Stationary entanglement and nonlocality of two qubits or qutrits collectively interacting with the thermal environment: The role of Bell singlet state

Shang-Bin Li\(^1\),*, and Jing-Bo Xu\(^1\)

1. Zhejiang Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou 310027, People’s Republic of China and 2. Shanghai research center of Amertron-global, Zhangjiang High-Tech Park, 299 Lane, Bisheng Road, No. 3, Suite 202, Shanghai, 201204, P.R. China

We investigate the stationary entanglement and stationary nonlocality of two qubits collectively interacting with a common thermal environment. We assume two qubits are initially in Werner state or Werner-like state, and find that thermal environment can make two qubits become stationary nonlocality. The analytical relations among average thermal photon number of the environment, entanglement and nonlocality of two qubits are given in details. It is shown that the fraction of Bell singlet state plays a key role in the phenomenon that the common thermal reservoir can enhance the entanglement of two qubits. Moreover, we find that the collective decay of two qubits in a thermal reservoir at zero-temperature can generate a stationary maximally entangled mixed state if only the fraction of Bell singlet state in the initial state is not smaller than $\frac{2}{3}$. It provides us a feasible way to prepare the maximally entangled mixed state in various physical systems such as the trapped ions, quantum dots or Josephson Junctions. For the case in which two qubits collectively coupled with a common thermal reservoir at zero-temperature, we find that the collective decay can induce the entanglement of two qubits initially in the maximally mixed state. The collective decay of two qubits can also induce distillable entanglement from the initial conjectured negative partial transpose bound entangled states.

PACS numbers: 03.65.Ud, 03.67.-a, 05.40.Ca

I. INTRODUCTION

Quantum entanglement plays an important role in quantum information. It has been recognized as a useful resource in various quantum information processes. While entanglement can be destroyed by the interaction between the system of interest and its surrounding environment in most situations, there have been many works showing that the collective interaction with a common thermal environment can cause the entanglement of qubits. Beige et al. have analyzed ways in which entanglement could be established within a dissipative environments and shown that one could even utilize a strong interaction of the system with its environment to produce entanglement. Braun has also shown that two qubits with degenerate energy levels can be entangled via interaction with a common heat bath. Schneider and Milburn have studied the pairwise entanglement in the steady state of theDicke model and revealed how the steady state of the ion trap with all ions driven simultaneously and coupled collectively to a heat bath could exhibit quantum entanglement. Kim et al. have investigated the interaction of the thermal field and a quantum system composed of two qubits and found that such a chaotic field with minimal information could entangle qubits that were prepared initially in a separable state. Kraus and Cirac have shown how one could entangle distant atoms by using squeezed light. Clark and Parkins have proposed a scheme to controllably entangle the internal states of two atoms trapped in a high-finesse optical cavity by employing quantum-reservoir engineering. For generating multipartite entanglement, Duan and Kimble have proposed an efficient scheme to engineer multi-atom entanglement by detecting cavity decay through single-photon detectors. More recently, it has been shown that white noise may play a constructive role in generating the controllable entanglement in some specific situations.

In this paper, we investigate the system of two qubits or two qutrits collectively interacting with a common thermal reservoir. For two-qubit case, we analyze the role of the fraction of Bell singlet state and the average photon number of thermal reservoir in the stationary state entanglement and Bell violation. It is shown that the fraction of Bell singlet state in the initial state is a key fact determining whether the common thermal reservoir can enhance the entanglement or Bell violation of two qubits or not.

For two-qutrit case, we find that two qutrits initially in the conjectured bound entangled Werner state can become distillable due to the collective decay caused by the common thermal reservoir at zero-temperature. Even if two qutrits are initially in the maximally mixed state, they can evolve into a stationary entangled state under the collective decay. The distinct aspect of collective decay of two qutrits is that a pure Bell singlet state may be generated from an initial mixed state.

\*Electronic address: stephenli74@yahoo.com.cn
In the last year, much attention has been paid to the preparation of the maximally entangled mixed state \cite{12, 13}. The properties of maximally entangled mixed state have been studied by many authors \cite{14, 15, 16}. Here, we show that collective decay of two qubits initially in the standard Werner state in a common thermal reservoir at zero-temperature can generate a stationary maximally entangled mixed state if only the fraction of Bell singlet state in the initial state is not smaller than $\frac{3}{8}$. It is found that stationary state $\rho_1$ of two qubits initially in the standard Werner state in the common thermal reservoir builds a bridge across the Werner state with $r \geq \frac{3}{8}$ and the maximally entangled mixed state if the temperature of the reservoir can be adiabatically varied from zero to infinite or vice versa.

This paper is organized as follows: In Sec.II we investigate the stationary state entanglement of two qubits collectively interacting with a common thermal reservoir and find that the fraction of the Bell singlet state in the initial state plays a key role in the question whether the common thermal reservoir can enhance the entanglement of two qubits or not. In Sec.III, the Bell violation of the stationary state of two qubits is investigated and it is shown that, in certain situation, the common thermal reservoir may drive two qubits initially satisfying Bell-CHSH inequality into a stationary state which violates the Bell-CHSH inequality. In Sec.IV, we investigate the concurrence versus the linear entropy of the stationary state and find that the common thermal reservoir at zero-temperature can make two qubits initially in the standard Werner state become a maximally entangled mixed state if only the fraction of the Bell singlet state in the initial state is not smaller than $\frac{3}{8}$. In Sec.V, we turn to consider the case in which two qutrits collectively interacting with a common thermal reservoir at zero-temperature and find that the initial maximally mixed state of two qutrits can become a stationary entangled state. Furthermore, we show that two qutrits initially in the conjectured bound entangled Werner state can become free entangled due to the collective decay caused by the common thermal reservoir at zero-temperature. In Sec.VI, there are some concluding remarks.

II. THE STATIONARY ENTANGLEMENT OF TWO QUBITS COLLECTIVELY INTERACTING WITH A THERMAL RESERVOIR

Up to date, much attention has been paid to the environment-induced entanglement \cite{2, 3, 4, 5, 6, 8, 10, 11}. Here, we consider such a situation in which two qubits collectively interacting with a common thermal reservoir. Two qubits are assumed initially in the Werner state or Werner-like states. Under the Markovian approximation, the dynamical behavior of two qubits in this case can be described by the following master equation

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{(N+1)\gamma}{2}(2\hat{J}_-\hat{\rho}\hat{J}_+ - \hat{J}_+\hat{J}_-\hat{\rho} - \hat{\rho}\hat{J}_+\hat{J}_-),$$  \hspace{1cm} (1)

where $\gamma$ characterizes the coupling strength between two qubits and the thermal reservoir. $N$ is the mean phonon number of the thermal environment. $\hat{J}_\pm$ are the collective atomic operators defined by

$$\hat{J}_\pm = \sum_{i=1}^{2} \hat{\sigma}^{(i)}_\pm,$$

$$\hat{\sigma}^{(i)}_+ = |1_i\rangle\langle 0_i|, \hspace{0.5cm} \hat{\sigma}^{(i)}_- = |0_i\rangle\langle 1_i|,$$  \hspace{1cm} (2)

where $|1_i\rangle$ and $|0_i\rangle$ are up and down states of the $i$th qubit, respectively. Recently, the Werner or Werner-like states \cite{16, 17, 18, 19} has intrigued many interests for the applications in quantum information processes. Lee and Kim have discussed the entanglement teleportation via the Werner states \cite{20}. Hiroshima and Ishizaka have studied the entanglement of the so-called Werner derivative, which is the state transformed by nonlocal unitary operations from a Werner state \cite{21}. Miranowicz has examined the Bell violation and entanglement of Werner states of two qubits in independent decay channels \cite{22}. The experimental preparation and characterization of the Werner states have also been reported. An experiment for preparing a Werner state via spontaneous parametric down-conversion has been put forward \cite{23}. Barbieri et al. have presented a novel technique for generating and characterizing two-photon polarization Werner states \cite{13}, which is based on the peculiar spatial characteristics of a high brilliance source of entangled pairs. If the two qubits are initially prepared in the Werner state or Werner-like state, how does the external common thermal reservoir affect their entanglement and nonlocality properties? Here, we address this question and show that both the stationary state entanglement and nonlocality properties heavily depend on the fraction of Bell singlet state in the initial state and the intensity of the thermal reservoir. The standard two-qubit Werner state is defined by \cite{17}

$$\rho_W = r|\Phi^-\rangle\langle \Phi^-| + \frac{1-r}{4}I \otimes I,$$  \hspace{1cm} (3)

where $r \in [0, 1]$, and $|\Phi^-\rangle$ is the singlet state of two qubits. $I$ is the identity operator of a single qubit. Recently, definition (3) is generalized to include the following states of two qubits \cite{16, 18, 19}

$$\rho_W = r|\Phi^+\rangle\langle \Phi^+| + \frac{1-r}{4}I \otimes I,$$  \hspace{1cm} (4)

where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. Both the Werner state (3) and the Werner-like state (4) are very important in quantum information. The Werner state (3) is highly symmetric and $SU(2) \otimes SU(2)$ invariant \cite{17, 24}. The mixedness, entanglement and nonlocality of both the Werner state and the Werner-like state are uniquely determined by the parameter $r$. In the following, we consider two different
cases. In the case 1, two qubits are initially in the state $(r \in [0, 1]) \ r |\Phi^-\rangle \langle \Phi^-| + \frac{1-r}{2} I \otimes I$; In the case 2, two qubits are initially $r |\Phi^+\rangle \langle \Phi^+| + \frac{1-r}{2} I \otimes I$. The fractions of the Bell singlet state defined by $\text{Tr}(|\Phi^-\rangle \langle \Phi^-| \rho)$ in both cases are $f_1(r) = \frac{1+r}{2}$ and $f_2(r) = \frac{1-r}{2}$ respectively. We will show that the fraction of the Bell singlet state plays a crucial role in the stationary entanglement and stationary Bell violation. In the case 1, as the time $t \to \infty$, the stationary state of the master equation (1) can be obtained as follows:

$$\rho_1 = a_1 |11\rangle\langle 11| + a_2 |10\rangle\langle 10| + a_3 |01\rangle\langle 01| + a_4 |00\rangle\langle 00| + a_5 |10\rangle\langle 01| + a_6^* |01\rangle\langle 10|, \quad (5)$$

where

$$a_1 = \frac{(3 - 3r)N^2}{4L},$$

$$a_2 = a_3 = \frac{r - 1 + (2 + 2r)L}{8L},$$

$$a_4 = 1 - a_1 - a_2 - a_3,$$

$$a_5 = \frac{r - 1 - 4r L}{8L},$$

$$L = 1 + 3N(N + 1). \quad (6)$$

In the case 2, the stationary state of the master equation (1) can be obtained as follows:

$$\rho_2 = b_1 |11\rangle\langle 11| + b_2 |10\rangle\langle 10| + b_3 |01\rangle\langle 01| + b_4 |00\rangle\langle 00| + b_5 |10\rangle\langle 01| + b_6^* |01\rangle\langle 10|, \quad (7)$$

where

$$b_1 = \frac{(3 + r)N^2}{4L},$$

$$b_2 = b_3 = \frac{-r - 3 + (6 - 2r)L}{24L},$$

$$b_4 = 1 - b_1 - b_2 - b_3,$$

$$b_5 = \frac{-r - 3 + 4r L}{24L}. \quad (8)$$

In order to quantify the degree of entanglement, we adopt the concurrence $C$ defined by Wooters [22]. The concurrence varies from $C = 0$ for an unentangled state to $C = 1$ for a maximally entangled state. For two qubits, in the "Standard" eigenbasis: $|1\rangle \equiv |11\rangle$; $|2\rangle \equiv |10\rangle$; $|3\rangle \equiv |01\rangle$; $|4\rangle \equiv |00\rangle$, the concurrence may be calculated explicitly from the following:

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \quad (9)$$

where the $\lambda_i (i = 1, 2, 3, 4)$ are the square roots of the eigenvalues in decreasing order of magnitude of the "spin-flipped" density matrix operator $R = \rho_s (\sigma^y \otimes \sigma^y) \rho_s^* (\sigma^y \otimes \sigma^y)$, where the asterisk indicates complex conjugation. It is straightforward to compute analytically the concurrence $C_1$ and $C_2$ for the density matrices $\rho_1$ and $\rho_2$, respectively,

$$C_1 = \max[0, \frac{1 + 3r + (18r - 6)(N^2 + N)}{4(3N^2 + 3N + 1)}], \quad (10)$$

and

$$C_2 = \max[0, \frac{1 - r - (6 + 6r)(N + N^2)}{4(3N^2 + 3N + 1)}], \quad (11)$$

which implies the stationary state $\rho_1$ is entangled for the case with a fixed $N$ if and only if $r > \frac{4N^2 + 6N - 1}{18N^2 + 18N + 3}$. Meanwhile, the stationary state $\rho_2$ is entangled if and only if two inequalities $0 \leq r < \frac{1 - 6N - 6N^2}{1 + 6N + 6N^2}$ and $0 \leq N < \frac{\sqrt{13} - 3}{6}$ are simultaneously satisfied. In Fig.1, the concurrences $C_1$ is plotted as the function of the parameter $r$ of the initial Werner state and the average phonon number $N$ of the thermal reservoir. It is shown that, if two qubits are initially in the standard Werner state, the stationary entanglement of two qubits increases with the fraction of the Bell singlet state in their initial state, and decreases with $N$. In Fig.2, we plot the concurrence $C_2$ as the function of the parameter $r$ of the initial Werner-like state in Eq.(4) and the average photon number $N$ of the thermal reservoir. We can see that the stationary state entanglement decreases both with the increase of $r$ and $N$, which implies the higher initial entanglement does not result in the higher stationary state entanglement. The reason is that the fraction of the Bell singlet state in the Werner-like state $r |\Phi^+\rangle \langle \Phi^+| + \frac{1-r}{2} I \otimes I$ decreases with $r$. From Fig.1 and Fig.2, we can observe that, in the case with $N = 0$, namely two qubits collectively interact with a thermal reservoir at zero-temperature, the stationary state is always entangled if only the initial state of two qubits is not absolutely symmetric, i.e. the fraction of Bell singlet state in the initial state is not zero. When $r \leq \frac{1}{4}$, the initial Werner state or Werner-like state are separable. Surprisingly, the external thermal reservoir can enhance the entanglement of two qubits even if two qubits are initially in a separable state. For example, when $r = 0$, two qubits are initially in the maximally mixed state. However, the collective interaction between two qubits and the low-temperature thermal reservoir can drive two initial maximally mixed qubits into a stationary entangled mixed state. Comparing the stationary state entanglement with the entanglement of the initial Werner or Werner-like states, we can obtain the condition that the entanglement can be enhanced by external common thermal environment. This condition closely relates to the fraction $F$ of Bell singlet state and the average photon number of the thermal environment, which can be expressed by the following inequalities:

$$1 > F > \frac{1}{6} \left(\frac{3N^2 + 3N}{6N^2 + 6N + 1}\right), \quad (12)$$

or

$$\frac{1}{6} > F > \frac{9N^2 + 9N + 2}{36N^2 + 36N + 14} \quad (13)$$

The increment of entanglement between the stationary state entanglement and its initial entanglement of the Werner state can be obtained as follows:

$$\Delta C = \max[0, C_1 - \max(0, \frac{3r - 1}{2})] \quad \text{For case 1,}$$
\[ \Delta C = \max[0, C_2 - \max(0, \frac{3r-1}{2})] \quad \text{For case 2.} \]

In Fig. 3, we plot the increment of entanglement between the stationary state entanglement and its initial entanglement of the Werner state as the function of \( N \) and \( r \). It is shown that the thermal reservoir can enhance the entanglement of two qubits in the range indicated by the inequalities (12) and (13). When the parameter \( r \) of the initial Werner state is larger than \( \frac{1}{3} \), the increment of entanglement \( \Delta C \) decreases both with the increase of \( r \) and \( N \). When \( r \leq \frac{1}{3} \), the initial state is separable; nevertheless, the stationary state may be entangled only if \( r > \frac{6N^2+6N-1}{18N^2+18N+3} \). From Eq.(10) or Eq.(14), we get a conclusion that the common thermal reservoir with any large intensity can enhance the entanglement of qubits collectively coupled with the reservoir if only fraction \( F \) of Bell singlet state in the initial Werner state is smaller than 1 and not smaller than \( \frac{1}{2} \). In Fig. 4, increment of entanglement between the stationary state entanglement and its initial entanglement of the Werner-like state in Eq.(4) is plotted as the function of \( N \) and \( r \). We can see that, in this case, the increment of entanglement decreases both with \( r \) and \( N \). In the following section, we shall investigate how a common thermal reservoir affects the Bell violation of two qubits in the Werner state.

**III. BELL VIOLATION OF TWO QUBITS INTERACTING WITH A COMMON THERMAL ENVIRONMENT**

In this section, we attempt to discuss the nonlocality of two qubits in their stationary states. The nonlocal property of two qubits can be characterized by the maximal violation of Bell inequality. Recently, it has been argued that entanglement and nonlocality of two qubits are different resources \[26\]. We also find that the stochastic-resonance-like behavior of entanglement can not be observed in the Bell violation of two qubits during the evolution \[27\]. The concurrence, one of the good entanglement measures, is not monotonic function of the maximal violation of Bell inequality for some entangled mixed states of two qubits \[31\]. So it is interesting to investigate how the collective decay of two qubits can affect their maximal violation of Bell inequality. The most commonly discussed Bell inequality is the CHSH inequality \[28, 29\]. The CHSH operator reads

\[ \hat{B} = \hat{a} \cdot \hat{\sigma} \otimes (\hat{b} + \hat{b}') \cdot \hat{\sigma} + \hat{a}' \cdot \hat{\sigma} \otimes (\hat{b} - \hat{b}') \cdot \hat{\sigma}, \quad (15) \]
where \( \vec{a}, \vec{a}', \vec{b}, \vec{b}' \) are unit vectors. In the above notation the Bell inequality reads

\[
|\langle \vec{B} \rangle| \leq 2.
\]

The maximal amount of Bell violation of a state \( \rho \) is given by

\[
B = 2\sqrt{\lambda + \lambda'},
\]

where \( \lambda \) and \( \lambda' \) are the two largest eigenvalues of \( T_{\rho} \). The matrix \( T_{\rho} \) is determined completely by the correlation functions being a \( 3 \times 3 \) matrix whose elements \( (T_{\rho})_{nm} = \text{Tr}(\rho \sigma_n \otimes \sigma_m) \). Here, \( \sigma_x \equiv \sigma_y \equiv \sigma_z \) and \( \sigma_3 \equiv \sigma_e \) denote the usual Pauli matrices. We call quantity \( B \) the maximal violation measure, which indicates the Bell violation when \( B > 2 \) and the maximal violation when \( B = 2\sqrt{2} \). In what follows, we focus attention on the role of the common thermal reservoir the Bell violation of two qubits. If two qubits are initially in the Werner or Werner-like states, we find that the Bell violation of stationary state of two qubits heavily depends on the fraction of the Bell singlet state in initial state.

For the density operator \( \rho_1 \) in Eqs.(5-6) and \( \rho_2 \), Eqs.(7-8) characterizing the two stationary states of qubits governed by the master equation (1) corresponding to two kinds of different initial states, the maximal Bell violation \( |B_1|_{\text{max}} \) and \( |B_2|_{\text{max}} \) can be written as follows

\[
|B_1|_{\text{max}} = 2\sqrt{4|a_5|^2 + \max |4|a_5|^2, (1 - 4a_2)^2}}
\]

\[
|B_2|_{\text{max}} = 2\sqrt{4|b_5|^2 + \max |4|b_5|^2, (1 - 4b_2)^2}}
\]

The sufficient and necessary condition for \( |B_1|_{\text{max}} > 2 \) can be derived as follows:

\[
r > \frac{2\sqrt{2} - 1 + 6\sqrt{2}N(N+1)}{3 + 12N(N+1)},
\]

and \( |B_2|_{\text{max}} \) can be easily verified that it cannot be larger than 2. Since the initial standard Werner state can not violate any Bell-CHSH inequality when \( r \leq \sqrt{2}/2 \), it is interesting that the corresponding stationary state may achieve the Bell violation even if \( \frac{\sqrt{2}}{2} \geq r > \frac{2\sqrt{2} - 1 + 6\sqrt{2}N(N+1)}{3 + 12N(N+1)} \), which means the common thermal reservoir can induce the stationary Bell violation of two qubits. This may be important for the experimental verification of Bell violation in the quantum dots in which...
the collective decay may be caused by the common thermal phonon background. In Fig.5, we plot the maximal value of the Bell violation of the stationary state as the function of \( r \) and \( N \) for the case in which two qubits are initially in the standard Werner state. It is shown that Bell violation of the stationary state decreases with the decrease of the parameter \( r \). The Bell violation also decreases with \( N \). However, if the initial standard Werner state is very pure, the Bell violation of its stationary state is robust against the collective decay.

IV. CONCURRENCE VERSUS MIXEDNESS OF TWO QUBITS INTERACTING WITH A COMMON THERMAL ENVIRONMENT

In this section, we pay our attention to investigate how the common thermal reservoir affects the relation between the concurrence and the mixedness of the stationary state. We find the common thermal reservoir can drive two qubits to exceed the curve of the initial standard Werner state in the figure labelled by the concurrence and the linear entropy if only the fraction of Bell singlet state is not smaller than a threshold value. Ordinarily, the mixedness of a state can be characterized by the linear entropy which is defined by

\[
M = \frac{1}{3}(1 - \text{Tr}\rho^2).
\]

For the stationary states \( \rho_1 \) and \( \rho_2 \) in Eqs.(5-6) and Eqs.(7-8) respectively, the mixedness can be calculated as follows:

\[
M_1(\rho_1) = \frac{4}{3}(1 - \sum_{i=1}^{4} a_i^2 - 2a_3^2),
\]

\[
M_2(\rho_2) = \frac{4}{3}(1 - \sum_{i=1}^{4} b_i^2 - 2b_3^2).
\]

In Fig.6, we display the concurrence and the mixedness of the stationary state \( \rho_1 \) of two qubits which is initially in the standard Werner state. From Fig.6, it can be observed that in the situations with \( r \geq 0.4 \), the corresponding stationary state can go beyond the curve of the concurrence and linear entropy of the original Werner state. When the intensity of the common thermal reservoir decreases, the concurrence of the corresponding stationary state \( \rho_1 \) increases and its mixedness characterized by the linear entropy decreases. This implies that the common thermal reservoir not only can enhance the concurrence of the mixed state initially with large fraction of Bell singlet state but also can decrease the mixedness. This counterintuitive phenomenon may be helpful for the entanglement purification or distillation.

From Eqs.(5-6), we can immediately know that the stationary state is as the same as the initial Werner state in the case of infinite high temperature thermal reservoir, i.e. \( N \to \infty \). In the case with \( N = 0 \), i.e. a thermal reservoir at zero-temperature, the corresponding stationary states \( \rho_1 \) with \( r \geq \frac{5}{9} \) become the maximally entangled mixed state, i.e. the frontier of the concurrence versus the linear entropy of two qubits \([10]\). So we can achieve a conclusion that the common thermal reservoir at zero-temperature can make two qubits initially in the standard Werner state become a maximally entangled mixed state if only the fraction of the Bell singlet state in the initial state is not smaller than \( \frac{5}{9} \). It provides us a feasible way to prepare the maximally entangled mixed state in various physical systems such as the trapped ions, quantum dots or Josephson Junctions. In these systems, the collective decay has been extensively studied both theoretically and experimentally. One may see that stationary state \( \rho_1 \) of two qubits in the common thermal reservoir builds a bridge across the Werner state with \( r \geq \frac{5}{9} \) and the maximally entangled mixed state if the temperature of the reservoir can be adiabatically varied.

We conjecture that this property is closely related to the fact that the frontier of the concurrence versus the linear entropy of two qubits contains two different branches \([10]\). Interestingly, If we adopt negativity to measure the entanglement of two qubits, the Werner state becomes the frontier in the sense that these states have the maximal negativity for a given linear entropy or Von-Neumann entropy \([10]\). So roughly speaking, the stationary states \( \rho_1 \) with \( r \geq 5/9 \) of two qubits in the common thermal reservoir in two extreme situations, i.e. the zero temperature and the infinite high temperature, become part of the frontier of the concurrence versus linear entropy and the whole frontier of the negativity versus linear entropy, respectively. Therefore, it is very necessary to study the relation among the negativity of the stationary state, the intensity of the common reservoir and the fraction of Bell singlet state.

The negativity for a bipartite state \( \rho \) is defined as

\[
\mathcal{N}(\rho) = 2 \sum_i |\mu_i|,
\]

where \( \mu_i \) is the negative eigenvalues of partial transpose \( \rho^T \) of the density matrix \( \rho \). We can easily obtain the

FIG. 7: Concurrence versus mixedness for the stationary states \( \rho_2 \) is depicted for any possible values of \( N \) and different values of \( r \). The dash dot line represents the Werner state.
negativity $\mathcal{N}_1$ and $\mathcal{N}_2$ of the stationary state $\rho_1$ in Eqs. (5-6) and $\rho_2$ in Eqs. (7-8) respectively as follows:

$$\mathcal{N}_1 = \frac{1}{2}|a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4|a_5|^2}| - \frac{1}{2}|a_1 + a_4 - \sqrt{(a_1 - a_4)^2 + 4|a_5|^2}|,$$

and

$$\mathcal{N}_2 = \frac{1}{2}|b_1 + b_4 - \sqrt{(b_1 - b_4)^2 + 4|b_5|^2}| - \frac{1}{2}|b_1 + b_4 - \sqrt{(b_1 - b_4)^2 + 4|b_5|^2}|.$$  (23)

We find that the negativity $\mathcal{N}_1$ decreases with $N$ and increases with $r$, and the negativity $\mathcal{N}_2$ decreases with $N$ and $r$. In Fig.7, we display the concurrence and the mixedness of the stationary state $\rho_2$ of two qubits. It is shown that, the entanglement of the stationary state for a given value of mixedness is much smaller than the entanglement of the Werner state with the same value of mixedness.

V. THE COLLECTIVE DECAY OF TWO QUTRITS

In this section, we turn to consider the collective decay of two qutrits in the common thermal reservoir at zero-temperature. Under the Markovian approximation, the collective decay of two qutrits can be described by the following master equation:

$$\frac{\partial \rho}{\partial t} = \gamma (2L_- \rho L_+ - L_+ L_- \rho - \rho L_+ L_-),$$

where $L_\pm = \sum_{k=1}^{3} J_{k \pm}^{(i)}$ and $J_{k}^{(i)}$ is the qutrit down (up) operator of the $i$th qutrit. The representation of $J_{k}$ in the space spanned by the three orthogonal vector $\{|1\rangle, |2\rangle, |3\rangle\}$ of a qutrit can be written as

$$J_- = \sqrt{2}|1\rangle\langle 2| + \sqrt{2}|2\rangle\langle 3|$$

$$J_+ = \sqrt{2}|2\rangle\langle 1| + \sqrt{2}|3\rangle\langle 2|.$$  (26)

If we assume that two qutrits are initially in the maximally mixed state, i.e. $\rho_0 = \frac{1}{9} \sum_{i,j=1}^{3} |i,j\rangle\langle i,j|$, then the corresponding stationary state of the master equation (24) can be obtained as

$$\rho_s = \frac{5}{9}|1,1\rangle\langle 1,1| + \frac{1}{6}(|1,2\rangle - |2,1\rangle)(|1,2\rangle - |2,1\rangle)$$

$$+ \frac{1}{27}(|3,1\rangle + |1,3\rangle - |2,2\rangle)(|3,1\rangle + |1,3\rangle - |2,2\rangle).$$  (27)

It is easy to verify that the $\rho_s$ is an entangled state of two qutrits and its negativity defined by Eq. (22) can be calculated as $\sqrt{\frac{57}{27}}$. In what follows, we further show that two qutrits initially in some conjectured negative partial transpose bound entangled states can become distillable.

In Ref. [32, 33], the authors presented the following conjecture: Given is the class of Werner state in $\mathcal{H}_3 \otimes \mathcal{H}_3$

$$\rho_W(\eta) = \frac{1}{8\eta - 1} (\eta I_3 \otimes I_3 - \frac{\eta + 1}{3} \sum_{i,j=1}^{3} |i,j\rangle\langle j,i|)$$  (28)

where $I_3$ is the identity operator in $\mathcal{H}_3$. The state $\lim_{\eta \to \infty} \rho_W(\eta)$ is separable and for any finite $\eta \geq \frac{1}{2}$ $\rho_W(\eta)$ is entangled and violates the Peres-Horodecki criterion [34, 35]. It has been shown that there is convincing evidence in support of the conjecture that for all $\eta \geq 2$ the state $\rho_W(\eta)$ is undistillable. We will show that two qutrits initially in $\rho_W(\eta)$ can become a stationary free entangled state.

Substituting $\rho_W(\eta)$ into Eq. (25), and the corresponding stationary state can be easily obtained

$$\rho_W^{(s)}(\eta) = \frac{1}{24\eta - 3} [(10\eta - 5)|1,1\rangle\langle 1,1| + (2\eta - 1)|S_1\rangle\langle S_1|$$

$$+(12\eta + 3)|A_1\rangle\langle A_1|)$$  (29)

where $|S_1\rangle = \sqrt{2}(|1,3\rangle + |1,3\rangle - |2,2\rangle)$, and $|A_1\rangle = \sqrt{2}(|1,2\rangle - |2,1\rangle)$. The negativity of the stationary state $\rho_W^{(s)}(\eta)$ can be calculated as

$$N = 2(\sqrt{\frac{c_4^2 + 4c_2^2 - c_3^2}{c_3^2}} + 2|\kappa|),$$  (30)

where $\kappa$ is the negative root of the polynomial

$$x^3 - (\zeta_1 + \zeta_2)x^2 + (\zeta_1 \zeta_2 - \zeta_3^2)x + \zeta_3^2 = 0,$$  (31)

and

$$\zeta_1 = \frac{10\eta - 5}{24\eta - 3},$$

$$\zeta_2 = \frac{2\eta - 1}{72\eta - 9},$$

$$\zeta_3 = \frac{4\eta + 1}{16\eta - 2}.\quad (32)$$

In Fig.8, we plot the negativity $N$ of the stationary state $\rho_W^{(s)}(\eta)$ as the function of $\eta$. It is shown that the negativity decreases with $\eta$ and eventually converges to about 0.21. In the case with $\eta = \frac{1}{2}$, we can see that the stationary state $\rho_W^{(s)}(\frac{1}{2})$ is a maximally entangled state $|A_1\rangle\langle A_1|$ in the space spanned by $\{|1,1\rangle, |1,2\rangle, |2,1\rangle, |2,2\rangle\}$. This implies that the common zero-temperature thermal reservoir plays a similar role with the entanglement purifier in this case. In the limit $\eta \to \infty$, $\rho_W^{(s)}(\infty)$ is also entangled. We will demonstrate that $\rho_W^{(s)}(\eta)$ ($\eta \in [2, \infty)$) is distillable even though the initial state $\rho_W(\eta)$ is conjectured to be undistillable in this region. By applying the local projection $\Pi_{1,2} \otimes \Pi_{1,2} = \{|1\rangle\langle 1| + |2\rangle\langle 2|\} \otimes \{|1\rangle\langle 1| + |2\rangle\langle 2|\}$ on $\rho_W^{(s)}(\eta)$, we can immediately obtain the resulted density operator

$$\rho_W^{(r)}(\eta) = \frac{\Pi_{1,2} \otimes \Pi_{1,2} \rho_W^{(s)}(\eta) \Pi_{1,2} \otimes \Pi_{1,2}}{\text{Tr}(\Pi_{1,2} \otimes \Pi_{1,2} \rho_W^{(s)}(\eta) \Pi_{1,2} \otimes \Pi_{1,2})}$$
thermal reservoir. It is shown that the fraction of Bell singlet state in the initial state is a key fact determining whether the collective decay can enhance the stationary state entanglement or Bell violation of two qubits or not. We have also found that collective decay of two qubits or two qutrits can induce stationary entanglement from their initial maximally mixed state. The detailed analytical relations among average thermal phonon number of the thermal reservoir, entanglement and Bell violation of two qubits have been obtained. The common thermal reservoir with any large intensity can enhance the entanglement of two qubits initially in a mixed Werner state collectively coupled with the reservoir if only the fraction $F$ of Bell singlet state in the initial state is not smaller than $\frac{1}{2}$. If the fraction $F$ of Bell singlet state in the initial Werner state is not smaller than $\frac{2}{3}$, two qubits in a common zero-temperature thermal reservoir can evolve into a stationary maximally entangled mixed state as the time $t \to \infty$. The corresponding stationary states of two qubits initially in the standard Werner state with $r \geq 5/9$ in the common thermal reservoirs with two extreme situations, i.e. the zero temperature and the infinite high temperature, become part of the frontier of the concurrence versus linear entropy and the whole frontier of the negativity versus linear entropy, respectively.

For two-qutrit case, we have found that two qutrits initially in the conjectured bound entangled Werner state can become distillable due to the collective decay caused by the common zero-temperature thermal reservoir. In addition, we obtained a more striking result that a pure stationary Bell singlet state in the initial state is not smaller than $F_{\text{W}}$. The common thermal reservoir with any large intensity can enhance the stationary state entanglement or Bell violation of two qubits or not.

VI. CONCLUSION

In this paper, we have investigated the systems of two qubits or qutrits collectively interacting with a common zero-temperature thermal reservoir. It is shown that the fraction of Bell singlet state in the initial state is a key fact determining whether the collective decay can enhance the stationary state entanglement or Bell violation of two qubits or not. We have also found that collective decay of two qubits or two qutrits can induce stationary entanglement from their initial maximally mixed state. The detailed analytical relations among average thermal phonon number of the thermal reservoir, entanglement and Bell violation of two qubits have been obtained. The common thermal reservoir with any large intensity can enhance the entanglement of two qubits initially in a mixed Werner state collectively coupled with the reservoir if only the fraction $F$ of Bell singlet state in the initial state is not smaller than $\frac{1}{2}$. If the fraction $F$ of Bell singlet state in the initial Werner state is not smaller than $\frac{2}{3}$, two qubits in a common zero-temperature thermal reservoir can evolve into a stationary maximally entangled mixed state as the time $t \to \infty$. The corresponding stationary states of two qubits initially in the standard Werner state with $r \geq 5/9$ in the common thermal reservoirs with two extreme situations, i.e. the zero temperature and the infinite high temperature, become part of the frontier of the concurrence versus linear entropy and the whole frontier of the negativity versus linear entropy, respectively.

For two-qutrit case, we have found that two qutrits initially in the conjectured bound entangled Werner state can become distillable due to the collective decay caused by the common zero-temperature thermal reservoir. In addition, we obtained a more striking result that a pure stationary Bell singlet state in the initial state is not smaller than $F_{\text{W}}$. The common thermal reservoir with any large intensity can enhance the stationary state entanglement or Bell violation of two qubits or not.

VI. CONCLUSION

In this paper, we have investigated the systems of two qubits or qutrits collectively interacting with a common zero-temperature thermal reservoir. It is shown that the fraction of Bell singlet state in the initial state is a key fact determining whether the collective decay can enhance the stationary state entanglement or Bell violation of two qubits or not. We have also found that collective decay of two qubits or two qutrits can induce stationary entanglement from their initial maximally mixed state. The detailed analytical relations among average thermal phonon number of the thermal reservoir, entanglement and Bell violation of two qubits have been obtained. The common thermal reservoir with any large intensity can enhance the entanglement of two qubits initially in a mixed Werner state collectively coupled with the reservoir if only the fraction $F$ of Bell singlet state in the initial state is not smaller than $\frac{1}{2}$. If the fraction $F$ of Bell singlet state in the initial Werner state is not smaller than $\frac{2}{3}$, two qubits in a common zero-temperature thermal reservoir can evolve into a stationary maximally entangled mixed state as the time $t \to \infty$. The corresponding stationary states of two qubits initially in the standard Werner state with $r \geq 5/9$ in the common thermal reservoirs with two extreme situations, i.e. the zero temperature and the infinite high temperature, become part of the frontier of the concurrence versus linear entropy and the whole frontier of the negativity versus linear entropy, respectively.

For two-qutrit case, we have found that two qutrits initially in the conjectured bound entangled Werner state can become distillable due to the collective decay caused by the common zero-temperature thermal reservoir. In addition, we obtained a more striking result that a pure stationary Bell singlet state in the initial state is not smaller than $F_{\text{W}}$. The common thermal reservoir with any large intensity can enhance the stationary state entanglement or Bell violation of two qubits or not.

VI. CONCLUSION

In this paper, we have investigated the systems of two qubits or qutrits collectively interacting with a common zero-temperature thermal reservoir. It is shown that the fraction of Bell singlet state in the initial state is a key fact determining whether the collective decay can enhance the stationary state entanglement or Bell violation of two qubits or not. We have also found that collective decay of two qubits or two qutrits can induce stationary entanglement from their initial maximally mixed state. The detailed analytical relations among average thermal phonon number of the thermal reservoir, entanglement and Bell violation of two qubits have been obtained. The common thermal reservoir with any large intensity can enhance the entanglement of two qubits initially in a mixed Werner state collectively coupled with the reservoir if only the fraction $F$ of Bell singlet state in the initial state is not smaller than $\frac{1}{2}$. If the fraction $F$ of Bell singlet state in the initial Werner state is not smaller than $\frac{2}{3}$, two qubits in a common zero-temperature thermal reservoir can evolve into a stationary maximally entangled mixed state as the time $t \to \infty$. The corresponding stationary states of two qubits initially in the standard Werner state with $r \geq 5/9$ in the common thermal reservoirs with two extreme situations, i.e. the zero temperature and the infinite high temperature, become part of the frontier of the concurrence versus linear entropy and the whole frontier of the negativity versus linear entropy, respectively.

For two-qutrit case, we have found that two qutrits initially in the conjectured bound entangled Werner state can become distillable due to the collective decay caused by the common zero-temperature thermal reservoir. In addition, we obtained a more striking result that a pure stationary Bell singlet state in the initial state is not smaller than $F_{\text{W}}$. The common thermal reservoir with any large intensity can enhance the stationary state entanglement or Bell violation of two qubits or not.

ACKNOWLEDGMENT

This project was supported by the National Natural Science Foundation of China (Project NO. 10174066).

[1] P. W. Shor, Phys. Rev. A 52, 2493 (1995); D. P. DiVincenzo, Science 270, 255 (1995); L. K. Grover, Phys. Rev. Lett. 79, 325 (1997); J. I. Cirac, and P. Zoller, Nature 404, 579 (2000); M. A. Nielsen, and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] A. Beige et al., J. Mod. Opt. 47, 2583 (2000).
[3] D. Braun, Phys. Rev. Lett. 89, 277901 (2002).
[4] S. Schneider, and G. J. Milburn, Phys. Rev. A 65, 042107 (2002).
[5] M. S. Kim, J. Lee, D. Ahn, and P. L. Knight, Phys. Rev. A 65, 040101(R) (2002).
[6] B. Kraus, J. I. Cirac, Phys. Rev. Lett. 92, 013602 (2004).
[7] L. M. Duan, H. J. Kimble, Phys. Rev. Lett. 90, 253601
[8] M. B. Plenio, and S. F. Huelga, Phys. Rev. Lett. 88, 197901 (2002).
[9] P. Zanardi, Phys. Rev. A 63, 012301 (2001).
[10] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett 91, 070402 (2003); Phys. Rev. A 67, 042110 (2003).
[11] S. G. Clark, and A. S. Parkins, Phys. Rev. Lett 90, 047905 (2003).
[12] N. A. Peters et al., Phys. Rev. Lett. 92, 133601 (2004).
[13] M. Barbieri et al., Phys. Rev. Lett. 92, 177901 (2004).
[14] S. Ishizaka and T. Hiroshima, Phys. Rev. A 62, 022310 (2000).
[15] F. Verstraete, K. Audenaert, and B. De Moor, Phys. Rev. A 64, 012316 (2001).
[16] T. C. Wei et al., Phys. Rev. A 67, 022110 (2003).
[17] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[18] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, Phys. Rev. A 64, 030302(R) (2001).
[19] S. Ghosh et al., Phys. Rev. A 64, 044301 (2001).
[20] J. Lee, M. S. Kim, Phys. Rev. Lett. 84, 4236 (2000).
[21] T. Hiroshima, S. Ishizaka, Phys. Rev. A 62, 044302 (2000).
[22] A. Miranowicz, Phys. Lett. A 327, 272 (2004).
[23] Y. S. Zhang et al., Phys. Rev. A 66, 062315 (2002).
[24] C. H. Bennett et al., Phys. Rev. Lett 76, 722 (1996).
[25] W. K. Wootters, Phys. Rev. Lett 80, 2245 (1998).
[26] N. Brunner, N. Gisin, and V. Scarani, New J. Phys. 7, 88 (2005).
[27] J.-B. Xu, S.-B. Li, New J. Phys. 7, 72 (2005).
[28] J. S. Bell, Physics (N. Y.) 1, 195 (1965).
[29] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[30] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 200, 340 (1995).
[31] F. Verstrete and M. M. Wolf, Phys. Rev. Lett. 89, 170401 (2002).
[32] D. P. DiVincenzo et al., Phys. Rev. A 61, 062312 (2000).
[33] W. Dür et al., Phys. Rev. A 61, 062313 (2000).
[34] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[35] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[36] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[37] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003).