Rivalry of Two Families of Algorithms for Memory-Restricted Streaming PCA

Chun-Liang Li  
Carnegie Mellon University  
chunlial@cs.cmu.edu

Hsuan-Ten Lin  
National Taiwan University  
htlin@csie.ntu.edu.tw

Chi-Jen Lu  
Academia Sinica  
cjlu@iis.sinica.edu.tw

Abstract

We study the problem of recovering the subspace spanned by the first $k$ principal components of $d$-dimensional data under the streaming setting, with a memory bound of $O(kd)$. Two families of algorithms are known for this problem. The first family is based on stochastic gradient descent, but no convergence proof of its existing algorithms was previously known when $k > 1$. The second family is based on the power method over blocks of data, but setting the block size for its existing algorithms is not an easy task. In this paper, we analyze the convergence rate of a representative algorithm in the first family [1] for the general $k > 1$ case. Moreover, we propose a novel algorithm for the second family that sets the block sizes automatically and dynamically with faster convergence rate. We then conduct empirical studies that fairly compare the two families on real-world data. The studies reveal the advantages and disadvantages of these two families.

1 Introduction

For data points in $\mathbb{R}^d$, the goal of principal component analysis (PCA) is to find the first $k \ll d$ eigenvectors (principal components) that correspond to the top-$k$ eigenvalues of the $d \times d$ covariance matrix. For a batch of stored data points with a moderate $d$, efficient algorithms like the power method [2] can be directly run on the empirical covariance matrix to compute the solution.

In addition to the batch algorithms, the stream setting (streaming PCA) is attracting much research attention in recent years [3, 4, 5]. Streaming PCA assumes that each data point $x \in \mathbb{R}^d$ arrives sequentially and it is not feasible to store all data points. If $d$ is moderate, the empirical covariance matrix can again be computed and fed to an eigenproblem solver to compute the streaming PCA solution. When $d$ is huge, however, it is not feasible to store the $O(d^2)$ empirical covariance matrix.

The situation arises in many modern applications of PCA. Those applications call for memory-restricted streaming PCA, which will be the main focus of this paper. We shall consider restricting to only $O(kd)$ memory usage, which is of the same order as the minimum amount needed for the PCA solution.

There are two measurements for the goodness of the streaming PCA solution. One is the reconstruction error that measures the expected squared error when projecting a data point to the solution, which comes from the fact that the actual principal components should result in the lowest reconstruction error. The other is the spectral error that measures the difference between the subspace spanned by the solution and the subspace spanned by the actual principal components, which will be formally defined in Section 2. Note that when the $k$-th and $(k+1)$-th eigenvalues are close, the solution that wrongly includes the $(k+1)$-th eigenvector instead of the $k$-th one can still reach a...
small reconstruction error, but the spectral error can be large. That is, the spectral error is somewhat harder to knock down. We will focus on the spectral error in this paper.

There are several existing streaming PCA algorithms, but not all of them focus on the spectral error and meet the memory restriction. For instance, [5] proposed an online algorithm which considers the spectral error, but its space complexity is at least $\Omega(kd \log n)$, where $n$ is the number of data points received. Based on [7], [9] proposed an online PCA algorithm along with regret guarantee on the reconstruction error, but not the spectral error, and its space complexity grows in the order of $d^2$. [9] extended [4] to derive convergence rate analysis for minimizing the reconstruction error along with a memory-efficient implementation, but the space complexity is not precisely guaranteed to meet the bound $O(kd)$. That is, those works do not match the focus of this paper.

There are two families of algorithms that tackle the spectral error while respecting the memory restriction, the family of stochastic gradient descent (SGD) PCA algorithms, and the family of block power methods. The SGD family solves a non-convex optimization problem that minimizes the reconstruction error, and applies SGD [11] under the memory restrictions to design streaming PCA algorithms. Interestingly, although the non-convex problem does not match standard convergence assumptions of SGD [10], minimizing the reconstruction error for the special case of $k = 1$ allows [11] to derive spectral-error guarantees on the classic stochastic-gradient-descent PCA (SPCA) algorithm [1]. One conjecture is that the analysis of [11] can be similarly extended to the general case of $k > 1$. The proof of the conjecture turns out to be non-trivial, and will be one major contribution of this paper.

The other family, namely the block power methods [4], extends the batch power method [2] for the memory-restricted streaming PCA by defining blocks (periods) on the time line. The key of the block power methods is to efficiently compute the product of the estimated covariance matrices in different blocks. The product serves as an approximation to the power of the empirical covariance matrix, which is a core element of the batch power method. This family could also be treated as the mini-batch SGD algorithms but with different update rule from the above SGD family. The original block-power-method PCA (BPCA; [4]) is proved to converge under the generative and spike-model assumption of Gaussian distribution. [5] generalize the analysis to a broader class of distributions without spike-model limitation. In both works, the convergence proof of BPCA depends on determining the block size from the total number of data points or a pre-specified error goal, neither of which is generally known in real-world streaming PCA applications. The dependence makes it difficult to choose proper block sizes for the block power methods in practice.

From the theoretical perspective, SPCA lacks the convergence proof for the general $k$; from the algorithmic perspective, it is difficult to determine the block size for BPCA; from the practical perspective, it is not clear which family should be preferred in real-world applications. This paper makes contributions on all the three perspective. We first prove the convergence of SPCA for $k > 1$ in Section [3]. Then in Section [4] we propose a dynamic block power method (DBPCA) that automatically decides the block size to not only allow easier algorithmic use but also guarantee better convergence rate. Finally, we conduct experiments on real-world datasets and provide concrete recommendations in Section [5].

## 2 Preliminaries

Let us first introduce some notations which will be used later. First, let $x \leq O(y)$ and $x \geq \Omega(y)$ denote that for some universal constant $c$, independent of all our parameters, $x \leq cy$ and $x \geq cy$, respectively, for a large enough $y$. Next, let $\lfloor x \rfloor$ denote the smallest integer that is at least $x$. Finally, for a vector $x$, we let $\|x\|$ denote its $\ell_2$-norm, and for a matrix $M$, we let $\|M\| = \max_x \frac{\|Mx\|}{\|x\|}$, which is the spectral norm.

In this paper, we study the PCA problem in the streaming model, in which input data points are received in a stream, with a point $x_n \in \mathbb{R}^d$ received at step $n$. Following previous works [4][11], we make the following assumption on the data distribution.

**Assumption 1.** Assume that each $x_n$ is sampled independently from some distribution $\mathcal{X}$ with mean zero and covariance matrix $A$, which has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, with $\lambda_i > \lambda_{i+1}$.
Moreover, assume that \( \|x\| \leq 1 \) for any \( x \) in the support of \( X \) which implies that \( \|A\| \leq 1 \) and \( \|x_n x_n^\top\| \leq 1 \) for each \( n \).

Our goal is to find a \( d \times k \) matrix \( Q_n \) at each step \( n \), with its column-space quickly approaching that spanned by the first \( k \) eigenvectors of \( A \). For convenience, we let \( \lambda = \lambda_k \) and \( \hat{\lambda} = \lambda_{k+1} \), and moreover, let \( U \) denote the \( d \times k \) matrix with the first \( k \) eigenvectors of \( A \) as its columns. One common way to measure the distance between such two spaces is

\[
\Phi_n = \max_{v \in \mathbb{R}^k} \left( 1 - \frac{\|U^\top Q_n v\|^2}{\|v\|^2} \right),
\]

which can be used as an error measure for \( Q_n \). It is known that \( \Phi_n = \sin \theta_k(U, Q_n)^2 \), where \( \theta_k(U, Q_n) \) is the \( k \)-th principle angle between these two spaces. For simplicity, we will denote \( \sin \theta_k(U, Q_n) \) by \( \sin(U, Q_n) \). Moreover, \( \cos(U, Q_n) = \sqrt{1 - \sin(U, Q_n)^2} \) and \( \tan(U, Q_n) = \sin(U, Q_n)/\cos(U, Q_n) \). It is also known that \( \cos(U, Q_n) \) equals the smallest singular value of the matrix \( U^\top Q_n \).

Our algorithms will generate an initial matrix \( S_0 \in \mathbb{R}^{d \times k} \) by sampling each of its entries independently from the normal distribution \( \mathcal{N}(0, 1) \). Let \( S_0 \sim \mathcal{N}(0, 1)^{d \times k} \) denote this process, and we will rely on the following guarantee.

**Lemma 1.** \([4]\) Suppose we sample \( S_0 \sim \mathcal{N}(0, 1)^{d \times k} \) and let \( S_0 = Q_0 R_0 \) be its QR decomposition. Then for a large enough constant \( c \), there is a small enough constant \( \delta_0 \) such that

\[
\Pr \left[ \cos(U, Q_0) \leq \sqrt{c/(dk)} \right] \leq \delta_0.
\]

**3 Stochastic Gradient Descent**

In this section, we study a generalization of the classic PCA algorithm of \([1]\) from the rank-one case to the rank-\( k \) case, which can be seen as performing stochastic gradient descent. Our algorithm, called SPCA, is given in Algorithm \([1]\) where we choose the step size at step \( n \) as

\[
\gamma_n = \frac{c}{n}, \quad \text{with} \quad c = \frac{c_0}{\lambda - \hat{\lambda}} \quad \text{for a constant} \ c_0 \geq 12.
\]

The algorithm has a space complexity of \( \mathcal{O}(kd) \), by noting that the computation of \( x_n x_n^\top Q_{n-1} \) can be done by first computing \( x_n^\top Q_{n-1} \) and then multiplying the result by \( x_n \). The sample complexity of our algorithm is guaranteed by the following, which we prove in Subsection 3.1. Our analysis is inspired by and follows closely that of \([1]\) for the rank-one case, but there are several new hurdles which we need to overcome in the general rank-\( k \) case.

**Theorem 1.** For any \( \rho \in (0, 1) \), there is some \( N \leq (2^{1/(\lambda - \hat{\lambda})}kd)^{\mathcal{O}(1)} + \mathcal{O} \left( \frac{k \log(1/\rho)}{\rho(\lambda - \hat{\lambda})^2} \right) \), such that our algorithm with high probability can achieve \( \Phi_n \leq \rho \) for any \( n \geq N \).

Let us remark that we did not attempt to optimize the first term in the bound above, as it is dominated by the second term for a small enough \( \rho \).

**3.1 Proof of Theorem [1]**

The analysis of \([1]\) works for the rank-one case by using a potential function \( \Psi_n = 1 - (U^\top Q_n)^2 \), where \( U \) and \( Q_n \) are both vectors instead of matrices. To work in the general rank-\( k \) case, we choose the function \( \Phi_n \) defined in \([1]\) as a generalization of their \( \Psi_n \), and our goal is to bound \( \Phi_n \).

Following \([1]\), we divide the steps into epochs, with epoch \( i \) ranging from step \( n_{i-1} \) to step \( n_i - 1 \), where we choose \( n_0 = c^0 k^3 d^2 \log d \), for a large enough constant \( c \), and \( n_i = \left[ e^{5/c_0} \right] (n_{i-1} + 1) - 1 \) for \( i \geq 1 \).

**Remark 1.** This gives us \( n_i + 1 \geq e^{5/c_0} (n_{i-1} + 1) \) and \( n_i \leq c_1 n_{i-1} \) for some constant \( c_1 \).

\[\footnote{We can relax this condition to that of having a small \( \|x\| \) with high probability as in \([5]\), but we choose this stronger one to simplify our presentation.}\]
As in (11), we also use the convention of starting from step \( n_0 \). For each epoch \( i \), we would like to establish an upper bound \( \rho_i \) on \( \Phi_n \) for each step \( n \) in that epoch. To start with, we know the following from Lemma 1, using the fact that \( \Phi_0 = \sin(U,Q_0)^2 = 1 - \cos(U,Q_0)^2 \).

**Lemma 2.** Let \( \Gamma_0 \) denote the event that \( \Phi_0 \leq \rho_0 \), where \( \rho_0 = 1 - \bar{\epsilon}/(kd) \) for the constant \( \bar{\epsilon} \) in Lemma 2. Then we have \( \Pr[\neg \Gamma_0] \leq \delta_0 \).

Next, for each epoch \( i \geq 1 \), we consider the event

- \( \Gamma_i : \sup_{n_i-1 \leq n < n_i} \Phi_n \leq \rho_i \),

for some \( \rho_i \) to be specified later. Then our goal is to show that \( \Pr[\neg \Gamma_{i+1} | \Gamma_i] \) is small, for \( i \geq 0 \). This can be done for the rank-one case, but it relies crucially on the property that the potential function \( \Phi_n \) of \( \Pi \) satisfies a nice recurrence relation. Unfortunately, this does not appear so for our function \( \Phi_n \), mainly because it takes an additional maximization over \( v \in \mathbb{R}^k \). To overcome this problem, we take the following approach.

Consider an epoch \( i \) and a step \( n \) in the epoch. Let us define a new matrix \( Y_n = (I + \gamma_n x_n x_n^\top) Y_{n-1} = Q_n R_n R_{n-1} \cdots R_{n_i-1} \), with \( Y_{n_i-1} = Q_{n_i-1} \). Let \( \mathcal{S} = \{ v \in \mathbb{R}^k : \|v\| = 1 \} \). Then for any \( v \in \mathcal{S} \), define

\[
\Phi^{(v)}_n = 1 - \frac{\|U^\top Y_n v\|^2}{\|Y_n v\|^2},
\]

and note that \( \Phi_n = \max_{v \in \mathcal{S}} \Phi^{(v)}_n \). Now for each such new function \( \Phi^{(v)}_n \), with a fixed \( v \), we can establish a similar recurrence relation as follows, but for our purpose later we show a better upper bound on \( |Z_n| \) than that in (11). We give the proof in Appendix A.

**Lemma 3.** For any \( n > n_0 \) and any \( v \in \mathcal{S} \), we have \( \Phi^{(v)}_n \leq \Phi^{(v)}_{n-1} + \beta_n - Z_n \), where (i) \( \beta_n = 5 \gamma_n^2 + 2 \gamma_n^3 \), (ii) \( |Z_n| \leq 2 \gamma_n \Phi^{(v)}_{n-1} \), and (iii) \( \mathbb{E} [Z_n | F_{n-1}] \geq 2 \gamma_n \phi \Phi^{(v)}_{n-1}(1 - \Phi^{(v)}_{n-1}) \geq 0 \).

With this lemma, the analysis of (11) can be used to show that \( \mathbb{E} [\Phi^{(v)}_n] \) decreases as \( n \) grows, but only for each individual \( v \) separately. This alone is not sufficient to guarantee the event \( \Gamma_{i+1} \) as it requires small \( \Phi^{(v)}_n \) for all \( v \)’s simultaneously. To deal with this, a natural approach is to show that each \( \Phi^{(v)}_n \) is large with a small probability, and then apply a union bound, but an apparent difficulty is that there are infinitely many \( v \)’s. We will overcome this difficulty by showing how it is possible to apply a union bound only over a finite set of \( \text{“}\varepsilon\text{-net”} \) for these infinitely many \( v \)’s. Still, for this approach to work, we need the probability of having a large \( \Phi^{(v)}_n \) to be small enough, compared to the size of the \( \varepsilon \)-net. However, the beginning steps of the first epoch seem to have us in trouble already as the probability of their \( \Phi^{(v)}_n \) values exceeding \( \Phi^{(v)}_{n_0} \) is not small. This seems to prevent us from having an error bound \( \rho_1 < \rho_0 \), and without this to start, it is not clear if we could have smaller and smaller error bounds for later epochs. To handle this, we sacrifice the first epoch by using an error bound \( \rho_1 \) slightly larger than \( \rho_0 \), but still small enough. The hope is that once \( \Gamma_1 \) is established, we then have a period of small errors, and later epochs could then start to have decreasing \( \rho_i \)’s. More precisely, we have the following for the first epoch, which we prove in Appendix B.

**Lemma 4.** Let \( \rho_1 = 1 - \bar{\epsilon}/(c^1 kd) \), for the constant \( c_1 \) given in Remark 2. Then \( \Pr[\neg \Gamma_1 | \Gamma_0] = 0 \).

It remains to set the error bounds for later epochs appropriately so that we can actually have small \( \Pr[\neg \Gamma_{i+1} | \Gamma_i] \) for \( i \geq 1 \). We let the error bounds decrease in three phases as follows. In the first phase, we let \( \rho_i = 1 - 2(1 - \rho_{i-1}) \), so that \( \eta_i = 1 - \rho_i \) doubles each time. It ends at the first epoch \( i \), denoted by \( \pi_1 \), such that \( \eta_i < 3/4 \). Note that \( \pi_1 \leq O(\log d) \) and at this point, \( \rho_{\pi_1} \) is still much larger than \( 1/n_{\pi_1} \). Then in the second phase, we let \( \rho_i = \rho_{\pi_1}/(c \gamma_i) \), which decreases in a faster rate than \( n_i \) increases. It ends at the first epoch \( i \), denoted by \( \pi_2 \), such that \( \rho_i \leq c_2 (c^2 k \log n_{\pi_1})/(n_{\pi_1} + 1) \), for some constant \( c_2 \)\(^2\) Note that \( \pi_2 \leq O(\log d) \) and at this point, \( \rho_{\pi_2} \) reaches about the order of \( 1/n_{\pi_2} \). Finally in phase three, we let \( \rho_i = c_2 (c^2 k \log n_{\pi_1})/(n_{\pi_1} + 1) \), which decreases in about the rate as \( n_i \) increases.

With these choices, the events \( \Gamma_i \)’s are now defined, and our key lemma is the following, which we prove in Appendix C. In the proof, we handle the difficulty mentioned before by showing how a

\(^2\)As in (11), \( F_{n-1} \) here denotes the \( \sigma \)-field of all outcomes up to and including step \( n - 1 \).

\(^3\)Determined later in the proof of Lemma 5 in Appendix C for the bound \( \delta \) there to hold.
Lemma 5. For any $\Phi$ $Q \in \mathcal{S}$, and our $\rho_i$’s are chosen in the ways above to guarantee that each $\Phi_k^{(v)}$ is large with a small enough probability.

From these lemmas, we can bound the failure probability of our algorithm as

$$\Pr[\exists i \geq 0: -\Gamma_j] \leq \Pr[-\Gamma_0] + \sum_{i \geq 0} \Pr[-\Gamma_{i+1} | \Gamma_i] \leq \delta_0 + \sum_{i \geq 0} \frac{\delta_0}{2((i+1)^2)} ,$$

which is at most $2\delta_0$ using the fact that $\sum_{i \geq 1} 1/i^2 \leq 2$.

To complete the proof, it remains to determine the number of samples needed by our algorithm to achieve an error bound $\rho$. This amounts to determine the number $n_i$ of an epoch $i$ with $\rho_i \leq \rho$. With $n_{\pi_2} \geq n_0$, it is not hard to check that $\rho_{\pi_2} \leq 1/(2^d kd)^{O(1)}$ and $n_{\pi_2} \leq (2^d kd)^{O(1)}$. Then if $\rho \geq \rho_{\pi_2}$, we can certainly use $n_{\pi_2}$ as an upper bound. If $\rho \leq \rho_{\pi_2}$, it is not hard to check that with $n_i \leq O(c^3 k(1/\rho) \log(1/\rho))$, we can have $\rho_i \leq \rho$. As $c = c_0/(\lambda - \hat{\lambda})$, this proves Theorem 1.

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4 Block-Wise Power Method

In this section, we turn to study a different approach, which is based on block-wise power methods. Our algorithm is modified from that of [4] (referred as BPCA), which updates the estimate $Q_n$ with a more accurate estimate of $A$ using a block of samples, instead of one single sample as in our first algorithm. Our algorithm differs from BPCA by allowing different block sizes, instead of a fixed size, for different iterations. More precisely, we divide the steps into blocks, with block $i$ consisting of steps from some interval $I_i$, and we use this block of $|I_i|$ samples to update our estimate from $Q_{i-1}$ to $Q_i$. We will specify $|I_i|$ later in (3), which basically grows exponentially after some initial blocks. We call our algorithm DBPCA, which is described in Algorithm 2.

This algorithm, as our first algorithm, also has a space complexity of $O(kd)$. The sample complexity is guaranteed by the following, which we will prove in Subsection 4.1. To have a easier comparison with the results of [4] and [5], we use $\sqrt{\Phi_n} = \sin(U, Q_n)$ as the error measure in this section.

**Theorem 2.** Given any $\varepsilon \leq 1/\sqrt{kd}$, our algorithm can achieve an error $\varepsilon$ with high probability after $L$ iterations with a total of $N$ samples, for some $L \leq O\left(\frac{\lambda}{\lambda - \hat{\lambda}} \log \frac{d}{\varepsilon} \right)$ and $N \leq O\left(\frac{\lambda \log(dL)}{\varepsilon^2(\lambda - \hat{\lambda})^3} \right)$.

Let us make some remarks about the theorem. First, the error $\rho$ in Theorem 1 corresponds to the error $\varepsilon^2$ here, and one can see that the bound in Theorem 2 is better than those in Theorem 1 and [4,5] in general. In particular, when focusing on the dependence on $\varepsilon$, the sample complexity is $O((1/\varepsilon^2) \log(1/\varepsilon))$ in Theorem 2 but is $O((1/\varepsilon^2) \log(1/\varepsilon))$ in Theorem 1 and [4,5]. Next, the condition $\varepsilon \leq 1/\sqrt{kd}$ in the theorem is only used to simplify the error bound. One can check that our analysis also works for any $\varepsilon \leq 1$, but the resulting bound for $N$ has the factor $\varepsilon^2$ replaced by $\min(1/(kd), \varepsilon^2)$. Finally, from Theorem 2, one can also express the error in terms of the number of samples $n$ as $\varepsilon(n) \leq O\left(\sqrt{\frac{\lambda}{n(\lambda - \hat{\lambda})^2}} \log \frac{\lambda \log(n)}{\lambda - \hat{\lambda}} \right)$.

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Algorithm 1 SPCA

1: $S_0 \sim N(0, 1)^{d \times k}$
2: $S_0 = Q_0 R_0$ (QR decomposition)
3: $n \leftarrow 1$
4: while receiving data do
5: $S_n \leftarrow Q_{n-1} + \gamma_n x_n x_n^T Q_{n-1}$
6: $S_n = Q_n R_n$ (QR decomposition)
7: $n \leftarrow n + 1$
8: end while

Algorithm 2 DBPCA

1: $S_0 \sim N(0, 1)^{d \times k}$
2: $S_0 = Q_0 R_0$ (QR-decomposition)
3: $i \leftarrow 1$
4: while receiving data do
5: $S_i \leftarrow 0$
6: for $n \in I_i$ do
7: $S_i \leftarrow S_i + \frac{1}{|I_i|} x_n x_n^T Q_{n-1}$
8: end for
9: $S_i = Q_i R_i$ (QR-decomposition)
10: $i \leftarrow i + 1$
11: end while
4.1 Proof of Theorem 2

Recall that after the $i$-th block, we have the estimate $Q_i$, and we would like it to be close to $U$, with a small error $\sin(U, Q_i)$. To bound this error, we follow \cite{5} and work on bounding a surrogate error $\tan(U, Q_i)$, which suffices as $\sin(U, Q_i) \leq \tan(U, Q_i)$.

To start with, we know from Lemma 1 that for $\varepsilon_0 = \sqrt{\epsilon/(kd)}$ with some constant $\bar{\varepsilon}$, $\Pr[\tan(U, Q_0) > \varepsilon_0] \leq \delta_0$, using the fact that $\tan(U, Q_0)^2 = 1/\cos(U, Q_0)^2 - 1$.

Next, we would like to bound each $\tan(U, Q_i)$ in terms of the previous $\tan(U, Q_{i-1})$. For this, recall that with $F_i = \sum_{n \in I_i} x_n x_n^\top / |I_i|$, we have $Q_i R_i = F_i Q_{i-1}$, which can be rewritten as $A Q_{i-1} + (F_i - A) Q_{i-1}$. Using the notation $G_i = (F_i - A) Q_{i-1}$, we have $Q_i R_i = A Q_{i-1} + G_i$, where $G_i$ can be seen as the noise arising from estimating $A$ by $F_i$ using the $i$-th block of samples. Then, we rely on the following lemma from \cite{5}, with the parameters:

$$\bar{\lambda} = \max(\lambda, \lambda/4), \gamma = (\bar{\lambda}/\lambda)^{1/4} \text{ and } \triangle = (\lambda - \bar{\lambda})/4. \quad (2)$$

Lemma 6. \cite{5} Suppose $\|G\| \leq \Delta \cdot \min(\cos(U, Q), \beta)$, for some $\beta > 0$. Then

$$\tan(U, A Q + G) \leq \max(\beta, \max(\beta, \gamma) \tan(U, Q)).$$

From this, we can have the following lemma, proved in Appendix D which provides an exponentially-decreasing upper bound on $\tan(U, Q_i)$, for the parameters:

$$\varepsilon_i = \varepsilon_0 \gamma^i \text{ and } \beta_i = \min \left( \gamma/\sqrt{1 + \varepsilon_{i-1}^2}, \gamma \varepsilon_{i-1} \right)$$

where $\varepsilon_0 = \sqrt{\epsilon/(dk)}$ with the constant $\bar{\epsilon}$ in Lemma 1.

Lemma 7. Suppose $\tan(U, Q_{i-1}) \leq \varepsilon_{i-1}$ and $\|G_i\| \leq \Delta \beta_i$. Then $\tan(U, Q_i) \leq \varepsilon_i$.

The key which sets our approach apart from that of \cite{4,5} is the following observation. According to Lemma 7, for earlier iterations, one can in fact tolerate a larger $\|G_i\|$ and thus a larger empirical error for estimating $A$. This allows us to have smaller blocks at the beginning to save the number of samples, while still keeping the failure probability low. More precisely, we have the following lemma, proved in Appendix E with the parameters:

$$\delta_i = \delta_0/(2\beta^2) \text{ and } |I_i| = (c/(\Delta \beta_i)^2) \log(d/\delta_i), \quad (3)$$

where $\delta_0$ is the error probability given in Lemma 1 and $c$ is a large enough constant.

Lemma 8. For any $i \geq 1$, given $|I_i|$ samples in iteration $i$, we have $\Pr[\|G_i\| > \Delta \beta_i] \leq \delta_i$.

With this, we can bound the failure probability of our algorithm as

$$\Pr[3i \geq 0 : \tan(U, Q_i) > \varepsilon_i] \leq \Pr[\tan(U, Q_0) > \varepsilon_0] + \sum_{i \geq 1} \Pr[\|G_i\| > \Delta \beta_i]$$

which by Lemma 1 and Lemma 5 is at most $\delta_0 + \sum_{i \geq 1} \delta_i = \delta_0 + \sum_{i \geq 1} \frac{\delta_0}{2\beta^2} \leq 2\delta_0$.

To complete the proof of Theorem 2 it remains to bound the number of samples needed for achieving error $\varepsilon$. For this, we rely on the following lemma which we prove in Appendix F.

Lemma 9. For some $L \leq O\left( \frac{\beta^2 \log(dL)}{\lambda^2(\lambda - \bar{\lambda})^3} \right)$, we have $\varepsilon_L \leq \varepsilon$ and $\sum_{i=1}^L |I_i| \leq O\left( \frac{\lambda \log(dL)}{\varepsilon^2(\lambda - \bar{\lambda})^3} \right)$.

Finally, as $\bar{\lambda} = \max(\bar{\lambda}, \lambda/4)$, we have $\lambda - \bar{\lambda} \geq \Omega(\lambda - \bar{\lambda})$, and putting this into the bound above yields the sample complexity bound stated in the theorem.

5 Experiment

We conduct experiments on two large real-world datasets NYTImes and PubMed \cite{12} as used by \cite{4}. The dimension $d$ of the data points in the datasets are 102 and 141 thousands, respectively, which match our memory-restricted setting. The features of both datasets are normalized into $[0, 1]$. 

6
Parameter tuning is generally difficult for streaming algorithms. Instead of tuning the parameters extensively and reporting with the most optimistic (but perhaps unrealistic) parameter choice for each algorithm, we consider four parameter choices per algorithm to understand each algorithm more deeply. For SPCA, we follow its existing work \cite{dhillon1} to fix $n_0 = 0$ while considering $c \in \{10^4, 10^5, 10^6, 10^7\}$. For BPCA, we follow its existing works \cite{dhillon2,dhillon3} and let the block size be $\lfloor N/T \rfloor$, where $N$ is the size of the dataset and $T$ is the number of blocks. Theoretical results of BPCA \cite{dhillon2} suggest $T = \mathcal{O}\left(\frac{1}{\lambda \hat{\lambda}} \log d\right)$. Because $\lambda$ and $\hat{\lambda}$ are unknown in practice, we set $T = \lfloor L \log d \rfloor$ with $L \in \{1, 5, 25, 125\}$. For the proposed DBPCA, we set the initial block size as $2k$ to avoid being rank-insufficient in the first block. Then, we consider the ratio $\gamma^2 \in \{0.6, 0.7, 0.8, 0.9\}$ for enlarging the block size.

We run each algorithm 50 times by randomly generating data streams from the dataset. We consider $\sin(U_n Q_n)^2$, which is the error function used for the convergence analysis, as the performance evaluation criterion. The average performance on the two datasets for $k = 4$ and $k = 10$ are shown in Figure 1 and Figure 2 respectively. Our experiments on other $k$ values lead to similar observations and are not included here because of the space limit. To visualize the results clearly, we crop the figures up to $n = 200,000$, which is sufficient for checking the convergence of most of the parameter choices on the algorithms.

5.1 Comparison between DBPCA and BPCA

We first compare our proposed DBPCA with the original BPCA. From Figure 1, DBPCA outperforms BPCA under most parameter choices when $k = 4$, and is competitive to BPCA when $k = 10$. The edge of DBPCA over BPCA is even more remarkable in Figure 2.

As can be observed from Figures 1 and 2, if $L$ is too small (larger block), BPCA only sees one or two blocks of data within $n = 200,000$, and cannot reduce the error much. BPCA typically needs $L > 1$ (smaller blocks) to achieve lower error in the end. $L = 125$ gives the best performance of BPCA in Figure 2. However, sometimes large $L$ (small blocks) in BPCA allows reducing the error in the beginning, the error cannot converge to a competitive level in the long run. For instance, in Figure 1(d), $L = 125$ converges fast but cannot improve much after $n = 50,000$; $L = 25$ converges slower but keeps going towards the lowest error after $n = 200,000$. Also, using smaller blocks cannot ensure reducing the error after each update, and hence BPCA with larger $L$ results in less stable curves even after averaging over 50 runs. The results shows the difficulty of setting parameters of BPCA by the strategy proposed in \cite{dhillon2,dhillon3}.

\footnote{Note that \cite{dhillon3} suggests $\gamma^2 \geq 0.5$.}
On the other hand, DBPCA achieves better results by using a smaller block in the beginning to make improvements and a larger block later to further reduce the error. Also, in both datasets and under all parameter choices, DBPCA stably reduces the error after each update, which matches our theoretical analysis that guarantees error reduction with a high probability. In addition, DBPCA is quite stable with respect to the choice of $\gamma^2$ across the two datasets, making it easier to tune in practice. The properties make DBPCA favorable over BPCA in the family of block power methods.

### 5.2 Comparison between DBPCA and SPCA

As observed, DBPCA is less sensitive to the parameter $\gamma$ that corresponds to the theoretical suggestion of $\max(\hat{\lambda}/\lambda, 1/4)^{1/2}$. Somehow SPCA is rather sensitive to the parameter $c$ that corresponds to the theoretical suggestion of $\frac{c_0}{\lambda - \hat{\lambda}}$. For instance, setting $c = 10^4$ results in strong performance when $k = 4$ in Figure 1(c), but the worst performance when $k = 10$ in Figure 1(f). Similar results can be observed in Figure 2(c) and Figure 2(f) when $c = 10^5$. Furthermore, the parameter $c$ in SPCA directly affects the step size of each gradient descent update. Thus, compared with the best parameter choice, larger $c$ leads to less stable performance curve, while smaller $c$ sometimes results in significantly slower convergence. The results suggest that SPCA needs a more careful tuning and/or some deeper studies on proper parameter ranges.

The empirical study shows that DBPCA is competitive to SPCA in NYTimes when $k = 10$ and outperforms SPCA in other cases under the best choice of parameters. The result supports the theoretical study that DBPCA has better converges rate guarantee than SPCA.

However, the benefit of SPCA is its immediate use of new data point. DBPCA, as a representative of the block-power-method family, cannot update the solution until the end of the growing block. Then, the latter points in the larger blocks may be effectively unused for a long period of time.

### 6 Conclusion

We strengthen two families of streaming PCA algorithms, and compare the two strengthened families fairly from both theoretical and empirical sides. For the SGD family, we analyze the convergence rate of the famous SPCA algorithm for the multiple-principal-component cases; for the family of block power methods, we propose a dynamic-block algorithm DBPCA that enjoys faster convergence rate than the original BPCA. Then, the empirical studies demonstrate that DBPCA not only outperforms BPCA often by dynamically enlarging the block sizes, but also converges to competitive results more stably than SPCA in many cases. Both the theoretical and empirical studies thus
justify that DBPCA is the best among the two families, with the caveat of stalling the use of data points in larger blocks.

Our work opens some new research directions. Empirical results seem to suggest SPCA is competitive to or slightly worse than DBPCA. It is worth studying whether it is resulted from the substantial difference between \( \log 1/\epsilon \) and \( \log \log 1/\epsilon \) or caused by the hidden constants in the bounds. So one conjecture is that the bound in Theorem 1 can be further improved. On the other hand, although (2) suggests \( \gamma^2 \geq 0.5 \), the empirical results show that larger \( \gamma^2 \) generally results in better performance. Hence, it is also worth studying whether the lower bound could be further improved.

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A Proof of Lemma 3

Using the notation \( \hat{v} = Y_{n-1}v/\|Y_{n-1}v\| \) and \( A_n = x_n x_n^T \), one can follow the analysis in [11] to show that \( \Phi_n^{(v)} \leq \Phi_{n-1}^{(v)} + \beta_n - Z_n \), with

- \( \beta_n = 5\gamma_n^2 + 2\gamma_n^3 \),
- \( Z_n = 2\gamma_n (\hat{v}^T U U^T A_n \hat{v} - \|U^T \hat{v}\|^2 \|\hat{v}\|) \), and
- \( \mathbb{E}[Z_n, \mathcal{F}_{n-1}] \geq 2\gamma_n (\lambda - \hat{\lambda}) \Phi_{n-1}(1 - \Phi_{n-1}) \geq 0 \).

We omit the proof here as the adaptation is straightforward. It remains to show our better bound on \( |Z_n| \). For this, note that

\[
|Z_n| \leq 2\gamma_n \|\hat{v}^T U U^T - \|U^T \hat{v}\|^2 \|\| \cdot \|A_n \hat{v}\|,
\]
Therefore, assuming where \( \| A_n \| \leq 1 \) and

\[
\| \hat{v}^\top UU^\top - \| U^\top \hat{v} \|^2 \| v \|^2 \|
= \| U^\top \hat{v} \|^2 - 2\| U^\top \hat{v} \|^4 + \| U^\top \hat{v} \|^4
= \| U^\top \hat{v} \|^2 \left( 1 - \| U^\top \hat{v} \|^2 \right).
\]

As \( \| U^\top \hat{v} \|^2 \leq 1 \) and \( 1 - \| U^\top \hat{v} \|^2 \) = \( \Phi_{n-1}^{(v)} \), we have

\[
|Z_n| \leq 2\gamma_n \sqrt{\Phi_{n-1}}.
\]

## B Proof of Lemma 4

Assume that the event \( \Gamma_0 \) holds and consider any \( n \in [n_0, n_1) \). We need the following, which we prove in Appendix B.1.

### Proposition 1

For any \( n > m \) and any \( v \in \mathbb{R}^k \),

\[
\frac{\| U^\top Y_n v \|}{\| Y_n \|} \geq \left( \frac{m}{n} \right)^{3c} \frac{\| U^\top Y_m v \|}{\| Y_m \|}.
\]

From Proposition 1 we know that for any \( v \in S \),

\[
\frac{\| U^\top Y_n v \|}{\| Y_n v \|} \geq \frac{\| U^\top Y_n v \|}{\| Y_n \|} \geq \left( \frac{n_0}{n} \right)^{3c} \frac{\| U^\top Y_0 v \|}{\| Y_0 \|},
\]

where \( (n_0/n)^{3c} \geq (n_0/n_1)^3 \geq (1/c_1)^3 \) for the constant \( c_1 \) given in Remark 1. As \( Y_0 = Q_0 \) and \( \| Q_0 \| = 1 = \| Q_0 v \| \), we obtain

\[
\frac{\| U^\top Y_n v \|}{\| Y_n v \|} \geq \frac{\| U^\top Q_0 v \|}{\| Y_0 v \|} \geq \frac{\sqrt{1 - \rho_0}}{c_1^{3c}} \frac{\| Q_0 v \|}{\| Y_0 v \|} = \frac{c^c}{c_1^{3c} kd}.
\]

Therefore, assuming \( \Gamma_0 \), we always have

\[
\Phi_n = \max_v \left( 1 - \frac{\| U^\top Y_n v \|^2}{\| Y_n v \|^2} \right) \leq 1 - \frac{c^c}{c_1^{3c} kd} = \rho_1.
\]

### B.1 Proof of Proposition 1

Recall that for any \( n, Y_n = Y_n - n + n\gamma_n x_n x_n^\top Y_n - n \) and \( \| x_n x_n^\top \| \leq 1 \). Then for any \( v \in \mathbb{R}^k \),

\[
\frac{\| U^\top Y_n v \|}{\| Y_n \|} \geq \frac{\| U^\top Y_n - n + n\gamma_n U^\top Y_n, - n \|}{\| Y_n \| + n\gamma_n \| Y_n \|},
\]

which is

\[
\frac{1 - \gamma_n}{1 + \gamma_n} \cdot \frac{\| U^\top Y_n - n \|}{\| Y_n \|} \geq e^{-3\gamma_n} \frac{\| U^\top Y_n \|}{\| Y_n \|},
\]

using the fact that \( 1 - x \geq e^{-2x} \) for \( x \leq 1/2 \) and \( \gamma_n \leq 1/2 \). Then by induction, we have

\[
\frac{\| U^\top Y_n v \|}{\| Y_n \|} \geq e^{-3\sum_{t=m}^{n} \gamma_t} \cdot \frac{\| U^\top Y_m v \|}{\| Y_m \|}.
\]

The Proposition follows as

\[
e^{-3\sum_{t=m}^{n} \gamma_t} = e^{-3c \sum_{t=m}^{n} \frac{1}{t}} \geq \left( \frac{m}{n} \right)^{3c}
\]

using the fact that \( \sum_{t=m}^{n} \frac{1}{t} \leq \int_{m}^{n} \frac{1}{x} dx = \ln(n/m) \).
C Proof of Lemma 5

According to Lemma 3, our \( \Phi_1^{(v)} \)'s satisfy the same recurrence relation as the functions \( \Psi_n \)'s of [11]. We can therefore have the following, which we prove in Appendix C.1.

**Lemma 10.** Let \( \hat{\rho}_i = \rho_i / \lceil e^{5/c_0} \rceil c_0 (1 - \rho_i) \). Then for any \( \mathbf{u} \in S \) and \( \alpha_i \geq 12c^2 / n_i - 1 \),

\[
\Pr \left[ \sup_{n \geq n_i} \Phi_n^{(u)} \geq \hat{\rho}_i + \alpha_i \mid \Gamma_i \right] \leq e^{-\Omega((\alpha_i^2 / (c^2 \rho_i)) n_i - 1)}.
\]

Our goal is to bound \( \Pr [-\Gamma_{i+1} | \Gamma_i] \), which is

\[
\Pr \left[ \exists \mathbf{v} \in S : \sup_{n \leq n_i < n_{i+1}} \Phi_n^{(v)} \geq \hat{\rho}_{i+1} | \Gamma_i \right].
\]

As discussed before, we cannot directly apply a union bound on the bound in Lemma 10 as there are infinitely many \( \mathbf{v} \)'s in \( S \). Instead, we look for a small "\( \epsilon \)-net" \( D_i \) of \( S \), with the property that any \( \mathbf{v} \in S \) has some \( \mathbf{u} \in D_i \) with \( \| \mathbf{v} - \mathbf{u} \| \leq \epsilon \). Such a \( D_i \) with \( |D_i| \leq (1/\epsilon) \Omega(k) \) is known to exist (see e.g. [13]). Then what we need is that when \( \mathbf{v} \) and \( \mathbf{u} \) are close, \( \Phi_n^{(v)} \) and \( \Phi_n^{(u)} \) are close as well. This is guaranteed by the following, which we prove in Appendix C.2.

**Lemma 11.** Suppose \( \Gamma_i \) happens. Then for any \( n \in [n_i, n_{i+1}) \), any \( \epsilon \leq \sqrt{1 - \hat{\rho}_i / (2c^2 \epsilon)} \), and any \( \mathbf{u}, \mathbf{v} \in S \) with \( \| \mathbf{u} - \mathbf{v} \| \leq \epsilon \), we have

\[
|\Phi_n^{(v)} - \Phi_n^{(u)}| \leq 16 c^6 \epsilon / \sqrt{1 - \hat{\rho}_i}.
\]

According to this, we can choose \( \alpha_i = (\rho_{i+1} - \hat{\rho}_i) / 2 \) and \( \epsilon = \alpha_i \sqrt{1 - \hat{\rho}_i} / (16c^6) \) so that with \( \| \mathbf{u} - \mathbf{v} \| \leq \epsilon \), we have \( |\Phi_n^{(v)} - \Phi_n^{(u)}| \leq \alpha_i \). This means that given any \( \mathbf{v} \in S \) with \( \Phi_n^{(v)} \geq \rho_{i+1} \), there exists some \( \mathbf{u} \in D_i \) with \( \Phi_n^{(u)} \geq \rho_{i+1} - \alpha_i = \hat{\rho}_i + \alpha_i \). As a result, we can now apply a union bound over \( D_i \) and have

\[
\Pr [-\Gamma_{i+1} | \Gamma_i] \leq \sum_{\mathbf{u} \in D_i} \Pr \left[ \sup_{n \geq n_i} \Phi_n^{(u)} \geq \hat{\rho}_i + \alpha_i | \Gamma_i \right].
\]

(4)

To bound this further, consider the following two cases.

First, for the case of \( i < \pi_1 \), we have \( \rho_i \geq 3/4 \) and \( \eta_i = 1 - \rho_i \leq 1/4 \), so that

\[
\hat{\rho}_i = \rho_i e^{-5(1 - \rho_i)} = (1 - \eta_i) e^{-5\eta_i} \leq e^{-6\eta_i} \leq 1 - 3\eta_i.
\]

Then \( \alpha_i \geq ((1 - 2\eta_i) - (1 - 3\eta_i)) / 2 = \eta_i / 2 \), which is at least \( 12c^2 / n_{i-1} \), as \( \eta_i \geq \eta_i \geq \hat{\rho}_i/2 = \hat{\rho}_i + \alpha_i \). As a result, we can now apply Lemma 10 and the bound in (4) becomes

\[
(c^5_{\epsilon} / \eta_i)^\Omega(k) e^{-\Omega((\alpha_i^2 / c^2 \rho_i) n_{i-1})} \leq \frac{\delta_0}{2(t + 1)^2}.
\]

Next, for the case of \( i \geq \pi_1 \), we have \( \rho_i \leq 3/4 \) so that

\[
\hat{\rho}_i \leq \rho_i / \lceil e^{5/c_0} \rceil c_0 / 4 \leq \rho_i / \lceil e^{5/c_0} \rceil^3,
\]

as \( c_0 \geq 12 \) by assumption. Since \( \rho_{i+1} \geq \rho_i / \lceil e^{5/c_0} \rceil^2 \), this gives us \( \alpha_i \geq \rho_i \lceil e^{5/c_0} \rceil^{-2} - \lceil e^{5/c_0} \rceil^{-3} / 2 \), which is at least \( 12c^2 / n_{i-1} \), as \( \rho_i \), according to our choice, is about \( c_2(e^3k \log n_{i-1}) / (n_{i-1} + 1) \) for a large enough constant \( c_2 \). Thus, we can apply Lemma 10 and the bound in (4) becomes

\[
(c^5_{\epsilon} / \rho_i)^\Omega(k) e^{-\Omega((\alpha_i^2 / c^2 \rho_i) n_{i-1})} \leq \frac{\delta_0}{2(t + 1)^2}.
\]

(5)

This completes the proof of Lemma 5.
C.1 Proof of Lemma 10

By Lemma 3, the random variables $\Phi_n^{(v)}$’s satisfy the same recurrence relation of (11) for their random variables $\Phi_n$’s. Thus, we can follow their analysis but use our better bound on $|Z_n|$, and have the following.

First, when given $\Gamma_i$, we have $|Z_n| \leq 2\gamma_n\sqrt{p_i}$ for $n_i - 1 \leq n < n_i$. Then one can easily modify the analysis in (11) to show that for any $t \geq 0$,

$$\mathbb{E} \left[ e^{t\Phi_n^{(v)}} | \Gamma_i \right] \leq \exp \left( t\hat{\rho}_i + c^2(6t + 2t^2\rho_i) \left( \frac{1}{n_i - 1} - \frac{1}{n_i} \right) \right),$$

by noting that $(n_i - 1)/(n_i - 1 + 1) = [e^{5/c}]$ and $n \geq n_0 = \hat{c}e^k d^2 \log d$ according to our choice of parameters.

Next, following (11) and applying Doob’s martingale inequality, we obtain

$$\Pr \left[ \sup_{n \geq n_i} \Phi_n^{(v)} \geq \hat{\rho}_i + \alpha_i | \Gamma_i \right] \leq \mathbb{E} \left[ e^{t\Phi_n^{(v)}} | \Gamma_i \right] \exp \left( -t\alpha_i + \frac{c^2}{n_i - 1}(6t + 2t^2\rho_i) \right) \leq \exp \left( -\frac{t\alpha_i}{2} + \frac{2c^2t^2\rho_i}{n_i - 1} \right),$$

as $\alpha_i \geq \frac{12c^2}{n_i - 1}$. Finally, by choosing $t = \frac{\alpha_i n_i - 1}{8c^2\rho_i}$, we have the lemma.

C.2 Proof of Lemma 11

Assume without loss of generality that $\Phi_n^{(v)} \leq \Phi_n^{(u)}$ (otherwise, we switch $v$ and $u$), so that

$$\left| \Phi_n^{(v)} - \Phi_n^{(u)} \right| = \frac{\|U^TY_n v\|^2}{\|Y_n v\|^2} - \frac{\|U^TY_n u\|^2}{\|Y_n u\|^2}.$$

As $\|v - u\| \leq \epsilon$, we have

$$\frac{\|U^TY_n v\|}{\|Y_n v\|} \leq \frac{\|U^TY_n u\| + \epsilon\|U^TY_n\|}{\|Y_n u\| - \epsilon\|Y_n\|}. \tag{6}$$

To relate this to $\|U^TY_n u\|^2/\|Y_n u\|^2$, we would like to express $\|U^TY_n\|$ in terms of $\|U^TY_n u\|$ and $\|Y_n\|$ in terms of $\|Y_n u\|$. For this, note that both $\|U^TY_n u\|/\|Y_n\|$ and $\|Y_n u\|/\|\|Y_n\|\|$ are at least $\|U^TY_n u\|/\|\|Y_n\|\|$, which by Proposition 1 is at least

$$\left( \frac{n_i - 1}{n} \right)^{3c} \frac{\|U^TY_{n-1} u\|}{\|Y_{n-1}\|} \geq c_1^{-6c} \frac{\|U^TY_{n-1} u\|}{\|Y_{n-1}\|}, \tag{7}$$

using the fact that $n_i - 1/n \geq n_i - 1/n_i + 1 \geq 1/c_1^2$. Then as $Y_{n-1} = Q_{n-1}$ and $\|Q_{n-1} u\| = \|Q_{n-1} u\|$, the right hand side of (7) becomes

$$c_1^{-6c} \frac{\|U^TY_{n-1} u\|}{\|Q_{n-1} u\|} = c_1^{-6c} \sqrt{1 - \Phi_{n-1}^{(u)}} \geq c_1^{-6c} \sqrt{1 - \rho_i},$$

given $\Gamma_i$. What we have obtained so far is a lower bound for both $\|U^TY_n u\|/\|\|Y_n\|\|$ and $\|Y_n u\|/\|\|Y_n\|\|$. Plugging this into (6), with $\hat{\epsilon} = \hat{c}c_1^{-6c}/\sqrt{1 - \rho_i}$, we get

$$\frac{\|U^TY_n v\|}{\|Y_n v\|} \leq \frac{\|U^TY_n u\|}{\|Y_n u\|}(1 + \hat{\epsilon}).$$

As a result, we have

$$\left| \Phi_n^{(v)} - \Phi_n^{(u)} \right| \leq \frac{\|U^TY_n u\|^2}{\|Y_n u\|^2} \left( \frac{(1 + \hat{\epsilon})^2}{(1 - \hat{\epsilon})^2} - 1 \right) \leq 16\hat{\epsilon},$$

since $(1 + \hat{\epsilon})^2/(1 - \hat{\epsilon})^2 - 1 \leq 16\hat{\epsilon}$ for $\hat{\epsilon} \leq 1/2$.

3In particular, their proofs for Lemma 2.9 and Lemma 2.10.
D Proof of Lemma 7

As \( \cos(U, Q_{i-1})^2 = \frac{1}{1 + \tan(U, Q_{i-1})^2} \geq \frac{1}{1 + \varepsilon_{i-1}^2} \geq \beta_i^2 \), we have \( \|G_i\| \leq \Delta \beta_i \leq \Delta \cos(U, Q_{i-1}) \). Thus, we can apply Lemma 6 and have

\[
\tan(U, AQ_{i-1} + G_i) \leq \max(\beta_i, \max(\beta_i, \gamma \varepsilon_{i-1})),
\]

which is at most \( \max(\beta_i, \gamma \varepsilon_{i-1}) \leq \gamma \varepsilon_{i-1} = \varepsilon_i \). The lemma follows as \( \tan(U, AQ_{i-1} + G_i) = \tan(U, AQ_{i-1} + G_i) \).

E Proof of Lemma 8

Let \( \rho = \Delta \beta_i \) and note that \( \|G_i\| \leq \|A - F_i\| \), where \( F_i \) is the average of \( |I_i| \) i.i.d. random matrices, each with mean \( A \). Recall that \( \|A\| \leq 1 \) by Assumption 1. Then from a matrix Chernoff bound, we have

\[
\Pr \left( \|G_i\| > \rho \right) \leq \Pr \left( \|A - F_i\| > \rho \right) \leq d e^{-\Omega(\rho^2 |I_i|)} \leq \delta_i,
\]

for \( |I_i| \) given in (3).

F Proof of Lemma 9

Let \( L \) be the iteration number such that \( \varepsilon_{L-1} > \varepsilon \) and \( \varepsilon_L \leq \varepsilon \). Note that with \( \varepsilon_L = \varepsilon_0 \gamma^L = \varepsilon_0 (1 - (\lambda - \bar{\lambda})/\lambda)^{L/4} \leq \varepsilon_0 e^{-L(\lambda - \bar{\lambda})/(4\lambda)} \), we can have

\[
L \leq O \left( \frac{\lambda}{\lambda - \bar{\lambda}} \log \frac{\varepsilon_0}{\varepsilon} \right) \leq O \left( \frac{\lambda}{\lambda - \bar{\lambda}} \log \frac{d}{\varepsilon} \right).
\]

As the number of samples in iteration \( i \) is

\[
|I_i| = O \left( \frac{\log(d/\varepsilon_i)}{(\lambda - \bar{\lambda})^2 \varepsilon_i^2} \right) \leq O \left( \frac{\log(d)}{(\lambda - \bar{\lambda})^2 \varepsilon_i^2} \right),
\]

the total number of samples needed is

\[
\sum_{i=1}^{L} |I_i| \leq O \left( \frac{\log(dL)}{(\lambda - \bar{\lambda})^2} \right) \cdot \sum_{i=1}^{L} \frac{1}{\beta_i^2}.
\]

With \( \beta_i = \min(\gamma/\sqrt{1 + \varepsilon_{i-1}^2}, \gamma \varepsilon_{i-1}) \), one sees that for some \( i_0 \leq O(\log d) \), \( \beta_i = \gamma/\sqrt{1 + \varepsilon_{i-1}^2} \) when \( i \leq i_0 \) and \( \beta_i = \gamma \varepsilon_{i-1} = \varepsilon_i \) when \( i > i_0 \). This implies that

\[
\sum_{i=1}^{L} \frac{1}{\beta_i^2} = \sum_{i=1}^{i_0} \frac{1}{\gamma^2} + \sum_{i=i_0+1}^{L} \frac{1}{\varepsilon_i^2},
\]

(8)

where the first sum in the righthand side of (8) is

\[
\frac{i_0}{\gamma^2} + \sum_{i=1}^{i_0} \varepsilon_0^2 \gamma^{2i-4} \leq O(\log d) \frac{\varepsilon_0^2}{\gamma^2} + \frac{\varepsilon^2}{\gamma^2 (1 - \gamma^2)},
\]

while the second sum is

\[
\sum_{i=i_0+1}^{L} \frac{\gamma^{2(L-i)}}{\varepsilon_L^2} \leq \frac{1}{(1 - \gamma^2)\varepsilon_L^2} \leq \frac{1}{\gamma^2 (1 - \gamma^2)\varepsilon^2}
\]

using the fact that \( \varepsilon_L = \gamma \varepsilon_{L-1} \geq \gamma \varepsilon \). Since \( \gamma^2 = \left( 1 - \frac{\lambda - \bar{\lambda}}{\lambda} \right)^{1/2} \leq 1 - \frac{\lambda - \bar{\lambda}}{2\lambda} \), we have \( \frac{\lambda}{\gamma^2} \leq \frac{2\lambda}{\lambda - \bar{\lambda}} \), and since \( \lambda \leq O(\bar{\lambda}) \), we also have \( \frac{1}{\gamma^2} \leq O(1) \). Moreover, as we assume that \( \varepsilon \leq 1/\sqrt{kd} \), we can conclude that the total number of samples needed is at most

\[
\sum_{i=1}^{L} |I_i| \leq O \left( \frac{\log(dL)}{(\lambda - \bar{\lambda})^2} \right) \cdot O \left( \frac{\lambda}{(\lambda - \bar{\lambda}) \varepsilon^2} \right) \leq O \left( \frac{\lambda \log(dL)}{\varepsilon^2 (\lambda - \bar{\lambda})^2} \right).
\]