Mixing of Quantum Walk on Circulant Bunkbeds

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Abstract

We give new observations on the mixing dynamics of a continuous-time quantum walk on circulants and their bunkbed extensions. These bunkbeds are defined through two standard graph operators: the join $G + H$ and the Cartesian product $G \oplus H$ of graphs $G$ and $H$. Our results include the following:

- The quantum walk is average uniform mixing on circulants with bounded eigenvalue multiplicity. This extends a known fact about the cycles $C_n$.
- Explicit analysis of the probability distribution of the quantum walk on the join of circulants. This explains why complete partite graphs are not average uniform mixing, using the fact $K_n = K_1 + K_{n-1}$ and $K_{n,\ldots,n} = K_n + \ldots + K_n$.
- The quantum walk on the Cartesian product of a $m$-vertex path $P_m$ and a circulant $G$, namely, $P_m \oplus G$, is average uniform mixing if $G$ is. This highlights a difference between circulants and the hypercubes $Q_n = P_2 \oplus Q_{n-1}$.

Our proofs employ purely elementary arguments based on the spectra of the graphs.

Keywords: Quantum walks, Circulant graphs, Average mixing, Join, Cartesian product.

1 Introduction

The study of continuous-time quantum walks on graphs has important potential applications in quantum computation [15]. First, as an algorithmic technique, it was used to devise efficient quantum search algorithms with considerable speedup over classical algorithms [7]. Second, it may provide a simpler physical implementation of a quantum computer, given that there is an abundance of physical processes that simulate quantum walk on graphs [9]. In the physics literature, continuous-time quantum walks is mainly studied over infinite constant-dimensional lattices, such as the one-dimensional line (see [10], Chapters 13,16). On the other hand, the study of random walks on general graphs is a topic of broad interest in the mathematics and computer science community [6] [13].

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In this paper, we study the mixing dynamics of continuous-time quantum walks on circulant graphs. More particularly, we consider the average or limiting probability distribution of a quantum walk. This notion was introduced in [1] and is the quantum analogue of a stationary distribution of classical random walks. On the circulant graphs, our goal was to characterize the graphs for which the continuous-time quantum walk reaches (almost) uniform average probability distribution. It was previously known that cycles are near uniform mixing, whereas the complete graphs and hypercubes are not (see [2, 14]). Our other goal in this paper is to discover graph theoretic structures that may explain this polarized phenomena.

First, we show that circulants with bounded eigenvalue multiplicity are almost uniform mixing. This generalization explains why cycles are uniform mixing. Second, we consider bunkbed graphs constructed using the join and the Cartesian product operators. By analyzing the join of two circulants, we observe an interesting mixing phenomena on the cone $K_1 + G$ of a circulant $G$, that is dependent on the density of $G$. If the quantum walk starts on $K_1$, a dense graph $G$ repels the probability away from the copy of $G$. This explains why the limiting distribution of a quantum walk on the complete graph is not near the uniform distribution. We extend this investigation to the homogeneous join of circulants, namely, $G + \ldots + G$, for a circulant $G$. We show that this bunkbed graph is uniform mixing if $G$ is uniform mixing and the join is over a constant number of copies of $G$. A corollary of this transference property explains the non-uniform mixing of the complete multipartite graphs $\overline{K}_n + \ldots + \overline{K}_n$.

We also analyze bunkbed graphs obtained from the Cartesian product $P_m \oplus G$ of a path $P_m$ and a circulant $G$. On this bunkbed structure, we observe another transference property: the quantum walk on $P_m \oplus G$ is uniform mixing if it is uniform mixing on $G$ and the path is of constant size. This highlights a striking difference with the hypercube $Q_n$, since the hypercube is also a bunkbed $Q_n = P_2 \oplus Q_{n-1}$, but it is known that they are not uniform mixing [14]. It is interesting to note that both classes of graphs are group-theoretic circulants (see [8]), since our circulants are the $\mathbb{Z}_n$-circulants while hypercubes are the $(\mathbb{Z}_2)^n$-circulants. This suggests a group theoretic investigation into the mixing phenomena of generalized circulants, which we leave for future work.

In this paper, we focus exclusively on continuous-time quantum walks. We refer the reader to [12] for a survey of other models of quantum walks. As a final remark, we mention that most of the graphs we consider have the standard stationary distributions in the classical random walks where the limiting probability of a vertex is proportional to its degree [4].

2 Preliminaries

We consider simple, undirected graphs that are connected, and mostly regular. For a graph $G = (V, E)$, let $A_G$ be the adjacency matrix of $G$, where $A_G[j, k] = 1$ if $(j, k) \in E$. Here and throughout, we will use $[\Psi]$ to denote the characteristic function of a logical statement $\Psi$, that is, 1 if $\Psi$ is true, and 0 if it is false. The set of eigenvalues of $A_G$ is denoted $Sp(G)$, and the (algebraic) multiplicity of an eigenvalue $\lambda$ is denoted $m(\lambda)$. The spectral type $\tau(G)$ of a graph $G$ is the number of distinct eigenvalues of the adjacency matrix $A_G$ of $G$. We will denote the maximum (algebraic) multiplicity of any eigenvalue of graph $G$ by $\mu(G)$. Some
We will assume that $S$ specified by a subset $K$ indices of the families of graphs that we will consider include the complete multipartite graphs $K_n^{(m)}$, where there are $m$ partitions with a partition size of $n$, the cycles $C_n$ and paths $P_n$, and the hypercubes $Q_n$. Relevant background on graphs and their spectral properties can be found in [5].

A graph $G$ is called circulant if its adjacency matrix $A_G$ is circulant. A circulant matrix $A$ is specified by its first row, say $(a_0, a_1, \ldots, a_{n-1})$, and is defined as $A_{j,k} = a_{k-j} \pmod{n}$, where $j, k \in \mathbb{Z}_n$. Here $\mathbb{Z}_n$ denotes the group of integers $\{0, \ldots, n-1\}$ under addition modulo $n$. Note that $a_0 = 0$, since our graphs are simple, and $a_j = a_{n-j}$, since our graphs are undirected. Connectivity is guaranteed if the greatest common divisor of $Q$ and the hypercubes $A$ is specified by its first row, say $(F)$ as a unit vector, with $\langle F \rangle = \sqrt{n} \sum_{k=1}^{n} 2 \cos \left( \frac{2\pi j k}{n} \right) + \left\lceil n \text{ even} \right\rceil a_{n/2} (-1)^j$.

A continuous-time quantum walk on a graph $G = (V,E)$ is defined using the Schrödinger equation with the real symmetric matrix $A_G$ as the Hamiltonian (see [4]). If $|\psi(t)\rangle \in \mathbb{C}^{|V|}$ is a time-dependent amplitude vector on the vertices of $G$, then the evolution of the quantum walk is given by

$$|\psi(t)\rangle = e^{-itA_G}|\psi(0)\rangle,$$

where $i = \sqrt{-1}$ and $|\psi(0)\rangle$ is the initial amplitude vector. We usually assume that $|\psi(0)\rangle$ is a unit vector, with $\langle x|\psi(0)\rangle = [x = 0]$, for some vertex $0$. The amplitude of the quantum walk on vertex $j$ at time $t$ is given by $\psi_j(t) = \langle j|\psi(t)\rangle$, while the probability of being on vertex $j$ at time $t$ is $p_j(t) = |\psi_j(t)|^2$. The average (or limiting) probability of being on vertex $j$ is defined as

$$\overline{p}_j = \lim_{T \to \infty} \frac{1}{T} \int_0^T p_j(t) \, dt.$$ 

This notion appeared in [1] in the context of discrete-time quantum walks. The limiting probability distribution of the quantum walk will be denoted $\overline{P}$.

Figure 1: Examples of circulants of order 8. From left to right: (i) the empty graph $K_8$. (ii) the cycle $C_8$. (iii) a strongly-regular circulant: clique minus a perfect matching. (iv) the complete graph $K_8$. Of the families of graphs that we will consider include the complete multipartite graphs $K_n^{(m)}$, where there are $m$ partitions with a partition size of $n$, the cycles $C_n$ and paths $P_n$, and the hypercubes $Q_n$. Relevant background on graphs and their spectral properties can be found in [5].
Definition 1 (Average Uniform Mixing)
The average mixing of a continuous-time quantum walk on a graph $G = (V,E)$ is called uniform if $\mathbb{P}_j = O(1/|V|)$, for each vertex $j$ of $G$.

Remark Note that in the above definition, we only require that each limiting probability be linearly proportional to the uniform probability value. This is less stringent than requiring that the quantum walk achieves exactly uniform probability distribution (see [3, 2]). When the graph $G$ is not regular, the limiting probability distribution $\mathbb{P}$ may depend on the initial state $|\psi(0)\rangle$. We will specify carefully the effect of the initial states in these cases, but suppress this dependence for vertex-transitive graphs.

3 Mixing and Bounded Multiplicities

Theorem 1 Let $G$ be a circulant graph. If $\mu(G)$ is bounded, then the continuous-time quantum walk on $G$ is average uniform mixing.

Proof Let $n$ be the order of $G$ and let $A$ be the adjacency matrix of $G$. Since $|0\rangle = \sum_{k=0}^{n-1} \frac{1}{\sqrt{n}} |F_k\rangle$, if $|\psi(0)\rangle = |0\rangle$, we have $|\psi(t)\rangle = e^{-itA}|\psi(0)\rangle = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-it\lambda_k} |F_k\rangle$. This yields $\langle j|\psi(t)\rangle = \frac{1}{n} \sum_{k=0}^{n-1} e^{-it\lambda_k} \omega^{jk}$. Thus,

$$p_j(t) = \frac{1}{n^2} \sum_{k,\ell} e^{-it(\lambda_k-\lambda_{\ell})} \omega^{j(k-\ell)} = \frac{1}{n} + \frac{1}{n^2} \sum_{k\neq \ell} e^{-it(\lambda_k-\lambda_{\ell})} \omega^{j(k-\ell)}. \quad (4)$$

Using the above, the average (limiting) probabilities are

$$\left|p_j - \frac{1}{n}\right| \leq \frac{1}{n^2} \sum_{\lambda \in \text{Sp}(A)} \left( \frac{m(\lambda)}{2} \right) \leq \frac{1}{n} \left( \frac{\mu(G)}{2} \right). \quad (5)$$

So, if $\mu(G) = O(1)$, we have uniform mixing.

Theorem 2 The continuous-time quantum walk on a constant-degree $n$-vertex circulant of the form $(1,n/k_1,\ldots,n/k_d)$, where, for each $j = 1,\ldots,d$, $k_j$ is a constant which divides $n$, is average uniform mixing.

Proof The eigenvalues of $G$ are given by

$$\lambda_j = 2 \cos \left( \frac{2\pi j}{n} \right) + 2 \sum_{\ell=1}^{d} \cos \left( \frac{2\pi j}{k_\ell} \right), \quad (6)$$
for \( j = 1, \ldots, n - 1 \). Since the sum \( \sum_{j=1}^d \cos(2\pi j/k) \) can have at most \( \prod_{j=1}^d k_j = O(1) \) distinct values, each eigenvalue must have a constant multiplicity. By Theorem 1, we have the claimed result.

\[ \square \]

**Corollary 3** The continuous-time quantum walk on the 3-regular circulant "wheel" \( V_n = (1, n/2) \) of even order \( n \) is uniform mixing.

### 4 Mixing on Join Bunkbeds

In this section, we study a circulant bunkbed structure obtained by the join of circulants. Formally, the \textit{join} \( G + H \) of two graphs \( G \) and \( H \) is defined as to satisfy \( G + H = G \cup H \) (see [16]). It is easy to see that this is a graph obtained by connecting each vertex of \( G \) to each vertex of \( H \), while maintaining the internal structures of \( G \) and \( H \). For a graph \( G \), the \textit{cone} of \( G \) will denote the graph \( K_1 + G \).

**Lemma 4** Let \( G \) and \( H \) be circulants of degrees \( k \) and \( \ell \), respectively. Suppose that the eigenvalues of \( G \) and \( H \) are \( k = \mu_0 > \mu_1 \geq \ldots \geq \mu_{|G|-1} \) and \( \ell = \nu_0 > \nu_1 \geq \ldots \geq \nu_{|H|-1} \), respectively. Then, the eigenvalues and (orthonormal) eigenvectors of \( G + H \) are found in three separate sets \( \{ (\mu_a, |z_a^G|) : 1 \leq a \leq |G| - 1 \} \), \( \{ (\nu_b, |z_b^H|) : 1 \leq b \leq |H| - 1 \} \), and \( \{ (\lambda_{\pm}, |z_{\pm}|) \} \), where, for \( x = 0, \ldots, |G||H| - 1 \), we have

\[
\begin{align*}
\langle x | z_a^G \rangle &= \frac{1}{\sqrt{|G|}} \omega_{|G|}^{ax} [x] & a = 1, \ldots, |G| - 1 & \quad (7) \\
\langle x | z_b^H \rangle &= \frac{1}{\sqrt{|H|}} \omega_{|H|}^{bx} [x] & b = 1, \ldots, |H| - 1 & \quad (8) \\
\langle x | z_{\pm} \rangle &= \frac{1}{L_{\pm}} (\beta_{\pm})^{[x]} [x] & \quad (9)
\end{align*}
\]

where \( \beta_{\pm} = (\lambda_{\pm} - k) / |H| \), \( L_{\pm} = \sqrt{|G| + |H| \beta_{\pm}^2} \), and \( \lambda_{\pm} \) are the roots of \( \lambda^2 - (k + \ell) \lambda - (|G||H| - k\ell) = 0 \).

**Proof** Note that the adjacency matrix of \( G + H \) is given by

\[
A = \begin{bmatrix}
A_G & J_{|G| \times |H|} \\
J_{|H| \times |G|} & A_H
\end{bmatrix}
\]

(10)

It is easy to see that \( |z_a^G| \) are eigenvectors of \( A \) with eigenvalues \( \mu_a \), for \( a = 1, \ldots, |G| - 1 \), and \( |z_b^H| \) are eigenvectors of \( A \) with eigenvalues \( \nu_b \), for \( b = 1, \ldots, |H| - 1 \). The last two eigenvectors are obtained by noting that the eigenvectors have the form \([ a \ldots a \ b \ldots b]^T \). This gives the equations \( ka + b|H| = \lambda a \) and \( \ell b + a|G| = \lambda b \), whose solutions yield the eigenvalues \( \lambda_{\pm} = \frac{1}{2}((k + \ell)^2 \pm \sqrt{\Delta}) \), where \( \Delta = (k - \ell)^2 + 4|G||H| \), and eigenvectors with \( a = 1 \) and \( b = (\lambda_{\pm} - k) / |H| \). \[ \square \]

**Theorem 5** Suppose that \( G \) and \( H \) are circulants of degrees \( k \) and \( \ell \), respectively. Let \( \Delta = (k - \ell)^2 + 4|G||H| \) and \( \lambda_{\pm} = 1/2((k + \ell)^2 \pm \sqrt{\Delta}) \). Consider a continuous-time quantum
walk on \( G + H \) starting at some vertex of \( G \). Let \( \overline{\rho}_x(G) \) denote the limiting probability of \( x \in G \) over the subgraph \( G \). Assume that

\[
\lambda - \notin (Sp(G) \setminus \{ k \}) \cup (Sp(H) \setminus \{ \ell \}).
\]  

(11)

Then, the limiting probabilities of the vertices of \( G + H \) are

\[
\overline{\rho}_x(G + H) = \left\{ \left( \overline{\rho}_x(G) - \frac{1}{|G|} \right) + \frac{1}{|G|} \left( \frac{|H| - 2H}{\Delta} \right) \right\} \mathbb{1}[x \in G] + \frac{2}{\Delta} \mathbb{1}[x \in H]
\]

(12)

Proof Let the initial state be \( |\psi(0)\rangle = |0\rangle \) where the quantum walk starts at a vertex of \( G \). By Lemma 4, we have

\[
|\psi(0)\rangle = \frac{1}{\sqrt{|G|}} \sum_{a=1}^{G-1} |z_a^G\rangle + \sum_{\pm} \frac{1}{L_\pm^2} |z_\pm\rangle,
\]

(13)

and, thus,

\[
|\psi(t)\rangle = \frac{1}{\sqrt{|G|}} \sum_{a=1}^{G-1} e^{-it\mu_a} |z_a^G\rangle + \sum_{\pm} e^{-it\lambda_\pm} \omega_{L_\pm^2} |z_\pm\rangle.
\]

(14)

The amplitude on vertex \( x \) at time \( t \) is given by

\[
\langle x|\psi(t)\rangle = \frac{1}{|G|} \sum_{a=1}^{G-1} e^{-it\mu_a} \omega_{\alpha x} + \sum_{\pm} e^{-it\lambda_\pm} \beta_{x}^{[x \in H]},
\]

(15)

where \( \beta_\pm = (\lambda_\pm - k)/|H| \), and we obtain

\[
\overline{\rho}_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \ |\langle x|\psi(t)\rangle|^2 = \overline{\rho}_x(G) - \frac{1}{|G|} + \sum_{\pm} \left( \frac{\beta_\pm^{[x \in H]}}{L_\pm^2} \right)^2,
\]

(16)

where \( \overline{\rho}_x(G) \) is the limiting probability on the subgraph \( G \). After some calculations, we get

\[
\sum_{\pm} \left( \frac{1}{L_\pm^2} \right)^2 = \frac{1}{|G|} \left( \frac{1}{|G|} - \frac{2|H|}{\Delta} \right), \quad \sum_{\pm} \left( \frac{\beta_\pm}{L_\pm^2} \right)^2 = \frac{2}{\Delta},
\]

(17)

which completes the stated claim.

The single theorem above implies the following various known and new facts about mixing on the family of complete and related graphs. First, we obtain a perfect uniform mixing behavior on \( K_2 \), but not on \( K_n \), for \( n > 2 \).

**Corollary 6** The continuous-time quantum walk on \( K_2 \) is average exactly uniform mixing.

Proof By Theorem 5 we have \( |G| = |H| = 1 \) and \( k = \ell = 0 \). Thus, \( \Delta = 4 \), and therefore, \( \overline{\rho}_0 = \overline{\rho}_1 = 1/2 \).

**Corollary 7** The continuous-time quantum walk on \( K_n \) is not average uniform mixing, as \( n \to \infty \).
Proof By Theorem 5, we have $K_n = K_1 + K_{n-1}$. We have $|G| = 1$, $|H| = n - 1$, $k = 0$, and $\ell = n - 1$. Then, $\Delta = (n-1)^2 + 4n$, with $\overline{p}_0 = 1 - 2n/\Delta \sim 1$ and $\overline{p}_j = 2/\Delta \sim 0$, as $n \to \infty$. □

Next, we consider the cone of circulants. The following corollary provides a simple explanation why $K_n$ is not average uniform mixing, for large $n$; it is because $K_n$ is a cone of a dense circulant.

**Corollary 8** The continuous-time quantum walk on the cone of any circulant $C$, namely, $K_1 + C$, is not average uniform mixing.

Proof Let $C$ be a $\ell$-regular circulant of order $n$. By Theorem 5, we have $|G| = 1$, $|H| = n$, $k = 0$. Then, $\overline{p}_0 = 1 - (1/2)(1 + (\ell/2)^2/n)]$. Thus, $\overline{p}_0 = \Omega(1)$, regardless of $\ell$. □

**Homogeneous Joins of Circulants** Consider the unbounded $m$-fold homogeneous join of a circulant $G$, namely, $G^{(m)} = G + \ldots + G$, where there are $m$ terms in the summation.

The following theorem shows that the uniform mixing property of $G$ transfers into its unbounded homogeneous join if $m$ is a constant.

**Theorem 9** Let $G$ be a circulant of order $n$. Let $m \geq 2$ is a constant and $n > 2\lambda_0(G)$. In the continuous-time quantum walk, $G^{(m)} = \sum_{k=1}^m G$ is average uniform mixing if $G$ is.

Proof The adjacency matrix of $G^{(m)} = \sum_{k=1}^m G$ is given by

$$
A = I_m \otimes G + K_m \otimes J_n,
$$

where $I_m$ is the $m \times m$ identity matrix, $K_m$ is a complete graph on $m$ vertices, and $J_n$ is the $n \times n$ all-one matrix. Since $G$ is a circulant, both summands share the same set of the following orthonormal eigenvectors

$$
\{ |F_{j,k}(m)\rangle = |F_{j,k}^{(m)}\rangle \otimes |F_k^{(n)}\rangle : 0 \leq j \leq m - 1, 0 \leq k \leq n - 1 \},
$$

where $|F_j^{(m)}\rangle$ denotes the $j$-th column of the $m \times m$ Fourier matrix, and similarily for $|F_k^{(n)}\rangle$.

Let $\lambda_k(G)$, for $0 \leq k \leq n - 1$, be the eigenvalues of $G$ in descending order. The corresponding eigenvalues of $G^{(m)}$ are given by

$$
\lambda_{j,k} = \begin{cases} 
\lambda_0(G) + (m - 1)n & \text{if } j = k = 0 \\
\lambda_0(G) - n & \text{if } j \neq 0 \text{ and } k = 0 \\
\lambda_k(G) & \text{if } j, k \neq 0
\end{cases}
$$

If $|\psi(0)\rangle = |0\rangle \otimes |0\rangle$ then $|\psi(0)\rangle = 1/\sqrt{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |F_{j,k}\rangle$. Thus,

$$
|\psi(t)\rangle = \frac{1}{\sqrt{mn}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} e^{-it\lambda_{j,k}} |F_{j,k}\rangle.
$$

Thus, for $x \in \mathbb{Z}_m$ and $y \in \mathbb{Z}_n$, we have

$$
\psi_{x,y}(t) = \langle x, y | \psi(t) \rangle = \frac{1}{mn} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} e^{-it\lambda_{j,k}} \exp\left(\frac{2\pi ijx}{m}\right) \exp\left(\frac{2\pi kxy}{n}\right).
$$
Note that the three types of eigenvalues of $G^{(m)}$ are mutually distinct, since
\[ \lambda_0(G) - n < \lambda_k(G) < \lambda_0(G) + (m - 1)n. \]  
(23)

Therefore, we have
\[
\left| \vec{p}_{x,y} - \frac{1}{mn} \right| \leq \frac{1}{(mn)^2} \left( \frac{m-1}{2} \right) \left[ 1 + n \left( \frac{\mu(G)}{2} \right) \right] = O \left( \frac{\mu^2(G)}{mn} \right),
\]
(24)
since $m$ is a constant.

The above theorem also explains why the complete graph $K_N$ is not uniform mixing, since $K_N$ can be viewed as a homogeneous $m$-fold join of $K_{N/m}$, for some constant $m$ that divides $N$. The theorem also implies the following claim about the multipartite complete graphs.

**Corollary 10** The continuous-time quantum walk on the complete multipartite graph $K_n^{(m)}$ is not average uniform mixing if $m \geq 2$ is a constant.

**Proof** Since a continuous-time quantum walk is not average uniform mixing on the empty graph $K_n$ and $K_n^{(m)} = K_n + \ldots + K_n$, we have our claim.

\[ \square \]

## 5 Mixing on Cartesian Bunkbeds

In this section, we consider a circulant bunkbed structure obtained by the Cartesian product $P_2 \oplus C$, where $C$ is a circulant graph.

**Lemma 11** Let $G$ be a circulant of degree $d$ and order $n$, whose eigenvalues are $d = \mu_0 > \mu_1 \geq \ldots \geq \lambda_{n-1}$. Then, the eigenvalues of $P_2 \oplus G$ are $\lambda_j^\pm = \mu_j \pm 1$ with the following (orthonormal) set of eigenvectors
\[
|z_j^\pm \rangle = |\pm \rangle \otimes |z_j \rangle,
\]
(25)
where $|\pm \rangle = \frac{1}{\sqrt{2}}(|0 \rangle \pm |1 \rangle)$ and $\langle x|z_j \rangle = (1/\sqrt{|G|})\omega^{jx}$, for $j, x \in [n]$, with $\omega = e^{2\pi i/n}$.

**Proof** Note that the adjacency matrix of $P_2 \oplus G$ is given by $P_2 \otimes I_n + I_2 \otimes A_G$. Since $P_2$ is a circulant, both $P_2 \otimes I_n$ and $I_2 \otimes A_G$ are simultaneously diagonalizable by $|z_j^\pm \rangle$. This implies the stated claim on the spectra of $P_2 \oplus G$.

\[ \square \]

**Theorem 12** Let $G$ be a circulant of order $n$. In the continuous-time quantum walk, $P_2 \oplus G$ is average uniform mixing if $G$ is.

**Proof** Assume that $|\psi(0)\rangle = |0 \rangle \otimes |0 \rangle$. Thus, $|\psi(0)\rangle = \sum_{\pm} \frac{1}{\sqrt{2}} |\pm \rangle \otimes \sum_j \frac{1}{\sqrt{n}} |z_j \rangle$, and
\[
|\psi(t)\rangle = e^{-itA} |\psi(0)\rangle = \frac{1}{\sqrt{2n}} \sum_{\pm, j} e^{-it\lambda_j^\pm} |\pm \rangle \otimes |z_j \rangle.
\]
(26)

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This implies that

\[
(b, x|\psi(t)) = \frac{1}{\sqrt{2n}} \sum_{\pm j} e^{-it(\lambda_j \pm 1)}\langle b|\pm\rangle\langle x|z_j \rangle
\]

(27)

\[
= \frac{1}{2n} \sum_{\pm j} e^{-it(\lambda_j \pm 1)}(\pm 1)^b \omega^j x
\]

(28)

\[
= \frac{1}{n} \sum_j e^{-it\lambda_j \omega^j x} \sum_{\pm} e^{-it(\pm 1)}(\pm 1)^b
\]

(29)

\[
= \frac{1}{n} \sum_j e^{-it\lambda_j \omega^j x}[(1 - b) \cos(t) + b(-i \sin(t))]
\]

(30)

Let \( p_{b,x}(t) = |\langle b, x|\psi(t)\rangle|^2 \). Thus,

\[
p_{b,x}(t) = \frac{1}{n^2} \sum_{j,k} [(1 - b) \cos^2(t) + b \sin^2(t)] e^{-it(\lambda_j - \lambda_k) \omega^{(j-k)x}}.
\]

(31)

Note that \( p_{0,x}(t) + p_{1,x}(t) = p_x(t) \), where \( p_x(t) \) is the (instantaneous) probability on vertex \( x \) at time \( t \) of a quantum walk on \( G \) alone. Then,

\[
\overline{p}_{0,x} = \frac{1}{n^2} \sum_{j,k} \omega^{(j-k)x} \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos^2(t)e^{-it(\lambda_j - \lambda_k)} \ dt = \frac{1}{2} \overline{p}_x,
\]

(32)

since \( \lim_{T \to \infty} \frac{1}{T} \int_0^T \cos^2(t)e^{-it\Delta} \ dt = \frac{1}{2} [\Delta = 0] \). Similarly, we obtain \( \overline{p}_{1,x} = \frac{1}{2} \overline{p}_x \). This yields the claim.

**Corollary 13** The continuous-time quantum walk on a Cartesian bunkbed \( P_2 \oplus G \), where \( G \) is a \( d \)-degree circulant of the form \( \langle 1, n/k_1, \ldots, n/k_{d-1} \rangle \), where \( d \) and \( k_1, \ldots, k_{d-1} \) are constants, is average uniform mixing.
Circulant Cylinders  To extend our Cartesian bunkbeds over paths with more than two vertices, we provide, for completeness, an analysis of the quantum walk on paths. This problem is well-known in the physics literature, but is normally done on the infinite paths using different techniques [10]. The eigenvalues $\lambda_j$ and eigenvectors $|Q_j\rangle$ of the path $P_m$ (see [17]), for $j = 1, \ldots, m$, are defined as

$$
\lambda_j = 2 \cos \left( \frac{j \pi}{m+1} \right) \tag{33}
$$

$$
\langle x | Q_j \rangle = \frac{1}{\sqrt{(m+1)/2}} \sin \left( \frac{j x \pi}{m+1} \right), \quad x = 1, \ldots, m \tag{34}
$$

If the quantum walk starts with the initial state $|\psi(0)\rangle = |1\rangle$, where the basis states are $|1\rangle, \ldots, |m\rangle$, then Thus, we have

$$
|\psi(t)\rangle = \sum_{j=1}^{m} \frac{e^{-it\lambda_j}}{\sqrt{(m+1)/2}} \sin \left( \frac{j \pi}{m+1} \right) |Q_j\rangle. \tag{35}
$$

Since $P_m$ has $m$ distinct eigenvalues, the limiting probabilities are given by

$$
\mathbb{P}_x = \frac{4}{(m+1)^2} \sum_{j=1}^{m} \sin^2 \left( \frac{j \pi}{m+1} \right) \sin^2 \left( \frac{j x \pi}{m+1} \right). \tag{36}
$$

Note that, since $\int_0^\pi \sin^2(t)dt = \pi/2$, we get an upper bound of

$$
\mathbb{P}_x \leq \frac{4}{(m+1)^2} \sum_{j=1}^{m} \sin^2 \left( \frac{j \pi}{m+1} \right) \tag{37}
$$

$$
\leq \frac{4}{(m+1)\pi} \left( \int_0^\pi \sin^2(t) \, dt + \frac{\pi}{(m+1)} \right) \tag{38}
$$

$$
\leq \frac{2}{(m+1)} + \frac{4}{(m+1)^2} = O \left( \frac{1}{m} \right), \tag{39}
$$

which implies that the quantum walk on $P_m$ is average uniform mixing.

The eigenvalues of a circulant cylinder $T = P_m \oplus G$, where $G$ is a circulant of order $n$, are given by

$$
\lambda_{j,k} = \mu_j + \nu_k, \quad \text{where} \quad 1 \leq j \leq m, \quad 0 \leq k \leq n - 1, \tag{40}
$$

where $\mu_j = 2 \cos(j\pi/(m+1))$ and $\nu_k$ are the eigenvalues of $P_m$ and $G$, respectively. Since the adjacency matrix of $T$ is defined as $P_m \otimes I_n + I_m \otimes G$, the eigenvectors of $T$ are

$$
|T_{j,k}\rangle = |Q_j\rangle \otimes |F_k\rangle, \quad \text{where} \quad 1 \leq j \leq m, \quad 0 \leq k \leq n - 1, \tag{41}
$$

where $|Q_j\rangle$ and $|F_k\rangle$ are the eigenvectors of $P_m$ and the circulant $G$, respectively. Recall that $\langle x | Q_j \rangle = \sqrt{\frac{2}{m+1}} \sin \left( \frac{j x \pi}{m+1} \right)$, for $1 \leq j, x \leq m$, and $\langle y | F_k \rangle = \frac{1}{\sqrt{n}} \exp \left( \frac{2\pi i k y}{n} \right)$, for $0 \leq k, y \leq n - 1$. 

10
If the initial state is \( |\psi(0)\rangle = |1\rangle \otimes |0\rangle \), we have

\[
|\psi(0)\rangle = \sum_{j=1}^{m} (Q_j|1\rangle \langle Q_j| \otimes \sum_{k=0}^{n-1} (F_k|0\rangle \langle F_k|).
\]

(42)

The adjacency matrix of \( P_m \oplus G \) is given by \( A = P_m \otimes I_n + I_m \otimes G \), where the two summands commute with each other. Thus, \( e^{-itA} = e^{-it(P_m \otimes I_n)}e^{-it(I_m \otimes G)} \), and

\[
|\psi(t)\rangle = \sum_{j=1}^{m} (Q_j|1\rangle \langle \sum_{k=0}^{n-1} (F_k|0\rangle \langle F_k|) e^{-it \mu_j} \langle Q_j| \otimes \sum_{k=0}^{n-1} (F_k|0\rangle \langle F_k|) e^{-it \nu_k} |F_k\rangle.
\]

(43)

The amplitudes of \( |\psi(t)\rangle \) at vertex \( x \) on the path \( P_m \) and vertex \( y \) within the circulant \( G \) is given by

\[
\langle x, y|\psi(t)\rangle = \sum_{j=1}^{m} e^{-it \mu_j} \langle x|Q_j\rangle \langle Q_j|1\rangle \sum_{k=0}^{n-1} e^{-it \nu_k} \langle y|F_k\rangle \langle F_k|0\rangle
\]

(44)

**Corollary 14** Let \( G \) be a Cartesian product \( P_m \oplus C \), where \( C \) is a circulant of order \( n \). The continuous-time quantum walk on \( G \) is uniform mixing if \( m \) is constant or \( n \) is constant.

6 Conclusions

It was known that a continuous-time quantum walk is uniform average mixing on the cycles \( C_n \), but is not uniform average mixing on the complete graphs \( K_n \) and on the hypercubes \( Q_n \). Our goal in this work was to provide a graph-theoretic explanation for this polarized phenomena.

First, we extend the phenomenon of the cycles, by showing that uniform mixing is achieved on circulants with bounded eigenvalue multiplicity. We also gave other explicit examples of circulants meeting this criteria. Second, we consider two graph-theoretic bunkbed structures over circulants in order to study the non-uniform mixing on \( K_n \) and \( Q_n \). Our analysis on the join bunkbed sheds some light on the non-uniform mixing of the complete multipartite graphs (which includes \( K_n \)). Our analysis of the Cartesian bunkbed of circulants highlights a difference between the \( Z_n \)-circulants and the \((Z_2)^n\)-circulants (see [3]). We leave a similar investigation of general group-theoretic circulants and Cayley graphs for future work (see [11]).

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