Polyakov-Alvarez type comparison formulas for determinants of Laplacians on Riemann surfaces with conical metrics

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Abstract

We prove Polyakov-Alvarez type comparison formulas for the determinants of Friederichs extensions of Laplacians corresponding to conformally equivalent conical metrics on compact Riemann surfaces. We illustrate our results by obtaining new and recovering known explicit formulas for determinants of Laplacians on surfaces with conical singularities.

1 Introduction

Investigation of determinants of Laplacians as functions of metrics on compact Riemann surfaces is motivated by the needs of geometric analysis and quantum field theory. For smooth metrics the determinants have been comprehensively studied, see e.g. [29, 42, 43, 30]. The Polyakov formula [31, 32] and a similar formula for surfaces with boundary due to Alvarez [1] often appears as the key of an argument, e.g. [29, 42, 43]. In the last decade significant progress was achieved for flat (curvature zero) conical metrics, see e.g. [2, 3, 35, 22, 21, 15, 16]. Here, for instance, results in [2, 3] can be interpreted as a generalization of Polyakov-Alvarez formula to the case of flat conical metrics on a disk and on a sphere, the main result in [21] is a simple consequence of an analog of Polyakov formula for two conformally equivalent flat conical metrics and the results in [22]. Some results were also obtained for determinants of Laplacians in constant positive curvature (spherical) [10, 36, 23, 18, 19] and other conical metrics [17], but no Polyakov-Alvarez type formulas for metrics other than smooth or conical flat were available until now.

In the first part of this paper we prove Polyakov-Alvarez type formulas relating the determinants of Friederichs selfadjoint extensions of Laplacians for a pair of conformally equivalent conical metrics on compact Riemann surfaces. In the case of smooth metrics our formulas reduce to the classical Polyakov-Alvarez formulas [31, 32, 1]. In the second part of the paper we demonstrate how our results can be used to obtain new and recover known explicit formulas for determinants of Laplacians on surfaces with conical singularities.

The paper is organized as follows. Section 1.1 contains preliminaries and the main results of the first part of the paper. In Subsection 1.2 we formulate two important

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corollaries: a formula for the value of spectral zeta function at zero and Polyakov-Alvarez type formulas for two conformally equivalent conical metrics. The first corollary is an immediate consequence of our main result and the Gauss-Bonnet theorem. The proof of other results is carried out in Section 2. Thus in Subsection 2.1 we obtain an asymptotic estimate for the determinant of the Friederichs Dirichlet Laplacian on a shrinking conical metric disk. In Subsection 2.2 we prove BFK-type decomposition formulas. These decomposition formulas allow to cut shrinking conical metric disks out of a singular surface. In Subsection 2.3 we finalize the proof of Polyakov-Alvarez type formulas. This completes the first part of the paper.

In the second part of the paper we demonstrate how the results of the first part can be used to obtain explicit formulas for determinants of Laplacians on singular surfaces with or without boundary, this occupies Section 3. In Subsection 3.1 we consider the constant curvature spheres with two conical singularities: we recover and generalize the corresponding results in [36, 23, 17, 18] and discuss extremal properties of the determinant for the metrics of area 4π. In Subsection 3.2 we consider polyhedral surfaces with spherical topology and obtain an analog of the Aurell-Salomonson formula in [3]. In Subsection 3.3 we deduce a formula for the determinant of Friederichs Dirichlet Laplacian on the constant curvature metric disks with a conical singularity at the center (for the spherical and hyperbolic metrics the result is new, the case of flat conical metrics was studied in [35]). In Subsection 3.4 we study the determinant of Friederichs Laplacian on hyperbolic spheres: we obtain a new explicit formula for the determinant, recover a variational formula from [17] and find the corresponding undetermined constant. Finally, in Subsection 3.5 we present a general explicit formula for the determinant of Friederichs Laplacian on singular genus g > 1 surface without boundary, this is a generalization of the results in [21, 22].

Genus one examples will be considered elsewhere. We only note that explicit formulas for determinants of Laplacians on genus one surfaces can be obtained by using results of this paper together with known explicit formulas for the determinant of Laplacian on the (smooth) flat tori [30, 29] and the flat annulus [42]; in particular, one can expect to recover the variational formula in [19] and to find the corresponding undetermined constant. Let us also mention that conical metrics with cylindrical and conical ends can be included into consideration by pairing results of this paper with the BFK-type decomposition formulas in [15, Theorem 1] and [16, Theorem 1], however we do not discuss this here.

1.1 Preliminaries and main results

Let M be a compact Riemann surface (perhaps with smooth boundary ∂M). We say that m is a (conformal Riemannian) conical metric on M if for any point P ∈ M there exist a neighbourhood U of P, a local (holomorphic) parameter x ∈ C centred at P (i.e. x(P) = 0), and a real-valued function φ ∈ L^1(U) such that m = |x|^{2β}e^{2φ}|dx|^2 in U with some β > −1, and ∂x∂xφ ∈ L^1(U). If β = 0, then the point P is regular. If β ≠ 0, then P is a conical singularity of order β and total angle 2π(β + 1). A function K : M → R defined by

\[ K = |x|^{-2β}e^{-2φ}(-4∂x∂xφ) \]
is the (regularized) Gaussian curvature of $m$ in the neighbourhood $U$ ($K$ does not depend on the choice of $x$).

The information about all conical singularities of $m$ is encoded in a divisor: a metric with conical singularities of order $\beta_1, \ldots, \beta_n$ at (distinct) points $P^1, \ldots, P^n \in M$ is said to represent the divisor $\mathbf{\beta} = \sum_j \beta_j P^j$, which is a formal sum. By definition, the set $\text{supp} \mathbf{\beta} := \{P^1, \ldots, P^n\}$ is the support and the number $|\mathbf{\beta}| := \sum_j \beta_j$ is the degree of the divisor $\mathbf{\beta}$. We assume that the curvature $K$ is a smooth function on $M$, if $M$ has a boundary $\partial M$, then $\text{supp} \mathbf{\beta} \cap \partial M = \emptyset$ (i.e. there are no conical singularities on the boundary) and the geodesic curvature $k$ is a well defined continuous function on $\partial M$.

Let $m_\varphi$ be a conical metric conformally equivalent to a smooth metric $m_0$ on $M$, i.e. $m_\varphi$ represents a divisor $\mathbf{\beta}$ and $m_\varphi = e^{2\varphi} m_0$ with some function $\varphi \in C^\infty(M \setminus \text{supp} \mathbf{\beta})$. In a local parameter centred at $P^j \in \text{supp} \mathbf{\beta}$ we have $m_\varphi = |x|^{2\beta_j} e^{2\varphi_j} |dx|^2$ and $m_0 = e^{2\varphi_j} |dx|^2$, where $\phi_j$ is a continuous and $\psi_j$ is a smooth function in a small neighbourhood of $x = 0$. (In particular, $\varphi(x) = \beta_j \log |x| + \phi_j(x) - \psi_j(x)$. ) In what follows it is important that the value $\frac{\phi_j(0)}{\beta_j+1} - \psi_j(0)$ does not depend on the choice of local parameter $x$, $x(P^j) = 0$.

As is known, for any flat in a neighbourhood of $P^j$ conical metric $m_\varphi$ there exists a local holomorphic parameter $x$ such that $m_\varphi = |x|^{2\beta_j} |dx|^2$, see e.g. [39, Lemma 3.4]. The metric $|x|^{2\beta_j} |dx|^2$ is homogeneous of degree 2 with respect to the dilation $x \mapsto e^{\frac{1}{\beta_j+1}} x$. It is natural to consider a non-flat in a neighborhood of $P^j$ conical metric $m_\varphi = |x|^{2\beta_j} e^{2\varphi_j} |dx|^2$ as a local perturbation of the flat metric $|x|^{2\beta_j} |dx|^2$. In this paper we consider dilation analytic perturbations. Namely, we assume that $m_\varphi$ is dilation analytic in the following sense: for any $P^j \in \text{supp} \mathbf{\beta}$ there exists a centred at $P^j$ local holomorphic parameter $x$ such that $m_\varphi = |x|^{2\beta_j} e^{2\varphi_j} |dx|^2$ and the function $\epsilon \mapsto \phi_j(\epsilon^{\frac{1}{\beta_j+1}} x)$ extends by analyticity to a neighbourhood of zero for any $x$ with $|x| < c$, where $c > 0$ is sufficiently small. In particular, recent results on regularity of spherical (positive constant curvature) and hyperbolic (negative constant curvature) conical metrics [7, 8] show that any constant curvature conical metric $m_\varphi$ is dilation analytic, see also Remark 2.1 in Sec. 2.1.

Let $\Delta^\varphi$ stand for the Friederichs extension of the Laplacian on $(M, m_\varphi)$ initially defined on the functions in $\mathcal{C}^\infty_0(M \setminus \text{supp} \mathbf{\beta})$. The spectrum $\sigma(\Delta^\varphi)$ of $\Delta^\varphi$ consists of isolated eigenvalues $\lambda_k$ of finite multiplicity. If $\partial M = \emptyset$, then the first eigenvalue $\lambda_0 = 0$ of the nonnegative selfadjoint operator $\Delta^\varphi$ is of multiplicity 1 (and the eigenspace consists of constant functions). If $\partial M \neq \emptyset$, then $\Delta^\varphi$ is the Friederichs Dirichlet Laplacian, it is a positive selfadjoint operator. From results in [4, 33, 39] it follows that the spectral zeta function
\[ \zeta(s) = \sum_{\lambda_k \in \sigma(\Delta^\varphi), \lambda_k \neq 0} \lambda_k^{-s}, \quad \Re s > 1, \]
extends by analyticity to a neighborhood of $s = 0$. The zeta regularized determinant of $\Delta^\varphi$ is defined by $\det \Delta^\varphi = e^{-\zeta'(0)}$ (if $\partial M = \emptyset$, then it is a modified determinant, i.e. with zero eigenvalue excluded). For a smooth metric $m_0$ on $M$ the determinant $\det \Delta^0$ can be defined via the spectral zeta function of the selfadjoint Laplacian $\Delta^0$ on $(M, m_0)$ in exactly the same way, e.g. [29].

The main result of the first part of this paper is the following generalization of Polyakov and Polyakov-Alvarez formulas.
Let $m_0$ be a smooth conformal metric on a compact Riemann surface $M$. Denote the Gaussian curvature of $m_0$ by $K_0$. Let $K_\varphi$ stand for the Gaussian curvature of a dilation analytic conical metric $m_\varphi = e^{2\varphi}m_0$ representing a divisor $\beta = \sum_{j=1}^n \beta_j P_j$. By $\phi_j(x)$ and $\psi_j(x)$ we denote the functions in the representations $m_\varphi = |x|^{2\beta} e^{2\varphi_j}|dx|^2$ and $m_0 = e^{2\varphi_0}|dx|^2$ in a local holomorphic parameter $x$ centred at $P_j \in \text{supp}\, \beta$. Let also $A_\varphi$ (resp. $A_0$) stand for the total area of $M$ in the metric $m_\varphi$ (resp. $m_0$).

1. If $\partial M = \emptyset$, then for the modified zeta regularized determinants of the Friederichs Laplacian $\Delta^\varphi$ on $(M,m_\varphi)$ and the selfadjoint Laplacian $\Delta^0$ on $(M,m_0)$ we have

$$\log \frac{(\det \Delta^\varphi)/A_\varphi}{(\det \Delta^0)/A_0} = -\frac{1}{12\pi} \left( \int_M K_\varphi \varphi \, dA_\varphi + \int_M K_0 \varphi \, dA_0 \right)$$
$$+ \frac{1}{6} \sum_{j=1}^n \beta_j \left( \frac{\phi_j(0)}{\beta_j + 1} - \psi_j(0) \right) - \sum_{j=1}^n C(\beta_j).$$

(1.1)

Here

$$C(\beta) = 2\zeta_B(0; \beta + 1, 1, 1) - 2\zeta_R(-1) - \frac{\beta^2}{6(\beta + 1)} \log 2 - \frac{\beta}{12} + \frac{1}{2} \log(\beta + 1),$$

(1.2)

where $\zeta_B$ is the Barnes double zeta function

$$\zeta_B(s; a, b, x) = \sum_{m,n=0}^{\infty} (am + bn + x)^{-s},$$

(1.3)

the prime stands for the derivative with respect to $s$, and $\zeta_R(s)$ is the Riemann zeta function.

2. If $\partial M \neq \emptyset$ and $\text{supp}\, \beta \cap \partial M = \emptyset$, then for the zeta regularized determinants of the Friederichs Dirichlet Laplacian $\Delta^\varphi$ on $(M,m_\varphi)$ and the selfadjoint Dirichlet Laplacian $\Delta^0$ on $(M,m_0)$ we have

$$\log \frac{\det \Delta^\varphi}{\det \Delta^0} = -\frac{1}{12\pi} \left( \int_M K_\varphi \varphi \, dA_\varphi + \int_M K_0 \varphi \, dA_0 + \int_{\partial M} \varphi \partial_{\vec{n}} \varphi \, ds_0 \right)$$
$$- \frac{1}{6\pi} \int_{\partial M} k_0 \varphi \, ds_0 - \frac{1}{4\pi} \int_{\partial M} \partial_{\vec{n}} \varphi \, ds_0$$
$$+ \frac{1}{6} \sum_{j=1}^n \beta_j \left( \frac{\phi_j(0)}{\beta_j + 1} - \psi_j(0) \right) - \sum_{j=1}^n C(\beta_j),$$

(1.4)

where $k_0$ is the geodesic curvature of the boundary $\partial M$ of $M$, $s_0$ is the arc length, and $\vec{n}$ is the outward unit normal to the boundary $\partial M$ (all are with respect to the metric $m_0$); the function $C(\beta)$ is the same as in (1.2).

The non-integral terms in the right hand side of (1.1) and (1.4) are responsible for the inputs from the conical singularities and do not depend on the choice of local parameters. If the function $\varphi$ in the equality $m_\varphi = e^{2\varphi}m_0$ is smooth (or, equivalently, the metric $m_\varphi$ is smooth, $\text{supp}\, \beta = \emptyset$, and $n = 0$), then the non-integral terms (the
last lines in (1.1) and (1.4)) disappear. As a result the f-las (1.1) and (1.4) become the well-known Polyakov-Alvarez formulas written in a slightly different “regularized” form, cf. e.g. [31, 32, 1, 29, 42]. This regularization keeps the integrals in (1.1) and (1.4) finite for the conical metrics $m_\varphi$.

The assumption on dilation analyticity of $m_\varphi$ allows us to rely only on known short time heat trace asymptotic expansions of elliptic cone differential operators [4, 12], which significantly simplifies the proof of Theorem 1.1 making it accessible to a larger audience. However, there are good grounds to believe that the formulas (1.1) and (1.4) remain valid under weaker assumptions on regularity of conical metrics.

For the conical singularities of rational orders $\beta$ the values of $C(\beta)$ in (1.1) and (1.4) can be expressed in terms of $\zeta'(R)(-1)$ and gamma functions. Namely, the following equality is valid for the derivative of the Barnes double zeta function in (1.2):

$$
\zeta'_B(0; p/q, 1, 1) = \frac{1}{pq} \zeta'_R(-1) - \frac{1}{12pq} \log(q) + \left( \frac{1}{4} + S(q, p) \right) \log \frac{q}{p} + \sum_{k=1}^{p-1} \left( \frac{1}{2} - \frac{k}{p} \right) \log \Gamma \left( \left( \frac{kq}{p} \right) + \frac{1}{2} \right) + \sum_{j=1}^{q-1} \left( \frac{1}{2} - \frac{j}{q} \right) \log \Gamma \left( \left( \frac{jp}{q} \right) + \frac{1}{2} \right),
$$

(1.5)

where $p$ and $q$ are coprime natural numbers, $S(q, p) = \sum_{j=1}^{p} \left( \frac{j}{p} \right) \left( \frac{1}{p} \right)$ is the Dedekind sum, and the symbol $\langle x \rangle$ is defined so that $\langle x \rangle = x - \lfloor x \rfloor - 1/2$ for $x$ not an integer and $\langle x \rangle = 0$ for $x$ an integer (here $\lfloor x \rfloor$ is the floor of $x$, i.e. the largest integer not exceeding $x$). In particular, $\zeta'_B(0; 1, 1, 1) = \zeta'_R(-1)$ and (1.2) gives $C(0) = 0$; recall that $\beta = 0$ for a regular point $P \in M$. Similarly, for a singularity of order $\beta = 1$ (i.e. of angle $4\pi$) we have

$$
C(1) = -\zeta'_R(-1) - \frac{1}{12} \log 2 - \frac{1}{12},
$$

for a singularity of order $\beta = -1/2$ (i.e. of angle $\pi$) we have

$$
C(-1/2) = -\zeta'_R(-1) - \frac{1}{6} \log 2 + \frac{1}{24},
$$

and etc. A proof of (1.5) and some particular values of $\zeta'_B(0; p/q, 1, 1)$ can be found in Appendix A. Note that available asymptotics of the Barnes double zeta function (e.g. [27, 37], [23, A.6]) imply that $C(\beta) \to +\infty$ as $\beta \to -1^+$ and $C(\beta) \to -\infty$ as $\beta \to +\infty$, see Fig. 1.1 for a graph of $C(\beta)$.

### 1.2 A formula for $\zeta(0)$ and comparison formulas for two conical metrics

In this subsection we discuss two corollaries of Theorem 1.1. First we find the value of the spectral zeta function of $\Delta^\varphi$ at zero. Then we present a generalization of Theorem 1.1 to the case of two conical metrics.

Let $\zeta(s) = \sum_k \lambda_k^{-s}$ stand for the spectral zeta function of the Friederichs Laplacian $\Delta^\varphi$. Then $\zeta_R(s) = \sum_k (R^{-2}\lambda_k)^{-s}$ is the zeta function of the operator $R^{-2}\Delta^\varphi$ corresponding to the metric $R^2m^\varphi$. On the one hand, differentiating $\zeta_R(s)$ with respect to $s$ and
evaluating the result at \( s = 0 \) we arrive at the standard rescaling property

\[
\zeta'_R(0) = (2 \log R) \zeta(0) + \zeta'(0), \quad R > 0.
\] (1.6)

On the other hand, the Polyakov f-la (1.1) gives

\[
\zeta'_R(0) - \log(R^2 A_\varphi) - \zeta'(0) + \log A_\varphi = -\frac{\log R}{12\pi} \left( \int_M K_\varphi dA_\varphi + \int_M K_0 dA_0 \right) + \frac{1}{6} \sum_{j=1}^n \beta_j \log R, \quad R > 0.
\]

where \( 2\pi \int_M K_\varphi dA_\varphi = \chi(M, \beta) \) and \( 2\pi \int_M K_0 dA_0 = \chi(M) \) by the Gauss-Bonnet theorem [38]; here \( \chi(M, \beta) = \chi(M) + |\beta| \) is the Euler characteristic of \( M \) with topological Euler characteristic \( \chi(M) \) and divisor \( \beta \). This implies

\[
\zeta(0) = \frac{\chi(M, \beta)}{6} - \frac{1}{12} \sum_{j=1}^n \left( \beta_j + 1 - \frac{1}{\beta_j + 1} \right) - \dim \ker \Delta^\varphi
\] (1.7)

in the case \( \partial M = \emptyset \). Similarly, the Polyakov-Alvarez f-la (1.4) implies (1.7) in the case \( \partial M \neq \emptyset \). We formulate this result as a corollary of Theorem 1.1.

**Corollary 1.2.** Let \( \chi(M, \beta) = \chi(M) + |\beta| \) stand for the Euler characteristic of the Riemann surface \( M \) with topological Euler characteristic \( \chi(M) \) and divisor \( \beta \) of degree \( |\beta| = \sum_{j=1}^n \beta_j \). Let \( m_\varphi \) be a conformal dilation analytic conical metric on \( M \) representing the divisor \( \beta \). Then the spectral zeta function \( \zeta(s) \) of the Friederichs Laplacian \( \Delta^\varphi \) on \( (M, m_\varphi) \) satisfies (1.7), where \( \dim \ker \Delta^\varphi = 1 \) in the case \( \partial M = \emptyset \) and \( \dim \ker \Delta^\varphi = 0 \) in the case \( \partial M \neq \emptyset \).

We also note that the f-la (1.7) for \( \zeta(0) \) allows to find the constant term in the asymptotic expansion of the heat trace \( \text{Tr} e^{-t\Delta^\varphi} \) as \( t \to 0^+ \) (see Remark 2.5 in Sec. 2.2).

The next corollary presents comparison formulas for the determinants of Laplacians in two conical metrics.

**Corollary 1.3.** Let \( m_0 \) and \( m_\varphi = e^{2\varphi} m_0 \) be two dilation analytic conformal conical metrics on \( M \) representing divisors \( \alpha \) and \( \beta \) respectively. Let \( \{P_1, \ldots, P_n\} \) be the set of
all distinct points in the union supp $\alpha \cup$ supp $\beta$. In a local holomorphic parameter $x$ centred at $P_i$ we have $m_0 = |x|^{2\alpha_j}e^{2\phi_j}|dx|^2$ and $m_\varphi = |x|^{2\beta_j}e^{2\phi_j}|dx|^2$, where $\alpha_j = 0$ if $P_i \notin$ supp $\alpha$ and $\beta_j = 0$ if $P_i \notin$ supp $\beta$.

1. If $\partial M = \emptyset$, then the determinants of the Friederichs Laplacians $\Delta^\varphi$ and $\Delta^0$ satisfy

$$
\log \frac{\det \Delta^\varphi}{\det \Delta^0} = -\frac{1}{12\pi} \left( \int_M K_\varphi \varphi dA_\varphi + \int_M K_0 \varphi dA_0 \right) - \frac{1}{6\pi} \int_{\partial M} k_\varphi d\bar{n}_\varphi ds_0 - \frac{1}{4\pi} \int_{\partial M} \partial n_\varphi d\bar{n}_\varphi ds_0 
+ \frac{1}{6} \sum_{j=1}^n \left\{ \beta_j \left( \frac{\phi_j(0)}{\beta_j + 1} - \psi_j(0) \right) - \alpha_j \left( \frac{\psi_j(0)}{\alpha_j + 1} - \phi_j(0) \right) \right\} 
- \sum_{j=1}^n \left( C(\beta_j) - C(\alpha_j) \right).
$$

(1.8)

2. If $\partial M \neq \emptyset$, supp $\alpha \cap \partial M = \emptyset$, and supp $\beta \cap \partial M = \emptyset$, then the determinants of the Friederichs Dirichlet Laplacians $\Delta^\varphi$ and $\Delta^0$ satisfy

$$
\log \frac{\det \Delta^\varphi}{\det \Delta^0} = -\frac{1}{12\pi} \left( \int_M K_\varphi \varphi dA_\varphi + \int_M K_0 \varphi dA_0 + \int_{\partial M} \varphi \partial n_\varphi ds_0 \right) - \frac{1}{6\pi} \int_{\partial M} k_0 \varphi d\bar{n}_\varphi ds_0 
- \frac{1}{4\pi} \int_{\partial M} \partial n_\varphi d\bar{n}_\varphi ds_0 
+ \frac{1}{6} \sum_{j=1}^n \left\{ \beta_j \left( \frac{\phi_j(0)}{\beta_j + 1} - \psi_j(0) \right) - \alpha_j \left( \frac{\psi_j(0)}{\alpha_j + 1} - \phi_j(0) \right) \right\} 
- \sum_{j=1}^n \left( C(\beta_j) - C(\alpha_j) \right),
$$

where $k_\beta$ is the geodesic curvature of the boundary $\partial M$, $s_\beta$ is the arc length, and $\bar{n}$ is the outward unit normal to the boundary $\partial M$ (with respect to the metric $m_0$).

In [21, Prop.1] a Polyakov type formula was obtained for a pair of flat conformally equivalent conical metrics $\{m_0, m_\varphi\}$ on a surface without boundary under the additional assumption supp $\alpha \cap$ supp $\beta = \emptyset$. In this case the f-la (1.8) returns the same result.

The proof of Corollary 1.3 is postponed to Section 2.3.

## 2 Proof of Polyakov-Alvarez type comparison formulas

### 2.1 Dirichlet Laplacian on a shrinking conical metric disk

Let $P$ be a conical singularity of dilation analytic metric $m_\varphi$ representing a divisor $\beta$. We pick a centred at $P$ local holomorphic parameter $x \in \mathbb{C}$ such that $m_\varphi = |x|^{2\beta_j}e^{2\phi_j}|dx|^2$ and for any $x$, $|x| < c$ with sufficiently small $c > 0$, the function

$$
\epsilon \mapsto \phi(\epsilon, x) := \phi(e^{\frac{1}{\epsilon+1}}x)
$$

(2.1)

is analytic in a neighborhood of zero.
Consider the Friederichs Dirichlet Laplacian $\Delta^\varphi_{D_\epsilon}$ on the disk $D_\epsilon = \{ x \in \mathbb{C} : |x| \leq \epsilon \}$ endowed with conical metric $|x|^{2\beta}e^{2\varphi}dx^2$. More precisely, $\Delta^\varphi_{D_\epsilon}$ is the Friederichs selfadjoint extension of the operator $-\varphi|x|^{-2\beta}e^{-2\varphi}4\partial_x\partial_x$ on $C_0^\infty(0 < |x| \leq \epsilon)$ in the $L^2$-space with the norm

$$
\|f\| = \left( \int_{|x| \leq \epsilon} |f(x,\bar{x})|^2 |x|^{2\beta}e^{2\varphi}dx \right)^{1/2}.
$$

In this section we obtain an asymptotic estimate for $\det \Delta^\varphi_{D_\epsilon}$ as $\epsilon \to 0+$ (Lemma 2.2 below). For the Dirichlet Laplacian $\Delta^\psi_{D_\epsilon}$ on the smooth metric disk $(D_\epsilon, e^{2\psi}|dx|^2)$ the corresponding result can be easily obtained from the usual Polyakov-Alvarez formula [1, 29, 42] and the explicit formula [42, f-la (28)] for the determinant of Dirichlet Laplacian on the flat metric disk $(D_\epsilon, |dx|^2)$ (see Lemma 2.3 at the end of this section).

**Remark 2.1.** For any constant curvature $K_\varphi$ metric $m_\varphi$ on $M$ and any point $P \in M$ there exists a local holomorphic parameter $x$ such that $x(P) = 0$ and in a neighborhood of $P$ we have

$$
m_\varphi = |x|^{2\beta}e^{2\varphi}|dx|^2, \quad \phi(x) = \log(2\beta + 2) - \log(1 + K_\varphi|x|^{2\beta+2});
$$

see e.g. [39, Lemma 3.4] for the case $K_\varphi = 0$, [7] for the case $K_\varphi > 0$, and [8] for the case $K_\varphi < 0$. Thus any constant curvature conical metric $m_\varphi$ is dilation analytic and $|x|^{2\beta}e^{2\phi(x)}|dx|^2$ with $\epsilon \geq 0$ (and $\phi(x)$ defined in (2.1)) is a metric of curvature $\epsilon^2 K_\varphi$.

Let $\zeta_<(s, \beta)$ stand for the spectral zeta function of the Friederichs Dirichlet Laplacian on the unit disk $D_1$ with flat conical metric $4|x|^{2\beta}|dx|^2$. As is known [35], the function $s \to \zeta_<(s, \beta)$ admits an analytic continuation to $s = 0$ and

$$
\zeta_<(0, \beta) = \frac{1}{12} \left( \beta + 1 + \frac{1}{\beta + 1} \right), \quad (2.2)
$$

$$
\zeta'_<(0, \beta) = 2\zeta'_B(0; \beta + 1, 1, 1) + \frac{5}{12}(\beta + 1) + \frac{1}{2}\log(\beta + 1) + \frac{1}{2}\log 2\pi, \quad (2.3)
$$

where the prime stands for the derivative with respect to $s$ and $\zeta_B$ is the Barnes double zeta function (1.3); see also [23].

**Lemma 2.2.** For the spectral determinant of the Friederichs Dirichlet Laplacian $\Delta^\varphi_{D_\epsilon}$ on the disk $D_\epsilon = \{ x \in \mathbb{C} : |x| \leq \epsilon \}$ with dilation analytic metric $|x|^{2\beta}e^{2\varphi}|dx|^2$ we have

$$
\log \det \Delta^\varphi_{D_\epsilon} = 2\left( \log(2\epsilon^{-\beta-1}) - \phi(0) \right)\zeta_<(0, \beta) - \zeta'_<(0, \beta) + O\left( -\epsilon^{\beta+1} \log \epsilon \right) \quad \text{as } \epsilon \to 0+, \quad (2.4)
$$

where $\zeta_<(0, \beta)$ and $\zeta'_<(0, \beta)$ are the same as in (2.2) and (2.3).

**Proof.** Denote $\epsilon = \epsilon^{\beta+1}$. The metric disks $(D_\epsilon, m_\varphi)$ and $(D_1, \epsilon^2|x|^{2\beta}e^{2\phi(x)}|dx|^2)$ are isometric and hence we can replace $\Delta^\varphi_{D_\epsilon}$ by the Friederichs extension $\Delta^\varphi_{D_\epsilon}$ of the Dirichlet Laplacian $-\epsilon^{-2}|x|^{-2\beta}e^{-2\phi(x)}4\partial_x\partial_x$ on the unit disk $D_1$. 

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Let $\hat{\Delta}_s^\varphi = \frac{1}{4}e^{2\phi(0)}\Delta_s^\varphi$. In a small neighbourhood of zero $\varepsilon \mapsto \hat{\Delta}_s^\varphi$ is a type A [20] analytic family of operators in the space $L^2(|x| \leq 1, |x|^{2\beta}|dx|^2)$. In particular, the selfadjoint operator $\hat{\Delta}_s^\varphi$ corresponds to the flat conical metric $4|x|^{2\beta}|dx|^2$ and $\zeta_c(s)$ is its spectral zeta function (in this proof $\beta$ is fixed and for brevity of notations we do not list it as an argument of the zeta functions). It is known that the spectrum $\sigma(\hat{\Delta}_0^\varphi)$ of $\hat{\Delta}_0^\varphi$ consists of isolated eigenvalues $\lambda_k$, 

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty,$$

and $(\hat{\Delta}_0^\varphi)^{-2}$ is a trace class operator. Since 

$$\| (\hat{\Delta}_0^\varphi - \lambda)^{-1} \| = 1 / \text{dist}\{\lambda, \sigma(\hat{\Delta}_0^\varphi)\},$$

the first resolvent identity together with inequality $\|AB\|_1 \leq \|A\|\|B\|_1$ implies 

$$\| (\hat{\Delta}_0^\varphi - \lambda)^{-2} \|_1 \leq \left( 1 + |\lambda| \| (\hat{\Delta}_0^\varphi - \lambda)^{-1} \| \right)^2 \| (\hat{\Delta}_0^\varphi)^{-2} \|_1 \leq C$$

uniformly in $\lambda \in C$, where $C$ is a set such that $\text{dist}\{\lambda, \sigma(\hat{\Delta}_0^\varphi)\} \geq c|\lambda|$ for any $\lambda \in C$ and some $c > 0$. Denote 

$$T(\varepsilon, \lambda) = (\hat{\Delta}_s^\varphi - \hat{\Delta}_0^\varphi)(\hat{\Delta}_0^\varphi - \lambda)^{-1}$$

and observe that $\|T(\varepsilon, \lambda)\| \to 0$ uniformly in $\lambda \in C$ and $\varepsilon$ as $|\varepsilon| \to 0$. We have 

$$\| (\hat{\Delta}_s^\varphi - \lambda)^{-2} \|_1 = \| (\text{Id} + T^*(\varepsilon, \bar{\lambda}))^{-1}(\hat{\Delta}_0^\varphi - \lambda)^{-2}(\text{Id} + T(\varepsilon, \lambda))^{-1} \|_1 \leq C$$

for all $\lambda \in C$ and $|\varepsilon| < \delta \ll 1$. Introduce the spectral zeta function 

$$\zeta_c(s, \varepsilon) = \frac{1}{2\pi i} \int_C \lambda^{1-s} \text{Tr}(\hat{\Delta}_s^\varphi - \lambda)^{-2} d\lambda,$$

where $C$ is a contour running clockwise at a sufficiently close distance around the cut $(-\infty, 0]$ and $\lambda^s = |\lambda|^s e^{i 2\pi \arg \lambda}$ with $|\arg \lambda| \leq \pi$. Then (2.5) implies that $(s, \varepsilon) \mapsto \zeta_c(s, \varepsilon)$ is an analytic function of $s$ for $\Re s > 2$ and $\varepsilon$ for $|\varepsilon| < \delta \ll 1$. One of the ways to see analyticity in $\varepsilon$ is to make the substitution 

$$\text{Tr}(\hat{\Delta}_s^\varphi - \lambda)^{-2} = \frac{1}{2\pi i} \sum_{k \geq 0} \oint \frac{((\hat{\Delta}_s^\varphi - \lambda)^{-2}\psi_k, \psi_k)}{\mu - \varepsilon} d\mu,$$

where $\psi_k$ is an orthonormal basis in $L^2(|x| \leq 1, |x|^{2\beta}|dx|^2)$ and 

$$\sum |((\hat{\Delta}_s^\varphi - \lambda)^{-2}\psi_k, \psi_k)| \leq \| (\hat{\Delta}_s^\varphi - \lambda)^{-2} \|_1 \leq C$$

because of (2.5). After the substitution one can change the order of integration and summation to obtain the Cauchy’s integral formula for $\varepsilon \mapsto \zeta_c(s, \varepsilon)$.

In the remaining part of this proof we show that $(s, \varepsilon) \mapsto \zeta_c(s, \varepsilon)$ continues analytically to $(0, 0)$. Then thanks to $\zeta_c(s, 0) = \zeta_c(s)$ we conclude that 

$$\zeta_c(0, \varepsilon) = \zeta_c(0) + O(\varepsilon), \quad \zeta_c'(0, \varepsilon) = \zeta_c'(0) + O(\varepsilon),$$

(2.6)
where the prime stands for the derivative with respect to $s$. The standard rescaling argument guarantees that multiplication of a metric by $R^2$ adds $\zeta(0) \log R^2$ to the corresponding value of $\zeta'(0)$; see Sec. 1.2. Since $\tilde{\Delta}_\varepsilon^\varphi = \frac{1}{4} \varepsilon^2 e^{2\phi(0)} \Delta_\varepsilon^\varphi$, the rescaling argument and (2.6) lead to
\[
\log \det \Delta_\varepsilon^\varphi = 2 \left( \log(2\varepsilon^{-1}) - \phi(0) \right) \zeta_\varepsilon(0) - \zeta'_\varepsilon(0) + O(-\varepsilon \log \varepsilon) \quad \text{as } \varepsilon \to 0+.
\]
Taking into account the equality $\varepsilon = \varepsilon^{\beta+1}$ we arrive at (2.4).

It suffices to show that $s \mapsto \zeta_\varepsilon(s,\varepsilon)$ continues analytically from $\Re s > 2$ to $s = 0$ for each $\varepsilon, |\varepsilon| < \delta < 1$. We will rely on the representation
\[
\zeta_\varepsilon(s,\varepsilon) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\Delta_\varepsilon^\varphi}) \, dt \tag{2.7}
\]
with short time heat trace asymptotics $[4, 12]$. (For the large values of $t$ the estimate $|\text{Tr}(e^{-t\Delta_\varepsilon^\varphi})| = O(e^{-ct})$ with some $c > 0$ immediately follows from
\[
e^{-t\Delta_\varepsilon^\varphi} = \frac{i}{2\pi t} \int_{\mathfrak{C}} e^{-\lambda t} (\hat{\Delta}_\varepsilon^\varphi - \lambda)^{-2} d\lambda
\]
with a suitable contour $\mathfrak{C}$ in the right half-plane $\Re \lambda > 0$ and (2.5).)

In the polar coordinates $(r, \theta) = ((\beta + 1)|x|^\beta+1, \arg x)$ the operator $\hat{\Delta}_\varepsilon^\varphi$ takes the form
\[
\hat{\Delta}_\varepsilon^\varphi = -e^{2(\phi(0) - \phi(\varepsilon r, \theta))} (2r)^{-2} \left( r \partial_r^2 + (\beta + 1)^{-2} \partial_\theta^2 \right),
\]
where the function $(r, \theta) \mapsto \phi(\varepsilon r, \theta) = \phi(\varepsilon, x)$ is smooth up to $r = 0$ due to dilation analyticity of $m_\varphi$. Therefore $\hat{\Delta}_\varepsilon^\varphi$ falls into the class of elliptic cone differential operators with stationary domains studied in [12]; we recall that the domain of $\hat{\Delta}_\varepsilon^\varphi$ coincides with the domain of the Friederichs extension $\hat{\Delta}_0^\varphi$ and the domains of Friederichs extensions are always stationary. Let $\chi(r, \theta) = \chi(r)$ with a cutoff function $\chi \in C_\infty^\infty((0, \frac{1}{\varepsilon^{\beta+1}}))$ that equals 1 in a neighborhood of $r = 0$. A direct application of the main result in [12] implies that the short time asymptotics of the heat trace $\text{Tr}(\chi e^{-t\hat{\Delta}_\varepsilon^\varphi})$ has the form
\[
c_{-1} t^{-1} + c_{-1/2} t^{-1/2} + c_0 + c_{01} \log t + \sum_{k=1}^{\infty} \sum_{\ell=0}^{m_k} c_{k\ell} t^{\frac{\varepsilon}{2}} \log^{\ell} t \quad \text{as } t \to 0+,
\tag{2.8}
\]
where the coefficients $c_k$ and $c_{k\ell}$ depend on $\varepsilon$. There are no conical singularities on the support of $(1 - \chi)$ and hence the short time asymptotic expansion
\[
\text{Tr}((1 - \chi) e^{-t\hat{\Delta}_\varepsilon^\varphi}) \sim \sum_{j > -2} C_j(\varepsilon) t^{j/2} \quad \text{as } t \to 0+.
\]

---

$^1$Proof of Thm. 4.4 in [12] requires some corrections [13]: it should be $J = N + n + 1$ (instead of $J = N + 1$) on both places where the choice is relevant, the statement $a(y, \lambda) = 0$ for $k < n$ on page 6511 is incorrect, on the same page the estimate $\epsilon_{x,N+n}(y, \lambda) = O(|\lambda|^{-N/m-t} \log |\lambda|)$ (that follows from the last equality on page 6510) is needed in addition to (4.11) and (4.12). I would like to thank Juan B. Gil and Thomas Krainer for responding promptly to my inquiries about the proof and for sending me the corrected version of the paper.
can be obtained in the standard well-known way, e.g. [14, 33]. In total we have
\[
\text{Tr}(e^{-t\Delta^s}) = a_{-1}(\varepsilon)t^{-1} + a_{-1/2}(\varepsilon)t^{-\frac{1}{2}} + a_0^0(\varepsilon) + a_0^1(\varepsilon)\log t + O(t^{1/2}\log^{m_1} t) \quad \text{as } t \to 0+
\]
with some coefficients \(a_{-1}, a_{-1/2}, a_0^0, \text{ and } a_0^1\). The representation (2.7) gives
\[
\zeta(s, \varepsilon) = \frac{1}{\Gamma(s)} \left( \frac{a_{-1}(\varepsilon)}{s-1} + \frac{a_{-1/2}(\varepsilon)}{s-1/2} + \frac{a_0^0(\varepsilon)}{s} - \frac{a_0^1(\varepsilon)}{s^2} + R(\varepsilon, s) \right), \quad (2.9)
\]
where \(R(\varepsilon, s)\) is analytic in \(s\) for \(\Re s > -1/2\); recall that \(1/\Gamma(s) = s + \gamma_s s^2 + O(s^3)\).

Thus \((s, \varepsilon) \mapsto s\zeta(s, \varepsilon)\) continues analytically from \(\Re s > 2, |\varepsilon| < \delta \ll 1\) to a neighbourhood of \((0, 0)\). Moreover, \(s\zeta(s, \varepsilon)\big|_{s=0} = -a_0^1(\varepsilon)\). But results in [4, 39] guarantee that \(a_0^1(\varepsilon) = 0\) (first for all \(\varepsilon \geq 0\), and then, by analyticity, for all \(\varepsilon\) with \(|\varepsilon| < \delta \ll 1\)).

Indeed, if \(\varepsilon \geq 0\), then \(4|x|^{2\beta}e^{2(\phi(x) - \phi(0))}|dx|^2\) is a conical metric on the disk \(|x| \leq 1\). By [39, Theorem 4.1] in a small neighborhood of \(x = 0\) there exist smooth local geodesic polar coordinates \((\rho, \theta)\) such that
\[
4|x|^{2\beta}e^{2(\phi(x) - \phi(0))}|dx|^2 = d\rho^2 + h^2(\rho, \theta)d\theta^2, \quad \theta \in [0, 2\pi(\beta + 1)),
\]
\[
\lim_{\rho \to 0} \frac{h(\rho, \theta)}{\rho} = 1, \quad h(0, \theta) = 1, \quad h_{\rho\rho}(0, \theta) = 0,
\]
where \(h_{\rho} = \partial_{\rho}h\) and \(h_{\rho\rho} = \partial_{\rho}^2h\). Let \(\chi(\rho, \theta) = \chi(\rho)\) be a smooth cutoff function supported in a small neighborhood of \(\rho = 0\) and such that \(\chi(\rho) = 1\) for all \(\rho\) sufficiently close to 0. Then \(\chi\Delta^s\chi\) can be considered as the operator \(\chi h^{-1/2}(-\partial^2_{\rho} + \rho^{-2}A(\rho))h^{1/2}\) in the space \(L^2(h(\rho, \theta) d\rho d\theta)\), where
\[
\rho \mapsto A(\rho) = -\rho^2 \left( \frac{h^2}{4h^2} - \frac{h_{\rho\rho}}{2h} + h^{1/2} \left( \frac{1}{h} \partial_{\theta} \right)^2 h^{-1/2} \right)
\]
is a smooth family of operators on the circle \(\mathbb{R}/2\pi(\beta + 1)\mathbb{Z}\). As a consequence, by [4, Theorem 5.2 and Theorem 7.1] we have
\[
\text{Tr} \chi e^{-t\Delta^s} \sim \sum_{j=0}^{\infty} A_j t^{\frac{j-\alpha_j}{2}} + \sum_{j=0}^{\infty} B_j t^{-\frac{j+\alpha_j}{2}} + \sum_{j: \alpha_j \in \mathbb{Z}_-} C_j t^{-\frac{j+\alpha_j}{2}} \log t \quad \text{as } t \to 0+ \quad (2.10)
\]
with some coefficients \(A_j, B_j, \text{ and } C_j, \) and an infinite sequence of numbers \(\{\alpha_j\} \) with \(\Re \alpha_j \to -\infty\). The coefficient \(C_j\) before \(t^0\log t\) is given by \(\frac{1}{3} \text{Res} \zeta(-1), \) where \(\zeta\) is the spectral zeta function of \((A(0) + 1/4)^{1/2}; \) see [4, f-la (7.24)]. Since \(A(0) = -\partial^2_{\rho} - 1/4, \) we obtain
\[
\zeta(s) = 2 \sum_{j \geq 1} (j/(2\beta + 2))^{-s} = 2(2\beta + 2)^{s}\zeta(s).
\]
Thus \(\text{Res} \zeta(-1) = 0\) and the coefficient \(C_j\) before \(t^0\log t\) is zero. This together with \(\text{Tr} (1 - \chi) e^{-t\Delta^s} \sim \sum_{j \geq -2} C_j t^{j/2}\) implies that the coefficient \(a_0^1(\varepsilon)\) in (2.9) is zero. Hence \(s \mapsto \zeta(s, \varepsilon)\) continues analytically from \(\Re s > 2\) to \(s = 0\) for each \(\varepsilon, |\varepsilon| < \delta \ll 1\). This completes the proof. \(\square\)
Lemma 2.3. Let $\Delta^\psi_{D_\epsilon}$ be the selfadjoint Dirichlet Laplacian on the metric disk $(D_\epsilon, e^{2\psi}|dx|^2)$, where $\psi$ is smooth. Then

$$\log \det \Delta^\psi_{D_\epsilon} = \frac{1}{3} \left( \log(2\epsilon^{-1}) - \psi(0) \right) - \zeta'_<(0,1) + O(\epsilon) \ 	ext{as} \ \epsilon \to 0+ \quad (2.11)$$

with $\zeta'_<(0,\beta)$ given in (2.3).

**Proof.** For the selfadjoint Dirichlet Laplacian $\Delta^0_{D_\epsilon} = -4\partial_x\partial_x$ in the disk $|x| \leq \epsilon$ we have

$$\log \det \Delta^0_{D_\epsilon} = -\frac{1}{3} \log \epsilon + \frac{1}{3} \log 2 - \zeta'_<(0,1); \quad (2.12)$$

see (2.2) and (2.3) with $\beta = 0$ or [42, f-la (28)]. Since $\psi$ is smooth, we can use the classical Polyakov-Alvarez f-la [1, 29, 42], which gives

$$\log \det \Delta^\psi_{D_\epsilon} = \frac{1}{6\pi} \left( \frac{1}{2} \int_{|x|\leq \epsilon} |\nabla_0 \psi|^2 dA_0 + \int_{|x|\epsilon} k_0 \psi ds_0 \right) - \frac{1}{4\pi} \int_{|x|\epsilon} \partial_n \psi ds_0.$$ 

Here $\nabla_0$ is the gradient, $k_0 = 1/\epsilon$ is the geodesic curvature of the circle $|x| = \epsilon$, and $n$ is the outward unit normal to the disk $|x| \leq \epsilon$ (all with respect to the metric $|dx|^2$). Therefore

$$\log \det \Delta^\psi_{D_\epsilon} = -\frac{1}{6\pi} \left( O(\epsilon) + \int_0^{2\pi} \left( \psi(0) + O(\epsilon) \right) d\theta \right) - O(\epsilon) = -\frac{1}{3} \psi(0) + O(\epsilon).$$

This together with (2.12) completes the proof. \(\square\)

2.2 BFK decomposition formulas

By $D^j_\epsilon \subset M$ we denote the $\epsilon$-neighborhood of conical point $P^j \in \text{supp}\beta$ of $m_\varphi$ such that $P \in D^j_\epsilon$ if and only if $|x(P)| \leq \epsilon$, where $x(P^j) = 0$ and $x$ is a local holomorphic parameter in which the metric $m_\varphi = |x|^{2\beta} e^{2\varphi |dx|^2}$ is dilation analytic. For sufficiently small $\epsilon > 0$ the disks $D^1_\epsilon, \ldots, D^N_\epsilon$ are disjoint and do not touch the boundary $\partial M$ of $M$. Let $M_\epsilon = M \setminus \{D^1_\epsilon \cup \cdots \cup D^N_\epsilon\}$ and let $\partial M_\epsilon$ stand for the boundary of $M_\epsilon$.

In this section we prove BFK-type decomposition formulas for $\Delta^\varphi$ along the boundary $\partial M_\epsilon \setminus \partial M$ (Proposition 2.7 below). This is an analog of the BFK decomposition formula in [5, Theorem B'] if $\partial M = \emptyset$ and of the one in [25, Corollary 1.3] if $\partial M \neq \emptyset$. As is known, for the conical metrics that are flat near the conical points the BFK decomposition formulas and their proofs remain valid provided that one considers the Friederichs extensions of the Laplacians and the decomposition is done along a smooth closed curve that does not contain any singularity of the metric; see e.g. [15, 16, 21, 22] and [26] for a more general result. In our case the decomposition formulas are still valid but their proof requires some minor modifications due to appearance of logarithmic terms in the short time heat trace asymptotics.

As before, let $\Delta^\varphi$ stand for the Friederichs extension of the Laplacian on $(M, m_\varphi)$ ($\Delta^\varphi$ is the Dirichlet Laplacian if $\partial M \neq \emptyset$). Consider also the Friederichs extension $\Delta^\varphi_{M_\epsilon}$ of the Laplacian on $(M, m_\varphi)$ with Dirichlet boundary condition on $\partial M_\epsilon$; more precisely, $\Delta^\varphi_{M_\epsilon}$ is the Friederichs extension in $L_2(M, m_\varphi)$ of the operator $\Delta^\varphi$ defined on the functions $u \in C_0^\infty(M \setminus \text{supp}\beta)$ satisfying $u|_{\partial M_\epsilon} = 0$.  

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Lemma 2.4. The heat traces $\text{Tr}(e^{-t\Delta^\varphi})$ and $\text{Tr}(e^{-t\Delta^\varphi_{Mc}})$ have short time asymptotic expansions of the form

$$a_{-1}t^{-1} + a_{-1/2}t^{-1/2} + a_0 + \sum_{k=1}^{\infty} \sum_{l=0}^{m_k} a_{kl} t^{k/2} \log^l t \quad \text{as } t \to 0^+,$$  

where $a_k$ and $a_{kl}$ are some coefficients. If, in addition, the metric $m_\varphi$ is flat in a neighborhood of $\text{supp} \beta$, then there are no logarithmic terms in the asymptotic expansions (i.e. $m_k = 0$ or, equivalently, $a_{k1} = 0$ for all $k = 1, 2, 3, \ldots$).

Remark 2.5. As is known, the constant term $a_0$ in the short time asymptotic expansion (2.13) of the heat trace $\text{Tr}(e^{-t\Delta^\varphi})$ is related to the value of the spectral zeta function of $\Delta^\varphi$ at zero by $a_0 = \zeta(0) + \dim \ker \Delta^\varphi$. As a consequence of Corollary 1.2, we thus obtain

$$a_0 = \frac{\chi(M, \beta)}{6} - \frac{1}{12} \sum_{j=1}^{n} \left( \beta_j + 1 - \frac{1}{\beta_j + 1} \right).$$

Proof of Lemma 2.4. Introduce the local polar coordinates

$$(r, \theta) = \left( (\beta_j + 1)^{-1} |x|^{\beta_j + 1}, \arg x \right)$$

centred at a conical point $P^j \in \text{supp} \beta$; here $x(P^j) = 0$ and $x$ is a local holomorphic parameter in which the metric $m_\varphi = |x|^{2\beta_j} e^{2\phi_j} |dx|^2$ is dilation analytic. In these coordinates the Laplacian $\Delta^\varphi$ takes the form

$$\Delta^\varphi = -e^{\varphi_j(r, \theta)} r^{-2} \left( (r \partial_r)^2 + (\beta_j + 1)^{-2} \partial_{\theta_j}^2 \right)$$

and thus falls into the class of elliptic cone operators with stationary domains studied in [12]: 1) The potential $(r, \theta) \mapsto \phi_j(r, \theta) = \phi_j(x)$ is smooth up to $r = 0$ due to dilation analyticity of the metric $m_\varphi$; 2) The domain of $\Delta^\varphi$ is stationary because $\Delta^\varphi$ is the Friederichs selfadjoint extension in $L^2(M, m_\varphi)$.

Let $\chi_j(r, \theta) = \chi_j(r)$ with a cutoff function $\chi_j \in C^\infty_c \left( [0, \frac{\beta_j + 1}{\beta_j}] \right)$ that equals 1 in a neighborhood of $r = 0$; we extend $\chi_j$ from $D_j^j$ to $M$ by zero. Then a direct application of [12, Theorem 1.1] implies that the heat trace $\text{Tr} \left( \chi_j e^{-t\Delta^\varphi} \right)$ has a short time asymptotic expansion of the form (2.8). Moreover, if the potential $\phi_j$ does not depend on $r$ for all sufficiently small values of $r$, then in the expansion (2.8) we have $m_k = 0$ for all values of $k$ (note that for a flat near $P^j$ metric $m_\varphi$ we can always archive $\phi_j(r, \theta) = 0$ by taking a suitable local holomorphic parameter $x$, e.g. [39, Lemma 3.4]). This together with the standard well know expansion

$$\text{Tr} \left( (1 - \sum_j \chi_j) e^{-t\Delta^\varphi} \right) \sim \sum_{k \geq -2} C_k t^{k/2} \quad \text{as } t \to 0^+$$

implies (2.13) with extra term $a_0^1 \log t$. Relying on [4, 39] and using the same argument as in the proof of Lemma 2.2, one can verify that $a_0^1 = 0$, we omit the details. This proof can also be repeated verbatim with $\Delta^\varphi$ replaced by $\Delta^\varphi_{\partial Mc}$. $$\square$$
For $\lambda > 0$ the operator $\Delta^\varphi + \lambda$ is positive and hence $e^{-t\lambda} \text{Tr}(e^{-t\Delta^\varphi}) = O(e^{-t\lambda})$ as $t \to +\infty$. Based on this and Lemma 2.4 we conclude that the spectral zeta function

$$\zeta(s, \lambda) = \text{Tr}(\Delta^\varphi + \lambda)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t\lambda} \text{Tr}(e^{-t\Delta^\varphi}) \, dt$$

(2.14)

is holomorphic in $s$ for $\Re s > 1$ and admits an analytic continuation to $s = 0$ given by the right hand side of (2.14). Therefore we can set $\det(\Delta^\varphi + \lambda) = e^{-\partial_s \zeta(0, \lambda)}$. Similarly we define $\det(\Delta^\varphi_{\partial M_\epsilon} + \lambda)$.

Now we are in position to introduce the Neumann jump operator on $\partial M_\epsilon \setminus \partial M$. For $\lambda > 0$ and any $f \in C^\infty(\partial M_\epsilon \setminus \partial M)$ there exists a unique solution to the Dirichlet problem

$$(\Delta^\varphi + \lambda) u(\lambda) = 0 \text{ on } M \setminus \partial M_\epsilon, \quad u(\lambda) = f \text{ on } \partial M_\epsilon \setminus \partial M, \quad u(\lambda) = 0 \text{ on } \partial M,$$

(2.15)

such that

$$u(\lambda) = \hat{f} - (\Delta^\varphi_{\partial M_\epsilon} + \lambda)^{-1}(\Delta^\varphi + \lambda)\hat{f},$$

where $\hat{f} \in C^\infty(M \setminus \text{supp } \beta)$ is an extension of $f$. Introduce the Neumann jump operator $R^\varphi_\epsilon(\lambda) : C^\infty(\partial M_\epsilon \setminus \partial M) \to C^\infty(\partial M_\epsilon \setminus \partial M)$ that acts by the formula

$$R^\varphi_\epsilon(\lambda)f = \partial_{\bar{n}}(u(\lambda)|_{M \setminus M_\epsilon}) - \partial_{\bar{n}}(u(\lambda)|_{M_\epsilon}),$$

where $\bar{n}$ is the outward (for $M_\epsilon$) unit normal to $\partial M_\epsilon \setminus \partial M$ with respect to $m_\varphi$; note that there are no conical singularities of $m_\varphi$ on $\partial M_\epsilon$ since $\epsilon > 0$ is sufficiently small. The operator $R^\varphi_\epsilon(\lambda)$ is an invertible first order elliptic classical pseudodifferential operator on $\partial M_\epsilon \setminus \partial M$. In particular, on each component $\partial D_\ell^j$ of $\partial M_\epsilon \setminus \partial M$ the principal symbol of $R^\varphi_\epsilon(\lambda)$ is given by $\sigma(x, \xi) = 2e^{i\beta_j(x)}|\xi|$, which can be easily seen from the representation

$$R^\varphi_\epsilon(\lambda)^{-1} = ((\Delta^\varphi + \lambda)^{-1}(\cdot \otimes \delta_{\partial M_\epsilon \setminus \partial M})|_{\partial M_\epsilon \setminus \partial M}$$

(2.16)

where $\delta_{\partial M_\epsilon \setminus \partial M}$ is the Dirac $\delta$-function along $\partial M_\epsilon \setminus \partial M$, the action of the resolvent is understood in the sense of distributions, and $\Delta^\varphi = -|x|^{2}\beta_j e^{-2\beta_j(x)}\partial_x \partial_\beta$ in the local parameter $x$ centred at $P^j$; cf. [6, Thm 2.1] and [5, Sec. 4.4]. As a consequence, for $s \in \mathbb{C}, \Re s > 1$, the operator $R^\varphi_\epsilon(\lambda)^{-s}$ in $L^2(\partial M_\epsilon \setminus \partial M)$ is trace class and its zeta function $s \mapsto \zeta(s, \lambda) = \text{Tr} R^\varphi_\epsilon(\lambda)^{-s}$ is holomorphic. Moreover, $s \mapsto \zeta(s, \lambda)$ admits a meromorphic continuation from the half-plane $\Re s > 1$ to $\mathbb{C}$ with no pole at $s = 0$; see e.g. [34]. We set

$$\det R^\varphi_\epsilon(\lambda) = e^{-\partial_s \zeta(0, \lambda)}.$$

Lemma 2.6. The formula

$$\det(\Delta^\varphi + \lambda) = C \det(\Delta^\varphi_{\partial M_\epsilon} + \lambda) \det R^\varphi_\epsilon(\lambda)$$

is valid, where $C$ is independent of $\lambda > 0$.

Proof. The assertion is an analogue of [5, Theorem A].

The relation (2.16) also implies that $\lambda \mapsto R^\varphi_\epsilon(\lambda)$ is an analytic family of pseudodifferential operators and the order of $\partial_\lambda R^\varphi_\epsilon(\lambda)^{-1}$ is $-1 - 2\ell$. Thus the order of $\partial_\lambda R^\varphi_\epsilon(\lambda)^{-1}$ is
1−2ℓ and \( [\partial_\lambda R_\xi^\varphi(\lambda)] R_\xi^\varphi(\lambda)^{-1} \) is a trace class operator in \( L^2(\partial M_\epsilon \setminus \partial M) \). As a consequence we have
\[
\partial_\lambda \log \det R_\xi^\varphi(\lambda) = \operatorname{Tr} \left( [\partial_\lambda R_\xi^\varphi(\lambda)] R_\xi^\varphi(\lambda)^{-1} \right),
\]
see [9, Prop. 1.1]. By writing the Schwartz kernel of \((\partial_\lambda R_\xi^\varphi(\lambda)) R_\xi^\varphi(\lambda)^{-1}\) in terms of those of \((\Delta^\varphi + \lambda)^{-1}\) and \((\Delta_{\partial M_\epsilon}^\varphi + \lambda)^{-1}\) it is not hard to verify that
\[
\operatorname{Tr} \left( [\partial_\lambda R_\xi^\varphi(\lambda)] R_\xi^\varphi(\lambda)^{-1} \right) = \operatorname{Tr} \left( (\Delta_{\partial M_\epsilon}^\varphi + \lambda)^{-1} - (\Delta^\varphi + \lambda)^{-1} \right);
\]
the corresponding calculation can be found in [6, Proof of Thm 2.2], we omit the details.

It remains to show that
\[
\partial_\lambda \left[ \log \det(\Delta^\varphi + \lambda) - \log \det(\Delta_{\partial M_\epsilon}^\varphi + \lambda) \right] = \operatorname{Tr} \left( (\Delta_{\partial M_\epsilon}^\varphi + \lambda)^{-1} - (\Delta^\varphi + \lambda)^{-1} \right);
\]
here we closely follow [6, Proof of Thm 4.2]. By the Krein theorem (see e.g. [44, Ch. 8.9]) there exists a spectral shift function \( \xi \in L^1(\mathbb{R}_+, (1 + \mu)^{-2} \, d\mu) \) such that
\[
\operatorname{Tr} \left( (\Delta_{\partial M_\epsilon}^\varphi + \lambda)^{-s} - (\Delta^\varphi + \lambda)^{-s} \right) = \int_0^\infty \xi(\mu) \frac{s \, d\mu}{(\mu + \lambda)^{s+1}},
\]
where \( \Re s > 1 \) or \( s = 1 \). In fact, in our case the spectrum of selfadjoint operator \( \Delta^\varphi \) (resp. \( \Delta_{\partial M_\epsilon}^\varphi \)) in \( L^2(\mathbb{M}, m_\varphi(\cdot)) \) consists of isolated eigenvalues \( 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) and hence \( \xi(\mu) = N_{\Delta^\varphi}(\mu) - N_{\Delta_{\partial M_\epsilon}^\varphi}(\mu) \), where \( N_{\Delta^\varphi}(\mu) = \# \{ k : \lambda_k < \mu, \lambda_k \in \sigma(\Delta^\varphi) \} \) is the spectral counting function of \( \Delta^\varphi \) and \( N_{\Delta_{\partial M_\epsilon}^\varphi}(\mu) \) is the spectral counting function of \( \Delta_{\partial M_\epsilon}^\varphi \). We have
\[
\partial_\lambda \left[ \log \det(\Delta^\varphi + \lambda) - \log \det(\Delta_{\partial M_\epsilon}^\varphi + \lambda) \right] = -\partial_\lambda \left[ \partial_s \int_0^\infty \xi(\mu) \frac{s \, d\mu}{(\mu + \lambda)^{s+1}} \right]_{s=0}
\]
\[
= \int_0^\infty \xi(\mu) \frac{d\mu}{(\mu + \lambda)^2} = \operatorname{Tr} \left( (\Delta_{\partial M_\epsilon}^\varphi + \lambda)^{-1} - (\Delta^\varphi + \lambda)^{-1} \right).
\]
This proves (2.19). Now the assertion of lemma follows from (2.17), (2.18), and (2.19).

In the same way as before we define the Neumann jump operator \( R_\xi^\varphi(\lambda) \) for \( \lambda = 0 \) and denote \( R_\xi^\varphi = R_\xi^\varphi(0) \). If \( \partial M \neq \emptyset \), then the operators \( \Delta^\varphi \) and \( R_\xi^\varphi \) are still invertible and we define \( \det \Delta^\varphi \) and \( \det R_\xi^\varphi \) by setting \( \lambda = 0 \) in the definitions for \( \det(\Delta^\varphi + \lambda) \) and \( \det R_\xi^\varphi(\lambda) \). If \( \partial M = \emptyset \), then both \( \Delta^\varphi \) and \( R_\xi^\varphi \) have zero as a simple eigenvalue (and the corresponding kernels consist of constant functions on \( \mathbb{M} \) and \( \partial M_\epsilon \) respectively). In this case we introduce the modified determinant (i.e. with zero eigenvalue excluded). Namely, we set \( \det \Delta^\varphi = e^{-\partial_\lambda \zeta^* (\lambda)} \), where for \( \zeta^*(s) \) we may write
\[
\zeta^*(s) = \sum_{k: 0 < \lambda_k \in \sigma(\Delta^\varphi)} \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \operatorname{Tr} (e^{-t \Delta^\varphi}) - 1 \right) dt;
\]
similarly, \( \det R_\xi^\varphi \) is defined via \( \zeta^*(s) = \operatorname{Tr}(R_\xi^\varphi P^\perp)^{-s} \), where \( P^\perp \) is the orthogonal projection onto \( (\ker R_\xi^\varphi)^\perp \) in \( L^2(\partial M_\epsilon, m_\varphi) \).
Proposition 2.7 (BFK formulas). 1. If $\partial M = \emptyset$, then
\[
\det \Delta^\varphi = A_\varphi \det \Delta^\varphi_{\partial M_\epsilon} \frac{\det R^\varphi_\epsilon}{L_\varphi(\partial M_\epsilon)},
\]  
(2.20)

where $A_\varphi$ is the total area of $M$ and $L_\varphi(\partial M_\epsilon)$ is the length of the boundary $\partial M_\epsilon$ in the metric $m_\varphi$.

2. If $\partial M \neq \emptyset$ and $\partial M \cap \text{supp } \beta = \emptyset$, then
\[
\det \Delta^\varphi = \det \Delta^\varphi_{\partial M_\epsilon} \det R^\varphi_\epsilon.
\]
(2.21)

The proof is preceded by Lemma 2.8 (After L. Friedlander & A. Voros). The functions $\lambda \mapsto \log \det(\Delta^\varphi + \lambda)$ and $\lambda \mapsto \log \det(\Delta^\varphi_{\partial M_\epsilon} + \lambda)$ admit asymptotic expansions with zero constant terms as $\lambda \to +\infty$.

Proof. If the metric $m_\varphi$ representing the divisor $\beta$ is flat in a neighborhood of $\text{supp } \beta$, then there are no logarithms in the asymptotic expansion (2.13) and the assertion is due to Friedlander & Voros [11, 41]. Here we adapt the Voros’ argument, cf. [24, Prop. 2.7].

Consider, for instance, the spectral zeta function $\zeta(s, \lambda) = \text{Tr}(\Delta^\varphi + \lambda)^{-s}$, which is well defined for $\Re s > 1$ and $\Re \lambda > 0$. Let $\eta(s, \lambda) = \zeta(s, \lambda)\Gamma(s)$, then
\[
\eta(s, \lambda) = \int_0^\infty t^{s-1} e^{-\lambda t} \text{Tr} e^{-t\Delta^\varphi} dt.
\]
(2.22)

For $\Re s > 1$, $\eta(s, \lambda)$ can be expanded in $\lambda$ as $|\lambda| \to \infty$ by formally substituting the asymptotic expansion (2.13) into (2.22). After the change of variable $t \mapsto t/\lambda$ we get
\[
\eta(s, \lambda) \sim a_{-1} \lambda^{1-s} \int_0^\infty t^{s-2} e^{-t} dt + a_{-1/2} \lambda^{\frac{1}{2}-s} \int_0^\infty t^{s-rac{3}{2}} e^{-t} dt + a_0 \lambda^{-s} \int_0^\infty t^{s-1} e^{-t} dt
\]
\[
+ \sum_{j=1}^\infty \sum_{k=0}^{m_j} \sum_{\ell=0}^{k} a_{jk} \lambda^{-s-j} \binom{k}{\ell} (-\log \lambda)^{\ell} \int_0^\infty t^{\frac{s}{2}+j-\ell} e^{-t} \log^{k-\ell} t dt
\]
\[
= a_{-1} \lambda^{1-s} \Gamma(s-1) + a_{-1/2} \lambda^{\frac{1}{2}-s} \Gamma \left( s - \frac{1}{2} \right) + a_0 \lambda^{-s} \Gamma(s)
\]
\[
+ \sum_{j=1}^\infty \sum_{k=0}^{m_j} \sum_{\ell=0}^{k} a_{jk} \lambda^{-s-j} \binom{k}{\ell} (-\log \lambda)^{\ell} \Gamma(k-\ell) \left( s + \frac{j}{2} \right).
\]

Thus
\[
\zeta(s, \lambda) \sim \frac{\lambda^{-s}}{\Gamma(s)} \left( a_{-1} \lambda \Gamma(s-1) + a_{-1/2} \lambda^{\frac{1}{2}} \Gamma \left( s - \frac{1}{2} \right) + a_0 \Gamma(s) \right)
\]
\[
+ \sum_{j=1}^\infty \sum_{k=0}^{m_j} \sum_{\ell=0}^{k} a_{jk} \lambda^{-j} \binom{k}{j} (-\log \lambda)^{\ell} \Gamma(k-\ell) \left( s + \frac{j}{2} \right).
\]
All functions involved are meromorphic functions of \( s \). Moreover, \( s = 0 \) is a regular point of \( \zeta(s, \lambda) \) and thus \( \zeta'(0, \lambda) \) admits an asymptotic expansion in \( \lambda \) of the form

\[
\zeta'(0, \lambda) \sim a_{-1}(\log \lambda - 1) - 2a_{-1/2}\sqrt{\pi}\lambda^{3/2} - a_0 \log \lambda
\]

\[
+ \sum_{j=1}^{\infty} \sum_{k=0}^{m_j} \sum_{\ell=0}^{k} a_{jk\ell} \lambda^{-\ell} \binom{k}{\ell} (-\log \lambda)^\ell \Gamma(k-\ell) \left( \frac{j}{2} \right),
\]

where there is no constant term. \( \square \)

**Proof of Proposition 2.7.** Following \([5]\), we evaluate the constant \( C \) in Lemma 2.6 by considering the asymptotic expansion of all determinants involved in

\[
\log \det(\Delta^\varphi + \lambda) = \log C + \log \det(\Delta^\varphi_{\partial M_e} + \lambda) + \log \det R^\varphi_{\xi}(\lambda)
\]  

(2.23)

as \( \lambda \to +\infty \). It is known that the function \( \lambda \mapsto \log \det R^\varphi_{\xi}(\lambda) \) admits an asymptotic expansion with zero constant term; for the proof in the case \( \partial M = \emptyset \) we refer to \([5, \text{Sec. 4.7}]\), the case \( \partial M \neq \emptyset \) is studied in \([25, \text{Sec. 2 & 3}]\). For the other two determinants in (2.23) we have proved the same fact in Lemma 2.8. Thus we conclude that \( C = 1 \) and hence

\[
\det(\Delta^\varphi + \lambda) = \det(\Delta^\varphi_{\partial M_e} + \lambda)\det R^\varphi_{\xi}(\lambda), \quad \lambda > 0.
\]  

(2.24)

It remains to pass in (2.24) to the limit as \( \lambda \to 0+ \). In the case \( \partial M = \emptyset \) only \( \Delta^\varphi_{\partial M_e} \) is positive and thus \( \det(\Delta^\varphi_{\partial M_e} + \lambda) \to \det \Delta^\varphi_{\partial M_e} \) as \( \lambda \to 0+ \). From the definition of the modified determinant it immediately follows that

\[
\log \det(\Delta^\varphi + \lambda) = \log \lambda + \log \det \Delta^\varphi + o(1), \quad \text{as} \lambda \to 0+.
\]

Clearly, \( \Delta^\varphi A^1/2_{\varphi} = 0 \) and \( \|A^1/2_{\varphi}\|_{L^2(M_m, m_\varphi)} = 1 \); recall that \( A_{\varphi} \) stands for the total area of \( M \) in the metric \( m_\varphi \). Hence for any \( F \in L^2(M, m_\varphi) \) we have

\[
(\Delta^\varphi + \lambda)^{-1} F = \frac{1}{A_{\varphi} \lambda} (F, 1)_{L^2(M,m_\varphi)} + (\Delta^\varphi + \lambda)^{-1} \left( F - \frac{1}{A_{\varphi}} (F, 1)_{L^2(M,m_\varphi)} \right),
\]  

(2.25)

where the second term in the right hand side is holomorphic in \( \lambda, \ |\lambda| \ll 1 \). The relation (2.16) implies that for \( \lambda \geq 0 \) the operator \( R^\varphi_{\xi}(\lambda) \) in \( L^2(\partial M_e, m_\varphi) \) is selfadjoint and nonnegative, together with (2.25) it also gives

\[
R^\varphi_{\xi}(\lambda)^{-1} = \frac{1}{\lambda A_{\varphi}(\cdot, 1)_{L^2(\partial M_e, m_\varphi)}} + h(\lambda),
\]

where \( \|h(\lambda)\|_{B(L^2(\partial M_e, m_\varphi))} = O(1) \) as \( \lambda \to 0 \). Therefore, as \( \lambda \to 0+ \) the first eigenvalue \( \mu_0(\lambda) = 1/\|R^\varphi_{\xi}(\lambda)^{-1}\|_{B(L^2(\partial M_e, m_\varphi))} \) of \( R^\varphi_{\xi}(\lambda) \) goes to zero, while the others satisfy \( \mu_k(\lambda) \geq \delta \) with some \( \delta > 0 \). Finally, for the determinant of \( R^\varphi_{\xi}(\lambda) \) we obtain

\[
\log \det R^\varphi_{\xi}(\lambda) = \log \mu_0(\lambda) + \log \det R^\varphi_{\xi} + o(1)
\]

(2.20)

as \( \lambda \to 0+ \); here \( L_\varphi(\partial M_e) \) is the norm of the operator \( (\cdot, 1)_{L^2(\partial M_e, m_\varphi)} \) in the space of bounded operators acting in \( L^2(\partial M_e, m_\varphi) \). Thus passing in (2.24) to the limit we get (2.20).

In the case \( \partial M \neq \emptyset \) the operators \( \Delta^\varphi, \Delta^\varphi_{\partial M_e}, \text{ and } R^\varphi_{\xi} \) are positive and hence the determinants in (2.24) tend to the corresponding determinants in (2.21) as \( \lambda \to 0+ \). \( \square \
2.3 Proof of Theorem 1.1 and Corollary 1.3

Recall that $m_\varphi = e^{2\varphi}m_0$, where $m_\varphi$ is a conical dilation analytic metric representing a divisor $\beta$ and $m_0$ is a smooth conformal metric on $M$. In addition to the BFK decomposition formulas obtained in Proposition 2.7 we will also be using similar decomposition formulas for $\det \Delta^0$. The latter formulas can be formally obtained by setting $\varphi = 0$ in (2.20), (2.21), and the definitions for $\Delta^0_{\partial M}$, $\det R^\varphi_\epsilon$, and $L_\varphi(\partial M_\epsilon)$ in Section 2.2.

We only notice that the corresponding results are well known: since the metric $m_0$ is smooth, the formula (2.20) (resp. (2.21)) with $\varphi = 0$ is a particular case of [5, Theorem B*] (resp. [25, Corollary 1.3]).

**Lemma 2.9.** Let $\epsilon > 0$ be sufficiently small.

1. If $\partial M = \emptyset$, then $\frac{\det R^\varphi_\epsilon}{L_\varphi(\partial M_\epsilon)} = \frac{\det R^\varphi_0}{L_\varphi(\partial M_0)}$.

2. If $\partial M \neq \emptyset$ and $\partial M \cap \text{supp} \beta = \emptyset$, then $\det R^\varphi_\epsilon = \det R^\varphi_0$.

**Proof.** For all sufficiently small $\epsilon > 0$ the disks $D^1_\epsilon, \ldots, D^n_\epsilon$ are disjoint and do not touch the boundary $\partial M$. In each disk $D^j_\epsilon$ we replace $\varphi(x) = \beta_j \log |x| + \phi_j(x) - \psi_j(x)$ by the smooth potential

$$\tilde{\varphi}(x) = \chi(|x|/\epsilon) (\beta_j \log |x| + \phi_j(x)) - \psi_j(x),$$

where $\chi \in C^\infty(\mathbb{R}_+)$ is a cutoff function with properties: $\chi(|x|) = 0$ for $|x| \leq 1/3$ and $\chi(|x|) = 1$ for $|x| \geq 1/2$. We also set $\tilde{\varphi} = \varphi$ on $M_\epsilon$. As a result we obtain $\tilde{\varphi} \in C^\infty(M)$ such that $\varphi = \tilde{\varphi}$ on $M_{\epsilon/2} \supset M_\epsilon$. Hence $L_{\tilde{\varphi}}(\partial M_\epsilon) = L_\varphi(\partial M_\epsilon)$ and $R^\varphi_\epsilon = R^\tilde{\varphi}_\epsilon$ (recall that $\Delta^\varphi_{\partial M_\epsilon}$ is the Friederichs extension of the Dirichlet Laplacian and hence the solution $u = \check{f} - (\Delta^\varphi_{\partial M_\epsilon})^{-1}\Delta^\varphi \check{f}$ to (2.15) with $\lambda = 0$ is bounded and thus coincides with $\check{u} = \check{f} - (\Delta^\tilde{\varphi}_{\partial M_\epsilon})^{-1}\Delta^\tilde{\varphi} \check{f}$, where $\Delta^\tilde{\varphi}_{\partial M_\epsilon}$ is the selfadjoint Dirichlet Laplacian).

As is known [43], the invariance of $\frac{\det R^\varphi_0}{L_\varphi(\partial M_\epsilon)}$ in the case $\partial M = \emptyset$ (resp. of $\det R^\varphi_0$ in the case $\partial M \neq \emptyset$) under the conformal transformations $m_0 \mapsto e^{2\tilde{\varphi}}m_0$ with smooth $\tilde{\varphi}$ can be easily seen from the BFK formula (2.20) (resp. (2.21)) together with Polyakov/Polyakov-Alvarez f-las on $M, M_\epsilon$, and $D^j_\epsilon$.

**Proof of Theorem 1.1.** BFK f-las in Proposition 2.7 and Lemma 2.9 imply

$$\log \frac{(\det \Delta^\varphi/A^\varphi)}{(\det \Delta^0/A^0)} = \log \frac{\det \Delta^\varphi_{\partial M_\epsilon}}{\det \Delta^0_{\partial M_\epsilon}} \quad \text{if } \partial M = \emptyset, \quad (2.26)$$

$$\log \frac{\det \Delta^\varphi}{\det \Delta^0} = \log \frac{\det \Delta^\varphi_{\partial M_\epsilon}}{\det \Delta^0_{\partial M_\epsilon}} \quad \text{if } \partial M \neq \emptyset. \quad (2.27)$$

Note that $\Delta^\varphi_{\partial M_\epsilon}$ can be decomposed into the direct sum of operators:

$$\Delta^\varphi_{\partial M_\epsilon} = \Delta^\varphi_{M_\epsilon} \oplus_{j=1}^n \Delta^\varphi_{D^j_\epsilon}, \quad (2.27)$$

where $\Delta^\varphi_{M_\epsilon}$ is the selfadjoint Dirichlet Laplacian on $(M_\epsilon, m_\varphi)$ and $\Delta^\varphi_{D^j_\epsilon}$ is the Friederichs extension of the Dirichlet Laplacian on the metric disk $(D^j_\epsilon, |x|^{2\beta_j} e^{2\tilde{\varphi}} |dx|^2)$ studied in Section 2.1. As a consequence we have

$$\det \Delta^\varphi_{\partial M_\epsilon} = \det \Delta^\varphi_{M_\epsilon} \prod_{j=1}^n \det \Delta^\varphi_{D^j_\epsilon}. \quad (2.28)$$
Similarly we decompose $\Delta^0_{\partial M_{\epsilon}}$ into the corresponding direct sum and obtain (2.27) and (2.28) with $\varphi$ replaced by 0. This together with Lemma 2.2, f-la (2.2) for $\zeta_{<}(0, \beta)$, and Lemma 2.3 implies

$$\log \frac{\det \Delta^\varphi_{\partial M_{\epsilon}}}{\det \Delta^0_{\partial M_{\epsilon}}} = \log \frac{\det \Delta^\varphi_{M_{\epsilon}}}{\det \Delta^0_{M_{\epsilon}}} - \sum_{j=1}^{n} \left( \frac{1}{6} (\beta_j^2 + 2\beta_j) \log \epsilon + 2\phi_j(0) \zeta_{<}(0, \beta_j) \right)$$

$$- \frac{1}{3} \psi_j(0) + C(\beta_j) + \frac{\beta_j}{2} + o(1), \quad \epsilon \to 0+, \quad (2.29)$$

where we introduced the notation

$$C(\beta) = \zeta_{<}'(0, \beta) - \left( 2\zeta_{<}(0, \beta) - \frac{1}{3} \right) \log 2 - \zeta_{<}'(0, 1) - \beta \frac{\beta}{2}.$$

The formula (1.2) for $C(\beta)$ now follows from (2.2), (2.3), and Lemma A.1 in Appendix A.

The classical Polyakov-Alvarez f-la on $M_{\epsilon}$ reads

$$\log \frac{\det \Delta^\varphi_{M_{\epsilon}}}{\det \Delta^0_{M_{\epsilon}}} = -\frac{1}{6\pi} \left( \frac{1}{2} \int_{M_{\epsilon}} (|\nabla_0^\varphi|^2 + 2K^0_0 \varphi) \, dA_0 + \int_{\partial M_{\epsilon}} k_0 \varphi \, ds_0 \right)$$

$$- \frac{1}{4\pi} \int_{\partial M_{\epsilon}} \partial n \varphi \, ds_0; \quad (2.30)$$

see e.g. [1, 29, 42]. Let us rewrite the right hand side of (2.30) in the form

$$-\frac{1}{6\pi} \left( \frac{1}{2} \int_{M_{\epsilon}} (\varphi \Delta^0 \varphi + 2K^0_0 \varphi) \, dA_0 + \frac{1}{2} \int_{\partial M_{\epsilon}} \varphi \partial n \varphi \, ds_0 + \int_{\partial M_{\epsilon}} k_0 \varphi \, ds_0 \right)$$

$$- \frac{1}{4\pi} \int_{\partial M_{\epsilon}} \partial n \varphi \, ds_0. \quad (2.31)$$

Let $x$ be the same local holomorphic parameter centred at $P_j$ as in Sec. 2.1. We have $m_\varphi = |x|^{2\beta} e^{2\phi_j}|dx|^2$, $m_0 = e^{2\psi_j}|dx|^2$, and

$$\varphi(x) = \beta_j \log |x| + \phi_j(x) - \psi_j(x).$$

Taking into account the equalities

$$k_0 = -e^{-\psi}(e^{-1} + \partial_x |\psi_j|), \quad z = e^{i\theta}, \quad ds_0 = e^{i\theta} \, d\theta,$$

on $\partial D^j_{\epsilon}$, for the integrals along the $j$-th component $\partial D^j_{\epsilon}$ of the boundary $\partial M_{\epsilon} \setminus \partial M$ in (2.31) we get

$$\int_{\partial D^j_{\epsilon}} \partial n \varphi \, ds_0 = -\int_0^{2\pi} e^{-\psi_j(x)} \partial_x \left( \beta_j \log |x| + \phi_j(x) - \psi_j(x) \right) \bigg|_{x=e^{i\theta}} \, \epsilon \, d\theta$$

$$= -\int_0^{2\pi} \left( \frac{\beta_j}{\epsilon} + O(1) \right) \epsilon \, d\theta = -2\pi \beta_j + O(\epsilon),$$

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\[
\int_{\partial D_L^1} k_0 \varphi \, ds_0 \\
= - \int_0^{2\pi} e^{-\psi_j(x)} (\epsilon^{-1} + \partial_x \psi_j(x)) \left( (\beta_j \log |x| + \phi_j(x) - \psi_j(x)) e^{\psi_j(x)} \right)_{x=e^{i\theta}} \epsilon \, d\theta \\
= -2\pi \beta_j \log \epsilon - 2\pi (\phi_j(0) - \psi_j(0)) + o(1),
\]

\[
\int_{\partial D_L^1} \varphi \partial_{\bar{\eta}} \varphi \, ds_0 = - \int_0^{2\pi} \left( (\beta_j \log |x| + \phi_j(x) - \psi_j(x)) e^{-\psi_j(x)} \times \partial_x (\beta_j \log |x| + \phi_j(x) - \psi_j(x)) e^{\psi_j(x)} \right)_{x=e^{i\theta}} \epsilon \, d\theta \\
= - \int_0^{2\pi} (\beta_j \log |x| + \phi_j(x) - \psi_j(x))_{x=e^{i\theta}} \left( \frac{\beta_j}{\epsilon} + O(1) \right) \epsilon \, d\theta \\
= -2\pi \beta_j^2 \log \epsilon - 2\pi \beta_j (\phi_j(0) - \psi_j(0)) + o(1).
\]

In (2.31) we also use the identities \( K_\varphi = e^{-2\varphi}(K_0 + \Delta^0 \varphi) \) on \( M_e \) and \( dA_\varphi = e^{2\varphi} dA_0 \) and finally obtain from (2.30) the following:

\[
\log \frac{\det \Delta^\varphi_{M_e}}{\det \Delta^0_{M_e}} = - \frac{1}{2\pi} \left( \int_M K_\varphi \varphi \, dA_\varphi + \int_M K_0 \varphi \, dA_0 + \int_{\partial M} \varphi \partial_{\bar{\eta}} \varphi \, ds_0 \right) \\
- \frac{1}{6\pi} \int_{\partial M} k_0 \varphi \, ds_0 - \frac{1}{4\pi} \int_{\partial M} \partial_{\bar{\eta}} \varphi \, ds_0 \\
+ \sum_{j=1}^n \left( \frac{1}{6} (\beta_j^2 + 2\beta_j) \log \epsilon + \frac{\beta_j + 2}{6} (\phi_j(0) - \psi_j(0)) + \frac{\beta_j}{2} \right) + o(1).
\]

This together with (2.29) and (2.26) implies the desired formulas (1.1) and (1.4) (if \( \partial M = \emptyset \), then the integrals along \( \partial M \) do not appear).

**Proof of Corollary 1.3.** Consider for instance the case \( \partial M = \emptyset \). Let us change the notation: by \( m_\alpha \) and \( m_\beta \) denote the metrics \( m_0 \) and \( m_\varphi \) representing the divisors \( \alpha \) and \( \beta \) respectively. Let also \( m_0 \) be a smooth metric in the same conformal class, \( m_\alpha = e^{2\alpha} m_0 \) and \( m_\beta = e^{2\beta} m_0 \) with some functions \( \alpha \in C^\infty(M, \text{supp} \alpha) \) and \( \beta \in C^\infty(M \setminus \text{supp} \beta) \), then \( \varphi = \beta - \alpha \) (\( m_0 \) can be constructed by smoothing a conical metric potential in the same way as in the proof of Lemma 2.9). In a local parameter centered at \( P^j \in \text{supp} \alpha \cup \text{supp} \beta \) we write \( m_\alpha = |x|^{2\alpha_j} e^{2\alpha_j(x)} |dx|^2, m_\beta = |x|^{2\beta_j} e^{2\beta_j(x)} |dx|^2, \) and \( m_0 = e^{2\psi_j(x)} |dx|^2 \). In the same way as in the proof of Theorem 1.1 the integral over \( M \) below reduces to a sum of line integrals with contours shrinking to the conical singularities of \( m_\alpha \) and \( m_\beta \). As a result we obtain

\[
- \frac{1}{12\pi} \int_M [(\Delta^0 \alpha) \beta - \alpha (\Delta^0 \beta)] \, dA_0 = \frac{1}{6} \sum_{k=1}^n \alpha_k (\psi_k(0) - \hat{\beta}_k(0)) - \frac{1}{6} \sum_{j=1}^n \beta_j (\psi_j(0) - \hat{\alpha}_j(0)).
\]

This together with identity \( K_0 = e^{2\varphi} (K_\varphi - \Delta^\varphi \varphi) \) and the Polyakov f-la (1.1) for \( \varphi = \alpha \) and \( \varphi = \beta \) leads to (1.8). The case \( \partial M \neq \emptyset \) is similar. \( \square \)
3 Explicit formulas for determinants of Laplacians

In this section we obtain new and recover known explicit formulas for determinants of Laplacians on surfaces with conical singularities.

3.1 Constant curvature spheres with two conical points

By the uniformisation theorem a Riemann surface with two conical singularities homeomorphic to a sphere is conformally equivalent to the Riemann sphere $\mathbb{C}P^1$; we can assume that the conical points are at $z = 0$ and $z = \infty$. Let $m_\varphi$ be a corresponding conformal metric on $\mathbb{C}P^1$ with constant curvature $K_\varphi$. Then, by [40, Theorem II], $K_\varphi$ is positive and there exist $\mu \in [0, \infty)$ and $\beta > -1$, such that either $\beta$ is an integer or $\mu = 0$, and (up to a change of coordinates $z \mapsto pz$ with a constant $p \in \mathbb{C}$) we have

$$m_\varphi = \frac{(2\beta + 2)^2|z|^{2\beta}|dz|^2}{(|1 + \mu z^{\beta+1}|^2 + K_\varphi|z|^{2\beta+2})^2}. \quad (3.1)$$

The distance between $z = 0$ and $z = \infty$ in the metric (3.1) is $d = \frac{2}{\sqrt{K_\varphi}} \arctan \left( \frac{\sqrt{K_\varphi}}{\mu} \right)$. If $\beta \notin \mathbb{N}$, then $\mu = 0$, $d = \pi / \sqrt{K_\varphi}$, and the two conical singularities are antipodal.

**Proposition 3.1.** Let $K_\varphi > 0$, $\mu \in [0, \infty)$, and $\beta > -1$. Then for the determinant of (the Friedrichs extension of) the Laplacian $\Delta_\varphi$ on the Riemann sphere $\mathbb{C}P^1$ endowed with metric (3.1) we have

$$\log \det \Delta_\varphi = -\frac{1}{6} \left( \beta + 1 - \frac{1}{\beta + 1} \right) \log \left( 1 + \frac{\mu^2}{K_\varphi} \right) + \frac{\beta + 1}{2} - \frac{1}{3} \left( \beta + 1 + \frac{1}{\beta + 1} \right) \log \frac{\beta + 1}{\sqrt{K_\varphi}} - 4\zeta'(0; \beta + 1, 1, 1) - \log K_\varphi, \quad (3.2)$$

where either $\beta$ is an integer or $\mu = 0$.

The case $\mu = 0$ (a sphere with two antipodal conical singularities of angle $2\pi(\beta + 1)$, or, equivalently, a spindle or an american football) was previously studied in [36] by an approach based on separation of variables; see also [23]. A variational formula for $\zeta'(0)$ with respect to $\mu$ (for $\beta = 1$) was recently obtained in [17, 18]. In the case $\beta = K_\varphi = 1$ the formula (3.2) simplifies to

$$\det \Delta_\varphi = \sqrt[6]{2}e^{1-2\zeta'_B(-1)}(1 + \mu^2)^{-1/4}; \quad (3.3)$$

see Lemma A.1 in Appendix A. Thus we find that the undetermined in [17, f-la (1.2)] with $\rho(z, \bar{z}) = 4(1 + |z|^2)^{-2}$ and [18, f-la (1)] constant $C$ equals $\sqrt[6]{2}e^{1-2\zeta'_B(-1)}$. (In order to compare (3.3) with the result in [17, 18], set there $z_1 = 0$ and $z_2 = 1/\mu$.)

**Proof of Proposition 3.1.** We will rely on (1.1), where as $m_0 = e^{2\psi}|dz|^2$ we take the standard curvature one metric on $\mathbb{C}P^1$, i.e. $\psi(z) = \log 2 - \log(1 + |z|^2)$. Consider the map

$$w = f(z) = \frac{z^{\beta+1}}{1 + \mu z^{\beta+1}} : \mathbb{C}P^1_z \to \mathbb{C}P^1_w.$$
It is a ramified covering with ramification divisor $\beta \cdot 0 + \beta \cdot \infty$. In the case $K_\phi = 1$ we have

$$m_\phi = f^* \left( e^{2\varphi(w)|dw|^2} \right) = |z|^{2\beta} e^\varphi(z) dz^2, \quad \varphi(z) = \psi \circ f(z) + \log |z^{-\beta} f'(z)|,$$

and $A_\phi = 4\pi(\beta + 1)$. The integral part of Polyakov type f-la (1.1) takes the form

$$\int \varphi |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = \int \varphi e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i}$$

$$= \int \left( \beta \log |z| + \psi \circ f + \log |z^{-\beta} f'| - \psi \right) |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i}$$

$$+ \int \left( \beta \log |z| + \phi - \psi \right) e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i}.$$

We remove the parenthesis and calculate the integrals term by term. For the first one we use the Liouville equation $|z|^{2\beta} e^{2\phi} = -4\partial_z \partial_{\bar{z}} \varphi$ and then integrate by parts to get

$$\int \log |z| |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = \lim_{\epsilon \to 0^+} \left( \int_{|z|=1/\epsilon} + \int_{|z|=\epsilon} \right) \left( \log |z| \partial_{\bar{z}} \varphi(z) - \varphi(z) \partial_{\bar{z}} \log |z| \right) dz$$

$$= 2\pi \left( \log \frac{2\beta + 2}{1 + |\mu|^2} - \log(2\beta + 2) \right) = -2\pi \log(1 + |\mu|^2). \tag{3.4}$$

For the second term we first observe that

$$\int \psi e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i} = 4\pi(\log 2 - 1) \tag{3.5}$$

(e.g. by passing to the polar coordinates $(r, \theta) = (|z|, \arg z)$) and then by changing the variable $z \mapsto f(z)$ we obtain

$$\int \psi \circ f |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = 4\pi(\log 2 - 1)(\beta + 1).$$

In the third term we represent $|z^{-\beta} f'|$ as $|f|^2(\beta + 1)|z|^{-2\beta - 2}$ and get

$$\int \log |z^{-\beta} f'| |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = 2 \int \left( \log |f| \right) |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i}$$

$$+ 4\pi(\beta + 1) \log(\beta + 1) + 4\pi(\beta + 1) \log(1 + |\mu|^2),$$

where the integral in the right hand side is zero (as it follows e.g. from (3.4) with $\mu = \beta = 0$ after the change of variables $z \mapsto f(z)$).

Next we use the Liouville equation for $\phi$ and $\psi$ and integrate by parts to evaluate fourth and sixth terms together. We have

$$- \int \psi |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} + \int \phi e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i} = - \lim_{\epsilon \to 0^+} \int_{|z|=1/\epsilon} \left( \psi \partial_{\bar{z}} \phi - \phi \partial_{\bar{z}} \psi \right) dz$$

$$= -2\pi \left( 2(\beta + 1) \log 2 - 2 \log \frac{2\beta + 2}{1 + |\mu|^2} \right).$$
The fifth term can be integrated as in (3.4) and gives zero. For the seventh term see (3.5).

In total for the integral part we get
\[
\int_{C} \varphi |z|^{2\beta} e^{\varphi} dz \wedge d\bar{z} = 2\pi \beta \log(1 + |\mu|^2) - 4\pi \beta + 4\pi (\beta + 2) \log(\beta + 1).
\]

For the non-integral part of (1.1) we obtain
\[
\frac{1}{6} \sum_{j=1}^{2} \beta \left( \phi_j(0) - \psi_j(0) \right) - 2C(\beta) = \frac{1}{6} \beta \left( \log(2\beta + 2) \right) - \log \frac{2\beta + 2}{1 + |\mu|^2} - 2 \log 2 - 2C(\beta).
\]

It remains to notice that for the standard (smooth) curvature one sphere of area \( A_0 = 4\pi \) one has
\[
\log \det \Delta_0 = 1/2 - 4\zeta_R'(1); \quad (3.6)
\]
see e.g. [29, p. 204]. This together with (1.1) and (1.2) leads to (3.2) with \( K_{\varphi} = 1 \).

In order to include into consideration the case \( 0 < K_{\varphi} \neq 1 \) we do the change of variables \( z \mapsto (K_{\varphi})^{-1/2} z \) in the curvature one metric \( m_{\varphi} \), then divide the resulting metric by \( K_{\varphi} \). In accordance with the rescaling argument this decreases the value of \( \zeta'(0) \) by \( \zeta(0) \log K_{\varphi} \). It remains to note that for a sphere with two conical singularities of order \( \beta \) Corollary 1.2 gives \( \zeta(0) = \frac{1}{6} \left( \beta + 1 + \frac{1}{\beta + 1} \right) - 1 \).

In the remaining part of this section we discuss some extremal properties of the determinant \( \det \Delta_{\varphi}^{\text{Area}} \) on the constant curvature spheres \( (\mathbb{C}P^1, m_{\varphi}) \) with two conical singularities as a function of \( \mu \) and \( \beta \) while the area \( A_{\varphi} = 4\pi \) remains fixed. The Gauss-Bonnet theorem [38] reads \( K_{\varphi} = \beta + 1 \) and from (3.2) we obtain
\[
\log \det \Delta_{\varphi}^{\text{Area}} = -\frac{1}{6} (\beta + 1) \log(\beta + 1) - \frac{1}{6} \left( \beta + 1 \right) - 4\zeta_B(0; \beta + 1, 1, 1) + \frac{\beta + 1}{2}. \quad (3.7)
\]

Clearly \( \det \Delta_{\varphi}^{\text{Area}} \) monotonically goes to zero as \( |\mu| \) increases and \( \beta \in \mathbb{N} \) remains fixed. If \( \mu = 0 \) and \( \beta \to -1^+ \) (or \( \mu \in [0, \infty) \) and \( \beta \to +\infty \)), then the value of \( \det \Delta_{\varphi}^{\text{Area}} \) increases without any bound (this can be easily seen from (3.7) and available asymptotic expansions of the Barnes zeta function, see e.g. [27], [37] or [23, A.6]). Namely, we have
\[
\log \det \Delta_{\varphi}^{\text{Area}} = -\frac{1}{6(\beta + 1)} \log(\beta + 1) - \frac{1}{6} \left( \log \frac{\beta + 1}{1 + |\mu|^2} \right) - \log \frac{\beta + 1}{2\pi} - \frac{1}{6} (\beta + 1) \log(\beta + 1) + O(\beta + 1) \text{ as } \beta \to -1^+,
\]
\[
\log \det \Delta_{\varphi}^{\text{Area}} = -\frac{1}{6} \left( \beta + 1 \right) - \frac{1}{6} \left( \beta + 1 \right) \log(\beta + 1) + \frac{1}{6} \log 2\pi + O(1/\beta) \text{ as } \beta \to +\infty.
\]
Figure 2: Constant curvature sphere of area $4\pi$ with two antipodal conical points of angle $2\pi(\beta + 1)$: a graph of $\det \Delta^\varphi_{\text{Area } 4\pi}$ as a function of $\beta > -1$ when $\mu = 0$.

In particular this demonstrates that for conical metrics the assertion [29, Corollary 1.(a)], saying that

$$\det \Delta^\varphi_{\text{Area } 4\pi} \leq \exp(1/2 - 4\zeta'_R(-1))$$

for all smooth metrics $m_\varphi$ on $\mathbb{C}P^1$ of area $4\pi$ with equality iff $\varphi(z) = \log 2 - \log(1+|z|^2)$ (i.e. iff $(\mathbb{C}P^1, m_\varphi)$ is isometric to the standard curvature one sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$, cf. (3.6)), is no longer valid.

It is interesting to note however that $\beta = 0$ corresponds to $\varphi(z) = \log 2 - \log(1+|z|^2)$ and provides $\det \Delta^\varphi_{\text{Area } 4\pi}$ with local maximum $\det \Delta^\varphi_{\text{Area } 4\pi}_{|\beta=0} = \exp(1/2 - 4\zeta'_R(-1))$; see Fig. 3.1 for a graph of $\det \Delta^\varphi_{\text{Area } 4\pi}$. Indeed, we have

$$\log \det \Delta^\varphi_{\text{Area } 4\pi} = \frac{1}{2} - 4\zeta'_R(-1) - \left(\frac{\gamma}{3} + \frac{1}{9}\right) \beta^2 + \left(\frac{\gamma}{3} + \frac{7}{36}\right) \beta^3 + O(\beta^4) \text{ as } \beta \to 0,$$

where $\gamma = -\Gamma'(1)$. Here we rely on the asymptotic expansion

$$\zeta_B(0; \beta + 1, 1, 1) = \zeta_R(-1) - \frac{5}{24} \beta + \left(\frac{\gamma}{12} + \frac{7}{36}\right) \beta^2 - \left(\frac{\gamma}{12} + \frac{29}{144}\right) \beta^3 + O(\beta^4)$$

obtained by means of the representation

$$\zeta'_B(0; a, 1, 1) = \frac{1}{12} \left(a + \frac{1}{a}\right) \gamma - \frac{1}{12} \left(\frac{1}{a} + 3 + a\right) \log a + \frac{5}{24} a + \frac{1}{4} \log(2\pi) + J(a),$$

where

$$J(a) = \int_0^\infty \frac{1}{e^x - 1} \left[\frac{1}{2x} \coth \frac{x}{2a} - \frac{a}{4} \csch^2 \frac{x}{2} - \frac{1}{12} \left(a + \frac{1}{a}\right)\right] dx, \quad (3.8)$$

$$J'(a) = -\frac{1}{36} (a - 1) + \frac{1}{16} (a - 1)^2 + O((a - 1)^3);$$

see (3.14) and [2, f-las (54) and (85)–(88)].
3.2 Flat conical metrics

Consider the Riemann sphere \( \mathbb{CP}^1 \) with flat (curvature zero) metric

\[
m_{\varphi} = \prod_{j=1}^{n} |z - p_j|^{2\beta_j} |dz|^2
\]  

(3.9)

with \( n \geq 3 \) distinct conical points \( p_j \in \mathbb{C} \) of order \( \beta_j > -1 \), \( \sum \beta_j = |\beta| = -2 \).

**Proposition 3.2.** For the determinant (of the Friederichs selfadjoint extension) of the Laplacian \( \Delta^\varphi \) on the Riemann sphere \( \mathbb{CP}^1 \) with flat conical metric (3.9) we have

\[
\log \frac{\det \Delta^\varphi}{A_\varphi} = \frac{1}{6} \sum_{j=1}^{n} \sum_{i=1, i \neq j}^{n} \frac{\beta_i \beta_j}{\beta_j + 1} \log |p_i - p_j|
\]

\[
- \sum_{j=1}^{n} C(\beta_j) - 4\zeta_R'(-1) - \frac{4}{3} \log 2 + \frac{1}{6} - \log \pi,
\]

(3.10)

where \( A_\varphi = \int_\mathbb{C} \prod_{j=1}^{n} |z - p_j|^{2\beta_j} \frac{dz \wedge d\bar{z}}{2i} < \infty \) is the total area of \( (\mathbb{CP}^1, m_\varphi) \).

**Proof.** Let \( m_\varphi = e^{2\chi}|dz|^2 \), where \( \chi(z) = \sum \beta_j \log |z - p_j| \). As \( m_0 = e^{2\psi}|dz|^2 \) we take the standard curvature one metric on \( \mathbb{CP}^1 \), i.e. \( \psi(z) = \log 2 - \log(1 + |z|^2) \). Thus \( A_0 = 4\pi \), \( K_0 = 1 \), and \( \det \Delta^0 \) is given by (3.6). The f-la (1.1) in Theorem 1.1 takes the form

\[
\log(\det \Delta^\varphi/A_\varphi) + 4\zeta_R'(-1) - 1/2 + \log(4\pi) = -\frac{1}{12\pi} \int_\mathbb{C} (\chi - \psi) e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i}
\]

\[
+ \frac{1}{6} \sum_{j=1}^{n} \beta_j \left( \frac{1}{\beta_j + 1} \sum_{i=1, i \neq j}^{n} \beta_i \log |p_i - p_j| - \log \frac{2}{1 + |p_j|^2} \right) - \sum_{j=1}^{n} C(\beta_j).
\]

(3.11)

We remove the parentheses in the integral and evaluate the first term:

\[
\frac{1}{12\pi} \int_\mathbb{C} \chi e^{2\psi} \frac{dz \wedge d\bar{z}}{-2i} = \frac{1}{12\pi} \int_\mathbb{C} \chi(-4\partial_z \partial_{\bar{z}} \psi) \frac{dz \wedge d\bar{z}}{-2i}
\]

\[
= \frac{1}{12\pi} \lim_{\epsilon \to 0} \left( \int_{|z| = 1/\epsilon} + \sum_{j=1}^{n} \int_{|z - p_j| = \epsilon} \right) \left( \chi \partial_{\bar{z}} \psi - \psi \partial_{\bar{z}} \chi \right) |dz|
\]

\[
= \frac{1}{12\pi} \sum_{j=1}^{n} \beta_j \lim_{\epsilon \to 0} \left( \int_{0}^{2\pi} \left( \epsilon \log 2 + o(\epsilon) \right) \frac{1}{\epsilon} d\theta - \int_{0}^{2\pi} \left( \psi(p_j) \frac{1}{\epsilon} + O(1) \right) d\theta \right)
\]

\[
= -\frac{1}{3} \log 2 - \frac{1}{6} \sum_{j=1}^{n} \beta_j \log \frac{2}{1 + |p_j|^2}.
\]

(3.12)

Now (3.11), (3.12), and (3.5) imply (3.10).

**Remark 3.3.** The formula (1.2) for \( C(\beta) \) together with the identity \( \sum \beta_j = -2 \) allows to write (3.10) in the form

\[
\log \frac{\det \Delta^\varphi}{A_\varphi} = \frac{1}{6} \sum_{j=1}^{n} \sum_{i=1, i \neq j}^{n} \frac{\beta_i \beta_j}{\beta_j + 1} \log |p_i - p_j|
\]

\[
- \sum_{j=1}^{n} \left( 2\zeta_{\beta_j+1}(0) + \frac{1}{2} \log(\beta_j + 1) \right) - \log 2,
\]

(3.13)
where $Z_{\beta+1}(0)$ is given by

$$Z_{\beta+1}(0) = \zeta_B'(0; \beta+1, 1, 1) - (\beta+1)\zeta_R(-1) + \frac{1}{12}(\beta+1 - \frac{1}{\beta+1}) \log 2 - \frac{\beta}{4} \log 2\pi. \quad (3.14)$$

We note that (3.13) coincides with Aurell-Salomonson formula [3, f-la (50), where $\beta_j$ (resp. $p_j$) is denoted by $-\beta_j$ (resp. $w_j$), and $\text{Area} = A(M) = A_\phi$]. One of equivalent definitions for $Z_{\beta+1}(0)$ in [2, 3] reads

$$Z_a(0) = \frac{1}{12} \left( \frac{1}{a} - a \right) (\gamma - \log 2) - \frac{1}{12} \left( \frac{1}{a} + 3 + a \right) \log a + J(a) - a \left( -\frac{1}{6} \gamma - \frac{5}{24} + \frac{1}{4} \log(2\pi) + \zeta'_R(-1) \right), \quad (3.15)$$

where $\gamma = -\Gamma'(1)$ and $J(a)$ is the same as in (3.8); see [2, f-las (53), (54) and (85)–(87)]. One can easily check that for all rational numbers $a$ the values of $Z_a(0)$ defined by (3.14) and (3.15) coincide (we use Lemma A.1 in Appendix A to evaluate (3.14); for the evaluation of (3.15) we refer to [2, f-la (102)]). Thus due to analytic regularity of $\mathbb{R}_+ \ni a \mapsto Z_a(0)$ the definitions (3.14) and (3.15) are equivalent. To the best of my knowledge, this is the first rigorous mathematical proof of the Aurell-Salomonson formula.

Let us also note that if $m_0$ is the smooth hyperbolic (curvature $K_0 = -1$) metric on a surface $M$ (of genus greater than one) and $m_\phi$ is a conformally equivalent flat conical metric, then Theorem 1.1.1 returns the result of [28, Cor. 6.2].

### 3.3 Constant curvature conical metric disks

Consider the unit disk $|z| \leq 1$ endowed with the metric

$$m_\phi = |z|^{2\beta}e^{2\phi}|dz|^2, \quad \phi(z) = \log 2 - \log(1 + K|z|^{2\beta+2}), \quad (3.16)$$

where $K_\phi = (\beta + 1)K > -1$ is the curvature and $2\pi(\beta + 1) > 0$ is the angle of conical point at $z = 0$.

**Proposition 3.4.** Let $K > -1$ and $\beta > -1$. Then for the determinant of (the Friederichs extension of) the Dirichlet Laplacian $\Delta^\phi$ on the disk $|z| \leq 1$ with metric (3.16) we have

$$\log \det \Delta^\phi = -2\zeta_B'(0; \beta+1, 1, 1) - \frac{1}{2} \log(\beta + 1) + \frac{11K - 5}{12(1 + K)}(\beta + 1) - \frac{1}{2} \log(2\pi). \quad (3.17)$$

In the curvature zero case this result was obtained in [42] (if there is no conical singularity, i.e. $K = \beta = 0$) and in [35] (if there is a conical singularity at $z = 0$, i.e. $\beta > -1$ and $K = 0$), cf. (2.2) and (2.3). We also note that in the case $K = 1$ and $\beta = 0$ the metric disk is isometric to the unit hemisphere and the f-las (1.7), (3.17) return the corresponding result, cf. [42, f-las (24)]. In all other cases the result is new.
Proof. Let us take the flat metric \(|dz|^2\) as \(m_0\) and do the calculations for

\[ \phi(z) = -\log(1 + K|z|^{2\beta + 2}). \]

Then one can use the standard rescaling argument in order to add \(\log 2\) to \(\phi\) and obtain (3.17) for the determinant corresponding to the metric in (3.16); this will only decrease \(\log \det \Delta^\varphi\) by \(\zeta(0)2\log 2\), where \(\zeta(0) = \frac{1}{12} \left(\beta + 1 + \frac{1}{\beta+1}\right)\) cf. (1.7).

From (1.4) we get

\[ -\zeta'(0) - (1/3) \log 2 + (1/2) \log(2\pi) + 5/12 + 2\zeta'_R(-1) \]

\[ = -\frac{1}{12\pi} \left( \int_{|z|\leq1} K_{\varphi}|z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} + \int_{|z|=1} \varphi \partial_{|z|} \varphi \ ds_0 \right) \]

\[ = -\frac{1}{6\pi} \int_{|z|=1} k_0 \varphi \ ds_0 - \frac{1}{4\pi} \int_{|z|=1} \partial_{|z|} \varphi \ ds_0 - C(\beta), \]

where \(\varphi(z) = \beta \log |z| + \phi(z)\) and \(K_{\varphi} = (2\beta + 2)^2 K\); see (2.12) for the value of \(\log \det \Delta^0\). We have

\[ \int_{|z|\leq1} K_{\varphi} |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = \beta \int_{|z|\leq1} \log |z| (-4\partial_z \partial_{\bar{z}} \phi) \frac{dz \wedge d\bar{z}}{-2i} + (2\beta + 2)^2 K \int_{|z|\leq1} \phi |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i}. \]

Here

\[ \int_{|z|\leq1} \log |z| (-4\partial_z \partial_{\bar{z}} \phi) \frac{dz \wedge d\bar{z}}{-2i} = \oint_{|z|=1} \frac{\phi(z)}{|z|} \ |dz| - \lim_{\epsilon \to 0^+} \oint_{|z|=\epsilon} \frac{\phi(z)}{|z|} \ |dz| = -2\pi \log(1 + K) \]

and

\[ \int_{|z|\leq1} \phi |z|^{2\beta} e^{2\phi} \frac{dz \wedge d\bar{z}}{-2i} = -\frac{\pi}{\beta + 1} \int_0^1 (1 + K u)^{-2} \log(1 + K u) \ du \]

\[ = \frac{\pi}{\beta + 1} \frac{\log(1 + K) - K}{K(1 + K)}. \]

Next we evaluate the integrals along the circle \(|z| = 1\) and get

\[ \int_{|z|=1} \varphi \partial_{|z|} \varphi \ ds_0 = 2\pi \log(1 + K) \left( \frac{(2\beta + 2) K}{1 + K} - \beta \right), \]

\[ \int_{|z|=1} \varphi \ ds_0 = -2\pi \log(1 + K), \]

\[ \int_{|z|=1} \partial_{|z|} \varphi \ ds_0 = 2\pi \left( \beta - \frac{(2\beta + 2) K}{1 + K} \right). \]

These calculations together with f-la (1.2) for \(C(\beta)\) imply

\[ \zeta'(0) = 2\zeta'_B(0; \beta + 1, 1, 1) - \frac{1}{6} \left( \beta + 1 + \frac{1}{\beta + 1} \right) \log 2 - \frac{11K - 5}{12(1 + K)} (\beta + 1) + \frac{1}{2} \log(2\pi). \]

\[ \square \]
### 3.4 Hyperbolic spheres

As is known, there exists a unique hyperbolic (curvature $K_\varphi = -1$) conformal metric $m_\varphi = e^{2\varphi}|dz|^2$ on the Riemann sphere $\mathbb{C}P^1$ with conical singularities of order $\beta_j > -1$ at $p_j \in \mathbb{C}P^1$, $j = 1, \ldots, n$, provided $n \geq 3$ and $\sum_{j=1}^{n} \beta_j = |\beta| < -2$; see e.g. [38]. We shall assume that $p_n = \infty$ (and then $\beta_n = 0$ and $n \geq 4$ if there is no conical point at infinity). The corresponding metric potential $\varphi$ has the following asymptotics in a neighborhood of each $p_j$:

$$
\varphi(z) = \beta_j \log |z - p_j| + \phi_j + O(|z - p_j|^{2\beta_j+2}), \quad z \to p_j, \quad 0 < j < n,
$$

$$
\varphi(z) = - (\beta_n + 2) \log |z| + \phi_n + O(|z|^{-2\beta_n-2}), \quad z \to \infty;
$$

the asymptotics can be differentiated. We first express the determinant of Laplacian on $(\mathbb{C}P^1, e^{2\varphi}|dz|^2)$ in terms of the metric potential $\varphi$.

**Proposition 3.5.** For the spectral determinant $\det \Delta^\varphi$ of the Friederichs extension of Laplacian $\Delta^\varphi$ on the hyperbolic (curvature $K_\varphi = -1$) sphere $(\mathbb{C}P^1, e^{2\varphi}|dz|^2)$ we have

$$
\log \det^* \Delta^\varphi = \log (-2 - |\beta|) + \frac{1}{2\pi} \int_{\mathbb{C}} \varphi e^{2\varphi} \frac{dz \wedge d\bar{z}}{-2i} - \frac{1}{6} \left( 1 + \frac{1}{\beta_n + 1} \right) \phi_n + \frac{1}{6} \sum_{j=1}^{n-1} \frac{\beta_j}{\beta_j + 1} \phi_j - \sum_{j=1}^{n} C(\beta_j) - \frac{1}{3} \log 2 + \frac{1}{6} - 4\zeta'(1) - 4\zeta'(1),
$$

where $\phi_j$ stands for the constant term in the asymptotic expansion (3.18), and $C(\beta)$ is defined in (1.2).

**Proof.** By using the BFK f-la we cut the Riemann sphere into two pieces along the circle $|z| = 1/\epsilon$ as $\epsilon \to 0+$. For the Neumann jump operator on this circle we have

$$
\frac{\det R^\varphi_{|z|=1/\epsilon}}{L^\varphi_{|z|=1/\epsilon}} = \frac{1}{2},
$$

this can be easily seen from the conformal invariance of the left hand side (see Lemma 2.9) together with BFK formula (2.20) and formulas for the determinant of Laplacian on the unit sphere and hemisphere (see [42] or (3.6) and (3.17) with $K = 1$ and $\beta = 0$). The Gauss-Bonnet theorem [38] implies that the total area $A_\varphi$ of hyperbolic sphere $(\mathbb{C}P^1, e^{2\varphi}|dz|^2)$ is $A_\varphi = -2\pi \chi(\mathbb{C}P^1, \beta) = -2\pi (2 + |\beta|)$. Thus we have

$$
\det \Delta^\varphi = -\pi \left( 2 + |\beta| \right) \lim_{\epsilon \to 0+} \left( \det \Delta^\varphi_{|z|<1/\epsilon} \det \Delta^\varphi_{|z|>1/\epsilon} \right),
$$

where

$$
\log \det \Delta^\varphi_{|z|>1/\epsilon} = -\frac{1}{6} \left( (\beta_n + 1)^2 + 1 \right) \log \epsilon - \frac{1}{6} \left( \beta_n + 1 + \frac{1}{\beta_n + 1} \right) \phi_n - C(\beta_n) - \zeta'(0; 1) - \beta_n \frac{\bar{\beta}_n}{2} + o(1).
$$

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The last formula for the determinant of Dirichlet Laplacian in the disk $|w| \leq \epsilon$, $w = 1/z$, immediately follows from (1.4), (3.18), and (2.12) (it can also be obtained from (3.17) if in the local parameter $w$ the metric $e^{2\varphi}|dz|^2$ takes the form (3.16)).

Let $m_0 = |dz|^2$. Then thanks to the Polyakov-Alvarez type f-la (1.4) we get

\[
\frac{\log \det \Delta^0_{|z| \leq 1/\epsilon}}{\det \Delta^0_{|z| \leq 1/\epsilon}} = \frac{1}{12\pi} \int_{|z| = 1/\epsilon} \varphi dA - \frac{1}{12\pi} \int_{|z| = 1/\epsilon} \varphi \partial_{\overline{z}} \varphi ds_0
\]

\[
- \frac{1}{6\pi} \int_{|z| = 1/\epsilon} \epsilon \varphi ds_0 - \frac{1}{4\pi} \int_{|z| = 1/\epsilon} \partial_{\overline{z}} \varphi ds_0 + \frac{1}{6} \sum_{j=1}^{n-1} \beta_j + 1 \phi_j - \sum_{j=1}^{n-1} C(\beta_j),
\]

where

\[
\frac{\log \det \Delta^0_{|z| \leq 1/\epsilon}}{\det \Delta^0_{|z| \leq 1/\epsilon}} = \frac{1}{3} \log \epsilon + \frac{1}{3} \log 2 - \frac{1}{2} \log 2\pi - \frac{5}{12} - 2\zeta_R'(1) + o(1);
\]

cf. (2.12).

It is easy to verify that

\[
- \frac{1}{4\pi} \int_{|z| = 1/\epsilon} \partial_{\overline{z}} \varphi ds_0 = \frac{\beta_n + 2}{2} + o(1);
\]

\[
- \frac{1}{6\pi} \int_{|z| = 1/\epsilon} \epsilon \varphi ds_0 = - \frac{\beta_n + 2}{3} \log \epsilon - \frac{1}{3} \phi_n + o(1),
\]

\[
- \frac{1}{12\pi} \int_{|z| = 1/\epsilon} \varphi \partial_{\overline{z}} \varphi ds_0 = \frac{1}{6} \left( (\beta_n + 2)^2 \log \epsilon + (\beta_n + 2)\phi_n \right) + o(1).
\]

This together with (3.19)–(3.21) completes the proof. \qed

Consider the mapping

\[
w = f(z) = \frac{z^2}{1 + \mu z^2} : \mathbb{C}P^1 \to \mathbb{C}P^1
\]

with $\mu \in [0, \infty)$. It is a ramified covering with ramification divisor $1 \cdot 0 + 1 \cdot \infty$. The pull back of $m_\varphi$ by $f$ is a hyperbolic (curvature $-1$) metric on $\mathbb{C}P^1$ with potential $f^* \varphi = \varphi \circ f + \log |f'|$. For brevity we assume that 0 and $1/\mu$ are not among the conical points of $m_\varphi$ (for any $\mu \in [0, \infty)$). Then $e^{2f^*\varphi}|dz|^2$ has conical singularities of order $\beta_j$ at the pre-images $z = \pm \sqrt{\frac{p_j}{1-\mu p_j}}$ of the conical points $p_1, \ldots, p_n$ as well as conical singularities of order 1 at $z = 0$ and $z = \infty$. By setting $\rho = e^{2\varphi}$, $z_1 = 0$, and $z_2 = 1/\mu$ in [17, f-la (1.2)] one obtains the variational formula

\[
\det \Delta f^* \varphi = C \mu^{-1/2} e^{2\varphi(0)} e^{2\varphi(1/\mu)},
\]

where $\Delta f^* \varphi$ is the Friederichs extension of Laplacian on $(\mathbb{C}P^1, e^{2f^*\varphi}|dz|^2)$ and $C$ is an undetermined constant that does not depend on the parameter $\mu \in [0, \infty)$. In Proposition 3.6 below we independently deduce (3.23) and find that

\[
C = -\frac{2^{2/3} e^{6\zeta_R(-1)}}{2 + |\beta|} (\det \Delta^\varphi)^2,
\]

where the divisor $\beta$ and $\det \Delta^\varphi$ are the same as in Proposition 3.5.
Proposition 3.6. Let \( f^* \varphi \) stand for the potential of the pull back of the hyperbolic metric \( e^{2\varphi(z)}|dz|^2 \) by the mapping (3.22). Then the spectral determinant of the Friederichs extension of Laplacian \( \Delta^f \varphi \) on \((\mathbb{C}P^1, e^{2f^*\varphi}|dz|^2)\) satisfies (3.23) with \( C \) specified in (3.24).

**Proof.** We will rely on Proposition 3.5 with \( \varphi \) replaced by \( f^* \varphi \). Notice that for the corresponding integral we have

\[
\int_{\mathbb{C}} f^* \varphi e^{2f^* \varphi} \frac{dz \wedge d\bar{z}}{-2i} = \int_{\mathbb{C}} (\log |f'|) e^{2f^* \varphi} \frac{dz \wedge d\bar{z}}{-2i} + 2 \int_{\mathbb{C}} \varphi e^{2\varphi} \frac{dz \wedge d\bar{z}}{-2i},
\]

where the last integral can be expressed in terms of \( \det \Delta^f \varphi \). Let \( \mathbb{C}^\varepsilon \) stand for the annulus \( \{ z \in \mathbb{C} : \varepsilon \leq |z| \leq 1/\varepsilon \} \), the union of the epsilon neighborhoods of all pre-images \( z = \pm \sqrt{\frac{p_j}{1-\mu p_j}} \) of conical points \( p_1, \ldots, p_n \). Then the Liouville equation together with Stokes’ theorem gives

\[
\int_{\mathbb{C}} (\log |f'|) e^{2f^* \varphi} \frac{dz \wedge d\bar{z}}{-2i} = \lim_{\varepsilon \to 0^+} \int_{\mathbb{C}^\varepsilon} (\log |f'|)(-4\partial_z \partial_{\bar{z}}(\varphi \circ f)) \frac{dz \wedge d\bar{z}}{-2i}
\]

\[
= -\lim_{\varepsilon \to 0^+} \int_{\mathbb{C}^\varepsilon} \left[(\varphi \circ f)\partial_{\bar{z}}(\log |f'|) - (\log |f'|)\partial_{\bar{z}}(\varphi \circ f)\right]|dz|,
\]

where the right hand side can be easily evaluated based on (3.18) and the explicit expression for \( f \) (it suffices to take into account only logarithmic and constant terms of asymptotics for \( \varphi \circ f \) and \( \log |f'| \) as \( z \) approaches a conical point of \( e^{2f^*\varphi}|dz|^2 \); we omit the details). This together with \( f^* \varphi = \varphi \circ f + \log |f'| \) allows to write the formula from Proposition 3.5 in the form

\[
\log \det \Delta^f \varphi = \log(-2 - |\beta|) + \frac{1}{6\pi} \int_{\mathbb{C}} \varphi e^{2\varphi} \frac{dz \wedge d\bar{z}}{-2i} - \frac{1}{3} \left(1 + \frac{1}{\beta_n + 1}\right) \phi_n
\]

\[
+ \frac{1}{3} \sum_{j=1}^{n-1} \frac{\beta_j}{\beta_j + 1} \phi_j - 2 \sum_{j=1}^{n} C(\beta_j) - 2C(1) - \frac{1}{6} \log 2 + \frac{1}{6} - 4\zeta'(-1)
\]

\[
- \frac{1}{2} \log \mu + \frac{1}{4}(\varphi(0) + \varphi(1/\mu)),
\]

where \( C(1) = -\zeta'(-1) - \frac{1}{12} \log 2 - \frac{1}{12} \) (see (1.2) and Lemma A.1). This together with Proposition 3.5 completes the proof. \( \Box \)

### 3.5 Genus \( g > 1 \) surfaces without boundary

Here we present a general explicit formula for the determinant of Friederichs Laplacian on genus \( g > 1 \) Riemann surface \( M \) without boundary. The result is based on the formula for the determinant in a flat conical metric with trivial holonomy [22] and Corollary 1.3. This is a straightforward generalization of the scheme in [21]: Corollary 1.3.1 together with the particular values \( C(0) = 0 \) and \( C(1) = -\zeta'(-1) - \frac{1}{12} \log 2 + 1 \) of the function \( C(\beta) \) in (1.2) should be used instead of [21, Prop.1]. Therefore we only formulate the result and omit the proof; for details we refer to [21].
Proposition 3.7. Let \( \omega \) be a holomorphic one-form on \( M \) with \( 2g - 2 \) simple zeros and let \( m_0 = |\omega|^2 \) be the corresponding flat conical metric (with trivial holonomy and first order conical singularities at the zeros of \( \omega \)). Consider a dilation analytic conical metric \( m_{\varphi} = e^{2\varphi} m_0 \) of (smooth) curvature \( K_\alpha \). Let \( \{ P_1, \ldots, P_n \} \) be the set of all distinct points in the union \( \text{supp } \alpha \cup \text{supp } \beta \), where \( \beta \) (resp. \( \alpha \)) is the divisor of \( m_{\varphi} \) (resp. \( m_0 \) and \( \omega \)). Pick a local holomorphic parameter \( x \) centred at \( P_j \) such that \( m_0 = |x|^{2\alpha_j} |dx|^2 \) and \( m_{\varphi} = |x|^{2\beta_j} e^{2\varphi_j} |dx|^2 \), where \( \alpha_j = 0 \) if \( P_j \notin \text{supp } \alpha \), \( \alpha_j = 1 \) if \( P_j \in \text{supp } \alpha \), and \( \beta_j = 0 \) if \( P_j \notin \text{supp } \beta \). Then the determinant of the Friederichs Laplacian \( \Delta^\varphi \) on \((M, m_{\varphi})\) satisfies

\[
\det \Delta^\varphi = (2\pi)^{-4/3} \kappa_0^{\beta - 1} A_{\varphi} (\det \Im \mathcal{B}) |\tau_g(M, \omega)|^2 \exp \left\{ -\frac{1}{12\pi} \int_M K_{\varphi} \varphi \, dA_{\varphi} \right\} + \sum_{j=1}^n \left( \frac{\phi_j(0)}{6} \left( \frac{\beta_j}{\beta_j + 1} + \alpha_j \right) - C(\beta_j) \right) - (2g - 2) \left( \zeta'_{R}(1) + \frac{1}{12} (\log 2 + 1) \right),
\]

where \( \kappa_0 \) is an absolute constant that can be expressed in terms of spectral determinants of some model operators, \( A_{\varphi} \) is the total area of \( M \) in the metric \( m_{\varphi} \), \( \mathcal{B} \) is the matrix of \( b \)-periods of the Riemann surface \( M \), and the Bergman tau-function \( \tau \) is a holomorphic function that admits explicit expression through theta-functions, prime forms, and the divisor \( \alpha \).

In the particular case of a flat metric \( m_{\varphi} \) (i.e. \( K_{\varphi} = 0 \)) Proposition 3.7 is a reformulation of the main result in [21]; recall that any constant curvature metric is dilation analytic.

A Derivative \( \zeta'_B(0; \beta + 1, 1, 1) \) of the Barnes zeta function for rational values of \( \beta \)

Lemma A.1. Let \( p \) and \( q \) be coprime natural numbers. Then

\[
\zeta'_B(0; p/q, 1, 1) = \frac{1}{pq} \zeta'_R(1) - \frac{1}{12pq} \log q + \left( \frac{1}{4} + S(q, p) \right) \log \frac{q}{p} + \sum_{k=1}^{p-1} \left( \frac{1}{2} - \frac{k}{p} \right) \log \Gamma \left( \left( \frac{kq}{p} \right) + \frac{1}{2} \right) + \sum_{j=1}^{q-1} \left( \frac{1}{2} - \frac{j}{q} \right) \log \Gamma \left( \left( \frac{jp}{q} \right) + \frac{1}{2} \right),
\]

where \( S(q, p) = \sum_{j=1}^{p-1} \left( \frac{j}{p} \right) \left( \frac{jq}{p} \right) \) is the Dedekind sum, and the symbol \( (\cdot) \) is defined so that \( (x) = x - \lfloor x \rfloor - 1/2 \) for \( x \) not an integer and \( (x) = 0 \) for \( x \) an integer.

In particular, for natural numbers \( p \) and \( q \) one has

\[
\zeta'_B(0; p, 1, 1) = \frac{1}{p} \zeta'_R(1) - \left( \frac{p}{12} + \frac{1}{4} + \frac{1}{6p} \right) \log p - \sum_{j=1}^{p-1} \frac{j}{p} \log \Gamma \left( \frac{j}{p} \right) + \frac{p-1}{4} \log 2\pi,
\]

\[
\zeta'_B(0; 1/q, 1, 1) = \frac{1}{q} \zeta'_R(1) - \frac{1}{12q} \log q - \sum_{j=1}^{q-1} \frac{j}{q} \log \Gamma \left( \frac{j}{q} \right) + \frac{q-1}{4} \log 2\pi.
\]
\[
\zeta_B'(0; 1, 1, 1) = \zeta_R'(-1),
\]
\[
\zeta_B'(0; 2, 1, 1) = \frac{1}{2} \zeta_R'(-1) - \frac{1}{4} \log 2, \quad \zeta_B'(0; 1/2, 1, 1) = \frac{1}{2} \zeta_R'(-1) + \frac{5}{24} \log 2,
\]
\[
\zeta_B'(0; 3, 1, 1) = \frac{1}{3} \zeta_R'(-1) + \frac{1}{6} \log 2 - \frac{7}{18} \log 3 - \frac{1}{3} \log \Gamma \left( \frac{2}{3} \right) + \frac{1}{6} \log \pi,
\]
\[
\zeta_B'(0; 1/3, 1, 1) = \frac{1}{3} \zeta_R'(-1) + \frac{1}{6} \log 2 + \frac{5}{36} \log 3 - \frac{1}{3} \log \Gamma \left( \frac{2}{3} \right) + \frac{1}{6} \log \pi,
\]
\[
\zeta_B'(0; 4, 1, 1) = \frac{1}{4} \zeta_R'(-1) - \frac{5}{8} \log 2 - \frac{1}{2} \log \Gamma \left( \frac{3}{4} \right) + \frac{1}{4} \log \pi,
\]
\[
\zeta_B'(0; 1/4, 1, 1) = \frac{1}{4} \zeta_R'(-1) + \frac{7}{12} \log 2 - \frac{1}{2} \log \Gamma \left( \frac{3}{4} \right) + \frac{1}{4} \log \pi, \ldots
\]

Proof. Let us first prove (A.2). Notice that
\[
\zeta_B(s; 1, 1, 1) = \sum_{m,n=0}^{\infty} (m+n+1)^{-s} = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\ell} (\ell+1)^{-s} = \zeta_R(s-1),
\]
where the left hand side can also be represented as the sum \(\sum_{k=1}^{p} \zeta_B(s; p, 1, k)\). For each term of this sum we have
\[
\zeta_B(s; p, 1, k) = \sum_{m=0}^{\infty} \sum_{n=k-1}^{\infty} (pm+n+1)^{-s}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{\infty} (pm+n+1)^{-s} - \sum_{n=0}^{k-2} (pm+n+1)^{-s} \right) = \zeta_B(s; p, 1, 1) - p^{-s} \sum_{j=1}^{k-1} \zeta_H(s; j/p),
\]
where
\[
\zeta_H(s; x) = \sum_{m=0}^{\infty} (m+x)^{-s}
\]
is the Hurwitz zeta function. Solving the resulting equation for \(\zeta_B(s; p, 1, 1)\) we obtain
\[
\zeta_B(s; p, 1, 1) = \frac{1}{p} \zeta_R(s-1) + p^{-s-1} \sum_{j=1}^{k-1} \zeta_H(s; j/p)
\]
\[
= \frac{1}{p} \zeta_R(s-1) + p^{-s-1} \sum_{j=1}^{p-1} \sum_{k=j+1}^{p} \zeta_H(s; j/p)
\]
\[
= \frac{1}{p} \zeta_R(s-1) + p^{-s-1} \sum_{j=1}^{p-1} (p-j) \zeta_H(s; j/p).
\]
Now we differentiate with respect to \(s\) and use the well-known identities
\[
\zeta_H(0; x) = \frac{1}{2} - x, \quad \zeta_H'(0; x) = \log \Gamma(x) - \frac{1}{2} \log 2\pi.
\]

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As a result we obtain
\[ \zeta_B(0; p, 1, 1) = \frac{1}{p} \zeta_R(-1) + \sum_{j=1}^{p-1} \left( 1 - \frac{j}{p} \right) \left( \log \Gamma \left( \frac{j}{p} \right) + \frac{j}{p} \log p \right) - \frac{p-1}{4} \left( \log 2\pi + \log p \right). \]

Taking into account that
\[ \sum_{j=1}^{p-1} \frac{j^2}{p^2} = \frac{p}{3} - \frac{1}{2} + \frac{1}{6p}, \quad \prod_{j=1}^{p-1} \left( \frac{j}{p} \right) = (2\pi)^{p-1/2} p^{-1/2}, \]
we arrive at (A.2).

In order to prove (A.3) we use the relation \( \zeta_B(s; 1/q, 1, 1) = q^s \zeta_B(s; q, 1, q) \) and obtain
\[ \zeta_B(s; 1/q, 1, 1) = q^s \left( \zeta_B(s; q, 1, 1) - q^{-s} \sum_{j=1}^{q-1} \zeta_H(s; j/q) \right) \]
\[ = q^{s-1} \zeta_R(s - 1) - \sum_{j=1}^{q-1} \frac{j}{q} \zeta_H(s; j/q). \]

Since \( \zeta_R(-1) = -1/12 \), this implies (A.3).

Let \( p \) and \( q \) be coprime. Bézout’s identity reads \( xp + yq = 1 \). Without loss of generality we can assume that \( x \leq 0 \) and \( y \geq 0 \) (otherwise take \( x := x - q \) and \( y := y + p \)). We start with
\[ \zeta_B(s; p/q, 1, 1) = q^s \zeta_B(s; p, q, q). \]

We have
\[ \zeta_B(s; p, q, q + k) = \sum_{n=0}^{\infty} \left( \sum_{m=-nk}^{\infty} \sum_{m=0}^{-nk} \right) (pm + qn + q + k)^{-s} \]
\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (pm + qn + q)^{-s} + q^{-s} \sum_{m=0}^{-nk} \zeta_H(s; (pm + q + k)/q) \]
\[ = \zeta_B(s; p, q, q) - p^{-s} \sum_{n=0}^{yq-1} \zeta_H(s; (qn + q)/p) + q^{-s} \sum_{m=0}^{-nk} \zeta_H(s; (pm + q + k)/q). \]

Therefore
\[ \zeta_B(s; p, q, q) = \zeta_B(s; p, q, q + k) + p^{-s} \sum_{n=0}^{yq-1} \zeta_H(s; (qn + q)/p) - q^{-s} \sum_{m=0}^{-nk} \zeta_H(s; (pm + q + k)/q) \]
\[ = \frac{1}{p} \left( \zeta_B(s; 1, q, q) + p^{-s} \sum_{k=0}^{p-1} \sum_{n=0}^{yk-1} \zeta_H(s; (qn + q)/p) - q^{-s} \sum_{k=0}^{p-1} \sum_{m=0}^{-nk} \zeta_H(s; (pm + q + k)/q) \right). \]
Here
\[
\zeta_B(s; 1, q, q) = \zeta_B(s; 1, q, q + j) + q^{-s} \sum_{m=0}^{j-1} \zeta_H(s; (m + q)/q)
\]
\[
= \frac{1}{q} \left( \zeta_B(s; 1, q) + q^{-s} \sum_{j=0}^{q-1} \sum_{m=0}^{j-1} \zeta_H(s; (m + q)/q) \right)
\]
\[
= \frac{1}{q} \left( \zeta_R(s - 1) - \sum_{m=1}^{q-1} \zeta_H(s; m) + q^{-s} \sum_{j=0}^{q-1} \sum_{m=0}^{j-1} \zeta_H(s; (m + q)/q) \right).
\]
Finally we get
\[
\zeta_B(s; p/q, 1, 1) = \frac{q^s}{pq} \left( \zeta_R(s - 1) - \sum_{m=1}^{q-1} \zeta_H(s; m) \right) + \frac{1}{pq} \sum_{j=1}^{q-1} \sum_{m=0}^{j-1} \zeta_H(s; (m + q)/q)
\]
\[
+ q^s p^{-s-1} \sum_{k=1}^{p-1} \sum_{n=0}^{qk-1} \zeta_H(s; (qn + q)/p) - \frac{1}{p} \sum_{k=1}^{p-1} \sum_{m=0}^{qk-1} \zeta_H(s; (pm + q + k)/q).
\]
This together with (A.4) gives a representation of \( \zeta_B(0; p/q, 1, 1) \) in terms of \( \zeta'_R(-1) \) and gamma functions. Then Gauss’ multiplication formula allows to write the result in the form (A.1). A similar computation can be found in [10, Section 4].

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