DISPERSE ESTIMATES FOR THE DIRAC EQUATION IN AN AHARONOV-BOHM FIELD

F. CACCIAFESTA AND L. FANELLI

ABSTRACT. We prove local smoothing and weighted Strichartz estimates for the Dirac equation with an Aharonov-Bohm potential. The proof, inspired by [9], relies on an explicit representation of the solution built in terms of spectral projections.

1. Introduction

The Dirac Hamiltonian in the Aharonov-Bohm magnetic field (in the units with $\hbar = c = 1$) is

\[
\mathcal{D}_A^m = \begin{pmatrix} m & D^* \\ D & -m \end{pmatrix}, \quad D = (p_1 + A^1) + i(p_2 + A^2)
\]

where $p_j = i\partial_j$ and the magnetic potential $A$ reads as, in the radial gauge,

\[ A_r = 0, \quad A_\phi = \frac{\alpha}{2\pi r} \Phi_0. \]

In terms of Pauli matrices, we can rewrite Hamiltonian (1.1) as

\[
\mathcal{D}_A^m(A) = \sigma_3 m + \sigma_1(p_1 + A^1) + \sigma_2(p_2 + A^2)
\]

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and the magnetic potential $A(x) = (A_1(x), A_2(x))$ is given by

\[
A(x) = \alpha \left( -\frac{x_2}{|x|^2}, -\frac{x_1}{|x|^2} \right).
\]

We recall that the Pauli matrices satisfy the following relations of anticommutations

\[
\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}1_2, \quad j, k = 1, 2.
\]
The Aharonov-Bohm effect was firstly predicted in [1]: it occurs when electrons propagate in a domain with a zero magnetic field but with a nonzero vector potential $A$. The potential magnetic field is totally confined within a cylindrical tube of infinitesimal radius (see [27] and references therein for greater details). In what follows we will restrict our attention to the massless case, i.e. $m = 0$, and we will denote the corresponding Hamiltonian with $\mathcal{D}_A$.

In the present paper we are interested in studying dispersive properties of the flow associated to the Hamiltonian $\mathcal{D}_A$: we thus mean to study the Cauchy problem

$$\begin{cases}
i \partial_t u = \mathcal{D}_A u, \\ u(t,x) : \mathbb{R} \times \mathbb{R}^2_x \to \mathbb{C}^2 \\ u(0,x) = f(x).\end{cases}$$

We stress the fact that the magnetic potential $A$ is critical with respect to the scaling of the massless Dirac operator; as it is well known, the study of dispersive estimates for flows perturbed by scaling-critical potentials represents a particularly interesting and challenging problem, as perturbation arguments typically do not work in this setting. In this framework we mention [5]-[6] in which smoothing and Strichartz estimates for the Schrödinger and wave equations with inverse square potential are proved, then [14] in which the $L^1 \to L^\infty$ time decay is proved for a wide class of Schrödinger flows with critical electromagnetic potentials, later [9] in which local smoothing estimates for the Dirac-Coulomb equation is discussed, and finally [8] which is devoted to the study of weak dispersion for fractional Aharonov-Bohm-Schrödinger groups. On the other hand, the study of dispersive estimates for the Dirac equation perturbed with small magnetic potential has been developed e.g. in [4] and [7].

The first problem to be addressed in order to study the dynamics of equation (1.3) is the self-adjointness of the Hamiltonian $\mathcal{D}_A$. As $\mathcal{D}_A$ commutes with the angular momentum operator, we can resort on the classical partial wave decomposition

$$L^2(\mathbb{R}^2)^2 \cong L^2((0,\infty), dr) \bigotimes_{l \in \mathbb{Z}} h_l$$

and reduce the problem to study the self-adjointness of "radial" Dirac operators $\mathcal{D}_{A,l}$ (see forthcoming Proposition 2.1 and formula (2.2) for details). By resorting on general theory (see Remark 2.1) it is possible to show that the operators $\mathcal{D}_{A,l}$, which can be originally defined on the space $C_0^\infty(\mathbb{R}^+)$, are essentially self-adjoint as long as $l$ is not in the range $-1 < l + \alpha < 0$. This case needs to be discussed separately:
it can be shown that the corresponding operator admits a one parameter family of self adjoint extensions, which can be distinguished by imposing different boundary conditions.

Before stating our main result, let us fix some useful notations. We shall denote as usual with $L^p_t L^q_x = L^p_t(\mathbb{R}_t; L^q_x(\mathbb{R}^n_x))$ the mixed space-time Strichartz spaces; with $L^2_{rdr}$ we will denote the $L^2$ space with respect to the measure $rdr$. The presence of the subscript $L^p_T$ will be a shortcut for $L^p_t([0,T])$, while we denote with $\|f\|_{L^2_{|x|\geq A}} := \|\hat{f} \chi_{|x|\geq A}\|_{L^2}$ where $\chi_A$ is the standard characteristic function of the set $A$. We denote with $\Omega^s$ the operator

$$(\Omega^s\phi)(x) = |x|^s \phi(x)$$

and with a little abuse of notation we will use the same symbol to indicate the operators which are pointwise equal for all times,

$$(\Omega^s\psi)(t, x) = |x|^s \psi(t, x).$$

We will also make use of the Gauss hypergeometric function, that we recall to be

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}$$

where here $(q)_n$ denotes the Pochhammer symbol. We refer to [11] as a general reference for details and properties of special functions.

We are now ready to state the first result of this paper, that is a local smoothing estimate for solutions of (1.3); we prefer to postpone the precise definition of the angular spaces $H^{\geq l}$ to forthcoming Proposition 2.1.

**Theorem 1.1.** Let $\alpha \in \mathbb{R}$ and $u(t, x)$ be a solution of (1.3). Then for any

$$1/2 < \gamma < 1 + |l + \alpha|.$$

and any $f \in L^2((0, \infty)rdr) \otimes H^{\geq l}$ there exists a constant $c = c(\alpha, \gamma, l)$ such that the following estimate holds

$$\|\Omega^{-\gamma} D_A^{1/2-\gamma} u\|_{L^2_t L^2_x} \leq c\|f\|_{L^2}.$$

In addition, in the endpoint case $\gamma = 1/2$ the following estimate holds

$$\sup_{R>0} R^{-1/2}\|e^{-itD_A} f\|_{L^2_t L^2_x|_{|x|\leq R}} \lesssim \|f\|_{L^2},$$

From the point of view of the range in (1.7), the worst frequency $l$ is the closest one to the circulation $\alpha \in \mathbb{R}$. Indeed, as an immediate consequence of (1.5), we obviously obtain the following result for a generic non-localized $L^2$-function.
Corollary 1.2. Let $\alpha \in \mathbb{R}$, denote by $\mu_0 := \text{dist}(\alpha, \mathbb{R})$ and let $u(t, x)$ be a solution of (1.3). Then for any
\begin{equation}
1/2 < \gamma < 1 + \mu_0.
\end{equation}
and any $f \in L^2(\mathbb{R}^2)$ there exists a constant $c = c(\alpha, \gamma, l)$ such that the following estimate holds
\begin{equation}
\left\| \Omega^{-\gamma} D_{\lambda}^{1/2-\gamma} u \right\|_{L^2_t L^2_x} \leq c \| f \|_{L^2}.
\end{equation}

Remark 1.1. Notice that estimate (1.8) for $\gamma = 1$ fails in the free case (as it fails the corresponding one for the wave equation); on the other hand, as it is often the case, the presence of the Aharonov-Bohm magnetic field improves the range of the admissible exponents and allows us to include it, since $\mu_0 \neq 0$, as soon as $\alpha \notin \mathbb{Z}$. This is a kind of diamagnetic effect, which is in general not expected in the relativistic setting, but which is known to be possible (see [3]).

Remark 1.2. It is interesting to compare the range of the admissible exponents (1.7) with its analogous for the local smoothing estimates for the Dirac-Coulomb model, as given in Theorem 1.1 in [9]. In that case indeed, for generic initial data, the wideness of the range is a decreasing function of the (modulus of the) charge, meaning that the bigger the charge, the smaller the admissible range is, and it shrinks to the empty set as the charge tends to 1, which we recall to be the threshold value for the charge in order to define a self-adjoint operator. Here instead, the range is in fact wider than the one of the free case. Also, estimate (1.6) could not (yet) be proved for Dirac-Coulomb: the main ingredient in our argument is indeed, as it will be clear, an estimate for the radial components of the generalized eigenstates $\chi_l(r)$ of the operator, namely
\begin{equation}
\frac{1}{R} \int_0^R \chi_l(r)^2 r dr < C.
\end{equation}
uniform in $R$ and $l$. In the contest of the Dirac equation with Aharonov-Bohm fields, the functions $\chi_l(r)$ are nothing but standard Bessel functions $J_\lambda(r)$; property (1.9) with this choice has been originally proved in [31] for integers $k$ and then for generic $k$ in [30]. It should also be noticed the fact that the stronger estimate
$$\sup_{r, \nu} |\sqrt{r} J_\lambda(r)| \leq C$$
has been misproved in [25], where in particular the author shows that $\sup_{r>0} |\sqrt{r} J_\lambda(r)|$ strictly increases to infinity as $\lambda$ increases from $1/2$ to infinity. The case of Coulomb potential is more difficult, and this is
due to the more complicated structure of generalized eigenstates, which involve confluent hypergeometric functions: the proof of estimate (1.9) in this setting will be object of forthcoming work.

Remark 1.3. The problem, natural, of including a mass term in equation (1.3) and thus in estimate (1.5) looks to be non trivial, as in our proof we strongly rely on scaling properties of the operator. A possible solution might be to exploit the Kato-Smoothing theory and then rely on the machinery developed in [10], and sucessfully applied for Schrödinger flows in Aharonov-Bohm fields in [8], to go from wave to Klein-Gordon smoothing estimates; nevertheless, the argument needs to be adapted to the delicate Dirac structure. This will be topic of forthcoming works.

Remark 1.4. By combining this result with [9], in which an analogous estimate is proved for the massless Dirac-Coulomb system, it is possible to give an all-comprehensive estimate for an electromagnetic dynamic with Coulomb electric and Aharonov-Bohm magnetic potential. We mention that the explicit structure of the generalized eigenstates for this model can be found e.g. in [24].

Next, we are able to prove the following local in time estimate, which in the contest of the wave equation is known as KSS estimate (see [23]).

Theorem 1.3. Let $\alpha \in \mathbb{R}$, $\mu \leq 0$ and $u(t,x)$ be a solution of (1.3).
For any $f \in L^2$ the following estimate is satisfied
\begin{equation}
\| (x)^\mu e^{itDA} f \|_{L^2_t L^2_x} \lesssim A_\mu(T) \| f \|_{L^2}
\end{equation}
where
\[
A_\mu(T) = \begin{cases}
T^{1/2-\mu}, & \text{if } -1/2 < \mu \leq 0, \\
(\log((2 + T))^{1/2}, & \text{if } \mu = -1/2, \\
C & \text{if } \mu < -1/2.
\end{cases}
\]
where $C$ is an absolute constant. Moreover, as a consequence of (1.6), for $-1/2 < \mu \leq 0$ we have
\begin{equation}
\| x^\mu e^{itDA} f \|_{L^2_t L^2_x} \lesssim T^{1/2+\mu} \| f \|_{L^2}.
\end{equation}

A natural application of Theorem (1.1) would be proving Strichartz estimates, which have been a relevant research topic in the last decades, due to their applications to nonlinear problems. The Aharonov-Bohm potential is known to be critical for their validity in this setting (see [2] and also [17, 20] for the non-relativistic counterpart). In order to apply a standard perturbation argument based on the Duhamel formula, one would need to use estimate (1.5) with $\gamma = 1/2$, and this estimate seems
to fail (surely we are not able to include it in our proof), as it fails in
the free case. Anyway, as an application of Theorems 1.1 and 1.3, it
is still possible to prove some weighted Strichartz estimates. In the
following, we use the polar coordinates $x = r\omega$, $r \geq 0$, $\omega \in S^1$, and
given a measurable function $F = F(t, x) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{C}$ we denote by

$$\|F\|_{L^q_t L^q_r L^2_\omega} := \int_{-\infty}^{+\infty} \left( \int_0^{+\infty} \left( \int_{S^1} |F(t, r, \omega)|^2 \, d\sigma \right)^q \, r \, dr \right) \, dt,$$

being $d\sigma$ the surface measure on the sphere. Then we can prove the following

**Theorem 1.4.** Let $\alpha \in \mathbb{R}$ and $u(t, x)$ be a solution of (1.3). For any $f \in L^2$ the following estimates are satisfied

$$(1.12) \quad \|r^{\frac{1}{2} - \varepsilon - \frac{d}{2}} u\|_{L^q_t L^q_r L^2_\omega} \leq C \|D_A^{\frac{1}{2} - \varepsilon - \frac{d}{2}} \Lambda_\omega^{-\varepsilon} f\|_{L^2}, \quad q \in [2, \infty]$$

for some $C > 0$, where $\Lambda_\omega = \sqrt{1 - \Delta_\omega}$ and $\Delta_\omega$ is the Laplace-Beltrami
operator on $S^1$.

**Remark 1.5.** In this result we have used the fact that

$$\|f\|_{\dot{H}^s} \leq C_1 \|D_A f\|_{L^2}$$

for $s \in (0, 1)$ (notice that fractional powers of $D_A$ commute with the
flow of (1.3)). This will be proved in forthcoming Lemma 2.5.

**Remark 1.6.** By interpolating estimate (1.10) with the endpoint trace
lemma (see e.g. [19]), it is possible to obtain a family of local in time
Strichartz estimates; anyway this would need a much careful insight in
order to proper defined Besov spaces in this contest, and we prefer not
to deal with this problem here.

The proof of estimate (1.12) only requires to interpolate between
estimate (1.5) and the 2D Sobolev inequality

$$\sup_{r > 0} r^{\frac{1}{2} - \varepsilon} \|f(r\omega)\|_{L^2_\omega} \leq C \|D^{\frac{1}{2} + \varepsilon} \Lambda_\omega^{-\varepsilon} f\|_{L^2} \leq C \|D_A^{\frac{1}{2} + \varepsilon} \Lambda_\omega^{-\varepsilon} f\|_{L^2}.$$

2. The setup: spectral theory and integral transform.

In this section we build the necessary setup needed to prove our main
result.
2.1. **Spectral theory of the Dirac operator in Aharonov-Bohm field.** We devote this subsection to briefly review the spectral theory of the Dirac Hamiltonian in an Aharonov-Bohm field, as the explicit form of the generalized eigenstates will play a crucial role in what follows; for further details we refer to [12].

First of all, we recall the following classical result, which can be found in [32] (we will limit ourselves to the massless case as it is the one we need, but the theory can be extended to positive mass as well).

**Proposition 2.1.** We can define a unitary isomorphism between Hilbert spaces

\[ L^2(\mathbb{R}^2) \cong L^2((0, \infty), dr) \bigotimes_{l \in \mathbb{Z}} h_l \]

by means of the following decomposition

\[ \Phi(x) = \sum_{l \in \mathbb{Z}} \frac{1}{2\sqrt{\pi}} \begin{pmatrix} f_l(r) \\ g_l(r) e^{i\phi} \end{pmatrix} e^{il\phi}, \]

which holds for any \( \Phi \in L^2(\mathbb{R}^2, \mathbb{C}^2) \). Moreover, the operator \( D_A^0 \) defined in (1.1) leaves invariant the partial wave subspaces \( C^\infty_0((0, \infty)) \otimes h_l \) and, with respect to the basis \( \{ e^{i\phi}, e^{i(l+1)\phi} \} \) is represented by the matrix

\[ D_{A,l} = \begin{pmatrix} 0 & -i (\partial_r + \frac{l+\alpha+1}{r}) \\ -i (\partial_r - \frac{l+\alpha}{r}) & 0 \end{pmatrix} \]

The operator \( D_A^0 \) on \( C^\infty_0(\mathbb{R}^2)^2 \) is unitary equivalent to the direct sum of \( D_{A,l} \) that is

\[ D_A^0 \cong \bigoplus_{l \in \mathbb{Z}} D_{A,l}. \]

For fixed \( l \in \mathbb{Z} \) we denote with

\[ H_{\geq l} = \bigoplus_{l: |l| \geq |l|} h_l. \]

Therefore, by defining for positive energies \( E > 0 \)

\[ \Psi_{E,l}(t, r, \phi) \cong \begin{pmatrix} f_{l,E}(r) \\ g_{l,E}(r) e^{i\phi} \end{pmatrix} e^{i\phi} e^{-iEt}, \]

the radial eigenvalue problem for a fixed value of \( l \in \mathbb{Z} \) reads as

\[ D_{A,l} \chi_{l,E}(r) = E \chi_{l,E}(r), \]

which gives the solution

\[ \chi_{l,E}(r) = \begin{pmatrix} f_{l,E}(r) \\ g_{l,E}(r) \end{pmatrix} = \sqrt{\frac{\pi}{2}} \begin{pmatrix} (\epsilon_l)^l J_{|l+\alpha|}(Er) \\ i(\epsilon_l)^{l+1} J_{|l+2+\alpha|}(Er) \end{pmatrix} \]
with
\[
\epsilon_l = \begin{cases} 
1 & \text{if } l + \alpha \geq 0 \\
-1 & \text{if } l + \alpha < 0.
\end{cases}
\]

Direct calculations show that the generalized eigenfunctions for negative values of the energy can be written as
\[
(2.7) \quad \chi_{l,-E}(r) = \sqrt{\frac{\pi}{2}} \left( \frac{\epsilon_l}{2} J_{|l+\alpha|}(|E|r) \right),
\]
so that in particular
\[
f_{l,-E}(r) = f_{l,E}(r), \quad g_{l,-E}(r) = -g_{l,E}(r).
\]
Notice that the wave functions above can be normalized by the condition
\[
(2.8) \quad \int \chi_{l,E}^*(x)\chi_{l',E'}(x)dx^2 = 2\pi\delta_{l,l'}\delta(E-E').
\]

Remark 2.1. Let us briefly comment on the formulas above. System (2.5) admits indeed, for a fixed value of $l$, another family of solutions that can be obtained by replacing the couple $(J_{|l+\alpha|}, J_{|l+\alpha+1|})$ by $(J_{-|l+\alpha|}, J_{-|l+\alpha+1|})$ in (2.6). On the other hand, this second couple can not be considered, as long as $|l + \alpha| > 1$, as generalized eigenstates are required to be square-integrable in the origin; we recall indeed the well known asymptotic behavior of Bessel functions in the origin given by
\[
\lim_{x \to 0} J_{\nu}(x) \sim \frac{x^\nu}{2^{\nu} \Gamma(1 + \nu)}.
\]

General theory therefore ensures that the operators $D_{A,l}$ are essentially selfadjoint as long as $|l + \alpha| > 1$. When $-1 < l + \alpha < 0$, the situation becomes more delicate: both the couples $(J_{|l+\alpha|}, J_{|l+\alpha+1|})$ satisfy indeed the condition of square integrability in the origin. This is enough to guarantee that the corresponding operator $D_{A,l}$ is not essentially selfadjoint (recall that a corollary of the Theorem proven in [33] states that, for the partial Dirac Hamiltonian to be essentially self-adjoint, it is necessary and sufficient that a non-square-integrable at $r \to 0$ solution exists). On the other hand, by taking advantage of the classical Von Neumann deficiency indices theory, it is possible to show that the operator $D_{A,l}$ in the "critical case" $-1 < l + \alpha < 0$ admits a one-parameter family of self adjoint extensions; to fix one, suitable further boundary conditions on the eigenfunctions need to be imposed. We refer to [29], [12] and [13] for a detailed analysis of the topic.
2.2. **The integral transform.** We now introduce the crucial integral transform that will be used in the proof of the main result, that essentially consists in a projection on the continuous spectrum. Throughout this subsection, we are fixing a value of \( l \in \mathbb{Z} \) and working only on radial functions.

**Definition 2.2.** Let \( \varphi(r) = (\varphi_1(r), \varphi_2(r)) \in L^2((0, \infty), rdr)^2 \). We define, for \( l \in \mathbb{Z} \), the following integral transform

\[
\mathcal{P}_l \varphi(E) = \begin{pmatrix} \mathcal{P}_l^+ \varphi(E) \\ \mathcal{P}_l^- \varphi(E) \end{pmatrix} = \begin{pmatrix} \int_0^{+\infty} \chi_{l,E}(r) \varphi(r) r dr \\ \int_0^{+\infty} -\chi_{l,-E}(r) \varphi(r) r dr \end{pmatrix} = \int_0^{+\infty} H_l(\varepsilon r) \cdot \varphi(r) r dr
\]

where we have introduced the matrix

\[
H_l = \begin{pmatrix} f_{l,E}(r) & g_{l,E}(r) \\ -f_{l,-E}(r) & -g_{l,-E}(r) \end{pmatrix}
\]

We collect in the following proposition some crucial properties of the operator \( \mathcal{P}_l \).

**Proposition 2.3.** For any \( \varphi \in L^2((0, \infty), dr)^2 \) the following properties hold:

1. \( \mathcal{P}_l \) is an \( L^2 \)-isometry.
2. \( \mathcal{P}_l \mathcal{D}_{A,l} = \Omega \mathcal{P}_l \).
3. The inverse transform of \( \mathcal{P}_l \) is given by

\[
\mathcal{P}_l^{-1} \varphi(r) = \int_0^{+\infty} H^*_l(\varepsilon r) \cdot \varphi(E) E dE
\]

where the matrix \( H^*_l \) is the transpose conjugate of \( H_l \).
4. For every \( \gamma \in \mathbb{R} \) we can formally define the fractional operators

\[
\mathcal{D}_{A,l}^\gamma \varphi(r) = \mathcal{P}_l \Omega^\gamma \mathcal{P}_l^{-1} \varphi(r) = \int_0^{+\infty} S_l^\gamma(r, s) \cdot \varphi_l(s) s ds.
\]

where the integral kernel \( S_l(r, s) \) is the \( 2 \times 2 \) matrix given by

\[
S_l^\gamma(r, s) = \int_0^{+\infty} H_l(\varepsilon r) \cdot H^*_l(\varepsilon s) E^{1+\gamma} dE
\]

**Remark 2.2.** When summing on \( l \), property (2.12) defines in a standard way fractional powers of the Hamiltonian \( \mathcal{D}_{A} \), which are used in the statement of Theorem (1.1).
Remark 2.3. Let us notice that $P_l^+$ and $P_l^-$ are (the radial part of) a sum of Hankel transforms: indeed, due to (2.6), we have that (some factors $\pi$ will be neglected)

$$P_l^+(\varphi)(E) = \int_0^\infty (J_{|l+\alpha|}(Er)\phi_1(r) + J_{|l+1+\alpha|}(Er)\phi_2(r))rdr$$

and, due to (2.7), a similar one for $P_l^-$. In Proposition above we are thus just transferring to our framework several important properties that are well known for Hankel (see e.g. [5]).

Proof. Property (1) is a standard feature of Hankel transform.

Property (2) comes from the definition of $P_l$, once noticed that

$$P_l^+(D_{A,l}\varphi)(E) = \langle \chi_l,E \rangle_{L^2_{rdv}} = \langle D_{A,l}\chi_l,E \rangle_{L^2_{rdv}} = E \langle \chi_l,E \rangle_{L^2_{rdv}}$$

where we have used the fact that $D_{A,l}$ is selfadjoint with respect to the scalar product above (an analogous calculation can be developed for $P^-_l$). This shows that

$$P_l^\pm(D_{A,l}\varphi)(E) = EP_l^\pm\varphi(E),$$

and thus

$$P_l(D_{A,l}\varphi)(E) = \left( \frac{P_l^+D_{A,l}\varphi(E)}{EP_l^+\varphi(E)}, \frac{P_l^-D_{A,l}\varphi(E)}{EP_l^-\varphi(E)} \right) = \Omega \varphi(E)$$

which proves Property (2).

Property (3) is a direct calculation.

To prove property (4) we write

$$D_{A,l}^t\varphi_l(r) = P_l\Omega^\gamma P_l\varphi_l(r)$$

$$= \int_0^{+\infty} H_l(Er)E^{1+\gamma} \left( \int_0^{+\infty} H_l^*(Es)\varphi_l(s)ds \right) dE.$$

Exchanging the order of the integrals yields (2.12)-(2.13) and thus (4).

Remark 2.4. Notice that, due to (2.7),

$$H_l^* = \left( \begin{array}{cc} f_{l,E}(r) - f_{l,E}(r) & g_{l,E}(r) \\ g_{l,E}(r) & g_{l,E}(r) \end{array} \right).$$

By calculating the integrals, we are able to write down explicitly the single components of the $2 \times 2$ integral kernel $S_l^\gamma(r,s)$.

Proposition 2.4. Let $l \in \mathbb{Z}$, $\gamma > 0$, $0 < r < s$. Then,

$$S_l^\gamma = \left( \begin{array}{cc} F_l^\gamma(r,s) & G_l^\gamma(r,s) \\ G_l^\gamma(r,s) & F_l^\gamma(r,s) \end{array} \right)$$
Remark 2.5. The representation in the region $0 < s < r$ can be obtained by exchanging the roles of $r$ and $s$ in formulas above.

Remark 2.6. This result should be compared with formula (10) in [5] (see also [28]).

Proof. We rely on explicit formulas to calculate the integrals in (2.14), which have already been object of detailed analysis in the literature. We recall indeed the general results (see e.g. [26] pag. 49)

\begin{equation}
\int_0^\infty J_\nu(rt)J_\mu(st)t^{-\lambda}dt = r^\nu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) \frac{2^\lambda s^{\nu-\lambda+1} \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) \Gamma(\nu+1)}{2^{\lambda}s^{\nu-\lambda+1} \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right) \Gamma(\nu+1)}
\end{equation}

provided $\text{Re}(\nu+\mu-\lambda+1) > 0$, $\text{Re}(\lambda) > -1$ and $0 < r < s$. We rely on this formula to evaluate our integrals. We have indeed that

\begin{align*}
F_i^\gamma(r, s) &= \int_0^\infty (f_{i,E}(r) f_{i,E}(s) + g_{i,E}(r) g_{i,E}^*(s)) E^{1+\gamma} dE \\
&= \int_0^\infty (J_{|l+\alpha|}(Er) J_{|l+\alpha|}(Es) + J_{|l+\alpha|}(Er) J_{|l+\alpha|}(Es)) E^{1+\gamma} dE.
\end{align*}

Applying (2.16) with the choice $\nu = \mu = |l+\alpha|$ (resp. $\nu = \mu = |l+1+\alpha|$) and $\lambda = -1-\gamma$ (notice that our assumptions on the parameters allow us to rely on such a formula), gives (2.15). Analogously one obtains that

\begin{align*}
G_i^\gamma(r, s) &= \int_0^\infty (-f_{i,E}(r) f_{i,E}(s) + g_{i,E}(r) g_{i,E}^*(s)) E^{1+\gamma} dE \\
&= \int_0^\infty (-J_{|l+\alpha|}(Er) J_{|l+\alpha|}(Es) + J_{|l+\alpha|}(Er) J_{|l+\alpha|}(Es)) E^{1+\gamma} dE.
\end{align*}

Applying (2.16) concludes the proof. \qed
2.3. The norm induced by $\mathcal{D}_A$. We conclude this section with the following Lemma, which is a key ingredient for proving Theorem 1.4.

**Lemma 2.5.** Let $\alpha \in \mathbb{R}$. For any $s \in [0, 1]$

$$\|f\|_{\dot{H}^s} \leq C_1 \|\mathcal{D}_A f\|_{L^2}. \tag{2.17}$$

**Proof.** We prove the equivalence $\|\nabla A f\|_{\dot{H}^s} \cong \|\mathcal{D}_A f\|_{L^2}$, where we are denoting with

$$\nabla A = (\partial_1^A, \partial_2^A) = (\partial_1 + iA_1(x), \partial_2 + iA_2(x))$$

the magnetic gradient; then estimate (2.17) will be a consequence of diamagnetic inequality (see e.g. [15, 16, 18]). Moreover, it is enough to prove the case $s = 1$, as the full range of exponents can then be obtained by interpolation (the case $s = 0$ is obvious). But this is an immediate consequence of the relation of anticommutations of Pauli matrices as indeed

$$\|\mathcal{D}_A f\|_{L^2}^2 = \int \left| (\sigma_3 \partial_1^A + \sigma_2 \partial_2^A) f \right|^2$$

$$= \int \left| \left[ (\sigma_1 \partial_1^A)^2 + (\sigma_2 \partial_2^A)^2 + (\sigma_1 \partial_1^A \sigma_2 \partial_2^A + \sigma_2 \partial_2^A \sigma_1 \partial_1^A) \right] f \right|^2$$

which concludes the proof. \qed

3. Proof of the main results

We devote this section to proving our main Theorems.

3.1. **Proof of Theorem 1.1.** Our proof is a combination of the arguments used in [9] with the ones in [8], and follows the strategy originally developed in [5], the idea being use decomposition (2.1) to reduce equation (1.3) to a much simpler problem, use Propositions (2.3) and (2.4) to prove the local smoothing estimate for a fixed value of $l \in \mathbb{Z}$ and then sum back. We thus set an initial condition $f \in L^2$ with angular part in $h_l$ and denote with $L_l f$ the solution to the Cauchy problem

$$\begin{aligned}
  \begin{cases}
    i\partial_t u = \mathcal{D}_{A,l} u, \\
    u(0, x) = f(x)
  \end{cases}
\end{aligned} \tag{3.1}$$

where $\mathcal{D}_{A,l}$ is given by (2.2). Then, by applying operator $\mathcal{P}_l$ and using its properties, the LHS of estimate (1.5) is equivalent to (notice that the application of the matrix $\sigma_3$ does not alter the $L^2$ norm)

$$\|\mathcal{P}_l \Omega^{-\gamma} \mathcal{D}_{A,l}^{1/2 - \gamma} L_l f\|_{L^2_l L^2_x} = \|\mathcal{D}_{A,l}^{-\gamma} \Omega^{1/2 - \gamma} \mathcal{P}_l L_l f\|_{L^2_l L^2_x}$$
The function $\mathcal{P}_L L f$ solves now
\begin{equation}
\begin{cases}
  i\partial_t \mathcal{P}_L L f = \Omega \mathcal{P}_L L f, \\
  \mathcal{P}_L L f(0, \xi) = \mathcal{P}_L f(\xi),
\end{cases}
\end{equation}
so that the solution to this problem is explicitly given by
\[ \mathcal{P}_k L_k f(t, \xi) = e^{it\xi} \mathcal{P}_k f(\xi). \]
We now Fourier transform in time (which does not alter the $L^2$ norm) to have
\[ (\mathcal{F}_t \mathcal{P}_L L f)(\tau, \xi) = (\mathcal{P}_L f)(\rho) \delta(\tau + \xi). \]
Therefore, we can write
\begin{equation}
(\mathcal{D}^{\gamma \Omega}_{\mathcal{A}})^{1/2-\gamma} \mathcal{F}_t \mathcal{P}_L L f)(\tau, \xi) = \int_0^{+\infty} S^{-\gamma}_l(\xi, s) \delta(\tau + s) \mathcal{P}_L f(s)s^{3-2\gamma} ds
= -S^{-\gamma}_l(\xi, \tau) \mathcal{P}_L f(\tau) \tau^{3-2\gamma}.
\end{equation}
We now take the $L^2$ norm in time and space of quantity above (notice that, as the angular part in decomposition (2.1) is $L^2$-unitary, we only need to consider the radial integrals)
\begin{equation}
\int_0^{+\infty} \int_0^{+\infty} (\mathcal{P}_L f)^*(\tau) S^{-\gamma}_l(\rho, \tau)^T \cdot (S^{-\gamma}_l(\rho, \tau)(\mathcal{P}_L f)(\tau)) \tau^{3-2\gamma} \rho^2 d\rho.
\end{equation}
Since $S^\rho_l(\rho, \tau)^T = S^\rho_l(\tau, \rho)$, the integral in $d\rho$ yields $S^{-2\gamma}_l(\tau, \tau)$ and we are therefore left with
\begin{equation}
\int_0^{+\infty} (\mathcal{P}_L f)^*(\tau) S^{-2\gamma}_l(\tau, \tau)(\mathcal{P}_L f)(\tau) \tau^{3-2\gamma} d\tau
\leq \int_0^{+\infty} \text{Tr}(S^{-2\gamma}_l(\tau, \tau)) |(\mathcal{P}_L f)(\tau)|^2 \tau^{3-2\gamma} d\tau
\end{equation}
where in the last step we have used the fact that the matrix $S^{-\gamma}_l$ is positive definite in the diagonal values, as it is the integral kernel of a positive definite operator. To conclude the proof we thus need to estimate the integral above: in view of Proposition (2.4) we recall the explicit formula of Gauss hypergeometric functions with argument 1, to be
\begin{equation}
{2F_1}(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}
\end{equation}
provided $\text{Re}(c - a - b) > 0$ (notice that this restriction forces the bound $\gamma > 1/2$); we also stress that when $\tau := r = s$ we have the equivalence
of the ratios in (2.15)

$$\frac{r^{l+\alpha}}{s^{l+\alpha-2\gamma+2}} = \frac{r^{l+\alpha+1}}{s^{l+\alpha-2\gamma+3}} = r^{2\gamma-2}$$

which is the right weight which allows to recover the $L^2$-norm in (3.4).

Therefore, we eventually have

$$(3.4) \leq C_{\gamma,l} \int_0^{+\infty} |(P_l f)(\tau)|^2 \tau \, d\tau \leq C_{\gamma,l} \int_0^{+\infty} |f(\tau)|^2 \tau \, d\tau = C_{\gamma,l} \|f\|_{L^2}$$

where the constant

$$(3.6) \quad C_{\gamma,l} = \frac{\pi \Gamma(2\gamma - 1)}{2^{2\gamma} \Gamma(\gamma)^2} \left[ \frac{\Gamma(|l+\alpha| - \gamma + 1)}{\Gamma(|l+\alpha| + \gamma)} + \frac{\Gamma(|l+\alpha| - \gamma + 2)}{\Gamma(|l+\alpha| + \gamma + 1)} \right]$$

Notice that our assumption on the range of $\gamma$ is now necessary to guarantee the constant $C_{\gamma,l}$ to be finite, as indeed we are forced to assume $\gamma > 1/2$ and $\gamma < |l+\alpha| + 1$. This proves the inequality for a fixed level $l$. Notice also that within our range $C_{\gamma,l}$ is in fact a decreasing function of $l$; this allows us to rely on decomposition (2.1) and use triangle inequality to conclude the proof.

We now turn to the proof of estimate (1.6).

In analogy with what has been done above, we resort on decomposition (2.1) and prove it for a fixed value of $l$, showing that in fact the estimate holds with a constant independent on $l$: this will allow to sum back and obtain (1.6). We denote for a fixed value of $l \in \mathbb{Z}$

$$F_l(r, \phi) = \left( \begin{array}{c} f_l(r) e^{i\phi} \\ ig_l(r) e^{i(l+1)\phi} \end{array} \right), \quad F_l(r) = \left( \begin{array}{c} f_l(r) \\ ig_l(r) \end{array} \right), \quad Y_l(\phi) = \left( \begin{array}{c} e^{i\phi} \\ e^{i(l+1)\phi} \end{array} \right)$$

Writing indeed

$$\|e^{itDA} f\|_{L_t^2 L_x^2; t \leq R} = \| \sum_{l \in \mathbb{Z}} e^{itDA} F_l(r, \phi) \|_{L_t^2 L_x^2; t \leq R}^2 \leq \sum_{l \in \mathbb{Z}} \|e^{itDA} F_l(r, \phi)\|_{L_t^2 L_x^2; t \leq R}^2 = \sum_{l \in \mathbb{Z}} \|e^{itDA} F_l(r)\|_{L_t^2 L_x^2; t \leq R}^2 \leq \sum_{l \in \mathbb{Z}} \|e^{itDA} F_l(r)\|_{L_t^2 L_x^2; t \leq R}^2 \leq \sum_{l \in \mathbb{Z}} \|e^{itDA} F_l(r)\|_{L_t^2 L_x^2; t \leq R}^2 \leq C_l \|F_l(r)\|_{L_t^2 L_x^2; t \leq R}$$

so that, by the unitarity of the angular term, it will be enough to prove

$$\sup_{R > 0} \|e^{itDA} F_l(r)\|_{L_t^2 L_x^2; t \leq R}^2 \leq C_l \|F_l(r)\|_{L_t^2 L_x^2; t \leq R}^2$$
and show that the constant $C_l$ is bounded with respect to $l$. We start by writing
\[ e^{itD_A} F_l = \mathcal{P}_l \left[ e^{it|\xi|} \mathcal{P}_l f \right] = \int_0^{+\infty} e^{it|\xi|} H_l(r\xi) \cdot \mathcal{P}_l f(\xi) \xi \, d\xi = \mathcal{F}_{|\xi|\rightarrow t} \{ H_l(r\xi) \cdot \mathcal{P}_l f(\xi) \xi \chi_{\mathbb{R}^+} \}. \]

Taking the $L^2_{(x|\leq R)}$ norm then gives, by Plancherel,
\[ \| e^{itD_A} F_l \|_{L^2_{(x|\leq R)}}^2 = \| \mathcal{F}_{|\xi|\rightarrow t} \{ H_l(r\xi) \cdot \mathcal{P}_l f(\xi) \xi \chi_{\mathbb{R}^+} \} \|_{L^2_{(\xi|\leq R)}}^2 \]
\[ = \| H_l(r\xi) \cdot \mathcal{P}_l f(\xi) \xi \chi_{\mathbb{R}^+} \|_{L^2_{(\xi|\leq R)}}^2 \]
\[ = \int_0^{+\infty} \left( \int_0^R (H_l(r\xi) \cdot \mathcal{P}_l f(\xi))^2 r \, dr \right) \xi^2 \, d\xi. \]

We now consider the double integral above componentwise ($H_k$ is a matrix), and deal with each component separately. The first one (the second one is analogous) reads as
\[ \int_0^{+\infty} \left( \int_0^R (\chi_l(r\xi))^2 r \, dr \right) |\mathcal{P}_l f(\xi)|^2 \xi^2 \, d\xi \]
where we recall that $\chi_l$ denotes the radial component of the generalized eigenfunctions of the operator $D_{A,l}$. To estimate inner integral in (3.8) we rely on the following estimate (see [31])
\[ \int_0^R J_k(|\xi|) r^2 \, dr < C R \frac{1}{|\xi|} \]
which holds with a constant $C$ independent on $R$ and $k$. Thus, we eventually obtain
\[ (3.8) \lesssim CR \int_0^{+\infty} |\mathcal{P}_l f(\xi)|^2 \xi \, d\xi = R \| f \|_{L^2}^2 \]
as $\mathcal{P}_l$ is an isometry on $L^2$. Notice that the constant $C$ does not depend on $l$; therefore we can sum in decomposition (2.1) and by using the triangle inequality we obtain (1.6).

3.2. Proof of Theorem 1.3. The argument here turns out to be only a slight modification of the original one for the wave equation in [23] (see also [22]) as, in fact, the only tools needed are the local smoothing estimate given by (1.6) and a standard energy estimate. We include the proof here anyway for the sake of completeness.
We start from the case $\mu < -1/2$: in this range we have, by applying estimate (1.6),
\[
\|e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}} \lesssim \sum_{j \geq 0} 2^{j\mu} \|e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{|x| \leq 2}} \lesssim \sum_{j \geq 0} 2^{j(\mu+1/2)} \|f\|_{L^{2}_{x}} \leq \|f\|_{L^{2}_{x}}.
\]
Now we deal with the case $-1/2 \leq \mu \leq 0$; we start by considering the case $T \leq 1$. Here, estimate (1.10) is in fact weaker than the energy estimate
\[
\|e^{itD_{A}} f\|_{L^{\infty}_{t}L^{2}_{x}} \leq \|f\|_{L^{2}_{x}},
\]
so that we can immediately write
\[
\|e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}} \lesssim T^{1/2} \|e^{itD_{A}} f\|_{L^{\infty}_{t}L^{2}_{x}} \lesssim A_{\mu}(T) \|f\|_{L^{2}_{x}}
\]
as $\mu \geq -1/2$. In the region $T \geq 2$ we can use energy estimate to have a control on the region $\{x : |x| \geq T\}$ as follows
\[
\|\langle x \rangle^{\mu} e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}} \lesssim T^{\mu} \|e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}} \leq T^{1/2+\mu} \|f\|_{L^{2}_{x}} \lesssim A_{\mu}(T) \|f\|_{L^{2}_{x}}.
\]
For the remaining region, i.e. $T \in (1, 2)$, we rely again on (1.6) to write
\[
\|\langle x \rangle^{\mu} e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}}^{2} \leq \sum_{0 \leq j \leq \ln(T)} 2^{2j\mu} \|e^{itD_{A}} f\|_{L^{2}_{t}L^{2}_{x}}^{2} \lesssim \sum_{0 \leq j \leq \ln(T)} 2^{2j(\mu+1/2)} \|f\|_{L^{2}_{x}}^{2} \leq A_{\mu}(T)^{2} \|f\|_{L^{2}_{x}}^{2}.
\]
Eventually, we give a proof of (1.11), which is a simple scaling argument from (1.10). If we considered for some $\lambda > 0$ the rescaled function $f_{\lambda}(x) = \lambda^{\frac{\gamma}{2}} f(\lambda x)$ so that $(e^{itD_{A}} f_{\lambda})(x) = \lambda^{\frac{\gamma}{2}} (e^{itD_{A}} f)(\lambda x)$, we have that estimate (1.10) reads as
\[
\|\langle x \rangle^{\mu} e^{itD_{A}} f_{\lambda}\|_{L^{2}_{t}L^{2}_{x}} \leq \lambda^{-\frac{(1/2)+\mu}{2}} T^{\frac{1}{2}+\mu} \|f\|_{L^{2}_{x}}.
\]
Now we rewrite the (square of the) LHS of (3.9) by exploiting the change of variables $s = \lambda t$ and then $y = \lambda x$ to have
\[
\int_{0}^{T/\lambda} \|e^{itD_{A}} f_{\lambda}(x)\|_{L^{2}_{x}}^{2} dt = \lambda^{-\frac{1}{2}} \int_{0}^{T} \|\lambda^{\frac{\gamma}{2}} (1 + |x|^{2})^{\frac{\mu}{2}} (e^{itD_{A}} f)(\lambda x)\|_{L^{2}_{x}}^{2} ds
\]
\[
= \lambda^{-\frac{(1/2)+\mu}{2}} \int_{0}^{T} \int_{\mathbb{R}^{3}} (\lambda^{2} + |y|^{2})^{\mu} (e^{ixD_{A}} f)^{2}(y) dy ds.
\]
Plugging identity above into (3.9) and sending $\lambda \to 0$ will give (1.11), by dominated convergence.
References

[1] Y. Aharonov, D. Bohm. Significance of electromagnetic potentials in the quantum theory. Phys. Rev. (2) 115 485-491 (1959).
[2] N. Arrizabalaga, L. Fanelli, and A. García. On the lack of dispersion for a class of magnetic Dirac flows. J. Evol. Eq. 13
[3] J. Avron, and B. Simon. A counterexample to the paramagnetic conjecture. Phys. Lett. A 79 (1979/80), no. 172, 417-42.
[4] N. Boussaid, P. D’Ancona and L. Fanelli. Virial identity and weak dispersion for the magnetic Dirac equation. J. Math. Pures et App., 95 (2011), 137-150.
[5] N. Burq, F. Planchon, J.G. Stalker and A. Tahvildar-Zadeh Shadi. Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential. J. Funct. Anal. 203 (2), 519-549 (2003).
[6] N. Burq, F. Planchon, J.G. Stalker and A. Tahvildar-Zadeh Shadi. Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, Indiana Univ. Math. J. 53 (6), 1665-1680 (2004).
[7] F. Cacciafesta. Virial identity and dispersive estimates for the n-dimensional Dirac equation. J. Math. Sci. Univ. Tokyo 18, 1-23 (2011).
[8] F. Cacciafesta and L. Fanelli. Weak dispersion and weighted Strichartz inequalities for fractional Schrödinger equations in Aharonov-Bohm magnetic fields. http://arxiv.org/abs/1604.02726.
[9] F. Cacciafesta and Eric Séré. Local smoothing estimates for the Dirac Coulomb equation in 2 and 3 dimensions. J. Funct. Anal. 239 no.8, 2339-2358 (2016)
[10] P. D’Ancona. Kato smoothing and Strichartz estimates for wave equations with magnetic potentials. Commun. Math. Phys. 335, No. 1, 1-16 (2015)
[11] A. Erdelyi et al. Higher transcendental functions. McGraw-Hill Book Company, Inc., New York, 1953, Vol. 1.
[12] Ph. de Sousa Gerbert. Fermions in an Aharonov-Bohm field and cosmic strings. Phys. Rev. D 40, 1346 (1989).
[13] H. Falomir and P. A. G. Pisani. Hamiltonian self-adjoint extensions for (2+1)-dimensional Dirac particles. J. Phys. A 34, no. 19, 4143-4154 (2001).
[14] L. Fanelli, V. Felli, M. Fontelos and A. Primo. Time decay of scaling critical electromagnetic Schrödinger flows, Comm. Math. Phys. 324 (3), 1033–1067 (2013).
[15] L. Fanelli, V. Felli, M. Fontelos and A. Primo. Time decay of scaling invariant electromagnetic Schrödinger equations on the plane. Comm. Math. Phys. 337, 1515-1533 (2015).
[16] L. Fanelli, V. Felli, M. Fontelos and A. Primo. Frequency-dependent time decay of Schrödinger flows. To appear in J. Spectral Theory.
[17] L. Fanelli, and A. García. Counterexamples to Strichartz estimates for the magnetic Schrödinger equation, Comm. Cont. Math. 13 (2011) no. 2, 213–234.
[18] L. Fanelli, G. Grillo, and H. Kovářík. Improved time-decay for a class of scaling critical electromagnetic Schrödinger flows, J. Func. Anal. 269 (2015), 3336–3346.
[19] D. Fang and C. Wang. Weighted Strichartz estimates with angular regularity and their applications. Forum Math. 23, no. 1, 181-205 (2011).
[20] M. Goldberg, L. Vega, and N. Visciglia. Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials, Int. Math Res Not., 2006 Vol. 2006: article ID 13927.
[21] G. Grillo, and H. Kovařík. Weighted dispersive estimates for two-dimensional Schrödinger operators with Aharonov-Bohm magnetic field. *Journal of Differential Equations* **256** (2014), 3889–3911.

[22] J. C. Jiang, C. Wang, and X. Yu. Generalized and weighted Strichartz estimates. *Commun. Pure Appl. Anal.* 11 no. 5, 1723-1752 (2012).

[23] M. Keel, H. Smith and C. D. Sogge. Almost global existence for some semilinear wave equations, Dedicated to the memory of Thomas H. Wolff. *J. Anal. Math.* 87, 265-279 (2002).

[24] V. R. Khalilov and K. E. Lee. Fermions in scalar Coulomb and Aharonov-Bohm potentials in 2+1 dimensions. *J. Phys. A: Math. Theor.* 44 205303 (2011).

[25] L. J Landau. Bessel functions: monotonicity and bounds. *J. London Math. Soc.* (2) 61, no. 1, 197-215 (2000).

[26] W. Magnus and F. Oberhettinger. Formeln und Sätze für die speziellen Funktionen der mathematischen Physik. Springer-Verlag, Berlin, (1948).

[27] M. Peshkin and A. Tonomura. The Aharonov-Bohm Effect. *Lect. Notes Phys.* 340 (1989).

[28] F. Planchon, J. Stalker and A. Tahvildar-Zadeh Shadi.. $L^p$ estimates for the wave equation with the inverse-square potential. *Discrete Contin. Dynam. Systems*, 9(2):427-442, (2003).

[29] Y. A. Sitenko. Self-adjointness of the two-dimensional massless Dirac Hamiltonian and vacuum polarization effects in the background of a singular magnetic vortex. *Ann. Physics* 282, no. 2, 167-217 (2000).

[30] K. Stempak. A weighted uniform $L^p$-estimate of Bessel functions: a note on a paper of Guo. *Proc. Amer. Math. Soc.* 128, 2943-2945 (2000).

[31] R. Strichartz. Harmonic analysis as spectral theory of the Laplacians. *J. Func. Anal.* 87, 51-148 (1989).

[32] B. Thaller. The Dirac Equation. Springer-Verlag, Texts and Monographs in Physics (1992).

[33] J. Weidmann. Oszillationsmethoden für Systeme gewöhnlicher Differentialgleichungen. *Math. Z.* 119 349-373 (1971).