CATEGORIFICATION OF TWO-DIMENSIONAL COHOMOLOGICAL HALL ALGEBRAS

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ABSTRACT. In the present paper, we provide a full categorification, at the level of stable ∞-categories, of two-dimensional cohomological Hall algebras of curves and surfaces. This is achieved by producing a suitable derived enhancement of the relevant moduli stacks entering in the constructions of such algebras. This method categorifies the cohomological Hall algebra of Higgs sheaves on a curve and the cohomological Hall algebra of coherent sheaves on a surface. Furthermore, it applies also to several other moduli stacks, such as the moduli stack of vector bundles with flat connections on a curve X and the moduli stack of finite-dimensional representations of the fundamental group of X.

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1. INTRODUCTION

In the present paper, we provide a full categorification, at the level of stable \( \infty \)-categories, of two-dimensional cohomological Hall algebras of curves and surfaces. This is achieved by constructing a suitable derived enhancement of the relevant moduli stacks entering in the constructions of such algebras. This method categorifies the cohomological Hall algebra of Higgs sheaves on a curve \([\text{Min18}, \text{SS18}]\), and the cohomological Hall algebra of coherent sheaves on a surface \([\text{Zha19}, \text{KV19}]\). Furthermore, it applies also to several other moduli stacks, such as the moduli stack of vector bundles with flat connections on a curve \(X\) and the moduli stack of finite-dimensional representations of the fundamental group of \(X\). By using a similar method, we are able to categorify the cohomological Hall algebra of finite-dimensional representations of the preprojective algebra of a quiver \([\text{SV17a}]\), as we will show in a separate paper.

Before providing precise statements of our results, we shall briefly recall the literature about cohomological Hall algebras.

1.1. Background. Let \(A\) be an abelian category and denote by \(M_A\) the corresponding moduli stack of objects: \(M_A\) is a geometric stack over \(\mathbb{C}\) whose groupoid of \(\mathbb{C}\)-points \(\text{Map}(\text{Spec}(\mathbb{C}), M_A)\) naturally coincides with the groupoid of objects of \(A\). By abuse of language, we shall call a “cohomological Hall algebra” (CoHA in the following) associated to \(A\) a convolution algebra structure à la Hall on either the Borel-Moore homology or any oriented Borel-Moore homology\(^1\) of \(M_A\).

In this paper we are interested in two-dimensional CoHAs, which are CoHAs associated to abelian categories of global dimension two. The first instances\(^2\) of two-dimensional CoHAs can be traced back to the works of Schiffmann and Vasserot \([\text{SV13b}, \text{SV12}]\). As a “geometric Langlands dual algebra” of the (classical) Hall algebra of a curve\(^3\), the authors introduced a convolution algebra structure on the (equivariant) \(G_0\)-theory of the cotangent stack \(T^* \text{Rep}(Q_g)\) of the stack \(\text{Rep}(Q_g)\) of finite-dimensional representations of the quiver \(Q_g\) with one vertex and \(g\) loops. When \(g = 1\), the corresponding associative algebra is isomorphic to a positive part of the elliptic Hall algebra. A study of the representation theory of the elliptic Hall algebra by using its CoHA description was initiated in \([\text{SV13b}]\) and pursued by Negut \([\text{Neg18a}]\) in connection with gauge theory and deformed vertex algebras.

---

\(^1\)Examples of oriented Borel-Moore homology theories are the \(G_0\)-theory (i.e., the Grothendieck group of coherent sheaves), Chow groups, elliptic cohomology.

\(^2\)To be best of the author’s knowledge, the first circle of ideas around two-dimensional CoHAs can be found in an unpublished manuscript by Grojnowski \([\text{Gro94}]\).

\(^3\)By (classical) Hall algebra of a curve we mean the Hall algebra associated with the abelian category of coherent sheaves on a smooth projective curve defined over a finite field. As explained in \([\text{Sch12}]\), one can reinterpret such an algebra as an algebra coming from the “Bun” side of the geometric Langlands correspondence.
A similar construction of the two-dimensional CoHA works for any quiver and at the same time in Borel-Moore homology theory and more generally in any oriented Borel-Moore homology theory (as shown e.g. in [YZ18a]). Note that $\text{T}^* \text{Rep}(Q)$ is equivalent to the stack of finite-dimensional representations of the preprojective algebra $\Pi_Q$ of $Q$. For this reason, sometimes this CoHA is called the CoHA of the preprojective algebra of $Q$.

In the Borel-Moore homology case, one can give a characterization of the generators of the CoHA of the preprojective algebra of $Q$ as done in [SV17a] and establish a relation with the (Maulik-Okounkov) Yangian [SV17b, DM16, YZ18b]. Again, when $Q = Q_1$, one can establish a connection between the corresponding two-dimensional CoHA and vertex algebras [SV13a, Neg16] (see also [RSYZ18]).

Another CoHA has been introduced by Kontsevich and Soibelman [KS11], in order to provide a mathematical definition of Harvey and Moore’s algebra of BPS states [HM98]. It goes under the name of three-dimensional CoHA since it is associated with Calabi-Yau categories of global dimension three (such as the category of representations of the Jacobi algebra of a quiver with potential, the category of coherent sheaves on a CY 3-fold, etc). As shown by Davison in [RS17, Appendix] (see also [YZ16]), using a dimensional reduction argument, the CoHA of the preprojective algebra of a quiver described above can be realized as a Kontsevich-Soibelman one.

For certain choices of the quiver $Q$, the cotangent stack $\text{T}^* \text{Rep}(Q)$ is a stack parameterizing coherent sheaves on a surface. Thus the corresponding algebra can be seen as an example of a CoHA associated to a surface. This is the case when the quiver is the one-loop quiver $Q_1$: indeed, $\text{T}^* \text{Rep}(Q_1)$ coincides with the stack $\text{CoH}_0(C^2)$ parameterizing zero-dimensional sheaves on the complex plane $C^2$. In particular, the elliptic Hall algebra can be seen as an algebra attached to zero-dimensional sheaves on $C^2$.

Other examples of CoHAs of a surface came from algebras of two-dimensional abelian categories associated to a smooth curve $X$. For example, if we consider the category of Higgs sheaves on $X$, then the Borel-Moore homology of the stack $\text{Higgs}(X)$ of Higgs sheaves on $X$ is endowed with the structure of a convolution algebra. It goes under the name of the cohomological Hall algebra of Higgs sheaves\footnote{A Higgs sheaf is a pair $(E, \phi; E \to \Omega_X^1 \otimes E)$, where $E$ is a coherent sheaf on $X$ and $\phi$ a morphism of $\mathcal{O}_X$-modules, called a Higgs field. Here, $\Omega_X^1$ is the sheaf of 1-forms of $X$.} on $X$ (Dolbeaut CoHA of $X$ in the following). Such an algebra has been introduced by the second-named author and Schiffmann in [SS18]. In [Min18], independently Minets has introduced the Dolbeaut CoHA in the rank zero case. Thanks to the Beauville-Narasimhan-Ramanan correspondence, the Dolbeaut CoHA can be interpreted as the CoHA of torsion sheaves on $\text{T}^* X$ such that their support is proper over $X$. In particular, Minets’ algebra is an algebra attached to zero-dimensional sheaves on $\text{T}^* X$. Such an algebra coincides with Negut’s shuffle algebra [Neg17] of a surface $S$ when $S = \text{T}^* X$.

Negut’s algebra of a smooth surface $S$ is defined by means of Hecke correspondences depending on zero-dimensional sheaves on $S$, and its construction comes from a generalization of the realization of the elliptic Hall algebra in [SV13b] via Hecke correspondences. Zhao [Zha19] has constructed the cohomological Hall algebra of the moduli stack of zero-dimensional sheaves on a smooth surface $S$ and fully established the relation between this CoHA and Negut’s algebra of $S$. Finally, Kapranov and Vasserot [KV19] have defined the CoHA associated to a category of coherent sheaves on a smooth surface $S$ with proper support of a fixed dimension, generalizing the work of Zhao.

Now, we would like to describe the way that the convolution product has been constructed in the examples above. Roughly speaking, the product is always induced by a convolution diagram...
of the form
\[
\mathcal{M}_A \times \mathcal{M}_A \xleftarrow{p} \mathcal{M}_A^{\text{ext}} \xrightarrow{q} \mathcal{M}_A,
\]
(1.1)

\[
(E_2, E_1) \xrightarrow{0} E_1 \xrightarrow{E} E_2 \xrightarrow{0} E
\]

where $\mathcal{A}$ is an abelian category, $\mathcal{M}_A$ is its moduli stack of objects, and $\mathcal{M}_A^{\text{ext}}$ is the moduli stack parameterizing extensions of objects belonging to $\mathcal{A}$.

Naively, the product should be given by the composition $q_* \circ p^*$ in one of the homology theories considered above, but of course one needs to have the correct functoriality properties for the homology theories he is interested in. In the situation we are considering, the map $q$ is a proper and representable map, and therefore defining the pushforward $q_*$ does not pose any significant problem. On the other hand, the map $p$ is typically not flat, and therefore it is more subtle to define a functorial version of the $p^*$ functor. All the solutions to this problem we are aware of have a common leitmotiv: they all pass from finding a suitable additional structure on the map $p$ that allows to produce a well defined and functorial version of the $p^*$ functor.

Let us sketch the three main approaches to this problem in order of increasing sophistication. For simplicity, we replace the map $p$ with a map $f: X \to Y$ between geometric stacks and we limit ourselves to discuss the case of the $G$-theory.

- If $f$ fits in a pullback square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{u} & V
\end{array}
\]

where $v$ is a local complete intersection, then it is possible to consider the refined Gysin pullback $v^!$, that is a good replacement for the $f^*$. For simplicity, we refer to this situation by saying that $u$ is an lci extension of $f$. We refer to [Ful98] for a detailed account of this technique. This method works when $\mathcal{M}_A$ is either $T^*\text{Rep}(Q)$ or $\text{Higgs}(X)$ (cf. [SV17a, SS18]).

- When dealing with the more general situation of moduli stacks of coherent sheaves on a smooth surface or flat bundles on a curve, the previous method does not apply. It is possible to circumvent the problem by means of Behrend-Fantechi’s perfect obstruction theories. In the case of of coherent sheaves on a surface this has been done in [Zha19, KV19]. The associated pullback functor is now referred to as virtual pullback and it depends a priori on the choice of the perfect obstruction theory, which is a map $E \to \mathbb{L}_{X/Y}$ where $E$ is in particular a perfect complex on $X$, as well as on the choice of a global resolution of $E$. It is then possible to show a posteriori that the virtual pullback is independent of the choice of the global resolution of $E$.

- The main drawback of the two previous approaches is that it is difficult to formulate satisfactory functoriality properties for the refined Gysin or the virtual pullback functors. Derived geometry can be used to produce a more robust way of dealing with these issues. In this case, what one is looking for is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
\]

\footnote{A different construction of the cohomological Hall algebra of character varieties has been suggested by B. Davison in [Dav17] via the Kontsevich-Soibelman CoHA formalism.}
where $\tilde{X}$ and $\tilde{Y}$ are derived geometric stacks whose truncations are $X$ and $Y$, and where $\tilde{f}$ is lci in the derived sense. In this case, $\tilde{f}^*$ induces a well defined map

$$\tilde{f}^* : \text{Coh}^b(\tilde{Y}) \to \text{Coh}^b(\tilde{X}).$$

The $K$-theory of these categories coincides with the $G$-theory of $X$ and $Y$ respectively, so $\tilde{f}^*$ induces a cohomological operation that substitutes the refined Gysin or the virtual pullback.

The goal of this paper is to provide a categorification of the cohomological Hall algebra construction. In other words, we seek a method to attach to $\mathcal{C}_A$ a category $\mathcal{C}_A$ together with an additional structure which can be after be used to induce the convolution algebra structures on Borel-Moore homology theories of additional structure which can be after be used to induce the convolution algebra structures on more refined invariants, such as the full spectrum of $G$-theory, Hochschild and periodic homology etc. Notice that this would be impossible if we contented ourselves with $\mathcal{C}_A$ being a triangulated category — see e.g. [Schl02, TV04].

The most natural candidate to achieve such a goal would be $\text{Coh}^b(\mathcal{C}_A)$. However, the same difficulties forcing the use of refined or virtual Gysin pullback arise in this setting as well. From this point of view, the approach via perfect obstruction theories presents a serious drawback. Indeed, every choice of a global presentation for the perfect obstruction theory provides a different derived lci enhancement for the map $p$ we want to consider. After passing to $G$-theory, all different choices of global presentations disappear, but this is not true at the categorified level. For this reason, it is important to work with a sufficiently natural derived enhancement $R\mathcal{M}_A$ of $\mathcal{M}_A$ and of its convolution diagram. Our method gives rise to an $E_1$-monoidal structure on the stable $\infty$-category $\text{Coh}^b(R\mathcal{M}_A)$ of complexes with bounded coherent cohomology. As it arises from a derived enhancement of the convolution diagram for $\mathcal{M}_A$, we refer to $\text{Coh}^b(R\mathcal{M}_A)$ it the two-dimensional categorified Hall algebra of $\mathcal{M}_A$.

1.2. Main results. Let $X$ be a smooth proper complex scheme. We introduce the following derived enhancement $\text{Coh}(X)$ of the (classical) geometric stack of coherent sheaves on $X$: informally, as a functor of points, it assigns to any derived affine $C$-scheme $S \in \text{dAff}$ the $\infty$-groupoid $\text{Coh}_S(X \times S)\simeq$ of perfect complexes on $X \times S$ that are flat over $S$. We show in Proposition 2.9 that $\text{Coh}(X)$ is a derived geometric stack locally of finite presentation.

Similarly, we introduce the derived stack $\text{Coh}^\text{ext}(X)$ which, roughly speaking, parameterizes extensions of $S$-families of perfect complexes on $X$ which are flat over $S$. Thus, we have a convolution diagram

$$\text{Coh}(X) \times \text{Coh}(X) \longrightarrow \text{Coh}(X) \longrightarrow \text{Coh}(X)$$

of the form (1.1). The above diagram can be encoded in the $\infty$-categorical Waldhausen construction $\mathcal{S}\text{Coh}(X)$ of $\text{Coh}(X)$. A direct check shows the following:

**Proposition 1.1.** Let $X$ be a smooth proper complex scheme. Then the simplicial object $\mathcal{S}\text{Coh}(X)$ is a 2-Segal object, in the sense of Dyckerhoff-Kapranov [DK12], in the $\infty$-category $\mathcal{dSt}$ of derived stacks.

---

\textsuperscript{6}This means that the relative cotangent complex $L_{S/X,Y}$ is perfect and in tor-amplitude $[1, 0]$.  
\textsuperscript{7}More precisely, in the main body of the paper we will construct directly stable $\infty$-categories, without passing through explicit dg-enhancements.  
\textsuperscript{8}The reader unfamiliar with little disks operads can simply think of an associative monoidal structure.  
\textsuperscript{9}For us, the categorification will be always a dg-category rather than a triangulated category, since we want also to construct a convolution algebra for any (co)homological theory associated with dg-categories.
Any 2-Segal space in $\text{dSt}$ has an associated associative algebra (more precisely, an $E_1$-algebra) in $\text{Corr}^\times(\text{dSt})$. Here, $\text{Corr}^\times(\text{dSt})$ is the $(\infty,1)$-category of correspondences on derived stacks\textsuperscript{10}. Moreover, we consider $\text{Corr}^\times(\text{dSt})$ endowed with the monoidal structure induced by the cartesian product on $\text{dSt}$.

**Corollary 1.2** (cf. Corollary 4.5). Let $X$ be a smooth proper complex scheme. Then the 2-Segal object $\text{SCoh}(X)$ endows $\text{Coh}(X)$ with the structure of an $E_1$-algebra in $\text{Corr}^\times(\text{dSt})$.

In order to provide a categorification of two-dimensional cohomological Hall algebras, we need to restrict ourselves to a smaller category of correspondences. We let $\text{dGeom}$ denote the full $\infty$-subcategory of the category $\text{dSt}$ of derived stacks spanned by those which are geometric, and we let $\text{Corr}^\times(\text{dGeom})_{\text{lc},\text{rps}}$ be the $(\infty,1)$-category of correspondences generated by the following choice of horizontal and vertical morphisms:

- the class horiz of horizontal morphisms to be the collection of morphisms representable by proper schemes (for short, rps morphisms);
- the class vert of vertical morphisms to be the collection of derived l.c.i. morphisms (for short, lci morphisms).

The first non-trivial thing to observe is that the convolution diagram (1.2) is a morphism in $\text{Corr}^\times(\text{dGeom})_{\text{lc},\text{rps}}$. Indeed, we prove in §3 that the map $p$ is derived locally complete intersection. Thus, it follows the following:

**Theorem 1.3** (cf. Theorem 4.6). Let $X$ be a smooth proper complex surface. Then the stack $\text{Coh}(X)$ is an $E_1$-algebra in $\text{Corr}^\times(\text{dGeom})_{\text{lc},\text{rps}}$.

Thanks to the work of Gaitsgory-Rozenblyum [GaR17a, GaR17b], one obtains a right-lax monoidal functor

$$\text{Coh}^b : \text{Corr}^\times(\text{dGeom})_{\text{lc},\text{rps}} \longrightarrow \text{Cat}_\odot^\infty.$$ 

Using this functor, we obtain:

**Corollary 1.4.** Let $X$ be a smooth proper complex surface. Then $\text{Coh}^b(\text{Coh}(X))$ is an $E_1$-algebra in $\text{Cat}_\odot^\infty$.

Since $E_1$-algebras in $\text{Cat}_\odot^\infty$ are (by definition) the same as $E_1$-monoidal categories in $\text{Cat}_\odot^\infty$, we refer to the corresponding tensor structure as the CoHA tensor structure on $\text{Coh}^b(\text{Coh}(X))$. We denote this monoidal structure by $\otimes$.

Let $X$ be a smooth proper complex surface. Similar results hold for the derived stack $\text{Bun}(X) := \coprod_{n \geq 0} \text{Map}(X, B\text{GL}_n)$. If $X$ is projective, similar results hold for the stack $\text{Coh}^{\text{sh}, p(m)}(X)$ of Gieseker-semistable coherent sheaves on $X$ with reduced Hilbert polynomial equals a fixed monic polynomial $p(m) \in \mathbb{Q}[m]$. Finally, if $X$ is only quasi-projective, the results above hold for the stack $\text{Coh}^{\text{c}, d}_{\text{prop}}(X)$ of coherent sheaves on $X$ with proper support and dimension of the support less or equals an integer $d$.

Thus the results above categorify the two-dimensional cohomological Hall algebras of surfaces introduced in [Zha19, KV19]. Finally, if the surface is toric, minimal variations on our construction (discussed in §4.3) allow to consider the toric-equivariant setting. In particular, we obtain the following:

**Proposition 1.5.** The CoHA tensor structure on the stable $\infty$-category $\text{Coh}^b(\text{Coh}(\mathbb{C}^2))$ is a categorification of a positive nilpotent part of the elliptic Hall algebra of Burban and Schiffmann [BSc12]. Here we set $\text{Coh}(\mathbb{C}^2) := \text{Coh}^{\text{c}, 0}_{\text{prop}}(\mathbb{C}^2)$ and the $C^* \times C^*$-action on $\text{Coh}(\mathbb{C}^2)$ is induced by the torus action on $\mathbb{C}^2$.

\textsuperscript{10}Using the notations of [GaR17a], we are considering $\text{Corr}^\times(\text{dSt})^{\text{equiv}}$. 

In [Neg18b], by means of (smooth) Hecke correspondence, Negut defined functors on the bounded derived category of the smooth moduli space of Gieseker-stable sheaves on a smooth projective surface, which after passing to K-theory, give rise to an action of the elliptic Hall algebra on the K-theory of such smooth moduli spaces. In order to make a clear comparison with Negut’s approach, one has to obtain first a “whole” categorified quantum group starting from $\text{Coh}^b_{\text{C}^* \times \text{C}}(\text{Coh}_0(\mathbb{C}^2))$. This could be achieved by categorifying the common procedure of obtaining the whole quantum group in the CoHAS theory: usually, one adds a Cartan subalgebra (as in [SV17b, §3.5.2] or in [YZ17, §5]), then defines a coproduct structure, which induces a bialgebra structure, and a non-degenerate pairing compatible with the bialgebra structure; thus the “whole” quantum group is obtained by applying the (reduced) Drinfeld double construction. An alternative method, suggested by the work of Joyce [Joy18], consists of working directly with the stack of perfect complexes by changing the definition of the convolution product: in this way one avoids the use of Drinfeld double and gets directly the “whole” algebra. In this case, it is not clear how to define the bialgebra structure. Both approaches are challenging at the categorified level and they are subjects of future investigations.

In the present paper we are also interested in categorifying the two-dimensional CoHAS associated with a smooth projective complex curve $X$, as the Dolbeaut CoHA of $X$. Since we can also consider other two-dimensional categories attached to $X$, such as the category of vector bundles with flat connections on $X$ and the category of finite-dimensional representations of the fundamental group $\pi_1(X)$, we follow a unified approach thanks to Simpson’s theory of the shapes of a curve. We consider the following shapes (cf. §A for their precise definitions): the Betti shape $X_B$, the de Rham shape $X_{\text{dR}}$, the Dolbeaut shape $X_{\text{Dol}}$. As their names suggest, the category of coherent sheaves on $X_B$ is equivalent to the category of vector bundles with flat connections on $X$, and similarly for the other shapes. In §2.2, we introduce a derived enhancement $\text{Coh}(X_s)$ of the (classical) geometric stack of coherent sheaves on $X_s$ as the fiber product $\text{Coh}(X) \times_{\text{Perf}(X)} \text{Perf}(X_s)$ for $s \in \{B, \text{dR}, \text{Dol}\}$.

Let $* \in \{B, \text{dR}, \text{Dol}\}$. Thanks to the computations carried in §3.1, we prove that the map $p$ is derived locally complete intersection, hence the convolution diagram (1.2) for $X_s$ is an element in $\text{Corr}^\otimes(\text{dGeom})_{\text{cl,rps}}$. As before, it follows the following:

**Theorem 1.6** (cf. Theorem 4.6). Let $X$ be a smooth projective complex curve and let $* \in \{B, \text{dR}, \text{Dol}\}$. Then the stack $\text{Coh}(X_s)$ is an $\mathbb{E}_1$-algebra in $\text{Corr}^\otimes(\text{dGeom})_{\text{cl,rps}}$. In particular, $\text{Coh}^B(\text{Coh}(X_s))$ is an $\mathbb{E}_1$-algebra in $\text{Cat}^{\text{st}}$.

We call these algebra structures the Betti, de Rham, and Dolbeaut categorified Hall algebras. We can also consider the natural $\mathbb{C}^*$-action on $\text{Coh}(X_{\text{Dol}}) \simeq \text{Coh}(X)^{*}$ “scaling the fibers”, and we get a “categorification” of the Dolbeaut CoHA of $X$.

**Corollary 1.7.** The CoHA tensor structure on the stable $\infty$-category $\text{Coh}^B(\text{Coh}(X_{\text{Dol}}))$ is a categorification of the cohomological Hall algebra of Higgs sheaves on $X$ introduced in [SS18, Min18].

One can observe that given a curve, we have introduced three – a priori distinct – two-dimensional categorified Hall algebras of the same curve. It is natural to wonder what are the relations among them. The first result is the following:

**Theorem 1.8** (CoHA version of the derived Riemann-Hilbert correspondence). Let $X$ be a smooth projective complex curve. Then:

1. the analytifications $\text{Coh}(X_{\text{dR}})^{\text{an}}$ and $\text{Coh}(X_{\text{B}})^{\text{an}}$ carry a canonical $\mathbb{E}_1$-algebra structure in the $\infty$-category $\text{Corr}^\otimes(\text{dAnSt})$ of derived analytic stacks. Moreover, this induces categorifications $(\text{Coh}^B(\text{Coh}(X_{\text{dR}})^{\text{an}}), \oplus^{\text{an}}_{\text{dR}})$ and $(\text{Coh}^B(\text{Coh}(X_{\text{B}})^{\text{an}}), \oplus^{\text{an}}_{\text{B}})$.

2. There are natural morphisms of stable $\mathbb{E}_1$-monoidal $\infty$-categories

$$(\text{Coh}^B(\text{Coh}(X_{\text{dR}})), \oplus_{\text{dR}}) \longrightarrow (\text{Coh}^B(\text{Coh}(X_{\text{dR}})^{\text{an}}), \oplus^{\text{an}}_{\text{dR}})$$
and

$$(\text{Coh}^b(\text{Coh}(X_B)), \oplus_B) \longrightarrow (\text{Coh}^b(\text{Coh}(X_B)^{an}, \oplus_B)).$$

(3) There is a natural equivalence of stable $\mathbb{E}_1$-monoidal $\infty$-categories

$$(\text{Coh}^b(\text{Coh}(X_{\text{dR}})^{an}, \oplus_{\text{dR}}^{an}), \oplus_{\text{dR}}^{an}) \simeq (\text{Coh}^b(\text{Coh}(X_{\text{B}})^{an}), \oplus_{\text{B}}^{an}).$$

Here, we work in the derived analytic setting as introduced by J. Lurie in [Lur11b] and further expanded by the first-named author in [Por15, PY17, HP18]. In addition, $\text{Coh}(X_*)^{an}$ is the underlying analytic derived stack of $\text{Coh}(X_*)$ for $* \in \{\text{dR}, \text{B}\}$. Having the framework of derived analytic geometry at our disposal, the above theorem builds on two main ideas: the first is to provide a description for the analytifications of the two-dimensional categorified Hall algebras. In the Dolbeaut case, we expect that such a stack is intrinsically analytic. This is achieved by introducing an analytic version of the derived stack of coherent sheaves and proving that

$$\text{Coh}(X_{\text{dR}})^{an} \simeq \text{AnCoh}(X_{\text{dR}})^{an}, \quad \text{Coh}(X_{\text{B}})^{an} \simeq \text{AnCoh}(X_{\text{B}})^{an}.$$  

The second main idea is to employ the morphism

$$\eta_{\text{RH}}: X_{\text{dR}}^{an} \longrightarrow X_{\text{B}}^{an},$$

which has been shown in [Por17] to induce the Riemann-Hilbert correspondence.

The relation between the de Rham and the Dolbeaut categorified Hall algebras is more subtle. In order to state it, one has to use a further Simpson’s shape, the Deligne shape $X_{\text{Del}} \rightarrow \mathbb{A}^1$. Then the derived stack $\text{Coh}(X_{\text{Del}}) = (\text{Coh}(X) \times \mathbb{A}^1) \times_{\text{Perf}(X) \times \mathbb{A}^1} \text{Perf}(X_{\text{Del}})$ is the derived moduli stack of Deligne’s $\lambda$ connections on $X$. Such a stack interpolates the de Rham moduli stack with the Dolbeaut moduli stack:

$$\text{Coh}(X_{\text{Del}}) \times_{\mathbb{A}^1} \{0\} \simeq \text{Coh}(X_{\text{Del}}) \quad \text{and} \quad \text{Coh}(X_{\text{Del}}) \times_{\mathbb{A}^1} \{1\} \simeq \text{Coh}(X_{\text{dR}}).$$

We restrict ourselves to the open substack $\text{Coh}^*(X_{\text{Del}}) \subset \text{Coh}(X_{\text{Del}})$ for which the fiber at zero is the derived moduli stack $\text{Coh}^{ss,0}(X_{\text{Del}})$ of semistable Higgs bundles on $X$ of degree zero. Thus, we have the following:

**Theorem 1.9.** Let $X$ be a smooth projective complex curve. Then $\text{Coh}^*_C(X_{\text{Del}})$ is an $\mathbb{E}_1$-algebra. In addition, it is a module over $\text{Perf}^{\text{filt}} := \text{Perf}([\mathbb{A}^1_C/G_m])$ and we have $\mathbb{E}_1$-algebra morphisms:

$$\Phi: \text{Coh}^b_C(\text{Coh}^*(X_{\text{Del}})) \otimes_{\text{Perf}^{\text{filt}}} \text{Perf}_C \longrightarrow \text{Coh}^b(\text{Coh}(X_{\text{dR}})),$$

$$\Psi: \text{Coh}^b(\text{Coh}^*(X_{\text{Del}})) \otimes_{\text{Perf}^{\text{filt}}} \text{Perf}_C^{gr} \longrightarrow \text{Coh}^b(\text{Coh}^{ss,0}(X_{\text{Del}})),$$

where $\text{Perf}_C^{gr} := \text{Perf}((\mathbb{B}G_m)).$

**Conjecture 1.10** (CoHA version of the non-abelian Hodge correspondence). The morphisms $\Phi$ and $\Psi$ are equivalences.

Let us finish this part by mentioning what happens in the quiver setting. In the main body of the paper, we do not deal with quivers at all, although one can prove similar results than Theorem 1.3 and Corollary 1.4 for the abelian category of finite-dimensional representations of the preprojective algebra of a quiver. The correct derived enhancement to consider is introduced by Yeung in [Yeu18]. We leave the investigation of such a categorification to a separate paper.

Finally, let us mention that it would be interesting to construct in a geometric way representations of the two-dimensional categorified Hall algebras. In the Dolbeaut case, we expect that such representations should be given by the smooth moduli spaces introduced in [BrS15, BPSS16].
1.3. Outline. In §2 we introduce our derived enhancement of the classical stack of coherent sheaves on a smooth complex scheme. We also define derived moduli stacks of coherent sheaves on the Betti, de Rham, and Dolbeaut shapes of a smooth scheme. In §3 we introduce the derived enhancement of the classical stack of extensions of coherent sheaves on both a smooth complex scheme and on a Simpson’s shape of a smooth complex scheme. In addition, we define the convolution diagram (1.2) and provide the tor-amplitude estimates for the map p. §4 is devoted to the construction of the categorified Hall algebra associated with the moduli stack of coherent sheaves on either a smooth scheme or a Simpson’s shape of a smooth scheme: in §4.1 we endow such a stack of the structure of a 2-Segal space à la Dyckerhoff-Kapranov, while in §4.2 by applying the functor Cohb, we obtain one of our main results, i.e., a $E_1$-algebra structure on Cohb$(\text{Coh}(Y))$ when either $Y$ is a smooth curve or surface, or a Simpson’s shape of a smooth curve; finally, §4.3 is devoted to the equivariant case of the construction of the categorified Hall algebra. In §5 and §6 we discuss CoHA versions of the non-abelian Hodge correspondence and of the Riemann-Hilbert correspondence, respectively. In particular, in §6 we develop the construction of the categorified Hall algebra in the analytic setting and we compare the two resulting categorified Hall algebras.

Finally, there are two appendices: §A is a review of the Simpson’s shapes of varieties and stacks, while §B provides a version of the Beauville-Narasimhan-Ramanan correspondence for perfect complexes.

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1.4. Notations and convention. In this paper we freely use the language of $\infty$-categories. Although the discussion is often independent of the chosen model for $\infty$-categories, whenever needed we identify them with quasi-categories and refer to [Lur09] for the necessary foundational material.

The notations $\mathcal{S}$ and $\text{Cat}_{\infty}$ are reserved to denote the $\infty$-categories of spaces and of $\infty$-categories, respectively. If $\mathcal{C} \in \text{Cat}_{\infty}$ we denote by $\mathcal{C}^{\infty}$ the maximal $\infty$-groupoid contained in $\mathcal{C}$. We let $\text{Cat}_{\infty}^{st}$ denote the $\infty$-category of stable $\infty$-categories with exact functors between them. We also let $\mathcal{P}_{\text{st}}^{L}$ denote the $\infty$-category of presentable $\infty$-categories with left adjoints between them. Similarly, we let $\mathcal{P}_{\text{st}}^{L}$ denote the $\infty$-categories of stably presentable $\infty$-categories with left adjoints between them. Finally, we set

$$\text{Cat}_{\infty}^{st} := \text{CAlg}(\text{Cat}_{\infty}^{st}), \quad \mathcal{P}_{\text{st}}^{L} := \text{CAlg}(\mathcal{P}_{\text{st}}^{L}).$$

Given an $\infty$-category $\mathcal{C}$ we denote by $\text{PSh}(\mathcal{C})$ the $\infty$-category of $\mathcal{S}$-valued presheaves. We follow the conventions introduced in [PY16, §2.4] for $\infty$-categories of sheaves on an $\infty$-site.

Since we only work over the field of complex numbers $\mathbb{C}$, we reserve the notation $\text{CAlg}$ for the $\infty$-category of simplicial commutative rings over the field of complex numbers $\mathbb{C}$. We often refer to objects in $\text{CAlg}$ simply as derived commutative rings. We denote its opposite by $\text{dAff}$, and we refer to it as the $\infty$-category of derived affine schemes.

In [Lur18, Definition 1.2.3.1] it is shown that the étale topology defines a Grothendieck topology on $\text{dAff}$. We denote by $\text{dSt} := \text{Sh}(\text{dAff}, \tau_{\text{ét}})$ the hypercompletion of the $\infty$-topos of sheaves
on this site. We refer to this ∞-category as the ∞-category of derived stacks. For the notion of derived geometric stacks, we refer to [PY16, Definition 2.8].

Let $A \in \text{CAlg}$ be a derived commutative ring. We let $A\text{-Mod}$ denote the stable ∞-category of $A$-modules, equipped with its canonical symmetric monoidal structure provided by [Lur17, Theorem 3.3.3.9]. Furthermore, we equip it with the canonical $t$-structure whose connective part is its smallest full subcategory closed under colimits and extensions and containing $A$. Such a $t$-structure exists in virtue of [Lur17, Proposition 1.4.4.11]. Notice that there is a canonical equivalence of abelian categories $A\text{-Mod}^\heartsuit \simeq \pi_0(A)\text{-Mod}^\heartsuit$.

We say that an $A$-module $M \in A\text{-Mod}$ is perfect if it is a compact object in $A\text{-Mod}$. We denote by $\text{Perf}(A)$ the full subcategory of $A\text{-Mod}$ spanned by perfect complexes\footnote{It is shown in [Lur17, Proposition 7.2.4.2] that an $\text{Perf}(A)$ coincides with the smallest full stable subcategory of $A\text{-Mod}$ closed under retracts and containing $A$. In particular, $\text{Perf}(A)$ is a stable ∞-category which is furthermore idempotent complete.}. On the other hand, we say that an $A$-module $M \in A\text{-Mod}$ is almost perfect\footnote{Suppose that $A$ is almost of finite presentation over $C$. In other words, suppose that $\pi_0(A)$ is of finite presentation in the sense of classical commutative algebra and that each $\pi_i(A)$ is coherent over $\pi_0(A)$. Then [Lur17, Proposition 7.2.4.17] shows that an $A$-module $M$ is almost perfect if and only if $\pi_i(M) = 0$ for $i < 0$ and each $\pi_i(M)$ is coherent over $\pi_0(A)$.} if $\pi_i(M) = 0$ for $i \ll 0$ and for every $n \in \mathbb{Z}$ the object $\tau_{\leq n}(M)$ is compact in $A\text{-Mod}^{= n}$. We denote by $\text{APerf}(A)$ the full subcategory of $A\text{-Mod}$ spanned by almost perfect complexes.

Given a morphism $f : A \to B$ in $\text{CAlg}$ we obtain an ∞-functor $f^* : A\text{-Mod} \to B\text{-Mod}$, which preserves perfect and almost perfect complexes. We can assemble these data into an ∞-functor

$$\text{Qcoh} : \text{dAff}^{\text{op}} \to \text{Cat}^{\heartsuit, \heartsuit}_t.$$

Since the functors $f^*$ preserve perfect and almost perfect complexes, we obtain well defined subfunctors

$$\text{Perf}, \ A\text{Perf} : \text{dAff}^{\text{op}} \to \text{Cat}^{\heartsuit, \heartsuit}_t.$$

Given a derived stack $X \in \text{dSt}$, we denote by $\text{Qcoh}(X)$, $\text{APerf}(X)$ and $\text{Perf}(X)$ the stable ∞-categories of quasi coherent, almost perfect, and perfect complexes respectively. One has

$$\text{Qcoh}(X) \simeq \lim_{\text{Spec}(A) \to X} \text{Qcoh}(\text{Spec}(A)), \ A\text{Perf}(X) \simeq \lim_{\text{Spec}(A) \to X} \text{APerf}(\text{Spec}(A)), \ \text{and} \ \text{Perf}(X) \simeq \lim_{\text{Spec}(A) \to X} \text{Perf}(\text{Spec}(A)).$$

The ∞-category $\text{Qcoh}(X)$ is presentable. In particular, using [Lur17, Proposition 1.4.4.11] we can endow $\text{Qcoh}(X)$ with a canonical $t$-structure.

Let $f : X \to Y$ be a morphism in $\text{dSt}$. We say that $f$ is flat if the induced functor $f^* : \text{Qcoh}(Y) \to \text{Qcoh}(X)$ is $t$-exact.

Let $X \in \text{dSt}$. We denote by $\text{Coh}(X)$ the full subcategory of $\mathcal{O}_X\text{-Mod}$ spanned by $\mathcal{F} \in \mathcal{O}_X\text{-Mod}$ for which there exists an atlas $\{f_i : U_i \to X\}_{i \in I}$ such that for every $i \in I$, $n \in \mathbb{Z}$, the $\mathcal{O}_{U_i}$-modules $\pi_n(f_i^*\mathcal{F})$ are coherent sheaves. We denote by $\text{Coh}^\heartsuit(X)$ (resp. $\text{Coh}^+X$, $\text{Coh}^-X$) the full subcategory of $\text{Coh}(X)$ spanned by objects cohomologically concentrated in degree 0 (resp. locally cohomologically bounded, bounded below, bounded above).

2. Derived moduli stacks of coherent sheaves

We let

$$\text{Perf} : \text{dAff}^{\text{op}} \to \mathcal{S}$$

be the derived moduli stack of perfect complexes, defined by

$$\text{Perf}(\text{Spec}(A)) := \text{Perf}(A)^\heartsuit,$$
where \((-\otimes_{\mathbf{S}})^{\circ} \colon \text{Cat}_{\infty, \mathbf{S}} \to \mathbf{S}\) is the maximal \(\infty\)-groupoid functor. Recall from [TVa07, Proposition 3.7] that it is a locally geometric derived stack, locally of finite presentation.

Let \(X \in \text{dSt}\) be a derived stack. We let
\[
\text{Perf}(X) := \text{Map}(X, \text{Perf})
\]
be the derived stack of perfect complexes on \(X\). Recall from [Lur12, Proposition 3.3.8] that if \(X\) is a proper flat scheme then \(\text{Perf}(X)\) is a locally geometric stack, locally of finite presentation.

**Definition 2.1.** Let \(f \colon X \to Y\) be a morphism of derived geometric stacks. We say that an almost perfect complex \(F \in \text{APerf}(X)\) has tor-amplitude \(\leq n\) relative to \(Y\) if for every map \(x \colon \text{Spec}(A) \to Y\) there exists an affine atlas \(U = \text{Spec}(B)\) of \(X_A := \text{Spec}(A) \times_Y X\) such that \(u^*(F)\) has tor-amplitude \(\leq n\) relative to \(A\).\(^{13}\) where \(u : U \to X_A \to X\) is the induced map.

Let \(X\) be a derived geometric stack. We let \(\text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A))\) denote the full subcategory of \(\text{APerf}(X \times \text{Spec}(A))\) spanned by those almost perfect complexes \(F\) having tor-amplitude \(\leq 0\) relative to \(\text{Spec}(A)\).

**Remark 2.2.** Let \(A \in \text{CAlg}_{\mathbf{C}}\) be a derived commutative ring and let \(M \in A\text{-Mod}\). Then \(M\) has tor-amplitude \(\leq n\) if and only if \(M \otimes_A \pi_0(A)\) has tor-amplitude \(\leq n\). In particular, if \(A\) is underived and \(M \in A\text{-Mod}^{\text{fd}}\), then \(M\) has tor-amplitude \(0\) if and only if \(M\) is flat in the sense of usual commutative algebra.

The following lemma is a variation of the local criterion for flatness.

**Lemma 2.3.** Let \(X\) be a derived geometric stack and let \(S = \text{Spec}(A) \in \text{dAff}\) be a derived affine scheme. Let \(p : X \to S\) be a morphism and let \(F \in \text{APerf}(X)\) be an almost perfect complex. Suppose furthermore that

1. \(F\) is flat relative to \(S\);
2. for every geometric point \(s : \text{Spec}(K) \to S\), let \(X_s := \text{Spec}(K) \times_S X\) and let \(j_s : X_s \to X\) be the natural map. Then the pullback \(j_s^*(F) \in \text{APerf}(X_s)\) is in tor-amplitude \([a, b]\).

Then \(F\) is in tor-amplitude \([a, b]\), and in particular it belongs to \(\text{Perf}(X)\).

**Proof.** The question is local on \(X\), and we can therefore assume \(X\) to be a derived affine scheme, say \(X = \text{Spec}(B)\). Given a geometric point \(x : \text{Spec}(K) \to X\), we let \(B_{(x)}\) denote the localization
\[
B_{(x)} := \lim_{\substack{\longrightarrow \ U \in \mathcal{U}_X}} \mathcal{O}_X(U),
\]
where the colimit ranges over all the open Zariski neighborhoods of the image of \(x\) inside \(X\). It is then enough to prove that for each such geometric point, \(F \otimes_B B_{(x)}\) is contained in tor-amplitude \([a, b]\).

Given \(x : \text{Spec}(K) \to X\) let \(s := p \circ x : \text{Spec}(K) \to S\). By assumption \(j_s^*(F) \in \text{APerf}(X_s)\) is in tor-amplitude \([a, b]\). Let \(\pi : X \to \text{Spec}(B_{(x)})\) be the induced point. Then \(x = j_s \circ \pi\), and therefore \(x^*(F) \simeq \pi^*(j_s^*(F))\) is in tor-amplitude \([a, b]\). Let \(\kappa\) denote the residue field of the local ring \(\pi_0(B_{(x)})\). Since the map \(\kappa \to K\) is faithfully flat, we can assume without loss of generality that \(K = \kappa\). Let \(m\) be the maximal ideal of \(\pi_0(B_{(x)})\) and let \(\hat{B}_m\) denote the formal completion of \(B_{(x)}\) at \(m\) (see [Lur18, Notation 7.3.1.5 & Remark 8.1.2.4]). Then [Lur18, Corollary 7.3.6.9] implies that the map \(B_{(x)} \to \hat{B}_m\) is faithfully flat. Thus, we are reduced to check that \(F \otimes_B \hat{B}_m\) is in tor-amplitude \([a, b]\). This follows directly from [Lur18, Corollary 8.3.5.8].

**Lemma 2.4.** Let \(X \in \text{dSt}\) be a derived geometric stack and let \(f : \text{Spec}(B) \to \text{Spec}(A)\) be a morphism of derived affine schemes. Let \(f_X := \text{id}_X \times f : X \times \text{Spec}(B) \to X \times \text{Spec}(A)\). If \(F \in \text{APerf}(X \times \text{Spec}(A))\) has tor-amplitude \(\leq n\) relative to \(\text{Spec}(A)\), then \(f_X^*(F) \in \text{APerf}(X \times \text{Spec}(B))\) has tor-amplitude \(\leq n\) relative to \(\text{Spec}(B)\).

\(^{13}\)Cf. [Lur17, Definition 7.2.4.21] for the definition of tor-amplitude \(\leq n\).
Proof. Let \( q: U \to X \) be a smooth morphism with \( U \) an affine scheme. Consider the derived pullback square

\[
\begin{array}{c}
U \times \text{Spec}(B) \xrightarrow{f_U} U \times \text{Spec}(A) \\
\downarrow p_B \quad \quad \quad \quad \quad \quad \downarrow p_A \\
\text{Spec}(B) \xrightarrow{f} \text{Spec}(A)
\end{array}
\]

We have to prove that \( p_B f_U^*(q^*F) \) has tor-amplitude \( \leq n \) relative to \( B \). The base change formula shows that \( p_B f_U^*(q^*F) \simeq f^* p_A (q^*F) \). As \( p_A (q^*F) \) has tor-amplitude \( \leq n \) relative to \( A \) by assumption, the conclusion follows. \( \square \)

Let \( X \in \text{dSt} \) be a derived geometric stack. The above lemma allows us to define a derived stack

\[
\text{Coh}(X): \text{dAff}^{\text{op}} \to \mathcal{S}
\]

informally defined by sending \( \text{Spec}(A) \in \text{dAff} \) to

\[
\text{Coh}(X)(\text{Spec}(A)) := \text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A))^{\simeq}.
\]

\textbf{Remark 2.5.} Note that there exists a natural map \( \text{Coh}(X) \to \text{Perf}(X) \) which is formally étale. \( \triangle \)

When \( X \) is smooth and proper, the stack \( \text{Coh}(X) \) is a geometric stack locally almost of finite presentation. However the proof of this statement requires some work. We will apply Lurie’s representability theorem [Lur18, Theorem 18.1.0.2].

\textbf{Lemma 2.6.} Let \( X \) be a smooth complex scheme. Then the truncation \( ^d \text{Coh}(X) \) coincides with the usual stack of coherent sheaves on \( X \).

\textbf{Proof.} Let \( \text{Spec}(A) \) be an underived affine scheme. By definition,

\[
\text{Coh}(X)(\text{Spec}(A)) \simeq \text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A))^{\simeq}.
\]

Let \( F \in \text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A)) \). By definition, \( F \) is an element in \( \text{Perf}(X \times \text{Spec}(A)) \), which is furthermore flat relative to \( \text{Spec}(A) \). As \( A \) is underived, this shows that the image of \( F \) in \( \text{APerf}(X \times \text{Spec}(A)) \) belongs to \( \text{APerf}^\circ (X \times \text{Spec}(A)) \), hence that \( F \simeq \pi_0(F) \) is a discrete coherent sheaf on \( X \times \text{Spec}(A) \) which is flat relative to \( \text{Spec}(A) \).

Suppose vice-versa that \( F \) is a discrete coherent sheaf on \( X \times \text{Spec}(A) \) which is flat relative to \( \text{Spec}(A) \). We have to prove that it belongs to \( \text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A)) \). This amounts to check that \( F \) actually belongs to \( \text{Perf}(X \times \text{Spec}(A)) \). As \( X \) is smooth, we see that for every closed point \( p: \text{Spec} \to \text{Spec}(A) \), the pullback \(( \text{id}_X \times p )^* (F) \in \text{APerf}(X) \) actually belongs to \( \text{Perf}(X) \). The conclusion now follows from Lemma 2.3. \( \square \)

\textbf{Lemma 2.7.} Let \( X \in \text{dSt} \) be a derived geometric stack. Then derived stack \( \text{Coh}(X) \) is infinitesimally cohesive and nilcomplete.

\textbf{Proof.} We start by proving that \( \text{Coh}(X) \) is infinitesimally cohesive. Let \( S = \text{Spec}(A) \) be an affine derived scheme and let \( M \in \text{QCo}(S)^{\geq 1} \) be a quasi-coherent complex. Let \( S[M] := \text{Spec}(A \oplus M) \) and let \( d: S[M] \to S \) be a derivation. Finally, let \( S_d[M] \) be the pushout

\[
\begin{array}{c}
S[M] \xrightarrow{d} S \\
\downarrow d_0 \quad \quad \quad \downarrow f_0 \\
S \xrightarrow{f} S_d[M],
\end{array}
\]
where \( d_0 \) denotes the zero derivation. Since the functor \((-)^\ast: \text{Cat}_\infty \to S\) commutes with limits, it is enough to prove that the natural map
\[
\text{Coh}_{S_d[M]}(X \times S_d[M]) \to \text{Coh}_S(X \times S) \times_{\text{Coh}_{S_d[M]}(X \times S_d[M])} \text{Coh}_S(X \times S)
\]
is an equivalence. Using [Lur18, Theorem 16.2.0.1] we see that the natural map
\[
\text{A Perf}(X \times S) \times_{\text{A Perf}(X \times S_d[M])} \text{A Perf}(X \times S)
\]
is an equivalence. Let \( \varphi, \varphi_0: X \times S_d[M] \to X \) be the two morphisms induced by \( f \) and \( f_0 \), respectively. Lemma 2.4 guarantees that if \( \mathcal{F} \in \text{Coh}_{S_d[M]}(X \times S_d[M]) \), then both \( \varphi^\ast(\mathcal{F}) \) and \( \varphi_0^\ast(\mathcal{F}) \) belong to \( \text{Coh}_S(X \times S) \). Vice-versa, let \( \mathcal{F} \in \text{A Perf}(X \times S_d[M]) \) be such that \( \varphi^\ast(\mathcal{F}), \varphi_0^\ast(\mathcal{F}) \in \text{Coh}_S(X \times S) \). We want to prove that \( \mathcal{F} \) belongs to \( \text{Coh}_{S_d[M]}(X \times S_d[M]) \). This question is local on \( X \), and we can therefore assume that \( X \) is affine. Let \( p: X \times S \to S \) and \( q: X \times S_d[M] \to S_d[M] \) be the natural projections. Then
\[
f^\ast q_\ast(\mathcal{F}) \simeq p_\ast \varphi^\ast(\mathcal{F}) \quad \text{and} \quad f_0^\ast q_\ast(\mathcal{F}) \simeq p_\ast \varphi_0^\ast(\mathcal{F})
\]
have tor-amplitude zero. The conclusion now follows from [Lur18, Proposition 16.2.3.1-(3)].

We now prove that \( \text{Coh}(X) \) is nilcomplete. Let therefore \( S = \text{Spec}(A) \) be an affine derived scheme and let \( S_n := \text{Spec}(\tau_{\leq n}(A)) \). We know that the natural map
\[
\text{A Perf}(X \times S) \to \text{lim}_n \text{A Perf}(X \times S_n)
\]
is an equivalence. Lemma 2.4 implies that this map restricts to a functor
\[
\text{Coh}_S(X \times S) \to \text{lim}_n \text{Coh}_{S_n}(X \times S_n) .
\]
Given \( \mathcal{F} \in \text{A Perf}(X \times S) \) denote by \( \mathcal{F}_n \) its image in \( \text{A Perf}(X \times S_n) \). We wish to show that if each \( \mathcal{F}_n \) belongs to \( \text{Coh}_{S_n}(X \times S_n) \). Since the squares
\[
x \times S \to X \times S_n \\
\downarrow \quad \downarrow \\
S \to S_n
\]
are derived pullback, we can use the derived base change to reduce ourselves to check that the equivalence
\[
\text{Q Coh}(S) \to \text{lim}_n \text{Q Coh}(S_n)
\]
respects tor-amplitude 0. This follows at once from [Lur18, Proposition 2.7.3.2-(c)]. \( \square \)

**Lemma 2.8.** Let \( X \) be a smooth and proper complex scheme. Then the derived stack \( \text{Coh}(X) \) admits a global cotangent complex.

**Proof.** Let \( S = \text{Spec}(A) \) be a derived affine scheme and let \( x: S \to \text{Coh}(X) \) be a morphism. Let \( \mathcal{F} \in \text{Coh}_{\text{Spec}(A)}(X \times \text{Spec}(A)) \) be corresponding coherent complex on \( X \times S \) relative to \( S \). Notice that since \( X \) is smooth, Lemma 2.3 implies that \( \mathcal{F} \in \text{Perf}(X \times S) \). Let
\[
F := S \times_{\text{Coh}(X)} S
\]
be the loop stack based at \( x \) and let \( \delta_x: S \to F \) be the induced morphism. Since \( \text{Coh}(X) \) is infinitesimally cohesive thanks to Lemma 2.7, we see that [TV08, Proposition 1.4.1.11] implies that \( \text{Coh}(X) \) admits a cotangent complex at \( x \) if and only if \( F \) admits a cotangent complex at \( \delta_x \). We have to prove that the functor
\[
\text{Der}_F(A; -): A\text{-Mod} \to S
\]
defined by
\[
\text{Der}_F(A; M) := \text{fib}(F(S[M]) \to F(S))
\]
is representable by an eventually connective module. Here $S[M] := \text{Spec}(A \oplus M)$, and the fiber is taken at the point $x$. We observe that

$$F(S[M]) \simeq \text{fib}(\text{Map}_{\text{Coh}(X \times S)}(d_0^0(F), d_0^0(F)) \to \text{Map}_{\text{Coh}(X \times S)}(F, F)),$$

the fiber being taken at the identity of $F$. Unraveling the definitions, we therefore see that

$$\text{Der}_F(A; M) \simeq \text{Map}_{\text{Coh}(X \times S)}(F, F \otimes p^*M),$$

where $p: X \times S \to S$ is the canonical projection. Since $F \in \text{Perf}(X \times S)$, we can rewrite the above mapping space as

$$\text{Map}_{\text{Coh}(X \times S)}(F \otimes F^\vee, p^*M).$$

Finally, since $X$ is smooth and proper, [Lur18, Proposition 6.4.5.3] shows that the functor $p^*: \text{QCoh}(S) \to \text{QCoh}(X \times S)$ admits a left adjoint

$$p_+: \text{QCoh}(X \times S) \longrightarrow \text{QCoh}(S).$$

Therefore,

$$\text{Der}_F(A; M) \simeq \text{Map}_{\text{Coh}(S)}(p_+(F \otimes F^\vee), M).$$

Therefore, $F$ admits a cotangent complex at $\delta_x$, and therefore $\text{Coh}(X)$ admits a cotangent complex at the point $x$, which is given by the formula

$$\mathbf{L}_{\text{Coh}(X,x)} \simeq p_+(F \otimes F^\vee)[1].$$

Finally, we see that $\text{Coh}(X)$ admits a global cotangent complex: this is a straightforward consequence of the derived base change theorem for the functor $p_+$ (see [Lur18, Proposition 6.4.5.4]).

**Proposition 2.9.** Let $X$ be a smooth and proper complex scheme. Then the derived stack $\text{Coh}(X)$ is geometric and locally of finite presentation.

**Proof.** Lemma 2.6 implies that $\mathcal{d}\text{Coh}(X)$ coincides with the usual stack of coherent sheaves on $X$, which we know being a geometric stack (cf. [LMB00, Théorème 4.6.2.1]). On the other hand, combining Lemmas 2.7 and 2.8 we see that $\text{Coh}(X)$ is infinitesimally cohesive, nilcomplete and admits a global cotangent complex. Therefore the assumptions of Lurie’s representability theorem [TV08, Theorem C.0.9] are satisfied and the conclusion follows.

Let $X$ be a smooth and proper scheme over $\mathbb{C}$. Informally, $\text{Coh}(X)$ is the derived stack parameterizing $S$-families of perfect complexes on $X$ of tor-amplitude $\leq 0$ relative to $S$. We shall call $\text{Coh}(X)$ the derived stack of coherent sheaves on $X$. Such a terminology will be justified by the following results.

**Corollary 2.10.** Let $X$ be a smooth and proper complex scheme of dimension $n$. Then the cotangent complex $\mathbf{L}_{\text{Coh}(X)}$ is perfect and has tor-amplitude $[-1, n - 1]$. In particular, $\text{Coh}(X)$ is smooth when $X$ is a curve and derived l.c.i. when $X$ is a surface.

**Proof.** It is enough to check that for every point $x: \text{Spec}(A) \to \text{Coh}(X)$, $x^*\mathcal{T}_{\text{Coh}(X)}$ is perfect and in tor-amplitude $[1 - n, 1]$. Let $F \in \text{Perf}(X \times \text{Spec}(A))$ be the perfect complex classified by $x$ and let $p: X \times \text{Spec}(A) \to \text{Spec}(A)$ be the canonical projection. Then Lemma 2.8 shows that

$$x^*\mathcal{T}_{\text{Coh}(X)} \simeq p_*\text{End}(F)[1].$$

Since $p$ is proper and smooth, the pushforward $p_*$ preserves perfect complexes (see [Lur18, Theorem 6.1.5.2]). As $\text{End}(F) \simeq F \otimes F^\vee$ is perfect, we therefore can conclude that $x^*\mathcal{T}_{\text{Coh}(X)}$ is perfect.

We are then left to check that it is in tor-amplitude $[1 - n, 1]$. Let $j: \mathcal{d}(\text{Spec}(A)) \to \text{Spec}(A)$ be the canonical inclusion. It is enough to prove that $j^*x^*\mathcal{T}_{\text{Coh}(X)}$ has tor-amplitude $[1 - n, 1]$. In
other words, we can assume \text{Spec}(A) to be underived. In this case, for \(m \gg 0\) we can represent \(\mathcal{F}\) as a \((n+1)\)-term complex of vector bundles (see, e.g., the proof of [LMB00, Théorème 4.6.2.1])

\[
\mathcal{O}(-m)^{\oplus k_0} \longrightarrow \ldots \longrightarrow \mathcal{O}(-m)^{\oplus k_1} \longrightarrow \mathcal{O}(-m)^{\oplus k_n},
\]

and therefore

\[
\text{End}(\mathcal{F}) \cong \mathcal{F}(m)^{\oplus k_0} \longrightarrow \mathcal{F}(m)^{\oplus k_1} \longrightarrow \ldots \longrightarrow \mathcal{F}(m)^{\oplus k_n}.
\]

Serre’s vanishing theorem guarantees that \(R^l p_* (\mathcal{F}(m)^{\oplus k_i}) = 0\) for \(m \gg 0, \ell \geq 1, i = 0, \ldots, n\). So, we deduce that

\[
\pi_{-h}(p_* \text{End}(\mathcal{F})[1]) = 0,
\]

for \(h \geq n\). This completes the proof.

We are also interested in relaxing the properness assumption on the scheme \(X\). For such a reason, we want to consider almost perfect complexes with proper support.

**Definition 2.11.** Let \(X\) be a complex scheme. We say that an almost perfect complex \(\mathcal{F} \in \mathcal{A}_\text{Perf}(X)\) has proper support if there exists a proper scheme \(Z \subset X\) such that \(i^* (\mathcal{F}) \cong 0\), where \(i: X \hookrightarrow Z \rightarrow X\) is the natural inclusion morphism.

By arguing as above, one can prove the following.

**Proposition 2.12.** Let \(X\) be a smooth complex scheme. Then there exists a derived stack \(\text{Coh}_{\text{prop}}(X)\), parameterizing \(S\)-families of almost perfect complexes of tor-amplitude \(\leq 0\) relative to \(S\) and with proper support. \(\text{Coh}_{\text{prop}}(X)\) is geometric and locally of finite presentation.

**Proof.** Since \(\mathcal{C}_{\text{Coh}_{\text{prop}}}(X)\) is a geometric stack by [KV19, Proposition 4.1.1], the same arguments as in the proof of Proposition 2.9 applies in this case.

\[\square\]

### 2.1. Other examples of moduli stacks.

Let \(X \in \mathcal{dSt}\) be a derived geometric stack. We introduce the derived stack of vector bundles on \(X\) as

\[
\text{Bun}(X) := \bigsqcup_{n \geq 0} \text{Map}(X, B\text{GL}_n).
\]

It is an open substack of \(\text{Coh}(X)\).

Let \(X\) be a smooth and proper complex scheme. Then, \(\text{Bun}(X)\) is geometric and locally of finite presentation thanks to Proposition 2.9. In addition, the truncation \(\mathcal{d}\text{Bun}(X)\) of \(\text{Bun}(X)\) is the underived stack of vector bundles on \(X\).

Assume that \(X\) is projective over \(\mathbb{C}\). Recall that for any polynomial \(P(m) \in \mathbb{Q}[m]\) there exists an open substack \(\mathcal{d}\text{Coh}^{P(m)}(X)\) of \(\mathcal{C}_{\text{Coh}}(X)\) parameterizing flat families of coherent sheaves on \(X\) with fixed Hilbert polynomial \(P(m)\); we denote by \(\text{Coh}^{P(m)}(X)\) its canonical derived enhancement\(^{14}\). Similarly, we define \(\text{Bun}^{P(m)}(X)\).

For any nonzero polynomial \(P(m) \in \mathbb{Q}[m]\) of degree \(d\), we denote by \(P(m)^\text{red}\) its reduced polynomial, which is given as \(P(m)/a_d\), where \(a_d\) is the leading coefficient of \(P(m)\). Given a monic polynomial \(P(m) \in \mathbb{Q}[m]\), define

\[
\text{Coh}^{P(m)}(X) := \bigsqcup_{P(m)/\mathbb{Q}[m]} \text{Coh}^{P(m)}(X) \quad \text{and} \quad \text{Bun}^{P(m)}(X) := \bigsqcup_{P(m)/\mathbb{Q}[m]} \text{Bun}^{P(m)}(X).
\]

\(^{14}\) The construction of such a derived enhancement follows from [STV15, Proposition 2.1].
Assume that deg(p(m)) = dim(X). Recall that the Gieseker semistability is an open property\footnote{Cf. [HL10, Definition 1.2.4] for the definition of semistability of coherent sheaves on projective schemes and [HL10, Proposition 2.3.1] for the openness property in families of the semistability.}. Thus there exists an open substack \( \text{Coh}^{ss, p(m)}(X) \) of \( \text{Coh}^{p(m)}(X) \) parameterizing families of semistable coherent sheaves on \( X \) with fixed reduced polynomial \( p(m) \); we denote by \( \text{Coh}^{ss, p(m)}(X) \) its canonical derived enhancement. Similarly, we define \( \text{Bun}^{ss, p(m)}(X) \).

Finally, let \( 0 \leq d \leq \dim(X) \) be an integer and define
\[
\text{Coh}^{\leq d}(X) := \coprod_{\substack{p(m) \in \mathbb{Q}[m] \\
\deg(p(m)) \leq d}} \text{Coh}^{p(m)}(X).
\]

**Remark 2.13.** Let \( X \) be a smooth projective complex curve. Then the assignment of a monic polynomial \( p(m) \in \mathbb{Q}[m] \) of degree two is equivalent to the assignment of a slope \( \mu \in \mathbb{Q} \). In addition, in the one-dimensional case we have \( \text{Bun}^{ss, \mu}(X) \simeq \text{Coh}^{ss}(X) \). \( \triangle \)

Assume that \( X \) is only quasi-projective. As above, we can define the derived moduli stack \( \text{Coh}^{\leq d}_{\text{prop}}(X) \) of coherent sheaves on \( X \) with proper support and dimension of the support less or equal \( d \).

### 2.2. Coherent sheaves on the shapes.

In the previous section we introduced the stack of coherent sheaves on a geometric stack. In this paper however we will be also concerned with coherent sheaves over Simpson’s shapes \( X_{d\text{R}}, X_B, X_{d\text{Vr}} \) and \( X_{d\text{Del}} \), where \( X \) is a smooth and proper scheme over \( \mathbb{C} \) (cf. §A for a recollection of Simpson’s shapes). For this reason we need a slightly more general version of Definition 2.1.

**Definition 2.14.** Let \( Y \in \text{dSt} \) be a derived stack and let \( u: U \to Y \) be a flat effective epimorphism from a derived geometric stack \( U \). Let \( S = \text{Spec}(A) \in \text{dAff} \) be a derived affine scheme. We say that an almost perfect complex \( F \in \text{APerf}(S \times Y) \) has tor-amplitude \( \leq n \) relative to \( S \) with respect to the map \( u \) if \((\text{id}_S \times u)^*(F)\) has tor-amplitude \( \leq n \) relative to \( S \).

Given \( Y \in \text{dSt} \) and \( u: U \to X \) as in the above definition, we define
\[
\text{Coh}(Y, u) := \text{Perf}(Y) \times_{\text{Perf}(U)} \text{Coh}(U).
\]

When the map \( u: U \to Y \) is clear from the context, we will often abuse notations and write \( \text{Coh}(Y) \) instead of \( \text{Coh}(Y, u) \). Thus, \( \text{Coh}(Y, u) \) is the derived stack parameterizing \( S \)-families of almost perfect complexes on \( Y \) which have tor-amplitude \( \leq 0 \) relative to \( S \) with respect to the map \( u \).

Similarly, we define
\[
\text{Bun}(Y, u) := \text{Perf}(Y) \times_{\text{Perf}(U)} \text{Bun}(U).
\]

#### 2.2.1. Betti shape.

Let \( K \in S \) be a finite space. Let \( I := \pi_0(K) \) and choose a section \( x: I \to K \) of the natural map \( K \to \pi_0(K) \). We set \( Y := K_B \) and \( U := \text{Spec}(C)^I \simeq I_B \). We let \( u: U \to Y \) be the map induced by \( x \). Notice that it is a flat effective epimorphism by Lemma A.1-(2). We consider the stack \( \text{Coh}(K_B, u) \).

**Lemma 2.15.** There is a canonical equivalence
\[
\text{Coh}(K_B, u) \simeq \text{Bun}(K_B, u).
\]

**Proof.** We can review both \( \text{Coh}(K_B, u) \) and \( \text{Bun}(K_B, u) \) as full substacks of \( \text{Map}(K_B, \text{Perf}) \). It is therefore enough to show that they coincide as substacks of \( \text{Map}(K_B, \text{Perf}) \). Suppose first that \( K \) is discrete. Then it is equivalent to a disjoint union of finitely many points, and therefore
\[
K_B \simeq \text{Spec}(C)^I \simeq \text{Spec}(C) \sqcup \text{Spec}(C) \sqcup \cdots \sqcup \text{Spec}(C).
\]
In this case
\[
\text{Map}(K_B, \text{Perf}) \simeq \text{Perf} \times \text{Perf} \times \cdots \times \text{Perf}.
\]
If \(S \in \text{dAff}\), an \(S\)-point of \(\text{Map}(K_B, \text{Perf})\) is therefore identified with an object in \(\text{Fun}(I, \text{Perf}(S))\). Being of tor-amplitude 0 with respect to \(S\) is equivalent to being of tor-amplitude 0 on \(S^I\), and therefore the conclusion follows in this case. Using the equivalence \(S^{k+1} \simeq \Sigma(S^k)\), we deduce that the same statement is true when \(K\) is a sphere. We now observe that since \(K\) is a finite space, we can find a sequence of maps
\[
K_0 = \emptyset \to K_1 \to \cdots \to K_\ell = K,
\]
such that each map \(K_i \to K_{i+1}\) fits in a pushout diagram
\[
\begin{array}{ccc}
S^m & \to & \ast \\
\downarrow & & \downarrow \\
K_i & \to & K_{i+1}
\end{array}
\]
The conclusion therefore follows by induction. \(\square\)

Notice that the above lemma shows that \(\text{Coh}(K_B, u)\) is independent of the choice of the map \(x: I \to K\).

Now let \(X\) be a smooth and proper complex scheme. Define the stacks
\[
\text{Coh}_B(X) := \text{Coh}(X_B, u) \quad \text{and} \quad \text{Bun}_B(X) := \text{Bun}(X_B, u).
\]
These stacks are geometric and locally of finite presentation since \(\text{Perf}(X_B)\) is so. By Lemma 2.15, we have \(\text{Coh}_B(X) = \text{Bun}_B(X)\). We call it the derived Betti moduli stack of \(X\). In addition, we shall call
\[
\text{Bun}^n_B(X) := \text{Map}(X_B, \text{BGL}_n)
\]
the derived stack of \(n\)-dimensional representations of the fundamental group \(\pi_1(X)\) of \(X\). The terminology is justified by Lemma A.1-(3).

**Remark 2.16.** Assume that \(X\) is a smooth projective complex curve. Then \(\text{Bun}^n_B(X)\) can be obtained as a quasi-Hamiltonian derived reduction\(^{16}\). Indeed, let \(X'\) be the topological space \(X^{\text{top}}\) minus a disk \(D\). Then one can easily see that \(X'\) deformation retracts onto a wedge of \(2g_X\) circles, where \(g_X\) is the genus of \(X\). We get
\[
\text{Bun}^n_B(X) \simeq \text{Bun}^n_B(X') \times_{\text{Bun}^n_B(S^1)} \text{Bun}^n_B(D).
\]
Since \(\text{Bun}^n_B(S^1) \simeq [\text{GL}_n/\text{GL}_n]\) (see, e.g., [Cal14, Example 3.8]), and \(\text{Bun}^n_B(D) \simeq \text{Bun}^n_B(\text{pt})\), we obtain
\[
\text{Bun}^n_B(X) \simeq \text{Bun}^n_B(X') \times_{[\text{GL}_n/\text{GL}_n]} [\text{pt}/\text{GL}_n].
\]
Thus, \(\text{Bun}^n_B(X)\) is the quasi-Hamiltonian derived reduction of \(\text{Bun}^n_B(X')\). By further using \(\text{Bun}^n_B(X') \simeq \text{Bun}^n_B(S^1)^{\times 2g_X}\), the derived stack \(\text{Bun}^n_B(X)\) reduces to
\[
\text{Bun}^n_B(X) \simeq [\text{GL}_n^{\times 2g_X} \times \text{GL}_n \text{pt}/\text{GL}_n].
\]
\(\triangle\)

\(^{16}\)Cf. [Saf16] for the notion of Hamiltonian reduction in the derived setting.
2.2.2. de Rham shape. Let $X$ be a smooth, proper and connected scheme over $\mathbb{C}$. Thanks to Lemma A.2, we take $Y := X_{\text{dr}}$, $U := X$ and $u := \lambda_X$ be the canonical map. Define the stacks

$$\text{Coh}_{\text{dr}}(X) := \text{Coh}(X_{\text{dr}}, \lambda_X) \quad \text{and} \quad \text{Bun}_{\text{dr}}(X) := \text{Bun}(X_{\text{dr}}, \lambda_X).$$

These stacks are geometric and locally of finite presentation since $\text{Perf}(X_{\text{dr}})$ is so$^{17}$.

**Lemma 2.17.** There is a natural equivalence

$$\text{Coh}_{\text{dr}}(X) \simeq \text{Bun}_{\text{dr}}(X).$$

**Proof.** We can see both derived stacks as full substacks of $\text{Map}(X_{\text{dr}}, \text{Perf})$. Let $S \in \text{dAff}$ and let $x : S \to \text{Map}(X_{\text{dr}}, \text{BGL}_n)$. Then $x$ classifies a perfect complex $\mathcal{F} \in \text{Perf}(X_{\text{dr}} \times S)$ such that $\mathcal{G} := (\lambda_X \times \text{id}_S)^*(\mathcal{F}) \in \text{Perf}(X \times S)$ has tor-amplitude $0$ and rank $n$. Since the map $X \times S \to S$ is flat, it follows that $\mathcal{G}$ has tor-amplitude $0$ relative to $S$, and therefore that $x$ determines a point in $\text{Coh}_{\text{dr}}(X)$.

Vice-versa, let $x : S \to \text{Coh}_{\text{dr}}(X)$. Let $\mathcal{F} \in \text{Perf}(X_{\text{dr}} \times S)$ be the corresponding perfect complex and let $\mathcal{G} := (\lambda_X \times \text{id}_S)^*(\mathcal{F})$. Then by assumption $\mathcal{G}$ has tor-amplitude $0$ relative to $S$. We wish to show that it has tor-amplitude $0$ on $X \times S$. Using Lemma 2.3, we see that it is enough to prove that for every geometric point $s : \text{Spec}(K) \to S$, the perfect complex $j^*(\mathcal{G}) \in \text{Perf}(X_K)$ has tor-amplitude $0$. Here $X_K := \text{Spec}(K) \times X$ and $j : X_K \to X$ is the natural morphism. Consider the commutative diagram

$$\begin{array}{ccc}
X_K & \xrightarrow{j} & X \times S \\
\downarrow{\lambda_X} & & \downarrow{\lambda_X \times \text{id}_S} \\
(X_K)_{\text{dr}} & \xrightarrow{j_{\text{dr}}} & X_{\text{dr}} \times S
\end{array}$$

Then

$$j^{*}_{\text{dr}} \mathcal{G} \simeq \lambda_X^*j^{*}_{\text{dr}} \mathcal{F}.$$ 

We therefore see that $j^* \mathcal{G}$ comes from a $K$-point of $\text{Coh}_{\text{dr}}(X)$. By [HTT08, Theorem 1.4.10], $j^* \mathcal{G}$ is a vector bundle on $X$, i.e. it has tor-amplitude $0$. The conclusion follows.

We shall call $\text{Coh}_{\text{dr}}(X)$ the derived de Rham moduli stack of $X$.

2.2.3. Dolbeault shape. Let $X$ be a smooth, proper and connected complex scheme. We take $Y := X_{\text{Dol}}$ (resp. $Y := X_{\text{Dol}n}$), $U := X$ and $u := \kappa_X$ (resp. $u := \kappa_X^n$). Then by Lemma A.3, we define

$$\text{Coh}_{\text{Dol}}(X) := \text{Coh}(X_{\text{Dol}}, \kappa_X) \quad \text{and} \quad \text{Coh}_{\text{Dol}n}(X) := \text{Coh}(X_{\text{Dol}n}, \kappa_X^n),$$

and

$$\text{Bun}_{\text{Dol}}(X) := \text{Bun}(X_{\text{Dol}}, \kappa_X) \quad \text{and} \quad \text{Bun}_{\text{Dol}n}(X) := \text{Bun}(X_{\text{Dol}n}, \kappa_X^n).$$

These stacks are geometric and locally of finite presentation since $\text{Perf}(X_{\text{Dol}})$ and $\text{Perf}(X_{\text{Dol}n})$ are so$^{18}$.

We call $\text{Coh}_{\text{Dol}}(X)$ the derived Dolbeaut moduli stack of $X$, while $\text{Coh}_{\text{Dol}n}(X)$ is the derived nilpotent Dolbeaut moduli stack of $X$. The truncation $c^{\text{Dol}} \text{Coh}_{\text{Dol}}(X)$ (resp. $c^{\text{Dol}n} \text{Coh}_{\text{Dol}}(X)$) coincides with the moduli stack of Higgs sheaves (resp. nilpotent Higgs sheaves) on $X$.

We denote by $j_X : \text{Coh}_{\text{Dol}n}(X) \to \text{Coh}_{\text{Dol}}(X)$ and $j_{\text{Dol}}^n : \text{Bun}_{\text{Dol}n}(X) \to \text{Bun}_{\text{Dol}}(X)$ the canonical maps induced by $i_X : X_{\text{Dol}} \to X_{\text{Dol}n}$.

$^{17}$A possible way to prove the geometricity of $\text{Perf}(X_{\text{dr}})$ is to combine Simpson’s proof of the geometricity of the corresponding undervived stack [Sim09], Lemma 3.16 which implies the existence of the cotangent complex for $\text{Perf}(X_{\text{dr}})$, and Lurie’s representability theorem [TV08, Theorem C.0.9].

$^{18}$By following similar arguments as in the previous footnote, one proves the geometricity of $\text{Perf}(X_{\text{Dol}})$. By using Lemma 3.20, also one gets the geometricity of $\text{Perf}(X_{\text{Dol}n})$. 
Remark 2.18. Let $X$ be a smooth and proper complex scheme. Define the derived geometric stack
\[ \text{Higgs}^{\text{naïf}}(X) := \mathbb{T}^*\![0] \text{Coh}(X) = \text{Spec}_{\text{Coh}(X)}(\text{Sym}(\mathbb{T}_{\text{Coh}(X)})). \]
There is a natural morphism
\[ \text{Coh}_{\text{Dol}}(X) \to \text{Higgs}^{\text{naïf}}(X), \]
which is an equivalence when $X$ is a smooth and projective curve (see, e.g., [GiR18]). In higher dimension, this morphism is no longer an equivalence. This is due to the fact that in higher dimensions the symmetric algebra and the tensor algebra on $\mathbb{T}_{\text{Coh}(X)}$ differ. \(\triangle\)

Let $X$ be a smooth projective complex scheme. For any monic polynomial $p(m) \in \mathbb{Q}[m]$, we set
\[ \text{Coh}^{\text{Dol}}_{\text{nil}, p(m)}(X) := \text{Perf}(X_{\text{Dol}}) \times_{\text{Perf}(X)} \text{Coh}^{p(m)}(X), \]
\[ \text{Coh}^{\text{Dol}}_{\text{nil}, p(m)}(X) := \text{Perf}(X_{\text{nil}}) \times_{\text{Perf}(X)} \text{Coh}^{p(m)}(X), \]
and
\[ \text{Bun}^{\text{Dol}}_{\text{nil}, p(m)}(X) := \text{Perf}(X_{\text{nil}}) \times_{\text{Perf}(X)} \text{Bun}^{p(m)}(X), \]
\[ \text{Bun}^{\text{Dol}}_{\text{nil}, p(m)}(X) := \text{Perf}(X_{\text{nil}}) \times_{\text{Perf}(X)} \text{Bun}^{p(m)}(X), \]
These are geometric stacks locally of finite presentation.

As shown by Simpson [Sim94a, Sim94b], the higher dimensional analogue of the semistability condition for Higgs bundles on a curve (introduced, e.g., in [Nit91]) is an instance of the Gieseker stability condition for modules over a sheaf of rings of differential operators, when such a sheaf is induced by $\Omega^1_X$ with zero symbol (see [Sim94a, §2] for details). This semistability condition is an open property for flat families (cf. [Sim94a, Lemma 3.7]). Thus, there exists an open substack $\text{Coh}^{\text{Dol}}_{\text{Dol}}(X)$ of $\text{Coh}^{\text{Dol}}_{\text{Dol}}(X)$ parameterizing families of semistable Higgs sheaves on $X$ with fixed reduced polynomial $p(m)$; we denote by
\[ \text{Coh}^{\text{Dol}}_{\text{Dol}}(X) \]
its canonical derived enhancement. Similarly, we define $\text{Coh}^{\text{Dol}}_{\text{Dol}}(X)$, $\text{Bun}^{\text{Dol}}_{\text{Dol}}(X)$ and $\text{Bun}^{\text{Dol}}_{\text{Dol}}(X)$. These are geometric stacks locally of finite presentation.

Finally, for any integer $0 \leq d \leq \dim(X)$, set
\[ \text{Coh}^{\text{Dol}}_{\text{Dol}}(X) := \text{Perf}(X_{\text{Dol}}) \times_{\text{Perf}(X)} \text{Coh}^{\leq d}(X), \]
\[ \text{Coh}^{\text{Dol}}_{\text{Dol}}(X) := \text{Perf}(X_{\text{Dol}}) \times_{\text{Perf}(X)} \text{Coh}^{\leq d}(X). \]
These are geometric stacks locally of finite presentation.

Remark 2.19. Let $X$ be a smooth projective complex curve and let $\mu \in \mathbb{Q}$ (which corresponds to a choice of a reduced Hilbert polynomial). Then one has $\text{Coh}^{\text{Dol}}_{\text{Dol}}(X) \simeq \text{Bun}^{\text{Dol}}_{\text{Dol}}(X)$ and $\text{Coh}^{\text{Dol}}_{\text{Dol}}(X) \simeq \text{Bun}^{\text{Dol}}_{\text{Dol}}(X).$ \(\triangle\)

3. Derived Moduli Stack of Extensions of Coherent Sheaves

Let $\Delta^1$ be the 1-simplex, and define the functor
\[ \text{Perf}^{\Delta^1 \times \Delta^1} : \text{dAff}^{op} \to \mathcal{S} \]
by
\[ \text{Perf}^{\Delta^1 \times \Delta^1}(\text{Spec}(A)) := \text{Fun}(\Delta^1 \times \Delta^1, \text{Perf}(A))^{\simeq}. \]
We let $\text{Perf}^{\text{ext}}$ denote the full substack of $\text{Perf}^{\Delta^1 \times \Delta^1}$ whose $\text{Spec}(A)$-points correspond to diagrams

\[
\begin{array}{ccc}
\mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\
\downarrow & & \downarrow \\
\mathcal{F}_4 & \longrightarrow & \mathcal{F}_3
\end{array}
\]

in $\text{Perf}(A)$ which are pullbacks and where $\mathcal{F}_4 \simeq 0$. Notice that

\[
\text{Perf}^{\Delta^1 \times \Delta^1} \simeq \mathcal{M}_{\text{Fun}(\Delta^1 \times \Delta^1, \text{Perf}(k))}.
\]

Since $\text{Fun}(\Delta^1 \times \Delta^1, \text{Perf}(k))$ is of finite type, [TVa07, Theorem 3.6] implies that $\text{Perf}^{\Delta^1 \times \Delta^1}$ is locally geometric and locally of finite presentation. Observe that the natural map $\text{Perf}^{\text{ext}} \rightarrow \text{Perf}^{\Delta^1 \times \Delta^1}$ is representable by Zariski open immersions. Therefore $\text{Perf}^{\text{ext}}$ is itself a locally geometric stack locally of finite presentation. There are three natural morphisms

\[ev_i: \text{Perf}^{\text{ext}} \longrightarrow \text{Perf}, \quad i = 1, 2, 3,
\]

which at the level of functor of points send a fiber sequence

\[\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3\]

to $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, respectively.

Let $X \in \text{dSt}$ be a derived geometric stack. We define

\[
\text{Perf}^{\Delta^1 \times \Delta^1}(X) := \text{Map}(X, \text{Perf}^{\Delta^1 \times \Delta^1}),
\]

and

\[
\text{Perf}^{\text{ext}}(X) := \text{Map}(X, \text{Perf}^{\text{ext}}).
\]

Once again, the morphism

\[
\text{Perf}^{\text{ext}}(X) \longrightarrow \text{Perf}^{\Delta^1 \times \Delta^1}(X) \tag{3.1}
\]

is representable by Zariski open immersions. Moreover, the morphism $ev_i$ induce a morphism $\text{Perf}^{\text{ext}}(X) \rightarrow \text{Perf}(X)$, which we still denote $ev_i$. We define $\text{Coh}^{\text{ext}}(X)$ as the pullback

\[
\begin{array}{ccc}
\text{Coh}^{\text{ext}}(X) & \longrightarrow & \text{Perf}^{\text{ext}}(X) \\
\downarrow & & \downarrow_{ev_1 \times ev_2 \times ev_3} \\
\text{Coh}(X) \times 3 & \longrightarrow & \text{Perf}(X) \times 3
\end{array}
\tag{3.2}
\]

Since the natural map $\text{Coh}(X) \rightarrow \text{Perf}(X)$ is formally étale, the same goes for

\[
\text{Coh}^{\text{ext}}(X) \longrightarrow \text{Perf}^{\text{ext}}(X).
\]

Similarly, we define $\text{Bun}^{\text{ext}}(X)$ as the pullback with respect to a diagram of the form (3.2), where we have substituted $\text{Coh}(X) \times 3$ with $\text{Bun}(X) \times 3$.

In the remaining of this section, we assume that $X$ is a smooth and proper complex scheme.

**Proposition 3.1.** Let $x: \text{Spec}(A) \rightarrow \text{Perf}^{\text{ext}}(X)$ be a morphism and let

\[\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3\]
be the fiber sequence in $\text{Perf}(X \times \text{Spec}(A))$ corresponding to $x$. Then $x^* \mathbb{L}_{\text{Perf}^\Delta(X)}[1]$ coincides with the colimit in $\text{Perf}(A)$ of the diagram

$$
p_+(F_2 \otimes F_3^\vee) \longrightarrow p_+(F_3 \otimes F_3^\vee) \quad , \quad (3.3)
$$

$$
p_+(F_1 \otimes F_2^\vee) \longrightarrow p_+(F_2 \otimes F_2^\vee) \quad , \\
p_+(F_1 \otimes F_1^\vee)
$$

where $p: X \times \text{Spec}(A) \to \text{Spec}(A)$ is the natural projection and $p_+: \text{QCoh}(X \times \text{Spec}(A)) \to \text{QCoh}(\text{Spec}(A))$ is the left adjoint to $p^*$ introduced in [Lur18, Proposition 6.4.5.3].

**Proof.** First, since the morphism (3.1) is a Zariski open immersion, we can instead compute the cotangent complex of $\text{Perf}^\Delta_1 \times \Delta_1^!(X)$ at the induced point, which we still denote by $x: \text{Spec}(A) \to \text{Perf}^\Delta_1 \times \Delta_1^!(X)$.

Write

$$F := \text{Spec}(A) \times _{\text{Perf}^\Delta_1 \times \Delta_1^!(X)} \text{Spec}(A),$$

and let $\delta_x: \text{Spec}(A) \to F$ be the diagonal morphism induced by $x$. In virtue of [TV08, Proposition 1.4.1.11] (see also [PY17, Proposition 7.9]) we know that $x^* \mathbb{L}_{\text{Perf}^\Delta_1 \times \Delta_1^!(X)} \simeq \delta_x^* \mathbb{L}_f[-1]$.

We therefore focus on the computation of $\delta_x^* \mathbb{L}_f$. Given $f: \text{Spec}(B) \to \text{Spec}(A)$, write $f_X$ for the induced morphism

$$f_X: X \times \text{Spec}(B) \longrightarrow X \times \text{Spec}(A).$$

We can identify $F(\text{Spec}(B))$ with the $\infty$-groupoid of commutative diagrams

$$f_X^* F_1 \longrightarrow f_X^* F_2 \longrightarrow f_X^* F_3 \quad ,$$

$$f_X^* F_1 \longrightarrow f_X^* F_2 \longrightarrow f_X^* F_3$$

in $\text{Perf}(X \times \text{Spec}(B))$, where $\alpha_1$, $\alpha_2$ and $\alpha_3$ are equivalences. In other words, $F(\text{Spec}(B))$ fits in the following limit diagram:

$$F(\text{Spec}(B)) \longrightarrow \square \longrightarrow \text{Aut}(f_X^* F_3) \quad ,$$

$$\square \longrightarrow \text{Aut}(f_X^* F_2) \longrightarrow \text{Map}(f_X^* F_2, f_X^* F_3) \quad ,$$

$$\text{Aut}(f_X^* F_1) \longrightarrow \text{Map}(f_X^* F_1, f_X^* F_2)$$

Here the mapping and automorphism spaces are taken in $\text{Perf}(X \times \text{Spec}(B))$. We have to represent the functor

$$\text{Der}_F(A; -): A\text{-Mod} \longrightarrow \mathcal{S}$$

which sends $M \in A\text{-Mod}$ to the space

$$\text{fib}_{\delta_x}(F(\text{Spec}(A \oplus M)) \longrightarrow F(\text{Spec}(A))).$$
Write $X_A := X \times \text{Spec}(A)$ and let $p : X_A \to \text{Spec}(A)$ be the natural projection, so that

$$(X_A)[p^*M] \simeq X \times \text{Spec}(A \oplus M).$$

Let $d_0 : X_A[p^*M] \to X_A$ be the zero derivation. Observe now that

$$\{\text{id}_{\mathcal{F}_1}\} \times_{\text{Map}(\mathcal{F}_1, \mathcal{F}_1)} \text{Map}(d_0^0 \mathcal{F}_1, d_0^0 \mathcal{F}_1) \simeq \{\text{id}_{\mathcal{F}_1}\} \times_{\text{Map}(\mathcal{F}_1, \mathcal{F}_1)} \text{Aut}(d_0^0 \mathcal{F}_1).$$

We are therefore free to replace $\text{Aut}(d_0^0 \mathcal{F}_1)$ by $\text{Map}(d_0^0 \mathcal{F}_1, d_0^0 \mathcal{F}_1)$ in the diagram computing $F(\text{Spec}(A \oplus M)).$ Unraveling the definitions, we can thus identify $\text{Der}_F(A; M)$ with the pullback diagram

$$
\begin{array}{ccc}
\text{Der}_F(A; M) & \longrightarrow & \square \\
\downarrow & & \downarrow \\
\square & \longrightarrow & \text{Map}(\mathcal{F}_2, \mathcal{F}_2 \otimes p^*M) \\
\downarrow & & \downarrow \\
\text{Map}(\mathcal{F}_1, \mathcal{F}_1 \otimes p^*M) & \longrightarrow & \text{Map}(\mathcal{F}_1, \mathcal{F}_2 \otimes p^*M)
\end{array}
$$

Since $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ are perfect, they are dualizable. Moreover, since $X$ is smooth and proper, [Lur18, Proposition 6.4.5.3] guarantees the existence of a left adjoint $p_+$ for $p^*.$ We can therefore rewrite the above diagram as

$$
\begin{array}{ccc}
\text{Der}_F(A; M) & \longrightarrow & \square \\
\downarrow & & \downarrow \\
\square & \longrightarrow & \text{Map}(p_+(\mathcal{F}_3 \otimes \mathcal{F}_3^\vee), M) \\
\downarrow & & \downarrow \\
\text{Map}(p_+(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee), M) & \longrightarrow & \text{Map}(p_+(\mathcal{F}_2 \otimes \mathcal{F}_2^\vee), M) \\
\downarrow & & \downarrow \\
\text{Map}(p_+(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee), M) & \longrightarrow & \text{Map}(p_+(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee), M)
\end{array}
$$

where now the mapping spaces are computed in $\text{Perf}(X \times \text{Spec}(A)).$ Therefore, the Yoneda lemma implies that $\text{Der}_F(A; M)$ is representable by the colimit of the diagram (3.3) in $\text{Perf}(X \times \text{Spec}(A)).$ \hfill $\Box$

**Remark 3.2.** There are two natural morphisms

$$\text{fib}, \text{cofib} : \text{Perf}^{A^1}(X) \longrightarrow \text{Perf}^{\text{ext}}(X),$$

which send a morphism $\beta : \mathcal{F} \to \mathcal{G}$ to the fiber sequence

$$\text{fib}(\beta) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \quad (\text{resp. } \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \text{cofib}(\beta)).$$

Applying [Lur09, Proposition 4.3.2.15] twice, we see that these morphisms are equivalences.

Let $y : \text{Spec}(A) \to \text{Perf}^{A^1 \times A^1}(X)$ be a morphism classifying a diagram

$$
\begin{array}{ccc}
\mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}_3
\end{array}
$$

Let $x : \text{Spec}(A) \to \text{Perf}^{A^1}(X)$ is the point corresponding to $\mathcal{F}_1 \to \mathcal{F}_2.$ Then we have a canonical morphism

$$x^* \mathbb{L}_{\text{Perf}^{A^1}(X)}[1] \longrightarrow y^* \mathbb{L}_{\text{Perf}^{A^1 \times A^1}(X)}[1],$$

which in general is not an equivalence. When the point $y$ factors through the open substack $\text{Perf}^{\text{ext}}(X),$ then the above morphism becomes an equivalence. \hfill $\triangle$
Lemma 3.3. Let $X$ be a smooth and proper complex scheme of dimension $n$. Then the cotangent complex $\mathfrak{L}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}$ is perfect and has tor-amplitude $[-1, n - 1]$. In particular, $\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)$ is smooth when $X$ is a curve and derived lci when $X$ is a surface.

Remark 3.4. Notice that $\operatorname{Perf}_{\operatorname{ext}}^{\operatorname{ext}}(X)$ is not smooth, even if $X$ is a smooth projective complex curve. △

Proof of Lemma 3.3. Let $\Spec(A) \in \mathbf{dAff}$ and let $x : \Spec(A) \to \operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)$ be a point. We have to check that $x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}$ is perfect and in tor-amplitude $[1 - n, 1]$. Since the map $\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X) \to \operatorname{Perf}_{\operatorname{ext}}^{\operatorname{ext}}(X)$ is formally etale, we can use Proposition 3.1 to compute the cotangent complex, and hence the tangent one. Let

$$F_1 \to F_2 \to F_3$$

be the fiber sequence in $\operatorname{Perf}(X \times \Spec(A))$ corresponding to the point $x$. Let $p : X \times \Spec(A) \to \Spec(A)$ be the canonical projection. Using Remark 3.2 we see that $x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}$ fits in the pullback diagram

$$x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)} \longrightarrow p_*(F_2^\vee \otimes F_2)[1] \quad \downarrow \quad \longrightarrow p_*(F_1^\vee \otimes F_1)[1]$$

Since $X$ is smooth and proper, $p_*$ preserves perfect complexes. Therefore, $x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}$ is perfect.

In order to check that it has tor-amplitude $[1 - n, 1]$, it is sufficient to check that its pullback to $\Spec(\pi_0(A))$ has tor-amplitude $[1 - n, 1]$. In other words, we can suppose from the very beginning that $A$ is discrete. In this case, $F_1$, $F_2$ and $F_3$ are discrete as well and the map $F_1 \to F_2$ is a monomorphism. Since $X$ is an $n$-dimensional scheme, the functor $p_*$ has cohomological dimension $n$. It is therefore sufficient to check that $\pi_{-n}(x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}) = 0$. We have a long exact sequence

$$\mathcal{E}xt^n_p(F_1, F_1) \oplus \mathcal{E}xt^n_p(F_2, F_2) \to \mathcal{E}xt^n_p(F_1, F_2) \to \pi_{-n}(x^* \mathbb{T}_{\operatorname{Coh}_{\operatorname{ext}}^{\operatorname{ext}}(X)}) \to \mathcal{E}xt^{n+1}_p(F_1, F_1) \oplus \mathcal{E}xt^{n+1}_p(F_2, F_2).$$

Choose an integer $m \gg 0$ such that both sheaves $F_1(m)$ and $F_2(m)$ are globally generated and we have $R^ip_*(F_i(m)) = 0$ for $i \leq 1$. We can therefore choose a resolution

$$0 \to F(m)^{\oplus k_n} \to \ldots \to F(m)^{\oplus k_1} \to F(m)^{\oplus k_0} \to F_1 \to 0.$$

Then $\mathcal{H}om_{X \times \Spec(A)}(F_1, F_1)$ is equivalent to the complex

$$F(m)^{\oplus k_n} \to F(m)^{\oplus k_1} \to \ldots \to F(m)^{\oplus k_0}.$$

Since $R^ip_*(F_1(m)^{\oplus k_i}) = 0$ for $i \geq 1$ and $i = 0, \ldots, n$, the cohomological descent spectral sequence shows that

$$\mathcal{E}xt^{n+1}_p(F_1, F_1) := R^{n+1}p_*\mathcal{H}om_{X \times \Spec(A)}(F_1, F_1) = 0.$$

Similarly, we see that $\mathcal{E}xt^{n+1}_p(F_2, F_2) = 0$. We are thus left to check that the map

$$\mathcal{E}xt^n_p(F_1, F_1) \oplus \mathcal{E}xt^n_p(F_2, F_2) \to \mathcal{E}xt^n_p(F_1, F_2)$$

is surjective. It is enough to prove that

$$\mathcal{E}xt^n_p(F_2, F_2) \to \mathcal{E}xt^n_p(F_1, F_2)$$

is surjective. We have a long exact sequence

$$\mathcal{E}xt^n_p(F_2, F_2) \to \mathcal{E}xt^n_p(F_1, F_2) \to \mathcal{E}xt^{n+1}_p(F_3, F_2).$$

The same argument given above shows that $\mathcal{E}xt^{n+1}_p(F_3, F_2) = 0$. The proof is therefore complete. □
We now study the regularity properties of the map
\[ ev_3 \times ev_1 : \text{Coh}^{\text{exf}}(X) \to \text{Coh}(X). \]

Let us start by recalling the following definition:

**Definition 3.5.** Let \( F : \text{dAff}^{\text{op}} \to S \) be a derived stack and let \( S \) be a derived geometric stack. Let \( p : F \to S \) be a morphism between them.

1. We say that \( p \) has **cohomological dimension** \( \leq n \) if there exists \( n \geq 0 \) such that for every \( F \in \text{QCoh}^n(F) \), the quasi-coherent complex \( p_*(F) \) belongs to \( \text{QCoh}^{\geq -n}(S) \);
2. We say that \( p \) is **categorically proper** if it is of finite cohomological dimension and the functor \( p_* : \text{QCoh}(F) \to \text{QCoh}(S) \) restricts to a functor
   \[ p_* : \text{Coh}^{-}(F) \to \text{Coh}^{-}(S). \]
3. We say that \( p \) has **global tor-amplitude** \( \leq a \) if the functor \( p_* : \text{QCoh}(F) \to \text{QCoh}(S) \) takes quasi-coherent complexes in tor-amplitude \( \leq a \) to objects in tor-amplitude \( \leq a + n \).

We say that \( F \) has finite cohomological dimension (resp. is categorically proper, has finite tor-amplitude) if the map \( F \to \text{Spec}(C) \) has the same property. We say that \( F \) has universally finite cohomological dimension (resp. is universally categorically proper, has universally finite global tor-amplitude) if for every \( S \in \text{dAff} \) the projection \( p_S : Y \times S \to S \) has the same property.

**Remark 3.6.**

1. Let \( p : F \to S \) be a morphism of derived stacks. If \( F \) is geometric and \( S = \text{Spec}(A) \), then \( p \) has tor-amplitude \( \leq n \) if there exists an atlas \( U \to F \) with \( U = \text{Spec}(B) \) such that \( B \) has tor-amplitude \( \leq n \) as an \( A \)-module (cf. [Lur18, Definition 6.1.1.1.]). If \( p \) has finite cohomological dimension, then \( p \) has tor-amplitude \( \leq n \) if and only if it has global tor-amplitude \( \leq n \).

2. In [PTVV13, Definition 2.1], the authors introduced the notion of \( S \)-compact stack. A morphism \( p : F \to S \) is said to be strictly \( S \)-compact if the functor \( p_* : \text{QCoh}(F) \to \text{QCoh}(S) \) commutes with filtered colimits and preserves perfect complexes. We say that \( p \) is \( S \)-compact if it is universally strictly \( S \)-compact. We remark that if \( p \) has finite cohomological dimension, then it commutes with filtered colimits. Moreover, if \( p \) is categorically proper and has finite global tor-amplitude, then it preserves perfect complexes. Thus, if \( p \) has finite cohomological dimension and finite global tor-amplitude, and it is categorically proper, then \( p \) is strictly \( S \)-compact.

**Lemma 3.7.** Let \( Y \in \text{dSt} \) be a derived stack. If \( Y \) is categorically proper and has universally finite global tor-amplitude then for every \( S \in \text{dAff} \) the functor
\[ p_{S*} : \text{QCoh}(Y \times S) \to \text{QCoh}(S) \]
restricts to a functor
\[ p_{S*} : \text{Perf}(Y \times S) \to \text{Perf}(S). \]
Moreover, in this case the functor \( p_{S*}^\top : \text{Perf}(S) \to \text{Perf}(Y \times S) \) admits a left adjoint
\[ p_{S*}^\leftadj : \text{Perf}(Y \times S) \to \text{Perf}(S), \]
which satisfies the base change property.

**Proof.** It follows from [HLP14, Corollary B.16] that \( Y \) is universally categorically proper and that for every \( S \in \text{dAff} \) the projection \( p_S : Y \times S \to S \) satisfies the projection formula. The functor \( p_{S*} \) preserves perfect complexes because perfect complexes can be characterized as those almost perfect complexes that have finite tor-amplitude. Finally, we set
\[ p_{S*}(\mathcal{F}) := (p_{S*}(\mathcal{F}^\vee))^\vee \]
for \( F \in \text{Perf}(Y \times S) \). The projection formula implies that \( p_{S+} \) defines a left adjoint for \( p_S^* \). The base change property is a direct consequence of the base change for \( p_{S+} \).

**Proposition 3.8.** Let \( Y \in \text{dSt} \) be a derived stack which is categorically proper and has universally finite global tor-amplitude. Let \( q: Y \times \text{Perf}(Y) \times \text{Perf}(Y) \to \text{Perf}(Y) \times \text{Perf}(Y) \) be the canonical projection and \( F \in \text{Perf}(Y \times \text{Perf}(Y)) \) be the universal perfect complex on \( Y \). Let \( p_1, p_2: Y \times \text{Perf}(Y) \times \text{Perf}(Y) \to Y \times \text{Perf}(Y) \) be the two projections and set \( F_i := p_i^* F \). Then the map

\[
e_{V} \times e_{V} : \text{Perf}^{\text{ext}}(Y) \longrightarrow \text{Perf}(Y) \times \text{Perf}(Y)
\]

is equivalent to \( \mathcal{V}_{\text{Perf}(Y) \times \text{Perf}(Y)}(q_+ \text{Hom}_{Y \times \text{Perf}(Y) \times \text{Perf}(Y)}(F_2, F_1)[-1]) \).

**Proof.** Set

\[
\mathcal{G} := \text{Hom}_{Y \times \text{Perf}(Y) \times \text{Perf}(Y)}(F_2, F_1)[-1].
\]

Then for any \( S \in \text{dAff} \) and any point \( x: S \to \text{Perf}(Y) \times \text{Perf}(Y) \), we can identify the fiber at \( x \) of the morphism

\[
\text{Map}_{\text{dSt}}(S, \mathcal{V}_{\text{Perf}(Y) \times \text{Perf}(Y)}(q_+ \mathcal{G})) \longrightarrow \text{Map}_{\text{dSt}}(S, \text{Perf}(Y) \times \text{Perf}(Y))
\]

with the mapping space

\[
\text{Map}_{\text{Perf}}(S, x^* q_+ (\mathcal{G}), O_S).
\]

Consider the pullback square

\[
\begin{array}{ccc}
Y \times S & \xrightarrow{y} & Y \times \text{Perf}(Y) \times \text{Perf}(Y) \\
\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
S & \xrightarrow{x} & \text{Perf}(Y) \times \text{Perf}(Y)
\end{array}
\]

The base change for the plus pushforward allows us to rewrite

\[
x^* q_+ (\mathcal{G}) \simeq q_S + y^* (\mathcal{G}).
\]

Therefore, we have

\[
\text{Map}_{\text{Perf}}(S, x^* q_+ (\mathcal{G}), O_S) \simeq \text{Map}_{\text{Perf}}(S, (q_S + y^* (\mathcal{G})), O_S)
\]

\[
\simeq \text{Map}_{\text{Perf}}((Y \times S), (y^* (\mathcal{G})), O_{Y \times S})
\]

\[
\simeq \text{Map}_{\text{Perf}}((Y \times S), (y^* (\mathcal{G})))
\]

\[
\simeq \tau_{\ge 0} \Gamma (Y \times S, \text{Hom}_{Y \times S} (y^* F_1, y^* F_2)[1]).
\]

We therefore see that any choice of a fiber sequence

\[
y^* F_1 \longrightarrow F \longrightarrow y^* F_2
\]

in \( \text{Perf}(Y \times S) \) gives rise to a point \( S \to \mathcal{V}_{\text{Perf}(Y) \times \text{Perf}(Y)}(q_+ \mathcal{G}) \). This provides us with a canonical map

\[
\text{Perf}^{\text{ext}}(Y) \longrightarrow \mathcal{V}_{\text{Perf}(Y) \times \text{Perf}(Y)}(q_+ \mathcal{G}),
\]

which induces, for every point \( x: S \to \text{Perf}(Y) \times \text{Perf}(Y) \), an equivalence

\[
\text{Map}_{\text{dSt}}(S, \text{Perf}^{\text{ext}}(Y)) \simeq \text{Map}_{\text{dSt}}(S, \mathcal{V}_{\text{Perf}(Y) \times \text{Perf}(Y)}(q_+ \mathcal{G})).
\]

The conclusion follows.

**Corollary 3.9.** Let \( Y \) be a derived stack satisfying the same assumptions of Proposition 3.8. Then the cotangent complex of the map

\[
e_{V} \times e_{V} : \text{Perf}^{\text{ext}}(Y) \longrightarrow \text{Perf}(Y) \times \text{Perf}(Y)
\]

is computed as

\[
(e_{V} \times e_{V})^* \left( q_+ \left( \text{Hom}_{Y \times \text{Perf}(Y) \times \text{Perf}(Y)}(F_2, F_1)[-1] \right) \right).
\]
Proof. This is an immediate consequence of Proposition 3.8 and of [Lur17, Proposition 7.4.3.14].

Corollary 3.10. Let $X$ be a smooth and proper complex scheme of dimension $n$. Then the relative cotangent complex of the map

$$\text{ev}_3 \times \text{ev}_1 : \text{Coh}^{\text{ext}}(X) \longrightarrow \text{Coh}(X) \times \text{Coh}(X) \tag{3.5}$$

is perfect and has tor-amplitude $[-1, n - 1]$. In particular, it is smooth when $X$ is a curve and derived l.c.i. when $X$ is a surface.

Remark 3.11. When $X$ is a curve, Corollary 2.10 and Lemma 3.3 imply that $\text{Coh}^{\text{ext}}(X)$ and $\text{Coh}(X)$ are smooth. This immediately implies that $\text{ev}_3 \times \text{ev}_1$ is derived l.c.i., hence the above corollary improves this result.

Proof of Corollary 3.10. Let $S \in \text{dAff}$ and let $x : S \to \text{Perf}^{\text{ext}}(X)$ be a point classifying a fiber sequence

$$\mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3$$

in $\text{Perf}(X \times S)$. If $\mathcal{F}_1$ and $\mathcal{F}_3$ have tor-amplitude 0 relative to $S$, then the same goes for $\mathcal{F}_2$. This implies that the diagram

$$\begin{array}{ccc}
\text{Coh}^{\text{ext}}(X) & \longrightarrow & \text{Perf}^{\text{ext}}(X) \\
\text{ev}_3 \times \text{ev}_1 \downarrow & & \downarrow \text{ev}_3 \times \text{ev}_1 \\
\text{Coh}(X) \times \text{Coh}(X) & \longrightarrow & \text{Perf}(X) \times \text{Perf}(X)
\end{array}$$

is a pullback square. Notice that smooth and proper schemes are categorically proper and universally of tor-amplitude 0. Therefore the assumptions of Proposition 3.8 are satisfied. Since the horizontal maps in the above diagram are formally étale, we can therefore use Corollary 3.9 to compute the relative cotangent complex of the morphism (3.5). This immediately implies that this relative cotangent complex is perfect, and we are left to prove that it has tor-amplitude $[-1, n - 1]$. For this reason, it is enough to prove that for any (underived) affine scheme $S \in \text{Aff}$ and any point $x : S \to \text{Coh}^{\text{ext}}(X)$, the perfect complex $x^* \mathcal{L}_{\text{ev}_3 \times \text{ev}_1}$ has tor-amplitude $[-1, n - 1]$. Let $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3$ be the extension classified by $x$ and let $q_S : Y \times S \to S$ be the canonical projection. The base change for the plus pushforward reduces us to compute the tor-amplitude of

$$q^+_S(\text{Hom}_{X \times S}(\mathcal{F}_3, \mathcal{F}_1)[-1]) \simeq (q^+_S(\text{Hom}_{X \times S}(\mathcal{F}_1, \mathcal{F}_3)[1]))^\vee.$$

Moreover, since $S$ is generic, it is enough to prove that

$$\pi_i(q^+_S(\text{Hom}_{X \times S}(\mathcal{F}_1, \mathcal{F}_3)[1])) \simeq 0$$

for $i \leq 1 - n$. However

$$\pi_i(q^+_S(\text{Hom}_{X \times S}(\mathcal{F}_1, \mathcal{F}_3)[1])) \simeq \mathcal{E}xt^i_q(\mathcal{F}_1, \mathcal{F}_3).$$

Since $S$ is underived, $\mathcal{F}_1$ and $\mathcal{F}_3$ belong to $\text{QCoh}^\vee(X \times S)$. Since $X$ has dimension $n$, it follows that $\mathcal{E}xt^j_q(\mathcal{F}_1, \mathcal{F}_3) \simeq 0$ for $j > n$. The conclusion follows.

Corollary 3.12. Let $X$ be a smooth and proper complex scheme of dimension $n$. Then the relative cotangent complex of the map

$$\text{ev}_3 \times \text{ev}_1 : \text{Bun}^{\text{ext}}(X) \longrightarrow \text{Bun}(X) \times \text{Bun}(X)$$

is perfect and has tor-amplitude $[-1, n - 1]$. 
Proof. The assertion follows by noticing that the diagram

\[
\begin{array}{ccc}
\text{Bun}^\text{ext}(X) & \longrightarrow & \text{Coh}^\text{ext}(X) \\
\text{ev}_3 \times \text{ev}_1 & & \text{ev}_3 \times \text{ev}_1 \\
\text{Bun}(X) \times \text{Bun}(X) & \longrightarrow & \text{Coh}(X) \times \text{Coh}(X)
\end{array}
\]

is a pullback square. □

3.1. Extensions of coherent sheaves on shapes. We now introduce the analogue of the stack \( \text{Coh}^\text{ext}(X) \) for the shapes of \( X \).

Let \( Y \in \text{dSt} \) be a derived stack and let \( u : U \to Y \) be a flat effective epimorphism from a derived geometric stack \( U \). We define

\[
\begin{align*}
\text{Coh}^\text{ext}(Y, u) & := \text{Perf}^\text{ext}(Y) \times_{\text{Perf}^\text{ext}(U)} \text{Coh}^\text{ext}(U), \\
\text{Bun}^\text{ext}(Y, u) & := \text{Perf}^\text{ext}(Y) \times_{\text{Perf}^\text{ext}(U)} \text{Bun}^\text{ext}(U).
\end{align*}
\]

When the map \( u : U \to Y \) is clear from the context, we will often abuse notations and write \( \text{Coh}^\text{ext}(Y) \) instead of \( \text{Coh}^\text{ext}(Y, u) \). Notice that since the map \( \text{Coh}^\text{ext}(U) \to \text{Perf}^{\Delta^1 \times \Delta^1}(U) \) is formally étale, the same goes for the natural map

\[
\text{Coh}^\text{ext}(Y, u) \longrightarrow \text{Perf}^{\Delta^1 \times \Delta^1}(Y).
\]

We now study the regularity properties of the map

\[
\text{ev}_3 \times \text{ev}_1 : \text{Coh}^\text{ext}(Y, u) \longrightarrow \text{Coh}(Y, u).
\]

We start by the following general consideration: if \( u : U \to Y \) is a flat effective epimorphism, then it follows from the definitions of \( \text{Coh}(Y, u) \) and \( \text{Coh}^\text{ext}(Y, u) \) that the diagram

\[
\begin{array}{ccc}
\text{Coh}^\text{ext}(Y, u) & \longrightarrow & \text{Perf}^\text{ext}(Y) \\
\text{ev}_3 \times \text{ev}_1 & & \text{ev}_3 \times \text{ev}_1 \\
\text{Coh}(Y, u) \times \text{Coh}(Y, u) & \longrightarrow & \text{Perf}(Y) \times \text{Perf}(Y)
\end{array}
\]

is a pullback square. In particular, if \( Y \) satisfies the assumptions of Proposition 3.8, then we can compute the relative cotangent complex of the map \( \text{ev}_3 \times \text{ev}_1 : \text{Coh}^\text{ext}(Y, u) \to \text{Coh}(Y, u) \times \text{Coh}(Y, u) \) via Corollary 3.9. Our goal is to write explicit estimates for the tor-amplitude of this map for \( Y \) being one of the Simpson’s shapes \( X_{\text{B}}, X_{\text{dR}}, X_{\text{Dol}}, \) and \( X_{\text{Del}} \), where \( X \) is a derived geometric stack. For this, we first verify that these stacks satisfy the assumptions of Proposition 3.8.

3.1.1. Betti shape. Let \( K \in S \) be a finite space. The following lemma is already implicit in [TV08, PTVV13]. We include the proof only for sake of completeness:

Lemma 3.13. The derived stack \( K_{\text{B}} \) is categorically proper and universally of global tor-amplitude \( \leq 0 \).

Proof. Since \( K \) is a finite space, we can write it as a composition

\[
K_0 = \emptyset \xrightarrow{a_0} K_1 \xrightarrow{a_1} \cdots \xrightarrow{a_n} K_n = K,
\]

where each map \( a_i : K_i \to K_{i+1} \) fits in a pushout diagram

\[
\begin{array}{ccc}
S^{m_i} & \xrightarrow{a_i} & * \\
\downarrow{b_i} & & \downarrow{x_i} \\
K_i & \xrightarrow{a_i} & K_{i+1}
\end{array}
\]

\]
Denote by \( p_i : (K_i)_B \to \text{Spec}(C) \) the canonical map. The above pushout diagram implies that we have an equivalence
\[
\text{Qcoh}((K_{i+1})) \cong \text{Qcoh}((K_i)_B) \times \text{Qcoh}((S^m)_B) k\text{-Mod}.
\]

It follows from the discussion following [PY16, Corollary 8.6] that for \( F \in \text{Qcoh}(K_{i+1}) \) one has a pullback diagram
\[
\begin{array}{ccc}
(p_{i+1})_*(F) & \longrightarrow & (p_i)_*(q_i^*F) \\
\downarrow & & \downarrow \\
q_i^*(F) & \longrightarrow & (q_i)_*(b_i^*a_i^*F)
\end{array}
\]

Since \( k\text{-Mod}^+, \text{Coh}^- (\text{Spec}(k)) \) and the full subcategory of \( k\text{-Mod} \) spanned by objects of tor-amplitude \( \leq 0 \) are closed under finite limits, we see that if the derived stacks \((K_i)_B \) and \((S^m)_B \) are categorically proper and universally of global tor-amplitude \( \leq 0 \), then the same goes for \((K_{i+1})_B \). We can therefore argue by induction: the statement for \( K_0 \) is trivial, and we are thus immediately reduced to check it for the spheres. Since \( S^{m+1} \cong \Sigma(S^n) \), we can once more proceed by induction, and reduce ourselves to the case of \( S^0 \). In this case, \((S^0)_B = \text{Spec}(k) \amalg \text{Spec}(k) \), and here the statement is obvious.

It follows from this lemma that the relative cotangent complex of the map
\[
ev_3 \times ev_1 : \text{Coh}^\text{ext}(K_B, u) \longrightarrow \text{Coh}(K_B, u) \times \text{Coh}(K_B, u) \tag{3.6}
\]
at a point \( S \to \text{Coh}^\text{ext}(K_B, u) \) classifying an extension \( F_1 \to F \to F_2 \) in \( \text{Perf}(K_B \times S) \) is computed by the pullback along the projection \( S \times \text{Coh}(K_B, u) \times \text{Coh}(K_B, u) \to S \) of
\[q_S^+ (\text{Hom}_{K_B \times S} (F_2, F_1)[-1]).\]

Here \( q_S : K_B \times S \to S \) is the natural projection. In particular, we obtain:

**Proposition 3.14.** Suppose that \( K_B \) has cohomological dimension \( \leq m \). The relative cotangent complex of the map (3.6) has tor-amplitude contained in \([-1, m - 1]\). Furthermore, if \( K \) is the space underlying a complex scheme \( X \) of complex dimension \( n \), then we can take \( m = 2n \).

**Proof.** It is enough to prove that for every underived affine scheme \( S \in \text{Aff} \) and every point \( x : S \to \text{Coh}^\text{ext}(K_B, u) \) classifying an extension \( F_1 \to F \to F_2 \) in \( \text{Perf}(K_B \times S) \) of perfect complexes of tor-amplitude \( 0 \) relative to \( S \), the complex \( q_S^+ (\text{Hom}_{K_B \times S} (F_2, F_1)[-1]) \) is contained in cohomological amplitude \([-1, m - 1]\). Unraveling the definitions, this is equivalent to check that the complex \( q_S^+ (\text{Hom}_{K_B \times S} (F_1, F_2)) \) is contained in cohomological amplitude \([-m, 0]\). This follows from the assumption on the cohomological dimension of \( K_B \) and from Lemma 2.15.

Now let \( X \) be a smooth and proper complex scheme. Define the stacks
\[
\text{Coh}^\text{ext}_B(X) := \text{Coh}^\text{ext}(X^\text{top}, u) \quad \text{and} \quad \text{Bun}^\text{ext}_B(X) := \text{Bun}^\text{ext}(X^\text{top}, u).
\]

These stacks are geometric and locally of finite presentation since \( \text{Perf}^\text{ext}(X^\text{top}) \) is so. In addition, by Lemma 2.15 we get \( \text{Coh}^\text{ext}_B(X) \cong \text{Bun}^\text{ext}_B(X) \).

**Corollary 3.15.** If \( X \) is a smooth projective complex curve and \( K := X^\text{top} \), then the map (3.6) is derived locally complete intersection.

### 3.1.2. De Rham shape

Let \( X \) be a smooth and proper complex scheme of dimension \( n \).

**Lemma 3.16.** The derived stack \( X_{\text{dR}} \) is categorically proper and universally of global tor-amplitude \( \leq 0 \).

**Proof.** Let us first prove that \( X_{\text{dR}} \) has finite cohomological dimension. Let \( q : X_{\text{dR}} \to \text{Spec}(C) \) and \( p : X \to \text{Spec}(C) \) be the natural morphisms. We can identify \( F \in \text{Qcoh}^{\text{V}}(X_{\text{dR}}) \) with a left
$D_X$-module. Using [Bha12, Corollary 4.30] and [CPT+17, Proposition 2.2.3] we obtain a canonical equivalence

$$q_*(\mathcal{F}) \simeq p_*(\lambda_X^*\mathcal{F} \otimes_{\mathcal{O}_X} |\text{DR}(\mathcal{O}_X)|).$$

Here $|\text{DR}(\mathcal{O}_X)|$ is the realization of the mixed de Rham algebra of $X$. Since $X$ is smooth, we can simply identify it with the complex

$$\mathcal{O}_X \xrightarrow{d_{\text{DR}}} \Omega^1_X \xrightarrow{d_{\text{DR}}} \cdots \xrightarrow{d_{\text{DR}}} \Omega^n_X.$$

Since $X$ is of dimension $n$, the spectral sequence for descent implies that $X_{\text{dr}}$ has finite cohomological dimension. Moreover, since $X$ is proper, we see that $p_*(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^n_X)$ has coherent cohomology. Therefore, the spectral sequence for descent implies once again that $X_{\text{dr}}$ is categorically proper. Observe now that since $X$ is underived, $\lambda_X^*\mathcal{F}$ belongs to $\text{Coh}^b(X)$. This together with the derived base change immediately implies that $X_{\text{dr}}$ has universally global tor-amplitude $\leq 0$. □

Define the stacks

$$\text{Coh}^\text{ext}_{\text{dr}}(X) := \text{Coh}^\text{ext}(X_{\text{dr}}, \lambda_X) \quad \text{and} \quad \text{Bun}^\text{ext}_{\text{dr}}(X) := \text{Bun}^\text{ext}(X_{\text{dr}}, \lambda_X).$$

These stacks are geometric and locally of finite presentation since $\text{Perf}^\text{ext}(X_{\text{dr}})$ is so. In addition, by Lemma 2.17 we get $\text{Coh}^\text{ext}_{\text{dr}}(X) \simeq \text{Bun}^\text{ext}_{\text{dr}}(X)$.

As in the case of the Betti shape, we deduce that the relative cotangent complex of the map

$$\text{ev}_3 \times \text{ev}_1 : \text{Coh}^\text{ext}_{\text{dr}}(X) \rightarrow \text{Coh}^\text{ext}_{\text{dr}}(X) \times \text{Coh}^\text{ext}_{\text{dr}}(X) \quad (3.7)$$

at a point $x : S \rightarrow \text{Coh}^\text{ext}_{\text{dr}}(X)$ classifying an extension $\mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2$ in $\text{Perf}(X_{\text{dr}} \times S)$ is computed by the pullback along the projection $S \times_{\text{Coh}^\text{ext}_{\text{dr}}(X) \times \text{Coh}^\text{ext}_{\text{dr}}(X)} \text{Coh}^\text{ext}_{\text{dr}}(X) \rightarrow S$ of

$$q_S : (\text{Hom}_{\text{dr} \times S}(\mathcal{F}_2, \mathcal{F}_1)[−1]).$$

Here $q_S : X_{\text{dr}} \times S \rightarrow S$ is the natural projection. In particular, we obtain:

**Proposition 3.17.** Suppose that $X$ is connected and of dimension $n$. The the relative cotangent complex of the map (3.7) has tor-amplitude contained in $[−1, 2n − 1]$.

**Proof.** It is enough to prove that for every underived affine scheme $S \in \text{Aff}$ and every point $x : S \rightarrow \text{Coh}^\text{ext}_{\text{dr}}(X)$ classifying an extension $\mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2$ in $\text{Perf}_{\text{dr}}(X \times S)$ of perfect complexes of tor-amplitude $0$ relative to $S$, the complex $q_S^+(\text{Hom}_{\text{dr} \times S}(\mathcal{F}_2, \mathcal{F}_1)[−1])$ is contained in cohomological amplitude $[−1, 2n − 1]$. Unraveling the definitions, this is equivalent to check that the complex $q_S^+(\text{Hom}_{\text{dr} \times S}(\mathcal{F}_1, \mathcal{F}_2))$ is contained in cohomological amplitude $[−2n, 0]$. In other words, we have to check that

$$\text{Ext}^{2i}_{\text{dr} \times S}(\mathcal{F}_1, \mathcal{F}_2) = 0$$

for $i > 2n$. This follows from [HTT08, Theorem 2.6.11] and [Ber83, §11]. □

**Corollary 3.18.** If $X$ is a smooth projective complex curve, then the map (3.7) is derived locally complete intersection.

3.1.3. **Dolbeault shape.** Let $X$ be a smooth and proper complex scheme.

**Lemma 3.19.** The derived stack $X_{\text{Dol}}$ is categorically proper and universally of tor-amplitude $\leq 0$.

**Proof.** Using the BNR correspondence for perfect complexes proven in Proposition B.3, we can identify $\text{Perf}(X_{\text{Dol}})$ with the $\infty$-category $\text{Perf}_{\text{prop}}(X)$ of perfect complexes properly supported with respect to the projection $p : T^+X \rightarrow X$. Under this equivalence, the functor

$$q_* : \text{QCoh}(X_{\text{Dol}}) \rightarrow \text{QCoh}(X)$$

is identified with the global section functor on $T^+X$. Since $X$ is smooth, $T^+X$ is smooth as well, and therefore we conclude that $X_{\text{Dol}}$ has universally tor-amplitude $\leq 0$. Finite cohomological
dimension follows immediately. Finally, categorical properness is consequence of the properness of \( X \), Lemma B.2 and the BNR correspondence.

By using similar arguments as above and Corollary B.4, one can prove the following.

**Lemma 3.20.** The derived stack \( \mathcal{X}^{nil}_{\text{Dol}} \) is categorically proper and universally of tor-amplitude \( \leq 0 \).

Define the stacks
\[
\text{Coh}^{\text{ext}}_{\text{Dol}}(X) := \text{Coh}^{\text{ext}}(\text{Dol}_{\text{Dol}}(X), \kappa_X) \quad \text{and} \quad \text{Bun}^{\text{ext}}_{\text{Dol}}(X) := \text{Bun}^{\text{ext}}(\text{Dol}_{\text{Dol}}(X), \kappa_X),
\]
\[
\text{Coh}^{\text{nil,ext}}_{\text{Dol}}(X) := \text{Coh}^{\text{ext}}(\text{Dol}_{\text{Dol}}(X), \kappa_{\text{Dol}}^{\text{nil}}) \quad \text{and} \quad \text{Bun}^{\text{nil,ext}}_{\text{Dol}}(X) := \text{Bun}^{\text{ext}}(\text{Dol}_{\text{Dol}}(X), \kappa_{\text{Dol}}^{\text{nil}}).
\]

These stacks are geometric and locally of finite presentation since \( \text{Perf}^{\text{ext}}(\text{Dol}_{\text{Dol}}) \) and \( \text{Perf}^{\text{ext}}(\text{X}^{\text{nil}}_{\text{Dol}}) \) are so.

As in the case of the Betti and de Rham shapes, we thus deduce that the relative cotangent complex of the map
\[
ev_3 \times \ev_1 : \text{Coh}^{\text{ext}}_{\text{Dol}}(X) \longrightarrow \text{Coh}_{\text{Dol}}(X) \times \text{Coh}_{\text{Dol}}(X)
\]
(3.8) at a point \( x: S \rightarrow \text{Coh}^{\text{ext}}_{\text{Dol}}(X) \) classifying an extension \( \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \) in \( \text{Perf}(\text{Dol}_{\text{Dol}} \times S) \) is computed by the pullback along the projection \( S \times_{\text{Coh}_{\text{Dol}}(X) \times \text{Coh}_{\text{Dol}}(X)} \text{Coh}^{\text{ext}}_{\text{Dol}}(X) \rightarrow S \) of
\[
q_{S^+}(\text{Hom}_{\text{Dol}_{\text{Dol}} \times S}(\mathcal{F}_2, \mathcal{F}_1)[-1]).
\]
Here \( q_S: \text{Dol}_{\text{Dol}} \times S \rightarrow S \) is the natural projection. In particular, we obtain:

**Proposition 3.21.** Suppose that \( X \) is connected and of dimension \( n \). Then the relative cotangent complex of the map (3.8) has tor-amplitude in \([-1, 2n - 1]\).

**Proof.** It is enough to check that for every underived affine scheme \( S \in \text{Aff} \) and every point \( x: S \rightarrow \text{Coh}^{\text{ext}}_{\text{Dol}}(X) \) classifying an extension \( \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \) in \( \text{Perf}(\text{Dol}_{\text{Dol}} \times S) \) of perfect complexes of tor-amplitude 0 relative to \( S \), the complex \( q_{S^+}(\text{Hom}_{\text{Dol}_{\text{Dol}} \times S}(\mathcal{F}_2, \mathcal{F}_1)[-1]) \) is contained in cohomological amplitude \([-1, 2n - 1]\). Unraveling the definitions, this is equivalent to check that the complex \( q_{S^+} \text{Hom}_{\text{Dol}_{\text{Dol}} \times S}(\mathcal{F}_1, \mathcal{F}_2) \) is contained in cohomological amplitude \([-2n, 0]\). In other words, we have to check that
\[
\text{Ext}_{\text{Dol}_{\text{Dol}} \times S}(\mathcal{F}_1, \mathcal{F}_2) = 0
\]
for \( i > 2n \). This follows from the BNR correspondence \([\text{Sim}94b, \text{Lemma} \, 6.8]\) (cf. also \([\text{GK}05, \text{§4}]\) and \([\text{SS}18, \text{§2.3}]\)).

**Corollary 3.22.** If \( X \) is a smooth projective complex curve, then the map (3.8) is derived locally complete intersection.

4. Two-Dimensional Categorified Hall Algebras

4.1. Convolution algebra structure for the stack of perfect complexes. Most of the results in this section are due to T. Dyckerhoff and M. Kapranov \([\text{DK}12]\). For the convenience of the reader we briefly recall their constructions.

Let
\[
T := \text{Hom}_\Delta([1], -) : \Delta \longrightarrow \text{Cat}_\infty,
\]
where \( \Delta \) is the simplicial category.

We write \( T_n \) instead of \( T([n]) \). Given any \( \text{C} \)-linear stable \( \infty \)-category \( \mathcal{C} \), we let
\[
S\mathcal{C} := \text{Fun}(T(\_), \mathcal{C}) : \Delta^{op} \longrightarrow \text{Cat}_\infty.
\]
We refer to \( S\mathcal{C} \) as the \( \infty \)-categorical Waldhausen construction on \( \mathcal{C} \). We also write \( S_n \mathcal{C} \) for \( \text{Fun}(T_n, \mathcal{C}) \). It follows from \([\text{DK}12, \text{Theorem} \, 7.3.3]\) that \( S\mathcal{C} \) is a 2-Segal object in \( \text{Cat}_\infty \).
Assume furthermore that $\mathcal{C}$ is smooth and proper. Then for any integer $n \geq 0$, the category $T_n$ is finite and therefore $\text{Fun}(T_n, \mathcal{C})$ is again smooth and proper. Applying the moduli of objects construction of Toën-Vaquie [TVa07] we obtain a simplicial derived stack

$$\mathcal{M}_{\text{SC}}: \Delta^{\text{op}} \to \text{dSt}.$$ 

Since the functor $(-)^{\text{op}}: \text{Cat}_{\infty} \to \mathcal{S}$ commutes with limits, we immediately deduce that $\mathcal{M}_{\text{SC}}$ is a 2-Segal object in $\text{dSt}$ (cf. [DK12, Theorem 7.4.18]). Applying this construction when $\mathcal{C} = \text{Perf}_C$ we obtain a simplicial derived stack

$$\text{SPerf}: \Delta^{\text{op}} \to \text{dSt},$$ 

which is a 2-Segal object. We write $\text{S}_n\text{Perf}$ for $\mathcal{M}_{\text{S}_n\text{Perf}_C}$.

Let now $X \in \text{dSt}$ be a derived stack. Since the functor $\text{Map}(X, -): \text{dSt} \to \text{dSt}$ commutes with limits, we obtain a 2-Segal object

$$\text{Map}(X, \text{SPerf}): \Delta^{\text{op}} \to \text{dSt}.$$ 

To simplify the notation, we will denote this 2-Segal object by $\text{SPerf}(X)$. As before, we write $\text{S}_n\text{Perf}(X)$ instead of $\text{Map}(X, \text{S}_n\text{Perf})$.

Recall now from [DK12, Theorem 11.1.6] that if $\mathcal{T}$ is a presentable $\infty$-category then there is a canonical functor

$$2\text{-Seg}(\mathcal{T}) \to \text{Alg}_{\mathcal{E}_1}(\text{Corr}^\times(\mathcal{T})).$$

Here $\text{Corr}^\times(\mathcal{T})$ denotes the $(\infty,2)$-category of correspondences equipped with the symmetric monoidal structure induced from the cartesian structure on $\mathcal{T}$. See [GaR17a, §7.2.1 & §9.2.1]. As $\mathcal{E}_1$-algebras in correspondences play a significant role for us, we introduce the following terminology:

**Definition 4.1.** Let $\mathcal{T}$ be an $\infty$-category with finite products and let $0^\odot$ be an $\infty$-operad. We define the $\infty$-category of $0^\odot$-convolution algebras in $\mathcal{T}$ as the $\infty$-category $\text{Alg}_{0^\odot}(\text{Corr}^\times(\mathcal{T}))$.

Taking $\mathcal{T} = \text{dSt}$, we therefore obtain the following result:

**Proposition 4.2.** Let $X \in \text{dSt}$ be a derived stack. The 2-Segal object $\text{SPerf}(X)$ endows $\text{Perf}(X)$ with the structure of an $\mathcal{E}_1$-convolution algebra in $\text{dSt}$.

Suppose now that $X$ is a smooth derived geometric stack. For every integer $n \geq 0$ we let $\text{S}_n\text{Coh}(X)$ to be the full substack of $\text{S}_n\text{Perf}(X)$ whose $\text{Spec}(A)$ points correspond to morphisms $f: T_n \to \text{Perf}(X \times \text{Spec}(A))$ such that for each $(i, j) \in T_n$ the object $F_{i,j} \coloneqq f(i, j) \in \text{Perf}(X \times \text{Spec}(A))$ has tor-amplitude $\leq 0$ relative to $\text{Spec}(A)$. Lemma 2.4 implies that this condition defines indeed a substack of $\text{S}_n\text{Perf}(X)$. It is furthermore easy to verify that if $[n] \to [m]$ is a morphism in $\Delta$ then the induced map

$$\text{S}_m\text{Perf}(X) \to \text{S}_n\text{Perf}(X)$$

restricts to a morphism $\text{S}_m\text{Coh}(X) \to \text{S}_n\text{Coh}(X)$. In other words, we obtain a simplicial object

$$\text{SCoh}(X): \Delta^{\text{op}} \to \text{dSt}.$$ 

**Lemma 4.3.** The simplicial object $\text{S}\text{Coh}(X)$ is a 2-Segal object.

**Proof.** Using [DK12, Proposition 2.3.2(3)], we are reduced to check that for every $n \geq 3$ and every $0 \leq i < j \leq n$, the natural morphism

$$\text{S}_n\text{Coh}(X) \to \text{S}_{n-j+i+1}\text{Coh}(X) \times \text{S}_{i}\text{Coh}(X) \text{S}_{j-i}\text{Coh}(X)$$

is an equivalence. Here the morphism is induced by the maps $[n-j+i+1] \to [n]$ and $[j-i] \to [n]$ corresponding to the inclusions

$$\{0,1,\ldots,i,j,j+1,\ldots,n\} \subset \{0,\ldots,n\} \quad \{i,i+1,\ldots,j\} \subset \{0,\ldots,n\}.$$
We have the following commutative diagram:

\[
\begin{array}{ccc}
S_n \text{Coh}(X) & \longrightarrow & S_{n-j+i+1} \text{Coh}(X) \times_{S_i \text{Coh}(X)} S_{j-i} \text{Coh}(X) \\
\downarrow & & \downarrow \\
S_n \text{Perf}(X) & \longrightarrow & S_{n-j+i+1} \text{Perf}(X) \times_{S_i \text{Perf}(X)} S_{j-i} \text{Perf}(X).
\end{array}
\]

The bottom horizontal map is an equivalence. After evaluating on \( S \in \text{dAff} \), we see that the vertical maps are induced by fully faithful functors. It is therefore enough to check that the image of \( a \), which can informally be described as follows:

\[
\text{Fun}^{\langle G \rangle} \rightarrow \text{dSt}
\]

is a pullback square in \( \text{Perf}(X \times S) \). Assume that \( G_0 \), \( G_2 \) and \( G_3 \) belong to \( \text{Coh}_S(X \times S) \). Then \( G_1 \) belongs to \( \text{Coh}_S(X \times S) \) as well. Since \( G_0 \) and \( G_3 \) have tor-amplitude 0 relative to \( S \), we see that, locally on \( X \), for every \( G \in \text{Coh}^+(S) \) one has

\[
\pi_k(p_\ast(G_1 + G_2) \otimes G) \simeq 0
\]

for \( k \geq 1 \), where \( p : X \times S \rightarrow S \) is the canonical projection. However, \( \pi_k(p_\ast(G_2) \otimes G) \simeq 0 \) because \( G_2 \) has tor-amplitude 0 relative to \( S \). Therefore \( \pi_k(p_\ast(G_1) \otimes G) \simeq 0 \) as well. The proof is therefore complete. \( \square \)

**Corollary 4.4.** Let \( X \) be a derived smooth geometric stack. Then 2-Segal derived stack \( \mathcal{S} \text{Coh}(X) \) endows \( \text{Coh}(X) \) with the structure of an \( E_1 \)-convolution algebra in \( \text{dSt} \).

Let now \( Y \in \text{dSt} \) be a derived stack and let \( u : U \rightarrow Y \) be a flat effective epimorphism, with \( U \) a derived smooth geometric stack. We define \( \mathcal{S} \text{Coh}(Y, u) \) as the following fiber product in \( \text{Fun}(\Delta^{op}, \text{dSt}) \):

\[
\mathcal{S} \text{Coh}(Y, u) \longrightarrow \mathcal{S} \text{Coh}(U) \\
\downarrow & & \downarrow \\
\mathcal{S} \text{Perf}(Y) & \longrightarrow & \mathcal{S} \text{Perf}(U).
\]

Notice now that if \( C \) is an \( \infty \)-category with finite products, then 2-Seg(\( C \)) \( \hookrightarrow \text{Fun}(\Delta^{op}, C) \) is closed under limits. In particular, we deduce that \( \mathcal{S} \text{Coh}(Y, u) \) is a 2-Segal object. As a consequence, we obtain:

**Corollary 4.5.** Let \( Y \) be a derived stack and let \( u : U \rightarrow Y \) be a flat effective epimorphism, with \( U \) a derived smooth geometric stack. Then 2-Segal derived stack \( \mathcal{S} \text{Coh}(Y, u) \) endows \( \text{Coh}(Y, u) \) with the structure of an \( E_1 \)-convolution algebra in \( \text{dSt} \).

### 4.2. Categorification of CoHASs.

The construction performed in [GaR17a, §3.1] provides us with a right-lax symmetric monoidal functor

\[
\text{QCoh} : \text{Corr}^X(\text{dSt}) \longrightarrow \text{Cat}_{\text{st}}^{\text{nl}},
\]

which can informally be described as follows:

- it sends a derived stack \( F \in \text{Corr}^X(\text{dSt}) \) to \( \text{QCoh}(F) \);
it sends a 1-morphism

\[
\begin{array}{c}
X_0 \\ p \\
\downarrow q \\
X_1
\end{array}
\]

\[\xrightarrow{p} X \xrightarrow{\eta} q \xrightarrow{\eta_*} Qcoh(X_1);\]

the right-lax symmetric monoidal structure is given by

\[\bigotimes: Qcoh(X) \otimes Qcoh(Y) \rightarrow Qcoh(X \times Y).\]

Denoting by \(pr_X: X \times Y \rightarrow X\) and \(pr_Y: X \times Y \rightarrow Y\) the two natural projections, then

\[\mathcal{F} \boxtimes \mathcal{G} := pr_X^* \mathcal{F} \boxtimes_{\mathcal{O}_{X \times Y}} pr_Y^* \mathcal{G}.\]

We now consider the full subcategory \(\text{Corr}^\times(d\text{Geom})_{\text{lci},\text{rps}}\) of \(\text{Corr}^\times(d\text{St})\) characterized as follows:

- the objects of \(\text{Corr}^\times(d\text{Geom})_{\text{lci},\text{rps}}\) are derived geometric stacks;
- the 1-morphisms from \(X_0\) to \(X_1\) are correspondences

\[
\begin{array}{c}
X_0 \\ p \\
\downarrow q \\
X_1
\end{array}
\]

where \(X\) is a derived geometric stack, \(p\) is derived lci and \(q\) is representable by proper schemes.

The right-lax symmetric monoidal functor \(Qcoh: \text{Corr}^\times(d\text{St}) \rightarrow \text{Cat}_{\infty}\) restricts to a right-lax symmetric monoidal functor over \(\text{Corr}^\times(d\text{Geom})_{\text{lci},\text{rps}}\).

Moreover, recall that the proof of [Toë12, Lemma 2.2] implies that if \(f: X \rightarrow Y\) is a derived lci morphism between derived geometric stacks, then the functor

\[f^*: Qcoh(Y) \rightarrow Qcoh(X)\]

restricts to

\[f^*: \text{Coh}^b(Y) \rightarrow \text{Coh}^b(X).\]

On the other hand, if \(f\) is proper and of finite cohomological dimension then the proper direct image theorem implies that

\[f_*: Qcoh(X) \rightarrow Qcoh(Y)\]

restricts to

\[f_*: \text{Coh}^b(X) \rightarrow \text{Coh}^b(Y).\]

These considerations imply that

\[Qcoh: \text{Corr}^\times(d\text{Geom})_{\text{lci},\text{rps}} \rightarrow \text{Cat}_{\infty}^{st}\]

admits a right-lax monoidal subfunctor

\[\text{Coh}^b: \text{Corr}(d\text{Geom})_{\text{lci},\text{rps}} \rightarrow \text{Cat}_{\infty}^{st}.\]

Applying the tor-amplitude estimates obtained in §3, we obtain the following result:

**Theorem 4.6.** Let \(Y\) be one of the following derived stacks:

1. a smooth proper complex scheme of dimension either one or two;
(2) the Betti, de Rham or Dolbeault stack of a smooth projective curve.

Then the composition

\[
\text{Coh}^b(\text{Coh}(Y)) \times \text{Coh}^b(\text{Coh}(Y)) \xrightarrow{\ominus} \text{Coh}^b(\text{Coh}(Y) \times \text{Coh}(Y)) \xrightarrow{q \circ p^*} \text{Coh}^b(\text{Coh}(Y)),
\]

where the map on the right-hand-side is induced by the 1-morphism in correspondences:

\[
\begin{array}{ccc}
\text{Coh}^\text{ext}(Y) & \\
\downarrow p & \downarrow q & \\
\text{Coh}(Y) \times \text{Coh}(Y) & \text{Coh}(Y)
\end{array}
\]

ends \(\text{Coh}^b(\text{Coh}(Y))\) with the structure of an \(E_1\)-monoidal stable \(\infty\)-category.

\textbf{Proof.} By Corollaries 4.4 and 4.5 we know that the 2-Segal object \(\mathcal{S}\text{Coh}(Y)\) endows \(\text{Coh}(Y)\) with the structure of an \(E_1\)-algebra in \(\text{Corr}^\otimes(\text{dSt})\). By Corollary 3.10 we have a tor-amplitude estimate of \(p\) when \(Y\) is a smooth proper complex scheme of dimension either one or two, while by Corollaries 3.15, 3.18 and 3.22 we have a tor-amplitude estimate of \(p\) when \(Y\) is the Betti, de Rham or Dolbeault stack of a smooth projective curve. Now, by using it together with the 2-Segal condition, we conclude that \(\mathcal{S}\text{Coh}(Y)\) actually endows \(\text{Coh}(Y)\) with the structure of an \(E_1\)-algebra in \(\text{Corr}(\text{dGeom})_{\text{lci, rps}}\). Applying the right-lax monoidal functor \(\text{Coh}^b : \text{Corr}(\text{dGeom})_{\text{lci, rps}} \rightarrow \text{Cat}^\text{st}_{\infty}\), we conclude that \(\text{Coh}^b(\text{Coh}(Y))\) inherits the structure of an \(E_1\)-algebra in \(\text{Cat}^\text{st}_{\infty}\). \(\square\)

Since \(E_1\)-algebras in \(\text{Cat}^\text{st}_{\infty}\) are (by definition) the same as \(E_1\)-monoidal categories in \(\text{Cat}^\otimes_{\infty}\), we refer to the corresponding tensor structure as the \textit{CoHA tensor structure on Coh}^b(\text{Coh}(Y)). We denote this monoidal structure by \(\ominus\).

\textbf{Remark 4.7.} Let \(Y\) be smooth proper complex scheme of dimension either one or two. Then the stack \(\text{Bun}(Y)\) is an \(E_1\)-algebra in \(\text{Corr}^\otimes(\text{dGeom})_{\text{lci, rps}}\). Assume that \(Y\) is projective, then the same holds for all the moduli stacks introduced in \(\S 2.1\). If \(Y\) is quasi-projective, then \(\text{Coh}_{\text{prop}}(Y)\) is an \(E_1\)-algebra in \(\text{Corr}^\otimes(\text{dGeom})_{\text{lci, rps}}\) for any integer \(d \leq \dim(Y)\).

Similarly, for the Dolbeaut shape, a statement similar to that of Theorem 4.6 holds for all the moduli stacks introduced in \(\S 2.2.3\). \(\triangle\)

\textbf{4.2.1. Applications.} Let \(S\) be a smooth (quasi-)projective complex surface and let \(0 \leq d \leq 2\) be an integer. A convolution algebra structure on the Grothendieck group \(K_0(\text{Coh}^d_{\text{prop}}(S))\) of the truncation of the derived geometric stack \(\text{Coh}_{\text{prop}}^d(S)\) has been defined in [Zha19, KV19].

The main difficulty in defining the convolution product in \textit{loc. cit.} consisted of having a refined \textit{Gysin pullback} \(p^!\) for the map \(p\) in \(K\)-theory. This is achieved by using Behrend-Fantechi perfect obstruction theory, which concretely boils down to prove that both \(\text{Coh}^d_{\text{prop}}(S) \times \text{Coh}^d_{\text{prop}}(S)\) and the truncation of the corresponding derived geometric stack of extensions fit into a cartesian diagram with a map between smooth stacks such that \(p\) is obtained as a pullback. Such smooth stacks are defined by using an explicit presentation of the complex of extensions (3.4).

As explained in [MR18, \S 4.3], a perfect obstruction theory gives rise to a derived enhancement: in our case, we obtain derived enhancements of the classical stack of coherent sheaves and of the classical stack of extensions of coherent sheaves which are different from those defined by us in \(\S 2\) and \(\S 3\), since the former ones depend on a choice of an explicit presentation of the complex (3.4). It is important to remark that different presentations for the complex (3.4) give rise to different derived enhancements. However, it is possible to show that the \(G\)-theory and the cohomological Hall algebra product are in fact independent on such choices. The same cannot be said for the stable \(\infty\)-category of bounded coherent sheaves on these derived enhancements.
Anyhow, by using deformation to the normal cone of the inclusion $\mathcal{C}^{d}\mathcal{C_{prop}} \to \mathcal{C}_{prop}$ together with $\mathbb{A}^{1}$-invariance of $G$-theory (cf. the proof of [MR18, Proposition 4.3.2]), it is possible to show the following:

**Theorem 4.8.** Let $S$ be a smooth (quasi-)projective complex surface and let $0 \leq d \leq 2$ be an integer. The convolution algebra product on $K_{0}(\mathcal{C}_{prop}^{d})$, induced by the $\mathcal{C}_{prop}$ tensor structure on the stable $\infty$-category $\mathcal{C}_{prop}^{d}$, coincides with that defined in [Zha19, KV19]. Thus, $(\mathcal{C}_{prop}^{d}, \mathcal{C}_{prop}, \mathcal{C}_{prop}^{d})$ is a categorification of the cohomological Hall algebra of a smooth surface.

We will give more details in a subsequent version of this paper.

### 4.3. The equivariant case

The main results of §4.1 and of §4.2 carry over without additional difficulties in the equivariant setting. Let us sketch how to modify the key constructions.

Let $X \in \mathcal{D}_{St}$ be a derived stack and let $G \in \text{Mon}_{E_{1}}(\mathcal{D}_{St})$ be a group-like $E_{1}$-monoid in derived stacks acting on $X$. Typically, $X$ will be geometric and $G$ will be an algebraic group. Since the monoidal structure on $\mathcal{D}_{St}$ is cartesian, we can use [Lur17, Proposition 4.2.2.9] to reformulate the datum of the $G$-action on $X$ as a diagram

$$A_{G, X} : \Delta^{op} \times \Delta^{1} \to \mathcal{D}_{St}$$

satisfying the relative 1-Segal condition. Informally speaking, $A_{G, X}$ is the diagram

$$\cdots \xymatrix{ \cdots \ar[d] \ar[r] & G^{2} \times X \ar[d] \ar[r] & G \times X \ar[d] \ar[r] & X \ar[d] \ar[r] & \cdots }$$

which encodes at the same time the $E_{1}$-structure on $G$ and the action on $X$. We denote the geometric realization of the top simplicial object by $[X/G]$, while it is customary to denote the geometric realization of the bottom one by $BG$.

We now define

$$\mathcal{S}\text{Perf}_{G}(X) : \Delta^{op} \to \mathcal{D}_{St/BG}$$

by setting

$$\mathcal{S}\text{Perf}_{G}(X) := \text{Map}_{/BG}([X/G], \mathcal{S}\text{Perf} \times BG).$$

We also write $\text{Perf}_{G}(X)$ for $\mathcal{S}\text{Perf}_{G}(X)$. Notice that

$$\text{Spec}(k) \times_{BG} \mathcal{S}\text{Perf}_{G}(X) \simeq \text{Map}(X, \mathcal{S}\text{Perf}).$$

We can therefore unpack the datum of the map $\mathcal{S}\text{Perf}_{G}(X) \to BG$ by saying that $G$ acts canonically on $\mathcal{S}\text{Perf}(X)$. From this point of view, we have a canonical equivalence\(^{19}\)

$$\mathcal{S}\text{Perf}_{G}(X) \simeq [\mathcal{S}\text{Perf}(X)/G].$$

As an immediate consequence we find that

$$\mathcal{C}_{h}(\mathcal{S}\text{Perf}_{G}(X)) \simeq \mathcal{C}_{h}(\mathcal{S}\text{Perf}(X)).$$

The right hand side denotes the $G$-equivariant stable $\infty$-category of bounded coherent complexes on $\mathcal{S}\text{Perf}(X)$. Since the functor

$$\text{Map}_{/BG}([X/G], (-) \times BG) : \mathcal{D}_{St} \to \mathcal{D}_{St/BG}$$

commutes with limits, we deduce:

**Proposition 4.9.** The simplicial derived stack $\mathcal{S}\text{Perf}_{G}(X) : \Delta^{op} \to \mathcal{D}_{St/BG}$ is a 2-Segal object.

\(^{19}\)This is nothing but a very special case of the descent for $\infty$-topoi, see [Lur09, Theorem 6.1.3.9 and Proposition 6.1.3.10].
Suppose now that $X$ is a derived geometric stack. Then the morphism $[X/G] \to BG$ is smooth. We define $\mathcal{S}\text{Coh}_G(X) \in \text{Fun}(\Delta^{op}, \text{dSt}^s_{/BG})$ as follows. Given a derived affine scheme $S = \text{Spec}(A)$ and a morphism $x: S \to BG$, we set

$$\text{Map}_{/BG}(S, \mathcal{S}\text{Coh}_G(X)) := (\mathcal{S}\text{Coh}_S(S \times_{BG} [X/G]))^{\ast} \in \text{Fun}(\Delta^{op}, S).$$

We immediately obtain:

**Corollary 4.10.** Let $X$ be a derived geometric stack. Then the simplicial derived stack $\mathcal{S}\text{Coh}_G(X) : \Delta^{op} \to \text{dSt}^s_{/BG}$ is a 2-Segal object.

The above 2-Segal object endows $\text{Coh}(X)$ with the structure of a $G$-equivariant $E_1$-convolution algebra in dSt.

**Corollary 4.11.** Let $X$ be a derived geometric stack such that the 2-Segal object $\mathcal{S}\text{Coh}(X)$ endows $\text{Coh}(X)$ with the structure of an $E_1$-algebra in $\text{Coh}^s_{\text{ext}}(\text{dSt})_{\text{gp}, \text{pcfd}}$. If $X$ admits an action of a smooth algebraic group $G$, then the $G$-equivariant 2-Segal object $\mathcal{S}\text{Coh}_G(X)$ induces an $E_1$-monoidal structure on $\text{Coh}^B_G(X) \simeq \text{Coh}^E_B(\text{Coh}(X))$.

**Proof.** All we need to check is that the map $\text{ev}_1 \times \text{ev}_1 : \text{Coh}^\text{ext}_G(X) \to \text{Coh}_G(X) \times \text{Coh}_G(X)$ is derived lci and that the map $\text{ev}_2 : \text{Coh}^\text{ext}_G(X) \to \text{Coh}_G(X)$ is proper and of finite cohomological dimension. We now observe that for $i = 1, 2, 3$ the right and the outer squares in the commutative diagram

$$\begin{array}{ccc}
\text{Coh}^\text{ext}_G(X) & \xrightarrow{\text{ev}_i} & \text{Coh}(X) \\
\downarrow & & \downarrow \\
\text{Coh}^\text{ext}_G(X) & \xrightarrow{\text{ev}_i} & \text{Coh}_G(X) \longrightarrow BG
\end{array}$$

are pullback squares. Therefore the same goes for the left one. The conclusion now follows because $\text{Spec}(k) \to BG$ is a smooth atlas. \(\square\)

Let $Y \in \text{dSt}$ be a derived stack and let $u : U \to Y$ be a flat effective epimorphism from a derived geometric stack $U$. Assume that both $U$ and $Y$ have an action of grouplike $E_1$-monoid $G \in \text{Mor}_{E_1}(\text{dSt}^s)$ such that $u$ is $G$-equivariant. Then the equivariant construction above extends also to this case with minor modifications (similarly to those in §3.2 and §3.1).

### 4.3.1. Applications

Let $\text{Coh}^0_0(C^2) := \text{Coh}^0_{\text{zer}}(C^2)$ be the derived geometric stack of zero-dimensional coherent sheaves on $C^2$. The construction performed above shows that the $C^* \times C^*$-action on $C^2$ lifts to an action on $\text{Coh}^0_0(C^2)$, so one can consider both the non-equivariant and the equivariant versions of the cohomological Hall algebra construction for $\text{Coh}^0_0(C^2)$.

A convolution algebra structure on the Grothendieck group $K_0(\text{dCoh}_0(C^2))$ of the truncation of $\text{Coh}^0_0(C^2)$ has been defined in [SV13b, SV12]. In loc. cit., the convolution product is defined by using an explicit presentation of $\text{Coh}^0_0(C^2)$ as disjoint union of quotient stacks. In [KV19, Proposition 6.1.5], the authors showed that the convolution product defined by using Behrend-Fantechi perfect obstruction theory coincides with the convolution product defined by using the explicit description of $\text{Coh}^0_0(C^2)$ in terms of quotient stacks. Thanks to this result (which holds also equivariantly), by arguing as in §4.2.1, one can show the following.

**Theorem 4.12.** The convolution algebra product on $K^0_0(C^* \times C^* (\text{dCoh}_0(C^2)))$, induced by the CoHA tensor structure on the stable $\infty$-category $\text{Coh}_{C^* \times C^*}^\text{prop}(\text{Coh}_0(C^2))$, coincides with that defined in [SV13b, SV12]. Thus, $(\text{Coh}_{C^* \times C^*}^\text{prop}(\text{Coh}_0(C^2)), \odot)$ is a categorification of the cohomological Hall algebra of zero-dimensional sheaves on $C^2$. 

As proved in [SV13b, SV12], the convolution algebra on $K^C \times C^\ast \cdot (\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{H} \text{O} (C^2))$ is isomorphic to a positive nilpotent part of the elliptic Hall algebra of Burban and Schiffmann [BSc12]. Thus, $(\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} (C^2), \circ)$ is a categorification of such a part of the elliptic Hall algebra.

Let $X$ a smooth projective complex curve and let $\text{Higgs}^{\text{naif}}(X) := T^\ast [0] \text{Coh}(X)$ (cf. Remark 2.18). $C^\ast$ acts by “scaling the Higgs fields”, so we can consider both the non-equivariant and the equivariant versions of the cohomological Hall algebra construction for $\text{Higgs}^{\text{naif}}(X)$.

The Grothendieck group $K^C_0(\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O}^\text{Coh}(X_{\text{Del}}))$ of the truncation of $\text{Higgs}^{\text{naif}}(X)$ is endowed with a convolution algebra structure as constructed in [SS18] and in [Min18] for the rank zero case. The construction of the convolution product induced by the construction of the product for the convolution algebra discussed above (in the higher rank case, such a construction of the product is performed locally and then one glues suitably to get a global convolution product). For similar arguments as above and thanks to Remark 2.18, we have the following.

**Theorem 4.13.** The convolution algebra product on $K_0^C(\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O}^\text{Coh}(X_{\text{Del}}))$, induced by the CoHA tensor structure on the stable $\infty$-category $\text{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O}^\text{Coh}(\text{Spec}(\text{C}))$, coincides with that defined in [SS18, Min18]. Thus, $(\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O}^\text{Coh}(\text{Spec}(\text{C})), \circ)$ is a categorification of the Dolbeaut cohomological Hall algebra.

5. A CoHA Version of the Hodge Filtration

In this section, we shall present a relation between the de Rham categorified Hall algebra and the Dolbeaut categorified Hall algebra, which is induced by the Deligne categorified Hall algebra $(\mathcal{C} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O} \text{H} \text{O}^\text{Dol}(\text{Spec}(\text{C})), \circ)$. Deligne’s $\lambda$-connections interpolate Higgs bundles with vector bundles with flat connections, and they were used by Simpson [Sim97] to prove the non-abelian Hodge correspondence. For such a reason, the relation we prove in this section can be interpreted as a version of the Hodge filtration in the setting of categorified Hall algebras.

5.1. Categorical filtrations. We let

$$\text{Perf}^{\text{filt}} := \text{Perf}(\mathbb{A}^1_C / G_m), \quad \text{Perf}^{\text{gr}} := \text{Perf}(B \text{G}_m).$$

The two morphisms

$$B \text{G}_m \longrightarrow [\mathbb{A}^1_C / G_m] \overset{i}{\rightarrow} \text{Spec}(\text{C}) \simeq [G_m / G_m]$$

induce canonical morphisms

$$j^* : \text{Perf}^{\text{filt}} \longrightarrow \text{Perf}^{\text{gr}}, \quad \text{Perf}^{\text{filt}} \longrightarrow \text{Perf}.$$

The group structure on $B \text{G}_m$ endows $\text{Perf}^{\text{gr}}$ with a Kunneth monoidal structure. The same goes for $\text{Perf}^{\text{filt}}$. With respect to these monoidal structures, the above functors are symmetric monoidal.

**Definition 5.1.** Let $\text{C}$ be a stable $\text{C}$-linear $\infty$-category. A lax filtered structure on $\text{C}$ is the given of an $\infty$-category $\text{C}^\ast \in \text{Perf}^{\text{filt}}\text{-Mod}(\text{Cat}^{\text{gr}}_\text{cat})$ equipped with a functor

$$\Phi : \text{C}^\ast \otimes_{\text{Perf}^{\text{filt}}} \text{Perf} \longrightarrow \text{C}.$$ 

We refer to the datum $(\text{C}, \text{C}^\ast, \Phi)$ as the datum of a lax filtered stable (C-linear) $\infty$-category. We say that a lax filtered $\infty$-category is filtered if $\Phi$ is an equivalence.

**Definition 5.2.** Let $(\text{C}, \text{C}^\ast, \Phi)$ be a lax filtered stable $\infty$-category. A lax associated graded category is the given of an $\infty$-category $\text{G} \in \text{Perf}^{\text{gr}}\text{-Mod}(\text{Cat}^{\text{gr}}_\text{cat})$ together with a morphism

$$\Psi : \text{C}^\ast \otimes_{\text{Perf}^{\text{gr}}} \text{Perf}^{\text{gr}} \longrightarrow \text{G}.$$ 

We say that $(\text{G}, \Psi)$ is the associated graded if the morphism $\Psi$ is an equivalence.
5.2. Hodge filtration. Let $X$ be a smooth projective complex curve. We will apply the formalism in the previous section with $\mathcal{C} = \text{Coh}^b(\text{Coh}(X_{\text{dR}}))$ and $\mathcal{G} = \text{Coh}^b(\text{Coh}^{ss,0}(X_{\text{Del}}))$.

Let $X_{\text{Del}}$ and $X_{\text{Del},G_m}$ be the Deligne’s shape and the equivariant Deligne’s shape of $X$, respectively. Define the derived geometric stacks

$$
\text{Coh}(X_{\text{Del}}) := (\text{Coh}(X) \times \mathbb{A}^1) \times_{\text{Perf}(X) \times \mathbb{A}^1} \text{Perf}(X_{\text{Del}}),
$$

and

$$
\text{Coh}(X_{\text{Del},G_m}) := (\text{Coh}(X) \times [\mathbb{A}^1/G_m]) \times_{\text{Perf}(X) \times [\mathbb{A}^1/G_m]} \text{Perf}(X_{\text{Del}}).
$$

We have canonical maps $\text{Coh}(X_{\text{Del}}) \to \mathbb{A}^1$ and $\text{Coh}(X_{\text{Del},G_m}) \to [\mathbb{A}^1/G_m]$. In particular,

$$
\text{Coh}(X_{\text{Del}}) \times \mathbb{A}^1 \{0\} \simeq \text{Coh}(X_{\text{Del}}) \quad \text{and} \quad \text{Coh}(X_{\text{Del}}) \times \mathbb{A}^1 \{1\} \simeq \text{Coh}(X_{\text{dR}}),
$$

while

$$
\text{Coh}(X_{\text{Del},G_m}) \times [\mathbb{A}^1/G_m] \text{B}G_m \simeq \text{Coh}_{\text{C}}(X_{\text{Del}}) \quad \text{Coh}(X_{\text{Del}}) \times \mathbb{A}^1 [G_m/G_m] \simeq \text{Coh}(X_{\text{dR}}) \times \text{B}G_m.
$$

(5.1)

We also consider the open substack $\text{Coh}^*(X_{\text{Del}}) \subset \text{Coh}(X_{\text{Del}})$ for which the fiber at zero is the derived moduli stack $\text{Coh}^{ss,0}(X_{\text{Del}})$ of semistable Higgs bundles on $X$ of degree zero (cf. [Sim09, §7]).

Similarly, we can define the derived moduli stacks of extensions of Deligne’s $\lambda$-connections. Thus, we have the convolution diagram in $\text{dSt}_{/\mathbb{A}^1}$:

$$
\begin{array}{ccc}
\text{Coh}^{\text{ext}}(X_{\text{Del}}) & \xrightarrow{p} & \text{Coh}(X_{\text{Del}}) \\
\text{Coh}(X_{\text{Del}}) \times \mathbb{A}^1 & \xleftarrow{q} & \text{Coh}(X_{\text{Del}}) \\
\end{array}
$$

and the convolution diagram in $\text{dSt}_{/[\mathbb{A}^1/G_m]}$:

$$
\begin{array}{ccc}
\text{Coh}^{\text{ext}}(X_{\text{Del},G_m}) & \xrightarrow{p} & \text{Coh}(X_{\text{Del},G_m}) \\
\text{Coh}(X_{\text{Del},G_m}) \times [\mathbb{A}^1/G_m] & \xleftarrow{q} & \text{Coh}(X_{\text{Del},G_m}) \\
\end{array}
$$

Because of Corollaries 3.18 and 3.22, it follows that the map $p$ above is derived locally complete intersection. A similar result holds when we restrict to the open substack $\text{Coh}^*(X_{\text{Del}})$ and the corresponding open substack of extensions. Following the same arguments as in the previous section, we can encode such convolution diagrams into 2-Segal objects, and obtain the following:

**Proposition 5.3.** Let $X$ be a smooth projective complex curve. Then

- there exists a 2-Segal object $\mathcal{S} \text{Coh}(X_{\text{Del}})$ which endows $\text{Coh}(X_{\text{Del}})$ with the structure of an $\mathcal{E}_1$-algebra in $\text{Corr}^* (\text{dGeom}_{/\mathbb{A}^1})_{\text{lc, rps}}$;
- there exists a 2-Segal object $\mathcal{S} \text{Coh}(X_{\text{Del},G_m})$ which endows $\text{Coh}(X_{\text{Del},G_m})$ with the structure of an $\mathcal{E}_1$-algebra in $\text{Corr}^* (\text{dGeom}_{/\mathbb{A}^1/G_m})_{\text{lc, rps}}$.

A similar result holds for $\text{Coh}^*(X_{\text{Del}})$ and $\text{Coh}^*(X_{\text{Del},G_m})$.

**Corollary 5.4.** $\text{Coh}^b(\text{Coh}(X_{\text{Del}}))$ and $\text{Coh}^b(\text{Coh}(X_{\text{Del},G_m}))$ are $\mathcal{E}_1$-algebras in $\text{Cat}^*_{\text{lc}}$. A similar result holds for $\text{Coh}^*(X_{\text{Del}})$ and $\text{Coh}^*(X_{\text{Del},G_m})$.

By combining the results above with (5.1), we get:
Theorem 5.5. Let $X$ be a smooth projective complex curve. Then
\[ \text{Coh}^b_{\text{gr}}\left(\text{Coh}^*(X_{\text{Del}})\right) \simeq \text{Coh}^b\left(\text{Coh}^*(X_{\text{Del}}, C_m)\right) \]
is a module over $\text{Perf}^\text{filt}$ and we have $E_1$-algebra morphisms:
\[
\Phi^*(\cdot): \text{Coh}^b_{\text{gr}}\left(\text{Coh}^*(X_{\text{Del}})\right) \otimes_{\text{Perf}^\text{filt}} \text{Perf}_C \rightarrow \text{Coh}^b\left(\text{Coh}^*(X_{\text{dR}})\right),
\]
\[
\Psi^*(\cdot): \text{Coh}^b_{\text{gr}}\left(\text{Coh}^*(X_{\text{Del}})\right) \otimes_{\text{Perf}^\text{filt}} \text{Perf}^\text{fr} \rightarrow \text{Coh}^b_{\text{gr}}\left(\text{Coh}_{\text{ss},0}^*(X_{\text{Del}})\right).
\]

Following Simpson [Sim09, §7], we expect the following to be true:

Conjecture 5.6 (CoHA version of the non-abelian Hodge correspondence). The morphisms $\Phi^*$ and $\Psi^*$ are equivalences, i.e., $\text{Coh}^b\left(\text{Coh}^*(X_{\text{dR}})\right)$ is filtered by $\text{Coh}^b_{\text{gr}}\left(\text{Coh}^*(X_{\text{Del}})\right)$ with associated graded $\text{Coh}^b_{\text{gr}}\left(\text{Coh}_{\text{ss},0}^*(X_{\text{Del}})\right)$.

6. A CoHA VERSION OF THE RIEMANN-HILBERT CORRESPONDENCE

In this section we briefly consider a complex analytic analogue of the theory developed so far. Thanks to the foundational work on derived analytic geometry [Lur11b, PY16, Por15, HP18] most of the constructions and results obtained so far carry over in the analytic setting. After sketching how to define the derived analytic stack of coherent sheaves, we focus on two main results. The first, is the construction of a monoidal functor between the algebraic and the analytic higher categorification of the CoHAS coming from nonabelian Hodge theory. The second, is to provide an equivalence between the analytic higher categorification of the Betti CoHA and the de Rham one. This equivalence is an instance of the Riemann-Hilbert correspondence, and it is indeed induced by the main results of [Por17, HP18].

6.1. The analytic stack of coherent sheaves. We refer to [HP18, §2] for a review of derived analytic geometry. Using the notations introduced there, we denote by $\text{AnPerf}$ the complex analytic stack of perfect complexes (see §4 in loc. cit.). Similarly, given derived analytic stacks $X$ and $Y$, we let $\text{AnMap}(X, Y)$ be the derived analytic stack of morphisms between them.

Fix a derived geometric analytic stack $X$. We wish to define a substack of $\text{AnPerf}(X) := \text{AnMap}(X, \text{AnPerf})$ classifying families of coherent sheaves on $X$. The same ideas of §2 apply, but as usual some extra care to deal with the notion of flatness in analytic geometry is needed.

Definition 6.1 (cf. [PY18, Definitions 7.1 & 7.2]). Let $f: X \rightarrow S$ be a morphism of derived analytic stacks. Assume that $S$ is a derived Stein space and $X$ is a geometric stack. We say that an almost perfect complex $F \in \text{APerf}(X)$ has tor-amplitude $[a, b]$ relative to $S$ if there exists a derived Stein atlas $\{u_i: U_i \rightarrow X\}$ such that each $\Gamma(U_i; u_i^*F)$ has tor-amplitude $[a, b]$ as $\Gamma(S; \mathcal{O}^\text{alg}_S)$-module. We say that $F \in \text{APerf}(X)$ is a family of coherent sheaves relative to $S$ if $F$ has tor-amplitude $[0, 0]$ relative to $S$.

The above definition has the disadvantage that the modules $\Gamma(U_i; u_i^*F)$ are not almost perfect as $\Gamma(S; \mathcal{O}^\text{alg}_S)$-modules. This makes this definition hard to manipulate in practice. It is therefore useful to rephrase it as follows:

Lemma 6.2. Let $f: X \rightarrow S$ be a morphism of derived analytic stacks. Suppose that $S$ is derived Stein and $X$ is geometric. Then an almost perfect complex $F \in \text{APerf}(X)$ has tor-amplitude $[a, b]$ relative to $S$ if and only if for every $G \in \text{APerf}^\text{ss}(S)$ one has
\[ \pi_i(F \otimes_{\mathcal{O}_X} f^*G) = 0 \]
for $i \notin [a, b]$. 
The question is local on $X$ and we can therefore assume that $X$ is a derived Stein space. Notice that $\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G} \in \text{APerf}(X)$. Therefore, Cartan’s theorem B applies and shows that $\pi_i(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}) = 0$ if and only if $\pi_i(f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})) = 0$. Observe now that there is a canonical morphism

$$\eta_{\mathcal{F}, \mathcal{G}} : f_*(\mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{G} \to f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G}).$$

When $\mathcal{G} = \mathcal{O}_S$ this morphism is obviously an equivalence. We claim that it is an equivalence for any $\mathcal{G} \in \text{APerf}(S)$.

This question is local on $S$. Write $A_S := \Gamma(S; \mathcal{O}_S^{\text{alg}})$. Using [HP18, Lemma 4.12] we can reduce ourselves to the case where $\mathcal{G} \simeq \epsilon_S^*(M)$ for some $M \in \text{APerf}(A_S)$. Here $\epsilon_S^* : A_S \text{-Mod} \to \mathcal{O}_S \text{-Mod}$ is the functor introduced in [HP18, §4.2]. In this case, we see that since $\eta_{\mathcal{F}, \mathcal{G}}$ is an equivalence whenever $\mathcal{G} = \mathcal{O}_S$, it is also an equivalence whenever $M$ (and hence $\mathcal{G}$) is perfect. In the general case, we use [Lur17, 7.2.4.11(5)] to find a simplicial object $P^* \in \text{Fun}(\Delta^{op}, \text{APerf}(A_S))$ such that

$$|P^*| \simeq M.$$

Write $P^* := \epsilon_S^*(P^*)$. Reasoning as in [PY18, Corollary 3.5], we deduce that

$$|P^*| \simeq \epsilon_S^*(M) \simeq \mathcal{G}.$$

It immediately follows that

$$\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G} \simeq |\mathcal{F} \otimes_{\mathcal{O}_X} f^* P^*|,$$

and the question of proving that $\eta_{\mathcal{F}, \mathcal{G}}$ is an equivalence is reduced to check that $f_*$ preserves the above colimit. Since the above diagram as well as its colimit takes values in $\text{APerf}(X)$, we can apply Cartan’s theorem B. The descent spectral sequence degenerates, and therefore the conclusion follows. $\square$

Corollary 6.3. Let $f : X \to S$ be a morphism as in the previous lemma. Let $j : \text{cl} S \to S$ be the canonical morphism and consider the pullback diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{i} & X \\
\downarrow{f_0} & & \downarrow{f} \\
\text{cl} S & \xrightarrow{j} & S
\end{array}
$$

Then an almost perfect complex $\mathcal{F} \in \text{APerf}(X)$ has tor-amplitude $[a, b]$ relative to $S$ if and only if $i^* \mathcal{F}$ has tor-amplitude $[a, b]$ relative to $\text{cl} S$.

Proof. The map $j$ is a closed immersion and therefore the same goes for $i$. In particular, for any $\mathcal{G} \in \text{APerf}(\text{cl} S)$ the canonical map

$$f^* j_*(\mathcal{G}) \to i_* f_0^* (\mathcal{G})$$

is an equivalence. Moreover, the projection formula holds for $i$ and $i_*$ is $t$-exact. Suppose that $\mathcal{F}$ has tor-amplitude $[a, b]$ relative to $S$. Let $\mathcal{G} \in \text{APerf}^\lor(\text{cl} S)$. Then

$$i_*(i^* \mathcal{F} \otimes_{\mathcal{O}_X} f_0^* \mathcal{G}) \simeq \mathcal{F} \otimes_{\mathcal{O}_X} i_* f_0^* \mathcal{G} \simeq \mathcal{F} \otimes_{\mathcal{O}_X} f^* j_* \mathcal{G}.$$

Since $j_*$ is $t$-exact, $j_* \mathcal{G} \in \text{APerf}^\lor(S)$, and therefore the above tensor product is concentrated in homological degree $[a, b]$. In other words, $i^* \mathcal{F}$ has tor-amplitude $[a, b]$ relative to $\text{cl} S$. For the vice-versa, it is enough to observe that $j_*$ induces an equivalence $\text{APerf}^\lor(\text{cl} S) \simeq \text{APerf}^\lor(S)$. $\square$

\text{[20]} Ultimately, this can be traced back to the unramifiedness of the analytic pregeometry $T_m(C)$. See [PY18, Lemma 6.1] for an argument in the non-archimedean case.
Corollary 6.4. Let $X$ be an underived geometric stack and let $S$ be a derived Stein space. Let $f : S' \to S$ be a morphism of derived Stein spaces and consider the pullback

$$
\begin{array}{ccc}
X \times S' & \xrightarrow{g} & X \times S \\
\downarrow g & & \downarrow p \\
S' & \xrightarrow{f} & S
\end{array}
$$

If $\mathcal{F} \in \text{APerf}(X \times S)$ has tor-amplitude $[0, 0]$ relative to $S$, then $g^* \mathcal{F}$ has tor-amplitude $[0, 0]$ relative to $S'$.

Proof. Using Corollary 6.3, we can reduce ourselves to the case where $S$ and $S'$ are underived. Since the question is local on $X$, we can furthermore assume $X$ to be a Stein space. At this point, the conclusion follows directly from [Dou66, §8.3, Proposition 3].

Using the above corollary, we can therefore define a derived analytic stack $\text{AnCoh}(X)$, which is a substack of $\text{AnPerf}(X)$.

In what follows, we will often restrict ourselves to the study of $\text{AnCoh}(X^{\text{an}})$, where now $X$ is an algebraic variety. Combining [HP18, Proposition 5.2 & Theorem 5.5] we see that if $X$ is a proper complex scheme, then there is a natural equivalence$^{21}$

$$
\text{Perf}(X)^{\text{an}} \simeq \text{AnPerf}(X^{\text{an}}).
$$

We wish to extend this result to $\text{Coh}(X)^{\text{an}}$ and $\text{AnCoh}(X^{\text{an}})$. Let us start by constructing the map between them. The map $\text{Perf}(X)^{\text{an}} \to \text{AnPerf}(X^{\text{an}})$ is obtained by adjunction from the map

$$
\text{Perf}(X) \longrightarrow \text{AnPerf}(X^{\text{an}}) \circ (-)^{\text{an}},
$$

which, for $S \in \text{dAff}^{\text{afp}}$, is induced by applying $(-)^{\text{an}} : \text{Cat}_{\text{an}} \to \text{S}$ to the analytification functor

$$
\text{Perf}(X \times S) \longrightarrow \text{Perf}(X^{\text{an}} \times S^{\text{an}}).
$$

It is therefore enough to check that this functor respects the two subcategories of families of coherent sheaves relative to $S$ and $S^{\text{an}}$, respectively.

Lemma 6.5. Let $f : X \to S$ be a morphism of derived complex stacks locally almost of finite presentation. Suppose that $X$ is geometric and that $S$ is affine. Then $\mathcal{F} \in \text{APerf}(S)$ has tor-amplitude $[a, b]$ relative to $S$ if and only if $\mathcal{F}^{\text{an}} \in \text{APerf}(X^{\text{an}})$ has tor-amplitude $[a, b]$ relative to $S^{\text{an}}$.

Proof. Suppose first that $\mathcal{F}^{\text{an}}$ has tor-amplitude $[a, b]$ relative to $S^{\text{an}}$. Let $\mathcal{G} \in \text{APerf}^{\text{an}}(S)$. Then we have to check that $\pi_i((\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})^{\text{an}}) = 0$ for $i \notin [a, b]$. As the analytification functor $(-)^{\text{an}}$ is $t$-exact and conservative, this is equivalent to check that $\pi_i((\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})^{\text{an}}) = 0$. On the other hand,

$$
(\mathcal{F} \otimes_{\mathcal{O}_X} f^* \mathcal{G})^{\text{an}} \simeq \mathcal{F}^{\text{an}} \otimes_{\mathcal{O}^{\text{an}}_X} \mathcal{G}^{\text{an}}.
$$

The conclusion now follows from the fact that $\mathcal{G}^{\text{an}} \in \text{APerf}^{\text{an}}(S^{\text{an}})$.

Suppose now that $\mathcal{F}$ has tor-amplitude $[a, b]$ relative to $S = \text{Spec}(A)$. We can check that $\mathcal{F}^{\text{an}}$ has tor-amplitude $[a, b]$ relative to $S^{\text{an}}$ locally on $S^{\text{an}}$. For every derived Stein open subspace $j_U : U \subset S^{\text{an}}$, write

$$
A_U := \Gamma(U; \mathcal{O}^{\text{an}}_{S^{\text{an}}})_{|U}.
$$

$^{21}$The derived analytification functor has been firstly introduced in [Lur11b, Remark 12.26] and studied extensively in [Por15, §4]. For a review, see [HP18, §3.1].
Write $a_U: \text{Spec}(A_U) \to S$ for the morphism induced by the canonical map $A \to A_U$. Consider the two pullback squares

$$
\begin{array}{ccc}
X_U & \xrightarrow{b_U} & X \\
\downarrow{f_U} & & \downarrow{f} \\
\text{Spec}(A_U) & \xrightarrow{a_U} & S
\end{array}
\quad
\begin{array}{ccc}
X^\text{an}_U & \xrightarrow{i_U} & X^\text{an} \\
\downarrow{j_U} & & \downarrow{j} \\
U & \xrightarrow{f} & S^\text{an}
\end{array}
.$$}

There is a natural analytification functor relative to $U$

$$(-)^\text{an}_U: \text{APerf}(X_U) \to \text{APerf}(X^\text{an}_U).$$

Moreover, the canonical map

$$i^*_U(\mathcal{H}^\text{an}) \to (b^*_U(\mathcal{H}))^\text{an}_U$$

is an equivalence for every $\mathcal{H} \in \text{APerf}(X)$.

Fix now $\mathcal{G} \in \text{APerf}(S^\text{an})$. If $\mathcal{G} \simeq (\check{\mathcal{G}})^\text{an}$ for some $\check{\mathcal{G}} \in \text{APerf}(S)$, then the equivalence (6.2) shows that

$$\pi_i(F^\text{an} \otimes_{\mathcal{O}_{X^\text{an}}} f^\text{an}_*(\mathcal{G})) = 0$$

for $i \notin [a, b]$. In the general case, we choose a double covering $\{V_i \subseteq U_i \subseteq S^\text{an}\}$ by relatively compact derived Stein open subspaces of $S^\text{an}$. Using [HP18, Lemma 4.12] we can find $\check{\mathcal{G}}_i \in \text{APerf}(A_{V_i})$ such that $\mathcal{G}|_{V_i} \simeq \check{\iota}_{V_i}(\check{\mathcal{G}}_i)$. Here $\check{\iota}_{V_i}$ is the functor introduced in [HP18, §4.2]. At this point, we observe that Lemma 2.4 guarantees that $b^*_U(F)$ has tor-amplitude $[a, b]$ relative to $\text{Spec}(A_U)$. The conclusion therefore follows from the argument given in the first case.

As a consequence, we find a morphism

$$\text{Coh}(X) \to \text{AnCoh}(X^\text{an}) \circ (-)^\text{an},$$

which by adjunction induces

$$\mu_X: \text{Coh}(X)^\text{an} \to \text{AnCoh}(X^\text{an})$$

which is compatible with the morphism $\text{Perf}(X)^\text{an} \to \text{AnPerf}(X^\text{an})$.

**Proposition 6.6.** If $X$ is a proper complex scheme, the natural transformation

$$\mu_X: \text{Coh}(X)^\text{an} \to \text{AnCoh}(X^\text{an})$$

is an equivalence.

**Proof.** Reasoning as in the proof of the equivalence (6.1) in [HP18, Proposition 5.2], we reduce ourselves to check that for every derived Stein space $U \in \text{dStn}_C$ and every compact derived Stein subspace $K$ of $U$, the natural morphism

$$\text{“colim”} \text{Coh}_{\text{Spec}(A_U)}(\text{Spec}(A_V) \times X) \to \text{“colim”} \text{Coh}_V(V \times X^\text{an})$$

is an equivalence in $\text{Ind}(\text{Cats}^\text{st}_{\infty})$. Here the colimit is taken over the family of open Stein neighborhoods $V$ of $K$ inside $U$. Using [HP18, Lemma 5.13] we see that for every $V$, the functor

$$\text{Coh}_{\text{Spec}(A_U)}(\text{Spec}(A_V) \times X) \to \text{Coh}_V(V \times X^\text{an})$$

is fully faithful. The conclusion now follows combining [HP18, Proposition 5.15] and the “only if” direction of Lemma 6.5. □
6.2. Cohomological Hall algebras in the C-analytic setting. Let \( X \in \mathsf{dAnSt} \) be a derived analytic stack. In the previous section, we have introduced the analytic stack \( \mathsf{AnCoh}(X) \) parameterizing families of almost perfect complexes over \( X \) of tor-amplitude \( \leq 0 \) relative to the base. Similarly, we can define the derived analytic stacks \( \mathsf{AnPerf}^\text{ext} \), \( \mathsf{AnPerf}^\text{ext}(X) \), and \( \mathsf{AnCoh}^\text{ext}(X) \). We deal directly with the Waldhausen construction.

We define the simplicial derived analytic stack
\[
\mathcal{S} \mathsf{AnPerf} : \mathsf{dSt}_C^{op} \longrightarrow \mathsf{Fun}(\Delta^{op}, \mathcal{S})
\]
by sending a derived Stein space \( S \) to\(^{22}\)
\[
\mathsf{Fun}(T_n, \mathsf{Perf}(S)).
\]
Since each \( T_n \) is a finite category, [HP18, Corollary 7.2] implies that the canonical morphism
\[
(\mathcal{S} \mathsf{Perf})^\text{an} \longrightarrow \mathcal{S} \mathsf{AnPerf}
\]
is an equivalence. Moreover, Proposition 7.3 in loc. cit. implies that the analytification commutes with the limits appearing in the 2-Segal condition. We can therefore deduce that \( \mathcal{S} \mathsf{AnPerf} \) is a 2-Segal object in \( \mathsf{dAnSt} \). From here, we deduce immediately that for every derived analytic stack \( X \), \( \mathsf{AnMap}(X, \mathcal{S} \mathsf{AnPerf}) \) is again a 2-Segal object. At this point, the same reasoning of Lemma 4.3 yields:

**Proposition 6.7.** Let \( X \in \mathsf{dAnSt} \) be a derived geometric analytic stack. Then \( \mathcal{S} \mathsf{AnCoh}(X) \) is a 2-Segal object in \( \mathsf{dAnSt} \), and therefore it endows the derived analytic stack \( \mathsf{AnCoh}(X) \) with the structure of an \( \mathbb{E}_1 \)-convolution algebra.

The morphism (6.2) can be naturally upgraded to a natural transformation
\[
\mathcal{S} \mathsf{Coh}(X) \longrightarrow \mathcal{S} \mathsf{AnCoh}(X^\text{an}) \circ (-)^\text{an}
\]
in \( \mathsf{Fun}(\Delta^{op}, \mathsf{dSt}) \). By adjunction, we therefore find a morphism of simplicial objects
\[
(\mathcal{S} \mathsf{Coh}(X))^\text{an} \longrightarrow \mathcal{S} \mathsf{AnCoh}(X^\text{an})).
\]

**Remark 6.8.** Suppose that \( X \) is such that each \( \mathcal{S} \mathsf{anCoh}(X) \) is geometric. Then [HP18, Proposition 7.3] implies that \( (\mathcal{S} \mathsf{Coh}(X))^\text{an} \) is a 2-Segal object in \( \mathsf{dAnSt} \). \( \triangle \)

Let \( Y \in \mathsf{dAnSt} \) be a derived analytic stack and let \( u : U \rightarrow Y \) be a flat effective epimorphism from a derived geometric analytic stack \( U \). We define
\[
\mathsf{AnCoh}(Y, u) := \mathsf{AnPerf}(Y) \times_{\mathsf{AnPerf}(U)} \mathsf{AnCoh}(U).
\]
Similarly, we can define \( \mathsf{AnCoh}^\text{ext}(Y, u) \) and \( \mathsf{AnBun}^\text{ext}(Y, u) \) and more generally their Waldhausen analogues \( \mathcal{S} \mathsf{AnCoh}(Y, u) \) and \( \mathcal{S} \mathsf{AnBun}(Y, u) \). As before, we shall omit \( u \) from the notation when its choice is clear from the context. We immediately obtain:

**Proposition 6.9.** Let \( Y \in \mathsf{dAnSt} \) be a derived analytic stack and let \( u : U \rightarrow Y \) be a flat effective epimorphism from a derived geometric analytic stack \( U \). Then \( \mathcal{S} \mathsf{AnCoh}(Y, u) \) is a 2-Segal object and it endows \( \mathsf{AnCoh}(Y, u) \) with the structure of an \( \mathbb{E}_1 \)-convolution algebra in \( \mathsf{dAnSt} \).

As a particular case, let \( X \) be a smooth proper connected analytic space. The Simpson’s shapes \( X_B \), \( X_{dR} \), \( X_{dol} \), and \( X_{Dol} \) exist also in derived analytic geometry (as introduced e.g. in [HP18, § 5.2]). We have the following analytic analog of Corollary 4.5.

**Corollary 6.10.** Let \( X \in \mathsf{dAnSt} \) be a derived geometric analytic stack and let \( Y \) be one of the following stacks: \( X_B \), \( X_{dR} \), and \( X_{Dol} \). Then \( \mathcal{S} \mathsf{AnCoh}(Y) \) is a 2-Segal object in \( \mathsf{dAnSt} \), and therefore it endows the derived analytic stack \( \mathsf{AnCoh}(Y) \) with the structure of an \( \mathbb{E}_1 \)-convolution algebra.

\(^{22}\)See §4.1 for the notations used here.
Our next step is to construct the categorification in the analytic setting. In §4.2 the starting input has been the right-lax monoidal functor

\[ \text{QCoh}: \text{Corr}^\times(\text{dSt}) \to \text{Cat}^{\text{st}}_\infty, \]

which in turn is obtained in [GaR17a, §3.1] out of the strong monoidal functor

\[ \text{QCoh}: \text{dAff} \to \text{Cat}^{\text{st}}_\infty. \]

In the analytic setting there are two main difficulties: on one hand, quasi-coherent sheaves are not available and have to be replaced with \( O_X \)-modules; on the other hand, even after restricting to derived Stein spaces, the functor

\[ \text{AnCoh}^b: \text{dSt}^\text{C} \to \text{Cat}^{\text{st}}_\infty, \]

is not strong monoidal, only right-lax. Therefore, we need some extra care in dealing with this construction in the analytic setting. For sake of future applications, we perform a construction slightly more general than needed.

**Construction 6.11.** Let \( k \) be a field and let \( \text{CAlg}_k \) denote the \( \infty \)-category of derived commutative \( k \)-algebras. Let \( T_{\text{disc}}(k) \) be the full subcategory of \( \text{Sch}_k \) spanned by finite dimensional affine spaces \( \mathbb{A}_k^n \). Given an \( \infty \)-topos \( X \), sheaves on \( X \) with values in \( \text{CAlg}_k \) can be canonically identified with product preserving functors \( T_{\text{disc}}(k) \to X \). We let \( R \text{Top}(T_{\text{disc}}(k)) \) denote the \( \infty \)-category of \( \infty \)-topoi equipped with a sheaf of derived commutative \( k \)-algebras. The construction performed in [Lur11a, Notation 2.2.1] provides us with a functor

\[ \Gamma: (R \text{Top}(T_{\text{disc}}(k)))^\text{op} \to \text{CAlg}_k. \]

Equipping both \( \infty \)-categories with the coCartesian monoidal structure, we see that \( \Gamma \) can be upgraded to a right-lax symmetric monoidal structure. Composing with the symmetric monoidal functor \( \text{QCoh}: \text{CAlg}_k \to \text{Cat}^{\text{st}}_\infty \) we therefore obtain a right-lax symmetric monoidal functor

\[ (R \text{Top}(T_{\text{disc}}(k)))^\text{op} \to \text{Cat}^{\text{st}}_\infty. \]

Finally, we sheafify this functor with respect to the étale topology on \( R \text{Top}(T_{\text{disc}}(k)) \) (see [Lur11a, Definition 2.3.1]). The so obtained functor canonically coincides with

\[ \mathcal{O} \text{-Mod}: (R \text{Top}(T_{\text{disc}}(k)))^\text{op} \to \text{Cat}^{\text{st}}_\infty, \]

which in this is canonically endowed with a right-lax symmetric monoidal structure.

We now consider the natural functor

\[ \text{dAnC} \hookrightarrow R \text{Top}(T_{\text{an}}(C)) \xrightarrow{(\_)^{\text{NLG}}} R \text{Top}(T_{\text{disc}}(k)). \]

This functor is right-lax monoidal. Composing with the functor obtained in the above construction we therefore obtain a right-lax symmetric monoidal functor

\[ \mathcal{O} \text{-Mod}: \text{dAnC} \to \text{Cat}^{\text{st}}_\infty. \]

At this point, the same method applied in [GaR17a, §3.1] provides us with a right-lax monoidal functor

\[ \mathcal{O} \text{-Mod}: \text{Corr}^\times(\text{dAnSt}) \to \text{Cat}^{\text{st}}_\infty. \]

**Lemma 6.12.** Let \( f: X \to Y \) be a morphism of derived geometric analytic stacks. If \( f \) is lci\(^{23}\) then it has finite tor-amplitude and in particular it induces a functor

\[ f^*: \text{Coh}^b(Y) \to \text{Coh}^b(X). \]

**Proof.** Essentially the same argument of [PY19, Corollary 2.6] applies. \( \square \)

\(^{23}\)In this setting, it means that the analytic cotangent complex \( L_{X/Y}^{\text{an}} \) introduced in [PY17] is perfect and has tor-amplitude \([0,1]\).
As a consequence, we obtain a right-lax monoidal functor

\[ \text{AnCoh}^b : \text{Corr} \times (\text{dAnSt})_{\text{li}, \text{rps}} \to \text{Cat}^\text{st}_\infty. \]

Finally, we want to restrict ourselves to derived geometric analytic stacks. In particular, we need that \( \text{AnCoh}(Y, u) \) and the corresponding 2-Segal space to be geometric. So, first note that if \( Y \in \text{dSt} \) is a derived stack and \( u : U \to Y \) is a flat effective epimorphism from a derived geometric stack \( U \), then we obtain as before a natural transformation

\[ S\text{Coh}(Y, u)^{\text{an}} \to S\text{AnCoh}(Y, u) \quad (6.3) \]

in \( \text{Fun}(\Delta^{op}, \text{dAnSt}) \).

Let \( X \) be a smooth and proper complex scheme. By [HP18, Proposition 5.2], \( \text{AnPerf}(X) \) is equivalent to the analytification \( \text{Perf}(X)^{\text{an}} \) of the derived stack \( \text{Perf}(X) = \text{Map}(X, \text{Perf}) \). Thus, \( \text{AnPerf}(X) \) is a locally geometric derived stack, locally of finite presentation.

**Lemma 6.13.** The map (6.3) induces an equivalence \( S\text{Coh}(X)^{\text{an}} \simeq S\text{AnCoh}(X^{\text{an}}) \). In particular, for each \( n \geq 0 \) the derived analytic stack \( S_n \text{AnCoh}(X^{\text{an}}) \) is locally geometric and locally of finite presentation.

**Proof.** When \( n = 1 \), this is exactly the statement of Proposition 6.6. The proof of the general case is similar, and there are no additional subtleties than the ones discussed there. \( \square \)

Let \( X \) be a smooth proper connected complex scheme. As proved in [HP18, §5.2], the analytification functor commutes with the Simpson’s shape functor, i.e., we have the following canonical equivalences:

\[ (X^{\text{dR}})^{\text{an}} \simeq (X^{\text{dR}})^{\text{an}}, \quad (X^{\text{B}})^{\text{an}} \simeq (X^{\text{B}})^{\text{an}}, \quad (X^{\text{Dol}})^{\text{an}} \simeq (X^{\text{Dol}})^{\text{an}}. \]

**Lemma 6.14.** Let \( * \in \{ \text{B}, \text{dR}, \text{Dol} \} \). Then the map (6.3) induces an equivalence \( S\text{Coh}(X^*)^{\text{an}} \simeq S\text{AnCoh}(X^{\text{an}})^* \). In particular, for each \( n \geq 0 \) the derived analytic stack \( S_n \text{AnCoh}((X^{\text{an}})^*) \) is locally geometric and locally of finite presentation.

**Proof.** The same proof of Proposition 6.6 applies, with only the following caveat: rather than invoking [HP18, Lemma 5.13 & Proposition 5.15], we should use instead Propositions 5.26 (for the de Rham case), 5.28 (for the Betti case) and 5.32 (for the Dolbeault case) in loc. cit. \( \square \)

Finally, we are able to give the analytic counterpart of Theorem 4.6:

**Theorem 6.15.** Let \( Y \) be one of the following derived stacks:

1. a smooth proper complex scheme of dimension either one or two;
2. the Betti, de Rham or Dolbeault stack of a smooth projective curve.

Then the composition

\[ \text{Coh}^b(\text{AnCoh}(Y^{\text{an}}) \times \text{Coh}^b(\text{AnCoh}(Y^{\text{an}}))) \xrightarrow{\text{ext}} \text{Coh}^b(\text{AnCoh}(Y^{\text{an}}) \times \text{AnCoh}(Y^{\text{an}})) \]

\[ \xrightarrow{q \circ p} \text{Coh}^b(\text{AnCoh}(Y^{\text{an}})), \]

where the map on the right-hand-side is induced by the 1-morphism in correspondences:

\[ \xymatrix{ \text{AnCoh}^\text{ext}(Y^{\text{an}}) \ar[rd]_p \ar[rd]_q & \text{AnCoh}(Y^{\text{an}}) \ar[ld]^p \ar[ld]^q \ar[rd]_{\text{AnCoh}} \ar[lld] & \text{AnCoh}(Y^{\text{an}}) \ar[ll]^p \ar[lld]_q \ar[lld] \ar[ll]_{\text{AnCoh}} \}
\]

ends \( \text{Coh}^b(\text{AnCoh}(Y)) \) with the structure of an \( \mathbb{E}_1 \)-monoidal stable \( \infty \)-category.
**Proof.** The only main point to emphasize is how to use the tor-amplitude estimates for the map $p$ in the algebraic case (i.e., Corollaries 3.10, 3.15, 3.18, and 3.22) in the analytic setting. First of all, we use Lemmas 6.13 and 6.14 to identify the 2-Segal object $S\text{AnCoh}(Y^\text{an})$ with $(S\text{Coh}(Y))^\text{an}$. Then we are reduced to check that $p^\text{an}$ is derived lci, where now $p$ is the map appearing in (4.1). This follows combining Lemma 6.5 and [PY17, Theorem 5.21].

**Corollary 6.16.** Let $Y$ be as in Theorem 6.15. Then the derived analytification functor induces a morphism in $\text{Alg}_{\mathcal{E}_1}(\text{Cat}^\text{st}_{\mathcal{E}_1})$

$$\text{Coh}^b(\text{Coh}(Y)) \longrightarrow \text{Coh}^b(\text{AnCoh}(Y^\text{an})) .$$

**Proof.** By using Lemmas 6.13 and 6.14, we have $\text{Coh}^b(\text{Coh}(Y)^\text{an}) \simeq \text{Coh}^b(\text{AnCoh}(Y^\text{an}))$ as $\mathcal{E}_1$-algebras. The analytification functor $(-)^\text{an}$ promotes to a symmetric monoidal functor

$$(-)^\text{an}: \text{Corr}^x(\text{dSt}) \longrightarrow \text{Corr}^x(\text{dAnSt}) .$$

Combining Lemma 6.5 and [PY17, Theorem 5.21], we conclude that $(-)^\text{an}$ preserves lci morphisms. Moreover, [PY17, Lemma 3.1(3)] and [PY16, Proposition 6.3], we see that $(-)^\text{an}$ also preserves proper morphisms. Finally, using the derived GAGA theorems [Por15, Theorems 7.1 & 7.2] we see that $(-)^\text{an}$ takes morphisms which are representable by proper schemes to morphisms which are representable by proper analytic spaces. Therefore, it restricts to a symmetric monoidal functor

$$(-)^\text{an}: \text{Corr}^x(\text{dSt})_{\text{lic},\text{rps}} \longrightarrow \text{Corr}^x(\text{dAnSt})_{\text{lic},\text{rps}} .$$

The analytification functor for coherent sheaves induces a natural transformation of right-lax symmetric monoidal functors

$$\text{Coh}^b \longrightarrow \text{AnCoh}^b \circ (-)^\text{an} .$$

Here both functors are considered as functors $\text{dSt} \rightarrow \text{Cat}_{\mathcal{E}_1}$. Using the universal property of the category of correspondences, we can extend this natural transformation of right-lax symmetric monoidal functors defined over the category of correspondences. The key point is to verify that if $p: X \rightarrow Y$ is a proper morphism of derived geometric stacks locally almost of finite presentation, then the diagram

$$\begin{array}{ccc}
\text{Coh}^b(X) & \xrightarrow{(-)^\text{an}} & \text{Coh}^b(X^\text{an}) \\
\downarrow{\rho_\ast} & & \downarrow{\rho_\ast^\text{an}} \\
\text{Coh}^b(Y) & \xrightarrow{(-)^\text{an}} & \text{Coh}^b(Y^\text{an})
\end{array}$$

commutes. This is a particular case of [Por15, Theorem 7.1]. The conclusion follows. □

6.3. The derived Riemann-Hilbert correspondence. Let $X$ be a smooth proper connected complex scheme. In [Por17, §3] it is constructed a natural transformation

$$\eta_{\text{RH}}: X^\text{an}_{d\text{R}} \longrightarrow X^\text{an}_{B} ,$$

which induces for every derived analytic stack $Y \in \text{dAnSt}$ a morphism

$$\eta_{\text{RH}^Y}: \text{AnMap}(X^\text{an}_{d\text{R}}, Y) \longrightarrow \text{AnMap}(X^\text{an}_{B}, Y) .$$

It is then shown in [Por17, Theorem 6.11] that this map is an equivalence when $Y = \text{AnPerf}$. Taking $Y = S\text{AnPerf}$, we see that $\eta_{\text{RH}}$ induces a morphism of 2-Segal objects

$$\eta_{\text{RH}}: S\text{AnPerf}(X^\text{an}_{d\text{R}}) \longrightarrow S\text{AnPerf}(X^\text{an}_{B}) .$$

\[24\text{Using [PY16, Proposition 6.3] it is enough to prove that the analytification takes representable morphisms with geometric target to representable morphisms. This immediately follows from [PY16, Proposition 2.25].}

\[25\text{See [HP18, Corollary 7.6] for a discussion of which other derived analytic stacks $Y$ see $\eta_{\text{RH}}$ as an equivalence.}]}
By applying the functor $2\text{-Seg}(d\text{AnSt}) \to \text{Alg}_{E_1}(\text{Corr}^\times(d\text{AnSt}))$, we therefore conclude that

$$\eta_{\text{RH}}^* : \text{AnPerf}(X_{\text{dR}}^{\text{an}}) \to \text{AnPerf}(X_B^{\text{an}})$$

acquires a natural structure of morphism between $E_1$-convolution algebras. We have:

**Proposition 6.17.** The morphism

$$\eta_{\text{RH}}^* : S\text{AnPerf}(X_{\text{dR}}^{\text{an}}) \to S\text{AnPerf}(X_B^{\text{an}})$$

is an equivalence. Moreover, it restricts to an equivalence

$$\eta_{\text{RH}}^*: S\text{Coh}_{\text{dR}}(X) \to S\text{Coh}_B(X).$$

**Proof.** Fix a derived Stein space $S \in \text{dStn}_C$. Then [Por17, Theorem 6.11] provides an equivalence of stable $\infty$-categories

$$\text{Perf}(X_{\text{dR}}^{\text{an}} \times S) \simeq \text{Perf}(X_B^{\text{an}} \times S).$$

Therefore, for every $n \geq 0$ we obtain an equivalence

$$S_n\text{AnPerf}(X_{\text{dR}}^{\text{an}})(S) \simeq \text{Fun}(T_n, \text{Perf}(X_{\text{dR}}^{\text{an}} \times S)) \simeq \text{Fun}(T_n, \text{Perf}(X_B^{\text{an}} \times S)) \simeq S_n\text{AnPerf}(X_B^{\text{an}})(S).$$

The first statement follows at once. The second statement follows automatically given the commutativity of the natural diagram

\[
\begin{array}{ccc}
X_{\text{dR}}^{\text{an}} & \xrightarrow{\lambda_X} & X_B^{\text{an}} \\
\downarrow{\eta_{\text{RH}}} & & \downarrow{\eta_{\text{RH}}} \\
X_B^{\text{an}} & \xrightarrow{\lambda_X} & X_B^{\text{an}}
\end{array}
\]

\[\square\]

**Theorem 6.18** (CoHA version of the derived Riemann-Hilbert correspondence). There is an equivalence of stable $E_1$-monoidal $\infty$-categories

$$(\text{Coh}^b(\text{AnCoh}_{\text{dR}}(X)), \otimes_{\text{dR}}^{\text{an}}) \simeq (\text{Coh}^b(\text{AnCoh}_B(X)), \otimes_B^{\text{an}}).$$

**Appendix A. Shapes of stacks and varieties**

In this section we briefly review the various shapes of Hodge theory, which have been first introduced by Simpson (cf. [Sim96, Sim97]).

**A.1. Betti shape.** Given a space $K \in \mathcal{S}$ we let

$$K_B : \text{dAff}^{\text{op}} \to \mathcal{S}$$

be the (étale) sheafification of the constant presheaf associated to $K$. We refer to $K_B$ as the Betti shape of $K$. When $X$ is a $\mathbb{C}$-scheme of finite type, we abuse of notation and let $X_B := (X^{\text{top}})_B$, where $X^{\text{top}}$ denotes space underlying the analytification $X^{\text{an}}$.

**Lemma A.1.**

1. There is a canonical equivalence $\text{Spec}(\mathbb{C})_B \simeq \text{Spec}(\mathbb{C})$.

2. Let $X$ be a connected $\mathbb{C}$-scheme of finite type and let $x : \text{Spec}(\mathbb{C}) \to X$ be a closed point. Then the induced map $\text{Spec}(\mathbb{C}) \to X_B$ is flat and an effective epimorphism.\footnote{See [Lur09, §6.2.3] for the definition of effective epimorphism in an $\infty$-topos $\mathcal{X}$.}
(3) Let $X$ be a $\mathbb{C}$-scheme of finite type. There is an equivalence
\[ \text{Map}_{\text{dSt}}(X_B, \text{BGL}_n) \simeq \text{Rep}_n(\tau_{\leq 1}(X_{\text{top}})) \simeq \text{Rep}_n(\tau_{\leq 1}(X)) \]
where the right hand side denotes the groupoid of representations of the fundamental groupoid $\tau_{\leq 1}(X_{\text{top}})$ of $X$.

Proof. The first point is immediate from the definition. For the second point, the fact that the morphism $\text{Spec}(\mathbb{C}) \to X_B$ is an effective epimorphism is a direct consequence of [Lur09, Proposition 7.2.1.14]. Let us show the flatness. We have
\[ \text{QCoh}(X_B) \simeq \text{Map}_{\text{Cat}^{\infty}}(X_{\text{top}}, \text{C-Mod}) \].
The $t$-structure is induced object-wise. In particular, the restriction along a map $* \to X_{\text{top}}$ induces a $t$-exact functor. Finally, for point (3) we observe that there is a canonical equivalence
\[ \text{Map}_{\text{dSt}}(X_B, \text{BGL}_n) \simeq \text{Map}_{\text{S}}(X_{\text{top}}, \text{BGL}_n(\mathbb{C})) \]
where $X_{\text{top}}$ denotes the space underlying the analytification $X_{\text{an}}$. Notice that $\text{BGL}_n(\mathbb{C})$ is the 1-groupoid of $\mathbb{C}$-vector spaces of rank $n$. Therefore, the conclusion follows from [Lur09, Proposition 5.5.6.18]. □

By (3) above, it follows that the derived category $D_{\text{qcoh}}(X_B)$ is canonically equivalent to $D_{\text{loc}}(X_{\text{top}}, \mathbb{C})$, which is the full sub-category of the derived category of sheaves of $\mathbb{C}$-vector spaces on the topological space $X_{\text{top}}$ consisting of complexes with locally constant cohomology sheaves. Perfect complexes on $X_B$ correspond to objects in $D_{\text{loc}}(X_{\text{top}}, \mathbb{C})$ locally quasi-isomorphic to bounded complexes of constant sheaves of projective modules of finite type.

A.2. De Rham shape. Given a derived stack $F: \text{dAff}^{\text{op}} \to \mathcal{S}$ we define its de Rham shape $F_{\text{dR}}$ by setting
\[ F_{\text{dR}}(\text{Spec}(A)) := F(\text{Spec}(\pi_0(A)_{\text{red}})). \]
Notice that there is a natural morphism
\[ \lambda_F: F \to F_{\text{dR}}. \]

Lemma A.2. If $X$ is a smooth scheme over $\mathbb{C}$. Then the map $\lambda_X: X \to X_{\text{dR}}$ is flat and an effective epimorphism.

Proof. We start by proving that $\lambda_X$ is an effective epimorphism. Let $\text{Spec}(A) \in \text{dAff}$ and let $f: \text{Spec}(A) \to X_{\text{dR}}$ be a morphism. We have to prove that up to an étale cover we can factor $f$ through $\lambda_X$. The map $f$ corresponds to a map $\tilde{f}: \text{Spec}(\pi_0(A)_{\text{red}}) \to X$. Since $X$ is smooth, this map extends first to a map $\text{Spec}(\pi_0(A)) \to X$ and then to a map $\text{Spec}(A) \to X$.

As for the flatness statement, we let $\mathcal{C}_* \to \pi_0$ be the Čech nerve of $\lambda_F$. Since $\lambda_F$ is an effective epimorphism, we have
\[ X_{\text{dR}} \simeq |\mathcal{C}_*|, \]
and in particular
\[ \text{QCoh}(X_{\text{dR}}) \simeq \lim_{n \in \Delta} \text{QCoh}(\mathcal{C}_n). \]
Using [GaR14b, Proposition 6.5.5] we can identify $\mathcal{C}_n$ with the completion of $X^n$ along the small diagonal. In particular, the transition maps $\mathcal{C}_{n+1} \to \mathcal{C}_n$ are flat. This shows that the $t$-structure on $\text{QCoh}(X_{\text{dR}})$ is characterized by the fact that the projection functors
\[ \text{QCoh}(X_{\text{dR}}) \to \text{QCoh}(\mathcal{C}_n) \]
are $t$-exact. □
Let $X$ be a smooth scheme over $C$. Then $D_{\text{coh}}(X_{\text{dR}})$ is canonically equivalent to the derived category of left quasi coherent $D$-modules on $X$ (cf. [GaR14a, §5.5]). This is a generalization of a result of Grothendieck [Gro68] relating the category of crystals of quasi coherent sheaves on $X$ with the category of left quasi coherent $D$-modules on $X$: as pointed out in [GaR14a, §5.5.4], the former category corresponds to the heart of $\text{QCoh}(X_{\text{dR}})$.

A.3. Dolbeault shape. Let $X$ be a derived geometric stack. The Dolbeault stack of $X$ is defined as follows: let

$$TX := \text{Spec}_X(\text{Sym}_{\mathcal{O}_X}(\mathcal{I}_X))$$

be the derived tangent bundle to $X$. Let $\hat{TX} := X_{\text{dR}} \times_{(TX)_{\text{dR}}} TX$ be the formal completion of $TX$ along the zero section. Using [Lur17, 4.2.2.9] we can convert the natural commutative group structure of $TX$ relative to $X$ (seen as an associative one) into a simplicial diagram $T^*X: \Delta^{op} \to (\text{dSt})_X$. Unwinding the definitions, we see that $T^*X$ can be identified with the $n$-fold product $TX \times_X \cdots \times_X TX$. The zero section $X \to TX$ allows to promote $T^*X$ to a simplicial diagram

$$T^*X: \Delta^{op} \to (\text{dSt})_X/\mathcal{X}.$$ 

Formal completion along the natural maps $X \to T^nX$ provides us with a new simplicial object

$$\hat{T^*X}: \Delta^{op} \to (\text{dSt})_X/\mathcal{X}.$$ 

The Dolbeault shape of $X$ is the geometric realization

$$X_{\text{Dol}} := |\hat{T^*X}| \in (\text{dSt})_X/\mathcal{X},$$

while the nilpotent Dolbeault shape of $X$ is the geometric realization

$$X_{\text{Dol}}^{\text{nil}} := |T^*X| \in (\text{dSt})_X/\mathcal{X}.$$ 

We let

$$\kappa_X: X \to X_{\text{Dol}} \quad \text{and} \quad \kappa_X^{\text{nil}}: X \to X_{\text{Dol}}^{\text{nil}}$$

be the natural maps. In addition, $\kappa_X^{\text{nil}} = i_C \circ \kappa_X$, where $i_C: X_{\text{Dol}} \to X_{\text{Dol}}^{\text{nil}}$ is the canonical map induced by $\hat{T^*X} \to T^*X$. We have:

**Lemma A.3.** Let $X$ be a derived geometric stack over $C$. The map $\kappa_X$ is flat and an effective epimorphism. The same properties hold for $\kappa_X^{\text{nil}}$.

**Proof.** The map $\kappa_X$ is an effective epimorphism by construction. The flatness follows from the fact that the transition maps in the diagram $\hat{T^*X}$ are flat. By arguing similarly, one proves the same statement for $\kappa_X^{\text{nil}}$. $\square$

Let $X$ be a smooth scheme over $C$. Then the category of (quasi) coherent sheaves on $X_{\text{Dol}}$ is canonically equivalent to the category of (quasi) coherent Higgs sheaves on $X$ (cf. [Sim96, Sim97, Sim02]).

A.4. Deligne shape. Following ideas of Deligne and Simpson,\textsuperscript{27} we consider the cosimplicial affine scheme

$$\text{Del}^*: \Delta \to \text{dAff}_{/\mathcal{A}^1},$$

given by

$$\text{Del}^n := \text{Spec}(C[X, Y]/(X^n - Y^n)), $$

where the structural map to $\mathcal{A}^1 := \text{Spec}(C[T])$ is given by $T \mapsto Y$. Moreover $G_m$ naturally acts on $\text{Del}^n$ in an equivariant way with respect to $\mathcal{A}^1$. This gives rise to a cosimplicial stack

$$\text{Del}^n_{/G_m} := [\text{Del}^*_{/G_m}: \Delta \to \text{dSt}_{/\mathcal{A}^1_{/G_m}}].$$

\textsuperscript{27}The reader might want to compare with the general construction performed in [GaR17b, §9.1.6].
Let now $X$ be a smooth scheme over $\mathbb{C}$. Then

$$\text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times [A^1/G_m])$$

is a simplicial object over $[A^1/G_m]$. Pulling back along the atlas $\mathbb{A}^1 \to [A^1/G_m]$ (that is, forgetting the $G_m$-action) we obtain the $\mathbb{A}^1$-cosimplicial object

$$\text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times \mathbb{A}^1),$$

which can explicitly be described as follows: over the open $\mathbb{A}^1 \setminus \{0\}$, it is canonically equivalent to the simplicial object

$$[n] \mapsto X^n \times (\mathbb{A}^1 \setminus \{0\}),$$

while over the point $0 \in \mathbb{A}^1$ it becomes the simplicial object

$$[n] \mapsto T^n X$$

described in the previous section. The canonical map

$$\text{Del}_{G_m}^\bullet \longrightarrow [A^1, G_m]$$

gives rise to a map

$$\delta: X \times [A^1, G_m] \longrightarrow \text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times [A^1/G_m]).$$

Given $[n] \in \Delta$, the induced family of morphisms

$$\begin{array}{ccc}
X \times \mathbb{A}^1 & \longrightarrow & \text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times \mathbb{A}^1) \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times \mathbb{A}^1)
\end{array}$$

coincides with the deformation to the normal cone of the diagonal embedding $X \hookrightarrow X^n$. We now define the simplicial object $X_{\text{Del}G_m}^\bullet$ as the fiber product

$$X_{\text{Del}G_m}^\bullet \longrightarrow \text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times [A^1/G_m])$$

$$\downarrow$$

$$(X \times [A^1/G_m])_{dR} \longrightarrow \text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times [A^1/G_m])_{dR}$$

In other words, $X_{\text{Del}G_m}^\bullet$ is the formal completion of $\text{Map}_{/A^1}(\text{Del}_{G_m}^\bullet, X \times [A^1/G_m])$ along the diagonal morphism $\delta$. Finally, we let

$$X_{\text{Del}G_m} := |X_{\text{Del}G_m}| \in \text{dSt}_{/[A^1/G_m]}.$$  

We also let $X_{\text{Del}}$ be the pullback of $X_{\text{Del}G_m}$ along the atlas $\mathbb{A}^1 \to [A^1/G_m]$.

Let $X$ be a smooth scheme over $\mathbb{C}$. Then for any (quasi)-coherent sheaf $E$ on $X_{\text{Del}}$, its fiber $E|_{\lambda}$ at $\lambda \in \mathbb{A}^1$ is a (quasi) coherent $\lambda$-connection on $X$ (see e.g. [Sim09, §7]).

**APPENDIX B. THE BNR CORRESPONDENCE FOR PERFECT COMPLEXES**

In this section, we shall provide a version of the Beauville-Narasimhan-Ramanan correspondence, in the sense of Simpson [Sim94b, Lemma 6.8], for perfect complexes.

**Definition B.1.** Let $p: X \to Y$ be a morphism of derived schemes. Let $F \in \text{Perf}(X)$ be a perfect complex. We say that $F$ is properly supported with respect to $p$ if there exists a closed subscheme $i: Z \to X$ such that:

1. the composition $Z \overset{i} \to X \overset{p} \to Y$ is proper;
2. let $j: X \setminus Z \to X$ be inclusion of open complementary of $Z$. Then $j^* F \simeq 0$. 

We let $\text{Perf}_{p-\text{prop}}(X)$ denote the full subcategory of $\text{Perf}(X)$ spanned by perfect complexes properly supported with respect to $p$.

**Lemma B.2.** Let $p : X \to Y$ be a quasi-compact and quasi-separated morphism of derived schemes of finite tor-amplitude. Let $\mathcal{F} \in \text{Perf}(X)$ be a perfect complex which is properly supported with respect to $p$. Then $p_*(\mathcal{F})$ is perfect.

**Proof.** Since $p$ has finite tor-amplitude and $X$ and $Y$ are schemes, a Čech cohomology argument shows that $p_*(\mathcal{F})$ has finite tor-amplitude. It is therefore enough to prove that $p_*(\mathcal{F})$ is almost perfect. Since $j^*\mathcal{F} \simeq 0$, we see that each $\pi_*(\mathcal{F})$ is set-theoretically supported on $\mathbb{Z}$. Therefore, the cohomological descent spectral sequence

$$R^i p_*(\pi_*(\mathcal{F})) \Rightarrow R^{i+j} p_*(\mathcal{F})$$

implies that each $R^i p_*(\mathcal{F})$ is coherent and that $R^i p_*(\mathcal{F}) = 0$ for $i \gg 0$. The conclusion follows. 

**Proposition B.3.** Let $X$ be a smooth and proper scheme. Let $T^*X := \text{Spec}_X(\text{Sym}_{O_X}(T_X))$ and let $p : T^*X \to X$ be the natural projection. Then the functor $p_* : \text{QCoh}(T^*X) \to \text{QCoh}(X)$ restricts to an equivalence

$$\text{Perf}_{p-\text{prop}}(T^*X) \simeq \text{Perf}(X_{\text{Dal}}).$$

**Proof.** The functor $p_* : \text{QCoh}(T^*X) \simeq \text{QCoh}(X)$ induces an equivalence

$$\text{QCoh}(T^*X) \simeq \text{Mod}_{\text{Sym}_{O_X}(T_X)}(\text{QCoh}(X)) \simeq \text{QCoh}(X_{\text{Dal}}).$$

Lemma B.2 implies that the functor $p_*$ restricts to a functor

$$\text{Perf}_{p-\text{prop}}(T^*X) \longrightarrow \text{Perf}(X_{\text{Dal}}).$$

On the other hand, let $\mathcal{F} \in \text{QCoh}(T^*X)$ be such that $p_*(\mathcal{F}) \in \text{Perf}(X_{\text{Dal}})$. We want to prove that it is properly supported with respect to $p$. Since $X$ is smooth, we have that $\pi_*(\mathcal{F}) \neq 0$ for only finitely many integers $i$. It is therefore enough to check that each $\pi_*(\mathcal{F})$ is properly supported with respect to $p$. This follows from the classical BNR correspondence (cf. [Sim94b, Lemma 6.8]).

By using the same arguments as above and [Sim94b, Lemma 6.10], one can prove:

**Corollary B.4.** Let $X$ be a smooth and proper scheme. Then the functor $p_* : \text{QCoh}(T^*X) \to \text{QCoh}(X)$ restricts to an equivalence

$$\text{Perf}_X(H^*X) \simeq \text{Perf}(X_{\text{Dal}}),$$

where $\text{Perf}_X(H^*X)$ is the full subcategory of $\text{Perf}(H^*X)$ of perfect complexes set-theoretically supported at $X$, seen as the zero-section of $T^*X$.

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