WELL-POSEDNESS OF FULLY NONLINEAR KDV-TYPE EVOLUTION EQUATIONS

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Abstract. We study the well-posedness of the initial value problem for fully nonlinear evolution equations, $u_t = f[u]$, where $f$ may depend on up to the first three spatial derivatives of $u$. We make three primary assumptions about the form of $f$: a regularity assumption, a dispersivity assumption, and an assumption related to the strength of backwards diffusion. Because the third derivative of $u$ is present in the right-hand side and we effectively assume that the equation is dispersive, we say that these fully nonlinear evolution equations are of KdV-type. We prove the well-posedness of the initial value problem in the Sobolev space $H^7(\mathbb{R})$. The proof relies on gauged energy estimates which follow after making two regularizations, a parabolic regularization and mollification of the initial data.

1. Introduction

We study the question of well-posedness in Sobolev spaces of the initial value problem for the fully nonlinear evolution equation

$$u_t = f(u_{xxx}, u_{xx}, u, x, t),$$

under suitable conditions on the function $f$ and its partial derivatives. Chief among these conditions will be a condition which ensures that the equation is dispersive, so that the contribution of $u_{xxx}$ to the function $f$ is in a sense dominant and nonvanishing. An explicit calculation in Fourier space shows that the equation $u_t = u_{xxx} - u_{xx}$ is ill-posed in $L^2$-based Sobolev spaces because of the presence of backwards diffusion; another principal condition for well-posedness therefore must be a balance between the effects of dispersion and backwards diffusion.

There are several papers treating existence theory for semilinear dispersive equations, especially the case in which the leading-order term has a constant coefficient. Kenig, Ponce, and Vega show local well-posedness of the initial value problem for

$$u_t + \partial_x^{2j+1}u + P(\partial_x^{2j+1}u, \ldots, \partial_x u, u) = 0,$$

with $P$ a polynomial and $j \in \mathbb{N}$, using weighted Sobolev spaces [KPV94]. Kenig and Staffilani treated the generalization of (2) to systems, again proving local well-posedness in weighted Sobolev spaces [KS97]. Kenig and Pilod in [KPT10] studied some special cases of (2) which include the integrable KdV hierarchy. Recently, Harrop-Griffiths has treated the $j = 1$ case of (2). In [HG15a], local well-posedness is proved in certain translation-invariant subspaces of Sobolev spaces on the real line. Under a further assumption on the polynomial $F$, Harrop-Griffiths also proves well-posedness in Sobolev spaces [HG15b]. In another recent work, Germain, Harrop-Griffiths, and Marzuola prove existence of spatially localized solutions for a particular quasilinear KdV-type equation [GHGM18].

As mentioned briefly above, all of these results must contend with the fact that in some cases, equations of the form (1) or (2) can be ill-posed; ill-posedness results have been proved in [Pil08, Akh14, AW13]. The choice of spaces other than the $L^2$-based Sobolev spaces $H^s$ in [KPV94, KS97, HG15a] allows the ill-posedness to be avoided. Alternatively, in [HG15b], the additional condition on the polynomial $F$ (that there is no term of the form $uu_{xx}$) removes the ill-posedness.

In the non-constant coefficient, semilinear case, Cai shows dispersive smoothing properties for solutions of $u_t - a(x, t)\partial_x^2 u + P(x, t, \partial_x^2 u, \partial_x u, u) = 0$ [Cai97]. Here, the coefficient $a$ is required to
be bounded away from zero, so that the dispersion does not vanish. The result of [Cai97] is related to results of [CKS92]: there, Craig, Kappeler, and Strauss proved well-posedness and dispersive smoothing for solutions of (1), under some fairly strong assumptions. Both [Cai97] and [CKS92] use weights to ensure certain rates of decay at infinity; in the semilinear case, Cai is able to weaken the assumptions of Craig, Kappeler, and Strauss. The assumptions of these papers, as in the present work, include a condition that controls the effect of backward diffusion. In the present work, as in [CKS92] and unlike [Cai97], we treat the fully nonlinear evolution equation (1); similarly to [Cai97], we are interested in significantly weakening the conditions imposed in [CKS92]. Further differentiating our work from [CKS92], we do not use weights, and instead only use the $L^2$-based Sobolev spaces $H^s$.

The authors have previously established well-posedness results for initial value problems in some special cases of the equation (1), including some quasilinear equations. Akhunov has shown well-posedness of quasilinear systems [Akh13] and linear equations [Akh14] on the real line. In [AW13], Ambrose and Wright studied linear equations on periodic intervals, as well as certain specific quasilinear equations such as the $K(2,2)$ Rosenau-Hyman compacton equation [RH93] and the Harry Dym equation [Kru75]. The results of the present paper are given on the real line; the authors may treat the spatially periodic case in a future work.

The conditions that we presently require on $f$ are similar to the conditions assumed by the authors in [Akh14] and [AW13], adapted to the fully nonlinear evolution equation (1), and allowing for as much generality as possible. These conditions will be specified more technically in what follows, but they are, roughly: (a) the function $f$ is sufficiently smooth, (b) the partial derivative of $f$ with respect to $u_{xxx}$ does not vanish, so that the dispersion does not degenerate, and (c) the “modified diffusion ratio,” to be defined, but which balances the effects of dispersion and backwards diffusion, must either be integrable or be the derivative of a smooth function (this condition is closely related to the “Mizozaha” condition needed in [HG15a]). With these conditions satisfied, we are able to use a gauged energy estimate and a parabolic regularization to prove well-posedness of initial value problems in Sobolev spaces.

Allowing for the most general $f$ possible requires us to study solutions at somewhat high regularity, $H^7$. We discuss certain special cases in which we are able to lower this regularity requirement, such as the case of quasilinear equations. We mention that by disallowing the occurrence of terms of the form $uu_{xx}$ in the nonlinearity, Harrop-Griffiths was able to use Sobolev spaces $H^s$ for $s \geq \frac{2}{3}$ for semilinear equations; furthermore, Harrop-Griffiths includes a discussion of when lower regularity results are possible, depending on whether certain terms are or are not present in the nonlinearity. In [AW13], in the spatially periodic case, well-posedness of the initial value problem with strictly positive data for the $K(2,2)$ equation $u_t + (u^2)_{xxx} + (u^2)_x = 0$, among other quasilinear problems, was demonstrated in $H^4$.

The plan of the paper is as follows: in Section 2, we state the main result. In Section 3, we introduce a regularized problem. In Section 4, we prove a useful estimate for a related linear evolution equation. In Section 5, we use this linear estimate to find bounds for the solution of the nonlinear regularized problem. In Section 6, we pass to the limit, finding solutions to the original, non-regularized nonlinear problem. We close with some discussion in Section 7, and as an appendix we provide an explicit calculation of the third spatial derivative of (1).

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2. Statement of the result

Suppose that the following assumption on the function $f(\tilde{z}, x, t)$ from (1) holds:

Condition (A1): Regularity. Assume $f(\tilde{z}, x, t) \in C^1_t C^{11}_\tilde{z} W^{11, \infty}_x$.

That is, the nonlinear function can grow in the dependent variable $\tilde{z}$ and is bounded in the
independent space-time variables. Stated more precisely, for any $k \geq 0$ and \( |(z_0, \ldots, z_3)| \leq k \), we have for each choice of $\alpha$, $\beta$, and $\gamma$ satisfying $\alpha \leq 1$, $\max\{\beta, \gamma\} \leq 11$, there exists a positive increasing function $s \mapsto C_{\alpha, \beta, \gamma}(s)$ such that

\[
\|\partial_t^3 \partial_x^2 \partial_z^2 f(\bar{z}, x, t)\|_{L^\infty_{x, [\bar{z}, 1]}} \leq C_{\alpha, \beta, \gamma}(|\bar{z}|) \leq C(k).
\]

**Remark 1.** To simplify the accounting of functions depending on derivatives of $u$, we introduce the following notation:

\[
a(\partial_z^k u) := a(\partial_z^k u, \ldots, \partial_x^3 u, u, x, t)
\]

i.e. we bold the highest derivative on which the function in question depends and hide all other factors. Often we also introduce a variable $\bar{z}(x, t) = (\partial_z^k u, \ldots, \partial_x^3 u, u, x, t)$ for the same purpose.

Furthermore, we define $f(z_1, \bar{z}, f(z_3, \ldots, z_0, x, t)$ to be derivatives with respect to the derivatives of the unknown solution $u$.

**Condition (A2):** Suppose that the “modified diffusion ratio” $g_M$, defined below, has the following form

\[
g_M := \frac{f_{z_2}}{f_{z_3}}(\partial_z^3 u) = \partial_x [g_D(\partial_z^2 u)] + g_H(\partial_z^3 u),
\]

where $g_D \in C^1_{\alpha} C_x^0 C^\infty_{\bar{z}}$ and $g_H \in C^1_{\alpha} C_x^2 C^\infty_{\bar{z}}$, i.e. satisfying bounds similar to (3) with lower regularity. Moreover,

\[
g_H(\bar{0}, x, t) = 0 = \partial_{z_j} g_H(\bar{0}, x, t) \text{ for } j = 0, \ldots, 3.
\]

**Condition (A3):** Assume $f(\bar{0}, x, t) = 0$.

Before stating the next result, we comment on the meaning of these conditions.

**Remark 2.** Unlike (A1) and (A2), condition (A3) is done for simplicity. All the arguments below are valid, but are lengthier in the presence of an additional forcing term $\lambda$. We define $f_{z_1}, \bar{z}, f(z_3, \ldots, z_0, x, t)$ to be derivatives with respect to the derivatives of the unknown solution $u$.

Likewise, [1] can be thought of as Taylor expansion of $f_{z_2}$ with respect to the vector $\partial_z^3 u$, with additional assumptions. By ruling out degeneracy of the dispersion coefficient $f_{z_3}$ (to be justified below), we observe that $f_{z_2}$ is a smooth function and hence the most general Taylor expansion is

\[
\frac{f_{z_2}}{f_{z_3}} = g_C(x, t) + \sum_{j=0}^{3} g_j(x, t) \partial_z^j u + g_H(x, t, \partial_z^3 u),
\]

where $g_H$ has no constant or linear terms in $\partial_z^3 u$, like $g_H$ in [3]. The technical framework of the paper allows the $g_C$ term, as in [Akh14]. We choose to omit it for simplicity. However, the argument breaks down unless the “linear coefficient” terms $g_j \partial_z^j u$ have the total derivative form of (1). We address the necessity of this below, after stating our main theorem.

We define

\[
\lambda(\partial_z^3 u(t)) := \left\| \frac{1}{f_{z_3}(\partial_z^3 u(t))} \right\|_{L^\infty_x}
\]

**Remark 3.** Note that the quantity $\lambda$ measures non-degeneracy of the dispersion in (1). In the case when $\lambda(\partial_z^3 u_0) > 0$, we demonstrate in section 5 that this condition remains valid for a small time determined by the size of the solution.

**Theorem 4.** Suppose $f$ from (1) satisfies (A1)–(A3). Let $u_0 \in H^7$, be such that

\[
\lambda_0 := \lambda(\partial_z^3 u_0) < \infty \text{ for } \lambda \text{ from (6).}
\]

Then there exists $T = T(\|u_0\|_{H^7}, \lambda_0)$, such that (1) is wellposed. That is
We will use Sobolev Spaces. In particular, by $C$ and the size of data is of paramount importance and will be kept, e.g. $f$ nonlinear function on $H$. $\partial\[Pil08\]$ has demonstrated that for the evolution equation $A$ mean $2.1$. Notation. We expect to extend these ill-posedness results to show a lack of continuous dependence on data and hence demonstrate the sharpness of Theorem 4 with respect to the "modified diffusion ratio."

2.1. Notation. When estimating with multiplicative constants, we will often write $A \lesssim_{x,y} B$, to mean $A \leq C \cdot B$, where the constant $C = C(x, y)$ may increase from line to line. By an equivalence $A \approx_{x,y} B$, we mean $A \lesssim_{x,y} B \lesssim_{x,y} A$. In most of this work, the constants $C$ will depend on the nonlinear function $f$, i.e. $C = C(f, s, k)$. The functional dependence of the constants on dispersion and the size of data is of paramount importance and will be kept, e.g. $C = C(\lambda(t), \|u(t)\|_{H^s})$.

Sobolev Spaces. We will use $L^2$-based Sobolev spaces extensively and will define them here for reference. In particular, by $H^s$ we mean the set of tempered distributions $f$, such that

$$\|f\|_{H^s} := \|(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)\|_{L^2} < \infty.$$  

When $s$ is a non-negative integer the Fourier Transform turns derivatives into multiplication, hence

$$\|f\|_{H^s} \approx \|f\|_{L^2} + \|\partial_x^s f(x)\|_{L^2}.$$  

For $s > \frac{1}{2}$ we often choose a version of $f \in H^s(\mathbb{R})$ that is in addition a smooth function in $C^{[s-\frac{1}{2}]}(\mathbb{R})$, where $[\cdot]$ is a lower integer part of a number. Choosing such a version is well-defined by the Sobolev embedding.

Space-time norms. We will use the following space-time norms:

$$\|f(x, t)\|_{L^2_{[t_1,t_2]} L^2_x} \equiv \sup_{t \in [t_1,t_2]} \left( \int_{\mathbb{R}} |f(x, t)|^2 dx \right)^{\frac{1}{2}},$$  

$$\|f(x, t)\|_{L^1_{[t_1,t_2]} L^2_x} \equiv \int_{[t_1,t_2]} \left( \int_{\mathbb{R}} |f(x, t)|^2 dx \right)^{\frac{1}{2}} dt.$$  

Most of the time $[t_1, t_2]$ will stand for $[0, T]$ or $[-T, 0]$ or $[-T, T]$ for some $T > 0$.

3. Regularizations

To prove wellposedness, we will use two types of regularization – one on the data, and one on the equation. We first regularize the data, then the equation. The solution of (11) will be constructed as a limit of such regularized solutions.

3.1. Data regularization. Let $0 \leq \phi(\xi) \leq 1$ be a radial smooth bump function satisfying

$$\phi(|\xi|) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases}$$  

For all $0 < \delta \leq 1$, define $\mathcal{F}((u_0)_\delta)(\xi) = \hat{u}_0(\xi) \cdot \phi(\delta|\xi|)$. Hence

$$\begin{equation} (u_0)_\delta = \left[ u_0 \ast \frac{1}{\delta} \phi \left( \frac{1}{\delta} \right) \right](x). \end{equation}(8)$$
The structure of $\phi$ allows us to give quantitative bounds on the convergence of $(u_0)_\delta$ to $u_0$ in $L^2$ and in $H^2$ as $\delta \to 0$. To do this, we restate Lemma 4.1 from \[\text{Akh13}\], as we need the details in the present manuscript.

**Lemma 5.** Let $K \in H^2$ be a compact set. Then $\forall \delta > 0$, and any $u_0 \in K$, $(u_0)_\delta \in \mathcal{S}$ defined above satisfies

\[
\begin{align}
\|(u_0)_\delta\|_{H^{2+j}} &\leq C_j \|u_0\|_{H^2} \delta^{-j}, \quad \text{for all } j \geq 0, \\
\|(u_0)_\delta - u_0\|_{L^2} &\approx o(\delta^j) \quad \text{and} \quad \|(u_0)_\delta - u_0\|_{H^2} = o(1),
\end{align}
\]

with the convergence rate dependent on $K$.

**Proof.** Let $j \geq 0$ and $0 < \delta < 1$ be given. We calculate as follows:

\[
\|(u_0)_\delta\|_{H^{2+j}} = \int_{|\xi| \leq \frac{\delta}{4}} (1 + |\xi|^2) |\hat{u}_0(\xi)|^2(1 + |\xi|^2)^j |\phi(\delta \xi)|^2 d\xi \leq \left( \frac{3}{\delta} \right)^{2j} \|u_0\|^2_{H^2}.
\]

This proves (10). For (9b) it suffices to show the first estimate, with the second done identically. By the Fundamental Theorem of Calculus, we have

\[
\|(u_0)_\delta - u_0\|_{L^2} = \int (\phi(\delta |\xi|) - 1)^2 |\hat{u}_0(\xi)|^2 d\xi \leq \left( \sup_{\eta \in B_{\delta|\xi|}(0)} |\phi'(\eta)| \right)^2 \delta^2 |\hat{u}_0|^2 d\xi,
\]

where we have used $\phi(0) = 1$ and defined $B_{\delta|\xi|}(0) = [-\delta|\xi|, \delta|\xi|]$. As $\partial_\xi \phi(0) = 0$ for all $j > 0$, we can continue with the Taylor expansion of $|\phi(j)(\eta)|$ seven times and then use $\partial_\xi \phi \equiv 0$ on $B_1(0)$ for $j > 0$ to conclude

\[
\|(u_0)_\delta - u_0\|_{L^2} \leq \delta^{14} \|\partial_\xi^7 \phi\|^2_{L^\infty} \int_{|\xi| \geq \frac{\delta}{4}} |\xi|^7 |\hat{u}_0|^2 d\xi.
\]

The $o(1)$ rate that is uniform for $u_0$ in a compact set $K$ comes by the Lebesgue Dominated Convergence Theorem.

**3.2. Parabolic regularization of the main equation.** We now add both the data regularization and the parabolic regularization to (11):

\[
\begin{cases}
\partial_t u_{\varepsilon, \delta} + f(\partial_3^3 u_{\varepsilon, \delta}, \ldots, u_{\varepsilon, \delta}, x, t) = -\varepsilon \partial_1^4 u_{\varepsilon, \delta}, \\
u_{\varepsilon, \delta}(x, 0) = (u_0)_\delta(x).
\end{cases}
\]

Note that for $\varepsilon = 0$, this equation is (11). Whereas for $\varepsilon > 0$, this equation is a semilinear parabolic equation and is well-understood. We quote the following wellposedness result.

**Proposition 6.** Suppose $(u_0)_\delta \in H^s$ for $s \geq 4$ and let $\varepsilon > 0$ be given. There exists a maximal interval of existence $[0, T_\varepsilon)$, such that $T_\varepsilon \geq 1/C(\frac{1}{\varepsilon}, \|u_0\|_{H^4}, \delta)$, so that (11) is well-posed in $H^s$. In particular, (11) has a unique solution in $C([0, T_\varepsilon), H^s \cap C^4_{[0, T_\varepsilon)} H^{s-4}) \cap L^2_{[0, T_\varepsilon)} H^{s+2}$ and this solution $u_{\varepsilon, \delta}$ depends continuously on data, i.e., $(u_0)_\delta \mapsto u_{\varepsilon, \delta}(t)$ is a continuous map from $H^s \to H^s$. Moreover, if the maximal time of existence satisfies $T_\varepsilon < \infty$, then (11) has the following blow up criterion:

\[
\lim_{t \nearrow T_\varepsilon} \|u_{\varepsilon, \delta}(t)\|_{H^4} = \infty.
\]

**Sketch of proof.** The proof of the proposition follow the outline of a standard semilinear parabolic problem, e.g. \[\text{Akh13}\]. The linear semi-group gains 3 derivatives in $L^2$-based spaces and allows the $f(\partial_3^2 u^s)$ terms to be treated as a lower order perturbation for a contraction mapping argument. □
3.3. Main functional norms. We define

\[
M_\varepsilon(t) = \|u_{\varepsilon, \delta}\|_{L^\infty_t H^s} + \varepsilon \|u_{\varepsilon, \delta}\|_{L^\infty_t H^{s+1}},
\]

and

\[
k(t) := \sup_{0 \leq t' \leq t} \left\{ \|u_{\varepsilon, \delta}(t')\|_{H^s} : \lambda(\partial_3^2 u_{\varepsilon, \delta}(t')) \right\}.
\]

The main technical ingredient of the paper is the simultaneous control of the high Sobolev norm \(M_\varepsilon\) and this function \(k(t)\), which measures dispersion and low Sobolev norms of the solution.

Remark 7. Note that upon combining Lemma 5 with the definition of \(M_\varepsilon(t)\) we obtain

\[
M_\varepsilon(0) \leq \|u_0\|_{H^s(1 + \varepsilon C\delta^{-1})}.
\]

Thus we can ensure that there exists \(0 < \delta_0 \ll 1\) such that

\[
M_\varepsilon(0) \approx \|u_0\|_{H^s} \text{ for } \varepsilon \leq \delta^2, \text{ for } \delta \leq \delta_0.
\]

4. A Linear Estimate

In this section, we develop an estimate for solutions of a linear problem. In the subsequent sections, we will prove estimates for the nonlinear problem by applying this linear estimate. The linear problem we consider now is as follows:

\[
\begin{aligned}
\partial_t w + \sum_{j=0}^3 a_j(x, t) \partial_j^2 w &= -\varepsilon \partial_2^2 w + h, \\
w(0, x) &= w_0(x).
\end{aligned}
\]

Let

\[
k_G(t) := \left\| \int_0^t \frac{a_2(x', t)}{a_3(t)} \, dx' \right\|_{L^\infty_x} + \left\| \frac{1}{a_3}(t) \right\|_{L^\infty_x} + \|a_3(t)\|_{L^\infty_x},
\]

\[
\tilde{M}(t) := \sup_{0 \leq t' \leq t} \left( \sum_{j=0}^3 \|a_j(t')\|_{L^\infty_x} + \left\| \frac{1}{a_3(t')} \right\|_{L^\infty_x} + \left\| \int_0^t \frac{a_2(y, t)}{a_3(y, t)} \, dy \right\|_{L^\infty_x} \right) + \sup_{0 \leq t' \leq t} \left( \left\| \partial_t \int_0^t \frac{a_2(y, t')}{a_3(y, t')} \, dy \right\|_{L^\infty_x} + \|\partial_t a_3(t')\|_{L^\infty_x} \right).
\]

The following theorem is the main result of the present section:

Theorem 8. Assume \(k_G(t)\) and \(\tilde{M}(t)\) are bounded for \(t \in [0, T]\) for some \(T > 0\), and \(h \in L^1_t L^2_x\). Then any classical solution \(w\) of (16) with \(w \in C^1_t H^{-2} \cap C^0_t H^3\) satisfies

\[
\|w(t)\|_{L^2} + \varepsilon \|w\|_{L^2 H^2} \leq C(k_G(t)) \exp \left( \int_0^t \tilde{M}(t') \, dt' \right) \left( \|w_0\|_{L^2} + \|h\|_{L^1 L^2} \right).
\]

In [Akh14] the first author established a similar linear estimate with a constant \(C(\|\tilde{M}(t')\|_{L^\infty_x})\) without the exponential. However, the \(H^7\) wellposedness we are proving in Theorem 8 as compared to the \(H^{12}\) result of [Akh14], demands more delicate accounting of constants than the one given in [Akh14]. In particular, constant dependence on \(k_G(t)\) and \(\tilde{M}(t)\) is done separately, which was not the case in [Akh14]. We also confirm that the additional \(-\partial_2^4\) term is harmless for energy estimates.

The proof is based on the energy method, attempting to estimate \(\partial_t \|w\|_{L^2}^2\) by \(\|w\|_{L^2}^2\). As this method does not apply directly we modify the solution \(w\) first.
**Definition 9.** A function \( \phi \in C^0_{[0,T]} W^3_x \cap C^1_{[0,T]} L^\infty \) is called a gauge, if the following bounds hold:

\[
\text{(19)} \quad \|\phi(t)\|_{L^\infty_x} + \left\| \frac{1}{\phi(t)} \right\|_{L^\infty_x} \leq C(k_G(t)),
\]

\[
\text{(20)} \quad \|\phi\|_{W^3_x} + \|\partial_t \phi\|_{L^\infty_x} \leq C(\dot{M}(t)),
\]

with \( k_G(t) \) and \( \dot{M}(t) \) as defined in (17).

Given a gauge, \( \phi \), we define

\[
\text{(21)} \quad v = \phi^{-1} w.
\]

A substitution of \( v \) into (16) gives:

\[
\text{(22)} \quad \begin{cases}
\partial_t v + L_\phi v = \phi^{-1} h - \varepsilon \phi^{-1} \partial_x^4 (\phi v), \\
v(x,0) = \phi^{-1} u_0,
\end{cases}
\]

where

\[
\text{(23)} \quad L_\phi = a_3 \partial_x^4 + (a_2 + \phi^{-1} 3a_3 \partial_x \phi) \partial_x^2 + (a_1 + \phi^{-1} (2a_2 \partial_x \phi + 3a_3 \partial_x^2 \phi)) \partial_x
\]

\[+ (a_0 + \phi^{-1} (\partial_t \phi + a_1 \partial_x \phi + a_2 \partial_x^2 \phi + a_3 \partial_x^3 \phi)) .
\]

**Lemma 10.** Let \( \alpha \) and \( \beta \) be related by (21), e.g. \( \alpha = \phi^{-1} \beta \). Then \( \alpha \in C^0_{[0,T]} H^{-2} \cap C^1_{[0,T]} H^2 \) if and only if \( \beta \in C^0_{[0,T]} H^{-2} \cap C^1_{[0,T]} H^2 \) with comparability constants of the form \( C(\dot{M}(t)) \). Moreover,

\[
\text{(24)} \quad \|\beta\|_{L^\infty_x L^2} \approx k_G(t) \|\alpha\|_{L^\infty_x L^2}.
\]

**Proof.** From (21) and (19) we immediately conclude (24). Similarly, differentiating (21) twice, we find

\[
\|\beta\|_{H^2} \approx \|\phi\|_{W^2_x, \infty} \|\phi\|_{L^\infty} \|\alpha\|_{H^2}.
\]

We use these observations to justify comparability of norms for \( H^{-2} \) using duality, where we apply the estimate above to a test function \( \gamma = \phi^{-1} \cdot (\phi \gamma) \):

\[
\|\beta\|_{H^{-2}} = \sup_{\|\gamma\|_{H^2} \leq 1} \left| \int \phi \alpha \cdot \gamma \right| \leq \sup_{\|\gamma\|_{H^2} \leq 1} \|\alpha\|_{H^{-2}} C(\|\phi\|_{W^2_x, \infty}) \|\gamma\|_{H^2} \lesssim \|\phi\|_{W^2_x, \infty} \|\alpha\|_{H^{-2}}.
\]

Similarly,

\[
\|\alpha\|_{H^{-2}} \lesssim \|\phi\|_{W^2_x, \infty} \|\phi\|_{L^\infty} \|\beta\|_{H^{-2}}.
\]

Finally, using the fact that \( L^2 \subseteq H^{-2} \), we have

\[
\|\partial_t \beta\|_{H^{-2}} \leq \|\partial_t \phi \cdot \alpha\|_{H^{-2}} + \|\phi \cdot \partial_t \alpha\|_{H^{-2}}
\]

\[\leq \|\partial_t \phi \cdot \alpha\|_{L^2} + \|\phi \cdot \partial_t \alpha\|_{H^{-2}} \leq C(\dot{M}(t))(\|\alpha\|_{L^2} + \|\partial_t \alpha\|_{H^{-2}}).
\]

Hence \( \|\partial_t \beta\|_{H^{-2}} + \|\beta\|_{H^2} \lesssim \dot{M}(t) \|\partial_t \alpha\|_{H^{-2}} + \|\alpha\|_{H^2} \) and similarly for the other comparability direction. \( \square \)

**Remark 11.** Observe, that by Lemma 10 applied to \( v \) and \( \phi \) from (21), the proof of Theorem 8 is reduced to (18) for \( v \) satisfying (22).

The energy method involves multiplying (22) by \( v \) to estimate \( \partial_t \|v\|_{L^2}^2 \) by \( \|v\|_{L^2}^2 \). We begin as follows:

\[
\text{(25)} \quad \partial_t \int |v|^2 = -2(L_\phi v, v) - (\varepsilon \partial_x^4 (\phi v) - h, \phi^{-1} v),
\]

where \((u, v)\) is an \( L^2 \) pairing. We quote the following integration by parts argument:
Lemma 13. Let $\phi(x,t)$ be a solution of the ODE

$$
(Lv, v) = \left( -b_2 + \frac{3}{2} \partial_x b_3 \right) \partial_x v, \partial_x v \right) + (c_0 v, v)
$$

As can be seen in (23), the $\partial^2_x$ coefficient in $L$ includes both $a_2$ as well as a term involving $\phi$. Hence applying Lemma 12 to (25), we see that a choice of $\phi$ can be made to eliminate derivative terms in $(Lv, v)$. In particular, applying Lemma 12 to (25), the choice of $\phi$ we need is the one to satisfy the following identity:

$$
\left( \frac{1}{2} \left[ 2a_2 + \frac{6a_3 \partial_x \phi}{\phi} - 3\partial_x a_3 \right] \partial_x v, \partial_x v \right) = 0.
$$

The Lemma below justifies that such a choice of $\phi$ is indeed a gauge.

Lemma 14. Let $\phi(x, t)$ be a function satisfying Definition 9 and let $w \in H^2$. Then there exists a constant $C(k_G)$, such that

$$
I_w := -\langle \phi^{-2} \partial^2_x w, w \rangle \leq -\frac{1}{C(k_G)} \|w\|_{H^2}^2 + C(\tilde{M}(t))\|w\|_{L^2}^2
$$

Proof. Integrating by parts twice gives

$$
I_w = -\left[ (\phi^{-2} \partial^2_x w, \partial^2_x w) + (\partial^2_x (\phi^{-2}) \partial^2_x w, w) - 2(\partial^2_x (\phi^{-2}) \partial_x w, \partial_x w) \right],
$$

where we have used $\partial^2_x w \cdot \partial_x w = \frac{1}{2} \partial_x [\partial_x w^2]$ for one more integration by parts in the last term. Using Cauchy-Schwarz implies

$$
I_w \leq -\langle \phi^{-2} \partial^2_x w, \partial^2_x w \rangle + \|\phi^{-2}\|_{W^{2,\infty}} (\|w\|_{H^1}^2 + \|w\|_{H^2}^2 \|w\|_{L^2})].
$$

Interpolating $\|w\|_{H^1}^2 \leq \|w\|_{H^2}^2 \|w\|_{L^2}$ and using (20) gives

$$
I_w \leq -\langle \phi^{-2} \partial^2_x w, \partial^2_x w \rangle + C(\tilde{M}(t))\|w\|_{H^2}^2 \|w\|_{L^2}.
$$

A Cauchy-Schwarz estimate gives, for $\alpha > 0$,

$$
I_w \leq -\langle \phi^{-2} \partial^2_x w, \partial^2_x w \rangle + \alpha \|w\|_{H^2}^2 + \alpha^{-1}C(\tilde{M}(t))\|w\|_{L^2}^2.
$$

Using the upper bound for $\phi$ from (19), we estimate

$$
\langle \phi^{-2} \partial^2_x w, \partial^2_x w \rangle \geq \frac{1}{C(k_G(t))} \|w\|_{H^2}^2.
$$

Making the choice $\alpha = \frac{1}{C(k_G(t))}$ completes the proof. \qed
Finally, note that

\[ III \]  

where \( I, II \)  

Next we use (19) to estimate the term \( I \)  

Indeed, if (28) were true, we ignore the \( H^2 \) term and use Grownwall's lemma and Cauchy-Schwarz to get

\[
\sup_{0 \leq t \leq T} \|v(t')\|_{L^2}^2 \leq e^{\int_0^t C(\tilde{M}(t'))dt'} \left[ \|v_0\| + \int_0^t \|h(t')\|_{L^2} \right]^2.
\]

Applying (24) demonstrates (18) except for the \( H^2 \) term on the left. To get it, we rearrange (28):

\[
\frac{\varepsilon}{C(k_G)} \|w\|_{H^2}^2 \leq C(\tilde{M}(t))(\|v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|h\|_{L^2}) - \partial_t \|v\|_{L^2}^2.
\]

Integrating in time and using (18) (without the \( H^2 \) term) we get:

\[
\frac{\varepsilon}{C(k_G)} \|w\|_{H^2}^2 \leq C(k_G) \left[ \frac{1}{2} \int_0^t C(\tilde{M}(t'))(\|v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|h\|_{L^2}) dt' - \|v(t)\|_{L^2}^2 + \|v_0\|_{L^2}^2 \right]
\]

\[
\leq C(k_G(t)) \exp \left( \int_0^t C(\tilde{M}(t'))dt' \right) \left( \|w_0\|_{L^2}^2 + \|h\|_{L^2}^2 \right).
\]

It remains to establish the estimate (28). To do so we return to (26) with \( \phi \) from Lemma 13. Using Lemma 12 implies

\[
\partial_t \int |v|^2 = -2(e_0 v, v) + (h, \phi^{-1} v) - (\varepsilon \partial_t^k (\phi v), \phi^{-1} v) =: I + II + III,
\]

where \( e_0 \) is defined by Lemma 12 applied to \( L = L_\phi \). In particular,

\[
\|e_0\|_{L^\infty} \leq C(\tilde{M}(t)),
\]

which gives an estimate of

\[
I \leq C(\tilde{M}(t)) \|v\|_{L^2}^2.
\]

Next we use (19) to estimate the term \( II \):

\[
II = \| (h, \phi^{-1} v) \| \leq C(k_G(t)) \|v\|_{L^2}^2 \|h\|_{L^2}.
\]

Finally, note that \( III = \varepsilon I_w \) for \( I_w \) from Lemma 14 and \( w \) from (21). Hence from Lemma 14 and (24):

\[
III \leq \varepsilon C(\tilde{M}(t)) \|v\|_{L^2}^2 - \frac{\varepsilon}{C(k_G)} \|w\|_{H^2}^2.
\]

Combining the estimates for \( I, II, \) and \( III \) establishes (28) and concludes the proof. \qed

5. Nonlinear \textit{a priori} estimates

To construct solutions of (11), we begin with the solutions of the parabolically regularized equation (11). The goal of this section is to establish a uniform \textit{a priori} estimate on the dispersion and on the high norms of the solutions of (11). Namely, we show that the solution cannot grow too fast for a certain time, with this time depending on the size of the initial data, dispersion and a balance of parameters \( \varepsilon \) and \( \delta \). Our main nonlinear estimate is summarized in the following propositions, proofs of which will occupy most of this section.
5.1. Main propositions.

**Proposition 15.** Let \( T' > 0 \) be given. There exists increasing functions \( C_1(\cdot, \cdot) \) and \( C_3 \) with \( C_1 \geq 1 \) and \( C_3 \geq 1 \) such that the following inequalities hold. Let \( u_{\epsilon, \delta} \in C_t^1, H^8 \cap C_t^0, H^{12} \) be a solution of (11). Let \( M_\epsilon(t), k(t) \) be as in (13) and (14), respectively. Then for \( t \leq T' \),

\[
(29) \quad k(t) \leq k(0) \frac{1}{1 - tC_1(M_\epsilon(t), k(0))} + tC_3(M_\epsilon(t)).
\]

**Proposition 16.** For \( T' > 0 \) as before, there exist functions \( C_2 \) and \( C_4 \) both increasing and bounded below by 1, so that for \( u_{\epsilon, \delta} \in C_t^1, H^8 \cap C_t^0, H^{12} \) as in Proposition 15,

\[
(30) \quad M_\epsilon(t) \leq C_2(k(t), tC_4(M_\epsilon(t))) [M_\epsilon(0) + tC_4(M_\epsilon(t))].
\]

**Remark 17.** Note that we demand that all the functions \( C_1, C_2, C_3, \) and \( C_4 \) are only dependent on \( \epsilon \geq 0 \) through \( M_\epsilon \) and not in any other way.

Before proving these propositions, we discuss their implications. Essentially, we want to obtain a lower bound on the lifespan of the solution \( u_{\epsilon, \delta} \), independent of the values of \( \epsilon \) and \( \delta \). One way to achieve this is to find \( M > M_\epsilon(0) \), so that whenever the solution norm is trapped in the region \( M_\epsilon(t) \in (M, 2M) \), a “substantial” amount of time must have passed. When combined with the existence result for a parabolic regularization, Proposition 5 these propositions allow us to create solutions to (11), whose \( H^7 \)-norm and time of existence are independent of the regularizations. We provide the details of this informal outline before giving the proof of the Propositions.

We also want to discuss an interesting technical challenge here. The linear estimate (13) applied to (11) may naively suggest that it may be possible to prove an energy estimate

\[
\|u_{\epsilon, \delta}\|_{H^s} \leq O(\|u_{0, \epsilon}\|_{H^s}),
\]

and of course such estimates are valid for KdV and other semilinear equations. However, the validity of such an estimate, even on the linear level, relies on the non-vanishing of dispersion and finiteness of “antidiffusion” (as captured in coefficient \( k_G(t) \) in (17) and \( k(t) \) in the nonlinear problem (11)). For this reason, our argument requires the combination of Propositions 15 and 16. In some form Proposition 16 estimates the \( H^7 \) Sobolev norm for the regularized problem and is based on the refined linear estimate (Theorem 5), while Proposition 15 gives an estimate of dispersion and its proof is cruder, but just as essential.

**Corollary 18.** Define

\[
(31) \quad M = 2C_2(4k(0), 4k(0))[M_\epsilon(0) + 2k(0)], \text{ for the function } C_2 \text{ from Proposition 16}
\]

Then there exists \( T = T(M, k(0)) \), so that if \( t > 0 \) and

\[
(32) \quad M < M_\epsilon(t),
\]

then \( t > T \).

**Proof.** Let \( T = \min \left( \frac{2k(0)}{C_3(2M)}, \frac{1}{2C_1(2M, k(0))}, \frac{2k(0)}{C_4(2M)} \right) \) for \( C_1, C_3 \) and \( C_4 \) from (29) and (30). Without loss of generality, since the solution is continuous in time, we may assume more than (32) holds:

\[
M < M_\epsilon(t) \leq 2M.
\]

Now, assume for the sake of contradiction that \( t \leq T \). From (29) for \( t \leq T \) we obtain:

\[
(33) \quad k(t) \leq k(0) \frac{1}{1 - TC_1(M_\epsilon(t), k(0))} + TC_3(M_\epsilon(t)) \leq 4k(0).
\]

With this bound on \( k(t) \) we apply (30) and use \( t \leq T \):

\[
M_\epsilon(t) \leq C_2(4k(0), 2k(0))[M_\epsilon(0) + 2k(0)] < M.
\]
The last estimate contradicts (32), thus the only way to avoid the contradiction is to conclude that $t > T$. □

Note that the estimate (33) in Corollary 18 only depends on $T$ and the size of $M_\varepsilon(t)$. In particular, we can conclude the following.

**Remark 19.** With the choice of $M$ and $T$ from the prior Corollary 18, we get that if $M_\varepsilon(t) \leq 2M$ and $0 \leq t \leq T$, then

$$k(t) \leq 4k(0).$$

**Corollary 20.** Let $T > 0$ be as in Corollary 18. Suppose $u_{\varepsilon, \delta} \in C_{T'} H^{12}$ is a solution of (11) for $T' \leq T$. Then

$$M_\varepsilon(T') \leq 2M. \quad (34)$$

**Proof.** By continuity of $\|u_{\varepsilon, \delta}(t)\|_{H^s}$, $M_\varepsilon(t)$ is a continuous function. Let $0 \leq t^* \leq T'$ be the smallest value so that $\|u_{\varepsilon, \delta}(t^*)\|_{H^s} \geq 2M$ (if such $t^*$ exists). Recalling that the function $C_2$ from Proposition 16 satisfies the bound $C_2 \geq 1$, we deduce from the definition of $M$ in Corollary 18 that

$$M \geq 2C_2 M_\varepsilon(0) > M_\varepsilon(0).$$

We conclude that $t^* > 0$ and $\|u_{\varepsilon, \delta}(t^*)\|_{H^s} = 2M$. Hence by Corollary 18 applied to $u_{\varepsilon, \delta}$, we conclude $t^* \geq T$. This is a contradiction, as $t^* \leq T' \leq T$ by the hypothesis. The only alternative is $M_\varepsilon(t) \leq 2M$ for all $t \in [0, T')$. □

**Corollary 21.** Let $\delta_0$ be as in Remark 7 and $0 < \varepsilon \leq \delta^2 \leq \delta_0^2$. Then there exists a $T = T \left(\frac{1}{\|u_0\|_{H^7}}, \|u_0\|_{H^7}, \lambda(0)\right) > 0$, such that the maximal interval for wellposedness of (11), as stated in Proposition 6, contains the interval $[0, T]$ for all $\varepsilon$ specified above. Furthermore, $M_\varepsilon(T) \leq 2M$ for $M$ from (31).

**Proof.** Note that by the Proposition 6 the proof reduced to an estimate of $\|u_{\varepsilon, \delta}\|_{H^4}$ or higher. We set $s = 12$ in the Proposition 6, so that Corollary 20 applies (and by Lemma 5 $(u_0)_\delta \in H^{12}$ for $\delta > 0$).

We combine Remark 7 with Corollary 18 to conclude that the bound $M$ and a time interval $T > 0$ from that Corollary are increasing in the following parameters:

$$M = M(\|u_0\|_{H^7}, \lambda(0)),$$

$$T = T \left(\frac{1}{\|u_0\|_{H^7}}, \lambda(0)\right),$$

and independent of $\varepsilon$. Note, that we have used (14) to relate $k(0)$ with $\lambda$.

We now conclude using Corollary 20 that for all $T' \leq T$,

$$\|u_{\varepsilon, \delta}\|_{C_{T'} H^4} \leq M_\varepsilon(T') \leq 2M.$$

Hence up to time $T$, the $H^4$ norm cannot blow up and (11) must be wellposed. □

The rest of the section is organized as follows. Proposition 15 can be proved directly from (11) and we prove it first. The estimate (30) is more involved and is done in Sections 5.3 through 5.5.
5.2. Proof of Proposition 15. The content of Proposition 15 is a bound for the dispersion and for a low norm of the solution. To estimate the dispersion and the $H^4$-norm, we first estimate the time derivative of the solution via the evolution equation in (11).

Proposition 22. Suppose $u_{\varepsilon, \delta} \in C_1^1 H^4 \cap C_0^2 H^8$ satisfies (11). Then

\begin{equation}
\| \partial_t u_{\varepsilon, \delta} \|_{H^4} \leq C (\| u_{\varepsilon, \delta} \|_{H^7}, \varepsilon \| u_{\varepsilon, \delta} \|_{H^8}).
\end{equation}

We omit the proof as it is fairly obvious, since the evolution equation involves at most three derivatives inside the function $f$, and the parabolic term has an $\varepsilon$ and a fourth derivative.

We next set down some slightly unconventional notation in order to simplify chain rule computations.

Remark 23. Denote $z_{-1} = x$ and $z_{-2} = t$. so that $(\bar{z}, z_{-1}, z_{-2}) = (\partial_x^3 u^\varepsilon, x, t)$. This way

\begin{equation}
\partial_{\bar{z}} z = (\partial_x [\partial_x^2 u^\varepsilon], 1, 0)
\end{equation}

Proof of Proposition 15. From (14), we need to analyze $\| u_{\varepsilon, \delta}(t) \|_{H^4}$ and $\| \partial_t u_{\varepsilon, \delta}(t) \|_{H^4}$, which we do separately.

From the Fundamental Theorem of Calculus

\begin{equation}
\begin{aligned}
\partial_t u_{\varepsilon, \delta}(t) &= (u_0)_t + \int_0^t \partial_t u_{\varepsilon, \delta}(t')dt'.
\end{aligned}
\end{equation}

Taking $H^4$-norms implies

\begin{equation}
\| u_{\varepsilon, \delta}(t) \|_{H^4} \leq \| (u_0)_t \|_{H^4} + \int_0^t \| \partial_t u_{\varepsilon, \delta}(t') \|_{H^4}dt'.
\end{equation}

Hence from Proposition 22

\begin{equation}
\| u_{\varepsilon, \delta}(t) \|_{H^4} \leq \| (u_0)_t \|_{H^4} + C(M_\varepsilon(t))t.
\end{equation}

We now proceed to an estimate of $\lambda(u_{\varepsilon, \delta}(t))$, which is deduced by a similar method. First, from the definition [90] and the Fundamental Theorem of Calculus,

\begin{equation}
f_{z_3}(\partial_x^3 u^\varepsilon(x, t)) = f_{z_3}(\partial_x^3 (u_0)_t(x)) + \int_0^t \partial_t f_{z_3}(\partial_x^3 u^\varepsilon)(x, t')dt'.
\end{equation}

Expanding the time derivative above yields

\begin{equation}
f_{z_3}(\partial_x^3 u^\varepsilon) \geq f_{z_3}(\partial_x^3 (u_0)_t) - t \sum_{j=-2}^{3} \| f_{z_3, z_j}(\partial_x^3 u^\varepsilon) \partial_t [z_j(x, t)] \|_{L_3^\infty},
\end{equation}

where we have used the notation from Remark 24. Thus

\begin{equation}
f_{z_3}(\partial_x^3 u^\varepsilon) \geq f_{z_3}(\partial_x^3 (u_0)_t) - tC(\| u_{\varepsilon, \delta} \|_{L_3^\infty W_3^{1, \infty}})(1 + \| \partial_t u_{\varepsilon, \delta} \|_{L_3^\infty W_3^{1, \infty}}).
\end{equation}

Hence by Sobolev embedding we obtain

\begin{equation}
\frac{1}{f_{z_3}(\partial_x^3 u^\varepsilon)} \leq \frac{1}{f_{z_3}(\partial_x^3 (u_0)_t)} \left[ 1 - \frac{TC(\| u_{\varepsilon, \delta} \|_{L_3^\infty W_3^{1, \infty}}, f_{z_3}(\partial_x^3 (u_0)_t))}{1} \right].
\end{equation}

Using Proposition 22 to estimate $\partial_t u_{\varepsilon, \delta}$, and using [90], allows us to express the above inequality as

\begin{equation}
\lambda(t) \leq \lambda(0) \frac{1}{1 - TC(M_\varepsilon(t), k(0))}.
\end{equation}

Combining this result with [90] concludes the proof. □
5.3. Reduction to the linear estimate. To prove Proposition [10] we aim to use linear estimate [8] from Theorem [8] for the equation that \( \partial_t \partial_x^n u_{\varepsilon, \delta} \) satisfies for \( n \geq 3 \). We begin with differentiation of [11].

Remark 24. Note that we distinguish between \( f_x(\partial_x^3 u^x) = [\partial_x f](\vec{z}, x, t) |_{\vec{z}=(\partial_x^3 u_{\varepsilon, \delta}, \ldots, u_{\varepsilon, \delta})} \) and

\[
\partial_x[f(\partial_x^3 u^x)] = \sum_{i=0}^{3} f_{z_i}((\partial_x^3 u^x)\partial_x^{i+1}u_{\varepsilon, \delta} + f_x(\partial_x^3 u^x) = \sum_{i=-1}^{3} f_{z_i}((\partial_x^3 u^x),
\]

where we use the notation from Remark 23 (i.e. \( z_{-1} = x \)).

Occasionally, to be more efficient we may omit the variable \( z \) and denote derivatives of \( f \) with just the indices, i.e. \( f_{-2} = f_t \) and \( f_{-1,2} = \frac{\partial^2 f}{\partial x \partial z_2} \), for example.

Proposition 25. Suppose \( u_{\varepsilon, \delta} \) solves [11]. Then

\[
(37) \quad \partial_t \partial_x^2 u_{\varepsilon, \delta} + f_{z_2} \partial_x^2 u_{\varepsilon, \delta} + (f_{z_2} + 3\partial_x[f_x(\partial_x^3 u^x)]) \partial_x^2 u_{\varepsilon, \delta} + \tilde{f}(\partial_x^4 u^x) = -\varepsilon \partial_x^2 u_{\varepsilon, \delta},
\]

where

\[
\tilde{f} \in C_1^0 C_7^\infty W_{x}^{\infty, \infty}.
\]

Remark 26. For completeness, we present the structure of \( \tilde{f} \) in the proof.

Proof. Differentiating \( f \) once we obtain

\[
\partial_x[f] = \sum_{i=-1}^{3} f_i \partial_x z_i.
\]

In the next step we have a product rule in addition to the chain rule:

\[
\partial_x^2[f] = \sum_{i,j=-1}^{3} f_{i,j} \partial_x z_i \partial_x z_j + \sum_{i=0}^{3} f_i \partial_x^2 z_i.
\]

Note that we eliminated terms that are obviously equal to zero, such as \( \partial_x^2 z_{-1} = \partial_x^2 (x) = 0 \). We then claim that applying a third derivative produces the following expression:

\[
(38) \quad \partial_x^3[f] = \sum_{i=0}^{3} f_i \partial_x^3 z_i + \sum_{i,j \leq 3} f_{i,j} \partial_x^2 z_i \partial_x z_j + 2 \sum_{i,j \leq 3} f_{i,j} \partial_x^2 z_i \partial_x z_j + \sum_{i,j,k \leq 3} f_{i,j,k} \partial_x z_i \partial_x z_j \partial_x z_k.
\]

The first two terms on the right-hand side of (38) include derivatives of the term \( \sum_{i=0}^{3} f_i \partial_x^3 z_i \), while in the third term on the right-hand side we have introduced a coefficient 2 obtained from changing the index of terms with \( \partial_x^2 z_i \partial_x z_i \) from \( i \) to \( j \). Further note that the second and third terms on the right-hand side may be combined.

From now on, most terms end up in the \( \tilde{f} \) function that we will define below. We make a couple of observations. First, the only term with 6 derivatives on \( u_{\varepsilon, \delta} \) is \( f_{z_2} \partial_x^2 z_3 \).

Second, we inspect the terms with \( \partial_x^2 u_{\varepsilon, \delta} \). They are

\[
f_{z_2} \partial_x^2 z_2 + 3 \sum_{j=-1}^{3} f_{z_3,j} \partial_x^2 z_3 \partial_x z_j = f_{z_2} \partial_x^3 z_2 + 3 \partial_x[f_x(\vec{z})].
\]

Finally, all the remaining terms combine into the term we call \( \tilde{f} \), written explicitly as follows:

\[
(39) \quad \tilde{f} = \sum_{i,j,k=-1}^{3} f_{i,j,k} \partial_x z_i \partial_x z_j \partial_x z_k + 3 \sum_{i<3, j \leq 3} f_{i,j} \partial_x^2 z_i \partial_x z_j + \sum_{i=0}^{1} f_i \partial_x^3 z_i.
\]

\( \square \)
From now on the coefficient of the second highest space derivative \( \partial^2_x u_{\epsilon, \delta} \) above will change in the “linear fashion” (i.e., similar in structure to differentiation of (16)). We continue with differentiation until this pattern appears for lower order terms by considering \( n \geq 7 \):

**Lemma 27.** Let \( u_{\epsilon, \delta} \) solve (11). Then \( \partial^n_x u_{\epsilon, \delta} \) satisfies the following expression for \( n \geq 7 \):

\[
\begin{align*}
\partial_t \partial^n_x u_{\epsilon, \delta} + f_{z_2}(\partial^3_x u^\epsilon) \partial^{n+3}_x u_{\epsilon, \delta} + \left( f_{z_2}(\partial^3_x u^\epsilon) + n \partial_z [f_{z_2}(\partial^3_x u^\epsilon)] \right) \partial^{n+2}_x u_{\epsilon, \delta} \\
+ a_{1,n}(\partial^{n+1}_x u^\epsilon) \partial^{n+1}_x u_{\epsilon, \delta} + a_{0,n}(\partial^n_x u^\epsilon) \partial^n_x u_{\epsilon, \delta} = - \epsilon \partial^{n+4}_x u_{\epsilon, \delta} + \tilde{f}_n(\partial^{n-1}_x u^\epsilon),
\end{align*}
\]

where the functions \( a_{0,n}, a_{1,n} \) and \( \tilde{f}_n \) are in \( C^1_1 C^{11-n}_1 W^{11-n}_x \).

**Remark 28.** Note that the number 11 in the index \( 11 - n \) in the regularity above comes from the condition (A1) in (40). Therefore a smoother nonlinear function \( f \) would allow \( n \geq 11 \).

Essentially, Lemma 27 is a tedious application of the chain rule, a.k.a. Faa Di Bruno formula. We do this in full detail, but we first make some reductions.

Once the equation (40) is known to be valid for some \( n \), say \( n = 7 \), we may differentiate for higher \( n \) inductively getting the following formulas:

\[
\begin{align*}
a_{1,n+1}(\partial^{n}_x u^\epsilon) &= a_{1,n}(\partial^{n-1}_x u^\epsilon) + \partial_z[f_{z_2} + n \partial_z[f_{z_2}(\partial^3_x u^\epsilon)]] \\
a_{0,n+1}(\partial^n_x u^\epsilon) &= a_{0,n}(\partial^{n-1}_x u^\epsilon) + \partial_z[a_{1,n}(\partial^n_x u^\epsilon)], \\
\tilde{f}_{n+1}(\partial^{n}_x u^\epsilon) &= \partial_z[\tilde{f}_n(\partial^{n-1}_x u^\epsilon)] + \partial_z[a_{0,n}(\partial^n_x u^\epsilon)]\partial^n_x u_{\epsilon, \delta}.
\end{align*}
\]

However, to establish the structure of (40) initially for \( n = 7 \) requires doing a full expansion of terms of order \( n \) similar to (38). Before doing this we make some calculations to prepare.

For the terms of order \( n + 3 \) and the terms of order \( n + 2 \), the structure in (40) appears as early as \( n = 3 \), which is why we have established Proposition 25 first. We thus establish such a lemma first by differentiating (47) \( n - 3 \) times and focusing on higher order terms.

**Lemma 29.** Let \( u_{\epsilon, \delta} \) solve (11). Then \( \partial^n_x u_{\epsilon, \delta} \) satisfies the following expression for \( n \geq 3 \):

\[
\begin{align*}
\partial_t \partial^n_x u_{\epsilon, \delta} + f_{z_2}(\partial^3_x u^\epsilon) \partial^{n+3}_x u_{\epsilon, \delta} + \left( f_{z_2}(\partial^3_x u^\epsilon) + n \partial_z [f_{z_2}(\partial^3_x u^\epsilon)] \right) \partial^{n+2}_x u_{\epsilon, \delta} \\
+ a_{1,n}(\partial^{n+1}_x u^\epsilon) \partial^{n+1}_x u_{\epsilon, \delta} + a_{0,n}(\partial^n_x u^\epsilon) \partial^n_x u_{\epsilon, \delta} = - \epsilon \partial^{n+4}_x u_{\epsilon, \delta} + \tilde{f}_n(\partial^{n-1}_x u^\epsilon)
\end{align*}
\]

where the functions \( a_{0,n}, a_{1,n} \) and \( \tilde{f}_n \) are in \( C^1_1 C^{11-n}_1 W^{11-n}_x \).

Note that the main difference between this lemma and Lemma 27 is the structure of the coefficients \( a_{1,n} \) and \( a_{0,n} \). In this lemma we simply collect all terms of order \( n + 1 \), \( n \) and below into \( a_{1,n} \partial^{n+1}_x u_{\epsilon, \delta}, a_{0,n} \partial^n_x u_{\epsilon, \delta} \) and \( \tilde{f}_n \), respectively, without excluding impossible terms.

**Proof.** Differentiate the equation (38) \( n - 3 \) times aiming to get a result similar to (47). First, the only way to obtain the term \( \partial^{n+3}_x u_{\epsilon, \delta} \) when differentiating (47) \( n + 3 \) times is to keep the \( f_{z_2} \) term and place all derivatives on \( \partial^{n}_x u_{\epsilon, \delta} \).

Second, for the \( \partial^{n+2}_x u_{\epsilon, \delta} \) term, a part of the coefficient comes from differentiating the \( f_{z_2} + 3 \partial_z[f_{z_2}(u^\epsilon)] \partial^n_x u^\epsilon \) term and the remaining \( (n-3) \partial_n[f_{z_2}(u^\epsilon)] \) comes from differentiating the \( f_{z_2} \partial^{n}_x u_{\epsilon, \delta} \) term.

Third, note that all the coefficients in the equation (40) are obtained from \( n \) derivatives placed on the nonlinear function \( f \) in (11). By the condition (A1) we have assumed \( f(\bar{z}, x, t) \in C^1_1 C^{11}_1 W^{11}_x \) and hence \( \partial^n_x \partial^\beta f(\bar{z}, x, t) \in C^1_1 C^{11-n}_1 W^{11-n}_x \) for \( |\alpha| + |\beta| = n \).

Finally, we arrange all terms of order \( n + 1 \) in the form \( a_{1,n} \partial^{n+1}_x u_{\epsilon, \delta} \), terms of order \( n \) as \( a_{0,n} \partial^n_x u_{\epsilon, \delta} \) and terms of order less than \( n \) as \( \tilde{f}_n \). □

We now return to the explicit analysis of \( a_{1,n}, a_{0,n} \) and \( \tilde{f}_n \). In order to track the dependence of the coefficients of order \( 7 \) and higher, we need an expansion of all the possible terms that arise in \( \partial^n_x [f] \) similar to (38) that works for all \( n \geq 1 \).
Since the expansion is messier, we will change the notation and explain this change in the case of \( n = 3 \). In order to keep track of terms of the type \( \partial_x^\alpha \partial_z^\beta f \), we organize them by the index \( k = |\alpha| + |\beta| \).

The terms containing \( \partial_x^\alpha \partial_z^\beta f \) will be paired with a polynomial of degree \( k \) in derivatives of \( u \), which we need to label. For each of the \( \beta \) derivatives, which we enumerate with a label \( l \), there are 4 choices that each \( z \)-derivative produces: \( z_{-1} = x \), \( z_0 = u \), \( z_2 = \partial_z^2 u \) (by the notation of the Remarks 23 and 24). We reserve the subscript \( j \) for each of those choices, i.e. \( j = 0 \) will lead to a \( u \) term, rather than a \( \partial_z u \) term from \( j = 1 \).

Finally, if we include \( z_{-1} = x \) each of the \( n \) derivatives hits a \( z \) term. For example a term

\[
f_{1,0,-1}(z)\partial_x^2 z_1 \partial_x z_0 \partial_x z_{-1}
\]

is obtained by differentiating the nonlinear function \( f \) three times in direction \( z_1, z_0 \) and \( z_{-1} = x \), followed by one more derivative of the function \( z_{-1} = \partial_z u \) to get to \( \partial_x^2 z_1 \). We label the index \( \partial_x^2 z_2 \) as \( i_1 = 2 \), the index of \( z_0 \) as \( i_2 = 1 \) and the index of \( z_{-1} \) as \( i_3 = 1 \) proceeding from highest index to lowest. With this language in mind, \( \partial_x^\alpha f \) can be expressed as:

\[
\partial_x^\alpha[f] = \sum_{k=1}^{n} \sum_{i_1 \geq i_2 \geq \ldots \geq i_k \geq 1} \sum_{l=1}^{\min(i_2,3)} \sum_{k=1}^{\min(i_1,\min(i_2,3)-\min(i_2,3)-1)} C_{i_1,i_2,\ldots,i_k} f_{i_1,i_2,\ldots,i_k}(z) \partial_x^{i_1} z_{i_1} \cdots \partial_x^{i_k} z_{i_k}. \tag{44}
\]

Note that the \( C_{i_1,i_2} \) parameters appear from the rearrangement of terms and product rule.

We demonstrate the change of notation from the case of \( n = 3 \) in (38). The terms for \( k = 1 \) are \( \sum_{i=0}^{3} f_i \partial_x^3 z_i \), which we relabel as

\[
3 \sum_{i=0}^{3} f_i \partial_x^3 z_i = \sum_{j_1 = -1}^{3} f_{j_1} \partial_x^3 z_{j_1}.
\]

The terms for \( k = 2 \) are relabeled as

\[
3 \sum_{i,j \leq 3} f_{i,j} \partial_x^2 z_i \partial_x z_j = \sum_{j_1 = -1}^{3} \sum_{j_2 = -1}^{3} 3f_{j_1,j_2} \partial_x^2 z_{j_1} \partial_x z_{j_2}.
\]

and the terms for \( k = 3 \) become

\[
\sum_{i,j,k \leq 3} f_{i,j,k} \partial_x z_i \partial_x z_j \partial_x z_k = \sum_{j_1 = -1}^{3} \sum_{j_2 = -1}^{3} \sum_{j_3 = -1}^{3} f_{j_1,j_2,j_3} \partial_x z_{j_1} \partial_x z_{j_2} \partial_x z_{j_3}.
\]

**Proof of Lemma 27** By Lemma 29 and the inductive formula (44) the proof reduces to the analysis of the coefficients \( a_{1,7} \) and \( a_{0,7} \).

We now return to use the notation in (44) to confirm that \( a_{1,7} \) depends on no more than five derivatives of \( u_{\epsilon,\delta} \). It is easier to do the analysis for \( k = 1, k = 2 \) and greater separately:

\[
\partial_x^2[f] = \sum_{j_1 = 0}^{3} f_{j_1} \partial_x^3 z_{j_1} + \sum_{l=1}^{6} \sum_{i_1 = 4}^{3} \sum_{j_1,j_2 \leq 3} C_{i_1,j_1,j_2} f_{i_1,j_1,j_2} \partial_x^{i_1} z_{j_1} \partial_x^{i_2} z_{j_2} + \sum_{l=1}^{7} \sum_{i_1 \geq i_2 \geq \ldots \geq i_l \geq 1} \sum_{l=1}^{\min(i_2,3)-\min(i_2,3)-1} \sum_{k=1}^{\min(i_1,\min(i_2,3)-\min(i_2,3)-1)} C_{i_1,j_1,\ldots,j_l}(z) \partial_x^{i_1} z_{j_1} \cdots \partial_x^{i_l} z_{j_l} |_{z = (\partial_x u_{\epsilon,\delta}, x, t)}.
\]

In the sum above we have excluded some of the vanishing terms, like \( j_1 = -1 \) for \( k = 1 \). We also reorganized the sum for \( k = 2 \), relabeling \( i_2 = 7 - i_1 \) and exploiting \( \sum_{i_3 \geq i_2 \geq \ldots \geq i_l \geq 1} \sum_{l=1}^{\min(i_2,3)-\min(i_2,3)-1} \sum_{k=1}^{\min(i_1,\min(i_2,3)-\min(i_2,3)-1)} C_{i_1,j_1,\ldots,j_l}(z) \partial_x^{i_1} z_{j_1} \cdots \partial_x^{i_l} z_{j_l} |_{z = (\partial_x u_{\epsilon,\delta}, x, t)} \).

The terms for \( i_1 + j_1 = 10 \), as well as \( i_1 + j_1 = 9 \), are already accounted for in (40) as coefficients of \( \partial_x^{10} u_{\epsilon,\delta} \) and \( \partial_x^{9} u_{\epsilon,\delta} \) for \( n = 7 \). Therefore, to obtain \( a_{7,1} \), the coefficient for \( \partial_x^7 u_{\epsilon,\delta} \), we need to
focus on terms with $i_1 + j_1 = 8$. Note that as $i_2 + j_2 = 7 - i_1 \leq 3$ if $k \geq 2$, it is impossible to obtain $i_2 + j_2 = 8$ for $n = 7$.

We do an enumeration of all terms of order 8 using $i_1 + j_1 = 8$. If $i_1 = 7$, then $k = 1$ to satisfy $i_1 + i_2 + \ldots + j_3 = 7$ and all $i$ terms being positive. If $i_1 = 6$, $j_1 = 2$ then $k = 2$ with $i_2 = 2$. If $i_1 = 5$, $j_1 = 8 - i_1 = 3$ then either $k = 3$ and $i_2 = k_3 = 1$ or $k = 2$ and $i_2 = 2$. Explicitly, all terms of order 8 can be listed as follows:

\[ f_1 \partial_x^2 z_1 + \sum_{i=5}^{6} \sum_{j_2=-1}^{3} C_{i,j} f_{j_1} = z_{i_2-j_2} \partial_x^2 z_{j_2} + \sum_{j_2,j_3=-1}^{3} C_{i,j} f_{j_2,j_3} z_{j_3} (z) \partial_x^2 z_{j_3}. \]

We now substitute $\tilde{z} = (\partial_x^2 u_{\epsilon,\delta}, \ldots, u_{\epsilon,\delta}, x, t)$ to get:

\[
\partial_x^8 u_{\epsilon,\delta} \cdot a_{1,7} := \partial_x^8 u_{\epsilon,\delta} \cdot [f_1 + \sum_{j_2=-1}^{3} C_{i,j} f_{j_2,j_3} \partial_x^{j_2+1} u_{\epsilon,\delta} + \sum_{j_2=0}^{3} C_{i,j} f_{j_2,j_3} \partial_x^{j_2+2} u_{\epsilon,\delta} + \sum_{j_2,j_3=-1}^{3} C_{i,j} f_{j_2,j_3} (z) \partial_x^{j_2+1} u_{\epsilon,\delta} \partial_x^{j_2} u_{\epsilon,\delta}].
\]

We now claim that $a_{1,7} = a_{1,7} (\partial_x^3 u^\epsilon)$. Indeed, the nonlinear function $f$ depends on at most $\partial_x^3 u_{\epsilon,\delta}$, and the highest derivative of $u_{\epsilon,\delta}$ possible inside is the term $\partial_x^{j_2+2} u_{\epsilon,\delta}$, where $j_2 = 3$ is possible.

The analysis of the $\partial_x^7 u_{\epsilon,\delta}$ coefficient, $a_{0,7}$ is similar. \(\square\)

**Corollary 30.** By restricting inadmissible indices in the expansion of $\partial_x^2 [f]$ in (44) we can also get an explicit description of $\tilde{f}_n$:

\[
\tilde{f}_n = \sum_{k=2}^{n} \sum_{\tilde{i}, \tilde{j} \in \mathcal{S}_k^n} C_{\tilde{i}, \tilde{j}} \cdot f_{j_1, \ldots, j_k} (z) \partial_x^{i_1} z_{j_1} \cdot \ldots \cdot \partial_x^{i_k} z_{j_k},
\]

where the $k$-tuples $\tilde{i}, \tilde{j}$ of admissible indices $\mathcal{S}_k^n$ are defined by

\[
\mathcal{S}_k^n = \left\{ 1 \leq i \leq k; i_1 \geq i_2 \geq \ldots \geq i_k \geq 1; \sum_{i=1}^{k} i_i = n; -1 \leq j_i \leq 3; i_i + j_i < n; \max \{ j_i, 1 - i_i \} \geq 0 \right\}.
\]

Note that we will use this description for $n = 11$ later in the paper.

**Proof.** From (44) we remove all terms involving more than $n - 1$ derivatives of $u_{\epsilon,\delta}$. In the language of (44) this means $i_i + j_i < n$. The indices in $\mathcal{S}_k^n$ are those that remain organized by the parameter $k$ that counts the number of $z$ derivatives on the nonlinear function $f$.

Note that $k = 1$, for example, leads to $i_1 = n$ and hence only contain terms of order higher than $n - 1$. Similarly the condition max $\{ j_i, 1 - i_i \} \geq 0$ is just a statement that more than two derivatives annihilate $z_{-1} = x$. \(\square\)

We aim to consider a linear problem (50) with coefficients from (40):

\[
\begin{cases}
    \partial_t w + \sum_{j=0}^{3} a_j (x, t) \partial_x^j w = -\varepsilon \partial_x^{j_2} w + \tilde{f}_n (\partial_x^{j_2 - 1} u^\epsilon), \\
    w(x, 0) = \partial_x^0 (u_0)^{\delta}(x),
\end{cases}
\]

where $a_3 := f_{z_3} (\partial_x^3 u^\epsilon)$, $a_2 := \{ f_{z_2} + n \partial_x [f_{z_3} (\partial_x^{j_2} u^\epsilon)] \}$, and $a_j := a_j, n (\partial_x^{j_2} u^{\epsilon})$.

Note that $w = \partial_x^2 u_{\epsilon,\delta}$ is a solution of that linear equation and we can apply Theorem 3 to it to find the following estimate:

\[
\| \partial_x^2 u_{\epsilon,\delta} \|_{L^2} \leq C(k_G (t)) \exp \left( \int_0^t \bar{C}(M (t')) dt' \right) \left( \| \partial_x^0 (u_0)^{\delta} \| + \| \tilde{f}_n (\partial_x^{j_2 - 1} u^\epsilon) \|_{L^1 L^2} \right).
\]
Remark 31. To emphasize the dependence of the constants $k_G$ and $\tilde{M}$ upon the coefficients of $\delta^7 u^7$ and hence implicitly on $\partial^6_x u^6$ we will add superscripts of $n$, such as $k^n_G$ and $\tilde{M}^n$. That is, we will denote the coefficient norms in Theorem 3 for (46) as $k^n_G$ and $\tilde{M}^n$.

5.4. Remainder terms and coefficient estimates. Estimate (17) is a crucial ingredient for the proof of Proposition 16. We have essentially reduced the estimate of the $H^7$-norm and the $H^8$-norm to a proper estimate of coefficients captured by $k_G$ and $\tilde{M}$, as well as by the lower order terms that we denote by $\tilde{f}_n$. We begin with the estimate of the lower order terms $\tilde{f}_n$ for $n = 7$ and $n = 8$.

Lemma 32. Let $\tilde{f}_n$ be as in (46). Then the following bounds are satisfied:

\begin{align}
\|\tilde{f}_7\|_{L^\infty T^\infty L_x^2} &\leq C(\|u_{\varepsilon, \delta}\|_{H^7}), \\
\|\tilde{f}_8\|_{L^\infty T^\infty L_x^2} &\leq C(\|u_{\varepsilon, \delta}\|_{H^7})\|u_{\varepsilon, \delta}\|_{H^8}.
\end{align}

Proof. Since $\tilde{f}_7(0, x, t) = 0$, we can use the Fundamental Theorem of Calculus as follows:

$$\tilde{f}_7 = \sum_{j=0}^6 \partial^j_x u_{\varepsilon, \delta} \int_0^1 \partial_{x,j} \tilde{f}_7(s \partial^j_x u^7) ds.$$ 

Since $\tilde{f}_7 \in C^1_1 C^4_2 W^{4,\infty}_x$, we may apply (3) to the integrands, concluding

$$\left| \int_0^1 \partial_{x,j} f(s \partial^j_x u^7) ds \right| \leq C(\|u_{\varepsilon, \delta}\|_{W^{6,\infty}}).$$

Therefore

$$\|\tilde{f}_7\|_{L^\infty T^\infty L_x^2} \leq C(\|u_{\varepsilon, \delta}\|_{W^{6,\infty}})\|u_{\varepsilon, \delta}\|_{H^8},$$

and Sobolev embedding, i.e. the inclusion $H^1(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, allows us to conclude (18) for $n = 7$.

Now for $n = 8$, we put fewer derivatives in $L^\infty$ by using finer structure of $\tilde{f}_8$. Namely, from (12) we obtain

$$\tilde{f}_8 = \sum_{j=-1}^5 \partial_{x,j} \tilde{f}_7 + \partial_{x, j} a_0 \partial^j_x u_{\varepsilon, \delta} \|\tilde{f}_7\|_{L^\infty} \|u_{\varepsilon, \delta}\|_{H^7}.$$

Estimating the $L^2$-norm of the expression above, we have

$$\|\tilde{f}_8\|_{L^2_x} \leq \sum_{j=-1}^5 \|\partial_{x,j} \tilde{f}_7 + \partial_{x, j} a_0 \partial^j_x u_{\varepsilon, \delta}\|_{L^2_x} \|\partial^j_x u_{\varepsilon, \delta}\|_{L^\infty} \|u_{\varepsilon, \delta}\|_{H^7}.$$

Since both $\tilde{f}_7$ and $a_0$ are in $C^1_1 C^4_2 W^{4,\infty}_x$, we may further estimate coefficients by (3), yielding

$$\|\tilde{f}_8\|_{L^2_x} \leq C(\|u_{\varepsilon, \delta}\|_{W^{6,\infty}}) \cdot (1 + \|u_{\varepsilon, \delta}\|_{W^{7,\infty}})\|u_{\varepsilon, \delta}\|_{H^7},$$

where we have used that no more than one factor of $\partial^j_x u_{\varepsilon, \delta}$ is present in the $L^\infty$ terms. Using Sobolev embedding we conclude (18) for $n = 8$. \hfill \Box

We now estimate the coefficients for (46).

Lemma 33. Let $u_{\varepsilon, \delta} \in C^0_1 H^8$ for $\varepsilon > 0$ or $u_{\varepsilon, \delta} \in C^0_1 H^7$ for $\varepsilon = 0$ satisfy (11) and consider the linear equation (16). Then the coefficient norms from (17) for the equation (46) can be estimated as follows (where we follow the convention of the Remark 31):

\begin{align}
k^n_G(t) &\leq C(k(t), n), \\
\tilde{M}^n(t) &\leq C(M_\varepsilon(t), k(t), n),
\end{align}

where $k(t)$ and $M_\varepsilon$ are as in (14) and (13) respectively.
Remark 34. Note that Lemma \(\text{X3}\) and Theorem \(\text{X8}\) determine the regularity we pursue in Theorem \(\text{X9}\); that is, these are the steps of our proof which cause us to work in the space \(H^7\). Knowing more precise structure of the function \(f\) in \(\text{X1}\), e.g. if \(f\) is “less nonlinear,” would lower the regularity needed in our proof. In particular, our argument could utilize the spaces \(H^4\) for \(K(2,2)\)-type equations and \(H^{7+\varepsilon}\) for \(\text{KIV}\).

Proof. We first estimate the lower-order norms for \(k_G\). The estimates of the dispersive coefficient follow from the coefficient hypothesis \((A1)\) and the lower bound on the nonlinear dispersion \((i)\).

More precisely, from the definition of the coefficient \(a_3 = f_{z_3} (\partial_x^3 u^\varepsilon, ...), \) we see
\[
\|a_3\|_{L^\infty} + \left\| \frac{1}{a_3} \right\|_{L^\infty} \leq \|f_{z_3}\|_{L^\infty} + \left\| \frac{1}{f_{z_3}} \right\|_{L^\infty}.
\]

Using \((3)\) for \(f_{z_3} (\partial_x^3 u^\varepsilon)\), the definition of \(\lambda(t)\) in \((6)\), and Sobolev embedding implies
\[
\|a_3\|_{L^\infty} + \left\| \frac{1}{a_3} \right\|_{L^\infty} \leq C(\|u^\varepsilon\|_{W^{3,\infty}}) + \frac{1}{\lambda(t)} \\
\leq C\left(\|u^\varepsilon\|_{H^4}, \frac{1}{\lambda(t)}\right).
\]

It remains to estimate \(\int_0^x \frac{a_3}{a_3} dx'\|_{L^\infty}\) to finish \((19)\) for \(\hat{k}_G\). From the definition of the coefficients in \((13)\), we have
\[
\int_0^x \frac{a_3}{a_3} dx' = n \log f_{z_3} (\partial_x^3 u^\varepsilon, ...). \]

Observe that a logarithm is dominated by its argument:
\[
\log y \leq y + \frac{1}{y^2} \quad \text{for} \quad y > 0.
\]

Hence the logarithm is comparable to the norm previously estimated above:
\[
\| \log f_{z_3} (\partial_x^3 u^\varepsilon) \| \leq \|a_3\|_{L^\infty} + \left\| \frac{1}{a_3} \right\|_{L^\infty}.
\]

Meanwhile, from \((11)\) we have
\[
\int_0^x g_M dx' = g_D (\partial_x^3 u^\varepsilon) + \int_0^x g_H (\partial_x^3 u^\varepsilon) dx'.
\]

The term \(g_D\) is controlled by \((3)\):
\[
\|g_D\|_{L^\infty} \leq C(\|u^\varepsilon\|_{W^{2,\infty}}).
\]

Continuing, the Taylor expansion of \(g_H\) to the quadratic terms using \((15)\) implies
\[
g_H (\partial_x^3 u^\varepsilon) = \sum_{i,j=0}^3 \partial_x^i u_{\varepsilon, \delta} \partial_x^j u_{\varepsilon, \delta} \int_0^1 \int_0^1 \partial_{z_i, z_j} g_H (s_1 s_2 \partial_x^2 u^\varepsilon) s_1 ds_2 ds_1.
\]

We then use Cauchy-Schwarz, \((3)\), and Sobolev embedding:
\[
\left\| \int_0^x g_H (\partial_x^3 u^\varepsilon) dx' \right\|_{L^\infty} \leq \sum_{i,j=0}^3 \|\partial_x^i u_{\varepsilon, \delta}\|_{L^2} \|\partial_x^j u_{\varepsilon, \delta}\|_{L^2} \sup_{\|s\| \leq 1} \|\partial_{z_i, z_j} g_H (s \partial_x^2 u^\varepsilon)\|_{L^\infty} \leq C(\|u_{\varepsilon, \delta}\|_{H^4}).
\]

Estimates of \(\hat{M}(t)\) are quite similar to estimates of \(\hat{k}_G(t)\). The relevant coefficients in \((40)\) can be written in the form \(a_j = a_{j,n} (\partial_x^3 u_{\varepsilon, \delta})\) for \(a_j\) satisfying \(a_j \in C^1(\varepsilon, \varepsilon^3 W^{7+\varepsilon, \infty}\) by \((11)\) and \((A1)\). Thus
\[
\|a_j\|_{W^{3-j, \infty}} \leq C(\|\partial_x^3 u^\varepsilon\|_{W^{2-j, \infty}}) \leq C(\|u_{\varepsilon, \delta}\|_{H^7}) = C(M_\varepsilon(t)).
\]
We then observe that \( \partial_t[a_3(x,t)] = f_{z_3,t} + \sum_{j=0}^{3} f_{z_3,z_j} (\partial_x^2 u^e) \partial_i \partial_{x}^2 u_{\varepsilon,\delta} \). Estimating as above, and using Proposition \[P22\]
\[
\| \partial_t a_3 \|_{L^\infty} \leq C(\| \partial_t u_{\varepsilon,\delta} \|_{H^4}, \| u_{\varepsilon,\delta} \|_{H^4}) \leq C(M_e(t)).
\]
Meanwhile, differentiation yields the following:
\[
\partial_t \int_0^x \frac{a_2}{a_3} dx' = n \partial_t [ \log f_{z_3}(\partial_x^2 u^e)] + \int_0^x \partial_t [g_M] dx'.
\]
For the first term, \( \partial_t [ \log f_{z_3}(\partial_x^2 u^e)] \leq C(\| \partial_t a_3 \|_{L^\infty}, \| a_3 \|_{L^\infty}, \| \frac{1}{a_3} \|_{L^\infty}) \). Then \( \partial_t [g_M] \) is estimated similarly to \( \int_0^x \frac{a_2}{a_3} dx' \), with the additional \( \partial_i \partial_{x}^2 u_{\varepsilon,\delta} \) terms estimated with Proposition \[P22\]. These considerations yield the following:
\[
\left\| \partial_t \int_0^x \frac{a_2}{a_3} dx' \right\|_{L^\infty_x} \leq C \left( \| u_{\varepsilon,\delta} \|_{H^7}, \varepsilon \| u_{\varepsilon,\delta} \|_{H^5}, \frac{1}{\lambda(t)} \right).
\]
This completes the proof. \( \square \)

5.5. Proof of Proposition \[P16\]

Proof. By applying Lemma \[P33\] we conclude from \[P47\] for \( n \geq 7 \) that
\[
\| \partial_x^2 u_{\varepsilon,\delta}(t) \|_{L^2_x} \leq C(k(t), n) \exp \left( \int_0^t C(M_e(t'), k(t')) dt' \right) \left( \| \partial_x^n u_0 \|_{L^2_x} + t \| \tilde{f}_n(\partial_x^{n-1} u^e) \|_{L^\infty_x L^2_t} \right).
\]
Adding this estimate to \[P50\] we obtain
\[
\| u_{\varepsilon,\delta}(t) \|_{H^7} \leq C(k(t)) \exp \left( C(\sup_{t' \leq t} [M_e(t'), k(t')]) \right) \left( \| u_0 \|_{H^7} + t C(M_e(t)) + t \| \tilde{f}_n(\partial_x^{n-1} u^e) \|_{L^\infty_x L^2_t} \right).
\]
Using Lemma \[P32\] for \( \tilde{f}_5 \) implies
\[
\| u_{\varepsilon,\delta}(t) \|_{H^7} \leq C(k(t)) \exp \left( C(\sup_{t' \leq t} [M_e(t'), k(t')]) \right) \left( \| u_0 \|_{H^7} + t C(M_e(t)) + t C(\| u_{\varepsilon,\delta}(t') \|_{L^\infty_x H^7}) \right).
\]
Meanwhile the use of \[P48\] for \( \tilde{f}_8 \) gives
\[
(51) \quad \varepsilon \| u_{\varepsilon,\delta}(t) \|_{H^8} \leq C(k(t)) \exp \left( C(\sup_{t' \leq t} [M_e(t'), k(t')]) \right)
\]
\[
\cdot \left( \| u_0 \|_{H^8} + \varepsilon t C(M_e(t)) + t C(\| u_{\varepsilon,\delta}(t') \|_{L^\infty_x H^7} \cdot \| u_{\varepsilon,\delta}(t') \|_{L^\infty_x H^8}) \right).
\]
Adding the last two estimates concludes the proof. \( \square \)

5.6. Refined boundedness. The following lemma is not needed for the proof of the Proposition \[P16\] for which the estimates of \( \tilde{f}_7 \) and \( \tilde{f}_8 \) are enough. However, in order to justify the \( C_1H^7 \) regularity of the solution we need a more precise estimate of \( \tilde{f}_{11} \). We can see this effect quite well in \[P51\], where an estimate \( \| \tilde{f}_5 \|_{L^2} \leq C(\| u_{\varepsilon,\delta} \|_{H^8}) \) would not be sufficient. Thus, the lemma below can be thought of as a refinement of Lemma \[P32\].

Lemma 35. Let \( u_{\varepsilon,\delta} \) be a solution of \[P11\] and let \( \tilde{f}_{11} \) be as in \[P40\] for \( n = 11 \). Then
\[
(52) \quad \| \tilde{f}_{11} \|_{L^2} \leq C(\| u_{\varepsilon,\delta} \|_{H^7}) \| u_{\varepsilon,\delta} \|_{H^{11}}.
\]

This lemma would allow us to show persistence of regularity, i.e. a solution with \( H^{11} \) data has an \( H^{11} \) solution on the same time interval. We defer the proof of Lemma \[P59\] until we prove the following corollary, which is the main motivation for the lemma.

Corollary 36. For \( u_{\varepsilon,\delta} \) as before and \( M, T \) from Corollary \[P21\] there exists a constant \( C = C(M,k(0)) \), such that
\[
\| u_{\varepsilon,\delta} \|_{L^\infty_{|t|} H^{11}} \leq C \delta^{-4}.
\]
Proof. As $u_{\varepsilon, \delta}$ satisfies (11), $\partial_{x}^{11} u_{\varepsilon, \delta}$ satisfies (10) for $n = 11$. We now apply the linear estimate, Theorem 8 with the coefficients for $\partial_{x}^{11} u_{\varepsilon, \delta}$ as in (16). First, observe that coefficients in (10) for $n = 11$ satisfy the same bounds as those for $\partial_{x}^{11} u_{\varepsilon, \delta}$ by Lemma 35. Second, observe that we can use Theorem 8 on an interval $[t_0, t_1] \subset [0, T]$ rather than $[0, t_1]$. With those two observations in mind we get, for $0 \leq t' \leq 1$,
\[ \| \partial_{x}^{11} u_{\varepsilon, \delta}(t') \|_{L^2} \leq C(M(t), k(t))(\| u_{\varepsilon, \delta}(t_0) \|_{H^{11}} + \| \hat{f}_{11} \|_{L_{[t_0, t_1]}^{11} L^2}). \]
We extract the time factor and estimate $\hat{f}_{11}$ in $L^\infty$ in time. Furthermore, Lemma 35 allows us to estimate the remainder $\hat{f}_{11}$ with no more than a single factor of $\partial_{x}^{11} u_{\varepsilon, \delta}$:
\[ \| \partial_{x}^{11} u_{\varepsilon, \delta}(t) \|_{L^2} \leq C(M(t), k(t))(\| u_{\varepsilon, \delta}(t_0) \|_{H^{11}} + (t_1 - t_0)C(\| u_{\varepsilon, \delta} \|_{L_{[t_0, t_1]}^\infty H^T})\| u_{\varepsilon, \delta}(t_0) \|_{L_{[t_0, t_1]}^\infty H^{11}}). \]
As $[t_0, t_1] \subset [0, T]$, $\| u_{\varepsilon, \delta}(t') \|_{H^T} \leq M$ for $t' \in [t_0, t_1]$. Incorporating this estimate, after adding the $L^2$ norm, we get
\[ \| u_{\varepsilon, \delta}(t') \|_{H^{11}} \leq C(\| u_{\varepsilon, \delta} \|_{L^2} + \| \partial_{x}^{11} u_{\varepsilon, \delta} \|_{L^2}) \leq C(M) + C(M, k(t'))\| u_{\varepsilon, \delta}(t_0) \|_{H^{11}} + (t_1 - t_0)C(M, k(t'))\| u_{\varepsilon, \delta}(t_0) \|_{L_{[t_0, t_1]}^\infty H^{11}}. \]
By Remark 13 $k(t') \leq 4k(0)$. We can thus eliminate the dependence of the bound on $t'$ at a cost of a larger constant:
\[ \| u_{\varepsilon, \delta}(t') \|_{H^{11}} \leq C(M) + C(M, k(0))\| u_{\varepsilon, \delta}(t_0) \|_{H^{11}} + (t_1 - t_0)C(M, k(0))\| u_{\varepsilon, \delta}(t_0) \|_{L_{[t_0, t_1]}^\infty H^{11}}. \]
Furthermore, we let $t_1 = t_0 + \Delta t$ and we make the width of the interval $\Delta t$ small enough so that
\[ (t_1 - t_0)C(M, k(0)) = \frac{1}{2}. \]
This choice allows us to eliminate the $H^{11}$ term on the right hand side:
\[ \| u_{\varepsilon, \delta}(t_0) \|_{L_{[t_0, t_0 + \Delta t]}^\infty H^{11}} \leq C(M) + C(M, k(0))\| u_{\varepsilon, \delta}(t_0) \|_{H^{11}}. \]
We can now iterate this estimate for $t_0 = 0$, $\Delta t$, $2\Delta t$, ..., $j\Delta t$, where
\[ j\Delta t \geq T \text{ for } T \text{ from Corollary 38} \]
so that we get
\[ \| u_{\varepsilon, \delta}(t_0) \|_{L_{[t_0, t_0 + j\Delta t]}^\infty H^{11}} \leq C(M, k(0))^j(1 + \| u_{\varepsilon, \delta}(0) \|_{H^{11}}). \]
Using Lemma 5 implies that $\| u_{\varepsilon, \delta}(0) \|_{H^{11}} \leq C(M)\delta^{-4}$ and concludes the proof. □

We now return to the proof of Lemma 35. In the proof we need the following variation of a basic interpolation result.

Lemma 37. Let $w \in H^{11}$. Then for $0 \leq \theta \leq 4$,
\[ \| w \|_{H^{7+\theta}} \leq \| w \|_{H^{11}}^{\frac{\theta}{7}} \| w \|_{H^7}^{\frac{7}{\theta}}. \]
Proof. Use the Plancherel Theorem and use Hölder’s inequality for the function $\hat{\psi}(\xi) = (\xi)^{7} \hat{\psi}(\xi)$:
\[ \int |\hat{\psi}(\xi)|^{2\theta} |\xi|^{2\theta} d\xi \leq \left( \int |\hat{\psi}(\xi)|^{2+\theta} |\xi|^{2+\theta} d\xi \right)^{\frac{\theta}{7}} \left( \int |\hat{\psi}(\xi)|^{2} d\xi \right)^{\frac{7}{\theta}}. \]
This concludes the proof. □

Proof of Lemma 35. We use the precise variant from the Corollary 30
\[ \hat{f}_{11} = \sum_{k=2}^{11} \sum_{\mathcal{U}_{k,j} \in \mathcal{S}_k^1} C_{i,j} \cdot f_{j_1, ..., j_k}(z) \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \cdots \partial_{x_d}^{j_k} \z_{j_k} |z=(\partial_{x}^t u^x, x, t). \]
where the $k$-tuples $\tilde{i}, \tilde{j}$ of admissible indices $S_k$ are defined by the following:

$$S_k^{11} = \left\{ 1 \leq l \leq k; i_1 \geq i_2 \geq \ldots \geq i_k \geq 1; \sum_{i=1}^{k} i_l = 11; -1 \leq j_l \leq 3; i_l + j_l < 11; \max\{j_l, 1-i_l\} \geq 0 \right\},$$

i.e. we include all terms in $\partial_x^{11}f$ that are of order less than 11 in $u$. We then place the highest-order term in $L^2$, remaining terms in $L^\infty$, and analyze four different scenarios:

$$\|\tilde{f}_{11}(\partial_x^{10}u^r)\|_{L^2} \leq \sum_{k=2}^{11} \sum_{i_1, j_1 \in S_k} C_{i_1}^1 \|f_{j_1, \ldots, j_k}(z)\|_{L^\infty} \|\partial_x^{i_1}z_{j_1}\|_{L^2} \|\partial_x^{i_2}z_{j_2}\|_{L^\infty} \cdots \|\partial_x^{i_k}z_{j_k}\|_{L^\infty}$$

$$:= I_{\geq 8} + I_7 + I_6 + I_{\leq 5}.$$  

Here, the sum is separated by the largest number of derivatives $i_1$. That is, $I_{\geq 8}$ includes all the terms where $i_1 \geq 8; I_7$, where $i_1 = 7$, etc... We estimate the new $I_l$ sums term by term.

For any sum $I_{\leq 5}$ through $I_{\geq 8}$, the fact that the number of $i_l$ derivatives adds up to 11 means that $k - 1 \leq \sum_{l=2}^{k} i_l \leq 11 - i_1$. In particular, for $I_{\geq 8}$, $k \leq 4$ and $i_2 \leq 3$. Therefore all the $L^\infty$ terms have at most 3 derivatives. We also estimate $f_{j_l}$ via (3):

$$I_{\geq 8} \leq \sum_{i_1=8}^{10} \sum_{j_1 < 11} C(\|z\|_{L^\infty}) (1 + \|z\|_{W^3, \infty}^4) \|\partial_x^{i_1}z_{j_1}\|_{L^2}.$$  

Using Sobolev embedding and $z = (\partial_x^3u^r, x, t)$ we get

$$I_{\geq 9} \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{11}}.$$  

For $I_7$, $i_1 = 7$, which allows $i_2 \leq 4$ and $i_2 \leq 11 - i_1 - i_2 \leq 3$ for $l \geq 3$. Hence by (3) as before,

$$I_7 \leq C(\|z\|_{L^\infty}) \cdot \|\partial_x^{i_2}z\|_{L^2} \cdot (1 + \|z\|_{W^4, \infty}) (\|z\|_{W^3, \infty}).$$

Here, we have estimated terms beyond $i_2$ with $W^{3, \infty}_x$ norm. Hence by Sobolev embedding,

$$I_7 \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{10}} \cdot \|u_{x, \delta}\|_{H^{8}}.$$  

We now use Lemma 37 to conclude

$$I_7 \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{10}} \cdot \|u_{x, \delta}\|_{H^{11}} \cdot (\|u_{x, \delta}\|_{H^{11}} \|u_{x, \delta}\|_{H^{11}}) \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{11}}.$$  

The remaining terms are similar. For $I_6$, we have $i_2 \leq 5$. If $i_2 = 5$, then $k = 2$ as we do not have any derivatives left for $i_3, \ldots$. If $i_2 \leq 4$, then $i_3 \leq \min(5 - i_2, i_2) \leq 2$. Note that we used the non-increasing arrangement $i_1 \geq i_2 \geq i_3 \cdots$. Hence

$$I_6 \leq C(\|z\|_{L^\infty}) \cdot \|\partial_x^{i_2}v\|_{L^2} \cdot \|z\|_{W^5, \infty} \cdot C(\|z\|_{W^2, \infty}).$$

Hence by Sobolev embedding and Lemma 37 we can conclude with

$$I_6 \leq C(\|u_{x, \delta}\|_{H^7}) \cdot \|u_{x, \delta}\|_{H^9} \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{11}}.$$  

For $I_{\leq 5}$, we have $1 \leq i_2 \leq i_1 \leq 5$, thus

$$1 \leq i_3 \leq \min\{6 - i_1 - i_2, i_2\} \leq 3.$$  

We estimate

$$I_{\leq 5} \leq C(\|z\|_{L^\infty}) \|z\|_{H^8} \|z\|_{W^3, \infty} C(\|z\|_{W^3, \infty})$$

$$\leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^9} \|u_{x, \delta}\|_{H^1} \leq C(\|u_{x, \delta}\|_{H^7}) \|u_{x, \delta}\|_{H^{11}}.$$  

Adding the estimates for $I_{\geq 7}, \ldots, I_{\leq 5}$ completes the proof.
6. Passage to the Limit

**Proposition 38.** Suppose that $u_\varepsilon$ and $u_{\varepsilon'}$ are in $C_T H^8$ (or $H^7$ for $\varepsilon, \varepsilon' = 0$) and each solve the evolution equation (11), with initial data $u^0$ and $u'^0$, respectively. Then for $M = \sup_{\tau \in [0,t]} (M_\varepsilon(\tau) + k(\tau))$, there is a constant $C(M)$ such that

$$\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_{L^\infty_T H^3} \leq C(M)T \|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_{L^\infty_T H^3} + (\varepsilon + \varepsilon')C(M) + C(M)\|u^0 - u'^0\|_{H^3}.$$  

Therefore, there exists $T_1 > 0$ such that

$$\|u_\varepsilon - u_{\varepsilon'}\|_{L^\infty_T \tau^1 H^3} \leq 2(\varepsilon + \varepsilon')C(M) + C(M)\|u^0 - u'^0\|_{H^3}.$$  

**Remark 39.** The data $u^0$ and $u'^0$ will be taken to depend on smoothing parameters $\delta$ and $\delta'$, but for the present proposition, it is not necessary to be that explicit about the nature of the data.

**Remark 40.** If we take $\varepsilon = \varepsilon' = 0$ and $u^0 = u'^0$ in (54), then we see that solutions are unique. This proves one of the claims of Theorem 4.

**Remark 41.** By iterating (54) as in the proof of Corollary 10, we can replace $T_1$ with $T$ from Corollary 21. This reiterates that size of the solution and dispersion (i.e., $M_0(t)$ and $k(t)$ from (13) and (14)) determine the time of wellposedness.

**Proof.** We let $0 \leq \varepsilon' \leq \varepsilon$, and we consider the solutions $u_\varepsilon$ and $u_{\varepsilon'}$ which we have shown above to exist. We treat the difference $u_\varepsilon - u_{\varepsilon'}$ in $H^3$ by treating the difference first in $L^2$, and then by treating three spatial derivatives in $L^2$. We first note that the inequality

$$\|u_\varepsilon - u_{\varepsilon'}\|_{L^\infty_T L^2} \leq C(M)T \|u_\varepsilon - u_{\varepsilon'}\|_{L^\infty_T H^3} + (\varepsilon + \varepsilon')C(M) + C(M)\|u^0 - u'^0\|_{H^3},$$

follows immediately from the Fundamental Theorem of Calculus (integrating $(u_\varepsilon - u_{\varepsilon'})_t$ with respect to time) and a Lipschitz estimate for the function $f$. We therefore are free to move on to considering three derivatives of the difference.

We define $w$ to be three $x$-derivatives of the difference of the solutions,

$$w = \partial^3_x (u_\varepsilon - u_{\varepsilon'}).$$

We can write the evolution equation for $w$ in the framework of (16); to confirm this, we will identify explicitly, more or less, all of the coefficients and the forcing. Some additional notation will help with this task, so we will have the following decompositions:

$$h = h - h_0 + h_1 + h_2 + h_3 + h_4,$$

$$a_1 = b_1 + b_2,$$

$$a_0 = d_0 + d_1 + d_2.$$  

The coefficients $a_2$ and $a_3$ will be more straightforward to calculate, and no such decomposition will be necessary for them. Furthermore, we introduce the following notation:

$$\partial_t w = \partial_t \partial^3_x u_\varepsilon - \partial_t \partial^3_x u_{\varepsilon'} = A_2 + A_3 + A_4 + A_5 + A_6 + A_7.$$  

Here, $A_7$ will consist of terms which involve seventh derivatives of $u$; this term simply comes from the parabolic regularization. Continuing, $A_6$ will consist of terms from the right-hand side of (55) which involve sixth derivatives of $u$, $A_5$ will consist of terms which involve fifth derivatives of $u$, $A_4$ will consist of terms which involve fourth derivatives of $u$ but no higher derivatives, $A_3$ will consist of terms which involve third derivatives of $u$ but no higher derivatives, and $A_2$ will be the remaining terms, which involve at most second derivatives of $u$.

To begin, we may write $A_7$ simply as

$$A_7 = \varepsilon' \partial^7_x u_{\varepsilon'} - \varepsilon \partial^7_x u_\varepsilon.$$
We add and subtract, to form the fourth-derivative term on the right-hand side of (19):

\[ A_7 = -\varepsilon(\partial_x^2 u_x - \partial_x^2 u_{x'}) - \varepsilon\partial_x^2 u_x + \varepsilon'\partial_x^2 u_{x'} = -\varepsilon\partial_x^2 (w + (\varepsilon' - \varepsilon)u_x). \]

The second term on the right-hand side makes up the contribution \( h_4 : \)

\[ h_4 = (\varepsilon' - \varepsilon)\partial_x^2 u_{x'}. \]

We next note that on the right-hand side of (55), there is only one term that is a sixth derivative of \( w \); this term contributes the following to the evolution equation for \( w : \)

\[ A_6 = f_{zz}(\partial_x^3 u_x)(\partial_x^4 u_x) - f_{zz}(\partial_x^3 u_x)(\partial_x^4 u_{x'}) = f_{zz}(\partial_x^3 u_{x'})(\partial_x^4 u_x) + f_{zz}(\partial_x^3 u_x)(\partial_x^4 u_{x'}). \]

\[ = \left\{ (f_{zz}(\partial_x^3 u_x) - f_{zz}(\partial_x^3 u_{x'}))(\partial_x^4 u_x) + f_{zz}(\partial_x^3 u_{x'})(\partial_x^4 w) \right\}. \]

We define \( a_3 \) and \( h_3 \) as follows:

\[ a_3 = f_{zz}(\partial_x^3 u_x), \]

\[ h_3 = (f_{zz}(\partial_x^3 u_x) - f_{zz}(\partial_x^3 u_{x'}))(\partial_x^4 u_x). \]

To identify \( a_2 \), we must consider the six terms on the right-hand side of (55) which involve fifth spatial derivatives of the unknown, \( u : \)

\[ A_5 = \left(3f_{xzz}(\partial_x^3 u_x) + 3f_{zoz}(\partial_x^3 u_x)(\partial_x u_x) + 3f_{zoz}(\partial_x^3 u_x)(\partial_x^2 u_x) + 3f_{zzzz}(\partial_x^3 u_x)(\partial_x^3 u_x)ight) \]

\[ + 3f_{zzz}(\partial_x^3 u_x)(\partial_x^4 u_x) + f_{zz}(\partial_x^3 u_{x'})(\partial_x^4 u_x) \]

\[ - \left(3f_{xzz}(\partial_x^3 u_{x'}) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x u_x) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x^2 u_x) + 3f_{zzzz}(\partial_x^3 u_{x'})(\partial_x^3 u_{x'})ight) \]

\[ + 3f_{zzz}(\partial_x^3 u_{x'})(\partial_x^4 u_{x'}) + f_{zz}(\partial_x^3 u_{x'})(\partial_x^4 u_{x'}). \]

After some adding and subtracting, we can write this as

\[ A_5 = a_2(\partial_x^4 w) + b_2(\partial_x w) + d_2 w + h_2, \]

where we have the following formulas:

\[ a_2 = 3f_{xzz}(\partial_x^3 u_{x'}) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x u_x) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x^2 u_x) + 3f_{zzzz}(\partial_x^3 u_{x'})(\partial_x^3 u_{x'}) \]

\[ + 3f_{zzz}(\partial_x^3 u_{x'})(\partial_x^4 u_{x'}) + f_{zz}(\partial_x^3 u_{x'})(\partial_x^4 u_{x'}), \]

\[ b_2 = 3f_{xzzz}(\partial_x^3 u_{x'}), \]

\[ d_2 = 3f_{zzzz}(\partial_x^3 u_{x'}), \]

and

\[ h_2 = \left\{ \left(3f_{xzz}(\partial_x^3 u_x) + 3f_{zoz}(\partial_x^3 u_x)(\partial_x u_x) + 3f_{zoz}(\partial_x^3 u_x)(\partial_x^2 u_x) + f_{zz}(\partial_x^3 u_x) \right) \right\} \]

\[ - \left(3f_{xzz}(\partial_x^3 u_{x'}) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x u_x) + 3f_{zoz}(\partial_x^3 u_{x'})(\partial_x^2 u_x) + f_{zz}(\partial_x^3 u_{x'}) \right) \right\} \]

\[ + (3f_{xzz}(\partial_x^3 u_x) - 3f_{xzz}(\partial_x^3 u_{x'}))(\partial_x^4 u_x) + (3f_{xzz}(\partial_x^3 u_x) - 3f_{xzz}(\partial_x^3 u_{x'}))(\partial_x^4 u_{x'}). \]

We may continue in this way with \( A_4 \), noting that there are 23 terms from the right-hand side of (55) which contribute to \( A_4 \). We can write

\[ A_4 = b_1(\partial_x w) + d_1 w + h_1. \]
We may then treat $A_3$ in the same manner, noting that there are 16 terms from the right-hand side of (55) which contribute to $A_3$. We may write

$$A_3 = d_0 v + h_0.$$  

Finally, we note that the remaining terms comprising $A_2$ all contribute to $h$:

$$A_2 = h_{-1}.$$  

Now that we have established the formulas 50 and 57, we can see the following form of the ratio:

$$\frac{a_2}{a_3} = \partial_x \left( 3 \ln \left( f_{z_3} (\partial_x^2 u') \right) \right) + \frac{f_{z_2} (\partial_x^3 u')}{f_{z_2} (\partial_x^3 v')}.$$  

We seek to apply Theorem 8 (the linear estimate), and as such, we use the definitions of $k_G(t)$ and $\hat{M}(t)$ as given in (17). By Lemma 33 as well as the definition of $a_3$ in (50), we see that $\|a_3\|_{L^\infty}$ and $\|1/a_3\|_{L^\infty}$ are bounded. To conclude that $k_G$ is bounded, we then need to conclude that the antiderivative of $a_2/a_3$ is bounded; the antiderivative we must consider, using (50), is

$$\int_0^x \frac{a_2}{a_3} (x', t) \, dx' = 3 \ln(f_{z_3} (\partial_x^2 u'))(x, t) - 3 \ln(f_{z_3} (\partial_x^2 u'))(0, t) + \int_0^x \frac{f_{z_2} (\partial_x^3 u')}{f_{z_2} (\partial_x^3 v')}(x', t) \, dx'.$$

Again using the definition of $a_3$ in (50), and using Lemma 33 and knowing that $a_3 \in C([0, T])$ (this fact also uses Condition (A1)), we see that $a_3$ is positive and bounded away from zero for all $x$ and for all $t \in [0, T]$. The properties of the natural logarithm function and the bound of Lemma 33 imply a uniform bound for $\ln(f_{z_3} (\partial_x^2 u'))(x', t)$, for any $x'$ and $t$. For the other term (the antiderivative of $f_{z_2}/f_{z_3}$) we simply again apply Lemma 33. These considerations yield the desired bound for $k_G$.

We still must estimate $\hat{M}$ and $h$. To begin with $\hat{M}$, we must have an estimate for $a_3 \in W^{3, \infty}$, for $a_2 \in W^{2, \infty}$, for $a_1 \in W^{1, \infty}$, and for $a_0 \in W^{0, \infty}$. We have already given exact formulas for $a_2$ and $a_3$, so we will begin now with a description of $a_1$; this is in lieu of being fully explicit with a formula for $a_1$. We have decomposed $a_1$ previously as $a_1 = b_1 + b_2$, and we have given the formula for $b_2$ in (58). For $b_1$, inspection of (55), together with the definition of $b_1$, shows that the regularity of $b_1$ is like four derivatives of $u$. Thus, $b_2$ is the most singular part, and if we can bound $b_2$, then we have the requisite bound for $a_1$. By Corollary 20 each of $u_e$ and $u_{e'}$ are uniformly bounded in $H^3$; here, when we say “uniformly,” we refer to a bound independent of $\varepsilon$ or $\varepsilon'$, and also independent of $t$. Together with assumption (A1), this implies that $b_2$ (and, as per our discussion, $a_1$ as well) is bounded in $H^2$ and thus in $W^{1, \infty}$, as desired. The bounds in $W^{k, \infty}$ for the other coefficients $a_k$ are similar, so we omit further details.

To complete our estimate of $M$, we must estimate the terms on the right-hand side of (17), which involve time derivatives. For both of these terms, which are $\partial_t a_3$ and $\partial_t \int_0^1 \frac{a_2}{a_3} \, dy$, that they are uniformly bounded in time in $L^\infty_{x, t}$ may be demonstrated identically as in the proof of Lemma 43.

All that remains is the estimate for $h$ in $L^1_t L^2_x$. We can, in fact, bound $h$ in $L^\infty_t L^2_x$, and this clearly implies a bound in $L^1_t L^2_x$ over our finite time interval. We treat $h_4$ differently from $h_j$ for $j \notin \{-1, 0, 1, 2, 3\}$. From the definition of $h_4$, we see that

$$\|h_4\|_{L^\infty_t L^2_x} \leq \varepsilon \sup_t \|\partial_x^2 u_e\|_{L^2_x} \leq \varepsilon M.$$

We have given detailed formulas for $h_3$ and $h_2$ above. From these formulas, it is clear that, because $f$ is smooth and thus Lipschitz in its arguments, we have the following bound:

$$\|h_2 + h_3\|_{L^\infty_t L^2_x} \leq C(M) \|u_e - u_{e'}\|_{L^\infty_t H^2_x}.$$  

While we have not written out $h_{-1}$, $h_0$, and $h_1$ fully explicitly, the completely analogous estimate to (60) is available for them. Our conclusion for $h$ is then

$$\|h\|_{L^1_t L^2_x} \leq C(M) T \|u_e - u_{e'}\|_{L^\infty_t H^2_x} + \varepsilon MT.$$
This completes the proof.

**Proposition 42.** Let \( \delta > 0 \) and \( \delta' > 0 \) be given. Let \( \varepsilon \) and \( \varepsilon' \) be given, such that \( 0 < \varepsilon < \varepsilon' \). Let \( u^{\varepsilon, \delta} \) and \( u^{\varepsilon', \delta'} \) solve the evolution equation

\[
\partial_t u + f(\partial_x^3 u, \ldots, u, x, t) = -\varepsilon \partial_x^4 u,
\]

on the common time interval \([0, T]\), with \( T \) as above, with parameter \( \varepsilon \) equal to \( \varepsilon \) or \( \varepsilon' \), respectively, and with initial data \( u^{\varepsilon, \delta}(\cdot, 0) = (u_0)_\delta \) and \( u^{\varepsilon', \delta'}(\cdot, 0) = (u_0)_{\delta'} \), for given \( u_0 \in H^7 \). (Recall the definition of the regularized data in \( \text{[S]} \).) Then there exists \( C > 0 \), depending on \( M \), such that

\[
\| u^{\varepsilon, \delta} - u^{\varepsilon', \delta'} \|_{L^\infty_t H^7_x} \leq \frac{C\varepsilon'}{\delta'^4} + \frac{C(\varepsilon + \delta')}{\delta'^4} + o(1),
\]

where the \( o(1) \) notation indicates a function which vanishes as \( \varepsilon \to 0 \) and \( \delta' \to 0 \).

**Proof.** The proof of this proposition is similar to the proof of the previous proposition, but we estimate some terms differently (making use of properties of mollifiers). We let \( w = \partial_t^2 (u - u') \), and we write

\[
w_t = -\varepsilon \partial_x^4 w + a_3 \partial_x^2 w + a_2 \partial_x^3 w + a_1 \partial_x w + a_0 w + h.
\]

We are again using Theorem 8 so we again need to check the bounds for the induced \( k_G \) and \( \tilde{M} \) as defined in \( \text{[L]} \). As we defined the coefficients earlier in the system \( \text{[L]} \), we again have the same formulas for the coefficients \( a_i \), especially

\[
a_3 = f_{z_3}(\partial_x^2 u^{\varepsilon, \delta}), \quad a_2 = f_{z_3}(\partial_x^3 u^{\varepsilon, \delta}) + 7\partial_x (f_{z_3}(\partial_x^3 u^{\varepsilon, \delta})).
\]

Together with the uniform bound of Corollary 20, most of the required estimates for \( k_G \) and \( \tilde{M} \) are then routine to check. We focus now on the estimates for \( u_0 \) and \( h \).

We do not provide full details, but instead focus on the most interesting terms. To begin, we consider the initial data:

\[
\| u_0 \|_{L^2} = \| \partial_t^2 (u_0)_\delta - (u_0)_{\delta'} \|_{L^2}.
\]

We then use \( \text{[L]} \) to bound this as

\[
\| u_0 \|_{L^2} \leq o(1).
\]

Otherwise, we now focus on the two most singular terms; we write

\[
h = h_1 + h_2 + h_{\text{rest}},
\]

with \( h_1 \) and \( h_2 \) defined by

\[
h_1 = (\varepsilon - \varepsilon') \partial_x^{11} u^{\varepsilon, \delta},
\]

and

\[
h_2 = \left( \partial_x^{10} u^{\varepsilon', \delta'} \right) \left( f_{z_3}(\partial_x^3 u^{\varepsilon, \delta}) - f_{z_3}(\partial_x^3 u^{\varepsilon', \delta'}) \right)
\]

Recalling that we must bound \( h \) in \( L^1_t L^2_x \), we estimate \( h_1 \) and \( h_2 \) in this space. We begin with \( h_1 \); we find the following by making use of Corollary 30

\[
\| h_1 \|_{L^1_t L^2_x} \leq T \| h_1 \|_{L^\infty_t L^2_x} \leq T \max\{ \varepsilon, \varepsilon' \} \| u^{\varepsilon', \delta'} \|_{L^\infty_t H^1_x} \leq \frac{C \max\{ \varepsilon, \varepsilon' \}}{\delta'^4} = \frac{C \varepsilon'}{\delta'^4}.
\]

We next consider \( h_2 \), and begin as we did for \( h_1 \):

\[
\| h_2 \|_{L^1_t L^2_x} \leq T \| h_2 \|_{L^\infty_t L^2_x} \leq T \| \partial_x^{10} u^{\varepsilon', \delta'} \|_{L^\infty_t L^2_x} \| f_{z_3}(\partial_x^3 u^{\varepsilon, \delta}) - f_{z_3}(\partial_x^3 u^{\varepsilon', \delta'}) \|_{L^2_t L^2_x}.
\]

We use Sobolev embedding and we again use Corollary 30, finding the following:

\[
\| h_2 \|_{L^1_t L^2_x} \leq \frac{C}{\delta'^4} \| f_{z_3}(\partial_x^3 u^{\varepsilon, \delta}) - f_{z_3}(\partial_x^3 u^{\varepsilon', \delta'}) \|_{L^2_t L^2_x}.
\]
A Lipschitz estimate for \( f \) (as in the proof of Proposition 38) allows us to bound this as follows:
\[
\|h_2\|_{L^1_t L^2_x} \leq \frac{C}{\delta^4} \|u^{\varepsilon, \delta} - u^{\varepsilon', \delta'}\|_{L^\infty_t H^2_x}.
\]

We use Proposition 38 and in particular (54), to bound the right-hand side in terms of the initial data:
\[
\|h_2\|_{L^1_t L^2_x} \leq \frac{C\varepsilon'}{\delta^4} + \frac{C\varepsilon}{\delta^4} \|(u_0)_\delta - (u_0)_{\delta'}\|_{H^2_x}.
\]

Interpolating in (9b), we have Corollary 43.

Let \( \varepsilon > 0 \) and \( \varepsilon' > 0 \) be given such that \( \varepsilon' > \varepsilon \). Let \( \delta = \delta^5 \) and \( \delta' = \delta^0 \). Let \( u^\delta \) and \( u^{\delta'} \) solve the initial value problem (11) on the common time interval \([0, T]\), with \( T \) as above, with parameter values \((\varepsilon, \delta)\) and \((\varepsilon', \delta')\), respectively. Then
\[
\|u^\delta - u^{\delta'}\|_{L^\infty_t H^2_x} = o(1),
\]
where the \( o(1) \) notation indicates a function which vanishes as \( \delta' \to 0 \). This convergence is uniform with respect to the choice of \( u_0 \) if \( u_0 \) is taken from a compact set.

**Proof.** The only statement which requires justification is the final statement, about uniformity of the convergence when the initial data is taken from a compact set. This uniformity is provided by Lemma 5 for the term on the right-hand side of (61) denoted as \( o(1) \). \( \Box \)

We are now able to conclude that our original system, (11), has a solution in \( H^7 \). This is the first conclusion of Theorem 3.

**Corollary 44.** Under the assumptions of Theorem 3, there exists \( T = T(\|u_0\|_{H^7}, \lambda_0) \) such that there exists a classical solution \( u \in C_{[-T,T]} H^7 \) of (11) with \( u(\cdot, 0) = u_0 \).

**Proof.** From Corollary 43, the sequence \( u^\delta \) is Cauchy. It has a limit, \( u \in C_{[-T,T]} H^7 \). Integrating (11) with respect to time, choosing \( \varepsilon = \delta^5 \) as in Corollary 43 and using the Fundamental Theorem of Calculus, we get a formula for \( u^\delta \) which involves at most fourth derivatives of itself inside a time integral:
\[
u^\delta(\cdot, t) = (u_0)_\delta + \int_0^t f(\partial_\tau^3 u^\delta(\cdot, \tau)) + \delta^5 \partial_\tau^4 u^\delta(\cdot, \tau) \, d\tau.
\]

The convergence of \( u^\delta \) to \( u \) in \( C_{[-T,T]} H^7 \) implies that the convergence of up to fourth derivatives is uniform in \( t \), and this uniform convergence allows us to pass to the limit as \( \delta \to 0 \) in the integral representation formula (64). After taking the limit, we take the time derivative again to find that (11) is satisfied. \( \Box \)

**Remark 45.** Again, the convergence of \( u^\delta \) to \( u \) is uniform with respect to the choice of initial data \( u_0 \) if this data is chosen from a compact subset of \( H^7 \). To state this more precisely, let \( \mathcal{K} \) be a compact subset of \( H^7 \). Recall that the time of existence of solutions to the initial value problem, \( T \),
guaranteed by our existence theorem can be taken to be independent of the initial data \( u_0 \in \mathcal{K} \). Let \( \eta > 0 \) be given. There exists \( D > 0 \) such that for all \( \delta \in (0, D) \), for all \( u_0 \in \mathcal{K} \),

\[
\sup_{t \in [-T,T]} \| u^\delta(\cdot, t) - u(\cdot, t) \|_{H^7} < \eta,
\]

where \( u^\delta \) and \( u \) are the solutions of the unregularized and regularized problems, respectively, corresponding to the unregularized initial data \( u_0 \). This uniformity on compact sets stems from the uniformity in Lemma 5 as in the proof of Corollary 45.

We are now able to state our continuous dependence result for initial data in \( H^7 \).

**Corollary 46.** Let \( u_0 \in H^7 \) and let \( u_n \) be a sequence in \( H^7 \) such that \( u_n \to u_0 \). Note that since \( \{ u_n \in H^7 : n \in \{0,1,2,\ldots\} \} \) is compact in \( H^7 \), the time of existence of solutions guaranteed by Corollary 44 can be taken to be independent of \( n \); we therefore let \( U_n \in C([-T,T];H^7) \) be the solution of the initial value problem (7) with data \( u_n \), for all \( n \in \{0,1,2,\ldots\} \), with this \( T \) independent of \( n \). Then,

\[
\lim_{n \to \infty} \sup_{t \in [-T,T]} \| U_n - U_0 \|_{H^7} = 0.
\]

**Proof.** Let \( \eta > 0 \) be given. Given any \( \delta > 0 \), we begin by adding and subtracting:

\[
U_n - U_0 = (U_n - U_\delta_n) + (U_\delta_n - U_0) + (U_0 - U_0).
\]

As in Remark 45, we may take a particular \( \delta > 0 \) such that for all \( n \in \{0,1,2,\ldots\} \),

\[
\sup_{t \in [-T,T]} \| U_\delta_n(\cdot, t) - U_m(\cdot, t) \|_{H^7} < \frac{\eta}{3}.
\]

To make \( U_n - U_0 \) small, then, it is only necessary to focus on the difference \( U_\delta_n - U_0 \). These are solutions of the approximate problem (11) for fixed parameter \( \delta \). This is a parabolic problem, and as such, has the continuous dependence result of Proposition 6. Therefore there exists \( N \in \mathbb{N} \) such that for all \( n > N \), we have

\[
\sup_{t \in [-T,T]} \| U_\delta_n(\cdot, t) - U_0(\cdot, t) \|_{H^7} < \frac{\eta}{3}.
\]

This completes the proof. \( \square \)

7. **Fully explicit calculation of third derivative**

Here, we give a complete calculation of \( \partial_x f \), \( \partial_{xx} f \), and \( \partial_{xxx} f \). We begin with simply the first derivative:

\[
u_{xt} = f_x + f_{z_0} u_x + f_{z_1} u_{xx} + f_{z_2} u_{xxx} + f_{z_3} (\partial_x^3 u).
\]

We apply another derivative with respect to \( x \), finding the following:

\[
u_{xxt} = f_{xx} + 2 f_{xxz_0} u_x + 2 f_{xz_1} u_{xx} + 2 f_{zz_2} u_{xxx} + 2 f_{zz_3} (\partial_x^3 u) + f_{z_0} u_{xx} + f_{z_2} u_{xxx}^2 + 2 f_{z_0} u_{xx}^2 + 2 f_{z_2} u_{xxx} u_x + 2 f_{z_3} (\partial_x^3 u) + f_{z_1} u_{xx} + f_{z_2} u_{xxx}^2 + 2 f_{z_3} u_{xx} (\partial_x^3 u) + f_{z_3} (\partial_x^3 u) + f_{z_3} (\partial_x^3 u)^2.
\]
We differentiate once more. The formula for the third derivative uses 59 terms on the right-hand side:

\[
(65) \quad u_{xxxx} = f_{xxxx} + 3f_{xxx}u_x + 3f_{xxz}u_{xx} + 3f_{xzz}u_{xxx} + 3f_{xxxz}(\partial^3_x u) + 3f_{xxz}u_{xx} \\
+ 3f_{xxz}u_x^2 + 6f_{xxz}u_x u_{xx} + 6f_{xzz}u_{xx} u_{xxx} + 6f_{xzzz}u_x (\partial^3_x u) + 3f_{xxz}u_{xxx} \\
+ 3f_{xxz}u_x^2 + 6f_{xxz}u_x u_{xx} + 6f_{xzz}u_{xx} u_{xxx} + 6f_{xzzz}u_x (\partial^3_x u) + 3f_{xxz}u_{xxx} + 3f_{xxz}u_{xx}^2 \\
+ 6f_{xxz}u_x (\partial^3_x u) + 3f_{xxz}u_x (\partial^3_x u)^2 + 3f_{xxz}u_{xx} (\partial^3_x u)^2 + f_{xxz}u_{xxx} + 3f_{xzz}u_x u_{xx} \\
+ 3f_{xzz}u_x u_{xx} + u_{xx}^2 + 3f_{xzzz}u_x^2 (\partial^3_x u) + 3f_{xzzz}u_{xx} (\partial^3_x u) + u_{xxx} + 3f_{xzzz}u_x (\partial^3_x u) + u_{xx} (\partial^3_x u) \\
+ f_{xzz}u_{xx} + 3f_{xzz}u_x u_{xx} + 3f_{xzz}u_x u_{xx} + 3f_{xzz}u_x u_{xx} + 3f_{xzz}u_x u_{xx} + 3f_{xzz}u_x u_{xx} + 3f_{xzz}u_x u_{xx} \\
+ 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} + 6f_{xzz}u_x u_{xx} \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) \\
+ 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) + 3f_{xzz}u_x (\partial^3_x u) \\
+ 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 \\
+ 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 + 3f_{xzz}u_x (\partial^3_x u)^2 .
\]

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