A \log^4 \frac{1}{s} + B \log^3 \frac{1}{s} + C \log^2 \frac{1}{s} + D \log \frac{1}{s} + E)$$

$$+ s^{-\frac{1}{2}} \left\{ \sum_{j=1}^{\infty} \alpha_j H_j^3(\frac{1}{2}) \left( s^{-i\kappa_j} R(\kappa_j) \Gamma(\kappa_j) + s^{i\kappa_j} R(-\kappa_j) \Gamma(-\kappa_j) \right) \right\} + G_2(s),$$

where the constants $A, \ldots, E$ are as in (6.3), $R(y)$ is given by (6.2) and in the above region $G_2(s)$ is a regular function satisfying $(C > 0$ is a suitable constant)

$$G_2(s) \ll |s|^{-\frac{3}{4}} \exp \left\{ -\frac{C \log(|s|^{-1})}{(\log \log(|s|^{-1}))^{\frac{3}{2}} (\log \log \log(|s|^{-1}))^{\frac{3}{2}}} \right\}. $$

In [14] the author proved: There exist constants $A \neq 0$ and $B > 1$ such that, for $T \geq T_0 > 0$, every interval $[T, BT]$ contains points $t_1, t_2$ for which

$$\int_0^{t_1} E_2(t) \, dt > A t_1^{3/2}, \quad \int_0^{t_2} E_2(t) \, dt < -A t_2^{3/2}.$$ 

This result implies that

$$\int_0^{T} E_2(t) \, dt = \Omega_\pm (T^{3/2}),$$

and that

$$\limsup_{n \to \infty} \frac{\log(u_{n+1} - u_n)}{\log u_n} \leq 1,$$

where $u_n$ is the $n$-th zero of $E_2(T)$. It can also be used to prove the lower bound result

$$\int_0^T E_2^2(t) \, dt \gg T^2,$$
which complements the earlier result with Motohashi (cf. [18]) that

\[ \int_0^T E_2^2(t) \, dt \ll T^2 \log^C T \quad (C = 22). \]

Another recent result of the author [15], which provides an asymptotic formula for the integral of \( E_2(t) \), is

\[ \int_0^T E_2(t) \, dt = O\{T^{3/2} \exp(-C(\log T)^{5/3}(\log \log T)^{-1/3})}\} +
\]

\[ + 2T^{3/2} \Re \left\{ \sum_{j=1}^\infty \alpha_j H_j^3(\frac{1}{2}) \frac{T \kappa_j}{(\frac{1}{2} + i \kappa_j)(\frac{3}{2} + i \kappa_j)} R(\kappa_j) \right\}. \]

Here \( C > 0 \) is a suitable constant, and the error term depends on the best known zero-free region for \( \zeta(s) \) (see [9, Chapter 6]).

In spite of significant recent results on \( E_2(T) \), many open problems remain. Here are four of them.

**Problem 1.** Does there exist \( A > 0 \) such that, as \( T \to \infty \),

\[ \int_0^T E_2^2(t) \, dt \sim AT^2? \]

**Problem 2.** Is it true that for every \( \varepsilon > 0 \)

\( E_2(T) = O_{\varepsilon}(T^{3/2 + \varepsilon})? \)

**Problem 3.** Does one have

\[ \limsup_{T \to \infty} |E_2(T)|T^{-\frac{1}{2}} = \infty? \]

**Problem 4.** Does one have

\[ \limsup_{n \to \infty} \frac{\log(u_{n+1} - u_n)}{\log u_n} < 1? \]
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