GENERALIZED LIPSCHITZ NUMBERS, FINE DIFFERENTIABILITY, AND QUASICONFORMAL MAPPINGS

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Abstract. We introduce a generalized version of the local Lipschitz number $\text{lip}_u$, and show that it can be used to characterize Sobolev functions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, $1 \leq p \leq \infty$, as well as functions of bounded variation. This concept turns out to be fruitful for studying, and for establishing new connections between, a wide range of topics including fine differentiability, Rademacher’s theorem, Federer’s characterization of sets of finite perimeter, regularity of maximal functions, quasiconformal mappings, Alberti’s rank one theorem, as well as generalizations to metric measure spaces.

1. Introduction

According to Rademacher’s theorem [51], Lipschitz functions $u \in \text{Lip}(\mathbb{R}^n)$ are differentiable almost everywhere. In the scale of Sobolev functions, locally Lipschitz functions are exactly $u \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$, up to a choice of pointwise representative. In fact, all functions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, with $p > n$, are differentiable almost everywhere, see e.g. [16, Theorem 6.5], while in the case $p \leq n$ this is not true.

Instead of the global or local Lipschitz condition, one can consider the pointwise infinitesimal Lipschitz number defined by

$$\text{Lip}_u(x) := \limsup_{r \to 0} \sup_{y \in B(x, r)} \frac{|u(y) - u(x)|}{r}.$$ 

Stepanoff [53] showed that a function $u$ is differentiable almost everywhere in the set where $\text{Lip}_u < \infty$. Several authors including Cheeger [12] and Keith [31] have studied extensively also the Lipschitz number $\text{lip}_u(x)$, which is defined similarly but with $\limsup$ replaced by $\liminf$. This smaller number is not sufficient in Stepanoff’s result, but it can be shown that if $u$ is continuous, $\text{lip}_u < \infty$ outside a set of $\sigma$-finite $n - 1$-dimensional Hausdorff measure, and also $\text{lip}_u \in L^p_{\text{loc}}(\mathbb{R}^n)$, then $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$, see Balogh–Csörnyei [6] and Zürcher [58].

Of course, a converse result does not hold: a function $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ for $1 \leq p \leq n$ can have singularities in a dense set, and then $\text{lip}_u = \infty$ everywhere. On the other hand, because such Sobolev functions are not continuous either, one often considers the weaker notions of
quasicontinuity or fine continuity. We wish to consider, in a similar vein, relaxed versions of Lipschitz numbers or of differentiability, which would be satisfied also by functions $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ when $1 \leq p \leq n$ and which would in fact characterize these classes.

Lipschitz and Sobolev functions are part of a broader class of functions of bounded variation (BV). These can be highly discontinuous, and an important special case is given by the characteristic functions of sets of finite perimeter. For such sets, there are various “Federer-style” characterizations which state that a set has finite perimeter if and only if the $n-1$-dimensional Hausdorff measure of a suitable boundary is finite. These quantities seem unrelated to Lipschitz continuity. However, we will be able to use our generalized Lipschitz numbers to characterize also BV functions, and in the special case of sets of finite perimeter, we recover a Federer-style characterization.

To define the generalized Lipschitz number at a point $x$, our idea is to “cut out” a thin set and consider the oscillation of the function only in the complement. In order to deal with discontinuities and other irregular behavior, we also add a suitable scaling factor, as follows. Given a function $w: \Omega \to [-\infty, \infty]$ in an open set $\Omega \subset \mathbb{R}^n$, a positive Radon measure $\nu$ in $\Omega$, and $\delta \geq 0$, for every $x \in \Omega$ we define

$$Lip^{\nu,\delta}_{w}(x) := \frac{1}{2} \inf \left\{ \limsup_{r \to 0} \frac{\text{osc}_{U_r} w}{r} \frac{\mathcal{L}^n(B(x,r))}{\nu(B(x,r))} : \limsup_{r \to 0} \frac{\text{Cap}_1(B(x,r) \setminus U_r) r^{n-1}}{r^{n-1}} \leq \delta \right\}; \tag{1.1}$$

the infimum is taken over collections of sets $U_r \subset B(x,r)$, which at every $x \in \Omega$ need to be defined for all sufficiently small $r > 0$. Here $\text{Cap}_1$ is the Sobolev 1-capacity; we give definitions in Section 2. Often we will let $\delta$ take the constant value

$$c(n) := \frac{\min\{\omega_{n-1}, \omega_n\}}{2^{13}n^2 C_I}, \tag{1.2}$$

where $C_I \geq 1$ is the constant from the relative isoperimetric inequality and $\omega_n$ is the volume of the $n$-dimensional unit ball. If $d\nu = a d\mathcal{L}^n$, we denote $Lip^{\nu,\delta}_{w} = Lip^{a,\delta}_{w}$. If $a \equiv 1$, we denote simply $Lip^{1,\delta}_{w} = Lip^{\delta}_{w}$. The case $Lip^{0,\delta}_{w}(x) < \infty$ is very close to the concept of fine differentiability at $x$, as we will see.

We denote by $\mathcal{M}(\Omega, \cdot)$ a suitable class of positive Radon measures, given in Definition 2.35. Now we can characterize Sobolev and BV functions as follows. We prove the following theorems in somewhat more general form in Section 3.

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^1(\Omega)$. Then the following are equivalent:

1. $u \in BV(\Omega)$;
2. There exists $\nu \in \mathcal{M}(\Omega, 2^{-5n}c(n)^{-1})$ such that $\nu(\Omega) < \infty$ and $Lip^{\nu, c(n)}_{u}(x) \leq 1$ for every $x \in \Omega$.

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^p(\Omega)$, where $1 \leq p \leq \infty$. Then the following are equivalent:
(1) \( u \in W^{1,p}(\Omega) \);
(2) There exists a nonnegative \( a \in L^p(\Omega) \) such that \( \text{Lip}_u^{a,c(n)}(x) \leq 1 \) for every \( x \in \Omega \).

As a corollary, we will obtain that sets of finite perimeter can be characterized by the condition that a suitable “capacitary boundary” has finite \( n-1 \)-dimensional Hausdorff measure. The key idea conveyed by the above theorems is that while the quantities \( \text{lip}_u^* \) and \( \text{Lip}_u^* \) are very sensitive to oscillations and singularities, the generalized Lipschitz numbers have just the required flexibility in the context of Sobolev and BV functions. We will discover a similar phenomenon with other quantities as well, and we will explore related topics in the ensuing sections. Each of these requires several new definitions, so we choose to give only a brief overview here and leave the details as well as more extensive literature references to the later sections.

In Section 4 we show that Sobolev functions \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) with \( 1 \leq p \leq n \), while not differentiable as we noted above, are 1-finely differentiable almost everywhere. In fact we show that Sobolev functions can, roughly speaking, be characterized as finely differentiable functions. We also prove a Stepanoff-type theorem stating that any measurable function \( w \) is 1-finely differentiable a.e. in the set \( \{ \text{Lip}_w^0 < \infty \} \). We use this to show that the Hardy-Littlewood maximal function of a Sobolev function is also 1-finely differentiable almost everywhere; this is related to the so-called \( W^{1,1} \)-problem for maximal operators.

In Section 5 we expand our study to quasiconformal mappings, which are defined by means of the distortion number \( H_f \) or \( h_f \). Starting from Gehring \([21, 22]\), there are many results in the literature of the following form: if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a homeomorphism, and \( h_f \) is finite outside a set of \( \sigma \)-finite \( H^{n-1} \)-measure as well as bounded by a given constant a.e., then \( f \) is a Sobolev mapping. Note that such results involve two different exceptional sets, of dimensions \( n-1 \) and \( n \), and also that they are analogous with the previously mentioned theorem of Balogh–Csörnyei \([6]\) involving \( \text{lip}_u \). We show that in fact these results arise from a much more general theorem written in terms of a generalized distortion number \( h_f^\nu \), when we choose a particular measure \( \nu \) with an \( n \)-dimensional and \( n-1 \)-dimensional part.

In Section 6 we show that for a function of bounded variation \( f \in BV(\mathbb{R}^n; \mathbb{R}^n) \), the following concepts are closely related: the rank of \( \frac{dDf}{d|Df|}(x) \), fine differentiability at \( x \), and the condition \( H_f^{\text{fine}}(x) < \infty \), where \( H_f^{\text{fine}} \) is another generalized distortion number. Based on this, we discuss an interpretation of Alberti’s rank one theorem in terms of the quasiconformal behavior of a mapping, which allows to conjecture a version of the theorem in more general metric measure spaces. Throughout the paper, also various other open problems arise from the concepts that we introduce.

2. Preliminaries

2.1. Basic notation and definitions. Our definitions and notation are standard, and the reader may consult e.g. the monograph Evans–Gariepy \([16]\) for more background. We
will always work in the Euclidean space $\mathbb{R}^n$, $n \geq 1$. We denote the $n$-dimensional Lebesgue outer measure by $\mathcal{L}^n$. We denote the $s$-dimensional Hausdorff measure by $\mathcal{H}^s$, $s \geq 0$, and it is obtained as a limit of the Hausdorff pre-measures $\mathcal{H}^s_R$ as $R \to 0$. If a property holds outside a set of Lebesgue measure zero, we say that it holds almost everywhere, or “a.e.”. With other measures, we write more explicitly e.g. “$\mathcal{H}^{n-1}$-a.e.”.

We denote the characteristic function of a set $E \subset \mathbb{R}^n$ by $\chi_E : \mathbb{R}^n \to \{0, 1\}$. We write $B(x, r)$ for an open ball in $\mathbb{R}^n$ with center $x$ and radius $r$, that is, $\{y \in \mathbb{R}^n : |y - x| < r\}$. We always work with the Euclidean norm $| \cdot |$ for vectors $v \in \mathbb{R}^n$ as well as for matrices $A \in \mathbb{R}^{k \times n}$. When we consider closed balls, we always specify this by the bar $\overline{B}(x, r)$. We will often use the notation $2B(x, r) := B(x, 2r)$.

By a “measurable” set we mean $\mathcal{L}^n$-measurable, unless otherwise specified. If a function $u$ is in $L^1(D)$ for some measurable set $D \subset \mathbb{R}^n$ of nonzero and finite Lebesgue measure, we write

$$u_D := \int_D u(y) \, d\mathcal{L}^n(y) := \frac{1}{\mathcal{L}^n(D)} \int_D u(y) \, d\mathcal{L}^n(y)$$

for its mean value in $D$.

We will always denote by $\Omega \subset \mathbb{R}^n$ an open set, and we consider $1 \leq p \leq \infty$. Let $k \in \mathbb{N}$. The Sobolev space $W^{1,p}(\Omega; \mathbb{R}^k)$ consists of functions $u \in L^p(\Omega; \mathbb{R}^k)$ whose first weak partial derivatives $(Du_j)_{l=1,\ldots,k}$, $j = 1, \ldots, n$, belong to $L^p(\Omega)$. The Dirichlet space $D^p(\Omega)$ is defined in the same way, except that the integrability requirement for the function itself is relaxed to $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^k)$. The Sobolev norm is

$$\|u\|_{W^{1,p}(\Omega; \mathbb{R}^k)} := \|u\|_{L^p(\Omega; \mathbb{R}^k)} + \|Du\|_{L^p(\Omega; \mathbb{R}^{k \times n})}.$$  

We will need the following Vitali-Carathéodory theorem; for a proof see e.g. [28, p. 108]. By a positive Radon measure we mean a nonnegative measure which is defined for all Borel sets, and is Borel regular.

**Theorem 2.1.** Let $\mu$ be a positive Radon measure on $\Omega$, and let $\rho \in L^1(\Omega, \mu)$, with values in $[0, \infty]$. Then there exists a sequence $\{\rho_i\}_{i=1}^{\infty}$ of lower semicontinuous functions on $\Omega$ such that $\rho \leq \rho_{i+1} \leq \rho_i$ for all $i \in \mathbb{N}$, and $\rho_i \to \rho$ in $L^1(\Omega, \mu)$.

Let $S \subset \mathbb{R}^n$ be an $\mathcal{H}^{n-1}$-measurable set. We say that $S$ is countably $\mathcal{H}^{n-1}$-rectifiable if there exist countably many Lipschitz functions $f_j : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that

$$\mathcal{H}^{n-1}\left(S \setminus \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^{n-1})\right) = 0.$$

Let $H \subset \mathbb{R}^n$ be an $n - 1$-dimensional hyperplane, let $\pi$ be the orthogonal projection onto $H$, and let $\pi^\perp$ be the orthogonal projection onto the orthogonal complement $H^\perp$. Let $f : H \to H^\perp$ be an $L$-Lipschitz function, and let

$$S := \{x \in \mathbb{R}^n : f(\pi(x)) = \pi^\perp(x)\}.$$
be the graph of $f$. We call this an $L$-Lipschitz $n-1$-graph, also if it is only defined on a subset of $H$. Every countably $\mathcal{H}^{n-1}$-rectifiable set can be presented, modulo an $\mathcal{H}^{n-1}$-negligible set, as an at most countable union of disjoint $1$-Lipschitz $n-1$-graphs, see [4, Proposition 2.76]. We can also assume the $1$-Lipschitz $n-1$-graphs to be $\mathcal{H}^{n-1}$-measurable, since a graph defined on an entire $n-1$-dimensional hyperplane is a closed set.

2.2. Functions of bounded variation. The theory of BV functions presented here can be found in the monograph Ambrosio–Fusco–Pallara [4]. Let $k \in \mathbb{N}$, and as before let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f \in L^1(\Omega; \mathbb{R}^k)$ is a function of bounded variation, denoted $f \in BV(\Omega; \mathbb{R}^k)$, if its weak derivative is an $\mathbb{R}^k \times \mathbb{R}^n$-valued Radon measure with finite total variation. This means that there exists a (unique) Radon measure $Df$ such that for all $\varphi \in C^1_c(\Omega)$, the integration-by-parts formula

$$\int_{\Omega} f_j \frac{\partial \varphi}{\partial y_l} \, d\mathcal{L}^n = - \int_{\Omega} \varphi \, d(Df_j)_l, \quad j = 1, \ldots, k, \ l = 1, \ldots, n,$$

holds. The Dirichlet space $D^{BV}(\Omega)$ is defined in the same way, except that the integrability requirement for the function itself is relaxed to $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^k)$. The BV norm is defined by

$$\|f\|_{BV(\Omega; \mathbb{R}^k)} := \|f\|_{L^1(\Omega; \mathbb{R}^k)} + |Df|(\Omega).$$

If we do not know a priori that a function $f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^k)$ is a BV function, we consider

$$\text{Var}(f, \Omega) := \sup \left\{ \sum_{j=1}^k \int_{\Omega} f_j \, \text{div} \varphi_j \, d\mathcal{L}^n, \varphi \in C^1_c(\Omega; \mathbb{R}^{k \times n}), |\varphi| \leq 1 \right\}. \quad (2.2)$$

If $\text{Var}(f, \Omega) < \infty$, then the $\mathbb{R}^{k \times n}$-valued Radon measure $Df$ exists and $\text{Var}(f, \Omega) = |Df|(\Omega)$ by the Riesz representation theorem, and $f \in BV(\Omega)$ provided that $f \in L^1(\Omega; \mathbb{R}^k)$. If $E \subset \mathbb{R}^n$ with $\text{Var}(\chi_E, \mathbb{R}^n) < \infty$, we say that $E$ is a set of finite perimeter.

The coarea formula states that for a function $u \in BV(\Omega)$ and a Borel set $A \subset \Omega$, we have

$$|Du|(A) = \int_{-\infty}^{\infty} |D\chi\{u>t\}|(A) \, dt. \quad (2.3)$$

Here we abbreviate $\{u > t\} := \{x \in \Omega : u(x) > t\}$.

The relative isoperimetric inequality states that for every measurable set $E \subset \mathbb{R}^n$ and every ball $B(x, r)$, we have

$$\min \{ \mathcal{L}^n(B(x, r) \cap E), \mathcal{L}^n(B(x, r) \setminus E) \} \leq C_I r \, \text{Var}(\chi_E, B(x, r)), \quad (2.4)$$

where the constant $C_I \geq 1$ only depends on $n$.

Let $f \in L^1_{\text{loc}}(\Omega)$. We will often consider a particular pointwise representative, namely the precise representative

$$f^*(x) := \limsup_{r \to 0} \int_{B(x, r)} f \, d\mathcal{L}^n, \quad x \in \Omega.$$
This is easily seen to be a Borel function. For \( f \in L^1_{\text{loc}}(\Omega; \mathbb{R}^k) \), we also define \( f^* := (f_1^*, \ldots, f_k^*) \). It is worth noting that Sobolev and BV functions, usually denoted by \( u \) or \( f \), are understood to be defined only almost everywhere, but in many cases we need pointwise defined functions such as \( u^* \) and \( f^* \). The symbol \( w \) will often denote a pointwise defined function, for example in the definition (1.1) of \( \text{Lip}_{\nu,\delta}^\nu \). The oscillation of \( w: U \to \mathbb{R}^k \) in a set \( U \subset \mathbb{R}^n \) is defined by

\[
\text{osc}_U w := \sup\{|w(x) - w(y)|, \ x, y \in U\}.
\]

If \( |w(x)| = \infty \) for some \( x \in U \), we interpret \( \text{osc}_U w = \infty \).

We say that \( x \in \Omega \) is a Lebesgue point of \( f \) if

\[
\lim_{r \to 0} \int_{B(x,r)} |f(y) - \tilde{f}(x)| \, d\mathcal{L}^n(y) = 0
\]

for some \( \tilde{f}(x) \in \mathbb{R}^k \). We denote by \( S_f \subset \Omega \) the set where this condition fails and call it the approximate discontinuity set.

Given a unit vector \( \nu \in \mathbb{R}^n \), we define the half-balls

\[
B^+_{\nu}(x, r) := \{y \in B(x, r): \langle y - x, \nu \rangle > 0\},
\]

\[
B^-_{\nu}(x, r) := \{y \in B(x, r): \langle y - x, \nu \rangle < 0\},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product. We say that \( x \in \Omega \) is an approximate jump point of \( f \) if there exist a unit vector \( \nu \in \mathbb{R}^n \) and distinct vectors \( f^+(x), f^-(x) \in \mathbb{R}^k \) such that

\[
\lim_{r \to 0} \int_{B^+_{\nu}(x, r)} |f(y) - f^+(x)| \, d\mathcal{L}^n(y) = 0
\]

and

\[
\lim_{r \to 0} \int_{B^-_{\nu}(x, r)} |f(y) - f^-(x)| \, d\mathcal{L}^n(y) = 0.
\]

The set of all approximate jump points is denoted by \( J_f \). For \( f \in \text{BV}(\Omega; \mathbb{R}^k) \), we have that \( \mathcal{H}^{n-1}(S_f \setminus J_f) = 0 \), see [4, Theorem 3.78].

The lower and upper approximate limits of a function \( u \in \text{BV}_{\text{loc}}(\Omega) \) are defined respectively by

\[
u(\cdot) \in \text{BV}_{\text{loc}}(\Omega), \quad u^\nu(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \cap \{u > t\})}{\mathcal{L}^n(B(x, r))} = 0 \right\},
\]

for all \( x \in \Omega \). We interpret the supremum and infimum of an empty set to be \( -\infty \) and \( \infty \), respectively. Note that for \( x \in \Omega \setminus S_u \), we have \( \tilde{u}(x) = u^\wedge(x) = u^\vee(x) \). Also, for \( x \in J_u \), we have \( u^\wedge(x) = \min\{u^-(x), u^+(x)\} \) and \( u^\vee(x) = \max\{u^-(x), u^+(x)\} \).
Given \( f \in \text{BV}(\Omega; \mathbb{R}^k) \), for every \( x \in J_f \) we have
\[
f^*(x) = \frac{f^-(x) + f^+(x)}{2}.
\]
(2.7)

We write the Radon-Nikodym decomposition of the variation measure of \( f \) into the absolutely continuous and singular parts with respect to \( \mathcal{L}^n \) as \( Df = D^a f + D^s f \). Furthermore, we define the Cantor and jump parts of \( Df \) by
\[
D^c f := D^s f \mathbb{1}(\Omega \setminus S_f), \quad D^j f := D^s f \mathbb{1} J_f.
\]
(2.8)

Here 
\[
D^s f \mathbb{1} J_f(A) := D^s f (J_f \cap A), \quad \text{for } D^s f \text{-measurable } A \subset \mathbb{R}^n.
\]

Since \( \mathcal{H}^{n-1}(S_f \setminus J_f) = 0 \) and \( |Df| \) vanishes on \( \mathcal{H}^{n-1} \)-negligible sets, we get the decomposition (see [4, Section 3.9])
\[
Df = D^a f + D^c f + D^j f.
\]
(2.9)

For the jump part, we know that 
\[
d|D^j f| = |f^+ - f^-| d\mathcal{H}^{n-1} \mathbb{1} J_f.
\]
(2.10)

Note that when \( f \in W^{1,1}_\text{loc}(\Omega; \mathbb{R}^k) \), necessarily \( \mathcal{H}^{n-1}(J_f) = 0 \), and since we also have \( \mathcal{H}^{n-1}(S_f \setminus J_f) = 0 \) as noted after (2.6), we know that \( \mathcal{H}^{n-1} \)-a.e. point is a Lebesgue point of \( f \).

For basic results in the one-dimensional case \( n = 1 \), see [4, Section 3.2]. If \( \Omega \subset \mathbb{R} \) is open and connected, we define the pointwise variation of \( f : \Omega \to \mathbb{R}^k \) by
\[
pV(f, \Omega) := \sup \sum_{j=1}^{N-1} |f(x_j) - f(x_{j+1})|,
\]
(2.11)

where the supremum is taken over all collections of points \( x_1 < \ldots < x_N \) in \( \Omega \). For a general open \( \Omega \subset \mathbb{R} \), we define \( pV(f, \Omega) \) to be \( \sum pV(f, I) \), where the sum runs over all connected components \( I \) of \( \Omega \). For every \( f \in L^1_\text{loc}(\Omega; \mathbb{R}^k) \), we have \( \text{Var}(f, \Omega) \leq pV(f, \Omega) \).

Denote by \( \pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1} \) the orthogonal projection onto \( \mathbb{R}^{n-1} \): for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \),
\[
\pi_n((x_1, \ldots, x_n)) := (x_1, \ldots, x_{n-1}).
\]
(2.12)

For \( z \in \pi_n(\Omega) \), we denote the slices of \( \Omega \) by
\[
\Omega_z := \{ t \in \mathbb{R} : (z, t) \in \Omega \}.
\]
We also denote \( f_z(t) := f(z, t) \) for \( z \in \pi_n(\Omega) \) and \( t \in \Omega_z \). For any pointwise defined \( f \in L^1_\text{loc}(\Omega; \mathbb{R}^k) \), we know that \( \text{Var}(f, \Omega) \) is at most the sum of (the integral can be understood as an upper integral if necessary)
\[
\int_{\pi_n(\Omega)} pV(f_z, \Omega_z) d\mathcal{L}^{n-1}(z)
\]
(2.13)
and the analogous quantities for the other $n-1$ coordinate directions, see [4, Theorem 3.103].

2.3. Capacities and fine topology. We will often use the notion of capacity, and in this subsection we record relevant background facts and some elementary lemmas that will be needed. Recall that the capacity is needed already in the definition (1.1) of $\text{Lip}^{\nu, \delta}_w$. Apart from relying heavily on the facts given in this subsection, otherwise our proofs throughout the paper will be fairly elementary, though often technical.

Consider $1 \leq p < \infty$. The (Sobolev) $p$-capacity of a set $A \subset \mathbb{R}^n$ is defined by

$$\text{Cap}_p(A) := \inf \|u\|_{W^{1,p}(\mathbb{R}^n)}^p,$$

where the infimum is taken over Sobolev functions $u \in W^{1,p}(\mathbb{R}^n)$ satisfying $u \geq 1$ in a neighborhood of $A$.

It is straightforward to check that for any set $A \subset \mathbb{R}^n$ and some constant $C$ depending only on $n$, we have

$$\text{Cap}_1(A) \leq C\mathcal{H}^{n-1}(A). \quad (2.14)$$

We say that a set $V \subset \mathbb{R}^n$ is $p$-quasiopen if for every $\varepsilon > 0$ there exists an open set $G \subset \mathbb{R}^n$ such that $V \cup G$ is open. We say that a function $w$ is $p$-quasicontinuous on $\Omega$ if for every $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $\text{Cap}_p(G) < \varepsilon$ and $w|_{\Omega \setminus G}$ is finite and continuous. We know that

if $u \in W^{1,p}_{\text{loc}}(\Omega)$, then $u^*$ is $p$-quasicontinuous on $\Omega$, \quad (2.15)

see e.g. [16, Theorem 4.19]. Given sets $A \subset W \subset \mathbb{R}^n$, where $W$ is open, the relative $p$-capacity is defined by

$$\text{cap}_p(A, W) := \inf \int_W |\nabla u|^p \, d\mathcal{L}^n,$$

where the infimum is taken over functions $u \in W^{1,p}_0(W)$ satisfying $u \geq 1$ in a neighborhood of $A$. The class $W^{1,p}_0(W)$ is the closure of $C_c^1(W)$ in the $W^{1,p}(\mathbb{R}^n)$-norm.

By [11, Theorem 3.3], given a function $u \in \text{BV}(\Omega)$, there is a sequence $\{u_j\}_{j=1}^\infty$ of functions in $W^{1,1}(\Omega)$ such that

$$u_j \rightharpoonup u \quad \text{in} \quad L^1(\Omega), \quad |Du_j|(\Omega) \to |Du|(\Omega), \quad \text{and} \quad u_j^\vee(x) \geq u^\vee(x) \quad \text{for} \quad \mathcal{H}^{n-1}\text{-a.e.} \quad x \in \Omega. \quad (2.16)$$

If $B(x, r)$ is a ball with $0 < r \leq 1$, and $F$ is a measurable set with $\mathcal{L}^n(F \cap B(x, r)) \leq \frac{1}{2} \mathcal{L}^n(B(x, r))$ and $|D\chi_F|(B(x, r)) < \infty$, then by combining e.g. Theorem 5.6 and Theorem 5.15(iii) of [16], we get

$$|D\chi_{B(x, r) \cap F}|(\mathbb{R}^n) \leq C\|\chi_F\|_{\text{BV}(B(x, r))}.$$


for some constant $C$ depending only on $n, r$. On the other hand, by the relative isoperimetric inequality (2.4), we have

$$
\|\chi_F\|_{BV(B(x,r))} = \mathcal{L}^n(F \cap B(x,r)) + |D\chi_F|(B(x,r)) \leq (C_I r + 1)|D\chi_F|(B(x,r)) \\
\leq 2C_I|D\chi_F|(B(x,r)),
$$

since $r \leq 1$ and $C_I \geq 1$. Combining these, we get

$$
|D\chi_{B(x,r) \cap F}|(\mathbb{R}^n) \leq C|D\chi_F|(B(x,r)),
$$

(2.17)

and by a scaling argument we see that in fact $C$ only depends on $n$, not on $r$.

We denote by $\omega_n$ the measure of the $n$-dimensional unit ball.

**Lemma 2.18.** Suppose $x \in \mathbb{R}^n$, $0 < r < 1$, and $A \subset B(x,r)$. Then we have

$$
\frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B(x,r))} \leq \frac{2C_I \text{Cap}_1(A)}{\omega_n r^{n-1}} \quad \text{and} \quad \text{cap}_1(A, B(x,2r)) \leq C \text{Cap}_1(A),
$$

where $C_I$ is the constant in the relative isoperimetric inequality (2.4), and $C$ is a constant depending only on $n$.

Throughout the paper, we will use $C$ to denote various constants that only depend on $n$ and whose exact value is not important for us. On the other hand, in many cases we track the value of certain constants, such as $2C_I/\omega_n$ above.

**Proof.** For both inequalities, we can assume that $\text{Cap}_1(A) < \infty$. Let $\varepsilon > 0$. We can choose a function $u \in W^{1,1}(\mathbb{R}^n)$ such that $u \geq 1$ in a neighborhood of $A$, and

$$
\|u\|_{W^{1,1}(\mathbb{R}^n)} < \text{Cap}_1(A) + \varepsilon.
$$

By the coarea formula (2.3), we then find $0 < t < 1$ such that $\{u > t\}$ contains a neighborhood of $A$, and

$$
|D\chi_{\{u > t\}}|(\mathbb{R}^n) \leq |Du|(\mathbb{R}^n) \leq \|u\|_{W^{1,1}(\mathbb{R}^n)} < \text{Cap}_1(A) + \varepsilon.
$$

Denote $F := \{u > t\}$.

Case 1: Suppose $\mathcal{L}^n(F \cap B(x,r)) \geq \frac{1}{2}\mathcal{L}^n(B(x,r))$. We find $R \geq r$ such that $\mathcal{L}^n(F \cap B(x,R)) = \frac{1}{2}\mathcal{L}^n(B(x,R))$. By the relative isoperimetric inequality (2.4), we have

$$
\text{Cap}_1(A) + \varepsilon > |D\chi_F|(\mathbb{R}^n) \geq |D\chi_F|(B(x,R)) \geq C_I^{-1}\frac{1}{2}R^{-1}\mathcal{L}^n(B(x,R)) \\
= \frac{\omega_n}{2C_I} R^{n-1} \\
\geq \frac{\omega_n}{2C_I} r^{n-1} \\
\geq \frac{\omega_n}{2C_I} r^{n-1} \frac{\mathcal{L}^n(F \cap B(x,r))}{\mathcal{L}^n(B(x,r))} \\
\geq \frac{\omega_n}{2C_I} r^{n-1} \frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B(x,r))}.
$$

(2.19)
Letting $\varepsilon \to 0$, we get the first result. Defining the cutoff function

$$\eta(y) := \max \left\{ 0, 1 - \frac{1}{r} \operatorname{dist}(y, B(x, r)) \right\}, \quad y \in \mathbb{R}^n,$$

(2.20)

for which $\eta = 1$ in $B(x, r)$ and $\eta = 0$ in $\mathbb{R}^n \setminus B(x, 2r)$, we get

$$\operatorname{cap}_1(A, B(x, 2r)) \leq \int_{\mathbb{R}^n} |\nabla \eta| \, d\mathcal{L}^n \leq \frac{\omega_n (2r)^n}{r} \leq 2^{n+1} C_I (\operatorname{Cap}_1(A) + \varepsilon)$$

by the first three lines of (2.19). Letting $\varepsilon \to 0$, we get the second result with $C = 2^{n+1} C_I$.

Case 2: Suppose $\mathcal{L}^n(F \cap B(x, r)) < \frac{1}{2} \mathcal{L}^n(B(x, r))$. By the relative isoperimetric inequality,

$$\operatorname{Cap}_1(A) + \varepsilon \geq |D\chi_F|([\mathbb{R}^n]) \geq |D\chi_F|(B(x, r))$$

(2.20)

$$\geq \frac{1}{C_I r} \mathcal{L}^n(F \cap B(x, r)) \geq \frac{1}{C_I r} \mathcal{L}^n(A) \geq \frac{\omega_n \mathcal{L}^n(A)}{C_I} \frac{r^{n-1}}{\mathcal{L}^n(B(x, r))}.$$

Letting $\varepsilon \to 0$, we get the first result.

By (2.17), we get

$$|D\chi_{B(x,r)\cap F}|([\mathbb{R}^n]) \leq C |D\chi_F|(B(x, r)) \leq C \operatorname{Cap}_1(A) + C \varepsilon.$$  

(2.21)

By (2.16), we find a sequence $\{u_j\}_{j=1}^\infty$ in $W^{1,1}([\mathbb{R}^n])$ such that $u_j \to \chi_{B(x,r)\cap F}$ in $L^1([\mathbb{R}^n])$, $|Du_j|([\mathbb{R}^n]) \to |D\chi_{B(x,r)\cap F}|([\mathbb{R}^n])$, and $u_j \geq 1$ a.e. in a neighborhood of $A$. Consider the cutoff function $\eta$ from (2.20). We have $u_j \eta \to \chi_{B(x,r)\cap F}$ in $L^1([\mathbb{R}^n])$, $|D(u_j \eta)|([\mathbb{R}^n]) \to |D\chi_{B(x,r)\cap F}|([\mathbb{R}^n])$, and $u_j \eta \geq 1$ a.e. in a neighborhood of $A$. Thus

$$\operatorname{cap}_1(A, B(x, 2r)) \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} |\nabla (u_j \eta)| \, d\mathcal{L}^n = |D\chi_{B(x,r)\cap F}|([\mathbb{R}^n])$$

(2.22)

$$\leq C \operatorname{Cap}_1(A) + C \varepsilon \quad \text{by (2.21)}.$$

Letting $\varepsilon \to 0$, we get the second result. \qed

It is straightforward to show that

$$\operatorname{cap}_1(B(x, r), B(x, 2r)) = s_{n-1} r^{n-1} \leq \operatorname{Cap}_1(B(x, r)),$$  

(2.22)

where $s_{n-1}$ is the $\mathcal{H}^{n-1}$-measure of the unit sphere in $\mathbb{R}^n$. By [9, Proposition 6.16], we know that for a ball $B(x, r)$ and $A \subset B(x, r)$, and for all $1 \leq p < \infty$, we have

$$\frac{\operatorname{Cap}_p(A)}{C'(1 + r^p)} \leq \operatorname{cap}_p(A, B(x, 2r)) \leq 2^p \left(1 + \frac{1}{r^p}\right) \operatorname{Cap}_p(A),$$  

(2.23)

where $C'$ is a constant depending only on $n, p$. It follows that for $A \subset B(x, 1)$ and $1 \leq p < q < \infty$, we have $\operatorname{Cap}_p(A)^{1/p} \leq C'' \operatorname{Cap}_q(A)^{1/q}$ for another constant $C''$ depending only on $n, p, q$, and so using a covering argument, we can show for any function $w$ that

if $w$ is $q$-quasicontinuous on $\Omega$, then $w$ is $p$-quasicontinuous on $\Omega$.  

(2.24)
**Definition 2.25.** We say that $A \subset \mathbb{R}^n$ is $1$-thin at the point $x \in \mathbb{R}^n$ if
\[
\lim_{r \to 0} \frac{\text{Cap}_1(A \cap B(x,r))}{r^{n-1}} = 0.
\]
We also say that a set $U \subset \mathbb{R}^n$ is $1$-finely open if $\mathbb{R}^n \setminus U$ is $1$-thin at every $x \in U$. Then we define the $1$-fine topology as the collection of $1$-finely open sets on $\mathbb{R}^n$.

We denote the $1$-fine interior of a set $H \subset \mathbb{R}^n$, i.e. the largest $1$-finely open set contained in $H$, by $\text{fine-int} H$. We denote the $1$-fine closure of $H$, i.e. the smallest $1$-finely closed set containing $H$, by $\overline{H}$. The $1$-fine boundary of $H$ is $\partial H := \overline{H} \setminus \text{fine-int} H$. The $1$-base $b_{1}H$ is defined as the set of points where $H$ is not $1$-thin.

For a function $w$, we then get the concepts of $1$-fine continuity and $1$-fine limit at a point $x \in \mathbb{R}^n$. The latter is denoted by $\text{fine-lim}_{y \to x} w(y)$ and defined using punctured neighborhoods.

See [37, Section 4] for discussion on Definition 2.25, and for a proof of the fact that the $1$-fine topology is indeed a topology. In fact, in [37] the criterion
\[
\lim_{r \to 0} \frac{\text{cap}_1(A \cap B(x,r), B(x,2r))}{\text{cap}_1(B(x,r), B(x,2r))} = 0
\]
for $1$-thinness was used, in the context of more general metric measure spaces. By the second inequality of Lemma 2.18, the first inequality of (2.23), and (2.22), this is equivalent with our current definition in the Euclidean setting.

We note that the analogous concepts when $1 < p < \infty$ are well known in the literature. In this case, the definition of $p$-thinness is formulated using a suitable integral, and in fact the $p$-fine topology is known to be the coarsest topology that makes all $p$-superharmonic functions continuous. We will not discuss these concepts here, however see e.g. the monographs [9, 46]. The case $p = 1$ has been studied much less, but now we will record certain results that are known in this case.

By [40, Corollary 6.12] we know that for an arbitrary set $U \subset \mathbb{R}^n$,
\[
U \text{ is } 1\text{-quasiopen} \iff U = V \cup N \text{ where } V \text{ is } 1\text{-finely open and } \mathcal{H}^{n-1}(N) = 0. \tag{2.26}
\]
According to [35, Corollary 3.5], the $1$-fine closure of $A \subset \mathbb{R}^n$ can be characterized as:
\[
\overline{A} = A \cup b_1 A. \tag{2.27}
\]
By [35, Proposition 3.3] we know that
\[
\text{Cap}_1(\overline{A}) = \text{Cap}_1(A) \quad \text{for every } A \subset \mathbb{R}^n. \tag{2.28}
\]

**Lemma 2.29.** Let $u \in BV_{\text{loc}}(\Omega)$. For every $t \in \mathbb{R}$, we have
\[
\text{Cap}_1(\{u^\wedge > t\}) = \text{Cap}_1(\{u^\vee > t\}).
\]

**Proof.** Fix $t \in \mathbb{R}$. Obviously we have $\text{Cap}_1(\{u^\wedge > t\}) \leq \text{Cap}_1(\{u^\vee > t\})$, so we need to consider the opposite inequality. There is a $\mathcal{H}^{n-1}$-negligible set $N \subset \Omega$ such that every
$x \in \Omega \setminus N$ is either a Lebesgue point of $u$, or else (2.5) or (2.6) holds. Consider $x \in \Omega \setminus N$ with $u^\vee(x) > t$. Using (2.5), (2.6), we get
\[
\liminf_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \cap \{u^\vee > t\})}{\mathcal{L}^n(B(x, r))} = \liminf_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \cap \{u > t\})}{\mathcal{L}^n(B(x, r))} \geq \frac{1}{2^2}.
\]
By the first part of Lemma 2.18, we get
\[
\liminf_{r \to 0} \frac{\text{Cap}_1(B(x, r) \cap \{u^\vee > t\})}{r^{n-1}} > 0.
\]
Thus $x \notin \overline{\{u^\vee > t\}}^1$, and so $\{u^\vee > t\} \setminus N \subset \overline{\{u^\vee > t\}}^1$. By (2.28), we now get
\[
\text{Cap}_1(\{u^\vee > t\}) = \text{Cap}_1(\overline{\{u^\vee > t\}}^1) \geq \text{Cap}_1(\{u^\vee > t\} \setminus \{u^\vee > t\}) = \text{Cap}_1(\{u^\vee > t\}),
\]
with the constant depending only on $n$. Hence by (2.14) we know that also $\text{Cap}_1(N) = 0$.

**Lemma 2.30.** Let $A_1 \subset A_2 \subset \ldots$ be subsets of $\mathbb{R}^n$, and denote $A := \bigcup_{j=1}^\infty A_j$. Then
\[
\lim_{j \to \infty} \text{Cap}_1(A_j) = \text{Cap}_1(A).
\]

**Proof.** Clearly $\lim_{j \to \infty} \text{Cap}_1(A_j) \leq \text{Cap}_1(A)$. To prove the other direction, we can assume that $\lim_{j \to \infty} \text{Cap}_1(A_j) < \infty$. For a given $\varepsilon > 0$, we find $u \in \text{BV}(\mathbb{R}^n)$ such that $u \geq 1$ in a neighborhood of $A$ and
\[
\|u\|_{\text{BV}(\mathbb{R}^n)} \leq \lim_{j \to \infty} \text{Cap}_1(A_j) + \varepsilon;
\]
see the proof of Theorem 3.4 in [25]. Now (2.16) gives the result. \hfill \Box

For any $u \in W^{1,1}(B(x, r))$ with $0 < r \leq 1$, we have the following Maz’ya’s inequality from [50, Theorem 10.1.2]:
\[
\int_{B(x, r)} |u| \, d\mathcal{L}^n \leq \frac{C_{\text{maz}}}{\text{Cap}_1(B(x, r) \cap \{u^* = 0\})} \int_{B(x, r)} |\nabla u| \, d\mathcal{L}^n \tag{2.31}
\]
for some constant $C_{\text{maz}}$ depending only on $n$.

For BV functions, we have the following version.

**Lemma 2.32.** For any nonnegative $u \in \text{BV}(B(x, r))$ with $0 < r \leq 1$, we have
\[
\int_{B(x, r)} u \, d\mathcal{L}^n \leq \frac{C_{\text{maz}}}{\text{Cap}_1(B(x, r) \cap \{u^\vee = 0\})} |Du|(B(x, r)). \tag{2.33}
\]

**Proof.** From (2.16), we find a sequence $\{u_j\}_{j=1}^\infty$ of functions in $W^{1,1}(B(x, r))$ such that
\[
u_j \to u \quad \text{in } L^1(B(x, r)) \quad \text{and} \quad |Du_j|(B(x, r)) \to |Du|(B(x, r)).
\]
By [39, Theorem 3.2], after passing to a subsequence (not relabeled), we also have that
\[
\limsup_{j \to \infty} u_j^*(y) \leq u^*(y) \quad \text{for every } y \in B(x, r) \setminus N, \quad \text{where } \mathcal{H}^{n-1}(N) = 0.
\]
We can also include in $N$ all points that are not Lebesgue points of some $u_j$; recall the comment after (2.10). Fix $\varepsilon > 0$. Denote
\[
A_j := \{y \in B(x, r) : u_j^*(y) \leq \varepsilon \text{ for all } k \geq j\}, \quad j \in \mathbb{N}.
\]
Then by Lemma 2.30, we have
\[
\liminf_{j \to \infty} \text{Cap}_1(B(x, r) \cap \{u_j^* \leq \varepsilon\}) \geq \liminf_{j \to \infty} \text{Cap}_1(A_j)
\geq \text{Cap}_1(B(x, r) \cap \{u^\vee = 0\} \setminus N)
= \text{Cap}_1(B(x, r) \cap \{u^\vee = 0\}),
\]
using also (2.14). We denote the positive and negative parts of a function by \(v_+ := \max\{v, 0\}\) and \(v_- := -\min\{v, 0\}\). Now we have
\[
\int_{B(x, r)} u \, d\mathcal{L}^n - \varepsilon = \limsup_{j \to \infty} \int_{B(x, r)} (u_j - \varepsilon)_+ \, d\mathcal{L}^n
\leq \limsup_{j \to \infty} \frac{C_{\text{max}}}{\text{Cap}_1(B(x, r) \cap \{u_j^* \leq \varepsilon\})} \int_{B(x, r)} |\nabla u_j| \, d\mathcal{L}^n
\leq \frac{C_{\text{max}}}{\text{Cap}_1(B(x, r) \cap \{u^\vee = 0\})} |Du|(B(x, r)).
\]
Letting \(\varepsilon \to 0\), we get the claim. \(\Box\)

It will be convenient to consider the following class of measures that are, in a suitable sense, absolutely continuous with respect to \(\text{Cap}_1\).

**Definition 2.35.** Given a constant \(C_* \geq 1\), we denote by \(\mathcal{M}(\Omega, C_*)\) the class of those positive Radon measures \(\nu\) on \(\Omega\) for which the following holds:

For every open \(W \subset \Omega\) and every \(\delta > 0\), there exists \(R = R(\nu, W, \delta) > 0\) such that if \(0 < r \leq R\) and \(A \subset \mathbb{R}^n\) is a Borel set with
\[
\frac{\text{Cap}_1(B(x, r) \cap A)}{r^{n-1}} < \delta
\]
for all \(x \in W\), then \(\nu(A \cap W) \leq C_* \delta \nu(W)\).

**Remark 2.36.** As noted previously, the theory of 1-finely open sets that we rely on has been developed in more general metric measure spaces. Throughout the paper, though we work exclusively in the Euclidean setting, we are interested in the possibility of eventually generalizing many of our results to metric measure spaces, and in the possibility of the Euclidean and the metric space theories providing insight into each other. Note that Theorems 1.3 and 1.4 characterize BV and Sobolev functions without any reference to the linear structure of Euclidean space, and so they could be formulated in metric measure spaces, or even taken as a definition of BV or Sobolev functions in such spaces.

Of course, the definition of the 1-capacity needed in the definition of the generalized Lip-

schitz number already relies on Sobolev functions, which is problematic from the viewpoint of characterizing or potentially even defining Sobolev functions using Theorem 1.4. However, the 1-capacity is a very geometric quantity, and could be replaced by the Hausdorff content \(\mathcal{H}_1^\infty\). See e.g. [32, Theorem 3.5] for this type of result.
3. Proof of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4.

3.1. “Only if” direction. In this subsection we prove the “only if” direction of Theorems 1.3 and 1.4, in a slightly stronger form.

First we record the following “Borel regularity”.

**Lemma 3.1.** Let $D \subset \mathbb{R}^n$. Then there exists a Borel set $D^* \supset D$ such that

$$\mathcal{L}^n(D \cap B(x, r)) = \mathcal{L}^n(D^* \cap B(x, r))$$

and

$$\text{Cap}_1(D \cap B(x, r)) = \text{Cap}_1(D^* \cap B(x, r))$$

for all balls $B(x, r) \subset \mathbb{R}^n$.

**Proof.** See the proof of [40, Lemma 4.3]. □

Then we consider a few basic lemmas concerning rectifiable sets. In these lemmas (until and including Lemma 3.4), we assume $n \geq 2$. Recall from (2.12) that $\pi_n$ denotes the orthogonal projection onto $\mathbb{R}^{n-1}$.

**Lemma 3.2.** Let $S \subset \mathbb{R}^n$. Then we have

$$2\mathcal{H}^{n-1}(\pi_n(S)) \leq \text{Cap}_1(S).$$

**Proof.** We can assume that the right-hand side is finite. Consider $u \in W^{1,1}(\mathbb{R}^n)$ with $u \geq 1$ in a neighborhood of $S$. For $z \in \mathbb{R}^{n-1}$, denote by $l_z$ the line in the $n$:th coordinate direction intersecting $(z,0)$. By the known behavior of Sobolev functions on lines, see e.g. [16, Theorem 4.21], for $\mathcal{L}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$ we have

$$\int_{l_z} \left| \frac{\partial u^*}{\partial x_n} \right| ds \geq 2\chi_{\pi_n}(z).$$

Integrating over $\mathbb{R}^{n-1}$, we get

$$\int_{\mathbb{R}^n} |\nabla u| d\mathcal{L}^n \geq 2\mathcal{H}^{n-1}(\pi_n(S)).$$

Thus $\|u\|_{W^{1,1}(\mathbb{R}^n)} \geq 2\mathcal{H}^{n-1}(\pi_n(S))$, and by taking the infimum over all such $u$, we get the result. □

**Lemma 3.3.** Let $D \subset \mathbb{R}^{n-1}$, let $h: D \to \mathbb{R}$ be 1-Lipschitz, and let $S \subset \mathbb{R}^n$ be the graph of $h$, that is, $S = \{x = (x',t) \in D \times \mathbb{R}: t = h(x')\}$. Then

$$\mathcal{H}^{n-1}(S) \leq (2\sqrt{2})^{n-1} \text{Cap}_1(S).$$

**Proof.** By Lemma 3.2, we have $2\mathcal{H}^{n-1}(D) \leq \text{Cap}_1(S)$. On the other hand, since $h$ is 1-Lipschitz, it is easy to check from the definition of Hausdorff measures that $\mathcal{H}^{n-1}(S) \leq (2\sqrt{2})^{n-1}\mathcal{H}^{n-1}(D)$, and so we get the result. □
We say that a set $N \subset \mathbb{R}^n$ is purely $\mathcal{H}^{n-1}$-unrectifiable if $\mathcal{H}^{n-1}(N \cap S) = 0$ for every countably $\mathcal{H}^{n-1}$-rectifiable set $S \subset \mathbb{R}^n$.

**Lemma 3.4.** Suppose $N \subset \mathbb{R}^n$ is a Borel set such that
\[
\liminf_{r \to 0} \frac{\text{Cap}_1(N \cap B(x,r))}{r^{n-1}} < \frac{\omega_{n-1}}{(2\sqrt{2})^{n-1}}.
\]
for every $x \in N$. Then $N$ is purely unrectifiable.

**Proof.** Consider a countably $\mathcal{H}^{n-1}$-rectifiable set $S \subset \mathbb{R}^n$. We can assume that $S$ is a 1-Lipschitz $n-1$-graph
\[ S = \{ x = (x', t) \in D \times \mathbb{R} : t = h(x') \}, \]
where $D \subset \mathbb{R}^{n-1}$. By Lemma 3.3, we get
\[
\liminf_{r \to 0} \frac{\mathcal{H}^{n-1}(N \cap S \cap B(x,r))}{r^{n-1}} \leq (2\sqrt{2})^{n-1} \liminf_{r \to 0} \frac{\text{Cap}_1(N \cap S \cap B(x,r))}{r^{n-1}} < \omega_{n-1}
\]
for every $x \in N \cap S$. On the other hand, $N \cap S$ is also countably $\mathcal{H}^{n-1}$-rectifiable, and so by [4, Theorem 2.83] we know that
\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(N \cap S \cap B(x,r))}{\omega_{n-1}r^{n-1}} = 1
\]
for $\mathcal{H}^{n-1}$-a.e. $x \in N \cap S$. Thus necessarily $\mathcal{H}^{n-1}(N \cap S) = 0$. \qed

Recall that we always assume $\Omega \subset \mathbb{R}^n$ to be an open set. Recall also the definition of the class of Radon measures $\mathcal{M}(\Omega, C_*)$ from Definition 2.35. In the following two lemmas we show that this class is quite broad.

**Lemma 3.5.** Let $S \subset \Omega$ be a countably $\mathcal{H}^{n-1}$-rectifiable set with $\mathcal{H}^{n-1}(S) < \infty$, and let $u \in \text{BV}(\Omega)$. Then $\mathcal{H}^{n-1}[S]$, $|Du|$, and $\mathcal{L}^n$ all belong to
\[ \mathcal{M} \left( \Omega, \frac{80^n C_I}{\min \{\omega_{n-1}, \omega_n\}} \right) \subset \mathcal{M}(\Omega, 2^{-5n-1}c(n)^{-1}). \]

The inclusion above follows simply from the definition of the number $c(n)$ in (1.2).

**Proof.** In the case $n = 1$, we have $\text{Cap}_1(\{y\}) = 2$ for every $y \in \mathbb{R}$, and so it is obvious that every positive Radon measure belongs to $\mathcal{M}(\Omega, 1)$. Thus we can assume $n \geq 2$.

**Part 1:** $\mathcal{H}^{n-1}[S]$.

First we consider the countably $\mathcal{H}^{n-1}$-rectifiable set $S \subset \Omega$, with $\mathcal{H}^{n-1}(S) < \infty$. Fix $\delta > 0$ and an open set $W \subset \Omega$. We have that $S \cap W = \bigcup_{j=1}^{\infty} S_j \cup N$, where the $S_j$'s are disjoint $\mathcal{H}^{n-1}$-measurable 1-Lipschitz $n-1$-graphs and $\mathcal{H}^{n-1}(N) = 0$. Fix $j \in \mathbb{N}$. We can assume that
\[ S_j = \{ x = (x', t) \in D_j \times \mathbb{R} : t = h(x') \}, \]
where $D_j \subset \mathbb{R}^{n-1}$. 
By e.g. \[4, \text{Theorem 2.83}\], we have
\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap S_j)}{\omega_{n-1} r^{n-1}} = 1
\]
for \(\mathcal{H}^{n-1}\)-a.e. \(x \in S_j\). For such \(x\), we find arbitrarily small \(r > 0\) such that
\[
\frac{\mathcal{H}^{n-1}(B(x, r) \cap S_j)}{\omega_{n-1} r^{n-1}} \geq \frac{1}{2} \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B(x, r) \cap S_j) = 0.
\]
By the Vitali covering theorem, see e.g. \[4, \text{Theorem 2.19}\], we find an at most countable collection of pairwise disjoint balls \(B(x_l, r_l)\) with
\[
\frac{\mathcal{H}^{n-1}(B(x_l, r_l) \cap S_j)}{\omega_{n-1} r_l^{n-1}} \geq \frac{1}{2} \quad \text{and} \quad \mathcal{H}^{n-1}(\partial B(x_l, r_l) \cap S_j) = 0,
\]
covering \(\mathcal{H}^{n-1}\)-all of \(S_j\). Then the open balls \(B(x_l, r_l)\) also cover \(\mathcal{H}^{n-1}\)-all of \(S_j\).

Thus we can in fact further assume that \(D_j \subseteq B(z_j, s_j)\) for some \(z_j \in \mathbb{R}^{n-1}\) and \(s_j > 0\), with
\[
\frac{\mathcal{H}^{n-1}(S_j)}{\omega_{n-1} s_j^{n-1}} \geq \frac{1}{2}. \tag{3.6}
\]
Then consider \(0 < r \leq s_j\), and a Borel set \(A \subseteq \mathbb{R}^n\) such that
\[
\frac{\operatorname{Cap}_1(B(x, r) \cap A)}{r^{n-1}} < \delta \quad \text{for all} \ x \in W. \tag{3.7}
\]
By Lemma 3.3, we have
\[
\mathcal{H}^{n-1}(B(x, r) \cap S_j \cap A) \leq (2\sqrt{2})^{n-1} \operatorname{Cap}_1(B(x, r) \cap A \cap S_j). \tag{3.8}
\]
Consider the covering \(\{B(x, r/5)\}_{x \in S_j \cap A}\). By the 5-covering theorem (see e.g. \[16, \text{Theorem 1.24}\]), we can pick an at most countable collection of pairwise disjoint balls \(B(x_k, r/5)\) such that the balls \(B(x_k, r)\) cover \(S_j \cap A\). Recall from (2.12) that \(\pi_n\) denotes the orthogonal projection onto \(\mathbb{R}^{n-1}\). Since \(h\) is 1-Lipschitz, the \(n - 1\)-dimensional balls
\[
B_{n-1}(\pi_n(x_k), r/(5\sqrt{2})) \quad \text{are disjoint and contained in the ball} \ B_{n-1}(z_j, 2s_j). \tag{3.9}
\]
Thus
\[
\mathcal{H}^{n-1}(S_j \cap A) \leq \sum_k \mathcal{H}^{n-1}(S_j \cap A \cap B(x_k, r))
\leq (2\sqrt{2})^{n-1} \sum_k \operatorname{Cap}_1(S_j \cap A \cap B(x_k, r)) \quad \text{by (3.8)}
\leq (2\sqrt{2})^{n-1} \delta \sum_k r^{n-1} \quad \text{by (3.7)} \tag{3.10}
\leq 20^{n-1} \delta \sum_k (r/(5\sqrt{2}))^{n-1}
\leq 20^{n-1} \delta (2s_j)^{n-1} \quad \text{by (3.9)}
\leq 40^n \omega_{n-1} \delta \mathcal{H}^{n-1}(S_j) \quad \text{by (3.6)}.
Now we return to the entire set $S$. We can assume that $\mathcal{H}^{n-1}(S \cap W) > 0$; choose $N \in \mathbb{N}$ such that
\[ \sum_{j=N+1}^{\infty} \mathcal{H}^{n-1}(S_j) < 40^n \omega_{n-1}^{-1} \delta \mathcal{H}^{n-1}(S \cap W). \]
Now for $0 < r \leq \min\{s_1, \ldots, s_N\}$ and a Borel set $A \subset \mathbb{R}^n$ such that
\[ \frac{\text{Cap}_1(B(x,r) \cap A)}{r^{n-1}} < \delta \quad \text{for all } x \in W, \]
we have
\[ \mathcal{H}^{n-1}(S \cap A) \leq \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(S_j \cap A) \]
\[ \leq 40^n \omega_{n-1}^{-1} \delta \sum_{j=1}^{N} \mathcal{H}^{n-1}(S_j) + \sum_{j=N+1}^{\infty} \mathcal{H}^{n-1}(S_j) \quad \text{by (3.10)} \]
\[ \leq 2 \times 40^n \omega_{n-1}^{-1} \delta \mathcal{H}^{n-1}(S \cap W). \]

We conclude that $\mathcal{H}^{n-1}(S \in M(\Omega, 2 \times 40^n \omega_{n-1}^{-1}))$, proving the claim.

**Part 2: $|Du|$**

Fix $\delta > 0$ and an open set $W \subset \Omega$. For each $r > 0$, consider a Borel set $A_r \subset \mathbb{R}^n$ such that
\[ \frac{\text{Cap}_1(B(x,r) \cap A_r)}{r^{n-1}} < \delta \quad \text{for all } x \in W. \]
Assume by contradiction that there is a choice of these Borel set $A_r$ and a sequence $r_j \to 0$ such that
\[ |Du|(A_{r_j} \cap W) > \frac{80^n C_I}{\min\{\omega_{n-1}, \omega_n\}} \delta |Du|(W). \quad (3.11) \]

By [4, Theorem 3.59, Theorem 3.61] we know that for any set $E \subset \mathbb{R}^n$ with $\text{Var}(\chi_E, \Omega) < \infty$, we have $|D\chi_E| = \mathcal{H}^{n-1}\big|_{\partial^* E \cap \Omega}$ in $\Omega$, and that $\partial^* E \cap \Omega$ is an $\mathcal{H}^{n-1}$-rectifiable set with finite $\mathcal{H}^{n-1}$-measure; see the definition of the measure-theoretic boundary below (3.49). Thus by the coarea formula (2.3), we get
\[ |Du|(A_{r_j} \cap W) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap A_{r_j} \cap W) \, dt. \]

By Part 1, we know that for a.e. $t \in \mathbb{R}$,
\[ \limsup_{j \to \infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap A_{r_j} \cap W) \leq 2 \times 40^n \omega_{n-1}^{-1} \delta \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap W). \quad (3.12) \]
It follows that
\[
\limsup_{j \to \infty} |Du|(A_{r_j} \cap W) = \limsup_{j \to \infty} \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap A_{r_j} \cap W) \, dt \\
\leq 2 \times 40^n \omega_{n-1}^{-1} \delta \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial^* \{u > t\} \cap W) \, dt \quad \text{by (3.12)} \\
= 2 \times 40^n \omega_{n-1}^{-1} \delta |Du|(W).
\]

This contradicts (3.11), and so in fact we have \( \mathcal{H}^{n-1}(S \in \mathcal{M}(\Omega, \frac{80^n C_I}{\min(\omega_{n-1}, \omega_n)}) \).

**Part 3: \( L^n \).**

Fix \( \delta > 0 \) and an open set \( W \subset \Omega \). We can assume that \( L^n(W) < \infty \). Define
\[
W_R := \{ x \in W : \text{dist}(x, \mathbb{R}^n \setminus W) > R \}.
\]

For sufficiently small \( R > 0 \), we have \( L^n(W \setminus W_R) \leq \frac{2C_I}{\omega_n} \delta L^n(W) \). Consider \( 0 < r \leq R \) and a Borel set \( A \subset \mathbb{R}^n \) such that
\[
\frac{\text{Cap}_1(A \cap B(x, r))}{r^{n-1}} < \delta \quad \text{for all } x \in W.
\]

By Lemma 2.18, we also have
\[
\frac{L^n(A \cap B(x, r))}{L^n(B(x, r))} \leq \frac{2C_I}{\omega_n} \delta \quad \text{for all } x \in W.
\]

Consider the covering \( \{ B(x, r/5) \}_{x \in W_R} \). By the 5-covering theorem (see e.g. [16, Theorem 1.24]), we can pick an at most countable collection of pairwise disjoint balls \( B(x_k, r/5) \) such that the balls \( B(x_k, r) \) cover \( W_R \). Thus
\[
L^n(A \cap W_R) \leq \sum_k L^n(A \cap B(x_k, r)) \\
\leq \frac{2C_I}{\omega_n} \delta \sum_k L^n(B(x_k, r)) \\
\leq 5^n \frac{2C_I}{\omega_n} \delta \sum_k L^n(B(x_k, r/5)) \\
\leq 5^n \frac{2C_I}{\omega_n} \delta L^n(W).
\]

In total, we get
\[
L^n(A \cap W) \leq L^n(A \cap W_R) + L^n(W \setminus W_R) \leq 5^n \frac{4C_I}{\omega_n} \delta L^n(W).
\]

**Lemma 3.13.** Let \( C_* \geq 1 \) and suppose \( \nu \in \mathcal{M}(\Omega, C_*) \), and that \( h \in L^1(\Omega, \nu) \) is nonnegative. Then \( h \, d\nu \in \mathcal{M}(\Omega, 2C_*) \).
Proof. Fix $\delta > 0$ and an open set $W \subset \Omega$. We can assume that $\int_W h \, d\nu > 0$. Choose a simple nonnegative function

$$a = \sum_{k=1}^N \chi_{U_k}$$

such that $\|a - h\|_{L^1(W, \nu)} < (\delta/2) \int_W h \, d\nu$. By the Borel regularity of $\nu$, we can assume that the sets $U_k \subset W$ are open. Choose $R = \min_{k=1, \ldots, N} R(\nu, U_k, \delta)$, and consider $0 < r \leq R$ and a Borel set $A \subset \mathbb{R}^n$ such that

$$\frac{\text{Cap}_1(B(x, r) \cap A)}{r^{n-1}} < \delta \quad \text{for all } x \in W.$$

We get

$$\int_{A \cap W} h \, d\nu - (\delta/2) \int_W h \, d\nu \leq \int_{A \cap W} a \, d\nu = \sum_{k=1}^N \int_{A \cap W} \chi_{U_k} \, d\nu$$

$$= \sum_{k=1}^N \nu(A \cap U_k)$$

$$\leq C\delta \sum_{k=1}^N \nu(U_k)$$

$$= C\delta \int_W a \, d\nu$$

$$\leq C\delta \int_W h \, d\nu + (\delta/2) \int_W h \, d\nu.$$

Thus

$$\int_{A \cap W} h \, d\nu \leq 2C\delta \int_W h \, d\nu.$$

□

The Hardy–Littlewood maximal function of a function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is defined by

$$M_u(x) := \sup_{r>0} \int_{B(x, r)} |u| \, d\mathcal{L}^n, \quad x \in \mathbb{R}^n. \quad (3.14)$$

We also define a restricted version $M_{Ru}(x)$, with $R > 0$, by requiring $0 < r \leq R$ in the supremum.

The following weak-type estimate is standard, see e.g. [16, Theorem 4.18]; in this reference a slightly different definition for capacity is used, but a small modification of the proof gives the following result.

Lemma 3.15. Let $u \in \text{BV}(\mathbb{R}^n)$. Then for some constant $C$ depending only on $n$, we have

$$\text{Cap}_1(\{M_u > t\}) \leq C \frac{\|u\|_{\text{BV}(\mathbb{R}^n)}}{t} \quad \text{for all } t > 0.$$

Now we prove the main result of this subsection.
Proposition 3.16. Suppose $u \in BV_{\text{loc}}(\Omega)$, and let $x \in \Omega$ and $0 < \delta < s_{n-1}$. Then we have
\[ \text{Lip}_{u}^{Du|\delta}(x) \leq C\delta^{-1} \]
for a constant $C$ depending only on $n$.

In the definition of the generalized Lipschitz number (1.1), we consider the quantity
\[ \frac{\text{osc}_{B(x,r)} w}{r} \frac{\mathcal{L}^n(B(x,r))}{\nu(B(x,r))} \]
to be zero if $\text{osc}_{B(x,r)} w = 0$, and $\infty$ if only $\nu(B(x,r))$ is zero.

Proof. Consider $0 < r \leq 1$ sufficiently small such that $\overline{B}(x,r) \subset \Omega$. Let $M \in \mathbb{R}$ be the median of $u$ in $B(x,r)$, that is,
\[ \frac{\mathcal{L}^n(B(x,r) \cap \{u \geq M\})}{\mathcal{L}^n(B(x,r))} \geq \frac{1}{2} \quad \text{and} \quad \frac{\mathcal{L}^n(B(x,r) \cap \{u \leq M\})}{\mathcal{L}^n(B(x,r))} \geq \frac{1}{2}. \]
There is a $\mathcal{H}^{n-1}$-negligible and thus by (2.14) also $\text{Cap}_1$-negligible set $N \subset \Omega$ such that every $y \in \Omega \setminus N$ is either a Lebesgue point of $u$, or else (2.5) or (2.6) holds. It is easy to see that for all $y \in \Omega \setminus N$, we have
\[ u^\wedge(y) \leq u^*(y) \leq u^\vee(y) \quad \text{and} \quad ((u - M)^+)^*(y) \geq \frac{1}{2}((u - M)^+)^\vee(y). \quad (3.17) \]
Let $v \in BV(\mathbb{R}^n)$ be an extension of $(u - M)^+$ from $B(x,r)$ to the whole space (see e.g. [4, Proposition 3.21]). We clearly have $Mv \geq v^*$ everywhere, and then by (3.17) in fact
\[ Mv \geq v^* = ((u - M)^+)^* \geq \frac{1}{2}((u - M)^+)^\vee = \frac{1}{2}(u^\vee - M)^+ \quad \text{in } B(x,r) \setminus N. \]
By Lemma 3.15, we now see that
\[ \lim_{t \to \infty} \text{Cap}_1(\{(u^\vee - M)^+ \geq t\} \cap B(x,r)) = 0. \]
Thus we can choose $t_+ \in [0, \infty)$ such that
\[ \text{Cap}_1(\{(u^\vee - M)^+ \geq t\} \cap B(x,r)) \leq \frac{\delta}{2} r^{n-1} \quad \text{for all } t > t_+, \]
and (recall (2.22))
\[ \text{Cap}_1(\{(u^\vee - M)^+ \geq t\} \cap B(x,r)) \geq \frac{\delta}{2} r^{n-1} \quad \text{for all } t < t_+. \]
By Lemma 2.30, we also have
\[ \text{Cap}_1(\{(u^\vee - M)^+ > t_+\} \cap B(x,r)) \leq \frac{\delta}{2} r^{n-1}, \quad (3.18) \]
and by Lemma 2.29, we also have
\[ \text{Cap}_1(\{(u^\wedge - M)^+ > t\} \cap B(x,r)) \geq \frac{\delta}{2} r^{n-1} \quad \text{for all } t < t_+. \quad (3.19) \]
By the fact that $M$ is the median, for any $t \geq 0$ we have
\[
\int_{B(x,r)} (t - \min\{(u - M)_+, t\}) d\mathcal{L}^n \geq t - \int_{B(x,r)} \min\{(u - M)_+, t\} d\mathcal{L}^n \geq \frac{t}{2}.
\] (3.20)

Note that for every $t \in \mathbb{R}$,
\[
(t - \min\{(u - M)_+, t\})^{\ast} = t - \min\{(u^{\ast} - M)_+, t\}. \tag{3.21}
\]

By (3.20), we get
\[
\frac{t_+}{2} \leq \int_{B(x,r)} (t_+ - \min\{(u - M)_+, t_+\}) d\mathcal{L}^n
\]
\[
= \lim_{t \searrow t_+} \int_{B(x,r)} (t - \min\{(u - M)_+, t\}) d\mathcal{L}^n
\]
\[
\leq \limsup_{t \searrow t_+} \frac{C_{\text{max}}}{C_{\text{Cap}}(\{(u^{\ast} - M)_+ \geq t\} \cap B(x, r))} |D(u - M)_+|(B(x, r)) \quad \text{by (2.33), (3.21)}
\]
\[
\leq \frac{2C_{\text{max}}}{\delta^{n-1}} |D(u - M)_+|(B(x, r)) \quad \text{by (3.19)}.
\]

Denote $V_+ := B(x, r) \setminus \{(u^{\ast} - M)_+ > t_+\} \cup N$. Now
\[
(u^{\ast} - M)_+ \leq t_+ \quad \text{in } V_+ \quad \text{and} \quad |D(u - M)_+|(B(x, r)) \geq \frac{\delta}{4C_{\text{max}}} t_+ r^{n-1},
\]
and by (3.18), also
\[
\text{Cap}_1(B(x, r) \setminus V_+) \leq \frac{\delta}{2} r^{n-1}.
\]

We can perform the analogous reasoning with $(u^{\ast} - M)_-$. Thus we find a number $t_- \in [0, \infty)$ such that the set $V_- := B(x, r) \setminus \{(u^{\ast} - M)_- > t_-\} \cup N$ satisfies
\[
(u^{\ast} - M)_- \leq t_- \quad \text{in } V_- \quad \text{and} \quad |D(u - M)_-|(B(x, r)) \geq \frac{\delta}{4C_{\text{max}}} t_- r^{n-1},
\]
and
\[
\text{Cap}_1(B(x, r) \setminus V_-) \leq \frac{\delta}{2} r^{n-1}.
\]

Define $U_r := V_+ \cap V_-$. Then by (3.17),
\[
(u^{\ast} - M)_+ = (u^{\ast} - M)_+ \leq t_+ \quad \text{and} \quad (u^{\ast} - M)_- \leq (u^{\ast} - M)_- \leq t_- \quad \text{in } U_r,
\]
and so
\[
\text{osc}_{U_r} u^{\ast} \leq t_+ + t_-.
\]

Using the coarea formula (2.3), we see that
\[
|Du|(B(x, r)) = |D(u - M)_+|(B(x, r)) + |D(u - M)_-|(B(x, r)) \geq \frac{\delta}{4C_{\text{max}}} (t_+ + t_-) r^{n-1},
\]
and we also have
\[
\text{Cap}_1(B(x, r) \setminus U_r) \leq \text{Cap}_1(B(x, r) \setminus V_+) + \text{Cap}_1(B(x, r) \setminus V_-) \leq \delta r^{n-1}.
\]
If \(t_+ + t_- > 0\), then we obtain
\[
\frac{\text{osc}_{U_r} u^* \mathcal{L}^n(B(x, r))}{r} \leq \frac{t_+ + t_-}{4C_{\text{max}} \omega_n} \frac{4C_{\text{max}} \omega_n}{\delta} r(t_+ + t_-)^{-1} = C \delta^{-1}
\]
with \(C = 4C_{\text{max}} \omega_n\). If \(t_+ + t_- = 0\), then \(\text{osc}_{U_r} u^* = 0\) and recall that we interpret the left-hand side of (3.22) to be zero. Since we can do this for all sufficiently small \(r > 0\), we get
\[
\text{Lip}_{u^*}^{\mathcal{L}^n}(x) \leq C \delta^{-1}.
\]
\(\square\)

**Proof of (1) \(\Rightarrow\) (2) in Theorem 1.3.** By assumption \(u \in \text{BV}(\Omega)\), and now by Proposition 3.16 with the choice \(\delta = c(n) < s_{n-1}\), for every \(x \in \Omega\) we have
\[
\text{Lip}_{u^*}^{\mathcal{L}^n,c(n)}(x) \leq Cc(n)^{-1}.
\]
Then clearly
\[
\text{Lip}_{u^*}^{\mathcal{L}^n,c(n)}(x) \leq 1
\]
with the choice \(\nu = Cc(n)^{-1}|Du|\), which belongs to \(\mathcal{M}(\Omega, 2^{-5n}c(n)^{-1})\) by Lemma 3.5. \(\square\)

**Proof of (1) \(\Rightarrow\) (2) in Theorem 1.4.** By assumption \(u \in W^{1,p}(\Omega)\), and now by Proposition 3.16 with the choice \(\delta = c(n) < s_{n-1}\), for every \(x \in \Omega\) we have
\[
\text{Lip}_{u^*}^{\mathcal{L}^n,c(n)}(x) \leq Cc(n)^{-1}.
\]
Then clearly
\[
\text{Lip}_{u^*}^{\mathcal{L}^n,c(n)}(x) \leq 1
\]
with the choice \(a = Cc(n)^{-1}|\nabla u| \in L^p(\Omega)\). \(\square\)

### 3.2. “If” direction.

In this subsection we prove the “if” direction of Theorems 1.3 and 1.4, again in a slightly stronger form.

The following lemma will be used to analyze \(u^*\) on lines. Note that if \(|w(x)| = \infty\) for some \(x \in W \subset \mathbb{R}\), we interpret \(\text{osc}_W w = \infty\).

**Lemma 3.23.** Let \(w: [a, b] \to [-\infty, \infty]\) for some finite interval \([a, b] \subset \mathbb{R}\). Suppose there is an at most countable set \(E \subset [a, b]\) such that \(w\) is finite and continuous at every point \(x \in E\). Suppose also that there is a sequence of at most countable unions of sets \(W_j = \bigcup_k U_{j,k}\), \(j \in \mathbb{N}\), where each \(U_{j,k} \subset \mathbb{R}\) is open and bounded, and such that \(\chi_{W_j}(x) \to 1\) as \(j \to \infty\) for all \(x \in [a, b] \setminus E\). Then
\[
|w(a) - w(b)| \leq \lim_{j \to \infty} \inf \sum_k \text{osc}_{U_{j,k}} w.
\]

By a slight abuse of notation, we write \(\text{osc}_{U_{j,k}} w\) for \(\text{osc}_{U_{j,k} \cap [a,b]} w\).
Proof. First assume that \( w \) is bounded. Then, without loss of generality, we can also assume that \( w(a) < w(b) \). Define
\[
h(x) := \sup_{a \leq t \leq x} w(t), \quad a \leq x \leq b,
\]
and let \( h(x) := h(a) \) for \( x \leq a \) and \( h(x) := h(b) \) for \( x \geq b \). Now \( h \) is an increasing function, and so \( h \in \text{BV}_{\text{loc}}(\mathbb{R}) \). Consider a bounded open set \( U \subset \mathbb{R} \). We can represent \( U \) as a disjoint union of open intervals \( U = \bigcup_{l=1}^{\infty} U_l \) with \( U_l = (a_l, b_l) \). Since \( h \) is increasing, in each interval we can consider the one-sided limits \( h(a_l^+) \) and \( h(b_l^-) \). Moreover, in each interval \( U_l \), define the truncation
\[
w_l := \min\{h(b_l^-), \max\{h(a_l^+), w\}\}.
\]
Since the intervals \( (h(a_l^+), h(b_l^-)) \) are disjoint, we have
\[
\sum_{l=1}^{\infty} \text{osc} w_l \leq \text{osc}_U w.
\]
Since \( h \) is increasing, for every interval \( U_l \) we clearly have
\[
|Dh|(U_l) = h(b_l^-) - h(a_l^+).
\]
On the other hand, we also have (in fact equality holds)
\[
\text{osc} w_l \geq h(b_l^-) - h(a_l^+),
\]
because \( \inf_{U_l} w \leq h(a_l^+) \), and either \( h(b_l^-) = h(a_l^+) \) or \( h(b_l^-) = \sup\{w(y) : y \in U_l\} \).
Then since \( |Dh| \) is a Radon measure,
\[
|Dh|(U) = \sum_{l=1}^{\infty} |Dh|(U_l) \leq \sum_{l=1}^{\infty} \text{osc} w_l \leq \text{osc}_U w. \tag{3.24}
\]
Fix \( \varepsilon > 0 \). We have \( E = \{x_j\}_{j=1}^{\infty} \) with \( x_j \in (a, b) \), and by the continuity, for every \( j \in \mathbb{N} \) we can choose \( r_j > 0 \) such that
\[
|Dh|(B(x_j, r_j)) = \text{osc}_{B(x_j, r_j)} h < 2^{-j} \varepsilon.
\]
Thus
\[
|Dh|(E) < \varepsilon. \tag{3.25}
\]
Noting that \( w(a) = h(a) \), and then using basic properties of the Radon measure \(|Dh|\), we estimate

\[
|w(b) - w(a)| \leq h(b) - h(a) \\
= |Dh([a, b])| \\
= \left[ |Dh([a, b] \setminus E) + |Dh(E)| \right] \\
\leq \liminf_{j \to \infty} \left[ \sum_{k} |Dh(U_{j,k} \cap [a, b]) + |Dh(E)| \right] \\
\leq \liminf_{j \to \infty} \sum_{k} \text{osc } w_{M} \leq \liminf_{j \to \infty} \sum_{k} \text{osc } w \\
\text{by (3.24), (3.25).}
\]

Since \( \varepsilon > 0 \) was arbitrary, we get the result.

In the general case, note that if \( |w(a)| = \infty \), then necessarily \( a \notin E \) and then the sum on the right-hand side will be \( \infty \) for all \( j \in \mathbb{N} \) sufficiently large that \( a \in W_{j} \). The same applies to the point \( b \). If \( w(a) \) and \( w(b) \) are both finite, then we consider the truncations \( w_{M} := \min \{ M, \max \{ -M, w \} \} \). For sufficiently large \( M \), we have

\[
|w(a) - w(b)| = |w_{M}(a) - w_{M}(b)| \leq \liminf_{j \to \infty} \sum_{k} \text{osc } w_{M} \leq \liminf_{j \to \infty} \sum_{k} \text{osc } w.
\]

**Lemma 3.26.** Let \( U \subset \mathbb{R}^{n} \) be a 1-finely open set. Then for almost every line \( L \) in the direction of a coordinate axis, the set \( L \cap U \) is a relatively open subset of \( L \).

**Proof.** In the case \( n = 1 \), we have \( \text{Cap}_{1}(\{ y \}) = 2 \) for every \( y \in \mathbb{R} \), and so 1-finely open sets and open sets coincide. Thus the result is clear, and we can assume \( n \geq 2 \).

By (2.26), \( U \) is also 1-quasifinely open. Thus, given \( \varepsilon > 0 \) we find an open set \( G \subset \mathbb{R}^{n} \) such that \( \text{Cap}_{1}(G) < \varepsilon \) and \( U \cup G \) is open. It is enough to consider lines \( L \) in the \( n \)th coordinate direction. By Lemma 3.2, we have that the orthogonal projection \( \pi_{n}(G) \) to \( \mathbb{R}^{n-1} \) has \( \mathcal{H}^{n-1} \)-measure less than \( \varepsilon \). For a line \( L \) passing through \( (z, 0) \) for any \( z \in \mathbb{R}^{n-1} \setminus \pi_{n}(G) \), the set \( L \cap (U \cup G) = L \cap U \) is relatively open. Since \( \varepsilon \) can be chosen arbitrarily small, the result follows.

We will use Whitney-type coverings consisting of balls.

**Lemma 3.27.** Let \( A \subset W \), where \( W \subset \mathbb{R}^{n} \) is an open set. Given a scale \( 0 < R < \infty \), there exists a finite or countable Whitney-type covering \( \{ B_{l} = B(x_{l}, r_{l}) \}_{l} \) of \( A \) in \( W \), with \( x_{l} \in A, r_{l} \leq R \), and the following properties:

1. \( B_{l} \subset W \) and \( A \subset \bigcup_{l} \frac{1}{2} B_{l} \), and the balls \( \frac{1}{20} B_{l} \) are disjoint;
2. If \( 2B_{l} \cap 2B_{m} \neq \emptyset \), then \( r_{l} \leq 2r_{m} \);
3. \( \sum_{l} \chi_{2B_{l}}(x) \leq 280^{n} \) for all \( x \in W \);
(4) If \( y \in \frac{1}{2}B_l \) for some \( l \), then
\[
\frac{r_l}{16} \leq \min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\} \leq \frac{r_l}{4},
\]
(3.28)

If \( y \in W \) is in no ball \( \frac{1}{2}B_l \), then
\[
B(y, \min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\}) \cap A = \emptyset.
\]

Proof. For every \( x \in A \), let \( r_x := \min\{R, \frac{1}{8}\text{dist}(x, \mathbb{R}^n \setminus W)\} \). Consider the covering \( \{B(x, \frac{1}{20}r_x)\}_{x \in A} \). By the 5-covering theorem (see e.g. [16, Theorem 1.24]), we can pick an at most countable collection of pairwise disjoint balls \( B(x_l, \frac{1}{4}r_l) \) such that the balls \( B(x_l, \frac{1}{4}r_l) \) cover \( A \). Denote \( B_l = B(x_l, r_l) \). We have established property (1).

Suppose \( 2B_l \cap 2B_m \neq \emptyset \). If \( r_m = \frac{1}{8}\text{dist}(x_m, \mathbb{R}^n \setminus W) \), then
\[
8r_m = \text{dist}(x_m, \mathbb{R}^n \setminus W) \geq \text{dist}(x_l, \mathbb{R}^n \setminus W) - 2r_m - 2r_l \geq 8r_l - 2r_m - 2r_l = 6r_l - 2r_m,
\]
and so we get \( 2r_m \geq r_l \). If \( r_m = R \), then \( r_l \leq R = r_m \). Thus we get property (2).

If \( x \in 2B_l \) for some \( l \), denote by \( I \) the set of those indices \( m \in I \) such that \( x \in 2B_m \). For all \( m \in I \), by (2) we have \( \frac{1}{20}B_m \subset 7B_l \), and so
\[
\sum_{m \in I} 40^{-n}\omega_n r_m^n \leq \sum_{m \in I} 20^{-n}\omega_n r_m^n = \sum_{m \in I} \mathcal{L}^n(\frac{1}{20}B_m) \leq \mathcal{L}^n(7B_l) = 7^n\omega_n r_l^n,
\]
and so the cardinality of \( I \) is at most \( 280^n \), and we obtain (3).

To prove the first part of (4), suppose \( y \in \frac{1}{2}B_l \) for some \( l \). Recall that
\[
r_l = \min\{R, \frac{1}{8}\text{dist}(x_l, \mathbb{R}^n \setminus W)\}.
\]
We estimate
\[
\text{dist}(y, \mathbb{R}^n \setminus W) \geq \text{dist}(x_l, \mathbb{R}^n \setminus W) - r_l \geq \text{dist}(x_l, \mathbb{R}^n \setminus W) - \text{dist}(x_l, \mathbb{R}^n \setminus W)/8 \geq \text{dist}(x_l, \mathbb{R}^n \setminus W)/2.
\]
Analogously, we have \( \text{dist}(y, \mathbb{R}^n \setminus W) \leq 2\text{dist}(x_l, \mathbb{R}^n \setminus W)/2 \). From these, (3.28) easily follows.

To prove the second part of (4), suppose instead that \( B(y, \min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\}) \cap A \neq \emptyset \). Thus there exists a point \( x \in A \) with
\[
|x - y| < \min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\}.
\]
By (1), we have \( x \in \frac{1}{4}B_l \) for some \( l \), and so we have
\[
|x_l - y| < \min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\} + \frac{1}{4}r_l.
\]
(3.29)
We have either
\[
r_l = R \geq 8\min\{R/8, \text{dist}(y, \mathbb{R}^n \setminus W)/64\} > 8|x_l - y| - 2r_l,
\]
so that \( y \in \frac{1}{2} B_1 \), or

\[
\begin{align*}
    r_l &= \frac{1}{8} \text{dist}(x_l, \mathbb{R}^n \setminus W) \\
    &\geq \frac{1}{8}(\text{dist}(y, \mathbb{R}^n \setminus W) - |x_l - y|) \\
    &\geq \frac{1}{8}(64|x_l - y| - 16r_l - |x_l - y|) \quad \text{by (3.29)} \\
    &= \frac{63}{8}|x_l - y| - 2r_l.
\end{align*}
\]

We conclude that \( y \in \frac{1}{2} B_1 \). \( \square \)

**Lemma 3.30.** Let \( C_s \geq 1 \) and let \( \nu \in \mathcal{M}(\Omega, C_s) \). Let \( \delta > 0 \), and let \( W \subset \Omega \) be an open set. There exists \( R_* = R_*(\nu, W, \delta) > 0 \) such that the following holds. If \( A \subset \mathbb{R}^n \) is Borel, \( 0 < r \leq R_* \), and denoting \( r_x := \min\{r, \text{dist}(x, \mathbb{R}^n \setminus W)/64\} \) we have

\[
\frac{\text{Cap}_1(A \cap B(x, r_x))}{r_x^{n-1}} < \delta \quad \text{for all} \quad x \in W,
\]

(3.31)

then \( \nu(D) \leq 2C_s \delta \nu(W) \).

**Proof.** We can assume that \( \nu(W) > 0 \). For sufficiently small \( \varepsilon > 0 \), we have \( \nu(W \setminus W_\varepsilon) < \delta \nu(W) \); recall that

\[
W_\varepsilon := \{x \in W : \text{dist}(x, \mathbb{R}^n \setminus W) > \varepsilon\}.
\]

Choose \( R_* = \min\{\varepsilon/64, \text{R}(\nu, W_\varepsilon, \delta)\} \). If \( A \subset \mathbb{R}^n \) is Borel, \( 0 < r \leq R_* \), and (3.31) holds, then in fact

\[
\frac{\text{Cap}_1(A \cap B(x, r))}{r^{n-1}} < \delta \quad \text{for all} \quad x \in W_\varepsilon.
\]

It follows that \( \nu(A \cap W_\varepsilon) \leq C_s \delta \nu(W_\varepsilon) \). In total, we get

\[
\nu(A) \leq \nu(A \cap W_\varepsilon) + \nu(W \setminus W_\varepsilon) \leq 2C_s \delta \nu(W).
\]

\( \square \)

**Theorem 3.32.** Let \( w: \Omega \to [-\infty, \infty] \) be measurable. Suppose there is a set \( E \subset \Omega \) such that for \( \mathcal{H}^{n-1} \)-almost every direction \( v \in \partial B(0,1) \), we have for a.e. line \( L \) parallel to \( v \) that \( w|_{L \cap \Omega} \) is continuous at every point in \( E \), and \( E \cap L \) is at most countable. Suppose also that there exists \( \nu \in \mathcal{M}(\Omega, 2^{-5n}c(n)^{-1}) \) such that \( \nu(\Omega) < \infty \), \( \text{Lip}_w^{\nu_c(n)}(x) < \infty \) for every \( x \in \Omega \setminus E \), and \( \int_{\Omega} h \, dv < \infty \) for some nonnegative \( h \in L^1(\Omega) \) with \( h \geq \text{Lip}_w^{\nu_c(n)} \) in \( \Omega \setminus E \). Then \( w \in D^{BV}(\Omega) \) with

\[
|Dw| \leq Chv.
\]

Recall that \( D^{BV}(\Omega) \) denotes the Dirichlet space, that is, the BV space without the requirement \( w \in L^1(\Omega) \).

**Proof.** We can assume that \( \Omega \) is nonempty and bounded; the unbounded case then easily follows. Assume first also that \( \nu(B(x,r)) > 0 \) for every ball \( B(x,r) \subset \Omega \), and that \( h \) is lower semicontinuous.

Fix \( 0 < \varepsilon \leq \min\{1, \mathcal{L}^n(\Omega)\} \). In the definition (1.1) of the generalized Lipschitz number, for convenience we can interpret the set \( U_r \) to be defined for every \( r > 0 \). Thus to every
\( x \in \Omega \setminus E \) and \( r > 0 \), there corresponds a set \( U_{x,r} \subset B(x,r) \) with
\[
\frac{\text{Cap}_1(B(x,r) \setminus U_{x,r})}{r^{n-1}} \leq 2c(n),
\]
and
\[
\frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{U_{x,r}} w \mathcal{L}^n(B(x,r))}{r} \frac{\nu(B(x,r))}{\nu(B(x,r))} < \text{Lip}_w^{n,c}(x) + \epsilon.
\]
Note that
\[
B(x,r) \setminus \text{fine-int } U_{x,r} = B(x,r) \cap (X \setminus \text{fine-int } U_{x,r})
\]
\[
= B(x,r) \cap X \setminus U_{x,r}^{-1}
\]
\[
\subset B(x,r) \setminus U_{x,r}^{-1} \quad \text{by (2.27).}
\]
Combining this with (2.28) and (3.33), we see that
\[
\frac{\text{Cap}_1(B(x,r) \setminus \text{fine-int } U_{x,r})}{r^{n-1}} \leq 2c(n),
\]
and so we can actually assume that every \( U_{x,r} \) is 1-finely open. For each \( j \in \mathbb{N} \), let \( A_j \) consist of points \( x \in \Omega \setminus E \) for which
\[
\frac{1}{2} \sup_{0 < r \leq 1/j} \frac{\text{osc}_{U_{x,r}} w \mathcal{L}^n(B(x,r))}{r} \frac{\nu(B(x,r))}{\nu(B(x,r))} < \text{Lip}_w^{n,c}(x) + \epsilon
\]
and also
\[
\text{Lip}_w^{n,c}(x) \leq h(y) + \epsilon \quad \text{for all } y \in B(x,1/j).
\]
We have \( \Omega = \bigcup_{j=1}^{\infty} A_j \cup E \). Fix \( j \in \mathbb{N} \) and define \( D_{j,1} := A_j \) and \( W_{j,1} := \Omega \).

Let \( k \geq 1 \). From Lemma 3.13, we know that \( h \, d\nu \in \mathcal{M}(\Omega, 2^{-5n+1}c(n)^{-1}) \). Inductively, we do the following. From Lemma 3.30, we obtain a parameter \( R_\ast = R_\ast(h \, d\nu, W_{j,k}, 2^{4n-3}c(n)) \). Using Lemma 3.27, take a Whitney-type covering \( \{B_{j,k,l} = B(x_{j,k,l}, r_{j,k,l})\}_l \) of \( D_{j,k} \) in \( W_{j,k} \) at scale
\[
R_{j,k} := \min\{1/(j + k), R_\ast\}.
\]
For each ball \( B_{j,k,l} \), there is the corresponding 1-finely open set \( U_{j,k,l} \subset B_{j,k,l} \). Define the new “bad” set
\[
D_{j,k+1} := D_{j,k} \setminus \bigcup_{l} U_{j,k,l}.
\]
By (3.33), we have
\[
\frac{\text{Cap}_1(D_{j,k+1} \cap B_{j,k,l})}{r_{j,k,l}^{n-1}} \leq \frac{\text{Cap}_1(B_{j,k,l} \setminus U_{j,k,l})}{r_{j,k,l}^{n-1}} \leq 2c(n) \quad \text{for all } l.
\]
Consider \( y \in W_{j,k} \). Suppose \( y \) is in no ball \( \frac{1}{2} B_{j,k,l} \). Let \( r_y := \min\{R_{j,k}/8, \text{dist}(y, \mathbb{R}^n \setminus W_{j,k})/64\} \). Then by Lemma 3.27(4), we have
\[
B(y,r_y) \cap D_{j,k} = \emptyset \quad \text{and so also } \quad B(y,r_y) \cap D_{j,k+1} = \emptyset.
\]
If we have \( y \in \frac{1}{2}B_{j,k,l} \) for some \( l \), then by Lemma 3.27(4) we have
\[
\frac{r_{j,k,l}}{16} \leq r_y \leq \frac{r_{j,k,l}}{4},
\]
and so we get
\[
\operatorname{Cap}_1(D_{j,k+1} \cap B(y, r_y)) \leq \operatorname{Cap}_1(D_{j,k+1} \cap B_{j,k,l}) \leq 2^{4n-3}c(n) r_y^{n-1} \quad \text{by (3.37)}.
\]
Note that \( \operatorname{Cap}_1(D_{j,k+1} \cap B(y, r_y)) \leq 2^{4n-3}c(n) r_y^{n-1} \) thus holds for every \( y \in W_{j,k} \). By Lemma 3.1, we can find a Borel set \( D^*_{j,k+1} \supset D_{j,k+1} \), with \( D^*_{j,k+1} \subset W_{j,k} \), such that we still have
\[
\operatorname{Cap}_1(D^*_{j,k+1} \cap B(y, r_y)) \leq 2^{4n-3}c(n) r_y^{n-1} \quad \text{for every } y \in W_{j,k}.
\]
(3.38)
Thus from Lemma 3.30, we get
\[
\int_{D^*_{j,k+1}} h \, d\nu \leq 4 \times 2^{-5n}c(n)^{-1} 2^{4n-3}c(n) \int_{W_{j,k}} h \, d\nu \leq \frac{1}{4} \int_{W_{j,k}} h \, d\nu.
\]
(3.39)
Since \( \nu(W_{j,k}) > 0 \), we can finally choose a nonempty open set \( W_{j,k+1} \subset W_{j,k} \) with \( W_{j,k+1} \supset D^*_{j,k+1} \), such that
\[
\int_{W_{j,k+1}} (h + 2\varepsilon) \, d\nu \leq \int_{D^*_{j,k+1}} (h + 2\varepsilon) \, d\nu + \frac{1}{4} \int_{W_{j,k}} (h + 2\varepsilon) \, d\nu.
\]
(3.40)
Define \( N_j := \bigcap_{k=1}^{\infty} D^*_{j,k} \). Recall that \( R_{j,k} \leq 1/(j + k) \to 0 \) as \( k \to \infty \). By (3.38), for all \( x \in N_j \) we now have
\[
\liminf_{r \to 0} \frac{\operatorname{Cap}_1(N_j \cap B(x, r))}{r^{n-1}} \leq 2^{4n-3}c(n) < \frac{\omega_{n-1}}{(2\sqrt{2})^{n-1}} \quad \text{by (1.2)}.
\]
By Lemma 3.4, \( N_j \) is purely unrectifiable (empty in the case \( n = 1 \), since \( \operatorname{Cap}_1(\{y\}) = 2 \) for every \( y \in \mathbb{R} \)).

For each \( j \in \mathbb{N} \), define
\[
g_j := \sum_{k,l} \frac{\text{osc}_{U_{j,k,l}} w}{r_{j,k,l}} \chi_{2B_{j,k,l}}.
\]
Recall that \( D_{j,1} = A_j \). Now by (3.36), we have
\[
A_j \setminus N_j \subset \bigcup_{k,l} U_{j,k,l}.
\]
(3.41)
By rotating the coordinate axes if necessary, almost every line in the direction of a coordinate axis has empty intersection with the purely unrectifiable set \( \bigcup_{j=1}^{\infty} N_j \); see [18, 3.3.13]. We can also do this so that each coordinate axis is parallel to one of the permissible directions \( v \) from the statement of the Theorem. Now, for almost every line \( L \) in the direction of a coordinate axis, its intersection with \( \bigcup_{j=1}^{\infty} N_j \) is empty, \( w|_{L \cap \Omega} \) is continuous at every point in \( E \) and \( E \cap L \) is at most countable, and each \( U_{j,k,l} \cap L \) is a relatively open subset of \( L \) by Lemma 3.26. Take a line segment \( \gamma: [0, \ell] \to L \cap \Omega \) in such a line \( L \), with length \( \ell > 0 \). We denote also the image of \( \gamma \) by the same symbol. Note that the sets \( A_j \)
are increasing, and
\[ \gamma \subset E \cup \bigcup_{j=1}^{\infty} A_j \] and thus
\[ \lim_{j \to \infty} \chi_{\bigcup_{k,l} U_{j,k,l}}(x) = 1 \text{ for every } x \in \gamma \setminus E \quad \text{by (3.41).} \]

When \( j \geq 1/\ell \), we have
\[ \int_{\gamma} g_j \, ds \geq \sum_{k,l, \gamma \cap U_{k,l} \neq \emptyset} \int_{\gamma} \frac{\text{osc}_{U_{j,k,l}} w}{r_{j,k,l}} \chi_{2B_{j,k,l}} \, ds \geq \sum_{k,l, \gamma \cap U_{k,l} \neq \emptyset} \text{osc}_{U_{j,k,l}} w. \]

By (3.42) and Lemma 3.23 we get
\[ |w(\gamma(0)) - w(\gamma(\ell))| \leq \liminf_{j \to \infty} \sum_{k,l, \gamma \cap U_{k,l} \neq \emptyset} \text{osc}_{U_{j,k,l}} w \leq \liminf_{j \to \infty} \int_{\gamma} g_j \, ds. \] (3.43)

We estimate
\[ \frac{\text{osc}_{U_{j,k,l}} w}{r_{j,k,l}} \mathcal{L}^n(2B_{j,k,l}) \leq 2\mathcal{L}^n(2B_{j,k,l}) \frac{\nu(B_{j,k,l})}{\mathcal{L}^n(B_{j,k,l})} (\text{Lip}_{\nu,c}(n)(x_{j,k,l}) + \varepsilon) \quad \text{by (3.34)} \]
\[ \leq 2^{n+1} \int_{B_{j,k,l}} (h + 2\varepsilon) \, d\nu \quad \text{by (3.35).} \] (3.44)

We also have
\[ \int_{W_{j,k+l}} (h + 2\varepsilon) \, d\nu \leq \int_{D_{j,k+l}} (h + 2\varepsilon) \, d\nu \leq \frac{1}{4} \int_{W_{j,k}} (h + 2\varepsilon) \, d\nu \quad \text{by (3.40)} \]
\[ \leq \frac{1}{4} \int_{W_{j,k}} (h + 2\varepsilon) \, d\nu \leq \frac{1}{4} \int_{W_{j,k}} (h + 2\varepsilon) \, d\nu \quad \text{by (3.39)} \]
\[ \leq (1/2)^k \int_{\Omega} (h + 2\varepsilon) \, d\nu \quad \text{by induction.} \] (3.45)

It follows that for every \( j \in \mathbb{N} \),
\[ 1120^{-n} \int_{\Omega} g_j \, d\mathcal{L}^n \leq 1120^{-n} \sum_{k,l} \frac{\text{osc}_{U_{j,k,l}} w}{r_{j,k,l}} \mathcal{L}^n(2B_{j,k,l}) \]
\[ \leq 280^{-n} \sum_{k,l} \int_{B_{j,k,l}} (h + 2\varepsilon) \, d\nu \quad \text{by (3.44)} \]
\[ \leq \sum_{k} \int_{W_{j,k}} (h + 2\varepsilon) \, d\nu \quad \text{by Lemma 3.27(3)} \]
\[ \leq 2 \int_{\Omega} (h + 2\varepsilon) \, d\nu \quad \text{by (3.45).} \] (3.46)

Recall the definition of pointwise variation from (2.11). By (3.43), for a.e. \( z \in \pi_n(\Omega) \) we get
\[ \text{pV}(w_z, \Omega_z) \leq \liminf_{j \to \infty} \int_{\Omega_z} g_j \, ds. \]
We estimate (the first integral can be understood as an upper integral if necessary)
\[
\int_{\pi_n(\Omega)} pV(w, \Omega) d\mathcal{L}^{n-1}(z) \\
\leq \int_{\pi_n(\Omega)} \liminf_{j \to \infty} \int_{\Omega_z} g_j \, ds \, d\mathcal{L}^{n-1}(z) \\
\leq \liminf_{j \to \infty} \int_{\pi_n(\Omega)} \int_{\Omega_z} g_j \, ds \, d\mathcal{L}^{n-1}(z) \quad \text{by Fatou's lemma} \quad (3.47) \\
= \limsup_{j \to \infty} \int_{\Omega} g_j \, d\mathcal{L}^n \quad \text{by Fubini} \\
\leq 2 \times 1120^n \int_{\Omega} (h + 2\varepsilon) \, d\nu \quad \text{by (3.46)}.
\]
Recall (2.13). Since we can do the above calculation also in other coordinate directions, we obtain
\[
\text{Var}(w, \Omega) \leq 2n \times 1120^n \int_{\Omega} (h + 2\varepsilon) \, d\nu.
\]
Letting $\varepsilon \to 0$, and since the above holds also with $\Omega$ replaced by any open subset of $\Omega$, we get $w \in D^{BV}(\Omega)$ with $|Dw| \leq C\nu$, where $C = 2n \times 1120^n$.

Now we remove the extra assumptions made at the beginning of the proof. If $h$ is not lower semicontinuous, then using the Vitali-Carathéodory theorem (Theorem 2.1) we find a sequence $\{h_i\}_{i=1}^\infty$ of lower semicontinuous functions in $L^1(\Omega)$ such that $h \leq h_{i+1} \leq h_i$ for all $i \in \mathbb{N}$, and $h_i \to h$ in $L^1(\Omega, \nu)$. Thus we get
\[
|Dw| \leq \lim_{i \to \infty} Ch_i \nu = Ch \nu.
\]
Finally, if we do not have $\nu(B(x, r)) > 0$ for every ball $B(x, r) \subset \Omega$, we can fix $\kappa > 0$ and then
\[
h \geq \text{Lip}_{\nu, c(n)}^{\nu, \kappa \mathcal{L}^n} \geq \text{Lip}_{\nu}^{\nu, c(n)} \quad \text{in } \Omega \setminus E,
\]
and hence $|Dw| \leq Ch(\nu + \kappa \mathcal{L}^n)$; note that $\nu + \kappa \mathcal{L}^n \in \mathcal{M}(\Omega, 2^{-5n}c(n)^{-1})$ by Lemma 3.5.

Letting $\kappa \to 0$, we get the result.

**Remark 3.48.** Proving and making use of the pure unrectifiability of $N$ in the above proof is one of the relatively few places where we strongly rely on the Euclidean structure. Instead of lines, line segments, and Sobolev functions, most of the above proof could be written in metric measure spaces using curves and Newton–Sobolev functions, see [52].

**Proof of Theorem 1.3.** (1) $\Rightarrow$ (2): This was shown at the end of Subsection 3.1.

(2) $\Rightarrow$ (1): Assume that $\Omega$ is bounded; generalization to the unbounded case is then easy. By assumption, we have $\text{Lip}_{\nu, c(n)}^{\nu, c(n)}(x) \leq 1$ for some $\nu \in \mathcal{M}(\Omega, 2^{-5n}c(n)^{-1})$ with $\nu(\Omega) < \infty$ and for every $x \in \Omega$, and so by applying Theorem 3.32 with $E = \emptyset$, $w = u^*$, and $h = 1$, we get $u \in D^{BV}(\Omega)$ with $|Du| \leq C\nu$. Since $u \in L^1(\Omega)$ by assumption, we get $u \in BV(\Omega)$. □

**Proof of Theorem 1.4.** (1) $\Rightarrow$ (2): This was shown at the end of Subsection 3.1.
(2) ⇒ (1): As above, assume that Ω is bounded. By assumption, we have \( \text{Lip}_{u^*}^{a,c(n)}(x) \leq 1 \) for some nonnegative \( a \in L^p(\Omega) \subset L^1(\Omega) \) and for every \( x \in \Omega \). By Lemmas 3.5 and 3.13, we know that \( a d\mathcal{L}^n \in M(\Omega, 2^{-5n}c(n)^{-1}) \), and so by applying Theorem 3.32 with \( E = \emptyset \), \( dv := a d\mathcal{L}^n \), \( w = u^* \), and \( h = 1 \), we get \( u \in D^1(\Omega) \) with \( |\nabla u| \leq Ca \in L^p(\Omega) \). Thus in fact \( u \in W^{1,p}(\Omega) \).

3.3. Federer’s characterization of sets of finite perimeter. We have seen that while the Lipschitz number \( \text{Lip}_{u^*} \) is excessively sensitive to oscillations, the generalized Lipschitz number \( \text{Lip}_{a,c(n)}^{u^*} \) can be used to characterize Sobolev functions. An analogous observation can be made in the setting of sets of finite perimeter. It is well known that if \( F \subset \mathbb{R}^n \) and \( \mathcal{H}^{n-1}(\partial F) < \infty \), then \( F \) is a set of finite perimeter, but the converse does not generally hold. This is because \( \partial F \) is also very sensitive to irregularities, and so it can be a very big set even when \( F \) has finite perimeter.

On the other hand, one defines the measure-theoretic boundary \( \partial^* F \) of a set \( F \subset \mathbb{R}^n \) as the set of points \( x \in \mathbb{R}^n \) where

\[
\limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap F)}{\mathcal{L}^n(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \setminus F)}{\mathcal{L}^n(B(x,r))} > 0.
\]

Federer’s characterization states that a set \( F \) has finite perimeter if and only if \( \mathcal{H}^{n-1}(\partial^* F) < \infty \); see Federer [18, Section 4.5.11] or alternatively [16, p. 222–]. There are various other “Federer-style” characterizations as well. In [15, Section 6] it is shown that one can replace the zero on the right-hand sides of (3.49) with a positive constant, and in [36, Theorem 1.1] it is shown that one can also replace the “\( \limsup \)” on the left-hand sides by “\( \liminf \)”. In [35, Theorem 1.1] it is shown that instead of the measure-theoretic boundary, one can use a fine boundary.

It is well known that a set of finite perimeter \( F \subset \mathbb{R}^n \) can also be characterized by the condition that there is an approximating sequence \( \{u_j\}_{j=1}^\infty \) in \( \text{Lip}_{loc}(\mathbb{R}^n) \) such that \( u_j \to \chi_F \) in \( L^1_{loc}(\mathbb{R}^n) \) and

\[
\liminf_{j \to \infty} \int_{\mathbb{R}^n} \text{Lip}_{u_j} \, d\mathcal{L}^n < \infty;
\]

see e.g. [4, Theorem 3.9]. By contrast, the quantities appearing in Federer-style characterizations, namely boundaries and Hausdorff measures, seem to have nothing as such to do with the concept of Lipschitz continuity. However, we will show that our characterization of BV functions by means of generalized Lipschitz numbers in fact reduces to a Federer-style characterization in the special case of sets of finite perimeter, thus serving to unify the picture.

Just as we consider the representative \( u^* \) throughout the paper, for a measurable set \( F \subset \mathbb{R}^n \) we need to consider a suitable pointwise representative, which we choose to be...
$F^* := \{ \chi^*_F = 1 \}$. We define a boundary $\partial_n F$ as the set of points $x \in \mathbb{R}^n$ where

$$\limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \cap F^*)}{r^{n-1}} \geq c(n) \quad \text{and} \quad \limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus F^*)}{r^{n-1}} \geq c(n).$$

As usual, $\Omega \subset \mathbb{R}^n$ is an open set.

**Corollary 3.50.** Let $F \subset \mathbb{R}^n$ be measurable. Then $\text{Var}(\chi_F, \Omega) < \infty$ if and only if $\mathcal{H}^{n-1}(\partial_n F \cap \Omega) < \infty$.

**Proof.** $\Rightarrow$: We assume $\text{Var}(\chi_F, \Omega) < \infty$. By Proposition 3.16, we have $\text{Lip}_{\chi^*_F}^{[D\chi^*_F; c(n)/2]}(x) \leq 2C(c(n))^{-1}$ for all $x \in \Omega$. At a given point $x \in \Omega$, this means that for all sufficiently small $r > 0$, we find a set $U_r \subset B(x, r)$ such that

$$\frac{\text{Cap}_1(B(x, r) \setminus U_r)}{r^{n-1}} \leq \frac{2c(n)}{3} \quad (3.51)$$

and

$$\frac{\text{osc}_{U_r} \chi^*_F}{r} \frac{\mathcal{L}^n(B(x, r))}{|D\chi^*_F|(B(x, r))} \leq 3Cc(n)^{-1}. \quad (3.52)$$

By a standard density result, see e.g. [4, Theorem 2.56], there exists a set $S \subset \Omega$ such that $\mathcal{H}^{n-1}(S) < \infty$ and

$$\limsup_{r \to 0} \frac{|D\chi^*_F|(B(x, r))}{r^{n-1}} \leq \frac{\omega_n c(n)}{4C} \quad \text{for all } x \in \Omega \setminus S.$$

Combining with (3.52), we conclude that for every $x \in \Omega \setminus S$, there is $R_x > 0$ such that for all $0 < r \leq R_x$, we have that $\chi^*_F$ is constant in $U_r$. By Lemma 2.18, we have that

$$\frac{\mathcal{L}^n(B(x, r) \setminus U_r)}{\mathcal{L}^n(B(x, r))} \leq \frac{2C_I \text{Cap}_1(B(x, r) \setminus U_r)}{r^{n-1}} \leq \frac{2c(n)}{3} \frac{2C_I}{\omega_n} \text{ by (3.51)}$$

$$< 2^{-n} \quad \text{by (1.2),}$$

for all $0 < r \leq R_x$, and so clearly $\chi^*_F$ is the same constant in $U_r$ for all $0 < r \leq R_x$. If this constant is 1, then $\chi^*_F = 1$ in $U_r$, and so $U_r \subset F^*$. Otherwise $\chi^*_F < 1$ in $U_r$, and so $U_r \cap F^* = \emptyset$. Thus by (3.51), $x \notin \partial_n F$. We conclude that $\partial_n F \cap \Omega \subset S$, and so $\mathcal{H}^{n-1}(\partial_n F \cap \Omega) < \infty$.

$\Leftarrow$: Using Lemma 2.30, one can show that $\partial_n F$ is a Borel set. We can decompose $\partial_n F \cap \Omega = S \cup S_0$, where $S$ is countably $\mathcal{H}^{n-1}$-rectifiable and $S_0$ is purely $\mathcal{H}^{n-1}$-unrectifiable, see [4, p. 83]. Choose $\nu := \mathcal{L}^n + \mathcal{H}^{n-1}|_S$. By Lemma 3.5, we have $\nu \in \mathcal{M}(\Omega, 2^{-5n}c(n)^{-1})$. Consider $x \in \Omega \setminus \partial_n F$.

$$\limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \cap F^*)}{r^{n-1}} < c(n) \quad \text{or} \quad \limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus F^*)}{r^{n-1}} < c(n).$$

Supposing the former, we can choose $U_r = B(x, r) \setminus F^*$ to get

$$\text{Lip}_{\nu}^{c(n)}(x) = \limsup_{r \to 0} \frac{\text{osc} \chi^*_F}{r} \frac{\mathcal{L}^n(B(x, r))}{\nu(B(x, r))} \leq \limsup_{r \to 0} \frac{0 \mathcal{L}^n(B(x, r))}{r \mathcal{L}^n(B(x, r))} = 0.$$
In the latter case, we choose \( U_r = B(x, r) \cap F^* \). There exists a \( \mathcal{H}^{n-1} \)-negligible set \( N \subset S \) such that for every \( x \in S \setminus N \), we have
\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap S)}{\omega_{n-1} r^{n-1}} = 1,
\]
see e.g. [4, Theorem 2.83]. Thus at every \( x \in S \setminus N \), we can choose \( U_r = B(x, r) \) to get
\[
\text{Lip}_{\chi_{F^*}}(x) \leq \limsup_{r \to 0} \frac{\text{osc}_{U_r} \chi_{F^*} \mathcal{L}^n(B(x, r))}{\nu(B(x, r))} \leq \limsup_{r \to 0} \frac{\mathcal{L}^n(B(x, r))}{r} \frac{1}{\mathcal{H}^{n-1}(B(x, r) \cap S)} \leq \frac{\omega_n}{\omega_{n-1}} < \infty.
\]
(3.53)

For \( \mathcal{H}^{n-1} \)-almost every direction \( v \in B(0,1) \), almost every line parallel to \( v \) has empty intersection with \( S_0 \cup N \), see [18, 3.3.13]. By applying Theorem 3.32 with the choices \( E = S_0 \cup N \), \( w = \chi_{F^*} \), and \( h = \omega_n/\omega_{n-1} \), we get \( \chi_{F^*} \in D^{BV}(\Omega) \), that is, \( \text{Var}(\chi_{F^*}, \Omega) = \text{Var}(\chi_{F^*}, \Omega) < \infty \).

**Remark 3.54.** In the usual Federer’s characterization, one uses the measure-theoretic boundary that is defined by means of measure densities, rather than capacitary densities. Thus it is natural to ask:

- Can one obtain analogs of Theorems 1.3 and 1.4 via using a version of generalized Lipschitz numbers that is defined by using the Lebesgue measure \( \mathcal{L}^n \) in place of the capacity \( \text{Cap}_1 \)?

4. Fine differentiability and the Hardy–Littlewood maximal function

In this section we study the fine differentiability of Sobolev and BV functions, as well as the Hardy–Littlewood maximal function. For some previous studies of fine differentiability and finely harmonic functions, see Gardiner [19, 20] and Lávička [43, 44, 45]. These references study the case \( p = 2 \), whereas we will keep working with the 1-fine topology.

It is known that for a Sobolev function \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \), \( 1 \leq p < \infty \), the representative \( u^* \) is \( p \)-finely continuous at \( \text{Cap}_p \)-a.e. point, see [46]. In the case \( p > n \) we know that \( u^* \) is actually continuous, and differentiable a.e., whereas when \( 1 \leq p \leq n \) it only seems to be known that \( u^* \) is approximately differentiable a.e. We say that \( u^* \) is approximately differentiable at \( x \in \mathbb{R}^n \) if there is \( v \in \mathbb{R}^n \) such that for every \( \varepsilon > 0 \), the set
\[
A := \left\{ y \in \mathbb{R}^n : \frac{|u(y) - u^*(x) - \langle v, y - x \rangle|}{|y - x|} > \varepsilon \right\}
\]
has measure density zero at \( x \), meaning that
\[
\lim_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \cap A)}{\mathcal{L}^n(B(x, r))} = 0.
\]
Then we say that \( v \) is the approximate gradient of \( u \) at \( x \). In this section we improve on this result by showing that in fact \( u^* \) is 1-finely differentiable a.e.

Note that for a function \( u \in \text{BV}_{\text{loc}}(\mathbb{R}^n) \), the approximate gradient agrees a.e. with the density \( \frac{dDu}{d\mathcal{L}^n} \), for this, see e.g. [4, Theorem 3.83]. Note also that in the case \( u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n) \),
\[ \frac{dDu}{dx} \] is of course the weak gradient. Thus we use the notation \( \nabla u \) for each of these. When we talk about \( \nabla u(x) \) for a given point \( x \), we mean the approximate gradient which is well-defined unlike the density or the weak gradient, which are interpreted to be defined only a.e.

As before, \( \Omega \subset \mathbb{R}^n \) is always an open set. Recall Definition 2.25.

**Definition 4.1.** We say that a function \( w: \Omega \to [-\infty, \infty] \) is 1-finely differentiable at \( x \in \Omega \) if there exists \( v \in \mathbb{R}^n \) such that

\[
\lim_{y \to x} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} = 0.
\]

We then call \( v \) the fine derivative of \( w \) at \( x \), and denote it by \( \nabla_{\text{fine}} w(x) \).

Note that by the first part of Lemma 2.18, fine differentiability is stronger than approximate differentiability. So if it exists, the fine derivative agrees with the approximate gradient, and so the subscript “fine” will rarely be needed. We do not use the term “fine gradient”, since it has been used previously to describe a weak gradient of Sobolev functions in finely open sets, see [10].

First we show the following lemma.

**Lemma 4.2.** Suppose \( w: \Omega \to [-\infty, \infty] \) and let \( x \in \Omega \). Then the following are equivalent:

1. The function \( w \) is 1-finely differentiable at \( x \).
2. There exists \( v \in \mathbb{R}^n \) and a 1-finely open set \( U \ni x \) such that

\[
\lim_{y \to x} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} = 0.
\]

3. There are sets \( U_k \), vectors \( v_k \in \mathbb{R}^n \), and numbers \( \beta_k \to 0 \) such that

\[
\limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus U_k)}{r^{n-1}} < \beta_k
\]

and

\[
\limsup_{U_k \ni y \to x} \frac{|w(y) - w(x) - \langle v_k, y - x \rangle|}{|y - x|} < \beta_k
\]

for all \( k \in \mathbb{N} \).

**Proof.** (2) \(\Rightarrow\) (1): Given \( \varepsilon > 0 \), there exists \( r > 0 \) such that

\[
\sup_{y \in U \cap B(x, r) \setminus \{x\}} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} < \varepsilon.
\]

The set \( U \cap B(x, r) \) is 1-finely open, and so we have the result.

(1) \(\Rightarrow\) (3): Choose numbers \( \beta_k = 1/k \). By the assumption of fine differentiability, for every \( k \in \mathbb{N} \) we find a 1-finely open set \( U_k \ni x \) such that

\[
\sup_{y \in U_k \setminus \{x\}} \frac{|w(y) - w(x) - \langle \nabla_{\text{fine}} w(x), y - x \rangle|}{|y - x|} < \beta_k.
\]
Now
\[ \limsup_{r \to 0} \frac{\text{Cap}_1(B(x,r) \setminus U_k)}{r^{n-1}} = 0 < \beta_k, \]
and we can choose \( v_k = \nabla_{\text{fine}} w(x) \) for all \( k \in \mathbb{N} \).

(3) \implies (2): We can assume \( n \geq 2 \); in the case \( n = 1 \) the proof is easy since \( \text{Cap}_1(\{y\}) = 2 \) for every \( y \in \mathbb{R} \).

By passing to a subsequence (not relabeled), we can assume that \( \beta_k = 2^{-k} \). Choose radii \( r_k > 0 \) with \( r_{k+1} \leq r_k/2 \) and such that
\[ \sup_{0 < r \leq r_k} \frac{\text{Cap}_1(B(x,r) \setminus U_k)}{r^{n-1}} = 2^{-k}, \tag{4.3} \]
and
\[ \sup_{y \in B(x,r_k) \cap U_k \setminus \{x\}} \left| \frac{w(y) - w(x) - \langle v_k, y-x \rangle}{|y-x|} \right| \leq 2^{-k}. \tag{4.4} \]
Define the annuli
\[ A_k := B(x,r_k) \setminus B(x,r_{k+1}), \]
and then let
\[ U' := \bigcup_{k=2}^{\infty} (A_k \cap U_k \cap U_{k-1}) \cup \{x\}. \]
Then for \( r_{k+1} \leq r < r_k \) with \( k \geq 2 \),
\[ \frac{\text{Cap}_1(B(x,r) \setminus U')}{r^{n-1}} \leq \frac{\text{Cap}_1(B(x,r) \cap A_k \setminus U')}{r^{n-1}} + \sum_{j=k+1}^{\infty} \frac{\text{Cap}_1(A_j \setminus U')}{r_j^{n-1}} \]
\[ \leq \frac{\text{Cap}_1(B(x,r) \setminus U_k)}{r^{n-1}} + \frac{\text{Cap}_1(B(x,r) \setminus U_{k-1})}{r^{n-1}} \]
\[ + \sum_{j=k+1}^{\infty} \frac{\text{Cap}_1(A_j \setminus U_j)}{r_j^{n-1}} + \sum_{j=k+1}^{\infty} \frac{\text{Cap}_1(A_j \setminus U_{j-1})}{r_j^{n-1}} \]
\[ \leq 2^{-k} + 2^{-k+1} \quad \text{by (4.3)} \]
\[ + \sum_{j=k+1}^{\infty} \frac{\text{Cap}_1(B(x,r_j) \setminus U_j)}{r_j^{n-1}} + \sum_{j=k+1}^{\infty} \frac{\text{Cap}_1(B(x,r_j) \setminus U_{j-1})}{r_j^{n-1}} \]
\[ \leq 2^{-k} + 2^{-k+1} + \sum_{j=k+1}^{\infty} 2^{-j} + \sum_{j=k+1}^{\infty} 2^{-j+1} \]
\[ \leq 2^{-k+3}. \]
Hence \( \mathbb{R}^n \setminus U' \) is 1-thin at \( x \). Denote the basis vectors of \( \mathbb{R}^n \) by \( e_1, \ldots, e_n \). By Lemma 3.2, we now have
\[ \mathcal{H}^{n-1}(\pi_n(\partial B(x,r_{k+1}) \setminus U')) \leq \text{Cap}_1(\partial B(x,r_{k+1}) \setminus U') \leq 2^{-k+3} r_{k+1}^{n-1}. \]
Note that this is very small compared to \( H^{n-1}(\pi_n(\partial B(x, r_{k+1}))) \) when \( k \) is large, and so we find a point in \( U' \cap \partial B(x, r_{k+1}) \) which is almost in the \( e_n \)-direction from \( x \). More precisely, considering similarly projections in the other coordinate directions, we find points

\[ y_{k,j} \in U' \cap \partial B(x, r_{k+1}) = U_k \cap U_{k-1} \cap \partial B(x, r_{k+1}), \]

with \( j = 1, \ldots, n \), such that for \( V_{k,j} := y_{k,j} - x \), we have

\[ |V_{k,j}| |e_j| < C 2^{-k/(n-1)} \]

for some constant \( C' \) depending only on \( n \). When \( y \in U \cap A_k \subset U_k \cap U_{k-1} \), by (4.4) we get

\[ \frac{|w(y) - w(x) - \langle v_k, y - x \rangle|}{|y - x|} \leq 2^{-k} \quad \text{and} \quad \frac{|w(y) - w(x) - \langle v_{k-1}, y - x \rangle|}{|y - x|} \leq 2^{-k+1}. \]

In particular,

\[ \langle v_k - v_{k-1}, V_{k,j} \rangle \leq |w(y) - w(x) - \langle v_k, V_{k,j} \rangle| + |w(y) - w(x) - \langle v_{k-1}, V_{k,j} \rangle| \leq 2^{-k+2}|V_{k,j}|. \]

Since this holds for all \( j = 1, \ldots, n \), from (4.5) we get for all sufficiently large \( k \in \mathbb{N} \)

\[ |v_k - v_{k-1}| \leq C 2^{-k} \]

for another constant \( C \) depending only on \( n \). Thus \( \{v_k\}_{k=1}^{\infty} \) is Cauchy and then in fact for large \( k \in \mathbb{N} \),

\[ |v - v_k| \leq C 2^{-k+1}. \]

Now by (4.6), we get

\[ \sup_{y \in U' \cap \mathbb{B}_k} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} \leq C 2^{-k+2}, \]

and so

\[ \lim_{U' \ni y \to x} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} = 0. \]

Then also

\[ \lim_{\text{fine-int } U' \ni y \to x} \frac{|w(y) - w(x) - \langle v, y - x \rangle|}{|y - x|} = 0, \]

and since \( \mathbb{R}^n \setminus U' \) was shown to be 1-thin at \( x \), by (2.27) we have that \( x \in \text{fine-int } U' \). Thus we can let \( U := \text{fine-int } U' \).

We will need the following version of Lemma 3.15; recall also the definition of \( \mathcal{M}_R u \) from above that lemma.

**Lemma 4.7.** Let \( u \in L^1(\mathbb{R}^n) \). Then for some constant \( C \) depending only on \( n \), we have

\[ \text{Cap}_1(\{\mathcal{M}_1 u > t\} \cap B(x, 1)) \leq C \frac{\|u\|_{BV(B(x, 2))}}{t} \quad \text{for all } t > 0. \]

**Proof.** We can assume that \( \|u\|_{BV(B(x, 2))} \) is finite. Denote by \( Eu \) an extension of \( u \) from \( B(x, 2) \) to \( \mathbb{R}^n \) with \( \|Eu\|_{BV(\mathbb{R}^n)} \leq C' \|u\|_{BV(B(x, 2))} \), for some \( C' \) depending only on \( n \); see
e.g. [4, Proposition 3.21]. We estimate
\[
\text{Cap}_1(\{M_1u > t\} \cap B(x, 1)) = \text{Cap}_1(\{M_1Eu > t\} \cap B(x, 1)) \\
\leq \text{Cap}_1(\{M_1Eu > t\}) \\
\leq C\|Eu\|_{\text{BV}(\mathbb{R}^n)} \quad \text{by Lemma 3.15} \\
\leq C'C\|u\|_{\text{BV}(B(x, 2))}.
\]

\[\square\]

**Theorem 4.8.** Let \(u \in \text{BV}_{\text{loc}}(\Omega)\). Then \(u^*\) is 1-finely differentiable at a.e. \(x \in \Omega\).

Of course, the same is then true for every \(u \in N^{1,p}_{\text{loc}}(\Omega)\), with \(1 \leq p \leq \infty\). Moreover, a generalization to \(\mathbb{R}^k\)-valued BV or Sobolev functions is obvious, by considering separately the component functions.

**Proof.** Since the issue is local, we can assume that \(\Omega = \mathbb{R}^n\). At a.e. \(x \in \mathbb{R}^n\), we have
\[
\lim_{r \to 0} \frac{\int_{B(x, r)} |u(y) - u^*(x) - (\nabla u(x), y - x)| \, d\mathcal{L}^n(y)}{r} = 0,
\]
see [4, Theorem 3.83], as well as
\[
\lim_{r \to 0} \frac{|\nabla u(y) - \nabla u(x)| \, d\mathcal{L}^n(y)}{r^n} = 0 \quad \text{and} \quad \lim_{r \to 0} \frac{|D^s u|(B(x, r))}{r^n} = 0.
\]
Consider such \(x\). Define \(L(z) := (\nabla u(x), z)\). Thus for the scalings
\[
u_{x,r}(z) := \frac{u(x + rz) - u^*(x)}{r}, \quad z \in B(0, 2),
\]
we have
\[
u_{x,r}(\cdot) \to L(\cdot) \quad \text{in} \ L^1(B(0, 2)) \quad \text{as} \ r \to 0 \quad \text{and} \quad \nabla \nu_{x,r}(z) = \nabla u(x + rz), \ z \in B(0, 2).
\]

Then
\[
|D(u_{x,r} - L)|(B(0, 2)) = \int_{B(0, 2)} |\nabla u_{x,r}(z) - \nabla u(x)| \, d\mathcal{L}^n(z) + |D^s u_{x,r}|(B(0, 2)) \\
= \frac{1}{r^n} \int_{B(x, 2r)} |\nabla u(y) - \nabla u(x)| \, d\mathcal{L}^n(y) + \frac{|D^s u|(B(x, 2r))}{r^n} \\
\to 0 \quad \text{as} \ r \to 0.
\]

In conclusion, we have the norm convergence
\[
u_{x,r} \to L \quad \text{in} \ BV(B(0, 2)).
\]

Note that \((u^*)_{x,r} = (u_{x,r})^*\) in \(B(0, 2)\), so we simply use the notation \(u_{x,r}^*\). Note also that
\[
|u_{x,r}^* - L| = |(u_{x,r} - L)^*| \leq |u_{x,r} - L|^* \leq M_1|u_{x,r} - L|,
\]
and so for every $j \in \mathbb{N}$ and $t > 0$ we get
\[
\text{Cap}_1(\{ z \in B(0,1) : |u^*_{x,2^{-j}}(z) - L(z)| > t \}) \\
\leq \text{Cap}_1(\{ z \in B(0,1) : \mathcal{M}_1u_{x,2^{-j}} - L(z) > t \}) \\
\leq C \frac{\|u_{x,2^{-j}} - L\|_{BV(B(0,2))}}{t} \text{ by Lemma 4.7} \\
\to 0 \text{ as } j \to \infty \text{ by (4.10)}.
\]
Thus we can choose numbers $t_j \searrow 0$ such that for the sets
\[
D_j := \{ z \in B(0,1) : |u^*_{x,2^{-j}}(z) - L(z)| > t_j \},
\]
we get $\text{Cap}_1(D_j) \to 0$ as $j \to \infty$. Define $A_j := D_j \setminus B(0,1/2)$ and $A := \bigcup_{j=1}^{\infty} 2^{-j}A_j + x$. Now for all $k \in \mathbb{N}$, we have
\[
\text{Cap}_1(A \cap B(x,2^{-k})) \leq \sum_{j=k}^{\infty} \text{Cap}_1(2^{-j}A_j + x) \\
= \sum_{j=k}^{\infty} 2^{-j(n-1)} \text{Cap}_1(A_j) \\
\leq \sum_{j=k}^{\infty} 2^{-j(n-1)} \text{Cap}_1(D_j) \\
\leq 2^{-k(n-1)+1} \max_{j \geq k} \text{Cap}_1(D_j).
\]
Since $\text{Cap}_1(D_j) \to 0$, we obtain
\[
\frac{\text{Cap}_1(A \cap B(0,2^{-k}))}{2^{-k(n-1)}} \to 0 \text{ as } k \to \infty,
\]
and so clearly $A$ is 1-thin at $x$. By (2.27), the 1-finely open set $U := \mathbb{R}^n \setminus \overline{A}$ contains $x$.

For any $j \in \mathbb{N}$ and $y \in U \cap B(x,2^{-j}) \setminus B(x,2^{-j-1})$, we have
\[
\frac{|u^*(y) - u^*(x) - L(y-x)|}{|y-x|} \leq 2 \frac{|u^*(y) - u^*(x) - L(y-x)|}{2^{-j}} \\
= 2 |u^*_{x,2^{-j}}((y-x)/2^{-j}) - L((y-x)/2^{-j})| \\
\leq 2t_j \to 0 \text{ as } j \to \infty,
\]
and so
\[
\lim_{U \ni y \to x} \frac{|u^*(y) - u^*(x) - (\nabla u(x), y-x)|}{|y-x|} = 0.
\]
Thus Condition (2) of Lemma 4.2 is satisfied, and we have fine differentiability at $x$. □

**Remark 4.11.** The definition of $p$-thinness when $1 < p < \infty$ has quite a different form compared to the definition of 1-thinness, and thus in the case $u \in N^{1,p}_{loc}(\Omega)$ for $1 < p < \infty$,
the above proof still only gives differentiability with respect to the 1-fine topology, and not the $p$-fine topology.

Note that if the fine derivative exists at a point $x$, by Lemma 4.2(2) we get

$$\text{Lip}_{u_0}^0(x) \leq |\nabla_{\text{fine}} w(x)|,$$

as we can choose $U_r = U \cap B(x, r)$ in the definition (1.1) of $\text{Lip}_{u_0}^0$. Soon we will note that in fact equality holds.

A natural question related to Theorem 1.4 is: what is the relationship between $a$ and $|\nabla u|$? Now we show that $|\nabla u|$ can be seen as essentially the “minimal $a$”.

**Proposition 4.13.** Suppose $\Omega$ is bounded and let $u \in L^p(\Omega)$, with $1 \leq p \leq \infty$. Then the following are equivalent:

1. $u \in W^{1,p}(\Omega)$;
2. There exists $a \in L^p(\Omega)$ such that $\text{Lip}_{u_0}^{a,0}(x) \leq 1$ for a.e. $x \in \Omega$; and for all $0 < \delta < s_{n-1}$, we have $\text{Lip}_{u_0}^{a,\delta}(x) \leq C\delta^{-1}$ for every $x \in \Omega$ and for a constant $C$ depending only on $n$.

For any $\varepsilon > 0$, we can choose $a = |\nabla u| + \varepsilon$, and for any $a$ as in (2), we have $a(x) \geq |\nabla u(x)|$ for a.e. $x \in \Omega$.

**Proof.** (1) $\Rightarrow$ (2): The second part is given by Proposition 3.16, with $a = |\nabla u|$. For the first part, note that from (4.12) and Theorem 4.8 we get

$$\text{Lip}_{u_0}^{a}(x) \leq |\nabla_{\text{fine}} u(x)| = |\nabla u(x)|$$

for a.e. $x \in \Omega$.

Now for each such $x$ that is also a Lebesgue point of $|\nabla u|$, we have $\text{Lip}_{u_0}^{a}(x) \leq 1$. Thus we obtain (2) with the choice $a = |\nabla u| + \varepsilon$.

(2) $\Rightarrow$ (1): By applying Theorem 3.32 with the choices $E = \emptyset$, $dv = a d\mathcal{L}^n$ (recall Lemmas 3.5 and 3.13), and $h = Cc(n)^{-1}$, we get (merging the constants $C$) $|\nabla u| \leq Cc(n)^{-1}a$, and so $u \in W^{1,p}(\Omega)$.

For the last claim, suppose $u \in W^{1,p}(\Omega)$ and suppose $a \in L^p(\Omega)$ such that $\text{Lip}_{u_0}^{a,0}(x) \leq 1$ for a.e. $x \in \Omega$. Consider a point $x$ that is a Lebesgue point of $a$, and where $u^*$ is finely differentiable and $\text{Lip}_{u_0}^{a,0}(x) \leq 1$; a.e. point $x \in \Omega$ satisfies these conditions. By Lemma 4.2(2), there exists a 1-finely open set $U \ni x$ such that

$$\lim_{U \ni y \to x} \frac{|u^*(y) - u^*(x) - \langle \nabla u(x), y - x \rangle|}{|y - x|} = 0.$$ 

Let $\kappa > 0$. On the other hand, we find sets $U_r \subset B(x, r)$ with

$$\lim_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus U_r)}{r^{n-1}} = 0.$$
and
\[
\frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{U_r} u}{r} \left( \int_{B(x,r)} (a + \kappa) \, d\mathcal{L}^n \right)^{-1} \leq \frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{U_r} u}{r} \left( \int_{B(x,r)} a \, d\mathcal{L}^n \right)^{-1} \leq 1 + \kappa.
\]
(4.14)

Note that by Lemma 2.18, also
\[
\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \setminus (U_r \cap U))}{\mathcal{L}^n(B(x,r))} = 0,
\]
and so for a linear mapping \( L(y) = \langle v, y - x \rangle \) with \( v \in \mathbb{R}^n \), we clearly have
\[
\frac{1}{2} \lim_{r \to 0} \frac{\text{osc}_{U_r \cap U} L}{r} = |v|.
\]
(4.15)

Then by (4.14), we have
\[
1 + \kappa \geq \frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{U_r} u}{r} \left( \int_{B(x,r)} (a + \kappa) \, d\mathcal{L}^n \right)^{-1} \geq \frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{U_r \cap U} u}{r} ((a + \kappa)(x))^{-1} = |\nabla u(x)|((a + \kappa)(x))^{-1}
\]
by (4.15) combined with the fine differentiability. Hence \( |\nabla u(x)| \leq (1 + \kappa)(a(x) + \kappa) \) for a.e. \( x \in \Omega \), and letting \( \kappa \to 0 \) we get the result. \( \square \)

**Proposition 4.16.** Let \( w: \Omega \to [-\infty, \infty] \). If \( \nabla_{\text{fine}} w(x) \) exists at \( x \in \Omega \), then \( \text{Lip}_0^w(x) = |\nabla_{\text{fine}} w(x)| \).

**Proof.** We recall from (4.12) that inequality “\( \leq \)" holds. The opposite inequality follows from (4.15). \( \square \)

Now we give the following formulation which shows that Sobolev functions can, roughly speaking, be characterized as finely differentiable functions.

**Corollary 4.17.** Let \( u: \Omega \to [-\infty, \infty] \) be measurable and let \( 1 \leq p \leq \infty \).

1. If \( u \in D^p(\Omega) \), then \( \nabla_{\text{fine}} u^*(x) \) exists for a.e. \( x \in \Omega \), and \( \nabla_{\text{fine}} u^* \in L^p(\Omega) \).
2. If \( \nabla_{\text{fine}} u^*(x) \) exists for \( \mathcal{H}^{n-1}\text{-a.e.} \) \( x \in \Omega \), and \( \nabla_{\text{fine}} u^* \in L^p(\Omega) \), then \( u \in D^p(\Omega) \).

**Proof.** Claim (1) is given by Theorem 4.8. To prove claim (2), we can assume that \( \Omega \) is bounded. Note that from Proposition 4.16 we get \( \text{Lip}_{u^*}^0 < \infty \) for every \( x \in \Omega \setminus E \) with \( E \subset \Omega \) \( \mathcal{H}^{n-1}\)-negligible, and \( \text{Lip}_{u^*}^0 \in L^p(\Omega) \). Applying Theorem 3.32 with the choices \( \nu = \mathcal{L}^n \) and \( h = \text{Lip}_{u^*}^0 \) then gives \( u \in D^p(\Omega) \). \( \square \)

Note that in the case \( n = 1 \), we have \( \text{Cap}_1(\{x\}) = 2 \) for every \( x \in \mathbb{R} \), and so the fine derivative coincides with the usual derivative. Thus any Sobolev function on the real line that fails to be differentiable at some point demonstrates that we cannot have “\( \mathcal{H}^{n-1}\text{-a.e.} \)” in Claim (1). The usual Cantor–Vitali function \( u: [0,1] \to [0,1] \) demonstrates that we cannot take “a.e.” in Claim (2).
We say that a set $A$ is porous at $x \in A$ if there exists $\delta > 0$ and a sequence of points $x_j \to x$ such that $B(x_j, \delta|x - x_j|) \cap A = \emptyset$ for all $j \in \mathbb{N}$.

**Lemma 4.18.** Let $A \subset \mathbb{R}^n$. Then at a.e. $x \in A$, $A$ is not porous.

*Proof.* The set $\overline{A}$ is measurable, and so a.e. $x \in \overline{A}$ is a point of density one. This is then true also of a.e. $x \in A$. Clearly $A$ is not porous at such $x$. \hfill $\square$

The point of Lemma 4.18 is simply that even for a nonmeasurable set $A$, a.e. point is a “density point” in a weak sense, which will be sufficient for us.

Similarly to the definition of approximate differentiability, we say that a function $w$ on $\Omega$ is approximately continuous at $x \in \Omega$ if for every $\varepsilon > 0$, the set
\[ \{ y \in \Omega : |w(y) - w(x)| > \varepsilon \} \]
has measure density zero at $x$.

Recall that Stepanoff [53] showed that a function $w$ is differentiable almost everywhere in the set where $\text{Lip}_w(x) < \infty$. We show the following analog of this for fine differentiability.

**Theorem 4.19.** Let $w$ be a measurable function on $\Omega$, finite a.e. Then $w$ is 1-finely differentiable a.e. in the set
\[ H := \{ x \in \Omega : \text{Lip}_w^0(x) < \infty \}. \]

*Proof.* In the case $n = 1$, this reduces to the usual Stepanoff’s theorem, and so we assume $n \geq 2$.

For every $x \in H$, we have
\[ \frac{1}{2} \limsup_{r \to 0} \frac{\text{osc}_{B(x,r)} w}{r} < \text{Lip}_w^0(x) + 1 < \infty \]
for a choice of sets $U_{x,r} \subset B(x, r)$ with
\[ \lim_{r \to 0} \frac{\text{Cap}_1(B(x,r) \setminus U_{x,r})}{r^{n-1}} = 0. \tag{4.20} \]
Consider a point $x \in H$ that is an approximate continuity point of $w$; a.e. point $x \in H$ is such a point, see e.g. [16, Theorem 1.37]. Denote annuli by $A_k := B(x, 2^{-k}) \setminus B(x, 2^{-k-1})$, and $U_k := U_{x,2^{-k}}$. For all sufficiently large $k \in \mathbb{N}$, we have
\[ \frac{1}{2} \frac{\text{osc}_{U_k} w}{2^{-k}} < \text{Lip}_w^0(x) + 1. \tag{4.21} \]
Then define
\[ U' := \bigcup_{j=1}^{\infty} A_j \cap U_j \cap U_{j-1}. \]
Now for all \( k \in \mathbb{N} \), we have

\[
\text{Cap}_1(B(x, 2^{-k}) \setminus U') \leq \sum_{j=k}^{\infty} \text{Cap}_1(A_j \setminus U')
\]

\[
\leq \sum_{j=k}^{\infty} \left[ \text{Cap}_1(A_j \setminus U_j) + \text{Cap}_1(A_j \setminus U_{j-1}) \right]
\]

\[
\leq \sum_{j=k}^{\infty} \left[ \text{Cap}(B(x, 2^{-j}) \setminus U_j) + \text{Cap}(B(x, 2^{-j+1}) \setminus U_{j-1}) \right]
\]

\[
\leq 2 \max_{j \geq k-1} \frac{\text{Cap}(B(x, 2^{-j}) \setminus U_j)}{2^{-j(n-1)}} \times \sum_{j=k}^{\infty} 2^{-j(n-1)}
\]

\[
\leq 2^{-k(n-1)+2} \max_{j \geq k} \text{Cap}(B(x, 2^{-j}) \setminus U_j).
\]

By (4.20), we get

\[
\lim_{k \to \infty} \frac{\text{Cap}(B(x, 2^{-k}) \setminus U')}{2^{-k(n-1)}} = 0.
\]

Thus

\[
\lim_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus U')}{r^{n-1}} = 0 \quad \text{and then also} \quad \lim_{r \to 0} \frac{\mathcal{L}^n(B(x, r) \setminus U')}{\mathcal{L}^n(B(x, r))} = 0
\]

by Lemma 2.18. For all sufficiently large \( k \geq 2 \), choose \( x_k \in U' \cap A_k \), and by the approximate continuity we can choose the points such that \( w(x_k) \to w(x) \). Now \( x_k \in U_k \cap U_{k-1} \), and so \( x_k \) and \( x_{k+1} \) are both in \( U_k \). Thus by (4.21), we have

\[
|w(x_k) - w(x_{k+1})| \leq 2^{-k+1}(\text{Lip}_w^0(x) + 1)
\]

(4.22)

for large \( k \in \mathbb{N} \). On the other hand, since \( w(x_k) \to w(x) \), we have in fact

\[
|w(x_k) - w(x)| \leq 2^{-k+2}(\text{Lip}_w^0(x) + 1).
\]

Combining this with (4.21), we get

\[
\frac{\text{osc}(A_k \cup U_{k} \cap U_{k-1} \cup \{x\})}{2^{-k}} \leq \frac{\text{osc}_{U_k \cup \{x\} \cap B(x, r)} w}{2^{-k}} \leq 2^3(\text{Lip}_w^0(x) + 1)
\]

for large \( k \in \mathbb{N} \), and so

\[
\limsup_{r \to 0} \frac{\text{osc}(U' \cup \{x\} \cap B(x, r))}{r} \leq 2 \limsup_{k \to \infty} \frac{\text{osc}(U' \cup \{x\} \cap B(x, 2^{-k}))}{2^{-k}} \leq 2^5(\text{Lip}_w^0(x) + 1) < \infty.
\]

Denote \( U' \cup \{x\} := U_x \). In total, for points \( x \in H \) of approximate continuity of \( w \) we have

\[
\limsup_{r \to 0} \frac{\text{osc}_{U_x \cap B(x, r)} w}{r} < \infty \quad \text{and} \quad \lim_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus U_x)}{r^{n-1}} = 0.
\]
Let $N \subset H$ be the $\mathcal{L}^n$-negligible set where $w$ is not approximately continuous. We can write $H$ as

$$H = \bigcup_{j=1}^{\infty} \bigcup_{i=j}^{\infty} H_{i,j} \cup N,$$

where each $H_{i,j}$ is the set of points $x \in H$ for which

$$\sup_{0 < r \leq 2/j} \frac{\text{Osc}_{U \cap B(x,r)} w}{r} \leq j \quad \text{and} \quad \sup_{0 < r \leq 2/i} \frac{\text{Cap}_1(B(x,r) \setminus U_x)}{r^{n-1}} \leq \frac{1}{j^{2n+1}}. \quad (4.24)$$

First we claim that $|w(x) - w(y)| \leq 2j|x - y|$ for all $x, y \in H_{i,j}$ with $|x - y| < 2/i$ and $i \geq j$ for sufficiently large $j \geq 2$. For such points $x$ and $y$, by (4.24) we have

$$\text{Cap}_1(B(x, |x-y|) \cap B(y, |x-y|) \setminus (U_x \cap U_y)) \leq \frac{\text{Cap}_1(B(x, |x-y|) \setminus U_x)}{|x-y|^{n-1}} + \frac{\text{Cap}_1(B(y, |x-y|) \setminus U_y)}{|x-y|^{n-1}} \leq \frac{2}{j^{2n+1}}.$$

By Lemma 2.18, it follows that

$$\frac{\mathcal{L}^n(B(x, |x-y|) \cap B(y, |x-y|) \setminus (U_x \cap U_y))}{\mathcal{L}^n(B(x, |x-y|))} \leq \frac{2C_1}{\omega_n} \frac{2}{j^{2n+1}},$$

and so when $j \geq J$ for some $J$ depending only on $n$, there exists a point

$$z \in B(x, |x-y|) \cap B(y, |x-y|) \cap (U_x \cap U_y).$$

Now

$$|w(x) - w(y)| \leq |w(x) - w(z)| + |w(z) - w(y)| \leq j|x-z| + j|z-y| \quad \text{by} \quad (4.24) \leq 2j|x-y|,$$

proving the claim for $j \geq J$.

In the intersection of $H_{i,j}$ with any ball of diameter at most $1/i$, when $i \geq j \geq J$, we now know that $w$ is $2j$-Lipschitz, and so we can extend it to a $2j$-Lipschitz function on the entire $\mathbb{R}^n$ (see e.g. [16, Theorem 3.1]) and then apply Rademacher’s theorem to obtain that $w|_{H_{i,j}}$ is differentiable a.e. Going over indices $i \geq j \geq J$, this gives countably many exceptional sets, so we can denote their union by $N'$ and we have $\mathcal{L}^n(N') = 0$. By Lemma 4.18, if we denote the union of nonporous points of the sets $H_{i,j}$ by $N''$, we also have $\mathcal{L}^n(N'') = 0$.

Now consider an arbitrary $x \in H \setminus (N \cup N' \cup N'')$. Note that in the representation (4.23), both the outer union and the inner union is increasing, and so we have $x \in H_{i,j}$ for some
\( i \geq j \geq J \). Now by the conclusion of Rademacher’s theorem, we have
\[
\lim_{H_{i,j} \ni y \to x} \frac{|w(y) - w(x) - \langle \nabla w |_{H_{i,j}}(x), y - x \rangle|}{|y - x|} = 0. \tag{4.25}
\]
As before, consider the annuli \( A_k := B(x, 2^{-k}) \setminus B(x, 2^{-(k+1)}) \), for all \( k \) sufficiently large that \( B(x, 2^{-k}) \subset \Omega \) and \( 2^{-k} < 1/i \). By the nonporosity, for all sufficiently large \( k \in \mathbb{N} \) we find \( Cj^{2n} \) many points \( y_{k,l} \in A_k \cap H_{i,j} \) such that for every \( z \in A_k \) there is a point \( y_{k,l} \) with \( |z - y_{k,l}| < j^{-2}2^{-k} \). As usual, \( C \) depends only on \( n \). Define
\[
U := \bigcup_k \left( U_{y_{k,l}} \cap A_k \right),
\]
where the union is over sufficiently large \( k \in \mathbb{N} \), as we required above. Then
\[
\text{Cap}_1(A_k \setminus U) \leq \sum_l \text{Cap}_1(A_k \setminus U_{y_{k,l}}) \leq \sum_l \text{Cap}_1(B(y_{k,l}, 2^{-k+1}) \setminus U_{y_{k,l}}) \leq Cj^{2n} \cdot 2^{-n-1} \times 2^{(-k+1)(n-1)} \quad \text{by (4.24)} \]
\[
= Cj^{-1} \times 2^{(-k+1)(n-1)}.
\]
Thus for large \( k \),
\[
\frac{\text{Cap}_1(B(x, 2^{-k}) \setminus U)}{2^{-k(n-1)}} \leq \sum_{m=k}^{\infty} \frac{\text{Cap}_1(A_m \setminus U)}{2^{-k(n-1)}} \leq C \sum_{m=k}^{\infty} \frac{j^{-1} \times 2^{(-m+1)(n-1)}}{2^{-k(n-1)}} \leq 2^n Cj^{-1},
\]
and so
\[
\limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \setminus U)}{r^{n-1}} \leq 2^n Cj^{-1}.
\]
Now for every \( z \in U \) sufficiently close to \( x \), we have \( z \in A_k \) for some \( k \in \mathbb{N} \) as large as required in the above estimates, and we find \( y_{k,l} \in A_k \cap H_{i,j} \) with \( |z - y_{k,l}| \leq j^{-2}2^{-k} \) and then
\[
|w(z) - w(x)| \leq |w(z) - w(y_{k,l})| + |w(y_{k,l}) - w(x)|,
\]
where by (4.24),
\[
|w(z) - w(y_{k,l})| \leq j|z - y_{k,l}| \leq jj^{-2}2^{-k}. \tag{4.26}
\]
We can assume \( J \geq 2 \), so that \( |y_{k,l} - x| \leq 2|z - x| \), and now
\[
\frac{|w(z) - w(x) - \langle \nabla w |_{H_{i,j}}(x), z - x \rangle|}{|z - x|} \leq \frac{|w(y_{k,l}) - w(x) - \langle \nabla w |_{H_{i,j}}(x), y_{k,l} - x \rangle|}{|z - x|} + \frac{|w(z) - w(y_{k,l})|}{|z - x|} + \frac{|\nabla w |_{H_{i,j}}(x)||y_{k,l} - z|}{|z - x|} \leq 2 \frac{|w(y_{k,l}) - w(x) - \langle \nabla w |_{H_{i,j}}(x), y_{k,l} - x \rangle|}{|y_{k,l} - x|} + jj^{-2}2^{-k} \frac{2}{2-k-1} + jj^{-2}2^{-k} \frac{2}{2-k-1}.
by (4.26) and since $|\nabla w|_{H_{i,j}}(x) \leq 2j$ due to $w|_{H_{i,j}}$ being locally $2j$-Lipschitz. Combining this with (4.25), we get
\[
\limsup_{U \ni z \to x} \frac{|w(z) - w(x) - (w|_{H_{i,j}}(x), z - x)|}{|z - x|} \leq 6j^{-1}.
\]
Recall that this holds for $i \geq j \geq J$, and so we can choose $i, j$ arbitrarily large. Thus Condition (3) of Lemma 4.2 is satisfied, and we get 1-fine differentiability at $x$. \hfill \Box

Recall the definition of the Hardy–Littlewood maximal function of a locally integrable function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ from (3.14). The so-called $W^{1,1}$-problem, posed in [24] by Hajlasz and Onnin, asks: do we have $M \in W^{1,1}(\mathbb{R}^n)$ when $u \in W^{1,1}(\mathbb{R}^n)$, and is the operator $u \mapsto |\nabla M u|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$? Toward answering this question, partial results have been proved by many authors in e.g. [2, 34, 38, 55]; often one considers the non-centered maximal function, where the supremum is taken over balls containing $x$. In particular, Hajlasz–Malý [23] showed the Hardy–Littlewood maximal function to be approximately differentiable a.e. Now we will show that it is 1-finely differentiable a.e.; as we have seen, this property is stronger and much closer to characterizing Sobolev functions.

We start with the following standard estimate.

**Lemma 4.27.** Let $u \in BV(\mathbb{R}^n)$, $x, y \in \mathbb{R}^n$, and $r > 0$. Then
\[
\left| \int_{B(y,r)} u \, d\mathcal{L}^n - \int_{B(x,r)} u \, d\mathcal{L}^n \right| \leq \frac{|y - x|}{\mathcal{L}^n(B(x,r))} |Du|(B(x, r + |y - x|)). \tag{4.28}
\]

**Proof.** First consider $v \in C^\infty(\mathbb{R}^n)$. We have
\[
\left| \int_{B(y,r)} v \, d\mathcal{L}^n - \int_{B(x,r)} v \, d\mathcal{L}^n \right| = \left| \int_{B(x,r)} (v(z + (y - x)) - v(z)) \, d\mathcal{L}^n(z) \right|
= \left| \int_{B(x,r)} \int_0^1 \frac{d}{dt} v(z + t(y - x)) \, dt \, d\mathcal{L}^n(z) \right|
\leq \frac{|y - x|}{\mathcal{L}^n(B(x,r))} \int_{B(x, r + |y - x|)} |\nabla v| \, d\mathcal{L}^n. \tag{4.29}
\]
We find a sequence of functions $\{v_i\}_{i \in \mathbb{N}}$ in $C^\infty(\mathbb{R}^n)$ such that $v_i \to u$ in $L^1(\mathbb{R}^n)$ and the measures $|Du_i|$ converge in the weak* sense to $|Du|$, see [4, Theorem 3.9, Proposition 1.80]. Writing (4.29) with $v = v_i$ and taking the limit $i \to \infty$, by the weak* convergence we get (see [4, Example 1.63])
\[
\left| \int_{B(y,r)} u \, d\mathcal{L}^n - \int_{B(x,r)} u \, d\mathcal{L}^n \right| \leq \frac{|y - x|}{\mathcal{L}^n(B(x,r))} |Du|(B(x, r + |y - x|)).
\]
\hfill \Box
We also define the Hardy–Littlewood maximal function of positive Radon measure \( \nu \) by
\[
\mathcal{M}_\nu(x) := \sup_{r > 0} \frac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))}.
\]

**Theorem 4.30.** Let \( u \in BV(\mathbb{R}^n) \). Then \( M_u \) is 1-finely differentiable a.e.

**Proof.** We also have \( |u| \in BV(\mathbb{R}^n) \), and so we can assume that \( u \geq 0 \). Take a point \( x \in \mathbb{R}^n \) where \( u^* \) is 1-finely differentiable; this is true of a.e. point by Theorem 4.8. By Lemma 4.2, this means that there is a 1-finely open set \( U \) containing \( x \) such that
\[
limit_{U \ni y \to x} \frac{|u^*(y) - u^*(x) - \langle \nabla u(x), y - x \rangle|}{|y - x|} = 0.
\]
(4.31)

We can also assume that \( x \) is a Lebesgue point of \( \nabla u \) (the density of the absolutely continuous part of \( Du \)) and that \( \mathcal{M}|Du|(x) < \infty \); note that \( \mathcal{M}\left|Du\right|(x) = \infty \) exactly when
\[
\limsup_{r \to 0} \left\frac{|Du|(B(x,r))}{\mathcal{L}^n(B(x,r))}\right. = \infty,
\]
which can happen only in a set of Lebesgue measure zero.

For simplicity, we can assume that \( x = 0 \).

**Step 1.** Here we estimate \( Mu(0) - Mu(y) \). The first possibility is that \( Mu(0) = u^*(0) \). Then we can estimate
\[
Mu(0) - Mu(y) \leq u^*(0) - u^*(y).
\]
The second possibility is that \( Mu(0) = u_{B(0,r)} \) for some \( 0 < r < \infty \). Then
\[
Mu(0) - Mu(y) \leq \int_{B(0,r)} u d\mathcal{L}^n - \int_{B(y,r)} u d\mathcal{L}^n
\]
\[
\leq \frac{|y|}{\mathcal{L}^n(B(0,r))} |Du|(|B(0, r + |y|)|) \quad \text{by (4.28)}
\]
\[
\leq \frac{\mathcal{L}^n(B(0, r + |y|))}{\mathcal{L}^n(B(0, r))} |y| \mathcal{M}\left|Du\right|(0).
\]
Combining the two cases and using (4.31) in the first case, we get
\[
\limsup_{U \ni y \to 0} \frac{Mu(0) - Mu(y)}{|y|} \leq \max\{|\nabla u(0)|, \mathcal{M}\left|Du\right|(0)\} = \mathcal{M}\left|Du\right|(0) < \infty,
\]
using also the fact that 0 is a Lebesgue point of \( \nabla u \).

**Step 2.** Here we estimate \( Mu(y) - Mu(0) \). Note that
\[
Mu(y) = \max\left\{ \sup_{0 < r \leq |y|} \int_{B(y,r)} u d\mathcal{L}^n, \sup_{r \geq |y|} \int_{B(y,r)} u d\mathcal{L}^n \right\}.
\]

**Step 2a.** Consider the first option in the maximum; recall that it is denoted by \( \mathcal{M}_{|y|}u(y) \).

We define \( L(z) := \langle \nabla u(0), z \rangle \) and the scalings
\[
u_r(z) := \frac{u(rz) - u^*(0)}{r}, \quad z \in B(0, 2).
\]
From (4.10), initially excluding another set of Lebesgue measure zero, we can also assume that
\[ u_r \to L \text{ in } \text{BV}(B(0, 2)). \]  
(4.32)
Suppose \( u^*(0) = 0 \); then by (4.31) necessarily \( \nabla u(0) = 0 \) since \( u \geq 0 \). Now by the easily verified scaling properties of the maximal function, for \( \varepsilon > 0 \) we estimate
\[
2^{j(n-1)} \text{Cap}_1(\{ y \in B(0, 2^{-j}) : M_{2^{-j}}u(y) > 2^{-j}\varepsilon \}) \\
= \text{Cap}_1(\{ z \in B(0, 1) : M_1 u_{2^{-j}}(z) > \varepsilon \}) \\
\leq C \frac{\|u_{2^{-j}}\|_{\text{BV}(B(0, 2))}}{\varepsilon} \text{ by Lemma 4.7} \\
\to 0 \text{ as } j \to \infty \text{ by (4.32)}.
\]

Then assume \( u^*(0) > 0 \), and also define \( L'(y) := (\nabla u(0), y) + u^*(0) \). Now we have \( L'(y) > 0 \) in a neighborhood of 0, and so for large \( j \) we estimate
\[
2^{j(n-1)} \text{Cap}_1(\{ y \in B(0, 2^{-j}) : M_{2^{-j}}u(y) - L'(y) > 2^{-j}\varepsilon \}) \\
= 2^{j(n-1)} \text{Cap}_1(\{ y \in B(0, 2^{-j}) : M_{2^{-j}}u(y) - M_{2^{-j}}L'(y) > 2^{-j}\varepsilon \}) \\
\leq 2^{j(n-1)} \text{Cap}_1(\{ y \in B(0, 2^{-j}) : M_{2^{-j}}|u - L'|(y) > 2^{-j}\varepsilon \}) \\
= \text{Cap}_1(\{ z \in B(0, 1) : M_1 |u_{2^{-j}} - L|((z) > \varepsilon \}) \\
\leq C \frac{\|u_{2^{-j}} - L\|_{\text{BV}(B(0, 2))}}{\varepsilon} \text{ by Lemma 4.7} \\
\to 0 \text{ as } j \to \infty \text{ by (4.32)}.
\]

Thus, whether \( u^*(0) = 0 \) or \( u^*(0) > 0 \), in both cases we can choose a sequence \( \varepsilon_j \to 0 \) and sets \( V_j \subset B(x, 2^{-j}) \) such that
\[
\frac{M_{2^{-j}}u(y) - L'(y)}{2^{-j}} \leq \varepsilon_j \text{ for all } y \in V_j \text{ and } \lim_{j \to \infty} \text{Cap}_1(B(0, 2^{-j}) \setminus V_j) = 0. \]  
(4.33)
Define the annuli \( A_j := B(0, 2^{-j}) \setminus B(0, 2^{-j-1}) \). Then defining \( V' := \bigcup_{j=1}^{\infty} (A_j \cap V_j) \cup \{x\} \), we have that
\[
\text{Cap}_1(B(0, 2^{-k}) \setminus V') \leq \sum_{j=k}^{\infty} \text{Cap}_1(A_j \setminus V_j) \\
\leq \sum_{j=k}^{\infty} 2^{-j(n-1)} \times \max_{j \geq k} \frac{\text{Cap}_1(B(0, 2^{-j}) \setminus V_j)}{2^{-j(n-1)}} \\
\leq 2^{-k(n-1)+1} \times \max_{j \geq k} \frac{\text{Cap}_1(B(0, 2^{-j}) \setminus V_j)}{2^{-j(n-1)}}.
\]

Thus
\[
\lim_{k \to \infty} \frac{\text{Cap}_1(B(0, 2^{-k}) \setminus V')}{2^{-k(n-1)}} = 0,
\]
and so \( \mathbb{R}^n \setminus V' \) is 1-thin at 0. By (2.27), \( V := \text{fine-int}(V') \) is a 1-finely open set containing 0, and by (4.33),
\[
\limsup_{V \ni y \to 0} \frac{\mathcal{M}|y|u(y) - L'(y)}{|y|} \leq 0.
\]
Thus
\[
\limsup_{V \ni y \to 0} \frac{\mathcal{M}|y|u(y) - \mathcal{M}u(0)}{|y|} \leq \limsup_{V \ni y \to 0} \frac{L'(y) - \mathcal{M}u(0)}{|y|} \leq \limsup_{V \ni y \to 0} \frac{L'(y) - u^*(0)}{|y|} = |\nabla u(0)|.
\]

**Step 2b.** Now consider the latter quantity in the maximum at the beginning of Step 2. Note that for every \( y \neq 0 \), we find \( r_y \geq |y| \) such that
\[
\sup_{r \geq |y|} \int_{B(y,r)} u \, d\mathcal{L}^n = \int_{B(y,r_y)} u \, d\mathcal{L}^n.
\]
We estimate
\[
\sup_{r \geq |y|} \int_{B(y,r)} u \, d\mathcal{L}^n - \mathcal{M}u(0) \leq \int_{B(y,r_y)} u \, d\mathcal{L}^n - \int_{B(0,r_y)} u \, d\mathcal{L}^n \leq \frac{|y|}{\mathcal{L}^n(B(0,r_y))} |Du| (B(0,r_y) + |y|) \text{ by (4.28)} \leq \frac{\mathcal{L}^n(B(0,r_y) + |y|)}{\mathcal{L}^n(B(0,r_y))} |y| |\nabla u(0)| \leq 2^n |y| |\mathcal{M}|Du|(0).
\]
Combining Steps 2a and 2b, we have
\[
\limsup_{V \ni y \to 0} \frac{\mathcal{M}u(y) - \mathcal{M}u(0)}{|y|} \leq 2^n \max\{|\nabla u(0)|, \mathcal{M}|Du|(0)\} = 2^n \mathcal{M}|Du|(0).
\]
Combining Steps 1 and 2, we have that
\[
\limsup_{U \cap V \ni y \to 0} \frac{|\mathcal{M}u(y) - \mathcal{M}u(0)|}{|y|} \leq 2^n \mathcal{M}|Du|(0).
\]
The set \( U \cap V \ni 0 \) is 1-finely open. Thus \( \text{Lip}_x^0 \mathcal{M}u(x) \leq 2^n \mathcal{M}|Du|(x) < \infty \) for a.e. \( x \in \mathbb{R}^n \). Now by Theorem 4.19, \( \mathcal{M}u \) is 1-finely differentiable a.e.

The above proof works, with small changes, also for the non-centered maximal function, where the supremum is taken over balls containing \( x \). We chose to give the proof for the ordinary Hardy–Littlewood maximal function because in general, the non-centered version tends to have the better regularity of the two.
5. The number \( \text{lip}_f \) and quasiconformal mappings

As mentioned in the introduction, there are certain results that can be formulated quite analogously on the one hand with the Lipschitz number \( \text{lip}_f \), on the other hand with the distortion number \( h_f \). Thus, having studied generalized Lipschitz numbers in the previous sections, in this section we will study a generalized version of the distortion number.

As before, \( \Omega \subset \mathbb{R}^n \) is always an open set. Moreover, we assume \( 1 \leq p \leq \infty \).

First consider the Lipschitz number

\[
\text{lip}_w(x) := \liminf_{r \to 0} \sup_{y \in B(x, r)} \frac{|w(y) - w(x)|}{r}.
\]

Balogh–Csörnyei [6, Theorem 1.2] showed that if \( w : \Omega \to \mathbb{R} \) satisfies the following:

- \( w \) is continuous;
- \( \text{lip}_w < \infty \) outside a set of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure; and
- \( \text{lip}_w \in L^p(\Omega) \),

then \( w \in W^{1,p}_{\text{loc}}(\Omega) \). Generalizations of this result to metric measure spaces have been shown by Wildrick and Zürcher [57, 58].

Of course, none of the converses hold: when \( 1 \leq p \leq n \), a Sobolev function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) may be discontinuous and have \( \text{lip}_{u^*}(x) = \infty \) for every \( x \in \Omega \). Precisely because Sobolev functions are not generally continuous, one often uses the fact that they are nonetheless quasicontinuous, and so it was asked in [58, Remark 3.15] whether it would be enough to assume quasicontinuity in the above type of theorem. As a special case of our Theorem 1.4, we get the following proposition that is much closer to an if and only if result.

**Proposition 5.1.** Suppose \( \Omega \) is bounded and \( u : \Omega \to [-\infty, \infty] \) is measurable such that:

1. \( u^* \) is \( p \)-quasicontinuous in \( \Omega \);
2. \( \text{Lip}^0_{u^*} < \infty \) outside a set of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure; and
3. \( \text{Lip}^0_{u^*} \in L^p(\Omega) \).

Then \( u \in D^p(\Omega) \).

Conversely, if \( u \in D^p(\Omega) \), then (1) and (3) are satisfied.

**Proof.** To prove the first claim, denote by \( E \) the subset of \( \Omega \) where \( \text{Lip}^0_{u^*} = \infty \). Since \( u^* \) is \( p \)-quasicontinuous, it is also \( 1 \)-quasicontinuous by (2.24), and thus also measurable. Then by Lemma 3.2, given any direction \( v \in \partial B(0, 1) \), \( u^* \) is continuous on a.e. line parallel to \( v \), and the intersection of \( E \) with almost every line \( l \) parallel to \( v \) is at most countable, see e.g. [54, p. 103]. Now by applying Theorem 3.32 with the choices \( \nu = \mathcal{L}^n \) and \( h = \text{Lip}^0_{u^*} \), we get \( u \in D^p(\Omega) \) with \( d|Du| \leq C \text{Lip}^0_{u^*} \ d\mathcal{L}^n \), so that in fact \( u \in D^p(\Omega) \).

For the converse claim, note that \( D^p(\Omega) \subset W^{1,p}_{\text{loc}}(\Omega) \) and recall the \( p \)-quasicontinuity from (2.15). By Theorem 4.8, \( u^* \) is \( 1 \)-finely differentiable at a.e. \( x \in \Omega \). At such points, by Proposition 4.16 we have \( \text{Lip}^0_{u^*}(x) = |\nabla_{\text{fine}} u^*(x)| = |\nabla u(x)| \). Thus \( \text{Lip}^0_{u^*} \in L^p(\Omega) \). \( \square \)
The next example shows that we do not in general have a converse for (2).

**Example 5.2.** Let \( g \in L^1(\mathbb{R}) \) be a nonnegative, lower semicontinuous function on the real line with \( g = \infty \) in an uncountable set \( A \) (with \( L^1(A) = 0 \), of course). Let

\[
u(x) := \int_{-\infty}^{x} g(t) \, dt, \quad x \in \mathbb{R}.
\]

Now \( \nu \in D^1(\mathbb{R}) \), and \( \nu \) (as well as \( \nu^* \)) is continuous and \( \text{Lip}_{\nu^*}^0 \nu = g \) a.e., so that \( \text{Lip}_{\nu^*}^0 \nu \in L^1(\mathbb{R}) \). But \( \text{Lip}_{\nu^*}^0 \nu = \infty \) in \( A \), which is not a set of \( \sigma \)-finite \( \mathcal{H}^0 \)-measure.

On the other hand, the usual Cantor–Vitali function \( u : [0, 1] \to [0, 1] \) is a continuous function for which \( \text{lip}^0 u^* = 0 \) a.e. but \( u \notin W^{1,1}_{\text{loc}}((0, 1)) \).

This example demonstrates that the quantity \( \text{Lip}_{\nu^*}^0 \nu \) is insufficient for characterizing Sobolev functions. This is the main motivation for considering the function \( a \) and the generalized Lipschitz number \( \text{Lip}_{a,\delta}^0 u^* \).

Now we study quasiconformal mappings; for the rest of the section we will assume \( n \geq 2 \).

Consider a mapping \( f : \Omega \to \mathbb{R}^n \). For every \( x \in \Omega \) and \( r > 0 \), we define

\[
L_f(x, r) := \sup \{|f(y) - f(x)| : y \in \Omega, |y - x| \leq r}\]

and

\[
l_f(x, r) := \inf \{|f(y) - f(x)| : y \in \Omega, |y - x| \geq r}\],
\]

and then

\[
H_f(x, r) := \frac{L_f(x, r)}{l_f(x, r)}; \quad (5.5)
\]

we interpret this to be \( \infty \) if the denominator is zero or the numerator is \( \infty \). A homeomorphism \( f : \mathbb{R}^n \to \mathbb{R}^n \) is said to be (metric) quasiconformal if there is a number \( 1 \leq H < \infty \) such that

\[
H_f(x) := \limsup_{r \to 0} H_f(x, r) \leq H
\]

for all \( x \in \mathbb{R}^n \). We also define

\[
h_f(x) := \liminf_{r \to 0} H_f(x, r). \quad (5.7)
\]

For \( 1 \leq p \leq n \), we denote the Sobolev conjugate by \( p^* = np/(n - p) \) when \( p < n \), and \( p^* = \infty \) when \( p = n \).

Quasiconformal mappings are in the class \( W^{1,n}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n) \), but with relaxed requirements on \( H_f \) or \( h_f \), it is possible to show at least weaker regularity. The following theorem is known to hold; for a proof, see e.g. Koskela–Rogovin [33, Corollary 1.3] and Kallunki–Martio [30, Theorem 2.2]. Note the analogy with the above result by Balogh–Csörgyevi involving \( \text{lip}_w \).

**Theorem 5.8.** Let \( f : \Omega \to f(\Omega) \) be a homeomorphism, where \( f(\Omega) \) is open, and let \( 1 \leq p \leq n \). Suppose \( h_f \in L_{\text{loc}}^{n/(n-1)}(\Omega) \) and \( h_f < \infty \) outside a set \( E \) with \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure. Then \( f \in W^{1,1}_{\text{loc}}(\Omega; f(\Omega)) \). If also \( h_f \in L_{\text{loc}}^{p^*/(n-1)/n}(\Omega) \), then \( f \in W^{1,p}_{\text{loc}}(\Omega; f(\Omega)) \).
The literature studying this type of result as well as previous weaker versions is extensive, see e.g. Gehring [21, 22], Margulis–Mostow [47], Fang [17], Balogh–Koskela [7], Kallunki–Koskela [29], Heinonen–Koskela–Shanmugalingam–Tyson [27], Kallunki–Martio [30], Koskela–Rogovin [33], Balogh–Koskela–Rogovin [8], and Lahti–Zhou [41, 42]. Essentially, in this type of result there are always two exceptional sets. One exceptional set is the \( n-1 \)-dimensional set \( E \), whereas the condition \( h_f \in L^{n/(n-1)}_{\text{loc}}(\Omega) \) fails to give control in a larger exceptional set of zero \( L^n \)-measure. The sharpness of the requirement on the size of \( E \) has been studied by Hanson [26] and Williams [56, Remark 1.9].

It is then natural to ask, why are there specifically these two exceptional sets, of dimensions \( n-1 \) and \( n \)? The crux of the proof is usually showing absolute continuity on lines or \( W^{1,1}_{\text{loc}} \)-regularity, which also raises the question of whether there is a BV version of the result. In fact, the sizes of the two different exceptional sets are the same as the sizes of the sets where a BV function fails to be approximately continuous, and where it fails to be approximately differentiable.

Towards resolving these questions, consider our definition of generalized Lipschitz numbers. For a mapping \( f: \Omega \to \mathbb{R}^n \) and a positive Radon measure \( \nu \) on \( \Omega \), similarly to (1.1) we define

\[
\text{lip}_\nu^f(x) := \frac{1}{2} \liminf_{r \to 0} \frac{\text{osc}_{B(x,r)} f \mathcal{L}^n(B(x,r))}{r \nu(B(x,r))}, \quad x \in \Omega.
\]

Analogously, we define a generalized version of \( h_f \) as follows:

\[
h^\nu_f(x) := \liminf_{r \to 0} \frac{L_f(x,r)}{l_f(x,r)} \left( \frac{\mathcal{L}^n(B(x,r))}{\nu(B(x,r))} \right)^{(n-1)/n}, \quad x \in \Omega.
\]

For convenience, assume that the entire \( \Omega \) is in the support of \( \nu \), so that always \( \nu(B(x,r)) > 0 \) above. It is straightforward to show that \( \text{lip}_\nu^f, h^\nu_f \) are Borel functions when \( f \) is continuous, which guarantees that various integrals below are well defined.

**Remark 5.9.** Note that unlike with \( \text{Lip}_\nu^{\delta} \), here we do not work with the 1-finely open sets \( U_r \subset B(x,r) \), but rather we consider the entire ball \( B(x,r) \). The reason for this is the extra difficulty caused by the fact that we have “\( \lim\inf \)” in the definitions instead of “\( \lim\sup \)”. Overall, there are many possible variations and alternative formulations of our definitions, which may prove interesting to study, but in this paper we limit ourselves to certain natural choices, which often allow easy comparison with previous results in the literature.

Now we prove the following theorem which includes the BV case. We will find that Theorem 5.8 can be deduced as a special case, but Theorem 5.10 is also able to detect more Sobolev functions, as we will see in Example 5.28.

**Theorem 5.10.** Suppose \( \Omega \) is nonempty and bounded, \( f: \Omega \to f(\Omega) \subset \mathbb{R}^n \) is injective and continuous such that \( f(\Omega) \) is open and \( \mathcal{L}^n(f(\Omega)) < \infty \), and there exists a finite Radon
measure \( \nu \geq L^n \) on \( \Omega \) such that \( \min \{ \lip_f(y), h_f(y) \} < \infty \) for \( \mathcal{H}^{n-1} \)-a.e \( x \in \Omega \), and
\[
\int_{\Omega} \min \{ \lip_f, (h_f)^{n/(n-1)} \} \, d\nu < \infty. \tag{5.11}
\]

Then \( f \in \BV_{\text{loc}}(\Omega; \mathbb{R}^n) \) with
\[
\Var(f, \Omega) \leq C \left( \int_{\Omega} \min \{ \lip_f, (h_f)^{n/(n-1)} \} \, d\nu + L^n(f(\Omega)) \right),
\]
where \( C = 2^{n^2+2} n N_n \). If \( \min \{ \lip_f, (h_f)^{n/(n-1)} \} \, d\nu \) is absolutely continuous with respect to the Lebesgue measure, then \( f \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^n) \).

Proof. By assumption, we find a partition of \( \Omega \) into disjoint sets \( A_1, A_2, \) and \( N \), where \( \min \{ \lip_f, (h_f)^{n/(n-1)} \} = \lip_f < \infty \) in \( A_1 \), \( \min \{ \lip_f, (h_f)^{n/(n-1)} \} = (h_f)^{n/(n-1)} < \infty \) in \( A_2 \), and \( \mathcal{H}^{n-1}(N) = 0 \). Fix \( \epsilon > 0 \). By the Vitali-Carathéodory theorem (Theorem 2.1), we can take a lower semicontinuous function \( h \geq 0 \) on \( \Omega \) such that \( h \geq \min \{ \lip_f, (h_f)^{n/(n-1)} \} \) and
\[
\int_{\Omega} h \, d\nu \leq \int_{\Omega} \min \{ \lip_f, (h_f)^{n/(n-1)} \} \, d\nu + \epsilon. \tag{5.12}
\]

For every \( x \in A_1 \), we can choose a radius \( 0 < r_x \leq \epsilon \) such that \( \overline{B}(x, r_x) \subset \Omega \) and
\[
\frac{1}{2} \frac{\osc_{\overline{B}(x, r_x)} f}{r_x} \frac{L^n(\overline{B}(x, r_x))}{\nu(\overline{B}(x, r_x))} \leq \frac{1}{2} \frac{\osc_{\overline{B}(x, r_x)} f}{r_x} \frac{L^n(B(x, r_x))}{\nu(B(x, r_x))} < \lip_f(x) + \epsilon, \tag{5.13}
\]
and also, by the lower semicontinuity of \( h \),
\[
\lip_f(x) \nu(\overline{B}(x, r_x)) \leq \int_{\overline{B}(x, r_x)} (h + \epsilon) \, d\nu. \tag{5.14}
\]

By the Besicovitch covering theorem (see e.g. [16, Theorem 1.27]), we can select collections \( \{ \overline{B}_{j,k} \in \overline{B}(x, r_x) \}_{k=1}^{\infty} \), consisting of disjoint balls for each \( j = 1, \ldots, N_n \), and such that \( A_1 \subset \bigcup_{j=1}^{N_n} \bigcup_{k=1}^{\infty} \overline{B}_{j,k} \).

Analogously, for every \( y \in A_2 \), we can choose a radius \( 0 < s_y \leq \epsilon \) such that \( \overline{B}(y, s_y) \subset \Omega \) and \( L_f(y, s_y) \subset f(\Omega) \) (using the continuity of \( f \)), and
\[
\frac{L_f(y, s_y)}{l_f(y, s_y)} \left( \frac{L^n(\overline{B}(y, s_y))}{\nu(\overline{B}(y, s_y))} \right)^{(n-1)/n} \leq \frac{L_f(y, s_y)}{l_f(y, s_y)} \left( \frac{L^n(B(y, s_y))}{\nu(B(y, s_y))} \right)^{(n-1)/n} < h_f(y) + \epsilon, \tag{5.15}
\]
and also
\[
h_f(y)^{n/(n-1)} \nu(\overline{B}(y, s_y)) \leq \int_{\overline{B}(y, s_y)} (h + \epsilon) \, d\nu. \tag{5.16}
\]

By the Besicovitch covering theorem, we can select collections \( \{ \overline{B}_{j,k} \in \overline{B}(y, s_y) \}_{k=1}^{\infty} \), consisting of disjoint balls for each \( j = 1, \ldots, N_n \), and such that \( A_2 \subset \bigcup_{j=1}^{N_n} \bigcup_{k=1}^{\infty} \overline{B}_{j,k} \).
Denote $L_{j,k} := L_f(y_{j,k}, s_{j,k})$ and $l_{j,k} := l_f(y_{j,k}, s_{j,k})$, and also write the sum $\sum_{j=1}^{N_n} \sum_{k=1}^{\infty}$ in the abbreviated form $\sum_{j,k}$. Define

$$g := \sum_{j,k} \frac{\text{osc} B_{j,k}}{r_{j,k}} f_{2B_{j,k}} + 2 \sum_{j,k} \frac{L_{j,k}}{s_{j,k}} \chi_{2\widehat{B}_{j,k}}.$$  

Almost every line in the direction of a coordinate axis has empty intersection with $N$. Take a line segment $\gamma : [0, \ell] \to \Omega$ in such a line, of length $\ell$. Assume $\ell \geq \varepsilon$. The collections $\{B_{j,k}\}_{j,k}$ and $\{\widehat{B}_{j,k}\}_{j,k}$ together cover $\gamma$. If $\gamma \cap B_{j,k} \neq \emptyset$, then

$$\int_{\gamma} \frac{\text{osc} B_{j,k}}{r_{j,k}} f_{2B_{j,k}} \, ds \geq \text{osc} B_{j,k} f.$$  

Similarly if $\gamma \cap \widehat{B}_{j,k} \neq \emptyset$, then

$$2 \int_{\gamma} \frac{L_{j,k}}{r_{j,k}} \chi_{2\widehat{B}_{j,k}} \, ds \geq 2L_{j,k} \geq \text{osc} \widehat{B}_{j,k} f.$$  

It follows that

$$\int_{\gamma} g \, ds \geq \sum_{j,k : \gamma \cap B_{j,k} \neq \emptyset} \text{osc} B_{j,k} f + \sum_{j,k : \gamma \cap \widehat{B}_{j,k} \neq \emptyset} \text{osc} \widehat{B}_{j,k} f \geq |f(\gamma(0)) - f(\gamma(\ell))| \quad (5.17)$$  

by the continuity of $f$. On the other hand, we estimate

$$\sum_{j,k} \frac{\text{osc} B_{j,k}}{r_{j,k}} f_{2B_{j,k}} \leq 2^{n+1} \sum_{j,k} \nu(2B_{j,k})(\text{lip}_f(x_{j,k}) + \varepsilon) \quad \text{by (5.13)}$$  

$$\leq 2^{n+1} \sum_{j,k} \int_{B_{j,k}} (h + 2\varepsilon) \, d\nu \quad \text{by (5.14)} \quad (5.18)$$

$$\leq 2^{n+1} N_n \int_{\Omega} (h + 2\varepsilon) \, d\nu.$$  

Next, we will use a generalized Young’s inequality: with $1/n + (n - 1)/n = 1$, by the usual Young’s inequality we have for any $\alpha, \beta \geq 0$ and any choice of $0 < \delta \leq 1$ that

$$\alpha \beta = \delta^{1/n} \alpha^{1-n/\beta} \leq \frac{1}{n} \delta \alpha^n + \frac{n-1}{n} \delta^{-1/(n-1)} \beta^{n/(n-1)} \leq \delta \alpha^n + \delta^{-1/(n-1)} \beta^{n/(n-1)}. \quad (5.19)$$

Note also that $B(f(y_{j,k}), l_{j,k}) \subset B(f(y_{j,k}), L_{j,k}) \subset f(\Omega)$, and then by the definition of $l_f(\cdot, \cdot)$, necessarily

$$B(f(y_{j,k}), l_{j,k}) \subset f(\widehat{B}_{j,k}). \quad (5.20)$$
Now we estimate
\[
\sum_{j,k} \frac{L_{j,k}}{s_{j,k}} \mathcal{L}^n(2\hat{B}_{j,k})
\]
\[
\leq \sum_{j,k} \frac{l_{j,k}}{s_{j,k}} \frac{\mathcal{L}^n(2\hat{B}_{j,k})}{\mathcal{L}^n(\hat{B}_{j,k})} \left( \frac{\nu(\hat{B}_{j,k})}{\mathcal{L}^n(\hat{B}_{j,k})} \right)^{(n-1)/n} (h_f^r(y_{j,k}) + \varepsilon) \quad \text{by (5.15)}
\]
\[
= 2^n \omega_1^{1/n} \sum_{j,k} l_{j,k} \nu(\hat{B}_{j,k})^{(n-1)/n} (h_f^r(y_{j,k}) + \varepsilon)
\]
\[
\leq 2^n \omega_1 \sum_{j,k} l_{j,k} + \delta^{-1/(n-1)} \sum_{j,k} (h_f^r(y_{j,k}) + \varepsilon)^{n/((n-1))} \nu(\hat{B}_{j,k}) \quad \text{by (5.19)}
\]
\[
\leq 2^n \delta \sum_{j,k} \mathcal{L}^n(f(\hat{B}_{j,k})) \quad \text{by (5.20)}
\]
\[
+ \delta^{-1/(n-1)} 2^{n/((n-1))} \sum_{j,k} \left( h_f^r(y_{j,k})^{n/((n-1))} + \varepsilon^{n/((n-1))} \right) \nu(\hat{B}_{j,k})
\]
\[
\leq 2^n \delta N_n \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} 2^{n/((n-1))} N_n \int_{\Omega} (h + \varepsilon + \varepsilon^{n/((n-1))}) d\nu
\]
by the injectivity of \( f \) and by (5.16). By combining (5.18) and (5.21), we get
\[
\int_{\Omega} g d\mathcal{L}^n \leq \sum_{j,k} \frac{\text{Osc}_{B_j,k} f}{r_{j,k}} \mathcal{L}^n(2B_{j,k}) + 2 \sum_{j,k} \frac{L_{j,k}}{s_{j,k}} \mathcal{L}^n(2\hat{B}_{j,k})
\]
\[
\leq 2^{n+1} \delta N_n \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} 2^n N_n \int_{\Omega} (h + 2\varepsilon) d\nu.
\]
(5.22)

Recall that \( \varepsilon > 0 \) has been fixed. Now, with the choices \( \varepsilon = 1/i \), we get sequences \( \{g_i\}_{i=1}^{\infty} \) and \( \{h_i\}_{i=1}^{\infty} \). Recall the definition of pointwise variation from (2.11). By (5.17), for every \( z \in \pi_n(\Omega) \) such that the line in the \( n:\text{th} \) coordinate direction intersecting \( (z,0) \) does not intersect \( N \), we get
\[
pV(f_z, \Omega_z) \leq \liminf_{i \to \infty} \int_{\Omega_z} g_i ds.
\]
We estimate
\[
\int_{\pi_n(\Omega)} pV(f, \Omega) \, d\mathcal{L}^{n-1}(z) \\
\leq \int_{\pi_n(\Omega)} \liminf_{i \to \infty} \int_{\Omega} g_i \, ds \, d\mathcal{L}^{n-1}(z) \\
\leq \liminf_{i \to \infty} \int_{\pi_n(\Omega)} \int_{\Omega} g_i \, ds \, d\mathcal{L}^{n-1}(z) \text{ by Fatou’s lemma} \\
= \limsup_{i \to \infty} \int_{\Omega} g_i \, d\mathcal{L}^n \text{ by Fubini} \\
\leq \limsup_{i \to \infty} \left[ 2^{n^2+1} \delta N_n \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} 2^{n+2} N_n \int_{\Omega} (h_i + 2/i) \, d\nu \right] \text{ by (5.22)} \\
\leq 2^{n^2+1} \delta N_n \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} 2^{n+2} N_n \int_{\Omega} \min \{ \text{lip}_f, (h_f)_{n/(n-1)} \} \, d\nu \text{ by (5.12)} \\
< \infty \text{ by (5.11)}.
\]

Recall (2.13). Since we can do the above calculation also in other coordinate directions, we obtain
\[
\text{Var}(f, \Omega) \leq 2^{n^2+1} \delta N_n \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} 2^{n+2} N_n \int_{\Omega} \min \{ \text{lip}_f, (h_f)_{n/(n-1)} \} \, d\nu.
\]

With the choice \( \delta = 1 \), this proves the first claim.

Now assume that \( \min \{ \text{lip}_f, (h_f)_{n/(n-1)} \} \, d\nu \) is absolutely continuous with respect to the Lebesgue measure. Note that by replacing \( \Omega \) with an open set \( W \subset \Omega \), we have in fact
\[
|Df|(W) \leq C \left( \delta \mathcal{L}^n(f(\Omega)) + \delta^{-1/(n-1)} \int_W \min \{ \text{lip}_f, (h_f)_{n/(n-1)} \} \, d\nu \right)
\]
for every open set \( W \subset \Omega \) and every \( 0 < \delta \leq 1 \). From this, it easily follows that \( |Df| \) is absolutely continuous with respect to the Lebesgue measure, and so in fact \( f \in D^1(\Omega; \mathbb{R}^n) \subset \mathcal{W}^{1,1}_\text{loc}(\Omega; \mathbb{R}^n) \).

Given a positive \( a \in L^1(\Omega) \), denote \( h^a_f := h^a_f \mathcal{L}^n \).

**Corollary 5.23.** Suppose \( \Omega \) is bounded, \( f: \Omega \to f(\Omega) \subset \mathbb{R}^n \) is injective and continuous such that \( f(\Omega) \) is open and \( \mathcal{L}^n(f(\Omega)) < \infty \), and there exists \( a \in L^1(\Omega) \) with \( a \geq 1 \), and a set \( E \) of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure such that \( h^a_f(x) < \infty \) for every \( x \in \Omega \setminus E \). Assume also that
\[
\| h^a_f \mathcal{L}^n(\Omega) \|_{L^{n/(n-1)}(\Omega)} < \infty.
\]
Then \( f \in D^1(\Omega; \mathbb{R}^n) \).

**Proof.** We can represent \( E \) as a union \( E = \bigcup_{j=1}^\infty E_j \cup N \), with \( 0 < \mathcal{H}^{n-1}(E_j) < \infty \) for all \( j \in \mathbb{N} \), and \( \mathcal{H}^{n-1}(N) = 0 \). Since \( \mathcal{H}^{n-1} \) is Borel regular, we can assume each \( E_j \) to be a
Borel set. Define $\nu$ by
\[
d\nu := a \, d\mathcal{L}^n + \sum_{j=1}^{\infty} 2^{-j} \mathcal{H}^{n-1}(E_j)^{-1} d\mathcal{H}^{n-1} \mathcal{L}^n_{E_j}.
\] (5.24)

By a standard density result, see e.g. [4, Theorem 2.56], for $\mathcal{H}^{n-1}$-a.e. $x \in E$ we have
\[
\limsup_{r \to 0} r \, \nu(B(x, r)) > 0.
\]

At these points, by the continuity of $f$ we get
\[
\text{lip}_{\nu} f(x) = \liminf_{r \to 0} \frac{\text{osc}_{B(x, r)} f}{r} \frac{\mathcal{L}^n(B(x, r))}{\nu(B(x, r))} = 0.
\]

Thus we obtain $\min\{\text{lip}_{\nu} f, (h_{\nu} f)^{n/(n-1)}\} = (h_{\nu} f)^{n/(n-1)} < \infty$ $\mathcal{H}^{n-1}$-a.e. in $\Omega \setminus E$. Moreover, now
\[
\min\{\text{lip}_{\nu} f, (h_{\nu} f)^{n/(n-1)}\} \, d\nu \leq (h_{\nu} f)^{n/(n-1)} a \, d\mathcal{L}^n,
\]
which is absolutely continuous with respect to $\mathcal{L}^n$, and
\[
\int_{\Omega} \min\{\text{lip}_{\nu} f, (h_{\nu} f)^{n/(n-1)}\} \, d\nu \leq \int_{\Omega} (h_{\nu} f)^{n/(n-1)} a \, d\mathcal{L}^n < \infty
\]
by assumption. Now by Theorem 5.10 we get $f \in D^1(\Omega; \mathbb{R}^n)$. \hfill $\square$

**Proof of Theorem 5.8.** To obtain the first claim, apply Corollary 5.23 with $a = 1$. The second claim is then straightforward to obtain, see the proof of [30, Theorem 2.2]. \hfill $\square$

**Remark 5.25.** Recalling (5.24), and as we further choose $a = 1$ in the proof of Theorem 5.8, we get
\[
\nu = \mathcal{L}^n + \sum_{j=1}^{\infty} 2^{-j} \mathcal{H}^{n-1}(E_j)^{-1} \mathcal{H}^{n-1} \mathcal{L}^n_{E_j}.
\] (5.26)

So in effect the “standard” result in the literature, Theorem 5.8, utilizes this very specific form of $\nu$! But there is no reason why this should always be the optimal choice. For example, one can consider a weight $a$ that is not identically 1. In Example 5.28 below we do precisely this, and as a result we are able to detect many more Sobolev functions.

Recall also that in Theorem 5.10, we showed that
\[
|Df|(\Omega) \leq C \left( \int_{\Omega} \min\{\text{lip}_{\nu} f, (h_{\nu} f)^{n/(n-1)}\} \, d\nu + \mathcal{L}^n(f(\Omega)) \right).
\] (5.27)

The measure $\nu$ given by (5.26) charges the set $E$, which is $\sigma$-finite with respect to $\mathcal{H}^{n-1}$, just like the approximate discontinuity set $S_f$ of a BV function. Thus (5.27) appears to give an upper bound for the jump part $d|D^1 f| = |f^+ - f^-| \, d\mathcal{H}^{n-1} \mathcal{L}^n_{S_f}$ (recall (2.10)), and so the exceptional set $E$ can be interpreted to correspond to the approximate discontinuity set of
the BV function $f$, as was alluded to previously. But since $f$ is continuous by assumption, there can be no approximate discontinuity set and so we actually got $f \in W^{1,1}_{\text{loc}}(\Omega; \mathbb{R}^n)$.

On the other hand, this suggests that there might be an interesting version of Theorem 5.10 where continuity would not be assumed.

The condition on the size of $E$ in Theorem 5.8 has been investigated in the literature, and it is well known that $E$ can be at most $n-1$-dimensional, see e.g. [26]. In particular, Cantor sets of higher dimension are not allowed. However, we can now consider the following example.

**Example 5.28.** Let $\Omega = (0,1) \times (0,1)$ and for $(x_1, x_2) \in \Omega$ let

$$f(x_1, x_2) := (f_1(x_1), f_2(x_2)),$$

where

$$f_1(x_1) := x_1 \quad \text{and} \quad f_2(x_2) := \int_0^{x_2} g(s) \, ds$$

for a function $g \in L^1((0,1))$, $g \geq 1$. For all $(x_1, x_2) \in \Omega$, we have

$$h_f(x_1, x_2) = \liminf_{r \to 0} \int_{(x_2-r, x_2+r)} g \, ds. \quad (5.29)$$

Let $C \subset (0,1)$ be the 1/3-Cantor set minus $\{0,1\}$ (some other Cantor set would work as well). Define the open sets $U_1 := (0,1)$ and

$$U_j := \{x \in (0,1): d(x, C) < b_j\}, \quad j = 2, 3, \ldots$$

for numbers $b_j$ to be chosen shortly. Suppose $g$ is given by

$$g := \sum_{j=1}^{\infty} \chi_{U_j}.$$  

Denote $\tilde{g}(x_1, x_2) := g(x_2)$. Choose the numbers $b_j \searrow 0$ to decrease sufficiently fast that we get $\tilde{g} \in L^2(\Omega)$ and also

$$\sum_{k=j+1}^{\infty} L^1(U_k) \leq b_j \quad \text{for all } j \in \mathbb{N}. \quad (5.30)$$

From (5.29) we obtain $h_f = \infty$ in $(0,1) \times C$. Thus the “standard” Theorem 5.8 is not applicable. On the other hand, let $a := \tilde{g}^2$, so that $a \in L^1(\Omega)$. For every $x = (x_1, x_2) \in \Omega$,
we get
\[ h_f^a(x) = \liminf_{r \to 0} \int_{(x_2-r,x_2+r)} g \, ds \left( \int_{B(x,r)} a \, d\mathcal{L}^2 \right)^{-1/2} \]
\[ \leq \liminf_{r \to 0} \int_{(x_2-r,x_2+r)} g \, ds \left( \int_{B(x,r)} \tilde{g} \, d\mathcal{L}^2 \right)^{-1} \text{ by Hölder} \]
\[ \leq \liminf_{j \to \infty} \int_{(x_2-b_j,x_2+b_j)} g \, ds \left( \int_{B(x,b_j)} \tilde{g} \, d\mathcal{L}^2 \right)^{-1} \]
\[ \leq \liminf_{j \to \infty} \left( j + \frac{1}{2} \sum_{k=j+1}^\infty \mathcal{L}^1(U_k) \right) j^{-1} \]
\[ \leq \liminf_{j \to \infty} \left( j + \frac{b_j}{2b_j} \right) j^{-1} \text{ by (5.30)} \]
\[ = 1. \]

Thus
\[ \int_\Omega (h_f^a)^2 a \, d\mathcal{L}^2 \leq \int_\Omega a \, d\mathcal{L}^2 < \infty. \]

Now Corollary 5.23 gives \( f \in D^1(\Omega; \mathbb{R}^2) \).

Recall that in the definition of quasiconformal mappings, one requires \( H_f \leq H < \infty \) everywhere. The conditions in Theorem 5.8, namely \( h_f^a \in L^{n/(n-1)}(\mathbb{R}^n) \) and \( h_f < \infty \) outside a set of \( \sigma \)-finite \( \mathcal{H}^{n-1} \)-measure, amount to a weaker, more quantitative requirement. But the requirements that \( h_f^a a^{(n-1)/n} \in L^{n/(n-1)}(\Omega) \) and \( h_f^a(x) < \infty \) for \( \mathcal{H}^{n-1} \)-a.e \( x \in \Omega \) in Corollary 5.23 quantify the distortion requirement on \( f \) in an even more flexible way. Thus in the above example we were able to deduce Sobolev regularity purely from the asymptotic behavior of \( L_f(x,r)/l_f(x,r) \), even though there was \( h_f = \infty \) in a large set.

6. On the Rank of \( \frac{dDf}{d|Df|} \)

The central theme of Sections 3 and 4 was that via relaxing the notions of differentiability and the local Lipschitz number by means of the fine topology, we can obtain quantities that are better adapted to the context of Sobolev and BV functions. As an application, in Subsection 3.3 we saw that sets of finite perimeter can be characterized by means of a “capacitary boundary”.

In this section we will show that a similar phenomenon emerges also in another context related to BV functions. Alberti’s rank one theorem [1], which was conjectured by Ambrosio and De Giorgi and then proved by Alberti, states that for a BV function \( f \in BV(\mathbb{R}^n; \mathbb{R}^k) \), with \( k \in \mathbb{N} \), the rank of the matrix \( \frac{dDf}{d|Df|}(x) \) is 1 for \( |D^s f| \)-a.e. \( x \in \mathbb{R}^n \). The theorem has many applications in the calculus of variations, and other proofs and generalizations have been given in [13, 14, 48].
Now consider \( f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \). We assume \( n \geq 2 \). We will show that there is a close connection between the rank of \( \frac{dDf}{d|Df|} (x) \) and a relaxed notion of quasiconformality of \( f \). Recall the definitions (5.3) to (5.7). We define the relaxed versions of these quantities as follows.

**Definition 6.1.** Let \( f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \). For an arbitrary set \( U \subset \mathbb{R}^{n} \) containing \( x \) and \( r > 0 \), we let
\[
L_{f,U}(x,r) := \sup\{|f(x) - f(z)|: |z - x| \leq r, z \in U\}
\]
and
\[
l_{f,U}(x,r) := \inf\{|f(x) - f(z)|: |z - x| \geq r, z \in U\},
\]
and then
\[
H_{f,U}(x,r) := \frac{L_{f,U}(x,r)}{l_{f,U}(x,r)} \quad \text{and} \quad H_{f}^{\text{fine}}(x) := \inf_{U} \limsup_{r \to 0, |y_r - x| = o(r)} H_{f,U}(y_r, r),
\]
where the infimum is taken over 1-finely open sets \( U \) containing \( x \). Explicitly, by the “lim sup” we mean
\[
\sup \left\{ \limsup_{j \to \infty} H_{f,U}(y_j, r_j): r_j \to 0, U \ni y_j \to x, |y_j - x|/r_j \to 0 \right\}.
\]
Also, we again interpret \( H_{f,U}(x,r) \) to be \( \infty \) if the denominator is zero or the numerator is \( \infty \). If \( f(x) \) or \( f(z) \) is not in \( \mathbb{R}^{n} \), we interpret \( |f(x) - f(z)| \) to be \( \infty \).

Note that quite analogously with our definition of generalized Lipschitz numbers, we take an infimum over 1-finely open sets, and we do not place special emphasis on the point \( x \), instead considering all points \( y \) close to \( x \) in the “lim sup”. This kind of fine-tuned version of \( H_{f} \) has the ability to capture the rank of \( \frac{dDf}{d|Df|} \), as we will soon show. The last sentence of the definition will be needed because the precise representative \( f^{*} \) of a function \( f \in L^{1}_{\text{loc}}(\mathbb{R}^{n};\mathbb{R}^{n}) \) may take values in \( \mathbb{R}^{n} \) only a.e.; we use the same interpretation with the previously defined quantities \( h_{f}, H_{f} \).

We start with the following lemma.

**Lemma 6.2.** Let \( S \) be a countably \( \mathcal{H}^{n-1} \)-rectifiable set with \( \mathcal{H}^{n-1}(S) < \infty \). Then for \( \mathcal{H}^{n-1} \)-a.e. \( x \in S \), given any 1-finely open set \( U \ni x \) we have
\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S \cap U)}{\omega_{n-1}r^{n-1}} = 1.
\]

**Proof.** We have \( S = \bigcup_{j=1}^{\infty} S_{j} \cup N \), where each \( S_{j} \) is a \( \mathcal{H}^{n-1} \)-measurable 1-Lipschitz \( n - 1 \)-graph and \( \mathcal{H}^{n-1}(N) = 0 \). Consider \( x \in S_{j} \); we can assume that \( S_{j} \) is the graph of \( h: D \to \mathbb{R} \) where \( D \subset \mathbb{R}^{n-1} \) and \( h \) is 1-Lipschitz. Excluding \( \mathcal{H}^{n-1} \)-negligible sets, we can also assume that
\[
\lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S_{j})}{\omega_{n-1}r^{n-1}} = 1 \quad \text{and} \quad \lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x,r) \cap S \setminus S_{j})}{\omega_{n-1}r^{n-1}} = 0,
\]
see [4, Theorem 2.56, Theorem 2.83]. By Lemma 3.3,
\[
\limsup_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap S_j \setminus U)}{\omega_{n-1} r^{n-1}} \leq (2\sqrt{n})^{n-1} \limsup_{r \to 0} \frac{\text{Cap}_1(B(x, r) \cap S_j \setminus U)}{\omega_{n-1} r^{n-1}} = 0,
\]
since $U$ is 1-finely open. Combining these facts, we get the result. \hfill \Box

Note that if $f \in \text{BV}(\mathbb{R}^n; \mathbb{R}^n)$ has singularities in a dense subset of $\mathbb{R}^n$, then $H_f^* = h_f^* = \infty$ everywhere, so in this sense quasiconformality and the rank of $\frac{dDf}{|Df|}$ are unrelated. However, for $H_f^\text{fine}$ we get the following.

**Theorem 6.3.** Let $f \in \text{BV}(\mathbb{R}^n; \mathbb{R}^n)$. Then for $|Df|$-a.e. $x \in \mathbb{R}^n$, we have $H_f^\text{fine}(x) < \infty$ if and only if $\frac{dDf}{|Df|}(x)$ is of full rank.

**Proof.** Recall the decomposition $Df = D^a f + D^c f + D^j f$ from (2.8). We will show that the claim holds for $|D^a f|$-a.e., $|D^c f|$-a.e., and $|D^j f|$-a.e. $x \in \mathbb{R}^n$.

**The absolutely continuous part.** We prove the claim for $|D^a f|$-a.e. $x \in \mathbb{R}^n$. Suppose $\frac{dD^a f}{|D^a f|}(x)$ is of full rank. We need to show that $H_f^\text{fine}(x) < \infty$. Excluding a set of $\mathcal{L}^n$-measure zero, which is also a set of $|D^a f|$-measure zero, we have $\frac{dD^a f}{|D^a f|}(x) = \nabla f(x)$ (the approximate gradient), and the fine differentiability from Theorem 4.8 holds at $x$. Consider the quantities
\[
\|\nabla f\|_{\max} := \max_{|v|=1} |\nabla f(x)v| \quad \text{and} \quad \|\nabla f\|_{\min} := \min_{|v|=1} |\nabla f(x)v|,
\]
which are both in $(0, \infty)$ since $\nabla f(x)$ has full rank. By Theorem 4.8 we find a 1-finely open set $V$ containing $x$ such that
\[
\lim_{V \ni y \to x} \frac{|f^*(y) - f^*(x) - \nabla f(x)(y - x)|}{|y - x|} = 0.
\]
Now letting $U := V \cap B(x, r)$ for a sufficiently small $r > 0$, we still have
\[
\lim_{U \ni y \to x} \frac{|f^*(y) - f^*(x) - \nabla f(x)(y - x)|}{|y - x|} = 0,
\]
and also
\[
\frac{|f^*(y) - f^*(x) - \nabla f(x)(y - x)|}{|y - x|} < \frac{\|\nabla f\|_{\min}}{2} \quad \text{for all} \ y \in U \setminus \{x\}. \quad (6.5)
\]
Choose any sequences $r_j \to 0$, $r_j > 0$, and $U \ni y_j \to x$ with $|y_j - x|/r_j \to 0$. For every $j \in \mathbb{N}$, also consider $z_j \in U$ with $|z_j - y_j| \leq r_j$. Then we have
\[
\limsup_{j \to \infty} \frac{|f^*(z_j) - f^*(y_j)|}{r_j} \leq \limsup_{j \to \infty} \frac{|f^*(z_j) - f^*(x)|}{r_j} + \limsup_{j \to \infty} \frac{|f^*(x) - f^*(y_j)|}{r_j} \leq \|\nabla f\|_{\max} + 0 \quad \text{by (6.4)}.
\]

Thus
\[ \limsup_{j \to \infty} \frac{L_{f^*,U}(y_j, r_j)}{r_j} = \sup_{j \to \infty} \frac{\sup\{|f^*(z) - f^*(y_j)|: |z - y_j| \leq r_j, z \in U\}}{r_j} \leq \|\nabla f\|_{\text{max}}. \]

On the other hand, for every \( j \in \mathbb{N} \) consider \( \tilde{z}_j \in U \) with \( |\tilde{z}_j - y_j| \geq r_j \) (if it exists, as it does for all sufficiently large \( j \)). We have
\[
\liminf_{j \to \infty} \frac{|f^*(\tilde{z}_j) - f^*(y_j)|}{r_j} \geq \liminf_{j \to \infty} \frac{|f^*(\tilde{z}_j) - f^*(x)|}{r_j} - \limsup_{j \to \infty} \frac{|f^*(x) - f^*(y_j)|}{r_j}
\]
\[
= \liminf_{j \to \infty} \frac{|f^*(\tilde{z}_j) - f^*(x)|}{|\tilde{z}_j - x|} \cdot \frac{|\tilde{z}_j - x|}{r_j} \geq \liminf_{j \to \infty} \frac{|f^*(\tilde{z}_j) - f^*(x)|}{|\tilde{z}_j - x|}
\]
\[
\geq \frac{\|\nabla f\|_{\text{min}}}{2} \quad \text{by (6.5).}
\]
Thus
\[
\liminf_{j \to \infty} \frac{L_{f^*,U}(y_j, r_j)}{r_j} = \liminf_{j \to \infty} \frac{\inf\{|f^*(z) - f^*(y_j)|: |z - y_j| \geq r_j, z \in U\}}{r_j} \geq \frac{\|\nabla f\|_{\text{min}}}{2}.
\]

In total, we obtain
\[
\limsup_{j \to \infty} H_{f^*,U}(y_j, r_j) = \limsup_{j \to \infty} \frac{L_{f^*,U}(y_j, r_j)}{L_{f^*,U}(y_j, r_j)} \leq 2 \frac{\|\nabla f\|_{\text{max}}}{\|\nabla f\|_{\text{min}}}.
\]

We conclude
\[ H_{f^*,x}^\text{fine} \leq 2 \frac{\|\nabla f\|_{\text{max}}}{\|\nabla f\|_{\text{min}}} < \infty. \]

Then suppose \( \frac{dDf}{dDf}(x) \) is not of full rank. We need to show that \( H_{f^*,x}^\text{fine} = \infty \). Fix an arbitrary 1-finely open set \( U \) containing \( x \). Again excluding a set of \( \mathcal{L}^n \)-measure zero, which is also a set of \( |D^a f| \)-measure zero, we have \( \frac{dDf}{dDf}(x) = \nabla f(x) \) and by Theorem 4.8 we can assume that \( f^* \) is 1-finely differentiable at \( x \). This means that by making the 1-finely open set \( U \) smaller, which only decreases \( H_{f,U}(x, r) \), we have that
\[
\lim_{U \ni y \to x} \frac{|f^*(y) - f^*(x) - \nabla f(x)(y - x)|}{|y - x|} = 0. \quad (6.6)
\]
But now the \( \nabla f_j(x) \)'s do not span \( \mathbb{R}^n \); we can assume that they do not span \( e_n \). By Lemma 3.2, we have
\[
\mathcal{H}^{n-1}(\partial \pi_n(U \setminus U)) \leq \frac{\text{Cap}_1(\partial B(x, r) \setminus U)}{r^{n-1}} \to 0 \quad \text{as } r \to 0,
\]
since $U$ is 1-finely open. Let $r_j \searrow 0$. Now we can choose points $z_j \in \partial B(x, r_j) \cap U$ such that the angle between $z_j - x$ and $e_n$ goes to zero. Thus
\[
\limsup_{j \to \infty} \frac{l_{f^*, U}(x, r_j)}{r_j} = \limsup_{j \to \infty} \frac{\inf\{|f^*(x) - f^*(z)|: |z - x| \geq |z_j - x|, z \in U\}}{r_j} \\
\leq \limsup_{j \to \infty} \frac{|f^*(x) - f^*(z_j)|}{r_j} \\
= \limsup_{j \to \infty} \frac{\nabla f(x)(z_j - x)}{r_j} \quad \text{by (6.6)} \\
= 0.
\]
Similarly, we can choose points $\tilde{z}_j \in \partial B(x, r_j) \cap U$ such that the angle between $\tilde{z}_j - x$ and $\nabla f_k(x)$ goes to zero, where we choose any nonzero $\nabla f_k(x)$. It follows that
\[
\liminf_{j \to \infty} \frac{L_{f^*, U}(x, r_j)}{r_j} = \liminf_{j \to \infty} \sup\{|f^*(x) - f^*(z)|: |z - x| \leq |\tilde{z}_j - x|, z \in U\} \\
\geq \liminf_{j \to \infty} \frac{|f^*(x) - f^*(\tilde{z}_j)|}{r_j} \\
= \liminf_{j \to \infty} \frac{\nabla f(x)(\tilde{z}_j - x)}{r_j} \quad \text{by (6.6)} \\
\geq |\nabla f_k(x)|.
\]
Thus we get
\[
\limsup_{r \to 0} H_{f^*, U}(x, r) \geq \limsup_{j \to \infty} \frac{L_{f^*, U}(x, r_j)}{l_{f^*, U}(x, r_j)} = \infty.
\]
We conclude
\[
\limsup_{r \to 0, |y_r - x| = o(r)} H_{f^*, U}(y_r, r) \geq \limsup_{r \to 0} H_{f^*, U}(x, r) = \infty,
\]
and so $H_{f^*, U}(x) = \infty$.

**The Cantor part.** By Alberti’s rank one theorem, $\frac{dDf}{|Df|}(x)$ is of rank one for $|D^c f|$-a.e. $x \in \mathbb{R}^n$, so we consider such a point. Fix an arbitrary 1-finely open set $U$ containing $x$. We need to show that
\[
\limsup_{r \to 0, |y_r - x| = o(r)} H_{f^*, U}(y_r, r) = \infty.
\]
For all $r > 0$, define the scalings
\[
 f_{x,r}(y) = \frac{f(x + ry) - f(B(x,r))}{|Df|(B(x,r))^{-1} y_{n-1}}, \quad y \in B(0, 1).
\]
Excluding another $|D^c f|$-negligible set, the following blowup behavior is known, see e.g. [4, Theorem 3.95]. For a suitable sequence $r_j \to 0$, we get $f_{x,r_j} \to w$ in $L^1(B(0,1); \mathbb{R}^n)$, where
\[
w(y) = \eta h((y, v)),
\]
where \( \eta, v \in \mathbb{R}^n \) are unit vectors and \( h \) is an increasing, nonconstant function on \((-1, 1)\). We can assume \( v = e_n \). As an increasing function, \( h \) has one-sided limits at 0, denoted by \( h(0-), h(0+) \in \mathbb{R} \). Because \( h \) is not constant, we can find and fix \( 0 < \varepsilon < 1/4 \) such that
\[
|w(y) - \eta h(0-)| \geq \varepsilon \quad \text{whenever the } n: \text{th coordinate of } y \text{ is at most } -1 + 4\varepsilon, \quad (6.7)
\]
assume without loss of generality the latter.

Denote
\[
r_j^{-1}(U - x) := \{ r_j^{-1}(y - x) : y \in U \}.
\]
We have
\[
\text{Cap}_1(B(0, 1) \setminus r_j^{-1}(U - x)) = \frac{\text{Cap}_1(B(x, r_j) \setminus U)}{r_j^{-n}} \to 0 \quad \text{as } j \to \infty,
\]
as \( j \to \infty \), and then by Lemma 2.18, also
\[
\mathcal{L}^n(B(0, 1) \setminus r_j^{-1}(U - x)) \to 0 \quad \text{as } j \to \infty.
\]
Passing to further subsequences (not relabeled), we get the pointwise convergences
\[
\chi_{B(0,1)\setminus r_j^{-1}(U-x)}(y) \to 0 \quad \text{and} \quad f_{x,r_j}^*(y) \to w(y) \quad \text{for } \mathcal{L}^n\text{-a.e. } y \in B(0,1).
\]
Note that \( (f^*)_{x,r_j} = (f_{x,r_j})^* \) in \( B(0, 1) \), so we simply use the notation \( f_{x,r_j}^* \). Fix \( 0 < \delta < \varepsilon \).

We find a point \( y \in B(0, \delta) \cap r_j^{-1}(U - x) \) for all sufficiently large \( j \in \mathbb{N} \), with \( n: \text{th coordinate negative} \) and
\[
|w(y) - \eta h(0-)| < \delta \quad \text{and} \quad \lim_{j \to \infty} f_{x,r_j}^*(y) = w(y). \quad (6.8)
\]
Secondly, we find a point \( \tilde{y} \in (B(0, 1) \setminus B(0, 1 - \varepsilon)) \cap r_j^{-1}(U - x) \) for all sufficiently large \( j \in \mathbb{N} \), whose \( n: \text{th coordinate is between } -\delta \text{ and } 0 \) and
\[
|w(\tilde{y}) - \eta h(0-)| < \delta \quad \text{and} \quad \lim_{j \to \infty} f_{x,r_j}^*(\tilde{y}) = w(\tilde{y}). \quad (6.9)
\]
Thirdly, we find a point \( \hat{y} \in B(0, 1 - 3\varepsilon) \cap r_j^{-1}(U - x) \) for all sufficiently large \( j \in \mathbb{N} \), with \( n: \text{th coordinate at least } 1 - 4\varepsilon \), such that
\[
f_{x,r_j}^* (\hat{y}) \to w(\hat{y}). \quad (6.10)
\]
Now
\[
\liminf_{j \to \infty} |f_{x,r_j}^*(\hat{y}) - f_{x,r_j}^*(y)| \geq \liminf_{j \to \infty} \left[ -|f_{x,r_j}^*(\hat{y}) - w(\hat{y})| + |w(\hat{y}) - \eta h(0-)| - |\eta h(0-) - w(y)| - |w(y) - f_{x,r_j}^*(y)| \right] > \varepsilon - \delta \quad \text{by } (6.10), (6.7), (6.8),
\]
and
\[
\limsup_{j \to \infty} \left| f_{x,r_j}^*(\hat{y}) - f_{x,r_j}^*(y) \right|
\leq \limsup_{j \to \infty} \left[ \left| f_{x,r_j}^*(\hat{y}) - w(\hat{y}) \right| + \left| w(\hat{y}) - \eta h(0-) \right| + \left| \eta h(0-) - w(y) \right| + \left| w(y) - f_{x,r_j}^*(y) \right| \right]
< 2\delta \text{ by (6.9), (6.8).}
\]
Note that \( |\hat{y} - y| \leq 1 - 2\varepsilon \) and \( |\hat{y} - y| \geq 1 - 2\varepsilon \). Thus
\[
\liminf_{j \to \infty} H_{f_{x,r_j}^*}^{-1}(U-x)(y,1-2\varepsilon) \geq \liminf_{j \to \infty} \frac{|f_{x,r_j}^*(\hat{y}) - f_{x,r_j}^*(y)|}{|f_{x,r_j}^*(\hat{y}) - f_{x,r_j}^*(y)|} > \frac{\varepsilon - \delta}{2\delta},
\]
Recall that \( y \in B(0,\delta) \). Now we can choose sequences \( \delta = 1/k \) and \( y_k \in B(0,1/k) \) and then strictly increasing, sufficiently large \( j(k) \) depending on \( k \), to get
\[
H_{f_{x,r_j(k)}^*}^{-1}(U-x)(y_k,1-2\varepsilon) \geq \frac{\varepsilon - 1/k}{2/k} \quad \text{for all } k \in \mathbb{N}.
\]
Now
\[
H_{f_{x,U}^*}(x + r_{j(k)}y_k, r_{j(k)}(1-2\varepsilon)) = H_{f_{x,r_j(k)}^*}^{-1}(U-x)(y_k,1-2\varepsilon) \to \infty \quad \text{as } k \to \infty,
\]
and here \( r_{j(k)}(1-2\varepsilon) \to 0 \) as \( k \to \infty \), and \( r_{j(k)}y_k/r_{j(k)} = y_k \to 0 \) as \( k \to \infty \). Thus
\[
\limsup_{r \to 0, |y_r - x| = o(r)} H_{f_{x,U}^*}(y_r, r) = \infty,
\]
which is what we needed to show.

**The jump part.** By Alberti’s rank one theorem, \( \frac{d|Df|}{|Df|} \) is of rank one for \( |D^j f| \)-a.e. \( x \in \mathbb{R}^n \), and so we need to show that \( H_{f_{x,U}^*}^\text{fine}(x) = \infty \). Consider the sets
\[
J_f \cap \{ f^- \in B(q_j,r_k) \} \cap \{ f^+ \in B(q_l,r_m) \},
\]
for \( q_j, q_l \in \mathbb{Q}^n \) and \( r_k, r_m \in \mathbb{Q}^+ \), \( |q_j - q_l| > r_k + r_m \). Each of these sets has finite \( \mathcal{H}^{n-1} \)-measure due to (2.10), and is countably \( \mathcal{H}^{n-1} \)-rectifiable (see [4, Theorem 3.78]). Thus for each of these sets, from Lemma 6.2 we obtain an exceptional set of \( \mathcal{H}^{n-1} \)-measure zero. Since there are countably many sets, corresponding to different choices of \( q_j, q_l, r_k, r_m \), we get countably many exceptional sets of \( \mathcal{H}^{n-1} \)-measure zero.

Fix \( x \in J_f \) which is outside all of the exceptional sets; this is true of \( \mathcal{H}^{n-1} \)-a.e. \( x \in J_f \) and thus of \( |D^j f| \)-a.e. \( x \in \mathbb{R}^n \). Fix \( 0 < \varepsilon < |f^-(x) - f^+(x)|/2 \). Choose \( q_j, q_l, r_k, r_m \) such that \( r_k + r_m < \varepsilon \) and \( r_k + r_m < |q_j - q_l| \), and
\[
f^-(x) \in B(q_j,r_k) \quad \text{and} \quad f^+(x) \in B(q_l,r_m).
\]
Now \( x \) is contained in
\[
S := J_f \cap \{ f^- \in B(q_j,r_k) \} \cap \{ f^+ \in B(q_l,r_m) \}.
\]
Let $U$ be an arbitrary 1-finely open set containing $x$. From Lemma 6.2, we have
\[ \lim_{r \to 0} \frac{\mathcal{H}^{n-1}(B(x, r) \cap S \cap U)}{\omega_{n-1} r^{n-1}} = 1. \]
In particular, we find points $y_i \in S \cap U$ with $y_i \to x$, $y_i \neq x$. By the definition of $S$ and by (2.7), we have
\[ |f^*(y_i) - f^*(x)| = \left| \frac{f^-(y_i) + f^+(y_i)}{2} - \frac{f^-(x) + f^+(x)}{2} \right| \leq \frac{1}{2}|f^-(y_i) - f^-(x)| + \frac{1}{2}|f^+(y_i) - f^+(x)| \leq r_k + r_m \leq \varepsilon. \]
On the other hand, from the asymptotic behavior given in (2.5) and (2.6), for arbitrarily small $r > 0$ we find points $\hat{y}_i \in B_r^+(x, r) \cap U$ with $|f^*(\hat{y}_i) - f^+(x)| < \varepsilon$ and so
\[ |f^*(\hat{y}_i) - f^*(x)| = |f^*(\hat{y}_i) - (f^-(x) + f^+(x))/2| \geq \frac{1}{2}|f^+(x) - f^-(x)| - |f^*(\hat{y}_i) - f^+(x)| \geq \frac{1}{2}|f^+(x) - f^-(x)| - \varepsilon. \]
Hence for all sufficiently small $r > 0$,
\[ H_{f^*,U}(x, r, y_r) = \frac{L_{f^*,U}(x, r)}{f^*,U(x, r)} \geq \frac{\frac{1}{2}|f^+(x) - f^-(x)| - \varepsilon}{\varepsilon}. \]
We get
\[ \limsup_{r \to 0, |y_r - x| \to 0} H_{f^*,U}(y_r, r) \geq \limsup_{r \to 0} H_{f^*,U}(x, r) \geq \frac{\frac{1}{2}|f^+(x) - f^-(x)| - \varepsilon}{\varepsilon}. \]
Since $\varepsilon > 0$ was arbitrary and so was the 1-finely open set $U$, we get $H_{f^*,U}(x) = \infty$. □

In the proof of Theorem 6.3 we handled the Cantor and jump parts by showing that
\[ H_{f^*,U}^{\text{fine}}(x) = \infty \quad \text{for } |D^s f|\text{-a.e. } x \in \mathbb{R}^n. \tag{6.11} \]
Note that while the formulation of Alberti’s rank one theorem of course relies on the linear structure of Euclidean space, equation (6.11) makes sense also in metric measure spaces, where the theory of BV functions was first developed in [3, 5, 49]. Thus (6.11) could be seen as an alternative formulation of Alberti’s rank one theorem in spaces where we do not have access to partial derivatives.

As for trying to prove (6.11) in metric spaces, note that equivalently it states the absolute continuity
\[ |Df|_{(H_{f^*,U}^{\text{fine}} < \infty)} \ll \mathcal{L}^n, \tag{6.12} \]
basically saying that $f$ is Sobolev in the set where $\{H_{f}^{\text{fine}} < \infty\}$; in metric measure spaces one needs to replace $\mathcal{L}^n$ with the ambient measure of the space. As we discussed in Section 5, there is a wide literature stating that if one of the quantities $h_f, H_f, \text{lip}_f, \text{Lip}_f$ is “not too large”, then $f$ is Sobolev. This indicates the possibility of proving (6.12) also in quite general metric measure spaces. We will return to this and related questions in future work.

References

[1] G. Alberti, Rank one property for derivatives of functions with bounded variation, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), no. 2, 239–274.
[2] J. M. Aldaz and F. J. Pérez Lázaro, Regularity of the Hardy-Littlewood maximal operator on block decreasing functions, Studia Math. 194 (2009), no. 3, 253–277.
[3] L. Ambrosio, Fine properties of sets of finite perimeter in doubling metric measure spaces, Calculus of variations, nonsmooth analysis and related topics, Set-Valued Anal. 10 (2002), no. 2-3, 111–128.
[4] L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000. 5, 6, 7, 8, 15, 16, 17, 20, 31, 32, 33, 37, 45, 56, 60, 62, 64
[5] L. Ambrosio, M. Miranda, Jr., and D. Pallara, Special functions of bounded variation in doubling metric measure spaces, Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004.
[6] Z. Balogh and M. Csörnyei, Scaled-oscillation and regularity, Proc. Amer. Math. Soc. 134 (2006), no. 9, 2667–2675.
[7] Z. Balogh and P. Koskela, Quasiconformality, quasisymmetry, and removability in Loewner spaces. With an appendix by Jussi Väisälä. Duke Math. J. 101 (2000), no. 3, 554–577.
[8] Z. Balogh, P. Koskela, and S. Rogovin, Absolute continuity of quasiconformal mappings on curves, Geom. Funct. Anal. 17 (2007), no. 3, 645–664.
[9] A. Björn and J. Björn, Nonlinear potential theory on metric spaces, EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zürich, 2011. xii+403 pp.
[10] A. Björn, J. Björn, and V. Latvala, Sobolev spaces, fine gradients and quasicontinuity on quasiregular mappings, Ann. Acad. Sci. Fenn. Math. 41 (2016), no. 2, 551–560.
[11] M. Carriero, G. Dal Maso, A. Leaci, and E. Pascali, Relaxation of the nonparametric plateau problem with an obstacle, J. Math. Pures Appl. 67 (1988), no. 4, 359–396.
[12] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999), no. 3, 428–517.
[13] C. De Lellis, A note on Alberti’s rank-one theorem, Transport equations and multi-D hyperbolic conservation laws, 61–74, Lect. Notes Unione Mat. Ital., 5, Springer, Berlin, 2008.
[14] G. De Philippis and F. Rindler, On the structure of $A$-free measures and applications, Ann. of Math. (2) 184 (2016), no. 3, 1017–1039.
[15] S. Eriksson-Bique, A New Hausdorff Content Bound for Limsup Sets, preprint 2022. https://arxiv.org/abs/2201.13412
[16] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions. Revised edition. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015. xiv+299 pp.
[17] A. Fang, The ACL property of homeomorphisms under weak conditions, Acta Math. Sinica (N.S.) 14 (1998), no. 4, 473–480.
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[18] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer-Verlag New York Inc., New York 1969 xiv+676 pp. 28, 31, 33

[19] S. Gardiner, *Finely continuously differentiable functions*, Math. Z. 266 (2010), no. 4, 851–861. 33

[20] S. Gardiner, *Recent progress on fine differentiability and fine harmonicity*, Complex analysis and potential theory, 283–291, CRM Proc. Lecture Notes, 55, Amer. Math. Soc., Providence, RI, 2012. 33

[21] F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. 103 (1962), 353–393. 3, 51

[22] F.W. Gehring, *The \(L^p\)-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–277. 3, 51

[23] P. Hajlasz and J. Malý, *On approximate differentiability of the maximal function*, Proc. Amer. Math. Soc. 138 (2010), no. 1, 165–174. 45

[24] P. Hajlasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. 29 (2004), no. 1, 167–176. 45

[25] H. Hakkarainen and J. Kinnunen, *The BV-capacity in metric spaces*, Manuscripta Math. 132 (2010), no. 1-2, 51–73. 12

[26] B. Hanson, *Linear dilatation and differentiability of homeomorphisms of \(\mathbb{R}^n\)*, Proc. Amer. Math. Soc. 140 (2012), no. 10, 3541–3547. 51, 57

[27] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*, J. Anal. Math. 85 (2001), 87–139. 51

[28] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson, *Sobolev spaces on metric measure spaces. An approach based on upper gradients*, New Mathematical Monographs, 27. Cambridge University Press, Cambridge, 2015. xii+434 pp. 4

[29] S. Kallunki and P. Koskela, *Exceptional sets for the definition of quasiconformality*, Amer. J. Math. 122 (2000), no. 4, 735–743. 51

[30] S. Kallunki and O. Martio, *ACL homeomorphisms and linear dilatation*, Proc. Amer. Math. Soc. 130 (2002), no. 4, 1073–1078. 50, 51, 56

[31] S. Keith, *A differentiable structure for metric measure spaces*, Adv. Math. 183 (2004), no. 2, 271–315. 1

[32] J. Kinnunen, R. Korte, N. Shanmugalingam, H. Tuominen, *Lebesgue points and capacities via the boxing inequality in metric spaces*, Indiana Univ. Math. J. 57 (2008), no. 1, 401–430. 13

[33] P. Koskela and S. Rogovin, *Linear dilatation and absolute continuity*, Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 2, 385–392. 50, 51

[34] O. Kurka, *On the variation of the Hardy-Littlewood maximal function*, Ann. Acad. Sci. Fenn. Math. 40 (2015), no. 1, 109–133. 45

[35] P. Lahti, *A Federer-style characterization of sets of finite perimeter on metric spaces*, Calc. Var. Partial Differential Equations 56 (2017), no. 5, Paper No. 150, 22 pp. 11, 31

[36] P. Lahti, *A new Federer-type characterization of sets of finite perimeter*, Arch. Ration. Mech. Anal. 236 (2020), no. 2, 801–838. 31

[37] P. Lahti, *A notion of fine continuity for BV functions on metric spaces*, Potential Anal. 46 (2017), no. 2, 279–294. 11

[38] P. Lahti, *On the regularity of the maximal function of a BV function*, J. Differential Equations 300 (2021), 53–79. 45

[39] P. Lahti, *Strict and pointwise convergence of BV functions in metric spaces*, J. Math. Anal. Appl. 455 (2017), no. 2, 1005–1021. 12
[40] P. Lahti, *The Choquet and Kellogg properties for the fine topology when p = 1 in metric spaces*, J. Math. Pures Appl. (9) 126 (2019), 195–213.

[41] P. Lahti and X. Zhou, *Quasiconformal and Sobolev mappings in non-Ahlfors regular metric spaces*, preprint 2021. https://arxiv.org/abs/2106.03602

[42] P. Lahti and X. Zhou, *Quasiconformal and Sobolev mappings in non-Ahlfors regular metric spaces when p > 1*, preprint 2021. https://arxiv.org/abs/2109.01260

[43] R. Lávička, *A remark on fine differentiability*, Adv. Appl. Clifford Algebr. 17 (2007), no. 3, 549–554.

[44] R. Lávička, *Finely continuously differentiable functions*, Expo. Math. 26 (2008), no. 4, 353–363.

[45] R. Lávička, *Finely differentiable monogenic functions*, Arch. Math. (Brno) 42 (2006), suppl., 301–305.

[46] J. Malý and W. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp.

[47] G. A. Margulis and G. D. Mostow, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. 5 (1995), no. 2, 402–433.

[48] A. Massaccesi and D. Vittone, *An elementary proof of the rank-one theorem for BV functions*, J. Eur. Math. Soc. (JEMS) 21 (2019), no. 10, 3255–3258.

[49] M. Miranda, Jr., *Functions of bounded variation on "good" metric spaces*, J. Math. Pures Appl. (9) 82 (2003), no. 8, 975–1004.

[50] V. Maz’ya, *Sobolev spaces*, Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985. xix+486 pp.

[51] H. Rademacher, *Über partielle und totale differenzierbarkeit von Funktionen mehrerer Variablen und über die Transformation der Doppelintegrale*, Math. Ann. 79 (1919), no. 4, 340–359.

[52] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana 16(2) (2000), 243–279.

[53] W. Stepnoff, *Über totale Differenzierbarkeit*, Math. Ann. 90 (1923), no. 3-4, 318–320.

[54] J. Väisäliä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971. xiv+144 pp.

[55] J. Weigt, *Variation of the uncentered maximal characteristic function*, preprint 2020. https://arxiv.org/abs/2004.10485

[56] M. Williams, *Dilatation, pointwise Lipschitz constants, and condition N on curves*, Michigan Math. J. 63 (2014), no. 4, 687–700.

[57] K. Wildrick and T. Zürcher, *Sharp differentiability results for the lower local Lipschitz constant and applications to non-embedding*, J. Geom. Anal. 25 (2015), no. 4, 2590–2616.

[58] T. Zürcher, *Local Lipschitz numbers and Sobolev spaces*, Michigan Math. J. 55 (2007), no. 3, 561–574.

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