Twisting Functors and Generalized Verma modules.

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Abstract

Let \( g \) be a reductive Lie algebra. We give a condition that ensures that the character of a generalized Verma module is well-behaved under a twisting functor. We show that a similar result holds for basic classical simple Lie superalgebras, and generalize a result from [CM16] about twisting Verma modules.

1 Introduction and Notation.

1.1 Introduction.

Let \( k \) be an algebraically closed field of characteristic zero, and \( g \) a reductive Lie algebra over \( k \). Our current understanding of the BGG category \( O \) of \( g \)-modules owes much to several well known endofunctors of the category \( O \) and its blocks see [Hum08], [Maz12]. An important class of functors are the twisting functors introduced by Arkhipov [Ark04] and studied further by Andersen in collaborations with Lauritzen [AL03], and Stroppel [AS03]. We postpone some definitions until later, but remark here that there is such a functor \( T_w \) for any Weyl group element \( w \), and for any Verma module \( M(\lambda) \), the modules \( T_w M(\lambda) \) and \( M(w \cdot \lambda) \) have the same character.

Suppose \( p \) is a parabolic subalgebra of \( g \) with Levi factor \( l \). Let \( O_l \) be the analog of the category of \( O \) for \( l \). Consider the functor \( F : O_l \rightarrow O \) defined on an object \( L \) as follows. First make \( L \) into a \( p \)-module by allowing the radical \( m^+ \) of \( p \) to act trivially and then let \( FL = \text{Ind}_p^g L \), be the induced \( g \)-module. If \( L \) is finite dimensional simple, we call \( \text{Ind}_p^g L \), a generalized Verma module. An analog of the above mentioned result on twisted Verma modules does not always hold for generalized Verma modules. However if \( w\alpha \) is a positive root of \( g \) for every positive root \( \alpha \) of \( l \), there is a suitable analog for the character of \( T_w \text{Ind}_p^g L \). The result is most clearly expressed using partition functions. Suppose that \( X \) is any set of positive

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roots, and define \( p_X = \prod_{\alpha \in \Delta^+ \setminus X} (1 - e^{-\alpha})^{-1} \). Then the character of \( \mathcal{F}L = \text{Ind}_p^g L \) is given by

\[
\text{ch} \text{Ind}_p^g L = \text{ch} L p_X.
\]

Recall that the Verma module \( M(\lambda) \) with highest weight \( \lambda \) has character \( e^\lambda p \) where \( p \) is the Kostant partition function. We can think of \( p_X \) as the generating function for the number of partitions with support disjoint from \( X \). When \( X \) is empty we have \( p_X = p \). Under the above assumption on \( v \) we show that \( \text{ch} T_v \mathcal{F}L = v \cdot \text{ch} L p_{vX} \).

Furthermore the result extends easily to the case where \( g \) is a classical simple Lie superalgebra, with a simple change in the definition of \( p_X \) to accommodate odd roots. In [Mus] we showed in the contragredient case, that if \( X \) is an isotropic set of (necessarily odd) roots, and \( \lambda \in h^* \) satisfies \( \lambda([l,l] \cap h) = 0 \), then \( \text{ch} T_v \mathcal{F}L = e^{v\lambda} p_{vX} \).

1.2 Notation.

We collect the main notation that will be used in this paper. We assume that \( g \) is a reductive Lie algebra or a contragredient Lie superalgebra. Fix a Cartan subalgebra \( h \) of \( g \). Let

\[
g = n^- \oplus h \oplus n^+
\]

be a triangular decomposition of \( g \). Then let \( p \) be a parabolic subalgebra of \( g \). We assume \( b = h \oplus n^+ \subseteq p \). There are subalgebras \( m^\pm \) such that

\[
g = m^- \oplus l \oplus m^+
\]

and

\[
p = l \oplus m^+.
\]

Set \( h_l = [l,l] \cap h \). For any \( h \) stable, \( \mathbb{Z}_2 \) graded subalgebra \( c \) of \( g \), let \( \Delta^+_0(c) \) and \( \Delta^+_1(c) \) be the set of even and odd positive roots of \( c \) respectively. We define
\( \Delta^+(c) = \Delta^+_0(c) \cup \Delta^+_1(c) \), and set \( \Delta^+ = \Delta^+_0(g), \Delta^+_0 = \Delta^+_0(g), \Delta^+_1 = \Delta^+_1(g) \). Set
\[
\rho_0 = \sum_{\alpha \in \Delta^+_0} \alpha, \quad \rho_1 = \sum_{\alpha \in \Delta^+_1} \alpha
\]
and \( \rho = \rho_0 - \rho_1 \). For any root \( \alpha \), we can choose a root vector \( e_\alpha \in \mathfrak{g}^\alpha \) such that 
\( \mathfrak{g}^\alpha = ke_\alpha \). For an element \( w \) of the Weyl group \( W \), set \( N(w) = \{ \alpha \in \Delta^+_0 | w\alpha < 0 \} \).
We define translated actions of the Weyl group \( W \) on \( \mathfrak{h}^* \) by
\[
w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \circ \lambda = w(\lambda + \rho_0) - \rho_0.
\]
If \( X \) is a set of positive roots, let \( X_0 \) (resp. \( X_1 \)) be the set of even (resp. odd) roots contained in \( X \). Then define
\[
\tau_X = \prod_{\alpha \in X_0} (1 - e^{-\alpha}) \text{ resp. } s_X = \prod_{\alpha \in X_1} (1 + e^{-\alpha}).
\]
Clearly for \( w \in W \),
\[
wr_X = r_wX \text{ and } ws_X = s_wX.
\]
We also set \( r = r_\emptyset \), and \( p_X = r_X/r \). Note that \( pr = 1 \). If \( X = \Delta^+(l) \), then since
\[
\text{ch } U(m^-) = \prod_{\alpha \in \Delta^+(m)} (1 - e^{-\alpha})^{-1} = p_X,
\]
we have
\[
\text{ch } \text{Ind}_\mathfrak{g}^\mathfrak{g} L = \text{ch } L \frac{r_X}{r} = \text{ch } L p_X.
\]
If \( L_\ell(\lambda) \) is one dimensional, then \( \lambda(\mathfrak{h}_l) = 0 \). Then we denote \( L \) by \( k_\lambda \), and if \( h \in \mathfrak{h}, v \in k_\lambda \), we have \( hv = \lambda(h)v \). In this case, if \( X = \Delta^+(l) \) we have
\[
\text{ch } \text{Ind}_\mathfrak{g}^\mathfrak{g} k_\lambda = e^\lambda p_X.
\]
Let \( \Lambda \) be the lattice of functions on \( \mathfrak{h}^* \) such that \((\sigma, \alpha^\vee) \in \mathbb{Z} \) for all simple roots \( \alpha \) As in [Hum72] 22.5, we use the group ring \( \mathbb{Z}[\Lambda] \) with \( \mathbb{Z} \)-basis the symbols \( e^\sigma \) with \( \sigma \in \Lambda \). The circle action of \( w \in W \) on \( \mathbb{Z}[\Lambda] \) is defined by \( w \circ e^\sigma = e^{w_0\sigma} \). (There is a similar dot action defined using \( \rho \) in place of \( \rho_0 \)). We warn the reader that the circle action is not an action by algebra automorphisms. Instead we have for \( a, b \in \mathbb{Z}[\Lambda] \),
\[
v \circ (ab) = (va)(v \circ b).
\]
With this notation, we can now state the main result on Lie algebras.

**Theorem 1.1.** Suppose \( \mathfrak{g} \) is reductive and set \( X = \Delta^+(l) \). If \( w \in W \) is such that \( N(w) \cap X = 0 \), the module \( T_w \text{Ind}_\mathfrak{g}^\mathfrak{g} L \) has character \((w \circ \text{ch } L)p_wX \). In particular
\[
\text{ch } T_w \text{Ind}_\mathfrak{g}^\mathfrak{g} k_\lambda = e^{w_0\lambda} p_wX.
\]
This result extends to classical simple Lie superalgebras. In particular this gives a new proof of a result of Coulembier and Mazorchuk about twisting Verma modules.

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2 Reductive Lie algebras.

2.1 Basics on Twisting Functors.

We assume that \( g \) is a reductive Lie algebra. For \( w \in W \), the twisting functor \( T_w \) for semisimple Lie algebras was introduced by Arkhipov \cite{Ark04} in his work on the semi-infinite BGG resolution. Twisting functors for Lie superalgebras are studied in \cite{CM16}. Until further notice, we assume that \( g \) is reductive. The twisting functor on the category \( \text{Mod-}U(g) \) of \( U(g) \)-modules is defined as follows: Let \( n_w = n^- \cap w^{-1}(n^+) \), and let \( N_w = U(n_w) \). We make \( g \) into a \( \mathbb{Z} \)-graded Lie superalgebra \( g = \bigoplus_{i \in \mathbb{Z}} g(i) \) such that \( g(0) = h \) and \( g(\pm 1) = \oplus g_{\pm \alpha} \), where the sum runs over all simple roots \( \alpha \). This grading induces a grading on \( N_w \). Let \( (N_w^*)_i = \text{Hom}_k(N_w^{-i}, k) \).

Then \( N_w^* = \bigoplus_{i \in \mathbb{Z}} (N_w^*)_i \) is the graded dual of \( N_w \). Set \( U = U(g) \). Then the corresponding semi-infinite \( U \)-bimodule \( S_w \) is defined as \( S_w = U \otimes_{N_w^*} N_w^* \). For a proof that \( S_w \) is a \( U \)-bimodule see \cite{Ark04}. As a \( U \)-module we have, see \cite{AL03} 6.1,

\[
S_w \cong N_w^* \otimes_{N_w} U. \quad (2.1)
\]

The twisting functor \( T_w : \text{Mod-}U(g) \rightarrow \text{Mod-}U(g) \) corresponding to \( w \in W \) is defined by \( T_w(?) = S_w \otimes_U (?) \). The functor \( T_w \) restricts to an endofunctor on \( \mathcal{O} \).

**Lemma 2.1.** The functor \( T_w \) is right exact, and has left derived functors \( \mathcal{L}_i T_w \) given by

\[
\mathcal{L}_i T_w(?) = \text{Tor}_i^{N_w^*}(N_w^*, ?). \quad (2.2)
\]

**Proof.** This follows since by \((2.1)\),

\[
T_w(?) = S_w \otimes_U (?) = N_w^* \otimes_{N_w^*} U \otimes_U (?) = N_w^* \otimes N_w(?) \nonumber.
\]

When \( g \) is reductive and \( w = s_{\alpha} \) is a simple reflection, \( T_w \) has the following easy description. Let \( U_s \) denote the localization of \( U = U(g) \) at the negative root vector \( e_{-\alpha} \). Then \( U(s) = U_s/U \) is a \( U-U \)-bimodule, and \( T_s M \cong U(s) \otimes U M \). There is an inner automorphism \( \phi = \phi_{\alpha} \) of \( g \) such that \( \phi(g_{\beta}) = g_{s_{\alpha} \beta} \) for all roots \( \beta \), and \( \phi(h) = h \). The action of \( g \) on \( U(s) \) is twisted by \( \phi \), i.e. the action is given by \((x, u) \rightarrow \phi(x)u\).

It was shown in \cite{AL03}, Remark 6.1 ii) that

\[
T_{ws} \cong T_w T_s \text{ if } ws > w \text{ and } s \text{ is a simple reflection}. \quad (2.3)
\]

2.2 Twisting Generalized Verma Modules.

It was shown in \cite{AL03}, Proposition 6.1 ii), that for any Verma module \( M(\lambda) \) the modules \( T_w M(\lambda) \) and \( M(w \circ \lambda) \) have the same character. Equivalently by \cite{Jan79} Satz 1.11, we have in the Grothendieck group \( K(O) \) of the category \( \mathcal{O} \) that

\[
[T_w M(\lambda)] = [M(w \circ \lambda)]. \quad (2.4)
\]

Our goal is to obtain an analog of this result for generalized Verma modules \( M^X(\lambda) \).
Proposition 2.2. If \( N(w) \cap \Delta^+(l) = \emptyset \), then

(a) \( n_w \) is subalgebra of \( m^- \).

(b) If \( L \) is a \( \ell \)-module and \( M = \mathcal{F}L \), then \( \mathcal{L}_iT_wM = 0 \) for \( i > 0 \).

Proof. By definition
\[
n_w = \text{span}\{e_{-\alpha} | \alpha \in N(w)\},
\]
and
\[
m^- = \text{span}\{e_{-\alpha} | \alpha \in \Delta^+(g) \setminus \Delta^+(l)\}.
\]
Hence (a) follows from the hypothesis. With \( L \) as in (b), \( M = \text{Ind}_p^g L = U(m^-) \otimes L \) is a free \( U(m^-) \)-module. By the PBW Theorem and (a) \( U(m^-) \) is a free \( N_w = U(n_w) \)-module. Thus \( M \) is free and hence flat over \( N_w \), and (b) follows from (2.2).

Lemma 2.3. Suppose \( w \in W \) is such that \( N(w) \cap \Delta^+(l) = \emptyset \), and suppose that the sequence of \( g \)-modules
\[
0 \to Q_1 \to Q_2 \to Q_3 \to 0,
\]
is exact. Then provided that \( Q = Q_3 \) is a free (or even flat) \( N_w \)-module, the sequence
\[
0 \to T_wQ_1 \to T_wQ_2 \to T_wQ_3 \to 0,
\]
is also exact. This holds for example if \( Q \) is induced from a \( p \)-module.

Proof. The twisting functor \( T_w \) is right exact, so it suffices to show that its left derived functor \( \mathcal{L}_1T_w \) satisfies \( \mathcal{L}_1T_wQ = 0 \). This follows from Lemma 2.1.

Lemma 2.4. Suppose
\[
0 \to M_n \overset{f_{n-1}}\to M_{n-1} \to \cdots \to M_1 \overset{f_0} \to M_0 \to M \to 0,
\]
is a long exact sequence such that \( \mathcal{L}_1T_wM_j = 0 \) all \( j \) and and \( \mathcal{L}_1T_wM = 0 \). Then the sequence
\[
0 \to T_wM_n \overset{Twf_{n-1}}\to T_wM_{n-1} \to \cdots \to T_wM_1 \overset{Twf_0} \to T_wM_0 \to T_wM \to 0,
\]
is also exact.

Proof. Let \( K_i = \ker f_i \). Then from the exact sequence \( 0 \to K_0 \to M_0 \to M \to 0 \), and the long exact sequence for Tor, we see that \( 0 \to T_wK_0 \to T_wM_0 \to T_wM \to 0 \) is exact and \( \mathcal{L}_1T_wK_0 = 0 \). The same reasoning applied to the exact sequences \( 0 \to K_i \to M_i \to K_{i-1} \to 0 \) shows that \( 0 \to T_wK_i \to T_wM_i \to T_wK_{i-1} \to 0 \) is exact and \( \mathcal{L}_1T_wK_i = 0 \) for all \( i \). Assembling all the sequences involving \( T_w \) we get the result.
Let $C$ be the full subcategory of $O$ consisting of $g$-modules that have finite resolutions by (direct sums of) Verma modules, and let $C(l)$ be the analogous category of $l$-modules. The Verma module for $l$ with highest weight $\lambda \in h^*$ will be denoted by $M_l(\lambda)$. It is clear that $\mathcal{F}M_l(\lambda) \cong M(\lambda)$, so the functor $\mathcal{F}$ takes $C(l)$ to $C$.

**Lemma 2.5.** If $L$ is a finite dimensional $l$-module, then $\mathcal{F}L \in C$.

**Proof.** Begin with the BGG-resolution of $L$ as an $l$-module and apply $\mathcal{F}$ to get a resolution of $\mathcal{F}L$. By transitivity of induction, Verma's induce to Verma's.

**Lemma 2.6.** Suppose $M \in C$, $w \in W$ and $\mathcal{L}_1 T_w M = 0$. Then

(a) $\text{ch } M = \sum_{\mu} b_{\mu} \text{ch } M(\mu)$ implies $\text{ch } T_w M = \sum_{\mu} b_{\mu} \text{ch } M(w \circ \mu)$.

(b) If $\text{ch } M = ap$ for $a \in Z[\Lambda]$, then $\text{ch } T_w M = (w \circ a)p$.

**Proof.** Suppose that

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M \to 0,$$

is a resolution of $M$ by direct sums of Vermas, and suppose the multiplicity of the Verma $M(\mu)$ in $M_i$ is $a_{i,\mu}$, then clearly

$$\text{ch } M = \sum_{i=0}^{n} (-1)^i \text{ch } M_i = \sum_{i=0}^{n} (-1)^i a_{i,\mu} \text{ch } M(\mu).$$

$$= \sum_{\mu} b_{\mu} \text{ch } M(\mu),$$

where $b_{\mu} = \sum_{i=0}^{n} (-1)^i a_{i,\mu}$. Now assuming that $\mathcal{L}_1 T_w M = 0$, we have a resolution

$$0 \to T_w M_n \to T_w M_{n-1} \to \cdots \to T_w M_1 \to T_w M_0 \to T_w M \to 0,$$

and by (2.4) this implies that

$$\text{ch } T_w M = \sum_{i=0}^{n} (-1)^i a_{i,\mu} \text{ch } M(w \circ \mu).$$

This proves (a), and (b) follows since $\text{ch } M(\lambda) = e^\lambda p$. □

**Proof of Theorem 1.1.** If $L$ is a finite dimensional $l$-module then $M = \text{Ind}_g^p L \in C$ by Lemma 2.5. Also $\mathcal{L}_1 T_w M = 0$ by Proposition 2.2. Therefore from (1.3) and Lemma 2.6 we obtain the first equality below. For the second we use (1.2) and (1.4). We obtain

$$\text{ch } T_w \text{Ind}_g^p L = \frac{w \circ (r X \text{ch } L)}{r} = p_w X w \circ \text{ch } L,$$

as claimed. □
3 Lie Superalgebras.

From now on we fix a contragredient Lie superalgebra \( \mathfrak{g} \), and apply the results of the previous Subsection to the reductive algebra \( \mathfrak{g}_0 \). In \([CM16]\) the twisting functor \( T_s \), for \( \mathfrak{g} \) is defined as follows. We assume that \( s \) is a reflection corresponding to the simple non-isotropic root \( \alpha \). Then as before denote the localization \( U(\mathfrak{g}) \) at \( e_{-\alpha} \) by \( U_s \) and set \( U(s) = U_s/U \). Since \( \phi \) is inner, it extends to \( \mathfrak{g} \) and then to \( U(\mathfrak{g}) \). Then the action of \( U \) is twisted by \( \phi \).

This definition can be extended to \( W \) because if we use (2.3) to define an action of the free group generated by symbols \( T_s \) for \( s \) a simple reflection, then the Braid relations are satisfied \([CM16]\) Lemma 5.3. Thus (2.3) yields a well-defined twisting functor (which we also denote by \( T_w \)) for all \( w \in W \). Also the restriction functor \( \text{Res}^\mathfrak{g}_{\mathfrak{g}_0} \) intertwines \( T_w \), that is we have an isomorphism of functors, see \([CM16]\) Lemma 5.1.

\[
\text{Res}^\mathfrak{g}_{\mathfrak{g}_0} \circ T_w \cong T_w \circ \text{Res}^\mathfrak{g}_{\mathfrak{g}_0}.
\]

(3.1)

There is a similar result for the induction functor \( \text{Ind}^\mathfrak{g}_{\mathfrak{g}_0} \), but we will not need it.

**Lemma 3.1.** Suppose \( E \) is a finite dimensional simple \( \mathfrak{l} \)-module with highest weight \( \lambda \), and make \( E \) into a \( \mathfrak{p} \)-module by allowing \( \mathfrak{m}^+ \) to act trivially.

(a) If \( M = \text{Ind}^\mathfrak{g}_{\mathfrak{g}_0} E \) there is a finite chain of \( U(\mathfrak{g}_0) \)-submodules

\[
M = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = 0,
\]

and \( \mathfrak{l}_0 \)-modules \( E_i \) with \( M_i/M_{i-1} \cong \text{Ind}^\mathfrak{g}_{\mathfrak{p}_0} E_i \) and \( \dim E_i < \infty \) for \( 1 \leq i \leq s \).

(b) We have \( \sum_{i=1}^s \dim E_i = \dim \Lambda(\mathfrak{m}^-) \otimes E \).

**Proof.** Let \( \Lambda(\mathfrak{m}^-) \) be the exterior algebra on \( \mathfrak{m}^- \). Then \( \mathfrak{l}_0 \) acts on \( \Lambda(\mathfrak{m}^-) \) via the adjoint action, and we have as a \( \mathfrak{l}_0 \)-module, (compare \([Mus12]\) Corollary 6.4.5),

\[
M = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E = U(\mathfrak{m}^-) \otimes E = U(\mathfrak{m}_0^-) \otimes \Lambda(\mathfrak{m}^-) \otimes E.
\]

Note that \( N = \Lambda(\mathfrak{m}^-) \otimes E \) is an \( \mathfrak{p}_0 \)-module, and \( M = \text{Ind}^\mathfrak{g}_{\mathfrak{p}_0} N \). Write \( N \) as a direct sum of finite dimensional simple \( \mathfrak{l}_0 \)-modules. Say

\[
N = \Lambda(\mathfrak{m}^-) \otimes E = \bigoplus_{i=1}^s E_i
\]

with \( E_i \) simple. Since \( M \) is an object of \( \mathcal{O} \), \( \mathfrak{n}_0^+ \) acts nilpotently on \( M \), so some power of \( \mathfrak{m}_0^- \) in \( U(\mathfrak{m}^-) \) annihilates \( N \). To construct the chain (3.2) we construct a similar chain of \( U(\mathfrak{p}_0) \)-submodules

\(^1\)We have excluded the cases where \( \mathfrak{g} = P(n) \) or \( \mathfrak{g} = Q(n) \) for technical reasons. The main results however can be adapted to these Lie superalgebras.
\[ N = N_s \supset N_{s-1} \supset \cdots \supset N_1 \supset N_0 = 0 \]  \hspace{1cm} (3.4)

such that \( N_i/N_{i-1} \cong E_i \) for \( 1 \leq i \leq s \). Suppose we have constructed \( N_i \) and set \( M_i = \text{Ind}_{p_0}^{g_0} N_i \) and \( N_i = (M_i + N)/M_i \). Then consider

\[ \text{ann}_{\Lambda^i} m_0^+ = \{ x \in N_i | m_0^+ x = 0 \} \]

This is a nonzero \( p_0 \)-submodule of \( N_i \) killed by \( m_0^+ \). By renumbering the \( E_j \) we can assume that \( N_{i+1}/N_i \cong E_{i+1} \) is a simple submodule of \( \text{ann}_{\Lambda^i} m_0^+ \). Finally applying the induction functor \( \text{Ind}_{p_0}^{g_0} \) to the exact sequence

\[ 0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow E_i \rightarrow 0, \]

we obtain

\[ 0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow \text{Ind}_{p_0}^{g_0} E_i \rightarrow 0. \]  \hspace{1cm} (3.5)

Hence (a) follows and (b) holds by (3.3). \( \square \)

Note that

\[ \text{ch} \Lambda(m_1^-) \otimes E = s_X \text{ch} E \]  \hspace{1cm} (3.6)

**Theorem 3.2.** Assume \( X = \Delta^+(1) \) and \( w \in W \) is such that \( N(w) \cap X = \emptyset \). Then if \( M = \text{Ind}_p^{g_0} E \), we have

\[ \text{ch} T_w M = \frac{r_w X s_w X (w \circ \text{ch} E)}{r} = p_{w,X}(w \circ \text{ch} E). \]  \hspace{1cm} (3.7)

**Proof.** By (3.1) the character of \( T_w M \) is the same as its character when regarded as a \( g_0 \)-module. By Lemma 2.3, the functor \( T_w \) is exact on the sequence (3.3), so \( T_w M_i/T_w M_{i-1} \cong T_w \text{Ind}_{p_0}^{g_0} E_i \). These remarks justify the first two equalities below. For the third we use (3.3)

\[
\begin{align*}
\text{ch} T_w M &= \sum_{i=1}^s \text{ch} T_w (M_i/M_{i-1}) \\
&= \sum_{i=1}^s \text{ch} T_w \text{Ind}_{p_0}^{g_0} E_i \\
&= \text{ch} T_w \text{Ind}_{p_0}^{g_0} (\Lambda(m_1^-) \otimes E),
\end{align*}
\]

but

\[ \text{ch} \text{Ind}_{p_0}^{g_0} \Lambda(m_1^-) \otimes E = \frac{r_X s_X \text{ch} E}{r}. \]  \hspace{1cm} (3.8)

Hence using Lemma 2.6, we obtain the result. \( \square \)
We specialize to the case where $E = k\lambda$ is one dimensional. To do this we need some more notation. Let $\Gamma$ be the set of sums of distinct odd positive roots, and for $\gamma \in \Gamma$, let $K(\gamma)$ be the number of partitions of $\gamma$ into distinct odd positive roots. By [Mus97] Lemma 2.3, $W$ acts on $\Gamma$ by

$$w \ast \gamma = \rho_1 - w(\rho_1 - \gamma)$$

for $w \in W$ and $\gamma \in \Gamma$. This action is related to those in (1.1) by

$$w \circ (\lambda - \gamma) = w \cdot \lambda - w \ast \gamma.$$  

Now for $\gamma \in \Gamma$, define $K_X(\gamma)$ by

$$\prod_{\alpha \in \Delta^+_i \setminus X} (1 + e^{-\alpha}) = \sum_{\gamma \in \Gamma} K_X(\gamma)e^{-\gamma}$$

and let $\Gamma_X = \{ \gamma \in \Gamma | K_X(\gamma) > 0 \}$. We have $s_X = \sum_{\gamma \in \Gamma_X} K_X(\gamma)e^{-\gamma}$. So (3.8) yields

$$\text{ch Ind}_{\text{p}^0_\lambda(M^i)}^{\text{p}^0_\lambda} = p_{wX}e^{w\cdot \lambda}.$$ 

**Theorem 3.3.** Suppose $w \in W$ is such that $N(w) \cap \Delta^+(t) = \emptyset$. Then

$$\text{ch} T_w M^X(\lambda) = p_{wX}e^{w\cdot \lambda}.$$ 

First we isolate a key step in the proof.

**Lemma 3.4.** We have

$$\sum_{\gamma \in \Gamma_X} K_X(\gamma)e^{-w\ast \gamma} = \sum_{\gamma \in \Gamma_{wX}} K_{wX}(\gamma)e^{-\gamma}.$$ 

**Proof.** Note that

$$e^{\rho_1} \prod_{\alpha \in X} (1 + e^{-\alpha}) \sum_{\gamma \in \Gamma_X} K_X(\gamma)e^{-\gamma} = \prod_{\alpha \in \Delta^+_i} (e^{\alpha/2} + e^{-\alpha/2}),$$

and this expression is $W$-invariant and independent of $X$. Using (3.9), we apply $w$ to (3.13) to get the first equality below, and replace $X$ by $wX$ for the second,

$$e^{\rho_1} \prod_{\alpha \in X} (1 + e^{-w\alpha}) \sum_{\gamma \in \Gamma_X} K_X(\gamma)e^{-w\ast \gamma} = \prod_{\alpha \in \Delta^+_i} (e^{\alpha/2} + e^{-\alpha/2})$$

$$= e^{\rho_1} \prod_{\beta \in wX} (1 + e^{-\beta}) \sum_{\gamma \in \Gamma_{wX}} K_{wX}(\gamma)e^{-\gamma}.$$

The result follows since

$$e^{\rho_1} \prod_{\alpha \in X} (1 + e^{-w\alpha}) = e^{\rho_1} \prod_{\beta \in wX} (1 + e^{-\beta}).$$

\qed
Proof of Theorem 3.3. By Lemma 2.6 applied to (3.12),

\[
\text{ch } T_w M^X(\lambda) = \text{pr}_{wX} \sum_{\gamma \in \Gamma_X} K_X(\gamma) e^{w(\lambda - \gamma)}. \tag{3.14}
\]

Using (3.10) and then Lemma 3.4, we see that \( T_w M^X(\lambda) \) has character

\[
\text{pr}_{wX} e^{w\lambda} \sum_{\gamma \in \Gamma_X} K_X(\gamma) e^{-w\gamma} = \text{pr}_{wX} e^{w\lambda} \sum_{\gamma \in \Gamma_{wX}} K_{wX}(\gamma) e^{-\gamma} = e^{w\lambda} \text{pr}_{wX}. \tag{3.15}
\]

This completes the proof.

Remarks 3.5. When \( g = \mathfrak{gl}(m, n) \) Theorem 3.3 is used in the Jantzen sum formula from [Mus]. For Verma modules i.e. the case where \( X \) is empty, the Theorem reduces to [CM16] Lemma 5.5.

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