Faraday Induction and the Current Carriers in a Circuit

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Abstract

In this article, it is pointed out that Faraday induction can be treated from an untraditional, particle-based point of view. The electromagnetic fields of Faraday induction can be calculated explicitly from approximate point-charge fields derived from the Liénard-Wiechert expressions or from the Darwin Lagrangian. Thus the electric fields of electrostatics, the magnetic fields of magnetostatics, and the electric fields of Faraday induction can all be regarded as arising from charged particles. Some aspects of electromagnetic induction are explored for a hypothetical circuit consisting of point charges which move frictionlessly in a circular orbit. For a small number of particles in the circuit (or for non-interacting particles), the induced electromagnetic fields depend upon the mass and charge of the current carriers while energy is transferred to the kinetic energy of the particles. However, for an interacting multiparticle circuit, the mutual electromagnetic interactions between the particles dominate the behavior so that the induced electric field cancels the inducing force per unit charge, the mass and charge of the individual current carriers become irrelevant, and energy goes into magnetic energy.
I. INTRODUCTION

When students are asked what causes the electric field in a parallel-plate capacitor, the response involves charges on the capacitor plates. Also, students say that the magnetic field in a solenoid is due to the currents in the solenoid winding. But when asked for the cause of the Faraday induction fields in a solenoid with changing currents, the student response is that the induction field is due to a changing magnetic field, not that it is due to the acceleration fields of the charges in the solenoid winding. The student view reflects what is emphasized in the standard electromagnetism textbooks. Indeed, although C. G. Darwin computed induction fields from accelerating charges in the 1930s, today it is rare to have a physicist report that the Faraday induction field arises from the acceleration of charges.

In this article, we wish to broaden the perspective on Faraday induction by reviewing some aspects of the particle point of view. We treat the induction fields as arising from the electromagnetic fields of point charges as derived from the Liénard-Wiechert expressions or from the Darwin Lagrangian. First we mention the Liénard-Wiechert form taken by the electric and magnetic fields of a point charge in general motion. Then we turn to the low-velocity-small-distance approximation derived in the 1940 textbook by Page and Adams. This approximate form for the electromagnetic fields is the same as that obtained from the Darwin Lagrangian of 1920. Here in the present article, the approximated fields are used to treat Faraday induction in detail for a hypothetical circuit consisting of point charged particles moving frictionlessly on a circular ring. This circuit provides a rough approximation to that of a thin wire of finite thickness which is bent into a circular loop. For small numbers of particles (or for noninteracting particles), we see that the magnitudes of the masses and charges of the charge carriers are important, that the induced electric fields can be small, and that energy goes into the mechanical kinetic energy of the charge carriers. However, for large numbers of interacting charges, the mutual interactions make the magnitudes of the masses and charges unimportant, the induced electric fields balance the inducing fields, and the energy goes into magnetic energy of the circuit.
II. POINT-CHARGE FIELDS

A. Point-Charge Fields for General Motion

Although students are familiar with the Coulomb electric field of a point charge, many do not study electromagnetism to the point that they see the full retarded point-charge fields from the Liénard-Wiechert potentials:

\[
E(r, t) = e \left[ \left(1 - \frac{v_e^2}{c^2}\right)(n - v_e/c) \right] \frac{n \cdot (r - r_e)}{|r - r_e|^2} + e \left[ \frac{n \times \{n - v_e/c \times a_e\}}{(1 - n \cdot v_e/c)^3|r - r_e|} \right] t_{\text{ret}}
\]

and

\[
B(r, t) = en_{\text{ret}} \times E(r, t)
\]

where the unit vector \( n = (r - r_e)/|r - r_e| \), and where the position \( r_e(t) \), velocity \( v_e(t) \), and acceleration \( a_e(t) \) of the charge \( e \) must be evaluated at the retarded time \( t_{\text{ret}} \) such that \( c|t - t_{\text{ret}}| = |r - r_e(t_{\text{ret}})| \). These field expressions, together with Newton’s second law for the Lorentz force, give the causal interactions between point charges. The only thing missing from this formulation of classical electrodynamics is the possible existence of a homogeneous solution of Maxwell’s equations (such as, for example, a plane wave) which might interact with the charged particles. However, electromagnetic induction fields are distinct from homogeneous radiation fields, and therefore we expect that the fields of Faraday induction can be treated as having their origin from charged particle motions. The exploration of this point-charge point of view in connection with Faraday induction is the subject of the present article.

B. Low-Velocity-Small-Distance Approximation without Retardation

Since the electric field of Faraday induction is distinct from the Coulomb field of a stationary charge, we expect this induction field to involve the additional velocity- and acceleration-dependent terms of Eqs. (1). The use of the full expressions involving retardation in Eqs. (1) and (2) can be a formidable task. For the radiation fields which fall off as \( 1/r \) at large distances, the presence of retardation cannot be avoided as the signal travels from the source charge to the distant field point. However, for field points which are near point charges which are moving at low velocities, it is possible to derive from Eqs. (1) and (2) approximate...
expressions for the electric and magnetic fields which involve no retardation. The task of approximation is not trivial and is carried out in the textbook *Electrodynamics* by Page and Adams giving

\[ E_j(r, t) = e_j \frac{(r - r_j)}{|r - r_j|^3} \left[ 1 + \frac{v_j^2}{2c^2} - \frac{3}{2} \left( \frac{v_j \cdot (r - r_j)}{c|r - r_j|} \right)^2 \right] \]

\[ - \frac{e_j}{2c^2} \left( \frac{a_j}{|r - r_j|} + \frac{a_j \cdot (r - r_j)(r - r_j)}{|r - r_j|^3} \right) + O(1/c^3) \]  

(3)

and

\[ B_j(r, t) = e_j \frac{v_j}{c} \times \left( \frac{r - r_j}{|r - r_j|^3} \right) + O(1/c^3) \]

(4)

where in Eq. (3) the quantity \( a_j \) refers to the acceleration of the \( j \)th particle. These approximate expressions are power series in \( 1/c \) where \( c \) is the speed of light in vacuum.

The approximate fields in Eqs. (3) and (4) can be quite useful; they were used by Page and Adams to discuss “Action and Reaction Between Moving Charges,” and by the current author when interested in Lorentz-transformation properties of energy and momentum and questions of mass-energy equivalence. Here the approximate expressions for the fields are exactly what is needed to understand Faraday induction from a particle point of view.

The approximate fields in Eqs. (3) and (4) also correspond to those which arise from the Darwin Lagrangian

\[ \mathcal{L} = \sum_{i=1}^{i=N} m_i c^2 \left( -1 + \frac{v_i^2}{2c^2} + \frac{(v_i^2)^2}{8c^4} \right) - \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{|r_i - r_j|} \]

\[ + \frac{1}{2} \sum_{i=1}^{i=N} \sum_{j \neq i} \frac{e_i e_j}{2c^2} \left[ \frac{v_i \cdot v_j}{|r_i - r_j|} + \frac{v_i \cdot (r_i - r_j)v_j \cdot (r_i - r_j)}{|r_i - r_j|^3} \right] \]

\[ - \sum_{i=1}^{i=N} e_i \Phi_{ext}(r_i, t) + \sum_{i=1}^{i=N} e_i \frac{v_i}{c} \cdot A_{ext}(r_i, t) \]

(5)

where the last line includes the scalar potential \( \Phi_{ext} \) and vector potential \( A_{ext} \) associated with the external electromagnetic fields. The Darwin Lagrangian omits radiation but expresses accurately the interaction of charged particles through order \( 1/c^2 \). The Darwin Lagrangian continues to appear in advanced textbooks, but the approximate expressions (3) and (4) seem to have disappeared from the consciousness of most contemporary physicists. The Lagrangian equations of motion from the Darwin Lagrangian can be rewritten in the form of Newton’s second law \( \frac{dp}{dt} = d(m\gamma v)/dt = F \) with \( \gamma = (1 - v^2/c^2)^{-1/2} \). In this Newtonian
form, we have
\[
\frac{d}{dt} \left[ \frac{m_i \mathbf{v}_i}{(1 - v_i^2/c^2)^{1/2}} \right] \approx \frac{d}{dt} \left[ m_i \left( 1 + \frac{v_i^2}{2c^2} \right) \mathbf{v}_i \right] = e_i \mathbf{E} + e_i \frac{\mathbf{v}_i}{c} \times \mathbf{B}
\]
\[
= e_i \left( \mathbf{E}_{ext}(\mathbf{r}_i, t) + \sum_{j \neq i} \mathbf{E}_j(\mathbf{r}_i, t) \right) + e_i \frac{\mathbf{v}_i}{c} \times \left( \mathbf{B}_{ext}(\mathbf{r}_i, t) + \sum_{j \neq i} \mathbf{B}_j(\mathbf{r}_i, t) \right)
\]
(6)

with the Lorentz force on the \(i\)th particle arising from the external electromagnetic fields and from the electromagnetic fields of the other particles. The electromagnetic fields due to the \(j\)th particle are given through order \(v^2/c^2\) by exactly the approximate expressions appearing in Eqs. (3) and (4).

### III. ELECTROMAGNETIC INDUCTION

Electromagnetic induction was discovered by Michael Faraday, not as a motion-dependent modification of Coulomb’s law, but rather in terms of emfs producing currents in circuits. This circuit-based orientation remains the way that electromagnetic induction is discussed in textbooks today. Now the emf in a circuit is the closed line integral around the circuit of the force per unit charge \(\mathbf{f}\) acting on the charges of the circuit, \(\text{emf} = \oint \mathbf{f} \cdot d\mathbf{r}\). Faraday’s emfs were associated with changing magnetic fluxes, and Faraday’s law of electromagnetic induction in a circuit is given by
\[
\text{emf}_F = -\frac{1}{c} \frac{d\Phi}{dt}
\]
(7)
where \(\Phi\) is the magnetic flux through the circuit.

As correctly emphasized in some textbooks, electromagnetic induction in a circuit can arise in two distinct aspects. The “motional emf” in a circuit which is moving through an unchanging magnetic field can be regarded as arising from the magnetic Lorentz force acting on the mobile charges of the moving circuit. On the other hand, when the circuit is stationary in space but the current in the circuit is changing, new electric fields arise in space. These new electric fields can cause an emf in an adjacent circuit (mutual inductance) or in the original circuit itself (self-inductance); the new electric fields are precisely those appearing due to the motions of the charges of the circuit as given in Eq. (4), or, through order \(v^2/c^2\), as given in Eq. (3). It is these electric fields which are the subject of our discussion of Faraday induction.
It should be noted that for steady-state currents in a multipartic le circuit with large numbers of charges where the charge density and current density are time-independent, all the complicating motion-dependent terms in Eqs. (1)-(2) or (3)-(4) beyond the first leading term in $1/c$ actually cancel, so that the electromagnetic fields can be calculated simply using Coulomb’s law and the Biot-Savart Law. However, for time-varying charge densities and/or current densities, the motion-dependent terms in Eqs. (1)-(2) or (3)-(4) do not cancel and indeed provide the Faraday induction fields.

If an external emf $emf_{ext}$ is present in a continuous circuit with a self-inductance $L$ and resistance $R$, the current $i$ in the circuit is given by the differential equation

$$emf_{ext} = L\frac{di}{dt} + iR$$

where the term $Ldi/dt$ corresponds to the negative of the Faraday-induced emf associated with the changing current in the circuit. Here in traditional electromagnetic theory, the self-inductance $L$ of a rigid circuit is a time-independent quantity which depends only upon the geometry of the circuit. The energy balance for the circuit is found by multiplying Eq. (8) by the current $i$

$$emf_{ext} \times i = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) + i^2R$$

corresponding to a power $emf_{ext} \times i$ delivered by the external emf going into the time-rate-of-change of magnetic energy $(1/2)Li^2$ stored in the inductor and the power $i^2R$ lost in the resistor.

Although the energy analysis for Eq. (9) seems natural, the differential equation (8) presents some unusual aspects if we consider the circuit from the particle point of view. If at time $t = 0$ the constant external emf $emf_F$ is applied to the circuit and the current is zero, $i(0) = 0$, then the Faraday induced emf, $emf_F = -Ldi/dt$, must exactly cancel the external emf $emf_{ext}$ so that $emf_{ext} - Ldi/dt = 0$ at time $t = 0$. Phrased in terms of forces per unit charge applied to the circuit, the Faraday induced electric field $E_F$ must exactly cancel the external force per unit charge $f_{ext}$ associated with the external emf. Indeed, if the resistance $R$ of the circuit becomes vanishingly small, $R \rightarrow 0$, then this canceling balance of the Faraday electric field $E_F$ against the force per unit charge $f_{ext}$ associated with the external emf holds at all times, and yet the current increases at a constant rate following $di/dt = emf_{ext}/L$. But if the net force per unit charge goes to zero, why do the charges accelerate so as to produce a changing current $di/dt$? After all, in classical mechanics,
\( \mathbf{F}_R = m \mathbf{a} \); it is the resultant force \( \mathbf{F}_R \) on a particle which determines the acceleration \( \mathbf{a} \) of the particle of mass \( m \). Thus we expect that if the resultant force on a particle of mass \( m \) is zero, then the mass does not accelerate. However, electromagnetism involves some aspects which are different from what is familiar in nonrelativistic mechanics, and electromagnetic circuit theory involves some unmentioned approximations. In this article, we wish to explore these differences and unmentioned approximations by using a particle model in connection with Faraday induction. We will note the approximations involved in Eq. (8) which lead to the troubling apparent contradiction with Newton’s second law.

IV. DETAILED DISCUSSION OF FARADAY INDUCTION IN A SIMPLE HYPOTHETICAL CIRCUIT

A. Model for a Detailed Discussion

Here we would like to explore Faraday induction in some detail for the simplest possible circuit in hopes of obtaining some physical insight. Accordingly, we will discuss a hypothetical circuit consisting of \( N \) equally-spaced particles of mass \( m \) and charge \( e \) which are constrained by centripetal forces of constraint to move in a circular orbit of radius \( R \) in the \( xy \)-plane centered on the origin. A balancing negative charge to give the circuit neutrality can be thought of as a uniform line charge in the orbit or else as a single compensating charge at the center of the orbit. The choice does not influence the analysis to follow. The system may be thought of as consisting of charged beads sliding on a frictionless ring. There is no frictional force and hence no resistance \( \mathcal{R} \) in the model. The model is intended as a rough approximation to a circular loop of wire of small cross-section.

We now imagine that a constant external force per unit charge \( \mathbf{f}_{ext} \) is applied in a circular pattern in the tangential \( \hat{\phi} \)-direction, \( \mathbf{f}_{ext} = \hat{\phi} f_{ext} \), to all the charges of the ring. One need not specify the source of this external force per unit charge \( f_{ext} \), but one example of such a situation involves an axially-symmetric magnetic field applied perpendicular to the plane of the circular orbit in the \( -\hat{z} \)-direction which is increasing in magnitude at a constant rate. The external emf around the circular orbit is given by

\[
emf_{ext} = \oint \mathbf{f}_{ext} \cdot d\mathbf{r} = 2\pi R f_{ext}. \tag{10}
\]

The external force per unit charge \( \mathbf{f}_{ext} \) places a tangential force \( \mathbf{F}_i = e_i \hat{\phi}_i f_{ext} \) on the \( i \)th
A particle located at \( r_i \). The Faraday inductance of the charged-particle system is determined by the response of all the particles \( e_i \) in the circular orbit.

**B. One-Particle Model for a Circuit**

1. Motion of the Charged Particle

We start with the case when there is only one charged particle of mass \( m \) and charge \( e \) in the circular orbit. In this case, the tangential acceleration \( a_\phi \) of the charged particle \( e \) in the circular orbit arises from the (tangential) force of only the external force per unit charge \( f_{ext} \), since the centripetal forces of constraint are all radial forces. From Eq. (6), written for a single particle and with \( \frac{d(m\gamma v)}{dt} = m\gamma^3 a_\phi \) where \( \gamma = (1 - v^2/c^2)^{-1/2} \), we have

\[
a_\phi = \frac{e f_{ext}}{m\gamma^3}
\]

where \( f_{ext} \) is the magnitude of the tangential force per unit charge due to the external emf \( emf_{ext} \) at the position of the charge \( e \).

2. Magnetic Field of the Charged Particle

The magnetic field \( B_e \) at the center of the circular orbit due to the accelerating charge \( e \) is given by Eq. (4)

\[
B_e(0, t) = \hat{k}e \frac{v}{cR^2}
\]

where the velocity \( v \) is increasing since the external force per unit charge \( f_{ext} \) gives a positive charge \( e \) a positive acceleration in the \( \hat{\phi} \)-direction. This magnetic field \( B_e \) produced by the orbiting charge \( e \) is increasing in the \( \hat{z} \)-direction, which is in the opposite direction from the increasing external magnetic field which could have created the external force per unit charge \( f_{ext} \) and the external emf \( emf_{ext} \) in Eq. (10).

3. Induced Electric Field from Faraday’s Law

Associated with this changing magnetic field \( B_e \) created by the orbiting charge \( e \), there should be an induced electric field \( E_e(\mathbf{r}, t) \) according to Faraday’s law. Thus averaging over the circular motion of the charge \( e \), we expect an average induced tangential electric field
\( \langle E_{e\phi}(r) \rangle \) at a distance \( r \) from the center of the circular orbit (where \( r \ll R \) so that the magnetic field \( B_e \) has approximately the value \( B_e(0, t) \) at the center) given from Eq. (12) by

\[
2\pi r \langle E_{e\phi}(r) \rangle = \frac{e}{c} \frac{d\Phi_e}{dt} = \frac{1}{c} \frac{d}{dt}[B_e(0, t) \pi r^2]
\]

\[
= -\frac{1}{c} \frac{d}{dt} \left[ e \frac{v}{cR^2} \pi r^2 \right] = -\frac{1}{c} \left( e \frac{a_\phi}{cR^2} \right) \pi r^2
\]

since \( dv/dt = a_\phi \). Using Eq. (11), the average tangential electric field follows from Eq. (13) as

\[
\langle E_{e\phi}(r, t) \rangle = -\hat{\phi} \frac{e^2 r f_{ext}}{2mc^2\gamma^3 R^2}
\]

This derivation of the Faraday induction field corresponds to that familiar from the textbook approach.

4. Induced Electric Field from the Approximate Point-Charge Fields

We will now show that this induced average tangential electric field \( \langle E_{e\phi}(r, t) \rangle \) is exactly the average electric field due to the charge \( e \) obtained by use of the approximate electric field expression given in Eq. (3). Thus we assume that the charge \( e \) is located momentarily at \( r_e = \hat{x}R \cos \phi_e + \hat{y}R \sin \phi_e \), and we average the electric field \( E_e(r, t) \) due to \( e \) over the phase \( \phi_e \). Since the entire situation is axially symmetric when averaged over \( \phi_e \), we may take the field point along the \( x \)-axis at \( r = \hat{x}r \), and later generalize to cylindrical coordinates. The velocity fields given in the first line of Eq. (3) point from the charge \( e \) to the field point. Also, the velocity fields are even if the sign of the velocity \( v_e \) is changed to \( -v_e \). Thus the velocity fields when averaged over the circular orbit can point only in the radial direction. The acceleration fields arising from the centripetal acceleration of the charge will also point in the radial direction. Since we are interested in the average tangential component of the field \( E_e \), we need to average over only the tangential acceleration terms in the second line of Eq. (3). If the field point is close to the center of the circular orbit so that \( r \ll R \), then we may expand in powers of \( r/R \); we retain only the first-order terms, giving \( |\hat{x}r - r_e|^{-1} \approx R^{-1}(1 + \hat{x}r \cdot r_e/R^2) \) and \( |\hat{x}r - r_e|^{-3} \approx R^{-3}(1 + 3\hat{x}r \cdot r_e/R^2) \). Then the
average tangential component of the electric field due to the charge $e$ can be written as

$$\langle E_{\phi}(r, t) \rangle = \left\langle -\frac{e}{2c^2} \left( \frac{a_{e\phi} \cdot (\hat{x}r - r_e) (\hat{x}r - r_e)}{|\hat{x}r - r_e|^3} \right) \right\rangle = \left\langle -\frac{e}{2c^2} \left[ \frac{a_{e\phi}}{R} \left( 1 + \frac{\hat{x}r \cdot r_e}{R^2} \right) + \frac{a_{e\phi} \cdot (\hat{x}r - r_e) (\hat{x}r - r_e)}{R^3} \left( 1 + \frac{3\hat{x}r \cdot r_e}{R^2} \right) \right] \right\rangle .$$

(15)

Now we average over the phase $\phi_e$ with $r_e = \hat{x}R \cos \phi_e + \hat{y}R \sin \phi_e$ and $a_{e\phi} = a_{e\phi}(-\hat{x} \sin \phi_e + \hat{y} \cos \phi_e)$. We note that $\langle a_{e\phi} \rangle = 0$, $a_{e\phi} \cdot r_e = 0$, $\langle a_{e\phi} (\hat{x} \cdot r_e) \rangle = \hat{y}a_{e\phi}R/2 = -\langle (a_{e\phi} \cdot \hat{x}) r_e \rangle$.

After averaging and retaining terms through order $r/R$, equation (15) becomes

$$\langle E_{e\phi}(\hat{x}r, t) \rangle = -\hat{y} \frac{e a_{e\phi} r}{2c^2 R^2}$$

(16)

which is in agreement with our earlier results in Eqs. (13) and (14). Thus indeed the electric field of Faraday induction in this case arises from the acceleration of the charged current carrier of the circuit.

5. **Limit on the Induced Electric Field**

We are now in a position to comment on the average response of our one-particle circuit to the applied external emf. If the source of the external $emf_{ext}$ is a changing magnetic field, then this situation corresponds to the traditional example for diamagnetism within classical electromagnetism. For this one-particle example, the response depends crucially upon the mass $m$ and charge $e$ of the particle. When the mass $m$ is large, the acceleration of the charge is small; therefore the induced tangential electric field $E_{e\phi}$ in Eq. (14) is small. This large-mass situation is what is usually assumed in examples of charged rings responding to external emfs. On the other hand, if we try to increase the induced electromagnetic field $E_{e\phi}$ by making the mass $m$ small, we encounter a fundamental limit of electromagnetic theory. The allowed mass $m$ is limited below by considerations involving the classical radius of the electron $r_{cl} = e^2/(mc^2)$. Classical electromagnetic theory is valid only for distances large compared to the classical radius of the electron. Thus in our example where the radius $R$ of the orbit is a crucial parameter, we must have $R >> r_{cl}$. This means we require the mass $m >> e^2/(Rc^2)$ and so $e^2/(mc^2R) << 1$. Combining this limit with $r/R < 1$, and $1 < \gamma$ leads to a limit on the magnitude of the induced electric field in Eq. (14)

$$\langle E_{e\phi}(r, t) \rangle << f_{ext} \text{ for } r < R .$$

(17)
The induced electric field of a one-particle circuit is small compared to the external force per unit charge associated with the external emf.

6. Energy Balance

We also note that the power delivered by the external force per unit charge goes into kinetic energy of the orbiting particle. Thus if we take the Newton’s-second-law equation giving Eq. (11) and multiply by the speed $v$ of the particle, we have

$$\frac{d}{dt} (m\gamma v)v = \frac{d}{dt} (m\gamma c^2) = m\gamma^3 a\phi v = e f_{ext} v$$

so that the power $ef_{ext}v$ delivered to the charge $e$ by the external force goes into kinetic energy of the particle.

The situation of a one-particle circuit can be summarized as follows. For the one-particle circuit, the induced electric field is small compared to the external electric field and depends explicitly upon the particle’s mass and charge, while the energy transferred by the external field goes into kinetic energy of the one charged particle. Clearly this is not the situation which we usually associate with electromagnetic induction for circuit problems.

C. Multiparticle Model for a Circuit

1. Motion of the Charged Particles

In order to make contact with the usual discussion of Faraday inductance in a circuit, we must go to the situation of many particles, each one of charge $e$ and mass $m$. However, if we take the one-particle circuit above and simply superimpose the fields corresponding to $N$ equally-spaced charges while maintaining the acceleration appropriate for the single-particle case, we arrive at a completely false result. Thus if we take Eq. (11) for the Faraday-induced average electric field $\langle E_{\phi}(r, t) \rangle$ due to a single particle of charge $e$ and mass $m$ accelerating in the external force per unit charge $f_{ext}$ and then simply multiply by the number of charges, we have a result which is linear in $N$ and increases without bound. Thus merely extrapolating the one-particle circuit suggests that the Faraday induced electric field arising from many charges might far exceed the inducing force per unit charge $f_{ext}$. In order to obtain a correct understanding of the physics, we must include the mutual interactions between the
accelerating charges of the circuit. Now with these mutual interactions, the force on any charge in the circular orbit is not just the external force \( e f_{ext} \) due to the original external electric field, and the acceleration of any charge is not given by \( a_{\phi} = e f_{ext} / (m \gamma^3) \). Now the force on any charge is a sum of the force due to the original external force per unit charge plus the forces due to the fields of all the other charged particles in the circular orbit as given in Eq. (6). The magnetic force \( e_i v_i \times B / c \) is simply a deflection and does not contribute to the the tangential acceleration of the charge \( e_i \). Thus the equation of motion for the \( i \)th particle becomes
\[
\frac{d}{dt}(m_i \gamma_i v_i) \cdot \hat{\phi} = m_i \gamma_i^3 a_{\phi} = m_i \gamma_i^3 R \frac{d^2 \phi_i}{dt^2}
\]
\[
= \hat{\phi}_i \cdot e_i \left\{ f_{ext}(r_i) + \sum_{j \neq i} e_j \frac{(r_i - r_j)}{|r_i - r_j|^3} - \sum_{j \neq i} \frac{e_j}{2c^2} \left( \frac{a_j \cdot (r_i - r_j)(r_i - r_j)}{|r_i - r_j|^3} \right) \right\}
\]
where it is understood that the factor \( \gamma_i = (1 - v_i^2 / c^2)^{-1/2} \) should be expanded through second order in \( v_i / c \) so as to be consistent with the remaining terms arising from the approximate field expression (3). Since the particles are equally spaced around the circular orbit and all have the same charge \( e \) and mass \( m \), the situation is axially symmetric. The equation of motion for every charge takes the same form, and the angular acceleration of each charge is the same, \( d^2 \phi_i / dt^2 = d^2 \phi / dt^2 \). For simplicity of calculation, we will take the \( N \)th particle along the \( x \)-axis so that \( \phi_N = 0 \), \( r_N = \hat{x} R \), and \( \phi_N = \hat{y} \). The other particles are located at \( r_j = \hat{x} R \cos(2\pi j/N) + \hat{y} R \sin(2\pi j/N) \), corresponding to an angle \( \phi_j = 2\pi j/N \) for \( j = 1, 2, ..., N - 1 \). The tangential acceleration of the \( j \)th particle is given by \( a_{j\phi} = \left( d^2 \phi / dt^2 \right)[-\hat{x} R \sin(2\pi j/N) + \hat{y} R \cos(2\pi j/N)] \). By symmetry, it is clear that the electrostatic fields, the velocity fields, and the centripetal acceleration fields of the other particles can not contribute to the tangential electric field at particle \( N \). The equation of motion for the tangential acceleration for each charge in the circular orbit is the same as that for the \( N \)th particle, which from Eq. (19) is
\[
\gamma^3 R \frac{d^2 \phi}{dt^2} = \left\{ e f_{ext} - \sum_{j=1}^{j=N-1} \frac{e^2}{2c^2} \left( \frac{\hat{y} \cdot a_{j\phi}}{|\hat{x} R - r_j|} + \frac{a_{j\phi} \cdot (\hat{x} R - r_j)\hat{y}}{|\hat{x} R - r_j|^3} \right) \right\}.
\]
Now we evaluate the distance between the \( j \)th particle and the \( N \)th particle in the circular orbit as
\[
|\hat{x} R - r_j| = \left( 2R^2 - 2R^2 \cos(2\pi j/N) \right)^{1/2} = |4R^2 \sin^2(\pi j/N) |^{1/2} = |2R \sin(\pi j/N)|
\]
(21)
while

$$\hat{y} \cdot a_{j\phi} = (d^2\phi/dt^2) R \cos(2\pi j/N)$$  \hspace{1cm} (22)

and

$$a_j \cdot (\hat{x} R - \mathbf{r}_j) \hat{y} \cdot (\hat{x} R - \mathbf{r}_j) = (d^2\phi/dt^2) R[-R \sin(2\pi j/N)][-R \sin(2\pi j/N)] .$$  \hspace{1cm} (23)

Then equation (20) becomes

$$m\gamma^3 R \frac{d^2\phi}{dt^2} = ef_{\text{ext}} - \frac{d^2\phi}{dt^2} \sum_{j=1}^{N-1} \frac{e^2}{2c^2} \left( \frac{R \cos(2\pi j/N)}{|2R \sin(\pi j/N)|} + \frac{R[R \sin(2\pi j/N)][R \sin(2\pi j/N)]}{|2R \sin(\pi j/N)|^3} \right)$$

or, solving for \(d^2\phi/dt^2\),

$$\frac{d^2\phi}{dt^2} = ef_{\text{ext}} \left[ m\gamma^3 R + \sum_{j=1}^{N-1} \frac{e^2}{2c^2} \left( \frac{2 - 3 \sin^2(\pi j/N)}{2 \sin(\pi j/N)} \right) \right]^{-1} .$$  \hspace{1cm} (24)

If there is only one particle on the frictionless ring so that \(N = 1\), the sum disappears, and the tangential acceleration corresponds to the result obtained earlier in Eq. (11) above with \(R d^2\phi/dt^2 = a_\phi\). We note that the mass term in Eq. (25) remains unchanged by the number of particles while the sum increases with each additional particle. Thus if there are many particles, then the electric field at particle \(i\) due to the other particles \(j\) can lead to so large a sum in Eq. (25) that the mass contribution \(m\gamma^3 R\) becomes insignificant. In this case, the common angular acceleration of each particle becomes from Eq. (25)

$$\frac{d^2\phi}{dt^2} \approx \frac{2c^2}{e} f_{\text{ext}} \left[ \sum_{j=1}^{N-1} \left( \frac{2 - 3 \sin^2(\pi j/N)}{2 \sin(\pi j/N)} \right) \right]^{-1} .$$  \hspace{1cm} (26)

We see that in this multiparticle situation the angular acceleration no longer depends upon the mass \(m\) of the charge carriers. This is the situation envisioned in the usual textbook treatment of Faraday induction.

2. *Induced Electric Field*

Furthermore, in this multiparticle situation where the particle mass becomes insignificant, the left-hand side of Eq. (20) is negligible, so that the sum \(\sum_j \mathbf{E}_{\text{ej}}(\mathbf{r}_i)\) of the acceleration

\[\text{(25)}\]
fields of all the other charges $e_j$ cancels the external force per unit charge $f_{ext}(r_i)$ of the external emf at the position $r_i$ of each charge in the circular orbit

$$-f_{ext}(r_i) \approx E_e(r_i) = \sum_{j \neq i} E_{ej}(r_i). \quad (27)$$

This situation is analogous to that in electrostatics where the fields of the charges in a conductor move to new positions so as to cancel the external force per unit charge at the position of each charge; here the charges accelerate so that the acceleration fields cancel the external force per unit charge at the position of each charge. Now the induced electric field $E_e(r) = \sum_i E_{ei}(r)$ at a general field point due to the orbit particles is independent of the charge $e$ of the charge carriers, since the angular acceleration in Eq. (26) depends inversely on the charge $e$, and this inverse dependence upon $e$ cancels with the $e$ appearing in Eq. (3) so as to give an induced electric field which is independent of the charge on the charge carriers. Again, this situation corresponds to that treated in the textbooks when there is no resistance in the circuit; if there is no resistance, the self-induced emf cancels the external emf and the total force per unit charge is zero.

Most physicists find surprising this idea that the net force on each particle is zero and yet the particles are accelerating. Indeed, the zero-force result from Eq. (27) is only an approximation; the full equation is given in Eq. (24) where indeed Newton’s second law holds. There is a self-consistent relation between the common acceleration of each charge and the back electric force on each charge due to the accelerations of the other charges. In the limit of a large number of charges, we can compute the common acceleration simply by insisting that the acceleration field back at any charge due to the acceleration of the other charges should (approximately) balance the external field. The situation is perhaps easier to grasp if we imagine multiplying Eq. (24) by the common velocity of each particle. Then the equation states that the rate-of-change of a particle’s kinetic energy equals the difference between the power delivered to the particle by the external force and the power absorbed from the particle as magnetic field energy. Since the acceleration of the particle is very small, the rate of change of the particle kinetic energy is very small and most of the energy delivered by the external force goes into magnetic field energy. If one neglects the very small amount of energy going into particle kinetic energy, then one simply states that the energy delivered by the external force goes into magnetic field energy. This last statement is the approximation which appears in the textbooks of electromagnetism.
In nonrelativistic classical mechanics (which forms the physical ideas of most physicists), we expect to be able to store potential energy of relative position; however, there is no such thing as potential energy of velocity. In classical mechanics, the energy associated with velocity is always mechanical kinetic energy based on mass times velocity squared. In complete contrast to this situation, electromagnetic theory contains magnetic energy associated with the velocity of charges and yet not associated with particle mass. For consistency, the electromagnetic theory requires that accelerating charges cause fields which produce forces on other charges. The mutual interaction of the charges through the Faraday induction fields assures that the magnetic energy stored indeed required work by some external forces. For a multiparticle system, the net force on each particle may become tiny compared to the external force on each charge because of the acceleration fields due to the other charges. The charges have a tiny acceleration and gain a tiny amount of kinetic energy while the work done by the external force goes into the large amount of energy stored in the magnetic field.

3. Self-Inductance of the Circuit

The value for the self-inductance of the multiparticle circuit can be obtained from Eq. (8) when the circuit resistance vanishes. If the resistance vanishes, the external emf \(emf_{\text{ext}} = 2\pi R f_{\text{ext}}\) around the ring equals the self-inductance \(L\) multiplied by the time-rate-of-change of the current \(i = Ne(d\phi/dt)/(2\pi)\)

\[
emf_{\text{ext}} = 2\pi RE_{\text{ext}}(R) = L \frac{di}{dt} = L \left( \frac{Ne \frac{d^2\phi}{dt^2}}{2\pi} \right).
\] (28)

Now traditional classical electromagnetism ignores the mass contribution to the inertia which appears in Eq. (25) and regards the self-inductance as a geometrical quantity associated with the approximation given in Eq. (26). Thus the self-inductance \(L\) of the multi-charge ring system from Eq. (26) is

\[
L = \frac{2\pi R E_e(R)}{[Ne(d^2\phi/dt^2)/(2\pi)]} = \frac{(2\pi)^2 R}{2e^2 N} \sum_{j=1}^{N-1} \left( \frac{2 - 3 \sin^2(\pi j/N)}{2 \sin(\pi j/N)} \right) \quad .
\] (29)

We see that the self-inductance of this multiparticle, circular-orbit circuit is now independent of the mass \(m\) or the charge \(e\) of the current carriers. As with all expressions for the self-inductance \(L\) of a circuit, here Eq. (29) has dimensions of length divided by \(c^2\). In the
appendix, the self-inductance $L$ in Eq. (29) for our hypothetical circuit is connected to the self-inductance of a circular wire of finite cross-section.

4. Energy of the Current Carriers

The change in energy $\Delta U$ of the system associated with non-zero velocity $v$ for the charges of the circuit includes both mechanical kinetic energy $\Delta U_{\text{mechanical}}$ and magnetic field energy $\Delta U_{\text{mag}}$

$$\Delta U = \Delta U_{\text{mechanical}} + \Delta U_{\text{mag}} = \sum_{i=1}^{N} m_i c^2 (\gamma_i - 1) + \frac{1}{8\pi} \int d^3r B^2 .$$

The self-inductance $L$ of the circuit is associated solely with the magnetic energy stored in the circuit $\Delta U_{\text{mag}} = (1/2) Li^2$. The magnetic energy $U_{\text{mag}} = \int d^3r B^2/(8\pi)$ stored in the circular-orbit circuit is given by the cross-terms (but not the self-terms) when the magnetic field is squared, and corresponds to the velocity-dependent double sum in the Darwin Lagrangian Eq. (5). Thus we have

$$\Delta U_{\text{mag}} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{e^2}{2c^2} \left( \vec{v}_i \cdot \vec{v}_j + \vec{v}_i \cdot \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \vec{v}_j \cdot \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \right)$$

$$= N \sum_{j=1}^{N-1} \frac{e^2}{2c^2} \left( \frac{v\hat{y} \cdot v_j}{|\hat{x}R - \vec{r}_j|^3} + \frac{v\hat{y} \cdot (-\vec{r}_j) v_j \cdot \hat{x}R)}{|\hat{x}R - \vec{r}_j|^3} \right)$$

where in the last line of Eq. (31) we have used $\vec{v}_i \cdot \vec{r}_i = 0$ and have taken advantage of the symmetry to evaluate the magnetic energy when the $N$th particle is located on the $x$-axis at $\vec{r}_N = \hat{x}R$ and is moving with velocity $\vec{v}_N = \hat{y}v = \hat{y}Rd\phi/dt$. The $j$th particle is located at $\vec{r}_j = R[\hat{x} \cos(2\pi j/N)+\hat{y} \sin(2\pi j/N)]$ with velocity $\vec{v}_j = R(d\phi/dt)[-\hat{x} \sin(2\pi j/N)+\hat{y} \cos(2\pi j/N)]$. Introducing these expressions along with the distance given in Eq. (21), the magnetic energy of Eq. (31) is

$$\Delta U_{\text{mag}} = \frac{N}{2} \sum_{j=1}^{N-1} \frac{e^2}{2c^2} \left( \frac{R(d\phi/dt)}{2\pi \sin^2(\pi j/N)} \left( \frac{\cos(2\pi j/N)}{2\sin(\pi j/N)} + \frac{\sin^2(2\pi j/N)}{2\sin^2(\pi j/N)} \right) \right)$$

$$= \frac{1}{2} \left( \frac{(2\pi)^2 R}{2c^2 N} \sum_{j=1}^{N-1} \left( \frac{2 - 3\sin^2(\pi j/N)}{2\sin^2(\pi j/N)} \right) \left( eN \frac{d\phi}{2\pi dt} \right)^2 . \right.$$
in Eq. (29). Again for an interacting multiparticle system, the mechanical kinetic energy increases linearly with the number of particles $N$ while the magnetic energy increases as $N^2$. In this multiparticle limit, the power $P_i = e f_{ext} v_i$ delivered to the $i$th charge by the external force per unit charge $f_{ext}$ associated with the original external emf goes only slightly into particle kinetic energy but mainly is converted into the magnetic energy associated with the motion of the charges on the ring. This stored magnetic energy is exactly what is given by the expression $(1/2)L i^2$.

For our example of an external emf acting on a circular charged-particle circuit, the situation with many particles is totally transformed from the situation with only one particle. As charged particles of mass $m$ and charge $e$ are added to the circuit, the mechanical inertia increases linearly with the number of particles $N$ while the inertia associated with the mutual electromagnetic interactions increases quadratically with $N$. In the multiparticle case, the mutual interaction between the particles overwhelms the single-particle behavior so that the mass and charge of the individual charge carriers is no longer of significance. The particles move so that the sum of the induced electric fields at each particle approximately cancels the external electric field. The energy of the charge carriers of the ring includes both mechanical kinetic energy and also magnetic energy, but the magnet energy is dominant in the multiparticle limit. For the situation envisioned in the textbook discussions of Faraday induction, the mechanical energy is so small compared to the magnetic energy that the mechanical kinetic energy is never mentioned.

We notice that in Eq. (8) if the resistance $R$ of the circuit vanishes, then the current increase at a steady rate $di/dt = em f_{ext} / L$ without limit. Of course, equation (8) involves the self-inductance $L$ which has no role for the inertia or kinetic energy of the charge carriers because these are assumed miniscule compared to the electromagnetic mutual interactions and magnetic energy. Since our analysis has used the low-velocity approximation of Eq. (3), our treatment assumes $(v/c)^2 << 1$ and cannot be extrapolated to velocities comparable to the speed of light $c$.

V. DISCUSSION

In the analysis above, we have discussed the Faraday induction from an unfamiliar particle point of view. The main interest of our calculations involves a detailed treatment of a
simple hypothetical circuit. We have evaluated the electric fields of individual electric charges and shown how a system involving a single charge is transformed over to a familiar electromagnetic system when the number of charges is increased. Our simple example involves charges under centripetal constraining forces giving a circular orbit but allowing tangential acceleration along the circular orbit. In the one-particle example, the induced electric field depends upon both the charge $e$ and the mass $m$ of the charge carrier with the induced field proportional to $e^2/m$. In the one-particle case, the induced electric field is small, and the energy transferred to the particle goes into the kinetic energy of the ring particle. The multiparticle circuit has a completely different behavior from that given by a summation over many one-particle circuits with the one-particle acceleration. When we deal with an interacting multiparticle case, then the electromagnetic forces between the charges are such as to transform the behavior over to the familiar behavior of a conducting circuit where the charge and mass of the current carriers are of no significance. When there are a large number of charged particles, the acceleration of each charge is determined by essentially the requirement that the sum of the acceleration fields of all the other charges should cancel the external force per unit charge which produces the external emf around the circuit. This crucial mutual interaction of the current carriers is sometimes not appreciated in the traditional textbook treatment.

VI. ACKNOWLEDGEMENTS

I wish to thank Dr. Hanno Essén for sending me copies of the work of C. G. Darwin and of his own work, listed now in references 3 and 4. I had been unaware of these contributions. Also, I wish to thank a referee for many helpful comments on earlier versions of this article which directed my attention toward needed clarifications.

VII. APPENDIX

The self-inductance of our hypothetical circuit of $N$ charges moving in a circle of radius $R$ was given in Eq. (29). In order to compare this expression with that of a continuous wire, we approximate the sum by an integral. Using the trapezoid rule $\int_{a}^{b} dx f(x) =$
\[
\frac{(b - a)/n}{\left\{ \sum_{k=1}^{n-1} f(a + k(b - a)/n) + f(a)/2 + f(b)/2 \right\}},
\]
we have
\[
\sum_{j=1}^{j=N-1} \left( \frac{2 - 3 \sin^2(\pi j/N)}{2 \sin(\pi j/N)} \right) = \left[ \frac{N}{\pi} \int_{\pi/N}^{\pi} dx \left( \frac{2 - 3 \sin^2(x)}{2 \sin(x)} \right) \right] + \frac{1}{2} \left( \frac{2 - 3 \sin^2(\pi/N)}{2 \sin(\pi/N)} \right) + \frac{1}{2} \left( \frac{2 - 3 \sin^2(\pi - \pi/N)}{2 \sin[\pi - \pi/N]} \right).
\] (33)

The integral can be evaluated analytically from \(\int dx/\sin x = (1/2)[\ln(1 - \cos x) - \ln(1 + \cos x)]\) and \(\int dx \sin x = -\cos x\). Then noting \(\sin(\pi - \pi/N) = \sin(\pi/N)\) and \(\cos(\pi - \pi/N) = -\cos(\pi/N)\), and expanding in powers of \(1/N\), the sum on the first line of Eq. (33) becomes
\[
\sum_{j=1}^{j=N-1} \left( \frac{2 - 3 \sin^2(\pi j/N)}{2 \sin(\pi j/N)} \right) = \frac{N}{\pi} \left[ \ln \left( \frac{1 + \cos(\pi/N)}{1 - \cos(\pi/N)} \right) - 3 \cos \left( \frac{\pi}{N} \right) \right] + \left( \frac{2 - 3 \sin^2(\pi/N)}{2 \sin(\pi/N)} \right)
\] + \left( \frac{2 - 3 \sin^2[\pi - \pi/N]}{2 \sin[\pi - \pi/N]} \right)
= \frac{N}{\pi} \left[ 2 \ln \left( \frac{2N}{\pi} \right) \right] - \left( \frac{2N}{\pi} \right) + O(1/N). \quad (34)

Accordingly the self-inductance of our hypothetical circuit is
\[
L \approx \frac{(2\pi)^2}{2c^2} R \left\{ \frac{N}{\pi} \left[ 2 \ln \left( \frac{2N}{\pi} \right) \right] - \left( \frac{2N}{\pi} \right) \right\}
= \frac{4\pi c^2}{c^2} R \left[ \ln \left( \frac{8R}{a} \right) - \frac{7}{4} \right]. \quad (35)
\]

The self-inductance of a circular loop of wire of radius \(R\) and circular cross-section of radius \(a\) is given by\(^{20}\)
\[
L = \frac{4\pi}{c^2} R \left[ \ln \left( \frac{8R}{a} \right) - \frac{7}{4} \right]. \quad (36)
\]

The separation between the charges of our hypothetical circuit is \(\Delta s = 2\pi R/N\). Evidently if we take the effective radius \(a\) of the cross-section of the hypothetical circuit as \(a \approx 2\Delta s\), then there is agreement for the logarithmic terms between Eqs. (35) and (36). For large
values of $N$, the logarithmic term should be the dominant contribution.

1 See, for example, the undergraduate textbook by D. J. Griffiths, Introduction to Electrodynamics 3rd edn (Prentice-Hall, Upper Saddle River, NJ 1999).
2 See, for example, the graduate-level textbook by J. D. Jackson, Classical Electrodynamics 3rd edn (John Wiley & Sons, New York, 1999).
3 C. G. Darwin, “The Inertia of Electrons in Metals,” Proc. Roy. Soc. London A 154, 61-66 (1936).
4 One of the rare instances is provided by the work of H. Essén, “From least action in electrodynamics to magnetomechanical energy—a review,” Eur. J. Phys. 30, 515-539 (2009). In an appendix, Essén evaluates the self-inductance of the same hypothetical circuit as used in the present article.
5 Attention to a current-source point of view is encouraged by S. E. Hill, “Rephrasing Faraday’s Law,” The Physics Teacher 48, 410-412 (2010) who refers to Jeffimenko’s equations when seeking the source of the Faraday induction field. Hill’s reminder is mentioned in a footnote by D. J. Griffiths, Introduction to Electrodynamics 4th edn (Pearson, New York 2013) on p. 313.
6 See, for example, ref. 2, p. 664. When treating relativistic aspects of electromagnetism, gaussian units are far more natural than S.I. In S.I. units one would have a factor of $1/(4\pi\varepsilon_0)$ multiplying the right-hand side of Eq. (1) and a factor of $\mu_0 c/(4\pi)$ multiplying the right-hand side of Eq. (2).
7 L. Page and N. I. Adams, Electrodynamics (D. Van Nostrand, New York 1940), p. 175. In 1965, this text was reprinted by Dover Publications, New York. The book no longer seems to be in print.
8 C. G. Darwin, “The Dynamical Motions of Charged Particles,” Phil. Mag. 39, 537-551 (1920). The Darwin Lagrangian appears in some modern textbooks. See, for example, ref. 2, pp. 596-598 or L. D. Landau and E. M. Lifshitz, “The Classical Theory of Fields 4th ed.,” (Pergamon, New York 1975), pp. 165-168.
9 See, for example, ref. 1, pp. 458-459.
10 L. Page and N. I. Adams, “Action and reaction between moving charges,” Am. J. Phys. 13, 141-147 (1945).
11 T. H. Boyer, “Lorentz-transformation properties of energy and momentum in electromagnetic
systems,” Am. J. Phys. 53, 167-171 (1985).

12 T. H. Boyer, “Example of mass-energy relation: Classical hydrogen atom accelerated or supported in a gravitational field,” Am. J. Phys. 66, 872-876 (1998).

13 See, for example, ref. 1, p. 303.

14 See, for example, ref. 2, problem 14.24 on pp. 705-706. There is no radiation emitted, and the fields of the circuit are those of electrostatics and magnetostatics.

15 Essén in his appendix discusses the necessary approximations to connect the self-inductance of this hypothetical circuit back to that of a wire of finite thickness which is bent in a circle. See the appendix of ref. 4. We have followed Essén’s basic analysis in the appendix to the present article.

16 See, for example, ref. 1, pp. 271-272.

17 The situation of our one-particle example contains elements of the same physical system as “the Feynman disk paradox.” See R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading MA, 1964), Vol. II, p. 17-5 and p. 27-6. See also, Ref. 1, pp. 359-361. In the present article, we evaluate the induced electromagnetic field arising from the accelerating disk charges by use of the approximate electric field expression (3).

18 A continuous circular wire of finite cross-section provides the analogue within traditional electromagnetism texts of our hypothetical circuit. Our hypothetical circuit is not continuous but rather involves a finite number $N$ of charges with spaces between the charges. It seems interesting that for our discrete circular charge arrangement, the typical multiparticle behavior requires at least four charges. The summation in Eq. (29) is steadily increasing with increasing particle number $N$. However, the summation starts out negative ($-0.5$) for $N = 2$, and is still negative ($-0.289$) for $N = 3$. Only for $N = 4$, does the summation become positive ($+0.207$). It should be emphasized that even for small numbers of charges, provided that we observe the restriction that $MR >> e^2/c^2$, the total inertia is positive. For a few charges, the mass of the current carriers is crucial; for large $N$, the mass of the current carriers is unimportant. The negative value of $L$ for a small numbers of charges corresponds to total energy in the combined magnetic fields which is smaller than that of the individual magnetic fields of the charges when located far from each other. The Darwin-Lagrangian approximation contains some of the run-away aspects which are seen elsewhere in classical electromagnetic theory.

19 See, for example, M. D. Greenberg, *Advanced Engineering Mathematics* (Prentice Hall, Upper
Saddle River, NJ 1998), p. 312.

20 See, for example, ref. 2, p. 234.