A stochastic Gronwall inequality and applications to moments, strong completeness, strong local Lipschitz continuity, and perturbations

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Abstract

There are numerous applications of the classical (deterministic) Gronwall inequality. Recently, Michael Scheutzow discovered a stochastic Gronwall inequality which provides upper bounds for $p$-th moments, $p \in (0,1)$, of the supremum of nonnegative scalar continuous processes which satisfy a linear integral inequality. In this article we complement this with upper bounds for $p$-th moments, $p \in [2,\infty)$, of the supremum of general Itô processes which satisfy a suitable one-sided affine-linear growth condition. As example applications, we improve known results on strong local Lipschitz continuity in the starting point of solutions of stochastic differential equations (SDEs), on (exponential) moment estimates for SDEs, on strong completeness of SDEs, and on perturbation estimates for SDEs.

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1 Introduction

Recently, Scheutzow [41] discovered a powerful stochastic Gronwall inequality. More precisely, Scheutzow [41, Theorem 4] proves that if \( Z, \alpha, \eta : [0, \infty) \times \Omega \to [0, \infty) \) are adapted processes on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, \infty)})\) with continuous sample paths and if \( M : [0, \infty) \times \Omega \to \mathbb{R}\) is a continuous local martingale with \( M_0 = 0 \) which satisfy that \( \mathbb{P}\)-a.s. it holds for all \( t \in [0, \infty) \) that

\[
Z_t \leq \int_0^t \alpha_s Z_s \, ds + M_t + \eta_t, \tag{1}
\]

then for all \( t \in [0, \infty) \), \( q_1, q_3 \in (0, 1) \), \( q_2 \in (0, \infty) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \) it holds that

\[
\left\| \sup_{s \in [0, t]} Z_s \right\|_{L^q(\mathbb{P}; \mathbb{R})} \leq \left( \min\{4, \frac{1}{q_2}\} \|\alpha\|_{L^{q_2}(\mathbb{P}; \mathbb{R})} + 1 \right)^{\frac{1}{q_3}} \exp \left( \int_0^t \alpha_u \, du \right) \left\| \sup_{s \in [0, t]} \eta_s \right\|_{L^{q_3}(\mathbb{P}; \mathbb{R})}. \tag{2}
\]

In this article we complement Scheutzow [41, Theorem 4] with upper bounds for \( p \)-th moments, \( p \in [2, \infty) \), of general multidiimensional Itô processes which satisfy the one-sided affine-linear growth condition \( \langle 3 \rangle \) below. Our contribution is as follows:

- Marginal estimate \( \langle 3 \rangle \): We observe that the proof of Hutzenthaler and Jentzen [23, Theorem 2.10] transfers to the more general setting of Theorem 1.1. The estimate resulting from this approach turns out to be suboptimal if \( \beta \) in \( \langle 3 \rangle \) is non-zero due to an estimate with Young’s inequality. We show how to avoid the use of Young’s inequality by applying the Gronwall-Bellman-Opial inequality in Lemma 2.3 below.

- Uniform estimate \( \langle 5 \rangle \): We observe that Itô’s formula applied for fixed \( p \in [2, \infty) \) to \( (\|X_t\|_{L^p})_{t \in [0, \infty)} \) together with the one-sided affine-linear growth condition \( \langle 3 \rangle \) and Young’s inequality results in inequality \( \langle 11 \rangle \) with \( Z = (\|X\|_{L^p})_{t \in [0, \infty)} \); see the proof of Theorem 2.3 below for details. Thus Theorem 4 in Scheutzow [41] can be applied to \( (\|X_t\|_{L^p})_{t \in [0, \infty)} \). The estimate resulting from this approach turns out to be suboptimal if \( \beta \) in \( \langle 3 \rangle \) is non-zero. We show how to avoid the use of Young’s inequality and how to get a better upper bound.

- We observe that the one-sided affine-linear growth condition \( \langle 3 \rangle \) (or its analog \( \langle 15 \rangle \) for more general Lyapunov-type functions) is satisfied in many applications; see Section 3 for a few examples.

**Theorem 1.1 (A stochastic Gronwall inequality for multi-dimensional Itô processes).** Let \( d, m \in \mathbb{N}, T \in (0, \infty), p \in [2, \infty), \) let \( \| \cdot \|_{\mathbb{R}^d}, \| \cdot \|_{\mathbb{R}^m}, \) and \( \langle \cdot, \cdot \rangle_{\mathbb{R}^d} \) denote Euclidean norm respectively scalar product, let \( \| \cdot \|_{L^{d \times m}} : \mathbb{R}^{d \times m} \to [0, \infty) \) satisfy for all \( A = (A_{i,j})_{i \in \{1, \ldots, d\}, j \in \{1, \ldots, m\}} \in \mathbb{R}^{d \times m} \) that

\[
\|A\|^2_{L^{d \times m}} = \sum_{i=1}^d \sum_{j=1}^m |A_{i,j}|^2, \tag{3}
\]

let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with a normal filtration \((\mathcal{F}_t)_{t \in [0, T]}\), let \( W : [0, T] \times \Omega \to \mathbb{R}^d \) be a standard Brownian motion, let \( X, \alpha : [0, T] \times \Omega \to \mathbb{R}^d, b : [0, T] \times \Omega \to \mathbb{R}^{d \times m}, \alpha, \beta : [0, T] \times \Omega \to [0, \infty) \) be \( \mathcal{B}([0, T]) \otimes \mathcal{F}\)-measurable and adapted stochastic processes which satisfy \( \mathbb{P}\)-a.s. that \( \int_0^T \|a_s\|_{\mathbb{R}^d} + \|b_s\|^2_{L^{d \times m}} + |\alpha_s| \, ds < \infty, \) which satisfy that \( X \) has continuous sample paths, which satisfy for all \( t \in [0, T] \) that it holds \( \mathbb{P}\)-a.s. that \( X_t = X_0 + \int_0^t a_s \, ds + \int_0^t b_s \, dW_s \), and which satisfy that \( \mathbb{P}\)-a.s. it holds for Lebesgue-almost all \( t \in [0, T] \) that

\[
\langle X_t, a_t \rangle_{\mathbb{R}^d} + \frac{1}{2} \|b_t\|^2_{L^{d \times m}} + \frac{p-2}{2} \frac{\langle X_t, b_t \rangle_{\mathbb{R}^d \times m}^2}{\|X_t\|^2_{L^p}} \leq \alpha_t \|X_t\|_{L^p}^2 + \frac{1}{2} |\beta_t|^2. \tag{3}
\]

Then
(i) it holds for all \(q_1, q_2 \in (0, \infty]\), \(t \in [0, T]\) with \(\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p}\) that
\[
\|X_t\|_{L^{q_1}(P; \mathbb{R}^d)} \leq \left\| \exp \left( \int_0^t \alpha_u \, du \right) \right\|_{L^{q_2}(P; \mathbb{R})} \left( \|X_0\|^2_{L^p(P; \mathbb{R}^d)} + \int_0^t \left\| \exp \left( \frac{1}{\beta_u} \right) \right\|_{L^p(P; \mathbb{R})}^2 \, ds \right)^{\frac{1}{2}} \tag{4}
\]
and
\[
\|X_t\|_{L^{q_3}(P; \mathbb{R}^d)} \leq \left( \frac{p}{q_3} \int_0^\infty \left( \frac{s}{(s+1)^2} \right)^{\frac{p}{3}} \, ds + 1 \right)^{\frac{1}{q_3}} \cdot \left\| \exp \left( \int_0^t \alpha_u \, du \right) \right\|_{L^{q_2}(P; \mathbb{R})} \left( \|X_0\|^2_{L^p(P; \mathbb{R}^d)} + \int_0^T \left\| \exp \left( \frac{1}{\beta_u} \right) \right\|_{L^p(P; \mathbb{R})}^2 \, ds \right)^{\frac{1}{2}} \tag{5}
\]

(ii) it holds for all \(q_1, q_2, q_3 \in (0, \infty]\) with \(q_3 < p\) and \(\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}\) that
\[
\left\| \sup_{s \in [0, T]} \|X_s\|_{L^p(P; \mathbb{R}^d)} \right\|_{L^{q_3}(P; \mathbb{R})} \leq \left( \frac{p}{q_3} \int_0^\infty \left( \frac{s}{(s+1)^2} \right)^{\frac{p}{3}} \, ds + 1 \right)^{\frac{1}{q_3}} \cdot \left\| \exp \left( \int_0^t \alpha_u \, du \right) \right\|_{L^{q_2}(P; \mathbb{R})} \left( \|X_0\|^2_{L^p(P; \mathbb{R}^d)} + \int_0^T \left\| \exp \left( \frac{1}{\beta_u} \right) \right\|_{L^p(P; \mathbb{R})}^2 \, ds \right)^{\frac{1}{2}} \tag{6}
\]

Theorem 1.1 is an immediate consequence of Corollary 2.5 below and the proof of Theorem 1.1 is therefore omitted. Corollary 2.5 follows from Theorem 2.4 below which is the main result of this article and which proves \(L^p\)-estimates for \(p \in [1, \infty)\) of more general functions of Itô processes in a separable Hilbert space. Since Theorem 2.4 below essentially generalizes both the differential form of the Gronwall inequality and Lyapunov’s second method for stability of equilibria of ordinary differential equations, we refer to Theorem 2.4 below as a stochastic Gronwall-Lyapunov inequality.

In the literature, there are numerous results which assume that \(\alpha\) in condition (3) is deterministic. In that case one can first take \(L^p\)-norm, \(p \in [1, \infty)\) and then apply the classical Gronwall inequality. This approach imposes strong assumptions (e.g., global monotonicity which is condition (58) in the special case \(p = 2\), \(V_0 \equiv 0, \dot{V} \equiv 0\)) on the problem under consideration which are not satisfied by numerous interesting stochastic differential equations. To illustrate the power of the stochastic Gronwall-Lyapunov inequality in Theorem 2.4 below, we discuss in Section 3 the impact of Theorem 2.4 on the following problems:

(i) (Exponential) moment estimates for stochastic differential equations (SDEs); see Subsection 3.1 and Subsection 3.2

(ii) Strong local Lipschitz continuity in the initial value; see Subsection 3.3

(iii) Strong completeness of SDEs; see Subsection 3.4

(iv) Perturbation estimates for SDEs; see Subsection 3.5

In particular, we considerably improve existing results in the literature on these applications; see Section 3 below for details. Moreover, in the subsequent article Hudde et al. [22] we apply Theorem 2.4 and Corollary 2.5 to derive versions of solutions of SDEs which are twice continuously differentiable in the initial value without assuming the coefficients of the SDE to satisfy a global monotonicity condition.

1.1 Notation

Throughout this article we frequently use the following notation. For every topological space \((E, \mathcal{E})\) we denote by \(\mathcal{B}(E)\) the Borel-sigma-algebra on \((E, \mathcal{E})\). For all measurable spaces \((A, \mathcal{A})\) and \((B, \mathcal{B})\) we denote by \(\mathcal{M}(A, \mathcal{B})\) the set of \(\mathcal{A}/\mathcal{B}\)-measurable functions from \(A\) to \(B\). For every probability space \((\Omega, \mathcal{A}, \mathbb{P})\), real number \(p \in (0, \infty]\), and normed vector space \((V, \| \cdot \|_V)\) we denote by \(\| \cdot \|_{L^p(P; V)} : \mathcal{M}(A, \mathcal{B}(V)) \to [0, \infty)\) the function that satisfies for all \(X \in \mathcal{M}(A, \mathcal{B}(V))\) that \(\|X\|_{L^p(P; V)} = (E[\|X\|^p])^{\frac{1}{p}}\) if \(p < \infty\) and \(\|X\|_{L^\infty(P; V)} = \inf\{c \in [0, \infty) : \|X\|_V \leq c \mathbb{P}\text{-a.s.}\}\)
otherwise. For every $a \in (0, \infty)$ we denote by $\frac{a}{\infty}$ and $\infty^a$ the extended real numbers given by $\frac{a}{\infty} = \infty$ and $\infty^a = \infty$. We denote by $\frac{0}{\infty} = 0$, $0^0 = 0$, and $0^0 = 1$. For two separable $\mathbb{R}$-Hilbert spaces $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ and an orthonormal basis $\mathcal{U}$ of $(U, \langle \cdot, \cdot \rangle_U)$ we denote $L(U, H)$ the set of continuous linear functions, by $\|\cdot\|_{\text{HS}(U, H)}: L(U, H) \to [0, \infty]$ the function satisfying for all $A \in L(U, H)$ that $\|A\|_{\text{HS}(U, H)}^2 = \sum_{u \in \mathcal{U}} \|Au\|_H^2$, and by $\text{HS}(U, H) = \{A \in L(U, H): \|A\|_{\text{HS}(U, H)} < \infty\}$. Stochastic integrals with respect to Wiener processes over product measurable and adapted integrands are defined, e.g., in Weizäcker & Winkler [13, Definition 6.3.4].

2 A stochastic Gronwall-Lyapunov inequality

In this section we derive the main result of this paper: the stochastic Gronwall-Lyapunov inequality in Theorem 2.2 below. First we prove in Lemma 2.2 an almost sure identity for functions of Itô processes with an exponential integrating factor. Moreover, in Lemma 2.3 we provide an analog of Gronwall’s inequality where the exponential function is replaced by a monomial. Throughout this section we use the notation from Subsection 1.1.

Setting 2.1. Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t \in [0, T]}$, let $(W_t)_{t \in [0, T]}$ be an $\mathbb{F}^U$-cylindrical $(\mathcal{F}_t)_{t \in [0, T]}$-Wiener process, let $O \subseteq H$ be an open set, let $\tau: \Omega \to [0, T]$ be a stopping time, let $X: [0, T] \times \Omega \to O$, $a: [0, T] \times \Omega \to H$, $b: [0, T] \times \Omega \to \text{HS}(U, H)$ be $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and adapted stochastic processes which satisfy that it holds $\mathbb{P}$-a.s. that $\int_0^\tau \|a_s\|_H + \|b_s\|_{\text{HS}(U, H)}^2 ds < \infty$, which satisfy that $X$ has continuous sample paths, and which satisfy that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that $X_{\text{min}(t, \tau)} = X_0 + \int_0^t 1_{[0, \tau]}(s) a_s ds + \int_0^t 1_{[0, \tau]}(s) b_s dW_s$.

2.1 Almost sure identity with an exponential integrating factor

The following lemma, Lemma 2.2, slightly generalizes Lemma 2.1 in Hutzenthaler & Jentzen [23] to time-dependent test functions.

Lemma 2.2 (Exponential integrating factor). Assume Setting 2.1. Let $V = (V(t, x))_{t \in [0, T], x \in O} \in C^{1,2}([0, T] \times O, \mathbb{R})$, and let $\varphi: [0, T] \times \Omega \to \mathbb{R}$ be $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and adapted stochastic processes which satisfy $\mathbb{P}$-a.s. that $\int_0^\tau |\varphi_s| ds < \infty$. Then it holds for all $t \in [0, T]$ that $\mathbb{P}$-a.s.

\[
\frac{V(\varphi_{\text{min}(t, \tau)}, X_{\text{min}(t, \tau)})}{\exp(\int_0^t 1_{[0, \tau]}(s) \chi_s \varphi_s \|\varphi_s\|_{\text{HS}(U, H)}^2 ds + \int_0^t 1_{[0, \tau]}(s) \eta_s ds)} = V(0, X_0) + \int_0^t 1_{[0, \tau]}(s) \exp \left( \int_0^s \left( \frac{d}{dt} V(s, X_s) a_s - V(s, X_s) \eta_s \right) ds \right) dW_s + \int_0^\tau \left( \frac{d}{dt} V(s, X_s) a_s + \frac{1}{2} \text{trace}(b_s^* \text{Hess} V(s, X_s) b_s) + \frac{1}{2} \text{trace}(b_s^* \text{Hess} V(s, X_s) b_s) - V(s, X_s) \varphi_s \right) dW_s.
\]

Proof. Itô’s formula proves that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

\[
\frac{V(\varphi_{\text{min}(t, \tau)}, X_{\text{min}(t, \tau)})}{\exp(\int_0^t 1_{[0, \tau]}(s) \chi_s \varphi_s \|\varphi_s\|_{\text{HS}(U, H)}^2 ds + \int_0^t 1_{[0, \tau]}(s) \eta_s ds)} = V(0, X_0) + \int_0^t 1_{[0, \tau]}(s) \exp \left( \int_0^s \left( \frac{d}{dt} V(s, X_s) a_s - V(s, X_s) \eta_s \right) ds \right) dW_s + \int_0^t 1_{[0, \tau]}(s)\left( \frac{d}{dt} V(s, X_s) a_s + \frac{1}{2} \text{trace}(b_s^* \text{Hess} V(s, X_s) b_s) \right) ds + \int_0^t 1_{[0, \tau]}(s)\left( \frac{d}{dt} V(s, X_s) \|\varphi_s\|_{\text{HS}(U, H)}^2 - \frac{1}{2} \text{trace}(b_s^* \text{Hess} V(s, X_s) b_s) \right) ds.
\]
Combining this with the fact that for all $s \in [0, T]$ it holds that
\[
V(s, X_s)\|\eta_s\|_{\text{HS}(U, R)}^2 - \langle \eta_s^*(\frac{\partial}{\partial x^*}) V(s, X_s) b_s \rangle
= \text{trace}(\eta_s^* V(s, X_s) \eta_s) - \text{trace}(\eta_s^* (\frac{\partial}{\partial x^*}) V(s, X_s) b_s)
= \text{trace}(\eta_s^* [V(s, X_s) \eta_s - (\frac{\partial}{\partial x^*}) V(s, X_s) b_s])
\]
proves (7) and finishes the proof of Lemma 2.2.

\[\square\]

### 2.2 A nonlinear Gronwall-Bellman-Opial inequality

The following Gronwall-Bellman-Opial lemma (see Opial [37]) is well-known under additional continuity assumptions; see, e.g., Beesack [3] or Hutzenthaler & Jentzen [24, Lemma 2.11].

**Lemma 2.3.** Let $t \in [0, \infty)$, $p \in (1, \infty)$, let $x, \beta: [0, t] \to [0, \infty]$ be Borel-measurable functions, and assume for all $s \in [0, t]$ that
\[
(x_s)^p \leq (x_0)^p + p \int_0^s (x_r)^{p-1} \beta_r \, dr < \infty.
\]
Then it holds that
\[
x_t \leq x_0 + \int_0^t \beta_r \, dr.
\]

**Proof of Lemma 2.3.** Nonnegativity of $p, x, \beta$, the fact that $(\varepsilon/2, \infty) \ni y \mapsto y^{\frac{1}{p}} \in \mathbb{R}, \varepsilon \in (0, \infty)$, are continuously differentiable functions, the chain rule for absolutely continuous functions, the fact that $V \in [0, \infty) \ni y \mapsto y^{\frac{1}{p}-1} \in \mathbb{R}$ is decreasing, the fact that $V \in [0, \infty) \ni y \mapsto y^{\frac{p-1}{p}} \in \mathbb{R}$ is increasing, and assumption (10) imply for all $\varepsilon \in (0, \infty)$ that
\[
\left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, dr\right)^{\frac{1}{p}}
= \left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, dr\right)^{\frac{1}{p}}
\leq \left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, ds\right)^{\frac{1}{p}}
\leq \left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, ds\right)^{\frac{1}{p}}
= \left(\varepsilon + (x_0)^p + p \int_0^t \beta_r \, ds\right)^{\frac{1}{p}}
\leq \lim_{(0, \infty) \ni \varepsilon \to 0} \left(\varepsilon + (x_0)^p + p \int_0^t \beta_r \, ds\right)^{\frac{1}{p}}
= x_0 + \int_0^t \beta_r \, ds.
\]

This, the fact that the function $(0, \infty) \ni y \mapsto y^{\frac{1}{p}} \in \mathbb{R}$ is increasing and continuous, and assumption (10) imply that
\[
x_t \leq \left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, dr\right)^{\frac{1}{p}}
= \lim_{(0, \infty) \ni \varepsilon \to 0} \left(\varepsilon + (x_0)^p + p \int_0^t (x_r)^{p-1} \beta_r \, dr\right)^{\frac{1}{p}}
\leq \lim_{(0, \infty) \ni \varepsilon \to 0} \left(\varepsilon + (x_0)^p + p \int_0^t \beta_r \, ds\right)^{\frac{1}{p}}
= x_0 + \int_0^t \beta_r \, ds.
\]

This completes the proof of Lemma 2.3.

\[\square\]

### 2.3 A new stochastic Gronwall-Lyapunov inequality

The following theorem, Theorem 2.4, is the main result of this article and states our stochastic Gronwall-Lyapunov inequality for Itô processes. A central step in the proof of the uniform
moment estimate (17) below is a strong observation of Scheutzow [41] namely – suitably adapted to our situation – that inequality [20] implies (27) and then a maximal $L^p$-inequality for local martingales of Burkholder [7] can be applied which together with (27) eliminates the involved local martingale inequality (20). Here instead of Burkholder [7] we apply Theorem 1.4 of Bañuelos & Osękowski [1] which provides the optimal constants

$$
\left( \frac{1}{p} - 1 \right)^p + \frac{\int_{p-1}^{\infty} \frac{\partial_s}{s+1} \ ds}{\frac{1}{p}} = \left( \frac{1}{p} \right) \int_{p-1}^{\infty} \frac{\partial_s}{(s+1)^2} \ ds, \quad p \in (0, 1),
$$

in Burkholder’s result.

**Theorem 2.4** (A stochastic Gronwall-Lyapunov inequality). Assume Setting 2.7 let $p \in [1, \infty)$, let $V = (V(s,x))_{s \in [0,T], x \in O} \in C^{1,2}([0,T] \times O, [0, \infty))$, let $\alpha, \beta: [0,T] \times \Omega \to [0, \infty)$ be $\mathcal{B}([0,T]) \otimes \mathcal{F}/\mathcal{B}([0, \infty])$-measurable and adapted stochastic processes which satisfy $\mathbb{P}$-a.s. that $\int_0^\tau \alpha_u \ du < \infty$ and which satisfy that $\mathbb{P}$-a.s. it holds for Lebesgue-almost all $s \in [0, \tau]$ that

$$
\frac{\partial_s}{s+1} V(s, X_s) + (\frac{\partial_x}{R} V)(s, X_s) a_s + \frac{1}{2} \text{trace} \left( b_s^* b_s^* (\text{Hess}_x V)(s, X_s) \right) + \frac{1}{2} \frac{\left( \frac{\partial_s}{s+1} V(s, X_s) b_s \right)^2_{HS(U, U)}}{V(s, X_s)} \leq \alpha_s V(s, X_s) + \beta_s.
$$

Then

(i) it holds for all $q_1, q_2 \in (0, \infty]$ with $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p}$ that

$$
\left\| V(\tau, X_\tau) \right\|_{L^{n_1}(\mathbb{P}; \mathbb{R})} \leq \left\| \exp \left( \int_0^\tau \alpha_u \ du \right) \right\|_{L^{n_2}(\mathbb{P}; \mathbb{R})} \left( \left\| V(0, X_0) \right\|_{L^p(\mathbb{P}; \mathbb{R})} + \int_0^\tau \left\| \frac{1}{\exp(\int_0^s \alpha_u \ du)} \right\|_{L^p(\mathbb{P}; \mathbb{R})} \ ds \right)
$$

and

(ii) it holds for all $q_1, q_2, q_3 \in (0, \infty]$ with $q_3 < p$ and $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}$ that

$$
\left\| \sup_{s \in [0, \tau]} V(s, X_s) \right\|_{L^{n_1}(\mathbb{P}; \mathbb{R})} \leq \left( \frac{p}{q_3} \int_{\frac{q_3}{p-q_3}}^{\infty} \frac{\tau}{(s+1)^2} \ ds + 1 \right)^{\frac{p}{q_3}} \left\| \exp \left( \int_0^\tau \alpha_u \ du \right) \right\|_{L^{n_2}(\mathbb{P}; \mathbb{R})} \left( \left\| V(0, X_0) \right\|_{L^p(\mathbb{P}; \mathbb{R})} + \int_0^\tau \frac{\partial_s}{\exp(\int_0^s \alpha_u \ du)} \ ds \right)
$$

Proof of Theorem 2.4. Throughout this proof let $\tau_n : \Omega \to [0, T], n \in \mathbb{N}$ be the functions which satisfy for all $n \in \mathbb{N}$ that $\tau_n = \inf \left\{ \{ s \in [0, T] : V(s, X_s) + \int_s^\tau \left\| \frac{\partial_x}{R} V(u, X_u) b_u \right\|_{HS(U, U)}^2 \ du \geq n \} \cup \{ \tau \} \right\}$ and for all $x, y \in \mathbb{R}$ we denote by $x \wedge y \in \mathbb{R}$ the real number which satisfies that $x \wedge y = \min\{x, y\}$. Lemma 2.2 (applied for every $\varepsilon \in (0, \infty)$ with $V = ([0, T] \times O \ni (t, x) \mapsto (\varepsilon + V(t, x)))^p \in \mathbb{R}$), $\chi_s(\omega) = p a_s 1_{[0, \tau_s]}(s)$, and $\eta_s(\omega) = 0$ for all $s \in [0, T]$, $\omega \in \Omega$ in the notation of Lemma 2.2 yields that for all $t \in [0, T], \varepsilon \in (0, \infty)$ it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
&\left( \varepsilon + V(\tau(T, X_{\tau(T)})) \right)^p_{\exp(\int_0^\tau \alpha_u \ du, dr)} = \left( \varepsilon + V(0, X_0) \right)^p_{\exp(\int_0^\tau \alpha_u \ du, dr)} + \int_0^\tau \left( \varepsilon + V(s, X_s) \right)^{p-1}(\frac{\partial_s}{s+1} V(s, X_s) b_s 1_{[0, \tau_s]}(s)) dW_s \\
&+ \int_0^\tau \partial_s(\varepsilon + V(s, X_s))^{p-1}(\frac{\partial_x}{R} V(s, X_s) + p(\varepsilon + V(s, X_s))^{p-1}(\frac{\partial_s}{s+1} V(s, X_s) a_s + p(\varepsilon + V(s, X_s))^{p-1} \frac{1}{2} \text{trace} (b_s^* (\text{Hess}_x V)(s, X_s) b_s) dS \right. \\
&+ \int_0^\tau \frac{\partial_s}{s+1} \left( \varepsilon + V(s, X_s) \right)^{p-2} \left( \frac{\partial_x}{R} V(s, X_s) b_s \right)_{HS(U, U)}^2 - (\varepsilon + V(s, X_s))^{p-1} \frac{\partial_{ss}}{s+1} \right) dW_s.
\end{align*}
$$

(18)
The growth assumption \[ (15) \] and nonnegativity of \( \alpha \) yield for all \( \varepsilon \in (0, \infty) \) that \( \mathbb{P} \)-a.s. it holds for Lebesgue-almost all \( s \in [0, \tau] \), that

\[
p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V(s, X_s) + p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V(s, X_s) \right) a_s \right) + p(\varepsilon + V(s, X_s))^{p-1} \frac{1}{2} \text{trace} \left( b_s^* (\text{Hess}_s V)(s, X_s) b_s \right) + \frac{1}{2} p(p - 1) (\varepsilon + V(s, X_s))^{p-2} \left\| \left( \frac{\partial}{\partial x} V(s, X_s) b_s \right) \right\|_{\text{HS}(\mathbb{T}, \mathbb{R})}^2 - (\varepsilon + V(s, X_s)) p \alpha_s
\]

\[
= p(\varepsilon + V(s, X_s))^{p-1} \left[ \frac{\partial}{\partial x} V(s, X_s) + \frac{1}{2} \text{trace} \left( b_s b_s^* (\text{Hess}_s V)(s, X_s) \right) \right] + \frac{p-1}{2} \left\| \left( \frac{\partial}{\partial x} V(s, X_s) b_s \right) \right\|_{\text{HS}(\mathbb{T}, \mathbb{R})}^2 - \alpha_s (\varepsilon + V(s, X_s))
\]

\[
\leq p(\varepsilon + V(s, X_s))^{p-1} \beta_s.
\]

Then \[(18)\] and \[(19)\] imply that for all \( t \in [0, T] \), \( \varepsilon \in (0, \infty) \), \( n \in \mathbb{N} \) it holds \( \mathbb{P} \)-a.s. that

\[
\frac{(\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge}))^p}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \leq (\varepsilon + V(0, X_0))^p + \int_0^t \frac{p(\varepsilon + V(s, X_s))^{p-1} \left( \frac{\partial}{\partial x} V(s, X_s) b_s \right) 1_{\{0 \leq \tau_n \wedge \leq s\}}}{\exp(\int_0^s \alpha_u \, du)} \, dW_s + \int_0^t \frac{p(\varepsilon + V(s, X_s))^{p-1} \beta_s 1_{\{0 \leq \tau_n \wedge \leq s\}}}{\exp(\int_0^s \alpha_u \, du)} \, ds.
\]

Now we prove item \[(1)\]. Without loss of generality we assume that \( \mathbb{E}[|V(0, X_0)|^p] < \infty \) and that \( \int_0^T \left\| \frac{1_{\{0 \leq \tau_n \wedge \leq s\}}}{\exp(\int_0^s \alpha_u \, du)} \right\|_{L^p(\mathbb{P}; \mathbb{R})} \, ds \leq \infty \) (otherwise the assertion is trivial). Note for every \( n \in \mathbb{N} \) that \( \tau_n \) is a stopping time and that the stochastic integral on the right-hand side of \[(20)\] stopped at \( \tau_n \) is integrable with vanishing expectation. This, \[(20)\], linearity, Tonelli’s theorem, and Hölder’s inequality yield for all \( t \in [0, T] \), \( \varepsilon \in (0, \infty) \), \( n \in \mathbb{N} \) that

\[
\left| \frac{\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge})}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \right|^p \leq \mathbb{E} \left[ \left( \frac{\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge})}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \right)^p \right] = \mathbb{E} \left[ \left( \frac{\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge})}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \right)^p \right] + \int_0^t \mathbb{E} \left[ \left( \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] \, ds
\]

\[
\leq \mathbb{E} \left[ \left( \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] + \int_0^t \mathbb{E} \left[ \left( \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] \, ds
\]

\[
\leq \mathbb{E} \left[ \left( \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] + \int_0^t \mathbb{E} \left[ \left( \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] \, ds
\]

\[
= \mathbb{E} \left[ \left( \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] + \int_0^t \mathbb{E} \left[ \left( \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right)^p \right] \, ds
\]

Observe for all \( t \in [0, T], \varepsilon \in (0, \infty), n \in \mathbb{N} \) that

\[
\left| \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right|^p \leq \int_0^t \left| \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right|^p \, ds
\]

\[
\leq \left| \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right|^p + \int_0^t \left| \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right|^p \, ds
\]

\[
\leq \left| \frac{\varepsilon + V(0, X_0)}{\exp(\int_0^s \alpha_u \, du)} \right|^p + \int_0^t \left| \frac{\varepsilon + V(s, X_s)}{\exp(\int_0^s \alpha_u \, du)} \right|^p \, ds
\]

This, \[(21)\], and the nonlinear Gronwall-Bellman-Opial inequality in Lemma \[2.3\] (applied in the case \( p > 1 \) for all \( n \in \mathbb{N}, \varepsilon \in (0, \infty) \) with \( t = T, p = p, x_s = \frac{\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge})(s)}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \), \( \beta_s = \frac{1_{\{0 \leq \tau_n \wedge \leq s\}}}{\exp(\int_0^s \alpha_u \, du)} \) for all \( s \in [0, T] \) in the notation of Lemma \[2.3\]) imply for all \( n \in \mathbb{N}, \varepsilon \in (0, \infty) \) that

\[
\left| \frac{\varepsilon + V(\tau_n \wedge X_{\tau_n \wedge})}{\exp(\int_0^{\tau_n \wedge} \alpha_u \, du)} \right|_{L^p(\mathbb{P}; \mathbb{R})} \leq \varepsilon + V(0, X_0) \left| \int_0^T \left| \frac{1_{\{0 \leq \tau_n \wedge \leq s\}}}{\exp(\int_0^s \alpha_u \, du)} \right|_{L^p(\mathbb{P}; \mathbb{R})} \, ds.
\]
Moreover the fact that \( (V(s, X_s))_{s \in [0,T]} \) and \( ((\frac{1}{2} \sigma V)(s, X_s))_{s \in [0,T]} \) have continuous sample paths and the fact that \( \mathbb{P}(\int_0^T \|b_s\|_{\mathcal{H}(U; H)}^2 \, ds < \infty) = 1 \) imply that \( \mathbb{P}(\tau = \lim_{n \to \infty} \tau_n) = 1 \). This, continuity of \( V \), nonnegativity of \( V \) and \( \beta \), Fatou’s lemma, (23), and the dominated convergence theorem together with \( \mathbb{E}[1 + V(0, X_0)] < \infty \) ensure that

\[
\mathbb{E} \left[ \frac{(V(\tau, X_\tau))^p}{\exp(\int_0^\tau \frac{1}{p} \alpha_u \, du)} \right] = \mathbb{E} \left[ \lim_{n \to \infty} \frac{1}{\exp(\int_0^\tau \frac{1}{p} \alpha_u \, du)} \right] \leq \lim_{n \to \infty} \lim_{\epsilon \to 0} \mathbb{E} \left[ \frac{1}{\exp(\int_0^\tau \frac{1}{p} \alpha_u \, du)} \right]
\]

(24)

This, the assumption \( \mathbb{P}(\int_0^\tau \alpha_u \, du < \infty) = 1 \), and Hölder’s inequality yield that for all \( q_1, q_2 \in (0, \infty) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p} \) it holds that

\[
\|V(\tau, X_\tau)\|_{L^q(\mathbb{P}; \mathbb{R})} = \left\| \exp \left( \int_0^\tau \alpha_u \, du \right) \frac{V(\tau, X_\tau)}{\exp(\int_0^\tau \alpha_u \, du)} \right\|_{L^q(\mathbb{P}; \mathbb{R})}
\]

\[
\leq \left\| \exp \left( \int_0^\tau \alpha_u \, du \right) \right\|_{L^{q_2}(\mathbb{P}; \mathbb{R})} \left\| \frac{V(\tau, X_\tau)}{\exp(\int_0^\tau \alpha_u \, du)} \right\|_{L^p(\mathbb{P}; \mathbb{R})}
\]

(25)

This proves item (ii).

Next we prove item (iii). Throughout the proof of item (ii) let \( q_1, q_2, q_3 \in (0, \infty) \) satisfy that \( q_3 < p \) and \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \) and let \( M^\varepsilon : [0, T] \times \Omega \to \mathbb{R}, \varepsilon \in (0, \infty) \), be stochastic processes with continuous sample paths such that for all \( t \in [0, T], \varepsilon \in (0, \infty) \) it holds \( \mathbb{P}\)-a.s. that

\[
M^\varepsilon = \int_0^t \frac{p(\varepsilon + V(s, X_s))^{p-1} \frac{1}{2} \sigma V(s, X_s)b_s 1_{[0, \tau]}(s)}{\exp(\int_0^\tau \frac{1}{p} \alpha_u \, du)} \, dW_s.
\]

(26)

Such a continuous process exists since Itô-integrals admit continuous versions. Throughout the proof of item (iii) we assume without loss of generality that \( \|V(0, X_0) + \int_0^\tau \frac{\beta_s}{\exp(\int_0^\tau \alpha_u \, du)} \, ds\|_{L^{q_3}(\mathbb{P}; \mathbb{R})} < \infty \) (otherwise the assertion is trivial). Now (24), and nonnegativity of \( V \) and \( \beta \) imply that for all \( \varepsilon \in (0, \infty), n \in \mathbb{N} \) it holds \( \mathbb{P}\)-a.s. for all \( t \in [0, T] \) that

\[
\max \{ -M^\varepsilon_{\tau_n \wedge t}, 0 \} \leq (\varepsilon + V(0, X_0))^p + \int_0^t \frac{p(\varepsilon + V(s, X_s))^{p-1} \frac{1}{2} \sigma V(s, X_s)b_s 1_{[0, \tau]}(s)}{\exp(\int_0^\tau \frac{1}{p} \alpha_u \, du)} \, ds.
\]

(27)

Then (20), the triangle inequality, Theorem 1.4 in Bañuelos & Osękowski [1] (applied for all \( \varepsilon \in (0, \infty), n \in \mathbb{N} \) with continuous martingale \( (M^\varepsilon_{\tau_n \wedge t})_{t \in [0, \infty)} \)) together with (ii), and (27) yield
for all $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, $q \in (0,1)$ that

\[
E \left[ \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^q}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right] = E \left[ \sup_{t \in [0,T]} \frac{(\varepsilon + V(t \wedge \tau_n \wedge t))^q}{\exp(f_0^{\varepsilon + V(t \wedge \tau_n \wedge t)} \rho \alpha u_\varepsilon du)} \right]^{\frac{q}{p}}
\]

\[
\leq E \left[ \sup_{t \in [0,\infty)} M_{\tau_n \wedge t} + (\varepsilon + V(0,X_0))^p + \int_0^{\tau_n} \frac{p(\varepsilon + V(s,X_s))^{p-1} \beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right]^{\frac{q}{p}}
\]

\[
\leq E \left[ \left( \sup_{t \in [0,\infty)} \max \{M_{\tau_n \wedge t}, 0\} \right)^q \right] + E \left[ (\varepsilon + V(0,X_0))^p + \int_0^{\tau_n} \frac{p(\varepsilon + V(s,X_s))^{p-1} \beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right]^{\frac{q}{p}}
\]

\[
\leq \left( \frac{1}{q} \int \frac{q^q}{(s+1)^q} \frac{ds}{\rho \alpha u_\varepsilon} + 1 \right) \left[ (\varepsilon + V(0,X_0))^p + \int_0^{\tau_n} \frac{p(\varepsilon + V(s,X_s))^{p-1} \beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right]^{\frac{q}{p}}.
\] (28)

The fact that $\forall n \in \mathbb{N}$: $\tau_n \leq \tau$, nonnegativity of $\beta$, and Hölder’s inequality prove that for all $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, $q \in (0,1)$ it holds that

\[
E \left[ (\varepsilon + V(0,X_0))^p + \int_0^{\tau_n} \frac{p(\varepsilon + V(s,X_s))^{p-1} \beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right]^{\frac{q}{p}}
\]

\[
\leq E \left[ \left( \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^p}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right) \left( \varepsilon + V(0,X_0) + \int_0^{\tau_n} \frac{\beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right) \right]^{\frac{q}{p}}
\]

\[
\leq \left( E \left[ \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^p}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right]^{\frac{p-1}{p}} \left( E \left[ (\varepsilon + V(0,X_0) + \int_0^{\tau_n} \frac{\beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right)^{pq} \right) \right)^{\frac{1}{p}}
\] (29)

Combining (29) and (28) proves for all $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, $q \in (0,1)$ that

\[
E \left[ \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^p}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right]^{\frac{p-1}{p}} \left( E \left[ (\varepsilon + V(0,X_0) + \int_0^{\tau_n} \frac{\beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right)^{pq} \right)^{\frac{1}{p}}
\]

\[
\leq \left( \frac{1}{q} \int \frac{q^q}{(s+1)^q} \frac{ds}{\rho \alpha u_\varepsilon} + 1 \right)^{\frac{1}{p}}.
\] (30)

Note that $q_3/p < 1$. Next dividing for every $n \in \mathbb{N}$, $\varepsilon \in (0, \infty)$, $q \in (0,1)$ with $pq \leq q_3$ both sides of (30) by

\[
\left( E \left[ \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^p}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right]^{pq} \right)^{\frac{p-1}{p}} \in \left[ (\varepsilon^{p-1} q, \left( E \left[ (\varepsilon + V(0,X_0) + \int_0^{\tau_n} \frac{\beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right)^{pq} \right) \right] \subseteq (0, \infty)
\]

shows for all $q \in (0, q_3/p)$ that

\[
\left( E \left[ \sup_{t \in [0,\tau_n]} \frac{(\varepsilon + V(t,X_t))^p}{\exp(f_0^p \rho \alpha u_\varepsilon du)} \right]^{pq} \right)^{\frac{p-1}{p}} \left( \frac{1}{q} \int \frac{q^q}{(s+1)^q} \frac{ds}{\rho \alpha u_\varepsilon} + 1 \right)^{\frac{1}{q}}
\]

\[
\leq \left( E \left[ \varepsilon + V(0,X_0) + \int_0^{\tau_n} \frac{\beta_s}{\exp(f_0 \rho \alpha u_\varepsilon du)} \, ds \right]^{pq} \right)^{\frac{1}{pq}}.
\] (31)
Then the monotone convergence theorem, the fact that \( \| V(0, X_0) + \int_0^\tau \frac{\beta_s}{\exp(J_0^a \alpha_u du)} ds \|_{L^q(P; \mathbb{R})} < \infty \), and the dominated convergence theorem prove that

\[
\left\| \sup_{t \in [0, \tau]} \frac{V(t, X_t)}{\exp(J_0^a \alpha_u du)} \right\|_{L^q(P; \mathbb{R})} \leq \liminf_{n \to \infty} \left( \mathbb{E} \left[ \sup_{t \in [0, \tau]} \left\| \frac{V(t, X_t)}{\exp(J_0^a \alpha_u du)} \right\|_{L^q(P; \mathbb{R})} \right] \right)^{\frac{1}{q}}
\]

\[
= \lim_{(0, \infty) \in \to 0} \lim_{n \to \infty} \left( \mathbb{E} \left[ \sup_{t \in [0, \tau]} \left| V(t, X_t) \right| \frac{\exp(J_0^a \alpha_u du)}{1} \right] \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{q_3} \int_0^\infty \frac{\alpha(t) + 1}{\alpha(t)} ds + 1 \right) \left( \mathbb{E} \left[ \left| V(0, X_0) + \int_0^\tau \frac{\beta_s}{\exp(J_0^a \alpha_u du)} \right| \right] \right)^{\frac{1}{q_3}}
\]

\[
= \left( \frac{p}{q_3} \int_0^\infty \frac{s^p}{(s+1)^p} ds + 1 \right) \left( \mathbb{E} \left[ \left| V(0, X_0) + \int_0^\tau \frac{\beta_s}{\exp(J_0^a \alpha_u du)} \right| \right] \right)^{\frac{1}{q_3}}.
\]

Finally, this nonnegativity of \( \alpha \), and Hölder’s inequality prove that

\[
\left\| \sup_{t \in [0, \tau]} V(t, X_t) \right\|_{L^q(P; \mathbb{R})} \leq \left\| \exp \left( \int_0^\tau \alpha_u du \right) \sup_{t \in [0, \tau]} \frac{V(t, X_t)}{\exp(J_0^a \alpha_u du)} \right\|_{L^q(P; \mathbb{R})}
\]

\[
\leq \left\| \exp \left( \int_0^\tau \alpha_u du \right) \right\|_{L^q(P; \mathbb{R})} \left\| \sup_{t \in [0, \tau]} \frac{V(t, X_t)}{\exp(J_0^a \alpha_u du)} \right\|_{L^q(P; \mathbb{R})}
\]

\[
\leq \left( \frac{p}{q_3} \int_0^\infty \frac{s^p}{(s+1)^p} ds + 1 \right) \left( \mathbb{E} \left[ \left| V(0, X_0) + \int_0^\tau \frac{\beta_s}{\exp(J_0^a \alpha_u du)} \right| \right] \right)^{\frac{1}{q_3}}.
\]

This establishes item (i) and completes the proof of Theorem 2.4.

\[
\square
\]

**Corollary 2.5** (A stochastic Gronwall inequality for Itô processes). Assume Setting 2.1, let \( p \in [2, \infty) \), and let \( \alpha, \beta : [0, T] \times \Omega \to [0, \infty] \) be \( \mathcal{B}([0, T]) \otimes \mathcal{F}/\mathcal{B}([0, \infty]) \)-measurable and adapted stochastic processes that satisfy \( \mathbb{P} \)-a.s. that \( \int_0^\tau |\alpha_u| du < \infty \) and which satisfy that \( \mathbb{P} \)-a.s. it holds for Lebesgue-almost all \( t \in [0, T] \) that

\[
\langle X_t, \alpha_t \rangle_H + \frac{1}{2} \frac{1}{2} ||b_t||_H^2 \leq \alpha_t \|X_t\|_H^2 + \frac{1}{2} |\beta_t|^2.
\]

Then

(i) it holds for all \( q_1, q_2 \in (0, \infty) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p} \) that

\[
\| X_t \|_{L^q(P; H)} \leq \left\| \exp \left( \int_0^\tau \alpha_u du \right) \right\|_{L^2(P; H)} \left( \| X_0 \|_{L^p(P; H)} + \int_0^T \left\| \frac{1}{1+\tau} |\beta_s| \exp(J_0^a \alpha_u du) \right\|_{L^p(P; H)} \right)^{\frac{1}{q}}.
\]

and

(ii) it holds for all \( q_1, q_2, q_3 \in (0, \infty) \) with \( q_3 < p \) and \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \) that

\[
\left\| \sup_{s \in [0, \tau]} \| X_s \|_H \right\|_{L^q(P; \mathbb{R})} \leq \left( \frac{p}{q_3} \int_0^\infty \frac{s^p}{(s+1)^p} ds + 1 \right)^{\frac{p}{q_3}}.
\]

\[
\cdot \left\| \exp \left( \int_0^\tau \alpha_u du \right) \right\|_{L^2(P; H)} \left( \| X_0 \|_{L^p(P; H)} + \int_0^T \left\| \frac{1}{1+\tau} |\beta_s| \exp(J_0^a \alpha_u du) \right\|_{L^p(P; H)} \right)^{\frac{1}{q}}.
\]
Proof of Corollary 2.5. Assumption (33) implies that \( \mathbb{P}\text{-a.s. it holds for Lebesgue-almost all } t \in [0, \tau] \) that

\[ 2 \langle X_t, a_t \rangle_H + \frac{1}{p} \text{trace} (b_s b_s^*) 2 \text{Id}_H = \frac{p}{2} \left( 2 \langle X_t, b_t \rangle_H + \frac{2}{p} \frac{\|X_t\|_{H^2}^2}{\|X_t\|_H^2} \right) \leq 2 \alpha_t \|X_t\|_H^2 + |\beta_t|^2. \tag{39} \]

Theorem 2.4 (applied with \( p = \frac{q}{2} \), \( V(s, x) = \|x\|_H^2 \), \( \alpha_s = 2 \alpha_q \), \( \beta_s = |\beta_q|^2 \), \( q_1 = \frac{q}{2}, q_2 = \frac{q}{2}, q_3 = \frac{q}{2} \) for all \( s \in [0, T], x \in O \) in the notation of Theorem 2.4) yields that it holds for all \( q_1, q_2 \in (0, \infty) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p} \) that

\[
\|X_t\|_{L^{q_1}(P; H)}^2 = \left\| \langle X_t, H \rangle \right\|_{L^{q_2}(P; R)}^2 \leq \exp \left( \int_0^T \left( \left\| \langle X_0, H \rangle \right\|_{L^{q_2}(P; R)} + t \right) \exp \left( \int_0^T \left( \frac{\|X_0\|_{H^2}^2 + |\beta_t|^2}{\|X_0\|_H^2} \right) ds \right) \right)
\]

and yields that it holds for all \( q_1, q_2, q_3 \in (0, \infty) \) with \( q_3 < p \) and \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \) that

\[
\sup_{s \in [0, \tau]} \|X_s\|_H^2 \leq \sup_{s \in [0, \tau]} \|X_0\|_H^2 \left( \exp \left( \int_0^T \frac{\|X_0\|_{H^2}^2 + |\beta_t|^2}{\|X_0\|_H^2} ds \right) \right) \leq \left( \frac{q_1}{q_3} \int \frac{|\beta_t|^2}{\|X_0\|_H^2} ds + 1 \right)^{\frac{p/2}{q_3/2}}
\]

This completes the proof of Corollary 2.5. \( \square \)

3. Applications of the stochastic Gronwall-Lyapunov inequality

In this section we apply the stochastic Gronwall-Lyapunov inequality, i.e. Theorem 2.4, to improve existing results on moment estimates for SDEs in Subsection 3.1, exponential moment estimates for SDEs in Subsection 3.2, strong local Lipschitz continuity in the initial value in Subsection 3.3, strong completeness for SDEs in Subsection 3.4, and strong perturbation estimates in Subsection 3.5. Throughout this section we use the notation from Subsection 1.1 and we frequently use the following setting.

Setting 3.1. Let \((H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)\) and \((U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)\) be separable \(\mathbb{R}\) Hilbert spaces, let \(T \in (0, \infty)\), let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with a normal filtration \((\mathbb{F}_t)_{t \in [0, T]}\), let \((W_t)_{t \in [0, T]}\) be an \(\text{Id}_U\)-cylindrical \((\mathbb{F}_t)_{t \in [0, T]}\)-Wiener process, let \(O \subseteq H \) be an open set, let \(\mathcal{O} \in \mathcal{B}(O), \) let \(\mu: [0, T] \times O \to H\) and \(\sigma: [0, T] \times O \to HS(U, H)\) be Borel measurable functions, let \(\tau: \Omega \to [0, T]\) be a stopping time, and let \(X: [0, T] \times \Omega \to O\) be an adapted stochastic process with continuous sample paths which satisfies that \(\mathbb{P}\text{-a.s. it holds that} \int_0^T \|\mu(s, X_s)\|_H + \|\sigma(s, X_s)\|_{HS(U, H)} ds < \infty\) and which satisfies that for all \(t \in [0, T]\) it holds \(\mathbb{P}\text{-a.s. that} \)

\[
X_{\min(t, \tau)} = X_0 + \int_0^t \mathbb{1}_{[0, \tau]}(s) \mu(s, X_s) ds + \int_0^t \mathbb{1}_{[0, \tau]}(s) \sigma(s, X_s) dW_s. \tag{42}
\]
3.1 Moment estimates for SDEs

The following corollary, Corollary 3.2, provides marginal and uniform Lyapunov-type estimates for solutions of SDEs. The marginal Lyapunov-type estimate (14) below is essentially known in the literature; see, e.g., Cox et al. [10, Lemma 2.2] or Gyöngy & Krylov [17, Lemma 2.2]. To the best of our knowledge, the uniform Lyapunov-type estimate (45) below is new. In the literature, uniform moment estimates are derived with the help of a Burkholder-Davis-Gundy inequality. In a number of situations, finiteness of uniform $L^p$-moments could not be established for all $p \in (1, \infty)$ for which marginal $L^p$-moments are finite; e.g. for the 3/2-model of Heston [24] and Platen [38] or for the 4/2-model of Grasselli [14]. In many situations where upper bounds for uniform moments could be established, these are less sharp than (45); see, e.g., Proposition 2.27 in Cox et al. [10] (with $V$ depending only on the first component). Corollary 3.2 follows directly from Corollary 2.5 (applied with $\mu(t, x) = s, \sigma(t, x) = \alpha(t, x)$, $a_s = \mu(s, X_s)$, $b_s = \sigma(s, X_s)$, $\alpha_s = \alpha$, $\beta_s = \beta$, $\eta_2 = \infty$ for all $s \in [0, T]$ in the notation of Corollary 2.5).

Corollary 3.2 (Moment estimates for SDEs). Assume Setting 3.1, let $p \in [2, \infty)$, $\alpha, \beta \in [0, \infty)$, and assume that for all $t \in [0, T]$, $x \in O$ it holds that

$$\langle x, \mu(t, x) \rangle_H + \frac{1}{2} \| \sigma(t, x) \|^2_{HS(U, H)} + \frac{p-2}{2} \frac{\| (x, \sigma(t, x))_H \|^2_{HS(U, R)}}{\| x \|^2_H} \leq \alpha \| x \|^2_H + \frac{1}{2} \beta^2. \quad (43)$$

Then

(i) it holds that

$$\| X_t \|_{L^p(P, H)} \leq e^{\alpha T} \left( \| X_0 \|^2_{L^p(P, H)} + \int_0^T \beta^2 e^{\alpha s} \, ds \right)^{\frac{1}{2}} \quad (44)$$

and

(ii) it holds for all $q \in (0, p)$ that

$$\left\| \sup_{t \in [0, T]} \| X_t \|_{H} \right\|_{L^q(P, \mathbb{R})} \leq e^{\alpha T} \left( \| X_0 \|^2_H + \int_0^T \beta^2 e^{\alpha s} \, ds \right)^{\frac{1}{2}} \left\| \frac{p-q}{\frac{q}{2} - \frac{1}{2}} \int_0^T \frac{e^{\alpha s}}{(s+1)^2} \, ds + 1 \right\|^\frac{q}{2}. \quad (45)$$

3.2 Exponential moment estimates for SDEs

For a number of problems involving SDEs with non-globally monotone coefficients, it is useful to estimate exponential moments; see, e.g., Subsection 3.3 or Subsection 3.5 below. The following corollary, Corollary 3.3, applies Theorem 2.4 to derive suitable exponential moment estimates. Condition (46) and the marginal exponential moment estimate (47) have been found and derived in Cox et al. [10, Proposition 2.3 and Corollary 2.4]. To the best of our knowledge, the uniform exponential moment estimate (48) is new.

Corollary 3.3 (Exponential moment estimates for SDEs). Assume Setting 3.1, let $U : [0, T] \times O \to \mathbb{R}$ be Borel-measurable and satisfy $\int_0^T \| U(s, X_s) \| ds < \infty$, let $U = (U(s, x))_{s \in [0, T], x \in O} \in C^{1,2}([0, T] \times O, [0, \infty))$, let $\alpha \in \mathbb{R}$, and assume that for all $(s, x) \in \cup_{\omega \in \Omega} \{ (t, X_t(\omega)) \in [0, T] \times O : t \in [0, \tau(\omega)] \}$ it holds that

$$\left( \frac{\partial}{\partial s} U \right)(s, x) + \left( \frac{\partial}{\partial x} U \right)(s, x) \mu(s, x) + \frac{1}{2} \text{trace} \left( (\sigma \cdot \sigma^*)(s, x) \text{ (Hess}_x U)(s, x) \right)$$

$$\quad + \frac{1}{2 \sigma^2} \left\| (\frac{\partial}{\partial x} U)(s, x) \sigma(s, x) \right\|^2_{HS(U, R)} + \bar{U}(s, x) \leq \alpha U(s, x). \quad (46)$$

Then
Proof of Corollary \[\text{Corollary 3.3.}\] Throughout this proof let \( \Omega \rightarrow \mathbb{R} \rightarrow [0, \infty) \) and \( \Omega \rightarrow \mathbb{R} \rightarrow \mathbb{R} \) satisfy for all \( t \in [0, T] \), \( x \in O \), \( y \in \mathbb{R} \) that \( V(t,x,y) = \exp(U(t,x)e^{-at} + y) \) and \( Y_t = \int_{0}^{\min(t,\tau)} \hat{U}(s,X_s)e^{-a s}ds \). Assumption \[\text{(46)}\] yields for all \( s \in [0, \tau] \) that

\[
\left( \frac{\partial}{\partial s} V \right)(s, X_s, Y_s) + \left( \frac{\partial}{\partial (x,y)} V \right)(s, X_s, Y_s) \left( \mu(s, X_s), \hat{U}(s, X_s) e^{-a s} \right) \\
+ \frac{1}{2} \text{trace} \left( (\sigma(s, X_s), 0)(\sigma(s, X_s), 0)^T \left( \text{Hess}_{(x,y)} V \right)(s, X_s, Y_s) \right) \\
= V(s, X_s, Y_s)e^{-as} \left( \frac{\partial}{\partial s} U \right)(s, X_s) - V(s, X_s, Y_s)e^{-as} \alpha U(s, X_s) \\
+ V(s, X_s, Y_s)e^{-as} \left( \frac{\partial}{\partial x} U \right)(s, X_s) \mu(s, X_s) + V(s, X_s, Y_s)e^{-as} \hat{U}(s, X_s) \\
+ V(s, X_s, Y_s)e^{-as} \frac{1}{2} \text{trace} \left( (\sigma(s, X_s)(\sigma(s, X_s))^T \left( \text{Hess}_x U \right)(s, X_s) \right) \\
+ V(s, X_s, Y_s)e^{-2as} \frac{1}{2} \left\| \left( \frac{\partial}{\partial x} U \right)(s, X_s) \sigma(s, X_s) \right\|^2_{\text{Hess}(U,\mathbb{R})} \\
\leq 0.
\]

This and Theorem \[\text{Theorem 2.4.}\] (applied with \( H = H \times \mathbb{R} \), \( \langle (x, r), (y, s) \rangle_H = \langle x, y \rangle_H + r s \), \( X_t = (X_t, Y_t) \), \( a_t = 1_{[0,\tau]}(t)(\mu(t, X_t), \hat{U}(t, X_t)e^{-at}) \), \( b_t = 1_{[0,\tau]}(t)(\sigma(t, X_t), 0) \), \( V = V \), \( a_t = 0 \), \( \beta_t = 0 \), \( p = 1 \), \( q_2 = \infty \) for all \( x, y \in H \), \( r, s \in \mathbb{R} \), \( t \in [0, T] \) in the notation of Theorem \[\text{Theorem 2.4.}\] yield item \[\text{(i)}\] and item \[\text{(ii)}\]. This finishes the proof of Corollary \[\text{Corollary 3.3.}\]

Clearly if the solution of an SDE has finite exponential moments, then it has finite moments if the starting point has sufficient exponential moments. The marginal moment estimate \[\text{(51)}\] and the uniform moment estimate \[\text{(52)}\] below show that it suffices that the starting point has suitable finite moments if a suitable special case of the exponential moment condition \[\text{(46)}\] is satisfied. In particular, we show in the proof of Corollary \[\text{Corollary 3.4.}\] below that the exponential moment condition \[\text{(50)}\] below implies the moment condition \[\text{(53)}\] below.

Corollary 3.4 (Exponential moment condition implies moments). Assume Setting \[\text{Setting 3.7.}\] let \( U = \left( U(s, x) \right)_{s \in [0, T], x \in \mathcal{O}} \in C^{1,2}([0, T] \times \mathcal{O}; [0, \infty)) \), let \( \alpha \in \mathbb{R} \), \( \beta \in [0, \infty) \), and assume that for all \( (s, x) \in \cup_{t \in \mathcal{O}} \{(t, X_t(\omega)) \in [0, T] \times \mathcal{O} : t \in [0, \tau(\omega)] \} \) it holds that

\[
\left( \frac{\partial}{\partial s} U \right)(s, x) + \left( \frac{\partial}{\partial x} U \right)(s, x) \mu(s, x) + \frac{1}{2} \text{trace} \left( (\sigma \cdot \sigma^T)(s, x)(\text{Hess}_x U)(s, x) \right) \\
+ \frac{1}{2e^{as}} \left\| \left( \frac{\partial}{\partial x} U \right)(s, x) \sigma(s, x) \right\|^2_{\text{Hess}(U,\mathbb{R})} \leq \alpha U(s, x) + \beta.
\]

Then
(i) it holds for all \( p \in [1, \infty) \) that

\[
\mathbb{E}\left[ |p + e^{-\alpha s}u(\tau, X) - p| \right] \leq \mathbb{E}\left[ |p + U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds| \right] \leq p^p \mathbb{E}\left[ \exp(U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds) \right]
\]

and

(ii) it holds for all \( p \in [1, \infty), q \in (0, p) \) that

\[
\mathbb{E}\left[ \sup_{t \in [0, \tau]} |p + e^{-\alpha t}u(t, X_t)|^q \right] \leq \mathbb{E}\left[ |p + U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds| \right]^q \left( \frac{p^2}{q} \int_0^T \frac{s^2}{(s+1)^2} \, ds + 1 \right)^{p-q}. \tag{52}
\]

**Proof of Corollary 3.4.** Throughout this proof let \( p \in [1, \infty) \) and let \( V : [0, T] \times O \to [0, \infty) \) satisfy for all \( s \in (0, T], x \in O \) that \( V(s, x) = p + U(s, x)e^{-\alpha s} + \beta \int_s^T e^{-\alpha u} \, du \). Note that \((50)\) implies that for all \( s \in [0, T], x \in O \) it holds that

\[
\begin{align*}
(\frac{\partial}{\partial s} V)(s, x) + (\frac{\partial}{\partial x} V)(s, x) &\mu(s, x) + \frac{1}{2} \text{trace}\left( (\sigma \cdot \sigma^*) (s, x) \text{Hess}_x V(s, x) \right) \\
&+ \frac{p-1}{2} \left\| (\frac{\partial}{\partial s} V)(s, x) \sigma(s, x) \right\|^2_{\text{Hess}(U, \mathbb{R})} \\
= \left( \frac{\partial}{\partial s} U \right)(s, x)e^{-\alpha s} - U(s, x)\alpha e^{-\alpha s} - \beta e^{-\alpha s} + (\frac{\partial}{\partial x} U)(s, x) \mu(s, x) e^{-\alpha s} \\
&+ \frac{1}{2} \text{trace}\left( (\sigma \cdot \sigma^*) (s, x) \text{Hess}_x U(s, x) \right) e^{-\alpha s} + \frac{1}{2e^{\alpha s}} \left\| (\frac{\partial}{\partial x} U)(s, x) \sigma(s, x) \right\|^2_{\text{Hess}(U, \mathbb{R})} \\
&\leq e^{-\alpha s} \left( (\frac{\partial}{\partial s} U)(s, x) - U(s, x)\alpha - \beta + (\frac{\partial}{\partial x} U)(s, x) \mu(s, x) \right) \\
&+ \frac{1}{2} \text{trace}\left( (\sigma \cdot \sigma^*) (s, x) \text{Hess}_x U(s, x) \right) + \frac{1}{2e^{\alpha s}} \left\| (\frac{\partial}{\partial x} U)(s, x) \sigma(s, x) \right\|^2_{\text{Hess}(U, \mathbb{R})} \leq 0. \tag{53}
\end{align*}
\]

This, Theorem 2.4 (applied with \( a_t = \mu(t, X_t), b_t = \sigma(t, X_t), \alpha_t = 0, \beta_t = 0, q_2 = \infty \) for all \( t \in [0, T] \) in the notation of Theorem 2.4, and the fact that \( \forall \in (0, \infty) : 1 + x \leq e^x \) yield that

\[
\begin{align*}
\mathbb{E}\left[ |p + e^{-\alpha s}u(\tau, X) - p| \right] \leq \mathbb{E}\left[ |V(\tau, X)| \right] \leq \mathbb{E}\left[ |V(0, X)| \right] \\
= \mathbb{E}\left[ |p + U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds| \right] \leq p^p \mathbb{E}\left[ \exp(U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds) \right] \\
\leq p^p \mathbb{E}\left[ \exp \left( \frac{1}{p} U(0, X) + \frac{1}{p} \int_0^T \frac{\beta}{e^{\alpha s}} \, ds \right) \right] \leq p^p \mathbb{E}\left[ \exp(U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds) \right]
\end{align*}
\]

and yield for all \( q \in (0, p) \) that

\[
\left( \mathbb{E}\left[ \sup_{t \in [0, \tau]} |p + e^{-\alpha t}u(t, X_t)|^q \right] \right)^{\frac{1}{q}} \leq \left( \mathbb{E}\left[ |V(t, X_t)|^q \right] \right)^{\frac{1}{q}} \leq \left( \mathbb{E}\left[ |V(0, X)|^q \right] \right)^{\frac{1}{q}} \left( \frac{p^2}{q} \int_0^\infty \frac{s^2}{(s+1)^2} \, ds + 1 \right)^{\frac{1}{q}} \leq \left( \mathbb{E}\left[ |p + U(0, X) + \int_0^T \frac{\beta}{e^{\alpha s}} \, ds| \right] \right)^{\frac{1}{q}} \left( \frac{p^2}{q} \int_0^\infty \frac{s^2}{(s+1)^2} \, ds + 1 \right)^{\frac{1}{q}}. \tag{55}
\]

This finishes the proof of Corollary 3.4. \qed
3.3 Strong local Lipschitz continuity in the initial value

In this subsection we derive strong local Lipschitz continuity in the initial value of solutions of SDEs. We do not assume that the coefficients of the SDE satisfy a global monotonicity condition since this condition is not satisfied for most example SDEs from applications; cf., e.g., Cox et al. [10, Chapters 4, 5]. Establishing such a strong local Lipschitz continuity is nontrivial since there exist even SDEs with globally bounded and smooth coefficients which do not have this property due to a loss of regularity phenomenon; see Hairer et al. [18]. The marginal local Lipschitz estimate (59) below improves existing results in [10, 13, 31, 42, 44]. To the best of our knowledge, the uniform local Lipschitz estimate (60) below is new.

Setting 3.5. Assume Setting 3.3, let \( Y : [0, T] \times \Omega \to \mathcal{O} \) be an adapted stochastic process with continuous sample paths which satisfies that \( \mathbb{P}\text{-a.s.} \) it holds that \( \int_0^t \| \mu(s, Y_s) \|_H + \| \sigma(s, Y_s) \|_{\text{HS}(U,H)}^2 \, ds < \infty \) and which satisfies that for all \( t \in [0, T] \) it holds \( \mathbb{P}\text{-a.s.} \) that

\[
Y_{\min(t,\tau)} = Y_0 + \int_0^t 1_{[0,\tau]}(r) \mu(r, Y_r) \, dr + \int_0^t 1_{[0,\tau]}(r) \sigma(r, Y_r) \, dW_r,
\]

(56)

let \( \alpha_0, \alpha_1, \beta_0, \beta_1 \in [0, \infty) \), \( V_0, V_1 \in C^2(\mathcal{O}, [0, \infty)) \), let \( \bar{V} : [0, T] \times \mathcal{O} \to [0, \infty) \) be a Borel measurable function which satisfies that \( \mathbb{P}\text{-a.s.} \) it holds that \( \int_0^T |\bar{V}(r, X_r)| + |\bar{V}(r, Y_r)| \, dr < \infty \) and that for all \( i \in \{0, 1\} \), \( t \in [0, T] \), \( x \in \mathcal{O} \) it holds that

\[
\langle \mu(t, x), (\nabla V_i)(x) \rangle_H + \frac{1}{2} \text{trace} \left( \sigma(t, x)[\sigma(t, x)]^*(\text{Hess} V_i)(x) \right)
+ \frac{1}{2^{q_i-1}} \| \sigma(t, x)^*(\nabla V_i)(x) \|_{L^2}^2 + 1_{\{1\}}(i) \cdot \bar{V}(t, x) \leq \alpha_i V_i(x) + \beta_i,
\]

(57)

let \( \phi : [0, T] \to [0, \infty) \) be a Borel measurable function which satisfies that \( \int_0^T \phi(r) \, dr < \infty \), let \( p \in [2, \infty) \), \( q, q_0, q_1 \in (0, \infty) \) satisfy that \( \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{q} \), and assume that for all \( t \in [0, T] \), \( x, y \in \mathcal{O} \) it holds that

\[
\langle x - y, \mu(t, x) - \mu(t, y) \rangle_H + \frac{1}{2} \| \sigma(t, x) - \sigma(t, y) \|_{\text{HS}(U,H)}^2 + \frac{p-2}{2} \left( \frac{\|x - y\|_{H}^2}{\|x - y\|_{H}^2} \right) 
\leq \| x - y \|_H^2 \cdot \left( \phi(t) + \frac{V_0(x) + V_0(y)}{2q_0 e^{q_0 t}} + \frac{\bar{V}(t,x) + \bar{V}(t,y)}{2q_1 e^{q_1 t}} \right).
\]

(58)

Lemma 3.6 (Strong local Lipschitz continuity in the initial value). Assume Setting 3.3 and let \( x, y \in \mathcal{O} \). Then

(i) it holds for all \( t \in [0, T] \) that

\[
\left\| X_{\min(t,\tau)} - Y_{\min(t,\tau)} \right\|_{L^{\frac{2}{\alpha_i}}(\mathbb{P}; H)} \leq \left\| X_0 - Y_0 \right\|_{L^{\frac{2}{p}}(\mathbb{P}; H)} \exp \left( \int_0^t \phi(r) + \frac{\beta_0 (1 - \frac{q}{q_0 e^{q_0 t}})}{q_0 e^{q_0 t}} + \frac{\beta_1}{q_1 e^{q_1 t}} \, dr \right)
\cdot \prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{1/2q_i} \prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(Y_0) \right) \right] \right)^{1/2q_i},
\]

(59)

and

(ii) it holds for all \( \delta \in (0, 1) \) that

\[
\sup_{t \in [0, T]} \left\| X_t - Y_t \right\|_{L^{\frac{2}{p}}(\mathbb{P}; H)} \leq \left\| X_0 - Y_0 \right\|_{L^{\frac{2}{p}}(\mathbb{P}; H)} \exp \left( \int_0^T \phi(r) + \frac{\beta_0 (1 - \frac{q}{q_0 e^{q_0 t}})}{q_0 e^{q_0 t}} + \frac{\beta_1}{q_1 e^{q_1 t}} \, dr \right)
\cdot \left( \frac{1}{\delta} \int_0^{\infty} \left( \frac{\delta^{2}}{(s+1)^2} ds + 1 \right) \prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{1/2q_i} \prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(Y_0) \right) \right] \right)^{1/2q_i} \right),
\]

(60)
Proof of Lemma 3.6. It follows from (42) and (56) that for all \( t \in [0, T] \) it holds \( \mathbb{P} \)-a.s. that

\[
X_{\min\{t, \tau\}} - Y_{\min\{t, \tau\}} = x - y + \int_0^t \mathbb{1}_{[0, \tau]}(r) (\mu(r, X_r) - \mu(r, Y_r)) \, dr + \int_0^t \mathbb{1}_{[0, \tau]}(r) (\sigma(r, X_r) - \sigma(r, Y_r)) \, dW_r.
\]  
(61)

Assumption (58) implies for all \( t \in [0, T] \) that

\[
\langle X_t - Y_t, \mu(t, X_t) - \mu(t, Y_t) \rangle_H + \frac{1}{2} \left\| \sigma(t, X_t) - \sigma(t, Y_t) \right\|_{HS(U, H)}^2 + \frac{1}{2} \left\| X_t - Y_t \right\|_H^2 \leq \left\| X_t - Y_t \right\|_H^2 \cdot \left( \phi(t) \right) + \frac{\bar{V}(X_t) + \bar{V}(Y_t)}{2q_0\epsilon^q_{t, r}} + \frac{\bar{V}(X_t) + \bar{V}(Y_t)}{2q_1\epsilon^q_{t, r}}.
\]  
(62)

This, (61), and item (i) in Corollary 2.5 (applied for every \( s \in [0, T] \) with \( T = s, \tau = \min\{s, \tau\} \), \( X_t = X_t - Y_t, \alpha_t = \mu(t, X_t) - \mu(t, Y_t), \beta_t = \sigma(t, X_t) - \sigma(t, Y_t), \) \( \alpha_t = \phi(t) + \frac{\bar{V}(X_t) + \bar{V}(Y_t)}{2q_0\epsilon^q_{t, r}} + \frac{\bar{V}(X_t) + \bar{V}(Y_t)}{2q_1\epsilon^q_{t, r}} \)) for all \( t \in [0, s] \) in the notation of Corollary 2.5 imply for all \( t \in (0, T] \) that

\[
\left\| X_{\min\{t, \tau\}} - Y_{\min\{t, \tau\}} \right\|_{L^p(P, H)} \leq \left\| X_0 - Y_0 \right\|_{L^p(P, H)} \left\| \exp \left( \int_0^{\min\{t, \tau\}} \phi(r) + \frac{\bar{V}(X_r) + \bar{V}(Y_r)}{2q_0\epsilon^q_{t, r}} + \frac{\bar{V}(X_r) + \bar{V}(Y_r)}{2q_1\epsilon^q_{t, r}} \, dr \right) \right\|_{L^q(P; \mathbb{R})}.
\]  
(63)

Hölder’s inequality together with \( \frac{1}{q} = 2 \frac{1}{q_0} + 2 \frac{1}{q_1}, \) the fact that \( \beta_0, \beta_1 \geq 0, \) the fact that \( \forall t \in (0, T] : \int_0^t \frac{\beta_0(1-t)}{q_0\epsilon^q_{t, r}} \, dr = \int_0^t \frac{\beta_0}{q_0\epsilon^q_{t, r}} \, dr \geq \int_0^{\min\{t, \tau\}} \frac{\beta_0}{q_0\epsilon^q_{t, r}} \, dr, \) Jensen’s inequality, Tonelli’s theorem, nonnegativity of \( V_1, (57), \) and Corollary 3.3 (applied for every \( r \in (0, T] \) with \( \tau = \min\{r, \tau\}, U(s, x) = \bar{V}(x), \tau = \bar{V}(x) - \beta_0, \alpha = \alpha_0, X = X \) (resp. \( X = Y \)) for all \( s \in (0, T], x \in \mathcal{O} \) and applied for every \( t \in (0, T] \) with \( \tau = \min\{t, \tau\}, U(s, x) = \bar{V}(x), \tau = \bar{V}(x) - \beta_1, \alpha = \alpha_1, X = X \) (resp. \( X = Y \)) for all \( s \in (0, T], x \in \mathcal{O} \) in the notation of Corollary 3.4 show for
all \( t \in (0, T] \) that

\[
\left\| \exp \left( \int_0^{\min \{t, \tau \}} \frac{V_0(X_s) + V_0(Y_s)}{2q_1 e^{q_1 \tau}} + \frac{V(r, X_s) + V(r, Y_s)}{2q_1 e^{q_1 \tau}} \, ds \right) \right\|_{L^1(\mathbb{P}; \mathbb{R})} \leq \left\| \exp \left( \int_0^{\min \{t, \tau \}} \frac{\bar{\beta}_i (1 - \alpha_1) \beta_i \bar{\alpha}_i}{q_1 e^{q_1 \tau}} \, ds \right) \right\|_{L^1(\mathbb{P}; \mathbb{R})}.
\]

This and inequality (63) yield for all \( t \in (0, T] \) that

\[
\left\| X_{\min \{t, \tau \}} - Y_{\min \{t, \tau \}} \right\|_{L^{p_0}(\mathbb{P}; H)} \leq \left\| X_0 - Y_0 \right\|_{L^{p_0}(\mathbb{P}; H)} \exp \left( \int_0^t \phi(r) + \frac{\beta_0 (1 - \alpha_1) \beta_i \bar{\alpha}_i}{q_1 e^{q_1 \tau}} \, dr \right) \cdot \prod_{i=0}^{\infty} \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{1/2q_i} \prod_{i=0}^{\infty} \left( \mathbb{E} \left[ \exp \left( V_i(Y_0) \right) \right] \right)^{1/2q_i}.
\]

This proves item (i). Next, (61), (62), item (ii) in Corollary 2.5 (applied for every \( \tau \in (0, 1) \) with \( \tau = \tau, X_t = X_t - Y_t, a_t = \mu(t, X_t) - \mu(t, Y_t), b_t = \sigma(t, X_t) - \sigma(t, Y_t), \alpha_t = \phi(t) + \frac{V_0(X_t) + V_0(Y_t)}{2q_1 e^{q_1 \tau}} + \frac{V(t, X_t) + V(t, Y_t)}{2q_1 e^{q_1 \tau}}, \beta_t = 0, q_1 = \frac{p+q}{p+b+q}, q_2 = q, q_3 = \delta p \) for all \( t \in [0, T] \) in the notation of Corollary 2.5), and (64) imply for all \( \delta \in (0, 1) \) that

\[
\left\| \sup_{t \in [0, \tau]} \left\| X_t - Y_t \right\|_{H^{\delta}} \right\|_{L^{p_0}(\mathbb{P}; \mathbb{R})} \leq \left( \frac{1}{\delta} \int_0^T \frac{s^\delta}{(s+1)^{\delta+1}} \, ds + 1 \right) \left\| X_0 - Y_0 \right\|_{L^{p_0}(\mathbb{P}; H)} \exp \left( \int_0^T \phi(r) + \frac{\beta_0 (1 - \alpha_1) \beta_i \bar{\alpha}_i}{q_1 e^{q_1 \tau}} \, dr \right) \cdot \prod_{i=0}^{\infty} \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{1/2q_i} \prod_{i=0}^{\infty} \left( \mathbb{E} \left[ \exp \left( V_i(Y_0) \right) \right] \right)^{1/2q_i}.
\]

This proves item (ii) and completes the proof of Lemma 3.6. \( \square \)
The following lemma, Lemma 3.7, is essentially well-known and is included for the convenience of the reader.

**Lemma 3.7** (Temporal regularity). Assume Setting 3.7, let \( \gamma \in \left[ \frac{1}{p}, \infty \right) \), \( c \in [0, \infty) \) satisfy for all \( t \in [0, T] \), \( x \in \mathcal{O} \) that

\[
\max \left\{ \|\mu(t, x)\|_H, \|\sigma(t, x)\|_{\text{HS}(U, H)} \right\} \leq c(1 + V_0(x))^\gamma,
\]

and let \( s \in [0, T] \). Then it holds that

\[
\left\| \sup_{t \in [s, T]} \left\| X_t - X_{\min(s, T)} \right\|_H \right\|_{L^p(\mathbb{P}; \mathbb{R})} \leq c e^{\alpha_0 T} \left\| p \gamma + V_0(X_0) + \int_0^T \frac{\beta_0}{\varepsilon^{\alpha_0}} \, du \right\|_{L^p(\mathbb{P}; \mathbb{R})}^\gamma \left( \sqrt{T} + p \right) \sqrt{T - s}.
\]

**Proof of Lemma 3.7**. Equation (12), the triangle inequality, the Burkholder-Davis-Gundy type inequality in Da Prato & Zabczyk [1], Lemma 7.2 and Lemma 7.7, \( (67) \), the fact that \( p \gamma \geq 1 \), the fact that \( \beta_0 \geq 0 \), and Corollary 3.4 (applied for every \( r \in [s, T] \) with \( U(t, x) = V_0(x) \), \( \alpha = \alpha_0 \), \( \beta = \beta_0 \) for all \( t \in [s, T] \), \( x \in \mathcal{O} \) in the notation of Corollary 3.4) ensure that

\[
\left\| \sup_{t \in [s, T]} \left\| X_t - X_s \right\|_H \right\|_{L^p(\mathbb{P}; \mathbb{R})} \leq \int_s^T \left( \left[ \frac{1}{2} \left( 1 + V_0(X_{\min(r, T)}) \right) \right] \right) \left( T - s + \sqrt{T - s} \frac{p^3}{2(p-1)} \right) \right)^{\frac{1}{2}}
\]

The proof of Lemma 3.7 is thus completed.

**Lemma 3.8** (Strong local Hölder estimate). Assume Setting 3.8, let \( \gamma \in \left[ \frac{1}{p}, \infty \right) \), \( c \in [0, \infty) \) satisfy for all \( t \in [0, T] \), \( x \in \mathcal{O} \) that

\[
\max \left\{ \|\mu(t, x)\|_H, \|\sigma(t, x)\|_{\text{HS}(U, H)} \right\} \leq c(1 + V_0(x))^\gamma,
\]

assume that \( \frac{pq}{p+q} \in [2, \infty) \cap \left[ \frac{1}{\gamma}, \infty \right) \) and let \( t_1, t_2 \in [0, T] \), \( x_1, x_2 \in \mathcal{O} \). Then it holds that

\[
\|X_{t_1} - X_{t_2}\|_{L^{pq}(\mathbb{P}, H)} \leq \sqrt{|t_1 - t_2| e^{\alpha_0 T}} \left\| \frac{pq}{p+q} \gamma + V_0(X_0) + \int_0^T \frac{\beta_0}{\varepsilon^{\alpha_0}} \, ds \right\|_{L^{pq}(\mathbb{P}; \mathbb{R})}^\gamma \left( \sqrt{T} + \frac{pq}{p+q} \right)
\]

\[
+ \left\| X_0 - Y_0 \right\|_{L^p(\mathbb{P}; H)} \exp \left( \int_0^T \phi(r) + \frac{\beta_0}{q \varepsilon^{\alpha_0}} + \frac{\beta_1}{q_1 \varepsilon^{\alpha_1}} \, dr \right)
\]

\[
\prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{\frac{1}{2q_i}} \prod_{i=0}^{1} \left( \mathbb{E} \left[ \exp \left( V_i(Y_0) \right) \right] \right)^{\frac{1}{2q_i}}.
\]

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Proof of Lemma 3.8. Without loss of generality we assume that $t_1 + t_2 > 0$. The triangle inequality, Lemma 3.7 (applied with $T = \max\{t_1, t_2\}$, $\tau = \max\{t_1, t_2\}$, $p = \frac{pq}{p+q}$, $s = \min\{t_1, t_2\}$) in the notation of Lemma 3.7, and Lemma 3.6 (applied with $\tau = T$, $x = x_1$, $y = x_2$, $t = t_2$ in the notation of Lemma 3.6) yield that

$$
\|X_{t_1} - Y_{t_2}\|_{L^{pq/\tau}(\mathbb{P}; H)} \leq \|X_{t_1} - X_{t_2}\|_{L^{pq/\tau}(\mathbb{P}; H)} + \|X_{t_2} - Y_{t_2}\|_{L^{pq/\tau}(\mathbb{P}; H)}
$$

\[
\leq \sqrt{t_1 - t_2}\left|e^{\alpha t_1 T} \right| \left| \frac{pq}{p+q} \right| + V_0(X_0) + \int_0^T \frac{\beta_0}{e^{\alpha sT}} ds \left( \sqrt{T + \frac{pq}{p+q}} \right)
\]

\[+ \|X_0 - Y_0\|_{L^p(\mathbb{P}; H)} \exp \left( \int_0^T \phi(r) + \frac{\beta_0}{q e^{\alpha sT}} + \frac{\beta_1}{q e^{\gamma sT}} dr \right) \cdot \prod_{i=0}^1 \left( \mathbb{E} \left[ \exp \left( V_i(X_0) \right) \right] \right)^{1/2n_i}. \tag{72}
\]

This completes the proof of Lemma 3.8. \hfill \square

3.4 Strong completeness

In this subsection we derive conditions on the coefficients of an SDE which ensure strong completeness of the SDE. For this we first derive a version of the Kolmogorov-Chentsov theorem. More precisely, the following proposition, Proposition 3.9, provides a method which allows to obtain a continuous version of a mapping $X: D \times \Omega \to F$ from a subset $D$ of a finite-dimensional Hilbert space to a closed subset of a Banach space if there exist $p \in (\dim(H), \infty)$ and $\alpha \in \left(\frac{\dim(H)}{p}, 1\right]$ such that the mapping $D \ni x \mapsto X(x) \in L^p(\mathbb{P}; F)$ is locally bounded and locally $\alpha$-Hölder continuous. In the case where $F = E$, $H = \mathbb{R}^d$, and (73) below holds for $n = \infty$, the proof of Proposition 3.9 is provided in Theorem 2.1 in Mittmann & Steinwart [34]. Proposition 3.9 slightly generalizes Cox et al. [11, Theorem 3.5] and Grohs et al. [13, Lemma 2.19].

Proposition 3.9 (Existence of a continuous version). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(H, \langle \cdot, \cdot \rangle_H, \| \cdot \|_H)$ be a finite-dimensional $\mathbb{R}$-Hilbert space, let $D \subseteq H$ be a set, let $(E, \| \cdot \|_E)$ be a Banach space, let $F \subseteq E$ be a closed subset, let $p \in (\dim(H), \infty)$, $\alpha \in \left(\frac{\dim(H)}{p}, \infty\right)$, and let $X: D \times \Omega \to F$ be a random field which satisfies for all $n \in \mathbb{N}$ that

$$
\sup \left\{ \mathbb{E} \left[ \left\| X(x) \right\|_E^p : x \in D, \|x\|_H \leq n \right\} \cup \left\{ \mathbb{E} \left[ \left\| x - y \right\|_H^p : x, y \in D, \|x\|_H \leq n, \|y\|_H \leq n, x \neq y \right\} \cup \{0\} \right\} < \infty. \tag{73}
$$

Then there exists a function $\mathcal{X}: \overline{D} \times \Omega \to F$ which satisfies

(i) that $\mathcal{X}$ is $\mathcal{B}(\overline{D}) \otimes \mathcal{F} / \mathcal{B}(F)$-measurable,

(ii) that for all $\omega \in \Omega$ it holds that $(\overline{D} \ni x \mapsto \mathcal{X}(x, \omega) \in F) \in C(\overline{D}, F),

(iii) for all $n \in \mathbb{N}$, $\beta \in (0, \alpha - \frac{\dim(H)}{p})$ that $\mathcal{X}|_{\{x \in \overline{D}: \|x\|_H \leq n\}} \in \mathcal{L}^p(\mathbb{P}; C^\beta_b(\{x \in \overline{D}: \|x\|_H \leq n\}, F)),

(iv) that for all $x \in D$ it holds $\mathbb{P}$-a.s. that $\mathcal{X}(x) = X(x).

Proof of Proposition 3.9. Without loss of generality we assume that $D \neq \emptyset$ (otherwise the assertion is trivial) and that $H \neq \{0\}$ (if $H = \{0\}$, then $D = \{0\}$ and $X$ itself satisfies (i)-(iii)). Throughout this proof we choose for every $n \in \mathbb{N}$ that $D_n \subseteq D$ be the set which satisfies that $D_n = \{x \in D: \|x\|_H \leq n\}$, let $d \in \mathbb{N}$ satisfy that $d = \dim(H)$, let $\{h_1, \ldots, h_d\} \subseteq H$ be an orthonormal basis of $H$, and let $D \subseteq D$ be the set $D = \{x \in D: \langle (x, h_i)_H \rangle_{i=1}^d \in \mathbb{Q}^d\}$. By assumption it holds for all $n \in \mathbb{N}$ that $X|_{D_n} \in C^\beta_b(D_n, \mathcal{L}^p(\mathbb{P}; F))$. Then Theorem 3.5 in
Cox et al. [10] shows that for all \( n \in \mathbb{N} \) there exists \( X^n \in \bigcap_{\beta \in (0, \alpha - \frac{1}{2})} L^p(\mathbb{P}; C^0_b([\tau, \tau] + F)) \) such that for every \( x \in D_n \) it holds \( \mathbb{P}\text{-a.s.} \) that \( X^n(x) = X(x) \). Let \( \Omega_0 = \cdots \Omega \) be the set satisfying that \( \Omega_0 = \bigcap_{n \in \mathbb{N}} \bigcap_{x \in D_n \cap D} \bigcap_{\beta \in \mathbb{N}[n, \infty)} \{ \mathcal{X}^n(x) = X(x) \} \). Then, this, the fact that \( X, (\mathcal{X}^n)_{n \in \mathbb{N}} \) are random fields, and the fact that \( \mathbb{N} \times D \times \mathbb{N} \) is a countable set imply that \( \Omega_0 \in \mathcal{F} \) and that \( \mathbb{P}(\Omega_0) = 1 \). Continuity yields that for all \( \omega \in \Omega_0, n, m \in \mathbb{N} \) with \( m \geq n \) it holds that \( \mathcal{X}^m(\omega)|_{D_n} = \mathcal{X}^n(\omega) \). Note that \( \mathcal{D} = \bigcup_{n=1}^\infty D_n \). Now let \( \mathcal{X} : \mathcal{D} \times \Omega \to F \) be the function satisfying for all \( x \in \mathcal{D}, \omega \in \Omega \) that \( \mathcal{X}(x, \omega) = \mathcal{X}_n(\omega) \lim_{n \to \infty} \mathcal{X}^n(x, \omega) \). Then it holds for all \( n \in \mathbb{N} \) that \( \mathcal{X}_n|_{\mathcal{D}_n \times \Omega} = \mathcal{X}_n \in \bigcap_{\beta \in (0, \alpha - \frac{1}{2})} L^p(\mathbb{P}; C^0_b([\tau, \tau] + F)) \) and that for all \( x \in D \) it holds \( \mathbb{P}\text{-a.s.} \) that \( \mathcal{X}(x) = X(x) \). The fact that \( \mathcal{D} = \bigcup_{n=1}^\infty D_n \) finally yields for all \( \omega \in \Omega \) that \( \mathcal{X}(\omega) \in C(\mathcal{D}, F) \). Path continuity also implies that \( \mathcal{X} \) is \( \mathcal{B}(\mathcal{D}) \otimes \mathcal{F}/\mathcal{B}(F) \)-measurable. The proof of Proposition 3.7 is thus completed.

We emphasize that strong completeness may fail to hold even in the case of smooth and globally bounded coefficients; see Li & Scheutzow [32]. The following theorem, Theorem 3.10 essentially generalizes the results in [10, 13, 31, 40, 42, 46].

**Theorem 3.10 (Strong completeness).** Assume Setting 3.5, assume that \( \tau = T \) and that \( V_0, V_1 \) are bounded on every bounded subset of \( \mathcal{O} \), let \( \gamma \in (0, \infty) \), \( c \in [0, \infty) \) assume that for all \( t \in [0, T] \), \( x \in \mathcal{O} \) it holds that

\[
\max \{ ||\mu(t, x)||_H, ||\sigma(t, x)||_{H_{SU(H)}(T)} \} \leq c(1 + V_0(x))^\gamma,
\]

assume that \( \dim(H) < \infty \), let \( X^x : [0, T] \times \Omega \to \mathcal{O}, x \in \mathcal{O} \), be adapted stochastic processes with continuous sample paths satisfying that for all \( t \in [0, T] \), \( x \in \mathcal{O} \) it holds \( \mathbb{P}\text{-a.s.} \) that

\[
X^x_t = x + \int_0^t \mu(r, X^x_r) \, dr + \int_0^t \sigma(r, X^x_r) \, dW_r,
\]

and assume that

\[
\frac{p_1 q_1}{p_1 + q_1} \in (\dim(H), \infty) \cap \left[ \frac{1}{c}, \infty \right).
\]

Then there exists a function \( \mathcal{X} : [0, T] \times \mathcal{O} \to \mathcal{O} \) such that

(i) \( \mathcal{X} \) is \( \mathcal{B}([0, T] \times \mathcal{O}) \otimes \mathcal{F}/\mathcal{B}(\mathcal{O}) \)-measurable,

(ii) it holds for every \( \omega \in \Omega \) that \( ([0, T] \times \mathcal{O} ) \ni (t, x) \mapsto \mathcal{X}_t(\omega) \in \mathcal{O} \) in \( C([0, T] \times \mathcal{O}, \mathcal{O}) \), and

(iii) for all \( x \in \mathcal{O} \) it holds \( \mathbb{P}\text{-a.s.} \) that \( (\mathcal{X}^x_t)_{t \in [0, T]} = (X^x_t)_{t \in [0, T]} \).

**Proof of Theorem 3.10**. Throughout this proof let \( \delta \in (0, 1) \) satisfy \( \frac{p q_1 q_1}{p_1 + q_1} \in (\dim(H), \infty) \) and let \( D_n \subseteq H, n \in \mathbb{N} \), be the sets which satisfy for all \( n \in \mathbb{N} \) that \( D_n = \{ v \in \mathcal{O} : ||v||_H \leq n \} \). The triangle inequality, Lemma 3.7 (applied for every \( x \in \mathcal{O}, n \in \mathbb{N} \) with \( s = 0 \) in the notation of Lemma 3.7), and boundedness of \( V_0 \) on the bounded subsets \( D_n, n \in \mathbb{N} \), of \( \mathcal{O} \) show for all \( n \in \mathbb{N} \) with \( D_n \neq \emptyset \) that

\[
\sup_{x \in D_n} \sup_{t \in [0, T]} \| X^x_t \|_H \leq \sup_{x \in D_n} \| X^x_t - x \|_H \| X^x_t \|_H + \sup_{x \in D_n} \| x \|_H \leq \sqrt{T e^{\alpha n^\gamma}} \left( \frac{p_1 q_1}{p_1 + q_1} \gamma + \sup_{y \in D_n} V_0(y) \right) \left( \sqrt{T} + \frac{p_1 q_1}{p_1 + q_1} \right) n < \infty.
\]

Moreover, item (ii) in Lemma 3.7 and boundedness of \( V_0 \) on the bounded subsets \( D_n, n \in \mathbb{N} \), of \( \mathcal{O} \) imply for all \( n \in \mathbb{N} \) with \#\( D_n \in [2, \infty) \) that

\[
\sup_{x_1, x_2 \in D_n} \sup_{x_1 \neq x_2} \| X^{x_1}_t - X^{x_2}_t \|_H \leq \left( \frac{1}{2} \int_0^T \frac{\phi(s)}{(s+1)^2} \, ds + 1 \right)^{\frac{1}{2}} \left( \frac{1}{2} \int_0^T \frac{\phi(s)}{(s+1)^2} \, ds + 1 \right)^{\frac{1}{2}} \cdot \exp \left( \int_0^T \phi(r) + \frac{\beta_0}{\eta_0 r^{\eta_0}} + \frac{\beta_1}{\eta_1 r^{\eta_1}} \, dr + \sum_{i=0}^{\#D_n} \sup_{y \in D_n} V_{0}(y) + \sum_{i=0}^{\#D_n} \sup_{y \in D_n} V_{0}(y) \right) < \infty.
\]
Proposition 3.9 (applied with $H = H$, $D = O$, $E = C([0, T], H)$, $F = C([0, T], \Omega)$, $p = \frac{\nu \delta}{\rho \gamma + q}$, $\alpha = 1$ in the notation of Proposition 3.9) finally yields the assertion. This completes the proof of Theorem 3.10

\[ \square \]

3.5 Perturbation estimates for SDEs

Many problems can be formulated as perturbations of SDEs, e.g.: time discretizations of SDEs, spatial discretizations of stochastic partial differential equations (SPDEs), or small noise approximations of ordinary differential equations. We follow here the principal perturbation approach of Hutzenthaler & Jentzen [23]. The following corollary, Corollary 3.11, applies Theorem 2.4 to derive a suitable perturbation estimate. The marginal perturbation estimate (79) below is a minor improvement of Hutzenthaler & Jentzen [23, Theorem 1.2]. To the best of our knowledge, the uniform perturbation estimate (80) is new.

Corollary 3.11 (Perturbation estimate for SDEs). Assume Setting 3.1, let $a: [0, T] \times \Omega \to H$ and $b: [0, T] \times \Omega \to \text{HS}(U, H)$ be $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable and adapted stochastic processes which satisfy $\mathbb{P}$-a.s. that $\int_0^T \|a_s\|_H + \|b_s\|^2_{\text{HS}(U, H)} ds < \infty$ and which satisfy that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

\[ Y_{\min(t, \tau)} = Y_0 + \int_0^t 1_{[0, \tau]}(s) a_s ds + \int_0^t 1_{[0, \tau]}(s) b_s dW_s, \quad (78) \]

let $p \in [2, \infty)$, $\varepsilon \in (0, \infty)$, and assume that $\mathbb{P}$-a.s. it holds that $\int_0^T \max \{ \|X_s - Y_s, \mu(s, X_s) - \mu(s, Y_s)\|_H + \frac{1 + \varepsilon}{2} \|\sigma(s, X_s) - \sigma(s, Y_s)\|^2_{\text{HS}(U, H)} + \frac{(p-2)(1+\varepsilon)}{2} \|X_s - Y_s, \sigma(s, X_s) - \sigma(s, Y_s)\|_{\text{HS}(U, U)} \} ds < \infty$. Then

(i) it holds for all $q_1, q_2 \in (0, \infty]$ with $\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{p}$ that

\[ \|X_T - Y_T\|_{L_{q_1}(\mathbb{P}; H)} \leq \left( \|X_0 - Y_0\|_{L_{p}(\mathbb{P}; H)}^2 + 2 \int_0^T \|1_{[0, \tau]}(t) (\delta \|\mu(t, Y_t) - a_t\|_H^p + \frac{p-1}{2} (1 + \varepsilon) \|\sigma(t, Y_t) - b_t\|^2_{\text{HS}(U, H)} \right) \|_{L_{q_1}(\mathbb{P}; \mathbb{R})} dt \right)^{\frac{1}{p}} \]

and

\[ \exp \left( \int_0^T \max \left\{ \frac{(X_t - Y_t, \mu(t, X_t) - \mu(t, Y_t))_H + \frac{1 + \varepsilon}{2} \|\sigma(t, X_t) - \sigma(t, Y_t)\|^2_{\text{HS}(U, H)}}{\|X_t - Y_t\|_H^p} + \frac{(p-2)(1+\varepsilon)}{2} \|X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)\|_{\text{HS}(U, U)} \} dt \left\|_{L_{q_2}(\mathbb{P}; \mathbb{R})} \geq \frac{1}{4\delta}, 0 \right\} \right) \right) \]
Moreover, it holds for all \( q_1, q_2, q_3 \in (0, \infty) \) with \( q_3 < p \) and \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \) that

\[
\sup_{s \in [0, \tau]} \left\| X_s - Y_s \right\|_{H}^{q_3} \leq \left( \left\| X_0 - Y_0 \right\|_{H}^2 + 2 \int_0^\tau \left| \delta \| \mu(t, Y_t) - \alpha_t \|_{H}^{q_3} + \frac{p-2}{2} \left( 1 + \frac{1}{\epsilon} \right) \| \sigma(t, Y_t) - b_t \|_{HS(U,H)}^2 \right) dt \right) \frac{p}{q_3} \]

Proof of Corollary 3.11 Without loss of generality we assume that

\[ P(\left\{ t \in [0, T] \mid \| \sigma(t, X_t) - \sigma(t, Y_t) \|_{HS(U,H)}^2 + \frac{1}{\epsilon} \| \sigma(t, Y_t) - b_t \|_{HS(U,H)}^2 + \delta \mu(t, Y_t) - \alpha_t \|_{H}^2 dt < \infty \} \cap \{ P \text{-a.s.} \}) = 1 \]

(81)

(otherwise the assertion is trivial). First, (42) and (78) imply that for all \( t \in [0, T] \) it holds \( P \text{-a.s.} \) that

\[ X_{\min(t, \tau)} - Y_{\min(t, \tau)} = \int_0^t 1_{[0, \tau]}(s) (\mu(s, X_s) - a_s) ds + \int_0^t 1_{[0, \tau]}(s) (\sigma(s, X_s) - b_s) dW_s. \]

Moreover, it holds for all \( t \in [0, T] \) that

\[
\frac{(X_t - Y_t, \mu(t, X_t) - \alpha_t)_H + \frac{1}{2} \| \sigma(t, X_t) - b_t \|_{HS(U,H)}^2 + \frac{p-2}{2} \left( \frac{p}{q_3} \right) \frac{\| (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) \|_{HS(U,H)}^2}{\| X_t - Y_t \|_H^2}}{\| X_t - Y_t \|_H^2} \leq \frac{\frac{(X_t - Y_t, \mu(t, X_t) - \alpha_t)_H + \frac{1}{2} \| \sigma(t, X_t) - b_t \|_{HS(U,H)}^2 + \frac{p-2}{2} \left( \frac{p}{q_3} \right) \frac{\| (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) \|_{HS(U,H)}^2}{\| X_t - Y_t \|_H^2}}{\| X_t - Y_t \|_H^2}} \leq \left( \frac{p}{q_3} \right) \frac{\frac{(X_t - Y_t, \mu(t, X_t) - \alpha_t)_H + \frac{1}{2} \| \sigma(t, X_t) - b_t \|_{HS(U,H)}^2 + \frac{p-2}{2} \left( \frac{p}{q_3} \right) \frac{\| (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) \|_{HS(U,H)}^2}{\| X_t - Y_t \|_H^2}}{\| X_t - Y_t \|_H^2}} \leq \frac{\frac{(X_t - Y_t, \mu(t, X_t) - \alpha_t)_H + \frac{1}{2} \| \sigma(t, X_t) - b_t \|_{HS(U,H)}^2 + \frac{p-2}{2} \left( \frac{p}{q_3} \right) \frac{\| (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) \|_{HS(U,H)}^2}{\| X_t - Y_t \|_H^2}}{\| X_t - Y_t \|_H^2}} + \frac{1}{48} \right)
\]

(84)

This, (82), (81), and Corollary 2.5 (applied with \( X_t = X_t - Y_t, \alpha_t = \mu(t, X_t) - \alpha_t, b_t = \sigma(t, X_t) - b_t, \))

\[
\alpha_t = \max \left\{ \frac{(X_t - Y_t, \mu(t, X_t) - \alpha_t)_H + \frac{1}{2} \| \sigma(t, X_t) - b_t \|_{HS(U,H)}^2 + \frac{p-2}{2} \left( \frac{p}{q_3} \right) \frac{\| (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) \|_{HS(U,H)}^2}{\| X_t - Y_t \|_H^2}}{\| X_t - Y_t \|_H^2}, \frac{1}{48}, 0 \right\}
\]

(85)
\[ \beta_t = \sqrt{2}(\|\mu(t, Y_t) - a_t\|_H^2 + \frac{p-1}{2}(1 + \frac{1}{q_t})\|\sigma(t, Y_t) - b_t\|_{HS(U, H)}^2)^{\frac{1}{2}} \text{ for all } t \in [0, T] \] in the notation of Corollary \[23\] imply that it holds for all \(q_1, q_2 \in (0, \infty), \delta \in (0, \infty)\) with \(\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}\) that

\[ \|X_t - Y_t\|_{L^2(P; H)} \leq \left( \|X_0 - Y_0\|_{L^p(P; H)}^2 + 2\int_0^T \|\mu(t, Y_t) - a_t\|_H^2 + \frac{p-1}{2}(1 + \frac{1}{q_t})\|\sigma(t, Y_t) - b_t\|_{HS(U, H)}^2 \, dt \right)^{\frac{1}{2}} \]

(86)

and it holds for all \(q_1, q_2, q_3 \in (0, \infty)\) with \(q_3 < p\) and \(\frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3}\) that

\[ \left\| \sup_{s \in [0, T]} \|X_s - Y_s\|_H \right\|_{L^{q_1}(P; \mathbb{R})} \leq \left\| \left( \|X_0 - Y_0\|_H^2 + 2\int_0^T \|\mu(t, Y_t) - a_t\|_H^2 + \frac{p-1}{2}(1 + \frac{1}{q_t})\|\sigma(t, Y_t) - b_t\|_{HS(U, H)}^2 \, dt \right)^{\frac{1}{2}} \right\|_{L^{q_2}(P; \mathbb{R})} \]

(87)

This completes the proof of Corollary 3.11. \( \square \)

For example, Corollary 3.11 can be applied to prove strong convergence rates for implementable approximations of solutions of SDEs. The classical (exponential) Euler approximations diverge in the strong and weak sense for most one-dimensional SDEs with super-linearly growing coefficients (see \[23, 27\]) and also for some SPDEs (see Becari et al. \[2\]). It was shown in \[26, 24\] that minor modifications of the Euler method – so called tamed Euler methods – avoid this divergence problem; see also the Euler-type methods, e.g., in \[1, 5, 6, 8, 9, 12, 16, 19, 21, 29, 30, 33, 35, 38, 39, 44, 45, 47, 48\]. Now, analogously to Hutzenthaler & Jentzen \[23\], Corollary 3.11 is a powerful tool to establish uniform strong convergence rates (in combination with exponential moment estimates for suitably tamed Euler approximations, e.g., Hutzenthaler et al. \[28\]).

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