Lower Bound for the Geometric Type from a Generalized Estimate in the $\overline{\partial}$-Neumann Problem – a New Approach by Peak Functions

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1. Introduction

In a series of seminal papers in the Annals of Mathematics [Cat83; Cat87], Catlin proved the equivalence of the finite type of a boundary (cf. [D’A82]) with the existence of a subelliptic estimate for the $\overline{\partial}$-Neumann problem by triangulating through the $t^\varepsilon$-property (see below)

(i) finite type $m \Rightarrow t^\varepsilon$-property with $\varepsilon = m^n - m^2$;  
(ii) $t^\varepsilon$-property $\Rightarrow \varepsilon$-subelliptic estimate;  
(iii) $\varepsilon$-subelliptic estimate $\Rightarrow$ finite type $m$ for $m \leq \frac{1}{\varepsilon}$.

Here, the $t^\varepsilon$-property of a boundary $b\Omega_1$ is a special case of a more general “$f$-property” defined as follows. For a smooth strictly increasing function $f : (1 + \infty) \rightarrow [1, +\infty)$ with $f(t) \leq t^1/2$, the $f$-property at $z_o$ means the existence of a neighborhood $U$ of $z_o$, of constants $C_1, C_2$, and of a family of functions $\{\phi_\delta\}$ such that

1) $\phi_\delta$ are plurisubharmonic and $C^2$ on $U$, and $-1 \leq \phi_\delta \leq 0$;  
2) $\partial \partial \phi_\delta \geq C_1 f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \leq C_2\delta^{-1}$ for any $z \in U \cap \{z \in \Omega: -\delta < r(z) < 0\}$, where $r$ is a defining function of $\Omega$.

The results in steps (ii) and (iii) were generalized in [KZ10; KZ12]. In particular, in [KZ10] it was shown that the $f$-property implies an $f$-estimate for any $f$, and in [KZ12] that an $f$-estimate with $\frac{f}{\log} \rightarrow \infty$ at $\infty$ implies that the type along a complex analytic variety has a lower bound with the rate $G$ with

$$G(\delta) = \left( \frac{f}{\log} \right)^* \left( \delta^{-1} \right)^{-1}, \tag{1.1}$$

where the superscript $^*$ denotes the inverse function. Combining the above results, we obtain the following:

**Theorem 1.1** (Catlin [Cat83; Cat87]; Khanh and Zampieri [KZ10; KZ12]). *Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^n$ with $C^\infty$-smooth boundary $b\Omega$, and $z_o$ be a boundary point. Assume that the $f$-property holds at $z_o$ with $\frac{f}{\log} \nearrow \infty$ as $t \rightarrow \infty$.***

Received March 1, 2013. Revision received June 10, 2013.  
This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2012.16.
Then, if $b\Omega$ has type $\leq F$ along a one-dimensional complex analytic variety $Z$ at $z_0$, that is,
\[
|r(z)| \leq F(|z - z_0|), \quad z \in Z, z \to z_0,
\]
then $F(\delta) \geq \alpha G(\delta)$ for a suitable constant $\alpha > 0$ and for any $\delta$ small.

The purpose of this note is to give a short proof of Theorem 1.1, which has also the advantage of requiring only a minimal smoothness of $b\Omega$ if a slightly stronger assumption on $f$ is given. More precisely, we prove the following:

**Theorem 1.2.** Let $\Omega$ be a pseudoconvex domain of $\mathbb{C}^n$ with $C^2$-smooth boundary $b\Omega$, and $z_0$ be a boundary point. Assume that the $f$-property holds at $z_0$ with $f$ satisfying $(g(t))^{-1} := \int_t^{\infty} \frac{da}{af(a)} < \infty$ for some $t \geq 1$, and set $G(\delta) = (g^*(\delta^{-1}))^{-1}$. Then, if $b\Omega$ has type $\leq F$ along a one-dimensional complex analytic variety $Z$ at $z_0$, then $F(\delta) \geq \alpha G(\beta \delta)$ for suitable constants $\alpha, \beta > 0$ and for any $\delta$ small.

Some remarks are in order. First, the $C^\infty$-smoothness of the boundary in the results of Catlin and Khanh–Zampieri (Theorem 1.1) is required since they applied the regularity of the $\bar{\partial}$-Neumann problem. In Theorem 1.2, the condition of smoothness is reduced because of the use of a plurisubharmonic peak function. However, in the construction of the family of the plurisubharmonic peak functions, we need a slightly stronger hypothesis on $f$ (e.g., $f(t) = \log t \cdot \log^\varepsilon (\log t)$ with $0 < \varepsilon \leq 1$), which fulfills the hypothesis in Theorem 1.1 but does not in Theorem 1.2. Finally, the statements of the two theorems are equivalent in the cases $f(t) = \log^\beta t$ for $\beta > 1$ or $f(t) = t^\varepsilon$ for any $0 < \varepsilon \leq \frac{1}{2}$.

**2. Proof of Theorem 1.2**

The proof of Theorem 1.2 follows immediately from Theorems 2.1 and 2.2. In [Kha13], we showed that there exists a family of plurisubharmonic functions with good estimates.

**Theorem 2.1.** Under the assumptions of Theorem 1.2, for a fixed constant $0 < \eta \leq 1$, there are a neighborhood $V$ of $z_0$ and positive constants $c_1, c_2, c_3$ such that the following holds. For any $w \in V \cap \Omega$, there is a plurisubharmonic function $\psi_w$ on $V \cap \Omega$ verifying
\begin{align*}
(1) \quad |\psi_w(z) - \psi_w(z')| &\leq c_1|z - z'|^\eta, \\
(2) \quad \psi_w(z) &\leq -G^\eta(e_2|z - w|), \quad \text{and} \\
(3) \quad \psi_\pi(z) &\geq -c_3\delta_{\hat{\Omega}}(z)^\eta
\end{align*}
for any $z$ and $z'$ in $V \cap \hat{\Omega}$ (where $\delta_{\hat{\Omega}}(z)$ and $\pi(z)$ denote the distance and projection of $z$ to the boundary, respectively).

Using Theorem 2.1 for $w = z_0$, we get the following:
Theorem 2.2. Let $\Omega$ be a $C^2$-smoothly pseudoconvex domain in $\mathbb{C}^n$, and $z_o$ be a boundary point. Assume that there are a neighborhood $V$ of $z_o$ and a plurisubharmonic function $\psi$ on $V \cap \Omega$ such that

$$-c_1|z - z_o|^\eta \leq \psi(z) \leq -G^\eta(c_2|z - z_o|), \quad z \in V \cap \Omega,$$

(2.1)

for suitable $c_1, c_2 > 0$ and $\eta \in (0, 1]$. If $b\Omega$ has type $\leq F$ along a one-dimensional complex analytic variety $Z$, then $F(\delta) \geq \alpha G(\beta \delta)$ for some $\alpha, \beta > 0$ and for any small $\delta$.

Proof. Let $\Omega$ be a domain in $\mathbb{C}^n$ and assume that there is a function $F$ and an one-dimensional complex analytic variety $Z$ passing through $z_o$ such that (1.2) is satisfied for $z \in Z$. Then, in any neighborhood $U$ of $z_o$, there are constants $c_3, c_4 > 0$ and a family $\{Z_\delta\}$ of one-dimensional complex manifolds $Z_\delta \subset U$ defined by $h_\delta : \Delta \to U$ with $h_\delta(0) = z_o$ such that

$$\delta = \sup_{t \in \Delta} |h_\delta(t) - z_o| \geq |h'_\delta(0)| \geq c_3 \delta$$

(2.2)

and

$$\sup_{t \in \Delta} |\delta_{b\Omega}(h_\delta(t))| < c_4 F(\delta),$$

(2.3)

where $\Delta$ denotes the unit disc in $\mathbb{C}$.

Let $v$ be the outward normal vector to $b\Omega$ at $z_o$. From (2.3) we have $h_\delta(t) - c_4 F(\delta)v \in \Omega \cap U$ for any $t \in \Delta$. Applying the submean value inequality to the subharmonic function $\psi(h_\delta(t) - c_4 F(\delta)v)$ on $\Delta$, we get

$$\psi(z_o - c_4 F(\delta)v) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(h_\delta(e^{i\theta}) - c_4 F(\delta)v) d\theta.$$

(2.4)

Now, we use the first inequality in (2.1) for the left-hand side term of (2.4):

$$-\psi(z_o - c_4 F(\delta)v) \leq c_1 c_4^\eta F^\eta(\delta)^\eta.$$

For the right-hand side term of (2.4), we use the second inequality of (2.1):

$$-\frac{1}{2\pi} \int_0^{2\pi} \psi(h_\delta(e^{i\theta}) - c_4 F(\delta)v) d\theta \geq \frac{1}{2\pi} \int_0^{2\pi} G^\eta(c_2|h_\delta(e^{i\theta}) - c_4 F(\delta)v - z_o|) d\theta.$$

(2.5)

Using (2.2) and the Jensen inequality for the increasing, convex function $G^\eta$, we get

$$G^\eta(c_2c_3\delta) \leq G^\eta(c_2|h'_\delta(0)|)$$

$$\leq G^\eta\left(\frac{1}{2\pi} \int_0^{2\pi} c_2|h_\delta(e^{i\theta}) - CF(\delta)v - z_o| d\theta\right)$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} G^\eta(c_2|h_\delta(e^{i\theta}) - CF(\delta)v - z_o|) d\theta.$$

(2.6)
Combining (2.4), (2.5), and (2.6), we obtain
\[ F(\delta) \geq \alpha G(\beta \delta) \]
with \( \alpha = (c_1^{1/\eta} c_4)^{-1} \) and \( \beta = c_2 c_3 \). The proof of Theorem 1.2 is completed. \( \square \)

**Remark 2.3.** In the case \( G(t) = t^m \), the result was obtained by Fornaess and Sibony [FS89].

**Acknowledgments.** This article was written while the author was a visiting member at the Vietnam Institute for Advanced Study in Mathematics (VIASM). He would like to thank the institution for its hospitality and support.

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