Revisiting on-shell renormalization conditions in theories with flavour mixing

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Abstract

In this review, we present a derivation of the on-shell renormalization conditions for scalar and fermionic fields in theories with and without parity conservation. We also discuss the specifics of Majorana fermions. Our approach only assumes a canonical form for the renormalized propagators and exploits the fact that the inverse propagators are non-singular in $\varepsilon = p^2 - m_n^2$, where $p$ is the external four-momentum and $m_n$ is a pole mass. In this way, we obtain full agreement with commonly used on-shell conditions. We also discuss how they are implemented in renormalization.
1 Introduction

On-shell renormalization conditions in theories with inter-family or flavour mixing \[1, 2, 3, 4, 5, 6\] are quite important in view of the experimentally established quark and lepton mixing matrices \[7\] and the mixing between the photon and the \(Z\) boson in the Standard Model—see for instance \[8, 9\]. In theories beyond the Standard Model, mixing of new fermions and scalar mixing might occur as well. However, we feel that several aspects of the derivation of the on-shell renormalization conditions remain a bit vague for the general reader of the relevant literature and should be discussed in more detail. In this review, we present a consistent way of deriving the on-shell conditions, first for scalar, then for fermionic fields in theories with and without parity conservation and finally for Majorana fields, for the mixing of \(N\) fields. Our approach is solely based on the pole structure of the \(N \times N\) propagator matrix and relies on the fact that the inverse propagator has no singularity in \(p^2\), where \(p\) is the external momentum. We also review the counting of the number of renormalization conditions and demonstrate that this number coincides with the number of degrees of freedom in the counterterms, except for overall phase factors that remain free for the fermionic fields in the case of Dirac fermions. Our results agree, of course, with the ones derived in \[2\]. Explicit formulas for the field strength renormalization constants and mass counterterms in the above-mentioned theories are given for the lowest non-trivial order.

Our discussion is based on two assumptions:

1. All \(N\) physical masses \(m_i\) are different.

2. The \(N\) poles in the propagator matrix are located in a region where absorptive parts are absent or can be neglected.

Some remarks relating to these assumptions are in order. Equality of two or more masses would require a “flavour” symmetry, but dealing with on-shell renormalization in the presence of such a symmetry is beyond the scope of this review.\[1\] The on-shell renormalization in this review deals with the dispersive parts of the scalar and fermionic propagators. If large imaginary parts appear in the higher order corrections to the propagators, a treatment of their absorptive parts becomes relevant, but lies beyond the scope of this review as well. Still, the renormalization conditions derived here are fully applicable in the regions of \(p^2\) where the absorptive parts vanish. Elsewhere, they can be used by simply inserting only the dispersive parts of the self-energy functions into the conditions. Another possible approach is to use complex masses as well as complex counterterms in the so-called complex-mass scheme. A treatment of this approach can \(e.g.\) be found in \[10, 11\]. Since our paper is intended as a pedagogical review and the distinction between dispersive an absorptive parts in the propagator matrix plays an important role in our presentation, we have included, for the sake of completeness, a discussion of this issue as an appendix.

The treatment of on-shell conditions in our paper is based on an expansion in \(p^2 - m_n^2\) around each pole mass \(m_n\), for both propagator and inverse propagator, whereas the

\[\text{footnote}1\text{This would, for instance, change the aforementioned counting of number of renormalization conditions and degrees of freedom in the counterterms.}\]
authors of [6] make use of exact matrix relations between the propagator matrix and its inverse. In this sense, our treatment of fermions is complementary to that of [6].

The plan of the paper is as follows. In section 2 we discuss mixing of real scalar fields, whereas fermions are treated in section 3 in the case of parity conservation. The complications which arise when parity is violated are elaborated in section 4. This section contains also a subsection on the on-shell renormalization of Majorana fermions. After a summary in section 5, the emergence of dispersive and absorptive contributions to the propagator is covered in appendix A in the framework of the Källén–Lehmann representation. Some computational details in the treatment of fermions are deferred to appendix B. The condition on the propagator matrix which arises in the case of Majorana nature of the fermions is derived in appendix C.

In the following, we will use \( k, l \) as summation indices whereas \( i, j, n \) do not imply summation.

## 2 Scalar propagator

### 2.1 On-shell conditions

We first study the scalar propagator for \( N \) real scalar fields, which is a simple and instructive case to begin with. Here the propagator is an \( N \times N \) matrix

\[
\Delta(p^2) = \left( \Delta_{ij}(p^2) \right),
\]

where \( p^2 \) is the Minkowski square of the four-momentum \( p \). We assume that all masses \( m_n \) of the scalars are different. We stress that \( \Delta(p^2) \) is the renormalized propagator. Defining

\[
\varepsilon \equiv p^2 - m_n^2,
\]

the on-shell renormalization conditions consist of the requirement [2]

\[
\Delta_{ij}(p^2) \xrightarrow{\varepsilon \to 0} \frac{\delta_{in} \delta_{nj}}{\varepsilon} + \Delta_{ij}^{(0)} + O(\varepsilon)
\]

for all \( n = 1, \ldots, N \). In this formula and in the following, the symbol \( \delta_{rs} \) always signifies the Kronecker symbol. The coefficients \( \Delta_{ij}^{(0)} \) are of order one \( \varepsilon \). In the following, the superscripts \( (0) \) and \( (1) \) will always indicate order \( \varepsilon^0 \) and \( \varepsilon^1 \), respectively.

The complication comes from the fact that we actually want to impose on-shell renormalization conditions on the inverse propagator, which we denote by \( A = (A_{ij}) \). Accordingly, we have to translate equation (3) into conditions on \( A \). The inverse propagator fulfills

\[
\Delta_{ik} A_{kj} = A_{ik} \Delta_{kj} = \delta_{ij} \quad \text{with} \quad A_{ij} = A_{ij}^{(0)} + \varepsilon A_{ij}^{(1)} + O(\varepsilon^2),
\]

where the latter relation states that \( A \) has no singularity in \( \varepsilon \). Equation (4) is reformulated as

\[
\Delta_{ik} A_{kj} = \frac{1}{\varepsilon} \delta_{in} A_{nj}^{(0)} + \delta_{in} A_{nj}^{(1)} + \Delta_{ik} A_{kj}^{(0)} + O(\varepsilon) = \delta_{ij},
\]

\footnote{From now on, for the sake of simplicity of notation, we skip the dependence on \( p^2 \) in all quantities, whenever this dependence is obvious.}
\[ A_{ik} \Delta_{kj} = \frac{1}{\varepsilon} A_{in}^{(0)} \delta_{nj} + A_{in}^{(1)} \delta_{nj} + A_{ik}^{(0)} \Delta_{kj}^{(0)} + \mathcal{O}(\varepsilon) = \delta_{ij}. \quad (5b) \]

Avoiding the singularity in \(1/\varepsilon\) requires
\[ A_{in}^{(0)} = 0 \quad \forall \ i = 1, \ldots, N \quad \text{and} \quad A_{nj}^{(0)} = 0 \quad \forall \ j = 1, \ldots, N. \quad (6) \]

For \(i = j = n\) we obtain the further condition
\[ A_{nn}^{(1)} = 1. \quad (7) \]

Note that, because of equation (6), \(\Delta_{nk}^{(0)}\) and \(\Delta_{kn}^{(0)}\) do not occur in the second condition.

The remaining coefficients in equation (5), which have not yet been fixed by equations (6) and (7), are determined by the orthogonality conditions at order \(\varepsilon^0\):
\[
\begin{align*}
&i \neq n, j \neq n : \quad \Delta_{ik}^{(0)} A_{kj}^{(0)} = A_{ik}^{(0)} \Delta_{kj}^{(0)} = \delta_{ij}, \\
&i = n, j \neq n : \quad A_{nj}^{(1)} + \Delta_{nk}^{(0)} A_{nk}^{(0)} = 0, \\
&i \neq n, j = n : \quad A_{in}^{(1)} + A_{ik}^{(0)} \Delta_{kn}^{(0)} = 0.
\end{align*}
\]

However, these conditions have nothing to do with on-shell renormalization.

In summary, for the inverse propagator we have derived the on-shell conditions
\[ A_{in}(m_n^2) = A_{nj}(m_n^2) = 0 \quad \forall \ i, j = 1, \ldots, N \quad \text{and} \quad \frac{dA_{mn}(p^2)}{dp^2} \bigg|_{p^2=m_n^2} = 1, \quad (9) \]
in agreement with the result of [2]. Of course, for every \(n\) there is such a set of conditions.

### 2.2 Renormalization and parameter counting

The dispersive part of the inverse propagator is a real symmetric matrix, \(i.e.\)
\[ (\Delta^{-1}(p^2))^T = \Delta^{-1}(p^2). \quad (10) \]

A derivation of this symmetry relation in terms of the Källén–Lehmann representation of the scalar propagator is given in appendix A.1. Using the fact that on-shell renormalization conditions are imposed on the dispersive part, we can reformulate them as
\[
\begin{align*}
&i \neq j : \quad A_{ij}(m_j^2) = 0, \quad i = j : \quad A_{ii}(m_i^2) = 0, \quad \frac{dA_{ii}(p^2)}{dp^2} \bigg|_{p^2=m_i^2} = 1. \quad (11)
\end{align*}
\]

The number of on-shell conditions is thus
\[ 2 \binom{N}{2} + 2N = N^2 + N. \quad (12) \]

The factor 2 in front of the binomial coefficient originates from the two pairs \((i, j)\) and \((j, i)\) for \(i \neq j\).
In terms of the self-energy, the renormalized inverse propagator has the form
\[ A(p^2) = p^2 - \hat{m}^2 - \Sigma^{(r)}(p^2). \] (13)

The bare fields \( \varphi_i^{(b)} \) are related to the renormalized ones via the field strength renormalization matrix:
\[ \varphi_i^{(b)} = \left( Z^{(1/2)} \right)_{ik} \varphi_k. \] (14)

In this way, the renormalized self-energy \( \Sigma^{(r)}(p^2) \) is related to the unrenormalized one by
\[ \Sigma^{(r)}(p^2) = \Sigma(p^2) + \left( 1 - (Z^{(1/2)})^T Z^{(1/2)} \right) p^2 + \left( Z^{(1/2)} \right)^T \left( \hat{m}^2 + \delta \hat{m}^2 \right) Z^{(1/2)} - \hat{m}^2. \] (15)

For real scalar fields, \( Z^{(1/2)} \) is a general real \( N \times N \) matrix. Therefore, it contains \( N^2 \) real parameters. Assuming to be in a basis where
\[ \hat{m}^2 = \text{diag} \left( m_1^2, \ldots, m_N^2 \right), \] (16)

then \( \delta \hat{m}^2 \) is diagonal too. Thus, there are \( N \) real parameters in \( \delta \hat{m}^2 \). In summary, we have \( N^2 + N \) free parameters at disposal for implementing \( N^2 + N \) on-shell renormalization conditions. The condition \( A_{ii}(m_i^2) = 0 \) is imposed with the help of \( \delta \hat{m}^2 \), while for the rest the field strength renormalization matrix \( Z^{(1/2)} \) is responsible.

It is instructive to perform the renormalization of the inverse propagator at the lowest non-trivial order in \( Z^{(1/2)} \). In this case we write
\[ Z^{(1/2)} = 1 + \frac{1}{2} z \] (17)

and
\[ \Sigma^{(r)}(p^2) = \Sigma(p^2) - \frac{1}{2} \left( z + z^T \right) p^2 + \frac{1}{2} \left( \hat{m}^2 z + z^T \hat{m}^2 \right) + \delta \hat{m}^2. \] (18)

Then it is straightforward to derive
\[ i \neq j: \quad \frac{1}{2} z_{ij} = -\frac{\Sigma_{ij}(m_i^2)}{m_i^2 - m_j^2}, \quad i = j: \quad \delta \hat{m}^2_i = -\Sigma_{ii}(m_i^2), \quad z_{ii} = \frac{d\Sigma_{ii}(p^2)}{dp^2} \bigg|_{p^2 = m_i^2} \] (19)

from equation (11).

### 3 Fermion propagator with parity conservation

#### 3.1 On-shell conditions

For fermions the situation is more involved, but we can nevertheless proceed in analogy to the scalar case. We write the propagator as
\[ S = C\phi - D \quad \text{with} \quad C = \left( C_{ij}(p^2) \right), \quad D = \left( D_{ij}(p^2) \right) \] (20)

being \( N \times N \) matrices. It is now expedient to formulate for fermions the condition analogous to equation (3). Using again \( \varepsilon \) of equation (2), we have now
\[ S_{ij} \xrightarrow{\varepsilon \to 0} \frac{\delta_{ij} \delta_n}{p - m_n} + S_{ij}, \] (21)
where $\tilde{S}_{ij}$ is non-singular in $\varepsilon$. First of all, we have to work out what equation (21) means for $C$ and $D$. Multiplying $S$ by $\bar{p} - m_n$ and exploiting equation (21), we find

$$ (\bar{p} - m_n) S_{ij} = \delta_{in} \delta_{nj} + (\bar{p} - m_n) \tilde{S}_{ij} = \varepsilon C_{ij} - (\bar{p} - m_n) (D_{ij} + m_n C_{ij}), \quad (22) $$

whence we conclude

$$ C_{ij} = \frac{\delta_{in} \delta_{nj}}{\varepsilon} + C_{ij}^{(0)} + O(\varepsilon), \quad D_{ij} = -\frac{m_n \delta_{in} \delta_{nj}}{\varepsilon} + D_{ij}^{(0)} + O(\varepsilon). \quad (23) $$

The second relation follows from the non-singularity of $\tilde{S}_{ij}$. We see that in the fermion case we have two relations instead of one, equation (3), in the scalar case.

Now we have to formulate the conditions of equation (23) for the inverse propagator

$$ S^{-1} = A\bar{p} - B. \quad (24) $$

The relation between $A$, $B$ and $C$, $D$ is given by

$$ (SS^{-1})_{ij} = \delta_{ij} \Rightarrow C_{ik} A_{kj} p^2 + D_{ik} B_{kj} = \delta_{ij}, \quad C_{ik} B_{kj} + D_{ik} A_{kj} = 0, \quad (25a) $$

$$ (S^{-1}S)_{ij} = \delta_{ij} \Rightarrow A_{ik} C_{kj} p^2 + B_{ik} D_{kj} = \delta_{ij}, \quad B_{ik} C_{kj} + A_{ik} D_{kj} = 0. \quad (25b) $$

Since the inverse propagator is non-singular for $\varepsilon \to 0$, we have the expansion

$$ A_{ij} = A_{ij}^{(0)} + \varepsilon A_{ij}^{(1)} + O(\varepsilon^2), \quad B_{ij} = B_{ij}^{(0)} + \varepsilon B_{ij}^{(1)} + O(\varepsilon^2) \quad (26) $$

with

$$ A_{ij}^{(0)} = A_{ij}(m_n^2), \quad A_{ij}^{(1)} = \frac{dA_{ij}(p^2)}{dp^2} \bigg|_{p^2 = m_n^2} \quad (27) $$

and the analogous relations for $B$. We have to plug the relations of equation (26) into equation (25) and invoke the expansion for the propagator presented in equation (23). For the details, we refer the reader to appendix B.1.

Summarizing the computation in appendix B.1 the on-shell renormalization conditions on the renormalized inverse propagator are given by

$$ B_{in}(m_n^2) = m_n A_{in}(m_n^2) \quad \forall \ i = 1, \ldots, N; \quad (28a) $$

$$ B_{nj}(m_n^2) = m_n A_{nj}(m_n^2) \quad \forall \ j = 1, \ldots, N; \quad (28b) $$

$$ A_{nn}(m_n^2) + 2m_n^2 \left. \frac{dA_{nn}(p^2)}{dp^2} \right|_{p^2 = m_n^2} - 2 m_n \left. \frac{dB_{nn}(p^2)}{dp^2} \right|_{p^2 = m_n^2} = 1. \quad (28c) $$

They are in agreement with the conditions derived in [2]. Relations (28a) and (28b) follow from the cancellation of the terms with $1/\varepsilon$, whereas relation (28c) stems from the terms with zeroth power in $\varepsilon$. 

6
3.2 Renormalization and parameter counting

As in the scalar case, we can make use of a symmetry of the dispersive part of the propagator, namely

\[ \gamma_0 (S^{-1}(p))^{\dagger}_{\text{disp}} \gamma_0 = S^{-1}(p)_{\text{disp}}, \]  

(29)

which shows us that

\[ A^{\dagger} = A, \quad B^{\dagger} = B \]  

(30)

is valid for the dispersive parts. A derivation of equation (29) is given in appendix A.2. Therefore, the on-shell conditions can be rewritten as

\[ i \neq j: \quad B_{ij}(m^2_j) = m_j A_{ij}(m^2_j), \]  

(31a)

\[ i = j: \quad B_{ii}(m^2_i) = m_i A_{ii}(m^2_i), \]  

(31b)

\[ A_{ii}(m^2_i) + 2m_i \frac{dA_{ii}(p^2)}{dp^2}\bigg|_{p^2=m^2_i} - 2m_i \frac{dB_{ii}(p^2)}{dp^2}\bigg|_{p^2=m^2_i} = 1. \]  

(31c)

For each pair \( i \neq j \) we have four conditions, because relation (31a) is complex and contains \( i < j \) and \( i > j \). For every \( i = j \) there are two real conditions, one from equation (31b) and one from equation (31c). Thus, the number of on-shell conditions amounts to

\[ 4 \left( \frac{N}{2} \right) + 2N = 2N^2. \]  

(32)

In analogy to the scalar case, we write

\[ S^{-1}(p) = \phi - \hat{m} - \Sigma^{(r)}(p). \]  

(33)

We introduce the field strength renormalization via

\[ \psi^{(b)}_i = (Z^{(1/2)})_{ik} \psi_k \]  

(34)

with bare and renormalized fields \( \psi^{(b)}_i \) and \( \psi_k \), respectively. Then, the renormalized self-energy \( \Sigma^{(r)}(p) \) is written as

\[ \Sigma^{(r)}(p) = \Sigma(p) + \left( 1 - (Z^{(1/2)})^{\dagger} Z^{(1/2)} \right) \phi + \left( Z^{(1/2)} \right)^{\dagger} (\hat{m} + \delta \hat{m}) Z^{(1/2)} - \hat{m} \]  

(35)

with the unrenormalized self-energy \( \Sigma(p) \). From this equation it is obvious that the transformation

\[ Z^{(1/2)} \rightarrow e^{i\alpha} Z^{(1/2)} \]  

(36)

by a diagonal matrix \( e^{i\alpha} \) of phase factors leaves \( \Sigma^{(r)}(p) \) invariant. Though \( Z^{(1/2)} \) is a general complex \( N \times N \) matrix, we only have \( 2N^2 - N \) parameters in this matrix at our disposal. Adding to this number the \( N \) parameters in \( \delta \hat{m} \), we arrive at \( 2N^2 \) renormalization parameters, which equals the number of renormalization conditions in equation (32).

To perform the renormalization, we decompose the unrenormalized self-energy into

\[ \Sigma(p) = \Sigma^{(A)}(p^2) \phi + \Sigma^{(B)}(p^2). \]  

(37)
This leads to
\[ A = (Z^{(1/2)})^\dagger Z^{(1/2)} - \Sigma^{(A)} \quad \text{and} \quad B = \Sigma^{(B)} + (Z^{(1/2)})^\dagger (\hat{m} + \delta \hat{m}) Z^{(1/2)}. \] (38)

Let us—just as in the scalar case—do the one-loop renormalization to illustrate the above discussion. Again we set \( Z^{(1/2)} = 1 + \frac{1}{2} z \) and obtain
\[ A = 1 - \Sigma^{(A)} + \frac{1}{2} (z + z^\dagger), \] (39a)
\[ B = \hat{m} + \Sigma^{(B)} + \frac{1}{2} (\hat{m} z + z^\dagger \hat{m}) + \delta \hat{m}. \] (39b)

We assume a diagonal mass matrix
\[ \hat{m} = \text{diag} (m_1, \ldots, m_N). \] (40)

Inserting these quantities \( A, B \) and \( \delta \hat{m} \) into equation (31), it is straightforward to find the solution
\[ \frac{1}{2} z_{ij} = -\frac{1}{m_i - m_j} \left( m_j \Sigma^{(A)} (m_j^2) + \Sigma^{(B)} (m_j^2) \right) \] (41)
for \( i \neq j \) and
\[ \delta m_i = -m_i \Sigma^{(A)} (m_i^2) - \Sigma^{(B)} (m_i^2), \] (42a)
\[ \text{Re} z_{ii} = \Sigma^{(A)} (m_i^2) + 2m_i^2 \frac{d \Sigma^{(A)} (p^2)}{dp^2} \bigg|_{p^2 = m_i^2} + 2m_i \frac{d \Sigma^{(B)} (p^2)}{dp^2} \bigg|_{p^2 = m_i^2} \] (42b)
for \( i = j \). We notice that \( \text{Im} z_{ii} \) is not determined as a consequence of the phase freedom expressed in equation (36).

### 3.3 Formal derivation of the fermionic on-shell conditions

An interesting aspect of the on-shell conditions for fermions is that they can be derived by formally considering \( \phi \) as a variable [2] and expanding in
\[ \eta \equiv \phi - m_n. \] (43)

Then one can exactly imitate the computation for scalars. We begin with the condition
\[ S_{ij} \xrightarrow{\eta \to 0} \frac{\delta m_i \delta m_j}{\eta} + S^{(0)}_{ij} \] (44)
for the propagator. Then with
\[ T = S^{-1} \quad \text{and} \quad T_{ij} = T^{(0)}_{ij} + \eta T^{(1)}_{ij} + \mathcal{O}(\eta^2) \] (45)
we obtain
\[ S_{ik} T_{kj} = \frac{1}{\eta} \delta_{ij} T^{(0)}_{nj} + \delta_{in} T^{(1)}_{nj} + S^{(0)}_{ik} T^{(0)}_{kj} + \mathcal{O}(|\eta|) = \delta_{ij}, \] (46a)
\[ T_{ik}S_{kj} = \frac{1}{\eta} T_{in}^{(0)} \delta_{nj} + T_{in}^{(1)} \delta_{nj} + T_{ik}^{(0)} S_{kj}^{(0)} + \mathcal{O}(\eta) = \delta_{ij}. \]  

(46b)

In this way we arrive at conditions completely analogous to the scalar conditions:

\[ T_{in}^{(0)} = 0 \quad \forall i = 1, \ldots, N, \quad T_{nj}^{(0)} = 0 \quad \forall j = 1, \ldots, N, \quad T_{nn}^{(1)} = 1. \]  

(47)

But we know that the inverse propagator has the decomposition

\[ T_{ij}(\phi) = A_{ij}(p^2)\phi - B_{ij}(p^2), \]  

(48)

where \( p^2 = \phi^2 \). Therefore, equation (47) translates into

\[ T_{in}(\phi = m_n) = A_{in}(m_n^2)m_n - B_{in}(m_n^2) = 0, \]  

(49a)

\[ T_{nj}(\phi = m_n) = A_{nj}(m_n^2)m_n - B_{nj}(m_n^2) = 0. \]  

(49b)

In order to tackle \( T_{nn}^{(1)} = 1 \), we note that

\[ \frac{dp^2}{d\phi} = 2\phi. \]  

(50)

Therefore, we end up with

\[ \left. \frac{dT_{nn}(\phi)}{d\phi} \right|_{\phi=m_n} = A_{nn}(m_n^2) + 2m_n \left. \frac{dA_{nn}(p^2)}{dp^2} \right|_{p^2=m_n^2} - 2m_n \left. \frac{dB_{nn}(p^2)}{dp^2} \right|_{p^2=m_n^2} = 1. \]  

(51)

Thus we recover relation (28c).

\section{Fermion propagator without parity conservation}

\subsection{On-shell conditions}

With the chiral projectors

\[ \gamma_L = \frac{1 - \gamma_5}{2}, \quad \gamma_R = \frac{1 + \gamma_5}{2}, \]  

(52)

the propagator is now given by

\[ S = \phi \left( C_L \gamma_L + C_R \gamma_R \right) - \left( D_L \gamma_L + D_R \gamma_R \right). \]  

(53)

The on-shell condition (21) is again valid. Since

\[ (\phi - m_n) \, S_{ij} = \varepsilon \left( C_L \gamma_L + C_R \gamma_R \right)_{ij} - (\phi - m_n) \left[ (D_L + m_n C_L) \gamma_L + (D_R + m_n C_R) \gamma_R \right]_{ij}, \]  

(54)

we have the following behaviour for \( \varepsilon \to 0 \):

\[ (C_L)_{ij} = \frac{\delta_{in} \delta_{nj}}{\varepsilon} + (C_L^{(0)})_{ij} + \mathcal{O}(\varepsilon), \quad (D_L)_{ij} = -\frac{m_n \delta_{in} \delta_{nj}}{\varepsilon} + (D_L^{(0)})_{ij} + \mathcal{O}(\varepsilon), \]  

(55a)
\[(C_R)_{ij} = \frac{\delta_{in}\delta_{nj}}{\varepsilon} + (C_R^{(0)})_{ij} + O(\varepsilon), \quad (D_R)_{ij} = -\frac{m_n\delta_{in}\delta_{nj}}{\varepsilon} + (D_R^{(0)})_{ij} + O(\varepsilon). \quad (55b)\]

We define the inverse propagator as
\[S^{-1} = \phi (A_L\gamma_L + A_R\gamma_R) - (B_L\gamma_L + B_R\gamma_R). \quad (56)\]

Since the set of matrices
\[\{\gamma_L, \gamma_R, \psi\gamma_L, \psi\gamma_R\}\]

is linearly independent for \(p \neq 0\), we obtain the relations
\[(SS^{-1})_{ij} = \delta_{ij} \Rightarrow (C_RA_Lp^2 + DLBL)_{ij} = \delta_{ij}, \quad (C_RBL + DRAL)_{ij} = 0, \quad (58a)\]
\[(C_Rp^2 + DBR)_{ij} = \delta_{ij}, \quad (C_RB_R + DLAR)_{ij} = 0, \quad (58b)\]
\[(S^{-1}S)_{ij} = \delta_{ij} \Rightarrow (ARCLp^2 + DLDL)_{ij} = \delta_{ij}, \quad (BRCL + ALDL)_{ij} = 0, \quad (58c)\]
\[(ALC_Rp^2 + DBRD)_{ij} = \delta_{ij}, \quad (BLCR + ARTD)_{ij} = 0. \quad (58d)\]

Because the inverse propagator is non-singular in \(\varepsilon\), we are allowed to assume the expansion
\[(A_L)_{ij} = (A_L^{(0)})_{ij} + \varepsilon(A_L^{(1)})_{ij} + O(\varepsilon^2), \quad (B_L)_{ij} = (B_L^{(0)})_{ij} + \varepsilon(B_L^{(1)})_{ij} + O(\varepsilon^2), \quad (59a)\]
\[(A_R)_{ij} = (A_R^{(0)})_{ij} + \varepsilon(A_R^{(1)})_{ij} + O(\varepsilon^2), \quad (B_R)_{ij} = (B_R^{(0)})_{ij} + \varepsilon(B_R^{(1)})_{ij} + O(\varepsilon^2). \quad (59b)\]

For the details of evaluating equations \((55), (58)\) and \((59)\), we refer the reader to appendix \([3, 2]\).

In summary, the on-shell conditions on the inverse propagator are given by equations \((39)\) and \((13)\). The first equation reads
\[(B_L)_{in}(m_n^2) = m_n(A_R)_{in}(m_n^2), \quad (B_R)_{in}(m_n^2) = m_n(A_L)_{in}(m_n^2) \quad \forall i, \quad (60a)\]
\[(B_L)_{nj}(m_n^2) = m_n(A_L)_{nj}(m_n^2), \quad (B_R)_{nj}(m_n^2) = m_n(A_R)_{nj}(m_n^2) \quad \forall j. \quad (60b)\]

Equation \((13)\) can be brought into the form
\[\left. (A_L)_{nn}(m_n^2) + m_n^2 \frac{d}{dp^2} ((A_L)_{nn}(p^2) + (A_R)_{nn}(p^2)) \right|_{p^2=m_n^2} = 1, \quad (61a)\]
\[\left. -m_n \frac{d}{dp^2} ((B_L)_{nn}(p^2) + (B_R)_{nn}(p^2)) \right|_{p^2=m_n^2} = 1, \quad (61b)\]

Note that in these two equations the parts with the derivatives are identical, therefore, we are lead to the consistency condition
\[(A_L)_{nn}(m_n^2) = (A_R)_{nn}(m_n^2). \quad (62)\]

Plugging this into equation \((61)\), the further consistency condition
\[(B_L)_{nn}(m_n^2) = (B_R)_{nn}(m_n^2) \quad (63)\]

follows.
4.2 Renormalization and parameter counting

Turning to the counting of the number of renormalization conditions, we note that the relations analogous to the ones of equation (30) are now

\[ A_L^\dagger = A_L, \quad A_R^\dagger = A_R, \quad B_L^\dagger = B_R. \]  

(64)

These relations cut the number of independent relations in equation (60) in half and it suffices to consider only equation (60a). Written in indices \(i, j\), this equation reads

\[(B_L)_{ij}(m_i^2) = m_j(A_R)_{ij}(m_j^2), \quad (B_R)_{ij}(m_j^2) = m_j(A_L)_{ij}(m_i^2).\]  

(65)

In counting the number of independent on-shell conditions we proceed as in section 3.2. We first consider \(i \neq j\). The relations in equation (65) are not symmetric under the exchange \(i \leftrightarrow j\). Furthermore, each of the two independent relations in equation (65) delivers one complex condition. Therefore, each pair of indices gives eight real conditions. For \(i = j\), the problem is more subtle. We know from equation (64) that \(\text{Im } (A_L)_{ii} = (A_R)_{ii} = 0\) and \((B_R)_{ii}(m_i^2) = ((B_L)_{ii}(m_i^2))^\dagger\). Clearly, these are no renormalization conditions. Discounting them, there remain the three real renormalization conditions

\[(A_L)_{ii}(m_i^2) = (A_R)_{ii}(m_i^2), \quad \text{Im } (B_L)_{ii}(m_i^2) = 0, \quad \text{Re } (B_L)_{ii}(m_i^2) = m_i(A_R)_{ii}(m_i^2)\]  

(66)

plus one condition containing the derivatives. Actually, only the second and third relation in equation (66) follow from equation (65) with \(i = j\), whereas the first one ensues from that equation only for \(m_i \neq 0\). However, the first one also derives from equation (61) without the necessity of assuming \(m_i \neq 0\) and, because of this, the same equation gives only one real condition containing the derivatives. Thus we end up with a total of

\[8 \binom{N}{2} + 4N = 4N^2\]  

(67)

real on-shell conditions.

Next we switch to renormalization. Equation (83) holds again, but now we have to introduce a field strength renormalization matrix for each set of chiral fields, \textit{i.e.}

\[\psi^{(b)}_{iL} = \left(Z_{L}^{(1/2)}\right)_{ik} \psi_{kL}, \quad \psi^{(b)}_{iR} = \left(Z_{R}^{(1/2)}\right)_{ik} \psi_{kR}.\]  

(68)

Then, the renormalized self-energy \([\Sigma^{(r)}(p)\) is written as

\[\Sigma^{(r)}(p) = \Sigma(p) + \left(\mathbb{1} - \left(Z_{L}^{(1/2)}\right)^\dagger Z_{L}^{(1/2)}\right) \psi_{L} + \left(\mathbb{1} - \left(Z_{R}^{(1/2)}\right)^\dagger Z_{R}^{(1/2)}\right) \psi_{R}\]  

\[+ \left(Z_{L}^{(1/2)}\right)^\dagger (\hat{m} + \delta \hat{m}) Z_{L}^{(1/2)} \gamma_{L} + \left(Z_{L}^{(1/2)}\right)^\dagger (\hat{m} + \delta \hat{m}) Z_{R}^{(1/2)} \gamma_{R} - \hat{m}\]  

(69)

with the unrenormalized self-energy \([\Sigma(p)\). In order to satisfy the \(4N^2\) renormalization conditions, we have \(Z_{L}^{(1/2)}, Z_{R}^{(1/2)}\) and \(\delta \hat{m}\) at our disposal. Just as in the case of parity conservation, we have to take the phase freedom

\[Z_{L}^{(1/2)} \to e^{i\alpha} Z_{L}^{(1/2)}, \quad Z_{R}^{(1/2)} \to e^{-i\alpha} Z_{R}^{(1/2)}\]  

(70)
into account, because \( \Sigma^{(r)}(p) \) is invariant under this rephasing. Note that both field strength renormalization matrices are multiplied with the same diagonal matrix of phase factors. Each field strength renormalization matrix is a general complex matrix and thus has \( 2N^2 \) real parameters. Taking into account the \( N \) parameters in \( \delta \hat{m} \), we end up with total number of \( (4N^2 - N) + N = 4N^2 \) real parameters, which we have at our disposal for renormalization, which exactly matches the number of independent real renormalization conditions in equation (67).

For renormalization, we decompose the bare self-energy into

\[
\Sigma(p) = \Phi \left( \Sigma_L^{(A)}(p^2)\gamma_L + \Sigma_R^{(A)}(p^2)\gamma_R \right) + \Sigma^{(B)}(p^2)\gamma_L + \Sigma_R^{(B)}(p^2)\gamma_R, \tag{71}
\]

which in turn gives

\[
A_L = (Z_L^{(1/2)})^\dagger Z_L^{(1/2)} - \Sigma_L^{(A)}, \quad B_L = \Sigma_L^{(B)} + (Z_L^{(1/2)})^\dagger (\hat{m} + \delta \hat{m}) Z_L^{(1/2)}, \tag{72a}
\]
\[
A_R = (Z_R^{(1/2)})^\dagger Z_R^{(1/2)} - \Sigma_R^{(A)}, \quad B_R = \Sigma_R^{(B)} + (Z_L^{(1/2)})^\dagger (\hat{m} + \delta \hat{m}) Z_L^{(1/2)}. \tag{72b}
\]

Note that

\[
\left( \Sigma_L^{(A)} \right)^\dagger = \Sigma_L^{(A)}, \quad \left( \Sigma_R^{(A)} \right)^\dagger = \Sigma_R^{(A)}, \quad \left( \Sigma_L^{(B)} \right)^\dagger = \Sigma_R^{(B)} \tag{73}
\]
due to the discussion in appendix A.

Let us now perform explicit renormalization at the lowest non-trivial order. To this end, we assume that \( \hat{m} \) is diagonal, with the pole masses on its diagonal. Then \( \delta \hat{m} \) is diagonal too. We write \( Z_L^{(1/2)} = 1 + \frac{1}{2} z_L \) and \( Z_R^{(1/2)} = 1 + \frac{1}{2} z_R \) and obtain

\[
A_L = 1 - \Sigma_L^{(A)} + \frac{1}{2} z_L + \frac{1}{2} z_L^\dagger, \tag{74a}
\]
\[
A_R = 1 - \Sigma_R^{(A)} + \frac{1}{2} z_R + \frac{1}{2} z_R^\dagger, \tag{74b}
\]

and

\[
B_L = \hat{m} + \delta \hat{m} + \Sigma_L^{(B)} + \frac{1}{2} \hat{m} z_L + \frac{1}{2} z_L^\dagger \hat{m}, \tag{75a}
\]
\[
B_R = \hat{m} + \delta \hat{m} + \Sigma_R^{(B)} + \frac{1}{2} \hat{m} z_R + \frac{1}{2} z_R^\dagger \hat{m}. \tag{75b}
\]

With this identification, it is straightforward to compute \( (z_L)_{ij} \) and \( (z_R)_{ij} \) for \( i \neq j \) from equation (60a). The result is (see, for instance, [4, 12])

\[
\frac{1}{2} (z_L)_{ij} = \frac{1}{m_i^2 - m_j^2} \left[ m_j^2 \left( \Sigma_L^{(A)} \right)_{ij} + m_i m_j \left( \Sigma_R^{(A)} \right)_{ij} + m_j \left( \Sigma_L^{(B)} \right)_{ij} + m_i \left( \Sigma_R^{(B)} \right)_{ij} \right]_{p^2 = m_i^2}, \tag{76a}
\]
\[
\frac{1}{2} (z_R)_{ij} = \frac{1}{m_i^2 - m_j^2} \left[ m_i m_j \left( \Sigma_L^{(A)} \right)_{ij} + m_j^2 \left( \Sigma_R^{(A)} \right)_{ij} + m_i \left( \Sigma_R^{(B)} \right)_{ij} + m_j \left( \Sigma_L^{(B)} \right)_{ij} \right]_{p^2 = m_j^2}. \tag{76b}
\]
For $i = j$, we have the three conditions in equation (66) plus one of the conditions of equation (61), i.e. there are four equations for five unknowns, which are $\text{Re} (z_L)_{ii}$, $\text{Re} (z_R)_{ii}$, $\text{Im} (z_L)_{ii}$, $\text{Im} (z_R)_{ii}$ and $\delta m_i$. This reflects the rephasing invariance of equation (70). To solve for $\delta m_i$, the conditions in equation (66) are sufficient, leading to the result

$$2 \delta m_i = -m_i \left((\Sigma^i_L)_{ii}(m^2_i) + (\Sigma^i_R)_{ii}(m^2_i)\right) - \left((\Sigma^i_L)_{ii}(m^2_i) + (\Sigma^i_R)_{ii}(m^2_i)\right).$$

(77)

Note that, due to equation (73), the second expression on the right-hand side is real. The remaining task is the determination of the diagonal entries of the field strength renormalization constants. The real parts can be completely fixed:

$$\text{Re}(z_L)_{ii} = (\Sigma^i_L)_{ii}(m^2_i) + m_i^2 \frac{d}{dp^2} \left((\Sigma^i_L)_{ii}(p^2) + (\Sigma^i_R)_{ii}(p^2)\right) \bigg|_{p^2 = m_i^2},$$

$$+ m_i \frac{d}{dp^2} \left((\Sigma^i_B)_{ii}(p^2) + (\Sigma^i_R)_{ii}(p^2)\right) \bigg|_{p^2 = m_i^2},$$

(78a)

$$\text{Re}(z_R)_{ii} = (\Sigma^i_R)_{ii}(m^2_i) + m_i^2 \frac{d}{dp^2} \left((\Sigma^i_L)_{ii}(p^2) + (\Sigma^i_R)_{ii}(p^2)\right) \bigg|_{p^2 = m_i^2},$$

$$+ m_i \frac{d}{dp^2} \left((\Sigma^i_B)_{ii}(p^2) + (\Sigma^i_B)_{ii}(p^2)\right) \bigg|_{p^2 = m_i^2}.$$  

(78b)

Only $\text{Im} (B_L)_{ii}(m^2_i) = 0$ in equation (66) involves a non-trivial imaginary part, from which we deduce [12]

$$m_i \text{Im} (z_L - z_R)_{ii} = -2 \text{Im} (\Sigma^i_L)_{ii}(m^2_i) = 2 \text{Im} (\Sigma^i_R)_{ii}(m^2_i).$$

(79)

The latter equality results from equation (73).

### 4.3 Comparison with Aoki et al.

In [2], the inverse propagator is parameterized as

$$S^{-1} = K_1 \mathbb{1} + K_5 \gamma_5 + K_\gamma \not{\psi} + K_5 \gamma_5 \not{\psi}.$$  

(80)

Comparison with equation (56) leads to the translation table

$$K_1 = -\frac{1}{2} (B_L + B_R),$$

$$K_5 = \frac{1}{2} (B_L - B_R),$$

$$K_\gamma = \frac{1}{2} (A_L + A_R),$$

$$K_{5\gamma} = -\frac{1}{2} (A_L - A_R).$$

(81)

Inverting it, we have

$$A_L = K_\gamma - K_{5\gamma},$$

$$A_R = K_\gamma + K_{5\gamma},$$

$$B_L = -K_1 + K_5,$$

$$B_R = -K_1 - K_5.$$  

(82)
With this information, equation (60) is translated into

\[(K_1)_{in}(m^2_n) + m_n(K_\gamma)_{in}(m^2_n) = 0, \quad (K_5)_{in}(m^2_n) - m_n(K_{5\gamma})_{in}(m^2_n) = 0 \quad \forall i,\]

\[(K_1)_{nj}(m^2_n) + m_n(K_\gamma)_{nj}(m^2_n) = 0, \quad (K_5)_{nj}(m^2_n) + m_n(K_{5\gamma})_{nj}(m^2_n) = 0 \quad \forall j.\]  

(83)

For \(i = j = n\), we find

\[(K_1)_{nn}(m^2_n) + m_n(K_\gamma)_{nn}(m^2_n) = 0, \quad (K_5)_{nn}(m^2_n) = 0, \quad (K_{5\gamma})_{nn}(m^2_n) = 0, \quad (84a)\]

\[(K_\gamma)_{nn}(m^2_n) + 2m_n^2 \frac{dK_\gamma(p^2)}{dp^2} \bigg|_{p^2=m^2_n} + 2m_n \frac{dK_1(p^2)}{dp^2} \bigg|_{p^2=m^2_n} = 1. \quad (84b)\]

Therefore, we have full agreement with the on-shell conditions in [2].

4.4 On-shell renormalization of Majorana fermions

In the case of Majorana fermions, the propagator matrix is severely restricted because each fermion field is its own charge-conjugate field—see appendix C. For instance, the bare fields can represented as

\[\psi_i^{(b)} = \psi_{Li}^{(b)} + (\psi_{Li}^{(b)})^c,\]

(85)

where the superscript \(c\) denotes charge conjugation. This formula expresses the fact that for each field only one chiral component is independent. Therefore,

\[Z_{R}^{(1/2)} = (Z_{L}^{(1/2)})^* \]

(86)

holds [5]. The freedom of rephasing expressed by equation (70) is lost, except for those indices \(i\) for which \(m_i + \delta m_i = 0\). However, in the following discussion, we will exclude such cases. Therefore, real parameters we have at disposal for renormalization are those in \(Z_{L}^{(1/2)}\) and \(\delta \hat{m}\). This amounts to \(2N^2 + N\) parameters.

The Majorana condition

\[S^{-1}(p) = C \left( S^{-1}(-p) \right)^T C^{-1} \]

(87)

on the propagator matrix is derived in appendix C. Applying this condition to the inverse propagator as parameterized in equation (56), one readily finds

\[A_L^T = A_R, \quad B_L^T = B_L, \quad B_R^T = B_R.\]

(88)

These conditions hold, in the Majorana case, in addition to those of equation (64). Therefore, in summary we have here

\[A_R = A_L^* \quad \text{with} \quad A_L^T = A_L \quad \text{and} \quad B_R = B_L^*, \quad \text{with} \quad B_L^T = B_L.\]

(89)

It is important not to confuse these relations with renormalization conditions.

In order to count the number of the latter ones in the case of Majorana fermions, we first consider \(i \neq j\). As discussed in section 4.2, it suffices to consider equation (65).
However, using equation (89) to eliminate $B_R$ and $A_L$ in the second relation of this equation, we find that in the Majorana case the two relations are equivalent. Thus, for $i \neq j$ there are $4\binom{N}{2}$ independent real renormalization conditions. For $i = j$, we reconsider equation (66). The first relation in this equation now follows from equation (89) and must not be considered as a renormalization condition. Therefore, in summary, together with the residue condition of equation (61), we have

$$4\binom{N}{2} + 3N = 2N^2 + N$$

(90)

renormalization conditions, which matches the number of independent renormalization constants determined above.

Equation (89) holds for the unrenormalized self-energies as well, in particular, the relations

$$\Sigma_R^{(A)} = (\Sigma_L^{(A)})^* \quad \text{and} \quad \Sigma_R^{(B)} = (\Sigma_L^{(B)})^*$$

(91)

are valid. Using these to eliminate the self-energy parts $\Sigma_R^{(A)}$ and $\Sigma_R^{(B)}$ in the one-loop renormalization conditions of section 4.2, it is easy to see their compatibility with the Majorana condition. Indeed, one finds that equation (76b) is the complex conjugate of equation (76a) and the same is true for equations (78a) and (78b). Equation (79) now reads

$$m_i \text{Im} (z_L)_{ii} = -\text{Im} (\Sigma_L^{(B)})_{ii} (m_i^2)$$

(92)

because of $\text{Im} (z_R)_{ii} = -\text{Im} (z_L)_{ii}$. As discussed above, there is no phase freedom in the Majorana case.

5 Summary

In this review we have given a very explicit presentation of on-shell renormalization. In particular, we have taken pains to dispel any unclear point in the derivation of the on-shell renormalization conditions imposed on the inverse propagator matrix. We have extensively discussed mixing of $N$ fields in the case of real scalar fields and fermion fields in parity-conserving and parity-violating theories. We have also distinguished between Dirac and Majorana fermions and described how the renormalization scheme gets modified in the latter case as compared to the more familiar Dirac case. We have not treated here mixing of complex scalar fields, but with the methods explained in this review this should pose no problem. Moreover, we have omitted mixing of vector fields, in order to avoid the complications of gauge theories—for photon–$Z$ boson mixing we refer the reader to [8, 9], for instance. The main motivation for this review originates from extensions of the Standard Model in the fermion and scalar sector, with special focus on Majorana neutrinos. For self-consistency and because of the important role in our discussion, we have also supplied a derivation of the emergence of dispersive and absorptive parts in the propagator matrix and a derivation of the restrictions on the propagator matrix in the case of Majorana fermions.
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A Dispersive and absorptive parts in the propagator

In this review we stick to hermitian counterterms in the Lagrangian. In the present appendix, for the sake of completeness, we want to demonstrate that the dispersive parts of the propagator matrix fulfill hermiticity conditions matching those of the counterterms. This is a necessary prerequisite for imposing on-shell conditions. Dispersive and absorptive parts arise through the well-known relation in distribution theory

\[ \frac{1}{p^2 - \mu^2 + i\epsilon} = \text{P} \frac{1}{p^2 - \mu^2} - i\pi\delta(p^2 - \mu^2), \]  

(A1)

from the principle value and the delta function, respectively. This can be verified quite easily in the Källén–Lehmann representation of the propagator matrix. By and large we follow the derivation in [13].

A.1 Real scalar fields

The total propagator matrix is defined by

\[ i (\Delta'(x - y))_{ij} = \langle 0 | T \varphi_i(x) \varphi_j(y) | 0 \rangle, \]  

(A2)

where T denotes time ordering. The propagator \( \Delta' \) could be renormalized or unrenormalized. Inserting a complete system of four-momentum eigenstates and exploiting energy momentum conservation leads to

\[ i (\Delta'(x - y))_{ij} = \sum_n \left\{ \Theta(x^0 - y^0) \langle 0 | \varphi_i(0) | n \rangle \langle n | \varphi_j(0) | 0 \rangle e^{-ip_n \cdot (x - y)} \\
+ \Theta(y^0 - x^0) \langle 0 | \varphi_j(0) | n \rangle \langle n | \varphi_i(0) | 0 \rangle e^{ip_n \cdot (x - y)} \right\}. \]  

(A3)

It is then useful to define the density

\[ (2\pi)^3 \sum_n \delta^{(4)}(q - p_n) \langle 0 | \varphi_i(0) | n \rangle \langle n | \varphi_j(0) | 0 \rangle \equiv \rho_{ij}(q^2)\Theta(q^0), \]  

(A4)

where the Heaviside function \( \Theta \) indicates that only \( q^0 \geq 0 \) gives a contribution.

Next we invoke CPT invariance, which holds in any local, Lorentz-invariant field theory. The transformation of the real scalar fields is given by

\[ (CPT)\varphi_i(x)(CPT)^{-1} = \varphi_i(-x). \]  

(A5)

Since (CPT) is an antiunitary operator and (CPT)\( |0\rangle = |0\rangle \), we have the relation

\[ \langle 0 | \varphi_i(0) | n \rangle = \langle 0 | \varphi_i(0)(CPT) | n \rangle^* \]  

(A6)
Clearly, if $|n\rangle$ is a complete system, then $(\mathcal{CPT}|n\rangle$ is a complete system as well. Application of equation (A6) to $\rho_{ij}(q^2)$ gives the two relations
\[ \rho_{ij}(q^2) = (\rho_{ij}(q^2))^* = \rho_{ji}(q^2). \] (A7)

The second one allows us to write the propagator matrix as
\[ i\Delta'(x-y) = \frac{1}{(2\pi)^3} \int d^4 q \Theta(q^0) \rho(q^2) \left( \Theta(x^0 - y^0)e^{-iq\cdot(x-y)} + \Theta(y^0 - x^0)e^{iq\cdot(x-y)} \right). \] (A8)

By insertion of $1 = \int_0^\infty d\mu^2 \delta(q^2-\mu^2)$ one ends up with the Källén–Lehmann representation
\[ i\Delta'(x-y) = i \int_0^\infty d\mu^2 \rho(\mu^2) \Delta_F(x-y;\mu), \] (A9)

where
\[ \Delta_F(x-y;\mu) = -i \int \frac{d^3 q}{(2\pi)^3 2\sqrt{\mu^2 + q^2}} \left( \Theta(x^0 - y^0)e^{-iq\cdot(x-y)} + \Theta(y^0 - x^0)e^{iq\cdot(x-y)} \right) \] (A10)
is the free Feynman propagator of a single real scalar field with mass $\mu$ and $q^2 = \mu^2$. In momentum space, this formula reads
\[ \Delta'(p^2) = \int_0^\infty d\mu^2 \rho(\mu^2) \frac{1}{p^2 - \mu^2 + i\epsilon}. \] (A11)

If we identify $\Delta'(p^2)$ with the renormalized Feynman propagator, we see that $\Delta_{ij}(p^2)$ is real and symmetric in the region of $p^2$ where the absorptive part vanishes. This justifies the assumption made in section 2.2.

### A.2 Fermion fields

In the fermionic case we have
\[ i (S'(x-y))_{ia,jb} = \langle 0|\mathcal{T} \psi_{ia}(x)\bar{\psi}_{jb}(y)|0\rangle \]
\[ \equiv \Theta(x^0 - y^0)\langle 0|\psi_{ia}(x)\bar{\psi}_{jb}(y)|0\rangle - \Theta(y^0 - x^0)\langle 0|\bar{\psi}_{jb}(y)\psi_{ia}(x)|0\rangle, \] (A12)
where for the sake of clarity we have also introduced Dirac indices $a, b$ for each field. With the first term on the right-hand side of equation (A12) one can, just as in the scalar case, define a density:
\[ (2\pi)^3 \sum_n \delta^{(4)}(q - p_n) \langle 0|\psi_{ia}(0)|n\rangle \langle n|\bar{\psi}_{jb}(0)|0\rangle \equiv \rho_{ia,jb}(q)\Theta(q^0). \] (A13)

Lorentz-invariance permits the decomposition
\[ \rho(q) = \slashed{q} (c_L(q^2)\gamma_L + c_R(q^2)\gamma_R) + d_L(q^2)\gamma_L + d_R(q^2)\gamma_R. \] (A14)

In this formula, $c_{L,R}$, $d_{L,R}$ are $N \times N$ matrices in family space, while $\slashed{q}\gamma_{L,R}$ carry the Dirac indices.
In order to relate the second term on the right-hand side of equation (A12) to the density \( \rho \), CPT-invariance has to be invoked \([13]\). To be as explicit as possible, we present all our conventions. We use the defining relation

\[
C^{-1}\gamma_\mu C = -\gamma_\mu^T
\]  
(A15)

for the charge conjugation matrix, whence \( C^T = -C \) ensues. As for the Dirac gamma matrices, we assume the hermiticity conditions \( \gamma^\dagger_0 = \gamma_0 \) and \( \gamma^\dagger_i = -\gamma_i \) for \( i = 1, 2, 3 \). In this basis, \( C^i = C^{-1} \) without loss of generality. Charge conjugation, parity and time reversal transformation for fermion fields are formulated as

\[
C\psi_i(x)C^{-1} = C\gamma_0^T \psi^*_i(x), \quad \mathcal{P}\psi_i(x)\mathcal{P}^{-1} = \gamma_0\psi_i(x^0, \vec{x}), \quad \mathcal{T}\psi_i(x)\mathcal{T}^{-1} = iC^{-1}\gamma_5\psi_i(-x^0, \vec{x}),
\]  
(A16)

respectively. Combination of the three discrete transformations gives the CPT transformation property \([13, 14]\)

\[
(C\mathcal{PT})\psi_i(x)(C\mathcal{PT})^{-1} = -i\gamma_5^T \psi^*_i(-x) \quad \text{and} \quad (C\mathcal{PT})\bar{\psi}_i(x)(C\mathcal{PT})^{-1} = i\psi^T_i(-x)(\gamma_5\gamma_0)^*,
\]  
(A17)

where the overall sign is convention, but the \( i \) is necessary in Yukawa couplings.\(^3\)

Now we apply equation (A17) to the second term on the right-hand side of the propagator (A12). A straightforward computation, taking into account antiunitary of \( (C\mathcal{PT}) \) and the above conventions for \( C \) and the gamma matrices, leads to the simple result \([13]\)

\[
\langle 0|\bar{\psi}_{jb}(y)\psi_{ia}(x)|0\rangle = -\langle 0|\psi_{ia}(-x)\bar{\psi}_{jc}(-y)|0\rangle \gamma_5 \langle jb|c\rangle.
\]  
(A18)

The minus sign is used to cancel the minus from the time ordering when \( y^0 > x^0 \). Thus, for the contribution to the propagator of equation (A18), we obtain the density

\[
\gamma_5 \rho(q) \gamma_5 = -\slashed{q} \left( c_L(q^2)\gamma_L + c_R(q^2)\gamma_R \right) + d_L(q^2)\gamma_L + d_R(q^2)\gamma_R,
\]  
(A19)

with \( \rho(q) \) being identical to that of equation (A14).

The remaining steps are the same as in the scalar case. We finally arrive at

\[
iS'(x - y) = i\int_0^\infty d\mu^2 \left[ i\phi_x \left( c_L(\mu^2)\gamma_L + c_R(\mu^2)\gamma_R \right) + d_L(\mu^2)\gamma_L + d_R(\mu^2)\gamma_R \right] \Delta_F(x - y; \mu),
\]  
(A20)

where the subscript \( x \) indicates derivative with respect to \( x \). In momentum space the result is

\[
S'(p) = \int_0^\infty d\mu^2 \left[ p \left( c_L(\mu^2)\gamma_L + c_R(\mu^2)\gamma_R \right) + d_L(\mu^2) + d_R(\mu^2) \right] \frac{1}{p^2 - \mu^2 + i\epsilon}.
\]  
(A21)

\(^3\) The reason is that the transformation (A5) together with (A17) leaves the most general Yukawa coupling

\[
\sum_{r,s,k} \Gamma^k_{rs} \psi^T_s C^{-1} \psi_k \varphi_k + \text{H.c.} \quad \text{with} \quad \Gamma^k_{rs} = \Gamma^k_{sr} \forall r, s
\]

invariant; in this Yukawa coupling, all fermion fields are, for instance, left-chiral. Therefore, equations (A5) and (A17) together form a true CPT transformation.
From the definition of the density $\rho(q)$, equation (A14), the following property is easy to prove:

$$\gamma_0 \rho^\dagger(q) \gamma_0 = \rho(q) \iff c^\dagger_L(q^2) = c_L(q^2), \quad c^\dagger_R(q^2) = c_R(q^2), \quad d^\dagger_L(q^2) = d_R(q^2).$$  \hfill (A22)

Transferring the latter relations to the dispersive part of $S'$, we obtain

$$\gamma_0 (S'(p))^\dagger_{\text{disp}} \gamma_0 = (S'(p))_{\text{disp}}.$$  \hfill (A23)

This is the justification of equation (29).

## B Computational details

### B.1 Theories with parity conservation

We first consider the second relation in each line of equation (25). The expansion in $\varepsilon$ furnishes

$$\int \varepsilon \left( B_{nj}^{(0)} - m_n A_{nj}^{(0)} \right) + \int \varepsilon \left( B_{nj}^{(1)} - m_n A_{nj}^{(1)} \right) + C_{ik}^{(0)} B_{kj}^{(0)} + D_{ik}^{(0)} A_{kj}^{(0)} + \mathcal{O}(\varepsilon) = 0,$$  \hfill (B1a)

$$\left( B_{in}^{(0)} - m_n A_{in}^{(0)} \right) \delta_{nj} + \left( B_{in}^{(1)} - m_n A_{in}^{(1)} \right) \delta_{nj} + B_{ik}^{(0)} C_{kj}^{(0)} + A_{ik}^{(0)} D_{kj}^{(0)} + \mathcal{O}(\varepsilon) = 0.$$  \hfill (B1b)

Removing the singularity, we obtain the conditions

$$B_{nj}^{(0)} = m_n A_{nj}^{(0)} \forall \ i = 1, \ldots, N \quad \text{and} \quad B_{nj}^{(0)} = m_n A_{nj}^{(0)} \forall \ j = 1, \ldots, N.$$  \hfill (B2)

Next we consider the first relation in each line of equation (25). Taking into account $p^2 = \varepsilon + m_n^2$, we find

$$-\frac{\delta_{in} m_n}{\varepsilon} \left( B_{nj}^{(0)} - m_n A_{nj}^{(0)} \right) + \delta_{in} \left( A_{nj}^{(0)} + m_n^2 A_{nj}^{(1)} - m_n B_{nj}^{(1)} \right) + m_n^2 C_{ik}^{(0)} A_{kj}^{(0)} + D_{ik}^{(0)} B_{kj}^{(0)} + \mathcal{O}(\varepsilon) = \delta_{ij},$$  \hfill (B3a)

$$-\delta_{in} \left( B_{in}^{(0)} - m_n A_{in}^{(0)} \right) m_n \delta_{nj} + \left( A_{in}^{(0)} + m_n A_{in}^{(1)} - m_n B_{in}^{(1)} \right) \delta_{nj} + m_n^2 A_{ik}^{(0)} C_{kj}^{(0)} + B_{ik}^{(0)} D_{kj}^{(0)} + \mathcal{O}(\varepsilon) = \delta_{ij}.$$  \hfill (B3b)

The singularity in these relations leads again to equation (B2). Up to now we have only considered the singular terms. In order to obtain a renormalization condition from the terms of order $\varepsilon^0$, we choose the indices $i = j = n$. In this case, equation (B3a) leads to

$$A_{nn}^{(0)} + m_n^2 A_{nn}^{(1)} - m_n B_{nn}^{(1)} + m_n^2 C_{nk}^{(0)} A_{kn}^{(0)} + D_{nk}^{(0)} B_{kn}^{(0)} = 1,$$  \hfill (B4a)

$$A_{nn}^{(0)} + m_n^2 A_{nn}^{(1)} - m_n B_{nn}^{(1)} + m_n^2 A_{nk}^{(0)} C_{kn}^{(0)} + B_{nk}^{(0)} D_{kn}^{(0)} = 1.$$  \hfill (B4b)

Invoking equation (B2), we rewrite this equation as

$$A_{nn}^{(0)} + m_n^2 A_{nn}^{(1)} - m_n B_{nn}^{(1)} + \left( m_n C_{nk}^{(0)} + D_{nk}^{(0)} \right) A_{kn}^{(0)} m_n = 1.$$  \hfill (B5a)
\[ A_{nn}^{(0)} + m_n^2 A_{nn}^{(1)} - m_n B_{nn}^{(1)} + m_n A_{nk}^{(0)} \left( m_n C_{kn}^{(0)} + D_{kn}^{(0)} \right) = 1. \quad \text{(B5b)} \]

In the same way equation \((B1)\) gives

\[ B_{nn}^{(1)} - m_n A_{nn}^{(1)} + \left( m_n C_{nk}^{(0)} + D_{nk}^{(0)} \right) A_{kn}^{(0)} = 0, \quad \text{(B6a)} \]
\[ B_{nn}^{(1)} - m_n A_{nn}^{(1)} + A_{nk}^{(0)} \left( m_n C_{kn}^{(0)} + D_{kn}^{(0)} \right) = 0. \quad \text{(B6b)} \]

These relations allow us to eliminate in equation \((B5)\) the terms in parentheses. We obtain one further renormalization condition

\[ A_{nn}^{(0)} + 2m_n^2 A_{nn}^{(1)} - 2m_n B_{nn}^{(1)} = 1. \quad \text{(B7)} \]

### B.2 Theories without parity conservation

Inserting the expansions \((B5)\) and \((B9)\) of \(S\) and \(S^{-1}\), respectively, into equation \((B8)\), right column, leads to

\[
\begin{align*}
\frac{\delta \ln}{\varepsilon} \left( (B_L^{(0)})_{nj} - m_n(A_L^{(0)})_{nj} \right) + \frac{\delta \ln}{\varepsilon} \left( (B_L^{(1)})_{nj} - m_n(A_L^{(1)})_{nj} \right) & \quad + (C_L^{(0)})_{ik}(B_L^{(0)})_{kj} + (D_L^{(0)})_{ik}(A_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) = 0, \\
\frac{\delta \ln}{\varepsilon} \left( (B_R^{(0)})_{nj} - m_n(A_R^{(0)})_{nj} \right) + \frac{\delta \ln}{\varepsilon} \left( (B_R^{(1)})_{nj} - m_n(A_R^{(1)})_{nj} \right) & \quad + (C_R^{(0)})_{ik}(B_R^{(0)})_{kj} + (D_L^{(0)})_{ik}(A_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) = 0,
\end{align*}
\]

\[
\begin{align*}
(B_R^{(0)})_{in} - m_n(A_L^{(0)})_{in} & \quad \frac{\delta n}{\varepsilon} + (B_R^{(1)})_{in} - m_n(A_L^{(1)})_{in} \frac{\delta n}{\varepsilon} \quad + (B_L^{(1)})_{in} - m_n(A_R^{(1)})_{in} \frac{\delta n}{\varepsilon} + \mathcal{O}(\varepsilon) = 0, \\
(B_L^{(0)})_{in} - m_n(A_R^{(0)})_{in} & \quad \frac{\delta n}{\varepsilon} + (B_R^{(1)})_{in} - m_n(A_L^{(1)})_{in} \frac{\delta n}{\varepsilon} + \mathcal{O}(\varepsilon) = 0. 
\end{align*}
\]

The poles in \(\varepsilon\) must vanish, which gives the conditions

\[
\begin{align*}
(B_L^{(0)})_{in} = m_n(A_R^{(0)})_{in}, & \quad (B_R^{(0)})_{in} = m_n(A_L^{(0)})_{in} \quad \forall \ i = 1, \ldots, N, \\
(B_L^{(0)})_{nj} = m_n(A_R^{(0)})_{nj}, & \quad (B_R^{(0)})_{nj} = m_n(A_L^{(0)})_{nj} \quad \forall \ j = 1, \ldots, N.
\end{align*}
\]

Now we can insert the same expansions into equation \((B8)\), left column, set \(p^2 = \varepsilon + m_n^2\) and arrive at

\[
\begin{align*}
- \frac{\delta \ln m_n}{\varepsilon} \left( (B_L^{(0)})_{nj} - m_n(A_L^{(0)})_{nj} \right) + \frac{\delta \ln}{\varepsilon} \left( (A_L^{(0)})_{nj} + m_n^2 (A_L^{(1)})_{nj} - m_n(B_L^{(1)})_{nj} \right) & \quad + m_n^2 (C_L^{(0)})_{ik}(A_L^{(0)})_{kj} + (D_L^{(0)})_{ik}(B_L^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij}, \\
- \frac{\delta \ln m_n}{\varepsilon} \left( (B_R^{(0)})_{nj} - m_n(A_R^{(0)})_{nj} \right) + \frac{\delta \ln}{\varepsilon} \left( (A_R^{(0)})_{nj} + m_n^2 (A_R^{(1)})_{nj} - m_n(B_R^{(1)})_{nj} \right) & \quad + m_n^2 (C_L^{(0)})_{ik}(A_R^{(0)})_{kj} + (D_R^{(0)})_{ik}(B_R^{(0)})_{kj} + \mathcal{O}(\varepsilon) = \delta_{ij},
\end{align*}
\]

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The terms with $1/\varepsilon$ lead again to equation (B9). Next, we investigate equation (B10) for $i = j = n$ at order $\varepsilon^0$ and make use of equation (B9). In this way we obtain

$$
(A_L^{(0)})_{nn} + m_n^2 (A_L^{(1)})_{nn} - m_n (B_L^{(1)})_{nn} + m_n^2 (C_R^{(0)})_{nk} (A_L^{(0)})_{kn} + m_n (D_L^{(0)})_{nk} (A_R^{(0)})_{kn} = 1, 
$$
(B11a)

$$
(A_R^{(0)})_{nn} + m_n^2 (A_R^{(1)})_{nn} - m_n (B_R^{(1)})_{nn} + m_n^2 (C_R^{(0)})_{ik} (A_R^{(0)})_{kj} + m_n (D_R^{(0)})_{nk} (A_L^{(0)})_{kn} = 1, 
$$
(B11b)

$$
(A_R^{(0)})_{nn} + m_n^2 (A_R^{(1)})_{nn} - m_n (B_R^{(1)})_{nn} + m_n^2 (C_L^{(0)})_{ck} (A_R^{(0)})_{kj} + m_n (D_R^{(0)})_{nk} (D_L^{(0)})_{kn} = 1, 
$$
(B11c)

$$
(A_L^{(0)})_{nn} + m_n^2 (A_L^{(1)})_{nn} - m_n (B_L^{(1)})_{nn} + m_n^2 (A_R^{(0)})_{nk} (C_R^{(0)})_{kn} + m_n (A_L^{(0)})_{nk} (D_R^{(0)})_{kn} = 1. 
$$
(B11d)

Via equation (B8) for $i = j = n$ at order $\varepsilon^0$, i.e.

$$
(B_L^{(1)})_{nn} - m_n (A_L^{(1)})_{nn} + m_n (C_L^{(0)})_{nk} (A_L^{(0)})_{kn} = 0, 
$$
(B12a)

$$
(B_R^{(1)})_{nn} - m_n (A_R^{(1)})_{nn} + m_n (C_R^{(0)})_{nk} (A_R^{(0)})_{kn} = 0, 
$$
(B12b)

$$
(B_R^{(1)})_{nn} - m_n (A_R^{(1)})_{nn} + m_n (A_R^{(0)})_{nk} (C_L^{(0)})_{kn} + (A_L^{(0)})_{nk} (D_L^{(0)})_{kn} = 0, 
$$
(B12c)

$$
(B_L^{(1)})_{nn} - m_n (A_L^{(1)})_{nn} + m_n (A_L^{(0)})_{nk} (C_R^{(0)})_{kn} + (A_R^{(0)})_{nk} (D_R^{(0)})_{kn} = 0, 
$$
(B12d)

we can eliminate the terms with $C_{L,R}$ and $D_{L,R}$ in equation (B11). Eventually, this leads to the final two on-shell conditions

$$
(A_L^{(0)})_{nn} + m_n^2 \left( (A_L^{(1)})_{nn} + (A_R^{(1)})_{nn} \right) - m_n \left( (B_L^{(1)})_{nn} + (B_R^{(1)})_{nn} \right) = 1, 
$$
(B13a)

$$
(A_R^{(0)})_{nn} + m_n^2 \left( (A_L^{(1)})_{nn} + (A_R^{(1)})_{nn} \right) - m_n \left( (B_L^{(1)})_{nn} + (B_R^{(1)})_{nn} \right) = 1. 
$$
(B13b)

### C  Majorana condition for the propagator matrix

In the case of Majorana fields, a condition on the propagator matrix arises from the fact that each fermion field $\psi_n$ is identical to its charge-conjugate field $(\psi_n)^c$. Dealing with four-component spinors, this reads

$$
C \gamma_0^T \psi_n^* (x) = \psi_n (x), 
$$
(C1)

where $C$ is the charge-conjugation matrix. The star means that one has to take the hermitian conjugate of each component $\psi_{na}$ ($a = 1, \ldots, 4$) of $\psi_n$ such that $\psi_n^*$ is a column vector.
The starting point for the derivation of a propagator condition in the case of Majorana fermions is the identity

\[ \langle 0 | T \psi_{ia}(x) \psi_{jb}(y) | 0 \rangle = -\langle 0 | T \psi_{jb}(y) \psi_{ia}(x) | 0 \rangle. \]

(C2)

With \( \psi_j^T = -\bar{\psi}_j C \) and \( \psi_i = C \bar{\psi}_i^T \), which follow from equation (C1), this identity is rewritten as

\[ 4 \sum_{c=1}^{4} \langle 0 | T \psi_{ia}(x) \bar{\psi}_{jc}(y) | 0 \rangle (-C_{cb}) = -\sum_{d=1}^{4} C_{ad} \langle 0 | T \psi_{jb}(y) \bar{\psi}_{id}(x) | 0 \rangle. \]

(C3)

Therefore, the Majorana condition on the propagator matrix is

\[ S'(x - y) = CS'^T(y - x)C^{-1}. \]

(C4)

Note that the transposition refers to both Dirac and family indices. In momentum space, equation (C4) reads [2, 15]

\[ S'(p) = CS'^T(-p)C^{-1}. \]

(C5)

There is the analogous condition for the inverse propagator.

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