Quantifying algebraic asymmetry of Hamiltonian systems

Hui-Hui Qin, Shao-Ming Fei and Chang-Pu Sun

1 Beijing Computational Science Research Center, Beijing 100193, People’s Republic of China
2 School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China
3 Max-Planck-Institute for Mathematics in the Sciences, Leipzig 04103, Germany
4 Graduate School of China Academy of Engineering Physics, Beijing 100193, People’s Republic of China

E-mail: cpsun@csrc.ac.cn and feishm@mail.cnu.edu.cn

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Abstract

Symmetries play important roles in physical systems. We study the symmetries of a Hamiltonian system by investigating the asymmetry of the Hamiltonian with respect to certain algebras. We define the asymmetry of an operator with respect to an algebraic basis in terms of their commutators. Detailed analysis is given to the Lie algebra $su(2)$ and its $q$-deformation. The asymmetry of the $q$-deformed integrable spin chain models is calculated. The corresponding geometrical pictures with respect to such asymmetry is presented.

Keywords: asymmetry degree of Hamiltonian systems, quantum harmonic and deformed harmonic systems, integrable spin chain

(Some figures may appear in colour only in the online journal)

1. Introduction

The symmetry of a physical system plays a central role in a broad range of problems in physics [1–3]. In Nöthers theorem [4] symmetries are used to find the corresponding conservation laws of dynamics. The symmetries of a Hamiltonian also help to obtain the solutions of the evolution equations, as well as the degeneracy of the energy levels of the physical systems [2, 5]. Some important internal symmetries such as the identity principle of micro-particles have been also revealed [6]. In studying the integrable models the theory of Lie groups and Lie algebras [2, 7, 8], as well as the quantum groups [9–11] and quantum algebras [12–14] has been effectively used.
The quantification of the symmetry or asymmetry of a quantum system is of significance. The asymmetry of quantum states with respect to groups has been studied in the view of resource, such as coherence [15–21]. In [22] the cost of symmetrization of quantum states has been investigated. In [23–25] the authors considered the asymmetry measure of a Hamiltonian with respect to Lie group transformations, aiming at revealing some phenomena such as symmetry and spontaneous symmetry breaking, and quantum phase transition caused by the symmetry changes of Hamiltonian systems.

It is said that an operator $H$ is symmetric with respect to a given algebra $\mathfrak{g}$, if it commutates with all the generators of $\mathfrak{g}$. An example of such $H$ is the Casimir operator of $\mathfrak{g}$. However, if $H$ does not commutate with all the generators of $\mathfrak{g}$, one needs to assure the degree of asymmetry of $H$ with respect to $\mathfrak{g}$. Geometrically, one may ask what the degree of asymmetry of a deformed sphere, say ellipsoid, with respect to the standard sphere is.

In this paper we study the asymmetry of an operator with respect to algebras including the usual Lie algebras, Heisenberg algebras [27], Lorentz algebras [28] and Hopf algebras [26] etc.. We give a definition of the asymmetry degree, and analyze in detail the systems with $SU(2)$ symmetry and the deformed $SU_q(2)$ symmetry. The harmonic oscillator systems and the integrable spin chain systems are also taken into account. The explicit geometrical pictures associated with the asymmetries are presented.

2. Asymmetry of operators with respect to an algebraic basis

Denote $\{X_j\}$ the basis of an algebra $\mathfrak{g}$. The symmetry of an operator $H$ with respect to $\mathfrak{g}$ is judged by the commutation relations between $H$ and the basis $\{X_j\}$: $[H, X_j] = 0, \forall j$. If there exists $X_j$ such that $[H, X_j] \neq 0$, $H$ is not symmetric with respect to $\mathfrak{g}$. We quantify the non-commutative extent between $H$ and $X_j$ to characterize the degree of asymmetry of $H$ with respect to $\mathfrak{g}$. Denote $\|O\|_2 = \text{tr}(O^\dagger O)$ the Frobenius norm of operator $O$. We define the asymmetry degree of the operator $H$ with respect to $\mathfrak{g}$ as follows,

$$A(\mathfrak{g}, H) = \frac{1}{\|H\|_2^2} \sum_j \|[H, X_j]\|_2^2,$$

where $\tilde{H} = H - \frac{\text{tr}(H)}{d} I$, $d$ is the order of the operator $H$ and $I$ is the identity operator of $d$ orders.

From the definition (1), $A(\mathfrak{g}, H)$ has the following properties:

(I). $A(\mathfrak{g}, H) \geq 0$, the equality holds if and only if $[H, X_j] = 0, \forall j$. Namely, $A(\mathfrak{g}, H) = 0$ if and only if $H$ commutates with $X_j$ for all $j$. We call $H$ symmetric with respect to $\mathfrak{g}$, or $H$ is $\mathfrak{g}$-symmetric in this case.

(II). $A(\mathfrak{g}, H) \geq A(\mathfrak{g}, THT^{-1})$, where $T$ is an invertible transformation which commutes with $\mathfrak{g}$, $[T, X_j] = 0 \forall j$. This property can be seen as following,

$$A(\mathfrak{g}, THT^{-1}) = \sum_j \text{tr}([X_j, THT^{-1}]^\dagger [X_j, THT^{-1}])$$

$$= \sum_j \text{tr}([X_j, H]^\dagger (T^\dagger T)[X_j, H](T^\dagger T)^{-1})$$

$$\leq \sum_j \text{tr}([X_j, H]^\dagger [X_j, H]) = A(\mathfrak{g}, H).$$
Therefore, the asymmetry degree of $H$ is monotonically non-increasing under $g$-symmetric transformations.

(III). $\mathcal{A}(g, H + \lambda I) = \mathcal{A}(g, H)$ for arbitrary parameter $\lambda \in \mathbb{R}$, which implies that $\mathcal{A}(g, H)$ is independent of constant potential energy, i.e. the asymmetry keeps invariant under shifting the ground energy of the Hamiltonian $H$ of the system.

We now compute several examples in detail.

2.1. The harmonic and deformed harmonic systems

For a two linear oscillator system, the Hamiltonian has the form,

$$H = a_1^\dagger a_1 + a_2^\dagger a_2,$$

where $a_j$ and $a_j^\dagger$, $j = 1, 2$, are the creation and annihilation operators [3], satisfying the commutation relations $[a_j, a_k^\dagger] = \delta_{jk} I$, $j, k = 1, 2$.

Underlying the Hamiltonian system (3) is the $\mathfrak{su}(2)$ algebra, with the generators $J_1, J_2$ and $J_3$ satisfying

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad i, j, k = 1, 2, 3$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol. In terms of the creation and annihilation operators, $J_+ = J_1 + iJ_2$, $J_- = J_1 - iJ_2$, and $J_3$ satisfy

$$J_+ = J_1 + iJ_2 = a_1^\dagger a_2, \quad J_- = J_1 - iJ_2 = a_2^\dagger a_1,$$

$$J_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2) = \frac{1}{2}(N_1 - N_2),$$

where $N_j = a_j^\dagger a_j$ is the number operator defined by the creation and annihilation operators $a_j^\dagger$ and $a_j$, $(j = 1, 2)$. It is easily verified that the system $H$ has $\mathfrak{su}(2)$ symmetry, $[H, J_i] = 0$ for $i = 1, 2, 3$.

In addition to the usual Lie algebras, one also has the universal enveloping algebras, the quantum (q-deformed) Lie algebras [29, 30]. With respect to the Lie algebra $\mathfrak{su}(2)$, the quantum algebra $\mathfrak{su}_q(2)$ is given by the generators $J_\pm = J_i \pm iJ_i^\prime$ and $J_3 = J_3^\prime$, satisfying the following algebraic relations,

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_\pm, J_\mp] = [2J_3]_q,$$

where $[O]_q = \frac{q^O - q^{-O}}{q - q^{-1}}$.

Correspondingly, one has the $q$-deformed quantum harmonic oscillators [11] and the irreducible representation of $\mathfrak{su}_q(2)$ [12–14, 31, 32]. The $q$-deformed quantum harmonic oscillators are given by $b_j^\dagger$ and $b_j$ satisfying

$$b_j b_k^\dagger - q^{-1} b_j^\dagger b_k = q^N \delta_{jk}, \quad j, k = 1, 2,$$

where $N_j = a_j^\dagger a_j$, $j = 1, 2$.

From (7) one has the representations,

$$J_\pm = b_1^\dagger b_2, \quad J_- = b_2^\dagger b_1,$$

$$J_3 = \frac{1}{2}(N_1 - N_2).$$

$$J_\pm = b_1^\dagger b_2, \quad J_- = b_2^\dagger b_1,$$
It is straightforward to verify that the q-deformed Hamiltonian $H'$,
\begin{equation}
H' = b_1^\dagger b_1 + b_2^\dagger b_2,
\end{equation}
is $su_q(2)$-symmetric, i.e. $[H', J_i^\pm] = 0$ for $i = 1, 2, 3$.

However, the q-deformed harmonic Hamiltonian $H'$ is no longer $su(2)$ symmetric. In order to calculate the asymmetry degree of $H'$ with respect to $su(2)$, consider the Fock representations of $\{a^\dagger, a, N\}$ and $\{b^\dagger, b, N\}$ [12–14], respectively,
\begin{align*}
|m\rangle &= |m_1, m_2\rangle = \left(\frac{a_1^{m_0} a_2^{m_2}}{\sqrt{m_1! \sqrt{m_2!}}}\right)|0\rangle, \\
|m'\rangle &= |m_1, m_2\rangle' = \left(\frac{b_1^{m_0} b_2^{m_2}}{\sqrt{m_1! \sqrt{m_2!}}}\right)|0\rangle,
\end{align*}
where the vacuum state $|0\rangle$ satisfies $a_j|0\rangle = b_j|0\rangle = 0, \quad j = 1, 2$, and $|m_i\rangle = |m_i|_q = (|m_i|_q - 1)\cdots |2|_q [1]_q$.

The generators of $su(2)$ have the following representations,
\begin{align}
J_+ |m\rangle &= a_1^\dagger a_2 |m\rangle = \sqrt{(m_1 + 1)m_2}|m_1 + 1, m_2 - 1\rangle; \\
J_- |m\rangle &= a_1^\dagger a_2 |m\rangle = \sqrt{(m_2 + 1)m_1}|m_1 - 1, m_2 + 1\rangle; \\
J_3 |m\rangle &= \frac{1}{2}(N_1 - N_2)|m\rangle = \frac{1}{2}(m_1 - m_2)|m_1, m_2\rangle.
\end{align}

Since $N_j |m\rangle = m_j |m\rangle$ and $N_j |m\rangle' = m_j |m\rangle'$, the Hamiltonian $H'$ can be also represented as
\begin{equation}
H' |m\rangle = (b_1^\dagger b_1 + b_2^\dagger b_2)|m\rangle = (|m_1\rangle + |m_2\rangle)|m_1, m_2\rangle.
\end{equation}

On a finite dimensional subspace with basis $\{|m| = |m_1, m_2\rangle|m_1 + m_2 = M|1 < M < \infty\}$, we have that
\begin{align*}
[H', J_+] |m\rangle &= \begin{cases} \sqrt{(m_1 + 1)m_2}|m_1 + 1, m_2 - 1\rangle & m_2 > 0, m_1 = 0, \\
0 & m_1 = m_2 = 0, \\
\sqrt{m_1(m_2 + 1)}|m_1 - 1, m_2 + 1\rangle & m_1 > 0, m_2 = 0, \\
\end{cases} \\
[H', J_-] |m\rangle &= \begin{cases} \sqrt{m_1(m_2 + 1)}|m_1 - 1, m_2 + 1\rangle & m_1 > 0, m_2 = 0, \\
0 & m_1 = m_2 = 0, \\
\sqrt{(m_1 + 1)m_2}|m_1 + 1, m_2 - 1\rangle & m_2 > 0, m_1 = 0, \\
\end{cases} \\
[H', J_3] |m\rangle &= 0.
\end{align*}

Then the asymmetry degree of $H'$ with respect to $su(2)$ is then given by,
\begin{align}
\mathcal{A}(su(2), H') &= \frac{1}{||H'||^2} \sum_{\alpha = 1, 2, 3} \sum_{m_1 + m_2 = M} \sum_{m_1 + m_2 = M} \langle m_1, m_2 ||H', J_\alpha||^2 H', J_\alpha ||m_1, m_2\rangle \\
&= \frac{2}{R} \sum_{m_1 + m_2 = M} (m_1 + 1)m_2(|m_1 + 1\rangle + |m_2 - 1\rangle - |m_1\rangle - |m_2\rangle)^2 \\
&= \frac{2}{R} \sum_{j=1}^M j(M + 1 - j) \frac{\cosh \gamma(M - j + \frac{1}{2}) - \cosh \gamma(j - \frac{1}{2})}{\cosh^2 \gamma},
\end{align}
where $R = ||H'||^2 = \sum_{j=1}^M (j_\alpha + |M - j_\alpha|^2 - \frac{4\gamma^2}{M+1}|j_\alpha|^2)$ and $\gamma = \log q$.

Figure 1 shows the asymmetry of the operator $H'$ with respect to the algebra $su(2)$ for $M = 2$. From (12) and figure 1 it is obvious that when $q \to 1 (\gamma \to 0), \mathcal{A}(su(2), H') \to 0$, i.e. $H' \to H$ is $su(2)$ symmetric.
2.2. The geometrical characterization of asymmetry: SU(2) versus deformed SUq(2)

The Casimir operator $J^2$ of $su(2)$ is given by

$$J^2 = J_1^2 + J_2^2 + J_3^2.$$  \hspace{1cm} (13)

The q-deformed Casimir operator $J'^2$ with respect to $su_q(2)$ has the form,

$$J'^2 = J_1'^2 + J_2'^2 + \frac{(\sinh \gamma J_3')^2}{\gamma \sinh \gamma}.$$  \hspace{1cm} (14)

Taking $J_i$ ($J'_i$), $i = 1, 2, 3$, as three real variables, and the Casimir operator $J^2$ ($J'^2$) as a constant, the equations (13) and (14) can be viewed as a three dimensional sphere (deformed sphere). In fact, from symplectic geometry it has been shown in [32] that $J_i$ ($J'_i$), $i = 1, 2, 3$, constitute exactly the Poisson algebra of $su(2)$ ($su_q(2)$), which give rise further to the algebra $su(2)$ ($su_q(2)$) by geometrical quantization. Therefore, hidden in the symmetry deformation is the deformation from a sphere (13) to deformed sphere (14) which could be ellipsoid like, drum like, or even cylinder like depending on the deformation parameter $q = e^\gamma$, see figure 2.

An interesting question is what the asymmetry degree of such ellipsoid like, drum like, or cylinder like geometries with respect to a perfect sphere is. This kind of asymmetries can be characterized by asymmetries of $su_q(2)$ with respect to $su(2)$. To calculate the asymmetry of the operator $J^2$ with respect to the algebra $su(2)$, we consider the co-product representations [11, 33] of $su(2)$

$$J_\pm = \sigma_\pm \otimes I + I \otimes \sigma_\pm,$$

$$J_3 = \sigma_z \otimes I + I \otimes \sigma_z,$$  \hspace{1cm} (15)

and $su_q(2)$

$$J'_\pm = \sigma_\pm \otimes q^{\sigma_z} + q^{-\sigma_z} \otimes \sigma_\pm,$$

$$J'_3 = J_3 = \sigma_z \otimes I + I \otimes \sigma_z,$$  \hspace{1cm} (16)
where $\sigma_x = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \frac{i}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Pauli matrices, $I$ is the identity and $\sigma_{\pm} = \sigma_x \pm i\sigma_y$. With the co-product representations (15) and (16) we have
\[
J^2 = \begin{pmatrix}
\left[\frac{1}{2}\right]_q^2 & 0 & 0 & 0 \\
0 & e^{\gamma} + \left[\frac{1}{2}\right]_q^2 & 1 & 0 \\
0 & 1 & e^{-\gamma} + \left[\frac{1}{2}\right]_q^2 & 0 \\
0 & 0 & 0 & \left[\frac{1}{2}\right]_q^2
\end{pmatrix},
\]
where $[x]_q = \frac{e^{xq} - e^{-xq}}{q - q^{-1}}$ and $\gamma = \log q$. 

Figure 2. The $q$-deformation of the sphere: sphere figure (a) without deformation, ellipsoid like figure (b) for $\gamma = 1$, drum like figure (c) for $\gamma = 2$, cylinder like figure (d) for $\gamma = 5$. (a) $q = 1 (\gamma = 0)$. (b) $q = e (\gamma = 1)$. (c) $q = e^2 (\gamma = 2)$. (d) $q = e^5 (\gamma = 5)$. 

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Then the asymmetry degree of $J^2$ with respect to $\text{su}(2)$ is derived as
\[
A(\text{su}(2), J^2) = \frac{1}{\|J^2\|^2} \sum_{\alpha=\pm} ||[J', J_\alpha]||^2 \leq 16(\cosh \gamma - 1) \cdot \frac{3}{\cosh \gamma}.
\]

In figure 3 we show the asymmetry degree of $J^2$ with respect to $\text{su}(2)$ as a function of the deformation parameter $\gamma$. One can see that under the deformation $A(\text{su}(2), J^2)$ increases quickly and finally tends to be stable as $|\gamma|$ increases.

### 2.3. The integrable spin chain with boundary conditions

The Heisenberg spin chain models play important roles in the study of critical points and phase transitions of magnetic systems. The Hamiltonian of the standard $\text{XXX} \frac{1}{2}$-spin chain model with nearest-neighbor interactions has the form
\[
H_{\text{XXX}} = \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z).
\]

Generalizing the co-product representation (15) of $\text{su}(2)$ to $N$-spin case, one obtains
\[
J_\pm = \frac{1}{2} \sum_{j=1}^{N} \sigma_j^\alpha, \quad J_3 = \frac{1}{2} \sum_{j=1}^{N} \sigma_j^z,
\]
where the operator $\sigma_j^\alpha$ means acting $\sigma^\alpha$ on the space of the $j$th spin, $\sigma_j^\alpha = I \otimes I ... \otimes \sigma_j^\alpha \otimes ... \otimes I$ ($\alpha = \pm, 3$). It is easily verified that $H_{\text{XXX}}$ is $\text{su}(2)$-symmetric, $[H_{\text{XXX}}, J_\pm] = [H_{\text{XXX}}, J_3] = 0$.

Nevertheless, the following integrable spin chain Hamiltonian $H_q$ with an extra boundary term is no longer $\text{su}(2)$-symmetric,
\[
H_q = \sum_{j=1}^{N} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} \sigma_j^z \sigma_{j+1}^z)
+ \frac{q - q^{-1}}{2} (\sigma_1^z - \sigma_N^z).
\]

This Hamiltonian, derived from the solution of Yang–Baxter equations [34], reduces to $H_{\text{XXX}}$ when $q$ goes to 1.
Generally, $H_q$ is symmetric with respect to the $q$-deformed algebra $su_q(2)$. From the co-product representation (16) of $su_q(2)$, for the $N$-spin case one has the generators of $su_q(2)$,

$$J'_\pm = \frac{1}{2} \sum_{j=1}^N \sigma'_j \pm, \quad J'_3 = J_3 = \frac{1}{2} \sum_{j=1}^N \sigma'_j,$$

where $\sigma'_j = q^{-\sigma_j} \otimes \ldots \otimes q^{-\sigma_1} \otimes \sigma_j \otimes q^{\sigma_1} \otimes \ldots \otimes q^{\sigma_N}$. It is easily verified that $H_q$ is $su_q(2)$-symmetric, $[H_q, J'_\pm] = [H_q, J'_3] = 0$.

According to the representation (20) of $su(2)$, we get

$$[H_q, J_+] = \frac{1}{2} \left( \cosh \gamma - 1 \right) \sum_{k=1}^N (\sigma_k^+ \sigma_{k+1}^- + \sigma_k^- \sigma_{k+1}^+ + \sinh \gamma (\sigma_k^+ - \sigma_k^-) \right),$$

$$[H_q, J_-] = -\frac{1}{2} \left( \cosh \gamma - 1 \right) \sum_{k=1}^N (\sigma_k^- \sigma_{k+1}^+ + \sigma_k^+ \sigma_{k+1}^- + \sinh \gamma (\sigma_k^- - \sigma_k^+) \right),$$

$$[H_q, J_3] = 0.$$ (22)

Then the asymmetry of $H_q$ with respect to $su(2)$ can be obtained,

$$A(su(2), H_q) = \frac{1}{||H_q||^2} \sum_{\beta = \pm, 3} ||[H_q, J_\beta]||^2$$

$$= \frac{N(\cosh \gamma - 1)^2 + 4 \sinh^2 \gamma}{N(2 + \cosh^2 \gamma) + 8 \sinh^2 \gamma}. \quad (23)$$

From the formula (23) we have that the asymmetry degree of $H_q$ with respect to $su(2)$ depends on the number of spines as well. In particular, when $N \to \infty$, we have

$$A(su(2), H_q) \to \frac{(\cosh \gamma - 1)^2}{\cosh^2 \gamma + 2}.$$ 

The asymmetry $A(su(2), H_q)$ versus the deformation parameter $\gamma$ is shown in figure 4 for different spin number $N$. From figure 4 one sees that the asymmetry of $H_q$ with respect to $su(2)$
The asymmetry degree $A(G,H)$ of a Hamiltonian $H$ with respect to a Lie group $G$ has been defined in terms of the commutators of the Hamiltonian and the group elements. Let $g$ be the Lie algebra associated with the Lie group $G$. In an infinitesimal neighborhood of the unit element $e$ of $G$, one can show that $A(G,H)$ is proportional to $A(g,H)$ defined in (1). Nevertheless, the asymmetry $A(g,H)$ of an operator $H$ with respect to an algebra $g$ defined in (1) is for any algebras $g$ including Lie algebras and quantum algebras etc.. The relation between $A(G,H)$ and $A(g,H)$ for general $g$ could be more complicated.

We have studied the asymmetry degree of operators with respect to certain algebraic representations. The asymmetry has been defined and its properties have been investigated. As detailed applications, the asymmetries of the q-deformed harmonic Hamiltonian system (9) with respect to the harmonic operator representation (5) of $su(2)$, the q-deformed Casimir operator (14) with respect to the co-product representation (15) of $su(2)$, and the integrable spin chain Hamiltonian (21) with respect to the $su(2)$ representation (20) have been computed analytically. In particular, the asymmetry (1) presents a qualitative characterization of the degree of geometrical deformation of sphere. As symmetry plays crucial roles in physics, our results may highlight investigations on some important topics such as critical phenomena and phase transitions in physical systems. In addition, the $\mathcal{PT}$-symmetric quantum mechanics has been extensively studied recently [35–37]. In such pseudo-Hermitian quantum mechanical systems, the Hamiltonian are no longer necessarily Hermitian, but may still have real eigenvalues [38–40]. Despite the original motivation to build a new framework of quantum theory, researchers are also aware of the importance of simulating the $\mathcal{PT}$-symmetric systems with conventional quantum mechanics [41]. Adapting our proposed asymmetry degree to non-Hermitian Hamiltonians would be also an interesting open question.

3. Discussions and conclusions

In [23] the asymmetry degree $A(G,H)$ of a Hamiltonian $H$ with respect to a Lie group $G$ has been defined in terms of the commutators of the Hamiltonian and the group elements. Let $g$ be the Lie algebra associated with the Lie group $G$. In an infinitesimal neighborhood of the unit element $e$ of $G$, one can show that $A(G,H)$ is proportional to $A(g,H)$ defined in (1). Nevertheless, the asymmetry $A(g,H)$ of an operator $H$ with respect to an algebra $g$ defined in (1) is for any algebras $g$ including Lie algebras and quantum algebras etc.. The relation between $A(G,H)$ and $A(g,H)$ for general $g$ could be more complicated.

We have studied the asymmetry degree of operators with respect to certain algebraic representations. The asymmetry has been defined and its properties have been investigated. As detailed applications, the asymmetries of the q-deformed harmonic Hamiltonian system (9) with respect to the harmonic operator representation (5) of $su(2)$, the q-deformed Casimir operator (14) with respect to the co-product representation (15) of $su(2)$, and the integrable spin chain Hamiltonian (21) with respect to the $su(2)$ representation (20) have been computed analytically. In particular, the asymmetry (1) presents a qualitative characterization of the degree of geometrical deformation of sphere. As symmetry plays crucial roles in physics, our results may highlight investigations on some important topics such as critical phenomena and phase transitions in physical systems. In addition, the $\mathcal{PT}$-symmetric quantum mechanics has been extensively studied recently [35–37]. In such pseudo-Hermitian quantum mechanical systems, the Hamiltonian are no longer necessarily Hermitian, but may still have real eigenvalues [38–40]. Despite the original motivation to build a new framework of quantum theory, researchers are also aware of the importance of simulating the $\mathcal{PT}$-symmetric systems with conventional quantum mechanics [41]. Adapting our proposed asymmetry degree to non-Hermitian Hamiltonians would be also an interesting open question.
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ORCID iDs

Hui-Hui Qin https://orcid.org/0000-0002-3597-4855

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