Revisiting $f_B$ and $\bar{m}_b(\bar{m}_b)$ from HQET spectral sum rules

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Abstract
Using recent values of the QCD (non-) perturbative parameters given in Table 1 we reconsider the extraction of $f_B$ and the on-shell mass $M_b$ from HQET Laplace spectral sum rules known to N2LO PT series and including dimension 7 condensates in the OPE. We especially study the convergence of the PT series, the effects on “different spectral sum rules data” of the continuum threshold and subtraction point varied in a larger range than in the existing literature and include in the error an estimate of the N3LO PT series based on a geometric growth of the PT series. We obtain the Renormalization Group Invariant (RGI) universal coupling: $f^\mu_B = 0.416(60) \text{ GeV}^{3/2}$ in the static limit $M_b \to \infty$ and the physical decay constant including $1/M_b$ corrections: $f^\mu_B^{\text{ptet}} = 199(29) \text{ MeV}$. Using the ratio of sum rules, we obtain, to order $\alpha_s^2$, the running mass $\bar{m}_b(\bar{m}_b) = 4213(59) \text{ MeV}$. The previous results are in good agreement with the ones from QCD spectral sum rules (QSSR) in full QCD to the same order from the same channel $[^1]: f^\mu_B \sim 206(7) \text{ MeV}$ and $\bar{m}_b(\bar{m}_b)^\text{ptet} = 4236(69) \text{ MeV}$. Keywords: QCD spectral sum rules, meson decay constants, heavy quark masses, heavy quark effective theory.

1. Introduction
The (pseudo)scalar meson decay constants $f_P$ are of prime interests for understanding the realizations of chiral symmetry in QCD. In addition to the well-known values of $f_K = 130.4(2) \text{ MeV}$ and $f_K = 156.1(9) \text{ MeV}$ which control the light flavour chiral symmetries, it is also desirable to extract the ones of the heavy-light charm and bottom quark systems with high-chiral symmetries, it is also desirable to extract the ones of the heavy-light charm and bottom quark systems with high-accuracy. These decay constants are normalized through the matrix element:

$$\langle 0|P^\mu_{PQ}(x)|P \rangle = f_P M_P^\alpha_\mu \ ,$$

where:

$$P^\mu_{PQ}(x) \equiv (m_Q + M_Q)\bar{q}(i\gamma_5)Q \ ,$$

is the local heavy-light pseudoscalar current; $Q \equiv d, s$; $Q \equiv c, b; \ P \equiv D_{ij}, B_{ij}$ and where $f_P$ is related to the leptonic width:

$$\Gamma(P^\mu \rightarrow l^\nu l) = \frac{G_F^2}{8\pi} |V_{Ql}|^2 f_P^2 m_l^2 M_P \left(1 - \frac{m_l^2}{M_P^2}\right)^2 \ ,$$

where $m_l$ is the lepton mass and $|V_{Ql}|$ the CKM mixing angle. In a recent analysis $[^1]$, we have revised the extraction of these heavy-light decay constants in full QCD $[^1]$ using QCD spectral sum rules $[^4][^9]$. Here, we pursue the analysis by revisiting the determination of $f_B$ from HQET spectral sum rules. In so doing, we shall explicitly analyze the influence on the results of the subtraction point $\mu$ and of the continuum threshold $t_c$. We shall also use (besides recent determinations of the QCD input parameters) the new precise value of $m_b$ from the $\gamma$ sum rules $[^2]$. In addition, we shall re-extract the meson-quark mass difference using HQET sum rules from which we shall deduce the running $b$-quark mass.

2. HQET preliminaries
HQET spectral sum rules have been initially used by Shuryak $[^11]$ using a non-relativistic $[^7]$ version of the NSVZ $[^12]$ sum rules in the large $M_b$ limit. Shuryak’s sum rule has been applied later on in HQET $[^13]$ by several authors $[^14][^20]$. The most important input in the analysis of HQET sum rule is the local heavy-light quark axial-vector current of the full QCD theory which can be expressed as an OPE of the HQET operators $\hat{O}_a$ in the inverse of the heavy quark mass:

$$F_a(x, M_b) = C_b \left(\frac{M_b}{\mu} \alpha_s(\mu)\right) F_a^\mu(x, M_b = \infty) + \sum_{n=1} \frac{C_n}{M_b^{2n}} \hat{O}_n (M_b, \infty, \mu) \ ,$$

where: $F_a^\mu = \bar{q} \gamma^\mu \gamma^5 h_s$ is the quark current in HQET built from a light antiquark field $\bar{q}$ and a properly normalized heavy quark field $h_s$ $[^13]$. $C_{2,3}$ are Wilson coefficients and $M_b$ is the on-shell $b$-quark mass. Using a non-covariant normalization of hadronic states $[^21]$, one can define an universal coupling in the static limit:

$$\langle 0|F_a^\mu(0, \nu) \rangle = \frac{i}{\sqrt{2}} \delta_{\nu \mu} \hat{f}_{\text{tot}} \ .$$

The coefficient function $C_b (M_b/\mu, \alpha_s(\mu))$ is obtained by requiring that HQET reproduces the full QCD theory at $\mu = M_b$. It has been obtained to order $\alpha_s^2$ in $[^7]$ and to order $\alpha_s^2$ in $[^22]$. It reads in the $\overline{MS}$-scheme:

$$C_b (M_b) = 1 - \frac{2}{3} \alpha_s (M_b) + \alpha_s^2 (M_b) \left( - \frac{1871}{1729} \ - \frac{17\pi^2}{72} \right)$$

$[^1]$ Some earlier attempt to use a non-relativistic approach for estimating $f_0$ can e.g. be found in $[^10]$.

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\[
- \frac{\alpha_s^2}{18} \ln 2 - \frac{11}{36} \zeta(3) + n_l \left( \frac{478 + \alpha_s^2}{36} \right).
\]

with: \( \alpha_s \equiv \alpha_s/\pi \). The HQET current \( \bar{J}_\beta \) acquires anomalous dimension, which reads to \( O(\alpha_s^2) \) \cite{13,22,24} (in our normalizations) for \( n_l \) flavours:

\[
\gamma \equiv \gamma_1 \alpha_s + \gamma_2 \alpha_s^2 + \ldots, \quad \gamma_1 = 1, \quad \gamma_2 = \frac{127}{72} + \frac{7n_l^2}{54} - \frac{5}{36} n_l.
\]

Therefore, the universal coupling scales as:

\[
f_{\text{sw}}(\mu) = \frac{R_0(M_b)}{B_\beta(\mu)} f_{\text{sw}}(M_b), \tag{8}
\]

with:

\[
R_0(\mu) = (\alpha_s(\mu))^{-\gamma_1/\beta_1} \left[ 1 - \left( \frac{\gamma_2}{\beta_1} - \beta_2 \right) a_r(\mu) \right], \tag{9}
\]

where the two first coefficients of the \( \beta \) functions are:

\[
\beta_1 = -\frac{1}{2} \left( 11 - \frac{2}{3} n_l \right), \quad \beta_2 = -\frac{1}{4} \left( 51 - \frac{19}{3} n_l \right). \tag{10}
\]

The universal coupling is connected to the physical decay constant as:

\[
f_B \sqrt{M_B} = C_B(M_b) f_{\text{sw}}(M_b) + O(1/M_b). \tag{11}
\]

It is also convenient to introduce the universal Renormalization Group Invariant (RGI) current and the associated coupling:

\[
\bar{J}_\beta^R = R_\beta(\mu) \bar{J}_\beta(\mu), \quad \hat{f}_R = R_0(\mu) \hat{f}_{\text{sw}}(\mu), \tag{12}
\]

which we shall estimate in the following.

3. HQET spectral rules for \( \hat{f}_R \) in the static limit

We shall be concerned with the universal RGI 2-point-function:\footnote{\( \bar{J}_\beta \) is HQET Lorentz structure is unimportant and it is only the parity which counts such that we shall omit it in the following.}

\[
\hat{\Pi}(q^2 \equiv -Q^2) = i \int d^4x e^{i q x} \langle 0| \bar{J}(x) \hat{J}(0)|0 \rangle \tag{13}
\]

for determining the coupling \( \hat{f}_R \) using QCD spectral sum rules (QSSR). Like in the case of the full theory, we can use either the Laplace (LSR) \cite{4,26,27}:\footnote{Some attempts to use \( Q^2 = 0 \)-moment sum rules can be found in \cite{20}.}

\[
\mathcal{L}(\tau, \mu) \equiv \lim_{Q^2, n \to \infty} \frac{(-Q^2)^n}{n!} \frac{d^n \hat{\Pi}}{d(Q^2)^n} \equiv \tau \int_0^{\infty} dt e^{-\tau t} \frac{1}{\tau} \text{Im} \hat{\Pi}(t, \mu), \tag{14}
\]

or the \( Q^2 = 0 \) Moments sum rules (MSR) \cite{4}:

\[
M^{(n)}(\mu) \equiv \frac{(-1)^n}{n!} \frac{d^n \hat{\Pi}}{d(Q^2)^n} |_{Q^2=0} = \int_0^{\infty} \frac{dt}{e^{\tau t}} \frac{1}{\tau} \text{Im} \hat{\Pi}(t, \mu). \tag{15}
\]

However, the use of the \( Q^2 = 0 \)-moment sum rules is rather delicate as they do not have a proper infinite heavy quark mass limit. We shall not consider these sum rules here.\footnote{Some attempts to use \( Q^2 = 0 \)-moment sum rules can be found in \cite{20}.} For the present analysis, it is convenient to introduce respectively the soft scale, the meson-quark mass-difference and the HQET Laplace sum rule variable:

\[
\omega = \frac{(q^2 - M_b^2)}{M_b}, \quad \Delta M = \frac{(M_b^2 - M_\tau^2)}{M_b}, \quad \tau_H = \tau M_b, \tag{16}
\]

where \( M_b \) is the on-shell quark mass and \( \tau \) is the usual LSR variable used in the full QCD theory and has the dimension of GeV\(^{-2} \). As usual, we parametrize the spectral function using the Minimal Duality Ansatz (MDA):

\[
\frac{1}{\pi} \text{Im} \hat{\Pi}(t) \approx \hat{f}_R^2 \delta(t - M_B^2) + \text{"QCD cont."} \delta(t - t_0), \tag{17}
\]

where the accuracy for the sum rule approaches has been explicitly tested from heavy quarkonia systems in \cite{11}. The perturbative (PT) expression of the spectral function related to the current \( \bar{J}_R^0(\mu) \) has been evaluated to order \( O(\alpha_s) \) (NLO) in \cite{17} and to \( O(\alpha_s^2) \) (N2LO) in \cite{28}. It reads:

\[
\text{Im} \hat{\Pi}_{\text{PT}}(\omega) = \frac{3\alpha_s^2}{8\pi} \left[ 1 + a_\mu(\mu) \left( \frac{17}{3} + \frac{4\alpha_s^2}{9} + L_\omega \right) + a_\mu^2(\mu) \left( 99(15) + \frac{1657}{72} + \frac{97\alpha_s}{54} \right) L_\omega + \frac{15}{8} L_\omega^2 + \frac{1}{12} \right], \tag{18}
\]

with: \( L_\omega \equiv \ln(\mu^2/\omega^2) \) and \( \hat{f}_{\text{PT}}(\omega) \equiv \hat{R}^2_\beta(\mu) \hat{\Pi}_{\text{PT}}(\omega) \). We estimate the \( O(\alpha_s^2) \) (N3LO) by assuming the geometric growth of the PT series \cite{29} as a dual to the effect of a \( 1/q^2 \) term \cite{30,31} which parametrizes the UV renormalon contributions. NP corrections up to dimension 7 condensates has been obtained in \cite{16,32}. Including all the previous corrections, the sum rule reads for \( M_b \to \infty \):

\[
\hat{f}_R^2 = e^{\gamma_E M_B^2} R_\beta^2(\mu) \left[ \frac{1}{\pi} \int_0^{\infty} dt e^{-\omega t} \text{Im} \hat{\Pi}_{\text{PT}}(\omega) \right. \left. + NP \right], \tag{19}
\]

where: \( \omega_c = (t_c - M_\tau^2)/M_b \) and:

\[
NP(\mu) = -\langle \bar{u}u(\mu) \rangle \left[ 1 + 2a_\mu(\mu) - \frac{M_b^2}{4} \tau_H^2 + \frac{\pi(\alpha_s G^2)}{18} \tau_H \right] + \left( \frac{\mu}{\mu} \right)^2 \left( \frac{3g(\mu)}{81 \alpha_s^2} \right) \frac{2}{12} \tau_H^2. \tag{20}
\]

4. The QCD input parameters

The QCD parameters which shall appear in the following analysis will be the on-shell bottom quark mass \( M_b \) (we shall
neglect the light quark masses $q \equiv u$ here and in the following), the light quark condensate $\langle \bar{q}q \rangle$, the gluon condensates $\langle g G^2 \rangle \equiv \langle g^2 G_{\mu \nu}^a G_{\mu \nu}^a \rangle$ and $\langle g^2 G^2 \rangle \equiv \langle g^3 f_{abc} G_{\mu \nu}^a G_{\mu \nu}^b G_{\mu \nu}^c \rangle$, the mixed condensate $\langle \bar{q}g\sigma Gq \rangle \equiv \langle \bar{q}g\sigma_{\mu \nu}(\lambda_3/2)G_{\mu \nu}q \rangle = M_0^2 \langle \bar{q}q \rangle$ and the four-quark condensate $\rho(\bar{q}q)^2$, where $\rho \approx 2$ indicates the deviation from the four-quark vacuum saturation. Their values are given in Table I. We shall work with the running light quark parameters known to order $a_s^3$ [6,7,33]:

$$\langle \bar{q}q(\mu) \rangle = \bar{m}_{q,0}(\mu) (\beta_0 a_s)^{-2/3} \times C(a_s)$$

$$\langle \bar{q}g\sigma Gq(\mu) \rangle = -M_0^2 \bar{q}_0 (\beta_0 a_s)^{1/3} / C(a_s),$$

(21)

$\bar{m}_{q,0}$ is the RGI quark mass, $\bar{q}_0$ is spontaneous RGI light quark condensate [34]. The QCD correction factor $C(a_s)$ in the previous expressions is numerically:

$$C(a_s) = 1 + 0.8951 a_s + 1.3715 a_s^2 + \ldots ; n_f = 3,$$

$$= 1 + 1.755 a_s + 1.5008 a_s^2 + \ldots ; n_f = 5,$$

(22)

which shows a good convergence. We shall use:

$$\alpha_s(M_z) = 0.325(8) \Rightarrow \alpha_s(M_2) = 0.1192(10)$$

(23)

from $\tau$-decays [35,36], which agree perfectly with the world average 2012 [37,38]:

$$\alpha_s(M_2) = 0.1184(7).$$

(24)

The value of the running $\langle \bar{q}q \rangle$ condensate is deduced from the value of $\langle m_b + m_\omega(\bar{u}u + \bar{d}d) \rangle$ at the PT scale. We shall use the value of the RGI spontaneous mass to order $a_s$ for consistency with the known $\alpha_s$ correction in the OPE:

$$\bar{m}_B = 251(6) \text{ MeV.}$$

(25)

The values of the running $\bar{M}S$ mass $\bar{m}_B(m_b)$ recently obtained in Ref. 2 from bottomonium sum rules, will also be used$^7$. Using the relation between the running $\bar{m}_B(\bar{m}_B) = 4177(11) \text{ MeV}$ from the $\Upsilon$-systems [2] and the on-shell (pole) $M_B$ masses (see e.g. [6,7,28], one can deduce to order $a_s^2$:

$$M_B = 4804(50) \Rightarrow \alpha_s(M_B) = 0.2326(22),$$

(26)

where the error is mainly due to the one of $\alpha_s$. This large error has to be contrasted with the precise value of the running mass, and can be an obstacle for a precise determination of $\bar{f}_B$ and $\bar{f}_{\omega}$ from HQET at a given $\mu$. On can see in Section 8 that a direct extraction of the on-shell mass from the HQET at the same $a_s^2$ order leads to about the same value and error which is an (a posteriori) self-consistency of the value and error used in Eq. (26) for the analysis. We are aware that the inclusion of the known $a_s^2$-correction and an estimate of the PT higher order terms using a geometric growth of the PT coefficients à la Ref. 29,30 increase the value of $M_B$ by about (100 ~ 200) MeV, which could only be considered if one works to higher order in

$\alpha_s^4$ ($n \geq 3$). These large order terms are expected to be dual to the $1/q^2$-term which mimics the UV renormalon contribution. On the other, some eventual IR renormalon contributions are usually expected to be absorbed by the ones of the QCD condensates when the mass terms are included into the OPE. However, the use of the pole mass $M_B$ is only a convenient numerical step in our numerical analysis, as we could have worked from the beginning with the running mass. Therefore, our results on the running mass $\bar{m}_B$ truncated at a given PT order are not (a priori) affected by the IR renormalon contributions.

5. The LSR determination of the RGI $\bar{f}_B$ in the static limit

Analysis of the convergence of the PT series

We study the effect of the truncation of the PT series on the value of $\bar{f}_B$ from Eqs. (19) and (20). For a given value of $\mu = M_B$ and $\omega_c = 3$ GeV, we show the result of the analysis in Fig. 1. At the minimum (optimal) value, $\bar{f}_B$ moves from 0.365 (LO+NLO) to 0.414 (+N2LO) to 0.454 (+N3LO) GeV$^3/2$, i.e. a change of about 13% from LO+NLO to N2LO and of about 8.8% from N2LO to N3LO, which indicates a slow convergence of the PT series. We shall consider the N3LO contribution as a systematic error from the truncation of the PT series.

![Figure 1: $\tau_H$-behaviour of $\bar{f}_B$ for $\mu = M_B$ and $\omega_c = 3$ GeV for different truncations of the PT series.](image)

Analysis of the $\tau_H$ and $\omega_c$ stabilities

We show in Fig. 2 the $\tau_H$-behaviour of the result for a given value of $\mu$ and for different values of $\omega_c$ where the PT series is known to N2LO. The $\tau_H$-stabilities are:

$$\tau_H = (1, 1.5) \text{ GeV}^{-1},$$

(27)
where $\tau_H \approx 1$ GeV$^{-1}$ is obtained for $\omega_\tau \approx 2$ GeV (beginning of $\tau_H$-stability), while $\tau_H \approx 1.5$ GeV$^{-1}$ corresponds to the beginning of the $\omega_\tau$-stability which is $\omega_\tau \approx 4$ GeV. We consider, as optimal and conservative values, the ones obtained in the previous range of $\omega_\tau$ values. For, e.g $\mu = M_b$, we obtain in this way:

$$f_B(M_b) = (0.373 - 0.427) \text{ GeV}^{3/2} . \tag{28}$$

This range of values is much larger range than the one used in the current literature which appears to be an ad hoc choice.

![Figure 2: a) $\tau_B$-behaviour of $\hat{f}_B$ for $\mu = M_b$ and for different values of $\omega_\tau$; b) The same as a) but for $\mu = \tau_H^2$.](image)

**Summary of the results for $f_B$ and error calculations**

At a given value of $\mu$, we estimate the errors induced by the QCD parameters compiled in Table 1. We summarize the results of the analysis in Table 2. We show in Fig 3 the “sum rules data points” at different values of the subtraction point $\mu$. The error is large at small $\mu$ due to the bad behaviour of the PT series at low scale which confirms the scepticism of the authors of Ref. [24] on the reliable extraction of $\hat{f}_B$ at a such low scale. However, as we have shown in previous section, the convergence of the PT series improves obviously at larger scale which enables to extract $\hat{f}_B$ with a reasonable accuracy of about 16-15% for $\mu \geq 3$ GeV. Fitting the previous data by an horizontal line or taking their average, we deduce the final value of the RGI universal coupling:

$$\hat{f}_B = 0.416(25) \text{ mean}(48)_{\text{var}} \text{ GeV}^{3/2} = 0.416(54) \text{ GeV}^{3/2} . \tag{29}$$

The estimate of the error is more delicate as there are not (a priori) any rigorous ways for obtaining it due to the unclear eventual correlations among these different points. Here, we have deduced the error by adding the one 25 GeV$^{3/2}$ from the weighted average (assuming uncorrelated data points) which is dominated by the most accurate prediction to the one 48 GeV$^{3/2}$ obtained from the square root of the usual unbiased estimator for $n$-number of data points [25]:

$$\Delta x = \sqrt{\text{var}(x) \equiv \frac{1}{n-1} \sum_{i=1}^{n}(x_i - \langle x \rangle)^2} , \tag{30}$$

where $\langle x \rangle$ is the mean value. The size of the error is comparable with the one of about 60 GeV$^{3/2}$ from the best determination at $\mu \approx M_b$. A frequently used estimate of the error induced by the truncation of the PT series would be obtained by varying the scale $\mu^{-1}$ from 1/2 to 2 times the value of $\tau_B^0$ at which the sum rule is optimized instead of estimating the size of the $a_i^2$-term like done above. Besides the fact that this choice of range is arbitrary, the value of $\tau_B^0$ at which the sum rule is optimized is quite large [see Eq. (27)] such that at $\mu^{-1} = 2\tau_B^0$ the PT series breaks down rendering the approach inadequate. Instead, we can consider the value $\mu = (3.5 \pm 2)$ GeV where the central value would correspond to the average obtained in Eq. (29). Keeping only terms to order $a_i^2$ and adding the different errors quadratically, one would obtain a final error of about 51 GeV$^{3/2}$.
which is slightly lower than the one in Eq. (29). Due to the arbitrariness of the choice of the range variation of \( \mu \), we shall only consider this result as an informative value and we shall not retain it in the final prediction.

Here and in the following, we shall alternatively estimate the final error which does not suffer from the previous drawbacks by taking the one coming from the most accurate measurement here at \( \mu = M_b \). Then, we obtain:

\[
\bar{f}_B = 0.416(60) \text{ GeV}^{3/2}
\]

(31)

from which, we can deduce the value of the static coupling evaluated at, e.g., \( M_B = 5.28 \text{ GeV} \):

\[
\bar{f}_{\text{stat}}(M_B) = 0.603(2) \alpha_s(87) f_B \text{ GeV}^{3/2}
\]

(32)

The corresponding decay constant from Eq. (11):

\[
f_B^{\text{N3LO}} = 234(1) \alpha_s(33) f_B \text{ MeV}
\]

(33)

which is relatively large compared to the value of \( f_B = 206(7) \) MeV obtained from the full QCD theory [11] suggests some large \( 1/M_b \) corrections which we shall analyze in the next section. We consider the previous results does [29 to 33] as improvements of previous results in the literature [15 to 20]. Here, we have used updated values of the QCD input parameters. We have varied the continuum threshold and subtraction point \( \mu \) we have used updated values of the QCD input parameters. We have included NP contributions of higher dimensions \( (d \geq 7) \), though small, which are important for controlling the convergence of the OPE at a relatively large value of \( \tau_H \) where the sum rule is optimized. We have also included in the error an estimate of the N3LO contribution based on a geometric growth of the PT series which controls the convergence of the PT series.

6. \( 1/M_b \) corrections and value of the decay constant \( f_B \)

Taking into account the mass-difference between the meson \( M_B \) and the on-shell quark mass \( M_b \), in the relation in Eq. (11) expressed in terms of the RGI coupling in Eq. (12) becomes:

\[
f_B^{\text{N3LO}} = \left( \frac{M_B}{M_B} \right)^3 \left( \frac{C_b(M_b)}{R_b(M_b)} f_B^{\text{N3LO}} + \delta f_B^{\text{N3LO}} \right),
\]

(34)

where \( M_B = 5.279 \text{ GeV} \) and we shall use the value of \( M_b \) in Eq. (26).

**LSR expression of \( 1/M_b \) correction \( \delta f_B^{\text{N3LO}} \)**

The \( 1/M_b \) corrections \( \delta f_B^{\text{N3LO}} \) to the HQET two-point correlator can be obtained by subtracting its expression in the full theory with the one of HQET in the limit \( M_B \rightarrow \infty \). The \( 1/M_b \) correction to the physical decay constant \( f_B \) reads (see e.g. [20]):

\[
\delta f_B^{\text{N3LO}} = \frac{e^{\tau_H \Delta M}}{M_B} \left[ \frac{1}{\pi} \int_0^{\tau_H} d\omega \ e^{-\tau_H \omega} \text{Im} \Pi_{\text{PT}}(\omega) + \delta_{NP} \right],
\]

(35)

where:

\[
\text{Im} \Pi_{\text{PT}} = \text{Im} \Pi_{\text{PT}} - C_b(M_b) \frac{R_b(\mu)}{R_b(M_b)} \text{Im} \Pi_{\text{PT}}.
\]

(36)

Up to order \( \alpha_s \), it reads:

\[
\text{Im} \Pi_{\text{PT}}(x) = \frac{-3}{8\pi} \frac{M_B^2}{1+x} \left[ \frac{1}{1+x} \left( 1 + \alpha_s \right) + \frac{13}{4} \left( \frac{\pi^2}{3} - \frac{3}{2} \ln x - F(x) \right) \right]
\]

(37)

where \( : x \equiv \omega/M_b \) and:

\[
F(x) = 2 Li_2(-x) + \ln(x) \ln(1+x) - \frac{x}{1+x} \ln(x)
\]

+ \frac{1}{x} \ln(1+x) - 1.

(38)

The order N2LO \( \alpha_s^2 \) PT correction to the spectral function can be numerically obtained by subtracting the complete expression in full QCD obtained in [23] with the HQET asymptotic result in Eq. (19) and by using the relation in Eq. (35). In the same way, we estimate the N3LO PT corrections assuming a geometric growth of the PT series both in full QCD and HQET theories. The NP corrections read up to \( d=5 \) condensates [17 to 20]:

\[
\delta_{NP}(\mu) = 2 \alpha_s(\mu) \langle \bar{u} u \rangle(\mu) \int_0^{\infty} \frac{d\omega}{M_b} \frac{e^{-\tau_H \omega}}{1 + \omega/M_b}
\]

+ \langle \alpha_s G^2 \rangle \frac{\tau_H}{2 M_b} \left( \frac{\tau_H}{2 M_b} \right) \left( \bar{q} \gamma \sigma G q \right)(\mu).

(39)

**Analysis of the convergence of the PT series**

Like in the case of \( f_B \), we study the convergence of the PT series. We notice that the \( \alpha_s^2 \) and \( \alpha_s^3 \) corrections are very small for \( \tau_H \leq 1 \text{ GeV}^{-1} \), which can then be neglected. The analysis is shown in Fig. 4.

![Figure 4](image_url)

**Figure 4**: \( \tau_H \)-dependence of \( \delta f_B \) for \( \mu = M_b \) and \( \omega_c = 3 \text{ GeV} \) for different truncations of the PT series where the contributions of condensates up to \( d=5 \) have been included.

**Analysis of the \( \tau_H \) and \( \omega_c \) stabilities**

We show in Fig. 5 the \( \tau_H \)-behaviour of \( \delta f_B \) for different values of \( \omega_c \) and including the \( d = 5 \) condensates. We study the effects of the \( d = 5 \) condensates on the \( \tau_H \)-stability for given two extremal values of \( \omega_c \) (beginning of \( \tau_H \) and of \( \omega_c \)-stabilities). The analysis is shown in Fig. 6 from which we consider as optimal results the ones corresponding to the range:

\[
\tau_H \approx (1.6 \sim 2.2) \text{ GeV}^{-1},
\]

(40)

considering the fact that the inflexion point is not precisely localized.
Results for $\delta f_B^2$

The “sum rules data” of $\delta f_B^2$ for different values of $\mu$ are shown Table 3 and in Fig. 7 where the main errors come from the localization of $\tau_H$ from 1.8 to 2.2 GeV$^{-2}$ and the one induced from its corresponding $\omega_c$ values. The errors from the QCD parameters are negligible. Taking the average of different values, we deduce the $1/M_b$ corrections due to $\delta f_B^2$:

$$\delta f_B^2 = -2.2(1.6) \times 10^{-3} \text{ GeV}^2,$$

where the error comes from the most accurate measurement at $\mu = \tau_H^{-1}$.

7. $f_B$ from HQET and from full QCD

Combining the result in Eq. (41) with the one in Eq. (29) with the help of Eq. (34), one obtains:

$$f_B^{\text{QCD}} = 199(28.6)_{f_b}(3)\delta f_B^2(3)\lambda_c(0.3)\lambda_{\text{q}} \text{ MeV}$$
$$= 199(29) \text{ MeV}, \quad (42)$$

where we have added the errors quadratically. Notice that, unlike the full QCD case [1], we have not tried to extract an upper bound on $f_B$ from the positivity of the spectral function because of the undefined sign of $\delta f_B^2$ in Eq. (34). We can compare this result with the one obtained in the static limit in Eq. (33), where one can see that the main corrections are to the $(M_b/M_B)^{1/2}$ ratio in Eq. (33). This HQET result is consistent with the one from the average of LSR and Moment sum rules in full QCD [1]:

$$f_B^{\text{QSSR}} = 206(7) \text{ MeV}. \quad (43)$$

8. Extraction of the $b$-quark mass from HQET

One can extract the meson-quark mass-difference $\Delta M$ using the ratio of the LSR obtained from Eq. (34):

$$\mathcal{R}_H \equiv \frac{-\partial}{\partial \tau_H} \left( \frac{\rho_B^2 e^{-\tau_H\Delta M}}{f_B^2 e^{-\tau_H\Delta M}} \right) = \Delta M \equiv \frac{M_b^2 - M_B^2}{M_b}, \quad (44)$$

where $M_b$ is the on-shell $b$-quark mass. We show the $\tau_H$-behaviour of $\Delta M$ in Fig. 8. $\tau_H$-stability is obtained for $\tau_H$ about the values in Eq. (27). We show in Table 3 the different sources of errors on $\Delta M$, where one can notice that the most important ones come from $\omega_c$, the estimated $\lambda_c^3$ and the mixed condensate contributions. We show in Fig. 2 the $\mu$-behaviour of different “QSSR data points” from which we deduce the average:

$$\Delta M = 907(89) \text{ MeV}. \quad (45)$$
Table 4: Central values and corresponding errors for $\Delta M$ in units of MeV from the LSR at different values of the subtraction point $\mu$ in units of GeV. The $+$(resp. $-$) sign means that the values of $\Delta M$ increase (resp. decreases) when the input increases (resp. decreases). The total error comes from a quadratic sum.

| $\mu$ | $\Delta M$ | $t_c$ | $\alpha_s$ | $\alpha_s^2$ | $M_0$ | $\langle \bar{u}u \rangle$ | $(G^2)$ | $M_0^2$ | $\text{Total}$ |
|-------|----------|-------|-----------|------------|-------|----------------|---------|-------|----------|
| $t_H$ | 981      | +69   | +9        | +64        | 1     | 0              | 0       | +18   | 96       |
| 1     | 964      | +102  | +8        | +22        | +1    | -6             | -3      | +15   | 106      |
| 2     | 918      | +109  | +8        | +44        | +1    | -9             | -4      | +16   | 122      |
| 3     | 890      | +109  | +7        | +51        | +1    | -10            | -4      | +17   | 122      |
| 4     | 872      | +108  | +6        | +52        | +1    | -11            | -5      | +17   | 122      |
| $M_0$ | 862      | +108  | +7        | +54        | +1    | -11            | -5      | +16   | 122      |
| 5     | 858      | +107  | +6        | +55        | +1    | -12            | -5      | +16   | 122      |
| 6     | 847      | +106  | +6        | +57        | +2    | -12            | -5      | +16   | 122      |

where the error comes from the most accurate measurement at $\mu = t_H^{-1}$. Using the previous value of $\Delta M$, one can extract the on-shell $b$-quark mass to order $\alpha_s^2$:

$$M_{b,\text{on-shell}} = 4846(41) \text{MeV}.$$  

Using the known relation between the on-shell and running quark mass to order $\alpha_s^2$ (see e.g. [6–8, 33]), we deduce:

$$\overline{m}_b(\overline{m}_b)_{\text{on-shell}} = 
\begin{align*}
4213(47)\alpha_s(36)_{\text{QSSR}} \text{MeV} \\
= 4213(59) \text{MeV}.
\end{align*}$$  

9. $\overline{m}_b(\overline{m}_b)$ from HQET and from full QCD

The previous value of the running mass is in good agreement with the one from heavy-light QCD spectral sum rules in full QCD to order $\alpha_s^2$ [1]:

$$\overline{m}_b(\overline{m}_b)^{\text{QCD}} = 4236(69) \text{MeV},$$

and with the more accurate result from the $\Upsilon$ sum rules to order $\alpha_s^3$ [2]:

$$\overline{m}_b(\overline{m}_b)^{\Upsilon} = 4177(11) \text{MeV}.$$  

Table 5: Results for $f_B$ and $\overline{m}_b(\overline{m}_b)$ in units of MeV and comparison with lattice simulations using $n_f = 2$ [57, 58] and $n_f = 3$ [57, 58] dynamical quarks. $f_B$ are normalized as $f_B = 130.4 \text{MeV}$.  

| Observables | Methods | Refs. |
|-------------|---------|-------|
| $f_B$ | QSSR | 199(29) = 1.53(23)f_{\pi} | HQET (this work) |
| | Lattice | 206(7) = 1.58(5)f_{\pi} | full QCD [1] |
| | | $\leq 235.3(3.8) = 1.80(3)f_{\pi}$ | full QCD [1] |
| $\overline{m}_b(\overline{m}_b)$ | QSSR | 4213(59) | $B$-meson - HQET (this work) |
| | Lattice | 4236(69) | $B$-meson - full QCD [1] |
| | | 4177(11) | $\Upsilon$ - full QCD [2] |

10. Summary and conclusions

We have re-estimated $f_B$ and $M_0$ from HQET Laplace spectral sum rules to order $\alpha_s^2$ by including an estimate of the $\alpha_s^4$ and non-perturbative terms up to dimension $d = 7$ condensates. We

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5We could have also extracted $\overline{m}_b(\overline{m}_b)$ directly by replacing the on-shell mass $M_0$ with $\overline{m}_b(\overline{m}_b)$ using their PT relation known to order $\alpha_s^2$, in the QCD expression. However, this procedure is not convenient in the numerical analysis.
have also taken larger ranges of $\omega$, $\tau_H$ and $\mu$ values for extracting our optimal results. Most of these analyses have not been done in previous literature [16-20]. Our results in Eqs. (42) and (47) are in good agreement with the ones from full QCD in Eqs. (45) and (48). These results are comparable with some other $\gamma$ sum rule determinations [53] and with lattice results including $\Upsilon(43)$ and (48). These results are in good agreement with the ones from full QCD in Eqs. (47) and (49) are in good agreement with the ones from full QCD in Eqs. (45) and (46).

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