Research Article

Characterization and Stability of Multimixed Additive-Quartic Mappings: A Fixed Point Application

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In this article, we introduce the multi-additive-quartic and the multimixed additive-quartic mappings. We also describe and characterize the structure of such mappings. In other words, we unify the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also show that under what conditions, a multimixed additive-quartic mapping can be multiadditive, multiquartic, and multi-additive-quartic. Moreover, by using a fixed point technique, we prove the Hyers-Ulam stability of multimixed additive-quartic functional equations thus generalizing some known results.

1. Introduction

Let $V$ be a commutative group, $W$ be a linear space over rational numbers, and $n$ be an integer with $n \geq 2$. A mapping $f: V^n \rightarrow W$ is called

(i) **Multiadditive** if it satisfies the Cauchy’s functional equation $A(x + y) = A(x) + A(y)$ in each variable [1]

(ii) **Multiquadratic** if it fulfills quadratic functional equation $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ in each variable [2, 3]

(iii) **Multicubic** if it satisfies the cubic equation $C(2x + y) + C(2x - y) = 2C(x + y) + 2C(x - y) + 12C(x)$ in each variable [4, 5]

(iv) **Multiquartic** if it satisfies the quartic equation

\[ Q(x + 2y) + Q(x - 2y) = 4Q(x + y) + 4Q(x - y) - 6Q(x) + 24Q(y), \]  \hspace{1cm} (1)

in each variable [6, 7].

We have the following observations about a several variables mapping $f: V^n \rightarrow W$.

(i) $f$ is multiadditive [1] if and only if it satisfies

\[ f(x_1 + x_2) = \sum_{j_1, \ldots, j_n \in \{1,2\}} f(x_{j_1}, \ldots, x_{j_n}). \]  \hspace{1cm} (2)

(ii) $f$ is multiquadratic [8] if and only if it satisfies
where \( x_i = (x_{1j}, x_{2j}, \ldots, x_{nj}) \in V^n \) with \( j \in \{1, 2\} \). More information about the structure of multiaffine and multiquadratic mappings, we refer for instance to \([9, 10]\).

Bodaghi et al. \([4]\) (resp., \([6]\)) provided a characterization of multicharacteristic (resp., multiquadratic) mappings, and they showed that every multicharacteristic (resp., multiquadratic) mapping can be shown a single functional equation and vice versa.

Lee et al. \([11]\) introduced and obtained the general solution of the quartic functional equation which somewhat different from (1) as follows:

\[
\sum_{s \in \{-1, 1\}^n} f(x_1 + sx_2) = 2^n \sum_{i,j,\ldots,\delta \in \{1,2\}} f(x_{i1}, \ldots, x_{nj}),
\]

(3)

For the generalized forms of the quartic functional, equations (1) and (4) refer to \([12, 13]\). Recently, in \([14]\), we firstly define multiaffine-quartic mappings and include a characterization of such mappings. In fact, we prove that every multi-affine-quartic mapping can be shown a single functional equation and vice versa (under some extra conditions). Section 3 is devoted to the study of structure of multiaffine-quartic mappings. In other words, motivated by equation (6), we introduce the multiaffine-quartic mappings and reduce the system of \( n \) equations defining the multiaffine-quartic mappings to a single equation, namely, the multiaffine-quartic functional equation. In Section 4, we prove the Hyers-Ulam stability for the multi-affine-quartic and the multiaffine-quartic mappings in the setting of Banach spaces by applying a fixed point method \([31]\). As an application of this result, we establish the stability of multi-affine-quartic mappings. Finally, we show that under some mild conditions every multiaffine and multi-quartic functional equations are \( \delta \)-stable for a small positive number \( \delta \).

2. Characterization of Multi-Additive-Quartic Mappings

Throughout this paper, \( N \) and \( Q \) stand for the set of all positive integers and the rational numbers, respectively, \( N_0 = \mathbb{N} \cup \{0\}, R_+ = [0, \infty) \). For any \( l \in N_0, m \in \mathbb{N}, t = (t_1, \ldots, t_m) \in \{-1, 1\}^m \), and \( x = (x_1, \ldots, x_m) \in V^m \), we write \( lx = (lx_1, \ldots, lx_m) \) and \( tx = (t_1x_1, \ldots, t_nx_n) \), where \( ra \) stands, as usual, for the \( r \)-th power of an element \( a \) of the commutative group \( V \).

Let \( V \) and \( W \) be linear spaces, \( n \in \mathbb{N} \) and \( k \in \{0, \ldots, n\} \). A mapping \( f : V^n \rightarrow W \) is called \( k \)-additive and \( n-k \)-quartic (briefly, multi-additive-quartic) if \( f \) is additive in each of some \( k \) variables and satisfies (4) in each of the other variables. In what follows, for simplicity, it is assume that \( f \) is additive in each of the first \( k \) variables. Moreover, for \( k = n \) (\( k = 0 \)), the above definition leads to the so-called multiadditive (multiquartic) mappings.

In the sequel, we assume that \( V \) and \( W \) are vector spaces over \( Q \). Moreover, we identify \( x = (x_1, \ldots, x_n) \in V^n \) with \((x^k, x^{n-k}) \in V^k \times V^{n-k} \), where \( x^k = (x_1, \ldots, x_k) \) and \( x^{n-k} = (x_{k+1}, \ldots, x_n) \). Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( x^i = (x_{i1}, x_{i2}, \ldots, x_{im}) \in V^m \), where \( i \in \{1, 2\} \). Throughout, we shall denote \( x^i \) by \( x_i \) if there is no risk of mistake. Put also \( x^i_j = (x_{i1}, \ldots, x_{ij}) \in V^k \) and \( x^{n-k}_j = (x_{k+1}, \ldots, x_{nj}) \in V^{n-k} \). For \( x_1, x_2 \in V^n \) and \( p, q \in N_0 \) with \( 0 \leq p \leq n \) and \( 0 \leq k \leq n - 1 \), set \( x^{p-k}_j = \{x^{p-k}_j \mid N_j \in \{x_{1j}, x_{2j}, \ldots, x_{nj}\} \} \), where \( j \in \{k+1, \ldots, n\} \). Consider the subset \( N^{p-k}_j \) of \( N \) as follows:

\[
f(x + 2y) - 4f(x + y) - 4f(x - y) + f(x - 2y) = \frac{12}{7} (f(2y) - 2f(y)) - 6f(x).
\]

(6)

It is easily verified that the function \( f(x) = ax^2 + bx \) is a solution of equations (5) and (6); the generalized version of equation (6) can be found in \([30]\).

This paper is organized as follows: In the second section, we firstly define multi-affine-quartic mappings and include a characterization of such mappings. In fact, we prove that every multi-affine-quartic mapping can be shown a single functional equation and vice versa (under some extra conditions). Section 3 is devoted to the study of structure of multiaffine-quartic mappings. In other words, motivated by equation (6), we introduce the multiaffine-quartic mappings and reduce the system of \( n \) equations defining the multiaffine-quartic mappings to a single equation, namely, the multiaffine-quartic functional equation. In Section 4, we prove the Hyers-Ulam stability for the multi-affine-quartic and the multiaffine-quartic mappings in the setting of Banach spaces by applying a fixed point method \([31]\). As an application of this result, we establish the stability of multi-affine-quartic mappings. Finally, we show that under some mild conditions every multiaffine and multi-quartic functional equations are \( \delta \)-stable for a small positive number \( \delta \).
\[ A_{n-k}^{p_1, p_2} := \{ R_{n-k} \in A_{n-k} \mid \text{Card}\{ N_j : N_j = x_{ij} \} = p_i (i \in \{1, 2\}) \}. \tag{7} \]

To achieve our aims, for the multi-additive-quartic mappings, we use the oncoming notations:

\[ f\left( A_{n-k}^{p_1, p_2} \right) = \sum_{R_{n-k} \in A_{n-k}^{p_1, p_2}} f(R_n), \tag{8} \]

\[ f(z, A_{n-k}^{p_1, p_2}) = \sum_{R_{n-k} \in A_{n-k}^{p_1, p_2}} f(z, R_{n-k})(z \in V). \tag{9} \]

For each \( x_1, x_2 \in V^n \), we consider the equation

\[
\sum_{r \in (-1,1)^n} f(x_1^r + x_2^r, 2x_1^n + tx_2^k) = \sum_{p_i=0}^{n-k} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(x_1^n, A_{n-k}^{p_1, p_2}\right), \tag{10}
\]

for all \( x^r_i = (x_{i1}, \ldots, x_{ik}) \in V^k \) and \( x^n_i = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k} \) where \( i \in \{1, 2\} \).

It is shown in Proposition 2.2 in [14] that if a mapping \( f : V^n \longrightarrow W \) is multi-quartic, then it satisfies the equation

\[
\sum_{r \in (-1,1)^n} f(2x_1 + tx_2) = \sum_{p_1=0}^{n-p_0} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(A_{n-k}^{p_1, p_2}\right). \tag{11}
\]

The next proposition shows that the system of \( n \) equations defining a multi-additive-quartic mapping can be reduced to (10).

**Proposition 1.** Let \( n \in \mathbb{N} \) and \( k \in \{0, \ldots, n\} \). Suppose that a mapping \( f : V^n \longrightarrow W \) is \( k \)-additive and \( n-k \)-quartic (multi-additive-quartic) mapping. Then, \( f \) fulfills equation (10).

**Proof.** For \( k \in \{0, n\} \), the result follows from Proposition 2.2 in [14] and Theorem 2 in [1], and so we prove the assertion for the case that \( k \in \{1, \ldots, n-1\} \). For any \( x^n_i \in V^{n-k} \), consider the mapping \( g_{x^n_i} : V^k \longrightarrow W \) defined by \( g_{x^n_i}(x^k_i) := f(x^k_i, x^n_i) \) for \( x^k_i \in V^k \). The assumption shows that \( g_{x^n_i} \) is \( k \)-additive, and thus, we can obtain from Theorem 2 in [1] that

\[
g_{x^n_i}(x_1^k + x_2^k) = \sum_{j, \ell = 1}^{k} g_{x^n_i}(x_{j1}^k, x_{j2}^k, \ldots, x_{\ell k}^k), \quad (x_1^k, x_2^k \in V^k). \tag{12}
\]

The above equality implies that

\[
f\left(x_1^n, x_2^n \right) = \sum_{j, \ell = 1}^{k} f(x_{j1}^k, \ldots, x_{\ell k}^k), \tag{13}
\]

for all \( x_1^n, x_2^n \in V^k \) and \( x^n \in V^{n-k} \). Repeat the above method, and for any \( x^k \in V^k \), define the mapping \( h_{x^k} : V^{n-k} \longrightarrow W \) via \( h_{x^k}(x_{i1}^k, \ldots, x_{ik}^k) := f(x^k, x_{i,k+1}^n, \ldots, x_{in}^n) \). This mapping is \( n-k \)-quartic, and hence, by Proposition 2.2 from [14], we have

\[
\sum_{r \in (-1,1)^n} h_{x^k}(2x_1^k + tx_2^k) = \sum_{p_i=0}^{n-k-p_0} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(A_{n-k}^{p_1, p_2}\right), \tag{14}
\]

for all \( x_1^k, x_2^n \in V^{n-k} \). On the other hand, by the definition of \( h_{x^k} \), relation (14) converts to

\[
\sum_{r \in (-1,1)^n} f(x^n_i, x_1^k + tx_2^k) = \sum_{p_1=0}^{n-k-p_0} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(x^n_i, A_{n-k}^{p_1, p_2}\right), \tag{15}
\]

for all \( x_1^k, x_2^n \in V^{n-k} \) and \( x^k \in V^k \). It now follows between (13) and (15) that

\[
\sum_{r \in (-1,1)^n} f(x_1^n, x_2^n) = \sum_{p_1=0}^{n-k-p_0} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(x_1^n, A_{n-k}^{p_1, p_2}\right)
\]

\[
= \sum_{p_1=0}^{n-k-p_0} \sum_{p_2=0}^{n-p_1} 4^{n-k-p_1-p_2} 24^p_1(-6)^pf\left(x_1^n, A_{n-k}^{p_1, p_2}\right), \tag{16}
\]

for all \( x_1^k = (x_{i1}, \ldots, x_{ik}) \in V^k \) and \( x_{i,k+1}^n = (x_{i,k+1}, \ldots, x_{in}) \in V^{n-k} \). This finishes the proof. \( \square \)

By Proposition 6, it is easily verified that the mapping \( f(z_1, \ldots, z_n) = \prod_{i=1}^{k} z_i \prod_{j=k+1}^{n} z_j^2 \) satisfies (10), and so this equation is said to be multi-additive-quartic functional equation.

**Definition 2.** Let \( r \in \mathbb{N} \). Consider a mapping \( f : V^n \longrightarrow W \). We say \( f \)
Theorem 4. Suppose that a mapping \( f \) satisfies equation (10). Under one of the following assumptions, for all \( z_1, \ldots, z_n \in V^n \), sometimes 4-power condition is called quartic condition.

(i) Satisfies (has) the \( r \)-power condition in the \( j \)th variable if
\[
f(z_{1j}, z_{2j}, \ldots, z_{nj}) = 2^r f(z_{1j}, z_{2j}, \ldots, z_{nj}),
\]
for all \( z_1, \ldots, z_n \in V^n \). Sometimes 4-power condition is called quartic condition.

(ii) Has zero condition if \( f(x) = 0 \) for any \( x \in V^n \) with at least one component which is equal to zero

We remember that the binomial coefficient for all \( n, r \in \mathbb{N}_0 \) with \( n \geq r \) is defined and denoted by \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \).

We wish to show that if a mapping satisfies equation (10), then it is multi-additive-quartic. For doing it, we need the upcoming lemma. The method of the proof of Lemma 3 is similar to the proof of ([14], Lemma 2.5) and so we include lemma without the proof.

Lemma 3. Suppose that a mapping \( f : V^n \rightarrow W \) satisfies equation (10). Under one of the following assumptions, \( f \) satisfying zero condition.

(i) \( f \) satisfies the quartic condition in the last \( n-k \) variables

(ii) \( f \) is even in the last \( n-k \) variables.

Theorem 4. Suppose that a mapping \( f : V^n \rightarrow W \) fulfilling equation (10). Under one of the hypothesis of Lemma 3, \( f \) is multi-additive-quartic.

Proof. It follows from Lemma 3; \( f \) satisfies zero condition. Putting \( x_2^{n-k} = (0, \cdots, 0) \) in the left side of (10) and applying the hypothesis, we obtain
\[
2^{n-k} \times 2^{4(n-k)} f(x_1^k, x_2^n, x_1^{n-k}) = 2^{5(n-k)} f(x_1^k, x_2^n, x_1^{n-k}).
\]

On the other hand, by using Lemma 3, the right side of (10) converts to
\[
\sum_{p_1=0}^{n-k} \binom{n-k}{p_1} \binom{n-k}{p_1} 24^{p_1} 2^{n-k-p_1} f(x_{j_1}, x_{j_2}, \ldots, x_{j_k}, x_1^{n-k}),
\]
for all \( x_i \in V^n \) and \( x_2^{n-k} \in V^{n-k} \). In light of Theorem 2 in [1], we see that \( f \) is additive in each of the \( k \) first variables. In addition, by considering \( x_2^2 = (0, \cdots, 0) \) in (10) and applying again Lemma 3, we have
\[
\sum_{p_1=0}^{n-k} \sum_{p_2=0}^{n-k-p_1} 4^{n-k-p_1} 24^{p_2} (-6)^{p_2} f(x_1^k, x_{j_1}, x_{j_2}, \ldots, x_{j_k}, x_1^{n-k}),
\]
for all \( x_i \in V^k \) and \( x_2^{n-k} \in V^{n-k} \), and thus, by Theorem 2.6 in [14], \( f \) is quartic in each of the last \( n-k \) variables.

The proof of second part is similar.

\[ \square \]

3. Characterization of Multimixed Additive-Quartic Mappings

In this section, we introduce the multimixed additive-quartic mappings and then characterize them as an equation. We start this section with the definition of such mappings.

Definition 5. Let \( V \) and \( W \) be vector spaces over \( \mathbb{Q}, n \in \mathbb{N} \). A mapping \( f : V^n \rightarrow W \) is called \( n \)-multimixed additive-quartic if \( f \) satisfies mixed additive-quartic equation (6) in each variable.

Let \( n \in \mathbb{N} \) with \( n \geq 2 \) and \( x_i^q = (x_{i1}, x_{i2}, \ldots, x_{iq}) \in V^n \), where \( i \in \{1, 2\} \). For \( x_1, x_2 \in V^n \) and \( q \in \mathbb{N}_0 \) with \( 0 \leq q \leq n \), put
\[
\mathcal{M} = \{ \mathcal{M}_n = (M_1, \cdots, M_n) \mid M_j \in \{ x_{j1} \pm 2x_{j2}, 2x_{j2} \}, j \in \{1, \cdots, n\} \}.
\]

Consider the subset \( \mathcal{M}_q^n \) of \( \mathcal{M} \) as follows:
\[
\mathcal{M}_q^n = \{ \mathcal{M}_n \in \mathcal{M} : |\{ M_j : M_j = 2x_{j2} \}| = q \}.
\]

Hereafter, for the multimixed additive-quartic mappings, we use the following notations:
\[
f(\mathcal{M}_q^n) = \sum_{\mathcal{M}_n \in \mathcal{M}_q^n} f(\mathcal{M}_n),
\]
\[
f(\mathcal{M}_q^n, z) = \sum_{\mathcal{M}_n \in \mathcal{M}_q^n} f(\mathcal{M}_n, z)(z \in V).
\]

Next, we reduce the system of \( n \) equations defining the multimixed additive-quartic mapping to obtain a single functional equation.
Proposition 6. If a mapping \( f : V^n \to W \) is multimixed additive-quartic, it satisfies the equation

\[
\sum_{q=0}^{n+1} \left( -\frac{12}{7} \right)^q f (\mathcal{M}^n_q) = \sum_{p_i=0}^{n} \sum_{p_j=0}^{n} 4^{n-p_i-p_j} (-6)^{p_i} \left( -\frac{24}{7} \right)^{p_j} f (\mathcal{A}^n_{(p_i, p_j)}),
\]

(26)

where \( f (\mathcal{M}^n_q) \) and \( f (\mathcal{A}^n_{(p_i, p_j)}) \) are defined in (24) and (8), respectively.

Proof. The proof is based on induction for \( n \). For \( n = 1 \), it is obvious that \( f \) satisfies (6). Assume that (26) holds for some positive integer \( n > 1 \). Then

\[
\sum_{q=0}^{n+1} \left( -\frac{12}{7} \right)^q f (\mathcal{M}^n_q) = \sum_{p_i=0}^{n} \sum_{p_j=0}^{n} \left( -\frac{12}{7} \right)^q f (\mathcal{A}^n_{(p_i, p_j)}),
\]

(27)

The assertion is now proved.

Since the mapping \( f (z_1, \ldots, z_n) = \prod_{j=1}^{n} (a_j z_j^4 + b_j z_j) \) is multimixed additive-quartic, it satisfies (26) by proposition above, and so this equation is called multimixed additive-quartic functional equation.

Here, we bring an elementary lemma without proof.

Lemma 7. Let \( n, k, p_i \in \mathbb{N}_o \) such that \( k + \sum_{i=1}^{m} p_i \leq n \), where \( l \in \{1, \ldots, m\} \). Then

\[
\begin{pmatrix}
    n-k \\
    \sum_{i=1}^{m} p_i \\
    \sum_{i=1}^{m} p_i
\end{pmatrix}
\]

(28)

Similar to Lemma 2.1 from [6], we need the following lemma in obtaining our goal in this section. The proof is similar, but we include some parts for the sake of completeness.

Lemma 8. If a mapping \( f : V^n \to W \) satisfies equation (26), then it has zero condition.

Proof. Putting \( x_1 = x_2 = (0, \ldots, 0) \) in (26), we have

\[
\left[ \sum_{q=0}^{n} \left( -\frac{12}{7} \right)^q 2^{-n-2q} \right] f (0, \ldots, 0)
\]

(29)

Using Lemma 7 for \( k = 0 \) and \( p_1, p_2 \), the right side of (29) will be as follows:

\[
\left[ \sum_{p_1=0}^{n} \sum_{p_2=0}^{n} \left( -\frac{12}{7} \right)^{n-p_1-p_2} 2^{n-p_1-p_2} f (0, \ldots, 0) \right]
\]

(30)

On the other hand, by a simple computation, the left side of (29) is

\[
\left( \frac{2}{7} \right)^n f (0, \ldots, 0).
\]

(31)

It follows from relations (29), (30), and (31) that \( f (0, \ldots, 0) = 0 \). One can continue this method to show that \( f \) has zero condition.

\[\square\]
Definition 9. A mapping \( f : V^n \rightarrow W \) is

(iii) Odd in the \( j \)th variable if

\[
f(z_1, \ldots, z_{j-1}, -z_j, z_{j+1}, \ldots, z_n) = -f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).
\]

(iv) Even in the \( j \)th variable if

\[
f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n) = f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).
\]

Proposition 10. Suppose that a mapping \( f : V^n \rightarrow W \) satisfies equation (26). Then, it is multimixed additive-quartic. Moreover,

(i) If \( f \) is odd in a variable, then it is additive in the same variable

(ii) If \( f \) is even in a variable, then it is quartic in the same variable

Proof. Let \( j \in \{1, \ldots, n\} \) be arbitrary and fixed. Set

\[
f_j^*(z) := f(z_1, \ldots, z_{j-1}, z_j, z_{j+1}, \ldots, z_n).
\]

Putting \( x_{2k} = 0 \) for all \( k \in \{1, \ldots, n\} \setminus \{j\} \) in (26) and using Lemma 8, we get

\[
2^{n-1} \left[ f_j^*(z + 2w) + f_j^*(z - 2w) - \frac{12}{7} f_j^*(2w) \right]
= \sum_{p=0}^{n-1} \left[ \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) \right] 4^{n-1-p} (-6)^p 2^{n-p-1} \left( f_j^*(z + w) + f_j^*(z - w) \right)
+ \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) 4^{n-1-p} (-6)^p 2^{n-p-1} f_j^*(z)
+ \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) 4^{n-1-p} (-6)^p \left( \frac{24}{7} \right) 2^{n-p-1} f_j^*(w)
= 4 \left[ 2^{n-1} \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) 4^{n-1-p} (-3)^p \right] f_j^*(z + w) + f_j^*(z - w)
- 6 \left[ 2^{n-1} \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) 4^{n-1-p} (-3)^p \right] f_j^*(z) - \frac{24}{7}
+ 2 \left[ 2^{n-1} \sum_{p=0}^{n-1} \left( \frac{n-1}{p} \right) 4^{n-1-p} (-3)^p \right] f_j^*(w)
= 4 \times 2^{n-1} \left[ f_j^*(z + w) + f_j^*(z - w) \right] - 6 \times 2^{n-1} f_j^*(z)
- \frac{24}{7} \times 2^{n-1} f_j^*(w).
\]

The above equalities show that

\[
f_j^*(z + 2w) + f_j^*(z - 2w) - \frac{12}{7} f_j^*(2w)
= 4 \left[ f_j^*(z + w) + f_j^*(z - w) \right] - 6 f_j^*(z) - \frac{24}{7} f_j^*(w).
\]

In other words, (6) is true for \( f_j^* \). Since \( j \) is arbitrary, \( f \) is a multimixed additive-quartic mapping.

(i) Repeating the proof of Lemma 2.1 (i) from [29] for \( f_j^* \), we see that \( f_j^*(z + w) = f_j^*(z) + f_j^*(w) \). This means that \( f \) is additive in the \( j \)th variable

(ii) Similar to the previous part, it follows from the proof of part (ii) of Lemma 2.1 in [29] that

\[
f_j^*(2z + w) + f_j^*(2z - w) = 4 \left[ f_j^*(z + w) + f_j^*(z - w) \right] + 24 f_j^*(z) - 6 f_j^*(w).
\]

Therefore, \( f \) is quartic in the \( j \)th variable.

\( \square \)

Corollary 11. Suppose a mapping \( f : V^n \rightarrow W \) satisfies equation (26).

(i) If \( f \) is odd in each variable, then it is multiadditive. Moreover, it satisfies (2)

(ii) If \( f \) is even in each variable, then it is multiquartic. In particular, it fulfills (11)

(iii) If \( f \) is odd in each of some \( k \) variables and is even in each of the other variables, then it is multi-additive-quartic. In addition, (10) is valid for \( f \)

4. Various Stability Results

In this section, we prove some Hyers-Ulam stability results by a fixed point method in the setting of Banach spaces. In what follows, we denote the set of all mappings from \( E \) to \( F \) by \( F^E \). We remember the following theorem which is an essential result in fixed point theory ([23], Theorem 1). This achievement is a key tool in obtaining our aim in this section.

Theorem 12. Let the hypotheses

(A1) \( Y \) is a Banach space, \( E \) is a nonempty set, \( j \in \mathbb{N} \), \( g_1, \ldots, g_j : E \rightarrow E \), and \( L_1, \ldots, L_j : E \rightarrow \mathbb{R} \).

(A2) \( T : Y^E \rightarrow Y^E \) is an operator satisfying the inequality

\[
\| T \lambda(x) - T \mu(x) \| \leq \sum_{i=1}^j L_i(x) \| \lambda(g_i(x)) - \mu(g_i(x)) \|, \lambda, \mu \in Y^E, x \in E,
\]

(38)
(A3) $\Lambda : \mathbb{R}^E_{>0} \rightarrow \mathbb{R}^E_{>0}$ is an operator defined through

$$\Lambda \delta(x) = \sum_{i=1}^{j} L_i(x) \delta(g_i(x)) \delta \in \mathbb{R}^E_{>0}, x \in E, \quad (39)$$

hold, and a function $\theta : E \rightarrow \mathbb{R}_+$ and a mapping $\phi : E \rightarrow Y$ fulfill the following two conditions:

$$||\mathcal{T} \phi(x) - \phi(x)|| \leq \theta(x), \theta^*(x) = \sum_{i=0}^{\infty} \Lambda^i \theta(x) < \infty (x \in E). \quad (40)$$

Then, there exists a unique fixed point $\psi$ of $\mathcal{T}$ such that

$$||\phi(x) - \psi(x)|| \leq \theta^*(x)(x \in E). \quad (41)$$

Moreover, $\psi(x) = \lim_{l \rightarrow \infty} \mathcal{T}^l \phi(x)$ for all $x \in E$.

For the rest of this paper and for each mapping $f : V^n \rightarrow W$, we consider the difference operator $\Gamma_{A\phi} : V^n \times V^n \rightarrow W$ defined via

$$\Gamma_{A\phi}(x_1, x_2) = \sum_{q=0}^{n} \left(-\frac{12}{7}\right)^q f\left(\Lambda_q^n\right) - \sum_{p_i \neq 0} \sum_{p_i} 4^{n-p_i} \phi\left(x_{(\overline{p_i})}\right) \cdot \left(-\frac{24}{7}\right)^p f\left(\Lambda_p^n(x_{(\overline{p_i})})\right), \quad (42)$$

where $f(\Lambda_q^n)$ and $f(\Lambda_p^n(x_{(\overline{p_i})}))$ are defined in (24) and (8), respectively. In the sequel, all mappings $f : V^n \rightarrow W$ are assumed to satisfy zero condition. With this assumption, we have the next stability result for functional equation (26) in the odd case.

**Theorem 13.** Let $\beta \in \{-1, 1\}$ be fixed, $V$ be a linear space, and $W$ be a Banach space. Suppose that $\phi : V^n \times V^n \rightarrow \mathbb{R}_+$ is a mapping satisfying

$$\lim_{l \rightarrow \infty} \left(\frac{1}{2^{n}}}\right)^l \phi\left(\beta^{n}\lambda_{j}, \beta^{n}\lambda_{k}\right) = 0, \quad (43)$$

for all $x_j, x_k \in V^n$ and

$$\Phi(x) = \left(\frac{7}{12}\right)^n \frac{1}{2^{n/(\beta+1)/2}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{n}}}\right)^l \phi\left(0, 2^{\beta+((\beta-1)/2)}x\right) < \infty, \quad (44)$$

for all $x \in V^n$. Assume also $f : V^n \rightarrow W$ is a mapping fulfilling the inequality

$$\|\Gamma_{A\phi}(x_1, x_2)\| \leq \Phi(x_1, x_2), \quad (45)$$

for all $x_1, x_2 \in V^n$. If $f$ is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \rightarrow W$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \Phi(x), \quad (46)$$

for all $x \in V^n$.

**Proof.** Replacing $(x_1, x_2)$ by $(0, x_1)$ in (45) and using the assumptions, we have

$$\left\|\left(-\frac{12}{7}\right)^{n}\left(\frac{-24}{7}\right)^{n}f(x)\right\| \leq \phi(0, x), \quad (47)$$

for all $x = x_1 \in V^n$ (here and the rest of the proof) and so

$$\left\|f(x) - \frac{1}{2\theta}(f(2x))\right\| \leq \left(\frac{7}{24}\right)^{n}\phi(0, x). \quad (48)$$

Set

$$\theta(x) = \left(\frac{7}{12}\right)^n \frac{1}{2^{n/(\beta+1)/2}} \phi\left(0, 2^{\beta+((\beta-1)/2)}x\right), \text{ and } \mathcal{T} \theta(x) = \frac{1}{2\theta} \mathcal{T} \phi\left(2^{\beta}x\right)(\theta \in W^{V^n}). \quad (49)$$

Then, relation (48) can be modified as

$$\|f(x) - \mathcal{T} f(x)\| \leq \theta(x)(x \in V^n). \quad (50)$$

Define $\Lambda \eta(x) = (1/2^{n/2}) \eta(2^{\beta}x)$ for all $\eta \in \mathbb{R}^V_{>0}$. It is seen that $\Lambda$ has the form (A3) of Theorem 12 for which $E = V^n$, $g_1(x) = 2^{\beta}x$, and $L_1(x) = 1/2^{n\beta}$. Furthermore, for each $\lambda, \mu \in W^{V^n}$, we get

$$\|\mathcal{T} \lambda(x) - \mathcal{A} \mu(x)\| = \left\|\frac{1}{2\theta} \left[\lambda\left(2^{\beta}x\right) - \mu\left(2^{\beta}x\right)\right]\right\| \leq L_1(x)||\lambda(g_1(x)) - \mu(g_1(x))||. \quad (51)$$

The above relation portrays that the hypothesis (A2) holds. By induction on $l$, one can check that for any $l \in \mathbb{N}_0$, we have

$$\Lambda^i \theta(x) = \left(\frac{7}{12}\right)^n \frac{1}{2^{n/(\beta+1)/2}} \phi\left(0, 2^{\beta+((\beta-1)/2)}x\right). \quad (52)$$

Now, relations (44) and (52) necessitate that all assumptions of Theorem 12 are satisfied. Hence, there exists a unique mapping $\mathcal{A} : V^n \rightarrow W$ such that

$$\mathcal{A}(x) = \lim_{l \rightarrow \infty} \left(\mathcal{T}^l f\right)(x) = \frac{1}{2\theta}(2^{\beta}x)(x \in V^n), \quad (53)$$
and (46) holds. In continuation, we prove that

\[
\|\Gamma_{AQ}(\mathcal{T}^1 f)(x_1, x_2)\| \leq \left(\frac{1}{2^{nB}}\right)^I \phi\left(2^B x_1, 2^B x_2\right),
\]  

(54)

for all \(x_1, x_2 \in V^n\) and \(I \in \mathbb{N}_0\). We argue by induction on \(I\). Clearly, inequality (54) is valid for \(I = 0\) by (45). Assume that (54) is true for an \(I \in \mathbb{N}_0\). Then

\[
\|\Gamma_{AQ}(\mathcal{T}^{I+1} f)(x_1, x_2)\|
\]

\[
= \left|\sum_{q=0}^{n} \left(-\frac{12}{7}\right)^q \left(\mathcal{T}^{I+1} f\right)(\mathcal{M}^q) - \sum_{p_1=0}^{n-p_1} \sum_{p_2=0}^{n-p_2} 4^{p_1-p_2} \left(\mathcal{N}(p_1, p_2)\right)\right| \cdot \left(\frac{24}{7}\right)^{p_2} \left(\mathcal{T}^{I+1} f\right)(\mathcal{M}(p_1, p_2))\| \cdot \left(\frac{24}{7}\right)^{p_2} \left(\mathcal{T}^{I+1} f\right)(\mathcal{M}(p_1, p_2))\|
\]

\[
\leq \left(\frac{1}{2^{nB}}\right)^{I+1} \phi\left(2^{B(I+1)} x_1, 2^{B(I+1)} x_2\right),
\]

(55)

for all \(x_1, x_2 \in V^n\). Letting \(I \to \infty\) in (54) and applying (43), we arrive at \(\Gamma_{AQ}(\mathcal{T}^{I+1} f)(x_1, x_2) = 0\) for all \(x_1, x_2 \in V^n\). This means that the mapping \(\mathcal{T}\) satisfies (26), and so it is multiadditive by Corollary 11. This finishes the proof. \(\square\)

Here, in analogy with Theorem 13, we bring the next stability result for functional equation (26) in the even case.

**Theorem 14.** Let \(\beta \in (-1, 1)\) be fixed, \(V\) be a linear space, and \(W\) be a Banach space. Suppose that \(\phi : V^n \times V^n \to \mathbb{R}_+\) is a mapping satisfying

\[
\lim_{l \to \infty} \left(\frac{1}{2^{lB}}\right)^l \phi\left(2^l x_1, 2^l x_2\right) = 0,
\]

(56)

for all \(x_1, x_2 \in V^n\) and

\[
\Psi(x) = \left(\frac{7}{2}\right)^n \frac{1}{2^{4n(B+1/2)}} \sum_{l=0}^{\infty} \left(\frac{1}{2^{2lB}}\right)^l \phi\left(2^{B+1/2} x, 0\right) < \infty,
\]

(57)

for all \(x \in V^n\). Assume also \(f : V^n \to W\) is a mapping fulfilling the inequality

\[
\|\Gamma_{AQ} f(x_1, x_2)\| \leq \phi(x_1, x_2),
\]

(58)

for all \(x_1, x_2 \in V^n\). If \(f\) is even in each variable, then there exists a unique solution \(\hat{g} : V^n \to W\) of (26) such that

\[
\|f(x) - \hat{g}(x)\| \leq \Psi(x),
\]

(59)

for all \(x \in V^n\). In particular, if \(\hat{g}\) is even mapping in each variable, then it is multiquartic.

**Proof.** Replacing \((x_1, x_2)\) by \((0, x_1)\) in (58) and applying the hypotheses, we obtain

\[
\|\sum_{q=0}^{n} \left(-\frac{12}{7}\right)^q 2^{n-q} f(2x) - \sum_{p_2=0}^{n} \left(\frac{n}{p_2}\right) \left(-\frac{24^n}{7}\right)^{p_2} \cdot 2^{n-p_2} f(x)\|
\]

\[
\leq \phi(0, x),
\]

(60)

where \(x = x_1 \in V^n\) (here and the rest of the proof). On the other hand

\[
\sum_{p_2=0}^{n} \left(\frac{n}{p_2}\right) \left(-\frac{24^n}{7}\right)^{p_2} \cdot 2^{n-p_2} \leq \left(\frac{24^n}{7} + 8\right)^n = \left(\frac{32^n}{7}\right).
\]

(61)

By the relations above (60) will be

\[
\left\|\left(\frac{2}{7}\right)^n f(2x) - \left(\frac{32^n}{7}\right) \cdot f(x)\right\| \leq \phi(0, x),
\]

(62)

and so

\[
\left\|f(x) - \frac{1}{2^{nB}} f(2x)\right\| \leq \left(\frac{7}{32}\right)^n \phi(0, x).
\]

(63)

One can rewrite (63) as

\[
\|f(x) - \mathcal{T} f(x)\| \leq \theta(x)(x \in V^n),
\]

(64)

where

\[
\theta(x) = \left(\frac{7}{2}\right)^n \frac{1}{2^{4n(B+1/2)}} \phi\left(0, 2^{(B-1/2)} x, 0\right),
\]

(65)

\[
\mathcal{T} \theta(x) = \left(\frac{7}{2}\right)^n \frac{1}{2^{4n(B+1/2)}} \phi\left(0, 2^{(B-1/2)} x, 0\right).
\]

Similar to the proof of Theorem 13, consider \(\Delta\eta(x) = (1/2^{4nB}) \eta(2^B x)\) for all \(\eta \in \mathbb{R}_+\), and hence, \(A\) satisfies (A3) of Theorem 12 with \(E = V^n\), \(G_1(x) = 2^B x\), and \(L_1(x) = 1/2^{4nB}\). Moreover, for each \(\lambda, \mu \in W^{3n}\), we obtain

\[
\||\mathcal{T} \lambda(x) - \mathcal{T} \mu(x)\| = \left\|\frac{1}{2^{4nB}} \left(\lambda(2^B x) - \mu(2^B x)\right)\right\|
\]

\[
\leq L_1(x)||\lambda(g_1(x)) - \mu(g_1(x))||.
\]

(66)
The last relation implies that the hypothesis (A2) is true. It is easily checked by induction on $l$ that for any $l \in \mathbb{N}_0$ and $x \in V^n$

$$\Lambda^l \theta(x) = \left( \frac{1}{2^k} \right)^l \theta \left( 2^l \beta x \right) = \left( \frac{7}{32} \right)^n \left( \frac{1}{2^{4n(\beta+1)/2}} \right)^l \psi \left( 0, 2^{2^l+\beta} \right).$$  

(67)

It now follows between (57) and (67) that all assumptions of Theorem 12 hold, and thus, there exists a unique mapping $\varnothing : V^n \to W$ such that

$$\varnothing(x) = \lim_{l \to \infty} \left( \mathcal{F}^l f \right)(x) = \frac{1}{2^k} \theta \left( 2^l \beta x \right) (x \in V^n),$$  

and (46) is valid. Similar to the proof of Theorem 13, one can show that

$$\left\| \Gamma_{AQ} \left( \mathcal{F}^l f \right) (x_1, x_2) \right\| \leq \left( \frac{1}{2^k} \right)^l \psi \left( 2^l \beta x_1, 2^l \beta x_2 \right),$$  

(69)

for all $x_1, x_2 \in V^n$ and $l \in \mathbb{N}_0$. Letting $l \to \infty$ in (69) and applying (56), we arrive at $\Gamma_{AQ} \varnothing(x_1, x_2) = 0$ for all $x_1, x_2 \in V^n$, and therefore, the mapping $\varnothing$ satisfies (26). The last part follows from part (ii) of Corollary 11.

Here and subsequently, it is assumed that $V$ is a normed space and $W$ is a Banach space unless otherwise stated explicitly. In the following corollary, we show that the multiadditive and multiquartic mappings are stable. Since the proof is routine, we include it without proof.

**Corollary 15.** Given $\alpha \in \mathbb{R}$. Suppose that $f : V^n \to W$ is a mapping satisfying the inequality

$$\left\| \Gamma_{AQ} f (x_1, x_2) \right\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|x_{ij}\|^\alpha,$$  

(70)

for all $x_1, x_2 \in V^n$.

(i) If $\alpha \neq n$ and $f$ is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \to W$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \left( \frac{7}{12} \right)^n \frac{1}{2^n - 2^n} \sum_{j=1}^{n} \|x_{ij}\|^\alpha,$$  

(71)

for all $x = x_1 \in V^n$. Moreover, if $\varnothing$ is even mapping in each variable, then it is multiquartic.

The upcoming corollaries are direct consequences of Theorems 13 and 14 when the functional equation (26) is controlled by a small positive number $\delta$.

**Corollary 16.** Let $\delta > 0$ and $f : V^n \to W$ be a mapping satisfying the inequality

$$\left\| \Gamma_{AQ} f (x_1, x_2) \right\| \leq \delta,$$  

(73)

for all $x_1, x_2 \in V^n$.

(i) If $f$ is odd in each variable, then there exists a unique multiadditive mapping $\mathcal{A} : V^n \to W$ such that

$$\|f(x) - \mathcal{A}(x)\| \leq \left( \frac{7}{12} \right)^n \delta,$$  

(74)

for all $x \in V^n$.

(ii) If $f$ is even in each variable, then there exists a unique solution $\varnothing : V^n \to W$ of (26) such that

$$\|f(x) - \varnothing(x)\| \leq \left( \frac{7}{2} \right)^n \delta \frac{1}{2^n - 1},$$  

(75)

for all $x \in V^n$. Furthermore, if $\varnothing$ is even mapping in each variable, then it is multiquartic.

**Proof.** Letting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ and using Theorem 13 and Theorem 14 in the case $\beta = 1$, one can obtain the desired result.

Given the mapping $f : V^n \to W$, we define the operator $\Gamma f : V^n \times V^n \to W$ through

$$\Gamma f (x_1, x_2) := \sum_{i \in \{1, 2\}^{n-k}} f \left( x_{i1} + x_{i2}, 2x_{i1} - t x_{i2} \right)$$

$$- \sum_{i=1}^{n-k} \sum_{p_1=0}^{n-k-p_1} \sum_{p_2=0}^{p_1} \sum_{i=1}^{24} 4^{n-k-p_1-p_2} \cdot (-6)^{p_2} \cdot f \left( x_{i1}, x_{i2} - t x_{i2} \right),$$  

(76)

for all $x = (x_{11}, \ldots, x_{ik}) \in V^k$ and $x^{n-k} = (x_{ik+1}, \ldots, x_{i2}) \in V^{n-k}$ where $i \in \{1, 2\}$ and $f(A^{n-k})$ are defined in (8).

In the next result, we show that the functional equation (10) can be stable.
Theorem 17. Let $\beta \in \{-1, 1\}$ be fixed, $V$ be a linear space, and $W$ be a Banach space. Suppose that $\phi : V^n \times V^n \to \mathbb{R}$ is a mapping satisfying the inequality
\[
\sum_{l=0}^{\infty} \left( \frac{1}{2^{(m-3)l}\beta} \right)^l \phi \left( 2^{\beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor} x_1, 2^{\beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor} x_2 \right) < \infty, \tag{77}
\]
for all $x_1, x_2 \in V^n$. Assume also $f : V^n \to W$ is a mapping satisfying the inequality
\[
\|f(x_1, x_2)\| \leq \phi(x_1, x_2), \tag{78}
\]
for all $x_1, x_2 \in V^n$. Then, there exists a unique solution $F : V^n \to W$ of (10) such that
\[
\|f(x) - F(x)\| \leq \Phi(x), \tag{79}
\]
for all $x = (x^k, x^{n-k}) \in V^n$, where
\[
\Phi(x) = \frac{1}{2^{(m-3)l}\beta} \phi \left( 2^{\beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor} x, 2^{\beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor} x, 0 \right). \tag{80}
\]
Proof. Putting $x_1^k = x^k$ and $x_1^{n-k} = x^{n-k}, x_2^k = 0$ in (79), we have
\[
\left\|2^{n-k} f(2x) - \sum_{p=0}^{n-k} \binom{n-k}{p} 2^n 2^{n-k-p} f(x)\right\| \leq \Phi(x, \left( x^k, 0 \right)), \tag{81}
\]
in which $x = (x^k, x^{n-k})$. A computation shows that (81) can be rewritten as follows:
\[
\left\|2^{n-k} f(2x) - 2^{5n-4k} f(x)\right\| \leq \Phi(x, \left( x^k, 0 \right)), \tag{82}
\]
and so
\[
\left\|f(2x) - 2^{4n-3k} f(x)\right\| \leq \frac{1}{2^n} \Phi(x, \left( x^k, 0 \right)). \tag{83}
\]
Set
\[
\xi(x) = \frac{1}{2^{(m-3)l}\beta} \phi \left( \frac{x}{2^{\left\lfloor \left\lfloor \beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor \right\rfloor}^{\frac{\beta l}{2}}}, \left( \frac{x^k}{\beta l - \left\lfloor \frac{\beta l}{2} \right\rfloor} \right), 0 \right), \tag{84}
\]
and $\mathcal{F} \xi(x) = (1/2^{(4n-3)l}\beta) \xi(2^\beta x)$ where $\xi \in W^v$. Then, relation (83) can be modified as
\[
\|f(x) - \mathcal{F} f(x)\| \leq \xi(x) (x \in V^n). \tag{85}
\]
Define $\Lambda \eta(x) = (1/2^{(4n-3)l}\beta) \eta(2^\beta x)$ for all $\eta \in \mathbb{R}^v, x = (x^k, x^{n-k}) \in V^n$. The rest of the proof is similar to the proof of Theorem 13. \hfill \square

Corollary 18. Let $\delta > 0$. If $f : V^n \to W$ is a mapping satisfying the inequality
\[
\|f(x_1, x_2)\| \leq \delta, \tag{86}
\]
for all $x_1, x_2 \in V^n$, then there exists a unique solution $F : V^n \to W$ of (10) such that
\[
\|f(x) - F(x)\| \leq \delta \frac{\left(2^n\left(2^{4n-3k} - 1\right)\right)}{2^{4n-k}}, \tag{87}
\]
for all $x \in V^n$.

Proof. Setting the constant function $\phi(x_1, x_2) = \delta$ for all $x_1, x_2 \in V^n$ and applying Theorem 17 in the case $\beta = 1$, the result can be found. \hfill \square

5. Conclusion

In the present paper, we introduced the multi-additive-quartic and multimixed additive-quartic mappings. Indeed, we characterized the mentioned mappings and then unified the system of functional equations defining a multi-additive-quartic or a multimixed additive-quartic mapping to a single equation. We also showed that under which conditions a multimixed additive-quartic mapping is multiadditive, multi-quartic, and multi-additive-quartic. Finally, we applied a fixed point theorem to establish the Hyers-Ulam stability of multi-additive-quartic mappings and multimixed additive-quartic functional equations.

Data Availability

All results are obtained without any software and found by manual computations.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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