Foundations

Logics of involutive Stone algebras

Sérgio Marcelino · Umberto Rivieccio

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Abstract
An involutive Stone algebra (IS-algebra) is simultaneously a De Morgan algebra and a Stone algebra (i.e., a pseudo-complemented distributive lattice satisfying the Stone identity $\sim x \lor \sim \sim x \approx 1$). IS-algebras have been studied algebraically and topologically since the 1980s, but a corresponding logic (here denoted $\mathcal{IS}_\leq$) has been introduced only very recently. This logic is the departing point of the present study, which we then extend to a wide family of previously unknown logics defined from IS-algebras. We show that $\mathcal{IS}_\leq$ is a conservative expansion of the Belnap-Dunn four-valued logic (i.e., the order-preserving logic of the variety of De Morgan algebras), and we give a finite Hilbert-style axiomatization for it. More generally, we introduce a method for expanding conservatively every super-Belnap logic (i.e., every strengthening of the Belnap-Dunn logic) so as to obtain an extension of $\mathcal{IS}_\leq$. We show that every logic thus defined can be axiomatized by adding a fixed finite set of multiple-conclusion rule schemata to the corresponding super-Belnap base logic. Our results entail that the lattice of super-Belnap logics (which is known to be uncountable) embeds into the lattice of extensions of $\mathcal{IS}_\leq$. In fact, as in the super-Belnap case, we establish that the finitary extensions of $\mathcal{IS}_\leq$ are already uncountably many. When the base super-Belnap logic possesses a disjunction, we show that we can reduce the multiple-conclusion calculus to a traditional one; some of the multiple-conclusion axiomatizations so introduced are analytic and are thus of independent interest from a proof-theoretic standpoint. We also consider a few extensions of $\mathcal{IS}_\leq$ that cannot be obtained in the above-described way, but can nevertheless be axiomatized finitely by other methods.

1 Introduction

Involutive Stone algebras (from now on, IS-algebras) were first considered in the papers (Cignoli and de Gallego 1981, 1983) within a study of finite-valued Łukasiewicz logics and, more specifically, in connection with the algebraic structures nowadays known as Łukasiewicz-Moisil algebras. The term ‘involutive’ is due to the observation that every IS-algebra has a primitive negation operation $\sim$ that satisfies the involutive law ($\sim \sim x \approx x$), whereas ‘Stone’ refers to the existence of a term-definable pseudo-complement operation $\neg$ that satisfies the well-known Stone identity $\neg x \lor \neg \neg x \approx 1$. From an algebraic point of view, IS-algebras are a variety of De Morgan algebras endowed with an additional unary operation (here denoted by $\nabla$); alternatively, IS-algebras can be viewed as the subclass of De Morgan algebras that satisfy certain structural properties ensuring the definability of $\nabla$. The connection between De Morgan and IS-algebras will indeed play a prominent role in the present paper.

De Morgan algebras form a variety that is well known in non-classical logic as the algebraic counterpart of $\mathcal{B}$, the four-valued Belnap-Dunn logic (Belnap 1977; Font 1997). From a logical point of view, $\mathcal{B}$ can be viewed as a weakening of classical two-valued logic designed to allow for both para-consistency (in that $\mathcal{B}$ does not validate the rule $p \land \sim p \vdash q$, known as ex contradictione quodlibet) and para completeness (in that $\mathcal{B}$ does not validate the principle of excluded middle, $\vdash p \lor \sim p$). De Morgan algebras can thus be viewed as a generalization of Boolean algebras on which the operation $\sim$ (that interprets the negation connective) need not be a Boolean complement, i.e., the classical laws $x \land \sim x \approx 0$ and $x \lor \sim x \approx 1$ need not be satisfied. The involutive and the De Morgan laws are however valid on every De Morgan algebra (see Definition 3.1).
As mentioned above, involutive Stone algebras may be regarded as a subclass of De Morgan algebras characterized by certain structural properties. From this perspective, an involutive Stone algebra may be viewed as a De Morgan algebra \( A \) having an additional unary operation \( \nabla \) that receives an arbitrary element \( a \in A \) as argument and outputs a certain ‘classical’ element \( \nabla a \) (i.e., an element that possesses a Boolean complement in \( A \)). Just as not every distributive lattice can be equipped with a Boolean complement operation, so not every De Morgan algebra can be endowed with an operation \( \nabla \) meeting the above requirement. However, if such an operation is definable, then it is unique.

One might say that, on every De Morgan algebra \( A \), the behavior of \( \nabla \) provides a measure of how far \( A \) is from being Boolean: the limit cases being, at one end of the spectrum, Boolean algebras themselves (on which \( \nabla \) is the identity map) and, at the other, the algebras (such as those depicted in Fig. 2) where the only Boolean elements are the top and the bottom. These structural requirements on \( \nabla \) can be completely captured by means of identities (Cignoli and de Gallego 1983, Thm. 2.1). Therefore, regardless of the preceding considerations, IS-algebras can be simply introduced as a variety of De Morgan algebras having an extra unary operation \( \nabla \) that is required to satisfy four additional identities (see Definition 3.2).

A logic associated to IS-algebras has been considered for the first time in Cantú (2019), Cantú and Figallo (2020). In the present paper, we shall denote this logic by \( \mathcal{IS}_\leq \), suggesting that \( \mathcal{IS}_\leq \) is the order-preserving logic canonically associated to the variety of IS-algebras (see Sect. 2 for the relevant definitions). As we will show, \( \mathcal{IS}_\leq \) is a conservative expansion of the Belnap-Dunn logic \( B \), which is itself the order-preserving logic of the variety of De Morgan algebras. We are moreover going to prove that, between the logics extending \( B \) (known as super-Belnap logics after the paper (Rivieccio 2012)) and the extensions of \( \mathcal{IS}_\leq \), a connection can be established and exploited in order to obtain a number of non-trivial results.

The background facts we shall need on the Belnap-Dunn logic can be found in Font (1997), including a complete Hilbert-style axiomatization and a characterization of the reduced matrix models (see Sect. 2). For further information on super-Belnap logics, we refer the reader to the papers (Rivieccio 2012; Albuquerque et al. 2017; Pfenosil 2021), from which we shall also import a few results as needed.

The rest of the paper is organized as follows. Sect. 2 introduces the generic algebraic and logical notions that will be used in the following ones. Sect. 3 contains the basic algebraic results on De Morgan and involutive Stone algebras. In Sect. 4 we look at the logic \( \mathcal{IS}_\leq \) of involutive Stone algebras from a semantical point of view. We observe that \( \mathcal{IS}_\leq \) is non-protoalgebraic (Proposition 4.2) and can be characterized by a single finite matrix (Proposition 4.1). We further introduce a simple operation on logical matrices that allows us to associate, to any given super-Belnap logic, a logic extending \( \mathcal{IS}_\leq \) in such a way that the latter is a conservative expansion of the former (Lemma 4.10). This entails that the lattice of super-Belnap logics is embeddable into the lattice of extensions of \( \mathcal{IS}_\leq \) (Corollary 4.12), which in turn tells us that the latter must have at least the cardinality of the continuum (Corollary 4.13).

In Sect. 5 we take a little detour through the realm of multiple-conclusion logics, presenting a uniform method for axiomatizing all the extensions of \( \mathcal{IS}_\leq \) that are defined from super-Belnap logics via the construction introduced in Sect. 4 (Theorem 5.4, Corollaries 5.5 and 5.8). Via Corollary 5.10 and Proposition 5.12 we obtain standard single-conclusion axiomatizations for a wide class of logics, which include those obtained by adding the \( \nabla \) connective to well-known extensions of \( B \) such as G. Priest’s Logic of Paradox and the two three-valued logics due to S. C. Kleene. An axiomatization for \( \mathcal{IS}_\leq \) is obtained as a special application of the general method (Example 5.11). In Sect. 5.3 we axiomatize a few extensions of \( \mathcal{IS}_\leq \) that are not obtained in this way from a super-Belnap logic, among which we find the three-valued Łukasiewicz(-Moisil) logic. For the latter results we cannot apply the above-mentioned method, so we need to take a longer detour through multiple-conclusion logics and analytical calculi (Sect. 5.4). Finally, Sect. 6 contains a few concluding remarks and suggestions for future research.

### 2 Algebraic and logical preliminaries

In this Section we recall the main algebraic and logical notions that will be needed in the following ones. We assume familiarity with basic results of lattice theory (Davey and Priestley 1990), universal algebra (Burris and Sankappanavar 2000) and the general theory of logical calculi (Wójcicki 1988; Font and Jansana 2009; Font 2016).

We shall denote by \( A, B \) etc. algebras over a given algebraic similarity type \( \Sigma \). The set of \( \Sigma \)-homomorphisms between two algebras \( A \) and \( B \) will be denoted by \( \text{Hom}(A, B) \). Given \( \Sigma \)-algebras \( A, B \) and a sub-signature \( \Sigma' \subseteq \Sigma \), we denote by \( \text{Hom}_{\Sigma'}(A, B) \) the set of functions \( h : A \rightarrow B \) that are only required to preserve the operations in \( \Sigma' \). The algebra of formulas over a signature \( \Sigma \), freely generated by a countable set of variables (denoted \( p, q \) etc.), will be denoted by \( \text{Fm}_{\Sigma} \) (or simply by \( \text{Fm} \), if \( \Sigma \) is clear from the context), and its elements by \( \varphi, \psi \) etc. Given a class \( K \) of similar algebras, we denote by \( \mathcal{I}(K), \mathcal{H}(K), \mathcal{S}(K), \mathcal{P}(K), \mathcal{P}_{\Sigma}(K) \) the classes formed by closing \( K \) under (respectively) isomorphisms, homomorphisms, subalgebras, direct products and subdirect products. A variety is a class \( K \) of algebras that is closed under \( \mathcal{H}, \mathcal{S}, \mathcal{P}, \) or, equivalently, that is definable by means of algebraic identities. A quasi-variety is a class \( K \) of
algebras that is definable by means of quasi-identities, that is, implications having the conjunction of a finite number of identities as premiss and a single identity as conclusion. The variety, resp. quasivariety, generated by \( K \) will be denoted by \( \forall(K), \) resp. \( \forall(Q)(K) \). Every variety \( V \) is generated by the class \( V_{si} \) of its subdirectly irreducible members, defined as follows: an algebra \( A \) is subdirectly irreducible if \( A \) has a minimum congruence above the identity relation (as a special case, we say that \( A \) is simple if \( A \) has exactly two congruences). For every variety \( V \), we have \( V = \forall(V_{si}) = \Pi_{S}(V_{si}). \)

We view a (propositional, single-conclusion) logic as a structural consequence relation on \( \phi(Fm) \times Fm \) (see e.g., Font 2016, Def. 1.5). The symbol \( \vdash \) will be used to denote arbitrary logics.

We say that a logic \( \vdash \) is an extension of \( \vdash \) when both logics share the same propositional language \( \Sigma \) and \( \vdash \subseteq \vdash \). The family of all extensions of a logic \( \vdash \) forms a complete lattice (in which the meet is the intersection); in this paper we shall be concerned with the lattice of extensions of the logic \( IS_{\subseteq} \) of involutive Stone algebras, and will relate it to the lattice of extensions of the Belnap–Dunn logic \( B \) (i.e., the lattice of super-Belnap logics). We say that a logic \( \vdash_{2} \) over a language \( \Sigma_{2} \) is an expansion of a logic \( \vdash_{1} \) over \( \Sigma_{1} \) when \( \Sigma_{1} \subseteq \Sigma_{2} \) and \( \vdash_{1} \subseteq \vdash_{2} \). We speak of a conservative expansion when both consequence relations coincide on the formulas over \( \Sigma_{1} \).

A (logical) matrix is a pair \( M = \langle A, D \rangle \) where \( A \) is an algebra (with universe \( A \)) and \( D \subseteq A \) is a set of designated elements. One defines the notions of isomorphism, homomorphism, submatrix and product of matrices as straightforward extensions of the corresponding universal algebraic constructions (see Wójcicki 1988; Font and Jansana 2009 for details). Given a matrix \( M = \langle A, D \rangle \) with \( A \) a \( \Sigma \)-algebra, we let \( Val(M) = \text{Hom}(FM_{\Sigma}, A) \). The elements of \( Val(M) \) are called valuations. We denote by \( \text{Log} \) the mapping that associates a logic to a class of matrices in the standard fashion. Indeed, each matrix determines a logic (denoted \( \text{Log} M \) or \( \vdash_{M} \)) as follows: for all \( \Gamma \cup \{ \phi \} \subseteq Fm \), one lets \( \Gamma \vdash_{M} \phi \) iff, for every valuation \( v \in Val(M) \), we have that \( v(\Gamma) = \{ v(\gamma) : \gamma \in \Gamma \} \subseteq D \) entails \( v(\phi) \in D \). To a class of matrices \( M = \{ M_{i} : i \in I \} \), we associate the logic \( \text{Log} M = \vdash_{M} := \bigcap_{i \in I} \vdash_{M_{i}} \). We say that a matrix \( M \) is a model of a logic \( \vdash \) when \( \vdash \subseteq \vdash_{M} \), that is, when \( \vdash \) entails \( \vdash_{M} \), for all \( \Gamma \cup \{ \phi \} \subseteq Fm \).

Every matrix \( M = \langle A, D \rangle \) has an associated Leibniz congruence \( \mathcal{O}_{A}(D) \), which is the greatest congruence on \( A \) that is compatible with \( D \) in the following sense: for all \( a \in D \) and \( b \in A \), if \( (a, b) \in \mathcal{O}_{A}(D) \), then \( b \in D \). This property allows one to define the quotient matrix \( M^{*} = \langle A/\mathcal{O}_{A}(D), D/\mathcal{O}_{A}(D) \rangle \), which is known as the reduction of \( M \). A matrix \( M \) is reduced if \( \mathcal{O}_{A}(D) \) is the identity relation, so no further reduction is possible. Reduced matrices are important in the study of algebraic models of logics, because, for every matrix \( M \), one has \( \text{Log} M = \text{Log} M^{*} \). It follows that every logic coincides with the logic determined by the class of all its reduced matrix models; we shall denote by \( \text{Matr}^{*}(\vdash) \) the class of all reduced matrices of a logic \( \vdash \).

In algebraic logic, two classes of algebras, \( \text{Alg}^{*}(\vdash) \) and \( \text{Alg}(\vdash) \), are traditionally associated to a given logic \( \vdash \). The former is defined as follows: \( \text{Alg}^{*}(\vdash) := \{ A : \text{there is } D \subseteq A \text{ such that } (A, D) \in \text{Matr}^{*}(\vdash) \} \). By the characterization of Font and Jansana (2009, Thm. 2.23), the class \( \text{Alg}(\vdash) \) may be defined as the closure of \( \text{Alg}^{*}(\vdash) \) under subdirect products, that is, \( \text{Alg}(\vdash) := \text{F}_{S}(\text{Alg}^{*}(\vdash)) \).

Let \( K \) be a class of algebras such that each \( A \in K \) has a meet-semilattice reduct with top element 1. From \( K \) one can obtain a finitary logic \( \vdash_{K}^{\subseteq} \) as follows. One lets \( \emptyset \vdash_{K}^{\subseteq} \phi \) if and only if the identity \( \phi \approx 1 \) is valid in \( K \) and, for all \( \Gamma \cup \{ \phi \} \subseteq Fm \), one lets \( \Gamma \vdash_{K}^{\subseteq} \phi \) iff there are \( \gamma_{1}, \ldots, \gamma_{n} \in \Gamma \) such that the identity \( \gamma_{1} \wedge \ldots \wedge \gamma_{n} \wedge \phi \approx \gamma_{1} \wedge \ldots \wedge \gamma_{n} \) is valid in \( K \). We shall call \( \vdash_{K}^{\subseteq} \) the order-preserving logic associated to \( K \). Observe that, by definition, \( K \) and \( \forall(K) \) define the same logic; thus, the order-preserving logics considered in the literature are usually associated to varieties (of semilattice-based algebras). We note that, if \( V \) is a variety of algebras having a meet-semilattice reduct with top element 1 (as will always be the case in the present paper), then \( \vdash_{V}^{\subseteq} \) coincides with the logic defined by the class of matrices \( \{ (A, D) : A \in V, D \subseteq A \} \) is a (non-empty) semilattice filter of \( A \).

### 3 De Morgan and involutive Stone algebras

In this Section we recall the main definitions and basic results on the classes of algebras involved.

**Definition 3.1** A De Morgan lattice is an algebra \( A = (A; \wedge, \vee, \sim) \) of type \( (2, 2, 1) \) such that \( (A; \wedge, \vee) \) is a distributive lattice and the following identities are satisfied:

- (DM1) \( \sim(x \vee y) \approx \sim x \wedge \sim y \).
- (DM2) \( \sim(x \wedge y) \approx \sim x \vee \sim y \).
- (DM3) \( x \approx \sim \sim x \).

A De Morgan algebra is a De Morgan lattice whose lattice reduct is bounded (thus we include constant symbols \( \top \) and \( \bot \) in the algebraic signature) and satisfies the following identities; \( \sim \top \approx \bot \approx \sim \top \).

Figure 1 depicts (all) the subdirectly irreducible De Morgan algebras. On each algebra, the lattice operations are determined by the diagram. The negation is defined on \( \text{DM}_{4} \) by \( \sim 0 = 1, \sim 1 = 0, \sim a = a \) and \( \sim b = b \). These prescriptions apply to \( K_{3} \) and \( B_{2} \) as well, viewed as sub-algebras of \( \text{DM}_{4} \). Obviously \( B_{2} \) is the two-element Boolean...
algebra, and \( K_3 \) is the three-element Kleene algebra associated to the three-valued logics originating from the work of S.C. Kleene\(^1\).

**Definition 3.2** An involutive Stone algebra (IS-algebra) is an algebra \( A = (\wedge, \vee, \sim, \neg, \bot, \top) \) of type \((2, 2, 1, 0, 0)\) such that \((A; \wedge, \vee, \sim, \bot, \top)\) is a De Morgan algebra and the following identities are satisfied:

\[
\begin{align*}
(\text{IS1}) & \quad \neg \bot \approx \bot, \\
(\text{IS2}) & \quad x \approx x \land \neg x, \\
(\text{IS3}) & \quad \neg(x \land y) \approx \neg x \land \neg y, \\
(\text{IS4}) & \quad \sim \neg x \land \neg y \approx \neg (x \lor y).
\end{align*}
\]

The class of IS-algebras will be denoted IS. The name ‘involutive Stone algebras’ is motivated by the following observation. For every IS-algebra \( A = (A; \wedge, \vee, \sim, \neg, \bot, \top) \), the operation \( \neg \) that realizes the term \( \neg x := \sim \neg x \) is a pseudo-complement; moreover, \( A \) satisfies the so-called Stone identity \( \neg x \lor \neg \neg x \approx \top \). Hence, \( (A; \wedge, \vee, \sim, \neg, \bot, \top) \) is a Stone algebra\(^2\). Conversely, given an algebra \( (A; \wedge, \vee, \sim, \neg, \bot, \top) \) that has both a De Morgan negation and a pseudo-complement operation, upon defining \( \neg x := \neg \neg x \), one has that \( (A; \wedge, \vee, \sim, \neg, \bot, \top) \) is an IS-algebra if and only if the following identity is satisfied: \( \sim x \approx \sim \neg x \) (Cignoli and de Gallego 1983, Remark 2.2).

The variety of IS-algebras is generated by the six-element algebra \( IS_6 \), which is shown in Fig. 2 together with its subalgebras \( IS_5 \), \( IS_4 \), \( IS_3 \) and \( IS_2 \). Our notation reflects the observation that the De Morgan algebra \( DM_4 \) is obtained by adjoining a new top \( \hat{1} \) and a new bottom \( \hat{0} \) element to the De Morgan algebra \( DM_4 \), and by extending the De Morgan operations in the obvious way (in particular, \( \sim \hat{1} = \hat{0} \) and \( \sim \hat{0} = \hat{1} \)).

\[\text{Fig. 2 (All the) subdirectly irreducible IS-algebras}\]

\[\text{Fig. 1 (All the) subdirectly irreducible De Morgan algebras}\]

1 Formally, a Kleene lattice (algebra) is defined as a De Morgan lattice (algebra) that satisfies \( x \land \sim x \leq y \lor \sim y \). It is well known that the variety of Kleene lattices (algebras) is \( V(K_3) \).

2 Formally, a Stone algebra can be defined as a bounded distributive lattice \( (A; \wedge, \vee, \neg, \bot, \top) \) endowed by an extra unary operation \( \neg \) that satisfies, for all \( a, b \in A \), the following requirements: (i) \( a \land b = 0 \) iff \( a \sim b \), and (ii) \( \neg a \lor \neg \neg a = \top \).

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\[ \mathcal{V} = (A \cup \{\hat{0}, \hat{1}\}; \wedge, \vee, \sim, \veevee, \top, \bot) \] as follows:

\[
\begin{align*}
\nabla^A x := & \begin{cases} 
\hat{0} & \text{if } x = \hat{0} \\
\hat{1} & \text{otherwise}
\end{cases} \\
\sim_A x := & \begin{cases} 
\sim x & \text{if } x \in A \\
\hat{0} & \text{if } x = \hat{1} \\
\hat{1} & \text{if } x = \hat{0}
\end{cases} \\
x \wedge_A y := & \begin{cases} 
x \wedge y & \text{if } x, y \in A \\
\hat{1} & \text{if } x = \hat{1} \text{ or } y = \hat{1} \\
\hat{0} & \text{if } x = \hat{0} \text{ or } y = \hat{0} \\
z & \text{if } (x, y) = (\hat{1}, z), z \in A
\end{cases} \\
x \vee_A y := & \begin{cases} 
x \vee y & \text{if } x, y \in A \\
\hat{1} & \text{if } x = \hat{1} \text{ or } y = \hat{1} \\
z & \text{if } (x, y) = (\hat{0}, z), z \in A
\end{cases} \\
\top_A^A := & \hat{1} \quad \bot_A := \hat{0}.
\end{align*}
\]

If \( A \) is a De Morgan algebra, it is clear that the \( \nabla \)-free reduct of \( \mathcal{V} \) is also a De Morgan algebra. Moreover, it is very easy to check that the above-defined \( \vee \) operation satisfies all the properties required by Definition 3.1. Thus, we have the following.

**Proposition 3.4** For every De Morgan algebra \( A \), the above-defined algebra \( \mathcal{V} \) is an IS-algebra.

## 4 Semantical considerations on IS-logics

It is shown in Cantú and Figallo (2020, Thm 5.2) that the order-preserving logic of the variety IS (which we denote by \( \mathcal{IS}_{\leq} \)) coincides with the logic determined by the closure system of all lattice filters on the generating algebra IS\(_6\) (these are the principal up-sets \( \uparrow 0, \uparrow a, \uparrow b, \uparrow 1, \) and \( \{1\} \)). Since the matrices \( IS_6, \uparrow a \) and \( IS_6, \uparrow b \) define the same logic (Cantú and Figallo 2020, Lemma 5.4), we have that \( \mathcal{IS}_{\leq} \) is determined by the following set of matrices: \( \{IS_6, \uparrow 0\}, \{IS_6, \uparrow a\}, \{IS_6, \uparrow 1\}, \{IS_6, \{1\}\} \). This result can be further sharpened, as the following Proposition shows.

**Proposition 4.1** \( \mathcal{IS}_{\leq} = \text{Log} \{IS_6, \uparrow a\} \).

**Proof** Observe that the matrices \( IS_6, \uparrow 1 \) and \( IS_6, \uparrow a \) are reduced, while \( IS_6, \{1\} \) and \( IS_6, \uparrow 0 \) are not. The reduction of \( IS_6, \{1\} \) is isomorphic to \( IS_3, \{1\} \), and is therefore isomorphically reduced to a submatrix of \( IS_6, \uparrow a \). The reduction of \( IS_6, \uparrow 0 \) is isomorphic to \( IS_3, \{0, \hat{1}\} \), which in turn is isomorphic to a submatrix of \( IS_6, \uparrow a \). Thus \( \text{Log} \{IS_6, \uparrow a\}, \{IS_6, \uparrow 1\}, \{IS_6, \{1\}\}, \{IS_6, \uparrow 0\} = \text{Log} \{IS_6, \uparrow a\}, \{IS_6, \uparrow 1\} \). To conclude the proof, it suffices to show that \( \text{Log} \{IS_6, \uparrow a\} \subseteq \text{Log} \{IS_6, \uparrow 1\} \). To see this, notice that \( \uparrow 1 = \uparrow a \cap \uparrow b \). This easily entails that \( \text{Log} \{IS_6, \uparrow a\}, \{IS_6, \uparrow b\} \subseteq \text{Log} \{IS_6, \uparrow 1\} \), and we have seen that \( \text{Log} \{IS_6, \uparrow a\}, \{IS_6, \uparrow b\} = \text{Log} \{IS_6, \uparrow a\} \).

In our study, it will be useful to be able to work with reduced matrix models of IS-logics. Proposition 4.2 suggests that these cannot be characterized by simply applying the Blok-Pigozzi algebraization process, but Proposition 4.4 provides sufficient information for our purposes. (For the definitions of selfextensional, protoalgebraic and algebraizable logic, we refer the reader to Font (2016), respectively, Defs. 5.25, 6.1 and 3.11).

**Proposition 4.2** \( \mathcal{IS}_{\leq} \) is selfextensional and non-protoalgebraic (hence, non-algebraizable).

**Proof** Selfextensionality simply follows from the observation that two formulas \( \varphi, \psi \) are inter-derivable in \( \mathcal{IS}_{\leq} \) if and only if the identity \( \varphi \equiv \psi \) is valid in the variety of IS-algebras. To show that our logic is not protoalgebraic, we verify that the Leibniz operator \( \Omega \) is not monotone on matrix models (Font 2016, Thm. 6.13). To see this, observe that the algebra \( IS_6 \) has (exactly) one non-trivial congruence \( \theta \), which identifies the elements \( \{0, a, b, 1\} \). It is then easy to check that \( \Omega IS_6(\hat{1}) = \theta \). On the other hand, the matrix \( IS_6, \uparrow 1 \) is reduced. Hence, the Leibniz operator is not monotone on the matrix models based on \( IS_6 \).

**Remark 4.3** Proposition 4.2 can in fact be slightly strengthened. If we consider the matrices \( IS_4, \{1\} \) and \( IS_4, \uparrow 1 \), where \( IS_4 \) is the four-element subalgebra of \( IS_6 \) with universe \( \{0, 0, 1, \hat{1}\} \), we can observe that \( IS_4, \uparrow 1 \) is reduced while \( IS_4, \{1\} \) is not. Hence the logic determined by these two submatrices (which is obviously stronger than \( \mathcal{IS}_{\leq} \)) is also non-protoalgebraic. This, in turn, entails that \( \mathcal{IS}_{\leq} \) cannot be protoalgebraic.

The following observation is an instance of a general result on order-preserving logics (see e.g., Albuquerque et al. (2017, Thm. 2.13.iii)).

**Proposition 4.4** \( \text{Alg}(\mathcal{IS}_{\leq}) = IS \).

The next Proposition characterizes the logic determined by the class of matrices \( \{A, \{1A\} : A \in IS\} \). The latter (denoted by \( \text{Log}_I \)) is known in the algebraic literature as the \( I \)-assertional logic of the class \( IS \).

**Proposition 4.5** \( \text{Log}_I IS_3 = \text{Log}_I IS_3, \{\hat{1}\} \).

**Proof** Obviously \( \text{Log}_I IS_3 \subseteq \text{Log}_I IS_3, \{\hat{1}\} \), so it suffices to verify the inclusion \( \text{Log}_I IS_3, \{\hat{1}\} \subseteq \text{Log}_I IS_3 \). Assume \( \Gamma \vdash_{\text{Log}_I IS_3, \{\hat{1}\}} \varphi \). Observe that, since \( \text{Log}_I IS_3, \{\hat{1}\} \) is finitary, we can assume \( \Gamma \) to be finite. Then \( \gamma \vdash_{\text{Log}_I IS_3} \varphi \).
for \( \gamma := \bigwedge \Gamma \). The latter is equivalent to \( \vdash_{\log(\text{IS}_6,\{1\})} \psi \), because, as observed earlier, \( (\text{IS}_6,\{1\}) \approx = (\text{IS}_3,\{1\}) \). In turn, \( \vdash_{\log(\text{IS}_6,\{1\})} \psi \) entails that \( \text{IS}_6 \) satisfies the quasi-identity \( \gamma \approx \top \Rightarrow \phi \approx \top \). By Proposition 3.3, this entails that \( \gamma \approx \top \Rightarrow \phi \approx \top \) is satisfied by every \( A \in \text{IS} \). Hence, \( \vdash \phi \) holds in every matrix in the class \( \{(A,\{1\}_A) : A \in \text{IS}\} \). Thus we have \( \Gamma_1^{\text{IS}} \equiv \phi \), as required. \( \square \)

Recalling that the algebra \( \text{IS}_3 \) is isomorphic to the three-element Łukasiewicz-(Moisil) algebra, Proposition 4.5 tells us that \( \vdash_1 \) is (term equivalent) to three-valued Łukasiewicz logic. This logic is axiomatized, relatively to \( \text{IS}_\leq \), in Theorem 5.16 (i).

We now return to the construction introduced at the end of Sect. 3 and illustrate its remarkable logical consequences. Let \( M = \langle A, D \rangle \) be a matrix, with \( A \) an algebra in the language of De Morgan algebras. Then \( A^\vee \) is in the language of \( \text{IS} \).

**Lemma 4.6** If \( M = \langle A, D \rangle \) is a non-trivial model of \( B \) (i.e., \( D \neq A \)), then \( M^* \equiv (M^*)^* \).

**Proof** Let \( M = \langle A, D \rangle \) be a model of \( B \), so that \( \widehat{M} = \langle \widehat{A}, D \cup \{1\} \rangle \). Let \( h: \widehat{A} \to A \) be the map that is the identity on the elements of \( A \), while \( h(\hat{1}) = 1 \) and \( h(\hat{0}) = 0 \). It is easy to verify that \( h \) is a (surjective) homomorphism, and that \( h^{-1}[D] = D \cup \{1\} \); the latter holds because \( D \neq A \). As is well known (see e.g., Font (2016, Prop. 4.32)), the existence of such a homomorphism entails \( M^* \equiv (M^*)^* \), as required. \( \square \)

Two matrices whose reductions are isomorphic determine the same logic. Hence, the preceding lemma give us the following result.

**Corollary 4.7** For every non-trivial matrix model \( M \) of \( B \), we have \( \log M = \log \widehat{M}^\vee \). Hence, \( \log \widehat{M}^\vee \) is a conservative expansion of \( \log M \).

**Corollary 4.8** \( \text{IS}_\leq \) is a conservative expansion of the Belnap-Dunn logic \( B \).

**Proof** Recall that \( \text{IS}_\leq = \log (\text{IS}_6, \uparrow a) \) (Proposition 4.1), and observe that the matrix \( (\text{IS}_6, \uparrow a) \) can be obtained as \( M^\vee \) from the four-element matrix \( M = (\text{DM}_4, \uparrow a) \) that defines \( B \). Then the result follows from Corollary 4.7. \( \square \)

Corollary 4.7 also gives us the following important result.

**Corollary 4.9** Let \( M_1 \) and \( M_2 \) be matrices for the Belnap-Dunn logic. If \( \log M_1^\vee = \log M_2^\vee \), then \( \log M_1 = \log M_2 \).

Recall that all reduced matrices for the Belnap-Dunn logic (hence, the reduced matrices for super-Belnap logics as well) have the form \( (A, D) \), with \( A \) a De Morgan algebra and \( D \) a lattice filter (Font 1997, Thm. 3.14). Given a super-Belnap logic \( \vdash \), let \( \vdash^\vee = \log \{M^\vee : M \in \text{Matr}^*(\vdash)\} \), where each \( M^\vee = (A^\vee, D \cup \{1\}) \) is defined as before. By Proposition 3.4, we have that \( A^\vee \) is an IS-algebra, and it is clear that \( D \cup \{1\} \) is a lattice filter of \( A^\vee \). Then \( M^\vee \) is a model of \( \text{IS}_\leq \) (Proposition 4.4). Therefore, each logic \( \vdash^\vee \) is an extension of \( \text{IS}_\leq \), and the following lemma shows that \( \vdash^\vee \) is conservative over \( \vdash \).

**Lemma 4.10** Let \( \vdash \) be a super-Belnap logic. Then \( \vdash^\vee \) is a conservative expansion of \( \vdash \).

**Proof** Suppose, in view of a contradiction, that there exist formulas \( \Gamma, \phi \) in the \( \vee \)-free language such that \( \Gamma \vdash^\vee \phi \) but \( \Gamma \not\vdash \phi \). Then there is \( M \in \text{Matr}^*(\vdash) \) such that \( \Gamma \nvdash M \phi \). Then, from \( B \subseteq \vdash \) we have that \( M \) is a non-trivial model of \( B \) and by Corollary 4.7, we obtain that \( \Gamma \nvdash M^\vee \phi \). By definition, \( \vdash^\vee \subseteq \log M^\vee \). Hence, \( \Gamma \not\vdash^\vee \phi \). \( \square \)

**Corollary 4.11** Let \( \vdash_1, \vdash_2 \) be super-Belnap logics. Then \( \vdash_1 \subseteq \vdash_2 \) if and only if \( \vdash_1^\vee \subseteq \vdash_2^\vee \).

**Proof** Assuming \( \vdash_1 \subseteq \vdash_2 \), we have \( \text{Matr}^*(\vdash_2) \subseteq \text{Matr}^*(\vdash_1) \). Hence, \( \{M^\vee : M \in \text{Matr}^*(\vdash_2)\} \subseteq \{M^\vee : M \in \text{Matr}^*(\vdash_1)\} \), which entails \( \vdash_1^\vee \subseteq \vdash_2^\vee \). Conversely, let \( \vdash_1^\vee \subseteq \vdash_2^\vee \), and let \( \Gamma, \phi \) be formulas (in the language of \( B \)) such that \( \Gamma \vdash \phi \).

The latter assumption gives us that \( \Gamma \vdash^\vee \phi \), and, therefore, also \( \Gamma \vdash_2^\vee \phi \). Then, by Lemma 4.10, we conclude \( \Gamma \vdash_2 \phi \). \( \square \)

**Corollary 4.12** The map given by \( \vdash \mapsto \vdash^\vee \) is an embedding of the lattice of super-Belnap logics into the lattice of extensions of \( \text{IS}_\leq \).

**Corollary 4.13** The lattice of extensions of \( \text{IS}_\leq \) has (at least) the cardinality of the continuum.

**Proof** By Corollary 4.12 and the observation that the lattice of super-Belnap logics contains continuum many logics [Albuquerque et al. 2017, Thm. 4.13]. \( \square \)

In fact, in the light of the results of Sect. 5, we shall be able to prove that there are at least continuum many finitary extensions of \( \text{IS}_\leq \).

## 5 Axiomatizing IS-logics

In Cantú (2019), Cantú and Figallo (2020), the logic \( \text{IS}_\leq \) is axiomatized by means of a Gentzen-style calculus. In this Section we tackle the problem of axiomatizing \( \text{IS}_\leq \) and its extensions by means of Hilbert-style calculi. From a technical point of view, we shall take profit from the theory of multiple-conclusion calculi, a generalization of traditional Hilbert-style calculi in which the inference rules can have more than one conclusion (with a disjunctive reading). In these calculi proofs are typically ramified instead of
sequential. Multiple-conclusion calculi can be used to study single-conclusion logics, but also correspond to a generalized notion of logic due to D. Scott and developed by D.J. Shoesmith and T.J. Smiley. We recall some of the basic definitions and results below; for further details see (Shoesmith and Smiley 1978; Marcelino and Caleiro 2021).

A multiple-conclusion consequence relation \( R \) is a relation \( \triangleright \subseteq \varphi(Fm) \times \varphi(Fm) \) satisfying the following conditions. For every \( \Gamma, \Delta, \Delta', \Lambda, T, F \subseteq Fm \),

\begin{enumerate}[(i)]
  \item \( \Gamma \triangleright \Delta \) whenever \( \Gamma \cap \Delta \neq \emptyset \) (overlap),
  \item \( \Gamma, \Gamma' \triangleright \Delta, \Delta' \) whenever \( \Gamma \triangleright \Delta \) (dilution),
  \item \( \Gamma \triangleright \Delta \) whenever \( \Gamma, T \triangleright \Delta, F \) for every partition \( (T, F) \) of \( \Delta \) (cut for sets),
  \item \( \Gamma^\sigma \triangleright \Delta^\sigma \) for every substitution \( \sigma \) whenever \( \Gamma \triangleright \Delta \) (substitution invariance).
\end{enumerate}

We say that a multiple conclusion logic \( \triangleright \) is finitary whenever \( \Gamma \triangleright \Delta \) implies that there are finite \( \Gamma' \subseteq \Gamma \) and \( \Delta' \subseteq \Delta \) such that \( \Gamma' \triangleright \Delta' \).

Given a set of multiple-conclusion rules \( R \subseteq \varphi(Fm) \times \varphi(Fm) \), we denote by \( \triangleright_R \) the smallest multiple-conclusion consequence relation containing \( R \) (hence, \( R \) axiomatizes \( \triangleright_R \)). It is easy to verify that a consequence relation \( \triangleright \) is finitary if and only if \( \triangleright \) is axiomatized by some set \( R \) such that, for every \( \Gamma \subseteq \Delta \), the sets \( \Delta \) and \( \Delta' \) are finite. In such a case we shall say that \( R \) is finitary.

From a proof-theoretic perspective, we have \( \Gamma \triangleright_R \Delta \) whenever there is a labelled tree-proof whose root is labelled by \( \Gamma \) and the leaf of every non-discontinued branch is labelled with a formula in \( \Delta \). Every class of matrices \( \mathcal{M} \) determines a multiple-conclusion logic defined as follows: we let \( \Gamma \triangleright_{\mathcal{M}} \Delta \) whenever, for every valuation \( v \in \text{Val}(\mathcal{M}) \) over a matrix \( \mathcal{M} = (A, D) \in \mathcal{M} \), we have that \( v(\Gamma) \subseteq D \) implies \( v(\Delta) \cap D \neq \emptyset \).

Multiple-conclusion logics smoothly generalize Tarski logics and their proof-theoretic and semantical definitions. Indeed, for every multiple-conclusion logic \( \triangleright \), we have that \( \triangleright_{\mathcal{M}} = \triangleright \cap (\varphi(Fm) \times Fm) \) is a Tarski consequence relation (Font 2016, Def. 1.5). We call \( \triangleright_{\mathcal{M}} \) the single-conclusion companion of \( \triangleright \) and, given a set of multiple-conclusion rules \( R \), we shall write \( \triangleright_R \) instead of \( \triangleright_{\mathcal{M}} \). A single-conclusion logic is said to be finitary whenever \( \triangleright \triangleright \psi \) implies there is some finite \( \Gamma' \subseteq \Gamma \) such that \( \Gamma' \triangleright \psi \). Note that, if \( \triangleright \) is finitary, then \( \triangleright_{\mathcal{M}} \) is finitary as well. The following remark collects a few useful facts that can be easily deduced from Sections 5.2 to 17.3 of Shoesmith and Smiley (1978).

**Remark 5.1** The sign of a multiple-conclusion relation \( \triangleright \) is negative if \( Fm \triangleright \emptyset \) and is positive otherwise. We denote by \( \simeq \) the equivalence relation that identifies two logics \( \triangleright_1 \) and \( \triangleright_2 \) that may differ only in the sign, that is, we let \( \triangleright_1 \simeq \triangleright_2 \) whenever \( \triangleright_1 \cup \{(Fm, \emptyset)\} = \triangleright_2 \cup \{(Fm, \emptyset)\} \). Let \( \triangleright_1 \simeq \triangleright_2 \).

Then \( \triangleright_{\mathcal{M}} = \triangleright_R \) and also, if \( \triangleright_1 \subseteq \triangleright \subseteq \triangleright_2 \), then \( \triangleright_1 \simeq \triangleright_2 \). Let \( \mathcal{P}(\mathcal{M}) \) be the closure under products of the class \( \mathcal{M} \) (products among matrices are defined as usual for first-order structures; see e.g., Font (2016, p. 225-6)). The following observations are well known:

\begin{enumerate}[(i)]
  \item \( \triangleright_{\mathcal{M}} = \mathcal{P}(\mathcal{M}) \).
  \item If \( \mathcal{P}(\mathcal{M}) = \mathcal{P}(\mathcal{M}) \) for \( \mathcal{R} \subseteq \varphi(Fm) \times Fm \), then \( \triangleright_{\mathcal{M}} \simeq \triangleright_{\mathcal{P}(\mathcal{M})} \). Therefore, if \( \mathcal{M}_1 = \mathcal{M}_2 \), then \( \triangleright_{\mathcal{P}(\mathcal{M}_1)} \simeq \triangleright_{\mathcal{P}(\mathcal{M}_2)} \).
\end{enumerate}

Under certain conditions, a finitary single-conclusion axiomatization can be obtained algorithmically from a finitary multiple-conclusion axiomatization; moreover, if the former is finite, then the latter will be finite as well.\(^3\) The next result covers the case of some of the logics that interest us here.

**Remark 5.2** If \( \vdash \) is a single-conclusion \( \Sigma \)-logic and \( \lor \) is a connective in \( \Sigma \), we say that \( \lor \) is a disjunction if, for all \( \Gamma \cup \{\varphi, \psi, \xi\} \subseteq Fm \), we have that \( \Gamma, \varphi \lor \psi \vdash_R \xi \) if and only if \( \Gamma, \varphi \vdash_R \xi \) and \( \Gamma, \psi \vdash_R \xi \). If \( \vdash \) is defined by matrices of type \((A, D)\) whose underlying algebra \( A \) has a lattice reduct (where \( \lor \) is interpreted as the lattice join), then requiring \( \lor \) to be a disjunction amounts to saying that the set of designated elements \( D \) of each non-trivial matrix \((A, D)\) is a prime filter (recall that a lattice filter \( D \subseteq A \) is prime if \( D \neq A \) and \( a \lor b \in D \) entails either \( a \in D \) or \( b \in D \)).

Given a finite set \( \Phi = \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm \) and \( \psi \in Fm \), let \( \Delta := (\varphi_1 \lor \varphi_2 \lor \ldots \lor \varphi_n) \lor \psi \), and let \( \Phi \lor \psi = \{\varphi \lor \psi : \varphi \in \Phi\} \).

**Theorem 5.3** Shoesmith and Smiley (1978, Thm. 5.37) Let \( \mathcal{R} \) be a set of finitary multiple-conclusion rules. If \( \lor \) is a disjunction for \( \vdash_{\mathcal{R}} \), then \( \vdash_{\mathcal{R}} \) is axiomatized by the set \( \mathcal{R}^\lor \) consisting of the following rules:

\begin{enumerate}[(i)]
  \item \( r^\lor = r \) for each \( r \in R \in \mathcal{R} \),
  \item \( \frac{\Gamma \lor \psi}{\Gamma^\lor} \) for each \( r \in \mathcal{R} \),
  \item \( \frac{\Gamma^\lor \lor \psi}{\Gamma^\lor} \) and \( \frac{\Gamma^\lor \lor \psi}{\Gamma^\lor \lor \psi} \) when \( \psi \lor \psi \)
\end{enumerate}

where \( p_0 \) is a variable not occurring in \( R \).

In the next Section, we proceed to explain how the results of Sect. 4 together with the general considerations on multiple-conclusion logics introduced above help us deal with extensions of \( \mathcal{IS}_\Sigma \).

\(^3\) This recipe is applicable as long as every multiple-conclusion rule being translated has a finite conclusion-set.
5.1 Adding $\nabla$ to the Belnap-Dunn logic

Let $\Sigma = \{\land, \lor, \neg, \top, \bot\}$ be the language of $\mathcal{B}$, and let $\Sigma^V$ be the expansion of $\Sigma$ with the unary connective $\nabla$ (thus $\Sigma^V$ is the language of $\mathcal{L}_{\Sigma^V}$). Given a $\Sigma$-matrix $\mathbb{M} = (A, D)$, let $\mathbb{M}^V = (A^V, D \cup \{\top\})$ be the $\Sigma^V$-matrix with underlying algebra $A^V$ defined as in Sect. 3 (cf. Proposition 3.4). Let us denote by $\mathbb{M}$ the $\Sigma$-fragment of $\mathbb{M}^V$. Observe that, if $\mathbb{M} = (A, D)$ with $A$ a De Morgan algebra, then $\mathbb{M}$ is precisely the matrix considered in Corollary 4.7; moreover, if $D$ was a prime filter, then $D \cup \{\top\}$ will also be a prime filter (cf. Remark 5.2). Given a class of $\Sigma$-matrices $\mathcal{M}$, we let $\mathcal{M}^V := \{\mathbb{M}^V : \mathbb{M} \in \mathcal{M}\}$ and $\mathcal{M}^v := \{\mathbb{M}^V : \mathbb{M} \in \mathcal{M}\}$.

The following Theorem contains a generic recipe for axiomatizing the multiple-conclusion logic determined by the class $\mathcal{M}^V$, assuming we have a set of rules $R$ that axiomatizes the multiple-conclusion logic determined by $\mathcal{M}$.

**Theorem 5.4** Let $\mathcal{M}$ be a class of $\Sigma$-matrices. If $\mathcal{M}^V \simeq \mathcal{M}_R$, then $\mathcal{M}^v = \mathcal{M}^v_R \cup \mathcal{M}_R$, where $\mathcal{M}_R$ consists of the following rules:

- $\nabla \varphi \vdash \nabla \varphi$ (rule r1)
- $\neg \nabla \varphi \vdash \neg \nabla \varphi$ (rule r2)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r3)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r4)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r5)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r6)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r7)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r8)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r9)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r10)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r11)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r12)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r13)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r14)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r15)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r16)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r17)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r18)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r19)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \lor \psi)$ (rule r20)
- $\nabla \varphi, \nabla \psi \vdash \nabla (\varphi \land \psi)$ (rule r21)

**Proof** Checking the soundness of the new rules is the usual way. We give only a couple of examples. Let $v$ be a valuation over a matrix $M^V$. The rule r1 is sound in $M^V$, for either $v(\nabla \varphi) = 1$ (if $v(\varphi) = 0$) or $v(\neg \nabla \varphi) = 1$ (if $v(\varphi) = 0$). Regarding rule r2, we have that, if $v(\nabla \varphi) = 1$ then $v(\neg \nabla \varphi) = v(\neg \varphi) = 0$, so $v(\neg \nabla \varphi) = 1$.

For completeness, assume $\Gamma \not\vdash_{R_{\Sigma^V}} \Delta$. Then, by cut for sets, there is a partition $(T, F)$ of $\text{FM}_{\Sigma^V}$ such that $\Gamma \subseteq T$ and $\Delta \subseteq D$ and $T \not\vdash_{R_{\Sigma^V}} F$. Note that (by r1 and r4) for each $\varphi$, we have either $\nabla \varphi \in T$ or $\nabla \varphi \in T$, but never both. In particular, $D$ is never empty. Also, by r5 we must have either $\nabla \varphi \in T$ or $\nabla \varphi \in T$. Hence, each $\varphi$ must be exactly in one of three cases: (i) $\nabla \varphi \in T$, (ii) $\nabla \varphi \in T$, or (iii) $\nabla \varphi \in T$.

Since $R \subseteq R \cup R_{\Sigma^V}$, we also have $T \not\vdash_{R_{\Sigma^V}} F$. From the fact that $\nabla \varphi \not\in \mathcal{M}^v$ and $F \not\in \mathcal{M}$ we know that $\nabla \varphi \not\in \mathcal{M}^v$.

We can therefore pick $v \in \text{Hom}_{\Sigma} (\text{FM}_{\Sigma^V}, \mathcal{M}^v)$, for some $\mathbb{M} \in \mathcal{M}$ such that $v(T) \subseteq D$ and $v(F) \cap D = \emptyset$. The function $v$ may not be a valuation over $M^V$, for $v$ treats formulas of the form $\nabla \varphi$ as variables, but this can be fixed. Consider $v' : \text{FM}_{\Sigma^V} \rightarrow M^V$ defined by:

$$
v'(\varphi) := \begin{cases} 
\hat{1} & \text{if } \nabla \varphi \in T \\
\hat{0} & \text{if } \nabla \varphi \in T
\end{cases}
$$

We are going to show that $v' \in \text{Val}(M^V) = \text{Hom}_{\Sigma^V} (\text{FM}_{\Sigma^V}, M^V)$.

1. $v'(\nabla \varphi) = \nabla v'(\varphi)$. From $r_1$ we have that either (ii) $\nabla \varphi \in T$ or (iii) $\nabla \varphi \in T$.

2. Given $r_3$ if $\nabla \varphi \in T$ we have that $\nabla \varphi \in T$ and so $v'(\varphi) = \hat{1}$. Further, by $r_2$ we obtain that $\nabla \varphi \in T$, hence $v'(\varphi) = \hat{1} = \nabla v'(\varphi)$.

3. If (ii) $\nabla \varphi \in T$ then $v'(\varphi) = 0$ and $v'(\varphi) = \hat{1}$, thus we immediately obtain $v'(\varphi) = \nabla v'(\varphi)$.

4. $v'(\varphi \land \psi) = v'(\varphi) \land v'(\psi)$. If (ii) we have that $v'(\varphi \land \psi) = \nabla v'(\varphi \land \psi) \subseteq T$. From $v'(\varphi \land \psi) \in T$, by $r_3$ and $r_4$ we obtain that $\nabla \varphi, \nabla \psi \in T$ if $(\varphi \land \psi) \in T$ (so $v'(\varphi) = \hat{1}$ or $v'(\psi) \in T$). Also, from $\nabla \varphi \land \psi \in T$, by $r_{10}$, either $\nabla \varphi \in T$ or $\nabla \psi \in T$ and hence $v'(\varphi \land \psi) = v'(\varphi) \land v'(\psi)$. Otherwise, if $\nabla \varphi \not\in T$, then by $r_1$ we conclude that $\nabla \varphi \not\in T$ and hence $v'(\varphi \land \psi) = v'(\varphi) \land v'(\psi)$. The case $\nabla \psi \not\in T$ is similar.

5. $v'(\varphi \lor \psi) = v'(\varphi) \lor v'(\psi)$.
Let $R$ be a set of single-conclusion rules. Then $RV$ is a set of multiple-conclusion rules and we abbreviate $\vdash_{R,RV}$ to $\vdash_{R,RV}$. Note that if all the rules in $R$ are finitary, then so are all the rules in $RV$.

**Corollary 5.8** Let $M$ be a class of $\Sigma$-matrices, and let $R$ be a set of single-conclusion rules. If $\vdash_R = \vdash_M = \vdash_{R,M}$, then $\vdash_{R,RV} = \vdash_{M,V}$. 

**Proof** By Remark 5.1 (ii), from $\vdash_M = \vdash_{R,M}$ we have that $\vdash_{R,M}$ is the set of multiple-conclusion rules and we abbreviate $\vdash_{R,M}$ to $\vdash_{R,M}$. By Theorem 5.4 we know that $\vdash_{R(M)} \subseteq \vdash_{R,RV}$. Finally, from Lemma 5.7 (ii) we conclude that $\vdash_{M,V}$ is axiomatized by $R \cup RV$. 

The preceding result implies that $\vdash_{M,V}$ is finitary when $\vdash_{M}$ is. Since the lattice of super-Belnap logics contains continuum many finitary logics (Prenosil 2021, Cor. 8.17), by Lemma 4.10, we obtain the following sharpening of Corollary 4.13.

**Proposition 5.9** There are (at least) continuum many finitary extensions of $\mathcal{IS}_\Sigma$. 

By joining Theorem 5.3 and Corollary 5.8 we obtain a recipe for capturing the effect of adding $\bigvee$ to single-conclusion axiomatizations of logics determined by matrices having prime filters as designated elements. The latter requirement stems from the observation that, on every matrix whose designated elements form a prime filter, the connective $\bigvee$ will be a *disjunction* in the sense of Remark 5.2; if this is the case, then we are able to translate, *salva veritate*, the set $\Delta$ appearing in a multiple-conclusion rule $\Gamma \vdash \Delta$ as the formula $\bigvee \Delta$.

**Corollary 5.10** Let $M$ be a class of $\Sigma$-matrices that are models of $\mathcal{B}$, whose designated sets are prime filters, and let $R$ be a set of single-conclusion rules. If $\vdash_R = \vdash_M$, then $\vdash_{R,RV} = \vdash_{M,V}$. 

**Proof** Recall that every prime filter is proper, so all matrices in $M$ are non-trivial. By Corollary 4.6 we have that $\vdash_M = \vdash_{R,M}$. Since the designated sets are prime filters, the result follows by Corollary 5.8 and Theorem 5.3. 

**Example 5.11** Let $M_4 = (\mathcal{DM}_4, \uparrow a)$ be the four-element matrix that defines the Belnap-Dunn logic $\mathcal{B}$, and observe that the filter $\uparrow a$ is prime. From Corollary 5.10 we can obtain a Hilbert-style axiomatization for $\mathcal{IS}_\Sigma = \vdash_{M_4}$. Let $R_B$ be the Hilbert-style calculus used in Font (1997) to axiomatize $\mathcal{B}$ (expanded with the rules introduced in Albuquerque et al. (2017, p. 1065) to account for
the constants:

\[
\begin{array}{cccc}
  p \land q & p \land q & p \land q \\
  p & q & p & q \\
  p & p \lor q & p \lor p & p \lor (q \land r) \\
  p \lor (q \land r) & (p \lor q) \land (p \land r) & p & (p \lor q) \land r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
  (p \lor q) \land r & (p \lor q) \lor r & (p \lor q) \lor r \\
\end{array}
\]

It follows that \( IS\leq \) is axiomatized by \((R_B \cup R_V)^V\), which is the result of adding to \( R_B \) the following rules:

\[
\begin{align*}
  \vdash p \lor \neg p & \quad \text{(DS)} \\
  \vdash (p \lor \neg p) \lor q & \quad \text{(K)} \\
  \vdash (p \lor \neg p) \lor q & \quad \text{(K)} \\
  \vdash \emptyset \vdash p \lor \neg p & \quad \text{(EM)}
\end{align*}
\]

(Regarding the names of the above rules, the \( K \)'s are suggestive of Kleene's logics, (DS) stands for Disjunctive Sylogism and (EM) for Excluded Middle.)

**Proposition 5.12** Let \( M \) be a class of models of \( B \) having prime filters as designated elements. If \( \log M \) is axiomatized relative to \( B \) by a set of single-conclusion rules \( R \), then \( \log M^V \) is also axiomatized by \( R \) relative to \( B^V \).

The super-Belnap logics considered below are G. Priest's Logic of Paradox \( \mathcal{LP} \), the two logics \( K_\leq \) and \( K_1 \) named after S. C. Kleene, and classical logic \( \mathcal{C\mathcal{L}} \). \( K_\leq \) is the order-preserving logic of the variety of Kleene algebras, and \( K_1 \) is the 1-assertional logic associated to the same variety (see Albuquerque et al. 2017 for further details). Proposition 5.13 shows that each of these logics can be axiomatized, relative to \( B \), by a combination of the following rules:

\[
\begin{align*}
  p \land (p \lor q) & \vdash q \\
  (p \land q) \lor q & \vdash q \\
  p \land (p \lor q) & \vdash q \\
\end{align*}
\]

As observed earlier, \( IS\leq \vdash M \), where \( M \) is the following class of matrices:

\[
M := \{(A, D) : A \in IS, D \subseteq A \}
\]

is a (non-empty) lattice filter).

Thus, each subclass \( M' \subseteq M \) (it suffices to consider those \( M' \) consisting of reduced matrices) defines a logic \( \vdash_{M'} \) which is an extension of \( IS\leq \). We have seen with Corollary 4.13 that there are at least continuum many of these, and Corollary 4.12 suggests that the structure of the lattice of extensions of \( IS\leq \) is quite complex (see Albuquerque et al. 2017; Pfenosil 2021) for analogous considerations on the lattice of super-Belnap logics). A systematic study of this lattice lies outside the scope of the present paper and even beyond our present grasp on IS-logics; however, in this subsection we consider a few extensions of \( IS\leq \) that are defined by substructures of \((IS_5, \uparrow a)\), illustrating how our methods can be used to axiomatize them.

The following result, which is an immediate consequence of Corollary 5.10, shows that Example 5.11 smoothly generalizes to all super-Belnap logics that are determined by matrices having prime filters as designated elements.

**Proposition 5.12** Let \( M \) be a class of models of \( B \) having prime filters as designated elements. If \( \log M \) is axiomatized relative to \( B \) by a set of single-conclusion rules \( R \), then \( \log M^V \) is also axiomatized by \( R \) relative to \( B^V \).

The super-Belnap logics considered below are G. Priest's Logic of Paradox \( \mathcal{LP} \), the two logics \( K_\leq \) and \( K_1 \) named after S. C. Kleene, and classical logic \( \mathcal{C\mathcal{L}} \). \( K_\leq \) is the order-preserving logic of the variety of Kleene algebras, and \( K_1 \) is the 1-assertional logic associated to the same variety (see Albuquerque et al. 2017 for further details). Proposition 5.13 shows that each of these logics can be axiomatized, relative to \( B \), by a combination of the following rules:

\[
\begin{align*}
  p \land (p \lor q) & \vdash q \\
  (p \land q) \lor q & \vdash q \\
  p \land (p \lor q) & \vdash q \\
\end{align*}
\]

(Regarding the names of the above rules, the \( K \)'s are suggestive of Kleene's logics, (DS) stands for Disjunctive Sylogism and (EM) for Excluded Middle.)

**Proposition 5.13** (Albuquerque et al. 2017, Thm. 3.4)

(i) \( \mathcal{LP} = \log (K_3, \uparrow a) = B+(EM) \).

(ii) \( K_1 = \log (K_3, \{1\}) = B+(K_1) \).

(iii) \( K_\leq = \log (K_3, \uparrow a), (K_3, \{1\}) \} = B+(K_\leq) \).

(iv) \( \mathcal{C\mathcal{L}} = \log (B_2, \{1\}) = B+(DS)+(EM) \).

Notice that all the above matrices are defined over linearly-ordered Kleene algebras, where all filters are prime; hence Proposition 5.12 applies.

**Theorem 5.14** For logics above \( IS\leq \) we have the following relative axiomatizations:

(i) \( \log (IS_3, \uparrow a) = \mathcal{LP}^V = IS\leq + (EM) \).

(ii) \( \log (IS_5, \uparrow 1) = K_\leq^V = IS\leq + (K_1) \).

(iii) \( \log (IS_5, \uparrow a), (IS_5, \uparrow 1) \} = K_\leq^V = IS\leq + (K_\leq) \).
(iv) \( \text{Log } (\mathcal{IS}_4, \uparrow 1) = CL^\lor = IS_{\leq} + (DS) + (EM). \)

**Proof** The statement follows directly from Proposition 5.12 and Proposition 5.13, having noticed that, for \( x \in \{1, a\} \), we have \( (\mathcal{DM}_1, \uparrow x) = (\mathcal{IS}_5, \uparrow x) \), \( (\mathcal{K}_3, \uparrow x) = (\mathcal{IS}_5, \uparrow x) \) and \( (\mathcal{B}_2, \uparrow 1) = (\mathcal{IS}_4, \uparrow 1). \)

5.3 Other extensions of \( IS_{\leq} \)

In this Subsection we consider a few examples of extensions of \( IS_{\leq} \) (defined by substructures of the matrix \( (\mathcal{IS}_6, \uparrow a) \)) that do not have the form \( L^\lor \) for some super-Belnap logic \( L \).

Given a \( \Sigma \)-matrix \( M = (A, D) \), a set of axioms \( A \subseteq Fm_\Sigma \) and a set of rules \( R \subseteq \varphi(Fm_\Sigma) \times Fm_\Sigma \), we write \( \text{Val}^M \) for the set of valuations on \( M \) such that \( \varphi^\sigma = D \) for every \( \varphi \in A \) and every substitution \( \sigma \), and \( \text{Val}^R \) for the set of valuations on \( M \) such that \( \varphi^\sigma = D \) implies \( \varphi^\sigma \in D \) for every \( \varphi \in R \) and substitution \( \sigma \).

The following result (whose simple proof we omit) is a corollary of (Caleiro et al. 2019, Lemma 2.7) and will be very useful to show relative axiomatization results (this technique is used in Caleiro and Marcellino (2021) to obtain general modular semantics for axiomatic extensions of a given logic).

In item (ii), \( M^\omega \) is a shorthand for \( \prod_{i : < \omega} M_i \).

**Proposition 5.15** Let \( M = (A, D) \) be a \( \Sigma \)-matrix. Given \( A \subseteq Fm_\Sigma \) and \( R \subseteq \varphi(Fm_\Sigma) \times Fm_\Sigma \). We have that:

(i) \( \text{Val}^M \) is a complete semantics for \( \text{Log } M + A \).

(ii) \( \text{Val}^R \) is a complete semantics for \( \text{Log } M + R \).

We are now ready to give an axiomatization relative to \( IS_{\leq} \), the 1-assertional logic of the class \( IS \) (i.e., three-valued Lukasiewicz-Moisil logic; cf. Proposition 4.5). This is the first item of Theorem 5.16. The logic axiomatized by the second item is the order-preserving logic of the variety \( V(IS_3) \) of three-valued Lukasiewicz-Moisil algebras, which is also \( \text{Log } (IS_3, \{1\}), (IS_3, \uparrow 0) \). Recall that all the logics mentioned below are given by matrices over the linearly-ordered algebra \( IS_3 \), on which all filters are prime.

**Theorem 5.16**

(i) \( \text{Log } (IS_6, \{1\}) = \text{Log } (IS_5, \{1\}) = \text{Log } (IS_4, \{1\}) = \text{Log } (IS_3, \{1\}) = \vdash_{IS} \equiv IS_{\leq} + p \vdash \sim \lor \sim p. \)

(ii) \( \text{Log } (IS_6, \{1\}), (IS_3, \uparrow 0) = \vdash_{IS} \equiv IS_{\leq} + (K_\leq) + \sim p \lor r, \sim \lor p \vdash r \sim r + \sim \lor p \lor r \vdash \sim p \lor r. \)

(iii) \( \text{Log } (IS_6, \{1\}) = \text{Log } (IS_5, \{1\}) = \text{Log } (IS_4, \{1\}) = \text{Log } (IS_3, \{1\}) = \vdash_{IS} \equiv IS_{\leq} + p \vdash \lor \sim p. \)

**Problem 5.17** Is it possible (by some other ad hoc method) to axiomatize using single-conclusion rules \( IS \)-logics arising by adding \( \lor \) to super-Belnap logics that are not determined by matrices having prime filters as designated elements? An example of the latter is every logic \( \mathcal{E}T \mathcal{L}_n \) for \( n < \omega \) considered in Albuquerque et al. (2017).

5.4 Analytic calculi

Let \( A \subseteq Fm \) and let \( R \) be a set of multiple-conclusion rules. We write \( \Gamma >^A_\Phi \Delta \) when there exists an \( R \)-proof of \( \Delta \) from \( \Gamma \) where only formulas in \( A \) occur. Let \( \Phi \subseteq Fm \). We say that \( R \) is \( \Phi \)-analytic if when \( \Gamma >^A_\Phi \Delta \) then \( \Gamma >^A_\Phi \Delta \) with \( \forall \gamma = \text{sub} (\Gamma \cup \Delta) \) and \( \forall \gamma = \gamma \cup \gamma \). \( A \) : \( A \in \Phi \). \( \sigma : P \rightarrow \gamma \). Intuitively, this means that an \( R \)-proof of \( \Delta \) from \( \Gamma \) needs only to use formulas which are subformulas of \( \Gamma \cup \Delta \) or, instances of \( \Phi \) with such subformulas. Hence, formulas in \( \gamma \) can be seen as ‘generalized subformulas’.

Given \( \Phi \subseteq Fm \), let \( \Phi^\lor = \Phi \cup \{ \lor p, \sim \lor p, \lor \sim p, \sim \lor \sim p \}. \)

\footnote{Note that in general \( >^A_\Phi \) is not a multiple-conclusion consequence relation. It still satisfies dilution and cut for set properties, but only weaker versions of overlap and substitution invariance.}
Theorem 5.18 Let $\mathcal{M}$ be a class of $\Sigma$-matrices. If $\mathcal{R}$ is an $\Phi$-analytic axiomatization of $\mathcal{S}_{\mathcal{M}}$, then $\mathcal{R} \cup \mathcal{R}_\Phi$ is an $\Phi^\top$-analytic axiomatization of $\mathcal{S}_{\mathcal{M}^\top}$.

Proof The proof can be easily obtained by adapting the proof of Theorem 5.4. Let $\mathcal{Y} = \text{sub}(\mathcal{H} \cup \Delta)$ and $\Lambda = \mathcal{Y}_{\Phi^\top}$. Assume $\Gamma \not\vdash_{\mathcal{R}_\Phi} \Delta$. Then, by cut for sets, there is a partition $(T, F)$ of $\Lambda$ such that $\Gamma \subseteq T$, $\Delta \subseteq D$ and $T \not\vdash_{\mathcal{R}_\Phi} F$. Since $\mathcal{R}_\Phi \subseteq \mathcal{Y}_{\Phi^\top}$, by $\Phi$-analyticity of $\mathcal{R}$ we know that $T \not\vdash_{\mathcal{R}_\Phi} F$. Therefore, since $\mathcal{Y}_{\Phi^\top}$, by $\Phi$-analyticity of $\mathcal{R}$, we have that $T \not\vdash_{\mathcal{M}} D$ and we can pick $v \in \text{Hom}_\Sigma(FM_{\mathcal{M}^\top}, M)$ for some $M \in \mathcal{M}$ such that $v(T) \subseteq V$ and $v(F) \cap V = \emptyset$. Noting that for every $A \in \mathcal{Y}$ we have $\nabla A, \nabla \sim A, \nabla \nabla A, \nabla \nabla \nabla A \in T_{\mathcal{M}^\top} = \Lambda$, we can define $v': Y \rightarrow M^V$ as in Theorem 5.4. That $v'$ respects all the connectives (and is therefore a partial $M^V$ valuation) follows from the fact that in the proof of Theorem 5.4 we only used instances of the rules using formulas in $\mathcal{Y}$ yielding formulas in $T_{\mathcal{M}^\top} = \Lambda$. As $M$ is a matrix, $v'$ can be extended to a total valuation and therefore $\Gamma \not\vdash_{\mathcal{M}^\top} \Delta$, thus concluding the proof. □

The papers (Marcelino and Caleiro 2019, 2021) introduced a general method for obtaining analogical calculi for logics given by (partial non-deterministic) matrices whenever a certain expressiveness requirement is met. In particular, for the logic determined by a single matrix $M = \langle A, D \rangle$, it suffices that $M$ be monadic (Shoesmith and Smiley 1978, p. 265). This means that, for all $x, y \in A$ with $x \neq y$, there is a one-variable separating formula, that is, a formula $\varphi(p)$ such that $\varphi(x) \in D$ and $\varphi(y) \notin D$ (or vice versa).

From now on, let us fix the separating set $S := \{p, \sim p\}$. Applying the above-described method, we obtain the following axiomatization for $\mathcal{B}$.

Example 5.19 The matrix $(\text{DM}_4, \uparrow a)$ is monadic with set of separators $S$. We can therefore apply the method introduced in Marcelino and Caleiro (2021) to obtain the following $\mathcal{S}$-analytic axiomatization for $\mathcal{B} = \text{Log}(\text{DM}_4, \uparrow a)$:

\[
\begin{array}{lll}
\hline
p & \sim p & p \\
\hline
p & \sim p & p \\
\hline
\hline
p & \sim p & p \\
\hline
\hline
p & \sim p & p \\
\hline
p & \sim p & p \\
\hline
\hline
p & \sim p & p \\
\hline
\hline
\end{array}
\]

Note that this axiomatization coincides with the one presented in Pfenrisl (2021, Section 9). Theorem 5.18 then tells us that we can obtain an $S_{\mathcal{V}}$-analytic axiomatization of $\mathcal{S}_{\mathcal{S}_{\mathcal{V}}}$ by adding $R_{\mathcal{V}}$ to the above rules.

Example 5.20 In Marcelino and Caleiro (2019, Example 5) we showed that the following rules provide an $\mathcal{S}$-analytic axiomatization of Kleene’s logic of order $\mathcal{K}_{\leq}$:

- $p, q \vdash p \land q$,
- $q, p \vdash p \land q$,
- $p, q \vdash p \lor q$,
- $p \vdash p$,
- $q \vdash q$,
- $p \vdash p$,
- $p \vdash p$,
- $q \vdash q$.

Since $\{\langle IS_5, \uparrow 1 \rangle, \langle IS_5, \uparrow a \rangle\} = \{\langle K_3, \{1\} \rangle, \langle K_3, \uparrow a \rangle\}^\top$, by Theorem 5.18, we have that adding $R_{\mathcal{V}}$ to the above rules gives us an $S_{\mathcal{V}}$-analytic axiomatization of

\[\text{Log}(\langle IS_5, \uparrow 1 \rangle, \langle IS_5, \uparrow a \rangle)\]

The method of Marcelino and Caleiro (2021) can also be applied directly to obtain an $\mathcal{S}$-analytic axiomatization of $\mathcal{S}_{\mathcal{V}}$-isomorphic to $\text{Log}(\langle IS_3, \{1\} \rangle, \langle IS_3, \uparrow 0 \rangle)$. Indeed, the $\mathcal{V}$-free fragment of $\text{Log}(\langle IS_3, \{1\} \rangle, \langle IS_3, \uparrow a \rangle)$ is $\text{Log}(\langle K_3, \{1\} \rangle, \langle K_3, \uparrow a \rangle)$. The latter set of matrices can be viewed as a partial matrix (Marcelino and Caleiro 2021, Section 2.2) and is monadic with separating set $S$ (in which $\mathcal{V}$ does not occur). Therefore, the modularity of the method of Marcelino and Caleiro (2021) tells us just have to add the rules corresponding to $\mathcal{V}$. Hence, it suffices to add the rules:

\[
\begin{array}{llllllllllllllllllll}
\hline
\sim p, \nabla p & \vdash_{s_{16}} & \sim p & \vdash_{s_{17}} & p & \vdash_{s_{18}} \\
\hline
p, \nabla p & \vdash_{s_{19}} & \nabla p & \vdash_{s_{20}} & \nabla p & \vdash_{s_{21}} \\
\hline
\hline
\end{array}
\]

which the single-conclusion axiomatization relative to $\mathcal{S}_{\mathcal{S}_{\mathcal{V}}}$ mentioned in Theorem 5.16 (ii).

6 Conclusions and future work

As we have shown, the lattice of super-Belnap logics is embeddable in the lattice of extensions of $\mathcal{S}_{\mathcal{S}_{\mathcal{V}}}$. This con-
nection provides significant insight, but it also suggests that fully describing the latter is at least as complex as describing the former, whose structure is still not completely understood (see Albuquerque (2017)). Obviously, in the present study we have only scratched the surface of the general problem. A reasonable starting point for a systematic account of the extensions of $\mathcal{IS}_{\leq}$ would be to adapt the various results and strategies in Rivieccio (2012), Albuquerque et al. (2017), Pfenosil (2021) to the richer setting of involutive Stone algebras. We mention, in particular, the issues of characterizing the reduced models of extensions of $\mathcal{IS}_{\leq}$, and that of providing a general semantical description of the explosive extensions (in the sense of Albuquerque et al. (2017), Pfenosil (2021)) of logics over $\mathcal{IS}_{\leq}$.

An altogether different perspective on extensions of $\mathcal{IS}_{\leq}$, which has not been considered in the present paper, comes from the observation made in Cantú and Figallo (2020, Section 6) that $\mathcal{IS}_{\leq}$ may be viewed as a paraconsistent logic, more precisely as a Logic of Formal Inconsistency (LFIs) in the sense of da Costa (1963). Indeed, $\mathcal{IS}_{\leq}$ (and so its extensions) can be equivalently presented in a language that replaces the $\forall$ connective with either the consistency operator ($\circ$) or the inconsistency operator ($\bullet$) that are usually considered in the literature on LFIs. One possible definition is $\forall \varphi := \neg \circ \varphi \lor \varphi$, and, conversely, one may define $\circ \varphi := \neg \forall (\varphi \land \neg \varphi)$ and $\bullet \varphi := \forall (\varphi \land \neg \varphi)$.

From a philosophical logic point of view, the advantage of the latter presentation is that the operators $\circ$ and $\bullet$ have a clearer logical interpretation than $\forall$, namely, $\circ \varphi$ means ‘$\varphi$ is consistent’ and $\bullet \varphi$ means ‘$\varphi$ is inconsistent’; on the other hand, $\forall$ behaves very well from the points of view of algebraic logic (and duality theory), for it satisfies the usual axioms for modal operators. A more interesting observation is that, in the setting of LFIs, the (in)consistency operators are usually required to satisfy much weaker axioms than those that result from the definitions $\circ \varphi := \neg \forall (\varphi \land \neg \varphi)$ and $\bullet \varphi := \forall (\varphi \land \neg \varphi)$ within $\mathcal{IS}_{\leq}$. This suggests a potentially fruitful project for future research: namely, a systematic study of more general algebraic structures (e.g. De Morgan algebras endowed with a consistency operator) corresponding to weaker logics (viewed as LFIs) that approximate $\mathcal{IS}_{\leq}$ from below.

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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