Quantum Geometry and Wild embeddings as quantum states

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In this paper we discuss wild embeddings like Alexanders horned ball and relate them to fractal spaces. We build a $C^\ast$-algebra corresponding to a wild embedding. We argue that a wild embedding is the result of a quantization process applied to a tame embedding. Therefore quantum states are directly the wild embeddings. Then we give an example of a wild embedding in the 4-dimensional spacetime. We discuss the consequences for cosmology.

Keywords: wild embeddings, Alexanders horned sphere, $C^\ast$ algebras, deformation quantization

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1. Introduction

General relativity (GR) has changed our understanding of spacetime. In parallel, the appearance of quantum field theory (QFT) has modified our view of particles, fields and the measurement process. The usual approach for the unification of QFT and GR, to a quantum gravity, starts with a proposal to quantize GR and its underlying structure, spacetime. There is an unique opinion in the community about the relation between geometry and quantum theory: The geometry as used in GR is classical and should emerge from a quantum gravity in the limit (Planck’s constant tends to zero). Most theories went a step further and try to get a spacetime from quantum theory. But what happens if this prerequisite is wrong? Is it possible to derive the quantization procedure from the structure of space and time? What is the geometrical representation of a quantum state? This paper try to answer partly these questions.

All approaches of quantum gravity have problems with the construction of the state space. If we assume that the spacetime has the right properties for a spacetime picture of quantum gravity then the quantum state must be part of the spacetime or must be geometrically realized in the spacetime. Consider (as in geometrodynamics) a 3-sphere $S^3$ with metric $g$. This metric (as state of GR) is modeled on $S^3$ at every 3-dimensional subspace. If $g$ is a metric of a homogeneous space then one can choose a small coordinate patch. But if $g$ is inhomogeneous then one can use a diffeomorphism to ”concentrate” the inhomogeneity at a chart. Now one combines these infinite charts (we consider only metrics up to diffeomorphisms) into a 3-sphere but without destroying the infinite charts by a diffeomorphism. Then we obtain a model of a quantum state. But as we argue in this paper, wild embeddings are the right structure to realize this idea. A wild embedding cannot be undone by a diffeomorphism of the embedding space. But it is non-trivial and (in most cases) determined by its complement. If we assume that wild embeddings are the quantum states then we must obtain the wild embedding by a quantization process. In this paper we will construct a $C^*$-algebra and its enveloping von Neumann algebra in the next section. Then we will show that closed curves (used to construct the $C^*$-algebra) are the observables forming a Poisson algebra for a tame embedding. The deformation quantization (so-called Drinfeld-Turaev-quantization) of this Poisson algebra leads to skein spaces used in knot theory. For the example of Alexander’s horned ball we are able to show that the corresponding skein space and the enveloping von Neumann algebra are the same. Therefore the wild embedding (Alexander’s horned ball) can be seen as a quantization of a tame embedding, i.e. it is a quantum state. Wild embeddings occur generically in non-compact 4-manifolds with an exotic smoothness structure like $S^3 \times \mathbb{R}$. In the last section we discuss also the cosmological consequences of this approach.
2. From wild embeddings to fractal spaces

In this section we define wild and tame embeddings and construct a $C^*$–algebra associated to a wild embedding. The example of Alexanders horned ball is discussed.

2.1. Wild and tame embeddings

We call a map $f: N \to M$ between two topological manifolds an embedding if $N$ and $f(N) \subset M$ are homeomorphic to each other. From the differential-topological point of view, an embedding is a map $f: N \to M$ with injective differential on each point (an immersion) and $N$ is diffeomorphic to $f(N) \subset M$. An embedding $i: N \hookrightarrow M$ is tame if $i(N)$ is represented by a finite polyhedron homeomorphic to $N$. Otherwise we call the embedding wild. There are famous wild embeddings like Alexanders horned sphere or Antoine’s necklace. In physics one uses mostly tame embeddings but as Cannon mentioned in his overview [8], one needs wild embeddings to understand the tame one. As shown by us [4], wild embeddings are needed to understand exotic smoothness. As explained in [8] by Cannon, tameness is strongly connected to another topic: decomposition theory (see the book [11]).

Two embeddings $f, g: N \to M$ are said to be isotopic, if there exists a homeomorphism $F: M \times [0,1] \to M \times [0,1]$ such that

1. $F(y,0) = (y,0)$ for each $y \in M$ (i.e. $F(\cdot ,0) = id_M$)
2. $F(f(x),1) = g(x)$ for each $x \in N$, and
3. $F(M \times \{t\}) = M \times \{t\}$ for each $t \in [0,1]$.

If only the first two conditions can be fulfilled then one call it concordance. Embeddings are usually classified by isotopy. An important example is the embedding $S^1 \to \mathbb{R}^3$, known as knot, where different knots are different isotopy classes of the embedding $S^1 \to \mathbb{R}^3$.

2.2. Wild embeddings and perfect groups

Wild embeddings are important to understand usual embeddings. Consider a closed curve in the plane. This curve divides the plane into an interior and an exterior area (by the Jordan curve theorem). But what about one dimension higher, i.e. consider the embedding $S^2 \to \mathbb{R}^3$? Alexander was the first who constructed a counterexample to a generalized Jordan curve theorem, Alexanders horned sphere [1], as wild embedding $I: D^3 \to \mathbb{R}^3$. The main property of this wild object $D^3_W = I(D^3)$ is the non-simple connected complement $\mathbb{R}^3 \setminus D^3_W$. This property is a crucial point of the following discussion. Given an embedding $I: D^3 \to \mathbb{R}^3$ which induces a decomposition $\mathbb{R}^3 = I(D^3) \cup (\mathbb{R}^3 \setminus I(D^3))$. In case, the embedding is tame, the image $I(D^3)$ is given by a finite complex and every part of the decomposition is contractable, i.e. especially $\pi_1(\mathbb{R}^3 \setminus I(D^3)) = 0$. For a wild embedding, $I(D^3)$ is an infinite complex (but contractable). The complement $\mathbb{R}^3 \setminus I(D^3)$ is given by a sequence of spaces so that $\mathbb{R}^3 \setminus I(D^3)$ is non-simple connected (otherwise the
embedding must be tame). If \( \mathbb{R}^3 \setminus I(D^3) \) has the homology of a point (that is true for every embedding) then \( \pi_1(\mathbb{R}^3 \setminus I(D^3)) \) is non-trivial whereas its abelianization \( H_1(\mathbb{R}^3 \setminus I(D^3)) = 0 \) vanishes. Therefore \( \pi_1 \) is generated by the commutator subgroup \([\pi_1, \pi_1] \) with \([a, b] = aba^{-1}b^{-1} \) for two elements \( a, b \in \pi_1 \), i.e. \( \pi_1 \) is a perfect group.

As a warm up we will consider wild embeddings of spheres \( S^n \) into spheres \( S^m \) equivalent to embeddings of \( \mathbb{R}^n \) into \( \mathbb{R}^m \) relative to the infinity \( \infty \) point or to relative embeddings of \( D^n \) into \( D^m \) (relative to its boundary). From the physical point of view, we have the embedding of branes (seen as topological objects of a trivial type like \( \mathbb{R}^n, S^n \) or \( D^n \)) into the Euclidean space \( \mathbb{R}^m \). Let’s start with the case of a finite \( k \)-dimensional polyhedron \( K^k \) (i.e. a piecewise-linear version of a \( k \)-disk \( D^k \)). Consider the wild embedding \( i : K \to S^n \) with \( 0 \leq k \leq n - 3 \) and \( n \geq 7 \). Then, as shown in [12], the complement \( S^n \setminus i(K) \) is non-simple connected with a countable generated (but not finitely presented) fundamental group \( \pi_i(S^n \setminus i(K)) = \pi \). Furthermore, the group \( \pi \) is perfect (i.e. generated by the commutator subgroup \([\pi, \pi] = \pi \) implying \( H_1(\pi) = 0 \)) and \( H_2(\pi) = 0 \) (\( \pi \) is called a superperfect group). Again, \( \pi \) is a group where every element \( x \in \pi \) can be generated by a commutator \( x = [a, b] = aba^{-1}b^{-1} \) (including the trivial case \( x = a, b = e \)). By using geometric group theory, one can represent \( \pi \) by a grope (or generalized disk, see Cannon [9]), i.e. a hierarchical object with the same fundamental group as \( \pi \). This group is not finite in case of a wild embedding.

2.3. \( C^* \)-algebras associated to wild embeddings and idempotents

Let \( I : K^n \to \mathbb{R}^{n+k} \) be a wild embedding of codimension \( k \) with \( k = 0, 1, 2 \). In the following we assume that the complement \( \mathbb{R}^{n+k} \setminus I(K^n) \) is non-trivial, i.e. \( \pi_1(\mathbb{R}^{n+k} \setminus I(K^n)) = \pi \neq 1 \). Now we define the \( C^* \)-algebra \( C^*(\mathcal{G}, \pi) \) associated to the complement \( \mathcal{G} = \mathbb{R}^{n+k} \setminus I(K^n) \) with group \( \pi = \pi_1(\mathcal{G}) \). If \( \pi \) is non-trivial then this group is not finitely generated. The construction of wild embeddings is usually given by an infinite construction (see Antoine’s necklace or Alexanders horned sphere). From an abstract point of view, we have a decomposition of \( \mathcal{G} \) by an infinite union

\[
\mathcal{G} = \bigcup_{i=0}^{\infty} C_i
\]

of ’level sets’ \( C_i \). Then every element \( \gamma \in \pi \) lies (up to homotopy) in a finite union of levels.

The basic elements of the \( C^* \)-algebra \( C^*(\mathcal{G}, \pi) \) are smooth half-densities with compact supports on \( \mathcal{G} \), \( f \in C_c(\mathcal{G} ; \Omega^{1/2}) \), where \( \Omega^{1/2} \) for \( \gamma \in \pi \) is the one-dimensional complex vector space of maps from the exterior power \( \Lambda^k L \) (\( \dim L = k \)), of the union of levels \( L \) representing \( \gamma \), to \( \mathbb{C} \) such that

\[
\rho(\lambda \nu) = |\lambda|^{1/2} \rho(\nu) \quad \forall \nu \in \Lambda^2 L, \lambda \in \mathbb{R}.
\]

*This infinite construction is necessary to obtain an infinite polyhedron, the defining property of a wild embedding.*
For \( f, g \in C_c^\infty(\mathcal{G}, \Omega^{1/2}) \), the convolution product \( f * g \) is given by the equality

\[
(f * g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2)
\]

with the group operation \( \gamma_1 \circ \gamma_2 \) in \( \pi \). Then we define via \( f^*(\gamma) = f(\gamma^{-1}) \) a \(*\)operation making \( C_c^\infty(\mathcal{G}, \Omega^{1/2}) \) into a \(*\)algebra. Each level set \( C_i \) consists of simple pieces (in case of Alexanders horned sphere, we will explain it below) denoted by \( T \). For these pieces, one has a natural representation of \( C_c^\infty(\mathcal{G}, \Omega^{1/2}) \) on the \( L^2 \) space over \( T \). Then one defines the representation

\[
(\pi_x(f))\xi(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2) \quad \forall \xi \in L^2(T), \forall x \in \gamma.
\]

The completion of \( C_c^\infty(\mathcal{G}, \Omega^{1/2}) \) with respect to the norm

\[
\|f\| = \sup_{x \in \mathcal{G}} ||\pi_x(f)||
\]

makes it into a \(*\)algebra \( C_c^\infty(\mathcal{G}, \pi) \). Finally we are able to define the \(*\)—algebra associated to the wild embedding:

**Definition 1.** Let \( j : K \rightarrow S^n \) be a wild embedding with \( \pi = \pi_1(S^n \setminus j(K)) \) as fundamental group of the complement \( M(K, j) = S^n \setminus j(K) \). The \(*\)—algebra \( C_c^\infty(K, j) \) associated to the wild embedding is defined to be \( C_c^\infty(K, j) = C_c^\infty(\mathcal{G}, \pi) \) the \(*\)—algebra of the complement \( \mathcal{G} = S^n \setminus j(K) \) with group \( \pi \).

Among all elements of the \(*\) algebra, there are distinguished elements, idempotent operators or projectors having a geometric interpretation. For later use, we will construct them explicitly (we follow [10] sec. I.8.β closely). Let \( j(K) \subseteq S^n \) be the wild submanifold. A small tubular neighborhood \( N \) of \( j(K) \) in \( S^n \) defines the corresponding \(*\)algebra \( C_c^\infty(N', \pi_1(S^n \setminus N')) \) is isomorphic to \( C_c^\infty(\mathcal{G}, \pi_1(S^n \setminus j(K))) \otimes \mathcal{K} \) with \( \mathcal{K} \) the \(*\) algebra of compact operators. In particular it contains an idempotent \( e = e^2 = e^* \), \( e = 1_N \otimes f \in C_c^\infty(\mathcal{G}, \pi_1(S^n \setminus j(K))) \otimes \mathcal{K} \), where \( f \) is a minimal projection in \( \mathcal{K} \). It induces an idempotent in \( C_c^\infty(\mathcal{G}, \pi_1(S^n \setminus j(K))) \). By definition, this idempotent is given by a closed curve in the complement \( S^n \setminus j(K) \).

### 2.4. Example: Alexanders horned ball as fractal space

In this subsection we will construct Alexanders horned ball (originally described in [1]) as an example of a wild embedding \( D^3 \rightarrow S^3 \). The construction needs an infinite number of levels \( C_i \). The 0th level \( C_0 \) is a solid cylinder (or annulus) \( D^2 \times [0,1] \) with the two ends \( D^2 \times \{0\} \) and \( D^2 \times \{1\} \). For the first level \( C_1 \), consider a pair of \( Y = (D^2 \times [0,1]) \setminus D^2 \) (the thicken letter 'Y') with three ends \( D^2 \times \{0\}, D^2 \times \{1\} \) and \( D^2 \times \{1/2\} \). Now glue \( D^2 \times \{1/2\} \) of one \( Y \) to one end of \( C_0 \) and do the same for the other \( Y \). Then link the two \( Y \) (as shown in fig. [1] which is repainted from [1]) and repeat the procedure for every pair of ends \( D^2 \times \{0\} \) and \( D^2 \times \{1\} \) of \( Y \). As
explained above, the infinite union

\[ C = \bigcup_{i=0}^{\infty} C_i \]

of all levels is the image \( I(D^3) \) of the wild embedding \( I : D^3 \to S^3 \), known as Alexander’s horned ball \( A = I(D^3) \).

Wild embeddings are known to be fractals [17]. It is also possible to give an estimate for the fractal dimension for the boundary of Alexander’s horned ball, i.e. for a wildly embedded 2-sphere. First assume that the next level is twice smaller then the previous one (scale \( 1 : 2 = 1 : m \)). Secondly, in every of the three direction, one has two copies of the whole object, i.e. one obtains \( n = 6 \) disjoint objects. Then one obtains for the Hausdorff dimension

\[ D = \frac{\ln n}{\ln m} = \frac{\ln 6}{\ln 2} \approx 2.584925... \]

a greater value then the usual dimension of the 2-sphere. We remark that this simple calculation is only a rough estimate to demonstrate the fact that wild embeddings are fractals.

For an impression of Alexander’s horned ball \( A \), we have to consider the complement \( S^3 \setminus A \). Main instrument in this context is the fundamental group \( \pi_1(S^3 \setminus A) \)
Fig. 2. schematic picture for the first three levels

of the complement. It was constructed in [5] by using a schematic picture of \( A \) (the spine in modern notation), see Fig. 2 (repainted from [5]). Let \( \alpha \) be a finite sequence \( \alpha = \alpha_1 \alpha_2 \cdots \alpha_l \) of length \( l(\alpha) = l \) over the alphabet \( \alpha_i = 1, 2 \). Then the fundamental group is given by

\[
\pi_1(S^3 \setminus A) = \{ z_\alpha, l(\alpha) \geq 0 \mid z_\alpha = [z_{\alpha_1}, z_{\alpha_2}] \}
\]

with the generators \( z_\alpha \) by using the group commutator \([a, b] = aba^{-1}b^{-1}\). The group \( \pi_1(S^3 \setminus A) \) is a locally free group of infinite rank which is perfect. But the last property implies that this group has the infinite conjugacy class property (icc), i.e. only the identity element has a finite conjugacy class. This property has a tremendous impact on the \( C^* \)-algebra [10] and its enveloping von Neumann algebra:

The enveloping von Neumann algebra \( W(C, \pi_1(S^3 \setminus A)) \) of the \( C^* \)-algebra

\[
C_c^\infty(C, \pi_1(S^3 \setminus A))
\]

for the wild embedding \( A \) is the hyperfinite factor \( II_1 \) algebra.

3. Quantum states from wild embeddings

In this section we will describe a way from a (classical) Poisson algebra to a quantum algebra by using deformation quantization. Therefore we will obtain a positive answer to the question: Does the \( C^* \)-algebra of a wild (specific) embedding comes from a (deformation) quantization? Of course, this question cannot be answered in most generality, i.e. we use the example of Alexanders horned ball. But for this example we will show that the enveloping von Neumann algebra of this wild embedding (Alexanders horned ball) is the result of a deformation quantization using the classical Poisson algebra (of closed curves) of the tame embedding. This result shows two things: the wild embedding can be seen as a quantum state and the classical state is a tame embedding. We conjecture that this result can be generalized to all other wild embeddings.
3.1. **Embeddings and observables**

Let $i : K \rightarrow S^n$ be an embedding of a polyhedron $K$. If we have a fixed geometry of $S^n$ then we obtain also an induced geometry of $i(K)$ by using the embedding. Without loss of generality, we can assume a connection $\Gamma_{S^n}$ of constant curvature on $S^n$. By a pullback $\Gamma_K = i^* \Gamma_{S^n}$ we obtain a connection of $i(K) \subset S^n$. Using the Ambrose-Singer theorem we obtain the curvature via the holonomy

$$\oint \gamma \Gamma_K$$

along a closed curve $\gamma \in i(K)$. In case of a constant curvature, this integral depends only on the homotopy class of the curve $\gamma$. But also for every piece having a constant curvature, we can choose a closed curve (in this piece) up to homotopy. Therefore, the closed curve is an important ingredient to obtain the curvature.

Now we consider the complement $S^n \setminus i(K)$ together with the fundamental group $\pi = \pi_1(S^n \setminus i(K))$. An element of $\pi$ is a closed curve surrounding $i(K)$. The curve can be projected to $i(K)$. In most cases, the group $\pi$ is the trivial group and we obtain the algebra $C^\infty_c(S^n \setminus i(K), \pi) = \mathbb{C}$, the center of the enveloping von Neumann algebra. The generator of $\pi$ is a contractible closed curve used to determine the curvature (and therefore the geometry). So, if we have a fixed geometry of the embedding space (preferable constant curvature) then we can construct the curvature (up to diffeomorphisms) by choosing a closed curve. In this case, the closed is an observable. This concept agrees with the construction of an idempotent in the $C^*$-algebra (see subsection 2.3) seen as closed curve surrounding the complement of the wild embedding.

In the example of Alexanders horned ball, we also obtain a wild embedding of a 2-sphere (as boundary of the ball) into the 3-space. This case, the wild embedding of a surface into the 3-space, will be the generic case for the following. We will motivate the choice in section 4.

3.2. **Intermezzo 1: The observable algebra and its Poisson structure**

In this section we will describe the formal structure of a classical theory coming from the algebra of observables using the concept of a Poisson algebra. In quantum theory, an observable is represented by a hermitean operator having the spectral decomposition via projectors or idempotent operators. The coefficient of the projector is the eigenvalue of the observable or one possible result of a measurement. At least one of these projectors represent (via the GNS representation) a quasi-classical state. Thus to construct the substitute of a classical observable algebra with Poisson algebra structure we have to concentrate on the idempotents in the

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bThis approach is not completely new. The spin network of Loop quantum gravity is also an expression to obtain the geometry using the holonomy of a connection along the network.
$C^*$ algebra. Now we will see that the set of closed curves on a surface has the structure of a Poisson algebra. Let us start with the definition of a Poisson algebra.

**Definition 2.**

Let $P$ be a commutative algebra with unit over $\mathbb{R}$ or $\mathbb{C}$. A Poisson bracket on $P$ is a bilinear form $\{,\} : P \otimes P \to P$ fulfilling the following 3 conditions:

- anti-symmetry $\{a, b\} = -\{b, a\}$
- Jacobi identity $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$
- derivation $\{ab, c\} = a\{b, c\} + b\{a, c\}$.

Then a Poisson algebra is the algebra $(P, \{,\})$.

Now we consider a surface $S$ together with a closed curve $\gamma$. Additionally we have a Lie group $G$ given by the isometry group. The closed curve is one element of the fundamental group $\pi_1(S)$. From the theory of surfaces we know that $\pi_1(S)$ is a free abelian group. Denote by $Z$ the free $\mathbb{K}$-module ($\mathbb{K}$ a ring with unit) with the basis $\pi_1(S)$, i.e. $Z$ is a freely generated $\mathbb{K}$-module. Recall Goldman’s definition of the Lie bracket in $Z$ (see [14]). For a loop $\gamma : S^1 \to S$ we denote its class in $\pi_1(S)$ by $\langle \gamma \rangle$. Let $\alpha, \beta$ be two loops on $S$ lying in general position. Denote the (finite) set $\alpha(S^1) \cap \beta(S^1)$ by $\alpha \# \beta$. For $q \in \alpha \# \beta$ denote by $\epsilon(q; \alpha, \beta) = \pm 1$ the intersection index of $\alpha$ and $\beta$ in $q$. Denote by $\alpha_q \beta_q$ the product of the loops $\alpha, \beta$ based in $q$. Up to homotopy the loop $(\alpha_q \beta_q)(S^1)$ is obtained from $\alpha(S^1) \cup \beta(S^1)$ by the orientation preserving smoothing of the crossing in the point $q$. Set

$$[[\alpha], [\beta]] = \sum_{q \in \alpha \# \beta} \epsilon(q; \alpha, \beta)(\alpha_q \beta_q) \quad .$$

According to Goldman [14], Theorem 5.2, the bilinear pairing $[,,] : Z \times Z \to Z$ given by (1) on the generators is well defined and makes $Z$ to a Lie algebra. The algebra $\text{Sym}(Z)$ of symmetric tensors is then a Poisson algebra (see Turaev [21]).

The whole approach seems natural for the construction of the Lie algebra $Z$ but the introduction of the Poisson structure is an artificial act. From the physical point of view, the Poisson structure is not the essential part of classical mechanics. More important is the algebra of observables, i.e. functions over the configuration space forming the Poisson algebra. Thus we will look for the algebra of observables in our case. For that purpose, we will look at geometries over the surface. By the uniformization theorem of surfaces, there is three types of geometrical models: spherical $S^2$, Euclidean $\mathbb{E}^2$ and hyperbolic $\mathbb{H}^2$. Let $\mathcal{M}$ be one of these models having the isometry group $\text{Isom}(\mathcal{M})$. Consider a subgroup $H \subset \text{Isom}(\mathcal{M})$ of the isometry group acting freely on the model $\mathcal{M}$ forming the factor space $\mathcal{M}/H$. Then one obtains the usual (closed) surfaces $S^2, \mathbb{R}P^2, T^2$ and its connected sums like the surface of genus $g$ ($g > 1$). For the following construction we need a group $G$ containing the isometry groups of the three models. Furthermore the surface $S$ is part of a 3-manifold and for later use we have to demand that $G$ has to be also a isometry group of 3-manifolds. According to Thurston [19] there are 8 geometric models in dimension 3 and the largest isometry group is the hyperbolic...
group $PSL(2, \mathbb{C})$ isomorphic to the Lorentz group $SO(3, 1)$. It is known that every representation of $PSL(2, \mathbb{C})$ can be lifted to the spin group $SL(2, \mathbb{C})$. Thus the group $G$ fulfilling all conditions is identified with $SL(2, \mathbb{C})$. This choice fits very well with the 4-dimensional picture.

Now we introduce a principal $G$ bundle on $S$, representing a geometry on the surface. This bundle is induced from a $G$ bundle over $S \times [0, 1]$ having always a flat connection. Alternatively one can consider a homomorphism $\pi_1(S) \to G$ represented as holonomy functional

$$hol(\omega, \gamma) = \mathcal{P} \exp \left( \int_\gamma \omega \right) \in G \quad (2)$$

with the path ordering operator $\mathcal{P}$ and $\omega$ as flat connection (i.e. inducing a flat curvature $\Omega = d\omega + \omega \wedge \omega = 0$). This functional is unique up to conjugation induced by a gauge transformation of the connection. Thus we have to consider the conjugation classes of maps

$$hol : \pi_1(S) \to G$$

forming the space $X(S, G)$ of gauge-invariant flat connections of principal $G$ bundles over $S$. Now (see [18]) we can start with the construction of the Poisson structure on $X(S, G)$. The construction based on the Cartan form as the unique bilinear form of a Lie algebra. As discussed above we will use the Lie group $G = SL(2, \mathbb{C})$ but the whole procedure works for every other group too. Now we consider the standard basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $sl(2, \mathbb{C})$ with $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$. Furthermore there is the bilinearform $B : sl_2 \otimes sl_2 \to \mathbb{C}$ written in the standard basis as

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Now we consider the holomorphic function $f : SL(2, \mathbb{C}) \to \mathbb{C}$ and define the gradient $\delta f(A)$ along $f$ at the point $A$ as $\delta f(A) = Z$ with $B(Z, W) = df_A(W)$ and

$$df_A(W) = \frac{d}{dt} f(A \cdot \exp(tW)) \bigg|_{t=0}.$$

The calculation of the gradient $\delta tr$ for the trace $tr$ along a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$
is given by
\[ \delta_{tr}(A) = -a_{21}Y - a_{12}X - \frac{1}{2}(a_{11} - a_{22})H. \]

Given a representation \( \rho \in X(S, SL(2, \mathbb{C})) \) of the fundamental group and an invariant function \( f : SL(2, \mathbb{C}) \to \mathbb{R} \) extendable to \( X(S, SL(2, \mathbb{C})) \). Then we consider two conjugacy classes \( \gamma, \eta \in \pi_1(S) \) represented by two transversal intersecting loops \( P, Q \) and define the function \( f_{\gamma} : X(S, SL(2, \mathbb{C})) \to \mathbb{C} \) by \( f_{\gamma}(\rho) = f(\rho(\gamma)) \). Let \( x \in P \cap Q \) be the intersection point of the loops \( P, Q \) and \( c_x \) a path between the point \( x \) and the fixed base point in \( \pi_1(S) \). Then we define \( \gamma_x = c_x \gamma c_x^{-1} \) and \( \eta_x = c_x \eta c_x^{-1} \). Finally we get the Poisson bracket

\[ \{ f_{\gamma}, f'_{\eta} \} = \sum_{x \in P \cap Q} \text{sign}(x) B(\delta f(\rho(\gamma_x)), \delta f'(\rho(\eta_x))) \]

where \( \text{sign}(x) \) is the sign of the intersection point \( x \). Thus,

The space \( X(S, SL(2, \mathbb{C})) \) has a natural Poisson structure (induced by the bilinear form \( \{ , \} \) on the group) and the Poisson algebra \( (X(S, SL(2, \mathbb{C})), \{ , \}) \) of complex functions over them is the algebra of observables.

3.3. Intermezzo 2: Drinfeld-Turaev Quantization

Now we introduce the ring \( \mathbb{C}[[h]] \) of formal polynomials in \( h \) with values in \( \mathbb{C} \). This ring has a topological structure, i.e. for a given power series \( a \in \mathbb{C}[[h]] \) the set \( a + h^n \mathbb{C}[[h]] \) forms a neighborhood. Now we define

**Definition 3.**

A Quantization of a Poisson algebra \( P \) is a \( \mathbb{C}[[h]] \) algebra \( P_h \) together with the \( \mathbb{C} \)-algebra isomorphism \( \Theta : P_h/hP \to P \) so that

1. the module \( P_h \) is isomorphic to \( V[[h]] \) for a \( \mathbb{C} \) vector space \( V \)
2. let \( a, b \in P \) and \( a', b' \in P_h \) be \( \Theta(a) = a', \Theta(b) = b' \) then

\[ \Theta \left( \frac{a'b' - b'a'}{h} \right) = \{a, b\} \]

One speaks of a deformation of the Poisson algebra by using a deformation parameter \( h \) to get a relation between the Poisson bracket and the commutator. Therefore we have the problem to find the deformation of the Poisson algebra \( (X(S, SL(2, \mathbb{C})), \{ , \}) \). The solution to this problem can be found via two steps:

(1) at first find another description of the Poisson algebra by a structure with one parameter at a special value and

(2) secondly vary this parameter to get the deformation.

Fortunately both problems were already solved (see [20,21]). The solution of the first problem is expressed in the theorem:

The Skein module \( K_{-1}(S \times [0, 1]) \) (i.e. \( t = -1 \)) has the structure of an algebra isomorphic to the Poisson algebra \( (X(S, SL(2, \mathbb{C})), \{ , \}) \). (see also [7,6])
Then we have also the solution of the second problem:

The skein algebra $K_t(S \times [0,1])$ is the quantization of the Poisson algebra $(X(S, SL(2, \mathbb{C})), \{ , \})$ with the deformation parameter $t = \exp(h/4)$. (see also [7])

To understand these solutions we have to introduce the skein module $K_t(M)$ of a 3-manifold $M$ (see [16]). For that purpose we consider the set of links $\mathcal{L}(M)$ in $M$ up to isotopy and construct the vector space $\mathbb{C}\mathcal{L}(M)$ with basis $\mathcal{L}(M)$. Then one can define $\mathbb{C}\mathcal{L}[[t]]$ as ring of formal polynomials having coefficients in $\mathbb{C}\mathcal{L}(M)$. Now we consider the link diagram of a link, i.e. the projection of the link to the $\mathbb{R}^2$ having the crossings in mind. Choosing a disk in $\mathbb{R}^2$ so that one crossing is inside this disk. If the three links differ by the three crossings $L_{oo}, L_o, L_{\infty}$ (see figure 3) inside of the disk then these links are skein related. Then in $\mathbb{C}\mathcal{L}[[t]]$ one writes the skein relation $L_{\infty} - tL_o - t^{-1}L_{oo}$. Furthermore let $L \sqcup O$ be the disjoint union of the link with a circle then one writes the framing relation $L \sqcup O + (t^2 + t^{-2})L$.

Let $S(M)$ be the smallest submodul of $\mathbb{C}\mathcal{L}[[t]]$ containing both relations, then we define the Kauffman bracket skein module by $K_t(M) = \mathbb{C}\mathcal{L}[[t]]/S(M)$. We list the following general results about this module:

- The module $K_{-1}(M)$ for $t = -1$ is a commutative algebra.
- Let $S$ be a surface then $K_t(S \times [0,1])$ caries the structure of an algebra.

The algebra structure of $K_t(S \times [0,1])$ can be simple seen by using the diffeomorphism between the sum $S \times [0,1] \cup S \times [0,1]$ along $S$ and $S \times [0,1]$. Then the product $ab$ of two elements $a, b \in K_t(S \times [0,1])$ is a link in $S \times [0,1] \cup S \times [0,1]$ corresponding to a link in $S \times [0,1]$ via the diffeomorphism. The algebra $K_t(S \times [0,1])$ is in general non-commutative for $t \neq -1$. For the following we will omit the interval $[0,1]$ and denote the skein algebra by $K_t(S)$.

\[\text{The relation depends on the group } SL(2, \mathbb{C}).\]
3.4. Temperley-Lieb algebra and Alexanders horned ball

Now we will present the relation between skein spaces and wild embeddings (in particular to its $C^*$–algebra). For that purpose we will concentrate on the wild embedding $i : D^3 \to S^3$ or equivalently $i : D^2 \times [0,1] \to S^3$ of Alexanders horned ball. We will explain now, that the complement $S^3 \setminus i(D^2 \times [0,1])$ and its fundamental group $\pi_1 \left( S^3 \setminus i(D^2 \times [0,1]) \right)$ can be described by closed curves around tubes (or annulus) $S^1 \times [0,1]$.

Let $C$ be the image $C = i(D^2 \times [0,1])$ decomposed into components $C_i$ so that $C = \bigcup_i C_i$. Furthermore, let $C_i$ be the decomposition of $i(D^2 \times [0,1])$ at ith level (i.e. a union of $D^2 \times [0,1]$). The complement $S^3 \setminus C_i$ of $C_i$ with $n_i$ components (i.e. $C_i = \bigcup_i^{n_i} (D^2 \times [0,1])$) has the same (isomorphic) fundamental group like $\pi_1 \left( \bigcup_i^{n_i} (S^1 \times [0,1]) \right)$ of $n_i$ components of $S^1 \times [0,1]$. Therefore, instead of studying the complement we can directly consider the annulus $S^1 \times [0,1]$ replacing every $D^2 \times [0,1]$ component.

Let $C'$ be the boundary of $C$, i.e. in every component we have to replace every $D^2 \times [0,1]$ by $S^1 \times [0,1]$. The skein space $K_i(S^1 \times [0,1])$ is a polynomial algebra (see the previous subsection) $\mathbb{C}[\alpha]$ in one generator $\alpha$ (a closed curve around the annulus). Let $TL_n$ be the Temperley-Lieb algebra, i.e. a complex $*$–algebra generated by $\{e_1, \ldots, e_n\}$ with the relations

$$
e_i^2 = \tau e_i, \ e_i e_j = e_j e_i : |i - j| > 1, \\
e_i e_{i+1} e_i = e_i, \ e_{i+1} e_i e_{i+1} = e_{i+1}, \ e_i^* = e_i$$

and the real number $\tau$. If $\tau$ is the number $\tau = a_0^2 + a_0^{-2}$ with $a_0$ a 4th root of unity ($a_0^{4k} \neq 1$ for $k = 1, \ldots, n - 1$) then there is an element $f^{(n)}$ with

$$f^{(n)} A_n = A_n f^{(n)} = 0$$
$$1_n - f^{(n)} \in A_n$$
$$f^{(n)} f^{(n)} = f^{(n)}$$

in $A_n \subset TL_n$ (a subalgebra generated of $\{e_1, \ldots, e_n\}$ missing the identity $1_n$), called the Jones-Wenzl idempotent. The closure of the element $f^{(n+1)} \in TL_{n+1}$ in $K_i(S^1 \times [0,1])$ is given by the image of the map $TL_{n+1} \to K_i(S^1 \times [0,1])$ which maps $f^{(n+1)}$ to some polynomial $S_{n+1}(\alpha)$ in the generator $\alpha$ of $K_i(S^1 \times [0,1])$.

Therefore we obtain a relation between the generator $\alpha$ and the element $f^{(n)}$ for some $n$.

Alexanders horned ball is homeomorphic to $D^3 = D^2 \times [0,1]$ (by definition of an embedding). The wilderness is given by a decomposition of $D^2 \times [0,1]$ into an infinite union of $D^2 \times [0,1]$–components $C_i$ (in the notation above). By the calculation in subsection 2.4 we have an infinite fundamental group where every generator is represented by a curve around one $(D^2 \times [0,1])$–components $C_i$. This decomposition can be represented by a decomposition of a square (as substitute for $D^2$) into (countable) infinite rectangles (see Fig. 3b). Every closed curve surrounding $C_i$ is a pair of opposite points at the boundary (see Fig. 3b), the starting point of the curve and one passing point (to identify the component). Every $C_i$ gives one
pair of points. Motivated by the discussion above, we consider the skein algebra $K_t(D^2, 2n)$ with $2n$ marked points (representing $n$ components). This algebra is isomorphic (see [16]) to the Temperley-Lieb algebra $TL_n$. As Jones [15] showed: the limit case $\lim_{n \to \infty} TL_n$ (considered as direct limit) is the factor $II_1$. Thus we have constructed the factor $II_1$ algebra as skein algebra.

Therefore we have shown that the enveloping von Neumann algebra

$$W(C, \pi_1(S^3 \setminus A))$$

(=the hyperfinite factor $II_1$ algebra) is obtained by deformation quantization of a classical Poisson algebra (the tame embedding). But then, a wild embedding can be seen as a quantum state.

4. Wild embeddings and 4-manifolds

This section is a kind of motivation that wild embeddings should be considered. We start with the physically significant non-compact examples of a spacetime: $S^3 \times \mathbb{R}$. This non-compact 4-manifold has the usual form used in general relativity (GR). There is a global foliation along $\mathbb{R}$, i.e. $S^3 \times \{t\}$ with $t \in \mathbb{R}$ are the (spatial) leafs. $S^3 \times \mathbb{R}$ with this foliation is called the "standard $S^3 \times \mathbb{R}$". But this choice is not unique. In dimension 4, there is a plethora (uncountable many) of exotic smoothness structures (see [3]). In the following we will denote an exotic version by $S^3 \times_\theta \mathbb{R}$. The construction of $S^3 \times_\theta \mathbb{R}$ is rather complicated (see [13]). As a main ingredient one needs a homology 3-sphere $\Sigma$ (i.e. a compact, closed 3-manifold with the homology groups of the 3-sphere) which does not bound a contractable 4-manifold (i.e. a 4-manifold which can be contracted to a point by a smooth homotopy). Interestingly, this homology 3-sphere $\Sigma$ is smoothly embedded in $S^3 \times_\theta \mathbb{R}$ (as cross section, i.e. $\Sigma \times \{0\} \subset S^3 \times_\theta \mathbb{R}$). What about the foliation of $S^3 \times_\theta \mathbb{R}$? There is no foliation along $\mathbb{R}$ but there is a codimension-one foliation of the 3-sphere $S^3$ (see [4] for the construction). So, $S^3 \times_\theta \mathbb{R}$ is foliated along $S^3$ and the leafs are $S_1 \times \mathbb{R}$ with the surfaces $\{S_i\}_{i \in I} \subset S^4$. But what happens with the 3-spheres in $S^3 \times_\theta \mathbb{R}$? There is no smoothly embedded $S^3$ in $S^3 \times_\theta \mathbb{R}$ (otherwise it would have the standard smoothness structure). But there is a wildly embedded $S^3$! This case is generic, i.e.
also all known exotic smoothness structures of non-compact 4-manifolds have this property. Therefore, wild embeddings are generic for our 4-dimensional spacetime.

5. Conclusion

In this paper we discussed a spacetime realization of a quantum state. We conjectured that the quantum state is given by a wild embedding like Alexander's horned ball. If this conjecture is true then we must obtain a wild embedding by some quantization process from a classical state. This program was done in this paper. At first we identify the classical state with (the expected) tame embedding. Using Drinfeld-Turaev quantization, we constructed a (deformation) quantum space and relate them to a wild embedding. We demonstrate the process by the example of Alexander's horned ball. Wild embeddings are generic constructions in the theory of 4-manifolds with exotic smoothness structures. Therefore, we conjecture that our technique must be useful in a future quantum gravity theory.

At the end we will discuss a cosmological consequence of our approach. Assume an exotic $S^3 \times \mathbb{R}$ as a model of our spacetime. A scaling of the $\mathbb{R}$ parameter leads to a beginning at the $-\infty$-"point". If we further assume a finite size of the spatial component $S^3$ at the beginning (using a big bounce effect, see [2]), then we have to look for 3-sphere at the $-\infty$-"point". But exotic smoothness enforce us: there is no smoothly embedded 3-sphere and we obtain a wildly embedded 3-sphere again. Therefore we obtain as a result of this paper: the cosmos started in a quantum state (represented by a wildly embedded 3-sphere).

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