ON THE HILBERT SERIES OF VERTEX COVER ALGEBRAS OF UNMIXED BIPARTITE GRAPHS

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Abstract. We compute the reduced Gröbner basis of the toric ideal with respect to a suitable monomial order and we study the Hilbert series of the vertex cover algebra $A(G)$, where $G$ is an unmixed bipartite graph without isolated vertices.

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Introduction

Let $G = (V,E)$ be a simple (i.e., finite, undirected, loopless and without multiple edges) graph with the vertex set $V = [n]$ and the edge set $E = E(G)$. For $k \in \mathbb{N}$, a $k$-vertex cover of $G$ is a vector $c = (c_1, c_2, ..., c_n) \in \mathbb{N}^n$ such that $c_i + c_j \geq k$ for every edge $\{i,j\}$ of $G$.

Let $R = K[x_1, x_2, ..., x_n]$ be the polynomial ring over a field $K$. The vertex cover algebra $A(G)$ is defined as the subalgebra of the one variable polynomial ring $R[t]$ generated by all monomials $x_1^{c_1}x_2^{c_2}...x_n^{c_n}t^k$, where $c = (c_1, c_2, ..., c_n)$ is a $k$-vertex cover of $G$. This algebra was introduced and studied in [7]. Let $\mathfrak{m}$ be the maximal graded ideal of $R$. $\bar{A}(G) = A(G)/\mathfrak{m}A(G)$ is called the basic cover algebra and it was studied in [6], [1] and [2]. In [8], the Hilbert series of $A(G)$, $H_{A(G)}(z)$, for a Cohen-Macaulay bipartite graph $G$ is studied and several consequences are derived.

Our main aim in this paper is to extend the study of $H_{A(G)}(z)$ for unmixed bipartite graphs. It will turn out that many of the results concerning the Cohen-Macaulay case extend naturally to the larger class of unmixed bipartite graphs. The first step in getting the formula for $H_{A(G)}(z)$ is to compute the toric ideal of $A(G)$. This is done in Section 2.

In the last section we state the main theorem which relates $H_{A(G)}(z)$ to the Hilbert series of the basic cover algebras $\bar{A}(G_F)$, for all $F \subset [n]$. Based on this formula we derive sharp bounds for the multiplicity of $A(G)$.

1. The lattice associated to an unmixed bipartite graph

Let $G$ be an unmixed bipartite graph without isolated vertices. By [9, Theorem 1.1] we may assume that $G$ admits a bipartition of its vertices $V_n = W \cup W'$, where $W = \{x_1, ..., x_n\}$ and $W' = \{y_1, ..., y_n\}$, $n \geq 1$, such that:

(a) $\{x_i, y_i\} \in E(G)$, for all $i \in [n]$;
Corollary 1.3. Let $l$.

Throughout this paper, whenever we refer to an unmixed bipartite graph we assume it is given with its above bipartition.

For $\emptyset \neq F \subset [n]$ we denote by $G_F$ the subgraph of $G$ induced by the subset $V_F = \{x_i | i \in F\} \cup \{y_i | i \in F\}$.

Remark 1.1. $G_F$ also satisfies (a) and (b), hence $G_F$ is an unmixed bipartite graph on $V_F$ and each minimal vertex cover of $G_F$ has the cardinality equal to $|F|$.

Let $K_{i,j}$, $1 \leq i < j \leq n$, be the complete bipartite graph on $\{x_i, x_j\} \cup \{y_i, y_j\}$.

Lemma 1.2. Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 2$. Suppose that $G$ has an induced subgraph $K_{i,j}$ with $1 \leq i < j \leq n$. Let $H$ be the subgraph of $G$ induced by the subset $V_n \setminus \{x_i, y_j\}$. Then there exists a one-to-one correspondence between the sets $\mathcal{M}(G)$, respectively $\mathcal{M}(H)$, of minimal vertex covers of $G$, respectively $H$. More precisely, for all subsets $C \subset V_n \setminus \{x_i, y_j\}$ we have:

(i) if $x_i \in C$, then $C \in \mathcal{M}(H) \iff C \cup \{x_i\} \in \mathcal{M}(G)$;

(ii) if $x_i \notin C$, then $C \in \mathcal{M}(H) \iff C \cup \{y_j\} \in \mathcal{M}(G)$.

Proof. Let $C \in \mathcal{M}(H)$. If $x_i \in C$, put $B = C \cup \{x_i\}$. We show that $B \in \mathcal{M}(G)$. $B \cap \{x_k, y_i\} \neq \emptyset$, for all $\{x_k, y_i\} \in E(H)$ and $B \cap \{x_j, y_i\} \neq \emptyset$, for all $\{x_j, y_i\} \in E(G)$. Let $\{x_k, y_j\} \in E(G)$ with $k \notin \{i, j\}$. Since $\{x_k, y_j\} \in E(G)$ and $\{x_j, y_i\} \in E(G)$, it follows, by (b), that $\{x_k, y_i\} \in E(H)$. Hence $\{x_k, y_i\} \in E(H)$ and $C \cap \{x_k, y_i\} \neq \emptyset$.

$C \in \mathcal{M}(H)$ implies that $|C \cap \{x_i, y_j\}| = 1$ and, since $x_i \in C$, we have that $y_j \notin C$. On the other hand, $C \cap \{x_k, y_i\} \neq \emptyset$, hence $x_k \in C$. Thus $B \cap \{x_k, y_i\} \neq \emptyset$ and $B \in \mathcal{M}(G)$. If $y_i \in C$, put $B = C \cup \{y_j\}$. Similarly, it can be proved that $B \in \mathcal{M}(G)$.

Conversely, let $B \in \mathcal{M}(G)$. Then $|B \cap \{x_j, y_j\}| = 1$, which implies that either $B \cap \{x_j, y_j\} = \{x_j\}$ or $B \cap \{x_j, y_j\} = \{y_j\}$. If $B \cap \{x_j, y_j\} = \{x_j\}$, then, since $B \cap \{x_i, y_j\} \neq \emptyset$, it follows that $x_i \in B$. Put $C = B \cap (V_n \setminus \{x_i, y_j\})$. For all $\{x_k, y_i\} \in E(H)$ we have $\emptyset \neq B \cap \{x_k, y_i\} \subset B \setminus \{x_j\} = C$, hence $C$ is a vertex cover of $H$. Since $|C| = |B| - 1 = n - 1$, we get $C \in \mathcal{M}(H)$. Similarly, if $B \cap \{x_j, y_j\} = \{y_j\}$, then $x_j \notin B$ and $C = B \setminus \{y_j\} \in \mathcal{M}(H)$. 

Herzog and Hibi proved in [6] Theorem 1.2] that each unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, can be uniquely associated to a sublattice $\mathcal{L}_G$ of the Boolean lattice $\mathcal{L}_n$ on $\{p_1, p_2, ..., p_n\}$ such that $\emptyset \in \mathcal{L}_G$ and $\{p_1, p_2, ..., p_n\} \in \mathcal{L}_G$. The lattice $\mathcal{L}_G$ is defined as $\{\alpha \subset \{p_1, p_2, ..., p_n\} | C \in \mathcal{M}(G), p_k \in \alpha \iff x_k \in C\}$.

Corollary 1.3. In the hypothesis of Lemma 1.2 and with the above notation, we have $\mathcal{L}_H \simeq \mathcal{L}_G$.

Proof. Let $\nu : \mathcal{L}_H \to \mathcal{L}_G$ defined by $\nu(\alpha) = \begin{cases} \alpha \cup \{p_j\}, & \text{if } p_i \in \alpha, \\ \alpha, & \text{if } p_i \notin \alpha. \end{cases}$ By Lemma 1.2 $\nu$ is well defined and bijective. It can be easily checked that $\nu$ is a lattice isomorphism.

We show that $G$ has a unique Cohen-Macaulay bipartite subgraph, up to a graph isomorphism, such that the lattices associated to $G$ and $G'$ are isomorphic.
Proposition 1.4. Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices. Then there exists a Cohen-Macaulay bipartite subgraph $G'$ of $G$, unique up to a graph isomorphism, such that $\mathcal{L}_{G'} \simeq \mathcal{L}_G$. In particular, all maximal chains of $\mathcal{L}_G$ have the same length.

Proof. We proceed by induction on $n$. If $n = 1$, then $G$ is Cohen-Macaulay. Put $G' = G$ and the assertion trivially holds.

Let us suppose that $n > 1$. If $G$ is Cohen-Macaulay, then put $G' = G$. If $G$ is not Cohen-Macaulay, then, by [4, Theorem 3.4], $G$ has an induced subgraph $K_{\{i,j\}}$ with $1 \leq i < j \leq n$. Let $H$ be the subgraph of $G$ induced by the subset $V_n \{x_j, y_j\} \subset V_n$. By the induction hypothesis there exists a unique Cohen-Macaulay bipartite subgraph $G''$ of $H$, up to a graph isomorphism, such that $\mathcal{L}_{G''} \simeq \mathcal{L}_H$. Obviously, $G'$ is also a subgraph of $G$. By Corollary [3, Corollary 1.3] $\mathcal{L}_H \simeq \mathcal{L}_G$, hence $\mathcal{L}_{G''} \simeq \mathcal{L}_G$. Since $G'$ is Cohen-Macaulay, it follows, by [6, Theorem 2.2], that $\mathcal{L}_{G''}$ is a full sublattice of the Boolean lattice on $\{p_i | x_i \in V(G')\}$, which implies that all maximal chains of $\mathcal{L}_{G''}$ have the same length, hence the conclusion.

Remark 1.5. Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices. One may derive a procedure to compute a Cohen-Macaulay bipartite subgraph $G'$ of $G$ such that the lattices $\mathcal{L}_G$ and $\mathcal{L}_{G'}$ are isomorphic. In fact, $G' = G_F$, where $F$ is a maximal subset of $[n]$ such that $K_{\{i,j\}}$ is not an induced subgraph of $G_F$, for all distinct $i, j \in F$.

Remark 1.6. If $G' = G_F$, $F \subset [n]$, is a Cohen-Macaulay bipartite subgraph of $G$ from Proposition [3, Proposition 1.4] then, by Corollary [3, Corollary 1.3] the lattice isomorphism $\nu : \mathcal{L}_{G'} \to \mathcal{L}_G$ is defined by $\nu(\alpha') = \alpha' \cup \{p_j | j \in [n] \setminus F, p_i \in \alpha', K_{\{i,j\}} \subset G\}$, for all $\alpha' \in \mathcal{L}_{G'}$.

2. A Gröbner basis of the toric ideal of the vertex cover algebra of an unmixed bipartite graph

Let $S = K[x_1, ..., x_n, y_1, ..., y_n]$ and let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices. In this case $A(G)$ is the Rees algebra of the cover ideal $I_G$, which is generated by all monomials $x_1^{c_1} ... x_n^{c_n} y_1^{c_{n+1}} ... y_n^{c_{2n}}$, where $c = (c_1, ..., c_{2n})$ is a 1-vertex cover of $G$ ([7]). Thus

$$A(G) = S \oplus I_G t \oplus ... \oplus I_G^{k} t^k \oplus ...$$

Let $\mathcal{L}_G$ be the lattice associated to $G$. Put $B_G = K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}, \{u_\alpha\}_{\alpha \in \mathcal{L}_G}]$. For each $\alpha \in \mathcal{L}_G$ we denote $m_\alpha = (\prod_{p_i \in \alpha} x_i) \cdot (\prod_{p_j \notin \alpha} y_j)$. The toric ideal $Q_G$ of $A(G)$ is the kernel of the surjective ring homomorphism $\xi : B_G \to A(G)$, $\xi(x_i) = x_i$, $\xi(y_i) = y_i$, $1 \leq i \leq n$, $\xi(u_\alpha) = m_\alpha t$, $\alpha \in \mathcal{L}_G$.

Let $\langle \text{lex} \rangle$ the lexicographic order on $S$ induced by the ordering of the variables $x_1 > ... > x_n > y_1 > ... > y_n$. Let $\langle \# \rangle$ the reverse lexicographic order on the polynomial ring $K[\{u_\alpha\}_{\alpha \in \mathcal{L}_G}]$ induced by an ordering of the variables $u_\alpha$’s such that $u_\alpha > u_\beta$ if $\beta < \alpha$ in $\mathcal{L}_G$. Let $\langle \text{ex} \rangle$ the monomial order on $B_G$ defined as the product of the monomial orders $\langle \text{lex} \rangle$ and $\langle \# \rangle$ from above. This monomial order was introduced in [5].
Next, inspired by [3] Theorem 1.1, we compute the reduced Gröbner basis of the toric ideal of the vertex cover algebra of an unmixed bipartite graph $G$ on $V_n = W \cup W'$, $n \geq 1$, with respect to the monomial order $<_\text{lex}$. For $\alpha \in \mathcal{L}_G$ let $V(\alpha)$ be the set of all upper neighbours of $\alpha$ in $\mathcal{L}_G$. We denote $x_{\beta \setminus \alpha} = \prod_{p_i \in \beta \setminus \alpha} x_i$, $y_{\beta \setminus \alpha} = \prod_{p_i \in \beta \setminus \alpha} y_i$, where $\alpha \in \mathcal{L}_G$, $\alpha \neq \{p_1, p_2, \ldots, p_n\}$ and $\beta \in V(\alpha)$.

**Theorem 2.1.** Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices. Then the reduced Gröbner basis of the toric ideal $Q_G$ of the vertex cover algebra $A(G)$ with respect to $<_\text{lex}$ is:

$$G_{<_\text{lex}}(Q_G) = \{x_{\beta \setminus \alpha}u_\alpha - y_{\beta \setminus \alpha}u_\beta \mid \alpha \in \mathcal{L}_G, \alpha \neq \{p_1, p_2, \ldots, p_n\}, \beta \in V(\alpha)\}
\cup \{u_\alpha u_\beta - u_{\alpha \cup \beta}u_{\alpha \cap \beta} \mid \alpha, \beta \in \mathcal{L}_G, \alpha \not\subset \beta, \beta \not\subset \alpha\},$$

where the initial monomial of each binomial is the first monomial.

**Proof.** We essentially follow the proof of [3] Theorem 1.1 with a slight modification in its last part. As it was shown there, we only need to consider a primitive binomial $g$ of the reduced Gröbner basis of $Q_G$ with respect to $<_\text{lex}$. Let $g \in G_{<_\text{lex}}(Q_G)$,

$$g = \left(\prod_{i=1}^{n} x_i^{a_i} y_i^{b_i}\right) (u_{\alpha_1} u_{\alpha_2} \ldots u_{\alpha_r}) - \left(\prod_{i=1}^{n} x_i^{a'_i} y_i^{b'_i}\right) (u_{\alpha'_1} u_{\alpha'_2} \ldots u_{\alpha'_r}),$$

with $\alpha_1 \subseteq \alpha_2 \subseteq \ldots \subseteq \alpha_r$ and $\alpha'_1 \subseteq \alpha'_2 \subseteq \ldots \subseteq \alpha'_r$, chains in $\mathcal{L}_G$ and $\text{in}_{<_\text{lex}}(g)$ equal to the first monomial of $g$.

We assume that $g \notin K[\{u_\alpha\}_{\alpha \in \mathcal{L}_G}]$. As in [3] Theorem 1.1] we get that there exists some $1 \leq j \leq r$ such that $\alpha_j \not\subset \alpha_j$. Let $\beta$ be an upper neighbour of $\alpha_j$ with $\alpha_j \subset \beta \subset \alpha_j \cup \alpha_j'$ and let $p_i \in \beta \setminus \alpha_j$. Then $p_i \in \alpha_j'$ for all $k \geq j$ and $p_i \notin \alpha_l$ for all $l < j$. This implies that $a_i > 0$ for all $i$ for which $p_i \notin \beta \setminus \alpha_j$. Then the binomial $h = x_{\beta \setminus \alpha_j}u_{\alpha_j} - y_{\beta \setminus \alpha_j}u_{\beta} \in Q_G$ and $\text{in}_{<_\text{lex}}(h) = x_{\beta \setminus \alpha_j}u_{\alpha_j} \mid \text{in}_{<_\text{lex}}(g)$. Hence $\text{in}_{<_\text{lex}}(g)$ must coincide with $x_{\beta \setminus \alpha_j}u_{\alpha_j}$, and, moreover, $h = g$. \[\square\]

### 3. The Hilbert Series of the Vertex Cover Algebra of Unmixed Bipartite Graphs

Let $\{m_1, m_2, \ldots, m_l\}$ be the minimal system of generators of $I_G$. We view $A(G)$ as a standard graded $K$-algebra by assigning to each $x_i$ and $y_j$, $1 \leq i, j \leq n$ and to each $m_k$, $1 \leq k \leq l$, the degree 1.

The Hilbert function and the Hilbert series of the vertex cover algebra $A(G)$ are invariant to a certain class of graph isomorphisms.

**Remark 3.1.** Let $\sigma$ be a permutation of $[n]$ and let $^G\sigma G$ denote the bipartite graph on $V_n = W \cup W'$ with the edge set $E(^G\sigma G) = \{(x_{\sigma(i)}, y_{\sigma(j)}) \mid (x_i, y_j) \in E(G)\}$. The graph isomorphism $h : V(G) \to V(^G\sigma G)$, $h(x_i) = x_{\sigma(i)}$ and $h(y_j) = y_{\sigma(j)}$, $i, j \in [n]$, induces a $K$-automorphism of $S$ which maps $I_G$ onto $I_{^G\sigma G}$. Therefore, $A(G)$ and $A(^G\sigma G)$ have the same Hilbert function and series.
Let $\Delta(\mathcal{L}_G)$ be the order complex of the lattice $\mathcal{L}_G$. (We refer the reader to [3, §5.1] for the definition and properties of the order complex associated to a poset.) Let $S_G = K[\{u_\alpha\}_{\alpha \in \mathcal{L}_G}]$ be the polynomial ring in $|\mathcal{L}_G|$ variables over $K$. The toric ideal $\bar{Q}_G$ of the basic cover algebra $\bar{A}(G)$ is the kernel of the surjective ring homomorphism $\pi : S_G \rightarrow \bar{A}(G)$ defined by $\pi(u_\alpha) = m_\alpha$, for all $\alpha \in \mathcal{L}_G$.

**Proposition 3.2.** The graded $K$-algebra $\bar{A}(G)$ and the order complex $\Delta(\mathcal{L}_G)$ have the same vector $h$-vector.

**Proof.** By [3] Proposition 3.1 $\bar{Q}_G$ is a graded ideal and the initial ideal in $\langle \bar{Q}_G \rangle$ of the toric ideal $\bar{Q}_G$ coincides with the Stanley-Reisner ideal $I_{\Delta(\mathcal{L}_G)}$, hence $S_G/\bar{Q}_G$ and $K[\Delta(\mathcal{L}_G)]$ have the same $h$-vector. Since $S_G/\bar{Q}_G$ and $\bar{A}(G)$ are isomorphic as graded $K$-algebras, the conclusion follows. \qed

**Remark 3.3.** Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices and let $G'$ be a Cohen-Macaulay bipartite subgraph of $G$ with $\mathcal{L}_G \simeq \mathcal{L}_{G'}$. If $h$, respectively $h'$, are the $h$-vectors of $\bar{A}(G)$, respectively $\bar{A}(G')$, then, by using Proposition 3.2 and the fact that the lattices $\mathcal{L}_G$ and $\mathcal{L}_{G'}$ are isomorphic, it follows that $h = h'$. Moreover, by [3] Remark 1.3, $h_i \geq 0$, for all $0 \leq i \leq r + 1$ and $h_r = h_{r+1} = 0$, where $r = \text{rank}(\mathcal{L}_G)$.

In order to prove the main theorem we need some preparatory results. They are closely related to those for the Cohen-Macaulay case which were proved in [3].

Let $\emptyset \neq F \subseteq \mathcal{P}(n)$, $P_n(F) = \{p_i | i \in F\}$ and let $\alpha \in \mathcal{L}_{G,F}$, where $\bar{F}$ denotes the complement set of $F$ in $[n]$. We denote by $\delta_\alpha$ the maximal subset of $P_n(F)$ such that $\alpha \cup \delta_\alpha \in \mathcal{L}_G$. Note that

$$\delta_\alpha = \cup \{\gamma | \gamma \subset P_n(F), \alpha \cup \gamma \in \mathcal{L}_G\}.$$ 

If we set $\beta = \alpha \cup \delta_\alpha$, then, by the definition of $\delta_\alpha$, $\beta$ has the following property: there exists no subset $\emptyset \neq A \subset F$ such that $\beta \cup \{p_i | i \in A\}$ is an upper neighbour of $\beta$ in $\mathcal{L}_G$.

**Lemma 3.4.** Let $\emptyset \neq F \subseteq \mathcal{P}(n)$ and let $\mathcal{S}$ be the set of all $\beta \in \mathcal{L}_G$ with the property that there exists no subset $\emptyset \neq A \subset F$ such that $\beta \cup \{p_i | i \in A\}$ is an upper neighbour of $\beta$ in $\mathcal{L}_G$. Then the map $\varphi : \mathcal{L}_{G,F} \rightarrow \mathcal{S}$ defined by $\alpha \mapsto \beta = \alpha \cup \delta_\alpha$, is an isomorphism of posets.

**Proof.** We follow the proof of Lemma 1.4 in [3]. We show that $\varphi$ is invertible. Indeed, the map $\psi : \mathcal{S} \rightarrow \mathcal{L}_{G,F}$ defined by $\psi(\beta) = \beta \cap P_n(F)$ is the inverse of $\varphi$ since $\alpha = \beta \cap P_n(F)$ and $\alpha \in \mathcal{L}_{G,F}$.

Let $\alpha_1, \alpha_2 \in \mathcal{L}_{G,F}$ with $\alpha_1 \subseteq \mathcal{P} \alpha_2$ and $\beta_i = \varphi(\alpha_i) = \alpha_i \cup \delta_i$, $i = 1, 2$. We only need to show that $\beta_1 \subseteq \beta_2$ since the strict inclusion follows from the hypothesis $\alpha_1 \subseteq \mathcal{P} \alpha_2$. Let us assume that $\beta_1 \not\subset \beta_2$ and let $p_{r_1} \in \beta_1 \setminus \beta_2$ with $r_1 \in F$. Since $p_{r_1} \not\in \delta_2$, it follows that $\beta_2 \cup \{p_{r_1}\} \notin \mathcal{L}_G$.

We claim that $\{u \in F | p_u \in \beta_1 \setminus \{p_{r_1}\}\} \neq \emptyset$. Let us suppose, on the contrary, that $p_u \notin \beta_1 \setminus \{p_{r_1}\}$, for all $u \in F$. Then $\beta_1 = \alpha_1 \cup \{p_{r_1}\}$. Since $\beta_1, \beta_2 \in \mathcal{L}_G$, it follows that $\beta_1 \cup \beta_2 \in \mathcal{L}_G$. On the other hand, we have $\beta_1 \cup \beta_2 = \beta_2 \cup \{p_{r_1}\}$, which implies that $\beta_1 \cup \beta_2 \notin \mathcal{L}_G$, a contradiction.
By repeated application of this argument we get the sequence \( r_1, r_2, \ldots, r_k, r_{k+1}, \ldots \)
with \( r_{k+1} \in F \setminus \{r_1, \ldots, r_k\} \) and \( p_{r_{k+1}} \in \beta_1 \setminus (\beta_2 \cup \{p_{r_1}, \ldots, p_{r_k}\}) \), for all \( k \geq 0 \). Therefore, the set \( F \) is infinite, which is impossible. Hence \( \beta_1 \subseteq \beta_2 \).

Now let \( \beta_1, \beta_2 \in S \) with \( \beta_1 \subseteq \beta_2 \) and assume that \( \alpha_1 = \alpha_2 \), where \( \alpha_1 = \beta_1 \cap P_n(\bar{F}) \), and \( \alpha_2 = \beta_2 \cap P_n(\bar{F}) \). Then \( \delta_1 = \beta_1 \setminus P_n(\bar{F}) \subseteq \bar{\delta}_2 = \beta_2 \setminus P_n(\bar{F}) \). But this is impossible since \( \delta_1 \) is maximal among the subsets \( \gamma \subset P_n(\bar{F}) \) such that \( \alpha_1 \cup \gamma \in \mathcal{L}_G \). \( \square \)

The next result relates the Hilbert series of the vertex cover algebras \( \bar{A}(G) \) to the Hilbert series of the basic covers algebras \( A(G_F) \), for all \( F \subset [n] \). If \( F = \emptyset \), we put by convention \( H_{A(G_F)}(z) = \frac{1}{1-z} \).

**Theorem 3.5.** Let \( G \) be an unmixed bipartite graph on \( V_n = W \cup W' \), \( n \geq 1 \), without isolated vertices. For \( F \subset [n] \) let \( r_F = \text{rank}(\mathcal{L}_{G_F}) \), let \( H_{A(G_F)}(z) \) be the Hilbert series of \( A(G_F) \), and \( H_{A(G)}(z) \) be the Hilbert series of \( A(G) \). Then:

\[
H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{A(G_F)}(z) \left( \frac{z}{1-z} \right)^{|F|}.
\]

In particular, if \( h(z) = \sum_{j \geq 0} h_j z^j \), respectively \( h_F(z) = \sum_{j \geq 0} h^F_j z^j \), where \( h = (h_j)_{j \geq 0} \), respectively \( h_F = (h^F_j)_{j \geq 0} \), are the \( h \)-vectors of \( A(G) \), respectively of \( A(G_F) \), then

\[
h(z) = \sum_{F \subset [n]} h^F(z) (1-z)^{|F| - r_F} z^{|F|}.
\]

**Proof.** [1] can be proved exactly as in [3] Theorem 1.5.

It is known that \( H_{A(G)}(z) = \frac{h(z)}{(1-z)^{2n+1}} \) (since \( \text{dim} A(G) = \text{dim} S + 1 = 2n + 1 \) [3]) and \( H_{A(G_F)} = \frac{h^F(z)}{(1-z)^{|F|+1}} \) (since \( \text{dim} \bar{A}(G_F) = r_F + 1 \) [2]), for all \( F \subset [n] \), hence

\[
h(z) = \sum_{F \subset [n]} h^F(z) (1-z)^{|F| - r_F} z^{|F|}.
\]

**Remark 3.6.** By using [2] we get

\[
h_{n+1} = \sum_{F \subset [n]} (-1)^{|F| - r} h^F_{r+1} \quad \text{and} \quad h_n = \sum_{F \subset [n]} (-1)^{|F| - r} [h^F_r - (|F| - r) h^F_{r+1}],
\]

where \( r = r_F = \text{rank}(\mathcal{L}_{G_F}) \). By Remark 3.3, \( h^F_r = h^F_{r+1} = 0 \), for all \( \emptyset \neq F \subset [n] \). Hence \( h_{n+1} = h^0_1 \), \( h_n = h^0_0 = 1 \) and the \( a \)-invariant of \( A(G) \) is \( a = -n - 1 \).

In [7] Corollary 4.4 it was proved that \( A(G) \) is a Gorenstein ring, therefore, by [3] Corollary 4.3.8 (b) and Remark 4.3.9 (a), \( h_i = h_{n-i} \), for all \( 0 \leq i \leq n \).

**Corollary 3.7.** Let \( G \) be an unmixed bipartite graph on \( V_n = W \cup W' \), \( n \geq 1 \), without isolated vertices. Then

\[
e(A(G)) = \sum_{F \subset [n]} e(\bar{A}(G_F)),
\]

where, by convention, \( G_\emptyset \) is considered a Cohen-Macaulay subgraph of \( G \).
Proof. By [3] Theorem 2.2] $G_F$ is a Cohen-Macaulay bipartite graph if and only if 
$\text{rank}(\mathcal{L}_{G_F}) = |F|$, for all $\emptyset \neq F \subset [n]$. Thus (3) follows immediately from (2). \hfill \Box

We compute the Hilbert series of the vertex cover algebra of unmixed complete bipartite graphs $K_{n,n}$, $n \geq 1$.

**Proposition 3.8.** For all $n \geq 1$ $H_{A(K_{n,n})}(z) = \frac{1+z+...+z^n}{(1-z)^{2n+1}}$. In particular, the multiplicity $e(A(K_{n,n})) = n + 1$.

**Proof.** $\mathcal{L}_{K_{n,n}} = \{\emptyset, \{p_1, p_2, ..., p_n\}\}$, therefore, by Theorem [2.1] $Q_{K_{n,n}}$ is a principal ideal generated by $b = x_1...x_n u_2 - y_1...y_n u_1$, where $u_1 = u_{\{p_1,...,p_n\}}$ and $u_2 = u_{\emptyset}$. Then $A(K_{n,n}) \simeq B_{K_{n,n}}/Q_{K_{n,n}}$ and the minimal graded free resolution of $A(K_{n,n})$ is given by the exact short sequence $0 \rightarrow B_{K_{n,n}}(-(n+1)) \rightarrow B_{K_{n,n}} \rightarrow A(K_{n,n}) \rightarrow 0$. Hence $H_{A(K_{n,n})}(z) = \frac{1+z+...+z^n}{(1-z)^{2n+1}}$. In particular, the multiplicity $e(A(K_{n,n})) = n + 1$. \hfill \Box

Let $P_n = \{p_1, p_2, ..., p_n\}$ be a poset with a partial order $\leq$. We denote by $G(P_n)$ the bipartite graph on $V_n = W \cup W'$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i, y_j\}$ with $p_i \leq p_j$. It is said that a bipartite graph $G$ on $V_n = W \cup W'$ comes from a poset, if there exists a finite poset $P_n$ on $\{p_1, p_2, ..., p_n\}$ such that $p_i \leq p_j$ implies $i \leq j$, and after relabeling of the vertices of $G$ one has $G = G(P_n)$.

**Corollary 3.9.** Let $G$ be an unmixed bipartite graph on $V_n = W \cup W'$, $n \geq 1$, without isolated vertices. Then

$$n + 1 \leq e(A(G)) \leq n! \sum_{l=0}^{n} \frac{1}{l!}.$$ 

The left equality holds if and only if $G = K_{n,n}$ and the right equality holds if and only if $G$ comes from an antichain with $n$ elements.

**Proof.** By [3] and [11] Proposition 3.4(3), $e(A(G)) = \sum_{F \subseteq [n]} f^F_r$, where $r = \text{rank}(\mathcal{L}_{G_F})$ and $f^F_r$ is the last component of the $f$-vector of the order complex $\Delta(\mathcal{L}_{G_F})$. Then $e(A(G)) = n + 1 + \sum_{F \subseteq [n]} f^F_r$, which implies that $e(A(G)) \geq n + 1$. The equality holds if and only if $\text{rank}(\mathcal{L}_{G_F}) < |F|$ for all $F \subseteq [n]$ with $|F| \geq 2$, which is equivalent to $G = K_{n,n}$. On the other hand, $e(A(G)) = 1 + \sum_{F \subseteq [n]} f^F_r$. If $\text{rank}(\mathcal{L}_{G_F}) = |F|$, $\emptyset \neq F \subseteq [n]$, then $\mathcal{L}_{G_F}$ is a full sublattice of a Boolean lattice on a set with $|F|$ elements, hence $f^F_r \leq |F|!$ and $e(A(G)) \leq 1 + \sum_{F \subseteq [n]} |F|! = n! \sum_{l=0}^{n} \frac{1}{l!}$. The equality holds if and only if $\mathcal{L}_{G_F}$ is a Boolean lattice on a set with $|F|$ elements, for all $\emptyset \neq F \subseteq [n]$, which is equivalent to saying that $G$ comes from an antichain. \hfill \Box

**Remark 3.10.** In general, unmixed bipartite graphs are not uniquely determined, up to an isomorphism, by the $h$-vector of their corresponding vertex cover algebras. Let $G_3$ be the bipartite graph on $V_3$ with the edge set: $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$,
\{x_2, y_3\}, \{x_3, y_2\} and \(G'_3\) be the bipartite graph on \(V_3\) that comes from the chain \(P'_3 = \{p'_1, p'_2, p'_3\}\) with \(p'_1 \leq p'_2 \leq p'_3\). \(G_3\) and \(G'_3\) are unmixed and they are not isomorphic, and by computation we get \(H_{A(G_3)}(z) = H_{A(G'_3)}(z) = 1 + 3z + 3z^2 + z^3 \). However, unmixed complete bipartite graphs and bipartite graphs that come from chains and antichains are uniquely determined (up to a graph isomorphism) by the \(h\)-vector of their corresponding vertex cover algebras. The statement for bipartite graphs that come from chains and antichains was proved in [8, Proposition 2.3].

**Corollary 3.11.** Let \(G\) be an unmixed bipartite graph on \(V_n = W \cup W'\), \(n \geq 1\), without isolated vertices. Then \(G = K_{n,n}\) if and only if \(H_{A(G)}(z) = \frac{1+z+\ldots+z^n}{(1-z)^{2n+1}}\).

**Proof.** ("If") By using (2) we get \(h_1 = h_1^{[n]} + \text{rank}(\mathcal{L}_G)\). Since \(h_1^{[n]}\) is the component of rank 1 in the \(h\)-vector of \(\bar{A}(G)\), by using the formula which relates the \(h\)-vector to the \(f\)-vector of the order complex \(\Delta(\mathcal{L}_G)\), we get \(h_1^{[n]} = |\mathcal{L}_G| - \text{rank}(\mathcal{L}_G) - 1\), which implies that \(h_1 = |\mathcal{L}_G| - 1\). By hypothesis, \(h_1 = 1\), hence \(|\mathcal{L}_G| = 2\). \(G\) is an unmixed bipartite graph on \(V_n\), therefore, \(\mathcal{L}_G = \{\emptyset, \{p_1, \ldots, p_n\}\}\) and \(G = K_{n,n}\).

("Only if") It follows from Proposition 3.8. □

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