Uniqueness among scalar-flat Kähler metrics on non-compact toric 4-manifolds

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Abstract

In [Abreu and Sena-Dias, Ann. Global Anal. Geom. 41 (2012) 209–239], the authors construct two distinct families of scalar-flat Kähler non-compact toric metrics using Donaldson’s rephrasing of Joyce’s construction in action-angle coordinates. In this paper and using the same set-up, we show that these are the only scalar-flat Kähler metrics on any given strictly unbounded toric surface. We also show that the asymptotic behaviour of such a metric determines it uniquely if the metric satisfies some extra mild assumptions.

1. Introduction

Based on work of Donaldson’s (see [7]) which in turn builds on an important construction of Joyce giving local models for scalar-flat Kähler metrics with torus symmetry (see [10]), Abreu and the author have constructed two distinct types of scalar-flat Kähler toric metrics on strictly unbounded toric symplectic fourfolds. Namely:

- an ALE scalar-flat Kähler toric metric which had been previously written down in [5];
- a family of so-called Donaldson generalised Taub-NUT metrics which are all complete scalar-flat Kähler metrics. Some of these metrics are Ricci-flat and were previously known (see [12]). Others are actually not Ricci-flat and are new.

The construction uses action-angle coordinates and gives the so-called symplectic potentials of the metrics. In [7], Donaldson explains how to establish a local correspondence between:

1. solutions to a non-linear PDE describing symplectic potentials of scalar-flat Kähler metrics;
2. pairs of solutions to a familiar linear PDE describing axi-symmetric harmonic functions on $\mathbb{R}^3$ which give a local diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$.

[3] uses only one side of this correspondence namely that a pair of axi-symmetric harmonic functions on $\mathbb{R}^3$ determines the local symplectic potential of a scalar-flat Kähler metric. It is natural to ask if the above correspondence can be used to prove some uniqueness results for scalar-flat Kähler metrics on strictly unbounded symplectic fourfolds. The goal of this paper is to show that indeed one can use the correspondence to prove that the metrics constructed in [3] are essentially the only scalar-flat Kähler metrics one can construct on strictly unbounded toric fourfolds. More precisely we prove the following theorems.

Theorem 1.1. Let $(X, \omega)$ be a strictly unbounded toric 4-manifold and $J$ a compatible complex structure which is torus invariant. Let $g$ be the corresponding toric metric determined by $\omega$ and $J$. If $g$ is scalar-flat, then $g$ is equivariantly isometric to either
(1) the ALE toric scalar-flat Kähler metrics on $X$; or
(2) a Donaldson generalised Taub-NUT metric.

The theorem ensures that the metrics constructed in [3], are the only possible scalar-flat Kähler toric metrics up to equivariant isometry. It follows in particular that any scalar-flat Kähler toric metric on a strictly unbounded toric 4-manifold is automatically complete. The metrics in [3] which are not ALE are referred to as Donaldson generalised Taub-NUT metrics. We also prove that the asymptotic behaviour for a toric scalar-flat Kähler metric on a strictly unbounded toric fourfold determines it uniquely. Recall that a strictly unbounded toric 4-manifold carries a canonical complex structure determined by its fan (or its moment polytope) which we will call the Guillemin complex structure. We will review this structure in the next section.

**Theorem 1.2.** Let $(X, \omega)$ be a strictly unbounded toric 4-manifold and $J$ a compatible complex structure which is torus invariant and biholomorphic to the Guillemin complex structure via a diffeomorphism $\Psi$. Let $g$ be the toric metric determined by $\Psi^* \omega$ and the Guillemin complex structure. If $g$ is scalar-flat and asymptotic to one of the Donaldson generalised Taub-NUT metrics from [3] with parameter $\nu$, then $g$ is the Donaldson generalised Taub-NUT metric with parameter $\nu$ on $X$ up to equivariant isometry.

We will give a more precise statement of the above theorem ahead. We will in particular explain what we mean by ‘asymptotic to a Taub-NUT metric’.

The idea of using Donaldson’s action-angle coordinates version of Joyce’s construction to tackle uniqueness questions first appeared in [15]. Wright even suggests a strategy to approach the question and proves some partial results. This is essentially the strategy we pursue here. Another approach to this question is developed by Weber in [14], using very different techniques.

**Remarks 1.3.** A strictly unbounded toric 4-manifold is one whose moment polytope is unbounded and whose unbounded edges are not parallel. The notion appears in [3] but we will give a more precise definition ahead.

Part of Theorem 1.2, namely the part that corresponds to ALE metrics and $\nu = 0$, was proved in [16] using Twistors.

2. Unbounded toric 4-manifolds

In this section, we give a quick review of some facts about unbounded toric manifolds. For more details, see [3].

**Definition 2.1.** A symplectic 4-manifold $(X, \omega)$ is said to be toric if it admits an effective Hamiltonian $\mathbb{T}^2$-action whose moment map is proper.

In this setting, the moment map image $P$ is the closure of a convex polygonal region in $\mathbb{R}^2$ of the form

$$P = \{ x \in \mathbb{R}^2 : x \cdot \nu_i - \lambda_i > 0, \ i = 1 \cdots d \},$$

where:

- $d$ is an integer (the number of facets of $P$);
- for each $i$, $\nu_i$ is a vector in $\mathbb{R}^2$ the primitive interior normal to the $i$th facet;
- $\lambda_i$ is a real number.
The functions $\nu_i$ satisfy the Delzant condition. Namely,
\[
\det(\nu_i, \nu_{i+1}) = -1,
\]
for all $i = 1, \ldots, d - 1$. It is now easy to define an unbounded toric manifold.

**Definition 2.2.** A symplectic toric 4-manifold $(X, \omega)$ is said to be unbounded if its moment map image is unbounded. It is said to be strictly unbounded if it is unbounded and the unbounded edges in its moment map image are not parallel.

It is an important result of Guillemin’s that every toric symplectic manifold actually admits a Kähler structure.

**Theorem 2.3 (Guillemin).** Let $(X, \omega)$ be a symplectic toric manifold with moment map image $\bar{P}$. Then $(X, \omega)$ admits an integrable, compatible, torus invariant complex structure. This structure is completely determined by $P$.

We will call the resulting metric Guillemin’s metric and denote the Kähler structure by $(X, \omega_P, J_P, g_P)$. We will use integrable, compatible, torus invariant complex structures to parametrise Riemannian metrics on $X$. The advantage of toric manifolds is that there is a particularly nice way to parametrise such complex structures as we will see ahead.

At this point, we would like to give some important examples. We start with a very simple and familiar one.

**Example 2.4.** Let $X = \mathbb{C}^2 = \mathbb{R}^4$ with coordinates $(z_1, z_2)$ with symplectic form
\[
\omega = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).
\]
Let $T^2 = S^1 \times S^1$ act via the usual $S^1$-action on $\mathbb{C}$. Then it is easy to see that $X$ is toric and the moment map of the torus action is
\[
\phi(z_1, z_2) = (|z_1|^2, |z_2|^2),
\]
so that the moment map image is the closure of $P = (\mathbb{R}^+)^2$. This has two edges both of which are unbounded.

Another important example is the following.

**Example 2.5.** Let $\Gamma$ be a finite cyclic subgroup of $U(2)$ whose action on $\mathbb{C}^2$ has finite isolated singularities. Then the orbifold $\mathbb{C}^2/\Gamma$ is toric (although we have not defined toric orbifold, it is straightforward to generalise the above definition for manifolds). More importantly this orbifold admits a minimal resolution $X_\Gamma$ which is itself a toric 4-manifold. There is a complex structure on $X_\Gamma$ coming from the usual complex structure on $\mathbb{C}^2$. One can check that this is equivariantly biholomorphic to the Guillemin complex structure.

Once we have a symplectic toric manifold, we can perform symplectic blow-ups of any finite number of fixed points to obtain other symplectic toric manifolds. It is well known that such an operation corresponds to ‘corner chopping’ on the moment map image.

These examples essentially give all possible strictly unbounded toric 4-manifolds up to equivariant biholomorphism. In [3], we prove the following proposition.

**Proposition 2.6.** Let $X$ be a strictly unbounded toric 4-manifold endowed with the Guillemin complex structure. Then $X$ is equivariantly biholomorphic to a finite iterated blow-up of some $X_\Gamma$ for some subgroup $\Gamma$ of $U(2)$. 
3. Symplectic potential

Symplectic potentials are the way in which we parametrise toric Kähler metrics on toric manifolds. In this section, we briefly explain what these are and why they are useful. For more details, see [1, 3, 4]. Let \((X, \omega)\) be a symplectic toric 4-manifold. Denote its moment map by \(\phi : X \to \mathbb{R}^2\) and its moment map image \(\phi(X)\) by \(P\), where

\[
P = \{ x \in \mathbb{R}^2 : x \cdot \nu_i - \lambda_i > 0, \ i = 1 \cdots d \}\]

Associated with \(P\) is the following convex function called the Guillemin potential of \(P\):

\[
u_P = \sum_{i=1}^{d} l_i \log l_i - l_i,
\]

where \(l_i(x) = x \cdot \nu_i - \lambda_i\). This function plays a crucial role in what follows.

Any \(J\) which is an integrable, compatible, torus invariant complex structure corresponding to a Kähler metric determines a function \(u : P \to \mathbb{R}\) such that:

- \(u\) is convex;
- the function \(u - u_P\) extends smoothly to a neighbourhood of \(P\), where
  \[u_P = \sum_{i=1}^{d} l_i \log l_i - l_i;\]
- the function \((\prod_{i=1}^{d} l_i) \det \text{Hess} u\) extends to \(\bar{P}\) and is nowhere vanishing.

The function \(u\) is called the symplectic potential of the Kähler metric and it is such that the corresponding Riemann metric is equivariantly isometric to

\[
g = \sum_{i,j=1}^{2} u_{ij} dx_i dx_j + u^{ij} d\theta_i d\theta_j,
\]

where \((x_1, x_2, \theta_1, \theta_2)\) are the usual action-angle coordinates on \((X, \omega)\) and \(u_{ij}\) are the entries of Hessian of \(u\) and \(u^{ij}\) the entries of its inverse.

Following [4], we are going to sketch how to associate such a symplectic potential to any toric Kähler metric. Let \(\phi\) denote the moment map of the toric manifold and let \(P\) be the interior of the moment map image. Let \(X_1, X_2\) be linearly independent vectors fields on \(X\) generated by the torus action, say

\[
X_i(p) = \frac{d}{dt|_{t=0}} e^{\sqrt{-1} t e_i} p,
\]

where \(e_1, e_2\) form a basis for the Lie algebra of the torus and consider the tangent bundle frame \((X_1, X_2, JX_1, JX_2)\) over \(\phi^{-1}(P)\) and its dual frame which we denote by \((\beta_1, \beta_2, \alpha_1, \alpha_2)\). We can choose \(\beta_i\) so that \(\beta_i = d\theta_i + f_i(x)\) for some functions \(f_1, f_2\) on \(P\). Because \(J\) is an isometry, we can write

\[
g = \sum_{i,j=1}^{2} H_{ij}(\alpha_i \otimes \alpha_j + \beta_i \otimes \beta_j),
\]

where \(H_{ij}\) are the entries of a positive definite symmetric matrix. We also have

\[
\omega = \sum_{i=1}^{2} dx_i \wedge \beta_i.
\]
To see this, simply consider that by definition $X_i \omega = -dx_i$. On the other hand, we must be able to write

$$g = \sum_{ij=1}^{2} G_{ij} dx_i \otimes dx_j + H_{ij} \beta_i \otimes \beta_j,$$

for some symmetric 2 by 2 matrix $(G_{ij})$. We want to show that $G_{ij}$ are the entries of the inverse of the matrix $(H_{ij})$. It is enough to compare

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \text{ and } g(X_i, X_j).$$

Now

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \omega\left(\frac{\partial}{\partial x_i}, J \frac{\partial}{\partial x_j}\right) = \beta_i \left(J \frac{\partial}{\partial x_j}\right).$$

We have

$$\frac{\partial}{\partial x_j} = \sum_{k=1}^{2} A_{jk} J X_k$$

for some coefficients $A_{jk}$ and

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -A_{ji}.$$ 

On the other hand,

$$g(X_i, X_j) = \omega(X_i, J X_j) = -dx_i(J X_j) = -A_{ji} \alpha_i,$$

Now because $\alpha_i$ is closed for $i = 1, 2$, there are functions on $P$, $\xi_1, \xi_2$ such that $\alpha_i = d\xi_i$. We have $J \beta_i = \alpha_i$ and it follows from the expression for $g$ and $\omega$ that $J \beta_i = \omega_{ij} dx_i$, therefore $G_{ij} = \frac{d\xi_i}{dx_j}$ for $i = 1, 2$. Consider the form $\sum_{i=1}^{2} \xi_i dx_i$. Because $G_{ij}$ are the entries of a symmetric matrix, this form is closed and thus exact on $P$. Let $u$ be its primitive. We have

$$g = \sum_{i,j=1}^{2} u_{ij} dx_i \otimes dx_j + u^{ij} \beta_i \otimes \beta_j.$$ 

As we have seen, $\beta_i = d\theta_i + f_i(x)$. The map $(x, \theta) \to (x, \theta + f(x))$ is defined on $\phi^{-1}(P)$ and yields an isometry between the metric $\sum_{i,j=1}^{2} u_{ij} dx_i \otimes dx_j + u^{ij} \beta_i \otimes \beta_j$ and $\sum_{i,j=1}^{2} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j$. To see that the isometry extends to $X$, pick a ball $B$ centred at a boundary point on the polytope. The closure of its pre-image via the moment map in $X$ is a compact subset of $X$. Both metrics $g$ and $\sum_{i,j=1}^{2} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j$ are defined and complete on this compact set $\phi^{-1}(B)$. Therefore the map must extend as an isometry to this pre-image.

Since the point on the boundary of $P$ we are considering is arbitrary, we see that the isometry extends to the whole $X$. It remains to check that the obtained function satisfies the boundary conditions above. We refer the reader to [4, Section 1.4].
Example 3.1. Guillemin showed that the symplectic potential for the Guillemin metric is \( u_P \).

This example is particularly important because of the boundary behaviour of symplectic potentials.

Conversely, any function \( u \) satisfying the three conditions above defines a toric Kähler metric on the toric manifold via the formula

\[
\sum_{i,j=1}^{2} u_{ij} dx_i dx_j + u^{ij} d\theta_i d\theta_j
\]

for the metric on the dense open set \( \phi^{-1}(P) \). In \([1]\), Abreu wrote down a formula for scalar curvature in terms of symplectic potential.

**Theorem 3.2 (Abreu).** The scalar curvature of a torus invariant metric determined by the symplectic potential \( u \) is given by

\[
s = \frac{1}{2} \sum_{j,k} \frac{\partial^2 u^{jk}}{\partial x_i \partial x_j},
\]

where \( u^{ij} \) denote the entries of the matrix \( \text{Hess}(u)^{-1} \).

Hence the relevant PDE for symplectic potentials of scalar-flat Kähler toric metrics is

\[
\frac{\partial^2 u^{jk}}{\partial x_i \partial x_j} = 0,
\]  (1)

where we have used Einstein’s convention for sums of repeated indices. This is a non-linear second-order PDE.

**Remark 3.3.** We end this section by pointing out that even though we think of symplectic potential as functions on \( P \), symplectic potentials for distinct complex structures describe the associated Riemannian metrics at different points in the toric manifold. For example, fixing the underlying complex toric manifold \((X,J_P)\) and given \( x \in P \), \( u_P(x) \) determines the metric associated with \( J_P \) at a point with complex coordinates \((\partial u_P/\partial x, \theta)\) whereas for a different \( J \), \( u(x) \) determines the associated metric at the point with complex coordinates \((\partial u/\partial x, \theta)\).

4. **Joyce’s construction in action-angle coordinates**

In \([7]\), Donaldson translates Joyce’s construction of local scalar-flat Kähler metrics with torus symmetries into the language of symplectic potentials and action-angle coordinates. In this section, our aim is to describe Donaldson’s construction. For more details and proofs, see \([7]\).

Consider the following linear PDE

\[
\frac{\partial^2 \xi}{\partial H^2} + \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} = 0,
\]  (2)

on \( \mathbb{H} = \{(H,r) \in \mathbb{R}^2 : r > 0\} \).

**Theorem 4.1** \([7]\). Let \( \xi_1 \) and \( \xi_2 \) be two solutions of equation (2) on an open subset \( B \) of \( \mathbb{H} \). Let

\[
\epsilon_1 = r \left( \frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right)
\]
and
\[ \epsilon_2 = -r \left( \frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right). \]

Then these two 1-forms are closed. Let \( x_1 \) and \( x_2 \) denote their primitives, well defined up to a constant. Then \( (x_1, x_2) \) are local coordinates in \( \mathbb{R}^2 \). Let
\[ \epsilon = \xi_1 dx_1 + \xi_2 dx_2. \]

This 1-form is also closed. Let \( u \) be a primitive of \( \epsilon \) and write \( \xi = (\xi_1, \xi_2) \). Then, if \( \det D\xi > 0 \), where
\[ D\xi = \begin{pmatrix} \frac{\partial \xi_1}{\partial H} & \frac{\partial \xi_1}{\partial r} \\ \frac{\partial \xi_2}{\partial H} & \frac{\partial \xi_2}{\partial r} \end{pmatrix}, \]
the function \( u \) is a local symplectic potential for some toric Kähler metric on \( \mathbb{R}^4 \) whose scalar curvature is 0, that is,

1. \( \text{Hess}(u) \) is positive definite on the interior of the polytope and when restricted to \( i \)th facet of the moment polytope \( l_i^{-1}(0) \), the non-singular part of \( u \), that is, \( u - l_i \log(l_i) \) is convex on the interior of \( l_i^{-1}(0) \);
2. \( u \) solves equation (1).

The above construction is local but one can use it with the appropriate boundary conditions on the solutions of equation (2) to construct global metrics. In [7], Donaldson also explains that the above construction is reversible. That is, starting from an \( u \) satisfying both conditions in Theorem (4.1), we can find \( (H, r) \) in terms of \( (x_1, x_2) \) as well as a pair of solutions \( (\xi_1, \xi_2)(H, r) \) of equation (2) such that \( \det D\xi > 0 \). More precisely, let \( r = (\det \text{Hess } u)^{-1/2} \) and \( H \) so that
\[ \frac{\partial H}{\partial x_1} = -\frac{u^{2j}}{r} \frac{\partial r}{\partial x_j}, \]
\[ \frac{\partial H}{\partial x_2} = \frac{u^{1j}}{r} \frac{\partial r}{\partial x_j}. \]

Donaldson shows that \( r \) and \( H \) as functions on the polytope \( P \) endowed with the metric \( g_r = u_{ij} dx_i \otimes dx_j \) are harmonic and harmonic conjugates. The way we will want to interpret this is the following. The metric \( g_r \) on \( P \) induces a complex structure \( J_r \) via its Hodge star. By definition this complex structure is such that \( a \circ J_r = \ast a \) for any 1-form \( a \) on \( P \). We have \( \ast dr = -dH \). Therefore
\[ d(H + ir) \circ J_r = \ast d(H + ir) = i d(H + ir). \]

Set \( z = H + ir \). This is a \( J_r \)-holomorphic local coordinate on \( P \).

5. The map \( z = H + ir \)

The goal of this section is to prove the following proposition.

**Proposition 5.1.** Let \( (X, \omega) \) be a strictly unbounded toric 4-manifold and \( J \) a compatible complex structure which is torus invariant and consider its symplectic potential \( u \) as in Section 3. Let \( r = (\det \text{Hess } u)^{-1/2} \) and \( H \) so that
to the pre-image of the boundary. Because the extension of \( r \) itself does not admit such a complex structure this is why we need to use \( X \) sets \( z \).

The map \( z = H + ir : P \to \mathbb{H} \) is a bijection.

We start by noticing that because the boundary behaviour of \( u \) is determined on \( \partial P \), the behaviour of \( z \) is also determined on \( \partial P \) (for more details on this see, [13]). We can see that \( z \) extends to \( \partial P \) as a continuous function which we denote by \( \tilde{z} \) and that because \( r \) vanishes on the boundary, \( \tilde{z}(\partial P) \subset \partial \mathbb{H} \). It is not hard to see that \( \tilde{z}_{|\partial P} : \partial P \to \partial \mathbb{H} \) is a bijection as it coincides with the extension of the map \( z \) for the ALE metric which we can calculate explicitly.

5.1. Injectivity of \( z \)

We start by showing injectivity. We will make use the real sub-manifold associated with \((X, \omega)\) and we start by quickly reviewing its definition. In [6], the real manifold described and referred to as real form of the toric Kähler manifold. It follows from [11] that even in our non-compact setting such a manifold exists. This is because when the moment map is proper, a toric manifold is symplectomorphic to the quotient of some complex plane \( \mathbb{C}^N \) by a sub-torus of the standard torus (see [11, Theorem 6.7]; as in the proof of [11, Theorem 6.5], properness of the moment map implies that the orbital moment map is an embedding). The sub-manifold corresponding to the quotient of \( \mathbb{R}^N \) by the real sub-torus is the sought real sub-manifold which we denote \( X_{\mathbb{R}} \).

The restriction of \( \phi \) to \( X_{\mathbb{R}} \) is denoted by \( \phi_{\mathbb{R}} \). The map \( \phi_{\mathbb{R}} : X_{\mathbb{R}} \to \mathbb{R}^2 \) is a 4 to 1 branched cover branched along \( \phi_{\mathbb{R}}^{-1}(\partial P) \) and \( \tilde{X}_{\mathbb{R}} = \phi_{\mathbb{R}}^{-1}(P) \) can be written as the disjoint union of four open sets \( P_j, j = 0, \ldots, 3 \). For each such open subset, there is a map \( \sigma_j : P_j \to P_0 \) given by the action of one of the four elements of \( T^2 \) preserving \( X_{\mathbb{R}} \) namely \((\pm 1, \pm 1)\). Any toric Kähler metric \( g \) on \( X \) induces a metric \( g_{\mathbb{R}} \) on \( X_{\mathbb{R}} \) and on the open set \( P_0 \). This metric can be identified with \( u_{ij} dx_i \otimes dx_j \), more precisely

\[
\frac{\partial H}{\partial x_1} = -\frac{u^{2j}}{r} \frac{\partial r}{\partial x_j}, \\
\frac{\partial H}{\partial x_2} = \frac{u^{1j}}{r} \frac{\partial r}{\partial x_j}.
\]

Because \( T^2 \) acts by isometries on \( X \), each \( \sigma_j \) is an isometry for \( g_{\mathbb{R}} \).

Note that unless \( P \) is a rectangle or \((\mathbb{R}^+)^2 \), \( X_{\mathbb{R}} \) is not orientable. It admits an orientable double cover \( X_{\mathbb{R}}^o \to X_{\mathbb{R}} \) which can be thought of as a gluing of eight copies of \( P \), \( \tilde{P}_j^k, j = 0, \ldots, 3 \) and \( k = 0, 1 \), where each \( \tilde{P}_j^k \) is the pre-image under the double cover of \( P_j \) discussed above.

The metric \( g_{\mathbb{R}} \) induces a metric on \( X_{\mathbb{R}}^o \) and the \( \sigma_i \) can be lifted as isometries. We will also need the lift of \( \phi_{\mathbb{R}} \) to \( X_{\mathbb{R}}^o \) which we denote by \( \phi_{\mathbb{R}}^o : X_{\mathbb{R}}^o \to \mathbb{R}^2 \). Because \( X_{\mathbb{R}}^o \) is orientable, a metric on it determines a complex structure via the Hodge star which we denote \( J_{\mathbb{R}} \). The manifold \( X_{\mathbb{R}} \) itself does not admit such a complex structure this is why we need to use \( X_{\mathbb{R}}^o \) rather than \( X_{\mathbb{R}} \). This structure also induces a complex structure on \( P_j, J_r \).

**Lemma 5.2.** Let \( w \in \partial P \) be not a vertex of \( P \) and \( p \) one of the four elements in \((\phi_{\mathbb{R}}^o)^{-1}(w) \subset X_{\mathbb{R}}^o \). There is a neighbourhood \( \mathcal{V}_p \) of \( p \) in \( X_{\mathbb{R}}^o \) such that \( z \circ \phi_{\mathbb{R}}^o \) extends to \( \mathcal{V}_p \) as a holomorphic function for \( J_{\mathbb{R}} \).

**Proof.** The function \( r \) is defined and smooth on \( X \) and thus on \( X_{\mathbb{R}} \) and on its double cover \( X_{\mathbb{R}}^o \). As we have seen, it is harmonic on the open subset of \( X_{\mathbb{R}} \), \( P_0 \) and therefore because the \( \sigma_i \) are isometries and \( r \) is invariant via the moment map, it is also harmonic on the other open sets \( P_i \) and the corresponding pre-images in the double cover. One can then extend it by zero to the pre-image of the boundary. Because the extension of \( r \) is continuous, it is harmonic on
X_0^p. In a sufficiently small neighbourhood of p, there is a harmonic conjugate coinciding with the lift of H where this lift is defined. Together with the lift of r, this yields the extension. □

**Example 5.3.** To illustrate the above lemma, consider the case when P = (R^+)^2 corresponding to X = C^2 and X_R = R^2. Assume X is endowed with the flat metric. Then z(x, y) = x - y + 2i\sqrt{xy}. On the other hand, \phi(z_1, z_2) = (|z_1|^2, |z_2|^2) and \phi_R(x_1, x_2) = ((x_1)^2, (x_2)^2) where z_j = x_j + iy_j for j = 1, 2 are the coordinates in C^2 and therefore z \circ \phi_R(x_1, x_2) = x_1^2 - y_1^2 + 2i|x_1x_2| on P = (R^+)^2. But this clearly extends as

\[(x_1 + ix_2) \mapsto (x_1 + ix_2)^2,\]

which is holomorphic.

With this extension and given the fact \tilde{z} is bijective on the boundary, we will be able to prove that z is injective.

**Proof of injectivity of z.** Because z : (P, J_z) \rightarrow (\mathbb{H}, J_0) is holomorphic, it has a well-defined notion of degree. We need to show that the degree of z is 1. Consider w_0 \in \partial \mathbb{H} and let w \in \partial P be the single pre-image of w_0 via \tilde{z}. From the lemma, there is an extension of z to a neighbourhood of p \in (\phi_R^p)^{-1}(w) \subset X_R^p, \mathcal{V}_p. We can choose p to be in the closure of P_0^0 and P_1^0.

Informally, the idea is that the number of pre-images of a point in R should be 1. Consider a lift \xi = z \circ \phi_R \in \mathcal{V}_p. This sequence is bounded in R and let \{w_k\} \subset P with |z(w_k) - w_0| \leq 1/k and |w_k - w| > \epsilon. This sequence is bounded in P (otherwise infinity would be in the pre-image of w_0 \in \partial \mathbb{H}) and admits a convergent subsequence in P. The limit w_\infty satisfies \tilde{z}(w_\infty) = w_0 which as \tilde{z}^{-1}(\partial H) = \partial P then forces w_\infty = w. But this contradicts |w_k - w| > \epsilon.

Given w_0 \in B_\epsilon(w_0) \cap \mathbb{H}, z^{-1}(w_0') \subset B_\epsilon(w),

\[
\#z^{-1}(w_0') = \int_\gamma \frac{dz}{\xi(z) - w_0'},
\]

where \gamma is a closed curve close to \partial(B_\epsilon(w_0)) and z is the complex coordinate. Consider a lift of \gamma to \phi_R^{-1}(w_0) \subset X_R^p containing p in its closure. If \epsilon is small enough, it is contained in a coordinate chart for X_R^p. Now modify this lift slightly to enclose p \in (\phi_R^p)^{-1}(w_0) while remaining within \mathcal{V}_p and call this modification \tilde{\gamma}. Then for the extension \zeta of z \circ \phi_R to \mathcal{V}_p,

\[
\#\zeta^{-1}(w_0') = \int_{\tilde{\gamma}} \frac{dz}{\zeta(z) - w_0'},
\]

where z denotes the complex coordinate on \mathcal{V}_p. The quantity

\[
\int_{\tilde{\gamma}} \frac{dz}{\zeta(z) - w_0'},
\]

depends continuously on w_0' and is always an integer. By making w_0' tend to w_0, we see that

\[
1 = \#\zeta^{-1}(w_0) = \int_{\tilde{\gamma}} \frac{dz}{\zeta(z) - w_0'}.
\]
and conclude \( \# \zeta^{-1}(w'_0) = 1 \) which yields \( \# z^{-1}(w'_0) = 1 \) as \( \zeta \) takes values in the lower half plane on \( P^0_1 \). The degree of \( z \) is 1 and we are done. \( \square \)

### 5.2. Surjectivity of \( z \)

\((P, J_\rho)\) cannot be biholomorphic to \( S^2 \) as it is not compact nor to \( \mathbb{C} \) as it admits a positive harmonic function. By the uniformisation theorem, there is a holomorphic map \( \kappa : (P, J_\rho) \to (\mathbb{H}, J_0) \).

**Lemma 5.4.** The map \( \kappa \) is extendable to \( \partial P \). The extension is bijective.

We will denote the extension by \( \tilde{\kappa} : \bar{P} \to \mathbb{H} \).

**Proof.** Let \( w \in \partial P \), \( \epsilon > 0 \). Let \( V_p \) be a neighbourhood of \( p \in (\phi^c_\mathbb{R})^{-1}(w) \subset X^c_\mathbb{R} \) and \( \varphi_p : \mathbb{D} \to V_p \) a complex chart. Assume that \( \varphi_p \) is actually defined in a larger open set with image a neighbourhood of \( V_p \) but we want to consider only its restriction to \( \mathbb{D} \). The point \( p \) is on the intersection of the closure of 2 open subsets of \( X^c_\mathbb{R} \) say \( P^0_0 \) which we identify with \( P \) and \( P^0_1 \). Set \( \mathbb{D}^+ = \varphi_p^{-1}(V_p \cap P^0_0) \). Then

\[
\kappa \circ \phi^c_\mathbb{R} \circ \varphi_p : \mathbb{D}^+ \to \kappa(\phi^c_\mathbb{R}(V_p \cap P^0_0)).
\]

Our first goal is to show \( \mathbb{D}^+ \) and \( \kappa(\phi^c_\mathbb{R}(V_p \cap P^0_0)) \) have boundaries which are Jordan curves.

- The boundary of \( \mathbb{D}^+ \) has two portions.

\[
\partial \mathbb{D}^+ = \partial \mathbb{D} \cap \varphi_p^{-1}(V_p \cap P^0_0) \cup \varphi_p^{-1}(\partial P^0_0 \cap V_p).
\]

The set \( \partial \mathbb{D} \cap \varphi_p^{-1}(V_p \cap P^0_0) \) is a segment in \( \partial \mathbb{D} \) as \( \partial V_p \cap P^0_0 \) is a Jordan segment and \( \varphi_p^{-1} \) is a homeomorphism there. As for \( \varphi_p^{-1}(\partial P^0_0 \cap V_p) \), this is a Jordan segment as \( \partial P^0_0 \cap V_p \) is.

- The boundary of \( \kappa(\phi^c_\mathbb{R}(V_p \cap P^0_0)) \) also has two portions:

\[
\partial \kappa(\phi^c_\mathbb{R}(V_p \cap P^0_0)) = \kappa(\phi^c_\mathbb{R}(\partial V_p \cap P^0_0) \cup s),
\]

where \( s \) is a subset of \( \partial \mathbb{H} \) which we will show is an interval. Note that \( \kappa(\phi^c_\mathbb{R}(\partial V_p \cap P^0_0)) \) is a Jordan segment because \( \partial P^0_0 \cap P^0_0 \) and \( \kappa \circ \phi^c_\mathbb{R} \) is a homeomorphism on \( P^0_0 \). Showing that \( s \) is an interval turns out to be rather technical. The details are as follows.

**Lemma 5.5.** The subset \( s \) of \( \partial \mathbb{H} \) defined above is an interval.

**Proof.** For the sake of completeness and because we will use the details here, we recall Donaldson’s proof the uniformisation theorem from [8]. Let \( o \in P \) and consider a holomorphic coordinate \( z \) around \( o \) as well as a bump function around \( o \) whose support is contained in a ball in the coordinate neighbourhood. We let \( z_0 \) denote the coordinate of \( o \). Let

\[
A = \partial \left( \frac{\beta}{z - z_0} \right).
\]

This is a smooth \((0,1)\) form. There exists \( h \), a smooth function on \( P \) such that \( \partial \bar{\partial} h = \partial A \). The existence of such an \( h \) is the core of the uniformisation theorem and we simply make use of \( h \).

From [8, proposition 32], we know:

**Proposition 5.6** (Donaldson). There is a sequence of smooth functions \( h_i \) on \( P \) such that:

- \( h_i = c_i \) a constant outside a compact set \( B_i \) in \( P \);
- \( \|dh - dh_i\|_{\infty} \) tends to zero as \( i \) tends to \( \infty \).
Set \( a = A + \overline{\partial h} + A - \overline{\partial h} \) so that \( da = 0 \) and let \( \psi \) be a primitive of \( a \), that is, \( d\psi = a \). Consider as in [8]
\[
F = \frac{\beta}{z - z_0} - h - \psi.
\]
This is a holomorphic map to \( S^2 \) by design and bijective onto \( S^2 \setminus I \), where \( I \) is some closed interval in \( \mathbb{R} \). We may as well assume that \( I = [0, 1] \). The map \( \kappa \) is then given by
\[
\kappa(z) = \sqrt{1 - \frac{1}{F(z)}}.
\]
Although as we will see \( F \) extends by continuity to \( \partial P \), the square root does not extend by continuity from \( \mathbb{C} \setminus [0, +\infty[ \) to \( \mathbb{C} \). So, it does not follow from the above that \( \kappa \) extends to the boundary although we will eventually prove that in our case, that is, for \( J_r \) on \( P \), it does. Let \( \{ w_k \} \) be a sequence in \( P \) tending to \( w \in \partial P \). We show that \( \{ F(w_k) \} \) converges.

- First note that \( dF \) is bounded in a neighbourhood of \( w \). In fact, we may assume that \( F = -h - \psi \) so that \( dF = \partial(h - h) \) which uniformly bounded from Proposition (5.6)
\[
|dF| \leq 2|h| \leq 2(1 + |dh_{w_0}|)
\]
for \( i_0 \) sufficiently large which is bounded by a constant \( K \) in a neighbourhood of \( w \).

- Next note
\[
F(w_{k'}) - F(w_k) = F_1(1) - F_1(0) + i(F_2(1) - F_2(0)),
\]
where \( F_1 \) and \( F_2 \) are the restrictions of the real and imaginary parts of \( F \) to the segment \( w_k + t(w'_{k'} - w_k) \). By Taylor’s theorem,
\[
F_1(t) - F_1(0) = F_1'(t_1)t, \quad F_2(t) - F_1(0) = F_1'(t_2)t,
\]
for some \( t_1 \) and \( t_2 \) in [0,1] and
\[
F_1'(1) = \text{Re} \left( \frac{dF}{dz} \right)(w_{k'} - w_k), \quad F_2'(2) = \text{Im} \left( \frac{dF}{dz} \right)(w_{k'} - w_k),
\]
so that
\[
|F_1'(t_1)| \leq K|w_{k'} - w_k|, \quad |F_2'(t_2)| \leq K|w_{k'} - w_k|
\]
as both \( w_k + t_1(w_{k'} - w_k) \) and \( w_k + t_2(w_{k'} - w_k) \) are in the small ball around \( w \) where \( dF \) is bounded by \( K \). We thus have \( |F(w_{k'}) - F(w_k)| \leq K|w_{k'} - w_k| \) and \( \{ F(w_k) \} \) converges to an element in \( S^2 \).

Because it is known that the inverse of \( F \) is well defined and continuous on \( S^2 \setminus I \), this limit cannot be in \( S^2 \setminus I \) (otherwise this would force \( w_k \) to converge to \( F^{-1} \) of the limit). Hence we have constructed an extension \( \tilde{F} : \tilde{P} \to S^2 \) of \( F : P \to S^2 \setminus I \).

It follows from the fact that \( F \) extends to the boundary of its domain as a continuous function that \( \tilde{F}(\phi_0^R(V_p) \cap \partial P) \) is an interval in [0,1] say \( [a, b] \) with \( 0 < a < b < 1 \). In fact, because it is connected, \( F(\phi_0^R(V_p)) \) must be on ‘one side’ of the interval [0,1] in \( S^2 \) and
\[
s = \left[ \sqrt{1 - \frac{1}{b}}, \sqrt{1 - \frac{1}{a}} \right] \text{ or } \left[ -\sqrt{1 - \frac{1}{a}}, -\sqrt{1 - \frac{1}{b}} \right],
\]
so \( s \) is a segment.

We have thus proved that \( \kappa \circ \phi_0^R \circ \varphi_p : \mathbb{D}^+ \to \kappa(\phi_0^R(V_p \cap P_0^0)) \) is an bijective map from two simply connected domains whose boundary is a Jordan curve. \( \square \)
Now it follows from Carathéodory’s theorem that this map can be extended (as a homeomorphism) to the boundary. Because the point we were considering in $\partial P$ is arbitrary (the reasoning applies to vertices of $P$ with a small modification), we are done.

Let $\tilde{\kappa} : P \to \mathbb{H}$ be the extension. We proceed to show that it is injective. This is almost exactly as Carathéodory’s original argument. Suppose the extension is not injective. Let $w$ and $v$ be two points in $\partial P$ such that $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Let $o$ be a point in the interior of $P$. Denote the segments $ow$ and $ov$ as $W$ and $V$, respectively. The Jordan curve $\kappa(W \cup V)$ goes through $\kappa(o)$ and $\tilde{\kappa}(w) = \tilde{\kappa}(v)$. Let $A$ be the interior of this curve and $B = \kappa^{-1}(A)$. Consider $Z$ a (maybe broken) segment in $\partial P$ connecting $w$ to $v$. Then

$$
\tilde{\kappa}(Z) \subset \partial A \cap \partial \mathbb{H} = \{\tilde{\kappa}(w)\},
$$

so that $\tilde{\kappa}$ is constant on $Z$. We may assume the constant is 1. Now consider $V_p$ a neighbourhood of $p \in (\phi_R^o)^{-1}(w) \subset X^o_R$, $\varphi_p : \mathbb{D} \to V_p$ a complex chart and

$$
\kappa \circ \phi_R^o \circ \varphi_p : \mathbb{D}^+ \to \kappa(\phi_R^o(V_p \cap P_0^0)),
$$

where as before $\mathbb{D}^+ = \varphi_p^{-1}(V_p \cap P_0^0)$. Now consider $u : \mathbb{D} \to \mathbb{D}^+$ given by Riemann’s mapping theorem and

$$
\kappa \circ \phi_R^o \circ \varphi_p \circ u : \mathbb{D} \to \kappa(\phi_R^o(V_p \cap P_0^0)),
$$

which by Carathéodory’s theorem extends to $\partial \mathbb{D}$. The function $\phi_R^o \circ \varphi_p \circ u$ also extends because by Carathéodory’s theorem $u$ extends. Therefore, $(\phi_R^o \circ \varphi_p \circ u)^{-1}(Z_p)$, where $Z_p$ is the portion of $Z$ in $\phi_R^o(V_p)$, and $(\phi_R^o \circ \varphi_p \circ u)$ denotes the extension of $\phi_R^o \circ \varphi_p \circ u$, is actually an arc in $\mathbb{D}$. On this arc, the extension of $\kappa \circ \phi_R^o \circ \varphi_p \circ u$ is constant and equal to 1. By Schwartz reflection principle, it thus extends further to a neighborhood of the given arc. But this arc has accumulation points and this brings about a contradiction as the zero sets of holomorphic functions are isolated in their domains of definition. We could similarly prove that the inverse of the extension is injective so the extension is bijective.

We are now in a position to prove the surjectivity of $z$. Let $\mathcal{U} = z(P)$. We have that

$$
 \partial \mathcal{U} = \partial \mathcal{U} \cap \partial \mathbb{H} \cup \partial \mathcal{U} \cap \mathbb{H}
$$

and we denote the second portion of the boundary as $\partial_2 \mathcal{U} = \partial \mathcal{U} \cap \mathbb{H}$. The goal is to show that $\partial_2 \mathcal{U} = \emptyset$. Let $z_k = z \circ \kappa^{-1} : \mathbb{H} \to \mathbb{H}$. This map is holomorphic, injective and can be extended to the boundary $\partial \mathbb{H}$ as a bijection $\partial \mathbb{H} \to \partial P$ as both $z$ and $\kappa$ can be extended to the boundaries of their domains of definition. Let

$$
 f(w) = \frac{1}{z \circ \kappa^{-1}(\frac{1}{w})} : \mathbb{H} \to \mathbb{H}.
$$

This map is holomorphic and maps $\mathbb{R}^+ \subset \mathbb{C}$ to $\mathbb{R}$. By Schwartz reflection principle, the formula $f(w) = \overline{f(\overline{w})}$ extends $f$ as a holomorphic function on $\mathbb{C}^*$ which we still denote by $f$ and which is still injective. The point 0 is an isolated singularity of $f$.

- This singularity cannot be essential as by Picard’s theorem $f$ could not be injective.
- If the singularity is a pole, then it has to be a simple pole so as to not violate the injectivity of $f$.
- The singularity may be removable.

But $\mathcal{U} = z_\kappa(\mathbb{H})$. The first thing to note is that

$$
\partial_2 \mathcal{U} = \{\lim_k z_{\kappa}(w_{n_k}), \ (w_k) \text{ unbounded}, \ (w_{n_k}) \text{ subsequence of } (w_k)\}.
$$

This is because if $z \in \partial_2 \mathcal{U}$, then there is a sequence $(w_k)$ in $\mathbb{H}$ such that $z_{\kappa}(w_k) \to z$. Assuming the sequence $(w_k)$ to be bounded would lead to a contradiction. Namely, if it were bounded, it
would have a convergent subsequence \((w_{n_k})\) in \(\tilde{H}\) so that \(z_n(w_{n_k}) \to \tilde{z}_n(w)\), where \(w\) denotes the limit of the considered subsequence \((w_{n_k})\) in \(\tilde{P}\). If \(w \in \tilde{P}\), then \(\tilde{z}_n(w) = z_n(w) \in U\) but because \(U\) is open this is incompatible with the assumption that \(z \in \partial U\). If \(w \in \partial P\), then \(\tilde{z}_n(w) \in \partial \tilde{H}\), which is incompatible with the assumption that \(z \in \tilde{H}\). It follows that

\[
\partial \tilde{U} = \{\lim_k f(w_{n_k}), \{w_k\} \subset \tilde{H}, w_k \to 0, (w_{n_k})\text{subsequence of}(w_k)\}.
\]

We have shown that

\[
f(0) := \{\lim_k f(w_{n_k}), \{w_k\} \subset \tilde{H}, w_k \to 0, (w_{n_k})\text{subsequence of}(w_k)\}
\]

is either \(\infty\) or a single point. Assume the latter. Let \(a\) be this single point. Then \((z_n)^{-1} : U \to \mathbb{C}\), which is holomorphic, has an isolated pole at \(a\). No such function can have image contained in \(\tilde{H}\) as one can see by considering the image of a neighbourhood of the pole.

6. Uniqueness of the two families of scalar-flat Kähler toric metrics

In this section, the aim is to prove Theorem 1.1. Note that the isothermal coordinates \((H, r)\) can also be described as satisfying

\[
u_{ij}dx_i \otimes dx_j = V(dH \otimes dH + dr \otimes dr),
\]

where \(\nu_{ij}dx_i \otimes dx_j\) is the metric induced by \(g\) on \(X_\mathbb{R}\). We want to think of \((H, r)\) as a map of \((x_1, x_2)\). From what we showed in the previous section, this is a bijection from \(P\) to \(\tilde{H}\).

**Proof of Theorem 1.1.** Without loss of generality we may assume that the unbounded edges of \(X\) meet at the origin. The interior of the moment map image of \(X\), \(P\) is contained in the region

\[
\{x \in \mathbb{R}^2 : x \cdot \nu_1 > 0, x \cdot \nu_d > 0\},
\]

where \(\nu_1\) and \(\nu_d\) are the normals to the unbounded edges. Suppose we are given any Kähler toric metric on \(X\) associated with a complex structure \(J\). We can further associate to \(g\) the following quantities.

- A symplectic potential \(u\).
- \(\eta = (u_{x_1}, u_{x_2})\).

\(X\) admits an ALE metric which we will denote by \(g_{ALE}\) as before. In this case, we write:

- \(u_{ALE}\) for the symplectic potential of \(g_{ALE}\);
- \(\eta_{ALE}\) for the derivative of \(u\) with respect to \((x_1, x_2)\).

By reversing the construction in Section 4, we see that there are isothermal coordinates \((H, r) \in \tilde{H}\) depending on \(g\) and a map \(\mu\) on \(\tilde{H}\) which gives the coordinates change between \((H, r)\) and symplectic coordinates \((x_1, x_2)\). We will sometimes refer to this map as the moment map for the torus action but it really is the moment map expressed in the coordinates \((H, r)\). This is the inverse of the map \(z\) studied in the previous section and it follows from what we did there that \(\mu\) is defined on the whole of \(\tilde{H}\). There is also a function \(\xi(H, r) = \eta \circ \mu(H, r)\) which is a solution of equation (2) and can be seen as a harmonic, axi-symmetric function on \(\mathbb{R}^3\). We will use the following notation:

- \(u_0 = u_{ALE} - u\).
- \(\eta_0 = \eta_{ALE} - \eta\).
- \(\mu_0 = \mu_{ALE} - \mu\).
- \(\xi_0 = \xi_{ALE} - \xi\).
Lemma 6.1. There is a smooth function $f$ on $\mathbb{H}$ such that

$$\mu_0 = r^2 f.$$  

Proof of the Lemma. We start by proving that $\mu_0$ extends as an analytic function to $\mathbb{H}$. From the boundary conditions in Section 3, it follows that $u_0$ is smooth on a neighbourhood of $\partial P$, thus $\eta_0$ is bounded on a neighbourhood of each point in $\partial P$. We have

$$D\mu_0 = r \begin{pmatrix} \frac{\partial \xi_{0,2}}{\partial r} & -\frac{\partial \xi_{0,2}}{\partial H} \\ -\frac{\partial \xi_{0,1}}{\partial r} & \frac{\partial \xi_{0,1}}{\partial H} \end{pmatrix}.$$  

This follows from

$$dx_1 = r \left( \frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right),$$

$$dx_2 = -r \left( \frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right),$$

which holds for both $\mu$ and $\mu_{ALE}$. Thus, we need only to prove that $\xi_0$ extends as an analytic function to $\mathbb{H}$. Since $\xi_0$ is harmonic outside the $H$ axis, a standard argument using the mean value property for harmonic functions ensures it is enough to see that $\xi_0$ is bounded in a neighbourhood of each point in the $H$ axis. We have

$$\xi_0 = \eta_{ALE} \circ \mu_{ALE} - \eta \circ \mu.$$  

This is the sum of the terms

1. $\eta_{ALE} \circ \mu_{ALE} - \eta_{ALE} \circ \mu$;
2. $\eta_{ALE} \circ \mu - \eta \circ \mu$.

The second term is indeed bounded because $\eta_{ALE} - \eta$ is and $\mu$ extends as a continuous function to $\mathbb{H}$. So what we need to do is show that the first term is bounded. We start by setting up some notation. Let $0 < a_1 < \ldots < a_{d-1}$ be positive real numbers (which are in fact determined by $P$ as explained in [3]). Set $a_0 = -\infty$ and $a_d = +\infty$. For $i = 1, \ldots, d - 1$, we define:

- $H_i = H + a_i$;
- $\rho = \sqrt{H^2 + r^2}$;
- $\rho_i = \sqrt{H_i^2 + r^2}$.

Rewrite the first term as $\xi_{ALE} - \xi_{ALE} \circ (\mu_{ALE}^{-1} \circ \mu)$. We have an explicit expression for $\xi_{ALE}$ from [3]. Namely

$$\xi_{ALE,1} = \alpha_1 \log (r) + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) \log (H_i + \rho_i) + \alpha H,$$

$$\xi_{ALE,2} = \beta_1 \log (r) + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) \log (H_i + \rho_i) + \beta H.$$  

Close to $\partial \mathbb{H}$, $\xi_{ALE}$ has a singularity and its singular behaviour is as follows:

$$H \in [-a_{i+1}, a_i], r \to 0 \Rightarrow \xi_{ALE} = \nu_i \log r + O(1).$$

This can be checked using the above formula but in fact, this needs to hold for any scalar-flat Kähler toric metric. Since we will use this fact, we explain why it holds. The symplectic
potential $u$ satisfies $u = l_i \log l_i + O(1)$ near the $i$th facet of the moment map image. This implies that $\eta = \nu_i \log l_i + O(1)$ near that facet. But we have

$$r^2 = \text{det}(\text{Hess } u)^{-1} = \gamma \prod_{i=1}^{d} l_i.$$ 

The Guillemin boundary condition for $u$ shows that there is a nowhere vanishing smooth function $\gamma$ defined near $\partial P$ such that

$$\text{det} \text{Hess } u)^{-1} = \gamma \prod_{i=1}^{d} l_i,$$

and thus, near the interior of the $i$th facet of $P$, $\log l_i = \log r^2 + O(1)$, where $O(1)$ denotes a function that is bounded in a neighbourhood of every point in $\partial P$. It follows that

$$H \in ] - a_{i+1}, a_i [ \implies H' \in ] - a_{i+1}, a_i [,$$

where $H'$ is the $H$ coordinate of the map $\mu^{-1} = z$.

(1) We need to show that $H' \in ] - a_{i+1}, a_i [$ implies $H = \nu_i \log r + O(1)$.

(2) We also need to show that $\log r / (\mu_{ALE}^{-1} \circ \mu)$ is bounded.

First we prove (1). Let $a'_i$ be such that $\mu(a'_i, 0)$ is the $i$th vertex of $P$ for $i = 1, \ldots, d - 1$. What we want to show is that $a'_i = a_i$ for all $i = 1, \ldots, d - 1$. We may assume that $a_1 = a'_1$. We claim that $a'_{i+1} - a'_i$ does not depend on the metric $g$. It is in fact given by

$$a'_{i+1} - a'_i = \frac{\text{length}(e_{i+1})}{2\pi |\nu_i|^2}, \quad i = 1, \ldots, d - 1,$$

where $\text{length}(e_{i+1})$ is the length of the $i + 1$ edge of $P$. We prove the following lemma:

**Lemma 6.2.** Consider any toric metric on $X$ and let $(H, r)$ be given as before by the map $\mu^{-1} = z$. Let $(a_i, 0) = \mu^{-1}(p_i)$ where $p_i$ is the $i$th vertex of $P$, for $i = 1, \ldots, d - 1$. Then

$$a_{i+1} - a_i = \frac{\text{length}(e_{i+1})}{2\pi |\nu_i|^2}, \quad i = 1, \ldots, d - 1,$$

where $\text{length}(e_{i+1})$ is the length of the $i + 1$ edge of $P$.

**Proof.** Let $E_i$ be the pre-image via the moment map in $X$ of the $i$th edge of $P$. For $i = 2, \ldots, d - 1$, the set $E_i$ is an $S^2$ and its volume is precisely the length of $e_i$. On the other hand, we can calculate this volume using the $(H, r)$ coordinates. In fact,

$$\text{vol}(E_i) = \int_{E_i} \omega$$

and

$$\omega = dx_1 \wedge d\theta_1 + dx_2 \wedge d\theta_2$$
where \((\theta_1, \theta_2)\) are coordinates in \(T^2\). We can rewrite \(dx_1\) and \(dx_2\) in \((H, r)\) coordinates and it follows that
\[
\omega = r \left( \frac{\partial \xi_2}{\partial r} dH - \frac{\partial \xi_2}{\partial H} dr \right) \wedge d\theta_1 - r \left( \frac{\partial \xi_1}{\partial r} dH - \frac{\partial \xi_1}{\partial H} dr \right) \wedge d\theta_2.
\]
In these coordinates, we see that
\[
E_i = \{(H, r, \theta_1, \theta_2) : r = 0, H \in ] - a'_i + 1, a'_i[, (\theta_1, \theta_2) \cdot \nu_i = 0\}.
\]
Hence restricted to \(E_i\), \(\omega\) becomes
\[
\omega|_{E_i} = r \frac{\partial \xi_2}{\partial r} dH \wedge d\theta_1 - r \frac{\partial \xi_1}{\partial r} dH \wedge d\theta_2.
\]
Now we can use the fact that \(H \in ] - a'_i + 1, a'_i[, r \to 0 \Rightarrow \xi = \nu_i \log r + O(1)\), to rewrite the above as
\[
\omega|_{E_i} = \alpha_i dH \wedge d\theta_1 - \beta_i dH \wedge d\theta_2,
\]
where \(\nu_i = (\beta_i, \alpha_i)\). The direction in \(T^2\) which is not fixed in \(E_i\) is the direction perpendicular to \(\nu_i\) so that we write \((\theta_1, \theta_2) = t(\alpha_i, -\beta_i)\). Substituting in \(\omega|_{E_i}\), we see that
\[
\omega|_{E_i} = |\nu_i|^2 dH \wedge dt.
\]
Integrating the above over \(] - a'_i + 1, a'_i[\times]0, 2\pi[\) yields the desired result. \(\Box\)

Now we proceed to prove (2). Composing with \(\mu^{-1}\) we see that we may instead show that
\[
\log \frac{r \circ \mu^{-1}}{r \circ \mu^{-1}_{ALE}}
\]
is bounded. As we have seen, for both metrics \(g_{ALE}\) and \(g\), the \(r\) factor in \(\mu^{-1}_{ALE}\) and \(\mu^{-1}\) are given by
\[
r \circ \mu^{-1} = \gamma \prod_{i=1}^{d} t_i^{1/2},
\]
and
\[
r \circ \mu^{-1}_{ALE} = \gamma_{ALE} \prod_{i=1}^{d} t_i^{1/2},
\]
respectively, for some smooth nowhere vanishing functions \(\gamma\) and \(\gamma_{ALE}\) in a neighbourhood of \(\partial P\). Thus our function is
\[
\log \frac{\gamma}{\gamma_{ALE}},
\]
which is clearly bounded as \(\gamma_g\) and \(\gamma_{ALE}\) are nowhere vanishing. \(\Box\)

The function \(f\) defined in the above lemma satisfies a linear PDE. Indeed it turns out that \(f\) is an axi-symmetric harmonic function on \(\mathbb{R}^5\). By this we mean that \(f\) is a harmonic function on \(\mathbb{R}^5\) which only depends on \(H\) and the distance to the \(H\) axis, \(r\), where \(H\) is one of the coordinates in \(\mathbb{R}^5\).

**Lemma 6.3** (Wright). Let \(f\) be such that \(\mu = \mu_{ALE} - r^2 f\), then
\[
f_{HH} + f_{rr} + \frac{3f_r}{r} = 0.
\]
Proof. Since
\[ f = \frac{\mu_0}{r^2}, \]
we have
\[ f_r = \frac{\mu_{0,r}}{r^2} - \frac{2\mu_0}{r^3} \]
and
\[ f_{rr} = \frac{\mu_{0,rr}}{r^2} - \frac{4\mu_{0,r}}{r^3} + \frac{6\mu_0}{r^4}. \]
Also
\[ f_{HH} = \frac{\mu_{0,HH}}{r^2}. \]
We see that
\[ f_{HH} + f_{rr} + \frac{3f_r}{r} = \frac{1}{r^2} \left( \mu_{0,HH} + \mu_{0,rr} - \frac{\mu_0}{r} \right). \]
But we also have
\[ \mu_{0,H} = r(\xi_{0,2,r},-\xi_{0,1,r}) \]
and
\[ \mu_{0,r} = -r(\xi_{0,2,H},-\xi_{0,1,H}) \]
therefore
\[ \mu_{0,HH} = r(\xi_{0,2,Hr},-\xi_{0,1,Hr}) \]
and
\[ \mu_{0,rr} = -(\xi_{0,2,H},-\xi_{0,1,H}) - r(\xi_{0,2,Hr},-\xi_{0,1,Hr}). \]
Replacing in equation (3) the result of the lemma follows. \qed

Because \( \mu(X) = P \), we must have \( \mu \cdot \nu_1 \geq 0 \) and therefore defining \( f_1 = f \cdot \nu_1 \), we have
\[ f_1 \leq \frac{\mu_{ALE} \cdot \nu_1}{r^2}. \]
Now \( \mu_{ALE} \) was explicitly calculated in [3] and it is easy to see that \( |\mu_{ALE}| \leq C \sqrt{H^2 + r^2} \). It follows that there is a constant \( C \) such that for any \( w \in \mathbb{R}^5 \),
\[ f_1(w) \leq \frac{C|w|}{r^2}. \]
Since \( f_1 \) is harmonic on \( \mathbb{R}^5 \), the mean value theorem states that
\[ f_1(z) = \frac{1}{\text{vol}(\partial B(z,R))} \int_{\partial B(z,R)} f_1(w) dw. \]
Here \( B(z,R) \) denotes the ball with center \( z \) and radius \( R \) in \( \mathbb{R}^5 \) and \( dw \) the induced Euclidean volume on \( \partial B(z,R) \). Therefore
\[ f_1(z) = \frac{1}{AR^2} \int_{\partial B(z,R)} f_1(w) dw. \]
for a universal constant \( A \). We have
\[ f_1(w) \leq C \frac{|w|}{r^2} \leq C' \frac{R}{r^2}, \]
where we think of \( r \) as the distance to the \( H \) axis in \( \mathbb{R}^5 \). Therefore

\[
f_1(z) \leq \frac{C''}{R^3} \int_{\partial B(z,R)} \frac{dw}{r^2}.
\]

When \( R \) is very large, the integral

\[
\int_{\partial B(z,R)} \frac{dw}{r^2}
\]

is of the same order of magnitude as

\[
\int_{\partial B(0,R)} \frac{dw}{r^2}
\]

and a straightforward calculation shows that this integral is \( O(R^2) \).

**Lemma 6.4.** Let \( B(R) \) be a ball of radius \( R \) in \( \mathbb{R}^5 \). Then there is a constant \( C \) such that

\[
\int_{\partial B(R)} \frac{1}{r^2} dw \leq CR^2.
\]

**Proof.** A straightforward but perhaps tedious way to check the above is to parametrise the sphere of radius \( R \) in \( \mathbb{R}^5 \). This can be done in the following way. Use the parameters \((\alpha, \alpha_1, \alpha_2, \alpha_3)\),

\[
\phi(\alpha, \alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} R \cos \alpha \\ R \sin \alpha \cos \alpha_1 \\ R \sin \alpha \sin \alpha_1 \cos \alpha_2 \\ R \sin \alpha \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 \\ R \sin \alpha \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 \sin \alpha_3 \end{pmatrix}
\]

In this parametrisation, \( r = R \sin \alpha \) and the integral becomes

\[
\int_{\mathcal{R}} \frac{R^4 \sin^3 \alpha \sin^2 \alpha_1 \sin \alpha_2}{R^2 \sin^2 \alpha} d\alpha d\alpha_1 d\alpha_2 d\alpha_3.
\]

on some bounded region \( \mathcal{R} \). The result follows. \( \square \)

This then implies that \( f_1 \) must be bounded from above. Any bounded harmonic function on \( \mathbb{R}^5 \) is constant, thus \( f_1 \) is constant. In the same way, we show that \( f \cdot \nu_2 \) is constant, and since \( \nu_1 \) and \( \nu_2 \) are linearly independent, we see that \( f \) must be constant which in turn implies that

\[
\mu_0 = r^2 \nu,
\]

for some constant vector \( \nu \), and given the relation between \( \mu \) and \( \xi \) this implies that

\[
\xi_0 = H \nu,
\]

where \( \nu \) is a constant vector. Therefore either the metric \( g \) is the ALE metric if \( \nu = 0 \) or it is a Donaldson generalised Taub-NUT metric.

It only remains to see that the constant vector \( \nu \) must be in the cone determined by \( \nu_1 \) and \( -\nu_d \). Write \( \nu = (\alpha, \beta) \). Then

\[
\mu = \mu_{ALE} + r^2 \left( \frac{-\beta}{\alpha} \right).
\]
As we have seen $|\mu_{ALE}| \leq C\sqrt{H^2 + r^2}$. Now fix a given $H$ and make $r$ tend to infinity, we see that

$$\mu \simeq r^2 \left( -\frac{\beta}{\alpha} \right),$$

as $r$ tends to infinity and therefore

$$\mu \cdot \nu_1 \simeq -r^2 \det(\nu_1, \nu).$$

The fact that $\mu \cdot \nu_1 > 0$ in the interior of $P$ implies $\det(\nu, \nu_1) > 0$. In the same way, we see that $\det(\nu, \nu_d) > 0$ and we are done.

\[\Box\]

7. Uniqueness given asymptotic behaviour

The aim of this section is to show that the asymptotic behaviour of a scalar-flat Kähler toric metric on an unbounded toric manifold determines the metric completely at least if the corresponding complex structure is biholomorphic to the Guillemin complex structure. Such complex structures were called complete complex structures in [3] as they can be characterised by the completeness of the vector fields $J\frac{\partial}{\partial r^i}$, $i = 1, 2$. Completeness of the complex structure is a priori unrelated to completeness of the metric.

We start by setting up some notation. Let $(X, \omega)$ be a strictly unbounded toric 4-manifold with moment polytope $P$. Let $P_0$ be the convex polygonal region with only two unbounded sides coinciding with the unbounded sides of $P$, that is, $P_0$ has only two normal vectors $\nu_1$ and $\nu_d$. We will assume further that these two unbounded edges meet at 0 in $P_0$.

The convex set $P_0$ is the moment polytope of a toric orbifold $X_0$. This orbifold is of the form $\mathbb{C}^2/\Gamma$ for $\Gamma$ some finite subgroup of $U(2)$. Since we are assuming the complex structure on $X$ to be biholomorphic to the Guillemin complex structure, up to equivariant biholomorphism, we have $X = (\mathbb{C}^2 \setminus B)/\Gamma \sqcup \Delta$, where $\Delta$ is some open set in $X$ whose closure is compact and $B$ is a ball centred at the origin in $\mathbb{C}^2$. We can endow $X$ and $X_0$ with Kähler structures. The manifold $X$ is a resolution of $X_0$ and there is a holomorphic map $\pi : X \to X_0$ which is $T^2$ invariant.

In [3], the authors show that for any $\nu \in \mathbb{R}^2$ satisfying

$$\det(\nu_1, \nu), \det(\nu_d, \nu) > 0, \quad (4)$$

there is a scalar-flat Kähler metric on $X$, the Donaldson generalised Taub-NUT metric with parameter $\nu$. We will denote this metric by $g^P_\nu$. The same parameter value $\nu$ defines a Donaldson generalised Taub-NUT orbifold metric with parameter $\nu$ on $X_0$ which we denote by $g^P_{\nu_0}$.

**Theorem 7.1.** Let $(X, \omega)$ be a strictly unbounded toric 4-manifold endowed with a compatible complex structure $J$ which is torus invariant and for which there is $\Psi : (X, J) \to (X, J_P)$, an equivariant biholomorphism. Let $g$ be the toric metric determined by $\Psi^*\omega$ and $J_P$. The manifold $X$ is the resolution of a toric compact orbifold $X_0 = \mathbb{C}^2/\Gamma$. Let $\pi : X \to X_0$ be an equivariant resolution map. If $g$ is scalar-flat and if we have

$$|z|^2|\pi_*g - g^P_{\nu_0}|_{g_{\flat\flat}}(z \mod \Gamma) \to 0, \quad |z| \to \infty,$$

outside $B/\Gamma \subset X_0$, where $g_{\flat\flat}$ is the metric induced on $X_0$ from the flat metric on $\mathbb{C}^2$ and $z$ is the complex coordinate on $\mathbb{C}^2$, then $g = g^P_{\nu_0}$.

**Remark 7.2.** The condition that the complex structure on $X$ be biholomorphic to $J_P$ ensures that our manifold is a resolution of $\mathbb{C}^2/\Gamma$ and that we have a map $\pi$ allowing us to compare the metrics. The special form of the metrics we are considering, namely the fact that they arise from $J_P$ and a pullback of $\omega$ gives a way to ‘choose’ the coordinates $\theta$.
Proof. We can use $\pi$ to pick complex coordinates on $X$. In fact, outside a compact set $B$ in $\mathbb{C}^2/\Gamma$, the map $\pi$ is a biholomorphism and $(\mathbb{C}^2 \setminus B)/\Gamma$ admits coordinates $(z_1, z_2)$. By considering the composition 

$$(z_1, z_2) \circ \pi^{-1}((\mathbb{C}^2 \setminus B)/\Gamma),$$

we get coordinates on the complement of the compact set $\pi^{-1}((\mathbb{C}^2 \setminus B)/\Gamma)$ in $X$. For convenience, we write $z = (z_1, z_2) = (e^{\xi_1 + i\theta_1}, e^{\xi_2 + i\theta_2})$. On $\pi^{-1}((\mathbb{C}^2 \setminus B)/\Gamma)$, we have 

$$\Psi^* \omega = i\partial \bar{\partial} f,$$

and equivariance implies $f$ only depends on $\xi$ so that 

$$g = \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} (d\xi_i \otimes d\xi_j + d\theta_i \otimes d\theta_j).$$

Therefore on $(\mathbb{C}^2 \setminus B)/\Gamma$

$$\pi_* g = \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} (d\xi_i \otimes d\xi_j + d\theta_i \otimes d\theta_j).$$

Similarly, there is $f_0$ such that 

$$g_{P\nu_0} = \sum_{i,j=1}^{2} \frac{\partial^2 f_0}{\partial \xi_i \partial \xi_j} (d\xi_i \otimes d\xi_j + d\theta_i \otimes d\theta_j).$$

Now

$$d\xi_i \otimes d\xi_j + d\theta_i \otimes d\theta_j = \Re(d\log z_i \otimes d\log \bar{z}_j) = \frac{dz_i \otimes d\bar{z}_j}{z_i \bar{z}_j},$$

so that with respect to the flat metric on $\mathbb{C}^2/\Gamma$,

$$|d\xi_i \otimes d\xi_j + d\theta_i \otimes d\theta_j|_{\text{flat}} = \frac{1}{2|z_i \bar{z}_j|}$$

and

$$|\pi_* g - g_{P\nu_0}|_{\text{flat}}^2 = \sum_{i,j=1}^{2} \left| \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} - \frac{\partial^2 f_0}{\partial \xi_i \partial \xi_j} \right|^2 1 \frac{1}{4|z_i \bar{z}_j|^2}.$$

Action-angle coordinates $(x, \theta)$ for $\Psi^* \omega$ are potentially different from those for the symplectic form on $\mathbb{C}^2/\Gamma$, which we denote by $(x, \theta)$. This is why we have used complex coordinates $(\xi, \theta)$ so far. 

Still there is a relation between complex and action-angle coordinates via Legendre transform. We know from theorem (1.1) that there is some $\nu \in \mathbb{R}^2$ satisfying condition 4 such that $g$ is isometric to $g^P_\nu$. We show we must have $g = g^P_\nu$. As usual in toric geometry, we can define $x = \frac{\partial f}{\partial \xi}$ and the Legendre dual of $f$ via 

$$u(x) + f(\xi) = x \cdot \xi, \quad x = \frac{\partial f}{\partial \xi}, \quad \xi = \frac{\partial u}{\partial x}.$$
see [2] for more details. This allows us to express $g$ in coordinates $(x, \theta)$ as $\text{Hess}(u)(x) = \text{Hess}^{-1}(f)(\xi)$. We have

$$g = \sum_{i,j=1}^{2} u_{ij} dx_i \otimes dx_j + u^{ij} d\theta_i \otimes d\theta_j,$$

where $u_{ij}$ and $u^{ij}$ denote the entries of $\text{Hess}(u)$ and its inverse, respectively. Therefore $g$ is itself a metric admitting a symplectic potential which satisfies Abreu’s equation and the right boundary conditions. As functions on the polytope, we have actually shown that such functions are restricted to be one of the functions $u$ coming from the construction is [3] which in turn define one of the metrics $g^P_\nu$ for some $\nu$.

What remains to be seen is that $\nu = \nu_0$. Namely if $u_{P,\nu}$ denotes the symplectic potential of $g = g^P_\nu$, then $u_{P,\nu}$ is Legendre dual to $f$:

$$x = \frac{\partial f}{\partial \xi}, \quad \xi = \frac{\partial u_{P,\nu}}{\partial x}, \quad u_{P,\nu}(x) + f(\xi) = x \cdot \xi.$$

This implies that the Hessian of $f$ at $\xi$ is the inverse of the Hessian of $u_{P,\nu}$ at $x(\xi)$ so that

$$\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f_0}{\partial \xi_i \partial \xi_j} = u^{ij}_{P,\nu}(x(\xi)) - u_{P_0,\nu_0}(\xi(\xi)),$$

where $u_{P_0,\nu_0}$ denotes the symplectic potential of $g^{P_0}_{\nu_0}$. Our condition therefore implies

$$\left| u^{ij}_{P,\nu}(x(\xi)) - u^{ij}_{P_0,\nu_0}(\xi(\xi)) \right| \to 0,$$

for all $i, j \in \{1, 2\}$ as

$$\left| u^{ij}_{P,\nu}(x(\xi)) - u^{ij}_{P_0,\nu_0}(\xi(\xi)) \right|^2 = \frac{4|z_i|^2 |z_j|^2 |z|^4}{4|z_i z_j|^2} \left| u^{ij}_{P,\nu}(x(\xi)) - u^{ij}_{P_0,\nu_0}(\xi(\xi)) \right|^2 \leq 4 \sum_{i,j=1}^{2} \frac{|z|^4}{4|z_i z_j|^2} \left| u^{ij}_{P,\nu}(x(\xi)) - u^{ij}_{P_0,\nu_0}(\xi(\xi)) \right|^2 = 4|z|^2 \pi_* g - g^{P_0}_{\nu_0} \to 0.$$

We will use our explicit knowledge of $u_{P,\nu}$ and $u_{P_0,\nu_0}$ to prove the above can only occur if $\nu = \nu_0$. We start by showing the following lemma.

**Lemma 7.3.** Let $g$ be a Kähler toric metric on a symplectic toric fourfold $X$ with symplectic potential $u$. Let $(H, r) \in \mathbb{H}$ be isothermal coordinates for $g$ and $\mu : \mathbb{H} \to P$ be the coordinate change map and $\xi = (u_{x_1}, u_{x_2}) \circ \mu$ be defined as before. Then

$$(\text{Hess } u(x))^{-1} = (V((D\xi)^{-1} D\xi^{-1})(\mu^{-1}(x)),$$

where $V = r \det D\xi$.

**Proof.** Let $\eta = (u_{x_1}, u_{x_2})$ so that $\eta = \xi \circ \mu^{-1}$. We have

$$(\text{Hess } u) = D\eta = D\xi D\mu^{-1}.$$
Now as we have seen before
\[ D\mu = r \begin{pmatrix} \xi_{2,r} & -\xi_{2,H} \\ -\xi_{1,r} & \xi_{1,H} \end{pmatrix}, \]
hence
\[ D\mu^{-1} = \frac{D\xi^t}{r \det D\xi} \]
and the result follows. \( \square \)

We want to use this lemma to show the following.

**Lemma 7.4.** Let \((H, r) \in \mathbb{H}\) be isothermal coordinates for \(g^P, \nu = (\alpha, \beta)\). Then if \(x\) tends to infinity in \(P\) with \(r(x) \to \infty\):

- if \(\nu \neq 0\)
  \[ \lim_{x \to \infty} \frac{u_{P,\nu}^{ij}(x)}{r^2(x)} = \lambda \nu^+ \nu^-, \nu \neq 0, \]
  where \(\nu^+ = (\beta, -\alpha)\) and \(\lambda\) is positive constant;
- if \(\nu = 0\) and \(\rho/r^2(x) \to 0\), then
  \[ \lim_{x \to \infty} \frac{u_{P,\nu}^{ij}(x)}{r^2(x)} = 0. \]

**Proof.** We start with the \(\nu \neq 0\) case. From [3], we have an explicit formula for the function \(\xi = (u_{x_1}, u_{x_2}) \circ \mu\). Namely
\[ \xi_1 = \alpha_1 \log(r) + \frac{1}{2} \sum_{i=1}^{d-1} (\alpha_{i+1} - \alpha_i) \log(H_i + \rho_i) + \alpha H \]
and
\[ \xi_2 = \beta_1 \log(r) + \frac{1}{2} \sum_{i=1}^{d-1} (\beta_{i+1} - \beta_i) \log(H_i + \rho_i) + \beta H, \]
where \(\nu = (\alpha, \beta)\) and we use the same notation as before for \(H_i\) and \(\rho_i\). This implies
\[ D\xi = \begin{pmatrix} \frac{\alpha_1}{r} + \sum_{i=1}^{d-1} \frac{(\alpha_{i+1} - \alpha_i)r}{2(H_i + \rho_i)\rho_i} \sum_{i=1}^{d-1} \frac{(\alpha_{i+1} - \alpha_i)}{2\rho_i} + \alpha \\ \frac{\beta_1}{r} + \sum_{i=1}^{d-1} \frac{(\beta_{i+1} - \beta_i)r}{2(H_i + \rho_i)\rho_i} \sum_{i=1}^{d-1} \frac{(\beta_{i+1} - \beta_i)}{2\rho_i} + \beta \end{pmatrix} \]
The map \(\mu\) is proper and therefore when \(x\) tends to infinity in \(P\), \(\rho\) also tends to infinity. Because
\[ \frac{1}{H_i + \rho_i} - \frac{1}{H + \rho} = O\left(\frac{1}{\rho^2}\right) \quad \text{and} \quad \frac{1}{\rho_i} - \frac{1}{\rho} = O\left(\frac{1}{\rho^2}\right), \]
$D\xi$ can be written as
\[
\begin{pmatrix}
0 & \alpha \\
0 & \beta
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
\alpha_1 + \frac{(\alpha_d - \alpha_1)r^2}{2p(H + \rho)} & \frac{(\alpha_d - \alpha_1)r}{2p} \\
\beta_1 + \frac{(\beta_d - \beta_1)r^2}{2p(H + \rho)} & \frac{(\beta_d - \beta_1)r}{2p}
\end{pmatrix} + \frac{r}{\rho} O\left(\frac{1}{\rho^2}\right),
\]
and $D\xi^{-1}$ can be written as
\[
\frac{1}{\det(D\xi)} \begin{pmatrix}
\beta & -\alpha \\
0 & 0
\end{pmatrix} + \frac{1}{r} \begin{pmatrix}
\frac{(\beta_d - \beta_1)r}{2p} & -\frac{(\alpha_d - \alpha_1)r}{2p} \\
-\left(\beta_1 + \frac{(\beta_d - \beta_1)r^2}{2p(H + \rho)}\right) & \alpha_1 + \frac{(\alpha_d - \alpha_1)r^2}{2p(H + \rho)}
\end{pmatrix} + \frac{r}{\rho} O\left(\frac{1}{\rho^2}\right).
\]
Using the above, we can also write an expression for $V = r \det D\xi$ and study its asymptotic behaviour.
\[
V = \det(\nu_1, \nu) \left(1 - \frac{r^2}{2p(H + \rho)}\right) + \frac{\det(\nu_d, \nu)^2}{2p(H + \rho)} + \frac{\det(\nu_1, \nu_d)}{2p} + O\left(\frac{1}{\rho^2}\right).
\]
The upshot of the above expressions is that
\[
0 \leq \frac{r^2}{2p(H + \rho)} \leq \frac{1}{2}.
\]
So this quantity tends to some value $a$ and it follows that, if $\nu \neq 0$, $V$ tends to $\det(v, \nu)$, where $v = (1 - a)\nu_1 + a\nu_d$. In this case, the asymptotic expression for $\text{Hess}^{-1} u$ follows from
\[
\text{Hess}^{-1} u(x) = V((D\xi)^t)^{-1} D\xi^{-1} (\mu^{-1}(x)).
\]
We have
\[
\text{Hess}^{-1} u(x) = \frac{r^2}{\det(v, \nu)} \left(\begin{pmatrix}
\beta^2 & -\alpha \beta \\
-\alpha \beta & \alpha^2
\end{pmatrix} + O\left(\frac{1}{r}\right) + O\left(\frac{1}{\rho^2}\right)\right).
\]
Because
\[
\begin{pmatrix}
\beta^2 & -\alpha \beta \\
-\alpha \beta & \alpha^2
\end{pmatrix} = \nu^\perp (\nu^\perp)^t,
\]
the result follows in the case $\nu \neq 0$.

In the case where $\nu = 0$, $D\xi$ can be written as
\[
\frac{1}{r} \begin{pmatrix}
\alpha_1 + \frac{(\alpha_d - \alpha_1)r^2}{2p(H + \rho)} & \frac{(\alpha_d - \alpha_1)r}{2p} \\
\beta_1 + \frac{(\beta_d - \beta_1)r^2}{2p(H + \rho)} & \frac{(\beta_d - \beta_1)r}{2p}
\end{pmatrix} + \frac{r}{\rho} O\left(\frac{1}{\rho^2}\right),
\]
and $D\xi^{-1}$ can be written as
\[
\frac{1}{V} \begin{pmatrix}
\frac{(\beta_d - \beta_1)r}{2p} & -\frac{(\alpha_d - \alpha_1)r}{2p} \\
-\left(\beta_1 + \frac{(\beta_d - \beta_1)r^2}{2p(H + \rho)}\right) & \alpha_1 + \frac{(\alpha_d - \alpha_1)r^2}{2p(H + \rho)}
\end{pmatrix} + \frac{r}{\rho} O\left(\frac{1}{\rho^2}\right),
\]
and $V$ is asymptotic to
\[
\frac{\det(\nu_1, \nu_d)}{2p}.
and therefore \( \text{Hess}^{-1} u \) is given by
\[
\frac{2\rho}{\det(\nu_1, \nu_d)} \left( (\nu^+ + a(\nu^+ - \nu^+_1))(\nu^- + a(\nu^- - \nu^-_1))' + b^2(\nu^+ - \nu^+_1)(\nu^- - \nu^-_1)' + O\left(\frac{1}{\rho}\right) \right),
\]
where
\[
a = \lim_{r \to 0} \frac{r^2}{2\rho(H + \rho)}, \quad b = \lim_{r \to 0} \frac{r}{2\rho}, \quad \nu^+_i = (\beta_i, -\alpha_i), \ i = 1, d.
\]
Assuming \( \rho/r^2 \to 0 \), we are done. \( \Box \)

We are now ready to prove Theorem (7.1).

- Assume that \( \nu \) and \( \nu_0 \) are both different from zero. We know that
\[
\left| u^{ij}_{\nu'}(x(\xi)) - u^{ij}_{\nu_0}(x(\xi)) \right| \to 0
\]
and that
\[
\lim_{x \to \infty} \frac{u^{ij}_{\nu'}(x)}{r^2(x)} = \lambda \nu^+_i \nu^+_j, \quad \nu^+ = 0, \ \lim_{r \to \infty} \frac{u^{ij}_{\nu_0}(x)}{r^2_0(x)} = \lambda_0 \nu_0^+ \nu_0^+, \quad \nu_0^+ \neq 0.
\]
Note that as \( \xi \) tends to infinity, both \( x(\xi) \) and \( x(\xi) \) tend to infinity because we are considering only proper moment maps. Let us limit ourselves to regions in \( \mathbb{C}^2/\Gamma \) such that \( r(x(\xi)) \) tends to infinity. Then,
\[
r^2(x(\xi)) \left| \frac{u^{ij}_{\nu'}(x(\xi))}{r^2(x(\xi))} - \frac{r^2_0(x(\xi))}{r^2(x(\xi))} \frac{u^{ij}_{\nu_0}(x(\xi))}{r^2_0(x(\xi))} \right| \to 0
\]
implies
\[
\left| \frac{u^{ij}_{\nu'}(x(\xi))}{r^2(x(\xi))} - \frac{r^2_0(x(\xi))}{r^2(x(\xi))} \frac{u^{ij}_{\nu_0}(x(\xi))}{r^2_0(x(\xi))} \right| \to 0.
\]
If \( r^2_0(x(\xi)) \) is unbounded, we immediately get a contradiction. Otherwise the above implies that \( \nu^+_i (\nu^+_j)' \) is a scalar multiple of \( \nu_0^+ (\nu_0^+_j)' \) which in turn implies that \( \nu^+ = \pm \nu_0^+ \) and \( \nu = \pm \nu_0 \). In this case, we are done.

- Let us now consider the case when \( \nu = 0 \) but \( \nu_0 = 0 \). Again let us limit ourselves to regions in \( \mathbb{C}^2/\Gamma \) such that \( r(x(\xi)) \) tends to infinity. We have
\[
\left| \frac{u^{ij}_{\nu'}(x(\xi))}{r^2(x(\xi))} - \frac{r^2_0(x(\xi))}{r^2(x(\xi))} \frac{u^{ij}_{\nu_0}(x(\xi))}{r^2_0(x(\xi))} \right| \to 0.
\]
We have seen that if \( \xi \to \infty \), our hypothesis imply
\[
\left| \text{Hess}^{-1} u(x(\xi)) - \text{Hess}^{-1} u_0(x(\xi)) \right| \to 0.
\]
Now one can easily see that
\[
\left| \det \text{Hess}^{-1} u - \det \text{Hess}^{-1} u_0 \right| \leq \left| \text{Hess}^{-1} u - \text{Hess}^{-1} u_0 \right| \left| \text{Hess}^{-1} u \right|,
\]
and since \( \text{Hess}^{-1} u \) is bounded (it tends to \( a^\nu (\nu^+)' \)), it follows that
\[
\left| \det \text{Hess}^{-1} u(x(\xi)) - \det \text{Hess}^{-1} u_0(x(\xi)) \right|\]
tends to zero. Now we know from the Donaldson’s version of Joyce’s construction in action-angle coordinates that
\[ r = (\det \text{Hess} \ u)^{-1/2} \]
\[ r_0 = (\det \text{Hess} \ u_0)^{-1/2}. \]
Therefore \(|r^2(x(\xi)) - r_0^2(x(\xi))| \to 0\) and \(|r^2(x(\xi)) - 1| \to 0\). On the other hand, we know that if \(\xi\) tends to infinity in the region where \(r_0(x(\xi))\) tends to infinity, then \(r(x(\xi))\) also tends to infinity and
\[ \lim_{\xi \to \infty} \frac{u_{ij}(x(\xi))}{\rho_0(x(\xi))} = \lambda_0 \nu_{i,0}^+ \nu_{0,j}^+, \]
\[ \lim_{\xi \to \infty} \frac{\iota^+}{r^2(x(\xi))} = \lambda \nu^+ \nu^+, \]
for a given vector \(\nu^+\). We can choose a sequence \(\xi_n \to \infty\) such that \(r_0(x(\xi_n)) \to \infty\) and \(H_0(x(\xi_n)) = c\) for some constant \(c\) and for such a sequence
\[ \frac{\rho_0(x(\xi_n))}{r_0^2(x(\xi_n))} = \frac{\sqrt{r_0^2(x(\xi_n)) + c^2}}{r_0^2(x(\xi_n))} \to 0. \]
Replacing in equation (5), we find that \(\lambda \nu^+(\nu^+)^t = 0\), thus implying \(\nu^+ = 0\) and \(\nu = 0\). By symmetry, we conclude that it is impossible for \(\nu = 0\) and \(\nu_0 \neq 0\).

Acknowledgements. I would to thank Simon Donaldson for his support and Gustavo Granja for interesting conversations. I would also like to express my gratitude to the anonymous referee for his careful reading, for spotting a mistake in a previous write up of the proof of Theorem 1.2 and for suggesting the removal of unnecessary conditions in an initial version of the main theorem.

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