Symmetry Breaking in UCA-Based Vortex Waves

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Abstract. In literature, it is often said that a uniform circular array (UCA) composed of a sufficient number of radiating elements is able to generate a vortex wave with arbitrary topological charge. While providing a very simple and intuitive solution without the need for any synthesis method, the excitation of UCAs by means of a progressive phase shift does not always guarantee that vortex waves are correctly generated, since a major role is played by the symmetry and radiation properties of each individual element which constitutes the array. When the UCAs are employed in vortex communication links, these symmetry properties strongly affect the orthogonality of the channel modes and, therefore, the communication efficiency. In this work, the symmetry breaking associated with the radiation of vortex waves by UCAs and its impact on the mode orthogonality are explored in detail.

1. Introduction
In the last years, electromagnetic waves characterized by a vortex term \( \Phi_m(\varphi) = \exp(im\varphi) \), where \( \varphi \) is the azimuthal angle and the integer \( m \) represents the so-called topological charge, have become extremely popular, especially by virtue of their orbital angular momentum (OAM) content (for a review of the applications, see [1–4] and references therein). Vortex waves are usually described as solutions of the homogeneous Helmholtz equation, often in the paraxial regime. Since all these source-free modes just correspond to mathematical idealizations, the problem of generating realistic vortex waves has been largely addressed at both optical and radio frequencies.

In optics, vortex waves are synthesized in a variety of possible ways, ranging from computer generated holograms [5] to cylindrical lenses [6] and spiral phase plates [7]. Most of the techniques involve the transformation of conventional laser radiation in waves carrying OAM by means of suitable optical devices and mode converters, the former being able to imprint the characteristic phase topology on the incoming standard beam through an azimuthal variation of the refractive index, the latter exploiting the completeness of the modes (see also [8,9]). Recently, many researchers have also dealt with the conversion of the electromagnetic angular momentum from spin to orbital, bringing to light the concept of q-plate as a further technique for the generation of OAM beams [10,11]. Besides their utility in a wide series of optical experiments, q-plates...
represent indeed one of the most intriguing applications of the well-known Pancharatnam-Berry phase phenomenon, presented in [12].

Several of the above mentioned techniques have been extended to the radio frequency domain, where many other methods have also been proposed, leading to an explosive growth in OAM-related publications ([13–20], just to cite a few). As originally proposed in [21], antenna arrays can be employed to generate vortex radio waves upon proper choice of the feeding currents/voltages. The use of uniform circular arrays (UCAs), which consist of a circle of equally spaced radiators, probably represents one of the best solutions, exploiting the natural vortex waves circular symmetry for the benefit of a reduced number of antennas. Despite these advantages, in this work it is shown that the circular symmetry of the antenna array is not enough to attain good mode purity and also the radiation and symmetry properties of each constituent element must be taken into account. This, in turn, strongly impacts on the efficiency of a vortex communication link based on UCAs, an issue that seems to have been rather neglected so far.

In Section 2 the vortex waves generation by means of UCAs is reviewed and the problem of symmetry breaking of the field profiles is addressed with the help of several examples. The vortex content of the channel modes relative to some communication links between UCAs is explored in Section 3, where the effect of symmetry breaking on the mode orthogonality is presented. The most important results are briefly summarized in the conclusions. As a support to the main document, four detailed appendices are devoted to theoretical insights and additional calculations.

2. Generation of vortex waves via phased antenna arrays

Antenna arrays are configurations of multiple connected antennas, distributed according to a specific geometry, in which the excitation of each radiator can be controlled in amplitude and phase [22, 23]. As the electromagnetic waves radiated by the constituent elements combine coherently, affecting the spatial distribution of the transmitted power, antenna arrays represent one of the most manageable and versatile solutions to deal with the synthesis of complex waveforms at the radio frequencies. Furthermore, unlike the majority of other systems involving a single radiator, they are well suited for the simultaneous transmission of several independent beams, each of which can be encoded with a different information content and electronically controlled in direction, thus ensuring a high flexibility for multiplexing [24–26]. In the theoretical model that will be developed, the presence of an ideal beamforming network which provides the array excitations is assumed: noise, energy losses and couplings among the antennas will not be considered. Actually, in view of a practical implementation, many of the unbalances that may come into play in a realistic scenario could be treated with the help of the electronics or digitally controlled systems.

According to the literature, a UCA composed of \( N \) arbitrary elements can generate a vortex wave with topological charge \( m \) in the integer range \((-N/2, N/2)\) when the input excitation to the \( n \)-th antenna terminals is expressed as:

\[
\Phi_m^n = \frac{1}{\sqrt{N}} \exp \left( im \phi_n \right) = \frac{1}{\sqrt{N}} \exp \left[ 2\pi i m \left( \frac{n - 1}{N} \right) \right],
\]

being \( \phi_n \) the angular position of the \( n \)-th element of the UCA, as shown in Figure 1. The set \( \{ \Phi_m^n \} \) can be interpreted as the discretized version of the function \( \Phi_m (\varphi) / \sqrt{2\pi} \), where \( \Phi_m (\varphi) \) is the azimuthal phase term characterizing vortex waves. Orthogonality and completeness relations (to be compared with their continuous counterparts) are also derived:

\[
\sum_{n=1}^{N} \Phi_j^n * \Phi_m^n = \delta_{jm}; \quad \sum_{m} \Phi_m^n (\Phi_m^n)^* = \delta_{np},
\]

(2)
Figure 1. Graphical representation of a UCA composed of $N = 12$ elements and of the excitation coefficients $\Phi_{n}^{m}$, defined in (1), for some values of $m$.

where the azimuthal indices $m$ and $j$ are required to span a set of $N$ integer values, that is $[-N/2, N/2]$ or $(-N/2, N/2]$ for $N$ even, in which case the extreme values $\pm N/2$ lead to degenerate configurations with spurious vortex contribution$^{1}$ and $[-N/2, N/2]$ for $N$ odd. Both equations in (2) directly follow from the summation formula of the geometric progression to $N$ terms $^{27}$.

Let us consider a UCA of $N$ ideal isotropic radiators lying in the $xy$ plane; the frequency-domain analytic expression of the scalar vortex wave generated by such distribution reads:

$$f_{m}(r) = \int d^{3}r' g_{+}(r-r') \sum_{n=1}^{N} \Phi_{n}^{m} \delta^{3}(r'-r_{n})$$

where $g_{+}$ is the outgoing-wave Green function defined in (A.1), $m$ is the topological charge, $|m| < N/2$, $r_{n}$ identifies the position of the $n$-th spherical wave source and all dimensional constants have been neglected for simplicity. Equation (3) is derived from (A.2) upon proper choice of the source term. Although approximated closed-form expressions could be obtained from (3) under specific assumptions (for example, a very large number of array elements), these will not be taken into consideration for brevity. Instead, we can resort to the useful far-field approximation $r \gg a$, being $a$ the radius of the array, which leads to:

$$f_{m}(r, \theta, \phi) \approx -\frac{\exp(-ikr)}{4\pi \sqrt{Nr}} \sum_{n=1}^{N} \exp[ika \sin \theta \cos(\phi - \varphi_{n}) + im\varphi_{n}],$$

where $\{r, \theta, \phi\}$ represents the spherical coordinate system and the expression $r_{n} = a \{\cos \varphi_{n}, \sin \varphi_{n}, 0\}$ has been employed.

Figure 2 displays some of the transverse intensity and phase profiles of the function $f_{m}(r)$ at varying $m$ in a fixed $z$-plane for a UCA like that reported in Figure 1. The mode patterns are

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$^{1}$ Sampling arguments can be used to show that for $m = \pm N/2$ and $N$ even the two corresponding phase weighting configurations $\{\Phi_{n}^{m}\}$ coincide and therefore cannot excite distinct fields. The pattern relative to such limiting cases does not represent a pure vortex wave but rather a superposition of the two.
Relative intensity (upper row) and phase (lower row) profiles of the scalar wave $f_m$ in (4) for $a = \lambda = 1$ m, $N = 12$ and $z = 10 \lambda$. To better highlight fainter details, the $m = 0$ intensity maximum has been clipped.

found to be in good agreement with those expected for a scalar vortex solution but, of course, neither (3) nor (4) suffices to describe a realistic vortex waveform.

Let us now move to the case of an arbitrary $N$-element antenna array composed of more realistic radiators. The knowledge of the current term $J(r)$ enables the estimation of the electric and magnetic fields via (A.5) and (A.6), from which the corresponding far-field expressions are easily derived [23]. Neglecting all mutual couplings among the antennas, the total electric field radiated by the considered array in a region where $r$ is much greater than the spatial extent of the antennas can be written as:

$$E(r, \theta, \phi) = ik\eta I_0 \sum_{n=1}^{N} \xi_n \exp\left(\frac{-ik |r - r_n|}{4\pi |r - r_n|}\right) h_n(\theta, \phi),$$

being $\eta$ the vacuum impedance, $I_n = I_0 \xi_n$ the supply current to the $n$-th radiator, where $\xi_n \in \mathbb{C}$ is the relative synthesis coefficient and $I_0$ a constant current term associated to the total input power, whereas $h_n(\theta, \phi)$ represents the $n$-th antenna vector effective height (see Appendix A, equation (A.8)).

In deriving equation (5), no assumption was made on the feeding, arrangement and type of antennas employed. By specializing the array geometry to the case of a UCA in the $xy$ plane, it is possible to analyze the properties of the electric field (5) associated with a set of vortex weighting coefficients $\xi_n = \Phi_m^n$. Under this hypothesis, and for $r \gg a$, the following far-field expression holds:

$$E_m(r, \theta, \phi) \approx \frac{ik\eta I_0 \exp\left(-ikr\right)}{4\pi \sqrt{N}r} \sum_{n=1}^{N} h_n(\theta, \phi) e^{ika \sin \theta \cos(\phi - \phi_n) + im\varphi_n}.$$ 

Some of the transverse intensity and phase profiles of the $x$-component of $E_m(r)$ computed in the plane $z = 10 \lambda$ are reported in Figure 3 for the case of a UCA composed of $N = 12$
Figure 3. Relative intensity (upper row) and phase (lower row) profiles of the $x$-component of the electric field $\vec{E}$ radiated by a UCA composed of $N = 12$ $x$-polarized half-wave dipoles, for $a = \lambda = 1$ m and $z = 10\lambda$.

$x$-directed half-wave dipoles. It is fundamental to note how the use of linear sources breaks the degenerate symmetry of the isotropic case, causing a deviation of the patterns from the expected ones. In particular, the intensity distributions are no longer perfectly circular and the phase profiles witness a corruption of the on-axis vortex with charge $m$ in $m$ distinct off-axis vortices with unitary charge.

As shown in Figure 4 and in Figure 5, the standard vortex profiles can be recovered by using low-directive circular symmetric array elements such as crossed dipoles with a $\pi/2$ phase delay, in which case also a spin angular momentum (SAM) contribution will be present. It is worth emphasizing that circular polarization alone is not enough to completely solve the symmetry breaking of vortex waves, as confirmed by the fact that higher order profiles obtained with crossed Hertzian dipoles are better than those produced by crossed half-wave dipoles, the former type of antennas being closer to the ideal spherical wave source.

More generally, owing to the vector nature of the above equations, it is possible to reproduce approximate versions of the vector vortex solutions [28]. For instance, the azimuthally and radially polarized vector beams are found by considering UCAs of dipoles arranged in the $\vec{u}_\phi$-direction and in the $\vec{u}_\rho$-direction, respectively. As an alternative, any physical rotation of the dipoles in the circumference plane can be replaced by a rotation of the current vectors implemented through a UCA of properly fed crossed dipoles displaced along two fixed orthogonal directions.

The electric field norm and polarization plots of some vector waves generated by UCAs of azimuthally and radially polarized half-wave dipoles are displayed in Figure 6, showing some of the interesting features that characterize the theoretical vector vortex solutions. Not surprisingly, the possibility of synthesizing vector vortex beams by means of circular arrays of sources oriented along $\vec{u}_\phi$ and $\vec{u}_\rho$ has also been explored in optics as a theoretical tool for modeling optical vortex beams emitters [30,31].

It is interesting to note that such configurations belong to the class of the so-called ring quasi-arrays, originally proposed long ago and analyzed in connection with their supposed super-gain properties [29].
Figure 4. Relative intensity (upper row) and phase (lower row) profiles of the \( \mathbf{u}_x - i\mathbf{u}_y \) component of the right circularly polarized electric field \( \mathbf{E} \) radiated by a UCA composed of \( N = 12 \) crossed half-wave dipoles arranged along the \( x \) and \( y \) directions, for \( a = \lambda = 1 \) m and \( z = 10\lambda \).

Figure 5. Phase profiles of the electric field \( \mathbf{E} \) radiated by a UCA composed of \( N = 12 \) half-wave dipoles (\( x \)-component, first row), right circularly polarized crossed half-wave dipoles (\( \mathbf{u}_x - i\mathbf{u}_y \) component, second row) and right circularly polarized crossed Hertzian dipoles of length \( \lambda/20 \) (\( \mathbf{u}_x - i\mathbf{u}_y \) component, third row), for \( a = \lambda = 1 \) m and \( z = 10\lambda \).
Figure 6. Norm and polarization plots of the electric field \( \mathbf{E} \) radiated by a UCA composed of \( N = 25 \) half-wave dipoles arranged along the \( u_\varphi \)-direction (upper row) and the \( u_\rho \)-direction (lower row), for \( a = 1 \text{ m}, \lambda = 0.4 \text{ m} \) and \( z = 10\lambda \). Some of the intensity maxima have been clipped for better clarity.

As a further point, one may wonder if it is possible to generalize the set \( \{ \Phi^m_n \} \) in (1) beyond the UCA case without the need for a complete array synthesis procedure (that usually requires to solve a linear inverse problem). For instance, the new set of vortex excitation coefficients could be heuristically defined via sampling of some orthogonal basis functions which contain the sought term \( \exp (\im \varphi) \) and are supported on the reference volume for the transmitting array.

Limiting to the case of a uniform disk antenna array of radius \( a \), a complete set of functions that are orthogonal in the interior of the disk is provided by the well-known generalized Zernike polynomials [32]:

\[
Z^m_q (\varrho, \varphi) = R^m_q (\varrho) \exp (\im \varphi),
\]

where \( \varrho = \rho/a \) represents the scaled radial coordinate, \( m \) is the azimuthal index, \( q \leq |m| \) corresponds to a non-negative integer such that \( q - |m| \) is even and the radial function is expressed by:

\[
R^m_q (\varrho) = \sum_{s=0}^{q-|m|} \frac{(-1)^s (q-s)!}{s! \left( \frac{q+|m|}{2} - s \right)! \left( \frac{q-|m|}{2} - s \right)!} \varrho^{q-2s}.
\]

For each of the \( N \) antennas regularly distributed over the transmitting disk, \( Z^m_q (\varrho_n, \varphi_n) \) identifies the local value of the Zernike function with indices \( q \) and \( m \) and can be used to build a corresponding column vector of vortex excitations for the array. It should be remembered that the discretization introduces bounds on the possible values of the azimuthal and radial indices and limits the number of effectively orthogonal column vectors. For the sake of simplicity, let us neglect the radial degrees of freedom by fixing \( q = |m| \) and consider the set made of the \( N \)-length column vectors \( \{ Z^m_{|m|} (\varrho_n, \varphi_n) \} \).

Figure 7 and Figure 8 display the electric field profiles relative to some of the effectively orthogonal sampled coefficients for a disk array composed of uniformly spaced \( x \)-directed
Figure 7. Synthesis of some vortex waves via sampling of the Zernike polynomials on a uniform disk array of $x$-polarized Hertzian dipoles of length $\lambda/20$: sketch of the geometry (left), relative intensity and phase profiles of the $x$-component of the synthesized electric field evaluated at $z = 10\lambda$ (right), for $N = 133$ and $a = \lambda = 1$ m.

Figure 8. Synthesis of some vortex waves via sampling of the Zernike polynomials on a uniform disk array of right circularly polarized crossed Hertzian dipoles of length $\lambda/20$: sketch of the geometry (left), relative intensity and phase profiles of the $u_x - iu_y$ component of the synthesized electric field evaluated at $z = 10\lambda$ (right), for $N = 133$ and $a = \lambda = 1$ m.

Hertzian dipoles and circularly polarized crossed Hertzian dipoles, respectively. The results are easily interpreted in accordance with what seen above and could be generalized to other geometries.
3. Basis sets and circular symmetry

As suggested by David A. B. Miller in [33], the orthogonal spatial information channels, also referred to as the communications modes or simply the channel modes, can be interpreted as the pairs of functions that diagonalize the mathematical operator formalizing the communications between the transmitting and the receiving volumes. In Appendix B the abstract operator is specialized to the case of a finite-dimensional channel matrix which describes the communications between two arbitrary antenna arrays. The same concepts are applied to some specific examples in the present section. Within the proposed framework, the singular vectors of the channel matrix are analyzed in relation to the standard vortex modes basis, shedding light onto some of the interesting features that arise from the intimate connection between system geometry and symmetry properties of the channel modes.

In the following, transmitting and receiving arrays are considered identical in spatial disposition, number of elements, labeling order and type of constituent antennas, for simplicity. Under this assumption, the Cartesian coordinates \( r^R_p = \{ x^R_p, y^R_p, z^R_p \} \) defining the locations of the receiving elements are linked to those relative to the transmitting ones \( r^T_n = \{ x^T_n, y^T_n, z^T_n \} \) via the matrix equation:

\[
r^R_p = R r^T_n + t, \tag{9}
\]

where \( R \) represents a three-dimensional rotation matrix and \( t \) a translation vector (for instance, \( R = I \) and \( t = \{0, 0, d\} \) in the case of perfectly aligned antenna arrays separated by a distance \( d \)). From simple geometrical arguments, the set of unit vectors \( v = \{ u_x, u_y, u_z \} \) employed to describe the antenna orientation transforms accordingly:

\[
v^R_p = R^{-1} v^T_p. \tag{10}\]

As a straightforward example, let \( \mathcal{V}_T \) and \( \mathcal{V}_R \) be two identical facing circumferences of radius \( a \) at a distance \( d \) apart and suppose two UCAs composed of \( N \) \( x \)-polarized Hertzian dipoles of length \( L \) to be arranged along such circumferences. Since the effective height of a \( z \)-directed Hertzian antenna placed in the origin reads \( l \sin \theta u_\theta \), the channel matrix (B.3) for this aligned communication system is simply given by:

\[
H_{pn} = i k \eta I_0 \frac{\exp \left( -i k \left| r^R_p - r^T_n \right| \right)}{4 \pi \left| r^R_p - r^T_n \right|} \left( l \sin \theta_{np} \right)^2, \tag{11}
\]

being \( \theta_{np} \) the angle between the vector \( \left( r^R_p - r^T_n \right) \) and the \( x \)-axis. On using the explicit expressions for \( r^R_p \) and \( r^T_n \), equation (11) reduces to:

\[
H_{pn} = i k \eta I_0 \frac{\exp \left( -i k \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left( \frac{2\pi(p-n)}{N} \right) \right\}} \right)}{4 \pi \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left( \frac{2\pi(p-n)}{N} \right) \right\}}} \left( l \sin \theta_{np} \right)^2. \tag{12}
\]

In Figure [9] are reported some density plots of the electric field radiated in the transverse plane \( z = d = 10\lambda \) by the transmitting UCA when the first five right singular vectors of the channel matrix (12) are employed as excitation coefficients. Analogous profiles can be found for the left singular vectors. Despite no one of the wavefields excited by the considered channel modes presents vortex character, some important subtleties are involved and will now be discussed.

In order to minimize the notation, let \( |m| = \{ \Phi^m \} \) be a column vector in \( \mathbb{C}^N \) and suppose the azimuthal index value \( m \) to be specified via the following rule:

\[
m_j = \begin{cases} \frac{N}{2} + j - 1 & \text{N even;} \\ \frac{N-1}{2} + j - 1 & \text{N odd,} \end{cases} \tag{13}\]
Figure 9. Norm (upper row) and phase (x-component, lower row) profiles of the electric field radiated by a UCA composed of $N = 25$ $x$-polarized Hertzian dipoles with excitation coefficients given by the right singular vectors $\left\{\xi_n^{(i)}\right\}$ of the channel matrix (12), for $a = \lambda = 1$ m, $l = \lambda/20$, $d = 10\lambda$. A clipping of the first three intensity maxima has been introduced to avoid faint plots.

with $j$ running from 1 to $N$. By this convention, the collection $\left\{\left|m_j\right\rangle\right\}$ provides a basis set for the space of all the possible UCA excitations $\left|\xi\right\rangle = \{\xi_n\}$ and the orthogonality and completeness relations (2) can be rewritten in Dirac notation:

$$\langle m_i | m_j \rangle = \delta_{ij}; \quad \sum_{j=1}^{N} |m_j\rangle \langle m_j| = I,$$

where $I$ is the $N \times N$ identity matrix. Now, labeling the $i$-th right singular vector of the channel matrix $\left|e_i\right\rangle = \{\xi_n^{(i)}\}$, the scalar product:

$$c_{ji} = \langle m_j | e_i \rangle$$

can be understood as the $j$-th expansion coefficient of $\left|e_i\right\rangle$ over the vortex modes basis $\left\{\left|m_j\right\rangle\right\}$, namely:

$$\left|e_i\right\rangle = \sum_{j=1}^{N} |m_j\rangle \langle m_j| e_i \rangle = \sum_{j=1}^{N} c_{ji} |m_j\rangle,$$

as follows from the completeness condition.

Figure 10 (left) displays the projection (15) for the right singular vectors of the channel matrix (12), highlighting the presence of spurious vortex contributions in the matrix spectrum. This circumstance can be traced back to the use of non-isotropic antennas, as clarified by considering a simplified version of (12):

$$H_{pn}^{(e)} = ik\eta l_0^2 \frac{\exp \left( -ik \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\} } \right)}{4\pi \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\} }},$$

(17)
where the directivity of the array elements has been neglected and the label \((\circ)\) means “isotropic case”. The projection of the right singular vectors relative to (17) on the vortex modes basis is reported in Figure 11 (left), from which it is apparent that each singular vector represents an exact superposition of two vortex modes with opposite topological charges, i.e. \(\pm m\).

As it is clear from both Figure 10 and Figure 11, the singular values of the channel matrix for the transmission between facing UCAs are two-fold degenerate due to the fact that, when resolved in terms of the vortex modes, there is no difference between two configurations with opposite \(m\) (more on this in Appendix D). In the isotropic case (17), pure vortex modes can be directly obtained by implementing a rotation of the singular vectors in their corresponding eigenspaces so as to select one of the two directions along which there is no coexistence of opposite azimuthal indices; in this sense, vortex modes constitute a legitimate basis of singular vectors. This is not surprising, since it has been shown that \(H_{pn}^{(\circ)}\) is diagonalized by the discrete Fourier transform (DFT) matrix \(T_{pi} = \Phi_{p,m}^{n}\), as it only depends on the difference \((p - n)\), and is therefore circulant [34].

While ensuring a pure vortex spectrum of the channel matrix, the removal of the antennas directivity is nonphysical and does not prevent that the radiated vortex waves still reveal some of the inaccuracies shown in Figure 3 unless also the polarization constraint is removed (in which case the electric field must be replaced by a less realistic scalar function).

The above considerations naturally lead to question whether it is possible to get results close to that obtained for the isotropic case without compromising antennas directivity and vector character of the fields. When complex array layouts are involved, synthesis techniques must be employed to determine a proper set of beamforming coefficients able to yield approximations of the sought vortex fields; in this context, the synthesized set might not represent a basis of singular vectors (especially whenever the circular symmetry is not fully guaranteed) or, if it were, there could be no reason to prefer the vortex set over any other. On the contrary, the possibility of relying on the well-known \(a\ priori\) set of coefficients \(\{\Phi_{n,m}\}\) as the very basis which diagonalizes the channel matrix in a scenario involving UCAs would be of great benefit for some applications, where it may bring technological simplifications.

To generate pure vortex beams when only simple UCAs are concerned, we replace the \(x\)-directed elements in the system above with circularly polarized crossed dipoles that provide a major degree of rotational symmetry. Assuming the dipoles of each pair to be oriented along
the $x$-axis and the $y$-axis, respectively, the channel matrix can be written as the scalar product:

$$ H_{pn}^{(\times)} = \left( \frac{E_n^x + s_T E_n^y}{\sqrt{2}} \right) \cdot \left( \frac{h_p^x + s_R h_p^y}{\sqrt{2}} \right), \quad (18) $$

with $E_n^x$ and $E_n^y$ representing the electric fields radiated by the $x$-directed and the $y$-directed elements of the $n$-th pair, respectively, and $h_p^x, h_p^y$ the effective heights of the $p$-th receiving pair. In (18), where the label $(\times)$ stands for “crossed dipoles case”, the complex term $s_T$ corresponds to the ratio between the supply currents of the $y$-polarized and $x$-polarized elements, which is supposed to be the same for all the $N$ pairs; similarly, a further term $s_R$ has been introduced in the receiving pairs which accounts for a possible beamforming phase shift between the signals collected along the two orthogonal directions. A fifty-fifty power splitting between the two elements of each pair is enforced at both the transmitting and the receiving sides through the factors $1/\sqrt{2}$. When $s_T = \pm i$, a circularly polarized electric field is generated along the propagation axis and the choice $s_R = s_T^*$ correctly implements the required chirality inversion upon reception.

For the considered case of facing UCAs, equation (18) reduces to:

$$ H_{pn}^{(\times)} = \left( \sin \theta_{np}^x \right)^2 + s_T s_R \left( \sin \theta_{np}^y \right)^2 + \sin \theta_{np}^x \sin \theta_{np}^y (s_T + s_R) \left( \theta_{np}^x \cdot \theta_{np}^y \right) $$

$$ \quad \times \frac{ik\eta I_0}{2} \exp \left( -ik \sqrt{d^2 + 2a^2} \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\} \right), \quad (19) $$

where the following conventions have been introduced:

$$ r_{np} = \{x_{np}, y_{np}, z_{np}\} = \vec{r}_p - \vec{r}_n; \quad (20) $$

$$ \theta_{np}^x = \arccos \left( \frac{x_{np}}{r_{np}} \right); \quad \phi_{np}^x = \arctan \left( \frac{z_{np}}{y_{np}} \right); \quad (21) $$

$$ \theta_{np}^x = \{ -\sin \theta_{np}^x, \cos \theta_{np}^x, \cos \phi_{np}^x, \sin \phi_{np}^x \}; \quad (22) $$

$$ \theta_{np}^y = \arccos \left( \frac{y_{np}}{r_{np}} \right); \quad \phi_{np}^y = \arctan \left( \frac{x_{np}}{z_{np}} \right); \quad (23) $$

$$ \theta_{np}^y = \{ \cos \theta_{np}^y, \sin \phi_{np}^y, -\sin \theta_{np}^y, \cos \theta_{np}^y, \cos \phi_{np}^y \}. \quad (24) $$
Expressions (21) to (24) identify the spherical coordinates and polar unit vectors relative to \( r_{np} \) in the reference frames fixed to the \( x \)-directed and \( y \)-directed element of the \( n \)-th transmitting pair and oriented accordingly; under relabeling \( n \leftrightarrow p \), the same expressions are used to represent the vector \( r_{pn} = -r_{np} \) in the reference frames relative to the \( p \)-th receiving pair. It is important to emphasize that equation (19) is correct only under the assumption that the two UCAs are perfectly aligned, in which case \( \theta_{x,y}^{x,y} = \pi - \theta_{x,y}^{x,y} \), \( \phi_{x,y}^{x,y} = \pi + \phi_{x,y}^{x,y} \), \( \theta_{x}^{x} = \theta_{x}^{x} \), \( \theta_{y}^{y} = 1 \).

With the choice \( s_T = \pm i \), \( s_R = \mp i \), (19) becomes:

\[
H^{(\pm)}_{pn} = \left[ 1 + \frac{d^2}{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\}} \right] \times \frac{i k n \sqrt{\frac{I_0}{2} \ell^2} \exp \left( -ik \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\}} \right)}{4\pi \sqrt{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\}}},
\]

which is again a circulant matrix, as also evidenced by the strong similarity between Figure 11 and Figure 12. This fundamental result confirms that vortex modes are well-suited to implement multiple communications between perfectly aligned UCAs when circular polarization and low-directive elements are implied: in this context, \( \{|m_i\} \) represents a natural basis set of right channel modes yielding vortex waves with good purity. Unfortunately, when expressed in terms of such basis, the channel matrix spectrum is strongly influenced by the presence of misalignments between the two arrays, as can be seen in Figure 13.

The use of circularly-polarized crossed Hertzian dipoles does not represent the only way to enforce the symmetry condition so as to attain a pure vortex spectrum of the channel matrix in communication links between facing UCAs. For instance, azimuthally or radially polarized dipoles are equally good candidates in this respect, the corresponding channel matrices being expressed as in Appendix C formulas (C.1) and (C.2). The projection of the right singular vectors relative to (C.1) on the vortex modes basis is reported in Figure 14 (left) and differs from the previous ones in Figure 11 and Figure 12 owing to the atypical intensity profile of the \( m = 0 \) and \( m = \pm 1 \) azimuthally polarized waves (compare with Figure 6). Analogous results are obtained for the radially polarized case (C.2).
Figure 13. Histograms of the projection of the second right channel mode ($i = 2$) relative to a transmission link between two UCAs composed of $N = 25$ crossed Hertzian dipoles over the vortex modes, for $a = \lambda = 1$ m, $l = \lambda/20$, $d = 10\lambda$, $s_R = s_T^* = -i$ and four different rotation angles $\alpha$ of the receiving array around the $y$-axis passing through its center. When $\alpha = 0$ the two UCAs are perfectly aligned.

Figure 14. Spectral analysis of the channel operator (C.1) relative to a transmission link between perfectly aligned UCAs composed of azimuthally polarized Hertzian dipoles: matrix plot of the scalar product $c_{ji}$ (left) and singular values of $H_{pn}^{(A)}$ as a function of the spectral index (right), for $N = 25$, $a = \lambda = 1$ m, $l = \lambda/20$ and $d = 10\lambda$. 
Figure 15. On-axis link budget relative to a communication link between two eight-element UCAs as a function of the transmitting and receiving OAM excitation coefficients for three different choices of the constituent antennas: from left to right, $x$-directed half-wave dipoles, circularly polarized crossed half-wave dipoles and circularly polarized crossed Hertzian dipoles, respectively, for $\lambda = 1.46$ m, $a = 1.5$ m and $d = 40$ m.

Finally, in Appendix B it is explained how the knowledge of the channel matrix and of the sets of transmitting and receiving coefficients enables the estimation of the link budget, which represents a useful means to check the mode orthogonality, as shown in Figure 15.

4. Conclusions

Vortex waves are found to represent a natural set of solutions for all the problems in which the circular symmetry is preserved, while being very susceptible to departures from such condition. Contrary to what is commonly believed, the use of UCAs does not represent a sufficient constraint to generate vortex waves with good purity. The presence of linear antennas is responsible for the symmetry breaking of the vortex profiles and leads to a deterioration of the on-axis vortex with charge $m$ in $m$ distinct off-axis vortices with unitary charge. Owing to this symmetry breaking, the set of vortex excitation coefficients (i.e., the columns of the DFT matrix) do no longer represent a natural basis of channel modes for the communication links between UCAs. A possible solution to restore the lost circular symmetry requires the use of low-directive circularly, azimuthally or radially polarized sources.

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Appendix A. Green functions and the radiation problem

In principle, when the source distribution is known, the inhomogeneous wave equations can be solved via the Green function approach, enabling the determination of the radiated fields. In the frequency domain, the outgoing-wave Green function is given by:

$$g_+ (\mathbf{R}) = -\frac{\exp (-ikR)}{4\pi R}, \quad (A.1)$$
where \( k = 2\pi/\lambda \) is the wavenumber and \( \lambda \) the wavelength. The radiated scalar field can be formally written as:

\[
f (\mathbf{r}) = f_0 (\mathbf{r}) + \int d^3 r' g_+ (\mathbf{r} - \mathbf{r}') q (\mathbf{r}') ,
\]

with \( f_0 (\mathbf{r}) \) representing an arbitrary solution of the homogeneous Helmholtz equation and \( q (\mathbf{r}) \) the source term. Similarly, an integral representation for the frequency-domain electromagnetic potentials can be expressed in convolutional form:

\[
\Phi (\mathbf{r}) = - \int d^3 r' \frac{1}{\varepsilon} \rho (\mathbf{r}') g_+ (\mathbf{r} - \mathbf{r}') ;
\]

\[
A (\mathbf{r}) = - \int d^3 r' \mu \mathbf{J} (\mathbf{r}') g_+ (\mathbf{r} - \mathbf{r}') ,
\]

where \( \rho \) and \( \mathbf{J} \) stand for charge and current density distributions and the principal value of the integrals is assumed. The corresponding expressions for the electric and magnetic fields in the Lorenz gauge read:

\[
\mathbf{E} (\mathbf{r}) = i \omega \varepsilon \int d^3 r' \left\{ [\mathbf{J} (\mathbf{r}') \cdot \nabla] \nabla g_+ (\mathbf{r} - \mathbf{r}') + k^2 \mathbf{J} (\mathbf{r}') g_+ (\mathbf{r} - \mathbf{r}') \right\} ;
\]

\[
\mathbf{H} (\mathbf{r}) = \int d^3 r' \mathbf{J} (\mathbf{r}') \times \nabla g_+ (\mathbf{r} - \mathbf{r}') .
\]

Whenever it is necessary to determine the directional properties of the radiated field, the following integral, known as the radiation vector, plays a leading role:

\[
\mathbf{F} (\mathbf{k}) = \int d^3 r' \exp (i \mathbf{k} \cdot \mathbf{r}') .
\]

Indeed, it is possible to show that the polarization of the fields radiated by an antenna depends on the transverse component of this radiation vector, \( \mathbf{F}_\perp (\mathbf{k}) \), and a vector effective height of the antenna can be defined as:

\[
h (\theta, \phi) = - \frac{1}{I_0} \mathbf{F}_\perp (\mathbf{k}) = - \frac{1}{I_0} (\mathbf{u}_\theta \mathbf{u}_\theta + \mathbf{u}_\phi \mathbf{u}_\phi) \cdot \mathbf{F} (\mathbf{k}) ,
\]

being \( I_0 \) the input current to the antenna terminals, \( \theta, \phi, \mathbf{u}_\theta, \mathbf{u}_\phi \) the spherical angles and corresponding unit vectors and \( \mathbf{u} \mathbf{u} \) the outer product \( \mathbf{u} \otimes \mathbf{u} \). As an example, for a \( z \)-directed half-wave dipole placed in the origin one gets:

\[
h (\theta, \phi) = \frac{2}{k \sin \theta} \cos \left( \frac{\pi}{2} \cos \theta \right) \mathbf{u}_\theta ,
\]

whereas for a square microstrip antenna laid in the \( yz \) plane and with feeding line parallel to the \( y \)-axis a little bit more tedious calculation gives:

\[
h (\theta, \phi) = - 4L \cos \left( \frac{\pi L}{\lambda} \sin \theta \sin \phi \right) \sin \left( \frac{\pi L}{\lambda} \cos \theta \right) \sin \theta \mathbf{u}_\phi ,
\]

where \( L \) represents the microstrip antenna side.

The evaluation of the fields radiated by a given source distribution in free space and subject to the requirement of causality, often referred to as “the radiation problem”, can also be systematized through an alternative approach involving the use of the dyadic Green functions \([22,37]\). For instance, if we define:

\[
g_+^E (\mathbf{R}) = \left( \mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y + \mathbf{u}_z \mathbf{u}_z \right) + \frac{\nabla \nabla}{k^2} \exp (- i k R) \frac{4 \pi R}{4 \pi R} ,
\]
the electric field generated by an arbitrary current distribution reads:

\[ E(\mathbf{r}) = -i\omega \mu \int d^3 r' \mathbf{J}(\mathbf{r}') \cdot \mathbf{g}_E^+ (\mathbf{r} - \mathbf{r}') , \tag{A.12} \]

which is easily traced back to (A.5). In both the Green function approaches, the source of the radiated field is assumed to be specified and to be independent of the field that it produces. However, a similar derivation can be obtained for both the electric and magnetic fields under the hypothesis of fictitious sources that appear as a result of the application of equivalence theorems. Within this framework, the radiation problem can be extended to more complex scenarios, such as in the presence of physical obstacles, dispersive media, anisotropic and inhomogeneous regions of space, by properly generalizing the above defined free-space Green functions.

Appendix B. Communication by orthogonal modes

Let us consider a set of \( N \) distinct sources arranged over the region \( \mathcal{V}_T \) and \( P \) receiving elements distributed within \( \mathcal{V}_R \). Suppose the sources and the receiving elements to be characterized by vector effective heights \( \mathbf{h}_T^n \) and \( \mathbf{h}_R^p \), respectively, and spatial extent much smaller than the distance \( d \) between the two volumes. In correspondence with a given set of excitation coefficients \( \{\xi_n\} \in \mathbb{C}^N \) for the transmitting array, the expression of the radiated electric field is reported in (5).

Assuming no coupling, the voltage induced on a \( p \)-th receiving element in the absence of noise can be expressed approximately as the dot product between the incoming electric field and the \( p \)-th effective height [23]:

\[ V_p = ik\eta I_0 \sum_{n=1}^{N} \xi_n \exp \left( -i k \frac{\| \mathbf{r}_R^p - \mathbf{r}_T^n \|}{4\pi \| \mathbf{r}_R^p - \mathbf{r}_T^n \|} \right) \mathbf{h}_T^n \cdot \mathbf{h}_R^p. \tag{B.1} \]

In (B.1), \( \mathbf{r}_R^p \) identifies the position of the \( p \)-th receiving element and the functions \( \mathbf{h}_T^n \) and \( \mathbf{h}_R^p \) are evaluated at the angular coordinates which define the direction from the \( n \)-th source to the \( p \)-th receiving element, and vice versa, according to the relative reference frame. Equation (B.1) can be expressed in matrix form:

\[ V_p = \sum_{n=1}^{N} H_{pn} \xi_n , \tag{B.2} \]

with \( H \) representing the so-called channel matrix for the considered system:

\[ H_{pn} = ik\eta I_0 \frac{\exp \left( -i k \frac{\| \mathbf{r}_R^p - \mathbf{r}_T^n \|}{4\pi \| \mathbf{r}_R^p - \mathbf{r}_T^n \|} \right) \mathbf{h}_T^n \cdot \mathbf{h}_R^p. \tag{B.3} \]

The matrix elements \( H_{pn} \) can of course be interpreted as coupling coefficients between the \( n \)-th radiator and the \( p \)-th receiving element, whose specific value depends on the type of antennas used (i.e., the explicit form of the effective heights) and on their geometrical arrangement and orientation within the two volumes \( \mathcal{V}_T \) and \( \mathcal{V}_R \).

If we suppose the transmitting array to be excited by means of an ideal beamforming network, the total input power is given by:

\[ P_{in} = \frac{1}{2} R I_0^2 \sum_{n=1}^{N} |\xi_n|^2 = \frac{1}{2} R I_0^2 , \tag{B.4} \]
being $R$ the resistance of the radiators and having required the normalization condition for the synthesis coefficients. Similarly, the total power collected by the receiving array under ideal circumstances can be written as:

$$P_{\text{out}} = \frac{|V_{\text{out}}|^2}{8R} = \frac{1}{8R} \left| \sum_{p=1}^{P} \mathcal{Y}_p V_p \right|^2 = \frac{1}{8R} \left| \sum_{p=1}^{P} \sum_{n=1}^{N} \mathcal{Y}_p H_{pn} \xi_n \right|^2,$$

where the elements of the set $\{\mathcal{Y}_p\} \in \mathbb{C}^P$ are the beamforming weights introduced at the receiver and the network is supposed to be connected to a matched load $[23]$. The link budget of a communication system is defined as the ratio between the received and the transmitted powers and plays a fundamental role for the characterization of the link; in the present case, from (B.4) and (B.5), the explicit expression of the link budget reads:

$$\mathcal{L}B = \frac{P_{\text{out}}}{P_{\text{in}}} = \left| \frac{1}{2RI_0} \sum_{p=1}^{P} \sum_{n=1}^{N} \mathcal{Y}_p H_{pn} \xi_n \right|^2.$$

To find the transmitting and receiving column vectors $\{\xi_n\}$ and $\{\mathcal{Y}_p\}$ which define the sought channel modes for the considered link between antenna arrays, it is possible to resort to the singular value decomposition (SVD) of the channel matrix, thus rewriting it in terms of the product $H = U\Sigma W^\dagger$, where $U$ and $W$ are unitary square matrices containing the left and right singular vectors of $H$, respectively, and $\Sigma$ represents a real diagonal rectangular matrix whose entries correspond to the non-null singular values of $H$ sorted in decreasing order $[38,39]$. Such diagonal entries $\sigma^{(i)} = \Sigma_{ii}$ determine the gains of the orthogonal subchannels identified by the couples $\{\xi^{(i)}_n, \mathcal{Y}^{(i)}_p\} = \{W_{ni}, U_{pi}\}$.

It is straightforward to verify that, when the transmitting and receiving beamforming weights are chosen to be the so-derived channel modes, the link budget formula (B.6) for a communication by the $i$-th subchannel reduces to:

$$\mathcal{L}B^{(i)} = \left( \frac{\sigma^{(i)}}{2RI_0} \right)^2.$$

Equation (B.7) is just the proof that channel modes represent the best coupled pairs of transmitting and receiving beamforming coefficients, since the link budgets for the corresponding communications are the optimal ones, as follows from the SVD definition.

A sum rule is easily derived by considering the superpositions of all the possible link budgets from the $n$-th source to the $p$-th receiver, obtained from (B.6) with $\xi$ and $\mathcal{Y}$ replaced by a vector of $N-1$ zeros and a one at position $n$ and by a vector of $P-1$ zeros and a one at position $p$, respectively:

$$\sum_{p=1}^{P} \sum_{n=1}^{N} \mathcal{L}B^{(n \rightarrow p)} = \sum_{p=1}^{P} \sum_{n=1}^{N} \left\| \frac{H_{pn}}{2RI_0} \right\|^2 = \text{tr} \left[ \frac{H^\dagger H}{(2RI_0)^2} \right] = \sum_i \left( \frac{\sigma^{(i)}}{2RI_0} \right)^2 = \sum_i \mathcal{L}B^{(i)}.$$

The voltage induced on the receiving antenna by the incoming wave has been expressed in (B.1) as the dot product between the unperturbed electric field and the antenna effective height. This is of course an approximation, the scattered field being neglected under the assumption of a single plane wave incidence. Although such approximation proves to be quite convenient to deal with antenna arrays, it is reasonable to expect that it gives accurate results to the extent that the receiving antennas do not alter consistently the background electric field and can be
considered almost point-like, two conditions which are hardly met in a realistic situation. The link budget concept can be extended beyond this simplified scenario with the help of some fundamental results of the electromagnetic theory, enabling a rigorous definition which applies, in principle, to the majority of communication systems, including those based on aperture or reflector antennas, spiral phase plates, lenses and analogous optical devices.

Let us consider an arbitrary receiving antenna together with its surrounding structure and a background electromagnetic field \( \{ \mathbf{E}_{\text{inc}}(\mathbf{r}), \mathbf{H}_{\text{inc}}(\mathbf{r}) \} \) impinging on it. A more precise formulation of the received voltage is given by:

\[
V_{\text{out}} = -\int d\mathbf{l} \cdot \mathbf{E}_a(\mathbf{l}),
\]

where \( \mathbf{E}_a \) represents the electric field in the presence of the open circuited antenna and its surroundings and the integral is performed over a line connecting the antenna terminals. Making use of the well-known electromagnetic equivalence principles \([22]\), a generalization of equation \((A.12)\) is now provided by expressing the total electric field inside an arbitrary finite volume \( V \) which contains the antenna and any material objects in its proximity as if it were generated by electric \( \mathbf{j}_S \), in units of A/m, and magnetic \( \mathbf{m}_S \), in V/m, equivalent surface currents distributed over the volume boundary \( S = \partial V \):

\[
\mathbf{E}_a(\mathbf{r}_V) = \int_S d\mathbf{r}_S \left[ \mathbf{j}_S(\mathbf{r}_S) \cdot \mathbf{g}^{E_a}_j(\mathbf{r}_V, \mathbf{r}_S) + \mathbf{m}_S(\mathbf{r}_S) \cdot \mathbf{g}^{E_a}_m(\mathbf{r}_V, \mathbf{r}_S) \right].
\]

In \((B.10)\), \( \mathbf{g}^{E_a}_j \) and \( \mathbf{g}^{E_a}_m \) represent the electric field dyadic Green functions relative to the radiation from the equivalent electric and magnetic sources in the presence of the antenna and its surroundings, \( \mathbf{r}_V \) is a generic point inside \( V \) and \( \mathbf{r}_S \) the integration variable over \( S \). Furthermore, all the dimensional constants are now incorporated in the Green functions definitions, in contrast to the convention employed for the free-space equation \((A.12)\).

The use of equivalent surface currents permits to postulate the existence of a different electromagnetic field \( \{ \mathbf{E}_{\text{ext}}, \mathbf{H}_{\text{ext}} \} \) outside \( V \). In other words, the original problem, where the electromagnetic field \( \{ \mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}} \} \) generated by some distant sources scatters the antenna and its structure giving rise to a total resultant field \( \{ \mathbf{E}_a, \mathbf{H}_a \} \), has been replaced by an equivalent one in which the fictitious sources \( \mathbf{j}_S \) and \( \mathbf{m}_S \) over \( S \) radiate \( \{ \mathbf{E}_a, \mathbf{H}_a \} \) inside \( V \) (in the presence of the antenna and its surroundings) and \( \{ \mathbf{E}_{\text{ext}}, \mathbf{H}_{\text{ext}} \} \) in the free space outside \( V \). In order that the so defined field be a valid solution to the Maxwell equations throughout the whole space, the equivalent sources on the volume boundary must ensure the discontinuous change in the tangential components of the electric and magnetic fields across \( S \):

\[
\mathbf{j}_S = \mathbf{u}_n \times (\mathbf{H}_{\text{ext}} - \mathbf{H}_a); \quad \mathbf{m}_S = -\mathbf{u}_n \times (\mathbf{E}_{\text{ext}} - \mathbf{E}_a),
\]

where \( \mathbf{u}_n \) is the unit vector associated with the outward pointing normal to the surface \( S \).

By choosing \( \{ \mathbf{E}_{\text{ext}}, \mathbf{H}_{\text{ext}} \} \) to be the scattered electromagnetic field \( \{ \mathbf{E}_a - \mathbf{E}_{\text{inc}}, \mathbf{H}_a - \mathbf{H}_{\text{inc}} \} \), the equivalent sources are then easily expressed via the incident electromagnetic field in the absence of the antenna and its structure:

\[
\mathbf{j}_S = -\mathbf{u}_n \times \mathbf{H}_{\text{inc}}; \quad \mathbf{m}_S = \mathbf{u}_n \times \mathbf{E}_{\text{inc}}.
\]

Several approaches can be adopted to express the received voltage \((B.9)\) as a reaction integral \([40]\); among those, the following one is of particular interest:

\[
V_{\text{out}} = \frac{1}{I_0} \int_S d\mathbf{r}_S \left[ \mathbf{H}^0_0(\mathbf{r}_S) \cdot \mathbf{m}_S(\mathbf{r}_S) - \mathbf{E}^0_0(\mathbf{r}_S) \cdot \mathbf{j}_S(\mathbf{r}_S) \right],
\]
where the electric current term $j$ and the electromagnetic field $\{E_0, H_0\}$ radiated by an impressed efficient antenna:

$$P_0 = \frac{1}{2} R I_0^2 = \frac{1}{2} \int_S d\mathbf{S} \left[ \text{Re} \left( \mathbf{E}_a^0 \times \mathbf{H}_a^0 \right) \right] \cdot \mathbf{u}_n. \quad (B.17)$$

An extended link budget formula is then simply obtained by performing the ratio between (B.16) and the total power delivered to the source, which in turn can be described as a power outflow at the transmitting side. The importance of equation (B.16) lies in the fact that it is completely general and can be computed from the only knowledge of the incident electromagnetic field and of the field radiated by the receiving antenna when operating in transmission mode.

Appendix C. Azimuthally and radially polarized dipoles

In Section 3, some channel matrices relative to a communication link between facing UCAs have been presented for different choices in the disposition of the arrays elements. When azimuthally and radially polarized Hertzian dipoles are involved, the corresponding channel matrices read:

$$H_{pn}^{(A)} = H_{pn}^{(o)} \sin \theta_{np}^r \sin \varphi_{np}^r \theta_{np}^r \cdot \theta_{pn}^o; \quad (C.1)$$

$$H_{pn}^{(R)} = H_{pn}^{(o)} \sin \theta_{np}^o \sin \varphi_{pn}^o \theta_{np}^o \cdot \theta_{pn}^o, \quad (C.2)$$

where $H_{pn}^{(o)}$ is defined in (17) and we now have:

$$\mathbf{r}_{np} = \{x_{np}, y_{np}, z_{np}\} = \mathbf{r}_{p}^{R} - \mathbf{r}_{n}^{T}; \quad (C.3)$$

$$\mathbf{v}_{n}^{o} = \{x_{n}^{o}, y_{n}^{o}, z_{n}^{o}\} = R_{z}^{-1} \left( \varphi_{n} + \frac{\pi}{2} \right) \{u_x, u_y, u_z\}; \quad (C.4)$$

$$\varphi_{np}^o = \arccos (\mathbf{x}_{n}^{o} \cdot \mathbf{r}_{np} / r_{np}); \quad (C.5)$$

$$\varphi_{np}^r = \arctan (\mathbf{z}_{n}^{o} \cdot \mathbf{r}_{np} / \mathbf{y}_{n}^{o} \cdot \mathbf{r}_{np}); \quad (C.6)$$

$$\theta_{np}^o = R_{z} \left( \varphi_{n} + \frac{\pi}{2} \right) \left\{ - \sin \theta_{np}^o, \cos \theta_{np}^o \cos \phi_{np}^o, \cos \theta_{np}^o \sin \phi_{np}^o \right\}; \quad (C.7)$$
\[ \mathbf{v}_p^\circ = \{ x_p^\circ, y_p^\circ, z_p^\circ \} = R_z^{-1} \left( \varphi_p + \frac{\pi}{2} \right) \{ u_x, u_y, u_z \}; \]  
(C.8)

\[ \theta_{pn}^\circ = \arccos (x_p^\circ \cdot r_{pn}/r_{pn}) ; \]  
(C.9)

\[ \phi_{pn}^\circ = \arctan (z_p^\circ \cdot r_{pn}/y_p^\circ \cdot r_{pn}) ; \]  
(C.10)

\[ \Theta_{pn}^\circ = R_z \left( \varphi_p + \frac{\pi}{2} \right) \left\{ -\sin \theta_{pn}^\circ, \cos \theta_{pn}^\circ \cos \phi_{pn}^\circ, \cos \theta_{pn}^\circ \sin \phi_{pn}^\circ \right\} ; \]  
(C.11)

\[ \mathbf{v}_n^\rho = \{ x_n^\rho, y_n^\rho, z_n^\rho \} = R_z^{-1} \left( \varphi_n \right) \{ u_x, u_y, u_z \}; \]  
(C.12)

\[ \theta_{np}^\rho = \arccos (x_n^\rho \cdot r_{np}/r_{np}) ; \]  
(C.13)

\[ \phi_{np}^\rho = \arctan (z_n^\rho \cdot r_{np}/y_n^\rho \cdot r_{np}) ; \]  
(C.14)

\[ \Theta_{np}^\rho = R_z (\varphi_n) \left\{ -\sin \theta_{np}^\rho, \cos \theta_{np}^\rho \cos \phi_{np}^\rho, \cos \theta_{np}^\rho \sin \phi_{np}^\rho \right\} ; \]  
(C.15)

\[ \mathbf{v}_p^\rho = \{ x_p^\rho, y_p^\rho, z_p^\rho \} = R_z^{-1} (\varphi_p) \{ u_x, u_y, u_z \}; \]  
(C.16)

\[ \theta_{pn}^\rho = \arccos (x_p^\rho \cdot r_{pn}/r_{pn}) ; \]  
(C.17)

\[ \phi_{pn}^\rho = \arctan (z_p^\rho \cdot r_{pn}/y_p^\rho \cdot r_{pn}) ; \]  
(C.18)

\[ \Theta_{pn}^\rho = R_z (\varphi_p) \left\{ -\sin \theta_{pn}^\rho, \cos \theta_{pn}^\rho \cos \phi_{pn}^\rho, \cos \theta_{pn}^\rho \sin \phi_{pn}^\rho \right\} ; \]  
(C.19)

with \( R_z (\varphi_n) \) representing the rotation matrix by an angle \( 2\pi (n-1)/N \) around the \( z \)-axis. After some lengthy calculations, we get:

\[ H_{pn}^{(A)} = H_{pn}^{(o)} \frac{a^2 \left\{ 3 + \cos \left[ \frac{4\pi(p-n)}{N} \right] \right\}}{4a^2 \cos \left[ \frac{2\pi(p-n)}{N} \right]} - 2 \left( 2a^2 + d^2 \right) \cos \left[ \frac{2\pi(p-n)}{N} \right]; \]  
(C.20)

\[ H_{pn}^{(R)} = H_{pn}^{(o)} \frac{d^2 \cos \left[ \frac{2\pi(p-n)}{N} \right] + a^2 \sin^2 \left[ \frac{2\pi(p-n)}{N} \right]}{d^2 + 2a^2 \left\{ 1 - \cos \left[ \frac{2\pi(p-n)}{N} \right] \right\}}. \]  
(C.21)

Since both \( H_{pn}^{(A)} \) and \( H_{pn}^{(R)} \) depend on their indices only through the difference \( (p-n) \), they are circulant and can be diagonalized via DFT matrix.

It is worth mentioning that a possible alternative to the above approach consists in the use of properly fed crossed dipoles, in which case:

\[ H_{pn}^{(A)} = (- \sin \varphi_n \mathbf{E}_n^x + \cos \varphi_n \mathbf{E}_n^y) \cdot \left( - \sin \varphi_p \mathbf{h}_p^x + \cos \varphi_p \mathbf{h}_p^y \right); \]  
(C.22)

\[ H_{pn}^{(R)} = (\cos \varphi_n \mathbf{E}_n^x + \sin \varphi_n \mathbf{E}_n^y) \cdot \left( \cos \varphi_p \mathbf{h}_p^x + \sin \varphi_p \mathbf{h}_p^y \right), \]  
(C.23)

with the same convention as in \( (18) \). When explicitly rearranged, equations \( (C.22) \) and \( (C.23) \) still lead to \( (C.20) \) and \( (C.21) \), respectively.

**Appendix D. Channel matrix degeneracy**

When the number of transmitting and receiving antennas is the same, a diagonalization procedure can be considered instead of the full SVD (whenever possible), in which case both the right and left singular vectors are reinterpreted in terms of the matrix eigenvectors. An
Figure D1. Comparison between the eigenvalues of (25), expressed in terms of the spectral index $i$ (left), and the link budget (B.6) relative to $H_{pn}^{(±)}$ for the couple of transmitting and receiving beamforming coefficients defined in (D.2) as a function of the OAM charge $m$ (right), for $N = 25$, $a = \lambda = 1$ m, $l = \lambda/20$, $d = 10\lambda$.

Instructive and straightforward example in this sense is provided by the channel matrix (25): it can be shown that $H_{pn}^{(±)}$ is normal, which means that a rigorous connection exists between its diagonalization and the SVD [39]. As mentioned above, $H_{pn}^{(±)}$ is diagonalized by the DFT matrix $T_{pi} = \Phi_{pm}$, and, therefore, equation (B.6) can be rewritten as:

$$\text{LB} = \frac{1}{2RI_0} \left| \sum_{p,i,j,n=1}^{N} T_{pi} T_{pj} D_{ij} T_{jn}^{\dagger} \xi_n \right|^2,$$

(D.1)

where $D_{ij}$ corresponds to the diagonal matrix containing the eigenvalues of $H_{pn}^{(±)}$. Since the $N$ columns of the DFT matrix are nothing but the vectors $|m_i\rangle$ defined according to (13), the best pair of transmitting and receiving coefficients is given by the conjugate couple of vortex excitations:

$$\{\xi_{m}^p, \gamma_{m}^p\} = \{T_{ni}, T_{ip}^{\dagger}\} = \{\Phi_{m_i}^p, (\Phi_{m_i}^p)^*\} = \{\Phi_{m_i}^p, \Phi_{-m_i}^p\},$$

(D.2)

a result which is intuitively consistent with the request for a mode-matched reception. Substituting expression (D.2) into (D.1) and making use of the Dirac notation, we get:

$$\text{LB}^{(m_i)} = \left| \frac{\langle m_i | H_{pn}^{(±)} | m_i \rangle}{2RI_0} \right|^2 = \left| \frac{1}{2RI_0} \sum_j \lambda^{(j)} \langle m_i | m_j \rangle \langle m_j | m_i \rangle \right|^2$$

$$= \left| \frac{1}{2RI_0} \left( \sum_j \lambda^{(j)} \langle m_i | m_j \rangle \langle m_j | m_i \rangle \right) \right|^2 = \left( \frac{1}{2RI_0} \lambda^{(i)} \right)^2,$$

(D.3)

where $\lambda^{(i)}$ represents the $i$-th eigenvalue of the channel matrix and the orthogonality relation in [14] has been exploited. Upon proper rearrangement of the eigenvalues list, (D.3) is still the same as (B.7), being the singular values of a normal matrix equal to the modulus of its eigenvalues [39].

In Figure D1, a comparison between the eigenvalues distribution of $H_{pn}^{(±)}$ and the corresponding link budget for the communication by vortex waves as a function of $m$ is proposed.
for the same choice of the parameters introduced in the main text. From the comparison we infer how the two-fold degeneracy in the channel matrix spectrum can be traced back to the fact that two communications by vortex waves with opposite charges $\pm m$ share the same power level.

Equation (D.2) tells us that, at least when identical facing UCAs are concerned, the best beamforming weights that should be introduced at the receiver to properly collect an incoming vortex wave are just the complex conjugates of those employed for generating such wave at the transmitting side. The validity of this result extends far beyond the proposed scenario, in that it represents a natural consequence of the reciprocity theorem.

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