On the information-theoretic approach to Gödel’s incompleteness theorem

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Abstract

In this paper we briefly review and analyze three published proofs of Chaitin’s theorem, the celebrated information-theoretic version of Gödel’s incompleteness theorem. Then, we discuss our main perplexity concerning a key step common to all these demonstrations.

1 Introduction

The notion of algorithmic (or program-size) complexity, suggested for the first time in the 1960s, identifies the complexity of a binary string with the size of the shortest computer program that computes that string as output. In the 1970s, mathematician G. J. Chaitin made use of this notion to reformulate in terms of information-theoretic arguments the first theorem of Gödel on the incompleteness of formal axiomatic systems. Chaitin’s main result states that a formal axiomatic system coded in a computer program of \( n \) bits is unable to prove that a binary sequence exists with a complexity greater than \( n \) bits, whereas it can be easily proved that there are actually infinitely many strings with this property. The proof of Chaitin’s result is based on the logical paradox which would arise if a program of \( n \) bits were able to prove the existence, and therefore to generate a specific string of a complexity greater than \( n \) bits. In this paper we briefly analyze three published proofs
of Chaitin’s theorem and argue that they rely on a questionable, \textit{a priori} paradoxical assumption which seems to invalidate from the beginning the demonstration \textit{ab absurdo}: as a matter of fact, the logical paradox reached at the end of the proofs seems to have been tacitly inserted at the beginning of the proofs and therefore only a tautological, useless \textit{absurdum} is obtained.

2 A questionable aspect of Chaitin’s approach

In order to explain what represents, according to the author, a dubious point in Chaitin’s approach to the incompleteness of formal systems, we give here a more detailed description of his result.

In Chaitin \cite{1} and Gardner \cite{2} a formal axiomatic system is described as a program for an idealized computer, a routine of \( c \) bits which makes use of \( n \) bits of axioms and rules of inference to generate systematically and check all possible proofs in order of their size: first all proofs of one bit, then all proofs of two bits, and so forth (this kind of computation is known as the \textit{British Museum algorithm}). After having generated a proof, according to Chaitin \cite{1} and Gardner \cite{2}, the routine tests whether it is the first one proving that a specific binary sequence is of a complexity greater than the number of bits in the program, \( n+c \). When the routine finds such a proof, it prints the specific binary sequence and then halts. Now, the logical paradox arises quite clearly: a program of \( n+c \) bits supposedly calculates a number that no program its size should be able to calculate.

This is the way in which Chaitin \cite{1} and Gardner \cite{2} show why a formal axiomatic system of \( n+c \) bits is unable to prove the existence of any specific number with algorithmic complexity greater than \( n+c \) bits. Actually, there are infinitely many number of complexity greater than \( n+c \) bits. As a matter of fact, with a number of bits less than or equal to \( n+c \) it is possible to create at the most \( 2^{n+c+1} - 2 \) different binary strings\footnote{That is, we add up the number of all possible strings of one bit \((2)\), that of all possible strings of two bits \((2^2)\), and so on up to \( n+c \) bits:}, each one of them is a program that might generate a number. Thus, at the most \( 2^{n+c+1} - 2 \) numbers have algorithmic complexity less than or equal to \( n+c \) bits, while

\[
\sum_{i=1}^{n+c} 2^i = 2 + 2^2 + 2^3 + ... + 2^{n+c} = 2^{n+c+1} - 2.
\]
there are infinitely many integer numbers.

Chaitin [3, 4] shows that there is a program of length nearly equal to \( \log_2 N + k \) bits to calculate a number which supposedly cannot be calculated by a program shorter than \( N \) bits. Here \( k \) is the dimension of the proof-checking routine, essentially the formal axiomatic system, and \( \log_2 N \) is nearly equal to the dimension of the binary expansion of decimal number \( N \). Since \( \log_2 N + k \) becomes much smaller than \( N \) for sufficiently large \( N \), we have again that such formal axiomatic system is unable to prove the existence of any number of a complexity greater than \( N \) bits.

Here we quote a passage from Chaitin [4] in which the working scheme of such program is described:

\begin{quote}
You start running through all possible proofs in the formal axiomatic system in size order. You apply the proof-checking algorithm to each proof. And after filtering out all the invalid proofs, you search for the first proof that a particular positive integer requires at least an \( N \)-bit program.

The algorithm that I’ve just described is very slow but it is very simple. It’s basically just the proof-checking algorithm, which is \( k \) bits long, and the number \( N \), which is \( \log_2 N \) bits long. So the total number of bits is just
\[
\log_2 N + k
\]

as was claimed. That concludes the proof of my incompleteness result that I wanted to discuss with Gödel.
\end{quote}

Let us now come to what appears to be a rather unsatisfactory aspect of the approaches mentioned above. For the sake of simplicity, we concentrate on the last proof, but our point holds also for proofs [1], [2] and [3] with small changes.

It is not clear how the sole decimal number \( N \) is enough to the formal axiomatic system to identify univocally and unequivocally the desired proof. Keeping in mind the argument in Chaitin [4], let us divide the main algorithm of \( \log_2 N + k \) bits into two programs of \( k_1 \) and \( \log_2 N + k_2 \) bits respectively, such that \( k_1 + k_2 \simeq k \) (meaning “not too much greater than \( k \)”), and such that the concatenation of the first and the second program works as the main algorithm. The first one is the pure proof-checking algorithm, namely
that which generates all strings in increasing order of size and checks their formal correctness within the formal axiomatic system. The second one accomplishes the pure searching task; it mechanically runs through all valid proofs provided by the first algorithm and tries to find a particular binary string. In its searching task it uses as a guide only what is coded in its own log₂ N + k₂ bits. Like every pure searching algorithm, this one is able to find univocally and unequivocally only what is exactly expressible or compressible in at most log₂ N + k₂ bits, not more. A successful search occurs in the following two cases:

a) when there is an exact matching between the string memorized in the searching algorithm (and thus, a string necessarily less than log₂ N + k₂ bits long) and one of the strings provided by the first algorithm.

b) when there is an exact matching between a string longer than log₂ N + k₂ bits, but generated by a sub-program of the searching algorithm (and thus, again less complex than log₂ N + k₂ bits), and one of the strings provided by the first algorithm.

In both cases, the searching algorithm is able to find only strings less complex than log₂ N + k₂ bits.

But we already know that what we are searching is not compressible in less than N bits. The algorithmic complexity of the desired theorem (or, equivalently, the algorithmic complexity of its proof, since the proof provides the theorem) is necessarily greater than N bits because it generates a string s more complex than N bits as output. Therefore, we know from the beginning that the searching algorithm will not be able to spot it. An easy way to show that fact ironically follows from the very notion of algorithmic complexity, as the Chaitin’s theorem does.

Let us forget for a while the subject of this paper. According to the notion of algorithmic complexity, any string l of algorithmic complexity greater than N bits could not be unequivocally recognized using an algorithm of a complexity of log₂ N + k₂ bits, if log₂ N + k₂ is less than N. As if it were the case, it would be possible to write an algorithmic procedure of complexity ∼ log₂ N + k₂ which systematically generates all the strings in increasing size order until it recognizes the desired one. But, in this case there would be a contradiction since the algorithmic complexity of the generated string would necessarily be not greater than ∼ log₂ N + k₂ bits. Now, returning to our
point, what difference does it make if \( l \) is a whatever string more complex than \( N \) bits or is the string coding our desired theorem? And this limit affects the *searching algorithm*, not the *proof-checking one*. Indeed, all the above does not eliminate *a priori* the possibility that among all the valid proofs generated and checked by the first algorithm there is one which states that a binary string \( s \) is more complex than \( N \) bits.

It is known from the work of A. M. Turing that algorithms coding formal axiomatic systems have inescapable limits in proof-checking. These limits are strictly related to the undecidability of the *halting problem*, namely the non-existence of a general algorithmic procedure to establish in a finite number of steps whether a computer program halts or not, simply by analyzing its code. Indeed, it can be demonstrated that the unsolvability of the halting problem implies the uncomputability of the program-size complexity.

Instead, in the case of Chaitin [4] the claimed limit seems to be not a fundamental one. What can be rigorously deduced from his arguments is only a trivial limit in the mechanical recognition of the proof, not in the capability of the system in checking the formal correctness of such proof.

Substantially, when Chaitin [4] states that it is possible to write a program only \( \log_2 N + k \) bits long to generate a number of a complexity greater than \( N \) bits, he seems to assume from the beginning that it is always possible to compress any theorem of this kind to \( k \) bits plus the bits which give the binary expansion of \( N \). Hence, no wonder if he gets a logical paradox then.

### 3 Conclusions

From what is outlined above it seems that an obvious, clear relation between the size of a formal axiomatic system and its capabilities/limitations in checking proofs can not be claimed. For instance, a formal axiomatic system of \( N \) bits could be able in principle to check the formal correctness of the proof that a specific sequence is of a complexity of \( C \) bits, with \( C \gg N \). The point is that this proof must be provided as input to the program which codes the proof-checking system. All this is by no means font of logical paradox. For instance, in the case such proof exists, and in the case such existence means that the formal axiomatic system of \( N \) bits is able to partially resolve the halting problem for every string/program in increasing size order up to \( C \) bits, then we have to tell anyway to the formal axiomatic system at which point to stop its analysis of the halting problem. Namely, we have to furnish
the ordinal number of the last string/program to be checked, i.e. an extra $C$ bits$^2$. Otherwise, the system will never halt, vainly trying to establish the halting problem on the first undecidable program. Therefore, we simply have a program of $N+C$ bits which generates a string as complex as $C$ bits.

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**References**

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$^2$Let us list all strings of 1, 2, 3, ..., $n$ bits, in increasing (lexicographical) order:

| $N_\#$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ...
|-------|---|---|---|---|---|---|---|---|---|---
| $p$   | 0 | 1 | 00| 10| 000|100|101|111|0000|...

where $N_\#$ is the ordinal number of the string, while $p$ denotes the binary string itself. As it can be seen $p \simeq (N_\#)_2$ and thus the size of $p$ in bits is approximately equal to $\log_2 N_\#$. 

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