Universal properties of anyon braiding on one-dimensional wire networks

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We demonstrate that anyons on wire networks have fundamentally different braiding properties than anyons in 2D. Our analysis reveals an unexpectedly wide variety of possible non-abelian braiding behaviours on networks. The character of braiding depends on the topological invariant called the connectedness of the network. As one of our most striking consequences, particles on modular networks can change their statistical properties when moving between different modules. However, sufficiently highly connected networks already reproduce braiding properties of 2D systems. Our analysis is fully topological and independent on the physical model of anyons.

Introduction. – Studies of anyon braiding on one-dimensional wire networks are at the forefront of research of architectures for topological quantum computers. Such a computer would be capable of performing computational tasks using topological states of matter (describing anyons) that are intrinsically robust against different types of noise and decoherence [1]. Anyons arise in quantum systems that are effectively one- or two-dimensional. Braiding of anyons transforms a state of the corresponding quantum system by a unitary operator which is a topological quantum gate. A robust realisation of controlled braiding of anyons is one of the major challenges in this field. Recently developed experimental and theoretical proposals address this challenge by exploring the possibility of braiding of anyons on junctions of one-dimensional wire networks [2, 3]. Such networks are believed to provide a platform for engineering anyonic braiding most easily.

This paper shows that braiding of anyons on networks provides a wider range of possibilities for the resulting topological quantum operations in comparison to 2D architectures. This suggests that there may exist quantum systems where computational universality can be accomplished more easily than in currently known proposals. Our paper also provides a mathematical justification for a widely assumed fact that braiding rules in 2D are consistent with braiding rules on 1D networks. Our purpose here is to describe the above new results, whose mathematical details will be spelled elsewhere [4].

Of particular importance in this context is Kitaev’s superconducting chain that supports so-called Majorana edge modes. Such a chain can be experimentally realised as semiconductor nanowires coupled to superconductors [3] as well as in other solid state [5–8] and photonic systems [9]. Braiding of edge modes is then realised by coupling endpoints of wires so that they form a network or, in the simplest case, a trijunction [3]. Importantly, Majorana edge modes braid in a non-abelian way making them useful for quantum computation. This proposal has been recognised as one of the most robust candidates for an architecture of a topological quantum computer. Experimental proposals of the above mentioned trijunction have been made so far including photonic systems [9] and Josephson junctions [10, 11]. We would also like to mention in this context effective hopping models for anyons that have been studied in [16] and that have led to the classification of abelian quantum statistics on networks [17].

Despite the significant interest in problems related to braiding of anyons on networks, relatively little is known about their topological braiding properties. In this work, we fill this gap by studying relations coming from continuous deformations of paths corresponding to braiding of anyons on a network. Because quantum statistics is a topological property, any physical model that supports anyonic braiding on a network has to respect such relations. In other words, any topological transformation of the quantum system related to an exchange of anyons remains invariant under a continuous deformation of the corresponding braid [12–14]. In the standard 2D setting, an example of such a relation is shown on Fig.2. It relates two ways of exchanging a triple of anyons. To see this, consider firstly the so-called simple braid from Fig.1a that exchanges two neighbouring anyons. Fig.1a also explains the origin of term braiding as the world lines of anyons form braids in space-time. Let us denote such • a simple braid that exchanges ith and (i + 1)th anyon by \( \sigma_i \). Any exchange of anyons in 2D can be written as a composition of simple braids. More formally, simple

\[ \sigma_i \sigma_j = \sigma_{j-i} \quad \text{for } i \neq j, \quad \sigma_i \sigma_i = 1 \]

We will now turn to the question of how braiding of anyons on networks relates to braiding in 2D. We will show that such a relation is always satisfied, and that 1D networks provide a very rich set of braiding rules that are not present in 2D. Our analysis is fully topological and independent on the physical model of anyons.
braids generate the planar braid group. However, they do not generate the braid group freely, as they are subject to the following braid relation: \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \).

This can be seen by drawing both braids (left and right hand side of Fig. 2 respectively) and noting that they differ by a deformation of the world line of the middle anyon (blue line on Fig. 2). Simple braids also satisfy a commutative relation where exchanges of disjoint sets of anyons commute with each other, \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( |j - i| \geq 2 \).

**Braiding on junctions.** In order to see if the braid relation is satisfied by braids on the trijuncton, we first define the network counterpart of the simple braid. It is shown on Fig. 2 – the trijunction stretched in a time interval makes up three rectangles. Anyon 1 is first transported to the right branch (the bottom branch in the picture) of the trijunction, then anyon 2 travels to the left branch making space for anyon 1 to go back to the original initial position of anyon 2. The exchange is completed by the return of anyon 2 from the right branch to the original initial position of anyon 1. In order to track all moves of anyons on an arbitrary junction, we set up the following notation. For a d-junction (d incident branches), we fix the initial position of anyons to align one after another on a fixed branch. Having drawn the junction on a plane, we enumerate the remaining branches in a clockwise fashion by labels from 1 to \( d - 1 \). The exchange of ith and \((i + 1)\)th anyon will be unambiguously encoded by a sequence of integers \( a := (a_1, a_2, \ldots, a_{i+1}) \) with \( 1 \leq a_j \leq d - 1 \) and \( a_i \neq a_{i+1} \). Elements of \( a \) denote i) labels of branches where first \((i - 1)\) anyons were distributed – these are \( a_1, \ldots, a_{i-1} \) ii) labels of branches where anyon \( i \) and \((i + 1)\) exchange – these are \( a_i \) and \( a_{i+1} \). Note, that swapping the order of \( a_i \) and \( a_{i+1} \) reverses the direction of the exchange. Going back to the concrete example of two particles on a trijunction from Fig 1b, the depicted braid would be denoted by \( \sigma_1^{(2,1)} \), i.e. \( a = (2, 1) \).

In order to visualise the counterparts of braids in the braid relation from Fig 2 we need to define the counterpart of \( \sigma_2 \) - the simple braid exchanging anyons 2 and 3. To this end, anyon 1 has to be moved to the right or left branch of the junction so that anyons 2 and 3 can carry on and exchange as on Fig. 1b. Let us choose the braid where anyon 1 moves to the right branch, which we denote \( \sigma_2^{(2,2,1)} \). Composition \( \sigma_1^{(2,1)} \sigma_2^{(2,2,1)} \sigma_1^{(2,1)} \) is shown on Fig. 3. Strikingly, the world line of the middle anyon (blue line on Fig. 3) is now blocked by world lines of the other particles and it cannot be deformed freely. Consequently, \( \sigma_1^{(2,1)} \sigma_2^{(2,2,1)} \sigma_1^{(2,1)} \) and \( \sigma_2^{(2,2,1)} \sigma_1^{(2,1)} \sigma_2^{(2,2,1)} \) describe topologically different moves. In fact, the three-particle braid group of the trijunction is freely generated by \( \sigma_1^{(2,1)} \), \( \sigma_2^{(2,2,1)} \) and \( \sigma_2^{(1,2,1)} \) – there are no relations between these generators. However, when one considers a bigger junction or a larger number of anyons, some relations appear. Their precise form is as follows.

1. For \( n \geq 4 \), pseudo-commutative relations appear. For \( j - i \geq 2 \),
\[
\sigma_j^{a_1 \ldots a_{j+1}} \sigma_i^{a_1 \ldots a_{i+1}} = \sigma_i^{a_1 \ldots a_{i-1} a_{i+1} a_i a_{i+2} \ldots a_{j+1}} \sigma_j^{a_1 \ldots a_{i-1} a_{i+1} a_i a_{i+2} \ldots a_{j+1}}.
\]

2. For \( d \geq 4 \) and \( n \geq 3 \), pseudo-braid relations appear. For \( 1 \leq i \leq n - 2 \),
\[
\sigma_{i+1}^{a_1 \ldots a_{i-1} a_i a_{i+1} a_i a_{i+2} \ldots a_{n-1}} \sigma_i^{a_1 \ldots a_{i-1} a_i a_{i+1} a_i a_{i+2} \ldots a_{n-1}} = \sigma_i^{a_1 \ldots a_{i-1} a_i a_{i+1} a_i a_{i+2} \ldots a_{n-1}} \sigma_{i+1}^{a_1 \ldots a_{i-1} a_i a_{i+1} a_i a_{i+2} \ldots a_{n-1}}.
\]

Minimal settings where the above relations appear are the following. A pseudo-commutative relation on a trijunction for \( n = 4 \): \( \sigma_1^{(2,1)} \sigma_3^{(2,1,2,1)} = \sigma_3^{(1,2,2,1)} \sigma_1^{(2,1)} \).

A pseudo-braid relation for \( n = 3 \) on a tetrajunction \((d = 4)\):
\[
\sigma_2^{(5,2,1)} \sigma_1^{(3,1)} \sigma_2^{(3,1)} = \sigma_1^{(3,2)} \sigma_2^{(2,3,1)} \sigma_1^{(2,1)}.
\]

Let us emphasise that although there are pseudo-braid and pseudo-commutative relations between generators, the braid group of a junction has more generators than the planar braid group and hence it imposes fewer topological constraints on the unitary braiding operators that are assigned to simple braids in a physical model. This is
perhaps most striking in the case of three anyons on a tri-
junction where we had three generators and no relations
between them.

General (planar) network architectures. – In order to
relate braiding relations for anyons on general networks
with the braiding of anyons in 2D, we first have to con-
sider a slightly different presentation of the braid group
in 2D. Namely, we will consider the total braid, \( \delta \),
which is a product of all simple braids, \( \delta := \sigma_1 \sigma_2 \ldots \sigma_{n-1} \). Braid \( \delta \)
corresponds to the move where the last anyon exchanges
with anyons and jumps in front of the line. Using 2D
braiding relations one can show that any simple braid
can be expressed by \( \sigma_1 \) and \( \delta \) as \( \sigma_i = \delta^{i-1} \sigma_1 \delta^{1-i} \).

We will next show how the above relations are recov-
ered on networks. To this end, we fix a spanning tree
of our network, \( T \), which is a connected tree that
contains all vertices of the network. Moreover, we choose
the root of \( T \) to be a vertex of degree two that lies on the bound-
dary of the network. The initial configuration of anyons
is such that the anyons are assembled on the edge of \( T \)
which is incident to the root. The above choice of a span-
ning tree unambiguously defines all possible exchanges
on junctions. To see this, note that for every essential
vertex \( v \) of the network (i.e. vertex at which three or
more edges are incident), we have a unique path in \( T \),
denoted by \([v, *] \), that connects this vertex with the root.
Such a path implies labelling of branches of the junction
at \( v \) with branch 0 being the one that is contained in
\([v, *] \) and the remaining branches labelled clockwise
as described in the previous section. Consequently, simple
braids at \( v \) will be denoted by additional superscript –
v. The counterpart of total braid \( \delta \) is realised by util-
is ing a loop containing the root of \( T \) (effectively consid-
ering a lollipop-shaped subnetwork, see Fig. 4) – anyon 1
is transported along the loop to the end of the line. It
is straightforward to check that up to some backtracking
moves, in the lollipop setting from Fig. 4 we have
\[
\sigma_i^{v(1, \ldots, 1, 2, 1)} = \delta^{i-1} \sigma_1^{v(2,1)} \delta^{1-i}.
\]

Moreover, braid \( \delta \) can be expressed in terms of simple
braids at the junction of the lollipop and a one-particle
move \( \gamma \) defined as the move where anyons 1 through \( n-1 \)
are transported to branch 2 of the junction and anyon \( n \)
travels alone around the lollipop loop. The precise
relation reads
\[
\gamma = \sigma_{n-1}^{v(2, \ldots, 2, 1, 1)} \ldots \sigma_1^{v(2,1)} \delta.
\]

Let us pause for a moment to analyse the role of one-
particle moves. Such moves do not describe any ex-
change, hence assigning unitary operators to these moves
can only come from the existence of some external gauge
fields puncturing the plane where the considered network
is confined. For instance, the presence of a delta-like
magnetic flux flowing perpendicularly through the mid-
dle of the lollipop loop would result with multiplication
of the anyonic wave function by a phase factor due to
the Aharonov-Bohm effect. From now on, we will al-
ways assume that there are no such external gauge fields
present in the system. Consequently, we will equate all
one-particle loops to identities.

By putting \( \gamma \) to identity, we obtain \( \delta = \sigma_1^{v(1, 2)} \ldots \sigma_{n-1}^{v(2, \ldots, 2, 1, 2)} \). Note that at this point we have
almost recovered the presentation of the planar braid
group that we considered at the beginning of this sec-
tion. The only difference is that expression for \( \delta \) involves
different simple braids than expression (5). As we show
in the next section, this problem disappears for a wide
class of networks that are sufficiently connected.

We say that a network is \( k \)-connected when any two
of its essential vertices can be connected by at least \( k \)
paths that are mutually internally disjoint. By Menger’s
theorem, this is equivalent to the fact that after removing
at most \( k-1 \) vertices, the network remains connected.

Braiding on 2-connected networks. – The key feature
of 2-connected networks that simplifies their braid groups
is that for every trijunction in the network we can find
suitable lollipop subnetworks that allow us to reduce
the number of generators. In particular, if \( v \) is the first
essential vertex from the root \( * \) (i.e. there are no essen-
tial vertices on path \([v, *] \), as in Fig. 4), then for every
branch \( a_1 \) at junction \( v \), there exists a path connecting
\( v \) and \( * \) that contains branch \( a_1 \) and is independent of
\([v, *] \). Consequently, we have a lollipop where, for any
\( a = (a_1, a_2, \ldots, a_{i+1}) \), we obtain
\[
\sigma_{i}^{v[a]} = \delta \sigma_{i-1}^{v[a']} \delta^{-1},
\]
where \( a' = (a_2, \ldots, a_{i+1}) \). The above expression allows
us to inductively reduce any simple braid at \( v \) to a braid
of the form \( \delta^{i-1} \sigma_1^{v[a,b]} \delta^{1-i} \) with \( a > b \). This in turn

FIG. 4. a) Total braid \( \delta \) on a lollipop network. b) One-particle
move \( \gamma \) on a lollipop network. The rooted spanning tree with
root \( * \) is drawn by solid lines.
means that for simple braids taking place at a fixed trijunction spanned on branches \((a, b)\) at vertex \(v\), we indeed obtain a set of 2D braiding relations. However, braids at different trijunctions are still \textit{a priori} independent of each other. One can show that a similar situation concerns simple braids at junctions that are further away from the root.

To recapitulate, 2-connected networks indeed support genuine 2D braiding relations. However, the relations are valid only within certain sets of braids that are restricted to fixed trijunctions of the network. Relations between braids on different trijunctions can still have a very complicated form. This is strikingly different from the 2D anyon braiding where braiding is ruled by only one type of simple braids. Note that this feature of braiding can be utilised in quantum computing to design networks consisting of different modules where quantum statistics can be easily changed when moving anyons from module to module. An example of such a modular network is shown on Fig. 5. We will analyse properties of this network in a closer detail in further parts of this paper. Interestingly, such a network has also been proposed in \cite{3} as an architecture that is suitable for adiabatic braiding of Majorana fermions on semiconductor wire networks. Utilising this feature of modular 2-connected networks in a concrete physical system can lead to very robust proposals in quantum computing where different quantum particles loops to identity, we have a set of braiding relations

\[
\delta \gamma = \gamma' \delta, \quad \sigma_1^v;\gamma = \gamma' \sigma_1^v;\gamma.
\]  

Because \(\gamma\) is a one-particle cycle, according to our assumption about the non-existence of external fields, we put it to identity. Then, the left relation in (6) implies that \(\gamma'\) is identity as well. This in turn applied to the right relation yields \(\sigma_1^w;\gamma = \sigma_1^w;\gamma\).

To sum up, relations (6) for lollipops together with relation (6) for \(\Theta\)-subnetworks enabled us to identify the \textit{a priori} complicated braiding relations on networks with the well-known 2D braiding when the considered network is 3-connected. The converse fact that 2D braiding relations are satisfied by braids on any network is very natural and could be seen by identifying simple braids on junctions with their corresponding simple braids in 2D. In terms of quantum operations this means assigning the same unitary operator \(U_i\) to every simple braid \(\sigma_i^v;\gamma\). However, let us emphasise that it is a highly nontrivial fact that merely by putting all one-particle loops on a 3-connected network to identity, we recover the 2D braid group.

\textit{Example: a modular 2-connected network.} – A simple realisation of a modular network that admits different non-abelian quantum statistics in different modules is shown on Fig. 5. Because one can span a \(\Theta\)-connection between trijunctions at \(v\) and \(v'\), simple braids at these junctions are identified with each other. Similarly, one identifies simple braids at trijunctions at \(w\) and \(w'\). However, simple braids at \(v\) and \(w\) are independent of each other. This in turn means that upon putting all one-particle loops to identity, we have a set of braiding relations for each of the trijunctions:

\[
\sigma_i^w;\gamma = \delta^{i-1} \sigma_i^v;\gamma, \quad \sigma_i^w;\gamma = \delta^{i+1} \sigma_i^v;\gamma.
\]

In the above equations, sequences \(a, a'\) can be arbitrary as lollipop relations ensure that \(\sigma_i^v;\gamma = \delta^{i-1} \sigma_i^v;\gamma = \delta^{i+1} \sigma_i^v;\gamma\).
and $\sigma_{i}^{w, a} = \delta^{-1} \sigma_{i}^{w(2,1)} \sigma^{1-i}$ for any $a$. Therefore, there are no topological constraints that would forbid utilising simple braids at $v$ and $w$ in such a way that they would realise different topological quantum gates $\sigma_{i}^{w, a} \rightarrow U_{i}$, $\sigma_{i}^{w, a} \rightarrow V_{i}$. In such a system, braiding of anyons $i$ and $(i+1)$ at junctions $v$ or $v'$ would realise gate $U_{i}$, while braiding at junctions $w$ or $w'$ would realise gate $V_{i}$. This proposal shows that conducting quantum computations on a topological quantum computer based on such a modular network would be a relatively easy task provided that one could manipulate anyons efficiently.

Finally, let us remark that the above desired features of the modular network are lost when one adds just a single edge to the network in the way shown on Fig. 6b. By a visual inspection, one can check that network from Fig 6b is now 3-connected. This means that there exists an additional $\Theta$-subnetwork which identifies simple braids on junction $v$ with simple braids on junction $w$. Analogous properties of quantum statistics on networks have been observed for abelian anyons [17 20].

**Relation to the braiding of Majorana fermions in quantum wires** – Let us consider a particular model that supports braiding of anyons on networks. It describes a network of spinless fermionic quantum chains that are coupled together at their endpoints. Hamiltonian of a single quantum chain is given by

$$H = -\mu \sum_{k=1}^{N} c_{k}^\dagger c_{k} - \sum_{k=1}^{N-1} \left( t c_{k}^\dagger c_{k+1} + |\Delta| e^{i\phi} c_{k} c_{k+1} + \text{h.c.} \right)$$  \(7\)

where parameters $\mu$ and $t > 0$ are respectively the chemical potential and hopping amplitude $t$. Parameter $|\Delta| e^{i\phi}$ is called the superconducting gap, lending itself to the role of the model in the description of spinless $p$-wave superconductors [21]. When $|\mu| < 2t$, Hamiltonian [7] for $N \rightarrow \infty$ has one zero-energy eigenmode $d_{0}$ which can be represented in terms of two Majorana modes $\gamma_{1}$ and $\gamma_{2}$, $d_{0} = \gamma_{1} + \gamma_{2}$, that are localised at the beginning and at the end of the chain respectively [22]. This property persists for finite chains with mode $d_{0}$ having energy which is exponentially small in the size of the chain. As Majorana modes $\gamma_{1}$ and $\gamma_{2}$ are localised at the endpoints of the chain, one can consider their braiding on chains that are connected into trijunctions by adiabatically tuning parameters of Hamiltonian [6] in a local fashion [3]. As authors of [3] point out, braiding is well-defined provided that i) $\gamma_{1}$ and $\gamma_{2}$ remain sufficiently separated throughout the process and ii) the energy gap remains open at all times. Schematically, one can visualise adiabatic transport of Majorana fermions as a transport of two point-like anyons that are connected by a string of topological region where parameters of the Hamiltonian satisfy $|\mu| < 2t$. Performing quantum computations with Majorana edge modes would require creating a network with multiple well-separated topological regions on it. Creating $n$ topological regions results with $2n$ edge modes which can be subsequently braided on junctions. It has been shown in [3] that exchange of two edge modes $\gamma_{1}$ and $\gamma_{i+1}$ gives quantum gate $U_{i} = \exp(\pi \gamma_{i} \gamma_{i+1}/4)$. Moreover any one-particle move where just one Majorana fermion is being adiabatically transported, results with the multiplication of the wavefunction by a global phase factor. Hence, in terms of anyon braiding, this model has the following properties i) all one-particle moves do not change the quantum state of the system (are effectively put to identity) and ii) all simple braids are represented by the same quantum gate, i.e. $\sigma_{i}^{w, a} \rightarrow U_{i}$ for any $v$ and $a$. Therefore, braiding of Majorana fermions on any network is exactly the same as braiding in 2D and it seems not to exploit the full potential of modular networks outlined in previous sections. That said, this model is definitely of utmost importance and it shows that one can hope to find more physical models for anyon braiding on networks that would not be directly equivalent do 2D braiding.

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