A QUANTITATIVE VERSION OF STEINHAUS’ THEOREM FOR COMPACT, CONNECTED, RANK-ONE SYMMETRIC SPACES

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Abstract. Let $d_1, d_2, \ldots$ be a sequence of positive numbers that converges to zero. A generalization of Steinhaus’ theorem due to Weil implies that, if a subset of a homogeneous Riemannian manifold has no pair of points at distances $d_1, d_2, \ldots$ from each other, then it has to have measure zero. We present a quantitative version of this result for compact, connected, rank-one symmetric spaces, by showing how to choose distances so that the measure of a subset not containing pairs of points at these distances decays exponentially in the number of distances.

1. Introduction

A well-known fact in measure theory is Steinhaus’ theorem (cf. Steinhaus [11]; see also Halmos [8], page 68). It states that if $A$ is a Lebesgue measurable subset of $\mathbb{R}^n$ that has positive measure, then the origin lies in the interior of the difference set

$$A - A = \{ x - y : x, y \in A \}.$$

Let $H$ be a set of positive real numbers having infimum 0. Steinhaus’ theorem implies that the measure of a subset of $\mathbb{R}^n$ has to be zero, if it avoids all the distances in $H$, i.e., if it does not contain any pair of points whose distance is in $H$.

The upper density (or simply density) of a measurable set $A \subseteq \mathbb{R}^n$ is defined as

$$\overline{d}(A) = \limsup_{T \to \infty} \frac{\mu(A \cap [-T, T]^n)}{\mu([-T, T]^n)},$$

where $[-T, T]^n$ denotes the regular cube in $\mathbb{R}^n$ with side $2T$ centered at the origin and where $\mu$ denotes the Lebesgue measure in $\mathbb{R}^n$. For a given set $H$ of positive real numbers we let

$$m_H(\mathbb{R}^n) = \sup \{ \overline{d}(A) : A \subseteq \mathbb{R}^n \text{ is measurable and avoids all distances in } H \}.$$

In words, $m_H(\mathbb{R}^n)$ is the maximum density a measurable subset of $\mathbb{R}^n$ can have if it avoids all distances in $H$. Steinhaus’ theorem implies that $m_H(\mathbb{R}^n) = 0$ if $\inf H = 0$.

Székely [14] asked whether Steinhaus’ theorem has the following continuous version:

Question 1. Let $d_1, d_2, \ldots$ be a sequence of positive real numbers that converges to zero. Is it true that

$$\lim_{N \to \infty} m_{\{d_1, \ldots, d_N\}}(\mathbb{R}^n) = 0?$$
The following example of Falconer [6] shows that Question 1 has a negative answer in dimension one: We consider the set 
\[ A = \bigcup_{k \in \mathbb{Z}} (2k, 2k + 1). \]
Set \( A \) has density \( 1/2 \) and avoids all odd distances. The scaled set \( 3^{-N} A \) also has density \( 1/2 \), and in particular it avoids distances \( 1, 3^{-1}, \ldots, 3^{-N} \). So the statement of Question 1 fails for the zero-convergent sequence \( 1, 3^{-1}, 3^{-2}, \ldots \) when \( n = 1 \).

It turns out that the real line is an exceptional case: Falconer [6] showed that the answer to Question 1 is affirmative for \( n \geq 2 \).

Now we aim at transforming Question 1 into a question in hard analysis: Is it possible to turn Question 1 into a quantitative statement? Or, can one say anything about the rate of convergence? So we are led to ask:

**Question 2.** How fast does \( m_{\{d_1, \ldots, d_N\}}(\mathbb{R}^n) \) converge to zero if one is allowed to choose the distances \( d_1, \ldots, d_N \)?

One answer was given by Bukh [4] based on a version of Szemerédi’s regularity lemma for measurable functions (see also Chapter 2.4 in Tao [15]). He showed that the density decays exponentially, as long as the distances space out exponentially.

Oliveira and Vallentin [10] prove a slightly weaker result that however already shows exponential decay. They show that, for all \( N \geq 1 \), there is a number \( r = r(N) \) such that, if the sequence \( d_1, \ldots, d_N \) satisfies
\[
d_1/d_2 > r, \quad d_2/d_3 > r, \ldots, \quad d_{N-1}/d_N > r,
\]
then
\[
m_{\{d_1, \ldots, d_N\}}(\mathbb{R}^n) \leq 2^{-N}.
\]

The simplest Riemannian manifolds are, next to the Euclidean space \( \mathbb{R}^n \), the connected, compact, rank-one symmetric spaces. They are also called continuous, compact, two-point homogeneous spaces because the isometry group acts transitively on pairs of points at distance \( d \) for every \( d > 0 \). Such spaces (with more than one point) were classified by Wang [16], the complete list being: the unit sphere \( S^{n-1} \), the real projective space \( \mathbb{R}P^{n-1} \), the complex projective space \( \mathbb{C}P^{n-1} \), the quaternionic projective space \( \mathbb{H}P^{n-1} \), and the octonionic projective plane \( \mathbb{O}P^2 \).

Here observe that \( S^{n-1} \subseteq \mathbb{R}^n \) and that \( \mathbb{R}P^{n-1} \) is the set of all lines passing through the origin of \( \mathbb{R}^n \); the spaces \( \mathbb{C}P^{n-1}, \mathbb{H}P^{n-1}, \) and \( \mathbb{O}P^2 \) are defined similarly. Notice also that the real dimension of \( S^{n-1} \), that is, its dimension as a real manifold, is \( n - 1 \). Similarly, the real dimension of \( \mathbb{R}P^{n-1} \) is \( n - 1 \), and the real dimensions of \( \mathbb{C}P^{n-1}, \mathbb{H}P^{n-1}, \) and \( \mathbb{O}P^2 \) are \( 2(n-1), 4(n-1), \) and \( 16 \), respectively.

In this paper we consider quantitative versions of Steinhaus’ theorem for these spaces, providing answers to the analogues of Questions 1 and 2 for them. It turns out that, as in the case of the Euclidean space, one is able to show exponential decay when the real dimension of the space considered is at least 2, and when the real dimension is 1 it is possible to show that the answer to the analogue of Question 1 is negative.

To make ideas precise, we start by considering a generalization of Steinhaus’ theorem to locally compact groups due to Weil (cf. Weil [18], page 50). Let \( G \) be a locally compact group and let \( A \) be a measurable subset of \( G \) having positive measure (here we consider the Haar measure for \( G \)). Weil’s result then says that the set
\[
AA^{-1} = \{ xy^{-1} : x, y \in A \}
\]
contains the identity in its interior. A short and elementary proof is due to Stromberg [12]. As in the case of \( \mathbb{R}^n \), this theorem implies that subsets of a homogeneous space avoiding a set of distances with infimum zero need to have measure zero. More precisely, let \( M \) be a Riemannian manifold with isometry group \( G \) acting transitively on \( M \), i.e., \( M \) is a homogeneous \( G \)-space whose measure is induced by the Haar measure of \( G \). Let \( H \) be a set of positive real numbers having infimum zero. Weil’s result then says that the measure of a subset of \( M \) has to be zero, if it avoids all distances given in \( H \).

If \( M \) is compact we normalize the measure \( \mu \) on \( M \) so that \( \mu(M) = 1 \). For a set of distances \( H \) we let
\[
m_H(M) = \sup \{ \mu(A) : A \subseteq M \text{ is measurable and avoids all distances in } H \}.
\]
Then by Weil’s generalization of Steinhaus’ theorem, we have that \( m_H(M) = 0 \) if \( \inf H = 0 \).

The main result of this paper is the following theorem:

**Theorem 1.1.** Let \( M \) be a compact, connected, rank-one symmetric space of real dimension at least 2. If distances \( d_1, \ldots, d_N \) are given so that \( d_1 \gg d_2 \gg \cdots \gg d_N \),
\[
\text{then } m_{\{d_1,\ldots,d_N\}}(M) \leq 2^{-N}.
\]

In the statement of the theorem we are purposely vague: The essence of the statement is that, as long as the distances \( d_1, \ldots, d_N \) are sufficiently spaced out, then the maximum density is exponentially small. In Section 4 we will give a precise statement of Theorem 1.1 in which we specify how the distances should space out in order for the conclusion of the theorem to hold.

Notice that this theorem implies that, if \( d_1, d_2, \ldots \) is a sequence of distances converging to zero and the real dimension of the space is at least 2, then
\[
\lim_{N \to \infty} m_{\{d_1,\ldots,d_N\}}(M) = 0,
\]
thus giving a positive answer to the analogue of Question 1 for the space \( M \). We will show in Section 2 that if the real dimension of \( M \) is 1, then there are zero-convergent sequences \( d_1, d_2, \ldots \) of distances for which the limit in 1 is nonzero, thus showing that Question 1 has a negative answer in dimension 1. So the big picture is similar to the one of \( \mathbb{R}^n \) discussed earlier, and also the example that shows lack of decay in dimension one is similar to the one we gave before for the real line.

We prove Theorem 1.1 in Section 3. For the proof of this theorem we will use the generalization of the Lovász theta number for distance graphs defined over compact metric spaces given by Bachoc, Nebe, Oliveira, and Vallentin [3]. This generalization will provide upper bounds for \( m_H(M) \) by means of the solution of an infinite-dimensional semidefinite programming problem, which reduces to an infinite-dimensional linear programming problem when the Riemannian manifold \( M \) is a symmetric space. We recall the necessary background for this approach in Section 3.

## 2. Counter-examples in dimension one

There are only two compact, connected, rank-one symmetric spaces \( M \) of real dimension one: The unit circle \( S^1 \) and the real projective line \( \mathbb{RP}^1 \). For each of these two spaces we show that there is a zero-convergent sequence of distances \( d_1, d_2, \ldots \) and measurable subsets \( C_1, C_2, \ldots \) of \( M \) all having measure at least \( L \) for some positive constant \( L \), such that \( C_k \) avoids distances \( d_1, \ldots, d_k \). This implies a negative answer to the analogue of Question 1 for the space \( M \).

We aim at presenting a single construction that works for both the unit circle and the real projective line. To this end, for \( k = 1, 2, \ldots \), write \( \theta_k = (\pi/2)/3^k \).
and $N_k = [3^k/2]$. Let $E_k = \{e_{k,1}, \ldots, e_{k,N_k}\}$ be the set of vectors in $\mathbb{R}^2$ given as follows:

1. $e_{k,1} = (1,0)$;
2. for $i > 1$, vector $e_{k,i}$ is equal to vector $e_{k,i-1}$ rotated counterclockwise by an angle of $2\theta_k$.

Figure 1 illustrates this construction. Notice that all vectors in $E_k$ belong to the unit circle $S^1$ and moreover they all lie in the nonnegative quadrant of $\mathbb{R}^2$.

When $M$ is the real projective line, we represent its elements by unit vectors in $\mathbb{R}^2$ by identifying antipodal vectors. So we can see $E_k$ as a subset of $M$. Moreover, notice that by construction the points in $E_k$ are all distinct in $M$.

Let $d$ be the distance function of $M$. If $M = S^1$, then for $x, y \in S^1$ we have $d(x, y) = \arccos(x \cdot y)$. If $M = \mathbb{R}P^1$, then $d(x, y) = \arccos(2(x \cdot y)^2 - 1)$. So write $d_k = d(e_{k,1}, e_{k,2})/2$. From the formula for the distance we see that

$$d(e_{k,i}, e_{k,j}) = 2|i - j|d_k \quad \text{for } i, j = 1, \ldots, N_k.$$ 

Moreover, one has $d_k = 3d_{k+1}$.

For $i = 1, \ldots, N_k$ consider the set $B_{k,i} = \{ x \in M : d(x, e_{k,i}) < d_k/2 \}$ and let

$$C_k = \bigcup_{i=1}^{N_k} B_{k,i}.$$ 

Figure 1 illustrates the construction of set $C_k$. We claim that set $C_k$ avoids all distances of the form $3^i d_k$ for $i = 0, \ldots, k$.

Indeed, let $x, y \in C_k$ and suppose $x \in B_{k,i}$ and $y \in B_{k,j}$. Then

$$d(x, y) \leq d(x, e_{k,i}) + d(e_{k,i}, e_{k,j}) + d(y, e_{k,j}) < d_k + 2|i - j|d_k = (2|i - j| + 1)d_k.$$ 

On the other hand,

$$d(e_{k,i}, e_{k,j}) \leq d(e_{k,i}, x) + d(x, y) + d(e_{k,j}, y) < d_k + d(x, y),$$

whence it follows that $d(x, y) > (2|i - j| - 1)d_k$.

So we see that $d(x, y)$ is never of the form $(2l + 1)d_k$ for $l = 0, \ldots, N_k - 1$. In particular, it is never of the form $3^i d_k$ for $i = 0, \ldots, k$, proving the claim.

Now consider the sequence $d_1, d_2, \ldots$ of distances. This is a zero-convergent sequence. Moreover, by our construction, and since also $d_k = 3d_{k+1}$, set $C_k$ avoids distances $d_1, \ldots, d_k$.
Finally, notice that all sets $B_{k,i}$ for fixed $k$ and $i = 1, \ldots, N_k$ have the same positive measure. Moreover, since $M$ has real dimension 1, and since $d_k = 3d_{k+1}$, we have
\[ \mu(B_{k,i}) = 3\mu(B_{k+1,i}). \]
So we have
\[ \mu(C_k) = N_k3^{1-k}\mu(B_{1,1}) \geq (3/2)\mu(B_{1,1}), \]
and we see that the zero-convergent sequence of distances $d_1, d_2, \ldots$ and the sequence $C_1, C_2, \ldots$ of subsets of $M$ have the properties claimed in the beginning of the section.

3. The theta number

Let $M$ be a compact Riemannian manifold with distance function $d$ and $G$ be its isometry group. Suppose $M$ is a homogeneous $G$-space and let $\mu$ be the measure on $M$ induced by the Haar measure on $G$, so that $\mu$ is invariant under the action of $G$. Moreover, normalize $\mu$ so that $\mu(M) = 1$.

A kernel is a continuous function $K : M \times M \to \mathbb{R}$. We say that $K$ is positive if the matrix $(K(x_i, x_j))_{i,j=1}^N$ is positive semidefinite for all $N$ and all choices of points $x_1, \ldots, x_N \in M$. We say that $K$ is invariant if
\[ K(\sigma x, \sigma y) = K(x, y) \quad \text{for all } \sigma \in G \text{ and } x, y \in M. \]

Now, given positive distances $d_1, \ldots, d_N$, consider the optimization problem
\begin{align*}
\vartheta(M, \{d_1, \ldots, d_N\}) = \sup & \int_M \int_M K(x, y) \, d\mu(x) \, d\mu(y) \\
\int_M K(x, x) \, d\mu(x) = 1, & \\
K(x, y) = 0 & \text{if } d(x, y) \in \{d_1, \ldots, d_N\}, \\
K : M \times M \to \mathbb{R} & \text{is a positive and invariant kernel},
\end{align*}
which is an infinite-dimensional semidefinite programming problem.

Bachoc, Nebe, Oliveira, and Vallentin [3] proved that
\[ \vartheta(M, \{d_1, \ldots, d_N\}) \geq m_{\{d_1, \ldots, d_N\}}(M), \]
so that by solving problem (2) one obtains an upper bound for $m_H(M)$.

Now, when $M$ is a compact, connected, rank-one symmetric space, one may decompose a positive and invariant kernel $K : M \times M \to \mathbb{R}$ in terms of Jacobi polynomials, as stated in the theorem below (cf. Askey [2], page 65). We denote the Jacobi polynomial of degree $k$ with parameters $\alpha$ and $\beta$ by $P_k^{(\alpha, \beta)}$, and we normalize it so that $P_k^{(\alpha, \beta)}(1) = 1$. Notice that this is not the normalization commonly found in the literature on Jacobi polynomials (for background on Jacobi polynomials, see e.g. the book by Szegő [13]).

**Theorem 3.1.** Let $M$ be a compact, connected, rank-one symmetric space. A kernel $K : M \times M \to \mathbb{R}$ is positive and invariant if and only if there are numbers $\alpha$ and $\beta$ such that
\begin{equation}
K(x, y) = \sum_{k=0}^\infty f_k P_k^{(\alpha, \beta)}(\cos d(x, y))
\end{equation}
for some nonnegative numbers $f_0, f_1, \ldots$ such that $\sum_{k=0}^\infty f_k$ converges, in which case the series in (3) converges absolutely and uniformly over $M \times M$. 

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The parameters \( \alpha \) and \( \beta \) depend on the space \( M \); they are summarized in Table 1.

Using Theorem 3.1 one may rewrite the optimization problem (2), obtaining that \( \vartheta(M, \{d_1, \ldots, d_N\}) \) is equal to the optimal value of the following infinite-dimensional linear programming problem in variables \( f_k \):

\[
\begin{align*}
\sup_{f_0} & \quad f_0 \\
\text{s.t.} & \quad \sum_{k=0}^{\infty} f_k = 1, \\
& \quad \sum_{k=0}^{\infty} f_k P_k^{(\alpha, \beta)}(\cos d_i) = 0 \quad \text{for } i = 1, \ldots, N, \\
& \quad f_k \geq 0 \quad \text{for } k = 0, 1, \ldots.
\end{align*}
\]

In this reformulation, we used the fact that \( K \) is invariant and \( M \) is homogeneous, so that all the diagonal entries of \( K \) are the same, and therefore equal to 1. Moreover, we also use our normalization \( P_k^{(\alpha, \beta)}(1) = 1 \) for the Jacobi polynomials, and the orthogonality property of the Jacobi polynomials.

A possible dual for problem (4) is the following linear programming problem on variables \( z_0, z_1, \ldots, z_N \) with infinitely many constraints:

\[
\begin{align*}
\inf_{z_0} & \quad z_0 \\
\text{s.t.} & \quad z_0 + z_1 + \cdots + z_N \geq 1, \\
& \quad z_0 + z_1 P_k^{(\alpha, \beta)}(\cos d_1) + \cdots + z_N P_k^{(\alpha, \beta)}(\cos d_N) \geq 0 \quad \text{for } k = 1, 2, \ldots.
\end{align*}
\]

It is easy to show that weak duality holds between (4) and (5), that is, that any feasible solution of (5) provides an upper bound for the optimal value of (4). Indeed, if \( f_0, f_1, \ldots \) is a feasible solution of (4) and \( (z_0, z_1, \ldots, z_N) \) is a feasible solution of (5), then

\[
f_0 \leq \sum_{k=0}^{\infty} f_k (z_0 + z_1 P_k^{(\alpha, \beta)}(\cos d_1) + \cdots + z_N P_k^{(\alpha, \beta)}(\cos d_N)) = z_0,
\]

as we wanted.

The optimization problem (5) is the tool we use to prove Theorem 1.1. Notice that, because of weak duality, it is not necessary to solve (5) to optimality in order to get an upper bound for \( m_H(M) \), any feasible solution will provide such a bound.

4. A proof of the main theorem

In this section we make the statement of Theorem 1.1 precise by giving a condition that, if satisfied by the distances, implies exponential density decay. More specifically we prove the following theorem.

**Theorem 4.1.** Let \( M \) be a compact, connected, rank-one symmetric space of real dimension at least 2 and let \( N \geq 1 \). There is a number \( 0 < d_0 < \pi/2 \) and a
function \( r : (0, d_0) \to (0, d_0) \), depending on \( N \), such that if numbers \( d_1 > \cdots > d_N \) are picked from the interval \((0, d_0)\) so as to satisfy
\[
d_{i+1} \leq r(d_i) \quad \text{for } i = 1, \ldots, N - 1,
\]
then \( m(d_1, \ldots, d_N)(M) \leq 2^{-N} \).

Recall that we denote by \( P_k^{(\alpha, \beta)} \) the Jacobi polynomial of degree \( k \) and parameters \( \alpha \) and \( \beta \), and that we normalize it so that \( P_k^{(\alpha, \beta)}(1) = 1 \). This normalization differs from the one normally adopted in the literature about Jacobi polynomials, where one usually sets
\[
P_k^{(\alpha, \beta)}(1) = \left( \frac{k + \alpha}{k} \right).
\]

One important property of our normalization is the following:

\[ (7) \quad \text{For } \alpha \geq 0 \text{ and all intervals } [a, b] \subseteq (-1,1), \text{ we have that } P_k^{(\alpha, \beta)}(u) \to 0 \text{ as } k \to \infty \text{ uniformly in the interval } [a, b]. \]

This follows from the asymptotic formula for the Jacobi polynomials given in Theorem 8.21.8 of Szegő [13], together with the fact that for \( \alpha \geq 0 \) we have \((k+\alpha)/k \geq 1\). This fact will play an important role in the rest of this section.

For \( \alpha, \beta > -1 \), if \( -1 < t < 1 \), let
\[
l^{(\alpha, \beta)}(t) = \inf \{ P_k^{(\alpha, \beta)}(t) : k = 0, 1, \ldots \}.
\]

Note that \( l^{(\alpha, \beta)}(t) < 0 \) for all \(-1 < t < 1\). This follows from a straightforward application of the interlacing property for Jacobi polynomials (cf. Theorem 3.3.2 in Szegő [13]), or alternatively from the asymptotic formula for Jacobi polynomials (cf. Theorem 8.21.8 in Szegő [13]).

In our proof of Theorem 4.1, we will use the following rather technical result about the behavior of \( l^{(\alpha, \beta)}(t) \).

**Lemma 4.2.** For all \( \alpha \geq 0 \) and \( \beta \geq -1/2 \) with \( \alpha \geq \beta \), there is a number \( t_0 \) with \( 0 < t_0 < 1 \) such that \( l^{(\alpha, \beta)}(t) \geq -1/2 \) for all \( t_0 < t < 1 \).

Before proving this lemma, we first prove Theorem 4.1.

**Proof of Theorem 4.1.** Let \( t_0 \) be given as in Lemma 4.2 and set \( d_0 = \arccos t_0 \). We now analyze the linear programming problem (5) for distances \( d_1, \ldots, d_N \), with \( \alpha \) and \( \beta \) given as in Table 1, according to the space \( \mathcal{M} \) considered. Here it is important to observe that, since the real dimension of \( \mathcal{M} \) is at least 2, we have \( \alpha \geq 0 \).

If \( N = 1 \), then since \( l^{(\alpha, \beta)}(t) \geq -1/2 \) for all \( t_0 < t < 1 \), it is easy to see from (5) that \( m(\{d_1\})(M) \leq 1/2 \) for all \( d_0 > d_1 > 0 \), just by setting \( z_0 = 1/2 \) and \( z_1 = 1 \).

So suppose \( N > 1 \). Let
\[
\lambda = \inf \{ l^{(\alpha, \beta)}(u) : t_0 < u < 1 \}.
\]

By the choice of \( t_0 \), and since \( l^{(\alpha, \beta)}(t) < 0 \) for \(-1 < t < 1\), we must have \( \lambda \leq 1/2 \).

Write \( \varepsilon = \lambda^{N+1}/((1-\lambda)(N-1)) \).

We now define function \( r \). So let \( d \in (0, d_0) \) be given. We show how to pick a number \( r(d) \in (0, d_0) \) having the following property:

\[ (8) \quad \text{For every } 0 < s \leq r(d), \text{ if } P_k^{(\alpha, \beta)}(\cos s) \leq 1 - \varepsilon, \text{ then } |P_k^{(\alpha, \beta)}(\cos d')| < \varepsilon \text{ for all } d' \leq d' \leq \pi/2. \]

To prove that we may pick such a number \( r(d) \), we use the fact that \( P_k^{(\alpha, \beta)}(t) \to 0 \) as \( k \to \infty \) uniformly in the interval \([0, \cos d]\), as observed in (7). So there is a \( k_0 \) such that \( |P_k^{(\alpha, \beta)}(t)| < \varepsilon \) for all \( 0 \leq t \leq \cos d \) and \( k > k_0 \). Now, since each \( P_k^{(\alpha, \beta)} \) is continuous and such that \( P_k^{(\alpha, \beta)}(1) = 1 \), we may pick a number \( u_0 \)
such that $P^{(\alpha,\beta)}_k(u) > 1 - \varepsilon$ for all $u_0 \leq u \leq 1$ and $k \leq k_0$. But then we may set $r(d) = \arccos u_0$ and (5) is satisfied.

Suppose now that numbers $d_1, \ldots, d_N \in (0, d_0)$ with $d_1 > \cdots > d_N$ are given which satisfy (3). We claim that, for $1 \leq j \leq N$,

$$(9) \quad \sum_{i=1}^{j} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) \geq -\lambda^j - \varepsilon (j-1) \quad \text{for all } k \geq 0.$$

Before proving the claim, let us show how to apply it in order to prove the theorem. Taking $j = N$ we see that

$$\sum_{i=1}^{N} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) \geq -\lambda^N - \varepsilon (N - 1) \quad \text{for all } k \geq 0.$$

So, letting $S = 1 + \lambda + \cdots + \lambda^N + \varepsilon (N - 1)$, we may set

$$z_0 = \frac{\lambda^N + \varepsilon (N - 1)}{S} \quad \text{and} \quad z_i = \frac{\lambda^{i-1}}{S} \quad \text{for } i = 1, \ldots, N$$

and check that this is a feasible solution of (6). But then, since $\lambda \leq 1/2$ and also $\varepsilon = \lambda^{N+1}/((1-\lambda)(N-1))$, from the weak duality relation between (4) and (5) we get that

$$m_{\{d_1, \ldots, d_N\}}(M) \leq \frac{\lambda^N + \varepsilon (N - 1)}{1 + \lambda + \cdots + \lambda^N + \varepsilon (N - 1)} \leq \frac{\lambda^N + \varepsilon (N - 1)}{(1 - \lambda^{N+1})/(1 - \lambda) + \lambda^{N+1}/(1 - \lambda)} = \lambda^N (1 - \lambda) + \lambda^{N+1} \leq 2^{-N},$$

where we use that $\lambda(1 - \lambda) \leq 1/4$, and the theorem follows.

We finish by proving (9). For $j = 1$, the statement is obviously true. Now suppose the statement is true for some $1 \leq j < N$; we show that it is also true for $j + 1$. To this end, let $k \geq 0$ be an integer. If $P^{(\alpha,\beta)}_k(\cos d_{j+1}) > 1 - \varepsilon$, then by using the induction hypothesis and since $\lambda \leq 1$ we get

$$\sum_{i=1}^{j+1} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) = \lambda^j P^{(\alpha,\beta)}_k(\cos d_{j+1}) + \sum_{i=1}^{j} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) \geq \lambda^j (1 - \varepsilon) - \lambda^j - \varepsilon (j - 1) \geq -\varepsilon j.$$

If, on the other hand, $P^{(\alpha,\beta)}_k(\cos d_{j+1}) \leq 1 - \varepsilon$, we know from the choice of the $d_i$ and from (6) that $|P^{(\alpha,\beta)}_k(\cos d_i)| < \varepsilon$ for $i = 1, \ldots, j$. But then we have

$$\sum_{i=1}^{j+1} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) = \lambda^j P^{(\alpha,\beta)}_k(\cos d_{j+1}) + \sum_{i=1}^{j} \lambda^{i-1} P^{(\alpha,\beta)}_k(\cos d_i) \geq -\lambda^{j+1} - \varepsilon j,$$

proving (9). \hfill \Box

Let $t^{(\alpha,\beta)}_{k,1} < \cdots < t^{(\alpha,\beta)}_{k,k}$ be the zeros of $P^{(\alpha,\beta)}_k$ in ascending order. In the proof of Lemma 4.2 we will use the following two facts proven by Wong and Zhang [19].

Let $\alpha \geq 0$ and $\beta \geq -1/2$. The first fact is that

$$\lim_{k \to \infty} P^{(\alpha,\beta)}_k(t^{(\alpha+1,\beta+1)}_{k-1,k-1}) = \Gamma(\alpha + 1)\left(\frac{2}{\alpha + 1}\right)\alpha J_\alpha(j_{\alpha+1}),$$
where \( J_\nu \) is the Bessel function of the first kind of order \( \nu \) and \( j_\nu \) is its first positive zero. Note that the right-hand side of the expression above is negative for all \( \alpha \geq 0 \).

The second fact we use is that, for all large enough \( k \),
\[
(11) \quad P_{k+1}(\alpha,\beta)\left(t_{k,k}^{(\alpha+1,\beta+1)}\right) > P_k(\alpha,\beta)\left(t_{k-1,k-1}^{(\alpha+1,\beta+1)}\right).
\]

Both these facts follow from (1.6) in Wong and Zhang \([19]\). In their paper, however, formula (1.6) is proven under the condition that \( \alpha > \beta > -1/2 \). One may verify, however that the same proof holds when \( \alpha \geq 0 \) and \( \beta \geq -1/2 \). Indeed, the only step of the proof in which \( \alpha > \beta > -1/2 \) is assumed is in the use of formula (2.3), and this formula (given by Frenzen and Wong \([7]\)) is valid under more general conditions that include all the cases in which \( \alpha \geq 0 \) and \( \beta \geq -1/2 \).

**Proof of Lemma \([4,2]\)** In our proof we use the following claim: a stronger version of this claim is stated in (6.4.24) of Andrews, Askey, and Roy \([1]\) for the case of the ultraspherical polynomials.

**Claim A.** For every \( \alpha \geq 0 \) and \( \beta > -1 \) with \( \alpha \geq \beta \), there is a number \( k_0 \geq 2 \) such that for all \( k \geq k_0 \) we have
\[
P_k(\alpha,\beta)\left(t_{k-1,k-1}^{(\alpha+1,\beta+1)}\right) = \min\{ P_k(\alpha,\beta)(u) : 0 \leq u \leq 1 \}.
\]

**Proof.** Consider the function
\[
g(u) = (P_k(\alpha,\beta)(u))^2 + \frac{1-u^2}{k(k+\alpha+\beta+1)}(dP_k(\alpha,\beta)(u))^2.
\]

We use the identity
\[
(1-u^2)\frac{d^2P_k(\alpha,\beta)(u)}{du^2} + (\beta - \alpha)(\alpha + \beta + 2)u\frac{dP_k(\alpha,\beta)(u)}{du} + k(k+\alpha+\beta+1)P_k(\alpha,\beta)(u) = 0
\]
(cf. (6.3.9) in Andrews, Askey, and Roy \([1]\)) to obtain
\[
g'(u) = \frac{2((\alpha + \beta + 1)u + \alpha - \beta)}{k(k+\alpha+\beta+1)}(dP_k(\alpha,\beta)(u))^2.
\]

From this it is at once clear that, since \( \alpha \geq 0 \), \( \beta > -1 \), and \( \alpha \geq \beta \), we will have \( g'(u) \geq 0 \) for all \( 0 \leq u \leq 1 \). This means that \( g \) is nondecreasing in the interval \((0,1)\). So, if for some \( k \) and \( i \) we have \( t_{k-1,i}^{(\alpha+1,\beta+1)} > 0 \), then \( t_{k-1,i}^{(\alpha+1,\beta+1)} \leq t_{k-1,k-1}^{(\alpha+1,\beta+1)} < 1 \) and we also have
\[
g(t_{k-1,i}^{(\alpha+1,\beta+1)}) \leq g(t_{k-1,k-1}^{(\alpha+1,\beta+1)}),
\]
and since the zeros of \( P_k^{(\alpha+1,\beta+1)}(u) \) correspond to the extrema of \( P_k^{(\alpha,\beta)}(u) \) (cf. (6.3.8) in Andrews, Askey, and Roy \([1]\)), with \([12]\) this amounts to
\[
(P_k^{(\alpha,\beta)}(t_{k-1,i}^{(\alpha+1,\beta+1)}))^2 \leq (P_k^{(\alpha,\beta)}(t_{k-1,k-1}^{(\alpha+1,\beta+1)}))^2.
\]

Now, we know that there is a \( k_0 \) such that \( t_{k-1,k-1}^{(\alpha+1,\beta+1)} > 0 \) for all \( k \geq k_0 \) (cf. Theorem 6.1.1 of Szegö \([23]\)). Suppose also \( k_0 \) is large enough so that \( P_k^{(\alpha,\beta)}(t_{k-1,k-1}^{(\alpha+1,\beta+1)}) \) is close to the right-hand side of \([10]\) for all \( k \geq k_0 \). From \((7)\), we may also assume that \( k_0 \) is such that \( |P_k^{(\alpha,\beta)}(u)| \) is close to zero in some interval \([-\varepsilon,0]\) for some \( 0 < \varepsilon < 1 \) chosen arbitrarily. But then it is clear that the minimum of \( P_k^{(\alpha,\beta)}(u) \) in the interval \([0,1]\) is achieved in the interior of the interval, and then since the \( t_{k-1,k-1}^{(\alpha+1,\beta+1)} \) are the extrema of \( P_k^{(\alpha,\beta)}(u) \), and since \( t_{k-1,k-1}^{(\alpha+1,\beta+1)} \) is a local minimum of \( P_k^{(\alpha,\beta)}(u) \), together with \([13]\) we are done.

**Claim B.** For \( \alpha \geq 0 \) the expression on the right-hand side of \([10]\) is always at least \(-0.45\).
Proof. For $\alpha \geq 0$ and $t > 0$ write
\[ \Omega_\alpha(t) = \Gamma(\alpha + 1) \left( \frac{2}{t} \right)^\alpha J_\alpha(t). \]
The global minimum of $\Omega_\alpha$ is attained at $j_{\alpha+1}$, the first positive zero of $J_{\alpha+1}$, and it is negative (cf. (4.6.2) in Andrews, Askey, and Roy [1] and Section 15.31 of Watson [17]).

We first show that $\Omega_\alpha(t) \geq -0.45$ for all $0 \leq \alpha < 1$. Indeed, the zero $j_{\alpha+1}$ is increasing as a function of $\alpha$ (cf. (2) in Section 15.6 of Watson [17]) and one may verify numerically that $j_1 = 3.8317 \ldots \geq 2$. So from the definition of $\Omega_\alpha$ and since the global minimum of $\Omega_\alpha$ is attained at $j_{\alpha+1}$ we see that
\[ \Omega_\alpha(t) \geq J_\alpha(j_{\alpha+1}) \]
whenever $0 \leq \alpha < 1$. Landau [9] has shown that $J_\alpha(j_{\alpha+1})$ is increasing as a function of $\alpha$. But then for $0 \leq \alpha < 1$ we have
\[ \Omega_\alpha(t) \geq J_0(j_1) = -0.402759 \ldots \geq -0.45, \]
as we wanted.

Suppose now $\alpha \geq 1$. We use the formula
\[ J_{\alpha-1}(t) + J_{\alpha+1}(t) = \frac{2\alpha}{t} J_\alpha(t) \]
(cf. (4.6.5) in Andrews, Askey, and Roy [1]) to see that
\[ \Omega_\alpha(t) = \Gamma(\alpha + 1) \left( \frac{2}{t} \right)^\alpha \frac{t}{2\alpha} (J_{\alpha-1}(t) + J_{\alpha+1}(t)) \]
\[ = \Omega_{\alpha-1}(t) + \Gamma(\alpha) \left( \frac{2}{t} \right)^\alpha J_{\alpha+1}(t). \]

Since the global minimum of $\Omega_\alpha$ is attained at $j_{\alpha+1}$, this implies that the global minimum of $\Omega_\alpha$ is at least that of $\Omega_{\alpha-1}$. Since we have shown that $\Omega_{\alpha}(t) \geq -0.45$ for all $0 \leq \alpha < 1$, the result then follows.

Now we may prove the lemma. Let $k_0$ be given as in Claim A. Suppose that $k_0$ is large enough so that $P_k^{(\alpha,\beta)}(t_{k-1,k-1}^{(\alpha+1,\beta+1)})$ is close enough to the right-hand side of (10) for all $k \geq k_0$, in such a way that from Claim B we see that
\[ P_k^{(\alpha,\beta)}(t_{k-1,k-1}^{(\alpha+1,\beta+1)}) \geq -1/2 \quad \text{for all } k \geq k_0. \]

Suppose also that $k_0$ is large enough so that (11) holds for all $k \geq k_0$. Now take some $k \geq k_0$ such that $t_0 = t_{k-1,k-1}^{(\alpha+1,\beta+1)} > 0$. We show that $f^{(\alpha,\beta)}(t) \geq -1/2$ for all $t_0 < t < 1$.

So fix some $t$ such that $t_0 < t < 1$. We begin by showing that the sequence
\[ P_0^{(\alpha,\beta)}(t), P_1^{(\alpha,\beta)}(t), \ldots, P_k^{(\alpha,\beta)}(t) \]
is decreasing. To this end we shall use the formula
\[ P_j^{(\alpha,\beta)}(u) - P_{j+1}^{(\alpha,\beta)}(u) = \frac{2j + \alpha + \beta + 2}{2(\alpha + 1)} (1 - u) P_j^{(\alpha+1,\beta)}(u), \]
which is adapted to our normalization of $P_j^{(\alpha,\beta)}$ from (6.4.20) in Andrews, Askey, and Roy [1].

Now, for $j < k$ we have
\[ f_j^{(\alpha,\beta)} < f_{k-1,k-1}^{(\alpha+1,\beta+1)} < t. \]
Here, the first inequality comes from the interlacing property. The second inequality is a consequence of Theorem 6.21.1 of Szegö [13]. So $t$ lies to the right of the
rightmost zero of $P_j^{(\alpha,\beta)}$ and hence $P_j^{(\alpha+1,\beta)}(t) > 0$. So it is clear from (15) that $P_j^{(\alpha,\beta)}(t) > P_{j+1}^{(\alpha,\beta)}(t)$, proving that (14) is decreasing.

The fact that (14) is decreasing, together with Claim A, implies that

$$P_j^{(\alpha,\beta)}(t) = P_k^{(\alpha,\beta)}(t) \geq P_k^{(\alpha,\beta)}(t_0)$$

for all $j < k$. Now suppose $j > k$. We then have that

$$P_j^{(\alpha,\beta)}(t) \geq P_j^{(\alpha,\beta)}(t_j^{(\alpha+1,\beta+1)}) > P_k^{(\alpha,\beta)}(t_k^{(\alpha+1,\beta+1)}) = P_k^{(\alpha,\beta)}(t_0),$$

where the first inequality follows from Claim A and the second inequality follows from (11). Now, since we have by construction that $P_k^{(\alpha,\beta)}(t_0) \geq -1/2$, together with (10) we see that for all $j \neq k$ we have $P_j^{(\alpha,\beta)}(t) \geq P_k^{(\alpha,\beta)}(t) \geq -1/2$, and we are done. \qed

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