Matrix String Theory As A Generalized Quantum Theory

D.MINIC
Physics Department
Pennsylvania State University
University Park, PA 16802
and
Enrico Fermi Institute
University of Chicago
Chicago, IL 60637
dminic@yukawa.uchicago.edu

Abstract

Matrix String Theory of Banks, Fischler, Shenker and Susskind can be understood as a generalized quantum theory (provisionally named "quansical" theory) which differs from Adler’s generalized trace quantum dynamics. The effective planar Matrix String Theory Hamiltonian is constructed in a particular fermionic realization of Matrix String Theory treated as an example of "quansical" theory.
1. Introduction

In this article I want to show how a generalized quantum-theoretical structure (in which $\hbar$ is kept finite, so that the classical limit is not defined by letting $\hbar \to 0$) naturally appears in connection with the problem of non-perturbative formulation of string theory. Matrix String Theory (the planar SUSY Yang-Mills quantum mechanics of Polchinski’s D0-branes [1] [2] [3], the suggested partonic constituents of the fundamental strings) presents one such formulation.

How can one go about formulating the classical limit of a quantum theory while keeping $\hbar$ finite? One possible approach was put forward by Adler [4]. The idea is to avoid "quantization" altogether and directly formulate Hamiltonian dynamics on a non-commuting phase space for general non-commuting degrees of freedom. Adler assumes that operator multiplication of non-commuting operator variables is associative and that there exists a graded trace $\text{Tr}$ obeying the fundamental property of cyclic permutation of non-commuting operator variables according to $\text{Tr}O_1O_2 = \pm \text{Tr}O_2O_1$, where $\pm$ corresponds to the situation when both $O_1$ and $O_2$ are bosonic/fermionic. Then it can be shown [4] that for a general trace functional $A$, defined as the graded trace $\text{Tr}$ of a bosonic polynomial function of operator variables $q_i$, one can uniquely define $\delta A/\delta q_i$, from $\delta A = \delta A/\delta q_i$. Then Adler shows how to define the generalized Poisson bracket, formulate generalized Hamilton equations of motion, etc. [4]. The whole structure nicely applies to Matrix String Theory.

Now, given the fact that Matrix String Theory is the $N = \infty$ limit of the supersymmetric quantum mechanics describing $D0$-branes, it is important to incorporate the basic features of the planar limit, such as factorization, in order to be able to discuss any dynamical issues. (Factorization by definition means that $\langle F_1F_2 \rangle = \langle F_1 \rangle \langle F_2 \rangle$ [5], given two observables $F_1$ and $F_2$.) Therefore, I propose to study a generalized quantum theory.
which naturally contains certain features of classical dynamics, implied by factorization. Such a theory can be induced starting from the Feynman-Schwinger differential variational principle \[6\][7]

\[
\langle F \delta_X S \rangle = i \hbar \langle \delta_X F \rangle. \tag{1}
\]

Here \( S \) is the classical action, given in terms of some variables \( X \), and \( F \) is some appropriate observable. As is well-known, one can deduce the canonical Heisenberg commutation relations (by choosing \( F = X \)), the quantum dynamical equations of motion etc., starting from (1) \[6\][7]. The correspondence principle is by definition manifested in the classical nature of \( S \). Bohr's complementarity principle is formally contained in the Heisenberg uncertainty relations, that follow from the canonical commutation relation.

Suppose that one changes the meaning of \( \delta_X \) in (1) in order to generate a generalized version of quantum theory in which the classical property of factorization holds when the dynamical variables that appear in the variational derivative happen to be non-commuting. That would imply a summation over the planar trajectories only in the path-integral, the integral version of (1). One would be then dealing with a quantum theory with certain classical features. (One might call such a generalization of quantum theory, planar quantum theory or "quansical" theory.)

In what follows I wish to show that Matrix String Theory of Banks, Fischler, Shenker and Susskind \[1\] can be understood as an example of "quansical" theory (see [8] for an application of the same set of ideas to the planar Yang-Mills theory). The Matrix String Theory Hamiltonian written in terms of the nine non-commutative coordinates \( X_i \), and their sixteen fermionic superpartners \( \Phi \) reads (\( R \to \infty \) defines the decompactified limit of M theory, as in [1])

\[
H = R tr (\frac{1}{2} P_i^2 + \frac{1}{4} [X_i, X_j]^2 + \Phi^T \gamma_i [\Phi, X_i]), \tag{2}
\]
where $P_i$ is canonically conjugate to $X_i$. In the following I also wish to present a particular heuristic realization of Matrix String Theory treated as an example of "quansical" theory which should be useful for further analyses of many important dynamical questions in Matrix String Theory. The emphasis is placed on the planar nature of the theory. Possible subtleties due to supersymmetry are not taken into consideration.

2. Planar Quantum Theory, Alias "Quansical" Theory

First let me state the definition of planar quantum theory or "quansical" theory. The Feynman-Schwinger differential variational principle (1) can be used to postulate the following Euclidean version of quantum dynamics that is consistent with factorization [9]. (I concentrate on the bosonic variables for the time being):

$$\left(\frac{\delta S}{\delta X_\mu} - 2\Pi_\mu\right)|0\rangle = 0,$$

where

$$[X_\mu, \Pi_\nu] = \delta_{\mu\nu} \hbar |0\rangle\langle 0|.$$

Equations (3) and (4) simply represent the original equation (1) written in such a way so that the property of factorization is valid, in other words, so that the only state that survives the planar limit is the ground state. Equations (3) and (4) define a Euclidean planar quantum theory, or Euclidean "quansical" theory.

Likewise, one could consider a Hamiltonian version of the same planar quantum theory (see again [9]). The "quansical" commutation relations are given by

$$[X_i, P_j] = i\delta_{ij} \hbar |0\rangle\langle 0|.$$

(In other words, in the expansion of unity that appears in the usual canonical commutation
relations \([X_i, P_j] = i\delta_{ij}\hbar\)

\[
1 = |0\rangle\langle 0| + \sum_{n=1}^{\infty} |n\rangle\langle n|, \tag{6}
\]

only the first term, which is a projection operator, is kept. Expression (5) can be taken as the starting point for a discussion of a generalized version of the Heisenberg uncertainty principle, and therefore, a generalized version of Bohr’s complementarity principle.) The dynamical equations of motion are given by the familiar expressions

\[
i[H_r, X_i] = \hbar \dot{X}_i, \tag{7}
\]

and

\[
i[H_r, P_i] = \hbar \dot{P}_i. \tag{8}
\]

It is important to note that \(H_r\) represents the reduced Hamiltonian (reduced onto the ground state of the theory). Equations (5), (7) and (8) define a Hamiltonian planar quantum theory, or Hamiltonian ”quansical” theory. I wish to adopt the Hamiltonian version of ”quansical” theory in the following discussion of Matrix String Theory.

3. Fermionic Realization

It seems apparent that it is absolutely crucial to come up with a suitable realization of the ground state in order to be able to use the above definiton of planar quantum theory. Consider then the following concrete realization of such generalized quantum Hamiltonian dynamics based on a very particular representation of the projection operator in the ”quansical” commutation relations (5):

\[
|0\rangle\langle 0| = \psi^\dagger \psi, \tag{9}
\]

where \(\psi^2 = \psi^\dagger^2 = 0\), \(\psi\psi^\dagger + \psi^\dagger\psi = 1\), i.e. \(\psi\) and \(\psi^\dagger\) are fermionic operators. This representation is suggestive of a fermionic ground state.
Using the commutation relation (5) one deduces that, at least formally,

\[ P_j = i\hbar X_j^{-1}(\delta_{ij}C + X_j^{-1}\delta_{ij}CX_i + (X_j^{-1})^2\delta_{ij}CX_i^2 + ...), \]  

(10)

where \( C^2 = C \) is the projection operator \(|0\rangle\langle 0| = \psi^\dagger\psi\). Likewise, the following formal expression for \( H_r \), in terms of \( X_i \) and \( P_i \), stems from (7) (\( P_i = \dot{X}_i \))

\[ H_r = i\hbar X_i^{-1}(P_i + X_i^{-1}P_iX_i + (X_i^{-1})^2P_iX_i^2 + ...). \]  

(11)

Therefore, given the particular realizations of the projection operator \(|0\rangle\langle 0|\) and \( X_i \) the relevant representations of \( P_i \) and \( H_r \) follow from (10) and (11). The reduced (or effective) planar Hamiltonian \( H_r \) completely defines the dynamics of the planar limit of a matrix theory under cosideration (in the present context, Matrix String Theory).

So, apart from choosing a suitable realization for the ground state (such as (9)) one has to pick a particular representation for \( X_i \), that is compatible with the already chosen realization for the ground state, and then deduce the reduced planar Hamiltonian.

Suppose the following dictionary is used to postulate a particular realization of \( X_i \) (and its supersymmetric partners \( \Phi \)):

\[ X_i \rightarrow x_i(\alpha, \beta), \Phi \rightarrow \phi(\alpha, \beta), \]  

(12)

where \( \alpha \) and \( \beta \) are two real parameters, and \( x_i \)'s and \( \phi \)'s are functions of \( \alpha, \beta \). (In other words, the matrix indices, which play the role of internal parameters and which in the planar limit run from zero to infinity, are replaced by two continuous, external parameters \( \alpha \) and \( \beta \).) Likewise, let the commutator bracket be replaced according to the following prescription:

\[ [X_i, X_j] \rightarrow \{x_i(\alpha, \beta), x_j(\alpha, \beta)\}, [\Phi, X_i] \rightarrow \{\phi(\alpha, \beta), x_i(\alpha, \beta)\}, \]  

(13)
where \( \{,\} \) denotes the ordinary Poisson bracket with respect to \( \alpha \) and \( \beta \), in other words
\[
\{ x_i(\alpha, \beta), x_j(\alpha, \beta) \} = (\partial_\alpha x_i(\alpha, \beta)\partial_\beta x_j(\alpha, \beta) - \partial_\beta x_i(\alpha, \beta)\partial_\alpha x_j(\alpha, \beta)).
\] (14)

Thus all expressions containing the non-commutative coordinates and the commutator are to be replaced with the "identically" looking ones, after the translation defined by (12) and (13) has been applied. Also, the operation of tracing should be replaced by the operation of integration over the extra continuous parameters \( \alpha \) and \( \beta \)
\[
Tr \to \int d\alpha d\beta.
\] (15)

(This is essentially the dictionary of [10], where the equivalence between the \( SU(\infty) \) Lie algebra and the algebra of area preserving diffeomorphisms of a two-dimensional sphere \( S^2 \), parametrized by \( \alpha \) and \( \beta \), was presented.)

The \( SU(\infty) \) structure constants translate according to
\[
X^c_i t^c \to \sum_{lm} x^l m Y^m (\alpha, \beta),
\] (16)
where \( t^c \) are the generators of \( SU(\infty) \) and \( Y^m (\alpha, \beta) \) are the \( S^2 \) spherical harmonics. The \( SU(\infty) \) structure constants are then identified with the structure constants of the area preserving diffeomorphisms of a two-sphere, defined in terms of the spherical harmonics basis [10]
\[
\{ Y^m l, Y^m l' \} = f^{m'' m'} f_{l'' l'} Y^m l' \] (17)

The expression defining the \( SU(\infty) \) gauge transformations (which generalizes the usual coordinate transformations of commuting coordinates) reads, for example, as follows
\[
\delta x_i(\alpha, \beta) = \{ x_i, \Omega \}.
\] (18)
The Matrix String Theory Hamiltonian, on the other hand, becomes

\[ H = \mathcal{R} \int d\alpha d\beta \left( \frac{1}{2} p_i^2 + \frac{1}{4} \{ x_i, x_j \}^2 + \phi^T \gamma_i \{ \phi, x_i \} \right), \]  

(19)

where \( p_i \to -i \frac{\delta}{\delta x_i} \).

The above realization of \( X_i \) is multiplicative, that is

\[ X_i f(x) = x_i f(x), \]  

(20)

where \( f(x) \) is an appropriate functional of \( x_i \). (Remember that \( x_i \)'s are functions of \( \alpha \) and \( \beta \).) Given that fact, it formally follows from (10) that

\[ P_i f(x) = \int Dz_j(\alpha, \beta) \delta_{ij}(z) \rho(z)f(z) \frac{\delta \rho(z)}{x_j - z_j}. \]  

(21)

Here, by definition, \( \delta_{ij}(z) \to \delta_{ij} \) for \( z \to x \). and the scalar product of two wave functionals is defined as

\[ \langle fg \rangle = \int Dx_i \rho(x)f^*(x)g(x), \]  

(22)

where the weight functional \( \rho \) satisfies the following constraint (as in [11] [12], even though the present \( \rho \) is not related to any density of eigenvalues)

\[ \int Dx_i \rho(x) = 1. \]  

(23)

Note that (23) implies that the ground wave functional is set to one, so that the ground state is completely described in terms of the weight functional \( \rho \). In other words, the physical nature of the weight functional \( \rho \) is that it could be taken as another realization of the ground state. (By taking \( \rho \) to be a constant equal to the inverse of \( \int Dx_i \) one can recover the canonical commutation relations and the canonical scalar product of two wave functionals. From this point of view the functional \( \rho \) is a "deformation" functional.
responsible for the described generalization of ordinary quantum theory into “quansical”
theory.) The action of the projector $|0\rangle\langle 0|$ is given by the following expression (in view of
(22))

$$\psi^\dagger \psi |f\rangle = \int Dx_i \rho(x) f(x).$$  \hfill (24)

This particular relation can be taken to define a fermionic ground state of the theory. (The
same statement is familiar from the original one-matrix model study [12], even though it is
clear that the present realization does not speak of any eigenvalues.) More precisely, there
exists a Fermi surface, the Fermi energy playing the role of a Lagrange multiplier due to
the constraint (23).

That can be seen from the ”quansical” commutation relations (5), given the fermionic
realization of the projector (9). First note that (5) implies $\langle 0|[X_i, P_j]|0\rangle = i\delta_{ij}$, which in
view of (12) implies $P_i \rightarrow p_i(\alpha, \beta)$. Then note that due to the fact that $\psi^\dagger \psi$ is a fermion
number operator, each phase cell $\Delta x_i \Delta p_i$ contains a single fermion. (The same fermion
number operator serves as a generator of the area preserving diffeomorphisms of $S^2$, which
is compatible with (12) and (13).) The ground state is then essentially characterized by a
certain region of the functional phase space $Dx_i Dp_i D\phi^T D\phi$ which possesses the property
of incompressibility according to the Liouville theorem. In other words the following
constraint is valid

$$\int Dx_i Dp_i D\phi^T D\phi \theta(e - H) = 1,$$  \hfill (25)

where $H$ denotes the Hamiltonian (19), $e$ stands for the characteristic Fermi energy and $\theta$
is the usual step function.

Equation (25) tells us that the volume of the functional phase space fluid is to be
normalized to one in such a way, as if there existed a single fermion placed at each phase
space cell, and consequently, taking into account the Pauli exclusion principle, as if there
existed, in the limit of a large number of cells, an incompressible fermionic fluid, with the Fermi energy $e$. By recalling that each phase space cell has a natural volume of the order of the Planck constant and that the planar limit corresponds to a situation where the number of cells goes to infinity, the product of the Planck constant and the number of cells can be adjusted to one (the reason being that the $1/N$ expansion formally corresponds to a "semiclassical" expansion, $1/N$ acting as an effective "Planck constant"). Then follows the relation (25), describing an incompressible drop of functional phase space of unit volume.

(Note that the appearance of fermions could be intuitively understood from the point of view of 't Hooft's double-line representation for the planar graphs [5]. The fact that such lines do not cross in the planar limit is achieved by attaching fermions to each line and using the exclusion principle.)

Therefore, equation (25) gives a rather natural, even though implicit, realization of the ground state of Matrix String Theory, that is compatible with the dictionary (12) and (13).

Now one can write down the reduced planar Hamiltonian. According to the fermionic-fluid picture of the ground state (25) the effective planar Matrix String Theory Hamiltonian is simply given by

$$H_r = \int Dx_i Dp_i D\phi^T D\phi (R \int d\alpha d\beta \left( \frac{1}{2} p_i^2 + \frac{1}{4} \{x_i, x_j\}^2 + \phi^T \gamma_i \{\phi, x_i\}\right) \theta(e - H),$$

or in terms of a fermionic functional $\Psi$ which describes the fermionic nature of the vacuum

$$H_r = R \int d\alpha d\beta \int Dx_i D\phi^T D\phi \left( \frac{1}{2} \frac{\delta \Psi^\dagger}{\delta x_i} \frac{\delta \Psi}{\delta x_i} + \left( \frac{1}{4} \{x_i, x_j\}^2 + \phi^T \gamma_i \{\phi, x_i\} - e \right) \Psi^\dagger \Psi \right).$$

This formula can be understood as the usual expression for the ground state energy, written in a second quantized manner, after taking into account the fact that the ground state of the planar theory is fermionic, as implied by (25).
Note that the above expressions for the effective Matrix String Theory Hamiltonian contain functional integrals, the fact which tells us that we are not dealing with an ordinary field theory.

One could use the bosonic weight functional $\rho = \Psi \dagger \Psi$ as a "collective functional" in the spirit of [11]. Then the expression (26) could be interpreted as the effective potential of the Das-Jevicki-Sakita collective-functional Hamiltonian. The minimum of the effective potential, which determines the ground state of Matrix String Theory, is in turn given by (25). Unfortunately, unlike in the one-matrix model case [12], a simple explicit expression for the collective functional $\rho$ cannot be readily obtained.

4. In Lieu Of Conclusion

Matrix String Theory presents a plausible non-perturbative formulation of string theory in which the number of degrees of freedom differs from ordinary quantum field theory, yet because of the nature of the planar limit, it also differs from ordinary quantum mechanics. Actually, as this article attempts to show, Matrix String Theory is not an ordinary quantum theory. That fact is indicated by equations (5), (7) and (8) which define a Hamiltonian version of "quansical" theory, or planar quantum theory. (The ground state of the theory being given by (25) and the effective planar Hamiltonian being given by (27).) This theory in turn is different from Adler’s generalized trace quantum dynamics, which can be also applied to the structure of Matrix String Theory [4].

The outlined "quansical"-theoretic structure is closely related to Connes’ non-commutative quantized calculus [13]. For example, consider the definition of Connes’ non-commutative ”quantum” derivative [13]

$$d_c O = [F, O],$$  \hspace{2cm} (28)
where $F$ is a self-adjoint operator such that $F^2 = 1$ or $F = 2C - 1$, where $C$ is a projection operator $C^2 = C$. (For example, $C = \psi^\dagger \psi$, or $F = \psi^\dagger + \psi$, where $\psi^\dagger$ and $\psi$ are fermionic as before.) Hence, $d_c O$ anticommutes with $F$. In principle, the "quansical" commutation relations (5) can be written as

$$[[X_i, P_j], X_k] = \frac{1}{2} i\hbar \delta_{ij} d_c X_k. \quad (29)$$

Then $P_i$ can be expressed in terms of $X_j$ and $d_c X_k$. The resulting expression turns out to be unfortunately rather complicated. (It is interesting to note, though, that in the zero-dimensional case the nature of the operator $F$ is uniquely determined [13], the operator $F$ being given by the familiar expression for the Hilbert transform, which appears in the usual treatment of the one-matrix model [12], and the non-commutative "quantum" derivative being given by the following intuitively appealing expression $d_c O f(s) = \int \frac{O(s) - O(t)}{s-t} f(t)dt$, where $f(s)$ is some suitable function.)

Perhaps even more important is the relation of (5) to non-commutative probability theory of Voiculescu [14], especially the idea of Free Fock spaces, defined by the action of "free" operators $a_i$, $a_j^\dagger$ ($a_i a_j^\dagger = \delta_{ij}$, and $a_i |0\rangle = 0$). As shown in [14], the "quansical" commutation relations (5) can be naturally obtained if appropriate operator representations of $X_i$ and $P_j$ are given in terms of free operators $a$ and $a^\dagger$. (In the present context this fact would imply that the non-perturbative Matrix String Theory Fock space is a Free Fock space.) Note that this operator representation is quite different from the one considered in section 3., which was an explicit fermionic representation of the ground state compatible with the particular representation of the fundamental non-commuting variables of Matrix String Theory.

A few words are in order about the nature of the above realization as compared to the old matrix model formulation of zero and one-dimensional non-perturbative string
physics [15]. The two are indeed very similar in spirit. (The above formulation is in some
sense an extension of [15].) One obvious difference is the use of functionals (as opposed
to functions [15]) in the present approach. Perhaps the most striking difference between
the two approaches is the appearance of two extra continuous parameters ($\alpha$ and $\beta$) in the
above formulation, compared to one extra parameter-dimension (the familiar eigenvalues
of the one-matrix model) of [15]. Unfortunately most of the above functional expressions
are still quite formal. The question of renormalization after appropriate regularization has
been completely ignored. One would expect the appearance of one extra parameter (the
renormalization scale $\lambda$) making the total number of parameters in the above Hamilto-
nian formulation equal to three. The true dynamics of Matrix String Theory would be
then governed by a Wilsonian non-perturbative RG equation describing a scale-by scale
evolution of the effective Matrix String Theory Hamiltonian (27).

I close this article with a sketch of a possible speculative interpretation/visualization
of the outlined realization of Matrix String Theory as a generalized quantum theory. The
"quansical" commutation relations (5) appear natural from the point of view of quantum
cosmology: the projection operator $|0\rangle\langle 0|$ nicely captures the classical features of a quan-
tum system, such as the Universe, that at some point during its evolution becomes very
large (that is - classical) as compared to some initial fundamental scale. The fermionic
ground state (the ground state of the Universe) could be interpreted from the point of view
of quantum information theory [16] and in accordance with Susskind’s holographic principle
[17]. A bit of information obtained through quantum measurements is created/destroyed
by the action of creation/annihilation fermionic operators that feature in the second quan-
tized description of the fermionic ground state. All information is stored at the boundary
of a region defining a fundamental "hole" (or "monad") of space (a gauge invariant bound
state of Matrix String Theory) inside which it is in principle impossible to make any observations, due to the non-commutative character of space within the ”hole”. The Fermi energy corresponds in this picture to that fundamental energy scale above which it is in principle impossible to obtain information through quantum measurements. Given such fermionic quantum-informational ground state of the Universe, the non-commutative coordinates that appear in (5) could be understood as being dynamically induced through wave functional overlaps, or in other words, as being the same infrared stable planar ”gauge connections” that feature in the geometric phase of the fermionic ground state.

I thank Shyamoli Chaudhuri, Joseph Polchinski and Branko Urošević for useful comments on the initial manuscript. I am also grateful to Bunji Sakita for drawing my attention to his work with Kavalov, and I thank V. P. Nair for initial discussions. I also wish to thank members of the University of Chicago Theory Group for their kind hospitality. This work was fully and generously supported by the Joy K. Rosenthal Foundation.

References
1. T. Banks, W. Fischler, S. Shenker, L. Susskind, Phys. Rev. D55 (1997) 5112.
2. E. Witten, Nucl. Phys. BB460 (1995) 335.
3. For a review of D-brane physics see: J. Polchinski, S. Chaudhuri and C. V. Johnson, ”Notes on D-branes”, NSF-ITP-96-003, hep-th/9602052.
4. S. L. Adler, Nucl. Phys. B415 (1994) 195; hep-th/9703053.
5. G. ’t Hooft, Nucl. Phys. B72 (1974) 461; E. Witten, Nucl. Phys. B160 (1979) 519; for a collections of papers on large N methods consult The large N Expansion in Quantum Field Theory and Statistical Mechanics, eds. E. Brezin and S. Wadia, World Scientific, 1994.
6. R. P. Feynman and A. R. Hibbs, Quantum mechanics and Path Integrals, McGraw-
Hill, 1965.

7. J. Schwinger, *Quantum Kinematics and Dynamics*, W. A. Benjamin, 1970.

8. D. Minic, *On The Planar Yang-Mills Theory*, May 1997.

9. K. Bardakci, Nucl. Phys. B178 (1980) 263; O. Haan, Z. Phys C6 (1980) 345; M. B. Halpern, Nucl. Phys. B188 (1981) 61; M. B. Halpern and C. Schwarz, Phys. Rev. D24 (1981) 2146; A. Jevicki and N. Papanicolaou, Nucl. Phys. B171 (1980) 363; A. Jevicki and H. Levine, Phys. Rev. Lett 44 (1980) 1443; Ann. Phys. 136 (1981) 113.

10. E. G. Floratos, J. Iliopoulos and G. Tiktopoulos, Phys. Lett. B217 (1989) 285; J. Hoppe, Ph. D. thesis (MIT, 1988); see also A. Kavalov and B. Sakita, hep-th/9603024.

11. B. Sakita, Phys. Rev. D21, (1980) 1067; A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511; Nucl. Phys. B185 (1981) 89; S. R. Das and A. Jevicki, Mod. Phys. Lett. A5 (1990) 1639.

12. E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, Comm. Math. Phys. 59 (1978) 35.

13. A. Connes, *Noncommutative Geometry*, Academic Press, 1994.

14. D. V. Voiculescu, K. J. Dykema and A. Nica, *Free Random Variables*, AMS, Providence 1992; M. Douglas, Phys. Lett. B344 (1995) 117; Nucl. Phys. Proc. Suppl. 41 (1995) 66; R. Gopakumar and D. Gross, Nuc. Phys. B451 (1995) 379.

15. D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635; E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144; for a highly inspiring review consult: J. Polchinski *What is String Theory?*. 1994 Les Houches lectures, hep-th/9411028.

16. See J. A. Wheeler and W. Zurek, eds., *Quantum Theory and Measurement*, Princeton University Press, 1983.

17. L. Susskind, J. Math. Phys. 36 (1995) 6377.