Pointwise Multipliers for Besov Spaces of Dominating Mixed Smoothness - II

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Abstract

We continue our investigations on pointwise multipliers for Besov spaces of dominating mixed smoothness. This time we study the algebra property of the classes $S_{r}^{p,q}B(\mathbb{R}^d)$ with respect to pointwise multiplication. In addition if $p \leq q$, we are able to describe the space of all pointwise multipliers for $S_{r}^{p,q}B(\mathbb{R}^d)$.

Key words: Pointwise multipliers; algebras with respect to pointwise multiplication, Besov spaces of dominating mixed smoothness; characterization by differences; localization property.

1 Introduction

The regularity concept related to Besov spaces of dominating mixed smoothness are standard in Approximation Theory [34], Numerical Analysis [5], [27] and Information-Based Complexity [20], [21], [22]. However, there is also some interest in Learning Theory in those classes, at least in $S_{r,2}^{p,q}B(\mathbb{R}^d)$, $r > 0$, see [31], [10].

Assertions on pointwise multipliers belong to the key problems in the modern theory of function spaces. In our previous paper [14] we investigated the set of all pointwise multipliers $M(S_{r,p}^{p,p}B(\mathbb{R}^d))$ for the classes $S_{r,p}^{p,p}B(\mathbb{R}^d)$. It turned out that under the natural restrictions $1 \leq p, q \leq \infty$ and $r > 1/p$ this set is given by $S_{r,p}^{r,p}B(\mathbb{R}^d)$unif. This assertion, formally, is completely parallel to the isotropic case where we have $M(B_{r,p}^{r,p}(\mathbb{R}^d)) = B_{r,p}^{r,p}(\mathbb{R}^d)$unif ($1 \leq p, q \leq \infty$, $r > d/p$). However, in reality the proof of the result in the dominating mixed case is much more involved than in the isotropic case. In the present paper our aim consists in an extension of the above characterization to the situation $p \leq q \leq \infty$. In [29], [15] we have shown for the isotropic case the characterization

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\[ M(B_{p,q}^r(\mathbb{R}^d)) = B_{p,q}^r(\mathbb{R}^d) \text{unif} \quad (1 \leq p \leq q \leq \infty, \ r > d/p). \] It turns out that this extension has a counterpart in the dominating mixed case as well; we shall prove below

\[ M(S_{p,q}^r B(\mathbb{R}^d)) = S_{p,q}^r B(\mathbb{R}^d) \quad \text{unif} \quad \text{if} \quad 1 \leq p \leq q \leq \infty \quad \text{and} \quad r > 1/p. \quad (1.1) \]

The extension from the isotropic case to the dominating mixed case is by no means straightforward. To our own surprise the dominating mixed case is much more sophisticated. The standard method in the isotropic situation, paramultiplication, seems to be not appropriate. We shall deal with the characterization by differences of the underlying spaces, sometimes mixed with the Fourier analytic description.

Let us mention that the restrictions in (1.1) are natural. In cases either \( q < p \) or \( r < 1/p \) the isotropic counterpart of the identity in (1.1) is not longer true. We refer to [29] and [15].

The paper is organized as follows. In Section 2 we collect what we need about the classes \( S_{p,q}^r B(\mathbb{R}^d) \) including some tools from Fourier analysis and few basic inequalities for differences. The next Section 3 is devoted to the multiplier problem. First we shall describe there some basics about pointwise multipliers. After that we list our main results. Finally, in Section 4 we collect all proofs.

### Notation

As usual \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{Z} \) denotes the integers, \( \mathbb{R} \) the real numbers, and \( \mathbb{C} \) the complex numbers. The letter \( d \in \mathbb{N}, \ d > 1, \) is always reserved for the underlying dimension in \( \mathbb{R}^d, \mathbb{Z}^d \) etc. By \( [d] \) we mean the set \( [d] := \{1, \ldots, d\} \). If \( k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d \), then we put

\[ |k|_1 := k_1 + \ldots + k_d \quad \text{and} \quad |k|_\infty := \max_{j=1,\ldots,d} k_j. \]

Further, by \( \langle x, y \rangle \) or \( x \cdot y \) we mean the usual Euclidean inner product in \( \mathbb{R}^d \). Let

\[ x \diamond y := (x_1 y_1, \ldots, x_d y_d) \in \mathbb{R}^d. \]

If \( X \) and \( Y \) are two normed spaces, the norm of an element \( x \) in \( X \) will be denoted by \( \|x\|_X \). The symbol \( X \hookrightarrow Y \) indicates that the identity operator is continuous. For two sequences \( a_n \) and \( b_n \) we will write \( a_n \lesssim b_n \) if there exists a constant \( c > 0 \) such that \( a_n \leq c b_n \) for all \( n \). We will write \( a_n \asymp b_n \) if \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \).

Let \( S(\mathbb{R}^d) \) be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^d \). The topological dual, the class of tempered distributions, is denoted by \( S'(\mathbb{R}^d) \) (equipped with the weak topology). The Fourier transform on \( S(\mathbb{R}^d) \) is given by

\[ \mathcal{F} \varphi(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} \varphi(x) \, dx, \quad \xi \in \mathbb{R}^d. \]

The inverse transformation is denoted by \( \mathcal{F}^{-1} \). We use both notations also for the transformations defined on \( S'(\mathbb{R}^d) \).
2 Besov spaces of dominating mixed smoothness

The history of Besov spaces has started in 1951 with a paper by Nikol’skij [16]. Nikol’skij had investigated the spaces $B_{s,\infty}^p(\mathbb{R}^d)$ there. Later, his Ph.D-studies Besov [3, 4] introduced the classes $B_{s,p,q}^r(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, $s > 0$. The dominating mixed counterparts $S_{s,p,q}^r(\mathbb{R}^d)$, $1 \leq p, q \leq \infty$, $r > 0$, have been introduced by Nikol’skij [17] ($q = \infty$), Amanov [1] and Dzabrailov [6, 7]. The main new feature of these classes consists in the cross-norm property, see Remark 2.2 below. Besov spaces of dominating mixed smoothness represent a quite different way to extend Besov spaces from $\mathbb{R}$ to $\mathbb{R}^d$, $d > 1$.

2.1 The definition and some basic properties

We introduce the spaces by using the Fourier analytic approach. Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\varphi_0 \equiv 1$ on $[-1, 1]$ and $\text{supp} \varphi_0 \subset [-\frac{3}{2}, \frac{3}{2}]$. For $j \in \mathbb{N}$ we define

$$
\varphi_j(\xi) = \varphi_0(2^{-j} \xi) - \varphi_0(2^{-j+1} \xi), \quad \xi \in \mathbb{R},
$$

and

$$
\varphi_k(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{N}_0^d.
$$

(2.1)

This implies

$$
\sum_{k \in \mathbb{N}_0^d} \varphi_k(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d,
$$

and

$$
\text{supp} \varphi_k \subset \left\{x \in \mathbb{R}^d : 2^{k_{\ell} - 1} \leq |x_\ell| \leq 3 \cdot 2^{k_{\ell} - 1}, \quad \ell = 1, \ldots, d\right\}, \quad k \in \mathbb{N}_0^d.
$$

With other words, $(\varphi_k)_{k \in \mathbb{N}_0^d}$ is a smooth dyadic decomposition of unity of tensor product type.

Definition 2.1. Let $(\varphi_k)_{k \in \mathbb{N}_0^d}$ be the above system. Let $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$. Then $S_{s,p,q}^r(\mathbb{R}^d)$ is the collection of all tempered distributions $f \in S'(\mathbb{R}^d)$ such that

$$
\| f \|_{S_{s,p,q}^r(\mathbb{R}^d)} := \left( \sum_{k \in \mathbb{N}_0^d} 2^{|k|/q} \| F^{-1}[\varphi_k Ff] \|_{L_p(\mathbb{R}^d)}^q \right)^{1/q} < \infty
$$

with the usual modifications if $q = \infty$.

Of course, $S_{s,p,q}^r(\mathbb{R}^d)$ are Banach spaces and they are independent from the chosen generator $\varphi_0$ of the smooth dyadic decomposition of unity $(\varphi_k)_{k \in \mathbb{N}_0^d}$ in the sense of equivalent norms. For those basic facts we refer to the monographs [2] and [26].

Remark 2.2. (i) If $d = 1$ we get $S_{s,p,q}^r(\mathbb{R}) = B_{s,p,q}^r(\mathbb{R})$.

(ii) One of the most remarkable properties of Besov spaces of dominating mixed smoothness consists in the following. If $f_i \in B_{s,p,q}^r(\mathbb{R})$, $i = 1, \ldots, d$, then its tensor product

$$
f(x) := (f_1 \otimes f_2 \otimes \ldots \otimes f_d)(x) = \prod_{i=1}^d f_i(x_i), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,
$$

is in $S_{s,p,q}^r(\mathbb{R}^d)$. 

3
2.2 Besov spaces of dominating mixed smoothness and differences

With other words, Besov spaces of dominating mixed smoothness have a cross-norm. Let $f \in \mathbb{C}$, $m \in \mathbb{N}$, $h \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we put

$$\Delta_h^m f(x) := \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{m}{\ell} f(x + \ell h)$$

and

$$\omega_m(f, t)_p := \sup_{|h|<t} \| \Delta_h^m f \|_{L_p(\mathbb{R}^d)}, \quad t > 0.$$ 

Let $1 \leq p, q \leq \infty$, $r > 0$ and $m \in \mathbb{N}$ such that $m-1 \leq r < m$. Then the (isotropic) Besov space $B_{p,q}^r(\mathbb{R}^d)$ is a collection of all $f \in L_p(\mathbb{R}^d)$ such that

$$\|f|B_{p,q}^r(\mathbb{R}^d)\| := \|f|L_p(\mathbb{R}^d)\| + \left( \sum_{j=0}^{\infty} (2^{jr} \omega_m(f, 2^{-j} p)^q) \right)^{1/q} < \infty.$$ 

We refer to the monographs [18] and [37].

Now we turn to Besov spaces of dominating mixed smoothness. Let $j \in [d] = \{1, 2, \ldots, d\}$, $m \in \mathbb{N}$, $h \in \mathbb{R}$ and $x \in \mathbb{R}^d$. We put

$$\Delta_{h,j}^m f(x) := \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{m}{\ell} f(x_1, \ldots, x_{j-1}, x_j + \ell h, x_{j+1}, \ldots, x_d).$$

This is the $m$-th order difference of $f$ in direction $j$. For $e \subset [d]$, $h \in \mathbb{R}^d$ and $m \in \mathbb{N}_0^d$ the mixed $(m,e)$-th difference operator $\Delta_h^{m,e}$ is defined to be

$$\Delta_h^{m,e} := \prod_{i \in e} \Delta_h^{m,e}_i \quad \text{and} \quad \Delta_h^{m,\emptyset} := \text{Id},$$

where Id $f = f$. An associated modulus of smoothness is given by

$$\omega_{m,e}^d(f, t)_p := \sup_{|h|<t, i \in e} \| \Delta_{h,i}^m f \|_{L_p(\mathbb{R}^d)}, \quad t \in [0, 1]^d,$$

where $f \in L_p(\mathbb{R}^d)$ (in particular, $\omega_{m,\emptyset}^d(f, t)_p = \|f|L_p(\mathbb{R}^d)\|$). Many times, e.g., in the Proposition below, we do not need to choose $m$ as a vector. For this reason, if $m \in \mathbb{N}$ we put $\bar{m} := (m, \ldots, m) \in \mathbb{N}_0^d$ and therefore

$$\Delta_h^{\bar{m},e} := \prod_{i \in e} \Delta_h^{m,\bar{e}}.$$ 

For a set $e \subset [d]$ we denote $e_0 := [d]\setminus e$ and

$$\mathbb{N}_0^d(e) := \{ k \in \mathbb{N}_0^d : k_i = 0 \text{ if } i \not\in e \}.$$ 

Let $k \in \mathbb{N}_0^d$. For brevity we write $2^{-k}$ instead of the vector $(2^{-k_1}, 2^{-k_2}, \ldots, 2^{-k_d})$. 

4
Proposition 2.3. Let $1 \leq p, q \leq \infty$, $r > 0$ and $m \in \mathbb{N}$ such that $m - 1 \leq r < m$. Then the Besov space of dominating mixed smoothness $S^r_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that

$$
\| f \|_{S^r_{p,q}B(\mathbb{R}^d)} := \sum_{\epsilon \in [d]} \left( \sum_{k \in \mathbb{N}^d_0(\epsilon)} 2^{r|k|} \| \phi_k^m(f, 2^{-k}) \|_p^q \right)^{1/q}
$$

is finite (with the usual modification if $q = \infty$). Furthermore, $\| \cdot \|_{S^r_{p,q}B(\mathbb{R}^d)}$ generates a norm equivalent to $\| \cdot \|_{S^r_{p,q}B(\mathbb{R}^d)}$ on $L_p(\mathbb{R}^d)$.

This can be generalized as follows.

Lemma 2.4. Let $1 \leq p, q \leq \infty$ and $r > 0$. Then $m \in \mathbb{N}_0^d$ such that $r < m_i$ for all $i \in [d]$. Then

$$
\| f \|_{S^r_{p,q}B(\mathbb{R}^d)} := \sum_{\epsilon \in [d]} \left( \sum_{k \in \mathbb{N}^d_0(\epsilon)} 2^{r|k|} \| \phi_k^m(f, 2^{-k}) \|_p^q \right)^{1/q}
$$

is an equivalent norm on the space $S^r_{p,q}B(\mathbb{R}^d)$.

For a proof of both assertions we refer to [26, 2.3.4] $(d = 2)$ and [39]. Sometime it is helpful to use the following characterization.

Lemma 2.5. Let $1 \leq p, q \leq \infty$ and $r > 0$. Let $m \in \mathbb{N}$ such that $m > r$. Then the Besov space of dominating mixed smoothness $S^r_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that

$$
\| f \|_{S^r_{p,q}B(\mathbb{R}^d)}^* := \sum_{\epsilon \in [d]} \left\{ \int_{[-1,1]^\epsilon} \prod_{i \in e} \| \Delta_{h_i} f(\cdot) \|_{L_p(\mathbb{R}^d)} q \prod_{i \in e} \frac{dh_i}{|h_i|} \right\}^{1/q}
$$

is finite (with the usual modification if $q = \infty$). Furthermore, $\| \cdot \|_{S^r_{p,q}B(\mathbb{R}^d)}^*$ generates a norm equivalent to $\| \cdot \|_{S^r_{p,q}B(\mathbb{R}^d)}$ on $L_p(\mathbb{R}^d)$.

Remark 2.6. A proof of a slightly modified statement (integration with respect to the components $t_i$ is taken on $(0, \infty)$, not on $(0, 1]$) can be found in [39]. The reduction to the case considered in Lemma 2.5 can be done by standard arguments, we omit details.

Later on we shall need also the following embedding result. By $C(\mathbb{R}^d)$ we denote the collection of all uniformly continuous and bounded functions $f : \mathbb{R}^d \to \mathbb{C}$, equipped with the sup-norm.

Lemma 2.7. Let $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$. Then the space $S^r_{p,q}B(\mathbb{R}^d)$ is continuously embedded into $C(\mathbb{R}^d)$ if and only if either $r > 1/p$ or $r = 1/p$ and $q = 1$.

For a proof we refer to [26, 2.4.1] $(d = 2)$, [42] and [9].

Remark 2.8. It is one of the remarkable observations that $S^r_{p,q}B(\mathbb{R}^d)$ many times behaves like a Besov space defined on $\mathbb{R}$. 
2.3 Tools from Fourier analysis

Next we will collect some required tools from Fourier analysis. We recall an adapted version of the famous Nikolskij inequality, see Uninskij [40, 41], Stöckert [32] or [26, Theorem 1.6.2].

**Proposition 2.9.** Let $1 \leq p_0 \leq p \leq \infty$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Let $\Omega = [-b_1, b_1] \times \cdots \times [-b_d, b_d]$, $b_i > 0$, $i = 1, \ldots, d$. Then there exists a positive constant $C$, independent of $(b_1, \ldots, b_d)$, such that

$$
\|D^\alpha f\|_{L^p(\mathbb{R}^d)} \leq C \left( \prod_{i=1}^{d} b_i^{\alpha_i + \frac{1}{p_0} - \frac{1}{p}} \right) \|f\|_{L^{p_0}(\mathbb{R}^d)}
$$

holds for all $f \in L^{p_0}((\mathbb{R}^d)^d)$ with $\text{supp } Ff \subset \Omega$.

The following construction of a maximal function is essentially due to Peetre, but based on earlier work of Fefferman and Stein. Let $a > 0$ and $b = (b_1, \ldots, b_d)$, $b_i > 0$, $i = 1, \ldots, d$ be fixed. Let $f$ be a regular distribution such that $Ff$ is compactly supported. We define the Peetre maximal function $P_{b,a}f$ by

$$
P_{b,a}f(x) := \sup_{z \in \mathbb{R}^d} \frac{|f(x-z)|}{\prod_{i=1}^{d} (1 + |b_iz_i|)^a}, \quad x \in \mathbb{R}^d.
$$

**Proposition 2.10.** Let $1 \leq p \leq \infty$ and $\Omega = [-b_1, b_1] \times \cdots \times [-b_d, b_d]$, $b_i > 0$, $i = 1, \ldots, d$. Let further $a > 1/p$. Then there exists a positive constant $C$, independent of $(b_1, \ldots, b_d)$, such that

$$
\|P_{b,a}f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}
$$

holds for all $f \in L^p(\mathbb{R}^d)$ with $\text{supp } (Ff) \subset \Omega$.

For a proof we refer to [26, Thm. 1.6.4]. A very useful relation between Peetre maximal function and differences is given by the following lemma, see [39] and [26, 2.3.3] (two-dimensional case).

**Lemma 2.11.** Let $a > 0$ and $m \in \mathbb{N}$. Then there exists a constant $C$ such that

$$
|\Delta^m h f(t)| \leq C \max \{1, |bh|^a \} \min \{1, |bh|^m \} P_{b,a}f(t).
$$

holds for all $b > 0$, all $h \neq 0$, all $t \in \mathbb{R}$ and all $f \in S'(\mathbb{R})$ satisfying $\text{supp } (Ff) \subset [-b, b]$.

Applying the above result iteratively with respect to components in $e \subset [d]$ we get the following modified version in the multivariate situation.

**Lemma 2.12.** Let $a > 0$, $e \subset [d]$, $m \in \mathbb{N}_0^d$ and $h = (h_1, \ldots, h_d) \in \mathbb{R}^d$. Let further $f \in S'(\mathbb{R}^d)$ with $\text{supp } (Ff) \subset Q_b$, where

$$
Q_b := [-b_1, b_1] \times \cdots \times [-b_d, b_d], \quad b_i > 0, \quad i = 1, \ldots, d.
$$

Then there exists a constant $C > 0$ (independent of $f$, $b$, $x$ and $h$) such that

$$
|\Delta^{m,e}_h f(x)| \leq C \left( \prod_{i \in e} \max \{1, |b_i h_i|^a \} \min \{1, |b_i h_i|^m \} \right) P_{b,a}f(x)
$$

holds for all $x \in \mathbb{R}^d$. 
Let \( m \in \mathbb{N} \). Then \( C^m_{\text{mix}}(\mathbb{R}^d) \) is the collection of all continuous functions \( f : \mathbb{R}^d \to \mathbb{C} \) such that all derivatives \( D^\alpha f \) with \( \max_{j=1,\ldots,d} \alpha_j \leq m \) are continuous and \( \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)| < \infty \).

**Lemma 2.13.** Let \( b = (b_1, \ldots, b_d) > 0 \), \( a > 0 \), \( e \subset [d] \), \( m \in \mathbb{N} \), \( \psi \in C^m_{\text{mix}}(\mathbb{R}^d) \) with \( k \geq m \) and \( h = (h_1, \ldots, h_d) \in \mathbb{R}^d \). Let further \( f \in \mathcal{S}'(\mathbb{R}^d) \) with \( \text{supp}(Ff) \subset Q_0 \), where

\[
Q_0 := [-b_1, b_1] \times \ldots \times [-b_d, b_d].
\]

Then there exists a constant \( C > 0 \) (independent of \( f, b \) and \( h \)) such that

\[
|\Delta_h^{m,e}(\psi \cdot f)(x)| \leq C_{m,a,\psi} \left( \prod_{i \in e} \max \left\{ 1, |b_i h_i|^a \right\} \min \left\{ 1, |b_i h_i|^m \right\} \right) P_{b,a} f(x)
\]

holds for all \( x \in \mathbb{R}^d \).

**Remark 2.14.** For a proof we refer to [19]. Note that the constant \( C_{m,a,\psi} \) depends on \( m, a \), and \( \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d}|D^\alpha \psi(x)| \) only.

## 3 Pointwise multipliers for Besov spaces of dominating mixed smoothness

### 3.1 Some generalities on pointwise multipliers

For a quasi-Banach space \( X \) of functions we shall call a function \( g \) a pointwise multiplier if \( g \cdot f \in X \) for all \( f \in X \) (this is includes, of course, that the operation \( f \mapsto g \cdot f \) must be well defined for all \( f \in X \)). If \( X \to L_p(\Omega) \) for some \( p \) (here \( \Omega \) is a domain in \( \mathbb{R}^d \)), as a consequence of the Closed Graph Theorem, we obtain that the linear operator \( T_g : f \mapsto g \cdot f \), associated to such a pointwise multiplier, must be continuous in \( X \), see [12] p. 33. By \( M(X) \) we denote the set of all pointwise multipliers for \( X \), i.e.,

\[
M(X) := \{ g : g \cdot f \in X \quad \forall f \in X \}
\]

and equip this set with the norm of the operator \( T_g \)

\[
\| g \|_{M(X)} := \| T_g : X \to X \| = \sup_{\| f \|_{X} \leq 1} \| g \cdot f \|_{X}.
\]

We shall call \( X \) an algebra with respect to pointwise multiplication (for short a multiplication algebra) if \( f \cdot g \in X \) for all \( f, g \in X \) and there exist a constant \( C > 0 \) such that

\[
\| f \cdot g \|_{X} \leq C \| f \|_{X} \cdot \| g \|_{X}
\]

holds for all \( f, g \in X \). It is obvious that if \( X \) is a multiplication algebra we have, \( X \hookrightarrow M(X) \).

**Lemma 3.1.** Let \( 1 \leq p, q \leq \infty \) and \( r \in \mathbb{R} \). Then we have \( C^{\infty}_0(\mathbb{R}^d) \subset M(S^{r}_{p,q} B(\mathbb{R}^d)) \).

Let \( \psi \) be a non-negative \( C^{\infty}_0(\mathbb{R}^d) \) function. We put \( \psi_{\mu}(x) = \psi(x - \mu), \mu \in \mathbb{Z}^d \), \( x \in \mathbb{R}^d \) and assume that

\[
\sum_{\mu \in \mathbb{Z}^d} \psi_{\mu}(x) = 1 \quad \text{for all } x \in \mathbb{R}^d.
\]
Definition 3.2. Let the Banach space $X$ be continuously embedded into $S'(^d\mathbb{R})$. Let $\psi$ be as in (3.1). Then $X_{\text{unif}}$ is the collection of all $f \in S'(^d\mathbb{R})$ such that

$$
\| f | X_{\text{unif}} \|_{\psi} := \sup_{\mu \in ^d\mathbb{Z}} \| \psi \mu \cdot f \| < \infty .
$$

Remark 3.3. The spaces $S_{r,p,q}^r B(^d\mathbb{R})_{\text{unif}}$ are independent of the special choice of $\psi$ (in the sense of equivalent norms). This is an immediate consequence of Lemma 3.1.

Lemma 3.4. Let $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$. Then the continuous embedding

$$
M(S_{p,q}^r B(^d\mathbb{R})) \hookrightarrow S_{p,q}^r B(^d\mathbb{R})_{\text{unif}}
$$

takes place.

3.2 Pointwise multipliers and algebras

Our first main result with respect to Besov spaces of dominating mixed smoothness reads as follows.

Theorem 3.5. Let $1 \leq p, q \leq \infty$ and $r \in \mathbb{R}$. Then $S_{p,q}^r B(^d\mathbb{R})$ is a multiplication algebra if and only if

- either $r > 1/p$
- or $1 \leq p < \infty$, $r = 1/p$ and $q = 1$.

Remark 3.6. There is a rich literature concerning this problem for the isotropic Besov spaces $B_{p,q}^s(^d\mathbb{R})$. We refer to Peetre [23], Triebel [35, 36, 2.6.2] and Mazy, Shaposnikova [11, 12]. The little supplement, that $B_{\infty,q}^0(^d\mathbb{R})$, $0 < q \leq 1$, is not an algebra, has been proved in [25, 4.6.4, 4.8.3].

With respect to the dominating mixed Besov spaces we refer to [14], where sufficient conditions in case $p = q$ are treated.

Our second main result consists in the description of the multiplier space under certain restrictions.

Theorem 3.7. Let $1 \leq p \leq q \leq \infty$ and $r > 1/p$. Then

$$
M(S_{p,q}^r B(^d\mathbb{R})) = S_{p,q}^r B(^d\mathbb{R})_{\text{unif}}
$$

(3.2)

holds in the sense of equivalent norms.

Remark 3.8. (i) In proving the characterization in (3.2) we partly follow the same strategy as in case of Theorem 3.5. However, the proof is much more sophisticated than the proof of Theorem 3.5.

(ii) In case $p = q$ the result (3.2) has been proved in [14].

(iii) The isotropic counterpart of Theorem 3.7, namely the identity

$$
M(B_{p,q}^s(^d\mathbb{R})) = B_{p,q}^s(^d\mathbb{R})_{\text{unif}}, \quad 1 \leq p \leq \infty, \quad s > d/p,
$$

holds in the sense of equivalent norms.
has been known for some years in the special case \( p = q \), we refer to Strichartz \[33\] \((p = q = 2)\), Peetre \[24\], page 151, \((1 \leq p = q \leq \infty)\), Maz’ya and Shaposnikova, see \[12\] Theorems 4.1.1, 5.3.1, 5.3.2, 5.4.1, \((1 \leq p = q < \infty)\). S. \[28\] \((1 \leq p = q < \infty)\) and Triebel \[38\] Proposition 2.22. The case \( p < q \) has been proved for the first time in S. and Smirnov \[29\]. Quite recently a different proof has been given by the authors \[15\].

By using duality arguments one can derive from Theorem 3.7 the following.

**Corollary 3.9.** Let \( 1 < q \leq p < \infty \) and \( r < \frac{1}{p} - 1 \). Then

\[
M(S_{p,q}^r B(\mathbb{R}^d)) = S_{p',q'}^{-r} B(\mathbb{R}^d)_{\text{unif}}
\]

(3.3)

holds in the sense of equivalent norms.

In the isotropic case it is well-known that Theorem 3.5 can be improved in the following way. Let \( 1 \leq p, q \leq \infty \) and \( s > 0 \). Then \( B^{s}_{p,q}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \) is a multiplication algebra and there exists a constant \( C \) such that

\[
\| f \cdot g | B_{p,q}^s(\mathbb{R}^d) \| \leq C \left( \| f | B_{p,q}^s(\mathbb{R}^d) \| \| g | L_\infty(\mathbb{R}^d) \| + \| f | L_\infty(\mathbb{R}^d) \| \| g | B_{p,q}^s(\mathbb{R}^d) \| \right)
\]

holds for all \( f, g \in B_{p,q}^s(\mathbb{R}^d) \). Inequalities of this type are sometimes called Moser inequalities. In the dominating mixed case those Moser-type inequalities are not true.

**Theorem 3.10.** Let \( d > 1, 1 \leq p, q \leq \infty \) and \( r > 0 \). Then there exists no constant \( C > 0 \) such that

\[
\| f \cdot g | S_{p,q}^r B(\mathbb{R}^d) \| \leq C \left( \| f | S_{p,q}^r B(\mathbb{R}^d) \| \| g | L_\infty(\mathbb{R}^d) \| + \| f | L_\infty(\mathbb{R}^d) \| \| g | S_{p,q}^r B(\mathbb{R}^d) \| \right)
\]

holds for all \( f, g \in S_{p,q}^r B(\mathbb{R}^d) \) \( \cap L_\infty(\mathbb{R}^d) \).

### 3.3 Pointwise multipliers and algebras - the local case

As a service for the reader we investigate the local situation as well, i.e., we consider pointwise multipliers for Besov spaces of dominating mixed smoothness defined on the cube \( \Omega := [0,1]^d \). For convenience we introduce the spaces under consideration by taking restrictions.

**Definition 3.11.** Let \( 1 \leq p, q \leq \infty \) and \( r \in \mathbb{R} \). Then \( S_{p,q}^r B(\Omega) \) is the space of all \( f \in D'(\Omega) \) such that there exists \( g \in S_{p,q}^r B(\mathbb{R}^d) \) satisfying \( f = g|\Omega \). It is endowed with the quotient norm

\[
\| f | S_{p,q}^r B(\Omega) \| = \inf \left\{ \| g | S_{p,q}^r B(\mathbb{R}^d) \| : g|\Omega = f \} \right.
\]

Our main results as listed in the previous subsection carry over to the local case.

**Theorem 3.12.** Let \( 1 \leq p, q \leq \infty \) and \( r \in \mathbb{R} \). Then \( S_{p,q}^r B(\Omega) \) is a multiplication algebra if and only if

- either \( r > 1/p \)
• or $1 \leq p < \infty$, $r = 1/p$ and $q = 1$.

In the local case Theorem 3.12 can be immediately turned into a satisfactory characterization of $M(S^r_{p,q}B(\Omega))$.

**Theorem 3.13.** Let $1 \leq p, q \leq \infty$ and $r > 1/p$. Then

$$M(S^r_{p,q}B(\Omega)) = S^r_{p,q}B(\Omega)$$

holds in the sense of equivalent norms.

Also in the local situation a Moser-type inequality does not hold.

**Theorem 3.14.** Let $d > 1$, $1 \leq p, q \leq \infty$ and $r > d$. Then there exists no constant $C > 0$ such that

$$\| f \cdot g |S^r_{p,q}B(\Omega)\| \leq C (\| f |S^r_{p,q}B(\Omega)\| \| g |L_\infty(\Omega)\| + \| f |L_\infty(\Omega)\| \| g |S^r_{p,q}B(\Omega)\|)$$

holds for all $f, g \in S^r_{p,q}B(\Omega) \cap L_\infty(\Omega)$.

4 Proofs

All proofs are collected in this section. We postpone the proof of Lemma 3.1 and Lemma 3.4 to the Subsection 4.2.

4.1 Proof of the algebra property

**Proof of Theorem 3.5** Step 1. Let $r < m \leq r + 1$. Since the norm $\| \cdot |S^r_{p,q}B(\mathbb{R}^d)\|_{(m)}$ does not depend on $m > r$ in the sense of equivalent norms, see Lemma 2.4, we shall prove that

$$\| f \cdot g |S^r_{p,q}B(\mathbb{R}^d)\| \leq C \| f |S^r_{p,q}B(\mathbb{R}^d)\| \| g |L_\infty(\mathbb{R}^d)\| + \| f |L_\infty(\Omega)\| \| g |S^r_{p,q}B(\Omega)\|$$

holds for all $f, g \in S^r_{p,q}B(\mathbb{R}^d)$. Taking into account Lemma 2.7 we obtain

$$\| f \cdot g |L_p(\mathbb{R}^d)\| \leq \| f |L_p(\mathbb{R}^d)\| \| g |C(\mathbb{R}^d)\| \leq c \| f |S^r_{p,q}B(\mathbb{R}^d)\| \| g |S^r_{p,q}B(\mathbb{R}^d)\|.$$  

This inequality should be interpreted as the estimate needed for the term with $e = \emptyset$. Next we need some identities for differences. Note that if $\psi, \phi : \mathbb{R} \to \mathbb{C}$ and $m \in \mathbb{N}$ we have

$$\Delta^m_h(\psi \phi)(x) = \sum_{j=0}^{m} \binom{m}{j} \Delta^{m-j}_h \psi(x + jh) \Delta^j_h \phi(x), \quad x, h \in \mathbb{R}, \quad (4.1)$$

which can be proved by induction on $m$. Let $e \subset [d]$, $e \neq \emptyset$ and recall the notation $e_0 = [d] \setminus e$,

$$x \cdot y = (x_1 \cdot y_1, \ldots, x_d \cdot y_d) \in \mathbb{R}^d$$

and

$$\mathbb{N}_0^d(e) = \{ k \in \mathbb{N}_0^d : k_i = 0 \text{ if } i \notin e \}.$$
Then we derive from (4.1) that
\[ \Delta_{h}^{2m,e}(f \cdot g)(x) = \sum_{u \in \mathbb{N}_0^d(e), |u|_{\infty} \leq 2m} \binom{2m}{u} \Delta_{h}^{2m-u,e}f(x + u \circ h) \Delta_{h}^{u,e}g(x), \quad x, h \in \mathbb{R}^d, \quad (4.2) \]
holds. Here \( 2m - u := (2m - u_1, \ldots, 2m - u_d) \) and
\[ \binom{2m}{u} = \prod_{i \in e} \binom{2m}{u_i}. \]

The main step of the proof will consist in the estimates of the terms
\[ S_{e,u} := \left\{ \sum_{k \in \mathbb{N}_0^d(e)} 2^{r[k]q} \left( \sup_{|h_i| < 2^{-k}, i \in e} \| \Delta_{h}^{2m-u,e}f(\cdot + u \circ h) \Delta_{h}^{u,e}g(\cdot) |L_p(\mathbb{R}^d)| \right)^q \right\}^{1/q} \quad (4.3) \]
e \neq \emptyset, u \in \mathbb{N}_0^d(e), |u|_{\infty} \leq 2m. \] Therefore we have to consider different cases.

**Step 2.** The case \( u_i \leq m \) for all \( i \in e. \) Obviously we have \( 2m - u_i \geq m \) for all \( i \in e. \) Using a change of variables in the \( L_p \)-integral in the second step we obtain for a certain constant \( c_1 \)
\[ \| \Delta_{h}^{2m-u,e}f(\cdot + u \circ h) \Delta_{h}^{u,e}g(\cdot) |L_p(\mathbb{R}^d)| \| \leq \| \Delta_{h}^{2m-u,e}f(\cdot + u \circ h) \Delta_{h}^{u,e}g(\cdot) |L_p(\mathbb{R}^d)| \| \sup_{x \in \mathbb{R}^d} \| \Delta_{h}^{u,e}g(x) \| \]
\[ \leq c_1 \| g |C(\mathbb{R}^d)| \| \| \Delta_{h}^{m,e}f(\cdot) |L_p(\mathbb{R}^d)| \|. \]
The embedding \( S^r_{p,q}B(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d), \) see Lemma 2.7, implies
\[ \sup_{|h_i| < 2^{-k}, i \in e} \| \Delta_{h}^{2m-u,e}f(\cdot + u \circ h) \Delta_{h}^{u,e}g(\cdot) |L_p(\mathbb{R}^d)| \| \leq c_1 \| g |C(\mathbb{R}^d)| \| \omega_{m}^{e}(f, 2^{-k})_p \]
\[ \leq c_2 \| g |S^r_{p,q}B(\mathbb{R}^d)| \| \omega_{m}^{e}(f, 2^{-k})_p \]
with an appropriate constant \( c_2. \) Consequently we have
\[ S_{e,u} \leq c_2 \| g |S^r_{p,q}B(\mathbb{R}^d)| \left( \sum_{k \in \mathbb{N}_0^d(e)} 2^{r[k]q} \omega_{m}^{e}(f, 2^{-k})_p \right)^{1/q} \]
\[ \leq c_2 \| g |S^r_{p,q}B(\mathbb{R}^d)| \left\| f |S^r_{p,q}B(\mathbb{R}^d)| \right\|. \]
The case \( u_i \geq m \) for all \( i \in e \) can be handled in the same way by interchanging the roles of \( f \) and \( g. \)

**Step 3.** The remaining cases. Without loss of generality we may assume that \( e = \{1, \ldots, N\} \) for some natural number \( N, N \leq d. \) In addition we assume
\[ u = (u_1, \ldots, u_L, u_{L+1}, \ldots, u_N, 0, \ldots, 0) \]
with
\[ m \leq u_i \leq 2m, \quad i = 1, \ldots, L, \quad 0 \leq u_i < m, \quad i = L + 1, \ldots, N \]
and \( 1 \leq L \leq N \) and \( L < d. \) For brevity we put
\[ e_1 := \{L + 1, \ldots, N\} \quad \text{and} \quad e_2 := \{1, \ldots, L\}. \]
We will estimate the sum on the right-hand side term by term. It follows in independent of $x$

Let $C$ with convergence in this yields with convergence in $\phi$ the univariate case

Substep 3.1. By assumption both sets are nontrivial. This covers all remaining cases up to an enumeration.

An application of the triangle inequality leads to

$$
\| \Delta_{h}^{2m-u,e} f(\cdot + u \circ h) \Delta_{h}^{u,e} g(\cdot) \|_{L_p(\mathbb{R}^d)} \leq \sum_{\ell} \| \Delta_{h}^{2m-u,e} f_{k+\ell}(\cdot + u \circ h) \Delta_{h}^{u,e} g_{k+\nu}(\cdot) \|_{L_p(\mathbb{R}^d)}.
$$

We will estimate the sum on the right-hand side term by term. It follows

$$
\| \Delta_{h}^{2m-u,e} f_{k+\ell}(\cdot + u \circ h) \Delta_{h}^{u,e} g_{k+\nu}(\cdot) \|_{L_p(\mathbb{R}^d)} \leq \left( \int_{\mathbb{R}^d} \sup_{x_i \in \mathbb{R}} |\Delta_{h}^{2m-u,e} f_{k+\ell}(x + u \circ h)|^p \prod_{i=L+1}^d dx_i \right)^{1/p} \left( \int_{\mathbb{R}^d} \sup_{x_i \in \mathbb{R}} |\Delta_{h}^{u,e} g_{k+\nu}(x)|^p \prod_{i=1}^L dx_i \right)^{1/p}.
$$

Let $F_L$ denote the Fourier transform with respect to $(x_1, \ldots, x_L)$. Observe, that for any $h \in \mathbb{R}^L$

$$
supp F_L(f_{k+\ell}(\cdot + h, x_{L+1}, \ldots, x_d)) \subset \{(\xi_1, \ldots, \xi_L) : |\xi_j| \leq 3 2^{k+\ell_j-1} , j = 1, \ldots, L\},$$

independent of $x_{L+1}, \ldots, x_d$. Consequently, Nikol’skijs inequality in Proposition 2.9 yields

$$
\left( \int_{\mathbb{R}^d} \sup_{x_i \in \mathbb{R}} |\Delta_{h}^{2m-u,e} f_{k+\ell}(x + u \circ h)|^p \prod_{i=L+1}^d dx_i \right)^{1/p} \leq c_3 \prod_{i=1}^L 2^{\frac{k+\ell_i}{p}} \left( \int_{\mathbb{R}^d} |\Delta_{h}^{2m-u,e} f_{k+\ell}(x + u \circ h)|^p dx \right)^{1/p}
$$

$$
\leq c_4 \prod_{i \in e_2} 2^{\frac{\ell_i}{p}} \left( \int_{\mathbb{R}^d} |\Delta_{h}^{m_c} f_{k+\ell}(x)|^p dx \right)^{1/p}
$$

12
with constants $c_3, c_4$ independent of $f$, $k$ and $\ell$, since $k^1 \in \mathbb{N}_0^d(e_1)$, i.e., $k^1_i = 0$ if $i \in e_0 \cup e_2$. A simple change of coordinates and an analogous argument with respect to $g_{k^2+\nu}$ results in

$$\|\Delta^2_{h_{k^1+\ell}} f_{k^1+\ell} (\cdot + u \circ h) \Delta^u_{h} g_{k^2+\nu}(\cdot) \|_{L_p(\mathbb{R}^d)} \leq c_5 \left( \prod_{i \in e_2} 2^{\ell_i^2} \right) \left( \prod_{i \in (e_0 \cup e_1)} 2^{|k^1_i|} \right) \|\Delta^m_{h_{k^1+\ell}} f_{k^1+\ell} \|_{L_p(\mathbb{R}^d)} \|\Delta^{m,e_1}_{h} g_{k^2+\nu} \|_{L_p(\mathbb{R}^d)} \|.$$

We need one more notation. For $\ell \in \mathbb{Z}^d$ we put

$$\omega(\ell) := \{ i \in \{1, \ldots, d\} : \ell_i < 0 \} \quad \text{and} \quad \overline{\omega}(\ell) := \{ i \in \{1, \ldots, d\} : \ell_i \geq 0 \}.$$ 

Since $k^1 \in \mathbb{N}_0^d(e_1)$ and $\varphi_{k^1+\ell} \equiv 0$ if $k^1_i + \ell_i < 0$, we can assume that

$$(e_0 \cup e_2) \subset \overline{\omega}(\ell) \quad \text{and therefore} \quad \omega(\ell) \subset e_1; \quad (4.4)$$

similarly

$$(e_0 \cup e_1) \subset \overline{\omega}(\nu) \quad \text{and} \quad \omega(\nu) \subset e_2. \quad (4.5)$$

Writing $\Delta^m_{h_{k^1+\ell}}$ as

$$\Delta^m_{h_{k^1+\ell}} = \left( \prod_{i \in \overline{\omega}(\ell) \cap e_1} \Delta^m_{h_{k^1_i}} \right) \left( \prod_{i \in \omega(\ell)} \Delta^m_{h_{i}} \right),$$

taking Lemma 2.12 and Proposition 2.10 into account, it is easily seen that

$$\sup_{|h_i| < 2^{-k^1_i}, i \in e_1} \|\Delta^m_{h_{k^1+\ell}} f_{k^1+\ell} \|_{L_p(\mathbb{R}^d)} \leq c_6 \prod_{i \in \omega(\ell)} 2^{\ell_i} \|f_{k^1+\ell} \|_{L_p(\mathbb{R}^d)} ,$$

where we used the second part in (4.3) and the definition of $e_1$ as well. Similarly

$$\sup_{|h_i| < 2^{-k^2_i}, i \in e_2} \|\Delta^m_{h} g_{k^2+\nu} \|_{L_p(\mathbb{R}^d)} \leq c_6 \prod_{i \in \omega(\nu)} 2^{\nu_i} \|g_{k^2+\nu} \|_{L_p(\mathbb{R}^d)} .$$

Altogether we have found the estimate

$$\sup_{|h_i| < 2^{-k^1_i}, i \in e_1} \|\Delta^2_{h_{k^1+\ell}} (\cdot + u \circ h) \Delta^u_{h} g_{k^2+\nu}(\cdot) \|_{L_p(\mathbb{R}^d)} \leq c_7 \prod_{i \in e_2} 2^{\ell_i^2} \prod_{i \in (e_0 \cup e_1)} 2^{\nu_i} \prod_{i \in \omega(\ell)} 2^{\ell_i} \prod_{i \in \omega(\nu)} 2^{\nu_i} \|f_{k^1+\ell} \|_{L_p(\mathbb{R}^d)} \|g_{k^2+\nu} \|_{L_p(\mathbb{R}^d)} \| . \quad (4.6)$$

For simplicity we denote by $P(f, g, k^1, k^2, \ell, \nu)$ the term on the right-hand side in (4.6). Hence, by applying triangle inequality we get

$$S_{e,u} \leq c_7 \left\{ \sum_{k^1 \in \mathbb{N}_0^d(e)} 2^{|k^1|} \left[ \sum_{\ell, \nu \in \mathbb{Z}^d} P(f, g, k^1, k^2, \ell, \nu) \right]^q \right\}^{1/q} \leq c_7 \sum_{\ell, \nu \in \mathbb{Z}^d} \left\{ \sum_{k^1 \in \mathbb{N}_0^d(e)} 2^{|k^1|} P(f, g, k^1, k^2, \ell, \nu)^q \right\}^{1/q}. \quad (4.7)$$
Observe that

\[
\sum_{k \in \mathbb{N}_0^d(e)} 2^{\ell_k |k+\ell| q} 2^{\ell_z |k+\ell| q} \| f_{k+\ell} (L_p(\mathbb{R}^d)) \|^{q} \| g_{k+\ell} (L_p(\mathbb{R}^d)) \|^{q} \\
= \left( \sum_{k \in \mathbb{N}_0^d(e)} 2^{\ell_k |k+\ell| q} \| f_{k+\ell} (L_p(\mathbb{R}^d)) \|^{q} \right) \left( \sum_{k \in \mathbb{N}_0^d(e)} 2^{\ell_z |k+\ell| q} \| g_{k+\ell} (L_p(\mathbb{R}^d)) \|^{q} \right) \\
\leq \| f \| S_{p,q}^r B(\mathbb{R}^d) \|^{q} \| g \| S_{p,q}^r B(\mathbb{R}^d) \|^{q}.
\] (4.8)

Recall, we only need to consider those terms when \(\min_i (k_i^1 + \ell_i) \geq 0\) and \(\min_i (k_i^2 + \nu_i) \geq 0\). Hence we get for any \(k \in \mathbb{N}_0^d(e)\), see (4.4) and (4.5),

\[
\sum_{i=1}^{d} |k_i| - \sum_{i=1}^{d} |k_i^1 + \ell_i| - \sum_{i=1}^{d} |k_i^2 + \nu_i| = \left( \sum_{i=1}^{L} k_i^1 - \sum_{i=1}^{L} \ell_i - \sum_{i=1}^{L} (k_i^2 + \nu_i) \right) \\
+ \left( \sum_{i=L+1}^{N} k_i^1 - \sum_{i=L+1}^{N} (k_i^1 + \ell_i) - \sum_{i=L+1}^{N} \nu_i \right) - \left( \sum_{i=N+1}^{d} \ell_i + \sum_{i=N+1}^{d} \nu_i \right) \\
= - \sum_{i=1}^{d} (\ell_i + \nu_i).
\]

Again in view of (4.4) and (4.5), this implies

\[
\left( 2^{\ell_1} \prod_{i \in \mathbb{R}^d} 2^{\ell_{i} \mu_i} \prod_{i \in \omega(\ell)} 2^{\ell_{i} m} \prod_{i \in \omega(\nu)} 2^{\nu_{i} m} \right) \left( 2^{-r|k+\ell| q} 2^{-r|k+\ell| q} \right) \\
= \left( \prod_{i \in \mathbb{R}^d} 2^{\ell_{i} \mu_i} \prod_{i \in \omega(\ell)} 2^{\ell_{i} m} \prod_{i \in \omega(\nu)} 2^{\nu_{i} m} \right) \left( \prod_{i \in \mathbb{R}^d} 2^{-r|\ell_{i}| \mu_i} \prod_{i \in \omega(\ell)} 2^{-r|\ell_{i}| m} \prod_{i \in \omega(\nu)} 2^{-r|\nu_{i}| m} \right) \\
\leq \left( \sum_{i=1}^{d} 2^{-|\ell_{i}|} \right) \left( \sum_{i=1}^{d} 2^{-|\nu_{i}|} \right)
\] (4.9)

where \(\delta := \min(r - 1/p, m - r, r) > 0\). Consequently we conclude that

\[
S_{e,u} \leq c_7 \sum_{\ell, \nu \in \mathbb{Z}^d} \left( \sum_{i=1}^{d} 2^{-|\ell_{i}|} \right) \left( \sum_{i=1}^{d} 2^{-|\nu_{i}|} \right) \| f \| S_{p,q}^r B(\mathbb{R}^d) \| \| g \| S_{p,q}^r B(\mathbb{R}^d) \|
\]

for an appropriate constant \(c_8\) independent of \(f\) and \(g\).

Step 3.2. The case \(1 \leq p < \infty, r = 1/p\) and \(q = 1\). Our point of departure is the first inequality in (4.7). This yields

\[
S_{e,u} \leq c_7 \sum_{k \in \mathbb{N}_0^d(e)} 2^{\ell_k |k| q} \sum_{\ell, \nu \in \mathbb{Z}^d} P(f, g, k^1, k^2, \ell, \nu).
\]
To continue we need another splitting of the summation as used in Substep 3.1. We observe that

$$
\sum_{\ell_i \in \mathbb{Z}} \sum_{i \in [d]} \sum_{k \in \mathbb{N}^d_{\text{even}}} 2^{r|k^1 + \ell^1|} \, 2^{r|k^2 + \nu^1|} \, \|f_{k^1 + \ell^1}|L_p(\mathbb{R}^d)\| \, \|g_{k^2 + \nu^1}|L_p(\mathbb{R}^d)\|
$$

$$
= \left( \sum_{\ell_i \in \mathbb{Z}} \sum_{i \in [d]} \sum_{k \in \mathbb{N}^d_{\text{even}}} 2^{r|k^1 + \ell^1|} \|f_{k^1 + \ell^1}|L_p(\mathbb{R}^d)\| \right) \left( \sum_{\nu_i \in \mathbb{Z}} \sum_{i \in [d]} \sum_{k \in \mathbb{N}^d_{\text{even}}} 2^{r|k^2 + \nu^1|} \|g_{k^2 + \nu^1}|L_p(\mathbb{R}^d)\| \right)
$$

$$
\leq \|f \, |S_{p,q}^r B(\mathbb{R}^d)|| \|g \, |S_{p,q}^r B(\mathbb{R}^d)||
$$

(as a replacement of (4.8)) and

$$
\left(2^{r|k^1|} \prod_{i \in e_2} 2^{\ell^1_i} \prod_{i \in (e_0 \cup e_1)} 2^{\ell^1_i} \prod_{i \in \omega(\ell)} 2^{r|\ell^1_i|} \prod_{i \in \omega(\nu)} 2^{r|\nu^1_i|} \right)
$$

$$
= \left( \prod_{i \in e_2} 2^{d_{\ell^1_i}} \prod_{i \in \omega(\ell)} 2^{d_{\ell^1_i}} \prod_{i \in \omega(\nu)} 2^{d_{\nu^1_i}} \right) \left( \prod_{i \in (e_0 \cup e_1)} 2^{d_{\ell^1_i}} \prod_{i \in \omega(\nu)} 2^{d_{\nu^1_i}} \right)
$$

$$
\leq \left( \prod_{i \in e_1} 2^{-|\ell^1_i|} \right) \left( \prod_{i \in e_2} 2^{-|\nu^1_i|} \right)
$$

with $\delta_1 := \min(m - r, r)$ (as a replacement of (4.9)). Now we can conclude as above that

$$
S_{u,c} \leq c_0 \|f \, |S_{p,q}^r B(\mathbb{R}^d)|| \|g \, |S_{p,q}^r B(\mathbb{R}^d)||
$$

holds as well in this case.

**Step 4. Necessity.** We shall work with tensor products of functions and the cross-norm property, see Remark 2.2. Let us assume that $S_{p,q}^r B(\mathbb{R}^d)$ is an algebra with respect to pointwise multiplication. Then all products of the form

$$
\left( f(x_1) \cdot \prod_{i=2}^d \psi(x_i) \right) \cdot \left( g(x_1) \cdot \prod_{i=2}^d \psi(x_i) \right)
$$

with $f, g \in B_{p,q}^r(\mathbb{R})$ and $\psi \in C_0^\infty(\mathbb{R})$ have to belong to $S_{p,q}^r B(\mathbb{R}^d)$. Again in view of the cross-norm property this implies that the product $f \cdot g$ has to belong to $B_{p,q}^r(\mathbb{R})$, which means that $B_{p,q}^r(\mathbb{R})$ itself has to be an algebra. But in this case it is well-known that the given restrictions are necessary and sufficient, we refer, e.g., to [35, 37] and [25]. The proof is complete. $\blacksquare$

### 4.2 Proofs of Lemma 3.1 and Lemma 3.4

We recall some results about the dual spaces of $S_{p,q}^r B(\mathbb{R}^d)$. For $1 \leq p \leq \infty$ the conjugate exponent $p'$ is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. It will be convenient to work with the closure of $S(\mathbb{R}^d)$ in these spaces.

**Definition 4.1.** By $\tilde{S_{p,q}^r B(\mathbb{R}^d)}$ we denote the closure of $S(\mathbb{R}^d)$ in $S_{p,q}^r B(\mathbb{R}^d)$. 

15
As in the isotropic case we have
\[ \tilde{S}_{p,q}^{r} B(\mathbb{R}^d) = S_{p,q}^{r} B(\mathbb{R}^d) \iff \max(p, q) < \infty. \]
Because of the density of \( \mathcal{S}(\mathbb{R}^d) \) in these spaces any element of the dual space can be interpreted as an element of \( \mathcal{S}'(\mathbb{R}^d) \). Hence, a distribution \( f \in \mathcal{S}'(\mathbb{R}^d) \) belongs to the dual space \( (\tilde{S}_{p,q}^{r} B(\mathbb{R}^d))' \) if and only if there exists a positive constant \( c \) such that
\[ |f(\varphi)| \leq c |\varphi|_{\tilde{S}_{p,q}^{r} B(\mathbb{R}^d)} \quad \text{holds for all } \varphi \in \mathcal{S}(\mathbb{R}^d). \]

**Proposition 4.2.** Let \( r \in \mathbb{R} \). If \( 1 \leq p \leq \infty \) and \( 1 \leq q \leq \infty \), then it holds
\[ [\tilde{S}_{p,q}^{r} B(\mathbb{R}^d)]' = S_{p',q'}^{r} B(\mathbb{R}^d). \]

We refer to Hansen \[8\] and \[13\] for most of the details. In case \( p = \infty \) we refer to Triebel \[36\] 2.5.1], in particular to Remark 7 there, where the isotropic case is treated. Essentially the arguments used in the isotropic case carry over to the dominating mixed case. We omit details.

Now we are in position to prove Lemma \[3.1\].

**Proof of Lemma 3.1.** Theorem \[3.5\] yields that \( C_0^\infty(\mathbb{R}^d) \) is a subset of \( M(\tilde{S}_{p,q}^{r} B(\mathbb{R}^d)) \), if \( 1 \leq p, q \leq \infty \) and \( r > 1/p \). Hence, it will be enough to deal with \( r \leq 1/p \).

**Step 1.** Let \( 0 < r \leq 1/p \). Therefore we proceed as in proof of Theorem \[3.5\]. Let \( g \in C_0^\infty(\mathbb{R}^d) \) and \( f \in \tilde{S}_{p,q}^{r} B(\mathbb{R}^d) \). Again we distinguish into the cases \( e = \emptyset \) and \( e \neq \emptyset \). Concerning the first one we may argue as above. Concerning the second one, we notice that we have to estimate again the quantities \( S_{e,u} \), see \[4.3\].

**Substep 1.1.** Let \( u_i \leq m \) for all \( i \in e \). Clearly
\[ \sup_{|h_i| < 2^{-k_i}, i \in e} \| \Delta_h^{2m-u,e} g(\cdot + u \circ h)\Delta_h^{u,e} f(\cdot) | L_p(\mathbb{R}^d) \| \lesssim \left( \prod_{i \in e} 2^{-k_i(2m-u_i)} \right) \sup_{|h_i| < 2^{-k_i}, i \in u} \| \Delta_h^{u,e} f | L_p(\mathbb{R}^d) \| \lesssim \left( \prod_{i \in e} 2^{-k_i m} \right) \| f | L_p(\mathbb{R}^d) \|. \]

Inserting this into the definition of the \( S_{e,u} \), we find
\[ \left\{ \sum_{k \in \mathbb{N}_0^e(e)} 2^{rk}q \left( \sup_{|h_i| < 2^{-k_i}, i \in e} \| \Delta_h^{2m-u,e} g(\cdot + u \circ h)\Delta_h^{u,e} f(\cdot) | L_p(\mathbb{R}^d) \| \right) q \right\}^{1/q} \]
\[ \leq \left\{ \sum_{k \in \mathbb{N}_0^e(e)} 2^{rk}q \left( \prod_{i \in e} 2^{-k_i m} \right) q \| f | L_p(\mathbb{R}^d) \| \right\}^{1/q} \]
\[ \lesssim \| f | S_{p,q}^{r} B(\mathbb{R}^d) \|, \]

since \( m > r \).

**Substep 1.2.** The case \( m \leq u_i \leq 2m \) for all \( i \in e \) is treated as Step 2 in the proof of Theorem \[3.5\].

**Substep 1.3.** The remaining cases. Let \( e = \{1, \ldots, N\} \) for some natural number \( N \), \( N \leq d \). In addition we assume \( u = (u_1, \ldots, u_L, u_{L+1}, \ldots, u_N, 0, \ldots, 0) \)
with
\[ m \leq u_i \leq 2m, \quad i = 1, \ldots, L, \quad 0 \leq u_i < m, \quad i = L + 1, \ldots, N \]
and \( 1 \leq L \leq N \) and \( L < d \). For brevity we put
\[ e_1 := \{ L + 1, \ldots, N\} \quad \text{and} \quad e_2 := \{1, \ldots, L\}. \]

By assumption both sets are nontrivial. Each \( k \in \mathbb{N}_0^d(e) \) can be written as a sum \( k = k^1 + k^2 \), \( k^1 \in \mathbb{N}_0^d(e_1), k^2 \in \mathbb{N}_0^d(e_2) \). Inserting this into the definition of the \( S_{e,u} \) we find
\[
\begin{align*}
&\left\{ \sum_{k^1 \in \mathbb{N}_0^d(e_1)} \sum_{k^2 \in \mathbb{N}_0^d(e_2)} 2^{\rho k^1} q \left( \prod_{i \in e_2} 2^{-k^1(2m-u_i)} \right)^q \left( \sup_{|h_i| < 2^{-k^1}, i \in e_1} \| \Delta_h \sum_{i \in e_1} \| L_p(\mathbb{R}^d) \| \right) \right\}^{1/q} \\
\leq &\left\{ \sum_{k^2 \in \mathbb{N}_0^d(e_2)} 2^{\rho k^2} q \left( \prod_{i \in e_2} 2^{-k^2(2m-u_i)} \right)^q \left( \sup_{|h_i| < 2^{-k^2}, i \in e_1} \| \Delta_h \sum_{i \in e_1} \| L_p(\mathbb{R}^d) \| \right) \right\}^{1/q} \\
= &\left\{ \sum_{k^1 \in \mathbb{N}_0^d(e_1)} 2^{\rho k^1} q \left( \prod_{i \in e_2} 2^{-k^1(2m-u_i)} \right)^q \left( \sup_{|h_i| < 2^{-k^1}, i \in e_1} \| \Delta_h \sum_{i \in e_1} \| L_p(\mathbb{R}^d) \| \right) \right\}^{1/q} \\
\leq &\| f \|_{S^r_{p,q} B(\mathbb{R}^d)}.
\end{align*}
\]

This proves the claim in case \( r > 0 \) (we do not need \( r \leq 1/p \)).

**Step 2.** Let \( r < 0 \). We shall argue by duality. Observe that the adjoint operator to \( T_g \) is given by \( T_g \) and \( g \in M(S^r_{p,q} B(\mathbb{R}^d)) \) if and only if \( g \in M(S^r_{p,q} B(\mathbb{R}^d)) \). Hence, if \( T_g \in \mathcal{L}(S^r_{p,q} B(\mathbb{R}^d)) \), then \( T_g \in \mathcal{L}(S^r_{p,q} B(\mathbb{R}^d)) \) follows.

**Substep 2.1.** Let \( \max(p,q) < \infty \). Then Proposition 4.2 and Step 1 yield
\[
C_0^\infty(\mathbb{R}^d) \subset M(S^r_{p,q} B(\mathbb{R}^d)).
\] (4.10)

**Substep 2.2.** Let \( \max(p,q) = \infty \). Then \( S^r_{p,q} B(\mathbb{R}^d) \) is a proper subspace of \( S^r_{p,q} B(\mathbb{R}^d) \). If \( g \in C_0^\infty(\mathbb{R}^d) \) and \( g \in M(S^r_{p,q} B(\mathbb{R}^d)) \) then \( g \in M(S^r_{p,q} B(\mathbb{R}^d)) \) as well. The same duality argument as in Substep 2.1 leads to (4.10) also in this case. Hence, (4.10) is valid for all \( 1 \leq p, q \leq \infty \) and all \( r > 0 \).

**Step 3.** The case \( r = 0 \). We proceed by complex interpolation. Let \( X \) be a quasi-Banach space of distributions. By \( \hat{X} \) we denote the closure in \( X \) of the set of all infinitely differentiable functions \( g \) such that \( D^\alpha g \in X \) for all \( \alpha \in \mathbb{N}_0^d \).

**Proposition 4.3.** Let \( \Theta \in (0, 1), r_i \in \mathbb{R} \) and \( 1 \leq p_i, q_i \leq \infty \), \( i = 1, 2 \).

(i) Suppose
\[
\min \left( \max(p_1, q_1), \max(p_2, q_2) \right) < \infty.
\]

If \( r_0, p_0 \) and \( q_0 \) are given by
\[
\frac{1}{p_0} = \frac{1 - \Theta}{p_1} + \frac{\Theta}{p_2}, \quad \frac{1}{q_0} = \frac{1 - \Theta}{q_1} + \frac{\Theta}{q_2}, \quad r_0 = (1 - \Theta)r_1 + \Theta r_2,
\] (4.11)

then
\[
S^r_{p_0,q_0} B(\mathbb{R}^d) = [S^r_{p_1,q_1} B(\mathbb{R}^d), S^r_{p_2,q_2} B(\mathbb{R}^d)]_{\Theta}.
\]
(ii) Let \( r_1 \neq r_2 \). If \( r_0 \) and \( q_0 \) are defined as in (4.11), then

\[
\hat{S}_{\infty,0}^{r_0} B(\mathbb{R}^d) = [S_{\infty,q_1}^{r_1} B(\mathbb{R}^d), S_{\infty,q_2}^{r_2} B(\mathbb{R}^d)]^c,
\]

(iii) Let \( 1 \leq p_1 = p_2 < \infty \) and \( r_1 \neq r_2 \). Let \( r_0, p_0 \) and \( q_0 \) be given by (4.11), then

\[
\hat{S}_{p_0,\infty}^{r_0} B(\mathbb{R}^d) = [S_{p_1,\infty}^{r_1} B(\mathbb{R}^d), S_{p_2,\infty}^{r_2} B(\mathbb{R}^d)]^c,
\]

We refer to Vybiral [42, Theorem 4.6] concerning part (i). The isotropic counterparts of parts (ii), (iii) may be found in Yuan, S., Yang [43, pp. 1857/1858]. The arguments carry over to the dominating mixed case.

Substep 3.1. Let \( 1 \leq p, q < \infty \). Combining Step 1, Step 2, Proposition 4.5(i) and the interpolation property of the complex method we conclude that

\[
C_0^\infty(\mathbb{R}^d) \subset M(S_{0,0}^{0} B(\mathbb{R}^d)).
\]

Substep 3.2. Let \( 1 < p \leq \infty \) and \( q = \infty \). We argue by duality as in Step 2. \( C_0^\infty(\mathbb{R}^d) \subset M(S_{p,1}^{0} B(\mathbb{R}^d)) \) yields \( C_0^\infty(\mathbb{R}^d) \subset M(S_{p,\infty}^{0} B(\mathbb{R}^d)).

Substep 3.3. Let \( p = 1 \) and \( q = \infty \). Again we use duality in combination with

\[
C_0^\infty(\mathbb{R}^d) \subset M(S_{0,1}^{0} B(\mathbb{R}^d)).
\]

The proof is complete.

Remark 4.4. A closer look to the proof yields

\[
S_{t,\infty,0}^{r} B(\mathbb{R}^d) \hookrightarrow M(S_{p,q}^{r} B(\mathbb{R}^d))
\]

if \( t > |r| \). This follows from the characterization of \( S_{t,\infty,0}^{r} B(\mathbb{R}^d) \) by differences, see Proposition 2.3.

Proof of Lemma 3.4. Since \( S_{p,q}^{r} B(\mathbb{R}^d) \) is translation invariant the associated multiplier space has this property as well. Because of \( \psi_{\mu} \in C_0^\infty(\mathbb{R}^d) \) Lemma 3.1 yields \( \psi_{\mu} \cdot f \in S_{p,q}^{r} B(\mathbb{R}^d) \) for all \( f \in S_{p,q}^{r} B(\mathbb{R}^d) \). Consequently

\[
\| \psi_{\mu} \cdot f \|_{S_{p,q}^{r} B(\mathbb{R}^d)} = \| \psi \cdot f(\cdot + \mu) \|_{S_{p,q}^{r} B(\mathbb{R}^d)} \leq c_{\psi} \| f(\cdot + \mu) \|_{S_{p,q}^{r} B(\mathbb{R}^d)}
\]

This proves the claim.

4.3 Proof of the characterization of the multiplier space

First, we recall the following two results. The first one deals with traces on hyperplanes.

Proposition 4.5. Let \( 1 \leq p, q \leq \infty \) and \( r > 1/p \). Let further \( L \in \mathbb{N} \) and \( L \leq d \). If \( f \in S_{p,q}^{r} B(\mathbb{R}^d) \) then the function

\[
g(x_1, \ldots, x_L) := f(x_1, \ldots, x_L, x_{L+1}, \ldots, x_d)
\]

of the \( L \) variables \( x_1, \ldots, x_L \) \((x_{L+1}, \ldots, x_d \) are considered as fixed) belongs to the space \( S_{p,q}^{r} B(\mathbb{R}^L)\).
Proof. For a proof we refer to [26, Theorem 2.4.2].

Next we recall the localization property of the spaces $S^r_{p,q}B(\mathbb{R}^d)$, proved in [14].

**Proposition 4.6.** Let $1 \leq p \leq \infty$ and $r > 1/p$. Let further $\psi_\mu$, $\mu \in \mathbb{Z}^d$, be the functions defined in (3.1). Then we have
\[
\|f|S^r_{p,p}B(\mathbb{R}^d)\| \asymp \left(\sum_{\mu \in \mathbb{Z}^d} \|\psi_\mu f|S^r_{p,p}B(\mathbb{R}^d)\|^p\right)^{1/p}
\]
holds for all $f \in S^r_{p,p}B(\mathbb{R}^d)$.

The heart of the matter consists in the following proposition.

**Proposition 4.7.** Let $1 \leq p \leq q \leq \infty$ and $r > 1/p$. Then there exists a constant $C$ such that
\[
\|f \cdot g|S^r_{p,q}B(\mathbb{R}^d)\| \leq C \|f|S^r_{p,q}B(\mathbb{R}^d)\| \|g|S^r_{p,q}B(\mathbb{R}^d)\|_{\text{unif}}
\]
holds for all $f \in S^r_{p,q}B(\mathbb{R}^d)$ and $g \in S^r_{p,q}B(\mathbb{R}^d)_{\text{unif}}$.

**Proof.** We follow the proof of Theorem 3.5. Again we make use of the characterizations by differences. Let $r < m \leq r + 1$. Then we shall prove that
\[
\|f \cdot g|S^r_{p,q}B(\mathbb{R}^d)\|_{(2m)} \leq C \|f|S^r_{p,q}B(\mathbb{R}^d)\| \|g|S^r_{p,q}B(\mathbb{R}^d)\|_{(2m)}
\]
holds for all $f \in S^r_{p,q}B(\mathbb{R}^d)$ and $g \in S^r_{p,q}B(\mathbb{R}^d)_{\text{unif}}$.

**Step 1.** Let $\psi$ be the function in Definition 3.2 and $\phi \in C^\infty_0(\mathbb{R}^d)$ chosen such that $\phi \equiv 1$ on the support of $\psi$. It follows that
\[
\|f \cdot g|S^r_{p,q}B(\mathbb{R}^d)\|_{(2m)} = \left\|\sum_{\mu \in \mathbb{Z}^d} \phi_\mu f |\psi_\mu g\right\|_{S^r_{p,q}B(\mathbb{R}^d)}\|_{(2m)}.
\]

In case $1 \leq p < \infty$ the series $\sum_{\mu \in \mathbb{Z}^d} \phi_\mu f |\psi_\mu g$ is convergent in $S^r_{p,q}B(\mathbb{R}^d)$, in case $p = \infty$ we use the fact, that the sum is locally finite. Clearly
\[
\left\|\sum_{\mu \in \mathbb{Z}^d} \phi_\mu f |\psi_\mu g\right\|_{L_p(\mathbb{R}^d)} \leq \left\|\sum_{\mu \in \mathbb{Z}^d} \phi_\mu f \cdot |\psi_\mu g|\right\|_{C(\mathbb{R}^d)}\|_{L_p(\mathbb{R}^d)}
\]
\[
\quad \leq \|f|_{L_p(\mathbb{R}^d)}\| \sup_{\mu \in \mathbb{Z}^d} \|\psi_\mu g|_{C(\mathbb{R}^d)}\|
\]
\[
\quad \leq \|f|_{S^r_{p,q}B(\mathbb{R}^d)}\| \|g|_{S^r_{p,q}B(\mathbb{R}^d)_{\text{unif}}}\|
\]
where we used Lemma 2.7 in the last step. For $e \subset [d]$, $e \neq \emptyset$, we have
\[
\Delta^u_{h} \left(\sum_{\mu \in \mathbb{Z}^d} \phi_\mu f \cdot |\psi_\mu g\right)(x) = \sum_{|u| \leq 2m} \sum_{\mu \in \mathbb{Z}^d} C_u \Delta^{2m-u,e}_{h}(\phi_\mu f)(x + u \circ h) \Delta^{u,e}_{h}(\psi_\mu g)(x),
\]
h $\in \mathbb{R}^d$, where $2m - u = (2m - u_1, \ldots, 2m - u_d)$, see (4.2). This makes clear that we have to estimate the terms
\[
S_{e,u} := \left\{\sum_{k \in \mathbb{N}_0(e)} 2^{k|l|q} \sup_{|h| \leq 2^{-k_i}, i \in e} \left\|\sum_{\mu \in \mathbb{Z}^d} \Delta^{2m-u,e}_{h}(\phi_\mu f)(\cdot + u \circ h) \Delta^{u,e}_{h}(\psi_\mu g)(\cdot)|_{L_p(\mathbb{R}^d)}\|^q\right\}^{1/q}
\]
(4.12)
For brevity we put
\[ P_k := \left\| \sum_{\mu \in \mathbb{Z}^d} |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f)(\cdot) + u \cdot h| \Delta^{u,e}_{\mu} (\psi_{\mu} g)(\cdot) \right\|_{L_p(\mathbb{R}^d)}. \]

**Step 2.** Estimate of \( S_{e,u} \) in case \( u_i < m \) for all \( i \in e \). We have
\[ P_k \lesssim \left\| \sum_{\mu \in \mathbb{Z}^d} |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f)(x)| \right\|_{L_p(\mathbb{R}^d)} \cdot \sup_{\mu \in \mathbb{Z}^d} \sup_{x \in \mathbb{R}^d} |\Delta^{u,e}_{\mu} (\psi_{\mu} g)(x)|. \] (4.13)

By Lemma 2.7 it is easily seen that
\[ \sup_{\mu \in \mathbb{Z}^d} \sup_{x \in \mathbb{R}^d} |\Delta^{u,e}_{\mu} (\psi_{\mu} g)(x)| \lesssim \sup_{\mu \in \mathbb{Z}^d} \sup_{x \in \mathbb{R}^d} |(\psi_{\mu} g)(x)| \]
\[ = \sup_{\mu \in \mathbb{Z}^d} \| \psi_{\mu} g | C(\mathbb{R}^d) \| \lesssim \| g | S_{p,q}^r B(\mathbb{R}^d)_{\text{unif}} \|. \]

We estimate the first term on the right-hand side of (4.13) by using the decomposition
\[ \phi_{\mu} f = \phi_{\mu} \sum_{\ell \in \mathbb{Z}^d} F^{-1} \varphi_{k+\ell} F f = \sum_{\ell \in \mathbb{Z}^d} \phi_{\mu} f_{k+\ell}, \]
see Substep 3.1 in the proof of Theorem 3.5. It follows that
\[ \left\| \sum_{\mu \in \mathbb{Z}^d} |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f)(\cdot)| \right\|_{L_p(\mathbb{R}^d)} \leq \sum_{\ell \in \mathbb{Z}^d} \left\| \sum_{\mu \in \mathbb{Z}^d} |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f_{k+\ell})(\cdot)| \right\|_{L_p(\mathbb{R}^d)}. \]

Again we shall use the notation
\[ \omega(\ell) := \{ i \in \{1, \ldots, d \} : \ell_i < 0 \} \quad \text{and} \quad \varpi(\ell) := \{ i \in \{1, \ldots, d \} : \ell_i \geq 0 \}. \]

Note that there exists a positive constant \( c \) such that \( |x - \mu| > c \) implies \( |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f_{k+\ell})(x)| \equiv 0 \) for all \( \mu \). In case \( |x - \mu| \leq c \) Lemma 2.13 yields
\[ |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f_{k+\ell})(x)| \lesssim \left( \prod_{\omega(\ell) \cap e} 2^{\ell_i (2m-u)} \prod_{\omega(\ell) \cap \bar{e}} 2^{\ell_i a} \right) P_{2^k+\ell,a} f_{k+\ell}(x) \]
\[ \lesssim \left( \prod_{\omega(\ell) \cap e} 2^{\ell_i m} \prod_{\omega(\ell) \cap \bar{e}} 2^{\ell_i a} \right) P_{2^k+\ell,a} f_{k+\ell}(x), \]
since \( \ell_i < 0 \) and \( u_i \leq m \) for \( i \in \omega(\ell) \cap e \). We choose \( a \) such that \( 1/p < a < r \). Hence
\[ \left\| \sum_{\mu \in \mathbb{Z}^d} |\Delta^{2\bar{m}-u,e}_{\mu} (\phi_{\mu} f)(\cdot)| \right\|_{L_p(\mathbb{R}^d)} \lesssim \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{\omega(\ell) \cap e} 2^{\ell_i m} \prod_{\omega(\ell) \cap \bar{e}} 2^{\ell_i a} \right) \| P_{2^k+\ell,a} f_{k+\ell} \|_{L_p(\mathbb{R}^d)} \]
\[ \lesssim \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{\omega(\ell) \cap e} 2^{\ell_i m} \prod_{\omega(\ell) \cap \bar{e}} 2^{\ell_i a} \right) \| f_{k+\ell} \|_{L_p(\mathbb{R}^d)}. \]

see Theorem 2.10 This implies
\[ P_k \lesssim \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{\omega(\ell) \cap e} 2^{\ell_i m} \prod_{\omega(\ell) \cap \bar{e}} 2^{\ell_i a} \right) \| f_{k+\ell} \|_{L_p(\mathbb{R}^d)} \cdot \| g | S_{p,q}^r B(\mathbb{R}^d)_{\text{unif}} \|. \]

20
Inserting this into (4.12), we obtain

\[ S_{e,u} \lesssim \left\{ \sum_{k \in \mathbb{N}^d_0(e)} 2^{r|k|_1} \left( \prod_{\ell \in \mathbb{Z}^d} \sum_{\omega(\ell) \cap e} 2^{\ell,m} \prod_{i \in \omega(\ell) \cap e} 2^{\ell,i} \| f_{k+\ell} | L_p(\mathbb{R}^d) \| \right)^q \right\}^{1/q} \| g | S_{p,q}^r B(\mathbb{R}^d)_{unif} \|.
\]

Observe that

\[ 2^{-r|k|_1} \left( \prod_{i \in \omega(\ell) \cap e} 2^{\ell,i} \right) \leq \prod_{i=1}^d 2^{-|\ell|_i} \delta,
\]

where \( \delta := \min(m - r, r - a, r) > 0 \). This leads to

\[ S_{e,u} \lesssim \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{i=1}^d 2^{-|\ell|_i} \delta \right) \| f | S_{p,q}^r B(\mathbb{R}^d) \| \cdot \| g | S_{p,q}^r B(\mathbb{R}^d)_{unif} \|.
\]

**Step 3.** Estimate of \( S_{e,u} \) in case \( u_i \geq m \) for all \( i \in \mathbb{N} \). We have

\[ \| \Delta_{h}^{2m-u} (\phi \mu f)(\cdot + uh) \|_{L_p(\mathbb{R}^d)} \leq \| \Delta_{h}^{2m-u} (\phi \mu g)(\cdot) \|_{L_p(\mathbb{R}^d)} \cdot \| \Delta_{h}^{u} (\psi \mu g)(\cdot) \|_{L_p(\mathbb{R}^d)} \]

Inserting this into \( S_{e,u} \) and applying the triangle inequality with \( q/p \geq 1 \) we have found

\[ S_{e,u} \lesssim \left\{ \sum_{k \in \mathbb{N}^d_0(e)} \left( \prod_{\ell \in \mathbb{Z}^d} \sup_{|h| < 2^{-k} \ell} \sum_{i \in \mathbb{Z}^d} \| \phi \mu f | C(\mathbb{R}^d) \| \cdot \| \Delta_{h}^{u} (\psi \mu g)(\cdot) | L_p(\mathbb{R}^d) \| \right)^{q/p} \right\}^{1/q} \]

Since \( r > 1/p \), there exists some \( \varepsilon > 0 \) such that \( r - \varepsilon > 1/p \). This implies \( S_{p,p}^{r-\varepsilon} B(\mathbb{R}^d) \subset C(\mathbb{R}^d) \), see Lemma 2.7. Hence, by means of the localization property of \( S_{p,p}^{r-\varepsilon} B(\mathbb{R}^d) \), see Proposition 4.3,

\[ \left( \sum_{\mu \in \mathbb{Z}^d} \| \phi \mu f | C(\mathbb{R}^d) \|^{p} \right)^{1/p} \lesssim \left( \sum_{\mu \in \mathbb{Z}^d} \| \phi \mu f | S_{p,p}^{r-\varepsilon} B(\mathbb{R}^d) \|^{p} \right)^{1/p} \leq \| f | S_{p,p}^{r-\varepsilon} B(\mathbb{R}^d) \|.
\]

Now the elementary embedding \( S_{p,q}^r B(\mathbb{R}^d) \hookrightarrow S_{p,p}^{r-\varepsilon} B(\mathbb{R}^d) \) implies

\[ S_{e,u} \lesssim \| f | S_{p,q}^r B(\mathbb{R}^d) \| \cdot \| g | S_{p,q}^r B(\mathbb{R}^d)_{unif} \|. \]
Step 4. Estimate of \( S_{e,u} \) for the remaining cases. We shall use the same notation as in proof of Theorem 3.5. Step 3, i.e., we assume that \( e = \{1, \ldots, N\} \) for some natural number \( N, N \leq d \),

\[
u = (u_1, \ldots, u_L, u_{L+1}, \ldots, u_N, 0, \ldots, 0)
\]

with

\[
m \leq u_i \leq 2m, \quad i = 1, \ldots, L, \quad 0 \leq u_i < m, \quad i = L + 1, \ldots, N
\]

and \( 1 \leq L \leq N \) and \( L < d \). Again we define

\[
e_1 := \{L + 1, \ldots, N\} \quad \text{and} \quad e_2 := \{1, \ldots, L\}.
\]

Both sets are nontrivial. This covers all remaining cases up to an enumeration. Again we make use of \( \mathbb{N}_0^d(e) = \mathbb{N}_0^d(e_1) \cup \mathbb{N}_0^d(e_2) \). For brevity we put

\[
T_{\mu,h}(f,g) := \|\Delta_{h}^{2m-u,e}(\phi_{\mu}f)(\cdot + u \circ h)\Delta_{h}^{u,e}(\psi_{\mu}g)(\cdot)\|_{L_p(\mathbb{R}^d)}.
\]

Then, because of \( q/p \geq 1 \), [1.12] yields

\[
S_{e,u} \leq \left\{ \sum_{k \in \mathbb{N}_0^d(e)} 2^{r|k|q} \sup_{|h| < 2^{-k,i},i \in e} \left( \sum_{\mu \in \mathbb{Z}^d} T_{\mu,h}(f,g)^p \right)^{q/p} \right\}^{1/q}
\]

\[
\leq \left\{ \sum_{k \in \mathbb{N}_0^d(e_1)} 2^{r|k|q} \left[ \left( \sum_{k^2 \in \mathbb{N}_0^d(e_2)} 2^{r|k^2|q} \sup_{|h| < 2^{-k,i},i \in e} T_{\mu,h}(f,g)^p \right)^{q/p} \right]^{q/p} \right\}^{1/q}
\]

\[
\leq \left\{ \sum_{k \in \mathbb{N}_0^d(e_1)} 2^{r|k|q} \left[ \sum_{\mu \in \mathbb{Z}^d} \left( \sum_{k^2 \in \mathbb{N}_0^d(e_2)} 2^{r|k^2|q} \sup_{|h| < 2^{-k,i},i \in e} T_{\mu,h}(f,g)^q \right)^{p/q} \right]^{q/p} \right\}^{1/q}.
\]

We consider the integral

\[
T_{\mu,h}(f,g)^p \leq \left( \int_{\mathbb{R}^d} \sup_{x_i \in L, i \leq L} \Delta_{h}^{2m-u,e}(\phi_{\mu}f)(x + u \circ h)^p dx_1 \right)^{d} \left( \int_{\mathbb{R}^L} \sup_{x_i \in L^1, i \leq L^1} \Delta_{h}^{u,e}(\psi_{\mu}g)(x)^{p} dx_1 \right)^{d}
\]

\[
\leq \left( \int_{\mathbb{R}^d} \sup_{x_i \in L, i \leq L} \Delta_{h}^{2m,e_1}(\phi_{\mu}f)(x)^p dx_1 \right)^{d} \left( \int_{\mathbb{R}^L} \sup_{x_i \in L^2, i \leq L^2} \Delta_{h}^{u,e_2}(\psi_{\mu}g)(x)^{p} dx_1 \right)^{d}.
\]

Let \( G : \mathbb{R}^d \to \mathbb{C} \) be a given function and \( a \in [d] \). When we write \( \|G|S_{p,p}^dB(\mathbb{R}^a)\| \), then we mean that the norm is taken with respect to the variables with indexes in \( e \), the remaining are considered as frozen. In addition we shall use the notation

\[
\Delta_{h,l}^{m,e_1,a} := \Delta_{h,l}^{m,a}(\Delta_{h,l}^{m,e_2}) \quad \text{and} \quad \omega_{m,a}^{e_1,ja}(G,2^{-k^2},2^{-l^2}) := \sup_{|h| < 2^{-k,i},i \in e_2} \|\Delta_{h,l}^{m,e_1,ja}(G)\|_{L_p(\mathbb{R}^d)}.
\]

Since \( r > 1/p \), there exists \( \varepsilon_1 > 0 \) such that \( r - \varepsilon_1 > 1/p \). From Lemmas 2.7, 2.5, Proposition 4.5.
and some monotonicity arguments we conclude

\[
\int \sup_{x_i \in \mathbb{R}^L} | \Delta_{h_i}^{m,e_2} (\psi \mu g)(x) |^p \prod_{i=1}^L dx_i \lesssim \int \left( \sum_{\alpha \in \{L+1, \ldots, d\}} \prod_{\ell \in \alpha} |t_{\ell}|^{-(r-\epsilon_1)p} \right) \left\| \Delta_{h,t}^{m,e_2(L+q)} (\psi \mu g)(\cdot) \right\|_{L_p(\mathbb{R}^{d-L})}^p \prod_{i \in \alpha} \frac{dt_i}{|t_i|} \prod_{i=1}^L dx_i
\]

\[
= \sum_{\alpha \in \{L+1, \ldots, d\}} \int \prod_{\ell \in \alpha} |t_{\ell}|^{-(r-\epsilon_1)p} \left\| \Delta_{h,t}^{m,e_2(L+q)} (\psi \mu g)(\cdot) \right\|_{L_p(\mathbb{R}^{d-L})}^p \prod_{i \in \alpha} \frac{dt_i}{|t_i|} \prod_{i=1}^L dx_i
\]

\[
\lesssim \sum_{\alpha \in \{L+1, \ldots, d\}} \sum_{\ell \in \mathbb{N}_0(a)} 2^{\ell_1 (r-\epsilon_1) p} \omega_{\ell}^{m,e_2(L+q)} (\psi \mu g, 2^{-k^2}, 2^{-\ell})_p^p. \tag{4.16}
\]

We need one more abbreviation

\[
F_{\mu}(k^1) := \sup_{|h_i| < 2^{-k_i}, i \in e_1} \int_{\mathbb{R}^{d-L}} \sup_{x_i \in \mathbb{R}^L} | \Delta_{h_i}^{m,e_1} (\phi \mu f)(x) |^p \prod_{i=L+1}^d dx_i.
\]

This leads to the estimate of the term in [\ldots] in (4.15)

\[
\sum_{\mu \in \mathbb{Z}^d} \left( \sum_{k^2 \in \mathbb{N}_0(e_2)} 2^{\ell^2 |q|} \sup_{|h_i| < 2^{-k_i}, i \in e} T_{\mu,h}(f, g)^q \right)^{p/q} \lesssim \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) \left\{ \sum_{k^2 \in \mathbb{N}_0(e_2)} 2^{\ell^2 |q|} \left( \sum_{\alpha \in \{L+1, \ldots, d\}} \sum_{\ell \in \mathbb{N}_0(a)} 2^{\ell_1 (r-\epsilon_1) p} \omega_{\ell}^{m,e_2(L+q)} (\psi \mu g, 2^{-k^2}, 2^{-\ell})_p^p \right) \right\}. \tag{4.17}
\]

Next we apply the elementary inequality

\[
\sum_{j \in \mathbb{N}_0} |a_j| \leq c \left( \sum_{j \in \mathbb{N}_0} 2^{j \varepsilon t} |a_j|^t \right)^{1/t},
\]

valid for all \( \varepsilon > 0 \) and all \( t \geq 1 \). This inequality, used with \( t = q/p \), yields

\[
\sum_{\mu \in \mathbb{Z}^d} \left( \sum_{k^2 \in \mathbb{N}_0(e_2)} 2^{\ell^2 |q|} \sup_{|h_i| < 2^{-k_i}, i \in e} T_{\mu,h}(f, g)^q \right)^{p/q} \lesssim \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) \left\{ \sum_{\alpha \in \{L+1, \ldots, d\}} \sum_{k^2 \in \mathbb{N}_0(e_2)} \sum_{\ell \in \mathbb{N}_0(a)} 2^{\ell^2 |q|} 2^{\ell_1 |r| q} \omega_{\ell}^{m,e_2(L+q)} (\psi \mu g, 2^{-k^2}, 2^{-\ell})_p^p \right\}^{p/q} \lesssim \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) || \psi \mu g || S_{s,q}^r B(\mathbb{R}^d) ||^p.
\]

since \( a \) and \( e_2 \) are disjoint. This can be inserted into the estimate of \( S_{u,e} \) to get

\[
S_{u,e} \lesssim \left\{ \sum_{k^1 \in \mathbb{N}_0(e_1)} 2^{\ell^1 |k^1|} \left[ \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) || \psi \mu g || S_{s,q}^r B(\mathbb{R}^d) ||^p \right]^{q/p} \right\}^{1/q} \lesssim \sum_{k^1 \in \mathbb{N}_0(e_1)} 2^{\ell^1 |k^1|} \left[ \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) \right] \left\{ \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) \right\}^{q/p} \approx (1/2) \left| g \right| S_{s,q}^r B(\mathbb{R}^d) ||^p || \psi \mu g || S_{s,q}^r B(\mathbb{R}^d) ||^p || \psi \mu g || S_{s,q}^r B(\mathbb{R}^d) ||^p.
\]
To finish the proof it will be sufficient to show that

\[ S_{\mu,e} := \left\{ \sum_{k^1 \in \mathbb{N}_0(e_1)} 2^{j_1} |\mu| q^{q/p} \left[ \sum_{\mu \in \mathbb{Z}^d} F_\mu(k^1) \right]^{q/p} \right\}^{1/q} \leq C \| f \|_{S_{p,q}^* B(\mathbb{R}^d)} \]

holds for some constant \( C \) independent of \( f \). Similar to (4.16) we conclude

\[ F_\mu(k^1) \lesssim \sum_{v \in [L]} \sum_{j \in \mathbb{N}_0(v)} 2^{j_1} |(r - \varepsilon_1) p| \sup_{|h_i| < 2^{-k^1_i}, i \in e_1} \left\| \Delta_{h,s}^{m,e_1 \cup v} (\phi_\mu f) \right\|_{L_p(\mathbb{R}^d)} . \]

Note that \( v \cap e_1 = \emptyset \). Again we have to decompose \( \phi_\mu f \). But this time we only split \( f \). This results in

\[ \phi_\mu f = \phi_\mu \sum_{\ell \in \mathbb{Z}^d} F^{-1} \varphi_{k^1 + j + \ell} F f = \sum_{\ell \in \mathbb{Z}^d} \phi_\mu f_{k^1 + j + \ell}, \]

where \( j \) and \( k^1 \) are at our disposal. With \( k^1 \in \mathbb{N}_0(e_1) \) and \( j \in \mathbb{N}_0(v) \), as in (4.4), we can assume

\[ [d] \setminus (e_1 \cup v) \subset \overline{\omega(\ell)} \quad \text{and} \quad \omega(\ell) \subset (e_1 \cup v). \quad (4.18) \]

Let \( c > 0 \) be chosen such that

\[ \Delta_{h,s}^{m,e_1 \cup v} (\phi \cdot f)(x) = 0 \quad \text{if} \quad |x|_\infty \geq c. \]

We put \( Q_\mu := \{ x \in \mathbb{R}^d : |x - \mu|_\infty \leq c \} \). Because of (4.18), Lemma 2.13 yields

\[ |\Delta_{h,s}^{m,e_1 \cup v} (\phi_\mu f_{k^1 + j + \ell})(x)| \leq \left( \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right) P_{2^{k^1 + j + \ell} a} f_{k^1 + j + \ell}(x), \quad x \in Q_\mu, \]

for all \( h, |h_i| < 2^{-k^1_i}, i \in e_1 \) and for all \( s, |s_i| < 2^{-j_i}, i \in v \). For those pairs \((h, s)\), applying the triangle inequality with respect to \( L_p(\mathbb{R}^d) \), it follows

\[ \left\| \Delta_{h,s}^{m,e_1 \cup v} (\phi_\mu f) \right\|_{L_p(\mathbb{R}^d)} \leq \left[ \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right) P_{2^{k^1 + j + \ell} a} f_{k^1 + j + \ell} \right\|_{L_p(Q_\mu)} . \]

Consequently we find

\[ F_\mu(k^1) \lesssim \sum_{v \in [L]} \sum_{j \in \mathbb{N}_0(v)} 2^{j_1} |(r - \varepsilon_1) p| \left[ \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right) P_{2^{k^1 + j + \ell} a} f_{k^1 + j + \ell} \right\|_{L_p(Q_\mu)} . \]
The final overlap property of the $Q_{\mu}$ leads to
\[
\left\{ \sum_{\mu \in \mathbb{Z}^d} F_{\mu}(k^1) \right\}^{1/p} \lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left\{ \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |j_1| (r - \varepsilon_1) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^p \times \sum_{\mu \in \mathbb{Z}^d} \| P_{2k^1 + j + \ell \cdot a} f_{k^1 + j + \ell} | L_p(Q_{\mu}) \|_p \right\}^{1/p}
\]
\[
\lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left\{ \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |j_1| (r - \varepsilon_1) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^p \| P_{2k^1 + j + \ell \cdot a} f_{k^1 + j + \ell} | L_p(\mathbb{R}^d) \|_p \right\}^{1/p}
\]
\[
\lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left\{ \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |j_1| (r - \varepsilon_1) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^p \| f_{k^1 + j + \ell} | L_p(\mathbb{R}^d) \|_p \right\}^{1/p},
\]
where in the last step we employed Theorem 2.10. The triangle inequality in $\ell_q$ yields
\[
S_{u,e}^* \lesssim \left\{ \sum_{k^1 \in \mathbb{N}_0^d(e_1)} \left[ \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left( \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |k^1| r_2 |j_1| (r - \varepsilon_1) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^p \times \| f_{k^1 + j + \ell} | L_p(\mathbb{R}^d) \|_p \right\}^{1/p} \right\}^{p/q}
\]
\[
\lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left\{ \sum_{k^1 \in \mathbb{N}_0^d(e_1)} \left[ \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |k^1| r_2 |j_1| (r - \varepsilon_1) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^p \times \| f_{k^1 + j + \ell} | L_p(\mathbb{R}^d) \|_p \right\}^{q/p} \right\}^{1/q}
\]
Next we apply the inequality (4.17) with $\varepsilon_2 > 0$ and $t = q/p$ to yield
\[
S_{u,e}^* \lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left\{ \sum_{k^1 \in \mathbb{N}_0^d(e_1)} \sum_{j \in \mathbb{N}_0^d(v)} \left( 2 |k^1| r_2 |j_1| (r - \varepsilon_1 + \varepsilon_2) \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \right)^q \times \| f_{k^1 + j + \ell} | L_p(\mathbb{R}^d) \|_q \right\}^{1/q}
\]
Since $\varepsilon_2 > 0$ is arbitrary we can choose $\varepsilon_2 < \varepsilon_1 < r - 1/p$ to get
\[
S_{u,e}^* \lesssim \sum_{v \in [L]} \sum_{\ell \in \mathbb{Z}^d} \left( \prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \prod_{i=1}^d 2^{-\ell_i r} \right): \| f \|_{S_{p,q}^* B(\mathbb{R}^d)}.
\]
Let $\delta_2 := \min(m - r, r - a, r) > 0$. Then, as in (4.14) (see (4.18)), we conclude
\[
\prod_{i \in \omega(\ell)} 2^{\ell_i m} \prod_{\omega(\ell) \cap (e_1 \cup v)} 2^{\ell_i a} \prod_{i=1}^d 2^{-\ell_i r} \leq \prod_{i=1}^d 2^{-|\ell_i| \delta_2},
\]
which finally implies \( S_{u,e}^* \lesssim \| f \|_{S_{p,q}^r(B(\mathbb{R}^d))} \) and hence

\[
S_{e,u} \lesssim \| f \|_{S_{p,q}^r(B(\mathbb{R}^d))} \| g \|_{S_{p,q}^r(B(\mathbb{R}^d))}.
\]

The proof is complete.

**Proof of Theorem 3.7** Theorem 3.7 is the direct consequence of Proposition 4.7 and Lemma 3.4.

**Proof of Theorem 3.10.** We may employ the same counterexamples as in case \( p = q \) which is treated in [14].

**Proof of Corollary 3.9.** The characterization of the multiplier space in (3.3) is an immediate consequence of Theorem 3.7 and the duality argument as used in proof of Lemma 3.1. We omit details.

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**4.4 Proof of the assertions in the local case**

**Proof of Theorem 3.12.** The if-part is obvious. To prove the only if-part we apply the arguments from the proof of Theorem 3.5 and conclude that there exists a constant \( C > 0 \) such that

\[
\| f \cdot g \|_{B_{p,q}^r(\mathbb{R})} \leq C \| f \|_{B_{p,q}^r(\mathbb{R})} \| g \|_{B_{p,q}^r(\mathbb{R})}
\]

holds for all \( f, g \in B_{p,q}^r(\mathbb{R}) \cap C^\infty(\mathbb{R}) \) satisfying \( \text{supp}\ f, g \subset (0,1)^d \). As in the proof of Theorem 2.6.2/1 in Triebel [36] we conclude that \( \tilde{B}_{p,q}^r([0,1]) \) must be embedded into \( C([0,1]) \). Again this is known to be equivalent to the given restrictions, see [30].

**Proof of Theorem 3.13.** Sufficiency follows from Theorem 3.5. Necessity is implied by the fact that the function \( g = 1 \) on \([0,1]^d\) belongs to all spaces \( S_{p,q}^r B([0,1]^d) \). Hence, a function \( f \in M(S_{p,q}^r B([0,1]^d)) \) has to satisfy \( f \cdot g \in S_{p,q}^r B([0,1]^d) \) for this \( g \) and therefore \( f \in S_{p,q}^r B([0,1]^d) \).

**Proof of Theorem 3.14.** It is enough to observe that the used counterexamples in the proof of Theorem 3.10 have compact support.

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