ON SOME ARITHMETIC PROPERTIES OF SIEGEL FUNCTIONS (II)

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Abstract. Let \( K \) be an imaginary quadratic field with discriminant \( d_K \leq -7 \). We deal with problems of constructing normal bases between abelian extensions of \( K \) by making use of singular values of Siegel functions. First, we show that a criterion achieved from the Frobenius determinant relation enables us to find normal bases of ring class fields of orders of bounded conductors depending on \( d_K \) over \( K \). Next, denoting by \( K(\mathcal{O}(N)) \) the ray class field modulo \( N \) of \( K \) for an integer \( N \geq 2 \) we consider the field extension \( K(\mathcal{O}(p^{2m})) \)/\( K(\mathcal{O}(pm)) \) for a prime \( p \geq 5 \) and an integer \( m \geq 1 \) relatively prime to \( p \) and then find normal bases of all intermediate fields over \( K(\mathcal{O}(pm)) \) by utilizing Kawamoto’s arguments (\[17\]). And, we further investigate certain Galois module structure of the field extension \( K(\mathcal{O}(p^{n+m})) \)/\( K(\mathcal{O}(p^{\ell m})) \) with \( n \geq 2 \ell \), which would be an extension of Komatsu’s work (\[19\]).

1. Introduction

Let \( F \) be a finite Galois extension of a field \( L \). Then there exists a normal basis of \( F \) over \( L \), namely a basis of the form \( \{ x^\gamma : \gamma \in \text{Gal}(F/L) \} \) for a single element \( x \in F \) by the normal basis theorem (\[23\]). After Okada (\[25\]) had constructed normal bases of the ray class fields over the Gaussian field \( \mathbb{Q}(\sqrt{-1}) \), several other people treated the problem of generating normal bases of abelian extensions of other imaginary quadratic fields by special values of elliptic functions or elliptic modular functions (\[1\], \[19\], \[27\], \[31\]). And, Jung-Koo-Shin (\[14\]) recently found normal bases of ray class fields over any imaginary quadratic field with discriminant \( \leq -7 \) by utilizing Siegel functions.

Let \( K \) be an imaginary quadratic field and \( H_\mathcal{O} \) be the ring class field of the order \( \mathcal{O} \) of conductor \( N \geq 2 \) in \( K \). In number theory, ring class fields over imaginary quadratic fields play an important role in the study of certain quadratic Diophantine equations. For example, let \( n \) be a positive integer and \( H_\mathcal{O} \) be the ring class field of the order \( \mathcal{O} = \mathbb{Z}[\sqrt{-n}] \) in \( K = \mathbb{Q}(\sqrt{-n}) \). If \( p \) is an odd prime not dividing \( n \), then we have the following assertions:

\[ p = x^2 + ny^2 \text{ is solvable for some integers } x \text{ and } y \iff p \text{ splits completely in } H_\mathcal{O} \iff \begin{cases} \text{the Legendre symbol } \left( \frac{-n}{p} \right) = 1 \text{ and } \\ f_n(X) \equiv 0 \pmod{p} \text{ has an integer solution} \end{cases} \]

where \( f_n(X) \) is the minimal polynomial of a real algebraic integer \( \alpha \) for which \( H_\mathcal{O} = K(\alpha) \) (\[5\]). It is a classical result by the main theorem of complex multiplication that for any proper fractional \( \mathcal{O} \)-ideal \( \mathfrak{a} \), the \( j \)-invariant \( j(\mathfrak{a}) \) is an algebraic integer and generates \( H_\mathcal{O} \) over \( K \) (\[23\] or \[28\]). Unlike the classical case, however, Chen-Yui (\[2\]) constructed a generator of the ring class field of certain conductor in terms of the singular value of the Thompson series which is a Hauptmodul for \( \Gamma_0(N) \)

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or \( \Gamma_0(N)^\dagger \). Here, \( \Gamma_0(N) = \{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma \equiv (\ast \ast) \pmod{N} \} \) and \( \Gamma_0^\dagger(N) \) is the subgroup of \( \text{SL}_2(\mathbb{R}) \) generated by \( \Gamma_0(N) \) and \( \left( \begin{array}{cc} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{array} \right) \). Similarly, Cox-McKay-Stevenhagen ([6]) showed that certain singular value of a Hauptmodul for \( \Gamma_0(N) \) or \( \Gamma_0(N)^\dagger \) with rational Fourier coefficients generates \( H_\mathcal{O} \) over \( K \). Furthermore, Cho-Koo ([3]) recently revisited and extended these results by using the theory of Shimura’s canonical models and his reciprocity law. On the other hand, as we see in the above example, it is essential to find the minimal polynomial of \( j(\mathcal{O}) \) over \( K \), namely, the class equation of \( \mathcal{O} \) in order to solve such quadratic equations. Although there are several known algorithms for finding the class equations ([2], [5], [16], [24]), we would like to adopt the idea of Gee and Stevenhagen ([9], [10] or [30]) because we could not claim with the formers that the conjugates of the ring class invariant form a normal basis of \( H_\mathcal{O} \) over \( K \).

In this paper we shall first construct a ring class invariant of \( H_\mathcal{O} \) under the condition

\[
d_K \leq -43 \quad \text{and} \quad 2 \leq N \leq \frac{-\sqrt{3\pi}}{\ln \left(1 - 2.16e^{-\frac{\pi}{2\sqrt{d_K}}}\right)} \tag{1.1}
\]

in terms of singular values of Siegel functions and also systematically find its minimal polynomial (Theorems 6.7 and 6.8) and Stevenhagen ([9], [10] or [30]) because we could not claim with the formers that the conjugates of \( j(\mathcal{O}) \) form a normal basis of \( H_\mathcal{O} \) over \( K \).

Next, we shall consider in Section 6 the extension \( K(p^{\alpha m})/K(pm) \) for a prime \( p \geq 5 \) and an integer \( m \geq 1 \) relatively prime to \( p \) and, by means of Kawamoto’s arguments (17), construct a normal basis of \( F \) over \( K(pm) \) for each intermediate field \( F \) via singular values of Siegel functions as algebraic integers (Theorems 5.7 and 6.8). And, we shall further discuss in Section 7 certain Galois module structure of the ring of \( p \)-integers of \( K(p^{\ell m}) \) over that of \( K(p^{\ell m}) \) where \( n \) and \( \ell \) are positive integers with \( n \geq 2\ell \), which is motivated by a relation between the existence of normal basis in \( \mathbb{Z}_p \)-extension and the Greenberg’s conjecture ([7], [8]).

2. Field of modular functions

In this section we briefly review some necessary arithmetic properties of Siegel functions as modular functions.

For a positive integer \( N \), let \( \zeta_N = e^{2\pi i/\sqrt{N}} \) and \( \mathcal{F}_N \) be the field of modular functions of level \( N \) which are defined over \( \mathbb{Q}(\zeta_N) \). Then \( \mathcal{F}_N \) is a Galois extension of \( \mathcal{F}_1 = \mathbb{Q}(j(\tau)) \) (\( j= \)the elliptic modular function) whose Galois group is isomorphic to \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm (1 \ 0) \} \). In order to describe the Galois action on the field \( \mathcal{F}_N \) we consider the decomposition of the group

\[
\text{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm (1 \ 0) \} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/N\mathbb{Z})^* \right\} \cdot \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm (1 \ 0) \}.
\]

Here, the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \) acts on \( \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \sigma_d}/N \in \mathcal{F}_N \) by

\[
\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \sigma_d}/N \mapsto \sum_{n=-\infty}^{\infty} c_n^d e^{2\pi i n \sigma_d}/N \tag{2.1}
\]

where \( \sigma_d \) is the automorphism of \( \mathbb{Q}(\zeta_N) \) induced by \( \zeta_N \mapsto \zeta_N^d \). And, for an element \( \gamma \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm (1 \ 0) \} \) let \( \gamma' \in \text{SL}_2(\mathbb{Z}) \) be a preimage of \( \gamma \) via the natural surjection \( \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{ \pm (1 \ 0) \} \).
as linear fractional transformation \((23)\) or \((28)\).

For any pair \(r_1, r_2 \in \mathbb{Q}^2 \setminus \mathbb{Z}^2\) we define a Siegel function \(g_{r_1, r_2}(\tau)\) on \(\mathcal{H}\) (=the complex upper half plane) by the following Fourier expansion

\[
g_{r_1, r_2}(\tau) = -q^{\frac{1}{2}}B_2(r_1) e^{\pi i r_2(r_1 - 1)}(1 - q)^\infty \prod_{n=1}^{\infty} (1 - q^n r_2)(1 - q^n q_2^{-1})
\]

where \(B_2(X) = X^2 - X + \frac{1}{12}\) is the second Bernoulli polynomial, \(q_r = e^{2\pi i r}\) and \(q_z = e^{2\pi i z}\) with \(z = r_1 \tau + r_2\). Then it is a modular unit which has no zeros and poles on \(\mathcal{H}\) \((21)\). For later use we introduce some arithmetic properties and a modularity condition of Siegel functions:

**Proposition 2.1.** Let \(r = (r_1, r_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2\). Then

(i) \(g_r(\tau)\) is integral over \(\mathbb{Z}[j(\tau)]\).

(ii) Let \(N\) be the smallest positive integer with \(Nr \in \mathbb{Z}^2\). If \(N\) has at least two prime factors, then \(1/g_r(\tau)\) is integral over \(\mathbb{Z}[j(\tau)]\). If \(N = p^s\) is a prime power, then \(1/g_r(\tau)\) is integral over \(\mathbb{Z}[\frac{1}{p}j(\tau)]\).

(iii) For \(\gamma \in \text{SL}_2(\mathbb{Z})\) we get

\[
g_r^\gamma(\tau) = g_r(\tau).
\]

(iv) For \(s = (s_1, s_2) \in \mathbb{Z}^2\) we have

\[
gr+s(\tau) = (-1)^{s_1 s_2 + s_1 + s_2} e^{-\pi i (s_1 r_2 - s_2 r_1)} gr(\tau).
\]

**Proof.** See \([20]\) Section 3 and Proposition 2.4. \(\square\)

**Proposition 2.2.** Let \(N \geq 2\). Let \(\{m(r)\}_{r \in \frac{1}{N} \mathbb{Z}^2 \setminus \mathbb{Z}^2}\) be a family of integers such that \(m(r) = 0\) except finitely many \(r\). Then a product of Siegel functions

\[
\prod_{r \in \frac{1}{N} \mathbb{Z}^2 \setminus \mathbb{Z}^2} g_r^{m(r)}(\tau)
\]

belongs to \(\mathcal{F}_N\), if \(\{m(r)\}\) satisfies

\[
\sum_r m(r)(Nr_1)^2 \equiv \sum_r m(r)(Nr_2)^2 \equiv 0 \pmod{\text{gcd}(2, N)}
\]

\[
\sum_r m(r)(Nr_1)(Nr_2) \equiv 0 \pmod{N}
\]

\[
\text{gcd}(12, N) \cdot \sum_r m(r) \equiv 0 \pmod{12}.
\]

**Proof.** See \([21]\) Chapter 3 Theorems 5.2 and 5.3. \(\square\)

**Corollary 2.3.** Let \(N \geq 2\). For \(r = (r_1, r_2) \in \frac{1}{N} \mathbb{Z}^2 \setminus \mathbb{Z}^2\) the function \(g_{r \cdot \text{gcd}(6, N)}^{12N}(\tau)\) satisfies

\[
g_{r_1, r_2}^{\text{gcd}(6, N)}(\tau) = g_{-r_1, r_2}^{\text{gcd}(6, N)}(\tau) = g_{(r_1, r_2)}^{\text{gcd}(6, N)}(\tau)
\]

where \(\langle X \rangle\) is the fractional part of \(X \in \mathbb{R}\) so that \(0 \leq \langle X \rangle < 1\). It belongs to \(\mathcal{F}_N\). And, \(\gamma \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}/\{\pm \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \}) \cong \text{Gal}(\mathcal{F}_N/\mathcal{F}_1)\) acts on the function by

\[
\left(g_{r \cdot \text{gcd}(6, N)}^{12N}(\tau)\right)^\gamma = g_{r_2}^{12N}(\tau).
\]

**Proof.** It is a direct consequence of Propositions \([21] 2.2\) and definition \([23]\). \(\square\)
3. Action of Galois groups

We shall investigate an algorithm for finding all conjugates of the singular value of a modular function, from which we can determine the conjugates of the singular values of certain Siegel functions.

Let $K(\not= \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}))$ be an imaginary quadratic field of discriminant $d_K$ and define

$$\theta = \begin{cases} \frac{\sqrt{d_K}}{2} & \text{for } d_K \equiv 0 \pmod{4} \\ \frac{1+\sqrt{d_K}}{2} & \text{for } d_K \equiv 1 \pmod{4} \end{cases}$$

which is a generator of the ring of integers $\mathcal{O}_K$ of $K$, that is, $\mathcal{O}_K = \mathbb{Z}[\theta]$. We denote by $H$ the Hilbert class field. Utilizing the Shimura’s reciprocity law Gee and Stevenhagen ([9], [10]) described the actions of $\text{Gal}(K(N)/H)$ and $\text{Gal}(H/K)$ explicitly. By extending their idea we shall examine $\text{Gal}(H\mathcal{O}/K)$ for the order $\mathcal{O}$ of conductor $N$.

Under the properly equivalent relation primitive positive definite quadratic forms $aX^2 + bXY + cY^2$ of discriminant $d_K$ determine a group $C(d_K)$, called the form class group of discriminant $d_K$. We identify $C(d_K)$ with the set of all reduced primitive positive definite quadratic forms, which are characterized by the conditions

$$-a < b \leq a < c \quad \text{or} \quad 0 \leq b = a = c$$

(3.2)

and the discriminant relation

$$b^2 - 4ac = d_K.$$  

(3.3)

Then from the above two conditions for reduced quadratic forms one can deduce

$$1 \leq a \leq \sqrt{-d_K}.$$ \hfill (3.4)

And, for a reduced quadratic form $Q = aX^2 + bXY + cY^2 \in C(d_K)$ we define a CM-point $\theta_Q$ by

$$\theta_Q = \frac{-b + \sqrt{d_K}}{2a}.$$ \hfill (3.5)

Furthermore, we define $\beta_Q = (\beta_p)_{p} \in \prod_p : \text{prime} \text{GL}_2(\mathbb{Z}_p)$ as

$$\beta_p = \begin{cases} \begin{pmatrix} \frac{a}{b} & \frac{1}{c} \\ 0 & 1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{1}{b} & \frac{1}{c} \\ 0 & 1 \end{pmatrix} & \text{if } p \mid a \text{ and } p \nmid c \text{ for } d_K \equiv 0 \pmod{4} \end{cases} \quad (3.6)$$

and

$$\begin{cases} \begin{pmatrix} \frac{a}{b} & \frac{1}{c} \\ 0 & -1 \end{pmatrix} & \text{if } p \nmid a \\ \begin{pmatrix} -\frac{1}{b} & \frac{1}{c} \\ -1 & 0 \end{pmatrix} & \text{if } p \mid a \text{ and } p \mid c \text{ for } d_K \equiv 1 \pmod{4}. \end{cases} \quad (3.7)$$

It is then well-known that $C(d_K)$ is isomorphic to $\text{Gal}(H/K)$ and the action of $Q$ on $H$ can be extended to that on $K(N)$ as

$$\text{Gal}(H/K) \cong C(d_K) \rightarrow \text{Gal}(K(N)/K) \quad Q \mapsto (h(\theta) \mapsto h^{\beta_Q(\theta_Q)}).$$

(3.8)
where $h$ is an element of $\mathcal{F}_N$, defined and finite at $\theta$. Note that the map (3.8) is not a homomorphism, just an injective map. And, observe that

$$K(N) = K(h(\theta) : h \in \mathcal{F}_N \text{ is defined and finite at } \theta)$$  \hspace{1cm} (3.9)

by the main theorem of complex multiplication (23 or 28) and there exists $\beta \in \text{GL}_2^+ (\mathbb{Q}) \cap \mathbb{M}_2(\mathbb{Z})$ such that $\beta \equiv \beta_p \pmod{NZ_p}$ for all primes $p$ dividing $N$ by the Chinese remainder theorem. Thus the action of $\beta_Q$ on $\mathcal{F}_N$ is understood as that of $\beta$ which is an element of $\text{GL}_2(\mathbb{Z}/NZ)/\{ \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \}$ (28, 9 or 10).

Let

$$\min(\theta, \mathbb{Q}) = X^2 + B_\theta X + C_\theta = \left\{ \begin{array}{ll} X^2 - \frac{d_K}{4} & \text{for } d_K \equiv 0 \pmod{4} \\ X^2 + X + \frac{1 - d_K}{4} & \text{for } d_K \equiv 1 \pmod{4} \end{array} \right.$$  \hspace{1cm} (3.10)

By the Shimura’s reciprocity law we have an isomorphism

$$W_N, \theta/\left\{ \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right\} \sim \text{Gal}(K(N)/H) \hspace{1cm} \gamma \mapsto \left( h(\theta) \mapsto h^\gamma(\theta) \right)$$

where $h \in \mathcal{F}_N$ is defined and finite at $\theta$, and

$$W_N, \theta = \left\{ \left( \begin{array}{cc} t - B_\theta s & -C_\theta s \\ s & t \end{array} \right) \in \text{GL}_2(\mathbb{Z}/NZ) : t, s \in \mathbb{Z}/NZ \right\} / \left\{ \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right\} \hspace{1cm} (28, 9 \text{ or } 10).$$

**Lemma 3.1.** Let $N \geq 2$. If the function $j(N\tau)$ satisfies $j(N\theta) = j(N\tau) \circ \alpha(\theta)$ for some $\alpha = \left( \begin{array}{ccc} x & \ast \\ \ast & \ast \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, then $z \equiv 0 \pmod{N}$, that is, $\alpha \in \Gamma_0(N)$. 

**Proof.** See [20] Lemma 9.2. \hfill $\square$

**Theorem 3.2.** Let $\mathcal{O}$ be the order of conductor $N \geq 2$ in $K$. Then we obtain

$$\text{Gal}(H_{\mathcal{O}}/H) \cong W_N, \theta/\left\{ \left( \begin{array}{cc} t & 0 \\ 0 & t \end{array} \right) : t \in (\mathbb{Z}/NZ)^* \right\}.$$ 

**Proof.** As is well-known, $H_{\mathcal{O}} = K(j(N\theta))$ (23 or 28). Let $\gamma$ be an element of $W_N, \theta$ which is of the form $\left( \begin{array}{cc} t & 0 \\ 0 & t \end{array} \right)$ for some $t \in (\mathbb{Z}/NZ)^*$. If we decompose $\gamma$ into $\left( \begin{array}{cc} 1 & 0 \\ 0 & d \end{array} \right) \beta$ for some $d \in (\mathbb{Z}/NZ)^*$ and $\beta \in \text{SL}_2(\mathbb{Z})$, then we obviously achieve $\beta \in \Gamma_0(N)$. Since the function $j(N\tau)$ is a modular function for $\Gamma_0(N)$ with rational Fourier coefficients, we deduce by (3.10) that

$$j(N\theta) = j(N\tau) = j(N\beta(\theta) = j(N\tau) \circ \beta(\theta) = j(N\theta).$$

Conversely, assume that an element $\gamma = \left( \begin{array}{cc} 1 - B_\theta s & -C_\theta s \\ s & t \end{array} \right) \in W_N, \theta$ fixes $j(N\theta)$. Decompose $\gamma$ into $\left( \begin{array}{cc} 1 & 0 \\ 0 & d \end{array} \right) \beta$ for some $d \in (\mathbb{Z}/NZ)^*$ and $\beta \in \text{SL}_2(\mathbb{Z})$. By the same reasoning as above we derive $j(N\theta) = j(N\tau) \circ \beta(\theta)$. On the other hand, we know $\beta \in \Gamma_0(N)$ by Lemma 3.1 and so $s \equiv 0 \pmod{N}$. Therefore $\gamma = \left( \begin{array}{cc} 1 & 0 \\ 0 & q \end{array} \right)$. This proves the theorem. \hfill $\square$

**Remark 3.3.** We have the degree formula

$$[K(N) : H] = \phi(N\mathcal{O}_K)w(N\mathcal{O}_K) \hspace{1cm} w_K$$  \hspace{1cm} (3.11)

where $\phi$ is the Euler function for ideals, namely

$$\phi(p^n) = (N_{K/Q}(p) - 1)N_{K/Q}(p)^{n-1}$$
for a power of prime ideal \( p \), \( w(N\mathcal{O}_K) \) is the number of roots of unity in \( K \) which are \( \equiv 1 \) (mod \( N\mathcal{O}_K \)) and \( w_K \) is the number of roots of unity in \( K \) \([22]\) Chapter VI Theorem 1). And, for the order \( \mathcal{O} \) of conductor \( N \) we know the formula

\[
[H_\mathcal{O} : H] = \frac{N}{[\mathcal{O}_K : \mathcal{O}]} \prod_{p|N} \left( 1 - \left( \frac{d_K}{p} \right) \frac{1}{p} \right)
\]

where \( \left( \frac{d_K}{p} \right) \) is the Legendre symbol for an odd prime \( p \) and \( \left( \frac{d_K}{p} \right) \) is the Kronecker symbol \([5]\) Chapter 2 Theorem 7.24). Thus the second part of the proof depending on Lemma 3.1 can be also established by showing that

\[
[K(N) : H_\mathcal{O}] = [\mathcal{O}_K : \mathcal{O}] \prod_{p|N} \left( 1 - \left( \frac{d_K}{p} \right) \frac{1}{p} \right)
\]

where \( \left( \frac{d_K}{p} \right) \) is the Legendre symbol for an odd prime \( p \) and \( \left( \frac{d_K}{p} \right) \) is the Kronecker symbol \([5]\) Chapter 2 Theorem 7.24). Thus the second part of the proof depending on Lemma 3.1 can be also established by showing that

\[
[K(N) : H_\mathcal{O}] = [\mathcal{O}_K : \mathcal{O}] \prod_{p|N} \left( 1 - \left( \frac{d_K}{p} \right) \frac{1}{p} \right)
\]

**Theorem 3.4.** Let \( \mathcal{O} \) be the order of conductor \( N \geq 2 \) in \( K \) and \( f \) be an element of \( \mathcal{F}_N \) such that

\[
\left\{ f^{\gamma \beta}Q(\theta_Q) : \gamma \in W_N, \theta \in \left\{ \left( \begin{array}{c} t \\ 0 \\ t \end{array} \right) : t \in (\mathbb{Z}/N\mathbb{Z})^* \right\} \text{ and } Q \in C(d_K) \right\}
\]

is the set of all conjugates of \( f(\theta) \) under the action of \( \text{Gal}(H_\mathcal{O}/K) \).

**Proof.** The assertion follows from the following diagram:

\[
\begin{array}{ccc}
H_\mathcal{O} & \overset{\text{Galois groups}}{\longrightarrow} & \text{Gal}(H_\mathcal{O}/H) \cong W_N, \theta/\left\{ \left( \begin{array}{c} t \\ 0 \\ t \end{array} \right) : t \in (\mathbb{Z}/N\mathbb{Z})^* \right\} \text{ by Theorem 3.2} \\
H & \bigg\Downarrow & \bigg\Downarrow \text{Gal}(H/K) = \left\{ \left( h(\theta) \mapsto h^{\beta}Q(\theta_Q) \right) : Q \in C(d_K) \right\} \text{ by (3.8)} \\
K & \bigg\Downarrow & \bigg\Downarrow
\end{array}
\]

where \( h \) is an element of \( \mathcal{F}_N \), defined and finite at \( \theta \). \( \square \)

**Remark 3.5.** Theorem 3.4 and transformation formulas in Corollary 2.3 enable us to find all conjugates of the singular value \( \prod_{1 \leq w \leq \frac{N}{d}, \gcd(w, N) = 1} g_{\left( \frac{w}{d}, \frac{N}{w} \right)}(\theta) \), which will be used to prove our first main theorem.

4. Normal bases of ring class fields

Let \( K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}) \) be an imaginary quadratic field with \( d_K(\leq -7) \) and let \( \theta \) be defined as in \([31]\) and \( N \geq 2 \). If we put

\[
D = \sqrt{-d_K} \quad \text{and} \quad A = |e^{2\pi i \theta}| = e^{-\pi \sqrt{-d_K}},
\]

then \( A^\frac{X}{2} = e^{-\sqrt{X}} \) which is independent of \( K \).

**Lemma 4.1.** We have the following inequalities:

(i) \( \left| \frac{1 - e^{-NX}}{1 - A^NX} \right| > 1. \)

(ii) \( \frac{1}{1 - A^X} < 1 + A^{\frac{X}{2}} \) for all \( X \geq \frac{1}{2} \).
Lemma 4.2. Assume the condition
\[ w < 1 + A^{\frac{X}{\alpha}} \text{ for all } X \geq \frac{1}{2}. \]

(iv) \[ 1 + X < e^X \text{ for all } X > 0. \]

Proof. See [15] Lemma 4.1.

Lemma 4.2. Assume the condition
\[ d_K \leq -43 \text{ and } 2 \leq N \leq \frac{-\sqrt{3} \pi}{\ln (1 - 2.16e^{-\frac{\pi \sqrt{-d_K}}{24}})} . \] (4.1)

Let \( Q = aX^2 + bXY + cY^2 \) be a reduced primitive positive definite quadratic form of discriminant \( d_K \) and \( \theta_Q \) as in \([7, 2]\). If \( \alpha \geq 2 \), then the inequality
\[ \left| \frac{g(0, \frac{s}{\alpha})}{g(s, \frac{s}{\alpha})(\theta_Q)} \right| < 1 \]
holds for \( w \in \mathbb{Z} \setminus N\mathbb{Z} \) and \( (s, t) \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2 \).

Proof. We may assume \( 0 \leq s \leq \frac{N}{2} \) by Corollary [2.3]. And, we have \( 2 \leq a \leq D \) by (3.4) because \( Q \) is a reduced primitive positive definite quadratic form. From the definition (2.3) we obtain that
\[ \left| \frac{g(0, \frac{s}{\alpha})}{g(s, \frac{s}{\alpha})(\theta_Q)} \right| \leq A^{\frac{1}{2}}(B_{2}(0)-\frac{1}{n}B_{2}(\frac{s}{\alpha})) \cdot \prod_{n=1}^{\infty} \frac{(1 + A^n)^2}{(1 - A^{\frac{1}{\alpha}}(n+\frac{s}{\alpha}))(1 - A^{\frac{1}{\alpha}}(n-\frac{s}{\alpha}))} \]
Now we see from the fact \( a \leq D \) and Lemma 4.1(i) that \( |1 - \zeta_n| < 2 \) and
\[ |1 - e^{2\pi i(\frac{n}{\alpha} - \frac{b+\sqrt{d_K}}{2n} + \frac{s}{\alpha})}| \geq \begin{cases} |1 - \zeta_n| & \text{if } s = 0 \\ |1 - A^{\frac{1}{\alpha}}| & \text{if } s \neq 0 \end{cases} \]
\[ \geq 1 - A^{\frac{1}{\alpha}}. \]

Therefore we achieve that
\[ \left| \frac{g(0, \frac{s}{\alpha})}{g(s, \frac{s}{\alpha})(\theta_Q)} \right| < A^{\frac{1}{2}}(B_{2}(0)-\frac{1}{n}B_{2}(\frac{s}{\alpha})) \prod_{n=1}^{\infty} \frac{(1 + A^n)^2}{(1 - A^{\frac{1}{\alpha}})(1 - A^{\frac{1}{\alpha}}(n-\frac{s}{\alpha}))} \]
by the facts \( 2 \leq a \leq D, 0 \leq s \leq \frac{N}{2} \)
\[ < \frac{2A^{\frac{1}{2}}}{1 - A^{\frac{1}{\alpha}}} \prod_{n=1}^{\infty} (1 + A^n)^2(1 + A^{\frac{1}{\alpha}}(n+\frac{s}{\alpha}))(1 + A^{\frac{1}{\alpha}}(n-\frac{s}{\alpha})) \]
by Lemma 4.1(ii)
\[ < \frac{2A^{\frac{1}{2}}}{1 - A^{\frac{1}{\alpha}}} \prod_{n=1}^{\infty} e^{2A^n + A^{\frac{n}{\alpha}}(n+\frac{s}{\alpha})} + A^{\frac{n}{\alpha}}(n-\frac{s}{\alpha}) \]
by Lemma 4.1(iv)
\[ = \frac{2A^{\frac{1}{2}}}{1 - A^{\frac{1}{\alpha}}} e^{2A^{\frac{1}{2}} + A^{\frac{1}{\alpha}}(n+\frac{s}{\alpha})} \leq 2e^{-\frac{\pi \sqrt{-d_K}}{24}} \cdot \frac{2e^{-\frac{\pi \sqrt{-d_K}}{24}} + e^{-\frac{\pi \sqrt{-d_K}}{24}} + e^{-\frac{\pi \sqrt{-d_K}}{24}}}{1 - e^{-\frac{\pi \sqrt{-d_K}}{24}}} \]
by the fact \( d_K \leq -43 \)
\[ < \frac{2.16e^{-\frac{\pi \sqrt{-d_K}}{24}}}{1 - e^{-\frac{\pi \sqrt{-d_K}}{24}}} < 1 \]
by the condition (4.1).

This proves the lemma. \( \square \)
Lemma 4.3. Assume the condition
\[ d_K \leq -43 \quad \text{and} \quad 2 \leq N \leq \sqrt{-d_K}. \] (4.2)

Let \( Q = X^2 + bXY + cY^2 \) be a reduced primitive positive definite quadratic form of discriminant \( d_K \). Then we get the inequality
\[ \left\lvert \frac{g(0, \frac{w}{N})(\theta)}{g(\frac{w}{N}, \frac{t}{N})(\theta_Q)} \right\rvert < 1 \]
for \( w \in \mathbb{Z} \setminus N\mathbb{Z} \) and \((s, t) \in \mathbb{Z}^2 \setminus N\mathbb{Z}^2 \) with \( s \not\equiv 0 \pmod{N} \).

Proof. We may assume \( 1 \leq s < \frac{N}{2} \) by Corollary 2.3. Then we establish that
\[
\left\lvert \frac{g(0, \frac{w}{N})(\theta)}{g(\frac{w}{N}, \frac{t}{N})(\theta_Q)} \right\rvert < A_{\frac{w}{N}}(\text{B}_2(0) - \text{B}_2(\frac{1}{N})) \left| 1 - \frac{\zeta_N}{1 - A_N} \right| \prod_{n=1}^\infty \frac{(1 + A^n)^2}{(1 - A^n)(1 - A^{n-\theta})} \quad \text{by (2.3)}
\]

\[
< A_{\frac{w}{N}}(\text{B}_2(0) - \text{B}_2(\frac{1}{N})) \frac{2}{1 - A_N} \prod_{n=1}^\infty \frac{(1 + A^n)^2}{(1 - A^n)(1 - A^{n-\theta})} \quad \text{by } 1 \leq s < \frac{N}{2}
\]

\[
< \frac{2A_{\frac{w}{N}}}{1 - A_N} \prod_{n=1}^\infty \frac{(1 + A^n)^2}{(1 + A^{n+\pi})(1 + A^{\frac{n}{2}}(n - \frac{1}{2}))} \quad \text{by Lemma 4.1(iii)}
\]

\[
< \frac{2A_{\frac{w}{N}}}{1 - A_N} \prod_{n=1}^\infty \frac{(1 + A^n)^2}{(1 + A^{n+\pi})(1 + A^{\frac{n}{2}}(n - \frac{1}{2}))} \quad \text{by the fact } N \geq 2 \text{ and Lemma 4.1(iv)}
\]

\[
= \frac{2A_{\frac{w}{N}}}{1 - A_N} e^{2\pi A_{\frac{w}{N}} + 4\pi A_{\frac{w}{N}} + 2\pi A_{\frac{w}{N}} - 2\pi} \leq \frac{2e^{-\sqrt{-d_K}}}{1 - e^{-\sqrt{-d_K}}} \quad \text{by the fact } d_K \leq -43
\]

\[
< 2\cdot 0001e^{-\frac{\sqrt{-d_K}}{4}} \leq 2\cdot 0001e^{-\frac{\pi}{2}} < 1 \quad \text{by the fact } N \leq \sqrt{-d_K},
\]

which proves the lemma. \( \square \)

Remark 4.4. Observe that the condition (4.1) is stronger than (4.2), namely
\[
\frac{-\sqrt{3\pi}}{\ln \left( 1 - 2.16e^{-\frac{\sqrt{-d_K}}{24}} \right)} < \sqrt{-d_K}.
\]

Now we are ready to prove our main theorem about primitive generators of ring class fields over \( K \).

Theorem 4.5. Assume the condition (4.1) and let \( \mathcal{O} \) be the order of conductor \( N \) in \( K \). Then the singular value
\[ \prod_{1 \leq w \leq \frac{N}{2}, \text{gcd}(w, N) = 1} g_{\frac{w}{N}}(0, \frac{\theta}{N})(\theta) \] (4.3)
generates \( \mathcal{H}_\mathcal{O} \) over \( K \). It is a real algebraic integer and its minimal polynomial has integer coefficients. In particular, if the conductor \( N \) has at least two prime factors, then it is a unit.
Proof. Let \( g(\tau) = \prod_{0 \leq w < N} g_{\frac{12N}{\text{gcd}(6, N)}}(\tau) \). By (3.10) and Theorem 3.2 we have \( \text{Gal}(K_N/H_O) \cong \{ (1, 0) : t \in (\mathbb{Z}/N\mathbb{Z})^* \}/\{ \pm (1, 0) \} \), and hence
\[
g(\theta) = \prod_w \left( g_{\frac{12N}{\text{gcd}(6, N)}}(0, \frac{6}{N}) (\theta) \right)^{w_0} = \prod_w \left( g_{\frac{12N}{\text{gcd}(6, N)}}(0, \frac{6}{N}) \right)^{w_0} = N_{K_N/H_O} \left( g_{\frac{12N}{\text{gcd}(6, N)}}(0, \frac{6}{N}) (\theta) \right)
\]
by Corollary 2.3 and (3.10). Thus \( g(\theta) \) belongs to \( H_O \). Now, if we show that the element of \( \text{Gal}(H_O/K) \) fixing the value \( g(\theta) \) is only the identity, then we can conclude by Galois theory that it generates \( H_O \) over \( K \).

It follows from Theorem 4.3 that any conjugate of \( g(\theta) \) is of the form
\[
g^{\gamma Q}(\theta_Q)
\]
for some \( \gamma = (t - B \sigma s - C \sigma s \tau) \in W_N, \theta \) and \( Q = aX^2 + bXY + cY^2 \in C(d_K) \). Assuming \( g(\theta) = g^{\gamma Q}(\theta_Q) \) we derive
\[
\prod_w g_{\frac{12N}{\text{gcd}(6, N)}}(\theta) = \prod_w g_{\frac{12N}{\text{gcd}(6, N)}}(\theta_Q)
\]
by Corollary 2.3. Since \( |g(\theta)| = |g^{\gamma Q}(\theta_Q)| \), Lemma 4.2 leads us to have \( a = 1 \). This yields
\[
Q = \text{id} = \left\{ \begin{array}{ll}
X^2 - \frac{4}{3}Y^2 & \text{for } d_K \equiv 0 \pmod{4} \\
X^2 + XY + \frac{1 - d_K}{4}Y^2 & \text{for } d_K \equiv 1 \pmod{4}
\end{array} \right.
\]
from the condition (3.2) and the relation (3.3); hence \( \beta_Q = (1, 0) \) as an element of \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) by definitions (3.6) and (3.7), and \( \theta_Q = \theta \) by definition (3.5). And we see from Corollary 2.3 that
\[
g(\theta) = g^{\gamma Q}(\theta_Q) = \prod_w g_{\frac{12N}{\text{gcd}(6, N)}}(\theta_Q) = \prod_w g_{\frac{12N}{\text{gcd}(6, N)}}(\theta)
\]
from which we get \( s \equiv 0 \pmod{N} \) by Lemma 4.3. Therefore the pair of \( \gamma = (1, 0) \) and \( Q = \theta \) represents the identity on \( H_O \) (see the tower in the proof of Theorem 4.4), and hence \( g(\theta) \) actually generates \( H_O \) over \( K \).

On the other hand, we derive from the definition (2.3)
\[
g(\theta) = \prod_w \left\{ \frac{1}{q_{\theta}^{12N}} \prod_{n=1}^{\infty} \left( 1 - q_{\theta}^{n} \zeta_{N}^{-w} (1 - q_{\theta}^{n} \zeta_{N}^{-w}) \right) \right\}^{12N \text{gcd}(6, N)}
\]
\[
= \prod_w \left\{ q_{\theta}^{N \text{gcd}(6, N)} (2 \sin \frac{w\pi}{N})^{12N \text{gcd}(6, N)} \prod_{n=1}^{\infty} \left( 1 - 2 \cos \frac{2\pi n}{N} q_{\theta}^{n} + q_{\theta}^{2n} \right)^{12N \text{gcd}(6, N)} \right\},
\]
and this claims that \( g(\theta) \) is a real number. Furthermore, we see from Proposition 2.11(i) that the function \( g(\tau) \) is integral over \( \mathbb{Z}[j(\tau)] \). Since \( j(\theta) \) is a real algebraic integer (23 or 28), so is the value \( g(\theta) \). And its minimal polynomial over \( K \) has integer coefficients. In particular, if \( N \) has at least two prime factors, the function \( 1/g(\tau) \) is also integral over \( \mathbb{Z}[j(\tau)] \) by Proposition 2.11(ii); hence \( g(\theta) \) becomes a unit.

Remark 4.6. Since the proof of Theorem 4.5 depends only on Lemmas 4.2 and 4.3 which do not include any power of singular values, any nonzero power of the value in (4.6) can be also a generator of \( H_O \) over \( K \).
Remark 4.7. We would like to present an example which cannot be covered by our method due to violation of the condition (4.1).

Let \( K = \mathbb{Q}(\sqrt{-5}) \) and \( N = 12 = 2^2 \cdot 3 \). Then \( d_K = -20, \theta = \sqrt{-5} \) and

\[
C(d_K) = \{ Q_1 = X^2 + 5Y^2, \quad Q_2 = 2X^2 + 2XY + 3Y^2 \}
\]

\[
\theta_Q = \sqrt{-5}, \quad \theta_{Q_2} = \frac{-1 + \sqrt{-5}}{2}
\]

\[
\beta_Q = \left( \frac{1}{\sqrt{5}} \right), \quad \beta_{Q_2} = \left( \frac{1}{\sqrt{2}} \right)
\]

\[
W_N, \theta/\left( \frac{1}{\sqrt{5}} \right) : t \in (\mathbb{Z}/N\mathbb{Z})^* = \{ \left( \frac{1}{1} \right), \left( \frac{1}{2} \right), \left( \frac{2}{2} \right), \left( \frac{3}{3} \right), \left( \frac{4}{4} \right), \left( \frac{5}{5} \right), \left( \frac{7}{7} \right), \left( \frac{9}{9} \right) \}
\]

Now, the conjugates of

\[
x = \prod_{1 \leq \theta \leq \frac{12N}{gcd(w, N)}} g_{(0, \frac{12}{w})}^{12N}(\theta) = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

are as follows:

\[
x_1 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_2 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_3 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_4 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_5 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_6 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_7 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_8 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_9 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_10 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_11 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_12 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_13 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_14 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

\[
x_15 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5}), \quad x_16 = g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})g_{(0, \frac{1}{12})}^{24}(\sqrt{-5})
\]

possibly with multiplicity by Theorem 3.3 and Corollary 2.3. Hence the minimal polynomial of \( x \) over \( K \) would be

\[
(X - x_1) \cdots (X - x_{16}) = X^{16} - 1597283771136X^{14} + 218685334974106886200X^{12} - 989798760399582851353280X^{10} + 16357932201311753339695900X^{8} - 147817040875368967782738383488X^{6} - 813690304957218006599231416378248X^{4} - 4647287791605145269746263247201600X^{2} + 167117715935951295057696524156063178310X^{0} - 9155763998650223557795196487031471321600X^{0} - 17410059883612682120508988571419246981752X^{0} - 31984181681760551803330979365226550023488X^{0} + 5677583625730635496465454293769775900X^{0} - 224910210064296546707613471293806200X^{0} + 23811058989356591012938086200X^{0} - 2550974942476760820051136X + 1.
\]

And this polynomial is irreducible over \( K \), so \( x \) is indeed a primitive generator of the ring class field of the order of conductor 12 in \( \mathbb{Q}(\sqrt{-5}) \). Moreover, \( x \) is a unit because the constant term is 1. Therefore, it would be worthwhile to check how much further one can release from the condition (4.1). On the other hand, in the next section we will find in a different way a ring class invariant as singular value of certain quotient of the \( \Delta \)-function without the condition (4.1) when the conductor of the extension \( H_K/K \) is a prime power.

From now on, we will investigate how the ring class invariant (4.3) is related to a normal basis of \( H_K \) over \( K \). Even though the following four lemmas were studied in [14], we will present their
proofs for the sake of completeness. Let $F$ be a finite abelian extension of a number field $L$ with $G = \text{Gal}(F/L) = \{\gamma_1 = \text{id}, \cdots, \gamma_n\}$.

**Lemma 4.8.** A set of elements $\{x_1, \cdots, x_n\}$ in $F$ is a $L$-basis of $F$ if and only if

\[ \det(x_k^{\gamma_{\ell}^{-1}})_{1 \leq k, \ell \leq n} \neq 0. \]

**Proof.** Straightforward. \(\square\)

By $\hat{G}$ we denote the character group of $G$. Then we have the Frobenius determinant relation:

**Lemma 4.9.** If $f$ is any $\mathbb{C}$-valued function on $G$, then

\[ \prod_{\chi \in \hat{G}} \sum_{1 \leq k \leq n} \chi(\gamma_k^{-1})f(\gamma_k) = \det(f(\gamma_k \gamma_{\ell}^{-1}))_{1 \leq k, \ell \leq n}. \]

**Proof.** See [23] Chapter 21 Theorem 5. \(\square\)

Combining Lemma 4.8 and Lemma 4.9 we derive the following lemma:

**Lemma 4.10.** The conjugates of an element $x \in F$ form a normal basis of $F$ over $L$ if and only if

\[ \sum_{1 \leq k \leq n} \chi(\gamma_k^{-1})x^k \neq 0 \quad \text{for all } \chi \in \hat{G}. \]

**Proof.** For an element $x \in F$, set $x_k = x^k$ for $1 \leq k \leq n$. Then we get that the conjugates of $x$ form a normal basis of $F$ over $L$

\[ \iff \{x_1, \cdots, x_n\} \text{ is a } L\text{-basis of } F \text{ by the definition of a normal basis} \]

\[ \iff \det(x_k^{\gamma_{\ell}^{-1}})_{1 \leq k, \ell \leq n} \neq 0 \quad \text{by Lemma 4.8} \]

\[ \iff \sum_{1 \leq k \leq n} \chi(\gamma_k^{-1})x_k \neq 0 \quad \text{for all } \chi \in \hat{G} \text{ by Lemma 4.9 with } f(\gamma_k) = x_k. \]

\(\square\)

Now we present a simple criterion which enables us to determine whether the conjugates of an element $x \in F$ form a normal basis of $F$ over $L$.

**Lemma 4.11.** Assume that there exists an element $x \in F$ such that

\[ \left| \frac{x^k}{x} \right| < 1 \quad \text{for } 1 < k \leq n. \quad (4.4) \]

Then the conjugates of a high power of $x$ form a normal basis of $F$ over $L$.

**Proof.** By the hypothesis \((4.4)\) we can take a suitably large integer $m$ such that

\[ \left| \frac{x^k}{x} \right|^m \leq \frac{1}{\#G} \quad \text{for } 1 < k \leq n \quad (4.5) \]

where $\#G$ is the cardinality of $G$. Then for $\chi \in \hat{G}$ we have

\[ \left| \sum_{1 \leq k \leq n} \chi(\gamma_k^{-1})x^m \right| \geq |x^m| \left( 1 - \sum_{1 \leq k \leq n} \left| \frac{x^m}{x^m} \gamma_k \right| \right) \quad \text{by the triangle inequality} \]

\[ \geq |x^m| \left( 1 - \frac{1}{\#G}(\#G - 1) \right) = \frac{|x^m|}{\#G} > 0 \quad \text{by } (4.5). \]

Therefore the conjugates of $x^m$ form a normal basis of $F$ over $L$ by Lemma 4.10. \(\square\)
Theorem 4.12. Assume the condition \([4.1]\) and let \(O\) be the order of conductor \(N\) in \(K\). Then the conjugates of a high power of
\[
\prod_{1 \leq z \leq N \atop z | d(w, N)} g_{(0, \frac{12N}{d(w, N)})}(\theta)
\]
(4.6)
form a normal basis of \(H_O\) over \(K\).

Proof. Let \(x\) be the value in (4.6). We then see from the proof of Theorem 4.10 that \(|x^\gamma/x| < 1\) for all \(\gamma \neq \text{id} \in \text{Gal}(H_O/K)\). Therefore, the result follows from Lemma 4.11. \(\square\)

5. Generators of class fields with conductors of prime power

Let
\[
\Delta(\tau) = (2\pi i)^{12} q \prod_{n=1}^{\infty} (1 - q^{\tau})^{24} \quad (\tau \in \mathfrak{A})
\]
be the \(\Delta\)-function (or, discriminant function). In this section we shall construct primitive generators of ring class fields with conductor of prime power by utilizing singular values of the \(\Delta\)-function.

Throughout this section we let \(K\) be an imaginary quadratic field with discriminant \(d_K\) and \(O_K = [\theta, 1]\) be its ring of integers with \(\theta \in \mathfrak{A}\). For a nonzero integral ideal \(f\) of \(K\) we denote by \(\text{Cl}(f)\) the ray class group of conductor \(f\) and write \(C_f\) for its unit class. If \(f \neq O_K\) and \(C \in \text{Cl}(f)\), then we take an integral ideal \(\gamma\) so that \(\gamma^{-1} = [\gamma_1, \gamma_2]\) with \(\gamma_1 = \gamma_2/\gamma_2 \in \mathfrak{A}\). Now we define the Siegel-Ramachandra invariant by
\[
g_f(C) = g_{\left(\frac{2\pi i}{\theta}, \frac{\gamma_1}{\gamma_2}\right)}(z)
\]
where \(N\) is the smallest positive integer in \(f\) and \(a, b \in \mathbb{Z}\) such that \(1 = \frac{a}{\theta} \gamma_1 + \frac{b}{\theta} \gamma_2\). This value depends only on the class \(C\) and belongs to the ray class field \(K_f\) modulo \(f\) of \(K\). Furthermore, we have a well-known transformation formula
\[
g_f(C_1)^{\sigma(C_2)} = g_f(C_1C_2)
\]
for \(C_1, C_2 \in \text{Cl}(f)\) where \(\sigma\) is the Artin map ([21] Chapter 11 Section 1).

Let \(\chi\) be a character of \(\text{Cl}(f)\). We then denote by \(f_\chi\) the conductor of \(\chi\) and let \(\chi_0\) be the proper character of \(\text{Cl}(f_\chi)\) corresponding to \(\chi\). For a nontrivial character \(\chi\) of \(\text{Cl}(f)\) with \(f \neq O_K\) we define
\[
S_f(\chi, g_f) = \sum_{C \in \text{Cl}(f)} \chi(C) \log |g_f(C)|
\]
\[
L_f(s, \chi) = \sum_{a \neq 0 : \text{integral ideals} \atop \gcd(a, f) = O_K} \chi(a) \frac{N_{K/Q}(a)^s}{N_{K/Q}(a)} \quad (s \in \mathbb{C}).
\]
If \(f_\chi \neq O_K\), then we see from the second Kronecker limit formula that
\[
L_{f_\chi}(1, \chi_0) = T_0 S_{f_\chi}(\chi_0, g_{f_\chi})
\]
where \(T_0\) is a nonzero constant depending on \(\chi_0\) ([24] Chapter 22 Theorem 2). Here we observe that the value \(L_{f_\chi}(1, \chi_0)\) is nonzero ([13] Chapter IV Proposition 5.7). Moreover, multiplying the above relation by the Euler factors we derive the identity
\[
\prod_{\mathfrak{p} \not| f, \mathfrak{p} \not| f_\chi} (1 - \chi_0(\mathfrak{p})) L_{f_\chi}(1, \chi_0) = TS_f(\chi, g_f)
\]
(5.3)
where \(T\) is a nonzero constant depending on \(f\) and \(\chi\) ([21] p. 244).
Theorem 5.1. Let \( L \) be an abelian extension of \( K \) with \( [L : K] > 2h_K \) where \( h_K \) is the class number of \( K \). Assume that the conductor of the extension \( L/K \) is a power of prime ideal, namely \( f = p^n \) (\( n \geq 1 \)). Then the value

\[
\varepsilon = N_{K_i/L}(g_f(C_0))
\]

generates \( L \) over \( K \).

Proof. We identify \( \text{Gal}(K_i/K) \) with \( \text{Cl}(f) \) via the Artin map. Letting \( F = K(\varepsilon) \) we deduce

\[
\# \{ \text{characters } \chi \text{ of } \text{Cl}(f) : \chi|_{\text{Gal}(K_i/L)} = 1 \text{ and } \chi|_{\text{Gal}(K_i/F)} \neq 1 \} = [L : K] - [F : K]. \tag{5.4}
\]

Furthermore, if we let \( H \) be the Hilbert class field of \( K \), then we have

\[
\# \{ \text{characters } \chi \text{ of } \text{Cl}(f) : f_\chi = \mathcal{O}_K \} = \# \{ \chi : \chi|_{\text{Gal}(K_i/H)} = 1 \} = h_K. \tag{5.5}
\]

Suppose that \( F \) is properly contained in \( L \). Then we deduce

\[
[L : K] - [F : K] = [L : K]
\left(1 - \frac{1}{[L : F]}\right) > 2h_K\left(1 - \frac{1}{2}\right) = h_K
\]

by the hypothesis \( [L : K] > 2h_K \). Thus there exists a character \( \psi \) of \( \text{Cl}(f) \) such that

\[
\psi|_{\text{Gal}(K_i/L)} = 1, \quad \psi|_{\text{Gal}(K_i/F)} \neq 1 \quad \text{and} \quad f_\psi \neq \mathcal{O}_K
\]

by (5.4) and (5.5). Moreover, since \( f = p^n \), we get \( f_\psi = p^m \) for some \( 1 \leq m \leq n \). Hence we obtain by (5.3) that

\[
0 \neq L_f(1, \psi_0) = TS_1(\overline{\psi}, g_1)
\]

for a nonzero constant \( T \) and the proper character \( \psi_0 \) of \( \text{Cl}(f_\psi) \) corresponding to \( \psi \). On the other hand, we get that

\[
S_1(\overline{\psi}, g_1) = \sum_{C \in \text{Cl}(f)} \overline{\psi}(C) \log |g_1(C)|
\]

\[
= \sum_{C_1 \in \text{Cl}(f)} \sum_{C_2 \in \text{Gal}(K_i/F)} \sum_{C_3 \in \text{Gal}(K_i/L)} \overline{\psi}(C_1C_2C_3) \log |g_1(C_1C_2C_3)|
\]

\[
= \sum_{C_1} \overline{\psi}(C_1) \sum_{C_2} \overline{\psi}(C_2) \log |\varepsilon(C_1C_2)| \quad \text{by the fact } \psi|_{\text{Gal}(K_i/L)} = 1 \text{ and } (5.2)
\]

\[
= \sum_{C_1} \overline{\psi}(C_1) \left( \sum_{C_2} \overline{\psi}(C_2) \right) \log |\varepsilon(C_1)| \quad \text{by the fact } \varepsilon \in F
\]

\[
= 0 \quad \text{by the fact } \psi|_{\text{Gal}(K_i/F)} \neq 1,
\]

which is a contradiction. Therefore \( L = F \) as desired. \( \square \)

Remark 5.2. Schertz achieved in [26] a similar result for generators of the ray class fields. However, there seems to be some defect in his argument. For instance, in the proof of [26] Lemma 1 he claimed that the conductor of a nontrivial character of \( \text{Cl}(p^n) \) is nontrivial. But one can see that his argument could be false if \( h_K \geq 2 \) because in this case the conductor of a character of \( \text{Cl}(p^n) \) induced from one of \( \text{Cl}(\mathcal{O}_K) \) is obviously trivial.

We apply this theorem to obtain ring class invariants in terms of singular values of the \( \Delta \)-function. To this end we are in need of certain relation between Siegel functions and the \( \Delta \)-function.
**Lemma 5.3.** Let $N \geq 1$. Then we have the relation

$$\prod_{w=1}^{N-1} g_{(0, \frac{N}{w})}^{12}(\tau) = N^{12} \frac{\Delta(N\tau)}{\Delta(\tau)}$$

where the left hand side is understood to be 1 when $N = 1$.

**Proof.** Note the identity

$$1 - X^N = 1 + X + \cdots + X^{N-1} = \prod_{w=1}^{N} (1 - e^{2\pi i w/N} X). \quad (5.6)$$

We then derive for $N \geq 2$ that

$$\prod_{w=1}^{N-1} g_{(0, \frac{N}{w})}^{12}(\tau) = \prod_{w=1}^{N-1} \left( q_\tau^{12} e^{\pi i w}(1 - e^{2\pi i w}) \prod_{n=1}^{\infty} (1 - q_\tau^n e^{2\pi i w})(1 - q_\tau^n e^{-2\pi i w}) \right)^{12} \quad \text{by (2.3)}$$

$$= q_\tau^{N-1} N^{12} \prod_{n=1}^{\infty} \left( \frac{1 - q_\tau^n}{1 - q_\tau^n} \right)^{24} \quad \text{by the identity (5.6)}$$

$$= N^{12} \frac{\Delta(N\tau)}{\Delta(\tau)} \quad \text{by definition (5.1)}.$$

\[\square\]

**Theorem 5.4.** Let $K$ be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. For a prime $p$ which is inert or ramified in $K/\mathbb{Q}$, let $O = \left[ p^\ell \theta, 1 \right] (\ell \geq 1)$. Then the real algebraic integer

$$p^{12} \frac{\Delta(p^\ell \theta)}{\Delta(p^{\ell-1} \theta)} \quad (5.7)$$

generates $H_O$ over $K$.

**Proof.** Let $\mathfrak{f} = p^\ell O_K$. Then the conductor of the extension $H_O/K$ is $\mathfrak{f}$ (for instance, see [5] Exercises 9.20~9.23) and

$$[H_O : K] = \begin{cases} 
  p^{\ell-1}(p + 1)h_K & \text{if } p \text{ is inert in } K/\mathbb{Q} \\
  p^{\ell}h_K & \text{if } p \text{ is ramified in } K/\mathbb{Q}
\end{cases}$$

by the class number formula ([5] Theorem 7.24).

If $p = 2$ and $\ell = 1$, then $K_\mathfrak{f} = H_O$ by Remark [3.3] and hence the real algebraic integer $g_{(0, \frac{1}{2})}^{24}(\theta)$ generates $H_O$ over $K$. And, $g_{(0, \frac{1}{2})}^{24}(\theta) = \left( 2^{12} \frac{\Delta(2\theta)}{\Delta(\theta)} \right)^2$ by Lemma 5.3.
As for the other cases, since \( f \) is a prime power and \( |H_\mathcal{O} : K| > 2h_K \), the value \( N_{K_{f}/H_\mathcal{O}}(g_f(C_0)) \) generates \( H_\mathcal{O} \) over \( K \) by Theorem 5.1. And we have

\[
N_{K_{f}/H_\mathcal{O}}(g_f(C_0))^2 = \prod_{1 \leq w \leq p^\ell-1 \atop \gcd(w, p) = 1} g_{(0, \frac{w}{p^\ell})}^{12p^\ell}(\theta) \quad \text{by (3.10) and Theorem 3.2}
\]

\[
= \prod_{w=1}^{p^\ell-1} g_{(0, \frac{w}{p^\ell})}^{12p^\ell}(\theta) / \prod_{w=1}^{p^\ell-1} g_{(0, \frac{w}{p^\ell})}^{12p^\ell}(\theta)
\]

\[
= \left( p^{12\ell} \frac{\Delta(p^\ell \theta)}{\Delta(\theta)} / p^{12(\ell-1)} \frac{\Delta(p^{\ell-1} \theta)}{\Delta(\theta)} \right)^{p^\ell} \quad \text{by Lemma 5.3}
\]

On the other hand, since both \( \frac{\Delta(p^{\ell-1} \theta)}{\Delta(\theta)} \) and \( \frac{\Delta(p^\ell \theta)}{\Delta(\theta)} \) are real algebraic numbers which belong to \( H_\mathcal{O} \) (Chapter 12 Corollary to Theorem 1), we get the assertion. \( \square \)

**Remark 5.5.** Unfortunately, however, we cannot guarantee the fact that the conjugates of the value in (5.7) (or, its inverse) constitute a normal basis of \( H_\mathcal{O} \) over \( K \).

6. CONSTRUCTION OF NORMAL BASES

Given an imaginary quadratic field \( K(\neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})) \) we consider the extension \( K(p^m)/K(p^m) \) for a prime \( p \geq 5 \) and an integer \( m \geq 1 \) relatively prime to \( p \). In this section we shall construct a normal basis of each intermediate field \( F \) over \( K(p^m) \) in a different way from Section 4 namely by using the idea of Kawamoto (17).

First we explicitly determine all intermediate fields \( F \) between \( K(p^m) \) and \( K(p^m) \). Let \( \theta \) be as in (3.1) and set \( \min(\theta, \mathbb{Q}) = X^2 + B\theta X + C\theta \). Then one can identify \( \Gamma = \text{Gal}(K(p^m)/K(p^m)) \) with

\[
\left\{ \gamma = \begin{pmatrix} t & -C\theta s \\ s & t \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod pm \right\} / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}
\]

by (3.10). Since \( |K(p^m) : K(p^m)| = p^2 \) by the formula (3.11), we readily know by inspection that

\[
\Gamma = \left\{ \begin{pmatrix} 1 + pm & 0 \\ 0 & 1 + pm \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} 1 - B\theta pm & -C\theta pm \\ pm & 1 \end{pmatrix} \right\},
\]

which shows that \( \Gamma \cong (\mathbb{Z}/p\mathbb{Z})^2 \). Hence an element of \( \Gamma \) is of the form

\[
\begin{pmatrix} 1 + pm & 0 \\ 0 & 1 + pm \end{pmatrix}^k \begin{pmatrix} 1 - B\theta pm & -C\theta pm \\ pm & 1 \end{pmatrix}^\ell = \begin{pmatrix} 1 + (k - B\theta \ell pm) & -C\theta \ell pm \\ \ell pm & 1 + k pm \end{pmatrix}
\]

for \( 0 \leq k, \ell \leq p - 1 \). Set

\[
\Gamma(k, \ell) = \left\{ \gamma(k, \ell) \right\} = \left\{ \begin{pmatrix} 1 + (k - B\theta \ell pm) & -C\theta \ell pm \\ \ell pm & 1 + k pm \end{pmatrix} \right\}
\]

for \( (k, \ell) \in \{(0, 1), (1, 0), (1, 1), \cdots, (1, p - 1)\} \), which represents all subgroups of \( \Gamma \) of order \( p \). And, let \( F(k, \ell) \) be its corresponding fixed field of \( \Gamma(k, \ell) \), namely

\[
F(k, \ell) = K_{(p^m)}^{\Gamma(k, \ell)} \quad \text{for} \quad (k, \ell) \in \{(0, 1), (1, 0), (1, 1), \cdots, (1, p - 1)\}.
\]
Then we have the field tower:

\[ K_{(p^2m)} \rightarrow K_{(pm)} \rightarrow K_{(0, \frac{1}{pm})} \rightarrow K_{(1, 0)} \rightarrow K_{(1, 1)} \rightarrow \cdots \rightarrow K_{(1, \frac{p-1}{pm})} \]

**Lemma 6.1.** \( \zeta_p, g^{12pm}_{(0, \frac{1}{pm})}(\theta) \in K_{(pm)} \) and \( \zeta_{p^2}, g^{12m}_{(0, \frac{1}{pm})}(\theta) \in K_{(p^2m)} \).

**Proof.** One can check by Proposition 2.2 that \( g^{12pm}_{(0, \frac{1}{pm})}(\tau) \in \mathcal{F}_{pm} \) and \( g^{12m}_{(0, \frac{1}{pm})}(\tau) \in \mathcal{F}_{p^2m} \). Hence we get the assertion by (6.1). \( \square \)

Let us investigate the action of \( \gamma(k, \ell) \) on \( \zeta_{p^2} \) and \( g^{12m}_{(0, \frac{1}{pm})}(\theta) \). To this end we decompose \( \gamma(k, \ell) \) into

\[
\gamma(k, \ell) = \alpha(k, \ell) \cdot \beta(k, \ell) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + (2k - B_0\ell)pm \end{pmatrix} \begin{pmatrix} 1 + (k - B_0\ell)pm & -C_0\ell pm \\ \ell pm & 1 + (B_0\ell - k)pm \end{pmatrix}
\]

\[
\in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : d \in (\mathbb{Z}/p^2m\mathbb{Z})^* \right\} \cdot \text{SL}_2(\mathbb{Z}/p^2m\mathbb{Z})/\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.
\]

We see directly from (2.3) that the function \( g^{12m}_{(0, \frac{1}{pm})}(\tau) \) has Fourier coefficients in \( \mathbb{Q}(\zeta_{pm}) \). Thus the action of \( \alpha(k, \ell) \) is described by (3.10) and (2.1) as

\[
\zeta_{p^2} \mapsto \zeta_{p^2}^{1+(2k-B_0\ell)pm} \\
g^{12m}_{(0, \frac{1}{pm})}(\theta) \mapsto g^{12m}_{(0, \frac{1}{pm})}(\theta).
\]

For some integers \( A, B, C, D \) let

\[
\beta'(k, \ell) = \begin{pmatrix} 1 + (k - B_0\ell)pm + p^2mA & -C_0\ell pm + p^2mB \\ \ell pm + p^2mC & 1 + (B_0\ell - k)pm + p^2mD \end{pmatrix}
\]

be a preimage of \( \beta(k, \ell) \) via the natural surjection \( \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/p^2m\mathbb{Z})/\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \). Then by (6.1) and (2.2) we derive that the action of \( \beta'(k, \ell) \) is given by

\[
\zeta_{p^2} \mapsto \zeta_{p^2}^{1+(2k-B_0\ell)pm} \\
g^{12m}_{(0, \frac{1}{pm})}(\theta) \mapsto g^{12m}_{(0, \frac{1}{pm})}(\theta) \circ \beta'(k, \ell)(\theta) = g^{12m}_{(0, \frac{1}{pm})}(\theta) \text{ by Proposition 2.1(iii)}
\]

\[
= g_{(\ell+pC, \frac{1}{pm}+B_0\ell-k+pD)}^{12m}(\theta) = \zeta_{p^2}^{-6p\ell} g^{12m}_{(0, \frac{1}{pm})}(\theta) \text{ by Proposition 2.1(iv)}.
\]

Hence \( \gamma(k, \ell) \) maps

\[
\zeta_{p^2} \mapsto \zeta_{p^2}^{1+(2k-B_0\ell)pm} \\
g^{12m}_{(0, \frac{1}{pm})}(\theta) \mapsto \zeta_{p^2}^{-6p\ell} g^{12m}_{(0, \frac{1}{pm})}(\theta).
\]
Lemma 6.2. Let $(k, \ell) \in \{(0, 1), (1, 0), (1, 1), \ldots, (1, p-1)\}$. Then $\Gamma(k, \ell)$ fixes $\zeta_p^x g_{(0, \frac{12m}{pm})}(\theta)$ for some $x, y \in \mathbb{Z}$ if and only if $x$ and $y$ satisfy
\[
\begin{align*}
-B_\theta mx & \equiv 6y \pmod{p} \quad \text{if } (k, \ell) = (0, 1) \\
x & \equiv 0 \pmod{p} \quad \text{if } (k, \ell) = (1, 0) \\
(2 - B_\theta \ell)mx & \equiv 6ly \pmod{p} \quad \text{otherwise.}
\end{align*}
\]

Proof. It follows from (6.1) and (6.2) that
\[
(\zeta_p^x g_{(0, \frac{12m}{pm})}(\theta))^{(1 + (2k - B_\theta \ell)p)x - 6p\ell y} = g_{(0, \frac{12m}{pm})}(\theta).
\]
Then this value is equal to $\zeta_p^x g_{(0, \frac{12m}{pm})}(\theta)$ if and only if
\[
(1 + (2k - B_\theta \ell)p)x - 6p\ell y \equiv x \pmod{p^2},
\]
which reduces to (6.3). And, this proves the lemma. \qed

Lemma 6.3. $K_{(p^{2m}m)} = K_{(pm)}(\zeta_p^x, g_{(0, \frac{12m}{pm})}(\theta)).$

Proof. Since $g_{(0, \frac{12m}{pm})}(\theta) \notin F_{(1, 0)}$ and $g_{(0, \frac{12m}{pm})}(\theta) \in F_{(1, 0)}$ by (6.2), we claim that $g_{(0, \frac{12m}{pm})}(\theta) \notin K_{(pm)}$ and $F_{(1, 0)} = K_{(pm)}(g_{(0, \frac{12m}{pm})}(\theta))$ owing to the fact $[F_{(1, 0)} : K_{(pm)}] = p.$ Furthermore, since $\zeta_p^x \notin F_{(1, 0)}$ by (6.1), we achieve by the fact $[K_{(p^{2m}m)} : F_{(1, 0)}] = p$ that
\[
K_{(p^{2m}m)} = F_{(1, 0)}(\zeta_p^x) = K_{(pm)}(\zeta_p^x, g_{(0, \frac{12m}{pm})}(\theta)).
\]
\qed

Theorem 6.4. Let $(k, \ell) \in \{(0, 1), (1, 0), (1, 1), \ldots, (1, p-1)\}$ and $y'$ be the integer such that $y \cdot y' \equiv 1 \pmod{p}$ and $0 < y' < p$ for an integer $y \not\equiv 0 \pmod{p}$. Then we have
\[
F_{(k, \ell)} = \begin{cases} 
K_{(pm)}(\zeta_p^x g_{(0, \frac{12m^26^6(p-B_\ell)}{pm})}(\theta)) & \text{if } (k, \ell) = (0, 1) \\
K_{(pm)}(g_{(0, \frac{12m}{pm})}(\theta)) & \text{if } (k, \ell) = (1, 0) \\
K_{(pm)}(\zeta_p^x g_{(0, \frac{12m^2(6p-B_\ell)}{pm})}(\theta)) & \text{otherwise.}
\end{cases}
\]

Proof. Take a solution of (6.3) as
\[
(x, y) = \begin{cases} 
(1, m6'(p-B_\theta)) & \text{if } (k, \ell) = (0, 1) \\
(0, 1) & \text{if } (k, \ell) = (1, 0) \\
(1, m(6l')'\cdot(2 + p - B_\ell)) & \text{otherwise}
\end{cases}
\]
which consists of nonnegative integers. We can then readily check that a solution $(x, y)$ in (6.4) does not satisfy two congruence equations in (6.3) simultaneously. This shows that for each $(x, y)$, $\zeta_p^x g_{(0, \frac{12m}{pm})}(\theta)$ belongs to a unique $F_{(k, \ell)}$; hence in particular, it is not in $K_{(pm)}$. Since $[F_{(k, \ell)} : K_{(pm)}] = p,$ we get the conclusion. \qed

To accomplish our goal we are in need of the following two lemmas:
Theorem 6.7. Let $L$ be a number field containing $\zeta_n$ and $F$ be a cyclic extension over $L$ of degree $n$. Then there exists an element $\xi$ of $L$ such that $F = L(\sqrt[n]{\xi})$. And, the conjugates of $\sum_{s=0}^{n-1} \sqrt[n]{\xi}^s$ over $L$ form a normal basis of $F$ over $L$.

Proof. See [17] p. 223.

Lemma 6.5. Let $L$ be a number field. Let $F_1$ and $F_2$ be finite Galois extensions of $L$ with $F_1 \cap F_2 = L$. If the conjugates of $\xi_s \in F_s$ over $L$ form a normal basis of $F_s$ over $L$ for $s = 1, 2$, then the conjugates of $\xi_1 \xi_2$ over $L$ form a normal basis of $F_1 F_2$ over $L$.

Proof. See [17] p. 227.

Now we are ready to prove our main theorem about normal bases.

Theorem 6.7. Let $(k, \ell)$ and $y'$ be as in Theorem 6.4. Then the conjugates of

$$
\begin{aligned}
\sum_{s=0}^{p-1} \left( \zeta_{p^2} g_{(0, \frac{1}{pm})}^{12m^2 \ell'}(\theta) \right)^s & \quad \text{if } (k, \ell) = (0, 1) \\
\sum_{s=0}^{p-1} g_{(0, \frac{1}{pm})}^{12ms}(\theta) & \quad \text{if } (k, \ell) = (1, 0) \\
\sum_{s=0}^{p-1} \left( \zeta_{p^2} g_{(0, \frac{1}{pm})}^{12m^2 \ell'}(2p-B_0 \ell)(\theta) \right)^s & \quad \text{otherwise}
\end{aligned}
$$

over $K_{(pm)}$ form a normal basis of $F(k, \ell)$ over $K_{(pm)}$. Moreover, the values in (6.5) are algebraic integers.

Proof. Since the function $g_{(0, \frac{1}{pm})}(\tau)$ is integral over $\mathbb{Z}[j(\tau)]$ by Proposition 2.1(i) and $j(\theta)$ is an algebraic integer (23 or 28), the values in (6.5) are all algebraic integers. Thus the theorem follows by applying Lemma 6.5 with the aid of Lemma 6.1 and Theorem 6.4.

Theorem 6.8. The conjugates of the algebraic integer

$$
\left( \sum_{s=0}^{p-1} \zeta_{p^2}^s \right) \left( \sum_{s=0}^{p-1} g_{(0, \frac{1}{pm})}^{12ms}(\theta) \right)
$$

over $K_{(pm)}$ form a normal basis of $K_{(p^2 m)}$ over $K_{(pm)}$.

Proof. If $F_1 = K_{(pm)}(\zeta_{p^2})$ and $F_2 = K_{(pm)}(g_{(0, \frac{1}{pm})}^{12ms}(\theta))$, then $[F_s : K_{(pm)}] = p$ for $s = 1, 2$ by Lemma 6.1. On the other hand, Lemma 6.3 shows $F_1 F_2 = K_{(p^2 m)}$, from which we get $F_1 \cap F_2 = K_{(pm)}$ and $[F_s : K_{(pm)}] = p$ for $s = 1, 2$. Hence the conjugates of $\sum_{s=0}^{p-1} \zeta_{p^2}^s$ and $\sum_{s=0}^{p-1} g_{(0, \frac{1}{pm})}^{12ms}(\theta)$ over $K_{(pm)}$ form normal bases of $F_1$ and $F_2$, respectively, by Lemmas 6.5 and 6.1. And, the theorem follows from Lemma 6.6.

7. Galois module structure

Let $L$ be a number field and $p$ be an odd prime. We say that an extension $L_\infty/L$ is a $\mathbb{Z}_p$-extension of $L$ if there exists a sequence of cyclic extensions of $L$

$$
L = L_0 \subset L_1 \subset \cdots \subset L_n \subset \cdots \subset L_\infty = \bigcup_{n=0}^{\infty} L_n
$$
with \( \text{Gal}(L_n/L) \cong \mathbb{Z}/p^n\mathbb{Z} \). Then \( \text{Gal}(L_\infty/L) \cong \mathbb{Z}_p \) and it is well-known that \( L_\infty/L \) is unramified outside \( p \) (§2 Proposition 13.2). And, Greenberg ([11]) has conjectured that if \( L \) is totally real, then the Iwasawa \( \lambda \)-invariant of \( L_\infty/L \) vanishes.

Denoting the ring of \( p \)-integers of \( L \) by \( \mathcal{O}_L[\frac{1}{p}] \) Kersten-Michaliček introduced in [18] that a finite Galois extension \( F \) of \( L \) has a normal \( p \)-integral basis over \( L \) if \( \mathcal{O}_F[\frac{1}{p}] \) is a free \( \mathcal{O}_L[\frac{1}{p}]\text{Gal}(F/L) \)-module of rank one. We then say that a \( \mathbb{Z}_p \)-extension \( L_\infty \) of \( L \) has a normal basis over \( L \) if each \( L_n \) has a normal \( p \)-basis over \( L \).

On the other hand, we see from [19] that there is a negative data for Greenberg’s conjecture. For instance, for a positive square free integer \( d \) with \( (\frac{-d}{p}) = -1 \), let \( L = \mathbb{Q}(\sqrt{d}) \) and \( L' = \mathbb{Q}(\sqrt{-d}) \). It was shown in [7] and [8] that if 3 divides the class number of \( L \) and if every \( \mathbb{Z}_3 \)-extension of \( L' \) has a normal basis, then the \( \lambda \)-invariant of the cyclotomic \( \mathbb{Z}_3 \)-extension of \( L' \) does not vanish. This suggests a relation between the existence of normal basis in \( \mathbb{Z}_p \)-extension and the Greenberg’s conjecture, which motivates this section.

Now, let \( K(\neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})) \) be an imaginary quadratic field, \( p \geq 5 \) be a prime and \( m \geq 1 \) be an integer relatively prime to \( p \). And, let \( n \) and \( \ell \) be positive integers with \( n \geq 2\ell \). Observe that the extension \( K(\zeta_{p^m})/K(\zeta_{p^m}) \) is unramified outside \( p \) ([4] Chapter 3) and \( \zeta_{pn} \in K(\zeta_{p^m}), \zeta_{p\ell} \in K(\zeta_{p^m}) \) but \( \zeta_{p^{n+1}} \notin K(\zeta_{p^m}), \zeta_{p^{\ell+1}} \notin K(\zeta_{p^m}) \) ([21] Chapter 9 Lemma 4.3). We shall prove in this section that \( K(\zeta_{p^m}) \) has a normal \( p \)-integral basis over \( K(\zeta_{p^m}) \). The special case for \( \ell = 1 \) and \( m = 1 \) has been done by Komatsu ([19]). However, we shall develop it in more comprehensive way by utilizing (3.10) and Proposition 2.1 as in the previous section unlike Komatsu’s method via class field theory. As a corollary we determine the existence of normal basis of the \( \mathbb{Z}_p \)-extension \( K_\infty K(\zeta_{p^m})/K(\zeta_{p^m}) \) for \( \ell \geq 1 \) where \( K_\infty \) is any \( \mathbb{Z}_p \)-extension of \( K \).

Let \( \theta \) be as in (3.1) and set \( \min(\theta, \mathbb{Q}) = X^2 + B_\theta X + C_\theta \). Then we can identify the Galois group \( \Gamma = \text{Gal}(K(\zeta_{2(n-\ell)m})/K(\zeta_{p^m})) \) with the group

\[
\left\{ \gamma = \begin{pmatrix} t - B_\theta s & -C_\theta s \\ s & t \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/p^{2(n-\ell)m}\mathbb{Z}) : \gamma \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{p^m} \right\} \left/ \left\{ \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right\} \right.
\]

by (3.10) and \( \#\Gamma = [K(\zeta_{2(n-\ell)m}) : K(\zeta_{p^m})] = p^{2(2(n-\ell) - \ell)} \) by the formula (3.11).

Lemma 7.1. There exists an element \( \beta_0 \) of \( \text{SL}_2(\mathbb{Z}) \) satisfying the property

\[
\beta_0^k = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \equiv \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{p^{\ell+k+m}} \quad (0 \leq k \leq 2(n-\ell) - \ell)
\]

for some integers \( q_k \neq 0 \pmod{p} \).

Proof. Consider an integral matrix \( \beta = \left( \begin{array}{cc} 1+p^mx - B_\theta p^m & -C_\theta p^m \\ p^m & 1+p^mx \end{array} \right) \) for an undetermined integer \( x \). Then the condition \( \det(\beta) \equiv 1 \pmod{p^{2(n-\ell)m}} \) is equivalent to

\[
f(x) = p^m x^2 + (2m - B_\theta p^m) x + C_\theta p^m \equiv 0 \pmod{p^{2(n-\ell)m}}.
\]

Since \( 2m - B_\theta p^m m \neq 0 \pmod{p} \), the equation \( f(x) \equiv 0 \pmod{p} \) has a solution. Furthermore, since the derivative \( f'(x) = 2m - B_\theta p^m m \neq 0 \pmod{p} \), we have an integer solution \( x = x_0 \) of the congruence equation (7.2) by Hensel’s lemma. Let \( \beta_0 \) be a preimage of \( \left( \begin{array}{cc} 1+p^mx_0 - B_\theta p^m & -C_\theta p^m \\ p^m & 1+p^mx_0 \end{array} \right) \) via the natural surjection \( \text{SL}_2(\mathbb{Z}) \rightarrow \text{SL}_2(\mathbb{Z}/p^{2(n-\ell)m}\mathbb{Z}) \). Then it is routine to check that \( \beta_0 \) satisfies the property (7.1). \qed
Set \( \alpha = \left( \begin{array}{cc} 1 + p^\ell m & 0 \\ 0 & 1 + p^\ell m \end{array} \right) \) and \( \beta = \beta_0 \) in Lemma 7.1. Now that they have the order \( p^{2(n-\ell)} \) in \( \Gamma \) and \( \# \Gamma = p^{2(n-\ell)} \), we derive

\[
\Gamma = \langle \alpha \rangle \times \langle \beta \rangle.
\]
as a direct product. And, we get

\[
\text{Gal}(K_{p^{2(n-\ell)}m}/K_{p^n m}) = \langle \alpha^{p^{n-\ell}} \rangle \times \langle \beta^{p^{n-\ell}} \rangle
\]
(7.3)

by (3.10). Let us define a function

\[
g(\tau) = \prod_{s=0}^{p^{n-2-1}} g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s(\tau).
\]

Since each factor \( g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s(\tau) \) lies in \( F_p^{2(n-\ell)}m \) by Proposition 2.2, the singular value \( g(\theta) \) belongs to \( K_{(p^{2(n-\ell)}m)} \) by (3.10).

**Lemma 7.2.** \( K_{p^n m} = K_{(p^{2m})} (\zeta_{p^n}, g(\theta)) \)

**Proof.** By the property (7.1) of \( \beta \), \( \beta^{p^{n-2-1}} \) is of the form \( \left( \begin{array}{cc} 1 + p^{n-\ell}m & 0 \\ 0 & 1 + p^{n-\ell}m \end{array} \right) \) for some integers \( A, B, C, D \) with \( C \neq 0 \mod p \). We then deduce by (3.10) and (2.2) that

\[
g(\theta)^\beta = \prod_{s=0}^{p^{n-2-1}} g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s(\theta) = \prod_{s=0}^{p^{n-2-1}} g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s \beta(\theta) \quad \text{by Proposition 2.1(iii)}
\]

\[
= g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s(\theta) \cdot g_{(0, \frac{1}{p^{n-\ell}m})}^{12m} \beta_s(\theta) = \zeta_{p^{n-\ell}} g(\theta) \quad \text{by Proposition 2.1(iv)}.
\]

(7.4)

In particular, \( g(\theta)^{p^{n-\ell}} \) is fixed by \( \beta \) and \( g(\theta) \) is fixed by \( \beta^{p^{n-\ell}} \) because \( \beta \) fixes \( \zeta_{p^{n-\ell}} \) by (3.10) and (2.2). Note that \( \alpha^{p^{n-2-1}} = \left( \begin{array}{cc} 1 + p^{n-\ell}mE & 0 \\ 0 & 1 + p^{n-\ell}mE \end{array} \right) \) for some integer \( E \). As an element of \( \Gamma \) we can decompose \( \alpha^{p^{n-2-1}} \) into

\[
\alpha^{p^{n-2-1}} = \alpha_1 \cdot \alpha_2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & (1 + p^{n-\ell}mE)^2 \end{array} \right) \left( \begin{array}{cc} 1 + p^{n-\ell}mE + p^{2(n-\ell)}mA' & p^{2(n-\ell)}mB' \\ p^{2(n-\ell)}mC' & E' + p^{2(n-\ell)}mD' \end{array} \right)
\]
for some integers $A'$, $B'$, $C'$, $D'$, $E'$ such that $(1 + p^{n-\ell}mE') \equiv 1 \pmod{p^{2(n-\ell)m}}$ and $\alpha_2 \in \text{SL}_2(\mathbb{Z})$. Hence, we get by (3.10), (2.1), (2.2) and Proposition 2.1(iii) that
\[
g(\theta)^{\alpha p^{n-2\ell}} = \prod_{s=0}^{p^{n-2\ell}-1} g_{(0, \frac{1}{p^{n-\ell}m})}^{12m}(\theta)^{\alpha p^{n-2\ell}} = \prod_{s=0}^{p^{n-2\ell}-1} \left( g_{(0, \frac{1}{p^{n-\ell}m})}^{12m}(\theta) \right)^{\beta^s \alpha p^{n-2\ell}}
\]
\[
= \prod_{s=0}^{p^{n-2\ell}-1} \left( g_{(0, \frac{1}{p^{n-\ell}m})}^{12m}(\theta) \right)^{\alpha 2 \beta^s}
\]
\[
= \prod_{s=0}^{p^{n-2\ell}-1} \left( g_{(0, \frac{1}{p^{n-\ell}m})}^{12m}(\theta) \right)^{\beta^s}
\]
by the fact $E' \equiv 1 \pmod{p^{n-\ell}m}$ and Proposition 2.1(iv)
\[
= \prod_{s=0}^{p^{n-2\ell}-1} g_{(0, \frac{1}{p^{n-\ell}m})}^{12m}(\theta) = g(\theta).
\] (7.5)

Observe that in particular, $g(\theta)$ is fixed by $\alpha^{p^{n-\ell}}$ and hence by $\langle \alpha^{p^{n-\ell}} \rangle \times \langle \beta^{p^{n-\ell}} \rangle$. Thus $g(\theta)$ belongs to $K_{(p^n m)}$ by (7.3).

On the other hand, we see from (3.10) that $\text{Gal}(K_{(p^n m)} / K_{(p^\ell m)}) = \langle \alpha \rangle \times \langle \beta \rangle$ in $\text{GL}_2(\mathbb{Z}/p^n m \mathbb{Z}) / \{ \pm (1 \ 0) \}$ with $\alpha$ and $\beta$ of order $p^{n-\ell}$. Suppose that $\alpha^A \beta^B$ fixes both $\zeta_{p^n}$ and $g(\theta)$ for some $0 \leq A, B < p^{n-\ell}$. Since $(\zeta_{p^n})^{\alpha^A \beta^B} = (\zeta_{p^n})^{\det(A)} = \zeta_{p^n}^{(1 + p^\ell m)^2A}$ by (3.10), (2.1) and (2.2), we have $A = 0$. It then follows $B = 0$ from (7.4). Therefore we conclude that $K_{(p^n m)} = K_{(p^\ell m)}(\zeta_{p^n}, g(\theta))$ by Galois theory.

Now we are in the following situation:

\[
K_{(p^n m)} = K_{(p^\ell m)}(\zeta_{p^n}, g(\theta))
\]
\[
K_{(p^\ell m)}(\zeta_{p^n}) \bigcap \text{cyclic of degree } p^{n-\ell}
\]
\[
K_{(p^\ell m)}(g(\theta)) \bigcap \text{cyclic of degree } p^{n-\ell}
\]
\[
K_{(p^\ell m)} \bigcap \text{cyclic of degree } p^{n-\ell}
\]

Here we see $K_{(p^\ell m)}(\zeta_{p^n}) \cap K_{(p^\ell m)}(g(\theta)) = K_{(p^\ell m)}$ by analyzing the actions of $\alpha$ and $\beta$ in the proof of Lemma 7.2. Then we are ready to attain our aim by means of the following two lemmas:
**Lemma 7.3.** Let $L$ be a number field and $F/L$ be a cyclic extension of a prime power degree $n = p^s$ which is unramified outside $p$.

(i) When $ζ_n \in L$, $F$ has a normal $p$-integral basis over $L$ if and only if $F = L(\sqrt[n]{ζ})$ for some $ξ \in O_L[\frac{1}{p}]^{*}$.

(ii) When $ζ_n \not\in L$, $F$ has a normal $p$-integral basis over $L$ if and only if $F(ζ_n)$ has a normal $p$-integral basis over $L(ζ_n)$.

**Proof.** See [12] Chapter 0 Proposition 6.5 and Chapter I Theorem 2.1.

**Lemma 7.4.** Let $L$ be a number field and $F_s/L$ be a cyclic extension which is unramified outside $p$ for $s = 1, 2$. If $F_s$ has a normal $p$-integral basis over $L$ for $s = 1, 2$ and $F_1 \cap F_2 = L$, then $F_1F_2$ has a normal $p$-integral basis over $L$.

**Proof.** See [17] p. 227.

**Theorem 7.5.** Let $K(\not\in \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}))$ be an imaginary quadratic field, $p \geq 5$ be a prime and $m \geq 1$ be an integer relatively prime to $p$. And, let $n$ and $ℓ$ be positive integers with $n \geq 2ℓ$. Then $K(p^n,m)$ has a normal $p$-integral basis over $K(p^{'m})$.

**Proof.** The extension $K(p^n,m)(ζ^n)/K(p^{'m})$ is cyclic of degree $p^{n-ℓ}$ and is unramified outside $p$. We consider the extension $K(p^{'m})(ζ^n)/K(p^{'m})(ζ^{n-ℓ})$. It is also a cyclic extension of degree $p^ℓ$ unramified outside $p$ by considering the action of $α$ on $ζ^n$. Since $K(p^{'m})(ζ^n) = K(p^{'m})(ζ^{n-ℓ})(\sqrt[ℓ]{ζ^{n-ℓ}})$, $K(p^{'m})(ζ^n)$ has a normal $p$-integral basis over $K(p^{'m})(ζ^{n-ℓ})$ by Lemma 7.3(i). If $n = 2ℓ$, then $ζ^{n-ℓ} = ζ^p$ and $K(p^{'m}) = K(p^{'m})(ζ^{n-ℓ})$. If $n > 2ℓ$, then $ζ^{n-ℓ} \not\in K(p^{'m})$, and hence $K(p^{'m})(ζ^n)$ has a normal $p$-integral basis over $K(p^{'m})$ by Lemma 7.3(ii).

Next we look into the cyclic extension $K(p^{'m})(g(θ))/K(p^{'m})$ of degree $p^{n-ℓ}$ unramified outside $p$. Let us examine the extension $K(p^{'m})(ζ^{n-ℓ}, g(θ))/K(p^{'m})(ζ^{n-ℓ})$. Then we see that it is also a cyclic extension of degree $p^{n-ℓ}$ unramified outside $p$ by considering the action of $β$ on $g(θ)$. Note that we can rewrite it as $K(p^{'m})(ζ^{n-ℓ}, g(θ)) = K(p^{'m})(ζ^{n-ℓ})(\sqrt[p^{n-ℓ}]{g(θ)^p^n})$. On the other hand, we know by [31], [24] and [22] that $\text{Gal}(K(p^{'m})(ζ^{n-ℓ}))/K(p^{'m})(ζ^{n-ℓ})) = (α^{p^{n-2ℓ}}) × (β)$. So $g(θ)p^{n-ℓ}$ belongs to $K(p^{'m})(ζ^{n-ℓ})$ by (7.4) and (7.5). Moreover, it belongs to $O_K(p^{'m})(ζ^{n-ℓ})(\frac{1}{p})^{*}$ by Proposition 2.1(i) and (ii). Hence $K(p^{'m})(ζ^{n-ℓ}, g(θ))$ has a normal $p$-integral basis over $K(p^{'m})(ζ^{n-ℓ})$ by Lemma 7.3(i). If $n = 2ℓ$, then $K(p^{'m}) = K(p^{'m})(ζ^{n-ℓ})$. If $n > 2ℓ$, then $ζ^{n-ℓ} \not\in K(p^{'m})$, and hence $K(p^{'m})(g(θ))$ has a normal $p$-integral basis over $K(p^{'m})$ by Lemma 7.3(ii).

Therefore the theorem follows from Lemma 7.3 because $K(p^{'m})(ζ^n) \cap K(p^{'m})(g(θ)) = K(p^{'m})$.

**Corollary 7.6.** Let $K(\not\in \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}))$ be an imaginary quadratic field, $p \geq 5$ be a prime and $m \geq 1$ be an integer relatively prime to $p$. Let $K_∞$ be any $\mathbb{Z}_p$-extension of $K$. Then the $\mathbb{Z}_p$-extension $K_∞K(p^{'m})$ has a normal basis over $K(p^{'m})$ for $ℓ \geq 1$.

**Proof.** It is a direct consequence of Theorem 7.5 by the well-known fact that if an extension $F/L$ of number fields has a normal $p$-integral basis over $L$, then so does $F'/L$ for each intermediate field $F'$.

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