SUBADJUNCTION THEOREM FOR PLURICANONICAL DIVISORS

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Abstract
We prove a subadjunction theorem which relates the multi-adjoint linear system of the ambient space and the linear system of the restricted bundle on a subvariety. MSC 32H25

1 Introduction
Let $M$ be a complex manifold and $L$ be a line bundle on $M$ and $S$ be a submanifold of $M$. It is a basic question whether the restriction map

$$H^0(M, \mathcal{O}_M(L)) \rightarrow H^0(S, \mathcal{O}_S(L))$$

is surjective.

In this paper we shall consider this question for multi-adjoint type line bundles under certain geometric conditions.

Let us state our result precisely. Let $M$ be a complex manifold of dimension $n$ and let $S$ be a closed complex submanifold of $M$. Then we consider a class of continuous function $\Psi : M \rightarrow [-\infty, 0)$ such that

1. $\Psi^{-1}(-\infty) \supset S$,
2. if $S$ is $k$-dimensional around a point $x$, there exists a local coordinate $(z_1, \ldots, z_n)$ on a neighbourhood of $x$ such that $z_{k+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |\Psi(z) - (n - k) \log \sum_{j=k+1}^n |z_j|^2| < \infty.$$ 

The set of such functions $\Psi$ will be denoted by $\sharp(S)$. 

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For each $\Psi \in \mathcal{Z}(S)$, one can associate a positive measure $dV_M[\Psi]$ on $S$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$\int_{S_k} f d\mu \geq \limsup_{t \to \infty} \frac{2(n-k)}{\sigma_{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} dV_M$$

for any nonnegative continuous function $f$ with $\text{supp } f \subset M$. Here $S_k$ denotes the $k$-dimensional component of $S$, $\sigma_m$ denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$, and $\chi_{R(\Psi,t)}$ denotes the characteristic function of the set

$$R(\Psi,t) = \{ x \in M \mid -t - 1 < \Psi(x) < -t \}.$$

**Theorem 1.1** Let $M$ be a complex manifold with a continuous volume form $dV_M$, let $L$ be a holomorphic line bundle over $M$ with a $C^\infty$-hermitian metric $h_L$, let $S$ be a compact complex submanifold of $M$, let $\Psi : M \to (-\infty,0)$ and let $K_M$ be the canonical bundle of $M$. Suppose that the followings are satisfied.

1. There exists a closed set $X \subset M$ such that
   (a) $X$ is locally negligible with respect to $L^2$-holomorphic functions, i.e.,
       for any local coordinate neighbourhood $U \subset M$ and for any $L^2$-holomorphic function $f$ on $U \setminus X$, there exists a holomorphic function $\tilde{f}$ on $U$ such that $\tilde{f} | U \setminus X = f$.
   (b) $M \setminus X$ is a Stein manifold which intersects with every component of $S$.

2. $\Psi \in \mathcal{Z}(S) \cap C^\infty(M \setminus S)$.

3. $\Theta_{h,e^{-(1+\epsilon)}\Psi} \geq 0$ for every $\epsilon \in [0,\delta]$ for some $\delta > 0$.

4. There is a positive line bundle on $M$.

Then every element of $H^0(S, \mathcal{O}_S(m(K_M+L)))$ extends to an element of $H^0(M, \mathcal{O}_M(m(K_M+L)))$.

One may think that the assumption on the existence of the function $\Psi$ is somewhat technical or restrictive. But as one see in the last section, this is not the case. In fact one may construct such a function by using an effective $\mathbb{Q}$-divisor on $M$. In the last section we shall formulate a variant of Theorem 1.1 which would be more useful but a little bit more complicated to formulate.

This paper is a continuation of [7] and the most of the ideas are transplanted from [6]. The results in this paper may be considered as a generalization of [6] to the case of nontrivial normal bundles. We also note that there exists another type of subadjunction theorem due to Y. Kawamata (\[8\]).

The key point of the proof is the extension of closed positive $(1,1)$-currents by using an algebraic approximation.
2 Preliminaries

2.1 \( L^2 \)-extension theorem

Let \( M \) be a complex manifold and let \( (E, h) \) be a holomorphic hermitian vector bundle over \( M \). Given a positive measure \( d\mu_M \) on \( M \), we shall denote \( A^2(M, E, h, d\mu_M) \) the space of \( L^2 \) holomorphic sections of \( E \) over \( M \) with respect to \( h \) and \( d\mu_M \). Let \( S \) be a closed complex submanifold of \( M \) and let \( d\mu_S \) be a positive measure on \( S \). The measured submanifold \( (S, d\mu_S) \) is said to be a set of interpolation for \( (E, h, d\mu_M) \), or for the space \( A^2(M, E, h, d\mu_M) \), if there exists a bounded linear operator\[ I : A^2(S, E \mid S, h, d\mu_S) \rightarrow A^2(M, E, h, d\mu_M) \]
such that \( I(f) \mid S = f \) for any \( f \). \( I \) is called an interpolation operator. Let \( M \) be a complex manifold and let \( S \) be a closed complex submanifold of \( M \). Let \( (L, h) \) be a singular hermitian line bundle on \( M \). Let \( dV \) be any continuous nowhere degenerate volume form on \( M \). The following theorem is crucial for the proof of Theorem 1.1.

**Theorem 2.1** (from [4], Theorem 4) Let \( M \) be a complex manifold with a continuous volume form \( dV_M \), let \( E \) be a holomorphic vector bundle over \( M \) with \( C^\infty \)-fiber metric \( h \), let \( S \) be a closed complex submanifold of \( M \), let \( \Psi \in \sharp(S) \) and let \( K_M \) be the canonical bundle of \( M \). Then \( (S, dV_M(\Psi)) \) is a set of interpolation for \( (E \otimes K_M, h \otimes (dV_M)^{-1}, dV_M) \), if the followings are satisfied.

1. There exists a closed set \( X \subset M \) such that
   - (a) \( X \) is locally negligible with respect to \( L^2 \)-holomorphic functions, i.e., for any local coordinate neighbourhood \( U \subset M \) and for any \( L^2 \)-holomorphic function \( f \) on \( U \setminus X \), there exists a holomorphic function \( \tilde{f} \) on \( U \) such that \( \tilde{f} \mid U \setminus X = f \).
   - (b) \( M \setminus X \) is a Stein manifold which intersects with every component of \( S \).
2. \( \Theta_h \geq 0 \) in the sense of Nakano,
3. \( \Psi \in \sharp(S) \cap C^\infty(M \setminus S) \),
4. \( e^{-(1+\epsilon)\Psi} \cdot h \) has semipositive curvature in the sense of Nakano for every \( \epsilon \in [0, \delta] \) for some \( \delta > 0 \).

Under these conditions, there exists a constant \( C \) and an interpolation operator from \( A^2(S, E \otimes K_M \mid S, h \otimes (dV_M)^{-1} \mid S, dV_M(\Psi)) \) to \( A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1} \cdot dV_M) \) whose norm does not exceed \( C\delta^{-3/2} \). If \( \Psi \) is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than \( 2^4 \pi^{1/2} \).

2.2 Analytic Zariski decomposition

In this subsection we shall introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like nef and big line bundles.
Definition 2.1 Let $M$ be a compact complex manifold and let $L$ be a holomorphic line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an analytic Zariski decomposition, if the followings hold.

1. $\Theta_h$ is a closed positive current,
2. for every $m \geq 0$, the natural inclusion
   $$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \rightarrow H^0(M, \mathcal{O}_M(mL))$$
   is an isomorphism.

Remark 2.1 If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above.

Theorem 2.2 (\cite[Theorem 1.5]{8}) Let $X$ be a smooth projective variety and let $L$ be a pseudoeffective line bundle on $X$. Then $L$ has an AZD.

In fact an AZD of $L$ can be constructed as follows. Let $E$ be the set of singular hermitian metric on $L$ defined by

$$E = \{ h; h : \text{lowersemicontinuous singular hermitian metric on } L, \Theta_h \text{ is positive}, \frac{h}{h_0} \geq 1 \}.$$

We set

$$h_{\text{min}} = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. Then it is easy to check that $h_{\text{min}}$ is an AZD of $L$.

The AZD $h_{\text{min}}$ constructed as above is said to be a canonical AZD of $L$. Also this construction can be generalized to the case of singular hermitian line bundles whose curvature current is bounded from below by some negative multiple of a Kähler form.

3 Inductive construction of metrics

Let $X$ be a smooth projective variety and let $K_X$ be the canonical line bundle of $X$. Let $h$ be a canonical AZD of $K_X + L$. Let $n$ denote the dimension of $X$.

Let $A$ be a sufficiently ample line bundle on $X$ such that for every pseudoeffective singular hermitian line bundle $(F, h_F)$

$$\mathcal{O}_X(A + F) \otimes \mathcal{I}(h_F)$$

and

$$\mathcal{O}_X(K_X + L + A + F) \otimes \mathcal{I}(h_F)$$

are globally generated. This is possible by \cite[p. 667, Proposition 1]{8}. Let $h_A$ be a $C^\infty$ hermitian metric on $A$ with strictly positive curvature.
For $m \geq 0$, let $h_m$ be the singular hermitian metrics on $A + m(K_X + L)$ constructed as follows. Let $h_0$ be a $C^\infty$-hermitian metric $h_A$ on $A$ with strictly positive curvature. Let $\{\sigma_0^{(m)}, \ldots, \sigma_{N(m)}^{(m)}\}$ be an orthonormal basis of $H^0(X, \mathcal{O}_X(A + m(K_X + L)) \otimes \mathcal{I}(h^{m-1}))$ with respect to the inner product:

$$
(\sigma, \sigma') := \int_X \bar{\sigma} \cdot \sigma' \cdot h_A \cdot h_m \cdot h_L
$$

$$
= \int_X \bar{\sigma} \cdot \sigma' \cdot (h_A \cdot h_m \otimes (dV)^{-1} \cdot h_L) \cdot dV,
$$

where $dV$ is an arbitrary nowhere degenerate $C^\infty$ volume form on $X$. We set

$$
K_m := \sum_{i=0}^{N(m)} |\sigma_i^{(m)}|^2,
$$

where $|\sigma_i^{(m)}|^2$ denotes $\sigma_i^{(m)} \cdot \bar{\sigma_i^{(m)}}$. We call $K_m$ the **Bergman kernel** of $A + m(K_X + L)$ with respect to $h_A \cdot h_m^{-1} \otimes (dV)^{-1} \cdot h_L$ and $dV$. Clearly it is independent of the choice of the orthonormal basis. And we define the singular hermitian metric $h_m$ on $A + m(K_X + L)$ by

$$
h_m := K_m^{-1}.
$$

It is clear that $K_m$ has semipositive curvature in the sense of currents. We note that for every $x \in X$

$$
K_m(x) = \sup\{| \sigma |^2 (x); \sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X)), \int_X h_A h_m \cdot | \sigma |^2 = 1\}
$$

holds by definition (cf. [1, p.46, Proposition 1.4.16]).

Let $dV$ be a $C^\infty$-volume form on $X$ with respect to a Kähler form $\omega$ on $X$. For a singular hermitian line bundle $(F, h_F)$ on $X$, let $A^2(M, F; h_F, dV)$ denote the Hilbert space of $L^2$ holomorphic sections of $F$ with respect to $h_F$ and $dV$

We may also assume that for any pseudoeffective singular hermitian line bundle $(F, h_F)$ and for any point $x \in X$, there exists an interpolation operator

$$
I_x : A^2(x, K_X \otimes A \otimes F, dV^{-1} h_A h_F, \delta_x) \rightarrow A^2(X, K_X \otimes A \otimes F, dV^{-1} h_A h_F, dV)
$$

such that the operator norm of $I_x$ is bounded from above by a positive constant independent of $x \in X$ and $(L, h_L)$, where $\delta_x$ denotes the Dirac measure at $x$. This is certainly possible, if we take $A$ to be sufficiently ample.

In fact let $x$ be a point on $X$ and let $(U, z_1, \ldots, z_n)$ be a local coordinate neighbourhood of $x$ which is biholomorphic to $\Delta^n$ and $z_i(x) = 0 (1 \leq i \leq n)$. Then by Theorem 2.1, we find an interpolation operator

$$
I_x^U : A^2(x, K_X \otimes A \otimes F, dV^{-1} h_A h_F, \delta_x) \rightarrow A^2(U, K_X \otimes A \otimes F, dV^{-1} h_A h_F, dV)
$$

such that the operator norm of $I_x^U$ is bounded from above by a positive constant $C_U$ independent of $(F, h_F)$. Now we note that the curvature of $h_A \cdot h_F$ is bounded from below by the Kähler form $\Theta_A$. Let $\rho$ be a $C^\infty$-function on $X$ such that
Supp $\rho \subset U$, $0 \leq \rho \leq 1$ and $\rho \equiv 1$ on a neighbourhood of $x$. Let $\sigma_x$ be an element of $A^2(x, K_X \otimes A \otimes F, dV^{-1}h_Ah_L, \delta_x)$. Then replacing $(A, h_A)$ by its sufficiently high positive multiple, we may assume that

$$\Theta_A + (n + 1)\sqrt{-1} \partial \bar{\partial} (\rho \cdot \log \sum_{i=1}^n |z_i|^2) \geq \omega$$

holds on $X$. We also note that there exists a positive constant $C_U'$ independent of $(F, h_F)$ and $\sigma_x$ such that

$$\int_X \exp(-(n+1)\rho \cdot \log \sum_{i=1}^n |z_i|^2) \cdot |\partial(\rho \cdot I^U_x \sigma_x)|^2 \, dV \leq C_U'(dV^{-1}h_A \cdot h_F)(\sigma_x, \sigma_x)$$

holds, where $|\partial(\rho \cdot I^U_x \sigma_x)|^2$ denotes the norm with respect to $h_A \cdot h_F$ and $\omega$. In fact $C_U'$ only depends on $C_U$ and the supremum of the norm of $\partial \rho$ with respect to $\omega$. Then by the usual $L^2$-estimate, we may assume that we can solve the $\partial$-equation

$$\partial u = \partial(\rho \cdot I^U_x \sigma_x)$$

with

$$u(x) = 0$$

so that

$$\int_X h_A \cdot h_F \, |u|^2 \leq C \cdot (dV^{-1}h_A \cdot h_F)(\sigma_x, \sigma_x)$$

holds for a positive constant $C$ independent of $(F, h_F)$ and $\sigma_x$. Then

$$\rho \cdot I^U_x \sigma_x - u \in H^0(X, O_X(K_X + A + L) \otimes \mathcal{I}(h_F))$$

is an extension of $\sigma_x$. Since $X$ is compact, moving $x$ and $U$, by the above estimates this implies the assertion.

**Lemma 3.1** Let $h$ be a canonical AZD of $K_X$ constructed as in the proof of Theorem 2.2. Then the inclusion :

$$\mathcal{I}(h^m) \subseteq \mathcal{I}(h_m)$$

holds for every $m \geq 0$.

By the choice of $A$ and Lemma 3.1, $h_m$ is well defined for every $m \geq 0$.

Now we shall make the above lemma quantitative.

**Lemma 3.2** There exists a positive constant $C$ such that

$$h_m \leq C \cdot h_A \cdot h^m$$

holds for every $m \geq 0$.

**Proof.** Let us denote the Bergman kernel of $A + m(K_X + L)$ with respect to a singular hermitian metric $H$ on $A + m(K_X + L)$ and the volume form $dV$ by $K(A+m(K_X+L), H, dV)$. In this notation $K_m$ is expressed as $K(A+m(K_X+L), h_A \cdot h^{m-1} \cdot h_L \otimes (dV)^{-1}, dV)$.  

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Then by the $L^2$-extension form a point to $X$ as above we have that

$$K(A + m(K_X + L), h_A \cdot h^{m-1} \cdot dV^{-1} \cdot h_L, dV) \geq C^{-1}(h_A \cdot h^m)^{-1}$$

hold for some positive constant $C$. This completes the proof of Lemma 3.2. Q.E.D.

On the other hand by the submeanvalue property of plurisubharmonic functions, we have :

**Lemma 3.3** *Let $x$ be a point on $M$ and let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a coordinate with center $x$. Then for every sufficiently small $r > 0$

$$h_A \cdot K_m \leq \sup_{|\zeta|<r} h^{-m} \frac{1}{\pi^{n} r^{2n}/n!}$$

hold, where we have trivialized $A$ and $K_M + L$ around $x$ by taking holomorphic frames.*

By Lemma 3.2 and Lemma 3.3

$$K_\infty := \text{the uppersemicontinuous envelope of } \limsup_{m \to \infty} \sqrt[7]{K_m}$$

is a well defined volume form on $X$ which does not vanish outside of a set of measure 0. We set

$$h_\infty := \frac{1}{K_\infty}.$$  

Then by Lemma 3.2, we see that

$$h_\infty \leq h$$

holds. By Lemma 3.3 we have the opposite estimate :

$$h_\infty \geq h$$

holds.

Hence we have the following theorem.

**Theorem 3.1** *$h_\infty = h$ holds. In particular $h_\infty$ is an AZD of $K_X + L$.*

### 4 Proof of Theorem 1.1

Let $M, S, L$ be as in Theorem 1.1. Let $h_S$ be a canonical AZD of $K_M + L |_S$. Let $A$ be a sufficiently ample line bundle on $M$. Let us define the singular hermitian metric on $m(K_M + L) |_S$ by

$$h_{m,S} := K(A + m(K_M + L) |_S, h_A \cdot h_S^{m-1} \cdot dV^{-1}_M \cdot h_L, d\Psi_S)^{-1}$$

Then as in the last section, we see that

$$h_{\infty,S} := \liminf_{m \to \infty} \sqrt[7]{h_{m,S}}$$
exists and an AZD of \( K_M + L \mid S \). We consider the Bergman kernel

\[
K(S, A + m(K_M + L) \mid S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S) = \sum_i |\sigma_i^{(m)}|^2,
\]

where \( \{\sigma_i^{(m)}\} \) is a complete orthonormal basis of \( A^2(S, A + m(K_M + L) \mid S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S) \). We note that (cf. [3, p.46, Proposition 1.4.16])

\[
K(S, A + m(K_M + L) \mid S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)(x) = \sup\{|\sigma|^2(x) \mid \sigma \in A^2(S, A+m(K_M+L) \mid S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S), \|\sigma\| = 1\}
\]

holds for every \( x \in S \). We note that as in Lemma 3.2 there exists a positive constant \( C_0 \) independent of \( m \) such that

\[
h_m, S \leq C_0 \cdot h_A \cdot h_S^m
\]

holds for every \( m \geq 1 \).

Inductively on \( m \), we extend each

\[
\sigma \in A^2(S, A + m(K_M + L) \mid S, h_A \cdot h_S^{m-1} \cdot dV_M^{-1} \cdot h_L, d\Psi_S)
\]

to a section

\[
\tilde{\sigma} \in A^2(M, A + m(K_M + L), dV^{-1} \cdot h_L \cdot \tilde{h}_{m-1}, dV)
\]

with the estimate

\[
\|\tilde{\sigma}\| \leq C_m \|\sigma\|
\]

where \( \|\| \)’s denote the \( L^2 \)-norms respectively, \( C_m \) is the positive constant depending only on \( m \) which will be specified later (the existence of \( C_m \) is assured by Theorem 2.1), we have defined

\[
\tilde{K}_m(x) := \sup\{|\tilde{\sigma}|^2(x) \mid \|\tilde{\sigma}\|_S = 1, \|\tilde{\sigma}\| \leq C_m\}
\]

and set

\[
\tilde{h}_m = \frac{1}{\tilde{K}_m}.
\]

Let us specify \( C_m \). Let \( X \) be the closed set as in Theorem 2.1. There exists a small Stein neighbourhood \( W \) of \( S \setminus X \) in \( M \) and a holomorphic retraction

\[
\rho : W \rightarrow S \setminus X.
\]

Then \( \rho \) defines a linear map

\[
I^m_\rho : A^2(S, m(K_M + L) + A, h_S^{m-1} \otimes dV^{-1} \otimes h_L \otimes h_A) \rightarrow A^2(W, m(K_M + L) + A, \tilde{h}_{m-1} \otimes dV^{-1} \otimes h_L)
\]

Then we may take \( C_m \) as

\[
C_m := C \cdot \|I^m_\rho\|
\]
where $C$ is a positive constant independent of $m$ and $\| I^m \|$ denotes the operator norm of $I^m$. This estimate follows from the proof of Theorem 2.1 (cf. [4, p.9]). By the above inductive estimates, we see that

$$\tilde{h}_\infty := \lim_{m \to \infty} \sqrt{\tilde{h}_m}$$

exists and gives an extension of $h_{\infty,S}$. Then by Theorem 2.1 we may extend every element of $A^2(m(K_M + L) \mid S, dV^{-1}_M \cdot h_L \cdot \tilde{h}_\infty \mid S, dV_M[\Psi])$ to $A^2(m(K_M + L), dV^{-1}_M \cdot h_L \cdot \tilde{h}_\infty, dV_M)$. This completes the proof of Theorem 1.1.

5 Generalization of Theorem 1.1

Let $M$ be a smooth projective variety and let $(L, h_L)$ be a singular hermitian line bundle on $M$ such that $\Theta_{h_L} \geq 0$ on $M$. Let $dV$ be a $C^\infty$-volume form on $M$. Let $\sigma \in \Gamma(M, \mathcal{O}_M(m_0L) \otimes \mathcal{I}(h))$ be a global section. Let $\alpha$ be a positive rational number $\leq 1$ and let $S$ be an irreducible subvariety of $M$ such that $(M, \alpha(\sigma))$ is logcanonical but not KLT (Kawamata log-terminal) on the generic point of $S$ and $(M, (\alpha - \epsilon)(\sigma))$ is KLT on the generic point of $S$ for every $0 < \epsilon << 1$. We set

$$\Psi = \alpha \log h_L(\sigma, \sigma).$$

We shall assume that $S$ is not contained in the singular locus of $h$, where the singular locus of $h$ means the set of points where $h$ is $+\infty$.

For the moment we shall consider the case that $S$ is smooth (when $S$ is not smooth, we just need to repeat the following argument after taking an embedded resolution of $S$). In this case $\Psi$ may not belong to $\sharp(S)$, since $\Psi$ may not have the prescribed singularity along $S$ as in the definition of $\sharp(S)$. But the proof of Theorem 2.1 and hence proof of Theorem 1.1 also works in this case except a minor difference. The difference is that $dV_M[\Psi]$ (which is defined similarly as above) may have singularities along some Zariski closed subset of $S$. Let $d\mu_S$ be a $C^\infty$-volume form on $S$ and let $\varphi$ be the function on $S$ defined by

$$\varphi := \log \frac{dV[\Psi]}{d\mu_S}.$$

According to the singularity of $\varphi$, the proof of Theorem 1.1 must be modified as follows.

Let $d$ be a positive integer such that $d > \alpha m_0$. Let $h_S$ be a canonical AZD of $(K_M + dL, e^{-\varphi} \cdot dV^{-1} \otimes h_L^d)$, i.e.,

$$h_S = \inf \{ H_S \mid \frac{H_S}{e^{-\varphi} \cdot dV^{-1} \otimes h_L^d} \geq 1, \Theta_{H_S} \geq 0 \},$$

where $H_S$ runs singular hermitian metrics on $K_M + dL$ satisfying the above conditions. Let $A$ be a sufficiently ample line bundle on $M$. We set

$$K_m := K(A + m(K_M + dL) \mid S, h_A \cdot h_S^{-1} \cdot dV^{-1} \cdot h_L^d, dV_M[\Psi]).$$
Then by Theorem 2.1 (with the modified $dV[Ψ]$), exactly as in the proof of Theorem 1.1, inductively we may extend

$$h_m := K_m^{-1}$$

to a singular hermitian metric $\hat{h}_m$ on $A + m(K_M + dL)$. Here we have used the factor $e^{-ϕ}$ to extend sections on $S$ inductively. Also by the same argument as in the proof of Theorem 1.1,

$$\hat{h}_∞ := \liminf_{m \to \infty} \sqrt[m]{\hat{h}_m}$$

exists and is a singular hermitian metric on $K_M + dL$ with semipositive curvature. Also as in Section 1, we see that $\hat{h}_∞|_S$ is an AZD of $(K_M + dL|_S, e^{-ϕ} \cdot dV^{-1} \otimes h_L^0)$. Hence applying Theorem 2.1, we obtain the following theorem:

**Theorem 5.1** Let $M, S, Ψ$ be as above. Suppose that $S$ is smooth. Then every element of $A^2(S, O_S(m(K_M + dL)), e^{-(m-1)ϕ} \cdot dV^{-m} \otimes h_L^m, dV[Ψ])$ extends to an element of $H^0(M, O_M(m(K_M + dL)))$.

As we mentioned as above the smoothness assumption on $S$ is just to make the statement simpler.

As an example of an application, we have:

**Corollary 5.1** ([7]) Let $\pi : X \to \Delta$ be a semistable degeneration of projective variety over the unit disk. Let $X_0 = π^{-1}(0) = \sum_i D_i$ be the irreducible decomposition. Then we have that

$$\sum_i P_m(D_i) \leq P_m(X_t)$$

holds where $t$ is any regular value of $\pi$ and $P_m$ denotes the $m$-th plurigenus.

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