General Existence Results for Reflected BSDE and BSDE

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Abstract

In this paper, we are concerned with the problem of existence of solutions for generalized reflected backward stochastic differential equations (GRBSDEs for short) and generalized backward stochastic differential equations (GBSDEs for short) when the generator $f_{ds} + g_{dAs}$ is continuous with general growth with respect to the variable $y$ and stochastic quadratic growth with respect to the variable $z$. We deal with the case of a bounded terminal condition $\xi$ and a bounded barrier $L$ as well as the case of unbounded ones. This is done by using the notion of generalized BSDEs with two reflecting barriers studied in [15]. The work is suggested by the interest the results might have in finance, control and game theory.

Keywords: Generalized reflected BSDE; generalized BSDE; stochastic quadratic growth; Itô-Tanaka formula.

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1 Introduction

Originally motivated by questions arising in stochastic control theory, backward stochastic differential equations have found important applications in fields as stochastic control, mathematical finance, Dynkin games and the second order PDE theory (see, for example, [11,17,27,26,8,9] and the references therein).

The particular case of linear BSDEs have appeared long time ago both as the equations for the adjoint process in stochastic control, as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. However the notion of nonlinear BSDEs has been introduced in 1990 by Pardoux and Peng [26]. A solution for such an equation is a couple of adapted processes $(Y, Z)$ with values in $\mathbb{R} \times \mathbb{R}^d$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T.$$  (1.1)

In [26], the authors have proved the existence and uniqueness of the solution under conditions including basically the Lipschitz continuity of the generator $f$.

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From the beginning, many authors attempted to improve the result of [26] by weakening the Lipschitz continuity of the coefficient \( f \), see e.g. [1] [2] [3] [10] [16] [18] [20] [24] [7], or the \( L^2\)-integrability of the initial data \( \xi \), see [11] [5].

When the generator \( f \) is only continuous there exists a solution to Equation (1.1) under one of the following group of conditions:

- \( \xi \) is square integrable and \( f \) has an uniform linear growth in \( y, z \) (see Lepeltier and San Martin [23]).
- \( \xi \) is bounded and \( f \) has a superlinear growth in \( y \) and quadratic growth in \( z \), i.e. there exist a positive constant \( C \) and a positive function \( \phi \), such that
  \[
  |f(t, \omega, y, z)| \leq \phi(y) + C|z|^2,
  \]
  where \( \int_0^{+\infty} \frac{ds}{\phi(s)} = \int_{-\infty}^0 \frac{ds}{\phi(s)} = \infty \) (see Lepeltier and San Martin [22]; Kobylanski [20]).
- \( \xi \) is bounded and \( f \) satisfies the following condition
  \[
  |f(t, \omega, y, z)| \leq C + R_t|z| + \frac{1}{2}|z|^2,
  \]
  where \( C \) is a positive constant and \( R \) is a square integrable process with respect to the measure \( dt dP \) (see Hamadène and El Karoui [13]).
  - There exist two constants \( \beta \geq 0 \) and \( \gamma > 0 \) together with a progressively measurable nonnegative stochastic process \( \{\alpha(t)\}_{t \leq T} \) and a deterministic continuous nondecreasing function \( \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \varphi(0) = 0 \) such that, \( P \)-a.s.,
    1. for each \((t, y, z)\), \( g(f(t, y, z) - f(t, 0, z)) \leq \beta|y|^2 \),
    2. for each \((t, y, z)\), \( |f(t, \omega, y, z)| \leq \alpha(t) + \varphi(|y|) + \frac{\gamma}{2} |z|^2 \),
    3. \( \mathbb{E}e^{\gamma t\alpha(T)}(\xi + \int_0^T \alpha(s)ds) < +\infty \),
   (see Briand and Hu [6]).

The notion of reflected BSDE has been introduced by El Karoui et al [12]. A solution of such an equation, associated with a coefficient \( f \); a terminal value \( \xi \) and a barrier \( L \), is a triple of processes \((Y, Z, K)\) with values in \( \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+ \) satisfying

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \quad Y_t \geq L_t \quad \forall t \leq T. \tag{1.2}
\]

Here the additional process \( K \) is continuous non-decreasing and its role is to push upwards the process \( Y \) in order to keep it above the barrier \( L \) and moreover it satisfies \( \int_0^T (Y_s - L_s) dK_s = 0 \), this means that the process \( K \) acts only when the process \( Y \) reaches the barrier \( L \). Once more under square integrability of the terminal condition \( \xi \) and the barrier \( L \) and Lipschitz property of the coefficient \( f \), the authors have proved that Equation (1.2) has a unique solution.

When the generator \( f \) is only continuous there exists a solution to Equation (1.2) under one of the following group of conditions:

- \( \xi \) and \( L \) are square integrable and \( f \) has an uniform linear growth in \( y, z \) (see Matoussi [24]).
- \( \xi \) and \( L \) are bounded and \( f \) has a superlinear growth in \( y \) and quadratic growth in \( z \), i.e. there exist a positive constant \( C \) and a positive function \( \phi \), such that
  \[
  |f(t, \omega, y, z)| \leq \phi(y) + C|z|^2,
  \]
where \( \int_{0}^{+\infty} \frac{ds}{\phi(s)} = \int_{-\infty}^{0} \frac{ds}{\phi(s)} = \infty \) (see Kobylanski, Lepeltier, Quenez and Torres [21]).

We should point out here that, in the previous works, the existence of a solution for RBSDE or BSDE has been proved in the case when the quadratic condition imposed on the coefficient \( f \) is uniform in \( \omega \) and hence those works can not cover, for example, a generator with stochastic quadratic growth of the form \( C_s(\omega)\psi(|y|) \cdot |z|^2 \). Moreover, most of the previous works require that the terminal condition \( \xi \) and the barrier \( L \) are bounded random variables in the case of GRBSDEs or \( \xi \) is bounded in the case of GBSDEs. These conditions on \( f, \xi \) and \( L \) seem to be restrictive and are not necessary to have a solution.

One of the main purpose of this work is to study the GRBSDE with one barrier \( L \) which is a reflected BSDE which involves an integral with respect to a continuous and increasing process \( A \) of the form :

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_{t}^{T} f(s, Y_s, Z_s)ds + \int_{t}^{T} g(s, Y_s)dA_s \\
& \quad \quad \quad + \int_{t}^{T} dK_s - \int_{t}^{T} Z_s dB_s, \quad t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t \leq Y_t, \\
(iii) & \quad \int_{0}^{T} (Y_t - L_t)dK_t = 0, \text{ a.s.}, \\
(iv) & \quad Y \in \mathcal{C}, \quad K \in \mathcal{K}, \quad Z \in \mathcal{L}^{2,d}.
\end{align*}
\]

We prove existence of solutions for GRBSDE (1.3) when the generator \( f ds + g dA_s \) is continuous with general growth with respect to the variable \( y \) and stochastic quadratic growth with respect to the variable \( z \). This allow us to cover some BSDEs having a generator satisfying, for example, the following condition : for each \( (s, \omega, y, z) \)

\[
|f(s, \omega, y, z)| \leq \alpha_s \phi(|y|) + \frac{C_s \psi(|y|)}{2} \cdot |z|^2 + R_s \cdot |z|, \\
|g(s, \omega, y)| \leq \beta_s \phi(|y|),
\]

where \( \alpha, \phi, C, \psi, R \) and \( \beta \) are given later. We deal with the case of a bounded terminal condition \( \xi \) and a bounded barrier \( L \) as well as the case of unbounded ones. We give some examples which are covered by our result and, in our knowledge, not covered by the previous works. Moreover, as we will see later, the existence of a solution for our GRBSDE (1.3) is related to the existence of a solution \((x, z, k)\) for the following BSDE :

\[
\begin{align*}
\begin{cases}
x_t = \xi \vee \sup_{s \leq T} L_s + \int_{t}^{T} \phi(x_s) d\eta_s + \int_{t}^{T} \frac{C_s \psi(x_s)}{2} \cdot |z_s|^2 ds \\
+ \int_{t}^{T} R_s \cdot |z_s| ds + \int_{t}^{T} dK_s - \int_{t}^{T} z_s dB_s, \\
x_s \geq 0, \quad \forall s \leq T, \quad k \in \mathcal{K}, \quad z \in \mathcal{L}^{2,d}.
\end{cases}
\end{align*}
\]

Roughly speaking, we prove that if the BSDE (1.4) has a solution and the coefficient \( f ds + g dA_s \) is continuous with general growth with respect to the variable \( y \) and stochastic quadratic growth with respect to the variable \( z \) (see condition (H.2) below), then the GRBSDE (1.3) has a solution. Therefore a natural question arises : under which condition on \((\xi, L, \phi, \psi, C, \eta)\), the BSDE (1.4) has a solution? This is the second purpose of this work.

The third purpose of this work is to prove the existence of solutions for the GRBSDE (1.3) when
the barrier \( L \equiv -\infty \) which is nothing else than a GBSDE of the form:

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & \quad Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d}.
\end{align*}
\]

As a very particular case of our result, when \( \xi \) is not bounded, we obtain that the following BSDE

\[
Y_t = \xi + \int_t^T \gamma_s |Z_s| ds - \int_t^T Z_s dB_s,
\]

has a solution if

\[
\mathbb{E}\left[ \frac{e^{C_T|\xi|}}{C_T} - 1_{\{C_T > 0\}} + |\xi|1_{\{C_T = 0\}} \right] < +\infty,
\]

where \( \gamma \) be a nonnegative process which is \( \mathcal{F}_t \)-adapted and \( C_t = \sup_{0 \leq s \leq t} |\gamma_s|, \ \forall t \in [0, T] \). Moreover

\[
|Y_t| \leq \frac{\ln(1 + C_t \mathbb{E}(\overline{X}|\mathcal{F}_t))}{C_t} 1_{\{C_t > 0\}} + \mathbb{E}(\overline{X}|\mathcal{F}_t) 1_{\{C_t = 0\}},
\]

where \( \overline{X} = \frac{e^{C_T|\xi|}}{C_T} - 1_{\{C_T > 0\}} + |\xi|1_{\{C_T = 0\}} \).

To prove our results, we will use an approach based upon the recent result obtained in the preprint of Essaky and Hassani \([15]\) where the authors have proved the existence of a solution for a generalized BSDE with two reflecting barriers when the generator \( fds + g dA_s \) is continuous with general growth with respect to the variable \( z \) and stochastic quadratic growth with respect to the variable \( z \) and without assuming any \( P \)-integrability conditions on the data. This result allows a simple treatment of the problem of existence of solutions for one barrier reflected BSDEs and also for BSDEs without reflection. This approach seems to be new.

Let us describe our plan. First, some notation is fixed in Section 2. In Section 3, we recall the existence of solutions for GBSDE with two reflecting barriers studied in \([15]\). Section 4 is devoted to the proof of a general existence result for GRBSDE and GBSDE when the coefficients \( fds + g dA_s \) is continuous with general growth with respect to the variable \( y \) and stochastic quadratic growth with respect to the variable \( z \). In section 5, we give sufficient conditions under which the BSDE \([14]\) has a solution. In section 6, we give some important consequences and examples of our results.

## 2 Notations

The purpose of this section is to introduce some basic notations, which will be needed throughout this paper.

Let \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, P) \) be a stochastic basis on which is defined a Brownian motion \((B_t)_{t \leq T}\) such that \((\mathcal{F}_t)_{t \leq T}\) is the natural filtration of \((B_t)_{t \leq T}\) and \( \mathcal{F}_0 \) contains all \( P \)-null sets of \( \mathcal{F} \). Note that \((\mathcal{F}_t)_{t \leq T}\) satisfies the usual conditions, i.e. it is right continuous and complete.

Let us now introduce the following notation. We denote:

- \( \mathcal{P} \) to be the sigma algebra of \( \mathcal{F}_t \)-progressively measurable sets on \( \Omega \times [0, T] \).
- \( \mathcal{C} \) to be the set of \( \mathbb{R} \)-valued \( \mathcal{P} \)-measurable continuous processes \((Y_t)_{t \leq T}\).
- \( \mathcal{L}^{2,d} \) to be the set of \( \mathbb{R}^d \)-valued and \( \mathcal{P} \)-measurable processes \((Z_t)_{t \leq T} \) such that

\[
\int_0^T |Z_s|^2 ds < \infty, \ P-a.s.
\]
The following assumptions will be needed throughout the paper:

- $K$ to be the set of $\mathcal{P}$-measurable continuous nondecreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $K_T < +\infty$, $P$-a.s.

The following assumptions will be needed throughout the paper:

- $\xi$ is an $\mathcal{F}_T$-measurable one dimensional random variable.
- $f : \Omega \times [0, T] \times \mathbb{R}^{1+d} \rightarrow \mathbb{R}$ is a function which to $(t, \omega, y, z)$ associates $f(t, \omega, y, z)$ which is continuous with respect to $(y, z)$ and $\mathcal{P}$-measurable.
- $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a function which to $(t, \omega, y)$ associates $g(t, \omega, y)$ which is continuous with respect to $y$ and $\mathcal{P}$-measurable.
- $A$ is a process in $\mathcal{K}$.
- $L := \{L_t, 0 \leq t \leq T\}$ is a real valued barrier which is $\mathcal{P}$-measurable and continuous process such that $\xi \geq L_T$.

3 Generalized BSDE with two reflecting barriers

In view of clarifying this issue, we recall some results concerning GRBSDEs with two barriers which shall play a central role in our proofs. Let us start by recalling the following definition of two singular measures.

**Definition 3.1.** Let $\mu_1$ and $\mu_2$ be two positives measures defined on a measurable space $(\Lambda, \Sigma)$, we say that $\mu_1$ and $\mu_2$ are singular if there exist two disjoint sets $A$ and $B$ in $\Sigma$ whose union is $\Lambda$ such that $\mu_1$ is zero on all measurable subsets of $B$ while $\mu_2$ is zero on all measurable subsets of $A$. This is denoted by $\mu_1 \perp \mu_2$.

Let us now define the notion of solution of the GRBSDE with two obstacles $L$ and $U$.

**Definition 3.2.** We call $(Y, Z, K^+, K^-) := (Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ a solution of the generalized reflected BSDE, associated with coefficient $fds + gdA_s$; terminal value $\xi$ and barriers $L$ and $U$, if the following hold:

\[
\begin{array}{l}
(i) \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dA_s \\
\quad \quad + \int_t^T dK^+_s - \int_t^T dK^-_s - \int_t^T Z_sdB_s, t \leq T; \\
(ii) \quad Y \text{ between } L \text{ and } U, \text{ i.e. } \forall t \leq T, L_t \leq Y_t \leq U_t; \\
(iii) \quad \text{the Skorohod conditions hold:} \int_0^T (Y_t - L_t)dK^+_t = \int_0^T (U_t - Y_t)dK^-_t = 0, \text{ a.s.}, \\
(iv) \quad Y \in \mathcal{C} \quad K^+, K^- \in \mathcal{K}, Z \in \mathcal{L}^{2,d}, \\
(v) \quad dK^+ \perp dK^-.
\end{array}
\]

We introduce also the following assumptions:

(A.0) $U_t := U_0 - V_t - \int_0^t \rho_s ds - \int_0^t \theta_s dA_s + \int_0^t \chi_s dB_s$, with $U_0 \in \mathbb{R}$, $V \in \mathcal{K}$, $\chi \in \mathcal{L}^{2,d}$, $\rho$ and $\theta$ are non-negatives predictable processes satisfying $\int_0^T \rho_s ds + \int_0^T \theta_s dA_s < +\infty$, $P$-a.s., such that $L_t \leq U_t$, $\forall t \in [0, T]$ and $\xi \leq U_T$.

(A.1) There exist two processes $\eta' \in L^0(\Omega, L^1([0, T], ds, \mathcal{B}_t))$ and $C' \in \mathcal{C}$ such that:

\[
\forall (s, \omega), \quad |f(s, \omega, y, z)| \leq \eta'_s(\omega) + \frac{C'_s(\omega)}{2}|z|^2, \quad \forall y \in [L_s(\omega), U_s(\omega)], \forall z \in \mathbb{R}^d.
\]
There exists a process $\eta'' \in L^0(\Omega, L^1([0, T], dA_s, \mathbb{R}_+))$ such that

$$
\forall (s, \omega), \quad |g(s, \omega, y)| \leq \eta''_s, \quad \forall y \in [L_s(\omega), U_s(\omega)].
$$

The following result is obtained by Essaky and Hassani [15] and it is related to the existence of maximal (resp. minimal) solution of (3.7), that is, there exists a quadruple $(Y_t, Z_t, K_t^+, K_t^-)_t \leq T$ which satisfies (3.7) and if in addition $(Y'_t, Z'_t, K'_t^+, K'_t^-)_t \leq T$ is another solution of (3.7), then $P$-a.s. holds, for all $t \leq T, Y'_t \leq Y_t$ (resp. $Y'_t \geq Y_t$).

**Theorem 3.1.** Let assumptions (A.0)–(A.2) hold true. Then there exists a maximal (resp. minimal) solution for GRBSDE with two barriers (3.7). Moreover for all solution $(Y, Z, K^+, K^-)$ of Equation (3.7) we have

$$
dK^-_s \leq \left( f(s, U_s, \chi_s) - \rho_s \right)_+ ds + \left( g(s, U_s) - \theta_s \right)_+ dA_s. \tag{3.8}
$$

Furthermore, if the following condition hold

$L_t := L_0 + \nabla_t + \int_0^t \overline{\nabla}_s ds + \int_0^t \overline{\chi}_s dB_s$, with $L_0 \in \mathbb{R}, \nabla, \overline{\chi} \in L^2, \overline{\nabla}$ and $\overline{\theta}$ are non-negatives predictable processes satisfying

$$
\int_0^T \overline{\nabla}_s ds + \int_0^T \overline{\theta}_s dA_s < +\infty \quad P\text{-a.s.,}
$$

then

$$
dK^+_s \leq -\left( f(s, L_s, \overline{\chi}_s) - \overline{\rho}_s \right)_+ ds + \left( -g(s, L_s) - \overline{\theta}_s \right)_+ dA_s. \tag{3.9}
$$

**Proof.** The existence result follows from Essaky and Hassani [15]. By applying Itô-Tanaka formula to $(U_t - Y_t)^+ = U_t - Y_t$, we find

$$
(\chi_s - Z_s)1_{\{U_s = Y_s\}} ds = 0,
$$

and

$$
dK^-_s \leq 1_{\{U_s = Y_s\}} \left( dK^+_s + (f(s, U_s, \chi_s) - \rho_s) ds + (g(s, U_s) - \theta_s) dA_s \right).
$$

Making use now the fact that $dK^+ \perp dK^-$, we obtain Inequality (3.8).

Inequality (3.9) follows by the same way by applying Itô-Tanaka formula to $(Y_t - L_t)^+ = Y_t - L_t$ and using the fact that $dK^+ \perp dK^-$. $lacksquare$

**Remark 3.1.** We should point out here that Theorem 3.1 does not involve any $P$-integrability conditions about the data.

### 4 General existence result for GRBSDE and GBSDE

The main objective of this section is to show an existence result of solutions of GRBSDEs and GBSDEs in assuming general conditions on the data. As we will see later, we prove that the existence of solutions for GRBSDE and BSDE is related to the existence of solutions for another BSDE.
4.1 One barrier generalized reflected BSDE

Let us introduce the definition of our GRBSDE with lower obstacle $L$.

**Definition 4.1.** We call $(Y, Z, K) := (Y_1, Z_1, K_1)_{t \leq T}$ a solution of the generalized reflected BSDE, associated with coefficient $fds + gdA_s$; terminal value $\xi$ and a lower barrier $L$, if the following hold:

\[
\begin{align*}
(i) & \quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s \\
& \quad \quad + \int_t^T dK_s - \int_t^T Z_s dB_s, \quad t \leq T, \\
(ii) & \quad \forall t \leq T, \quad L_t \leq Y_t, \\
(iii) & \quad \int_0^T (Y_t - L_t) dK_t = 0, \quad a.s., \\
(iv) & \quad Y \in \mathcal{C} \quad K \in \mathcal{K} \quad Z \in \mathcal{L}^{2, d}.
\end{align*}
\]

We are now given the following objects:

- an $\mathcal{F}_T$-measurable random variable $\Lambda : \Omega \to \mathbb{R}_+$,
- two positive predictable processes $\alpha$ and $\beta$ such that $\eta_T < +\infty$ $P$-a.s., where $\eta_T = \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s$,
- two continuous functions $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$,
- a nonnegative process $C \in \mathcal{C}$,
- a nonnegative process $R$ in $\mathcal{L}^{2, 1}$.

We will make the following assumptions:

(H.1) $\xi \leq \Lambda$ and $L_s \leq \Lambda$, $\forall s \in [0, T]$.

(H.2) There exists $(x, z, k) \in \mathcal{C} \times \mathcal{L}^{2, d} \times \mathcal{K}$ such that

\[
\begin{align*}
(i) & \quad (i) \quad x_t = \Lambda + \int_t^T \phi(x_s) d\eta_s + \int_t^T \frac{C_s \psi(x_s)}{2} | z_s |^2 ds + \int_t^T R_s | z_s | ds \\
& \quad \quad + \int_t^T dK_s - \int_t^T z_s dB_s, \\
(jj) & \quad x_s \geq 0, \forall s \leq T.
\end{align*}
\]

From now on, the above equation will be denoted by $E^+(\Lambda, \phi(x) d\eta + \frac{C_s \psi(x)}{2} | z |^2 ds + R_s | z | ds)$.

(ii) For all $(s, \omega) \in [0, T] \times \Omega$

\[
\begin{align*}
\alpha_s \phi(x_s) + \frac{C_s \psi(x_s)}{2} | z_s |^2 + R_s | z_s |, \\
g(s, \omega, x_s) \leq \beta_s \phi(x_s).
\end{align*}
\]

(iii) There exist two positive predictable processes $\overline{\alpha}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\alpha}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$ $P$-a.s., and $\overline{\psi} \in \mathcal{C}$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq L_s \forall x_s$

\[
\begin{align*}
| f(s, \omega, y, z) | & \leq \overline{\alpha}_s + \frac{\overline{\psi}_s}{2} | z |^2, \\
| g(s, \omega, y) | & \leq \overline{\beta}_s.
\end{align*}
\]
Remark 4.1. 1. By using a localization procedure and Fatou’s lemma one can prove easily that:
\[ x_t \geq E(\Lambda|F_t) \geq L_t, \forall t \in [0,T]. \]

2. It is worth noting that condition (H.2)(iii) holds true if the functions \( f \) and \( g \) satisfy the following:
\[ \forall (s,\omega), \ |f(s,\omega,y,z)| \leq \sigma_s \Phi(s,\omega,y) + \gamma_s \Psi(s,\omega,y)|z|^2, \forall y \in [L_s(\omega),x_s(\omega)], \forall z \in \mathbb{R}^d, \]
and
\[ \forall (s,\omega), \ |g(s,\omega,y)| \leq \delta_s \varphi(s,\omega,y), \forall y \in [L_s(\omega),x_s(\omega)], \]
where \( \Phi, \Psi \) and \( \varphi \) are continuous functions on \([0,T] \times \mathbb{R}^d \) and progressively measurable, \( \sigma \in L^0(\Omega,L^1([0,T],ds,\mathbb{R}_+)), \gamma \in C \) and \( \delta \in L^0(\Omega,L^1([0,T],dA_s,\mathbb{R}_+)) \). To do this, we just take \( \overline{\sigma}, \overline{\Psi} \) and \( \overline{\varphi} \) as follows:
\[
\overline{\sigma}_t(\omega) = \sigma_t(\omega) \sup_{s \leq t} \sup_{\alpha \in [0,1]} |\Phi(s,\omega,\alpha L_s + (1-\alpha)x_s)|,
\overline{\Psi}_t(\omega) = 2\gamma_t \sup_{s \leq t} \sup_{\alpha \in [0,1]} |\Psi(s,\omega,\alpha L_s + (1-\alpha)x_s)|,
\overline{\varphi}_t(\omega) = \delta_t(\omega) \sup_{s \leq t} \sup_{\alpha \in [0,1]} |\varphi(s,\omega,\alpha L_s + (1-\alpha)x_s)|.
\]

The following theorem is a consequence of Theorem 3.1.

Theorem 4.1. Let assumptions (H.1) – (H.2) hold. Then the following GRBSDE
\[
\begin{cases}
(i) & Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s \\
& \quad + \int_t^T dK_s - \int_t^T Z_s dB_s, \ t \leq T, \\
(ii) & \forall t \leq T, \ L_t \leq Y_t \leq x_t, \\
(iii) & \int_0^T (Y_t - L_t)dK_t = 0, \ a.s., \\
(iv) & \bar{Y} \in C, \ K \in \mathcal{K}, \ Z \in \mathcal{L}^{2,d}.
\end{cases}
\]

has a maximal (resp. minimal) solution. Moreover, if the following condition holds
\[ L_t := L_0 + \int_0^t \bar{\sigma}_s ds + \int_0^t \bar{\Psi}_s dA_s + \int_0^t \bar{\varphi}_s dA_s, \text{ with } L_0 \in \mathbb{R}, \bar{\Psi} \in \mathcal{K}, \bar{\varphi} \in \mathcal{L}^{2,d}, \bar{\sigma} \text{ and } \overline{\Psi} \text{ are non-negatives predictable processes satisfying }\]
\[ \int_0^T \bar{\sigma}_s ds + \int_0^T \bar{\Psi}_s dA_s < +\infty \text{ a.s.}, \text{ then for all solution } (Y,Z,K) \text{ of Equation (4.12)} \text{ we have }\]
\[ dK_s \leq \left( -f(s,L_s,x_s) - \frac{C_\varphi(x_s)}{2} |z_s|^2 - R_s |z_s| \right) + ds + \left( -g(s,L_s) - \bar{\sigma}_s \right) + dA_s. \]  \hspace{1cm} (4.13)

Proof. Let \((Y,Z,K^+,K^-)\) be the maximal (resp. minimal) solution of the Equation (4.12) with \( U_t = x_t \). By using Inequality \((5.8)\) of Theorem 4.1 we conclude that
\[
dK^- \leq \left( f(s,\omega,x_s,z_s) - \alpha_s \phi(x_s) - \frac{C_\varphi(x_s)}{2} |z_s|^2 - R_s |z_s| \right)^+ ds + \left( g(s,\omega,x_s) - \beta_s \phi(x_s) \right)^+ dA_s
\]
\[
= 0.
\]
Therefore \(dK^- = 0\) and then Equation (4.12) has a maximal (resp. minimal) solution. Inequality \((4.13)\) follows easily from Inequality \((5.8)\).
**Remark 4.2.** It is worth pointing out that the minimal solution of GRBSDE (4.12) is also the minimal solution of GRBSDE (4.10). This statement does not hold for maximal solution.

Once established the existence of solutions for GRBSDEs, we are now interested in proving the same result for GBSDEs.

### 4.2 Generalized BSDE without reflection

To begin with, let us introduce the definition of our GBSDE.

**Definition 4.2.** We call \((Y, Z) := (Y_t, Z_t)_{t \leq T}\) a solution of the generalized reflected BSDE, associated with coefficient \(fds + gdA_s\); terminal value \(\xi\), if the following hold:

\[
\begin{aligned}
\quad & (i) 
\quad Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)dA_s - \int_t^T Z_sdB_s, t \leq T, \\
\quad & (ii) 
\quad Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d}.
\end{aligned}
\]

For \(i = 1, 2\), we are given the following objects:
- an \(\mathcal{F}_T\)-measurable random variable \(\Lambda^i : \Omega \rightarrow \mathbb{R}_+\),
- two nonnegative predictable processes \(\alpha^i\) and \(\beta^i\) such that \(\eta^i_T < +\infty\) \(P\)-a.s., where \(\eta^i_T = \int_0^t \alpha^i_sds + \int_0^t \beta^i_sA_s\),
- two continuous functions \(\phi^i, \psi^i : \mathbb{R}_+ \rightarrow \mathbb{R}_+\),
- a nonnegative process \(C^i \in \mathcal{C}\),
- a nonnegative process \(R^i\) in \(\mathcal{L}^{2,1}\).

We will need the following assumptions:

(C.1) \(-\Lambda^1 \leq \xi \leq \Lambda^2\).

(C.2) There exists \((x^i, z^i, k^i) \in \mathcal{C} \times \mathcal{L}^{2,d} \times \mathcal{K}\) such that

(i)

\[
\begin{aligned}
\quad & (j) 
\quad x^i_t = \Lambda^i + \int_t^T \phi^i(x^i_s)ds + \int_t^T \frac{C^i_s \psi^i(x^i_s)}{2} |z^i_s|^2 ds + \int_t^T R^i_s |z^i_s| ds \\
\quad & \quad + \int_t^T d\Lambda^i_s - \int_t^T z^i_sdB_s, s \leq T, \\
\quad & (jj) 
\quad x^i_s \geq 0, \forall s \leq T.
\end{aligned}
\]

(ii) For all \((s, \omega) \in [0, T] \times \Omega\)

\[
\begin{aligned}
\quad & f(s, \omega, x^2_s, z^2_s) \leq \alpha^2_s \phi^2(x^2_s) + \frac{C^2_s \psi^2(x^2_s)}{2} |z^2_s|^2 + R^2_s |z^2_s|, \\
\quad & f(s, \omega, -x^1_s, -z^1_s) \geq -\alpha^1_s \phi^1(x^1_s) - \frac{C^1_s \psi^1(x^1_s)}{2} |z^1_s|^2 - R^1_s |z^1_s|.
\end{aligned}
\]

(iii) For all \((s, \omega) \in [0, T] \times \Omega\)

\[
\begin{aligned}
\quad & g(s, \omega, x^2_s) \leq \beta^2_s \phi^2(x^2_s), \\
\quad & g(s, \omega, -x^1_s) \leq -\beta^1_s \phi^1(x^1_s).
\end{aligned}
\]

(iv) There exist two positivenonnegative predictable processes \(\overline{\alpha}\) and \(\overline{\beta}\) such that \(\int_0^T \overline{\alpha}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty\) \(P\)-a.s., and \(\overline{\psi} \in \mathcal{C}\) such that \(\forall (s, \omega)\) and \(\forall (y, z)\) satisfying \(-x^1_s \leq y \leq x^2_s\)

\[
\begin{aligned}
\quad & |f(s, \omega, y, z)| \leq \overline{\alpha}_s + \frac{\overline{\psi}_s}{2} |z|^2, \\
\quad & |g(s, \omega, y)| \leq \overline{\beta}_s.
\end{aligned}
\]
The proof of the following Theorem follows easily from Theorem 4.1.

**Theorem 4.2.** Let assumptions (C.1) – (C.2) hold. Then the following GBSDE

\[
\begin{cases}
(i) & Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s)dA_s - \int_t^T Z_s dB_s, t \leq T, \\
(ii) & -x_1 \leq Y_s \leq x_2, \forall s \leq T, \\
(iii) & Y \in \mathcal{C}, \quad Z \in \mathcal{L}^{2,d},
\end{cases}
\] (4.15)

has a maximal (resp. minimal) solution.

The next section is devoted to give immediate consequences of Theorem 4.1 and 4.2 in the case where the terminal condition \( \xi \) and/or the barrier \( L \) are bounded.

5 First consequences of Theorem 4.1 and 4.2: the bounded case

5.1 One barrier GBSDE

In this subsection, we consider the same notations as in subsection 4.1 and we study only the existence of solutions for GBSDE (4.10) in the case of bounded terminal value \( \xi \) and barrier \( L \). The unbounded case is treated in the next sections. The following result is consequence of Theorem 4.1.

**Corollary 5.1.** Suppose that there exist two nonnegative real numbers \( D \) and \( a \) such that

1. \( \xi \leq D \) and \( L_t \leq D, \forall t \in [0,T] \).
2. \( \phi(y) > 0 \) for \( y \geq D \).
3. \( \eta_T = \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \leq a < \int_D^\infty \frac{dr}{\phi(r)} \).
4. For all \((s,\omega)\) \in \([0,T] \times \Omega \)

\[
f(s,\omega,H^{-1}(a-\eta_s),0) \leq \alpha_s \phi(H^{-1}(a-\eta_s)), \quad g(s,\omega,H^{-1}(a-\eta_s)) \leq \beta_s \phi(H^{-1}(a-\eta_s)),
\]

where \( H^{-1} \) denotes the inverse of the function \( H \) defined by:

\[
H : [D, +\infty] \rightarrow [0, \int_D^\infty \frac{dr}{\phi(r)}], \quad H(x) = \int_D^x \frac{dr}{\phi(r)}.
\]

5. There exist two nonnegative predictable processes \( \overline{\alpha} \) and \( \overline{\beta} \) satisfying

\[
\int_0^T \overline{\alpha}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty \quad P-a.s., \quad \text{and} \quad \overline{\alpha} \in \mathcal{C} \text{ such that } \forall (s,\omega) \text{ and } \forall (y,z) \text{ satisfying } L_s \leq y \leq H^{-1}(a-\eta_s)
\]

\[
|f(s,\omega,y,z)| \leq \overline{\alpha}_s + \overline{\alpha}_s |z|^2 \quad \text{and} \quad |g(s,\omega,y)| \leq \overline{\beta}_s.
\]

Then the GRBSDE (4.10) has a solution such that \( L_t \leq Y_t \leq H^{-1}(a-\eta_t) \).
Proof. Set $x_t = H^{-1}(a - \eta_t)$, for every $t \in [0, T]$. By Itô’s formula we have

$$x_t = H^{-1}(a - \eta_T) + \int_0^T \phi(x_s) d\eta_s.$$  

Set $\Lambda := H^{-1}(a - \eta_T)$. Since $H(D) = 0 \leq a - \eta_T$ and $H$ is increasing, it follows then from assumption 1. that $\xi \leq \Lambda$ and $L_t \leq \Lambda$, $\forall t \in [0, T]$. Hence assumption (H.1) is satisfied. Assumption (H.2)(i) is satisfied also with $(x, 0, 0)$. The result follows then form Theorem 4.1.

The following corollaries, with $\phi(x) = x \ln(x)$ and $\phi(x) = e^x$, assuring the existence of a solution for the GRBSDE (4.10). Their proofs follow easily from Corollary 5.1.

**Corollary 5.2.** Suppose that there exist two real numbers $D > 1$ and $a \geq 0$ such that

1. $\xi \leq D$ and $L_t \leq D$, $\forall t \in [0, T]$.
2. $\text{esssup}_w \left( \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \right) \leq a$.
3. For all $(s, \omega) \in [0, T] \times \Omega$

$$f(s, \omega, e^{\ln(D)} e^{a - \eta_s}, 0) \leq \alpha_s \ln(D) e^{\ln(D)} e^{a - \eta_s},$$

$$g(s, \omega, e^{\ln(D)} e^{a - \eta_s}) \leq \beta_s \ln(D) e^{\ln(D)} e^{a - \eta_s}.$$  

4. There exist two nonnegative predictable processes $\overline{\alpha}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\alpha}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$

$P$-a.s., and $\overline{\psi} \in C$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq e^{\ln(D)} e^{a - \eta_s}$

$$|f(s, \omega, y, z)| \leq \overline{\alpha}_s + \overline{\psi}_s \frac{|z|^2}{2} \quad \text{and} \quad |g(s, \omega, y)| \leq \overline{\beta}_s.$$  

Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq e^{\ln(D)} e^{a - \eta_t}$.

**Corollary 5.3.** Suppose that there exist two real nonnegative numbers $D$ and $a$ such that

1. $\xi \leq D$ and $L_t \leq D$, $\forall t \in [0, T]$.
2. $\text{esssup}_w \left( \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \right) \leq a < e^{-D}$.
3. For all $(s, \omega) \in [0, T] \times \Omega$

$$f(s, \omega, -\ln(e^{-D} - a + \eta_s), 0) \leq \frac{\alpha_s}{e^{-D} - a + \eta_s},$$

$$g(s, \omega, -\ln(e^{-D} - a + \eta_s)) \leq \frac{\beta_s}{e^{-D} - a + \eta_s}.$$  

4. There exist two nonnegative predictable processes $\overline{\alpha}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\alpha}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$

$P$-a.s., and $\overline{\psi} \in C$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq -\ln(e^{-D} - a + \eta_s)$

$$|f(s, \omega, y, z)| \leq \overline{\alpha}_s + \frac{\overline{\psi}_s}{2} |z|^2 \quad \text{and} \quad |g(s, \omega, y)| \leq \overline{\beta}_s.$$  

Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq -\ln(e^{-D} - a + \eta_t).$
5.2 GBSDE without reflection

In this subsection, we consider the same notations as in subsection 4.2 and we treat only the existence of solution in the case of bounded terminal value \( \xi \). The unbounded case is treated in the next subsections. The following result is a consequence of Theorem 4.2.

**Corollary 5.4.** Suppose that there exist four real numbers \( D^1 \geq 0, D^2 \geq 0, a^1 \) and \( a^2 \) such that

i) \(-D^1 \leq \xi \leq D^2\).

ii) For \( i = 1, 2 \), \( \phi^i(y) > 0 \) for \( y \geq D^i \).

iii) For \( i = 1, 2 \), \( \text{esssup}_w \int_0^T \alpha^i_s ds + \int_0^T \beta^i_s dA_s \leq a^1 < \int_{D^i}^{+\infty} \frac{dr}{\phi^i(r)} \).

iv) For all \( (s, \omega) \in [0, T] \times \Omega \)

\[
\begin{align*}
&f(s, \omega, (H^2)^{-1}(a^2 - \eta^2_s), 0) \leq \alpha_s^2\phi^2((H^2)^{-1}(a^2 - \eta^2_s)), \\
f(s, \omega, -(H^1)^{-1}(a^1 - \eta^1_s), 0) \geq -\alpha_s^1\phi^1((H^1)^{-1}(a^1 - \eta^1_s)), \\
g(s, \omega, (H^2)^{-1}(a^2 - \eta^2_s)) \leq \beta_s^2\phi((H^2)^{-1}(a^2 - \eta^2_s)), \\
g(s, \omega, -(H^1)^{-1}(a^1 - \eta^1_s)) \geq -\beta_s^1\phi((H^1)^{-1}(a^1 - \eta^1_s)),
\end{align*}
\]

where, for \( i = 1, 2 \), \( H^i(x) = \int_{D^i}^x \frac{dr}{\phi^i(r)} \), \( x \geq D^i \) and \( \eta^i_s = \int_0^t \alpha^i_r dr + \int_0^t \beta^i_r dA_r \).

v) There exist two nonnegative predictable processes \( \overline{\pi} \) and \( \overline{\mu} \) satisfying \( \int_0^T \overline{\pi}_s ds + \int_0^T \overline{\mu}_s dA_s < +\infty \)

\( P\)-a.s. and \( \overline{\psi} \in C \) such that \( \forall (s, \omega) \) and \( \forall (y, z) \) satisfying \( -(H^1)^{-1}(a^1 - \eta^1_s) \leq y \leq (H^2)^{-1}(a^2 - \eta^2_s) \)

\[
\begin{align*}
&|f(s, \omega, y, z)| \leq \overline{\pi}_s + \frac{\overline{\psi}_s}{2} |z|^2, \\
&|g(s, \omega, y)| \leq \overline{\mu}_s.
\end{align*}
\]

Then the GBSDE (4.12) has a solution such that \( -(H^1)^{-1}(a^1 - \eta^1_s) \leq \xi_s \leq (H^2)^{-1}(a^2 - \eta^2_s) \).

The following corollaries, with \( \phi^1(x) = \phi^2(x) = x \ln(x) \) and \( \phi^1(x) = \phi^2(x) = e^x \), assuring the existence of a solution for the GRBSDE 4.10. Their proofs follow from Corollary 5.4.

**Corollary 5.5.** Suppose that there exist two real numbers \( D > 1 \) and a such that

i) \( |\xi| \leq D \).

ii) \( \text{esssup}_w \left( \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \right) \leq a < +\infty \).

(iii) For all \( (s, \omega) \in [0, T] \times \Omega \)

\[
\begin{align*}
&f(s, \omega, e^{\ln(D)\alpha^1_s}, 0) \leq \alpha_s \ln(D)e^{\ln(D)\alpha^1_s} e^{\alpha^1_s}, \\
f(s, \omega, -e^{\ln(D)\alpha^2_s}, 0) \geq -\alpha_s \ln(D)e^{\ln(D)\alpha^2_s} e^{\alpha^2_s}, \\
g(s, \omega, e^{\ln(D)\beta^1_s}) \leq \beta_s \ln(D)e^{\ln(D)\beta^1_s} e^{\beta^1_s}, \\
g(s, \omega, -e^{\ln(D)\beta^2_s}) \geq -\beta_s \ln(D)e^{\ln(D)\beta^2_s} e^{\beta^2_s},
\end{align*}
\]

where \( \eta^i_s = \int_0^t \alpha^i_s ds + \int_0^t \beta^i_s dA_s \).

iv) There exist two nonnegative predictable processes \( \overline{\pi} \) and \( \overline{\mu} \) satisfying \( \int_0^T \overline{\pi}_s ds + \int_0^T \overline{\mu}_s dA_s < +\infty \)

\( P\)-a.s. and \( \overline{\psi} \in C \) such that \( \forall (s, \omega) \) and \( \forall (y, z) \) satisfying \( |y| \leq e^{\ln(D)\alpha^1_s} \)

\[
\begin{align*}
&|f(s, \omega, y, z)| \leq \overline{\pi}_s + \frac{\overline{\psi}_s}{2} |z|^2, \\
&|g(s, \omega, y)| \leq \overline{\mu}_s.
\end{align*}
\]
Then the GBSDE (4.14) has a solution such that \( |Y_t| \leq e^{\ln(D)e^{a-\eta_t}} \).

**Corollary 5.6.** Suppose that there exist two real numbers \( D \geq 0 \) and \( a \) such that

i) \( \text{esssup}_w |\xi| \leq D \).

ii) \( \text{esssup}_w \left( \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \right) \leq a < e^{-D} \).

(iii) For all \((s, \omega) \in [0, T] \times \Omega\)

\[
\begin{align*}
  f(s, \omega, -\ln(e^{-D} - a + \eta_s), 0) & \leq \frac{\alpha_s}{e^{-D} - a + \eta_s}, \\
  f(s, \omega, \ln(e^{-D} - a + \eta_s), 0) & \geq e^{-D} - a + \eta_s, \\
  g(s, \omega, -\ln(e^{-D} - a + \eta_s)) & \leq \frac{\beta_s}{e^{-D} - a + \eta_s}, \\
  g(s, \omega, \ln(e^{-D} - a + \eta_s)) & \geq e^{-D} - a + \eta_s,
\end{align*}
\]

where \( \eta_t = \int_0^t \alpha_s ds + \int_0^t \beta_s dA_s \).

Then the GBSDE (4.14) has a solution such that \( |Y_t| \leq -\ln(e^{-D} - a + \eta_t) \).

**Corollary 5.7.** Suppose that there exist a nonnegative real number \( D \) such that

i) \( \text{esssup}_w |\xi| \leq D \).

ii) \( \phi(x) = e^x \) for \( x \geq D \).

iii) \( \text{esssup}_w \left( \int_0^T \alpha_s ds + \int_0^T \beta_s dA_s \right) := a < e^{-D} \).

(iv) \( \forall (s, \omega) \) and \( \forall (y, z) \) satisfying \( |y| \leq -\ln(e^{-D} - a) \) we have

\[
\begin{align*}
  |f(s, \omega, y, z)| & \leq \frac{\alpha_s}{2} |z|^2 + R_s + |z|, \\
  |g(s, \omega, y)| & \leq \frac{\beta_s}{2} |y|^2.
\end{align*}
\]

Then the GBSDE (4.14) has a solution such that \( |Y_t| \leq -\ln(e^{-D} - a) \).

### 6 Existence of solutions for \( E^+(\Lambda, \phi(x))d\eta_s + \frac{C_s \psi(x)}{2} |z|^2 ds + R_s |z| ds \)

As we have seen, by using an approach based upon the recent result obtained in the preprint of Essaky and Hassani [13], Theorem 4.1 and Theorem 4.2 follow easily from Theorem 5.1 but there still an interesting and important question: under which conditions on \((\Lambda, \phi, \psi, C, \eta)\), Equation \( E^+(\Lambda, \phi(x))d\eta_s + \frac{C_s \psi(x)}{2} |z|^2 ds + R_s |z| ds \)
We use also the following notations:
• $\Gamma$ 
• $F$
• $\phi, \psi$
• $\eta$
• $G$

Proposition 6.1. $\sup_{t \in \mathbb{H}} \mathbb{E}^{T_{0,T}} \mathbb{X} < +\infty$ if and only if there exists $(x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}$ solution of the following BSDE

$$\left\{ \begin{array}{l}
\frac{C_s \psi(x^1)}{2} \mid z^1 \mid^2 ds + R_s \mid z^1 \mid ds \\
\xi_t = \Lambda + \int_0^T \phi(x_s) d\eta_s + \int_0^T \frac{C_s \psi(x_s)}{2} \mid z_s \mid^2 ds + \int_0^T R_s \mid z_s \mid ds \\
+ \int_0^T ds - \int_x^T z_s dB_s,
\end{array} \right.$$

has a solution $(x, z, k) \in \mathcal{C} \times \mathcal{L}^{2,d} \times K$? For that sake, we list all the notations that will be used throughout this section. We denote:
• $D$ to be a nonnegative constant.
• $\Lambda : \Omega \rightarrow [D, +\infty]$ to be an $\mathcal{F}_T$-measurable random variable.
• $\phi, \psi : [D, +\infty] \rightarrow \mathcal{R}_+$ to be two continuous functions such that $\phi$ is of class $C^1$.
• $\eta \in K$ to be a process such that $\eta_T < \int_\Lambda \frac{dr}{\phi(r)}$.
• $C$ to be a process in $\mathcal{R}_+ + K$.
• $R$ to be a nonnegative process in $\mathcal{L}^{2,1}$.

Further we define also the following functions:
• $H : [D, +\infty] \rightarrow [0, \int_D^\infty \frac{dr}{\phi(r)}]$. $H(x) = \int_0^x \frac{dr}{\phi(r)}$.
• $F : [D, +\infty] \times [0, +\infty] \rightarrow \mathcal{R}_+$, $F(x, c) = \int_D e^{c \int_0^r \psi(r) dr} dt$.
• $H^{-1} : [0, \int_D^\infty \frac{dr}{\phi(r)}] \rightarrow [D, +\infty]$, is such that $H^{-1}(y) = x$ if and only if $H(x) = y$.
• $F^{-1} : \mathcal{R}_+ \times [0, +\infty] \rightarrow [D, +\infty]$, is such that $F^{-1}(y, c) = x$ if and only if $F(x, c) = y$.
• $G : \mathcal{G} \rightarrow [D, +\infty]$, $G(x, c, \eta) = H^{-1} \left( H(F^{-1}(x, c)) - \eta \right)$, where $\mathcal{G}$ is the set defined by:

$$\mathcal{G} = \{ (x, c, \eta) \in (\mathcal{R}_+)^3 : H(F^{-1}(x, c)) \geq \eta \}. \quad (6.16)$$

We use also the following notations:
• $\mathbb{X} = F \left( H^{-1}(H(\Lambda) + \eta_T), \mathcal{F}_T \right)$
• $\mathbb{Y} := \{ \pi \in \mathcal{L}^{2,d} : \mid \pi_s \mid \leq 1, \ a.e. \}$
• $\Pi := \{ \pi \in \mathbb{Y} : \pi_\mathcal{F} \in \{ 0, 1 \} \ a.e. \text{ and } \sup_{t \in \mathcal{F}_T} \int_0^T R_s^2 \mid \pi_s \mid ds < +\infty \}$
• $\Gamma^T_{s,t} := e^{\int_s^t \int_R \pi_s dB_s - \frac{1}{2} \int_s^t R_s^2 \mid \pi_s \mid^2 du}$, for $\pi \in \mathbb{Y}$ and $s, t \in [0, T]$.

We are now ready to give necessary and sufficient conditions for the existence of a solution for a particular case of $\mathcal{E}^+(\Lambda, \phi(x) d\eta_s + \frac{C_s \psi(x)}{2} \mid z \mid^2 ds + R_s \mid z \mid ds$.

**Proposition 6.1.** $\sup_{t \in \mathbb{H}} \mathbb{E}^{T_{0,T}} \mathbb{X} < +\infty$ if and only if there exists $(x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}$ solution of the following BSDE

$$\left\{ \begin{array}{l}
\frac{C_s \psi(x^1)}{2} \mid z^1 \mid^2 ds + R_s \mid z^1 \mid ds \\
x_t = \mathbb{X} + \int_t^T R_s \mid z^1_s \mid ds - \int_t^T z^1_s dB_s, t \leq T \\
x^1_t \geq 0, \forall t \leq T.
\end{array} \right.$$

(6.17)
In this case, there exist $z \in \mathcal{L}^{2,d}$ and $\pi_t := \esssup_{t \in \mathbb{R}} \mathbb{E}(\Gamma_{t,T} \mathbb{1}[\mathcal{F}_t]) = \esssup_{t \in \mathbb{R}} \mathbb{E}(\Gamma_{t,T} \mathbb{1}[\mathcal{F}_t])$ such that $(\pi, z)$ is the minimal solution of Equation (6.17), that is, for all solution $(x_1, z_1)$ of Equation (6.17) we have $\pi_t \leq x_1^t$.

**Proof.** Let $(\tau_n)_{n \geq 2}$ be the sequence of stopping times defined by $\tau_n := \inf \{ t \geq 0 : \int_0^t R_2^2 ds \geq n \} \wedge T$. According to Theorem 4.2, there exists $(x^n, z^n) \in \mathcal{C} \times \mathcal{L}^{2,d}$ such that

$$
\begin{align*}
\begin{cases}
x^n_t = \mathbb{1}_{\{ \pi \leq n \}} + \int_t^T R_s \mathbb{1}_{\{ \tau \leq \tau_n \}} | z^n_s | ds - \int_t^T z^n_s dB_s, t \leq T \\
0 \leq x^n_t \leq n, \forall t \in [0, T].
\end{cases}
\end{align*}
$$

(6.18)

By using a localization procedure and Lebesgue’s convergence theorem we have that, for all stopping time $\nu$ and $n \geq 2$,

$$
x_0^\nu = \mathbb{E}(x_0^\nu + \int_0^\nu R_s \mathbb{1}_{\{ \tau \leq \tau_n \}} | z^n_s | ds).
$$

(6.19)

On other hand, it follows from Itô’s formula that, for all stopping times $\nu \leq \sigma \leq T$,

$$
\begin{align*}
\begin{cases}
x^n_\nu = \Gamma_{\nu,\sigma} x^n_\sigma - \int_\nu^\sigma \Gamma_{\nu,\sigma} (z^n_s + R_s x^n_s \pi^n_s) dB_s, t \leq T \\
0 \leq x^n_\nu \leq n,
\end{cases}
\end{align*}
$$

where

$$
\pi^n_s := \begin{cases}
\frac{z^n_s}{|z^n_s|} \mathbb{1}_{\{ \tau \leq \tau_n \}} & \text{if } z^n_s \neq 0 \\
0 & \text{elsewhere}.
\end{cases}
$$

Using standard localization procedure and Lebesgue’s convergence theorem we obtain that, for all stopping times $\nu \leq \sigma \leq T$ and for all $n \geq 2$,

$$
x_\nu^n = \mathbb{E}(\Gamma_{\nu,\sigma} x^n_\sigma | \mathcal{F}_\nu) = \mathbb{E}(\Gamma_{\nu,\sigma} x^n_\sigma | \mathcal{F}_\nu) = \mathbb{E}(\Gamma_{\nu,\sigma} \mathbb{1}_{\{ \pi \leq n \}} | \mathcal{F}_\nu) \leq \esssup_{t \in \mathbb{R}} \mathbb{E}(\Gamma_{\nu,\sigma} \mathbb{1}_{\{ \pi \leq n \}} | \mathcal{F}_\nu).
$$

(6.20)

where we have used the fact that $\Gamma_{\nu,\sigma}$ is a martingale on $[\nu, T]$. It follows from comparison theorem that $x^n_\nu \leq x^{n+1}_\nu$. Set then $\pi_t := \lim_{n \to +\infty} t^{n}_t$. Therefore, in view of (6.19) and (6.20) we get for all stopping time $0 \leq \nu \leq T$,

$$
\mathbb{E}(\pi_\nu) \leq \mathbb{E}(\pi_\nu) \leq \sup_{\pi \in \mathbb{R}} \mathbb{E}(\Gamma_{0,T} \mathbb{1}) \quad \text{and} \quad \pi_\nu \leq \esssup_{\pi \in \mathbb{R}} \mathbb{E}(\Gamma_{T} \mathbb{1}) | \mathcal{F}_\nu).
$$

(6.21)

Let us now define the sequences of stopping times $(\delta^n_i)_{i \geq 2}$ by $\delta^n_i := \inf \{ s \geq 0 : x^n_s \geq i \} \wedge T$ and $\delta_i := \inf n \delta^n_i = \lim \delta^n_i$. Note that $0 \leq \pi_t = \lim x^n_t \leq i$, for all $t \leq \delta_i$.

Define also $\alpha_{i} := \delta_i \wedge \tau_i$ and let

$$
\begin{align*}
\begin{cases}
\pi_t = \pi_{\alpha_i} + \int_{\alpha_i}^{\tau_i} R_s | \pi_s | ds - \int_{\alpha_i}^{\tau_i} \pi_s dB_s \\
0 \leq \pi_t \leq i, \forall t \in [0, \alpha_i].
\end{cases}
\end{align*}
$$

(6.22)

Applying Itô’s formula to $(\pi_t - x^n_t)^2 e^{\int_0^t R^2 ds}$ and using a localization procedure, we conclude that

$$
\mathbb{E}(\pi_{\alpha_i} - x^n_{\alpha_i})^2 \leq e^t \mathbb{E}(\pi_{\alpha_i} - x^n_{\alpha_i})^2, \forall n \geq i.
$$
By letting $n$ to infinity we get $\mathbf{x}_{t_i, \lambda_i} = \mathbf{x}_{t, \lambda_i}$ and then $\mathbf{x} = \mathbf{x}^{+1}$ on $[0, \lambda_i]$. Set $\mathbf{z}_s := \lim_n \mathbf{z}_s^1_{s \leq \lambda_i} = \mathbf{z}_s^j$ on $[0, \lambda_j]$. Hence, for all $i \geq 2$

$$\left\{ \begin{array}{l}
\mathbf{x}_t = \mathbf{x}_t + \int_t^{\lambda_i} R_s | \mathbf{z}_s | ds - \int_t^{\lambda_i} \mathbf{z}_s dB_s \\
0 \leq \mathbf{x}_t \leq i, \forall t \in [0, \lambda_i].
\end{array} \right.$$  

Suppose now that $\sup_{\pi \in \Pi} \mathbb{E}(\Gamma_0^\pi \Lambda) < +\infty$. Since $\lim_{n} x_{\delta_i, T}^{n} 1_{\{\delta_i < T\}} = i 1_{\{\delta_i < T\}}$, we have $i\mathbb{P}(\delta_i > T) \leq \mathbb{E}(\Gamma_{\delta_i} \mathbb{E}(\Gamma_0^\pi \Lambda) < +\infty$. Therefore $\mathbb{P}(\cup_{i \geq 2} (\delta_i = T)) = 1$, and then $\mathbb{P}(\cup_{i \geq 2} (\delta_i = T)) = 1$. Moreover, it is easy seen that $\mathbf{z} \in \mathcal{L}^{2,d}$. Now passing to the limit as $i$ goes to infinity in Equation (6.22) we obtain

$$\left\{ \begin{array}{l}
\mathbf{x}_t = \mathbf{X} + \int_t^{T} R_s | \mathbf{z}_s | ds - \int_t^{T} \mathbf{z}_s dB_s, t \leq T \\
0 \leq \mathbf{x}_t, \forall t \in [0, T].
\end{array} \right.$$  

Henceforth $(\mathbf{x}, \mathbf{z})$ is a solution of Equation (6.17) which satisfies $\mathbf{x}_\nu \leq \text{esssup}_{\pi \in \Pi} \mathbb{E}(\Gamma_{\nu, T} \Lambda | \mathcal{F}_\nu)$, for all stopping time $0 \leq \nu \leq T$. On other hand, let $(x^\pi, z^\pi) \in \mathcal{C} \times \mathcal{L}^{2,d}$ be a solution of Equation (6.17) and consider for all $\pi \in \Pi$, $(x^\pi, z^\pi) \in \mathcal{C} \times \mathcal{L}^{2,d}$ a solution of the following BSDE

$$\left\{ \begin{array}{l}
x^\pi_t = \mathbf{X} + \int_t^{T} R_s (\pi_s, z^\pi_s) ds - \int_t^{T} z^\pi_s dB_s, t \leq T \\
0 \leq x^\pi_t \leq x^\lambda_t, \forall t \in [0, T],
\end{array} \right.$$  

which is exists according to Theorem 4.2. It follows then from Itô’s formula that, for all stopping times $\nu \leq \sigma \leq T$,

$$\left\{ \begin{array}{l}
x^\pi_{\nu} = \Gamma_{\nu, \sigma} x^\pi_{\sigma} - \int_\nu^{\sigma} \Gamma_{\nu, s} (z^\pi_s + R_s x^\pi_s \pi_s) dB_s, t \leq T \\
0 \leq x^\pi_{\nu} \leq x^\lambda_{\nu}.
\end{array} \right.$$  

Consequently, for all stopping time $\nu \leq T$, we have by Fatou’s lemma and standard localization procedure

$$x^\pi_{\nu} \geq \mathbb{E}(\Gamma_{\nu, T} \Lambda | \mathcal{F}_\nu).$$  

Hence, for all stopping time $\nu \leq T$,

$$x^\pi_{\nu} \geq \text{esssup}_{\pi \in \Pi} \mathbb{E}(\Gamma_{\nu, T} \Lambda | \mathcal{F}_\nu) \quad (6.23)$$  

Hence $\sup_{\pi \in \Pi} \mathbb{E}(\Gamma_{0, T} \Lambda) \leq x^\pi_{\nu} < +\infty$.

By using inequalities (6.22) and (6.23) we get for all stopping time $\nu \leq T$,

$$\mathbf{x}_\nu = \text{esssup}_{\pi \in \Pi} \mathbb{E}(\Gamma_{\nu, T} \Lambda | \mathcal{F}_\nu) = \text{esssup}_{\pi \in \Pi} \mathbb{E}(\Gamma_{\nu, T} \Lambda | \mathcal{F}_\nu).$$

This completes the proof. 

\[\Box\]
The following remark play a crucial tool in our results.

**Remark 6.1.** Let \((x^1, z^1) \in \mathcal{C} \times \mathcal{L}^{2,d}\) be a solution of Equation \((6.17)\).

1. By using Fatou’s lemma, one can see that \(x^1\) satisfies the following inequality

\[
x_t^1 \geq \mathbb{E}(\mathcal{F}_t) = \mathbb{E} \left( F(H^{-1}(H(\Lambda + \eta_t), C_t)) \mid \mathcal{F}_t \right) \geq F(H^{-1}(\eta_t), C_t) \geq 0, \quad \forall t \in [0, T].
\]

This means that \((x_t^1, C_t, \eta_t) \in \mathcal{G}\), for all \((t, \omega) \in [0, T] \times \Omega\), where \(\mathcal{G}\) is defined by \((6.16)\).

2. For all \(t \in [0, T]\), let us set

\[
x_t := G(x_t^1, C_t, \eta_t).
\]

It is easy seen that

\[
(\mathbf{a}) \quad \frac{\partial G}{\partial x}(x, c, \eta) = \frac{\phi(G(x, c, \eta)) e^{-c \int_{0}^{t} \psi(r) dr}}{\phi(F^{-1}(x, c))}.
\]

\[
(\mathbf{b}) \quad \frac{\partial^2 G}{\partial x^2}(x, c, \eta) = \left( \frac{\partial G}{\partial x}(x, c, \eta) \right)^2 \left[ \phi'(G(x, c, \eta)) - \phi'(F^{-1}(x, c)) \right].
\]

\[
(\mathbf{c}) \quad \frac{\partial G}{\partial c}(x, c, \eta) = - \frac{\partial G}{\partial x}(x, c, \eta) \int_{D} F^{-1}(x, c) e^{c \int_{0}^{t} \psi(r) dr} \int_{D} \psi(r) dr dt.
\]

\[
(\mathbf{d}) \quad \frac{\partial G}{\partial \eta}(x, c, \eta) = - \phi(G(x, c, \eta)).
\]

Therefore, by using Itô’s formula, one can see that \(x\) satisfies the following BSDE

\[
x_t = \Lambda + \int_{t}^{T} \phi(x_s) d\eta_s + \int_{t}^{T} C_s \psi(x_s) \frac{z_s}{2} d s + \int_{t}^{T} R_s | z_s | d s + \int_{t}^{T} d k_s - \int_{t}^{T} z_s dB_s, \quad (6.25)
\]

where \((z, k)\) is given by :

\[
z_s = \frac{\phi(x_s) e^{-c_s \int_{0}^{s} \psi(r) dr}}{\phi(F^{-1}(x_s, C_s))} z_s^1
\]

\[
dk_s = - \frac{\partial G}{\partial c}(x_s^1, C_s, \eta_s) dC_s + \frac{1}{2} \frac{\phi(G(x_s^1, C_s, \eta_s)) e^{-c_s \int_{0}^{s} \psi(r) dr}}{\phi(F^{-1}(x_s^1, C_s))^2} |z_s|^2 M_s ds \quad (6.27)
\]

with

\[
M_s = \varphi(F^{-1}(x_s^1, C_s), C_s) - \varphi(G(x_s^1, C_s, \eta_s), C_s) \quad \text{and} \quad \varphi(x, c) = \phi'(x) + c \phi(x) \psi(x). \quad (6.28)
\]

We can now formulate our main results of this section.

**6.1 Main Results**

The following results give sufficient conditions for the solvability of \(\mathbf{E}^+(\Lambda, \phi(x) d\eta_s + \frac{C_s \psi(x)}{2} | z |^2 d s + R_s | z | d s)\). Their proofs follow easily by using Remark \(6.1\).

**Theorem 6.1.** Suppose that the following conditions hold :
1. \( \sup_{\pi \in \Pi} ET_{0,T}^\pi \bar{X} < +\infty \).

2. There exists a solution \((x^1, z^1)\) to Equation (6.17) such that, \(dk\) defined by (6.27), is a positive measure.

Then Equation \( E^+(\Lambda, \phi(x)d\eta_s + \frac{C_s \psi(x)}{2} \mid z \mid^2 ds + R_s \mid z \mid ds) \) has a solution \((x, z, k)\) given by (6.24), (6.26) and (6.27).

In particular, since \(-\frac{\partial G}{\partial c}(x^1, C_s, \eta_s) dC_s\) is a positive measure, we have the following corollary.

**Corollary 6.1.** Assume that

1. \( \sup_{\pi \in \Pi} ET_{0,T}^\pi \bar{X} < +\infty \).

2. There exists a solution \((x^1, z^1)\) to Equation (6.17) such that, \(M\), defined by (6.28), is positive.

Then Equation \( E^+(\Lambda, \phi(x)d\eta_s + \frac{C_s \psi(x)}{2} \mid z \mid^2 ds + R_s \mid z \mid ds) \) has a solution \((x, z, k)\) given by (6.24), (6.26) and (6.27).

An interesting corollary of Theorem 6.1 is the following.

**Corollary 6.2.** Suppose that the following assumptions hold:

1. \( \sup_{\pi \in \Pi} ET_{0,T}^\pi \bar{X} < +\infty \).

2. The function \( x \mapsto \varphi(x, C_s(\omega)) \), given by (6.28), is nondecreasing on \([D, +\infty]\) dsdP a.e. for \((s, \omega)\).

Then Equation \( E^+(\Lambda, \phi(x)d\eta_s + \frac{C_s \psi(x)}{2} \mid z \mid^2 ds + R_s \mid z \mid ds) \) has a solution \((x, z, k)\) given by (6.24), (6.26) and (6.27).

**Remark 6.2.** It follows from Hölder’s inequality that, for all stopping time \( \nu \leq T \),

\[
\text{esssup}_{\pi \in \Pi} \mathbb{E}(\Gamma_{\nu,T} \bar{X} \mid F_\nu) \leq \Delta_\nu \cdot
\]

where

\[
\Delta_\nu := \text{esssup}_{\nu \in \Pi} \text{essinf}_{q>1} \left( \mathbb{E} \left( e^{\frac{q}{2} \int_{\nu}^T R_s^2 ds} (\bar{X})^q \cdot 1_{\{\bar{X}+\int_{\nu}^T R_s^2 ds \leq n\}} \mid F_\nu \right) \right)^{\frac{1}{q}}.
\]

Indeed, for all \( \pi \in \Pi \), \( n \in \mathbb{N} \) and \( q > 1 \), we have

\[
\begin{align*}
\mathbb{E}(\Gamma_{\nu,T} \bar{X}^q \mid F_\nu) & \leq \mathbb{E} \left( e^{\frac{q}{2} \int_{\nu}^T R_s^2 ds} (\bar{X})^q \cdot 1_{\{\bar{X}+\int_{\nu}^T R_s^2 ds \leq n\}} \mid F_\nu \right)^{\frac{1}{q}} \cdot \\
& \leq \mathbb{E} \left( e^{\frac{q}{2} \int_{\nu}^T R_s^2 ds} (\bar{X})^q \cdot 1_{\{\bar{X}+\int_{\nu}^T R_s^2 ds \leq n\}} \mid F_\nu \right)^{\frac{1}{q}} \cdot \\
& \leq \mathbb{E} \left( e^{\frac{q}{2} \int_{\nu}^T R_s^2 ds} (\bar{X})^q \cdot 1_{\{\bar{X}+\int_{\nu}^T R_s^2 ds \leq n\}} \mid F_\nu \right)^{\frac{1}{q}}.
\end{align*}
\]

Hence \( \Delta_0 < +\infty \) is a sufficient condition to have \( \sup_{\pi \in \Pi} ET_{0,T}^\pi \bar{X} < +\infty \).
Remark 6.3. By taking into account the results of Corollary 6.2 and Remark 6.2, assumptions 1 and 2 of the Theorem 6.1 can be replaced by the following strong assumptions:

1. \( \Delta_0 < +\infty \).

2. The function \( x \mapsto \varphi(x, C_s(\omega)) \), given by (6.28), is nondecreasing on \([D, +\infty]\) \(d \sigma \) a.e. for \((s, \omega)\).

In order to justify the assumptions we introduce to prove the existence of solutions for both one barrier GBSDE and GBSDE we give the following consequences.

6.2 Second consequences of Theorem 4.1 and 4.2: the unbounded case

In this subsection, we apply the results from the above sections to study the problem of existence of solutions to the GRBSDE (4.10) and also to the GBSDE (4.14). We give various existence results dealing with the case of unbounded terminal condition \( \xi \) and unbounded barrier \( L \).

6.2.1 One barrier GBSDE

The following Corollary follows from Theorem 4.1 and Theorem 6.1.

Corollary 6.3. Suppose that the following assumptions hold:

1. \( \sup_{\pi \in \Pi} E \int_0^T \pi_t \bar{X}_t < +\infty \).

2. There exists a solution \((x^1, z^1)\) to Equation (6.17) such that \( dk, \) defined by (6.27), is a positive measure.

3. \( \xi \lor \sup_{t \leq T} L_t \leq \Lambda \).

4. For all \((s, \omega) \in [0, T] \times \Omega\)

   \[
   f(s, \omega, x_s, z_s) \leq \alpha_s \varphi(x_s) + \frac{C_s \psi(x_s)}{2} |z_s|^2 + R_s |z_s|, \\
g(s, \omega, x_s) \leq \beta_s \varphi(x_s).
   \]

5. There exist two nonnegative predictable processes \( \overline{\pi} \) and \( \overline{\beta} \) such that \( \int_0^T \overline{\pi}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty \) \(P\)-a.s., and \( \overline{\pi} \in C\) such that \( \forall (s, \omega)\) and \( \forall (y, z)\) satisfying \( L_s \leq y \leq x_s \)

   \[
   |f(s, \omega, y, z)| \leq \overline{\pi}_s + \frac{\overline{\beta}_s}{2} |z|^2 \quad \text{and} \quad |g(s, \omega, y)| \leq \overline{\beta}_s,
   \]

where \( x_t \) and \( z_t \) are given respectively by relations (6.24) and (6.26).

Then the GRBSDE (4.10) has a solution such that \( L_t \leq Y_t \leq x_t \).

The following corollaries are direct and interesting applications of Corollaries 6.2, 6.3 and Remark 6.3, since all the required assumptions are obviously satisfied.

Corollary 6.4. Suppose that there exists nonnegative real number \( D \) such that:

i) \( R = 0, \) \( \varphi(x) = x \) on \([D, +\infty[\) and \( \psi(x) = 1 \) on \([D, +\infty[\) and \( C \in \mathbb{R}_+ + K\).

ii) \( E \bar{X} < +\infty \), where \( \bar{X} = \frac{e^{C_T(\Lambda e^{\eta T} - D)} - 1}{C_T} 1_{(C_T > 0)} + (\Lambda e^{\eta T} - D) 1_{(C_T = 0)} \) and \( \Lambda = \xi \lor \sup_{t \leq T} L_t \lor D \).
(iii) There exist two nonnegative predictable processes $\overline{\pi}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\pi}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$ $P$-a.s. and $\overline{\psi} \in C$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq x_s$

$$-\overline{\pi}_s - \frac{\overline{\psi}_s}{2} |z|^2 \leq f(s, y, z) \leq \alpha_s \phi(|y|) + \frac{C_s \psi(|y|)}{2} |z|^2,$$

$$-\overline{\beta}_s \leq g(s, \omega, y) \leq \beta_s \phi(|y|),$$

where

$$x_s = G(\mathbb{E}(\overline{\Lambda} | \mathcal{F}_s)), C_s, \eta_s) = e^{-\eta_s} \left[ D + \frac{\ln(1 + C_s \mathbb{E}(\overline{\Lambda} | \mathcal{F}_s))}{C_s} 1_{\{C_s > 0\}} + \mathbb{E}(\overline{\Lambda} | \mathcal{F}_s) 1_{\{C_s = 0\}} \right].$$

Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq x_t$.

**Corollary 6.5.** Suppose that there exist two real numbers $D > 1$ and $m > 0$ such that:

i) $R = 0$, $\phi(x) = x \ln(x)$ on $[D, +\infty[$, $\psi(x) = 1$ on $[D, +\infty[ \text{ and } C_s = m$, $\forall s \in [0, T]$.

ii) $\mathbb{E} e^{m \ln(A \psi)} < +\infty$, where $\Lambda = \xi \vee \sup_{t \leq T} L_t \vee D$ and $\eta_t := \int_0^t \alpha_s ds + \int_0^t \beta_s A_s$.

(iii) There exist two nonnegative predictable processes $\overline{\pi}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\pi}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$ $P$-a.s. and $\overline{\psi} \in C$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq x_s$

$$-\overline{\pi}_s - \frac{\overline{\psi}_s}{2} |z|^2 \leq f(s, y, z) \leq \alpha_s \phi(|y|) + \frac{m \psi(|y|)}{2} |z|^2,$$

$$-\overline{\beta}_s \leq g(s, \omega, y) \leq \beta_s \phi(|y|),$$

where $x_s = G(\mathbb{E}(e^{m \ln(A \psi)} \overline{\Lambda} - \frac{1}{m} | \mathcal{F}_s)), C_s = m, \eta_s) = e^{-\eta_s} \ln[D + \mathbb{E}(e^{m \ln(A \psi)} \overline{\Lambda} | \mathcal{F}_s)]$.

Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq x_t$.

**Corollary 6.6.** Suppose that there exist two positive real numbers $D$ and $m$ such that:

i) $R = 0$, $\phi(x) = x$ on $[D, +\infty[$, $\psi(x) = x$ on $[D, +\infty[ \text{ and } C_s = m$, $\forall s \in [0, T]$.

ii) $\mathbb{E}(\int_0^T e^{\psi_s^2} ds) < +\infty$, where $\Lambda = \xi \vee \sup_{t \leq T} L_t \vee D$.

(iii) There exist two nonnegative predictable processes $\overline{\pi}$ and $\overline{\beta}$ satisfying $\int_0^T \overline{\pi}_s ds + \int_0^T \overline{\beta}_s dA_s < +\infty$ $P$-a.s. and $\overline{\psi} \in C$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq x_s$

$$-\overline{\pi}_s - \frac{\overline{\psi}_s}{2} |z|^2 \leq f(s, y, z) \leq \alpha_s \phi(|y|) + \frac{m \psi(|y|)}{2} |z|^2,$$

$$-\overline{\beta}_s \leq g(s, \omega, y) \leq \beta_s \phi(|y|),$$

where $x_s = e^{-\eta_s} F_0^{-1}(\mathbb{E}(F_0(\Lambda e^\psi) | \mathcal{F}_s))$ where the function $F_0$ is defined by $F_0(x) = \int_D^x e^{\psi_0(t^2 - D^2)} dt$ and $F_0^{-1}$ its inverse. Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq x_t$.

**Corollary 6.7.** Suppose that there exist two positive real numbers $D$ and $m$ such that:

i) $\phi(x) = x$ on $[D, +\infty[$, $\psi(x) = 0$ on $[D, +\infty[ \text{ and } C \in \mathbb{R}_+ + \mathbb{K}$.

ii) There exists $q > 1$ such that $\mathbb{E} \left( e^{\frac{1}{2} q \int_0^T R_s^2 ds} (\Lambda e^\psi - D)^q \right) < +\infty$ where $\Lambda = \xi \vee \sup_{t \leq T} L_t \vee D$. 20
(iii) There exist two nonnegative predictable processes $\overline{\pi}$ and $\overline{\gamma}$ such that $\int_0^T \overline{\pi}_s ds + \int_0^T \overline{\gamma}_s dA_s < +\infty$ $P$-a.s., and $\overline{\gamma} \in \mathcal{C}$ such that $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $L_s \leq y \leq x_s$

$$-\overline{\pi}_s - R \left| g(s, \omega, y) \right| \leq -\overline{\gamma}_s \leq g(s, \omega, y) \leq \overline{\gamma}_s \phi(y),$$

where $x_s = \text{esssup}_{\pi \in \Pi} \left( e^{-\overline{\gamma}_s} \mathbb{E} (\Gamma_{s,T} \Lambda e^{\overline{\gamma}_T} | \mathcal{F}_s) \right)$. Then the GRBSDE (4.10) has a solution such that $L_t \leq Y_t \leq x_t$.

6.2.2 GBSE without reflection

By combining Theorem 4.2 and Theorem 6.1 we obtain the following.

**Corollary 6.8.** Assume that the following hold:

1. $\sup_{\pi \in \Pi} \mathbb{E} T_{0,T} \overline{\Lambda} < +\infty$.
2. There exists a solution $(x^1, z^1)$ to Equation (6.17) such that $dk$, defined by (6.17), is a positive measure.
3. $|\xi| \leq \Lambda$.
4. $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $y \leq x_s$

$$|f(s, \omega, y, z)| \leq \alpha_s \phi(y) + \frac{C_s \psi(|y|)}{2} |z|^2 + R_s |z|,$$

$$|g(s, \omega, y)| \leq \beta_s \phi(|y|),$$

where $x_s$ is given by (6.24).

Then the GBSDE (4.17) has a solution such that $|Y_t| \leq x_t$.

**Corollary 6.9.** Suppose that there exists nonnegative real number $D$ such that:

i) $R = 0$, $\phi(x) = x$ on $[D, +\infty]$, $\psi(x) = 1$ on $[D, +\infty)$ and $C \in \mathbb{R}_+ + \mathbb{K}$.

ii) $\mathbb{E} \overline{\Lambda} < +\infty$, where $\overline{\Lambda} = \frac{e^{C_T (\Lambda e^{\overline{\gamma}_s} - D)}}{C_T 1_{\{C_T > 0\}}} + (\Lambda e^{\overline{\gamma}_s} - D) 1_{\{C_T = 0\}}$ and $\Lambda = |\xi| \vee D$.

(iii) $\forall (s, \omega)$ and $\forall (y, z)$ satisfying $y \leq x_s$

$$|f(s, y, z)| \leq \alpha_s \phi(y) + \frac{C_s \psi(|y|)}{2} |z|^2,$$

$$|g(s, \omega, y)| \leq \beta_s \phi(|y|),$$

where

$$x_s = G(\mathbb{E}(\overline{\Lambda} | \mathcal{F}_s), C_s, \eta_s) = e^{-\eta_s} \left[ D + \frac{\ln(1 + C_s \mathbb{E}(\overline{\Lambda} | \mathcal{F}_s))}{C_s} 1_{\{C_s > 0\}} + \mathbb{E}(\overline{\Lambda} | \mathcal{F}_s) 1_{\{C_s = 0\}} \right].$$

Then the GBSDE (4.14) has a solution such that $|Y_t| \leq x_t$.

The following remark gives a sufficient condition for the existence of solution for the BSDE (4.14) when $f(s, y, z) = \frac{\alpha_s}{2} |z|^2$ and $g(s, y) = 0$.  

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Remark 6.4. Let $\gamma$ be a nonnegative process which is $\mathcal{F}_t$-adapted and $C_t = \sup_{0 \leq s \leq t} \gamma_s$, $\forall t \in [0,T]$. We consider the following BSDE

$$Y_t = \xi + \int_t^T \frac{\gamma_s}{2} |Z_s|^2 \, ds - \int_t^T Z_s \, dB_s,$$

(6.29)

It follows from the Corollary 6.9 that if

$$E\left[ e^{C_T |\xi|} \frac{1}{C_T} \mathbf{1}_{\{C_T>0\}} + |\xi| \mathbf{1}_{\{C_T=0\}} \right] < +\infty,$$

then the BSDE (6.29) has a solution satisfying

$$|Y_t| \leq \ln(1 + C_t E(\Lambda |\mathcal{F}_t)) \mathbf{1}_{\{C_t>0\}} + E(\Lambda |\mathcal{F}_t) \mathbf{1}_{\{C_t=0\}},$$

where $\Lambda = e^{C_T |\xi|} \frac{1}{C_T} \mathbf{1}_{\{C_T>0\}} + |\xi| \mathbf{1}_{\{C_T=0\}}$.

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