On the generalized porous medium equation in Fourier-Besov spaces

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Abstract  We study a kind of generalized porous medium equation with fractional Laplacian and abstract pressure term. For a large class of equations corresponding to the form: $u_t + \nu \Lambda^\alpha u = \nabla \cdot (u \nabla P u)$, we get their local well-posedness in Fourier-Besov spaces for large initial data. If the initial data is small, then the solution becomes global. Furthermore, we prove a blowup criterion for the solutions.

Keywords  porous medium equation, well-posedness, blowup criterion, Fourier-Besov spaces

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1  Introduction

In this paper, we study the nonlinear nonlocal equation in $\mathbb{R}^n$ of the form

$$\begin{cases}
    u_t + \nu \Lambda^\alpha u = \nabla \cdot (u \nabla P u); \\
    u(0, x) = u_0.
\end{cases} \tag{1.1}$$

Usually, $u = u(t, x)$ is a real-valued function, represents a density or concentration. The dissipative coefficient $\nu > 0$ corresponds to the viscid case, while $\nu = 0$ corresponds to the inviscid case. In this paper we study the viscid case and take $\nu = 1$ for simplicity. The fractional operator $\Lambda^\alpha$ is defined by Fourier transform as $(\Lambda^\alpha u)^\wedge = |\xi|^\alpha \hat{u}$. $P$ is an abstract operator.

Equation (1.1) here comes from the same proceeding with that of the fractional porous medium equation (FPME) introduced by Caffarelli and Vázquez [5]. In fact, equation (1.1) comes into being by adding the fractional dissipative term $\nu \Lambda^\alpha u$ to the continuity equation $u_t + \nabla \cdot (u V) = 0$, where the velocity $V = -\nabla p$ and the velocity potential or pressure $p$ is related to $u$ by an abstract operator $p = Pu$.

The abstract form pressure term $P u$ gives a good suitability in many cases. The simplest case comes from a model in groundwater in filtration [1, 19]: $u_t = \Delta u^2$, that is: $\nu = 0, Pu =$

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u. A more general case appears in the fractional porous medium equation [5] when \( \nu = 0 \) and \( Pu = \Lambda^{-2s}u, 0 < s < 1 \). In the critical case when \( s = 1 \), it is the mean field equation first studied by Lin and Zhang [15]. Some studies on the well-posedness and regularity on those equations we refer to [1, 6, 7, 17, 18, 20, 23] and the references therein.

In the FPME, the pressure can also be represented by Riesz potential as \( Pu = \Lambda^{-2s}u = \mathcal{K} * u \), with kernel \( \mathcal{K} = c_n,s |y|^{2s-n} \). Replacing the kernel \( \mathcal{K} \) by other functions in this form: 

\[
P_u = \mathcal{K} * u,
\]

equation (1.1) also appears in granular flow and biological swarming, named aggregation equation. The typical kernels are the Newton potential \(|x|^\gamma\) and the exponent potential \(-e^{-|x|}\).

One of concerned problems on this equation is the singularity of the potential \(Pu\) which holds the well-posedness or leads to the blowup solution. Generally, smooth kernels at origin \( x = 0 \) lead to the global in time solution [3], meanwhile nonsmooth kernels may lead to blowup phenomenon [14]. Li and Rodrigo [13, 14] studied the well-posedness and blowup criterion of equation (1.1) with the pressure \( Pu = \mathcal{K} * u \), where \( \mathcal{K}(x) = e^{-|x|} \) in Sobolev spaces. Wu and Zhang [21] generalize their work to require \( \nabla \mathcal{K} \in W^{1,1} \) which includes the case \( \mathcal{K}(x) = e^{-|x|} \). They take advantage of the controllability in Besov spaces of the convolution \( \mathcal{K} * u \) under this condition, as well as the controllability of its gradient \( \nabla \mathcal{K} * u \).

In this article we study the well-posedness and blowup criterion of equation (1.1) in Fourier-Besov spaces under an abstract pressure condition

\[
\|\Delta_k(\nabla Pu)\|_{L^p} \leq C 2^{k\sigma} \|\Delta_k u\|_{L^p}.
\]

(1.2)

In Fourier-Besov spaces, it is the localization express of the norm estimate

\[
\|\nabla Pu\|_{F_{p,q}^{s-\beta}} \leq C \|u\|_{F_{p,q}^{s-\beta}}.
\]

(1.3)

Corresponding to the FPME, i.e. \( Pu = \Lambda^{-2s} \), we get \( \sigma = 1 - 2s \) obviously. And if \( Pu = \mathcal{K} * u \), \( \mathcal{K} \in W^{1,1} \) in the aggregation equation, we get \( \sigma = 1 \) when \( \mathcal{K} \in L^1 \) and \( \sigma = 0 \) when \( \nabla \mathcal{K} \in L^1 \).

The Fourier-Besov spaces we use here come from Konieczny and Yoneda [11] when deal with the Navier-Stokes equation (NSE) with Coriolis force. Besides, Fourier-Besov spaces have been widely used to study the well-posedness, singularity, self-similar solution, etc. of Fluid Dynamics in various of forms. For instance, the early pseudomeasure spaces \( PM^\alpha \) in which Cannone and Karch studied the smooth and singular properties of Navier-Stokes equations [8]. The Lei-Lin spaces \( \mathcal{X}^\sigma \) deal with global solutions to the NSE [12] and to the quasi-geostrophic equations (QGE) [2]. The Fourier-Herz spaces \( B_q^\sigma \) in the Keller-Segel system [9], in the NSE with Coriolis force [10] and in the magneto-hydrodynamic equations (MHD) [16].

In the case of Besov spaces, we gain some well-posedness and blow-up results of equation (1.1) under an corresponding condition to (1.3) in [24]. Due to the difficulty in deal with the nonlinear term \( \nabla \cdot (u \nabla Pu) \), in that case we need a little strict initial condition: \( u_0 \in B_{p,1}^{n/p+\sigma-\beta} \cap B_{p,1}^{n/p+\sigma-\beta+1} \). However, in this paper, we find that Fourier-Besov spaces are
powerful in dealing with the nonlinear term, by a very different method used in [24], we prove the following theorems:

**Theorem 1.1.** Let \( p, q \in [1, \infty] \), \( \max\{1, \sigma + 1\} < \alpha < n(1 - \frac{1}{p}) + \sigma + 2 \). Then for any \( u_0 \in FB_{p,q}^{n(1-\frac{1}{p})-\alpha+\sigma+1} \), equation (1.1) admits a unique mild solution \( u \) and

\[
\begin{align*}
  u &\in C([0,T); FB_{p,q}^{n(1-\frac{1}{p})-\alpha+\sigma+1} ) \cap \tilde{L}^1([0,T); FB_{p,q}^{n(1-\frac{1}{p})+\sigma+1}).
\end{align*}
\]

Moreover, there exists a constant \( C_0 = C_0(\alpha, p, q) \) such that for \( u_0 \) satisfying \( \|u_0\|_{FB_{p,q}^{n(1-\frac{1}{p})-\alpha+\sigma+1}} \leq C_0 \), the solution \( u \) is global, and

\[
\begin{align*}
  \|u\|_{L^\infty_T(FB_{p,q}^{\beta+\alpha})} + \|u\|_{\tilde{L}^1_T(FB_{p,q}^{\beta+\sigma+1})} &\leq 2C \|u_0\|_{FB_{p,q}^{n(1-\frac{1}{p})-\alpha+\sigma+1}}.
\end{align*}
\]

**Theorem 1.2.** Let \( T^* \) denote the maximal time of existence of \( u \) in \( L^\infty_T(FB_{p,q}^{\beta}) \cap \tilde{L}^1_T(FB_{p,q}^{\beta+\alpha}) \). Here \( \beta = n(1 - \frac{1}{p}) - \alpha + \sigma + 1 \). If \( T^* < \infty \), then

\[
\|u\|_{\tilde{L}^1([0,T^*); FB_{p,q}^{\beta+\alpha})} = \infty.
\]

### 2 Preliminaries

Let us introduce some basic knowledge on Littlewood-Paley theory and Fourier-Besov spaces. Let \( \varphi \in C^\infty_c(\mathbb{R}^n) \) be a radial positive function such that

\[
\text{supp} \ \varphi \subset \{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \neq 0.
\]

Define the frequency localization operators as follows:

\[
\begin{align*}
  \Delta_j u &= \varphi_j(D)u; \quad S_j u = \psi_j(D)u,
\end{align*}
\]

here \( \varphi_j(\xi) = \varphi(2^{-j}\xi) \) and \( \psi_j = \sum_{k \leq j-1} \varphi_j \).

By Bony’s decomposition we can split the product \( uv \) into three parts:

\[
uv = T_u v + T_v u + R(u, v),
\]

with

\[
T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u, v) = \sum_j \Delta_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j v = \Delta_{j-1} v + \Delta_j v + \Delta_{j+1} v.
\]

Let us now define the Fourier-Besov space as follows.
Definition 2.1. For $\beta \in \mathbb{R}, p, q \in [1, \infty]$, we define the Fourier-Besov space $\dot{F}B^\beta_{p,q}$ as

$$\dot{F}B^\beta_{p,q} = \{ f \in \mathcal{S}'/\mathbb{P} : \| f \|_{\dot{F}B^\beta_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{j\beta q} \| \varphi_j \hat{f} \|_{L^p}^q \right)^{1/q} < \infty \}.$$ 

Here the norm changes normally when $p = \infty$ or $q = \infty$, and $\mathbb{P}$ is the set of all polynomials.

Definition 2.2. In this paper, we need two kinds of mixed time-space norm defined as follows: For $s \in \mathbb{R}, 1 \leq p, q \leq \infty, I = [0, T), T \in (0, \infty]$, and let $X$ be a Banach space with norm $\| \cdot \|_X$,

$$\| f(t, x) \|_{L^r(I; X)} := \left( \int_I \| f(t, \cdot) \|_X \, dt \right)^{\frac{1}{r}},$$

$$\| f(t, x) \|_{\dot{L}^r(I; \dot{F}B^\beta_{p,q})} := \left( \sum_{j \in \mathbb{Z}} 2^{j\beta q} \| \varphi_j \hat{f} \|_{L^r(I; L^p)}^q \right)^{\frac{1}{q}}.$$ 

By Minkowski' inequality, there holds

$$L^r(I; \dot{F}B^\beta_{p,q}) \hookrightarrow \tilde{L}(I; \dot{F}B^\beta_{p,q}), \text{ if } r \leq q \text{ and } \tilde{L}^r(I; \dot{F}B^\beta_{p,q}) \hookrightarrow L^r(I; \dot{F}B^\beta_{p,q}), \text{ if } r \geq q. \ (2.1)$$

Lemma 2.1. Let $X$ be a Banach space with norm $\| \cdot \|_X$ and $B : X \times X \to X$ be a bounded bilinear operator satisfying

$$\| B(u, v) \|_X \leq \eta \| u \|_X \| v \|_X,$$

for all $u, v \in X$ and a constant $\eta > 0$. Then for any fixed $y \in X$ satisfying $\| y \|_X < \epsilon < \frac{1}{4\eta}$, the equation $x := y + B(x, x)$ has a solution $\bar{x}$ in $X$ such that $\| \bar{x} \|_X \leq 2\| y \|_X$. Also, the solution is unique in $\tilde{B}(0, 2\epsilon)$. Moreover, the solution depends continuously on $y$ in the sense: if $\| y' \|_X < \epsilon, x' = y' + B(x', x'), \| x' \|_X < 2\epsilon$, then

$$\| \bar{x} - x' \|_X \leq \frac{1}{1 - 4\epsilon\eta} \| y - y' \|_X.$$

Lemma 2.2. [21] If $\frac{1}{r} = \frac{1 - \theta}{r_1} + \frac{\theta}{r_2}$, then

$$\| u \|_{L^r_{\tau}(FB^{\beta+\theta\alpha}_p, \dot{F}B^\beta_{p,q})} \leq \| u \|^{1-\theta}_{L^1_{\tau}(FB^\beta_{p,q})} \| u \|^\theta_{L^2_{\tau}(FB^{\beta+\alpha}_p)}.$$

Now we prove a priori estimate which will be used in our proof. Consider the following dissipative equation:

$$\partial_t u + \Lambda^\alpha u = f(t, x), \quad u(0, x) = u_0(x), \quad t > 0, x \in \mathbb{R}^n. \quad \text{(2.2)}$$

Lemma 2.3. Let $0 < T < \infty$, $\beta \in \mathbb{R}$ and $1 \leq r \leq \infty$. Assume $u_0 \in \dot{F}B^\beta_{p,q}$, $f \in \tilde{L}^1(I; \dot{F}B^\beta_{p,q})$. Then the solution $u(t, x)$ to (2.2) satisfies

$$\| u \|_{\tilde{L}^r(I; \dot{F}B^{\beta+\alpha}_p)} \leq C(\| u_0 \|_{\dot{F}B^\beta_{p,q}} + \| f \|_{\tilde{L}^1(I; \dot{F}B^\beta_{p,q})}) \quad \text{(2.3)}$$
Proposition 3.1. The following estimate holds:

For the linear part,

\[ \| \varphi_j \mathcal{F}(Lu_0) \|_{L^p} \leq e^{-t(3/4)\alpha} \| \varphi_j \hat{u}_0 \|_{L^p}. \]

Hence

\[ \| Lu_0 \|_{L^p_t(FB^\beta_{p,q} \mathbb{R}^n)} \leq 2^{\frac{\alpha}{\gamma+\frac{\alpha}{2}}} e^{-t\frac{3}{4}\alpha} \| \varphi_j \hat{u}_0 \|_{L^p} \| u_0 \|_{FB^\beta_{p,q}}. \]

On the other hand, for the integral part,

\[ \| \varphi_j \mathcal{F}(Gf) \|_{L^p} \leq \int_0^t e^{-t(t-\tau)\frac{3}{4}\alpha} \| \varphi_j \hat{f} \|_{L^p} d\tau. \]

Taking \( L^r \)-norm with respect to time, by Minkowski’s inequality

\[ \| \varphi_j \mathcal{F}(Gf) \|_{L^r_t(L^p)} \leq C 2^{-\frac{\alpha}{4}} \| \varphi_j \hat{f} \|_{L^1_t(L^p)}. \]

Hence

\[ \| Gf \|_{L^r_t(FB^\beta_{p,q} \mathbb{R}^n)} = 2^{\frac{\alpha}{\gamma+\frac{\alpha}{2}}} \| \varphi_j \mathcal{F}(Gf) \|_{L^r_t(L^p)} \| f \|_{L^1_t(FB^\beta_{p,q} \mathbb{R}^n)} \leq C \| f \|_{L^1_t(FB^\beta_{p,q} \mathbb{R}^n)}. \]

Combine the above estimates, we obtain our desire inequality.

\[ \square \]

3 Local and global well-posedness

In this section we prove our main Theorem. First we know that the integral form of \( u \) is as follows

\[ u(t, x) = e^{-t\Lambda^\alpha} u_0 + \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot (u(\tau) \nabla P u(\tau)) d\tau := S(t)u_0 + H(u,u). \]

Here \( S(t)u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha} \hat{u}_0) \), and \( H(u,v) = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot (u(\tau) \nabla P v(\tau)) d\tau. \) Now we get the following estimate

**Proposition 3.1.** Let \( \gamma, p, q \geq 1, \epsilon > \max\{0, -\sigma\}, \beta > 0, \frac{1}{\gamma} = \frac{1}{\eta_1} + \frac{1}{\eta_2} \), there holds

\[
\| u \partial_t P v \|_{L^1_t(FB^\beta_{p,q} \mathbb{R}^n)} \leq C \| u \|_{L^1_t(FB^{\eta_1(1-\frac{1}{2})-\epsilon}_{p,q} \mathbb{R}^n)} \| v \|_{L^1_t(FB^{\eta_2(1-\frac{1}{2})-\epsilon}_{p,q} \mathbb{R}^n)} \left( \| u \|_{L^1_t(FB^{\beta+\sigma+\epsilon}_{p,q} \mathbb{R}^n)} + C \| v \|_{L^1_t(FB^{\eta_2(1-\frac{1}{2})-\epsilon}_{p,q} \mathbb{R}^n)} \right),
\]

where

\[
S(t)u_0 = \mathcal{F}^{-1}(e^{-t|\xi|^\alpha} \hat{u}_0), \quad H(u,v) = \int_0^t e^{-(t-\tau)\Lambda^\alpha} \nabla \cdot (u(\tau) \nabla P v(\tau)) d\tau.
\]
Proof. By the Bony’s decomposition, it is easy to get that

$$
\Delta_j(u \partial_t P v) = \sum_{|k-j| \leq 4} \Delta_j(S_{k-1} u \Delta_k(\partial_t P v)) + \sum_{|k-j| \leq 4} \Delta_j(S_{k-1}(\partial_t P v) \Delta_k u) + \sum_{k \geq j-2} \Delta_j(\Delta_k u \tilde{\Delta}_k(\partial_t P v)) \tag{3.2}
$$

We can get the following estimates:

$$
2^{j \beta} \left\| \sum_{|k-j| \leq 4} \varphi_j \mathcal{F}(S_{k-1} u \Delta_k(\partial_t P v)) \right\|_{L^1_t(L^p)} \leq C \sum_{|k-j| \leq 4} \left\| \mathcal{F}(S_{k-1} u) \ast \varphi_k \mathcal{F}(\partial_t P v) \right\|_{L^1_t(L^p)} \leq 2^{j \beta} \left\| \mathcal{F}(S_{k-1} u) \ast \varphi_k \mathcal{F}(\partial_t P v) \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \sum_{|k-j| \leq 4} \left\| \mathcal{F}(S_{k-1} u) \ast \varphi_k \mathcal{F}(\partial_t P v) \right\|_{L^1_t(L^p)} \leq C \left\| u \right\|_{L^{\gamma}_t(F B^{n(1-\frac{1}{p})}_{p,q} - \epsilon)} 2^{j(\beta+\sigma+\epsilon)} \left\| \varphi_j \tilde{v} \right\|_{L^{\gamma}_t(L^p)}.
$$

Similarly, we can estimate

$$
2^{j \beta} \left\| \sum_{|k-j| \leq 4} \varphi_j \mathcal{F}(S_{k-1}(\partial_t P v) \Delta_k u) \right\|_{L^1_t(L^p)} \leq C \left\| v \right\|_{L^{\gamma}_t(F B^{n(1-\frac{1}{p})}_{p,q} - \epsilon)} 2^{j(\beta+\sigma+\epsilon)} \left\| \varphi_k \tilde{v} \right\|_{L^{\gamma}_t(L^p)}.
$$

And when $\beta > 0$, we can also get that for some $\left\| a_j \right\|_{\nu} = 1$,

$$
2^{j \beta} \left\| \mathcal{F}(\sum_{k \geq j-2} \Delta_j(\Delta_k(\partial_t P v) \tilde{\Delta}_k u)) \right\|_{L^1_t(L^p)} \leq 2^{j \beta} \left\| \sum_{k \geq j-2} \varphi_j \mathcal{F}(\Delta_k(\partial_t P v)) \ast (\tilde{\varphi}_k \tilde{u}) \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \left\| \sum_{k \geq j-2} \varphi_j \mathcal{F}(\Delta_k(\partial_t P v)) \ast (\tilde{\varphi}_k \tilde{u}) \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \left\| \sum_{k \geq j-2} 2^{kn(1-\frac{1}{p})+k\sigma} \left\| \varphi_k \tilde{v} \right\|_{L^p} \left\| \tilde{\varphi}_k \tilde{u} \right\|_{L^p} \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \left\| \sum_{k \geq j-2} \varphi_j \mathcal{F}(\Delta_k(\partial_t P v)) \ast (\tilde{\varphi}_k \tilde{u}) \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \left\| \sum_{k \geq j-2} \varphi_j \mathcal{F}(\Delta_k(\partial_t P v)) \ast (\tilde{\varphi}_k \tilde{u}) \right\|_{L^1_t(L^p)} \leq C 2^{j \beta} \left\| \sum_{k \geq j-2} \varphi_j \mathcal{F}(\Delta_k(\partial_t P v)) \ast (\tilde{\varphi}_k \tilde{u}) \right\|_{L^1_t(L^p)} \leq C \left\| a_j \right\|_{\nu} \left\| v \right\|_{L^{\gamma}_t(F B^{n(1-\frac{1}{p})}_{p,q} - \epsilon)} \left\| u \right\|_{L^{\gamma}_t(F B^{\beta+\sigma+\epsilon}_{p,q} - \epsilon)}.
$$
Combine the above estimate, we obtain the following inequality
\[
\|u\partial_t Pu\|_{L_t^\infty(FB^{\beta}_{p,q})} \leq C\|u\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})-\epsilon}_{p,q})} \|v\|_{L_t^\infty(FB^{\beta+\epsilon}_{p,q})}
\]
\[
+ C\|v\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})-\epsilon}_{p,q})} \|u\|_{L_t^\infty(FB^{\beta+\epsilon}_{p,q})}.
\]

\[\square\]

**Proposition 3.2.** Let \( p, q \geq 1, \epsilon > \max\{0, -\sigma\}, \beta > -1 \). If \( u, v \in \bar{L}_T^2(FB^{n(1-\frac{1}{p})-\epsilon}_{p,q}) \cap \bar{L}_T^2(FB^{\beta+\epsilon}_{p,q}) \), we have
\[
\|H(u, v)\|_{L_t^\infty(FB^{\beta}_{p,q})} + \|H(u, v)\|_{L_t^\infty(FB^{\beta+\alpha}_{p,q})} \leq C\|u\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})-\epsilon}_{p,q})} \|v\|_{L_t^{\alpha}(FB^{\beta+\epsilon+1}_{p,q})} + C\|v\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})-\epsilon}_{p,q})} \|u\|_{L_t^{\alpha}(FB^{\beta+\epsilon+1}_{p,q})}.
\]

**Proof.** We note that \( H(u, v) \) is a solution to equation (2.2) with \( u_0 = 0, f = \nabla \cdot (u\nabla P v) \). So by Lemma 2.3 there holds
\[
\|H(u, v)\|_{L_t^\infty(FB^{\beta}_{p,q})} + \|H(u, v)\|_{L_t^\infty(FB^{\beta+\alpha}_{p,q})} \leq C\|\nabla \cdot (u\nabla P v)\|_{L_t^\infty(FB^{\beta}_{p,q})},
\]
Then Proposition 3.1 gives the estimate. \[\square\]

**Theorem 3.1.** Let \( p, q \in [1, \infty], 2\max\{1, \sigma + 1\} < \alpha < n(1-\frac{1}{p}) + \sigma + 2 \). Then for any 
\[u_0 \in FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q}, \text{ equation (1.1) admits a unique mild solution } u \text{ and}
\]
\[u \in C([0, T); FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q}) \cap L^1([0, T); FB^{n(1-\frac{1}{p})+\sigma+1}_{p,q}).
\]

Moreover, there exists a constant \( C_0 = C_0(\alpha, p, q) \) such that for \( u_0 \) satisfying \( \|u_0\|_{FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q}} \leq C_0 \), the solution \( u \) is global, and
\[
\|u\|_{L_t^\infty(FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q})} + \|u\|_{L_t^\infty(FB^{n(1-\frac{1}{p})+\sigma+1}_{p,q})} \leq 2C\|u_0\|_{FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q}}.
\]

**Proof.** First suppose \( t \in [0, T], T \) fixed. Let \( \epsilon = \frac{n}{2} - \sigma - 1, \beta = n(1-\frac{1}{p}) - \alpha + \sigma + 1 \), by the above proposition
\[
\|H(u, v)\|_{L_t^\infty(FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q})} + \|H(u, v)\|_{L_t^\infty(FB^{n(1-\frac{1}{p})+\sigma+1}_{p,q})} \leq C\|u\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})-\alpha+\sigma+1}_{p,q})} \|v\|_{L_t^{\alpha}(FB^{n(1-\frac{1}{p})+\sigma+1}_{p,q})}.
\]
Define \( X = L_T^2(FB^{n(1-\frac{1}{p})-\frac{\sigma}{2}+\sigma+1}_{p,q}) \), by Lemma 2.2
\[
\|H(u, v)\|_X \leq C\|u\|_X \|v\|_X.
\]
By Lemma 2.1 we know that if \( \|e^{-t\Lambda_{0}}u_0\|_{X} < \frac{1}{4C} \), then (3.1) has a unique solution in \( B(0, \frac{1}{4C}) \). Here \( B(0, \frac{1}{4C}) := \{ x \in X : \|x\|_{X} \leq \frac{1}{4C} \} \).

To conclude \( \|e^{-t\Lambda_{0}}u_0\|_{X} < \frac{1}{4C} \), we first note that \( e^{-t\Lambda_{0}}u_0 \) is the solution to (2.2) with \( f = 0, u_0 = u_0 \), by Lemma 2.3 we can obtain

\[
\|e^{-t\Lambda_{0}}u_0\|_{X} \leq C\|u_0\|_{\tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1}}. \tag{3.4}
\]

Hence if \( \|u_0\|_{\tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1}} \leq \frac{1}{4C^2} \), (3.1) has a unique global solution in \( X \). Moreover,\[
\|u\|_{X} \leq 2C\|u_0\|_{\tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1}}.
\]

On the other hand, denote \( u_0 = \mathcal{F}^{-1} \chi_{\{0 \leq \lambda\}} \hat{u}_0 + \mathcal{F}^{-1} \chi_{\{\lambda > 0\}} \hat{u}_0 := u_0^1 + u_0^2 \), where \( \lambda = \lambda(u_0) > 0 \) is a real number determined later. Since \( u_0^2 \) converges to 0 in \( \tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1} \) as \( \lambda \to +\infty \). By (3.4) there exists \( \lambda \) large enough such that

\[
\|e^{-t\Lambda_{0}}u_0^2\|_{X} \leq \frac{1}{8C}.
\]

For \( u_0^1 \),

\[
\|e^{-t\Lambda_{0}}u_0^1\|_{X} = \|2^{(n(1-\frac{1}{p})-\frac{\alpha}{2}+\sigma+1)}\|\varphi_{j}e^{-\frac{t}{2}}\chi_{\{0 \leq \lambda\}} \hat{u}_0\|_{L_{T}^{2}(L^{p})}\|_{L^{q}} \leq \|2^{(n(1-\frac{1}{p})-\frac{\alpha}{2}+\sigma+1)}\|\sup_{\lambda \leq \lambda} e^{-\frac{t}{2}}\|\varphi_{j}\|_{L_{T}^{2}}\|\hat{u}_0\|_{L^{q}} \leq C\lambda^{\frac{\alpha}{2}}T^{\frac{\alpha}{2}}\|u_0\|_{\tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1}}.
\]

Thus for arbitrary \( u_0 \) in \( \tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1} \), (3.1) has a unique local solution in \( X \) on \([0, T)\) where

\[
T \leq \left( \frac{1}{8C^2\lambda^{\frac{\alpha}{2}}\|u_0\|_{\tilde{F}_{B_{p,q}}^{n(1-\frac{1}{p})-\alpha+1}}} \right)^{2}.
\]

The continuity with respect to time is standard. \( \square \)

Next we give a blowup criterion as following:

**Theorem 3.2.** Let \( T^{*} \) denote the maximal time of existence of \( u \) in \( L_{T}^{\infty}(\tilde{F}_{B_{p,q}}^{\beta}) \cap \mathcal{L}_{T}^{1}(\tilde{F}_{B_{p,q}}^{\beta+\alpha}) \). Here \( \beta = n(1-\frac{1}{p})-\alpha+1 \). If \( T^{*} < \infty \), then

\[
\|u\|_{L_{1}([0, T^{*})};\tilde{F}_{B_{p,q}}^{\beta+\alpha}} = \infty. \tag{3.5}
\]

**Proof.** Supposing \( \|u\|_{L_{1}([0, T^{*})};\tilde{F}_{B_{p,q}}^{\beta+\alpha}} dt < \infty \), then we can find \( 0 < T_0 < T^{*} \) satisfying

\[
\|u\|_{L_{1}([T_0, T^{*})};\tilde{F}_{B_{p,q}}^{\beta+\alpha}} < \frac{1}{2}.
\]
For $t \in [T_0, T^*]$, $s \in [T_0, t]$, by Lemma 2.3:

$$
\|u(s)\|_{FB^\beta_{p,q}} + \|u\|_{L^1(T_0,s);FB^\beta_{p,q}} \leq \|u(0)\|_{FB^\beta_{p,q}} + \|u\|_{L^\infty(T_0,s);FB^\beta_{p,q}} \|u\|_{L^1(T_0,s);FB^\beta_{p,q}}
$$

So,

$$
\sup_{T_0 \leq s \leq t} \|u(s)\|_{FB^\beta_{p,q}} \leq \|u(0)\|_{FB^\beta_{p,q}} + \frac{1}{2}\|u\|_{L^\infty([T_0,t];FB^\beta_{p,q})}.
$$

Put

$$
M = \max(2\|u(0)\|_{FB^\beta_{p,q}}, \max_{t \in [0, T_0]} \|u\|_{FB^\beta_{p,q}}),
$$

we can get

$$
\|u(t)\|_{FB^\beta_{p,q}} \leq M, \quad \forall t \in [0, T^*].
$$

On the other side,

$$
u(t_2) - u(t_1) = e^{-t_2|D|}u_0 - e^{-t_1|D|}u_0 + \int_0^{t_2} e^{-(t_2-\tau)|D|} \nabla \cdot (uPu)(\tau)d\tau - \int_0^{t_1} e^{-(t_1-\tau)|D|} \nabla \cdot (uPu)(\tau)d\tau
$$

$$
= [e^{-t_2|D|}u_0 - e^{-t_1|D|}u_0] + \left[ \int_0^{t_2} e^{-(t_2-\tau)|D|}(e^{-(t_2-t_1)|D|} - 1) \nabla \cdot (uPu)(\tau)d\tau \right]
$$

$$
+ \left[ \int_0^{t_1} e^{-(t_1-\tau)|D|} \nabla \cdot (uPu)(\tau)d\tau \right]
$$

$$
:= L_1 + L_2 + L_3.
$$

$$
\|L_1\|_{FB^\beta_{p,q}} = \|2^{j\beta}\|\varphi_j(e^{-t_2|\xi|} - e^{-t_1|\xi|})\hat{u}_0\|_{L^p}\|_{l_q}
$$

$$
\leq \|2^{j\beta}\|\|\varphi_j(e^{-(t_2-t_1)|\xi|} - 1)\hat{u}_0\|_{L^p}\|_{l_q}
$$

$$
\leq \|u_0\|_{FB^\beta_{p,q}}.
$$

$$
\|L_2\|_{FB^\beta_{p,q}} \leq \|2^{j\beta}\| \int_0^{t_1} \|\varphi_j e^{-(t_2-\tau)|\xi|}(1 - e^{-(t_2-t_1)|\xi|})\mathcal{F}(uPu)(\tau)\|_{L^p}\|d\tau\|_{l_q}
$$

$$
\leq \|2^{j(\beta+1)}\| \int_0^{t_1} \|\varphi_j(e^{-(t_2-t_1)|\xi|} - 1)\mathcal{F}(uPu)(\tau)\|_{L^p}\|d\tau\|_{l_q}.
$$

$$
\|L_3\|_{FB^\beta_{p,q}} \leq \|2^{j\beta}\| \int_0^{t_2} \|\varphi_j e^{-(t_2-\tau)|\xi|} \mathcal{F}(uPu)(\tau)\|_{L^p}\|d\tau\|_{l_q}
$$
Thus by Lemma 2.1 we know that if
\[ \|H \|_{L^p(t_1, t_2)} \leq C \|u\|_{L^p(t_1, t_2)} \]
with
\[ \|u\|_{L^p(t_1, t_2)} \leq C \|u\|_{L^p(t_1, t_2)} \]
we can get
\[ \lim_{t \to T^*} \|u(t)\|_{FB_{p,q}} = u^* \]
Then there is an element \( u^* \) of \( FB_{p,q} \) such that
\[ \lim_{t \to T^*} \|u(t)\|_{FB_{p,q}} = u^* \]
Now set \( u(T^*) = u^* \) and consider the equation starting by \( u^* \), by the well-posedness we obtain a solution existing on a larger time interval than \([0, T^*)\), which is a contradiction. 

4 Improvement of the index

The index in Theorem 3.1 is not a nature one since \( \alpha > 2 \) is a very strong condition. However, the method used here can in fact gain some better index range by a slight modification. The key point is that we can seek the solution firstly in the space \( X := \tilde{L}^r_T(FB_{p,q}^{\beta+\sigma}) \) and consider the equation starting by \( \{r' \} \) instead of \( \{r\} \), with \( \beta = n(1 - \frac{1}{p}) - \alpha + \sigma + 1 \). Since the proof are very similar, we list the key steps.

**Step 1:** Taking respectively \( r \) and \( r' \) in Lemma 2.3 and using the proof of Proposition 3.2 we get for \( \epsilon > \max\{0, -\sigma\}, \beta > -1 \),
\[ \|H(u, v)\|_{L^r_T(FB_{p,q}^{\beta+\sigma})} + \|H(u, v)\|_{L^r_T(FB_{p,q}^{\beta+\sigma})} \leq C \|u\|_{L^r_T(FB_{p,q}^{\beta+\sigma})} + C \|v\|_{L^r_T(FB_{p,q}^{\beta+\sigma})} \]
Set \( \epsilon = \frac{\alpha}{n} - \sigma - 1, \beta = n(1 - \frac{1}{p}) - \alpha + \sigma + 1 \), we then gain the important bilinear estimate
\[ H(u, v) \leq C \|u\|_X \|v\|_X \]
under the condition:
\[ r' \max\{1, \sigma + 1\} < \alpha < n(1 - \frac{1}{p}) + \sigma + 2. \]
Thus by Lemma 2.3 we know that if \( \|e^{-t\Lambda^\alpha}u_0\|_X < \frac{1}{\lambda^\alpha} \), then equation (3.1) admits a unique solution in \( X \).

**Step 2:** Now we need to derive \( \|e^{-t\Lambda^\alpha}u_0\|_X < \frac{1}{\lambda^\alpha} \). Since \( e^{-t\Lambda^\alpha}u_0 \) is the solution to (2.2) with \( f = 0, u_0 = u_0 \), by Lemma 2.3 we gain the global solution in \( X \) for small initial data. On the other hand, we can also obtain the local solution on \([0, T)\) in \( X \) by the same method in Theorem 3.1 for arbitrary initial data, where
\[ T \leq \min\left\{ \left( \frac{1}{16C^2\lambda^\alpha \|u_0\|_{FB_{p,q}^{\beta}}} \right)^{\gamma'}, \left( \frac{1}{16C^2\lambda^\alpha \|u_0\|_{FB_{p,q}^{\beta}}} \right)^{\gamma'} \right\}. \]
**Step 3:** We have proved equation has an unique solution in $X$ under the condition:

$$r' \max\{1, \sigma + 1\} < \alpha < n(1 - \frac{1}{p}) + \sigma + 2, \ 1 < r < \infty.$$  

Using the integral form (3.1) and Lemma 2.3, we can deduce

$$\|u\|_{\tilde{L}^\infty_T (\dot{F}B_{p,q}^\beta)} + \|u\|_{\tilde{L}^1_T (\dot{F}B_{p,q}^{\beta+\alpha})} \leq C(\|u_0\|_{\dot{F}B_{p,q}^\beta} + \|u\|_X).$$

Hence $u$ is also belongs to $C([0,T); \dot{F}B_{p,q}^\beta) \cap \tilde{L}^1_T (\dot{F}B_{p,q}^{\beta+\alpha})$. Since $1 < r < \infty$ in $X$ can be chose to be a sufficiently large number, we in fact improve the index in Theorem 3.1 to

$$\max\{1, \sigma + 1\} < \alpha < n(1 - \frac{1}{p}) + \sigma + 2.$$

Besides, this improvement make no difference to the proof of Theorem 3.2.

**References**

[1] J. Bear, *Dynamics of Fluids in Porous Media*, Dover, New York, 1972.

[2] J. Benamour, M. Benhamed, *Global existence of the two-dimensional QGE with sub-critical dissipation*, J. Math. Anal. Appl., 423(2015), 1330-1347.

[3] A. L. Bertozzi, J. A. Carrillo, T. Laurent, *Blow-up in multidimensional aggregation equations with mildly singular interaction kernels*, Nonlinearity, 22 (2009), 683-710.

[4] P. Biler, C. Imbert, G. Karch, *The nonlocal porous medium equation: Barenblatt profiles and other weak solutions*, Arch. Ration. Mech. Anal. 215 (2015), 497-529.

[5] L. Caffarelli, J. L. Vázquez, *Nonlinear porous medium flow with fractional potential pressure*. Arch. Ration. Mech. Anal. 202 (2011), 537-565.

[6] L. Caffarelli, J. L. Vázquez, *Asymptotic behaviour of a porous medium equation with fractional diffusion*, Discrete Contin. Dyn. Syst. 29 (2011), 1393-1404.

[7] L. Caffarelli, F. Soria, J. L. Vázquez, *Regularity of solutions of the fractional porous medium flow*, J. Eur. Math. Soc. (JEMS) 15 (2013), 1701-1746.

[8] M. Cannone, K. Grzegorz, *Smooth or singular solutions to the Navier-Stokes system?* Journal of Differential Equations 197.2 (2004): 247-274.

[9] T. Iwabuchi, *Global well-posedness for Keller-Segel system in Besov type spaces*, J. Math. Anal. Appl. 379 (2011), 930-948.

[10] T. Iwabuchi, R. Takada, *Global well-posedness and ill-posedness for the Navier-Stokes equations with the Corilis force in function spaces of Besov type*, J. Funct. Anal. 267 (2014), 1321-1337.
[11] P. Konieczny, T. Yoneda, *On dispersive effect of the Coriolis force for the stationary Navier-Stokes equations*, J. Differential Equations, 250 (2011), 3859-3873.

[12] Z. Lei, F. Lin, *Global mild solutions of Navier-Stokes equations*, Commun. Pure Appl. Math. 64 (2011), 1297-1304.

[13] D. Li, J. Rodrigo, *Finite-time singularities of an aggregation equation in $\mathbb{R}^n$ with fractional dissipation*, Commun. Math. Phys., 287 (2009), 687-703.

[14] D. Li, J. Rodrigo, *Well-posedness and regularity of solutions of an aggregation equation*, Rev. Mat. Iberoam., 26 (2010), 261-294.

[15] F. Lin, P. Zhang, *On the hydrodynamic limit of Ginzburg-Landau wave vortices*, Comm. Pure Appl. Math. 55 (2002), 831-856.

[16] Q. Liu, J. Zhao, *Global well-posedness for the generalized magneto-hydrodynamic equations in the critical Fourier-Herz sapces*, J. Math. Anal. Appl. 420 (2014), 1301-1315.

[17] S. Serfaty, J. L. Vázquez, *A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators*, Calc. Var. Partial Differential Equations 49 (2014), 1091-1120.

[18] D. Stan, F. del Teso, J. L. Vázquez, *Finite and infinite speed of propagation for porous medium equations with fractional pressure*, C. R. Math. Acad. Sci. Paris 352 (2014), 123-128.

[19] J. L. Vázquez, *The Porous Medium Equation. Mathematical theory*. Oxford Mathematical Monographs. The Clarendon Press/Oxford University Press, Oxford, 2007.

[20] J. L. Vázquez, *Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators*, Discrete Contin. Dyn. Syst. Ser. S 7 (2014), 857-885.

[21] G. Wu, Q. Zhang, *Global well-posedness of the aggregation equation with supercritical dissipation in Besov spaces*, Z. Angew. Math. Mech. 93 (2013), 882-894.

[22] W. Xiao, J. Chen, D. Fan, X. Zhou, *Global Well-Posedness and Long Time Decay of Fractional Navier-Stokes Equations in Fourier-Besov Spaces*, Abstract and Applied Analysis., 2014 (2014).

[23] X. Zhou, W. Xiao, J. Chen, *Fractional porous medium and mean field equations in Besov spaces*, Electron. J. Differential Equations, 2014 (2014), 1-14.

[24] X. Zhou, W. Xiao, T. Zheng, *Well-posedness and blowup criterion of generalized porous medium equation in Besov spaces*, Electron. J. Differential Equations, 2015(2015), 1-14.