A random compiler for fast Hamiltonian simulation

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(Dated: November 21, 2018)

The dynamics of a quantum system can be simulated using a quantum computer by breaking down the unitary into a quantum circuit of one and two qubit gates. The most established method are the Trotter-Suzuki decompositions, for which rigorous bounds on the circuit size depend on the number of terms \( L \) in the system Hamiltonian and the size of the largest term in the Hamiltonian \( \Lambda \). Consequently, Trotter-Suzuki is only practical for sparse Hamiltonians. Trotter-Suzuki is a deterministic compiler but it was recently shown that randomised compiling offers lower overheads. Here we present and analyse a randomised compiler for Hamiltonian simulation where gate probabilities are proportional to the strength of a corresponding term in the Hamiltonian. This approach requires a circuit size independent of \( L \) and \( \Lambda \), but instead depending on \( \lambda \) the absolute sum of Hamiltonian strengths (the \( \ell_1 \) norm). Therefore, it is especially suited to electronic structure Hamiltonians relevant to quantum chemistry. Considering propane, carbon dioxide and ethane, we observe speed-ups compared to standard Trotter-Suzuki of between \( 306 \times \) and \( 1591 \times \) for simulation times relevant for estimating energy eigenvalues up-to chemical accuracy.

Quantum computers could be used to mimic the dynamics of other quantum systems, providing a computational method to understand physical systems beyond the reach of classical supercomputers. The operation of a quantum computer is broken down into a discrete sequence of elementary one and two qubit gates. To simulate the continuous unitary evolution of the Schrödinger equation, an approximation must be made into a finite sequence of discrete gates. The precision of this approximation can be improved by using more gates. The standard approaches are the Trotter and higher order Suzuki decompositions \([1,3]\). While Trotter-Suzuki methods are technically efficient, for many purposes the required number of gates is impractically large. Analytic methods of establishing the required number of gates are rigorous, but numerical studies \([4,6]\) give empirical evidence that fewer gates may suffice. This suggests room for improvement either in our understanding of existing Trotter-Suzuki decompositions or new variants where rigorous analysis proves more accurate.

In addition to simulating dynamics of physical systems, we are often interested in learning the energy spectra of Hamiltonians. Assuming a sufficiently good ansatz for the ground state, we can combine quantum simulation with phase estimation to find the ground state \([7]\). When the physical system is a molecule with unknown electronic configuration, this is known as the electronic structure problem \([8,9]\) and is crucially important in chemistry and related applications. Therefore, any advances in quantum simulation can also be applied to ground state finding problems.

In addition to Trotter-Suzuki decompositions, there are numerous other approaches with better proven asymptotic performance \([10,15]\). However, these typically use additional ancillary qubits and complex use of oracles that may make them more resource intensive for small computational tasks. Indeed, a significant revelation in recent years has been that the asymptotic performance of different approaches to Hamiltonian simulation is not a good indicator of performance and that Trotter-Suzuki is still preferable for simulating some modest size systems \([6]\).

It has been observed several times that stochastic noise can be less harmful than coherent noise (see e.g. \([16,17]\)), which hints that randomisation might be useful for washing out coherent errors in circuit design. Poulin et al \([18]\) showed that randomness is especially useful in simulation of time-dependent Hamiltonians as it allows us to average out rapid Hamiltonian fluctuations. Campbell \([19]\) and Hastings \([20]\) have shown that random compiling can actually help reduce errors below what is feasible with a deterministic compiler. Since optimisation of Hamiltonian simulation circuits is a special case of compilation, one expects random compilers to helpful in this setting. Indeed, it was shown by Childs, Ostrander and Su that randomly permuting the order of terms in Trotter-Suzuki decompositions and leads to improved performance \([21]\). However, both deterministic and randomly permuted Trotter-Suzuki decompositions share the feature that rigorous analysis points towards a gate count that grows rapidly with the number of terms in the Hamiltonian. One might hope that a Hamiltonian with many terms would not be expensive to simulate if most of the terms are small, which is often the case in electronic structure problems. There is also some hope that analytic analysis has been overly pessimistic, though here we restrict our scope to what is rigorously proven.

Here we propose a simple and elegant approach to Hamiltonian simulation. Our proposal is similar to Trotter-Suzuki in that we implement a sequence of small rotations, without any use of ancillary qubits or complex circuit gadgets. Our key idea is that we exploit random choice of gates, much like Childs, Ostrander and Su, but we go beyond permutations of Trotter-Suzuki decompositions. We generate inherently new gate sequences where the frequency, rather than the strength, of each gate is proportional to the corresponding interaction strength in the Hamiltonian. Our simulation scheme can be seen as
a Markovian process, which is inherently random but biased in such a way that we stochastically drift toward the correct unitary with high precision. For this reason, we call it the quantum stochastic drift protocol, or simply qDRIFT. Unlike randomly permuted Trotter-Suzuki protocols, qDRIFT weights the distribution of gates in such a way that the gate count depends much more modestly on the number of terms in the Hamiltonian. For systems where the Hamiltonian contains a broad range of weak and strong terms — as occur naturally in molecular systems — we find that our approach can speed-up quantum computations by several orders of magnitude within regimes of practical interest. In quantum chemistry, phase estimation is performed using a controlled Pauli gate to achieve a desired precision \(\epsilon\), and prior work [22] has identified that the Trotter decomposition is limited to simulations of quantum systems with sparse interactions. For a family of simulation problems to be formally efficient, \(L\) must scale polynomially with the system size \(n\). These resource costs can be improved by considering higher-order Suzuki decompositions and our later analysis will optimise over all such approaches.

The Hamiltonian simulation problem.- We begin by restating the problem more formally. Consider a Hamiltonian

\[
H = \sum_{j=1}^{L} h_j \mathbb{I}_j
\]

(1)
decomposed into a sum of \(H_j\) each of which is Hermitian and normalised (such that the largest singular value of \(H_j\) is 1). We can always choose \(H_j\) so that the weighting \(h_j\) are positive real numbers. Herein we denote \(\lambda = \sum_j h_j\) and remark that this upper bounds the largest singular value of \(H\). The decomposition of the Hamiltonian should be such that for each \(H_j\) the unitary \(e^{i\tau H_j}\) can be implemented on our quantum hardware for any \(\tau\). Our goal is then to find an approximation of \(e^{i\tau H}\) into a sequence of \(e^{i\tau H_j}\) gates up-to some desired precision. We use the number of \(e^{i\tau H_j}\) unitaries to quantify the cost of the quantum computation, and we aim to minimise the number of such unitaries used. The simplest and most well known approach to Hamiltonian simulation is to use the Trotter decomposition. One divides \(U = e^{i\tau H}\) into \(r\) segments so that \(U = U_r^r\) with \(U_r = e^{i\tau H/r}\) and uses that

\[
V_r = \prod_{j=1}^{L} e^{ih_j H_j / r},
\]

(2)
approaches \(U_r\) in the large \(r\) limit. Furthermore, \(r\) repetitions of \(V_r\) will approach \(U\) in the large \(r\) limit, so \(V_r \to U\). The gate count in this sequence will be \(N = Lr\), so we would like to know the smallest \(r\) that suffices to achieve a desired precision \(\epsilon\). Analytic work on this problem (we use the analysis of Refs. [6, 21]) shows that the Trotter error is no more than

\[
\epsilon = \frac{L^2 \lambda^2 \tau^2}{r} e^{\lambda L / r},
\]

(3)
where \(\lambda := \max_j h_j\) is the magnitude of the strongest term in the Hamiltonian. The Trotter decompositions are composed of \(r\) segments with \(L\) unitaries in each segment. Solving for \(r\) we find approximately \(r \sim L^2 \lambda^2 L^2 / 2\epsilon\) segments are needed, leading to a total gate count of \(N \sim L^3 \lambda^2 L^2 / 2\epsilon\). Recall that \(L\) denotes the number of terms in the Hamiltonian; what one might call the interaction complexity. The \(L\) dependence entails that Trotter decompositions are limited to simulations of quantum systems with sparse interactions. For a family of simulation problems to be formally efficient, \(L\) must scale polynomially with the system size \(n\). These resource costs can be improved by considering higher-order Suzuki decompositions and our later analysis will optimise over all such approaches.

The order that we label the terms \(H_j\) will affect the unitary \(V_r\) implemented. The analytical results hold for any ordering of terms, but in practice one may observe varying performance for different orderings. If any ordering is valid then why not then randomly choose the ordering within each segment! This was precisely the idea of Childs, Ostrander and Su, who showed that randomly permuted Trotter decompositions actually improve the gate count to approximately \(N_{\text{RT}} \sim \sqrt{L^7 \lambda^3 \epsilon^3 / 3\epsilon}\) and further improvements are also made by considering randomized variants of higher-order Suzuki decompositions [21].

The qDRIFT protocol.- Our full algorithm is given as pseudocode in Fig. 1. Our approach pursues a similar spirit of using randomization to speed-up quantum computers. However, our approach has several novel features that differentiate it from prior uses of randomization. First, each unitary in the sequence is selected independently from an identical distribution (i.i.d sampling). Second, the interaction strength \(\tau_j\) is fixed to a constant \(\tau_j = \tau := t\lambda / N\) independent of the strength of the term \(h_j\) so we implement gates of the form \(e^{i\tau H_j}\). Third, the probability of choosing unitary \(e^{i\tau H_j}\) is weighted by the interaction strength \(h_j\), with normalisation of the dis-

**FIG. 1.** Pseudocode for the qDRIFT protocol.
We have that to leading order in $\tau$,
\[ E(\rho) = \rho + i \sum_j \frac{h_j}{\lambda} (H_j \rho - \rho H_j) + O(\tau^2). \]
(8)

Using Taylor series expansions of the exponentials, we have that to leading order in $t/N$.
\[ U_N(\rho) = \exp\left(i\frac{t}{N}H\right)\rho \exp\left(-i\frac{t}{N}H\right) = \rho + i \frac{t}{N} (H \rho - \rho H) + O\left(\frac{t^2}{N^2}\right), \]
(9)
where we have expanded out to leading in $t/N$. Using that $H = \sum_j h_j H_j$, we have
\[ U_N(\rho) = \rho + i \sum_j \frac{h_j}{N} (H_j \rho - \rho H_j) + O\left(\frac{t^2}{N^2}\right). \]
(10)

Comparing $E$ and $U_N$, we see that the zeroth and first order terms match whenever $\tau = \lambda t/N$. The higher order terms will not typically match and more careful analysis (see App. B) shows that the channels $E$ and $U_N$ differ by an amount bounded by
\[ \delta \leq \frac{2\lambda^2 t^2}{N^2} e^{2\lambda t/N} \approx \frac{2\lambda^2 t^2}{N^2}, \]
(11)
where the first inequality is rigorous and the approximation on the far-right side is very accurate for even modest $N$. An important technicality is that standard measures of precision are only meaningful for unitary circuits, for a random circuit the appropriate measure of error is instead the diamond norm distance [23] that we review in App. A.

Since $\delta$ is the approximation error on a single random operation $E$, the error of $N$ repetitions $E^N$ relative to the target unitary $U$ is then
\[ \epsilon = N\delta \lesssim \frac{2\lambda^2 t^2}{N}. \]
(12)

We see the total error decreases as we increase $N$. Setting $N = 2\lambda^2 t^2/\epsilon$ (rounding up to nearest integer) suffices to ensure that $N\delta$ is less than the required precision $\epsilon$. The exact value of $N$ is easily calculated, but again the aforementioned approximation is very good.

It is crucial that in our analysis $\epsilon$ is the error of the random circuit as measured by diamond distance. If we instead consider a specific instance of a randomly chosen unitary $V_j$ in Eq. (1), then the error will typically (on average) be much larger than $\epsilon$, with standard statistical arguments (see e.g. [18]) suggesting it would be closer to
It is counter-intuitive that the random circuit error is considerably less than the error of any particular unitary, so let us elaborate. If we initialise the quantum computer in state $|\psi\rangle$, then qDRIFT leads to state $|\Psi_j\rangle = V_j |\psi\rangle$ with probability $P_j$. If our experimental setup forgets (erases from memory) which unitary was implemented, then it prepares the mixed state

$$\rho = E^N((\psi\langle \psi|) = \sum_j P_j V_j |\psi\rangle \langle \psi| V_j^\dagger + \sum_j P_j |\Psi_j\rangle \langle \Psi_j|. \tag{13}$$

Since this channel is $\epsilon$-close in diamond distance to the ideal channel $U$, it follows that $\rho$ is $\epsilon$-close in trace norm distance to the target state $U(|\psi\rangle \langle \psi|) = U|\psi\rangle \langle \psi|U^\dagger$. Trace norm distance is the relevant quantity because it ensures that if we perform a measurement, then the probabilities of the outcomes (on state $\rho$) do not differ by more than $2\epsilon$ from the ideal probability given by $U|\psi\rangle \langle \psi|U^\dagger$. Provided we estimate expectation values over several runs, each using a new and independent randomly generated unitary, the precision of our estimate will be governed by $\epsilon$ rather than $\sqrt{\epsilon}$.

**Numerics.** The qDRIFT approach needs approximately $2L^2t^2/e$ gates. Since it does not explicitly depend on $L$, there are no sparsity constraints on the Hamiltonian other than that more terms tend to increase $\lambda$. The bounds for other Trotter-Suzuki formulae are given in terms of $\Lambda = \max_j h_j$, whereas qDRIFT is most naturally described in terms of $L$ and $\lambda = \sum_j h_j$. These quantities are related by $\lambda \ll \Lambda L$ and we expect qDRIFT may have an advantage when the Hamiltonians terms have varying weights so that $\lambda \ll \Lambda L$. To the best of our knowledge, qDRIFT is the only Trotterization scheme with rigorously proven dependence on $\lambda$ rather than $\Lambda$, though quantum signal processing techniques do also have this property [14] [15] [21]. Using $N_{qD}$ and $N_T$ for the gate counts of qDRIFT and randomised Trotter, we find the ratio is

$$\text{speedup} = \frac{N_T}{N_{qD}} = \sqrt{\frac{L^2 \Lambda^4 \epsilon}{12 \lambda^4 t}} \tag{14}$$

For a fixed Hamiltonian and fixed precision $\epsilon$, we see that as $t$ increases we must eventually find that $N_T < N_{qD}$. However, in the $\lambda \ll \Lambda L$ regime there will be a significant advantage to qDRIFT over randomised Trotter provided $t$ is not too large. While other randomised Suzuki decompositions have a slightly different scaling, we shall see that comparing against randomised Trotter captures the salient details.

In two recent studies of Hamiltonian simulation [6] [21], they have considered the 1D nearest neighbour Heisenberg chain. For this model, almost all terms in the Hamiltonian have a similar magnitude and so $\lambda \sim \Lambda L$. We studied the performance of qDRIFT for this model and found it offered no advantage, but this was expected. Rather, we emphasise that real physical systems do not have nearest neighbour interactions but long-range interactions that gradually decrease. The coulomb interaction is such an interaction and is crucial to the electronic structure problem. We have generated electronic structure Hamiltonian using the openFermion library [25] for propane, carbon-dioxide and ethane. We find these natural systems satisfy $\lambda \ll \Lambda L$ and so qDRIFT should perform favourably. We present our results in Fig. 2 using target precision $\epsilon = 10^{-3}$. Observe that qDRIFT offers a significant advantage at low $t$, which is often several orders of magnitude better than any prior Trotter-Suzuki decomposition. We remarked in our introduction that $t = 6000$ has been identified as relevant for phase estimation in quantum chemistry problems [22] and here we see speed-ups of $591 \times$, $306 \times$ and $1006 \times$ for propane, carbon dioxide and ethane (respectively). In more recent work [26], they show that using improved phase estimation techniques mean they slightly smaller $t$ values could be used for quantum chemistry and qDRIFT offers a larger speed-up at for smaller $t$ values. For longer time simulations beyond those relevant to near-term phase estimation problems, our advantage decreases and we eventually observe a cross-over at times around $t = 10^7 - 10^8$ where prior methods perform better. But this cross-over does not occur until the simulation time is so long that $10^{23} - 10^{25}$ gates are required. This is an extremely high gate count. Quantum error correction would certainly be needed and it is well known that to implement this many non-Clifford gates would require many billions of physical qubits even with generous hardware assumptions [26] [29]. For these molecules, any foreseeable device performing Hamiltonian simulation would significantly benefit from using qDRIFT over standard Trotter-Suzuki.

**Discussion.** A common setting is where $H_j$ are taken as tensor product of Pauli spin operators, then $e^{i\tau H_j}$ can be realised using Clifford gates and a single-qubit Pauli Z rotation [30]. When performing quantum error correction, the resource overhead of Clifford gates is negligible [27] [28] whereas the single-qubit Pauli Z rotation must be decomposed into a large number of single-qubit $T$ and Clifford gates. For optimal unitary compilers [30], the number of gates in this decomposition is fairly constant with respect to $\tau$. This further justifies simply counting the number of exponential terms of the form $e^{i\tau H_j}$. One further advantage is that qDRIFT uses many copies of the same rotation, with the same magnitude $\tau = \lambda/\sqrt{N}$, which can further simplify the implementation of gates. This is especially true when $\tau = n\pi/2^k$ for some integers $n$ and $k$, as the Pauli-Z rotations then belong to the Clifford hierarchy [31]. Once within the Clifford hierarchy, one has the option of directly distilling magic states that inject the desired gate without any additional compilation cost [32] [35] potentially reducing the cost compared to using Clifford+$T$ compilation. Interestingly, Duclos-Cianci and Poulin [33] give a short discussion of how their magic state distillation protocol could be used in a Hamiltonian simulation scheme using a modified-Trotter decomposition where the gates all have the same $\tau$ value. While they allude to such a Hamiltonian simulation protocol, they do not provide any details or error analysis and nor did they suggest that randomi-
sation would be part of the protocol. Lastly, we note using higher powers of the Hamiltonians (e.g. $H^2, H^3, \ldots$) may no longer be prohibitive since we are not as sensitive to the number of Hamiltonian terms. This hints that qDRIFT may also be useful for learning the energy of excited states 30, 38.

Acknowledgements.- This work was supported by the EPSRC (grant no. EP/M024261/1). We thank Simon Benjamin, Xiao Yuan and Sam McArdle, for discussions on the electronic structure problem and providing molecular Hamiltonians taken from openFermion. For regular discussions on Hamiltonian simulation we thank John Clark, David White, Ben Jones and George O’Brien. We thank Yuan Su for sharing details regarding Ref. [21]. For comments on the manuscript, we thank Dominic Berry.

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Appendix A: Error measures

In these appendices, we switch to more mathematical notation. We use $|| \ldots ||$ to denote the operator norm or Schatten-$\infty$ norm, which is equal to the largest singular value of an operator. We use $|| \ldots ||_1$ for the trace norm or Schatten 1-norm, defined as $||Y||_1 := \text{Tr}[\sqrt{Y^\dagger Y}]$, which is equal to the sum of the singular values of an operator. Throughout, we use the diamond norm distance as a measure of error between two channels. The diamond distance is denoted

$$d_\diamond(\mathcal{E},\mathcal{N}) = \frac{1}{2}||\mathcal{E} - \mathcal{N}||_\diamond,$$  

where $|| \ldots ||_\diamond$ is the diamond norm

$$||\mathcal{P}||_\diamond := \sup_{||\rho||_1=1}||\mathcal{P} \otimes \mathbb{1}(\rho)||_1,$$
Similarly, we can define $L$. We have that $H$ can be found by employing the Hastings-Campbell mixing lemma \[19, 20\]. Each random operator of qDRIFT implements a single randomly chosen gate so that

$$
\mathcal{E} = \sum_j p_j e^{i\mathcal{L}_j} = \sum_j \frac{h_j}{\lambda} e^{i\mathcal{L}_j},
$$

(B6)

which expands out to

$$
\mathcal{E} = \mathbb{1} + \left( \sum_j \frac{h_j}{\lambda} \mathcal{L}_j \right) + \sum_j \frac{h_j}{\lambda} \sum_{n=2}^\infty \frac{\tau^n \mathcal{L}_j^n}{n!},
$$

(B7)

and

$$
= \mathbb{1} + \frac{\tau}{\lambda} \mathcal{L} + \sum_j \frac{h_j}{\lambda} \sum_{n=2}^\infty \frac{\tau^n \mathcal{L}_j^n}{n!},
$$

(B8)

where in the second line we have used Eq. (B4). This is to be compared against

$$
\mathcal{U}_N = e^{t\mathcal{L}/N}
$$

(B9)

We see the first two terms of $\mathcal{E}$ and $\mathcal{U}_N$ will match whenever $\tau = \lambda t/N$. Using this value for $\tau$, we have

$$
||\mathcal{U}_N - \mathcal{E}||_o = \left| \left| \sum_{n=2}^\infty \frac{1}{n! N^n} - \sum_j \frac{h_j}{\lambda} \sum_{n=2}^\infty \frac{\lambda^n t^n L_j^n}{n! N^n} \right| \right|_o
$$

$$
\leq \sum_{n=2}^\infty \frac{1}{n! N^n} + \sum_j \frac{h_j}{\lambda} \sum_{n=2}^\infty \frac{\lambda^n t^n |\mathcal{L}_j^n|}{n! N^n}
$$

The first inequality uses the triangle inequality and that all variables are positive real numbers. Next we use multiplicativity combined with Eq. (B3) and Eq. (B5) to conclude that $||\mathcal{L}_j^n||_o \leq ||\mathcal{L}_j||_o \leq (2\lambda)^n$ and $||\mathcal{L}_j^n||_o \leq n! N^n$, which leads to

$$
||\mathcal{U}_N - \mathcal{E}||_o \leq \sum_{n=2}^\infty \frac{1}{n! N^n} \left( \frac{2\lambda t}{N} \right)^n + \sum_j \frac{h_j}{\lambda} \sum_{n=2}^\infty \frac{1}{n! N^n} \left( \frac{2\lambda t}{N} \right)^n
$$

$$
= 2 \sum_{n=2}^\infty \frac{1}{n!} \left( \frac{2\lambda t}{N} \right)^n.
$$

The last equality uses that $\sum_j h_j = \lambda$ and collects together the pair of equal summations. Since our definition of diamond distance includes a factor 1/2, we have

$$
d(\mathcal{U}_N, \mathcal{E}) \leq \sum_{n=2}^\infty \frac{1}{n!} \left( \frac{2\lambda t}{N} \right)^n.
$$

(B10)

Next, we use the exponential tail bound (see Lemma F.2 of Ref [6]) that states that for all positive $x$ we have

$$
\sum_{n=2}^\infty \frac{x^n}{n!} \leq \frac{x^2}{2} e^x,
$$

(B11)

which we use with $x = 2\lambda t/N$ so that

$$
d(\mathcal{U}_N, \mathcal{E}) \leq \frac{2\lambda^2 t^2}{N^2} e^{2\lambda t/N} \approx \frac{2\lambda^2 t^2}{N^2}.
$$

(B12)
The approximation on the right is very accurate in the large $N$ limit. This gives Eq. (11) of the main text. Since the diamond distance is subadditive [23] under composition we have that

$$d_c(U, E_N) \leq N d(U_N, E) = \frac{2\lambda^2 t^2}{N} e^{2\lambda t/N} \approx \frac{2\lambda^2 t^2}{N}.$$  \hfill (B13)

Appendix C: Bounding higher order error terms

Here we reproduce for convenience some results on the Trotter and Suzuki decompositions. All these results are taken from Childs, Ostrander and Su [21].

First, we consider the Trotter decomposition, and begin by defining

$$a_{\text{TROTT}} := \frac{\lambda^2 t^2}{r^2} e^{\lambda t/r},$$  \hfill (C1)

$$b_{\text{TROTT}} := \frac{\lambda^3 t^3}{3r^3} e^{\lambda t/r}.$$  \hfill (C2)

From this, one can show that deterministic and randomised Trotter decompositions have errors

$$\epsilon_{\text{det \ TROTT}} \leq \frac{r}{2} a_{\text{TROTT}},$$  \hfill (C3)

$$\epsilon_{\text{random \ TROTT}} \leq \frac{r}{2} (a_{\text{TROTT}}^2 + 2b_{\text{TROTT}}).$$  \hfill (C4)

One can see that if $b_{\text{TROTT}} \ll a_{\text{TROTT}}$ there is a significant advantage to the randomised approach. To determine gate counts one must solve to find the smallest integer $r$ such that the errors are below some target $\epsilon$. Since the Trotter decomposition has $r$ segments and each segment contains $L$ gates, the total gate count is $Lr$.

Next, we consider the $2k$-order Suzuki decompositions, starting with the definitions

$$a_{2k-\text{SUZUKI}} := 2 \left( \frac{2 \cdot 5^{k-1} L}{n^{k+1}} \right)^{2k+1} e^{2 \cdot 5^{k-1} \Lambda t/r},$$  \hfill (C5)

$$b_{2k-\text{SUZUKI}} := 2 \left( \frac{2 \cdot 5^{k-1} L}{n^{k+1}} \right)^{2k+1} e^{2 \cdot 5^{k-1} \Lambda t/r}.$$  \hfill (C6)

From this, one can show that deterministic and randomised $2k$-order Suzuki decompositions have errors bounded by

$$\epsilon_{2k-\text{SUZUKI}} \leq \frac{r}{2} a_{2k-\text{SUZUKI}},$$  \hfill (C7)

$$\epsilon_{2k-\text{SUZUKI}} \leq \frac{r}{2} (a_{2k-\text{SUZUKI}}^2 + 2b_{2k-\text{SUZUKI}}).$$  \hfill (C8)

Again, if $b_{\text{TROTT}} \ll a_{\text{TROTT}}$ there is a significant advantage to the randomised approach. However, in the limit $k \to \infty$ both $b_{\text{TROTT}}$ and $a_{\text{TROTT}}$ approach a similar order of magnitude. As such, the advantage of randomised Suzuki decompositions disappears as $k$ increases, which was numerically reported by Childs, Ostrander and Su [21].

To determine gate counts one must solve to find the smallest integer $r$ such that the errors are below some target $\epsilon$. Since the $2k$-order Suzuki decompositions have $r$ segments and each segment contains $2 \cdot 5^{k-1} L$ gates, the total gate count is $2 \cdot 5^{k-1} Lr$. 