Uniform convergence criterion for non-harmonic sine series

K. A. Oganesyan

Abstract. We show that for a nonnegative monotonic sequence \( \{c_k\} \) the condition \( c_k k \to 0 \) is sufficient for the series \( \sum_{k=1}^{\infty} c_k \sin k^\alpha x \) to converge uniformly on any bounded set for \( \alpha \in (0, 2) \), and for any odd \( \alpha \) it is sufficient for it to converge uniformly on the whole of \( \mathbb{R} \). Moreover, the latter assertion still holds if we replace \( k^\alpha \) by any polynomial in odd powers with rational coefficients. On the other hand, in the case of even \( \alpha \) it is necessary that \( \sum_{k=1}^{\infty} c_k < \infty \) for the above series to converge at the point \( \pi/2 \) or at \( 2\pi/3 \). As a consequence, we obtain uniform convergence criteria. Furthermore, the results for natural numbers \( \alpha \) remain true for sequences in the more general class RBVS.

Bibliography: 17 titles.

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§1. Introduction

We consider the series

\[
\sum_{k=1}^{\infty} c_k \sin k^\alpha x, \quad c_k \downarrow 0,
\]

for \( \alpha > 0 \) and, for an odd natural number \( \alpha \), the more general series

\[
\sum_{k=1}^{\infty} c_k \sin f(k)x, \quad c_k \downarrow 0,
\]

where \( f(k) \) stands for a polynomial of degree \( \alpha \) in odd powers of \( k \) with rational coefficients. In the case of a natural number \( \alpha \) we also consider sequences \( \{c_k\} \) from a more general class. We are interested in conditions which are necessary and sufficient for the uniform convergence of the series (1.1) and (1.2).

For \( \alpha = 1 \) such conditions are well known (see [1]).

Theorem A (Chaundy and Jolliffe). If a nonnegative sequence \( \{c_k\}_{k=1}^{\infty} \) is non-increasing, then the series \( \sum_{k=1}^{\infty} c_k \sin kx \) converges uniformly on \( \mathbb{R} \) if and only if \( c_k k \to 0 \) as \( k \to \infty \).
In [2], the requirement of monotonicity is relaxed to the requirement of quasi-
monotonicity, that is, of the existence of a nonnegative number $\gamma$ such that $c_k k^{-\gamma}$
are nonincreasing, and the same criterion was extended to some more general
sequences in [3]. We can find one more generalization of Theorem A in [4], where
the corresponding criterion was proved for the sequences in the class RBVS, that
is, satisfying the following conditions:

$$\sum_{k=1}^\infty |c_k - c_{k+1}| \leq V cl, \quad c_k \to 0 \text{ as } k \to \infty, \quad (1.3)$$

for any $l$, where $V$ depends only on $\{c_k\}$.

For a more general class of sequences, containing all the classes mentioned above,
a result was obtained in [5].

**Theorem B** (Tikhonov). If a nonnegative sequence $\{c_k\}_{k=1}^\infty$ belongs to the class
$GM$, that is, if there exists a constant $A$ depending only on $\{c_k\}$ such that

$$\sum_{k=1}^{2l-1} |c_k - c_{k+1}| \leq Ac_l$$

for all $l$, then the series $\sum_{k=1}^\infty c_k \sin kx$ converges uniformly on $\mathbb{R}$ if and only if
$c_k k \to 0$ as $k \to \infty$.

Moreover, there are uniform convergence criteria for the series $\sum_{k=1}^\infty c_k \sin kx$ with coefficients satisfying various conditions of general monotonicity (see [6]
and [7]).

The cases $\alpha = 1/2$ and $\alpha = 2$ for the series (1.1) were considered in [8], where it
was shown that the condition $c_k k \to 0$ is necessary and sufficient for the series (1.1)
to converge uniformly on the interval $[0, \pi]$ for $\alpha = 1/2$, and for $\alpha = 2$ a necessary
and sufficient condition is $\sum_{k=1}^\infty c_k < \infty$.

We obtain the following.

**Theorem 1.** Let a nonnegative sequence $\{c_k\}_{k=1}^\infty$ be nonincreasing. Then

(a) if $\alpha$ is an even natural number, then the series (1.1) converges at the point
$\pi/2$ or at the point $2\pi/3$ only if $\sum_{k=1}^\infty c_k < \infty$;

(b) if $\alpha$ is an odd natural number, then for (1.2) to converge uniformly on $\mathbb{R}$ it
is sufficient that $c_k k \to 0$ as $k \to \infty$;

(c) if $\alpha \in (0, 2)$, then for (1.1) to converge uniformly on any bounded subset of
$\mathbb{R}$ it is sufficient that $c_k k \to 0$ as $k \to \infty$.

**Remark 1.** In particular, it follows from Theorem 1 that for odd $\alpha$ the sum of the series

$$\sum_{k=1}^\infty \frac{a_k \sin k^\alpha x}{k}, \quad a_k \to 0, \quad a_k \to 0,$$

represents a continuous function, whereas despite the function $\sum_{k=1}^\infty k^{-1} \sin k^\alpha x$
being bounded, it is discontinuous on a dense set in $\mathbb{R}$. More precisely, it has discontinuities at all points of the form $2\pi a/b$, $a, b \in \mathbb{Z}$, such that $\sum_{k=1}^b \exp\{2\pi ik^\alpha a/b\} \neq 0$.
At the same time it is known that for any natural numbers \( a, n > 2 \) and any prime \( p > n \) such that \((a, p) = 1\) we have

\[
\sum_{k=1}^{p^n} \exp\left\{ \frac{2\pi i ak^n}{p^n} \right\} = p^{n-1}
\]

(see [10], (72)), and for a fixed \( n \) the set of \( \pi \)-rational points of the form \( 2\pi a/p^n \), \((a, p) = 1\), is dense in \( \mathbb{R} \).

Theorem 1 represents an essential part of the uniform convergence criterion for the series (1.1), which we formulate later (in Theorem 2).

Remark 2. For part (a) of Theorem 1 we can also find other points with the same property, but it does not seem essential.

Remark 3. If instead of the sine series (1.1) we consider the corresponding cosine series, we can easily see that the condition \( \sum_{k=1}^{\infty} c_k < \infty \) is necessary for it to converge at the point 0, and for a natural number \( \alpha \), for convergence at the points of the form \( 2\pi m, m \in \mathbb{Z} \).

In the proof of Theorem 1 for the case (1.2) we deal with distributions of the fractional parts of the values of a polynomial (see [11] and [12], for example) and with estimates of Weyl sums (see also [13]–[15]). These play an important role in number theory, in particular, in solving Waring’s problem on the representation of a natural number as a sum of equal powers of natural numbers and in estimating sums appearing in the theory of the Riemann zeta function. The following well-known theorems provide bounds for Weyl sums at points of a special kind.

**Theorem C** (Weyl; see [10], Ch. II, §11, Theorem 14). Let \( n \geq 2 \), \( h(x) = \alpha_1 x + \cdots + \alpha_n x^n \) and

\[\alpha_n = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad |\theta| \leq 1.\]

If \( 0 < \varepsilon_1 < 1 \) and \( P^{\varepsilon_1} \leq q \leq P^{n-\varepsilon_1} \), then for any \( 0 < \varepsilon < 1 \)

\[\left| \sum_{k=1}^{P} e^{2\pi i h(k)} \right| \leq C(n, \varepsilon, \varepsilon_1) P^{1-(\varepsilon_1-\varepsilon)/2^{n-1}}.\]

**Theorem D** (Vinogradov; see [10], Ch. II, §14, Theorem 17). Let \( n > 2 \), \( h(x) = \alpha_1 x + \cdots + \alpha_n x^n \) and

\[\alpha_n = \frac{a}{q} + \frac{\theta}{q^2}, \quad (a, q) = 1, \quad |\theta| \leq 1.\]

If \( P \leq q \leq P^{n-1} \), then

\[\left| \sum_{k=1}^{P} e^{2\pi i h(k)} \right| \leq e^{3n} P^{1-1/(9n^2 \ln n)}.\]

However, in these theorems the length of the sum \( P \) is tied to the denominators of rational approximations of the leading coefficient of the polynomial \( h \).
We obtain estimates for Weyl sums depending on how well the leading coefficient of the polynomial is approximated by rational fractions whose denominators are less than some small power of \( P \). The best estimates are obtained at the points which are approximated rather badly in this way. Such estimates are of interest because if the leading coefficient is ‘close’ to a rational, then in a certain sense the Weyl sums behave similarly to rational sums, which are well studied and easier to deal with.

From the proof of Theorem 1, (b) it follows that when \( c_k k \to 0 \), the series

\[
\sum_{m=1}^{\infty} (c_m - c_{m+1}) \left| \text{Im} \sum_{k=0}^{m} e^{if(k)x} \right|
\]

converges uniformly. This means, in particular, that if we consider the coefficients \( c_m := m^{-1} \ln^{-1}(m + 1) \), the series

\[
\sum_{m=1}^{\infty} \frac{1}{m^2 \ln(m + 1)} \left| \text{Im} \sum_{k=0}^{m} e^{if(k)x} \right|
\]

converges uniformly, hence for any \( a > 0 \) the number of \( m \) such that \( |\text{Im} \sum_{k=0}^{m} e^{if(k)x}| \geq am \) is uniformly small.

It is worth mentioning that in [16] an estimate for symmetric partial sums of the series

\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi ih(k)}}{k}
\]

is given for a polynomial \( h \) with real coefficients. This result is used to establish lower estimates for Lebesgue constants and prove the following theorem which is strongly related to the present work.

**Theorem E** (Oskolkov). Let \( r \geq 2 \) and let \( P_r(y) = \alpha_0 + \alpha_1 y + \cdots + \alpha_r y^r \) be a polynomial with integer coefficients that assume different integer values for \( y \in \mathbb{N} \cup \{0\} \). Then \( \{P_r(n)\} \) is not a spectrum of uniform convergence.

Here by a spectrum of uniform convergence we mean a sequence \( \mathcal{K} = \{k_n\} \) of pairwise distinct integers such that the partial sums of the Fourier series of any continuous function whose Fourier coefficients are zero for \( k \notin \mathcal{K} \) converge uniformly.

In [17] it was shown that the symmetric partial sums

\[
\sum_{1 \leq |k| \leq m} \frac{e^{2\pi ih(k)}}{k}
\]

are uniformly bounded for \( m \in \mathbb{N} \) and \( \deg h \leq r \), for a fixed \( r \). In particular, this result leads to the following.

**Theorem F** (Arkhipov and Oskolkov). Let \( P^+(x) \) and \( P^-(x) \) be polynomials with real coefficients and let \( P^+(-x) \equiv P^+(x) \) and \( P^-(-x) \equiv -P^-(x) \). Then the series

\[
\sum_{n=1}^{\infty} \frac{e^{2\pi i P^+(n)} \sin 2\pi P^-(n)}{n}
\]

converges and the absolute values of its partial sums are bounded by a constant depending only on the degrees of \( P^+ \) and \( P^- \) but not on their coefficients.
Using the Abel transformation Theorem 1, (b) can be derived in the case when $c_k \downarrow 0$ from Theorem F.

In order to formulate the uniform convergence criterion for the series (1.1) we need to introduce the following definition.

For $\alpha > 0$ and $\gamma > 0$, by a discrete $(\alpha, \gamma)$-neighbourhood of zero we mean a sequence $\{x_j\}_{j=0}^{\infty}$ such that $|x_j| = \pi/(\gamma^{\alpha+1}(N+j)\alpha)$ for all $j \in \mathbb{Z}^+$ and some $N \in \mathbb{N}$.

Now we are ready to formulate the criterion.

**Theorem 2.** Let a nonnegative sequence $\{c_k\}_{k=1}^{\infty}$ be nonincreasing. Then the following holds:

(a) if $\alpha$ is an even natural number, then the series (1.1) converges uniformly on a set containing a point of the form $\pi/2 + 2\pi m$ or $2\pi/3 + 2\pi m$, $m \in \mathbb{Z}$, if and only if $\sum_{k=1}^{\infty} c_k < \infty$;

(b) if $\alpha$ is an odd natural number, then the series (1.1) converges uniformly on a set containing a discrete $(\alpha, \gamma)$-neighbourhood of zero for some $\gamma \geq 2$ if and only if $c_k k \to 0$ as $k \to \infty$;

(c) if $\alpha \in (0, 2)$, then the series (1.1) converges uniformly on a bounded set containing a discrete $(\alpha, \gamma)$-neighbourhood of zero for some $\gamma \geq 2$ if and only if $c_k k \to 0$ as $k \to \infty$.

**Remark 4.** Parts (a) and (b) of Theorems 1 and 2 remain true if we replace the condition that the coefficients $\{c_k\}$ be monotonic by their belonging to the class RBVS (see (1.3)).

**Remark 5.** In particular, in part (b) (part (c)) of Theorem 2 the condition $c_k k \to 0$ is necessary and sufficient for the uniform convergence of (1.1) on any (bounded) set containing a punctured neighbourhood of zero.

It will be easy to see that the criterion can be generalized slightly by adding some extra parameters in the definition of a discrete $(\alpha, \gamma)$-neighbourhood of zero. We will not do this to avoid making the formulation of Theorem 2 too complex.

§ 2. Weyl sum estimates depending on rational approximations of the leading coefficient of the polynomial

**Lemma 1.** Let $P \in \mathbb{N}$ and $1 \leq A \in \mathbb{R}$. Then for any natural number $k \geq 1$

$$\#\left\{(y_1, y_2, \ldots, y_k) \in \{1, 2, \ldots, P\}^k : y_1 y_2 \ldots y_k \leq \frac{P^k}{A}\right\} \leq \frac{k P^k}{A^{1/k}}.$$

**Proof.** The assertion follows from the successive inequalities

$$\#\left\{(y_1, y_2, \ldots, y_k) \in \{1, 2, \ldots, P\}^k : y_1 y_2 \ldots y_k \leq \frac{P^k}{A}\right\} \leq k \cdot \#\left\{(y_1, y_2, \ldots, y_k) \in \{1, 2, \ldots, P\}^k : y_1 \leq y_2, \ldots, y_k, y_1 y_2 \ldots y_k \leq \frac{P^k}{A}\right\} \leq k \cdot \#\left\{(y_1, y_2, \ldots, y_k) \in \{1, 2, \ldots, P\}^k : 1 \leq y_1 \leq \frac{P}{A^{1/k}}\right\} = \frac{k P^k}{A^{1/k}}.$$
Now we formulate a statement (see [10], Ch. II, §11, Lemma 13) which will be used on several occasions.

**Lemma A.** Let \( \lambda \) and \( x_1, \ldots, x_k \) be natural numbers and let \( \tau_k(\lambda) \) denote the number of the solutions of the equation \( x_1 \ldots x_k = \lambda \). Then for any \( \varepsilon \in (0,1) \) the following estimate holds:

\[
\tau_k(\lambda) \le C_k(\varepsilon)\lambda^\varepsilon, \tag{2.1}
\]

where \( C_k(\varepsilon) \) is a constant depending only on \( k \) and \( \varepsilon \).

For any number \( y \) we set

\[
\|y\| := \min(\{y\}, 1 - \{y\}),
\]

where \( \{y\} \) stands for the fractional part of \( y \).

Further, for any function \( \psi(y) \) and number \( y_1 \) we let

\[
\Delta \psi(y) = \psi(y + y_1) - \psi(y)
\]

denote the first order finite difference of the function \( \psi(y) \), and for \( k \geq 2 \) we define the \( k \)th order finite difference inductively by

\[
\Delta \psi(y) = \Delta \left( \Delta \left( \ldots \Delta \psi(y) \right) \right),
\]

where \( \alpha_k \) is the leading coefficient of \( \psi(y) \), and \( \eta \) depends only on the coefficients of \( \psi(y) \) and on the numbers \( y_1, \ldots, y_{k-1} \). Also, due to Lemma 12 in [10], Ch. II, §11, for any \( K, k \geq 1 \) we have

\[
\left| \sum_{y=1}^{K} e^{2\pi i h(y)} \right|^{2^k} \le 2^{2^k} K^{2^k-(k+1)} \left( \sum_{y_1=0}^{K-1} \cdots \sum_{y_k=0}^{K-1} \sum_{y=1}^{K+1} \exp \left\{ 2\pi i \Delta h(y) \right\} \right), \tag{2.3}
\]

where \( K_1 := K \) and \( K_{\nu+1} := K_\nu - y_\nu, \nu = 1, 2, \ldots, k \). Now, taking (2.2) and (2.3) into account, for any polynomial \( f \) of degree \( n \) with leading coefficient \( \alpha_n \) we obtain
\[ \leq 2^{2^{n-1}m^{2^{n-1}-1}}(n-1)m^{n-1} \]
\[ + \sum_{y_1,\ldots,y_{n-1}=1}^m \left| \sum_{y=1}^{m-y_1-\cdots-y_{n-1}} \exp\{in!yy\cdots y_{n-1}x\} \right| \]  \hspace{0.5cm} (2.4)

Note that for any \( t \) and any natural number \( l \) we have
\[
\left| \sum_{y=1}^l e^{iyt} \right| \leq \left| \sum_{y=1}^l \sin yt \right| + \left| \sum_{y=1}^l \cos yt \right| = \left| \frac{\cos \frac{t}{2} - \cos \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right| + \left| \frac{\sin \frac{t}{2} - \sin \frac{(2l+1)t}{2}}{2 \sin \frac{t}{2}} \right| \leq 2 \left| \frac{\sin \frac{t}{2}}{\sin \frac{t}{2}} \right| . \hspace{0.5cm} (2.5)
\]

Combining (2.4) and (2.5), we derive
\[
\left| \sum_{k=1}^m e^{if(k)x} \right|^{2^{n-1}} \leq 2^{2^{n-1}m^{2^{n-1}-1}}(n-1) \]
\[ + 2^{2^{n-1}+1}m^{2^{n-1}-n} \sum_{y_1,\ldots,y_{n-1}=1}^m \left| \frac{\sin \left(\frac{m-y_1-\cdots-y_{n-1}}{2}n!y_1\cdots y_{n-1}x\right)}{\sin \frac{n!y_1\cdots y_{n-1}x}{2}} \right| \]
\[ \leq 2^{2^{n-1}m^{2^{n-1}-1}}(n-1) \]
\[ + 2^{2^{n-1}+1}m^{2^{n-1}-n} \sum_{y_1,\ldots,y_{n-1}=1}^m \left| \min \left\{ m, \frac{1}{2\|n!y_1\cdots y_{n-1}x\|} \right\} \right| \]  \hspace{0.5cm} (2.6)

**Lemma 2.** Let \( 0 < y \in \mathbb{R} \setminus \mathbb{Q}, \varepsilon \in (0,1), 4 \leq P \in \mathbb{N} \) and \( 3 \leq n \in \mathbb{N} \). If there exists no pair of coprime natural numbers \( C \) and \( M \leq P \varepsilon \) such that
\[
\left| y - \frac{C}{M} \right| \leq P^{\varepsilon-1},
\]
then
\[
\# \{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : \|yy_1\cdots y_{n-1}\| \leq P^{\varepsilon-1}\} \leq 4C_{n-1} \left( \frac{\varepsilon}{2(n-1)} \right) P^{n-1-\varepsilon/2},
\]
where \( C_m(\gamma) \) is from (2.1), and also
\[
\sum_{y_1,\ldots,y_{n-1}=1}^P \min \left\{ P, \frac{1}{2\|yy_1\cdots y_{n-1}\|} \right\} \leq GP^{n-\varepsilon/2},
\]
where \( G \) depends only on \( n \) and \( \varepsilon \).

**Remark 6.** From now on we give our arguments for irrational (\( \pi \)-irrational) numbers, not because something really different happens for rational (\( \pi \)-rational) ones, but for the sake of simplicity.
Proof of Lemma 2. Let $T$ be the smallest number from $\{1, 2, \ldots, P^{n-1}\}$ such that $\|yT\| \leq P^{\varepsilon-1}$ (if there is no such $T$, then the assertion becomes trivial). Then we have $T \geq P^\varepsilon$. In this case, for any $0 \leq k \leq P^{n-1} - T$ there is at most one quantity among

$$\{y(k+1), \{y(k+2), \ldots, \{y(k+T)\}\}$$

which belongs to the half-open interval $(0, P^{\varepsilon-1}]$ (otherwise $\|y(i-j)\| \leq P^{\varepsilon-1}$ for some $1 \leq i < j \leq T$, which is impossible due to the minimality of $T$) and at most one value in the half-open interval $[1 - P^{\varepsilon-1}, 1)$. So, among the quantities $\{1, 2, \ldots, P^{n-1}\}$ there are at most

$$2\left[\frac{P^{n-1}}{T}\right] \leq \frac{4P^{n-1}}{T} \leq 4P^{n-1-\varepsilon}$$

integers $k$ satisfying $\|yk\| \leq P^{\varepsilon-1}$, and since for any $k$, according to (2.1),

$$\#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} = k\} \leq C_{n-1}\left(\frac{\varepsilon}{2(n-1)}\right)k^{\varepsilon/(2(n-1))},$$

we have

$$\#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: \|yy_1 \ldots y_{n-1}\| \leq P^{\varepsilon-1}\} \leq 4P^{n-1-\varepsilon}C_{n-1}\left(\frac{\varepsilon}{2(n-1)}\right)\left(P^{n-1}\right)^{\varepsilon/(2(n-1))} = 4C_{n-1}\left(\frac{\varepsilon}{2(n-1)}\right)P^{n-1-\varepsilon/2}.$$ 

Thus,

$$\sum_{y_1, \ldots, y_{n-1}=1}^P \min\left\{P, \frac{1}{2\|yy_1 \ldots y_{n-1}\|}\right\} \leq P \cdot 4C_{n-1}\left(\frac{\varepsilon}{2(n-1)}\right)P^{n-1-\varepsilon/2} + P^{n-1-\varepsilon} \leq GP^{n-\varepsilon/2},$$

where $G$ depends only on $n$ and $\varepsilon$. The proof is complete.

Corollary 1. Under the conditions of Lemma 2, for any real monic polynomial $f$ of degree $n$,

$$\left|\sum_{k=1}^P \exp\left\{\frac{2\pi if(k)y}{n!}\right\}\right| \leq DP^{1-\varepsilon/2^n},$$

where $D$ depends only on $n$ and $\varepsilon$.

Proof. From (2.6) and Lemma 2 it follows that

$$\left|\sum_{k=1}^P \exp\left\{\frac{2\pi if(k)y}{n!}\right\}\right|^{2^{n-1}} \leq 2^{2^{n-1}}P^{2^{n-1} - n}(n-1)P^{n-1} + 2GP^{n-\varepsilon/2}$$

$$\leq D'P^{2^{n-1}-\varepsilon/2},$$

where $D' > 0$ depends only on $\varepsilon$ and $n$. This leads to the required result with

$$D = (D')^{1/2^{n-1}}.$$ The corollary is proved.
Lemma 3. Let $0 < y \in \mathbb{R} \setminus \mathbb{Q}$, $\varepsilon \in (0, 1/6)$, $9 \leq P \in \mathbb{N}$ and $3 \leq n \in \mathbb{N}$. If there exists a pair of coprime natural numbers $C$ and $M \leq P^\varepsilon$ such that

$$P^{\varepsilon - n} < \left| y - \frac{C}{M} \right| =: |\beta| \leq P^{\varepsilon - 1},$$

then

$$\#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : \|yy_1 \ldots y_{n-1}\| \leq P^{\varepsilon - 1}\} \leq 6P^{n-3/2}C^{-1} \left(\frac{2\varepsilon}{n-1}\right)
+ (n-1)C^{-1} \left(\frac{1}{2(n-1)}\right)P^{n-1-(n-\varepsilon)/(n-1)}|\beta|^{-1/(n-1)}M^{-1/(2(n-1))},$$

and also

$$\sum_{y_1, \ldots, y_{n-1}=1}^P \min\left\{\frac{1}{P}, \frac{1}{2\|yy_1 \ldots y_{n-1}\|}\right\} \leq U\left(P^{n-\varepsilon} + P^{n-(n-\varepsilon)/(n-1)}|\beta|^{-1/(n-1)}M^{-1/(2(n-1))}\right),$$

where $U$ depends only on $n$ and $\varepsilon$.

Proof. Without loss of generality assume that $P^{\varepsilon - n} < y - C/M = \beta \leq P^{\varepsilon - 1}$. Suppose that there are coprime $C'$ and $M'$ distinct from $C$ and $M$ that satisfy the inequality

$$\left| y - \frac{C'}{M'} \right| =: |\beta'| \leq \frac{2}{M'P^{1-\varepsilon}}.$$

First, we have

$$\frac{1}{MM'} \leq \left| y - \frac{C}{M} \right| + \left| y - \frac{C'}{M'} \right| \leq \frac{3}{P^{1-\varepsilon}},$$

and hence,

$$M' \geq \frac{P^{1-\varepsilon}}{3M} \geq \frac{P^{1-2\varepsilon}}{3} \geq P^\varepsilon \geq M,$$

(2.8)

Second,

$$yMM' = C'M + \beta'M'M = CM' + \beta M'M,$$

so $\{\beta'M'M\} = \{\beta M'M\}$. Thus, if $\beta' > 0$, then since $\beta'M' \leq 2P^{\varepsilon - 1}$ we have $\beta'M'M \leq 2P^{2\varepsilon - 1} < 1$ and $\{\beta M'M\} = \beta M'M$. Hence, either $M' \geq \beta^{-1}M^{-1}$ or $\{\beta M'M\} = \beta M'M = \beta M'M$, from which we obtain $\beta M'M \geq \beta M = \beta M$.

If $\beta' < 0$, then $(-\beta' + \beta)M'M \geq 1$, which implies that

$$M' \geq \frac{1 + \beta' M'M}{\beta M} \geq (1 - 2P^{2\varepsilon - 1})\beta^{-1}M^{-1} \geq \frac{1}{2}\beta^{-1}M^{-1}.$$

(2.9)

So we have obtained that, independent of the sign of $\beta'$, either $\beta M' \geq \beta M$ or $M' \geq \frac{1}{2}\beta^{-1}M^{-1}$. 

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Now, let $T_1 < T_2 < \cdots < T_K$ be all the numbers $k$ from $\{1, 2, \ldots, P^{n-1}\}$ such that $\|y_k\| \leq P^{\varepsilon-1}$. Then, since $\{y(T_{i+1} - T_i)\} \in (0, 2P^{\varepsilon-1}] \cup [1 - 2P^{\varepsilon-1}, 1)$, from the argument above it follows that either $T_{i+1} - T_i \geq \frac{1}{2}\beta^{-1}M^{-1}$ or

$$
\{y(T_{i+1} - T_i) - (1 - P^{\varepsilon-1})\} \geq \beta M.
$$

But since

$$
\left(\|P^{\varepsilon-1}\beta^{-1}M^{-1}\| + 1\right)\beta M > P^{\varepsilon-1},
$$

among the numbers $i = 1, 2, \ldots, \|P^{\varepsilon-1}\beta^{-1}M^{-1}\| + 1$ there exists $i$ such that $T_{i+1} - T_i \geq \frac{1}{2}\beta^{-1}M^{-1}$. Note also that (2.10) implies the fact that among any $[2P^{\varepsilon-1}\beta^{-1}M^{-1}] + 1$ consecutive values of $i$ we can find one such that $T_{i+1} - T_i \geq \frac{1}{2}\beta^{-1}M^{-1}$.

Note that $\|P^{\varepsilon-1}\beta^{-1}M^{-1}\| M \leq P^{\varepsilon-1}P^{n-\varepsilon} = P^{n-1}$, hence

$$
\mathcal{I} := \{M, 2M, \ldots, \|P^{\varepsilon-1}\beta^{-1}M^{-1}\| M\} \subset \{T_k\}.
$$

Thus,

$$
P^{n-1} \geq T_K \geq \|P^{\varepsilon-1}\beta^{-1}M^{-1}\| M + \frac{K - \|P^{\varepsilon-1}\beta^{-1}M^{-1}\|}{2\|P^{\varepsilon-1}\beta^{-1}M^{-1}\| + 1} \left(2\|P^{\varepsilon-1}\beta^{-1}M^{-1}\| M + \frac{\beta^{-1}M^{-1}}{2}\right)
$$

which yields

$$
K - \|P^{\varepsilon-1}\beta^{-1}M^{-1}\| \leq 6P^{n-2+\varepsilon}.
$$

Then, taking (2.1) and Lemma 1 into account we can establish that

$$
\#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: \|yy_1 \ldots y_{n-1}\| \leq P^{\varepsilon-1}\}
$$

$$
= \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} \in \{T_k\}\}
$$

$$
= \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} \in \{T_k\} - \mathcal{I}\}
$$

$$
+ \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} \in \mathcal{I}\}
$$

$$
\leq 6P^{n-2+\varepsilon}C_{n-1}\left(\frac{2\varepsilon}{n-1}\right)(P^{n-1})^{2\varepsilon/(n-1)}
$$

$$
+ \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} \in \{M, 2M, \ldots, \|P^{\varepsilon-1}\beta^{-1}M^{-1}\| M\}\}
$$

$$
\leq 6P^{n-3/2}C_{n-1}\left(\frac{2\varepsilon}{n-1}\right)
$$

$$
+ \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} = M\}
$$

$$
\times \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1}: y_1 \ldots y_{n-1} \leq P^{\varepsilon-1}\beta^{-1}M^{-1}\}
$$

$$
\leq 6P^{n-3/2}C_{n-1}\left(\frac{2\varepsilon}{n-1}\right) + C_{n-1}\left(\frac{1}{2(n-1)}\right)M^{1/(2(n-1))}(n-1)^{n-1}\left(\beta MP^{n-\varepsilon}\right)^{1/(n-1)}.
$$
So,
\[
\sum_{y_1, \ldots, y_{n-1}=1}^{P} \min \left\{ P, \frac{1}{2\|yy_1 \ldots y_{n-1}\|} \right\} \leq P \left( 6P^{n-3/2}C_{n-1} \left( \frac{2\varepsilon}{n-1} \right) + (n-1)C_{n-1} \left( \frac{1}{2(n-1)} \right) P^{n-1-(n-\varepsilon)/(n-1)} \beta^{-1/(n-1)} M^{-1/(2(n-1))} \right) + P^{n-1} \frac{1}{2P^{\varepsilon-1}} \]
\[
\leq U \left( P^{n-\varepsilon} + P^{n-(n-\varepsilon)/(n-1)} \beta^{-1/(n-1)} M^{-1/(2(n-1))} \right),
\]
where \( U \) depends only on \( n \) and \( \varepsilon \). The proof is complete.

**Corollary 2.** Under the conditions of Lemma 3, for any real monic polynomial \( f \) of degree \( n \),
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi if(k)y}{n!} \right\} \right|^{2n-1} \leq D_1 \left( P^{1-\varepsilon/2^{n-1}} + P^{1-(n-\varepsilon)/(2^{n-1}(n-1))} M^{-1/(2^{n-1}(n-1))} \right),
\]
where \( D_1 \) depends only on \( n \) and \( \varepsilon \).

**Proof.** From Lemma 3 and (2.6) it follows in the same way as (2.7) that
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi if(k)y}{n!} \right\} \right|^{2n-1} \leq 2^{n-1} P^{2^{n-1}-1}(n-1)
\]
\[
+ 2^{n-1} + P^{2^{n-1}-n} \left( UP^{n-\varepsilon} + UP^{n-(n-\varepsilon)/(n-1)} \beta^{-1/(n-1)} M^{-1/(2(n-1))} \right)
\]
\[
\leq D_1 P^{2^{n-1}-\varepsilon} + D_1 P^{2^{n-1}-(n-\varepsilon)/(n-1)} \beta^{-1/(n-1)} M^{-1/(2(n-1))}.
\]
Therefore, in view of the inequality \( (a+b)^{1/(n-1)} \leq a^{1/(n-1)} + b^{1/(n-1)} \), which holds for any positive \( a \) and \( b \), we obtain the required assertion with \( D_1 = (D_1')^{1/2^{n-1}} \).

The corollary is proved.

**Lemma 4.** Let \( 0 < y \in \mathbb{R} \setminus \mathbb{Q} \), \( \varepsilon \in (0, \frac{1}{6}) \), \( 8 \leq P \in \mathbb{N} \) and \( 3 \leq n \in \mathbb{N} \). If there exists a pair of coprime natural numbers \( C \) and \( M \leq P^{\varepsilon} \) such that
\[
\left| y - \frac{C}{M} \right| = |\beta| \leq P^{\varepsilon-n},
\]
then
\[
\sum_{y_1, \ldots, y_{n-1}=1}^{P} \min \left\{ P, \frac{1}{2\|yy_1 \ldots y_{n-1}\|} \right\} \leq \frac{BP^n}{M^{1/(2(n-1))}},
\]
where \( B \) depends only on \( n \). Moreover, if \( P^n|\beta| \geq 2 \), then for any \( \delta > 0 \)
\[
\sum_{y_1, \ldots, y_{n-1}=1}^{P} \min \left\{ P, \frac{1}{2\|yy_1 \ldots y_{n-1}\|} \right\} \leq \frac{P^{n-\varepsilon}}{2} + \frac{A_\delta}{M^{1/(2(n-1))}|\beta|^{1/(n-1)-\delta}} P^{n-n/(n-1)+\delta n},
\]
where \( A_\delta \) depends only on \( \delta \) and \( n \).
Proof. Without loss of generality assume that $\beta > 0$. We show that the smallest $T \in \mathbb{N}$ such that \( \|y_T\| \leq P^{\varepsilon - 1} \) is $M$. If not, there exists $T < M$ such that $y_T = Z + \gamma$, where $Z \in \mathbb{N} \cup \{0\}$ and $\gamma \in (0, P^{\varepsilon - 1}] \cup [1 - P^{\varepsilon - 1}, 1)$, and in this case
\[
\frac{Z}{T} + \frac{\gamma}{T} = y = \frac{C}{M} + \beta.
\]

On the other hand
\[
\frac{1}{P^{2\varepsilon}} \leq \frac{1}{M^2} \leq \frac{1}{TM} \leq \left| \frac{C}{M} - \frac{Z}{T} \right| = \left| \beta - \frac{\gamma}{T} \right| < \frac{2}{P^{1 - \varepsilon}}, \tag{2.12}
\]
so $P^{1 - 3\varepsilon} < 2$, which is false since $P^{1 - 3\varepsilon} \geq \sqrt{P} > 2$. Let $T_1 = M < T_2 < T_3 < \cdots < T_K$ be all the natural numbers less than $P^{n - 1}$ such that $\|y_{T_k}\| \leq P^{\varepsilon - 1}$. We show that $K = \lfloor P^{\varepsilon - 1} \beta^{-1} M^{-1} \rfloor$ and $T_k = kM$ for each $k \leq K$. We prove the second part of the assertion using induction.

The basis is the case $k = 1$. Suppose that the assertion has been proved for $k = l < K$. We will prove it for $k = l + 1$. Assume the contrary, $lM < T_{l+1} < (l+1)M$.

As $y$ is irrational, we have $\{yT_{l+1}\} = l\beta$. The only two remaining cases are the following.

1) $l\beta < \{yT_{l+1}\} \leq P^{\varepsilon - 1}$. But then $0 < T_{l+1} - lM < M$ and $\{y(T_{l+1} - lM)\} \in (0, P^{\varepsilon - 1}]$, which is impossible since $T_1 = M$.

2) $\{yT_{l+1}\} \in [1 - P^{\varepsilon - 1}, 1)$ or $\{yT_{l+1}\} > 1$. Then $\{y(T_{l+1} - lM)\} \in (1 - 2P^{\varepsilon - 1}, 1)$, hence
\[
\{y(T_{l+1} - lM)M\} > 1 - 2MP^{\varepsilon - 1} > 1 - 2P^{2\varepsilon - 1} > 1 - 2P^{-2/3} > \frac{1}{2}. \tag{2.13}
\]

On the other hand
\[
\{yM(T_{l+1} - lM)\} \leq (T_{l+1} - lM)M \beta \leq M^2 \beta \leq P^{3\varepsilon - n} < P^{-2} \leq \frac{1}{64}. \tag{2.14}
\]

We have a contradiction. It only remains to show that $K = \lfloor P^{\varepsilon - 1} \beta^{-1} M^{-1} \rfloor$. Note that for $l$ satisfying $\lfloor P^{\varepsilon - 1} \beta^{-1} M^{-1} \rfloor + 1 \leq l \leq P^{n - 1}$ we have
\[
\{yM\} \leq \{yM\}l \leq P^{n - 1} \beta M \leq P^{n - 1 - n + \varepsilon + \varepsilon} = P^{2\varepsilon - 1}
\]
and
\[
\{yM\} > P^{\varepsilon - 1} \beta^{-1} M^{-1} \beta M = P^{\varepsilon - 1},
\]
that is, $lM \neq T_k$ for any $k$. Suppose there exists a natural number $T$ such that $lM < T < (l + 1)M$ for some $\lfloor P^{\varepsilon - 1} \beta^{-1} M^{-1} \rfloor \leq l \leq P^{n - 1}$ and $\|yT\| \leq P^{\varepsilon - 1}$. But then $\{y(T - lM)\} \geq 1 - 2P^{2\varepsilon - 1}$, and therefore, again
\[
\{y(T - lM)\} > 1 - 2MP^{2\varepsilon - 1} \geq 1 - 2P^{3\varepsilon - 1}
\]
\[
\geq 1 - 2P^{-1/2} \geq 1 - \frac{1}{\sqrt{2}} \geq \frac{1}{5}. \tag{2.15}
\]

In contrast,
\[
\{yM(T - lM)\} \leq (T - lM)M \beta \leq M^2 \beta \leq P^{3\varepsilon - 3} < P^{-2} \leq \frac{1}{64}.
\]
which leads to a contradiction. Thus,

\[ K = \lfloor P^{\varepsilon - 1} \beta^{-1} M^{-1} \rfloor \quad \text{and} \quad T_k = kM \quad \text{for all} \quad k \leq K. \quad (2.16) \]

Therefore,

\[ \sum_{\substack{y_1, \ldots, y_{n-1} = 1 \\text{for all} \quad y \in \{T_k\} \neq \{y \}} \min \left\{ P \cdot \frac{1}{2 \|yy_1 \cdots y_{n-1}\|} \right\} \leq P^{n-1} \frac{1}{2 \varepsilon - 1} \leq \frac{P^{n-\varepsilon}}{2}. \quad (2.17) \]

Note that using (2.1) we have, for an arbitrary natural number \( l \),

\[
\#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 y_2 \cdots y_{n-1} \in \{M, 2M, \ldots, lM\}\}
\leq \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 y_2 \cdots y_{n-1} \leq l\}
\times \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 y_2 \cdots y_{n-1} = M\}
\leq C_{n-1} \left( \frac{1}{2(n-1)} \right) M^{1/(2(n-1))}
\times \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 y_2 \cdots y_{n-1} \leq l\}. \quad (2.18)
\]

Now, according to (2.18) and Lemma 1, we can estimate

\[
\sum_{\substack{y_1, \ldots, y_{n-1} = 1 \\text{for all} \quad y \in \{T_k\} \neq \{y \}} \min \left\{ P \cdot \frac{1}{2 \|yy_1 \cdots y_{n-1}\|} \right\} \leq \sum_{\substack{y_1, \ldots, y_{n-1} = 1 \\text{for all} \quad y \in \{T_k\} \neq \{y \}}} P
\leq PC_{n-1} \left( \frac{1}{2(n-1)} \right) M^{1/(2(n-1))}
\times \#\{(y_1, y_2, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 y_2 \cdots y_{n-1} \leq \frac{P^{n-1}}{M}\}
\leq C_{n-1} \left( \frac{1}{2(n-1)} \right) \frac{(n-1)P^n}{M^{1/(2(n-1))}}. \quad (2.19)
\]

Combining estimates (2.17) and (2.19) we derive that

\[
\sum_{y_1, \ldots, y_{n-1} = 1} \min \left\{ P \cdot \frac{1}{2 \|yy_1 \cdots y_{n-1}\|} \right\}
\leq \frac{P^{n-\varepsilon}}{2} + C_{n-1} \left( \frac{1}{2(n-1)} \right) \frac{(n-1)P^n}{M^{1/(2(n-1))}} < \frac{BP^n}{M^{1/(2(n-1))}}.
\]

Now consider the case when \( P^n \beta \geq 2 \). Let

\[ F_t := \{(y_1, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 \cdots y_{n-1} \in \{M, 2M, \ldots, |t(P\beta M)^{-1}| M\}\}. \]

Note that for \( t > P^n \beta \) we have

\[ \left\lfloor \frac{t}{P\beta M} \right\rfloor M > \frac{P^n \beta}{P\beta M} M = P^{n-1} \geq y_1 y_2 \cdots y_{n-1}, \]
and hence for such $t$ we have $F_t = F_{[P^n \beta]}$. For any natural number $t \leq P^n \beta$, Lemma 1 gives

$$\#\left\{(y_1, \ldots, y_{n-1}) \in \{1, 2, \ldots, P\}^{n-1} : y_1 \ldots y_{n-1} \leq \frac{t}{P^n \beta M}\right\} \leq \frac{(n-1)P^{n-1}}{(P^n \beta M)^{1/(n-1)}}.$$  \hfill (2.20)

From (2.18) and (2.20) we obtain

$$|F_t| \leq C_{n-1}\left(\frac{1}{2(n-1)}\right)M^{1/(2(n-1))}\frac{(n-1)P^{n-1}}{(P^n \beta M)^{1/(n-1)}} = C_{n-1}\left(\frac{1}{2(n-1)}\right)\frac{(n-1)P^{n-1}}{(P^n \beta M)^{1/2}} \frac{1}{(n-1)}. \hfill (2.21)$$

If $y_1y_2 \ldots y_{n-1} \in \{T_k\}_{k=1}^K \setminus F_t$, then

$$\frac{1}{2\|yy_1 \ldots y_{n-1}\|} \leq \frac{1}{2\beta tP^{-1} \beta^{-1}} = \frac{P}{2t}. \hfill (2.22)$$

Using (2.22) we obtain

$$S(P,y) := \sum_{\substack{y_1, \ldots, y_{n-1} = 1 \\ y_1 \ldots y_{n-1} \in \{T_k\}}} \min \left\{ P, \frac{1}{2\|yy_1 \ldots y_{n-1}\|} \right\} \leq |F_2| \max p + (|F_3| - |F_2|) \frac{P}{2 \cdot 2} + (|F_4| - |F_3|) \frac{P}{2 \cdot 3} + \ldots$$

$$+ (|F_{[P^n \beta]} + 1| - |F_{[P^n \beta]}|) \frac{P}{2 \cdot [P^n \beta]}. \hfill (2.23)$$

Now, in the expression above every term $|F_i|$ appears with a nonnegative coefficient, and so with the help of (2.21) we derive that

$$S(P,y) \leq C_{n-1}\left(\frac{1}{2(n-1)}\right)P(n-1)P^{n-1-n/(n-1)} \beta^{-1/(n-1)}M^{-1/(2(n-1))}$$

$$\times \left(2^{1/(n-1)} + (3^{1/(n-1)} - 2^{1/(n-1)}) \frac{1}{2} + \ldots \right)$$

$$+ \left(\left([P^n \beta] + 1\right)^{1/(n-1)} - [P^n \beta]^{1/(n-1)}\right) \frac{1}{[P^n \beta]}.$$ \hfill (2.23)

For any $a \geq 2$, by Lagrange’s theorem, for some $\theta \in (0, 1)$ we have

$$(a + 1)^{1/(n-1)} - a^{1/(n-1)} = \frac{1}{n-1}(a + \theta)^{-(n-2)/(n-1)} \leq \frac{2}{n-1}a^{-(n-2)/(n-1)} \leq 1 \leq 2^{1/(n-1)},$$
and hence it follows from (2.23) that
\[
S(P, y) \leq C_{n-1} \left( \frac{1}{2(n-1)} \right) P(n-1) P^{n-1-n/(n-1)} \beta^{-1/(n-1)} M^{-1/(2(n-1))} 2^{1/(n-1)} \ln(P^n \beta).
\] (2.24)
Since for any \( \delta > 0 \) there exists a number \( B_\delta > 0 \) such that \( \ln(s + 1) \leq B_\delta s^\delta \) for \( s > 0 \), (2.24) implies that
\[
S(P, y) \leq B_\delta C_{n-1} \left( \frac{1}{2(n-1)} \right) (n - 1) P^{n-n/(n-1)+\delta n} \beta^{-1/(n-1)} M^{-1/(2(n-1))} 2^{1/(n-1)} = A_\delta P^{n-n/(n-1)+\delta n} \beta^{-1/(n-1)} M^{-1/(2(n-1))},
\] (2.25)
where \( A_\delta \) is a constant depending only on \( n \). Combining (2.17) and (2.25), we obtain the desired inequality in the case \( P^n \beta \geq 2 \).

Lemma 4 is proved.

**Corollary 3.** Under the conditions of Lemma 4, for any real monic polynomial \( f \) of degree \( n \),
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi i f(k) y}{n!} \right\} \right| \leq W \frac{P}{M^{1/(2^n(n-1))}},
\] (2.26)
where \( W \) depends only on \( n \). In addition, if \( P^n |\beta| \geq 2 \), then
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi i f(k) y}{n!} \right\} \right| \leq J \left( P^{1-\varepsilon/2^n-1} + \frac{P^{1-n/(2^n(n-1))}}{(M|\beta|)^{1/(2^n(n-1))}} \right),
\] (2.27)
where \( J \) depends only on \( n \).

**Proof.** From Lemma 4 and (2.6) we derive that
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi i f(k) y}{n!} \right\} \right|^{2^n-1} \leq 2^{2^n-1} P^{2^n-1-n} \left( (n-1) P^{n-1} + 2 \frac{BP^n}{M^{1/(2^n(n-1))}} \right),
\]
which implies (2.26).

If \( P^n |\beta| \geq 2 \), then, according to (2.6) and Lemma 4,
\[
\left| \sum_{k=1}^{P} \exp \left\{ \frac{2\pi i f(k) y}{n!} \right\} \right|^{2^n-1} \leq 2^{2^n-1} P^{2^n-1-1(n-1)}
\]
\[+ 2^{2^n-1} P^{2^n-1-n} \left( \frac{P^{n-1}}{2} + \frac{A_{1/(2^n(n-1))}}{M^{1/(2^n(n-1))}} |\beta|^{1/(2^n(n-1))} P^{n-n/(2^n(n-1))} \right). \] (2.28)
Using the inequality \((a + b)^{1/(n-1)} \leq a^{1/(n-1)} + b^{1/(n-1)}\) for \( a, b > 0 \) and (2.28), we obtain (2.27). The corollary is proved.

**Remark 7.** Lemmas 3 and 4, and also Corollaries 2 and 3, remain true for \( \varepsilon \in (0, 1/3) \) for large enough \( P \).

**Proof.** Only the following estimates would need to be changed in an obvious way: (2.8), (2.9), (2.12), (2.13), (2.14) and (2.15).
§ 3. The case of a natural power

Consider the case when $\alpha = n \in \mathbb{N}$. There are two fundamentally different situations: when $n$ is even and when $n$ is odd. We start with the even case.

Proof of Theorem 1, (a). Note that $k^2 \equiv 0 \pmod{4}$ for any even $k$ and $k^2 \equiv 1 \pmod{4}$ for any odd $k$. Therefore, for any $l < L$,

$$
\sum_{k=2l+1}^{2L} c_k \sin k^2 \frac{\pi}{2} = \sum_{k=1}^{L-l} c_{2l+2k-1} \geq \frac{1}{2} \sum_{k=2l+1}^{2L} c_k,
$$

which completes the proof.

A similar argument works for the point $2\pi/3$, due to the fact that $k^2 \equiv 1 \pmod{3}$ for any $k$ not divisible by 3.

Now we turn to the case of an odd power $n$ and the series (1.2). This case differs from the previous one because for any odd $l$ and any point of the form $x = 2\pi a/b \in 2\pi\mathbb{Q}$ we have $\sum_{k=1}^{b} \sin k^2 x = 0$. Therefore, there is no ‘accumulation’ such as we saw in the case of even $l$. We will see that due to this fact we manage to get suitable estimates for the corresponding imaginary parts of Weyl sums for points close enough to $\pi$-rationals. For other points effective estimates are provided by §2, and these estimates are still valid if we replace $f$ by any polynomial of the same degree.

Proof of Theorem 1, (b). Let $c_k \to 0$. Fix some $\varepsilon \in (0, 1/6)$ and $x \in \mathbb{R} \setminus \pi\mathbb{Q}$. Since we are going to prove the assertion for all $x \in \mathbb{R}$, we can assume that the polynomial $f$ is monic. Without loss of generality consider $x > 0$. Set

$$\mathcal{M} = \mathcal{M}_x := \left\{ M \in \mathbb{N} : \exists C_M \in \mathbb{N} \text{ such that } (C_M, M) = 1 \text{ and } \left| \frac{n! x}{2\pi} - \frac{C_M}{M} \right| \leq M^{(\varepsilon-1)/\varepsilon} \right\}.$$

Notice that $\mathcal{M}$ is a finite or infinite increasing sequence of natural numbers $\{M_i\}_{i \geq 1}$, and that

$$2M_{i+1} \geq M_i^4 \quad (3.2)$$

for any $i \geq 1$. In fact,

$$\frac{1}{M_i M_{i+1}} \leq \left| \frac{C_{M_i}}{M_i} - \frac{C_{M_{i+1}}}{M_{i+1}} \right| \leq \left| \frac{n! x}{2\pi} - \frac{C_{M_i}}{M_i} \right| + \left| \frac{n! x}{2\pi} - \frac{C_{M_{i+1}}}{M_{i+1}} \right| \leq M_i^{(\varepsilon-1)/\varepsilon} + M_{i+1}^{(\varepsilon-1)/\varepsilon} \leq 2M_i^{(\varepsilon-1)/\varepsilon},$$

and therefore,

$$2M_{i+1} \geq M_i^{(1-\varepsilon)/\varepsilon-1} \geq M_i^4.$$

We call a natural number $m$ inconvenient if there exists a pair of coprime natural numbers $C$ and $M \leq m^\varepsilon$ such that

$$\left| \frac{n! x}{2\pi} - \frac{C}{M} \right| \leq m^{\varepsilon-n}.$$
Otherwise, if there exists a pair of coprime natural numbers \( C \) and \( M \leq m^\varepsilon \) such that

\[
\left| \frac{n! x}{2\pi} - \frac{C}{M} \right| \leq m^{\varepsilon - 1},
\]

we say that \( m \) is almost convenient. In other cases we call \( m \) convenient. Let

\[
S_m(x) := \text{Im} \sum_{k=1}^{m} e^{if(k)x} = \sum_{k=1}^{m} \sin f(k)x.
\]

For any natural \( l < L \), using the Abel transformation we obtain

\[
\left| \sum_{m=l}^{L} c_m \sin f(m)x \right| \leq c_l + c_{L+1}L + \left| \sum_{m=l}^{L} (c_m - c_{m+1})S_m(x) \right| \\
\leq 2 \sup_{k \geq l} c_k k + \sum_{m=l}^{L} (c_m - c_{m+1})|S_m(x)|. \tag{3.3}
\]

From now on we assume that \( l \geq 9 \) in order to apply the estimates from §2. With the help of Corollaries 1 and 2 we will estimate \( S_m(x) \) for convenient and almost convenient \( m \), taking into account that in these cases \( \sin \frac{n! y_1 \ldots y_{n-1}x}{2\pi} \), roughly speaking, rarely assumes values close to zero. For such \( m \) we do not use the oddness of \( f \) and the fact that the coefficients of \( f \) are rational.

For convenient \( m \), using Corollary 1 we obtain

\[
\sum_{m \geq l}^{\text{convenient}} (c_m - c_{m+1})|S_m(x)| \leq D \sum_{m \geq l} (c_m - c_{m+1})m^{1-\varepsilon/2^n} \\
\leq Dc_l l^{1-\varepsilon/2^n} + 2D \sum_{m \geq l} c_m m^{-\varepsilon/2^n} \\
\leq D \left( 1 + 2 \sum_{m \geq l} m^{-1-\varepsilon/2^n} \right) \sup_{k \geq l} c_k k \\
\leq D \left( 1 + \frac{2^{n+1}}{\varepsilon} \right) \sup_{k \geq l} c_k k \leq D \frac{2^{n+1}}{\varepsilon} \sup_{k \geq l} c_k k =: D' \sup_{k \geq l} c_k k. \tag{3.4}
\]

Note that there is a natural number \( M = M(m) \leq m^\varepsilon \), \( M \in \mathcal{M} \), assigned to any almost convenient \( m \), and all the numbers \( m \) to which a certain \( M \) is assigned must satisfy the condition

\[
\left| \frac{n! x}{2\pi} - \frac{C}{M} \right|^{-1/(n-\varepsilon)} =: \beta^{-1/(n-\varepsilon)} \leq m \leq \beta^{-1/(1-\varepsilon)}. \tag{3.5}
\]
For almost convenient \( m \), according to (3.5) and Corollary 2, we have

\[
\sum_{m \geq l} (c_m - c_{m+1})|S_m(x)| \leq \sum_{M \in \mathfrak{M}} \sum_{m \geq l, M = M(m)} (c_m - c_{m+1})|S_m(x)| \\
\leq D_1 \sum_{m \geq l} (c_m - c_{m+1})m^{1-\varepsilon/2^{n-1}} + D_1 \beta^{-1/(2^{n-1}(n-1))} \\
\times \sum_{M \in \mathfrak{M}} M^{-1/(2^{n}(n-1))} \sum_{m \geq l} (c_m - c_{m+1})m^{1-(n-\varepsilon)/(2^{n-1}(n-1))}.
\]

We estimate the first sum as in (3.4); then

\[
\sum_{m \geq l} (c_m - c_{m+1})|S_m(x)| \leq \frac{D_1}{D} \sup_{k \geq l} c_k k + 2D_1 \beta^{-1/(2^{n-1}(n-1))} \\
\times \sum_{M \in \mathfrak{M}} M^{-1/(2^{n}(n-1))} \sum_{m \geq l} (c_m - c_{m+1})m^{-(n-\varepsilon)/(2^{n-1}(n-1))} \\
\leq \frac{D''_1}{D} \sup_{k \geq l} c_k k + 2D_1 \beta^{-1/(2^{n-1}(n-1))} \\
\times \sum_{M \in \mathfrak{M}} M^{-1/(2^{n}(n-1))} \frac{2^{n-1}(n-1)}{n-\varepsilon} (\lfloor \beta^{-1/(n-\varepsilon)} \rfloor - (n-\varepsilon)/(2^{n-1}(n-1))) \sup_{k \geq l} c_k k \\
\leq D''_1 \sup_{k \geq l} c_k k, \tag{3.6}
\]

where \( D''_1 \) depends only on \( n \) and \( \varepsilon \) due to (3.2).

Now we turn to inconvenient \( m \). In this case \( x \) is approximated by a \( \pi \)-rational fraction too well, and therefore the difference between \( S_n \) at the point \( x \) and \( S_m \) at the close \( \pi \)-rational point is sufficiently small. For some \( m \) we will use this, and it will be the unique case when we care about the fact that we only have to estimate the imaginary part of the Weyl sum, that \( f \) is an odd function and that the coefficients of \( f \) are rational. For the other \( m \) we will proceed according to one of the two inequalities in Corollary 3, and these estimates will remain true for the whole Weyl sums and for any monic polynomial of the same degree.

There is also a number \( M \leq m^\varepsilon, M \in \mathfrak{M} \), assigned to any inconvenient number \( m \). A fixed \( M \in \mathfrak{M} \) is assigned to all natural numbers \( m \) such that

\[
\beta^{-1/(n-\varepsilon)} = \beta_M^{-1/(n-\varepsilon)} = \left| \frac{n! x}{2\pi} - \frac{C}{M} \right|^{-1/(n-\varepsilon)} \geq m \geq M^{1/\varepsilon},
\]

where \( C = C_M \), and only for these numbers. Set \( m_1 := [M^{1/\varepsilon}] \) and \( m_2 := [\beta^{-1/(n-\varepsilon)}] \) (for a fixed \( M \)). Let \( K \) be a natural number such that

\[
\frac{1}{K^n \ln(2M)} \leq \beta \leq \frac{1}{(K-1)^n \ln(2M)}. \tag{3.7}
\]
We divide the interval \( m_1 \leq m \leq m_2 \) into three intervals: \( m_1 \leq m \leq K-1 \), \( K \leq m \leq [2K \ln(2M)] \) and \([2K \ln(2M)] + 1 \leq m \leq m_2 \). If \( m \) belongs to the second or third interval we estimate \( S_m(x) \) with the help of (2.26) and (2.27), respectively. Only on the interval \( m_1 \leq m \leq K-1 \) do we need the properties of the polynomial \( f \) and the fact that we are dealing with sines and not with cosines in estimating \( S_m(x) \).

Let \( Q \in \mathbb{N} \) be the smallest number such that \( Qf \in \mathbb{Z}[x] \). Note that for any odd \( l \), if \( k_1 \) and \( k_2 \) are integers such that \( k_1 \equiv -k_2 \) (mod \( QMn! \)), then

\[
\sin \left( k_1 \frac{2\pi C}{QMn!} \right) + \sin \left( k_2 \frac{2\pi C}{QMn!} \right) = 0.
\]

Hence for any \( g \in \mathbb{Z} \),

\[
\sum_{k=g+1}^{g+QMN!} \sin \left( f(k) \frac{2\pi C}{Mn!} \right) = 0,
\]

which implies that

\[
|S_m(x)| \leq \left| S_m \left( \frac{2\pi C}{Mn!} \right) \right| + \left| S_m(x) - S_m \left( \frac{2\pi C}{Mn!} \right) \right|
\]

\[
\leq \sum_{k=1}^{m} \sin \left( f(k) \frac{2\pi C}{Mn!} \right) + \frac{2\pi}{n!} \sum_{k=1}^{m} |f(k)| \beta
\]

\[
\leq \left\{ \frac{m}{QMn!} \right\} QMN! + 2C_f \beta \sum_{k=1}^{m} k^n
\]

\[
\leq QMN! + C_f m^{n+1} \beta \leq Qm^\varepsilon n! + C_f m^{n+1} \beta,
\]

where \( C_f \) is the sum of the absolute values of the coefficients of \( f \). From (3.7) and (3.8) we have

\[
\sum_{m=m_1}^{K-1} (c_m - c_{m+1}) |S_m(x)| \leq Qn! \sum_{m=m_1}^{K-1} (c_m - c_{m+1}) m^\varepsilon
\]

\[
+ C_f \beta \sum_{m=m_1}^{K-1} (c_m - c_{m+1}) m^{n+1} \leq Qn! c_{m_1} m_1^\varepsilon + Qn! \sum_{m=m_1+1}^{K-1} c_m (m - \theta_m)^{\varepsilon-1}
\]

\[
+ \frac{C_f}{(K-1)^n \ln(2M)} \sum_{m=m_1}^{K-1} (c_m - c_{m+1}) m^{n+1},
\]

where \( \theta_m \in (0, 1) \) by Lagrange’s theorem. Now, for any \( z \geq 2 \) we have the inequality

\[
(z-1)^{\varepsilon-1} \leq 2z^{\varepsilon-1},
\]

and so by (3.9)

\[
\sum_{m=m_1}^{K-1} (c_m - c_{m+1}) |S_m(x)| \leq Qn! m_1^{\varepsilon-1} \sup_{k \geq l} c_k k + 2zQn! \sum_{m=m_1+1}^{K-1} c_m m^{\varepsilon-1}
\]

\[
+ C_f c_{m_1} m_1^{\varepsilon-1} + C_f \frac{n+1}{(K-1)^n \ln(2M)} \sum_{m=m_1+1}^{K-1} c_m m^n
\]
Hence, using (2.27),

\[
\leq \sup_{k \geq l} c_k k \left( Qn! m_1^{\varepsilon^{-1}} + 2\varepsilon Qn! \sum_{m=m_1+1}^{K-1} m^{\varepsilon^{-2}} \right. \\
+ \frac{C_f}{\ln(2M)} + \frac{(n+1)C_f}{(K-1)^n \ln(2M)} \sum_{m=m_1+1}^{K-1} m^{n-1} \left. \right) \\
\leq \left( Qn! \left( m_1^{\varepsilon^{-1}} + \frac{2\varepsilon}{1-\varepsilon} m_1^{\varepsilon^{-1}} \right) \right. \\
\left. + \frac{C_f}{\ln(2M)} + \frac{(n+1)C_f}{(K-1)^n \ln(2M)} \frac{K^n}{n} \right) \sup_{k \geq l} c_k k,
\]

and now, because \( m_1^{\varepsilon^{-1}} \geq M \) and \((n+1)K^n/(n(K-1)^n) \leq 2^{n+1}\), we finally have

\[
\sum_{m=m_1}^{K-1} (c_m - c_{m+1})|S_m(x)| \leq A \left( M^{(\varepsilon^{-1})/\varepsilon} + \frac{1}{\ln(2M)} \right) \sup_{k \geq l} c_k k,
\]

where \( A > 0 \) depends only on \( n, \varepsilon \) and \( f \).

For \( 2K \ln(2M) \leq m \leq m_2 \) we have

\[
m^n \beta \geq (2K \ln(2M))^n (K^n \ln(2M))^{-1} = 2^n \ln^{n-1}(2M) \geq 2(2 \ln 2)^{n-1} \geq 2,
\]

Hence, using (2.27),

\[
\sum_{m=[2K \ln(2M)]+1}^{m_2} (c_m - c_{m+1})|S_m(x)| \\
\leq J \sum_{m=[2K \ln(2M)]+1}^{m_2} (c_m - c_{m+1}) \left( m^{1-\varepsilon/2^{n-1}} + \frac{m^{1-n/(2^n(n-1))}}{(M \beta)^{1/(2^n(n-1))}} \right).
\]

First, we estimate

\[
\sum_{m=[2K \ln(2M)]+1}^{m_2} (c_m - c_{m+1}) m^{1-\varepsilon/2^{n-1}} \\
\leq \frac{c_1 [2K \ln(2M)]+1 (2K \ln(2M)]+1) (1 + 2 \sum_{m=[2K \ln(2M)]+2}^{m_2} c_m m^{-\varepsilon/2^{n-1}}) \\
\leq \left( m_1^{1-\varepsilon/2^{n-1}} + 2 \sum_{m=[2K \ln(2M)]+2}^{m_2} m^{1-\varepsilon/2^{n-1}} \right) \sup_{k \geq l} c_k k \\
\leq \left( M^{-1/2^{n-1}} + 2 \sum_{m=[2K \ln(2M)]+2}^{m_2} m^{-\varepsilon/2^{n-1}} \right) \sup_{k \geq l} c_k k \\
\leq \left( \frac{2^{n+1}}{\varepsilon} m_1^{-\varepsilon/2^{n-1}} \right) \sup_{k \geq l} c_k k \leq \frac{2^{n+1}}{\varepsilon} M^{-1/2^{n-1}} \sup_{k \geq l} c_k k.
\]

Further, in view of (3.7),

\[
\sum_{m=[2K \ln(2M)]+1}^{m_2} (c_m - c_{m+1}) \frac{m^{1-n/(2^n(n-1))}}{(M \beta)^{1/(2^n(n-1))}} \\
\leq \sum_{m=[2K \ln(2M)]+1}^{m_2} (c_m - c_{m+1}) \frac{m^{1-n/(2^n(n-1))} K^n/(2^n(n-1)) (\ln(2M))^{1/(2^n(n-1))}}{M^{1/(2^n(n-1))}}
\]
Combining (3.11), (3.15) and (3.16) we obtain

\[
\leq \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} c_{[2K \ln(2M)]+1} \left( [2K \ln(2M)] + 1 \right) + 2 \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} \sum_{m=2K \ln(2M) + 2}^{m_2} c_m m^{-n/(2^n(n-1))} K^{n/(2^n(n-1))}
\]

\[
\leq \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} \times \left( 1 + 2K^{n/(2^n(n-1))} \right) \sum_{m=2K \ln(2M) + 2}^{m_2} m^{-1-n/(2^n(n-1))} \sup_{k \geq l} c_k k
\]

\[
\leq H \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} \sup_{k \geq l} c_k k,
\]

(3.14)

where \( H > 0 \) depends only on \( n \). Thus, from (3.12)–(3.14) we have

\[
\sum_{m=2K \ln(2M) + 1}^{m_2} (c_m - c_{m+1}) \vert S_m(x) \vert \leq H' \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} \sup_{k \geq l} c_k k,
\]

(3.15)

where \( H' > 0 \) also depends only on \( n \) and \( f \).

For \( K \leq m \leq 2K \ln(2M) \), according to (2.26),

\[
\sum_{m=K}^{[2K \ln(2M)]} (c_m - c_{m+1}) \vert S_m(x) \vert \leq W \sum_{m=K}^{[2K \ln(3M)]} (c_m - c_{m+1}) \frac{m}{M^{1/(2^n(n-1))}}
\]

\[
\leq \frac{W}{M^{1/(2^n(n-1))}} \left( c_K K + \sum_{m=K+1}^{[2K \ln(3M)]} c_m \right)
\]

\[
\leq \frac{W}{M^{1/(2^n(n-1))}} \left( 1 + 2 \ln \frac{2K \ln(3M)}{K + 1} \right) \sup_{k \geq l} c_k k \leq 3W \frac{\ln(3M)}{M^{1/(2^n(n-1))}} \sup_{k \geq l} c_k k.
\]

(3.16)

Combining (3.11), (3.15) and (3.16) we obtain

\[
\sum_{m=m_1}^{m_2} (c_m - c_{m+1}) \vert S_m(x) \vert \leq A \left( M^{(\epsilon-1)/\epsilon} + \frac{1}{\ln(2M)} \right) \sup_{k \geq l} c_k k
\]

\[
+ H' \left( \frac{\ln(2M)}{M} \right)^{1/(2^n(n-1))} + 3W \frac{\ln(3M)}{M^{1/(2^n(n-1))}} \sup_{k \geq l} c_k k \leq A \frac{\sup_{k \geq l} c_k k}{\ln(2M)},
\]

(3.17)
where \( A' > 0 \) depends only on \( \varepsilon, n \) and \( f \). Now, \( M \in \mathfrak{M} \) and so, using (3.2) we have
\[
\sum_{m=0}^{\infty} \left( c_m - c_{m+1} \right) |S_m(x)| \leq \sum_{i=1}^{\infty} \frac{A' \sup_{k \geq l} c_k k}{\ln(2M_i)} \leq A' \left( \frac{1}{\ln 2} + \sum_{i=2}^{\infty} \frac{1}{\ln M_i} \right) \sup_{k \geq l} c_k k
\]
\[
\leq A' \left( \frac{1}{\ln 2} + \frac{1}{\ln 2} \sum_{i=2}^{\infty} \frac{1}{3^{i-2}} \right) \sup_{k \geq l} c_k k \leq \frac{3}{\ln 2} A' \sup_{k \geq l} c_k k.
\]  
(3.18)

Thus, combining (3.3), (3.18), (3.4) and (3.6) we obtain
\[
\left| \sum_{m=l}^{L} c_m \sin f(m)x \right| \leq 2 \sup_{k \geq l} c_k k + \frac{3}{\ln 2} A' \sup_{k \geq l} c_k k + D'' \sup_{k \geq l} c_k k + D_1'' \sup_{k \geq l} c_k k
\]
\[
\leq A'' \sup_{k \geq l} c_k k,
\]
where \( A'' > 0 \) depends only on \( \varepsilon, n \) and \( f \). This means that the theorem is proved for \( x \in \mathbb{R} \setminus \pi \mathbb{Q} \).

Finally, if \( x \in \pi \mathbb{Q} \), we find \( x' \notin \pi \mathbb{Q} \) such that
\[
|x - x'| \leq L^{-n-1} c_1^{-1} \sup_{k \geq l} c_k k.
\]

Then
\[
\left| \sum_{m=l}^{L} c_m \sin f(m)x \right| \leq \left| \sum_{m=l}^{L} c_m \sin f(m)x' \right| + \left| \sum_{m=l}^{L} c_m \sin f(m)x - \sum_{m=l}^{L} c_m \sin f(m)x' \right|
\]
\[
\leq A'' \sup_{k \geq l} c_k k + C_f L^{-n} \sup_{k \geq l} c_k k \sum_{m=l}^{L} m^n
\]
\[
\leq (A'' + C_f) \sup_{k \geq l} c_k k,
\]  
(3.19)

which completes the proof of Theorem 1 for odd \( \alpha \).

§ 4. The case of a power from the interval \((1, 2)\)

The feature of this case is that for \( \alpha \in (1, 2) \) the differences \((k+1)^{\alpha} - k^{\alpha}\) increase, and increase quite slowly. The idea of the proof is the following: select blocks of \( k \) such that the differences \((k+1)^{\alpha} x - k^{\alpha} x\), taken modulo \(2\pi\), lie close to 0 or \(2\pi\). Then the ‘steps’ between \(k^{\alpha} x\) and \((k+1)^{\alpha} x\) in these blocks are small enough, and we can estimate the sums of the form \(\sum_{k=k_1}^{k+1} \sin k^{\alpha} x\) in them using Lemma 5. For other \(k\), a sum of the form \(\sum_{k=k_1}^{k+1} \sin k^{\alpha} x\) differs slightly from the sum
\[
\sum_{k=0}^{s} \sin(x_0 + k\gamma) = \frac{\cos(x_0 - \frac{\gamma}{2}) - \cos(x_0 + (2k + 1)\frac{\gamma}{2})}{2 \sin \frac{\gamma}{2}},
\]
where \(\gamma\) is separated from 0 and \(2\pi\), so that \(\sin \frac{\gamma}{2}\) is separated from zero. The main difficulty lies in choosing the lengths of these blocks: they should not be too
so that the differences \((k + 1)^\alpha x - k^\alpha x\) will change a lot in a long block.

**Proof of Theorem 1, (c) for the case \(\alpha \in (1, 2)\).** Let the condition \(c_k k \to 0\) be satisfied. We show that the series (1.1) converges uniformly on the set \(|x| \leq X < \infty\). Without loss of generality, from now on we assume that \(x > 0\) (the case \(x = 0\) is obvious). Fix some \(\delta\) in the interval \((0, (2 - \alpha)/3)\). Let \(l_0 \geq 2\) be a natural number such that the following conditions are fulfilled:

\[
\left( \frac{\pi}{\alpha(\alpha - 1)} - 1 \right) l_0^2 \geq \pi, \quad l_0^{1-\alpha/2} > 4\sqrt{\pi} \ln^2 l_0 \quad \text{and} \quad l_0^{(2-\alpha)/3-\delta} > 4\ln^2 l_0. \tag{4.1}
\]

Then for any \(l \geq l_0\) all these conditions are satisfied as well.

Consider

\[
\sum_{k=l}^{L} c_k \sin k^\alpha x,
\]

where \(l \geq l_0\) and \(x \in (0, X]\) is fixed. Let \(m := \lceil x^{-1/\alpha} \rceil\), so that

\[
\left| \sum_{k=l}^{m-1} c_k \sin k^\alpha x \right| \leq \left| \sum_{k=l}^{m-1} c_k \sin k^\alpha x \right| + \left| \sum_{k=m}^{L} c_k \sin k^\alpha x \right| =: |S_1| + |S_2|. \tag{4.2}
\]

If \(m = 1\), then \(S_1 = 0\). Otherwise \(2 \leq m \leq x^{-1/\alpha} + 1 \leq 2x^{-1/\alpha}\), and we have

\[
|S_1| \leq \sum_{k=l}^{m-1} c_k k^\alpha x \leq \sup_{k \geq l} c_k k \sum_{k=1}^{m-1} k^{\alpha-1} x \leq x \sup_{k \geq l} c_k k \int_{1}^{2x^{-1/\alpha}} y^{\alpha-1} dy \leq \frac{2\alpha}{\alpha} \sup_{k \geq l} c_k k. \tag{4.3}
\]

Further, set

\[
\Delta_k^1 := k^\alpha x - (k - 1)^\alpha x, \quad \Delta_k^2 := \Delta_k^1 - \Delta_{k-1}^1 \quad \text{and} \quad \Delta_k^1 := \Delta_k^1 \mod 2\pi,
\]

so that \(\Delta_k^1 \in [0, 2\pi]\). Note that \(\Delta_k^2\) decreases in \(k\). Indeed, by Lagrange’s theorem

\[
\left( \frac{\Delta_k^2}{x} \right)' = \alpha \left( k^{\alpha-1} - 2(k - 1)^{\alpha-1} + (k - 2)^{\alpha-1} \right) \]

\[
= \alpha(\alpha - 1) \left( (k - 1 + \theta_1)^{\alpha-2} - (k - 2 + \theta_2)^{\alpha-2} \right) \leq 0, \tag{4.4}
\]

where \(\theta_1, \theta_2 \in (0, 1)\). Note also that

\[
\Delta_k^2 = \alpha x \left( (k - 1 + \theta_3)^{\alpha-1} - (k - 2 + \theta_4)^{\alpha-1} \right) \leq 2\alpha(\alpha - 1)x(k - 2)^{\alpha-2}, \tag{4.5}
\]

where \(\theta_3, \theta_4 \in (0, 1)\), and that

\[
\Delta_k^2 \geq \frac{1}{2}(\Delta_{k+1}^2 + \Delta_k^2) = \frac{1}{2}(\Delta_{k+1}^1 - \Delta_{k-1}^1) \]

\[
= \frac{1}{2}\alpha x \left( (k + \theta_5)^{\alpha-1} - (k - 2 + \theta_6)^{\alpha-1} \right) \geq \frac{1}{2}\alpha(\alpha - 1)x(k + 1)^{\alpha-2}, \tag{4.6}
\]
where again \( \theta_5, \theta_6 \in (0, 1) \). Let
\[
K_1 := \{ k : \tilde{\Delta}^1_{k+1} \in [0, m^{-\delta}] \cup [2\pi - m^{-\delta}, 2\pi] \}
\]
and
\[
K_2 := \{ k : \tilde{\Delta}^1_{k+1} \in [m^{-\delta}, 2\pi - m^{-\delta}] \}.
\]
Then we have
\[
|S_2| \leq \left| \sum_{k=m}^{L} c_k \sin k^\alpha x \right| + \left| \sum_{k=m}^{L} c_k \sin k^\alpha x \right| =: |S'_2| + |S''_2|. \tag{4.7}
\]
First we estimate \( S'_2 \). According to (4.4) and (4.5), for \( k \geq m + 2 \) we have \( \Delta_k^2 \leq 2\alpha(\alpha - 1)xm^{\alpha - 2} \). Hence we can find \( p = p(m) \) such that
\[
p := \min \{ p' > 1 : |\Delta^1_{m+p'} - \Delta^1_{m+1} - 2\pi| \leq 2\alpha(\alpha - 1)xm^{\alpha - 2} \}.
\]
This yields
\[
2\alpha(\alpha - 1)xm^{\alpha - 2}p \geq 2\pi - 2\alpha(\alpha - 1)xm^{\alpha - 2},
\]
so
\[
p \geq -1 + \frac{2\pi}{2\alpha(\alpha - 1)xm^{2 - \alpha}} \geq m^{2 - \alpha}x^{-1}\tag{4.8}
\]
due to the first condition in (4.1).

Since \( \Delta_k^1 \) increases in \( k \) (for instance, see (4.6)), and we have chosen \( p \) to be minimal, we have \( 0 < \Delta^1_{m+p-1} - \Delta^1_{m+1} < 2\pi \). Hence, among \( \tilde{\Delta}^1_{m+1}, \tilde{\Delta}^1_{m+2}, \ldots, \tilde{\Delta}^1_{m+p} \), there are at most three blocks of consecutive \( \tilde{\Delta}^1 \), that is, blocks of the form \( \tilde{\Delta}^1_{i_1}, \tilde{\Delta}^1_{i_1+1}, \ldots, \tilde{\Delta}^1_{i_1+i_2} \) such that the numbers in each block are increasing and lie in one of the intervals \([0, m^{-\delta}]\) and \([2\pi - m^{-\delta}, 2\pi] \). We focus on the case of \([0, m^{-\delta}]\), the second can be treated similarly. Let our block be: \( \tilde{\Delta}^1_{s+1}, \tilde{\Delta}^1_{s+2}, \ldots, \tilde{\Delta}^1_{s+u} \). Without loss of generality we can assume that \( s^\alpha x \in [\pi u, \pi (u+1)] =: I_u \) for some even \( u \). Let \( t \) be such that
\[
s^\alpha x + \sum_{i=0}^{t} \tilde{\Delta}^1_{s+i} \in I_u \text{ and } s^\alpha x + \sum_{i=0}^{t+1} \tilde{\Delta}^1_{s+i} \notin I_u.
\]
Then the following inequality must be valid:
\[
\pi \geq (t - 1)\Delta^2_{s+2} + (t - 2)\Delta^2_{s+3} + \cdots + 1 \cdot \Delta^2_{s+t}. \tag{4.9}
\]
From (4.6) and (4.9) we have
\[
\pi \geq \frac{\alpha(\alpha - 1)}{2} \times x \left( (t - 1)(s + 3)^{\alpha - 2} + (t - 2)(s + 4)^{\alpha - 2} + \cdots + 1 \cdot (s + t + 1)^{\alpha - 2} \right). \tag{4.10}
\]
Note that the function \( \kappa(y) = y(a - y)^{-c} + (b - y)(a - b + y)^{-c} \) does not increase for \( c > 0 \) and \( a \geq b \geq 2y > 0 \). In fact,
\[
\kappa'(y) = (a - y)^{-c} - y(-c)(a - y)^{-c-1} - (a - b + y)^{-c} + (b - y)(-c)(a - b + y)^{-c-1} < 0,
\]
since \( a - y \geq a - b + y \) and \( y \leq b - y \). Therefore, for \( c = 2 - \alpha \), \( b = t \), \( a = s + t + 2 \) and \( y = t - i, \ i = 1, 2, \ldots, [(t - 1)/2] \), we have
\[
(t - i)(s + 2 + i)^{\alpha - 2} + i(s + t + 2 - i)^{\alpha - 2} \geq \frac{3}{2} \left( s + 2 + \frac{t}{2} \right)^{\alpha - 2},
\]
thus, using (4.10),
\[
\pi \geq \frac{\alpha(\alpha - 1)}{2} x(t - 1) \frac{t}{2} \left( s + 2 + \frac{t}{2} \right)^{\alpha - 2}
\geq \frac{\alpha(\alpha - 1)}{2} x(t - 1) \frac{t - 1}{2} \left( s + 2 + \frac{t - 1}{2} \right)^{\alpha - 2}.
\]
(4.11)
If \( t - 1 \geq 2(s + 2) \), then from (4.11) we have
\[
\pi \geq \frac{\alpha(\alpha - 1)}{2} x \left( \frac{t - 1}{2} \right)^{\alpha} \geq \frac{\alpha(\alpha - 1)}{4} (t - 1)^{\alpha} m^{-\alpha},
\]
from which
\[
t - 1 \leq \left( \frac{4\pi}{\alpha(\alpha - 1)} \right)^{1/\alpha} m \leq \frac{4\pi}{\alpha - 1} m \leq \frac{4\pi}{\alpha - 1} (s + 2).
\]
Thus, from (4.11) we obtain
\[
\pi \geq \frac{\alpha(\alpha - 1)}{2} x \left( \frac{t - 1}{2} \right)^{2} \left( \frac{4\pi}{\alpha - 1} + 1 \right)^{\alpha - 2} (s + 2)^{\alpha - 2}
\geq \frac{(\alpha - 1)^{2}}{20\pi} (t - 1)^{2} (s + 2)^{\alpha - 2} x,
\]
hence
\[
t < \frac{8}{\alpha - 1} (s + 2)^{1 - \alpha/2} x^{-1/2} + 1 \leq \frac{27}{\alpha - 1} s^{1 - \alpha/2} x^{-1/2}. \quad (4.12)
\]
Let \( t_{0} := t \). We define \( t_{i} \) for \( i \geq 1 \) in the following way:
\[
s^{\alpha} x + \sum_{j=1}^{t_{i}} \Delta_{s+j}^{1} \in I_{u+i} \quad \text{and} \quad s^{\alpha} x + \sum_{j=1}^{t_{i}+1} \Delta_{s+j}^{1} \notin I_{u+i}.
\]
We also denote by \( R \) the smallest even integer for which \( s^{\alpha} x + \sum_{j=1}^{v} \Delta_{s+j}^{1} < \pi(u + R + 2) \). Then using an argument similar to that in (4.12) we obtain
\[
t_{1} < \frac{54}{\alpha - 1} s^{1 - \alpha/2} x^{-1/2}. \quad (4.13)
\]
We also have
\[
\sum_{k=s}^{s+v} c_{k} \sin k^{\alpha} x = \sum_{k=s}^{s+t_{0}} c_{k} \sin k^{\alpha} x + \sum_{i=0}^{R/2 - 1} \sum_{k=s+i+1}^{s+t_{2i+1}} c_{k} \sin k^{\alpha} x + \sum_{k=s+t_{R}}^{s+v} c_{k} \sin k^{\alpha} x. \quad (4.14)
\]
Note that due to (4.12)
\[
\sum_{k=s}^{s+t_{0}} c_{k} \sin k^{\alpha} x \leq t c_{s} < \frac{27}{\alpha - 1} s^{1 - \alpha/2} x^{-1/2} c_{s} \leq \frac{27}{\alpha - 1} s^{-\alpha/2} x^{-1/2} \sup_{k \geq l} c_{k} k. \quad (4.15)
\]
Lemma 5. Let the points \(y_1, \ldots, y_k\) be such that \(0 < y_1 < y_2 - y_1 < y_3 - y_2 < \cdots < y_k - y_{k-1}\) and \(y_k < 2\pi\), and let the index \(q\) be such that \(y_q < \pi < y_{q+1}\) and \(y_{q+1} - y_q = a < \pi\). Then

\[
\sum_{i=1}^{k} \sin y_i \geq -\sin \frac{a}{2}.
\]

Proof. Let \(\mu\) be such that \(\sin y_\mu \geq \sin y_i\) for all \(i\), and \(\nu\) be such that \(\sin y_\nu \leq \sin y_i\) for all \(i\). Note that then

\[
y_1, \ldots, y_{\mu-1} \in \left[0, \frac{\pi}{2}\right], \quad y_\mu+1, \ldots, y_q \in \left[\frac{\pi}{2}, \pi\right],
\]

\[
y_{q+1}, \ldots, y_{\nu-1} \in \left[\pi, \frac{3\pi}{2}\right] \quad \text{and} \quad y_{\nu+1}, \ldots, y_k \in \left[\frac{3\pi}{2}, \pi\right].
\]

In this case, for \(1 \leq i \leq \mu - 1\) we have \(y_{i+1} - y_i \leq a\), hence

\[
\sin y_i \geq \max\{\sin(y_\mu - a(M - i)), 0\}.
\]

Thus,

\[
\sum_{i=1}^{\mu-1} \sin y_i \geq \sum_{j=1}^{\lceil y_\nu/a \rceil - 1} \sin(y_\mu - aj).
\]

Similarly, since for \(\mu + 1 \leq i \leq q\) we also have \(y_{i+1} - y_i \leq a\), it follows that

\[
\sum_{i=\mu+1}^{q} \sin y_i \geq \sum_{j=1}^{\lceil (\pi - y_\mu)/a \rceil - 1} \sin(y_\mu + aj).
\]

Further, since \(y_{i+1} - y_i \geq a\) for \(q + 1 \leq i \leq k - 1\), we have

\[
\sum_{i=q+1}^{\nu-1} \sin y_i \geq \sum_{j=1}^{\lceil (y_\nu - \pi)/a \rceil - 1} \sin(y_\nu - aj)
\]

and

\[
\sum_{i=\nu+1}^{k} \sin y_i \geq \sum_{j=1}^{\lceil (2\pi - y_\nu)/a \rceil - 1} \sin(y_\nu + aj).
\]

Then

\[
\sum_{i=1}^{q} \sin y_i \geq \sum_{j=-\lceil y_\nu/a \rceil + 1}^{\lceil (\pi - y_\mu)/a \rceil - 1} \sin(y_\mu + aj)
\]

\[
= \frac{\cos(y_\mu - \lceil y_\mu/a \rceil a + \frac{a}{2}) - \cos(y_\mu + \lceil \pi - y_\mu/a \rceil a - \frac{a}{2})}{2 \sin \frac{a}{2}} \geq \frac{\cos \frac{a}{2}}{\sin \frac{a}{2}}.
\]

At the same time

\[
\sum_{i=q+1}^{k} \sin y_i \geq \sum_{j=-\lceil (y_\nu - \pi)/a \rceil + 1}^{\lceil (2\pi - y_\nu)/a \rceil - 1} \sin(y_\nu + aj)
\]

\[
= \frac{\cos(y_\nu - \lceil y_\nu - \pi/a \rceil a + \frac{a}{2}) - \cos(y_\nu + \lceil 2\pi - y_\nu/a \rceil a - \frac{a}{2})}{2 \sin \frac{a}{2}} \geq \frac{-1}{\sin \frac{a}{2}}.
\]
So,
\[
\sum_{i=1}^{k} \sin y_i \geq \cos \frac{a}{2} - 1 = \frac{\cos^2 \frac{a}{2} - 1}{2} = -\sin \frac{a}{2}.
\]

Lemma 5 is proved.

**Corollary 4.** Let the points \( y_1, \ldots, y_k \) be as given in Lemma 5 and let the sequence \( \{a_j\} \) be nonincreasing. Then
\[
\sum_{i=1}^{k} a_i \sin y_i \geq -a_{q+1} \sin \frac{a}{2}.
\]

**Proof.** We have
\[
\sum_{i=1}^{k} a_i \sin y_i = \sum_{i=1}^{q} a_i \sin y_i + \sum_{i=q+1}^{k} a_i \sin y_i \\
\geq a_{q+1} \sum_{i=1}^{q} \sin y_i + a_{q+1} \sum_{i=q+1}^{k} \sin y_i \geq -a_{q+1} \sin \frac{a}{2}
\]
by Lemma 5. The corollary is proved.

By Corollary 4, for any \( 0 \leq i \leq R/2 - 1 \) we have
\[
\sum_{k=s+t_2i+2}^{s+t_2i+2} c_k \sin k^\alpha x \leq \frac{m-\delta}{2} c_{s+t_2i+1+1} + 1
\]
and
\[
\sum_{k=s+t_R}^{s+v} c_k \sin k^\alpha x \leq \frac{m-\delta}{2} c_{s+t_R+1+1}.
\]

Thus, from (4.14)–(4.17) we derive that
\[
\sum_{k=s}^{s+v} c_k \sin k^\alpha x \leq \left( \frac{R}{2} + 1 \right) \frac{m-\delta}{2} c_s + \frac{27}{\alpha - 1} s^{-\alpha/2} x^{-1/2} \sup_{k \geq l} c_k k.
\]

Note that \( v \) must fulfill the inequality
\[
\Delta^1_{s+v} - \Delta^1_{s+1} \leq m^{-\delta},
\]
and by Lagrange’s theorem, for some \( \theta_7, \theta_8 \in (0, 1) \) the left-hand side of (4.19) can be written as follows:
\[
\alpha x ((s + v - 1 + \theta_7)^{\alpha-1} - (s + \theta_8)^{\alpha-1}) \geq \alpha x ((s + v - 1)^{\alpha-1} - (s + 1)^{\alpha-1}),
\]
which yields
\[
(s + v - 1)^{\alpha-1} \leq m^{-\delta} x^{-1} + (s + 1)^{\alpha-1},
\]
hence
\[
\left(1 + \frac{v - 2}{s + 1}\right)^{\alpha - 1} \leq m^{-\delta} x^{-1}(s + 1)^{1-\alpha} + 1. \quad (4.20)
\]
By Bernoulli’s inequality, the left-hand side of (4.20) is not less than
\[
1 + (\alpha - 1)(v - 2)/(s + 1),
\]
so we obtain
\[
(\alpha - 1)\frac{v - 2}{s + 1} \leq m^{-\delta} x^{-1}(s + 1)^{1-\alpha},
\]
then
\[
v \leq \frac{1}{\alpha - 1} m^{-\delta} x^{-1}(s + 1)^{2-\alpha} + 2 \leq \frac{1 + 2X(\alpha - 1)}{\alpha - 1} m^{-\delta} x^{-1}(s + 1)^{2-\alpha}. \quad (4.21)
\]
Thus, it follows from (4.4), (4.5) and (4.21) that
\[
R \leq \frac{1}{2\pi} \Delta_{s+2}^2 \leq \frac{\alpha(\alpha - 1)}{\pi} s^{\alpha/2} x^{-1/2} \frac{1 + 2X(\alpha - 1)}{\alpha - 1} m^{-\delta} x^{-1}(s + 1)^{2-\alpha}
\]
\[
\leq \frac{2 \cdot 2^{-\alpha} \cdot (1 + 2X)}{\pi} m^{-\delta} \leq (2 + 4X)m^{-\delta}. \quad (4.22)
\]
From (4.18) and (4.22) we see that
\[
\sum_{k=0}^{s+v} c_k \sin k^\alpha x \leq (2 + 4X)m^{-\delta} \sum_{k=0}^{\infty} \frac{27}{\alpha - 1} s^{-\alpha/2} x^{-1/2} \sup_{k \geq l} c_k k
\]
\[
\leq \left((1 + 2X)m^{-1-2\delta} + \frac{27}{\alpha - 1} s^{-\alpha/2} x^{-1/2}\right) \sup_{k \geq l} c_k k,
\]
but hence
\[
\sum_{k=m}^{m+p} c_k \sin k^\alpha x \leq 3 \left((1 + 2X)m^{-1-2\delta} + \frac{27}{\alpha - 1} m^{-\alpha/2} x^{-1/2}\right) \sup_{k \geq l} c_k k.
\]
So, in view of (4.8),
\[
S'_2 \leq 3 \sum_{i=0}^{\infty} \left((1 + 2X)w_i^{-1-2\delta} + \frac{27}{\alpha - 1} w_i^{-\alpha/2} x^{-1/2}\right) \sup_{k \geq l} c_k k
\]
\[
\leq 3 \left((1 + 2X)\frac{m^{-2\delta}}{2\delta} + \frac{27}{\alpha - 1} \sum_{i=0}^{\infty} w_i^{-\alpha/2} x^{-1/2}\right) \sup_{k \geq l} c_k k, \quad (4.23)
\]
where \(w_0 := m\) and \(w_{i+1} := w_i + w_i^{2-\alpha} x^{-1} \geq w_i + 1\) for \(i \geq 0\), and therefore
\[
w_i \to \infty \quad \text{as} \quad i \to \infty. \quad (4.24)
\]
Recall that \(m \geq l \geq l_0 \geq 2\) and consider the function
\[
F(m) := \int_{m}^{\infty} \frac{dy}{y \ln^2 y} = \frac{1}{\ln m}.
\]
According to (4.24),

\[
F(m) = \sum_{j=0}^{\infty} \int_{w_j}^{w_{j+1}} \frac{dy}{y \ln^2 y} =: \sum_{j=0}^{\infty} W_j.
\] (4.25)

Suppose that for \( j = 0, \ldots, J \), and only for these values, we have \( x^{-1} > w_j^{\alpha-1} \); then for \( j = 0, \ldots, J - 1 \) we have \( w_{j+1} \geq 2w_j \), hence

\[
\sum_{i=0}^{J} w_i^{-\alpha/2} x^{-1/2} \leq m^{-\alpha/2} x^{-1/2} \sum_{i=0}^{\infty} 2^{-i\alpha/2} \leq \frac{1}{1 - 2^{-\alpha/2}} \leq 4. \] (4.26)

Furthermore, for \( j > J \) we have \( x^{-1} \leq w_j^{\alpha-1} \). Now, the inequality \( \ln(1 + y) \geq y/2 \) is valid for \( y \leq 1 \), and using this we obtain

\[
W_j = \frac{1}{\ln w_j} - \frac{1}{\ln(w_j + w_j^{2-\alpha} x^{-1})} = \frac{\ln(1 + w_j^{1-\alpha} x^{-1})}{\ln w_j \ln(w_j + w_j^{2-\alpha} x^{-1})} \geq \frac{w_j^{1-\alpha} x^{-1}}{2 \ln w_j \ln(2w_j)} \geq w_j^{-\alpha/2} x^{-1/2}. \] (4.27)

Here we have used the two-sided inequality

\[
w_j^{1-\alpha/2} x^{-1/2} \geq w_j^{1-\alpha/2} \pi^{-1/2} \geq 4 \ln^2 w_j,
\]

which is valid since \( w_j \geq m \geq l_0 \) by the second condition in (4.1). Thus, from (4.25) and (4.27),

\[
\sum_{i=J+1}^{\infty} w_i^{-\alpha/2} x^{-1/2} \leq \sum_{i=J+1}^{\infty} W_j \leq F(m) \leq \frac{1}{\ln 2}. \] (4.28)

Combining (4.26) and (4.28), from (4.23) we derive that

\[
S'_2 \leq 3 \left( \frac{1 + 2X}{2\delta} + \frac{27}{\alpha - 1} \left(4 + \frac{1}{\ln 2}\right)\right) \sup_{k \geq l} c_k k. \] (4.29)

Replacing (4.14) by

\[
\sum_{k=s}^{s+u} c_k \sin k^\alpha x = \sum_{k=s}^{s+t_1} c_k \sin k^\alpha x + \sum_{i=2}^{R/2-1} \sum_{k=s+t_{2i-1}+1}^{s+t_{2i}+1} c_k \sin k^\alpha x + \sum_{k=s+t_R}^{s+u} c_k \sin k^\alpha x
\]

and using the same argument, with the help of (4.13) we obtain

\[
S'_2 \geq -3 \left( \frac{1 + 2X}{2\delta} + \frac{54}{\alpha - 1} \left(4 + \frac{1}{\ln 2}\right)\right) \sup_{k \geq l} c_k k. \] (4.30)

From (4.29) and (4.30) we have finally

\[
|S'_2| \leq C(\alpha, X) \sup_{k \geq l} c_k k. \] (4.31)
Now consider $S'_2$. Let $m' = m'(m) \geq m$ be the first index such that $m' \in K_2$. Put $Q = Q(m) := \lceil m(2-\alpha)/3 \rceil$. Note that

$$\frac{m^\alpha}{\pi} \leq \frac{\tilde{\Delta}_{k+1}^1}{2} \leq \frac{m^\alpha}{\pi}$$

for $k \in K_2$. Applying the Abel transformation we obtain

$$\sum_{k=m'}^{m'+Q-1} c_k \sin k^\alpha x = \sum_{q=0}^{Q-1} (c_{m'+q} - c_{m'+q+1}) \sum_{k=m'}^{m'+q} \sin k^\alpha x + c_{m'+Q} \sum_{k=m'}^{m'+Q-1} \sin k^\alpha x.$$  \hspace{1cm} \text{(4.33)}

In addition,

$$(m' + q)^\alpha x \equiv (m')^\alpha x + \sum_{t=1}^{q} \tilde{\Delta}_{m'+t}^1,$$

and then from (4.4) and (4.5) we see that

$$|(m' + t)^\alpha - (m')^\alpha - \tilde{\Delta}_{m+1}^1| \leq \frac{t(t-1)}{2} \Delta_{m+2}^2 \leq t(t-1) \alpha(\alpha - 1) x m^{\alpha - 2}.\hspace{1cm} \text{(4.34)}$$

Since $|\sin(g + h) - \sin g| \leq |h|$ for arbitrary $g, h \in \mathbb{R}$, it follows from (4.34) that for $q \leq Q - 1$

$$\left| \sum_{k=m'}^{m'+q} \sin k^\alpha x - \sum_{t=0}^{q} \sin((m')^\alpha x + \tilde{\Delta}_{m+1}^1 t) \right| \leq \left(\frac{Q+1}{6} \right) \alpha(\alpha - 1) x m^{\alpha - 2} \leq Q^3 \alpha(\alpha - 1) x m^{\alpha - 2} \leq ((2m)^{\alpha - 3/3})^3 \alpha(\alpha - 1) x m^{\alpha - 2} = 2^{3-\alpha} \alpha(\alpha - 1) x \leq 4X.\hspace{1cm} \text{(4.35)}$$

Moreover, taking (4.32) into account,

$$\left| \sum_{t=0}^{q} \sin((m')^\alpha x + \tilde{\Delta}_{m+1}^1 t) \right| = \left| \cos((m')^\alpha x - \frac{\tilde{\Delta}_{m+1}^1}{2}) - \cos((m')^\alpha x + \frac{\tilde{\Delta}_{m+1}^1 + (2q+1)}{2}) \right| \leq \frac{2}{2 \pi} \frac{m^\delta}{2} = \pi m^\delta.\hspace{1cm} \text{(4.36)}$$

From (4.35) and (4.36) we have

$$\left| \sum_{k=m'}^{m'+q} \sin k^\alpha x \right| \leq \pi m^\delta + 4X \leq (\pi + 4X) m^\delta,\hspace{1cm} \text{(4.37)}$$

and from (4.37) and (4.33)

$$\left| \sum_{k=m'}^{m'+Q-1} c_k \sin k^\alpha x \right| \leq c_{m'}(\pi + 4X)m^\delta \leq c_{m}(\pi + 4X)m^\delta \leq (\pi + 4X)m^{\delta - 1} \sup_{k \geq l} c_k k.$$

\hspace{1cm} \text{(4.38)}
Let \( Q' = Q'(m) \geq Q(m) \) be the smallest number such that \( m' + Q' \in K_2 \). Set \( m_0 := m \) and \( m_{i+1} := m'(m_i) + Q'(m_i) \) for \( i \geq 0 \). Since
\[
Q' \geq Q \geq m^{(2-\alpha)/3},
\]
we have
\[
m_{i+1} \geq m_i + m_i^{(2-\alpha)/3}. \tag{4.39}
\]

Notice that in the sum on the left-hand side of (4.38) blocks of \( k \) can appear such that \( k \in K_1 \) and the quantities \( \Delta^1_{k+1} \) in a block increase and belong to an interval \([0, m^{-\delta}]\) or \([2\pi - m^{-\delta}, 2\pi]\). The sum over each of these blocks can be estimated as in (4.18), where we estimated the corresponding block in \( S'_2 \). So, from (4.31), (4.38) and (4.39), recalling that \( \delta < (2 - \alpha)/3 < 1 \), we obtain
\[
|S''_2| \leq C(\alpha, X) \sup_{k \geq l} c_k k + (\pi + 4X) \sup_{k \geq l} c_k k \sum_{i=0}^{\infty} c_{m_i} m_i^{\delta-1}
\]
\[
\leq C(\alpha, X) \sup_{k \geq l} c_k k + (\pi + 4X) \sup_{k \geq l} c_k k \sum_{i=0}^{\infty} z_i^{\delta-1}, \tag{4.40}
\]
where \( z_0 := m \), \( z_{i+1} := z_i + m_i^{(2-\alpha)/3} \geq z_i + 1 \) for any \( i \), and so \( z_i \to \infty \). Therefore,
\[
F(m) = \sum_{j=0}^{\infty} \int_{z_j}^{z_{j+1}} \frac{dy}{y \ln^2 y} =: \sum_{j=0}^{\infty} Z_j. \tag{4.41}
\]

For convenience set \( (2 - \alpha)/3 =: \gamma > \delta \). Using the inequality \( \ln(1 + y) \geq y/2 \), which is valid for \( y \leq 1 \), we have
\[
Z_j = \frac{1}{\ln z_j} - \frac{1}{\ln(z_j + z_j^\gamma)} = \frac{\ln(1 + z_j^{\gamma-1})}{\ln z_j \ln(z_j + z_j^\gamma)} \geq \frac{z_j^{\gamma-1}}{2 \ln z_j \ln(2z_j)} > z_j^\delta. \tag{4.42}
\]
The last inequality in (4.42) is due to the inequality
\[
z_j^{\gamma-\delta} > 4 \ln^2 z_j,
\]
which is true in view of the third condition in (4.1), since \( z_j \geq l_0 \). Thus, (4.40)–(4.42) imply that
\[
|S''_2| \leq C(\alpha, X) \sup_{k \geq l} c_k k + (\pi + 4X) \sup_{k \geq l} c_k k \sum_{i=0}^{\infty} Z_i
\]
\[
= C(\alpha, X) \sup_{k \geq l} c_k k + (\pi + 4X) F(m) \sup_{k \geq l} c_k k \leq \left( C(\alpha, X) + \frac{\pi + 4X}{\ln 2} \right) \sup_{k \geq l} c_k k. \tag{4.43}
\]
Finally, combining (4.2), (4.3), (4.7), (4.31) and (4.43) we obtain
\[
\left| \sum_{k=1}^{L} c_k \sin k^\alpha x \right| \leq \left( \frac{2^\alpha}{\alpha} + 2C(\alpha, X) + \frac{\pi + 4X}{\ln 2} \right) \sup_{k \geq l} c_k k,
\]
which ensures that if the condition \( c_k k \to 0 \) is satisfied, then our series converges uniformly.
§ 5. The case of a power from the interval (0,1)

Proof of Theorem 1, (c) for the case $\alpha \in (0, 1)$. Suppose that the condition $c_k k \to 0$ is satisfied. We show that then the series (1.1) converges uniformly on the set $|x| \leq X < \infty$. Without loss of generality, from now on we assume that $x > 0$. Take an odd integer $D \geq 3$ fulfilling the conditions

$$
(\pi X^{-1})^{1/\alpha} D^{1/\alpha-1} \geq 12\alpha, \quad \left(1 + \frac{1}{D}\right)^{1/\alpha-1} \leq \frac{4}{3},
$$

and let $E := D + 1$. Consider the sum $\sum_{k=l}^L c_k \sin k^\alpha x$ at an arbitrary point $x \in (0, X]$. If $x < \pi L^{-\alpha}$, then

$$
0 \leq \sum_{k=l}^L c_k \sin k^\alpha x \leq x \sum_{k=l}^L c_k k^\alpha \leq x \sup_{k \geq l} c_k k \sum_{k=l}^L k^{\alpha-1}
\leq \pi L^{-\alpha} \frac{(2L)^\alpha}{\alpha} \sup_{k \geq l} c_k k =: C_1 \sup_{k \geq l} c_k k.
$$

If $x \geq \pi l^{-\alpha}$ and $L^\alpha x - l^\alpha x \leq 6\pi$, then $L^\alpha - l^\alpha \leq 6\pi/x \leq 6l^\alpha$, hence $L \leq 7^{1/\alpha} l$, and therefore

$$
\left| \sum_{k=l}^L c_k \sin k^\alpha x \right| \leq \sum_{k=l}^L c_k \leq c_l (L - l + 1) < 7^{1/\alpha} lc_l
\leq 7^{1/\alpha} \sup_{k \geq l} c_k k =: C_2(\alpha) \sup_{k \geq l} c_k k.
$$

The remaining case is when $x \geq \pi l^{-\alpha}$ and $L^\alpha x - l^\alpha x > 6\pi$.

Let the odd integers $d_1$ and $d_2$ and even integers $e_1$ and $e_2$ be such that

$$
\pi(e_1 - 2) < x l^\alpha \leq \pi e_1, \quad \pi(d_1 - 2) < x l^\alpha \leq \pi d_1,
\pi e_2 \leq x L^\alpha < \pi(e_2 + 2) \quad \text{and} \quad \pi d_2 \leq x L^\alpha \leq \pi(d_2 + 2).
$$

Note that for any $\gamma > 0$ and $d \geq 3$ we have

$$
F(\gamma, d) = \frac{\left| \frac{\pi x^{-1} d_1 \sin k^\alpha} {\pi x^{-1} (d_1 - 2)^{1/\alpha}} \right| - \left| \frac{\pi x^{-1} d_1 \sin k^\alpha} {\pi x^{-1} (d_1 - 2)^{1/\alpha}} \right|}{\left( \pi x^{-1} (d_1 - 2)^{1/\alpha} \right)} \leq \frac{2 ((\gamma d)^{1/\alpha} - (\gamma d - 2)^{1/\alpha})}{(\gamma d - 2)^{1/\alpha}}
\leq \frac{2 ((\gamma d)^{1/\alpha} - (\gamma d - 2)^{1/\alpha})}{(\gamma d - 2)^{1/\alpha}} = 2 \left( \left( \frac{d}{d - 2} \right)^{1/\alpha} - 1 \right) \leq 2 (3^{1/\alpha} - 1) =: C.
$$

Thus, for $\gamma > 0$ and $d \geq 3$ we have

$$
\left| \sum_{k=l}^L c_k \sin k^\alpha x \right| \leq c_l \left| \sum_{k=l}^L \frac{1}{\pi x^{-1} (d_1 - 2)^{1/\alpha}} \right|
\leq c_l \left( \left| \frac{\pi x^{-1} (d_1 - 2)^{1/\alpha}} {\pi x^{-1} (d_1 - 2)^{1/\alpha}} \right| + 1 \right) F(\pi x^{-1}, d_1) \leq lc_l F(\pi x^{-1}, d_1) \leq C \sup_{k \geq l} c_k k.
$$
Similarly,
\[ \left| \sum_{k=l}^{[(\pi x^{-1}e_1)^{1/\alpha}]} c_k \sin k^\alpha x \right| \leq C \sup_{k \geq l} c_k k. \quad (5.5) \]

Further,
\[
\left| \sum_{k=[(\pi x^{-1}e_2)^{1/\alpha}]+1}^{L} c_k \sin k^\alpha x \right| \leq c_{[(\pi x^{-1}e_2)^{1/\alpha}]+1} \sum_{k=[(\pi x^{-1}e_2)^{1/\alpha}]+1}^{1} 1 \\
\leq c_{[(\pi x^{-1}e_2)^{1/\alpha}]+1} \left( \left\lfloor \left( \pi x^{-1}e_2 \right)^{1/\alpha} \right\rfloor + 1 \right) F(\pi x^{-1}, e_2 + 2) \leq C \sup_{k \geq l} c_k k. \quad (5.6)
\]

Similarly,
\[
\left| \sum_{k=[(\pi x^{-1}d_2)^{1/\alpha}]+1}^{L} c_k \sin k^\alpha x \right| \leq C \sup_{k \geq l} c_k k. \quad (5.7)
\]

Now consider the sum
\[
S(d) := \sum_{k=[(\pi x^{-1}d)^{1/\alpha}]+1}^{[(\pi x^{-1}(d+2))^{1/\alpha}]} \sin k^\alpha x = \sum_{k=[(\pi x^{-1}d)^{1/\alpha}]+1}^{[(\pi x^{-1}(d+1))^{1/\alpha}]} + \sum_{k=[(\pi x^{-1}(d+1))^{1/\alpha}]+1}^{[(\pi x^{-1}(d+2))^{1/\alpha}]} + \sum_{k=[(\pi x^{-1}(d+3/2))^{1/\alpha}]+1}^{[(\pi x^{-1}(d+2))^{1/\alpha}]} \sin k^\alpha x \\
= S_1(d) + S_2(d) + S_3(d) + S_4(d),
\]

where \( d \geq D \) is an odd integer.

First we show that the sum \( S_2(d) + S_3(d) \) cannot be too large, because most of the terms in the sums \( S_2(d) \) and \( S_3(d) \) can be combined into pairs so that the sum of any pair is close to zero and nonpositive. Let the quantities \( k \) in the \( s \)th pair be \( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor + s \) and \( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor - 1 - s \), where
\[
s = 0, 1, \ldots, \min \left\{ \left( \pi x^{-1} \left( d + \frac{3}{2} \right) \right)^{1/\alpha}, \left( \pi x^{-1} \left( d + \frac{1}{2} \right) \right)^{1/\alpha} \right\} - 1. \quad (5.8)
\]

Note that there is exactly one pair containing two terms in \( S_2(d) \) and every other pair consists of a term in \( S_2(d) \) and a term in \( S_3(d) \). The sum of the quantities in the \( s \)th pair is
\[
\sin \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor + s \right)^\alpha x + \sin \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor - 1 - s \right)^\alpha x \\
= 2 \sin \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor + s \right)^\alpha + \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor - 1 - s \right)^\alpha \frac{x}{2} \\
\times \cos \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor + s \right)^\alpha - \left( \left\lfloor \left( \pi x^{-1}(d+1) \right)^{1/\alpha} \right\rfloor - 1 - s \right)^\alpha \frac{x}{2}. \quad (5.9)
\]
According to (5.8), the argument of any cosine in (5.9) lies on the interval \([0, \pi/2]\), hence all the cosines are nonnegative. We show now that the arguments of all the sines in (5.9) lie in the half-open interval \([\pi d, \pi(d + 1))\), and so these sines are non-positive. Since the function \(\chi(y) = y^\alpha\) is convex on \(\mathbb{R}^+\) \((\chi''(y) = \alpha(\alpha - 1)y^{\alpha - 2} < 0\) for \(y > 0\)), the argument of the sine does not exceed

\[
2 \left( \left(\frac{\pi x^{-1}(d + 1)}{1/\alpha}\right)^{1/\alpha} - \frac{1}{2} \right)^{\alpha} \frac{x}{2} < \pi(d + 1).
\]

At the same time,

\[
\left(\pi x^{-1}\left(d + \frac{3}{2}\right)\right)^{1/\alpha} - \left(\pi x^{-1}(d + 1)\right)^{1/\alpha} + 1
\]

\[
< \left(\pi x^{-1}(d + 2)\right)^{1/\alpha} - \left(\pi x^{-1}(d + 1)\right)^{1/\alpha} - \frac{1}{2}, \tag{5.10}
\]

since by Lagrange’s theorem there exists \(\theta \in (0, 1/2)\) such that

\[
\left(\pi x^{-1}\right)^{1/\alpha} \left(d + 2\right)^{1/\alpha} - \left(d + \frac{3}{2}\right)^{1/\alpha} \geq \left(\pi x^{-1}\right)^{1/\alpha} \frac{1}{2\alpha} \left(d + \frac{3}{2} + \theta\right)^{1/\alpha - 1}
\]

\[
\geq \left(\pi x^{-1}\right)^{1/\alpha} \frac{1}{2\alpha} \left(d + \frac{3}{2} + \theta\right)^{1/\alpha - 1} \geq \left(\pi x^{-1}\right)^{1/\alpha} \frac{1}{2\alpha} D^{1/\alpha - 1} \geq 6 > \frac{3}{2}
\]

by the first condition in (5.1). Hence from (5.10) we obtain

\[
s \leq \left(\pi x^{-1}(d + 2)\right)^{1/\alpha} - \left(\pi x^{-1}(d + 1)\right)^{1/\alpha} - \frac{1}{2},
\]

and therefore

\[
\frac{s + 1/2}{\left(\pi x^{-1}(d + 1)\right)^{1/\alpha}} < \left(\frac{d + 2}{d + 1}\right)^{1/\alpha} - 1 \leq \frac{4}{3\alpha(d + 1)}, \tag{5.11}
\]

since the function \(t(y) = (1 + y)^{1/\alpha} - 1 - 4y/(3\alpha)\) vanishes at \(y = 0\) and \(t'(y) = ((1 + y)^{1/\alpha - 1} - 4/3)/\alpha \leq 0\) for \(y \leq 1/D\) due to the second condition in (5.1). Also, taking the first condition in (5.1) into account, along with the fact that \(d \geq D\), we have

\[
\frac{3}{\left(2\pi x^{-1}(d + 1)\right)^{1/\alpha}} \leq \frac{2}{\left(\pi x^{-1}\right)^{1/\alpha}(d + 1)^{1/\alpha}} \leq \frac{1}{6\alpha(d + 1)}. \tag{5.12}
\]

Thus, by (5.11) and (5.12), the argument of any sine on the right-hand side of (5.9) is not less than

\[
\left(\pi x^{-1}(d + 1)^{1/\alpha} - \frac{3}{2} + \max\left(s + \frac{1}{2}\right)\right)^{\alpha} \frac{x}{2}
\]

\[
+ \left(\pi x^{-1}(d + 1)^{1/\alpha} - \frac{3}{2} - \max\left(s + \frac{1}{2}\right)\right)^{\alpha} \frac{x}{2}.
\]
\[
\begin{align*}
\geq & \left(1 - \frac{3}{2(\pi x^{-1}(d+1))^{1/\alpha}} + \frac{4}{3\alpha(d+1)}\right)^\alpha \frac{\pi(d+1)}{2} \\
& + \left(1 - \frac{3}{2(\pi x^{-1}(d+1))^{1/\alpha}} - \frac{4}{3\alpha(d+1)}\right)^\alpha \frac{\pi(d+1)}{2} \\
\geq & \left(\left(1 - \frac{1}{6\alpha(d+1)} + \frac{4}{3\alpha(d+1)}\right)^\alpha + \left(1 - \frac{1}{6\alpha(d+1)} - \frac{4}{3\alpha(d+1)}\right)^\alpha\right) \frac{\pi(d+1)}{2} \\
\geq & \left(1 + \left(1 - \frac{3}{2\alpha(d+1)}\right)^\alpha\right) \frac{\pi(d+1)}{2}.
\end{align*}
\]

(5.13)

We show that the last expression is not less than \(\pi d\). It is sufficient to prove that the function
\[
g(y) = 1 + \left(1 - \frac{3}{2\alpha y}\right)^\alpha - 2 + 2y = \left(1 - \frac{3}{2\alpha y}\right)^\alpha - 1 + 2y
\]
is nonnegative at \(y = (d+1)^{-1}\). Note that \(g(0) = 0\) and
\[
g'(y) = -\frac{3}{2} \left(1 - \frac{3}{2\alpha y}\right)^{\alpha-1} + 2 \geq 0
\]
for \(y \leq 1/D\) by the third condition in (5.1). Thus, from (5.13) and the observation above it follows that the argument of any sine on the right-hand side of (5.9) is no less than \(\pi d\). It is also easy to see that these arguments are no greater than \((\pi + 1)d\), hence all the sines on the right-hand side of (5.9) are nonpositive, and this implies that the whole sum of the chosen pairs is nonpositive. If there is a term of \(S_2(d)\) which does not belong to any pair, we bound it above by zero.

We estimate the number of terms in \(S_3(d)\) which may be left without a pair. If there exist such terms, then since we have exactly one pair consisting of terms of \(S_2(d)\) and all other pairs consists of one term from \(S_2(d)\) and the other from \(S_3(d)\), the number of terms of \(S_3(d)\) left without a pair is exactly
\[
\left\lceil \left(\left(\pi x^{-1}\left(d + \frac{3}{2}\right)\right)^{1/\alpha} \right) - \left(\left(\pi x^{-1}(d+1)\right)^{1/\alpha}\right) \right\rceil - 2
\]
\[
\leq \left(\pi x^{-1}\left(d + \frac{3}{2}\right)\right)^{1/\alpha} - 2\left(\pi x^{-1}(d+1)\right)^{1/\alpha} + \left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha} + 4
\]
\[
\leq \frac{2}{\alpha} \left(\frac{1}{\alpha} - 1\right) d^{1/\alpha - 2} \left(\pi x^{-1}\right)^{1/\alpha} + 4.
\]

(5.14)

Here we have used Lagrange’s theorem for the function \(w(y) = y^{1/\alpha}\):
\[
w(y + 1) - 2w\left(y + \frac{1}{2}\right) + w(y)
\]
\[
= w'(y + \frac{1}{2} + \theta_1) - w'(y + \theta_2) = \left(\frac{1}{2} + \theta_1 - \theta_2\right)w''(y + \theta_0),
\]
where \( \theta_1, \theta_2 \in [0, 1/2] \) and \( \theta_0 \in [0, 1] \). Therefore,

\[
w(y + 1) - 2w\left(y + \frac{1}{2}\right) + w(y) \leq \sup_{[y+1/2, y+3/2]} w''(z) = \frac{1}{\alpha} \left( \frac{1}{\alpha} - 1 \right) \max\left\{ \left( d + \frac{1}{2} \right)^{1/\alpha - 2}, \left( d + \frac{3}{2} \right)^{1/\alpha - 2} \right\} \leq \frac{2}{\alpha} \left( \frac{1}{\alpha} - 1 \right) d^{1/\alpha - 2}
\]

according to the fourth condition in (5.1). Thus, estimate (5.14) is valid.

From the argument above it follows that

\[
S_2(d) + S_3(d) \leq \frac{2}{\alpha} \left( \frac{1}{\alpha} - 1 \right) d^{1/\alpha - 2} (\pi x^{-1})^{1/\alpha} + 4. \tag{5.15}
\]

We show now that the sum \( S_1(d) \) is little different from \( S_2(d) \), and \( S_4(d) \), from \( S_3(d) \). To each

\[
s = 1, 2, \ldots, \left\lfloor (\pi x^{-1}(d + 1))^{1/\alpha} \right\rfloor - \left\lfloor \left( \pi x^{-1}\left(d + \frac{1}{2}\right) \right)^{1/\alpha} \right\rfloor =: s_{\text{max}} \tag{5.16}
\]

we assign some

\[
k_s = \left\lfloor \left( \pi x^{-1}\left(d + \frac{1}{2}\right) \right)^{1/\alpha} \right\rfloor - \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + 1,
\]

\[
\ldots, \left\lfloor (\pi x^{-1}(d + 1))^{1/\alpha} \right\rfloor - \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor \tag{5.17}
\]

so that

\[
k_s := \min\{ k \in \mathbb{N} : \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + k^\alpha \geq \pi x^{-1}(2d + 1) - \left( \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + s \right)^\alpha \}. \tag{5.18}
\]

Then using the relations

\[
\pi d \leq \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + s^\alpha x \leq \pi \left(d + \frac{1}{2}\right) \leq \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + k_s^\alpha x \leq \pi (d + 1)
\]

and

\[
\pi \left(d + \frac{1}{2}\right) - \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + s^\alpha x \leq \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + k_s^\alpha x - \pi \left(d + \frac{1}{2}\right),
\]

we obtain

\[
\sin\left( \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + s^\alpha x \right) \leq \sin\left( \left\lfloor (\pi x^{-1}d)^{1/\alpha} \right\rfloor + k_s^\alpha x \right). \tag{5.19}
\]

Note that

\[
(\pi x^{-1}(2d + 1) - ((\pi x^{-1}d)^{1/\alpha} - 1 + s)^\alpha)^{1/\alpha} - (\pi x^{-1}d)^{1/\alpha} + 2 \leq (\pi x^{-1}(d + 1))^{1/\alpha} - 1, \tag{5.20}
\]
Then by Lagrange’s theorem

\[(\pi x^{-1}(2d+1) - \pi x^{-1}d)^{1/\alpha} - (\pi x^{-1}d)^{1/\alpha} + 3 - (\pi x^{-1}(d+1))^{1/\alpha} = 3 - (\pi x^{-1}d)^{1/\alpha} \leq 0\]

as \(d \geq D \geq 3\). Therefore, it follows from (5.20) that

\[\left(\pi x^{-1}(2d+1) - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + s\right)^{1/\alpha} - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + 1 \leq \left\lfloor (\pi x^{-1}(d+1))^{1/\alpha}\right\rfloor,\]

and hence for any \(s\) in (5.16) there exists \(k_s\) satisfying (5.17) and (5.18).

We also show that \(k_{s_1} \neq k_{s_2}\) for \(s_1 \neq s_2\). Since \(k_s\) does not increase as \(s\) increases, it suffices to show that \(k_s > k_{s+1}\). Indeed, we can see from (5.18) that

\[k_s \geq (\pi x^{-1}(2d+1) - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + s)^{1/\alpha} - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor > k_s - 1,\]

so it is sufficient to prove that

\[\left(\pi x^{-1}(2d+1) - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + s\right)^{1/\alpha} - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor > (\pi x^{-1}(2d+1) - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + s + 1)^{1/\alpha} - \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor + 1.\] (5.21)

For the sake of brevity we set \(a := \pi x^{-1}(2d+1)\) and \(b := \left\lfloor (\pi x^{-1}d)^{1/\alpha}\right\rfloor\) and consider the function

\[h_{a,b}(s) = (a - (b + s)^{\alpha})^{1/\alpha}.\]

Then by Lagrange’s theorem \(h_{a,b}(s) - h_{a,b}(s+1) = -h_{a,b}'(s_0)\), where \(s_0 \in (1, s_{\max})\). In addition,

\[h_{a,b}'(s) = -(a - (b + s)^{\alpha-1})(b + s)^{\alpha-1},\]

that is, \(|h_{a,b}'|\) decreases in \(b + s\), and hence, using the fact that according to (5.16)

\[b + s \leq \left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha},\]

on the interval \((1, s_{\max})\) we have

\[|h_{a,b}'(s)| > \left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha-1}\left(\left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha}\right)^{\alpha-1} = 1.\]

Thus, \(h_{a,b}(s) - h_{a,b}(s + 1) > 1\), which implies (5.21).

So, each \(s\) satisfying (5.16) corresponds injectively to \(k_s\) satisfying (5.17) and (5.18), so that for each \(s\) we have (5.19), that is, every term in \(S_1(d)\) is bounded above by the corresponding term of \(S_2(d)\). The number of terms in \(S_2(d)\) that are
not used in this estimate is
\[
[(\pi x^{-1}(d+1))^{1/\alpha}] - \left[\left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha}\right]
\]
\[-\left(\left[\left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha}\right] - [(\pi x^{-1}d)^{1/\alpha}]\right)
\]
\[\leq (\pi x^{-1}(d+1))^{1/\alpha} + (\pi x^{-1}d)^{1/\alpha} - 2\left(\pi x^{-1}\left(d + \frac{1}{2}\right)\right)^{1/\alpha} + 2
\]
\[\leq \frac{2}{\alpha} \left(\frac{1}{\alpha} - 1\right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2
\]
similarly to (5.14). Thus,
\[S_1(d) \leq S_2(d) + \frac{2}{\alpha} \left(\frac{1}{\alpha} - 1\right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2. \quad (5.22)
\]
Arguing similarly for \(S_3(d)\) and \(S_4(d)\), for \(d \geq D\) we obtain
\[S_4(d) \leq S_3(d) + \frac{2}{\alpha} \left(\frac{1}{\alpha} - 1\right) (d+1)^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2
\]
\[\leq S_3(d) + \frac{4}{\alpha} \left(\frac{1}{\alpha} - 1\right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2 \quad (5.23)
\]
due to the fourth condition in (5.1).

Finally, putting (5.4), (5.7), (5.15), (5.22) and (5.23) together, we derive that
\[
\sum_{k=l}^{L} c_k \sin k^\alpha x \leq 2C \sup_{k \geq l} c_k k + \sum_{d \geq d_1}^{D-2} \sum_{\substack{k \geq \lfloor (\pi x^{-1}d)^{1/\alpha} \rfloor + 1 \atop d \text{ is odd}}} c_k \sin k^\alpha x
\]
\[+ \sum_{d \geq D}^{d_2-2} \sum_{\substack{k \geq \lfloor (\pi x^{-1}d)^{1/\alpha} \rfloor + 1 \atop d \text{ is odd}}} c_k \sin k^\alpha x
\]
\[\leq 2C \sup_{k \geq l} c_k k + c_{\lfloor (\pi x^{-1}d_1)^{1/\alpha} \rfloor + 1} \left(\lfloor (\pi x^{-1}D)^{1/\alpha} \rfloor - \lfloor (\pi x^{-1}d_1)^{1/\alpha} \rfloor\right)
\]
\[+ \sum_{d \geq D}^{d_2-2} c_{\lfloor (\pi x^{-1}(d+1))^{1/\alpha} \rfloor} S(d)
\]
\[\leq 2C \sup_{k \geq l} c_k k + 2 \left(\left(\frac{D}{d_1}\right)^{1/\alpha} - 1\right) \sup_{k \geq l} c_k k
\]
\[+ 2 \sum_{d \geq D}^{d_2-2} c_{\lfloor (\pi x^{-1}(d+1))^{1/\alpha} \rfloor} \left(S_2(d) + S_3(d) + \frac{3}{\alpha} \left(\frac{1}{\alpha} - 1\right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2\right)
\]
\[\leq 2C \sup_{k \geq l} c_k k + 2(D^{1/\alpha} - 1) \sup_{k \geq l} c_k k + 2 \sum_{d \geq D}^{d_2-2} c_{\lfloor (\pi x^{-1}(d+1))^{1/\alpha} \rfloor}
\]
\begin{align*}
\times \left( \frac{2}{\alpha} \left( \frac{1}{\alpha} - 1 \right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 4 + \frac{3}{\alpha} \left( \frac{1}{\alpha} - 1 \right) d^{1/\alpha - 2}(\pi x^{-1})^{1/\alpha} + 2 \right) \\
\leq 2C \sup_{k \geq l} c_k k + 2(D^{1/\alpha} - 1) \sup_{k \geq l} c_k k \\
+ 2 \sup_{k \geq l} c_k k \sum_{d \geq d_1} \frac{5}{\alpha} \left( \frac{1}{\alpha} - 1 \right) d^{-2} + \frac{6}{(\pi x^{-1})^{1/\alpha}} d^{-1/\alpha} \\
- \left( 2C + 2(D^{1/\alpha} - 1) + \frac{10}{\alpha} \left( \frac{1}{\alpha} - 1 \right) D^{-1} + \frac{6}{(\pi x^{-1})^{1/\alpha}} \left( \frac{1}{\alpha} - 1 \right) D^{-1/\alpha} \right) \sup_{k \geq l} c_k k \\
\leq \left( 2C + 2(D^{1/\alpha} - 1) + \left( \frac{10}{\alpha} + 2X^2 \right) \left( \frac{1}{\alpha} - 1 \right) \right) \sup_{k \geq l} c_k k =: C_3 \sup_{k \geq l} c_k k. \quad (5.24)
\end{align*}

Similarly, using the same argument, with \( e_1, e_2 \) and \( E \) in place of \( d_1, d_2 \) and \( D \), and bounding \( S(e) \) below, from (5.5) and (5.6) we obtain

\begin{align*}
\sum_{k=l}^{L} c_k \sin k^\alpha x \geq -C_4 \sup_{k \geq l} c_k k . \quad (5.25)
\end{align*}

Combining (5.2), (5.3), (5.24) and (5.25) we have

\begin{align*}
\left| \sum_{k=l}^{L} c_k \sin k^\alpha x \right| &\leq \max\{C_1, C_2, C_3, C_4\} \sup_{k \geq l} c_k k ,
\end{align*}

which completes the proof of uniform convergence.

\section*{§ 6. Proof of Theorem 2}

\textbf{Proof of Theorem 2.} Part (a) of Theorem 2 follows clearly from Theorem 1, (a).

In view of Theorem 1, (b), (c), to prove the corresponding parts of Theorem 2 it suffices to show that for any \( \alpha > 0 \) the condition \( c_k k \to 0 \) is necessary and sufficient for the uniform convergence of the series (1.1) on a set containing a discrete \((\alpha, \gamma)-\)neighbourhood of zero for some \( \gamma \geq 2 \). Suppose that the series (1.1) converges uniformly on some set \( X \) containing a discrete \((\alpha, \gamma)-\)neighbourhood of zero and let \( \gamma \geq 2 \) and \( N \) be the numbers from the definition of such a neighbourhood. Take an arbitrary \( \varepsilon > 0 \). Then there exists \( l_0 = l_0(\varepsilon) \in \mathbb{N}, l_0 \geq N \), such that for any \( L > l \geq l_0 \) and any \( x \in X \) we have \( \left| \sum_{k=l}^{L} c_k \sin k^\alpha x \right| < \varepsilon \). So, taking any \( l \geq l_0 \) and setting \( x_0 = \pi/(\gamma^{\alpha+1}l) \) (either \( x_0 \) or \( -x_0 \) lies in \( X \)) we obtain

\begin{align*}
\varepsilon > \left| \sum_{k=l+1}^{2l} c_k \sin k^\alpha x_0 \right| = \left| \sum_{k=l+1}^{2l} c_k \sin \frac{k^\alpha \pi}{\gamma^{\alpha+1}l} \right| .
\end{align*}

Note that the argument of any sine here does not exceed \( \pi/2 \), hence,

\begin{align*}
\varepsilon > \frac{2}{\pi} \sum_{k=l+1}^{2l} c_k k^\alpha \frac{\pi}{\gamma^{\alpha+1}l} 2^\gamma \gamma^{-\alpha-1} \sum_{k=l+1}^{2l} c_k \geq 2\gamma^{-\alpha-1} l c_{2l} = \gamma^{-\alpha-1} c_{2l} 2l . \quad (6.1)
\end{align*}
that is, \( c_{2l}2l \leq \gamma^{\alpha+1}\varepsilon \). Furthermore,

\[
c_{2l+1}(2l + 1) \leq c_{2l}4l \leq 2\gamma^{\alpha+1}\varepsilon,
\]

which assures the necessity of the condition.

**Proof of Remark 4.** The estimates in the proof of Theorem 1, (a), (b) remain true up to a constant if we replace the differences \( c_m - c_{m+1} \) by their absolute values. In fact, this follows from the relations

\[
\sum_{k=1}^{L} |c_k - c_{k+1}| k^\xi = \sum_{k=l}^{L} |c_k - c_{k+1}| + \sum_{k=1}^{L} (k+1)^\xi - k^\xi \sum_{j=l}^{L} |c_k - c_{k+1}|
\]

\[
\leq Vc_l l^\xi + VC(\xi) \sum_{k=l}^{L} c_k k^{\xi-1},
\]

where \( \xi > 0, V \) is from (1.3), and

\[
c_k \leq c_m + \sum_{l=m}^{k-1} |c_l - c_{l+1}| \leq (V + 1)c_m
\]

for \( k > m \). Inequality (6.3) implies that (3.4), (3.6), (3.9), (3.10), (3.13), (3.14) and (3.16) hold with appropriate modifications, while inequality (6.4) implies (3.1), (3.19), (6.1) and (6.2).

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