Switching dynamics of reconfigurable perfect soliton crystals in dual-coupled microresonators

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Abstract: Dual-coupled structure is typically used to actively change the local dispersion of microresonator through controllable avoided mode crossings (AMXs). In this paper, we investigate the reconfigurability of perfect soliton crystals (PSCs) based on dual-coupled microresonators. The switching dynamics of PSCs are numerically simulated using perturbed Lugiato-Lefever equation (LLE). Nonlinear phenomena such as solitons rearranging, merging and bursting are observed in the switching process. Specially, for the first time, we have discovered an unexplored PSC region in the microcomb power-detuning phase plane. In PSC region, the soliton number (N) of PSC state can be switched successively and bidirectionally in a defect-free fashion, verifying the feasibility and advantages of our scheme. The reconfigurability of PSCs would further liberate the application potential of microcombs in a wide range of fields, including frequency metrology, optical communications, and signal-processing systems.

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1. Introduction

Dissipative Kerr solitons (DKSs) are self-organized pulses relying on double balance of dispersion and nonlinearity as well as parametric gain and dissipation in driven dissipative nonlinear systems [1]. The observation of DKSs in optical microresonators makes it possible to generate broadband and mode-locked microcombs with chip-scale footprint and high repetition rate ranging from GHz to THz [2]. Soliton microcombs have been proved to be a revolutionary fundamental technology in many important areas such as large-capacity optical communications [3,4], metrology [5–7], dual comb technology [8–10], and microwave photonics [11–16]. Moreover, optical microresonators provide an ideal test bench for studying nonlinear soliton physics such as soliton switching [17], crystallization [18] and breathing [19].

Soliton crystal (SC) state is a special kind of DKS states and shows a variety of complex soliton structures [18]. SCs have been generated in several platforms such as silica [18], Hydex glass [20–22], silicon nitride [23] and lithium niobate [24,25]. Among various soliton crystal states, perfect soliton crystal (PSC) state is characterized by a set of equally spaced soliton pulses in time domain and correspondingly enhanced comb lines (supermodes) separated by multiple free spectral ranges (FSRs) in frequency domain, which shows great application potential in a wide range of fields. In contrast to multiple-DKS state, PSC state could be generated in a deterministic fashion by standard forward tuning method [2] as long as the pump laser does not go through the spatiotemporal chaos (STC) and transient chaos (TC) regions in the microcomb power-detuning phase plane [23].

The formation of PSCs in microresonator was linked to the presence of avoided mode crossings (AMXs) [18], AMX induces a modulation on the intracavity background which
traps the soliton pulses [26] and leads them to distribute evenly on the microresonator circumference [23]. Moreover, the spectral position of AMX determines the soliton number $N$ of PSCs as well as the comb line space (CLS) of the supermodes. In a single microresonator, AMXs occur as a result of coupling between transverse mode families due to inevitable fabrication errors [27] and is therefore an inherent property which is hard to control and modify manually [28]. As a result, in order to generate PSCs with expected soliton numbers, the parameters of pump laser should be chosen carefully according to the spectral position of AMX. So, it is hard to deterministically change the repetition rate of PSCs in a single microresonator unless repumping another resonance [29] which is sophisticated to implement.

Although some works show that the spectral position and strength of AMXs are sensitive to temperature [23,25,30] and the switching phenomenon of PSCs has been observed [23,25], it is still unpredictable to some extent and cannot be controlled successively. Besides, although the dynamics of PSC with a fixed soliton number have been investigated [23], the switching dynamics between different PSC states remain mainly unexplored.

In this paper, a dual-coupled microresonator scheme is proposed for realizing reconfigurable PSCs taking advantage of its flexible control on the spectral position and strength of AMXs. The tuning procedure of AMX is modeled by introducing a time-variant local dispersion to the perturbed LLE. By solving the master equation using an adaptive step size Runge-Kutta algorithm, the switching dynamics of PSCs is numerically investigated. Specially, for the first time, we have discovered an unexplored PSC region in the microcomb power-detuning phase plane which enables the PSC state to recover from defective SC state through soliton bursting process. Based on PSC region, we can realize bidirectional and successive switch of PSCs in a defect-free fashion which verifies the feasibility and advantages of our proposed scheme.

2. Principle

The schematic diagram of the proposed dual-coupled microresonator model is shown in Fig. 1. The main microresonator ($\mathcal{P}$) is coupled to an auxiliary microresonator ($\mathcal{C}$) with a slightly different free spectral range (FSR). In contrast to a single microresonator, the AMXs are intentionally introduced by mode interactions between two microresonators. Moreover, the spectral position and strength of AMXs could be controlled by tuning the resonance of the auxiliary microresonator which has already been experimentally demonstrated to initiate the four-wave mixing (FWM) and facilitate comb generation in normal dispersion microresonators [31–33]. We adapt this tuning method here to realize successive and bidirectional switch of PSCs in optical microresonators with anomalous dispersion.

Fig. 1. Schematic diagram of the dual-coupled microresonator system.

The fields in dual-coupled microresonators obey the following coupled-mode equations [29,31,34]:
\[
\frac{d a^{(P)}}{dt} = \left( j \omega^{(P)} - \kappa^{(P)} / 2 \right) a^{(P)} + jg a^{(C)}
\]
\[
\frac{d a^{(C)}}{dt} = \left( j \omega^{(C)} - \kappa^{(C)} / 2 \right) a^{(C)} + jg^* a^{(P)}
\]
where \( P \) and \( C \) denote the main and auxiliary microresonator, respectively. \( a \), \( \omega \), \( \kappa \) and \( g \) are the field amplitude, resonant frequency, loss rate and coupling rate, respectively.

The eigenfrequencies of the coupled-microresonator system can be derived from Eqs. (1) and (2) as
\[
\omega = \left( \omega^{(P)} + \omega^{(C)} \right) / 2 + j(\kappa^{(P)} + \kappa^{(C)}) / 4
\]
\[
\pm \sqrt{\left( (\omega^{(P)} + \omega^{(C)}) / 2 + j(\kappa^{(P)} - \kappa^{(C)}) / 4 \right)^2 + G}
\]
The real parts of \( \omega \) are new resonant frequencies under AMX. \( G = g g^* = |g|^2 \) describes the coupling strength which affects the deviation between new resonant frequency and its origin counterpart. By tuning the resonant frequency \( \omega^{(C)} \), taking advantage of Vernier effect [35], the spectral position of AMXs can be successively switched from one longitude mode to another. Moreover, the coupling strength is also controllable because \( G \) is inversely related to the resonant frequency difference \( \omega^{(P)} - \omega^{(C)} \).

During the tuning process, we assume that only two adjacent modes of \( \omega^{(P)} \) (i.e. \( \omega_0^{(P)} \) and \( \omega_1^{(P)} \), the subscript denotes the mode number relative to pumped mode \( \omega_0^{(P)} \)) are involved in AMXs, which could be realized by designing the FSR difference between the main and auxiliary microresonators appropriately [31,32,35]. The new resonant frequencies of the main microresonator can be expressed by
\[
\bar{\omega}^{(P)}_{\mu}(t) = \omega^{(P)}_{\mu} + \delta^{(P)}_{\mu} \Delta_\mu(t) \quad , \quad \mu = N, N + 1
\]
where \( \Delta_\mu(t) \) denotes the tuning process of the AMXs and \( \delta^{(P)}_{\mu} \) is the Kronecker delta.

We simulated the tuning process by introducing the time-variant resonant frequencies Eq. (4) into standard LLE [36], the resulting perturbed LLE is written as
\[
\frac{\partial A^{(P)}_{\mu}(t)}{\partial t} = -\frac{\kappa^{(P)}}{2} + i(2\pi \delta) + iD^{(P)}_{in}(\mu,t) \bar{A}^{(P)}_{\mu}
\]
\[-ig_o F \left[ A^{(P)} \right] A^{(P)}_{\mu} + \delta^{(P)}_{\mu} \sqrt{\kappa_{es} s_{in}}
\]
\[D^{(P)}_{in}(\mu,t) = \delta^{(P)}_{\mu}(t) / 2 \]
\[D_{s}^{(P)} = -eD_{2}^{(P)} \beta_{s} \quad , \quad \mu = N, N + 1
\]
where \( \bar{A}^{(P)}_{\mu} \) and \( A^{(P)}_{\mu} \) are the spectral and temporal envelopes of intracavity field, respectively [related via \( \bar{A}^{(P)}_{\mu}(t) = \sum_{\mu} A^{(P)}_{\mu} e^{-i\omega^{(P)} \mu} \)], \( \mu \) is the relative mode number, \( \kappa^{(P)} \) is the loss rate of pumped mode, \( 2\pi \delta = \omega^{(P)}_{\mu} - \omega_0 \) is the laser frequency detuning, \( g_o \) denotes the nonlinearity coefficient (see Ref. [19] for the specific definition), \( F[] \) represents the \( \mu \)th component of the Fourier series, \( \kappa_{es} \) is the external coupling rate, \( s_{in} \) denotes the pump power, \( D_{s}^{(P)} \) is the FSR of the main resonator, \( D^{(P)}_{in} \) is the integrated dispersion and we only consider the second-order dispersion \( D_{s}^{(P)} \) which is related to the group velocity dispersion (GVD) \( \beta_{s} \) via \( D_{s} = -cD_{s}^{(P)} \beta_{s} / n_o \).

3. Numerical simulation and discussion
The switching dynamics of PSCs in dual-coupled microresonators is numerically simulated using the theory developed in Section 2 (Eq. (5) is solved via an adaptive step size Runge-Kutta algorithm [37]). The physical parameters used in the simulations are adapted from a typical Si$_3$N$_4$ microresonator [23] and the main parameters are listed in Table 1. First of all, we set the AMX as a fixed value $\Delta_{\text{AMX}}/2\pi = -140$ MHz and the pump power as $P_{\text{in}} = 0.13$ W which is below $P_{\text{sc}}$ (see Ref. [23]) for deterministic PSC generation. The PSC state with soliton number $N = 16$ is generated using the standard forward tuning procedure [2] initialized with one photon per mode noise. Fig. 2 presents the generation details, the intracavity waveform evolves from Turing patterns to PSC ($N = 16$) via chaotic state [see Fig. 2(b)]. Different from Turing patterns whose pulse number is related to the spectral location of maximum parametric gain [38,39], the soliton number of PSC is obviously determined by the spectral position of AMX $\Delta_{\text{AMX}}$ which leads to obvious modulation on intracavity background as shown in Fig. 2(a). Moreover, the intracavity energy of PSC is distributed in the equally spaced supermodes with CLS $N \times$ FSR as shown in Fig. 2(c).

### Table 1. Physical parameters of the main microresonator [23]

| Item                      | Symbol | Value                      |
|---------------------------|--------|----------------------------|
| Loss rate                 | $\kappa$ | $2\pi \times 200\text{MHz}$ |
| External coupling rate    | $\kappa_{\text{ex}}$ | $2\pi \times 100\text{MHz}$ |
| FSR                       | $\Delta_{\text{FSR}}$ | $2\pi \times 95.4\text{GHz}$ |
| Second-order dispersion   | $\beta_2$ | $2\pi \times 1.3\text{GHz}$ |
| Refractive index          | $n_0$ | 1.98                       |
| Nonlinearity              | $g_0$ | 1.86 rad / s               |

In the next subsections, we will start from a stable PSC solution as generated above and then investigate the switching dynamics during the process of soliton number decrease and increase respectively. In Section 3.A, the soliton number of PSC is decreased by tuning the
AMXs towards the pumped mode. In Section 3.B, we tuning the AMXs backwards for increasing the soliton number and found that the PSC state will evolve to defective SC state with a vacancy. For solving the vacancy problem, we implemented a back-and-forth tuning procedure on pump laser to recover the PSC state and discovered an unexplored PSC region for the first time. The PSC region is then fully simulated in the microcomb power-detuning phase plane and utilized to successively increase the soliton number of PSC in a defect-free fashion.

A. Soliton number decrease

In order to decrease the soliton number of PSC, the spectral position of AMX is gradually tuned from mode number \( \mu = 16 \) to \( \mu = 15 \) which could be realized by tuning the resonant frequency of auxiliary microresonator [31]. We simulate this tuning process by varying the strength of AMXs linearly with time as shown in Fig. 3(b), the \( \Delta_{\mu}/2\pi \) is tuned from -140 to 0 MHz while \( \Delta_{\mu}/2\pi \) is tuned from 0 to -140 MHz and then settled down. Fig. 3(c) shows the evolution of intracavity waveforms during the tuning process, we find that the switching process does not start immediately after tuning because the new modulation background is not completely formed. From a spectral point of view, as shown in Fig. 3(d), the extra comb lines between supermodes take time to grow up gradually from the noise floor and the modulation strength of intracavity background reaches its maximum when the optical spectrum is filled up by extra comb lines. After that, the new intracavity background mismatches original PSC with \( N = 16 \) and pushes soliton pulses to rearrange. During the rearrangement, all solitons adjust their relative positions to match the modulated background while two of them are pushed close to each other and the energy of one soliton is absorbed by the other as shown in the right panel of Fig. 3(g). Finally, two solitons merge into one and cause a wave splash as shown in the inset of Fig. 3(c) which has been experimentally observed in Ref. [40]. In the merging process, the overall intracavity power will increase firstly which may result from the resonance between merged solitons and the modulated intracavity background. After merging, the new soliton acts like a breather and experiences damped oscillations [see inset of Fig. 3(e)] before reaching stable state and form new PSC state with \( N = 15 \) eventually. In frequency domain [see Fig. 3(d)], the extra comb lines between new supermodes (CLS = 15×FSR) decay to noise floor and the spectrum evolves to the pattern of PSC with one less soliton pulse. The switching process could also be reflected in the stair-like pattern of intracavity power due to the annihilation of one soliton as plotted in Fig. 3(e) which is similar with the switching process of traditional multiple-DKS states [2,17,41], but it should be noted that there is no soliton rearrangement and merging in the switching process of multiple-DKS states.
Furthermore, to verify the feasibility of successively decreasing the soliton number, we implement the numerical experiment using the same strategy as described above. By tuning the AMXs from mode number $\mu = 16$ to $\mu = 12$, we realized successive decrease of soliton number as shown in Fig. 4. The final value of $\Delta'\mu$ in each switching process should be set appropriately as mentioned in Ref. [23], but the specific value range is not the point of this work. In our simulation, we set the value of $\Delta'\mu$ to eliminate the integrated dispersion of mode $\mu'$ (i.e. $D_{\text{int}}^{(P)}(\mu')$) in each switching process which could be experimentally realized by controlling the frequency difference between the resonances of two microresonators. As shown in Fig. 4(a), one step in the stair-like intracavity power represents a PSC state with specific soliton number and the switching from one step to another represents the merging of two solitons into one. It should be noted that we have increased the tuning speed of AMXs for reducing computation load which causes higher power spikes in Fig. 4(a) than in Fig. 3(e). We also find that the PSC with less soliton number takes longer to finish the switching process as shown in Fig. 4(b) and this phenomenon could be explained qualitatively as: the maximum strength of AMXs is decreased successively in each switching process so it is more time-consuming for the modulation background to completely form, and in PSC state with smaller $N$, the distance between adjacent soliton pulses is further so it would take longer for them to rearrange, interact with each and finally complete the switching process.

Fig. 4. Evolution process of successive soliton number decrease. (a) Intracavity power evolution. (b) Intracavity waveform evolution. (c) Intracavity optical spectrum evolution.

B. PSC region and soliton number increase

To explore the full reconfigurability of PSCs in the proposed dual-coupled microresonator system, we reverse the tuning direction described in Section 3.A. As shown in Fig. 5(b), we tune the spectral position of AMX from mode number $\mu = 12$ to $\mu = 13$, the DKS pulses start to rearrange after new modulated intracavity background is formed. Different from the soliton number decrease process, the 12 DKS pulses are rearranged at 13 potential traps introduced by modulated background and the SC loss its perfection with a vacancy [see Fig.
The defective SC state exhibits a “palm-like” optical spectrum [see Fig. 5(e)] which has been experimentally discovered in Ref. [18,20,42]. The “palm-like” spectrum is actually resulted from the superposition of PSC ( \( N = 13 \) ) and an out-of-phase single DKS pulse. From the perspective of application, the extra comb lines between the supermodes are always not wanted and need to be eliminated.

Fig. 5. Evolution details of PSC when the AMXs are tuned away from pumped mode. (a) Soliton intensity patterns sampled at two moments which correspond to a stable PSC state with \( N = 12 \) and a defective SC state with a vacancy (see red dashed circle). (b) Intracavity waveform evolution. Soliton rearrangement is shown in black dashed box. (c) Intracavity optical spectrum evolution. (d) Time domain details at the end of (b). (e) Frequency domain details at the end of (c).

In order to recover the PSC state from defective SC state, we attempted to melt the defective SC firstly and then recrystallize it [20,23], which can be experimentally implemented by simply tuning the frequency or power of pump laser in the microcomb power-detuning phase plane [39]. As shown in Fig. 6(a), the pump laser is started from stable DKS region (1 in Fig. 6(a)) and then tuned to MI region by change its frequency (1-2 in Fig. 6(a)) or power (1-3 in Fig. 6(a)). After entering the MI region and melting the SC, we tune the pump laser back to stable DKS region. As shown in Fig. 6(b) and (c), in both tuning routes, the chaotic MI pulses are successfully recrystallized to PSC state with \( N = 13 \) which verifies the feasibility of the back-and-forth tuning method for recovering the PSC states from defective SC states.

Unexpectedly, before entering the MI region, a soliton pulse has already burst on top of the SC’s vacancy as emphasized by the red dashed boxes in the middle panels of Fig. 6(b) and (c). As a result, the PSC state is directly recovered from defective SC state without entering the MI region. This unexpected phenomenon implies that there is an unexplored region in the microcomb power-detuning phase plane and we name it as the PSC region.
Fig. 6. Recovery process of PSC through soliton melting and recrystallization. (a) The microcomb power-detuning phase plane. Different stability regions are distinguished by different colors. The pump laser is tuned between (1-2) and (1-3) by controlling its frequency and power respectively. (b) and (c) The tuning process of pump laser (top panel), intracavity waveform evolution (middle panel), and intracavity optical spectrum evolution (bottom panel). The soliton bursting phenomena are emphasized by red dashed boxes.

In order to determine the specific boundary of PSC region, we initialize the perturbed LLE with a stable defective SC state (similar to Fig. 5(d)) at different pump power and then gradually decrease the laser detuning to find the edge of PSC region. The simulated PSC region is shown in Fig. 7, it is overlapped with traditional stable DKS region and breathers region. When the pump laser is operated in the PSC region, defective SCs will switch to stable PSC state (overlapped with stable DKS region) or breathing PSC state (overlapped with breathers region) through soliton bursting process as shown in Fig. 6, and we find that the PSC state can never be recovered once the pump power is set to be higher than $P_{sc}$. Moreover, it should be noted that the range of PSC region is related to the strength of AMX and the number of vacancies in defective SC state. Specifically, the right edge of PSC region will extend outwards a little when the strength of AMX is increased within an appropriate range or there are fewer vacancies in the defective SC state.

Fig. 7. The microcomb power-detuning phase plane. The PSC region is shaded with grey and overlapped with the stable DKSs and breathers region. Specially, we also find that the PSC state could be directly generated in PSC region from noise. As shown in Fig. 8, we set the parameters of pump laser in PSC region and initialize the simulation with one photon per mode noise, PSC state ($N = 13$) is directly generated without any tuning procedure. The result could explain why the PSC states can be directly generated by pumping at the red-detuned side of the resonance [25].
Fig. 8. Generation details of PSC \((N=13)\) state in PSC region. (a) Intracavity waveform evolution. (b) Intracavity optical spectrum evolution.

To verify the application of PSC region for increasing the soliton number of PSC in a defect-free fashion, we repeat the simulation as shown in Fig. 5 with pump laser in the PSC region. The simulated result is shown in Fig. 9, we can observe that the vacancy in defective SC state is occupied by a bursting soliton in a very short time after rearrangement and the PSC state with \(N=13\) is recovered.

Fig. 9. Switching details of soliton number increase. (a) Soliton intensity patterns sampled at two moments which correspond to soliton bursting and a stable PSC state with \(N=13\). (b) Intracavity waveform evolution. The inset shows the local detail of soliton bursting process. (c) Intracavity optical spectrum evolution.

Furthermore, in PSC region, we did the numerical experiment for successively increasing the soliton number of PSC. As shown in Fig. 10, the soliton number is successively switched from \(N=13\) to \(N=16\) and the dynamics in each switching process follow the same way in Fig. 9. The intracavity power evolves like upward steps and each step represents a specific
PSC state. As shown in the inset of Fig. 10(a), the soliton bursting process can be indicated from the bulge on evolution curve of intracavity power. After bursting, the soliton behaves like a damped breather before stabilizing down to form a new stable PSC state. Finally, combined with successive soliton number decrease demonstrated in Section 3.A, the full reconfigurability of PSC is verified in the proposed dual-coupled microresonator scheme, benefiting from the discovery of PSC region.

Fig. 10. Evolution process of successive soliton number increase. (a) Intracavity power evolution. The inset shows the local details of soliton bursting. (b) Intracavity waveform evolution. (c) Intracavity optical spectrum evolution.

4. Conclusion

In this paper, a dual-coupled microresonator structure is proposed and theoretically modeled to realize reconfigurable PSCs. The switching dynamics of PSC states are numerically simulated using perturbed LLE with time-variant AMXs. Pushed by newly generated intracavity modulation background, original PSC pulses are rearranged to switch to new PSC state accompanied with soliton merging, bursting and breathing. Importantly, we have discovered an unexplored PSC region in the microcomb power-detuning phase plane which enables the soliton number of PSC to be switched successively and bidirectionally in a defect-free fashion, verifying the feasibility and advantages of our scheme. From the perspective of theory, the presented simulation results could give some inspiration for nonlinear soliton dynamic research. From the perspective of application, the discovery of the PSC region will help to develop reconfigurable microcombs with bidirectionally tunable repetition rates which can meet different demands of applications such as optical communication, signal process and metrology.

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Disclosures.

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