BIRKHOFF SUMS AS DISTRIBUTIONS I: REGULARITY

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ABSTRACT. We study Birkhoff sums as distributions. We obtain regularity results on such distributions for various dynamical systems with hyperbolicity, as hyperbolic linear maps on the torus and piecewise expanding maps on the interval. We also give some applications, as the study of advection in discrete dynamical systems.

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1. Introduction

Consider a measurable dynamical system \( f : X \to X \), where \( X \) is a measure space endowed with a reference measure \( m \) and such that \( f \) has an invariant probability \( \mu \) that is absolutely continuous with respect to \( m \). Let \( \phi : X \to \mathbb{C} \) be a measurable observable. One of the main goals in the study of the ergodic theory is to study the statistical properties of the sequence of variables

\[
\phi, \phi \circ f, \phi \circ f^2, \ldots
\]

The Birkhoff ergodic theorem, for instance, says that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} \phi \circ f^k(x)
\]

converges for \( \mu \)-almost every point \( x \). On the other hand if we consider the Birkhoff sum

\[
\sum_{k=0}^{\infty} \phi \circ f^k.
\]

then in very regular situations (piecewise expanding maps and Anosov diffeomorphisms) this sum may not converge almost everywhere. Suppose now that \( X \) is a manifold with a measure \( m \). Then in many well-known cases we have that

\[
\sum_{k=0}^{\infty} \int \psi \phi \circ f^k \, dm
\]

does converge when \( \psi \) and \( \phi \) are regular enough. This allows us to define

\[
\int \psi \sum_{k=0}^{\infty} \phi \circ f^k \, dm = \sum_{k=0}^{\infty} \int \psi \phi \circ f^k \, dm
\]

for \( \psi \in C^\infty \), in such way that the Birkhoff sum induces a distribution. We should see the right hand-side of this equation as the definition of the left-hand side, which is not a proper Lebesgue integral.

Our goal is to study this distribution’s regularity for several dynamical systems with strong hyperbolicity: maps with exponential decay of correlations, linear Anosov maps, and piecewise expanding maps on the interval. We give three main motivations to this study.

1.1. Deformations of Dynamical Systems. A deformation of a dynamical system \( f_0 : M \to M \) is a smooth family \( f_t : M \to M \), with \( t \in (-\epsilon, \epsilon) \), such that \( f_t \) is conjugate to \( f_0 \) for every \( t \). That is, there exist homeomorphisms \( h_t : M \to M \) such that

\[
h_t \circ f_0 = f_t \circ h_t.
\]

For certain types of maps \( f_0 \) exhibiting hyperbolicity (e.g., expanding maps, piecewise expanding maps, and Anosov diffeomorphisms), the conjugacies \( h_t \) are rarely smooth. In fact, they are often singular with respect to the Lebesgue measure, meaning they map a set of full Lebesgue measure to a set of zero Lebesgue measure. This poses a significant challenge in studying such conjugacies. However, for one-dimensional dynamics (maps acting on an interval or the circle), and even for Anosov diffeomorphisms in arbitrary dimensions, the maps

\[
t \mapsto h_t(x)
\]

are smooth. This enables the study of infinitesimal deformations, defined by

\[
\alpha(x) = \partial_t h_t(x)|_{t=0}.
\]
Investigating the regularity of the function $\alpha$ provides precise insights into the regularity of the conjugacies $h_t$ and their dependence on the parameter $t$. For piecewise expanding maps on an interval, this approach has been developed in C-R and S. [16]. We believe, however, that it can be extended to much broader contexts. The connection to this paper lies in the fact that $\alpha$ is generally not differentiable. Nevertheless, it has derivatives in the sense of distributions, and its derivative is a Birkhoff sum. Consequently, $\alpha$ can be seen as a primitive of a Birkhoff sum. In this work, we conduct a detailed study of primitives of Birkhoff sums for piecewise expanding maps in one dimension. See Section 2.3 for the main results. In particular, we analyze the statistical properties of these primitives.

1.2. Invariant Distributions. An invariant distribution of a dynamical system $T$ is a distribution $\psi$ that satisfies

$$\langle \psi, \phi \circ T \rangle = \langle \psi, \phi \rangle$$

for all test functions $\phi$. A familiar example of an invariant distribution is a $T$-invariant probability measure. One might ask whether these are the only possible examples. Avila and Kocsard [1] (see also Navas and Triestino [25]) proved that for a $C^\infty$ diffeomorphism of the circle with irrational rotation, the unique invariant probability measure is also its only invariant distribution (up to multiplication by a constant).

For piecewise expanding maps on an interval, the situation is markedly different. By utilizing Birkhoff sums as distributions, we construct invariant distributions that are not (complex-valued) measures. See Section 2.4 for the main results.

1.3. Advection and cohomological equations in discrete dynamics. The mathematical problems to be considered in this paper are also motivated by the following physical question.

Let $M$ be a $C^\infty$ manifold that represents the state space of a discrete-time dynamical system $f : M \to M$, where $f$ is a $C^\infty$ diffeomorphism. The manifold is endowed with a $C^\infty$ volume form $m$, which may be that associated to a Riemannian metric on $M$. Let $J : M \to \mathbb{R}$ be the Jacobian determinant of $f$ with respect to $m$, $J$ is supposed positive ($f$ is orientation preserving). Notice that $m$ can be invariant under $f$, $J = 1$, or not, $J \neq 1$. Assume that a fluid lies on $M$ and that the dynamic changes the position of the fluid particles, the particle initially at $x_0 \in M$ moves to $x_1 = f(x_0)$. Several extensive physical quantities may be associated with a fluid: mass, internal energy, the mass of a diluted chemical substance, electric charge, etc. Each of these properties is characterized by a density function $\rho$ with respect to the volume form $m$ such that $\int_\Omega \rho \, dm$ gives the amount of the property inside the region $\Omega$. For simplicity we assume that $\rho$ is the electric charge density or just the charge density. Suppose that charge is advected by the dynamics (there is no charge diffusion between neighboring fluid particles). In this case if there is no charge input-output to the system then advection yields: if $\rho_j : M \to \mathbb{R}$ is the density at time $j$ then $f_* (\rho_j m) = \rho_{j+1} m$, for every $i \in \mathbb{Z}$, implies that the density at time $j+1$ is

$$\rho_{j+1} = L \rho_j$$

where $L : L^1(m) \to L^1(m)$ is the Ruelle-Perron-Frobenious operator

$$L \rho(x) = \frac{\rho o f^{-1}(x)}{\int f o f^{-1}(x)}.$$
Note that
\[(L^{-1} \rho)(x) = J \cdot (\rho \circ f).\]
More generally, at each time, charge can be added to or subtracted from the fluid by means of a $C^\infty$ distribution of sources and sinks that is supposed to be time-independent. In this case the form $\rho_j m$ at time $j$ is mapped to $\rho_{j+1} m = f_\ast(\rho_j m + R m)$ at time $j + 1$, where $R : M \to \mathbb{R}$ is the density function of sources (wherever $R > 0$) and of sinks (wherever $R < 0$). These relations imply
\[(1.3.4) \quad \rho_{j+1} = L(\rho_j + R)\]
Given a $C^\infty$ function $\phi : M \to \mathbb{R}$ with compact support, $\phi \in C^\infty_c(M)$, a measurement of charge at time $j$ is defined as
\[(1.3.5) \quad Q_j(\phi) = \int_M \phi \rho_j \, dm\]
If $\phi$ is positive and $\int_M \phi \, dm = 1$, then $Q_j(\phi)$ is the average charge density with respect to the measure $\phi m$. In this way, a measurement is a linear functional in $C^\infty_c(M)$ and define a distribution in the sense of Schwartz, $Q_j \in \mathcal{D}'(M)$ (see Hörmander [19]). The questions we are interested in are:

(i) Do the limits
\[
\lim_{j \to -\infty} Q_j(\phi) = u_a(\phi) \quad \text{and} \quad \lim_{j \to +\infty} Q_j(\phi) = u_\omega(\phi)
\]
exist for all functions $\phi \in C^\infty_c(M)$?

(ii) If the limits exist, then $u_a$ and $u_\omega$ define distributions in $\mathcal{D}(M)$. What is the regularity of $u_a$ and $u_\omega$?

It is easy to see that
\[(1.3.6) \quad \rho_j = L_j \rho_0 + \sum_{i=1}^{j} L^i \rho_i(R).
\]
\[\rho_{-j} = L^{-j} \rho_0 - \sum_{i=0}^{j-1} L^{-i} \rho_i(R)\]
Since we are mostly interested in the component associated to $R$, from now on we assume $\rho_0 = 0$. The fixed point equation associated to equation (1.3.4) is
\[(1.3.7) \quad L^{-1} \rho - \rho = R,\]
The solutions to that are invariant charge densities (if $R = 0$ then it defines an invariant measure for $f$). An integrable function $\rho$ is a weak solution to equation (1.3.7) if for every test function $\phi \in C^\infty(M)$ the following identity holds
\[
\int_M \rho \cdot \phi \circ f^{-1} \, dm - \int_M \rho \phi \, dm = \int_M R \phi \, dm
\]
This definition extends naturally to distributions. A distribution $u \in \mathcal{D}'(M)$ is a weak solution to equation (1.3.7) if for every test function $\phi \in C^\infty(M)$ the following identity holds
\[(1.3.8) \quad u(\phi \circ f^{-1}) - u(\phi) = \int_M R \phi \, dm
\]
Suppose that the limit $u_\omega(\phi)$ in question (i) exists for any $\phi$. Then, from (1.3.6)
\[
\lim_{j \to +\infty} \int_M L^j(R)\phi \, dm = 0.
\]
Using the definition of $u_\omega$, a computation shows that
\[ Q_j(\phi \circ f^{-1}) - Q_j(\phi) - \int_M R \phi \, dm = - \int_M L^j(R) \phi \, dm, \]
and taking the limit as $j \to \infty$ we conclude that $u_\omega$ is a weak solution of the cohomological equation. The same result holds for $u_\alpha$. Therefore, if the $\omega$-limit and the $\alpha$-limit (in a weak sense) exist, then these limits are weak solutions of the corresponding cohomological equation.

From now on, we suppose that $M$ is compact. Then integration of both sides of equation (1.3.4) over $M$ with respect to the volume form $\mu$ gives
\[ Q_{j+1}(1) = Q_j(1) + \int_M R \, dm. \]
Therefore, question (i) may have a positive answer only if
\[ (1.3.9) \int_M R \, dm = 0, \]
which means that the total amount of charge added to the system at each time is null.

The limits in question (i) do not exist unless the dynamics of $f$ is complex enough. For instance, if $f$ is the identity map then $u_\alpha$ and $u_\omega$ do not exist for any function $R \neq 0$. Suppose that $m$ is invariant under $f$. Then $J = 1$ and (1.3.7) is the classical Livsic cohomological equation
\[ \rho \circ f - \rho = R \]
and for $\rho_0 = 0$
\[ \rho_j = \sum_{i=1}^{j} R \circ f^{-i}. \]
\[ \rho_{-j} = - \sum_{i=0}^{j-1} R \circ f^i \]

If $u_\omega$ exists for every function $R \in C^\infty(M)$ satisfying $\int_M R \, dm = 0$ then
\[ \lim_{j \to \infty} \int_M (R \circ f^{-j}) \phi \, dm = 0 \]
for all functions $R$ and $\phi$ in $C^\infty(M)$. This implies the decay of correlations for any pair of functions in $C^\infty(M)$, which is equivalent to $(f, m)$ to be mixing. Therefore, if $m$ is invariant under $f$, the limits $u_\alpha$ and $u_\omega$ exists for any given function $R$ only if $(f, m)$ is mixing (this seems to be a natural physical condition for the existence of an equilibrium once we have neglected molecular diffusion). On the other hand, if $(f, \mu)$ is mixing and the decay of correlations is fast enough so that
\[ \sum_{j \in \mathbb{Z}} \left| \int_M (R \circ f^j) \phi \, dm \right| < \infty \]
for any functions $R$ and $\phi$ in $C^\infty(M)$ with $\int_M R \, dm = 0$, then the limits $u_\omega$ and $u_\alpha$ exist. Indeed we have that
\[ u_\alpha = \sum_{i=1}^{+\infty} R \circ f^{-i}. \]
\[ u_\omega = - \sum_{i=0}^{+\infty} R \circ f^i. \]
as distributions.
2. Main results

2.1. Dynamics with exponential decay of correlations. Our first result gives weaker regularity results than the ones we obtain later for hyperbolic linear maps on the torus and piecewise expanding maps on the interval. However, it is remarkable that its only assumption is exponential decay of correlations for Hölder observables, which has been proved for a wide variety of dynamical systems.

**Theorem A.** Let $M$ be a $C^r$ compact manifold, with $r \geq 1$, $f : M \to M$ be a measurable function with measurable inverse, and $\mu$ be a smooth volume form invariant under $f$. Suppose that for a $\mu$-integrable function $R \in L^\infty(M)$ satisfying $\int_M R \, d\mu = 0$ and for $\phi \in C^r(M)$ the exponential decay of correlations

\[
\left| \int_M R(f^j(x)) \phi(x) \, d\mu(x) \right| \leq C_1 e^{-C_2 |j| |\phi|_{C^r}}, \quad j \in \mathbb{Z}
\]

holds, where $C_1 > 0$ and $C_2 > 0$ are constants that depend neither on $j$ nor on $\phi$ and $|\phi|_{C^r}$, with $\gamma > 0$, is the usual Hölder norm of $\phi$. Then the Birkhoff sums $u_\alpha$ and $u_\omega$ given by

\[
u_\alpha = \sum_{j=-\infty}^{j=-1} R(f^j x) \quad \text{and} \quad u_\alpha = -\sum_{j=0}^{\infty} R(f^j x).
\]

belong to the logarithm Besov space $B_{\infty,1}^{0,-1}$. Moreover they are weak solutions of the cohomological equation

\[ R = u \circ f - u. \]

Logarithmic Besov spaces $B_{p,q}^{s,b}$ are a generalization classical Besov spaces (see Section 3.1).

**Remark 2.1.11.** If $f$ is not invertible, we can obtain a similar result for $u_\alpha$ assuming (2.1.10) for $j \geq 0$.

A volume-preserving linear Anosov on the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is a $C^\infty$-diffeomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ defined by $f x = M x$, where $M$ is a hyperbolic $n \times n$-unimodular matrix. The most famous example is the Arnold’s Cat map, an Anosov map obtained taking

\[ M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]

2.2. Hyperbolic Linear maps on the Torus. For hyperbolic linear maps we can use Fourier analysis methods to obtain

**Theorem B.** For every $R \in C^\beta(\mathbb{T}^n)$, with $\beta > n/2$, such that

\[ \int \mathbb{T}^n R \, dm = 0, \]

where $m$ is the Haar measure of $\mathbb{T}^n$. Consider the Birkhoff sums

\[ u_\alpha = -\sum_{j=0}^{\infty} R \circ f^j \]

and

\[ u_\omega = \sum_{j=1}^{\infty} R \circ f^{-j}. \]
Then \( u_α \) and \( u_ω \) are well-defined as distributions and \( u_ω, u_α \in Λ^0 \), where \( Λ^0 \) is a Zygmund space. Moreover they are both weak solutions of the cohomological equation

\[
(2.2.12) \quad R = u \circ f - u.
\]

Zygmund spaces \( Λ^s \) are introduced in Section 3.1.

2.3. **Piecewise expanding maps: Primitives of Birkhoff sums.** Here we give a fairly complete picture of the regularity of Birkhoff sums for piecewise expanding one-dimensional maps. One of the advantages of the one-dimensional setting is that one can easily define the primitive of a Birkhoff sum. Let \( I = [a, b] \) and \( f \) be a \( C^{1+BV} \) piecewise expanding map on \( I \). We are going to see that if \( φ \in L^∞(m) \) is orthogonal to the densities of all absolutely continuous invariant probability measures of \( f \) then

\[
ψ(x) = \int 1_{[a,x]} \left( \sum_{k=0}^{p-1} \sum_{j=0}^∞ \phi \circ f^{kp+j} \right) \, dm
\]

is a well-defined function (for an appropriated \( p \) that depends only on \( f \)) and

\[
ψ' = \sum_{k=0}^{p-1} \sum_{j=0}^∞ \phi \circ f^{kp+j}
\]

in the sense of distributions. Here \( BV \) is the space of bounded variation functions in \([a, b]\). This allows us study the regularity of Birkhoff’s sums in a far more effective way. Here \( p \) is related with the ergodic decomposition of the absolutely continuous invariant probabilities of \( f \).

There is a finite number of ergodic absolutely continuous \( f \)-invariant probabilities \( μ_ℓ \) and densities \( ρ_ℓ \), whose (pairwise disjoint) basins of attractions

\[
A_ℓ = \{ x \in I \text{ s.t. } \lim_{N \to ∞} \frac{1}{N} \sum_{k<N} \theta \circ f^k(x) = \int \theta \, dμ_ℓ, \text{ for every } θ \in C^0(I) \}.
\]

covers \( m \)-almost every point in \( I \). Let

\[
S_ℓ = \{ x \in I \text{ s.t. } ρ_ℓ(x) > 0 \} ⊂ A_ℓ
\]

By Boyarsky and Góra[8] the set \( S_ℓ \) is a finite union of intervals up to a zero \( m \)-measure set. Indeed

\[
A_ℓ = \bigcup_{n=0}^{∞} f^{-n} S_ℓ
\]

up to set of zero Lebesgue measure. Define

\[
Φ_1 : L^1(m) \to BV
\]

by

\[
Φ_1(γ) = \sum_ℓ \left( \int_{A_ℓ} γ \, dm \right) ρ_ℓ.
\]

**Theorem C** (Log-Lipchitz continuity). Let \( f : I \to I \) be a piecewise \( C^{1+BV} \) expanding map on the interval \( I = [a, b] \). There is \( p \in \mathbb{N}^* \) such that for every function \( φ \in L^∞(I) \) satisfying

\[
\int φΦ_1(γ) \, dm = 0
\]

for every \( γ \in BV \) we have \( ψ \) is Log-Lipchitz continuous.
See Theorem 5.1.30 for details. We can ask if the Log-Lipchitz regularity is sharp. Indeed, given an ergodic absolutely continuous probability \( \mu \) and complex valued functions \( \phi_1 \) and \( \phi_2 \) such that
\[
\int \phi_i \, dv = 0, \quad i = 1, 2
\]
we define
\[
\sigma_v(\phi_1, \phi_2) = \lim_{N \to \infty} \int \left( \frac{\sum_{i=0}^{N-1} \phi_1 \circ f^i}{\sqrt{N}} \right) \left( \frac{\sum_{i=0}^{N-1} \phi_2 \circ f^i}{\sqrt{N}} \right) \, dv
\]
whenever this limit exists, and
\[
\sigma_v^2(\phi) = \sigma_v(\phi, \phi).
\]
Note that \( \nu \) does not need to be \( f \)-invariant.

**Theorem D.** Let \( f \) be a piecewise \( C^{1+BV} \) expanding maps on the interval \( I = [0, 1] \) and let \( \phi \) be a piecewise \( C^\beta \) function on \( I \), with \( \beta \in (0, 1) \), such that
\[
\int \phi \Phi_1(\gamma) \, dm = 0
\]
for every \( \gamma \in BV \). Then the variance \( \sigma_\mu(\phi) \) is well-defined and finite for the Lebesgue measure \( m \) on \( I \) and for every ergodic absolutely continuous \( f \)-invariant probability \( \mu \). Indeed \( f \) has only a finite number of absolutely continuous ergodic \( f \)-invariant probabilities \( \{ \mu_\ell \} \) and
\[
\sigma_m^2(\phi) = \sum_\ell c_\ell \sigma_{\mu_\ell}^2(\phi),
\]
where \( c_\ell > 0 \) and \( \sum_\ell c_\ell = 1 \). If \( \sigma_m^2(\phi) = 0 \) then \( \psi \) is an absolutely continuous function and its derivative belongs to \( L^2(m) \).

See Theorem 5.2.46 for details. For \( \sigma_m^2(\phi) > 0 \) the regularity of \( \psi \) is quite bad.

**Theorem E** (Central Limit Theorem for the modulus of continuity). Let \( f \) be a piecewise \( C^{2+\beta} \) expanding map on the interval \( I = [0, 1] \) and let \( \phi \) be a piecewise \( C^\beta \) function on \( I \), with \( \beta \in (0, 1) \), such that
\[
\int \phi \Phi_1(\gamma) \, dm = 0
\]
for every \( \gamma \in BV \). Then the variance \( \sigma_\mu(\phi) \) is well-defined with respect to every ergodic absolutely continuous \( f \)-invariant probability \( \mu \). If \( \sigma_\mu(\phi) > 0 \) then
\[
\lim_{h \to 0} \mu(\{ x \in I : \frac{1}{\sigma_\mu(\phi) L \sqrt{-\log|h|}} \left( \frac{\psi(x+h) - \psi(x)}{|h|} \right) \leq y \}) = \frac{1}{2\pi} \int_{-\infty}^{y} e^{-x^2} \, dx.
\]
Here
\[
L = \left( \int |Df| \, d\mu \right)^{-1/2}.
\]
In particular \( \psi \) is not a Lipschitz function of any measurable subset of positive measure in the support of \( \mu \). In particular \( \psi \) does not have bounded variation on the support of \( \mu \).

See Theorem 5.3.58 for the precise statement. One can ask if \( \psi \) is in general a Zygymund function (all Zygmund functions are Log-Lipchitz continuous). That is not true. See Section 5.5.
2.4. Invariant distributions of piecewise expanding maps. Our results have applications in the study of the nature of invariant distributions of a piecewise expanding map. Invariant finite measures are an obvious example. However there is much more.

Theorem F. Let \( f \) be a piecewise \( C^{2+\beta} \) expanding map on the interval \( I = [0, 1] \), with \( \beta \in (0, 1) \). Choose \( \phi \in \mathcal{R}^\beta(C) \cap BV \) such that

\[
\int \phi d\nu = 0
\]

for every absolutely continuous \( f \)-invariant ergodic measure \( \nu \). Consider the distribution \( \Theta_\phi \in BV^* \) given by

\[
\Theta_\phi(g) = \sigma_m(g, \phi).
\]

Then \( \Theta_\phi \) is \( f \)-invariant. Moreover \( \Theta_\phi \) is a signed measure if and only if \( \Theta_\phi = 0 \). In particular if \( \sigma_m(\phi) > 0 \) then \( \Theta_\phi \) is not a signed measure.

See Theorem 5.6.84 for the precise statements.

3. Regularity under exponential decay of correlations

3.1. Zygmund and Logarithm Besov spaces. This section contains a series of definitions and results concerning the regularity properties of functions and distributions (in the sense of Schwartz) that we will use in the proof of Theorem A.

Let \( \mathcal{S} \) be the Schwartz space of complex-valued rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^n \) and \( \mathcal{S}' \) the space of continuous linear forms on \( \mathcal{S} \) (temperate distributions). For \( \phi \in \mathcal{S} \) let \( \mathcal{F}(\phi) \) and \( \mathcal{F}^{-1}(\phi) \) be the Fourier transform and its inverse (see Hörmander [19]).

The Fourier transform of \( u \in \mathcal{S}' \), denoted as \( \mathcal{F}u = \hat{u} \), is defined by \( \hat{u}(\phi) = u(\phi) \). The Fourier transform is an isomorphism of \( \mathcal{S}' \) (with the weak topology) with inverse given by \( \mathcal{F}^{-1}(\hat{u}(\phi)) = u(\mathcal{F}^{-1}(\phi)) \). If \( u \in \mathcal{S}' \) has compact support then \( u \) can be extended to the class of complex-valued infinitely differentiable functions, denoted as \( C^{\infty}_\mathbb{C} \), and \( \hat{u}(\xi) = u_x(\exp(-ix \cdot \xi)) \in \mathcal{C}^{\infty} \), where \( u_x \) denotes that \( u \) acts on the variable \( x \).

Let \( C^{\infty}_c(\mathbb{R}^n) \) denote the space of functions in \( C^{\infty}(\mathbb{R}^n) \) with compact support. Let \( \psi_0 \in C^{\infty}_c \) be a function that is radial, non increasing along rays, and such that: \( \psi_0(x) = 1 \) for \( |x| < 1 \), \( \psi_0(x) = 0 \) for \( |x| > 2 \). We define \( \psi(x) = \psi_0(x) - \psi_0(2x) \) and note that \( 0 \leq \psi(x) \leq 1 \) with \( \psi(x) = 0 \) for \( |x| < 1/2 \) and \( |x| > 2 \). We define \( \psi_\ell(x) = \psi(x/2^\ell) \). Note that

(3.1.13) \( \text{supp}(\psi_\ell) \subset \{x : 2^{\ell-1} \leq |x| \leq 2^{\ell+1}\} \)

and \( \sum_0^N \psi_\ell(x) = \psi_0(x/2^N) \rightarrow 1 \) as \( N \rightarrow \infty \), which implies that the set \( \{\psi_0, \psi_1, \ldots\} \) yields a partition of unit. If \( u \in \mathcal{S}' \) then \( \psi_\ell(D)u \) is defined by

(3.1.14) \( \psi_\ell(D)u(x) = \mathcal{F}^{-1}(\psi_\ell \hat{u}) \).

Since \( \psi_\ell \) has compact support, \( \psi_\ell(D)u \in C^{\infty}(\mathbb{R}^n) \). For any \( s \in \mathbb{R} \) we define the Zygmund class \( \Lambda^s \) as the set of all \( u \in \mathcal{S}' \) with the norm

\[
|u|_s = \sup_{\ell \geq 0} 2^{\ell s} \sup_{\ell \geq 0} |\psi_\ell(D)u| < \infty
\]

The Zygmund class has the following properties (see Hörmander [18, Section 8.6] and Triebel [28]):

(i) If \( s > 0 \) is not an integer then \( u \in \Lambda^s \) if, and only if, \( u \) is a Hölder function with exponent \( s \).
(ii) The Zygmund class $\Lambda^1$ consists of all bounded continuous functions such that
\[
\sup |u(x)| + \sup_{y \neq 0} \left| \frac{u(x + y) + u(x - y) - 2u(x)}{y} \right| < \infty
\]
and the norm $|u|_1$ is equivalent to the left-hand side. There exists analogous characterizations of $\Lambda^s$, for $s > 0$ integer.

(iii) If $u \in \Lambda^s$ then $\partial_{x_j} u \in \Lambda^{s-1}$ and, conversely, $u \in \Lambda^s$ if $\partial_{x_j} u \in \Lambda^{s-1}$, $j = 1, \ldots, n$.

(iv) If $u$ is bounded and continuous then $u \in \Lambda^0$.

Another class of spaces to be considered in this paper is a modification of the Zygmund class, the logarithm Besov spaces $B_{s,b}^{\infty,\infty}$. For any $s \in \mathbb{R}$, let $B_{s,b}^{\infty,\infty}$ be set of all $u \in \mathcal{S}'$ such that
\[
(3.1.15) \sup_{\ell \geq 0} 2^{\ell s} (1 + \ell)^b \sup_{\ell \geq 0} |\psi_{\ell}(D)u| < \infty
\]
This definition of logarithm Besov spaces can be found in Cobos, Domínguez, and Triebel [10, Eq. (4.1)]. Beware that there is another definition of (sometimes distinct) logarithm Besov spaces for $s \geq 0$ in this same reference, using the modulus of continuity instead of the Fourier transform approach.

In order to define $\Lambda^s$ and $B_{s,b}^{\infty,\infty}$ on compact $C^\infty$ manifolds, it is necessary to consider test functions with support in a coordinate domain. A partition of unity can be used to decompose test functions with support into functions with support in coordinate domains. However, in the particular case of the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$, it is more convenient to characterize those spaces of distributions using Fourier series instead.

The functions and distributions on $\mathbb{T}^n$ lift to periodic functions and periodic distributions on $\mathbb{R}^n$ (a distribution $u \in \mathcal{S}'$ is periodic if $u(\phi) = u_k(\phi(x + k))$ for every $\phi \in \mathcal{S}$ and $k \in \mathbb{Z}^n$). It is convenient to rewrite the definition of $\Lambda^s$ for periodic distributions (see Hörmander [19, Section 7.2]). Let $\Gamma \in C^\infty(\mathbb{R}^n)$ be such that $\sum_{k \in \mathbb{Z}^n} \Gamma(x + k) = 1$. It can be shown that any periodic $u \in \mathcal{S}'$ can be written as
\[
u = \sum_{k \in \mathbb{Z}^n} c_k e^{2\pi i x \cdot k} \quad \text{where} \quad c_k = u_k(\Gamma(x)e^{-2\pi i x \cdot k}).
\]

3.2. Proof of Theorem A. Let $f : M \to \mathbb{R}$ be a $C^\infty$ diffeomorphism and $\mu$ a smooth volume form preserved under $f$. Suppose that $M$ is compact so that it can be covered by a finite number of coordinate patches $U_1, U_2, \ldots$. The coordinates $x$ on each $U_i$ can be chosen such that $\mu = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n = dx$. Let $\chi_1, \chi_2, \ldots$ be a partition of unit such that $\text{supp } \chi_i \subset U_i$. If $\phi \in C^\infty(M)$ then $\phi = \phi_1 + \phi_2 + \ldots$ where $\phi_i = \chi_i \phi$ has its support in $U_i$. If $u$ is a distribution on $M$ then $u(\phi) = u(\sum \chi_i \phi) = u(\sum \chi_i \phi)$. So, $u$ can be decomposed into a sum of distributions $u_i = \chi_i u$, $i = 1, 2, \ldots$, such that $\text{supp } u_i \subset U_i$. We say that $u$ belongs to some class of regularity (for instance $\in \Lambda^s$) if $u_i$ belong to this class for all $i$. Since $\text{supp } u_i \subset U_i \subset \mathbb{R}^n$ the analysis of the regularity of $u_i$ can be made using the tools presented in section 3.1. Since there are finitely many coordinate patches and the analysis of regularity is similar in all of them we just choose a particular one and neglect the index $i$ associated to it.

The $u_\alpha$ and $u_\omega$ distributions restricted to a particular coordinate patch are given by
\[
u_\omega = \chi(x) \sum_{j=1}^{\infty} R(f^{-j}x) \quad \text{and} \quad u_\alpha = -\chi(x) \sum_{j=0}^{\infty} R(f^j x).
\]
The analysis of the regularity of $u_a$ and $u_w$ are similar, so we only consider $u_a$. Let $\Psi_\ell = \mathcal{F}^{-1} \psi_\ell \in \mathcal{F}$. Using that $\mathcal{F}(\Psi_\ell \ast (\chi u_a)) = \Psi_\ell \mathcal{F}(\chi u_a)$, where $\ast$ denotes the convolution, we obtain that

$$
\Psi_\ell(D)(\chi u_a)(x) = \mathcal{F}^{-1}[\Psi_\ell(\xi)\mathcal{F}(\chi u_a)](x) = \Psi_\ell \ast (\chi u_a)(x)
$$

$$
= u_{a_j}(\Psi_\ell(x-y)\chi(y)) = \sum_{j=0}^{\infty} \Psi_\ell(x-y)\chi(y)R(f^j y)dy.
$$

For $\ell = 0$, the decay of correlations (2.1.10) and the uniform boundness of

$$
|\Psi_0(x-\cdot)\chi(\cdot)|_{C^r}
$$

with respect to $x$ imply that $\sup |\psi_0(D)(\chi u_a)(x)| < \infty$. For $\ell \geq 1$ we claim that there exists a constant $C_3 > 0$ such that

$$
\sup |\psi_\ell(D)(\chi u_a)(x)| \leq C_3(\ell + 1).
$$

Indeed, the definition of $\psi_\ell$ for $\ell \geq 1$ implies

$$
\psi_\ell(x) = 2^{\ell n}\psi(2^\ell x) \quad \text{where} \quad \hat{\psi} = \psi
$$

This implies that $|\psi_\ell(x-\cdot)\chi(\cdot)|_{C^r} \leq C_42^{\ell(y+n)}$ where $C_4 > 0$ does not depend on $\ell$. So the decay of correlations (2.1.10) implies

$$
\left| \int_{\mathbb{R}^n} \Psi_\ell(x-y)\chi(y)R(f^j y)dy \right| \leq C_1C_4 \exp(-C_2 j + \ell(y + n) \ln 2)
$$

If $C_5 = (y + n) \ln 2/C_2$ then

$$
\left| \sum_{j \leq C_5} \int_{\mathbb{R}^n} \Psi_\ell(x-y)\chi(y)R(f^j y)dy \right| \leq \frac{C_1C_4}{1-e^{-C_2}} = C_6
$$

It remains to estimate

$$
\left| \sum_{j \leq C_5} \int_{\mathbb{R}^n} 2^{\ell n}\psi(2^\ell (x-y))\chi(y)R(f^j y)dy \right|
$$

$$
= \left| \sum_{j \leq C_5} \int_{\mathbb{R}^n} \psi(z)\chi(x-2^\ell y)R(f^j (x-2^\ell y))dy \right|
$$

$$
\leq |R|_{L^\infty(M)} \sum_{j \leq C_5} \int_{\mathbb{R}^n} |\psi(z)|dz \leq C_7 \ell
$$

where $C_7 > 0$ does not depend on $\ell$. Therefore

$$
\sup |\psi_\ell(D)(\chi u_a)(x)| \leq C_7 \ell + C_6 \leq C_3(\ell + 1).
$$

This completes the proof of the claim. Consequently

$$(3.2.16) \quad \sup_{\ell \geq 0} (1 + \ell)^{-1} \sup |\psi_\ell(D) u_a| < \infty$$

so $u_a \in B_{\infty, \infty}^{0, -1}$. 
4. HYPERBOLIC LINEAR MAPS ON THE TORUS

In order study its regularity we use the following.

**Proposition 4.1.17.** For every invertible $n \times n$ hyperbolic matrix $A$ there is $L > 0$ such that for every $\ell \in \mathbb{Z}$ and $p \in \mathbb{R}^n$ we have that

$$2^\ell \leq |A^j p| \leq 2^{\ell+1}$$

is verified for at most $L$ values of $j \in \mathbb{Z}$.

**Proof.** Let $E^s$ and $E^u$ be the stable and unstable spaces of $A$. Consider an adapted norm $\| \cdot \|$ and $\theta > 1$ such that for every $x \in \mathbb{R}^n$, if we denote $x = x_s + x_u$, with $x_s \in E^s$ and $x_u \in E^u$, we have

$$\|x\| = \|x_u + x_s\| = \|x_u\| + \|x_s\|,$$

for every $u \in E^u$ and $s \in E^s$, $\|Ax_u\| \geq \theta \|x_u\|$, for every $s \in E^u$, $\|A^{-1}x_s\| \geq \theta \|x_s\|$, for every $u \in E^u$.

The linearity of $A$ implies that it is enough to show that there is $k_0 \geq 1$ such that for every $p \in \mathbb{R}^n$

$$(4.1.18) \quad 1 \leq \|A^{k_0}j p\| \leq 2$$

is verified for at most two values $j \in \mathbb{Z}$.

Define

$$C^s = \{x \text{ s.t. } \|x_u\| \leq \|x_s\|\},$$
$$C^u = \{x \text{ s.t. } \|x_s\| \leq \|x_u\|\}.$$ 

Of course $\mathbb{R}^n = C^s \cup C^u$. Let $k_0$ be such that

$$\theta^{k_0} - \theta^{-k_0} > 6.$$ 

If $x \in C^u$ then $A^{k_0}x \in C^u$ and

$$\|A^{k_0}x\| \|A^n x_u\| \geq \|A^{k_0}x_u\| \|A^n x_s\| \geq \theta^{k_0} \|x_u\| - \theta^{-k_0} \|x_s\| \|x_u\| \geq \left(\theta^{k_0} - \theta^{-k_0}\right) \|x_u\| \|x_u\| \|x_u\| \geq \left(\theta^{k_0} - \theta^{-k_0}\right) \|x_u\| \|x_u\| \|x_u\| \geq \left(\theta^{k_0} - \theta^{-k_0}\right) \|x_u\|$$

$$(4.1.19) \quad \geq 3 \|x_u\|.$$ 

The $A^{k_0}$-forward invariance of $C^u$ and (4.1.19) implies that

$$\{j \in \mathbb{Z} : A^{k_0}j p \in C^u \text{ and } 1 \leq \|A^{k_0}j p\| \leq 2\}$$
contains at most one integer. By an analogous argument if $x \in C^s$ then $A^{-k_0}x \in C^s$ and

$$\|A^{-k_0}x\| \geq 3 \|x\|.$$ 

so

$$\{j \in \mathbb{Z} : A^{k_0}j p \in C^s \text{ and } 1 \leq \|A^{k_0}j p\| \leq 2\}$$
contains at most one integer. This completes the proof. \qed

**Proof of Theorem B.** The distributions we are interested in are of the form

$$u(\phi) = \lim_{j \to \mp \infty} Q_j(\phi),$$
where $Q_j(\phi)$ is as in (1.3.5), taking $\rho_0 = 0$ in (1.3.6). We will prove the theorem for $u_a$. The proof of the regularity of $u_a$ is analogous. Using the notation of Section 3.1

$$u_a(\phi) = \sum_{k \in \mathbb{Z}^n} c_k \int e^{2\pi i x \cdot k} \phi(x) dx$$

for every $\phi \in \mathcal{S}$, where

$$(4.1.20) \quad c_k = \lim_{j \to -\infty} \int_{\mathbb{R}^n} \rho_j(x) \Gamma(x) e^{-2\pi i x \cdot k} \ dm(x) = \lim_{j \to -\infty} \int_{\mathbb{R}^n} \rho_j(x) e^{-2\pi i x \cdot k} \ dm(x),$$

where $\rho_j$ is given in equation (1.3.6).

Let $\delta_z$ be the $\delta$-Dirac distribution with support at the point $z \in \mathbb{R}^n$. Using that $\hat{\delta}_z(\xi) = e^{-iz \cdot \xi}$ and $\mathcal{F} e^{ix \cdot \xi} = (2\pi)^n \hat{\delta}_z$ we obtain the Fourier transform of $u$:

$$\hat{u} = (2\pi)^n \sum_{k \in \mathbb{Z}^n} c_k \delta_{2\pi k}.$$

This and (3.1.14) imply

$$(4.1.21) \quad u_\ell(x) = \psi_\ell(D) u(x) = \mathcal{F}^{-1}(\psi_\ell(\xi) \hat{u}) = \sum_{k \in \mathbb{Z}^n} c_k \psi_\ell(2\pi k) e^{-2\pi i x \cdot k},$$

and

$$|u|_J = \sup_{\ell \geq 0} 2^{4\ell} \sup_{\ell \geq 0} |u_\ell(x)|.$$

We provide the proof for $u_a$. Let $R \in \Lambda^\delta$. We have

$$R(x) = \sum_{p \in \mathbb{Z}^n} b_p \exp(2\pi i p \cdot x).$$

Denote

$$u_{a,p}(x) = -\sum_{j=0}^{\infty} \exp(2\pi i p \cdot j \cdot x) = -\sum_{j=0}^{\infty} \exp(2\pi i x \cdot (M^*)^j p).$$

Consequently

$$u_a = \sum_{p \in \mathbb{Z}^n} b_p u_{a,p}.$$

From (4.1.20)

$$c_k = -\sum_{p \in \mathbb{Z}^n} b_p \int_{\mathbb{R}^n} e^{-2\pi i x \cdot (M^*)^j p} dx = -\sum_{p \in \mathbb{Z}^n} b_p \delta_{(M^*)^j p}$$

for every $k \in \mathbb{Z}^n$, where $\delta_{i,j} = 1$ if $i = j$, otherwise $\delta_{i,j} = 0$. From (4.1.21) and Proposition 4.1.17

$$|u_\ell(x)| = \left| \sum_{k \in \mathbb{Z}^n} b_p \sum_{p \in \mathbb{Z}^n} \sum_{j=0}^{\infty} \psi_\ell(2\pi k) e^{-2\pi i x \cdot k} \right| \leq L \sum_{p \in \mathbb{Z}^n} |b_p|.$$

Note that $\sum_{p \in \mathbb{Z}^n} |b_p|$ converges because $R \in C^\delta$ with $\beta > n/2$ (see for instance Grafakos [15, Theorem 3.2.16]).

**Remark 4.1.22.** Notice that

$$u = u_\omega - u_a = \sum_{j \in \mathbb{Z}} \exp(2\pi i p \cdot j \cdot x) \in \Lambda^\delta$$

is a weak solution of the equation $u \circ f - u = 0$, that is, $u$ is a $f$-invariant distribution.
4.1. Regularity on invariant foliations. Note that typically \( u_\alpha \) and \( u_\beta \) are not functions. Indeed it is well known since Livšic [23] that for a residual subset of functions \( R \in C^0 \) the cohomological equation \( R = u \circ f - u \) do not have a continuous solution \( u \) when \( f \) is an Anosov diffeomorphism. However, the distributions \( u_\omega \) and \( u_\alpha \) have directional derivatives with different regularity properties. Let \( E^s \) and \( E^u \) be the stable and unstable directions of \( M \). The weak derivative of \( u_\alpha \) with respect to \( s \in E^s \) is

\[
D_s u_\alpha (\phi) = -u_\alpha (D_s \phi) = \lim_{k \to +\infty} - \int_{\mathbb{T}^n} \sum_{j=0}^k R \circ f^j \cdot D_s \phi \, dm
\]

\[
= \lim_{k \to +\infty} - \int_{\mathbb{T}^n} \sum_{j=0}^k DR(f^j(x)) \cdot Df^j(x) \cdot s \, \phi(x) \, dm(x)
\]

If \( \lambda \in (0,1) \) satisfies \(|Df^j(x)\cdot s| \leq C\lambda^j|s|\), for every \( s \in E^s \) then

\[
|DR(f^j(x)) \cdot Df^j(x) \cdot s| = C\lambda^j|DR(f^j(x))||s|
\]

we obtain that

\[
\sum_{j=0}^\infty DR(f^j(x)) \cdot Df^j(x) \cdot s
\]

converges uniformly in \( x \) and therefore \( D_s u_\alpha \) is a continuous function. So \( u_\alpha \in \Lambda^0 \) is differentiable in the stable direction and its lack of regularity is related to the unstable direction (the "Wave-front set" of \( u_\alpha \) is in the unstable direction, see [19] chapter VIII for details). The same sort of analysis shows that \( D_w u_\omega \) is continuous, for every \( w \in E^u \). So, \( u_\omega \) is differentiable in the unstable direction and its lack of regularity is related to the stable direction.

Remark 4.1.23. Note that we are interested in the isotropic regularity of the Birkhoff sums. The logarithm Besov spaces \( B^{s}_{\infty,\infty} \) are isotropic spaces. All directions are treated in the same way. In the general case of an (nonlinear) Anosov diffeomorphisms on a compact manifold, the stable and unstable are typically just Hölder invariant foliations, so it is a more difficult setting, and it deserves further research. Anisotropic Banach spaces will certainly be quite useful here, since much more it is known on the regularity of the Koopman operator for many anisotropic Banach spaces in the literature, and consequently the "anisotropic" regularity of the solutions of the cohomological equation. See Baladi and Tsujii [6], Blank, Keller and Liverani [7], Gouëzel and Liverani [14], and Baladi [2].

4.2. Birkhoff sums as derivatives of infinitesimal conjugacies. The interest on Birkhoff sums as distributions can be motivated by the following problem. Let \( F_t \) be a \( C^\beta \)-smooth family of \( C^\beta \)-Anosov diffeomorphisms on \( \mathbb{T}^2 \), with \( \beta > 2 \), and such that \( F_0 \) is the Arnold’s cat map. Since Anosov maps are structurally stable there is a family of homeomorphisms \( H_t \) such that

\[
H_t \circ F_0 = F_t \circ H_t
\]

with \( H_0(x) = x \). It is not difficult to see that for each \( x \in \mathbb{T}^2 \) the map

\[
t \mapsto H_t(x)
\]

is smooth. If \( W = \partial F_t|_{t=0} \) and \( \alpha = \partial_t H_t|_{t=0} \) then

\[
W = \alpha \circ F_0 - DF_0 \cdot \alpha.
\]
From now on it is more convenient to consider $W$ and $\alpha$ as $\mathbb{Z}^2$-periodic functions on $\mathbb{R}^2$. Let $\pi_s: \mathbb{R}^2 \to E^s$ and $\pi_u: \mathbb{R}^2 \to E^u$ be linear projections on the stable and unstable directions of $F_0$ with $\pi_s(x) + \pi_u(x) = x$. Let $\lambda_s$ and $\lambda_u$ be the stable and unstable eigenvalues of $F_0$ (note that $\lambda_s \lambda_u = 1$) and by $\nu_s$ and $\nu_u$ the respective eigenvectors with $|\nu_s| = |\nu_u| = 1$. Using $B = (\nu_s, \nu_u)$ as a base, and $v = xv_s + yv_u = (x, y)_B$, we can write

$$\pi_s \circ W = \alpha_s \circ F_0 - DF_0 \cdot \alpha_s,$$

$$\pi_u \circ W = \alpha_u \circ F_0 - DF_0 \cdot \alpha_u,$$

that implies

$$\alpha_s(x, y) = - \sum_{k=0}^{\infty} DF_0^{k+1} \cdot \pi_s \circ W(F_0^{-k}(v)) = - \sum_{k=0}^{\infty} \lambda_s^k \pi_s \circ W(\lambda_s^{-k} x, \lambda_u^{-k} y),$$

$$\alpha_u(x, y) = - \sum_{k=0}^{\infty} DF_0^{-(k+1)} \cdot \pi_u \circ W(F_0^k(v)) = - \sum_{k=0}^{\infty} \lambda_u^{-k} \pi_u \circ W(\lambda_s^k x, \lambda_u^k y),$$

so $\alpha = \alpha_s + \alpha_u$. We call $\alpha$ the \textit{infinitesimal deformation} associated to $V$ and $F_0$. If we formally derive $\alpha$ we get

$$\partial_s \alpha(x, y) = - \sum_{k=0}^{\infty} \pi_s \circ \partial_s W(F_0^{-k}(x, y)) = - \sum_{k=0}^{\infty} \left( \frac{\lambda_s}{\lambda_u} \right)^k \pi_u \circ \partial_s W(F_0^{-k}(x, y)),$$

$$\partial_u \alpha(x, y) = - \sum_{k=0}^{\infty} \pi_u \circ \partial_u W(F_0^k(x, y)) = - \sum_{k=0}^{\infty} \left( \frac{\lambda_u}{\lambda_s} \right)^k \pi_s \circ \partial_u W(F_0^k(x, y)),$$

The first term in both expressions is a Birkhoff sum (in distinct time directions). The second term are continuous functions since $|\lambda_s/\lambda_u| < 1$. So the regularity of $\alpha$ depends on the regularity of Birkhoff sums. Theorem B implies $\partial_s \alpha, \partial_u \alpha \in \Lambda^0$, so $\alpha \in \Lambda^1$, that is, $\alpha$ is a Zygmund function.

It is an intriguing observation, up to this point limited to simple linear Anosov diffeomorphisms. One may ask if we can study the regularity of infinitesimal deformations of nonlinear Anosov diffeomorphisms using such methods. This poses new difficulties since the stable and unstable foliations are typically far less regular.

A similar study of deformations of one-dimensional piecewise expanding maps allows us to give a far more complete picture. See G.R. and S. [16] and previous results by Baladi and S. [3] [4] [5].

5. Piecewise expanding maps: Primitives of Birkhoff sums

We define $I = [a, b]$. Let $C = \{c_0, c_1, \ldots, c_n\}$, with $c_0 = a$, $c_n = b$, and $c_i < c_{i+1}$, for every $i < n$. Given $n \in \mathbb{N}$ and $\beta \in [0, 1) \cup \{BV\}$, let $\mathcal{B}^{n+\beta}(C)$ be the space of all functions

$$v: \cup_{i \in n} (c_i, c_{i+1}) \to \mathbb{C}$$

such that

- For each $i < n$, $v$ can be extended to a function $v_i: [c_i, c_{i+1}] \to \mathbb{C}$ which is $n-1$ times differentiable and $\partial^{n-1} v_i$ is absolutely continuous and its derivative is continuous for $\beta = 0$, it is $\beta$-Hölder, if $\beta \in (0, 1)$, and has bounded variation of $\beta = BV$.

Let

$$\hat{I} = [a^*, b^*] \cup \{x^*, x^- : x \in (a, b]\}.$$

Every $v \in \mathcal{B}^{n+\beta}(C)$ induces a function $v: \hat{I} \to \mathbb{C}$ defined by

$$v(x^*) = \lim_{z \to x^*} v(z),$$
where \( x \in I, \star \in \{+,-\} \) and \( x^* \in \hat{I} \).

We will denote \( \mathcal{B}^{n+0}(C) \) by \( \mathcal{B}^n(C) \). Let \( \mathcal{B}^{n+\beta}(C) \), with \( n \geq 1 \), be the set of all \( f \in \mathcal{B}^{n+\beta}(C) \) such that

- \( f \) is monotone on each interval \((c_i,c_{i+1})\), \( i < n \).
- For every \( i \) we have \( f_i|_{(c_i,c_{i+1})} \subset I \).
- There is \( \theta > 1 \) such that

\[
\min_{i \in \mathbb{N}, x \in (c_i,c_{i+1})} |Df_i(x)| \geq \theta.
\]

Note that \( f^k \in \mathcal{B}^{n+\beta}(C_k) \), for some set \( C_k \) and we can indeed define an extension \( f^k : \hat{I} \to \hat{I} \) using lateral limits. Moreover if \( v \in \mathcal{B}^{n+\beta}(C) \) then \( \nu f^k \in \mathcal{B}^{n+\beta}(C_k) \). Let \( f \in \mathcal{B}^{1+BV}(C) \).

Let \( \Lambda \) be the transfer operator of \( f \) associated with the Lebesgue measure \( m \) on \( I \). Then Lasota and Yorke \([21]\) proved that

- **(Lasota-Yorke inequality in BV)** There is \( C_9 \) and \( \lambda_1 \) such that

\[
|L\gamma|_{BV} \leq \lambda_1|\gamma|_{BV} + C_9|\gamma|_{L^1}.
\]

This implies

\[
|L^i\gamma|_{BV} \leq C_9\lambda_1^i|\gamma|_{BV} + C_{10}|\gamma|_{L^1}.
\]

for every \( i \). Let

\[
\Lambda = \{ \lambda \in \mathbb{S}^1 : \lambda \in \sigma(L) \},
\]

where \( \sigma(L) \) is the spectrum of \( L \) in \( BV \). Lasota-Yorke inequality implies that \( \Lambda \) is finite, \( 1 \in \Lambda \) and we can write

\[
L = \sum_{\lambda \in \Lambda} \lambda \Phi_\lambda + K
\]

where \( \Phi_\lambda^2 = \Phi_\lambda, \Phi_\lambda \Phi_{\lambda'} = 0 \) if \( j \neq j' \) and \( K\Phi_\lambda = \Phi_\lambda K = 0 \). Moreover

i. \( \Phi_\lambda \) is a finite rank operator.

ii. \( K \) is a bounded operator in \( BV \) whose spectral radius is smaller than one, that is, there is \( \lambda_2 \in (0,1) \) and \( C_{11} \) such that

\[
|K^j(\phi)|_{BV} \leq C_{11}\lambda_2^j|\phi|_{BV}.
\]

iii. there is \( p = p(f) \in \mathbb{N}^* \) such that for every \( \lambda \in \Lambda \) we have \( \lambda^p = 1 \).

We will use Lasota-Yorke result many times along this work. Let \( \lambda_3 = \sup_{x \in I}|Df(x)|^{-1} \).

**Lemma 5.1.27.** There is \( C_{10} \), that depends only the constants in the Lasota-Yorke inequality, such that

\[
|\Phi_\lambda(\gamma)|_{BV} \leq C_{10}|\gamma|_{L^1(m)}
\]

for every \( \gamma \in BV \).

**Proof.** It follows from (5.1.26) that

\[
\lim_{n \to \infty} \frac{1}{nP} \sum_{i=0}^{np-1} \frac{1}{\lambda_i} L^i \gamma = \Phi_\lambda(\gamma)
\]

in \( BV \). So (5.1.25) implies

\[
|\Phi_\lambda(\gamma)|_{BV} \leq C_{10}|\gamma|_{L^1}.
\]

\[\square\]
Lemma 5.1.28. Let $p = p(f)$. For every $\gamma \in BV$ and $\phi \in L^1(I)$ such that
\[
\int \phi \Phi_1(\gamma) \, dm = 0.
\]
we have
\[
\int_{j = p_l}^{p_{l+1}} \phi K^j(\gamma) \, dm = \sum_{j = p_l}^{p_{l+1}} \int \phi \sum_{j = 0}^{p_l-1} L^{k^p+j} \gamma \, dm
\]
\[
= \int \gamma \cdot \left( \sum_{j = p_l}^{p_{l+1}} \phi \circ f^{k^p+j} \right) \, dm.
\]
for every $j_1, j_2 \in \mathbb{N} \cup \{+\infty\}$, $j_1 \leq j_2$.

Proof. For every $\gamma \in BV$ we have that
\[
\int \phi \sum_{j = p_l}^{p_{l+1}} K^j(\gamma) \, dm
\]
converges, since the spectral radius of $K$ is smaller than one in $BV$. Since
\[
\sum_{j = 0}^{p_l-1} \lambda^j = 0
\]
for $\lambda \in \Lambda \setminus \{1\}$. We have
\[
\int \phi \sum_{j = p_l}^{p_{l+1}} K^j(\gamma) \, dm = \sum_{j = p_l}^{p_{l+1}} \int \phi \sum_{j = 0}^{p_l-1} K^{k^p+j}(\gamma) \, dm
\]
\[
= \sum_{j = 0}^{p_l-1} \int \phi \Phi_1(\gamma) + \sum_{j = 0}^{p_l-1} K^{k^p+j}(\gamma) \, dm
\]
\[
= \sum_{j = 0}^{p_l-1} \int \phi \sum_{j = 0}^{p_l-1} \sum_{\lambda \in \Lambda} k^{k^p+j} \Phi_1(\gamma) + \sum_{j = 0}^{p_l-1} K^{k^p+j}(\gamma) \, dm
\]
\[
= \sum_{j = 0}^{p_l-1} \int \phi \sum_{j = 0}^{p_l-1} \sum_{\lambda \in \Lambda} L^{k^p+j} \gamma \, dm
\]
\[
= \sum_{j = 0}^{p_l-1} \int \gamma \cdot \left( \sum_{j = 0}^{p_l-1} \phi \circ f^{k^p+j} \right) \, dm
\]
\[
= \int \gamma \cdot \left( \sum_{j = 0}^{p_l-1} \phi \circ f^{k^p+j} \right) \, dm.
\]

5.1. Log-Lipschitz regularity.

Theorem 5.1.30. Let $f : I \to I$ be a piecewise $C^1BV$ expanding map on the interval $I = [a,b]$. Let $p$ be a multiplier of $p(f)$. There are $C_{12}, C_{13}$ and $C_{14}$ with the following property. Let $\phi : I \to \mathbb{C}$ be a function in $L^\infty(I)$ such that
\[
\int \phi \Phi_1(\gamma) \, dm = 0
\]
for every $\gamma \in BV$. Then

**A.** For every $\gamma \in BV$ there is $i_0 \in \mathbb{N}$ such that

$$\frac{\ln|\gamma_{BV} - \ln|\gamma|_{L^1(m)}}{-\ln \lambda_1} - p(f) \leq i_0 \leq \frac{\ln|\gamma|_{BV} - \ln|\gamma|_{L^1(m)}}{-\ln \lambda_1} + p(f)$$

and

$$\left| \int \gamma \left( \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu \right| \leq C_{12} |\phi|_{L^\infty(m)} |\gamma|_{L^1}.$$

**B.** For every $\gamma \in BV$

$$\sum_{k=0}^{\infty} \left| \int \gamma \left( \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu \right| \leq C_{13} |\phi|_{L^\infty(m)} ((\ln|\gamma|_{BV} - \ln|\gamma|_{L^1(m)}) + C_{14}) |\gamma|_{L^1(m)}.$$

**C.** We have that

$$\psi(x) = \int_{1[a,x]} \left( \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu,$$

is well defined for every $x \in I$ and

$$|\psi(x) - \psi(y)| \leq C_{13} |x - y| |\ln |x - y|| + (C_{14} + \ln(2 + |I|)) |x - y|.$$

The constants $C_{13}$ and $C_{14}$ depend only on $C_8$, $m(I)$ and $|\phi|_{L^\infty}$.

**D.** Define

$$\psi_n(x) = \int_{1[a,x]} \left( \sum_{k=0}^{n} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu.$$

then for every $\beta \in (0,1)$ we have

$$\lim_{n} |\psi_n - \psi|_{C^\beta(I)} = 0.$$

**E.** For every $\gamma \in BV$ we have

$$\int \gamma d\psi = \int \gamma \left( \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu,$$

where the left-hand side is a (central) Young integral. Moreover for $\gamma \in C^\infty(I)$

$$- \int \psi \gamma d\mu = -\gamma(b) \psi(b) + \int \gamma \left( \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) d\mu$$

so we have

$$D\psi = -\psi(b) \delta_b + \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j}$$

in the sense of distributions.
Proof. By Lemma 5.1.28

\[ \int \gamma \left( \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) dm = \int \phi \sum_{j=0}^{p-1} K^{kp+j}(\gamma) dm = \int \phi \sum_{j=0}^{p-1} L^{kp+j}(\gamma) dm = \int \phi \sum_{j=0}^{p-1} L^{kp+j}(\gamma - \Phi_1(\gamma)) dm \]

for every \( k \in \mathbb{N} \). Due Lemma 5.1.27 we have

\[ |L^i(\gamma - \Phi_1(\gamma))|_{L^1(m)} \leq |\gamma - \Phi_1(\gamma)|_{L^1(m)} \leq C_{15}|\gamma|_{L^1(m)} \]

for every \( i \). Let \( i_0 = i_0(\gamma) \) be the largest \( i_0 \in p\mathbb{N} \) such that

\[ \lambda_{i_0} |\gamma|_{BV} \leq |\gamma|_{L^1(m)}. \]

Then

\[ \frac{\ln |\gamma|_{BV} - \ln |\gamma|_{L^1(m)}}{-\ln \lambda_1} - p(f) \leq i_0 \leq \frac{\ln |\gamma|_{BV} - \ln |\gamma|_{L^1(m)}}{-\ln \lambda_1} + p, \]

so

\[ \sum_{i=0}^{i_0-1} |L^i(\gamma - \Phi_1(\gamma))|_{L^1(m)} \leq C_{16}i_0 |\gamma - \Phi_1(\gamma)|_{L^1(m)} \leq C_{17} \frac{\ln |\gamma|_{BV} - \ln |\gamma|_{L^1(m)}}{-\ln \lambda_1} + p |\gamma|_{L^1(m)}. \]

and

\[ |\int \gamma \left( \sum_{k=0}^{i_0/p-1} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) dm| = \left| \int \phi \sum_{j=0}^{i_0/p-1} \sum_{j=0}^{p-1} L^{kp+j}(\gamma - \Phi_1(\gamma)) dm \right| \leq |\phi|_{L^\infty(m)} \sum_{j=0}^{i_0/p-1} \sum_{j=0}^{p-1} |L^{kp+j}(\gamma - \Phi_1(\gamma))|_{L^1(m)} \]

\[ \leq C_{17} |\phi|_{L^\infty(m)} \left( \frac{\ln |\gamma|_{BV} - \ln |\gamma|_{L^1(m)}}{-\ln \lambda_1} + p \right) |\gamma|_{L^1(m)}. \]

By (5.1.25)

\[ |L^{i_0}(\gamma - \Phi_1(\gamma))|_{BV} \leq C_9 \lambda_{i_0} |\gamma - \Phi_1(\gamma)|_{BV} + C_{10} |\gamma - \Phi_1(\gamma)|_{L^1} \leq C_{18} |\gamma|_{L^1(m)}. \]
so for every \( n \geq \frac{i_0}{p} \) and \( m \in \mathbb{N} \cup \{\infty\} \)

\[
| \int \gamma \cdot \left( \sum_{k=n}^{m} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) \, dm |
\]

\[
= | \int \phi \sum_{k=n}^{m} \sum_{j=0}^{p-1} L^{kp+j} (\gamma - \Phi_1(\gamma)) \, dm |
\]

\[
= | \int \phi \sum_{k=n}^{m} \sum_{j=0}^{p-1} L^{kp+j} \left( \gamma - \Phi_1(\gamma) \right) \, dm |
\]

\[
= \left| \int \phi \sum_{k=n}^{m} \sum_{j=0}^{p-1} K^{kp+j} L^{j} (\gamma - \Phi_1(\gamma)) \, dm \right|
\]

\[
\leq C_{11} |\phi|_{L^\infty(m)} |L^{j} (\gamma - \Phi_1(\gamma))|_{BV} \lambda_2^{n-\frac{i_0}{p}} \sum_{j=0}^{\infty} \sum_{j=0}^{p-1} \lambda_2^{kp+j}
\]

\[(5.1.37) \quad \leq C_{12} |\phi|_{L^\infty(m)} \lambda_2^{n-\frac{i_0}{p}} |\gamma|_{L^1}.
\]

Estimates (5.1.36) and (5.1.37) imply A. and B. If we choose \( \gamma = 1_{[a,x]} \) then B. implies C.

Let \( \psi_n \) be as in (5.1.33). Then

\[
\psi(x) - \psi_n(x) = \int \left| \sum_{k=n+1}^{m} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right| \, dm.
\]

and

\[
|\psi - \psi_n|_{C^0(I)} = \sup_{\delta < |I|} \sup_{x \in I} \frac{1}{\delta^p} \left| \int \left| \sum_{k=n+1}^{m} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right| \, dm \right|.
\]

Note that \( |1_{[x,x+\delta]}|_{BV} = \delta + 2 \leq 2 + |I| \) and \( |1_{[x,x+\delta]}|_{L^1(m)} = \delta \). If

\[(5.1.38) \quad np \geq \frac{i_0(1_{[x,x+\delta]})}{C_{13} \lambda_1^{i_0(1_{[x,x+\delta]})}},
\]

note that (5.1.35) gives us

\[
\delta \leq C_{19} \lambda_1^{i_0(1_{[x,x+\delta]})}
\]

for some \( C_{18} \). Let \( \lambda_4 = \max(\lambda_1^{\frac{1-\beta}{p}}, \lambda_2^{\frac{1}{p}}) \). Then (5.1.37) implies

\[
\frac{1}{\delta^p} \left| \int \left| \sum_{k=n+1}^{m} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right| \, dm \right| \leq C_{12} |\phi|_{L^\infty(m)} \lambda_2^{n-\frac{i_0}{p}} \lambda_4^{1-\beta}
\]

\[
\leq C_{20} |\phi|_{L^\infty(m)} \lambda_4^{np}.
\]

On the other hand, if (5.1.38) does not hold, then (5.1.36) and (5.1.37) imply

\[
\frac{1}{\delta^p} \left| \int \left| \sum_{k=n+1}^{m} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right| \, dm \right| \leq |\phi|_{L^\infty(m)} \delta^{1-\beta} (C_{17} \frac{\ln(2 + |I|) - \ln \delta}{-\ln \lambda_1} + p - np) + C_{12})
\]
and
\((5.1.39)\)
\[
\frac{\ln(2 + |I|) - \ln \delta}{-\ln \lambda_1} + p - np \geq 0.
\]
which is equivalent to
\[
\delta \leq (2 + |I|)\lambda_1^{p(n-1)}.
\]
So if we define \(h_n : (0, |I|] \rightarrow \mathbb{R}_+^\ast\) as
\[
h_n(\delta) = \begin{cases} 
\delta^{1-p}(C_{17}(\ln(2 + |I|)/\ln \delta) + p - np) + C_{12} + C_{20}\lambda_4^{np}, & \text{if } \delta \leq (2 + |I|)\lambda_1^{p(n-1)}, \\
C_{20}\lambda_4^{np}, & \text{otherwise.}
\end{cases}
\]
then
\[
|\psi - \psi_n|_{C^p(I)} \leq \sup_{\delta < |I|} h_n(\delta).
\]
Consequently it is easy to see that
\[
|\psi - \psi_n|_{C^p(I)} \leq C_{21}\lambda_4^{pn}.
\]
This proves D. In particular
\[
\lim_n |\psi - \psi_n|_{BV_{1/p}} = 0,
\]
for every \(\beta \in (0, 1)\), so Love-Young inequality (see Lyons, Caruana and Lévy [24, Theorem 1.16]) implies
\[
\lim_n \int \gamma \, d\psi_n = \int \gamma \, d\psi,
\]
where all integrals are central Young integrals. On the other hand, since \(\psi_n\) is absolutely continuous and \(D\psi_n \in BV\) we have
\[
\int \gamma \, d\psi_n = \int \gamma \left( \sum_{k=0}^{n} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) \, dm,
\]
and consequently
\[
\int \gamma \, d\psi = \lim_n \int \gamma \, d\psi_n = \int \gamma \left( \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) \, dm.
\]
If \(\gamma \in C^\infty(I)\) then \((5.1.34)\) follows from integration by parts for Young integrals (see Hildebrandt [17]). This concludes the proof of E. \(\square\)

**Remark 5.1.40.** Using Keller [20] generalised bounded variations spaces one could prove Theorem 5.1.30 assuming \(\gamma \in BV_q\), with \(q \geq 1\).

Define
\[
\tilde{\psi}_n(x) = \int_{1]_{x}} \left( \sum_{k=0}^{n} \phi \circ f^k \right) \, dm.
\]
If \(p(f) \neq 1\) then \(\lim_n \tilde{\psi}_n(x)\) may not exist. But their Cesàro mean does converge. For every \(\gamma \in BV\) denote
\[
T_u(\gamma) = \frac{1}{n} \sum_{u=0}^{n-1} \int \gamma \left( \sum_{k=0}^{n} \phi(f^k(u)) \right) \, du.
\]
Theorem 5.1.41. Let \( \phi : I \to \mathbb{C} \) be a function in \( L^\infty(I) \) such that
\[
\int \phi \Phi_1(\gamma) = 0
\]
for every \( \gamma \in BV \). Then for every \( \gamma \in BV \)
\[
\lim_u T_u(\gamma) = \int \gamma \sum_{n=0}^{p-1} \phi \circ f^{pn+j} \, dm + \int \phi \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1-\lambda} \Phi_\lambda(\gamma) \right) \, dm.
\]
In particular if
\[
\hat{\psi}_u(x) = \frac{1}{u} \sum_{n=0}^{u-1} \hat{\psi}_n(x),
\]
then
\[
\lim_u \hat{\psi}_u(x) = \int 1_{[a,x]} \sum_{n=0}^{p-1} \phi \circ f^{pn+j} \, dm + G(x),
\]
where
\[
G(x) = \int \phi \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1-\lambda} \Phi_\lambda(1_{[a,x]}) \right) \, dm
\]
is a Lipchitz function.

Proof. Note that for every \( \gamma \in BV \) (the manipulations with eigenvalues are as those in Broise [9])
\[
T_n(\gamma) = \int \gamma \sum_{n=0}^{u-1} \phi \circ f^k(u) \, du
\]
\[
= \int \phi \frac{1}{u} \sum_{n=0}^{u-1} \sum_{k=0}^{n} L^k \gamma \, dm
\]
\[
= \int \phi \left( \sum_{k=0}^{u-1} \left( 1 - \frac{k}{u} \right) \left( \sum_{\lambda \in \Lambda} \lambda^k \Phi_\lambda(\gamma) + K^k(1_{[a,x]}) \right) \right) \, dm
\]
\[
= \int \phi \left( \sum_{k=0}^{u-1} K^k(\gamma) \right) \, dm + \int \left( -\frac{1}{u} \sum_{k=0}^{u-1} k K^k(\gamma) + \sum_{\lambda \in \Lambda \setminus \{1\}} \left( \frac{1}{1-\lambda} + \frac{1}{u} \frac{1}{1-\lambda} - \frac{1}{u} \frac{1}{(1-\lambda)^2} \right) \Phi_\lambda(\gamma) \right) \, dm.
\]
So
\[
\lim_u T_u(\gamma) = \int \phi \left( \sum_{k=0}^{\infty} K^k(\gamma) + \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1-\lambda} \Phi_\lambda(\gamma) \right) \, dm
\]
\[
= \int \gamma \sum_{n=0}^{p-1} \phi \circ f^{pn+j} \, dm + G_\gamma,
\]
where
\[
G_\gamma = \int \phi \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1-\lambda} \Phi_\lambda(\gamma) \right).
\]
Taking \( \gamma = 1_{[a,x]} \) we conclude the proof. \( \square \)

5.2. Asymptotic variance and regularity of primitives.
5.2.1. Eigenspaces and spectral projections. Note that $f$ have a finite number of absolutely continuous ergodic probabilities $\mu_\ell = \rho_\ell m$, with $\ell \leq E$ and $\rho_\ell \in BV$, whose (pairwise disjoint) basins of attractions

$$A_\ell = \{x \in I \ s.t. \ \lim_{N \to \infty} \frac{1}{N} \sum_{k < N} \theta \circ f^k(x) = \int \theta \ d\mu_\ell, \ for \ every \ \theta \in C^0(I)\}.$$ 

covers $m$-almost every point in $I$. Let

$$S_\ell = \{x \in I \ s.t. \ \rho_\ell(x) > 0\} \subset A_\ell$$

By Boyarsky and Góra [8] the set $S_\ell$ is a finite union of intervals up to a zero $m$-measure set. So indeed

$$A_\ell = \bigcup_{n=0}^{\infty} f^{-n}S_\ell$$

up to set of zero Lebesgue measure. We have

$$\Phi_1(\theta) = \sum_{\ell \leq E} \left(\int_{A_\ell} \theta \ dm\right) \rho_\ell.$$ 

Let $\psi$ be in the image of $\Phi_A$, with $A \in \Lambda$. Then $L\psi = \lambda \psi$ and $L|\psi| = |\psi|$. So $|\psi|$ is a non negative linear combination of $\rho_\ell$, $\ell \leq E$. Replacing $\psi$ by $\psi 1_{S_\ell}$, we may assume that $|\psi|$ is a multiple of $\rho_\ell$.

Let $s(x) = \psi(x)/|\psi(x)|$ when $|\psi(x)| \neq 0$, or zero otherwise. One can see that $\lambda s(f(x)) = s(x)$ $m$-almost everywhere. Reciprocally if $s$ is a function such that either $|s(x)| = 1$ for $x \in S_\ell$, $s(x) = 0$ otherwise, and $\lambda s(f(x)) = s(x)$ almost everywhere, then $L(s\rho_\ell) = \lambda s\rho_\ell$. So define

$$E_{\lambda,\ell} = \{s: I \to \mathbb{C} \ s.t. \ supp \ s \subset S_\ell \ and \ \lambda s \circ f = s \ on \ S_\ell\}.$$ 

The ergodicity of $\mu_\ell$ implies that $E_{\lambda,\ell}$ is either zero or one-dimensional. Let $\Lambda^\ell \subset \Lambda$ be such that $\lambda \in \Lambda^\ell$ if and only dim $E_{\lambda,\ell} = 1$. We have that $\Lambda^\ell$ is a finite subgroup of $\mathbb{S}^1$.

By the previous considerations, if $\lambda \in \Lambda^\ell$, one can choose an element of $E_{\lambda,\ell}$, denoted $s_{\lambda,\ell}$, such that $|s_{\lambda,\ell}| = 1$ on $S_\ell$. Indeed if $\beta$ is a generator of the cyclic group $\Lambda^\ell$, we can choose $s_{\beta^m,\ell} = (s_{\beta,\ell})^m$ and $\{s_{\beta^m,\ell}\}_{\beta \in \Lambda^\ell}$ became a cyclic group isomorphic to $\Lambda^\ell$. In particular $s_{\lambda,\ell} s_{\lambda^2,\ell} = s_{\lambda^2,\ell}$ and $s_{1,\ell} = 1_{S_\ell}$.

If $s \in E_{\lambda,\ell}$ then $s \circ f^{[m]} = s$. So if $v$ is an ergodic component of $\mu_\ell$ for $f^{[m]}$ we have that $s$ is constant on $supp \ v$. By Boyarsky and Góra [8] the support of $v$ is a finite union of intervals up to a zero $m$-measure set. Since the support of the ergodic components of $\mu_\ell$ cover the support of $\mu_\ell$ we conclude that $s$ is piecewise constant on $S_\ell$ (and zero elsewhere) and consequently $E_{\lambda,\ell} \subset BV$. So we conclude that

$$\Phi_\lambda(BV) = \bigoplus_{\ell} (s\rho_\ell: s \in E_{\lambda,\ell}) = <s_{\lambda,\ell}\rho_\ell, \ell \leq E>.$$ 

It is convenient to consider a modification of $s_{\lambda,\ell}$. Define $\hat{s}_{\lambda,\ell}$ as equal to $s_{\lambda,\ell}$ on $S_\ell$, equals to zero outside $A_\ell$ and

$$\hat{s}_{\lambda,\ell}(x) = \lambda^n s_{\lambda,\ell}(f^n(x)),$$

where $n$ is some integer satisfying $f^n(x) \in S_\ell$. It is easy to see that $\hat{s}_{\lambda,\ell}$ is well-defined, $|\hat{s}_{\lambda,\ell}| = 1$ and $\lambda \hat{s}_{\lambda,\ell} \circ f = \hat{s}_{\lambda,\ell}$ on $I$. Of course $\hat{s}_{\lambda,\ell} \in L^\infty(m)$, but it may not belong to $BV$ anymore.

Consider the semi positive definite Hermitian form

$$<\gamma_1, \gamma_2>_{\ell} = \int_{I} \gamma_1 \overline{\gamma_2} \rho_\ell \ dm.$$
for every $\gamma_1, \gamma_2 \in BV$. Let $\phi \in BV$ be such that
\[ \int \phi \Phi_1(\gamma) \, dm = 0. \]
This is equivalent to
\[ \int \phi \rho_\ell \, dm = \int \phi \hat{s}_{1,\ell}\rho_\ell \, dm = 0 \]
for every $\ell$. Then one can find (using Gram–Schmidt process) constants $c_{\lambda,\ell}$ such that the function
\[ P_\ell(\phi) = \phi 1_{A_\ell} - \sum_{\lambda \in \Lambda \setminus \{1\}} c_{\lambda,\ell} \hat{s}_{\lambda,\ell} \]
is orthogonal to the subspace
\[ \bigoplus_{\lambda \in \Lambda} E_{\lambda,\ell} \]
with respect to this Hermitian form. So we obtained the decomposition
\[ (5.2.43) \quad \phi = \sum_{\ell} P_\ell(\phi) + \sum_{\lambda \in \Lambda \setminus \{1\}} c_{\lambda,\ell} \hat{s}_{\lambda,\ell}. \]
Let
\[ P(\phi) = \sum_{\ell} P_\ell(\phi). \]

**Lemma 5.2.44.** Let $\phi, \psi \in BV$ be such that
\[ \int \phi \Phi_1(\gamma) \, dm = 0 = \int \psi \Phi_1(\gamma) \, dm \]
for every $\gamma \in BV$. Then for every $\gamma \in BV$ the following holds.
A. For every $\phi, \gamma \in BV$
\[ \int P(\phi) \Phi_\lambda(\gamma) \, dm = 0. \]
B. For $\beta \neq \lambda$ we have
\[ \int \hat{s}_{\beta,\ell} \Phi_\lambda(\gamma) \, dm = 0. \]
C. We have
\[ \int \hat{s}_{\lambda,\ell} \Phi_\lambda(\gamma) \, dm = \int \hat{s}_{\lambda,\ell} \gamma \, dm. \]
D. For every $\phi, \psi, \gamma \in BV$ and $\lambda \in \Lambda \setminus \{1\}$ we have
\[ \int \phi \Phi_\lambda(\psi \Phi_\lambda(\gamma)) \, dm = 0. \]

**Proof.** By definition
\[ \int P_\ell(\phi) s \rho_\ell \, dm = 0 \]
for every $s \in E_{\lambda,\ell}, \lambda \in \Lambda \setminus \{1\}$. Due (5.2.42) this is equivalent to
\[ \int P(\phi) \Phi_\lambda(\gamma) \, dm = 0 \]
for every $\gamma \in BV$. This proves A.

Note that the support of $\Phi_\lambda(\gamma)$ is included in $\cup_\ell \mathcal{S}_\ell$, so
\[
\int \hat{s}_{\beta, \ell} \Phi_{\lambda}(\gamma) \, dm = \int s_{\beta, \ell} \Phi_{\lambda}(\gamma) \, dm = \lim_{N \to \infty} \frac{1}{Np} \int \sum_{i=0}^{Np-1} L^i(\gamma) \, dm
\]
\[
= \lim_{N \to \infty} \frac{1}{Np} \int \gamma \sum_{i=0}^{Np-1} s_{\beta, \ell} \circ f^i \, dm
\]
\[
= \lim_{N \to \infty} \frac{1}{Np} \int \gamma \sum_{i=0}^{Np-1} \frac{1}{(\beta \lambda)^i} \, dm.
\]
Since
\[
\lim_{N \to \infty} \frac{1}{Np} \sum_{i=0}^{Np-1} \frac{1}{(\beta \lambda)^i}
\]
is 1 if \( \beta = \lambda \) and 0 otherwise, we obtained \( B \) and \( C \).

To show \( D \), fix \( \gamma \in BV \) and \( \lambda \in \Lambda^\ell \). The function
\[
(\phi, \psi) \mapsto \int \phi \Phi_{\lambda}(\psi \Phi_{\lambda}(\gamma)) \, dm
\]
is bilinear. Applying the decomposition (5.2.43) to \( \phi \) one can see that is enough to show that the expressions

(1) \( \int P(\phi) \Phi_{\lambda}(\psi \Phi_{\lambda}(\gamma)) \, dm \),

(2) \( \int s_{\beta, \ell} \Phi_{\lambda}(\psi \Phi_{\lambda}(\gamma)) \, dm \),

with \( \beta \in \Lambda^\ell \),

are both zero. By \( A \), we have that (1) vanishes. \( B \) implies that (2) is also zero for \( \beta \neq \lambda \).

Let's consider the case \( \beta = \lambda \). Then \( C \) implies
\[
\int \hat{s}_{\lambda, \ell} \Phi_{\lambda}(\psi \Phi_{\lambda}(\gamma)) \, dm = \int \hat{s}_{\lambda, \ell} \psi \Phi_{\lambda}(\gamma) \, dm.
\]
Since \( \Phi_{\lambda}(\gamma) \) is a linear combination of elements of \( \{ s_{\lambda, j} \rho_j \}_{j \in E} \), it is enough to show that
\[
\int \hat{s}_{\lambda, \ell} \psi s_{\lambda, j} \rho_j \, dm = 0
\]
for every \( j \). This is obvious for \( j \neq \ell \), since in this case the support of \( \rho_j \) is disjoint from the support of \( \hat{s}_{\lambda, \ell} \). For \( j = \ell \) we have that \( \hat{s}_{\lambda, \ell} s_{\lambda, \ell} = 1 \) on \( S_{\ell} \), so
\[
\int \hat{s}_{\lambda, \ell} \psi s_{\lambda, \ell} \rho_\ell \, dm = \int \psi \rho_\ell \, dm = 0.
\]

5.2.2. Asymptotic variance and Livsic cohomological equation. Given a not necessarily invariant probability \( \mu \), define
\[
\sigma^2_{\mu}(\phi) = \lim_{N \to \infty} \int \left| \frac{\sum_{i=0}^{N-1} \phi \circ f^i}{\sqrt{N}} \right|^2 \, d\mu.
\]
whenever this limit exists. Note that \( \sigma^2_{\mu}(\phi) \) is similar to the usual asymptotic variance of \( \phi \), but it is not quite the same since \( m \) is not necessarily an invariant measure.

If \( \phi \) is a piecewise \( C^\beta \) function then it has finite \( 1/\beta \) bounded variation. In particular it belongs to the space of generalised bounded variations \( BV_{1/1/\beta} \) as defined by Keller [20, Theorem 3.3]. In our setting Keller proved that the transfer operator \( L \) satisfies the Lasota-Yorke inequality for the pair \( (L^1(m), BV_{1/1/\beta}) \). Consequently we can decompose \( L \) as in (5.1.26), so we keep this same notation for \( L \) acting on \( BV_{1/1/\beta} \).
By Keller [20, Theorem 3.3] (see also Broise [9]) we have that \( \sigma^2_{\mu_\ell}(\phi) \) is well-defined and

\[
\sigma^2_{\mu_\ell}(\phi) = \lim_{N \to \infty} \frac{1}{N} \int \left\{ 2 \text{Re} \left( \sum_{k=0}^{N-1} \phi f^k \phi^\ast - N|\phi|^2 \right) \right\} d\mu_\ell,
\]

\[
= 2 \text{Re} \left( \sum_{\lambda \in \Lambda \setminus \{1\}} \frac{1}{1 - \lambda} \int \phi \Phi_\lambda(x) d\mu_\ell + \sum_{i=0}^\infty \int \phi K^i(x) d\mu_\ell \right) - \int |\phi|^2 d\mu_\ell.
\]

The following result is a generalization of well-known results on asymptotic variance for invariant measures (see Broise [9]). The main difference is that we obtain regularity results in the whole phase space and not just on the support of an invariant measure.

**Theorem 5.2.46.** Let \( f \in \mathcal{B}^{1+BV}(C) \) expanding map on the interval \( I = [a, b] \). Let \( p \) be a multiplier of \( p(f) \). Let \( \phi \) and \( \psi \) be functions in \( \mathcal{B}_\beta(C) \), with \( \beta \in (0, 1) \), such that

\[
\int \phi \Phi_1(\gamma) d\mu = \int \psi \Phi_1(\gamma) d\mu = 0
\]

for every \( \gamma \in \text{BV} \). Then the limit

\[
\sigma_m(\phi, \psi) = \lim_{N \to \infty} \left\{ \frac{\sum_{i=0}^{N-1} \phi f^i}{\sqrt{N}} \right\} \left\{ \frac{\sum_{i=0}^{N-1} \psi f^i}{\sqrt{N}} \right\} d\mu
\]

exists. In particular \( \sigma^2_m(\phi) = \sigma_m(\phi, \phi) \) is well-defined and \( \sigma_m \) is a positive semidefinite Hermitian form. We also have

\[
\sigma^2_m(\phi) = \sum_\ell m(A_\ell) \sigma^2_{\mu_\ell}(\phi).
\]

Furthermore the following statements are equivalent

A. \( \sigma^2_m(\phi) = 0 \).

B. We have

\[
\sup_N \left| \sum_{i=0}^{N-1} \phi f^i \right|_{L^2(m)} < \infty.
\]

C. There is \( g \in L^2(m) \) such that \( g \) is the weak limit in \( L^2(m) \) of the sequence

\[
T_M(\phi) = -\frac{1}{M} \sum_{N=0}^{M-1} \sum_{k=0}^{N-1} \phi f^k.
\]

In particular the function

\[
\alpha(x) = \lim_{M \to \infty} -\frac{1}{M} \sum_{N=0}^{M-1} \int_{[a,x]} \left\{ \sum_{k=0}^{N-1} \phi f^k \right\} d\mu
\]

is absolutely continuous, \( 1/2 \)-Hölder continuous, and its derivative is \( g \).

D. There is \( g \in L^2(m) \) that satisfies

\[
\phi = g \circ f - g
\]

\( m \)-almost everywhere in \( I \).

E. For every \( \ell \leq E \) there is \( g_\ell \in L^\infty(m) \) such that

\[
\phi = g_\ell \circ f - g_\ell
\]

on \( S_\ell \).

Moreover A – E implies
For every periodic point $q \in \overline{I} \cap \mathcal{S}_\ell$, with $\ell \leq E$ and $f^m(q) = q$ we have

$$\sum_{j=0}^{m-1} \phi(f^j(q)) = 0.$$ 

Note that we need to consider lateral limits if $\phi$ is not continuous at some points in the orbit of $q$.

**Remark 5.2.49.** We believe that $F$ is indeed equivalent to $A-E$, but we did not manage to prove it.

**Proof.** To study $\sigma_m^2(\phi)$ we will use methods similar to the study of the usual asymptotic variance (see Broise [9]), however the non invariance of $m$ turns things a little more cumbersome. Note that

$$\int \phi \, d\mu_\ell = \int \psi \, d\mu_\ell$$

for every $\ell$. Denote

$$\sigma_{m,N}(\phi, \psi) = \int \left( \frac{\sum_{i=0}^{N-1} \phi \circ f^i}{\sqrt{N}} \right) \left( \frac{\sum_{i=0}^{N-1} \overline{\psi} \circ f^i}{\sqrt{N}} \right) \, dm.$$ 

Of course $\sigma_N$ is linear in $\phi$ and antilinear in $\psi$ and consequently it satisfies the polarization identity

$$\sigma_{m,N}(\phi, \psi) = \frac{1}{4}(\sigma_{m,N}(\phi + \psi) - \sigma_{m,N}(\phi - \psi) + i\sigma_{m,N}(\phi + i\psi) - i\sigma_{m,N}(\phi - i\psi),$$

so it is enough to show that $\sigma_m^2(\phi) = \lim_N \sigma_{m,N}(\phi)$ exists. Note that

$$\int \left| \frac{\sum_{i=0}^{N-1} \phi \circ f^i}{\sqrt{N}} \right|^2 \, dm$$

$$= \frac{1}{N} \int \sum_{i<j} \phi \circ f^i \overline{\phi} \circ f^j \, dm$$

$$= \frac{1}{N} \int \left( 2\text{Re} \left( \sum_{i<j} \phi \circ f^i \overline{\phi} \circ f^j \right) - \sum_{i<N} |\phi|^2 \circ f^i \right) \, dm$$

$$= \frac{1}{N} \int \left( 2\text{Re} \left( \sum_{i<j} \phi \circ L^j(\overline{\phi}L^i) \right) - \sum_{i<N} |\phi|^2 \circ L^i \right) \, dm$$

$$= \frac{1}{N} \int \left( 2\text{Re} \left( \sum_{i<N} \sum_{k=0}^{N-1-i} \phi \circ L^k \overline{\phi} \circ L^i \right) \right) \, dm + R_N$$

$$= \frac{1}{N} \int \sum_{i=1}^{E} m(A_\ell) \left( 2\text{Re} \left( \sum_{i<N} \sum_{k=0}^{N-1-i} \phi \circ L^k \overline{\phi} \circ L^i \right) \right) \, dm + R_N$$

(5.2.50) $$= \frac{1}{N} \sum_{i=1}^{E} m(A_\ell) \int \left( 2\text{Re} \left( \sum_{k=0}^{N-1-k} (N-k) \phi \circ f^k \overline{\phi} \right) \right) \, d\mu_\ell + R_N,$$
where

\[ NR_N = \int \left\{ 2Re \left( \sum_{i < N} \sum_{k=0}^{N-1-i} \phi \left[ \sum_{\lambda \in \Lambda^{(i)}} \tilde{\lambda}^k \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \right\} \, dm \]

\[ + \int \left\{ 2Re \left( \sum_{i < N} \sum_{k=0}^{N-1-i} \phi K^k \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right\} \, dm \]

\[ + \int \left\{ 2Re \left( \sum_{i < N} \sum_{k=0}^{N-1-i} \phi \left[ \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right] \right) \right\} \, dm \]

\[ + \int \left\{ 2Re \left( \sum_{i < N} \sum_{k=0}^{N-1-i} \phi K^k \left( \overline{\Theta} \cdot \left( K^i(1) \right) \right) \right) \right\} \, dm \]

\[ - \sum_{i < N} |\phi|^2 \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \, dm \]

\[ - \sum_{i < N} |\phi|^2 \cdot K^i(1) \, dm. \]

We have

\[ \int \left( \sum_{i < N} \sum_{k=0}^{N-1-i} \phi \left[ \sum_{\lambda \in \Lambda^{(i)}} \tilde{\lambda}^k \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \right\} \, dm \]

\[ = \int \left( \sum_{i < N} \sum_{\lambda \in \Lambda^{(i)}} \left( \sum_{k=0}^{N-1-i} \tilde{\lambda}^k \right) \phi \left[ \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \right\} \, dm \]

\[ = \int \left( \sum_{i < N} \sum_{\lambda \in \Lambda^{(i)}} \frac{1}{1-\Lambda} \left( \sum_{k=0}^{N-1-i} \tilde{\lambda}^k \right) \phi \left[ \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \right\} \, dm \]

\[ = \sum_{\lambda \in \Lambda^{(i)}} \sum_{\lambda \in \Lambda^{(i)}} \frac{1}{1-\Lambda} \left( \sum_{k=0}^{N-1-i} \tilde{\lambda}^k \right) \int \phi \left[ \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \, dm \]

\[ + \sum_{\lambda \in \Lambda^{(i)}} \sum_{\lambda \in \Lambda^{(i)}} \left( \sum_{k=0}^{N-1-i} \tilde{\lambda}^k \right) \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \, dm \]

\[ = -N \sum_{\lambda \in \Lambda^{(i)}} \frac{1}{1-\Lambda} \int \phi \left[ \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right] \, dm + O(1) \]

\[ = O(1). \]

The last passage follows from Lemma 5.2.44.D. A careful analysis of the (simpler) remaining terms of \( NR_N \) gives us

\[ NR_N = O(1). \]

Moreover note that

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{\ell=1}^{E} \sum_{\lambda \in \Lambda^{(i)}} \int \left( 2Re \left( \sum_{k=0}^{N-1} \lambda^i \Phi_{\lambda} \left( \overline{\Theta} \cdot \left( \sum_{\lambda \in \Lambda^{(i)}} \lambda^i \Phi_{\lambda}(1) \right) \right) \right) \right) \, dm = \sum_{\ell} m(A_{\ell}) |\sigma_{\mu_{\ell}}(\phi)|. \]
So (5.2.50) and (5.2.51) imply

\[
\sigma^2_m(\phi) = \lim_{M \to \infty} \frac{1}{M} \sum_{N<M} \left( \frac{\sum_{i=0}^{N-1} \phi \circ f^i}{\sqrt{N}} \right)^2 dm
\]

\[
= \sum \ell \cdot m(A_\ell) \sigma^2_{\mu_\ell}(\phi) + \lim_{M \to \infty} \frac{1}{M} \sum_{N<M} \frac{1}{1-\lambda} \sum_{\lambda \in \Lambda(1)} \int \phi \cdot \{ \Phi_\lambda(\phi \cdot (\Phi_\lambda(1))) \} dm
\]

This proves that \( \sigma^2(\phi) \) is well defined.

\( A \implies B. \) If \( A. \) holds, then \( \sigma^2_{\mu_\ell}(\phi) = 0 \) for every \( \ell \) and we can use the same method as in Broise [9, Lemma 6.2] to prove that \( B. \) holds.

\( B \implies C. \) We use the same methods as Broise [9]. Theorem 5.1.41 imply

\[
\lim_{M \to \infty} \frac{1}{M} \sum_{n=0}^{N-1} \phi \circ f^k \big( \sum_{k=0}^{N} \phi \circ f^k \big) dm
\]

for every \( \gamma \in BV. \) But \( B. \) implies

\[
\sup_M \left| \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{N} \phi \circ f^k \right|_{L^2(m)} < \infty.
\]

Since \( BV \) is dense in \( L^2(m) \) we conclude that there is \( g \in L^2(m) \) such that

\[ w - \lim_M - \frac{1}{M} \sum_{n=0}^{M-1} \sum_{k=0}^{N-1} \phi \circ f^k = g, \]

where \( w - \lim \) denotes the limit in the weak topology of \( L^2(m) \). Note that \( T_M(\phi \circ f) = T_M(\phi) \circ f. \) For every \( w \in BV \) we have

\[
\int (L \gamma - \gamma) \cdot T_M(\phi) dm = \int \gamma \cdot (T_M(\phi) \circ f - T_M(\phi)) dm
\]

\[
= \int \gamma \cdot (\frac{1}{M} \sum_{N<M} (\phi \circ f^N)) dm
\]

\[
= \int \gamma \phi dm - \frac{1}{M} \int \gamma \cdot (\sum_{N<M} \phi \circ f^N dm) dm
\]

Taking the limit on \( M \) we obtain

\[
\int \gamma (g \circ f - g) dm = \int (L \gamma - \gamma) \cdot g dm = \int \gamma \phi dm.
\]

For every \( \gamma \in BV. \) It easily follows that \( g \circ f - g = \phi. \)

\( D \implies E. \) Consider the spaces \( BV_{p,\beta} \) as in Keller [20]. Since \( \rho_\ell \in BV \) and \( \inf_{S_\ell} \rho_\ell > 0 \) (see
Boyarsky and Góra [8, Proposition 8.2.3] we have $1_{S_1} / \rho_\ell \in BV \subset BV_{1,1} \subset BV_{1,\beta}$. Since $BV_{1,\beta}$ is a Banach algebra (Saussol [27, Proposition 3.4]) we can consider the normalised transfer operator

$$P : BV_{1,\beta} \to BV_{1,\beta}$$

given by

$$P(w) = \frac{1_{S_1}}{\rho_\ell} L(w \rho_\ell).$$

Using the same argument as in Broise [9, Lemme 6.6] with the transfer operator acting on $BV_{1,\beta}$ instead of acting on $BV$ one can prove that $g \rho_\ell \in BV_{1,\beta}$. Since $1_{S_1} / \rho_\ell \in BV_{1,\beta}$ and $BV_{1,\beta}$ is a Banach algebra we get $1_{S_1} g \in BV_{1,\beta} \subset L^\infty(m)$. This completes the proof.

$E \implies A$. It is enough to show that $\sigma^2_{\rho_\ell}(\phi) = 0$ for every $\ell$. This is an easy and well-known argument.

$A - E \implies F$. Fix $\ell_0 \leq E$ and consider $g_{\ell_0}$ as in $E$. Define the function $\hat{g} : I \to \mathbb{R}$ as 1 outside $S_{\ell_0}$ and equals to $g_{\ell_0}$ inside $S_{\ell_0}$. Recall that $S_{\ell_0}$ is a finite union of intervals. Then

$$h(x) = \int 1_{(a,b)} e^g \, dm$$

is a Lipschitz function with a Lipschitz inverse. Define

$$\hat{f} : h(S_{\ell_0}) \to h(S_{\ell_0})$$

by

$$\hat{f} = h \circ f \circ h^{-1}$$

At first glance one can see that $\hat{f}$ is piecewise Lipschitz. We claim that $\hat{f}$ is indeed piecewise $C^{1+\beta}$. Note that

$$D \hat{f} (x) = Dh \circ f \circ h^{-1}(x) \cdot Df \circ h^{-1}(x) \cdot Dh^{-1}(x) = e^{g_{\ell_0} + g} h^{-1}(x) \cdot Df \circ h^{-1}(x) \cdot e^{-g_{\ell_0} + g} h^{-1}(x)$$

so $D \hat{f}$ is piecewise $C^\beta$ and its discontinuities belong to $h(C)$. Let $q \in S_{\ell_0}$ be a periodic point, $f^{m_0}(q) = q$. Choose $\delta > 0$ such that $Df^m$ do not have discontinuities on $I_0 = [q, q + \delta]$ (we can do the same argument for $I_0 = [q, q - \delta]$). Then $f^{m_0} : I_0 \to f^{m_0}(I_0)$ has an $C^{1+\beta}$ inverse, denoted by $T$. Let $I_j = T^j(I_0)$. By the mean value theorem and the expansion of $f$ there is $C_{22} > 0$ such that for every $j$

$$\frac{1}{C_{22}} |Df^{m_j}(q)| \leq \frac{|I_0|}{|I_j|} \leq C_{22} |Df^{m_j}(q)|,$$

so

$$|Df^{m_j}(q)| = \lim_{j} |I_j|^{-1/j}.$$  

Note that $\hat{f}^{m}(h(q)) = h(q)$. Since $\hat{f}$ is piecewise $C^{1+\beta}$ we can do the very same analysis considering $I_j = h(I_j)$ and conclude that

$$|D\hat{f}^{m_j}(h(q))| = \lim_{j} |h(I_j)|^{-1/j}.$$
Since $h$ and its inverse are Lipschitz there is $C_{23} > 1$ such that for every $j$
\[
\frac{1}{C_{23}} \leq \frac{h(I_j)}{I_j} \leq C_{23}.
\]
So $\lim_j |h(I_j)|^{-1/j} = \lim_j |I_j|^{-1/j}$ and consequently
$Df^{m_j}(h(q)) = Df^{m_j}(q)$. By (5.2.53) this implies
$Df^{m_j}(q) = Df^{m_j}(q)|I_j=0\exp(f^{j}(q))|$
and $F$ follows. \hfill \square

Let $\mathcal{P}^n$ be the partition of $I$ by the open intervals of monotonicity of $f^n$, that is, $J = (a, b) \in \mathcal{P}^n$ if $f^i(J) \cap C = \emptyset$ for every $i < n$ and there is $i_a, i_b < n$ such that $\{f^{i_a}(a^+), f^{i_b}(b^-)\} \subset \hat{C}$.

**Lemma 5.2.54.** There is $C_{24} > 0$ and $C_{25} > 0$ such that for every $n \in \mathbb{N}$ and $J \in \mathcal{P}^n$ we have
\[
\frac{1}{C_{24}} \leq \frac{Df^n(x)}{Df^n(y)} \leq C_{24}
\]
(5.2.55)
\[
|\ln|Df^n(x)| - \ln|Df^n(y)|| \leq C_{25}|f^n(x) - f^n(y)|.
\]
(5.2.56)
for all $x, y \in J$. Moreover if $\gamma$ is a bounded variation function with support contained in $J$ we have that for every $x \in J$
\[
\nu(L^p_1(\gamma)) \leq \frac{C_{24}}{|DF^n(x)|} \left(\nu(\gamma) + C_{25}|\gamma|_{L^\infty(m)}\right),
\]
(5.2.57)
where $\nu(g)$ denotes the variation of $g$.

### 5.3. Modulus of continuity: Statistical properties.
Indeed when $\sigma_m(\phi) > 0$ the behavior of primitives of Birkhoff sums can be very wild on at least some part of the phase space, as proved in de Lima and S. [12] for expanding maps of the circle. We extend some of those results to the setting of piecewise expanding maps.

**Theorem 5.3.58.** Let $f \in \mathcal{B}_{23p}^\beta(C)$ and $\phi \in \mathcal{B}_\beta(C)$, with $\beta \in (0, 1)$, such that
\[
\int \phi \Phi_1(\gamma) \, dm = 0
\]
for every $\gamma \in BV$. Suppose $\sigma_{\mu_\ell}(\phi) > 0$ for some $\ell \leq E$. Let
\[
\psi(x) = \int_{[a,x]} \left(\sum_{k=0}^{p-1} \sum_{j=0} h \circ f^{k+p+j}\right) \, dm.
\]
Then we have
\[
\lim_{h \rightarrow 0} \mu_\ell(x \in I: \frac{1}{\sigma_{\mu_\ell}(\phi) L_\ell \sqrt{-\log|h|}} \frac{\psi(x + h) - \psi(x)}{h} \leq y \} = \frac{1}{2\pi} \int_y e^{-x^2} \, dx.
\]
Here
\[
L_\ell = \left(\int |Df| \, d\mu_\ell\right)^{-1/2}.
\]
In particular $\psi$ is not a Lipschitz function of any measurable subset of positive measure in the support of $\mu_\ell$ and $\psi$ does not have bounded variation on the support of $\mu_\ell$. 

For $m$-almost every point $x \in I$ and $h > 0$ small we define $N(x, h)$ as the integer such that

$$\frac{1}{|Df^k(x)|} \leq |h| \leq \frac{1}{|Df^k(x)|}.$$  

We are going to need

**Proposition 5.3.59.** For every $\gamma > 0$ there is $\delta > 0$ with the following property. For every small $h_0 > 0$, $h' \leq h_0$ one can find sets $\Gamma^\delta_{h', h_0} \subset \Gamma^\delta_{h_0}$ such that

A. We have $m(\Gamma^\delta_{h_0}) \approx 1 - \gamma$.

B. If $h' \leq h$ then $\Gamma^\delta_{h', h_0} \subset \Gamma^\delta_{h, h_0}$.

C. $\lim_{h' \to 0} m(\Gamma^\delta_{h', h_0}) = m(\Gamma^\delta_{h_0})$.

D. There is $C_{26} > 1$ and $\mathcal{K} > 0$ such that for every $x \in \Gamma^\delta_{h, h_0}$ and $h < h'$ there is $N_1(x, h)$ satisfying

$$N(x, h) - \mathcal{K} \log N(x, h) \leq N_1(x, h) \leq N(x, h)$$

such that if $\omega_{x, h}$ is defined by $x \in \omega_{x, h} \in \mathcal{G}^k$ then

$$\frac{1}{C_{26} |Df^{N_1(x, h)}(y)|} \leq |\omega_{x, h}| \leq \frac{1}{|Df^{N_1(x, h)}(y)|},$$

$$|f^{N_1(x, h)}(x)|, \quad |Df^{N_1(x, h)}(x)| \geq \delta,$$

and moreover $[x, x + h]$ is in the interior of $\omega_{x, h}$.

**Proof.** The proof of this result it is quite similar to a related result in de Lima and S. [11, Proposition 4.5]. Indeed it is easier since we all dealing with the phase space instead of the parameter space as in that reference. \qed

**Proof of 5.3.58.** The proof is quite similar to the proof of the main results in de Lima and S. [11], so we detail only the main distinctions. Our setting here is actually easier (we deal with the phase space instead of the parameter space).

Let $x \in \Gamma^\delta_{h_0}$ and $h < h'$. Let $N_1(x, h)$ as in Proposition 5.3.59 and write $N_1(x, h) = ap(f) + r$, where $0 \leq r < p = p(f)$. Then $f^i|_{x, x + h} \cap C = \emptyset$ and $\phi$ is $\beta$-Hölder on $f^i|_{x, x + h}$ for every $i < ap(f)$. This implies

$$\int 1_{[x, x + h]} \sum_{M=0}^{a-1} \sum_{j=0}^{p-1} \phi \circ f^{M+p+j} \, dm = \left( \sum_{M=0}^{a-1} \sum_{j=0}^{p-1} \phi \circ f^{M+p+j} (x) \right) h + R(x, h),$$

where

$$|R_{x, h}| = \left| \int 1_{[x, x + h]}(y) \sum_{M=0}^{a-1} \sum_{j=0}^{p-1} \phi \circ f^{M+p+j}(y) - \phi \circ f^{M+p+j}(x) \, dm(y) \right|$$

$$\leq \int 1_{[x, x + h]}(y) \sum_{M=0}^{a-1} \sum_{j=0}^{p-1} |\phi \circ f^{M+p+j}(y) - \phi \circ f^{M+p+j}(x)| \, dm(y)$$

$$\leq C_{27} \left( \int 1_{[x, x + h]}(y) \sum_{M=0}^{a-1} \sum_{j=0}^{p-1} |f^{M+p+j}(y) - f^{M+p+j}(x)|^\beta \, dm(y) \right)$$

$$\leq C_{27} |h| \left( \sum_{j=0}^{ap-1} (\inf |Df^i|)^{-\beta_j} |l|^\beta \right) \leq C_{28} |h|.$$

(5.3.60)
Note that due Lemma 5.2.54 we have that there is a \( C_{29} \) such that
\[
\| v(L^{ap} 1_{[x,x+h]}) \| \leq \frac{C_{29}}{|D^f|^{ap}(x)},
\]
so we can use the same argument as in the proof of Theorem 5.1.30 to conclude that
\[
\| L^{ap} 1_{[x,x+h]} \|_{L^1(m)} = |h|,
\]
Let \( N_2(x,h) \) be the smallest integer divisible by \( p \) such that
\[
\frac{\lambda_{N_2(x,h)}}{|D^f|^{ap}(x)} \leq |h|.
\]
Then Lasota-Yorke inequality gives us
\[
|L^{ap+N_2(x,h)} 1_{[x,x+h]}| |Bv| \leq C_{30} |h|,
\]
so we can use the same argument as in the proof of Theorem 5.1.30 to conclude that
\[
\left| \int_{[x,x+h]} \sum_{j=0}^\infty \phi \circ f^{ap+N_2(x,h)} \right| = \left| \int \phi \sum_{j=0}^\infty L^{ap+N_2(x,h)} 1_{[x,x+h]} \ dm \right| \leq C_{31} |h|.
\]
(5.3.61)

Note that
\[
|h| \geq \frac{1}{|D^f N(x,h)+1(x)|} = \frac{1}{|D^f N(x,h)+1-ap f^{ap}(x)|} \|
\]
\[
1 \geq C_{32} \lambda_5 \frac{x \log N(x,h)}{|D^f|^{ap}(x)} \geq C_{32} \frac{\lambda_1}{|D^f|^{ap}(x)}.
\]

Here \( \lambda_5 = (\sup |Df|)^{-1} \). So
\[
N_2(x,h) \leq C_{33} \log N(x,h) + C_{34},
\]
and
\[
\left| \int \phi \sum_{j=0}^{N_2(x,h)-1} L^j 1_{[x,x+h]} \ dm \right| \leq (C_{33} \log N(x,h) + C_{34}) |h|.
\]
(5.3.62)

Finally note that
\[
\log N(x,h) \leq C_{35} \log |h|.
\]
(5.3.63)

Putting together the estimates (5.3.60), (5.3.61), (5.3.62) and (5.3.63) we obtain
\[
\frac{\psi(x+h) - \psi(x)}{h} = \frac{N_2(x,h)}{N} \sum_{j=0}^{N_2(x,h)-1} \phi \circ f^j (x) + O(\log \log(1/|h|))
\]
for every \( x \in \Gamma_{h/ blends}^{h'/ blends} \) and \( h < h' \). By Keller [20, Theorem 3.3] we have that \( \phi \) satisfies the Functional Central Limit Theorem, that is, if we define \( Y_\theta(x) \), with \( x \in I \), by
\[
Y_\theta(x) = \frac{1}{\sigma \mu \sqrt{N}} \sum_{j=0}^{[N\theta]-1} \phi(f^j (x)) + \frac{N\theta - [N\theta]}{\sigma \mu} \phi(f^{[N\theta]}(x)),
\]
then \( Y_\theta \) converges in distribution (considering the measure \( \mu_L \)) to the Wiener measure. Moreover one can easily verify that
\[
-\frac{N_1(x,h)}{\log |h|}
\]
converges in distribution (considering the measure $\mu_\ell$) to $L^{-1} = (\int \ln |Df| \, d\mu_\ell)^{-1}$. Using (5.3.64) and classical tools in Probability one can complete the proof of the central Limit Theorem for the modulus of continuity of $\psi$. See de Lima and S. [11, Section 5] for details. The fact that $\psi$ is not a Lipschitz function on any subset of positive measure in the support of $\mu_\ell$ also follows in the same way as a similar result there. See de Lima and S. [11, Section 9].

It is a simple exercise to show that if $\psi$ has bounded variation, then for every $\epsilon > 0$ there is a set $\Omega_\epsilon$ with $m(\Omega_\epsilon) > m(S_\ell) - \epsilon$ such that $\psi$ is Lipschitz in $\Omega_\epsilon$ (we have even a Lusin type result for bounded variation functions. See Goffman and Fon Che [13]). So $\psi$ does not have bounded variation. □

**Remark 5.3.65.** In an abstract setting of a compact metric space $X$ with a reference measure $m$ and a non-singular dynamics $f : X \to X$, one can ask if similar results holds. That is, if the "distribution" $\psi$ defined by a Birkhoff sum

$$\psi(A) = \lim_{N} \sum_{i=0}^{N-1} \int \phi \circ f^i(x) A(x) \, dm(x)$$

is well-defined when the function $A$ is the characteristic function of a ball, so can ask about the distributional limit of the random variables

$$\frac{\psi(1_{B(x,r)})}{m(1_{B(x,r)})}$$

considering the reference measure $m$, when $r$ goes to zero, after proper normalisation. Such study for the full shift and its Gibbs measures, for instance, would be interesting. This topic is somehow related (but not quite the same) to Leplaideur and Saussol [22].

### 5.4. Bounded variation regularity is rare.

One can ask if Birkhoff sums can be more regular than Log-Lipschitz. In this section, we are going to see that bounded variation regularity is very rare.

**Theorem 5.4.66.** Let $f \in \mathcal{B}^{2+\beta}(C)$, with $\beta \in (0, 1)$. Let $p$ be a multiplier of $p(f)$. Let $\phi \in \mathcal{B}^{\beta}(C)$ be such that

$$\int \phi \Phi_1(\gamma) = 0$$

for every $\gamma \in BV$. Consider

$$\psi(x) = \int 1_{[a,x]} \cdot \left( \sum_{k=0}^{p-1} \sum_{j=0}^{k} \phi \circ f^{k+j} \right) \, dm.$$

Then the following statements are equivalent

A. $\psi$ has bounded variation on each $S_\ell$, with $\ell \leq E$.

B. There is a function $g \in L^2(I)$ such that

$$\phi = g \circ f - g$$

and $g \in L^\infty(S_\ell)$ for every $\ell \leq E$.

C. $\psi$ is absolutely continuous, $1/2$-Hölder continuous on $I$, and it is Lipschitz on each $S_\ell$, with $\ell \leq E$.

Moreover if A–C hold then
D. for every periodic point \( q \in I \cap \overline{S}_\ell \), with \( \ell \leq E \) and \( f^m(p) = p \) we have

\[
(5.4.69) \quad \sum_{j=0}^{m-1} \phi(f^j(q)) = 0.
\]

Note that we need to consider lateral limits here if \( \phi \) is not continuous at some points in the orbit of \( q \).

**Remark 5.4.70.** The condition \((5.4.67)\) is satisfied in a subspace of \( \mathcal{B}(C) \) with *finite* codimension, but condition \((5.4.69)\) on all periodic points in \( \cup S_\ell \) is satisfied only in a subspace with *infinite* codimension. This justifies the claim that bounded variation regularity of \( \psi \) is "rare".

**Proof.** Of course \( C \implies A \).

\( A \implies B \) and \( C \). Suppose that \( \psi \) has bounded variation. We claim that \( \sigma_m^2(\phi) = 0 \). Indeed, suppose that this is not true. Then \( \sigma_{m_0}^2(\phi) > 0 \) for some \( m_0 \). By Theorem 5.3.58 we have that \( \psi \) does not have bounded variation on the support of \( \mu_{m_0} \). So we conclude that \( \sigma_m^2(\phi) \) must be zero. Theorem 5.2.46 says that \( \alpha \), as defined in \((5.2.48)\), is absolutely continuous on \( I \), and its derivative \( D\alpha \in L^2(m) \) satisfies \( \phi = D\alpha \circ f - D\alpha \) and \( D\alpha \in L^\infty(S_\ell) \) for every \( \ell \leq E \). Moreover \( \alpha \) is absolutely continuous, \( 1/2 \)-Hölder continuous on \( I \), and it is Lipschitz on each \( S_\ell \), with \( \ell \leq E \). By Theorem 5.4.41 we have that \( \alpha = \psi + G \) is a Lipschitz function, so \( D\alpha = D\psi + DG \in L^2(I) \) and \( D\alpha \in L^\infty(S_\ell) \) for every \( \ell \leq E \). This completes the proof.

\( B \implies C \). We have

\[
\psi(x) = \lim_{N \to \infty} \int_{1_{[a,x]}} \left( \sum_{k=0}^{N} \sum_{j=0}^{p-1} \phi \circ f^{kp+j} \right) \, dm
\]

\[
= \lim_{N \to \infty} \int_{1_{[a,x]}} (g \circ f^{(N+1)p} - g) \, dm
\]

\[
= - \int g1_{[a,x]} \, dm + \sum_{A \in \Lambda} \int g\Phi_A(1_{[a,x]}) \, dm,
\]

The boundness of \( \Phi_A : L^1(m) \to BV \) and the assumptions on \( g \) quickly implies that \( \psi \) is absolutely continuous and \( 1/2 \)-Hölder continuous on \( I \), and it is Lipschitz on each \( S_\ell \), with \( \ell \leq E \).

\( A, B \) and \( C \implies D \). We already saw that \( A \) implies \( \sigma_m^2(\phi) = 0 \). So \( D \) follows from Theorem 5.2.46.

5.5. **Zygmund regularity.** Theorem 5.1.30 tells us that the primitive \( \alpha \) of the Birkhoff sum of \( \phi \) is always Log-Lipschitz continuous. However, Theorem 5.4.66 says that it is very rare that \( \alpha \) has finite bounded variation. One can ask if Log-Lipschitz regularity is sharp. Note that for expanding maps on the circle \( \alpha \) is *always* Zygmund (see de Lima and S. [12]). If each break point is either eventually periodic or Misiurewicz, we can provide a definite answer for piecewise Hölder functions \( \phi \). Denote

\[
\mathcal{O}^*(f, y) = \{ f^n(y) : n \in \mathbb{N} \}.
\]
Theorem 5.5.71. Let \( f \in \mathcal{R}_{\exp}^k(C) \). Suppose that
\[
\inf_{c \in C} \inf_{x \in \partial^+(f; c) \setminus \hat{C}} \text{dist}(x, \hat{C}) > 0
\]
and let \( \phi \in \mathcal{H}(C) \), with \( \beta \in (0, 1) \), be such that
\[
\int \phi \Phi_1(y) = 0
\]
and for every \( c \in C \)
- either \( c \notin \partial I \) and there is \( N_{c^+}, M_{c^+} \in \mathbb{N} \) such that
  \[
  f^{M_{c^+}}(f^{N_{c^+}}(c^+)) = f^{N_{c^+}}(c^+)
  \]
  and
  \[
  \frac{1}{\ln |DF^{M_{c^+}}(f^{N_{c^+}}(c^+))|} \sum_{i=N_{c^+}+M_{c^+}}^{N_{c^+}+M_{c^+}+1} \phi(f^i(c^+))
  \]
  \[
  = \frac{1}{\ln |DF^{M_{c^+}}(f^{N_{c^+}}(c^+))|} \sum_{i=N_{c^+}}^{N_{c^+}+M_{c^+}+1} \phi(f^i(c^-))
  \]
- or \( f^i \) is continuous at \( c \) for every \( i \geq 0 \) and there is \( N_{c^+} = N_{c^-} \) such that \( f^i(c) \notin C \) for every \( i \geq N_c \).

Then
\[
\psi(x) = \int 1_{[a, x]} \cdot \left( \sum_{j=0}^{p-1} \sum_{k=0}^{N_j} \phi \circ f^{k+j} \right) \, dm
\]
is a Zygmund function, that is, there is \( C \) such that
\[
|\psi(x+h) + \psi(x-h) - 2\psi(x)| \leq C|h|
\]
for every \( x \) such that \( |x-h, x+h| \subset I \).

Proof. Let \( T \) be a multiple of the integers \( p(f), N_{c^+} \) and \( M_{c^+} \) for all \( c \in C \). Let \( F(x) = f^T(x) \) and \( G(x) = g^T(x) \). Then \( F \in \mathcal{R}_{\exp}^k(C_F) \) and
\[
\theta(x) = \sum_{i=0}^{T-1} \phi(f^i(x)).
\]
bELongs to \( \mathcal{R}_{\exp}^\beta(C_F) \) for some finite set \( C_F \), in such way that for every \( c \in C_F \)
- Type I, either we have \( F^2(c^+) = F(c^+) \) and
  \[
  \frac{\theta(F(c^+))}{\ln |D(F(F(c^+))|} = \frac{\theta(F(c^-))}{\ln |D(F(F(c^-))|},
  \]
- Type II, or \( F^i \) is continuous at \( c \) for every \( i \geq 0 \) and \( F^i(c) \notin C_F \) for every \( i \geq 1 \).

Moreover
\[
\psi(x) = \int 1_{[a, x]} \cdot \left( \sum_{k=0}^{\infty} \theta \circ F^k \right) \, dm.
\]

Denote
\[
d = \frac{1}{2} \inf_{c \in C_F} \inf_{x \in \partial^+(F; c) \setminus \hat{C}_c} \text{dist}(x, \hat{C}_c) > 0.
\]

Let \( x \in I \) and \( h \) be such that \( |x-h, x+h| \subset I \). We may assume \( |h| < d \). Then
\[
\psi(x+h) + \psi(x-h) - 2\psi(x) = \int \theta \left( \sum_{k=0}^{\infty} \lambda_k \left( \text{1}_{[x, x+h]} - 1_{[x-h, x]} \right) \right) \, dm
\]
Let \( q \) be the smallest integer satisfying
\[
F^q([x - h, x + h]) \cap C_F \neq \emptyset.
\]
We have
\[
|F^i([x - h, x + h])| \leq (\min_{z \in I} |DF(z)|)^{-i}
\]
and \( \text{supp} \ L^i(1_{[x,x+h]} - 1_{[x-h,x]}) \subset F^i([x - h, x + h]) \) for \( i < q \). Since
\[
\int |L^i(1_{[x,x+h]} - 1_{[x-h,x]})| \, dm \leq |2h|,
\]
\[
\int L^i(1_{[x,x+h]} - 1_{[x-h,x]}) \, dm = 0,
\]
and \( \theta \) is \( \beta \)-Hölder in \( I \setminus C_F \) it follows that
\[
\sum_{i=0}^{q-1} \int \theta \cdot L^i(1_{[x,x+h]} - 1_{[x-h,x]}) \, dm = O(|h|^{q-1} |F^i(x - h, x + h)|^\beta) = O(h).
\]
Define \( J = F^q([x - h, x + h]) \). Let \( c \in C_F \) be defined by
\[
|c| = F^q((x - h, x + h)) \cap C_F.
\]
Let \( f^1 \) and \( f^2 \) be the right and left connected components of \( F^q([x - h, x + h]) \setminus \{c\} \) and
\[
u(u_k) \leq \frac{C_{36}}{|DF^q(y)|}.
\]
and
\[
\frac{1}{|DF^q(y)|} \leq 2C_{24} \frac{|h|}{|J|}.
\]
Moreover
\[
|u_k| \leq 2C_{24} \frac{|h|}{|J|}.
\]
Let \( c_1 = c^+ \) and \( c_2 = c^- \).

**Case I. \( c \) is a type I point.** Let \( q_k \), with \( k = 1, 2 \) be the smallest integer such that
\[
|DF^{q_k}(c_k)| \frac{|J|^k}{} > \frac{d}{C_{24}}.
\]
This implies that $F^{q_k}$ is a diffeomorphism on $J^k$ and

$$|F^{q_k}J^k| \geq \frac{d}{C_{24}^2}.$$ 

Note also that $\text{supp} \, L^i_{F^i}(u_k) \subset F^i(J^k)$ and

$$|F^i(J^k)| \leq (\max_{z \in I} |DF(z)|)^{-i}.$$ 

for every $i \leq q_k$. It follows from (5.2.55) and (5.2.57) that

$$v(L^{q_k}(u_k)) \leq C \frac{1}{|DF^{q_k}(c_k)|} \frac{|h|}{|J|} \leq C_{37} \frac{|f^k|}{|J|} |h|.$$ 

and

$$|L^{q_k}(u_k)| \leq C \frac{1}{|DF^{q_k}(c_k)|} \frac{|h|}{|J|} \leq C_{37} \frac{|f^k|}{|J|} |h|.$$ 

for some $C_{37}$ that depends only on $f$. Consequently

$$|\sum_{i=0}^{\infty} \int \theta \cdot L^i(L^{q_k}(u_2)) \, dm| + |\sum_{i=0}^{\infty} \int \theta \cdot L^i(L^{q_k}(u_2)) \, dm| \leq C_{38} |h|.$$ 

Since $c$ is a type I critical point we have

$$q_k = -\frac{\ln |f^k|}{\ln |DF(F(c_k))|} + O(1).$$ 

We have that

$$\int L^i_{F^i}(u_k) \, dm = \int u_k \, dm$$ 

and

$$\int |L^i_{F^i}(u_k)| \, dm \leq \int |u_k| \, dm \leq 2|h|$$ 

holds for every $i \leq q_k$. Then

$$\int \theta \sum_{i=0}^{q_k} L^i_{F^i}(u_k) \, dm$$

$$= \sum_{i=0}^{q_k} \left( \theta(F^i(c_k)) \int L^i_{F^i}(u_k) \, dm + \int (\theta(x) - \theta(F^i(c_k))) L^i_{F^i}(u_k)(x) \, dm(x) \right)$$

$$= q_k \theta(F(c_k)) \int u_k \, dm + O(2|h|) \left( 1 + \sum_{i=0}^{q_k} |F^i(J^k)|^i \right)$$

$$= q_k \theta(F(c_k)) \int u_k \, dm + O(|h|)$$

$$= -\frac{\theta(F(c_k))}{\ln |DF(F(c_k))|} \ln |J^k| \int u_k \, dm + O(|h|),$$

(5.5.77)
consequently
\[
\int \theta \sum_{i=0}^{q_1} L_i^J(u_1) \, dm + \int \theta \sum_{i=0}^{q_2} L_i^J(u_2) \, dm
= -\frac{\theta(f^1(c_i))}{\ln|DF(F(c_i))|} \left[ \ln|f^1| \int u_1 \, dm + \ln|J^2| \int u_2 \, dm \right] + O(|h|).
\]
\[
= -\frac{\theta(f^1(c_i))}{\ln|DF(F(c_i))|} \left( \int u_1 \, dm \right) \left( \ln|f^1| - \ln \left( \frac{|f^1|}{|J|} \right) \right) + O(|h|).
\]
If \(Q_1\) and \(Q_2\) are the connected components of \(J \setminus \{F^q(x)\}\) then
\[
\frac{1}{C_{24}} \leq \frac{|Q_1|}{|Q_2|} \leq C_{24},
\]
so
\[
\frac{1}{1 + C_{24}} \leq \frac{|Q_1|}{|J|} \leq \frac{C_{24}}{1 + C_{24}}.
\]
In particular if
\[
\frac{1}{1 + C_{24}} \leq \frac{|f^1|}{|J|} \leq \frac{C_{24}}{1 + C_{24}}
\]
we have
\[
\left( \int u_1 \, dm \right) \left( \ln \frac{|f^1|}{|J|} - \ln \left( 1 - \frac{|f^1|}{|J|} \right) \right) = O(|h|).
\]
If (5.5.78) does not hold then
\[
\left| \left( \int u_1 \, dm \right) \left( \ln \frac{|f^1|}{|J|} - \ln \left( 1 - \frac{|f^1|}{|J|} \right) \right) \right|
\leq 2C_{24}|h| \min \left\{ \frac{|f^1|}{|J|}, 1 - \frac{|f^1|}{|J|} \right\} \left| \left( \ln \frac{|f^1|}{|J|} - \ln \left( 1 - \frac{|f^1|}{|J|} \right) \right) \right|
\leq 2C_{24}|h| \sup_{0 < t < 1} \min \{t, 1 - t\} |\ln t - \ln(1 - t)|,
\]
and (5.5.79) holds. So in every case
\[
\int \theta \cdot \sum_{i=0}^{q_1} L_i^J(u_1) \, dm + \int \theta \cdot \sum_{i=0}^{q_2} L_i^J(u_2) \, dm = O(h).
\]
\textit{Case II}. \(c\) is a type II point. Suppose \(|f^1| \geq |f^2|\) (the other case is analogous). Let \(q_1\) be the smallest integer such that
\[
\max_{z \in I} |DF(z)| |DF^{q_1}(F(c))| |f^1| > \frac{d}{C_{24}}.
\]
Then \(F^{q_1}\) is a diffeomorphism on \(f^1\) and \(f^2\). Let \(q_2 = q_1\). It follows from (5.2.55) and (5.2.57) that
\[
|nu^q_k(u_k)| \leq C \frac{|h|}{|DF^{q_k}(c_k)| |J|} \leq C_{39} \frac{|f^1|}{|J|} |h|.
\]
and
\[
|L^{q_k}(u_k)| \leq C \frac{|h|}{|DF^{q_k}(c_k)| |J|} \leq C_{39} \frac{|f^1|}{|J|} |h|.
\]
for some \(C_{39}\) that depends only on \(f\) and \(i = 1, 2\). We conclude that
\[
\sum_{i=0}^{\infty} \left| \int \theta \cdot L^J(u_1) \, dm \right| + \sum_{i=0}^{\infty} \left| \int \theta \cdot L^J(u_2) \, dm \right| \leq C_{40} |h|.
\]
Note also that $\text{supp } L^j_F(u_k) \subset F^i(J^k)$ and (5.5.74), (5.5.75), (5.5.76) hold for $i \leq q_k$. Consequently
\[
\int \theta \sum_{i=0}^{q_k} L^j_F(u_k) \, dm \\
= \sum_{i=0}^{q_k} \left\{ \theta(F^i(c_k)) \int L^j_F(u_k) \, dm + \int \left( \theta(x) - \theta(F^i(c_k)) \right) L^j_F(u_k)(x) \, dm(x) \right\} \\
= \theta(c_k) \int u_k \, dm + \sum_{i=1}^{q_k} \theta(F^i(c)) \int u_k \, dm + O(|h|) \\
= \sum_{i=1}^{q_k} \theta(F^i(c)) \int u_k \, dm + O(|h|).
\]
and (5.5.73) implies
\[
\int \theta \sum_{i=0}^{q_1} L^j_F(u_1) \, dm + \int \theta \sum_{i=0}^{q_2} L^j_F(u_2) \, dm \\
= \sum_{i=1}^{q_1} \theta(F^i(c))(\int u_1 \, dm + \int u_2 \, dm) + O(|h|). \\
= O(|h|).
\]
To conclude the proof of the theorem, note that
\[
(5.5.82) \quad \psi(x + h) + \psi(x - h) - 2\psi(x) \\
= \int \theta \cdot \left( \sum_{i=0}^{q_1-1} L^j_F(1_{[x,x+h]}) - 1_{[x-h,x]} \right) \, dm \\
+ \int \theta \cdot \sum_{i=0}^{q_1-1} L^j_F(u_1) \, dm + \int \theta \cdot \sum_{i=0}^{q_2-1} L^j_F(u_2) \, dm \\
+ \int \theta \cdot \sum_{i=0}^{q_1} L^j_F(L^{q_1} u_1) \, dm + \int \theta \cdot \sum_{i=0}^{q_2} L^j_F(L^{q_2} u_2) \, dm
\]
and apply the previous estimates. \hfill \Box

**Theorem 5.5.83.** Let $f \in \mathcal{R}_{\text{exp}}^0(C)$ be a piecewise expanding map. Suppose that there exists $c \in C \setminus \partial I$ satisfying
\[
\inf_{x \in \partial^{\gamma+}(f^c)} \text{dist}(x, \hat{C}) > 0
\]
and there are $N_{c^+}, M_{c^+} \in \mathbb{N}$ with $f^{M_{c^+}}(f^{N_{c^+}}(c^+)) = f^{N_{c^+}}(c^+)$. Let $\phi \in \mathcal{B}^0(C)$, with $\beta \in (0, 1)$, such that
\[
\int \phi \Phi_1(y) = 0
\]
for every $\gamma \in \text{BV}$ but
\[
\frac{1}{\ln |Df^{M_{c^+}}(f^{N_{c^+}}(c^+))|} \sum_{i=N_{c^+}}^{N_{c^+}+M_{c^+}-1} \phi(f^i(c^+)) \\
\neq \frac{1}{\ln |Df^{M_{c^-}}(f^{N_{c^-}}(c^-))|} \sum_{i=N_{c^-}}^{N_{c^-}+M_{c^-}-1} \phi(f^i(c^-))
\]
Then
\[ \psi(x) = \int 1_{[-1,1]} \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\mu-1} \phi \circ f^{k+p+j} \right) dm \]
is not a Zygmund function.

**Proof.** Let \( T \) be a common multiple of the integers \( p(f), N, \) and \( M \). Let \( F(x) = f^T(x) \). Define \( \theta \) as in the proof of Theorem 5.5.71. Then we have \( F^2(c^\pm) = F(c^\pm) \) and
\[
\frac{\theta(F(c^+))}{\ln|DF(F(c^+))|} \neq \frac{\theta(F(c^-))}{\ln|DF(F(c^-))|}.
\]
Let\[
d = \frac{1}{2} \inf_{x \in \mathbb{E}^+ \setminus \mathbb{E}^-} \text{dist}(x, \mathcal{L}^s) > 0.
\]
Using the same notation as in the proof of Theorem 5.5.71 take \( x = c \) and \( 0 < h < d \).
Then \( q = 0, f = [c - h, c + h], c_1 = c^+, c_2 = c^-, |f^1| = |c, c + h|, f^2 = [c - h, c], u_1 = 1_{[c, c + h]} \) and \( u_2 = -1_{[c - h, c]} \) we can write (5.5.82). Since (5.5.73), (5.5.81) and (5.5.77) holds
\[
\psi(c + h) - \psi(c) + 2\psi(c) = \left( \frac{\theta(F(c^-))}{\ln|DF(F(c^-))|} \right) |h| \ln|h| \ dm + O(|h|),
\]
so \( \psi \) is not Zygmund.

**5.6. Invariant distributions which are not measures.** An interesting application of Birkhoff sums as distributions is the construction of distributions in \( \Theta \in BV^* \) which are \textit{invariant} with respect to a certain piecewise expanding map \( f \in \mathcal{B}_f \exp(C) \), that is\[
\Theta(g) = \Theta(g \circ f)
\]
for every \( g \in BV \), but that are \textit{not} signed measures. Choose \( \phi \in \mathcal{B}_f^\beta(C) \cap BV \) such that\[
\int \phi \Phi_1(\gamma) \ dm = 0
\]
for every \( \gamma \in BV \). Consider the distribution \( \Theta_\phi \in BV^* \) given by\[
\Theta_\phi(g) = \sigma_\phi(g, \overline{\phi}) = \sum_\ell m(A_\ell) \sigma_{\mu_\ell}(g, \overline{\phi}),
\]
where \( \sigma_m \) is the hermitian form defined in Theorem 5.2.46.

The following result tells us that it is quite rare that \( \Theta_\phi \) is a signed measure.

**Theorem 5.6.84.** The functional \( \Theta_\phi \in BV^* \) is a \( f \)-invariant distribution. The following statements are equivalent
\begin{enumerate}
\item \( \Theta_\phi \) is a signed measure, that is, there is a signed regular measure \( \nu \) such that for every \( \psi \in BV \cap C^0(I) \)
\[ \Theta_\phi(\psi) = \int \psi \ dv. \]
\item \( \phi = \psi \circ f - \psi \), where \( \psi \in L^2(m) \) and \( \psi \in L^\infty(S_\ell) \) for every \( \ell \leq E \).
\item \( \Theta_\phi = 0. \)
\end{enumerate}

Moreover A. – C. implies
\begin{enumerate}
\item We have that
\[ \sum_{j=0}^{M-1} \phi(f^j(q)) = 0 \]
holds for every \( M \) and \( q \in \mathcal{S}_\ell \), with \( \ell \leq E, \) such that \( f^M(q) = q. \)
\end{enumerate}
Furthermore if $f$ is markovian, $p(f) = 1$ and it has an absolutely continuous ergodic invariant probability whose support is $I$ then $D$ is equivalent to $A$. – $C$.

Proof. Note that $\sigma^2_m(g \circ f - g) = 0$. By the Cauchy-Schwarz inequality

$$|\sigma_m(g \circ f, \phi) - \sigma_m(g, \phi)|^2 = |\sigma_m(g \circ f - g, \phi)|^2 \leq \sigma^2_m(g \circ f - g)\sigma^2_m(\phi) = 0,$$

so $\Theta_\phi$ is $f$-invariant. Using exactly the same argument as in Broise [9, Chapter 6] we obtain

$$\sigma_{\mu_{\ell_0}}(\gamma, \phi)$$

$$= \lim_{N \to \infty} \frac{1}{N} \int \left( \sum_{i=0}^{N-1} \gamma \circ f \right) \left( \sum_{i=0}^{N-1} \phi \circ f \right) \rho_{\ell_0} \, dm$$

$$= -\int \gamma \phi \rho_{\ell_0} \, dm + \lim_{N \to \infty} \frac{1}{N} \left( \int \gamma \phi \circ f \, dm + \int \gamma \circ f \phi \rho_{\ell_0} \, dm \right)$$

$$= -\int \gamma \phi \rho_{\ell_0} \, dm + \sum_{\lambda \in \Lambda_1(1)} \frac{1}{1 - \lambda} \int_{\lambda} \phi_{\lambda}(\gamma \rho_{\ell_0}) \rho_{\ell_0} \, dm$$

$$= \sum_{\lambda \in \Lambda_1(1)} \frac{1}{1 - \lambda} \int_{\lambda} \phi_{\lambda}(\gamma \rho_{\ell_0}) \rho_{\ell_0} \, dm$$

for every $\gamma \in BV$, $\ell_0 \leq E$. One can see that

$$\gamma \to \sigma_{\mu_{\ell_0}}(\gamma, \phi) - \sum_{i=0}^{\infty} \int K^i(\gamma \rho_{\ell_0}) \phi \, dm$$

is a bounded functional considering the $L^1(m)$ norm in its domain $BV$. Since $K$ is a contraction on $BV$ and $BV$ is a Banach algebra it follows that

$$\sum_{i=0}^{\infty} \int K^i(\gamma \rho_{\ell_0}) \phi \, dm$$

belongs to $BV^*$, so consequently

$$\gamma \to \sigma_m(\gamma, \phi)$$

is in $BV^*$. By Theorem 5.1.30.B we have that there are $C_{41}, C_{42}$ such that

$$\sigma_{\mu_{\ell_0}}(\gamma, \phi) \leq C_{41}(|\gamma|_{BV} - |\gamma|_{L^1(m)}) + C_{42} |\gamma|_{L^1(m)}$$

for every $\ell_0 \leq E$, so

$$\sigma_m(\gamma, \phi) \leq C_{41}(|\gamma|_{BV} - |\gamma|_{L^1(m)}) + C_{42} |\gamma|_{L^1(m)}.$$  

Of course $C. \implies A.$

$A. \implies B. and D$. Suppose that $\Theta_\phi$ is a signed measure, that is, there is a signed regular measure $\nu$ such that for every $\psi \in BV \cap C^0(I)$

$$\Theta_\phi(\psi) = \int \psi \, d\nu.$$  

Choosing $\gamma = 1_{[x,y]}$ it is easy to see that there is a sequence $\gamma_k \in BV \cap C^0(I)$ such that $\sup_k |\gamma_k|_{BV} < \infty$, $\lim_k \gamma_k(z) = 1_{[x,y]}(z)$ for every $z$. In particular $\lim_k |\gamma - \gamma_k|_{L^1(m)} = 0$ and consequently (5.6.86) implies

$$\Theta_\phi(1_{[x,y]}) = \lim_k \Theta_\phi(g_k) = \lim_k \int g_k \, d\nu = \int 1_{[x,y]} \, d\nu.$$
A similar argument shows that
\[ y \mapsto \Theta_\phi(1_{[x,y]}) \]
is continuous. So \( \nu \) does not have atoms and \( \Theta_\phi(1_{[x,y]}) \) is continuous and it has bounded variation with respect to \( \gamma \).

Given \( \gamma \in BV \) there is a sequence \( \gamma_k \in BV \cap C^0(I) \) such that \( \sup_k |\gamma_k|_{BV} < \infty \) and \( \lim_k \gamma_k(z) = \gamma(z) \) for every \( z \) except for \( z \) in the set of discontinuities of \( \gamma \), that is countable. Using an argument similar to the argument with \( \gamma = 1_{[x,y]} \), and using that \( \nu \) does not have atoms, we can conclude that
\[ \Theta_\phi(\gamma) = \int \gamma \, d\nu \]
for every \( \gamma \in BV \).

In particular for each \( x \in S_{\ell_0} \), with \( \ell_0 \leq E \) and \( y \) such that \( [x,y] \subset S_{\ell_0} \) we can choose \( \gamma = 1_{[x,y]} \rho_{\ell_0}^{-1} \in BV \) and we can prove that
\[ y \mapsto \Theta_\phi(1_{[x,y]} \rho_{\ell_0}^{-1}) \]
is continuous and it has bounded variation. We have
\[ \Theta_\phi(1_{[x,y]} \rho_{\ell_0}^{-1}) = m(A_{\ell_0}) \sigma_{\mu_{\ell_0}}(1_{[x,y]} \rho_{\ell_0}^{-1}, \phi), \]
so \( y \mapsto \sigma_{\mu_{\ell_0}}(1_{[x,y]} \rho_{\ell_0}^{-1}, \phi) \) is a bounded variation function on an interval. One can see that
\[ y \mapsto \sigma_{\mu_{\ell_0}}(1_{[x,y]} \rho_{\ell_0}^{-1}, \phi) = \sum_{i=0}^\infty K^i(1_{[x,y]}) \phi \, dm \]
is a Lipschitz function, so we conclude that
\[ y \mapsto u(y) = \sum_{i=0}^\infty K^i(1_{[x,y]}) \phi \, dm = \sum_{N=0}^p \sum_{j=0}^{p-1} L^{Np+j}(1_{[x,y]}) \phi \, dm \]
has bounded variation. By Theorem 5.4.66 this implies that (5.6.85) holds for every \( q \) and \( m \) such that \( f^m(q) = q \).

**B. \implies C.** Note that if \( B. \) holds, then \( \sigma_m^2(\phi) = \sigma_m^2(\overline{\phi}) = 0 \) and the Schwartz inequality for the hermitian form \( \sigma \) implies
\[ |\Theta_\phi(g)| = |\sigma_m(g, \overline{\phi})| \leq \sigma_m(g) \sigma_m(\overline{\phi}) = 0, \]
so \( C. \) holds.

**D. \implies A., B., C. under additional assumptions** The markovian property of \( f \) and the mixing property of the unique invariant absolutely continuous probability implies that if \( D. \) holds we can use the Livsic-type result for subshifts of finite type as in Parry and Pollicott [26, Proposition 3.7] to conclude that there is a piecewise Hölder continuous function \( \psi \) satisfying \( \phi = \psi \circ f - \psi \) on \( I \). So \( B. \) holds.

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