Maximum for Ginzburg-Landau fields

David Belius, Wei Wu

June 13, 2018

Abstract

We study two dimensional massless field in a box with potential $V(\nabla \phi(\cdot))$ and zero boundary condition, where $V$ is any symmetric and uniformly convex function. Naddaf-Spencer and Miller proved the macroscopic averages of this field converge to a continuum Gaussian free field. In this paper we prove the distribution of local marginal $\phi(x)$, for any $x$ in the bulk, has a Gaussian tail. We further characterize the leading order of the maximum and dimension of high points of this field, thus generalize the results of Bolthausen-Deuschel-Giacomin and Daviaud for the discrete Gaussian free field.

1 Introduction

1.1 Model

This paper studies the extreme values of certain two-dimensional (lattice) gradient Gibbs measures (also known as the Ginzburg-Landau model), with Hamiltonian given by convex gradient perturbation of the Gaussian free field. Take a nearest neighbor potential $V \in C^2(\mathbb{R})$ that satisfies:

$$V(x) = V(-x),$$
$$0 < c_- \leq V''(x) \leq c_+ < \infty,$$

where $c_-, c_+$ are positive constants.

Let $D_N := [-N, N]^2 \cap \mathbb{Z}^2$ and $\partial D_N$ consist of the vertices in $D_N$ that are connected to $\mathbb{Z}^2 \setminus D_N$ by some edge. The Ginzburg-Landau Gibbs measure on $D_N$ with zero boundary condition is given by

$$d\mu = Z_N^{-1}\exp \left[-\sum_{x \in D_N} \sum_{i=1}^2 V(\nabla_i \phi(x))\right] \prod_{x \in D_N \setminus \partial D_N} d\phi(x) \prod_{x \in \partial D_N} \delta_0(\phi(x)),$$

where $\nabla_i \phi(x) = \phi(x + e_i) - \phi(x)$ for $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and $Z_N$ is the normalization constant. The Ginzburg-Landau model is a natural generalization of the discrete Gaussian free field (DGFF). One particular non-Gaussian example of interest is $V(x) = x^2/2 + a \cos x$, which represents lattice dipole gas with activity $a$. 
1.2 Results

Our main result concerns the maximum of the Ginzburg-Landau field $\phi$ in $D_N$. For potential $V(\cdot)$ satisfying (1.1) and (1.2), the Brascamp-Lieb inequality (see Lemma 2.2) implies that with high probability, $\frac{\sup_{x \in D_N} \phi(x)}{\log N}$ is uniformly bounded above and below by two constants depending only on $c_+$ and $c_-$ (see [DG00]). We prove that this random variable indeed satisfies a law of large numbers, with tail bounds given by (1.4) and (1.5) below.

**Theorem 1.1** Let $\phi$ be sampled from the Gibbs measure (1.3). Assume the potential $V(\cdot)$ satisfies (1.1) and (1.2). Then there is a constant $g = g(c_+, c_-)$, such that

$$\frac{\sup_{x \in D_N} \phi(x)}{\log N} \rightarrow 2\sqrt{g} \text{ in probability.}$$

Moreover, we have the following tail bounds. For any $\delta > 0$, there is some $C = C(\delta) < \infty$, such that

a) $$\mathbb{P}\left( \sup_{x \in D_N} \phi(x) \geq (2\sqrt{g} + \delta) \log N \right) \leq C(\delta) N^{-\delta/\sqrt{g}}. \quad (1.4)$$

b) $$\mathbb{P}\left( \sup_{x \in D_N} \phi(x) \leq (2\sqrt{g} - \delta) \log N \right) \leq C(\delta) N^{-C\delta^{-1}}. \quad (1.5)$$

The reason that we define this limiting constant as $2\sqrt{g}$ will be clear from the explanation below. Indeed, $g$ is the so-called effective stiffness for the random surface model, and satisfies

$$\lim_{N \rightarrow \infty} \frac{\text{Var} \phi(0)}{\log N} = g. \quad (1.6)$$

The proof of (1.6) follows from the same (and simpler) argument as the proof of Proposition 1.3 below.

**Theorem 1.1** is known for the discrete Gaussian Free Field (see [BDG01]), but not for any other non-Gaussian random fields. We will summarize related results in Section 1.3 below.

Our next result studies the fractal structure of the sets where the Ginzburg-Landau field $\phi$ is unusually high. We say that $x \in D_N$ is an $\eta$–high point for the Ginzburg-Landau field if $\phi(x) \geq 2\sqrt{g}\eta \log N$. The next theorem generalizes the dimension of the high points for Gaussian free field, obtained by Daviaud in [Dav06].

**Theorem 1.2** Denote by $\mathcal{H}_N(\eta) = \{ x \in D_N : \phi(x) \geq 2\sqrt{g}\eta \log N \}$ the set of $\eta$–high points. Then for any $\eta \in (0, 1)$,

$$\frac{\log |\mathcal{H}_N(\eta)|}{\log N} \rightarrow 2 \left( 1 - \eta^2 \right) \quad (1.7)$$

in probability.
The main step to prove the upper bound (1.4) and the upper bound of (1.7) is the following pointwise tail bound for the Ginzburg-Landau field (1.3).

**Proposition 1.3** For all \( u > 0 \) and all \( x \in D_N \) we have

\[
P(\phi(x) \geq u) \leq \exp\left(-\frac{u^2}{2g \log \Delta} + o(u)\right),
\]

(1.8)

where \( \Delta = \text{dist}(x, \partial D_N) \).

For the class of potentials \( V(\cdot) \) that satisfy (1.1) and \( c_0 \leq V'' \leq 2c_0 \), for some \( c_0 > 0 \), the main result of [CS14] implies Proposition 1.3 for the infinite volume limit of the Gibbs measure (1.3).

Proposition 1.3 will be proved in Sections 3 and 4. Here we show how it directly implies the upper bound (1.4).

**Proof of (1.4).** If we pick \( \gamma_0 \) small enough then for \( x \in D_N \) such that \( \text{dist}(x, \partial D_N) \leq N^{\gamma_0} \) we have from the Brascamp-Lieb tail bound, Lemma 2.2, that

\[
P(\phi(x) \geq 2\sqrt{g} \log N) \leq \exp\left(-\frac{4g \log N}{\gamma_0 \log N}\right) \leq N^{-2}.
\]

Then a union bound shows that

\[
P\left(\max_{x: \text{dist}(x, \partial D_N) \leq N^{\gamma_0}} \phi(x) \geq 2\sqrt{g} \log N\right) \leq N^{-\gamma_0 - 1}.
\]

Fix this \( \gamma_0 \) and take any \( x \in D_N \) such that \( \text{dist}(x, \partial D_N) > N^{\gamma_0} \). Given any \( \delta > 0 \), applying Proposition 1.3 with \( u = (2\sqrt{g} + \delta) \log N \) yields

\[
P(\phi(x) \geq (2\sqrt{g} + \delta) \log N) \leq \exp\left(-\frac{2(\log N)^2}{\log \Delta} - \frac{2\delta (\log N)^2}{\sqrt{g} \log \Delta} + o(\log N)\right) \leq CN^{-2-2\delta/g + o(1)},
\]

for some \( C < \infty \). Therefore

\[
P\left(\max_{x: \text{dist}(x, \partial D_N) > N^{\gamma_0}} \phi(x) \geq 2\sqrt{g} \log N\right) \leq CN^{-2\delta/\sqrt{g} + o(1)},
\]

thus completing the proof of (1.4). 

1.3 Historical Survey

It is believed that the large scale behaviors of convex gradient fields resemble that of the Gaussian free field. Rigorous mathematical studies for convex perturbations of GFF (in particular, the special example called lattice dipole gas) were initiated by the renormalization group approach of [GK80], and further developed by [BY90], which confirm its correlation function behaves like a continuous GFF in the scaling limit. Renormalization group is a powerful tool to study gradient field models, but
it is only applicable in the perturbative case, i.e., when the potential is given by a small perturbation of Gaussian, and thus the Hessian of the Hamiltonian is close to identity. The non-perturbative approach allows one to study any convex gradient perturbations of GFF based on the Helffer-Sjostrand formula (see [HS94], [Hel02]) that represents the mean and covariance of such fields in terms of an elliptic operator (or, probabilistically, a random walk in dynamic random environment). We give here an incomplete list of references that study the scaling limits of gradient field models. The classification of the gradient Gibbs states on $\mathbb{Z}^d$ were proved by Funaki and Spohn [FS97]. Deuschel, Giacomin and Ioffe [DGI00] studied the large deviation principle of the macroscopic surface profile in a bounded domain, where they also introduce the random walk representation of the Helffer-Sjostrand formula. The central limit theorem for linear functionals of the gradient fields was first established by Naddaf and Spencer [NS97] for the infinite volume gradient Gibbs states with zero tilt (the corresponding dynamical CLT was proved in [GOS01]), and later by Miller [Mil10] for the gradient fields in bounded domains. It is also proved in [Mil11] that the level set for such gradient fields in a bounded domain (with certain Dirichlet boundary condition) converges to the chordal SLE(4), an example of the conformally-invariant random curve in the plane known as the Schramm-Loewner Evolution (for a survey on SLE, see e.g. [Law08]).

Although the macroscopic behavior of linear functionals of the gradient fields are now well understood, finer properties of the field, such as the behavior of its maximum, remain to be clarified. In the special case of GFF the knowledge of the maxima is now quite refined, because the Gaussian structure allows one to compare this directly with the maximum of a branching random walk, and also allows one to apply the Gaussian comparison theorems to reduce the maximum comparison to compare the variances. The best knowledge about the value of the recentered maximum is established by Bramson and Zeitouni [BZ12], where it is shown that

$$\mathbb{E} \sup_{x \in D_N} \phi(x) = 2\sqrt{g_0} \log N - \frac{3}{4} \sqrt{g_0} \log \log N + O(1), \text{ as } N \to \infty$$

with $g_0 = 2/\pi$. This improves an earlier result of Bolthausen, Deuschel and Giacomin [BDG01], which is analogous to our result. Further knowledge of the $O(1)$ term, and some spatial properties of the near maxima were shown by Ding and Zeitouni [DZ14], and Biskup and Louidor [BL13] further show that the random measure describing the location and height of the local maxima converges to a Poisson point process, with the marginal in space given by a version of the derivative martingale. Most results were extended to general log-correlated Gaussian fields by [DRZ15].

In the case of convex gradient fields that are non-Gaussian, the immediate branching structure and the comparison results fail, and very little is known about the behavior of their maximum. As mentioned in the beginning of the introduction, we find the leading order asymptotics of the maximum, thus generalizing the result in [BDG01] to convex potentials. We also give the volume of the high points in a bounded domain, that generalizing the result of Daviaud [Dav06] in the GFF case. These results are consistent with the conjecture that the level sets of the Ginzburg-Landau model with zero boundary condition converge to CLE(4), a collection of
conformally invariant random loops (see [SW12] for the definition of the CLE and how to construct a coupling with GFF).

Although the covariance of the Ginzburg-Landau model can be written as the (annealed) Green’s function for a random walk in dynamical random environment as in [DGI00], more general functionals of the fields are much more difficult to study. With additional bounded ellipticity assumption on $V$, it is proved by Conlon and Spencer [CS14] that for the infinite gradient-Gibbs states with zero slope, the moment generating function of $\phi(0) - \phi(x)$ is essentially contributed by the variance (up to a multiplicative constant). Their argument is based on the Helffer-Sjostrand formula and operator theory on weighted Hilbert space. This phenomenon is remarkable because it indicates the approximate Gaussianity holds in a strong (pointwise) sense. In this paper we remove the bounded ellipticity assumption, and we focus instead on the harmonic averages of the Ginzburg-Landau field. We apply the useful tool from [Mil11], that gives an approximate harmonic coupling of the Ginzburg-Landau field on a bounded domain with different boundary conditions. By a recursive decomposition argument, we manage to show that as we zoom in toward a point, the increment of the harmonic averaging field is distributed not far from Gaussian. Therefore, the height of the field at one site can be viewed as the end-point of a Markov random walk, such that in “most cases” the conditional increment of the walk is not far from Gaussian. This allows us to obtain the pointwise tail bound for the Ginzburg-Landau field, and thus the upper bound in Theorem 1.1. For the lower bound in Theorem 1.1 we use a modified second moment method based on harmonic coupling and the Gibbs property.

Finally we finish the introduction with the corresponding open question for dimer model. A (uniform) dimer model on $\mathbb{Z}^2$ can be thought of as an integer valued random surface $h(x), x \in \mathbb{Z}^2$. It is an integrable model with determinantal structure. It is shown in [Ken00] and [Ken01] that the height fluctuation $h(0) - h(x)$ has logarithmic variance, and moreover the rescaled height function converges weakly to GFF. A main conjecture in this field is that the level sets of the height function converges to CLE(4). Still, it would be very interesting to prove the maximum of the dimer height function satisfies Theorem 1.1. The method in the present paper does not apply directly because the harmonic coupling (see Section 2.3) have not yet been established for the dimer model.

### 1.4 Proof strategy

We now comment on the proofs of Theorems 1.1 and 1.2. We start by discussing the method used for the DGFF, that is for the special choice of potential

$$V(x) = x^2. \tag{1.9}$$

With this choice of $V$ the marginals of $\phi$ are exactly Gaussian. Furthermore the variance of $\phi(x)$ is at most $g_0 \log N + o(\log N)$ as $N \to \infty$ for all $x \in D_N$, where $g_0 = \sqrt{2/\pi}$ (see, e.g. [LL10]). One thus has the tail bound

$$P(\phi(x) \geq u) \leq c \exp \left( - \frac{u^2}{g_0 \log N + o(\log N)} \right), \forall x \in D_N, \text{ assuming } (1.9). \tag{1.10}$$
With \( u = (2\sqrt{g_0} + \delta) \log N \) one then has via a union bound over the \(|D_N| = (2N+1)^2\) points in \( D_N \) that
\[
P \left( \sup_{x \in D_N} \phi(x) \geq (2\sqrt{g_0} + \delta) \log N \right) \leq cN^2e^{-\frac{(2\sqrt{g_0} + \delta) \log N}{2g_0 \log N}} \to 0,
\]
again assuming (1.9). This proves (1.4), and a similar union bound gives the upper bound of Theorem 1.2 both under the assumption (1.9). Thus, for the Gaussian Free Field the upper bounds of Theorems 1.1 and 1.2 are almost trivial.

On the other hand, the lower bounds are not trivial even when (1.9) holds. A natural first try would be to apply the second moment method to the counting random variable
\[
Z = \sum_{x \in D_{N/2}} 1\{\phi(x) \geq (2\sqrt{g_0} - \delta) \log N\}.
\]
Since we only sum over points at macroscopic distance from the boundary it turns out that the left-hand side of (1.10) is actually of the order of the right hand side, so that an easy computation shows that \( E(Z) = N^{\delta+o(1)} \to \infty \). If the second moment \( E(Z^2) \) was asymptotic to \( E(Z)^2 \) then we could conclude that \( Z \geq cN^{\delta+o(1)} \) with high probability, thus proving the lower bound (1.5) in the case (1.9). However, the second moment explodes with respect to the first moment squared. By now it is well-understood that the origin of this phenomenon can be understood via an analogy to Branching Random Walk, where exactly the same phenomenon occurs. For branching random walks, it turns out the rightmost particle at time \( T \) stays slightly below a linear barrier during \( t \in [0, T] \). This indicates that we should replace \( Z \) by a truncated count
\[
\hat{Z} = \sum_{x \in D_{N/2}} 1\{X_l(x)-X_{l-1}(x) \in [\frac{1}{K}(2\sqrt{g_0} - \delta) \log N, \frac{1}{K}(2\sqrt{g_0} + \delta) \log N], \text{for } l=2,\ldots,K\}
\]
for some fixed large \( K \). For this \( \hat{Z} \) large enough \( K \), \( E(\hat{Z}^2) \) is indeed asymptotic to \( E(\hat{Z})^2 \) and a second moment argument yields \( \hat{Z} \geq 1 \) with high probability.

We would like to adapt the idea above to the case of a more general potential \( V \). For the upper bound the main challenge is when \( V \) is no longer quadratic, one loses the tail bound (1.10). The Brascamp-Lieb inequalities, which we recall below, do provide a tail bound but at the cost of replacing the constant \( g \) with a smaller one. Using such a bound with the union bound one can prove that the maximum is at most \( c \log N \) for some \( c > g \), but one can not obtain a sharp bound like in (1.4). For the thin region \( \{ x \in D_N : d(x, \partial D_N) < N^{\gamma} \} \) with sufficiently small \( \gamma \), one may hope that the Brascamp-Lieb inequalities, even though they lose a multiplicative constant in the variance, are strong enough to obtain an upper bound of \( 2\sqrt{g} \log N \) for the maximum, since the region has volume of order \( N^{1+\gamma} \ll |D_N| \).

Two main ingredients for our proof are the central limit theorem for macroscopic averages of \( \phi \) proved in [NS97] and [Mil11], and the approximate harmonic coupling stated in [Mil11] (see Theorem 2.5 below). The central limit theorem gives a right tail bound for macroscopic averages of \( \phi \), but it is not strong enough to give the distribution of \( \phi \) at a single point. Our solution to this difficulty is to consider mesoscopic harmonic averages, similar to the circle averages described above for
the GFF. We managed to obtain the Gaussian tail estimate for these harmonic averages at mesoscopic scales ($N^c$ for any $c > 0$, see Theorem 4.1). The proof is by a recursive decomposition argument repeatedly using the approximate harmonic coupling. By taking $c$ small enough this leads to the right tail bound for $\phi(x)$, for all $\{x \in D_N : d(x, \partial D_N) \geq N^\gamma\}$, where $\gamma$ can be an arbitrary positive number. A union bound combining the estimates above then yields the upper bound (1.4) for general potentials $V$.

For the proof of the lower bound, thanks to the Gaussian estimates Theorem 4.3, we use a second moment count for the increments of harmonic averages, similar to the argument for GFF.

2 Tools

2.1 Brascamp-Lieb inequality

One can bound the variances and exponential moments with respect to the Ginzburg-Landau measure by those with respect to the Gaussian measure, using the following Brascamp-Lieb inequality. Let $\phi$ be sampled from the Gibbs measure (1.3), with a nearest-neighbor potential $V \in C^2(\mathbb{R})$ that satisfies $\inf_{x \in \mathbb{R}} V''(x) \geq c_\gamma > 0$. Given $f \in \mathbb{R}^{D_N}$, we define

$$\langle \phi, f \rangle := \sum_{x \in D_N} \phi(x) f(x).$$

**Lemma 2.1 (Brascamp-Lieb inequalities [BL76])** Let $E_G$ and $\text{Var}_G$ denote the expectation and variance with respect to the DGFF measure (that is, (1.3) with $V(x) = x^2/2$). Then for any $f \in \mathbb{R}^{D_N}$,

$$\text{Var}_G \langle \phi, f \rangle \leq c_\gamma^{-1} \text{Var}_G \langle \phi, f \rangle,$$

$$E(\langle \phi, f \rangle - E \langle \phi, f \rangle)^{2k} \leq c_\gamma^{-k} E_G (\langle \phi, f \rangle - E \langle \phi, f \rangle)^{2k}, \text{ for } k \in \mathbb{N},$$

$$E[\exp(\langle \phi, f \rangle - E \langle \phi, f \rangle)] \leq \exp \left( \frac{1}{2} c_\gamma^{-1} \text{Var}_G \langle \phi, f \rangle \right).$$

The Brascamp-Lieb inequalities can be used to show the following a-priori tail bound for $\phi$.

**Lemma 2.2** There is a constant $c_{BL}$ such that

$$P(\phi(x) \geq u) \leq e^{-c_{BL} u^2 \log \text{dist}(x, \partial D_N)}, \text{ for } x \in D_N.$$

**Proof.** By Chebyshev’s inequality,

$$P(\phi(x) \geq u) \leq e^{-iu} E \exp(t \phi(x)).$$

Applying the Brascamp-Lieb inequality with $f(x) = \delta_x$, and using the fact that

$$\text{Var}_G \phi(x) = \sqrt{2/\pi} \log \text{dist}(x, \partial D_N) + O(1),$$

we obtain the desired tail bound.
we have
\[ P(\phi(x) \geq u) \leq \exp \left( -tu + \frac{t^2}{2} c^{-1} \sqrt{2/\pi} \log \text{dist}(x, \partial D_N) \right). \]
Optimizing over \( t \) then yields the result. ■

By doing a union bound over the \((2N + 1)^2\) points of \( D_N \) and setting \( u \gg \sqrt{1/c_{BL}} \log N \), so that the right-hand side in the bound is \( \ll N^{-2} \), one obtains an upper bound of the right order, but the constant in front of \( \log N \) is larger than the "true" one \( 2\sqrt{g} \).

2.2 Central limit theorem
We now state the central limit theorem for macroscopic averages of \( \phi \), proved in [Mil11] as a consequence of Theorem A in [NS97] and Theorem 2.5 below. For \( D \subset \mathbb{Z}^2 \), the Ginzburg-Landau measure on \( D \) with Dirichlet boundary condition \( f \) is defined by
\[
d\mu_f = Z^{-1} \exp \left[ -\sum_{x \in D} \sum_{i=1}^2 V(\nabla_i \phi(x)) \right] \prod_{x \in D \setminus \partial D} d\phi(x) \prod_{x \in \partial D} \delta_0(\phi(x) - f(x)). \tag{2.1}
\]

**Theorem 2.3** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^1 \) function, \( D \subset \mathbb{R}^2 \) be a smooth simply connected domain, \( D_N = D \cap \frac{1}{N}\mathbb{Z}^2 \) and \( \phi^f \) be sampled from the Ginzburg-Landau measure on \( D_N \) with boundary condition \( f \). Then for every \( \rho_N \) such that
\[
\sum_{x \in D_N} \rho_N(x) H(x) = 0, \quad \text{for any harmonic function } H : D_N \to \mathbb{R}, \tag{2.2}
\]
and \( \rho_N \to \rho \) for some \( \rho \in C^\infty_0(D) \), the linear functional
\[
N^{-1} \sum_{x \in D_N} \rho(x) \phi(x)
\]
converges in \( L^{2k} \), \( k \in \mathbb{N} \) to the random variable
\[
\int_D h^f(x) \rho(x) \, dx,
\]
where \( h^f \) is the (continuum) Gaussian free field on \( D \) with boundary condition \( f \).

A careful reader may notice the slight difference between the statement of Theorem 1.1 in [Mil11] and the one above. Indeed, (2.2) implies that for each \( N \), \( \rho_N \) can be written as the discrete Laplacian of some smooth function, and an integration by part yields Theorem 1.1 in [Mil11]. We choose to state the central limit theorem in this version because it can be directly applied in our proof.

We are mostly concerned with large deviation estimates, and therefore with moment generating functions. Thus we will actually use is the following Gaussian estimate on the moment generating function of macroscopic observables, which is proved by combining Theorem 2.3 with the Brascamp-Lieb inequality. Let \( B_r := \{ x \in \mathbb{R}^2 : |x| < r \} \).
We now claim
where
This yields (2.3), and thus finishes the proof.

**Proposition 2.4** For any \( \varepsilon, R, r > 0 \) such that \( \text{dist} (B_r, \partial D) > c_0 \cdot \text{diam} (D) \) for some \( c_0 > 0 \), and any smooth function \( f : \mathbb{R}^2 \to \mathbb{R} \) supported on \( A_{r, \varepsilon} = B_r \setminus B_{(1-\varepsilon)r} \), we have that

\[
\log E^{D_{R, 0}} \left[ \exp \left( t R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right) \right] = \frac{t^2}{2} \text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] + o_R (\varepsilon) + O (\varepsilon^2),
\]

where \( h \) is a zero boundary (continuum) GFF on \( D \), and \( g_D \) is the Dirichlet Green’s function on \( D \).

**Proof.** By Theorem 2.3 as \( R \to \infty \)

\[
\text{Var} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right] = \text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] + o_R \left( \text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] \right).
\]

A direct computation yields

\[
\text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] = \int_{A_{r, \varepsilon}} f (x) g_D (x - y) f (y) \, dy \, dx = O \left( \log (1 + \varepsilon) \right) = O (\varepsilon),
\]

therefore

\[
\text{Var} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right] = \text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] + o_R (\varepsilon).
\]

Using Taylor expansion and the fact that the distribution of \( \phi \) is symmetric, we can write

\[
E^{D_{R, 0}} \left[ \exp \left( t R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right) \right] = 1 + \frac{t^2}{2} \text{Var}_{\text{GFF}} \left[ \int_{A_{r, \varepsilon}} f (x) h (x) \, dx \right] + o_R (\varepsilon) + \sum_{k=2}^{\infty} \frac{t^{2k}}{(2k)!} \mathbb{E} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right]^{2k}.
\]

We now claim

\[
\sum_{k=2}^{\infty} \frac{t^{2k}}{(2k)!} \mathbb{E} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right]^{2k} = O (\varepsilon^2) \quad (2.3)
\]

By Brascamp-Lieb inequality for even moments, we have

\[
\mathbb{E} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right]^{2k} \leq c^{-k} \mathbb{E} \left[ R^{-1} \sum_{x \in D_R} \phi (x) f (x) \right]^{2k} \leq (2k - 1)!! c^{-k} \varepsilon^{2k} \quad (2.4)
\]

This yields (2.3), and thus finishes the proof. ■
2.3 Approximate harmonic coupling

All the Ginzburg-Landau fields satisfy the domain Markov property: conditioned on the values on the boundary of a domain, the field inside the domain is again a Discrete GFF resp. gradient field. For discrete GFF there is in addition a nice orthogonal decomposition. More precisely, the conditioned field inside the domain has the law of the discrete harmonic extension of the boundary value to the whole domain plus an independent copy of a zero boundary discrete GFF.

The Ginzburg-Landau measure is non-Gaussian, however, the result below by Jason Miller [Mil11] shows that this decoupling property still holds in an approximate sense.

Let $D \subset \mathbb{Z}^2$ be a simply connected domain of diameter $R$, and denote

$$D(r) = \{ x \in D : \text{dist} (x, \partial D) > r \}.$$

**Theorem 2.5 ([Mil11])**  Let $\Lambda$ be such that $f : \partial D \to \mathbb{R}$ satisfies $\max_{x \in \partial D} |f(x)| \leq \Lambda |\log R|^\Lambda$. Let $\phi$ be sampled from the Ginzburg-Landau measure (2.1) on $D$ with zero boundary condition, and $\phi^f$ be sampled from Ginzburg-Landau measure on $D$ with boundary condition $f$. Then there exist constants $c, \gamma, \delta > 0$, that only depend on $c_-, c_+$, so that if $r > cR^{1-\gamma}$ then the following holds. There exists a coupling $(\phi, \phi^f)$, such that if $\hat{\phi} : D(r) \to \mathbb{R}$ is discrete harmonic with $\hat{\phi}|_{\partial D(r)} = \phi^f - \phi|_{\partial D(r)}$, then

$$P(\phi^f - \phi \neq \hat{\phi} \text{ in } D(r)) \leq c(\Lambda) R^{-\delta}.$$

One immediate application of Theorem 2.5 that is also stated in [Mil11], shows the mean of Ginzburg-Landau field at one point in the bulk is approximately (discrete) harmonic.

**Theorem 2.6** Suppose the same conditions in Theorem 2.5 holds. Let $\phi^f, c, \gamma, \delta, D(r)$ be defined as in Theorem 2.5. For all $r > cR^{1-\gamma}$, and a discrete harmonic function $\hat{\phi} : D(r) \to \mathbb{R}$ with $\hat{\phi}|_{\partial D(r)} = E\phi^f|_{\partial D(r)}$, then

$$\max_{x \in D(r)} |E\phi^f(x) - \hat{\phi}(x)| \leq c(\Lambda) R^{-\delta}.$$

Theorem 2.5 allows to compare a field with non-zero boundary condition with one that has zero boundary condition. We will use repeatedly is the following consequence of the theorem. It applies to functions $\rho$ that are such that the integral of $\rho$ with respect to the harmonic extension of the boundary condition is always zero. Define

$$G = \left\{ \phi : \max_{x \in D} |\phi(x)| < (\log R)^2 \right\}.$$

**Lemma 2.7** There exists constants $\delta, \gamma > 0$ such that for any $r > R^{1-\gamma}$ and $\rho : D(r) \to \mathbb{R}$, that satisfies $\sum_{x \in D(r)} \rho(x) f(x) = 0$ for any $f$ harmonic in $D(r)$, we
have for $R$ large enough,
\[
\left| E^{D,f} \left[ \exp \left( R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right) 1_{g \cap C} \right] - E^{D,0} \left[ \exp \left( R^{-1} \sum_{x \in D_R} \rho(x) \phi(x) \right) 1_g \right] \right| \\
\leq 2 \exp \left( c \text{Var}_{GFF} \left( R^{-1} \sum_{x \in D_R} g(x) \phi(x) \right) \right) R^{-\delta},
\]
for some $c < \infty$.

**Remark 2.8** This lemma is useful if $\text{Var}_{GFF} \left( R^{-1} \sum_{x \in D_R} \rho(x) \phi(x) \right) \ll \delta \log R$.

**Proof.** Apply Theorem 2.5, there is an event $C$ with $P(C^c) \leq R^{-\delta_0}$, where $\delta_0$ is the constant $\delta$ in Theorem 2.5 and such that on $C$ we have $\phi^f - \phi = \hat{\phi}$ in $D(r)$. Therefore on $C$
\[
\sum_{x \in D_R} \rho(x) \phi^f(x) = \sum_{x \in D(r)} \rho(x) \phi^f(x) = \sum_{x \in D(r)} \rho(x) \phi(x) + \sum_{x \in D(r)} \rho(x) \hat{\phi}(x) \\
= \sum_{x \in D(r)} \rho(x) \phi(x).
\]
On $C^c$ we apply Holder inequality to obtain
\[
E^{D,f} \left[ \exp \left( R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right) 1_{g \cap C^c} \right] \\
\leq P(C^c)^{1/2} E^{D,f} \left[ \exp \left( 2R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right) \right]^{1/2} \\
\leq R^{-\delta_0/2} E^{D,f} \left[ \exp \left( 2R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) - E^{D,f} \left[ 2R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right] \right) \right]^{1/2} \\
\times \exp \left( E^{D,f} \left[ R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right] \right).
\]
By Brascamp-Lieb
\[
E^{D,f} \left[ \exp \left( 2R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) - E^{D,f} \left[ 2R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right] \right) \right] \\
\leq \exp \left( c \text{Var}_{GFF} \left( R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right) \right),
\]
and apply Theorem 2.6 yields
\[
\left| E^{D,f} \left[ R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right] \right| = \left| E^{D,f} \left[ R^{-1} \sum_{x \in D_R} \rho(x) \phi^f(x) \right] - R^{-1} \sum_{x \in D_R} \rho(x) \hat{\phi}(x) \right| \\
\leq R^{-\delta_0} \|\rho\|_{\infty}.
\]
Combining (2.5), (2.6) and (2.7), we have for $R$ large enough,

$$
\mathbb{E}^{D_R,f} \left[ \exp \left( R^{-1} \sum_{x \in D_R} \rho (x) \phi^f (x) \right) 1_{\mathcal{G}_\mathbb{C}^c} \right] \leq \exp \left( c \text{Var}_{\text{GFF}} \left( R^{-1} \sum_{x \in D_R} \rho (x) \phi^f (x) \right) \right) R^{-\delta_0 / 3}.
$$

And similarly,

$$
\mathbb{E}^{D_R,0} \left[ \exp \left( R^{-1} \sum_{x \in D_R} \rho (x) \phi (x) \right) 1_{\mathcal{G}_\mathbb{C}^c} \right] \leq \exp \left( c \text{Var}_{\text{GFF}} \left( R^{-1} \sum_{x \in D_R} \rho (x) \phi (x) \right) \right) R^{-\delta_0 / 3}.
$$

Take $\delta = \delta_0 / 3$ and notice the variance of linear functionals of Gaussian free field does not depend on boundary conditions, we finish the proof. ■

3 Harmonic averages

Our method to prove Proposition 1.3 is to apply Theorem 2.5 to study the bulk (harmonic) average of the Ginzburg-Landau field. Given $B \subset \mathbb{Z}^2$, $x \in B$ and $y \in \partial B$, we denote by $a_B (x, \cdot)$ the harmonic measure on $\partial B$ seen from $x$. In other words, let $S^x$ denote the simple random walk starting at $x$, and $\tau_{\partial B} = \inf \{ t > 0 : S [t] \in \partial B \}$, we have

$$
a_B (x, y) = \mathbb{P} (S^x [\tau_{\partial B}] = y).
$$

Given $v \in \mathbb{Z}^2$ and $R > r > 0$, let $B_R (v) = \{ x \in \mathbb{Z}^2 : |x_1 - v_1| + |x_2 - v_2| < R \}$, and $A_{r,R} (v) := B_R (v) \setminus B_r (v)$. And, the circle average of the Ginzburg-Landau field with radius $R$ at $v$ is given by

$$
C_R (v, \phi) = \sum_{y \in \partial B_R (v)} a_{B_R (v)} (v, y) \phi (y).
$$

For each $\epsilon, R > 0$, we take a smooth radial function $f^{R} : [(1-\epsilon) R, (1+\epsilon) R] \to \mathbb{R}$ such that $\sum_{r=(1-\epsilon) R}^{(1+\epsilon) R} f^{R} (r) = 1$. We further define

$$
X_R (v, \phi) = \sum_{r=(1-\epsilon) R}^{(1+\epsilon) R} f^{R} (r) C_r (v, \phi). \tag{3.1}
$$

The crucial object that we use below are the increments of the harmonic average process $X$. For $v \in D_N$, $R_1 > R_2 > 0$, we would like to study the increment

$$
X_{R_2} (v, \phi) - X_{R_1} (v, \phi) = \left( \sum_{r=(1-\epsilon) R_2}^{(1+\epsilon) R_2} f^{R_2} (r) - \sum_{r=(1-\epsilon) R_1}^{(1+\epsilon) R_1} f^{R_1} (r) \right) \sum_{y \in \partial B_r (v)} a_{B_r (v)} (v, y) \phi (y)
\geq \sum_{y \in D_N} \rho (y) \phi (y).
$$

Lemma 3.1 For any discrete harmonic function $h$ in $D_N$, we have $\sum_{y \in D_N} \rho (y) h (y) = 0$. 

12
Proof. Suppose \( h \) is defined up to \( \partial D_N \), and \( h|_{\partial D_N} = H \). We conclude the proof by showing for \( i = 1, 2 \)

\[
\sum_{r = (1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f^R_i (r) \sum_{y \in \partial B_r(v)} a_{B_r(v)} (v, y) h(y) = h(v).
\]

Indeed, since \( h \) is harmonic,

\[
h(y) = \sum_{z \in \partial D_N} a(y, z) H(z).
\]

Using the fact that

\[
\sum_{y \in \partial B_r(v)} a_{B_r(v)} (v, y) a(y, z) = a(v, z),
\]

we obtain

\[
\sum_{r = (1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f^R_i (r) \sum_{y \in \partial B_r(v)} a_{B_r(v)} (v, y) h(y) = \sum_{r = (1-\varepsilon)R_i}^{(1+\varepsilon)R_i} f^R_i (r) \sum_{z \in \partial D_N} a(v, z) H(z) = h(v).
\]

The following result is a consequence of Theorem 2.5 and the lemma above.

Lemma 3.2 Suppose the same conditions in Theorem 2.5 holds. Given \( v \in D_N \), \( R_2 > R_1 > 0 \), \( \varepsilon > 0 \) such that \( (1 + 2\varepsilon) R_1 < \text{dist}(v, \partial D_N) \), \( (1 + 2\varepsilon) R_2 < (1 - 2\varepsilon) R_1 \). Let \( \delta \) be the constants from Theorem 2.5. Let \( \phi^f \) be sampled from Ginzburg-Landau measure \( (2.1) \), and \( \phi^0 \) be sampled from the zero boundary Ginzburg-Landau measure on \( D_N \). Then, on an event with probability \( 1 - O(R_1^{-\delta}) \), we have

\[
X_{R_2} (v, \phi^f) - X_{R_1} (v, \phi^f) = X_{R_2} (v, \phi^0) - X_{R_1} (v, \phi^0).
\]

We sometimes omit the dependence of \( X \) on \( v \) and \( \phi \) when it is clear from the context.

4 Pointwise distribution for Ginzburg-Landau field

In this section we first employ an inductive decomposition argument to study the asymptotics of moment generating function of the harmonic average process. We then apply these Gaussian asymptotics to prove Proposition 1.3 (the Gaussian tail for the field at one site).

We first introduce the proper scales in order to carry out the inductive argument. Given any \( v \in D_N \), \( \varepsilon > 0 \) and \( c \in (0, 1) \), denote by \( \Delta = \text{dist}(v, \partial D_N) \) and \( M = M(c) = (1 - c) \log \Delta / \log (1 + \varepsilon) \). Define the sequence of numbers \( \{r_k\}_{k=1}^{\infty}, \{r_{k,+}\}_{k=0}^{\infty} \) and \( \{r_{k,-}\}_{k=0}^{\infty} \) by

\[
r_k = (1 + \varepsilon)^{-k} \Delta, \\
r_{k,+} = (1 + \varepsilon^3) r_k, \\
r_{k,-} = (1 - \varepsilon^3) r_k.
\]
We also define

\[ X_{r_{k,+}}(v) = \sum_{r=(1+\varepsilon)r_{k,+}}^{(1+\varepsilon)r_{k,+}} f^r_{\varepsilon} (r) C_r(v), \]

\[ X_{r_{k,-}}(v) = \sum_{r=(1-\varepsilon)r_{k,-}}^{(1-\varepsilon)r_{k,-}} f^r_{\varepsilon} (r) C_r(v), \]

for smooth \( f^r_{\varepsilon}, f^r_{\varepsilon} \) such that \( \sum_{r=(1-\varepsilon)r_{k,+}}^{(1+\varepsilon)r_{k,+}} f^r_{\varepsilon} (r) = \sum_{r=(1-\varepsilon)r_{k,-}}^{(1+\varepsilon)r_{k,-}} f^r_{\varepsilon} (r) = 1. \)

For \( r > 0 \), denote by \( \mathbb{P}^{r,0} \) the law of the Ginzburg-Landau field in \( B_r(0) \) with zero boundary condition (and denote by \( \mathbb{E}^{r,0} \) the corresponding expectation). The basic building block of all our large deviation and central limit estimates is the following.

**Theorem 4.1** Given \( C > 0, \ c \in (0,1) \), we have for all \( v \in D_N \) and \( t \leq C \),

\[ \log \mathbb{E}_{D_N,0}^{r}[\exp\left(tX_{r_{M,+}}(v)\right)] = \frac{t^2}{2} (1 - c) g \log \Delta + o_\Delta (\log \Delta) + O(1), \]

where the \( O(1) \) term depends on \( C \) and \( c \).

**Remark 4.2** The proof of Theorem 4.1 also yields

\[ \log \mathbb{E}_{D_N,0}^{r}[\exp\left(tX_{r_{M,+}}(v) - tX_{r_{0,-}}(v)\right)] = \frac{t^2}{2} (1 - c) g \log \Delta + o_\Delta (\log \Delta) + O(1). \]

This will be used in proving Theorem 4.4 below.

Roughly speaking, this theorem indicates that as long as \( r_M > \Delta^c \), the harmonic average \( X_{r_{M,+}} \) is near Gaussian. To prove this theorem we will first prove the following decoupling result. We denote

\[ W_j = \exp(t(X_{r_{j,+}} - X_{r_{j,-}})), \]

\[ Y_j = \exp(t(X_{r_{j,-}} - X_{r_{j,+}})), \]

\[ Z_j = \exp(t(X_{r_{j,+}})). \]

**Theorem 4.3** Given \( C > 0, \ c \in (0,1) \), we have for all \( v \in D_N \) and \( t \leq C \),

\[ \log \mathbb{E}_{D_N,0}^{r}[\exp(tX_{r_{M,+}})] = \sum_{j=1}^{M} \log \mathbb{E}_{D_N,0}^{r}[W_j] + \log \mathbb{E}_{D_N,0}^{r}[\exp(tX_{r_{0,-}})] + f(\varepsilon) \log \Delta + O(1), \]

where \( f(\varepsilon) \leq \varepsilon^2/\log(1+\varepsilon) \), and the \( O(1) \) term depends on \( C \) and constants from Lemma 2.7. More precisely, we have for and \( k = 1, \ldots, M, \)

\[ \log \mathbb{E}_{D_N,0}^{r}[\exp(tX_{r_{k,+}})] = \sum_{j=1}^{k} \log \mathbb{E}_{D_N,0}^{r}[W_j] + \log \mathbb{E}_{D_N,0}^{r}[\exp(tX_{r_{0,-}})] + f(\varepsilon) \log \frac{\Delta}{r_k} + O\left(\sum_{j=1}^{k} r_j^{-\delta}\right). \]

(4.2)
Notice that \( \sum_{j=1}^{k} r_j^{-\delta} \) is a geometric sum, and is thus finite.

**Proof of Theorem 4.1.** Apply Proposition 2.4, we see that as \( j \to \infty \),

\[
\log \mathbb{E}^{r_j^{-1.0}} [W_j] = \frac{t^2}{2} g \log (1 + \varepsilon) + O(\varepsilon^2) = \frac{t^2}{2} g \log \frac{r_j^{-1}}{r_j} + o_j \left( \log \frac{r_j^{-1}}{r_j} \right) + O(\varepsilon^2).
\]

Summing over \( j \) and apply Theorem 4.3, we have

\[
\log \mathbb{E}^{D_N,0} \left[ \exp \left( t X_{r_M,\pm} \right) \right] = \frac{t^2}{2} (1-c) g \log \Delta + o(\log \Delta) + O \left( \frac{\varepsilon^2}{\log(1+\varepsilon)} \right) \log \Delta + f(\varepsilon) \log \Delta + O(1).
\]

Sending \( \varepsilon \to 0 \) we conclude Theorem 4.1. \( \square \)

Our next result concerns the increment of the harmonic averages between some intermediate scales, defined as \( \{U_m\} \) below. For fixed \( K \geq 2 \) (which will be taking to infinity and the end), split “time” into \( K+1 \) intervals and consider the increments over these intervals

\[
U_m(v) = X_{r_{\left\lceil mK \right\rceil}^+}(v) - X_{r_{\left\lceil (m-1)K \right\rceil}^-}(v), \text{ for } m = 1, \ldots, K.
\]

(4.3)

Roughly speaking, when \( v \) is in the bulk of \( D_N \), \( \{U_m\}_{m=1}^{(1-a)K} \) are the differences between the harmonic average at scale \( N^{1-m/K} \) and the scale \( N^{1-(m-1)/K} \). The next lemma shows approximate joint Gaussianity of \( \{U_m\}_{m=1}^{K} \).

**Theorem 4.4** For all bounded sequence \( \{\lambda_m\}_{m=1}^{K} \) such that \( \max_m \lambda_m \leq C \) and \( v \in [-0.9N, 0.9N]^2 \), we have for all \( K \) sufficiently large,

\[
\mathbb{E}^{D_N,0} \left[ \exp \left( \sum_{m=1}^{K} \lambda_m U_m(v) \right) \right] = \exp \left( \frac{1}{2} \sum_{m=1}^{K} \lambda_m^2 \frac{1}{K} g \log N + o(\log N) + O(1) \right),
\]

(4.4)

where the \( O(1) \) term depends on \( K, \varepsilon, C, \) and the constant \( \delta \) from Theorem 2.3. Also, for \( v_1, v_2 \in [-N/2, N/2]^2 \) such that for some \( j \in \{1, \ldots, K\} \), \( N^{1-\frac{j}{K}} \leq |v_1 - v_2| \leq N^{1-\frac{j+1}{K}} \), and for bounded sequences \( \{\lambda_{m,i}\}_{i=1}^{2} \) such that \( \max_i \lambda_{m,i} \leq C \), we have for all \( K \) sufficiently large,

\[
\mathbb{E}^{D_N,0} \left[ \exp \left( \sum_{m=1}^{K} \lambda_{m,1} U_m(v_1) + \sum_{m=j+1}^{K} \lambda_{m,2} U_m(v_2) \right) \right] = \exp \left( \frac{1}{2} \sum_{m=1}^{K} \lambda_{m,1}^2 \frac{1}{K} g \log N + \frac{1}{2} \sum_{m=j+1}^{K} \lambda_{m,2}^2 \frac{1}{K} g \log N + o(\log N) \right).
\]

(4.5)

We will give the details of the proof of Theorem 4.3 and 4.4 in Sections 4.1 and 4.2 respectively. In Section 4.3 we apply Theorem 4.1 to finish the proof of Proposition 1.3.
4.1 Proof of Theorem 4.3

We write $X_{rM,+}$ as a telescoping sum

$$X_{rM,+} = (X_{rM,+} - X_{rM,-}) + (X_{rM,-} - X_{rM-2,+}) + \ldots + (X_{r,+} - X_{r0,}) + X_{r0,-},$$

and therefore

$$Z_M = W_M Y_{M-1} W_{M-1} \ldots Y_1 W_1 \exp \left(t X_{r0,-}\right).$$

Notice that

$$Z_k = W_k Y_{k-1} Z_{k-1} = W_k Z_{k-1} + W_k (Y_{k-1} - 1) Z_{k-1}.$$ 

Since $Z_{k-1} = W_{k-1} Y_{k-2} Z_{k-2}$, by iterating we obtain the decomposition

$$Z_k = \sum_{m=1}^{k-1} W_{m+1} Z_m \prod_{j=m+2}^{k} (W_j (Y_{j-1} - 1)) + Z_1 \prod_{j=2}^{k} W_j (Y_{j-1} - 1) \quad (4.6)$$

$$= W_k Z_{k-1} + E_Y, \quad (4.7)$$

where all the terms in $E_Y$ involve some of $Y_k$'s.

We will show that $\log \mathbb{E}^{D_{N,0}}[Z_k]$ is essentially contributed from the first term in the summation (4.6), i.e., $\log \mathbb{E}^{D_{N,0}}[W_k Z_{k-1}]$. Moreover, we show this can be further decoupled as

$$\log \mathbb{E}^{D_{N,0}}[W_k Z_{k-1}] = \log \mathbb{E}^{D_{N,0}}[Z_{k-1}] + \log \mathbb{E}^{r_{k-1},0}[W_k] + O \left(r_k^{-\delta}\right). \quad (4.8)$$

Keep iterating this will lead to the contribution $\sum_{j=1}^{k} \log \mathbb{E}^{r_{j-1},0}[W_j] + O \left(\sum_{j=1}^{k-1} r_{j-1}^{-\delta}\right)$.

We denote by $F_k = \sigma(\phi(x) : x \in B_{rk}(v)^c)$ and $G = \{ \max_{x \in B_{\Delta(v)}} |\phi(x)| \leq (c \log \Delta)^2 \}$.

**Lemma 4.5** There is some $c_1 = c_1(c) > 0$, such that $\mathbb{P}^{D_{N,0}}(G^c) \leq \exp \left(-c_1 (\log \Delta)^3\right)$.

**Proof.** By the union bound,

$$\mathbb{P}^{D_{N,0}}(G^c) \leq \sum_{x \in B_{\Delta(v)}} \mathbb{P}^{D_{N,0}}(|\phi(x)| > (c \log \Delta)^2).$$

We apply Lemma 2.2 to obtain

$$\mathbb{P}^{D_{N,0}}(|\phi(x)| > (c \log \Delta)^2) \leq \exp \left(- (4C)^{-1} (\log \Delta)^3 + O (\log \Delta)^2\right),$$

for some $C < \infty$, and summing over $x \in B_{\Delta(v)}$ then finishes the proof. □

Therefore, we can write

$$\mathbb{E}^{D_{N,0}}[Z_k] = \mathbb{E}^{D_{N,0}}[Z_k 1_G] + \mathbb{E}^{D_{N,0}}[Z_k 1_{G^c}].$$

By Holder and the exponential Brascamp-Lieb inequality,

$$\mathbb{E}^{D_{N,0}}[Z_k 1_{G^c}] \leq \left(\mathbb{E}^{D_{N,0}}[Z_k^2]\right)^{1/2} \mathbb{P}^{D_{N,0}}(G^c)^{1/2}$$

$$\leq \exp \left(2c_1^{-1} t^2 \text{Var}_G[X_{rM,+}]\right) \mathbb{P}^{D_{N,0}}(G^c)^{1/2}$$

$$\leq \exp \left(C t^2 \log \Delta - \frac{c_1}{2} (\log \Delta)^3\right) \leq \exp \left(\frac{c_1}{4} (\log \Delta)^3\right), \quad (4.9)$$

16
and similarly,
\[
E^{D,0}_N [W_k Z_{k-1} 1_{\mathcal{G}^r}] \leq \exp \left( -\frac{c_1}{4} (\log \Delta)^3 \right),
\]
\[
E^{r_{k-1},0} [W_k 1_{\mathcal{G}^r}] \leq \exp \left( -\frac{c_1}{4} (\log r_{k-1})^3 \right).
\]
(4.10)

Therefore it suffices to compute $E^{D,0}_N [Z_k 1_{\mathcal{G}^r}]$. Using Markov property,
\[
E^{D,0}_N [W_k Z_{k-1} 1_{\mathcal{G}^r}] = E^{D,0}_N [Z_{k-1} 1_{\mathcal{G}^r} E [W_k 1_{\mathcal{G}^r} | \mathcal{F}_{k-1}]]
\]
We apply Lemma 2.7 to obtain with probability one, there is some $C_1 < \infty$, such that
\[
|E [W_k 1_{\mathcal{G}^r} | \mathcal{F}_{k-1}] - E^{r_{k-1},0} [W_k 1_{\mathcal{G}^r}]| \leq 2 \exp (c \text{Var}_G W_k) r_{k-1}^{-\delta} \leq C_1 r_{k-1}^{-\delta},
\]
where $\delta$ is the constant from Theorem 2.5. Therefore,
\[
|E^{D,0}_N [W_k Z_{k-1}] - E^{D,0}_N [Z_{k-1}] E^{r_{k-1},0} [W_k]| \leq E^{D,0}_N [W_k Z_{k-1} 1_{\mathcal{G}^r}] + E^{r_{k-1},0} [W_k 1_{\mathcal{G}^r}] E^{D,0}_N [Z_{k-1}] + E^{D,0}_N [Z_{k-1} 1_{\mathcal{G}^r}] E^{r_{k-1},0} [W_k] + C_1 r_{k-1}^{-\delta}.
\]
(4.11)

Apply (4.9), (4.10) and Brascamp-Lieb inequality we see the above display can be bounded by $C_2 r_{k-1}^{-\delta}$, for some $C_2 < \infty$. This yields (4.8).

We now show all the other terms in (4.6), which involve at least one $Y_k$’s, are negligible. More precisely, assuming (4.11) holds for all $m < k$, we will prove
\[
E^{D,0}_N [E_Y] \leq O (\varepsilon^2) \prod_{j=1}^k E^{r_{j-1},0} [W_j].
\]
(4.12)

For $m = 1, \ldots, k-2$, using the Markov property and apply Lemma 2.7 again, we conclude that each term in the first summand of (4.6) (except for $W_k Z_{k-1}$) can be bounded by
\[
E^{D,0}_N \left[ W_{m+1} Z_m \prod_{j=m+2}^k (W_j (Y_{j-1} - 1)) 1_{\mathcal{G}^r} \right]
= E^{D,0}_N \left[ E \left[ W_{m+1} \prod_{j=m+2}^k (W_j (Y_{j-1} - 1)) 1_{\mathcal{G}^r} | \mathcal{F}_m \right] Z_m 1_{\mathcal{G}^r} \right]
\leq E^{D,0}_N \left[ \prod_{j=m+1}^k W_j^2 1_{\mathcal{G}^r} | \mathcal{F}_m \right]^{1/2} \left[ \prod_{j=m+1}^{k-1} (Y_{j-1} - 1)^2 1_{\mathcal{G}^r} | \mathcal{F}_m \right]^{1/2} Z_m 1_{\mathcal{G}^r}
\]
(4.12)

By Brascamp-Lieb inequality and Taylor expansion, there exist constants $C_3, C_4 < \infty$, such that
\[
E^{r_{j-1},0} [(Y_j - 1)^2] \leq C t^2 \text{Var}_G^{r_{j-1},0} [X_{r_{j-1}} - X_{r_{j+1}}] \leq C_3 \varepsilon^4,
\]
17
and
\[
\mathbb{E}^{r_1-1,0}[W_j^2] \leq \exp \left( 4c^{-1}t^2 \text{Var}_G^{r_1-1,0}[X_{r_1+} - X_{r_1-}] \right) 
\leq C_4
\]

We again using the Markov property and Lemma 2.7 to conclude
\[
\mathbb{E} \left[ \prod_{j=m+1}^{k-1} (Y_j - 1)^2 1_g|F_m \right] = \mathbb{E} \left[ \prod_{j=m+1}^{k-2} (Y_j - 1)^2 1_g|F_m \right] 
= (1 + O(r^{-\delta}_{k-2})) \mathbb{E}^{r_{k-2},0}[(Y_{k-1} - 1)^2] \mathbb{E} \left[ \prod_{j=m+1}^{k-2} (Y_j - 1)^2 1_g|F_m \right]
\]

Since \( \sum r_j^{-\delta} < \infty \), by iterating, there exists \( C_2' < \infty \), such that
\[
\mathbb{E} \left[ \prod_{j=m+1}^{k-1} (Y_j - 1)^2 1_g|F_m \right] \leq C_2' \left( C_3 \varepsilon^4 \right)^{k-m-1}.
\]

Therefore
\[
\mathbb{E} \left[ \prod_{j=m+1}^{k} W_j^2 1_g|F_m \right] \leq C_2' \mathbb{E}^{r_{j-1,0}}[W_j^2] \leq C_4' C_4^{k-m-1}.
\]

Similarly,
\[
\mathbb{E}^{D N,0} \left[ W_{m+1} Z_m \prod_{j=m+2}^{k} (W_j (Y_j - 1)) 1_g \right] \leq C_2' \left( C_5 \varepsilon^2 \right)^{k-m-1} \mathbb{E}^{D N,0}[Z_m]
\leq C_6 \varepsilon^2 \prod_{j=1}^{m} \mathbb{E}^{r_{j-1,0}}[W_j],
\]

where the last inequality follows from the induction hypothesis. Summing over \( m \), we then have
\[
\mathbb{E}^{D N,0} \left[ \sum_{m=1}^{k-2} W_{m+1} Z_m \prod_{j=m+2}^{k} (W_j (Y_j - 1)) 1_g \right] \leq C_7 \varepsilon^2 \prod_{j=1}^{k-2} \mathbb{E}^{r_{j-1,0}}[W_j],
\]

for some \( C_7 < \infty \). A similar argument yields
\[
\mathbb{E}^{D N,0} \left[ Z_k \prod_{j=2}^{k} W_j (Y_j - 1) 1_g \right] \leq C_7 \varepsilon^2 \prod_{j=1}^{k-2} \mathbb{E}^{r_{j-1,0}}[W_j].
\]

18
This finishes the proof of (4.11).

Finally, we prove Theorem 4.3 by induction and applications of (4.8) and (4.11). The base case \( k = 0 \) is trivial. Suppose Theorem 4.3 holds for \( k = l - 1 \). Since \( \mathbb{E}^{D_{N,0}}[\exp(t X_{r_{l,1}})] = \mathbb{E}^{D_{N,0}}[Z_l] = \mathbb{E}^{D_{N,0}}[W_l Z_{l-1}] + \mathbb{E}^{D_{N,0}}[E_Y] \), by (4.8), and the induction hypothesis

\[
\log \mathbb{E}^{D_{N,0}}[W_l Z_{l-1}] = \log \mathbb{E}^{D_{N,0}}[Z_{l-1}] + \mathbb{E}^{D_{l-1,0}}[W_l] + O(r_{l-1}^{-\delta})
\]

Combining with (4.11), we have

\[
\log \mathbb{E}^{D_{N,0}}[Z_l] = \sum_{j=1}^{l} \log \mathbb{E}^{D_{l-1,0}}[W_j] + \log \mathbb{E}^{D_{N,0}}[\exp(t (X_{r_{0,n}}))] + f(\varepsilon) \log \frac{\Delta}{r_{l-1}} + O\left(\sum_{j=1}^{l-1} r_j^{-\delta}\right).
\]

This finishes the proof of Theorem 4.3.

### 4.2 Proof of Theorem 4.4

We first prove (4.4). Using Brascamp-Lieb inequality and Lemma 4.5, it is easy to bound

\[
\mathbb{E}^{D_{N,0}}[\exp(\sum_{m=1}^{K} \lambda_m U_m(v)) 1_{G^c}] = o_N(1),
\]

therefore we only need to compute \( \mathbb{E}^{D_{N,0}}[\exp(\sum_{m=1}^{K} \lambda_m U_m(v)) 1_G] \).

Indeed, denote \( r \sim m / K \) as \( \tilde{r}_m \), and \( F_m = \sigma \{ \phi(x) : x \in B_{\tilde{r}_m}(v) \} \), by the Markov property we have

\[
\mathbb{E}^{D_{N,0}}[\exp(\sum_{m=1}^{K} \lambda_m U_m(v)) 1_G] = \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_m U_m \right) 1_G \mathbb{E}[e^{\lambda_K U_K} 1_G | F_{K-1}] \right].
\]

By Lemma 2.7, we can write

\[
|\mathbb{E}[e^{\lambda_K U_K} 1_G | F_{K-1}] - \mathbb{E}^{D_{K-1,0}}[e^{\lambda_K U_K} 1_G]| \leq \tilde{r}_K^{-\delta} \exp(c_1 \text{Var}_G(\lambda_K U_K)) \leq \tilde{r}_K^{-\delta} \exp(C^2 C_1 \frac{1}{K} \log N),
\]

19
for some $C_1 < \infty$ and $\delta > 0$, where $C = \max_m \lambda_m$. Take $K$ large enough such that
\[ C^2 C_1 \frac{1}{K} \leq \frac{1}{2} \epsilon \delta, \]
we thus have
\[ |\mathbb{E} [e^{\lambda K U_K 1_G | F_{K-1}}] - \mathbb{E}^{\tilde{r}_{K-1}} [e^{\lambda K U_K 1_G}]| \leq \tilde{r}_{K-1}^{-\delta/2}. \]
Therefore
\[ \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K} \lambda_m U_m(v) \right) 1_G \right] = \left( 1 + O \left( r_{K-1}^{-\delta/2} \right) \right) \mathbb{E}^{\tilde{r}_{K-1,0}} \left[ e^{\lambda K U_K 1_G} \right] \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_m U_m \right) 1_G \right]. \]
Keep iterating then yields
\[ \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K} \lambda_m U_m(v) \right) 1_G \right] = \prod_{m=1}^{K} \left( 1 + O \left( r_{m-1}^{-\delta/2} \right) \right) \mathbb{E}^{\tilde{r}_{m-1,0}} \left[ e^{\lambda m U_m 1_G} \right]. \]
By the Theorem 4.1 (and Remark 4.2),
\[ \mathbb{E}^{\tilde{r}_{m-1,0}} \left[ e^{\lambda m U_m} \right] = \exp \left( \frac{\lambda^2}{2} \frac{g}{K} \log N + o \log N + O \right), \quad (4.13) \]
and by Lemma 4.5 and Brascamp-Lieb inequality,
\[ \mathbb{E}^{\tilde{r}_{m-1,0}} \left[ e^{\lambda m U_m 1_{G^c}} \right] = o_N \left( 1 \right). \]
Since $\sum_{m=1}^{K} \tilde{r}_{m-1}^{-\delta/2} < \infty$, this finishes the proof of (4.4).
The proof of (4.5) is very similarly to that of (4.4). We define for $i = 1, 2$, $F_{m,i} = \sigma \{ \phi(x) : x \in B_{r_m}(v_i) \}$. Then by the same argument,
\[ \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K} \lambda_{m,1} U_m(v_1) + \sum_{m=1}^{K} \lambda_{m,2} U_m(v_2) \right) 1_G \right] \]
\[ = \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_{m,1} U_m(v_1) + \sum_{m=1}^{K} \lambda_{m,2} U_m(v_2) \right) 1_G \mathbb{E} [ \exp (\lambda_{K,1} U_K(v_1)) 1_G | F_{K-1,1}] \right] \]
\[ = \left( 1 + O \left( \tilde{r}_{K-1}^{-\delta/2} \right) \right) \mathbb{E}^{\tilde{r}_{K-1,0}} \left[ \exp (\lambda_{K,1} U_K(v_1)) 1_G \right] \]
\[ \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_{m,1} U_m(v_1) + \sum_{m=1}^{K} \lambda_{m,2} U_m(v_2) \right) 1_G \right] \]
\[ = \left( 1 + O \left( \tilde{r}_{K-1}^{-\delta/2} \right) \right) \mathbb{E}^{\tilde{r}_{K-1,0}} \left[ \exp (\lambda_{K,1} U_K(v_1)) 1_G \right] \]
\[ \times \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_{m,1} U_m(v_1) + \sum_{m=1}^{K-1} \lambda_{m,2} U_m(v_2) \right) 1_G \mathbb{E} [ \exp (\lambda_{K,2} U_K(v_2)) 1_G | F_{K-1,2}] \right] \]
\[ = \left( 1 + O \left( \tilde{r}_{K-1}^{-\delta/2} \right) \right) \mathbb{E}^{\tilde{r}_{K-1,0}} \left[ \exp (\lambda_{K,1} U_K(v_1)) 1_G \right] \mathbb{E}^{\tilde{r}_{K-1,0}} \left[ \exp (\lambda_{K,2} U_K(v_2)) 1_G \right] \]
\[ \times \mathbb{E}^{D_{N,0}} \left[ \exp \left( \sum_{m=1}^{K-1} \lambda_{m,1} U_m(v_1) + \sum_{m=1}^{K-1} \lambda_{m,2} U_m(v_2) \right) 1_G \right]. \]
Keep iterating, we obtain
\[
\mathbb{E}^{D_{N,0}}[\exp(\sum_{m=1}^{K} \lambda_{m,1} U_m(v_1) + \sum_{m=j+1}^{K} \lambda_{m,2} U_m(v_2)) 1_{\mathcal{G}}] \\
= \prod_{m=j+1}^{K} \left(1 + O\left(\epsilon_m^{-6/2}\right)\right) \mathbb{E}^{\mathcal{G}_{m-1,0}}[\exp(\lambda_{m,1} U_m(v_1)) 1_{\mathcal{G}}] \mathbb{E}^{\mathcal{G}_{m-1,0}}[\exp(\lambda_{m,2} U_m(v_2)) 1_{\mathcal{G}}] \\
\times \prod_{m=1}^{j} \left(1 + O\left(\epsilon_m^{-6/2}\right)\right) \mathbb{E}^{\mathcal{G}_{m-1,0}}[\exp(\lambda_{m,1} U_m(v_1)) 1_{\mathcal{G}}].
\]

Applying (4.13) we conclude the proof of (4.5).

4.3 Proof of Proposition 1.3

Given \(\delta > 0\) and \(x \in D_N\), take \(M = M(\delta) = (1 - \delta^6) \log \Delta\). Therefore
\[
\mathbb{P}^{D_{N,0}}(\phi(x) > u) \leq \mathbb{P}^{D_{N,0}}(X_{r_{M,+}}(x) > u - \delta \log \Delta) + \mathbb{P}^{D_{N,0}}(\phi(x) - X_{r_{M,+}}(x) > \delta \log \Delta).
\]

We apply Theorem 4.1 to obtain
\[
\mathbb{P}^{D_{N,0}}(X_{r_{M,+}} > u - \delta \log \Delta) \leq \exp(-t(u - \delta \log \Delta)) \mathbb{E}^{D_{N,0}}[\exp(tX_{r_{M,+}})] = \exp\left(-t(u - \delta \log \Delta) + \frac{t^2}{2}g(1 - \delta^6) \log \Delta + o(\log \Delta)\right).
\]

Minimize over \(t\) to obtain
\[
\mathbb{P}^{D_{N,0}}(X_{r_{M,+}} > u - \delta \log \Delta) \leq \exp\left(-\frac{(u - \delta \log \Delta)^2}{2g(1 - \delta^6) \log \Delta} + o(\log \Delta)\right) \leq \exp\left(-\frac{(u - \delta \log \Delta)^2}{2g \log \Delta} + o(\log \Delta)\right).
\]

Apply Lemma 2.2 to obtain
\[
\mathbb{P}^{D_{N,0}}(\phi(x) - X_{r_{M,+}} > \delta \log \Delta) \leq \exp\left(-c_{BL}(\delta \log \Delta)^2\right) = \exp\left(-c_{BL}\frac{\log \Delta}{g \delta^4}\right).
\]

Notice that for \(\delta\) small enough,
\[
2c_{BL}\frac{\log \Delta}{g \delta^4} > \frac{(u - \delta \log \Delta)^2}{2g \log \Delta},
\]
we send \(\delta \to 0\) to conclude the proof.

5 Proof of the Lower Bound

In this section we prove the lower bound (1.5). We first prove a weaker form of the lower bound in Section 5.1 and we then “bootstrap” to obtain the desired lower bound in Section 5.2. In what follows we denote by \(\mathbb{P}^{B,f}\) the law of gradient field in \(B \subset \mathbb{Z}^2\) with boundary condition \(f\) on \(\partial B\).
5.1 Second moment argument

Given $B \subset \mathbb{Z}^2$, $x \in B$ and $y \in \partial B$, we recall $a_B(x, y)$ is the harmonic measure on $\partial B$ seen from $x$. Also recall the harmonic averaged field $X_{r, +}(v)$ and $X_{r, -}(v)$ from the beginning of Section 4. Heuristically, the process $\{X_{r, +}(v)\}$ should behave like a random walk with increments of variance $g \log (1 + \varepsilon)$. We make this heuristics rigorous and show the following weak lower bound:

Proposition 5.1 For all $\beta > 0$, there is $N_0 = N_0(\beta)$ such that for $N > N_0(\beta)$

$$\mathbb{P} \left[ \exists v \in [-0.9N, 0.9N]^2 \text{ s.t. } \phi(v) - X_{r_0, -}(v) \geq (1 - 2\beta)2\sqrt{g} \log N \right] \geq N^{-17\beta}. \quad (5.1)$$

In fact, this probability tends to one as $N \to \infty$. This will be proved later by bootstrapping the weaker bound stated in Proposition 5.1. The proof of Proposition 5.1 is based on a second moment method studying the truncated count of the harmonic averaged process.

It suffices to prove Proposition 5.1 for small $\beta$. Given $v \in [-0.9N, 0.9N]^2$, take $c = \beta^3$ and $M = M(\beta^3) = (1 - \beta^3)\log N / \log (1 + \varepsilon)$, and define $r_k$ and $r_{k, \pm}$ as in (4.1). Then we have

$$\mathbb{P} \left[ \exists v \in [-0.9N, 0.9N]^2 \text{ s.t. } \phi(v) - X_{r_0, -}(v) \geq (1 - 2\beta)2\sqrt{g} \log N \right] \geq \mathbb{P} \left[ \exists v \in [-0.9N, 0.9N]^2 \text{ s.t. } X_{r_{M, +}}(v) - X_{r_0, -}(v) \geq (1 - 2\beta)2\sqrt{g} \log N \right] - \mathbb{P} \left[ \exists v \in [-0.9N, 0.9N]^2 \text{ s.t. } \phi(v) - X_{r_{M, +}}(v) \geq 2\sqrt{g} \log N \right].$$

The last term above can be bounded using Brascamp-Lieb inequality. Indeed, this implies

$$\mathbb{P} \left[ \exists v \in [-0.9N, 0.9N]^2 \text{ s.t. } \phi(v) - X_{r_{M, +}}(v) \geq 2\sqrt{g} \log N \right] \leq \sum_{v \in [-0.9N, 0.9N]^2} \mathbb{P} \left[ \phi(v) - X_{r_{M, +}}(v) \geq \frac{\beta}{2}2\sqrt{g} \log N \right] \leq N^2 \exp \left( -c_{BL} \frac{\beta^2 (\log N)^2}{\text{Var}_G(\phi(v) - X_{r_{M, +}}(v))} \right) \leq N^2 \exp \left( -c_{BL} \frac{\beta^2 (\log N)^2}{g \beta^3 \log N} \right) = N^{2 - c' \beta^{-1}}, \quad (5.2)$$

for some $c' > 0$. For small $\beta$ this is much smaller than $N^{-17\beta}$. Therefore it suffices to study $X_{r_{M, +}}(v) - X_{r_0, -}(v)$. 

22
Fix $K < \infty$ and recall the definition of $U_m$ in (4.3). Consider the events

$$J_m(v; \beta) = \left\{ U_m(v) \in \left[ \frac{1}{K}(1 - \beta)2\sqrt{g}\log N, \frac{1}{K}(1 + \beta)2\sqrt{g}\log N \right] \right\}.$$ 

and

$$J(v; \beta) = \bigcap_{m=1,...,K} J_m(v; \beta).$$

Define the counting random variable

$$Z(\beta) = \sum_{v \in [-0.9N, 0.9N]^2} 1_{J(v; \beta)}.$$

Note that if $Z(\beta) \geq 1$ then there exists a $v \in [-0.9N, 0.9N]^2$ such that

$$\sum_{m=1}^K U_m(v) \geq (1 - \beta)2\sqrt{g}\log N.$$

Furthermore, since

$$X_{rM,+}(v) - X_{r0,-}(v) = \sum_{m=1}^K U_m(v) + \sum_{m=1}^K (X_{[mM/K],-}(v) - X_{[mM/K],+}(v)),$$

and by direct computation

$$\text{Var}_G \left[ \sum_{m=1}^K (X_{[mM/K],-}(v) - X_{[mM/K],+}(v)) \right] = O(K),$$

the Brascamp-Lieb tail bound Lemma 2.2 implies

$$\mathbb{P} \left( \sum_{m=1}^K (X_{[mM/K],-}(v) - X_{[mM/K],+}(v)) > 2\sqrt{g}\beta \log N \right) \leq e^{-c(\beta,K)(\log N)^2}. \quad (5.4)$$

Combining (5.3) and (5.4), Proposition 5.1 will follow from

$$\mathbb{P} [Z(\beta) \geq 1] \geq N^{-17\beta}. \quad (5.5)$$

We will prove

**Lemma 5.2** For all $\beta > 0$ and $K \geq 2$ we have

$$E[Z(\beta)^2] \leq N^{17\beta} E[Z(\beta)]^2. \quad (5.6)$$

With additional work, the term $\exp(17\beta \log N)$ could be replaced $(1 + o(1))$, but for our purposes (5.6) is enough. Note that (5.6) is true only because $Z(\beta)$ is a truncated count of high points.

By the Paley-Zygmund inequality, Lemma 5.2 implies (5.5), and therefore yields Proposition 5.1.

**Lemma 5.2** follows from the following estimates:
Lemma 5.3 For all fixed $\beta > 0$ and $K \geq 2$ we have
\[ E[Z(\beta)] \geq cN^{-5\beta}. \]

Lemma 5.4 For all fixed $\beta > 0$ and $K \geq 2$ we have
\[ E[Z(\beta)^2] \leq N^{\frac{2}{\beta} + 6\beta}, \]
thus letting $K \to \infty$ we have
\[ E[Z(\beta)^2] \leq N^{6\beta}. \]

Lemma 5.3 is immediate from taking union bound from the following result.

Lemma 5.5 For all fixed $\beta > 0$ and $K \geq 2$ we have that
\[ \mathbb{P}^{D_N;0}[J(v;\beta)] \geq cN^{-2-5\beta}, \]
uniformly over $v \in [-0.9N, 0.9N]^2$.

Proof. Letting \( m \frac{dQ}{dP_{D_N;0}} = \frac{\exp(\lambda \sum_{m=1}^K U_m(v))}{\mathbb{E}(\exp(\lambda \sum_{m=1}^K U_m(v)))} \) we have
\[ \mathbb{P}^{D_N;0}[J(v, \beta)] = Q[J(v, \beta); e^{-\lambda \sum_{m=1}^K U_m(v)}] \mathbb{E}^{D_N;0}[\exp(\lambda \sum_{m=1}^K U_m(v))] \]
\[ \geq Q[J(v, \beta)] e^{-\lambda(1+\beta)2\sqrt{\beta} \log N} \mathbb{E}^{D_N;0}[\exp(\lambda \sum_{m=1}^K U_m(v))] \]

By Theorem 4.4 for all $\lambda \leq 2/\sqrt{g}$,
\[ \mathbb{E}^{D_N;0}[\exp(\sum_{m=1}^K \lambda U_m(v))] = \exp \left( \frac{1}{2} \sum_{m=1}^K \lambda^2 \frac{1}{K} g \log N + o(\log N) + O(1) \right), \quad (5.7) \]

Therefore
\[ \mathbb{P}^{D_N;0}[J(v, \beta)] \geq Q[J(v, \beta)] e^{\frac{1}{2} \lambda^2 g \log N - \lambda(1+\beta)2\sqrt{\beta} \log N + o(\log N)}. \]

Setting $\lambda = 2/\sqrt{g}$ we find that
\[ \mathbb{P}^{D_N;0}[J(v, \beta)] \geq Q[J(v, \beta)] e^{-2\log N - 5\beta \log N}. \]

It thus only remains to show that $Q[J(v)] \geq c$. Under $Q$ we have for each $j$ that
\[ Q[\exp(t(U_j(v) - \frac{1}{K} 2\sqrt{g} \log N))] = \frac{\mathbb{E}[\exp(\sum_{m=1}^K (\lambda + (1_{m=j}) t) U_m(v))] \exp(-2t \frac{1}{K} \sqrt{g} \log N)}{\mathbb{E}[\exp(\sum_{m=1}^K \lambda U_m(v))] \exp(-2t \frac{1}{K} \sqrt{g} \log N)}. \quad (5.8) \]

Thus applying Theorem 4.4 (with max $\lambda_i = 2/\sqrt{g} + 1$) we have that (5.8) equals
\[
\begin{align*}
\exp \left( \frac{1}{2} \lambda^2 g \log N + o(\log N) \right) & \cdot \exp \left( -2t \frac{1}{K} \sqrt{g} \log N \right) \\
& = \exp \left( \frac{1}{2} \lambda^2 \frac{1}{K} g \log N + o(\log N) \right),
\end{align*}
\]

24
where the last equality follows because \( \lambda = 2/\sqrt{g} \). Using the exponential Chebyshev inequality with \( t = \pm \beta/\sqrt{g} \) therefore shows that

\[
Q[|U_j - \frac{1}{K}\sqrt{g}\log N| \geq \beta\frac{1}{K}\sqrt{g}\log N] \leq \exp(-c\beta^2/K\log N),
\]

for some \( c > 0 \). Thus \( Q[J(v, \beta)] \geq 1 - K\exp(-c\beta^2/K\log N) \to 1 \), as \( N \to \infty \) for all \( K \) and \( \beta \).

Lemma 5.4 will follow from the following.

**Lemma 5.6** For all fixed \( \beta > 0 \) and \( K \geq 1 \) we have if \( N^{1-\frac{1}{K}} \leq |v_1 - v_2| \leq N^{1-\frac{j-1}{K}} \) for some \( j \in \{1, \ldots, K\} \), then

\[
\mathbb{P}^{D_N,0}[J(v_1, \beta) \cap J(v_2, \beta)] \leq \exp(-2K - \frac{1}{K}\log N + 5\beta\log N).
\]

**Proof.** Note that \( B_{N^{1-\frac{1}{K}}}(v_i) \) for \( i = 1, 2 \) are disjoint, but \( B_{N^{1-\frac{j}{K}}}(v_i) \) are not. Thus, roughly speaking, the increments \( U_{j+1}(v_i) \) for \( i = 1, 2 \) depend on disjoint regions but \( U_j(v_i) \) do not. Because of this we expect \( U_m(v_i), i = 1, 2 \) to be correlated for \( m = 1, \ldots, j \) (and essentially perfectly correlated if \( m \leq j - 1 \), but essentially independent for \( m = j + 1, \ldots, K \). With this in mind we in fact bound

\[
\mathbb{P}^{D_N,0}[J'],
\]

where

\[
J' = \cap_{m=1}^{K} J(v_1, \beta) \cap \cap_{m=j+1}^{K} J(v_2, \beta),
\]

i.e., we drop the condition on \( v_2 \) for \( m = 1, \ldots, j \).

Letting

\[
\frac{dQ}{d\mathbb{P}^{D_N,0}} = \frac{\exp(\sum_{m=1}^{K} \lambda U_m(v_1) + \lambda \sum_{m=j+1}^{K} U_m(v_2))}{\mathbb{E}^{J(v_1, \beta)}[\exp(\sum_{m=1}^{K} \lambda U_m(v_1) + \lambda \sum_{m=j+1}^{K} U_m(v_2))]} \]

we have

\[
\mathbb{P}^{D_N,0}[J'] \leq \mathbb{P}[J'; \exp\left(-\sum_{m=1}^{K} \lambda U_m(v_1) - \lambda \sum_{m=j+1}^{K} U_m(v_2)\right)]
\]

\[
\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K} \lambda U_m(v_1) + \lambda \sum_{m=j+1}^{K} U_m(v_2))]
\]

\[
\leq \exp\left(-\frac{2K-j}{K}(1-\beta)2\sqrt{g}\log N\right)
\]

By Theorem 4.4 for all \( \lambda \leq 2/\sqrt{g} \),

\[
\mathbb{E}^{D_N,0}[\exp(\sum_{m=1}^{K} \lambda U_m(v_1) + \sum_{m=j+1}^{K} \lambda U_m(v_2))]
\]

\[
= \exp\left(\frac{1}{2} \sum_{m=1}^{K} \lambda^2 \frac{1}{K} g \log N + \frac{1}{2} \sum_{m=j+1}^{K} \lambda^2 \frac{1}{K} g \log N + o(\log N)\right).
\]

Thus in fact \( \mathbb{P}^{D_N,0}[J'] \) is at most

\[
\exp\left(\frac{1}{2} \lambda^2 \frac{2K-j}{K} g \log N - \lambda \frac{2K-j}{K}(1-\beta)2\sqrt{g}\log N + o(\log N)\right).
\]

Setting \( \lambda = 2/\sqrt{g} \) we find that

\[
\mathbb{P}^{D_N,0}[J'] \leq \exp\left(-2\frac{2K-j}{K}\log N + 5\beta\log N\right).
\]
We can now prove the second moment estimate Lemma \[5.4\].

**Proof of Lemma \[5.4\].** We write the second moment as

\[
\mathbb{E}[Z^2] \leq \sum_{v_1, v_2 \in [-0.9N, 0.9N]^2} \mathbb{P}[J(v_1, \beta) \cap J(v_2, \beta)].
\]

Splitting the sum according to the distance \(|v_1 - v_2|\) we get that,

\[
\mathbb{E}[Z^2] = \sum_{j=1}^{K} \sum_{v_1, v_2 : N^{1-j/K} \leq |v_1 - v_2| \leq N^{1-(j-1)/K}} \mathbb{P}[J(v_1, \beta) \cap J(v_2, \beta)] + \sum_{|v_1 - v_2| \leq N^\eta} \mathbb{P}[J(v_1, \beta) \cap J(v_2, \beta)].
\]

Now using Lemma \[5.3\] and the fact that there are at most \(N^2 \times N^{2-2(j-1)/K}\) points at distance less than \(N^{1-(j-1)/K}\) we obtain an upper bound of

\[
\sum_{j=1}^{K} N^{4-2(j-1)/K} \times N^{-2 \frac{K}{K-1} + 5\beta} + N^2 N^{-2+5\beta} = N^{4+5\beta} \sum_{j=1}^{K} N^{-2(j-1)/K} \times N^{-2 \frac{K}{K-1} + 5\beta} + N^{5\beta} \leq [K + 1] N^{2+5\beta},
\]

which for \(N\) large enough is at most \(N^{2+6\beta} \). ■

**5.2 Bootstrapping**

We now use Proposition \[5.1\] to prove the desired lower bound (1.5). Proposition \[5.1\] shows the field reaches \((1 - 2\beta)2\sqrt{g} \log N\) with at least polynomially small probability. We will apply Theorem 2.5 to see that the field in different regions of \([-N, N]^2\) are essentially decoupled. Therefore applying Proposition \[5.1\] in each region one can show with high probability, there is some \(v \in [-N, N]^2\) such that \(\phi(v) - X_{\eta, m}(v) \geq (1 - 2\beta)2\sqrt{g} \log N\).

To carry out this argument, tile \([-N, N]^2\) by disjoint boxes \(D_1, D_2, \ldots, D_m\) of side-length \(N^{1-\eta}\), where \(m \approx N^{\eta}\), and \(\eta\) is a small number that will be chosen later. Let \(B\) be the union of all the \(\partial D_i\).

Consider the good event

\[
\mathcal{G} = \{ \max_{x \in [-N, N]^2} |\phi(x)| \leq (\log N)^2 \}. \quad (5.13)
\]

By Lemma \[4.3\] we have \(\mathbb{P}[\mathcal{G}^c] \leq e^{-c(\log N)^3}\), as \(N \to \infty\).

On the event \(\mathcal{G}\), for \(i = 1, \ldots, m\), let \(\tilde{D}_i\) be the box concentric to \(D_i\), but with side length \(\frac{N}{2}N^{1-\eta}\). Let \(R = \frac{1}{2} N^{1-\eta}\). We further define

\[
\tilde{Z}_i = \{ \forall v \in \tilde{D}_i : \phi(v) - X_{R, m}(v, \phi) < (1 - 2\beta)(1 - \eta)2\sqrt{g} \log N \}.
\]

Now

\[
\mathbb{P}[\tilde{Z}_i, i = 1, \ldots, m; \mathcal{G}] = \mathbb{P}[\mathbb{P}[\tilde{Z}_i, i = 1, \ldots, m|\phi(x), x \in B]; \mathcal{G}]. \quad (5.14)
\]
Using the Gibbs property of the measure \(1.3\), we have the conditional decoupling

\[
P[\tilde{Z}_i, i = 1, ..., m|\phi(x), x \in B] = \prod_{i=1}^{m} P_{D_i, \phi|\tilde{Z}_i|}[\phi(x), x \in \partial D_i].
\] (5.15)

Consider for each \(i\) the law \(P_{D_i, \phi|\tilde{Z}_i}\). Then on \(\mathcal{G}\) we can apply Lemma 5.2 to construct a coupling \(Q^i\) of a field \(\phi\) with law \(P_{D_i, \phi|\tilde{Z}_i}\) and a field \(\phi^{0,i}\) with law \(P_{D_i, 0}\) such that

\[
Q^i[\forall v \in D_i : \phi(v) - X_{R,-}(v, \phi) = \phi^{0,i}(v) - X_{R,-}(v, \phi^{0,i})] \geq 1 - N^{-\delta(1-\eta)},
\]

where the constant \(\delta > 0\) is from Theorem 2.5.

Thus

\[
P \left( \forall v \in [-0.9N, 0.9N]^2 : \phi(v) - X_{R,-}(v, \phi) < (1 - 2\beta)(1 - \eta) 2\sqrt{g} \log N; \mathcal{G} \right) \leq \prod_{i=1}^{m} \left( P_{D_i, 0}[\tilde{Z}_i] + N^{-\delta(\eta - 1)} \right)
\]

\[
\leq \prod_{i=1}^{m} \left( 1 - (N^{1-\eta})^{-17\beta} + N^{-\delta(\eta - 1)} \right),
\]

where we apply Proposition 5.1 to obtain the last inequality. Now let \(\beta\) and \(\eta\) be small enough, depending on \(\delta\), such that

\[
17\beta < \delta \text{ and } \eta > 17\beta / (1 + 17\beta).
\] (5.17)

Thus we have

\[
P \left( \forall v \in [-0.9N, 0.9N]^2 : \phi(v) - X_{R,-}(v, \phi) < (1 - 2\beta)(1 - \eta) 2\sqrt{g} \log N; \mathcal{G} \right) \leq e^{-N^{\varepsilon_1}},
\] (5.18)

for some \(\varepsilon_1 > 0\).

In view of (5.17), we can take \(\eta = 17\beta\). Then, on the complement of the event (5.18), there exists \(v_1 \in [-0.9N, 0.9N]^2\) such that

\[
\phi(v_1) - X_{R,-}(v_1, \phi) \geq (1 - 19\beta)2\sqrt{g} \log N.
\]

Notice that

\[
\text{Var}_G [X_{R,-}(v_1, \phi)] = g_0 \eta \log N + o(\log N) = 19\beta g_0 \log N + o(\log N).
\]

By Lemma 2.2

\[
P \left[ X_{R,-}(v_1, \phi) > \beta^{1/3} \log N \right] \leq \exp \left( -c_{BL} \frac{\beta^{2/3}(\log N)^2}{\beta \log N} \right) = N^{-c_{BL}\beta^{-1/3}}.
\] (5.19)

Combining (5.18) and (5.19), we see that

\[
P \left[ \max_{v \in [-0.9N, 0.9N]^2} \phi(v) < (1 - 2\beta^{1/3})2\sqrt{g} \log N \right] \leq N^{-c_{BL}\beta^{-1/3}} + e^{-N^{\varepsilon_1}}.
\]

And we conclude (1.3).
5.3 High points

We now sketch the proof of Theorem 1.2. The proof follows from the same argument as the proof of Theorem 1.1, for completeness we sketch the idea below.

It suffices to prove that for any \( \beta > 0 \),

\[
\mathbb{P}\left( |\mathcal{H}_N(\eta)| > N^{2(1-\eta^2)+\beta} \right) = o_N(1), \quad \text{and} \quad (5.20)
\]

\[
\mathbb{P}\left( |\mathcal{H}_N(\eta)| < N^{2(1-\eta^2)-\beta} \right) = o_N(1). \quad (5.21)
\]

Since

\[
\mathbb{P}\left( |\mathcal{H}_N(\eta)| > N^{2(1-\eta^2)+\beta} \right) \leq N^{-2(1-\eta^2)-\beta} \mathbb{E}[|\mathcal{H}_N(\eta)|] \leq N^{-2(1-\eta^2)-\beta} \sum_{v \in D_N} \mathbb{P}(\phi(v) \geq 2 \sqrt{g} \eta \log N),
\]

the upper bound (5.20) follows directly from applying Proposition 1.3 with \( u = 2 \sqrt{g} \eta \log N \).

We now focus on the lower bound (5.21). Recall the definition of \( U_m \) in (4.3).

For \( \eta \in (0,1) \) we define

\[
J_m(v; \eta; \beta) = \left\{ U_m(v) \in \left[ \frac{\eta}{K}(1 + \beta)2 \sqrt{g} \log N, \frac{\eta}{K}(1 + 2\beta)2 \sqrt{g} \log N \right] \right\},
\]

and

\[
J(v; \eta; \beta) = \bigcap_{m=1,\ldots,K} J_m(v; \eta; \beta).
\]

Also define the counting random variable

\[
Z(\eta, \beta) = \sum_{v \in [-0.9N, 0.9N]^2} 1_{J(v; \eta, \beta)}.
\]

By the same Brascamp-Lieb bounds as (5.3) and (5.4), to study the dimension of \( \mathcal{H}_N(\eta) \), it suffices to study \( \{ v : J(v; \eta; \beta) \text{ occurs} \} \). Indeed, the same first moment computation as Lemma 5.3 and Lemma 5.5 (but instead using the change of measure \( \frac{dQ}{d\mathbb{P}^{\mathcal{D}_N,0}} = \frac{\exp(\lambda_\eta \sum_{m=1}^K U_m(v))}{\mathbb{E}^{\mathbb{P}^{\mathcal{D}_N,0}}[\exp(\lambda_\eta \sum_{m=1}^K U_m(v))]} \) yields

\[
\mathbb{E}[Z(\eta, \beta)] \geq N^{2(1-\eta^2) - 8 \beta \eta^2},
\]

and the same first moment computation as Lemma 5.4 and Lemma 5.6 yields

\[
\mathbb{E}[Z^2(\eta, \beta)] \leq N^4(1-\eta^2) - 5 \beta \eta^2.
\]

Therefore

\[
\mathbb{E}[Z^2(\eta, \beta)] \leq N^{11 \beta \eta^2} \mathbb{E}[Z(\eta, \beta)]^2.
\]
Apply Payley-Zygmund inequality then yields

\[
P \left( \left| \{ v : J(v; \eta; \beta) \text{ occurs} \} \right| < \frac{1}{2} N^2 (1 - \eta^2)^{-\beta} \right) \\
\leq 1 - P \left( Z(\eta, \beta) > \frac{1}{2} \mathbb{E} [Z(\eta, \beta)] \right) \\
\leq 1 - c N^{-11\beta^2}.
\]

But to complete the proof of (5.21) we want \( P \left( Z(\eta, \beta) > \frac{1}{2} \mathbb{E} [Z(\eta, \beta)] \right) \) to be close to 1. This can be proved by carrying out the same bootstrapping in Section 5.2 obtaining the high probability by creating a large number \( N^\gamma \), where \( \gamma = \gamma(\beta, \delta) \), and \( \delta \) is the constant from Theorem decouple) of essentially independent trials with success probability \( N^{-11\beta^2} \).

**Acknowledgments:** We thank Ron Peled for very helpful discussions and helping us generalize the results in an earlier version of this paper; Ofer Zeitouni for helpful discussions; Thomas Spencer for discussions on gradient field models and the arguments in [CS14]; and Jason Miller for useful communications. The work of W.W. is supported in part by NSF grant DMS-1507019. Part of the work is done when the last author was visiting Weizmann Institute, Tel-Aviv University and NYU Shanghai, and we thank these institutes for their hospitality.

**References**

[BDG01] Erwin Bolthausen, Jean-Dominique Deuschel, and Giambattista Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *The Annals of Probability*, 29(4):1670–1692, 2001.

[BL76] Herm Jan Brascamp and Elliott H Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *Journal of Functional Analysis*, 22(4):366–389, 1976.

[BL13] Marek Biskup and Oren Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *arXiv preprint arXiv:1306.2602*, 2013.

[BY90] David Brydges and Horng-Tzer Yau. Grad \( \phi \) perturbations of massless Gaussian fields. *Communications in Mathematical Physics*, 129(2):351–392, 1990.

[BZ12] Maury Bramson and Ofer Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Communications on Pure and Applied Mathematics*, 65(1):1–20, 2012.
[CS14] Joseph G Conlon and Thomas Spencer. A strong central limit theorem for a class of random surfaces. *Communications in Mathematical Physics*, 325(1):1–15, 2014.

[Dav06] Olivier Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *The Annals of Probability*, 34(3):962–986, 2006.

[DG00] Jean-Dominique Deuschel and Giambattista Giacomin. Entropic repulsion for massless fields. *Stochastic processes and their applications*, 89(2):333–354, 2000.

[DG10] Jean-Dominique Deuschel, Giambattista Giacomin, and Dmitry Ioffe. Large deviations and concentration properties for $\nabla \phi$ interface models. *Probability Theory and Related Fields*, 117(1):49–111, 2000.

[DRZ15] Jian Ding, Rishideep Roy, and Ofer Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *arXiv preprint arXiv:1503.04588*, 2015.

[DZ14] Jian Ding and Ofer Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *The Annals of Probability*, 42(4):1480–1515, 2014.

[FS97] T Funaki and Herbert Spohn. Motion by mean curvature from the Ginzburg-Landau interface model. *Communications in Mathematical Physics*, 185(1):1–36, 1997.

[GK80] K Gawedzki and A Kupiainen. A rigorous block spin approach to massless lattice theories. *Communications in Mathematical Physics*, 77(1):31–64, 1980.

[GOS01] Giambattista Giacomin, Stefano Olla, and Herbert Spohn. Equilibrium fluctuations for $\nabla \phi$ interface model. *Annals of Probability*, pages 1138–1172, 2001.

[Hel02] Bernard Helffer. *Semiclassical analysis, Witten Laplacians, and statistical mechanics*, volume 1. World Scientific, 2002.

[HS94] Bernard Helffer and Johannes Sjöstrand. On the correlation for Kac-like models in the convex case. *Journal of Statistical Physics*, 74(1-2):349–409, 1994.

[Ken00] Richard Kenyon. Conformal invariance of domino tiling. *Annals of Probability*, pages 759–795, 2000.

[Ken01] Richard Kenyon. Dominos and the Gaussian free field. *Annals of Probability*, pages 1128–1137, 2001.

[Law08] Gregory F Lawler. *Conformally invariant processes in the plane*. Number 114. American Mathematical Soc., 2008.
[LL10] Gregory F Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123. Cambridge University Press, 2010.

[Mil10] Jason Miller. Universality for SLE (4). *arXiv preprint arXiv:1010.1356*, 2010.

[Mil11] Jason Miller. Fluctuations for the Ginzburg-Landau $\nabla \phi$ interface model on a bounded domain. *Communications in Mathematical Physics*, 308(3):591–639, 2011.

[NS97] Ali Naddaf and Thomas Spencer. On homogenization and scaling limit of some gradient perturbations of a massless free field. *Communications in Mathematical Physics*, 183(1):55–84, 1997.

[SW12] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Annals of Mathematics*, 176(3):1827–1917, 2012.