1 Introduction

1.1 Motivation

For some infinite-dimensional groups $G$ and suitable subgroups $K$ there exists a monoid structure on the set $K\backslash G/K$ of double cosets of $G$ with respect to $K$. This can be seen, for example, for the group $S_{\infty}$ of the finitely supported permutations of $\mathbb{N}$, for infinite-dimensional classical Lie groups, for groups of automorphisms of measure spaces and for $\text{Aut}(F_{\infty})$, a direct limit of the groups of automorphisms of the free groups $F_n$.

The study of these structures was pioneered by R. S. Ismagilov, followed by G. I. Olshanski, who used them in the representation theory of infinite-dimensional classical Lie groups ([15], [17]). More recently there is the work of Y. A. Neretin for the infinite tri-symmetric group and $\text{Aut}(F_{\infty})$ ([11], [8], [5]).

In this paper we show that the group $B_{\infty}$, of the finite braids on infinitely many strands, admits such a structure.

We also show how the multiplication defined for this group is related to the one defined in $\text{Aut}(F_{\infty})$, when $B_{\infty}$ is regarded as a subgroup of $\text{Aut}(F_{\infty})$ and the one defined in $GL(\infty)$. Furthermore we define a one-parameter monoid structure on $GL(\infty)$ which generalizes the usual structure (see [7]) and show that the Burau representation provides a functor between the categories of double cosets.

1.2 The infinite braid group and double cosets

The Artin braid group on $n$ strings $B_n$ has the presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the so-called braid relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2, \quad i, j \in \{1, \ldots, n-1\},$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n-2.$$

The generators $\sigma_i$ are called elementary braids. For each $n$, consider the monomorphism $i_n : B_n \to B_{n+1}$ sending the $k$-th elementary braid of $B_n$ to the $k$-th elementary braid of $B_{n+1}$. Geometrically this operation corresponds to adding a new string to the right of the braid, without creating any new crossings, as in the picture below:

![Braid Diagram](image)

Figure 1: The monomorphism $i_n$

The direct limit of this sequence of groups, with respect to the homomorphisms $i_n$, is the infinite braid group

$$B_{\infty} = \lim_{\longrightarrow} B_n,$$

consisting of braids with countably many strings and finitely many crossings. This group has the presentation:

$$B_{\infty} = \left\langle \sigma_i, i \in \mathbb{N} \left| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| \geq 2 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right. \right\rangle.$$

For each non-negative integer $\alpha$, let $B_{\infty}[\alpha]$ be the subgroup of $B_{\infty}$ given by

$$B_{\infty}[\alpha] = \langle \sigma_j | j > \alpha \rangle.$$
Definition 1.1. Let $G$ be a group, $g \in G$ and $K$ and $L$ be subgroups of $G$. The double coset on $G$ containing $g$ with respect to the pair $(K, L)$ is the set $KgL$. Denote by $K\setminus G/L$ the set of double cosets on $G$ with respect to the pair $(K, L)$.

1.3 The Burau representation of $B_\infty$

The Burau representation is the homomorphism $\eta_n : B_n \to GL(n, \mathbb{Z}[t, t^{-1}])$ given by

$$\eta_n(\sigma_i) = \begin{pmatrix} I_{i-1} & (1-t & t) \\ (1-t & 1 & 0) \\ 1_{n-i-1} \end{pmatrix}.$$

Consider the homomorphisms $j_n : GL(n) \to GL(n + 1)$ given by

$$j_n(T) = \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}.$$

The group $GL(\infty)$ is the direct limit of $GL(n)$ with respect to the homomorphisms $j_n$ and consists of infinite matrices that differ from the identity matrix only in finitely many entries. Due the commutativity of the diagram:

$$\begin{array}{c}
B_n \xrightarrow{\eta_n} GL(n) \\
\downarrow j_n \\
B_{n+1} \xrightarrow{\eta_{n+1}} GL(n + 1)
\end{array}$$

we can construct a representation $\eta : B_\infty \to GL(\infty)$ of $B_\infty$ by taking the limit of the representations $\eta_n$. More precisely, $\eta$ is given by the following formulas:

$$\eta(\sigma_i) = \begin{pmatrix} 1_{i-1} \\ (1-t & t) \\ 1_{\infty} \end{pmatrix}.$$

With this representation in mind we will define an operation on the double cosets of $GL(\infty)$ such that the Burau representation will be functorial between the categories of double cosets.

1.4 Main results

Consider the double cosets on $B_\infty$ with respect to the subgroups $B_\infty[\alpha]$. Given double cosets $p \in B_\infty[\alpha]\setminus B_\infty/B_\infty[\beta]$ and $q \in B_\infty[\beta]\setminus B_\infty/B_\infty[\gamma]$, we are going to define an element $p \circ q \in B_\infty[\alpha]\setminus B_\infty/B_\infty[\gamma]$. To this purpose we first introduce the following:

Definition 1.2. For integers $\beta \geq 0$ and $n > 0$, denote by $\tau_i^{(n)}$ the braid:

$$\tau_i^{(n)} = \sigma_n + \beta + \sigma_n + \beta + \ldots + \sigma_{\beta + i}.$$

Further we define the element $\theta_n[\beta] \in B_\infty[\beta]$ as:

$$\theta_n[\beta] = \tau_0^{(n)} \tau_1^{(n)} \cdots \tau_{n-1}^{(n)}.$$
Finally, the definition of the product of the double cosets is as follows:

**Definition 1.3.** Let \( p \in B_\infty[\alpha] \backslash B_\infty / B_\infty[\beta] \) and \( q \in B_\infty[\beta] \backslash B_\infty / B_\infty[\gamma] \) be double cosets. Consider \( p \in p \) and \( q \in q \) representatives of these double cosets. Then we define their product as

\[
p \circ q = B_\infty[\alpha] p \theta_n[\beta] q B_\infty[\gamma],
\]

for sufficiently large \( n \).

**Theorem 1.4.** The operation defined above does not depend on the choice of the representatives of the double cosets for \( n \) large enough. Moreover it is associative.

As a consequence we have that \( (B_\infty[\alpha] \backslash B_\infty / B_\infty[\alpha], \circ) \) is a monoid, for each non-negative integer \( \alpha \).

**Remark 1.5.** We will show that exists some \( n_0(\alpha, \gamma, p, q) \) such that \( B_\infty[\alpha] p \theta_n[\beta] q B_\infty[\gamma] = B_\infty[\alpha] p \theta_{n_0}[\beta] q B_\infty[\gamma] \) for all \( n \geq n_0 \). We can make \( n_0 \) more precise. In fact \( n_0 = \max\{\text{supp } p, \text{supp } q, \alpha, \gamma\} + 1 \), where \( \text{supp} \) is the support of a braid, defined in 2.1.

There is a natural one-to-one correspondence between the set \( B_\infty[\alpha] \backslash (B_\infty \times B_\infty[\alpha]) / B_\infty[\alpha] \) and the set of conjugacy classes \( B_\infty / B_\infty[\alpha] \) (here \( B_\infty[\alpha] \subset B_\infty^2 \) is the image of the subgroup \( B_\infty[\alpha] \) by the diagonal map). Therefore the conjugacy problem in \( B_\infty \) is equivalent to the word problem in \( B_\infty[\alpha] \backslash (B_\infty \times B_\infty[\alpha]) / B_\infty[\alpha] \). Furthermore, since it is a submonoid of \( B_\infty[\alpha] \backslash (B_\infty \times B_\infty[\alpha]) / B_\infty[\alpha] \), this correspondence gives a monoid structure to \( B_\infty / B_\infty[\alpha] \).

As a consequence of the existence of a solution for the conjugacy problem for the braid groups and the fact that the injections \( i_n \) do not merge conjugacy classes (see González-Meneses [3]) we have,

**Proposition 1.6.** The conjugacy problem for \( B_\infty \) has a solution.

Combining the observations above, we see that the word problem for \( B_\infty[0] \backslash B_\infty \times B_\infty[0] \) has a solution.

Consider the subgroup of \( GL(\infty) \) given by

\[
G[n] = \left\{ \begin{pmatrix} 1 & \ast \\ T \end{pmatrix} : T \in GL(\infty), v^T T = v^T, Tu = u \right\}
\]

where \( v = (1, t, t^2, \ldots) \) and \( u = (1, 1, 1, 1, \ldots) \) are vectors. It is easy to see that the image of \( B_\infty[n] \) by the Burau representation is contained in \( G[n] \).
Definition 1.7. Consider the matrix

\[ \Theta_j[k] = \begin{pmatrix} 1_k & 0 & 0 & 0 \\
0 & V_j & t^j 1_j & 0 \\
0 & 1_j & 0 & 0 \\
0 & 0 & 0 & 1_{\infty} \end{pmatrix} \]

where

\[ V_j = (1-t) \begin{pmatrix} 1 & t & \cdots & t^{j-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & t & \cdots & t^{j-1} \end{pmatrix} \]

Let \( p \in p \) and \( q \in q \) be representatives of the double cosets \( p \in G[n] \backslash GL(\infty)/G[k] \) and \( q \in G[k] \backslash GL(\infty)/G[m] \). Then we define their product as:

\[ p \ast q = G[n] p \Theta_j[k] q G[m], \]

for sufficiently large \( j \).

Theorem 1.8. The operation defined above does not depend on the choice of representatives of double cosets for \( j \) large enough. Moreover, it is associative.

Remark 1.9. We will show that exists some \( j_0(n, m, p, q) \) such that \( G[n] p \Theta_j[k] q G[m] = G[n] p \Theta_{j_0}[k] q G[m] \) for all \( j \geq j_0 \). We can make \( j_0 \) more precise. In fact let \( N \in \mathbb{N} \) be such that \( p \) and \( q \) can be written as diagonal block matrices

\[ \begin{pmatrix} A & 0 \\
0 & 1_{\infty} \end{pmatrix}, \]

where \( A \) is a square matrix of dimension \( k + N \). Then \( j_0 = \max\{m, n, k + N\} \).

Remark 1.10. The operation \( \ast \) generalizes the usual multiplication defined on the doubles cosets of \( GL(\infty) \) in the sense that setting the parameter \( t = 1 \) we recover the usual multiplication.

When there is a well defined operation on the set of double cosets of a group \( G \) with respect to a set of subgroups \( \{K[r]; s \in \mathbb{N}\} \), consider the category \( K(G, K) \) of double cosets, where the objects are nonnegative integers and the morphisms are given by \( Hom(r, s) = K[r]G/K[r] \). Then,

Proposition 1.11. The Burau representation \( \eta : B_{\infty} \rightarrow GL(\infty) \) induces a functor between the categories \( K(B_{\infty}, B_{\infty}[\ast]) \) and \( K(GL(\infty), G) \).

As a special case, when \( G \) is the bisymmetric group (the group that consists of pairs \((g, h)\) of permutations of \( \mathbb{N} \) such that \( gh^{-1} \) is a finite permutation) and \( K \) is its diagonal subgroup, we get a special category, called the train category of the pair \((G, K)\). This category admits a transparent combinatorial description and encode information about the representations of the bisymmetric group (see [16], [10]).

1.5 Comments

Let \( p \in B_{\infty}[\alpha] \backslash B_{\infty}/B_{\infty}[\beta] \) and \( q \in B_{\infty}[\beta] \backslash B_{\infty}/B_{\infty}[\gamma] \). Notice that the cosets \( B_{\infty}[\alpha] pq B_{\infty}[\gamma] \) do not always coincide for all choices of \( p \in p \) and \( q \in q \). For instance \( \sigma_2 \) and \( \sigma_3 \sigma_2 \) are representatives of the same double coset in \( B_{\infty}[2] \backslash B_{\infty}/B_{\infty}[2] \). But \( \sigma_2^2 \) and \( \sigma_3 \sigma_2 \sigma_3 \sigma_2 \) represent distinct cosets. To see this we consider the permutation associated to each braid. For the braid \( \sigma_2^2 \) it is the identity and for the braid \( \sigma_3 \sigma_2 \sigma_3 \sigma_2 \) it is \( (432) \). Since no braid in \( B_{\infty}[2] \) permutes the point 2 we see that these are in fact distinct double cosets.

2 Proofs of main results

2.1 Proof of Theorem 1.3

Before we proceed we introduce the notion of of support which will be needed later.

Definition 2.1. Let \( p \) be a braid in \( B_{\infty} \). The support of \( p \) is

\[ \text{supp} \, p = \min\{j \in \mathbb{N}; p \in \langle \sigma_1, \ldots, \sigma_j \rangle \}. \]
Notice that $p$ has no factors in $B_{\infty}[\text{supp } p]$, hence $p$ commutes with every element of $B_{\infty}[1 + \text{supp } p]$. Also, we can identify $p$ with an element of $B_{1 + \text{supp } p}$. We define $\text{supp } 1 = 0$.

Consider $p$ and $q$ representatives of the double cosets

$$p \in B_{\infty}[\alpha]\backslash B_{\infty}/B_{\infty}[\beta], \quad q \in B_{\infty}[\beta]\backslash B_{\infty}/B_{\infty}[\gamma].$$

Set $r_j = B_{\infty}[\alpha]p\theta_j[\beta]q B_{\infty}[\gamma]$ to be the sequence of double cosets in $B_{\infty}[\alpha]\backslash B_{\infty}/B_{\infty}[\gamma]$ defined in Definition 2.3.

**Proposition 2.2.** The sequence $(r_j)_{j \geq 1}$ defined above is eventually constant.

**Proof.** We are going to give a proof in several steps:

**Step 1.** Given $m > 0$ we have $r_i^{(m+1)} = \sigma_{m+i+1} \tau_i^{(m)}$ for all $0 \leq i \leq m - 1$.

Indeed, since $\text{supp } \tau_i^{(m)} = m + \beta + j$ and $\sigma_{m+i+2} \in B_{\infty}[1 + \beta + j]$, we find that $\sigma_{m+i+2}$ commutes with $\tau_i^{(m)}$.

**Step 2.** For all $j \leq i$ we have $\sigma_{m+i+2} r_j^{(m)} = r_i^{(m)} \sigma_{m+i+2}$.

**Step 3.** Define $u = (\sigma_{m+i+1} \sigma_{m+i+2} \cdots \sigma_{2m+i})^{-1}$ and $\ell^{-1} = \tau_i^{(m+1)}$. Then $\theta_m[\beta] = u \theta_{m+1}[\beta] \ell$.

**Step 4.** Let $M = \max\{\text{supp } p, \text{supp } q, \alpha, \gamma\} + 1$. Suppose that, for some $m \geq M$, we have $r_m = r_M$. We are going to show that $r_m = r_{m+1}$. Let $u$ and $\ell$ be like in step 3. Since $u, \ell \in B_{\infty}[m + \beta]$ it follows that $u \in B_{\infty}[\alpha]$, $\ell \in B_{\infty}[\gamma]$ and they commute with $p$ and $q$. Therefore:

$$u(p \theta_{m+1}[\beta] q) \ell = p(u \theta_{m+1}[\beta] \ell) q = p \theta_m[\beta] q.$$ 

Hence $r_m = r_{m+1}$. What proves Proposition 2.2. 

The following technical lemma will be used in Lemma 2.4 which in turn is used in Proposition 2.5 and more extensively in Theorem 2.6.

**Lemma 2.3.** Let $\{(v_j)_{j=1}^n\}_{i=1}^M$ be a family of sequences of positive integers such that $v_j < v_k$ whenever $i = n$ and $k < j$ or $i < n$ and $k = j$; in other words, the sequences $(v_j)_{j=1}^n$ are decreasing and the sequences $(v_j)_{j=1}^n$ are increasing. If $\mu_j = \prod_{k=1}^i \sigma_{v_k}$ and $\lambda_i = \prod_{k=1}^i \sigma_{v_k}$, then $\mu_1 \cdots \mu_n = P = \lambda_1 \cdots \lambda_n$.

**Proof.** We prove the lemma by induction on the pair $(g, \ell)$. The statement is trivial for $g = \ell = 1$. Assume it is true for $(g, \ell)$, we prove it is true for $(g + 1, \ell)$ and $(g, \ell + 1)$. 

(i) For \((g + 1, \ell)\), notice that:
\[
\prod_{s=1}^{g+1} \prod_{r=1}^{\ell} \sigma_{v^+_r} = \left( \prod_{s=1}^{g} \prod_{r=1}^{\ell} \sigma_{v^+_r} \right) \left( \prod_{r=1}^{\ell} \sigma_{v^{g+1}_r} \right) = \left( \prod_{s=1}^{g} \prod_{r=1}^{\ell} \sigma_{v^+_r} \right) \left( \prod_{r=1}^{g+1} \sigma_{v^{g+1}_r} \right).
\]
If \(x_r = \prod_{s=1}^{g} \sigma_{v^+_r}\) we have that \(x_r \sigma_{v^{g+1}_r} = \sigma_{v^{g+1}_r} x_r\) for \(r > t\), this follows from the inequalities \(v^+_r < v^{g+1}_r < v^{g+1}_t\) for \(s < g + 1\). Therefore:
\[
\left( \prod_{r=1}^{\ell} x_r \right) \left( \prod_{r=1}^{\ell} \sigma_{v^{g+1}_r} \right) = x_1 \cdots x_\ell \sigma_{v^{g+1}_1} \cdots \sigma_{v^{g+1}_\ell} = x_1 \sigma_{v^{g+1}_1} x_2 \sigma_{v^{g+1}_2} \cdots x_\ell \sigma_{v^{g+1}_\ell} = \prod_{r=1}^{\ell} x_r \sigma_{v^{g+1}_r} = \prod_{r=1}^{g+1} \prod_{s=1}^{\ell} \sigma_{v^+_r}.
\]

(ii) For \((g, \ell + 1)\) we have:
\[
\prod_{s=1}^{\ell+1} \prod_{r=1}^{g} \sigma_{v^+_r} = \left( \prod_{s=1}^{g} \prod_{r=1}^{\ell} \sigma_{v^+_r} \right) \left( \prod_{s=1}^{g} \sigma_{v^{\ell+1}_s} \right) = \left( \prod_{s=1}^{g} \prod_{r=1}^{\ell} \sigma_{v^+_r} \right) \left( \prod_{r=1}^{\ell+1} \sigma_{v^{\ell+1}_r} \right).\]
If \(y_s = \prod_{r=1}^{\ell} \sigma_{v^{\ell+1}_r}\) notice that \(y_s \sigma_{v^{\ell+1}_r} = \sigma_{v^{\ell+1}_r} y_s\) for \(s > t\), this follows from the inequalities \(v^{\ell+1}_r < v^{\ell+1}_t < v^+_r\) for \(r < \ell + 1\). Therefore:
\[
\left( \prod_{s=1}^{g} y_s \right) \left( \prod_{s=1}^{g} \sigma_{v^{\ell+1}_s} \right) = y_1 \cdots y_g \sigma_{v^{\ell+1}_1} \cdots \sigma_{v^{\ell+1}_g} = y_1 \sigma_{v^{\ell+1}_1} y_2 \sigma_{v^{\ell+1}_2} \cdots y_g \sigma_{v^{\ell+1}_g} = \prod_{s=1}^{g} y_s \sigma_{v^{\ell+1}_s} = \prod_{s=1}^{g} \prod_{r=1}^{\ell+1} \sigma_{v^+_r}.
\]

It will be useful to write the product \(P\) from Lemma 2.3 as a matrix, where the indices increase from right to left and from top to bottom.
\[
P = \begin{bmatrix}
v_1^1 & \cdots & v_1^g \\
\downarrow & \ddots & \downarrow \\
v_1^g & \cdots & v_1^g
\end{bmatrix}.
\]
In this way \(\lambda_1 \cdots \lambda_\ell\) is the column-wise product and \(\mu_1 \cdots \mu_g\) is the row-wise product.

Consider, for each positive integer \(m\), the homomorphism \(C_m : B_\infty \to B_\infty\) given by \(C_m(\sigma_j) = \sigma_{m+j}\). Then we have the following lemma.

**Lemma 2.4.** Let \(\beta\) and \(j\) be nonnegative integers with \(j > 1\). If \(d \in (\sigma_{\beta+1}, \ldots, \sigma_{\beta+j-1})\), then:

(i) \(d \theta_j[\beta] = \theta_j[\beta] C_j(d)\).

(ii) \(\theta_j[\beta] d = C_j(d) \theta_j[\beta]\).

**Proof.** Since \(C_j\) is a homomorphism, it is enough to prove both statements of the proposition for the case where \(d = \sigma_k\), for some \(\beta + 1 \leq k \leq \beta + j - 1\).

(i) Recall that \(\theta_j[\beta] = \tau_{i_0}^{(j)} \cdots \tau_{j-1}^{(j)}\). We claim that the following holds:
\[
\sigma_{k+i} \tau_i^{(j)} = \tau_{i_0}^{(j)} \sigma_{k+i+1}, \quad 0 \leq i \leq j - 1.
\]
Indeed, since \(\sigma_{k+i}\) is a letter of \(\tau_i^{(j)}\), but it is different from \(\sigma_{j+\beta+i}\), we have:
\[
\sigma_{k+i} \tau_i^{(j)} = \sigma_{k+i} (\sigma_{j+\beta+i} \cdots \sigma_{j+1+i}) = \\
= \sigma_{j+\beta+i} \cdots \sigma_{j+2+i} \sigma_{j+i+1} \sigma_{j+i+1} \sigma_{j+i+1} \cdots \sigma_{j+1+i} = \\
= \sigma_{j+\beta+i} \cdots \sigma_{j+2+i} \sigma_{j+i+1} \sigma_{j+i+1} \sigma_{j+i+1} \cdots \sigma_{j+1+i} = \\
= \sigma_{j+\beta+i} \cdots \sigma_{j+2+i} \sigma_{j+i+1} \sigma_{j+i+1} \tau_i^{(j)} \sigma_{k+i+1}.
\]
Therefore:
\[
\sigma_k \theta_j[\beta] = \sigma_k \tau_{i_0}^{(j)} \cdots \tau_{j-1}^{(j)} = \tau_{i_0}^{(j)} \sigma_{k+1} \tau_1^{(j)} \cdots \tau_{j-1}^{(j)} = \cdots = \tau_{i_0}^{(j)} \sigma_{k+j-1} \tau_{j-1}^{(j)} = \tau_{i_0}^{(j)} \cdots \tau_{i_{j-1}}^{(j)} \sigma_{k+j} = \theta_j[\beta] \sigma_{k+j}.
\]}
(ii) Let \( v^*_r = j + \beta - s - r \) for \( r \) and \( s \) positive integers. The family \( \{ (v^*_r) \}_{r,s=1}^j \) satisfies the hypothesis of Lemma 2.3 and therefore \( \mu_1 \cdots \mu_j = \lambda_1 \cdots \lambda_j \), where

\[
\mu_i = \sigma_{j+\beta-i+1} \cdots \sigma_{\beta+i} \quad \text{and} \quad \lambda_i = \sigma_{j+\beta-i+1} \cdots \sigma_{2j+\beta-i}.
\]

Since \( \mu_i = \gamma_{i-1}^{(j)} \) we see that \( \theta_{j}[\beta] = \lambda_1 \cdots \lambda_j \). As we saw in item (i), we have that

\[
\lambda_{j-i} \sigma_{k+i} = \sigma_{k+i+1} \lambda_{j-i}, \quad 0 \leq i \leq j - 1.
\]

What completes the proof. \( \Box \)

![Figure 3: \( \sigma_3 \theta_3[1] = \theta_3[1] \sigma_6 \)](image)

Our next step is to prove that the product does not depend on the chosen representatives.

**Proposition 2.5.** Let \( p' \) and \( q' \) be other two representatives of \( p \) and \( q \) respectively. Consider the sequence

\[
v'_j = B_\infty[\alpha] \; p' \theta_j[\beta] q' B_\infty[\gamma].
\]

Then there exists an integer \( N > 0 \) such that

\[v'_j = v_j, \quad \text{for all } j \geq N.\]

**Proof.** Since \( p \) and \( p' \) are representatives of the same double coset, there exist \( r \in B_\infty[\alpha] \) and \( h \in B_\infty[\beta] \) such that \( p' = rph \). In a similar way, there exist \( k \in B_\infty[\beta] \) and \( s \in B_\infty[\gamma] \) such that \( q' = kqs \). Therefore,

\[v'_j = B_\infty[\alpha] \; p' \theta_j[\beta] q' B_\infty[\gamma] = B_\infty[\alpha] \; rph \theta_j[\beta] kqs B_\infty[\gamma] = B_\infty[\alpha] \; ph \theta_j[\beta] kq B_\infty[\gamma].\]

Consider \( N = \max\{\supp p, \supp q, \supp h, \supp k, \alpha, \gamma\} + 1 \). Given \( j \geq N \), let \( \tilde{h} = C_j(h^{-1}) \) and \( \tilde{k} = C_j(k^{-1}) \). Then \( \tilde{h}, \tilde{k} \in B_\infty[j + \beta] \) and hence \( \tilde{h} \in B_\infty[\gamma] \) and \( \tilde{k} \in B_\infty[\alpha] \). Furthermore, \( \tilde{h} \) commutes with \( q \) and \( k \), and \( \tilde{k} \) commutes with \( p \) and \( h \). Now:

\[
\tilde{h} \tilde{k} \theta_j[\beta] kqh = \phi \tilde{k} \tilde{h} \theta_j[\beta] kqh = \phi \tilde{k} C_j(k) \theta_j[\beta] \tilde{h} q = \phi \tilde{k} C_j(k^{-1}) C_j(k) \theta_j[\beta] \tilde{h} q = \phi \tilde{h} \theta_j[\beta] \tilde{k} q = \phi \tilde{h} \theta_j[\beta] q.
\]

Therefore, for all pairs \( (p, q) \in B_\infty[\alpha] \setminus B_\infty[B_\infty[\beta]] \times B_\infty[B_\infty[\beta]] \setminus B_\infty[B_\infty[\gamma]] \) we have a well defined product \( p \circ q \in B_\infty[\alpha] \setminus B_\infty[B_\infty[\beta]] \) given by

\[p \circ q = B_\infty[\alpha] \; p \theta_j[\beta] q B_\infty[\gamma],
\]

\( p \in p, q \in q \) and \( j \) sufficiently large.

Finally, we are going to prove the associativity of the operation \( \circ \).
Proposition 2.6. The product of double cosets is associative.

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ and consider $a \in B_\infty[\alpha] \setminus B_\infty[\beta], b \in B_\infty[\beta] \setminus B_\infty[\gamma]$ and $c \in B_\infty[\gamma] \setminus B_\infty[\delta]$. Choose representatives $a \in a, b \in b$ and $c \in c$; put $k = \max\{\alpha, \beta, \gamma, \delta, \supp a, \supp b, \supp c\} + 1$. Then,

$$ab = B_\infty[\alpha] a \theta_k[\beta] b B_\infty[\gamma] \text{ and } bc = B_\infty[\beta] b \theta_k[\gamma] c B_\infty[\delta].$$

If $l = \supp\{a \theta_k[\beta] b\} + 1 = 2k + \beta$ and $l' = \supp\{b \theta_k[\gamma] c\} + 1 = 2k + \gamma$ we have

$$(ab)c = B_\infty[\alpha] a \theta_k[\beta] b \theta_l[\gamma] c B_\infty[\delta] \text{ and } a(bc) = B_\infty[\alpha] a \theta_l[\beta] b \theta_k[\gamma] c B_\infty[\delta].$$

To prove our claim we are going to show that both double cosets above are the same, by exhibiting two representatives that are equal (figures 4 and 5 give an example of the process involved). Here we are assuming $\beta \leq \gamma$, the case $\gamma < \beta$ is analogous.

Throughout the rest of the proof we will use the symbol $a \equiv b$ to signify that $a$ and $b$ are representatives of the same double coset of $B_\infty[\alpha] \setminus B_\infty[\beta]$ or $B_\infty[\beta] \setminus B_\infty[\gamma]$, that is, we can find elements $h \in B_\infty[\alpha]$ and $k \in B_\infty[\gamma]$ such that $hak = b$.

Using the notation of Lemma 2.3 we can write

$$a \theta_k[\beta] b \theta_l[\gamma] c = a \begin{bmatrix} k + \beta & \rightarrow & \beta + 1 \\ 2k + \beta - 1 & \rightarrow & k + \beta \\ 3k + \gamma & \rightarrow & 2k + \beta + \gamma \end{bmatrix} b \begin{bmatrix} 2k + \beta + \gamma & \rightarrow & \gamma + 1 \\ 4k + 2\beta + \gamma - 1 & \rightarrow & 2k + \gamma + \beta \end{bmatrix} c $$

$$a \theta_l[\beta] b \theta_k[\gamma] c = a \begin{bmatrix} 2k + \gamma + \beta & \rightarrow & \beta + 1 \\ 4k + 2\gamma + \beta - 1 & \rightarrow & 2k + \gamma + \beta \end{bmatrix} b \begin{bmatrix} k + \gamma & \rightarrow & \gamma + 1 \\ 2k + \gamma - 1 & \rightarrow & k + \gamma \end{bmatrix} c.$$

Using the same lemma, we can see that

$$\theta_k[\beta] = R_1 P; \quad P = \begin{bmatrix} k + 1 & \rightarrow & \beta + 1 \\ 2k & \rightarrow & k + \beta \end{bmatrix} \quad R_1 = \begin{bmatrix} k + \beta & \rightarrow & k + 2 \\ 2k + \beta - 1 & \rightarrow & 2k + 1 \end{bmatrix}$$

$$\theta_l[\beta] = R_2 P_2; \quad P_2 = \begin{bmatrix} k + 1 & \rightarrow & \beta + 1 \\ 3k + \gamma & \rightarrow & 2k + \beta + \gamma \end{bmatrix} \quad R_2 = \begin{bmatrix} 2k + \beta + \gamma & \rightarrow & k + 2 \\ 4k + 2\gamma + \beta - 1 & \rightarrow & 3k + \gamma + 1 \end{bmatrix}$$

$$\theta_k[\gamma] = P_3 R_3; \quad P_3 = \begin{bmatrix} k + \gamma & \rightarrow & \gamma + 1 \\ 2k & \rightarrow & k + 1 \end{bmatrix} \quad R_3 = \begin{bmatrix} 2k + 1 & \rightarrow & k + 2 \\ 2k + \gamma - 1 & \rightarrow & k + \gamma \end{bmatrix}$$

$$\theta_l[\gamma] = P_4 R_4; \quad P_4 = \begin{bmatrix} 2k + \beta + \gamma & \rightarrow & \gamma + 1 \\ 3k + \beta & \rightarrow & k + 1 \end{bmatrix} \quad R_4 = \begin{bmatrix} 3k + \beta & \rightarrow & k + 2 \\ 4k + 2\beta + \gamma - 1 & \rightarrow & 2k + \beta + \gamma \end{bmatrix}.$$

Since $R_i \in B_\infty[k + 1], 1 \leq i \leq 4$ we have

$$a R_1 P b P_4 R_4 c = a a P b P_4 c \quad a R_2 P_2 P_3 R_3 c = a P b P_3 c \equiv a P b P_3 c.$$

Notice also that $P_4 = R_5 W$, where

$$R_5 = \begin{bmatrix} 2k + \beta + \gamma & \rightarrow & 2k + 2 \\ 3k + \beta & \rightarrow & 3k - \gamma + 2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 2k + 1 & \rightarrow & \gamma + 1 \\ 3k - \gamma + 1 & \rightarrow & k + 1 \end{bmatrix}.$$
and notice that $P_2 = PF$. Since $F \in B_\infty[k]$ we see that $bF = Fb$.

Moreover, $F = EL$ where

$$
E = \left[ \begin{array}{c} 2k + 1 \\ \downarrow \\ 2k - \beta + \gamma \\ \downarrow \\ k + \gamma + 1 \\ \downarrow \\ k + \gamma \end{array} \right] \quad \text{and} \quad L = \left[ \begin{array}{c} 2k - \beta + \gamma + 1 \\ \downarrow \\ 3k + \gamma \\ \downarrow \\ k + \gamma + 1 \\ \downarrow \\ 2k + \beta + \gamma \end{array} \right].
$$

Step 2. $LP_3c \equiv CP_3c$ for some $C$. In fact, consider

$$
C = \left[ \begin{array}{c} 2k + \gamma - \beta + 1 \\ \downarrow \\ 3k - \beta + 1 \\ \downarrow \\ k + \gamma + 1 \\ \downarrow \\ 2k + 1 \end{array} \right] \quad \text{and} \quad D = \left[ \begin{array}{c} 3k - \beta + 2 \\ \downarrow \\ 3k + \gamma \\ \downarrow \\ 2k + 2 \\ \downarrow \\ 2k + \beta + \gamma \end{array} \right].
$$

Then $L = CD$ and, since $D \in B_\infty[2k + 1]$ and $supp P_3 = 2k$, we have $DP_3c = P_3cD \equiv P_3c$. Hence $LP_3c \equiv CP_3c$.

Step 3. $CP_3 = AW$ for some $A$. In fact, consider $A = \left[ \begin{array}{c} 2k + \gamma - \beta + 1 \\ \downarrow \\ 3k - \beta + 1 \\ \downarrow \\ 2k + 2 \\ \downarrow \end{array} \right] \quad \text{Then,}

$$
CP_3 = \left[ \begin{array}{c} 2k + \gamma - \beta + 1 \\ \downarrow \\ 3k - \beta + 1 \\ \downarrow \\ k + \gamma + 1 \\ \downarrow \\ 2k + 1 \end{array} \right] \left[ \begin{array}{c} k + \gamma \\ \downarrow \\ 3k - \beta + 1 \\ \downarrow \\ k + \gamma + 1 \\ \downarrow \\ 2k \end{array} \right] = AW
$$

Therefore, $aP_2bP_3c = aPbELP_3c \equiv aPbECP_3c = aPbEAWc$. At last, consider

$$
\tilde{W} = \left[ \begin{array}{c} 3k - \beta + 2 \\ \downarrow \\ 4k - 2\beta + \gamma + 1 \end{array} \right] \rightarrow \left[ \begin{array}{c} k + 2 \\ \downarrow \\ 2k - \beta + \gamma + 1 \end{array} \right].
$$

Then $A\tilde{W} = \theta_r[\gamma]$ with $r = 2k - \beta + 1$. Hence $aPbEAWc \equiv aPbE\theta_r[\gamma]c$ and, by Lemma \ref{lemma:lem1}, $E\theta_r[\gamma] = \theta_r[\gamma]C_r(E)$.

Therefore,

$$
aPbE\theta_r[\gamma]c = aPb\theta_r[\gamma]C_r(E)c = aPb\theta_r[\gamma]cC_r(E) \equiv aPb\theta_r[\gamma]c \equiv aPbAWc.
$$

Furthermore, since $A \in B_\infty[2k + 1]$ and $supp P = 2k$,

$$
aPbAWc = aPAbWc = AaPbWc \equiv aPbWc.
$$

This proves Theorem \ref{thm:thm2}.

\textbf{Example 2.7.} In this example we illustrate the method described in the proof of the theorem above. Here we used $a = \sigma_2^{-1}\sigma_1^{-1}, b = \sigma_1^2, c = \sigma_2^2\sigma_2^2, \alpha = \delta = 3, \beta = 1$ and $\gamma = 2$. 

\hfill $\Box$

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Figure 4: The equality $aθ_k[β|θ_1[γ]c = aPbWc$. 
2.2 Proof of Proposition 1.6

The conjugacy problem for the braid group has a solution (see [1],[2]). This fact yields a solution for the conjugacy problem in $B_\infty$. In fact, consider the monomorphisms $I_n : B_n \to B_\infty$ given by the direct limit (these homomorphisms consist of adding countably many strands to the right of the braid, without creating any new crossing).

Then, for two braids $p$ and $q$ in $B_\infty$, there exists $n \in \mathbb{N}$ such that $p = I_n(x)$ and $q = I_n(y)$ for some $x, y \in B_n$. If $x$ and $y$ are conjugate, there exists $z \in B_n$ such that $x = yz^{-1}$. Hence, $p = I_n(z)yI_n(z)^{-1}$, that is, $p$ and $q$ are conjugated. Now, suppose that $p$ and $q$ are conjugated. Then $p = rqr^{-1}$ for some $r \in B_\infty$. As before, there exists $m \in \mathbb{N}$, with $m \geq n$, and $w \in B_m$ such that $r = I_m(w)$. Then,

$$rqr^{-1} = I_m(w)I_n(y)I_m(w^{-1}) = I_n(x) = p.$$ 

But, since $I_n = I_m i_{m-1} i_{m-2} \cdots i_n$ we have

$$I_m(w)I_n(y)I_m(w^{-1}) = I_m(w)I_m(i_{m-1} i_{m-2} \cdots i_n(y))I_m(w^{-1}) = I_m(i_{m-1} i_{m-2} \cdots i_n(x)) = I_n(x),$$

which yields

$$w(i_{m-1} i_{m-2} \cdots i_n(y))w^{-1} = i_{m-1} i_{m-2} \cdots i_n(x).$$

Since the monomorphism $i_{m-1} i_{m-2} \cdots i_n$ does not merge conjugacy classes (see [3]) we conclude that $x$ and $y$ are conjugated in $B_n$.

2.3 Proof of Theorem 1.8

Let $p$ and $q$ be representatives of the double cosets $p \in G[n] \backslash \text{GL}(\infty)/G[k]$ and $q \in G[k] \backslash \text{GL}(\infty)/G[m]$, respectively. Define the sequence of double cosets

$$v_j = G[n] p \Theta_j[k] q G[m],$$

in $G[n] \backslash \text{GL}(\infty)/G[m]$.

We remark the following identity:
Lemma 2.8. For \( \eta : B_\infty \to GL(\infty) \) the Burau representation, the following holds
\[
\Theta_j[k] = \eta(\theta_j[k]), \quad \text{for all } j, k \in \mathbb{N}.
\]

Proposition 2.9. The sequence \( \tau_j \) above is eventually constant and its limit does not depends on the choice of representatives.

Proof. Let \( N \in \mathbb{N} \) be such that \( N > \max\{m, k, n\} \) and \( p \) and \( q \) can be written in the following block configuration:
\[
p = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \quad q = \begin{pmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.
\]

Where \( d \) and \( w \) are square matrices of dimension \( N \) and \( a \) and \( x \) are square matrices of dimension \( k \). Suppose that for \( i \geq N \) we have \( \tau_i = \tau_N \). We show that \( \tau_i = \tau_{i+1} \). As we saw in Proposition 2.2, there are elements \( u, l \in B_\infty \) such that \( \theta_i[k] = u\theta_{i+1}[k]l \). Hence, if \( U = \eta(u) \) and \( L = \eta(l) \) we have
\[
\Theta_i[k] = U\Theta_{i+1}[k]L.
\]

Furthermore, \( U \) and \( L \) have the following block configuration
\[
U = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & 1_i & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \quad L = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & 1_i & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.
\]

Thus,
\[
Up = pU \quad \text{and} \quad Lq = qL.
\]

Consequently,
\[
p\Theta_i[k]q = pU\Theta_{i+1}[k]Lq = Up\Theta_{i+1}[k]qL.
\]

Since \( U \) and \( L \) are elements of the image of the Burau representation \( \eta \) we have that \( U, L \in G[k] \) and therefore
\[
\tau_{i+1} = G[n] p\Theta_{i+1}[k]qG[m] = G[n] U p\Theta_{i+1}[k]qLG[m] = G[n] p\Theta_i[k]qG[m] = \tau_i
\]

To show that the limit of the sequence \( \tau_i \) does not depend on the choice of representatives, it suffices to show that for any \( H \) and \( J \) in \( G[k] \) we have
\[
\lim G[n] p\Theta_i[k]qG[m] = \lim G[n] pJ\Theta_i[k]HqG[m].
\]

Let \( N > 0 \) be as before. Consider \( M > N \) such that \( H \) and \( J \) have the block configuration:
\[
H = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \quad J = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1_\infty \end{pmatrix},
\]

Where \( j \) and \( h \) are square matrices of size \( M \). Now, since \( H \) preserves the vector \( v \), we have that \( V_M h = V_M \). Similarly, \( j V_M = V_M \). Therefore,
\[
J\Theta_M[k]H = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 \\ 0 & V_M & t^M 1_M \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j V_M h & t^M h \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j V_M & t^M h \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & V_M & t^M 1_M \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & V_M & t^M 1_M \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & 1_M & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.
\]
Call these new matrices containing the blocks $j$ and $h$, $J'$ and $H'$, respectively. Then we have
\[ pJ\Theta_M[k]Hq = pH'\Theta_M[k]J'q = H'p\Theta_M[k]qJ'. \]
Therefore, $p\Theta_M[k]q$ and $pJ\Theta_M[k]Hq$ belong to the same double coset, for $M$ sufficiently large, what completes the proof.

Therefore we have a well defined product of the double cosets $p$ and $q$ given by
\[ p \ast q = \lim_{n \to \infty} G[n]p\Theta_j[k]qG[n]. \]

Proposition 2.10. The operation defined above is associative. Furthermore, the Burau representation is functor between the categories of double cosets of $GL(\infty)$ and of $B_\infty$.

Proof. The proof of the associative property is analogous to the proof of Theorem 2.6 using Lemma 2.8. The functoriality follows from Lemma 2.8.

3 Further connexions and generalizations

We can extend the above constructions to the product $G^{[n]} = B_\infty \times \cdots \times B_\infty$ of $n$ copies of the infinite braid group. Let $K$ be the diagonal subgroup of $G^{[n]}$. Clearly, $K$ is naturally isomorphic to $B_\infty$. Let $K[\alpha]$ be the image of $B_\infty[\alpha]$ under this isomorphism. We define the product of double cosets componentwise.

Corollary 3.1. Consider two double cosets
\[ p \in K[\alpha]\backslash G^{[n]}/K[\beta], \quad q \in K[\beta]\backslash G^{[n]}/K[\gamma], \]
and let $p$ and $q$ be their respective representatives. Then the operation given by
\[ p \circ q = K[\alpha]p\Theta_j[\beta]qK[\beta], \]
for $j$ sufficiently large, is well defined and associative.

Proof. It follows from Propositions 2.2, 2.5 and 2.6.

Let $\psi : B_\infty \to G$ be a epimorphism and $G$ a group. Let $G[\alpha]$ be the image of $B_\infty[\alpha]$ by $\psi$, for $\alpha \in \mathbb{N}$. Then, the product of double cosets on $B_\infty$ induces a product on the double cosets of $G$ of the form $G[\alpha]\backslash G/G[\beta]$. In fact, this follows from the fact that in the definition of the product of double cosets, the sequence of double cosets defined not only converges, it becomes constant.

For each $n \in \mathbb{N}$, consider the symmetric group $S_n$ (of the permutations of $n$ elements). If $s_i$ is the permutation $(i, i+1)$ then we have the following presentation
\[ S_n = \left\{s_1, s_2, \ldots, s_{n-1} \mid s_is_j = s_js_i, |i - j| \geq 2, s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, s_i^2 = 1\right\}. \]
Therefore we can regard $S_n$ as the quotient group of $B_n$ by the relation $\sigma_i^2 = 1, 1 \leq i \leq n - 1$. Let $\xi_n : B_n \to S_n$ be the projection map, then this homomorphism gives a correspondence between a braid and the induced permutation of its endpoints. The kernel $P_n$ of $\xi_n$ is the subgroup of the pure braids in $n$ strands.

As we did for the braid group, consider the direct limit $S_\infty$ of the groups $S_n$ with relation to the monomorphisms $r_n : S_n \to S_{n+1}$, that take the permutation $(k, k+1) \in S_n$ to the permutation $(k, k + 1) \in S_{n+1}$. Since we have that
\[ \xi_n r_n = i_n \xi_{n+1}, \]
there exists an homomorphism $\xi : B_\infty \to S_\infty$.

Using the remarks above, we can define a multiplicative structure on the set of double cosets of $S_\infty$ (in fact, this structure coincide with the one defined by Neretin in [10, 8, 6]). Now, it is easy to see that $\xi$ is an epimorphism.
As a last remark, we point out some similarities between the multiplicative structure defined in $B_\infty$ and that of $\text{Aut}(F_\infty)$. The group $\text{Aut}(F_\infty)$ is defined as follows: Let $F_n$ be the free group with $n$ generators $x_1, \ldots, x_n$ and denote by $\text{Aut}(F_n)$ the group of automorphisms of $F_n$. Then

$$\text{Aut}(F_\infty) = \lim \text{Aut}(F_n).$$

The limit is taken with relation to the obvious inclusion $\text{Aut}(F_n) \to \text{Aut}(F_{n+1})$.

For each $\alpha \in \mathbb{N}$ consider the subgroup $H(\alpha)$ of $\text{Aut}(F_\infty)$ of automorphisms $h$ such that $h(x_i) = x_i$ for $i \leq \alpha$. In [11], it is defined a product on the double cosets of $\text{Aut}(F_\infty)$ in the following way: Consider the automorphism $\vartheta_j[\beta] \in \text{Aut}(F_\infty)$ given by

$$\vartheta_j[\beta](x_i) = \begin{cases} x_i, & i \leq \beta, i > 2j + \beta \\ x_{i+j}, & \beta < i \leq \beta + j \\ x_{i-j}, & \beta + j < i \leq 2\beta + j. \end{cases}$$

Then, for $p$ and $q$ in $\text{Aut}(F_\infty)$, the product of the double cosets $H(\alpha)\backslash p/H(\beta)$ and $H(\beta)\backslash q/H(\gamma)$ is the double coset limit of the sequence $p\vartheta_j[m]q$ in $H(\alpha)\backslash \text{Aut}(F_\infty)/H(\gamma)$.

For each $n \in \mathbb{N}$ we have a monomorphism $i_n : B_n \to \text{Aut}(F_n)$, given by

$$i_n(\sigma_j)(x_k) = \begin{cases} x_j, & k = j + 1 \\ x_jx_{j+1}x_j^{-1}, & k = j \\ x_k, & \text{otherwise}. \end{cases}$$

Therefore we can identify $B_n$ with the image of $i_n$ in $\text{Aut}(F_n)$. Consider the limit homomorphism $i_\infty : B_\infty \to \text{Aut}(F_\infty)$. The element $\vartheta_j[m]$ is related to the image of the element $i_n$ as we see in the following proposition.

**Proposition 3.2.** Let $\beta$ be a fixed positive integer. For each $k \in \mathbb{N}$, consider $y_k = x_{\beta+k}x_{\beta+k-1} \cdots x_{\beta+1} \in F_\infty$. Then

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} x_i, & i \leq \beta, i > 2k + \beta \\ y_k^{-1}x_{i+k}y_k, & \beta + 1 \leq i \leq k + \beta \\ x_{i-k}, & k + \beta < i \leq 2k + \beta. \end{cases}$$

In other words

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} y_k^{-1}\vartheta_k[\beta](x_i)y_k, & \beta + 1 \leq i \leq k + \beta \\ \vartheta_k[\beta](x_i), & \text{otherwise}. \end{cases}$$

**Proof.** For $k = 1$ we have that $\theta_1[\beta] = \sigma_{\beta+1}$ and therefore

$$i_\infty(\theta_1[\beta])(x_i) = i_\infty(\sigma_{\beta+1})(x_i) = \begin{cases} x_i, & i \leq \beta, i > \beta + 2 \\ x_i^{-1}x_{i+1}x_{i+1}, & i = 1 + \beta \\ x_{i-1}, & i = 2 + \beta. \end{cases}$$

We are going to show the truth of the identity by induction on $k$. Suppose the identity true for $k$. We can write $\theta_{k+1}[\beta]$ as

$$\theta_{k+1}[\beta] = \sigma_{k+\beta+1} \cdots \sigma_{2k+\beta+1} \theta_k[\beta] \sigma_{2k+\beta+1} \cdots \sigma_{k+\beta+1}.$$  

If we put $w = \sigma_{k+\beta+1} \cdots \sigma_{2k+\beta+1}$ and $s = \sigma_{2k+\beta+1} \cdots \sigma_{k+\beta+1}$ we can re-write the equation above as

$$\theta_{k+1}[\beta] = w\theta_k[\beta]s.$$  

We have five cases to analyse:

**Case 1** When $\beta + 1 \leq i \leq k + \beta$, notice that $i_\infty(s)(x_i) = x_i$ and $i_\infty(\theta_k[\beta])(x_i) = y_k^{-1}x_{i+k}y_k$, therefore $i_\infty(\theta_{k+1}[\beta])(x_i) = i_\infty(w)(y_k^{-1}x_{i+k}y_k)$. Now,

$$i_\infty(w)(x_{i+k}) = i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-1})i_\infty(\sigma_{i+k})(x_{i+k}) = i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-2})i_\infty(\sigma_{i+k-1})(x_{i+k}^{-1}x_{i+k+1}x_{i+k}) = i_\infty(\sigma_{k+\beta+1} \cdots \sigma_{i+k-3})i_\infty(\sigma_{i+k-2})(x_{i+k-1}^{-1}x_{i+k+1}x_{i+k-1}) = \cdots = i_\infty(\sigma_{k+\beta+1})(x_{k+\beta+2}x_{k+i+1}x_{k+i+2}) = x_{k+\beta+1}^{-1}x_{k+i+1}x_{k+\beta+1}.$$  

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Hence,
\[ i_{\infty}(w)(y_k^{-1}x_{i+k}y_k) = y_k^{-1}i_{\infty}(w)(x_{i+k})y_k = y_k^{-1}x_{k+\beta+1}x_k^{i+1}x_k^{\beta+1}y_k = y_k^{-1}x_{k+1}^{i+1}y_k. \]

Case 2 When \( i = k + \beta + 1 \), we have that
\[ i_{\infty}(s)(x_{k+i+\beta}) = x_{k+\beta+1}^{-1}\cdots x_{2k+\beta}^{-1}x_{2k+\beta+1}^{-1}\cdots x_{k+\beta+1}. \]

Hence,
\[ i_{\infty}(\theta_k[\beta]s)(x_{k+i+\beta}) = i_{\infty}(\theta_k[\beta])(x_{k+\beta+1}^{-1}\cdots x_{2k+\beta}^{-1}x_{2k+\beta+1}^{-1}\cdots x_{k+\beta+1}) = x_{k+1}^{-1}\cdots x_{2k+\beta}^{-1}x_{2k+\beta+1}^{-1}x_{k+\beta+1}^{-1} = y_k^{-1}x_{k+\beta+1}^{-1}y_k. \]

Furthermore, \( i_{\infty}(w)(x_{2k+\beta+1}) = x_{k+\beta+1}^{-1}x_{2k+\beta+2}^{-1}x_{k+\beta+1}^{-1} \) and hence
\[ i_{\infty}(\theta_k+1[\beta])(x_{k+\beta+1}) = i_{\infty}(w)(y_k^{-1}x_{2k+\beta+1}y_k) = y_k^{-1}x_{k+\beta+1}^{-1}x_{2k+\beta+2}^{-1}x_{k+\beta+1}^{-1} = y_k^{-1}x_{k+\beta+2}y_k. \]

Case 3 When \( k + \beta + 1 < i \leq 2k + \beta + 1 \), it is sufficient to notice that \( i_{\infty}(s)(x_i) = x_{i-1}, i_{\infty}(\theta_k[\beta])(x_{i-1}) = x_{i-k-1} \) and \( i_{\infty}(w)(x_{i-k-1}) = x_{i-k-1}. \)

Case 4 For the case \( i = 2k + \beta + 2 \) we have \( i_{\infty}(\theta_k[\beta]s)(x_{2k+\beta+2}) = x_{2k+\beta+2} \). Furthermore, \( i_{\infty}(w)(x_{2k+\beta+2}) = x_{k+\beta+1}^{-1} \).

Case 5 For \( i \leq \beta \) or \( i > 2k + \beta + 2 \), we have that \( i_{\infty}(w)(x_i) = i_{\infty}(\theta_k[\beta])(x_i) = i_{\infty}(s)(x_i) = x_i \) and the result follows.

Thus the elements \( \theta_j[\beta] \) and \( i_{\infty}(\theta_j[\beta]) \) are always conjugate in \( \text{Aut}(F_{\infty}) \) (in particular, by an element of \( H(\beta) \)). Nevertheless, \( i_{\infty} \) does not induce a homomorphism between the monoids of double cosets. In fact, consider the braid \( \omega = \sigma_2^{-1}\sigma_3\sigma_1\sigma_3\sigma_2 \) in \( B_{\infty} \) and its projection \( [\omega] \) in \( \langle B_2 \rangle \backslash B_{\infty}/B_2 \). Then \( i_{\infty}(\omega\theta_2[2]\omega) \) and \( i_{\infty}(\omega)\theta_2[2]i_{\infty}(\omega) \) do not belong to the same double coset of \( H(2) \backslash \text{Aut}(F_{\infty})/H(2) \).

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