Preorder-Based Triangle: A Modified Version of Bilattice-Based Triangle for Belief Revision in Nonmonotonic Reasoning

Kumar Sankar Ray
ECSU, Indian Statistical Institute, Kolkata

Sandip Paul
ECSU, Indian Statistical Institute, Kolkata

Diganta Saha
CSE Dept, Jadavpur University

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Abstract

Bilattice-based triangle provides an elegant algebraic structure for reasoning with vague and uncertain information. But the truth and knowledge ordering of intervals in bilattice-based triangle can not handle repetitive belief revisions which is an essential characteristic of nonmonotonic reasoning. Moreover the ordering induced over the intervals by the bilattice-based triangle is not sometimes intuitive. In this work, we construct an alternative algebraic structure, namely preorder-based triangle and we formulate proper logical connectives for this. It is an enhancement of the bilattice-based triangle to handle belief revision in nonmonotonic reasoning.

1 Introduction:

In many application domains, decision making and reasoning deal with imprecise and incomplete information. Fuzzy set theory is a formalism for representation of imprecise, linguistic information. A vague concept is described by a membership function, attributing to all members of a given universe X a degree of membership from the interval [0,1]. The graded membership value refers to many-valued propositions in presence of complete information. But this 'one-dimensional' measurement cannot capture the uncertainty present
in information. In absence of complete information the membership degree may not be assigned precisely. This uncertainty with respect to the assignment of membership degrees is captured by assigning a range of possible membership values, hence by assigning an interval. Interval-valued Fuzzy Sets (IVFSs) deal with vagueness and uncertainty simultaneously by replacing the crisp [0,1]-valued membership degree by intervals in [0,1]. The intuition is that the actual membership would be a value within this interval. The intervals can be ordered with respect to their degree of truth as well as with respect to their degree of certainty by means of a bilattice-based algebraic structure, namely Triangle (Arieli, Cornelis, Deschrijver, & Kerre, 2004, 2005; Cornelis, Arieli, Deschrijver, & Kerre, 2007). This algebraic structure serves as an elegant framework for reasoning with uncertain and imprecise information.

The truth and knowledge ordering of intervals as induced by the bilattice-based triangle are inadequate for capturing the repetitive revision and modification of belief in nonmonotonic reasoning and are not always intuitive. In this paper we address this issue and attempt to propose an alternate algebraic structure to eliminate the shortcomings of bilattice-based triangle. The major contributions of this paper are as follows:

- We demonstrate, with the help of proper examples (in section 3), that bilattice-based triangle is incapable of handling belief revision associated with nonmonotonic reasoning. In nonmonotonic reasoning, inferences are modified as more and more information is gathered. The prototypical example is inferring a particular individual can fly from the fact that it is a bird, but retracting that inference when an additional fact is added, that the individual is a penguin. Such continuous belief revision is not properly represented in bilattice-based triangle.

- We point that the truth ordering is unintuitive regarding the ordering of intervals when one interval lies completely within the other (section 3).

- Exploiting the discrepancies mentioned, we propose modifications for knowledge ordering and truth ordering of intervals so that the aforementioned shortcomings are removed (in section 4).

- Using the modified knowledge and truth ordering we construct an alternate algebraic structure, namely preorder-based triangle (in section 5). This structure can be thought of as a unification of bilattice-based triangle and default bilattice (Ginsberg, 1988). With this we come out of the realm of bilattice-based structures and explore a new algebraic structure based on simple linear pre-ordering.

- The proposed algebraic structure is then equipped with appropriate logical operators, i.e. negation, t-norms, t-conorms, implicators, in section
6. Most of the operators are in unison with those used for the bilattice-based structure. But the modified orderings offer additional flexibility.

2 Intervals as degree of belief:

This section addresses some of the basic definitions and notions that will ease the discussion in the forthcoming sections.

Uncertainty and incompleteness of information is unavoidable in real life reasoning. Hence, sometimes it becomes difficult and misleading, if not impossible, to assign a precise degree of membership to some fuzzy attribute or to assert a precise degree of truth to a proposition. Therefore, assigning an interval of possible truth values is the natural solution. Intervals are appropriate to describe experts’ degrees of belief, which may not be precise (Nguyen, Kreinovich, & Zuo, 1997). If an expert chooses a value, say 0.8, as his degree of belief for a proposition, actually we can only specify vaguely that his chosen value is around 0.8 and can be represented by an interval, say \([0.75, 0.85]\). Otherwise an interval can be thought of as a collection of possible truth values that a single or multiple rational experts would assign to a proposition in a scenario. Due to lack of complete knowledge the assertions made by different experts will be different and this lack of unanimity can be reflected by appropriate interval. The natural ordering of degree of memberships (\(\leq\)) can be extended to the set of intervals and that gives rise to IVFS (Sambuc, 1975).

An IVFS can be viewed as an L-fuzzy set (Goguen, 1967) and the corresponding lattice can be defined as (Deschrijver, Arieli, Cornelis, & Kerre, 2007):

**Definition 2.1.** \(L^I = (L^I, \leq_L)\), where \(L^I = \{[x_1, x_2] | (x_1, x_2) \in [0, 1] \times [0, 1] \text{ and } x_1 \leq x_2\}\) and \([x_1, x_2] \leq_L [y_1, y_2]\) iff \(x_1 \leq y_1\) and \(x_2 \leq y_2\).

In the definition, \(L^I\) is the set of all closed subintervals in \([0, 1]\). Figure 1 shows the set \(L^I\).

2.1 Bilattice-based Triangle:

Bilattices are ordered sets where elements are partially ordered with respect to two orderings, typically one depicts the degree of vagueness or truth (namely, truth ordering) and the other one depicting the degree of certainty (namely, knowledge ordering) (Arieli et al., 2004; Cornelis et al., 2007). A bilattice-based triangle, or simply Triangle, can be defined as follows:
Definition 2.2. Let $L = (L, \leq_L)$ be a complete lattice and let $I(L) = \{[x_1, x_2] \mid (x_1, x_2) \in L^2 \text{ and } x_1 \leq_L x_2\}$. A (bilattice-based) triangle is defined as a structure $B(L) = (I(L), \leq_t, \leq_k)$, where, for every $[x_1, x_2], [y_1, y_2]$ in $I(L)$:

1. $[x_1, x_2] \leq_t [y_1, y_2] \iff x_1 \leq_L y_1 \text{ and } x_2 \leq_L y_2$.
2. $[x_1, x_2] \leq_k [y_1, y_2] \iff x_1 \leq_L y_1 \text{ and } x_2 \geq_L y_2$.

This triangle $B(L)$ is not a bilattice, since, though the substructure $(I(L), \leq_t)$ is a complete lattice but $(I(L), \leq_k)$ is a complete semilattice.

When $L$ is the unit interval $[0,1]$, then $I(L)$ describes membership of IVFS, $L^I$, and the lattice $L^I$ becomes $(I(L), \leq_t)$. In knowledge ordering, the intervals are ordered by set inclusion, as was suggested by Sandewall (Sandewall, 1989). The knowledge inherent in an interval $[c,d]$ is greater than another interval $[a,b]$ if $[c,d] \subseteq [a,b]$.

Triangle $B(\{0,0.5,1\})$ is shown in Figure 2.

3 Inadequacy of Bilattice-based Triangle:

Intervals are used to approximate degree of truth of propositions in absence of complete knowledge. All values within an interval are considered to be equally probable to be the actual truth value of the underlying proposition. Thus considering intervals as truth status or epistemic state of propositions enables efficient representation of vagueness and uncertainty of information and reasoning. However, the Triangle structure suffers from the following shortcomings which must be eliminated.
3.1 Inadequacy in modeling belief revision in nonmonotonic reasoning:

One important aspect of human commonsense reasoning is that it is non-monotonic in nature (Brewka, 1991). In many cases conclusions are drawn in absence of complete information and we have to draw plausible conclusions based on the assumption that the world in which the reasoning is performed is normal and as expected. This is the best that can be done in contexts where the acquired knowledge is incomplete. But, these conclusions may have to be given up in light of further information. A proposition that was assumed to be true, may turn out to be false when new information is gathered. Such repetitive alterations of believes is an essential part of non-monotonic reasoning. This type of belief revision may not be adequately represented by Triangle. The following discussion will illuminate this issue.

3.1.1 An intuitive explanation:

Suppose the following information is given:

Rules:
\[ \text{Bird}(x) \rightarrow \text{Fly}(x), \ [\text{Birds Fly}] \]
\[ \text{Penguin}(x) \rightarrow \neg\text{Fly}(x), \ [\text{Penguin doesn’t Fly}] \]

Facts:
Bird (Tweety) [Tweety is a bird]

Given this information, suppose, multiple experts are trying to assess the
degree of truth of the proposition "Tweety Flies" [Fly (Tweety)]. The rule "Birds Fly" is not a universally true fact, rather it’s a general assumption that has several exceptions. Thus, being a Bird is not sufficient to infer that it will fly, since it may be a Penguin, an ostrich or some other non-flying bird. Since, nothing is specified about Tweety except for it is a bird, it is natural in human commonsense reasoning to "assume" that Tweety is not an exception and it will fly. Now, the confidence about this "assumption" will be different for different experts. An expert may bestow his complete faith on the fact that Tweety is not an exceptional bird and he will assign truth value 1 to "Tweety flies". Another expert may remain indecisive as he cannot ignore the chances that Tweety may be a non-flying bird and he will assign 0.5 (neither true nor false) to the proposition "Tweety flies". Others’ assignments may be at some intermediate level depending on their perception about the world. Thus, the experts’ truth assignments collectively construct an interval \([0, 1]\) as the epistemic state of the rule "Birds fly" as well as of the fact "Fly(Tweety)".

Now, suppose an additional information is acquired that: \(\text{Penguin(Tweety)}\). [Tweety is a penguin]

Then all the experts will unanimously declare Tweety doesn’t fly and assign an interval \([0, 0]\) as the revised epistemic state of the proposition "Tweety flies". The epistemic state of the proposition "Tweety flies" was first asserted by an interval \([0.5, 1]\) and later the experts retracted their previously drawn decision to assert another interval \([0, 0]\). From intuition it can be claimed that the interval \([0, 0]\) makes a more confident and precise assertion than \([0.5, 1]\), since in the former case all the experts were unanimous. But this is not reflected in the bilattice-based triangle (Figure 2); since in Triangle \([0.5, 1]\) and \([0, 0]\) are incomparable in knowledge ordering. Thus, given the two intervals, based on the triangle structure, we remain clueless about which interval has higher degree of knowledge and which interval we should take up as final assertion of "Tweety flies". This is counter-intuitive and unwanted.

This type of scenario can be efficiently taken care of in the default bilattice (Ginsberg, 1988). The general rule "Birds fly" will be assigned ‘dt’, i.e. true by default. Hence, 'Tweety flies' will also get dt. After acquiring the knowledge that Tweety is a penguin, 'Tweety flies' is asserted definitely false, i.e. f. In the default bilattice (Figure 3.a) \(f \geq_k dt\), expressing that the later conclusion is more certain than the earlier one.

The aforementioned example demonstrates that Triangle is incapable of
Figure 3: Default bilattices for Nonmonotonic Reasoning

3.1.2 Inadequacy of bilattice-based triangle in performing logical reasoning in application areas:

Bilattice-based structures are put to use for logical reasoning involving human detection and identity maintenance in visual surveillance systems by Shet et al. (Shet, 2007; Shet, Neumann, Ramesh, & Davis, 2007; Shet, Harwood, & Davis, 2006a).

Multi-valued default bilattice (also known as prioritized default bilattice (Figure 3.b) has been used for identity maintenance and contextual reasoning in visual surveillance system (Shet et al., 2006a). However in practice, logical facts are generated from vision analytics, which rely upon machine learning and pattern recognition techniques and in general have noisy values. Thus, in practical applications it would be more realistic to attach arbitrary amount of beliefs to logical rules rather than values such as dt, df etc that are allowed in multivalued default bilattices. For instance, similarity of different persons based on their appearances is a fuzzy attribute and may attain any degree over the [0,1] scale. But this cannot be captured by the multivalued default bilattice.
Bilattice-based square (Arieli et al., 2005) has been used for human detection in visual surveillance system. This algebraic structure is a better candidate than multivalued default bilattice as it provides continuous degrees of belief states.

The difference between bilattice-based square and bilattice-based triangle is that the former allows explicit representation of inconsistent information with different degrees inconsistency. But it is pointed out by Dubois (Dubois, 2008) that square-like bilattices, where explicit representations of unknown \((0,0)\) and inconsistent \((1,1)\) epistemic states are allowed, can not preserve classical tautologies and sometimes give rise to unintuitive results.

Hence, bilattice-based triangle seems to be the most dependable and suitable algebraic structure to be used in the aforementioned applications.

Now let’s apply the bilattice-based triangle to a slightly modified version of an example demonstrated by Shet et. al. (Shet et al., 2006a; Shet, Harwood, & Davis, 2006b) involving logical reasoning in identity maintenance. The example deals with determining whether two individuals observed in an image should be considered as being one and the same. The rules and facts along with the assigned epistemic states are as follows:

rules:
- r1: \(\phi[appear\_similar(P_1, P_2) \rightarrow equal(P_1, P_2)] = [0.5, 1]\)
- r2: \(\phi[distinct(P_1, P_2) \rightarrow \neg equal(P_1, P_2)] = [0.9, 1]\)

facts:
- f1: \(\phi[appear\_similar(a, b)] = [0.8, 0.8]\)
- f2: \(\phi[appear\_similar(c, d)] = [0.5, 0.5]\)
- f3: \(\phi[distinct(a, b)] = [1, 1]\)

The specified set of facts depicts that Individuals \(a\) and \(b\) are more similar than \(c\) and \(d\). Rules r1 and r2 encode the judgments of two different information sources or different algorithms, none of which present a confident, full-proof answer. However rule r2 (which may be based on some information of higher priority) gives greater assurance to the non-equality of two persons than the assertion of equality guaranteed by rule r1, which may have came from a simple appearance matching technique of low dependability.

Intuitively, from the given information, a rational agent would put more confidence to the fact that individuals \(a, b\) are not equal than on their equality; since the degree of distinction is more than the degree of similarity. The inference mechanism is specified in (Ginsberg, 1988). The closure operator over the truth assignment function \(\phi\) (\(cl(\phi)\))denotes the truth assignment
that labels information entailed from the given set of rules and facts. The operator \( cl_+ (\phi)(q) \) takes into account set of rules that entail \( q \) and \( cl_- (\phi)(q) \) considers set of rules that entail \( \neg q \). Here the conjunctor, disjunctor and negator used are min, max and 1− operators respectively.

\[
cl_+ (\phi)(equal(a, b)) = [0, 1] \lor ([0.8, 0.8] \land [0.5, 1]) = [0, 1] \lor [0.5, 0.8] = [0.5, 0.8]
\]

\[
cl_- (\phi)(equal(a, b)) = \neg([0, 1] \lor ([1, 1] \land [0.9, 1])) = \neg([0, 1] \lor [0.9, 1]) = [0, 0.1]
\]

Now the two intervals \([0.5, 0.8]\) and \([0, 0.1]\) are neither comparable with respect to \( \leq_k \) in Triangle nor they have a \( glb_k \) in the Triangle structure. Thus the two intervals cannot be combined to get a single assertion for \( equal(a, b) \). Hence, using Triangle it is not possible to achieve the intended inference that \( a \) and \( b \) don’t seem to be equal.

Thus the knowledge ordering in bilattice-based triangle must be modified in order to remove the aforementioned discrepancy. The modified knowledge ordering must incorporate within Triangle the ability to perform reasoning in presence of nonmonotonicity and demonstrate the repetitive belief revision as the default bilattice has.

3.2 Truth ordering is not always accurate:

In the bilattice-based triangle, for two intervals \([x_1, x_2]\) and \([y_1, y_2]\), \([x_1, x_2] \leq_t [y_1, y_2] \) iff \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \). According to this ordering any two intervals \( x \) and \( y \) are incomparable if \( x \) is a proper sub-interval of \( y \) or vice versa, i.e. if one interval lies completely within the other with no common boundary. The justification behind this incomparability is that, if an interval, say \( y \), is a proper sub-interval of \( x \) then the actual truth value approximated by interval \( x(\hat{x}) \) may be greater or less than that of \( y \). For instance, if \( x = [0.4, 0.8] \) and \( y = [0.5, 0.7] \) then \( \hat{x} \) can be less than \( \hat{y} \) (if \( \hat{x} \in [0.4, 0.5] \)) or \( \hat{x} \) can be greater than \( \hat{y} \) (if \( \hat{x} \in (0.7, 0.8] \)).

But similar situation may arise even when two intervals are not proper sub-interval of one another but just overlap, e.g. say \( x = [0.4, 0.8] \) and \( y = [0.6, 0.9] \). These intervals are t-comparable, i.e., \([0.4, 0.8] \leq_t [0.6, 0.9] \). Though, as the two intervals overlap, it is not ensured that the real truth value approximated by the lower interval will be smaller than the real truth value approximated by the higher interval (e.g. though \( x \leq_t y \) but it may be the case that \( \hat{x} = 0.75 \) and \( \hat{y} = 0.65 \)). In this respect the comparability of the two intervals is not justified. Therefore, it is not always the most accurate ordering and can be regarded as a ” weak truth ordering” (Esteva,
Garcia-Calvés, & Godo, 1994). The truth ordering must be modified in order to remove the anomaly, so that, if two overlapping intervals are (not) t-comparable, so would be two intervals - one lying inside the other.

### 4 Modification in Triangle structure:

Based on the discussions in the above two subsections the bilattice-based triangle is modified.

#### 4.1 Modification in knowledge ordering:

The knowledge ordering can be defined based on just the length of intervals and irrespective of the real truth values they attempt to approximate. Thus for two intervals \([x_1, x_2]\) and \([y_1, y_2] \in L^t\), where, \(L^t\) is the set of sub-intervals of \([0, 1]\) as shown in Figure 1

\[
[x_1, x_2] \leq_{k_p} [y_1, y_2] \iff (x_2 - x_1) \geq (y_2 - y_1).
\]

that is, wider the interval lesser is the knowledge content. Equality of the width of intervals is a necessary condition for \(x = y\), but not a sufficient condition; because two different intervals may have equal width, e.g. \([0.1, 0.2]\) and \([0.7, 0.8]\).

Using this modified knowledge ordering\((k_p)\) the shortcomings demonstrated in sections 3.1.1 and 3.1.2 can be overcome.

1. Interval \([0, 0]\) is placed higher in k-ordering than \([0.5, 1]\) (since \((1 - 0.5) > (0 - 0))\) and thus the new ordering prompts to choose the definite fact "Tweety doesn’t fly" (with the assigned interval \([0, 0]\)) over the default fact "Tweety flies" (with the assigned interval \([0.5, 1]\)).

2. Since, \([0.5, 0.8] \leq_{k_p} [0, 0.1]\), thus \(glb_k([0.5, 0.8], [0, 0.1]) = [0, 0.1]\), will be taken as the final assertion of \(equal(a, b)\) in the example shown in section 3.1.2. Hence using the modified knowledge ordering the intended inference that individuals \(a\) and \(b\) are not equal is achieved.

The algebraic structure for \((I(\{0.25, 0.5, 0.75, 1\}), \leq_t, \leq_{k_p})\) is shown in Fig. 4.

#### 4.2 Modification in Truth Ordering:

The truth ordering \((\leq_t)\) gives rise to certain discrepancies in ordering intervals, as discussed in section 3.2. The justification in support of this weak truth ordering is (Deschrijver, 2009)

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Figure 4: $I(\{0,0.25,0.5,0.75,1\})$ with modified knowledge ordering

"$x \leq_t y$ iff the probability that $\hat{x} \leq \hat{y}$ is larger than $\hat{x} \geq \hat{y}$" \hfill (*)

i.e. the basic intuition behind truth ordering of intervals lies in comparing the probabilities $\text{Prob}(\hat{x} \geq \hat{y})$ and $\text{Prob}(\hat{x} \leq \hat{y})$. Let's take this statement as a starting point to revisit the truth ordering, especially in case when one interval is a proper sub-interval of the other. In this respect the following theorem is stipulated.

**Theorem 4.1.** For two intervals $x = [x_1, x_2]$ and $y = [y_1, y_2] \in L^I$,

$$\text{Prob}(\hat{x} \geq \hat{y}) \leq \text{Prob}(\hat{x} \leq \hat{y}) \iff x_m \leq y_m$$

where, $\hat{x}(\hat{y})$ stands for the actual truth value approximated by the interval $x(y)$; and $x_m$ and $y_m$ are respectively the midpoints of intervals $x$ and $y$.

**Proof.** The proof is constructed by considering several cases depending on how intervals $x$ and $y$ are situated on the $[0,1]$ scale. Without loss of generality it is assumed that $x_1 \leq y_1$ for showing the proof. For the other case, i.e. $x_1 > y_1$ similar proof can be constructed which is not shown here.

Since any $x \in [x_1, x_2]$ is equally probable to be equal to $\hat{x}$ (i.e. there is a uniform probability distribution over $[x_1, x_2]$) then for a sub-interval $[a,b]$ of $[x_1, x_2]$ we have, $\text{Prob}(\hat{x} \in [a,b]) = \text{Prob}(\hat{x} \in (a,b)) = \text{Prob}(\hat{x} \in [a,b]) = \frac{b-a}{x_2-x_1}$.

**Case 1:**
Figure 5: y is a proper sub-interval of x

Suppose, \(x = [x_1, x_2]\) has \(y = [y_1, y_2]\) as a proper sub-interval (Figure 5). For these intervals \(x_1 < y_1\) and \(y_2 < x_2\), hence \(x\) and \(y\) can not be ordered using \(\leq\).

In this case,
1. \(\hat{x} \leq \hat{y}\) iff \(\hat{x} \in [x_1, y_1]\) or (\(\hat{x} \leq \hat{y}\) given \(\hat{x}, \hat{y} \in [y_1, y_2]\)),
2. \(\hat{x} \geq \hat{y}\) iff \(\hat{x} \in (y_2, x_2]\) or (\(\hat{x} \geq \hat{y}\) given \(\hat{x}, \hat{y} \in [y_1, y_2]\)).

Within the smaller interval \([y_1, y_2]\) the \(\hat{x} \leq \hat{y}\) and \(\hat{x} \geq \hat{y}\) are equally probable, i.e.

\[
Prob(\hat{x} \leq \hat{y}|\hat{x}, \hat{y} \in [y_1, y_2]) = Prob(\hat{x} \geq \hat{y}|\hat{x}, \hat{y} \in [y_1, y_2]).
\]

Now,

\[
Prob(\hat{x} \geq \hat{y}) \leq Prob(\hat{x} \leq \hat{y})
\]
\[
\equiv Prob(\hat{x} \in (y_2, x_2] \text{ or (} \hat{x} \leq \hat{y} \text{ given } \hat{x}, \hat{y} \in [y_1, y_2]\)) \leq Prob(\hat{x} \in [x_1, y_1])
\]
\[
\text{or (} \hat{x} \leq \hat{y} \text{ given } \hat{x}, \hat{y} \in [y_1, y_2]\))
\]
\[
\equiv Prob(\hat{x} \in (y_2, x_2] + Prob(\hat{x} \geq \hat{y}|\hat{x}, \hat{y} \in [y_1, y_2]) \leq Prob(\hat{x} \in [x_1, y_1]) +
\]
\[
Prob(\hat{x} \leq \hat{y}|\hat{x}, \hat{y} \in [y_1, y_2])
\]
\[
\equiv Prob(\hat{x} \in (y_2, x_2]) \leq Prob(\hat{x} \in [x_1, y_1])
\]
\[
\equiv \frac{x_2 - y_2}{x_2 - x_1} \leq \frac{y_1 - x_1}{x_2 - x_1}
\]
\[
\Rightarrow (x_2 - y_2) \leq (y_1 - x_1) \text{ (since } (x_2 - x_1) > 0)\)
\[
\Rightarrow (x_1 + x_2) \leq (y_1 + y_2)
\]
\[
\Rightarrow \frac{x_1 + x_2}{2} \leq \frac{y_1 + y_2}{2}
\]
\[
\Rightarrow \text{ the midpoint of interval } x \leq \text{ the midpoint of interval } y
\]
\[
\equiv x_m \leq y_m.
\]

Case 2:
Suppose two intervals \(x = [x_1, x_2]\) and \(y = [y_1, y_2]\) are overlapping, as shown in Figure 6. In this case, \(x_1 \leq y_1\) and \(x_2 \leq y_2\).

Here,
1. $\hat{x} \leq \hat{y}$ iff $\hat{x} \in [x_1, y_1)$ or $(\hat{x} \in [y_1, x_2]$ and $\hat{y} \in (x_2, y_2])$ or $(\hat{x} \leq \hat{y}$ given $\hat{x}, \hat{y} \in [y_1, x_2]$),

2. $\hat{x} \geq \hat{y}$ iff $(\hat{x} \geq \hat{y}$ given $\hat{x}, \hat{y} \in [y_1, x_2]$).

$$Prob(\hat{x} \geq \hat{y}) \leq Prob(\hat{x} \leq \hat{y})$$

$\equiv Prob(\hat{x} \geq \hat{y}|\hat{x}, \hat{y} \in [y_1, x_2]) \leq Prob(\hat{x} \in [x_1, y_1]) + Prob(\hat{x} \in [y_1, x_2] \text{ and } \hat{y} \in (x_2, y_2])$

$\equiv Prob(\hat{x} \in [x_1, y_1]) + Prob(\hat{x} \in [y_1, x_2]$ and $\hat{y} \in (x_2, y_2]) \geq 0$

(since, $Prob(\hat{x} \geq \hat{y}|\hat{x}, \hat{y} \in [y_1, x_2]) = Prob(\hat{x} \leq \hat{y}|\hat{x}, \hat{y} \in [y_1, x_2])$

$\equiv \frac{y_1 - x_1}{x_2 - x_1} + \frac{x_2 - y_1 y_2 - x_2}{x_2 - x_1 y_2 - y_1} \geq 0$

$\equiv (y_1 - x_1)(y_2 - y_1) + (y_2 - x_2)(x_2 - y_1) \geq 0$

$\equiv y_1 y_2 - x_1 y_2 - y_1^2 + x_1 y_1 + x_2 y_2 - x_2^2 - y_1 y_2 + x_2 y_1 \geq 0$

$\equiv x_1 y_1 + x_2 y_2 + x_2 y_1 - x_1 y_2 - y_1 - x_2^2 \geq 0$

(cancelling $y_1 y_2$ and $-y_1 y_2$ and rearranging terms)

$\equiv x_2(y_2 + y_1 - x_2) - y_1(y_1 - x_1) - x_1 y_2 \geq 0$

$\equiv x_2(y_2 + y_1 - x_2) - y_1(y_1 - x_1) - x_1 x_2 \geq 0$

(since $x_2 \leq y_2$)

$\equiv x_2(y_2 + y_1 - x_1 - x_2) - y_1(y_1 - x_1) \geq 0$

$\equiv x_2(y_2 + y_1 - x_1 - x_2) \geq 0$

(since $x_1 \leq y_1$)

$\equiv (y_2 + y_1 - x_1 - x_2) \geq 0$

$\equiv (y_1 + y_2) \geq (x_1 + x_2)$

$\equiv \frac{y_1 + y_2}{2} \geq \frac{x_1 + x_2}{2}$

$\equiv$ the midpoint of interval $y$ $\geq$ the midpoint of interval $x$
\( \equiv x_m \leq y_m. \)

**Case 3:**

We can have two subcases for disjoint intervals (Figure 7). For subcase a, the interval \( x \) is lower than the interval \( y \), i.e. \( \forall a \in [x_1, x_2], a \leq y_1 \) or in other words \( x_1 < x_2 \leq y_1 < y_2 \). Similarly, for subcase b, the interval \( y \) is lower than the interval \( x \), i.e. \( \forall b \in [y_1, y_2], b \leq x_1 \) or in other words \( y_1 < y_2 \leq x_1 < x_2 \).

In this case, since intervals are disjoint,
\[
\text{Prob}(\hat{x} \leq \hat{y}) = 1 \quad \text{and} \quad \text{Prob}(\hat{x} \geq \hat{y}) = 0 \quad \text{if} \quad x_2 \leq y_1 \quad \text{(Subcase a)};
\]
\[
\text{Prob}(\hat{x} \leq \hat{y}) = 0 \quad \text{and} \quad \text{Prob}(\hat{x} \geq \hat{y}) = 1 \quad \text{if} \quad y_2 \leq x_1 \quad \text{(Subcase b)};
\]

Now, \( \text{Prob}(\hat{x} \geq \hat{y}) < \text{Prob}(\hat{x} \leq \hat{y}) \)
\[
\Rightarrow \text{Prob}(\hat{x} \geq \hat{y}) = 0 \quad \text{and} \quad \text{Prob}(\hat{x} \leq \hat{y}) = 1
\]
\[
\Rightarrow \forall a \in [x_1, x_2], a \leq y_1
\]
\[
\Rightarrow x_2 \leq y_1
\]
\[
\Rightarrow x_1 + x_2 \leq y_1 + x_1
\]
\[
\Rightarrow x_1 + x_2 < y_1 + y_2 \quad \text{[since,} \quad x_1 < y_1]\]
\[
\Rightarrow x_m < y_m.
\]

Again:
\[
x_m < y_m
\]
\[
\Rightarrow x_1 + x_2 < y_1 + y_2
\]
\[
\Rightarrow x_1 < y_1 \quad \text{and} \quad x_2 < y_2 \quad \text{and} \quad x_2 \leq y_1 \quad \text{[since intervals are disjoint]}\]
\[
\Rightarrow \text{Prob}(\hat{x} \geq \hat{y}) = 0 \quad \text{and} \quad \text{Prob}(\hat{x} \leq \hat{y}) = 1
\]
\[
\Rightarrow \text{Prob}(\hat{x} \geq \hat{y}) < \text{Prob}(\hat{x} \leq \hat{y}).
\]
Thus \( \text{Prob}(\hat{x} \geq \hat{y}) < \text{Prob}(\hat{x} \leq \hat{y}) \equiv x_m < y_m. \)
Here the proof ends.

Hence, it is proved that the straightforward way to compare the probabilities \( \text{Prob}(\hat{x} \geq \hat{y}) \) and \( \text{Prob}(\hat{x} \leq \hat{y}) \) for two intervals \( x \) and \( y \) is to compare their midpoints. Case 1 in the above proof is particularly interesting, where one interval is a proper sub-interval of the other. Though the chosen intervals \( x \) and \( y \) are not comparable with respect to \( \leq_t \) ordering, but we can compare their midpoints and thus order the probabilities \( \text{Prob}(\hat{x} \geq \hat{y}) \) and \( \text{Prob}(\hat{x} \leq \hat{y}) \). Thus following statement (*) a truth ordering can be imposed on \( x \) and \( y \) based on the probabilistic comparison. The existing truth ordering (\( \leq_t \)) as shown in Definition 2.2, doesn’t allow this comparability of \( x \) and \( y \), and hence a new truth ordering is called for.

Now that we are able to estimate and order the probabilities, in light of statement (*) we are in a place to construct a generalised truth ordering (\( \leq_{tp} \)) as follows:

\[
x \leq_{tp} y \iff x_m \leq y_m.
\]

The equality of midpoints of two intervals \( x \) and \( y \), (i.e. \( \frac{x_1 + x_2}{2} = \frac{y_1 + y_2}{2} \)) is a necessary condition for \( x = y \), but not a sufficient condition; because two different intervals can have same midpoint, as shown in Figure 8.

Moreover, the discrepancy mentioned in section 3.2 is resolved, since cases where intervals are overlapped and when one interval is a proper sub-interval of the other are treated uniformly and in each case intervals are comparable with respect to \( \leq_{tp} \).

**Theorem 4.2.** For two intervals \( x = [x_1, x_2] \) and \( y = [y_1, y_2] \in L^I \), such that none is a proper subinterval of the other,

\[
x \leq_t y \Rightarrow x \leq_{tp} y.
\]

**Proof.** From the definition,
Figure 9: $I(\{0, 0.25, 0.5, 0.75, 1\})$ with modified truth ordering

\[
x \leq_t y \iff x_1 \leq x_2 \text{ and } y_1 \leq y_2
\]
\[
\Rightarrow x_1 + x_2 \leq y_1 + y_2
\]
\[
\Rightarrow x_m \leq y_m
\]
\[
\Rightarrow x \leq_{t_p} y.
\]

Thus, the probabilistic analysis gives a broader truth ordering of the intervals that can be achieved by comparing midpoints of intervals. For each pair of intervals if they are comparable with respect to $\leq_t$ they are also comparable with respect to the modified truth ordering $\leq_{t_p}$ and additionally $\leq_{t_p}$ can order intervals when one of them is a proper sub-interval of the other and hence are not $\leq_t$-comparable.

For instance, for two intervals $x = [0, 1]$ and $y = [0.8, 0.9]$ we have $[0, 1] <_{t_p} [0.8, 0.9]$ though $x$ and $y$ are not $t$-comparable w.r.t. $\leq_t$.

The algebraic structure for $(I(\{0, 0.25, 0.5, 0.75, 1\}), \leq_{t_p}, \leq_k)$ is shown in Figure 9.

5 An alternative algebraic structure

Based on these modifications we propose a modified and more intuitive algebraic structure for ordering intervals with respect to degree of truth and knowledge (or certainty).
Figure 10: Modified Triangle for $I(\{0,0.5,1\})$

Notation: For an interval $x \in L^I$, $x_m$ and $x_w$ will be used to denote the midpoint (or center) and the length of the interval respectively; i.e. $x_m = (x_1 + x_2)/2$ and $x_w = (x_2 - x_1)$. The pair $(x_m, x_w)$ uniquely specifies an interval $x$ and hence may be used instead of the traditional representation $[x_1, x_2]$.

Definition 5.1. A preorder-based triangle is a structure $P(L) = (L^I, \leq_{tp}, \leq_{kp})$, defined for every $[x_1, x_2]$ and $[y_1, y_2] \in L^I$ as:

1. $[x_1, x_2] \leq_{tp} [y_1, y_2] \iff x_m \leq y_m$,
2. $[x_1, x_2] \leq_{kp} [y_1, y_2] \iff x_w \geq y_w$,
3. $x = y \iff x_m = y_m$ and $x_w = y_w$.

Preorder-based triangle can be defined for any subset of $L^I$ as well. For instance, preorder-based triangle for $I(\{0, 0.5, 1\})$ and for $I(\{0, 0.25, 0.5, 0.75, 1\})$ are shown in Figure 10 and Figure 11 respectively. The dashed lines demonstrate the connections that were absent in the bilattice-based triangle.

With the truth and knowledge ordering presented in Definition 5.1 we step out of the realm of lattice-based structures. The substructure $(L^I, \leq_{t_p})$ is not a lattice since for any two intervals $a$ and $b$, existence of $lub_{t_p}(a,b)$ and $glb_{t_p}(a,b)$ are not guaranteed. For instance, suppose $L = \{0, 0.1, 0.2, .., 1\}$ i.e. the unit interval discretised with eleven equidistant points. Now, two intervals in $I(L)$, $[0.8, 0.8]$ and $[0.6, 1]$ are incomparable with respect to $\leq_{t_p}$. The upper bound of the two intervals is not a unique element, but a set of intervals $\{[0.7, 1], [0.8, 0.9]\}$. Hence $lub_{t_p}$ doesn’t exist. Lower bound of the two intervals is the set $\{[0.7, 0.8], [0.6, 0.9], [0.5, 1]\}$. 

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The ordering $\leq_{t_p}$ and $\leq_{k_p}$ over the set of intervals are not lattice-orders but pre-orders, i.e reflexive and transitive. The orderings are not symmetric clearly. Nor they are anti-symmetric, since $[0.5, 0.5] \leq_{t_p} [0, 1]$ and $[0, 1] \leq_{t_p} [0.5, 0.5]$ but $[0.5, 0.5] \neq [0, 1]$ and similar example holds for the k-ordering. Thus the substructure $(L^I, \leq_{t_p})$ and $(L^I, \leq_{k_p})$ form pre-ordered sets instead of lattices. Moreover, in $(L^I, \leq_{k_p})$ any set of intervals have lower bound but may not have upper bound. For instance, following the earlier example, intervals $[0.5, 0.5]$ and $[0.8, 0.8]$ doesn’t have an upper bound, but has the set of intervals of length 0.1 as its lower bound.

Because of the modified knowledge ordering, the preorder-based triangle can be thought of as a unification of the default bilattice and the bilattice-based triangle.

One point must be emphasized here is that, bilattice-based triangle is a more generalised algebraic structure that can be defined for any set of intervals over any complete lattice. However, preorder-based triangle can only be defined over subintervals of $[0, 1]$ (or any interval of real numbers), because intervals in general complete lattices may not have midpoints or lengths.

**Definition 5.2.** A set of intervals in $L^I$ is said to be an m-set for a specific value $a \in [0, 1]$ and is defined as:

$$m - set_a = \{x | x \in L^I \text{ and } x_m = a\}.$$
i.e. the set of intervals incomparable with the interval $[a, a]$ with respect to their degree of truth.

6 Logical Operators on $P(L)$

All the logical operators, e.g. conjunction, disjunction, implication and negation, defined for bilattice-based triangle ($B(L)$) (Cornelis et al., 2007; Deschrijver et al., 2007) are applicable for preorder-based triangle ($P(L)$) as well. But the modified truth and knowledge ordering will incorporate some modifications in the definition and properties of the connectives. The notations $0_{L^I}$ and $1_{L^I}$ stand for intervals $[0, 0]$ and $[1, 1]$ respectively.

6.1 Negator:

**Definition 6.1.** A negator on $(L^I, \leq_{tp})$ is a decreasing mapping $N : L^I \rightarrow L^I$, for which $N(0_{L^I}) = 1_{L^I}$ and $N(1_{L^I}) = 0_{L^I}$. If $N(N(x)) = x$, then $N$ is involutive.

**Theorem 6.2.** Suppose there exists an involutive negator $N$ on $([0, 1], \leq)$. Then for all $x = [x_1, x_2]$ in $L^I$ the mapping $N : L^I \rightarrow L^I$ defined as

$$N(x) = [N(x_2), N(x_1)]$$

is an involutive negator on $(L^I, \leq_{tp})$

**Proof.** $N$ to be an involutive negator it must satisfy the following criteria:

1. **Boundary Condition:**

   $N$ being an involutive negator on $([0, 1], \leq)$, $N(0) = 1$ and $N(1) = 0$.

   Therefore,

   $$N(0_{L^I}) = N([0, 0]) = [N(0), N(0)] = [1, 1] = 1_{L^I}.$$

   $$N(1_{L^I}) = N([1, 1]) = [N(1), N(1)] = [0, 0] = 0_{L^I}.$$

2. $N$ has to be decreasing on $(L^I, \leq_{tp})$.

   Let $x = [x_1, x_2]$ and $y = [y_1, y_2]$ are two intervals in $L^I$.

   Now suppose, without loss of generality, $x \geq_{tp} y$; which implies,

   $$\frac{x_1 + x_2}{2} \geq \frac{y_1 + y_2}{2} \text{ or, } x_1 + x_2 \geq y_1 + y_2.$$

   Case 1: If neither of $x$ and $y$ is a sub-interval of the other, i.e.

   $$x_1 \geq y_1 \text{ and } x_2 \geq y_2.$$
Hence, \( N(x_1) \leq N(y_1) \) and \( N(x_2) \leq N(y_2) \); since \( N \) is decreasing.

Therefore, \( N(x_1) + N(x_2) \leq N(y_1) + N(y_2) \),

or, \( \frac{N(x_1) + N(x_2)}{2} \leq \frac{N(y_1) + N(y_2)}{2} \),

or, \( N(x) \leq_{tp} N(y) \).

Hence, \( N \) is decreasing.

Case 2: When \( y \) is a sub-interval of \( x \). Thus,

\[ x_1 \leq y_1 \text{ and } y_2 \leq x_2 \]

Hence, \( N(x_1) \geq N(y_1) \) and \( N(y_2) \geq N(x_2) \).

Since, \( x \geq_{tp} y, \ x_1 + x_2 \geq y_1 + y_2 \).

or, \( x_2 - y_2 \geq y_1 - x_1 \).

Therefore, \( N(y_2) - N(x_2) \geq N(x_1) - N(y_1) \); since \( N \) is decreasing.

or, \( N(y_2) + N(y_1) \geq N(x_1) + N(x_2) \).

or, \( N(y) \geq_{tp} N(x) \).

Thus, \( N \) is decreasing.

Case 3: When \( x \) is a sub-interval of \( y \). Then;

\[ x_1 \geq y_1 \text{ and } y_2 \geq x_2 \]

Hence, \( N(x_1) \leq N(y_1) \) and \( N(y_2) \leq N(x_2) \).

Since, \( x \geq_{tp} y, \ x_1 + x_2 \geq y_1 + y_2 \).

or, \( x_1 - y_1 \geq y_2 - x_2 \).

Therefore, \( N(y_1) - N(x_1) \geq N(x_2) - N(y_2) \); since \( N \) is decreasing.

or, \( N(y_1) + N(y_2) \geq N(x_1) + N(x_2) \).

or, \( N(y) \geq_{tp} N(x) \).

Thus, \( N \) is decreasing.

Therefore, it is proved that \( N \) satisfies the boundary conditions and is a decreasing mapping on \( (L^I, \leq_{tp}) \). So \( N \) is a negator on \( (L^I, \leq_{tp}) \).

Since, \( N \) is involutive, we obtain that, \( \forall x \in [0, 1]; \)

\( N(N(x)) = N([N(x_2), N(x_1)]) \)

\( = [N(N(x_1)), N(N(x_2))] = [x_1, x_2] = x. \)

Hence, \( N \) is involutive.
6.1.1 A standard negator

For an element $x = [x_1, x_2]$ in $L^I$ the standard negation of $x$ is defined as:

**Definition 6.3.** $N_s(x) = [1 - x_2, 1 - x_1]$.

Thus the degree of knowledge is unaltered by negation, but the interval (and hence its midpoint) is reflected across the central line of $L^I$, i.e. the line joining points $[0.5, 0.5]$ and $[0, 1]$. This negation corresponds to classical negation.

**Properties:**

1. $N_s(0_{L^I}) = 1_{L^I}$.
2. $N_s$ is decreasing.
3. $N_s$ is continuous.
4. $N_s$ is involutive; i.e. $N_s(N_s(x)) = x$.

One point that must be emphasized is that involutive negators can be defined on $(L^I, \leq_{tp})$ that are not of the form stated in Theorem 6.2.

**Example:** Consider the lattice $L = \{0, 1/3, 2/3, 1\}$ and a mapping $N_1$ on $(I(L), \leq_{tp})$ defined as follows:

$$N_1([x_1, x_2]) = \begin{cases} 
[1/3, 2/3] & \text{if } [x_1, x_2] \text{ is } [0, 1] \\
[0, 1] & \text{if } [x_1, x_2] \text{ is } [1/3, 2/3] \\
[1 - x_2, 1 - x_1] & \text{otherwise.}
\end{cases}$$

$N_1$ is an involutive negator on $(I(L), \leq_{tp})$, but is not of the form specified in Theorem 6.2. This is the difference between negators on bilattice-based triangle (Cornelis et al., 2007) and preorder-based triangles.

6.2 T-norms and T-conorms:

The t-norms and t-conorms can be defined over the preorder-based triangle.

**Definition 6.4.** A conjunctor on $(L^I, \leq_{tp})$ is an increasing $L^I \times L^I \rightarrow L^I$ mapping $T$ satisfying $T(0_{L^I}, 0_{L^I}) = T(0_{L^I}, 1_{L^I}) = T(1_{L^I}, 0_{L^I}) = 0_{L^I}$ and $T(1_{L^I}, 1_{L^I}) = 1_{L^I}$.

A conjunctor is called a semi-norm if $(\forall x \in L^I)(T(1_{L^I}, x) = T(x, 1_{L^I}) = x)$ and a semi-norm is called a t-norm if it is commutative and associative.
Definition 6.5. A disjunctor on \( (L^I, \leq_{tp}) \) is an increasing \( L^I \times L^I \rightarrow L^I \) mapping \( S \) satisfying 
\[
S(1_{L^I}, 0_{L^I}) = S(0_{L^I}, 1_{L^I}) = S(1_{L^I}, 1_{L^I}) = 1_{L^I} \text{ and } S(0_{L^I}, 0_{L^I}) = 0_{L^I}.
\]
A disjunctor is called a semi-conorm if \( \forall x \in L^I \) \( (S(0_{L^I}, x) = S(x, 0_{L^I}) = x) \) and a semi-conorm is called a t-conorm if it is commutative and associative.

Two important classes of t-(co)norms defined for IVFS, namely t-representable and pseudo t-representable t-(co)norms (Deschrijver, 2008), can be defined over the preorder-based triangle structure.

Definition 6.6. A t-norm \( T \) on \( (L^I, \leq_{tp}) \) is called t-representable if there exist t-norms \( T_1 \) and \( T_2 \) on \( ([0, 1], \leq) \) such that \( T \preceq T_2 \) and such that \( T \) can be represented as, for all \( x, y \in L^I \):
\[
T(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)].
\]

\( T_1 \) and \( T_2 \) are called representants of \( T \).

Definition 6.7. A t-norm \( T \) on \( (L^I, \leq_{tp}) \) is called pseudo t-representable if there exists a t-norm \( T \) on \( ([0, 1], \leq) \) such that for all \( x, y \in L^I \):
\[
T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))].
\]

\( T \) is called the representant of \( T \).

6.2.1 Min t-norm and t-conorm

The Min t-norm \( (T_{Min}) \) is the greatest t-norm with respect to the \( \leq_t \) ordering and is defined as:
\[
T_{Min} = [\min(x_1, y_1), \min(x_2, y_2)].
\]

One property of this t-norm is that it doesn’t hold that \( \forall x, y \in L^I \) either \( T_{Min}(x, y) = x \) or \( T_{Min}(x, y) = y \); for instance, \( T_{Min}([0.1, 0.5], [0.2, 0.3]) = [0.1, 0.3] \). This phenomenon is not intuitive sometimes. Using the modified truth ordering \( (\leq_{tp}) \) a variant of the min t-norm can be defined over \( (L^I, \leq_{tp}) \) as follows:

Definition 6.8. For any two intervals \( x, y \in L^I \)
\[
T_{Min_p}(x, y) = \min_t\{x, y\} \quad \text{if} \quad x_m \neq y_m
= \max_k\{x, y\} \quad \text{if} \quad x_m = y_m
\]
Definition 6.9. For any two intervals \( x, y \in L^I \)

\[
\begin{align*}
S_{\text{Min}_p}(x, y) &= \max_{\mathcal{I}} \{x, y\} \quad \text{if } x_m \neq y_m \\
&= \max_k \{x, y\} \quad \text{if } x_m = y_m
\end{align*}
\]

In the above definition \( \min_{\mathcal{I}} \{x, y\} \) gives the interval having lower degree of truth irrespective of its knowledge content, i.e. \( \min_{\mathcal{I}} \{x, y\} = x \) if \( x \leq_{t_p} y \). Similar meaning can be ascribed to \( \max_{\mathcal{I}} \{x, y\} \). Whereas, \( \min_k \{x, y\} \) gives the interval which is lower with respect to the k-ordering, i.e. having higher degree of uncertainty. For instance, \( \min_k \{x, y\} = x \) if \( x \leq_{k_p} y \). Similarly \( \max_k \{x, y\} \) can be defined.

It is clear that \( T_{\text{Min}_p} \) and \( S_{\text{Min}_p} \) satisfies the conditions in Definition 6.4 and 6.5 respectively.

Example: \( T_{\text{Min}_p}(\{0.1, 0.5\}, \{0.2, 0.3\}) = [0.2, 0.3] \). Thus, for all \( x, y \in L^I \)
either \( T_{\text{Min}_p}(x, y) = x \) or \( T_{\text{Min}_p}(x, y) = y \).

Theorem 6.10. The t-norm \( T_{\text{Min}_p} \), t-conorm \( S_{\text{Min}_p} \) and negator \( N_s \) forms a De-Morgan triplet, i.e.

1. \( T_{\text{Min}_p}(x, y) = N_s(S_{\text{Min}_p}(N_s(x), N_s(y))) \)
2. \( S_{\text{Min}_p}(x, y) = N_s(T_{\text{Min}_p}(N_s(x), N_s(y))) \).

Proof. Consider two intervals \( x, y \in L^I \).

Part 1: First, suppose intervals \( x \) and \( y \) are comparable with respect to \( \leq_{t_p} \), and lets assume, without loss of generality \( x \geq_{t_p} y \). Thus \( T_{\text{Min}_p}(x, y) = y \) and \( S_{\text{Min}_p}(x, y) = x \). Since \( N_s \) is decreasing with respect to the degree of truth, then \( N_s(x) \leq_{t_p} N_s(y) \). So, from definition \( S_{\text{Min}_p}(N_s(x), N_s(y)) = N_s(y) \). Thus \( N_s(S_{\text{Min}_p}(N_s(x), N_s(y))) = y = T_{\text{Min}_p}(x, y) \).

Moreover, if \( x_m = y_m \) (i.e. \( x \) and \( y \) are incomparable with respect to their degree of truth), and say, \( x \leq_{k_p} y \) \( T_{\text{Min}_p}(x, y) = y \). Since the negator \( N_s \) preserves the degree of knowledge and reverses the degree of truth, \( N_s(x) \) and \( N_s(y) \) are incomparable in t-ordering and \( N_s(x) \leq_{k_p} N_s(y) \). Thus, from definition \( S_{\text{Min}_p}(N_s(x), N_s(y)) = N_s(y) \) and \( N_s(S_{\text{Min}_p}(N_s(x), N_s(y))) = y = T_{\text{Min}_p}(x, y) \).

Part 2: If \( x \leq_{t_p} y \), then \( N_s(x) \geq_{t_p} N_s(y) \) and \( T_{\text{Min}_p}(N_s(x), N_s(y)) = N_s(y) \). Thus \( N_s(T_{\text{Min}_p}(N_s(x), N_s(y))) = y = S_{\text{Min}_p}(x, y) \).

Moreover, if \( x_m = y_m \) and say, \( x \leq_{k_p} y \) \( S_{\text{Min}_p}(x, y) = y \). The negator \( N_s \) being order preserving for k-ordering, \( N_s(x) \leq_{k_p} N_s(y) \). Thus, from definition \( T_{\text{Min}_p}(N_s(x), N_s(y)) = N_s(y) \) and \( N_s(T_{\text{Min}_p}(N_s(x), N_s(y))) = y = S_{\text{Min}_p}(x, y) \).

Hence, the t-norm \( T_{\text{Min}_p} \), t-conorm \( S_{\text{Min}_p} \) and negator \( N_s \) forms a De-Morgan triplet.

□
6.2.2 Product t-norm and t-conorm

The product t-(co)norm is useful to model the conjunction of independent events in probabilistic semantics. The t-representable and pseudo t-representable product t-(co)norms on \((L^1, \leq_{t_p})\) are defined in the same way as defined on \((L^1, \leq_t)\).

**Definition 6.11.** For any two intervals \(x, y \in L^1\), the product t-norm is defined as follows:

- \(T_{pr}([x_1, x_2], [y_1, y_2]) = [x_1 y_1, x_2 y_2]\), (t-representable)
- \(T_{ppr}([x_1, x_2], [y_1, y_2]) = [x_1 y_1, \max(x_1 y_2, x_2 y_1)]\), (pseudo t-representable)

**Theorem 6.12.** For any \(x, y \in L^1\)

\[T_{pr} \geq_{t_p} T_{ppr}\]

The proof of the above theorem is straightforward.

**Definition 6.13.** The t-representable t-conorm can be defined as:

\(S_{pr}([x_1, x_2], [y_1, y_2]) = [1 - (1 - x_1) \times (1 - y_1), 1 - (1 - x_2) \times (1 - y_2)]\)

**Theorem 6.14.** The t-norm \(T_{pr}\), t-conorm \(S_{pr}\) and the standard negator \(N_s\) forms a De-Morgan triplet, i.e.

1. \(T_{pr}(x, y) = N_s(S_{pr}(N_s(x), N_s(y)))\)
2. \(S_{pr}(x, y) = N_s(T_{pr}(N_s(x), N_s(y)))\)

**Proof.** Consider any two intervals \([x_1, x_2], [y_1, y_2] \in L^1\).

1. \(S_{pr}(N_s(x), N_s(y))\)
   
   \[
   S_{pr}([1 - x_2, 1 - x_1], [1 - y_2, 1 - y_1]) = [1 - x_2 \times y_2, 1 - x_1 \times y_1].
   
   Now, \(N_s(S_{pr}(N_s(x), N_s(y))) = N_s([1 - x_2 \times y_2, 1 - x_1 \times y_1]) = [x_1 \times y_1, x_2 \times y_2] = T_{pr}([x_1, x_2], [y_1, y_2])\)

2. \(T_{pr}(N_s(x), N_s(y))\)

   \[
   T_{pr}([1 - x_2, 1 - x_1], [1 - y_2, 1 - y_1]) = [1 - x_2 \times (1 - y_2), (1 - x_1) \times (1 - y_1)].
   
   Now, \(N_s(T_{pr}(N_s(x), N_s(y))) = N_s((1 - x_2) \times (1 - y_2), (1 - x_1) \times (1 - y_1)) = [1 - (1 - x_1) \times (1 - y_1), 1 - (1 - x_2) \times (1 - y_2)] = S_{pr}([x_1, x_2], [y_1, y_2]); \text{ (from definition)}.
   
   The t-norm \(T_{pr}\), t-conorm \(S_{pr}\) and the standard negator \(N_s\) forms a De-Morgan triplet.

\[\square\]
Thus, the preorder-based triangle structure offers us the flexibility to choose t-norms and t-conorms already defined for bilattice-based triangle or to define new connectives in accordance to the newly defined t-ordering and k-ordering.

6.3 Implicators:

**Definition 6.15.** An implicator on \((L^I, \leq t_p)\) is a hybrid monotonous \(L^I \times L^I \to L^I\) mapping \(I\) (i.e. a mapping with decreasing first and increasing second partial mapping) that satisfies \(I(0_{L^I}, 0_{L^I}) = I(0_{L^I}, 1_{L^I}) = I(1_{L^I}, 1_{L^I}) = 1_{L^I}\) and \(I(1_{L^I}, 0_{L^I}) = 0_{L^I}\).

One of the common class of implicators are Strong-implicators or S-implicators in short.

**Definition 6.16.** For two intervals \(x, y \in L^I\) and any t-conorm \(S\) and negator \(N\) on \((L^I, \leq t)\) the S-implicator generated by \(S\) and \(N\) is

\[ I_{S,N}(x, y) = S(N(x), y). \]

The S-implicators defined for the structure \((L^I, \leq t_p)\) are similar to those defined for \((L^I, \leq t)\), and are not discussed further.

There is another important class of implicators, namely R-implicators, generated as residuum of some t-norms on \((L^I, \leq t)\). An R-implicator on \((L^I, \leq t)\) generated by a t-norm \(T\) is defined as:

\[ I_R(x, y) = \text{Sup}_{t_p}\{\gamma \in L^I | T(x, \gamma) \leq t_p y \text{ or } [T(x, \gamma)]_m = y_m\}. \]

Now, because the definition involves truth ordering, the modified definition of \(\leq t_p\) demands modification to the definition of R-implicator.

**Definition 6.17.** For a t-norm \(T\) defined on \((L^I, \leq t_p)\) an R-implicator generated from \(T\) is defined as:

\[ I_{R_{tp}}(x, y) = \text{Sup}_{t_p}\{\gamma \in L^I | T(x, \gamma) \leq t_p y \text{ or } [T(x, \gamma)]_m = y_m\} \]

where, \(\text{Sup}_{t_p}\) is the interval having maximum degree of truth. Sometimes, instead of a unique value, the operation \(\text{Sup}_{t_p}\) may give a set of intervals belonging to the same m-set and hence \(I_{R_{tp}}(x, y)\) may not be unique.

**Example:** Suppose \(L = ([0, 1], \leq)\) and the t-norm is \(T_{Min_p}\). Then the R-implicator generated from this t-norm is given by:

\[ I_{Min} = \text{Sup}_{t_p}\{\gamma \in L^I | T_{Min_p}(x, \gamma) \leq t_p y \text{ or } [T_{Min_p}(x, \gamma)]_m = y_m\}. \]
1. If \( x \leq_{tp} y \), for any \( \gamma \in L^I \), \( T_{Min_p}(x, \gamma) \leq_{tp} y \). Thus, \( I_{Min} = [1, 1] \).
2. If \( x_m = y_m \), then for any \( \gamma \geq_{tp} x \), \( [T_{Min_p}(x, \gamma)]_m = y_m \). Thus, \( I_{Min} = [1, 1] \).
3. If \( x \geq_{tp} y \), then for any interval \( \gamma \) with \( \gamma_m = y_m \), we have \( [T_{Min_p}(x, \gamma)]_m = y_m \). Thus, \( I_{Min} = \gamma \) s.t. \( \gamma_m = y_m \). Hence, the implicator does not give a unique element, but an \( m - set_a \) of intervals with \( a = y_m \).

**Example:** Suppose \( L = ([0, 1], \leq) \) and the t-norm is \( T_{pr} \).

Then the R-implicator generated from this t-norm is given by:

\[
I_{pr} = \sup_{tp} \{ \gamma \in L^I | T_{pr}(x, \gamma) \leq_{tp} y \text{ or } [T_{pr}(x, \gamma)]_m = y_m \}.
\]

or, in other words,

\[
I_{pr} = \sup_{tp} \{ \gamma \in L^I | (x_1 \times \gamma_1 + x_2 \times \gamma_2 \leq y_1 + y_2) \text{ or } (x_1 \times \gamma_1 = y_1 \text{ and } x_2 \times \gamma_2 = y_2) \}.
\]

Case 1: If \( x_1 + x_2 \leq y_1 + y_2 \); \( I_{pr} = [1, 1] \).

Case 2: When \( x_1 + x_2 > y_1 + y_2 \), i.e. \( x >_{tp} y \) and no interval resides completely in the other, i.e. \( y_1 \leq x_1 \) and \( y_2 \leq x_2 \):

\[
I_{pr} = \max_{tp} \left( \left[ \frac{y_1}{x_1}, \frac{y_2}{x_2} \right], \left[ \frac{y_1 + y_2}{x_1 + x_2}, \frac{y_1 + y_2}{x_1 + x_2} \right] \right).
\]

Note: \( \left[ \frac{y_1}{x_1}, \frac{y_2}{x_2} \right]_m - \left[ \frac{y_1 + y_2}{x_1 + x_2}, \frac{y_1 + y_2}{x_1 + x_2} \right]_m = \frac{(x_2 - x_1)(y_1x_2 - y_2x_1)}{2x_1x_2(x_1 + x_2)} \).

Thus,

\[
I_{pr} = \left[ \frac{y_1}{x_1}, \frac{y_2}{x_2} \right] \text{ if } \frac{y_1}{x_1} > \frac{y_2}{x_2},
\]

\[
= \left[ \frac{y_1 + y_2}{x_1 + x_2}, \frac{y_1 + y_2}{x_1 + x_2} \right] \text{ otherwise.}
\]

Case 3: When \( x_1 + x_2 > y_1 + y_2 \), i.e. \( x >_{tp} y \) and one interval resides completely in the other, i.e. either \( x_1 \leq y_1 \leq y_2 < x_2 \) or \( y_1 < x_1 \leq x_2 \leq y_2 \):

\( I_{pr} = [\gamma, \gamma] \) where, \( \gamma = \frac{y_1 + y_2}{x_1 + x_2} \), since, \( [x_1 \times \gamma_1, x_2 \times \gamma_2]_m = \gamma \times \frac{x_1 + x_2}{2} = y_m \).

### 7 Conclusion:

We conclude with a critical appreciation of the proposed structure with respect to the bilattice-based triangle. The structure, preorder-based triangle, together with the logical operators defined on it, provides a framework for reasoning with imprecise, uncertain and incomplete information. Unlike bilattice-based triangle, the preorder-based triangle is capable of handling repetitive belief revisions in nonmonotonic reasoning. Moreover the truth ordering in the new structure is more intuitive. The fact that the knowledge and truth order now become fully orthogonal does have a strong appeal in application areas involving nonmonotonic logical reasoning with vague and
incomplete information. As demonstrated here, all the operators defined for bilattice-based triangles are suitable for the proposed structure as well and the modified truth ordering invokes some new logical connectives with interesting properties. Thus, the proposed preorder-based structure can be considered as an enhancement to bilattice-based triangle.

This work is an preliminary analysis of the necessity of preorder-based triangle and its pros and cons, and leaves enough scope for further investigation and analysis.

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