APERIODIC HOMEOMORPHISMS APPROXIMATE
CHAIN MIXING ENDMORPHISMS
ON THE CANTOR SET

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Abstract. Let $f$ be a chain mixing continuous onto mapping from the Cantor set onto itself. Let $g$ be an aperiodic homeomorphism on the Cantor set. We show that homeomorphisms that are topologically conjugate to $g$ approximate $f$ in the topology of uniform convergence if a trivial necessary condition on periodic points is satisfied. In particular, let $f$ be a chain mixing continuous onto mapping from the Cantor set onto itself with a fixed point and $g$, an aperiodic homeomorphism on the Cantor set. Then, homeomorphisms that are topologically conjugate to $g$ approximate $f$.

1. Introduction

Let $(X, d)$ be a compact metric space. Let $\mathcal{H}^+(X)$ be the set of all continuous mappings from $X$ onto itself. In this manuscript, the pair $(X, f)$ ($f \in \mathcal{H}^+(X)$) is called a topological dynamical system. We mainly consider the case in which $X$ is homeomorphic to the Cantor set, denoted by $C$.

For any $f, g \in \mathcal{H}^+(X)$, we define $d(f, g) := \sup_{x \in X} d(f(x), g(x))$. Then, $(\mathcal{H}^+(X), d)$ is a metric space of uniform convergence. Let $\mathcal{H}(X)$ be the set of all homeomorphisms from $X$ onto itself. Let $X$ be homeomorphic to $C$. $\text{SFT}(X)$ denotes the set of all $f \in \mathcal{H}(X)$ that are topologically conjugate to some two-sided subshift of finite type. T. Kimura [3, Theorem 1] and I [4] have shown that elements in $\mathcal{H}(C)$ are approximated by expansive homeomorphisms with the pseudo-orbit tracing property. $\text{SFT}(C)$ coincides with the set of all expansive $f \in \mathcal{H}(C)$ with the pseudo-orbit tracing property (P. Walters [6, Theorem 1]). Therefore, $\text{SFT}(C)$ is dense in $\mathcal{H}(C)$. Fix $f \in \mathcal{H}(C)$. Homeomorphisms that are topologically conjugate to $f$ will approximate some other homeomorphisms. Let $(X, f)$ be a topological dynamical system. $x \in X$ is called a periodic point of period $n$ if $f^n(x) = x$. Let $\text{Per}(X, f) := \{n \in \mathbb{Z}_+ \mid f^n(x) = x \text{ for some } x \in X\}$, where $\mathbb{Z}_+$ denotes the set of all positive integers. Let $(X, f)$ and $(Y, g)$ be topological dynamical systems. In this manuscript, a continuous mapping $\phi : Y \to X$ is said to be commuting if $\phi \circ g = f \circ \phi$ holds. We write $(Y, g) \Rightarrow (X, f)$ if there exists a sequence of homeomorphisms $(\psi_k)_{k=1,2,\ldots}$ from $Y$ onto $X$ such that $\psi_k \circ g \circ \psi_k^{-1} \to f$ as $k \to \infty$. Suppose that $(Y, g) \Rightarrow (X, f)$ and that $g^n$
has a fixed point for some positive integer \( n \). Then, \( f^n \) must also have a fixed point. Therefore, we get \( \Per(Y, g) \subseteq \Per(X, f) \). Let \( \delta > 0 \). A sequence \( \{x_i\}_{i=0,1,\ldots,l} \) of elements of \( X \) is a \( \delta \) chain from \( x_0 \) to \( x_l \) if \( d(f(x_i), x_{i+1}) < \delta \) for all \( i = 0, 1, \ldots, l-1 \). Then, \( l \) is called the length of the chain. A topological dynamical system \((X, f)\) is chain mixing if for every \( \delta > 0 \) and for every pair \( x, y \in X \), there exists a positive integer \( N \) such that for all \( n \geq N \), there exists a \( \delta \) chain from \( x \) to \( y \) of length \( n \). Let \( \Lambda, \sigma \) be a two-sided subshift such that \( \Lambda \) is homeomorphic to \( C \). Let \( X \) be homeomorphic to \( C \) and \( f \), a chain mixing element of \( \mathcal{H}^+(X) \). In a previous paper [5, Theorem 1.1], it was shown that the following conditions are equivalent:

1. \( \Per(\Lambda, \sigma) \subseteq \Per(X, f) \);
2. \( (\Lambda, \sigma) \Rightarrow (X, f) \).

Let \( (Y, g) \) be a topological dynamical system and \( n \in \mathbb{Z}_+ \). In this manuscript, we say that \( g \) is periodic of period \( n \) if \( g^n = \text{id}_Y \), where \( \text{id}_Y \) denotes the identity mapping on \( Y \). We say that \( g \) is aperiodic if \( g \) is not periodic. Suppose that \( g \in \mathcal{H}(Y) \) is periodic of period \( n \) and that \( (Y, g) \Rightarrow (X, f) \) for some \( f \in \mathcal{H}^+(X) \). Then, it is easy to check that \( f \) is also periodic of period \( n \). Note that even if \( g \) is aperiodic, all the orbits of \( g \) may be periodic. This may happen if \( g \) has periodic points of least period \( n \) for infinitely many \( n \in \mathbb{Z}_+ \). In this manuscript, we shall show the following:

**Theorem 1.1.** Let \( X \) and \( Y \) be homeomorphic to \( C \); \( f \in \mathcal{H}^+(X) \), chain mixing; and \( g \in \mathcal{H}(Y) \), aperiodic. Then, the following conditions are equivalent:

1. \( \Per(Y, g) \subseteq \Per(X, f) \);
2. \( (Y, g) \Rightarrow (X, f) \).

In the previous theorem, suppose that \( f \) has a fixed point. Then, \( \Per(X, f) = \mathbb{Z}_+ \). Therefore, the following corollary is obtained:

**Corollary 1.2.** Let \( X \) and \( Y \) be homeomorphic to \( C \); \( f \in \mathcal{H}^+(X) \), chain mixing; and \( g \in \mathcal{H}(Y) \), aperiodic. Suppose that \( f \) has a fixed point. Then, \( (Y, g) \Rightarrow (X, f) \).

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**2. Preliminaries**

Although some lemmas in this section are listed in [5], we show the proof here for conveniences. A compact metrizable totally disconnected perfect space is homeomorphic to \( C \). Therefore, any non-empty closed and open subset of \( C \) is homeomorphic to \( C \). Let \( \mathbb{Z} \) denote the set of all integers. Let \( V = \{v_1, v_2, \ldots, v_n\} \) be a finite set of \( n \) symbols with discrete topology. Let \( \Sigma(V) := V^\mathbb{Z} \) with the product topology. Then, \( \Sigma(V) \) is a compact metrizable totally disconnected perfect space; hence, it is homeomorphic to \( C \). We define a homeomorphism \( \sigma : \Sigma(V) \rightarrow \Sigma(V) \) as

\[
(\sigma(t))(i) = t(i + 1) \quad \text{for all } i \in \mathbb{Z}, \quad \text{where } t = (t(i))_{i \in \mathbb{Z}} \in \Sigma(V).
\]
The pair \((\Sigma(V), \sigma)\) is known as a two-sided full shift of \(n\) symbols. If a closed set \(\Lambda \subseteq \Sigma(V)\) is invariant under \(\sigma\), i.e., \(\sigma(\Lambda) = \Lambda\), then \((\Lambda, \sigma|_\Lambda)\) is known as a two-sided subshift. In this manuscript, \(\sigma|_\Lambda\) is abbreviated to \(\sigma\).

A finite sequence \(u_1u_2 \cdots u_l\) of elements of \(V\) is called a word of length \(l\). For a word \(u\) of length \(l\) and \(m \in \mathbb{Z}\), we define the cylinder \(C_m(u) \subseteq \Lambda\) as

\[
C_m(u) := \{ t \in \Lambda \mid t(m+j-1) = u_j \text{ for all } 1 \leq j \leq l \}.
\]

Let \((X, f)\) be a topological dynamical system such that \(X\) is homeomorphic to \(C\). Let \(\mathcal{U}\) be a finite partition of \(X\) by non-empty closed and open subsets. In this manuscript, we consider partitions that are not trivial, i.e., they consist of more than one element. We define a directed graph \(G = G(f, \mathcal{U})\) as follows:

1. \(G\) has the set of vertices \(V(f, \mathcal{U}) = \mathcal{U}\)
2. \(G\) has the set of directed edges \(E(f, \mathcal{U}) \subseteq \mathcal{U} \times \mathcal{U}\) such that \((U, U') \in E(f, \mathcal{U})\) if and only if \(f(U) \cap U' \neq \emptyset\).

Note that all elements of \(V(f, \mathcal{U})\) have at least one outdegree and at least one indegree. Let \(G = (V, E)\) be a directed graph, where \(V\) is a finite set of vertices and \(E \subseteq V \times V\) is a set of directed edges. \(\Sigma(G)\) denotes the two-sided subshift defined as

\[
\Sigma(G) := \{ t \in V^\mathbb{Z} \mid (t(i), t(i+1)) \in E \text{ for all } i \in \mathbb{Z} \}.
\]

A two-sided subshift is said to be of finite type if it is topologically conjugate to \((\Sigma(G), \sigma)\) for some directed graph \(G\). Throughout this manuscript, unless otherwise stated, we assume that all the vertices appear in some element of \(\Sigma(G)\), i.e., all the vertices of \(G\) have at least one outdegree and at least one indegree. For the sake of conciseness, we write \((\Sigma(f, \mathcal{U}), \sigma)\) instead of \((\Sigma(G(f, \mathcal{U})), \sigma)\). The next lemma follows:

**Lemma 2.1.** Let \((X, f)\) be a topological dynamical system such that \(X\) is homeomorphic to \(C\). Let \(\mathcal{U}\) be a partition of \(X\) by non-empty closed and open subsets of \(X\). Then, \(\text{Per}(X, f) \subseteq \text{Per}(\Sigma(f, \mathcal{U}), \sigma)\).

**Proof.** Let \(x \in X\) be a periodic point of period \(n\) under \(f\). Then, there exists a sequence \(\{U_i\}_{i=0,1,\ldots,n}\) of elements of \(\mathcal{U}\) such that \(U_0 = U_n\) and \(f(U_i) \cap U_{i+1} \neq \emptyset\) for all \(i = 0, 1, \ldots, n - 1\). Thus, \((\Sigma(f, \mathcal{U}), \sigma)\) has a periodic point of period \(n\). \(\square\)

**Lemma 2.2** (Lemma 1.3 of R. Bowen [1]). Let \(G = (V, E)\) be a directed graph. Suppose that every vertex of \(V\) has at least one outdegree and at least one indegree. Then, \(\Sigma(G)\) is topologically mixing if and only if there exists an \(N \in \mathbb{Z}_+\) such that for any pair of vertices \(u\) and \(v\) of \(V\), there exists a path from \(u\) to \(v\) of length \(n \geq N\).

**Proof.** See Lemma 1.3 of R. Bowen [1]. \(\square\)

Let \(K \subseteq X\). The diameter of \(K\) is defined as \(\text{diam}(K) := \sup\{d(x, y) \mid x, y \in K\}\). We define \(\text{mesh}(\mathcal{U}) := \max\{\text{diam}(U) \mid U \in \mathcal{U}\}\).

**Lemma 2.3.** Let \((X, d)\) be a compact metric space and \(f : X \to X\), a continuous mapping. Then, for any \(\epsilon > 0\), there exists \(\delta = \delta(f, \epsilon) > 0\) such that

\[
\delta < \frac{\epsilon}{2}.
\]
if \( d(x, y) \leq \delta \), then \( d(f(x), f(y)) < \frac{\epsilon}{2} \) for all \( x, y \in X \).

\textbf{Proof.} This lemma directly follows from the uniform continuity of \( f \). \( \square \)

For two directed graphs \( G = (V, E) \) and \( G' = (V', E') \), \( G \) is said to be a subgraph of \( G' \) if \( V \subseteq V' \) and \( E \subseteq E' \).

\textbf{Lemma 2.4.} Let \( (X, d) \) be a compact metric space; \( f : X \to X \), a continuous mapping; and \( \epsilon > 0 \). Let \( \delta = \delta(f, \epsilon) \) be as in lemma \( 2.3 \) and \( \mathcal{U} \), a finite covering of \( X \) such that \( \text{mesh}(\mathcal{U}) < \delta \). Let \( g : X \to X \) be a mapping such that \( G(g, \mathcal{U}) \) is a subgraph of \( G(f, \mathcal{U}) \). Then, \( d(f, g) < \epsilon \).

\textbf{Proof.} Let \( x \in X \). Then, \( x \in U \) and \( g(x) \in U' \) for some \( U, U' \in \mathcal{U} \). Because \( G(g, \mathcal{U}) \) is a subgraph of \( G(f, \mathcal{U}) \), there exists a \( y \in U \) such that \( f(y) \in U' \). Therefore, from lemma \( 2.3 \) it follows that

\[
\frac{d(f(x), g(x))}{d(f(x), f(y)) + d(f(y), g(x)) < \frac{\epsilon}{2}} + \text{diam}(U') < \epsilon.
\]

\textbf{□}

From this lemma, the next lemma follows directly.

\textbf{Lemma 2.5.} Let \( (X, d) \) be a compact metric space; \( f : X \to X \), a continuous mapping; and \( \{\mathcal{U}_k\}_{k \in \mathbb{Z}_+} \), a sequence of coverings of \( X \) such that \( \text{mesh}(\mathcal{U}_k) \to 0 \) as \( k \to \infty \). Let \( \{g_k\}_{k=1,2,...} \) be a sequence of mappings from \( X \) to \( X \) such that \( G(g_k, \mathcal{U}_k) \) is a subgraph of \( G(f, \mathcal{U}_k) \) for all \( k \). Then, \( g_k \to f \) as \( k \to \infty \).

\textbf{Lemma 2.6.} Let \( (X_1, f_1) \) and \( (X_2, f_2) \) be topological dynamical systems such that both \( X_1 \) and \( X_2 \) are homeomorphic to \( C \). Let \( \{\mathcal{U}_k\}_{k \in \mathbb{Z}_+} \) be a sequence of finite partitions by non-empty closed and open subsets of \( X_1 \) such that \( \text{mesh}(\mathcal{U}_k) \to 0 \) as \( k \to \infty \). Let \( \{\pi_k\}_{k \in \mathbb{Z}_+} \) be a sequence of continuous commuting mappings from \( X_2 \) to \( X_1 \). Suppose that for all \( k \in \mathbb{Z}_+ \), \( \pi_k(X_2) \cap U \neq \emptyset \) for all \( U \in \mathcal{U}_k \). Then, \( (X_2, f_2) \Rightarrow (X_1, f_1) \).

\textbf{Proof.} Let \( k \in \mathbb{Z}_+ \). Let \( U \in \mathcal{U}_k \). Because \( \pi_k(X_2) \cap U \neq \emptyset \), \( \pi_k \) is a non-empty closed and open subset of \( X_2 \). Both \( \pi_k \) and \( U \) are homeomorphic to \( C \). Therefore, there exists a homeomorphism \( \psi_k : X_2 \to X_1 \) such that \( \psi_k(\pi_k^{-1}(U)) = U \) for all \( U \in \mathcal{U}_k \). Because \( \pi_k \) is commuting, \( \pi_k(f_2(\pi_k^{-1}(U))) \cap U' \neq \emptyset \) only if \( f_1(U) \cap U' \neq \emptyset \). Let \( g_k = \psi_k \circ f_2 \circ \psi_k^{-1} \). Then, from the construction of \( \psi_k \), \( G(g_k, \mathcal{U}_k) \) is a subgraph of \( G(f_1, \mathcal{U}_k) \). Because \( k \in \mathbb{Z}_+ \) is arbitrary, from lemma \( 2.5 \) we get the result. \( \square \)

\textbf{Lemma 2.7.} Let \( (X_1, f_1) \) and \( (X_2, f_2) \) be topological dynamical systems. Let \( (Y_k, g_k) \) be a sequence of topological dynamical systems. Suppose that there exists a sequence of homeomorphisms \( \psi_k : Y_k \to X_1 \) such that \( \psi_k \circ g_k \circ \psi_k^{-1} \rightarrow f_1 \) as \( k \to \infty \) and that \( (X_2, f_2) \Rightarrow (Y_k, g_k) \) for all \( k = 1,2,... \). Then, \( (X_2, f_2) \Rightarrow (X_1, f_1) \).

\textbf{Proof.} Let \( \epsilon > 0 \). There exists an \( N \in \mathbb{Z}_+ \) such that \( d(\psi_k \circ g_k \circ \psi_k^{-1}, f_1) < \epsilon/2 \) for all \( k > N \). Fix \( k > N \). Let \( \delta > 0 \) be such that if \( d(y, y') < \delta \), then \( d(\psi_k(y), \psi_k(y')) < \epsilon/2 \). Because \( (X_2, f_2) \Rightarrow (Y_k, g_k) \), there exists a homeomorphism \( \psi' : X_2 \to Y_k \) such that \( d(\psi' \circ f_2 \circ \psi'^{-1}, g_k) < \delta \). Then, we find that \( d(\psi_k \circ \psi' \circ f_2 \circ \psi_k^{-1}, f) < d(\psi_k \circ (\psi' \circ f_2 \circ \psi'^{-1}) \circ \psi_k^{-1}, \psi_k \circ g_k \circ \psi_k^{-1}) + d(\psi_k \circ g_k \circ \psi_k^{-1}, f_1) < \epsilon \). \( \square \)
Lemma 2.8. Let $G = (V,E)$ be a directed graph. Suppose that every vertex of $G$ has at least one outdegree and at least one indegree. Suppose that $\Sigma(G)$ is topologically mixing and that $\Sigma(G)$ is not a single point. Then, $\Sigma(G)$ is homeomorphic to $C$.

Proof. Suppose that $\Sigma(G)$ is topologically mixing. Then, by lemma 2.2 there exists an $N \in \mathbb{Z}_+$ such that for any pair $u$ and $v$ of vertices of $G$, there exists a path from $u$ to $v$ of length $n$ for all $n \geq N$. Then, it is easy to check that every point $t \in \Sigma(G)$ is not isolated. Hence, $\Sigma(G)$ is homeomorphic to $C$. □

Lemma 2.9 (Krieger’s Marker Lemma, (2.2) of M. Boyle [2]). Let $(\Lambda, \sigma)$ be a two-sided subshift. Given $k > N > 1$, there exists a closed and open set $F$ such that

1. the sets $\sigma^l(F), 0 \leq l < N$, are disjoint, and
2. if $t \in \Lambda$ and $t_{-k} \ldots t_k$ is not a $j$-periodic word for any $j < N$, then

$$t \in \bigcup_{-N < l < N} \sigma^l(F).$$

Proof. See M. Boyle [2] (2.2)]. □

The next lemma is essentially a part of the proof of the extension lemma given by M. Boyle [2] (2.4)]. Although the outlook of the next lemma seems slightly strengthened from Lemma 3.4 in [5], the proof is almost same. We show the proof only for completeness.

Lemma 2.10. Let $(\Sigma, \sigma)$ be a mixing two-sided subshift of finite type. Let $W$ be a finite set of words that appear in some elements of $\Sigma$. Then, there exists an $M \in \mathbb{Z}_+$ that satisfies the following condition:

- if $(\Lambda, \sigma)$ is a two-sided subshift such that $\text{Per}(\Lambda, \sigma) \subseteq \text{Per}(\Sigma, \sigma)$ and $\Lambda$ has either a non-periodic orbit or a periodic orbit of least period greater than $M$, then there exists a continuous shift-commuting mapping $\pi: \Lambda \to \Sigma$ such that there exists a $t \in \pi(\Lambda)$ in which all words in $W$ appear as segments of $t$.

Proof. $\Sigma$ is isomorphic to $\Sigma(G)$ for some directed graph $G = (V,E)$. Therefore, without loss of generality, we assume that $\Sigma = \Sigma(G)$. Because $(\Sigma(G), \sigma)$ is a mixing subshift of finite type, there exists an $n > 0$ such that for every pair of elements $v, v' \in V$ and every $m \geq n$, there exists a word of the form $v \ldots v'$ of length $m$. In addition, there exists an element $\bar{t} \in \Sigma(G)$ such that $\bar{t}$ contains all words of $W$ as segments. Let $w_0$ be a segment of $\bar{x}$ that contains all words of $W$. Let $n_0$ be the length of the word $w_0$. Let $N = 2n + n_0$. If $v, v' \in V$ and $m \geq N$, then there exists a word of the form $v \ldots w_0 \ldots v'$ of length $m$. Let $k > 2N$. Let $M > N$. Note that $N$ depends only on $\Sigma(G)$ and $W$. Therefore, $M$ also depends only on $\Sigma(G)$ and $W$. Let $\Lambda$ be a two-sided subshift such that $\text{Per}(\Lambda, \sigma) \subseteq \text{Per}(\Sigma, \sigma)$ and $\Lambda$ has either a non-periodic orbit or a periodic orbit of least period greater than $M$. Using Krieger’s marker lemma, there exists a closed and open subset $F \subseteq \Lambda$ such that the following conditions hold:

1. the sets $\sigma^l(F), 0 \leq l < N$, are disjoint;
(2) if \( t \in \Lambda \) and \( t \notin \bigcup_{-N<\ell<N} \sigma^\ell(F) \), then \( t(-k) \ldots t(k) \) is a \( j \)-periodic word for some \( j < N \);

(3) the number \( k \) is large enough to ensure that if \( j \) is less than \( N \) and a \( j \)-periodic word of length \( 2k + 1 \) occurs in some element of \( \Lambda \), then that word defines a \( j \)-periodic orbit that actually occurs in \( \Lambda \).

The existence of \( k \) follows from the compactness of \( \Lambda \). Let \( t \in \Lambda \). If \( \sigma^i(t) \in F \), then we mark \( t \) at position \( i \). There exists a large number \( L > 0 \) such that whether \( \sigma^i(t) \in F \) is determined only by the \( 2L + 1 \) block \( t(i-L) \ldots t(i+L) \). If \( t \) is marked at position \( i \), then \( t \) is unmarked for position \( l \) with \( i < l < i + N \). Suppose that \( t(i) \ldots t(i') \) is a segment of \( t \) such that \( t \) is marked at \( i \) and \( i' \) and \( t \) is unmarked at \( l \) for all \( i < l < i' \). Then, \( i' - i \geq N \). If \( t \in \bigcup_{-N<\ell<N} \sigma^\ell(F) \), then \( t \) is marked at some \( i \) where \( -N < i < N \). Suppose that \( t(-N+1) \ldots t(N-1) \) is an unmarked segment. Then, \( t \notin \bigcup_{-N<\ell<N} \sigma^\ell(F) \), and according to condition (2), \( t(-k) \ldots t(k) \) is a \( j \)-periodic word for some \( j < N \). Suppose that \( t(i) \ldots t(i') \) is an unmarked segment of length at least \( 2N - 1 \), i.e., \( i' - i \geq 2N - 2 \). Then, for each \( l \) with \( i + N - 1 \leq l \leq i' - N + 1 \), \( t(l-k) \ldots t(l+k) \) is a \( j \)-periodic word for some \( j < N \). Therefore, it is easy to check that \( t(i+N-1-k) \ldots t(i'-N+1+k) \) is a \( j \)-periodic word for some \( j < N \). In this proof, we call a maximal unmarked segment an interval. Let \( t \in \Lambda \). Let \( \ldots t(i) \) be a left infinite interval. Then, it is \( j \)-periodic for some \( j < N \). Similarly, a right infinite interval \( t(i) \ldots \) is \( j \)-periodic for some \( j < N \). If \( t \) itself is an interval, then it is a periodic point with period \( j < N \). If an interval is finite, then it has a length of at least \( N - 1 \). We call intervals of length less than \( 2N - 1 \) as short intervals. We call intervals of length greater than or equal to \( 2N - 1 \) as long intervals. If \( t \) has a long interval \( t(i) \ldots t(i') \), then \( t(i+N-1-k) \ldots t(i'-N+1+k) \) is \( j \)-periodic for some \( j < N \). We have to construct a shift-commuting mapping \( \phi : \Lambda \to \Sigma \). Let \( \Sigma \) be the set of symbols of \( \Lambda \). Let \( \Phi : \Sigma \to \Sigma \) be an arbitrary mapping. Let \( \Phi \) be a path in \( G \) of length \( l \) such that the word of the form \( v \Psi(v,v',l)v' \) is a path in \( G \).

A Coding for short interval: Let \( t(i) \ldots t(i') \) be a short interval. Then, \( t \) is marked at \( i - 1 \) and \( i' + 1 \). We have already defined a code for positions \( i - 1 \) and \( i' + 1 \) as \( \Phi(t(i-1)) \) and \( \Phi(t(i'+1)) \), respectively. The coding for \( \{i,i+1,i+2,\ldots,i'\} \) is defined by the path \( \Psi(\Phi(t(i-1)),\Phi(t(i'+1)),i'-i+1) \).

B Coding for periodic segment: For an infinite or long interval, there exists a corresponding periodic point of \( \Lambda \). The periodic points of \( \Lambda \) are already mapped to periodic points of \( \Sigma \). Therefore, an infinite or long periodic segment can be mapped to a naturally corresponding periodic segment.

C Coding for transition part: To consider a transition segment, let \( t(i) \ldots t(i') \) be a long interval. Then, \( t(i-1) \) has already been mapped to \( \Phi(t(i-1)) \), and \( t(i+N-1) \) is mapped according to periodic points. Assume that \( t(i+N-1) \) is mapped to \( v_0 \). The segment \( t(i-1) \ldots t(i+N-1) \) has length \( N+1 \). We map the segment \( t(i) \ldots t(i+N-2) \) to \( \Psi(\Phi(t(i-1)),v_0,N-1) \). In the same manner, the transition coding of the right-hand side of a
long interval is defined. Similarly, the transition coding of the left or right

infinite interval is defined.

It is easy to check that there exists a large number \(L' > 0\) such that the
coding of \((\phi(t))(i)\) is determined only by the block \(t(i - L') \ldots t(i + L')\).
Therefore, \(\phi : \Lambda \to \Sigma\) is continuous. Because \(\Lambda\) has either a \(t \in \Lambda\), which is

not a periodic point, or a \(t' \in \Lambda\), which is a periodic point of least period

greater than \(M\), there appears a short interval or transition segment in some
elements of \(\Lambda\). In the above coding, we can take \(\Psi\) such that both short

intervals and transition segments are mapped to words that involve \(w_0\). \(\square\)

3. PROOF OF THE MAIN RESULT

Lemma 3.1. Let \(X\) be homeomorphic to \(C\) and \(f\), a chain mixing element of
\(\mathcal{H}^+(X)\). Let \(\{\mathcal{W}_k\}_{k\in \mathbb{Z}_+}\) be a sequence of non-trivial finite partitions by non-

empty closed and open subsets of \(X\) such that \(\text{mesh}(\mathcal{W}_k) \to 0\) as \(k \to \infty\).

Then, there exists a sequence \(\{\psi_k\}_{k\in \mathbb{Z}_+}\) of homeomorphisms from \(\Sigma(f, \mathcal{W}_k)\)
to \(X\) such that \(\psi_k \circ \sigma \circ \psi_k^{-1} \to f\) as \(k \to \infty\). Furthermore, if \(f\) is chain

mixing, then all \((\Sigma(f, \mathcal{W}_k), \sigma)\) \((k \in \mathbb{Z}_+\) are mixing.

\begin{proof}
Consider a sequence \(\{\mathcal{W}_k\}_{k\in \mathbb{Z}_+}\) of non-trivial partitions of \(X\) by non-

empty closed and open subsets such that \(\text{mesh}(\mathcal{W}_k) \to 0\) as \(k \to \infty\). Assume

that \(k \in \mathbb{Z}_+\). Let \(G_k = G(f, \mathcal{W}_k)\). Let \(\delta > 0\) be such that if \(x, x' \in X\) satisfy
d\((x, x') < \delta\), then both \(x\) and \(x'\) are contained in the same element of \(\mathcal{W}_k\).

Let \(\{x_0, x_1\}\) be a \(\delta\) chain. Let \(U, U' \in \mathcal{W}_k\) be such that \(x_0 \in U\) and \(x_1 \in U'\).

Then, \(f(U) \cap U'' \neq \emptyset\). Therefore, \((U, U'')\) is an edge of \(G_k\). Let \(U, V \in \mathcal{W}_k\).

Let \(x \in U\) and \(y \in V\). Because \(f\) is chain mixing, there exists an \(N > 0\) such that for every \(n \geq N\), there exists a \(\delta\) chain from \(x\) to \(y\) of length \(n\).

Therefore, for every \(n \geq N\), there exists a path in \(G_k\) from \(U\) to \(V\) of length \(n\).

From lemma 2.2 \((\Sigma(G_k), \sigma)\) is topologically mixing. By lemma 2.3 \(\Sigma(G_k)\) is homeomorphic to \(C\). Therefore, there exists a homeomorphism

\(\psi_k : \Sigma(G_k) \to X\) such that for any vertex \(u\) of \(G_k\), \(\psi_k(\Sigma(u)) = u\). Let

\(g_k = \psi_k \circ \sigma \circ \psi_k^{-1}\). Then, by construction, we obtain \(G(g_k, \mathcal{W}_k) = G(f, \mathcal{W}_k)\).

Because \(\text{mesh}(\mathcal{W}_k) \to 0\) as \(k \to \infty\), we conclude that \(g_k \to f\) as \(k \to \infty\) by lemma 2.5.

\end{proof}

Proof of Theorem 1.1

\begin{proof}
Let \(X\) and \(Y\) be homeomorphic to \(C\). First, suppose that \((Y, g) \succeq (X, f)\). Then, it is easy to see that \(\text{Per}(Y, g) \subseteq \text{Per}(X, f)\). Conversely, suppose that \(f \in \mathcal{H}^+(X)\) is chain mixing; \(g \in \mathcal{H}(Y)\), aperiodic; and that \(\text{Per}(Y, g) \subseteq \text{Per}(X, f)\). Let \(\{\mathcal{W}_i\}_{i \in \mathbb{Z}_+}\) be a sequence of non-trivial finite partitions by non-empty closed and open subsets of \(X\) such that \(\text{mesh}(\mathcal{W}_i) \to 0\) as \(i \to \infty\). By lemma 3.1 there exists a sequence of homeomorphisms

\(\psi_i : \Sigma(f, \mathcal{W}_i) \to X\) such that \(\psi_i \circ \sigma \circ \psi_i^{-1} \to f\) as \(i \to \infty\) and that all

\((\Sigma(f, \mathcal{W}_i), \sigma)\) \((i \in \mathbb{Z}_+)\) are mixing. Fix \(i \in \mathbb{Z}_+\). Let \(\Sigma = \Sigma(f, \mathcal{W}_i)\). Let \(\{\mathcal{W}_k\}_{k \in \mathbb{Z}_+}\) be a sequence of finite partitions of \(\Sigma\) by non-empty closed and open subsets. Let \(\mathcal{W}_k = \{U_{k,j} : 1 \leq j \leq n_k\} \)for \(k \in \mathbb{Z}_+\). Then, there exists a sequence \(w_{k,j}\) \((k \in \mathbb{Z}_+, 1 \leq j \leq n_k)\) of words and a sequence \(m(k, j)\)

\((1 \leq j \leq n_k)\) of integers such that the following condition is satisfied:

\[ C_{m(k, j)}(w_{k,j}) \subseteq U_{k,j} \quad (k \in \mathbb{Z}_+, 1 \leq j \leq n_k). \]
Fix \( k \in \mathbb{Z}_+ \). Let \( W = \{u_{k,j} \mid 1 \leq j \leq n_k\} \). We shall show the following:

1. there exists a continuous commuting mapping \( \tilde{\phi}_k : Y \to \Sigma \) such that \( \tilde{\phi}_k(Y) \) contains an element \( t \in \Sigma \) that contains all words of \( W \).

Then, \( \tilde{\phi}_k(Y) \cap U \neq \emptyset \) for all \( U \in \mathcal{U}_k \). Because \( k \in \mathbb{Z}_+ \) is arbitrary, we conclude that \( (Y,g) \Rightarrow \Sigma \) by lemma 2.6. Then, by lemma 3.1 and lemma 2.7, we can conclude that \( (Y,g) \Rightarrow (X,f) \).

Let \( M \) be a positive integer that satisfies the condition in lemma 2.10. Let \( \mathcal{V} \) be a partition of \( Y \) by non-empty closed and open subsets. Then, for each \( y \in Y \), there exists a unique \( t_y \in \Sigma(g, \mathcal{V}) \) such that \( g^l(y) \in t_y(l) \in \mathcal{V} \) for all \( l \in \mathbb{Z} \). Therefore, there exists a commuting mapping \( \phi_\mathcal{V} : Y \to \Sigma(g, \mathcal{V}) \) such that \( \phi_\mathcal{V}(y) = t_y \) for all \( y \in Y \). Because all elements of \( \mathcal{V} \) are open, it is easy to see that \( \phi_\mathcal{V} \) is continuous. Let \( \Lambda = \phi_\mathcal{V}(Y) \). Then, \( \Lambda \) is a two-sided subshift. Because \( \Sigma \) is mixing, there exists an \( m \in \mathbb{Z}_+ \) such that for all integer \( n \geq m \), there exists a periodic point \( t_n \in \Sigma \) of period \( n \). If \( \mathcal{V} \) is sufficiently fine, then the period \( n \in \text{Per}(\Sigma(g, \mathcal{V}), \sigma) \), where \( n < m \), has a real periodic point of \( (Y,g) \) of period \( n \). Therefore, because \( \text{Per}(Y,g) \subseteq \text{Per}(X,f) \), we get \( \text{Per}(\Sigma(g, \mathcal{V}), \sigma) \subseteq \text{Per}(\Sigma, \sigma) \) for all sufficiently fine \( \mathcal{V} \). Let \( M > \max\{m, M\} \) be an arbitrary positive integer. Because \( g \) is aperiodic, if \( \mathcal{V} \) is sufficiently fine, then \( \Lambda \) is not a set of periodic points of period less than \( M \). Therefore, by lemma 2.10, there exists a continuous commuting mapping \( \pi_k : \Lambda \to \Sigma \) such that \( \pi_k(\Lambda) \) contains an element that contains all words of \( W \). Finally, let \( \tilde{\phi}_k = \pi_k \circ \phi_\mathcal{V} \); this concludes the proof.

\[ \square \]

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