ON THE GALOIS SYMMETRIES FOR THE CHARACTER TABLE OF AN INTEGRAL FUSION CATEGORY

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Abstract. In this paper we show that integral fusion categories with rational structure constants admit a natural group of symmetries given by the Galois group of their character tables. We also generalize a well known result of Burnside from representation theory of finite groups. More precisely, we show that any row corresponding to a non invertible object in the character table of a weakly integral fusion category contains a zero entry.

1. Introduction

Fusion categories can be regarded as a natural generalization of the category of representations finite groups. From this point of view, many results from group representations have been extended to the settings of fusion categories. For example, in [ENO11, Theorem 1.6] the author showed that any fusion category of Frobenius-Perron dimension $p^aq^b$ is semi-solvable extending the famous Burnside theorem for finite groups.

Recently, Shimizu introduced in [Shi17] the notion of conjugacy classes for fusion categories, extending Zhu’s work from [Zhu97] and some results for semisimple Hopf algebras obtained previously by Cohen and Westreich in [CW11]. In the same paper [Shi17] the author associated to each conjugacy class a central element called also conjugacy class sum. These class sums play the role of the sum of group elements of a conjugacy class in group theory and allow one to define character tables for pivotal fusion categories. In the same paper [Shi17], Shimizu proved the orthogonality relations for the character table, generalizing the famous orthogonality representations for finite groups.

The goal of this paper is to investigate some other properties of the character tables of finite groups that can be extended to the settings of fusion categories.

Let $\mathcal{C}$ be a pivotal fusion category. Shimizu has constructed in [Shi17] a ring of class function $\text{CF}(\mathcal{C})$ analogues to the character ring of a
semisimple Hopf algebra. He also associated to any object \( X \) of \( \mathcal{C} \) a class function \( \text{ch}(X) \in \text{CF}(\mathcal{C}) \). More details are given in Section \( \ref{sec:background} \) if \( X_0, X_1, \ldots, X_m \) are a complete set of representatives of the simple objects of \( \mathcal{C} \) we denote by \( \chi_i := \text{ch}(X_i) \).

Let \( F_0, F_1, \ldots, F_m \) be the central primitive idempotents of \( \text{CF}(\mathcal{C}) \). Recall that \( C_j := F_\lambda^{-1}(F_j) \in \text{CE}(\mathcal{C}) \) is called the conjugacy class sums corresponding to the conjugacy class \( C_j \) corresponding to \( F_j \), see Section \( \ref{sec:background} \). Here \( F_\lambda : \text{CE}(\mathcal{C}) \to \text{CF}(\mathcal{C}) \) is the usual Fourier transform associated to \( \mathcal{C} \), see also Section \( \ref{sec:background} \) for more details.

Since \( \{C_i\} \) form a basis for the set of central elements \( \text{CE}(\mathcal{C}) \) it follows that the product of two such class sums is a linear combination of the class sums. Therefore one can write that

\[
C_i C_j = \sum_{k=0}^{m} c_{ij}^k C_k
\]

where \( c_{ij}^k \in \mathbb{C} \) are complex numbers. It is well-known that for finite groups these structure constants are in fact integer numbers.

Cohen and Westreich in \cite{CohenWestreich} Theorem 2.6] proved that for a semisimple quasi-triangular Hopf algebra these structure constants are rational numbers (integers up to a factor of \( \dim(H)^{-2} \)). Recently in \cite{ZZ19} Theorem 5.6] the authors improved the above result by showing for a semisimple quasi-triangular Hopf algebra \( H \) that this factor can be replaced by \( \dim(H) \). Moreover, this result was generalized for pre-modular fusion categories in \cite{Bur20b}. We should mention that at this stage we are not aware of any integral fusion category whose structure constants are not rational numbers.

For any irreducible character \( \chi_i \) one can write \( \chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j \). Let \( \mathbb{K} \) the field obtained by adjoining \( \alpha_{ij} \) to \( \mathbb{Q} \). By \cite{ENO05} Corollary 8.53] there is a cyclotomic field \( \mathbb{Q}(\xi) \) such that \( \mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{Q}(\xi) \).

Our first main result is concerned with the action of the Galois group \( \text{Gal}(\mathbb{K}/\mathbb{Q}) \) on the set of irreducible characters of \( \mathcal{C} \). The notations are explained in details in Section \( \ref{sec:main_results} \).

**Theorem 1.** Suppose that \( \mathcal{C} \) is an integral fusion category with rational structure constants \( c_{ij}^k \in \mathbb{Q} \). Then \( \sigma_E \) is an algebra map for all \( \sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}) \). Moreover, there is a permutation \( \eta = \eta_\sigma \) such that

\[
\sigma_E(\chi_i) = \chi_{\eta(i)}, \quad \sigma_E(E_i) = E_{\eta^{-1}(i)}
\]

for all \( i \in \mathcal{I} \).

Recall that the character table of a fusion category \( \mathcal{C} \) with a commutative Grothendieck ring is defined as \( \chi_i(C_j) \) where \( \chi_i \) are the irreducible characters of \( \mathcal{C} \) and \( C_j \) are the class sums of \( \mathcal{C} \) mentioned above.
A classical result of Burnside in character theory of finite groups states that for any irreducible character $\chi$ of a finite group $G$ with $\chi(1) > 1$ there is some $g \in G$ such that $\chi(g) = 0$, see [BZ81, Chapter 21]. This result was generalized in [GN09, Appendix] to weakly integral modular categories.

Our second main result is the following generalization of the above result to weakly-integral fusion categories:

**Theorem 2.** Let $\mathcal{C}$ be a weakly-integral fusion category. Then any row corresponding to a non-invertible object in its character table contains a zero entry.

Shortly, the organization of this paper is the following. In Section 2 we recall the basic on fusion categories and the concept of conjugacy classes and class sums introduced in [Shi17]. In Section 3 we introduce the Galois group associated to a fusion category and investigate when this group produces symmetries on the irreducible characters of $\mathcal{C}$. In Section 4 we prove theorem 2. We also give a short comparison of our approach with the one from [GN09] for weakly integral modular categories.

All representations and fusion categories in this paper are considered over the ground filed $\mathbb{C}$ of complex numbers.

## 2. Preliminaries

For a finite abelian category $\mathcal{C}$ we denote by $\text{Gr}(\mathcal{C})$ its Grothendieck group and set $\text{Gr}_k(\mathcal{C}) := \text{Gr}(\mathcal{C}) \otimes_{\mathbb{Z}} k$. It is well known that for a finite tensor category $\text{Gr}_k(\mathcal{C})$ is a $k$-algebra with $[V] \cdot [W] = [V \otimes W]$ for any two objects $V, W$ of $\mathcal{C}$.

By a fusion category we mean a semisimple finite tensor category. We refer to [EGNO15] for the basic theory of tensor categories. For a fusion category $\mathcal{C}$ we denote by $\text{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$. It is well known that for a fusion category $\text{Gr}(\mathcal{C})$ is a based unital ring and therefore one can define Frobenius-Perron dimensions $\text{FPdim}(X)$ for any object $X$ of $\mathcal{C}$. The Frobenius-Perron dimension of the category $\mathcal{C}$ is defined as

$$\text{FPdim}(\mathcal{C}) := \sum_{X \in \text{Irr}(\mathcal{C})} \text{FPdim}(X)^2.$$  

A fusion category $\mathcal{C}$ is called *weakly integral* if $\text{FPdim}(\mathcal{C}) \in \mathbb{Z}$. Moreover a fusion category is called *integral* if $\text{FPdim}(X) \in \mathbb{Z}_{>0}$ for any simple object $X$ of $\mathcal{C}$.

Throughout this paper $\mathcal{C}$ denotes a fusion category and $\mathbf{1}$ the unit object of a $\mathcal{C}$. The *monoidal centre* (or Drinfeld centre) of $\mathcal{C}$ is a braided
fusion category $\mathcal{Z}(\mathcal{C})$ constructed as follows, see e.g. [Kas95, XIII.3] for details. The objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(V, \sigma_V)$ of an object $V \in \mathcal{C}$ and a natural isomorphism $\sigma_{V,X} : V \otimes X \to X \otimes V$ for all $X \in \mathcal{O}(\mathcal{C})$, satisfying a part of the hexagon axiom. A morphism $f : (V, \sigma_V) \to (W, \sigma_W)$ in $\mathcal{Z}(\mathcal{C})$ is a morphism in $\mathcal{C}$ such that $(id_X \otimes f) \circ \sigma_{V,X} = \sigma_{W,X} \circ (f \otimes id_X)$ for all objects $X$ of $\mathcal{C}$. The composition of morphisms in $\mathcal{Z}(\mathcal{C})$ is defined via the usual composition of morphisms in $\mathcal{C}$.

Let $\mathcal{C}$ be a finite tensor category and $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor, $F(V, \sigma_V) = V$. Then $F$ admits a right adjoint functor $R : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ such that $Z := FR : \mathcal{C} \to \mathcal{C}$ is a Hopf comonad. Moreover, as in [Shi17, Subsection 2.6] one has that

$$Z(V) \simeq \int_{X \in \mathcal{C}} X \otimes V \otimes X^*.$$ 

The counit $\epsilon : Z \to \text{id}_{\mathcal{C}}$ is given by $\epsilon_V := \pi_{V,1}$ where $\pi_{V,X} : Z(V) \to X \otimes V \otimes X^*$ are the universal dinatural transformation associated to the above end $Z(V)$. Moreover, using Fubini’s theorem for ends, the comultiplication $\delta : Z \to Z^2$ of $Z$ is also described in terms of the dinatural transformation $\pi$ associated to $Z$, see [Shi17, Subsection 3.2].

The object $A := Z(1) \in \mathcal{Z}(\mathcal{C})$ has the structure of a central commutative algebra in $\mathcal{Z}(\mathcal{C})$. It is called the adjoint algebra of $\mathcal{C}$.

The multiplication and the unit of $A$ are uniquely determined by by the universal property of the end $Z$ as:

$$\pi_{1,X} \circ m_A = (id_X \otimes ev_X \otimes id_{X^*}) \circ (\pi_{1,X} \otimes \pi_{1,X}), \quad \pi_{1,X} \circ u_A = \text{coev}_X.$$

Recall that a pivotal structure $j$ on $\mathcal{C}$ is a tensor isomorphism $j : \text{id}_C \to (\cdot)^\ast$. Given such a pivotal structure one can construct for any object $X$ of $\mathcal{C}$ a right evaluation $\text{ev}_X : X \otimes X^* \xrightarrow{j \otimes id} X^\ast \otimes X^* \xrightarrow{ev_{X^*}} 1$. For any morphism $f : A \otimes X \to B \otimes X$ in $\mathcal{C}$ one can define the right partial pivotal trace $\text{tr}^X_A : \text{Hom}_C(A, B)$ of $f$:

$$\text{tr}^X_{A,B} : A = A \otimes 1 \xrightarrow{id \otimes \text{coev}_X} A \otimes X \otimes X^* \xrightarrow{f \otimes id} B \otimes X \otimes X^* \xrightarrow{id \otimes \text{ev}_X} B.$$ 

Then the usual right pivotal trace of an endomorphism $f : X \to X$ is obtained as a particular case for $A = B = 1$. In particular, the right (quantum) dimension of $X$ with respect to $j$ is defined as the right trace of the identity of $X$.

A pivotal structure (or the underlying fusion category) is called spherical if $\text{dim}(X) = \text{dim}(X^*)$ for all objects $X$ of $\mathcal{C}$, see [ENO05, Definition 4.7.14].
Remark 3. By [ENO05, Proposition 29] one has that \( \dim(V^*) = \dim(V) \) for any object of a pivotal fusion category. In particular, in a spherical category \( \dim(V) \) is (totally) real number.

Recall also that the global dimension of the pivotal fusion category \( \mathcal{C} \) is defined as \( \dim(\mathcal{C}) := \sum_{i=0}^{m} d_i d_i^* \). It is well-known that over the complex numbers one has \( \dim(\mathcal{C}) \neq 0 \) for any pivotal fusion category.

A fusion category \( \mathcal{C} \) is called \textit{pseudo-unitary} if \( \text{FPdim}(\mathcal{C}) = \dim(\mathcal{C}) \). If this is the case, then by [ENO05, Proposition 8.23], \( \mathcal{C} \) admits a unique spherical structure with respect to which the categorical dimensions of simple objects are all positive. It is called the \textit{canonical spherical structure}. For this spherical structure, the categorical dimension of an object coincides with its Frobenius-Perron dimension, i.e. \( \text{FPdim}(X) = \dim(X) \) for any object \( X \) of \( \mathcal{C} \).

If \( \mathcal{C} \) is a weakly-integral fusion category then \( \mathcal{C} \) is pseudo-unitary by [ENO05, Proposition 8.24].

Remark 4. Let \( \mathcal{C} \) be a weakly integral fusion category. By [ENO05, Proposition 8.27] the dimensions of simple objects in \( \mathcal{C}_{\text{ad}} \) are integers. Since \( \mathcal{C}^j \) are sum of simple object of the adjoint subcategory it follows that \( \dim(\mathcal{C}^j) \) are integers.

The internal character \( \chi(X) \) of an object \( X \) of \( \mathcal{C} \) is defined as the partial pivotal trace

\[
\chi(X) := \text{tr}^X_{A,1}(\rho_X) : A \to 1.
\]

where \( \rho_X : A \otimes X \to X \) is the canonical action of \( A \) on \( X \), see [Shi17, Definition 3.3] for details.

The space \( \text{of class functions} \) of \( \mathcal{C} \) is defined as \( \text{CF}(\mathcal{C}) := \text{Hom}_{\mathcal{C}}(A, 1) \) and it is a \( \mathcal{C} \)-algebra. The multiplication of two class functions \( f, g \in \text{CF}(\mathcal{C}) \) is defined via \( f \ast g := f \circ Z(g) \circ \delta_1 \). Here the map \( \delta : Z \to Z^2 \) is the comultiplication structure of the Hopf comonad \( Z \) recalled above.

By [Shi17, Theorem 3.10] for a pivotal fusion category \( \mathcal{C} \) one has that \( \chi(X \otimes Y) = \chi(X)\chi(Y) \) for any two objects \( X \) and \( Y \) of \( \mathcal{C} \) and \( \text{Gr}_\mathcal{C}(\mathcal{C}) \to \text{CF}(\mathcal{C}), [X] \mapsto \chi(X) \) is an isomorphism of algebras. Moreover, the character of the unit object \( \chi(1) = \epsilon_1 \) is the unit of \( \text{CF}(\mathcal{C}) \). Recall from above that \( \epsilon : Z \to \text{id}_\mathcal{C} \) is the counit of the central Hopf comonad.

By [Shi17, Theorem 5.9] for a pivotal fusion category it follows that the map \( \chi : \text{Gr}_\mathcal{C}(\mathcal{C}) \to \text{CF}(\mathcal{C}), [X] \mapsto \chi(X) \) is an isomorphism of \( \mathbb{C} \)-algebras.

The space \( \text{CE}(\mathcal{C}) := \text{Hom}_{\mathcal{C}}(1, A) \) is called the \textit{space of central elements}. It is also a \( \mathbb{C} \)-algebra where the multiplication on CE(\( \mathcal{C} \)) is given
by \( a \cdot b := m \circ (a \otimes b) \) for any \( a, b \in \text{CE}(\mathcal{C}) \). There is a non-degenerate pairing \( \langle \ , \ \rangle : \text{CF}(\mathcal{C}) \times \text{CE}(\mathcal{C}) \to \mathbb{C} \), given by \( \langle f, a \rangle \text{id}_1 = f \circ a \), for all \( f \in \text{CF}(\mathcal{C}) \) and \( a \in \text{CE}(\mathcal{C}) \). We also denote \( f(a) := \langle f, a \rangle \). There is a right action of \( \text{CE}(\mathcal{C}) \) on \( \text{CF}(\mathcal{C}) \) denoted by \( \leftarrow \) given by \( f \leftarrow b = f \circ m \circ (b \otimes \text{id}_A) \) for all \( f \in \text{CF}(\mathcal{C}) \) and \( b \in \text{CE}(\mathcal{C}) \).

### 2.1. Orthogonality relations for pivotal fusion categories

For the rest of this paper we fix a pivotal fusion category \( \mathcal{C} \) with commutative Grothendieck ring and we use following notations. We denote by \( M_0, M_1, \ldots, M_m \) a complete set of representatives for the isomorphism classes of simple objects of \( \mathcal{C} \). For brevity, we also denote \( \chi_i := \text{ch}(M_i) \in \text{CF}(\mathcal{C}) \) and \( d_i := \chi_i(1_{\text{CE}(\mathcal{C})}) \) the categorical dimensions of the simple objects. We call \( \chi_i \) the irreducible character of the simple object \( M_i \). Without loss of generality we may suppose that \( M_0 = 1 \) and therefore \( d_0 = 1 \). In this case by [Shi17, Lemma 6.10] one has \( \chi_0 = \epsilon_1 \).

Moreover, we denote by \( i^* \) the unique index for which \( M_{i^*} \cong M_i \).

**Remark 5.** By [EGNO15, Proposition 4.8.4] it also follows that \( d_i \neq 0 \) for all \( i \), since the dimensions of simple objects are not zero.

Let \( \mathcal{C} \) be a pivotal fusion category with commutative Grothendieck ring. Recall that \( R : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \) is the right adjoint of the forgetful functor \( F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \). Note that in this case by [Shi17, Theorem 6.6] the object \( R(1) \in \mathcal{O}(\mathcal{Z}(\mathcal{C})) \) is multiplicity-free.

A conjugacy class of \( \mathcal{C} \) is defined as a simple subobject of \( R(1) \) in \( \mathcal{Z}(\mathcal{C}) \). Since the monoidal center \( \mathcal{Z}(\mathcal{C}) \) is also a fusion category we can write \( R(1) = \bigoplus_{j=0}^m C_j^i \) as a direct sum of simple objects in \( \mathcal{Z}(\mathcal{C}) \). Thus \( C_0, \ldots, C_m \) are the conjugacy classes of \( \mathcal{C} \). Since the unit object \( 1_{\mathcal{Z}(\mathcal{C})} \) is always a subobject of \( R(1) \), we may assume for the rest of this paper that \( C_0 = 1_{\mathcal{Z}(\mathcal{C})} \).

By [Shi17, Theorem 3.8] there is a canonical isomorphism

\[
\text{End}_{\mathcal{Z}(\mathcal{C})}(R(1)) \simeq \text{CF}(\mathcal{C}), \quad f \mapsto Z(f) \circ \delta_1.
\]

This isomorphism gives a natural bijection between \( F_j \) and \( C_j^i \). Then the \( C_0 \) corresponds \( F_0 = \lambda \).

A cointegral \( \lambda \) of \( \mathcal{C} \) is defined as the unique element (up to a scalar) of \( \text{CF}(\mathcal{C}) \) with the property that \( \chi \lambda = \chi(1) \lambda \) for all \( \chi \in \text{CF}(\mathcal{C}) \).

Following [Shi17] with the above notations one has

\[
\lambda = \frac{1}{\dim(\mathcal{C})} \left( \sum_{i=0}^m d_i \chi_i \right) \in \text{CF}(\mathcal{C}).
\]

Moreover, one has \( \langle \lambda, u_A \rangle = 1 \) and \( \lambda^2 = \lambda \) where \( u_A : 1 \to A \) is the unit of \( A \).
The Fourier transform of \( \mathcal{C} \) associated to \( \lambda \) is the linear map \( \mathcal{F}_\lambda : CE(\mathcal{C}) \to CF(\mathcal{C}) \) given by \( a \mapsto \lambda \leftarrow S(a) \) where \( S : CE(\mathcal{C}) \to CE(\mathcal{C}) \) is the antipodal map of \( CE(\mathcal{C}) \), see [Shi17, Definition 3.6].

Let also \( F_0, F_1, \ldots, F_m \) be the central primitive idempotents of \( CF(\mathcal{C}) \). Without loss of generality we may suppose that \( F_0 = \lambda \). We define \( C_j := \mathcal{F}_\lambda^{-1}(F_j) \in CE(\mathcal{C}) \) to be the conjugacy class sums corresponding to the conjugacy class \( C_j \). It follows as in [Shi17, Section 6] that \( C_0 = \mathcal{F}_\lambda^{-1}(\lambda) = 1_{CE(\mathcal{C})} = u_A \).

Since \( CF(\mathcal{C}) \) is commutative it follows that the set \( \{F_j\} \) of central primitive idempotents forms also a bases for \( CF(\mathcal{C}) \). Therefore for any irreducible character \( \chi_i \) one can write

\[
\chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j \tag{3}
\]

for some scalars \( \alpha_{ij} \in \mathbb{C} \).

For \( 0 \leq j \leq m \) let \( \mu_j : CF(\mathcal{C}) \to \mathbb{C} \) be the characters of \( CF(\mathcal{C}) \) corresponding to the primitive central idempotents \( F_j \). Since we assume \( CF(\mathcal{C}) \) is commutative the characters \( \mu_j \) are algebra morphisms.

By [Shi17, Equation (6.13)] one has that

\[
\alpha_{ij} = \mu_j(\chi_i) = \frac{\chi_i(C_j)}{\dim(C_j)}. \tag{4}
\]

Since in this case \( C^0 = 1_{Z(\mathcal{C})} \) it follows from the above equation that

\[
\mu_0(\chi_i) = \chi_i(1_{CE(\mathcal{C})}) = d_i \tag{5}
\]

for all \( i \). Note also that in this case \( \mu_0 = d \), the quantum dimension morphism.

Following [Shi17, Example 4.4] one has that

\[
A = \oplus_{i=0}^{m} M_i \otimes M_i^* \tag{6}
\]

In this case the irreducible characters \( \chi_i \) are given by

\[
\chi_i := A \xrightarrow{\text{projection}} M_i \otimes M_i^* \xrightarrow{\text{coev}_{M_i}} 1. \tag{7}
\]

As in [Shi17, Subsection 6.1] one has that the primitive central idempotents \( E_i \in CE(\mathcal{C}) \) of \( CE(\mathcal{C}) \) are given by

\[
1 \xrightarrow{\text{coev}_{M_i}} M_i \otimes M_i^* \hookrightarrow A. \tag{8}
\]

Moreover, in this case

\[
< \chi_i, E_j > = \delta_{i,j} \tag{9}
\]

for all \( i, j \).
By [Shi17, Corollary 6.11] the first orthogonality relation for $C$ can be written as

\begin{equation}
\sum_{k=0}^{m} \dim(C^k) \alpha_{ik} \alpha_{m^* k} = \delta_{i,m} \dim(C).
\end{equation}

and the second orthogonality relation as:

\begin{equation}
\sum_{i=0}^{m} \alpha_{ii} \alpha_{i^* k} = \delta_{i,k} \frac{\dim(C)}{\dim(C^k)}.
\end{equation}

Recall from [Bur20b, Equation (3.4)] that

\begin{equation}
\sum_{j=0}^{m} \frac{\dim(C)}{\dim(C^j)} F_j \otimes F_j = \sum_{i=0}^{m} \chi_i \otimes \chi_i^*.
\end{equation}

From here it follows that

\begin{equation}
F_j := \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \chi_i \right)
\end{equation}

where

\begin{equation}
n_j = \frac{\dim(C)}{\dim(C^j)}.
\end{equation}

For any two class functions $\chi = \sum_{i=0}^{m} \alpha_i \chi_i$ and $\mu = \sum_{i=0}^{m} \beta_i \chi_i$ of $\text{CF}(C)$ we define

\begin{equation}
m_C(\chi, \mu) := \sum_{i=0}^{m} \alpha_i \beta_i.
\end{equation}

**Lemma 6.** Let $C$ be a pivotal fusion category. With the above notations it follows that

\begin{equation}
m_C(\chi, \mu) = \sum_{j=0}^{m} \frac{\dim(C^j)}{\dim(C)} \mu_j(\chi) \mu_j(\mu)
\end{equation}

for any two class functions $\chi, \mu \in \text{CF}(C)$.

**Proof.** Indeed, for any two irreducible characters $\chi = \chi_s$ and $\mu = \chi_t$ the right hand side of the above equation can be written as

\[
\sum_{j=0}^{m} \frac{\dim(C^j)}{\dim(C)} \mu_j(\chi_s) \mu_j(\chi_t) \sum_{j=0}^{m} \frac{\dim(C^j)}{\dim(C)} \alpha_{s_j} \alpha_{t^* j} = \delta_{s,t}.
\]

then the Lemma follows by linearity. \qed
Note also that $\mu_j(\chi_i^*) = \overline{\mu_j(\chi_i)}$ for all $i, j$.
In particular, since $C^0 = 1_Z(C)$ it follows that $n_0 = \dim(C)$.
Since $\{C_j\}$ form a $\mathbb{C}$-linear basis for $\mathrm{CE}(C)$ one has that

\begin{equation}
C_{j_1}C_{j_2} = \sum_{l=0}^{m} c_{j_1,j_2}^l C_l
\end{equation}

form some scalars $c_{j_1,j_2}^l \in \mathbb{C}$. These scalars are called the structure constants of $C$.

By [Bur20b, Theorem 1.1] if $C$ is a pivotal fusion category with a commutative Grothendieck ring then

\begin{equation}
c_{ij}^k = \sum_{s=0}^{m} \frac{\chi_s(C_i)\chi_s(C_j)\chi_s^*(C_k)}{\dim(C)\dim(C^k)d_s}.
\end{equation}

Using Equation (12) this can be written as

\begin{equation}
c_{ij}^k = \left(\frac{\dim(C_{j_1}) \dim(C_{j_2})}{\dim(C)}\right) \left(\sum_{s=0}^{m} \frac{\alpha_{si}\alpha_{sj}\alpha_{sk}}{d_s}\right).
\end{equation}

3. Galois action for integral fusion categories

Let $C$ be a pivotal fusion category with a commutative Grothendieck ring $\mathrm{Gr}_k(C) \simeq \mathrm{CF}(C)$.

Let also $M_0, M_1, \ldots, M_m$ be a complete set of representatives for the isomorphism classes of simple objects of $C$. As above, without loss of generality we may assume that $M_0 = 1$ is the unit object of $C$. Moreover, we denote by $\chi_i := \text{ch}(M_i)$ the characters of the simple objects $M_i$. Recall that in this case $\chi_0 = \epsilon_1$ is the unit of the algebra $\mathrm{CF}(C)$.

As before, by $F_0, F_1, \ldots, F_m$ are denoted the central primitive idempotents of $\mathrm{CF}(C)$ and by $\mu_0, \mu_1, \ldots, \mu_m : \mathrm{CF}(C) \to \mathbb{C}$ their corresponding characters on $\mathrm{CF}(C)$. Therefore $\mu_i : \mathrm{CF}(C) \to \mathbb{C}$ are algebras maps and $\mu_i(F_j) = \delta_{i,j}$. We also denote by $C^0, C^1, \ldots, C^m$ the conjugacy classes of $C$ corresponding in this order to the primitive idempotents $F_0, F_1, \ldots, F_m$. Moreover, we let $C_0, C_1, \ldots, C_m$ denote their corresponding class sums.

Without loss of generality we may suppose that $\mu_0 = d$ is the quantum dimension homomorphism. Therefore $\mu_0(\chi_i) = d_i$ for all $i$. It follows as in the previous section that $C_0 = F^{-1}_\lambda(\lambda) = 1_{\mathrm{CE}(C)} = u_A$, the unit $u_A : 1 \to A$ of the adjoint algebra $A$.

Writing $\chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j$ we let $K$ be the field extension of $\mathbb{Q}$ by all the scalars $\alpha_{ij}$. By [ENO05, Corollary 8.53] there is a cyclotomic field $\mathbb{Q}(\xi)$ such that $\mathbb{Q} \subseteq K \subseteq \mathbb{Q}(\xi)$ for some root of unity $\xi$. 

Define \( G := \text{Gal}(\mathbb{K}/\mathbb{Q}) \) and note that is an abelian group as a quotient of the abelian group \( \text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) \).

**Remark 7.** Note that if \( \alpha_{ij} = \alpha_{ij'} \) for all \( i \) then \( \mu_j(\chi_i) = \mu_{j'}(\chi_i) \), thus \( \mu_j = \mu_{j'} \) and therefore \( j = j' \).

### 3.1. Action \( \hat{\sigma}_F \) on \( \widehat{\text{CF}(\mathcal{C})} \)

Recall that \( \widehat{\text{CF}(\mathcal{C})} \) is defined as the linear dual space of \( \text{CF}(\mathcal{C}) \). For \( \mu_j : \text{CF}(\mathcal{C}) \to \mathbb{C} \) and \( \sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}) \) define \( \sigma \cdot \mu_j \in \widehat{\text{CF}(\mathcal{C})} \) as the linear function on \( \text{CF}(\mathcal{C}) \) which on the basis given by the irreducible characters \( \{ \chi_i \}_i \) is given by

\[
[\sigma, \mu_j](\chi_i) = \sigma(\mu_j(\chi_i)) = \sigma(\alpha_{ij}).
\]

**Lemma 8.** For any pivotal fusion category \( \mathcal{C} \) with the above notations it follows that \( \sigma \cdot \mu_j : \text{CF}(\mathcal{C}) \to \mathbb{C} \) is an algebra map.

**Proof.** Indeed, suppose that \( \chi_{i_1} \chi_{i_2} = \sum_{k=0}^{m} N_{i_1,i_2}^k \chi_k \). Then one has

\[
[\sigma, \mu_j](\chi_{i_1} \chi_{i_2}) = [\sigma, \mu_j](\sum_{k=0}^{m} N_{i_1,i_2}^k \chi_k) = \sum_{k=0}^{m} N_{i_1,i_2}^k \sigma \cdot \mu_j(\chi_k) = \sum_{k=0}^{m} N_{i_1,i_2}^k \sigma(\mu_j(\chi_k)).
\]

On the other hand,

\[
[\sigma, \mu_j](\chi_{i_1}) [\sigma, \mu_j](\chi_{i_2}) = \sigma(\mu_j(\chi_{i_1})) \sigma(\mu_j(\chi_{i_2})) = \sigma(\mu_j(\chi_{i_1}) \mu_j(\chi_{i_2})) = \sigma(\mu_j(\sum_{k=0}^{m} N_{i_1,i_2}^k \chi_k))
\]

\[
= \sum_{k=0}^{m} N_{i_1,i_2}^k \sigma(\mu_j(\chi_k)).
\]

It follows that the group \( \text{Gal}(\mathbb{K}/\mathbb{Q}) \) acts on the set of all algebra homomorphisms \( \mu_j : \text{CF}(\mathcal{C}) \to \mathbb{C} \). It is easy to see that \( \sigma(\sigma', \mu_j) = (\sigma \sigma'), \mu_j \). Indeed one has

\[
[\sigma(\sigma', \mu_j)](\chi_i) = \sigma((\sigma', \mu_j)(\chi_i)) = \sigma\left( \sigma'(\mu_j(\chi_i)) \right) = \sigma \sigma'(\mu_j(\chi_i)) = [\sigma(\sigma', \mu_j)](\chi_i).
\]

We denote by \( \mathcal{J} \) the set of all indices \( j \in \{0, \ldots, m\} \) such that \( \mu_j \) is an algebra map.

Since \( \sigma \cdot \mu_j : \text{CF}(\mathcal{C}) \to \mathbb{C} \) is an algebra homomorphism it follows that there is an index \( \tau_\sigma(j) \in \mathcal{J} \) such that \( \sigma \cdot \mu_j = \mu_{\tau_\sigma(j)} \). It follows that

\[
(16) \quad \sigma(\alpha_{ij}) = \alpha_{i \tau_\sigma(j)}
\]

for any \( i, j \). Indeed, \( \sigma(\alpha_{ij}) = \sigma \cdot \mu_j(\chi_i) = \mu_{\tau_\sigma(j)}(\chi_i) = \alpha_{i \tau_\sigma(j)} \).
Proposition 9. For any \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) one has that \( \tau_\sigma \) is a permutation of \( \mathcal{J} \).

Proof. Indeed, if \( \tau_\sigma(j) = \tau_\sigma(j') \) then since \( \alpha_{i\tau_\sigma(j)} = \alpha_{i\tau_\sigma(j')} \) it follows that \( \sigma(\alpha_{ij}) = \sigma(\alpha_{ij'}) \) which shows that \( \alpha_{ij} = \alpha_{ij'} \) and therefore \( j = j' \) by Remark 7.

Define also a \( \mathbb{C} \)-linear map \( \widehat{\sigma}_F : \widehat{\text{CF}(C)} \to \widehat{\text{CF}(C)} \) written on the linear \( \mathbb{C} \)-basis \( \{ \mu_j \}_{j \in \mathcal{J}} \) by

\[
\widehat{\sigma}_F(\mu_j) = \sigma.\mu_j = \mu_{\tau_\sigma(j)}.
\]

Remark 10. Note that the map \( \widehat{\sigma}_F \) preserves the unit \( \mu_0 \) of \( \widehat{\text{CF}(C)} \) if and only if \( \sigma.\mu_0 = \mu_0 \iff d_i = \sigma(d_i) \) for all \( i \).

Then it follows that all the maps \( \widehat{\sigma}_F \) are unital (for any \( \sigma \in \text{Gal}(K/\mathbb{Q}) \)) if and only if \( C \) is an integral category.

Proposition 11. With the above notations one has that \( \tau_\sigma \tau_{\sigma'} = \tau_{\sigma\sigma'} \) for all \( \sigma, \sigma' \in \text{Gal}(K/\mathbb{Q}) \). This shows that we have a group embedding

\[
\text{Gal}(K/\mathbb{Q}) \hookrightarrow S_m, \quad \sigma \mapsto \tau_\sigma.
\]

Proof. Indeed one has

\[
\mu_{[\tau_\sigma \tau_{\sigma'}]}(j) = \sigma(\mu_{\tau_{\sigma'}(j)}) = \sigma.(\sigma'.(\mu_j)) = (\sigma \sigma').(\mu_j) = \mu_{\tau_{\sigma \sigma'}(j)}
\]

which implies \( \tau_\sigma \tau_{\sigma'}(j) = \tau_{\sigma \sigma'}(j) \), i.e. \( \tau_\sigma \tau_{\sigma'} = \tau_{\sigma \sigma'} \).

Remark 12. The above proposition implies that \( \tau_{\sigma^{-1}} = \tau_\sigma^{-1} \), i.e.

\[
(17) \quad \sigma^{-1}(\alpha_{ij}) = \alpha_{i\tau_\sigma^{-1}(j)}
\]

For the rest of the paper we shortly write \( \tau := \tau_\sigma \) if \( \sigma \) is implicitly understood.

Remark 13. Note that the permutation action \( \tau_\sigma \) holds also in the case of a non-commutative \( \text{CF}(C) \) since \( \sigma.\mu_j \) is a trace.

Proposition 14. Let \( C \) be a pivotal fusion category with a commutative \( \text{CF}(C) \). For any \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) with the above notations one has:

\[
(18) \quad \sigma \left( \frac{\dim(C)}{\dim(C^k)} \right) = \frac{\dim(C)}{\dim(C^{\tau(k)})}
\]

In particular if \( C \) is weakly integral one has

\[
(19) \quad \dim(C^k) = \dim(C^{\tau(k)}).
\]

Proof. Since \( C \) is pivotal, note that the size \( \dim(C^j) \) of \( C_j \) is non-zero since it is the quantum dimension of a simple object in a pivotal fusion
category $\mathcal{Z}(\mathcal{C})$. By the second orthogonality relation \cite[Cor. 6.11]{Shi17} one has:

\begin{equation}
\sum_{i=0}^{m} \alpha_{il} \alpha_{i\tau(k)} = \delta_{l,k} \frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^k)}
\end{equation}

Applying $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ to this Equation one obtains:

\begin{equation}
\sum_{i=0}^{m} \sigma(\alpha_{il}) \sigma(\alpha_{i\tau(k)}) = \delta_{l,k} \sigma\left(\frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^k)}\right)
\end{equation}

This implies that

\begin{equation}
\sum_{i=0}^{m} \alpha_{i\tau(l)} \alpha_{i\tau(k)} = \delta_{l,k} \sigma\left(\frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^\tau(k))}\right)
\end{equation}

On the other hand, by the same orthogonality relation we have:

\begin{equation}
\sum_{i=0}^{m} \alpha_{i\tau(l)} \alpha_{i\tau(k)} = \delta_{\tau(l),\tau(k)} \frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^\tau(k))}
\end{equation}

Therefore for $l = k$ it follows that

\begin{equation}
\sigma\left(\frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^k)}\right) = \frac{\dim(\mathcal{C})}{\dim(\mathcal{C}^\tau(k))}
\end{equation}

Moreover, if $\mathcal{C}$ is weakly integral it follows that

\begin{equation}
\sigma(\dim(\mathcal{C}^k)) = \dim(\mathcal{C}^\tau(k)).
\end{equation}

By Remark\[14 that both dimensions above are integers and therefore $\dim(\mathcal{C}^k) = \dim(\mathcal{C}^\tau(k))$. \hfill \square

**Remark 15.** From this it follows by Equation \eqref{eq:equation_10} that

\begin{equation}
\sigma(n_j) = n_{\tau(j)}.
\end{equation}

In particular, if $\mathcal{C}$ is weakly integral then $\dim(\mathcal{C}^l)$ is an integer and it follows that $n_j = n_{\tau(j)}$.

**Proposition 16.** If $\mathcal{C}$ is an integral fusion category then

\begin{equation}
\sigma(c^k_{ij}) = c^{\tau(k)}_{\tau(i)\tau(j)}
\end{equation}

for all $i, j, k$.

**Proof.** If $\mathcal{C}$ is integral then $\dim(\mathcal{C}^l)$ are integers by Remark\[14. On the other hand $\sigma(d_i) = d_i$ since $d_i = \text{FPdim}(X_i) \in \mathbb{Z}_{>0}$. Then applying $\sigma$
to the above formula it follows that
\[
\sigma(c_{ij}^k) = \left( \frac{\dim(C_j) \dim(C_{ij})}{\dim(C)} \right) \left( \sum_{s=0}^{m} \sigma(\alpha_{si}) \sigma(\alpha_{sk}) \sigma(d_s) \right) = \\
\left( \frac{\dim(C_j) \dim(C_{ij})}{\dim(C)} \right) \left( \sum_{s=0}^{m} \alpha_{s\tau(i)} \alpha_{s\tau(j)} \alpha_{s\tau(k)} \right) \\
= c_{\tau(j)}^{\tau(k)}.
\]

3.2. Action \( \sigma_F \) on CF(\( C \)). For any \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) we define \( \sigma(\chi_i) \in \text{CF}(\mathbb{C}) \) with \( \sigma.\chi_i := \sum_{j=0}^{m} \sigma(\alpha_{ij}) \chi_j \in \text{CF}(\mathbb{C}) \) for all irreducible characters \( \chi_i \).

Define also a \( \mathbb{C} \)-linear map \( \sigma_F : \text{CF}(\mathbb{C}) \rightarrow \text{CF}(\mathbb{C}) \) written on the linear \( \mathbb{C} \)-basis \( \{\chi_i\}_i \) by
\[
\sigma_F(\chi_i) = \sigma.\chi_i = \sum_{j=0}^{m} \sigma(\alpha_{ij}) F_j, \text{ if } \chi_i = \sum_{j=0}^{m} \alpha_{ij} F_j.
\]

Proposition 17. Let \( \mathcal{C} \) be a pivotal fusion category. With the above notations, for any \( \sigma \in \text{Gal}(K/\mathbb{Q}) \) one has that
\[
\sigma_F(F_j) = F_{\tau^{-1}(j)}, \quad \sigma_F^{-1}(F_j) = F_{\tau(j)}, \quad \sigma_F^{-1} = (\sigma^{-1})_F.
\]

Proof. Recall that \( \alpha_{i\tau(j)} = \mu_{i\tau}(\chi_i) = (\sigma.\mu_j)(\chi_i) = \sigma(\alpha_{ij}) \) and by Equation (20)
\[
\sum_{i=0}^{m} \alpha_{ij} \alpha_{i\tau(k)} = \delta_{i,k} \frac{\dim(C)}{\dim(C_k)} = \delta_{i,k} n_j.
\]

On the other hand, since by Equation (9) we have the formula \( F_j := \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \chi_i \right) \) it follows that,
\[
\sigma_F(F_j) = \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \sigma_F(\chi_i) \right) \\
= \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \left( \sum_{l=0}^{m} \sigma(\alpha_{il}) F_l \right) \right) \\
= \frac{1}{n_j} \left( \sum_{l=0}^{m} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \sigma(\alpha_{il}) \right) F_l \right) \\
= \frac{1}{n_j} \left( \sum_{l=0}^{m} \left( \sum_{i=0}^{m} \alpha_{ij} \sigma(\alpha_{i\tau(l)}) \right) F_l \right) \overset{20}{=} F_{\tau^{-1}(j)}.
\]
Since \( \tau \) is a permutation of \( \mathcal{J} \) it follows that \( \sigma_F \) is bijective. Moreover since \((\tau \sigma)^{-1} = \tau_{\sigma^{-1}} \) the last two equality also follow.

**Proposition 18.** Let \( \mathcal{C} \) be a pivotal fusion category. One has that \( \sigma_F : \mathcal{C}(\mathcal{C}) \rightarrow \mathcal{C}(\mathcal{C}) \) is an algebra map.

**Proof.** Indeed, suppose that \( \chi_{i_1} \chi_{i_2} = \sum_{k=0}^m N_{i_1i_2}^k \chi_k \). Expanding on the formula \( \chi_i = \sum_{j=0}^m \alpha_{ij} F_j \) it follows that

\[
\alpha_{i_1j} \alpha_{i_2j} = \sum_{k=0}^m N_{i_1i_2}^k \alpha_{kj} \tag{25}
\]

Applying \( \sigma \) to this Equation one has

\[
\sigma_F(\chi_{i_1} \chi_{i_2}) = \sum_{k=0}^m N_{i_1i_2}^k \sigma_F(\chi_k) = \sum_{k=0}^m N_{i_1i_2}^k \left( \sum_{j=0}^m \sigma(\alpha_{kj}) F_j \right)
\]

\[
= \sum_{j=0}^m \left( \sum_{k=0}^m N_{i_1i_2}^k \sigma(\alpha_{kj}) \right) F_j
\]

\[
= \sum_{j=0}^m \sigma(\alpha_{i_1j}) \sigma(\alpha_{i_2j}) F_j \]

\[
= \sigma_F(\chi_{i_1}) \sigma_F(\chi_{i_2}).
\]

Note that \( \chi_0 \), the character of the unit object is the identity of the \( \mathbb{C} \)-algebra \( \mathcal{C}(\mathcal{C}) \).

It follows that \( \sigma_F(\chi_0) = \sigma_F(\sum_{j=0}^m F_j) = \sum_{j=0}^m F_{\tau^{-1}(j)} = \chi_0 \) which shows that \( \sigma_F \) is unitary.

**Remark 19.** Since \( \sigma_F \) is linear it follows from above that

\[
\sigma_F(\sum_{j=0}^m \alpha_j F_j) = \sum_{j=0}^m \alpha_j F_{\tau^{-1}(j)}. \tag{26}
\]

for all scalars \( \alpha_j \in \mathbb{C} \).

**Corollary 20.** Let \( \mathcal{C} \) be a pivotal fusion category. For any \( \sigma, \sigma' \in \text{Gal}(\mathbb{K}/\mathbb{Q}) \) one has

\[ (\sigma' \sigma)_F = \sigma_F \circ \sigma'_F. \]

**Lemma 21.** If \( \mathcal{C} \) is a weakly integral fusion category then with the above notations one has that:

\[
m_C(\sigma(\chi_{i_1}), \sigma(\chi_{i_2})) = \delta_{i_1, i_2} \tag{27}
\]

for all \( i \).
Proof. Since $\mathcal{C}$ is weakly integral one has that $\dim(C^j) = \dim(C^{\tau(j)})$ for all $j$. On the other hand by Equation (11) one has that

$$m_{\mathcal{C}}(\sigma(\chi_{i_1}), \sigma(\chi_{i_2})) = m_{\mathcal{C}}(\sum_{j=0}^{m} \sigma(\alpha_{i_1,j})F_j, \sum_{j=0}^{m} \sigma(\alpha_{i_2,j})F_j) =$$

$$= \sum_{j=0}^{m} \frac{\dim(C^j)}{\dim(C)} \alpha_{i_1,j} \alpha_{i_2,\tau(j)} =$$

$$= \sum_{j=0}^{m} \frac{\dim(C^{\tau(j)})}{\dim(C)} \alpha_{i_1,j} \alpha_{i_2,\tau(j)} =$$

$$= \sum_{j=0}^{m} \frac{\dim(C^j)}{\dim(C)} \alpha_{i_1,j} \alpha_{i_2,j} =$$

$$= \delta_{i_1, i_2}. \square$$

3.3. When $\hat{\sigma_{F}}$ is an algebra map? Let $\mathcal{C}$ be a pivotal fusion category with a commutative character ring $CF(\mathcal{C})$. Recall [Bur20b] that $\hat{CF(\mathcal{C})}$ is defined as the space of all linear maps $f : CF(\mathcal{C}) \to \mathbb{C}$. It is a $\mathbb{C}$-algebra with multiplication defined linearly by

$$(\mu_{j_1} \star \mu_{j_2})(\frac{\chi_i}{d_i}) = \mu_{j_1}(\frac{\chi_i}{d_i})\mu_{j_2}(\frac{\chi_i}{d_i}).$$

where $d_i$ is the quantum dimension of the simple objects $X_i$. Since $CF(\mathcal{C})$ is a commutative ring it follows as above that the set of all algebra homomorphisms $\mu_j : CF(\mathcal{C}) \to \mathbb{C}$ is a $\mathbb{C}$-linear basis for $CF(\mathcal{C})$.

As above, since we assume $\mathbb{C}^0 = 1_{Z(\mathcal{C})}$ it follows that $\mu_0$ is the unit of $\hat{CF(\mathcal{C})}$. It follows that one can write that

$$(28) \quad \quad \mu_{j_1} \star \mu_{j_2} = \sum_{k=0}^{m} \hat{p}(j_1, j_2) \mu_k$$

for some scalars $\hat{p}(j_1, j_2) \in \mathbb{C}$.

From the proof [Bur20b, Theorem 1.1] one has that

$$(29) \quad \quad c_{i,j}^k = \frac{\dim(C^i)}{\dim(C^k)} \frac{\dim(C^j)}{\dim(C^k)} \hat{p}(i, j).$$

Note that as above the unit of the $\mathbb{C}$-algebra $CF(\mathcal{C})$ is given by the quantum dimension function $d : CF(\mathcal{C}) \to \mathbb{C}, \quad d(\chi_i) = d_i = \chi_i(1_{Z(\mathcal{C})}).$
For any $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ recall that we defined a linear map $\widehat{\sigma}_F : \widehat{\text{CF}}(\mathcal{C}) \to \widehat{\text{CF}}(\mathcal{C})$ which on the canonical basis given by the algebra homomorphisms $\mu_j$ is given by $\widehat{\sigma}_F(\mu_j) := \sigma \cdot \mu_j = \mu_{\tau(j)}$.

**Proposition 22.** Let $\mathcal{C}$ be a pivotal fusion category with a commutative Grothendieck ring.

1. Then $\widehat{\sigma}_F : \widehat{\text{CF}}(\mathcal{C}) \to \widehat{\text{CF}}(\mathcal{C})$ is an algebra map if and only if

$$\sum_{k=0}^m \sigma^{-1}(\widehat{p}_k(j_1, j_2)) \alpha_{ik} = \sum_{k=0}^m \widehat{p}_k(j_1, j_2) \frac{d_i}{\sigma^{-1}(d_i)} \alpha_{ik}$$

for all $i, k, j_1, j_2$.

2. If $\mathcal{C}$ is such that $d_i \in \mathbb{Z}$ for all $i$, then $\widehat{\sigma}_F$ is an algebra map if and only if $\widehat{p}_k(j_1, j_2)$ are rational numbers, for all $j_1, j_2, k$.

3. It follows that in the case $d_i \in \mathbb{Z}$ for all $i$, $\widehat{\sigma}_F$ are algebra maps for all $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$ if and only if $\widehat{p}_k(j_1, j_2)$ are rational numbers, for all $j_1, j_2, k$.

**Proof.** One has to show that

$$\widehat{\sigma}_F(\mu_{j_1} \star \mu_{j_2}) = \widehat{\sigma}_F(\mu_{j_1}) \star \widehat{\sigma}_F(\mu_{j_2}),$$

for all $j_1, j_2 \in \mathcal{J}$. This is equivalent to

$$\widehat{\sigma}_F(\mu_{j_1} \star \mu_{j_2}) \left[ \frac{\chi_i}{d_i} \right] = \widehat{\sigma}_F(\mu_{j_1}) \left[ \frac{\chi_i}{d_i} \right] \star \widehat{\sigma}_F(\mu_{j_2}) \left[ \frac{\chi_i}{d_i} \right],$$

for any irreducible character $\chi_i$.

As above we have $\mu_{j_1} \star \mu_{j_2} = \sum_{k=0}^m \widehat{p}_k(j_1, j_2) \mu_k$ and evaluating both sides at $\frac{\chi_i}{d_i}$ it follows that

$$\mu_{j_1} \left( \frac{\chi_i}{d_i} \right) \mu_{j_2} \left( \frac{\chi_i}{d_i} \right) = \sum_{k=0}^m \widehat{p}_k(j_1, j_2) \mu_k \left( \frac{\chi_i}{d_i} \right)$$

which can be written as

$$\frac{1}{d_i} \alpha_{i_{j_1}i_{j_2}} = \sum_{k=0}^m \widehat{p}_k(j_1, j_2) \alpha_{ik}.$$ 

Applying $\sigma^{-1}$ to the above Equation one obtains

$$\frac{\sigma^{-1}(\alpha_{i_{j_1}i_{j_2}})}{\sigma^{-1}(d_i)} = \sum_{k=0}^m \sigma^{-1}(\widehat{p}_k(j_1, j_2)) \sigma^{-1}(\alpha_{ik})$$
On the other hand, by Equation 28 note that

\[ \tilde{\sigma}_F(\mu_{j_1} \ast \mu_{j_2}) = \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \tilde{\sigma}_F(\mu_k) \]

and

\[ \tilde{\sigma}_F(\mu_j)(\frac{\chi_i}{d_i}) = \mu_{\tau(j)}(\frac{\chi_i}{d_i}) = \frac{\alpha_{i\tau(j)}}{d_i} = \frac{\sigma(\alpha_{ij})}{d_i} \]

Therefore

\[ \tilde{\sigma}_F(\mu_{j_1} \ast \mu_{j_2})(\frac{\chi_i}{d_i}) = \left( \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \tilde{\sigma}_F(\mu_k) \right)(\frac{\chi_i}{d_i}) = \]

\[ = \frac{1}{d_i} \left( \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \sigma(\alpha_{ik}) \right) = \]

\[ = \sigma \left( \sum_{k=0}^{m} \sigma^{-1}(\hat{p}_k(j_1, j_2)) \alpha_{ik} \frac{1}{\sigma^{-1}(d_i)} \right) \]

Moreover,

\[ \tilde{\sigma}_F(\mu_{j_1})(\frac{\chi_i}{d_i}) \tilde{\sigma}_F(\mu_{j_2})(\frac{\chi_i}{d_i}) = \frac{1}{d_i} \sigma(\alpha_{ij_1}) \sigma(\alpha_{ij_2}) = \]

\[ = \frac{1}{d_i^2} \sigma(\alpha_{ij_1} \alpha_{ij_2}) = \]

\[ \frac{1}{d_i^2} \sigma \left( \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \alpha_{ik} d_i \right) = \]

\[ = \sigma \left( \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \alpha_{ik} \frac{d_i}{\sigma^{-1}(d_i)^2} \right) \]

Therefore \( \tilde{\sigma}_F \) is an algebra map if and only if

\[ \sum_{k=0}^{m} \sigma^{-1}(\hat{p}_k(j_1, j_2)) \alpha_{ik} \frac{1}{\sigma^{-1}(d_i)} = \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \alpha_{ik} \frac{d_i}{\sigma^{-1}(d_i)^2} \]

which can be written

\[ \sum_{k=0}^{m} \sigma^{-1}(\hat{p}_k(j_1, j_2)) \alpha_{ik} = \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \frac{d_i}{\sigma^{-1}(d_i)} \alpha_{ik} \]

On the other hand if \( d_i \in \mathbb{Z} \) then \( \sigma^{-1}(d_i) = d_i \) and therefore \( \tilde{\sigma}_F \) is an algebra map if and only if

\[ \sum_{k=0}^{m} \sigma^{-1}(\hat{p}_k(j_1, j_2)) \alpha_{ik} = \sum_{k=0}^{m} \hat{p}_k(j_1, j_2) \alpha_{ik}. \]
Since \( \{\alpha_{ik}\} \) is an invertible matrix it follows that \( \widehat{\sigma}_F \) is an algebra map if and only if \( \sigma^{-1}(\hat{p}_k(j_1, j_2)) = \hat{p}_k(j_1, j_2) \).

**Remark 23.** For weakly integral premodular categories from Equation (4.5) of the proof of [Bur20b, Theorem 1.3] one has

\[
\hat{p}_k(i, j) = \sum_{v \in A_k} P_v(s, u) = \sum_{v \in A_k} \frac{N_{sv} \hat{d}_v}{d_v d_u}.
\]

This shows that in the case of \( \mathcal{C} \) is a weakly integral premodular category the scalars \( \hat{p}_k(j_1, j_2) \) are rational numbers and therefore \( \widehat{\sigma}_F \) is an algebra map for all \( \sigma \in \text{Gal}(K/Q) \).

**Proposition 24.** Let \( \mathcal{C} \) be an integral premodular category \( \mathcal{C} \). Then \( \widehat{\sigma}_F \) is an algebra map for all \( \sigma \in \text{Gal}(K/Q) \).

**Proof.** By the above remark, \( \hat{p}_k(i, j) \) are rational numbers and the result follows from Proposition 22. \( \square \)

### 3.4. Definition of \( \sigma_E \).

Let \( \mathcal{C} \) be a pivotal fusion category. Recall that there are dual bases for the canonical evaluation

\[
\langle , \rangle: \text{CF}(\mathcal{C}) \to \text{CE}(\mathcal{C}) \to \mathbb{C}, < \chi, z > \mapsto \chi \circ z
\]
given

\[
\{F_j, \frac{C_j}{\text{dim}(C_j)}\}.
\]

By [Bur20b, Theorem 3.12] there is a \( \mathbb{C} \)-algebra isomorphism

\[
\widehat{\text{CF}(\mathcal{C})} \cong \text{CE}(\mathcal{C}), \mu_j \mapsto \frac{C_j}{\text{dim}(C_j)}.
\]

**Lemma 25.** The isomorphism \( \alpha: \widehat{\text{CF}(\mathcal{C})} \to \text{CE}(\mathcal{C}), \mu_j \mapsto \frac{C_j}{\text{dim}(C_j)} \) verifies

\[
\langle \mu, \chi \rangle = \langle \chi, \alpha(\mu) \rangle
\]

**Proof.** Indeed, on the canonical bases one has:

\[
\langle \mu_j, \chi_i \rangle = \mu_j(\chi_i) = \alpha_{ij} = \langle \chi_i, \frac{C_j}{\text{dim}(C_j)} \rangle.
\]

\( \square \)

Transferring via \( \alpha \) the endomorphism \( \widehat{\sigma}_F \) on \( \text{CE}(\mathcal{C}) \) we obtain an endomorphism

\[
\sigma_E : \text{CE}(\mathcal{C}) \to \text{CE}(\mathcal{C}), \frac{C_j}{\text{dim}(C_j)} \mapsto \frac{C_{\tau(j)}}{\text{dim}(C_{\tau(j)})}.
\]
Indeed, \( \sigma_E(z) = \alpha(\hat{\sigma}_F(\alpha^{-1}(z))) \) for all \( z \in CE(\mathcal{C}) \). It follows that
\[
\sigma_E\left( \frac{C_j}{\dim(C_j)} \right) = \alpha(\hat{\sigma}_F(\alpha^{-1}(\frac{C_j}{\dim(C_j)}))) = \alpha(\hat{\sigma}_F(\mu_j)) = \alpha(\mu_{\tau(j)}) = \frac{C_{\tau(j)}}{\dim(C_{\tau(j)})}
\]

Since \( \sigma_E \) is linear it follows that
\[
\sigma_E(C_j) = \frac{\dim(C_j)}{\dim C_{\tau(j)}} C_{\tau(j)} n_{\tau(j)} C_{\tau(j)} n_j \sigma(n_j) C_{\tau(j)}
\]
for all \( j \).

Note that since \( \alpha \) is an algebra isomorphism it follows that \( \hat{\sigma}_F \) is an algebra map if and only if \( \sigma_E \) is an algebra endomorphism of \( CE(\mathcal{C}) \).

**Lemma 26.** If \( \mathcal{C} \) is weakly integral then
\[
\sigma_E(C_j) = C_{\tau(j)}
\]
for all \( j \).

**Proof.** In the weakly integral case one has that \( \dim(C_j) = \dim(C_{\tau(j)}) \) for all \( j \) by Equation (19). \( \square \)

Note that by Equation (14) it follows that \( c_{ij}^k \in \mathbb{K} \) for all \( i, j, k \).

**Lemma 27.** Let \( \mathcal{C} \) be a pivotal fusion category and \( \sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}) \). Then \( \sigma_E \) is an algebra map if and only if
\[
\frac{\sigma(n_k)}{n_k} c_{j_1, j_2}^k = \frac{\sigma(n_{j_1})}{n_{j_1}} \frac{\sigma(n_{j_2})}{n_{j_2}} c_{\tau(j_1)\tau(j_2)}^{\tau(k)}
\]

**Proof.** Note that \( \sigma_E \) is an algebra map if and only if
\[
\sigma_E(C_{j_1}C_{j_2}) = \sigma_E(C_{j_1})\sigma_E(C_{j_2}).
\]
On the other hand since \( C_{j_1}C_{j_2} = \sum_{k=0}^{m} c_{j_1, j_2}^k C_k \) it follows that
\[
\sigma_E(C_{j_1}C_{j_2}) = \sum_{k=0}^{m} c_{j_1, j_2}^k \frac{\sigma(n_{\tau(k)})}{n_k} C_{\tau(k)}.
\]
Note also that
\[
\sigma_E(C_{j_1})\sigma_E(C_{j_2}) = \frac{\sigma(n_{j_1})}{n_{j_1}} \frac{\sigma(n_{j_2})}{n_{j_2}} C_{\tau(j_1)} C_{\tau(j_2)} =
\]
\[
= \frac{\sigma(n_{j_1})}{n_{j_1}} \frac{\sigma(n_{j_2})}{n_{j_2}} \left( \sum_{k=0}^{m} c_{\tau(j_1), \tau(j_2)}^{\tau(k)} C_k \right) =
\]
\[
= \frac{\sigma(n_{j_1})}{n_{j_1}} \frac{\sigma(n_{j_2})}{n_{j_2}} \left( \sum_{k=0}^{m} c_{\tau(j_1), \tau(j_2)}^{\tau(k)} C_{\tau(k)} \right).
\]
Comparing the above two Equations we obtain that \( \sigma_E \) is an algebra map if and only if
\[
\frac{\sigma(n_k)}{n_k} c_{j_1, j_2}^k = \frac{\sigma(n_{j_1})}{n_{j_1}} \frac{\sigma(n_{j_2})}{n_{j_2}} c_{\tau(j_1)\tau(j_2)}^{\tau(k)}. \]
Corollary 28. If \( C \) is an integral fusion category then \( \sigma_E \) is an algebra map if and only if \( \sigma(c_{j_1,j_2}^k) = c_{j_1,j_2}^k \) for all \( j_1, j_2, k \).

Proof. If \( C \) is integral then \( \sigma(n_j) = n_j \) and the result follows from Proposition 16.

\[ \square \]

Lemma 29. Let \( C \) be a pivotal fusion category. For all \( \chi \in \text{CF}(C) \) and \( z \in \text{CE}(C) \) one has

\begin{equation}
< \chi, \sigma_E(z) >= < \sigma_F(\chi), z > .
\end{equation}

Proof. Since \(<, >\) is bilinear it is enough to verify the above identity on the basis \( \{ \chi_i \} \) of \( \text{CF}(C) \) and \( \{ \frac{C_j}{\dim(C_j)} \} \) of \( \text{CE}(C) \). Indeed, one has

\[ < \chi_i, \sigma_E \left( \frac{C_j}{\dim(C_j)} \right) >= < \chi_i, \sigma_F \left( \frac{C_{\tau(j)}}{\dim(C_{\tau(j)})} \right) >= \mu_{\tau(j)}(\chi_i) = \alpha_{ir(j)}. \]

On the other hand,

\[ < \sigma.F \chi_i, \frac{C_j}{\dim(C_j)} >= < \sum_{l=0}^{m} \sigma(\alpha_l) F_l, \frac{C_j}{\dim(C_j)} > = \sum_{l=0}^{m} \alpha_{l} \alpha_{ir(j)}. \]

\[ \square \]

Lemma 30. Let \( C \) be a pivotal fusion category. For any \( \chi_i \in \text{CF}(C) \) an irreducible character and all \( z, z' \in \text{CE}(C) \) one has

\begin{equation}
d_i \chi_i(z z') = \chi_i(z) \chi_i(z').
\end{equation}

Proof. If \( z = \sum_{l=0}^{m} z_l E_l \) and \( z' = \sum_{l=0}^{m} z'_l E_l \) then \( z z' = \sum_{l=0}^{m} z_l z'_l E_l \). Then \( \chi_i(z) \chi_i(z') = z_i z'_i d_{i}^2 \) and \( d(\chi_i) \chi_i(z z') = d^2_i z_i z'_i = \chi_i(z) \chi_i(z') \).

\[ \square \]

Lemma 31. Let \( C \) be a pivotal fusion category. Let \( \chi \in \text{CF}(C) \) and \( d \in \mathbb{C} \setminus \{0\} \). Then

\[ \chi(z z') = \frac{1}{d} \chi(z) \chi(z') \]

for all \( z, z' \in \text{CE}(C) \) if and only if

\[ \chi = \frac{d}{d_i} \chi_i \]

for some irreducible character \( \chi_i \) of \( C \).

Proof. Suppose that \( \chi = \sum_{i \in A} \alpha_i \chi_i \) where \( A \subseteq \{0, 1, \ldots, m\} \) is the set of all irreducible characters that have a non-zero coefficient \( \alpha_i \neq 0 \).

Let \( z = E_{i_1} \) and \( z' = E_{i_2} \). It follows that \( 0 = \sum_{i \in A} \alpha_i d_i \alpha_i d_i \) which is a contradiction. Therefore \( A \) is a single element and without loss of generality we may suppose \( A = \{i\} \). Therefore \( \chi = \alpha \chi_i \) and for \( z = z' = 1_{CE(C)} \) this gives that \( \alpha = \frac{d}{d_i} \).

\[ \square \]
Proposition 32. Let $\mathcal{C}$ be a pivotal fusion category and $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$. Then $\sigma_E$ is an algebra map if and only if there is a permutation $\eta$ of the indices $\{0, \ldots, m\}$ such that

$$\sigma_F(\chi_i) = \frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)}.$$  

(44)

for all $i \in \{0, \ldots, m\}$.

Proof. One has that $\sigma_E$ is multiplicative if and only if

$$\sigma_E(zz') = \sigma_E(z)\sigma_E(z')$$

for all $z, z' \in \text{CE}(\mathcal{C})$. Since the evaluation form is non-degenerate it follows that this is equivalent to

$$<\chi_i, \sigma_E(zz')> = <\chi_i, \sigma_E(z)\sigma_E(z')>,$$

for all $z, z' \in \text{CE}(\mathcal{C})$ and all $i$.

Note that

$$<\chi_i, \sigma_E(zz')> \Rightarrow <\chi_i, \sigma_F(\chi_i), zz'>$$

and

$$<\chi_i, \sigma_E(z)\sigma_E(z')> \Rightarrow \frac{1}{d_i} <\chi_i, \sigma_E(z)><\chi_i, \sigma_E(z')> = \frac{1}{d_i} <\sigma_F(\chi_i), z><\sigma_F(\chi_i), z'>$$

Therefore $\sigma_E$ is an algebra map if and only if

$$<\sigma_F(\chi_i), zz'> \Rightarrow \frac{1}{d_i} <\sigma_F(\chi_i), z><\sigma_F(\chi_i), z'>$$

(45)

for all $\chi_i \in \text{Irr}(\mathcal{C})$ and all $z, z' \in \text{CE}(\mathcal{C})$.

"$\Rightarrow$" Suppose that $\sigma_E$ is an algebra map. Equation (45) implies by Lemma 31 that that $\sigma_F(\chi_i)$ is a scalar of an irreducible character of $\mathcal{C}$. Therefore for all $i \in \{0, \ldots, m\}$ there is an index $\eta_i$ in $\{0, \ldots, m\}$ such that $\sigma_F(\chi_i) = \frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)}$. Since $\sigma_F$ is bijective it follows that $\eta$ is a permutation of $\{0, \ldots, m\}$.

"$\Leftarrow$" Conversely, if there is a permutation $\eta$ of the indices $\{0, \ldots, m\}$ such that $\sigma_F(\chi_i) = \frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)}$ then by Lemma 30 it follows that the Equation (45) is satisfied. Thus $\sigma_E$ is an algebra map in this case. Q.E.D.

Proposition 33. Let $\mathcal{C}$ be a pivotal fusion category and $\sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q})$. If $\sigma_E$ is an algebra map then $\sigma_E(E_i) = E_{\eta^{-1}(i)}$ for all $i$. 

Proof. Since $\sigma_E$ is an algebra isomorphism it permutes the central primitive $E_i$. Thus one can write $\sigma_E(E_i) = E_{\rho(i)}$ for some permutation $\rho = \rho_\sigma$ of $I$.

Recall that by Equation (42) one has

$$<\chi, \sigma_E(z)> = <\sigma_F(\chi), z>,$$

for all $\chi \in \text{CF}(C)$ and $z \in \text{CE}(C)$.

For $\chi = \chi_i$ and $z = E_s$ in the above equation it follows that

$$<\chi_i, \sigma_E(E_s)> = <\chi_i, E_{\rho(s)}> = \delta_{i, \rho(s)} d_i.$$

On the other hand, since $\sigma_E$ is an algebra map it follows by Equation (44) that

$$<\sigma_F(\chi_i), E_s> = <\frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)}, E_{s}> = \frac{d_i}{d_{\eta(i)}} <\chi_{\eta(i)}, E_s> = \delta_{\eta(i), s} d_i.$$

This shows that $\eta^{-1}(i) = \rho(i)$ for all $i$, i.e. $\eta^{-1} = \rho$. \qed

Proposition 34. Let $C$ be a pivotal fusion category. Suppose that $\sigma_E$ is an algebra map for some $\sigma \in \text{Gal}(K/\mathbb{Q})$. With the above notations it follows that

$$\eta(i^*) = \eta(i)^*$$

and

$$d_i d_i^* = d_{\eta(i)} d_{\eta(i)^*}$$

for all $i$.

Proof. Applying $\sigma_F \otimes \sigma_F$ to Equation (3) one obtains by Equations (22) and (24) that

$$\sum_{j=0}^{m} n_j F_j \otimes F_j = \sum_{i=0}^{m} \sigma_F(\chi_i) \otimes \sigma_F(\chi_i^*).$$

Since $\sigma_E$ is an algebra map it follows that Equation (48) can be written as

$$\sum_{j=0}^{m} n_j F_j \otimes F_j = \sum_{i=0}^{m} \frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)} \otimes \frac{d_i^*}{d_{\eta(i)^*}} \chi_{\eta(i)^*}.$$

Using again Equation (3) it follows that $\eta(i^*) = \eta(i)^*$ and $d_i d_i^* = d_{\eta(i)} d_{\eta(i)^*}$ for all $i$. \qed

Remark 35. In the weakly integral case the relation $d_i^2 = d_{\eta(i)^2}$ follows also from Equation (27).
Corollary 36. If $C$ is a spherical fusion category $\sigma_E$ is an algebra map then with the above notations one has $d_i = \pm d_{\eta(i)}$ for all $i$. If $C$ is a pseudo-unitary fusion category and $\sigma_E$ is an algebra map then with the above notations one has $d_i = d_{\eta(i)}$ for all $i$.

Proof. If $C$ is spherical then $d_i = d_i^*$ are real numbers and therefore $d_i^2 = d_{\eta(i)}^2$. Moreover, if $C$ is a pseudo-unitary fusion category then $d_i > 0$ in this case and therefore $d_i^2 = d_{\eta(i)}^2$ implies $d_i = d_{\eta(i)}$.

Corollary 37. Let $C$ be a pivotal fusion category and $\sigma \in \text{Gal}(K/Q)$ such that $\sigma_E$ is an algebra map. Then

$$\sigma(d_i) = d_i$$

for all $i \in \{0, \ldots, m\}$.

Proof. Note that by applying dimensions to Equation (44) it follows that $d(\sigma_F(\chi_i)) = d_i$ for all $i \in \{0, \ldots, m\}$. On the other hand, $d(\sigma_F(\chi_i)) = \sigma(\alpha_0) = \sigma(d_i)$ for all $i \in \{0, \ldots, m\}$.

Corollary 38. Let $C$ be a pivotal fusion category and suppose that $\sigma_E$ is an algebra map for all $\sigma$ and that the extension $Q \subseteq K$ is Galois. In this case $d_i \in \mathbb{Z}$ for all $i$.

Proof. $d_i$ are rational from previous Corollary and algebraic integers therefore they are integers.

Proposition 39. Let $C$ be a pivotal fusion category and $\sigma \in \text{Gal}(K/Q)$. If $\sigma_E$ is an algebra map then $\sigma_F(\lambda) = \lambda$, i.e. $\tau(0) = 0$.

Proof. By [Shi17, Equation (6.8)] one has

$$F_0 = \lambda = \frac{1}{\dim(C)} \left( \sum_{i=0}^{m} d_i^* \chi_i \right)$$

and therefore

$$\sigma_F(\lambda) = \frac{1}{\dim(C)} \left( \sum_{i=0}^{m} d_i^* \sigma_F(\chi_i) \right)$$

(50)

If $\sigma_E$ is an algebra map then there is a permutation $\eta = \eta_\sigma$ such that $\sigma_F(\chi_i) = \frac{d_i}{d_{\eta(i)}^*} \chi_{\eta(i)}$ for all $i$. Then Equation (50) implies that

$$\sigma_F(\lambda) = \frac{1}{\dim(C)} \left( \sum_{i=0}^{m} d_i^* \frac{d_i}{d_{\eta(i)}^*} \chi_{\eta(i)} \right) \frac{1}{\dim(C)} \left( \sum_{i=0}^{m} \frac{d_{\eta(i)} d_{\eta(i)}^*}{d_{\eta(i)}^*} \chi_{\eta(i)} \right) = \lambda$$

(51)

On the other hand by our assumption we have $\lambda = F_0$ and therefore $\sigma_F(\lambda) = F_{\tau^{-1}(0)}$. \qed
3.5. Proof of Theorem 11

Proof. Note that dim(\mathcal{C}) are integers since \mathcal{C} is integral. The proof of the theorem follows from Proposition 22 by noticing that \hat{p}_k(j_1, j_2) are rational numbers by Equation (29).

Proposition 40. Let \mathcal{C} be a pivotal fusion category and \sigma \in \text{Gal}(\mathbb{K}/\mathbb{Q}). If \sigma_E is an algebra map then one has that

\begin{equation}
\alpha_{ij} \left( \frac{d_i}{d_{\eta(i)}} \right) = \left( \frac{n_j}{\sigma^{-1}(n_j)} \right) \sigma^{-1}(\alpha_{\eta(i)j}).
\end{equation}

for all \(i, j\).

Proof. By Equation (9) one has that

\begin{equation}
F_j = \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i)\chi_i \right)
\end{equation}

Applying \(\sigma_E\) to this formula one has that

\begin{equation}
\sigma_E(F_j) = \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\sigma_E(\chi_i)) \sigma_E(\chi_i) \right) = \frac{1}{n_j} \left( \sum_{i=0}^{m} \mu_j(\chi_i) \frac{d_i}{d_{\eta(i)}} \chi_{\eta(i)} \right),
\end{equation}

On the other hand by Proposition 17 one has

\begin{equation}
\sigma_E(F_j) = F_{\tau^{-1}(j)} = \frac{1}{n_{\tau^{-1}(j)}} \left( \sum_{i=0}^{m} \mu_{\tau^{-1}(j)}(\chi_i)\chi_i \right) = \frac{1}{n_{\tau^{-1}(j)}} \left( \sum_{i=0}^{m} \mu_{\tau^{-1}(j)}(\chi_{\eta(i)})\chi_{\eta(i)} \right)
\end{equation}

Since \(n_j = \sigma(n_{\tau^{-1}(j)})\) by Equation (22), from the two formula for \(\sigma_E(F_j)\) one obtains that:

\begin{equation}
\frac{\mu_j(\chi_i)}{d_{\eta(i)}} = \frac{n_j}{\sigma^{-1}(n_j)} \mu_{\tau^{-1}(j)}(\chi_{\eta(i)})
\end{equation}

for all \(i, j\). This can also be written as:

\begin{equation}
\alpha_{ij} \left( \frac{d_i}{d_{\eta(i)}} \right) = \frac{n_j}{\sigma^{-1}(n_j)} \alpha_{\eta(i)\tau^{-1}(j)} = \frac{n_j}{\sigma^{-1}(n_j)} \sigma^{-1}(\alpha_{\eta(i)j})
\end{equation}

for all \(i, j\). □

Note that Equation (52) can also be written as

\begin{equation}
\sigma(\alpha_{ij}) = \alpha_{i\tau(j)} = \left( \frac{n_j}{\sigma^{-1}(n_j)} \right) \left( \frac{d_i}{d_{\eta(i)}} \right) \alpha_{\eta(i)j},
\end{equation}

which in the weakly-integral case can be written as

\begin{equation}
\sigma(\alpha_{ij}) = \alpha_{i\tau(j)} = \alpha_{\eta(i)j}
\end{equation}

for all \(i, j\).
4. Zeros in the character table

A classical result of Burnside in character theory states that for any irreducible character $\chi$ of a finite group $G$ with $\chi(1) > 1$ there is some $g \in G$ such that $\chi(g) = 0$, see [BZ81, Chapter 21]. This result was generalized in [GNN09] to weakly integral modular categories. Recall that a fusion category $\mathcal{C}$ is called weakly integral if its Frobenius-Perron dimension is an integer. In this case the Frobenius-Perron dimension of every simple object of $\mathcal{C}$ is the square root of an integer [ENO05].

In this section we extend further the above result to integral fusion categories with rational structure constants $c_{ij}^k$. The proof of this result goes along the same lines as the classical result proven in [BZ81, Chapter 21] with a slight modification concerning some AM-GM inequality. For the sake of completeness we include a sketch the proof of the result below.

We keep the same notations as in the previous sections.

Proof. For any $0 \leq i \leq m$ denote $T_i := T(\chi_i) = \{ j \mid \alpha_{ij} = 0 \}$ and $D_i := J \setminus (T_i \cup \{0\})$.

The first orthogonality relation from Equation (6) can be written as:

\[
\sum_{j=0}^{m} \frac{|\chi_i(C_j)|^2}{\dim(C_j)} = \dim(\mathcal{C}).
\]

Since for $j = 0$ one has $C_0 = 1_{\text{CE}(\mathcal{C})}$ this can be written as

\[
\dim(\mathcal{C}) = d_i^2 + \sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{\dim(C_j)}
\]

which gives that

\[
1 = \frac{\dim(\mathcal{C})}{d_i^2} - \sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{d_i^2 \dim(C_j)}
\]

On the other hand note that

\[
\dim(\mathcal{C}) = \sum_{j=0}^{m} \dim(C_j) = 1 + \sum_{j \in T_i} \dim(C_j) + \sum_{j \in D_i} \dim(C_j)
\]

Therefore Equation (54) can be written as:

\[
1 = \frac{1 + \sum_{j \in T_i} \dim(C_j)}{d_i^2} - \left(\sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{d_i^2 \dim(C_j)} - \sum_{j \in D_i} \frac{\dim(C_j)}{d_i^2}\right)
\]
Thus in order to finish the proof it is enough to show that
\[
\sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{d_i^2 \dim(C_j)} - \sum_{j \in D_i} \frac{\dim(C_j)}{d_i^2} \geq 0,
\]
since then it follows that \(1 + \sum_{j \in T_i} \dim(C_j) \geq d_i^2\). Since \(d_i > 1\) it follows that \(T_i \neq \emptyset\).

The inequality from Equation (56) can be written as
\[
\frac{1}{\sum_{j \in D_i} \dim(C_j)} \left( \sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{\dim(C_j)} \right) \geq 1.
\]
On the other hand the weighted AM-GM inequality gives that
\[
\frac{1}{\sum_{j \in D_i} \dim(C_j)} \left( \sum_{j \in D_i} \frac{|\chi_i(C_j)|^2}{\dim(C_j)} \right) \geq \left( \prod_{j \in D_i} \left( \frac{|\chi_i(C_j)|^2}{\dim(C_j)^2} \right)^{\dim(C_j)} \right)^{\frac{1}{|D_i|}},
\]
where \(|D_i| := \sum_{j \in D_i} \dim(C_j)\). Note that Equation (16) implies that the set \(D_i\) is stable under the Galois group \(\text{Gal}(\mathbb{K}/\mathbb{Q})\). This in turn implies that the product
\[
P_i := \prod_{j \in D_i} \left( \frac{|\chi_i(C_j)|^2}{\dim(C_j)^2} \right)^{\dim(C_j)}
\]
is fixed by the Galois group \(\text{Gal}(\mathbb{K}/\mathbb{Q})\) since \(\dim(C_{r(j)}) = \dim(C_j)\) by Equation (19). It follows that \(P_i\) is a rational number. On the other hand each factor of this product is an algebraic integer (since \(\dim(C_j) \in \mathbb{Z}_{\geq 0}\)) and therefore the entire product is an integer. Since it is positive it follows it is greater or equal to 1.

In analogy with group representations, we call a fusion category perfect if it has no non-trivial invertible objects. Next result generalizes a well-known result of Brauer from group representation theory to integral perfect braided fusion categories.

**Theorem 41.** Let \(C\) be a weakly integral braided fusion category. Then \(C\) is a perfect fusion category if and only if the following identity holds:
\[
\sum_{j=0}^{m} C_j = \frac{\dim(C)}{\prod_{j=0}^{m} \dim(C_j)} \left( \prod_{j=0}^{m} C_j \right).
\]

**Proof.** If \(C\) is perfect it is enough to show that
\[
\chi_i\left( \sum_{j=0}^{m} C_j \right) = \frac{\dim(C)}{\prod_{j=0}^{m} \dim(C_j)} \chi_i\left( \prod_{j=0}^{m} C_j \right).
\]
for any irreducible character $\chi_i \in \text{Irr}(\mathcal{C})$. For $\chi_i = \chi_0 = \epsilon_1$ one obtains equality by dimension argument. For $\chi_i \neq \chi_0$ both terms above are zero. Indeed note that $\lambda_\mathcal{C} = \frac{1}{\dim(\mathcal{C})}(\sum_{j=0}^{m} C_j)$ and therefore $\chi_i(\sum_{j=0}^{m} C_j) = 0$. On the other hand the right hand side is zero by the vanishing theorem since $\chi_i$ is the character of a non-invertible object. Conversely, note that if $M_i$ is an invertible object it follows that $\chi_i(C_j) \neq 0$ for any character $j$. Indeed one can write $\chi_i(\sum_{m} C_j) = 0$. On the other hand the right hand side is zero by the vanishing theorem since $\chi_i$ is the character of a non-invertible object. Conversely, note that if $M_i$ is an invertible object it follows that $\chi_i(C_j) \neq 0$ for any character $j$. Indeed one can write $\chi_i = \sum_{m} \alpha_j F_j$.

4.1. **The modular case.** Recall that the $S$-matrix of a braided pivotal fusion category is defined as $S_{XY} := \text{Tr}(C_{X,Y}C_{Y,X})$. Then if $S_{ij} := S_{M_i,M_j}$ it follows $S_{ij} = S_{ji}$, $S_{ij} = S_{ij}^*$. For all $i, j$.

As usually, by $\mathbb{Q}(S)$ is denoted the field obtained by adjoining the $S$-matrix entries $S_{ij}$ to $\mathbb{Q}$. By [EGNO15, Theorem 8.14.7] one has that $\mathbb{Q}(S)$ is also contained in a cyclotomic extension.

By [Bur20a] Theorem 6.2 in the modular case one has that $\dim(\mathcal{C}^k) = d_k^2$ and $\alpha_{ij} = S_{ij}/d_j$. The first orthogonality from Equation (6) can be written in this case as

\[
\sum_{k=0}^{m} S_{ik} S_{m^*k} = \delta_{i,m} \dim(\mathcal{C}).
\]

Since $S_{0j} = d_j$ one has that

$\mathbb{Q} \subseteq K = \mathbb{Q}(S_{ij}) \subseteq \mathbb{Q}(S) = \mathbb{Q}(S_{ij})$.

By the Fundamental Theorem of Galois theory, since the extension $\mathbb{Q} \subseteq K_{\mathcal{C}} = \mathbb{Q}(S_{ij})$ is Galois it follows that there is a group epimorphism $G \pi \rightarrow G_{\mathcal{C}}$. From here it follows that the two actions on $\mu_j$’s are compatible with the epimorphism $\pi$.

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