Staggered fermion approach to chiral gauge theories on the lattice

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The staggered fermion approach to build models with chiral fermions is briefly reviewed. The method is tested in a U(1) model with axial vector coupling in two and four dimensions.

1. INTRODUCTION

An important open problem in lattice field theory is to find a satisfactory method to describe chiral fermions. There exist several attempts and proposals, for an overview see [1], but none of these has so far been shown to work. Here we shall discuss and test an approach [2] based on the staggered fermion method. Note that in this approach the notorious species doublers need not be decoupled, since they are used as physical spin-flavor degrees of freedom.

Staggered fermion fields, denoted by $\chi_x$, do not carry explicit flavor and Dirac labels since these components are ‘spread out’ over the lattice. In the classical continuum limit one can see the usual Dirac and flavor structure in momentum space [3], and an equivalent method can also be given in position space. Thereto one introduces 4×4 matrices $\Psi^x_\alpha$ defined as,

$$\Psi_x = \frac{1}{N} \sum_b \gamma^{x+b} \chi_{x+b}^b, \quad \Psi_x^* = \frac{1}{N} \sum_b (\gamma^{x+b})^\dagger \bar{\chi}_{x+b}^b,$$

(1)

with $\gamma^x = \gamma^1 \cdots \gamma^4$ and the sum running over the corners of a hypercube, $b = 0, 1$. Note that these $\Psi_x$ are not independent, only their Fourier components in the restricted momentum range $\pi/2 < p_\mu \leq \pi/2$ may be considered independent. The index $\alpha(\kappa)$ on $\Psi$ acts like a Dirac (flavor) index and one can construct models involving arbitrary spin-flavor couplings to Higgs or gauge fields in a straightforward manner [4]. In the classical continuum limit (for smooth external fields and for the low momentum modes of $\Psi$) the desired symmetry properties of the model emerge. The important issue is if the same can be achieved when the effect of quantum fluctuations is taken into account.

This approach has been successfully applied to a study of a fermion-Higgs model [4]. Here we consider a U(1) model with axial-vector coupling in two and four dimensions, defined by the action in the $\Psi$ notation, $(d = 4)$,

$$S_F = -\sum_{x, \mu} \frac{1}{2} \text{Tr} \left[ \Psi_x^\dagger \gamma_\mu (U_{\mu x} P_L + U_{\mu x}^* P_R) \Psi_{x+\mu} \right]$$

$$- \Psi_{x+\mu}^\dagger \gamma_\mu (U_{\mu x} P_L + U_{\mu x}^* P_R) \Psi_x \right].$$

(2)

By working out the trace in (2) one finds the action for the staggered field,

$$S_F = -\frac{1}{N^2} \sum_{x, \mu} \left[ (\eta_{\mu x} \sum_{b} c^\mu_{x-b} \bar{\chi}_x \chi_{x+b} - h.c.)
$$

$$+ (i \eta_{\mu x} \eta_{\mu x+n} \sum_{b+c=n} s^\mu_{x-b} \bar{\chi}_x \chi_{x+c-b+\mu} + h.c.) \right]$$

(3)

where $c^\mu_{x} = \text{Re} U_{\mu x}$ and $s^\mu_{x} = \text{Im} U_{\mu x}$. The sign factor $\eta_{\mu x} = (-1)^{x_1 + \cdots + x_\mu - 1}$ represents the Dirac $\gamma_\mu$, $\eta_{0x} = \eta_{x_2 x_4} \cdots \eta_{x_4 x_1 + 1 + \cdots + 4} = (-1)^{x_1 + x_3}$ represents $\gamma_5$ and $n = (1, 1, 1, 1)$. In the classical continuum limit this action describes four (two) flavors of axially coupled Dirac fermions in four (two) dimensions.

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In the $\Psi$ form the action appears to be gauge invariant under $U \to U^g$, $\Psi \to \Psi^g$, with

$$U_{g}^{\mu x} = g_{x}U_{\mu x}g_{x+\hat{\mu}}, \quad g_{x} \in U(1),$$

$$\Psi^{g}_{x} = (g_{x}P_{L} + g_{x}P_{R})\Psi_{x}. \quad (5)$$

However, since the $\Psi_{x}$ are not independent, such a local transformation cannot be carried over to the $\chi$ field, and the action is not gauge invariant.

2. GAUGE (NON)INVARIANCE

Even though the action (3) is not gauge invariant, it is well-known [2] that the model is closely related to a gauge invariant model. Let $S = S_{F} + S_{\varphi}$ be the gauge non-invariant action, consisting of the term (3) plus additional counterterms, denoted by $S_{ct}$. The partition function can be written as,

$$Z = \int D\chi' D\chi DU e^{S(U',\chi',\overline{\chi})},$$

$$U'=U^{V1} = \int D\chi' D\chi DV DU e^{S(U^{V1},\chi',\overline{\chi})}. \quad (6)$$

Here we have used invariance of the Haar measure $DU = DU^{g}, \quad g = V^{1}$, and $\int DV = 1$. By explicitly writing the 'longitudinal gauge mode' $V$, the action $S(U^{V1},\chi',\overline{\chi})$ is seen to be gauge invariant under (3) and $V_{x} \to g_{x}V_{z}$.

In the scaling region this gauge mode $V$ may decouple from the physical states (if the fermion content is anomaly free), which is to be achieved by choosing the appropriate counterterms in $S_{ct}$. In particular we add a mass term for the gauge field,

$$S_{ct} = \kappa \sum_{x\mu}(U'_{\mu x} + h.c.),$$

$$= \kappa \sum_{x\mu}(V_{x}^{*}U_{\mu x}V_{x+\hat{\mu}} + h.c.). \quad (7)$$

The $V$ field can also be viewed as a radially frozen Higgs field. Then (3) acts like a kinetic term for the Higgs field. One expects a critical line $\kappa = \kappa_{c}(\beta)$ (with $\beta$ the bare gauge coupling) and for $\kappa \to \kappa_{c}$ the Higgs mode will be present as a physical state. By keeping $\kappa$ away from $\kappa_{c}$ in the symmetric phase, the scalar mass will be of the order of the cut-off and $V$ will decouple.

The attractive feature here is that gauge fixing is avoided. Alternatively one might try to fix the gauge and then remove undesired couplings of the $V$ field by adding suitable counterterms, as in the 'Rome approach' [5]. One then has to worry about technical and Gribov problems with non-perturbative gauge fixing.

Using $V$ as a Higgs field, we may think of $U'$ as the gauge field in the unitary gauge, and similar for $\Psi'$ and $\chi'$. It is natural to add a fermion mass term

$$S_{Y} = -y \sum_{x} Tr \overline{\Psi}_{x}\Psi'_{x}, \quad (9)$$

which would turn into a Yukawa coupling

$$-y \sum_{x} Tr \overline{\Psi}_{x}(V_{x}^{2}P_{R} + (V_{x}^{*})^{2}P_{L})\Psi_{x} \quad \text{if} \quad \Psi'_{x} = (V_{x}^{*}P_{L} + V_{x}P_{R})\Psi_{x},$$

could be implemented as a transformation of variables.

3. EXTERNAL GAUGE FIELDS

We shall first test our model for smooth external gauge fields, i.e. $U \approx 1$ everywhere, and $V = 1$, in two dimensions. Since our model has axial-vector coupling, the vector current has an anomaly. One therefore expects the relation,

$$\langle \theta'_{\mu} J_{\mu x} \rangle_{\chi} = \frac{1}{2\pi} F_{x}, \quad (10)$$

where the notation $(\bullet)_{\chi}$ means integration over the $\chi$ field. The vector current $J_{\mu}$ (given in $\Psi$ notation for simplicity) and field strength $F$ are defined as,

$$J_{\mu x}^{V} = iTr (\overline{\Psi}_{x}\gamma_{\mu}U_{\mu x}\Psi_{x+\hat{\mu}} + h.c.), \quad (11)$$

$$F_{x+n/2} = -i log U_{1x}U_{2x+1}U_{1x+2}^{*}U_{2x}^{*}, \quad (12)$$

As an example we consider a spatially constant $U$ field defined by,

$$U_{1x} = \exp(iA_{1} \sin(2\pi t/L)), \quad U_{2x} = 1. \quad (13)$$

Lattice sites are denoted by $x = (z, t)$ and the lattice size is $L^{2}$. Fig. 1 shows the divergence of the current (the left hand side of eq. (10), denoted by $\Box$) and the anomaly (full line). Clearly the correspondence is excellent and this remains the case for amplitudes $A_{1} \lesssim 0.2$. For larger values one begins to see deviations.
Figure 1. Current divergence (□) and anomaly (full line) for the smooth gauge field (13), $L = 24$. The (*) includes averaging over $V$ field fluctuations with $\kappa = 0.6$, $L = 8$.

Our second test case uses an external gauge field with non-trivial topology. For the field (13) the topological charge $Q = \sum_x F_x / 2\pi$ is zero. We can construct a gauge field with non-zero $Q$ and constant field strength $F_x = 2\pi Q / L^2$, by writing

$$U_{1x} = \exp(iFt), \quad U_{2x} = 1, \quad t = 1, \cdots, L - 1; \quad U_{2x} = \exp(iFLz), \quad t = L.$$  (14)

For small $Q/L$, the link field $U$ is close to one everywhere except on the time slice $t = L$ which contains a transition function. This transition function makes $U$ ‘rough’ (discontinuous) at $t = L$. With a gauge invariant model such a transition function is invisible, but since our action and the current (13) lack gauge invariance we may expect deviations from the anomaly equation (10) at this timeslice. This is illustrated in fig. 2, which is the same as fig. 1 but now for the gauge field (14). The presence of the slice with the transition function is clearly visible and its disturbance extends over roughly 6 time slices. We have checked that on a larger lattice the disturbance remains confined to this number of time slices and one might argue that in the infinite volume limit the effect of the transition function becomes an invisible surface effect.

Figure 2. Same as fig. 1, now for the gauge field (14) with $Q = 1$ and without $V$ field fluctuations.

These results show that for smooth fields the correct anomaly structure is reproduced by the staggered fermions. Also non-trivial topological sectors can presumably be incorporated in the proper way, but the strong effect of the roughness produced by the transition functions hints at difficulties for fields subject to quantum fluctuations.

4. FLUCTUATING $V$ FIELD

Suppose we decompose the gauge field in a ‘transverse’ and a ‘longitudinal’ part, $U_{\mu x} = V_x U_{\mu x}^{tr} V_x^*$, where $U_{\mu x}^{tr}$ satisfies the (lattice) Landau gauge condition. The fluctuations of $U_{\mu x}^{tr}$ are governed by a gauge coupling $\beta$ and get suppressed for large $\beta$. If not used as a physical Higgs field, the $V$ modes should be unobservable in the low energy physics, if the model becomes gauge invariant. This can be tested already for $U_{\mu x}^{tr} \to 1$.

We have investigated the resulting fermion-scalar model for $y \neq 0$. For $y \to 0$ and $\kappa$ strictly smaller than $\kappa_c$ the $V$ field should decouple. In the quenched approximation we have
computed the fermion propagators $\langle \Psi_x \overline{\Psi}_y \rangle$ and $\langle \Psi'_x \overline{\Psi'}_y \rangle$, where $\Psi_x \equiv (V_x P_R + V'_x P_L)\Psi'_x$. Only the first propagator should then show a physical particle pole (in the symmetric phase). We found encouraging results in the broken phase but ambiguous results, hampered by huge statistical fluctuations, in the symmetric phase.

As an alternative test we add the longitudinal modes to the smooth external gauge field (13), $U_{\mu x} \rightarrow U'_{\mu x} = V_x U_{\mu x} V'_{x+\mu}$, and compute the anomalous divergence relation (10) with the path integral average over $V$ included ($y = 0$). For large $\kappa > \approx 1$ we then reproduce the result for the fixed background, up to statistical errors. For smaller $\kappa$, approaching the phase transition at $\kappa_c \approx 0.5$ we find that the current gets renormalized. The result in fig. 1 (denoted by *) shows that the current strength is reduced by a factor $\approx 0.7$ at $\kappa = 0.6$. Such a renormalization is to be expected for staggered currents and has been studied previously in QCD, see e.g. [6]. For smaller values of $\kappa$, in particular inside the symmetric phase, the magnitude of the statistical fluctuations grows dramatically and prohibits computation of the average current divergence.

5. DYNAMICAL FERMIONS

Since the back reaction of the fermions on the $V$ field should reduce the troublesome fluctuations, we have also studied the scalar-fermion model in four dimensions with dynamical fermions. Here we focus on the phase diagram of the model in the $y - \kappa$ plane. For large $y$ one can use a hopping expansion to see that the fermion field $\Psi'$ in this unitary gauge has a pole whereas the fermion field $\Psi$ does not. If this persists also for small $y$ the model fails. If the model is viable we must see a cross over at some (small) value of $y$.

Unfortunately the problems which hampered the quenched computations appear to haunt us here as well: For small values of $y < \approx 0.4$ the model has extremely long living metastable states, reminiscent of spin glass dynamics and we have so far not been able to compute the phase structure in the region $y < \approx 0.4$. The phase diagram must be symmetric for $\kappa \rightarrow -\kappa$, which we can use to check our results. In fig. 3 we show the measured phase transitions for large $y$ and two speculations for the structure at small $y$. One possibility, the solid lines, represent first order transitions and would rule out our model. A second possibility is indicated by the dashed, second order lines and in this scenario our model is still alive.

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