A Faster Algorithm to Recognize Even-Hole-Free Graphs

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November 5, 2013

Abstract

We study the problem of determining whether an n-node graph $G$ has an even hole, i.e., an induced simple cycle consisting of an even number of nodes. Conforti, Cornuègeols, Kapoor, and Vušković gave the first polynomial-time algorithm for the problem, which runs in $O(n^{40})$ time. Later, Chudnovsky, Kawarabayashi, and Seymour reduced the running time to $O(n^{31})$. The best previously known algorithm for the problem, due to da Silva and Vušković, runs in $O(n^{19})$ time. In this paper, we solve the problem in $O(n^{11})$ time. Moreover, if $G$ has even holes, our algorithm also outputs an even hole of $G$ in $O(n^{11})$ time.

1 Introduction

A hole is an induced simple cycle consisting of at least four nodes. A hole is even (respectively, odd) if it consists of an even (respectively, odd) number of nodes. See Figure 1 for an illustration. Even-hole-free graphs have been extensively studied in the literature (see, e.g., [12, 13, 14, 19, 1, 37, 20, 29]). See Vušković [41] for a recent survey. This paper studies the problem of determining whether a graph has even holes. Let $n$ (respectively, $m$) be the number of nodes (respectively, edges) of the input graph. Conforti, Cornuègeols, Kapoor, and Vušković [11, 15] gave the first polynomial-time algorithm for the problem, which runs in $O(n^{40})$ time [6]. Later, Chudnovsky, Kawarabayashi, and Seymour [6] reduced the running time to $O(n^{31})$. Chudnovsky et al. [6] also observed that the running time can be further reduced to $O(n^{15})$ as long as prisms can be detected efficiently, but Maffray and Trotignon [30] showed that detecting prisms is NP-hard. The best previously known algorithm for the problem, due to da Silva and Vušković [20], runs in $O(n^{19})$ time. We solve the problem in $O(n^{11})$ time, as stated in the following theorem.

*The current version slightly improves upon the preliminary version [3] appeared in SODA 2012: (a) The time complexity for recognizing even-hole-free n-node m-edge graphs $G$ is reduced from $O(m^2 n^7)$ to $O(m^3 n^5)$, which is an improvement if $m = o(n^2)$; and (b) if $G$ has even holes, the current version shows how to output an even hole of $G$ also in $O(m^3 n^5)$ time.

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Figure 1: $C = v_1v_2v_3v_2v_7v_8v_1$ is a clean even hole of the 11-node graph $G$, since $M_G(C) = N_G^{2,2}(C) = \emptyset$. $C' = c_1c_2 \cdots c_9c_1$ is an odd hole with $M_G(C') = \{v_2\}$.

Theorem 1.1. It takes $O(m^3n^5)$ time to determine whether an $n$-node $m$-edge connected graph has even holes.

Technical overview Throughout the paper, a $k$-hole (respectively, $k$-cycle and $k$-path) is a $k$-node hole (respectively, cycle and path). Our recognition algorithm for even-hole-free graphs consists of two phases. The first phase (see Lemma 2.3) either (1) ensures that the input graph $G$ has even holes via the existence of a “beetle” (see §4 and Figure 2(a)) or a 4-hole in $G$ or (2) produces a set $T$ of “trackers” $(H, u_1u_2u_3)$, where $H$ is a beetle-free and 4-hole-free induced subgraph of $G$ that contains path $u_1u_2u_3$. $T$ has the following even-hole-preserving property (see Property 1): If $G$ has even holes, then $T$ has a “lucky” tracker $(H, u_1u_2u_3)$ in that $H$ has a shortest even hole $C$ of $G$ such that (a) $C$ contains path $u_1u_2u_3$ and (b) the neighborhood of $C$ in $H$ is “super clean” (i.e., $M_H(C) = N_H^{2,2}(C) = N_H^{1,2}(C) = N_H^4(C) = \emptyset$ using notation to be defined in [2]). The second phase applies an algorithm (see Lemma 2.4) on each tracker $(H, u_1u_2u_3) \in T$ to either ensure that $H$ has even holes or ensure that $(H, u_1u_2u_3)$ is not lucky. If all trackers in $T$ are not lucky, the even-hole-preserving property of $T$ implies that $G$ is even-hole-free. Otherwise, $G$ has an induced subgraph $H$ containing an even hole, implying that $G$ has even holes.

The recognition algorithm for beetle-free graphs (see the proof of Lemma 2.3) in the first phase is based on Chudnovsky and Seymour’s three-in-a-tree algorithm [9] (see Lemma 3.1). If $G$ has beetles or 4-holes, $G$ has even holes. Otherwise, if $G$ has even holes, the neighborhood of each shortest even hole $C$ of $G$ is “clean” (i.e., $N_G^{1,2}(C) = N_G^4(C) = \emptyset$, see the proof of Lemma 2.2). To further ensure that the neighborhood of $C$ is super clean, we generate a set $\mathcal{S}$ of “super cleaners” $(S, u_1u_2u_3)$, where $S$ is a node subset of $G$ and $u_1u_2u_3$ is a path of $G$, such that at least one super cleaner $(S, u_1u_2u_3) \in \mathcal{S}$ satisfies $u_1u_2u_3 \subseteq C \subseteq G \setminus S$ and $M_H(C) = N_H^{2,2}(C) = \emptyset$ for some shortest even hole $C$ of $G$ (see the proof of Lemma 2.3). The set $T$ consisting of the trackers $(G \setminus S, u_1u_2u_3)$ with $(S, u_1u_2u_2) \in \mathcal{S}$ has the required even-hole-preserving property.

The algorithm applied on each tracker $T = (H, u_1u_2u_3) \in T$ in the second phase is a decomposition algorithm (see, e.g., the categorization of Vušković [41, §4]) based upon an observation of da Silva and Vušković [20] (see Lemma 4.9) that if a connected graph $H$ is even-hole-free, star-cutset-free, and non-path 2-join-free, then $H$ is an extended clique tree. Since even holes can be efficiently detected in an extended clique tree (see Lemma 4.6), which is a slightly faster implementation of the algorithm of da Silva and Vušković [20], our algorithm performs two stages of even-hole-preserving decompositions on $H$, first via star-cutsets and then via non-path 2-joins, until each of the resulting graph either is an extended clique tree or has $O(1)$ nodes. If all of the resulting graphs are even-hole-free, $T$ is not lucky; otherwise, $H$ has even holes. An immediate
challenge for the first stage of decompositions is that there are no known polynomial-time detection algorithms for star-cutsets. Fortunately, as noted by Chvátal [10] (see Lemma 4.3), if $H$ is dominated-node-free, a star-cutset of $H$ has to be a full star-cutset of $H$, which can be efficiently detected. Thus, at the beginning of each decomposition in the first stage, we preprocess $(H, u_1u_2u_3)$ by deleting all dominated nodes of $H$ and carefully updating nodes $u_1$, $u_2$, and $u_3$ such that the luckiness of $(H, u_1u_2u_3)$ is preserved (see Lemma 4.4). The correctness of this preprocessing step relies on the fact that $H$ is beetle-free and the requirement for $(H, u_1u_2u_3)$ to be lucky that the neighborhood of some shortest even hole $C$ in $H$ with $u_1u_2u_3 \subseteq C$ is super clean. Path $u_1u_2u_3$ is crucial in the stage of decompositions via star-cutsets for the dominated-node-free graph $H$. Specifically, if $S$ is a star-cutset of $H$, by merely examining the neighborhood of path $u_1u_2u_3$ in $H$, we can efficiently identify a connected component $B$ of $H \setminus S$ such that $(H[S \cup B], u_1u_2u_3)$ preserves the luckiness of $(H, u_1u_2u_3)$ (see Step 3 in the proof of Lemma 4.1). We then let $H = H[C \cup B]$ and repeat the above procedure for $O(n)$ iterations until $H$ is star-cutset-free. The second stage, i.e., decompositions via non-path 2-joins for star-cutset-free graphs, is based upon the detection algorithm for non-path 2-joins of Charbit et al. [3] (see Lemma 4.5). This stage decomposes an $m$-edge star-cutset-free graph into a set of $O(m)$ smaller graphs, each of which either consists of $O(1)$ nodes or is an extended clique tree (see the proof of Lemma 4.2).

Related work  Even-hole-free planar graphs [33] can be recognized in $O(n^3)$ time. It is NP-complete to determine whether a graph has an even (respectively, odd) hole that contains a given node [2]. The strong perfect graph theorem of Chudnovsky, Robertson, Seymour, and Thomas [7] states that a graph $G$ is perfect if and only if both $G$ and the complement of $G$ are odd-hole-free. Although perfect graphs can be recognized in $O(n^9)$ time [5], the tractability of recognizing odd-hole-free graphs remains open (see, e.g., [25]). Polynomial-time algorithms for detecting odd holes are known for planar graphs [24], claw-free graphs [36, 28], and graphs with bounded clique numbers [16]. Graphs without holes (i.e., chordal graphs) can be recognized in $O(m + n)$ time [38, 34, 35]. Graphs without holes consisting of five or more nodes (i.e., weakly chordal graphs) can be recognized in $O(m^2 + n)$ time [31, 32]. It takes $O(n^2)$ time to detect a hole that contains any $o((\log n / \log \log n)^{2/3})$ given nodes in an $O(1)$-genus graph [26, 27]. See [8, 42, 21, 17] for more results on odd-hole-free graphs.

Road map  The rest of the paper is organized as follows. Section 2 gives the preliminaries and proves Theorem 1.1 by Lemmas 2.3 and 2.4. Section 3 proves Lemma 2.4. Section 4 proves Lemma 2.3. Section 5 concludes the paper by explaining how to augment Theorem 1.1 into an $O(m^3n^5)$-time algorithm that outputs an even hole of an $n$-node $m$-edge graph with even holes.

2 Preliminaries and the topmost structure of our proof

Unless clearly specified otherwise, all graphs throughout the paper are undirected and simple. Let $|S|$ be the cardinality of set $S$. Let $G$ be a graph. Let $V(G)$ consist of the nodes in $G$. For any subgraph $H$ of $G$, let $G[H]$ denote the subgraph of $G$ induced by $V(H)$. Subgraphs $H$ and $H'$ of graph $G$ are adjacent in $G$ if some node of $H$ and some node of $H'$ are adjacent in $G$. For any subset $U$ of $V(G)$, let $G \setminus U = G[V(G) \setminus U]$. For any subgraph $H$ of $G$, let $N_G(H)$ consist of the nodes of $V(G) \setminus V(H)$ that are adjacent to $H$ in $G$ and let $N_G[H] = N_G(H) \cup V(H)$. 


Let $C$ be a hole of $G$. A node $x \in V(G) \setminus V(C)$ is a major node of $C$ in $G$ if at least three distinct nodes of $N_C(x)$ are pairwise non-adjacent in $G$. Let $M_G(C)$ consist of the major nodes of $C$ in $G$. For instance, in Figure 1 $M_G(C) = \emptyset$ and $M_G(C') = \{v_2\}$.

**Lemma 2.1** (Chudnovsky et al. [6 Lemma 2.2]). If $C$ is a shortest even hole of graph $G$ and $x \in M_G(C)$, $|N_C(x)|$ is even.

If $x \in N_G(C) \setminus M_G(C)$, $1 \leq |N_C(x)| \leq 4$ and $C[N_C(x)]$ has at most two connected components. Moreover, if $C[N_C(x)]$ is not connected, each connected component of $C[N_C(x)]$ has at most two nodes. Let $N^i_G(C)$ with $1 \leq i \leq 4$ consist of the nodes $x \in N_G(C) \setminus M_G(C)$ such that $|N_C(x)| = i$ and $C[N_C(x)]$ is connected. Let $N^{i,j}_G(C)$ with $1 \leq i \leq j \leq 2$ consist of the nodes $x \in N_G(C) \setminus M_G(C)$ such that $C[N_C(x)]$ is not connected and the two connected components of $C[N_C(x)]$ has $i$ and $j$ nodes, respectively. We have

$$N_G(C) = N^1_G(C) \cup N^2_G(C) \cup N^3_G(C) \cup N^4_G(C) \cup N^{1,1}_G(C) \cup N^{1,2}_G(C) \cup N^{2,2}_G(C) \cup M_G(C).$$

We say that $C$ is a clean hole of $G$ if $M_G(C) = N^{2,2}_G(C) = \emptyset$. For any 3-path $u_1u_2u_3$ of $G$, a clean hole $C$ of $G$ that contains path $u_1u_2u_3$ is a $u_1u_2u_3$-hole of $G$ if $C$ is a shortest even hole of $G$. For instance, if $G$ is as shown in Figure 1 $C = v_1c_2c_3v_2c_7c_8v_1$ is a $v_1c_2c_3$-hole of $G$. If $H$ is an induced subgraph of $G$ and $u_1u_2u_3$ is a 3-path of $H$, we call $(H, u_1u_2u_3)$ a tracker of $G$. A tracker $(H, u_1u_2u_3)$ of $G$ is lucky if $H$ contains a $u_1u_2u_3$-hole of $G$. If $G$ has lucky trackers, $G$ has even holes. The following even-hole-preserving property of $T$ reduces the problem of determining whether $G$ is even-hole-free to the problem of determining whether all trackers in $T$ are not lucky.

**Property 1.** If $G$ has even holes, $T$ contains a lucky tracker of $G$.

An induced subgraph of $G$ is a beetle of $G$ if it consists of (1) a 4-cycle $b_1b_2b_3b_4b_1$ with exactly one chord $b_2b_4$ (i.e., a diamond of $G$) and (2) a tree $I$ of $G \setminus \{b_1\}$ having exactly three leaves $b_1$, $b_2$, and $b_3$ with the property that $I \setminus \{b_1, b_2, b_3\}$ is an induced tree of $G$ not adjacent to $b_4$. See Figure 2(a) for an illustration. At least one of the three cycles in $B \setminus \{b_2\}$, $B \setminus \{b_1, b_2\}$, and $B \setminus \{b_3, b_4\}$ is an even hole of $G$. Nodes $b_5$, $b_6$, $b_7$, and $b_8$ need not be distinct. For instance, as illustrated by Figure 2(b), if $C$ is a hole of $G$ and $x$ is a node of $N^1_G(C)$, then $G[C \cup \{x\}]$ is a beetle of $G$. 

Figure 2: (a) A beetle $B$, where $B[\{b_1, b_2, b_3, b_4\}]$ is a diamond. (b) If $x \in N^1_G(C)$, then $G[C \cup \{x\}]$ is a beetle $B$, where $B[\{u_1, u_2, u_3, x\}]$ is a diamond. (c) A node $x \in N^{1,2}_G(C)$. 
Lemma 2.2. If \( G \) is a beetle-free graph, \( N_G(C) \subseteq N_G^{1,1}(C) \cup N_G^{1}(C) \cup N_G^{2}(C) \cup N_G^{3}(C) \) holds for any clean shortest even hole \( C \) of \( G \).

Proof. By \( M_G(C) = N_G^{2,2}(C) = \emptyset \) and Equation (1), it suffices to show \( N_G^{1,2}(C) = N_G^{1}(C) = \emptyset \). If \( x \in N_G^{4}(C) \) as illustrated by Figure 2(b), then \( G[C \cup \{x\}] \) is a beetle of \( G \), a contradiction. If \( x \in N_G^{1,2}(C) \), then let \( u, v_1, \) and \( v_2 \) be the nodes of \( N_G(C) \) such that \( v_1 \) and \( v_2 \) are adjacent in \( C \), as illustrated by Figure 2(c). Let \( P_1 \) be the path of \( C \setminus \{v_2\} \) between \( u \) and \( v_1 \). Let \( P_2 \) be the path of \( C \setminus \{v_1\} \) between \( u \) and \( v_2 \). Either \( G[\{x\} \cup P_1] \) or \( G[\{x\} \cup P_2] \) is an even hole of \( G \) shorter than \( C \), a contradiction. The lemma is proved.

2.1 Proving Theorem 1.1

Lemma 2.3. It takes \( O(m^3 n^5) \) time to complete either one of the following tasks for any \( n \)-node \( m \)-edge graph \( G \). Task 1: Ensuring that \( G \) has even holes. Task 2: (a) Ensuring that \( G \) has no beetles and (b) obtaining a set \( T \) of \( O(m^2 n) \) trackers of \( G \) that satisfies Property 1.

Lemma 2.4. Given a tracker \( T = (H, u_1u_2u_3) \) of an \( n \)-node beetle-free graph \( G \), it takes \( O(mn^4) \) time to either ensure that \( H \) has even holes or ensure that \( T \) is not lucky.

Proof of Theorem 1.1 We apply Lemma 2.3 on \( G \) in \( O(m^3 n^5) \) time. If Task 1 is completed, then the theorem is proved. If Task 2 is completed, then \( G \) has no beetles and we have a set \( T \) of \( O(m^2 n) \) trackers of \( G \) that satisfies Property 1. By Property 1 of \( T \) and Lemma 2.4, one can determine whether \( G \) has even holes in time \( |T| \cdot O(mn^4) = O(m^3 n^5) \). The theorem is proved.

3 Proving Lemma 2.3

A clique of \( G \) is a complete subgraph of \( G \). A clique of \( G \) is maximal if it is not contained by other cliques of \( G \). We need the following four lemmas to prove Lemma 2.3.

Lemma 3.1 (Chudnovsky and Seymour [7]). It takes \( O(n^4) \) time to determine whether an \( n \)-node graph has an induced tree that contains three given nodes.

Lemma 3.2 (da Silva and Vušković [19] Section 2) and Farber [22] Proposition 2). If \( G \) is an \( n \)-node \( m \)-edge 4-hole-free graph, it takes \( O(mn^2) \) time to either ensure that \( G \) has even holes or obtain all \( O(n^2) \) maximal cliques of \( G \).

Lemma 3.3 (Chudnovsky, Kawarabayashi, and Seymour [6] Lemma 4.2]). Any shortest even hole \( C \) of a 4-hole-free graph \( G \) contains an edge \( v_1v_2 \) with \( N_G^{2,2}(C) \subseteq N_G(v_1) \cap N_G(v_2) \).

Lemma 3.4. For any shortest even hole \( C \) of a 4-hole-free graph \( G \), if \( G[M_G(C)] \) is not a clique of \( G \), there is a node \( u \) of \( C \) with \( M_G(C) \subseteq N_G(u) \).

Before proving Lemma 3.4 we first prove Lemma 2.3 using Lemmas 3.1 3.2 3.3 and 3.4.

Proof of Lemma 2.3 We claim that \( G \) has beetles if and only if at least one of the \( O(m^3 n) \) choices of node \( b_4 \) and three distinct edges \( b_1b_5, b_2b_6, \) and \( b_3b_7 \) of \( G \) satisfies the following conditions.

- \( G[\{b_1, b_2, b_3, b_4\}] \) is the 4-cycle \( b_1b_2b_3b_4b_1 \) with exactly one chord \( b_2b_4 \).
- The edges between \( \{b_1, b_2, b_3\} \) and \( \{b_5, b_6, b_7\} \) are exactly \( b_1b_5, b_2b_6, \) and \( b_3b_7 \).
• \( \{b_5, b_6, b_7\} \cap \{b_1, b_2, b_3, b_4\} = \emptyset \), but nodes \( b_5, b_6, \) and \( b_7 \) need not be distinct.
• An induced tree \( I' \) of \( G \setminus ((N_G[b_1] \cup \ldots \cup N_G[b_4]) \setminus \{b_5, b_6, b_7\}) \) contains \( \{b_5, b_6, b_7\} \).

The claim can be verified by seeing that if \( I'' \) is the minimal subtree of \( I' \) that contains \( \{b_5, b_6, b_7\} \), then \( I = I'' \cup \{b_1, b_2, b_6, b_3b_7\} \) is a tree of \( G \setminus \{b_4\} \) with leaf set \( \{b_1, b_2, b_3\} \) having the property that \( I \setminus \{b_1, b_2, b_3\} \) is an induced tree of \( G \) not adjacent to \( b_4 \). By the claim and Lemma 3.1 it takes \( O(m^3n^5) \) time to determine whether \( G \) has beetles. It takes \( O(n^4) \) time to determine whether \( G \) has 4-holes. If \( G \) has 4-holes or beetles, \( G \) has even holes. The lemma is proved by completing Task 1 in \( O(m^3n^5) \) time. The rest of the proof assumes that \( G \) is 4-hole-free and beetle-free.

By Lemma 3.2 it takes \( O(mn^2) \) time to either ensure that \( G \) has even holes or obtain the \( O(n^2) \) maximal cliques of \( G \). If \( G \) has even holes, the lemma is proved by completing Task 1 in \( O(mn^2) \) time. Otherwise, let \( T \) consist of the trackers of \( G \) that are in the form of \( (G \setminus S_1, u_1u_2u_3) \) or \( (G \setminus S_2, u_1u_2u_3) \) with

\[
S_1 = S_1(u_1, u_2, u_3, v_1, v_2) = (N_G(v_1) \cap N_G(v_2)) \cup (N_G(u_3) \setminus \{u_1, u_3\}); \\
S_2 = S_2(u_1, u_2, K) = (N_G(u_1) \cap N_G(u_2)) \cup V(K),
\]

where \( u_1u_2 \) and \( v_1v_2 \) are edges of \( G \) and \( K \) is a maximal clique of \( G \). We have \( |T| = O(m^2n) + O(mn^2) = O(mn^2) \). Since all \( O(n^2) \) maximal cliques of \( G \) are available, \( T \) can be computed in time \( O(m^2n) \cdot O(n + m) = O(m^3n) \) time. To ensure the completion of Task 2, it remains to show that \( T \) satisfies Property 1. Suppose that \( G \) has even holes. Let \( C \) be an arbitrary shortest even hole of \( G \). The following case analysis shows that \( T \) contains lucky trackers of \( G \).

**Case 1:** \( M_G(C) \subseteq N_G(u_2) \) holds for a node \( u_2 \) of \( C \). Let \( u_1 \) and \( u_3 \) be the neighbors of \( u_2 \) in \( C \). By \( M_G(C) \subseteq N_G(u_2) \setminus \{u_1, u_3\} \) and Lemma 3.3 there is an edge \( v_1v_2 \) of \( C \) with \( M_G(C) \cup N_G^2(C) \subseteq S_1 \). By the choices of \( u_1 \) and \( u_3 \), we have \( (N_G(u_2) \setminus \{u_1, u_3\}) \cap C = \emptyset \). Since \( v_1v_2 \) is an edge of hole \( C \), we have \( N_G(v_1) \cap N_G(v_2) \cap C = \emptyset \). Thus, \( S_1 \cap C = \emptyset \), implying that \( C \) is a clean hole of \( G \setminus S_1 \) containing path \( u_1u_2u_3 \). Since \( C \) is a shortest even hole of \( G \), \( C \) is also a shortest even hole of \( G \setminus S_1 \). Therefore, \( C \) is a \( u_1u_2u_3 \)-hole of \( G \setminus S_1 \).

**Case 2:** \( M_G(C) \not\subseteq N_G(u) \) holds for all nodes \( u \) of \( C \). By Lemma 3.4 \( G[M_G(C)] \) is a clique of \( G \). Let \( K \) be a maximal clique of \( G \) with \( M_G(C) \subseteq V(K) \). Combining with Lemma 3.3 there is an edge \( u_1u_2 \) of \( C \) with \( M_G(C) \cup N_G^2(C) \subseteq S_2 \). We have \( V(K) \cap C = \emptyset \) or else \( M_G(C) \cap C = \emptyset \) implies \( M_G(C) \subseteq V(K) \setminus \{u\} \subseteq N_G(u) \) for any node \( u \in V(K) \cap C \), a contradiction. Since \( u_1u_2 \) is an edge of \( C \), we have \( N_G(u_1) \cap N_G(u_2) \cap C = \emptyset \). Thus, \( S_2 \cap C = \emptyset \), implying that \( C \) is a clean hole of \( G \setminus S_2 \) containing path \( u_1u_2u_3 \), where \( u_3 \) is the neighbor of \( u_2 \) other than \( u_1 \). Since \( C \) is a shortest even hole of \( G \), \( C \) is also a shortest even hole of \( G \setminus S_2 \). Therefore, \( C \) is a \( u_1u_2u_3 \)-hole of \( G \setminus S_2 \).

The rest of the section proves Lemma 3.4. An edge \( u_1u_2 \) of hole \( C \) is a gate of \( C \) with respect to major nodes \( x_1 \) and \( x_2 \) of \( C \) if the following conditions hold:

**Condition G1:** There are two edges \( u_1x_2 \) and \( u_2x_1 \) and at least one of edges \( u_1x_1 \) and \( u_2x_2 \).

**Condition G2:** There is a node \( u_0 \) of \( C \setminus \{u_1, u_2\} \) such that \( x_1 \) (respectively, \( x_2 \)) is not adjacent to \( C \setminus \{P_1\} \) (respectively, \( C \setminus \{P_2\} \)), where \( P_1 \) (respectively, \( P_2 \)) is the path of \( C \) between \( u_2 \) (respectively, \( u_1 \)) and \( u_0 \) that contains \( u_1 \) (respectively, \( u_2 \)).

See Figure 3 for an illustration.
follows from Lemma 3.5(2). If both of $x$ node $x$ which contradicts with $G$ trivially if $|G| = 1$. Assume $u$ one of $x$ Figure 3(b), then Condition $G_2$ implies $N_u$ nodes $u$ of $C$ even hole 4-hole-free graph $C$. (a) (b) (c) Illustrations for the proof of Lemma 3.4.

Figure 3: (a) Edge $u_1u_2$ is a gate of the 8-hole $C$ induced by nodes other than $x_1$ and $x_2$, which are the major nodes of $C$. (b) and (c) Illustrations for the proof of Lemma 3.4.

Lemma 3.5 (Chudnovsky et al. [6] Lemmas 2.3 and 2.4]). The following statements hold for any shortest even hole $C$ of a 4-hole-free graph $G$.

1. If $x_1$ and $x_2$ are non-adjacent nodes of $M_G(C)$, there is a gate of $C$ with respect to $x_1$ and $x_2$ in $G$.
2. If $X$ is a subset of $M_G(C)$ with $|X| = 3$ such that $G[X]$ has at most one edge, $X \subseteq N_G(u)$ holds for some node $u$ of $C$.

Proof of Lemma 3.5: Let $x_1$ and $x_2$ be two non-adjacent nodes of $M_G(C)$. Let $U$ consist of the nodes $u$ of $C$ that are adjacent to both of $x_1$ and $x_2$. By Lemma 3.5[1], there is a gate $u_1u_2$ of $C$ with respect to $x_1$ and $x_2$. We have $\emptyset \neq U \subseteq \{u_1, u_2, u_0\}$, where $u_0$ is a node of $C$ ensured by Condition $G_2$. Assume $u_0 \in U$ for contradiction. By Condition $G_1$, $u_0$ is adjacent $u_1$ or $u_2$ in $G$ or else one of $u_1x_1u_0x_2u_1$ and $u_2x_1u_0x_2u_2$ would be a 4-hole of $G$. If $u_0$ is adjacent to $u_1$ as illustrated by Figure 3(b), then Condition $G_2$ implies $N_C(x_1) = \{u_0, u_1, u_2\}$, which contradicts with $x_1 \in M_G(C)$. If $u_0$ is adjacent to $u_2$ as illustrated by Figure 3(c), then Condition $G_2$ implies $N_G(x_2) = \{u_0, u_1, u_2\}$, which contradicts with $x_2 \in M_G(C)$. We have $u_0 \notin U$, and thus $U \subseteq \{u_1, u_2\}$. The lemma holds trivially if $|M_G(C)| = 2$. To prove the lemma for $|M_G(C)| \geq 3$, we first show the claim: “Each node $x \in M_G(C) \setminus \{x_1, x_2\}$ is adjacent to $U$. If one of $x_1$ and $x_2$ is not adjacent to $x$, the claim follows from Lemma 3.5[2]. If both of $x_1$ and $x_2$ are adjacent to $x$, each node $u \in U$ is adjacent to $x$ in $G$ or else $ux_1x_2u$ is a 4-hole, a contradiction. The claim is proved.

By the above claim, the lemma holds if $|M_G(C)| = 3$ or $|U| = 1$. It remains to consider the cases with $|M_G(C)| \geq 4$ and $U = \{u_1, u_2\}$ (thus, there are edges $u_1x_1$ and $u_2x_2$) by showing that either $u_1$
or \(u_2\) is adjacent to each node \(x \in M_G(C)\). Assume \(x_3 \in M_G(C) \setminus N_G(u_2)\) and \(x_4 \in M_G(C) \setminus N_G(u_1)\) for contradiction. By the above claim, \(G\) has edges \(u_1x_3\) and \(u_2x_4\). We know \(x_3 \notin N_G(x_4)\) or else \(u_1x_2u_4x_3u_1\) is a 4-hole. See Figure 4(a). Observe that \(x_4\) cannot be adjacent to both of \(x_1\) and \(x_2\) or else \(u_1x_1x_4x_2u_4\) is a 4-hole. Case 1: \(x_4\) is not adjacent to \(x_2\). By Lemma 3.5(2), a node \(u_3\) of \(C\) is adjacent to all of \(x_2, x_3,\) and \(x_4\). Since \(u_3\) is adjacent to both of \(x_3\) and \(x_4\), we have \(u_3 \notin \{u_1, u_2\}\). See Figure 4(b). If \(C\) has edge \(u_3u_4, u_1x_3u_3u_2u_1\) is a 4-hole; otherwise, \(u_2x_2u_3x_4u_2\) is a 4-hole, a contradiction. Case 2: \(x_4\) is not adjacent to \(x_1\). By Lemma 3.5(2), a node \(u_3\) of \(C\) is adjacent to all of \(x_1, x_3,\) and \(x_4\). Since \(u_3\) is adjacent to both of \(x_3\) and \(x_4\), we have \(u_3 \notin \{u_1, u_2\}\). See Figure 4(c). If \(C\) has edge \(u_2u_3, u_1x_3u_3u_2u_1\) is a 4-hole; otherwise, \(u_2x_1u_3x_4u_2\) is a 4-hole, a contradiction. The lemma is proved.

\[\square\]

## 4 Proving Lemma 2.4

Subset \(S\) of \(V(H)\) is a star-cutset \(^{10}\) of graph \(H\) if \(S \subseteq N_H[s]\) holds for some node \(s\) of \(S\) and the number of connected components of \(H \setminus S\) is more than that of \(H\).

**Lemma 4.1.** For any tracker \(T = (H, u_1u_2u_3)\) of an \(n\)-node \(m\)-edge beetle-free connected graph \(G\), it takes \(O(mn^3)\) time to complete one of the following three tasks. Task 1: Ensuring that \(H\) has even holes. Task 2: Ensuring that \(T\) is not lucky. Task 3: Obtaining a star-cutset-free induced subgraph \(H'\) of \(H\) such that if \(T\) is lucky, \(H'\) has even holes.

**Proof of Lemma 2.4** We apply Lemma 4.1 on the input tracker \(T = (H, u_1u_2u_3)\) of \(G\) in \(O(mn^3)\) time. If Task 1 or 2 is completed, the lemma is proved. If Task 3 is completed, since \(H'\) is star-cutset-free, Lemma 4.2 implies that it takes \(O(mn^4)\) time to determine whether \(H'\) has even holes. Since \(H'\) is an induced subgraph of \(H\), if \(H'\) has even holes, do so \(H\); otherwise, \(T\) is not lucky. The lemma is proved. \[\square\]

Subsection 4.1 proves Lemma 4.1. Subsection 4.2 proves Lemma 4.2.

## 4.1 Proving Lemma 4.1

A star-cutset \(S\) of graph \(H\) is full if \(S = N_H[s]\) holds for some node \(s\) of \(S\). No polynomial-time algorithms are known for detecting star-cutsets (see, e.g., \(^{15}\)), but full star-cutsets in an \(n\)-node \(m\)-edge graph can be detected in \(O(mn)\) time. Node \(x\) dominates node \(y\) in graph \(H\) if \(x \neq y\) and \(N_H[y] \subseteq N_H[x]\). Node \(y\) is dominated in \(H\) if some node of \(H\) dominates \(y\) in \(H\). We need the following three lemmas to prove Lemma 4.1.

**Lemma 4.3** (Chvátal \(^{10}\) Theorem 1). A graph without dominated nodes and full star-cutsets has no star-cutsets.

**Lemma 4.4.** If \(T = (H, u_1u_2u_3)\) is a tracker of an \(n\)-node \(m\)-edge beetle-free connected graph \(G\), it takes \(O(mn^2)\) time to obtain a tracker \(T' = (H', u'_1u'_2u'_3)\) of \(G\), where \(H'\) is a dominated-node-free induced subgraph of \(H\), such that if \(T\) is lucky, so is \(T'\).
Lemma 4.5. If \((H, u_1 u_2 u_3)\) is a lucky tracker of graph \(G\) and \(S\) is a full star-cutset of \(H\), one of the following two conditions holds.

Proof. We first prove the following claim for any beetle-free graph \(H\): “If a node \(x\) of \(H\) dominates a node \(y\) of a clean shortest even hole \(C\) of \(H\), then \(C' = H[C \cup \{x\} \setminus \{y\}]\) is a clean shortest even hole of \(H\).” Let \(u\) and \(v\) be the neighbors of \(y\) on \(C\). Since \(C\) is a hole and \(y \in C\), we know \(x \notin C\), implying \(x \in N_H(C)\). Since \(x\) dominates \(y\) and \(|N_C(y)| = 3\), there is a connected component of \(C[N_C(x)]\) with at least 3 nodes. By Lemma 2.2, we have \(x \in N_H^2(C)\), implying \(N_C(x) = \{u, y, v\}\).

Thus, \(C' = C\) is a shortest even hole of \(H\). Assume \(z \in M_H(C') \cup N_H^2(C')\) for contradiction. By \(y \in N_H^3(C')\), \(z \neq y\). By \(C \setminus \{y\} = C' \setminus \{x\}\), exactly one of \(x\) and \(y\) is adjacent to \(z\) in \(H\) or else \(z \in M_H(C) \cup N_H^2(C)\), contradicting to the fact that \(C\) is clean. Case 1: \(z \in N_H^2(C')\). If \(z \in N_H(y) \setminus N_H(x)\), we have \(z \in M_H(C)\), contradicting to the assumption that \(C\) is a clean hole of \(H\). If \(z \in N_H(x) \setminus N_H(y)\), we have \(z \in N_H^1(C)\), contradicting to Lemma 2.2. Case 2: \(z \in M_H(C')\).

By \(|N_{C'}(z)| \geq 3\) and Lemma 2.1, \(|N_{C'}(z)| \geq 4\). By \(M_H(C) = N_H^2(C) = \emptyset\) and Lemma 2.2, \(|N_C(z)| \leq 3\). By \(C \setminus \{x\} = C \setminus \{y\}\), we have \(z \in N_H(x) \setminus N_H(y)\), \(|N_C(z)| = 3\), and \(|N_{C'}(z)| = 4\). By Lemma 2.2, \(z \in N_H^3(C)\). See Figure 5 for an illustration. Thus, \(C[N_C(z)]\) is a 3-path, implying that \(H[C' \cup \{z\}]\) is a beetle \(B\) of \(H\) in which \(B[N_B(z)]\) is a diamond, a contradiction. The claim is proved.

The algorithm first iteratively updates \((H, u_1 u_2 u_3)\) by the following steps until \(H\) has no dominated nodes, and then outputs the resulting \((H, u_1 u_2 u_3)\) as \((H', u'_1 u'_2 u'_3)\).

Step 1: Let \(x\) and \(y\) be two nodes of \(H\) such that \(x\) dominates \(y\).
Step 2: If there is an \(i \in \{1, 2, 3\}\) with \(y = u_i\), then let \(u = x\).
Step 3: Let \(H = H \setminus \{y\}\).

It takes \(O(mn)\) time to detect nodes \(x\) and \(y\) such that \(x\) dominates \(y\). Each iteration of the loop decreases \(|V(H)|\) by one via Step 3. Therefore, the overall running time is \(O(mn^2)\). Graph \(H'\) is a dominated-node-free induced subgraph of the initial \(H\). It suffices to ensure that if the tracker \(T = (H, u_1 u_2 u_3)\) of \(G\) at the beginning of an iteration is lucky, the tracker at the end of the iteration, denoted \(T' = (H', u'_1 u'_2 u'_3)\), remains lucky. Let \(C\) be a \(u_1 u_2 u_3\)-hole of \(H\). If \(y \notin C\), \(C\) is a \(u'_1 u'_2 u'_3\)-hole of \(H' = H \setminus \{y\}\). If \(y \in C\), the above claim ensures that \(C' = H[C \cup \{x\} \setminus \{y\}]\) is a clean shortest even hole of \(H\). Since \(x\) dominates \(y\), hole \(C'\) contains path \(u'_1 u'_2 u'_3\). Thus, \(C' = u'_1 u'_2 u'_3\)-hole of \(H'\). Either way, \((H', u'_1 u'_2 u'_3)\) is lucky. The lemma is proved. 

Figure 5: An illustration for the proof of Lemma 4.4.
Condition B1: For each $u_1u_2u_3$-hole $C$ of $H$, there exists a connected component $B$ of $H \setminus S$ satisfying $C \subseteq H[B \cup S]$.

Condition B2: There are two non-adjacent nodes $s_1$ and $s_2$ of $S$ and two connected components $B_1$ and $B_2$ of $H \setminus S$ with $\{s_1, s_2\} \subseteq N_H(B_1)$ and $\{s_1, s_2\} \subseteq N_H(B_2)$.

Proof. Let $s$ be a node of $S$ with $N_H[s] = S$. Let $C$ be a $u_1u_2u_3$-hole of $H$. Assume that Condition B1 does not hold. There exist two distinct connected components $B_1$ and $B_2$ of $H \setminus S$ such that $V(C) \cap V(B_1) \neq \emptyset$ and $V(C) \cap V(B_2) \neq \emptyset$. Thus, $C[S]$ has at least two connected components. Let $s_1$ and $s_2$ be two nodes in distinct connected components of $C[S]$. By $\{s_1, s_2\} \subseteq N_H[s]$, we have $s \notin C$ or else $s, s_1$, and $s_2$ are in the same connected component of $C[S]$. By Lemma 2.2 we have $s \in N_{H}^{1,1}(C)$, implying $\{s_1, s_2\} = V(C) \cap S$. It follows that both $s_1$ and $s_2$ are adjacent to both $B_1$ and $B_2$. Let paths $P_1$ and $P_2$ be the two connected components of $C \setminus \{s_1, s_2\}$. $B_1$ has to contain one of $P_1$ and $P_2$ and $B_2$ has to contain other one of $P_1$ and $P_2$. Therefore, Condition B2 holds. The lemma is proved.

Proof of Lemma 4.7 Let $T_0$ be the initial given tracker $(H, u_1u_2u_3)$ of $G$. The algorithm iteratively updates $(H, u_1u_2u_3)$ by the following three steps until Task 1, 2, or 3 is completed.

Step 1: Apply Lemma 4.4 in $O(mn^2)$ time on tracker $T = (H, u_1u_2u_3)$ to obtain a tracker $T' = (H', u'_1u'_2u'_3)$ of $G$, where $H'$ is a dominated-node-free induced subgraph of $H$, such that if $T$ is lucky, so is $T'$. Determine in $O(mn)$ time whether $H'$ has full star-cutsets. If $H'$ has full star-cutsets, then let $(H, u_1u_2u_3) = (H', u'_1u'_2u'_3)$ and proceed to Step 2. Otherwise, complete Task 3 by outputting $H'$. 

Step 2: Let $S$ be a full star-cutset of $H$. If Condition B2 of Lemma 4.5 holds, then complete Task 1 by outputting that $G$ has even holes. Otherwise, proceed to Step 3.

Step 3: If either one of the following statements hold for $U = \{u_1, u_2, u_3\}$:

- $U \subseteq S$ and a connected component $B$ of $H \setminus S$ is adjacent to both $u_1$ and $u_3$;
- $U \not\subseteq S$ and $U \subseteq B \cup S$ holds for a connected component $B$ of $H \setminus S$,

then let $H = H[B \cup S]$ and proceed to the next iteration of the loop. Otherwise, complete Task 2 by outputting that $T_0$ is not lucky.

Step 1 does not increase $|V(H)|$. If Step 3 updates $H$, then $|V(H)|$ is decreased by at least one, since $H \setminus S$ has more than one connected component. The algorithm halts in $O(n)$ iterations. Step 1 takes $O(mn^2)$ time. Step 2 takes $O(mn^2)$ time: For any two non-adjacent nodes $s_1$ and $s_2$ in $S$, it takes $O(m)$ time to determine whether $s_1$ and $s_2$ have two or more common neighboring connected components of $H \setminus S$. Step 3 takes $O(m)$ time. The overall running time is $O(mn^3)$.

We first show the following claim for each iteration of the algorithm: “If the $(H, u_1u_2u_3)$ at the beginning of an iteration is a lucky tracker of $G$, then (1) the intermediate $(H, u_1u_2u_3)$ throughout the iteration remains a lucky tracker of $G$, and (2) Step 3 if reached, proceeds to the next iteration.” It suffices to consider the situation that Step 3 is reached and focus on the update operation that replaces $H$ with $H[B \cup S]$ via Step 3. By definition of Step 2 Condition B2 does not hold. By Lemma 4.5 Condition B1 holds. That is, there is a connected component $B^*$ of $H \setminus S$ such that $H[B^* \cup S]$ contains some $u_1u_2u_3$-hole $C$ of $H$. We prove the claim by showing that $B^*$ has to be the connected component $B$ of $H \setminus S$ in Step 3. Since $B^* = B$ holds trivially for the case $\{u_1, u_2, u_3\} \not\subseteq S$, we assume $\{u_1, u_2, u_3\} \subseteq S$. If $s \in C$, there are exactly two nodes of $C$ that are adjacent to $s$ in $H$;
for every four nodes \( u \) takes \( O(1) \) time. An even hole if and only if there are two non-adjacent neighbors \( is even and \( \text{P}_{H} \) \) whether \( \text{in} \) \( \text{H} \). By \( V(C) \subseteq \text{B}^{*} \cup S \) and \( |V(C) \cap S| \leq 3 \), we have \( (N_{C}(u_{1}) \cup N_{C}(u_{3})) \setminus \{u_{2}\} \subseteq \text{B}^{*} \), implying \( \text{B}^{*} = \text{B} \). The claim is proved.

For the correctness of the algorithm, we consider the three possible steps via which the algorithm halts. Step 1: Since \( \text{H}^{'} \) is dominated-node-free and full-star-cutset-free, Lemma 4.3 implies that \( \text{H}^{'} \) has no star-cutsets. By the above claim, Task 3 is completed. Step 2: Condition B2 holds. Let \( \text{P}_{1} \) be a shortest path between \( s_{1} \) and \( s_{2} \) in \( \text{H}[B_{1} \cup \{s_{1}, s_{2}\}] \). Let \( \text{P}_{2} \) be a shortest path between \( s_{1} \) and \( s_{2} \) in \( \text{H}[B_{2} \cup \{s_{1}, s_{2}\}] \). Since \( s_{1} \) and \( s_{2} \) are not adjacent, at least one of the three cycles of graph \( \text{P}_{1} \cup \text{P}_{2} \cup \{ss_{1}, ss_{2}\} \) is an even hole of \( \text{H} \). Since \( \text{H} \) is an induced subgraph of \( \text{G} \), \( \text{G} \) has even holes. See Figure 6 for an illustration. Task 1 is completed. Step 3: By the above claim, if \( \text{T}_{0} \) is lucky, Step 3 always proceeds to the next iteration of the loop. Thus, Task 2 is completed. The lemma is proved.

\[ \square \]

4.2 Proving Lemma 4.2

4.2.1 Extended clique trees

Graph \( \text{H} \) is an extended clique tree \([20]\) if there is a set \( \text{S} \) of two or less nodes of \( \text{H} \) such that each biconnected component of \( \text{H} \setminus \text{S} \) is a clique. da Silva and Vušković \([20, \S 2.3]\) described an \( O(n^{5}) \)-time algorithm to determine whether an \( n \)-node extended clique tree contains even holes, which can actually be implemented to run in \( O(n^{4}) \) time.

Lemma 4.6. It takes \( O(n^{4}) \) time to determine whether an \( n \)-node extended clique tree has even holes.

*Proof.* Let \( \text{H}_{0} \) be the \( n \)-node extended clique tree. Let \( x \) and \( y \) be two nodes of \( \text{H}_{0} \) such that each biconnected component of \( \text{H} = \text{H}_{0} \setminus \{x, y\} \) is a clique. For nodes \( u \) and \( v \) of \( \text{H} \), let \( P(u, v) \) be the shortest path of \( \text{H} \) between \( u \) and \( v \) and let \( p(u, v) \) be the number of edges in \( P(u, v) \). We spend \( O(n^{4}) \) time to store the following information in a table \( M_{1} \) for every two nodes \( u \) and \( v \) of \( \text{H} \): (i) \( p(u, v) \) and (ii) whether or not \( P(u, v) \setminus \{u, v\} \) is adjacent to \( x \) (respectively, \( y \)). With \( M_{1} \), it takes \( O(n^{2}) \) time to determine whether \( \text{H}_{0} \) has an even hole that contains \( y \) but not \( x \): \( \text{H}_{0} \setminus \{x\} \) has an even hole if and only if there are two non-adjacent neighbors \( u \) and \( v \) of \( y \) in \( \text{H} \) such that \( p(u, v) \) is even and \( P(u, v) \setminus \{u, v\} \) is not adjacent to \( y \). Similarly, with \( M_{1} \), it takes \( O(n^{2}) \) time to determine whether \( \text{H}_{0} \) has an even hole that contains \( x \) but not \( y \).

To determine whether \( \text{H}_{0} \) has an even hole containing both \( x \) and \( y \), we store in a table \( M_{2} \) for every four nodes \( u_{1}, v_{1}, u_{2}, v_{2} \) whether or not \( P(u_{1}, v_{1}) \) and \( P(u_{2}, v_{2}) \) are both disjoint and
non-adjacent. It takes $O(n^2)$ time to compute the connected components of $H \setminus N_H[P(u_1, v_1)]$. Paths $P(u_1, v_1)$ and $P(u_2, v_2)$ are both disjoint and non-adjacent if and only if $u_2$ and $v_2$ are in the same connected component of $H \setminus N_H[P(u_1, v_1)]$. Therefore, $M_2$ can also be computed in $O(n^4)$ time. With tables $M_1$ and $M_2$, it takes $O(n^4)$ time to determine whether $H_0$ has an even hole containing both $x$ and $y$: Case 1: $x$ and $y$ are adjacent in $H_0$. $H_0$ has an even hole containing both $x$ and $y$ if and only if there are nodes $u$ and $v$ such that (1) $H_0[\{u, x, y, v\}]$ is path $uxyv$, (2) $p(u, v)$ is odd, and (3) $P(u, v) \setminus \{u, v\}$ is not adjacent to $\{x, y\}$. Case 2: $x$ and $y$ are not adjacent in $H_0$. $H_0$ has an even hole containing both $x$ and $y$ if and only if there are nodes $u_x, v_x, u_y, v_y$ of $H$ such that (1) $H_0[\{u_x, x, v_x\}]$ is path $u_xxv_x$ and $H_0[\{u_y, y, v_y\}]$ is path $u_yyv_y$, (2) $p(u_x, u_y) + p(v_x, v_y)$ is even, and (3) $P(u_x, u_y)$ and $P(v_x, v_y)$ are both disjoint and non-adjacent. The lemma is proved.

4.2.2 2-joins and non-path 2-joins

We say that $V_1 \mid V_2$ is a 2-join \cite{18, 40} of a graph $H$ with split $(X_1, Y_1, X_2, Y_2)$ if (1) $V_1$ and $V_2$ form a disjoint partition of $V(H)$ with $|V_1| \geq 3$ and $|V_2| \geq 3$, (2) $X_1$ and $Y_1$ (respectively, $X_2$ and $Y_2$) are disjoint non-empty subsets of $V_1$ (respectively, $V_2$), and (3) each node of $X_1$ is adjacent to each node of $X_2$, each node of $Y_1$ is adjacent to each node of $Y_2$, and there are no other edges between $V_1$ and $V_2$. See Figure 7(a) for an example.

Lemma 4.7 \cite{40} Lemma 3.2). If $V_1 \mid V_2$ is a 2-join of a star-cutset-free graph $H$ with split $(X_1, Y_1, X_2, Y_2)$, then the following statements hold for each $i \in \{1, 2\}$.

1. Each connected component of $H[V_i]$ contains at least one node in $X_i$ and at least one node in $Y_i$.
2. Each node of $V_i$ has a neighbor in $V_i$.
3. Each node of $X_i$ has a non-neighbor in $Y_i$. Each node of $Y_i$ has a non-neighbor in $X_i$.
4. $|V_i| \geq 4$.

A 2-join $V_1 \mid V_2$ of $H$ with split $(X_1, Y_1, X_2, Y_2)$ is a non-path 2-join \cite{18} of $H$ if $H[V_1]$ is not a path between a node of $X_1$ and a node of $Y_1$ and $H[V_2]$ is not a path between a node of $X_2$ and a node of $Y_2$. For instance, the 2-join in Figure 7(a) is a non-path 2-join. (Non-path 2-joins are called 2-joins by da Silva and Vušković \cite{20} §1.3.)
Lemma 4.8 (Charbit et al. [41, Theorem 4.1]). Given an \( n \)-node connected graph \( H \), it takes \( O(n^4) \) time to either output a non-path 2-join of \( H \) together with a split or ensure that \( H \) has no non-path 2-joins.

Lemma 4.9 (da Silva and Vušković [20, Corollary 1.3]). A connected graph that is even-hole-free, star-cutset-free, and non-path-2-join-free is an extended clique tree.

Combining Lemmas 4.6, 4.8 and 4.9 we have the following lemma.

Lemma 4.10. Given an \( n \)-node star-cutset-free graph \( H \), it takes \( O(n^4) \) time to either (a) determine whether \( H \) has even holes or (b) obtain a non-path 2-join of \( H \) with a split.

Proof. It takes \( O(n^4) \) time to determine whether the graph \( H \) is an extended clique tree: For any set \( S \) of two or less nodes of \( H \), it takes \( O(n^2) \) time to obtain the biconnected components of subgraph \( H \setminus S \) [23] and determine whether all of them are cliques. If \( H \) is an extended clique tree, Lemma 4.6 implies that it takes \( O(n^4) \) time to determine whether \( H \) has even holes. If \( H \) is not an extended clique tree, Lemma 4.8 implies that it takes \( O(n^4) \) time to either obtain a non-path 2-join of \( H \) with a split or ensure that \( H \) has no non-path 2-joins. If \( H \) has no non-path 2-joins, Lemma 4.9 implies that \( H \) has even holes. \( \square \)

4.2.3 Parity-preserving blocks of decomposition for connected 2-joins

A 2-join \( V_1|V_2 \) with split \( (X_1, Y_1, X_2, Y_2) \) is connected [40] if, for each \( i \in \{1, 2\} \), there is an induced path \( P_i \) of \( H[V_i] \) between a node \( x_i \) of \( X_i \) and a node \( y_i \) of \( Y_i \) such that \( V(P_i) \setminus \{x_i, y_i\} \subseteq V_i \setminus (X_i \cup Y_i) \). For instance, the 2-join \( V_1|V_2 \) in Figure 7(a) is connected. By Lemma 4.7(1), any 2-join of a star-cutset-free graph is connected with respect to any split.

Let \( V_1|V_2 \) be a connected 2-join of graph \( H \) with split \( (X_1, Y_1, X_2, Y_2) \). For each \( i \in \{1, 2\} \), let \( P_i \) be a shortest induced path \( P_i \) of \( H[V_i] \) between a node \( x_i \) of \( X_i \) and a node \( y_i \) of \( Y_i \) with \( V(P_i) \setminus \{x_i, y_i\} \subseteq V_i \setminus (X_i \cup Y_i) \). If \( |V(P_i)| \) is even (respectively, odd), then let \( p_i = 4 \) (respectively, \( p_i = 5 \)). The parity-preserving blocks of decomposition [40] of \( H \) for 2-join \( V_1|V_2 \) with respect to split \( (X_1, Y_1, X_2, Y_2) \) are the following graphs \( H_1 \) and \( H_2 \).

- \( H_1 \) consists of (a) \( H[V_1] \), (b) a \( p_2 \)-path between nodes \( x_2 \) and \( y_2 \), (c) edges \( x_2x \) for all nodes \( x \) of \( X_1 \), and (d) edges \( y_2y \) for all nodes \( y \) of \( Y_1 \).
- \( H_2 \) consists of (a) \( H[V_2] \), (b) a \( p_1 \)-path between nodes \( x_1 \) and \( y_1 \), (c) edges \( x_1x \) for all nodes \( x \) of \( X_2 \), and (d) edges \( y_1y \) for all nodes \( y \) of \( Y_2 \).

See Figure 7(b) for an example of \( H_1 \) and \( H_2 \).

Lemma 4.11 (Troignon and Vušković [40, Lemma 3.8]). If \( V_1|V_2 \) is a connected 2-join of a star-cutset-free graph \( H \) with split \( (X_1, Y_1, X_2, Y_2) \), the parity-preserving blocks of decomposition \( H_1 \) and \( H_2 \) of \( H \) for \( V_1|V_2 \) with respect to \( (X_1, Y_1, X_2, Y_2) \) are star-cutset-free graphs such that \( H \) is even-hole-free if and only if both \( H_1 \) and \( H_2 \) are even-hole-free.

Lemma 4.12. Let \( H \) be an \( n \)-node \( m \)-edge star-cutset-free graph. Either one of the parity-preserving blocks \( H_1 \) and \( H_2 \) of decomposition for an arbitrary non-path 2-join of \( H \) with respect to an arbitrary split has at most \( n \) nodes and \( m - 1 \) edges.

Proof. We prove the lemma for \( H_1 \). The proof for \( H_2 \) is similar. Let \( V_1|V_2 \) be the non-path 2-join. Let \( (X_1, Y_1, X_2, Y_2) \) be the split. Let \( P_2 \) be a shortest path of \( H[V_2] \) between a node of \( X_2 \) and a node of \( Y_2 \). For the case that \( |V(P_2)| \) is even, we have \( p_2 = 4 \). By Lemma 4.7(4), \( |V_2| \geq 4 \), implying \( |V(H_1)| = n - |V_2| + p_2 \leq n \). By the following case analysis, \( H_1 \) has at most \( m - 1 \) edges.
\begin{itemize}
  \item $|V(P_2)| \geq 6$: By $P_2 \subseteq H[V_2]$, $H[V_2]$ has at least five edges. Thus, $H_1$ has at most $m - 2$ edges.
  \item $|V(P_2)| = 4$: Since $V_1[V_2]$ is a non-path 2-join of $H$, $P_2 \subseteq H[V_2]$. If $V(P_2) = V_2$, $H[V_2]$ has at least four edges. If $V(P_2) \subset V_2$, Lemma 4.7(2) implies that $H[V_2]$ has at least four edges. Either way, $H_1$ has at most $m - 1$ edges.
  \item $|V(P_2)| = 2$: Lemma 4.7(3) ensures $|X_2| \geq 2$ and $|Y_2| \geq 2$. Lemma 4.7(1) implies that $H[V_2]$ has at least two edges. By $|X_2| \geq 2$ and $|Y_2| \geq 2$, the number of edges between $V_1$ and $V_2$ in $H$ is at least two more than the number of edges between $V_1$ and $V(H_1) \setminus V_1$ in $H_1$. Therefore, $H_1$ has at most $m - 1$ edges.
\end{itemize}

As for the case that $|V(P_2)|$ is odd, we have $p_2 = 5$. The following case analysis shows that $H_1$ has at most $n$ nodes and at most $m - 1$ edges.

\begin{itemize}
  \item $|V(P_2)| \geq 5$: By $|V_2| \geq 5$, we have $|V(H_1)| \leq n$. $P_2$ has at least four edges. Since $V_1[V_2]$ is a non-path 2-join of $H$, $P_2 \subset H[V_2]$. If $V(P_2) = V_2$, then $H[V_2]$ has at least five edges. If $V(P_2) \subset V_2$, then Lemma 4.7(2) implies that $H[V_2]$ has at least five edges. Either way, $H_1$ has at most $m - 1$ edges.
  \item $|V(P_2)| = 3$: By Lemma 4.7(4), the proper subset $Z = V_2 \setminus V(P_2)$ of $V_2$ is non-empty. We know $Z \cap (X_2 \cup Y_2) \neq \emptyset$ or else $V(P_2)$ would be a star-cutset of $H$. Assume $Z \cap X_2 \neq \emptyset$ without loss of generality. Let $B$ be an arbitrary connected component of $H[Z]$ with $B \cap X_2 \neq \emptyset$. We know that $B$ is adjacent to $Y_2$ in $H$ or else $N_H[x] \cap Z$ would be a star-cutset of $H$, where $x$ is the endpoint of $P_2$ in $X_2$. Since $P_2$ is a shortest path between a node of $X_2$ and a node of $Y_2$, at least one node of $B$ is not in $X_2 \cup Y_2$. Therefore, $|V_2| \geq 5$, implying $|V(H_1)| \leq n$. Moreover, $H[V_2]$ has at least four edges. By $|X_2| \geq 2$, the number of edges between $V_1$ and $V_2$ in $H$ is at least one more than the number of edges between $V_1$ and $V(H_1) \setminus V_1$ in $H_1$. Thus, $H_1$ has at most $m - 1$ edges.
\end{itemize}

The lemma is proved.

4.2.4 Proving Lemma 4.2

We now prove Lemma 4.2 by Lemmas 4.10, 4.11 and 4.12.

Proof of Lemma 4.2. Assume without loss of generality that the given $n$-node $m$-edge star-cutset-free graph $H_0$ is connected. Let set $\mathcal{H}$ initially consist of a single graph $H_0$. We then repeat the following loop until $\mathcal{H} = \emptyset$ or we output that $H_0$ has even holes. Let $H$ be a graph in $\mathcal{H}$. Case 1: $H$ has at most 11 edges. It takes $O(1)$ time to determine whether $H$ has even holes. If $H$ has even holes, we output that $H_0$ has even holes. Otherwise, we delete $H$ from $\mathcal{H}$. Case 2: $H$ has at least 12 edges. We first delete $H$ from $\mathcal{H}$ and then apply Lemma 4.10 on $H$. If $H$ has even holes, we output that $H_0$ has even holes. If we obtain a non-path 2-join $V_1[V_2]$ of $H$ with split $(X_1, Y_1, X_2, Y_2)$, we add to $\mathcal{H}$ the parity-preserving blocks $H_1$ and $H_2$ of decomposition for $V_1[V_2]$ with respect to $(X_1, Y_1, X_2, Y_2)$. If the above loop stops with $\mathcal{H} = \emptyset$, we output that $H_0$ has no even holes.

The correctness of our algorithm follows immediately from Lemma 4.11. By Lemma 4.12 each graph ever in $\mathcal{H}$ throughout our algorithm has at most $n$ nodes. By Lemma 4.10 each iteration of the loop takes $O(n^4)$ time. It remains to show that the loop halts in $O(m)$ iterations. Observe that each iteration increases the overall number of edges of the graphs in $\mathcal{H}$ by no more than 10.
Let \( f(m) \) be the maximum number of iterations of the above loop in which Lemma 4.10 is applied. Lemma 4.12 implies

\[
f(m) \leq \begin{cases} 
0 & \text{if } m \leq 11 \\
\max\{1 + f(m_1) + f(m_2) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\} & \text{if } m \geq 12.
\end{cases}
\]

By induction on \( m \), we show \( f(m) \leq \max(m - 11, 0) \), which clearly holds for \( m = 1, 2, \ldots, 11 \). If \( m \geq 12 \), the inductive hypothesis implies

\[
f(m) \leq \max\{1 + \max(m_1 - 11, 0) + \max(m_2 - 11, 0) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\}
\leq \max\{\max(m_1 + m_2 - 21, m_1 - 10, m_2 - 10, 1) \mid m_1, m_2 \leq m - 1, m_1 + m_2 \leq m + 10\}
\leq \max(m - 11, m - 11, m - 11, 1)
= \max(m - 11, 0).
\]

By \( f(m) = O(m) \), the number of iterations of the above loop is \( O(m) \). The lemma is proved. \( \square \)

5 Concluding remarks

For any class \( \mathcal{G} \) of induced subgraphs, one can augment a recognition algorithm for \( \mathcal{G} \)-free graphs into a \( \mathcal{G} \)-detection algorithm for an \( n \)-node graph \( G \) with a factor-\( O(n) \) increase in the time complexity by a node-deletion method: (1) Let \( H = G \). (2) For each node \( v \) of \( G \), if \( H \setminus \{v\} \) is not \( \mathcal{G} \)-free, then let \( H = H \setminus \{v\} \). (3) Output the resulting graph \( H \). See, e.g., [41, §] for the case that \( \mathcal{G} \) consists of even holes. Thus, Theorem 1.1 immediately yields a detection algorithm that runs in time \( O(m^3n^6) = O(n^{12}) \). However, our \( O(m^3n^5) \)-time recognition algorithm can be augmented into an even-hole-detection algorithm without increasing the time complexity.

The combination of the proofs of Theorem 1.1 and Lemma 2.3 actually gives two algorithms. The first algorithm determines if \( G \) is both beetle-free and 4-hole-free. The second algorithm determines if a beetle-free and 4-hole-free graph \( G \) is also even-hole-free. We first describe how to augment the first algorithm into an \( O(m^3n^5) \)-time detection algorithm. Since it takes \( O(n^4) \) time to detect a 4-hole in \( G \), it suffices to show how to detect an even hole in a graph \( G \) with beetles in \( O(n^3n^5) \) time. As stated in the proof of Lemma 2.3 for each of the \( O(m^3n) \) choices of node \( b_4 \) and edges \( b_1b_5, b_2b_6, \) and \( b_3b_7 \), it takes \( O(n^4) \) time via Lemma 3.1 to determine if \( b_4, b_1b_5, b_2b_6, b_3b_7 \) are contained by a beetle \( B \) in which \( \{b_1, b_2, b_3, b_4\} \) induces a diamond. Once we know that there exists a beetle \( B \) containing a particular choice of \( b_4, b_1b_5, b_2b_6, b_3b_7, \) by an augmented version of Lemma 3.1 via the above node-deletion method, it takes \( O(n^5) \) time to actually detect such a beetle \( B \). Therefore, if \( G \) has beetles, it takes \( O(m^3n) \cdot O(n^4) + O(n^5) = O(m^3n^5) \) time to find a beetle of \( G \), in which an even hole of \( G \) can be detected in \( O(n) \) time.

The second algorithm can also be augmented into an \( O(m^3n^5) \)-time detection algorithm for a beetle-free graph \( G \) that has even holes. By Lemma 2.3, we obtain in \( O(m^3n^5) \) time a set \( T \) of \( O(m^2n) \) trackers that satisfies Property II. Since \( G \) has even holes, there must be a tracker \( (H, u_1u_2u_2) \) of \( T \) such that \( H \) contains an even hole of \( G \), which according to Lemma 2.4 can be found in time \( O(m^2n) \cdot O(mn^4) = O(m^3n^5) \). By the proof of Lemma 2.4 \( H \) is ensured to have even holes in two ways. (1) If it is ensured through completing Task 1 of Lemma 4.11 the proof of Lemma 4.11 actually gives a constructive proof for the existence of an even hole of \( H \), which is also an even hole of \( G \). (2) If it is ensured through completing Task 3 of Lemma 4.11 and then by
Lemma 4.2, we have a star-cutset-free induced subgraph $H'$ of $H$ that has even holes. We then apply the above node-deletion method on $H'$ using Lemma 4.2 to detect in $O(mn^5)$ time an even hole of $H'$, which is also an even hole of $H$ and $G$. Therefore, if $G$ is a 4-hole-free and beetle-free graph that has even holes, it takes time $O(m^3n^5) + O(mn^5) = O(m^3n^5)$ to output an even hole of $G$.Combining the above two detection algorithms, we have an $O(m^3n^5)$-time algorithm to output an even hole in an $n$-node $m$-edge graph with even holes.

**Acknowledgement**

We thank Gerard J. Chang for discussion.

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