Hasse Principle for Simply Connected Groups over Function Fields of Surfaces

Yong HU

Abstract

Let $K$ be the function field of a $p$-adic curve, $G$ a semisimple simply connected group over $K$ and $X$ a $G$-torsor over $K$. A conjecture of Colliot-Thélène, Parimala and Suresh predicts that if for every discrete valuation $v$ of $K$, $X$ has a point over the completion $K_v$, then $X$ has a $K$-rational point. The main result of this paper is the proof of this conjecture for groups of some classical types. In particular, we prove the conjecture when $G$ is of one of the following types: (1) $2A_n^*$, i.e. $G = \text{SU}(h)$ is the special unitary group of some hermitian form $h$ over a pair $(D, \tau)$, where $D$ is a central division algebra of square-free index over a quadratic extension $L$ of $K$ and $\tau$ is an involution of the second kind on $D$ such that $L^\tau = K$; (2) $B_n$, i.e., $G = \text{Spin}(q)$ is the spinor group of quadratic form of odd dimension over $K$; (3) $D_n^*$, i.e., $G = \text{Spin}(h)$ is the spinor group of a hermitian form $h$ over a quaternion $K$-algebra $D$ with an orthogonal involution.

Our method actually yields a parallel local-global result over the fraction field of a 2-dimensional, henselian, excellent local domain with finite residue field, under suitable assumption on the residue characteristic.

MSC classes: 11E72, 11E57

1 Introduction

Let $K$ be a field and $G$ a smooth connected linear algebraic group over $K$. The cohomology set $H^1(K, G)$ classifies up to isomorphism $G$-torsors over $K$, and a class $\xi \in H^1(K, G)$ is trivial if and only if the corresponding $G$-torsor has a $K$-rational point. Let $\Omega_K$ denote the set of (normalized) discrete valuations (of rank 1) of the field $K$. For each $v \in \Omega_K$, let $K_v$ denote the completion of $K$ at $v$. The restriction maps $H^1(K, G) \to H^1(K_v, G), v \in \Omega_K$ induce a natural map of pointed sets

$$H^1(K, G) \to \prod_{v \in \Omega_K} H^1(K_v, G).$$

If the kernel of this map is trivial, we say that the Hasse principle with respect to $\Omega_K$ holds for $G$-torsors over $K$. 
In the case of a $p$-adic function field, by which we mean the function field of an algebraic curve over a $p$-adic field (i.e., a finite extension of $\mathbb{Q}_p$), the following conjecture was made by Colliot-Thélène, Parimala and Suresh.

**Conjecture 1.1** \cite{6}. Let $K$ be the function field of an algebraic curve over a $p$-adic field and let $G$ be a semisimple simply connected group over $K$.

Then the kernel of the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G)$$

is trivial. In other words, if a $G$-torsor has points in all completions $K_v$, $v \in \Omega_K$, then it has a $K$-rational point.

(1.2) Let $K$ be a $p$-adic function field with field of constants $F$, i.e., $K$ is the function field of a smooth projective geometrically integral curve over the $p$-adic field $F$. Let $A$ be the ring of integers of $F$. It is in particular a henselian excellent local domain of dimension 1. By resolution of singularities, there exists a proper flat morphism $\mathcal{X} \to \text{Spec}A$, where $\mathcal{X}$ is a connected regular 2-dimensional scheme with function field $K$. We will say that $\mathcal{X}$ is a $p$-adic arithmetic surface with function field $K$, or that $\mathcal{X} \to \text{Spec}A$ is a regular proper model of the $p$-adic function field $K$.

An analog in the context of a 2-dimensional base is as follows. Let $A$ be a henselian excellent 2-dimensional local domain with finite residue field and let $K$ be the field of fractions of $A$. Again by resolution of singularities, there exists a proper birational morphism $\mathcal{X} \to \text{Spec}A$, where $\mathcal{X}$ is a connected regular 2-dimensional scheme with function field $K$. We will say that $\mathcal{X}$ is a local henselian surface with function field $K$ and that $\mathcal{X} \to \text{Spec}A$ is a regular proper model of $\text{Spec}A$.

Experts have also been interested in the following analog of Conjecture 1.1:

**Question 1.3.** Let $K$ be the function field of a local henselian surface $\text{Spec}A$ with finite residue field and let $G$ be a semisimple simply connected group over $K$.

Does the Hasse principle with respect to $\Omega_K$ hold for $G$-torsors over $K$?

Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field. For most quasi-split $K$-groups, the Hasse principle may be proved by combining an injectivity property of the Rost invariant map (cf. \cite[Thm. 5.3]{6}) and results from higher dimensional class field theory of Kato and Saito.

The goal of this paper is to prove the Hasse principle for groups of several types in the non-quasisplit case. To give precise statement of our main result, we will refine the usual classification of absolutely simple simply connected groups in some cases.

(1.4) Let $E$ be a field and let $G$ be an absolutely simple simply connected group over $E$. We say that $G$ is of type

(1) $^1A_n^*$, if $G = \text{SL}_n(A)$ is the special linear group of some central simple $E$-algebra $A$ of square-free index;

(2) $^2A_n^*$, if $G = \text{SU}(h)$ is the special unitary group of some nonsingular hermitian form $h$ over a pair $(D, \tau)$, where $D$ is a central division algebra of square-free index over
a separable quadratic field extension $L$ of $E$ and $\tau$ is an involution of the second kind on $D$ such that $L'=E$; when the index of division algebra $D$ is odd (resp. even), we say the group $G = \text{SU}(h)$ is of type $^2A_n^*$ of odd (resp. even) index;

(3) $C_n^*$, if $G = \text{U}(h)$ is the unitary group (also called symplectic group) of a nonsingular hermitean form $h$ over a pair $(D, \tau)$, where $D$ is quaternion algebra over $E$ and $\tau$ is a symplectic involution on $D$;

(4) $D_n^*$ (in characteristic $\neq 2$), if $G = \text{Spin}(h)$ is the spin group of a nonsingular hermitean form $h$ over a pair $(D, \tau)$, where $D$ is quaternion algebra over $E$ and $\tau$ is an orthogonal involution on $D$;

(5) $F_{4}^{\text{red}}$ (in characteristic different from 2, 3), if $G = \text{Aut}_{\text{alg}}(J)$ is the group of algebra automorphisms of some reduced exceptional Jordan $E$-algebra $J$ of dimension 27.

Recall also that $G$ is of type

(6) $B_n$ (in characteristic $\neq 2$), if $G = \text{Spin}(q)$ is the spin group of a nonsingular quadratic form $q$ of dimension $2n+1$ over $E$;

(7) $G_2$ (in characteristic $\neq 2$), if $G = \text{Aut}_{\text{alg}}(C)$ is the group of algebra automorphisms of a Cayley algebra $C$ over $E$.

(1.5) In the local henselian case, we shall exclude some possibilities for the residue characteristic. To this end, we define for any semisimple simply connected group $G$ a set $S(G)$ of prime numbers as follows (cf. [20, §2.2] or [9, p.44]):

$S(G) = \{2\}$, if $G$ is of type $G_2$ or of classical type $B_n, C_n$ or $D_n$ (trialitarian $D_4$ excluded);

$S(G) = \{2, 3\}$, if $G$ is of type $E_6, E_7, F_4$ or trialitarian $D_4$;

$S(G) = \{2, 3, 5\}$, if $G$ is of type $E_8$;

$S(G)$ is the set of prime factors of the index $\text{ind}(A)$ of $A$, if $G = \text{SL}_1(A)$ for some central simple algebra $A$;

$S(G)$ is the set of prime factors of $2.\text{ind}(D)$, if $G = \text{SU}(h)$ for some nonsingular hermitean form $h$ over a division algebra $D$ with an involution of the second kind.

In the general case, define $S(G) = \cup S(G_i)$, where $G_i$ runs over the almost simple factors of $G$.

When $G$ is absolutely simple, let $n_G$ be the order of the Rost invariant of $G$. Except for a few cases where $n_G = 1$, the set $S(G)$ coincides with the set of prime factors of $n_G$ (cf. [21, Appendix B] or [17, §31.B]).

We summarize our main results in the following two theorems.

**Theorem 1.6.** Let $K$ be the function field of a $p$-adic arithmetic surface and $G$ a semisimple simply connected group over $K$. Assume $p \neq 2$ if $G$ contains an almost simple factor of type $^2A_n^*$ of even index.

If every almost simple factor of $G$ is of type

$$^1A_n^*, \; ^2A_n^*, \; B_n^*, \; C_n^*, \; D_n^*, \; F_4^{\text{red}} \; \text{or} \; G_2,$$

* R. Preeti [26] has proved results on the injectivity of the Rost invariant which overlap with the results in this paper. Our work was carried out independently.
then the natural map

\[ H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G) \]

has a trivial kernel.

**Theorem 1.7.** Let \( K \) be the function field of a local henselian surface with finite residue field of characteristic \( p \). Let \( G \) be a semisimple simply connected group over \( K \). Assume \( p \notin \mathbb{S}(G) \).

If every almost simple factor of \( G \) is of type

\[ 1A_n^*, 2A_n^* \text{ of odd index, } B_n, C_n^*, D_n^*, F_{4\text{red}} \text{ or } G_2, \]

then the natural map

\[ H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^1(K_v, G) \]

has a trivial kernel.

If moreover the Hasse principle with respect to \( \Omega_K \) holds for quadratic forms \( q \) of rank 6 over \( K \) (i.e., \( q \) has a nontrivial zero over \( K \) if and only if it has a nontrivial zero over every \( K_v, v \in \Omega_K \)), then the same result is also true for an absolutely simple group of type \( 2A_n^* \) of even index.

In fact, it suffices to consider only divisorial discrete valuations in the above theorems.

**Remark 1.8.** Let \( K \) be as in Theorem 1.6 or 1.7. Assume the residue characteristic \( p \) is not 2.

1. By [28, Thm. 3.4] (the arithmetic case) and [13, Thm. 3.4] (the local henselian case), a central division algebra of exponent 2 over the field \( K \) is either a quaternion algebra or a biquaternion algebra. So for a group of type \( C_n \), say \( G = U(h) \) with \( h \) a hermitian form over a symplectic pair \((D, \tau)\), the only case not covered by our theorems is the case where \( D \) is a biquaternion algebra. Similarly, for a group of classical type \( D_n \), say \( D = \text{Spin}(h) \) with \( h \) a hermitian form over an orthogonal pair \((D, \tau)\), the only remaining case is the one with \( D \) a biquaternion algebra.

2. In Theorem 1.7, the hypothesis on the Hasse principle for quadratic forms of rank 6 is satisfied if \( K = \text{Frac}(O[[t]]) \) is the fraction field of a formal power series ring over a complete discrete valuation ring \( O \) (whose residue field is finite), by [12, Thm. 1.2]. In the arithmetic case, this is established in [6, Thm. 3.1].

In the rest of the paper, after some preliminary reviews in Section 2, we will prove our main theorems case by case: the cases \( 1A_n^*, C_n^*, F_{4\text{red}} \) and \( G_2 \) in Section 3, the cases \( B_n \) and \( D_n^* \) in Sections 4 and 5, and the case \( 2A_n^* \) in Section 6.

Our proofs use ideas from Parimala and Preeti’s paper [24]. In particular, two exact sequences of Witt groups, due to Parimala–Sridharan–Suresh and Suresh respectively, play a special role in some cases. Other important ingredients include Hasse principles for degree 3 cohomology of \( \mathbb{Q}/\mathbb{Z}(2) \) coming from higher dimensional class field theory of Kato and Saito (cf. [15] and [27]), as well as the work of Merkurjev and Suslin on reduced norm criterion and norm principles ([31], [20]). For spinor groups and groups of type \( 2A_n^* \) of even index, we also make use of results on quadratic forms over the base field \( K \) obtained in [25], [19] (see also [11]) in the \( p \)-adic case and in [13] in the local henselian case.
2 Some reviews and basic tools

In this section, we briefly review some basic notions which will be used frequently and we recall some known results that are essential in the proofs to come later.

Throughout this section, let $L$ denote a field of characteristic different from 2.

2.1 Hermitian forms and Witt groups

We will assume the readers have basic familiarity with the theory of involutions and hermitian forms over central simple algebras (cf. [29], [16], [17]). For later use, we recall in this subsection some facts on Witt groups, the “key exact sequence” of Parimala, Sridharan and Suresh and the exact sequence of Suresh. The readers are referred to [3, §3 and Appendix 2], [4, §3] and [23, §8] for more information.

Unless otherwise stated, all hermitian forms and skew-hermitian forms (in particular all quadratic forms) in this paper are assumed to be nonsingular.

(2.1) Let $L$ be a field of characteristic different from 2, $A$ a central simple algebra over $L$ and $\sigma$ an involution on $A$. Let $E = L^\sigma$. We say that $\sigma$ is an $L/E$-involution on $A$. To each hermitian or skew-hermitian form $(V, h)$ over $(A, \sigma)$, one can associate an involution on $\text{End}_A(V)$, called the adjoint involution on $\text{End}_A(V)$ with respect to $h$. This is the unique involution $\sigma_h$ on $\text{End}_A(V)$ such that

$$h(x, f(y)) = h(\sigma_h(f)(x), y), \quad \forall x, y \in V, \quad \forall f \in \text{End}_A(V).$$

For a fixed finitely generated right $A$-module $V$, define an equivalence relation $\sim$ on the set of hermitian or skew-hermitian forms on $V$ (with respect to the involution $\sigma$) by

$$h \sim h' \iff \text{there exists } \lambda \in E^* \text{ such that } h = \lambda h'.$$

Let $\mathcal{H}^+(V)$ (resp. $\mathcal{H}^-(V)$) denote the set of equivalence classes of hermitian (resp. skew-hermitian) forms on $V$ and let $\mathcal{H}^\pm(V) = \mathcal{H}^+(V) \cup \mathcal{H}^-(V)$. The assignment $h \mapsto \sigma_h$ defines a map from $\mathcal{H}^\pm(V)$ to the set of involutions on $\text{End}_A(V)$. If $\sigma$ is of the first kind, then the map $h \mapsto \sigma_h$ induces a bijection between $\mathcal{H}^\pm(V)$ and the set of involutions of the first kind on $\text{End}_A(V)$, and the involutions $\sigma_h$ and $\sigma$ have the same type (orthogonal or symplectic) if $h$ is hermitian and they have opposite types if $h$ is skew-hermitian. If $\sigma$ is of the second kind, then the map $h \mapsto \sigma_h$ induces a bijection between $\mathcal{H}^+(V)$ and the set of $L/E$-involutions on $\text{End}_A(V)$. (cf. [17] p.43, Thm. 4.2.)

If $A = L$ and $\sigma = \text{id}$, a hermitian (resp. skew-hermitian) form $h$ is simply a symmetric (resp. skew-symmetric) bilinear form $b$. In this case, $b \mapsto \sigma_b$ defines a bijection between equivalence classes of nonsingular symmetric or skew-symmetric bilinear forms on $V$ modulo multiplication by a factor in $L^*$ and involutions of the first kind on $\text{End}_L(V)$. If $q$ is the quadratic form associated to a symmetric bilinear form $b$, we also write $\sigma_q$ for the adjoint involution $\sigma_b$.

(2.2) Let $(A, \sigma)$ be a pair consisting of a central simple algebra $A$ over a field $L$ of characteristic $\neq 2$ and an involution (of any kind) $\sigma$ on $A$. The orthogonal sum
of hermitian forms defines a semigroup structure on the set of isomorphism classes of hermitian forms over \((A, \sigma)\). The quotient of the corresponding Grothendieck group by the subgroup generated by hyperbolic forms is called the Witt group of \((A, \sigma)\) and denoted \(W(A, \sigma) = W^1(A, \sigma)\). The same construction applies to skew-hermitian forms and the corresponding Witt group will be denoted \(W^{-1}(A, \sigma)\).

If \(A = L\) and \(\sigma = \text{id}\), then \(W(A, \sigma)\) is the usual Witt group \(W(L)\) of quadratic forms (cf. [18], [29]). One has a ring structure on \(W(L)\) induced by the tensor product of quadratic forms. The classes of even dimensional forms form an ideal \(I(L)\) of the ring \(W(L)\). For each \(n \geq 1\), we write \(I^n(L)\) for the \(n\)-th power of the ideal \(I(L)\). As an abelian group, \(I^n(L)\) is generated by the classes of \(n\)-fold Pfister forms.

(2.3) Let \(D\) be a quaternion division algebra over a field \(L\) of characteristic \(\neq 2\). Let \(\tau_0\) be the standard (symplectic) involution on \(D\). The Witt group \(W(D, \tau_0)\) has a nice description as follows (cf. [29, p.352]). If \(h : V \times V \to D\) is a hermitian form over \((D, \tau_0)\), then the map

\[ q_h : V \to L, \quad q_h(x) := h(x, x) \]

defines a quadratic form on the \(L\)-vector space \(V\), called the trace form of \(h\). If \(h\) is isomorphic to the diagonal form \(\langle \lambda_1, \ldots, \lambda_r \rangle\), then \(q_h\) is isomorphic to the form \(\langle \lambda_1, \ldots, \lambda_r \rangle \otimes n_D\), where \(n_D\) denotes the norm form of the quaternion algebra \(D\). By [29, p.352, Thm. 10.1.7], the assignment \(h \mapsto q_h\) induces an injective group homomorphism \(W(D, \tau_0) \to W(L)\), whose image is the principal ideal of \(W(L)\) generated by (the class of) the norm form \(n_D\) of \(D\). In particular, two hermitian forms over \((D, \tau_0)\) are isomorphic if and only if their trace forms are isomorphic.

(2.4) Let \(L/E\) be a quadratic extension of fields of characteristic different from 2. The nontrivial element \(\iota\) of the Galois group \(\text{Gal}(L/E)\) may be viewed as a unitary involution on the \(L\)-algebra \(A = L\). The Witt group \(W(L, \iota)\) can be determined as follows (cf. [29, pp.348–349]):

As in (2.3), to each hermitian form \(h : V \times V \to L\) over \((L, \iota)\), one can associate a quadratic form \(q_h\) on the \(E\)-vector space \(V\), called the trace form of \(h\), by defining

\[ q_h(x) := h(x, x) \in E, \forall x \in V. \]

One can show that \(h \mapsto q_h\) induces a group homomorphism \(W(L, \iota) \to W(E)\) which identifies \(W(L, \iota)\) with the kernel of the base change homomorphism \(W(E) \to W(L)\). In particular, two hermitian forms over \((L, \iota)\) are isomorphic if and only if their trace forms are isomorphic. (cf. [29, Thm. 10.1.2].)

Let \(\delta \in E\) be an element such that \(L = E(\sqrt{\delta})\). Then for \(a \in E^*\), the trace form of \(h = \langle a \rangle\) is isomorphic to \(\langle a, -a\delta \rangle = a \langle 1, -\delta \rangle\). So the image of the map

\[ W(L, \iota) \to W(E); \quad h \mapsto q_h \]

is the principal ideal generated by the form \(\langle 1, -\delta \rangle\) (cf. [29, Remark 10.1.3]).
(2.5) Let $A$ be a central simple algebra over a field $L$ of characteristic $\text{char}(L) \neq 2$. Let $\sigma$ be an involution on $A$ and let $E = L^\sigma$. For any invertible element $u \in A^*$, let $\text{Int}(u) : A \to A$ denote the inner automorphism $x \mapsto u.x.u^{-1}$. If $\sigma(u)u^{-1} = \pm 1$, then $\text{Int}(u) \circ \sigma$ is an involution on $A$ of the same kind as $\sigma$.

Conversely, let $\sigma, \tau$ be involutions of the same kind on $A$. If $\sigma$ and $\tau$ are of the first kind, then there is a unit $u \in A^*$, uniquely determined up to a scalar factor in $E^*$, such that $\tau = \text{Int}(u) \circ \sigma$ and $\sigma(u) = \pm u$. Moreover, the two involutions $\sigma$ and $\tau = \text{Int}(u) \circ \sigma$ are of the same type (orthogonal or symplectic) if and only if $\sigma(u) = u$. If $\sigma$ and $\tau$ are of the second kind, then there exists a unit $u \in A^*$, uniquely determined up to a scalar factor in $E^*$, such that $\tau = \text{Int}(u) \circ \sigma$ and $\sigma(u) = u$.

Let $\mathcal{H}(A, \sigma) = \mathcal{H}^1(A, \sigma)$ (resp. $\mathcal{H}^{-1}(A, \sigma)$) denote the category of hermitian (resp. skew-hermitian) forms over $(A, \sigma)$. Let $\varepsilon, \varepsilon' \in \{\pm 1\}$. Let $a \in A^*$ be an element such that $\sigma(a) = \varepsilon'a$. Then the functor

$$\Phi_a : \mathcal{H}^\varepsilon(A, \text{Int}(a^{-1}) \circ \sigma) \to \mathcal{H}^{\varepsilon\varepsilon'}(A, \sigma); \quad (V, h) \mapsto (V, a.h)$$

is an equivalence of categories, called a scaling. There is also an induced isomorphism of Witt groups

$$\phi_a : W^{\varepsilon}(A, \text{Int}(a^{-1}) \circ \sigma) \cong W^{\varepsilon\varepsilon'}(A, \sigma).$$

In particular, if $\sigma$ and $\tau$ are involutions of the same kind and type on $A$, then there is a scaling isomorphism of Witt groups $\phi_a : W(A, \tau) \cong W(A, \sigma)$.

(2.6) Let $A$ be a central simple algebra over a field $L$ of characteristic $\neq 2$ and $\sigma$ an involution of any kind on $A$. Let $(V, h)$ be a hermitian form over $(A, \sigma)$. Let $B = \text{End}_A(V)$ and let $\sigma_h$ be the adjoint involution with respect to $h$. There is an equivalence of categories, called the Morita equivalence,

$$\Phi_h : \mathcal{H}(B, \sigma_h) \to \mathcal{H}(A, \sigma)$$

defined as follows (cf. [3 §1.4], [16 §1.9]): For a hermitian form $(M, f)$ over $(B, \sigma_h)$, define a map

$$h \ast f : (M \otimes_B V) \times (M \otimes_B V) \to A$$

by

$$(h \ast f)(m_1 \otimes v_1, m_2 \otimes v_2) := h(v_1, f(m_1, m_2)(v_2)).$$

One verifies that $\Phi_h(M, f) := (M \otimes_B V, h \ast f)$ yields a well-defined functor $\mathcal{H}(B, \sigma_h) \to \mathcal{H}(A, \sigma)$, which can be shown to be an equivalence (cf. [16 p.56, Thm. I.9.3.5]). The Morita equivalence induces an isomorphism of Witt groups:

$$\phi_h : W(\text{End}_A(V), \sigma_h) \cong W(A, \sigma).$$

(2.7) We briefly recall the construction of the key exact sequence of Parimala, Sridharan and Suresh. The readers are referred to [3 §3 and Appendix 2] for more details.

Let $(A, \sigma)$ be a central simple algebra with involution over $L$. Let $E = L^\sigma$. Assume there is a subfield $M \subseteq A$ which is a quadratic extension of $L$ such that $\sigma(M) = M$. Suppose $\sigma|M = \text{id}_M$ if $\sigma$ is of the first kind. Let

$$\tilde{A} := \{a \in A \mid a.m = m.a, \forall m \in M\}$$

7
be the centralizer of $M$ in $A$. This is a central simple algebra over $M$. By [3, Lemma 3.1.1], there exists $\mu \in A^*$ such that $\sigma(\mu) = -\mu$ and that the restriction of $\text{Int}(\mu)$ to $M$ is the nontrivial element of the Galois group $\text{Gal}(M/L)$.

Set $\tau = \text{Int}(\mu) \circ \sigma$ and let $\tau_1, \tau_2$ be the restrictions of $\tau$ and $\sigma$ to $\wt{A}$ respectively. Then $\tau_1$ is an involution of the second kind, $\tau_2$ is of the same kind and type as $\sigma$, and $\tau$ is orthogonal (resp. symplectic) if and only if $\sigma$ is symplectic (resp. orthogonal).

One has a decomposition $A = \wt{A} \oplus \mu.A$ (as right $M$-modules). Let $\pi_1, \pi_2 : A \to \wt{A}$ be the $M$-linear projections

$$\pi_1(x + \mu y) = x, \quad \pi_2(x + \mu y) = y, \quad \forall x, y \in \wt{A}.$$ 

These induce well-defined group homomorphisms

$$\pi_1 : W(A, \tau) \longrightarrow W(\wt{A}, \tau_1) \quad \text{and} \quad \pi_2 : W^{-1}(A, \tau) \longrightarrow W(\wt{A}, \tau_2).$$

On the other hand, let $\lambda \in M$ be an element such that $\lambda^2 \in L$ and $M = L(\lambda)$. For a hermitian form $(\wt{V}, f)$ over $(\wt{A}, \tau_1)$, define $\rho(f)$ to be the unique skew-hermitian form on $V = \wt{V} \oplus \wt{V}\mu$ which extends $\lambda.f : \wt{V} \times \wt{V} \to \wt{A}$. This defines a group homomorphism

$$\rho : W(\wt{A}, \tau_1) \longrightarrow W^{-1}(A, \tau); \quad (\wt{V}, f) \mapsto (\wt{V} \oplus \wt{V}\mu, \rho(f)).$$

The sequence

$$(2.7.1) \quad W^{\varepsilon}(A, \tau) \xrightarrow{\pi_1} W^{\varepsilon}(\wt{A}, \tau_1) \xrightarrow{\rho} W^{-\varepsilon}(A, \tau) \xrightarrow{\pi_2} W^{\varepsilon}(\wt{A}, \tau_2)$$

turns out to be an exact sequence (cf. [3, Appendix 2]).

Since $\tau(\mu) = -\mu$, one has a scaling isomorphism (cf. (2.5))

$$\phi_{\mu}^{-1} : W^{-1}(A, \tau) \xrightarrow{\sim} W(A, \sigma).$$

We may thus replace $W^{-1}(A, \tau)$ in the exact sequence (2.7.1) by $W(A, \sigma)$ and rewrite it as

$$(2.7.2) \quad W(A, \tau) \xrightarrow{\pi_1} W(\wt{A}, \tau_1) \xrightarrow{\rho} W(A, \sigma) \xrightarrow{\pi_2} W(\wt{A}, \tau_2)$$

where $\rho = \phi_{\mu}^{-1} \circ \rho$ and $\pi_2 = \pi_2 \circ \phi_{\mu}$. This exact sequence is due to Parimala, Sridharan and Suresh and is referred to as the key exact sequence in [3].

We will only use the exact sequence (2.7.2) in the case where $A = D$ is a quaternion algebra and $\sigma$ is an orthogonal involution. This special case was already discussed by Scharlau in [29, p.359].

(2.8) Now let $D$ be a quaternion division algebra over a quadratic field extension $L$ of $E$ and let $\tau$ be a unitary $L/E$-involution on $D$ (i.e. a unitary involution such that $L^\tau = E$). There is a unique quaternion $E$-algebra $D_0$ contained in $D$ such that $D = D_0 \otimes_E L$ and $\tau = \tau_0 \otimes \iota$, where $\tau_0$ is the canonical (symplectic) involution on $D_0$ and $\iota$ is the nontrivial element of the Galois group $\text{Gal}(L/E)$. Write $L = E(\sqrt{d})$ with
Let \( d \in E^* \). Then \( D = D_0 \oplus D_0\sqrt{d} \). For any hermitian form \((V, h)\) over \((D, \tau)\), we may write

\[
h(x, y) = h_1(x, y) + h_2(x, y)\sqrt{d} \quad \text{with} \quad h_i(x, y) \in D_0, \quad \text{for} \quad i = 1, 2
\]

for any \( x, y \in V \).

The projection \( h \mapsto h_2 \) defines a group homomorphism

\[
p_2 : W(D, \tau) \longrightarrow W^{-1}(D_0, \tau_0).
\]

For a hermitian form \((V_0, f)\) over \((D_0, \tau_0)\), set

\[
V = V_0 \otimes_{D_0} D = V_0 \otimes_E L = V_0 \oplus V_0\sqrt{d}
\]

and let \( \bar{\rho}(f) : V \times V \to D \) be the map extending \( f : V_0 \times V_0 \to D_0 \) by \( \tau\)-sesquilinearity. One checks that this defines a group homomorphism

\[
\bar{\rho} : W(D_0, \tau_0) \longrightarrow W(D, \tau) ; \quad (V_0, f) \longmapsto (V_0 \oplus V_0\sqrt{d}, \bar{\rho}(f)).
\]

For any quadratic form \( q \) over \( L = E(\sqrt{d}) \), there are quadratic forms \( q_1, q_2 \) over \( k \) such that \( q(x) = q_1(x) + q_2(x)\sqrt{d} \). We have thus group homomorphisms

\[
\pi_i : W(L) \longrightarrow W(E) ; \quad q \longmapsto q_i, \quad i = 1, 2.
\]

We denote by \( \tilde{\pi}_1 : W(L) \to W(D_0, \tau_0) \) the composite map

\[
W(L) \xrightarrow{\tilde{\pi}_1} W(E) \longrightarrow W(D_0, \tau_0)
\]

where the map \( W(E) \to W(D_0, \tau_0) \) is induced by base change.

Suresh (cf. [23, Prop. 8.1]) proved that the sequence

\[
W(L) \xrightarrow{\tilde{\pi}_1} W(D_0, \tau_0) \xrightarrow{\bar{\rho}} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0)
\]

is exact. We will refer to this sequence as Suresh’s exact sequence in the sequel.

### 2.2 Invariants of hermitian forms

In this subsection, we recall the definitions of some invariants of hermitian forms. For more details, see [3, §2], [4, §3] and [23, §5, §7].

(2.9) Let \( (D, \sigma) \) be a central division algebra with involution over \( L \). Let \( E = L^\sigma \). Let \((V, h)\) be a hermitian form over \((D, \sigma)\). The rank of \((V, h)\), denoted \( \text{rank}(V, h) \) or simply \( \text{rank}(h) \), is by definition the rank of the \( D \)-module \( V \):

\[
\text{rank}(h) := \text{rank}_D(V).
\]
With notation as in (2.9), let $e_1, \ldots, e_n$ be a basis of the $D$-module $V$ (so that $\text{rank}(h) = \text{rank}_D(V) = n$). Let $M(h) := (h(e_i, e_j))$ be the matrix of the hermitian form $h$ with respect to this basis. The matrix algebra $A = M_n(D)$ has dimension
\[ \dim_L A = n^2 \dim_L D = (\text{rank}(h) \cdot \deg_L D)^2. \]

Put
\[ m = \sqrt{\dim_L A} = \text{rank}(h) \cdot \deg_L D = \frac{\dim_L V}{\deg_L D}. \]

We define the discriminant $\text{disc}(h) = \text{disc}(V, h)$ of the hermitian form $(V, h)$ by
\[ \text{disc}(h) = (-1)^{\frac{m(m-1)}{2}} \text{Nrd}_A(M(h)) \in \begin{cases} E^*/E^{*2} & \text{if } \sigma \text{ is of the first kind} \\ E^*/N_{L/E}(E^*) & \text{if } \sigma \text{ is of the second kind} \end{cases} \]

If $h$ is a hermitian form over $(D, \sigma)$, the image of the canonical map
\[ H^1(E, \text{SU}(h)) \rightarrow H^1(E, \text{U}(h)) \]
consists of classes $[h'] \in H^1(E, \text{U}(h))$ of hermitian forms $h'$ which have the same rank and discriminant as $h$.

(2.10) Let $D$ be a central division algebra over $L$ and let $\sigma$ be an orthogonal involution on $D$. Note that the Brauer class of $D$ in the Brauer group $\text{Br}(L)$ lies in the subgroup $\text{Br}(L) := \{ \alpha \in \text{Br}(L) \mid 2.\alpha = 0 \}$.

Let $h$ be a hermitian form over $(D, \sigma)$. Let
\[ \delta : H^1(L, \text{SU}(h)) \rightarrow H^2(L, \mu_2) = \text{Br}(L) \]
be the connecting map associated to the exact sequence of algebraic groups
\[ 1 \rightarrow \mu_2 \rightarrow \text{Spin}(h) \rightarrow \text{SU}(h) \rightarrow 1. \]

Let $h'$ be a hermitian form over $(D, \sigma)$ such that $\text{rank}(h') = \text{rank}(h)$ and $\text{disc}(h') = \text{disc}(h)$. Then there is an element $c(h') \in H^1(L, \text{SU}(h))$ which lifts $[h'] \in H^1(L, \text{U}(h))$. The class of $\delta(c(h'))$ in the quotient $2\text{Br}(L)/\langle [D] \rangle$ is independent of the choice of $c(h')$ (cf. [3, §2.1]). Following [2], we define the relative Clifford invariant $\mathcal{C} \ell_h(h')$ by
\[ \mathcal{C} \ell_h(h') := [\delta(c(h'))] \in \frac{2\text{Br}(L)}{\langle [D] \rangle}. \]

When $h$ has even rank $2n$ and trivial discriminant, the Clifford invariant $\mathcal{C} \ell(h)$ of $h$ is defined as
\[ \mathcal{C} \ell(h) := \mathcal{C} \ell_{H_{2n}}(h) \in \frac{2\text{Br}(L)}{\langle [D] \rangle}, \]
where $H_{2n}$ denotes a hyperbolic hermitian form of rank $2n = \text{rank}(h)$ over $(D, \sigma)$. If $D = L$ and $h = q$ is a nonsingular quadratic form over $L$, then $\mathcal{C} \ell(h)$ coincides with the usual Clifford invariant of the quadratic form $q$. 

10
(2.12) Let $(D, \sigma)$ be a central division algebra with an orthogonal involution over $L$. We denote by $U_{2n}(D, \sigma)$, $SU_{2n}(D, \sigma)$ and $Spin_{2n}(A, \sigma)$ respectively the unitary group, the special unitary group and the spin group of the hyperbolic form over $(D, \sigma)$ defined by the matrix $H_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$.

Let $h$ be a hermitian form of even rank $2n$, trivial discriminant and trivial Clifford invariant. There is an element $\xi \in H^1(L, Spin_{2n}(D, \sigma))$ which is mapped to the class $[h] \in H^1(L, U_{2n}(D, \sigma))$ under the composite map \[ H^1(L, Spin_{2n}(D, \sigma)) \rightarrow H^1(L, SU_{2n}(D, \sigma)) \rightarrow H^1(L, U_{2n}(D, \sigma)). \]

Let \[ R_{Spin_{2n}(D, \sigma)} : H^1(L, Spin_{2n}(D, \sigma)) \rightarrow H^3(L, \mathbb{Q}/\mathbb{Z}(2)) \] be the usual Rost invariant map of the simply connected group $Spin_{2n}(D, \sigma)$ (cf. [17 §31.B]). It is shown in [4, p.664] that the class of $R_{Spin_{2n}(D, \sigma)}(\xi)$ in the quotient \[ \frac{H^3(L, \mathbb{Q}/\mathbb{Z}(2))}{H^1(L, \mu_2) \cup (D)} \] is well-defined. The Rost invariant $\mathcal{R}(h)$ of the form $h$ is defined as \[ \mathcal{R}(h) := [R_{Spin_{2n}(D, \sigma)}(\xi)] \in \frac{H^3(L, \mathbb{Q}/\mathbb{Z}(2))}{H^1(L, \mu_2) \cup (D)}. \]

(2.13) Let $(D, \sigma)$ be a quaternion algebra with an orthogonal involution over $L$. We will need some further analysis on the map $\tilde{\rho} : W(\tilde{D}, \tau_1) \rightarrow W(D, \sigma)$ in the exact sequence \[ 2 \rightarrow 2 \rightarrow U_{2n}(M, \iota) \rightarrow SU_{2n}(M, \iota) \rightarrow U_{2n}(D, \sigma) \]. Note that in this case $\tilde{D} = M$ is a quadratic field extension of $L$ and $\tau_1$ is the nontrivial element $\iota$ of the Galois group $Gal(M/L)$. Let $U_{2n}(M, \iota)$ and $SU_{2n}(M, \iota)$ denote the unitary group and the special unitary group of the hyperbolic form over $(M, \iota)$ defined by the matrix $H_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. We have \[ U_{2n}(M, \iota)(L) = \{ A \in M_{2n}(M) \mid A.H_{2n}\iota(A)^t = H_{2n} \}. \]

Note that for $A \in M_{2n}(M)$, $\iota(A) = Int(\mu) \circ \sigma(A) = \mu A \mu^{-1}$ (cf. (2.7)) and \[ A.H_{2n}\iota(A)^t = H_{2n} \iff (A.H_{2n}\iota(A)^t)^t = (H_{2n})^t \iff \iota(A).H_{2n}.A^t = H_{2n}. \]

Therefore, for $A \in U_{2n}(M, \iota)(L)$, we have \[ A.\mu^{-1}\lambda H_{2n}.\sigma(A)^t = A.\mu^{-1}\lambda H_{2n}.A^t = \mu^{-1}(\mu A \mu^{-1})\lambda H_{2n}.A^t = \mu^{-1}\lambda \iota(A).H_{2n}.A^t = \mu^{-1}\lambda H_{2n} \]

inside $M_{2n}(D)$. So we have a natural inclusion \[ U_{2n}(M, \iota)(L) \subseteq U(\mu^{-1}\lambda H_{2n})(L) = \{ B \in M_{2n}(D) \mid B.\mu^{-1}\lambda H_{2n}.\sigma(B)^t = \mu^{-1}\lambda H_{2n} \}. \]
In fact, this defines an inclusion of algebraic groups over $L$:

$$\rho' : U_{2n}(M, \iota) \longrightarrow U(\mu^{-1}\lambda H_{2n}) ; \ A \mapsto A .$$

By [17, p.402, Example 29.19], any element $\xi$ of $H^1(L, U_{2n}(M, \iota))$ is represented by a matrix $S \in GL_{2n}(M)$ which is symmetric with respect to the adjoint involution $\iota H_{2n}$ on $M_{2n}(M)$, and $\xi$ is the isomorphism class of the hermitian form $H_{2n}S^{-1}$. The natural map

$$H^1(L, U_{2n}(M, \iota)) \longrightarrow H^1(L, U(\mu^{-1}\lambda H_{2n}))$$

induced by the homomorphism $\rho'$ maps $\xi$ to the class of the hermitian form $\mu^{-1}\lambda H_{2n}S^{-1}$. On the other hand, by the construction of the homomorphism $\tilde{\rho} : W(M, \iota) \rightarrow W(D, \sigma)$, the form $H_{2n}S'$ over $(M, \iota)$ is mapped to the form $\mu^{-1}\lambda H_{2n}S^{-1}$ over $(D, \sigma)$. Hence the natural map

$$H^1(L, U_{2n}(M, \iota)) \longrightarrow H^1(L, U(\mu^{-1}\lambda H_{2n}))$$

is compatible with the restriction of $\rho$ to forms of rank $2n$.

Clearly, the inclusion $\rho' : U_{2n}(M, \iota) \rightarrow U(\mu^{-1}\lambda H_{2n})$ induces an inclusion $SU_{2n}(M, \iota) \rightarrow SU(\mu^{-1}\lambda H_{2n})$ (cf. [4, p.671]). A choice of isomorphism of hermitian forms $\mu^{-1}\lambda H_{2n} \cong H_{2n}$ over $(D, \sigma)$ yields an injection

$$SU_{2n}(M, \iota) \longrightarrow SU(H_{2n}) = SU_{2n}(D, \sigma) .$$

This lifts to a homomorphism

$$\rho_0 : SU_{2n}(M, \iota) \longrightarrow Spin_{2n}(D, \sigma) .$$

The composition

$$SU_{2n}(M, \iota) \xrightarrow{\rho_0} Spin_{2n}(D, \sigma) \longrightarrow U_{2n}(D, \sigma)$$

induces a commutative diagram

$$\xymatrix{ H^1(L, SU_{2n}(M, \iota)) \ar[r]^{\rho_0} \ar[dr]^{\rho'} & H^1(L, Spin_{2n}(D, \sigma)) \ar[dl] \\
H^1(L, U_{2n}(D, \sigma)) }$$

such that the map $\tilde{\rho} : W(M, \iota) \rightarrow W(D, \sigma)$ restricted to forms of rank $2n$ and of trivial discriminant is compatible with the map $\rho'$ at the level of cohomology sets. Moreover, for any $\xi \in H^1(L, SU_{2n}(M, \iota))$, one has by [4, Prop. 3.20]

$$R_{Spin_{2n}(D, \sigma)}(\rho_0(\xi)) = R_{SU_{2n}(M, \iota)}(\xi) \in H^3(L, \mathbb{Q}/\mathbb{Z}(2)) ,$$

i.e., $\rho_0(\xi) \in H^1(L, Spin_{2n}(D, \sigma))$ has the same Rost invariant as $\xi$. If $h$ is a hermitian form over $(D, \sigma)$ representing the class $\rho'(\xi) \in H^1(L, U_{2n}(D, \sigma))$, then the Rost invariant of the form $h$ is

$$\mathcal{R}(h) = [R_{Spin_{2n}(D, \sigma)}(\rho_0(\xi))] = [R_{SU_{2n}(M, \iota)}(\xi)] \in \frac{H^3(L, \mathbb{Q}/\mathbb{Z}(2))}{H^1(L, \mu_2) \cup (D)} .$$
by definition (cf. (2.12)).

(2.14) We shall also use the notion of Rost invariant of hermitian forms over an algebra with unitary involution. The definition is as follows. Let $E$ be a field of characteristic $\neq 2$, $L/E$ a quadratic field extension and $(D, \tau)$ a central division algebra over $L$ with a unitary $L/E$-involution. Let $U_{2n}(D, \tau)$ and $SU_{2n}(D, \tau)$ denote respectively the unitary group and the special unitary group of the hyperbolic form $(0 \ 0 \ I_{n} \ I_{n} 0)$ over $(D, \tau)$. For a hermitian form $h$ of rank $2n$ and trivial discriminant over $(D, \tau)$, we may define its Rost invariant $\mathcal{R}(h)$ by

$$\mathcal{R}(h) := [R_{SU_{2n}(D, \tau)}(\xi)] \in H^3(E, \mathbb{Q}/\mathbb{Z}(2))/\text{Cores}_{L/E}((L^*) \cup (D)),$$

where $\xi \in H^1(E, SU_{2n}(D, \tau))$ is any lifting of the class $[h] \in H^1(E, U_{2n}(D, \tau))$ and

$$L^* = (R_{L/E}^{1} \mathbb{G}_m)(E) = \{a \in L^* | N_{L/E}(a) = 1\}.$$

Indeed, by [23, Appendix, Remark B], the class $[R_{SU_{2n}(D, \tau)}(\xi)]$ is independent of the choice of the lifting $\xi$, so that this Rost invariant $\mathcal{R}(h)$ is well defined. Note that if $D = D_0 \otimes_E L$ for some central division algebra $D_0$ over $E$, then

$$\text{Cores}_{L/E}((L^*) \cup (D)) = 0$$

and hence the Rost invariant of $h$ is simply the usual Rost invariant of any lifting $\xi \in H^1(E, SU_{2n}(D, \tau))$ of the isomorphism class of $h$.

2.3 Spinor norms

(2.15) Let $E$ be a field of characteristic different from 2, $A$ a central simple algebra over $E$ and $\sigma$ an orthogonal involution on $A$. Let $h$ be a nonsingular hermitian form over $(A, \sigma)$. The exact sequence of algebraic groups

$$1 \longrightarrow \mu_2 \longrightarrow \text{Spin}(h) \longrightarrow SU(h) \longrightarrow 1,$$

induces a connecting map

$$\delta : SU(h)(E) \longrightarrow H^1(E, \mu_2) = E^*/E^{*2}$$

which we call the spinor norm map. We will write

$$Sn(h_E) := \text{Im} (\delta : SU(h)(E) \longrightarrow E^*/E^{*2})$$

for the image of the above spinor norm map. If $A = E$, $\sigma = id$ and $h = q$ is a quadratic form, the spinor norm map $\delta : SO(q)(E) \to E^*/E^{*2}$ has an explicit description as follows (cf. [18, p.108]): Any element $\theta \in SO(q)(E)$ can be written as the product of an even number of hyperplan reflections associated with anisotropic vectors $v_1, \ldots, v_{2r}$. The spinor norm $\delta(\theta)$ is equal to the class of the product $q(v_1) \cdots q(v_{2r})$ in $E^*/E^{*2}$.

A deep theorem of Merkurjev is the following norm principle for spinor norms.
Theorem 2.16 (Merkurjev, [20, 6.2]). With notation as in (2.15), assume that \( \text{deg}(A) \cdot \text{rank}(h) \) is even and at least 4.

Then the image \( \text{Sn}(h_E) \) of the spinor norm map is equal to the subgroup of \( E^*/E^{*2} \) generated by the canonical images of the norm groups \( N_{L/E}(L^*) \) over all finite field extensions \( L/E \) such that \( A_L \) is split and \( h_L \) is isotropic.

The following corollary is immediate from the above theorem.

Corollary 2.17. With notation and hypotheses as in Theorem 2.16, for any finite field extension \( E'/E \), one has

\[
N_{E'/E}(\text{Sn}(h_{E'})) \subseteq \text{Sn}(h_E).
\]

(2.18) With notation and hypotheses as in Theorem 2.16, the well-known norm principle for reduced norms states that the subgroup \( \text{Nrd}(A^*) \subseteq E^* \) of reduced norms is generated by the norm groups \( N_{L/E}(L^*) \), where \( L/E \) runs over all finite field extensions such that \( A_L \) is split. So Theorem 2.16 implies that \( \text{Sn}(h_E) \) is contained in the canonical image of \( \text{Nrd}(A^*) \) in \( E^*/E^{*2} \).

(2.19) Let \((A, \sigma)\) be a central simple algebra with an orthogonal involution over a field \( E \) of characteristic \( \neq 2 \). Let \( L/E \) be a field extension which splits \( A \) and let \( \phi : (A, \sigma) \otimes_E L \cong (M_n(L), \sigma_{q_0}) \) be an isomorphism of \( L \)-algebras with involution, where \( \sigma_{q_0} \) is the adjoint involution of a quadratic form \( q_0 \) of rank \( n = \text{deg}(A) \) over \( L \). Let \( h \) be a hermitian form over \((A, \sigma) \otimes_E L\). Then by Morita theory (cf. (2.6)), \( h \) corresponds via the above isomorphism \( \phi \) to a quadratic form \( q \) of rank \( n \cdot \text{rank}(h) = \text{deg}(A) \cdot \text{rank}(h) \) over \( L \). The similarity class \( [q] \in W(L) \) of \( q \) is uniquely determined by \( h \) and is independent of the choice of \( \phi \) and \( q_0 \). The hermitian form \( h_L \) is isotropic if and only if the quadratic form \( q_L \) is isotropic. So, if \( \text{deg}(A) \cdot \text{rank}(h) \) is even and at least 4, one has \( \text{Sn}(q_L) = \text{Sn}(h_L) \) by Theorem 2.16.

3 Some easy cases

We shall now start the proofs of our main theorems. In a few cases, as may be already well-known to specialists, the results basically follow by combining a general injectivity result for the Rost invariant and a Hasse principle coming from higher dimensional class field theory.

(3.1) Recall that our base field \( K \) is the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field (cf. (1.2)). Namely, \( K \) is either

the case of \( p \)-adic arithmetic surface the function field \( F(C) \) of a smooth projective geometrically integral curve \( C \) over \( F \), where \( F \) is a \( p \)-adic field with ring of integers \( A \) and residue field \( k \);

or

the case of local henselian surface the field of fractions \( \text{Frac}(A) \) of a 2-dimensional, henselian, excellent local domain \( A \) with finite residue field \( k \) of characteristic \( p \).

In either case, by abuse of language we say \( k \) is the residue field of \( K \) and \( p = \text{char}(k) \) is the residue characteristic of \( K \).
In our proofs of the main theorems, we only use local conditions at divisorial valuations, i.e., valuations corresponding to codimension 1 points of regular proper models (cf. (1.2)). More precisely, the set $\Omega_A$ of divisorial valuations of the field $K$ is the subset of $\Omega_K$ defined as follows:

In $p$-adic arithmetic case, define

$$\Omega_A = \bigcup_{X \to \text{Spec} A} X^{(1)},$$

where $X \to \text{Spec} A$ runs over proper flat morphisms from a regular integral scheme $X$ with function field $K$ and $X^{(1)}$ denotes the set of codimension 1 points of $X$ identified with a subset of $\Omega_K$.

In the local henselian case, define

$$\Omega_A = \bigcup_{X \to \text{Spec} A} X^{(1)},$$

where $X \to \text{Spec} A$ runs over proper birational morphisms from a regular integral scheme $X$ with function field $K$ and $X^{(1)}$ denotes the set of codimension 1 points of $X$ identified with a subset of $\Omega_K$.

(3.2) Let $L/K$ be a finite field extension. Then $L$ is a field of the same type as $K$ if $K$ is the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field. In the $p$-adic arithmetic case, let $F'$ be the field of constants of $L$ and let $A'$ be the integral closure of $A$ in $F'$. In the local henselian case, let $A'$ be the integral closure of $A$ in $L$. Then the set $\Omega_{A'}$ of divisorial discrete valuations of $L$ is precisely the set of discrete valuations $w \in \Omega_L$ lying over valuations in $\Omega_A \subseteq \Omega_K$.

(3.3) By the general theory of semisimple groups (see e.g. [17, p.365, Thm. 26.8]), any semisimple simply connected group $G$ over $K$ is a finite product of groups of the form $R_{L/K}(G')$, where $L/K$ is a finite separable field extension, $G'$ is an absolutely simple simply connected group over $L$ and $R_{L/K}$ denotes the Weil restriction functor. For each $v \in \Omega_A$, one has $L \otimes_K K_v \cong \prod_{w | v} L_w$ and by Shapiro’s lemma,

$$H^1(K, R_{L/K}G') \cong H^1(L, G') \quad \text{and} \quad H^1(K_v, R_{L/K}G') \cong \prod_{w | v} H^1(L_w, G').$$

Therefore, to prove the Hasse principle for semisimple simply connected groups we may reduce to the case where $G$ is an absolutely simple simply connected group.

3.1 The quasi-split case

We recall the proof of the Hasse principle for quasi-split groups without $E_8$ factors (cf. [6, Thm. 5.4]).

The following theorem is of particular importance to us.
Theorem 3.4. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $\Omega_A$ be the set of divisorial discrete valuations of $K$ (as defined in (3.1)).

(i) (Kato, [13]) In the $p$-adic arithmetic case, the natural map

$$H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

(ii) (Saito, [27], cf. [13, Prop. 4.1]) In the local henselian case, let $n > 0$ be an integer prime to $p$. Then the natural map

$$H^3(K, \mu_n^{\otimes 2}) \to \prod_{v \in \Omega_A} H^3(K_v, \mu_n^{\otimes 2})$$

is injective.

The next result is an injectivity statement for the Rost invariant of quasi-split groups.

Theorem 3.5 (cf. [6, Thm. 5.3]). Let $E$ be a field of cohomological 2-dimension $\leq 3$ and let $G$ be an absolutely simple simply connected quasi-split group over $E$. Assume that $G$ is not of type $E_8$. Assume further the characteristic of $E$ is not $2$ if $G$ is of classical type $B_n$ or $D_n$.

Then the kernel of the Rost invariant map $R_G : H^1(E, G) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is trivial.

Proof. For a quasi-split group of type $^1A_n$ or $C_n$, it is well-known that $H^1(E, G) = 1$ over an arbitrary field $E$. For exceptional groups (not of type $E_8$), the kernel of the Rost invariant is trivial over an arbitrary field by the work of Chernousov, Garibaldi and Gille (cf. [9, Thm. 5.2], [5], [7] and [8]). If $G$ is of type $^2A_n$, $B_n$ or classical type $D_n$, the proof can be done as in [6, Thm. 5.3], by passing to a quadratic form argument. \qed

The $p$-adic case of the following result is [6, Thm. 5.4].

Theorem 3.6. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $G$ be an absolutely simple simply connected quasi-split group not of type $E_8$ over $K$. Assume $p \notin S(G)$ in the local henselian case (see (1.5) for the definition of $S(G)$).

Then the natural map

$$H^1(K, G) \to \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

$\dagger$See [26] for a recent improvement of this theorem.
Proof. The result follows from the following commutative diagram

\[
\begin{array}{ccc}
H^1(K, G) & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, G) \\
\downarrow & & \downarrow \\
H^3(K, \mathbb{Z}/2) & \longrightarrow & \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Z}/2)
\end{array}
\]

where the vertical maps have trivial kernel by Theorem 3.5 and the bottom horizontal map is injective by Theorem 3.4.

3.2 Groups of type $^{1}A_n^*$

For groups of inner type $A_n^*$, the proof is essentially the same as the quasi-split case.

**Theorem 3.7.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $A$ a central simple $K$-algebra of square-free index $n$ and $G = \text{SL}_1(A)$. Assume $p \nmid n$ in the local henselian case.

Then the natural map

\[
H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)
\]

is injective.

**Proof.** A well-known theorem of Suslin ([31, Thm. 24.4]) implies that under the assumptions of the theorem, the Rost invariant map

\[
H^1(E, \text{SL}_1(A)) = E^*/\text{Nrd}(A^*) \longrightarrow H^3(E, \mathbb{Z}/2) ; \quad \lambda \mapsto (\lambda) \cup (A)
\]

is injective for $E = K$ or $K_v$. An argument similar to the proof of Theorem 3.6 yields the result.

3.3 Groups of type $C_n^*$

**Lemma 3.8.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local henselian case.

Then the natural map

\[
I^3(K) \longrightarrow \prod_{v \in \Omega_A} I^3(K_v)
\]

is injective.

**Proof.** Consider the following commutative diagram

\[
\begin{array}{ccc}
I^3(K) & \longrightarrow & \prod_{v \in \Omega_A} I^3(K_v) \\
\downarrow & & \downarrow \\
H^3(K, \mathbb{Z}/2) & \longrightarrow & \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Z}/2)
\end{array}
\]
where the vertical maps are induced by the Arason invariants. Since \( \text{cd}_2(K) \leq 3 \), we have \( I^4(K) = 0 \). So the map
\[
e_3 : I^3(K) \longrightarrow H^3(K, \mathbb{Z}/2)
\]
is injective by [1, Prop. 3.1]. The map
\[
H^3(K, \mathbb{Z}/2) \longrightarrow \prod_{v \in \Omega_A} H^3(K_v, \mathbb{Z}/2)
\]
is injective by Theorem 3.4. The lemma then follows from the above commutative diagram.

**Theorem 3.9.** Let \( K \) be the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Let \( D \) be a quaternion division algebra over \( K \) with standard involution \( \tau_0 \) and \( h \) a nonsingular hermitian form over \((D, \tau_0)\). Assume \( p \neq 2 \) in the local henselian case. Let \( G = U(h) \) be the unitary group of the hermitian form \( h \).

Then the natural map
\[
H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)
\]
is injective.

**Proof.** The pointed set \( H^1(K, G) = H^1(K, U(h)) \) classifies up to isomorphism hermitian forms over \((D, \tau_0)\) of the same rank as \( h \). Let \( h_1 \) and \( h_2 \) be hermitian forms over \((D, \tau_0)\) of the same rank as \( h \). Put \( h' = h_1 \perp (-h_2) \). Note that \( h' \) has even rank, so the class of \( q_{h'} \) in the Witt group \( W(K) \) lies in the subgroup \( I^3(K) = I(K) \cdot I^2(K) \) (cf. (2.3)). Thus
\[
[q_{h_1}] - [q_{h_2}] = [q_{h'}] \in I^3(K).
\]
If \((h_1)_v \cong (h_2)_v\) for all \( v \in \Omega_A \), then by Lemma 3.8 \([q_{h'}] = 0 \in I^3(K)\). This implies that \( q_{h_1} \cong q_{h_2} \) over \( K \). Two hermitian forms over \((D, \tau_0)\) are isomorphic if and only if their trace forms are isomorphic as quadratic forms (cf. (2.3)). So we get from the above that \( h_1 \cong h_2 \), proving the theorem.

### 3.4 Groups of type \( G_2 \) or \( F^{\text{red}}_4 \)

**Theorem 3.10.** Let \( K \) be the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Let \( G \) be an absolutely simple simply connected group of type \( G_2 \) over \( K \). Assume \( p \neq 2 \) in the local henselian case.

Then the natural map
\[
H^1(K, G) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, G)
\]
has a trivial kernel.
Proof. The group $G$ is isomorphic to $\text{Aut}_{\text{alg}}(C)$ for some Cayley algebra $C$ over $K$. Let $\xi \in H^1(K, G)$ be a locally trivial class and let $C'$ be a Cayley algebra which represents $\xi$. We have $C_{K_v} \cong C'_{K_v}$ for every $v \in \Omega_A$ by hypothesis and we want to show $C \cong C'$ over $K$. Since two Cayley algebras are isomorphic if and only if their norm forms are isomorphic and since the norm form of a Cayley algebra is a 3-fold Pfister form (cf. [17, p.460]), the result follows easily from Lemma 3.8.

Theorem 3.11. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \nmid 6$ in the local henselian case. Let $G = \text{Aut}_{\text{alg}}(J)$ be the automorphism group of a reduced 27-dimensional exceptional Jordan algebra over $K$.

Then the natural map

$$H^1(K, G) \to \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

Proof. Recall that (cf. [30, $\S$9]) to each exceptional Jordan algebra $J'$ of dimension 27 over a field $F$ of characteristic not 2 or 3, one can associate three invariants

$$f_3(J') \in H^3(F, \mathbb{Z}/2), \quad f_5(J') \in H^5(F, \mathbb{Z}/2) \quad \text{and} \quad g_3(J') \in H^3(F, \mathbb{Z}/3).$$

One has $g_3(J') = 0$ if and only if $J'$ is reduced. Two reduced exceptional Jordan algebras are isomorphic if and only if their $f_3$ and $f_5$ invariants are the same.

Now our base field $K$ has cohomological 2-dimension $cd_2(K) = 3$. So the invariant $f_5(J')$ is always zero. Let $\xi \in H^1(K, G)$ correspond to the isomorphism class of an exceptional Jordan algebra $J'$ over $K$. Assume that $\xi$ is locally trivial in $H^1(K_v, G)$ for every $v \in \Omega_A$. By Theorem 3.4 we have $f_3(J) = f_3(J')$ and $g_3(J) = g_3(J')$. Since $J$ is reduced by assumption, we have $g_3(J') = 0$ and hence $J'$ is reduced. Thus it follows that $J \cong J'$ over $K$, showing that $\xi$ is trivial in $H^1(K, G)$ as desired.

4 Spin groups of quadratic forms

(4.1) Let $E$ be a field of characteristic different from 2 and $q$ a nonsingular quadratic form of rank $\geq 3$ over $E$. Recall that $\text{Sn}(q_E)$ denotes the image of the spinor norm map

$$\text{SO}(q)(E) \to E^*/E^{*2},$$

i.e., the connecting map associated to the cohomology of the exact sequence

$$1 \to \mu_2 \to \text{Spin}(q) \to \text{SO}(q) \to 1.$$
Then the natural map
\[
\frac{K^*/K^{*2}}{\text{Sn}(q_K)} \to \prod_{v \in \Omega_A} \frac{K_v^*/K_v^{*2}}{\text{Sn}(q_{K_v})}
\]
is injective.

Proof. If rank\(r(q) = 3\), we may assume \(q = \langle 1, a, b \rangle\) after scaling. Let \(D\) be the quaternion algebra \((-a, -b)_K\) over \(K\). Then \(\text{Sn}(q) = \text{Nrd}(D^*)\) modulo squares. The result then follows from Theorem 3.7.

Assume next rank\(r(q) = 4\). If \(\text{disc}(q) = 1\), we may assume after scaling \(q = \langle 1, a, b, ab \rangle\). Put \(D = (-a, -b)_K\). Then \(\text{Sn}(q) = \text{Nrd}(D^*)\) and the result follows again from Theorem 3.7. If \(d = \text{disc}(q)\) is nontrivial in \(K^*/K^{*2}\), we may assume \(q = \langle 1, a, b, abd \rangle\). Then \(\text{Sn}(q_K) = \text{Nrd}(D_{K'w}^*) \cap K^*\) modulo squares by [17, p.214, Coro. 15.11]. The field \(K(\sqrt{d})\) is a field of the same type as \(K\) (cf. (3.2)).

Let \(\Omega_{A'}\) denote the set of divisorial valuations of \(K' = K(\sqrt{d})\). If \(\alpha \in K^*\) lies in \(\text{Sn}(q_{K_v})\) for all \(v \in \Omega_{A'}\), then \(\alpha\) is a reduced norm from \(D_{K_v} = D_{K(\sqrt{d})}\). This finishes the proof.

Recall that the \(u\)-invariant \(u(E)\) of a field \(E\) of characteristic \(\neq 2\) is the supremum of dimensions of anisotropic quadratic forms over \(E\) (so \(u(E) = \infty\), if such dimensions can be arbitrarily large).

Proposition 4.3. Let \(E\) be a field of characteristic \(\neq 2\) and \(q\) a nonsingular quadratic form of rank \(r\) over \(E\). Assume \(u(E) < 2r\).

Then \(\text{Sn}(q_E) = E^*/E^{*2}\), i.e., the spinor norm map
\[
\text{SO}(q)(E) \to E^*/E^{*2}
\]
is surjective.

Proof. The image \(\text{Sn}(q_E)\) of the spinor norm map consists of elements of the form \(\prod_{i=1}^{2m} q(v_i)\), where \(v_i\) are anisotropic vectors for \(q\) (cf. (2.15)). If \(q\) is isotropic over \(E\), then for every \(\alpha \in E^*\), there is a vector \(v_\alpha\) such that \(q(v_\alpha) = \alpha\). Let \(v_1\) be a vector such that \(q(v_1) = 1\). Then we have \(\alpha = q(v_\alpha).q(v_1) \in \text{Sn}(q_E)\).

Assume next \(q\) is anisotropic. For any \(\alpha \in E^*\), the form \(q_{\perp}(\alpha.q)\) is isotropic over \(E\) by the assumption on the \(u\)-invariant. Hence there are vectors \(x, y\) such that \(q(x) - \alpha.q(y) = 0\). Since \(q\) is anisotropic, we have \(\lambda := q(y) \in E^*\) and \(q(x) \in E^*\). It follows that
\[
\alpha = q(x).q(y)^{-1} = \lambda^{-2}q(x).q(y) = q(x).q(\lambda^{-1}y) \in \text{Sn}(q_E)
\]
whence the desired result.
Corollary 4.4. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local henselian case. Let $q$ be a nonsingular quadratic form of rank $\geq 5$ over $K$.

Then $\text{Sn}(q_K) = K^*/K^{*2}$, i.e., the spinor norm map

$$\text{SO}(q)(K) \rightarrow K^*/K^{*2}$$

is surjective.

Proof. In the $p$-adic arithmetic case, we have $u(K) = 8$ by [25] (if $p \neq 2$) or [19] (see also [11]). In the local henselian case, it is proved in [13, Thm. 1.2] that $u(K) = 8$. The result then follows immediately from Proposition 4.3.

Theorem 4.5. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local henselian case. Let $q$ be a nonsingular quadratic form of rank $\geq 3$ over $K$ and $G = \text{Spin}(q)$.

(i) The natural map

$$H^1(K, G) \rightarrow \prod_{v \in \Omega_A} H^1(K_v, G)$$

has a trivial kernel.

(ii) The Rost invariant

$$R_G : H^1(K, G) \rightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

has a trivial kernel if $\text{rank}(q) \geq 5$.

Proof. Consider the exact sequence of algebraic groups

$$1 \rightarrow \mu_2 \rightarrow \text{Spin}(q) = G \rightarrow \text{SO}(q) \rightarrow 1$$

which gives rise to an exact sequence of pointed sets

$$(4.5.1) \quad \text{SO}(q)(K) \xrightarrow{\delta} K^*/K^{*2} \xrightarrow{\psi} H^1(K, \text{Spin}(q)) \xrightarrow{\eta} H^1(K, \text{SO}(q)).$$

The image of the map $\eta$ is in bijection with isomorphism classes of nonsingular quadratic forms $q'$ with the same rank, discriminant and Clifford invariant as $q$.

Let $\xi \in H^1(K, G) = H^1(K, \text{Spin}(q))$ with $\eta(\xi) \in H^1(K, \text{SO}(q))$ corresponding to a quadratic form $q'$. Then in the Witt group $W(K)$ the class of $q \perp (-q')$ lies in $I^3(K)$ by Merkurjev’s theorem (cf. [20] p.89, Thm. 2.14.3) and its Arason invariant $e_3([q \perp (-q')]) \in H^3(K, \mathbb{Z}/2)$ coincides with Rost invariant $R_G(\xi)$ of $\xi$ when $\text{rank}(q) \geq 5$ ([17] p.437).

For (i), assume the canonical image $\xi_v$ of $\xi$ in $H^1(K_v, G)$ is trivial for every $v \in \Omega_A$. We have

$$[q \perp (-q')]_v = 0 \in I^3(K_v), \quad \forall v \in \Omega_A.$$
By Lemma 3.8, we have $q \cong q'$ over $K$. This means that $\xi \in H^1(K, G)$ lies in the kernel of
\[ \eta : H^1(K, G) \to H^1(K, \text{SO}(q)). \]

By the exactness of the sequence (4.5.1), $\xi = \psi(\alpha)$ for some $\alpha \in \text{Coker}(\delta) = \frac{K^*/K^{*2}}{\text{Sn}(q_K)}$.

Consider now the following commutative diagram with exact rows
\[
\begin{array}{ccc}
1 & \longrightarrow & \frac{K^*/K^{*2}}{\text{Sn}(q_K)} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_A} \frac{K^*/K^{*2}}{\text{Sn}(q_{K_v})}. \\
\end{array}
\]

The canonical image $\alpha_v$ of $\alpha$ in $\frac{K^*/K^{*2}}{\text{Sn}(q_{K_v})}$ is trivial for all $v \in \Omega_A$. From Proposition 4.2 and Corollary 4.4, it follows that $\alpha = 1$ and hence $\xi = \psi(\alpha)$ is trivial.

For (ii), assume the Rost invariant $R_G(\xi)$ of $\xi$ is trivial. Then the Arason invariant $e_3([q, -q'])$ is zero. Since $\text{cd}_2(K) \leq 3$, the map $e_3 : I^3(K) \to H^3(K, \mathbb{Z}/2)$ is injective. So we get $q \cong q'$ over $K$ and therefore $\xi = \psi(\alpha)$ for some $\alpha \in \frac{K^*/K^{*2}}{\text{Sn}(q_K)}$. When the rank of $q$ is $\geq 5$, we have $K^*/K^{*2} = \text{Sn}(q_K)$ by Corollary 4.4. So $\alpha = 1$ and $\xi$ is trivial. \qed

**Remark 4.6.** Assertion (ii) of Theorem 4.5 may be compared with the following result, which was already known to experts (cf. [6, Prop. 5.2]): Let $E$ be a field of characteristic $\neq 2$ and of cohomological 2-dimension $\text{cd}_2(E) \leq 3$. Let $q$ be an *isotropic* quadratic form of rank $\geq 5$ over $E$. Then the Rost invariant
\[ H^1(E, \text{Spin}(q)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2)) \]

for the spinor group $\text{Spin}(q)$ has a trivial kernel.

## 5 Groups of type $D^n$

**Proposition 5.1.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local henselian case. Let $(D, \sigma)$ be a quaternion division algebra with an orthogonal involution over $K$ and let $h$ be a hermitian form of rank $\geq 2$ over $(D, \sigma)$.

Then the natural map
\[ \frac{K^*/K^{*2}}{\text{Sn}(h_K)} \to \prod_{v \in \Omega_A} \frac{K^*/K^{*2}}{\text{Sn}(h_{K_v})} \]

is injective.

**Proof.** First assume $\text{rank}(h) = 2$. Put $d = \text{disc}(h) \in K^*/K^{*2}$. If $d = 1 \in K^*/K^{*2}$, then $h$ is isotropic and $\text{Sn}(h) = \text{Nrd}(D^*)$ modulo squares by Merkurjev’s norm principle (Theorem 2.16). The result then follows from Theorem 3.7. Let us assume $d = \text{disc}(h) \in K^*/K^{*2}$ is nontrivial. Let $(A, \bar{\sigma}) = (M_2(D), \sigma_h)$, where $\sigma_h$ denotes
the adjoint involution of $h$ on $A = M_2(D)$. The even Clifford algebra $C = C_0(A, \hat{\sigma})$ of
the pair $(A, \hat{\sigma})$ (cf. [17 \S 8]) is a quaternion algebra over the field $K(\sqrt{d})$ and one has
\[ \text{Sn}(h_K) = \text{Nrd}(C^*) \cap K^* \quad (\text{mod } K^{*2}). \]
(cf. [17, p.94, Thm. 8.10 and p.214, Coro. 15.11]) As in the proof of Proposition 4.2, it follows from
Theorem 3.7 that an element $\lambda \in K^*/K^{*2}$ is a spinor norm for $h_K$ if and only if it is a spinor
norm for $h_{K_v}$ for all $v \in \Omega_A$.

Assume next $\text{rank}(h) \geq 3$. Let $\lambda \in K^*$ and assume $\lambda$ is a local spinor norm for $h_{K_v}$
for every $v \in \Omega_A$. Merkurjev’s norm principle (Theorem 2.16) implies that $\lambda \in \text{Nrd}(D_{K_v}^*)$ for
every $v \in \Omega_A$. Hence $\lambda \in \text{Nrd}(D^*)$ by Theorem 3.7. (Note that $K^{*2} \subseteq \text{Nrd}(D^*)$ since
$D$ is a quaternion algebra.) Let $K'/K$ be a field extension such that $D_{K'}$ is split and $\lambda = N_{K'/K}(\mu)$
for some $\mu \in (K')^*$. By Corollary 2.17, $N_{K'/K}(\text{Sn}(h_{K'})) \subseteq \text{Sn}(h_K)$. Since $\lambda \in K^*/K^{*2}$
lies in the image of $N_{K'/K} : (K')^*/(K')^{*2} \rightarrow K^*/K^{*2}$, to show $\lambda$ is a
spinor norm for $h_K$ it suffices to show that the map
\[ \delta' : \text{SU}(h)(K') \rightarrow (K')^*/(K')^{*2} \]
is surjective. Note that $D$ splits over $K'$ by the choice of $K'$. So we see from (2.19) that
$\text{Im}(\delta') = \text{Sn}(h_{K'}) = \text{Sn}(q_{K'})$, where $q_{K'}$ is a quadratic form of rank $2$.\text{rank}(h) \geq 6$ over
$K'$. Now the result follows immediately from Corollary 1.4.

**Proposition 5.2.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local
henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local
transcendental case. Let $(D, \sigma)$ be a quaternion division algebra with an orthogonal involution
over $K$. Let $h$ be a nonsingular hermitian form of even rank $\geq 2$ over $(D, \sigma)$. Assume
that $h$ has trivial discriminant, trivial Clifford invariant and trivial Rost invariant (cf. [2,2]).

If the form $h_{K_v}$ over $(D_{K_v}, \sigma) = (D \otimes_K K_v, \sigma)$ is hyperbolic for every $v \in \Omega_A$, then
the form $h$ over $(D, \sigma)$ is hyperbolic.

**Proof.** Let $L \subseteq D$ be a subfield which is a quadratic extension over $K$ such that $\sigma(L) = L$
and $\sigma|_L = \text{id}_L$. Such an $L$ exists since $\sigma$ is an orthogonal involution. Let $\mu \in D^*$ be
an element such that $\sigma(\mu) = -\mu$, $\text{Int}(\mu)(L) = L$ and $\text{Int}(\mu)|_L = \iota$, where $\iota$ denotes
the nontrivial element of the Galois group $\text{Gal}(L/K)$. The involution $\tau := \text{Int}(\mu) \circ \sigma$
is a symplectic involution on $D$ (and hence coincides with the canonical involution on the
quaternion algebra $D$). The “key exact sequence” of Parimala-Sridharan-Suresh (cf. [27,2]) yields the
following commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
W(D, \sigma) & \xrightarrow{\pi_1} & W(L, \iota) & \xrightarrow{\bar{\rho}} & W(D, \sigma) & \xrightarrow{\bar{\pi}_2} & W(L) \\
\prod_{v \in \Omega_A} W(D_v, \tau) & \xrightarrow{\pi_1} & \prod_{v \in \Omega_A} W(L_v, \iota) & \xrightarrow{\bar{\rho}} & \prod_{v \in \Omega_A} W(D_v, \sigma) & \xrightarrow{\bar{\pi}_2} & \prod_{v \in \Omega_A} W(L_v) \\
\end{array}
\]
where for any $K$-algebra $B$ we denote $B_v = B \otimes_K K_v$ for each $v \in \Omega_A$. (Here $L_v$ need
not be a field. It can be a Galois $K_v$-algebra of the form $L_{w_1} \times L_{w_2}$, where $w_1, w_2$ are
discrete valuations of $L$ lying over $v$. But this does not affect the construction of the key exact sequence for $D_v$. Indeed, the same choice of $\mu \in D^* \subseteq D_v^*$ satisfies the condition that $\operatorname{Int}(\mu)|_{L_v}$ is the nontrivial automorphism of the $K_v$-algebra $L_v$. It is not difficult to check that the key exact sequence for $D_v$ is still well defined.)

The form $\tilde{\pi}_2(h) \in W(L)$ has even rank, trivial discriminant and trivial Clifford invariant by [3, Prop. 3.2.2]. Hence $\tilde{\pi}_2(h) \in \mathcal{I}^3(L) \subseteq W(L)$. Let $\Omega_{A'}$ denote the set of divisorial valuations of $L$. Then for every $w \in \Omega_{A'}$ one has $\tilde{\pi}_2(h) = 0$ in $W(L_w)$. By Lemma 3.8 $\tilde{\pi}_2(h) = 0$ in $W(L)$. So by the exactness of the first row in the above diagram, there exists a hermitian form of even rank $h_0$ over $(L, \iota)$ such that $\tilde{\mu}(h_0) = h \in W(D, \sigma)$.

Let $\alpha = \operatorname{disc}(h_0) \in K^*/N_{L/K}(L^*)$ be the discriminant of $h_0$. One has

$$\mathcal{C} \ell(\tilde{\mu}(h_0)) = (L, \alpha) \in 2\operatorname{Br}(K)/(D)$$

by [3, Prop. 3.2.3]. Since $\mathcal{C} \ell(\tilde{\mu}(h)) = \mathcal{C} \ell(h) = 0$ by assumption, one has either $(L, \alpha) = 0$ or $(L, \alpha) = (D)$ in $\operatorname{Br}(K)$. If $(L, \alpha) = 0 \in \operatorname{Br}(K)$ then $\alpha$ is a norm for the extension $L/K$ so that $\operatorname{disc}(h_0) = 1 \in K^*/N_{L/K}(L^*)$. If $(L, \alpha) = D$, writing $L = K(\sqrt{a})$ such that $D = (a, K)$, one has $\operatorname{disc}(1, -\alpha) = \alpha \in K^*/N_{L/K}(L^*)$. By the construction of the map $\pi_1$, one has $\pi_1((1)) = (1, -\alpha) \in W(L, \iota)$ (since $D = L \oplus \mu L$ with $\mu^2 = \alpha$). Replacing $h_0$ by $h_0 - \pi_1((1))$, we may assume that $\operatorname{disc}(h_0) = 1 \in K^*/N_{L/K}(L^*)$. Let $2n = \operatorname{rank}(h_0)$ and let $\mathbf{SU}_{2n}(L, \iota)$ denote the special unitary group of the hyperbolic form $(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix})$ over $(L, \iota)$. The form $h_0$, having trivial discriminant, now determines a class in $H^1(K, \mathbf{SU}_{2n}(L, \iota))$.

Let $H_{2n}$ be the hyperbolic form $(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix})$ over $(D, \sigma)$ and let $\mathbf{U}_{2n}(D, \sigma), \mathbf{SU}_{2n}(D, \sigma)$ and $\mathbf{Spin}_{2n}(D, \sigma)$ denote respectively the unitary group, the special unitary group and the spin group of the form $H_{2n}$. By [2,13], there is a homomorphism

$$\rho_0 : \mathbf{SU}_{2n}(L, \iota) \longrightarrow \mathbf{Spin}_{2n}(D, \sigma)$$

which induces a commutative diagram

$$
\begin{array}{ccc}
H^1(K, \mathbf{SU}_{2n}(L, \iota)) & \xrightarrow{\rho_0} & H^1(K, \mathbf{Spin}_{2n}(D, \sigma)) \\
\rho' \downarrow & & \downarrow \\
H^1(K, \mathbf{U}_{2n}(D, \sigma)) & & \\
\end{array}
$$

such that the map $\tilde{\rho} : W(L, \iota) \rightarrow W(D, \sigma)$ in the “key exact sequence” (2.17.2) restricted to forms of rank $2n$ and of trivial discriminant is compatible with the map $\rho'$ at the level of cohomology sets.

By [4, Prop. 3.20], one has

$$R_{\mathbf{Spin}_{2n}(D, \sigma)}(\rho_0([h_0])) = R_{\mathbf{SU}_{2n}(L, \iota)}([h_0]) = H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

Thus by the definition of the Rost invariant $\mathcal{R}$ (cf. (2.12)),

$$0 = \mathcal{R}(h) = [R_{\mathbf{Spin}_{2n}(D, \sigma)}(\rho_0([h_0]))] = [R_{\mathbf{SU}_{2n}(L, \iota)}([h_0])] \in \frac{H^3(K, \mathbb{Q}/\mathbb{Z}(2))}{H^1(K, \mu_2) \cup (D)}.$$
Therefore, there is an element \( \beta \in K^*/K^{*2} = H^1(K, \mu_2) \) such that
\[
R_{SU_{2n}(L, \iota)}([h_0]) = (\beta) \cup (D) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)).
\]
A direct computation shows that the element \( \tilde{h}_0 := \pi_1(1, -\beta) \in W(L, \iota) \) has associated trace form \( q_{\tilde{h}_0} = \langle 1, -\beta \rangle \otimes n_D \), where \( n_D \) denotes the norm form of the quaternion algebra \( D \). By [17, p.438, Example 31.44], the class of \( \tilde{h}_0 \) has Rost invariant
\[
R_{SU_{4}(L, \iota)}([\tilde{h}_0]) = e_3(q_{\tilde{h}_0}) = (\beta) \cup (D) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2)).
\]
Modifying \( h_0 \) by \( \tilde{h}_0 = \pi_1(1, -\beta) \), we may further assume that the class \([h_0] \in H^1(K, SU_{2n}(L, \iota))\) has trivial Rost invariant, i.e., \( e_3(q_{h_0}) = 0 \). Since \( \text{cd}_2(K) \leq 3 \), the Arason invariant \( e_3 : I^3(K) \to H^3(K, \mathbb{Z}/2) \) is injective. Hence \([q_{h_0}] = 0 \in W(K)\) and \([h_0] = 0 \in W(L, \iota)\) by (2.4) (cf. [29, p.348, Thm. 10.1.1]). It then follows immediately that \([h] = \tilde{\rho}([h_0]) = 0 \in W(D, \sigma)\).

**Corollary 5.3.** Let \( K \) be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Assume \( p \neq 2 \) in the local henselian case. Let \((D, \sigma)\) be a quaternion division algebra with an orthogonal involution over \( K \). Let \( h_1, h_2 \) be hermitian forms over \((D, \sigma)\) with the same rank and discriminant such that
\[
\mathcal{C}(h_1 \perp (-h_2)) = 0 \in \text{Br}(K)/(D)
\]
and
\[
\mathcal{R}(h_1 \perp (-h_2)) = 0 \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))/H^1(K, \mu_2) \cup (D).
\]
Then \( h_1 \cong h_2 \) if and only if \((h_1)_v \cong (h_2)_v\) for every \( v \in \Omega_A \).

**Proof.** Apply Proposition 5.2 to the form \( h = h_1 \perp (-h_2) \) and use Witt’s cancellation theorem.

**Theorem 5.4.** Let \( K \) be the function field of a p-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Assume \( p \neq 2 \) in the local henselian case. Let \((D, \sigma)\) be a quaternion division algebra with an orthogonal involution over \( K \), \( h \) a nonsingular hermitian form of rank \( \geq 2 \) over \((D, \sigma)\) and \( G = \text{Spin}(h) \).

Then the natural map
\[
H^1(K, G) \to \prod_{v \in \Omega_A} H^1(K_v, G)
\]
has a trivial kernel.

**Proof.** Let \( \xi \in H^1(K, \text{Spin}(h)) \) be a class which is trivial in \( H^1(K_v, \text{Spin}(h)) \) for all \( v \in \Omega_A \). The image of \( \xi \) under the composite map
\[
H^1(K, G) = H^1(K, \text{Spin}(h)) \to H^1(K, SU(h)) \to H^1(K, U(h))
\]
is the class of a hermitian form \( h' \) which has the same rank and discriminant as \( h \) such that
\[
\mathcal{C}(h \perp (-h')) = 0 \in \text{Br}(K)/(D).
\]
Let \( n = \text{rank}(h) \). Let \( \text{Spin}_{2n}(D, \sigma) \) and \( U_{2n}(D, \sigma) \) denote respectively the spin group and the unitary group of the hyperbolic form \( \left( \begin{smallmatrix} 0 & \I_n \\ \I_n & 0 \end{smallmatrix} \right) \) over \((D, \sigma)\). Then the class \([h \perp (-h')] \in H^1(K, U_{2n}(D, \sigma))\) lifts to an element \( \xi \in H^1(K, \text{Spin}_{2n}(D, \sigma)) \). By \cite[Lemma 5.1]{23}, we have

\[
(R_G(\xi)) = R(h \perp (-h')) = R_{\text{Spin}_{2n}(D, \sigma)}(\xi') \in \frac{H^3(K, \Q/\Z(2))}{H^1(K, \mu_2 \cup D)}.
\]

Since \( \xi \) is locally trivial, the commutative diagram

\[
\begin{array}{ccc}
H^1(K, G) & \overset{R_G}{\longrightarrow} & H^3(K, \Q/\Z(2)) \\
\downarrow & & \downarrow \\
\prod_{v \in \Omega_A} H^1(K_v, G) & \overset{R_G}{\longrightarrow} & \prod_{v \in \Omega_A} H^3(K_v, \Q/\Z(2))
\end{array}
\]

shows that the Rost invariant \( R_G(\xi) \) is locally trivial. By Theorem 3.4 noticing that the Rost invariant \( R_G \) takes values in the subgroup \( H^3(K, \mu_4^{\otimes 2}) \), we get \( R_G(\xi) = 0 \in H^3(K, \Q/\Z(2)) \). Thus, by (5.4.1),

\[
\mathcal{R}(h \perp (h')) = 0 \in \frac{H^3(K, \Q/\Z(2))}{H^1(K, \mu_2 \cup D)}.
\]

Now Corollary 5.3 implies that \( h \cong h' \) and hence the image of \( \xi \in H^1(K, G) \) in \( H^1(K, U(h)) \) is trivial. By \cite[Lemma 7.11]{4}, the canonical image of \( \xi \) in \( H^1(K, SU(h)) \) is also trivial.

Now consider the following commutative diagram with exact rows

\[
\begin{array}{ccc}
1 & \longrightarrow & K^*/K^{*2} \quad \text{Sn}(h_K) \\
| & & | \\
1 & \longrightarrow & \prod_{v \in \Omega_A} K_v^*/K_v^{*2} \quad \text{Sn}(h_K) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, G) & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, SU(h))
\end{array}
\]

which is induced by the natural exact sequence of algebraic groups

\[
1 \longrightarrow \mu_2 \longrightarrow G = \text{Spin}(h) \longrightarrow SU(h) \longrightarrow 1.
\]

The exactness of the first row yields \( \xi = \varphi(\theta) \) for some \( \theta \in K^*/K^{*2} \text{Sn}(h_K) \). The commutative diagram then shows that \( \theta \) is locally trivial since \( \xi \) is locally trivial. From Proposition 5.1 it follows that \( \theta = 1 \in K^*/K^{*2} \text{Sn}(h_K) \) and hence \( \xi = \varphi(\theta) \) is trivial in \( H^1(K, G) \). This completes the proof.

6 Groups of type \( ^2A^*_n \)

6.1 Case of odd index

Proposition 6.1. Let \( K \) be the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Assume \( p \neq 2 \) in the local
henselian case. Let \( L/K \) be a quadratic field extension, \((D, \tau)\) a central division algebra of odd degree over \( L \) with an \( L/K \)-involution \( \tau \) (i.e., a unitary involution \( \tau \) such that \( L^\tau = K \)). Let \( h_1, h_2 \) be nonsingular hermitian forms over \((D, \tau)\) which have the same rank and discriminant.

If the forms \((h_1)_{K_v} \cong (h_2)_{K_v}\) over \((D_{K_v}, \tau) = (D \otimes_L L \otimes_K K_v, \tau)\) are isomorphic for all \( v \in \Omega_A \), then the forms \( h_1, h_2 \) over \((D, \tau)\) are isomorphic.

**Proof.** Let \( M/K \) be a field extension of odd degree such that \( D_M = D \otimes_L (L \otimes_K M) \) is split over the field \( LM = L \otimes_K M \). (Such an extension \( M/K \) exists by [3, Lemma 3.3.1].) The base extension \( \tau_M \) of \( \tau \) is a unitary involution on the central simple \((LM)\)-algebra \( D_M \) such that \( (LM)^{\tau_M} = M \). Let \( \iota \) denote the nontrivial element of the Galois group \( \text{Gal}(L/K) \) and regard \( \iota_M \in \text{Gal}(LM/M) \) as a unitary involution on \( LM \).

There is a nonsingular hermitian form \((V, f)\) over \((LM, \iota_M)\) such that \((D_M, \tau_M) \cong (\text{End}_{LM}(V), \iota_f)\), where \( \iota_f \) denotes the adjoint involution on \( \text{End}_{LM}(V) \) with respect to \( f \) (cf. [17, p.43, Thm. 4.2 (2)]). We have a Morita equivalence between the category of hermitian forms over \((D_M, \tau_M)\) and the category of hermitian forms over \((LM, \iota_M)\) (cf. [26]), which induces an isomorphism of Witt groups

\[
\phi_f : W(D_M, \tau_M) \xrightarrow{\sim} W(LM, \iota_M).
\]

Let \( h = h_1 \perp (-h_2) \) and let \( h_M \) be its base extension over \((D_M, \tau_M)\). Via the Morita equivalence mentioned above, \( h_M \) corresponds to a hermitian form \( h_M \) over \((LM, \iota_M)\). Let \( q_M := q_{h_M} \) be the trace form of \( h_M \) (which is a quadratic form over the field \( M \)). Since \( h \) has even rank and trivial discriminant, the class \([q_M] \in W(M)\) of the quadratic form \( q_M \) lies in \( I^3(M) \). The hypothesis on the local triviality (with respect to \( \Omega_A \)) of \([h] = [h_1 \perp (-h_2)]\) implies that \([q_M] \in I^3(M)\) is locally trivial (with respect to the set of discrete valuations of \( M \) defined in the same way as \( \Omega_A \)). By Lemma 3.8, we have \([q_M] = 0\) and hence \([h_M] = 0\) in \( W(LM, \iota_M)\). Since \( W(D_M, \tau_M) \cong W(LM, M), [h_M] = 0 \) in \( W(D_M, \tau_M)\). Since \( M/K \) is an odd degree extension, the natural map \( W(D, \tau) \to W(D_M, \tau_M) \) is injective by a theorem of Bayer-Fluckiger and Lenstra (cf. [17] p.80, Coro. 6.18). So we get \([h] = 0\) in \( W(D, \tau)\), thus proving the proposition. \( \square \)

**Lemma 6.2.** Let \( K \) be the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Let \( L/K \) be a separable quadratic field extension and \((D, \tau)\) a central division \( L \)-algebra of square-free index \( \text{ind}(D) \) with a unitary involution \( \tau \) such that \( L^\tau = K \). Assume \( p \nmid \text{ind}(D) \) in the local henselian case.

Then for any nonsingular hermitian form \( h \) over \((D, \tau)\), the natural map

\[
\frac{(R^1_{L/K} \mathbb{G}_m)(K)}{\text{Nrd}(U(h)(K))} \to \prod_{v \in \Omega_A} \frac{(R^1_{L/K} \mathbb{G}_m)(K_v)}{\text{Nrd}(U(h)(K_v))}
\]

is injective.

**Proof.** First assume \( \text{ind}(D) = 2 \) so that \( D \) is a quaternion division algebra over \( L \). By [17] p.202, Exercise III.12 (a)], we have

\[
\text{Nrd}(U(h)(K)) = \{ z \tau(z)^{-1} \mid z \in \text{Nrd}(D^\tau) \} = \text{Nrd}(U_2(D, \tau)(K)),
\]

27
where $U_2(D, \tau)$ denotes the unitary group of the rank 2 hyperbolic form \((\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix})\) over \((D, \tau)\). So we may assume that $h = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. The exact sequence of algebraic groups

\[
1 \rightarrow SU_2(D, \tau) \rightarrow U_2(D, \tau) \xrightarrow{Nrd} (R_{L/K}^1 \mathbb{G}_m) \rightarrow 1
\]
gives rise to the following commutative diagram with exact rows

\[
\begin{array}{c}
1 \rightarrow \frac{(R_{L/K}^1 \mathbb{G}_m)(K)}{Nrd(U(h)(K))} \xrightarrow{\varphi} H^1(K, SU_2(D, \tau)) \\
\downarrow \quad \downarrow \\
1 \rightarrow \prod_{v \in \Omega_A} \frac{(R_{L/K}^1 \mathbb{G}_m)(K_v)}{Nrd(U(h)(K_v))} \rightarrow \prod_{v \in \Omega_A} H^1(K_v, SU_2(D, \tau))
\end{array}
\]

We need only to show that the vertical map on the right in the above diagram is injective.

By [17, p.26, Prop. 2.22], there is a unique quaternion $K$-algebra $D_0$ contained in $D$ such that $D = D_0 \otimes_K L$ and $\tau = \tau_0 \otimes \iota$, where $\tau_0$ is the canonical involution on $D_0$ and $\iota$ is the nontrivial element in the Galois group $\text{Gal}(L/M)$. Write $L = K(\sqrt{d})$ and let $n_{D_0}$ be the norm form of the quaternion $K$-algebra $D_0$. Then by [17, p.229], we have $SU_2(D, \tau) = \text{Spin}(q)$, where $q = \langle 1, -d \rangle \otimes n_{D_0}$. Now the result follows from Theorem [13].

Assume next ind$(D)$ is odd (and square-free). By [17, p.202, Exercise III.12 (b)],

\[Nrd(U(h)(K)) = Nrd(D^*) \cap (R_{L/K}^1 \mathbb{G}_m)(K).\]

Let $\lambda \in (R_{L/K}^1 \mathbb{G}_m)(K) = \{ z \in L^* | N_{L/K}(z) = 1 \}$ be such that for every $v \in \Omega_A$, $\lambda \in Nrd(U(h)(K_v)) = Nrd((D \otimes_K K_v)^*) \cap (R_{L/K}^1 \mathbb{G}_m)(K_v)$. Since ind$(D)$ is square-free, it follows from Theorem [3.7] that $\lambda \in Nrd(D^*)$. Hence

\[\lambda \in Nrd(U(h)(K)) = Nrd(D^*) \cap (R_{L/K}^1 \mathbb{G}_m)(K).\]

Now assume ind$(D)$ is even such that ind$(D)/2$ is odd and square-free. In this case we have $D = H \otimes_L D'$ for some quaternion division algebra $H$ over $L$ and some central division algebra $D'$ of odd index over $L$. By [3, Lemma 3.3.1], there is an odd degree separable extension $K'/K$ such that $D' \otimes_K K' = D' \otimes_L LK'$ is split. By Morita theory, there is a unitary $LK'/K'$-involution $\sigma$ on $H \otimes_L LK'$ and a hermitian form $f$ over $(H \otimes_L LK', \sigma)$ such that the involution $\tau$ on $D \otimes_L LK'$ is adjoint to $f$, and moreover, the form $h_{K'}$ over $(D \otimes_L LK', \tau)$ corresponds to a hermitian form $h'$ over $(H \otimes_L LK', \sigma)$. Consider the commutative diagram

\[
\begin{array}{c}
\frac{R_{L/K}^1 \mathbb{G}_m(K)}{Nrd(U(h)(K))} \xrightarrow{\eta} \prod_{v \in \Omega_A} \frac{R_{L/K}^1 \mathbb{G}_m(K_v)}{Nrd(U(h)(K_v))} \\
\downarrow \quad \downarrow \\
\frac{R_{L/K'}^{L/K'} \mathbb{G}_m(K')}{Nrd(U(h')(K'))} \xrightarrow{\eta'} \prod_{v \in \Omega_A} \frac{R_{L/K'}^{L/K'} \mathbb{G}_m(K'_v)}{Nrd(U(h')(K'_v))}
\end{array}
\]

The map $\eta'$ is already shown to be injective. Let $\lambda \in R_{L/K}^1 \mathbb{G}_m(K) \subseteq L^*$ be an element which is a reduced norm for $U(h)(K_v)$ for every $v$. Then, considered as an element of $R_{L/K'}^{L/K'} \mathbb{G}_m(K') \subseteq (LK')^*$, $\lambda$ lies in $Nrd(U(h')(K'))$. By [23, Prop. 10.2], we have

\[N_{LK'/K'}(Nrd(U(h')(K'))) \subseteq Nrd(U(h)(K)).\]
Hence, $\lambda^{2r+1} \in \mathrm{Nrd}(U(h)(K))$, where $2r + 1 = [K' : K]$. It is sufficient to show that $\lambda^2 \in \mathrm{Nrd}(U(h)(K))$. For this, we choose a quadratic extension $M/K$ such that $H \otimes_K M = H \otimes_L LM$ is split. A similar argument as above, using the result in the case of odd index this time, shows that $\lambda \in \mathrm{Nrd}(U(h_M)(M))$. Thus,

$$\lambda^2 = N_{LM/M}(\lambda) \in N_{LM/M}(\mathrm{Nrd}(U(h_M)(M))) \subseteq \mathrm{Nrd}(U(h)(K)).$$

This completes the proof of the lemma. \hfill \Box

**Theorem 6.3.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $L/K$ be a separable quadratic field extension and $(D, \tau)$ a central division $L$-algebra with a unitary $L/K$-involution whose index $\text{ind}(D)$ is odd and square-free. Assume further that $p \nmid 2 \cdot \text{ind}(D)$ in the local henselian case.

Then for any nonsingular hermitian form $h$ over $(D, \tau)$, the natural map

$$H^1(K, \, SU(h)) \longrightarrow \prod_{v \in \Omega_A} H^1(K_v, \, SU(h))$$

has a trivial kernel.

**Proof.** Let $\xi \in H^1(K, \, SU(h))$ be a class that is locally trivial in $H^1(K_v, \, SU(h))$ for every $v \in \Omega_A$. Let $h'$ be a hermitian form whose class $[h'] \in H^1(K, \, U(h))$ is the image of $\xi$ under the natural map $H^1(K, \, SU(h)) \to H^1(K, \, U(h))$. The two forms $h'$ and $h$ have the same rank and discriminant, and they are locally isomorphic since $\xi$ is locally trivial. So by Proposition 6.1, $h' \cong h$ as hermitian forms over $(D, \tau)$. This means that $\xi \in H^1(K, \, SU(h))$ maps to the trivial element in $H^1(K, \, U(h))$.

Consider now the following commutative diagram with exact rows

$$
\begin{array}{ccc}
1 & \longrightarrow & (R_{L/K}^1G_m)(K) / \mathrm{Nrd}(U(h)(K)) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_A} (R_{L/K}^1G_m)(K_v) / \mathrm{Nrd}(U(h)(K_v)) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, \, SU(h)) \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \prod_{v \in \Omega_A} H^1(K_v, \, U(h))
\end{array}
$$

There is an element $\theta \in (R_{L/K}^1G_m)(K) / \mathrm{Nrd}(U(h)(K))$ such that $\varphi(\theta) = \xi$. The map $\eta$ is injective by Lemma 6.2. So we have $\theta = 1$ and $\xi = \varphi(\theta)$ is trivial. The theorem is thus proved. \hfill \Box

### 6.2 Some observations on Suresh’s exact sequence

**Lemma 6.2** (6.4) Let $E$ be a field of characteristic $\neq 2$. Let $D$ be a quaternion division algebra over a quadratic field extension $L$ of $E$. Let $\tau$ be a unitary $L/E$-involution on $D$. There is a unique quaternion $E$-algebra $D_0$ contained in $D$ such that $D = D_0 \otimes_E L$ and $\tau = \tau_0 \otimes \iota$, where $\tau_0$ is the canonical (symplectic) involution on $D_0$ and $\iota$ is the nontrivial element of the Galois group $\text{Gal}(L/E)$. Then we have Suresh’s exact sequence (cf. (2.8))

$$W(L) \xrightarrow{\hat{\pi}_1} W(D_0, \tau_0) \xrightarrow{\hat{\rho}} W(D, \tau) \xrightarrow{\hat{p}_2} W^{-1}(D_0, \tau_0).$$

29
The goal of this subsection is to analyze the image of the map \( \bar{\pi}_1 \) in this sequence.

(6.5) With notation as in (6.4), let \( h_0 \) be a hermitian form of rank \( m \) over \((D_0, \tau_0)\). Let \( M(h_0) \in A := M_m(D_0) \) be a representation matrix of \( h_0 \). One can define the pfaffian norm \( \text{Pf}(h_0) \) as the pfaffian norm of \( M(h_0) \in A \) with respect to the adjoint involution of \( h_0 \) on \( A \) (cf. [17] p.19]). This is a well defined element of the group \( E^*/\text{Nrd}(D_0^*) \). If \( h_0 = \langle \alpha_1, \ldots, \alpha_m \rangle \) with \( \alpha_i \in E^* \), then \( \text{Pf}(h_0) \) is represented by the discriminant of the quadratic form \( \langle \alpha_1, \ldots, \alpha_m \rangle \) over \( E \).

**Lemma 6.6.** With notation as in (6.4), write \( L = E(\sqrt{d}) \) with \( d \in E^* \). Let \( h_0 \) be a hermitian form of even rank over \((D_0, \tau_0)\).

(i) If the class \([h_0] \in W(D_0, \tau_0)\) lies in the image of \( \bar{\pi}_1 \), then its pfaffian norm \( \text{Pf}(h_0) \in E^*/\text{Nrd}(D_0^*) \) lies in the subgroup generated by \( \text{N}_{L/E}(L^*) \).

(ii) The converse of (i) is true if \( h_0 \) is of rank 2.

**Proof.** (i) For \( a + b\sqrt{d} \in L^* \) with \( a, b \in E \), the form \( \bar{\pi}_1(\langle a + b\sqrt{d} \rangle) \) is represented by the matrix

\[
\begin{pmatrix}
a & bd \\
bd & ad
\end{pmatrix}.
\]

One can then verify that

\[
\bar{\pi}_1(\langle a + b\sqrt{d} \rangle) = \begin{cases}
\langle a, ad(a^2 - b^2d) \rangle & \text{if } a \neq 0 \\
\langle 2bd, -2bd \rangle & \text{if } a = 0 \neq b \end{cases}.
\]

So it follows easily that \( \text{Pf}(\bar{\pi}_1(\langle a + b\sqrt{d} \rangle)) \) is represented by an element of \( \text{N}_{L/E}(L^*) \).

(ii) Conversely, let \( h_0 \) be a hermitian form of rank 2 whose pfaffian norm \( \text{Pf}(h_0) \) is represented by an element of \( \text{N}_{L/E}(L^*) \). We want to show \([h_0] \in \text{Im}(\bar{\pi}_1)\). By Suresh’s exact sequence, it suffices to show that the form \( \bar{\rho}(h_0) \) is hyperbolic over \((D_0, \tau)\).

We may assume \( h_0 = \langle \alpha, -\gamma \alpha \rangle \) with \( \alpha, \gamma \in E^* \). The assumption on the pfaffian norm implies that

\[
\tau_0(u)u\gamma = \text{Nrd}_{D_0}(u)\gamma = a^2 - b^2d
\]

for some \( u \in D_0^* \) and some \( a, b \in E \). Since

\[
\langle \alpha, -\gamma \alpha \rangle \cong \langle \alpha, -\gamma \alpha \tau_0(u)u \rangle \quad \text{over } (D_0, \tau_0),
\]

replacing \( \gamma \) by \( \gamma\tau_0(u) = \gamma.Nrd_{D_0}(u) \) if necessary, we may assume \( \gamma = a^2 - b^2d \) for some \( a, b \in E \). From the definition of the map \( \bar{\rho} \), it follows easily that the form \( \bar{\rho}(h_0) \) over \((D, \tau)\) is also represented by the diagonal matrix \( \langle \alpha, -\gamma \alpha \rangle \). But then for \( v = (a + b\sqrt{d}, 1) \in D^2 \), one has

\[
\bar{\rho}(h_0)(v, v) = (\tau(a + b\sqrt{d}), \tau(1)) \begin{pmatrix}
a & 0 \\
0 & -\gamma \alpha
\end{pmatrix} \begin{pmatrix}
a + b\sqrt{d} \\
1
\end{pmatrix} = \alpha(a^2 - b^2d - \gamma) = 0.
\]

This show that the rank 2 form \( \bar{\rho}(h_0) \) is isotropic and hence hyperbolic. \( \square \)
Lemma 6.7. With notation as above, assume that the field \( E \) has finite \( u \)-invariant \( u(E) = r \). Then for any hermitian form \( h_0 \) of rank \( m \geq r/3 \) over \((D_0, \tau_0)\), the form \( \tilde{\rho}(h_0) \) over \((D, \tau)\) is isotropic.

Proof. We may assume \( D_0^m \) consists of the underlying space of the form \( h_0 \) and \( h_0 = \langle \alpha_1, \ldots, \alpha_m \rangle \) with \( \alpha_i \in E^* \). Then the underlying space of \( \tilde{\rho}(h_0) \) is \( D_0^m = D_0^m \oplus D_0^m \sqrt{d} \). We fix a quaternion basis \( \{ 1, i, j, ij \} \) for the quaternion algebra \( D_0 \). The subspace \( \text{Sym}(D, \tau) \subseteq D \) consisting of \( \tau \)-invariant elements is a 4-dimensional \( E \)-vector space with basis

\[
1, i\sqrt{d}, j\sqrt{d}, ij\sqrt{d}.
\]

Let \( V \subseteq \text{Sym}(D, \tau) \) be the subspace generated by \( i\sqrt{d}, j\sqrt{d} \) and \( ij\sqrt{d} \). For \( w = x_1.i\sqrt{d} + x_2.j\sqrt{d} + x_3.ij\sqrt{d} \) with \( x_i \in E \), a straightforward calculation yields

\[
w^2 = d_{1} x_1^2 + d_{2} x_2^2 + d_{3} (ij)^2 x_3^2 \in E.
\]

So the map

\[\phi : V^m \rightarrow E ; \quad v = (v_1, \ldots, v_m) \longmapsto \tilde{\rho}(h_0)(v, v) = \sum \alpha_i v_i^2\]

defines a quadratic form of rank \( 3m \) over \( E \). By the assumption on the \( u \)-invariant of \( E \), the quadratic form \( \phi \) is isotropic and hence the hermitian form \( \tilde{\rho}(h_0) \) is isotropic. \( \square \)

Lemma 6.8. Assume that \( u(E) < 12 \). Then for any hermitian form \( h_0 \) of even rank \( 2n \) over \((D_0, \tau_0)\), one has

\[
[h_0] \in \text{Im}(\tilde{\tau}_1) \iff \text{Pf}(h_0) \in N_{L/E}(L^*).\text{Nrd}(D_0^*)
\]

Proof. In view of Lemma 6.3, we need only to prove that if \( \text{Pf}(h_0) \in N_{L/E}(L^*)\text{Nrd}(D_0^*) \), then \([h_0] \in \text{Im}(\tilde{\tau}_1) \).

To prove this, we use induction on \( n = \text{rank}(h_0) / 2 \), the case \( n = 1 \) being treated in Lemma 6.3. Now we assume \( \text{rank}(h_0) = 2n \geq 4 \) and \( h_0 \) is anisotropic. Let \( V_0 \) be the underlying space of \( h_0 \). Then the underlying space of the form \( \tilde{\rho}(h_0) \) is \( V = V_0 \oplus V_0 \sqrt{d} \).

By Lemma 6.7, the form \( \tilde{\rho}(h_0) \) is isotropic, that is, there is a nonzero vector \( x_1 + y_1 \sqrt{d} \in V = V_0 \oplus V_0 \sqrt{d} \) such that

\[
0 = \tilde{\rho}(h_0)(x_1 + y_1 \sqrt{d}, x_1 + y_1 \sqrt{d})
\]

\[
= (h_0(x_1, x_1) - h_0(y_1, y_1)d) + (h_0(x_1, y_1) - h_0(y_1, x_1)) \sqrt{d}.
\]

Thus

\[
h_0(x_1, x_1) = d.h_0(y_1, y_1) \quad \text{and} \quad h_0(x_1, y_1) = h_0(y_1, x_1).
\]

Since \( h_0 \) is anisotropic, \( h_0(x_1, x_1) \) and \( h_0(y_1, y_1) \) are both nonzero and hence lie in

\[
E^* = \{ x \in D_0^* \mid \tau_0(x) = x \}
\]

In particular, \( x_1 \neq 0, y_1 \neq 0 \) and

\[
h_0(x_1, y_1) = h_0(y_1, x_1) \in E = \{ x \in D_0 \mid \tau_0(x) = x \}.
\]

31
If \( x_1 = y_1 \lambda \) for some \( \lambda \in D_0^* \), then \((6.8.1)\) yields
\[
\tau_0(\lambda) \lambda = d \quad \text{and} \quad \tau_0(\lambda) = \lambda
\]
whence \( d = \lambda^2 \in E^{*2} \). Since \( d \) is not a square in \( E \), the two vectors \( x_1, y_1 \in V_0 \) generate a \( D_0 \)-submodule \( W_0 := x_1 D_0 + y_1 D_0 \subseteq V_0 \) of rank 2. Put \( a = h_0(y_1, y_1) \in E^* \) and \( bd = h_0(x_1, y_1) = h_0(y_1, x_1) \in E \). Then the restriction \( f_0 \) of \( h_0 \) to \( W_0 \) is represented by the matrix
\[
\begin{pmatrix}
ad & bd \\
bd & a
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}\begin{pmatrix}
a & bd \\
bd & ad
\end{pmatrix}\begin{pmatrix}
0 & 1
\end{pmatrix}.
\]
A direct computation then gives
\[
\tilde{\pi}_1([a + b\sqrt{d}]) = [f_0] \in W(D_0, \tau_0).
\]
This means that \( h_0 \) contains a subform \( f_0 \) of rank 2, which lies in the image of \( \tilde{\pi}_1 \). Writing \( h_0 = f_0 \perp g_0 \), we get \( Pf(g_0) \in N_{L/E}(L^*)\mathrm{Nrd}(D_0^*) \) since \( Pf(f_0) \) and \( Pf(h_0) \) lie in \( N_{L/E}(L^*)\mathrm{Nrd}(D_0^*) \). Now the induction hypothesis yields \([g_0] \in \mathrm{Im}(\tilde{\pi}_1)\), whence \([h_0] = [f_0] + [g_0] \in \mathrm{Im}(\tilde{\pi}_1)\).

\(\square\)

6.3 A Hasse principle for \( H^4 \) of function fields of conics

\textbf{Lemma 6.9.} Let \( F \) be a field of characteristic \( \neq 2 \), \( \overline{F} \) a separable closure of \( F \) and \( C \subseteq \mathbb{P}^2_F \) a smooth projective conic over \( F \). Put \( \overline{C} = C \times_F \overline{F} \) and let \( F(C), \overline{F}(C) \) denote the function fields of \( C \) and \( \overline{C} \) respectively.

Then the natural exact sequence
\[
0 \rightarrow \overline{F}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \rightarrow \text{Div}() \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \rightarrow \text{Pic}() \otimes \mathbb{Q}_2/\mathbb{Z}_2(2) \rightarrow 0
\]
induces an injection
\[
H^3(F, \overline{F}(C) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)) \rightarrow H^3(F, \text{Div}(C) \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)).
\]

\textbf{Proof.} Let \( C^{(1)} \) be the set of closed points of \( C \). For each \( P \in C^{(1)} \), let \( G_P \) be the absolute Galois group of the residue field \( F(P) \) of \( P \). This is an open subgroup of \( G = \text{Gal}(\overline{F}/F) \). Write \( M_P = \text{Hom}_{G_P}(\mathbb{Z}[G], \mathbb{Z}) \). We have an isomorphism of abelian groups \( M_P \cong \bigoplus_{Q \rightarrow P} \mathbb{Z} \), where the notation \( Q \rightarrow P \) means that \( Q \) runs over the closed points of \( \overline{C} \) lying over \( P \). On the other hand, we have an isomorphism of \( G \)-modules:
\[
\text{Div}(\overline{C}) \cong \bigoplus_{P \in C^{(1)}} M_P.
\]
Since \( C \) is a smooth projective conic, \( \text{Pic}(\overline{C}) \cong \mathbb{Z} \) as \( G \)-modules. The natural map \( \text{Div}(\overline{C}) \rightarrow \text{Pic}(\overline{C}) \) can be identified with the summation map
\[
\sigma : \bigoplus_{P \in C^{(1)}} \bigoplus_{Q \rightarrow P} \mathbb{Z} \rightarrow \mathbb{Z}
\]

32
So the exact sequence in the lemma may be identified with the following:

\[ 0 \rightarrow \frac{\mathcal{F}(C)^* \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)}{\bigoplus_{P \in C^{(1)}} M_P \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)} \rightarrow \frac{\mathbb{Q}_2/\mathbb{Z}_2(2)}{\mathbb{Q}_2/\mathbb{Z}_2(2)} \rightarrow 0. \]

For any \( i \geq 0 \),

\[ H^i(F, M_P \otimes \mathbb{Q}_2/\mathbb{Z}_2(2)) = H^i(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)). \]

It is thus sufficient to prove that the map

\[ \bigoplus_{P \in C^{(1)}} H^2(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)) \rightarrow H^2(F, \mathbb{Q}_2/\mathbb{Z}_2(2)) \]

is surjective. In fact, we can choose a closed point \( P \in C^{(1)} \) of degree 2 and consider the corresponding map

\[ \psi : H^2(F(P), \mathbb{Q}_2/\mathbb{Z}_2(2)) \rightarrow H^2(F, \mathbb{Q}_2/\mathbb{Z}_2(2)), \]

which coincides with the corestriction map. We claim that this map is already surjective.

To see this, consider for each \( n \in \mathbb{N} \) the corestriction map

\[ \psi_n : H^2(F(P), \mathbb{Z}/2^n(2)) \rightarrow H^2(F, \mathbb{Z}/2^n(2)). \]

By the Merkurjev–Suslin theorem, the map \( \psi_n \) may be identified with the norm map

\[ N_{F(P)/F} : K_2(F(P))/2^n \rightarrow K_2(F)/2^n \]

in Milnor’s \( K \)-theory. The cokernel of this norm map is killed by 2 = \( [F(P) : F] \). So taking limits yields the surjectivity of the map \( \psi \). This proves the lemma.

**Theorem 6.10.** Let \( K \) be the function field of a \( p \)-adic arithmetic surface or a local henselian surface with finite residue field of characteristic \( p \). Assume \( p \neq 2 \) in the local henselian case. Let \( C \) be a smooth projective conic in \( \mathbb{P}^2_K \).

Then the natural map

\[ H^4(K(C), \mathbb{Z}/2) \rightarrow \prod_{v \in \Omega_A} H^4(K_v(C), \mathbb{Z}/2) \]

is injective, where \( v \) runs over all divisorial valuations of \( K \).

**Proof.** By the Merkurjev–Suslin theorem, we may replace \( \mathbb{Z}/2 \) by \( \mathbb{Q}_2/\mathbb{Z}_2(3) \). Also, we may replace the completion \( K_v \) by the henselisation \( K_v \) for each \( v \) (cf. [14, Thm. 2.9 and its proof]). Let \( \overline{K} \) be a separable closure of \( K \). Then we have a diagram of field extensions

\[
\begin{array}{ccc}
\overline{K}(C) & \longrightarrow & \overline{K} \\
\downarrow & & \downarrow \\
K_v(C) & \longrightarrow & K_v \\
\downarrow & & \downarrow \\
K(C) & \longrightarrow & K
\end{array}
\]

33
which identifies the Galois groups

\[ \text{Gal}(\overline{K}/K) = \text{Gal}(\overline{K}(C)/K(C)) \quad \text{and} \quad \text{Gal}(\overline{K}/K_v) = \text{Gal}(\overline{K}(C)/K_v(C)) . \]

This induces Hochschild-Serre spectral sequences

\[ E_2^{pq}(K) = H^p(K, H^q(\overline{K}(C), Q_2/Z_2(3))) \Longrightarrow H^{p+q}(K(C), Q_2/Z_2(3)) \]

and

\[ E_2^{pq}(K_v) = H^p(K_v, H^q(\overline{K}(C), Q_2/Z_2(3))) \Longrightarrow H^{p+q}(K_v(C), Q_2/Z_2(3)) . \]

Using

\[ \text{cd}_2(\overline{K}(C)) \leq 1 \quad \text{and} \quad \text{cd}_2(K_v) \leq \text{cd}_2(K) \leq 3 , \]

one finds easily that the above spectral sequences induce canonical isomorphisms

\[ H^1(K(C), Q_2/Z_2(3)) \cong H^3(K(C), Q_2/Z_2(3)) \]

and

\[ H^1(K_v(C), Q_2/Z_2(3)) \cong H^3(K_v(C), Q_2/Z_2(3)) . \]

Since \( H^1(\overline{K}(C), Q_2/Z_2(3)) \cong \overline{K}(C)^* \otimes Q_2/Z_2(2) \), we need only prove the injectivity of the natural map

\[ H^3(K, \overline{K}(C)^* \otimes Q_2/Z_2(2)) \longrightarrow \prod_{v \in \Omega_A} H^3(K_v, \overline{K}(C)^* \otimes Q_2/Z_2(2)) \]

is injective.

By Lemma\textsuperscript{6.9} we have an injection

\[ H^3(K, \overline{K}(C)^* \otimes Q_2/Z_2(2)) \hookrightarrow \bigoplus_{p \in C^{(1)}} H^3(K(P), Q_2/Z_2(2)) . \]

For each \( v \), let \( C_v = C \times_K K_v \) be the base extension of \( C \) and let \( K_v(C) \) denote the function field of \( C_v \). By functoriality, we may reduce to proving the injectivity of the map

\[ \varphi : \bigoplus_{P \in C^{(1)}} H^3(K(P), Q_2/Z_2(2)) \longrightarrow \prod_{v \in \Omega_A} \bigoplus_{Q \in C_v^{(1)}} H^3(K_v(Q), Q_2/Z_2(2)) . \]

For fixed \( v \) and \( P \in C^{(1)} \), the corresponding component \( \varphi_{v,P} \) of the map \( \varphi \) is given by

\[ \varphi_{v,P} : H^3(K(P), Q_2/Z_2(2)) \longrightarrow \bigoplus_{Q \mid P} H^3(K_v(Q), Q_2/Z_2(2)) , \]

where \( Q \) runs over the points of the fiber \( C_v \times_K P = \text{Spec}(K_v \otimes_K K(P)) \). An element \( \alpha = (\alpha_P) \in \bigoplus_P H^3(K(P), Q_2/Z_2(2)) \) lies in ker(\( \varphi \)) if and only if for each \( P \in C^{(1)} \), \( \alpha_P \) lies in the kernel of

\[ \varphi_P = \prod_v \varphi_{v,P} : H^3(K(P), Q_2/Z_2(2)) \longrightarrow \prod_v H^3_{\text{ét}}(K_v \otimes_K K(P), Q_2/Z_2(2)) . \]
It suffices to prove that for every $P$, the map $\varphi_P$ is injective.

Replacing $K(P)$ by the separable closure of $K$ in $K(P)$ if necessary, we may assume that $K(P)/K$ is a finite separable extension. Then we have

$$K_v \otimes_K K(P) \cong \prod_{w \mid v} K(P)_{(w)},$$

(cf. [10, IV.18.6.8]). So the map $\varphi_P$ gets identified with the natural map

$$H^3(K(P), \mathbb{Q}_2/\mathbb{Z}_2(2)) \rightarrow \prod_{w} H^3(K(P)_{(w)}, \mathbb{Q}_2/\mathbb{Z}_2(2)),$$

where $w$ runs over divisorial valuations of $K(P)$. This map is injective by Theorem 3.3. The theorem is thus proved.

**Corollary 6.11.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Assume $p \neq 2$ in the local henselian case. Let $C$ a smooth projective conic in $\mathbb{P}^2_K$.

Then the natural map

$$I^4(K(C)) \rightarrow \prod_{v \in \Omega_A} I^4(K_v(C))$$

is injective, where $v$ runs over the set $\Omega_A$ of divisorial valuations of $K$.

**Proof.** For $F = K(C)$ or $K_v(C)$, we have $c_2(F) \leq 4$. By the degree 4 case of the Milnor conjecture (cf. [32] and [22]), we have an isomorphism $I^4(F) \cong H^4(F, \mathbb{Z}/2)$. (In the $p$-adic arithmetic case, we can also deduce this isomorphism from [11, p.655, Prop. 2] together with [19, Thm. 3.4].) The result then follows immediately from Theorem 6.10.

### 6.4 Case of even index

**Proposition 6.12.** Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $L/K$ be a quadratic field extension, $(D, \tau)$ a central division algebra over $L$ with a unitary $L/K$-involution whose index is not divisible by 4. Let $h$ be a nonsingular hermitian form over $(D, \tau)$ which has even rank, trivial discriminant and trivial Rost invariant (cf. (2.14)). Assume $p \neq 2$ if $\text{ind}(D)$ is even. In the local henselian case, assume further that the Hasse principle with respect to divisorial valuations holds for quadratic forms of rank 6 over $K$.

Then we have $[h] = 0 \in W(D, \tau)$ if and only if $[h \otimes_K K_v] = 0 \in W(D \otimes_K K_v, \tau)$ for every $v \in \Omega_A$.

**Proof.** If the index $\text{ind}(D)$ is odd, the result is already proved in Proposition 6.1. We assume next that $\text{ind}(D)$ is even and not divisible by 4.

We first consider the case where $D$ is a quaternion algebra. As in (6.4), we write $D = D_0 \otimes_K L$ with $D_0$ a quaternion division algebra over $K$ and $L = K(\sqrt{d})$ with $d \in K^*$, and we have Suresh’s exact sequence

$$W(L) \xrightarrow{\tilde{\eta}_1} W(D_0, \tau_0) \xrightarrow{\tilde{\rho}} W(D, \tau) \xrightarrow{p_2} W^{-1}(D_0, \tau_0)$$

(6.12.1)
Let $C \subseteq \mathbb{P}_K^2$ be the smooth projective conic associated to the quaternion algebra $D_0$. Then the algebra $D \otimes_K K(C) = D_0 \otimes_K L(C)$ is a split central simple algebra over $L(C)$ with a unitary $L(C)/K(C)$-involution $\tau$. By Morita theory, the hermitian form $h \otimes_K K(C)$ over $(D \otimes_K K(C), \tau)$ corresponds to a hermitian form $h'_C$ over $(L(C), \iota)$, where $\iota$ denotes the nontrivial element of the Galois group $\text{Gal}(L(C)/K(C))$. The trace form $q_{h,C}$ of $h'_C$ gives a quadratic form over $K(C)$. By [17, Example 31.44], the quadratic form $q_{h,C}$ has even rank, trivial discriminant, trivial Clifford invariant and trivial Rost invariant, since $h'_C$ has even rank, trivial discriminant and trivial Rost invariant (these invariants being invariant under Morita equivalence). Hence in the Witt group $W(K(C))$ we have $[q_{h,C}] \in I^3(K(C))$. Since $h$ is locally hyperbolic, it follows from Corollary 6.11 that $[q_{h,C}] = 0 \in W(K(C))$, whence $[h \otimes_K K(C)] = 0 \in W(D \otimes_K K(C), \tau)$. In the commutative diagram

$$
\begin{array}{ccc}
W(D, \tau) & \xrightarrow{p_2} & W^{-1}(D_0, \tau_0) \\
\downarrow & & \downarrow \\
W(D \otimes_K K(C), \tau) & \xrightarrow{p_2} & W^{-1}(D_0 \otimes_K K(C), \tau_0)
\end{array}
$$

the right vertical map is injective by [24]. So we have $p_2(h) = 0 \in W^{-1}(D_0, \tau_0)$. The exactness of the sequence (6.12.1) implies that $[h] = \tilde{\rho}(\tilde{h}_0)$ for some hermitian form $h_0$ over $(D_0, \tau_0)$ of even rank.

Let $\lambda = \text{Pf}(h_0) \in K^*/\text{Nrd}(D_0^\ast)$ be the pfaffian norm of $h_0$. Since $h$ is locally hyperbolic, by considering Suresh’s exact sequence locally, we see that $(h_0)_v$ lies in the image of $(\tilde{\pi}_1)_v$ for every $v$. By Lemma 6.6, this implies that $\lambda \in \text{Nrd}((D_0^\ast)_v^{-1}) \cdot N_{L_v/K_v}(L_0^\ast)$ for every $v$. In other words, the quadratic form

$$
\phi := \lambda \cdot n_{D_0} - (1, -d),
$$

where $n_{D_0}$ denotes the norm form of the quaternion algebra $D_0$, is isotropic over every $K_v$. By the assumption on the Hasse principle for quadratic forms of rank 6 (and [6, Thm. 3.1] in the $p$-adic arithmetic case), $\phi$ is isotropic over $K$, which shows $\lambda \in \text{Nrd}(D_0^\ast) \cdot N_{L/K}(L^\ast)$. As was mentioned in the proof of Corollary 4.4, the field $K$ has $u$-invariant 8. So by Lemma 6.8 we have $[h_0] \in \text{Im}(\tilde{\pi}_1)$. Hence $[h] = \tilde{\rho}(\tilde{h}_0) = 0 \in W(D, \tau)$ as desired.

Consider next the general case where $\text{ind}(D)$ is even and not divisible by 4. In this case we have $D = Q \otimes_L D'$ for some quaternion division algebra $Q$ over $L$ and some central division algebra $D'$ of odd index over $L$. By [3, Lemma 3.3.1], there is an odd degree separable extension $K'/K$ such that $D' \otimes_K K' = D' \otimes_L LK'$ is split. By Morita theory, there is a unitary $LK'/K'$-involution $\sigma$ on $H \otimes_L LK'$ and a hermitian form $f$ over $(H \otimes_L LK')$, such that the involution $\tau$ on $D \otimes LK'$ is adjoint to $f$, and moreover, the form $h_{K'}$ over $(D \otimes_L LK', \tau)$ corresponds to a hermitian form $h'$ over $(H \otimes_L LK', \sigma)$, which has even rank, trivial discriminant and trivial Rost invariant. The hypothesis that $h$ is locally hyperbolic over every $K_v$ implies that $h'$ is locally hyperbolic over every $K'_w$, where $w$ runs over the set of divisorial valuations of $K'$. By the previous case, $[h'] = 0 \in W(H \otimes LK', \sigma)$ and hence $[h] = 0 \in W(D \otimes LK', \tau)$. Since the degree $[LK': L] = [K': K]$ is odd, the natural map $W(D, \tau) \to W(D \otimes LK', \tau)$ is injective by a theorem of Bayer-Fluckiger and Lenstra (cf. [17, p.80, Coro. 6.18]). So we get $[h] = 0 \in W(D, \tau)$. This completes the proof.\[\Box\]
Theorem 6.13. Let $K$ be the function field of a $p$-adic arithmetic surface or a local henselian surface with finite residue field of characteristic $p$. Let $L/K$ be a quadratic field extension, $(D, \tau)$ a central division algebra over $L$ with a unitary $L/K$-involution whose index $\text{ind}(D)$ is square-free. Let $h$ be a nonsingular hermitian form over $(D, \tau)$. Assume $p \neq 2$ if $\text{ind}(D)$ is even. In the local henselian case, assume further that $p \nmid \text{ind}(D)$ and that the Hasse principle with respect to divisorial valuations holds for quadratic forms of rank 6 over $K$.

Then the natural map

$$H^1(K, \text{SU}(h)) \longrightarrow \prod_{v \in \Omega} H^1(K_v, \text{SU}(h))$$

has trivial kernel.

Proof. Let $\xi \in H^1(K, \text{SU}(h))$ be a class that is locally trivial. Let the image of $\xi$ in $H^1(K, \text{U}(h))$ correspond to a hermitian form $h'$. The form $h' \perp (-h)$ has even rank, trivial discriminant and is locally hyperbolic. We claim that the Rost invariant $R(h' \perp (-h))$ is trivial. Indeed, as $\xi$ is locally trivial, $R_{\text{SU}(h)}(\xi)$ is locally trivial in $H^3(K_v, \mathbb{Q}/\mathbb{Z}(2))$ for every $v$. By Theorem 3.4, $R_{\text{SU}(h)}(\xi) = 0$. There is a group homomorphism $\text{SU}(h) \longrightarrow \text{SU}(h' \perp (-h))$, $f \mapsto (f, \text{id})$ which induces a map

$$\alpha : H^1(K, \text{SU}(h)) \longrightarrow H^1(K, \text{SU}(h' \perp (-h))).$$

The image $\alpha(\xi)$ of $\xi$ lifts the class $[h' \perp (-h)] \in H^1(K, \text{U}(h' \perp (-h)))$. By general property of the (usual) Rost invariant, there is an integer $n_\alpha$ such that

$$R_{\text{SU}(h' \perp (-h))}(\alpha(\xi)) = n_\alpha R_{\text{SU}(h)}(\xi).$$

We have thus $R(h' \perp (-h)) = R_{\text{SU}(h' \perp (-h))}(\alpha(\xi)) = 0$ since $\xi$ has trivial Rost invariant.

Now Proposition 6.12 implies that the two forms $h', h$ over $(D, \tau)$ are isomorphic.

Consider the cohomology exact sequence

$$1 \longrightarrow \frac{R^1_{L/K}\mathbb{G}_m(K)}{\text{Nrd}(\text{U}(h)(K))} \xrightarrow{\varphi} H^1(K, \text{SU}(h)) \longrightarrow H^1(K, \text{U}(h))$$

arising from the exact sequence of algebraic groups

$$1 \longrightarrow \text{SU}(h) \longrightarrow \text{U}(h) \xrightarrow{\text{Nrd}} R^1_{L/K}\mathbb{G}_m \longrightarrow 1.$$

The fact that $h' \cong h$ implies that $\xi$ lies in the image of the map $\varphi$ in the above cohomology exact sequence (6.13.1). Considering the sequence (6.13.1) locally and using Lemma 6.2 we conclude that $\xi$ is trivial in $H^1(K, \text{SU}(h))$, thus proving the theorem. \qed

Acknowledgements. The author thanks Prof. Jean-Louis Colliot-Thélène for helpful discussions.
References

[1] J. Kr. Arason, R. Elman, B. Jacob, Fields of cohomological 2-dimension three, Math. Ann., 274 (1986), 649–657.

[2] H.-J. Bartels, Invarianten hermitescher Formen über Schiefkörpern, Math. Ann., 215 (1975), 269–288.

[3] E. Bayer-Fluckiger and R. Parimala, Galois cohomology of the classical groups over fields of cohomological dimension ≤ 2, Invent. math., 122 (1995), 195–229.

[4] E. Bayer-Fluckiger and R. Parimala, Classical groups and the Hasse principle, Ann. Math., 147 (1998), 651–693, Erratum, Ann. Math. 163 (2006), 381.

[5] V. Chernousov, The kernel of the Rost invariant, Serre’s conjecture II and the Hasse principle for quasi-split groups $^{3,6}D_4$, $E_6$, $E_7$, Math. Ann. 326 (2003), no. 2, 297–330.

[6] J.-L. Colliot-Thélène, R. Parimala, V. Suresh, Patching and local-global principles for homogeneous spaces over function fields of p-adic curves, Comment. Math. Helv., 87 (2012), 1011–1033.

[7] S. Garibaldi, The Rost invariant has trivial kernel for quasi-split groups of low rank, Comment. Math. Helv., 76 (2001), 684–711.

[8] P. Gille, Invariants cohomologiques des Rost en caractéristique positive, K-theory, 21 (2000), 57–100.

[9] P. Gille, Serre’s conjecture II: a survey, in: Quadratic Forms, Linear Algebraic Groups, and Cohomology, ed. J.-L. Colliot-Thélène and S. Garibaldi and R. Sujatha and V. Suresh, Developments in Math. 18, Springer, 2010.

[10] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie, Publ. Math. de l’Inst. Hautes Études Sci., 32, 1967.

[11] D. Harbater, J. Hartmann, D. Krashen, Applications of patching to quadratic forms and central simple algebras, Invent. math. 178 (2009), 231–263.

[12] Y. Hu, Local-global principle for quadratic forms over fraction fields of two-dimensional henselian domains, Ann. de l’Institut Fourier, 62 (2012), No.6, 2131–2143.

[13] Y. Hu, Division algebras and quadratic forms over fraction fields of two-dimensional henselian domains, Algebra & Number Theory, 7 (2013), No. 8, 1919–1952.

[14] U. Jannsen, Hasse principle for higher-dimensional fields, preprint, available at arXiv: 0910.2803.
[15] K. Kato, *A Hasse principle for two-dimensional global fields*, J. reine angew. Math. **366** (1986), 142–181.

[16] M.-A. Knus, *Quadratic and Hermitian Forms over Rings*, Springer-Verlag, 1991.

[17] M.-A. Knus and A. Merkurjev and M. Rost and J.-P. Tignol, *The book of involutions*, Amer. Math. Soc., 1998.

[18] T.Y. Lam, *Introduction to Quadratic Forms over Fields*. Grad. Studies in Math. **67**, Amer. Math. Soc., 2005.

[19] D. Leep, *The u-invariant of p-adic function fields*, J. reine angew. Math., to appear.

[20] A. Merkurjev, *Norm principle for algebraic groups*, St. Petersburg J. Math., **7** (1996), 243–264.

[21] A. Merkurjev, *Rost invariants of simply connected algebraic groups*, in: Cohomological invariants in Galois cohomology, University lecture series **28**, Amer. Math. Soc., 2003.

[22] D. Orlov, A. Vishik, and V. Voevodsky, *An exact sequence for $K^M_*/2$ with applications to quadratic forms*, Ann. of Math. (2), **165** No.1 (2007), 1–13.

[23] R. Parimala and R. Preeti, *Hasse principle for classical groups over function fields of curves over number fields*, J. Number Theory, **101** No.1 (2003), 151–184.

[24] R. Parimala, R. Sridharan and V. Suresh, *Hermitian analogue of a theorem of Springer*, J. Algebra, **243** No.2 (2001), 780–789.

[25] R. Parimala, V. Suresh, *The u-invariant of the function fields of p-adic curves*. Ann. Math. **172** (2010), 1391–1405.

[26] R. Preeti, *Classification theorems for hermitian forms, the Rost kernel and Hasse principle over fields with $cd_2(k) \leq 3$*, J. Algebra **385** (2013), 294–313.

[27] S. Saito, *Class field theory for two-dimensional local rings*, in *Galois Representation and Arithmetic Algebraic Geometry*, Advanced Studies in Pure Math., vol. **12** (1987), 343–373.

[28] D. Saltman, *Division algebras over p-adic curves*, J. Ramanujan. Math. Soc. **12**, No. 1 (1997), 25–47.

[29] W. Scharlau, *Quadratic and Hermitian Forms*, Springer-Verlag, 1985.

[30] J.-P. Serre, *Cohomologie galoisienne: progrès et problèmes*, Séminaire Bourbaki, exposé No.783, 1993/1994, Astérisque, **227** (1995), 229–257.

[31] A. A. Suslin, *Algebraic $K$-theory and the norm-residue homomorphism*, J. Soviet Math., **30** (1985), 2556–2611.

[32] V. Voevodsky, *Motivic cohomology with Z/2-coefficients*, Publ. Math. Inst. Hautes Études Sci., **98** (2003), 59–104.