THE CATEGORY OF FINITE CHOW-WITT CORRESPONDENCES

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Abstract. We introduce the category of finite Chow-Witt correspondences over a perfect field \( k \) with \( \text{char}(k) \neq 2 \). We then define for any essentially smooth scheme \( X \) and integers \( p, q \in \mathbb{Z} \) generalized motivic cohomology groups \( H^{p,q}(X, \mathbb{Z}) \). We finally prove that \( H^{n,n}(\text{Spec}(L), \mathbb{Z}) = \text{KMW}_n(L) \) for any finitely generated field extension \( L/k \) and any \( n \in \mathbb{Z} \).

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Introduction

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Let \( k \) be a perfect field and let \( \text{Sm}_k \) be the category of smooth separated schemes of finite type over \( k \). One of the central ideas of V. Voevodsky in his construction of motivic cohomology is the definition of the category of finite correspondences \( \text{Cor}_k \) (see for instance [14]). Roughly speaking, the category \( \text{Cor}_k \) is obtained from \( \text{Sm}_k \) by taking the smooth schemes as objects and formally adding transfer morphisms \( \tilde{f} : Y \to X \) for any finite surjective morphism \( f : X \to Y \) of schemes. There is an obvious functor \( \text{Sm}_k \to \text{Cor}_k \) and the presheaves (of abelian groups) on \( \text{Sm}_k \) endowed with transfer morphisms for finite surjective morphisms become naturally presheaves on \( \text{Cor}_k \), also called presheaves with transfers. Classical Chow groups or Chow groups with coefficients à la Rost are examples of such presheaves. Having the category of finite correspondences in hand, it is then relatively easy to define

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motivic cohomology, which is an algebro-geometric analogue of singular cohomology in topology. The analogy between topology and algebraic geometry hinted at above extends in various directions: algebraic $K$-theory is an analogue of topological $K$-theory, and there is a motivic Atiyah-Hirzebruch spectral sequence relating motivic cohomology and algebraic $K$-theory, as the classical Atiyah-Hirzebruch spectral sequence relates topological $K$-theory and singular cohomology [9].

However, there are also many examples of interesting (pre-)sheaves without transfers in the above sense. Our main examples here are the Chow-Witt groups [5] or the cohomology of the (stable) homotopy sheaves $\pi^A_1(X, x)$ of a pointed smooth scheme $(X, x)$, most notably the Milnor-Witt $K$-theory sheaves $K_{MW}^n$ for $n \in \mathbb{Z}$. Such sheaves naturally appear in the Gersten-Grothendieck-Witt spectral sequence computing higher Grothendieck-Witt groups, aka Hermitian $K$-theory [8] or in the unstable classification of vector bundles over smooth affine schemes [1, 2], and thus they are far from being exotic.

Although these sheaves don’t have transfers for general finite morphisms, they do have transfers for finite surjective morphisms with trivial relative canonical sheaf (depending on a trivialization of the latter), and one can hope to formalize this notion following Voevodsky’s idea. In his work on the Friedlander-Milnor conjecture, F. Morel introduced a notion of generalized transfers in order to deal with this situation [18]. Our approach in this article is a bit different in spirit. We enlarge the category of smooth schemes using Chow-Witt finite correspondences. Roughly speaking, we replace the Chow groups (or cycles) in Voevodsky’s definition by Chow-Witt groups (or cycles with extra quadratic information) and define in this way the category of finite Chow-Witt correspondences $\tilde{\text{Cor}}_k$. The obvious functor $\text{Sm}_k \to \text{Cor}_k$ factors through our category; namely there are functors $\text{Sm}_k \to \tilde{\text{Cor}}_k$ and $\text{Cor}_k \to \text{Cor}_k$ whose composite is the classical functor. Given $X, Y$ smooth, the homomorphism $\tilde{\text{Cor}}_k(X, Y) \to \text{Cor}_k(X, Y)$ is in general neither injective (by far) nor surjective (yet almost). We call the presheaves on $\tilde{\text{Cor}}_k$ presheaves with generalized transfers in analogy with Morel’s definition. It is easy to see that presheaves with generalized transfers in our sense are also such presheaves in Morel’s sense, and we believe that the two notions are the same. A presheaf on $\text{Cor}_k$ is also a presheaf on $\tilde{\text{Cor}}_k$, but the examples above are genuine presheaves with generalized transfers, so our notion includes many more examples than the classical one.

Having $\tilde{\text{Cor}}_k$ at hand, we define generalized motivic cohomology groups $H^{p,q}(X, \tilde{Z})$ for any smooth scheme $X$ and any integers $p, q \in \mathbb{Z}$. The main difference with the classical groups is that the generalized motivic cohomology groups are non trivial for $q < 0$, in which range they can be idenified with the cohomology of the Gersten-Witt complex defined in [3]. Once these groups are defined, one can ask numerous questions. What are the properties of these presheaves? Do the associated sheaves satisfy (strict) homotopy invariance? Are they representable in the stable $\mathbb{A}^1$-homotopy category? In this article, we compute the first non trivial cohomology group for finitely generated field extensions $L/k$. This can be seen as an analogue in our setting of the Nesterenko-Suslin-Totaro theorem in the framework of motivic cohomology and Bloch’s higher Chow groups (see [21, 26]). Our purpose is to show that these groups are computable in some situations and that they are indeed interesting. Our main result reads then as follows.
Theorem. Let $L/k$ be a finitely generated field extension with $\text{char}(k) \neq 2$. Then there is an isomorphism

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H^{n,n}(L, \mathbb{Z}).$$

The isomorphism in the theorem generalizes the result on (ordinary) motivic cohomology in the sense that the diagram commutes

$$\begin{array}{ccc}
\bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) & \xrightarrow{\Phi_L} & \bigoplus_{n \in \mathbb{Z}} H^{n,n}(L, \mathbb{Z}) \\
\downarrow & & \downarrow \\
\bigoplus_{n \in \mathbb{N}} K_n^{M}(L) & \xrightarrow{} & \bigoplus_{n \in \mathbb{N}} H^{n,n}(L, \mathbb{Z})
\end{array}$$

where the vertical homomorphisms are the “forgetful” homomorphisms and the bottom map is the isomorphism produced by Nesterenko-Suslin-Totaro. Unsurprisingly, our proof is quite similar to theirs but there are some essential differences. For instance, the complex in weight one, denoted by $\mathbb{Z}(1)$, admits a split epimorphism to $K_1^{MW}$ paralleling the split epimorphism $\mathbb{Z}(1) \to K_1^{M}$. However, the kernel of the epimorphism $\mathbb{Z}(1) \to K_1^{MW}$ is probably not acyclic. We are thus forced to compute by hand its cohomology at the right spot to prove our theorem.

We foresee many applications of our main theorem and more generally of generalized motivic cohomology. An immediate one being the computation of the groups $H^{2n,n}(X, \mathbb{Z})$ for any smooth scheme $X$. Namely, the Bloch-Ogus spectral sequence and a dévissage theorem should imply that $H^{2n,n}(X, \mathbb{Z}) \simeq CH^n(X)$, where the latter is the Chow-Witt group of codimension $n$ points on $X$. More seriously, it is expected that the generalized motivic cohomology groups will naturally appear in an Atiyah-Hirzebruch-type spectral sequence computing higher Grothendieck-Witt groups (aka Hermitian $K$-theory). Moreover, these groups should give a precise idea of the stable homology sheaves $H^{k,l}_{\mathbb{A}^1}(\mathbb{G}_m^n)$ of smash powers of $\mathbb{G}_m$. The ingredients here are quite deep. Let $D^B_{\mathbb{A}^1}(k)$ be the full subcategory of $\mathbb{A}^1$-local objects in the derived category of Nisnevich sheaves of abelian groups, and let $D_{\mathbb{A}^1}(k)$ be the category obtained from the latter by formally inverting the Tate object. By construction, there is a functor $D_{\mathbb{A}^1}(k) \to DM(k)$, where $DM(k)$ is obtained by inverting the Tate object in the full subcategory of $\mathbb{A}^1$-local objects in the derived category of (bounded below) Nisnevich sheaves with transfers. We believe that this functor should factorize through the category $\widehat{DM}(k)$ defined analogously using $\widehat{Cor}_k$ instead of $\text{Cor}_k$, and thus $\widehat{DM}(k)$ is in some sense closer to $D_{\mathbb{A}^1}(k)$ than $DM(k)$. It is quite possible that the functor $D_{\mathbb{A}^1}(k) \to \widehat{DM}(k)$ is an equivalence, and thus the generalized motivic cohomology sheaves actually compute the stable homology sheaves $H^{k,l}_{\mathbb{A}^1}(\mathbb{G}_m^n)$. Our theorem could be seen as a tiny piece of evidence that this is indeed the case.

To conclude this introduction, let us mention some works related to this one. First, the approach of the Friedlander-Milnor conjecture of F. Morel in [18] is the starting point of our definition of the category $\text{Cor}_k$. As discussed above, his sheaves with generalized transfers should coincide with ours. Second, there is work in progress by M. Schlichting and S. Markett on an Atiyah-Hirzebruch spectral sequence for higher Grothendieck-Witt groups. We hope to prove that the groups they obtain at page 2 are indeed generalized motivic cohomology groups. Finally,
let us mention a recent preprint of A. Neshitov [20] in which computations similar to ours are done in the category of framed correspondences defined by G. Garkusha and I. Panin following ideas of V. Voevodsky [10].

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Conventions. The schemes are separated of finite type over some perfect field \( k \) with \( \text{char}(k) \neq 2 \). If \( X \) is a smooth connected scheme over \( k \), we denote by \( \Omega_{X/k} \) the sheaf of differentials of \( X \) over \( \text{Spec}(k) \) and write \( \omega_{X/k} := \det \Omega_{X/k} \) for its canonical sheaf. In general we define \( \omega_{X/k} \) connected component by connected component. We use the same notation if \( X \) is the localization of a smooth scheme at any point. If \( k \) is clear from the context, we omit it from the notation. If \( f : X \to Y \) is a morphism of (localizations of) smooth schemes, we set \( \omega_f = \omega_{X/k} \otimes f^*\omega_{Y/k} \). In case \( f : X \times Y \to X \) is the projection, we often write \( \omega_{X \times Y/X} \) instead of \( \omega_f \). If \( X \) is a scheme and \( n \in \mathbb{N} \), we denote by \( X^{(n)} \) the set of codimension \( n \) points in \( X \).

1. Milnor-Witt K-theory

In this section, we recall first the definition of Milnor-Witt K-theory of a field and its associated sheaf following [19, §3]. We then recall the definition of Chow-Witt groups and spend some time on their functorial properties. Following F. Morel, we don’t make any assumption on the characteristic of the field.

For any field \( F \), let \( K^\text{MW}_*(F) \) be the \( \mathbb{Z} \)-graded associative (unital) ring freely generated by symbols \( [a] \), for each \( a \in F^\times \), of degree 1 and by a symbol \( \eta \) in degree \(-1\) subject to the relations

1. \( [a][1-a] = 0 \) for any \( a \neq 0, 1 \).
2. \( [ab] = [a] + [b] + \eta[a][b] \) for any \( a, b \in F^\times \).
3. \( \eta[a] = [a]\eta \) for any \( a \in F^\times \).
4. \( \eta(2 + \eta[-1]) = 0 \).

If \( a_1, \ldots, a_n \in F^\times \), we denote by \( [a_1, \ldots, a_n] \) the product \( [a_1] \cdot \ldots \cdot [a_n] \). Let \( GW(F) \) be the Grothendieck-Witt ring of non degenerate bilinear symmetric forms on \( F \). Associating to a form its rank yields a surjective ring homomorphism

\[
\text{rank} : GW(F) \to \mathbb{Z}
\]

whose kernel is the fundamental ideal \( I(F) \). We can consider for any \( n \in \mathbb{N} \) the powers \( I^n(F) \) and we set \( I^n(F) = W(F) \) for \( n \leq 0 \), where the latter is the Witt ring of \( F \). It follows from [19, Lemma 3.10] that we have a ring isomorphism

\[
GW(F) \to K^\text{MW}_0(F)
\]

defined by \( \langle a \rangle \mapsto 1 + \eta[a] \). We will thus identify \( K^\text{MW}_0(F) \) with \( GW(F) \) later on. In particular, we will denote by \( \langle a \rangle \) the element \( 1 + \eta[a] \) and by \( \langle a_1, \ldots, a_n \rangle \) the element \( \langle a_1 \rangle + \ldots + \langle a_n \rangle \).

If \( K^\text{M}_*(F) \) denotes the Milnor K-theory ring defined in [15, §1], we have a graded surjective ring homomorphism

\[
f : K^\text{MW}_*(F) \to K^\text{M}_*(F)
\]
defined by $f([a]) = \{a\}$ and $f(\eta) = 0$. In fact, ker $f$ is the principal (two-sided) ideal generated by $\eta$ [17, Remarque 5.2]. We sometimes refer to $f$ as the forgetful homomorphism. On the other hand, let

$$H : K^M_n(F) \rightarrow K^\text{MW}_n(F)$$

be defined by $H(\{a_1,\ldots,a_n\}) = (1, \ldots, 1)a_1, \ldots, a_n$ = $(2 + \eta[-1])a_1, \ldots, a_n]$. Using relation (i) above, it is easy to check that $H$ is a well-defined graded (group) homomorphism, that we call the hyperbolic homomorphism. As $f(\eta) = 0$, we see that $fH : K^M_n(F) \rightarrow K^M_n(F)$ is the multiplication by 2 homomorphism.

For any $a \in F^\times$, let $\langle\langle a\rangle\rangle := a - 1 \in I(F) \subset \text{GW}(F)$ and for any $a_1, \ldots, a_n \in F^\times$ let $\langle\langle a_1, \ldots, a_n\rangle\rangle$ denote the product $\langle\langle a_1\rangle\rangle \cdots \langle\langle a_n\rangle\rangle$ (our notation differs from [17, §2] by a sign). By definition, we have $\langle\langle a_1, \ldots, a_n\rangle\rangle \in I^m(F)$ for any $m \leq n$. In particular, we can define a map

$$K^\text{MW}_n(F) \rightarrow I^n(F)$$

by $\eta^s[a_1, \ldots, a_{n+s}] \mapsto \langle\langle a_1, \ldots, a_{n+s}\rangle\rangle$ for any $s \in \mathbb{N}$ and any $a_1, \ldots, a_{n+s} \in F^\times$. It follows from [19, Definition 3.3] and [17, Lemme 2.3] that this map is a well-defined homomorphism. Moreover, the diagram

$$\begin{array}{ccc}
K^\text{MW}_n(F) & \rightarrow & I^n(F) \\
\downarrow f & & \downarrow \\
K^M_n(F) & \rightarrow & I^n(F)/I^{n+1}(F)
\end{array}$$

where $s_n$ is the map defined in [15, Theorem 4.1] is Cartesian by [17, Théorème 5.3].

1.1. Residues. Suppose now that $F$ is endowed with a discrete valuation $v : F \rightarrow \mathbb{Z}$ with valuation ring $\mathcal{O}_v$, uniformizing parameter $\pi$ and residue field $k(v)$. The following theorem is due to F. Morel [19, Theorem 3.15].

**Theorem 1.1.** There exists a unique homomorphism of graded groups

$$\partial^\pi_v : K^\text{sMW}_*(F) \rightarrow K^\text{MW}_{*-1}(k(v))$$

commuting with the product by $\eta$ and such that $\partial^\pi_v([\pi, u_2, \ldots, u_n]) = [\overline{u}_2, \ldots, \overline{u}_n]$ and $\partial^\pi_v([u_1, \ldots, u_n]) = 0$ for any units $u_1, \ldots, u_n \in \mathcal{O}_v^\times$

This theorem allows to define the unruffled Milnor-Witt K-theory sheaves as follows. If $X$ is a smooth integral $k$-scheme, any point $x \in X^{(1)}$ defines a discrete valuation $v_x$ for which we can choose a uniformizing parameter $\pi_x$. We then set for any $n \in \mathbb{Z}$

$$K^\text{MW}_n(X) := \ker(K^\text{MW}_n(k(X)) \rightarrow \bigoplus_{x \in X^{(1)}} K^\text{MW}_{n-1}(k(x)))$$

where the map is induced by the residue homomorphisms $\partial^\pi_v$. Let $i : V \subset X$ be a codimension 1 closed smooth subvariety defining a valuation $v$ on $k(X)$ with uniformizing parameter $\pi$. We then have $k(V) = k(v)$ and we can define a map

$$s^\pi_v : K^\text{MW}_n(k(X)) \rightarrow K^\text{MW}_n(k(V))$$

by $s^\pi_v(\alpha) = \partial^\pi_v([\pi/\alpha])$. It turns out that $s^\pi_v$ induces a homomorphism $i^\ast : K^\text{MW}_n(X) \rightarrow K^\text{MW}_n(V)$ by [19, proof of Lemma 2.12]. Working inductively, the same method shows that we can define pull-back maps $j^\ast$ for any smooth closed immersion $j :
$Z \to X$ [19, p. 21]. On the other hand, it is easy to see that a smooth morphism $h : Y \to X$ induces a homomorphism $h^* : K_n^{MW}(X) \to K_n^{MW}(Y)$ and it follows from the standard graph factorization $X \to X \times_k Y \to Y$ that any morphism $f : X \to Y$ gives rise to a pull-back map $f^*$. Thus $X \mapsto K_n^{MW}(X)$ defines a presheaf $K_n^{MW}$ on $\text{Sm}_k$ which turns out to be a Nisnevich sheaf [19, Lemma 2.12]. We call it the (n-th) Milnor-Witt sheaf.

Recall that one can also define residues for Milnor K-theory [15, Lemma 2.1] and therefore an unramified Nisnevich sheaf $K_n^M$ on $\text{Sm}_k$. It is easy to check that both the forgetful and the hyperbolic homomorphisms commute with residue maps. As a consequence, we get morphisms of sheaves $f : K_n^{MW} \to K_n^M$ and $H : K_n^M \to K_n^{MW}$ for any $n \in \mathbb{Z}$ and the composite $f H$ is the multiplication by 2 map.

The multiplication map $K_n^{MW} \times K_m^{MW} \to K_{n+m}^{MW}$ induces for any $m, n \in \mathbb{Z}$ a morphism of sheaves

$$K_n^{MW} \times K_m^{MW} \to K_{n+m}^{MW}.$$  

1.2. Twisting by line bundles. We will also need a version of Milnor-Witt K-theory twisted by line bundles, which we now recall following [24, §1.2] and [19, §5].

Let $V$ be a one dimensional vector space over the field $F$. One can consider the group ring $\mathbb{Z}[F^\times]$ and the $\mathbb{Z}[F^\times]$-module $\mathbb{Z}[V^\times]$ where $V^\times = V \setminus 0$. Letting $a \in F^\times$ act by multiplication by $(a)$ defines a $\mathbb{Z}$-linear action of the group $F^\times$ on $\text{GW}(F) = K_n^{MW}$, which therefore extends to a ring morphism $\mathbb{Z}[F^\times] \to K_n^{MW}$. Thus, we get a $\mathbb{Z}[F^\times]$-module structure on $K_n^{MW}(F)$ for any $n$, and the action is central (since $K_0^{MW}(F)$ is central in $K_n^{MW}(F)$). We then define the n-th Milnor-Witt group of $F$ twisted by $V$ as

$$K_n^{MW}(F, V) = K_n^{MW}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[V^\times].$$

On Nisnevich sheaves, we perform a similar construction. Let $\mathbb{Z}[G_m]$ be the Nisnevich sheaf on $\text{Sm}_k$ associated to the presheaf $U \mapsto \mathbb{Z}[O(U)^\times]$. The morphism of sheaves of groups $G_m \to (K_n^{MW})^\times$ defined by $u \mapsto \langle u \rangle$ for any $u \in O(U)^\times$ extends to a morphism of sheaves of rings $\mathbb{Z}[G_m] \to K_n^{MW}$, turning $K_n^{MW}$ into a $\mathbb{Z}[G_m]$-module, the action being central.

Let now $\mathcal{L}$ be a line bundle over a smooth scheme $X$, and let $\mathbb{Z}[\mathcal{L}^\times]$ be the Nisnevich sheafification of $U \mapsto \mathbb{Z}[\mathcal{L}(U) \setminus 0]$. Following [19, Chapter 5], we define the Nisnevich sheaf on $\text{Sm}_X$, the category of smooth schemes over $X$, by

$$K_n^{MW}(\mathcal{L}) = K_n^{MW} \otimes_{\mathbb{Z}[G_m]} \mathbb{Z}[\mathcal{L}^\times].$$

(again, this is the sheaf tensor product).

2. Transfers in Milnor-Witt K-theory

A very important feature of Milnor-Witt K-theory is the existence of transfers for finite field extensions. They are more subtle than the transfers for Milnor K-theory, and we thus explain them in some details in this section. To avoid technicalities, we suppose that the fields are of characteristic different from 2.

Recall first that the geometric transfers in Milnor-Witt K-theory are defined, for a monogeneous finite field extension $F = L(x)$, using the split exact sequence [19,
Theorem 3.24]

\[
\begin{array}{c}
0 \rightarrow K_n^{\text{MW}}(L) \xrightarrow{\partial} K_n^{\text{MW}}(L(t)) \xrightarrow{\bigoplus_{x \in (A^1_L)^{(t)}}} K_{n-1}^{\text{MW}}(L(x)) \rightarrow 0
\end{array}
\]

where $\partial$ is defined using the residue homomorphisms associated to the valuations corresponding to $x$ and uniformizing parameters the minimal polynomial of $x$ over $K$. If $\alpha \in K_{n-1}^{\text{MW}}(L(x))$, its transfer is defined by choosing a preimage in $K_n^{\text{MW}}(L(t))$ and then applying the residue homomorphism $-\partial_\infty$ corresponding to the valuation at infinity (with uniformizing parameter $-\frac{1}{\omega}$). The corresponding homomorphism $K_{n-1}^{\text{MW}}(L(x)) \rightarrow K_n^{\text{MW}}(L)$ is denoted by $\tau_L^x(x)$. It turns out that the geometric transfers do not generalize well to arbitrary finite field extensions $F/L$, and one has to modify them in a suitable way as follows.

Let again $L \subset F$ be a field extension of degree $n$ generated by $x \in F$. Let $p$ be the minimal polynomial of $x$ over $L$. We can decompose the field extension $L \subset F$ as $L \subset F^{\text{sep}} \subset F$, where $F^{\text{sep}}$ is the separable closure of $L$ in $F$. If $\text{char}(L) = l \neq 0$, then the minimal polynomial $p$ can be written as $p(t) = p_0(t^m)$ for some $m \in \mathbb{N}$ and $p_0$ separable. Then $F^{\text{sep}} = L(x^{1/m})$ and $p_0$ is the minimal polynomial of $x^{1/m}$ over $L$. Following [19, Definition 4.26], we set $\omega_0(x) := p_0'(x^{1/m}) \in F^\times$ if $l = \text{char}(L) \neq 0$ and $\omega_0(x) = p'(x) \in L^\times$ if char$(L) = 0$. Morel then defines cohomological transfers as the composites

\[
K_n^{\text{MW}}(F) \xrightarrow{(\omega_0(x))} K_n^{\text{MW}}(F) \xrightarrow{\tau_L^x(x)} K_n^{\text{MW}}(L)
\]

and denotes them by $\text{Tr}_L^F(x)$. If now $F/L$ is an arbitrary finite field extension, we can write

\[L = F_0 \subset F_1 \subset \ldots \subset F_m = F\]

where $F_i/F_{i-1}$ is finite and generated by some $x_i \in F_i$ for any $i = 1, \ldots, m$. We then set $\text{Tr}_L^F := \text{Tr}_L^{F_1}(x_1) \circ \ldots \circ \text{Tr}_L^{F_m}(x_m)$. It turns out that this definition is independent of the choice of the subfields $F_i$ and of the generators $x_i$ [19, Theorem 4.27].

2.1. Transfers of bilinear forms. The definition of geometric and cohomological transfers can be recovered from the transfers in Milnor K-theory as well as the Scharlau transfers on bilinear forms as we now explain.

Recall that the Milnor K-theory group $K_n^{\text{MW}}(L)$ fits into a Cartesian square

\[
\begin{array}{ccc}
K_n^{\text{MW}}(L) & \xrightarrow{\iota^0(L)} & \Gamma^0(L) \\
\downarrow & & \downarrow \\
K_n^M(L) & \xrightarrow{\iota_n} & \Gamma^0(L)
\end{array}
\]

for any $n \in \mathbb{Z}$, where $\Gamma^0(L) = W(L)$ for $n < 0$, $K_n^M(L) = 0$ for $n < 0$ and $\Gamma^0(L) := \Gamma^0(L)/\Gamma^{n+1}(L)$ for any $n \in \mathbb{N}$.

If $L \subset F$ is a finite field extension, then any non-zero $L$-linear homomorphism $f : F \rightarrow L$ induces a transfer morphism $f_* : GW(F) \rightarrow GW(L)$. It follows from [16, Lemma 1.4] that this homomorphism induces transfer homomorphisms $f_* : \Gamma^0(F) \rightarrow \Gamma^0(L)$ for any $n \in \mathbb{Z}$ and therefore transfer homomorphisms $\overline{f_*} : \overline{\Gamma^0}(F) \rightarrow \overline{\Gamma^0}(L)$ for
any \( n \in \mathbb{N} \). Recall moreover that if \( g : F \to L \) is another non zero \( K \)-linear map, there exists a unit \( b \in F^\times \) such that the following diagram commutes

\[
\begin{array}{ccc}
GW(F) & \xrightarrow{(b)} & GW(F) \\
\downarrow g_* & & \downarrow f_* \\
GW(L) & & \\
\end{array}
\]

(2)

Using the split exact sequence of [15, Theorem 2.3] and the procedure described above, one can also define transfer morphisms \( N_{F/L} : K^M_n(F) \to K^M_n(L) \) (the notation reflects the fact that \( N_{F/L} \) coincides in degree 1 with the usual norm homomorphism).

**Lemma 2.1.** For any non-zero linear homomorphism \( f : F \to L \) and any \( n \in \mathbb{N} \), the following diagram commutes

\[
\begin{array}{ccc}
K^M_n(F) & \xrightarrow{N_{F/L}} & K^M_n(L) \\
\downarrow s_n & & \downarrow s_n \\
\Gamma^F(F) & \xrightarrow{\Gamma^F} & \Gamma^F(L) \\
\end{array}
\]

**Proof.** Observe first that for any \( b \in F^\times \), we have \((-1,b) : \Gamma^F(F) = 0\). It follows thus from Diagram (2) that \( s_n = \overline{\pi} \) for any non-zero linear homomorphisms \( f,g : L \to K \). Now both the transfers for Milnor K-theory and for \( \Gamma^F \) are functorial, and it follows that we can suppose that the extension \( L \subset F \) is monogeneous, say generated by \( x \in F \). If \( n := [F : L] \), then \( 1, x, \ldots, x^{n-1} \) is a \( L \)-basis of \( F \) and we define the \( L \)-linear map \( f : F \to L \) by \( f(x^i) = 0 \) if \( i = 0, \ldots, n-2 \) and \( f(x^{n-1}) = 1 \). The result now follows from [23, Theorem 4.1]. \( \square \)

As a consequence of the lemma, we see that any non-zero linear map \( f : F \to L \) induces a transfer homomorphism \( f_* : K^M_n(F) \to K^M_n(L) \) for any \( n \in \mathbb{Z} \).

**Lemma 2.2.** Let \( L \subset F \) be a monogeneous field extension of degree \( n \) generated by \( x \in F \). Then the geometric transfer is equal to the transfer \( f_* \), where \( f \) is the \( L \)-linear map defined by \( f(x^i) = 0 \) if \( i = 0, \ldots, n-2 \) and \( f(x^{n-1}) = 1 \).

**Proof.** Once again, this follows immediately from Scharlau’s reciprocity theorem [23, Theorem 4.1]. \( \square \)

**Lemma 2.3.** Suppose that \( L \subset F \) is a separable field extension, generated by \( x \in F \). Then the cohomological transfer coincides with the transfer obtained via the trace map \( F \to L \).

**Proof.** This is an immediate consequence of [25, III, §6, Lemme 2]. \( \square \)

If the extension \( L \subset F \) is purely inseparable, then the cohomological transfer coincides by definition with the geometric transfer. It follows that the cohomological transfer can be computed using trace maps (when the extension is separable), the other homomorphisms described in Lemma 2.2 (when the extension is inseparable) or a combination of both via the factorization \( L \subset F^{sep} \subset F \).
2.2. **Canonical orientations.** In order to properly define the category of finite Chow-Witt correspondences, we will have to use differential forms to twist Milnor-Witt K-theory groups. In this section, we collect a few useful facts about orientations of relative sheaves, starting with the general notion of an orientation of a line bundle.

Let $X$ be a scheme, and let $N$ be a line bundle over $X$. Recall from [19, Definition 4.3] that an orientation of $N$ is a pair $(\mathcal{L}, \psi)$, where $\mathcal{L}$ is a line bundle over $X$ and $\psi: \mathcal{L} \otimes \mathcal{L} \cong N$ is an isomorphism. Two orientations $(\mathcal{L}, \psi)$ and $(\mathcal{L}', \psi')$ are said to be equivalent if there exists an isomorphism $\alpha: \mathcal{L} \to \mathcal{L}'$ such that the diagram

$$
\begin{array}{c}
\mathcal{L} \otimes \mathcal{L} \\
\alpha \otimes \alpha \\
\downarrow \psi \\
\mathcal{L}' \otimes \mathcal{L}'
\end{array}
\begin{array}{c}
\mathcal{N} \\
\psi
\end{array}
$$

commutes. The set of equivalence classes of orientations of $N$ is denoted by $Q(N)$.

Any invertible element $x$ in the global sections of $\mathcal{L}$ gives a trivialization $O_X \cong \mathcal{L}$ sending $1$ to $x$. This trivialization can be considered as an orientation $(O_X, q_x)$ of $\mathcal{L}$ via the canonical identification $O_X \otimes O_X \cong O_X$ given by the multiplication. In other words, on sections, $q_x(a \otimes b) = abx$. Clearly, $q_x = q_{x'}$ if and only if $x = u^2x'$ for some invertible global section $u$.

Let $k \subset L \subset F$ be field extensions such that $F/L$ is finite and $F/k$ and $L/k$ are finitely generated and separable (possibly transcendental). Let $\omega_{F/L} := \omega_{F/k} \otimes_L \omega_{L/k}$ be its relative $F$-vector space (according to our conventions, this vector space is the same as $\omega_f$, where $f: \text{Spec}(F) \to \text{Spec}(L)$ is the morphism induced by $L \subset F$). Our goal is to choose for such an extension a canonical orientation of $\omega_{F/L}$.

Suppose first that the extension $L \subset F$ is purely inseparable. In that case, we have a canonical bijection of $\text{Q}(F)$-equivariant sets $\text{Fr}: \text{Q}(\omega_{L/k} \otimes_L F) \rightarrow \text{Q}(\omega_{F/k})$ induced by the Frobenius [23, §2.4]. Any choice of a non-zero element $x$ of $\omega_{L/k}$ yields an $L$-linear homomorphism $x^\times: \omega_{L/k} \rightarrow L$ defined by $x^\times(x) = 1$ and an orientation $\text{Fr}(q_x) \in \text{Q}(\omega_{F/k})$. We thus obtain a class in $\text{Q}(\omega_{F/k} \otimes_L \omega_{L/k}^\vee)$ represented by $\text{Fr}(x) \otimes q_{x^\vee}$.

If $x' = ux$ for some $u \in L^\times$, then $(x')^\vee = u^{-1}x^\vee$ and $\text{Fr}(q_{x'}) = q_u\text{Fr}(q_x) \in \text{Q}(\omega_{F/k})$. It follows that $\text{Fr}(q_x) \otimes q_{x^\vee} = \text{Fr}(q_{x'}) \otimes q_{(x')}^\vee \in \text{Q}(\omega_{F/k} \otimes_L \omega_{L/k}^\vee)$ and this class is thus independent of the choice of $x \in \omega_{L/k}$. By definition, we have $\omega_{F/L} := \omega_{F/k} \otimes_L \omega_{L/k}^\vee$ and we therefore get a canonical orientation in $\text{Q}(\omega_{F/L})$.

Suppose next that the extension $L \subset F$ is separable. In that case, the module of differentials $\Omega_{F/L} = 0$ and we have a canonical isomorphism $\omega_{F/k} \cong \omega_{L/k} \otimes_L F$. It follows that $\omega_{F/L} \cong F$ canonically and we choose the orientation $q_1$ given by $1 \in F$ under this isomorphism.

We have thus proved the following result.

**Lemma 2.4.** Let $k \subset L \subset F$ be field extensions such that $F/L$ is finite, $L/k$ and $F/k$ are finitely generated and separable (possibly transcendental). Then there is a canonical orientation of $\omega_{F/L}$.

**Proof.** It suffices to consider $L \subset F^{\text{sep}} \subset F$ and the two cases described above. \(\square\)
3. Chow-Witt groups

Recall the construction of Milnor-Witt K-theory sheaves twisted by line bundles from section 1.2.

**Definition 3.1.** For any smooth scheme $X$, any line bundle $L$ over $X$, any closed subset $Z \subset X$ and any $n \in \mathbb{N}$, we define the $n$-th Chow-Witt group (twisted by $L$) supported on $Z$ by $\overline{CH}_Z^n(X, L) := H^2_n(X, K^W_n(L))$.

If $L = O_X$ (resp. $Z = X$), we omit $L$ (resp. $Z$) from the notation. Provided the base field $k$ is perfect, the Rost-Schmid complex defined by Morel in [19, Chapter 5] provides a flabby resolution of $K^W_n(L)$ and we can use it to compute the cohomology of this sheaf. It follows from (1) above and [5, Remark 7.31] that this definition coincides with the one given in [5, Définition 10.2.14].

The groups $\overline{CH}_Z^n(X, L)$ are contravariant in $X$ (and $L$). If $f : X \to Y$ is a finite morphism between smooth schemes of respective (constant) dimension $d_X$ and $d_Y$, then there is a push-forward map

$$f_* : \overline{CH}_Z^n(X, \omega_{X/Y} \otimes f^* L) \to \overline{CH}_W^{n+d_Y-d_X}(Y, L)$$

for any line bundle $L$ over $Y$ [19, Corollary 5.30]. More generally, one can define a push-forward map as above for any proper morphism $f : X \to Y$ [5, Corollaire 10.4.5]. Actually, the push-forward map can be slightly generalized if one considers supports. If $f : X \to Y$ is a morphism of smooth schemes and $Z \subset X$ is a closed subscheme which is finite over $W \subset Y$, then we can define a push-forward map

$$f_* : \overline{CH}_Z^n(X, \omega_{X/Y} \otimes f^* L) \to \overline{CH}_W^{n+d_Y-d_X}(Y, L)$$

along the formula given in [19, p. 125]. Indeed, it suffices to check that the proof of [19, Corollary 5.30] holds in that case, which is easy.

Observe now that the forgetful morphism of sheaves $K^W_n(L) \to K^M_n$ yields homomorphisms $\overline{CH}_Z^n(X, L) \to \overline{CH}_Z^n(X)$ for any $n \in \mathbb{N}$, while the hyperbolic morphism $K^M_n \to K^W_n(L)$ yields homomorphisms $\overline{CH}_Z^n(X) \to \overline{CH}_Z^n(X, L)$ for any $n \in \mathbb{N}$. The composite $K^M_n \to K^W_n(L) \to K^M_n$ being multiplication by 2, the composite homomorphism

$$\overline{CH}_Z^n(X) \to \overline{CH}_Z^n(X, L) \to \overline{CH}_Z^n(X)$$

is also the multiplication by 2. Both the hyperbolic and forgetful homomorphisms are compatible with the pull-back and the push-forward maps. Moreover, the total Chow-Witt group of a smooth scheme $X$ is endowed with a ring structure refining the intersection product on Chow groups (i.e. the forgetful homomorphism is a ring homomorphism). More precisely, as for usual Chow groups, there is an external product

$$\overline{CH}_Z^n(X, L) \times \overline{CH}_W^m(X, N) \to \overline{CH}_Z^{m+n}(X \times X, p_1^* L \otimes p_2^* N)$$

commuting to pull-backs and push-forwards. By pulling back along the diagonal, it yields a cup-product

$$\overline{CH}_Z^n(X, L) \times \overline{CH}_W^m(X, N) \to \overline{CH}_Z^{m+n}(X, L \otimes N)$$

for any $m, n \in \mathbb{Z}$, any line bundles $L$ and $N$ over $X$ and any closed subsets $Z, W \subset X$. It is associative, and its unit is given by the pull-back to $X$ of $(1) \in K^W_n(k) = GW(k)$ [4, §6]. In general, the product is not commutative. If $\alpha \in \overline{CH}_Z^n(X, L)$ and
\[ \beta \in \widetilde{\text{CH}}^m(W(X,N)), \text{ we have} \quad \alpha \cdot \beta = (-1)^{mn} \beta \cdot \alpha \text{ (under the canonical identification } N \otimes \mathcal{L} \simeq \mathcal{L} \otimes N) \text{ by [4, Remark 6.7].} \]

For the sake of completeness, recall that Chow-Witt groups satisfy homotopy invariance by [5, Corollaire 11.3.2].

### 3.1. Some useful results

The goal of this section is to prove the analogues of some classical formulas for Chow-Witt groups. Most of them are “obvious” in the sense that their proofs are basically the same as for Chow groups. Before stating our first result, let us recall that two morphisms \( f : X \to Y \) and \( g : U \to Y \) are \emph{Tor-independent} if for every \( x \in X \), \( y \in Y \) and \( u \in U \) such that \( f(x) = g(u) = y \) we have \( \text{Tor}_n^{O_Y,y}(\mathcal{O}_{X,x}, \mathcal{O}_{U,u}) = 0 \) for \( n \geq 1 \).

**Proposition 3.2** (Base change formula). Let

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
g & & f \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

be a Cartesian square of smooth schemes with \( f \) proper. Suppose that \( f \) and \( u \) are Tor-independent. Then \( u^* f_* = g_* v^* \).

**Proof.** We first break the square into two Cartesian squares

\[
\begin{array}{ccc}
X' & \xrightarrow{(g,v)} & Y' \times X \\
g & & \downarrow 1 \times f \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{\Gamma_u} & Y' \times Y
\end{array}
\]

where \( \Gamma_u \) is the graph of \( u \) and the right-hand horizontal morphisms are the respective projections. By [5, Théorème 12.3.6], we know that the generalized base change formula holds for the right-hand square. Moreover, the morphisms \((1 \times f)\) and \( \Gamma_u \) are Tor-independent since \( f \) and \( u \) are. We are thus reduced to show that the formula holds if \( u : Y' \to Y \) is a closed embedding of smooth schemes. This follows from [6, Theorem 2.2] observing that the relative bundle \( E \) described there is trivial if \( f \) and \( u \) are Tor-independent. \( \square \)

**Remark 3.3.** There is no need for \( f \) to be proper in the above proposition, as long as we consider supports which are proper over the base. More precisely, suppose that we have a Cartesian square

\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
g & & f \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{u} & Y
\end{array}
\]

of smooth schemes with \( f \) and \( u \) Tor-independent. Let \( M \subset X \) be a closed subset such that the composite morphism \( M \subset X \xrightarrow{f} Y \) is proper (here \( M \) is endowed with its reduced scheme structure). Then the formula \( u^* f_* = g_* v^* \) holds for any \( \alpha \in \text{CH}^m(W(X, \omega_{X/Y} \otimes f^* \mathcal{L})) \). The proof is the same as the proof of the proposition, taking supports into account.
Corollary 3.4 (Projection formula). Let $m, n \in \mathbb{N}$ and let $L, N$ be line bundles over $Y$. Let $Z \subset X$ and $W \subset Y$ be closed subsets. If $f : X \to Y$ is a proper morphism, we have

$$f_*(\alpha) : \beta = f_*(\alpha \cdot f^*(\beta)) \quad \text{and} \quad \beta \cdot f_*(\alpha) = f_*(f^*(\beta) \cdot \alpha)$$

for any $\alpha \in \widetilde{CH}^m_a(X, \omega_{X/Y} \otimes f^*L)$ and $\beta \in \widetilde{CH}^n(Y, N)$.

Proof. Use the base change formula respectively on the following Cartesian squares:

$$
\begin{array}{ccc}
X & \xrightarrow{(1 \times f) \Delta_X} & X \times Y \\
\downarrow f & & \downarrow (f \times 1) \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
$$

$$
\begin{array}{ccc}
X & \xrightarrow{(f \times 1) \Delta_X} & X \times Y \\
\downarrow f & & \downarrow (1 \times f) \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
$$

together with the fact that the external product commutes to push-forwards. □

Lemma 3.5 (Flat excision). Let $f : X \to Y$ be a flat morphism of smooth schemes. Let $V \subset Y$ be a closed subset such that the morphism $f^{-1}(V) \to V$ induced by $f$ is an isomorphism. Then the pull-back morphism

$$f^* : \widetilde{CH}^i(Y, L) \to \widetilde{CH}^i_{f^{-1}(V)}(X, f^*L)$$

is an isomorphism for any $i \in \mathbb{N}$ and any line bundle $L$ over $Y$.

Proof. As said in Section 3, Chow-Witt groups can be computed using the flabby resolution provided by the Rost-Schmid complex of [19, Chapter 5], which coincide with the complex considered in [5, Définition 10.2.7]. Now Chow-Witt groups with supports are obtained by considering the subcomplex of points supported on a certain closed subset. The lemma follows now from the fact that in our case $f^*$ induces (by definition) an isomorphism of complexes. □

We now consider the problem of describing the cohomology of $X \times \mathbb{G}_m$ with coefficients in $K_j^{MW}$ (for $j \in \mathbb{Z}$) in terms of the cohomology of $X$. First observe that the pull-back along the projection $p : X \times \mathbb{G}_m \to X$ endows the cohomology of $X \times \mathbb{G}_m$ with the structure of a module over the cohomology of $X$. Let $t$ be a parameter of $\mathbb{G}_m$. The class $[t]$ in $K_t^{MW}(k(t))$ actually lives in its subgroup $K_1^{MW}(\mathbb{G}_m)$ since it clearly has trivial residues at all closed points of $\mathbb{G}_m$. Pulling back to $X \times \mathbb{G}_m$ along the projection to the second factor, we get an element in $K_1^{MW}(X \times \mathbb{G}_m)$ that we still denote by $[t]$.

Lemma 3.6. For any $i \in \mathbb{N}$, any $j \in \mathbb{Z}$ and any smooth scheme $X$ over $k$, we have

$$H^i(X \times \mathbb{G}_m, K_j^{MW}) = H^i(X, K_j^{MW}) \oplus H^i(X, K_{j-1}^{MW}) \cdot [t].$$

Proof. The long exact sequence associated to the open immersion $X \times \mathbb{G}_m \subset X \times \mathbb{A}_k^1$ reads as

$$\cdots \to H^i(X \times \mathbb{A}_k^1, K_j^{MW}) \to H^i(X \times \mathbb{G}_m, K_j^{MW}) \xrightarrow{\partial} H^{i+1}(X \times \mathbb{A}_k^1, K_j^{MW}) \to \cdots$$

By homotopy invariance, the pull-back along the projection to the first factor $X \times \mathbb{A}_k^1 \to X$ induces an isomorphism $H^i(X, K_j^{MW}) \to H^i(X \times \mathbb{A}_k^1, K_j^{MW})$. Pulling-back along the morphism $X \to X \times \mathbb{G}_m$ defined by $x \mapsto (x, 1)$ we get a retraction of the composite homomorphism

$$H^i(X, K_j^{MW}) \to H^i(X \times \mathbb{A}_k^1, K_j^{MW}) \to H^i(X \times \mathbb{G}_m, K_j^{MW})$$
and it follows that the long exact sequence splits into short split exact sequences
$$0 \to H^i(X \times \mathbb{A}^1_k, \mathbf{K}_j^{MW}) \to H^i(X \times \mathbb{G}_m, \mathbf{K}_j^{MW}) \overset{\partial}{\to} H^{i+1}_{X \times \{0\}}(X \times \mathbb{A}^1, \mathbf{K}_j^{MW}) \to 0$$

Now the push-forward homomorphism (together with the obvious trivialization of the normal bundle to $X \times \{0\}$ in $X \times \mathbb{A}^1_k$) yields an isomorphism $[5, \text{Remarque } 10.4.8]$

$$\iota : H^i(X, \mathbf{K}_j^{MW}) \to H^{i+1}_{X \times \{0\}}(X \times \mathbb{A}^1, \mathbf{K}_j^{MW})$$

and it suffices then to check that the composite

$$H^i(X, \mathbf{K}_j^{MW}) \xrightarrow{[t]} H^i(X, \mathbf{K}_j^{MW}) \to H^i(X \times \mathbb{G}_m, \mathbf{K}_j^{MW}) \overset{\iota^{-1}}{\to} H^i(X, \mathbf{K}_j^{MW})$$

is an isomorphism to conclude. This follows essentially from $[19, \text{Proposition } 3.17.2]$. 

\textbf{Remark 3.7.} By pull-back along the morphism $\text{Spec}(k) \to \mathbb{G}_m$ sending the point to 1, the element $[t] \in \mathbf{K}_j^{MW}(\mathbb{G}_m)$ maps to $[1] = 0$ in $\mathbf{K}_j^{MW}(k)$. Therefore this pull-back gives the splitting of $H^i(X, \mathbf{K}_j^{MW})$ in the decomposition above.

\section*{4. Finite Chow-Witt correspondences}

\textbf{4.1. Admissible subsets.} Let $X$ and $Y$ be smooth schemes over $\text{Spec}(k)$ and let $T \subset X \times Y$ be a closed subset. Any irreducible component of $T$ maps to an irreducible component of $X$ through the projection $X \times Y \to X$.

\textbf{Definition 4.1.} If, when $T$ is endowed with its reduced structure, this map is finite and surjective for every irreducible component of $T$, we say that $T$ is an \textit{admissible subset} of $X \times Y$. We denote by $\mathcal{A}(X, Y)$ the set of admissible subsets of $X \times Y$, partially ordered by inclusions. As usual, we sometimes consider $\mathcal{A}(X, Y)$ as a category.

\textbf{Remark 4.2.} Since the empty set has no irreducible component, it is admissible. An irreducible component of an admissible subset is clearly admissible, and the irreducible admissible subsets are minimal (non-trivial) elements in $\mathcal{A}(X, Y)$. Furthermore, any finite union of admissible subsets is admissible.

\textbf{Lemma 4.3.} If $f : X' \to X$ is a morphism between smooth schemes, then $T \mapsto (f \times \text{id}_Y)^{-1}(T)$ defines a map $\mathcal{A}(X, Y) \to \mathcal{A}(X', Y)$. Furthermore, the presheaf $U \mapsto \mathcal{A}(U, Y)$ thus defined is a sheaf for the Zariski topology.

\textbf{Proof.} Finiteness and surjectivity are stable by base change by $[11, 6.1.5]$ and $[12, 3.5.2]$, so the map is well-defined. The injectivity condition in the sheaf sequence is obvious. To prove the exactness in the middle, being closed is obviously a Zariski local property, so the union of the closed subsets in the covering defines a global closed subset. Both finiteness and surjectivity are properties that are Zariski local on the base, so this closed subset is admissible. 

If $Y$ is equidimensional and $d = \dim Y$, we define a covariant functor

$$\mathcal{A}(X, Y) \to \mathcal{A}b$$

by associating to each admissible subset $T \in \mathcal{A}(X, Y)$ the group $CH^d_T(X \times Y, \omega_{X \times Y/X})$ and to each morphism $T' \subset T$ the extension of support homomorphism

$$CH^d_{T'}(X \times Y, \omega_{X \times Y/X}) \to CH^d_T(X \times Y, \omega_{X \times Y/X})$$
and, using that functor, we set
\[ \widetilde{\text{Cor}}_k(X, Y) = \lim_{T \in \mathcal{A}(X, Y)} \widetilde{\text{CH}}_T^d(X \times Y, \omega_{X \times Y/X}). \]

If \( Y \) is not equidimensional, then \( Y = \bigsqcup_j Y_j \) with each \( Y_j \) equidimensional and we set
\[ \widetilde{\text{Cor}}_k(X, Y) = \prod_j \widetilde{\text{Cor}}_k(X, Y_j). \]

By additivity of Chow-Witt groups, if \( X = \bigsqcup_i X_i \) and \( Y = \bigsqcup_j Y_j \) are the respective decompositions of \( X \) and \( Y \) in irreducible components, we have
\[ \widetilde{\text{Cor}}_k(X, Y) = \prod_{i,j} \widetilde{\text{Cor}}_k(X_i, Y_j). \]

**Example 4.4.** Let \( X \) be a smooth scheme of dimension \( d \). Then
\[ \widetilde{\text{Cor}}_k(\text{Spec}(k), X) = \bigoplus_{x \in X^{(d)}} \widetilde{\text{CH}}_x^d(X, \omega_X) = \bigoplus_{x \in X^{(d)}} \text{GW}(k(x), \omega_{k(x)/k}). \]

On the other hand, \( \widetilde{\text{Cor}}_k(X, \text{Spec}(k)) = \widetilde{\text{CH}}^0(X) = K_0^{\text{MW}}(X) \) for any smooth scheme \( X \).

The group \( \widetilde{\text{Cor}}_k(X, Y) \) admits an alternate description which is often useful. Let \( X \) and \( Y \) be smooth schemes, with \( Y \) equidimensional. For any closed subscheme \( T \subset X \times Y \) of codimension \( d = \dim Y \), we have an inclusion
\[ \widetilde{\text{CH}}_T^d(X \times Y, \omega_{X \times Y/X}) \subset \bigoplus_{x \in (X \times Y)^{(d)}} K_0^{\text{MW}}(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x). \]

and it follows that
\[ \widetilde{\text{Cor}}_k(X, Y) = \bigcup_{T \in \mathcal{A}(X, Y)} \widetilde{\text{CH}}_T^d(X \times Y, \omega_{X \times Y/X}) \subset \bigoplus_{x \in (X \times Y)^{(d)}} K_0^{\text{MW}}(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x). \]

In general, the inclusion \( \widetilde{\text{Cor}}_k(X, Y) \subset \bigoplus_{x \in (X \times Y)^{(d)}} K_0^{\text{MW}}(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x) \) is strict as shown by Example 4.4. As an immediate consequence of this description, we see that the map
\[ \widetilde{\text{CH}}_T^d(X \times Y, \omega_{X \times Y/X}) \to \widetilde{\text{Cor}}_k(X, Y) \]
is injective for any \( T \in \mathcal{A}(X, Y) \).

If \( U \) is an open subset of a smooth scheme \( V \), since an admissible subset \( T \in \mathcal{A}(V, Y) \) intersects with \( U \times Y \) as an admissible subset by Lemma 4.3, the pull-backs along \( V \times Y \to U \times Y \) on Chow-Witt groups with support induce at the limit a map \( \text{Cor}_k(V, Y) \to \text{Cor}_k(U, Y) \).

**Lemma 4.5.** This map is injective.

**Proof.** We can assume \( Y \) equidimensional, of dimension \( d \). Let \( Z = (V \setminus U) \times Y \). Let moreover \( T \subset V \times Y \) be an admissible subset. Since \( T \) is finite and surjective over \( V \), the subset \( Z \cap T \) is of codimension at least \( d + 1 \) in \( V \times Y \), which implies that \( \widetilde{\text{CH}}_Z^{d+1}(V \times Y, \omega_{V \times Y/V}) = 0 \). The long exact sequence of localization with support then shows that the homomorphism
\[ \widetilde{\text{CH}}_T^d(V \times Y, \omega_{V \times Y/V}) \to \widetilde{\text{CH}}_{T \cap (U \times Y)}^d(U \times Y, \omega_{U \times Y/U}) \]
is injective. On the other hand, we have a commutative diagram
\[
\begin{array}{ccc}
\widetilde{\text{CH}}_d^d(V \times Y, \omega_{V \times Y/V}) & \longrightarrow & \widetilde{\text{CH}}_d^d(U \times Y, \omega_{U \times Y/U}) \\
\downarrow & & \downarrow \\
\widetilde{\text{Cor}}_k(V, Y) & \longrightarrow & \widetilde{\text{Cor}}_k(U, Y)
\end{array}
\]
with injective vertical maps. Since any \(\alpha \in \widetilde{\text{Cor}}_k(V, X)\) comes from the group \(\widetilde{\text{CH}}_d^d(V \times X, \omega_{V \times X/V})\) for some \(T \in \mathcal{A}(X, Y)\), the homomorphism \(\text{Cor}_k(V, X) \rightarrow \widetilde{\text{Cor}}_k(U, X)\) is injective.

\[\square\]

**Definition 4.6.** Let \(\alpha \in \widetilde{\text{Cor}}_k(X, Y)\), where \(X\) and \(Y\) are smooth. If \(Y\) is equidimensional, let \(d = \dim(Y)\). The support of \(\alpha\) is the closure of the set of points \(x \in (X \times Y)^{(d)}\) such that the component of \(\alpha\) in \(K^\text{MW}_0(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x)\) is nonzero. If \(Y\) is not equidimensional, then we define the support of \(\alpha\) as the union of the supports of the components appearing in the equidimensional decomposition.

**Lemma 4.7.** The support of an \(\alpha \in \widetilde{\text{Cor}}_k(X, Y)\) is an admissible subset, say \(T\), and \(\alpha\) is then in the image of the inclusion \(\text{CH}^d_T(X \times Y, \omega_{X \times Y/X}) \subseteq \widetilde{\text{Cor}}_k(X, Y)\).

**Proof.** By definition of \(\widetilde{\text{Cor}}_k(X, Y)\) as a direct limit, the support of \(\alpha\) is included in some admissible subset \(T \in \mathcal{A}(X, Y)\). Being finite and surjective over \(X\), any irreducible component \(T_i\) of \(T\) is of codimension \(\dim Y\) in \(X \times Y\). Therefore the support of \(\alpha\) is exactly the union of all \(T_i\) such that the component of \(\alpha\) on the generic point of \(T_i\) is non-zero. This is an admissible subset by Remark 4.2.

To obtain the last part of the statement, let \(S \subseteq T\) be the support of \(\alpha\) and let \(U\) be the open subscheme \(X \times Y \setminus S\). Consider the commutative diagram
\[
\begin{array}{ccc}
\widetilde{\text{CH}}_d^d(X \times Y, \omega_{X \times Y/X}) & \longrightarrow & \widetilde{\text{CH}}_d^d(U, (\omega_{X \times Y/X})|_U) \\
\downarrow & & \downarrow \\
\bigoplus_{x \in (X \times Y)^{(d)} \cap T} K^\text{MW}_0(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x) & \longrightarrow & \bigoplus_{x \in U^{(d)} \cap T} K^\text{MW}_0(k(x), \omega_x \otimes (\omega_{X \times Y/X})_x)
\end{array}
\]
with injective vertical maps (still for dimensional reasons). By definition of the support, \(\alpha\) maps to zero in the lower right group, so it maps to zero in the upper right one. Therefore, it comes from the previous group in the localization exact sequence for Chow groups with support, and this group is \(\text{CH}^d_S(X \times Y, \omega_{X \times Y/X})\). \(\square\)

Let \(\alpha \in \widetilde{\text{Cor}}_k(X, Y)\) with support \(T\) be restricted to an element denoted by \(\alpha|_U \in \text{Cor}_k(U, Y)\).

**Lemma 4.8.** The support of \(\alpha|_U\) is \(T \cap U\), in other words the image of \(T\) by the map \(\mathcal{A}(X, Y) \rightarrow \mathcal{A}(U, Y)\).

**Proof.** It is straightforward from the definition of the support. \(\square\)
4.2. Composition of finite Chow-Witt correspondences. Let $X$, $Y$ and $Z$ be smooth schemes of respective dimensions $d_X$, $d_Y$ and $d_Z$, with $X$ and $Y$ connected. Let $V \in \mathcal{A}(X,Y)$ and $T \in \mathcal{A}(Y,Z)$ be admissible subsets. Consider the following commutative diagram where all maps are canonical projections:

\[
\begin{array}{c}
\xymatrix{
 X \times Z \ar[r]^\pi & X \times Y \times Z \ar[r]^{q'} & Y \times Z \ar[r]^p & Z \\
 X \times Y \ar[u]^{q} \ar[r]^{t'} & Y \ar[u]^{t} \ar[r]^q & Y \\
 X \ar[u]^{p'} \ar[r]^p & Y \ar[u]_{q} &
}\end{array}
\]

We have homomorphisms

\[(t')^* : \widehat{CH}^{d_Y}_{/V}(X \times Y, \omega_{X \times Y/X}) \rightarrow \widehat{CH}^{d_Y}_{/(t')^{-1}V}(X \times Y \times Z, (t')^* \omega_{X \times Y/X})\]

and

\[(q')^* : \widehat{CH}^{d_Z}_{/Y}(X \times Y \times Z, \omega_{X \times Y \times Z/Y}) \rightarrow \widehat{CH}^{d_Z}_{/(q')^{-1}Y/X}(X \times Y \times Z, (q')^* \omega_{X \times Y \times Z/Y}).\]

Let $M = (t')^{-1}V \cap (q')^{-1}T$, endowed with its reduced structure. It follows from [14, Lemmas 1.4 and 1.6] that every irreducible component of $M$ is finite and surjective over $X$. As a consequence, the map $M \rightarrow \pi(M)$ is finite and the push-forward

\[\pi_* : \widehat{CH}^{d_Y+d_Z}_{/Y}(X \times Y \times Z, \omega_{X \times Y \times Z/k} \otimes \pi^* \mathcal{L}) \rightarrow \widehat{CH}^{d_Z}_{/\pi(M)}(X \times Z, \omega_{X \times Z/k} \otimes \mathcal{L})\]

is well-defined for any line bundle $\mathcal{L}$ over $X \times Z$. In particular for $\mathcal{L} = (p')^* \omega_{X/k}$, we get a push-forward map

\[\pi_* : \widehat{CH}^{d_Y+d_Z}_{/Y}(X \times Y \times Z, \omega_{X \times Y \times Z/k}) \rightarrow \widehat{CH}^{d_Z}_{/\pi(M)}(X \times Z, \omega_{X \times Z/k}).\]

**Lemma 4.9.** We have a canonical isomorphism

\[(t')^* \omega_{X \times Y/X} \otimes (q')^* \omega_{Y \times Z/Y} \simeq \omega_{X \times Y \times Z/X}.\]

**Proof.** Since the square in Diagram (3) is Cartesian, it follows that we have a canonical isomorphism $(q')^* \omega_{Y \times Z/Y} \simeq \omega_{X \times Y \times Z/X \times Y}$. Now

\[(t')^* \omega_{X \times Y/X} \otimes \omega_{X \times Y \times Z/X \times Y} = (t')^* \omega_{X \times Y/k} \otimes (q')^* \omega_{X \times Y \times Z/k} \otimes \omega_{Y \times Y/k} \simeq \omega_{X \times Y/k} \otimes \omega_{X \times Y/k} \simeq \omega_{X \times Y/k} \otimes \omega_{X \times Y/k} \simeq \omega_{X \times Y/k}

and it suffices to use the canonical isomorphism $\omega_{X \times Y/k} \otimes \omega_{X \times Y/k} \simeq O_{X \times Y}$ to conclude.

As a consequence, if we have cycles $\beta \in \widehat{CH}^{d_Y}_{/V}(X \times Y, q^* \omega_{Y})$ and $\alpha \in \widehat{CH}^{d_Z}_{/Y}(X \times Z, r^* \omega_{Z})$ the expression

\[\alpha \circ \beta := \pi_* [(t')^* \alpha \cdot (q')^* \beta]\]

is well-defined. Moreover, it follows from [14, Lemma 1.7] that $\pi(M)$ is an admissible subset of $X \times Z$. All the above homomorphisms commute with extension of supports, and therefore we get a well-defined composition

\[\circ : \widetilde{\text{Cor}}_k(X,Y) \times \widetilde{\text{Cor}}_k(Y,Z) \rightarrow \widetilde{\text{Cor}}_k(X,Z).\]

The intersection product being associative, the composition we just defined is also associative.
4.3. Embedding of morphisms. Let $X,Y$ be smooth schemes of respective dimensions $d_X$ and $d_Y$. Let $f : X \to Y$ be a morphism and let $\Gamma_f$ be its graph in $X \times Y$ with embedding $i$. Then $\Gamma_f$ is smooth over $k$, of codimension $d_Y$ in $X \times Y$, finite and surjective over $X$. If $p : X \times Y \to X$ is the projection map, then $pi$ is an isomorphism and it follows that $\omega_{\Gamma_f/X} \simeq \mathcal{O}_{\Gamma_f}$ canonically. Therefore we obtain a finite push-forward

$$i_* : K_0^{MW}(\Gamma_f) \to \widehat{CH}^{d_Y}_{\Gamma_f}(X \times Y, \omega_{X \times Y/X})$$

We denote by $\gamma_f$ the class of $i_*([1])$ in $\widehat{CH}^{d_Y}_{\Gamma_f}(X \times Y, p^*\omega_Y)$. In particular, when $X = Y$ and $f = id$, we set $1_X := \gamma_{id}$. Using [4, Proposition 6.8] we can check that $1_X$ is the identity for the composition defined in the previous section.

**Example 4.10.** Let $X$ be a smooth scheme over $k$. The diagonal morphism induces a push-forward homomorphism

$$K_0^{MW}(X) \to \widehat{CH}^{d_X}_{X}(X \times X, p^*\omega_X)$$

and a ring homomorphism $K_0^{MW}(X) \to \overline{\text{Cor}}_k(X,X)$. For any smooth scheme $Y$, composition of morphisms endows the group $\text{Cor}_k(Y,X)$ with the structure of a left $K_0^{MW}(X)$-module and a right $K_0^{MW}(Y)$-module.

**Definition 4.11.** Let $\overline{\text{Cor}}_k$ be the category whose objects are smooth schemes and whose morphisms are $\text{Cor}_k(X,Y)$ defined in Section 4.1. We call it the category of finite Chow-Witt correspondences over $k$.

We see that $\overline{\text{Cor}}_k$ is an additive category, with disjoint union as direct sum. Associating $\gamma_f$ to any morphism of smooth schemes $f : X \to Y$ gives a functor $\text{Sm}_k \to \overline{\text{Cor}}_k$.

**Remark 4.12.** The category of finite correspondences as defined by Voevodsky can be recovered by replacing Chow-Witt groups by Chow groups in our definition. Indeed, when $Y$ is equidimensional of dimension $d = \dim Y$ and $T \in A(X,Y)$,

$$\text{CH}^d_T(X \times Y) = \bigoplus_{x \in (X \times Y)^{(d)} \cap T} \mathbb{Z}$$

since the previous group in the Gersten complex is zero because $T$ is $d$-dimensional, and the following group is also zero because there are no negative $K$-groups. The composition of Voevodsky’s finite correspondences coincides with ours as one can easily see from Lecture 1 in [14].

By the same procedure, it is of course possible to define finite correspondences using other cohomology theories with support, provided that they satisfy the classical axioms used in the definition of the composition (base change, etc.).

The forgetful homomorphisms

$$\overline{\text{CH}}^d_T(X \times Y, \omega_{X \times Y/X}) \to \text{CH}^d_T(X \times Y)$$

yield a functor $F : \overline{\text{Cor}}_k \to \text{Cor}_k$ (use [4, Prop. 6.12]) which is additive, and the classical functor $\text{Sm}_k \to \text{Cor}_k$ is the composite functor $\text{Sm}_k \to \overline{\text{Cor}}_k \xrightarrow{F} \text{Cor}_k$.

On the other hand, the hyperbolic homomorphisms

$$\text{CH}^d_T(X \times Y) \to \overline{\text{CH}}^d_T(X \times Y, \omega_{X \times Y/X})$$
yield a homomorphism $H_{X,Y} : \text{Cor}_k(X,Y) \to \widetilde{\text{Cor}}_k(X,Y)$ for any smooth schemes $X,Y$ (but not a functor since $H_{X,X}$ doesn’t preserve the identity). The composite $F_{X,Y}H_{X,Y}$ is just the multiplication by 2, as explained in Section 3.

We now give two examples showing how to compose a finite Chow-Witt correspondence with a morphism of schemes.

**Example 4.13** (Pull-back). Let $X,Y,U \in \text{Sm}_k$ and let $f : X \to Y$ be a morphism. Let $(f \times 1) : (X \times U) \to (Y \times U)$ be induced by $f$ and let $T \in \mathcal{A}(Y,U)$ be an admissible subset. Then $F := (f \times 1)^{-1}(T)$ is an admissible subset of $X \times U$ by [14, Lemma 1.6]. It follows that the pull-back of cycles $(f \times 1)^* \omega$ induces a homomorphism $\widetilde{\text{Cor}}_k(Y,U) \to \widetilde{\text{Cor}}_k(X,U)$. We let the reader check that it coincides with the composition with $\gamma_f$.

**Example 4.14** (Push-forwards). Let $X$ and $Y$ be smooth schemes of dimension $d$ and let $f : X \to Y$ be a finite morphism such that any irreducible component of $X$ surjects to the irreducible component of $Y$ it maps to. Contrary to the classical situation, we don’t have a finite Chow-Witt correspondence $Y \to X$ associated to $f$ in general, however, we can define one if $\omega_f$ admits an orientation.

Let then $(L,\psi)$ be an orientation of $\omega_f$. We define a finite Chow-Witt correspondence $\alpha(f,L,\psi) \in \widetilde{\text{Cor}}_k(Y,X)$ as follows. Let $\Gamma_f^t : X \to Y \times X$ be the (transpose of the) graph of $f$. Then $X$ is an admissible subset and we have a transfer morphism

$$(\Gamma_f^t)_* : \mathbf{K}^{\text{MW}}_0(X,\omega_f) \to \overline{\text{CH}}_X^d(Y \times X,\omega_{Y \times X/X}).$$

Composing with the homomorphism

$$\overline{\text{CH}}_X^d(Y \times X,\omega_{Y \times X/X}) \to \overline{\text{Cor}}_k(Y,X),$$

we get a map $\mathbf{K}^{\text{MW}}_0(X,\omega_f) \to \overline{\text{Cor}}_k(Y,X)$. Now the isomorphism $\psi$ together with the canonical isomorphism $\mathbf{K}^{\text{MW}}_0(X) \simeq \mathbf{K}^{\text{MW}}_0(X,L \otimes L)$ yield an isomorphism $\mathbf{K}^{\text{MW}}_0(X) \to \mathbf{K}^{\text{MW}}_0(X,\omega_f)$. We define the finite Chow-Witt correspondence $\alpha(f,L,\psi)$ (or sometimes simply $(\alpha(f,\psi))$) as the image of $(1)$ under the composite

$$\mathbf{K}^{\text{MW}}_0(X) \to \mathbf{K}^{\text{MW}}_0(X,\omega_f) \to \overline{\text{Cor}}_k(Y,X).$$

If $(L',\psi')$ is equivalent to $(L,\psi)$, then it is easy to check that the correspondences $\alpha(f,L,\psi)$ and $\alpha(f,L',\psi')$ are equal. Thus any element of $Q(\omega_f)$ yields a finite Chow-Witt correspondence. In general, different choices of elements in $Q(\omega_f)$ yield different correspondences.

When $g : Y \to Z$ is another such morphism with an orientation $(\mathcal{M},\phi)$ of $\omega_g$, then $(L \otimes f^*\mathcal{M},\psi \otimes f^*\phi)$ is an orientation of $\omega_{g \circ f} = \omega_f \otimes f^*\omega_g$, and we have $\alpha(f,L,\psi) \circ \alpha(g,\mathcal{M},\phi) = \alpha(g \circ f,L \otimes f^*\mathcal{M},\psi \otimes f^*\phi)$.

Let now $U$ be a smooth scheme of dimension $n$ and let $T \in \mathcal{A}(X,U)$. The commutative diagram

$$\begin{array}{ccc}
X \times U & \xrightarrow{f \times 1} & Y \times U \\
\downarrow{p_V} & & \downarrow{p_V} \\
Y & & 
\end{array}$$

where $p_V : Y \times U$ is the projection on the first factor and [14, Lemma 1.4] show that $(f \times 1)(T) \in \mathcal{A}(Y,U)$ in our situation. Moreover, we have a push-forward morphism

$$(f \times 1)_* : \overline{\text{CH}}^n_T(X \times U,\omega_{X \times U/X} \otimes \omega_f) \to \overline{\text{CH}}^n_{(f \times 1)(T)}(Y \times U,\omega_{Y \times U/Y})$$
Using the trivialization $\psi$, we get a push-forward morphism

$$(f \times 1)_* : \widetilde{CH}^n_T(X \times U, \omega_{X \times U/X}) \to \widetilde{CH}^n_{(f \times 1)(T)}(Y \times U, \omega_{Y \times U/Y})$$

Now this map commutes to the extension of support homomorphisms, and it follows that we get a homomorphism

$$(f \times 1)_* : \widetilde{Cor}_k(X, U) \to \widetilde{Cor}_k(Y, U)$$

depending on $\psi$. We let the reader check that $(f \times 1)_* (\beta) = \beta \cdot \alpha(f, \psi)$ for any $\beta \in \widetilde{Cor}_k(X, U)$, using the base change formula as well as the projection formula.

In particular, when $U = Y$ and $\beta = \gamma_f$, using $\psi$ we can push-forward along $f$ as $K^\text{MW}_0(X) \simeq K^\text{MW}_0(X, \omega_f) \to K^\text{MW}_0(Y)$ to obtain an element $f_* (1)$ in $K^\text{MW}_0(Y)$, and we have $\gamma_f \circ \alpha(f, \psi) = f_* (1) \cdot \text{id}_Y$, using the action of Example 4.10.

Remark 4.15. Suppose that $X$ is connected, $f : X \to Y$ is a finite surjective morphism with relative bundle $\omega_f$ and that $\omega_f \neq 0$ in $\text{Pic}(X)/\mathbb{Z}$ (take for instance the map $\mathbb{P}^2 \to \mathbb{P}^2$ defined by $[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2]$). Consider the finite correspondence $Y \to X$ corresponding to (the transpose of) the graph of $f$. As in the previous example, we have a transfer homomorphism

$$(\Gamma_f)_* : K^\text{MW}_0(X, \omega_f) \to \widetilde{CH}^d_X(Y \times X, \omega_{Y \times X/X})$$

making the diagram

$$\begin{array}{ccc}
K^\text{MW}_0(X, \omega_f) & \xrightarrow{(\alpha_1 \times \alpha_2)} & \widetilde{CH}^d_X(Y \times X, \omega_{Y \times X/X}) \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{\sim} & \widetilde{CH}^d_X(Y \times X)
\end{array}$$

commutative, where the vertical homomorphisms are the forgetful maps and the horizontal ones are isomorphisms, as one clearly sees using the Milnor-Witt Gersten complex. Since $\omega_f$ is not a square in $\text{Pic}(X)$, the left-hand vertical map is not surjective: it is equal to the rank map. It follows that the map $\widetilde{Cor}_k(Y, X) \to \widetilde{Cor}_k(Y, X)$ is not surjective. Indeed, if we enlarge the support to a larger admissible set $X'$, one of its irreducible components will be $X$, and the (usual) Chow groups with support in $X'$ will be isomorphic to several copies of $\mathbb{Z}$, one for each irreducible component, and the forgetful map cannot surject to the copy of $\mathbb{Z}$ corresponding to $X$.

4.4. Tensor products. Let $X_1, X_2, Y_1, Y_2$ be smooth schemes over $\text{Spec}(k)$. Let $d_1 = \dim Y_1$ and $d_2 = \dim Y_2$.

Let $\alpha_1 \in \widetilde{CH}^{d_1}_{T_1}(X_1 \times Y_1, \omega_{X_1 \times Y_1/x_1})$ and $\alpha_2 \in \widetilde{CH}^{d_2}_{T_2}(X_2 \times Y_2, \omega_{X_2 \times Y_2/x_2})$ for some admissible subsets $T_1 \subset X_1 \times Y_1$ and $T_2 \subset X_2 \times Y_2$. The exterior product defined in [4, §4] gives a cycle

$$(\alpha_1 \times \alpha_2) \in \widetilde{CH}^{d_1 + d_2}_{T_1 \times T_2}(X_1 \times Y_1 \times X_2 \times Y_2, \omega_{Y_1} \otimes \omega_{Y_2}).$$

Let $\sigma : X_1 \times Y_1 \times X_2 \times Y_2 \to X_1 \times X_2 \times Y_1 \times Y_2$ be the transpose isomorphism. Applying $\sigma_*$, we get a cycle

$$\sigma_*(\alpha_1 \times \alpha_2) \in \widetilde{CH}^{d_1 + d_2}_{\sigma(T_1 \times T_2)}(X_1 \times X_2 \times Y_1 \times Y_2, \omega_{Y_1} \otimes \omega_{Y_2}).$$

It is straightforward to check that (the underlying reduced scheme) $\sigma(T_1 \times T_2)$ is finite and surjective over $X_1 \times X_2$. Thus $\sigma_*(\alpha_1 \times \alpha_2)$ defines a finite Chow-Witt correspondence between $X_1 \times X_2$ and $Y_1 \times Y_2$. 

Definition 4.16. If $X_1$ and $X_2$ are smooth schemes over $k$, we define their tensor product as $X_1 \otimes X_2 := X_1 \times X_2$. If $\alpha_1 \in \text{Cor}_k(X_1, Y_1)$ and $\alpha_2 \in \text{Cor}_k(X_2, Y_2)$, then we define their tensor product as $\alpha_1 \otimes \alpha_2 := \sigma_*(\alpha_1 \times \alpha_2)$.

Lemma 4.17. The tensor product $\otimes$, together with the obvious associativity and symmetry isomorphisms endows $\text{Cor}_k$ with the structure of a symmetric monoidal category.

Proof. Straightforward. □

5. Presheaves on $\widetilde{\text{Cor}}_k$

Definition 5.1. A presheaf with generalized transfers is a contravariant additive functor $\text{Cor}_k \to \mathcal{A}b$. We will denote by $\text{PSh}_k$ the category of presheaves with generalized transfers. Let $\tau$ be Zar, Nis and Et, respectively the Zariski, Nisnevich or étale topology on $\text{Sm}_k$. We say that a presheaf with generalized transfers is a $\tau$-sheaf with generalized transfers if when restricted to $\text{Sm}_k$, it is a sheaf in the $\tau$-topology. We denote by $\text{Sh}_{\tau,k}$ the category of $\tau$-sheaves with generalized transfers.

Remark 5.2. We use the terminology generalized transfers in analogy with [18, Definition 5.7]. Indeed, let $M$ be a Nisnevich sheaf with generalized transfers in the sense of the above definition. Then it is endowed with an action of $K_0^{\text{MW}}$ by Example 4.10. Following the procedure described in Section 5.1, one can define $M(F)$ for any finitely generated field extension $F/k$. If $F \subset L$ is a finite field extension, then the canonical orientation of Lemma 2.4 together with the push-forwards defined in Example 4.14 show that we have a homomorphism $\text{Tr}_L^F : M(L) \to M(F)$. One can then check that the axioms listed in [18, Definition 5.7] are satisfied. Conversely, we don’t know if a Nisnevich sheaf with generalized transfers in the sense of Morel yields a Nisnevich sheaf on $\widetilde{\text{Cor}}_k$ but this seems quite plausible.

Recall first that there is a forgetful additive functor $F : \text{Cor}_k \to \text{Cor}_k$. It follows that any (additive) presheaf on $\text{Cor}_k$ defines a presheaf on $\widetilde{\text{Cor}}_k$ by composition. We now give a more exotic example.

Lemma 5.3. For any $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, the contravariant functor $X \mapsto H^i(X, K_j^{\text{MW}})$ is a presheaf on $\text{Cor}_k$.

Proof. Let $X, Y$ be smooth schemes and $T \subset X \times Y$ be an admissible subset. Let $\beta \in H^i(Y, K_j^{\text{MW}})$ and $\alpha \in \text{CH}^{dv}_T(X \times Y, \omega_{X \times Y/X})$ be cycles. We set

$$\alpha^*(\beta) := (p_X)_*(p_Y^*(\beta) \cdot \alpha)$$

where $p_X$ and $p_Y$ are the respective projections. We let the reader check that $\alpha^*$ is additive. If $T \subset T'$, we have a commutative diagram

$$\xymatrix{ \text{CH}^{dv}_T(X \times Y, \omega_{X \times Y/X}) \ar[r] & \text{CH}^{dv}_{T'}(X \times Y, \omega_{X \times Y/X}) \ar[d]^{(p_X)_*} \ar[l]_{(p_X)_*} \ar[d]^{(p_X)_*} \ar[l]_{(p_X)_*} \ar[d]^{(p_X)_*} \ar[l]_{(p_X)_*} \ar[l]_{(p_X)_*} \ar[l]_{(p_X)_*} \ar[l]_{(p_X)_*} \ar[l]_{(p_X)_*} \ar[l]_{(p_X)_*} }$$

where the top horizontal morphism is the extension of support. It follows that $\alpha \mapsto \alpha^*$ defines a map $\text{Cor}_k(X, Y) \to \text{Hom}_{\mathcal{A}b}(H^i(Y, K_j^{\text{MW}}), H^i(X, K_j^{\text{MW}}))$. We now
check that this map preserves the respective compositions. Consider the diagram

Let \( \alpha_1 \in \text{CH}^{d_Y}_{T_Y}(X \times Y, \omega_{X \times Y/X}) \) and \( \alpha_2 \in \text{CH}^{d_Z}_{T_Z}(Y \times Z, \omega_{Y \times Z/Y}) \) be correspondences, with \( T_1 \subset X \times Y \) and \( T_2 \subset Y \times Z \) admissible. By definition, we have

\[
(\alpha_2 \circ \alpha_1)^*(\beta) = (p'_j)_*[(r')^*(\beta) \cdot \pi_*(\cdot (a_2) \cdot (t')^*(a_1))].
\]

Using the projection formula, we have

\[
(r')^*(\beta) \cdot \pi_*(\cdot (a_2) \cdot (t')^*(a_1)) = \pi_*((r')^*(\beta) \cdot (q')^*(\cdot (a_2) \cdot (t')^*(a_1)))
\]

\[
= \pi_*((q')^*(r^*(\beta) \cdot (q')^*(\cdot (a_2) \cdot (t')^*(a_1)))
\]

\[
= \pi_*((q')^*(r^*(\beta) \cdot (a_2) \cdot (t')^*(a_1))).
\]

Using the fact that \( (p')_*\pi_* = p_* t'_* \) and the projection formula once again, we get

\[
(\alpha_2 \circ \alpha_1)^*(\beta) = p_*(t'_*([q'_*(r^*(\beta) \cdot (a_2) \cdot (t')^*(a_1)]
\]

\[
= p_*([t'_*](q')^*(r^*(\beta) \cdot (a_2) \cdot (a_1)) = \alpha_2^*(\alpha_1^*(\beta)). \]

**Remark 5.4.** The presheaf \( X \mapsto H^*(X, \mathbf{K}^\text{MW}_j) \) doesn’t have transfers in the sense of Voevodsky, as the projective bundle formula doesn’t hold in that setting (see [7, Theorem 11.7]).

**5.1. Extending presheaves to limits.** We consider the category \( \mathcal{P} \) of filtered projective systems \((X_\lambda)_{\lambda \in I}, f_{\lambda j}\) of smooth quasi-compact schemes over \( k \), with affine étale transition morphisms \( f_{\lambda j} : X_\lambda \to X_\mu \). Morphisms in \( \mathcal{P} \) are defined by \( \text{Hom}((X_\lambda)_{\lambda \in I}, (X_\mu)_{\mu \in J}) = \lim_{\lambda \to \mu} \text{Hom}(X_\lambda, X_\mu) \), as in [13, 8.13.3]. The limit of such a system exists in the category of schemes by loc. cit. 8.2.3, and by 8.13.5, the functor sending a projective system to its limit defines an equivalence of categories from \( \mathcal{P} \) to the full subcategory \( \text{Sm}_k \) of schemes over \( k \) that are limits of such systems.

**Remark 5.5.** It follows from this equivalence of categories that such a projective system converging to a scheme that is already finitely generated (e.g. smooth) over \( k \) has to be constant above a large enough index.

Let now \( F \) be a presheaf of abelian groups on \( \text{Sm}_k \). We extend \( F \) to a presheaf \( \tilde{F} \) on \( \mathcal{P} \) by setting on objects \( \tilde{F}((X_\lambda)_{\lambda \in I}) = \lim_{\lambda \to \mu} F(X_\lambda) \). An element of the set \( \lim_{\lambda \to \mu} \text{Hom}(X_\lambda, X_\mu) \) yields a morphism \( \lim_{\mu \to \lambda} F(X_\mu) \to \lim_{\lambda \to j} F(X_\lambda) \), this respects composition, and thus \( \tilde{F} \) is well-defined. Using the above equivalence of categories, it follows immediately that \( \tilde{F} \) defines a presheaf on \( \text{Sm}_k \) which extends \( F \).
in the sense that $\tilde{F}$ and $F$ coincide on $\text{Sm}_k$ since a smooth scheme can be considered as a constant projective system.

Since any finitely generated field extension $L/k$ can be written as a limit of smooth schemes in the above sense, we can consider in particular $F(\text{Spec}(L))$ and to shorten the notation, we often write $F(L)$ instead of $F(\text{Spec}(L))$ in what follows.

We will mainly apply this limit construction when $F$ is $\text{Cor}_k(- \times X, Y), K^*_\text{MW}$ or the generalized motivic cohomology groups $H^{p,q}(-, \mathbb{Z})$, to be defined in 5.31.

We now slightly extend this equivalence of categories to a framework useful for Chow-Witt groups with support. We consider the category $\mathcal{T}$ of triples $(X, Z, \mathcal{L})$ where $X$ is scheme of finite type over $k$, with a closed subset $Z$ and a line bundle $\mathcal{L}$ over $X$. A morphism from $(X_1, Z_1, \mathcal{L}_1)$ to $(X_2, Z_2, \mathcal{L}_2)$ in this category is a pair $(f, i)$ where $f : X_1 \to X_2$ is a morphism of $k$-schemes such that $f^{-1}(Z_2) \subseteq Z_1$ and $i : f^*\mathcal{L}_2 \to \mathcal{L}_1$ is an isomorphism of line bundles over $X_1$. The composition of two such morphisms $(f, i)$ and $(g, j)$ is defined as $(f \circ g, j \circ g^*(i))$. Let $\mathcal{P}$ be the category of projective systems in that category such that:

- the objects are regular and the transition maps are affine étale;
- beyond any index, there is a $\mu$ such that for any $\lambda$ beyond $\mu$, we have $f^{-1}_\lambda(Z_{\mu}) = Z_{\lambda}$, with morphisms defined by a double limit of Hom groups in $\mathcal{T}$, as above. Let $X$ be the inverse limit of the $X_{\lambda}$. All the pull-backs of the various $\mathcal{L}_{\lambda}$ to $X$ are canonically isomorphic by pulling back the isomorphisms $i_{\lambda\mu}$, and by the last condition, the closed subsets $Z_{\lambda}$ stabilize to a closed subset $Z$ of $X$. In other words, the inverse limit of the system exists in $\mathcal{T}$.

**Proposition 5.6.** The functor sending a system to its inverse limit is an equivalence of categories from $\mathcal{P}$ to the full subcategory of $\mathcal{T}$ of objects $(X, Z, \mathcal{L})$ such that $X$ is a projective limit of regular schemes with affine étale transition morphisms.

**Proof.** It follows from [13], 8.13.5 for the underlying schemes, 8.3.11 for the closed subset (since our schemes are of finite type over a field, they are Noetherian and every closed subset is constructible), 8.5.2 and 8.5.5 for the line bundle. \qed

As previously, any presheaf of abelian groups or sets $F$ defined on the full subcategory of $\mathcal{T}$ with regular underlying schemes can be extended uniquely to the subcategory of $\mathcal{T}$ with underlying schemes that are limits.

If the limit scheme $X$ happens to be regular, for any nonnegative integer $i$, the pull-back induces a map

$$\lim_{\lambda \in I} \text{CH}_Z^i(X_{\lambda}, \mathcal{L}_{\lambda}) \to \text{CH}_Z^i(X, \mathcal{L}).$$

**Lemma 5.7.** This map is an isomorphism.

**Proof.** Since $(X, Z, \mathcal{L})$ can be considered as a constant projective system, it follows from the equivalence of categories of Proposition 5.6. \qed

**5.2. Representable presheaves.**

**Definition 5.8.** Let $X \in \text{Sm}_k$. We denote by $\tilde{\mathbb{Z}}_X$ the presheaf $\text{Cor}_k(-, X)$. If $x : \text{Spec}(k) \to X$ is a rational point, we denote by $\tilde{\mathbb{Z}}_{X,x}$ the cokernel of the morphism $\tilde{\mathbb{Z}}_{\text{Spec}(k)} \to \tilde{\mathbb{Z}}_X$ (which is split injective). More generally, let $X_1, \ldots, X_n$ be
smooth schemes pointed respectively by the rational points \( x_1, \ldots, x_n \). We define \( \hat{Z}(x_1, x_2, \ldots, x_n) \) as the cokernel of the split injective map
\[
\bigoplus_i \hat{Z}_{X_1 \times \cdots \times X_i} \rightarrow \hat{Z}_{X_1 \times \cdots \times X_n}.
\]

**Example 5.9.** It follows from Example 4.4 that \( \hat{Z}_{\text{Spec}(k)} = K_0^{\text{MW}} \), and we write it \( \hat{Z} \) when there is no possible confusion on the base field.

Our next goal is to check that for any smooth scheme \( X \), the presheaf \( \hat{Z}_X \) is actually a sheaf in the Zariski topology. We start with an easy lemma.

**Lemma 5.10.** Let \( Y \in \text{Sm}_k \) be connected, with function field \( k(Y) \). Then for any \( X \in \text{Sm}_k \), the homomorphism \( \text{Cor}_k(Y, X) \rightarrow \text{Cor}_k(k(Y), X) \) is injective.

**Proof.** It follows from Lemma 4.5 that all transition maps in the system converging to \( \text{Cor}_k(k(Y), X) \) are injective.

**Proposition 5.11.** For any \( X \in \text{Sm}_k \), the presheaf \( \hat{Z}_X \) is a sheaf in the Zariski topology.

**Proof.** It suffices to prove that for any cover of a scheme \( Y \) by two Zariski open sets \( U \) and \( V \), the equalizer sequence
\[
\text{Cor}_k(Y, X) \rightarrow \text{Cor}_k(U, X) \bigcap \text{Cor}_k(V, X) \rightarrow \text{Cor}_k(U \cap V, X)
\]
is exact in the well-known sense. Injectivity of the map on the left follows from Lemma 5.10. To prove exactness in the middle, let \( \alpha \) and \( \beta \) be elements in the middle groups, equalizing the right arrows and with respective supports \( E \in \mathcal{A}(U, X) \) and \( F \in \mathcal{A}(V, X) \) by Lemma 4.7. Since \( \alpha \) restricted to \( V \) is \( \beta \) restricted to \( U \), we must have \( E \cap V = F \cap U \) by Lemma 4.8. By Lemma 4.3, the closed set \( T = E \cup F \) is admissible, and the conclusion now follows from the long exact sequence of localization for Chow-Witt groups with support in \( T \). \( \square \)

**Example 5.12.** In general, the Zariski sheaf \( \hat{Z}_X = \text{Cor}_k(-, X) \) is not a Nisnevich sheaf (and therefore not an étale sheaf either). Set \( A^{1}_{a_1, \ldots, a_n} := A^1 \setminus \{a_1, \ldots, a_n\} \) for any rational points \( a_1, \ldots, a_n \in A^1(k) \) and consider the Nisnevich square
\[
\begin{array}{c}
A^{1}_{0, \ldots, 0} \longrightarrow A^{1}_{0} \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
A^{1}_{0, 1} \longrightarrow A^{1}_{0, 1}
\end{array}
\]
where the morphism \( f : A^1_{0,1} \rightarrow A^1_{0} \) is given by \( x \mapsto x^2 \), the horizontal maps are inclusions and \( g \) is the base change of \( f \).

We now show that \( \hat{Z}_{A^1_{0,1}} \) is not a Nisnevich sheaf. In order to see this, we prove that the sequence
\[
\hat{Z}_{A^1_{0,1}} \rightarrow \hat{Z}_{A^1_{0,1}} \oplus \hat{Z}_{A^1_{0,1}} \rightarrow \hat{Z}_{A^1_{0,1}}
\]
is not exact in the middle. Let \( \Delta : A^1_{0,1} \rightarrow A^1_{0,1} \times A^1_{0,1} \) be the diagonal embedding.

It induces an isomorphism
\[
K_0^{\text{MW}}(A^1_{0,1}) \rightarrow CH^1_{\Delta(A^1_{0,1})}(A^1_{0,1} \times A^1_{0,1} / A^1_{0,1})
\]
and thus a monomorphism $K^\mathrm{MW}_0(\mathbb{A}^1_{0,1}) \to \mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1})$. Since the restriction homomorphism $K^\mathrm{MW}_0(\mathbb{A}^1_0) \to K^\mathrm{MW}_0(\mathbb{A}^1_{0,1})$ is injective, it follows that the class $\eta \cdot [t] = -1 + (t)$ is non trivial in $K^\mathrm{MW}_0(\mathbb{A}^1_{0,1})$ and its image $\alpha_t$ in $\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1})$ is also non trivial. We claim that its restriction in $\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1,-1})$ is trivial. Indeed, consider the Cartesian square

$$
\begin{array}{c}
\mathbb{A}^1_{0,1,-1} \xrightarrow{g} \mathbb{A}^1_{0,1,-1} \times \mathbb{A}^1_{0,1} \\
\downarrow \quad \downarrow (g \times 1) \\
\mathbb{A}^1_{0,1} \quad \mathbb{A}^1_{0,1} \times \mathbb{A}^1_{0,1}
\end{array}
$$

From Example 4.13, we know that the restriction of $\alpha_t$ to $\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1,-1})$ is represented by $(g \times 1)\Delta(-1 + (t))$. By base change, the latter is $(\Gamma g)_*g^*(\alpha_t)$. Now we have $g^*(-1 + (t)) = -1 + (t^2) = 0$ in $K^\mathrm{MW}_0(\mathbb{A}^1_{0,1,-1})$. In conclusion, we see that the correspondence $(\alpha_t, 0)$ is in the kernel of the homomorphism

$$
\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1}) \oplus \mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1,-1}) \to \mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1,-1}).
$$

To conclude that $\mathcal{Z}_{\mathbb{A}^1_{0,1}}$ is not a sheaf, it then suffices to show that $\alpha_t$ cannot be the restriction of a correspondence in $\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_0)$. To see this, observe first that the restriction map $\mathcal{A}(\mathbb{A}^1_0, \mathbb{A}^1_{0,1}) \to \mathcal{A}(\mathbb{A}^1_{0,1}, \mathbb{A}^1_{0,1})$ is injective as well as the homomorphism $\mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_0) \to \mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_{0,1})$. Suppose then that $\beta \in \mathcal{Z}_{\mathbb{A}^1_{0,1}}(\mathbb{A}^1_0)$ is a correspondence whose restriction is $\alpha_t$, and let $T = \text{supp}(\beta)$. From the above observations, we see that $\Delta(\mathbb{A}^1_{0,1}) = \text{supp}(\alpha_t) = T \cap (\mathbb{A}^1_{0,1} \times \mathbb{A}^1_{0,1})$. In particular, we see that $T$ is irreducible and its generic point is the generic point of the (transpose of the) graph of the inclusion $\mathbb{A}^1_{0,1} \to \mathbb{A}^1_0$. But the graph is closed, but not finite over $\mathbb{A}^1_0$. It follows that $\beta$ doesn’t exist and thus that $\mathcal{Z}_{\mathbb{A}^1_{0,1}}$ is not a Nisnevich sheaf.

**Definition 5.13.** Let $Y \in \text{Sm}_k$ and $F \in \text{PSh}_k$. For any $X \in \text{Sm}_k$, we define $F^Y(X) := F(X \times Y)$.

Consider the unit $1_Y \in \overline{\text{Cor}}_k(Y, Y)$. Tensoring with $1_Y$ induces a map

$$
\overline{\text{Cor}}_k(X, Z) \to \overline{\text{Cor}}_k(X \times Y, Z \times Y)
$$

and it follows that $F^Y$ is also a presheaf on $\overline{\text{Cor}}_k$ for any smooth scheme $Y$. We see moreover that any morphism $f : Y \to Y'$ induces a morphism $Ff : F^Y \to F^{Y'}$. Given $F \in \text{PSh}_k$, we then get a contravariant functor

$$
\Theta_F : \text{Sm}_k \to \text{PSh}_k
$$

defined on objects by $\Theta_F(Y) := F^Y$ and on morphisms by $\Theta_F(f) = Ff$.

The proof of the next result is easy and we thus omit it.

**Lemma 5.14.** Suppose that $F \in \text{PSh}_k$ is a sheaf in the Zariski topology. Then $F^Y$ is also a sheaf for any $Y \in \text{Sm}_k$.

**Definition 5.15.** Let $Y$ be a scheme pointed by $y : \text{Spec}(k) \to Y$ and let $F \in \text{PSh}_k$. Following [19, Remark 2.33], we define $F^{(Y, w)}$ as the kernel of the map
\[ F^Y : F^Y \rightarrow F. \] Similarly, when \((Y_1, x_1), \ldots, (Y_n, x_n)\) are pointed schemes, we define 
\[ F(Y_1, y_1) \wedge \ldots \wedge (Y_n, y_n) \] as the kernel of the map 
\[ F^{Y_1 \times \ldots \times Y_n} \oplus F^{id_{X_1} \times \ldots \times id_{X_n}} \rightarrow \bigoplus_{i=1}^n F^{Y_1 \times \ldots \times Y_{i-1} \times Y_i \times \ldots \times Y_n}. \]

**Remark 5.16.** It is easy to check that 
\[ F(Y_1, y_1) \wedge (Y_2, y_2) = (F(Y_1, y_1))(Y_2, y_2) \] and so on with multiple factors.

It is immediate to check that \(F(X, y)\) is a sheaf (for some topology) whenever \(F\) is.

**Remark 5.17.** Let \(X\) be a smooth scheme and let \(X_+ := X \amalg \text{Spec}(k)\). The schemes \(X\) and \(Y\) can be considered as sheaves of sets on \(\text{Sm}_k\). As such it makes sense to consider the presheaf of sets \(\tilde{X} \wedge (Y, y) := (X \times Y)/(X \times y)\). Then \(F(X, y)(X)\) is precisely what \(F(X_+ \wedge (Y, y))\) would be if \(F\) was defined on presheaves of sets instead of only on \(\text{Sm}_k\) (and sent quotients to kernels).

**Definition 5.18.** Let \(E \in \widetilde{\text{PSh}_k}\) and \(n \in \mathbb{N}\). Then the \(n\)-th contraction of \(F\), denoted by \(F_{-n}\), is the presheaf \(F^{(\mathbb{G}_m, 1)^n}\). Thus, \(F_{-n} = (F_{-n+1})_{-1}\).

**Example 5.19.** If \(F = K^{\text{MW}}_j\) for some \(j \in \mathbb{Z}\), then it follows from Lemma 3.6 that \(F_{-1} = K^{\text{MW}}_{j-1}\).

Using Remark 5.17, we see that \(F_{-n}(X)\) is precisely what \(F(X_+ \wedge (\mathbb{G}_m, 1)^n)\) would be if \(F\) was defined on a broader category.

**Definition 5.20.** A presheaf \(F\) in \(\text{PSh}_k\) is said to be homotopy invariant if the map
\[ F(X) \rightarrow F(X \times \mathbb{A}^1) \]
induced by the projection \(X \times \mathbb{A}^1 \rightarrow X\) is an isomorphism for any \(X \in \text{Sm}_k\).

**Example 5.21.** We already know that \(\tilde{Z}_k\) coincides with the Nisnevich sheaf \(K^{\text{MW}}_0\). It follows from [5, Corollaire 11.3.3] that this sheaf is homotopy invariant.

### 5.3 Twisted sheaves

We define the pointed scheme \(\mathbb{G}_m, 1 = (\mathbb{G}_m, 1)\).

**Definition 5.22.** For any \(q \in \mathbb{Z}\), we define the Zariski sheaf \(\tilde{\mathbb{Z}}\{q\}\) by
\[ \tilde{\mathbb{Z}}\{q\} := \begin{cases} 
\mathbb{Z}\mathbb{G}^\wedge_{m, 1} & \text{if } q > 0, \\
\tilde{\mathbb{Z}}_k & \text{if } q = 0, \\
\mathbb{Z}\mathbb{G}^\wedge_{m, 1} & \text{if } q < 0. 
\end{cases} \]

Note that for \(q < 0\), the sheaf \(\tilde{\mathbb{Z}}\{q\}\) is the \((-q)\)-th contraction of \(\tilde{\mathbb{Z}}_k\).

**Lemma 5.23.** We have \(\tilde{\mathbb{Z}}\{q\} = \mathbb{W}\) for any \(q < 0\), where the latter is the Zariski sheaf associated to the presheaf \(X \mapsto \mathbb{W}(X)\) (Witt group).

**Proof.** \(\tilde{\mathbb{Z}}_{\text{Spec}(k)} = K^{\text{MW}}_0\) and \((K^{\text{MW}}_0)_q = K^{\text{MW}}_q = \mathbb{W}\) for \(q < 0\) by Example 5.19. \(\square\)

**Remark 5.24.** Of course, we could have defined \(\tilde{\mathbb{Z}}\{q\}\) to be \(\mathbb{W}\) when \(q < 0\) from the beginning. The advantage of our definition is to make the product structure defined below more explicit.
Our next goal is to define products $\mu_{p,q} : \tilde{\mathbb{Z}}\{p\} \otimes \tilde{\mathbb{Z}}\{q\} \to \tilde{\mathbb{Z}}\{p + q\}$ for any $p, q \in \mathbb{Z}$. We first make a few general comments.

Let $X, Y, Z$ be smooth schemes. Then the tensor product and the precomposition with the diagonal map induce a morphism

$$\widetilde{\text{Cor}}_k(X, Y) \otimes \widetilde{\text{Cor}}_k(X, Z) \to \widetilde{\text{Cor}}_k(X \times X, Y \times Z) \to \widetilde{\text{Cor}}_k(X, Y \times Z).$$

This construction is covariant in both $Y$ and $Z$ in the sense that a morphism $Z \to Z'$ yields a commutative diagram

$$\begin{array}{ccc}
\widetilde{\text{Cor}}_k(X, Y) \otimes \widetilde{\text{Cor}}_k(X, Z) & \to & \widetilde{\text{Cor}}_k(X \times X, Y \times Z) \\
\downarrow & & \downarrow \\
\widetilde{\text{Cor}}_k(X, Y) \otimes \widetilde{\text{Cor}}_k(X, Z') & \to & \widetilde{\text{Cor}}_k(X \times X, Y \times Z')
\end{array}$$

and similarly for $Y \to Y'$. Specializing at $Y = (\mathbb{A}^1_k \setminus 0)^{\times p}$ and $Z = (\mathbb{A}^1_k \setminus 0)^{\times q}$, keeping track of the base points and using the diagram above, we get a morphism

$$\mu_{p,q} : \tilde{\mathbb{Z}}\{p\} \otimes \tilde{\mathbb{Z}}\{q\} \to \tilde{\mathbb{Z}}\{p + q\}$$

provided $p, q \geq 0$.

Example 5.25. Let $X$ be a smooth scheme. Then the pairing

$$\mu_{p,0}(X) : \tilde{\mathbb{Z}}\{p\}(X) \otimes \tilde{\mathbb{Z}}\{0\}(X) \to \tilde{\mathbb{Z}}\{p\}(X)$$

corresponds to the right action of $K^\text{MW}_0(X) = \mathbb{Z}\{0\}(X)$ on $\widetilde{\text{Cor}}_k(X, \mathbb{G}_m^p) = \tilde{\mathbb{Z}}\{p\}(X)$ described in Example 4.10. Indeed, let $Y$ be any smooth connected scheme and let $p_X : X \times Y \to X$ be the projection. Then a direct computation shows that $(\alpha \otimes \beta) \circ \Delta_X = \alpha \cdot p_X^*(\beta)$ for any $\alpha \in \widetilde{\text{Cor}}_k(X, Y)$ and any $\beta \in \widetilde{\text{Cor}}_k(X, \text{Spec}(k)) = K^\text{MW}_0(k)$. On the other hand, the base change formula and the projection formula show that the composite $\alpha \circ ((\Delta_X)_*) = \alpha \cdot p_X^*(\beta)$ where $\Delta_X : X \to X \times X$ is the diagonal map and $(\Delta_X)_* : K^\text{MW}_0(X) \to \overline{\text{CH}}^\text{dim}(X)(X \times X, \omega_{X \times X/X})$ is the pushforward homomorphism. Now $\alpha \circ ((\Delta_X)_*)$ is precisely the action of $\beta$ on $\alpha$ and the observation follows.

In a similar fashion, the tensor product induces a morphism

$$g_{X,Y,Z} : \widetilde{\text{Cor}}_k(X \times Y, pt) \otimes \widetilde{\text{Cor}}_k(X \times Z, pt) \to \widetilde{\text{Cor}}_k(X \times Y \times X \times Z, pt).$$

On the other hand, the diagonal map and the switch of the second and third factors give a map

$$X \times Y \times Z \to X \times X \times Y \times Z \to X \times Y \times X \times Z$$

and thus a map $\widetilde{\text{Cor}}_k(X \times Y \times X \times Z, pt) \to \widetilde{\text{Cor}}_k(X \times Y \times Z, pt)$. Altogether, we have obtained a morphism

$$\widetilde{\text{Cor}}_k(X \times Y, pt) \otimes \widetilde{\text{Cor}}_k(X \times Z, pt) \to \widetilde{\text{Cor}}_k(X \times Y \times Z, pt).$$

covariant in both $Y$ and $Z$. Using this observation, specializing at $Y = (\mathbb{A}^1_k \setminus 0)^{\times p}$ and $Z = (\mathbb{A}^1_k \setminus 0)^{\times q}$ and keeping track of the base points, we see that we have defined a morphism

$$\mu_{p,q} : \tilde{\mathbb{Z}}\{p\} \otimes \tilde{\mathbb{Z}}\{q\} \to \tilde{\mathbb{Z}}\{p + q\}$$

for $p, q \leq 0$. 

We now consider a slightly more general situation. Let $X, Y, Z, T$ be smooth schemes. Tensoring with $1_{X \times T}$ on the left and with $1_Z$ on the right yields a map

$$\tilde{\text{Cor}}_k(X, Y \times Z) \otimes \tilde{\text{Cor}}_k(X \times T \times Y, pt)$$

while the composition and the map

$$X \times T \to X \times X \times T \to X \times T \times X$$

induce a morphism

$$\tilde{\text{Cor}}_k(X \times T \times X, X \times T \times Y \times Z) \otimes \tilde{\text{Cor}}_k(X \times T \times Y \times Z, Z)$$

$$\tilde{\text{Cor}}_k(X \times T, Z).$$

All in all, we have obtained a map

$$h_{X, Y, Z, T} : \tilde{\text{Cor}}_k(X, Y \times Z) \otimes \tilde{\text{Cor}}_k(X \times T \times Y, pt) \to \tilde{\text{Cor}}_k(X \times T, Z)$$

which is covariant in $Z$ and contravariant in $T$. The functoriality in $Y$ is a bit more complicated and we will need the following lemma.

**Lemma 5.26.** Let $f : Y' \to Y$ be a morphism of schemes. Let

$$C := \operatorname{coker}(\tilde{\text{Cor}}_k(X, Y' \times Z) \to \tilde{\text{Cor}}_k(X, Y \times Z))$$

and

$$K := \ker(\tilde{\text{Cor}}_k(X \times T \times Y, pt) \to \tilde{\text{Cor}}_k(X \times T \times Y', pt))$$

where the maps are induced by $f$. Then $h_{X, Y, Z, T}$ induces a morphism $C \otimes K \to \tilde{\text{Cor}}_k(X \times T, Z)$.

**Proof.** It is clear that $h_{X, Y, Z, T}$ induces a map

$$h'_{X, Y, Z, T} : \tilde{\text{Cor}}_k(X, Y \times Z) \otimes C \to \tilde{\text{Cor}}_k(X \times T \times Y \times Z),$$

and it suffices then to prove that $h'_{X, Y, Z, T}$ vanishes on cycles coming from the group $\tilde{\text{Cor}}_k(X, Y' \times Z)$ to conclude. If $\alpha \in \tilde{\text{Cor}}_k(X, Y' \times Z)$, then its image under the composite

$$\tilde{\text{Cor}}_k(X, Y' \times Z) \to \tilde{\text{Cor}}_k(X, Y \times Z) \to \tilde{\text{Cor}}_k(X \times T \times X, X \times T \times Y \times Z)$$

is

$$1_{X \times T} \otimes (f \times 1_Z) \circ (1_{X \times Z} \otimes \alpha).$$

The image of any $\beta \in \tilde{\text{Cor}}_k(X \times T \times Y, pt)$ is precisely $\beta \otimes 1_Z$ under the map

$$\tilde{\text{Cor}}_k(X \times T \times Y, pt) \to \tilde{\text{Cor}}_k(X \times T \times Y \times Z, Z).$$

Composing these two correspondences, we obtain

$$\beta \otimes 1_Z \circ (1_{X \times T} \otimes (f \times 1_Z)) \circ (1_{X \times Z} \otimes \alpha).$$
On the other hand, \((f \times 1_Z) = f \otimes 1_Z\) by definition of the tensor product, and thus \(1_{X \times T} \otimes (f \times 1_Z) = (1_{X \times T} \otimes f) \otimes 1_Z\). Thus the equality (4) above becomes
\[
(\beta \otimes 1_Z) \circ (1_{X \times T} \otimes f) \circ (1_{X \times Z} \otimes \alpha).
\]
Now \(\beta \circ (1_{X \times T} \otimes f) = 0\) if and only if \(\beta \in K\) and the result is proved. 

\[\square\]

**Corollary 5.27.** If \(p \geq q \geq 0\), the maps
\[
h_{X, G_{m, \bullet}} \otimes G_{m, \bullet} : \text{Cor}_k(X, G_{m, \bullet}^p) \otimes \text{Cor}_k(X \times G_{m, \bullet}^q, \text{pt}) \to \text{Cor}_k(X, G_{m, \bullet}^{p-q})
\]
for \(X \in \text{Sm}_k\) induce a map
\[
\mu_{p-q} : \hat{\mathbb{Z}}\{p\} \otimes \hat{\mathbb{Z}}\{-q\} \to \hat{\mathbb{Z}}\{p-q\}.
\]
If \(q \geq p \geq 0\), the maps
\[
h_{X, G_{m, \bullet}} \otimes G_{m, \bullet} : \text{Cor}_k(X, G_{m, \bullet}^p) \otimes \text{Cor}_k(X \times G_{m, \bullet}^q, \text{pt}) \to \text{Cor}_k(X \times G_{m, \bullet}^{p-q}, \text{pt})
\]
induce a map
\[
\mu_{p-q} : \hat{\mathbb{Z}}\{p\} \otimes \hat{\mathbb{Z}}\{-q\} \to \hat{\mathbb{Z}}\{p-q\}.
\]

**Proof.** We prove the result in case \(p \geq q \geq 0\), the other case being similar. We write \(G_{m, \bullet}^p = G_{m, \bullet}^{p-q} \times G_{m, \bullet}^q\). Any face \(G_{m, \bullet}^{p-1} \subset G_{m, \bullet}^q\) can be seen either as a face \(G_{m, \bullet}^{p-1} \subset G_{m, \bullet}^{p-q}\) or a face \(G_{m, \bullet}^{q-1} \subset G_{m, \bullet}^{p-q}\). In the first case, the map
\[
h_{X, G_{m, \bullet}} \otimes G_{m, \bullet} : \text{Cor}_k(X, G_{m, \bullet}^p) \otimes \text{Cor}_k(X \times G_{m, \bullet}^q, \text{pt}) \to \text{Cor}_k(X \times G_{m, \bullet}^{p-q}, \text{pt})
\]
factors through a map
\[
\mu_{p-q}(X) : \hat{\mathbb{Z}}\{p\}(X) \otimes \hat{\mathbb{Z}}\{-q\}(X) \to \hat{\mathbb{Z}}\{p-q\}(X).
\]
since \(h_{X, G_{m, \bullet}} \otimes G_{m, \bullet}\) is covariant in the factor \(G_{m, \bullet}^{p-q}\). We thus obtain the required morphism of sheaves. In the second case, we can apply Lemma 5.26 to get the result. 

Finally we define maps
\[
\mu_{-p,q}(X) : \hat{\mathbb{Z}}\{-p\}(X) \otimes \hat{\mathbb{Z}}\{q\}(X) \to \hat{\mathbb{Z}}\{-p+q\}(X)
\]
as the composite
\[
\hat{\mathbb{Z}}\{-p\}(X) \otimes \hat{\mathbb{Z}}\{q\}(X) \to \hat{\mathbb{Z}}\{q\}(X) \otimes \hat{\mathbb{Z}}\{-p\}(X) \to \hat{\mathbb{Z}}\{-p+q\}(X)
\]
where the first map is the switch of the factors.

### 5.4. Motivic sheaves.

Let \(\Delta^\bullet\) be the cosimplicial object on \(\text{Sm}_k\) defined by
\[
\Delta^n := \text{Spec}(k[x_0, \ldots, x_n]/(\sum_{i=0}^n x_i - 1)).
\]
and the usual faces and degeneracy maps. Given any \(F \in \text{PSh}_k\), we get a simplicial object \(\Theta_F(\Delta^\bullet)\) in \(\text{PSh}_k\).

**Definition 5.28.** The Suslin-Voevodsky singular construction on \(F\) is the complex associated to the simplicial object \(\Theta_F(\Delta^\bullet)\). Following the conventions, we denote it by \(C_F\).

The following lemma is well-known and we let the proof to the reader.

**Lemma 5.29.** Suppose that \(F\) is a homotopy invariant presheaf on \(\text{Cor}_k\). Then the natural homomorphism \(F \to C_F\) is a quasi-iso- morphism of complexes (here \(F\) is considered as a complex concentrated in degree 0).
Definition 5.30. For any integer \( q \in \mathbb{Z} \), we define \( \hat{\mathcal{Z}}(q) \) as the complex of Zariski sheaves \( C_* \hat{\mathcal{Z}}(q)[-q] \).

As opposed to [14], we use the homological notation and we then have by convention \( \hat{\mathcal{Z}}(q)_i = C_{q+i} \hat{\mathcal{Z}}(q) \). It follows that \( \hat{\mathcal{Z}}(q) \) is bounded below for any \( q \in \mathbb{Z} \).

Definition 5.31. The generalized motivic cohomology groups \( H^{p,q}(X, \hat{\mathcal{Z}}) \) are defined to be the hypercohomology groups

\[
H^{p,q}(X, \hat{\mathcal{Z}}) := H^p_{\text{Zar}}(X, \hat{\mathcal{Z}}(q)).
\]

The groups \( H^{p,q}(X, \hat{\mathcal{Z}}) \) are thus contravariant in \( X \in \text{Sm}_k \).

Our next aim is to define a ring structure on the total generalized motivic cohomology group. This is done exactly as in [14, Construction 3.11] replacing \( \mathbb{Z}(m) \) by \( \hat{\mathbb{Z}}(m) \). We recall the construction for the convenience of the reader. To obtain a morphism of complexes

\[
\hat{\mathcal{Z}}(m) \otimes \hat{\mathcal{Z}}(n) \to \hat{\mathcal{Z}}(m+n)
\]

it is sufficient to define a morphism of complexes

\[
\hat{\mathcal{Z}}(m)[m] \otimes \hat{\mathcal{Z}}(n)[n] \to \hat{\mathcal{Z}}(m+n)[m+n].
\]

Now \( \hat{\mathcal{Z}}(m)[m] = C_* \hat{\mathcal{Z}}(m) \) for any \( m \in \mathbb{Z} \) and the latter is the complex associated to the simplicial object \( \Theta_{\hat{\mathcal{Z}}(m)}(\Delta^\bullet) \). For any \( m, n \in \mathbb{Z} \), we obtain a bisimplicial sheaf \( \Theta_{\hat{\mathcal{Z}}(m)}(\Delta^\bullet) \otimes \Theta_{\hat{\mathcal{Z}}(n)}(\Delta^\bullet) \) and thus a simplicial sheaf \( \Theta_{\hat{\mathcal{Z}}(m)} \otimes \Theta_{\hat{\mathcal{Z}}(n)}(\Delta^\bullet) \) obtained by composing with the diagonal map \( \Delta^\bullet \to \Delta^\bullet \times \Delta^\bullet \). We denote by \( C_* \hat{\mathcal{Z}}(m,n) \) the associated complex. The Eilenberg-Zilber Theorem yields a quasi-isomorphism

\[
\nabla : C_* \hat{\mathcal{Z}}(m) \otimes C_* \hat{\mathcal{Z}}(n) \to C_* \hat{\mathcal{Z}}(m,n)
\]

Now the morphisms \( \mu_{m,n} : \hat{\mathcal{Z}}(m) \otimes \hat{\mathcal{Z}}(n) \to \hat{\mathcal{Z}}(m+n) \) defined in the previous section can be seen as morphisms of simplicial sheaves \( \Theta_{\hat{\mathcal{Z}}(m)} \otimes \Theta_{\hat{\mathcal{Z}}(n)}(\Delta^\bullet) \to \Theta_{\hat{\mathcal{Z}}(m+n)}(\Delta^\bullet) \) and thus induce morphisms \( C_* \hat{\mathcal{Z}}(m,n) \to C_* \hat{\mathcal{Z}}(m+n) \). Composing with \( \nabla \), we get the required morphisms

\[
\circ_{m,n} : C_* \hat{\mathcal{Z}}(m) \otimes C_* \hat{\mathcal{Z}}(n) \to C_* \hat{\mathcal{Z}}(m+n).
\]

Lemma 5.32. The morphism of complexes of sheaves

\[
\circ_{q,q'} : C_* \hat{\mathcal{Z}}(q) \otimes C_* \hat{\mathcal{Z}}(q') \to C_* \hat{\mathcal{Z}}(q + q').
\]

induces a pairing

\[
H^{p,q}(X, \hat{\mathcal{Z}}) \otimes H^{p',q'}(X, \hat{\mathcal{Z}}) \to H^{p+p',q+q'}(X, \hat{\mathcal{Z}})
\]

for any \( p, p', q, q' \in \mathbb{Z} \). This pairing is contravariant in \( X \in \text{Cor}_k \).

We now prove a series of results identifying the generalized motivic cohomology groups for \( q \leq 0 \).

Proposition 5.33. We have a canonical identification \( H^{p,0}(X, \hat{\mathcal{Z}}) = H^p(X, \mathcal{K}_0^{MW}) \).

Under this identification, the pairing

\[
H^{p,0}(X, \hat{\mathcal{Z}}) \otimes H^{p',0}(X, \hat{\mathcal{Z}}) \to H^{p+p',0}(X, \hat{\mathcal{Z}})
\]

coincides with the product

\[
H^p(X, \mathcal{K}_0^{MW}) \otimes H^{p'}(X, \mathcal{K}_0^{MW}) \to H^{p+p'}(X, \mathcal{K}_0^{MW}).
\]
and in particular, when \( p = 0 \), this gives a canonical identification of rings

\[
H^{0,0}(X, \tilde{Z}) = K^{\text{MW}}_0(X)
\]

**Proof.** By Example 4.4, we know that \( \tilde{Z}\{0\} = K^{\text{MW}}_0 \) and the latter is homotopy invariant by [5, Corollaire 11.3.3]. It follows that we have a canonical quasi-isomorphism \( C_* \tilde{Z}\{0\} \simeq K^{\text{MW}}_0 \) and thus a canonical quasi-isomorphism \( \tilde{Z}\{0\} \simeq K^{\text{MW}}_0 \). This yields the first claim.

For the second claim, recall that the product \( \tilde{Z}\{0\} \otimes \tilde{Z}\{0\} \to \tilde{Z}\{0\} \) is induced by the tensor product of Chow-Witt correspondences. In the case of \( \tilde{Z}\{0\} \), the tensor product is just the exterior product defined in [4, §4], and thus the product \( \tilde{Z}\{0\} \otimes \tilde{Z}\{0\} \to \tilde{Z}\{0\} \) is the usual product under the identification \( \tilde{Z}\{0\} = K^{\text{MW}}_0 \). The result follows immediately. \( \square \)

**Proposition 5.34.** For any \( q < 0 \), we have a canonical identification \( H^{p,q}(X, \tilde{Z}) = H^{p-q}(X, W) \). Under this identification, the pairing

\[
H^{p,q}(X, \tilde{Z}) \otimes H^{p',q'}(X, \tilde{Z}) \to H^{p+p',q+q'}(X, \tilde{Z})
\]

coincides with the product

\[
H^{p-q}(X, W) \otimes H^{p'-q'}(X, W) \to H^{p+p'-q-q'}(X, W)
\]

for any \( q' < 0 \).

**Proof.** The argument to prove the first claim is the same as in the above proposition, using Lemma 5.23 in place of Example 4.4 and [5, Théorème 11.2.4] (or [22]) for the homotopy invariance of \( W \).

For the second claim, recall that the product \( \tilde{Z}\{q\} \otimes \tilde{Z}\{q'\} \to \tilde{Z}\{q+q'\} \) is defined by using first the tensor product

\[
\overline{\text{Cor}}_k(X \times G^{-q}_m, pt) \times \overline{\text{Cor}}_k(X \times G^{-q'}_m, pt) \to \overline{\text{Cor}}_k(X \times G^{-q}_m \times X \times G^{-q'}_m, pt),
\]

which coincides with the exterior product

\[
K^{\text{MW}}_0(X \times G^{-q}_m) \otimes K^{\text{MW}}_0(X \times G^{-q'}_m) \to K^{\text{MW}}_0(X \times G^{-q}_m \times X \times G^{-q'}_m),
\]

and then the pull-back along the morphism \( X \times G^{-q}_m \times G^{-q'}_m \to X \times G^{-q}_m \times X \times G^{-q'}_m \).

Let \( t_1, \ldots, t_{-q} \) be parameters of \( G^{-q}_m \), \( u_1, \ldots, u_{-q'} \) be parameters of \( G^{-q'}_m \), \( \alpha \in K^{\text{MW}}_q(X) \) and \( \beta \in K^{\text{MW}}_{-q'}(X) \). Using the fact that elements of negative degrees in Milnor-Witt \( K \)-theory are central, we thus see that the image of \( (\alpha \cdot [t_1] \cdot \ldots \cdot [t_{-q}]) \otimes (\beta \cdot [u_1] \cdot \ldots \cdot [u_{-q'}]) \) under the map

\[
K^{\text{MW}}_0(X \times G^{-q}_m) \otimes K^{\text{MW}}_0(X \times G^{-q'}_m) \to K^{\text{MW}}_0(X \times G^{-q}_m \times G^{-q'}_m)
\]

is of the form \( \alpha \cdot \beta \cdot [t_1] \cdot \ldots \cdot [t_{-q}] \cdot [u_1] \cdot \ldots \cdot [u_{-q}] \). As a consequence, we see that the map

\[
\tilde{Z}\{q\}(X) \otimes \tilde{Z}\{q'\}(X) \to \tilde{Z}\{q+q'\}(X)
\]

coincides with the product \( K^{\text{MW}}_{-q'}(X) \otimes K^{\text{MW}}_{-q'}(X) \to K^{\text{MW}}_{-q-q'}(X) \). The result follows from the identification of all the relevant groups with \( W(X) \). \( \square \)

**Lemma 5.35.** The subring \( K^{\text{MW}}_0(X) = H^{0,0}(X, \tilde{Z}) \subset \bigoplus_{p,q \in \mathbb{Z}} H^{p,q}(X, \tilde{Z}) \) is central.
Proof: The products are induced by the morphisms of sheaves $\mu_{0,q} : \hat{\Z}(0) \otimes \hat{\Z}(q) \to \hat{\Z}(0)$ and $\mu_{q,0} : \hat{\Z}(q) \otimes \hat{\Z}(0) \to \hat{\Z}(q)$ for any $q \in \Z$. It suffices then to prove that the following diagram commutes

$$
\begin{array}{ccc}
\hat{\Z}(0) \otimes \hat{\Z}(q) & \xrightarrow{\mu_{0,q}} & \hat{\Z}(q) \\
\downarrow & & \downarrow \\
\hat{\Z}(q) \otimes \hat{\Z}(0) & \xrightarrow{\mu_{q,0}} & \hat{\Z}(q)
\end{array}
$$

where the left-hand vertical map is the switch of factors. In case $q < 0$, this is by definition of $\mu_{0,q}$ and $\mu_{q,0}$. If $q \geq 0$, we know that the map $\mu_{0,q}$ is induced by the exterior product

$$
\mu_{0,q} : \hat{\Z}(0) \otimes \hat{\Z}(q) \to \hat{\Z}(q)
$$

and the diagonal map. Let $\beta \in \Cor_k(X, pt) \otimes \Cor_k(X, G^q_m) \to \Cor_k(X \times X, G^q_m)$ together with the pull-back along the diagonal $\Delta : X \to X \times X$. The map $\mu_{q,0}$ is defined by considering this time the exterior product

$$
\mu_{q,0} : \hat{\Z}(q) \otimes \hat{\Z}(0) \to \hat{\Z}(q)
$$

and the diagonal map. Let $\beta \in \Cor_k(X, pt) = K_0^{\MW}(X)$ and $\alpha \in \Cor_k(X, G^q_m)$ be represented by a class in $\CH^q_m(X \times G^q_m, \omega_{X \times G^q_m/X})$, for $T \in A(X, G^q_m)$, that we still denote by $\alpha$. Let $\pi : X \times G^q_m \to X$ be the projection on the first factor. Using Remark 4.13, we see that $\mu_{0,q}(\beta \otimes \alpha)$ is represented by $\pi^*(\beta) \cdot \alpha$ in $\CH^q_m(X \times G^q_m, \omega_{X \times G^q_m/X})$. Similarly, $\mu_{q,0}(\alpha \otimes \beta)$ is represented by $\alpha \cdot \pi^*(\beta)$ and the result follows from the fact that $\beta$ is central in the total Chow-Witt ring (see Section 3).

Remark 5.36. At the moment, we don’t know exactly what is the commutation formula for elements $\alpha \in H^{p,q}(X, \hat{\Z})$ and $\beta \in H^{p',q'}(X, \hat{\Z})$. It should involve some power of $-1$ and some power of $(-1)$. For instance, a direct (but involved) adaptation of the proof of [14, Theorem 15.9] proves that $\alpha \cdot \beta = (-1)^{pp'}((-1)^{q'q}) \beta \cdot \alpha$ provided $q, q' \geq 0$.

5.5. Base change of the ground field. The various constructions considered until now are functorial with respect to the ground field $k$, and we now give details about some of the aspects of this functoriality. Let $L$ be a field extension of $k$, both $L$ and $k$ assumed to be perfect. For any scheme $X$ over $k$, let $X_L = \Spec(L) \times_{\Spec(k)} X$ be its extension to $L$. When $X$ is smooth, the canonical bundle $\omega_X$ pulls-back over $X_L$ to the canonical bundle $\omega_{X_L}$ of $X_L$, and thus all the relative canonical bundles of the type $\omega_{X \times Y/Y}$ are preserved by base-change to $L$.

By conservation of surjectivity and finiteness by base change, the pull-back induces a map $A(X, Y) \to A(X_L, Y_L)$. From the contravariant functoriality of Chow-Witt groups with support, passing to the limit, one then immediately obtains an extension of scalars functor

$$
\Cor_k \xrightarrow{\text{ext}_{L/k}} \Cor_L
$$

sending an object $X$ to $X_L$. This functor is monoidal, since $(X \times_k Y)_L \simeq X_L \times_L Y_L$ canonically.

When $L/k$ is finite, $L/k$ is automatically separable since $k$ is perfect (and $L$ is automatically perfect). Viewing an $L$-scheme $Y$ as a $k$-scheme $Y_k$ by composing its structural morphism with the smooth morphism $\Spec(L) \to \Spec(k)$ defines
a restriction of scalars functor $\text{res}_{L/K} : \text{Sm}_L \to \text{Sm}_k$. For any $L$-scheme $Y$ and $k$-scheme $X$, there are morphisms

$$\eta_Y : Y \to L \times_k Y = (Y_k)_L \quad \text{and} \quad \epsilon_X : (X_L)|_k = L \times_k X \to X$$

induced by the structural morphism of $Y$ and $\text{id}_Y$ for $\eta$, and the projection to $X$ for $\epsilon$.

**Lemma 5.37.** Let $X \in \text{Sm}_k$ and $Y, Y' \in \text{Sm}_L$.

1. The morphisms of functors $\eta$ and $\epsilon$ given on objects by $\eta_Y$ and $\epsilon_X$ define an adjunction $(\text{res}_{L/k}, \text{ext}_{L/k})$.
2. The natural morphism $(Y \times_L X)|_k \to Y_k \times_k X$ is an isomorphism (adjoint to the morphism $\eta_Y \times_L \text{id}_{X_k}$).
3. The morphism $\eta_Y$ is a closed embedding of codimension 0, identifying $Y$ with the connected components of $(Y_k)_L$ living over $\text{Spec}(L)$, diagonally inside $\text{Spec}(L) \times_k \text{Spec}(L)$.
4. The natural morphism $(Y \times_L Y')|_k \to Y_k \times_k Y'_k$ is a closed embedding of codimension 0, thus identifying the source as a union of connected components of the target.

**Proof.** Part (1) is straightforward. Part (2) can be checked locally, when $X = \text{Spec}(A)$ with $A$ a $k$-algebra and $Y = \text{Spec}(B)$ with $B$ an $L$-algebra, and the morphism considered is the canonical isomorphism $B \times_L L \times_k A \simeq B \times_k A$. To prove (3), note that the general case follows from base-change to $Y$ of the case $Y = \text{Spec}(L)$, which corresponds to $L \otimes_k L \to L$ via multiplication. Since the source is a product of separable extensions of $L$, one of which is isomorphic to $L$ by the multiplication map, the claim holds. Up to the isomorphism of part (2), the morphism $(Y \times_L Y')|_k \to Y_k \times_k Y'_k$ corresponds to the morphism $\eta_Y \times \text{id}_{Y'}$, so (4) follows from (3) by base change to $Y'$. All schemes involved being reduced, the claims about identifications of irreducible components are clear. \hfill $\square$

The functor $\text{res}_{L/k}$ extends to a functor $\Cor_L \xrightarrow{\text{res}_{L/k}} \Cor_k$, as we now explain. For any $X, Y \in \text{Sm}_L$, we have $\mathcal{A}(X, Y) \subseteq \mathcal{A}(X|_k, Y|_k)$ by part (4) of Lemma 5.37, so for any $T$ in it the push-forward induces a map

$$\overline{\text{CH}}^d_T(X \times Y, \omega_{X \times_L Y/X}) \to \overline{\text{CH}}^d_T(X|_k \times_k Y|_k, \omega_{X|_k \times_k Y|_k})$$

because $\omega_{L/k}$ is canonically trivial. Passing to the limit, it gives a map $\overline{\text{Cor}}_L(X, Y) \to \overline{\text{Cor}}_k(X|_k, Y|_k)$, and it is compatible with the composition of correspondences; this is an exercise using base change, where the only nontrivial input is that when $X, Y, Z \in \text{Sm}_L$, the push-forward from $X \times_L Y \times_L Z$ to $X|_k \times_k Y|_k \times_k Z|_k$ respects products, which follows from part (4) of Lemma 5.37.

The adjunction $(\text{res}_{L/k}, \text{ext}_{L/k})$ between $\text{Sm}_L$ and $\text{Sm}_k$ from part (1) of Lemma 5.37 extends to an adjunction between $\Cor_L$ and $\Cor_k$, using the same unit and counit, to which we apply the graph functors to view them as correspondences (see after Definition 4.11).

We are now going to define another adjunction $(\text{ext}_{L/k}, \text{res}_{L/k})$ (note the reversed order) that only exists at the level of correspondences. The unit and counit

$$\eta_X : X \to (X_L)|_k \quad \text{and} \quad \epsilon_Y : (Y|_k)_L \to Y$$
are defined respectively as the transpose of $\epsilon$ and $\eta$, using Example 4.14. By part (1) of Lemma 5.37 and by the composition properties of the transpose construction it is clear that they do define an adjunction $(\text{ext}_{L/k}, \text{res}_{L/k})$.

**Lemma 5.38.** The composition $\epsilon \circ \eta$ is the multiplication by the trace form of $L/k$ and the composition $\tilde{\epsilon} \circ \eta$ on $Y$ is the projection to the component of $(Y_{(k)})_{L}$ corresponding to $Y$ and mentioned in (3) of Lemma 5.37.

**Proof.** By Example 4.14, we obtain that $\epsilon \circ \eta$ is the multiplication by $\epsilon_{X}(1)$. Since $\epsilon_{X}$ is obtained by base change to $X$ of the structural map $\sigma : \text{Spec}(L) \to \text{Spec}(k)$, the element $\epsilon_{X}(1)$ is actually the pull-back to $X$ of $\epsilon_{(k)}(1)$, which is the trace form of $L/k$ by the definition of finite push-forwards for Chow-Witt groups (or the Milnor-Witt sheaf $K_{n}^{\text{MW}}$). Similarly, but this time by base change of the diagonal $\delta : \text{Spec}(L) \to \text{Spec}(L) \times \text{Spec}(L)$, we obtain that $\eta \circ \tilde{\epsilon}$ is the multiplication by $\delta_{(k)}(1)$, which is the projector to the component corresponding to $\text{Spec}(L)$, and thus to base change. \hfill $\square$

The functors $\text{ext}_{L/k}$ and $\text{res}_{L/k}$ between $\text{Sm}_{k}$ and $\text{Sm}_{L}$ are trivially continuous for the Zariski topology: they send a covering to a covering and they preserve fiber products: $(X \times_{Z} X')_{L} \simeq X_{L} \times_{Z} X'_{L}$ and $(Y \times_{T} Y')_{k} \simeq Y_{k} \times_{T_{k}} Y'_{k}$. Therefore, they induce functors between categories of Zariski sheaves with transfers

$$\text{res}_{L/k}^{*} : \tilde{\text{Sh}}_{\text{Zar}, k} \to \tilde{\text{Sh}}_{\text{Zar}, L} \text{ and } \text{ext}_{L/k}^{*} : \tilde{\text{Sh}}_{\text{Zar}, L} \to \tilde{\text{Sh}}_{\text{Zar}, k}.$$ 

In order to avoid confusion, we set $\text{bc}_{L/k} = \text{res}_{L/k}^{*}$ and $\text{tr}_{L/k} = \text{ext}_{L/k}^{*}$ in order to suggest the words “base change” and “transfer”, but we still use the convenient notation $F_{L}$ for $\text{bc}_{L/k}(F)$ and $G_{L/k}$ for $\text{tr}_{L/k}(G)$. We thus have, by definition

$$F_{L}(U) = F(U_{k}) \text{ and } G_{L/k}(V) = G(V_{L})$$

for any $F \in \tilde{\text{Sh}}_{\text{Zar}, k}$, $G \in \tilde{\text{Sh}}_{\text{Zar}, L}$, $U \in \text{Sm}_{L}$ and $V \in \text{Sm}_{k}$. It is formal that the adjunction $(\text{res}_{L/k}, \text{ext}_{L/k})$ induces an adjunction $(\text{bc}_{L/k}, \text{tr}_{L/k})$ with unit $\epsilon^{*}$ and counit $\eta^{*}$, while the adjunction $(\text{ext}_{L/k}, \text{res}_{L/k})$ induces an adjunction $(\text{tr}_{L/k}, \text{bc}_{L/k})$ with unit $\tilde{\epsilon}^{*}$ and counit $\tilde{\eta}^{*}$.

**Lemma 5.39.** For any $X \in \text{Sm}_{k}$ and $Y \in \text{Sm}_{L}$, we have natural isomorphisms

$$(\tilde{\mathbb{Z}}_{X})_{L} \simeq \tilde{\mathbb{Z}}_{X_{L}} \in \tilde{\text{Sh}}_{\text{Zar}, L} \text{ and } (\tilde{\mathbb{Z}}_{Y})_{k} \simeq \tilde{\mathbb{Z}}_{Y_{1}} \in \tilde{\text{Sh}}_{\text{Zar}, k}.$$ 

In the same spirit,

$$(\tilde{\mathbb{Z}}^{X})_{L} \simeq \tilde{\mathbb{Z}}^{X_{L}} \in \tilde{\text{Sh}}_{\text{Zar}, L} \text{ and } (\tilde{\mathbb{Z}}^{Y})_{k} \simeq \tilde{\mathbb{Z}}^{Y_{1}} \in \tilde{\text{Sh}}_{\text{Zar}, k}.$$ 

**Proof.** We have $(\tilde{\mathbb{Z}}_{X})_{L}(U) \simeq \text{Cor}_{k}(U_{k}, X) \simeq \text{Cor}_{L}(U_{k}, X_{L}) \simeq \tilde{\mathbb{Z}}_{X_{L}}(U)$, by the adjunction $(\text{res}_{L/k}, \text{ext}_{L/k})$ and $(\tilde{\mathbb{Z}}_{Y})_{k}(V) \simeq \text{Cor}_{k}(VY_{k}) \simeq \text{Cor}_{k}(V, Y_{k}) \simeq \tilde{\mathbb{Z}}_{Y_{1}}(V)$ by the transposed adjunction $(\text{ext}_{L/k}, \text{res}_{L/k})$.

Similarly, $(\tilde{\mathbb{Z}}^{X})_{L}(U) \simeq \text{Cor}_{k}(U_{k} \times X_{k}) \simeq \text{Cor}_{L}((U \times L X_{L})_{k}, k) \simeq \text{Cor}_{L}(U \times L X_{L}, L) \simeq \tilde{\mathbb{Z}}^{X_{L}}(U)$ and $(\tilde{\mathbb{Z}}^{Y})_{k}(V) \simeq \text{Cor}_{k}(VL \times Y, L) \simeq \text{Cor}_{k}((VL \times Y)_{k}, k) \simeq \text{Cor}_{k}(V \times Y_{k}, k) \simeq \tilde{\mathbb{Z}}^{Y_{1}}(V)$ \hfill $\square$

**Corollary 5.40.** For any sequence of pointed schemes $(X_{1}, x_{1}), \ldots, (X_{n}, x_{n})$, we have

$$(\tilde{\mathbb{Z}}((X_{1}, x_{1})_L \ldots (X_{n}, x_{n})_L) \simeq \tilde{\mathbb{Z}}((X_{1})_L \ldots (X_{n})_L) \in \tilde{\text{Sh}}_{\text{Zar}, L}.$$
and
\[(\tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n))_L \simeq \tilde{Z}(X_1)_L \land (x_1)_L \land \ldots \land (X_n)_L \land (x_n)_L) \in \tilde{\text{SH}_{\text{Zar},L}}.
\]

**Proof.** It immediately from the lemma applied to the split exact sequences defining the smash-products. \qed

The same type of result for smash products would hold for restrictions, but the restriction of an \(L\)-pointed scheme is an \(L/k\)-pointed scheme, not an \(k\)-pointed scheme. Nevertheless, the counit \(\epsilon^*\) of the adjunction \((\text{tr}_{L/k}, \text{bc}_{L/k})\) induces maps
\[(5) \quad ((\tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n))_L)_k \to \tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n)
\]
\[(6) \quad ((\tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n))_L)_k \to \tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n),
\]
while the unit \(\eta^*\) of the transposed adjunction \((bc_{L/k}, \text{tr}_{L/k})\) induces maps
\[(7) \quad \tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n) \to ((\tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n))_L)_k
\]
\[(8) \quad \tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n) \to ((\tilde{Z}(X_1,x_1) \land \ldots \land (X_n,x_n))_L)_k.
\]

**Lemma 5.41.** For any \(F \in \tilde{\text{SH}_{\text{Zar},k}}\) and any \(G \in \tilde{\text{SH}_{\text{Zar},L}}\), we have canonical isomorphisms \((C_*F)_L \simeq C_*(F_L)\) and \((C_*G)_k \simeq C_*(G_k)\).

**Proof.** It is straightforward, using \(Y_{[k]} \times \Delta^n \simeq (Y \times L \Delta^n_k)|_k\) for \(F\) and \(X_L \times X \Delta^n_k \simeq (X \times_k \Delta^n_k)_L\) for \(G\). \qed

To avoid confusion, let us write \(\tilde{Z}_k\{q\}, \tilde{Z}_L\{q\}\), etc. for the (complexes of) sheaves over \(k\), and \(\tilde{Z}_L\{q\}, \tilde{Z}_L\{q\}\) etc. for the same objects over \(L\).

For any \(q \in \mathbb{Z}\), using Corollary 5.40, we obtain, an isomorphism \(\tilde{Z}_k\{q\}_L \simeq \tilde{Z}_L\{q\}\). Using the maps (5) for \(q \geq 0\) and (6) for \(q < 0\), applied to copies of \(\mathbb{G}_{m,1}\), we obtain a morphism \((\tilde{Z}_L\{q\})_k \to \tilde{Z}_L\{q\}\). Symmetrically, using the maps (7) and (8) we obtain a morphism \(\tilde{Z}\{q\} \to (\tilde{Z}_L\{q\})_k\). Using Lemma 5.41, they induce an isomorphism and morphisms
\[(9) \quad \tilde{Z}_k\{q\}_L \simeq \tilde{Z}_L\{q\}, \quad (\tilde{Z}_L\{q\})_k \xrightarrow{\eta^*} \tilde{Z}_L\{q\} \quad \text{and} \quad \tilde{Z}\{q\} \xrightarrow{\epsilon^*} (\tilde{Z}_L\{q\})_k.
\]

**Lemma 5.42.** For any Zariski sheaf \(F\) on \(\text{Sm}_L\) and any \(X \in \text{Sm}_L\), we have \(H^i(X, F|_k) = H^i(X_L, F)\). For any Zariski sheaf \(G\) on \(\text{Sm}_k\) and any \(Y \in \text{Sm}_k\), we have \(H^i(Y, G_L) = H^i(Y_k, G)\). More generally, if \(F\) and \(G\) are complexes of Zariski sheaves, we have the same result for hypercohomology.

**Proof.** Let a Zariski sheaf \(F\) be flabby if restricted to the small site of any scheme, it gives a flabby sheaf in the usual sense: restrictions are surjective. The Zariski cohomology can then be computed out of resolutions by such flabby sheaves. Both functors \(\text{bc}_{L/k}\) and \(\text{tr}_{L/k}\) preserve such flabby resolutions. So, given a flabby resolution \(I^*\) of \(F\), we have
\[H^i(X_L, F) = H^i(I^*(X_L)) = H^i(\text{ext}^*_L/F^*(X)) = H^i(X, F|_k).
\] A similar proof holds for \(G\) and \(Y\). The claim about hypercohomology is proved similarly using flabby Cartan-Eilenberg resolutions. \qed

Using the morphisms (9) and Lemma 5.42, we obtain for any \(X \in \text{Sm}_k\) and any \(q\) two morphisms
\[H^p(X, \tilde{Z}_k\{q\}) \to H^p(X, (\tilde{Z}_L\{q\})_k) \simeq H^p(X_L, \tilde{Z}_L\{q\})\]
and
\[ \mathbb{H}^p(X, \tilde{Z}_L(q)) \simeq \mathbb{H}^p(X, (\tilde{Z}_L(q))_k) \rightarrow \mathbb{H}^p(X, \tilde{Z}_L(q)). \]
in other words:
\[ \text{(10)} \quad H^p_{\text{bc}_{L/k}}(X, \tilde{Z}) \xrightarrow{\text{bc}_{L/k}} H^p_{L}(X, \tilde{Z}) \quad \text{and} \quad H^p_{L}(X, \tilde{Z}) \xrightarrow{\text{tr}_{L/k}} H^p_{\text{bc}_{L/k}}(X, \tilde{Z}). \]

Using Lemma 5.38, we obtain

**Lemma 5.43.** On generalized motivic cohomology, the composition \( \text{tr}_{L/k} \circ \text{bc}_{L/k} \) is the multiplication by the trace form of the extension \( L/k \), when \( H^{p,q}(X, \tilde{Z}) \) is seen as a module over \( H^{0,0}(X, \tilde{Z}) \simeq K^W_0(X) \), itself over \( K^W_0(k) = GW(k) \).

We now compare the generalized cohomology groups when computed over \( k \) for a limit scheme that is of the form \( X_L \) and when computed over some extension \( L \) of \( k \).

If \( L \) is a finitely generated extension of \( k \), that we view as the inverse limit \( L = \varprojlim U \) of schemes \( U \in \text{Sm}_k \), we obtain a natural map
\[ \lim_{T' \in \mathcal{A}(X_L, Y_L)} \overline{\text{Cor}}_{T'}(U \times X \times Y, \omega(U_{X \times Y}/(U \times Y))) \rightarrow \lim_{T' \in \mathcal{A}(X_L, Y_L)} \overline{\text{Ch}}^d_{T'}(X_L \times Y_L, \omega_{X_L \times Y_L/X_L}) \]
induced by pull-backs between the groups where \( T' = T_U \). When \( d = \text{dim} Y \), this is a map
\[ \overline{\text{Cor}}_k(U \times X, Y) \rightarrow \overline{\text{Cor}}_k(X_L, Y_L) \]
functorial in \( U \). Taking the limit over \( U \), we therefore obtain a map
\[ \overline{\text{Cor}}_k(X_L, Y) = \lim_{(U,T)} \overline{\text{Cor}}_k(U \times X, Y) \xrightarrow{\Psi_{L/k}} \overline{\text{Cor}}_L(X_L, Y_L) \]
using the construction of section 5.1 to define the left hand side.

**Proposition 5.44.** The map \( \Psi_{L/k} \) is an isomorphism.

**Proof.** To shorten the notation, we drop the canonical bundles in the whole proof, since they behave as they should by pull-back. The source of \( \Psi_{L/k} \) can be defined using a single limit, as
\[ \lim_{(U,T)} \overline{\text{Ch}}^d_{T}(U \times X \times Y) \]
where the limit runs over the pairs \( (U, T) \) with \( (U, T) \leq (U', T') \) if there is a map \( U' \subseteq U \) and \( T \cap U' \subseteq T' \). The corresponding transition map on Chow-Witt groups is the restriction to \( U' \) composed with the extension of support from \( T \cap U' \) to \( T' \). Note that both of these maps are injective, as explained in the proof of Lemma 4.5 for the first one and right before that same Lemma for the second one. The maps \( f_{U,T} : \overline{\text{Ch}}^d_{T}(U \times X \times Y) \rightarrow \lim_{V \subseteq U} \overline{\text{Ch}}^d_{T \cap V}(V \times X \times Y) \) sending the initial group in the direct system to its limit are again injective, and their target can be identified with \( \overline{\text{Ch}}^d_{T}(X_L \times Y_L) \) by Lemma 5.7 (independently of \( U \), except that \( T \) lives on \( U \)). The \( f_{U,T} \) therefore induce an injective map
\[ \lim_{(U,T)} \overline{\text{Ch}}^d_{T}(U \times X \times Y) \rightarrow \lim_{(U,T)} \overline{\text{Ch}}^d_{T}(X_L \times Y_L) \]
This map is also surjective because any \( \overline{\text{Ch}}^d_{T \cap V}(V \times X \times Y) \) in the target of \( f_{U,T} \) is surjected by the same group on the source, after passing from \( (U, T) \) to \( (V, T \cap V) \). Finally, the isomorphism (11) actually maps to \( \overline{\text{Cor}}_L(X_L, Y_L) \) because any \( T' \in \).
that with the pull-back map

6. Comparison with Milnor-Witt K-theory

6.1. The homomorphism. In this section, we prove that there is a homomorphism of graded rings

$$\bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H_{n,n}(L, \hat{\mathbb{Z}}).$$

for any finitely generated field extension $L/k$.

Let $A(L)$ be the $\mathbb{Z}$-graded associative (unital) ring freely generated by the symbols $[a]$, for each $a \in L^\times$, of degree 1 and by the symbol $\eta$ in degree $-1$. Our first step is to define a graded homomorphism

$$s_L : A(L) \to \bigoplus_{n \in \mathbb{Z}} H_{n,n}(L, \hat{\mathbb{Z}}).$$

Observe that $H_{1,1}(\text{Spec}(L), \hat{\mathbb{Z}})$ is by definition a quotient of $\widetilde{\text{Cor}}_k(\text{Spec}(L), \mathbb{A}^1 \setminus 0)$. The latter is in turn a colimit of groups of the form $\widetilde{\text{Cor}}_k(U, \mathbb{A}^1 \setminus 0)$ where $U$ is a smooth connected scheme such that $k(U) = L$. Now any invertible element $a$ in $\mathcal{O}_U^*(U)$ yields a morphism $a : U \to \mathbb{A}^1 \setminus 0$ and thus an element $\Gamma_a$ of $\widetilde{\text{Cor}}_k(U, \mathbb{A}^1 \setminus 0)$. Passing to the colimit, we see that any $a \in L^\times$ yields a well-defined element in $\widetilde{\text{Cor}}_k(\text{Spec}(L), \mathbb{A}^1 \setminus 0)$ and in turn an element in $H_{1,1}(\text{Spec}(L), \hat{\mathbb{Z})}$, and we define $s_L([a])$ to be that element. Note right away that

$$s([1]) = 0$$

as $H_{1,1}(L, \hat{\mathbb{Z}})$ is defined using the complex $\hat{\mathbb{Z}}[1]$, which doesn’t see the point $1 \in \mathbb{G}_m$.

We now deal with the definition of $s(\eta)$. Recall from Lemma 3.6 that $K_0^{MW}(\mathbb{G}_m) = K_0^{MW}(k) \oplus K_{-1}^{MW}(k) \cdot [t]$. In particular, using $\eta \in K_{-1}^{MW}(k)$, we can consider $\eta : [t]$ in $K_0^{MW}(\mathbb{G}_m) = \widetilde{\text{Cor}}_k(\mathbb{G}_m, \text{pt})$. The precomposition with the unit map $\text{pt} \to \mathbb{G}_m$

$$\widetilde{\text{Cor}}_k(\mathbb{G}_m, \text{pt}) \to \widetilde{\text{Cor}}_k(\text{pt}, \text{pt})$$

identifies by example 4.13 with the pull-back map

$$K_0^{MW}(\mathbb{G}_m) \to K_0^{MW}(k)$$

and it sends $[t]$ to zero by Remark 3.7. Thus, $[t]$ defines an element in $H_{1,-1}(\text{pt}, \hat{\mathbb{Z}})$ and in $H_{1,-1}(\text{Spec}(L), \hat{\mathbb{Z}})$ by pull-back. We define $s_L(\eta)$ to be that element, and we claim:

**Proposition 6.1.** The ring map

$$s_L : A_L \to \bigoplus_{n \in \mathbb{Z}} H_{n,n}(\text{Spec}(L), \hat{\mathbb{Z}})$$

defined by the values $s_L([a])$, $a \in L^\times$ and $s_L(\eta)$ given above induces a graded homomorphism

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H_{n,n}(\text{Spec}(L), \hat{\mathbb{Z}}).$$
We have to prove that the relations (i)-(iv) defining Milnor-Witt $K$-theory are satisfied. This will involve a number of lemmas, in which we write $s$ instead of $s_L$ when no confusion can occur.

**Lemma 6.2.** Let $a \in L^\times$. Then
\[
s(\eta)s([a]) = s([a])s(\eta) = -(1) + \langle a \rangle
\]
in $K^M_W(L) = H^{0,0}(L, \mathbb{Z})$ under the canonical identification of Proposition 5.33.

**Proof.** The product $H^{-1,-1}(L, \mathbb{Z}) \times H^{1,1}(L, \mathbb{Z}) \rightarrow H^{0,0}(L, \mathbb{Z})$ is induced by the product
\[
\mu_{-1,1}(L) : \mathbb{Z}\{1\}(L) \otimes \mathbb{Z}\{1\}(L) \rightarrow \mathbb{Z}\{1\}(L) \otimes \mathbb{Z}\{1\}(L)
\]
which is the composite of the switch homomorphism $\mathbb{Z}\{1\}(L) \otimes \mathbb{Z}\{1\}(L) \rightarrow \mathbb{Z}\{1\}(L) \otimes \mathbb{Z}\{1\}(L)$ with
\[
\overline{Cor}_k(\text{Spec}(L), \mathbb{G}_m) \otimes \overline{Cor}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \xrightarrow{h_{\text{Spec}(L), \mathbb{G}_m, \text{pt}, \text{pt}}} \overline{Cor}_k(\text{Spec}(L), \text{pt}).
\]
In particular, we get $s(\eta)s([a]) = s([a])s(\eta)$ and thus $s(1 + [a] \eta) = s(1 + [a])$.

Now $s(\eta)$ is obtained by pull-back from $\overline{Cor}_k(\mathbb{G}_m, \text{pt})$ along the morphism $\text{Spec}(L) \times \mathbb{G}_m \rightarrow \mathbb{G}_m$. A straightforward computation shows that the composite
\[
\overline{Cor}_k(\text{Spec}(L), \mathbb{G}_m) \otimes \overline{Cor}_k(\mathbb{G}_m, \text{pt}) \quad \xrightarrow{h_{\text{Spec}(L), \mathbb{G}_m, \text{pt}, \text{pt}}} \quad \overline{Cor}_k(\text{Spec}(L), \text{pt})
\]
is the composition of correspondences. It follows that the product $s([a])s(\eta)$ equals $s(\eta) \circ s([a])$ which is precisely $-(1) + \langle a \rangle$. \hfill \Box

**Corollary 6.3.** For any $a, b \in L^\times$, we have
\[
s(\eta)s([a^2]) = 0
\]
\[
\langle a \rangle s([a^2]) = s([a^2]).
\]
**Proof.** We have $1 = \langle 1 \rangle = \langle a^2 \rangle = 1 + s(\eta)s([a^2])$, which proves (14). Then (15) follows by expanding $\langle a \rangle = 1 + s([a])s(\eta)$. \hfill \Box

**Lemma 6.4.** The hyperbolic relation (iv) is satisfied:
\[
s(\eta)(2 + s(\eta)s([-1])) = 0
\]
**Proof.** From the above lemma, we know that $s(\eta)s([-1]) + 2 = 1 + (-1) = (1, -1)$ in $GW(L) = \overline{Cor}_k(\text{Spec}(L), \text{pt})$. On the other hand, $s(\eta)$ is obtained from the group $\overline{Cor}_k(\mathbb{G}_m, \text{pt})$ by pull-back along $\mathbb{G}_m, L \rightarrow \mathbb{G}_m, k$. Now the multiplication
\[
\overline{Cor}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \otimes \overline{Cor}_k(\text{Spec}(L), \text{pt}) \rightarrow \overline{Cor}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt})
\]
is given by the composite map
\[ \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \otimes \overline{\text{Cor}}_k(\text{Spec}(L), \text{pt}) \]
\[ \overline{\text{Cor}}_k(\text{Spec}(L), \text{pt}) \otimes \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \]
\[ \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \]
and a direct verification shows that the composite
\[ \overline{\text{Cor}}_k(\mathbb{G}_m, \text{pt}) \otimes \overline{\text{Cor}}_k(\text{Spec}(L), \text{pt}) \]
\[ \overline{\text{Cor}}_k(\text{Spec}(L), \text{pt}) \otimes \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \]
\[ \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) \]
is just the tensor product of cycles, together with the pull-back along the switch map \( \mathbb{G}_m \times \text{Spec}(L) \to \text{Spec}(L) \times \mathbb{G}_m \). Thus \( s(\eta)(s(\eta)s([-1]) + 2) \) is represented by the class of the exterior product \( (-1 + \langle t \rangle) \times ((1, -1)) \) in \( \overline{\text{Cor}}_k(\text{Spec}(L) \times \mathbb{G}_m, \text{pt}) = \overline{\text{CH}}^0(\text{Spec}(L) \times \mathbb{G}_m) \). Now
\[ (-1 + \langle t \rangle) \times ((1, -1)) = -\langle 1, -1 \rangle + \langle t \rangle(1, -1) = -\langle 1, -1 \rangle + \langle 1, -1 \rangle = 0. \]
\[ \square \]

We now turn to relations (i) and (ii).

Lemma 6.5. For any finitely generated field extension \( F/L \) and any \( a \in F^\times \setminus \{1\} \), we have
\[ \langle 1, -1 \rangle s_F([a])s_F([1 - a]) = 0. \]

Proof. Recall from Section 4.3 that we have for any smooth schemes \( X, Y \) a map \( H_{X,Y} : \text{Cor}_k(X, Y) \to \text{Cor}_k(X, Y) \) sending a finite correspondence \( \beta \) to \( (1, -1)\beta \). In particular, when \( X = \text{Spec}(F) \) and \( Y = \mathbb{G}_m^\times \) we obtain a map
\[ \text{Cor}_k(\text{Spec}(F), (\mathbb{A}_k^1 \setminus 0)^\times) \to \text{Cor}_k(\text{Spec}(F), (\mathbb{A}_k^1 \setminus 0)^\times) \]
inducing a map
\[ H : H^{2,2}(\text{Spec}(F), \mathbb{Z}) \to H^{2,2}(\text{Spec}(F), \mathbb{Z}). \]
If \( a \in F^\times \setminus \{1\} \), we get \( H([a]) = \langle 1, -1 \rangle s([a])s([1 - a]) \). Now \( \{a\} \{1 - a\} = 0 \) and the result follows. \( \square \)
Lemma 6.6. For any $a, b \in L^\times$ with $a \neq b$ and $ab \neq 1$, we have

\begin{align*}
(18) & \quad s([a^2]) = \langle 1, -1 \rangle s([a]) \\
(19) & \quad s([a^{-1}]) = -\langle -1 \rangle s([a]) \\
(20) & \quad \langle ab - 1 \rangle s([ab]) = \langle b - a \rangle s([b]) + \langle a - b \rangle s([a])
\end{align*}

Proof. First note that all three relations are trivially true when $a$ or $b$ is 1, from (12). We therefore assume $a, b \neq 1$ in the rest of the proof.

Let $x$ be a coordinate of $\mathbb{A}^1_k$ and $t$ a coordinate of $G_m$. For $a, b \in L^\times \setminus \{1\}$, let $f := t^2 - x(a + b)t - (1 - x)(1 + ab)t + ab$. We can consider the class $[f] \otimes dt \in K^\MW_1(L(x, t), \omega_{\mathbb{A}^1_k \times G_m/\mathbb{A}^1_k})$ and its residue $\hat{f} = \partial([f] \otimes dt)$ under the map

$$
\partial : K^\MW_1(L(x, t), \omega_{\mathbb{A}^1_k \times G_m/\mathbb{A}^1_k}) \to \bigoplus_{y \in (G_m \times \mathbb{A}^1_k)^{(1)}} K^\MW_2(L(y), \omega_{G_m/k} \otimes \omega_{G_m/k}).
$$

Observe that $f = 0$ defines an admissible subset $V$ in $\mathcal{A}(\mathbb{A}^1_k, G_m)$ and that $\hat{f}$ is supported in $V$; it thus defines a class in $\tilde{CH}^1_k(\mathbb{A}^1_k \times G_m, \omega_{G_m \times \mathbb{A}^1_k/\mathbb{A}^1_k})$, and therefore an element in $\text{Cor}_k(\tilde{\text{Cor}}_k, G_m)$, still denoted by $\hat{f}$. Similarly, setting $x = i \in L$, we get correspondences $\hat{f}(i) \in \text{Cor}_k(L, G_m)$.

It is straightforward to check that the image of $\hat{f}$ under the restriction map $\tilde{\text{Cor}}_k(\mathbb{A}^1_k, G_m) \to \tilde{\text{Cor}}_k(L, G_m)$ induced by $i : L \to \mathbb{A}^1_k$ is precisely $\hat{f}(i)$ (see Example 4.13 and the explicit pull-backs $s^*_i$ of Section 1.1). By direct computation of the residue,

$$
\hat{f}(0) = \begin{cases} 
(1, -1)s([1]) & \text{if } ab = 1. \\
(1 - ab)s([1]) + \langle ab - 1 \rangle s([ab]) & \text{else.}
\end{cases}
$$

while

$$
\hat{f}(1) = \begin{cases} 
(1, -1)s([a]) & \text{if } a = b. \\
(b - a)s([b]) + \langle a - b \rangle s([a]) & \text{else.}
\end{cases}
$$

We now prove the lemma by equalizing $\hat{f}(0)$ and $\hat{f}(1)$, since $H^{1, 1}(L, \tilde{Z})$ is the cokernel of the difference of the two restriction maps for $i = 0, 1$. If $a \in L^\times \setminus \{\pm 1\}$, setting $b = a$, we obtain

$$
\langle a^2 - 1 \rangle s([a^2]) = \langle 1, -1 \rangle s([a]).
$$

Multiplying both terms by $\langle a^2 - 1 \rangle$, we get (18) for $a \neq -1$. For $a = -1$, setting $b = -1$ as well, we get precisely what we want: $\langle 1, -1 \rangle s([-1]) = 0$. To get (19), we first note that the case $a = -1$ follows from (18) and (12). Setting $b = a^{-1}$, we can thus assume $a \neq b$ and get

$$
\langle a^{-1} - a \rangle s([a^{-1}]) + \langle a - a^{-1} \rangle s([a]) = 0.
$$

Multiplying by $\langle a^{-1} - a \rangle$, we get the result. Finally, relation (20) is obtained by comparing the residues in the remaining case $a \neq b$ and $a \neq b^{-1}$. \hfill \Box

Corollary 6.7. For any $a, b, c \in L^\times$, we have

\begin{align*}
(21) & \quad s([a^2 b]) = s([a^2]) + s([b]) \\
(22) & \quad s([a^2])s([bc]) = s([a^2])s([b]) + s([a^2])s([c])
\end{align*}
Lemma 6.8. Let \( L/k \) be a field, \( a \in L^\times \setminus (L^\times)^2 \) and let \( K = L(\sqrt{a}) \). Let \( \text{Tr}_{K/L} \) be the transfer homomorphism obtained using the finite morphism \( \text{Spec}(K) \to \text{Spec}(L) \) and the canonical orientation of \( \omega_{K/L} \). Set \( b = \sqrt{a} \). Then \( \text{Tr}_{K/L}(\langle b \rangle s([1 - b])) = \langle 2 \rangle s([1 - a]) \).

Proof. A direct computation shows that the transfer of \( \langle b \rangle \{ 1 - b \} \) is the finite correspondence \( \alpha \in \text{Cor}_k(\text{Spec}(L), \mathbb{G}_m) \) defined by the admissible subset defined by the minimal polynomial \( t^2 - 2t + 1 - a \) of \( 1 - b \) over \( k \) and the form \( \langle b \rangle \) in \( GW(K) \). Consider the polynomial \( t^2 - (2 - au)t + 1 - a \in L[u, t, t^{-1}] \). As it is a monic polynomial with invertible constant term, it defines an admissible subset in \( \mathcal{A}(\mathbb{A}_L^1, \mathbb{G}_m) \). Arguing as in Lemma 6.6, we get a correspondence \( \beta \) in \( \text{Cor}_k(\mathbb{A}_L^1, \mathbb{G}_m) \). As \( \langle -2a \rangle \) is everywhere invertible, we can as well consider the correspondence \( \gamma := \langle -2a \rangle \beta \). The restriction of \( \gamma \) at \( u = 0 \) gives the correspondence \( \langle -2a \rangle \cdot (2(1 - b) - 2)s([1 - b]) \). Now \( a = b^2 \) is a square and thus the latter is just \( \langle b \rangle s([1 - b]) \). Observe next that the polynomial \( t^2 - (2 - au)t + 1 - a \) at \( u = 0 \) is just \( (t - (1 - a))(t - 1) \). Evaluating \( \gamma \) at \( u = 1 \) yields the correspondence \( \langle -2a \rangle \cdot \langle 1 - a \rangle s([1 - a]) = \langle 2 \rangle s([1 - a]) \). \( \square \)

Lemma 6.9. The Steinberg relation \( i \) holds:

\[
\langle a \rangle s([a]) s([1 - a]) = 0 \text{ for any } a \in L^\times \setminus \{ 1 \}.
\]
Proof. As a preliminary, let us note that for any $a \in L^\times \setminus \{-1\}$,
(24) \[(1, -1)s([a])s([1 + a]) = 0\]
Indeed \((1, -1)s([a]) = s([a^2]) = s([-a^2]) = (1, -1)s([-a]),\) and thus \((1, -1)s([a])s([1 + a]) = (1, -1)s([-a])s([1 + a]) \stackrel{(17)}{=} 0.\)

Now, suppose in (23) that $a = b^2$. In this case, we have
\[s([b^2])s([1 - b^2]) \stackrel{(22)}{=} s([b^2])(s([1 - b]) + s([1 + b]))\]
\[= (1, -1)s([b])s([1 - b]) + (1, -1)s([b])s([1 + b]) = 0\]
by (17) and (24). Assume on the contrary that $a$ is not a square in $L$. Let $K = L(\sqrt{a})$ and $b = \sqrt{a}$ in $K^\times$. Then
\[0 \stackrel{(17)}{=} (1, -1)s([b])s([1 - b]) \stackrel{(18)}{=} s([b^2])s([1 - b]) \stackrel{(15)}{=} \langle b \rangle s([b^2])s([1 - b]) = s([a])\langle b \rangle s([1 - b])\]
in $H^{2,2}(K, \mathbb{Z})$. Apply the transfer $\text{Tr}_{K/L}$, the result follows from the projection formula and Lemma 6.8, since (2) is invertible.

Corollary 6.10. For any $a, b \in L^\times$ such that $ab \neq 1$, we have
(25) \[(1 - a)s([a]) = s([a])\]
(26) \[\langle a^{-1} - b \rangle s([ab]) = \langle a \rangle s([ab])\]
Proof. By Lemma (13), we have \((1 - a) = 1 + s(\eta)s([1 - a])\) and (25) thus follows from the Steinberg relation (23). We then have
\[\langle a^{-1} - b \rangle s([ab]) \stackrel{(25)}{=} \langle a^{-1} - b \rangle \langle 1 - ab \rangle s([ab]) = \langle a^{-1} - b \rangle^2 s([ab]) = \langle a \rangle s([ab]) = (a)s([ab]).\]

Lemma 6.11. The product relation (ii) holds:
(27) \[s([ab]) = s([a]) + s([b]) + s(\eta)s([a])s([b])\]
Proof. Let $a, c \in L^\times$ be such that $c \neq a, a^{-1}$. Then
\[\langle c - a \rangle (s([c]) + (-1)s([a])) = (c - a)s([c]) + (a - c)s([a]) \stackrel{(20)}{=} (ac - 1)s([ac]) \stackrel{(25)}{=} (-1)s([ac]).\]
It follows that $s([c]) + (-1)s([a]) = \langle a - c \rangle s([ac])$, and therefore
\[s([c]) - s([a]) \stackrel{(19)}{=} s([c]) + (-1)s([a^{-1}])\]
\[= s([c]) + (-1)(s([a]) + s([a^{-2}])) \stackrel{(21)}{=} s([c]) + (-1)s([a - 2])\]
\[= s([c]) + (-1)s([a]) + s([a^{-2}]) \stackrel{(15)}{=} \langle a - c \rangle s([ac]) + s([a^{-2}])\]
\[= (a - c)s([ac]) + s([a^{-2}]) \stackrel{(15)}{=} (a - c)s([ac]) + s([a^{-2}])\]
\[= (a - c)s([a^{-1}c]) \stackrel{(21)}{=} (a - c)s([a^{-1}c])\]
\[= (a^{-1})s([a^{-1}c]) = (a)s([a^{-1}c]).\]
Setting $c = ab$ yields the result in case $b \neq 1, a^2$. The case $b = 1$ is obvious, while the case $b = a^2$ follows (21) and (14).

This last relation concludes the proof of Proposition 6.1.
6.2. A left inverse. In this section, we construct a left inverse to the homomorphism $\Phi_L$ of Proposition 6.6.1. By definition,

$$\tilde{Z}_{G_m^n}(L) := \bigoplus_{x \in (G_m^n)^{q}} CH_x^q(G_{m,L}^q, \omega_{G_m^n/L}) = \bigoplus_{x \in (G_m^n)^{q}} K_0^{MW}(L(x), \omega_{L(x)/L}).$$

Now any closed point $x$ in $(G_m^n)^{q}$ can be identified with a $q$-uple $(x_1, \ldots, x_q)$ of elements of $L(x)$. For any such $x$, we define a homomorphism

$$f_x : K_0^{MW}(L(x), \omega_{L(x)/L}) \to K_q^{MW}(L)$$

by $f_x(\alpha) = \text{Tr}_{L(x)/L}(\alpha \cdot [x_1, \ldots, x_q])$. We then obtain a homomorphism

$$f : \tilde{Z}_{G_m^n}(L) \to K_q^{MW}(L)$$

which is easily seen to factor through $\tilde{Z}_{G_m^n}(L)$ since $[1] = 0 \in K_1^{MW}(L)$.

We now check that this homomorphism vanishes on the image of $\tilde{Z}_{G_m^n}(A_L^1)$ in $\tilde{Z}_{G_m^n}(L)$. This will follow from the next lemma.

Lemma 6.12. Let $Z \in A(A_L^1, G_m^n)$. Let moreover $p : G_{m,L}^q \to \text{Spec}(L)$ and $p_{A_L^1} : A_L^1 \times G_{m,L}^q \to A_L^1$ be the projections and $Z_i := p(A_L^1(i)) \cap Z$ (endowed with its reduced structure) for $i = 0, 1$. Let $j_i : \text{Spec}(L) \to A_L^1$ be the inclusions in $i = 0, 1$ and let $g_i : G_{m,L}^q \to A_L^1 \times G_{m,L}^q$ be the induced maps. Then the homomorphisms

$$p_*(g_i)^* : CH^q_{Z}(A_L^1 \times G_{m,L}^q, \omega) \to CH^q_{Z_i}(G_{m,L}^q, \omega_{G_m^n/L}) \to K_0^{MW}(L)$$

are equal, where $\omega = \omega_{A_L^1 \times G_{m,L}^q}$.

Proof. For $i = 0, 1$, consider the Cartesian square

$$\begin{array}{ccc}
G_{m,L}^q & \xrightarrow{g_i} & A_L^1 \times G_{m,L}^q \\
\downarrow p & & \downarrow p_{A_L^1} \\
\text{Spec}(L) & \xrightarrow{j_i} & A_L^1
\end{array}$$

We have $(j_i)^*(p_{A_L^1})_* = p_*(g_i)^*$ by base change. The claim follows from the fact that $(j_0)^* = (j_1)^*$ by homotopy invariance. □

Proposition 6.13. The homomorphism $f : \tilde{Z}_{G_m^n}(L) \to K_q^{MW}(L)$ induces a homomorphism

$$\theta_L : H^{q,q}(L, \tilde{Z}) \to K_q^{MW}(L)$$

for any $q \geq 1$.

Proof. Observe first that the group $H^{q,q}(L, \tilde{Z})$ is the cokernel of the homomorphism

$$\partial_0 - \partial_1 : \tilde{Z}_{G_{m,q}^n}(A_L^1) \to \tilde{Z}_{G_{m,1}^n}(L)$$

It follows from Example 4.13 that $\partial_1 : \tilde{Z}_{G_{m}^n}(A_L^1) \to \tilde{Z}_{G_{m}^n}(L)$ is induced by $g_1^*$. We can use the above lemma to conclude. □

Corollary 6.14. The homomorphism

$$\Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H^{n,n}(L, \tilde{Z}).$$

is split injective.
Proof. It suffices to check that $\theta_L \Phi_L = \text{id}$, which is straightforward. \qed

6.3. Proof of the main theorem. In this section we prove our main theorem, namely that the homomorphism
\[ \Phi_L : \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(L) \to \bigoplus_{n \in \mathbb{Z}} H^{n,n}(L, \tilde{Z}) \]
is an isomorphism. We know from Corollary 6.14 that it is split injective, and we can deduce from Propositions 5.33 and 5.34 that it is indeed an isomorphism in degrees $\leq 0$. We now prove the result in positive degrees, starting with $n = 1$.

For any $d, n \geq 1$ and any field extension $L/k$ let $M_n^d(L) \subset \widetilde{\text{Cor}}_k(L, G_m^\times)$ be the subgroup of correspondences whose support is a finite union of field extensions $E/L$ of degree $\leq d$ (see Definition 4.6 for the notion of support of a correspondence). Let $H^{n,n}(L, \tilde{Z})^d \subset H^{n,n}(L, \tilde{Z})$ be the image of $M_n^d(L)$ under the surjective homomorphism
\[ \widetilde{\text{Cor}}_k(L, G_m^\times) \to H^{n,n}(L, \tilde{Z}). \]
Observe that
\[ H^{n,n}(L, \tilde{Z})^d \subset H^{n,n}(L, \tilde{Z})^{(d+1)} \quad \text{and} \quad H^{n,n}(L, \tilde{Z}) = \cup_{d \in \mathbb{N}} H^{n,n}(L, \tilde{Z})^d. \]

Lemma 6.15. The subgroup $H^{n,n}(L, \tilde{Z})^{(1)} \subset H^{n,n}(L, \tilde{Z})$ is the image of the homomorphism
\[ \Phi_L : K_n^{MW}(L) \to H^{n,n}(L, \tilde{Z}). \]

Proof. By definition, observe that the homomorphism $K_n^{MW}(L) \to H^{n,n}(L, \tilde{Z})$ factors through $H^{n,n}(L, \tilde{Z})^{(1)}$. Let $\alpha \in H^{n,n}(L, \tilde{Z})^{(1)}$. We may suppose that $\alpha$ is the image under the homomorphism $\widetilde{\text{Cor}}_k(L, G_m^\times) \to H^{n,n}(L, \tilde{Z})$ of a correspondence $a$ supported on a field extension $E/L$ of degree 1, i.e. $E = L$. It follows that $a$ is determined by a form $\phi \in GW(L)$ and an $n$-uple $a_1, \ldots, a_n$ of elements of $L$. This is precisely the image of $\Phi_L(a_1, \ldots, a_n)$ under the homomorphism $K_n^{MW}(L) \to H^{n,n}(L, \tilde{Z})$. \qed

Proposition 6.16. For any $d \geq 2$, we have $H^{1,1}(L, \tilde{Z})^d \subset H^{1,1}(L, \tilde{Z})^{(d-1)}$.

Proof. By definition, $H^{1,1}(L, \tilde{Z})^{(1)}$ is generated by correspondences whose supports are field extensions $E/L$ of degree at most $d$. Such correspondences are determined by an element $a \in E^\times$ given by the composite $\text{Spec}(E) \to G_{m,L} \to G_m$ together with a form $\phi \in GW(E, \omega_E/L)$ given by the isomorphism
\[ GW(E, \omega_E/L) \to \widetilde{\text{H}}^1_{\text{Spec}(E)}(G_{m,L}, \omega_{G_{m,L}/L}). \]
We denote this correspondence by the pair $(a, \phi)$. Recall from Lemma 2.4 that there is a canonical orientation $\xi$ of $\omega_{E/F}$ and thus a canonical element $\chi \in \widetilde{\text{Cor}}_k(\text{Spec}(L), \text{Spec}(E))$ yielding the transfer map
\[ \text{Tr}_{E/L} : \widetilde{\text{Cor}}_k(\text{Spec}(E), G_m) \to \widetilde{\text{Cor}}_k(\text{Spec}(L), G_m) \]
which is just the composition with $\chi$ (Example 4.14). Now $\phi = \psi \cdot \xi$ for $\psi \in GW(E)$, and it is straightforward to check that the Chow-Witt correspondence $(a, \psi)$ in $\widetilde{\text{Cor}}_k(\text{Spec}(E), G_m)$ determined by $a \in E^\times$ and $\psi \in GW(E)$ satisfies $\text{Tr}_{E/L}(a, \psi) = (a, \phi)$. Now $(a, \psi) \in H^{1,1}(E, \tilde{Z})^{(1)}$ and therefore belongs to the image of the homomorphism $K_1^{MW}(E) \to H^{1,1}(E, \tilde{Z})$. There exists thus $a_1, \ldots, a_n, b_1, \ldots, b_m \in E^\times$
(possibly equal) such that \((a, \psi) = \sum s(a_i) - \sum s(b_j)\). To prove the lemma, it suffices then to show that \(\text{Tr}_{E/L}(s(b)) \in H^{1,1}(L, \mathbb{Z})(d-1)\) for any \(b \in E^\times\).

Let thus \(b \in E^\times\). By definition, \(s(b) \in H^{1,1}(E, \mathbb{Z})\) is the class of the correspondence \(\Gamma_b\) associated to the morphism of schemes \(\text{Spec}(E) \to \mathbb{G}_m\) corresponding to \(b\). If \(F(b) \subset E\) is a proper subfield, we see that \(\text{Tr}_{E/L}(s(b)) \in H^{1,1}(F, \mathbb{Z})(d-1)\), and we may thus suppose that the minimal polynomial of \(b\) over \(F\) is of degree \(d\). By definition, \(\text{Tr}_{E/L}(s(b))\) is then represented by the correspondence associated to the pair \((b, (1) \cdot \xi)\). Consider the total residue homomorphism (twisted by \(\omega_{F[t]/F}\))

\[
\partial : K^\text{MW}_1(F(t), \omega_{F(t)/F}) \to \bigoplus_{x \in \mathbb{A}^{(1)}_{m,F}} K^\text{MW}_0(F(x), \omega_{F[t]/F} \otimes_{F[t]} (m_x/m^2_x)^\vee).
\]

where \(m_x\) is the maximal ideal corresponding to \(x\). Before working further with this homomorphism, we first identify the \(F(x)\)-vector space \(\omega_{F[t]/F} \otimes_{F[t]} (m_x/m^2_x)^\vee\).

Consider the exact sequence of \(F[t]\)-modules

\[
\Omega_{F/k} \otimes F[t] \to \Omega_{F[t]/k} \to \Omega_{F[t]/F} \to 0.
\]

Since \(F\) is the localization at the generic point of a smooth scheme of dimension \(d\) over \(k\), it follows that \(\Omega_{F[t]/k} \otimes F[t]\) is free of rank \(d\). For the same reason, \(\Omega_{F[t]/k}\) is free of rank \(d + 1\). Now \(\Omega_{F[t]/F}\) is of rank 1, and it follows that the above sequence is also exact on the left. We thus obtain a canonical isomorphism

\[
\omega_{F[t]/F} \simeq \omega_{F[t]/k} \otimes_F \omega_{F/k}^\vee.
\]

Consider next the exact sequence

\[
m_x/m^2_x \to \Omega_{F[t]/k} \otimes_{F[t]} F(x) \to \Omega_{F(x)/k} \to 0.
\]

A comparison of the dimensions shows that the sequence is also left exact (use the fact that \(F(x)\) is the localization of a smooth scheme of dimension \(d\) over the perfect field \(k\), and we thus get a canonical isomorphism

\[
\omega_{F[t]/k} \otimes_{F[t]} F(x) \simeq \omega_{F(x)/k} \otimes_{F[t]} m_x/m^2_x.
\]

Putting (29) and (30) together, we finally obtain a canonical isomorphism

\[
\omega_{F[t]/F} \otimes_{F[t]} (m_x/m^2_x)^\vee \simeq \omega_{F(x)/k} \otimes_F \omega_{F/k}^\vee = \omega_{F(x)/F}.
\]

We can thus rewrite the residue homomorphism (28) as a homomorphism

\[
\partial : K^\text{MW}_1(F(t), \omega_{F(t)/F}) \to \bigoplus_{x \in \mathbb{A}^{(1)}_{k,F}} K^\text{MW}_0(F(x), \omega_{F(x)/F})
\]

As in Section 2, write \(p(t) = p_0(t^m)\) with \(p_0\) separable and set \(\omega = p'_0(t^m) \in F[t]\) if \(\text{char}(k) = l\). If \(\text{char}(k) = 0\), set \(\omega = p'(t)\). It is easy to see that the element \(\langle \omega \rangle[p] \cdot dt\) of \(K^\text{MW}_1(F(t), \omega_{F(t)/F})\) ramifies in \(b \in \mathbb{G}^{(1)}_{m,F}\) and on (possibly) other points corresponding to field extensions of degree \(\leq d - 1\). Moreover, the residue at \(b\) is exactly \((1) \cdot \xi\), where \(\xi\) is the canonical orientation of \(\omega_{F(b)/F}\).

Write the minimal polynomial \(p \in F[t]\) of \(b\) as \(p = \sum_{i=0}^d \lambda_i t^i\) with \(\lambda_d = 1\) and \(\lambda_0 \in F^\times\), and decompose \(\omega = c \prod_{j=1}^m q_j^{\omega_j}\), where \(c \in F^\times\) and \(q_j \in F[t]\) are irreducible monic polynomials. Let \(f = (t-1)^{d-1}(t-(-1)^d\lambda_0) \in F[t]\). Observe that \(f\) is monic and satisfies \(f(0) = p(0)\). Let \(F(u, t) = (1-u)p + uf\). Since \(f\) and \(p\) are monic and have the same constant terms, it follows that \(F(u, t) = t^d + \ldots + \lambda_0\) and therefore \(F\) defines an element of \(\mathcal{A}(\mathbb{k}_F, \mathbb{G}_m)\). For the same reason, every \(q_j\) (seen
as a polynomial in \( F[u, t] \) constant in \( u \) defines an element in \( \mathcal{A}(\mathbb{A}^1_F, \mathbb{G}_m) \). The image of \( \langle \omega \rangle \cdot dt \in K_{1}^{\text{MW}}(F(u, t), \omega_{F(u, t)/F(u)}) \) under the residue homomorphism

\[
\partial : K_{1}^{\text{MW}}(F(u, t), \omega_{F(u, t)/F(u)}) \to \bigoplus_{x \in (\mathbb{A}^1_F \times_k \mathbb{G}_m)^{(1)}} K_{0}^{\text{MW}}(F(x), \omega_{F[u, t]/F[u]} \otimes F[t](m_x/m_x^2)^{\vee})
\]

is supported on the vanishing locus of \( F \) and the \( g_j \), and it follows that it defines a finite Chow-Witt correspondence \( \alpha \) in \( \text{Cor}^\sim_k(\mathbb{A}^1_F, \mathbb{G}_m) \). The evaluation \( \alpha(0) \) at \( u = 0 \) satisfies \( \alpha(0) = \langle 1 \rangle \cdot \xi \), while \( \alpha(1) \) is supported on the vanishing locus of \( f \) and the \( q_j \). Its class is then an element of \( H^{1,1}(L, \tilde{\mathbb{Z}})^{(d-1)} \).

**Corollary 6.17.** The homomorphism

\[
\Phi_L : K_{1}^{\text{MW}}(L) \to H^{1,1}(F, \tilde{\mathbb{Z}})
\]

is an isomorphism for any finitely generated field extension \( F/k \).

**Proof.** We know that the homomorphism is injective. The above proposition shows that it is also surjective.

One can then deduce from this corollary that \( \Phi_L \) respects transfers in degree 1. We now prove a slightly more general result that will be needed in the proof of Theorem 6.19

**Proposition 6.18.** Let \( n \in \mathbb{N} \) be such that \( 0 < n \). Then the following diagram commutes

\[
\begin{array}{ccc}
K_{n}^{\text{MW}}(F) & \xrightarrow{\Phi_F} & H^{n,n}(F, \tilde{\mathbb{Z}}) \\
\downarrow \text{Tr}_{F/L} & & \downarrow \text{Tr}_{F/L} \\
K_{n}^{\text{MW}}(L) & \xrightarrow{\Phi_L} & H^{n,n}(L, \tilde{\mathbb{Z}})
\end{array}
\]

for any finite field extensions \( F/L \).

**Proof.** Let \( \theta_L : H^{n,n}(L, \tilde{\mathbb{Z}}) \to K_{n}^{\text{MW}}(L) \) be the left inverse of \( \Phi_L \) defined in Proposition 6.13. We now claim that the following diagram

\[
\begin{array}{ccc}
H^{n,n}(F, \tilde{\mathbb{Z}}) & \xrightarrow{\theta_F} & K_{n}^{\text{MW}}(F) \\
\downarrow \text{Tr}_{F/L} & & \downarrow \text{Tr}_{F/L} \\
H^{1,1}(L, \tilde{\mathbb{Z}}) & \xrightarrow{\theta_F} & K_{1}^{\text{MW}}(L)
\end{array}
\]

commutes. Indeed, using Corollary 6.17 it is enough to check it on elements of the form \( \Phi_F([a_1, \ldots, a_n]) \) with \( a_i \in F^\times \), for which the result is easy. Now \( \theta_F \) and \( \theta_L \) are inverses of \( \Phi_F \) and \( \Phi_L \) and the result follows.

We can now prove our main theorem.

**Theorem 6.19.** The homomorphism

\[
\Phi_L : K_{n}^{\text{MW}}(L) \to H^{n,n}(L, \tilde{\mathbb{Z}})
\]

is an isomorphism for any \( n \in \mathbb{Z} \) and any finitely generated field extension \( L/k \).
Proof. We work by induction on \( n \), the case \( n = 1 \) being settled in Corollary 6.17. As in degree 1, it suffices to prove that \( H^\bullet_{\alpha,n}(\Lambda^\bullet_E,\tilde{\mathbb{Z}})_d \subset H^\bullet_{\alpha,n}(\Lambda^\bullet_E,\tilde{\mathbb{Z}})_{d-1} \) for any \( d \geq 2 \). Let then \( E/L \) be a degree \( d \) field extension, and let \( \alpha \in \text{Cor}_E(L,\mathbb{G}_m^\alpha) \) be a finite Chow-Witt correspondence supported on \( \text{Spec}(E) \subset (\mathbb{A}_L^1)^n \). Such a correspondence is determined by an \( n \)-uple \((a_1,\ldots,a_n) \in (E^\times)^n \) together with a bilinear form \( \phi \in GW(E_\omega E/L) \). Arguing as in Proposition 6.16, we see that we are reduced to prove that \( \text{Tr}_{E/L}(s(a_1) \cdot \ldots \cdot s(a_n)) \in H^\bullet_{\alpha,n}(\Lambda^\bullet_E,\tilde{\mathbb{Z}})_{d-1} \) for any \((a_1,\ldots,a_n) \in (E^\times)^n \). We may also assume that \( E = L(a_1,\ldots,a_n) \). By induction and Corollary 6.18, we know that the diagram

\[
\begin{array}{ccc}
K^\text{MW}_{m}(L(a_1,\ldots,a_n)) & \xrightarrow{\Phi_{L(a_1,\ldots,a_n)}} & H^m(L(a_1,\ldots,a_n),\tilde{\mathbb{Z}}) \\
\text{Tr}_{L(a_1,\ldots,a_n)/L(a_1,\ldots,a_{n-1})} & \downarrow & \text{Tr}_{L(a_1,\ldots,a_n)/L(a_1,\ldots,a_{n-1})} \\
K^\text{MW}_{m}(L(a_1,\ldots,a_{n-1})) & \xrightarrow{\Phi_{L(a_1,\ldots,a_{n-1})}} & H^m(L(a_1,\ldots,a_{n-1}),\tilde{\mathbb{Z}})
\end{array}
\]

(31)

is commutative for any \( m \leq n - 1 \). Now

\[
\text{Tr}_{E/L}(s(a_1) \cdot \ldots \cdot s(a_n)) = \text{Tr}_{L(a_1,\ldots,a_{n-1})/L}(\text{Tr}_{E/L}(a_1,\ldots,a_{n-1}))\cdot(s(a_1) \cdot \ldots \cdot s(a_n)).
\]

The projection formula shows that the right-hand term is equal to

\[
\text{Tr}_{L(a_1,\ldots,a_{n-1})/L}(s(a_1) \cdot \ldots \cdot s(a_{n-1}) \cdot \text{Tr}_{E/L}(a_1,\ldots,a_{n-1})\cdot(s(a_n))).
\]

By Diagram (31), \( \text{Tr}_{E/L}(s(a_1) \cdot \ldots \cdot s(a_n))(a_n) = \Phi_{L(a_1,\ldots,a_n)}(\text{Tr}_{E/L}(a_1,\ldots,a_{n-1})(a_n)) \). Using this, we see that if \( L(a_1,\ldots,a_{n-1}) \subset L(a_1,\ldots,a_n) \) is strict, then we get \( \text{Tr}_{E/L}(s(a_1) \cdot \ldots \cdot s(a_n)) \in H^\bullet_{\alpha,n}(\Lambda^\bullet_E,\tilde{\mathbb{Z}})_{d-1} \). We may thus suppose that we have \( L(a_1,\ldots,a_{n-1}) = L(a_1,\ldots,a_n) \). Arguing along the same lines with \( a_{n-1},\ldots,a_2 \), we are reduced to prove in the case \( E = L(a) \) that

\[
\text{Tr}_{E/L}(s(a) \cdot s(g_2(a)) \cdot \ldots \cdot s(g_n(a))) \in H^\bullet_{\alpha,n}(\Lambda^\bullet_E,\tilde{\mathbb{Z}})_{d-1}
\]

for any polynomials \( g_2,\ldots,g_n \) of degree \( \leq d - 1 \). Since the extension \( E/L \) is monogeneous, we can argue as in Proposition 6.16 mutatis mutandis. \( \square \)

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