$f_{D_{(s)}}$, $f_{B_{(s)}}$ and $f_{B_c}$ from QCD Laplace sum rules

Stephan Narison*
Laboratoire Univers et Particules de Montpellier, CNRS-IN2P3,
Case 070, Place Eugène Bataillon, 34095 - Montpellier, France.
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Anticipating future precise measurements of the $D$- and $B$-like (semi-)leptonic and hadronic
decays for alternative determinations of the CKM mixing angles, we pursue our program on the
$D$ and $B$-like mesons by improving the estimates of $f_{D_{(s)}}$ and $f_{B_{(s)}}$ (analogue to $f_s$) by using
the well-established (inverse) Laplace sum rules (LSR) and / or their suitable ratios less affected by
the systematics, which are known to N2LO pQCD and where the complete $d = 6$ non-perturbative
condensate contributions are included. The convergence of the PT series is analyzed by an estimate
of the N3LO terms based on geometric growth of the coefficients. In addition to the standard LSR
variable $\tau$ and the QCD continuum threshold $t_c$ stability criteria, we extract our optimal results
by also requiring stability on the variation of the arbitrary QCD subtraction point $\mu$. We complete
the analysis by a direct estimate of $f_{B_c}$. Our results summarized in Tables III and IV are compared
with some other recent estimates.

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I. INTRODUCTION

The meson decay constants $f_P, f_V$ are of prime interests
for understanding the realizations of chiral symmetry in
QCD and for controlling the meson (semi-)leptonic de-
cay widths, hadronic couplings and form factors . In
addition to the well-known values of $f_\pi=130.4(2)$ MeV
and $f_K=156.1(9)$ MeV which control the light flavour
chiral symmetries [2], it is also desirable to extract the
ones of the heavy-light charm and bottom quark systems
with high-accuracy. This program has been initiated
by the recent predictions of $f_{D_{(s)}}$, $f_{B_{(s)}}$ [1, 3] and their
scalars mesons analogue [4] from QCD spectral sum rules
(QSSR) [5] which are improved predictions of earlier esti-
mates [11–17] 2 since the pioneering work of Novikov et al. (NSVZ) [19]. Here, these decay constants are
normalized through the matrix element:

\[ \langle 0 | J_P(x) | P \rangle = f_P M_P^2 : J_P(x) \equiv (m_q + M_Q) \bar{q}(i\gamma_5)q , \]
\[ \langle 0 | J_V(x) | V \rangle = f_V M_V^2 : J_V^\mu(x) \equiv \bar{q}\gamma^\mu q \]

where: $J_P(x)$ (resp $J_V^\mu(x)$) are the local heavy-light pseudoscalar (resp. vector) current; $q \equiv d, c; Q \equiv c, b; P \equiv D, B, B_c, V \equiv D^*, B^*$
and where $f_P, f_V$ are related to the leptonic widths
$\Gamma[P(V) \rightarrow l^+\nu_l]$. The associated two-point correlators are:

\[ \psi_P(q^2) = i \int d^4x \, \epsilon^{aq,x} \langle 0 | T J_P(x) J_P(0) | 1 \rangle | 0 \rangle , \]
\[ \Pi_V^\mu(q^2) = i \int d^4x \, \epsilon^{aq,x} \langle 0 | T J_V^\mu(x) J_V^\mu(0) | 1 \rangle | 0 \rangle \]

where one notes that $\Pi^T(q^2)$ has more power of $q^2$ than
the transverse two-point function used in the current lit-
erature for $m_q = m_Q$ in order to avoid mass singularities
at $q^2 = 0$ if one of the quark masses goes to zero, while
$\Pi_V^\mu(q^2)$ (vector correlator) and $\psi^S(q^2)$ (scalar correlator)
are related each other through the Ward identities:

\[ q_\mu q_\nu \Pi_V^{\mu\nu}(q^2) = \psi_S(q^2) - \psi_S(0) , \]

where to lowest order the perturbative part of $\psi_S(0)$ reads:

\[ \psi_S(0)|_{PT} = \frac{3}{4\pi^2} (M_Q - m_q) (M_Q^2 Z_Q - m_q^3 Z_Q) , \]

with:

\[ Z_i = \left( 1 - \log \frac{M_i^2}{\mu^2} \right) \left( 1 + \frac{10}{3} a_s \right) + \frac{2}{3} a_s \log^2 \frac{M_i^2}{\mu^2} , \]

where $i \equiv Q, q; \mu$ is the QCD subtraction constant and
$a_s \equiv a_s / \pi$ is the QCD coupling. This PT contribution
which is present here has to be added to the well-known
non-perturbative contribution:

\[ \psi_S(0)|_{NP} = - (M_Q - m_q) \langle QQ - \bar{q}q \rangle , \]

for absorbing mass singularities appearing during the
evaluation of the PT two-point function, a point often
bypassed in the existing literature. Here, we extend the
previous analysis of [1, 3, 4] to the case of the $D_{(s)}$, $B_{(s)}$
and $B_c$ well observed mesons which have been respect-
ively estimated earlier in [7, 20] while $f_{B_c}$ has been also
re-estimated in [21, 22]. The method used here will be
similar to that in [1, 3] which are the companion papers
of this work. For improving the extraction of the decay

\[ * \text{Email address: snarison@yahoo.fr} \]
\[ 1 \text{ For reviews where complete references can be found, see e.g: [6–10].} \]
\[ 2 \text{ For a recent review, see e.g: [6, 18].} \]
of the well established (inverse) Laplace sum rules \(^3\):

\[
\mathcal{L}_P(\tau, \mu) = \int_{(m_q + M_Q)^2}^{t_c} dt \, e^{-t\tau} \frac{1}{\pi} \text{Im}\left[\psi(t, \Pi^V(t, \mu)\right],
\]

in order to minimize the systematics of the approach, the effects of heavy quark masses and the continuum threshold uncertainties which are one of the main sources of errors in the determinations of the decay constants. \(\tau_P\) denotes the value of \(\tau\) sum rule variable at which each individual sum rule is optimized (minimum or inflexion point). In general, \(\tau \neq \tau_P\) as we shall see later on which requires some care for a precise determination of the ratio of decay constants. This ratio of sum rule has lead to a successful prediction of the SU(3) breaking ratio of decay constants \(f_P/f_P\) [26] such that, from it, we expect to extract precise values of the ratio \(f_V/f_P\) in this paper.

II. QCD EXPRESSION OF THE (INVERSE) LAPLACE SUM RULE

The QCD expression of the Laplace sum rule \(\mathcal{L}_P(\tau, \mu)\) in the pseudoscalar channel has been already given in [1] for full QCD including N2LO perturbative QCD corrections and contributions of non-perturbative condensates up to the complete \(d = 6\) dimension condensates and will not be repeated here.\(^5\)

- To order \(\alpha_s^2\), the QCD theoretical side of the LSR for the vector channel reads, in terms of the on-shell heavy quark mass \(M_Q\) and for \(m_d = 0\):

\[
\mathcal{L}_V(\tau) = \int_{M_Q^2}^{\infty} dt \, e^{-t\tau} \frac{1}{\pi} \text{Im}\left[\psi(t, \Pi^V(t, \mu)\right]_{\mu_T} - \frac{\alpha_s G^2}{12\pi} e^{-z} \\
\quad - \left\{ 1 + 2\alpha_s \left[ 1 + (1 - \frac{\ln^2 \tau + 4}{3} \right) \right\} e^{-z} \\
\quad - 2\alpha_s \Gamma(0, z) \left( \frac{m_Q}{M_Q} \right)^2 m_Q \langle \bar{d}d \rangle + \\
\quad \tau e^{-z} \left( \frac{z}{4} M_Q M_Q^2 \langle \bar{d}d \rangle + \right. \\
\quad + \left. \frac{2}{9} \left( 1 + z - \frac{z^2}{8} \right) \langle \bar{d}d \rangle \right. \\
\quad + \left. (1 + 10z - 15z^2) \left( \frac{g^3 G^3}{8640\pi^2} \right) \right. \\
\quad + \left. \left[ 15\bar{L}(8 + 8z - z^2) - 11 + 124z \right. \right. \\
\quad - \left. 200z^2 \right] \left( \frac{j_1^2}{6480\pi^2} \right) \right),
\]

where:

\[
\frac{1}{\pi} \text{Im}\Pi^V(t)_{\mu_T} = \frac{1}{8\pi^2} \left[ t(1 - x)^2(2 + x) + a_4 R_{1v} + a_2^2 R_{2v} \right],
\]

with: \(z \equiv M_Q^2 \tau; x \equiv M_Q^2 / t; a_4 \equiv \alpha_s / \pi; \bar{L} \equiv \ln (\mu M_Q \tau) + \gamma_E; \gamma_E = 0.577215; \mu\) is an arbitrary subtraction point.

- \(R_{1v}\) is the \(\alpha_s\) corrections obtained by [9, 12, 27] and \(R_{2v}\) is the \(\alpha_s^2\)-term obtained semi-analytically in [27]. \(R_{1v}\) and \(R_{2v}\) are available as a Mathematica package program Rvs.m. We consider as a source of errors an estimate of the N3LO assuming a geometric growth of the PT series [28] which mimics the phenomenological \(1/q^2\) dimension-two term which parametrizes the large order terms of PT series [29, 30].

- The contribution up to the \(d = 4\) gluon condensate:

\[
\langle \alpha_s G^2 \rangle \equiv \langle \alpha_s G_{\mu\nu}^a G_{\mu\nu}^a \rangle,
\]

d of the \(d = 5\) mixed condensate:

\[
\langle \bar{q} G q \rangle \equiv \langle \bar{q} \sigma^{\mu\nu} (\lambda_a / 2) G_{\mu\nu}^a q \rangle = M_Q^2 \langle \bar{q} q \rangle,
\]

and \(d = 6\) quark condensates:

\[
\langle \bar{d} d \rangle \equiv \langle \bar{d} \gamma_\mu D^\mu G_{\mu\nu} \frac{\lambda_a}{2} d \rangle = g^2 \langle \bar{d} \gamma_\mu \frac{\lambda_a}{2} d \rangle \sum_q \bar{q} \gamma_\mu \frac{\lambda_a}{2} q \approx - \frac{16}{9} (\pi \alpha_s) \rho \langle \bar{d}d \rangle^2,
\]

after the use of the equation of motion have been obtained originally by NSVZ [19].

- The contribution of the \(d = 6\) gluon condensates:

\[
\langle g^3 f_{abc} G^3 \rangle \equiv \langle g^3 f_{abc} G_{\mu}^a G_{\nu}^b G_{\rho}^c \rangle,
\]

\[
\langle j^2 \rangle \equiv g^2 \langle \bar{D}_\mu G_{\mu\nu} \rangle^2 \approx g^4 \left( \sum_q \bar{q} \gamma_\nu \frac{\lambda_a}{2} q \right)^2
\]

\[
\approx - \frac{64}{3} (\pi \alpha_s)^2 \rho \langle \bar{d}d \rangle^2,
\]

after the use of the equation of motion which are not included in the expressions given by [34, 35] have been deduced from the expressions given by [12] (Eqs. II.4.28 and Table II.8) \(\rho \approx 3 - 4\) measures the deviation from the vacuum saturation estimate of the \(d = 6\) four-quark condensates [31–33].

- One can notice that the gluon condensate \(G^2\) and \(G^3\) contributions flip sign from the pseudoscalar to the vector channel while there is an extra \(m_Q M_Q^2 \langle \bar{d}d \rangle\) term with a

\(3\) We use the terminology: inverse Laplace sum rule instead of Borel sum rule as it has been demonstrated in [23] that its QCD radiative corrections satisfy these properties.

\(4\) One can also work with moment sum rules like in [1, 3] or with \(\tau\)-decay like finite energy sum rules [24] inspired from \(\tau\)-decay [25] but these different sum rules give approximately the same results as the one from (inverse) Laplace sum rules.

\(5\) Note a misprint of \(1/\pi\) in front of \(\text{Im}\psi(t)\) in [1].
positive contribution in the pseudoscalar channel. We shall see in Fig. 1 that these different signs transform the minimum in $\tau$ for the pseudoscalar channel into an inflexion point for the vector one.

- The $\alpha_s$ correction to $\langle dd \rangle$, in the $\overline{MS}$-scheme, comes from [34], where the running heavy quark mass $\overline{m}_Q$ enters into this expression.
- Using the known relation between the running $\overline{m}_Q(\mu)$ and on-shell mass $M_Q$ in the $\overline{MS}$-scheme to order $\alpha_s^2$ [36–40]:

$$M_Q = \overline{m}_Q(\mu) \left[ 1 + \frac{4}{3} a_s + (16.2163 - 1.0414 n_l)a_s^2 \right.$$  
$$+ \ln \left( \frac{\mu}{M_Q} \right)^2 (a_s + (8.8472 - 0.3611 n_l)a_s^2)$$  
$$+ \ln^2 \left( \frac{\mu}{M_Q} \right)^2 (1.7917 - 0.0833 n_l)a_s^2 + \ldots \right]$$ (15)

for $n_l$ light flavours, one can express all terms of the previous sum rules with the running mass $\overline{m}_Q(\mu)$. It is clear that, for some non-perturbative terms which are known to leading order of perturbation theory, one can use either the running or the pole mass. However, we shall see that this distinction does not affect, in a visible way, the present result, within the accuracy of our estimate, as the non-perturbative contributions are relatively small though vital in the analysis.

| Parameters          | Values              | Ref.                |
|---------------------|---------------------|---------------------|
| $\alpha_s(M_T)$    | 0.325(8)            | [25, 31, 41, 42]    |
| $\overline{m}_c(m_c)$ | 1261(12) MeV       | average [42–44]     |
| $\overline{m}_b(m_b)$ | 4177(11) MeV       | average [42, 43]    |
| $\mu_q$            | (253 ± 6) MeV       | [6, 24, 45, 46]     |
| $M_0^2$            | (0.8 ± 0.2) GeV$^2$ | [32, 47, 48]        |
| $\langle \alpha_sG^2 \rangle$ | $(7.3 ± 3) \times 10^{-2}$ GeV$^4$ | [31, 33, 43, 49–54] |
| $\langle g^2G^3 \rangle$ | $(8.2 ± 2.0) \times \langle \alpha_sG^2 \rangle$ | [43] |
| $\rho_{q\bar{q}}(\bar{q}q)^2$ | $(5.8 ± 1.0) \times 10^{-4}$ GeV$^6$ | [31–33] |
| $\tilde{m}_s$      | (0.114 ± 0.006) GeV | [6, 24, 45, 46, 55] |
| $\kappa \equiv \langle \bar{s}s \rangle/\langle \bar{d}d \rangle$ | $(0.74^{+0.34}_{-0.12})$ | [6, 56] |

TABLE I. QCD input parameters: the original errors for $\langle \alpha_sG^2 \rangle$, $\langle g^2G^3 \rangle$ and $\rho_{q\bar{q}}(\bar{q}q)^2$ have been multiplied by about a factor 3 for a conservative estimate of the errors (see also the text).

III. QCD INPUT PARAMETERS

The QCD parameters which shall appear in the following analysis will be the charm and bottom quark masses $m_{c,b}$ (we shall neglect the light quark masses $q \equiv u,d$), the light quark condensate $\langle \bar{q}q \rangle$, the gluon condensates $\langle \alpha_sG^2 \rangle$ and $\langle g^2G^3 \rangle$, the mixed condensate $\langle \bar{q}Gq \rangle$ defined in Eq. (11) to Eq. (14) and the four-quark condensate $\rho_{q\bar{q}}(\bar{q}q)^2$, where $\rho \approx 3 - 4$ indicates the deviation from the four-quark vacuum saturation. Their values are given in Table II.

- We shall work with the running light quark condensates and masses. They read:

$$\langle \bar{q}q \rangle(\tau) = -\tilde{\mu}_q^3 (-\beta_1 a_s)^{2/\beta_1} / C(a_s)$$  
$$\langle \bar{q}Gq \rangle(\tau) = -M_0^2 \tilde{\mu}_q^3 (-\beta_1 a_s)^{1/\beta_1} / C(a_s)$$,  
$$\overline{m}_s(\tau) = \frac{M_0^2}{(\text{Log} \sqrt{\tau A})^{2/\beta_1}} C(a_s)$$, (16)

where $\beta_1 = -(1/2)(11 - 2n_f/3)$ is the first coefficient of the $\beta$ function for $n_f$ flavours; $\alpha_s = \alpha_s(\tau)/\pi$; $\tilde{\mu}_q$ is spontaneous RGI light quark condensate [57]. The QCD correction factor $C(a_s)$ in the previous expressions is numerically [58]:

$$C(a_s) = 1 + 0.8951 a_s + 1.3715 a_s^2 + \ldots : n_f = 3 ,$$  
$$= 1 + 1.1755 a_s + 1.5008 a_s^2 + \ldots : n_f = 5, (17)$$

which shows a good convergence. We shall use:

$$\alpha_s(M_T) = 0.325(8) \Rightarrow \alpha_s(M_Z) = 0.1192(10) \quad (18)$$

from $\tau$-decays [25, 31], which agree perfectly with the world average 2012 [41, 42]:

$$\alpha_s(M_Z) = 0.1184(7). \quad (19)$$

The value of the running $\langle \bar{q}q \rangle$ condensate is deduced from the value of $\langle m_u + m_d \rangle(\bar{u}u + \bar{d}d) = (7.9 ± 0.6) \text{ MeV}$ obtained in [24] and the well-known GMOR relation:

$$\langle m_u + m_d \rangle(\bar{u}u + \bar{d}d) = -m_s^2 f^2_\pi,$$ (20)

where $f_\pi = 130.4(2) \text{ MeV}$ [2]. Then, we deduce the RGI light quark spontaneous mass $\tilde{\mu}_q$ given in Table II.

- For the heavy quarks, we shall use the running mass and the corresponding value of $\alpha_s$ evaluated at the scale $\mu$. These sets of correlated parameters are given in Table II for different values of $\mu$ and for a given number of flavours $n_f$.

- For the $\langle \alpha_sG^2 \rangle$ condensate, we have enlarged the original by a factor about 3 in order have a conservative result and to recover the original svz estimate and the alternative extraction in [44] from charmonium sum rules. We do not consider the inaccurate result from $\tau$-decay [59] allowing a negative value which may be due to a simultaneous fit of too many parameters and which contradicts the previous results and all extractions from some other channels [6, 7] including the same $\tau$-decay data [31]. We do not also use the too large value from FESR [60] coming from a cancellation of two large numbers.

- To be conservative, we have enlarged the original error on the value of the $SU(3)$ breaking condensate $\kappa \equiv \langle \bar{s}s \rangle/\langle \bar{d}d \rangle$ given in [56] to recover the central value 1.08 from lattice calculation [61].
IV. PARAMETRIZATION OF THE SPECTRAL FUNCTION AND STABILITY CRITERIA

We shall use the Minimal Duality Ansatz (MDA) for parametrizing the spectral function:

\[
\frac{1}{\pi} \text{Im} \psi_P(t) \simeq f_P^2 M_P^2 \delta(t - M_P^2) + \text{"QCD cont."} \theta(t - t_P^2), \\
\frac{1}{\pi} \text{Im} \psi_P^\tau(t) \simeq f_P^2 M_P^2 \delta(t - M_P^2) + \text{"QCD cont."} \theta(t - t_P^2),
\]

where \( f_{P,V} \) are the decay constants defined in Eq. (1) and the higher states contributions are smeared by the "QCD continuum" coming from the discontinuity of the QCD diagrams and starting from a constant threshold \( t_c^P, t_c^V \) which is independent on the subtraction point \( \mu \) in this standard minimal model. However, an eventual \( \mu \)-dependence of \( t_c \) as used in some model [35] should be included in the conservative range of \( t_c \) used our analysis. One should notice that this MDA with constant \( t_c \) describes quite well the properties of the lowest ground state as explicitly demonstrated in [1] and in various examples, while it has been also successfully tested in the large \( N_c \) limit of QCD in [62].

• In order to extract an optimal information for the lowest resonance parameters from this rather crude description of the spectral function and from the approximate QCD expression, one often applies the stability criteria (existence of an extremum or stability plateau or inflexion point versus the changes of the external sum rule variables \( \tau \) and \( t_c \) at which an optimal result can be extracted). This feature has been demonstrated in series of papers by Bell-Bertmann [54] in the case of \( \tau \) and has been extended to the case of \( t_c \) in [6, 7]. In this paper, we shall add to these well-known criteria, the one associated to the requirement of stability versus the arbitrary subtraction constant \( \mu \) often put by hand in the current literature and often a source of large errors from the PT series in the analysis. This \( \mu \)-stability procedure applied recently in [1, 3, 46, 63] will be used here\(^6\).

V. THE DECAY CONSTANT \( f_{D^*} \)

A. The ratio \( f_{D^*}/f_D \)

We start by showing in Fig. 1, the \( \tau \)-behaviour of the decay constants \( f_{D^*} \) and \( f_D \) at given value of the subtraction point \( \mu = m_c \) for different values of the continuum threshold \( t_c \). We have assumed that:

\[
\sqrt{t_c^{D^*}} - \sqrt{t_c^D} \simeq M_{D^*} - M_D = 140.6 \text{ MeV}. \quad (22)
\]

for the vector and pseudoscalar channels. For the pseudoscalar channel, we have used the expression in Eq. (20) of [1] consistently truncated at the same order of PT and NP series as the one in Eq. (9) for the vector channel. One can notice in Fig. 1 that working directly with the ratio in Eq. (7) by taking the same value \( \tau_P = \tau_D \) is inaccurate as the two sum rules \( \mathcal{L}_V(\tau) \) and \( \mathcal{L}_P(\tau) \) are not optimized at the same value of \( \tau \) (minimum for \( f_D \) and inflexion point for \( f_{D^*} \)). Therefore, for a given value of \( t_c \), we take separately the value of each sum rule at the corresponding value of \( \tau \) where they present minimum and/or inflexion point and then take their ratio. For a

\(^6\) Some other alternative approaches for optimizing the PT series can be found in [64].
given $\mu$, the optimal result corresponds to the mean obtained in range of values of $t_c$ where one starts to have a $\tau$-stability ($t_c \approx 5.6 - 5.7 \text{ GeV}^2$ for $\tau \approx 0.6 \text{ GeV}^{-2}$) and a $t_c$-stability ($t_c \approx 9.5 - 10.5 \text{ GeV}^2$ for $\tau \approx 0.8 \text{ GeV}^{-2}$).

Now, we look for the $\mu$-stability by plotting versus $\mu$ the previous optimal ratio $f_{D^*}/f_D$ in the variables $\tau$ and $t_c$.

The results are shown in Fig. 2. We obtain a minimum for $\mu = (1.5 \pm 0.1) \text{ GeV}$ which is about the average $1.5 \text{ GeV}$ of $[34]$ and $1.84 \text{ GeV}$ used in $[35]$. At this minimum, we deduce the final result:

$$f_{D^*}/f_D = 1.218(6)t_c(27)_\tau(23)_{svz}(2)_\mu$$

$$= 1.218(36)$$

(23)

where the error from the QCD expression within the SVZ expansion is the quadratic sum of

$$(23)_{svz} = (7)\alpha_s(2)\alpha_s^2(3)\langle \bar{d}d \rangle(18)_{\alpha_s G^2}$$

$$+ (12)\langle \bar{d}Gd \rangle(0)\langle g^2 G^2 \rangle(1)_{\langle \bar{d}d \rangle^2}.$$ (24)

One can notice that the largest error comes from $\tau$ and is due to the inaccurate localization of the inflexion point. The one due to $t_c$ is smaller as expected in the determination of the ratio unlike the direct extraction of the decay constant. The errors due to $\langle \alpha_s G^2 \rangle$ and $\langle \bar{d}Gd \rangle$ are large due to the opposite sign of their contributions in the vector and pseudoscalar channels which add when taking the ratio.

![Figure 1](image1.png)

**FIG. 1.** a) $\tau$-behaviour of $f_D$ from $\mathcal{L}_F$ for different values of $t_c$, at a given value of the subtraction point $\mu = m_c$; b) the same as in a) but for $f_{D^*}$ from $\mathcal{L}_V$.

**B. Estimate of $f_{D^*}$**

Using the value $f_D = 204(6) \text{ MeV}$ (Table 8 of [1]) obtained under a similar strategy and the ratio in Eq. (23), we deduce:

$$f_{D^*} = 248.5(10.4) \text{ MeV},$$

(25)

where the errors have been added quadratically.

One can also extract directly $f_{D^*}$ using the analysis of Fig. 1b and a similar strategy as the one used for extracting the ratio $f_{D^*}/f_D$. The $\mu$ behaviour of the optimal $\tau$ and $t_c$ result is given in Fig. 3. A minimum is also obtained for $\mu \approx 1.5 \text{ GeV}$ at which we deduce:

$$f_{D^*} = 253.5(11.5)t_c(5.7)_\tau(13)_{svz}(0.5)\mu \text{ MeV}$$

$$= 253.5(18.3) \text{ MeV},$$

(26)

with:

$$(13)_{svz} = (0.5)\alpha_s(12.3)\alpha_s^2(0.6)m_c(3.8)\langle \bar{d}d \rangle(1.8)_{\alpha_s G^2}$$

$$+ (1.4)\langle \bar{d}Gd \rangle(0.4)\langle g^2 G^2 \rangle(0.4)_{\langle \bar{d}d \rangle^2}.$$ (27)

If we move $\mu$ from 1.5 to 2.5 GeV, $f_{D^*}$ increases by about 12 MeV which is comparable with the value 17 MeV obtained in Ref. [34] by taking $\mu$ from 1.5 to 3 GeV. The small variation of $f_{D^*}$ around the $\mu$ stability point explains the small error in Eq. (28). Our final result will be the mean of the two determinations in Eqs. (25) and (28) which is:

$$\langle f_{D^*} \rangle = 249.7(10.5)(1.2)_{syst} \text{ MeV}$$

$$= 250(11) \text{ MeV},$$

(28)

where the 1st error comes from the most precise determination and the 2nd one from the distance of the mean value to it. This result is inside the range of the recent ones from [34] and [35] but lower than the alone available lattice value [65]: $f_{D^*} = 278(16) \text{ MeV}$ where an independent estimate from some other lattice groups is required.

![Figure 2](image2.png)

**FIG. 2.** Values of $f_{D^*}/f_D$ at different values of the subtraction point $\mu$. 
C. Upper bound on $f_{D^*}$

We derive an upper bound on $f_{D^*}$ by considering the positivity of the QCD continuum contribution to the spectral function and by taking the limit where $t_\tau \to \infty$ in Eq. (8) which corresponds to a full saturation of the spectral function by the lowest ground state contribution. The result of the analysis versus the change of $\tau$ for a given value of $\mu = 1.5$ GeV is given in Fig. 4 where one can observe like in the previous analysis the presence of a $\tau$-inflexion point. We also show in this figure the good convergence of the PT series by comparing the result at N2LO and the one including an estimate of the N3LO term based on the geometric growth of the PT coefficients. We show in Fig 5 the variation of the optimal bound versus the subtraction point $\mu$ where we find a region of $\mu$ stability from 1.5 to 2 GeV. We obtain:

\[ f_{D^*} \leq 267(10)\tau(14)_{svz}(0)\mu \text{ MeV}, \]  

(29)

with:

\[ (14)_{svz} = (1.4)_{a_s}(13)_{a_s^2}(1.3)_{m_\rho}(3)_{(d\bar{d})}(3)_{(G^2)} \]

\[ (2.5)_{(\bar{d}Gd)}(0)_{(G^3)}(0.7)_{(d\bar{d})^2} \]. \hspace{1cm} (30)

Alternatively, we combine the upper bound $f_{D^*} \leq 218.4(1.4)$ MeV obtained [1] with the previous ratio in Eq. (23) and deduce:

\[ f_{D^*} \leq 266(8) \text{ MeV}, \]  

(31)

where we have added the errors quadratically. The good agreement of the results in Eqs. (29) and (31) indicates the self-consistency of the approaches. This bound is relatively strong compared to the estimate in Eq. (28) while the recent lattice estimate $f_{D^*} \approx 278(16)$ MeV obtained in [65] is at its borderline.

VI. THE DECAY CONSTANT $f_{B^*}$

We extend the analysis to the case of the $B^*$ meson. We use the set of parameters in Table II and II. We show the $\tau$-behaviour of $f_B$ and $f_{B^*}$ in Fig. 6, where the shape is similar to the case of $f_{D^*}$ and $f_D$.

A. Estimate of $f_{B^*}$

Using the previous strategy, we estimate $f_{B^*}$ from the analysis in Fig. 6b where the $\tau$-stability is reached from $t_\tau = 34$ GeV$^2$ while the $t_c$ stability starts from $t_c = (55 - 60)$ GeV$^2$. We show the $\mu$-behaviour of the optimal result in Fig. 7 where we find a clear inflexion point for $\mu = (5.0 - 5.5)$ GeV at which we extract the optimal result:

\[ f_{B^*} = 239(38)_t(1)_{c}(2.7)_{svz}(0.7)_{\mu} \text{ MeV} \]

\[ = 239(38) \text{ MeV}, \]  

(32)

with:

\[ (2.7)_{svz} = (0.7)_{a_s}(2)_{a_s^2}(0.4)_{m_\rho}(1.6)_{(d\bar{d})}(0.4)_{(G^2)} \]

\[ (0.3)_{(\bar{d}Gd)}(0)_{(G^3)}(0)_{(d\bar{d})^2} \]. \hspace{1cm} (33)

where the error in $\mu$ comes by taking $\mu = (5.5 \pm 0.5)$ GeV.
5.5 GeV. We consider as a final result the mean of the ones from these two regions of $\mu$:

$$f_{B*}/f_B = 1.016(12)t_c(1)\tau_{svz}(9)_{svz}(3)_{\mu}$$

$$= 1.016(15),$$

with:

$$(9)_{svz} = (3)_{\alpha_s} (6)_{\alpha_s^2} (3)_{m_b} (4)_{dGd} (3)_{\langle \alpha_s G^2 \rangle}$$

$$(1)_{dGd} (1)_{(g^3 G^2)^2} (1)_{dGd^2} .$$

Our results in Eqs. (32) and (34) are comparable with the ones in [34] but with large errors due mainly to our conservative range of $t_c$ values. However, the value of $\mu$ at which our optimal results are obtained does not favour the choice $\mu = 3$ GeV adopted in [34]. Combining the results in Eq. (34) with the value $f_B = 206(7)$ MeV obtained in [1], we deduce:

$$f_{B*} = 209(8) \text{ MeV},$$

which is more accurate than the direct determination in Eq. (32). We consider the result in Eq. (36) which is also the mean of the results in Eqs. (36) and (32) as our final determination.

**B. The ratio $f_{B*}/f_B$**

Here, we extract directly the ratio $f_{B*}/f_B$ from the ratio of sum rules. We show in Fig. 8 its $\tau$ behaviour for different values of $t_c$ for $\mu = 5.5$ GeV from which we deduce as optimal value the mean of the $\tau$-minima obtained from $t_c = 34$ to 60 GeV$^2$. We show in Fig. 9 the $\mu$ behaviour of these optimal results where we find a minimum in $\mu$ around 3.8-4.5 GeV and a slight inflexion point around

![FIG. 6. a) $\tau$-behaviour of $f_B$ from $L_P$ for different values of $t_c$, at a given value of the subtraction point $\mu = m_b$; b) the same as in a) but for $f_{B*}$ from $L_V$.](image1)

![FIG. 7. Values of $f_{B*}$ at different values of the subtraction point $\mu$.](image2)

![FIG. 8. $\tau$-behaviour of $f_{B*}/f_B$ for different values of $t_c$, at a given value of the subtraction point $\mu = 5.5$ GeV.](image3)

![FIG. 9. Values of $f_{B*}/f_B$ at different values of the subtraction point $\mu$.](image4)
quark symmetry as expected from HQET \cite{66}, while the charm quark mass and QCD radiative corrections is still too low at which a such symmetry is broken by linear terms in $m_s$ obtained in \cite{34}. We show the $\tau$ behaviour of different results in Fig. 12 for a given value of $\mu = 1.5$ GeV and for different $t_c$. We study the $\mu$ dependence of these results in Fig. 13 where a nice $\mu$ stability is reached for $\mu \simeq 1.4 - 1.5$ GeV. We have used:

$$\sqrt{t_c^{D^*}} = \sqrt{t_c^{D^*}} = M_{D^*} - M_{D^*} = 102\text{ MeV}.$$ \hspace{1cm} (38)

Taking the conservative result ranging from the beginning of $\tau$-stability ($t_c \simeq 5.7$ GeV$^2$) until the beginning of $t_c$-stability of about (9–10) GeV$^2$, we obtain at $\mu = 1.5$ GeV:

$$f_{D^*} = 272 (24)\sqrt{\tau_c (2)_{svz} (1)}\mu \text{ MeV}$$
$$= 272 (30) \text{ MeV},$$ \hspace{1cm} (39)

We pursue the same analysis for studying the SU(3) breaking for $f_{D^*}$ and the ratio $f_{D^*}/f_{D^*}$. We work with the complete massive ($m_s \neq 0$) LO expression of the PT spectral function obtained in \cite{57} and the massless ($m_s = 0$) expression known to N2LO used in the previous sections. We include the NLO PT corrections due to linear terms in $m_s$ obtained in \cite{34}. We show the $\tau$ behaviour of different results in Fig. 12 for a given value of $\mu = 1.5$ GeV and for different $t_c$. We study the $\mu$ dependence of these results in Fig. 13 where a nice $\mu$ stability is reached for $\mu \simeq 1.4 - 1.5$ GeV. We have used:

$$\sqrt{t_c^{D^*}} = \sqrt{t_c^{D^*}} = M_{D^*} - M_{D^*} = 102\text{ MeV}.$$ \hspace{1cm} (38)

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We pursue the same analysis for studying the SU(3) breaking for $f_{D^*}$ and the ratio $f_{D^*}/f_{D^*}$. We work with the complete massive ($m_s \neq 0$) LO expression of the PT spectral function obtained in \cite{57} and the massless ($m_s = 0$) expression known to N2LO used in the previous sections. We include the NLO PT corrections due to linear terms in $m_s$ obtained in \cite{34}. We show the $\tau$ behaviour of different results in Fig. 12 for a given value of $\mu = 1.5$ GeV and for different $t_c$. We study the $\mu$ dependence of these results in Fig. 13 where a nice $\mu$ stability is reached for $\mu \simeq 1.4 - 1.5$ GeV. We have used:

$$\sqrt{t_c^{D^*}} = \sqrt{t_c^{D^*}} = M_{D^*} - M_{D^*} = 102\text{ MeV}.$$ \hspace{1cm} (38)

Taking the conservative result ranging from the beginning of $\tau$-stability ($t_c \simeq 5.7$ GeV$^2$) until the beginning of $t_c$-stability of about (9–10) GeV$^2$, we obtain at $\mu = 1.5$ GeV:

$$f_{D^*} = 272 (24)\sqrt{\tau_c (2)_{svz} (1)}\mu \text{ MeV}$$
$$= 272 (30) \text{ MeV},$$ \hspace{1cm} (39)
with:

\[(18)_{svz} = (0)\alpha_s(14)\alpha_s^2(1)m_s(2)\langle \bar{d}d \rangle(1.5)\langle \alpha_s G^2 \rangle

(0.8)\langle \bar{d}Gd \rangle(0)\langle g^3 G^2 \rangle(0)\langle \bar{d}d \rangle^2

(0.3)m_s(2)\kappa \right).

Taking the PT linear term in $m_s$ at lowest order and \( t_c = 7.4 \text{ GeV}^2 \), we obtain \( f_{D_s} = 281 \text{ MeV} \) in agreement with the one 293 MeV of [34] obtained in this way. The inclusion of the complete LO term decreases this result by about 5 MeV while the inclusion of the NLO PT $SU(3)$ breaking terms increase the result by about the same amount. However, we do not see any justification for choosing the value of \( t_c = 7.4 \text{ GeV}^2 \) used in [34]. Instead, one can consider that our result coming from the mean of the one at \( t_c = 5.7 \) and 10 GeV$^2$ is more conservative. Combining the result in Eq. (39) with the one in Eq. (28), we deduce the ratio:

\[ f_{D_s}/f_{D_s^*} = 1.09(7), \]

where we have added the relative errors quadratically.

![Fig. 14. $\tau$-behaviour of $f_{D_s}/f_{D_s^*}$ for different values of $t_c$ and for $\mu = 1.5 \text{ GeV}$.](image)

![Fig. 15. $\mu$-behaviour of $f_{D_s}/f_{D_s^*}$.](image)

Alternatively, we extract directly the previous ratio using the ratio of sum rules. We show the results in Fig. 14 versus $\tau$ and for different values of $t_c$ at $\mu = 1.5 \text{ GeV}$. $\tau$-stabilities occur from $\tau \simeq 1$ to 1.5 GeV$^{-2}$. We also show in Fig 15 the $\mu$ behaviour of the results where a good stability in $\mu$ is observed for $\mu \simeq (1.4 - 1.5) \text{ GeV}$ in the same way as for $f_{D_s^*}$. We deduce:

\[ f_{D_s}/f_{D_s^*} = 1.073(1)_t(16)_\tau(1)\mu (50)_{svz} \]

\[ = 1.073 \pm 0.052 , \]

with:

\[(50)_{svz} = (1)\alpha_s(45)\alpha_s^2(0)m_s(2)\langle \bar{d}d \rangle(16)\langle \alpha_s G^2 \rangle

(3)\langle \bar{d}Gd \rangle(1)\langle g^3 G^2 \rangle(2)\langle \bar{d}d \rangle^2

(4)m_s(13)\kappa , \]

where, for asymmetric errors, we have taken the mean of the two extremal values. The error associated to $\tau$ take into accounts the fact that, for some values of $t_c$, the $\tau$-minima for $f_{D_s^*}$ and $f_{D_s}$ do not coincide. Comparing the results in Eqs. (41) and (42), one can clearly see the advantage of a direct extraction from the ratio of moments due to the cancellation of systematic errors in the analysis. Taking the mean of the two different results in Eqs. (41) and (42), we deduce our final estimate:

\[ f_{D_s}/f_{D_s^*} = 1.08 \pm 0.06 \pm 0.01_{\text{syst}} , \]

where the 1st error comes from the most precise determination and the 2nd one from the distance of the mean value to the central value of this precise determination. This value is in better agreement with the lattice result [65]: $1.16 \pm 0.06$ than the one from the sum rules analysis $1.21 \pm 0.05$ in [34] and $1.21 \pm 0.07$ in [35]. The almost good agreement with the lattice result is due to the fact that both $f_{D_s^*}$ and $f_{D_s}$ are larger from the lattice than in the present paper while the ratio is less affected by this discrepancy. The disagreement with the sum rule result of [34] is due to a larger value of $f_{D_s}$ in [34] related to the choice of $t_c = 7.4 \text{ GeV}^2$ but to a value of of $f_{D_s^*}$ similar to ours because taking the same value of $t_c \simeq 5.6$ GeV$^2$. The discrepancy with the one in [35] is due to a larger value of subtraction scale $\mu = 1.94 \text{ GeV}$ which is outside the $\mu$ stability region shown in Fig. 13 for $f_{D_s}$. In fact, one would intuitively expect that, up to small $SU(3)$ breaking corrections, the value of $\mu$ is about the same for $f_{D_s^*}$ and $f_{D_s}$ as explicitly shown in Figs. 3 and 13. Using the value in Eq. (44) and the prediction for $f_{D_s^*}$ given in Eq. (28), we predict:

\[ f_{D_s} = 270 \pm 19 \text{ MeV} . \]

Combining the results in Eqs. (28) and (44), we deduce the upper bound:

\[ f_{D_s} \leq 287(8.6)(16) \text{ MeV} \]

\[ \leq 287(18) \text{ MeV} . \]

Future experimental measurements of $f_{D_s^*}$ and $f_{D_s}$ though most probably quite difficult should provide a decisive selection of these existing theoretical predictions.
VIII. SU(3) BREAKING FOR $f_{B_s^*}$ AND $f_{B_s^*/f_{B_s^*}}$

We extend the analysis done for the $D_s^*$ to the case of the $B_s^*$-meson. We show, in Fig. 16, the $\tau$-behaviour of the ratio $f_{B_s^*}/f_{B_s^*}$ at $\mu=5\ \text{GeV}$ for different values of $t_c$, where the $\tau$ stability starts from $t_c=40\ \text{GeV}^2$ while the $t_c$ one is reached for $t_c \approx (60-65)\ \text{GeV}^2$. Our optimal result is taken in this range of $t_c$. We study the $\mu$ behaviour in Fig. 17 where an inflexion point is obtained for $\mu=(5\pm0.5)\ \text{GeV}$. At this point, we obtain:

$$f_{B_s^*}/f_{B_s^*}=1.054(8)_{t_c}(0)_{\tau}(3)_{\mu}(4.6)_{svz} = 1.054 \pm 0.010 ,$$

with:

$$(4.6)_{svz} = (2)_{\alpha_s}(2.5)_{\alpha_s}(0)_{m_s}(2)_{\langle\bar{d}d\rangle}(1.5)_{\langle\alpha_s G^2\rangle}(0)_{\langle\bar{d}G\bar{d}\rangle}(0)_{\langle\bar{d}G\bar{d}\rangle}(0)_{\langle\bar{d}d\rangle}$$

$$= (0)_{m_s}(2)_{\kappa} .$$

We show, in Fig. 18, the $\tau$ behaviour of the result for $f_{B_s^*}$ at $\mu=5.5\ \text{GeV}$ and for different values of $t_c$. For $f_{B_s^*}$, $\tau$-stability starts from $t_c \approx 34\ \text{GeV}^2$ while $t_c$ stability is reached from $t_c \approx (50-65)\ \text{GeV}^2$. We show in Fig. 19 the $\mu$ behaviour of these optimal results. One can notice a slight inflexion point at $\mu=6\ \text{GeV}$ which is about the value $(5.0-5.5)\ \text{GeV}$ obtained for the ratio where $f_{B_s^*}/f_{B_s^*}$ obtained previously. At this value of $\mu$, we obtain:

$$f_{B_s^*} = 271(39)_{t_c}(0)_{\tau}(3)_{svz}(3)_{\mu} \ \text{MeV}$$

$$= (271 \pm 39) \ \text{MeV} .$$

with:

$$(3)_{svz} = (1)_{\alpha_s}(1.5)_{\alpha_s}(0.5)_{m_s}(2)_{\langle\bar{d}d\rangle}(0.5)_{\langle\alpha_s G^2\rangle}(0.5)_{\langle\bar{d}G\bar{d}\rangle}(0.5)_{\langle\bar{d}d\rangle}$$

$$= (0)_{m_s}(1)_{\kappa} .$$

Combining consistently this result with the one for $f_{B_s^*}$ in Eq. (32) obtained within the same approach and conditions, we deduce the ratio:

$$f_{B_s^*}/f_{B_s^*} = 1.134(17) .$$

We take as a final value of the ratio $f_{B_s^*}/f_{B_s^*}$ the mean of the results from Eqs. (47) and (51):

$$f_{B_s^*}/f_{B_s^*} = 1.075(10)/(21)_{\text{syst}}$$

$$= 1.075(23) .$$

Combining this result with the value of $f_{B_s^*}$ in Eq. (36), we deduce our final estimate:

$$f_{B_s^*} = 225(10) \ \text{MeV} .$$

Combining again this result of the ratio with the upper bound of $f_{B_s^*}$ in Eq. (37), we deduce the upper bound:

$$f_{B_s^*} \leq 317(17) \ \text{MeV} .$$
IX. THE DECAY CONSTANT \( f_{B_c} \)

We complete the analysis in this paper by the estimate and the bound of the decay constant \( f_{B_c} \) of the \( B_c(6277) \) meson \( bc \) bound state.

- The complete expression of the perturbative NLO spectral function has been obtained in [12] and explicitly written in [21]. We add to this expression the N2LO result obtained in [27] for \( m_c = 0 \). We consider as a source of errors an estimate of the N3LO contribution by assuming a geometric growth of the PT series [28] which mimics the phenomenological \( 1/q^2 \) dimension two term which parametrizes the large order terms of PT series [29, 30].

- The Wilson coefficients of the non-perturbative \( (\alpha_s G^2) \) and \( (g^3 G^3) \) contributions are also given in [21]. We transform the pole masses to the running masses using the previous expression in Eq. (15). We study the corresponding (inverse) Laplace sum rule versus \( \tau \) and for different values of \( t_c \) which we show in Fig. 20. We notice that the non-perturbative contributions are all negligible indicating that the dynamics of the \( B_c \) meson is dominated by the perturbative contributions. The optimal result is obtained from \( t_c = 44 \text{ GeV}^2 \) (beginning of \( \tau \) stability) until \( t_c = (50 - 60) \text{ GeV}^2 \) (beginning of \( t_c \) stability).

We show in Fig. 21 the \( \mu \) behaviour of different results, where one can notice that there is an inflexion point for \( \mu = (7.5 \pm 0.5) \text{ GeV} \) for both the estimate (Fig. 21a) and the upper bound (Fig. 21). At these optimal points, we deduce:

\[
f_{B_c} = 436(38)t_c(2)\alpha_s(2)\alpha_s^2(7)m_c(6)\mu \text{ MeV}
\]
\[
= 436(40) \text{ MeV}.
\]  

(55)

and

\[
f_{B_c} \leq 466(9)\alpha_s(2)\alpha_s^2(12)m_c(8)\mu \text{ MeV}
\]
\[
\leq 466(16) \text{ MeV}.
\]  

(56)

We consider the previous results as improvements of the earlier ones obtained in [7, 20–22]. Having in mind that, long time before the experimental discovery of \( B_c \), the correct prediction of \( M_{B_c} \) from QSSR has been given in [21] together with some potential model predictions, which was not the case of some early lattice results, the agreement of our results in Eq. (55) with the recent lattice value \( f_{B_c} = (427 \pm 6) \text{ MeV} \) in [67] (and to a lesser extent with the large range of potential model predictions \( f_{B_c} = (503 \pm 171) \text{ MeV} \) reviewed in [21]) can be considered as a strong support of our results and may question the validity of a recent estimate \( f_{B_c} = (528 \pm 19) \text{ MeV} \) from some variants of FESR [68] evaluated at a given \( \mu \). This upper bound in Eq. (56) and our previous estimate in Eq. (55) together with the recent lattice result will restrict the wide range of \( f_{B_c} \) values given the current literature.

**SUMMARY AND CONCLUSIONS**

Our main results are summarized in Table III and Table IV where a comparison with some other recent sum rules and lattice results is done.

- We have re-estimated \( f_D \) and \( f_{B^*} \) directly from the Laplace sum rule of vector current and indirectly by combining our previous results for \( f_D \) and \( f_B \) [1] with suitable ratios of Laplace sum rules for \( f_{D^*}/f_D \) and \( f_{B^*}/f_B \) known to N2LO of PT, including complete non-perturbative contributions of dimension 6 and an estimate of the N3LO PT-term where for the latter a geo-

---

**FIG. 21.** Values of \( f_{B_c} \) at different values of the subtraction point \( \mu \): a) estimate; b) upper bound

**FIG. 20.** \( \tau \)-behaviour of \( f_{B_c} \) from \( L^\rho \) for different values of \( t_c \), at a given value of the subtraction point \( \mu = 7.5 \text{ GeV} \).
metric growth of the PT coefficients has been assumed. Our results given in Eqs. (23), (28) and Eqs. (34), (36) agree and improve our earlier determinations in [7, 20] and agree with a recent estimate obtained at a particular value of the subtraction point [34]. These results indicate a good realization of heavy quark symmetry for the $B$ and $B^*$ mesons ($f_B \approx f_{B^*}$) as expected from HQET [66] but signal large charm quark mass and radiative QCD corrections for the $D$ and $D^*$ mesons.

- Our value of the $SU(3)$ breaking ratio of decay constants $f_{D^*}/f_{D^*}$ in Eqs. (44) disagree with the larger value given by [34] and [35] correlated to larger value of $f_{D^*}$ obtained there but is in a better agreement with the lattice result [65] though the absolute values of the decay constants from lattice are individually larger. The same feature is observed for our value of $f_{B^*}/f_{B^*}$ in Eq. (52) when compared with the result of [34]. We expect that future experimental measurements of these couplings may select among these theoretical predictions.

- Using the positivity of the spectral functions, we have also derived in Eqs. (29), (31), (??) and (56) upper bounds for $f_{D^*}$, $f_{B^*}$ and $f_{B^*}$. Combining these upper bounds with our estimate of the ratios $f_{D^*}/f_{D^*}$ and $f_{B^*}/f_{B^*}$, we have also derived, in Eqs.(29,31) and in Eq.(54), upper bounds on $f_{D^*}$ and $f_{B^*}$. We notice that the recent lattice result for $f_{D^*}$ [65] is at the borderline of the previous upper bound.

- For completing our present study and motivated by the wide range of predictions in the existing literature, we have re-estimated $f_{B^*}$ by working with NLO spectral function with massive quark. We have added N2LO terms with massless quark and an estimate of the N3LO contribution based on the geometric growth of the PT coefficients. The estimate in Eq. (55) and the upper bound in Eq. (56) may be considered as improvements of the ones obtained earlier from QCD spectral sum rules in [7, 20–22]. Comparing $f_{B^*}$ and $f_B$ which differs by about a factor two, we conclude a large $SU(4)$ breaking of the leptonic decay constant.

- It is informative to show the behaviour of the pseudoscalar and vector meson decay constants versus the corresponding meson masses in Fig. 22. The open circles correspond to $f_\pi$, $f_D$, $f_B$, and $f_{B^*}$. The triangles correspond to the one with $SU(3)$ breaking: $f_K$, $f_{D_s}$, and $f_{B_s}$. The boxes correspond to $f_\rho$, $f_{D^*}$ and $f_{B^*}$. SU(3) breaking in the vector channels are quite small. The values of $f_{D(\pi)}$ and $f_{B(\pi)}$ come from [1, 3] while $f_{\pi,K}$ comes

| Table III | Estimates and upper bounds of the decay constants and comparisons with recent sum rules and lattice results. |
|-----------|--------------------------------------------------------|
| $f_{D^*}$ [MeV] | $t_c$ [GeV]$^2$ | $\mu$ [GeV] | Sources | Refs. |
| 250(11) | 5.6 - 10.5 | 1.5 | Eq.(28) | This work |
| 242$^{+10}_{-12}$ | 6.2 | 1.5 | SR | [34] |
| 252(22) | 5.52 | 1.84 | SR | [35] |
| 278(16) | $\leq 266(8)$ | $\infty$ | $1.5 - 2.0$ | Eqs.(29,31) | This work |
| $f_{B^*}$ [MeV] | $f_{D^*}/f_D$ | $1.218(36)$ | 5.6 - 10.5 | 1.5 | Eq. (23) | This work |
| $1.20^{+0.10}_{-0.07}$ | 6.2 | 1.5 | SR | [34] |
| 1.221(80) | 5.52 | 1.84 | SR | [35] |
| $f_{B^*}/f_B$ | $1.016(15)$ | 34 - 60 | 5 - 5.5 | Eq. (34) | This work |
| $1.02^{+0.07}_{-0.03}$ | 34 - 36 | 3 | SR | [34] |
| $f_{B^*}$ [MeV] | $\leq 466(16)$ | $\infty$ | – | Eq. (56) | – |
| 427(6) | 44 - 60 | 7.5 ± 0.5 | Eq. (55) | This work |
| 503(171) | 50.6 | SR | [68] |
| Table IV | $SU(3)$ breaking effects on the estimates and on the upper bounds of the decay constants and comparisons with recent sum rules and lattice results. |
| $f_{D^*}$ [MeV] | $t_c$ [GeV]$^2$ | $\mu$ [GeV] | Sources | Refs. |
| 270(19) | 5.6 - 10.5 | 1.5 | Eq.(28) | This work |
| 293$^{+19}_{-14}$ | 7.4 | 1.5 | SR | [34] |
| 306(27) | 5.52 | 1.94 | SR | [35] |
| 311(9) | $\leq 287(18)$ | $\infty$ | $1.5 - 2.0$ | Eqs.(29,31) | This work |
| $f_{D^*}/f_D$ | $1.08(6)$ | 5.6 - 10.5 | 1.5 | Eq. (23) | This work |
| $1.21^{+0.06}_{-0.04}$ | 6.2 - 7.4 | 1.5 | SR | [34] |
| 1.211(61) | 5.52 | 1.84 | SR | [35] |
| 1.16(6) | Latt. | [65] |
| $f_{B^*}$ [MeV] | $f_{D^*}/f_B$ | $225(10)$ | 34 - 60 | 6 ± 0.5 | Eq.(39) | This work |
| $210^{+10}_{-12}$ | 34.1 | 3 | SR | [34] |
| $\leq 317(17)$ | $\infty$ | 6 ± 0.5 | Eq.(54) | This work |
| $\leq 296$ | $\infty$ | 3 | SR | [34] |
| $f_{B^*}/f_B$ | $1.075(23)$ | 34 - 60 | 5 ± 0.5 | Eq. (52) | This work |
| 1.20(4) | 34 - 36 | 3 | SR | [34] |
from [2]. We use $f_\rho = (221.6 \pm 1.0)$ MeV extracted from its electronic width compiled by [42]. One can remark similar $M$ behaviours of these different couplings. The results for the $D^{(*)}_s$ and $B^{(*)}_s$ mesons do not satisfy the $1/\sqrt{M}$ HQET relation.

FIG. 22. Behaviour of the meson decay constants versus the meson masses: open circle: $f_K$, $f_{D_s}$ and $f_{B_s}$; triangle: $SU(3)$ breaking: $f_K$, $f_{D_s}$ and $f_{B_s}$; boxes: $f_\rho$, $f_{D_s}$ and $f_{B_s}$.
[40] K.G. Chetyrkin and M. Steinhauser, Nucl. Phys. B573 (2000) 617; K. Melnikov and T. van Ritbergen, hep-ph/9912391.

[41] For a recent review, see e.g: S. Bethke, talk given at the 16th international QCD conference (QCD 12), 2-6th july 2012, Montpellier, arXiv:1210.0325 [hep-ex] (2012).

[42] J. Beringer et al. (PDG), Phys. Rev. D86 (2012) 010001.

[43] S. Narison, Phys. Lett. B693 (2010) 559; Erratum ibid 705 (2011) 544; ibid, Phys. Lett. B706 (2011) 412; ibid, Phys. Lett. B707 (2012) 259.

[44] B.L. Ioffe and K.N. Zyablyuk, Eur. Phys. J. C27 (2003) 229; B.L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006) 232.

[45] H.G. Dosch and S. Narison, Phys. Lett. B417 (1998) 173; S. Narison, Phys. Lett. B216 (1989) 191.

[46] B.L. Ioffe, Nucl. Phys. B188 (1981) 317; B.L. Ioffe, B191 (1981) 591; A.A.Ovchinnikov and A.A.Pivovarov, Yad. Fiz. 48 (1988) 1135.

[47] S. Narison, Phys. Lett. B605 (2005) 319.

[48] S. Narison, Phys. Lett. B300 (1993) 293; ibid B361 (1995) 121.

[49] S. Narison, Phys. Lett. B387 (1996) 162.

[50] F.J. Yndurain, hep-ph/9903457.

[51] S. Narison, Phys. Lett. B387 (1996) 162.

[52] S. Narison, Phys. Lett. B361 (1995) 121; S. Narison, Phys. Lett. B624 (2005) 223.

[53] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B227 (1983) 435; R.A. Bertlmann, Acta Phys. Austriaca 53 (1981) 305; R.A. Bertlmann and H. Neufeld, Z. Phys. C27 (1985) 437.

[54] [55] S. Narison, Phys. Lett. B673 (2009) 30 and references therein.

[56] R.M. Albuquerque, S. Narison, Phys. Lett. B694 (2010) 217; R.M. Albuquerque, S. Narison, M. Nielsen, Phys. Lett. B684 (2010) 236.

[57] E.G. Floratos, S. Narison and E. de Rafael, Nucl. Phys. B155 (1979) 155.

[58] K.G. Chetyrkin, J.H. Kühn and M. Steinhauser, hep-ph/0004189 and references therein.

[59] M. Davier et al., Eur. Phys. J. C56 (2008) 305.

[60] R.A. Bertlmann, G. Launer and E. de Rafael, Nucl. Phys. B250 (1985) 61; R.A. Bertlmann et al., Z. Phys. C39 (1988) 231; S. Bodenstein et al., JHEP 1307 (2013) 138.

[61] C. McNeile et al., Phys. Rev. 87 (2013) 034503.

[62] E. de Rafael, Nucl. Phys. Proc. Suppl. 96 (2001) 316; S. Peris, B. Phily and E. de Rafael, Phys. Rev. Lett. 86 (2001) 14.

[63] S. Narison, arXiv:1409.8148 [talk given at QCD 14 - Montpellier 30 June - 4 July 2014, to appear in Nucl. Phys. B (Proc. Suppl.)].

[64] P.M. Stevenson, Nucl. Phys. B868 (2013) 38; S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D28 (1983) 228; X.-G. Wu et al., arXiv:1405.3196 [hep-ph] (2014); A.L. Kataev and S.V. Mikhailov, arXiv:1408.0122 [hep-ph] (2014); J. -L. Kneur and A. Neveu, Phys. Rev. D88 (2013) 074025.

[65] D. Becirevic et al., JHEP 1202 (2012) 042.

[66] M. Neubert, Phys. Rept. 245 (1994) 259.

[67] C. McNeile et al., Phys. Rev. D86 (2012) 074503.

[68] M. Baker et al., arXiv:1310.0941 [hep-ph] (2013).