Stability of semi-wavefronts for delayed reaction–diffusion equations

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Abstract. This paper deals with the asymptotic behavior of solutions to the delayed monostable equation: 
\( u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t-h, x)), \) 
\( x \in \mathbb{R}, \ t > 0; \) here \( h > 0 \) and the reaction term \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is Lipschitz continuous and has exactly two fixed points (zero and \( \kappa > 0 \)). Under certain condition on the derivative of \( g \) at \( \kappa \) (without assuming classic KPP condition for \( g \)) the global stability of fast semi-wavefronts is proved. Also, when the Lipschitz constant \( L_g \) is equal to \( g'(0) \) the stability of all semi-wavefronts (e.g., critical, non-critical and asymptotically periodic semi-wavefronts) on each interval in the form \( (-\infty, N], N \in \mathbb{R}, \) to (1) is established, which includes classic equations such as the Nicholson’s model.

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1. Main results and discussion

In this work, the main object of study is the equation:
\( u_t(t, x) = u_{xx}(t, x) - u(t, x) + g(u(t-h, x)), \) \( x \in \mathbb{R}, \ t > 0, \) \( (1) \)
where \( h > 0 \) and the nonlinear reaction term \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is Lipschitz continuous with Lipschitz constant \( L_g \) which satisfies the monostability condition \( L_g \geq g'(0) > 1 \) and has exactly two fixed points: 0 and \( \kappa > 0 \). The Eq. (1) is frequently considered to model problems of population dynamics. In this case \( g \) stands for the birth rate function, \( h \) is the age when the individual reaches the sexual maturity and \( u(t, x) \) is the adult population at location \( x \) and time \( t \). The diffusion and death rates have been normalized. In this framework is relevant the stability properties of the positive equilibrium \( \kappa \) and the existence of colonization waves so-called wavefronts (see [2, 12, 21, 28, 34, 35] and references
Wavefronts with speed $c$ are non-negative entire bounded solutions $u(t,x) = \psi_c(x+ct)$ such that the profile $\psi_c : \mathbb{R} \to \mathbb{R}$ satisfies $\psi_c(-\infty) = 0$ and $\psi_c(+\infty) = \kappa$. It is well known that when $g$ is monotone and $h \geq 0$ then there exists a positive number $c_\# = c_\#(h)$ so-called critical speed or minimal speed such that (1) has wavefronts if and only if $c \geq c_\#$ [18,40,43]; wavefronts with speed $c_\#$ are called critical wavefronts. Moreover, these wavefronts are monotone and unique modulo translation. The main tool to obtain the stability and existence of wavefronts is to construct sub and super-solutions by using monotony arguments.

However, when $g$ is non-monotone the associated semi-flow is non-monotone in general and wavefronts are replaced by positive bounded solutions $u(t,x) = \psi_c(x+ct)$ such that $\psi_c(-\infty) = 0$ and $\liminf_{x \to +\infty} \psi_c(x) > 0$ which are called semi-wavefronts. For $g$ satisfying the subtangency condition $g(u) \leq g'(0)u$, for all $u \geq 0$, has been demonstrated the existence of a minimal speed $c_\# = c_\#(h)$ for the existence of semi-wavefronts to (1) for all $h \geq 0$ (see [37, Theorems 4.5 and 5.4] and [13,43]). Under the Diekmann- Kaper (D-K, for short) condition $L_g = g'(0)$ (see [11, Theorem 6.4]) Aguerrea, Gomez and Trofimchuk demonstrated the uniqueness modulo translation of all semi-wavefronts of (1). In the general case $L_g \geq g'(0)$ it is necessary to consider the following characteristic equation

$$E_c(\lambda) := \lambda^2 - c\lambda - 1 + L_g e^{-\lambda ch} = 0,$$

(2)

for which it has been showed that there exits a speed $c_* = c_*(L_g)$ defined as

$$c_* = c_*(L_g) := \inf\{c > 0 : E_c(\lambda) \text{ has a positive root}\},$$

(3)

such that $E_c$ has exactly two positive zeros $\lambda_1(c) \leq \lambda_2(c)$, also $\lambda_1(c) = \lambda_2(c)$ if and only if $c = c_*$ (for a more detailed study of (2) see [16, Lemma 22]). Thus, the authors in [1, Theorem 4] showed that for $c \geq c_*$ semi-wavefronts have the following representation

$$\psi_c(z) = A_{\psi_c}(-z)^j_c e^{\lambda_1(c)z} + e^{(\lambda_1(c)+\epsilon)z} r(z),$$

(4)

where $A_{\psi_c}, \epsilon \in \mathbb{R}_+$, $r \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ and $j_c = 0,1$ with $j_c = 1$ if and only if $L_g = g'(0)$, moreover, semi-wavefronts are unique (modulo translation) for all $c > c_*$ and $h \geq 0$ [1, Theorem 8]. We should mention that when $L_g$ in (2) is replaced by $g'(0)$ the speed $c_*$ in (3) coincides with the definition of the so-called linear speed $c_\#$ and

$$c_\# \leq c_* \leq c_\#,$$

(5)

(see [37, Theorem 4.5 and Theorem 5.4]), also when $g$ is subtangential then $c_\#$ coincides with the critical speed $c_\#$ for the existence of semi-wavefronts (see [43, Theorem 4.4]). In particular, if $g$ satisfies the D-K condition $L_g = g'(0)$ then $g$ is subtangential and $c_* = c_\# = c_\#$. One of the main results of this work is show the stability of semi-wavefronts (with unbounded exponential weights) for $c \geq c_\#(h)$ for all delay $h \geq 0$ (the same conditions for $c$ and $h$ to establish the uniqueness of semi-wavefronts in [1]).
In order to overcome the non-monotony of some reaction-diffusion equations with delay a \textit{quasi-monotonicity} condition is assumed which usually requires the monotony of the delayed argument. Indeed, in a pioneering work, Schaaf [29] considered the following parabolic functional differential equation

\begin{equation}
    u_t(t, x) = u_{xx}(t, x) + f(u(t, x), u(t - h, x)), \quad x \in \mathbb{R}, \ t > 0.
\end{equation}

Schaaf proved that for a concave nonlinearity $f$ with exactly two equilibria (0 and 1) satisfying a certain positivity condition (see [29, Section 2.1]) and

\begin{equation}
    \partial_2 f(u, v) \geq 0 \quad \text{for all } u, v \in \mathbb{R}_+ \quad \text{(quasi-monotonicity condition)}
\end{equation}

wavefronts are linearly stable for small delay $h$ [29, Theorem 4.13].

After the pioneering work by Schaaf, a series of other studies appeared where the KPP condition $|g'(u)| \leq g'(0)$, for all $u \geq 0$, or the concave condition $g''(u) \leq 0$, for all $u \geq 0$, was instrumental for the stability analysis. Among these studies, we would like to distinguish an important contribution [24] by Mei, Ou and Zhao where the authors proved the global stability of monotone wavefronts (critical as well as non-critical ones, see [24, Theorem 2.2]) of the following non-local equation

\begin{equation}
    u_t(t, x) = u_{xx}(t, x) - u(t, x) + \int_{\mathbb{R}} K(y)g(u(t - h, x - y))dy \quad x \in \mathbb{R}, \ t > 0,
\end{equation}

for monotone and concave $g$ and $K$ a heat kernel; here the perturbations are taken in weighted Sobolev spaces. At the same time, Lv and Wang [20] proved the global stability of non-critical wavefronts of (6) for monostable $f$ (with exactly two equilibria: 0 and $\kappa > 0$) satisfying (7) and the concavity condition: $\partial_{ij} f(u, v) \leq 0$ (i, j = 1, 2), for all $u, v \in [0, \kappa]$. The authors in [20] also study (6) with non-local reaction term (which includes (8), for monotone $g$) and demonstrated the stability of non-critical monotone wavefronts in Sobolev spaces with exponential weights; this result can also be obtained by our approach to (1) even to non-monotone wavefronts, see Remark 17 (for the non-local equation (8) see our recent work [31]).

With respect to non-monotone wavefronts, we should mention a work of Wu et al. [41, Theorem 2.4] where the authors take some type non-monotone $g \in C^2([0, \kappa], \mathbb{R})$ (`crossing monostable’ nonlinearity) satisfying $|g'(\kappa)| < 1$ and prove the local stability of wavefronts with speed $c$ for $c > 2\sqrt{2(L_g - 1)}$ and for all $h \geq 0$. Additionally, by assuming the KPP condition and $|g'(\kappa)|$ sufficiently small they prove the local stability of wavefronts with speed $c$ for all $c > c^*$ and $h \geq 0$ [41, Theorem 2.6]; here the existence of non-monotone wavefronts can be deduced, e.g., from [36] and [15]. Our second result for non-linear stability of wavefronts generalizes these results (see Remark 10). In this regard, for unimodal $g$ (i.e., $g$ has exactly one critical point which is the absolute maximum point) satisfying the KPP condition and $|g'(\kappa)| < 1$, Lin et al. [19] proved the local stability of non-critical wavefronts for all $h \geq 0$ (monotone or non-monotone) which includes well-known models
such as Nicholson’s model (see [15,19,33] and references therein) described by
\[ u_t(t, x) = u_{xx}(t, x) - \delta u(t, x) + \rho u(t-h,x) e^{-u(t-h,x)}, \quad t > 0, x \in \mathbb{R}, \quad (9) \]
where \( \rho, \delta > 0 \), or the Mackey–Glass model [1,3,19,23,24] given by
\[ u_t(t, x) = u_{xx}(t, x) - \tau u(t, x) + \frac{ab^n u(t-h,x)}{b^n + u^n(t-h,x)}, \quad t > 0, x \in \mathbb{R}. \quad (10) \]
where \( \tau, a, b > 0 \) and \( n \in \mathbb{Z}_+ \). These stability results were established in weighted Sobolev spaces to initial data with a suitable convergence to \( \kappa \) at \( x = +\infty \).

When \( \rho/\delta \in (1, e] \) in (9) wavefronts are monotone and by [24] they (critical as well as non-critical ones) are globally stable. The authors in [19] proved the local stability of (monotone and non-monotone) non-critical wavefronts to (9) when \( \rho/\delta \in (e, e^2) \) for all \( h \geq 0 \) and for small delay \( h \) when \( \rho/\delta \in (1, +\infty) \). Then, assuming \( |g'(\kappa)| < 1 \), Chern et al. [8, Theorem 2.3] have demonstrated the local stability of critical wavefronts (monotone or non-monotone) in the same Sobolev spaces.

Next, for Lipschitz continuous function \( g \) satisfying the D-K condition \( L_g = g'(0) \) and \( |g'(u)| < 1 \) in some neighborhood of \( \kappa \), Solar and Trofimchuk have established the global stability of (monotone or non-monotone) non-critical wavefronts [33, Corollary 3]. In particular, they obtained the global stability of non-critical wavefronts for (9) when \( \rho/\delta \in (1, e^2) \) for all \( h \geq 0 \). Here initial data are not required to convergence to \( \kappa \) at \( x = +\infty \) as above mentioned works. Then, in a recent work, for unimodal \( g \in \mathcal{C}^2[0, +\infty) \) satisfying the KPP condition, Mei et al. [22] have generalized the results in [8,19] for a global perturbation in the same Sobolev spaces.

On the other hand, non-subtangential models have recently attracted a lot of interest because of their connection to the so-called Allee effect in population dynamics [6,7,10,26]. More precisely, if we only consider as benefit to species a greater availability of resources then the per capita growth rate \( g(u)/u \) attains its maximum at \( u = 0 \), however if animal behavior is cooperative then individuals obtain benefits for intermediate densities \( u > 0 \) (individual fitness) which are not generated for low densities \( (u = 0) \), so that the per capita growth rate \( g(u)/u \) attains its maximum at some \( u_0 > 0 \). In this case model is said to have an Allee effect [9, Chapter 1] (since in our case the per capita growth birth rate \( g(u)/u \) is non-decreasing in a neighborhood of \( u = 0 \) model is said to have a weak Allee effect). In contrast to subtangential case, for a model with Allee effect it could occur \( c^* > c_\# \), critical wavefronts with speed \( c^* > c_\# \) are called pushed wavefronts. In this direction, for monotone \( g \) (necessarily non-subtangential), it has been possible to establish the stability of pushed wavefronts (see [32] and [42]) as well as that of non-critical wavefronts [33, Theorem 1]. These results show that pushed wavefronts are more attractive than critical wavefronts with speed \( c^* = c_\# \), for instance pushed wavefronts attract (orbitally) to the solution of (1) generated by the Heaviside step function while a critical wavefront (which is not a pushed wavefront) requires a logarithmic correction to attract this solution (see e.g. [12,39] for
$h = 0$ and [4] for $h > 0$). It is important to mention that the problem of the existence of semi-wavefronts for non-subtangential models is not completely solved (e.g., see [36, Corollary 4]): of course, in the available literature there are some partial results on the existence of semi-wavefronts for certain subclasses of equations, e.g., see [37, Theorem 2.4]).

Hence, in the above mentioned works, we can find stability results for Eq. (1) only when $g$ either is monotone or meets the sub-tangency condition. In this work we study the stability of semi-wavefronts without assuming the quasi-monotonicity nor the sub-tangency condition on $g$. Our approach uses ideas from [33] and a suitable Fourier analysis for partial functional differential equations. In the particular case when $g$ is unimodal and satisfies the KPP condition, i.e. $L_g = g(0)$ and $g \in C^1[0, +\infty)$, our estimates (in different spaces) are similar than [8,19,22,24] for perturbations of wavefronts, but our Fourier analysis (for wavefronts and proper semi-wavefronts) is different in many aspects, for instance by our approach Fourier transforms are estimated by means of a Halanay inequality on Banach spaces (see Lemma 12) instead of finite-dimensional spaces while the non-critical case $c > c_*$ (for wavefronts and proper semi-wavefronts) does not require Fourier analysis as in [19] (see Corollary 20 and Remark 17 below). However, approach used in [8,19,22,24] allows us to obtain stability results of wavefronts on the real line when $|g'(\kappa)| > 1$ for small $h$ whenever the initial datum $u_0(s, x)$ converges to $\kappa$ at $x = +\infty$.

In this regard, we obtain a general stability result for semi-wavefronts on each semi-infinite interval $(-\infty, N]$, $N \in \mathbb{R}$, without assume the restriction $|g'(\kappa)| < 1$ for all $h \geq 0$ (Theorem 3) which also includes critical semi-wavefronts and asymptotically periodic semi-wavefronts (see, e.g. [36, Theorem 3]). This kind of stability seems to be transversal to another models, indeed in a recent work [5] Benguria and Solar have established the stability of a class of non-monotone semi-wavefronts for the Hutchinson diffusive equation

$$u_t(t, x) = u_{xx}(t, x) + u(t, x)(1 - u(t - h, x)), \quad t > 0, \quad x \in \mathbb{R}, \quad (11)$$
on each semi-infinite interval $(-\infty, N]$.

More precisely, by a suitable Fourier analysis we can show that for $c \geq c_*$, an initial perturbation

$$u_0(s, x + cs) - \psi_c(x + cs) = e^{\lambda(x + cs)}r(s, x), \quad \text{for all} \quad (s, x) \in [-h, 0] \times \mathbb{R},$$

with $r \in C([-h, 0], L^1(\mathbb{R}))$ and $\lambda$ satisfying $E_c(\lambda) \leq 0$ [according to definition (2)], evolves as

$$u(t, x + ct) - \psi_c(x + ct) = O(t^{-1/2}e^{\gamma t}), \quad \text{for all} \quad t \geq -h,$$

uniformly for $x + ct \in (-\infty, N]$, $N \in \mathbb{R}$ and some $\gamma = \gamma(\lambda) \leq 0$. Moreover, $\gamma = 0$ if and only if $E(\lambda) = 0$, i.e. $\lambda = \lambda_1(c)$ or $\lambda = \lambda_2(c)$.

On the other hand, in our second main result we study the convergence of perturbations of wavefronts on the remaining domain $(N, +\infty)$, $N \in \mathbb{R}$. In this case it is necessary to assume the stability condition $|g'(\kappa)| < 1$ in order to establish (without assuming monotonicity or sub-tangency condition on $g$) the local stability of wavefronts with $c > c_*$ on whole the real line $(-\infty, +\infty)$ (Corollary 20). Additionally, assuming $|g'(u)| < 1$ for $u$ in a suitable
neighborhood of $\kappa$ we obtain the global stability of wavefronts with $c > c_*$ on whole real line $(-\infty, +\infty)$ (see Theorem 9 below).

In order to obtain these stability results we study the decay of solutions of the constant coefficient linear equation with delay,

$$u_t(t, x) = u_{xx}(t, x) + mu_x(t, x) + pu(t, x) + qu(t - h, x + d), \quad x \in \mathbb{R}, t > 0,$$

where the parameters $m, p, q, d$ are real numbers.

For an initial datum $u_0 \in C([-h, 0], L^1(\mathbb{R}))$, let us denote $C_{u_0} := \sup_{s \in [-h, 0]} ||u_0(s, \cdot)||_{L^1}$.

**Theorem 1.** Suppose that $-p \geq q \geq 0$ and $m, d \in \mathbb{R}$. Let $\gamma \leq 0$ be the only real solution of the following equation:

$$\gamma - p = qe^{-h\gamma}.$$  \hspace{1cm} (13)

If the initial datum $u_0$ belongs to $C([-h, 0]; L^1(\mathbb{R}))$ then the solution $u(t, x)$ of (12) satisfies the estimate:

$$\sup_{x \in \mathbb{R}} |u(t, x)| < A_0 \frac{e^{\gamma t}}{\sqrt{t}}, \quad \text{for all } t > h,$$

where $A_0 = C_{u_0}/2\sqrt{1 + h(\gamma - p)}$.

We note that in the special case $-p = q$ (which implies $\gamma = 0$) an exponential estimate is no longer available. In some cases, it can be established that the decay is not faster than that given by (14). For instance, if $d = 0$ for the evolution equation (12), the behavior of the solutions in the $L^1(\mathbb{R})$ phase space with an appropriate weight can be specified. In fact, we obtain the exact behavior which is embodied in Theorem below.

**Theorem 2.** (Asymptotic behavior) Let us consider (12) with $m, p \in \mathbb{R}$, $q \geq 0$ and $d = 0$. Let $u(t, x)$ be the solution generated by the initial data $u(s, \cdot) = e^{\sigma s}u_0$ where $u_0$ is such that $e^{\frac{m^2}{4}}u_0 \in L^1(\mathbb{R})$ and $\sigma$ is the only real solution of

$$qe^{-\sigma h} = \sigma + \frac{m^2}{4} - p,$$

then

$$\lim_{t \to \infty} \sqrt{t} e^{-\sigma t} u(t, x + o(\sqrt{t})) = \frac{\sqrt{1 + hqe^{-\sigma h}}}{2\sqrt{\pi}} e^{-\frac{m^2}{4}x} \int_{\mathbb{R}} e^{\frac{m^2}{2}y} u_0(y) dy,$$

for all $x \in \mathbb{R}$.

Now, for the study of the stability of semi-wavefronts with speed $c$, the following equation should be considered

$$v_t(t, z) = v_{zz}(t, z) - cv_z(t, z) - v(t, z) + g(v(t - h, z - ch)), \quad t > 0, z \in \mathbb{R}.$$  \hspace{1cm} (17)

For $c \geq c_*$ let us fix $\lambda_c \in [\lambda_1(c), \lambda_2(c)]$ and let us denote by $\xi_c(z) := e^{-\lambda_c z}$. Now, the first main result of this article can be set out.
Theorem 3. (Stability with weight) Assume that \( c \geq c_* \). Let \( v_0(s,z) \) and \( \psi_0(s,z) \) be two initial data to (17) such that \( v_0,\psi_0 \in C([-h,0];L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})) \), some \( \alpha \in (0,1) \), and

\[
\begin{align*}
  u_0(s,z) := \xi_c(z)|v_0(s,z) - \psi_0(s,z)| \in C([-h,0],L^1(\mathbb{R})),
\end{align*}
\]

then there are unique solutions \( v(t,z) \) and \( \psi(t,z) \) of (17) with initial data \( v_0 \) and \( \psi_0 \), respectively, and these solutions satisfies \( v(\cdot + kh,\cdot), \psi(\cdot + kh,\cdot) \in C([-h,0];L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})) \) for all \( k \in \mathbb{Z}_+ \). Moreover, if \( u(t,z) \) satisfies (12) with initial data \( u_0(s,z) \) and with parameters \( m = m(\lambda_c) = 2\lambda_c - c, \ p = p(\lambda_c) = \lambda_c^2 - c\lambda_c - 1, \ q = q(\lambda_c) = L_0e^{-\lambda_c ch} \) and \( d = -ch \), then

\[
\begin{align*}
  \xi_c(z)|v(t,z) - \psi(t,z)| \leq u(t,z), \quad \text{for all} \quad t \geq -h, z \in \mathbb{R},
\end{align*}
\]

in particular

\[
|v(t,z) - \psi(t,z)| \leq A_0 \xi_c(-z) \frac{e^{\gamma t}}{\sqrt{t}}, \quad \text{for all} \quad t > h, z \in \mathbb{R}
\]

where \( \gamma = \gamma(\lambda_c) \) is defined by (13) with \( p = p(\lambda_c) \) and \( q = q(\lambda_c) \).

Corollary 4. (Uniqueness) If \( \psi_c(z) \) and \( \phi_c(z) \) are two semi-wavefronts with speed \( c \geq c_* \) satisfying (18) then there exists \( z_* \in \mathbb{R} \) such that \( \psi_c(z + z_*) = \phi_c(z) \) for all \( z \in \mathbb{R} \).

Remark 5. If \( h = 0 \) in (1) then semi-wavefronts are monotone wavefronts and by taking a wavefront \( \psi(t,z) = \psi_c(z) \) in Theorem 3 we get the stability of the wavefront on the sets \( (-\infty,N], N \in \mathbb{R} \), which is comparable to a result obtained by Uchiyama [39, Theorem 4.1].

It has recently been showed that the estimation \( u(t,x) = O(t^{-1/2}) \) in (19)—(20) for critical semi-wavefronts, in the D-K case, is actually \( u(t,x) = o(t^{-1/2}) \) for all \( h \geq 0 \) (see [4, Corollary 1.2]). Also, since Theorem 3 does not assume some stability condition on \( \kappa \) then semi-wavefronts could be asymptotically periodic at \( +\infty \) [36, Theorem 3] and oscillations around \( \kappa \) can be approximated by the solution \( v(t,z) \) on each interval in the form \( (-\infty,N] \) with \( N \in \mathbb{R} \). The Corollary 4 refers essentially to the fact that semi-wavefronts are equal (up to translation) if they have the same one-order asymptotic terms at \( z = -\infty \), i.e., the condition (18).

By the change of variable \( t' := \delta^{-1}t \) and \( x' := \delta^{-1/2}x \) Eq. (9) can be reduced to (1) with delay \( h' := h\delta \) so that by Theorem 3 we obtain the stability of semi-wavefronts with speed \( c \) for the Nicholson’s model,

Corollary 6. (Nicholson Model) Let \( \rho/\delta \in [1, +\infty) \) be in (9). Consider \( N \in \mathbb{R} \) and the initial datum satisfying the conditions of Theorem 3, if \( c \geq c_* \) and \( \lambda_c \in [\lambda_1(c), \lambda_2(c)] \) then

\[
\begin{align*}
  \sup_{z \in (-\infty,N]} |v(t,z) - \psi_c(z)| = O(t^{-1/2}e^{\gamma t}),
\end{align*}
\]

where \( \gamma = \gamma(\lambda_c) \leq 0 \) is determined by (13).
It is well known that if \( \rho/\delta \in (e, e^2) \) then there are non-monotone wavefronts (see [15, Theorem 2.3]). Moreover, for some critical value \( \nu_0 = 2.808 \ldots \) and some delay \( h_0 \) if \( \rho/\delta \in [\nu_0, +\infty) \) then each minimal wavefront has oscillations around \( \kappa \) at \( +\infty \) and there exist a critical value \( \epsilon^* \) (a extended real number) such that each semi-wavefront with speed \( c > c_* \) has non-decaying slow oscillations [36, Theorem 3].

Nevertheless, the semi-wavefronts of Theorem 3 could exhibit a type of **convective instability** due to the positive equilibrium (e.g., see [27]), however by controlling the size of the slope of \( g \) at the positive equilibrium, the stability of the semi-wavefront on the remaining domain \([N, +\infty), N \in \mathbb{R}, \) can be obtained. In this framework, it is necessary to assume some additional hypotheses in order to establish the existence of semi-wavefronts, such as the following condition.

**Proposition 7.** (Existence of semi-wavefronts) Let \( g \) satisfy (M). Then, for each \( c > c_* (g_+^*) \) (according to definition (3)) Eq. (1) has semi-wavefronts with speed \( c \). Moreover, if \( 0 < \zeta_1 \leq \zeta_2 \) meet (B1)–(B4) then each semi-wavefront \( \psi_c \) satisfies:

\[
\zeta_1 \leq \liminf_{z \to -\infty} \psi_c(z) \leq \limsup_{z \to +\infty} \psi_c(z) \leq \zeta_2, \quad \text{for all } z \in \mathbb{R}.
\]

**Remark 8.** (Minimal speed for semi-wavefronts) Due [16, Theorem 18], in the case that \( g_+^* = g'(0) \) the number \( c_* (g_+^*) \) is actually the minimal speed for the existence of semi-wavefronts.

Now, let us introduce some notation. If \( I \subset \mathbb{R}^+ = \text{Dom}(g) \), let us denote by

\[
L_g(I) := \sup_{x \neq y, x, y \in I} \frac{|g(x) - g(y)|}{|x - y|},
\]

and for \( b \in \mathbb{R} \), let us denote by \( \eta_b(z) = \min \{1, e^{\lambda_c(z-b)}\} \), with \( \lambda_c \in [\lambda_1(c), \lambda_2(c)] \). With these notations, the second main result of this paper can be established

**Theorem 9.** (Global stability) Let \( c > c_* \) and \( \bar{g} \) be a non-decreasing function satisfying (M) with equilibrium \( K \) such that \( \bar{g}(u) \geq g(u) \) for all \( u \in \mathbb{R}^+ \) such
that $L_{\bar{g}} \leq L_g$. We denote by $m_K = \min_{u \in [\kappa,K]} g(u)$ and $\mathcal{I}_K := [m_K, K]$ and we suppose that $L_g(\mathcal{I}_K) < 1$. If for some $q_0 > 0$ and $z_0 \in \mathbb{R}$ the initial datum satisfies

$$v_0(s, z) \geq q_0 \text{ for all } (s, z) \in [-h, 0] \times [z_0, +\infty)$$  \hspace{1cm} (21)$$

and for some wavefront $\psi_c$, $b \in \mathbb{R}$ and $q > 0$

$$|v_0(s, z) - \psi_c(z)| \leq q\eta_h(z) \text{ for all } (s, z) \in [-h, 0] \times \mathbb{R},$$  \hspace{1cm} (22)

then there exists $C = C(\bar{g}, m_K, b) > 0$ and $\gamma_0 \geq 0$ satisfying

$$-\lambda_c^2 + c\lambda_c + 1 \geq \gamma_0 + L_g e^{\gamma_0 h} e^{-\lambda_c h} \text{ and } L_g(\mathcal{I}_K) \leq e^{-\gamma_0 h}(1 - \gamma_0),$$  \hspace{1cm} (23)

such that

$$|v(t, z) - \psi_c(z)| \leq C q e^{-\gamma_0 t}, \text{ for all } (t, z) \in [-h, +\infty) \times \mathbb{R}.$$  \hspace{1cm} (24)

**Remark 10.** (Crossing-monostable case) In [41] Wu et al. established the local stability for sufficiently fast wavefronts of the so-called crossing-monostable case. Theorem 9 generalizes those results by including global perturbations of wavefronts. More precisely, we suppose that for some positive number $K \geq \kappa$, the birth function $g$ is such that

(C1) \hspace{1cm} $g$ satisfies (M)

(C2) \hspace{1cm} $g(u) \leq \bar{g}(u) := \max\{g'(0)u, K\}$ for all $u \geq 0$ and

(C3) \hspace{1cm} $L_g(\mathcal{I}_K) < 1,$

then the non decreasing function $\bar{g}(u)$ clearly satisfies $L_{\bar{g}} = g'(0) \leq L_g$, therefore $g$ satisfies the conditions of Theorem 9. Hence if $g$ satisfies (C1)-(C3) then (22) implies (24). Note that the condition $(A_3)$ in [41, Theorem 2.4] is essentially our condition $(C_3)$. Moreover, by (2) and the definition of $c_*(h)$ we have $c_*(h)$ is a non-increasing function of $h$, therefore we have $c_*(h) \leq c_*(0) = 2\sqrt{L_g - 1}$ for all $h \geq 0$, so that we have improved the minimal speed $\tilde{c} := 2\sqrt{2(L_g - 1)}$ given in [41] for the local stability of wavefronts with speed $c > \tilde{c}$.

Now, if we take $\bar{g}(u) = \max_{s \in [0, \kappa]} g(s)$ then we have that $K = M_g := \max_{s \in [0, \kappa]} g(s)$ and by writing $m_g = \min_{u \in [\kappa, M_g]} g(u)$ and $\mathcal{I}_K = I_g := [m_g, M_g]$ the following global stability result is obtained.

**Corollary 11.** Let $g$ satisfy (M) such that $L_g(I_g) < 1$. If $\psi_c$ is a semi-wavefront with speed $c > c_*$, then $\psi_c$ is globally stable in the sense of Theorem 9.

Corollary 11 generalizes results for wavefronts which assume the D-K condition (see, e.g. [33]). In the Allee case with monotone $g$, Corollary 11 is an improvement, in terms of the globality of the disturbance, of [33, Theorem 2] for wavefronts with a speed greater than $c_*$ and it also gives us an exponential convergence rate for these waves. In this regard, exponential (in the time) stability as in (24) for pushed wavefronts was not studied in [32] but a recent work [42] by Wu, Niu and Hsu, has given a positive answer to this problem.

This paper is organized as follows. The linear theorems (Theorems 1 and 2) are proven in Sect. 2. Finally, results on the stability of semi-wavefronts are proven in Sect. 3.
2. Proof of Linear Theorems

In order to demonstrate both Theorems 1 and 2, the following two lemmas will be needed. The first one is an abstract version of the Halanay type inequalities [17]

**Lemma 12.** (Halanay Type Inequality) Let $X$ be a complex Banach space. Suppose that $\sigma, k \in \mathbb{C}$ and $h > 0$. If $r \in C([-h, \infty), X)$ is a function satisfying:

$$r_t(t) = \sigma r(t) + kr(t - h), \text{ a.e.},$$

then

$$|r(t)|_X \leq \sup_{s \in [-h, 0]} |r(s)|_X e^{\max\{0, \lambda\} h} e^{\lambda t}, \text{ for all } t > -h,$$

where $\lambda$ is the only real root of the equation:

$$\lambda = Re(\sigma) + |k| e^{-\lambda h}. \quad (26)$$

Moreover

(i) $\lambda \leq 0 \iff -Re(\sigma) \geq |k|.$  
(ii) $\lambda = 0 \iff -Re(\sigma) = |k|.$

**Proof.** It is clear that:

$$\frac{d}{dt}(r(t)e^{-\sigma t}) = ke^{-\sigma t}r(t - h) \text{ a.e.}$$

and from here, it is obtained that $|r(t)|_X$ meets the following inequality:

$$x(t) \leq |k| \int_0^t e^{Re(\sigma)(t-s)}x(s - h)ds + x(0)e^{Re(\sigma)t} \text{ for all } t > 0 \quad (27)$$

We note that for $A \in \mathbb{R}$ the function $e_A(t) = Ae^{\lambda t}$ meets (27) with equality. Now, for $A := \sup_{s \in [-h, 0]} |r(s)|_X e^{\max\{0, \lambda\} h}$ the function $\delta(t) = |r(t)|_X - e_A$ satisfies (27) for $t \in [0, h]$ and therefore $\delta(t) \leq 0$ for all $t \in [0, h]$. Similarly, it is concluded that $\delta(t) \leq 0$ for the intervals $[h, 2h], [2h, 3h] \ldots$ This proves (25).

Let us prove (i). If $-Re(\sigma) \geq |k|$ then: $\lambda \leq |k|(e^{-h\lambda} - 1)$ which necessarily implies that $\lambda \leq 0$. Otherwise, if $\lambda \leq 0$ let us suppose that $-Re(\sigma) < b$, then $\lambda > |k|(e^{-h\lambda} - 1)$ which is a contradiction.

In order to prove (ii) let us note that since the derivative of $f(\lambda) := \lambda - Re(\sigma) - |k|e^{-h\lambda}$ is always positive then $f(\lambda)$ has at most one zero. So, if $Re(\sigma) = |k|$ then $\lambda = 0$ is the only solution of (26), this proves (ii). \qed

Now, let us consider the function $\lambda : \mathbb{R} \to \mathbb{R}$ defined by

$$\lambda(\zeta) = -\zeta^2 + p + qe^{-h\lambda(\zeta)}, \quad (28)$$

where $q \geq 0$. Next, we proceed to estimate the even function $\lambda(\zeta)$.

For $\epsilon_h = \frac{1}{1 + h(\gamma - p)}$ we define the function

$$\alpha_h(\zeta) := -\frac{1}{h} \log(1 + h\epsilon_h \zeta^2).$$

Here $\gamma \in \mathbb{R}$ is defined by (13) for any $p \in \mathbb{R}$ and $q \geq 0.$
Lemma 13. If \( \lambda \) is defined by (28) then
\[
- \epsilon_h \zeta^2 + \gamma \leq \lambda(\zeta) \leq \alpha_h(\zeta) + \gamma \quad \text{for all } \zeta \in \mathbb{R}.
\]
Moreover, if \( q > 0 \) then
\[
\lim_{|\zeta| \to \infty} q^{-1} \zeta^2 e^{h\lambda(\zeta)} = 1.
\]

Remark 14. The function \( \alpha_h \) is a generalization of the function \( \alpha_0(\zeta) := -\zeta^2 = \lim_{h \to 0} \alpha_h(\zeta) \) for each \( \zeta \in \mathbb{R} \). Also, when \( h = 0 \) then \( \gamma = p + q \) [according to definition (13)] therefore \( \lambda(\zeta) = -\zeta^2 + \gamma \) in (28). Thus, by passing the limit \( h \to 0 \) in (29) we have the equality \( -\zeta^2 = \lambda(\zeta) - \gamma = \alpha_0(\zeta) \). In this regard, the estimates in (29) are sharp.

Proof. Let us denote \( \beta(\zeta) = \lambda(\zeta) - \alpha(\zeta) - \gamma \). Then \( \beta(\zeta) \) satisfies the following equation
\[
\beta(\zeta) = -\zeta^2 + \frac{1}{h} \log(1 + h \epsilon_h \zeta^2) - \gamma + p + q e^{-h\gamma}(1 + h \epsilon_h \zeta^2) e^{-h\beta(\zeta)}.
\]
From Lemma 12 we have that \( \beta(\zeta) \leq 0 \) if and only if:
\[
\zeta^2 - \frac{1}{h} \log(1 + h \epsilon_h \zeta^2) + \gamma - p \geq q e^{-h\gamma}(1 + h \epsilon_h \zeta^2).
\]
Now, using \( \log(1 + x) \leq x \), for all \( x \geq 0 \), in order to obtain (31) it is enough to have
\[
\zeta^2 - \epsilon_h \zeta^2 + \gamma - p \geq q e^{-h\gamma}(1 + h \epsilon_h \zeta^2) \quad \text{for all } \zeta \in \mathbb{R}
\]
\[
\iff (1 - \epsilon_h - q h \epsilon_h e^{-h\gamma}) \zeta^2 + \gamma - p - q e^{-h\gamma} \geq 0 \quad \text{for all } \zeta \in \mathbb{R},
\]
which is a consequence of definition of \( \gamma \) and \( \epsilon_h \). So that, this proves the upper estimate in (29)

To complete left hand side of (29) we note that due to (28), (13) and upper estimate in (29)
\[
\lambda(\zeta) \geq -\zeta^2 + \gamma - q^{-1} + q e^{-h\gamma} [1 + h \epsilon_h \zeta^2]
\]
\[
= -\epsilon_h \zeta^2 + \gamma.
\]
Next, by multiplying (28) by \( e^{h\lambda(\zeta)} \) and by using that \( \lambda(\zeta) \to -\infty \) as \( |\zeta| \to +\infty \) [which is obtained from upper estimation in (29)] we conclude
\[
\lim_{\zeta \to \pm \infty} e^{h\lambda(\zeta)} \zeta^2 = p
\]
which implies (30). \( \square \)

Consider the following equation
\[
u_t(t, z) = u_{zz}(t, z) + d_1 u_z(t, z) + d_2 u(t, z) + e^{-\lambda z} g(e^{\lambda(z-ch)} u(t-h, z-ch))
\]
where \( d_1, d_2, \lambda \in \mathbb{R} \)
Proposition 15. If \( u_0 \in C([-h,0]; L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})) \), some \( \alpha \in (0,1] \), then there is a unique solution \( u(t,z) \) of (32) with initial data \( u_0(t) \) and this solution satisfies \( u(\cdot + kh, \cdot) \in C([-h,0]; L^\infty(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})) \) for all \( k \in \mathbb{Z}_+ \). Moreover, if \( u_0 \in C([-h,0]; L^1(\mathbb{R})) \) and \( \lambda = 0 \) in (32) then \( u(t,\cdot), u_z(t,\cdot) \in L^1(\mathbb{R}) \) for all \( t \geq 0 \) and \( u_{zz}(t,\cdot) \in L^1(\mathbb{R}) \) for all \( t > h \).

Proof. By defining
\[
d_3(t,z) := e^{-\lambda ch}g(e^{(z-ch)}u(t-h, z - ch)) \frac{e^{(z-ch)}u(t-h, z - ch)}{u(t,z) - ch}
\]
we have \( d_3 \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \) and the function \( u \) satisfies
\[
u(t,z) = u_{zz}(t,z) + d_1 u_{zz}(t,z) + d_2 u(t,z) + d_3(t,z) u(t-h, z - ch) \tag{33}
\]
By making the change of variables \( \bar{u}(t,z) := u(t,z - d_1 t)e^{-d_2 t} \) the equation (33) is reduced to an inhomogeneous heat equation
\[
\bar{u}_t(t,z) = \bar{u}_{zz}(t,z) + f(t,z), \tag{34}
\]
where
\[
f(t,z) = e^{-d_2 h}d_3(t,z - d_1 t)\bar{u}(t-h, z - h(c + d_1)), \tag{35}
\]
Now, note that for \( 1 \leq p \leq \infty \)
\[
|f(t,\cdot)|_{L^p(\mathbb{R})} \leq e^{-d_2 h}|d_3|_{L^\infty} \max_{s \in [-h,0]} |\bar{u}_0(s,\cdot)|_{L^p(\mathbb{R})} \text{ for all } t \in [0,h]. \tag{36}
\]
Similarly, by using the definition of \( d_3 \), we get
\[
|f(t,\cdot)|_{C^{0,\alpha}(\mathbb{R})} \leq L_g e^{-d_2 h} \max_{s \in [-h,0]} |\bar{u}_0(s,\cdot)|_{C^{0,\alpha}(\mathbb{R})} \text{ for all } t \in [0,h]. \tag{37}
\]
So that, by [14, Chapter 1, Theorems 12 and 16] there exist a unique solution to (32) and this solution satisfies
\[
\bar{u}(t) := \Gamma_t * \bar{u}(0) + \int_0^t \Gamma_{t-s} * f(s) ds, \tag{38}
\]
where \( \Gamma_t \) is the one-dimensional heat kernel.

Now, we take \( 1 \leq p \leq \infty \). Then, for \( t \in [0,h] \) and \( t_n \to t \) we have
\[
|\bar{u}(t) - \bar{u}(t_n)|_{L^p} \leq |\Gamma_t - \Gamma_{t_n}|_{L^1} |\bar{u}(0)|_{L^p} + \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1} |f(s)|_{L^p} ds
\]
\[
+ \int_t^{t_n} |\Gamma_{t_n-s}|_{L^1} |f(s)|_{L^p} ds, \tag{39}
\]
and by using (36),
\[
|\bar{u}(t) - \bar{u}(t_n)|_{L^p} \leq (|\Gamma_t - \Gamma_{t_n}|_{L^1}
\]
\[
+ \int_0^t |\Gamma_{t-s} - \Gamma_{t_n-s}|_{L^1} ds + |t - t_n| \max_{s \in [-h,0]} |\bar{u}_0(s,\cdot)|_{L^p(\mathbb{R})}, \tag{40}
\]
where \( R = \max\{1, e^{-d_2 h}|d_3|_{L^\infty}\} \). Since \( |\Gamma_{t_n}|_{L^1} = |\Gamma_t|_{L^1} = 1 \) the last inequality implies \( |\bar{u}(t) - \bar{u}(t_n)| \to 0 \) as \( t_n \to t \), therefore if \( u_0(\cdot,\cdot) \in C([-h,0]; L^p(\mathbb{R})) \) then \( u(\cdot + h, \cdot) \in C([-h,0]; L^p(\mathbb{R})) \). Similarly, we get \( u(\cdot + h, \cdot) \in C([-h,0]; C^{0,\alpha}(\mathbb{R})) \) whenever \( u_0(\cdot,\cdot) \in C([-h,0]; C^{0,\alpha}(\mathbb{R})) \).
Analogously, by using the initial data $u(t+h, \cdot), u(t+2h, \cdot) \ldots$ we obtain $u(\cdot + kh, \cdot) \in C([-h,0]; L^p(\mathbb{R}) \cap C^{0,\sigma}(\mathbb{R}))$ for $k = 2, 3 \ldots$. Therefore, with $p = \infty$ we obtain the first assertion of the Proposition 15.

Otherwise, if $u_0 \in C([-h,0], L^1(\mathbb{R}))$ then with $p = 1$ we get $u(\cdot + kh, \cdot) \in C([-h,0]; L^1(\mathbb{R}))$ for all $k \in \mathbb{Z}_+$. Then, note that by (38) for $t > 0$ we get

$$
\bar{u}_z(t, z) = \int_{\mathbb{R}} \frac{(y-z)e^{-(z-y)^2/4t}}{4t^{3/2} \sqrt{\pi}} u_0(0, y) dy + \int_0^t \int_{\mathbb{R}} \frac{(y-z)e^{-(z-y)^2/4(t-s)}}{4(t-s)^{3/2} \sqrt{\pi}} f(s, y) dy ds,
$$

and using (36) with $p = 1$ for $t \in (0, h]$ we obtain

$$
|\bar{u}_z(t, \cdot)|_{L^1(\mathbb{R})} \leq \frac{|u_0(0, \cdot)|_{L^1(\mathbb{R})}}{\sqrt{\pi t}} \int_{\mathbb{R}} |y| e^{-y^2} dy + 2\sqrt{\frac{1}{\pi}} \int_{\mathbb{R}} |y| e^{-y^2} dy \max_{s \in [-h, 0]} |f(s, \cdot)|_{L^1(\mathbb{R})}
$$

and by using the initial data $\bar{u}(t+h, \cdot), \bar{u}(t+2h, \cdot) \ldots$, with $t \in (0, h]$, we obtain $\bar{u}_z(t+kh, \cdot) \in L^1(\mathbb{R})$ for $k \in \mathbb{Z}_+$ and $t \in (0, h]$. Moreover, if we differentiate in (32) and proceed as in (39) and (40) then we have $|\bar{u}_z(t, \cdot)|_{L^1(\mathbb{R})}$ continuously depends on $t \in \mathbb{R}_+$.

Finally, if $T > h$ then $\bar{u}(T+\cdot, \cdot) \in C([-h,0]; L^1(\mathbb{R}))$, by taking $\lambda = 0$, we obtain $\bar{u}_z(t, z)$ satisfies (33) with $d_3(t, z) = g'(\bar{u}(t-h, z-h))$ and taking as initial datum the function $\bar{u}(T+s, z)$ and using (42) (replacing $\bar{u}_z$ by $\bar{u}_{zz}$) we obtain $\bar{u}_{zz}(T+t, \cdot) \in L^1(\mathbb{R})$ for all $t \in (0, h]$. Similarly, by using the initial data $\bar{u}_z(t+h, \cdot), \bar{u}_z(t+2h, \cdot) \ldots$, with $t \in (0, h]$, we obtain $\bar{u}_{zz}(t+T+kh, \cdot) \in L^1(\mathbb{R})$ for $k \in \mathbb{Z}_+$ and $t \in (0, h]$, which completes the proof.

\textbf{Remark 16.} Since by Proposition 15 $u(\cdot + hk, \cdot) \in C([-h,0], L^\infty(\mathbb{R}))$ for all $k \in \mathbb{Z}$ then for each $t > 0$ we have $f(\cdot, \cdot) \in C([0,t], L^\infty(\mathbb{R}))$, therefore

$$
\left| \int_{\mathbb{R}} \frac{(y-z)e^{-(z-y)^2/4(t-s)}}{4(t-s)^{3/2} \sqrt{\pi}} f(s, y) dy \right| \leq \frac{C}{\sqrt{t-s}} \quad \text{for all} \quad t > s,
$$

for some constant $C > 0$ (which does not depend on $(z,t,s)$) so that from (41) we conclude $u(t, \cdot) \in C^1(\mathbb{R})$ for all $t > 0$.

\textbf{Proof of Theorem 1.} By using Proposition 15 with $d_1 = m$, $d_2 = p$, $\lambda = 0$ and $q(u) = qu$ we get $u(t, \cdot), u_z(t, \cdot), u_{zz}(t, \cdot) \in L^1(\mathbb{R})$ for all $t > h$. Next, by applying the Fourier transform, here

$$
\hat{u}(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-izy} u(y) dy,
$$

to Eq. (12) we have

$$
\hat{u}_t(t, \zeta) = \sigma(\zeta) \hat{u}(t, \zeta) + k(\zeta) \hat{u}(t-h, \zeta) \quad \text{for all} \quad t > h,
$$

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where \( \sigma(z) = -\zeta^2 + im\zeta + p \) and \( k(\zeta) = qe^{-id\zeta} \).

Since \(-\text{Re}(\sigma(\zeta)) \geq |k(\zeta)|\), by Lemma 12 we obtain \( \lambda(\zeta) \leq 0 \) for all \( \zeta \in \mathbb{R} \) and:

\[
|\hat{u}(t, \zeta)| \leq C_{u_0} e^{\lambda(\zeta)t} \quad \text{for all} \quad \zeta \in \mathbb{R}.
\]

If \( t > h \) then by the Fourier’s inversion formula (since by Remark 16 \( u(t, \cdot) \in C^1(\mathbb{R}) \) for \( t > 0 \)) and Lemma 13, we have

\[
|u(t, x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{u}(t, \zeta)| d\zeta \leq \frac{C_{u_0}}{2\pi} \int_{\mathbb{R}} e^{\lambda(\zeta)t} d\zeta \leq \frac{C_{u_0}}{2\pi} e^{\gamma t} \int_{\mathbb{R}} \frac{d\zeta}{(1 + \epsilon\zeta^2)^{\frac{1}{2}}}.
\]

Moreover, by Bernoulli’s inequality, we conclude that

\[
\int_{\mathbb{R}} \frac{d\zeta}{(1 + \epsilon\zeta^2)^{\frac{1}{2}}} \leq \int_{\mathbb{R}} \frac{d\zeta}{1 + \frac{4}{\pi^2} \zeta^2} = \frac{1}{\sqrt{t}} \left[ \sqrt{\frac{h}{\epsilon}} \int_{\mathbb{R}} \frac{d\zeta}{1 + \zeta^2} \right] = \frac{1}{\sqrt{t}} \sqrt{\frac{h}{\epsilon}} \pi.
\]

\[\square\]

**Proof of Theorem 2.** If we make the change of variable \( v(t, x) = e^{\frac{m^2}{4}x} u(t, x) \), then \( v(t, x) \) solves

\[
v_t(t, x) = v_{xx}(t, x) + \left( p - \frac{m^2}{4} \right) v(t, x) + qv(t - h, x).
\] (43)

By applying the Fourier transform to (43) we get

\[
\hat{v}_t(t, z) = \left( -z^2 + p - \frac{m^2}{4} \right) \hat{v}(t, z) + q\hat{v}(t - h, z) \quad \text{for all} \quad t > 0 \quad (44)
\]

Let us note that due to \( q \geq 0 \), we have that (44) satisfies the Comparison Principle; that is, if for each \( z \in \mathbb{R} \) we consider two solutions \( v(s) \) and \( w(s) \) of (44) defined on \([-h, +\infty)\) then, by denoting \( \Re(\hat{u}(t)) = u_1(t), \Im(\hat{u}(t)) = u_2(t) \), the inequality

\[
v_i(s) \leq w_i(s) \quad \text{for all} \quad s \in [-h, 0] \quad \text{and} \quad i = 1, 2.
\]

implies

\[
v_i(s) \leq w_i(s) \quad \text{for all} \quad s \in [-h, +\infty) \quad \text{and} \quad i = 1, 2
\]

Let us denote by \( e_A(t, z) = A e^{\lambda(z)t} \), where \( \lambda(z) \) satisfies

\[
\lambda(z) = -z^2 + p - \frac{m^2}{4} + qe^{-\lambda(z)h}. \quad (45)
\]

Let us note that \( e_A(t, z) \) satisfies (44) for all \( A \in \mathbb{C} \). Also, let us denote that

\[
m_i(z) = \min_{s \in [-h, 0]} (v_i(s, z)e^{-\lambda(z)s}) \quad \text{and} \quad M_i(z) = \max_{s \in [-h, 0]} (v_i(s, z)e^{-\lambda(z)s}) \quad i = 1, 2
\]

then we have that

\[
e_{m_i}(s, z) \leq v_i(s, z) \leq e_{M_i}(s, z) \quad \text{for all} \quad (s, z) \in [-h, 0] \times \mathbb{R}; i = 1, 2.
\]
By the comparison principle applied to real and imaginary part in (44), we have that
\[ e_{m_i}(t, z) \leq v_i(t, z) \leq e_{M_i}(t, z) \quad \text{for all} \quad (t, z) \in [-h, \infty) \times \mathbb{R}; \quad i = 1, 2 \]
(46)
or
\[ m_i(z)e^{\lambda(z)t} \leq v_i(t, z) \leq M_i(z)e^{\lambda(z)t} \quad \text{for all} \quad (t, z) \in [-h, \infty) \times \mathbb{R}; \quad i = 1, 2 \]
(47)
Now, by the Fourier inversion formula, we have that
\[ v(t, x) = \frac{1}{\sqrt{t}} \int_{\mathbb{R}} e^{\frac{xy}{\sqrt{t}}} \hat{v}(t, y/\sqrt{t}) dy. \]
(48)
However, if we apply Lemma 13 to (45) with \( \gamma = \sigma \) we have
\[ \lim_{t \to \infty} t[\lambda(y/\sqrt{t}) - \sigma] = -\frac{y^2}{1 + h\rho e^{-\sigma h}}. \]
(49)
and due to \( v(s, \cdot) \in L^1(\mathbb{R}) \) by the Lebesgue’s dominated convergence theorem
\[ \lim_{t \to \infty} M_1(y/\sqrt{t}) = \lim_{t \to \infty} m_1(y/\sqrt{t}) = \int_{\mathbb{R}} e^{\frac{m}{2}x} v(s, x) dx \]
and
\[ \lim_{t \to \infty} M_2(y/\sqrt{t}) = \lim_{t \to \infty} m_2(y/\sqrt{t}) = 0. \]
Therefore by (47)
\[ \lim_{t \to \infty} \hat{v}(t, y/\sqrt{t}) = e^{-\frac{y^2}{1 + h\rho e^{-\sigma h}}} \int_{\mathbb{R}} e^{\frac{m}{2}x} v(s, x) dx \]
(50)
However, by (25) there exists \( C(p, q, m) > 0 \) such that
\[ |\hat{v}(t, y/\sqrt{t})| \leq C \sup_{s \in [-h, 0]} |\hat{v}(s, y/\sqrt{t})|e^{\lambda(y/\sqrt{t})t} \]
but by (29) and Bernoulli’s Inequality
\[ |\hat{v}(t, y/\sqrt{t})| \leq \frac{Ce^{\lambda|\sigma|h}||e^{\frac{m}{2}u_0}||_{L^1(\mathbb{R})}}{1 + \epsilon y^2} \quad \text{for all} \quad t > 0. \]
(51)
Finally, by (48), (51), Lebesgue’s dominated convergence theorem and (50), the result obtained.

3. Proof of results of stability of semi-wavefronts

Proof of Theorem 3. The first assertion follows from Proposition 15 with \( \lambda = 0 \). Next, for a solution \( w(t, z) \) of (17), let us denote the function \( \tilde{w}(t, z) = \xi_c(z)w(t, z) \) which satisfies
\[ \tilde{w}_t(t, z) = \tilde{w}_{zz}(t, z) + mw_z(t, z) + p\tilde{w}(t, z) + \xi_c(z)g(\xi_c(-z + ch)\tilde{w}(t - h, z - ch)). \]
We consider the linear operator
\[ \mathcal{L}\delta(t,z) := \delta_{zz}(t,z) + m\delta_z(t,z) + p\delta(t,z) - \delta_t(t,z). \]
If \( \delta_{\pm}(t,z) := \pm[\tilde{v}(t,z) - \tilde{\psi}(t,z)] - u(t,z) \), then by (18): \( \delta_{\pm}(s,z) \leq 0 \) for \( (s,z) \in [-h,0] \times \mathbb{R} \). For \( (t,z) \in [0,h] \times \mathbb{R} \) by (17) and (18) we have
\[
\mathcal{L}\delta_{\pm}(t,z) = \mp\xi(z)[g(\xi(-z + ch)\tilde{\psi}(t-h,z-ch)) - g(\xi(-z + ch)\tilde{\psi}(t-h,z-ch))] - \mathcal{L}u(t,z)
\]
\[
\geq -L_ge^{-rch}[\tilde{v}(t-h,z-ch) - \tilde{\psi}(t-h,z-ch)] - \mathcal{L}u(t,z)
\]
\[
\geq -L_ge^{-rch}u(t-h,z-ch) - \mathcal{L}u(t,z) = 0.
\]
Now, by Proposition 15, \( \tilde{w}(\cdot + kh, \cdot) \in C([-h,0]; L^\infty(\mathbb{R})) \) for all \( k \in \mathbb{Z}_+ \) therefore by using the Phragmén–Lindelöf principle from [25, Chapter 3, Theorem 1], we have \( \delta_{\pm}(t,z) \leq 0 \) for \( (t,z) \in [0,h] \times \mathbb{R} \). The argument is repeated for intervals \([h,2h], [2h,3h], \ldots\) to conclude (19). Finally, the estimate in (20) is obtained using Theorem 1.

**Remark 17.** Note that in Proof of Theorem 3 it was only necessary to have an initial datum \( u_0 \) exponentially bounded to apply the Phragmén–Lindelöf principle in order to obtain estimate (20). So, we could use the elementary exponential solutions of (12) of the form \( u(t,z) = Be^{r+z} \), with \( r \) and \( \gamma \) satisfying
\[
q(\lambda_c)e^{-rch}e^{-\gamma h} = -r^2 - (2\lambda_c - c)r - p(\lambda_c) + \gamma.
\]
Here, \( \gamma \leq 0 \) if and only if
\[
-r^2 - (2\lambda_c - c)r - p(\lambda_c) \geq q(\lambda_c)e^{-rch},
\]
with \( \gamma = 0 \) if and only if (53) holds with equality. Thus, for \( c > c_* \) and \( \lambda_c \in (\lambda_1(c), \lambda_2(c)) \) we have \( -p(\lambda_c) > q(\lambda_c) \) and therefore by taking \( r = 0 \) in (53) we obtain \( \gamma < 0 \) in (52) and therefore the asymptotic stability of non-critical is obtained. However, when \( c = c_* \) we have \( \lambda_{c_*} = \lambda_1(c_*) = \lambda_2(c_*) \) and \( -p(\lambda_{c_*}) = q(\lambda_{c_*}) \) in (53), also due to the curves \(-\lambda^2 + \lambda + 1 + L_ge^{-rch} \) in (2) are tangent at \( \lambda = \lambda_{c_*} \). The function \( \Theta(r) := q(\lambda_{c_*})e^{-rch} + r^2 + (2\lambda_{c_*} - c_*)r + p(\lambda_{c_*}) \) holds \( \Theta'(0) = 0 \). Consequently, since \( \Theta \) is strictly convex and \( \Theta(0) = 0 \) we conclude \( r = 0 \) is the only solution in (53) and therefore \( -p(\lambda_{c_*}) = q(\lambda_{c_*}) \) implies \( \gamma = 0 \) in (52). Thus, this approach does not allow us to obtain the asymptotic stability of critical semi-wavefronts.

**Theorem 18.** Let \( v(t,z) \) and \( \psi(t,z) \) be solutions of equation (17) for \( c \geq c_* \). Assume that for some compact interval \( I \subset \mathbb{R} \), such that \( L_9(I) < 1 \), and \( b \in \mathbb{R} \) we have
\[
\psi(t,z), v(t,z) \in I \quad \text{for all} \quad (t,z) \in [-h,\infty) \times [b-bh,\infty),
\]
and for some \( q > 0 \) and \( \lambda_c \in [\lambda_1(c), \lambda_2(c)] \)
\[
|v_0(s,z) - \psi_0(s,z)| \leq q\eta(z) \quad \text{for all} \quad (s,z) \in [-h,0] \times \mathbb{R}.
\]
If \( \gamma_0 \geq 0 \) satisfies
\[
-\lambda_{c_*}^2 + c\lambda_{c_*} + 1 \geq \gamma_0 + L_9e^{\gamma_0h}e^{-\lambda_{c_*}ch} \quad \text{and} \quad L_9(I) \leq e^{-\gamma_0h}(1 - \gamma_0),
\]
then

\[ |v(t,z) - \psi(t,z)| \leq qe^{-\gamma_0 t} \eta_b(z) \quad \text{for all } (t,z) \in [-h, \infty) \times \mathbb{R}. \]  \hfill (57)

Proof. We define \( \eta(t,z) = qe^{-\gamma_0 t} \eta_b(z) \) and write the operator

\[ \mathcal{L}_0 \delta(t,z) := \delta_{zz}(t,z) - c \delta_z(t,z) - \delta(t,z) - \delta_t(t,z). \]

Note that by (55) if \( \delta_{\pm}(t,z) := \pm[v(t,z) - \psi(t,z)] - \eta(t,z) \) then \( \delta_{\pm}(s,z) \leq 0 \) for \( (s,z) \in [-h,0] \times \mathbb{R}. \) Now, for \( (t,z) \in [0,h] \times (-\infty,b) \) due to (17), (22) and (56) we have that

\[ \mathcal{L}_0 \delta_{\pm}(t,z) = \pm[-g(v(t-h, z-ch)) + g(\psi(t-h, z-ch))] - \mathcal{L}_0 \eta(t,z) \]

\[ \geq qe^{-\gamma_0 t + \lambda_c(z-b)}[-L_g e^{\gamma_0 h} e^{-\lambda ch} - (\lambda_c^2 - c\lambda_c - 1 + \gamma_0)] \geq 0. \]

Similarly, if \( (t,z) \in [0,h] \times [b,\infty) \) we obtain:

\[ \mathcal{L}_0 \delta_{\pm}(t,z) = \pm[-g(v(t-h, z-ch)) + g(\psi(t-h, z-ch))] - \mathcal{L}_0 \eta(t,z) \]

\[ \geq qe^{-\gamma_0 t}[-L_g(I)e^{\gamma_0 h}(z - ch) - (-1 + \gamma_0)] \]

\[ \geq qe^{-\gamma_0 t}[-L_g(I)e^{\gamma_0 h} + 1 - \gamma_0] \geq 0. \]

Now, as in the proof of the [33, Lemma 1], due to

\[ \frac{\partial \delta_{\pm}(t,b+)}{\partial z} - \frac{\partial \delta_{\pm}(t,b-)}{\partial z} > 0, \]  \hfill (58)

we have that \( \delta_{\pm}(t,z) \leq 0 \) for all \( t \in [0,h], \) \( z \in \mathbb{R}. \) Indeed, otherwise there exists \( r_0 > 0 \) such that \( \delta(t,z) \) restricted to any rectangle \( \Pi_r = [-r,r] \times [0,h] \) with \( r > r_0 \), reaches its maximum positive value \( M_r > 0 \) at some point \( (t',z') \in \Pi_r \).

We claim that \( (t',z') \) belongs to the parabolic boundary \( \partial \Pi_r \) of \( \Pi_r \). Indeed, suppose on the contrary, that \( \delta(t,z) \) reaches its maximum positive value at some point \( (t',z') \) of \( \Pi_r \setminus \partial \Pi_r \). Then clearly \( z' \neq z_s \) because of (58). Suppose, for instance that \( z' > z_s \). Then \( \delta(t,z) \) considered on the subrectangle \( \Pi = [z_s, r] \times [0,h] \) reaches its maximum positive value \( M_r \) at the point \( (t',z') \in \Pi \setminus \partial \Pi \). Then the classical results [25, Chapter 3, Theorems 5,7] show that \( \delta(t,z) \equiv M_r > 0 \) in \( \Pi \), a contradiction.

Hence, the usual maximum principle holds for each \( \Pi_r, r \geq r_0 \), so that we can appeal to the proof of the Phragmèn–Lindelöf principle from [25] (see Theorem 10 in Chapter 3 of this book), in order to conclude that \( \delta(t,z) \leq 0 \) for all \( t \in [0,h], \) \( z \in \mathbb{R}. \)

We can again repeat the above argument on the intervals \([h, 2h], [2h, 3h], \ldots\) establishing that the inequality \( w_-(t,z) \leq w(t,z) \leq w_+(t,z), \) \( z \in \mathbb{R}, \) holds for all \( t \geq -h \). \hfill \( \Box \)

Remark 19. We can generalize the function \( \eta_b(z) \) for \( b = +\infty \) and, thus, have \( \eta_\infty(z) = \xi_c(-z) \). In this proof, it was not necessary to use the condition (54) for \( z \leq b \) so by replacing \( \xi_c(-z) \) by \( \eta_b(z) \) it can be concluded that (55) implies (57).

Corollary 20. (Local stability) Suppose that there exist \( M, b \in \mathbb{R} \) and \( l_0 > 0, \) such that:

\[ \psi(t,z) \in [M-l_0, M+l_0] \quad \text{for all} \quad (t,z) \in [-h,\infty) \times [b-ch,\infty), \]  \hfill (59)
and that for some $l_1 > l_0$ the initial data satisfy
\[ |v_0(s, z) - \psi_0(s, z)| < (l_1 - l_0)e^{-\gamma_0 s}\eta_0(z) \quad \text{for all} \quad (s, z) \in [-h, 0] \times \mathbb{R}. \quad (60) \]
where $\gamma_0 \geq 0$ is defined by (56). If $L_g(I_1) < 1$, where $I_1 := [M - l_1, M + l_1]$, then
\[ |v(t, z) - \psi(t, z)| \leq (l_1 - l_0)e^{-\gamma_0 t}\eta_0(z) \quad \text{for all} \quad (t, z) \in [-h, \infty) \times \mathbb{R}. \quad (61) \]

Proof. Clearly, $\psi(t, z) \in I_1$ for all $(t, z) \in [-h, \infty) \times [b - ch, \infty)$.
Now if we suppose that the inequality in (60) is satisfied for $v_0(s, z) = v(hk + s, z)$ and $\psi_0(s, z) = \psi(hk + s, z)$, with $k \in \mathbb{Z}_+$, then $v(hk + t - h, z) \in I_1$ for all $(t, z) \in [0, h] \times \mathbb{R}$ and, arguing as in the proof of Theorem 18, we get
\[ L_0 \delta_k^b(t, z) \leq 0 \quad \text{for all} \quad (t, z) \in [0, h] \times \mathbb{R}, \]
where $\delta_k^b(t, z) = \pm[v(hk + t, z) - \psi(hk + t + z)] - (l_1 - l_0)e^{-\gamma_0(hk + t)}\eta_0(z)$ and from [33, Lemma 1] we conclude
\[ \delta_k^b(t, z) \leq 0 \quad \text{for all} \quad (t, z) \in [0, h] \times \mathbb{R}. \quad (62) \]
But (62) implies $v((k + 1)h + t - h, z) \in I_1$ for all $(t, z) \in [0, h] \times \mathbb{R}$ and, arguing as above, by using (62) we obtain $\delta_k^b(t, z) \leq 0$ for all $(t, z) \in [0, h] \times \mathbb{R}$. Therefore, it is sufficient to suppose (60) in order to conclude (61) for $(t, z) \in [0, h] \times \mathbb{R}$ and then we proceed inductively to obtain (61) for all $(t, z) \in [-h, \infty) \times \mathbb{R}$.

To prove Theorem 9, we will use the following lemma

**Lemma 21.** Suppose that functions $g_1, g_2 : D \subset \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy: $g_1(u) \leq g_2(u)$ for all $u \in D$. Let $v_1(t, z), v_2(t, z) : [-h, \infty) \times \mathbb{R} \rightarrow D$ be solutions to (17), with $g = g_1$ and $g = g_2$, respectively, such that: $v_1(s, z) \leq v_2(s, z)$ for $(s, z) \in [h, 0] \times \mathbb{R}$. If $g_1$ or $g_2$ is a non decreasing function, then we have:
\[ v_1(t, z) \leq v_2(t, z) \quad \text{for all} \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}. \]

Proof. We take $\delta(t, z) = v_1(t, z) - v_2(t, z)$. Let us note that if $(t, z) \in [0, h] \times \mathbb{R}$ then
\[ L_0 \delta(t, z) = g_2(v_2(t - h, z - ch)) - g_1(v_1(t - h, z - ch)) \geq 0, \]
because if $g_2$ is a non decreasing function we have that
\[ g_2(v_2(t - h, z - ch)) - g_1(v_1(t - h, z - ch)) \geq g_2(v_1(t - h, z - ch)) - g_1(v_1(t - h, z - ch)) \geq 0, \]
or if $g_1$ is a non decreasing function, we have
\[ g_2(v_2(t - h, z - ch)) - g_1(v_1(t - h, z - ch)) \geq g_2(v_2(t - h, z - ch)) - g_1(v_2(t - h, z - ch)) \geq 0. \]
Now, as $\delta(t, z) \leq 0$ for all $(t, z) \in [-h, 0] \times \mathbb{R}$ the Phragmén–Lindelöf principle from [25][Chapter 3, Theorem 10] implies that $\delta(t, z) \leq 0$ for $(t, z) \in [0, h] \times \mathbb{R}$. The argument is repeated for intervals $[h, 2h], [2h, 3h] \ldots$
Proof Theorem 9. Let us take $\epsilon > 0$ such that $L_g(I_\epsilon) < 1$, where $I_\epsilon := [m_K - \epsilon, K + \epsilon] \subset \mathbb{R}_+$. Then, there is an increasing function $\tilde{g}_\epsilon$ satisfying (M) with positive equilibrium $\kappa_+ \in (K, K + \epsilon)$, $c_*(L_{\tilde{g}_\epsilon}) \leq c_*(L_g)$ and $g \leq \tilde{g}_\epsilon$. Furthermore, there is also an increasing $\tilde{g}_\epsilon$ function meeting (M) with positive equilibrium $\kappa_- \in (m_K - \epsilon, m_K)$ and $c_*(L_{\tilde{g}_\epsilon}) \leq c_*(L_g)$ such that: $g_\epsilon(x) \leq g(x)$ for $x \in [0, K + \epsilon]$.

Now, if $v(t)$ is the homogenous solution of (17) replacing $g$ by $\tilde{g}_\epsilon$ with initial datum $v_0(s) = q, s \in [-h, 0]$, and $c > c_*(L_g)$ then by Lemma 21 and the global stability of $\kappa_+$ there is a number $T > 0$ such that

$$v(t, z) \leq \tilde{v}(t) \leq K + \epsilon \quad \text{for all} \quad (t, z) \in [T, +\infty) \times \mathbb{R}.$$  

(63)

Next, by (63)

$$\sup_{(t, z) \in \mathbb{R}_+ \times \mathbb{R}} v(t, z) =: \nu_\infty < \infty,$$

and by denoting $\kappa_* := \min_{u \in [\kappa, u_\infty]} g(u)$ we take an increasing function $g_\infty$ satisfying (M) with equilibrium $\kappa_\infty \in (\kappa_+ - \epsilon, \kappa_*)$, $c_*(L_{g_\infty}) \leq c_*(L_g)$ and $g_\infty(u) \geq g(u)$ for all $u \in [0, u_\infty]$. Next, without loss of generality we take $q_\infty := \kappa_\infty - q_0 > 0$. Then, by [1, Theorem 3], (21) and (22) there exist a monotone wavefront $\phi_\infty^c$ to (17) (with nonlinearity $g_\infty$) such that

$$\phi_\infty^c(z) - q_\eta_\psi(z) \leq v_0(s, z) \quad (s, z) \in [-h, 0] \times \mathbb{R}$$

(64)

thus by [32, Lemma 2.1] there are $C_0 > 0$ and $\gamma \geq 0$ such that

$$\phi_\infty^c(z - C_0q) - q^{-\gamma t} \eta_\psi(z) \leq v_\infty(t, z) \quad (t, z) \times [-h, \infty) \times \mathbb{R}$$

(65)

Now, by applying Lemma 21 with $D = [0, u_\infty]$

$$\phi_\infty^c(z - C_0q) - q^{-\gamma t} \eta_\psi(z) \leq v(t, z) \quad (t, z) \times [-h, \infty) \times \mathbb{R}.$$

So, there are $z_0'$ and $q_0' > 0$ such that

$$v(t, z) \geq q_0' > 0 \quad (t, z) \in [-h, \infty) \times [z_0', \infty).$$

(66)

Otherwise, denoting $v(t, z)$ the solution of (17) replacing $g$ by $\tilde{g}_\epsilon$ with initial data $v_0(s, z) = v(s + T + h, z)$. Due to (66) and Remark 19 the initial datum $v_0$ satisfies (21) and (22). Next, if we denote by $v(t)$ the homogenous solution of (17) replacing $g$ by $\tilde{g}_\epsilon$ with initial datum $v_0(s) = K + \epsilon, s \in [-h, 0]$, then by [30, Corollary 2.2, p.82] $v(t)$ converges monotonically to $\kappa_-$, therefore

$$v(t, z) \leq \nu(t) \leq K + \epsilon \quad \text{for all} \quad (t, z) \in [-h, +\infty) \times \mathbb{R}.$$  

So, for $c > c(L_g)$ by Lemma 21 (with $D = [0, K + \epsilon]$), Proposition 7 and [33, Theorem 1] there is a wavefront $\phi_\psi$ and $T_0 > 0$ such that

$$m_K - \epsilon \leq \phi_\psi(z) + \epsilon/2 \leq v(t, z) \leq v(t, z) \quad \text{for all} \quad (t, z) \in [T_0, \infty)^2.$$  

(67)

Thus there is $T_0$ such that the function $\hat{v}(t, z) := v(t + T + h, z)$ satisfies (54) with $b = t_v + ch$ and $I = I_\psi$. Analogously, for some $T_{\psi_\epsilon}$ we have $\psi_\epsilon(z) \in I_\psi$ for all $z \geq T_{\psi_\epsilon}$. Finally, by applying Theorem 18 we conclude (24) with $C := \max_{z \in \mathbb{R}} \eta_\psi(z)/\eta_{t_0 + ch}(z)$ where $t_0 := \max\{T_v, T_{\psi_\epsilon}\}$. □
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