THE ROSENBERG-STRONG PAIRING FUNCTION

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Abstract. This article surveys the known results (and not very well-known results) associated with Cantor’s pairing function and the Rosenberg-Strong pairing function, including their inverses, their generalizations to higher dimensions, and a discussion of a few of the advantages of the Rosenberg-Strong pairing function over Cantor’s pairing function in practical applications. In particular, an application to the problem of enumerating full binary trees is discussed.

1. Cantor’s pairing function

Given any set \( B \), a pairing function for \( B \) is a one-to-one correspondence from the set of ordered pairs \( B^2 \) to the set \( B \). The only finite sets \( B \) with pairing functions are the sets with fewer than two elements. But if \( B \) is infinite, then a pairing function for \( B \) necessarily exists.

For example, Cantor’s pairing function [3] for the positive integers is the function

\[
p(x, y) = \frac{1}{2} (x^2 + 2xy + y^2 - x - 3y + 2)
\]

that maps each pair \((x, y)\) of positive integers to a single positive integer \( p(x, y) \). Cantor’s pairing function serves as an important example in elementary set theory [10]. It is also used as a fundamental tool in recursion theory and in other related areas of mathematics [22, 24].

A few different variants of Cantor’s pairing function appear in the literature. First, given any pairing function \( f(x, y) \) for the positive integers, the function \( f(x+1, y+1) - 1 \) is a pairing function for the non-negative integers. Therefore, we refer to the function

\[
c(x, y) = p(x+1, y+1) - 1 = \frac{1}{2} (x^2 + 2xy + y^2 + 3x + y)
\]

as Cantor’s pairing function for the non-negative integers. And given any pairing function \( f(x, y) \) for a set \( B \), the function obtained by exchanging \( x \) and \( y \) in the definition of \( f \) is itself a pairing function for \( B \). Hence,

\[
c(x, y) = \frac{1}{2} (y^2 + 2yx + x^2 + 3y + x)
\]

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There is no general agreement on the definition of a pairing function in the published literature. For a given set \( B \), some publications [1, 35, 40] use a more general definition, and allow any one-to-one function (i.e. any injection) from \( B^2 \) to \( B \) to be regarded as a pairing function. Other publications [31, 36] are more restrictive, and require each pairing function to be a one-to-one correspondence (i.e. a bijection). We use the more restrictive definition in this paper.

The proof [10] for this claim relies on the axiom of choice. In fact, in Zermelo set theory, asserting that all infinite sets have pairing functions is equivalent to asserting the axiom of choice [14].
Figure 1. Cantor’s pairing function $c(x, y)$.

is another variant of Cantor’s pairing function for the non-negative integers. It has
been shown by Fueter and Polya [12] that there are only two quadratic polynomials
that are pairing functions for the non-negative integers, namely the polynomials $c(x, y)$ and $\bar{c}(x, y)$. But it is a longstanding open problem whether there exist any
other polynomials, of higher degree, that are pairing functions for the non-negative
integers. Partial results toward a resolution of this problem have been obtained by
Lew and Rosenberg [22, 23].

An enumeration of a countably infinite set $C$ is a one-to-one correspondence
from the set $\mathbb{N}$ of non-negative integers to the set $C$. We think of an enumeration $g: \mathbb{N} \to C$ as ordering the members of $C$ in the sequence

$$g(0), g(1), g(2), g(3), \ldots .$$

Given any pairing function $f: \mathbb{N}^2 \to \mathbb{N}$, its inverse $f^{-1}: \mathbb{N} \to \mathbb{N}^2$ is an enumeration of the set $\mathbb{N}^2$. For example, the inverse of Cantor’s pairing function $c(x, y)$ orders the points in $\mathbb{N}^2$ according to the sequence

$$(1.1) \quad (0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), \ldots .$$

This sequence is illustrated in Figure 1. Cantor’s pairing function is closely related to Cauchy’s product formula [14], which defines the product of two infinite series, $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$, to be the infinite series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} a_j b_{i-j} = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots .$$

In particular, the pairs of subscripts in this sum occur in exactly the same order as
the points in sequence (1.1).

Given any point $(x, y)$ in $\mathbb{N}^2$, we say that the quantity $x + y$ is the point’s shell number for Cantor’s pairing function. In Figure 1 any two consecutive points that share the same shell number have been joined with an arrow. The inverse of Cantor’s pairing function $c(x, y)$ is given by the formula

$$(1.2) \quad c^{-1}(z) = \left( z - \frac{w(w + 1)}{2}, \frac{w(w + 3)}{2} - z \right),$$

where $w = \lfloor \sqrt{2z + 1} \rfloor - 1$. This formula is illustrated in Figure 1.
Several different names for this representation appear in the published literature, including the combinatorial number system \[19\], the 8th Macaulay representation \[15\], the \(d\)-binomial expansion \[13\], and the \(d\)-cascade representation \[11\]. The earliest known description of the \(d\)-canonical representation is in a paper by Ernesto Pascal \[28\].

And similarly, the \(8\)th degree polynomial \(c(c(x_1, x_2), x_3)\) is a 4-tupling function. By repeatedly applying Cantor’s pairing function in this manner, one can obtain a \(2d \cdot 1\)-degree polynomial \(d\)-tupling function, given any positive integer \(d\). This method for generalizing Cantor’s pairing function to higher dimensions is commonly used in recursion theory \[32\].

Another higher-dimensional generalization of Cantor’s pairing function was identified by Skolem \[11\]. In particular, for each positive integer \(d\), the \(d\)-degree polynomial

\[
s_d(x_1, x_2, \ldots, x_d) = \sum_{i=1}^{d} \binom{x_1 + x_2 + \cdots + x_i + i - 1}{i}
\]

is a \(d\)-tupling function for the non-negative integers. Note that \(s_2(x_1, x_2)\) is Cantor’s pairing function for the non-negative integers. The inverse function \(s_d^{-1}(z)\) can be calculated using the \(d\)-canonical representation\[1\] of \(z\).

Lew \[21\] has shown that every polynomial \(d\)-tupling function for the non-negative integers has degree \(d\) or higher. That is, Skolem’s \(d\)-tupling function has the smallest possible degree for a polynomial \(d\)-tupling function from \(\mathbb{N}^d\) to \(\mathbb{N}\). But \(s_d(x_1, x_2, \ldots, x_d)\) is not necessarily the only polynomial \(d\)-tupling function with degree \(d\).

Let \(A_d\) denote the set of all \(d\)-tupling functions for the non-negative integers that can be expressed as polynomials of degree \(d\). Given any \(d\)-tupling function for a set \(B\), a function obtained by permuting the order of its arguments is also a \(d\)-tupling function for \(B\). If this permutation is not the identity, and if \(B\) contains at least

\[w = \left\lfloor \frac{-1 + \sqrt{1 + 8z}}{2} \right\rfloor\]

and where, for all real numbers \(t\), \(\lfloor t \rfloor\) denotes the floor of \(t\). In his derivation of this inverse formula, Davis \[8\] has shown that \(w\) is the shell number of \(c^{-1}(z)\). In fact, we can deduce this directly from equation (1.2), since the two components of \(c^{-1}(z)\) sum to \(w\).

In elementary set theory \[10\], an ordered triple \((x_1, x_2, x_3)\) is defined as an abbreviation for the formula \(((x_1, x_2), x_3)\). Likewise, an ordered quadruple \((x_1, x_2, x_3, x_4)\) is defined as an abbreviation for \(((x_1, x_2), (x_3, x_4))\), and so on. A similar idea can be used to generalize pairing functions to higher dimensions. Given any set \(B\) and any positive integer \(d\), we say that a one-to-one correspondence from \(B^d\) to \(B\) is a \(d\)-tupling function for \(B\). For example, if \(f\) is any pairing function for a set \(B\), then

\[g(x_1, x_2, x_3) = f(f(x_1, x_2), x_3)\]

is a 3-tupling function for \(B\). The inverse of this 3-tupling function is

\[g^{-1}(z) = (f^{-1}(u), v),\]

where \(u\) and \(v\) are defined so that \((u, v) = f^{-1}(z)\).
two elements, then the \( d \)-tupling function obtained in this manner is necessarily
distinct from the original function. Therefore, the set \( A_d \) contains at least \( d! \) many functions—one for each permutation of the arguments of \( s_d(x_1, x_2, \ldots, x_d) \). If \( d = 1 \) or \( d = 2 \), then these are the only members of \( A_d \), and \(|A_d| = d! \) in this case. But Chowla \([5]\) has shown that for all positive integers \( d \),

\[
\chi_d(x_1, x_2, \ldots, x_d) = \left( \frac{x_1 + \cdots + x_d + d}{d} \right) - 1 - \sum_{i=1}^{d-1} \left( \frac{x_{i+1} + \cdots + x_d + d - i - 1}{d - i} \right)
\]
is also a \( d \)-degree polynomial \( d \)-tupling function for the non-negative integers.\(^4\) Chowla’s function is identical to Skolem’s function for \( d = 1 \) and \( d = 2 \). When \( d \) is
greater than 2, the following theorem holds.

**Theorem 1.1.** Let \( d \) be any integer greater than 2. Then, \( \chi_d(x_1, x_2, \ldots, x_d) \) cannot
be obtained by permuting the arguments of \( s_d(x_1, x_2, \ldots, x_d) \).

**Proof by contradiction.** Assume that \( \chi_d(x_1, x_2, \ldots, x_d) \) can be obtained by permuting
the arguments of \( s_d(x_1, x_2, \ldots, x_d) \). Then, since \( s_d : \mathbb{N}^d \to \mathbb{N} \) is a one-to-one
correspondence and

\[
s_d(0, 1, 0, 0, \ldots, 0) = \chi_d(0, 1, 0, 0, \ldots, 0),
\]
the permutation must not move the argument \( x_2 \). But this contradicts the fact
that

\[
s_d(0, 2, 0, 0, \ldots, 0) \neq \chi_d(0, 2, 0, 0, \ldots, 0).
\]

Therefore, the assumption is false, and \( \chi_d(x_1, x_2, \ldots, x_d) \) cannot be obtained by
permuting the arguments of \( s_d(x_1, x_2, \ldots, x_d) \). \( \square \)

An immediate consequence is that if \( d \) is greater than 2, then \(|A_d| > d! \). A more
precise lower bound for \(|A_d| \) has been provided by Morales and Arredondo \([26]\). In
particular, they prove that

\[
|A_d| \geq d! a(d),
\]
where \( a(d) \) is defined recursively so that \( a(1) = 1 \), and so that

\[
a(d) = \sum_{\substack{i \in \mathbb{N}, i \neq 1 \text{ divides } d}} (i - 1)! a(d/i)
\]
for all integers \( d > 1 \). They also construct a set of polynomial \( d \)-tupling functions
for each positive integer \( d \). Morales and Arredondo conjecture that this is the set of
all polynomial \( d \)-tupling functions for the non-negative integers. If their conjecture
is correct, then the inequality \(^{14}\) is an equality. A proof of the Morales-Arredondo
conjecture would also imply that \( \bar{c}(x, y) \) and \( \bar{c}(x, y) \) are the only polynomial pairing
functions for the non-negative integers.

\(^4\) The function originally described by Chowla was a \( d \)-tupling function for the positive integers.
The variant given here is obtained by translating Chowla’s function from the positive integers to
the non-negative integers.
2. Other pairing functions

In addition to Cantor’s pairing function, a few other pairing functions are often encountered in the literature. Most notably, the Rosenberg-Strong pairing function for the non-negative integers is defined by the formula

\[ r_2(x, y) = (\max(x, y))^2 + \max(x, y) + x - y. \]  

In the context of the Rosenberg-Strong pairing function, the quantity \( \max(x, y) \) is said to be the shell number of the point \((x, y)\). Figure 2 contains an illustration of the Rosenberg-Strong pairing function. In this illustration, points that appear consecutively in the enumeration \( r_{2^{-1}} : \mathbb{N} \rightarrow \mathbb{N}^2 \) are joined by an arrow if and only if they share the same shell number. The inverse of the Rosenberg-Strong pairing function \( r_2(x, y) \) is given by the formula

\[
r_2^{-1}(z) = \begin{cases} 
(z - m^2, m) & \text{if } z - m^2 < m \\
(m, m^2 + 2m + z) & \text{otherwise} 
\end{cases},
\]

where \( m = \lfloor \sqrt{z} \rfloor \). Note that \( m \) is the shell number of the point \( r_2^{-1}(z) \).

A shell numbering can serve as a useful tool when one wishes to describe the properties of a \( d \)-tupling function. Formally, we make the following definition.

**Definition 2.1.** Let \( f : \mathbb{N}^d \rightarrow \mathbb{N} \) be a \( d \)-tupling function. A function \( \sigma : \mathbb{N}^d \rightarrow \mathbb{N} \) is said to be a shell numbering for \( f \) if and only if

\[ \sigma(x) < \sigma(y) \quad \text{implies} \quad f(x) < f(y) \]

for all \( x \) and \( y \) in \( \mathbb{N}^d \).

Given a shell numbering \( \sigma \), the quantity \( \sigma(x) \) is said to be the shell number of \( x \). The shell that contains \( x \) is the set of all points in \( \mathbb{N}^d \) that have the same shell number as \( x \). Therefore, a shell numbering \( \sigma \) partitions \( \mathbb{N}^d \) into shells according to

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\(^5\) The function originally described by Rosenberg and Strong was a pairing function for the positive integers. The variant defined here is obtained by translating the Rosenberg-Strong function from the positive integers to the non-negative integers, and by reversing the order of the arguments.
the equivalence relation \( \sigma(x) = \sigma(y) \). Notice that each \( d \)-tupling function for the non-negative integers has more than one shell numbering. In particular, given any \( d \)-tupling function \( f: \mathbb{N}^d \rightarrow \mathbb{N} \), the constant functions \( \sigma(x) = k \), where \( k \) is a non-negative integer, are all shell numberings for \( f \). But there is often one particular shell numbering that we prefer to use in the context of a given \( d \)-tupling function. We call this the standard shell numbering, or simply the shell numbering, for the function.

For Cantor’s pairing function \( c(x, y) \), the standard shell numbering is \( \sigma(x, y) = x + y \), and each shell is a set of points on a diagonal line. For this reason, \( d \)-tupling functions with the shell numbering \( \delta(x_1, x_2, \ldots, x_d) = x_1 + x_2 + \cdots + x_d \) are said to have *diagonal shells*. The pairing functions \( c(x, y) \) and \( \bar{c}(x, y) \) both have diagonal shells. For this reason, \( d \)-tupling functions \( s_d(x_1, x_2, \ldots, x_d) \) and \( \chi_d(x_1, x_2, \ldots, x_d) \) also have diagonal shells.

A \( d \)-tupling function with the shell numbering \( \max(x_1, x_2, \ldots, x_d) \) is said to have *cubic shells*. (Although in the \( d = 2 \) case, the term *square shells* is sometimes used, instead.) The Rosenberg-Strong pairing function \( r_2(x, y) \) has cubic shells. Other pairing functions with cubic shells have been described by Péter [29] and Szudzik [43]. In addition to cubic shells and diagonal shells, Rosenberg has also studied \( d \)-tupling functions with *hyperbolic shells* [35].

As another example of a popular pairing function, let \( q: \mathbb{N}^2 \rightarrow \mathbb{N} \) be defined so that

\[
q(x, y) = 2^y(2x + 1) - 1.
\]

Variants of this pairing function have been used by authors in computer science [25, 6, 9] and set theory [39, 15, 17], presumably because the inverse \( q^{-1}(z) \) is easily calculated from the binary representation of \( z + 1 \). In this context, it is convenient to define the shell number of \( (x, y) \) to be the number of bits in the binary representation of \( q(x, y) + 1 \). Notice that \( \lceil \log_2(x + 1) \rceil \) is the number of bits in the binary representation of the non-negative integer \( x \), where \( \lceil t \rceil \) denotes the ceiling of \( t \) for each real number \( t \). Therefore, the shell number of \( (x, y) \) is given by the formula

\[
y + 1 + \lceil \log_2(x + 1) \rceil.
\]

The shells of the pairing function \( q(x, y) \) are illustrated in Figure 3.

Regan [31] has investigated the computational complexity of pairing functions for the positive integers, and has described several pairing functions with low computational complexity. Among Regan’s results is a pairing function that can be computed in linear time and constant space.

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6. Of course, when \( d > 2 \) the points in each shell lie on a plane or hyperplane, rather than a line.

7. Morales and Lew [27] have devised a notation to describe the members in a certain family of \( d \)-tupling functions for the non-negative integers. The functions in this family all have diagonal shells. Using Morales and Lew’s notation,

\[
s_d(x_1, x_2, \ldots, x_d) = AAA \cdots A(x_d, x_{d-1}, \ldots, x_1)
\]

and

\[
\chi_d(x_1, x_2, \ldots, x_d) = BAA \cdots A(x_d, x_{d-1}, \ldots, x_1)
\]

for all integers \( d > 2 \).
3. ADDITIONAL RESULTS

Given a $d$-tupling function $f : \mathbb{N}^d \to \mathbb{N}$, we define

$$J_\sigma^{\leq n} = \{ i \in \mathbb{N} : \sigma(f^{-1}(i)) < n \}$$

for all functions $\sigma : \mathbb{N}^d \to \mathbb{N}$ and all non-negative integers $n$.

**Theorem 3.1.** Let $f : \mathbb{N}^d \to \mathbb{N}$ be any $d$-tupling function. A function $\sigma : \mathbb{N}^d \to \mathbb{N}$ is a shell numbering for $f$ if and only if for all points $x$ in $\mathbb{N}^d$,

$$|J_\sigma^{\leq n}| \leq f(x) < |J_\sigma^{\leq n+1}|,$$

where $n = \sigma(x)$.

**Remark.** In the statement of this theorem, if the cardinality of $J_\sigma^{\leq n}$ is infinite, then the inequality $|J_\sigma^{\leq n}| \leq i$ is false for all non-negative integers $i$. If the cardinality of $J_\sigma^{\leq n+1}$ is infinite, then $i < |J_\sigma^{\leq n+1}|$ is true for all non-negative integers $i$.

**Proof.** Let $\sigma$ be a function from $\mathbb{N}^d$ to $\mathbb{N}$. Note that if $\sigma(x) < \sigma(y)$ for any $x, y \in \mathbb{N}^d$, then inequality (3.1) implies that

$$f(x) < |J_\sigma^{\leq \sigma(x)+1}| \leq |J_\sigma^{\leq \sigma(y)}| \leq f(y).$$

Hence, $\sigma$ is a shell numbering for $f$ if inequality (3.1) holds.

Conversely, suppose that $\sigma$ is a shell numbering for $f$. Then by Definition 2.1,

$$\sigma(f^{-1}(i)) < \sigma(f^{-1}(j))$$

implies $i < j$ for all $i, j \in \mathbb{N}$. Now consider any $x \in \mathbb{N}^d$ and let $n = \sigma(x)$. Note that if $J_\sigma^{\leq n+1} = \mathbb{N}$, then $f(x) < |J_\sigma^{\leq n+1}|$ because the cardinality of $J_\sigma^{\leq n+1}$ is infinite. Otherwise, let $j$ be the smallest non-negative integer such that $j \notin J_\sigma^{\leq n+1}$. By the definition of $J_\sigma^{\leq n+1}$,

$$\sigma(f^{-1}(j)) \geq n + 1 > \sigma(f^{-1}(i))$$

for all $i \in J_\sigma^{\leq n+1}$. So, by condition (3.2), $i < j$ for all $i \in J_\sigma^{\leq n+1}$. We have shown that the smallest non-negative integer not in $J_\sigma^{\leq n+1}$ is larger than all the members

\[\text{Figure 3. The pairing function } q(x, y). \text{ Points connected by a sequence of dotted line segments have the same shell number.}\]
of \(J^{<n+1}_\sigma\). It immediately follows that
\[
J^{<n+1}_\sigma = \{0, 1, 2, \ldots, |J^{<n+1}_\sigma| - 1\}.
\]
But \(f(x) \in J^{<n+1}_\sigma\) because \(n = \sigma(x)\). Therefore,
\[
f(x) < |J^{<n+1}_\sigma|.
\]
And since \(n = \sigma(x)\), \(f(x) \notin J^{<n}_\sigma\). This implies that either \(J^{<\sigma}_\sigma\) is empty or, using the same argument as above,
\[
J^{<\sigma}_\sigma = \{0, 1, 2, \ldots, |J^{<\sigma}_\sigma| - 1\}.
\]
In either case,
\[
|J^{<\sigma}_\sigma| \leq f(x) < |J^{<n+1}_\sigma|.
\]
Hence, we have shown that inequality (3.1) holds if \(\sigma\) is a shell numbering for \(f\). □

Given a function \(\sigma: \mathbb{N}^d \to \mathbb{N}\) and a non-negative integer \(n\), \(|J^{<\sigma}_\sigma|\) is the number of points \(x\) such that \(\sigma(x) < n\). From this fact, simple combinatorial arguments are often sufficient to calculate \(|J^{<\sigma}_\sigma|\). For example, using the function \(\delta(x_1, x_2, \ldots, x_d) = x_1 + x_2 + \cdots + x_d\), we have that \(|J^{<\sigma}_\delta| = (w+d-1)^d\). And using \(\max(x_1, x_2, \ldots, x_d)\), we have that \(|J^{<\sigma}_{\max}| = m^d\). These two observations imply the following corollaries of Theorem 3.1.

**Corollary 3.2.** Let \(f: \mathbb{N}^d \to \mathbb{N}\) be any \(d\)-tupling function. The function \(f\) has diagonal shells if and only if for all points \((x_1, x_2, \ldots, x_d)\) in \(\mathbb{N}^d\),
\[
(w+d-1) \leq f(x_1, x_2, \ldots, x_d) < (w+d)^d,
\]
where \(w = x_1 + x_2 + \cdots + x_d\).

**Corollary 3.3.** Let \(f: \mathbb{N}^d \to \mathbb{N}\) be any \(d\)-tupling function. The function \(f\) has cubic shells if and only if for all points \((x_1, x_2, \ldots, x_d)\) in \(\mathbb{N}^d\),
\[
m^d \leq f(x_1, x_2, \ldots, x_d) < (m+1)^d,
\]
where \(m = \max(x_1, x_2, \ldots, x_d)\).

Next, we say that a function \(f: \mathbb{N}^d \to \mathbb{N}\) is **max-dominating** if and only if
\[
\max(x) \leq f(x) \quad \text{for all points } x \in \mathbb{N}^d.
\]

In particular, if \(f\) is max-dominating then
\[
x \neq (0, 0, \ldots, 0) \quad \text{implies} \quad 0 < \max(x) \leq f(x).
\]
It immediately follows that \(f(0, 0, \ldots, 0) = 0\) for every max-dominating \(d\)-tupling function \(f: \mathbb{N}^d \to \mathbb{N}\), since \(f(x)\) must be 0 for some \(x \in \mathbb{N}^d\). We also have the following lemma.

**Lemma 3.4.** If a \(d\)-tupling function \(f: \mathbb{N}^d \to \mathbb{N}\) has diagonal shells or cubic shells, then \(f\) is max-dominating.

**Proof.** Suppose that \(f: \mathbb{N}^d \to \mathbb{N}\) is a \(d\)-tupling function, and consider any non-negative integers \(x_1, \ldots, x_d\). Note that if \(f\) has diagonal shells, then by Corollary 3.2
\[
\max(x_1, \ldots, x_d) \leq x_1 + \cdots + x_d \leq \left(\frac{x_1 + \cdots + x_d + d - 1}{d}\right) \leq f(x_1, \ldots, x_d).
\]
Alternatively, if \(f\) has cubic shells, then
\[
\max(x_1, \ldots, x_d) \leq (\max(x_1, \ldots, x_d))^d \leq f(x_1, \ldots, x_d)
\]
by Corollary 3.3. In either case, \( f \) is max-dominating. □

A pairing function \( r_2(x, y) \) with cubic shells was described in the previous section. Rosenberg and Strong \[37, 33\] introduced a higher-dimensional generalization of this pairing function. In particular, the Rosenberg-Strong \( d \)-tupling function \( r_d \) for the non-negative integers is defined recursively so that \( r_1(x_1) = x_1 \), and so that for all integers \( d > 1 \),

\[
r_d(x_1, \ldots, x_d) = r_{d-1}(x_1, \ldots, x_{d-1}) + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}),
\]

where \( m = \max(x_1, \ldots, x_d) \). Note that equation (2.1) agrees with this definition when \( d = 2 \). It follows from Corollary 3.3 and Lemma 6.1 that \( r_d(x_1, \ldots, x_d) \) has cubic shells for each positive integer \( d \).

The inverse of the Rosenberg-Strong \( d \)-tupling function can also be defined recursively. In particular, \( r_1^{-1}(z) = z \). And for all integers \( d > 1 \),

\[
r_d^{-1}(z) = \left( (r_d^{-1}(z - m^d - (m - x_d)((m + 1)^{d-1} - m^{d-1}))), x_d \right),
\]

where

\[
x_d = m - \left\lfloor \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \right\rfloor
\]

and \( m = \lfloor \sqrt[3]{z} \rfloor \).

4. Applications

It is a common convention \[45\] that the vertices in a binary tree may have 0, 1, or 2 children. A binary tree where each vertex has either 0 or 2 children is said to be a full binary tree. We use \( o \) to denote the trivial binary tree that has only one vertex, and we use \( \tau(a, b) \) to denote the binary tree with left subtree \( a \) and right subtree \( b \). Then, the set \( T \) of all full binary trees is the smallest set that contains \( o \) and that is closed under the operation \( \tau \). Let the height of a binary tree be the length of the longest path from a leaf to the tree’s root. We use \( H(t) \) to denote the height of the binary tree \( t \). For all binary trees of the form \( \tau(a, b) \),

\[
H(\tau(a, b)) = 1 + \max(H(a), H(b)).
\]

The only binary tree of height zero is the trivial binary tree \( o \).

An important application of pairing functions is in the enumeration of full binary trees.

Theorem 4.1. Let \( f \) be any max-dominating pairing function for the non-negative integers. Let \( \phi_f(0) = o \), and for each pair \( (x, y) \) of non-negative integers let

\[
\phi_f(f(x, y) + 1) = \tau(\phi_f(x), \phi_f(y)).
\]

Then, \( \phi_f \) is an enumeration of the set \( T \) of full binary trees.

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*The function originally described by Rosenberg and Strong was a \( d \)-tupling function for the positive integers. The variant defined here is obtained by translating the Rosenberg-Strong function from the positive integers to the non-negative integers, and by reversing the order of the arguments.*
Proof. First, we show that \( \phi_f(n) \in T \) for each \( n \in \mathbb{N} \). The proof is by strong induction on \( n \). For the base step, notice that \( \phi_f(0) = o \in T \). Now consider any non-negative integer \( n \) and suppose, as the induction hypothesis, that \( \phi_f(i) \in T \) for all non-negative integers \( i \leq n \). Since \( f : \mathbb{N}^2 \to \mathbb{N} \) is a pairing function, there exist non-negative integers \( x \) and \( y \) such that \( f(x, y) = n \). Then, by the definition of \( \phi_f \),

\[
\phi_f(n + 1) = \phi_f(f(x, y) + 1) = \tau(\phi_f(x), \phi_f(y)).
\]

But \( x \leq f(x, y) = n \) and \( y \leq f(x, y) = n \) because \( f \) is max-dominating. Therefore, by the induction hypothesis, \( \phi_f(x) \in T \) and \( \phi_f(y) \in T \). We may conclude by equation (4.1) that \( \phi_f(n + 1) \in T \).

Next, we show that for each \( t \in T \) there exists a unique \( n \in \mathbb{N} \) such that \( \phi_f(n) = t \). The proof is by strong induction on the height of \( t \). For the base step, notice that \( 0 \) is the only non-negative integer \( n \) such that \( \phi_f(n) = o \). Then consider any non-negative integer \( k \) and suppose, as the induction hypothesis, that for each \( t \in T \) whose height is less than or equal to \( k \), there exists a unique \( n \in \mathbb{N} \) such that \( \phi_f(n) = t \). Now consider any \( t \in T \) with height \( k + 1 \). Since the height of \( t \) is greater than zero, it must be the case that \( t = \tau(a, b) \) for some binary trees \( a \in T \) and \( b \in T \) whose heights are each less than or equal to \( k \). By the induction hypothesis, there exists a unique \( x \in \mathbb{N} \) such that \( \phi_f(x) = a \) and a unique \( y \in \mathbb{N} \) such that \( \phi_f(y) = b \). Therefore, \( n = f(x, y) + 1 \) is the unique non-negative integer such that

\[
\phi_f(n) = \phi_f(f(x, y) + 1) = \tau(\phi_f(x), \phi_f(y)) = \tau(a, b) = t.
\]

We may conclude that \( \phi_f \) is a one-to-one correspondence from \( \mathbb{N} \) to \( T \). That is, \( \phi_f \) is an enumeration of the set \( T \). \( \square \)

It follows that any max-dominating pairing function for the non-negative integers can be used to construct an enumeration of the full binary trees\(^9\). For example, the enumeration \( \phi_o \) that is constructed using Cantor’s pairing function is illustrated in Figure 4 and the enumeration \( \phi_{r^2} \), constructed using the Rosenberg-Strong pairing function, is illustrated in Figure 5.

In these illustrations, one advantage of the Rosenberg-Strong pairing function over Cantor’s pairing function is apparent: the heights of trees never decrease in the enumeration \( \phi_{r^2} \). To formalize this observation, we extend the concept of a shell numbering so that it can be applied to any enumeration of a countably infinite set.

**Definition 4.2.** Let \( C \) be a countably infinite set, and let \( g : \mathbb{N} \to C \) be an enumeration of \( C \). A function \( \sigma : C \to \mathbb{N} \) is said to be a shell numbering for the enumeration \( g \) if and only if

\[
\sigma(g(i)) < \sigma(g(j)) \quad \text{implies} \quad i < j
\]

for all \( i \) and \( j \) in \( \mathbb{N} \).

---

\(^9\) Incidentally, Theorem 4.1 can also be used to construct an enumeration of all binary trees, including those binary trees that are not full binary trees. Let \( D \) denote the operation of defoliation. That is, for each non-trivial binary tree \( t \), let \( D(t) \) denote the tree that is obtained from \( t \) by deleting all of its leaves. The function \( D \) is a one-to-one correspondence from the set of non-trivial full binary trees to the set of all binary trees. As a consequence, \( D(\phi_f(n + 1)) \) is an enumeration of all binary trees, where \( f \) is any max-dominating pairing function for the non-negative integers.
The following theorem formalizes our observation that the heights of trees never decrease in the enumeration $\phi_{r_2}$.

**Theorem 4.3.** Let $f : \mathbb{N}^2 \to \mathbb{N}$ be any pairing function with cubic shells. Then, the height function $H$ is a shell numbering for the enumeration $\phi_f$.

**Proof.** We prove the contrapositive of the theorem. Namely, we prove that for all $i, j \in \mathbb{N}$,

$$i \geq j \implies H(\phi_f(i)) \geq H(\phi_f(j)).$$

The proof is by strong induction. For the base step, note that condition (4.2) is true for $i = 0$. Now consider any non-negative integer $n$ and suppose, as the induction hypothesis, that condition (4.2) is true for all $i, j \in \mathbb{N}$ such that $i \leq n$. Next, consider any positive integer $j \leq n + 1$, and let $(x, y) = f^{-1}(j - 1)$. By the definition of $\phi_f$,

$$\phi_f(j) = \phi_f(f(x, y) + 1) = \tau(\phi_f(x), \phi_f(y)).$$

Therefore,

$$H(\phi_f(j)) = 1 + \max(H(\phi_f(x)), H(\phi_f(y))).$$

But by Lemma 3.3, $f$ is max-dominating. So,

$$x \leq f(x, y) = j - 1 \leq n \quad \text{and} \quad y \leq f(x, y) = j - 1 \leq n.$$

It immediately follows from the induction hypothesis that if $x \geq y$ then $H(\phi_f(x)) \geq H(\phi_f(y))$. Similarly, if $y \geq x$ then $H(\phi_f(y)) \geq H(\phi_f(x))$. In either case,

$$\max(H(\phi_f(x)), H(\phi_f(y))) = H(\phi_f(\max(x, y))).$$

Therefore, by equation (4.3),

$$H(\phi_f(j)) = 1 + H(\phi_f(\max(x, y))).$$

And since this is true for all positive integers $j \leq n + 1$, it is true for $n + 1$. Hence,

$$H(\phi_f(n + 1)) = 1 + H(\phi_f(\max(u, v))).$$
where \((u, v) = f^{-1}(n)\). Furthermore,
\[
\max(u, v) < \max(x, y) \implies f(u, v) < f(x, y)
\]
because \(f\) has cubic shells. But \(f(u, v) = n \geq j - 1 = f(x, y)\), so it must be
the case that \(\max(u, v) \geq \max(x, y)\). And \(\max(u, v) \leq f(u, v) = n\) because \(f\) is
max-dominating. So, by the induction hypothesis,
\[
H(\phi_f(\max(u, v))) \geq H(\phi_f(\max(x, y))),
\]
\[
1 + H(\phi_f(\max(u, v))) \geq 1 + H(\phi_f(\max(x, y))),
\]
\[
H(\phi_f(n + 1)) \geq H(\phi_f(j)).
\]
This is also true if \(j = 0\), since \(H(\phi_f(0)) = 0\). Hence, we have shown that
\[
n + 1 \geq j \implies H(\phi_f(n + 1)) \geq H(\phi_f(j))
\]
for all \(j \in \mathbb{N}\). \(\square\)

Another important application, closely-related to the enumeration of full binary
trees, is the enumeration of finite-length sequences. For each positive integer \(d\),
the members of \(\mathbb{N}^d\) are said to be the \textit{length-}\(d\) \textit{sequences} of non-negative integers. Then,
\[
\mathbb{N}^* = \{(\)} \cup \mathbb{N}^1 \cup \mathbb{N}^2 \cup \mathbb{N}^3 \cup \ldots
\]
is the set of all finite-length sequences of non-negative integers, where \(()\) denotes the
\textit{empty sequence} of length zero. By convention, sequences of non-negative integers
that have different lengths are distinct members of \(\mathbb{N}^*\).

One simple way to construct an enumeration of \(\mathbb{N}^*\) is to choose a pairing function
\(f : \mathbb{N}^2 \rightarrow \mathbb{N}\), and to choose a \(d\)-tupling function \(g_d : \mathbb{N}^d \rightarrow \mathbb{N}\) for each positive integer \(d\). Then, the function \(\zeta_{f,g}\) that is defined so that \(\zeta_{f,g}(0) = ()\), and so that
\[
\zeta_{f,g}(f(x, y) + 1) = g_{y+1}(x)
\]
for each pair \((x, y)\) of non-negative integers, is an enumeration of \(\mathbb{N}^*\).

Another way to construct an enumeration of \(\mathbb{N}^*\) is described by the following
theorem. The proof of this theorem is similar to the proof of Theorem 4.1.
Theorem 4.4. Let $f$ be any max-dominating pairing function for the non-negative integers. Let $\xi_f(0) = ()$, and for each pair $(x, y)$ of non-negative integers let

$$
\xi_f(f(x, y) + 1) = \begin{cases} 
y & \text{if } x = 0 
(\xi_f(x), y) & \text{otherwise}
\end{cases}.
$$

Then, $\xi_f$ is an enumeration of $\mathbb{N}^*$.

Proof. First, we show that $\xi_f(n) \in \mathbb{N}^*$ for each $n \in \mathbb{N}$. Since $\xi_f(0) = () \in \mathbb{N}^*$, it suffices to prove that for each pair $(x, y)$ of non-negative integers there exists a positive integer $d$ such that $\xi_f(f(x, y) + 1) \in \mathbb{N}^d$. The proof is by strong induction on $x$. For the base step, notice that

$$
\xi_f(f(0, y) + 1) = y \in \mathbb{N}^1
$$

for all non-negative integers $y$. Now consider any non-negative integer $x$ and suppose, as the induction hypothesis, that for all non-negative integers $y$, and for each non-negative integer $i \leq x$ there exists a positive integer $d$ such that $\xi_f(f(i, y) + 1) \in \mathbb{N}^d$. Since $f : \mathbb{N}^2 \to \mathbb{N}$ is a pairing function, there exist non-negative integers $i$ and $j$ such that $f(i, j) = x$. Then, by the definition of $\xi_f$,

$$
(4.4) \quad \xi_f(f(x + 1, y) + 1) = \left(\xi_f(x + 1), y\right) = \left(\xi_f(f(i, j) + 1), y\right)
$$

for all non-negative integers $y$. But $i \leq f(i, j) = x$ because $f$ is max-dominating. Therefore, by the induction hypothesis, $\xi_f(f(i, j) + 1) \in \mathbb{N}^d$ for some positive integer $d$. We may conclude by equation (4.4) that $\xi_f(f(x + 1, y) + 1) \in \mathbb{N}^{d+1}$ for all non-negative integers $y$.

Next, we show that for each $u \in \mathbb{N}^*$ there exists a unique $n \in \mathbb{N}$ such that $\xi_f(n) = u$. The proof is by induction on the length of the sequence $u$. For the base step, notice that if $u = ()$ then $n = 0$ is the unique non-negative integer such that $\xi_f(n) = u$. And if $u \in \mathbb{N}^1$ then $n = f(0, u) + 1$ is the unique non-negative integer such that $\xi_f(n) = u$. Next, consider any positive integer $d$ and suppose, as the induction hypothesis, that for each $u \in \mathbb{N}^d$ there exists a unique $n \in \mathbb{N}$ such that $\xi_f(n) = u$. Now consider any $u \in \mathbb{N}^{d+1}$. It must be the case that $u = (v, y)$ for some unique $v \in \mathbb{N}^d$ and unique $y \in \mathbb{N}$. By the induction hypothesis, there also exists a unique $x \in \mathbb{N}$ such that $\xi_f(x) = v$. Note that $x \neq 0$ because $v \neq ()$. Therefore, $n = f(x, y) + 1$ is the unique non-negative integer such that

$$
\xi_f(n) = \xi_f(f(x, y) + 1) = (\xi_f(x), y) = (v, y) = u.
$$

We may conclude that $\xi_f$ is a one-to-one correspondence from $\mathbb{N}$ to $\mathbb{N}^*$. That is, $\xi_f$ is an enumeration of the set $\mathbb{N}^*$.

The Rosenberg-Strong pairing function was originally devised \cite{ref1, ref2} for applications to data storage in computer science. One of its key features is that if the binary representations of $x$ and $y$ each have $n$ or fewer bits, then the binary representation of $r_2(x, y)$ has $2n$ or fewer bits. This property is often useful when implementing the Rosenberg-Strong pairing function on a contemporary computer, and it is a consequence of the fact that, in Rosenberg’s words, $r_2$ “manages storage perfectly for square arrays” \cite{ref3}. In contrast, Cantor’s pairing function $c(x, y)$ does not possess this property. For example, the binary representation of $3 = (11)_2$ has two bits, and the binary representation of $2 = (10)_2$ also has two bits, but

$$
c(3, 2) = 18 = (10010)_2
$$
has more than 4 bits in its binary representation.

More generally, we have the following theorem.

**Theorem 4.5.** Let \( f : \mathbb{N}^d \to \mathbb{N} \) be any \( d \)-tupling function with cubic shells. If the non-negative integers \( x_1, x_2, \ldots, x_d \) each have a binary representation with \( n \) or fewer bits, then the binary representation of \( f(x_1, x_2, \ldots, x_d) \) has \( nd \) or fewer bits.

**Proof.** Consider any non-negative integers \( x_1, x_2, \ldots, x_d \) whose binary representations each have \( n \) or fewer bits. Then, for each positive integer \( i \leq d \),
\[
\left\lfloor \log_2(x_i + 1) \right\rfloor \leq n,
\]
\[
\log_2(x_i + 1) \leq n,
\]
\[
x_i + 1 \leq 2^n.
\]

Letting \( m = \max(x_1, x_2, \ldots, x_d) \), it immediately follows that
\[
m + 1 \leq 2^n,
\]
\[
(m + 1)^d \leq 2^{nd}.
\]

Then, by Corollary 3.3,
\[
f(x_1, x_2, \ldots, x_d) < (m + 1)^d \leq 2^{nd},
\]
\[
f(x_1, x_2, \ldots, x_d) + 1 \leq 2^{nd},
\]
\[
\log_2(f(x_1, x_2, \ldots, x_d) + 1) \leq nd,
\]
\[
\left\lceil \log_2(f(x_1, x_2, \ldots, x_d) + 1) \right\rceil \leq \lceil nd \rceil = nd.
\]

We may conclude that the binary representation of \( f(x_1, x_2, \ldots, x_d) \) has \( nd \) or fewer bits. \( \square \)

Historians [7, 18] regard Cantor’s 1878 paper [3] as the beginning of the study of pairing functions. In addition to describing a \( d \)-tupling function for the unit interval of the real line, Cantor’s paper was the first to provide an explicit formula for a pairing function for the positive integers. Variants of Cantor’s pairing function for the positive integers still dominate the literature, with variants of the Rosenberg-Strong pairing function tending to appear in more supplemental roles (for example, see References [17, 42]). But Rosenberg and Strong were motivated by practical concerns, and their pairing function has practical advantages over Cantor’s pairing function. Indeed, we have seen that an enumeration of full binary trees that is constructed from the Rosenberg-Strong pairing function orders the trees in a more intuitively convenient sequence than the corresponding enumeration that is constructed from Cantor’s pairing function. And in implementations of pairing functions on contemporary computers, where careful attention is paid to the number of bits in the binary representation of each number, the Rosenberg-Strong pairing function also provides a practical advantage. These advantages follow directly from the fact that the Rosenberg-Strong pairing function has cubic shells. And in this way, we see some merit in the study of pairing functions and their shell numberings.

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6. APPENDIX

In Section \ref{sec:6} it is claimed that the formula for $r_d^{-1}$, given in equations (3.3) and (3.4), is the inverse of the Rosenberg-Strong $d$-tupling function. This claim can be proved in the following manner.

**Lemma 6.1.** For all positive integers $d$ and all $(x_1, \ldots, x_d) \in \mathbb{N}^d$,

$$m^d \leq r_d(x_1, \ldots, x_d) < (m + 1)^d,$$

where $m = \max(x_1, \ldots, x_d)$.

**Proof.** The proof is by induction on $d$. For the base step, note that for all non-negative integers $x_1$, if $m = \max(x_1)$ then

$$m^1 \leq m = r_1(x_1) = x_1 = m < (m + 1)^1.$$

Now consider any integer $d > 1$ and suppose, as the induction hypothesis, that for all $(x_1, \ldots, x_{d-1}) \in \mathbb{N}^{d-1}$,

$$n^{d-1} \leq r_{d-1}(x_1, \ldots, x_{d-1}) < (n + 1)^{d-1},$$

where $n = \max(x_1, \ldots, x_{d-1})$. Next, consider any $(x_1, \ldots, x_{d-1}, x_d) \in \mathbb{N}^d$ and let $m = \max(x_1, \ldots, x_{d-1}, x_d)$. By the definition of $r_d$,

$$r_d(x_1, \ldots, x_d) = r_{d-1}(x_1, \ldots, x_{d-1}) + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}).$$

But $r_{d-1}(x_1, \ldots, x_{d-1})$ and $(m - x_d)((m + 1)^{d-1} - m^{d-1})$ are non-negative integers, so

$$r_d(x_1, \ldots, x_d) \geq m^d.$$

And it follows from the definition of $r_d$, together with the induction hypothesis, that

$$r_d(x_1, \ldots, x_d) < (n + 1)^{d-1} + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}),$$

$$r_d(x_1, \ldots, x_d) < (n + 1)^{d-1} + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}),$$

$$r_d(x_1, \ldots, x_d) < (m + 1)^{d-1} + m(m + 1)^{d-1} - x_d((m + 1)^{d-1} - m^{d-1}),$$

$$r_d(x_1, \ldots, x_d) < (m + 1)^d - x_d((m + 1)^{d-1} - m^{d-1}),$$

$$r_d(x_1, \ldots, x_d) < (m + 1)^d.$$

Hence, we have shown that

$$m^d \leq r_d(x_1, \ldots, x_d) < (m + 1)^d.$$

\[\square\]

**Corollary 6.2.** Let $d$ be a positive integer. For all $(x_1, \ldots, x_d) \in \mathbb{N}^d$,

$$\max(x_1, \ldots, x_d) = \lfloor \sqrt[d]{r_d(x_1, \ldots, x_d)} \rfloor.$$

**Remark.** A simple generalization of the following proof shows that a $d$-tupling function $f: \mathbb{N}^d \to \mathbb{N}$ has cubic shells if and only if $\max(x) = \lfloor \sqrt[d]{f(x)} \rfloor$ for all $x \in \mathbb{N}^d$. 

http://stackoverflow.com/questions/919612#answer-13871379
Proof. Consider any \((x_1, \ldots, x_d) \in \mathbb{N}^d\) and let \(m = \max(x_1, \ldots, x_d)\). By Lemma 6.1, 
\[
m^d \leq r_d(x_1, \ldots, x_d) < (m + 1)^d.
\]
Therefore, 
\[
m \leq \sqrt[d]{r_d(x_1, \ldots, x_d)} < m + 1,
\]
and we may conclude that \(m = \lfloor \sqrt[d]{r_d(x_1, \ldots, x_d)} \rfloor\). \(\square\)

Lemma 6.3. Given any positive integer \(d\) and any non-negative integer \(z\), let 
\(m = \lfloor \sqrt[z]{d} \rfloor\) and let \(x_d\) be defined by equation (3.3). Then, 
\[
0 \leq x_d \leq m.
\]
Proof. Given any positive integer \(d\) and any non-negative integer \(z\), let 
\(m = \lfloor \sqrt[z]{d} \rfloor\). Then, 
\[
\sqrt[z]{d} < m + 1,
\]
\[
z < (m + 1)^d,
\]
\[
z - (m + 1)m^{d-1} < (m + 1)^d - (m + 1)m^{d-1},
\]
\[
z - m^d - m^{d-1} < (m + 1)((m + 1)^{d-1} - m^{d-1}),
\]
\[
0 \leq \max(0, z - m^d - m^{d-1}) < (m + 1)((m + 1)^{d-1} - m^{d-1}).
\]
And dividing by \((m + 1)^{d-1} - m^{d-1}\), 
\[
0 \leq \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} < m + 1,
\]
\[
0 \leq \left\lfloor \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \right\rfloor < m + 1,
\]
\[
0 \leq \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \leq m,
\]
\[
0 \geq - \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \geq -m,
\]
\[
m \geq m - \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \geq 0.
\]
We may conclude from equation (3.3) that 
\[
m \geq x_d \geq 0.
\]
\(\square\)

Lemma 6.4. Let \(d\) be any positive integer. The formula for \(r_d^{-1}(z)\) that is given in equations (3.3) and (3.4) describes a function from \(\mathbb{N}\) to \(\mathbb{N}^d\). Moreover, 
\[
r_d(r_d^{-1}(z)) = z
\]
for all \(z \in \mathbb{N}\).

Proof. The proof is by induction on \(d\). For the base step, note that \(r_1^{-1}(z) = z\) is a function from \(\mathbb{N}\) to \(\mathbb{N}\), and 
\[
r_1(r_1^{-1}(z)) = r_1(z) = z
\]
for all \( z \in \mathbb{N} \). Next, consider any integer \( d > 1 \) and suppose, as the induction hypothesis, that \( r_{d-1}^{-1} \) is a function from \( \mathbb{N} \) to \( \mathbb{N}^{d-1} \) and

\[
r_{d-1}(r_{d-1}^{-1}(z)) = z
\]

for all \( z \in \mathbb{N} \). Now consider any non-negative integer \( z \) and let \( m = \left\lfloor \sqrt[2]{z} \right\rfloor \). By equation (3.3), (6.1)

\[
r_{d}^{-1}(z) = (r_{d-1}^{-1}(u), x_d),
\]

where \( x_d \) is defined by equation (3.4), and where

\[
u = z - m^d - (m - x_d)((m + 1)^{d-1} - m^{d-1}).
\]

Note that if \( u \in \mathbb{N} \), then it follows from the induction hypothesis that \( r_{d-1}^{-1}(u) \) is a member of \( \mathbb{N}^{d-1} \). Consequently, by equation (6.1) and Lemma 6.3, \( r_{d-1}^{-1}(z) \) is a member of \( \mathbb{N}^{d} \) if \( u \in \mathbb{N} \). And by Corollary 6.2,

\[
u \in \mathbb{N} \implies \max(r_{d-1}^{-1}(u)) = \left\lfloor \left( r_{d-1}^{-1}(u) \right)^{1/(d-1)} \right\rfloor.
\]

Hence, by the induction hypothesis,

(6.2) \( u \in \mathbb{N} \) implies \( \max(r_{d-1}^{-1}(u)) = \left\lfloor u^{1/(d-1)} \right\rfloor \).

There are now two cases to consider.

**Case 1:** If \( z - m^d - m^{d-1} < 0 \), then \( x_d = m \) by equation (3.4). Therefore, \( u = z - m^d \). But \( m = \left\lfloor \sqrt[2]{z} \right\rfloor \), so

\[
m \leq \sqrt[2]{z},
\]

\[
m^d \leq z.
\]

It immediately follows that \( u = z - m^d \) is a non-negative integer. And by condition (6.2),

\[
\max(r_{d-1}^{-1}(u)) = \left\lfloor (z - m^d)^{1/(d-1)} \right\rfloor.
\]

But because \( z - m^d - m^{d-1} < 0 \),

\[
z - m^d < m^{d-1},
\]

\[
(z - m^d)^{1/(d-1)} < m,
\]

\[
\left\lfloor (z - m^d)^{1/(d-1)} \right\rfloor < m.
\]

Hence, \( \max(r_{d-1}^{-1}(u)) < m \). And since \( x_d = m \), it follows from equation (6.1) that \( \max(r_{d}^{-1}(z)) = m \).

**Case 2:** If \( z - m^d - m^{d-1} \geq 0 \) then by equation (3.4),

\[
x_d = m - \left\lfloor \frac{z - m^d - m^{d-1}}{(m+1)^{d-1} - m^{d-1}} \right\rfloor.
\]

But by the Euclidean division theorem,

\[
z - m^d - m^{d-1} = \left\lfloor \frac{z - m^d - m^{d-1}}{(m+1)^{d-1} - m^{d-1}} \right\rfloor ((m+1)^{d-1} - m^{d-1}) + b
\]
for some integer $b$ such that $0 \leq b < (m + 1)^{d-1} - m^{d-1}$. Thus,
\[ z - m^d - m^{d-1} = (m - x_d)((m + 1)^{d-1} - m^{d-1}) + b. \]

It immediately follows that
\[ z - m^d - (m - x_d)((m + 1)^{d-1} - m^{d-1}) = b + m^{d-1}. \]

Therefore, $u = b + m^{d-1}$. And by condition (6.2),
\[ \max(r_{d-1}^{-1}(u)) = \left[ (b + m^{d-1})^{1/(d-1)} \right]. \]

Moreover, since
\[ 0 \leq b < (m + 1)^{d-1} - m^{d-1}, \]
we have that
\[ m^{d-1} \leq b + m^{d-1} < (m + 1)^{d-1}, \]
\[ m = (b + m^{d-1})^{1/(d-1)} < m + 1. \]

Hence,
\[ \max(r_{d-1}^{-1}(u)) = \left[ (b + m^{d-1})^{1/(d-1)} \right] = m. \]

It then follows from Lemma 6.3 and equation (6.1) that $\max(r_d^{-1}(z)) = m$. In either case, $u \in \mathbb{N}$ and $\max(r_d^{-1}(z)) = m$. So, by equation (6.1) and the definition of $r_d$,
\[ r_d(r_d^{-1}(z)) = r_d(r_d^{-1}(u), x_d) \]
\[ = r_{d-1}(r_{d-1}^{-1}(u) + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}). \]

Then, by the induction hypothesis,
\[ r_d(r_d^{-1}(z)) = u + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}). \]

And by the definition of $u$,
\[ r_d(r_d^{-1}(z)) = z - m^d - (m - x_d)((m + 1)^{d-1} - m^{d-1}) \]
\[ + m^d + (m - x_d)((m + 1)^{d-1} - m^{d-1}). \]

It immediately follows that
\[ r_d(r_d^{-1}(z)) = z. \]

Moreover, $u \in \mathbb{N}$ for all non-negative integers $z$. Therefore, $r_d^{-1}(z) \in \mathbb{N}^d$ for all non-negative integers $z$. We may conclude that $r_d^{-1}$ is a function from $\mathbb{N}$ to $\mathbb{N}^d$. \qedhere

**Lemma 6.5.** Let $d$ be a positive integer. For all $(x_1, x_2, \ldots, x_d) \in \mathbb{N}^d$,
\[ r_d^{-1}(r_d(x_1, x_2, \ldots, x_d)) = (x_1, x_2, \ldots, x_d). \]

**Proof.** The proof is by induction on $d$. For the base step, note that
\[ r_1^{-1}(r_1(x_1)) = r_1^{-1}(x_1) = x_1 \]
for all $x_1 \in \mathbb{N}$. Next, consider any integer $d > 1$ and suppose, as the induction hypothesis, that
\[ r_{d-1}^{-1}(r_{d-1}(x_1, \ldots, x_{d-1})) = (x_1, \ldots, x_{d-1}) \]
There are two cases to consider.

**Case 1:** If \( m = x_d > \max(x_1, \ldots, x_{d-1}) \) then it follows from equation (6.3) that
\[
z - m^d = r_{d-1}(x_1, \ldots, x_{d-1}).
\]
Now let \( n = \max(x_1, \ldots, x_{d-1}) \). By Lemma 6.1, \( z - m^d < (n+1)^{-d+1} \).

But \( m > n = \max(x_1, \ldots, x_{d-1}) \), so \( m \geq n+1 \). Therefore,
\[
z - m^d < m^{-d},
\]
\[
z - m^d - m^{-d+1} < 0.
\]
It immediately follows that
\[
x_d = m = m - \left\lfloor \frac{\max(0, z - m^d - m^{-d+1})}{(m+1)^{-d+1} - m^{-d+1}} \right\rfloor.
\]

**Case 2:** If \( m = \max(x_1, \ldots, x_{d-1}) \) then it follows from equation (6.3) and Lemma 6.1 that
\[
m^{-d+1} \leq z - m^d - (m - x_d)((m+1)^{-d+1} - m^{-d+1}).
\]
Therefore,
\[
-z + m^d + m^{-d+1} \leq -(m - x_d)((m+1)^{-d+1} - m^{-d+1})
\]
and
\[
\frac{z - m^d - m^{-d+1}}{(m+1)^{-d+1} - m^{-d+1}} \geq m - x_d.
\]
It also follows from Lemma 6.1 and equation (6.3) that
\[
z - m^d - (m - x_d)((m+1)^{-d+1} - m^{-d+1}) < (m+1)^{-d+1}.
\]
Hence,
\[
-(m - x_d)((m+1)^{-d+1} - m^{-d+1}) < -z + m^d + (m+1)^{-d+1}
\]
and
\[
m - x_d > \frac{z - m^d - (m+1)^{-d+1}}{(m+1)^{-d+1} - m^{-d+1}},
\]
\[
m - x_d > \frac{z - m^d - m^{-d+1} - ((m+1)^{-d+1} - m^{-d+1})}{(m+1)^{-d+1} - m^{-d+1}},
\]
\[
m - x_d > \frac{z - m^d - m^{-d+1}}{(m+1)^{-d+1} - m^{-d+1}} - 1.
\]
Combining inequalities (6.4) and (6.5),
\[
\frac{z - m^d - m^{d-1}}{(m + 1)^{d-1} - m^{d-1}} \geq m - x_d > \frac{z - m^d - m^{d-1}}{(m + 1)^{d-1} - m^{d-1}} - 1.
\]
It immediately follows that
\[
m - x_d = \left\lfloor \frac{z - m^d - m^{d-1}}{(m + 1)^{d-1} - m^{d-1}} \right\rfloor.
\]
But \(m - x_d\) cannot be negative because \(m = \max(x_1, \ldots, x_d)\). Therefore,
\[
m - x_d = \left\lfloor \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \right\rfloor.
\]
In either case,
\[
x_d = m - \left\lfloor \frac{\max(0, z - m^d - m^{d-1})}{(m + 1)^{d-1} - m^{d-1}} \right\rfloor.
\]
But \(m = \lfloor \sqrt{z} \rfloor\) by Corollary 6.2. Hence, by equation (3.3), equation (6.3), and the induction hypothesis,
\[
r_d^{-1}(r_d(x_1, \ldots, x_{d-1}, x_d)) = r_d^{-1}(z) = \left( r_d^{-1}\left( z - m^d - (m - x_d)((m + 1)^{d-1} - m^{d-1}) \right), x_d \right) = \left( r_d^{-1}(r_d^{-1}(x_1, \ldots, x_{d-1})), x_d \right) = (x_1, \ldots, x_{d-1}, x_d).
\]

**Theorem 6.6.** For each positive integer \(d\), \(r_d: \mathbb{N}^d \rightarrow \mathbb{N}\) is a \(d\)-tupling function that has \(r_d^{-1}: \mathbb{N} \rightarrow \mathbb{N}^d\) as its inverse.

**Proof.** By Lemma 6.4, \(r_d^{-1}\) is a right inverse of \(r_d\). And by Lemma 6.5 it is a left inverse of \(r_d\). Therefore, \(r_d^{-1}\) is the inverse of \(r_d\). And since invertible functions are one-to-one correspondences, \(r_d\) is a \(d\)-tupling function for the non-negative integers. \(\square\)

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