How to Describe Photons as (3+1)-Solitons?

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Abstract

This paper aims to present the pure field part of the newly developed nonlinear \textit{Extended Electrodynamics} [1]-[3] in non-relativistic terms, i.e. in terms of the electric and magnetic vector fields (\textbf{E}, \textbf{B}), and to give explicitly those (3+1)-soliton solutions of the new equations which have the integral properties of photons. The set of solutions to the new equations contains all solutions to Maxwell’s equations as a subclass, as well as, new solutions, called nonlinear. The important characteristics \textit{scale factor}, \textit{amplitude function}, and \textit{phase function} of a nonlinear solution are defined in a coordinate free way and effectively used. The nonlinear solutions are identified through the non-zero values of two appropriately defined vector fields \(\vec{F}\) and \(\vec{M}\), as well as, through the finite values of the corresponding scale factors. The intrinsic angular momentum (spin) is also defined. A limited superposition principle (interference of nonlinear solutions), yielding the well known classical \textit{coherence} conditions, is found to exist.
1 Introduction

The 19th century physics, due mainly to Faraday and Maxwell, created the theoretical concept of electromagnetic field, i.e. extended (in fact, infinite) object, having dynamical structure. The concepts of flux of a vector field through a 2-dimensional surface and circulation of a vector field along a closed curve were coined and used extensively. Maxwell’s equations in their integral form establish where the time-changes of the fluxes of the electric and magnetic fields go to, or come from, in both cases of a closed 2-surface and a 2-surface with a boundary. We note that these fluxes are specific to the continuous character of the physical object under consideration and it is important also to note that Maxwell’s field equations have not the sense of direct energy-momentum balance relations as the Newton’s law $\dot{p} = F$ has. Nevertheless, they are consistent with energy-momentum conservation, as is well known, the corresponding local energy-momentum quantities are quadratic functions of the electric and magnetic vectors.

Although very useful for considerations in finite regions with boundary conditions, the pure field Maxwell’s equations have time-dependent vacuum solutions (in the whole space) that give inadequate models of the real fields. As a rule, if these solutions are time-stable, they occupy the whole 3-space or an infinite subregion of it, and they do not go to zero at infinity, hence, they carry infinite energy and momentum. As an example we recall the plane wave solution, given by the electric and magnetic fields of the form (in a specially chosen coordinate system)

$$E = \{u(ct + \varepsilon z), p(ct + \varepsilon z), 0\}; \quad B = \{\varepsilon p(ct + \varepsilon z), -\varepsilon u(ct + \varepsilon z), 0\}, \quad \varepsilon = \pm 1,$$

where $u$ and $p$ are arbitrary differentiable functions. Even if $u$ and $p$ are soliton-like with respect to the coordinate $z$, they do not depend on the other two spatial coordinates ($x, y$). Hence, the solution occupies the whole $\mathbb{R}^3$, or its infinite subregion, and clearly it carries infinite integral energy (we use Gauss units)

$$W = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{E^2 + B^2}{2} dxdydz = \frac{1}{4\pi} \int_{\mathbb{R}^3} (u^2 + p^2) dxdydz = \infty.$$

In particular, the popular harmonic plane wave

$$u = U_0 \cos(\omega t \pm k_z z), \quad p = P_0 \sin(\omega t \pm k_z z), \quad c^2 k_z^2 = \omega^2, \quad U_0 = \text{const}, \quad P_0 = \text{const},$$

clearly occupies the whole 3-space, carries infinite energy

$$W = \frac{1}{4\pi} \int_{\mathbb{R}^3} (U_0 + P_0) dxdydz = \infty$$

and, therefore, could hardly be an adequate model of a really created field.

On the other hand, according to Cauchy’s theorem for the wave equation $\Box U = 0$ (which is necessarily satisfied by the components of $E$ and $B$ in the pure field case), every finite (and smooth enough) initial field configuration is strongly time-unstable [4]: the initial condition blows up radially and goes to infinity (see the next section). Hence, Maxwell’s equations cannot describe finite and time-stable localized fields. The contradictions between theory and experiment that became clear at the end of the last century were a challenge to theoretical physics. Planck and Einstein created the notion of elementary field quanta, named later by Lewis [5] the photon. The concept of photon proved to be very seminal and has been widely used in the 20th century physics. However, even now, after almost a century, we still do not have a complete and satisfactory self-consistent theory of single photons. It worths recalling at
this place Einstein’s remarks of unsatisfaction concerning the linear character of the vacuum 
Maxwell theory which makes it not able to describe the microstructure of radiation [6]. Along 
this line we may also note here some other results and oppinions [7]-[8].

In this paper we consider single photons as (3+1)-dimensional extended finite wave-like objects, or (3+1) solitary waves, moving as a whole in a consistent translational-rotational manner with the speed of light. Their integral characteristics like frequency, period and spin are considered to be intrinsically related to their periodic dynamical structure, and the most important integral quantity characterizing the single photon seems to be its intrinsic elementary action, this being put equal to Planck’s constant $h$. The usually made in quantum theory assumption for a point-like character of photons we consider as inadequate, it can never give a satisfactory explanation of where the characteristic frequency of a free photon comes from. Indeed, the point approximation means structurelessness, so, the frequency may appear only if an external force acts on the point particle, but this means that the particle is not free, hence, its velocity must not be constant and its spin momentum is not an intrinsic characteristic, which contradicts the usual understanding of Planck’s formula $E = h\nu$. In other words, all objects that obey the Planck’s formula $E = h\nu$ do not admit point approximation, hence, the solitary wave view, considered as a first step, seems much more natural. Extended Electrodynamics (EED) [1]-[3] was built mainly to meet this solitary wave view, i.e. to give a consistent field description of single photons, and this paper gives a completely non-relativistic approach to the pure field part of EED. This gives new insights into the problem.

Our assumption that single photons are objects of a finite solitary wave nature has as its mathematical representation the soliton concept. In accordance with this concept the components of the corresponding $E$ and $B$ at every moment $t$ have to be smooth nonsingular functions and different from zero only inside a finite 3-dimensional region $\Omega_t \subset R^3$. Moreover, a $t$-periodic process of constant frequency has to accompany the photon’s translational motion as a whole. Hence, we have to be ready to meet all difficulties coming from the unavoidable requirements for using nonlinear partial differential equations having finite with respect to the spatial variables ($x, y, z$) solutions. This goes along with the Einstein’s view that ”the whole theory must be based on partial differential equations and their singularity-free solutions”[9].

2 The new equations

We recall first Maxwell’s equations in the pure field case:

$$\text{rot} B - \frac{\partial E}{\partial \xi} = 0, \quad \text{div} E = 0,$$

(1)

$$\text{rot} E + \frac{\partial B}{\partial \xi} = 0, \quad \text{div} B = 0,$$

(2)

where $\xi$ denotes the product $ct$, $c$ is the velocity of light in vacuum and $t$ is the time variable. From these equations we get the well known Poynting relation

$$\frac{\partial}{\partial \xi} \left( \frac{E^2 + B^2}{8\pi} \right) = -\frac{1}{4\pi} \text{div} (E \times B).$$

(3)

We explain now why Maxwell’s equations (1)-(2) have no (3+1) soliton-like solutions. As we know from textbooks on Classical Electrodynamics (CED), (e.g. [10]) from (1) and (2) it
follows that every component $U$ of $E$ and $B$ necessarily satisfies the wave equation

$$\Box U \equiv U_{tt} - c^2 [U_{xx} + U_{yy} + U_{zz}] = 0. \tag{4}$$

We are interested in the behavior of $U$ at $t > 0$, if at $t = 0$ the function $U$ satisfies the initial conditions

$$U|_{t=0} = f(x, y, z), \quad \frac{\partial U}{\partial t}|_{t=0} = F(x, y, z),$$

where the functions $f(x, y, z)$ and $F(x, y, z)$ are finite, i.e. they are different from zero in some finite region $\Omega_o \subset \mathbb{R}^3$ and have no singularities. Besides, we assume also that $f$ is continuously differentiable up to third order, and $F$ is continuously differentiable up to the second order. Under these conditions Poisson proved (about 1818) that a unique solution $U(x, y, z; t)$ of the wave equation is defined, and it is expressed by the initial conditions $f$ and $F$ through the following formula (a good explanation is given in [4]):

$$U(x, y, z, t) = \frac{1}{4\pi c} \left\{ \frac{\partial}{\partial t} \left[ \int_{S_{ct}} \frac{f(P)}{r} d\sigma_r \right] + \int_{S_{ct}} \frac{F(P)}{r} d\sigma_r \right\}, \tag{5}$$

where $P$ is a point on the sphere $S$ of radius $r = ct$, centered at the point $(x, y, z)$, and $d\sigma_r$ is the surface element on $S_{r=ct}$.

The above formula (5) shows the following. In order to get the solution at the moment $t > 0$ at the point $(x, y, z)$, being at an arbitrary position with respect to the region $\Omega_o$, where the initial condition, defined by the two functions $f$ and $F$, is concentrated, we have to integrate $f/r$ and $F/r$ over a sphere $S_{r=ct}$, centered at $(x, y, z)$ and having a radius of $r = ct$, and then to form the expression (5). Clearly, the solution will be different from zero at the moment $t > 0$ only if the sphere $S_{r=ct}$ crosses the region $\Omega_o$ at this moment. Consequently, if $r_1 = ct_1$ is the shortest distance from $(x, y, z)$ to $\Omega_o$, and $r_2 = ct_2$ is the longest distance from $(x, y, z)$ to $\Omega_o$, then the solution at $(x, y, z)$ will be different from zero only inside the interval $(t_1, t_2)$.

From the point of view of an external observer this means the following. The initially concentrated in the region $\Omega_o$ perturbation begins to expand, it comes to an arbitrary point $(x, y, z)$ at the moment $t_1 > 0$, makes it "vibrate" (i.e. our devices show the availability of field at this point) during the time interval $\Delta t = t_2 - t_1$, after this the disturbed point goes back to its initial state and our devices find no more field there. Through every point out of $\Omega_o$ there will pass a wave, and its forefront reaches the point $(x, y, z)$ at some moment $t_1$ while its backfront leaves the same point at the moment $t_2 > t_1$. Roughly speaking, the initial condition "blows up radially" and goes to infinity with the velocity of light.

This rigorous mathematical result shows that every finite initial condition induces strongly time-unstable free finite time-dependant solution of Maxwell’s equations in vacuum, so these equations have no finite and smooth enough, i.e. nonsingular, time-dependent solutions, which could be used as models of real photons, as viewed by us.

Hence, if we want to describe 3-dimensional time-dependent soliton-like electromagnetic formations (or configurations) it is necessary to leave off Maxwell’s equations and to look for new equations for $E$ and $B$.

On the other hand we know that Maxwell’s theory is widely used in almost all natural sciences and electrical engineering, so, it does not seem reasonable to leave it off entirely and to look for a completely new theory. Moreover, in all energy computations for finite volumes, it gives very good results. This suggests, at this stage, to look for some extension of the theory, i.e. to extend in a nonlinear way the equations, keeping all solutions to Maxwell’s equations.
as solutions to the new equations and keeping the energy-momentum relations of Maxwell’s theory as relations of the new theory. In the same time we must incorporate new solutions with corresponding to our purpose properties.

The road we are going to follow in searching for the appropriate nonlinearization of (1)-(2) is suggested mainly by two ideas: the idea of local energy-momentum conservation, and the idea of invariance of Maxwell’s equations (1)-(2) with respect to the transformation

$$(E, B) \rightarrow (-B, E).$$

We begin by recalling the second Newton’s law in mechanics: $\dot{p}=F$, its true sense is local momentum balance, i.e. the momentum gained by the particle is lost by the external field. So, if the field is absent: $F = 0$, then the particle will not lose, or gain, momentum, and we get the evolution equation for a free particle: $\dot{p} = m\ddot{v} = 0$. On the other hand, if the particle is absent then the field will not change its momentum and we get the equation $F = 0$. Usually, $F$ depends on the characteristics of the particle, as well as, on the characteristics of the field. In order to make possible the interpretation of the relation $F = 0$ as a pure-field equation we have to express, if possible, this $F$ in terms of the corresponding field functions (and their derivatives) only. Fortunately, this can be done in electrodynamics, Maxwell’s equations in presence of sources make it possible. In this way we shall obtain explicitly in terms of the field functions and their derivatives one of the possible expressions describing how much momentum the field is potentially able to transfer locally to another physical system in case of its presence at the corresponding point and its ability to absorb this momentum, i.e. to interact with the field. Of course, the field may have various such momentum (and corresponding energy) exchanging abilities. We’d like to note at this moment that the following considerations (up to equation (7)) have a suggestive nature only, they do not prove what we are going to assume finally as the new equations.

In order to carry out the above idea we recall first that if the other system consists of charged particles, as it is in Maxwell theory, the corresponding force is the well known Lorentz’ force, acting on a particle of electric charge $e$: $F = eE + \frac{e}{c}(v \times B)$. In case of a continuous distribution of particles with charge density of $\rho$ and current $j = \rho v$ the Lorentz’ force is

$$F = \rho E + \frac{\rho}{c}(v \times B) = \rho E + \frac{1}{c}(j \times B).$$

The corresponding Maxwell’s equations with non-zero charge distribution $\rho$ and current $j = \rho v$ in this case read:

$$\text{rot}B - \frac{\partial E}{\partial \xi} = \frac{4\pi}{c}j, \quad \text{div}E = 4\pi\rho.$$

These last equations make possible to substitute $j$ and $\rho$ into the above given Lorentz’ force. Having this done we put $F = 0$ and then we forget about the character of the ”other system” (charged particles). We interpret now $F$, so obtained and expressed through the field functions and their derivatives only, as a definite quantity of momentum which the field is potentially able to transfer (locally) to any (continuous) physical system that is able to absorb it. This defines quantitatively one of the field’s momentum exchanging abilities.

The above suggestive considerations make us assume our first extended equation:

$$\left(\text{rot}B - \frac{\partial E}{\partial \xi}\right) \times B + E\text{div}E = 0.$$
This vector equation (7) extends Maxwell’s pure field equations (1) in the sense that (7) implies no more (1), i.e. (7) may have solutions which do not satisfy (1) and these new solutions, at least some of them, are considered as *admissible* from physical point of view, i.e. describing some physical reality. The physical sense of (7) is quite clear: *no field momentum is lost in this way*. At the same time relation (7) describes some internal, i.e. between \( \mathbf{E} \) and \( \mathbf{B} \), redistribution of the field energy-momentum during the field’s time evolution. The nonlinearity of (7) is also obvious. We’d like to stress once again the suggestive character of the above considerations, equation (7) is an assumption, we do not consider it as a consequence of Maxwell equations with non-zero current, on the contrary, we consider it as a pure field equation.

Now we look for another momentum exchanging ability of the field, different from the one given by the left-hand side of (7), and directed in general to new physical systems. In order to come to such an expression we make use of the above mentioned \((\mathbf{E}, \mathbf{B}) \rightarrow (-\mathbf{B}, \mathbf{E})\) invariance of the pure field Maxwell’s equations, known as electro-magnetic duality. This invariance is valid also for the energy \( w \) and momentum \( \vec{S} \) (i.e. Poynting’s vector) densities:

\[
 w = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} \rightarrow \frac{\mathbf{B}^2 + \mathbf{E}^2}{8\pi}, \quad \vec{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \rightarrow -\frac{c}{4\pi} \mathbf{B} \times \mathbf{E} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}.
\]

Moreover, the basic energy-momentum balance relation of Poynting (3), is also invariant with respect to this transformation (6). This suggests that, transforming (7), i.e. replacing in (7) \( \mathbf{E} \) by \(-\mathbf{B}\) and \( \mathbf{B} \) by \( \mathbf{E} \), we should obtain also a true and valuable relation, since (7) describes now intra-field local momentum balance. In this way we obtain our second vector equation:

\[
 \left( \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \text{Bdiv} \mathbf{B} = 0. \tag{8}
\]

The left-hand side of (8) defines explicitly another momentum exchanging ability of the field, and relation (8) defines another way of internal field energy-momentum redistribution with time. Note that (8) is obtained from (7) in the same way as (2) is obtained from (1).

We complete this process of extension of Maxwell’s equations by adding two new invariant with respect to the same transformation (6) equations, which also have the physical sense of intra-field local energy-momentum balance:

\[
 \left( \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} + \left( \text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{E} - \mathbf{E} \text{div} \mathbf{B} - \mathbf{B} \text{div} \mathbf{E} = 0, \tag{9}
\]

\[
 \mathbf{B} \left( \text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) - \mathbf{E} \left( \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) = 0. \tag{10}
\]

**Remark:** In the relativistic formulation of EED [3] the two relations (9) and (10) have a natural interpretation of energy-momentum transfers between \( F \) and \( *F \), where \( F \) is the conventional electromagnetic field tensor constructed by means of \( \mathbf{E} \) and \( \mathbf{B} \), and \( *F \) is its dual (constructed similarly by means of \(-\mathbf{B}\) and \( \mathbf{E} \)). This interpretation says that these transfers are mutual: \( F \Leftrightarrow *F \), and always in equal quantities. In other words, the relativistic formalism considers the couples \((\mathbf{E}, \mathbf{B})\) and \((-\mathbf{B}, \mathbf{E})\), or \( F \) and \( *F \), as two components of a new more general mathematical object [2]-[3].

Note that under the transformation (6) equations (7) and (8) transform into each other, while equations (9) and (10) are kept the same (up to a sign of the left-hand side).
Equations (7)-(10) constitute our new system of equations for the electromagnetic field in vacuum. Obviously, they do not introduce new parameters, they are non-linear and all solutions to Maxwell’s vacuum equations (1)-(2) are solutions to the new equations (7)-(10). We are going now to study those new solutions to (7)-(10), which satisfy the conditions

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \neq \mathbf{0}, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \neq \mathbf{0}, \quad \nabla \cdot \mathbf{E} \neq \mathbf{0}, \quad \nabla \cdot \mathbf{B} \neq \mathbf{0}. \quad (11)$$

For further convenience, all solutions to (7)-(10), which satisfy (11), will be called nonlinear.

## 3 Properties of the Nonlinear Solutions

The first two, almost obvious, properties of the nonlinear solutions follow directly from (7) and (8) and are given by the relations

$$\mathbf{E} \cdot \mathbf{B} = \mathbf{0}, \quad \quad \quad \quad (12)$$

$$\mathbf{B} \cdot \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) = \mathbf{0}, \quad \mathbf{E} \cdot \left( \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) = \mathbf{0}. \quad (13)$$

Relation (12) says that the electric and magnetic vectors of every nonlinear solution are necessarily orthogonal to each other at every point, the algebraic property that Maxwell’s (linear) plane wave solution has. From (13) it follows that the Poynting’s relation (3) is true for all nonlinear solutions, and this justifies the usage of all energy-momentum quantities and relations from Maxwell’s theory in the set of nonlinear solutions of the new equations. We can consider the left-hand sides of relations (13) as the energy quantities which the field is potentially able to transfer to some other physical object. We note also the obvious invariance of (12)-(13) with respect to transformation (6).

We are going to show now that all nonlinear solutions satisfy the relation

$$\mathbf{E}^2 = \mathbf{B}^2. \quad (14)$$

In order to prove (14) let’s take the scalar product of equation (8) from the left by \(\mathbf{B}\). We obtain

$$\mathbf{B} \cdot \left\{ \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} \right\} + \mathbf{B}^2 \nabla \cdot \mathbf{B} = \mathbf{0}. \quad (\ast)$$

Now, multiplying (9) from the left by \(\mathbf{E}\) and having in view (12), we obtain

$$\mathbf{E} \cdot \left\{ \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} \right\} - \mathbf{E}^2 \nabla \cdot \mathbf{B} = \mathbf{0}. \quad (**)$$

This last relation is equivalent to

$$-\mathbf{B} \cdot \left\{ \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} \right\} - \mathbf{E}^2 \nabla \cdot \mathbf{B} = \mathbf{0}. \quad (**)$$

Now, summing up (\ast) and (\ast\ast), in view of \(\nabla \cdot \mathbf{B} \neq \mathbf{0}\), we obtain (14).

Relation (14) is also true for the linear plane electromagnetic wave. It requires for all nonlinear solutions a permanent equality of the energy densities carried by the electric and
magnetic fields, although permanent mutual energy-momentum flows run between \( \mathbf{E} \) and \( \mathbf{B} \), which means that these two flows are always in equal quantities.

We introduce now the following two vector fields:

\[
\mathbf{F} = \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} + \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \text{div} \mathbf{B},
\]

\[
\mathbf{M} = \text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} - \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{E} \times \mathbf{B}|} \text{div} \mathbf{E}.
\]

It is obvious that on the solutions of Maxwell’s equations (1)-(2) \( \mathbf{F} \) and \( \mathbf{M} \) are equal to zero. Note also that under the transformation (6) we get \( \mathbf{F} \rightarrow -\mathbf{M} \) and \( \mathbf{M} \rightarrow \mathbf{F} \). We shall show now that on the non-zero nonlinear solutions of our equations (7)-(10) \( \mathbf{F} \) and \( \mathbf{M} \) are colinear to \( \mathbf{E} \) and \( \mathbf{B} \) respectively. Indeed, consider the products \( \mathbf{F} \times \mathbf{E} \) and \( \mathbf{M} \times \mathbf{B} \).

\[
(\mathbf{E} \times \mathbf{B}) \times \mathbf{E} = -\mathbf{E} \times (\mathbf{E} \times \mathbf{B}) = -[\mathbf{E} \cdot (\mathbf{E} \times \mathbf{B})] = \mathbf{B} |\mathbf{E}|^2
\]

and \( |\mathbf{E} \times \mathbf{B}| = \mathbf{E}^2 = \mathbf{B}^2 \), we obtain (see eqn.(8))

\[
\mathbf{F} \times \mathbf{E} = \left( \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{E} + \mathbf{B} \text{div} \mathbf{B} = 0.
\]

In the same way we get \( \mathbf{M} \times \mathbf{B} = 0 \). Hence, we can write the relations

\[
\mathbf{F} = f_1 \mathbf{E}, \quad \mathbf{M} = f_2 \mathbf{B},
\]

where \( f_1 \) and \( f_2 \) are two functions, and further we consider the interesting cases \( f_1 \neq 0, \infty; f_2 \neq 0, \infty \). Note that the physical dimension of \( f_1 \) and \( f_2 \) is the reciprocal to the dimension of coordinates, i.e. \([f_1] = [f_2] = [\text{length}]^{-1}\). Note also, that \( \mathbf{F} \) and \( \mathbf{M} \) are mutually orthogonal.

We shall prove now that \( f_1 = f_2 \). In fact, making use of the same formula for the double vector product, used above, we easily obtain (see eqn.(9))

\[
\mathbf{F} \times \mathbf{B} + \mathbf{M} \times \mathbf{E} = \left( \text{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \xi} \right) \times \mathbf{B} + \left( \text{rot} \mathbf{B} - \frac{\partial \mathbf{E}}{\partial \xi} \right) \times \mathbf{E} - \mathbf{E} \text{div} \mathbf{B} - \mathbf{B} \text{div} \mathbf{E} = 0.
\]

Therefore,

\[
\mathbf{F} \times \mathbf{B} + \mathbf{M} \times \mathbf{E} = f_1 \mathbf{E} \times \mathbf{B} + f_2 \mathbf{B} \times \mathbf{E} = (f_1 - f_2) \mathbf{E} \times \mathbf{B} = 0.
\]

The assertion follows. Now from (14) and (17) it follows also the relation \( |\mathbf{F}| = |\mathbf{M}| \).

**Definition 1.** The quantity

\[
L(\mathbf{E}, \mathbf{B}) = \frac{1}{|f_1|} = \frac{1}{|f_2|} = \frac{\mathbf{E}}{|\mathbf{F}|} = \frac{\mathbf{B}}{|\mathbf{M}|},
\]

will be called the **scale factor** for the nonlinear solution \((\mathbf{E}, \mathbf{B})\).

Obviously, \( L(\mathbf{E}, \mathbf{B}) = L(-\mathbf{B}, \mathbf{E}) \), and for all non-zero nonlinear solutions we have \( 0 < L < \infty \), while for the linear solutions \( L \rightarrow \infty \).
4 Photon-Like Solutions

As we mentioned earlier, we consider photons as finite nonsingular objects moving translationaly along straight lines in the 3-space with the velocity of light. The direction of motion is assumed to be that of \((E \times B)\). This means that the integral curves of \((E \times B)\) have to be straight lines. If we choose the coordinate system \((x, y, z)\) so that this direction of the translational motion to coincide with the direction of the coordinate line \(z\), in this coordinate system the vector fields \(E\) and \(B\) will have non-zero components only along \(x\) and \(y\):

\[
E = (u, p, 0); \quad B = (m, n, 0),
\]

so, \((E \times B) = (0, 0, un - pm)\). Now from \(E.B = 0\) and \(E^2 = B^2\) it follows \(m = \varepsilon p\) and \(n = -\varepsilon u\), \(\varepsilon = \pm 1\). Hence,

\[
E = (u, p, 0); \quad B = (\varepsilon p, -\varepsilon u, 0), \quad E \times B = [0, 0, -\varepsilon(u^2 + p^2)],
\]

and we have to determine just the two functions \(u\) and \(p\).

Let’s substitute these \(E\) and \(B\) into the left hand sides of equations (7)-(10). We obtain:

\[
\left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times E + B \text{div} B = [0, 0, \varepsilon p(p_\xi - \varepsilon p_z) + \varepsilon u(u_\xi - \varepsilon u_z)];
\]

\[
\left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times B + E \text{div} E = [0, 0, \varepsilon u(u_\xi - \varepsilon u_z) + \varepsilon p(p_\xi - \varepsilon p_z)];
\]

\[
\left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) \times B = [\varepsilon u(p_x - u_y), \varepsilon p(p_x - u_y), -u(p_\xi - \varepsilon p_z) + p(u_\xi - \varepsilon u_z)];
\]

\[
\left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) \times E = [\varepsilon p(u_x + p_y), \varepsilon u(u_x + p_y), -p(u_\xi - \varepsilon u_z) + u(p_\xi - \varepsilon p_z)];
\]

\[
-E. \left( \text{rot} E + \frac{\partial B}{\partial \xi} \right) = -\varepsilon u(p_\xi - \varepsilon p_z) + \varepsilon p(u_\xi - \varepsilon u_z);
\]

\[
B. \left( \text{rot} B - \frac{\partial E}{\partial \xi} \right) = -\varepsilon p(u_\xi - \varepsilon u_z) + \varepsilon u(p_\xi - \varepsilon p_z),
\]

where the indices of \(u\) and \(p\) mean the corresponding derivatives.

It is seen that equations (9)-(10) are satisfied identically, and equations (7)-(8) reduce to only one equation, namely

\[
u(u_\xi - \varepsilon u_z) + p(p_\xi - \varepsilon p_z) = \frac{1}{2}[(u^2 + p^2)_\xi - \varepsilon(u^2 + p^2)_z] = 0. \tag{19}\]

The solution to this equation is

\[
u^2 + p^2 = \phi^2(x, y, \xi + \varepsilon z), \tag{20}\]
where $\phi$ is an arbitrary differentiable function of its arguments. This relation means that the energy density
\[
\frac{1}{8\pi}(E^2 + B^2) = \frac{1}{4\pi}(u^2 + p^2)
\]
is a running wave along the coordinate $z$. Hence, our equations determine the field components $u$ and $p$ up to a bounded function $\varphi(x, y, z, \xi)$, $|\varphi| \leq 1$:
\[
u = \phi \varphi, \quad p = \pm \phi \sqrt{1 - \varphi^2}.
\] (21)

Reversely,
\[
\phi = \pm \sqrt{u^2 + p^2}, \quad \varphi = \pm \sqrt{u^2 + p^2}.
\] (22)

The above relations show that instead of $u$ and $p$ we can work with $\phi$ and $\varphi$. Equations (7)-(10) require only $\phi$ to be running wave along $z$ (in this coordinate system), and $\varphi$ to be bounded function. In all other respects these two smooth functions are arbitrary. Hence, they may be chosen finite with respect to the spatial coordinates $(x, y, z)$. Hence, the nonlinear equations (7)-(10) allow $(3+1)$ soliton-like solutions. Note that, since $\varphi$ is bounded, it is sufficient to choose just $\phi$ to be spatially finite.

We are going to show now that the two functions $\phi$ and $\varphi$ have a certain invariant sense and can be introduced in a coordinate free way.

First, let’s denote by $\alpha$ the invariant
\[
\alpha = \frac{1}{\sqrt{\frac{E^2 + B^2}{2}}}
\]
Since $E \cdot B = 0$ and $E^2 = B^2$ we have $\alpha = |E|^{-1} = |B|^{-1}$, and the local frame

\[
\chi = \begin{pmatrix} \alpha E, -\alpha \varepsilon B, -\alpha^2 \varepsilon E \times B \end{pmatrix}
\]
is orthonormal. This frame is defined at every point where the field is different from zero. At every point we have also the frame of unit vectors generated by the coordinate system chosen:

\[
\chi^o = \begin{bmatrix} e_x = \frac{\partial}{\partial x} = (1, 0, 0), \ e_y = \frac{\partial}{\partial y} = (0, 1, 0), \ e_z = \frac{\partial}{\partial z} = (0, 0, 1) \end{bmatrix}
\]

We represent now the frame vectors of $\chi$ through the frame vectors of $\chi^o$, and obtain the matrix
\[
\mathcal{A} = \begin{pmatrix} \frac{u}{\sqrt{u^2 + p^2}}, & -\frac{p}{\sqrt{u^2 + p^2}}, & 0 \\
\frac{p}{\sqrt{u^2 + p^2}}, & \frac{u}{\sqrt{u^2 + p^2}}, & 0 \\
0, & 0, & 1 \end{pmatrix}
\]
This matrix has three invariants: $I_1 = tr \mathcal{A}$, $I_2$=the sum of all principal minors of second order, $I_3 = det \mathcal{A}$. We find
\[
I_1 = I_2 = \frac{2u}{\sqrt{u^2 + p^2}} + 1; \quad I_3 = det \mathcal{A} = 1.
\]
Clearly, $\frac{1}{2}(I_1 - 1) \leq 1$.

Hence, we can define $\phi$ and $\varphi$ through $\alpha$ and these invariants in the following way:
\[
\phi = \pm \sqrt{\alpha^{-2} I_3(\mathcal{A})}, \quad \varphi = \frac{1}{2}(tr \mathcal{A} - 1).
\] (23)
Definition 2. The functions $\phi$ and $\varphi$, defined by (23) will be called *amplitude function* and *phase function* of the corresponding nonlinear solution, respectively. The function $\arccos(\varphi)$ will be called *phase* of the solution.

For $\vec{F}$ and $\vec{M}$ we obtain

$$
\vec{F} = \left[ \varepsilon(p_\xi - \varepsilon p_z), -\varepsilon(u_\xi - \varepsilon u_z), 0 \right],
$$

$$
\vec{M} = \left[ -(u_\xi - \varepsilon u_z), -(p_\xi - \varepsilon p_z), 0 \right].
$$

We shall express the scale factor $L$ through $\phi$ and $\varphi$. We obtain

$$
|\vec{F}| = |\vec{M}| = \frac{|\phi||\varphi_\xi - \varepsilon \varphi_z|}{\sqrt{1 - \varphi^2}}.
$$

Therefore, since $|\mathbf{E}| = |\mathbf{B}| = |\phi|$, the scale factor $L$ is obtained as function of $\varphi$ and its first derivatives only,

$$
L = \frac{|\mathbf{E}|}{|\vec{F}|} = \frac{|\mathbf{B}|}{|\vec{M}|} = \frac{\sqrt{1 - \varphi^2}}{|\varphi_\xi - \varepsilon \varphi_z|}.
$$

(24)

Now we shall separate a subclass of nonlinear solutions, called *almost photon-like*, through the following conditions on $\varphi$ and $L$:

$$
\begin{align*}
& u \frac{\partial \varphi}{\partial x} + p \frac{\partial \varphi}{\partial y} = 0, \\
& p \frac{\partial \varphi}{\partial x} - u \frac{\partial \varphi}{\partial y} = 0, \\
& (u^2 + p^2) \frac{\partial \varphi}{\partial z} = 0; \\
& \frac{\partial L}{\partial \xi} = 0.
\end{align*}
$$

(25)

The *invariant sense* of the first three equations of (25) is that the *phase function* $\varphi$ is a *first integral* of the vector fields $\mathbf{E}, \mathbf{B}, \mathbf{E} \times \mathbf{B}$. From the third equation of (25) is clearly seen that in this coordinate system $\varphi$ does not depend on $z$. The first two equations of (25), considered as an algebraic linear homogeneous system with respect to the two derivatives, yield $\varphi_x = \varphi_y = 0$ because the corresponding determinant is always nonzero: $u^2 + p^2 \neq 0$. Hence, $\varphi$ may depend only on $\xi$. In view of (24) and $\varphi_z = 0$ the fourth equation $\frac{\partial L}{\partial \xi} = 0$ means $L = constant$. Hence, relation (24) turns into equation for $\varphi$:

$$
L = \frac{\sqrt{1 - \varphi^2}}{|\varphi_\xi|} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial \xi} = \pm \frac{1}{L} \sqrt{1 - \varphi^2}.
$$

(26)

The obvious solution is (both signs on the right are admissible):

$$
\varphi = \cos \left( \kappa \frac{\xi}{L} + \beta_o \right) = \cos \left( \kappa \frac{c}{L} t + \beta_o \right),
$$

(27)

where $\kappa = \pm 1$, $\beta_o = \text{const}$. Since $\varphi$ is a periodic function with respect to $t$, then $c/L$ has the physical interpretation of *frequency*, and this frequency has nothing to do with the concept of frequency in classical electrodynamics since it is defined by $L$, and $L$ is not defined in Maxwell’s theory.

It is clearly seen the consistent translational-rotational behavior of the solution obtained: the electric and magnetic vectors

$$
\mathbf{E} = \left[ \phi \cos \left( \kappa \frac{t}{T} + \beta_o \right), \phi \sin \left( \kappa \frac{t}{T} + \beta_o \right), 0 \right],
$$

11
\[ \mathbf{B} = \left[ \varepsilon \phi \sin \left( \frac{\kappa t}{T} + \beta_0 \right), -\varepsilon \phi \cos \left( \frac{\kappa t}{T} + \beta_0 \right), 0 \right] \]

run along \( z \): \( \phi = \phi(x, y, \xi \pm z) \), and rotate (left or right: \( \kappa = \pm 1 \)) with the frequency \( \nu = (c/L) = 1/T \).

In order to separate the photon-like solutions we recall that the photon’s characteristic quantity is its integral intrinsic angular momentum, or spin, being equal to the Planck’s constant \( h \). Namely \( h \) represents quantitatively in a unified manner the rotational and translational aspects of its dynamical nature: for all photons the product \( WT \) has the same value \( h \), although \( W \) and \( T \) may be different for the different photons. That’s why every photon should be able to determine its own scale factor \( L = \text{const} \) in order to have a cosine periodic phase function and to obey the Planck’s law: \( h = WT = WL/c \). The photon’s intrinsic periodic process demonstrates itself in our approach through the (left or right) rotation of the pair \((\mathbf{E}, \mathbf{B})\).

Since these two vectors are orthogonal to each other and with equal modules: \( |\mathbf{E}| = |\mathbf{B}| \), the basic local quantity appears to be the area of the square defined by \( \mathbf{E} \) and \( \mathbf{B} \) at every point, and this area is equal to \( |\mathbf{E} \times \mathbf{B}| \). During one period \( T \) this square performs a full rotation around the direction of propagation and this gives the local action \( |\mathbf{E} \times \mathbf{B}| \cdot T \). In order to obtain the integral \( T \)-action of the solution we have to sum up all these local actions.

The above described idea is easily represented mathematically. In fact, let \( u, p \) and \( u^2 + p^2 \) be spatially finite functions. Then the integral energy

\[ W = \frac{1}{4\pi} \int_{\mathbb{R}^3} (u^2 + p^2) dxdydz < \infty \]

is finite. For every almost photon-like solution we define the local spin vector \( \mathbf{S} \) by

\[ \mathbf{S} = L^2 \frac{\mathbf{\tilde{F}} \times \mathbf{\tilde{M}}}{4\pi} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi}, \quad L = \text{const}, \quad 0 < L < \infty. \quad (28) \]

Now, the integral intrinsic action, or integral spin \( \mathcal{S} \) of the solution, is defined by

\[ \mathcal{S} = \int_{[0,T]} \int_{\mathbb{R}^3} |\mathbf{S}| dxdydzdt. \quad (29) \]

We obtain

\[ \mathcal{S} = WT. \quad (30) \]

We note once again that this approach works because \( W = \text{const} < \infty, \quad 0 < L = \text{const} < \infty \) and the solution is soliton-like, i.e. it is finite, it has periodic dynamical structure and is time-stable. Clearly, no solution of Maxwell’s equations (1)-(2) in the whole space has all these properties.

**Definition 3.** A nonlinear solution will be called photon-like if it is spatially finite, if it satisfies conditions (25) and if its integral spin \( \mathcal{S} \) is equal to the Planck constant \( h \): \( \mathcal{S} = h \).

Finally, we consider briefly the problem of interference of photon-like solutions: if we have two photon-like solutions

\[ \mathbf{E}_1 = \left[ \phi_1(x, y, \xi + \varepsilon_1 z) \cos \left( \frac{\kappa_1 \xi}{L_1} + \beta_1 \right), \phi_1(x, y, \xi + \varepsilon_1 z) \sin \left( \frac{\kappa_1 \xi}{L_1} + \beta_1 \right), 0 \right], \]

we ask: under what conditions their sum \( (E_1 + E_2, B_1 + B_2) \) will be again a nonlinear solution? Having done the corresponding elementary computations, we come to the following important conclusion: if

\[
\begin{align*}
\varepsilon_1 &= \varepsilon_2, \\
\kappa_1 &= \kappa_2, \\
L_1 &= L_2
\end{align*}
\tag{31}
\]

then the sum \( (E_1 + E_2, B_1 + B_2) \) is again a nonlinear solution.

These relations \( (31) \) coincide with the well known from CED coherence conditions. They say that the two photon-like solutions will interfere, i.e. their sum will be again a solution, if:

1. They propagate along the same direction: \( \varepsilon_1 = \varepsilon_2 \),
2. They have the same polarization: \( \kappa_1 = \kappa_2 \),
3. They have the same frequency: \( \nu_1 = \nu_2 \), i.e. the same scale factors.

Recall that in CED these coherence conditions do not follow directly from the theory as necessary conditions, e.g., condition 3 requires some time averaging. Moreover, CED is a linear theory and the sum of any two or more solutions is again a solution, for example, the sum “plane wave + a spherically symmetric field” is again a solution but no interference features are available.

In EED, which is a non-linear theory and there is no superposition principle in general, the interference of photon-like solutions is a very special case and it is a remarkable result that the experimentally found coherence conditions \( (31) \) appear from the nonlinear equations as necessary conditions, otherwise the sum will not be a solution.

The computation shows that the energy density \( w \) of the sum-solution is given by

\[
w = \frac{1}{4\pi} \left[ (\phi_1)^2 + (\phi_2)^2 + 2\phi_1\phi_2\cos(\beta_1 - \beta_2) \right],
\tag{32}
\]

and this relation \( (32) \) allows to talk about interference instead of superposition: in our approach the allowed superposition of photon-like solutions leads always to interference.

Of course, from the non-linear point of view, these interference phenomena are of some interest only for those soliton-like solutions which at a given moment occupy intersecting regions, otherwise the interference term \( 2\phi_1\phi_2\cos(\beta_1 - \beta_2) \) in \( (32) \) is equal to zero. Since the two summands follow the same direction of motion as a whole, then the sum-solution will be a time-stable solution, but it will not be photon-like one.

5 Conclusion

This paper presents a non-relativistic formulation of an extension of the pure field Maxwell equations. The main purpose of the extension is to give a mathematical description of the viewpoint that photons are extended but finite objects and that their existence is based on a
joint and consistent translational-rotational internal dynamics: a straight-line (translational) motion as a whole with the velocity of light and a rotation of the mutually orthogonal and perpendicular to the translation electric and magnetic vectors. The general mathematical concept of \((3 + 1)\)-solitary wave, or soliton, was found to be the most adequate one to this physical notion. Our nonlinear equations (7)-(10) for the couple of vector fields \((\mathbf{E}, \mathbf{B})\) realize directly the idea for local energy-momentum conservation, i.e. the idea that the free field dynamics (or time evolution) is determined by a definite and permanent intra-field local energy-momentum redistribution. Compare to Maxwell’s approach, this is a new look on the field’s evolution defining equations, it carries the idea of the second Newton’s law in mechanics to continuously distributed in the 3-space physical systems (electromagnetic fields). It is also a new moment that the field has more than one potential abilities to exchange energy-momentum with other physical systems. This nonlinear approach turned out to be successful in view of the existence of appropriate photon-like solutions. Every photon-like solution has finite integral energy \(W\), has its own scale factor \(L = \text{const}\), phase function \(\varphi\) of cosine type and corresponding to \(L\) and \(\varphi\) frequency \(\nu = c/L\), or period \(T = L/c\). Every photon-like solution carries intrinsic angular momentum of integral value equal to the Planck’s constant \(h = WT\), which is the famous Planck’s formula. The equations (7)-(10) and the additional conditions (25) do not determine the spatial structure of the solution and it should be so, because it is hardly believable that all photons must have the same shape, structure and extension. Moreover, if the amplitude function \(\phi\) consists of many non-overlapping appropriate bumps, we obtain a mathematical image of a number of coherent photons, i.e. a many-bump photon-like solution, and the corresponding integral spin momentum will be \(nh\), where \(n\) gives the number of bumps. Clearly, large enough parts of the classical plane wave can be considered as macro-pictures of such a many-bump photon-like solution. Nonlinear solutions with intrinsic angular momentum different from \(h\), or \(nh\), where \(n = 1, 2, \ldots\) are also, in principle, allowed. The remarkable limited superposition principle permits interference in the frame of photon-like solutions only if they are coherent in the sense of (31).

The existence of localized photon-like solutions in the pure field case suggests to make the corresponding extension of Maxwell’s equations in presence of external fields (media) and to look for \((3+1)\)-localized solutions with non-zero proper mass. Such an extension in relativistic terms was made and published [2], where a large family of variously shaped \((3 + 1)\) soliton-like solutions with non-zero mass and well defined conserved quantities is also given.

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REFERENCES

[1]. S.Donev, M.Tashkova, Proc.R.Soc.of Lond., A 443, (301), 1993.

[2]. S.Donev, M.Tashkova, Proc.R.Soc. of Lond., A 450, (281), 1995.

[3]. S.Donev, M.Tashkova, Annales de la Fondation Louis de Broglie, vol.23, No.No.2,3 (1998).

[4]. S.J.Farlow, "Partial Differential Equations for Scientists and Engineers", John Wiley & Sons, Inc., 1982.

[5]. G.N.Lewis, Nature 118, 874, 1926.

[6]. A.Einstein, Sobranie Nauchnih Trudov, vols.2,3, Nauka, Moskva, 1966.

[7]. M.Planck, J.Franklin Institute, 1927 (July), p.13.

[8]. J.J.Thomson, Philos.Mag.Ser. 6, 48, 737 (1924), and 50, 1181 (1925), and Nature, vol.137, 23 (1936); N.Rashevsky, Philos.Mag. Ser.7, 4, 459 (1927); W.Honig, Found.Phys. 4, 367 (1974); G.Hunter, R.Wadlinger, Phys.Essays, vol.2,158 (1989).

[9]. A.Einstein, J.Franklin Institute, 221 (349-382), 1936.

[10]. J.D.Jackson, CLASSICAL ELECTRODYNAMICS, John Wiley and Sons, Inc., New York-London, 1962.