On detection statistics in double-double-slit experiment

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In this paper, we analyze the statistics of detection data in a general double-double-slit experiment. The two particles are detected at random times which are not equal in general and because we do not have any constraint on the distances of left and right screens from their slits and the ratio as well, they can be detected in completely different timescales. As the detection of first particle leads to collapse of the wave function, there is no a straightforward and agreed method to study this problem in the orthodox formalism which lacks a clear prediction of these random events and therefore the quantum state afterwards. This is not the case in Bohmian framework which we implement in this paper and we can predict the system up to the end of experiment. The main result is the joint distribution of detection data including the arrival time and position of the particles on left and right screens. As one of the main consequences, we see, although the joint spatial distribution can be affected by a change to the relative location of screens, the marginals on each side remain intact compatible with signal-locality. At the end, we see how this result is very sensitive to quantum equilibrium condition.

PACS numbers: 03.65.Ta, 05.10.Gg

Introduction. — Double-slit experiment has always been one of the central experiments in quantum mechanics. Yet, if there is an experiment more mysterious, it is a double-double-slit experiment with entangled particle pairs [1]; see Fig(1). In this experiment, the two most important properties of quantum theory, i.e. superposition and entanglement, combine and lead to a non-local interference pattern. The coincident measurements of two particles make interference fringes in the configuration space, while the detection of one of the individual particles does not show that [2, 3]. This experiment has been realized experimentally using entangled photon pairs [4] or in another setup using entangled electron pairs on the molecular level [5]. It also can be carried out using pairs of metastable helium atoms as suggested in [6].

The full analysis of this experiment can be even more complex because of the third quantum effect, i.e. the collapse of the wave function when the first particle is detected in the middle of the experiment at random times. This, in principle, can change the detection statistics of the remaining particle. The analysis however is not completely straightforward in standard formalism. One of the reasons is that in the orthodox quantum mechanics we do not have a clear and agreed prediction of detection times distribution or the so-called arrival times of the first detected particle. This problem is actually very old [7–9], but it is still open [10–14]. The mathematical origin of this problem is that the time is a parameter in the standard formalism, not a self-adjoint operator, hence, there is no unique and unambiguous way to compute the temporal probability distribution of detection events from first principles (i.e. Born rule) [15]. Therefore, it is even not clear how one can determine the state of the system at a given time from initial two particle state; because the state and its equation of motion change at an undetermined time. Although some aspects of this experiment have been explored, for example in [2–6, 17–20], the analysis is not quite general and complete as they did not take the collapse effect into consideration and work in a regime where this effect has minimum and negligible consequences.

![Fig. 1. The source emits pairs of entangled particles, each of which passes through a double-slit. Two arrays of fast particle detectors are considered, on both sides, recording the detection events.](https://example.com/Fig1.png)

In this paper, we are going to study this experiment via Bohmian mechanics which leads to a clear prediction in this situation. In contrast to orthodox interpretation, the non-relativistic Bohmian mechanics describes a world in which particles move along realistic trajectories [21]. Nonetheless, it has been shown that this theory is experimentally equivalent to orthodox quantum mechanics [21, 22] insofar as the latter is unambiguous [23–25]; e.g.,
in usual position or momentum measurements at a specific time. Moreover, the Bohmian trajectories lead to a clear prediction for the temporal distribution of detection events [25–27]. In fact, here we explain how one can compute two particles position-time joint probability distribution \( P(r_1, t_1; r_2, t_2) \), the most general quantity that can be measured experimentally in this setup. Furthermore, we discuss how quantum statistics mask the non-locality of Bohmian dynamics, and signal-locality is preserved in this experiment.

**Theoretical framework and results.**—In the Bohmian mechanics, the state of a two-particle system is determined by the wave function \( \Psi(r_1, r_2) \) and actual positions of particles \((R_1, R_2)\). The time evolution of the wave function, in free space, is given by the following two-particle Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \Psi_t(r_1, r_2) = -\left(\frac{\hbar^2}{2m_1} \nabla_1^2 + \frac{\hbar^2}{2m_2} \nabla_2^2\right) \Psi_t(r_1, r_2), \tag{1}\]

where particle dynamics is given by the first-order differential equations in configuration space, the “guidance equation”,

\[
\frac{d}{dt} R_i(t) = v_i^\Psi(R_1(t), R_2(t)), \tag{2}
\]

as \( v_i^\Psi \) are the velocity fields associated with the wave function \( \Psi_t \); i.e. \( v_i^\Psi = (\hbar/m_i) \nabla_i \Psi_t/\Psi_t \) [29]. When the particle 1, for example, is detected at time \( t = t_c \), the two-particle wave function collapses **effectively** to a one-particle wave function as \( \Psi_{t_c}(r_1, r_2) \rightarrow \psi_{t_c}(r_2) \) where [22, 28]:

\[
\psi_{t_c}(r_2) = \Psi_{t_c}(R_1(t_c), R_2(t_c)) \tag{3}
\]

known as “conditional wave function” in Bohmian formalism [29, 30]. For \( t > t_c \), we can solve the one-particle Schrödinger equation

\[
i\hbar \frac{\partial}{\partial t} \psi_t(r_2) = -\frac{\hbar^2}{2m_2} \nabla_2^2 \psi_t(r_2), \tag{4}
\]

with the initial wave function \( \psi_{t_c} \), and also the associated one-particle guidance equations

\[
\frac{d}{dt} R_2(t) = v_2^\psi(R_2(t)), \tag{5}
\]

with the initial position \( R_2(t_c) \).

Here, in a double-double-slit setup, we restrict ourselves to propagation from the slits to detection screens. Hence, the initial wave function can be considered as follows [3, 17]:

\[
\Psi_{t_0}(r_1, r_2) = \left[ f_+^u(r_1) f_-^d(r_2) + f_+^d(r_1) f_-^u(r_2) \right] + 1 \leftrightarrow 2, \tag{6}
\]

where

\[
f_+^u(x, y) = G(x; \sigma_x, \pm l_x, \pm u_x)G(y; \sigma_y, +l_y, +u_y), \tag{7}
\]

\[
f_+^d(x, y) = G(x; \sigma_x, \pm l_x, \pm u_x)G(y; \sigma_y, -l_y, -u_y),
\]

and \( G \) is a Gaussian wave function

\[
G(x; \sigma_l, l, u) = Ne^{-(x-l)^2/4\sigma^2+im(x-l)/\hbar}. \tag{8}
\]

We also have symmetrized the wave function, as we have considered the particles as indistinguishable Bosons. This kind of entangled Gaussian state is practical for implementation in quantum technologies because such states can be readily produced and reliably controlled [17]. Furthermore, since the free two-particle Hamiltonian is separable, the time evolution of this wave function can be found from eq.(1) analytically as

\[
\Psi_t(r_1, r_2) = [f_+^u(r_1, t) f_-^d(r_2, t) + f_+^d(r_1, t) f_-^u(r_2, t)] + 1 \leftrightarrow 2, \tag{9}
\]

in which \( f_+^{\pm} \) functions are constructed out of time dependent Gaussian wave functions \( G_t \) as in (7) where

\[
G_t(x; \sigma_l, l, u) = (2\pi \sigma_l^2)^{-\frac{1}{4}} e^{\frac{-4l^2}{\sigma_l^2}} e^{i\frac{m}{\hbar} (x-l - \frac{t}{2})} \tag{9}
\]

and \( s_l = \sigma(1 + i \frac{\hbar}{2m_l e}) \).

In this paper, we run the simulation for \( 10^6 \) trajectories in the configuration space and the parameters have been chosen as \( \sigma_x = 10^{-6}(m) \), \( \sigma_y = 10^{-5}(m) \), \( u_x = 0.1(m/s) \), \( u_y = 0 \) (m/s), \( l_x = 5 \times 10^{-3}(m) \), \( l_y = 5 \times 10^{-5}(m) \) and \( m_1 = m_2 = 6.646 \times 10^{-27} \) (kg). These values are in agreement with the proposed setup in reference [6] in which colliding Helium-4 atoms have been used for producing initial entangled state [31].

![Fig. 2. The effect of collapse on trajectories. The green curves represent the case without considering collapse and the black ones are the modified trajectories due to the collapse effect. One can see that the right black trajectories are gradually deviate from green ones as the left screen detects the particle. Here, \( X_R = 0.5(m) \) (which is beyond the box of this figure) and \( X_L = -0.15(m) \) is represented by a vertical gray line.](image)

The detection time and position of the first observed particle are uniquely determined by solving equation (2).
Then, using equation (5), we can find the trajectories of the remaining particles. In figure 2, we have depicted some of the trajectories for some random initial positions \( \mathbf{R}^0 \equiv (\mathbf{R}_L(t_0), \mathbf{R}_R(t_0)) \) sampled from \(|\Psi_0(\mathbf{R}^0)|^2\) in accordance with the Born rule. In this figure the green trajectories are without considering the collapse effect and the black ones are with that. One can see that after the left particle detection, the right particle starts to deviate from the green paths as the conditional wave function now guides it. It is worth noting that, the ensemble of trajectories can be experimentally reconstructed using weak measurement techniques [3, 32–36], which can be used as a test for our result.

Using trajectories, we can find the joint detection data distribution in \((t_L, y_L; t_R, y_R)\) space where \(t_L/R\) is the detection time \(y_L/R\) is the detection position on the left/right screen. The probability density behind this distribution can be formally written as

\[
P(t_L, y_L; t_R, y_R) = \int d\mathbf{R}^0 |\Psi_0(\mathbf{R}^0)|^2 \prod_{i=L,R} \delta(t_i - T_i(\mathbf{R}^0)) \delta(y_i - Y_i(\mathbf{R}^0)),
\]

where \(T_{L,R}(\mathbf{R}_L^0, \mathbf{R}_R^0)\) and \(Y_{L,R}(\mathbf{R}_L^0, \mathbf{R}_R^0)\) are the arrival time and position of the particle with initial condition \((\mathbf{R}_L^0, \mathbf{R}_R^0)\) to the left and right screen, respectively. Note, how the above joint distribution and, therefore, any marginal distribution out of it, is sensitive to the Bohmian dynamics through functions \(T\) and \(Y\) and also to the Born rule by \(|\Psi_0(\mathbf{R}^0)|^2\). We have plotted \((y_L, y_R)\) and also the right and left marginal data distributions in figure 3 for two cases: with collapse in dark and without that in green.

For more understanding of what is physically happening, note that from equations (3), (4) and the fact that there is a negligible overlap between \(f^+\) and \(f^-\) wave packages, we can write the conditional wave function of the remaining system \(\psi_1(\mathbf{r}_{1,2})\) as particle 2, (particle 1) is detected first,

\[
\psi_1(\mathbf{r}_{1,2}) \approx \begin{cases} 
A_+ f_u^+(\mathbf{r}_{1,2}, t) + B_+ f_u^-(\mathbf{r}_{1,2}, t), & X_{2,1}(t_c) > 0 \\
A_- f_d^+(\mathbf{r}_{1,2}, t) + B_- f_d^-(\mathbf{r}_{1,2}, t), & X_{2,1}(t_c) < 0 
\end{cases}
\]

which is a superposition of the up and down slit states like an ordinary double-slit setup where \(A_\pm\) and \(B_\pm\) are random constants determined by time and position of the first particle detection \((t_c, \mathbf{R}(t_c))\),

\[
A_\pm = f_u^\pm(\mathbf{R}_{2,1}(t_c), t_c), \quad B_\pm = f_d^\pm(\mathbf{R}_{2,1}(t_c), t_c)
\]

As these numbers are random, the conditional wave function is random as well, and the system is really described by a density matrix which can be represented as,

\[
\hat{\rho}_1(\mathbf{r}_{1,2}, \mathbf{r}_{1,2}') = \sum_{\mathbf{R}^0} \psi_1(\mathbf{r}_{1,2}) \psi_1(\mathbf{r}_{1,2}').
\]

So despite the fact that for one instance of this sum, quantum superposition leads to interference patterns like in the double-slit experiment, mixing them washes out that pattern, as we see in figure 3.

In Bohmian formalism, non-locality is manifest due to non-local particle dynamics [29]. Moreover, here as we mentioned above, detection of the left moving particle makes instantaneous deviation in the trajectory of the right one. At the statistical level, this can affect the joint distribution in \((y_L, y_R)\)-space, which are represented by dark scatter plots in figure 3. Nevertheless, the right spatial marginal distribution, as a local observable quantity, turns out to be independent of whether there is any screen on the left or not and if there is any, it is not sensitive to the location of that detection screen. This is compatible with the well-known non-signaling theorem [39]. It is astonishing that not only the non-local

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**FIG. 4.** Distribution of arrival position on the right screen. The different panels represent the arrival position distributions for different values of \(X_L\). In all cases, the right screen surface is located at \(X_R = 0.5\) (m). As we expect the dark and green distribution coincide when \(X_L = X_R\) as there is not much for the remaining particle to be guided by the conditional wave function.
Bohmian dynamics is not leading to faster than light signaling, but on the contrary, it is vital for producing such fine-tuned mixing of superpositions and therefore saving signal-locality. It should also be noted, that the usual non-signaling theorem is proved for observable quantities which are described by self-adjoint operators in the standard formalism, while the marginal position distribution here is not of this type [37]. Nonetheless, our numerical study, based on Bohmian mechanics, approve signal-locality for this non-standard, but experimentally observable, distributions [40].

The signal-locality here is not however a result of the Bohmian dynamics alone and the Born rule for initial particle position distribution, i.e. quantum equilibrium condition, is critical as well [41, 42]. To test this fact here, we make a deviation from Born rule by generating the initial positions through the same mixture of Gaussian distributions as $|\Psi_0(R^0)|^2$ except with replacing $\sigma \to \sigma/2$. The result is shown in figure 4 where we have plotted the right screen marginal distributions for different locations of the left screen. First, note that with deviation from the Born rule, we should not expect that the right green marginal (even without considering the collapse effect) should be the same as before. But, the point we want to emphasize here is that how in contrast to the quantum equilibrium case the right true marginal (with considering collapse) is dependent on the location of left detection screen and so, deviation from Born rule leads to faster than light signaling.

Summary and outlook.—In any generic quantum experiment that contains some entangled particles and some particle detectors waiting for them to arrive, there is an obstacle to computing the time evolution of the quantum state in orthodox formalism from start to the end, when all the particles will be detected. The main reason is the absence of a unique prediction of the middle detection times in the orthodox formalism. The double-double-slit experiment here is one of the simplest examples of this kind which we have analyzed using the Bohmian formalism that can predict the quantum state to the end without any ambiguity. The main result is the joint detection distribution in a generic setup which is in fact a new experimentally testable prediction beyond orthodox quantum mechanics. One should note, however, that Bohmian mechanics is not the only framework that can predict this quantity and actually any interpretation with a proposal to arrival time such as decoherent histories [14, 43, 44]. Nelson stochastic mechanics.

FIG. 3. Here we have depicted joint distributions of $(y_L, y_R)$ data for different choices of $X_L$ where $X_R = 0.5$ (m) for all the cases. Green ones are for the cases where we do not take the collapse into account and the black ones are the correct distributions with collapse. The right marginal and the left one are also depicted in the top and right of the joint distribution, respectively. The important point here is that the right marginal is independent of whether there is a measurement on the left or not and if there is a measurement, it is independent of the location of the left detection screen. Here, for the sake of better visualization, we have only included $10^4$ data points in the joint distributions. However, the marginal ones are computed with our $10^6$ data points.
and other approaches [12, 13, 46, 47] can have predictions in principle. This fact opens a new window to compare different interpretations of quantum theory. In this regard, the study of double-double-slit experiment in these frameworks would be an interesting extension of the present work, which has been left for future studies. The other interesting improvement is to consider the back effect of presence of the detection screen on time evolution of the wave function. The arrival distributions computed here, should be considered as ideal or intrinsic distributions, since that effect is ignored [48]. The influence of a physical detector can be modeled in various phenomenological methods such as using a complex potential [49, 50], absorbing boundary condition [51] and other approaches [47]. In principle, these methods can lead to different predictions where the comparison would be interesting and has been left for future works, as well.

Acknowledgment.—The authors are grateful to Ali Ayat and Reinhard Werner for their useful discussions and comments and to Hamidreza Safari for his support of this work. The work of VH is supported in part by Iran Science Elites Federation under Grant No. 401138.

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