LONG TIME DECAY FOR GLOBAL SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN SOBOLEV-GEVERY SPACES

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Abstract. In this paper, we prove that if $u \in C([0, \infty), \dot{H}^{1/2}_{1/a}(\mathbb{R}^3))$ is a global solution of 3D incompressible Navier-Stokes equations, then $\|u\|_{\dot{H}^{1/2}_{1/a}}$ decays to zero as time approaches infinity.

Fourier analysis and standard techniques are used.

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1. INTRODUCTION

The 3D generalized Navier-Stokes system is given by:

\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u &= Q(u, u) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^3,
\end{align*}
\hspace{1cm}(GNS)

with $Q$ is the bilinear operator defined as:

\begin{align*}
Q^j(u, v) &= \sum_{i = 1}^{3} q_{k,j}^{i,m} \partial_m (u^i v^j), \quad j = 1, 2, 3
\end{align*}
\hspace{1cm}(1.1)

where

\begin{align*}
q_{k,j}^{i,m} &= \sum_{n,p=1}^{3} a_{k,j}^{i,m,n,p} F\left(\frac{\xi_n \xi_p}{|\xi|^2} u(\xi)\right)
\end{align*}

and $a_{k,j}^{i,m,n,p}$ are real numbers.

The particular case of the above system is the Navier-Stokes system for incompressible fluid:

\begin{align*}
\frac{\partial u}{\partial t} - \nu \Delta u + u. \nabla u &= -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^3,
\end{align*}
\hspace{1cm}(NS)

Here $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ is the velocity field of fluid, $\nu > 0$ is the viscosity coefficient of fluid, and $p = p(t, x) \in \mathbb{R}$ denotes the unknown pressure of the fluid at the point.
(t, x) ∈ R⁺ × R³ and u⁰ = (u¹⁰, u²⁰, u³⁰) is the initial velocity. If the initial condition is regular, then the pressure p determined.

Many works are interested to study the global well-posedness of strong solutions for small initial data and the local well posedness for any initial data in different spaces: L², H¹/², BMO⁻¹ and B˙₁⁻∞,... For more studies in this spaces the reader may refer to [8], [13], [4]. In this paper, we interested to study the non blow-up result of the global solution of Navier-Stokes equations, which is studied by several researches: Gallagher, Iftimie and Planchon (2002) [9] proved that if u is a global solution of 3D Navier-Stokes equation, then limₜ→∞ ∥u(t)∥H¹/² = 0. In (2016), Benamer and Jiali [5] showed that ∥u∥H¹,a, σ > 0 approaches to zero at infinity. The purpose of this article is to establish the above result in the limit space ˙H¹/a(ℝ³):

\[ \lim_{t \to +\infty} \|u(t)\|_{H¹/a} = \lim_{t \to +\infty} \lim_{N \to +\infty} \sum_{k=0}^{N} \frac{2a}{k!} \|u(t)\|_{H¹/²} = 0. \]

For simplicity, we take ν = 1 for the rest of the paper. Now we state our results:

**Theorem 1.1.** Let a > 0. If \( u⁰ \in ˙H¹/a(ℝ³) \) such that \( \text{div } u⁰ = 0 \), then there exists time \( T \) such that (NS) has unique solution

\[ u \in C([0, T^*], ˙H¹/a(ℝ³)) \cap L²([0, T^*], ˙H³/²(ℝ³)). \]

**Remark 1.2.** If \( u \) is a solution of (NS) system in \( C([0, T^*], ˙H¹/a(ℝ³)) \), then \( u \in L²_{loc}(0, T^*], ˙H³/²(ℝ³)) \).

In the second theorem, we give a result of blow-up if the maximal time is finite, precisely:

**Theorem 1.3.** Let a > 0 and \( u \in C([0, T], ˙H¹/a(ℝ³)) \cap L²([0, T], ˙H³/²(ℝ³)) \) be a maximal solution of (NS) given by theorem 1.1. Then:

(i) If \( \|u(0)\|_{H¹/a} < \frac{1}{a} \), then \( T^* = +\infty \).

(ii) If \( T^* \) is finite, then \( \int_{0}^{T^*} \|u(t)\|_{H³/²} dt = +\infty \).

In the next theorem, we show that the norm of the global solution in ˙H¹/a goes to zero at infinity.

**Theorem 1.4.** Let a > 0, \( u \in C(0, +\infty, ˙H¹/a(ℝ³)) \) be a global solution of (NS), then we have:

\[ \lim_{t \to +\infty} \|u(t)\|_{H¹/a} = 0. \]

**Remark 1.5.** This Theorem implies a result of polynomial decay in the homogeneous Sobolev spaces \( ˙H^s(ℝ³) \), for \( s \geq \frac{1}{2} \), precisely we have:

\[ \|u(t)\|_{H^s} = o(t^{-\frac{s-\frac{1}{2}}{2}}), \quad t \to +\infty. \]

For the proof see Appendix.

In the last result, we give the stability of global solution of (NS) system.

**Theorem 1.6.** Let \( u \in C(0, +\infty, ˙H¹/a(ℝ³)) \) be a global solution of (NS), then for all \( v⁰ \in ˙H¹/a(ℝ³) \) such that

\[ \|v⁰ - u⁰\|_{H¹/a} ≤ \frac{1}{4} \int_{0}^{\infty} \|u(z)\|_{H¹/a} dz. \]

Then, Navier Stokes system starting by \( v⁰ \) has a global solution. Moreover, if \( v \) is the corresponding global solution, then, for all \( t \geq 0 \), we have:

\[ \|v(t) - u(t)\|_{H¹/a}^2 + \frac{ν}{2} \int_{0}^{t} \|v(s) - u(s)\|_{H¹/a} ds \leq \|v⁰ - u⁰\|_{H¹/a}^2 + \frac{ν}{2}\int_{0}^{\infty} \|u(s)\|_{H¹/a} ds. \]
This article is organized as follows: In section 2, we give some important preliminary results. Section 3, is devoted to prove the existence of solution in the critical Sobolev-Gevery spaces $\dot{H}_{a,1}^{1/2}$. Section 4, we show the blow-up result of maximal solution in $L^2([0,T^*),\dot{H}_{a,1}^{1/2})$. Section 5, we prove the non-blowup result in $\dot{H}_{a,1}^{1/2}$. Finally, we give the proof of the stability result for global solution in section 6.

2. Notations and Preliminary results

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as
  \[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix\cdot\xi)f(x)dx, \quad \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3. \]

- The inverse Fourier formula is
  \[ \mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi\cdot x)g(\xi)\xi d\xi, \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3. \]

- The convolution product of a suitable pair of function $f$ and $g$ on $\mathbb{R}^3$ is given by
  \[ (f \ast g)(x) := \int_{\mathbb{R}^3} f(y)g(x-y)dy. \]

- If $f = (f_1,f_2,f_3)$ and $g = (g_1,g_2,g_3)$ are two vector fields, we set
  \[ f \otimes g := (g_1f,g_2f,g_3f), \]
  and
  \[ \text{div} (f \otimes g) := (\text{div} (g_1f),\text{div} (g_2f),\text{div} (g_3f)). \]

- Let $(B,||.||)$, be a Banach space, $1 \leq p \leq \infty$ and $T > 0$. We define $L^p_T(B)$ the space of all measurable functions $[0,t] \ni t \mapsto f(t) \in B$ such that $t \mapsto ||f(t)|| \in L^p([0,T])$.

- The homogeneous Sobolev space;
  \[ \dot{H}^s = \{ f \in \mathcal{S}'(\mathbb{R}^3); \hat{f} \in L^1_{\text{loc}}, \text{ and } ||\xi||^s \hat{f} \in L^2(\mathbb{R}^3) \}. \]

- The Sobolev-Gevery spaces as follows; for $a, s \geq 0$ and $|D| = (-\Delta)^{1/2}$,
  \[ \dot{H}_{a,1}^s(\mathbb{R}^3) = \{ f \in L^2(\mathbb{R}^3); e^{a|D|}f \in \dot{H}^s \}, \]
  with the norm
  \[ ||f(t)||_{\dot{H}_{a,1}^s} = \left( \int_{\mathbb{R}^3} |\xi|^{2s} e^{2a|\xi|} |\hat{f}(t,\xi)|^2 d\xi \right)^{1/2}. \]

- We define also the following spaces;
  \[ \tilde{L}^\infty(\dot{H}^{1/2}) = \{ f \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^3); \int_{\mathbb{R}^3} ||\xi|| \left( \sup_{0 \leq t < \infty} |\hat{f}(t,\xi)| \right)^2 d\xi < \infty \}, \]
  with the norm
  \[ ||f||_{\tilde{L}^\infty(\dot{H}^{1/2})} = \left( \int_{\mathbb{R}^3} ||\xi|| \left( \sup_{0 \leq t < \infty} |\hat{f}(t,\xi)| \right)^2 d\xi \right)^{1/2} \]
  and
  \[ L^2(\dot{H}^{3/2}) = \{ f \in \mathcal{S}'(\mathbb{R}_+ \times \mathbb{R}^3); \int_{\mathbb{R}^3} \int_0^\infty |\xi|^{3/2} |\hat{f}(t,\xi)|^2 dt d\xi < \infty \}, \]
  with the norm
  \[ ||f||_{L^2(\dot{H}^{3/2})} = \left( \int_{\mathbb{R}^3} \int_0^\infty |\xi|^{3/2} |\hat{f}(t,\xi)|^2 dt d\xi \right)^{1/2}. \]
2.2. Preliminary results. In this section, we recall some classical results and we give new technical lemmas.

It’s well to know that:

- The homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$ are Banach spaces if and only if $s < \frac{3}{2}$.
- The Sobolev-Gevery spaces $\dot{H}^s_{a,1}(\mathbb{R}^3)$ are Banach spaces if and only if $s < \frac{3}{2}$ (See [1]).

**Lemma 2.1.** [1] Let $E$ be a Banach space, $B$ a continuous bilinear map from $E \times E \to E$, and a positive real number such that $\alpha < \frac{1}{2|\partial E|}$ with

$$||B|| = \sup_{||u||<1, ||v||<1} ||B(u, v)||$$

For any $a$ in the ball $B(0, a)$ in $E$, there exists a unique $x$ in $B(0, 2a)$ such that

$$x = a + B(x, x).$$

**Lemma 2.2.** [7] Let $(s, t) \in \mathbb{R}^2$ such that $s < 3/2$, $t < 3/2$ and $s + t > 0$, then there exists a constant $C = C(s, t) > 0$, such that for all $u \in \dot{H}^s_{a,1}(\mathbb{R}^3)$ and $v \in \dot{H}^s_{a,1}(\mathbb{R}^3)$, we have

$$||uv||_{\dot{H}^{s-t-\frac{1}{2}}_{a,1}} \leq C ||u||_{\dot{H}^s_{a,1}} ||v||_{\dot{H}^s_{a,1}}.$$ 

The following Lemmas are inspired by [8].

**Lemma 2.3.** Let $Q$ be a bilinear form as defined in [1,7]. Then, there exists a constant $C > 0$ such that for all $u, v \in \dot{H}^s_{a,1}(\mathbb{R}^3)$ we have:

$$||Q(u, v)||_{\dot{H}^{s-\frac{1}{2}}_{a,1}} \leq C ||u||_{\dot{H}^s_{a,1}} ||v||_{\dot{H}^s_{a,1}}.$$ 

**Proof.** Thanks to the inequality (2.1), we get:

$$||Q(u, v)||_{\dot{H}^{s-\frac{1}{2}}_{a,1}} \leq C \sup_{k, l} (||u^k \partial v^l||_{\dot{H}^{s-\frac{1}{2}}_{a,1}} + ||v^k \partial u^l||_{\dot{H}^{s-\frac{1}{2}}_{a,1}})$$

$$\leq C (||u||_{\dot{H}^s_{a,1}} ||v||_{\dot{H}^s_{a,1}} + ||v||_{\dot{H}^s_{a,1}} ||v||_{\dot{H}^s_{a,1}})$$

$$\leq 2C ||u||_{\dot{H}^s_{a,1}} ||v||_{\dot{H}^s_{a,1}}.$$ 

**Lemma 2.4.** Let $u$ be the solution in $C([0, T[, S')]$ of the Cauchy problem

$$\begin{cases}
\partial_t u - \Delta u = f \\
u(0) = u^0
\end{cases}$$

with $f \in L^2([0, T], \dot{H}^{-\frac{1}{2}}_{a,1})$ and $u^0 \in \dot{H}^{1/2}_{a,1}$. Then

$$u \in \bigcap_{r=0}^{\infty} L^r([0, T], \dot{H}^{\frac{3}{2}+\frac{\alpha}{r}}_{a,1}) \cap C([0, T], \dot{H}^{1/2}_{a,1}).$$

Moreover, we have the following estimates:

$$||u(t)||^2_{\dot{H}^{1/2}_{a,1}} + \int_0^t \|\nabla u(s)\|^2_{\dot{H}^{1/2}_{a,1}} ds \leq ||u^0||^2_{\dot{H}^{1/2}_{a,1}} + \int_0^t \|f(s)\|^2_{\dot{H}^{1/2}_{a,1}} ds$$

(2.3)

$$\left[ \int |\xi|^2 e^{2\pi |\xi|(|\sup_{0\leq t' \leq t} |\hat{u}(t', \xi)||^2)} \right]^{1/2} \leq \sqrt{2} ||u^0||_{\dot{H}^{1/2}_{a,1}} + \|f\|_{L^2([0, T], \dot{H}^{-1/2}_{a,1})}$$

(2.4)

$$\|u\|_{L^p_T(\dot{H}^{1/2+\alpha/2}_{a,1})} \leq \|u^0\|_{\dot{H}^{1/2}_{a,1}} + \|f\|_{L^2([0, T], \dot{H}^{-1/2}_{a,1})}$$

(2.5)

**Proof.** First inequality is given by the energy estimate. The proof of the second one is based around writing Duhamel’s formula in Fourier space, namely,

$$\hat{u}(t, \xi) = e^{-|\xi|^2 t} \hat{u}^0 - \int_0^t e^{-|\xi|^2 (t-s)} \hat{f}(s, \xi) ds.$$
Thanks to Cauchy-Schwartz inequality, we have:
\[
|\tilde{w}(t, \xi)| \leq |\tilde{w}^0(\xi)| + \int_0^t e^{-(t-s)}|\xi|^2 |\tilde{f}(s, \xi)| ds
\]
\[
\leq |\tilde{w}^0(\xi)| + \int_0^t e^{-2(t-s)}|\xi|^2 ds \frac{1}{2} |\tilde{f}(s, \xi)|^2 ds \frac{1}{2}
\]
\[
\leq |\tilde{w}^0(\xi)| + \frac{1}{\sqrt{2}|\xi|} \| |\tilde{f}(\xi, \cdot)| \|_{L^2_t(\mathbb{R}, H^1)}.
\]

For any \(0 < t < T\), we get:
\[
\sup_{0 \leq t' \leq t} |\tilde{w}(t', \xi)| \leq |\tilde{w}^0(\xi)| + \frac{1}{\sqrt{2}|\xi|} \| |\tilde{f}(\xi, \cdot)| \|_{L^2_t(\mathbb{R}, H^1)}.
\]

Multiplying the obtained equation by \(|\xi|^{1/2}e^{a|\xi|}\), we obtain
\[
|\xi|^{1/2}e^{a|\xi|} \sup_{0 \leq t' \leq t} |\tilde{w}(t', \xi)| \leq |\xi|^{1/2}e^{a|\xi|}|\tilde{w}^0(\xi)| + \frac{|\xi|^{1/2}e^{a|\xi|}}{\sqrt{2}|\xi|} \| |\tilde{f}(\xi, \cdot)| \|_{L^2_t(\mathbb{R}, H^1)}
\]
Taking the \(L^2\) norm with respect to the frequency variable \(\xi\), we conclude that:
\[
\left( \int |\xi|^{2a|\xi|} \sup_{0 \leq t' \leq t} |\tilde{w}(t', \xi)|^2 dt \right)^{1/2} \leq \| |\tilde{w}^0(\xi)| \|_{H^1_{a,1} \left[ 0, T \right]} + \| f \|_{L^2(\mathbb{R}, H^1)}
\]

Since, for almost all fixed \(\xi \in \mathbb{R}^3\), the map \(t \mapsto \tilde{w}(t, \xi)\) is continuous over \([0, T]\), the Lebesgue dominated convergence theorem ensures that \(v \in C([0, T]; H^1_{a,1}(\mathbb{R}^3))\).

Similarly, we have:
\[
|\xi|^{3/2}e^{a|\xi|}|\tilde{w}| \leq |\xi|^{3/2}e^{a|\xi|}|\tilde{w}^0| + \int_0^t \| |\tilde{w}^0(\xi)| \|_{H^1_{a,1} \left[ 0, T \right]} + \| f \|_{L^2(\mathbb{R}, H^1)}
\]

Taking the \(L^2\) norm with respect to time and using Young inequality, we obtain:
\[
\left( \int_0^t |\xi|^{3/2}e^{a|\xi|} |\tilde{w}(s, \xi)|^2 ds \right)^{1/2} \leq \left( \int_0^t |\xi|^{3/2}e^{a|\xi|} ds \right)^{1/2} \left( \int_0^t |\tilde{w}^0(\xi)|^2 |\tilde{w}| \right)^{1/2} + \left( \int_0^t |\tilde{w}^0(\xi)|^2 |\tilde{w}| \right)^{1/2}
\]

which yields,
(2.6)  \quad \| u(s) \|_{L^2_t(\mathbb{R}^3)} \leq \| u^0 \|_{H^1_{a,1}} + \| f \|_{L^2_t(\mathbb{R}^3)}

Finally, the last inequality follows by interpolation:
\[
\| u \|_{H^\frac{1}{2} \left[ 0, T \right]} \leq \| u \|_{H^1_{a,1}} \| u \|_{H^{3/2}_{a,1}}^{1/2},
\]
and
\[
\| u \|_{H^{\frac{1}{2}+\frac{3}{4}} \left[ 0, T \right]} \leq \| u \|_{H^1_{a,1}} \| u \|_{H^{3/2}_{a,1}}^{1/2}.
\]

Taking the \(L^1\) norm with respect to time and using the two estimation (2.4) and (2.6) we can deduce the last inequality. This completes the proof of Lemma [2.4].

The regularizing effects of the critical space \(\dot{H}^{1/2}(\mathbb{R}^3)\) of \((NS)\) equations gives us \(u \in \dot{H}^{1/2}(\mathbb{R}^3)\):

**Lemma 2.5.** There exists a positive constant \(\epsilon_0 > 0\) such that for any initial data in \(\dot{H}^{1/2}\) with \(\|u^0\|_{H^{1/2}} < \epsilon_0\) there exists a unique global in time solution \(u \in \tilde{L}^\infty(\dot{H}^{1/2}) \cap \tilde{L}^\infty(\dot{H}^{3/2})\) which is analytic in the sense that:
(2.7)  \quad \| e^{\sqrt{\nabla} D} \| u \|_{L^\infty(\dot{H}^{1/2})} + \| e^{\sqrt{\nabla} D} \| u \|_{L^\infty(\dot{H}^{3/2})} \leq \epsilon_0 \| u^0 \|_{H^{1/2}}

where \(e^{\sqrt{\nabla} D}\) is a Fourier multiplier whose symbol is given by \(e^{\sqrt{|\xi|}}\) and \(\epsilon_0\) is a universal constant.
Proof. The proof of Lemma 2.3 are inspired from the work of Bae in [2]. This proof is done in three steps.

We first take the Fourier transform to the integral form of Navier-Stokes equation:

\[
\hat{u}(t, \xi) = e^{-t|\xi|^2} \hat{u}^0 - \int_0^t e^{-(t-s)|\xi|^2} \hat{f}(s, \xi) ds.
\]

**Step 1:** we start by estimating \( u \) in \( \tilde{L}^{\infty}(\tilde{H}^{1/2}) \). Multiplying (2.8) by \( |\xi| \), we get:

\[
|\xi|^{1/2} \hat{u}(t, \xi) \leq |\xi|^{1/2} |\hat{u}^0(\xi)| + \int_0^t e^{-(t-s)|\xi|^2} |\xi|^{3/2} |\hat{u} \otimes u(s, \xi)| ds
\]

\[
\leq |\xi|^{1/2} |\hat{u}^0(\xi)| + \sup_{0 \leq t < \infty} \int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} |\xi|^{-1/2} |\hat{u} \otimes u(s, \xi)| ds
\]

\[
\leq |\xi|^{1/2} |\hat{u}^0(\xi)| + \int_0^t |\xi|^2 e^{-(t-s)|\xi|^2} ds \sup_{0 \leq t < \infty} |\xi|^{-1/2} |\hat{u} \otimes u(t, \xi)|
\]

\[
\leq |\xi|^{1/2} |\hat{u}^0| + |\xi|^{-1/2} \sup_{0 \leq t < \infty} |\hat{u} \otimes u(t, \xi)|.
\]

Taking the \( L^2 \) norm with respect to the frequency variable \( \xi \), we obtain:

\[
\|u\|_{\tilde{L}^{\infty}(\tilde{H}^{1/2})} \leq \|u^0\|_{\tilde{H}^{1/2}} + C_{\tilde{\beta}, \tilde{\gamma}} \|u\|_{\tilde{L}^{\infty}(\tilde{H}^{1/2})}
\]

Now, we estimate \( u \) in \( \tilde{L}^2(\tilde{H}^{3/2}) \). Multiplying (2.8) by \( |\xi|^3 \), we get:

\[
|\xi|^{3/2} \hat{u}(t, \xi) \leq |\xi|^{3/2} |\hat{u}^0(\xi)| + \int_0^t |\xi|^{3/2} e^{-(t-s)|\xi|^2} \|\hat{u} \otimes u(s, \xi)\| ds
\]

Taking the \( L^2 \) norm with respect to time and using Young’s inequality, we deduce:

\[
\left( \int_0^t \|\xi|^{3/2} \hat{u}(t, \xi) \|^2 dt \right)^{1/2} \leq \left( \int_0^t |\xi|^2 e^{-t|\xi|^2} \|\hat{u}^0(\xi)\|^2 ds \right)^{1/2}
\]

\[
+ \left( \int_0^t \left( \int_0^t |\xi|^{3/2} e^{-(t-s)|\xi|^2} \|\hat{u} \otimes u(s, \xi)\| ds \right)^2 dt \right)^{1/2}
\]

\[
\leq |\xi|^{1/2} |\hat{u}^0(\xi)| + \int_0^t |\xi|^2 e^{-t|\xi|^2} ds \left( \int_0^t \|\hat{u} \otimes u(s, \xi)\|^2 ds \right)^{1/2}
\]

\[
\leq |\xi|^{1/2} |\hat{u}^0(\xi)| + \left( \int_0^\infty \|\hat{u} \otimes u(s, \xi)\|^2 ds \right)^{1/2}.
\]

Taking \( L^2 \) norm in \( \xi \) and using Lemma 2.2 and Young’s inequality, we obtain:

\[
\|u\|_{\tilde{L}^2(\tilde{H}^{1/2})} \leq \|u^0\|_{\tilde{H}^{1/2}} + \left( \int_0^\infty \int_{\mathbb{R}^3} |\xi| \|\hat{u}(s, \xi)\|^2 d\xi ds \right)^{1/2}
\]

\[
\leq \|u^0\|_{\tilde{H}^{1/2}} + C_{\tilde{\beta}, \tilde{\gamma}} \left( \int_0^\infty \|u(s)\|_{\tilde{H}^{3/2}} \|u(s)\|_{\tilde{H}^{1/2}} ds \right)^{1/2}
\]

\[
\leq \|u^0\|_{\tilde{H}^{1/2}} + C_{\tilde{\beta}, \tilde{\gamma}} \left( \int_0^\infty \|u(s)\|_{\tilde{H}^{3/2}} ds \sup_{0 \leq t < \infty} \int_{\mathbb{R}^3} |\hat{u}(t, \xi)| d\xi \right)^{1/2}
\]

\[
\leq \|u^0\|_{\tilde{H}^{1/2}} + C_{\tilde{\beta}, \tilde{\gamma}} \|u\|_{\tilde{L}^2(\tilde{H}^{3/2})} ds \left( \int_{\mathbb{R}^3} |\xi| \sup_{0 \leq t < \infty} |\hat{u}(t, \xi)|^2 d\xi \right)^{1/2},
\]

which yields,

\[
\|u\|_{\tilde{L}^2(\tilde{H}^{1/2})} \leq \|u^0\|_{\tilde{H}^{1/2}} + C_{\tilde{\beta}, \tilde{\gamma}} \|u\|_{\tilde{L}^2(\tilde{H}^{3/2})} \|u\|_{\tilde{L}^{\infty}(\tilde{H}^{1/2})}
\]

**Step 2:** Combining (2.9) and (2.10), we get:

\[
\|u\|_{\tilde{L}^{\infty}(\tilde{H}^{1/2})} + \|u\|_{\tilde{L}^2(\tilde{H}^{1/2})} \leq 2\|u^0\|_{\tilde{H}^{1/2}} + C(\|u\|_{\tilde{L}^{\infty}(\tilde{H}^{1/2})} + \|u\|_{\tilde{L}^2(\tilde{H}^{3/2})})^2,
\]
with \( C = C\frac{1}{4} + C\frac{1}{4} \).

Let \( 0 < \epsilon_0 < C_0 \) such that \( \| u^0 \|_{H^{1/2}} < \epsilon_0 \), with \( C_0 = \frac{3}{16C} \min \left( \frac{1}{4C}, \frac{1}{4} \right) \).

Let \( \frac{16C\epsilon_0}{3} < r < \min \left( \frac{1}{4C}, \frac{1}{4} \right) \), we take:

\[
B_r = \{ u \in \tilde{L}^\infty([0, \infty[, H^{1/2}) \cap L^2([0, \infty[, H^{3/2})/ \| u \|_{\tilde{L}^\infty(H^{1/2})} + \| u \|_{L^2(H^{3/2})} \leq r \}.
\]

We consider the application \( \psi \) defining by:

\[
\psi(u) = e^{t\Delta u} - \int_0^t e^{(t-s)\Delta} \mathcal{P}(\text{div}(u \otimes u)(s))ds.
\]

Then we have:

\[
\| \psi(u) \|_{\tilde{L}^\infty(H^{1/2})} + \| \psi(u) \|_{L^2(H^{3/2})} \leq \| u^0 \|_{H^{1/2}} + C(\| u \|_{\tilde{L}^\infty(H^{1/2})} + \| u \|_{L^2(H^{3/2})})^2,
\]

which yields

\[
\| u^0 \|_{H^{1/2}} < \epsilon_0 < \frac{3r}{16C}.
\]

Finally, we get:

\[
\| \psi(u) \|_{\tilde{L}^\infty(H^{1/2})} + \| \psi(u) \|_{L^2(H^{3/2})} \leq r
\]

then

\[
\psi(B_r) \subset B_r.
\]

We have, for all \( u_1, u_2 \in B_r \):

\[
\| \psi(u_1) - \psi(u_2)\|_{\tilde{L}^\infty(H^{1/2})} \leq \| B(u_1 - u_2, u_1) + B(u_2, u_1 - u_2)\|_{\tilde{L}^\infty(H^{1/2})}
\]

\[
\leq 2C(\| u_1 \|_{\tilde{L}^\infty(H^{1/2})} + \| u_2 \|_{\tilde{L}^\infty(H^{1/2})})\| u_1 - u_2\|_{\tilde{L}^\infty(H^{1/2})}
\]

\[
\leq 2Cr\| u_1 - u_2\|_{\tilde{L}^\infty(H^{1/2})}
\]

\[
\leq \frac{1}{2}\| u_1 - u_2\|_{\tilde{L}^\infty(H^{1/2})}
\]

Similarly, we have:

\[
\| \psi(u_1) - \psi(u_2)\|_{L^2(H^{3/2})} \leq \| B(u_1 - u_2, u_1) + B(u_2, u_1 - u_2)\|_{L^2(H^{3/2})}
\]

\[
\leq 2C(\| u_1 \|_{\tilde{L}^\infty(H^{1/2})} + \| u_2 \|_{\tilde{L}^\infty(H^{1/2})})\| u_1 - u_2\|_{L^2(H^{3/2})}
\]

\[
\leq 2Cr\| u_1 - u_2\|_{L^2(H^{3/2})}
\]

\[
\leq \frac{1}{2}\| u_1 - u_2\|_{L^2(H^{3/2})}
\]

Which implies the existence of a global solution in \( \tilde{L}^\infty(H^{1/2}) \cap L^2(H^{3/2}) \) for small initial data in \( H^{1/2}(\mathbb{R}^3) \), and we get:

\[
\| u \|_{\tilde{L}^\infty(H^{1/2})} + \| u \|_{L^2(H^{3/2})} \leq \| u^0 \|_{H^{1/2}}.
\]

**Step 3:** Multiplying (2.22) by \( e^{\sqrt{\eta}|\xi|} \) we obtain:

\[
e^{\sqrt{\eta}|\xi|} |\hat{\psi}(t, \xi)| \leq e^{\sqrt{\eta}|\xi| |\xi|^2 |\hat{\xi}|^2} + \int_0^t e^{\nu(t-s)}|\xi|^2 + e^{\sqrt{\eta}|\xi|} |\xi|^2 |\hat{\psi}(t, \xi)| \text{ds}
\]

\[
\leq e^{\sqrt{\eta}|\xi| |\xi|^2} e^{-\frac{1}{4}|\xi|^2 |\hat{\xi}|^2 |\hat{\psi}(t, \xi)|^2} + \int_0^t e^{\sqrt{\eta}|\xi| - \sqrt{\eta}|\xi| - \frac{1}{4}(t-s)|\xi|^2} e^{-\frac{1}{4}(t-s)|\xi|^2} e^{\sqrt{\eta}|\xi|} |\xi|^2 |\hat{\psi}(t, \xi)| \text{ds}.
\]

Since \( e^{\sqrt{\eta}|\xi|} |\hat{\psi}(t, \xi)| \) is uniformly bounded in time and \( \xi \), then we have:

\[
e^{\sqrt{\eta}|\xi|} |\hat{\psi}(t, \xi)| \leq c_0\left( e^{-\frac{1}{4}|\xi|^2 |\hat{\psi}(t, \xi)|^2} + \int_0^t e^{-\frac{1}{4}(t-s)|\xi|^2} |\xi| \int e^{\sqrt{\eta}|\xi| - \sqrt{\eta}|\xi| e^{-\frac{1}{4}(t-s)|\xi|^2} e^{\sqrt{\eta}|\xi|} |\xi|^2 |\hat{\psi}(t, \xi)| \text{dtds} \right)
\]

\[
\leq c_0\left( e^{-\frac{1}{4}|\xi|^2 |\hat{\psi}(t, \xi)|^2} + \int_0^t |\xi| e^{-\frac{1}{4}(t-s)|\xi|^2} |\hat{\psi}(t, \xi)| \text{dtds} \right).
\]
with \( V(t,.) = e^{\sqrt{t}D}v(t,.) \) and \( c_0 = \sqrt{c} \).

Then, by following the precedent steps, we get:

\[
\|V\|_{L^{\infty}(H^{1/2})} + \|V\|_{L^2(H^{3/2})} \leq c_0 \|v^0\|_{H^{1/2}},
\]

which yields,

\[(2.12)\]

\[
\|e^{\sqrt{t}D}v\|_{L^{\infty}(H^{1/2})} \leq c_0 \|v^0\|_{H^{1/2}}.
\]

3. Proof of Theorem 1.1

This proof is identical to the proof in [8], where Fujita and Kato proved the existence Navier-Stokes solution in the critical space \( \dot{H}^{1/2} \).

Let \( B(u, u) \) be the solution to the heat equation

\[
\begin{aligned}
\partial_t B(u,u) - \Delta B(u,u) &= Q(u,u) \\
\text{div} B(u,u) &= 0 \\
B(u,u)(0) &= 0
\end{aligned}
\]

with the bilinear operators \( Q \) defined as in (1.1) and

\[
B(u,u) = - \int_0^t e^{(t-s)\Delta} \bar{P}(\text{div}(u \otimes u)) ds
\]

Moreover,

\[
\int_0^T \|Q(u,v)(s)\|_{H^{1/2}_{-1}}^2 ds \leq C \int_0^T \|u(s)\|_{H^{1/2}_{-1}}^2 \|v(s)\|_{H^{1/2}_{-1}}^2 ds
\]

\[
\leq C \|u\|_{L^4(H^{1/2}_{-1})} \|v\|_{L^4(H^{1/2}_{-1})}.
\]

By Duhamel’s formula and the inequality (2.13), we get:

\[
\|B(u,v)\|_{L^4(H^{1/2}_{-1})} \leq \|B(u,u)(0)\|_{H^{1/2}_{-1}} + \|Q(u,v)\|_{L^4(H^{1/2}_{-1})}
\]

\[
\leq C \|v\|_{L^4(H^{1/2}_{-1})} \|v\|_{L^4(H^{1/2}_{-1})}
\]

which implies:

\[
\|B\|_{L^4(H^{1/2}_{-1})} \leq C.
\]

It is easy to check that

\[(3.2)\]

\[
\|e^{\Delta t}u^0\|_{L^4(H^{1/2}_{-1})} \leq \|u^0\|_{H^{1/2}_{-1}}
\]

thus, if \( \|u^0\|_{H^{1/2}_{-1}} \leq \frac{1}{4C} \), we get:

\[
\|e^{\Delta t}u^0\|_{L^4(H^{1/2}_{-1})} \leq \frac{1}{4C} < \frac{1}{4\|B\|}.
\]

According to Lemma 2.1, there exists a unique solution of (NS) in the ball with center 0 and radius \( \frac{1}{2C_0} \) in the space \( L^4([0, T]; \dot{H}^{1/2}_{-1}) \) such that \( u(t,x) = e^{\Delta t}u^0 + B(u,u) \)

We now consider the case of a large initial data \( u^0 \in H^{1/2}_{-1} \). Let \( \rho_{u^0} > 0 \) such that

\[
\left( \int_{|x| > \rho_{u^0}} |e^{\Delta t}u^0|^2 dx \right)^{1/2} < \frac{1}{8C_0}.
\]

Using the inequality (5.22) again and defining \( v_0 = F(1_{|\xi| < \rho_{u^0}} \hat{u}^0) \) we get:

\[
\|e^{\Delta t}u^0\|_{H^{1/2}_{-1}} \leq \|e^{\Delta t}F^{-1}(1_{|\xi| < \rho_{u^0}} \hat{u}^0)\|_{H^{1/2}_{-1}} + \|e^{\Delta t}v_0\|_{L^4(H^{1/2}_{-1})}
\]

\[
\leq \frac{1}{8C_0} + \|e^{\Delta t}v_0\|_{L^4(H^{1/2}_{-1})}.
\]
We note that,
\[
\|e^{t\Delta}u_0\|_{L^4_t(H^1_{u,1})}^4 \leq \int_0^T \|\xi|^2 |e^{2\xi}| |\hat{u}^0|^2 d\xi^2 ds \\
\leq \rho_u^2 \int_0^T \int_{|\xi|<\rho_u} |\xi|^2 |e^{2\xi}| |\hat{u}^0|^2 d\xi^2 dt \\
\leq T \rho_u^2 \|u^0\|_{H^{1/2}}^4,
\]
which yields
\[
\|e^{t\Delta}u^0\|_{L^4_t(H^1_{u,1})} \leq (\rho_u^2 T)^{1/4} \|u^0\|_{H^{1/2}}.
\]
Thus, if
\[ (3.3) \quad T \leq \left( \frac{1}{8C_0 \rho_u^2 \|u^0\|_{H^{1/2}}} \right)^4, \]
then we have the existence of a unique solution in the ball with center 0 and radius \( \frac{1}{2\rho_0} \) in the space \( L^4_t(H^1_{u,1}) \).

Finally, we observe that if \( u \) is a solution of \( (GNS) \) in \( L^4_t(H^1_{u,1}) \) then, by Lemma 2.3 \( Q(u, u) \) belongs to \( L^2_t(H^{-1/2}_{u,1}) \). Hence, Lemma 2.4 implies that the solution \( u \) belongs to \( C([0, T], H^{1/2}_{u,1}) \cap L^2([0, T], H^{3/2}_{u,1}). \)

4. PROOF OF THEOREM 1.3

Beginning by proving the blow-up result (ii): Suppose that
\[ \int_0^T \|u(t)\|_{H^{1/2}}^2 dt < \infty. \]

Let a time \( T \in (0, T^*) \) such that \( \int_0^T \|u(t)\|_{H^{1/2}}^2 dt < \frac{1}{4T} \). Lemma 2.2 gives, for all \( t \in [T, T^*] \) and \( z \in [T, t] \):
\[
\|u(z)\|_{H^{1/2}_{u,1}}^2 + 2 \int_T^z \|u(s)\|_{H^{1/2}_{u,1}}^2 ds \leq \|u(T)\|_{H^{1/2}_{u,1}}^2 + C \int_T^z \|u(s)\|_{H^{1/2}_{u,1}} \|u(s)\|_{H^{3/2}_{u,1}}^2 ds \\
\leq \|u(T)\|_{H^{1/2}_{u,1}}^2 + \frac{1}{2} \sup_{T \leq s \leq t} \|u(s)\|_{H^{1/2}_{u,1}}.
\]

Then
\[
\sup_{T \leq s \leq t} \|u(z)\|_{H^{1/2}_{u,1}}^2 \leq \|u(T)\|_{H^{1/2}_{u,1}}^2 + \frac{1}{2} \sup_{T \leq s \leq t} \|u(s)\|_{H^{1/2}_{u,1}},
\]
which implies that
\[
\sup_{T \leq s < t} \|u(s)\|_{H^{1/2}_{u,1}} \leq C_T,
\]
with \( C_T = \frac{1}{4} + \sqrt{\frac{1}{16} + \|u(T)\|_{H^{1/2}_{u,1}}^2}. \)

Let \( M = \max(\sup_{0 \leq t \leq T} \|u(t)\|_{H^{1/2}_{u,1}}, C_T) \), then for all \( t \in [0, T^*] \) we get:
\[
\|u(t)\|_{H^{1/2}_{u,1}} \leq M,
\]
which yields
\[
u \in L^4([0, T^*], H^{1/2}_{u,1}).
\]

Let \( 0 < t_0 < T^* \) such that
\[
\|u\|_{L^4([t_0, T^*], H^{1/2}_{u,1})} \leq \frac{1}{4C_0}.
\]
Now, consider the Navier Stokes system starting at $t = t_0$
\[
\begin{align*}
\partial_t v - \nu \Delta v + v \cdot \nabla v &= -\nabla q \\
\text{div} v &= 0 \\
v(0) &= u(t_0).
\end{align*}
\]
Then, we obtain:
\[
\|v(t)\|_{L^4([0,T^* - t_0),H^0_{a,1})} = \|u(t + t_0)\|_{L^4([0,T^* - t_0),H^0_{a,1})} = \|u(t)\|_{L^4([t_0,T^*),H^0_{a,1})} \leq \frac{1}{4C_0}.
\]
Which implies the existence of unique solution in $[0, T^* - t_0)$ which extends beyond to $T^*$, which is absurd.

Now, we shall prove the second result of theorem 1.3 we have:
\[
\partial_t u - \Delta u + u \cdot \nabla u = -\nabla p.
\]
Taking the inner product in $\dot{H}^{1/2}_{a,1}(\mathbb{R}^3)$ with $u$ and using lemma 2.2 we get:
\[
\frac{1}{2} \frac{dt}{dt} \|u\|^2_{\dot{H}^{1/2}_{a,1}} + \|\nabla u\|^2_{\dot{H}^{1/2}_{a,1}} \leq | u(0, u) R | \|u\|^2_{\dot{H}^{1/2}_{a,1}} \leq \|u \otimes u\|_{\dot{H}^{1/2}_{a,1}} \|\nabla u\|^2_{\dot{H}^{1/2}_{a,1}} \leq C \|u\|^2_{\dot{H}^{1/2}_{a,1}} \|\nabla u\|^2_{\dot{H}^{1/2}_{a,1}}.
\]
Let
\[
T = \sup \{0 \leq t, \sup_{0 \leq z \leq t} \|u(z)\|_{\dot{H}^{1/2}_{a,1}} < \frac{1}{C} \}.
\]
For all $0 < t \leq T$, we obtain:
\[
\|u(t)\|^2_{\dot{H}^{1/2}_{a,1}} + \nu \int_0^t \|\nabla u(z)\|^2_{\dot{H}^{1/2}_{a,1}} dz \leq \|u(0)\|^2_{\dot{H}^{1/2}_{a,1}} < \left(\frac{1}{C}\right)^2.
\]
Then $T = T^*$ and $\int_0^{T^*} \|\nabla u(z)\|^2_{\dot{H}^{1/2}_{a,1}} dz < \infty$, therefore $T^* = \infty$ and we get:
\[
\|u(t)\|^2_{\dot{H}^{1/2}_{a,1}} + \nu \int_0^t \|\nabla u(z)\|^2_{\dot{H}^{1/2}_{a,1}} dz \leq \|u(0)\|^2_{\dot{H}^{1/2}_{a,1}} \quad \forall t \geq 0.
\]
5. Proof of Theorem 1.3

In this section we prove that
\[
\lim_{t \to \infty} \|u(t)\|_{\dot{H}^{1/2}_{a,1}} = 0.
\]
For $0 < \epsilon < \frac{1}{8}$ and using the embedding $\dot{H}^{1/2}_{a,1}(\mathbb{R}^3) \hookrightarrow \dot{H}^{1/2}(\mathbb{R}^3)$, we can deduce that there exists a positive time $t_0 > 0$ such that
\[
\|u(t)\|^2_{\dot{H}^{1/2}_{a,1}} < \epsilon, \quad \forall t \geq t_0.
\]
Then, we get for all $t \geq t_0$,
\[
\|e^\sqrt{-\Delta} \nu D^* \nu(t)\|_{\dot{H}^{1/2}_{a,1}} \leq \|u(t_0)\|_{\dot{H}^{1/2}_{a,1}} < \epsilon.
\]
Consider the following system:
\[
\begin{align*}
\partial_t v - \Delta v + vv \cdot \nabla v &= -\nabla p_1 \\
\text{div} v &= 0 \\
v(0) &= u(t_0).
\end{align*}
\]
By the uniqueness of \((NS)\) solution in \(\tilde{L}^\infty(\dot{H}^{1/2}(\mathbb{R}^3))\), we obtain for all \(t \geq 0\):
\[
\|e^{\sqrt{\nabla}}v(t)\|_{\dot{H}^{1/2}} = \|e^{\sqrt{\nabla}}u(t + t_0)\|_{\dot{H}^{1/2}} \\
= \|e^{\sqrt{\nabla}}(t + t_0) - t_0 | D | u(t + t_0)\|_{\dot{H}^{1/2}} \\
< \epsilon
\]

Let a time \(t_1 > t_0 > 0\) such that \(\sqrt{t_1 - t_0} > a\). For all \(t \geq t_1 - t_0\), we get:
\[
\|\|e^{\sqrt{\nabla}}v(t)\|_{\dot{H}^{1/2}} = \|e^{\sqrt{\nabla}} | D | e^{\sqrt{\nabla}}v(t)\|_{\dot{H}^{1/2}} \\
\leq \|e^{\sqrt{\nabla}}v(t)\|_{\dot{H}^{1/2}} \\
\leq \epsilon
\]

Now, we consider the following system:
\[
\begin{aligned}
\partial_tw - \Delta w + w \cdot \nabla w &= -\nabla p \\
div w &= 0 \\
w(0) &= v(t_1)
\end{aligned}
\]
then we obtain:
\[
\|e^{\sqrt{\nabla}}w(t)\|_{\dot{H}^{1/2}} = \|e^{\sqrt{\nabla}}v(t + t_1 - t_0)\|_{\dot{H}^{1/2}} < \epsilon,
\]
which yields the result:
\[
\lim_{t \to \infty} \|e^{\sqrt{\nabla}}v(t)\|_{\dot{H}^{1/2}} = 0.
\]

6. Proof of Theorem 1.6

The proof of Theorem 1.6 is identical to the proofs in \([9, 3]\), where Gallager, Iftimie and Planchon proved the stability of global solutions in \(\dot{H}^{1/2}\) and Benameur showed the same result in Lei-Lin spaces. Let \(v \in C([0, T^*], \dot{H}^{1/2})\) be the maximal solution of \((NS)\) corresponding to the initial condition \(v^0\). We want to prove \(T^* = \infty\), if \(\|u(0) - v^0\|_{\dot{H}^{1/2}} < \epsilon\) (\(\epsilon\) is fixed later.)

Put \(w = v - u\) and \(w^0 = v^0 - u(0)\). We have
\[
\partial_tw - \Delta w + w \cdot \nabla w + \nabla w \cdot \nabla w = -\nabla P.
\]
Then we get
\[
\frac{dt}{dt}\|w\|_{\dot{H}^{1/2}}^2 + 2\|\nabla w\|_{\dot{H}^{1/2}}^2 \leq I_1 + I_2
\]
with
\[
I_1 = | <w, \nabla w, w >_{\dot{H}^{1/2}} |
\]
and
\[
I_2 = | <u, \nabla w, w >_{\dot{H}^{1/2}} | + | <w, \nabla u, w >_{\dot{H}^{1/2}} |.
\]
By using Cauchy-Shwartz inequality and Lemma 2.2 we get
\[
I_1 \leq \|w\|_{\dot{H}^{1/2}}\|\nabla w\|_{\dot{H}^{1/2}}^2
\]
\[
I_2 \leq (\|u \otimes w\|_{\dot{H}^{1/2}} + \|w \otimes u\|_{\dot{H}^{1/2}})\|\nabla w\|_{\dot{H}^{1/2}}^2
\]
\[
\leq 2\|u\|_{\dot{H}^{1/2}}\|w\|_{\dot{H}^{1/2}}\|\nabla w\|_{\dot{H}^{1/2}}^2
\]
\[
\leq 2C\|u\|_{\dot{H}^{1/2}}\|w\|_{\dot{H}^{1/2}}\|\nabla w\|_{\dot{H}^{1/2}}^3
\]
\[
\leq \frac{C}{2}\|u\|^4_{\dot{H}^{1/2}}\|w\|^2_{\dot{H}^{1/2}}^2 + \frac{3}{2}\|\nabla w\|^2_{\dot{H}^{1/2}}.
\]
Then we deduce:
\[
\frac{dt}{dt}\|w\|_{\dot{H}^{1/2}}^2 + 2\|\nabla w\|_{\dot{H}^{1/2}}^2 \leq \|w\|_{\dot{H}^{1/2}}\|\nabla w\|_{\dot{H}^{1/2}}^2 + \frac{C}{2}\|u\|^4_{\dot{H}^{1/2}}\|w\|^2_{\dot{H}^{1/2}} + \frac{3}{2}\|\nabla w\|^2_{\dot{H}^{1/2}}.
\]
which yields
\[
\frac{d}{dt} \|w\|^2_{H_t^{1/2}} + \frac{1}{2} \|\nabla w\|^2_{H_t^{1/2}} \leq \|w\|^2_{H_t^{1/2}} \|\nabla w\|^2_{H_t^{1/2}} + \frac{C}{2} \|u\|^4_{H_t^{1/2}} \|w\|^2_{H_t^{1/2}}.
\]
Suppose that \(\|w(0)\|_{H_t^{1/2}} < \frac{1}{4}\). Let
\[
T = \sup\{t \in [0, T^*], \sup_{0 \leq z \leq t} \|w(z)\|_{H_t^{1/2}} < \frac{1}{4}\}.
\]
For all \(t \in (0, T)\), we get:
\[
\|w(t)\|^2_{H_t^{1/2}} + \frac{1}{4} \int_0^t \|\nabla w(z)\|^2_{H_t^{1/2}} dz \leq \|w(0)\|^2_{H_t^{1/2}} + \frac{C}{2} \int_0^t \|u(z)\|^4_{H_t^{1/2}} \|w(z)\|^2_{H_t^{1/2}} dz
\]
Gronwall’s Lemma yields
\[
\|w(t)\|^2_{H_t^{1/2}} + \frac{1}{8} \int_0^t \|\nabla w(z)\|^2_{H_t^{1/2}} dz \leq \|w(0)\|^2_{H_t^{1/2}} \exp\left(\frac{C}{2} \int_0^\infty \|u(z)\|^4_{H_t^{1/2}} dz\right) < \frac{1}{16},
\]
then \(T = T^*\) and \(\int_0^{T^*} \|\nabla w\|^2_{H_t^{1/2}} < \infty\), then \(T^* = \infty\) and the proof is finished.

7. Appendix

In this section we prove the decreasing result of \(\|u(t)\|_{H^s}\) (See Remark 1.3)
\[
\|u(t)\|^2_{H_t^{1/2}} = \sum_{k=0}^{\infty} (2k)^k \|u(t)\|_{H_t^{1+k/2}} \to 0,\quad t \to +\infty
\]
then
\[
\|u(t)\|_{H^s} \to 0,\quad \forall s \geq \frac{1}{2}.
\]
Precisely:
\[
\|u(t)\|_{H^s} = o(t^{-\frac{s}{2}}),\quad t \to +\infty.
\]
Indeed:

First case: Let \(s = 1/2 + k\) where \(k\) is a positive integer. We solve the problem by induction:
If \(k = 0\) we have \(\|u(t)\|_{H_t^{1/2}} = o(1), t \to \infty\). (by [9]).
Suppose that, for some \(k \in \mathbb{N}\):
\[
\|u(t)\|_{H_t^{1/2+k}} = o(t^{-\frac{1}{2+k}}),\quad t \to +\infty.
\]
We have:
\[
\partial_t u - \Delta u + u, \nabla u = -\nabla p
\]
Taking the norm \(H_t^{1/2+k}\), we get:
\[
\|u(t)\|^2_{H_t^{1/2+k}} + 2 \int_{t_1}^{t_2} \|u(z)\|^2_{H_t^{1/2+k+1}} dz \leq \|u(t_1)\|^2_{H_t^{1/2+k}} + C_k \int_{t_1}^{t_2} \|u(z)\|_{H_t^{1/2}} \|u(z)\|^2_{H_t^{1/2+k+1}} dz.
\]
We have
\[
\lim_{t \to \infty} \|u(t)\|_{H_t^{1/2}} = 0,
\]
Then there exists a time \(t_k > 0\), such that for all \(t \geq t_k\) we get:
\[
\|u(t)\|_{H_t^{1/2}} \leq \frac{1}{C_k},
\]
By using the fact that,
\[
t \mapsto \|u(t)\|_{H_t^{1/2+k}} \text{ is decreasing in time for all } t \in [t_k, +\infty)
\]
and
\[
t \mapsto \|u(t)\|_{H_t^{1/2+k+1}} \text{ is decreasing in time for all } t \in [t_{k+1}, +\infty),
\]
we can deduce that for all $\frac{1}{2} > \max(t_k, t_{k+1})$:
\[
\|u(t)\|_{H^{k+1}}^2 + \int_0^t \|u(z)\|_{H^{k+1}}^2 dz \leq \|u(t)\|_{H^{k+1}}^2
\]
which yields,
\[
(t - \frac{t}{2})\|u(t)\|_{H^{k+1}}^2 \leq \|u(t)\|_{H^{k+1}}^2,
\]
then,
\[
\|u(t)\|_{H^{k+1}}^2 \leq \frac{2}{t}\|u(t)\|_{H^{k+1}}^2,
\]
and
\[
\|u(t)\|_{H^{k+1}} = o(t^{-\frac{1}{k+1}}); \ t \to \infty.
\]
**Second case:** Now, taking the case of $s \geq \frac{1}{2}$:
There exists $k \in \mathbb{N}$ such that for all $\frac{1}{2} + k \leq s < \frac{1}{2} + k + 1$, we have:
\[
s = \theta(\frac{1}{2} + k) + (1 - \theta)(\frac{1}{2} + k + 1), \ \text{with} \ \theta \in [0, 1].
\]
by interpolation we obtain:
\[
\|u(t)\|_{H^s} \leq \|u(t)\|_{H^{k+1}}^\theta \|u(t)\|_{H^{k+1}}^{1-\theta}.
\]
Then, we get:
\[
\|u(t)\|_{H^s} = o(t^{-\frac{1}{2}}(\theta(\frac{1}{2} + k) + (1 - \theta)(\frac{1}{2} + k + 1) - s + \frac{1}{2})), \ t \to \infty.
\]
which finish the proof.

**Acknowledgments**
I would like to thank Jamel Benameur for his kind advice and insightful comments.

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