Liouville Black Holes

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Abstract

The dynamics of Liouville fields coupled to gravity are investigated by applying the principle of general covariance to the Liouville action in the context of a particular form of two-dimensional dilaton gravity. The resultant field equations form a closed system for the Liouville/gravity interaction. A large class of asymptotically flat solutions to the field equations is obtained, many of which can be interpreted as black hole solutions. The temperature of such black holes is proportional to their mass-parameters. An exact solution to the back reaction problem is obtained to one-loop order, both for conformally coupled matter fields and for the quantized metric/Liouville system. Quantum effects are shown to map the space of classical solutions into one another. A scenario for the end-point of black-hole radiation is discussed.
1 Introduction

Two-dimensional gravity continues to be a subject of intense study, motivated both by string theory [1] and by a desire to study quantum gravitational effects in a mathematically tractable setting [2]. Black hole solutions [3, 4, 5, 6] have especially attracted attention as the simplicity of (1 + 1) dimensions permits significant clarification of the conceptual issues associated with black hole radiation.

In a spacetime of two dimensions, the construction of a gravitational theory necessarily departs from the Einstein-Hilbert formalism because of the triviality of the Einstein tensor: \( G_{\mu\nu}[g] \equiv 0 \) for all metrics \( g_{\mu\nu} \). As a consequence, the Einstein-Hilbert action is trivial, and new approaches must be taken. These have included non-local actions [1], higher-derivative theories [4], setting the Ricci scalar equal to a constant [5], setting the one-loop beta functions of the bosonic non-linear sigma model to zero in a two-dimensional target space [9], or setting the Ricci scalar equal to the trace of the 2D stress-energy tensor [4, 5]. In each approach the Ricci scalar becomes a non-vanishing function of the co-ordinates over some region of spacetime, permitting the spacetime to develop interesting features such as black-hole horizons and singularities.

The study of Liouville fields has been of particular interest, since the non-local action of ref. [1] becomes the (local) Liouville action for a certain choice of co-ordinates. In this approach, the conformal factor of the metric is the Liouville field. This has led most theorists to adopt the general paradigm of interpreting the Liouville action as the action for 2D (quantum) gravity [10].

In the present paper a different approach towards the relationship between the Liouville action and 2D gravity is developed. Rather than obtaining the Liouville action as a form of induced gravity [1] which arises from quantum effects (and in which the Liouville field is identified with the conformal factor of the metric [10]), the approach taken here is to consider an independent Liouville field in a curved (1 + 1)-dimensional spacetime in such a way that its stress-energy generates gravity. This may be done by taking the (flat-space) Liouville action and applying to it the principle of general covariance. The net effect of this is to set the Ricci scalar equal to the trace of the Liouville stress-energy, an approach which has been widely studied for other forms of two-dimensional stress-energy in a variety of previous contexts [5, 11].

More specifically, exact solutions to the field equations which follow from
the action of a Liouville field in a curved 2D spacetime are derived and their consequences investigated, extending the work of ref. [12]. The most general form of the action which is of lowest order in powers of the curvature depends upon three independent coupling constants; a choice of these three constants is equivalent to a choice of theory. An exact solution within a given theory then yields a spacetime which is a function of these parameters (along with the constants of integration that specify the particular solution). Conditions necessary for the existence of asymptotically flat black hole solutions of positive mass constrain the space of coupling constants, and it is shown that a wide class of theories exist which permit such solutions. One of these solutions [13] closely resembles that of the string-theoretic 2D black hole [6, 9]. However the thermodynamic properties of this (and the other) black hole solutions markedly differ from those of the string-theoretic case: the temperature is proportional to the ADM-mass parameter (as opposed to being a constant), and the entropy varies logarithmically with the mass.

Quantum corrections due to the presence of conformally coupled matter are then taken into account, and an exact solution to the field equations which includes the back-reaction is obtained. The effects of the back-reaction are to map the space of solutions into itself, so that a (classical) solution with a given set of parameters is mapped into one with another set of parameters that may or may not satisfy the criteria for the existence of asymptotically flat black holes of positive mass. When quantum effects due to the gravitational and Liouville fields are taken into account, a similar phenomenon happens, except that in this case the map of the space of solutions into itself is somewhat more complicated.

The paper is organized as follows. In section 2 the general form of the action is described and the field equations are derived. In section 3 the general form of the exact solutions is obtained, and in section 4 the criteria necessary for black hole solutions is discussed and applied to the solutions obtained in section 3. Section 5 is devoted to a consideration of the thermodynamic properties Liouville black holes, and section 6 to the incorporation of quantum effects. Section 7 contains some concluding remarks.
2 Two-Dimensional Gravity Coupled to a Liouville Field

The action considered here is taken to be \[ S = S_G + S_M = \int d^2x \sqrt{-g} \left( \frac{1}{8\pi G} \frac{1}{2} g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi + \psi R \right) + L_M \] (1)

where \( R \) is the Ricci scalar and \( S_M = \int d^2x \sqrt{-g} L_M \) is the two dimensional matter action which is independent of the auxiliary field \( \psi \). The action (1) is designed so that the auxiliary field \( \psi \) has no effect on the classical evolution of the gravity/matter system. This is easily seen by considering the field equations:

\[ \nabla^2 \psi - R = 0 \] (2)

\[ \frac{1}{2} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} (\nabla \psi)^2 \right) + g_{\mu\nu} \nabla^2 \psi - \nabla_\mu \nabla_\nu \psi = 8\pi G T_{\mu\nu} \] (3)

where \( T_{\mu\nu} \) is the stress-energy tensor of two-dimensional matter; it is easily seen to be conserved by taking the divergence of (3). Insertion of the trace of (3) into (1) yields

\[ R = 8\pi G T^\mu_\mu \quad \text{where} \quad \nabla_\nu T^{\mu\nu} = 0 \] (4)

the evolution of \( \psi \) then being determined only by the traceless part of (3).

The theory associated with the action (1) is of interest because it yields a two dimensional theory of gravity which closely resembles \((3+1)\) dimensional general relativity in that the (classical) evolution of the gravitational field is driven by the stress-energy and no other Brans-Dicke type fields \([14]\). Its classical and semi-classical properties and solutions are also markedly similar \([4, 14, 15, 16, 17]\). Indeed, the field equations which follow from (18) may be obtained from a reduction of the Einstein equations from \( D \) to 2 spacetime dimensions \([18]\).

Incorporation of the Liouville action

\[ S_L = \int d^2x \left[ b(\nabla \phi)^2 + \Lambda e^{-2a\phi} \right] \] (5)

is easily carried out by applying the principle of general covariance, so that the action which couples the Liouville field \( \phi \) to gravity is given by

\[ S = S_G + S_L = S_G + \int d^2x \sqrt{-g} \left[ b(\nabla \phi)^2 + \Lambda e^{-2a\phi} + \gamma \phi R \right] \] (6)
the Liouville action being viewed as a particular form of the matter action $S_M$. In (6) a non-minimal coupling $\gamma \phi R$ has been included (for a canonically normalized $\phi$, $b = -\frac{1}{2}$). Note that in other approaches to Liouville-gravity $S_G = 0$; the Liouville field in (6) is then interpreted as the conformal factor associated with the metric so that $g_{\mu \nu} = e^\phi \hat{g}_{\mu \nu}$, $\hat{g}_{\mu \nu}$ being a fixed background metric with respect to which the action is not varied. Here however, the Liouville field $\phi$ is considered a-priori to be an independent matter field coupled to gravity, so that the action in (5) is a functional of $\psi$, $g_{\mu \nu}$ and $\phi$ ($S = S[\psi, g, \phi]$). This action depends upon only three independent coupling constants, since one of $a$, $b$, or $\gamma$ may be absorbed by rescaling $\phi$. However in order to more easily take into account possible overall sign changes in each of the terms, all four coupling constants will be retained.

The field equations which follow from (6) are

$$\nabla^2 \psi - R = 0$$

and

$$-2b \nabla^2 \phi + \gamma R - 2ae^{-2a\phi} = 0$$

and

$$\frac{1}{2} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu \nu} (\nabla \psi)^2 \right) + g_{\mu \nu} \nabla^2 \psi - \nabla_\mu \nabla_\nu \psi$$

$$= 8\pi G \left[ -b \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} (\nabla \phi)^2 \right) - \gamma \left( g_{\mu \nu} \nabla^2 \phi + \nabla_\mu \nabla_\nu \phi \right) \right] + \frac{1}{2} g_{\mu \nu} \Lambda e^{-2a\phi}$$

It is easily checked that the left-hand-side of (9) is divergenceless when (7) holds and that the right-hand-side is divergenceless when (8) holds. Hence the system (7–9) is a system of 5 equations with two identities, allowing one to solve for the three unknown fields $\psi$, $\phi$ and the one independent degree of freedom in the metric.

Making use of the trace of (9), the field equations (7) and (8) become

$$(b + 4\pi G \gamma^2) \nabla^2 \phi = (4\pi G \gamma - a) \Lambda e^{-2a\phi}$$

and

$$(b + 4\pi G \gamma^2) R = 8\pi G \Lambda (a \gamma + b) e^{-2a\phi}$$

with the evolution of $\psi$ being determined by the traceless part of (9).
In conformal coordinates
\[ g_{+-} = -\frac{1}{2} e^{2\rho} \quad R = 8 e^{-2\rho} \partial_+ \partial_- \rho \quad \nabla^2 \phi = -4 e^{-2\rho} \partial_+ \partial_- \phi \] (12)
the field equations simplify to
\[ \partial_+ \partial_- (\psi + 2\rho) = 0 \] (13)
\[ \partial_+ \partial_- \rho = \frac{\pi G \Lambda (\alpha \gamma + b)}{b + 4\pi G \gamma^2} e^{2(\rho - a\phi)} \] (14)
\[ \partial_+ \partial_- \phi = \frac{\Lambda (a - 4\pi G \gamma)}{4(b + 4\pi G \gamma^2)} e^{2(\rho - a\phi)} \] (15)
\[ \frac{1}{2} (\partial_\pm \psi)^2 - \partial_\pm^2 \psi + 2\partial_\pm \rho \partial_\pm \psi + 8\pi G (b(\partial_\pm \phi)^2 - \gamma \partial_\pm^2 \phi + 2\gamma \partial_\pm \rho \partial_\pm \phi) = 0 \] (16)
where the last set of equations follows from the traceless part of (\[\text{[14]}\]).

3 Exact Solutions

Provided that \( b \neq -4\pi G \gamma^2 \) the field equations (14) and (15) are non-degenerate. Taking an appropriate linear combination of these two equations yields
\[ \partial_+ \partial_- (\rho - a\phi) = \frac{8\pi G a \gamma + 4\pi G b - a^2}{4\pi G (\alpha \gamma + b)} e^{2(\rho - a\phi)} \] (17)
\[ i.e. \rho - a\phi \text{ obeys the Liouville equation, provided } b \neq -4\pi G \gamma^2. \]
The solution to (17) is well-known
\[ \rho - a\phi = \frac{1}{2} \ln \left[ \frac{f_+ f_-}{(\Lambda^2 - K^2 f_+ f_-)^2} \right] \] (18)
where \( K \) is a constant of integration and \( f_\pm \) are arbitrary functions of the coordinates \( x_\pm \) and the prime refers to differentiation by the relevant functional argument. The constant \( A \) is
\[ A = \frac{\Lambda}{4(4\pi G \gamma^2 + b)} e^{-2a\phi_0} \] (19)
where from (13)
\[ \phi = \frac{a - 4\pi G\gamma}{4\pi G(a\gamma + b)} \rho + \frac{1}{a}(h_+ + h_-) + \phi_0 \]  

(20)

(provided \(a\gamma + b \neq 0\)) where \(h_{\pm}\) are arbitrary functions of the coordinates \(x_{\pm}\) respectively; for convenience an additional constant of integration \(\phi_0\) has been retained. From (14)
\[ \rho = \xi \ln \left[ \frac{f'_+ f'_-}{(\frac{A}{K^2} - K^2 f'_+ f'_-)^2} \right] + h_+ + h_- \]  

(21)

where
\[ \xi = \frac{2\pi G(a\gamma + b)}{4\pi Gb + 8\pi G a\gamma - a^2} \]  

(22)

yielding a metric
\[ ds^2 = -e^{2\rho} dx_+ dx_- = -\left[ \frac{f'_+ f'_-}{(\frac{A}{K^2} - K^2 f'_+ f'_-)^2} \right]^{2\xi} e^{2(h_+ + h_-)} dx_+ dx_- \]

\[ = -\frac{df_+ df_-}{(\frac{A}{K^2} - K^2 f'_+ f'_-)^{4\xi}} \]  

(23)

where the last term follows from a suitable choice of \(h_{\pm}\).

Equation (13) has the solution
\[ \psi = -2\rho + \chi_+(x_+) + \chi_-(x_-) \]  

(24)

and equations (16) constrain these functions to obey
\[ \frac{1}{8\pi G} \left( \frac{1}{2}(\chi'_\pm)^2 - \chi''_\pm \right) - \frac{b + 4\pi G\gamma^2}{2(4\pi G b + 8\pi G a\gamma - a^2)} \left[ \left( \frac{f''_\pm}{f'_\pm} \right)' - \frac{1}{2} \left( \frac{f''_\pm}{f'_\pm} \right)^2 \right] = 0 \]  

(25)

which has the solution
\[ \chi_\pm(x_\pm) = \ln \left[ f'_\pm \left( \sqrt{\frac{4\pi G(b + 4\pi G\gamma^2)}{4\pi G b + 8\pi G a\gamma - a^2}} \right)^x_\pm \right] . \]  

(26)

If \(A = 0\) the solution (17) becomes
\[ a\phi = \rho + \frac{1}{2} \ln(\hat{k}^2 g'_+ g'_-) + \phi_0 \]  

(27)
where \( g_\pm \) are arbitrary functions of the coordinates \( x_\pm \) respectively. From (14) and (15)

\[
\rho = -D \hat{k}^2 g_+ g_- + h_+ + h_-
\]

where \( h_\pm \) are again arbitrary functions of \( x_\pm \) respectively and

\[
D = \pi G \Lambda \frac{a}{4 \pi G \gamma - a}
\]

yielding a metric

\[
ds^2 = e^{2(h_+ + h_- - D \hat{k}^2 g_+ g_-)} dx_+ dx_- = e^{-2D \hat{k}^2 g_+ g_-} dg_+ dg_-
\]

where again the last relation follows by making a suitable choice of \( h_\pm \). The analogue of (25) for this choice is

\[
\frac{1}{8 \pi G} \left( \frac{1}{2} (\chi'_\pm)^2 - \chi''_\pm \right) + \left( \frac{g''_\pm}{g'_\pm} \right)' - \frac{1}{2} \left( \frac{g''_\pm}{g'_\pm} \right)^2 = 0
\]

with

\[
\chi_\pm(x_\pm) = \ln \left[ g'_\pm \left( \sqrt{4 \pi G x_\pm} \right) \right]
\]

being the solution.

If \( a \gamma + b = 0 \) then the spacetime is flat, \( \psi \) is a free scalar field and \( \phi \) obeys the flat-space Liouville equation. Alternatively, if \( a = 4 \pi G \gamma \), then \( \phi \) is a free scalar field and the spacetime has constant curvature.

Each of the metrics (23,30) may be transformed to a static system of coordinates

\[
ds^2 = -\alpha(x) dt^2 + \frac{dx^2}{\alpha(x)}
\]

under an inverse Kruskal-Szekeres transformation. There are three distinct classes of solutions.

(A) \( \xi \neq 1/4, A \neq 0 \)

In this case the solution (23) is

\[
\alpha(x) = 2Mx - \frac{A}{M^2(1-p)^2} (2Mx)^p = 2Mx - B x_0^2 \left( \frac{x}{x_0} \right)^p
\]

\[
\phi(x) = \frac{2-p}{2a} \ln(2Mx) + \phi_0 = \frac{2-p}{2a} \ln(x/x_0)
\]

\[
\psi(x) = -p \ln(2Mx) + \psi_0
\]
where \( K = M / (p - 1) \) and

\[
p = \frac{4\xi}{4\xi - 1} = 8\pi G \frac{a\gamma + b}{a^2 + 4\pi Gb}
\]

(35)

\[
B = \frac{4A}{(p - 1)^2} = \Lambda \frac{(a^2 + 4\pi Gb)^2}{(b + 4\pi G\gamma^2)(4\pi Gb + 8\pi Ga\gamma - a^2)}
\]

(36)

with \( \phi_0 = \frac{p - 2}{2a} \ln(2Mx_0) \).

(B) \( \xi = 1/4, A \neq 0 \)

In this case (23) is

\[
\alpha(x) = 1 - Ce^{-2M(x-x_0)}
\]
\[
\phi(x) = \frac{M}{a}(x-x_0)
\]
\[
\psi(x) = 2Mx + \psi_0
\]

(37)

where now

\[
b = -\frac{a^2}{4\pi G} \quad C \equiv \frac{2\pi G\Lambda a}{M^2(a + 4\pi G\gamma)}
\]

(38)

and \( K = M \).

(C) \( A = 0 \)

In this case \( p = 1 \) or

\[
b = \frac{a^2}{4\pi G} - 2a\gamma
\]

(39)

and

\[
\alpha(x) = 2Mx \ln(2Dx_0)
\]
\[
\phi(x) = \frac{1}{2a} \ln\left(\frac{4D}{M}x\right)
\]
\[
\psi(x) = -2M \ln\left(\frac{D}{M}x\right) + \psi_0
\]

(40)

where \( D \) is given by (29) and \( \hat{k}^2 = x_0/2 \).

In each of (34), (37) and (40) \( \psi_0, M \) and \( x_0 \) are constants of integration, with \( M \) being proportional to \( K \) for solution (A) and \( x_0 \) replacing the constant \( \phi_0 \) in (20, 27).
The solutions \((23, 30)\) (or alternatively \((34, 37, 40)\)) are, up to coordinate transformations, the most general set of solutions to the field equations for arbitrary \(a\), \(b\), \(\gamma\) and \(\Lambda\) in the absence of any other forms of stress-energy. The only restriction on the parameters is that \(b \neq -4\pi G\gamma^2\); otherwise the system \((10, 11)\) becomes underdetermined. Note that none of \((34, 37, 40)\) are invariant under \(x \to -x\). As such they may be matched onto a solution of collapsing matter which is moving either rightward or leftward. Alternatively, one may consider these solutions as exterior to a localized system of two-dimensional matter by making the replacement \(2Mx \to 2M_\pm|x| - X_\pm\), where the `+' refers to the rightward side of the localized stress-energy and the `−' to the leftward side. If the localized stress-energy is centered about the origin, then \(\pm\) refers to positive/negative \(x\).

Consider next the \(M \to 0\) limit of \((34, 37, 40)\). It will be shown in the next section that \(M\) is proportional to the ADM-mass. For \((34)\) this limit is straightforward:

\[
\begin{align*}
    ds^2 &= Bx_0^2\left(\frac{x}{x_0}\right)^p dt^2 + \frac{dx^2}{-Bx_0^2\left(\frac{x}{x_0}\right)^p} \\
    \phi(x) &= \frac{2-p}{2a} \ln(\frac{x}{x_0}) \\
    \psi(x) &= -p \ln(\frac{x}{x_0}) + \hat{\psi}_0
\end{align*}
\]

(where \(\psi_0\) has been shifted) and corresponds to

\[
\begin{align*}
    ds^2 &= -4x_0^2B \left[ \frac{Bx_0}{4\xi - 1} (x_+ + x_-) \right]^{-4\xi} dx_+ dx_- \\
    \phi(x) &= \frac{1 - 2\xi}{a} \ln\left( \frac{Ax_0}{4\xi - 1} (x_+ + x_-) \right)
\end{align*}
\]

in the maximally extended case. The metric \((42)\) may be obtained from \((23)\) by choosing \(f_+ = x_+\) and \(f_- = A/(K^4x_-)\), and absorbing the \(K \sim M\) factor into \(h_-\). Performing this operation on \((23)\) when \(\xi = 1/4\) yields

\[
\begin{align*}
    ds^2 &= -2 \frac{dx_+ dx_-}{\ell(x_+ + x_-)} \\
    \phi(x) &= \frac{1}{2a} \ln\left( \frac{2A}{\ell} (x_+ + x_-) \right)
\end{align*}
\]
or

\[
ds^2 = \frac{A}{\ell^2}e^{-2\ell x}dt^2 + \ell^2 \frac{dx^2}{-Ae^{-2\ell x}}
\]

\[
\phi(x) = \frac{\ell}{a}x
\]

\[
\psi(x) = 2\ell x + \psi_0
\]

in the static case. The solution (44) may be obtained from (37) by rescaling \( x \to \ell x/M, \ t \to Mt/\ell \) and then setting \( M = 0 \).

For the solution (40), there is no rescaling of \( x \) that permits one to take the \( M \to 0 \) limit in a non-singular way. It is possible to rescale the constant \( x_0 \) so that the \( \lim_{M \to 0} (\alpha(x)) \sim x \); the resultant spacetime is flat. However this limit is singular in that the dilaton field is shifted by an infinite constant.

### 4 Black Hole Solutions

In this section the circumstances under which the spacetimes associated with the solutions (34), (37), and (40) correspond to black holes will be investigated. This will be done by requiring these solutions to satisfy the following conditions.

1. The spacetime must be asymptotically flat, i.e. the curvature scalar \( R \) should vanish as \( x \to \infty \). This condition could be relaxed to allow \( R \) to approach a constant if one wished to consider cosmological black holes, i.e. black hole spacetimes which are asymptotic to (anti-)de Sitter space at large \( x \).
2. There must be an event horizon for a finite real value of \( x \). This will occur provided \( \alpha(x_H) = 0 \) for some \( x = x_H, |x_H| < \infty \).
3. The metric signature must be \((-,-)\) for large \( x \); otherwise the horizon is a cosmological horizon.
4. The ADM-mass must be real and positive. In a two dimensional spacetime that has a timelike Killing vector \( \xi^\mu \) the current \( \frac{\delta S}{\delta g_{\mu\nu}}\xi^\nu = T_{\mu\nu}\xi^\nu \) (where \( S \) is given by (4)) is covariantly conserved. In two dimensions this yields a
mass function $\nabla_\mu \hat{M} = -\epsilon_{\mu
u} T^\nu_\rho \xi^\rho$ from which one obtains

$$M = \left(\frac{\gamma}{4a} - \frac{1}{8\pi G}\right)\alpha' + (\gamma + \frac{b}{a})\alpha' - \int \left[\frac{1}{32\pi G} \left(\frac{\alpha'^2 - K^2}{\alpha}\right) - \frac{b}{2} (\phi')^2 - \frac{\gamma}{2} \alpha' \phi'\right]$$

as the ADM-mass \[13\]. The requirement then is that $M > 0$. Note that $M \sim M = 0$ spacetimes are not flat; rather they are spacetimes in which the stress-energies of the gravitational and Liouville fields cancel each other.

\[(A)\] $\alpha(x) = 2Mx - Bx_0^2\left(\frac{x}{x_0}\right)^p$

For this solution the Ricci scalar is

$$R = -\frac{d^2}{dx^2} \alpha = Bp(p-1)\left(\frac{x}{x_0}\right)^{p-2}$$

which, as $x \to \infty$, vanishes for $p < 2$ and approaches a constant for $p = 2$. The spacetime is singular at the origin $x = 0$. An event horizon is located at $x_H = \frac{M}{2B}\epsilon^{2a\phi_H}$, yielding from condition (2) $\frac{B}{2M} > 0$; $\phi_H$ is the value of the Liouville field at the horizon. Condition (3) implies that $B$ and $M$ are both positive for $p < 1$, and are both negative when $1 < p < 2$. Condition (4) implies that

$$M = \frac{M}{8\pi G} [p-2 - (2p-3)4\pi G^{-\gamma}\frac{\alpha'}{a'}] = M\frac{8\pi G\gamma(\pi Gb + 4\pi Ga\gamma) + a^3 - 10\pi G\gamma a^2}{4\pi Ga(a^2 + 4\pi Gb)}$$

is positive. Writing

$$\tilde{a} = \frac{a}{\pi G|\gamma|} \quad \tilde{b} = \frac{b}{\pi G|\gamma|^2}$$

and putting all the requirements together yields

$$\frac{\text{sgn}(\Lambda)}{M(4\tilde{b} - \tilde{a}^2 + 8s\tilde{a})(\tilde{b} + 4)} > 0 \quad M\frac{8s\tilde{b} + \tilde{a}^3 + 32\tilde{a} - 10s\tilde{a}^2}{\tilde{a}(\tilde{a}^2 + 4\tilde{b})} > 0$$

$$1 < \frac{\tilde{b} + s\tilde{a}}{\tilde{a}^2 + 4\tilde{b}} < 2, \quad M < 0 \quad 8\frac{\tilde{b} + s\tilde{a}}{\tilde{a}^2 + 4\tilde{b}} < 1, \quad M > 0$$

provided $\gamma \neq 0$; here $s \equiv |\gamma|/\gamma$. If $\gamma = 0$ then (50) becomes

$$\frac{\text{sgn}(\Lambda)}{M(4\tilde{b} - \tilde{a}^2)b} > 0 \quad M\frac{\tilde{a}^2 + 4\tilde{b}}{\tilde{a}(\tilde{a}^2 + 4\tilde{b})} > 0$$

$$1 < \frac{\tilde{b}}{\tilde{a}^2 + 4\tilde{b}} < 2, \quad M < 0 \quad or \quad 8\frac{\tilde{b}}{\tilde{a}^2 + 4\tilde{b}} < 1, \quad M > 0$$
where $\tilde{a}$ and $\tilde{b}$ are given by (48) with $\gamma = 1$. Several distinct cases arise for the allowed regions of parameter space.

(i) $M > 0$, $\gamma > 0$. For $\Lambda > 0$, (50) becomes

$$-\frac{\tilde{a}^2}{4} < \tilde{b} < -4 \quad \text{and} \quad \frac{10\tilde{a}^2 - \tilde{a}^3 - 32\tilde{a}}{8} < \tilde{b}$$

(52)

for $\tilde{a} > 0$; for $\tilde{a} < 0$ only the first inequality applies. For $\Lambda < 0$, the allowed regions are

$$-4 < \tilde{b} < \frac{\tilde{a}^2 - 8\tilde{a}}{4} \quad \text{and} \quad -\frac{\tilde{a}^2}{4} < \tilde{b}$$

(53)

in the $\tilde{a}$-$\tilde{b}$ plane.

(ii) $M < 0$, $\gamma > 0$. For $\Lambda > 0$ the allowed regions are

$$\frac{10\tilde{a}^2 - \tilde{a}^3 - 32\tilde{a}}{8} < \tilde{b} < \frac{\tilde{a}^2 - 8\tilde{a}}{4} \quad \text{and} \quad \tilde{b} > -4, \quad 0 < \tilde{a} < 4$$

(54)

whereas for $\Lambda < 0$,

$$\frac{10\tilde{a}^2 - \tilde{a}^3 - 32\tilde{a}}{8} < \tilde{b} \quad \text{and} \quad \tilde{a} < 0$$

(55)

The allowed regions of parameter space for the solution (34) are shown in Figures 1 ($M > 0$) and 2 ($M < 0$) for $s = 1$ (positive $\gamma$); the $s = -1$ case is easily obtained by reflecting $\tilde{a} \rightarrow -\tilde{a}$.

(iii) $\gamma = 0$. Only the $p < 1$ region is allowed, implying $M > 0$ always. $\Lambda$ and $b$ are of opposite sign, and $a^2 > 4\pi G|b|$.

In each of the above cases, if one adds the additional requirement that the Liouville field have positive kinetic energy, then all regions with $\tilde{b} > 0$ are excluded.

(B) $\alpha(x) = 1 - Ce^{-2Mx}$

The solution (37), where $b = -\frac{a^2}{4\pi G}$ is somewhat simpler to analyze. The Ricci scalar is

$$R = \frac{8\pi G\Lambda a}{a + 4\pi G\gamma} e^{-2M(x-x_0)}$$

(56)
and diverges for large negative $x$; the spacetime is asymptotically flat provided $M > 0$. In this case

$$\mathcal{M} = \frac{M}{8\pi G}(1 - 8\pi G\gamma/a)$$

and the horizon is at

$$x_H = \frac{1}{2M}\ln(C) + x_0 = \frac{1}{2M}\ln\left(\frac{2\pi G\Lambda a}{M^2(4\pi G\gamma + a)}\right) + x_0$$

so conditions (1)–(4) give

$$M > 0 \quad \frac{a\Lambda}{a + 4\pi G\gamma} > 0 \quad 1 > 8\pi G\gamma/a$$

which in terms of $\tilde{a}$ as given by (48) implies $-4 < \tilde{a} < 0$ for $\Lambda < 0$ and $\tilde{a} < -4$ or $\tilde{a} > 8$ for $\Lambda > 0$. In all cases the Liouville field $\phi$ has positive kinetic energy.

(C) $\alpha(x) = 2Mx \ln(2Dxx_0)$

For the solution (40), $b = \frac{a^2}{4\pi G} - 2a\gamma$. The Ricci scalar is

$$R = -2M/x$$

and diverges as $x \to 0$. The spacetime is asymptotically flat,

$$\mathcal{M} = \frac{M}{8\pi G}(1 - 4\pi G\gamma/a)$$

and $x_H = \frac{M(4\pi G\gamma - a)}{4\pi G\Lambda a}e^{2\alpha\phi_H}$, where $\phi_H$ is the value of the Liouville field at the horizon. Conditions (1)–(4) yield

$$M > 0 \quad \frac{4\pi G\gamma - a}{a\Lambda} > 0 \quad 1 > 8\pi G\gamma/a$$

and consequently $\Lambda$ and $a$ are of opposite sign. The Liouville field has positive kinetic energy only if $a > 8\pi G\gamma > 0$ or $a < 8\pi G\gamma < 0$. 

13
5 Thermodynamic Properties

The temperature of the black hole solutions (34), (37), and (40) may easily be computed by Euclideanizing them \((t = i\tau)\) and then requiring that the periodicity of \(\tau\) be such that no conical singularities are present. For a metric of the form (33), this yields a temperature \(T\):

\[
T = \frac{1}{4\pi} \left. \frac{d\alpha}{dx} \right|_{x_H}
\]

where \(x_H\) is the location of the horizon in the coordinate system (33), i.e. \(\alpha(x_H) = 0\).

For the solutions (34), (37), and (40) this yields

\[
\alpha = 2Mx - Bx_0^2(\frac{x}{x_0})^p \quad T = \frac{|M(1-p)|}{2\pi}
\]

\[
\alpha = 1 - Ce^{-2Mx} \quad T = M/2\pi
\]

\[
\alpha = 2Mx\ln(2Dxx_0) \quad T = M/2\pi
\]

These results may be confirmed using semiclassical methods which relate the trace anomaly to Hawking radiation [19]. Writing the metric in the form

\[
ds^2 = -C(u,v)du dv
\]

yields

\[
\langle O|T^c_{\mu\nu}|O \rangle = \Theta_{\mu\nu} - c_M\hbar R \frac{48\pi}{4\pi} g_{\mu\nu}
\]

for the expectation value of the stress-energy tensor \(T^c_{\mu\nu}\) for conformally coupled matter with central charge \(c_M\), where

\[
\Theta_{uu} = -\frac{1}{12\pi}\sqrt{C} \partial_u C^{-1/2} \quad \Theta_{vv} = -\frac{1}{12\pi}\sqrt{C} \partial_v C^{-1/2} \quad \Theta_{uv} = 0
\]

Evaluating these for the three types of black hole solutions found above yields in the \(c_M = 1\) case

\[
\alpha = 2Mx - Bx_0^2(\frac{x}{x_0})^p \quad p < 1 \quad \Theta_{uu} = \Theta_{vv} = -\frac{1}{12\pi}\left[(M(1-p))^2 - \frac{4M^2p(2-3p)}{e^{-2M(u-v)}}\right]
\]

\[
\alpha = 2Mx - Bx_0^2(\frac{x}{x_0})^p \quad 1 < p < 2 \quad \Theta_{uu} = \Theta_{vv} = -\frac{1}{12\pi}\left[(M(1-p))^2 - \frac{4M^2p(2-3p)}{e^{-2M(u-v)}}\right]
\]

\[
\alpha = 1 - Ce^{-2Mx} \quad \Theta_{uu} = \Theta_{vv} = -\frac{M^2}{12\pi}\left[1 - \frac{1}{1+Ce^{-2M(u-v)}}\right]
\]

\[
\alpha = 2Mx\ln(2Dxx_0) \quad \Theta_{uu} = \Theta_{vv} = -\frac{M^2}{12\pi}\left[1 + e^{-4M(u-v)}\right]
\]
and at $I^+$ ($v \to \infty$) these each approach a constant value. Extracting $\langle T_{tt} \rangle$ for each case in the usual manner and recalling that the energy density of $(1 + 1)$ dimensional radiation is $\pi T^2/6$, gives temperatures for each of these three cases consistent with (64).

Note that in each case the black hole radiation temperature is dependent upon the ADM-mass $M$, in contrast to the situation in string-inspired dilaton theories of gravity [20, 21, 22] in which the temperature is independent of this quantity. The entropy may be calculated using the thermodynamic relation $dM = TdS$. In all cases the entropy varies logarithmically with the mass, a feature of $(1 + 1)$-dimensional gravity pointed out previously [17]. Hence Liouville black holes have a positive specific heat (since the temperature decreases with the mass) and it is possible for several such black holes to have a larger entropy than one large black hole whose total mass is the sum of the smaller masses, quite unlike the situation in $(3 + 1)$ dimensions.

The detailed circumstances under which this can occur have been investigated in reference [23]. As an example consider two systems, one consisting of a black hole of mass $M$, and the other consisting of two black holes of mass $M$ and $M - m$. The difference in entropy between these two systems is

$$\Delta S = \hat{K}\left[\ln\left(\frac{M}{M_0}\right) - \ln\left(\frac{M - m}{M_0}\right) - \ln\left(\frac{m}{M_0}\right)\right]$$

$$= \hat{K}\ln\left(\frac{M - m}{M - m}\right)$$

(69)

where $M \geq m \geq M_0$ and $\hat{K}$ is a constant whose value changes depending upon which of the three solutions in (64) is under consideration. The two-hole system will have a smaller entropy than the one-hole system provided

$$M < \frac{m^2}{m - M_0}$$

(70)

The constant $M_0$ is a constant of integration which corresponds to the minimal mass a black hole system may have; it is a fundamental gravitational length scale somewhat analogous to the Planck mass [17].

Results for each of the solutions are given in the table below.
\[ \phi = \frac{2\pi}{\eta_0} \ln(x/x_0) + \phi_0 \]

\[ \psi = -p \ln(2Mx) + \psi_0 \]

Parameters

\[ p = \frac{4\pi}{\xi - 1} = 8\pi G \left(\frac{a \gamma + b}{a^2 + 4\pi G b}\right) \]

\[ B = \Lambda \left[ \frac{(b+4\pi G \gamma^2)/(4\pi G b + 8\pi G a \gamma - a^2)}{(a^2 + 4\pi G b)^2} \right] \]

\[ C = \frac{2\pi G \Lambda}{M^2(a + 4\pi G \gamma)} \]

\[ D = \pi G \Lambda \left( \frac{a}{4\pi G \gamma - a} \right) \]

\[ \mathcal{M} = \frac{8\pi G \gamma}{4G a(a^2 + 4\pi G b)} \left( 1 - \frac{8\pi G \gamma}{a} \right) \]

\[ \frac{M}{8\pi G \gamma} \left( 1 - 4\pi G \gamma/a \right) \]

Table I

| \( \alpha(x) \) | \( 2Mx - Bx_0^2 \left( \frac{1}{x_0^2} \right) \) | \( 1 - Ce^{-2M(x-x_0)} \) | \( 2Mx \ln(2Dx_0) \) |
| \( \phi \) | \( \frac{M}{\Lambda} (x - x_0) + \phi_0 \) | \( \frac{1}{2\pi} \ln \left( \frac{4\pi G}{M} x \right) \) | \( -2M \ln \left( \frac{4\pi G}{M} x \right) + \psi_0 \) |
| \( \psi \) | \( -p \ln(2Mx) + \psi_0 \) | \( 2Mx + \psi_0 \) | \( -2M \ln \left( \frac{4\pi G}{M} x \right) + \psi_0 \) |

Black hole Criteria

\( (\Lambda > 0) \)

\( (\gamma > 0) \)

If \( M > 0 \)

\[ -\frac{3G}{4} b < -4 \]

\[ 10\bar{a}^2 - \bar{a}^3 + 32\bar{a} < 8\bar{b} \]

\( M > 0 \) always

\( \bar{a} > 8 \) or

\( \bar{a} < -4 \)

\( 0 > a > 4\pi \gamma \)

If \( M < 0 \)

\[ 0 < \bar{a} < 4 \& \bar{b} > 4 \]

\[ 10\bar{a}^2 - \bar{a}^3 + 32\bar{a} < 8\bar{b} < 2\bar{a}^2 - 16\bar{a} \]

Horizon

\[ x_H = \frac{M}{\Lambda} \zeta_1 \]

\[ x_H = \frac{a - 8\pi G \gamma}{16\pi G \Lambda M a} \ln \left( \frac{A}{(4\pi G M)^2} \right) \zeta_2 \]

\[ x_H = -\frac{2M}{\Lambda} e^{2a \phi_H} \]

Temperature

\[ 2K\mathcal{M} \]

\[ 4G\mathcal{M} \left( \frac{a}{a - 8\pi G \gamma} \right) \]

\[ 4G\mathcal{M} \left( \frac{a}{a - 4\pi G \gamma} \right) \]

Entropy

\[ \frac{1}{2\pi} \ln \left( \frac{\mathcal{M}}{\mathcal{M}_0} \right) \]

\[ \frac{a - 8\pi G \gamma}{4G a} \ln \left( \frac{\mathcal{M}}{\mathcal{M}_0} \right) \]

\[ \frac{a - 4\pi G \gamma}{4G a} \ln \left( \frac{\mathcal{M}}{\mathcal{M}_0} \right) \]

where in the table

\[ \dot{K} = \frac{a(8\pi G a \gamma + 4\pi G b - a^2)}{a^3 - 10\pi G \gamma a^2 + 32(\pi G \gamma)^2 a + 8(\pi G)^2 b \gamma} \]

\[ \zeta_1 = 4\pi G \dot{K} e^{2a \phi_H} \frac{b + 4\pi G \gamma^2}{a^2 + 4\pi G b} \]

\[ \zeta_2 = \frac{(a - 8\pi G \gamma)^2 e^2}{4a(4\pi G \gamma + a)} \]
From table I one can construct the following scenario for radiating Liouville black holes. As the hole radiates its mass $M$ decreases at a rate

$$\frac{dM}{dt} = -\frac{\pi}{6}T^2 = -\kappa M^2$$

(71)

where $\kappa$ is a function of the coupling constants $a$, $b$ and $\gamma$ that depends upon which of the solutions (A)–(C) is under consideration. This implies that a black hole of initial mass $M_i$ decays at a rate

$$M(t) = \frac{M_i}{1 + \kappa M_i t}$$

(72)

lowering its temperature and entropy. Superficially it appears that the black hole is long-lived, since it takes infinitely long for the mass to radiate completely away. However at a finite time $t_f = (M_i/M_0 - 1)/(\kappa M_i)$ the mass approaches the constant $M_0$ at which point the entropy vanishes and this semi-classical picture no longer applies.

For solutions (A) and (C) the horizon moves closer to the singularity at $x = 0$; this (proper) distance eventually becomes comparable to the Compton wavelength of a body of mass $M$, after which the solution breaks down and quantum effects due to the Liouville field can no longer be ignored. Setting $M_0$ to be the mass at which the extremal proper length from the horizon to the singularity is equal to the Compton wavelength of a particle of mass $M_0$, i.e.

$$\int_0^{x_H} \frac{dx}{\sqrt{|\alpha(x)|}} = \frac{\hbar}{M_0}$$

(73)

yields

$$M_0 = \sqrt{G\hbar \left( \frac{h\Lambda}{\zeta} \right) \frac{\Gamma \left( \frac{p}{2(p-1)} \right)}{\Gamma \left( \frac{2p-1}{2(p-1)} \right)}}$$

for (A)

(74)

$$M_0 = \sqrt{G\hbar \left( \frac{4\hbar|\Lambda|a}{(a - 4\pi G\gamma)} \right) e^{-a\phi_H}}$$

for (C)

(75)

where

$$\hat{\zeta} = \left( 8\pi G\gamma a + 4\pi Gb - a^2 \right) \left( \pi Gb + (2\pi G\gamma)^2 \right) e^{2a\phi_H}$$

(76)
and the dependence on \( \hbar \) has explicitly been included. Note that \( G\hbar \) is dimensionless and units have been chosen so that the speed of light is unity.

Solution (B) has qualitatively different behaviour. For a given value of \( x_0 \), \( x_H \) is minimized at

\[
M_m = e \frac{a - 8\pi G\gamma}{8\pi Ga} \sqrt{\frac{2\pi Ga}{4\pi G\gamma + a}}
\]

(77)

where \( e = 2.71828... \). As the hole radiates, the mass \( M \) will decrease to this value, the location of the horizon also decreasing from its initial value. The extremal proper length varies inversely with \( M \) and so (73) implies

\[
a(1 - 32G\hbar) \geq 8\pi G\gamma
\]

(78)

which provides another constraint on the coupling parameters \( a \) and \( \gamma \), but does not provide a value for \( M_0 \). As \( M \) falls below \( M_m \), the location of the horizon then begins to increase without bound, and the black hole comes to dominate the entire spacetime. Presumably quantum gravitational effects come into play before this takes place, a condition which can be realized by requiring that the Compton wavelength of a body of mass \( M \) always be smaller than the proper length from the horizon to the singularity for a body of mass \( M_m \) so that

\[
\int_0^{x_H(M_m)} \frac{dx}{\sqrt{|a(x)|}} \geq \frac{\hbar}{M}
\]

(79)

or

\[
M \geq \frac{e\hbar}{\pi} \sqrt{\frac{2\pi G\Lambda a}{4\pi G\gamma + a}} \equiv M_0
\]

(80)

which permits a definition of \( M_0 \).

6 Quantum Corrections

The conservation of \( \langle O | T_{\mu\nu}^c | O \rangle \) combined with the trace anomaly

\[
\langle O | T_{\mu}^c \mu | O \rangle = -\frac{c_M h}{24\pi} R
\]

(81)
(which follows from (66)) yields

\[
\begin{align*}
\langle O | T^c_{+-} | O \rangle &= \frac{1}{12\pi} \partial_+ \partial_- \rho \\
\langle O | T^c_{\pm\pm} | O \rangle &= \frac{1}{12\pi} \left((\partial_{\pm}\rho)^2 \partial_{\mu}\rho + \hat{t}_{\pm}(x_{\pm})\right)
\end{align*}
\] (82)

where \( \hat{t}_{\pm}(x_{\pm}) \) depend upon the choice of boundary conditions.

Hence incorporation of the conformal stress-energy to one-loop order modifies (13 – 16) to

\[
\begin{align*}
\partial_+ \partial_- (\psi + 2\rho) &= 0 \quad (83) \\
\partial_+ \partial_- \rho &= \frac{\pi G \Lambda (a\gamma + b)}{b(1 - \frac{c_M h G}{3}) + 4\pi G \gamma^2} e^{2(\rho - a\phi)} \quad (84) \\
\partial_+ \partial_- \phi &= \frac{\Lambda (a(1 - \frac{c_M h G}{3}) - 4\pi G \gamma)}{4(b(1 - \frac{c_M h G}{3}) + 4\pi G \gamma^2)} e^{2(\rho - a\phi)} \quad (85)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} (\partial_{\pm}\psi)^2 - \partial_{\pm}^2 \psi + 2\partial_{\pm} \rho \partial_{\pm} \psi + 8\pi G (b(\partial_{\pm}\rho)^2 - \gamma \partial_{\pm}^2 \phi + 2\gamma \partial_{\pm} \rho \partial_{\pm} \phi) \\
+ 2 \frac{c_M h G}{3} \left((\partial_{\pm}\rho)^2 \partial_{\mu}\rho + \hat{t}_{\pm}\right) + T^c_{\pm\pm} = 0 \quad (86)
\end{align*}
\]

where \( T^c_{\mu\nu} \) is the classical part of the stress-energy of the conformal matter. There is also an equation of motion for the conformal matter fields; for example, if there are \( N \) scalar fields \{\( F_I \)\}, then

\[
T^c_{\mu\nu} = \frac{1}{8\pi} \sum_{I=1}^{N} \left( \nabla_\mu F_I \nabla_\nu F_I - \frac{1}{2} g_{\mu\nu} (\nabla F_I)^2 \right) \quad (87)
\]

and

\[
\partial_+ \partial_- F_I = 0 \quad (88)
\]

are the equations of motion of the \( F \)-fields, valid to one-loop. Solutions to equations (83) (86) incorporate the one-loop back-reaction of the conformal stress-energy into the spacetime geometry. The classical stress-energy tensor \( T^c_{\mu\nu} \) has no effect on the evolution of the metric/Liouville system; rather it modifies the evolution of the dilaton field \( \psi \) via (86). Its quantum corrections (from (81)) are easily deduced: a brief inspection of this system of equations
indicates that (83–86) are respectively identical to (13–16) but with the gravitational constant $G$ renormalized to

$$G_R = \frac{G}{1 - \frac{c_M h G}{3}}$$  \hspace{1cm} (89)$$

and the $\psi$-field renormalized to

$$\psi_R = \frac{1}{1 - \frac{c_M h G}{3}} \psi + 2 \rho \frac{c_M h G}{3 - c_M h G}.$$  \hspace{1cm} (90)$$

Hence quantum corrections due to conformally coupled matter map the metric

$$ds^2 = -\alpha(x; G) dt^2 + \frac{dx^2}{\alpha(x; G)}$$  \hspace{1cm} (91)$$

to

$$ds^2 = -\alpha(x; G_R) dt^2 + \frac{dx^2}{\alpha(x; G_R)}$$  \hspace{1cm} (92)$$

an expression which fully takes into account the back-reaction to one-loop order. As a consequence, the solutions (34), (37) and (40) are robust: the parameters $p, B, D$ and $\mathcal{M}$ become functions of $c_M$, and a solution with a given set of these parameters is mapped into solution with another set of these parameters.

Specifically, for the solution (34) (case (A))

$$p_R = \frac{8\pi G (a\gamma + b)}{a^2 (1 - \frac{c_M h G}{3}) + 4\pi G b}$$  \hspace{1cm} (93)$$

$$x_{HR} = M \frac{b (1 - \frac{c_M h G}{3}) + 4\pi G \gamma^2 (4\pi G b + 8\pi G a \gamma - a^2 (1 - \frac{c_M h G}{3})) e^{2a\phi_H}}{2\Lambda (a^2 (1 - \frac{c_M h G}{3}) + 4\pi G b)^2}$$

$$\mathcal{M}_R = M \frac{8\pi G \gamma (\pi G b + 4\pi G a \gamma) + a^3 (1 - \frac{c_M h G}{3})^2 - 10\pi G a^2 (1 - \frac{c_M h G}{3})}{4\pi G a (a^2 (1 - \frac{c_M h G}{3}) + 4\pi G b)}$$

and so (for Liouville fields with $b < 0$) the parameter $p$ increases, decreasing the parameter range for solutions obeying the flatness condition. Although $x_H$ increases as $c_M h \leq 3/G$ increases, the extremal proper length from the singularity to the horizon (slowly) decreases with increasing $c_M h < 3/G$; back-reaction effects 'shrink' the size of the black hole.
For the solution (37) (case (B)) (which now holds if \( b = -\frac{a^2}{4\pi G}(1 - \frac{c_M hG}{3}) \))

\[
x_{HR} = \frac{1}{2M} \ln \left( \frac{2\pi G\Lambda a}{M^2(4\pi G\gamma + a(1 - \frac{c_M hG}{3}))} \right) + x_0
\]

\[
\mathcal{M}_R = \frac{M}{8\pi Ga} (a(1 - \frac{c_M hG}{3}) - 8\pi G\gamma)
\]

and so the horizon moves outward and the mass decreases provided \( a > 4\pi G\gamma \). There is no effect on the extremal proper length from the singularity to the horizon as it varies like \( 1/M \). Note that the mass goes negative for \( c_M h > 3/(1 - \frac{8\pi G\gamma}{a}) \).

Finally for case (C) (eq. (40)), for which \( b = \frac{a^2}{4\pi G}(1 - \frac{c_M hG}{3}) - 2a\gamma \) (and \( \Lambda < 0 \))

\[
x_{HR} = \frac{M}{2\pi G|\Lambda|a} \left( a(1 - \frac{c_M hG}{3}) - 4\pi G\gamma \right) e^{2a\phi_H}
\]

\[
\mathcal{M}_R = \frac{M}{8\pi G} \left( 1 - \frac{c_M hG}{3} - \frac{4\pi G\gamma}{a} \right)
\]

and the effects of the back-reaction in this case are to decrease the location of both the horizon and the mass. The extremal proper length (which here varies as \( \sqrt{x_H} \)) therefore also decreases for positive mass black holes.

The case \( c_M hG = 3 \) may also be treated, provided that \( \gamma \neq 0 \). In this case

\[
p_R = 2(1 + a\gamma/b)
\]

\[
x_{HR} = \frac{\gamma^2 2a\gamma + b}{\Lambda b^2}
\]

\[
\mathcal{M}_R = \frac{M\gamma(b + 4a\gamma)}{2ab}
\]

In this case it is straightforward to show that asymptotically flat black hole solutions of positive mass exist provided

\[
p < 1 \quad \text{and} \quad M > 0
\]

If \( \Lambda > 0 \) : \( a\gamma < 0 \quad 4|a\gamma| > b > 2|a\gamma| \) \hspace{1cm} (98)

If \( \Lambda < 0 \) : \( a\gamma > 0 \quad -4a\gamma > b \)

\( a\gamma < 0 \quad -2a\gamma > b \)

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or

\[
1 < p < 2 \quad \text{and} \quad M < 0
\]

If \( \Lambda < 0 \)
\[
b < 0 \quad 4a\gamma > |b| > 2a\gamma
\]

If \( \Lambda > 0 \)
\[
a\gamma > 0 \quad b > -2a\gamma
\]
\[
a\gamma < 0 \quad b > -4a\gamma
\]

(99)

One can include quantum gravitational effects by considering a functional integral over the field configurations of the metric, \(\psi\), and \(\phi\) fields [24]. The path integral is

\[
Z = \int \frac{\mathcal{D}g}{V_{GC}} \mathcal{D}\psi \mathcal{D}\phi \mathcal{D}\Phi e^{-\left(S[\psi, g, \phi] + S_M[\Phi]\right)/\hbar}
\]

(100)

where \(S = S[\psi, g, \phi] + S_M[\Phi]\) is the Euclideanization of the action (8), with \(S_M\) being the part of the action incorporating additional matter fields \(\{\Phi\}\). The volume of the diffeomorphism group, \(V_{GC}\), has been factored out.

Making the same scaling assumption as in refs. [24, 10] about the functional measure yields

\[
Z = \int [\mathcal{D}\tau] \mathcal{D}g \mathcal{D}\phi \mathcal{D}b \mathcal{D}g \mathcal{D}\psi \mathcal{D}g \mathcal{D}\phi \mathcal{D}\Phi e^{-\left(S[\psi, \rho, \phi] + S_{gh} + S_M[\Phi] + \tilde{S}[\rho, \tilde{g}]\right)/\hbar}
\]

(101)

where \([\mathcal{D}\tau]\) represents the integration over the Teichmuller parameters and

\[
\tilde{S}[\rho, \tilde{g}] = \frac{\hbar}{8\pi} \int d^2x \sqrt{\tilde{g}} \left( \tilde{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - Q \rho \tilde{R} \right)
\]

(102)

is the Liouville action with arbitrary coefficients where \(g_{\mu\nu} = e^{\beta\rho} \hat{g}_{\mu\nu}\), and \(\hat{R}\) and \(\nabla^2\) are respectively the curvature scalar and Laplacian of the metric \(\hat{g}\).

\(S_{gh}\) is the action for the ghost fields \(b\) and \(c\).

The action \(S_{\text{TOT}} = \tilde{S} + S[\psi, g, \phi] + S_{gh} + S_M\) may be rewritten as

\[
S_{\text{TOT}} = S_{gh} + S_M + \frac{1}{8\pi G} \int d^2 x \sqrt{\hat{g}} \left[ -\frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - 8\pi G b \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right. \]

\[
- (\psi + 8\pi G^2 \gamma \phi)(\hat{R} - \beta \nabla^2 \rho) + G \hbar (\hat{g}^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - Q \rho \hat{R}) - 8\pi G A e^{\beta\rho - 2a\phi}
\]

(103)
where the cosmological constant has been renormalized to zero and the coupling $e^{-2a\phi}$ has been gravitationally dressed.

The approach is then to determine the parameters $\beta$ and $Q$ from the requirement that the conformal anomaly vanish and that $e^\beta \rho - 2a\phi$ is a conformal tensor of weight $(1,1)$. Upon rescaling the fields $\psi$, $\rho$ and $\phi$ so that

$$\tilde{\rho} = \sqrt{h + \frac{\beta^2}{2G} + 2\pi \frac{(\beta \gamma)^2}{b}} \rho$$
$$\tilde{\psi} = \frac{1}{\sqrt{2G}} (\psi + \beta \phi)$$
$$\tilde{\phi} = \sqrt{8\pi G |b|} (\phi + \frac{\beta \gamma}{2b} \rho)$$

(103) becomes

$$S_{\text{TOT}} = \frac{1}{8\pi} \int d^2x \sqrt{\tilde{g}} \left\{ \tilde{g}^{\mu\nu} \partial_\mu \tilde{\rho} \partial_\nu \tilde{\rho} - \tilde{g}^{\mu\nu} \partial_\mu \tilde{\psi} \partial_\nu \tilde{\psi} - \text{sgn}(b) \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \frac{2a}{\sqrt{8\pi |b|}} \right\} + S_{gh} + S_M .$$

The coefficients in front of the $\tilde{\psi}$ and $\tilde{\phi}$ terms are due to the signs of the kinetic energy terms in (1). They may be dealt with by rescaling $\tilde{\psi} \to i\tilde{\psi}$ and, if $b > 0$, $\tilde{\phi}$ to $i\tilde{\phi}$ so that the functional integral converges.

The action in (107) may now be analyzed using conformal field theoretic techniques. The fields $\tilde{\phi}$, $\tilde{\psi}$ and $\tilde{\rho}$ have propagators

$$< \tilde{\phi}(z) \tilde{\phi}(w) > = -\ln(z - w) = < \tilde{\psi}(z) \tilde{\psi}(w) >$$

(106)

and the contribution of the matter and ghost fields is as usual. Hence it is straightforward to compute the total central charge

$$c_{\text{TOT}} = 1 + 3 \left( \frac{Q - \frac{\beta}{2h} - 4\pi \sqrt{2\pi h}}{1 + \frac{\beta^2}{2h}} + 1 - \frac{6}{Gh} + 1 - 24 \frac{\pi \gamma^2}{b h} - 26 + c_M \right)$$

(107)
where \( \hat{G} \equiv \frac{bG}{b^2 + 4G\gamma} \). The requirement that \( e^{\beta \rho - 2a\phi} \) be a conformal tensor of weight (1,1) yields

\[
-\frac{1}{2} \beta \Upsilon \frac{1}{1 + \frac{\beta^2}{2G\hbar}} (Q - \frac{\beta}{G\hbar} + \beta \Upsilon) + \Delta_\phi - 1 = 0 \tag{108}
\]

where \( \Upsilon \equiv 1 - \frac{a\gamma}{2b} \) and

\[
\Delta_\phi = \frac{1}{2} \frac{a}{8\pi b} (8\pi \gamma + ah) \tag{109}
\]

is the conformal dimension of the Liouville field.

Equation (108) yields the constraint

\[
Q^2 \geq 8(1 - \Delta_\phi)(1 + \frac{1 - \Delta_\phi - \Upsilon}{\Upsilon^2 G\hbar}) \tag{110}
\]

since \( \beta \) must be real. Setting \( c_{\text{TOT}} = 0 \) and using (108) gives

\[
\beta^2 = \frac{6\Upsilon - 12 + \hat{G}\hbar(\Upsilon(23 - c_M) - 12) \pm \sqrt{\Omega}}{\Upsilon(c_M - 23 + 6\hat{G}\hbar\Sigma^2 + 12)} \tag{111}
\]

where

\[
\Omega \equiv (\Upsilon^2((c_M - 23)\hat{G}\hbar - 6) + 24\Upsilon(1 + \hat{G}\hbar\Sigma) - 24)((c_M - 23)\hat{G}\hbar - 6) \tag{112}
\]

and \( \Sigma \equiv \Upsilon - \frac{1 - \Delta_\phi - \Upsilon}{\Upsilon^2 G\hbar} \).

Equation (111) determines \( \beta = \beta(G; c_M) \) in terms of \( c_M, G \) and the coupling constants \( a, b \) and \( \gamma \). As \( \beta \) must be real and positive, this will put constraints on \( a, b \) and \( \gamma \) in terms of \( c_M \) and \( G \). Choosing \( \hat{g}_{+-} = -\frac{1}{2} \) (analogous to (12)) one finds that the classical equations of motion (13 – 16) become the same as (83 – 86) with the replacements

\[
\rho \rightarrow \hat{\rho} = \frac{2}{\beta} \rho \quad \text{and} \quad c_M \rightarrow -\frac{12}{\beta^2} \tag{113}
\]

and so quantum gravitational effects map the classical metric (11) to

\[
ds^2 = e^{2\hat{\rho}(G)} dx_+ dx_- \tag{114}
\]

where

\[
G_R = \frac{G}{1 + \frac{4G\hbar}{\beta^2}} \tag{116}
\]

is now the renormalized gravitational constant.
7 Discussion

Liouville fields coupled to two-dimensional gravity provide a rich theoretical laboratory for investigating the puzzles of black hole radiation. In the present paper the most general set of exact solutions to the field equations associated with the action (1) have been obtained; a wide class of these solutions correspond to asymptotically flat two-dimensional black holes of positive mass. In contrast to the situation for the string-theoretic black hole [25], the field equations yield an exact solution (to one-loop order) to the back-reaction problem for both conformally coupled matter and for the situation in which both the metric and Liouville field are quantized. In each case quantum effects map a solution with coupling constant $G$ to another solution with $G \rightarrow G_R$.

The above considerations suggest the following scenario for the evaporation of Liouville black holes. For definiteness, consider the situation when $b = -\frac{\beta^2}{4\pi G}$ and $\gamma = 0$ [12]. When quantum effects are taken into account, the classical spacetime described by (37) is modified to that described by (34) where

$$\alpha_Q = 2\bar{M}x - \frac{4\pi GA}{(1 + \frac{\beta^2}{4\bar{G}h})(1 + \frac{\beta^2}{2\bar{G}h})} \frac{2}{x_0} \left(\frac{x}{x_0}\right)^{-\frac{\beta^2}{2\bar{G}h}}$$

$$\phi_Q = \left(1 + \frac{\beta^2}{4\bar{G}h}\right) \ln\left(\frac{x}{x_0}\right) \quad \text{and} \quad \psi_B = \frac{\beta^2}{2\bar{G}h} \ln\left(\frac{x}{x_0}\right) + \psi_0$$

(117)

where $\beta$ is given by (111), $\bar{M} = \frac{M}{1 + \frac{\beta^2}{(2\bar{G}h)}}$ and the subscript “Q” denotes the fact that the fully quantized back-reaction has been included. The reality condition on $\beta$ yields, from (111)

$$6 > \bar{G}h(c_M - 23) > 6(1 - (\bar{G}h)^2)$$

(118)

thereby constraining the range of values of the matter central charge in terms of the dimensionless coupling $\bar{G}h$.

The spacetime described by the metric in (117) is that of an asymptotically flat positive mass black hole for large positive $x$. It has a singularity in the curvature, dilaton and Liouville fields at $x = 0$. The entropy is

$$S = \frac{(1 + \frac{4\bar{G}h}{\beta^2})^2}{1 + \frac{2\bar{G}h}{\beta^2}} \ln\left(\frac{\mathcal{M}_Q}{\mathcal{M}_0}\right)$$

(119)
and the temperature $T = \frac{M}{2\pi}$.

The ADM-mass is now

$$M_Q = \frac{M}{8\pi G} \frac{(1 + 4\frac{G\hbar}{\beta^2})^2}{1 + 2\frac{G\hbar}{\beta^2}}$$

(120)

and is manifestly positive. The inclusion of the quantum stress-energy results in a larger ADM-mass. This mass will decrease with time as $M_Q \sim 1/t$, decreasing both the location $x_H$ of the horizon and the maximal proper distance from the horizon to the singularity. Eventually this distance (the “size” of the event horizon) becomes comparable to the Compton wavelength as discussed in section 5. Since as $M_Q \to M_0$ the entropy tends to zero for finite temperature, at this point the thermodynamic description breaks down and higher-loop corrections become important.

What effect might these have? Previous work [4] suggests that quantum vacuum energies $\Lambda_Q$ (renormalized to zero to this order) will modify the temperature to $T \sim \sqrt{M^2 - \frac{\Lambda_Q}{2}}$, so that entropy becomes

$$S \sim \ln \left( \frac{\sqrt{M^2 - \frac{\Lambda_Q}{2}} + M}{\tilde{M}_0} \right)$$

where $\tilde{M}_0$ is a constant comparable in magnitude to $M_0$. In this case the black hole slowly cools off to a zero temperature remnant, leaving behind a global event horizon with its requisite loss of quantum coherence. Whether or not higher-loop effects will have similar consequences is a subject for further investigation.

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**Figure Captions**

**Fig. 1** Shaded areas correspond to the allowed regions in $(\tilde{a}, \tilde{b})$ parameter space for $M > 0$.

**Fig. 2** Shaded areas correspond to the allowed regions in $(\tilde{a}, \tilde{b})$ parameter space for $M < 0$. 