On the effect of self-steepening in modulation instability

S. M. Hernandez¹,*, P. I. Fierens²,³, J. Bonetti¹, and D. F. Grosz¹,³

¹Grupo de Comunicaciones Ópticas, Instituto Balseiro,Bariloche,
Río Negro 8400, Argentina
²Grupo de Optoelectrónica, Instituto Tecnológico de Buenos Aires,CABA 1106, Argentina
³Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina

*Corresponding author: shernandez@ib.edu.ar

May 9, 2016

Abstract

We revisit the problem of modulation instability (MI) in optical fibers, including higher-order dispersion terms, self-steepening, and Raman response. We derive expressions for the MI gain and use them to explore the role of self-steepening towards a high-power limit. We show that, contrary to common wisdom, there is a pump power level that maximizes the MI gain. Further increasing the power not only diminishes the gain, but eventually makes it disappear. We believe these findings to be of special relevance, for instance, when applied to the generation of supercontinuum in the mid and far infrared bands. Finally, numerical simulations confirming our analytical results are presented.

The phenomenon of modulation instability (MI) has been known and thoroughly studied for many years in a vast number of different areas of science. In the realm of optical fibers [1–7], MI plays a fundamental role as it is intimately connected to the appearance of optical solitons, which have had a strong impact on applications to high-capacity fiber-optics communication. Modulation instability also is at the heart of the occurrence of efficient parametric optical processes heavily relied upon to achieve bright and coherent light in the infrared range. These very same nonlinear processes are used to provide optical amplification and wavelength conversion in the telecommunication band, maybe one day enabling complete photonic control of optical data traffic. In recent years, nonlinear phenomena such as supercontinuum generation [8–12] and rogue waves [13–15] have rekindled the interest in MI.
Despite the abundant literature on the subject, to the best of our knowledge a complete analysis of MI including both the Raman response and the effect of self-steepening has only been presented by Béjot et al. [16]. In this paper, we derive expressions of the MI gain that coincide with those in Ref. [16] and focus on the role of self-steepening. By analyzing the dependence of the MI gain with the input pump power, we find that self-steepening plays a fundamental role as it yields an optimum power (in terms of maximizing the gain) and, surprisingly, makes the gain vanish above a certain threshold, which is obtained from the analytical model.

Wave propagation in a lossless optical fiber can be described by the generalized nonlinear Schrödinger equation [17],

$$\frac{\partial A}{\partial z} - i\hat{\beta}A = i\hat{\gamma}A(z, T) \int_{-\infty}^{+\infty} R(T') |A(z, T - T')|^2 dT',$$  

(1)

where $A(z, T)$ is the slowly-varying envelope, $z$ is the spatial coordinate, and $T$ is the time coordinate in a comoving frame at the group velocity ($= \beta^{-1}$). $\hat{\beta}$ and $\hat{\gamma}$ are operators related to the dispersion and nonlinearity, respectively, and are defined by

$$\hat{\beta} = \sum_{m \geq 2} \frac{i^m}{m!} \beta_m \frac{\partial^m}{\partial T^m}, \quad \hat{\gamma} = \sum_{n \geq 0} \frac{i^n}{n!} \gamma_n \frac{\partial^n}{\partial T^n}.$$  

The $\beta_m$'s are the coefficients of the Taylor expansion of the propagation constant $\beta(\omega)$ around a central frequency $\omega_0$. In the convolution integral in the right hand side of (1), $R(T)$ is the nonlinear response function that includes both the instantaneous (electronic) and delayed Raman response.

We shall analyze the effect of a small perturbation $a$ to the stationary solution $A_s$ of (1) (see [17])

$$A(z, T) = \left( \sqrt{P_0} + a \right) e^{i\gamma_0 P_0 z} = A_s + ae^{i\gamma_0 P_0 z}. $$  

(2)

If we keep only terms linear in the perturbation, after some manipulations, substitution of (2) into (1) leads to

$$\frac{\partial \tilde{a}(z, \Omega)}{\partial z} + \tilde{N}(\Omega)\tilde{a}(z, \Omega) = \tilde{M}(\Omega)\tilde{a}^*(z, -\Omega),$$  

(3)

where $\Omega = \omega - \omega_0$, $\tilde{a}$, $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{R}$ are the Fourier transforms of $a$, $\beta$, $\gamma$ and $R$, respectively. Moreover, for the sake of clarity we have defined

$$\tilde{N}(\Omega) = -i \left[ \tilde{\beta}(\Omega) + P_0 \tilde{\gamma}(\Omega) \left( 1 + \tilde{R}(\Omega) - P_0 \gamma_0 \right) \right],$$  

$$\tilde{M}(\Omega) = i P_0 \tilde{\gamma}(\Omega) \tilde{R}(\Omega).$$

After some straightforward algebra, (3) can be cast into a 2nd order ordinary differential equation

$$\frac{\partial^2 \tilde{a}(z, \Omega)}{\partial z^2} + 2i\tilde{B}(\Omega)\frac{\partial \tilde{a}(z, \Omega)}{\partial z} - \tilde{C}(\Omega)\tilde{a}(z, \Omega) = 0,$$  

(4)
where
\[
\tilde{\beta}_e(\Omega) = \sum_{n \geq 1} \frac{\beta_{2n}}{(2n)!} \Omega^{2n}, \quad \tilde{\beta}_o(\Omega) = \sum_{n \geq 1} \frac{\beta_{2n+1}}{(2n+1)!} \Omega^{2n+1},
\]
\[
\tilde{\gamma}_e(\Omega) = \sum_{n \geq 0} \frac{\gamma_{2n}}{(2n)!} \Omega^{2n}, \quad \tilde{\gamma}_o(\Omega) = \sum_{n \geq 0} \frac{\gamma_{2n+1}}{(2n+1)!} \Omega^{2n+1},
\]
\[
\tilde{B}(\Omega) = - \left[ \tilde{\beta}_o(\Omega) + P_0 \tilde{\gamma}_o(\Omega) \left( 1 + \tilde{R}(\Omega) \right) \right], \quad (5)
\]
\[
\tilde{C}(\Omega) = \tilde{\beta}_o^2(\Omega) - \tilde{\beta}_o^2(\Omega) +
+ P_0^2 \left( \tilde{\gamma}_c^2(\Omega) - \tilde{\gamma}_c^2(\Omega) \right) \left( 1 + 2\tilde{R}(\Omega) \right) - P_0^2 \tilde{\gamma}_o^2 +
+ 2P_0 \tilde{\gamma}_o \tilde{\beta}_e(\Omega) + 2P_0^2 \gamma_0 \gamma_e(\Omega) \left( 1 + \tilde{R}(\Omega) \right) +
+ 2P_0 \left( \tilde{\beta}_o \tilde{\gamma}_o - \tilde{\beta}_e \tilde{\gamma}_e \right) \left( 1 + \tilde{R}(\Omega) \right). \quad (6)
\]
Substitution of \( a(z, \Omega) = D \exp(iK(\Omega)z) \) in (4) leads to the dispersion relation
\[
K_{1,2}(\Omega) = - \tilde{B}(\Omega) \pm \sqrt{\tilde{B}^2(\Omega) - \tilde{C}(\Omega)}. \quad (7)
\]
A simple expression can be obtained by setting \( \gamma_n = 0 \) for \( n \geq 2 \) and \( \gamma_1 = \gamma_0 \tau_{sh} \) (accounting for the effect of self-steepening). In this case,
\[
K_{1,2}(\Omega) = \tilde{\beta}_o + P_0 \gamma_0 \tau_{sh} \Omega \left( 1 + \tilde{R} \right) \pm
\pm \sqrt{\left( \tilde{\beta}_e + 2\gamma_0 P_0 \tilde{R} \right) \tilde{\beta}_o + P_0^2 \gamma_0^2 \tau_{sh}^2 \Omega^2 \tilde{R}^2}. \quad (8)
\]
This expression agrees with a similar one presented in Ref. [16] and with the one with \( \tau_{sh} = 0 \) in Ref. [10]. As usual, the MI gain can be found as
\[
g(\Omega) = 2 \max\{-\text{Im}\{K_1(\Omega)\}, -\text{Im}\{K_2(\Omega)\}, 0\}, \quad (9)
\]
where the factor 2 is due to the fact that \( g(\Omega) \) is a power gain. The resulting equation exhibits many properties of the gain that have been thoroughly studied in the literature, for instance, the fact that it does not depend on odd terms of the dispersion relation (e.g., \( \beta_3 \)) [4,10]. However, the derived MI gain also reveals some novel aspects related to the self-steepening term \( \gamma_0 \tau_{sh} \). Indeed, it already has been noted that this term enables a gain even in a zero-dispersion fiber and that, in general, leads to a narrowing of the MI gain bandwidth [18,19].

For the case of a large input power, (8) shows that the MI gain spectrum is dominated by the Raman response, i.e.,
\[
|g(\Omega)| \approx 4P_0 \gamma_0 \tau_{sh} |\tilde{R}(\Omega)| \cdot |\text{Im}\left\{ \tilde{R}(\Omega) \right\}|. \quad (10)
\]
Then, in the large pump power limit, the MI gain is independent of the dispersion parameters \( \beta_m \). Although this particular scenario is worth mentioning,
very interesting properties are actually revealed when studying what happens as we tend towards this large-power regime. It is widely known [17] that, for the simplified model that only takes $\beta_2$ and $\gamma_0$ into account, and no self-steepening, as the pump power $P_0$ increases, the frequency $\Omega_{\text{max}}$ where the MI gain attains its maximum and the peak gain, both increase as, respectively,

$$\Omega_{\text{max}} = \pm \sqrt{\frac{2\gamma_0 P_0}{|\beta_2|}}, \quad g(\Omega_{\text{max}}) = 2\gamma_0 P_0. \quad (11)$$

Enter self-steepening and the dependence between the pump power and the MI gain changes drastically in a non-trivial way, for instance, there is an optimum pump power level for which a peak gain is attained; any further increase in pump power will make the MI gain decline. To see this, let us analyze the case of (8) considering only the electronic response (i.e., $R(\Omega) = 1$). We have

$$K_{1,2}(\Omega) = \tilde{\beta}_e + 2P_0\gamma_0\tau_{\text{sh}} + \sqrt{\Delta}, \quad \Delta = (\tilde{\beta}_e + 2\gamma_0 P_0)^2 + P_0^2\gamma_0^2\tau_{\text{sh}}^2. \quad (12)$$

We have gain whenever the imaginary part of (12) is negative, i.e., $\Delta < 0$. It is easily seen that, if we ‘turn off’ self-steepening, $\Delta = (\tilde{\beta}_e + 2\gamma_0 P_0)^2$ and if $\tilde{\beta}_e < 0$ (anomalous dispersion) there always will exist a sufficiently high $P_0$ such that $\Delta < 0$, namely, $P_0 > \frac{|\tilde{\beta}_e|}{2\gamma_0}$. It suffices to find where $\Delta = \Delta(\Omega)$ attains its maxima to find the maximum MI gain. Then,

$$\partial_\Omega \Delta = 2\tilde{\beta}_e \partial_\Omega \tilde{\beta}_e + 2\gamma_0 P_0 \partial_\Omega \tilde{\beta}_e = 0. \quad (14)$$

$\partial_\Omega \tilde{\beta}_e(\Omega)$ need not necessarily be nonzero for all $\Omega \neq 0$, but we assume so for the frequency range of interest. In addition, if $\tilde{\beta}_e(\Omega)$ is negative and decreasing, the sufficient condition for the maximum gain becomes $\tilde{\beta}_e(\Omega) = -\gamma_0 P_0$ and it is given by

$$g(\Omega_{\text{max}}) = 2\gamma_0 P_0, \quad \text{with } \Omega_{\text{max}} = \{\Omega : \tilde{\beta}_e(\Omega) = -\gamma_0 P_0\}, \quad (15)$$

which is the same expression as in (11), except that obtaining the desired frequencies is more involved. Let us now turn our attention to the effect of self-steepening. Consider (13), this time with $\tau_{\text{sh}} \neq 0$. The necessary condition for gain, namely $\Delta < 0$, becomes

$$(\tilde{\beta}_e + 2\gamma_0 P_0)^2 + P_0^2\gamma_0^2\tau_{\text{sh}}^2 < 0, \quad (16)$$

where the rightmost term in the l.h.s. assures that there will be a given pump power level above which gain vanishes. As in the case of (15), obtaining an explicit analytical formula for the MI frequencies is not possible for arbitrary dispersion profiles but, as before, we can write an implicit formula and obtain
the MI peak gain. By making $\partial_\Omega \Delta = 0$, assuming anomalous dispersion, and $\partial_\Omega \beta_e < 0$ in the frequency range of interest, we obtain

$$\Omega_{\text{max}} \in \{ \Omega : \beta_e(\Omega) + \gamma_0 P_0 + \Omega (\gamma_0 P_0 \tau_{sh})^2 = 0 \}. \quad (17)$$

Then, by direct substitution, the maximum gain is given by

$$g(\Omega_{\text{max}}) = 2 \sqrt{\left(\gamma_0 P_0 \tau_{sh}\right)^2 + 1}. \quad (18)$$

where

$$F_{sh} = 1 - \Omega_{\text{max}}^2 \tau_{sh}^2 \left[ \frac{(\gamma_0 P_0 \tau_{sh})^2}{(\partial_\Omega \beta_e(\Omega_{\text{max}}))^2} + 1 \right]. \quad (19)$$

Even though $F_{sh}$ is defined implicitly, we can draw some conclusions. If $\tau_{sh} = 0$, then $F_{sh} = 1$ and $g(\Omega_{\text{max}}) = 2 \gamma_0 P_0$, as expected. Otherwise, $F_{sh}$ provides a measure of how the gain is affected by self-steepening. Since $F_{sh} \leq 1$, gain can only be reduced, for $F_{sh} \in (0, 1)$, or vanish for $F_{sh} \leq 0$. As such, $F_{sh} \geq 0$ is an alternative condition to (16) for MI gain to exist.

As an example, let us solve (19) for the case $\beta_2 < 0$, $\beta_{n \geq 3} = 0$. An explicit computation can be carried out and

$$\Omega_{\text{max}} = \pm \sqrt{\frac{2|\beta_2| \gamma_0 P_0 - 2(\gamma_0 P_0 \tau_{sh})^2}{|\beta_2|}}. \quad (20)$$

Maxima or, equivalently, gain will occur if and only if $2|\beta_2| \gamma_0 P_0 - 2(\gamma_0 P_0 \tau_{sh})^2 > 0$, i.e.,

$$P_0 < \frac{|\beta_2|}{\gamma_0 \tau_{sh}^2}. \quad (21)$$

Any further pump power increase and the medium will exhibit no gain, a fundamental difference when compared to the case without self-steepening. In fact, there is an optimal pump power $P_0^* \in (0, |\beta_2|/(\gamma_0 \tau_{sh}^2))$ for which a maximum gain is achieved. $P_0^*$ can be obtained by solving $2F_{sh} + \partial_\Omega P_0 \partial P_0 F_{sh} = 0$. For the case $\beta_e(\Omega) = (\beta_2/2)\Omega^2$,

$$P_0^* = \frac{1}{2} \frac{|\beta_2|}{\gamma_0 \tau_{sh}^2}. \quad (22)$$

That is, (22) gives a peak MI gain right in the middle of the power range for which there is gain. This can be seen in Fig. 1 (bottom pane), where MI gain for a fiber with $\beta_2 = -1$ ps$^2$/km, $\beta_{n \geq 3} = 0$, $\gamma_0 = 100$ (W-km)$^{-1}$, is plotted at different pump power levels ($P_0$) centered at a wavelength of 5 μm. With these parameters (22) yields $P_0^* \approx 710$ W. For comparison, Fig. 1 also includes the case without self-steepening (top pane). We also show the effect of including a $\beta_4 = -0.0016$ ps$^4$/km in Fig. 2. As compared to Fig. 1, MI gain bands appear stretched and the power range changes, but in both cases there is a power level above which the MI gain vanishes. Both Figs. 1-2 include the delayed Raman response and $R(T) = (1 - f_R) \delta(T) + f_R h_R(T)$, with

$$h_R(T) = \frac{\tau_1^2 + \tau_2^2}{\tau_1 \tau_2^2} e^{-T/\tau_2} \sin(T/\tau_1) u(T), \quad (23)$$
Figure 1: MI gain versus pump power when only $\beta_2$ is considered, with (bottom pane) and without (top pane) self-steepening.

where $u(T)$ is the Heaviside step function, $f_R = 0.031$, $\tau_1 = 15.5$ fs, $\tau_2 = 230.5$ fs [20]. Although (12)-(22) do not include the delayed Raman response, since the Raman gain spectrum, as given by the Fourier transform of (23), provides a much lower and narrower gain in the lower-frequency limit of the MI gain spectrum, its influence is much weaker in comparison, as made apparent by the faint Raman gain bands in Figs. 1-2.

Figure 3 shows results of numerical simulations of (1) that confirm our observations. Simulation parameters are those of Fig. 2, with $\beta_3 = 0.04$ ps$^3$/km, at a propagated distance of 10 mm (for the sake of clarity, the pump was removed from the spectra). When the input pump power is 10 kW (top pane), with or without considering self-steepening, MI gain is produced. Nevertheless, when the input pump power level is 18 kW (bottom pane), and self-steepening
is included, MI gain nearly vanishes. This agrees with the analytical results depicted in Fig. 2 (bottom pane).

In summary, we have revisited the problem of modulation instability in optical fibers. We have included all relevant effects, i.e., higher-order dispersion terms, self-steepening, and the Raman response. We showed that self-steepening plays a fundamental role in the large-power limit, and an analytical expression for the pump power that maximizes the MI gain was derived. Also, we found that, contrary to common wisdom, increasing the pump power beyond the optimum leads to a decline in the MI gain and, eventually, to its disappearance. We believe this observation to be of particular relevance in the case of supercontinuum generation from CW and quasi-CW sources in the mid and far infrared, where the effect of self-steepening is expected to be more pronounced.
References

[1] A. Hasegawa and W. Brinkman. Tunable coherent ir and fir sources utilizing modulational instability. *IEEE Journal of Quantum Electronics*, 16(7):694–697, Jul 1980.

[2] D. Anderson and M. Lisak. Modulational instability of coherent optical-fiber transmission signals. *Opt. Lett.*, 9(10):468–470, Oct 1984.

[3] K. Tai, A. Hasegawa, and A. Tomita. Observation of modulational instability in optical fibers. *Phys. Rev. Lett.*, 56:135–138, Jan 1986.

[4] M. J. Potasek. Modulation instability in an extended nonlinear schrödinger equation. *Opt. Lett.*, 12(11):921–923, Nov 1987.

[5] M. J. Potasek and G. P. Agrawal. Self-amplitude-modulation of optical pulses in nonlinear dispersive fibers. *Phys. Rev. A*, 36:3862–3867, Oct 1987.

[6] Masataka Nakazawa, Kazunori Suzuki, Hirokazu Kubota, and Hermann A. Haus. High-order solitons and the modulational instability. *Phys. Rev. A*, 39:5768–5776, Jun 1989.

[7] G. P. Agrawal. Modulation instability in erbium-doped fiber amplifiers. *IEEE Photonics Technology Letters*, 4(6):562–564, June 1992.

[8] Ayhan Demircan and Uwe Bandelow. Supercontinuum generation by the modulation instability. *Optics communications*, 244(1):181–185, 2005.

[9] Michael H. Frosz, Ole Bang, and Anders Bjarklev. Soliton collision and raman gain regimes in continuous-wave pumped supercontinuum generation. *Opt. Express*, 14(20):9391–9407, Oct 2006.

[10] Michael H. Frosz, Thorkild Sørensen, and Ole Bang. Nanoengineering of photonic crystal fibers for supercontinuum spectral shaping. *J. Opt. Soc. Am. B*, 23(8):1692–1699, Aug 2006.

[11] J. M. Dudley, G. Genty, F. Dias, B. Kibler, and N. Akhmediev. Modulation instability, akhmediev breathers and continuous wave supercontinuum generation. *Opt. Express*, 17(24):21497–21508, Nov 2009.

[12] Martin E. Masip, A. A. Rieznik, Pablo G. König, Diego F. Grosz, Andrea V. Bragas, and Oscar E. Martinez. Femtosecond soliton source with fast and broad spectral tunability. *Opt. Lett.*, 34(6):842–844, Mar 2009.

[13] K. Hammani, C. Finot, B. Kibler, and G. Millot. Soliton generation and rogue-wave-like behavior through fourth-order scalar modulation instability. *Photonics Journal, IEEE*, 1(3):205–212, Sept 2009.

[14] Simon Toft Sørensen, Casper Larsen, Uffe Møller, Peter M. Moselund, Carsten L. Thomsen, and Ole Bang. Influence of pump power and modulation instability gain spectrum on seeded supercontinuum and rogue wave generation. *J. Opt. Soc. Am. B*, 29(10):2875–2885, Oct 2012.
[15] Shanti Toenger, Thomas Godin, Cyril Billet, Frédéric Dias, Miro Erkintalo, Goëry Genty, and John M Dudley. Emergent rogue wave structures and statistics in spontaneous modulation instability. *Scientific reports*, 5, 2015.

[16] P. Béjot, B. Kibler, E. Hertz, B. Lavorel, and O. Faucher. General approach to spatiotemporal modulational instability processes. *Phys. Rev. A*, 83:013830, Jan 2011.

[17] Govind Agrawal. *Nonlinear Fiber Optics*. Optics and Photonics. Academic Press, fifth edition, 2012.

[18] F.Kh. Abdullaev, S.A. Darmanyan, S. Bischoff, P.L. Christiansen, and M.P. Sørensen. Modulational instability in optical fibers near the zero dispersion point. *Optics Communications*, 108(1 - 3):60 – 64, 1994.

[19] F. Kh. Abdullaev, S. A. Darmanyan, S. Bischoff, and M. P. Sørensen. Modulational instability of electromagnetic waves in media with varying nonlinearity. *JOSA B*, 14(1):27–33, 1997.

[20] M. R. Karim, B. M. A. Rahman, and Govind P. Agrawal. Mid-infrared supercontinuum generation using dispersion-engineered ge11.5as24se64.5 chalcogenide channel waveguide. *Opt. Express*, 23(5):6903–6914, Mar 2015.
Figure 3: Numerical simulations for the fibers of Fig. 2 at a propagated distance of 10 mm. Pump power of 10 kW (top pane) and 18 kW (bottom pane) show the effect of (not) considering self-steepening.