Three dimensional finite temperature SU(3) gauge theory in the confined region and the string picture

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Abstract

We determine the correlation between Polyakov loops in three dimensional SU(3) gauge theory in the confined region at finite temperature. For this purpose we perform lattice calculations for the number of steps in the temperature direction equal to six. This is expected to be in the scaling region of the lattice theory. We compare the results to the bosonic string model. The agreement is very good for temperatures $T < 0.7 \ T_c$, where $T_c$ is the critical temperature. In the region $0.7 \ T_c < T < T_c$ we enter the critical region, where the critical properties of the correlations are fixed by universality to be those of the two dimensional three state Potts model. Nevertheless, by calculating the critical lattice coupling, we show that the ratio of the critical temperature to the square root of the zero temperature string tension, where the latter is taken from the literature, remains very near to the string model prediction.
1 Introduction

Three dimensional SU(3) gauge theory has many properties in common with the corresponding four dimensional theory. At zero temperature lattice calculations show the confinement of heavy quarks in a linear potential. Furthermore, they predict a mass gap and a nontrivial glueball spectrum. At finite temperature there is a deconfining phase transition. In contrast to SU(3) in four dimensions the transition is second order. Lattice calculations of the critical indices are consistent with the transition being in the universality class of the two dimensional three state Potts model [1], as expected from general arguments [2].

In a previous paper, we have studied in detail the equation of state in the high temperature phase by performing extensive lattice calculations, as is possible in three dimensions, and thus obtaining very precise results [3]. In this note we describe a study of the theory in the confined phase below the transition. We work at $N_\tau = 6$, which from the experience gathered in [3] we believe is in the scaling region. Thus our result should be relevant for the continuum limit. In particular we calculate the correlations between Polyakov loops. It has been proposed long ago that these can be described by an effective string model, the Nambu-Goto bosonic string [4, 5]. In this model the temperature dependent string tension becomes zero at a critical temperature $T_c$ corresponding to the Hagedorn temperature of the string theory. The critical index for the approach of the string tension to zero is $\nu = 1/2$. This index is different from that of the two dimensional three state Potts model, which has $\nu = 5/6$. In fact the above predictions from the string model are independent of the gauge group, but only depend on the dimensionality of space. Early investigations have shown that nevertheless the critical temperature is not far from the prediction of the Nambu-Goto model [1, 6, 7]. It has also been shown that for low enough temperature the correlations between Polyakov loops are very well described by the effective string model [8]. Finally, field theory investigations have proven that the first three terms of an expansion in $T^2$ of the string model are in fact universal results for a fluctuating bosonic string [9, 11]. But the expansion of course does not tell anything about the existence of a singularity at $T_c$.

Here, from the lattice, we extend the earlier results on the determination of the ratio between the critical temperature and the square root of the string tension. Furthermore we study in detail the behaviour of the Polyakov loop correlations in the region $T_c/2 < T < T_c$. For this purpose, we have taken data for a dense set of temperatures, varying the temperature by changing the lattice coupling constant. This method is complementary to the one used in [8], where the lattice extent in the temperature direction is varied. As will be shown, where our results overlap with those of [8], they agree with each other. It is also very interesting to study in detail the critical region, near $T_c$, where we do not expect the string model to describe the correlations. Instead, in fact, critical scaling is found to be
a very efficient tool to describe the correlation length between Polyakov loops. We postpone the discussion of these issues to a separate publication [12].

In the next section we present the lattice setup and define the quantities of interest in the string context. The simulations are described in section 3, together with the determination of the critical temperature for a lattice extent in the temperature direction equal to 6. Section 4 is devoted to the comparison of the results obtained with the string picture, by reference to the Nambu-Goto model and to field theoretic approaches. A summary and conclusions are given in a last section.

2 Polyakov loop correlations at finite temperature. Lattice setup.

We simulate the three dimensional $SU(3)$ theory, regularized on a finite euclidean lattice with lattice spacing $a$, $N_\tau$ points in the (inverse) temperature direction, defined as the 0 direction, $N_S$ points in the two space directions 1, 2 and periodic boundary conditions in all directions. We use the standard Wilson action:

$$S(U_P) = \beta \sum_P \left(1 - \frac{1}{3} \text{Re} \text{Tr} U_P\right),$$

(1)

where $P$ denotes one of the $3N_\tau \times N_S^2$ plaquettes on the lattice, $U_P$ is the product of the $U$-matrices around the plaquette, and $\beta$ is the lattice coupling constant. From the classical limit of the lattice action we may write

$$\beta = \frac{6}{ag^2},$$

(2)

where $g^2$ is the (dimensionful) gauge coupling constant of the continuum theory. We define the temperature and the volume of the lattice by

$$\frac{1}{T} = aN_\tau,$$

$$V = (aN_S)^2.$$

(3)

(4)

In the confined phase, we measure the correlations between Polyakov loops winding around the temperature direction:

$$L(x_1, x_2) = \text{Tr} \prod_{n=0}^{N_\tau-1} U_\tau (\vec{x} + n \vec{e}_\tau),$$

(5)
where $U_\tau$ denotes the link in the time direction, whose origin is located in space at $\vec{x} = (x_1, x_2)$ and $\vec{e}_\tau$ is the unit vector in the time direction. In 2-dimensional space, we define the correlation function $G(z)$ between two lines at distance $z$ from each other by

$$G(z) \equiv \frac{1}{2N_S^3} \left( \sum_{x_1, x_2, x'_2} \text{Re} \langle L(x_1, x_2) L^*(x_1 + z, x'_2) \rangle + \sum_{x_1, x'_1, x_2} \text{Re} \langle L(x_1, x_2) L^*(x'_1, x_2 + z) \rangle \right)$$  \hspace{1cm} (6)

Throughout the paper, the symbol $\langle ... \rangle$ denotes an average over a set of gauge configurations. Due to periodic boundary conditions, any coordinate is understood modulo $N_s$, and all sums run over intervals of length $N_S$. Using the discrete lattice symmetries, we may rewrite $G(z)$ as

$$G(z) = \frac{1}{N_s} \sum_{x_2} \text{Re} \langle L(0, 0) L^*(z, x_2) \rangle$$  \hspace{1cm} (7)

In the forthcoming analysis, it will be assumed that for $z$ large enough (see details in section 3), contributions from excited states of the 2-dimensional system of Polyakov loops can be neglected, and $G$ represented by the contribution of its ground state energy only. In this situation, we will use the parametrization

$$G(z) = b \cosh \left( m \left( \frac{N_S}{2} - z \right) \right) + c.$$  \hspace{1cm} (8)

Using (3), the ground state energy, here denoted $m$ in lattice units, is related to its physical value $M$ by

$$m = Ma = \frac{1}{N_\tau} \frac{M}{T}.$$  \hspace{1cm} (9)

In the right hand side of (5), $b$ and $c$ are assumed to be constants for given lattice parameters. The first term is the lattice expression of $G$ for a free bosonic field, whose normalization determines $b$. In our region of investigation the constant $c$ is essentially consistent with zero, and always negligible in practice.

From now on, we will interpret $M(T)$ as the ground state energy of a flux tube, writing

$$\frac{M(T)}{T} = \frac{\sigma(T)}{T^2}$$  \hspace{1cm} (10)

where $\sigma(T)$ is a temperature dependent string tension. We keep this definition of $\sigma(T)$ for any $T$, and from Eqs. (3,9) rewrite the latter equation as

$$\frac{\sigma(T)}{T^2} = mN_\tau.$$  \hspace{1cm} (11)
In order to study the temperature dependence of $\sigma(T)$, and compare our data with the string picture it is convenient to use the zero temperature string tension $\sigma_0$ as a scale, i.e. using Eqs. (3,11) to write

$$\frac{T}{\sqrt{\sigma_0}} = \frac{1}{N_r a \sqrt{\sigma_0}}$$

$$\frac{\sigma(T)}{\sigma_0} = \frac{m}{N_r (a \sqrt{\sigma_0})^2}$$

The quantity $a \sqrt{\sigma_0}$ has been recently measured from numerical simulations using large $N^3$ lattices [7, 8], leading to high accuracy results at $\beta = 8.1489, 11.3711, 14.7172, 18.131, 21$ and $40$. Because we need it for a dense set of $\beta$ values in the 10 to 22 range, we determined $a \sqrt{\sigma_0}$ for any $\beta$ by fits to the above numerical data. After some trials, we finally retained the following parametrization

$$F_{\sigma_0}(\beta) \equiv a \sqrt{\sigma_0} = \frac{h}{\beta} \frac{\beta - z}{\beta - p}.$$  

$$h = 3.3257 \quad z = 1.99 \quad p = 3.69$$

Not only all the data points are perfectly fitted, but it is interesting to notice that the zero and pole positions $z$ and $p$ in $F_{\sigma_0}$ suggest the existence of a cross-over from a weak to a strong coupling regime, for $\beta$ of the order of a few units, well below the lowest $\beta$ value needed. Moreover, we use this function only for interpolating accurate data. We do not quote the errors on the fitted parameters, as they are anyway highly correlated. What is important is the error in the interpolating function. We found that the absolute error on the inverse of the function $F_{\sigma_0}(\beta)$ is approximately constant and equal to 0.002.

### 3 Simulations and results

The simulations were done in a standard way using the Wilson action [1]. We used one heatbath sweep followed by four overrelaxation sweeps. The Polyakov loops correlations were measured according to formula (3). The measurements were performed every five combined (heatbath and overrelaxation) sweeps. At least 20,000 measurements were kept for every $\beta$ and $N_S$ combination, their average number being of the order of 60,000. These measurements are not independent, and the autocorrelation time for the correlation functions was found to be 10 in the worst case ($\beta = 20$), resulting in a minimum of 2,000 independent measurements ($\beta = 20, N_s = 96$). The ground state energies $m$ were obtained by fitting formula (3), and their errors estimated using the blocked bootstrap method as now described.

The total set of measurements for given values of $\beta$ and lattice size was first divided into blocks of 500 measurements, in order to take care of the autocorrelation time. Then 100 bootstrap samples were generated, each one resulting
from drawing a new set of blocks randomly from the original sample. For each
bootstrap sample we calculated the average value of the correlation function. The
average of those averages was taken as the final value of the correlation function,
and their standard deviation as the corresponding error.

The errors on the fitted parameters were obtained by repeating this procedure
once more. For each bootstrap sample, we fitted formula (8) for $z$ in an interval
$[z_0, N_s/2]$, using the $\chi^2$ value calculated from the errors found in the previous
step. We so obtained 100 sets of fitted parameters $m, b, c$ and used the standard
deviation between them as an estimate of their errors.

Eq. (8) retains the contribution from the ground state only. In order to control
the possible effect of higher states, we measured the dependence of the fitted
masses on the $z_0$ parameter, and found that the systematic error associated with
the choice of $z_0$ becomes smaller than the statistical error for $z_0 \geq 4$. In table 6
we list the final values and statistical errors of the parameters.

The value $\beta_c$ of the lattice coupling at which the transition takes place has
been estimated using the Binder cumulant approach [13], following the method
described in [14] and applied to the variable $|L|^2$, where

$$L = \frac{1}{N^2} \sum_{x_1, x_2} L(x_1, x_2).$$  

(16)

Using this method, we obtain

$$\beta_c = 21.36 \pm 0.01^{(st.)} \pm 0.05^{(sys.)}$$  

(17)

The systematic error quoted covers small differences observed if other methods
are investigated and uncertainties due to unknown subleading terms in the scaling
ansatz. These issues will be discussed in detail in the forthcoming paper on the
critical region [12]. Here, the present systematic error does not affect our result
on $T_c/\sqrt{\sigma(0)}$, which is the relevant quantity in the forthcoming comparison with
the string model. Indeed, from the result quoted above and Eq. (12) we obtain

$$\frac{T_c}{\sqrt{\sigma(0)}} = 0.976(15),$$  

(18)

where most of the error comes from the error on the function $F_\sigma(\beta)$.

4 The string picture

In a fundamental paper, Pisarski and Alvarez [4] long ago related the finite temper-
ature behaviour of gauge theories in the confined region up to the phase transition
to the finite temperature behaviour of a bosonic string randomly fluctuating in
the transverse directions. If the string is supposed to be the Nambu-Goto string,
the formula for the temperature dependent string tension is
where $D$ is the space-time dimension of the gauge theory. As we can see from the formula, the temperature dependent string tension is not dependent on the gauge group, but only on the number of transverse dimensions. The string tension goes to zero at a critical temperature

$$T_c = \sqrt{3\sigma_0/\pi}$$

which coincides with the Hagedorn temperature of the string model. The approach to this singularity has the mean field behaviour with the exponent $\nu = 1/2$.

Later, it was shown that in any dimension, the first term in the expansion in $T^2/\sqrt{\sigma_0}$ is universal in any dimension\[^9\], i.e.

$$\frac{\sigma(T)}{\sigma_0} = 1 - \frac{\pi(D - 2)T^2}{6\sigma_0} + ...$$

In three dimensions there is a special situation, as here the first three terms are universal and coincide with the development of the expression for the Nambu-Goto string truncated to this order\[^10, 11\]:

$$\frac{\sigma(T)}{\sigma_0} = 1 - \frac{\pi T^2}{6\sigma_0} - \frac{\pi^2 T^4}{72\sigma_0^2} - \frac{\pi^3 T^6}{432\sigma_0^3} + ...$$

Of course, this polynomial expression contains no information about a singularity where $\sigma(T)$ vanishes, such as that of the full Nambu-Goto expression.

In figure 1 we show the quantity $\sigma(T)/\sigma_0$ versus $T/\sqrt{\sigma_0}$ as measured on the lattice using Eqs. (12,13,14). The data points correspond to various $N_S$ values and cover the domain $0.5 < T/\sqrt{\sigma_0} < 1$. We do not extend the domain to smaller $T$, because we may enter the strong coupling region. In the figure we have also included the two points from \[^8\], which are in this region. As one can see our data agree with these points. From the figure one can also see that the NG-model is consistent with the lattice QCD results up to about $T/\sqrt{\sigma_0} = 0.7$, above which the corresponding curve is definitely above the data. We also show the two curves corresponding to Eqs. (21,22), both higher in the graph in the whole region (since any term of the full expansion of Eq. (19) are negative). The second curve cannot be distinguished from the full NG model up to $T/\sqrt{\sigma_0} = 0.7$. To fix the ideas, we just remark that if the last term in Eq. (22) were above 3 times larger, the agreement with the numerical results would be extended up to $T/\sigma_0$ about 0.8.

We checked that the spatial lattice sizes $N_S$ used in figure 1 were large enough for the above conclusions not being altered by any finite size effects for $T/\sqrt{\sigma_0} < 0.9$. In practice, below $T/\sqrt{\sigma_0} \approx 0.85$, $m$ is always larger than about 0.1 and the $N_S$ values used at least 6 times the correlation length $m^{-1}$. In turn, at the largest
Figure 1: Finite temperature string tension in units of the zero temperature string tension as a function of temperature for various $N_S$ values. Solid red squares come from [8]. The solid line represents the Nambu-Goto expression (19), and the dashed (dashed-dotted) line its expansion to the first (third) order in $T^2$. 
Table 1: The values of the parameters in equation (8) fitted to the data points shown in the figure.

| $\beta$  | $\frac{1}{\sqrt{\sigma_0}}$ | $N_s$ | $m$      | $b$      | $c$      |
|----------|-------------------------------|-------|----------|----------|----------|
| 12.5     | 0.5251                        | 96    | 0.5055(50)| 0.10(2)  | e-11     | −0.0000045(34) |
| 13       | 0.5509                        | 64    | 0.4540(27)| 0.26(2)  | e-7      | 0.0000031(58) |
| 13.5     | 0.5766                        | 64    | 0.4029(30)| 0.149(13)| e-6      | 0.000005(11)  |
| 14       | 0.6023                        | 64    | 0.3620(18)| 0.61(3)  | e-6      | 0.000001(1)  |
| 14.5     | 0.6279                        | 64    | 0.3207(20)| 0.248(14)| e-5      | 0.000008(20) |
| 14.7172  | 0.639                         | 64    | 0.3068(14)| 0.404(16)| e-5      | 0.000008(14) |
| 15       | 0.6535                        | 64    | 0.2866(12)| 0.808(28)| e-5      | 0.000000(16) |
| 15.5     | 0.679                         | 64    | 0.2564(14)| 0.234(10)| e-4      | −0.000008(27) |
| 16       | 0.7045                        | 96    | 0.22911(72)| 0.1054(34)| e-5 | −0.000019(11) |
| 16.5     | 0.73                          | 96    | 0.2023(11)| 0.42(2)  | e-5      | 0.000020(23) |
| 17       | 0.7555                        | 64    | 0.179(1)  | 0.374(12)| e-3      | −0.000022(82) |
| 17       | 0.7555                        | 96    | 0.17942(59)| 0.1393(37)| e-4 | −0.000018(25) |
| 17.5     | 0.7809                        | 96    | 0.157(1)  | 0.455(21)| e-4      | 0.0000057(46) |
| 18       | 0.8063                        | 64    | 0.13718(87)| 0.1783(49)| e-2 | −0.00005(17) |
| 18       | 0.8063                        | 96    | 0.13898(63)| 0.1218(35)| e-3 | −0.000006(48) |
| 18.5     | 0.8317                        | 96    | 0.11931(87)| 0.356(14)| e-3      | −0.000027(92) |
| 19       | 0.857                         | 64    | 0.1005(10)| 0.761(26)| e-2      | 0.00073(52)  |
| 19       | 0.857                         | 96    | 0.10025(46)| 0.1023(23)| e-2 | −0.00005(12) |
| 19.5     | 0.8824                        | 96    | 0.08206(66)| 0.2895(94)| e-2 | 0.00010(33)  |
| 20       | 0.9077                        | 84    | 0.0635(11)| 0.1476(86)| e-1      | 0.0017(18)  |
| 20       | 0.9077                        | 96    | 0.0642(13)| 0.846(60)| e-2      | 0.0033(15)  |
| 20       | 0.9077                        | 160   | 0.06332(7)| 0.699(37)| e-3      | 0.00015(24)  |
$T/\sigma_0$ value shown, the data points exhibit an $N_S$ dependence, to be associated with effects of the nearby critical temperature (see (18)).

At this point, it must be emphasized that this critical temperature is astonishingly close to the location of the Nambu-Goto singularity, which is from Eq. (20)

$$T_{c}^{NG} = \sqrt{\frac{3}{\pi}} = 0.977...$$

As a last remark, we point out that while the string model is often supposed to be relevant to $SU(N), N \to \infty$ [8], the transition is first order for $N \geq 5$ [2, 7], and thus cannot be associated with the Hagedorn temperature of the string model.

5 Conclusions

In this article we have described a numerical calculation on the lattice of the correlations between Polyakov loops in three dimensional $SU(3)$ gauge theory at finite temperature. We have chosen $N_T = 6$, which we believe is large enough to be in the scaling region in the three dimensional theory. We vary the temperature by varying the lattice coupling constant. By doing this we get a dense set of measurements between $T = T_c/2$ and $T_c$. We also calculated the susceptibility of the Polyakov loops in the neighborhood of the deconfinement transition, and the moments needed for the Binder method applied to the variable $|L|^2$. From this we obtain a determination of the critical lattice coupling constant. Using results from Teper et al. we construct an interpolating function and determine the dimensionless physical quantity $T_c/\sqrt{\sigma_0} = 0.976(15)$. It is compatible with the Hagedorn temperature of the Nambu-Goto string model. This is somewhat astonishing, because the string model has the mean field exponent near the phase transition $\nu = 1/2$, whereas from universality arguments we expect $\nu = 5/6$. To investigate this further, we use the correlation lengths from the correlation between Polyakov loops at temperatures below the phase transition, where there is confinement and the string model may be applicable. The correlation lengths are inversely proportional to the string tension within the string interpretation. We find a very good agreement with the string tension from the Nambu-Goto string model up to $T \approx 0.7T_c$. However, up to this temperature the universal terms for a bosonic string, which coincide with the expansion of the Nambu-Goto model also describe the data. Above this temperature we see a systematic deviation from the string behaviour. This is due to the fact that here we enter the critical region.

In a subsequent paper [12] we will study in detail the critical region of the deconfinement transition in the three dimensional SU(3) gauge theory.
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References

[1] C. Legeland et al, Nucl. Phys. B (Proc. Suppl.) 53 (1997)420; C. Legeland, PhD. Thesis, Bielefeld (1998).
[2] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B210(1982) 423
[3] P. Bialas, L. Daniel, A. Morel and B. Petersson, Nucl. Phys. B807 (2009) 547, [arXiv:0807.0855].
[4] R. D. Pisarski and O. Alvarez, Phys. Rev. D26 (1982) 3735a
[5] P. Olesen, Phys. Lett. B160(1985)408a
[6] M. Teper, Phys. Lett. B313(1998)417
[7] J. Liddle and M. Teper [arXiv:0803.2128]
[8] A. Anthenodorou, B. Bringholz and M. Teper, Phys. Lett. B656(2007)132 [arXiv:0709.0693]; PoS LAT2007(2007)288 [arXiv:0709.2981]
[9] M. Lüscher, Nucl. Phys. B180(1981)317
[10] M. Lüscher and P. Weisz, JHEP 0207(2002)049, JHEP 0407(2004)049
[11] O. Aharony and E. Karzbrun, JHEP 0906(2009)012 [arXiv:0903.1927]
[12] P. Bialas, L. Daniel, A. Morel and B. Petersson, in preparation.
[13] K. Binder, Z.fuer Physik, B43 (1981) 119; K. Binder, M, Nauenberg, V. Privman,A.P. Young, Phys. Rev. B31 (1985) 1498.
[14] A. Cucchieri, J. Engels, S. Holtmann, T. Mendes and T. Schulze, J. Phys. A 35, 6517 (2002) [arXiv:cond-mat/0202017].