ITERATIVE FINITE ELEMENT SOLUTION OF A
CONSTRAINED TOTAL VARIATION REGULARIZED
MODEL PROBLEM

Sören Bartels* and Marijo Milicevic

Department of Applied Mathematics, Mathematical Institute, University of Freiburg
Hermann-Herder-Str. 9, 79104 Freiburg i. Br., Germany

Dedicated to Professor T. Roubíček on the occasion of his 60th birthday.

Abstract. The discretization of a bilaterally constrained total variation minimization problem with conforming low order finite elements is analyzed and three iterative schemes are proposed which differ in the treatment of the non-differentiable terms. Unconditional stability and convergence of the algorithms is addressed, an application to piecewise constant image segmentation is presented and numerical experiments are shown.

1. Introduction. In this paper we investigate iterative schemes for the solution of the bilaterally constrained ROF problem which consists in the minimization of the functional

$$E(u) := |Du|(\Omega) + \frac{\alpha}{2} \|u - g\|_{L^2(\Omega)}^2 + I_K(u)$$

in the set of functions with bounded total variation $|Du|(\Omega)$, where $I_K$ is the indicator functional of a convex set $K \subset L^2(\Omega)$ characterized by two obstacle functions. While existence and uniqueness of a minimizer of $E$ can be established using the direct method in the calculus of variations, the non-differentiability of the total variation and the indicator functional are challenging from a numerical point of view. The minimization problem serves as a model problem for a wide class of functionals involving the total variation and the indicator functional which comprises a bilateral constraint on the variable $u$. For example, in [6, 14] the above minimization problem is considered in the context of image denoising and the bilateral constraint is imposed in order for the restored image to have pixel values lying in a certain range, e.g., in the interval $[0, 255]$. In [13] Chan et al. show the equivalence of a minimization problem occuring in image segmentation to the minimization of a functional that resembles $E$ with the quadratic fidelity term replaced by a linear functional in $u$. The application of a regularization term to a bilaterally constrained variable also occurs in the modelling of damage processes in continuum mechanics, where the bilaterally constrained variable is the damage variable and the regularization term consists of a differentiable $L^p$-norm $(p \geq 2)$ of the gradient of the damage variable (cf. [19]) or the total variation of the damage variable (cf. [21]) while the constraint ensures that the damage variable takes on values only in the interval $[0, 1]$.
In the recent years many PDE-based algorithms have been proposed for the classical ROF functional proposed by Rudin, Osher and Fatemi in [20], where the bilateral constraint is omitted, reaching from primal-dual methods, combinations of regularization and standard minimization techniques and dual methods to the alternate direction method of multipliers, augmented Lagrangian methods and split Brègman methods, see, e.g., [3, 23] for an overview of many of those algorithms. Regarding the bilaterally constrained ROF problem Casas et al. proposed in [10] an active set strategy using the anisotropic BV-seminorm. This ansatz works with the primal formulation of the problem and does not use regularization. However, a penalization approach is used to solve auxiliary subproblems. In [6], motivated by the dual approach of Chambolle in [11], Beck and Teboulle suggest a monotone fast iterative shrinkage/thresholding algorithm (MFISTA) for the dual of the bilaterally constrained ROF problem for which the convergence rate $O(1/k^2)$ with $k$ being the iteration counter is proven. Yet, as in [11], the constant entering the convergence rate grows like $O(1/h^2)$ with $h$ being the mesh size of the underlying grid. Chan et al. proposed in [14] an augmented Lagrangian method, which has already been successfully applied to the ROF problem by Wu and Tai in [22]. Let us remark that the primal-dual algorithm proposed in [12] can also be applied to solve the bilaterally constrained ROF problem.

The aim of this paper is to compare three different approaches for the iterative minimization of $E$ and to identify the most appropriate method. To this extent, we introduce a finite element discretization with globally continuous piecewise affine finite elements. The iterative algorithms differ in the treatment of the two challenging terms appearing in $E$. The first approach uses splitting techniques for both the total variation and the bilateral constraint and is a variant of the augmented Lagrangian method proposed in [14], where the involved norms are properly weighted in order to ensure unconditional stability of the method, see also [3] where the authors proposed a properly weighted augmented Lagrangian method for the classical ROF problem. The next ansatz uses regularization to handle the non-differentiability of the total variation and the indicator functional and employs a semi-implicit subgradient flow which turns out to be unconditionally stable and will be referred to as the Heron-penalty method. For the classical ROF problem the Heron method has already been introduced in [3]. The third algorithm uses a combination of the regularization and the splitting approach, i.e., using the observation made in [3] that a regularization approach seems to be the most appropriate to deal with the total variation, we employ a regularization for the total variation and a splitting technique for the bilateral constraint, which we call the Heron-split method. The primal-dual algorithm proposed in [12] has been compared to the Heron method and the augmented Lagrangian method in [3] for the classical ROF problem and it has been observed that the primal-dual algorithm is slow due to the restrictive condition on the step size.

The outline of the paper is as follows. In Section 2 we give some basic notation and introduce the space of functions with bounded variation as well as the globally continuous piecewise affine and the piecewise constant finite element space. We present the minimization problem in Section 3 and discuss existence and uniqueness of a minimizer. In Section 4 we consider the finite element discretization and prove convergence of discrete minimizers to the continuous solution before we present the three algorithms and address their stability in Section 5. In Section 6 we address
the minimization of the functional
\[ E^{\text{seg}}(u) := \int_{\Omega} |Du| + \alpha \int_{\Omega} fu \, dx + I_K(u), \]
prove convergence of discrete energies to the minimal energy and recall a minimization problem arising in an image segmentation model discussed by Chan et al. in \cite{cen}. We finally provide results of numerical experiments for the three iterative schemes applied to the bilaterally constrained ROF problem and for a particular choice of \( g \) and two different values of \( \alpha \) as well as for the image segmentation problem in Section \[7\]. We then conclude the paper in Section \[8\].

2. Preliminaries.

2.1. Notation. Let \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), be a bounded star-shaped polygonal Lipschitz domain. We denote by \( \| \cdot \| \) the \( L^2 \)-norm on \( \Omega \) induced by the scalar product
\[ (v, w) := \int_{\Omega} v \cdot w \, dx \]
for scalar functions or vector fields \( v, w \in L^2(\Omega; \mathbb{R}^\ell) \), \( \ell \in \{1, d\} \). The Euclidean norm will be denoted by \( |\cdot| \). We will work with the space of functions of bounded variation \( BV(\Omega) \subset L^1(\Omega) \) which contains all functions \( v \in L^1(\Omega) \) for which the total variation given by
\[ |Du|(\Omega) = \int_{\Omega} |Du| := \sup \left\{ -\int_{\Omega} u \, \text{div} \, q \, dx : q \in C^1_c(\Omega; \mathbb{R}^d), \, |q| \leq 1 \right\} \]
is finite. The space \( BV(\Omega) \) equipped with the norm \( \|u\|_{BV} := \|u\|_{L^1(\Omega)} + |Du|(\Omega) \) is a Banach space. We will also consider intermediate convergence on \( BV(\Omega) \), i.e., \((u_n)_{n \in \mathbb{N}} \subset BV(\Omega)\) converges to \( u \in BV(\Omega) \) with respect to intermediate convergence if
\[ \|u_n - u\|_{L^1(\Omega)} \to 0, \quad |Du_n|(\Omega) \to |Du|(\Omega). \]
For more details concerning the space \( BV(\Omega) \) we refer, e.g., to \cite{cen}. For a convex functional \( F : X \to \mathbb{R} \cup \{\infty\} \) on some Hilbert space \( X \) with inner product \( \langle \cdot, \cdot \rangle_X \) we define its subdifferential \( \partial F(u) \) at \( u \in X \) by
\[ \partial F(u) := \{u' \in X : (u', v - u)_X + F(u) \leq F(v) \text{ for all } v \in X\}. \]
If \( F \) is Fréchet-differentiable at \( u \), we denote by \( \delta F(u) \) its Fréchet-derivative at \( u \).

For a sequence \((a^j)_{j \in \mathbb{N}}\) and a step size \( \tau > 0 \) we let
\[ d_t a^{j+1} := \frac{a^{j+1} - a^j}{\tau} \]
be the backward difference quotient.

2.2. Finite element spaces. Throughout the paper we let \( c \) be a generic constant. We denote by \((T_h)_{h>0}\) a family of uniform triangulations of \( \Omega \) with mesh sizes \( h = \max_{T \in T_h} h_T \) where \( h_T \) is the diameter of the simplex \( T \). For a given triangulation \( T_h \) the set \( N_h \) contains the corresponding nodes and we consider the finite element spaces of continuous, piecewise affine functions
\[ S^1(T_h) := \{v_h \in C(\overline{\Omega}) : v_h|_T \text{ affine for all } T \in T_h\} \]
and of elementwise constant functions \((r = 1)\) or vector fields \((r = d)\)
\[ L^0(T_h)^r := \{q_h \in L^\infty(\Omega; \mathbb{R}^r) : q_h|_T \text{ constant for all } T \in T_h\}. \]
We also denote by $\mathcal{T}_\ell$, $\ell \in \mathbb{N}$, a triangulation of $\Omega$ generated from an initial triangulation $\mathcal{T}_0$ by $\ell$ uniform refinements and the integer $\ell$ being related to the mesh size $h$ by $h = c2^{-\ell}$. The set of nodes $\mathcal{N}_\ell$ is defined correspondingly.

We introduce on $\mathcal{S}^1(\mathcal{T}_h)$ the discrete norm $\| \cdot \|_h$ induced by the discrete scalar product

$$(v, w)_h := \int_\Omega I_h(v w) \, dx = \sum_{z \in \mathcal{N}_h} \beta_z v(z) w(z)$$

for $v, w \in \mathcal{S}^1(\mathcal{T}_h)$, where $I_h$ is the nodal interpolant, $\beta_z = \int_\Omega \varphi_z \, dx$ and $\varphi_z \in \mathcal{S}^1(\mathcal{T}_h)$ is the nodal basis function associated with the node $z \in \mathcal{N}_h$. The norm $\| \cdot \|_h$ and the $L^2$-norm are related on $\mathcal{S}^1(\mathcal{T}_h)$ as follows.

**Lemma 2.1.** For $v_h \in \mathcal{S}^1(\mathcal{T}_h)$ we have

$$\|v_h\| \leq \|v_h\|_h \leq (d + 1)^{1/2}\|v_h\|.$$  

The proof of **Lemma 2.1** can be found, e.g., in [21, Lemma 3.9].

3. **Model problem.** For a function $g \in L^\infty(\Omega)$ and a parameter $\alpha > 0$ we consider the constrained minimization problem defined by the functional

$$E(u) := \int_\Omega |Du| + \frac{\alpha}{2} \|u - g\|^2 + I_K(u)$$

in the set of functions $BV(\Omega) \cap L^2(\Omega)$. The convex set $K$ is given by

$$K := \{v \in L^2(\Omega) : \chi \leq v \leq \psi \text{ a.e.}\}$$

with $\chi, \psi \in BV(\Omega) \cap L^\infty(\Omega)$ and $\chi < \psi$ almost everywhere. We will refer to this minimization problem as the *bilaterally constrained ROF problem*.

**Proposition 1** (Existence and Uniqueness). *There exists a unique minimizer $u \in BV(\Omega) \cap L^2(\Omega)$ of $E$.**

**Proof.** The set of feasible functions is nonempty and the functional $E$ is bounded from below. Hence, there exists an infimizing sequence $(u_j)_j$ which is bounded in $BV(\Omega) \cap L^2(\Omega)$ due to the coercivity of $E$. The compact embedding of $BV(\Omega)$ into $L^1(\Omega)$ (cf. [P]) and the boundedness of the sequence imply the existence of a not relabelled - weakly convergent subsequence $u_j \rightharpoonup u$ in $BV(\Omega)$ and by passing to a further subsequence $u_j \rightarrow u$ in $BV(\Omega) \cap L^2(\Omega)$. Since $u_j \rightarrow u$ in $L^1(\Omega)$ there exists a further subsequence such that $u_j \rightarrow u$ almost everywhere in $\Omega$. This implies that $\chi \leq u \leq \psi$ almost everywhere in $\Omega$ since $\chi \leq u_j \leq \psi$ for all $j \in \mathbb{N}$. The weak lower semicontinuity of $E$ (cf. [P]) then yields that $u$ is a minimizer of $E$. Uniqueness follows by the strict convexity of $E$ on $K$.  

**Remark 1.** In fact, we have $u \in BV(\Omega) \cap L^\infty(\Omega)$ since $\chi, \psi \in L^\infty(\Omega)$.

We define the functionals $G : L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$, $H : L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ and $F : L^2(\Omega) \rightarrow \mathbb{R} \cup \{\infty\}$ by $G(v) := \frac{\alpha}{2} \|v - g\|^2$, $H(v) := I_K(v)$ and

$$F(v) := \begin{cases} \int_\Omega |Dv|, & \text{if } v \in BV(\Omega) \cap L^2(\Omega), \\ \infty, & \text{if } v \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

**Lemma 3.1.** Let $u$ be the unique minimizer of $E$ in $BV(\Omega) \cap L^2(\Omega)$. Then for any $v \in BV(\Omega) \cap L^2(\Omega)$ we have

$$\frac{\alpha}{2} \|u - v\|^2 \leq E(v) - E(u).$$
Proof. We note that $G$ is Fréchet-differentiable, i.e., we have $\partial G(v) = \{ \delta G(v) \}$ with $\delta G(v)[w] = \alpha(v - g, w)$ for $v, w \in L^2(\Omega)$. Since $u$ is a minimizer of $E$, we have

$$0 \in \partial E(u) = \partial(F + G + H)(u).$$

Standard arguments from convex analysis (cf. [1, Thm. 9.5.4]) yield

$$-\delta G(u) \in \partial(F + H)(u).$$

The strong convexity of $G$ and the subgradient inequality then imply that for arbitrary $v \in BV(\Omega) \cap L^2(\Omega)$

$$\frac{\alpha}{2} \|u - v\|^2 = -\delta G(u)[v - u] + G(v) - G(u) \leq F(v) + G(v) + H(v) - F(u) - G(u) - H(u) = E(v) - E(u),$$

which proves the lemma.

\[\square\]

4. Finite element discretization. We will need the following assumption for the error analysis which imposes restrictions on admissible domains and obstacles.

**Assumption 1.** The obstacle functions are such that there exists a family of regular triangulations $(T_h)_{h>0}$ and $h_0 > 0$ with $\chi, \psi \in \mathcal{S}^1(T_h)$ for $h \leq h_0$ such that for any $v \in BV(\Omega) \cap K$ and any $\varepsilon > 0$ there exists $v_\varepsilon \in C^\infty(\Omega) \cap BV(\Omega) \cap K$ with

$$\|v - v_\varepsilon\|_{L^1(\Omega)} \leq \varepsilon |Du|(\Omega), \quad \|\nabla v_\varepsilon\|_{L^1(\Omega)} \leq (1 + c\varepsilon)|Du|(\Omega),$$

which means that $C^\infty(\Omega) \cap BV(\Omega) \cap K$ is dense in $BV(\Omega) \cap K$ with respect to intermediate convergence, and with

$$\|D^2v_\varepsilon\|_{L^1(\Omega)} \leq c\varepsilon^{-1}|Du|(\Omega), \quad \|D^2v_\varepsilon\| \leq c\varepsilon^{-2}|Du|(\Omega).$$

Then, in particular, $C^\infty(\Omega) \cap K$ is dense in $BV(\Omega) \cap K$ with respect to intermediate convergence, cf. [1, Thm. 10.1.2].

**Remark 2.** Note that if, e.g., $\chi$ and $\psi$ are globally constant, then Assumption [1] is satisfied. Indeed, $\chi, \psi \in \mathcal{S}^1(T_h)$ for any regular triangulation of $\Omega$ with $T_{h_0} = \{ \Omega \}$, $h_0 = \text{diam}(\Omega)$ and $h \leq h_0$. Furthermore, taking into account that mollification does not increase the $L^\infty$-norm, [1, Thm. 10.1.2] ensures that $C^\infty(\Omega) \cap BV(\Omega) \cap K$ is dense in $BV(\Omega) \cap K$ with respect to intermediate convergence on the one hand and, on the other hand, the estimates stated in Assumption [1] can be shown as in [3, Prop. 2.1].

We consider a family of triangulations as required in Assumption [1] and $h \leq h_0$. Let us then remark that $I_K(v_h)$ is finite for $v_h \in \mathcal{S}^1(T_h)$ if and only if $\chi(z) \leq v_h(z) \leq \psi(z)$ at all nodes $z \in N_h$.

**Lemma 4.1.** Let $\varepsilon > 0$. For any $u \in BV(\Omega) \cap K$ there exists $u_{\varepsilon,h} \in \mathcal{S}^1(T_h) \cap K$ such that, if $d = 2$,

$$\|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)} \leq (1 + c\varepsilon^{-1} + c\varepsilon)|Du|(\Omega),$$

$$\|u_{\varepsilon,h} - u\|_{L^1(\Omega)} \leq c(h^2\varepsilon^{-1} + \varepsilon)|Du|(\Omega)$$

or, if $d = 3$,

$$\|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)} \leq (1 + c\varepsilon^{-2} + c\varepsilon)|Du|(\Omega),$$

$$\|u_{\varepsilon,h} - u\|_{L^1(\Omega)} \leq c(h^2\varepsilon^{-2} + \varepsilon)|Du|(\Omega).$$
Proof. For \( u \in BV(\Omega) \cap K \) and \( \varepsilon > 0 \) let \( u_\varepsilon \in C^\infty(\Omega) \cap K \) be as in Assumption \([4]\). Define \( u_{\varepsilon,h} = I_h u_\varepsilon \). Note that due to Assumption \([4]\) we have \( u_{\varepsilon,h} \in S^1(T_h) \cap K \). Standard interpolation estimates (cf. \([9, \text{Thm. 4.4.20}]\)) imply for \( d = 2 \)

\[
\|u_\varepsilon - u_{\varepsilon,h}\|_{L^1(\Omega)} + h \|\nabla (u_\varepsilon - u_{\varepsilon,h})\|_{L^1(\Omega)} \leq ch^2 \|D^2 u_\varepsilon\|_{L^1(\Omega)}
\]

and, for \( d = 3 \),

\[
\|u_\varepsilon - u_{\varepsilon,h}\| + h \|\nabla (u_\varepsilon - u_{\varepsilon,h})\| \leq ch^2 \|D^2 u_\varepsilon\|.
\]

Hence, by Assumption \([4]\) we have

\[
\|u_\varepsilon - u_{\varepsilon,h}\|_{L^1(\Omega)} \leq \|u_\varepsilon - u\|_{L^1(\Omega)} + \|u_\varepsilon - u_{\varepsilon,h}\|_{L^1(\Omega)}
\]

\[
\leq \varepsilon |Du|_{L^1(\Omega)} + ch^2 \|D^2 u_\varepsilon\|_{L^1(\Omega)}
\]

\[
\leq c(h^2 \varepsilon^{-1} + \varepsilon) |Du|_{L^1(\Omega)}.
\]

Furthermore, we obtain

\[
\|\nabla u_{\varepsilon,h}\|_{L^1(\Omega)} \leq \|\nabla (u_{\varepsilon,h} - u_\varepsilon)\|_{L^1(\Omega)} + \|\nabla u_\varepsilon\|_{L^1(\Omega)}
\]

\[
\leq ch \|D^2 u_\varepsilon\|_{L^1(\Omega)} + (1 + c \varepsilon) |Du|_{L^1(\Omega)}
\]

\[
\leq (1 + ch \varepsilon^{-1} + c \varepsilon) |Du|_{L^1(\Omega)}.
\]

The case \( d = 3 \) follows analogously using \( \|u_\varepsilon - u_{\varepsilon,h}\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|u_\varepsilon - u_{\varepsilon,h}\| \), \( \|\nabla (u_{\varepsilon,h} - u_\varepsilon)\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|\nabla (u_{\varepsilon,h} - u_\varepsilon)\| \), Assumption \([4]\) and the nodal interpolation estimate for \( d = 3 \). □

We seek a discrete minimizer \( u_h \in S^1(T_h) \) of

\[
E(u_h) = \int_\Omega |\nabla u_h| \, dx + \frac{\alpha}{2} \|u_h - g\|^2 + I_K(u_h)
\]

in \( S^1(T_h) \), where we used that \( |Dv|_{L^1(\Omega)} = ||\nabla v||_{L^1(\Omega)} \) for \( v \in W^{1,1}(\Omega) \). Existence and uniqueness can be proven following the arguments of the proof of Proposition \([4]\).

**Proposition 2.** Let \( u \) and \( u_h \) be the unique minimizers of \( E \) in \( BV(\Omega) \cap L^2(\Omega) \) and \( S^1(T_h) \), respectively. Then we have for \( d = 2 \)

\[
\frac{\alpha}{2} \|u - u_h\|^2 \leq ch^{1/2}.
\]

If \( d = 3 \), we have

\[
\frac{\alpha}{2} \|u - u_h\|^2 \leq ch^{1/3}.
\]

**Proof.** Let \( d = 2 \). Following the arguments in the proof of \([2, \text{Thm. 10.7}]\) we note that by Lemma \([3, \text{Lem. 3.1}]\) we have for \( v = u_h \) and for arbitrary \( v_h \in S^1(T_h) \) that

\[
\frac{\alpha}{2} \|u - u_h\|^2 \leq E(u_h) - E(u)
\]

\[
\leq E(v_h) - E(u).
\]

(1)
Choose \( v_h = u_{\varepsilon,h} \) as in Lemma 4.1. Using the binomial formula \( \|a-c\|^2 - \|b-c\|^2 = (a-b)(a+b-2c) \) we obtain
\[
E(u_{\varepsilon,h}) - E(u) = \int_{T_h} |\nabla u_{\varepsilon,h}| \, dx + \frac{\alpha}{2} \|u_{\varepsilon,h} - g\|^2 - \int_{T_h} |Du| - \frac{\alpha}{2} \|u - g\|^2
\]
\[
= \int_{T_h} |\nabla u_{\varepsilon,h}| \, dx - \int_{T_h} |Du| + \frac{\alpha}{2} \int_{T_h} (u_{\varepsilon,h} - u)(u_{\varepsilon,h} + u - 2g) \, dx
\]
\[
\leq c(h^{-1} + \varepsilon)|Du|(\Omega) + \frac{\alpha}{2} \|u_{\varepsilon,h} - u\|_{L^1(\Omega)} \|u_{\varepsilon,h} + u - 2g\|_{L^\infty(\Omega)}
\]
\[
\leq c(h^{-1} + \varepsilon)|Du|(\Omega) + c(h^2 \varepsilon^{-1} + \varepsilon)|Du|(\Omega).
\]

The choice \( \varepsilon = h^{1/2} \) yields the first estimate. If \( d = 3 \), we proceed as above and obtain by Lemma 4.1
\[
E(u_{\varepsilon,h}) - E(u) \leq c(h^{-2} + \varepsilon)|Du|(\Omega) + c(h^2 \varepsilon^{-2} + \varepsilon)|Du|(\Omega).
\]

The choice \( \varepsilon = h^{1/3} \) gives the second estimate. \( \square \)

**Remark 3.** The error estimate is suboptimal in the sense that for \( u \in BV(\Omega) \cap L^\infty(\Omega) \) we have the approximation property
\[
\inf_{v_h \in S^1(T_h)} \|u - v_h\| \leq ch^{1/2},
\]
cf. [2] Rmk. 10.9 (ii)]. On highly symmetric triangulations and for an anisotropic version of the total variation this quasioptimal convergence rate can be proved for related minimization problems, cf. [2] Thm. 6.5]. Numerical experiments for the classical ROF problem show that in general the convergence rate \( O(h^{1/2}) \) is not attained.

5. **Iterative solution.** The iterative minimization of \( E \) is difficult due to the non-differentiability of the seminorm and the occurrence of the constraints. In this section we will discuss three approaches to deal with these difficulties.

5.1. **Splitting.** We introduce for \( u_h \in S^1(T_h) \) the auxiliary variables \( p_h = \nabla u_h \in L^0(T_h)^d \) and \( s_h = u_h \in S^1(T_h) \) and incorporate Lagrange multipliers \( \lambda_h \in L^0(T_h)^d \) and \( \eta_h \in S^1(T_h) \) to enforce these relations. Note that since the sequence \( (\nabla u_h)_{h>0} \) of discrete minimizers \( u_h \) of \( E \) may not be bounded in \( L^2(\Omega) \) but only in \( L^1(\Omega) \) we have to measure these quantities in a weaker norm than the \( L^2 \)-norm. We therefore equip the space \( L^0(T_h)^d \) with the weighted \( L^2 \)-scalar product
\[
(p_h, q_h)_w := h^d \int_{T_h} p_h \cdot q_h \, dx
\]
and induced norm \( \| \cdot \|_w \). An inverse estimate shows that \( (\nabla u_h)_{h>0} \) is bounded with respect to \( \| \cdot \|_w \), cf. [3].

We then consider the consistently stabilized Lagrange functional
\[
L_h(u_h, p_h, s_h; \lambda_h, \eta_h)
\]
\[
:= \int_{T_h} |p_h| \, dx + \frac{\alpha}{2} \|u_h - g\|^2 + I_K(s_h)
\]
\[
+ (\lambda_h, p_h - \nabla u_h)_w + \frac{\sigma_1}{2} \|p_h - \nabla u_h\|_w^2 + (\eta_h, s_h - u_h)_h + \frac{\sigma_2}{2} \|s_h - u_h\|_h^2,
\]
The parameters $\sigma_1$ and $\sigma_2$ are assumed to be positive. We then have that $(u_h, p_h, s_h, \lambda_h, \eta_h)$ is a saddle point for $L_h$ if and only if $u_h$ minimizes $E$ (cf. [22] Thm. 4.1) for the same result for the ROF functional without bilateral constraint). Indeed, since

$$L_h(u_h, p_h, s_h; \mu_h, \nu_h) \leq L_h(u_h, p_h, s_h; \lambda_h, \eta_h)$$

for all $(\mu_h, \nu_h)$ we obtain $p_h = \nabla u_h$ and $s_h = u_h$. Then we have

$$E(u_h) = L_h(u_h, p_h, s_h; \lambda_h, \mu_h)$$

$$\leq L_h(v_h, \nabla v_h, v_h; \lambda_h, \mu_h)$$

$$= \int_\Omega |\nabla v_h| \, dx + \frac{\alpha}{2} \|v_h - g\|^2 + I_K(v_h) = E(v_h)$$

for all $v_h \in S^1(T_h)$, i.e., $u_h$ minimizes $E$. Conversely, if $u_h$ minimizes $E$ we choose $p_h = \nabla u_h$ and $s_h = u_h$. Since $u_h$ is the unique minimizer of $E$ we have

$$0 \in \partial E(u_h) = \partial F(u_h) + \partial G(u_h) + \partial H(u_h),$$

cf. [18] Thm. 23.8. In particular, there exist $-\lambda_h \in \partial \tilde{F}(\nabla u_h)$ and $-\eta_h \in \partial H(u_h)$ with

$$\alpha(v_h - g, v_h) = -\operatorname{div} \lambda_h + \eta_h, \quad (\text{2})$$

cf. [18] Thm. 23.9, where the operator $\operatorname{div}_h : L^0(T_h) \rightarrow S^1(T_h)$ is defined by $(-\operatorname{div}_h \mu_h, v_h) = (\mu_h, \nabla v_h)$ for all $v_h \in S^1(T_h)$, $\mu_h \in L^0(T_h)$, and $\tilde{F}(q_h) := \|q_h\|_{L^1(\Omega)}$. With our particular choice of $p_h$ and $s_h$ we have

$$L_h(u_h, p_h, s_h; \mu_h, \nu_h) = L_h(u_h, p_h, s_h; \lambda_h, \eta_h)$$

for all $(\mu_h, \nu_h)$. One can show that the function

$$(v_h, q_h, r_h) \mapsto L_h(v_h, q_h, r_h; \lambda_h, \eta_h)$$

has a minimizer $(v^*_h, q^*_h, r^*_h)$. The optimality conditions read

$$\alpha(v^*_h - g, v_h) = \sigma_1(q^*_h - \nabla v^*_h, \nabla v_h)_w + \sigma_2(r^*_h - v^*_h, v_h)_h + (\lambda_h, \nabla v_h)_w + (\eta_h, v_h)_h,$$

$$\partial \tilde{F}(q^*_h) \ni -\lambda_h + \sigma_1(\nabla v^*_h - q^*_h),$$

$$\partial H(r^*_h) \ni -\eta_h + \sigma_2(v^*_h - r^*_h).$$

Using (2) and the particular choice of functions $\lambda_h$ and $\eta_h$ we observe that the triple $(u_h, p_h, s_h)$ satisfies these optimality conditions. Altogether, we have

$$\min_{u_h} E(u_h) = \min_{(u_h, p_h, s_h)} \max_{(\lambda_h, \eta_h)} L_h(u_h, p_h, s_h; \lambda_h, \eta_h).$$

Due to the fact that $p_h$ is elementwise constant and that mass lumping is used for the Lagrange multiplier $\eta_h$ and the auxiliary variable $s_h$, the minimization with respect to $p_h$ and $s_h$ can be done explicitly on each element and in each node, respectively. We approximate a saddle-point with the following iterative scheme (cf. [13]).

**Algorithm 1** (Split-split method). Choose $(p_h^0, s_h^0) \in L^0(T_h)^d \times S^1(T_h)$ such that $\chi \leq s_h^0 \leq \psi$ and $(\lambda_h^0, \eta_h^0) \in L^0(T_h)^d \times S^1(T_h)$ and set $j = 0$.

1. Compute the minimizer $u^j_{h+1}$ of

$$u_h \mapsto L_h(u_h, p^j_h, s^j_h; \lambda^j_h, \eta^j_h).$$
i.e., find $u_{h}^{j+1} \in S^1(T_h)$ such that

$$0 = \alpha (u_{h}^{j+1} - g, v_h) - (\lambda_h^j, \nabla v_h)_w - \sigma_1(p_{h}^{j+1} - \nabla u_h^{j+1}, \nabla v_h)_w - (\eta_{h}^{j+1} - \sigma_2(u_{h}^{j+1} - v_h), v_h)_h$$

for all $v_h \in S^1(T_h)$.

(2) Compute the minimizer $(p_{h}^{j+1}, s_{h}^{j+1})$ of

$$(p_h, s_h) \mapsto L_h(u_{h}^{j+1}, p_h, s_h; \lambda_h^j, \eta_{h}^{j+1}),$$

i.e., define $(p_{h}^{j+1}, s_{h}^{j+1}) \in L^0(T_h)^d \times S^1(T_h)$ via

$$p_{h}^{j+1} = \frac{1}{\sigma_1} (|\sigma_1 \nabla u_h^{j+1} - \lambda_h^j| - h^{-d}) + \frac{\sigma_1 \nabla u_h^{j+1} - \lambda_h^j}{|\sigma_1 \nabla u_h^{j+1} - \lambda_h^j|},$$

where $(t)_+ = \max\{t, 0\}$, and

$$s_{h}^{j+1} = \max \left\{ \chi(z), \min \left\{ u_{h}^{j+1}(z) - \frac{\eta_{h}^{j+1}(z)}{\sigma_2}, \psi(z) \right\} \right\}.$$ 

for all $z \in N_h$.

(3) Update

$$\lambda_{h}^{j+1} = \lambda_h^j + \sigma_1(p_{h}^{j+1} - \nabla u_h^{j+1}),$$

$$\eta_{h}^{j+1} = \eta_h^j + \sigma_2(s_{h}^{j+1} - u_h^{j+1}),$$

and stop if $(\frac{1}{\sigma_1^2} \|\lambda_h^{j+1} - \lambda_h^j\|^2_w + \|p_{h}^{j+1} - p_h^j\|^2_w)^{1/2} \leq \varepsilon_{\text{stop}}^{(1)}$ and $(\frac{1}{\sigma_2^2} \|\eta_{h}^{j+1} - \eta_h^j\|^2_h + \|s_{h}^{j+1} - s_h^j\|^2_h)^{1/2} \leq \varepsilon_{\text{stop}}^{(2)}$. Increase $j \to j + 1$ and continue with (1) otherwise.

Note that step (3) in Algorithm 1 resembles ascent steps for $L_h$ with respect to $\lambda_h$ and $\eta_h$. The following stability estimate for the split-split method is a consequence of general results for splitting methods, see, e.g., [13, 15], and arguments exploiting only the (strong) convexity of the involved functionals, which have been used by Wu and Tai in [22], for the classical ROF problem. It implies the convergence $\|\lambda_{h}^{j+1} - \lambda_h^j\|_w + \|p_{h}^{j+1} - p_h^j\|_w \to 0$ and $\|\eta_{h}^{j+1} - \eta_h^j\|_h + \|s_{h}^{j+1} - s_h^j\|_h \to 0$, i.e., Algorithm 1 terminates.

**Proposition 3.** Let $(u_h, p_h, s_h; \lambda_h, \eta_h)$ be a saddle-point of $L_h$ with $p_h = \nabla u_h$ and $s_h = u_h$. For arbitrary initializations of Algorithm 1 we have for any $J \geq 0$

$$\frac{1}{2} (\|\eta_h - \eta_h^{j+1}\|_h^2 + \|\lambda_h - \lambda_h^{j+1}\|_w^2 + \sigma_1^2 \|p_h - p_h^{j+1}\|_w^2 + \sigma_2^2 \|s_h - s_h^{j+1}\|_h^2) + \frac{1}{2} \sum_{j=0}^J (\sigma_1 \|p_{h}^{j+1} - p_h^j\|_w^2 + \sigma_2 \|s_{h}^{j+1} - s_h^j\|_h^2$$

$$+ \frac{1}{\sigma_1} \|\lambda_h^{j+1} - \lambda_h^j\|_w^2 + \frac{1}{\sigma_2} \|\eta_{h}^{j+1} - \eta_h^j\|_h^2 + \alpha \|u_h - u_h^{j+1}\|_w^2) \leq \frac{1}{2} (\|\eta_h - \eta_h^0\|_h^2 + \|\lambda_h - \lambda_h^0\|_w^2 + \sigma_1^2 \|\nabla u_h - p_h^0\|_w^2 + \sigma_2^2 \|u_h - s_h^0\|_h^2).$$
5.2. Regularization and penalization. Another way to approximate the discrete minimizer $u_h$ of $E$ is to regularize the $BV$-seminorm, introduce an auxiliary variable $s_h = u_h$ and to penalize the difference $s_h - u_h$ with a penalization parameter $\delta > 0$ while allowing $u_h$ to penetrate the obstacles. We obtain the functional

$$E_{h,\varepsilon,\delta}(v_h, r_h) := \int_\Omega |\nabla v_h|_2 \, dx + \frac{\alpha}{2} \|v_h - g\|^2 + \frac{1}{2\delta} |r_h - v_h|^2_h + I_K(r_h),$$

where $|\nabla v_h|_2 = (|\nabla v_h|^2 + \varepsilon^2)^{1/2}$ and $\varepsilon, \delta > 0$. The introduction of $s_h$ decouples the two nonlinearities. As in the split-split method, we use mass lumping for the variable $s_h$ to update it node-wise. A similar approach has been used by Hintermüller et al. in [17] for the predual of the classical ROF problem and an alternate minimization technique has been employed for the numerical solution.

**Remark 4.** Note that in contrast to the other two methods, the constraint $u_h \in K$ is enforced only in the limit as $\delta \to 0$.

We can establish the existence of a discrete minimizer $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in S^1(\mathcal{T}_h) \times S^1(\mathcal{T}_h)$ of $E_{h,\varepsilon,\delta}$ using the direct method in the calculus of variations. Let $\Pi_{K,h}$ be the projection operator from $S^1(\mathcal{T}_h)$ onto $K$ with respect to $(\cdot, \cdot)_h$, i.e., $\Pi_{K,h}$ is defined by

$$\Pi_{K,h}v_h(z) := \begin{cases} v_h(z), & \text{if } \chi(z) \leq v_h(z) \leq \psi(z), \\ \psi(z), & \text{if } v_h(z) > \psi(z), \\ \chi(z), & \text{if } v_h(z) < \chi(z), \end{cases}$$

for $v_h \in S^1(\mathcal{T}_h)$ and all $z \in \mathcal{N}_h$. The optimality conditions for a minimizing pair $(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})$ then imply the relation $s_{h,\varepsilon,\delta} = \Pi_{K,h}u_{h,\varepsilon,\delta}$.

**Lemma 5.1.** Let $v_h \in S^1(\mathcal{T}_h)$. If $\mathcal{T}_h$ consists only of right triangles ($d = 2$) or of tetrahedra having three mutually orthogonal edges ($d = 3$) and if both $\chi$ and $\psi$ are globally constant, then we have for any $T \in \mathcal{T}_h$$$
|\nabla \Pi_{K,h}v_h|_T| \leq |\nabla v_h|_T|.$$

**Proof.** Consider $d = 2$, the case $d = 3$ is analogous. Without loss of generality we assume that the two edges $\text{conv}\{z_0, z_1\}$ and $\text{conv}\{z_0, z_2\}$ of $T$ forming the right angle are parallel to the $x$-axis and the $y$-axis, respectively. Otherwise, we rotate the triangle which preserves the length of the gradients. We abbreviate $v^K_h = \Pi_{K,h}v_h$ and calculate

$$|\nabla v^K_h|_T|^2 = \frac{(v^K_h(z_1) - v^K_h(z_0))^2}{h_x^2} + \frac{(v^K_h(z_2) - v^K_h(z_0))^2}{h_y^2} \leq \frac{(v_h(z_1) - v_h(z_0))^2}{h_x^2} + \frac{(v_h(z_2) - v_h(z_0))^2}{h_y^2} = |\nabla v_h|_T|^2.$$ 

\[\square\]

**Remark 5.** The assertion of Lemma 5.1 may not hold true for general partitions containing triangles which are not right-angled. To see this, consider $z_0 = [0, 1/2]^T$, $z_1 = [1, 0]^T$ and $z_2 = [0, 1]^T$ and let $T = \text{conv}\{z_0, z_1, z_2\}$. The associated affine nodal basis functions are given by

$$\varphi_0(x, y) = -2x - 2y + 2, \quad \varphi_1(x, y) = x, \quad \varphi_2(x, y) = x + 2y - 1.$$
Let \( \chi \equiv 0 \) and \( \psi \equiv 1 \) and define \( v_h \) to be affine-linear on \( T \) with \( v_h(z_0) = 1, v_h(z_1) = 2 \) and \( v_h(z_2) = 0 \). Then \( \Pi_{K,h} v_h(z_0) = 1, \Pi_{K,h} v_h(z_1) = 1 \) and \( \Pi_{K,h} v_h(z_2) = 0 \). This yields
\[
\nabla v_h = [-2, -2]^T + 2[1, 0]^T = [0, -2]^T,
\]
\[
\nabla \Pi_{K,h} v_h = [-2, -2]^T + [1, 0]^T = [-1, -2]^T,
\]
\[\text{i.e., } |\nabla \Pi_{K,h} v_h| = \sqrt{5} > 2 = |\nabla v_h|.
\]

**Proposition 4.** Let \( u_h \in S^1(T_h) \) and \((u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in S^1(T_h) \times S^1(T_h)\) be the unique minimizer of \( E \) and a minimizer of \( E_{h,\varepsilon,\delta} \), respectively, and let \((v_h, r_h) \in S^1(T_h) \times S^1(T_h)\) be arbitrary. Then we have
\[
\frac{\alpha}{2} \|v_h - u_{h,\varepsilon,\delta}\|^2 \leq E_{h,\varepsilon,\delta}(v_h, r_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}),
\]
and particularly
\[
\frac{\alpha}{2} \|u_h - u_{h,\varepsilon,\delta}\|^2 + \frac{1}{4\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2 \leq c(\varepsilon + \delta h^{-2}),
\]
\[\frac{\alpha}{2} \|u_h - s_{h,\varepsilon,\delta}\|^2 \leq c(\varepsilon + \delta h^{-2}).
\]

**Proof.** The proof of (3) is analogous to the proof of Lemma 3.1. Since \( u_h \in K \) we can choose \((v_h, r_h) = (u_h, u_h)\) in (3). Observing that \(|x| \leq |x_\varepsilon| \leq |x| + \varepsilon\) for \( x \in \mathbb{R}^d \), and using Lemma 2.1 and the minimality property of \( u_h \), we deduce
\[
\frac{\alpha}{2} \|u_h - u_{h,\varepsilon,\delta}\|^2
\]
\[\leq E_{h,\varepsilon,\delta}(u_h, u_h) - E_{h,\varepsilon,\delta}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})
\]
\[= \int_{\Omega} |\nabla u_h| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[= \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[\leq \varepsilon |\Omega| + \int_{\Omega} |\nabla u_h| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[= \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[\leq \varepsilon |\Omega| + \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[\leq \varepsilon |\Omega| + \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[= \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \varepsilon \, dx + \frac{\alpha}{2} \|u_h - g\|^2
\]
\[\leq \varepsilon |\Omega| + \|\nabla (s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta})\|_{L^1(\Omega)} - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|^2
\]
\[+ \frac{\alpha}{2} \int_{\Omega} (s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta})(s_{h,\varepsilon,\delta} + u_{h,\varepsilon,\delta} - 2g) \, dx
\]
Remark 6. Note that we only have convergence as \( h \to 0 \) if \( \delta = O(h^\gamma) \) with \( \gamma > 2 \).

With a view to recovering the convergence rate from Proposition 2 for \( d = 2 \) we choose \( \varepsilon = h^{1/2}/|\Omega| \) and \( \delta = h^{5/2}/\alpha \) (for \( d = 3 \): \( \varepsilon = h^{1/3}/|\Omega| \) and \( \delta = h^{7/3}/\alpha \)) or \( \varepsilon = h/|\Omega| \) and \( \delta = h^3/\alpha \) to recover the optimal convergence rate \( O(h^{1/2}) \), cf. [2] Rmk. 10.9 (ii) and [5] Thm. 6.5.

Remark 7. We can significantly relax the restriction on \( \delta \) if both \( \chi \) and \( \psi \) are globally constant and if \( \mathcal{T}_h \) consists only of right triangles or of tetrahedra having three mutually orthogonal edges. Indeed, by Lemma 5.1 we in this case have...
∥∇Π_{K,h}u_{h,ε,δ}∥_{L^1(Ω)} ≤ \|\nabla u_{h,ε,δ}\|_{L^1(Ω)} so that we do not need to use an inverse estimate in the proof of Proposition 4. This yields the improved estimates
\[
\frac{\alpha}{2}\|u_h - u_{h,ε,δ}\|^2 + \frac{1}{4\delta}\|s_{h,ε,δ} - u_{h,ε,δ}\|^2 \leq c(\epsilon + \delta),
\]
\[
\frac{\alpha}{2}\|u_h - s_{h,ε,δ}\|^2 \leq c(\epsilon + \delta).
\]
In this case, we may choose δ = h^{1/2}/α for d = 2 (for d = 3: δ = h^{1/3}/α) and δ = h/α to recover the convergence rate from Proposition 2 and the optimal convergence rate, respectively.

To define a stable scheme for the approximation of a minimizer (u_{h,ε,δ}, s_{h,ε,δ}) \in S^1(T_h) \times S^1(T_h) of E_{h,ε,δ} we consider for an auxiliary variable p_h = |\nabla u_{h,ε,δ}|^{1/2} the functional
\[
\bar{E}_{h,ε,δ}(v_h, q_h, r_h) := \int_{Ω} \frac{|\nabla v_h|^2 + \epsilon^2}{2q_h^2} + \frac{q_h^2}{2} dx + \frac{\alpha}{2}\|v_h - g\|^2 + \frac{1}{2\delta}\|r_h - v_h\|^2_h + I_K(r_h).
\]
Note that for fixed (v_h, r_h) it follows that
\[
\min_{q_h \in C^1(T_h), q_h > 0}\bar{E}_{h,ε,δ}(v_h, q_h, r_h) = E_{h,ε,δ}(v_h, r_h)
\]
with optimal q_h given by
\[
q_h = (|\nabla v_h|^2 + \epsilon^2)^{1/4}.
\]
Due to the separate convexity of the functional \bar{E}_{h,ε,δ} with respect to v_h, q_h and r_h we use a decoupled semi-implicit subgradient flow whose iterates can be determined explicitly.

**Algorithm 2** (Heron-penalty method). Choose \((u_{h,0}^0, p_{h,0}^0, s_{h,0}^0) \in S^1(T_h) \times L^0(T_h) \times S^1(T_h)\) such that \(\chi \leq s_{h,0}^1 \leq \psi\) and \(p_{h,0}^1 \geq \sqrt{\epsilon}\). Set \(j = 0\).

1. Compute the minimizer \(u_{h}^{j+1}\) of
\[
(u_h \mapsto \bar{E}_{h,ε,δ}(u_h, p_{h}^j, s_{h}^j) + \frac{1}{2\tau}\|u_h - u_h^j\|^2),
\]
i.e., \(u_{h}^{j+1} \in S^1(T_h)\) such that
\[
(-d_t u_{h}^{j+1}, v_h) = \int_{Ω} \frac{\nabla u_{h}^{j+1}, \nabla v_h}{(p_{h}^j)^2} dx + \alpha(u_{h}^{j+1} - g, v_h) - \frac{1}{\delta}(s_{h}^j - u_{h}^{j+1}, v_h)_h
\]
for all \(v_h \in S^1(T_h)\).

2. Compute the minimizing \((p_{h}^{j+1}, s_{h}^{j+1})\) of
\[
(p_h, s_h) \mapsto \bar{E}_{h,ε,δ}(u_{h}^{j+1}, p_h, s_h) + \frac{1}{2\tau}\|p_h - p_{h}^j\|^2 + \frac{1}{2\tau}\|s_h - s_{h}^j\|^2_h,
\]
i.e., define \((p_{h}^{j+1}, s_{h}^{j+1}) \in L^0(T_h)^d \times S^1(T_h)\) via
\[
(-d_t p_{h}^{j+1}, q_h) = \int_{Ω} \left( p_{h}^{j+1} - \frac{|\nabla u_{h}^{j+1}|^2 + \epsilon^2}{(p_{h}^{j+1})^3} \right) q_h dx
\]
for all \( q_h \in \mathcal{L}^0(T_h) \), and
\[
s_{h}^{j+1}(z) = \max \left\{ \chi(z), \min \left\{ \frac{\delta s_{h}^{j}(z) + \tau u_{h}^{j+1}(z)}{\tau + \delta}, \psi(z) \right\} \right\}
\]
for all \( z \in N_h \).

(3) Stop if \( \left( \|d_t u_{h}^{j+1} \|^2 + \|d_t p_{h}^{j+1} \|^2 + \|d_t s_{h}^{j+1} \|^2 \right) / \varepsilon_{\text{stop}} \leq \varepsilon_{\text{stop}} \). Otherwise, increase \( j \rightarrow j + 1 \) and continue with (1).

The update with respect to \( u_h, p_h \) and \( s_h \) can be regarded as proximal maps of \( \tilde{E}_{h,\varepsilon,\delta}(\cdot, p_h, s_h) \), \( \tilde{E}_{h,\varepsilon,\delta}(u_{h}^{j+1},\cdot, \cdot) \) and \( \tilde{E}_{h,\varepsilon,\delta}(u_{h}^{j+1}, p_{h}^{j+1}, \cdot) \), respectively. The following proposition guarantees stability and convergence of the algorithm.

Proposition 5. Let \( \tau > 0 \), \( u_0^h \in S^1(T_h) \), \( s_0^h \in S^1(T_h) \) with \( \chi \leq s_0^h \leq \psi \) and \( p_0^h \in \mathcal{L}^0(T_h) \) with \( p_0^h > 0 \) be arbitrary initializations of Algorithm 2 and let the sequences \( (u_h^j)_{j \in \mathbb{N}}, (s_h^j)_{j \in \mathbb{N}} \) and \( (p_h^j)_{j \in \mathbb{N}} \) be generated by Algorithm 2. Then for arbitrary \( J \in \mathbb{N} \) the stability estimate
\[
\tau \sum_{j=0}^{J} \|d_t u_{h}^{j+1} \|^2 + \|d_t p_{h}^{j+1} \|^2 + \|d_t s_{h}^{j+1} \|^2 \leq \tilde{E}_{h,\varepsilon,\delta}(u_0^h, p_0^h, s_0^h),
\]
holds. Particularly, Algorithm 2 terminates and \( \left( \tilde{E}_{h,\varepsilon,\delta}(u_0^h, p_0^h, s_0^h) \right)_{j \in \mathbb{N}} \) is monotonically decreasing. Furthermore, every convergent subsequence \( (u_{h}^{j})_{\ell \in \mathbb{N}} \subset S^1(T_h) \times S^1(T_h) \) converges to a minimizer \( (u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \in S^1(T_h) \times S^1(T_h) \) of \( E_{h,\varepsilon,\delta} \).

Proof. The proof of the stability estimate follows as in the proof of [3 Thm. 3.4] by choosing \( -d_t u_{h}^{j+1}, -d_t p_{h}^{j+1} \) and \( -d_t s_{h}^{j+1} \) in the optimality conditions defining the iterates in Algorithm 2 and using the separate convexity of \( \tilde{E}_{h,\varepsilon,\delta} \). The stability estimate (4) implies that the sequence \( (u_h^j) \) is bounded. We choose a subsequence \( (u_h^{j})_{\ell} \) which converges to some \( u_{h,\varepsilon,\delta} \in S^1(T_h) \). Since \( d_t u_{h}^{j} \rightarrow 0 \) it follows that \( u_{h}^{j+1} \rightarrow u_{h,\varepsilon,\delta} \) as well and by the equivalence of norms on finite-dimensional spaces we also have \( \nabla u_{h}^{j} \rightarrow \nabla u_{h,\varepsilon,\delta} \) and \( \nabla u_{h}^{j+1} \rightarrow \nabla u_{h,\varepsilon,\delta} \). Furthermore, since \( d_t p_{h}^{j+1} \rightarrow 0 \) the optimality condition for the iterates \( (p_h^{j+1})_{\ell} \) implies
\[
\frac{\|\nabla u_{h}^{j+1} \|^2 + \varepsilon^2}{(p_h^{j+1})^3} - p_h^{j+1} \rightarrow 0,
\]
which means that \( p_h^{j+1} \rightarrow (\|\nabla u_{h,\varepsilon,\delta} \|^2 + \varepsilon^2)^{1/4} \). The optimality condition for the iterate \( u_h^{j+1} \) and \( d_t u_{h}^{j+1} \rightarrow 0 \) then imply that
\[
\frac{1}{\delta} (s_h^{j} - u_h^{j+1}, v_h) \rightarrow \int_{\Omega} \frac{\nabla u_{h,\varepsilon,\delta} \cdot \nabla v_h}{\left( \|\nabla u_{h,\varepsilon,\delta} \|^2 + \varepsilon^2 \right)^{1/2}} \, dx + \alpha \int_{\Omega} (u_{h,\varepsilon,\delta} - g) v_h \, dx
\]
as \( \ell \rightarrow \infty \) for all \( v_h \in S^1(T_h) \), i.e., \( (s_h^{j})_{\ell} \) converges to some function \( s_{h,\varepsilon,\delta} \in S^1(T_h) \). Hence, \( \frac{1}{\delta} (s_h^{j} - u_h^{j+1} - v_h) \rightarrow \frac{1}{\delta} (s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}, v_h) \) for \( \ell \rightarrow \infty \) and we obtain
\[
0 = \int_{\Omega} \frac{\nabla u_{h,\varepsilon,\delta} \cdot \nabla v_h}{\left( \|\nabla u_{h,\varepsilon,\delta} \|^2 + \varepsilon^2 \right)^{1/2}} \, dx + \alpha \int_{\Omega} (u_{h,\varepsilon,\delta} - g) v_h \, dx - \frac{1}{\delta} \int_{\Omega} (s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}) v_h \, dx
\]
for all \( v_h \in S^1(T_h) \). The optimality condition for the iterate \( s_h^{j+1} \) reads
\[
(-d_t s_h^{j+1}, r_h - s_h^{j+1}) + \frac{1}{\delta} (s_h^{j+1} - s_h^{j+1}, r_h - s_h^{j+1}) = I_K(s_h^{j+1}) \leq I_K(r_h)
\]
for all \( q_h \in \mathcal{L}^0(T_h) \), and
for all \( r_h \in K \cap S^1(T_h) \). This means that for \( \ell \to \infty \) we have
\[
\frac{1}{\delta} (u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}, r_h - s_{h,\varepsilon,\delta})_h + I_K(s_{h,\varepsilon,\delta}) \leq I_K(r_h)
\]
for all \( r_h \in K \cap S^1(T_h) \), i.e., \( (u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \) satisfies the necessary and sufficient optimality conditions for a minimizer of \( E_{h,\varepsilon,\delta} \).

**Remark 8.** Note that due to Proposition 4 all minimizers \( (u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \) of \( E_{h,\varepsilon,\delta} \) are within the ball in \( L^2(\Omega) \) around the unique minimizer \( u_h \) of \( E \) with radius \( c(\varepsilon + \delta h^{-2}) \). Therefore, by Proposition 5 the iterates \( (u_h^j, s_h^j) \) generated by Algorithm 2 get arbitrarily close to this ball around \( u_h \).

### 5.3. Regularization and splitting.

Alternatively, we may consider only a regularization of the BV-seminorm and seek a minimizer \( u_{h,\varepsilon} \in S^1(T_h) \) of the functional
\[
E_{\varepsilon}(v_h) := \int_{\Omega} |\nabla v_h|_{\varepsilon} \, dx + \alpha \frac{1}{2} \|v_h - g\|^2 + I_K(v_h).
\]
In order to enforce the constraint strictly we can use an augmented Lagrangian ansatz, i.e., work with the regularized augmented Lagrangian functional
\[
L_{h,\varepsilon}(v_h, r_h; v_h) := \int_{\Omega} |\nabla v_h|_{\varepsilon} \, dx + \alpha \frac{1}{2} \|v_h - g\|^2
+ (v_h, r_h - v_h)_h + \frac{s}{2} \|r_h - v_h\|_h^2 + I_K(r_h)
\]
to approximate the unique minimizer \( u_{h,\varepsilon} \in S^1(T_h) \) of \( E_{\varepsilon} \). We can use the following iteration to approximate the unique minimizer \( u_{h,\varepsilon} \).

**Algorithm 3 (Regularized splitting method).** Choose \( \eta_{h,0} \in S^1(T_h) \) and set \( j = 0 \).

1. Compute a minimizer \( (u_h^{j+1}, s_h^{j+1}) \in S^1(T_h) \times S^1(T_h) \) of
\[
(u_h, s_h) \mapsto L_{h,\varepsilon}(u_h, s_h; \eta_{h}^{j}).
\]

2. Update
\[
\eta_{h}^{j+1} = \eta_{h}^{j} + \sigma(s_{h}^{j+1} - u_{h}^{j+1}).
\]
and stop if \( \frac{1}{\alpha} \|\eta_{h}^{j+1} - \eta_{h}^{j}\|_h \leq \varepsilon_{\text{stop}} \).

**Remark 9.** (i) The stability (and convergence) of Algorithm 3 follows from the theory of augmented Lagrangian methods (cf. [15][16]) using arguments from [22] which avoid differentiability assumptions on the involved functionals but exploit their (strong) convexity.

(ii) In contrast to Algorithm 2, we do not include the term \( \|s_{h}^{j+1} - s_{h}^{j}\|_h \) in the stopping criterion since the optimality conditions we obtain in step (1) are coupled, i.e., as soon as \( \eta_{h}^{j+1} - \eta_{h}^{j} = 0 \) the triple \( (u_{h}^{j+1}, s_{h}^{j+1}, \eta_{h}^{j+1}) \) satisfies the optimality conditions for a saddle-point of \( L_{h,\varepsilon} \).

Since the minimizer in step (1) of Algorithm 3 is not directly accessible we consider for \( j \in \mathbb{N} \) the augmented functional
\[
\tilde{L}_{h,\varepsilon}(u_h, p_h, s_h; \eta_{h}^{j}) := \int_{\Omega} \left[ \frac{\nabla u_h^2 + \varepsilon}{2 p_h^n} + \frac{p_h^2}{2} \right] \, dx + \frac{\alpha}{2} \|u_h - g\|^2 + (\eta_{h}^{j}, s_h - u_h)_h + \frac{s}{2} \|s_h - u_h\|_h^2 + I_K(s_h).
\]
Noting that \( \bar{L}_{h,\varepsilon} \) is separately convex in \( u_h, p_h \) and \( s_h \) we use as in the Heron-penalty method a semi-implicit subgradient flow to approximate a minimizer \( (u^{j+1}_h, s^{j+1}_h) \) of \( L_{h,\varepsilon}(u_h, s_h; \eta^0_h) \) in the \((j+1)\)-th iteration.

**Algorithm 4** (Decoupled regularized splitting method). Choose \((u^0_h, p^0_h, s^0_h) \in S^1(\mathcal{T}_h) \times L^0(\mathcal{T}_h) \times S^1(\mathcal{T}_h) \) and \( \eta^0_h \in S^1(\mathcal{T}_h) \) such that \( \chi_h \leq s^0_h \leq \psi_h \) and \( p^0_h \geq \sqrt{\varepsilon} \).

Set \( j = 0 \).

1. Set \( u^{j+1}_h = u^j_h, p^{j+1}_h = p^j_h \) and \( s^{j+1}_h = s^j_h \).
2. For \( \ell = 0, \ldots, M-1 \): first compute the minimizer \( u^{j+1,\ell+1}_h \) of
   \[
   u_h \mapsto \bar{L}_{h,\varepsilon}(u_h, p^{j+1}_h, s^{j+1}_h; \eta^{j+1}_h) + \frac{1}{2\tau} \| u_h - u^{j+1,\ell}_h \|^2,
   \]
   and then compute the minimizing \((p^{j+1,\ell+1}_h, s^{j+1,\ell+1}_h)\) of
   \[
   (p_h, s_h) \mapsto \bar{L}_{h,\varepsilon}(u^{j+1,\ell+1}_h, p_h, s_h; \eta^{j+1}_h) + \frac{1}{2\tau} \| p_h - p^{j+1,\ell}_h \|^2 + \frac{1}{2\tau} \| s_h - s^{j+1,\ell}_h \|^2;
   \]
3. Set \( u^{j+1}_h = u^{j+1,M}_h, p^{j+1}_h = p^{j+1,M}_h \) and \( s^{j+1}_h = s^{j+1,M}_h \).
4. Update \( \eta^{j+1}_h = \eta^j_h + \sigma(s^{j+1}_h - u^{j+1}_h) \),

and stop if \( \frac{1}{\sigma} \| \eta^{j+1}_h - \eta^j_h \|_h \leq \varepsilon_{\text{stop}} \). Otherwise, increase \( j \rightarrow j + 1 \) and continue with (1).

**Remark 10.** (i) Instead of the fixed number \( M \) of inner iterations in step (2) of Algorithm 4 one can also stop the inner iteration using the stopping criterion
\[
(\|d_t u^{j+1,\ell}_h\|^2 + \|d_t p^{j+1,\ell}_h\|^2 + \|d_t s^{j+1,\ell}_h\|^2)^{\frac{1}{2}} \leq \varepsilon_{\text{stop}}.
\]
(ii) In our experiments we will set \( M = 1 \) and use the stopping criterion
\[
(\|d_t u^{j+1}_h\|^2 + \|d_t p^{j+1}_h\|^2)^{\frac{1}{2}} \leq \varepsilon_{\text{stop}}^{(1)} \quad \text{and} \quad (\|d_t s^{j+1}_h\|^2 + \|d_t s^{j+1}_h - u^{j+1}_h\|^2)^{\frac{1}{2}} \leq \varepsilon_{\text{stop}}^{(2)}
\]
in step (4) of Algorithm 4. We will refer to this setting as the **Heron-split method**.

6. **Application to piecewise constant segmentation.** In this section we investigate the constrained minimization problem
\[
\min_{u \in BV(\Omega)} E^{reg}(u) := \min_{u \in BV(\Omega)} \int_{\Omega} |Du| + \alpha \int_{\Omega} fu \, dx + I_K(u)
\]
with \( f \in L^\infty(\Omega) \) and \( K := \{ v \in L^2(\Omega) : 0 \leq v \leq 1 \ \text{a.e.} \} \). This constrained minimization problem occurs, e.g., in the piecewise constant segmentation of images discussed in [13] where one aims at minimizing for a given image \( g \in L^\infty(\Omega) \) the nonconvex functional
\[
MS(\Sigma, c_1, c_2) := \int_{\Omega} |D\chi_\Sigma| + \alpha \int_{\Sigma} (c_1 - g)^2 \, dx + \alpha \int_{\Omega \setminus \Sigma} (c_2 - g)^2 \, dx
\]
with \( \chi_\Sigma \) being the characteristic function of the unknown set \( \Sigma \). The authors propose an alternating minimization with respect to the unknown variables \( c_1 \) and \( c_2 \). For a fixed set \( \Sigma \) the optimal \( c_1 \) and \( c_2 \) are given by
\[
c_1 = \frac{1}{|\Sigma|} \int_{\Sigma} g \, dx \quad \text{and} \quad c_2 = \frac{1}{|\Omega \setminus \Sigma|} \int_{\Omega \setminus \Sigma} g \, dx.
\]
However, the major difficulty amounts to solving the minimization problem defined by the functional \( MS(\cdot, c_1, c_2) \) for fixed \( c_1 \) and \( c_2 \). Berkels et al. considered a
Then, for almost every $\gamma \tilde{\in}$ ever, the construction of the functional $MS(\cdot, c_1, c_2)$ by computing a minimizer $u$ of $E^{seg}$ with $\chi \equiv 0$, $\psi \equiv 1$ and 
\[ f := (c_1 - g)^2 - (c_2 - g)^2. \]
Then, for almost every $\gamma \in [0, 1]$, the set 
\[ \Sigma := \{ x \in \mathbb{R}^d : u(x) \geq \gamma \} \]
is a minimizer of $MS(\cdot, c_1, c_2)$, cf. [13] Thm. 2. In [7] Thm. 1 Berkers has shown that this holds true even for all $\gamma \in [0, 1]$.

One can establish the existence of a minimizer $u \in BV(\Omega) \cap K$ of $E^{seg}$, however, the minimizer may not be unique. Restricting the minimization of $E^{seg}$ to $S^1(T_h)$ then also provides a minimizer $u_h \in S^1(T_h)$. Analogously to Proposition 2 we obtain the following estimate.

**Lemma 6.1.** Let $u$ and $u_h$ be minimizers of $E^{seg}$ on $BV(\Omega)$ and on $S^1(T_h)$, respectively. We then have for $d = 2$

\[ E^{seg}(u_h) - E^{seg}(u) \leq c h^{1/2}, \]
and, for $d = 3$,

\[ E^{seg}(u_h) - E^{seg}(u) \leq c h^{1/3}. \]

**Proof.** The proof is analogous to that of Proposition 2 and we therefore omit it.

The authors in [13] furthermore observed that the functional 
\[ \bar{E}^{seg}(u) := \int_{\Omega} |Du| + \int_{\Omega} \beta \theta(u) + \alpha f u \, dx, \]
where $\theta(\xi) := \max\{0, 2|\xi - \frac{1}{2}| - 1\}$, has the same minimizers as $E^{seg}$ provided that $\beta > \frac{1}{2}\|f\|_{L^\infty(\Omega)}$, cf. [13] Claim 1. They employ a regularization of the $BV$-seminorm and of the function $\theta$ and use an explicit gradient descent scheme to approximate a minimizer of $\bar{E}^{seg}$ and $E^{seg}$, respectively.

We remark that this procedure may be practical in this specific situation, however, the construction of the functional $\bar{E}^{seg}$ only enables one to deal with this particular constrained minimization problem defined by $E^{seg}$. Since we can adapt the iterative schemes presented in Section 5 to the minimization of $E^{seg}$ after making minor adjustments we will also present numerical experiments for the piecewise constant segmentation problem in the next section. We conclude this section with a result which is analogous to Proposition 4.

**Proposition 6.** Let $u_h \in S^1(T_h)$ and $(u_{h, \varepsilon, \delta}, s_{h, \varepsilon, \delta}) \in S^1(T_h) \times S^1(T_h)$ be minimizers of $E^{seg}$ and of the functional 
\[ E^{seg}_{h, \varepsilon, \delta}(v_h, r_h) := \int_{\Omega} |\nabla v_h| \, dx + \alpha \int_{\Omega} f v_h \, dx + \frac{1}{2\delta} \|v_h - r_h\|^2_h + I_K(r_h), \]
respectively. Then we have 
\[ -\varepsilon|\Omega| \leq E^{seg}(u_h) - E^{seg}_{h, \varepsilon, \delta}(u_{h, \varepsilon, \delta}, s_{h, \varepsilon, \delta}) + \frac{1}{4\delta} \|u_{h, \varepsilon, \delta} - s_{h, \varepsilon, \delta}\|^2 \leq c(\varepsilon + \delta h^{-2}). \]
Proof. Observing that for \( a \in \mathbb{R}^d \) we have \(|a| \leq |a|_\varepsilon \leq |a| + \varepsilon \) we obtain
\[
E_{h}^{seg}(u_h) - E_{h,\varepsilon,\delta}^{seg}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta}) \geq E_{h}^{seg}(u_h) - E_{h,\varepsilon,\delta}^{seg}(u_h, u_h)
\]
\[
= \int_{\Omega} |\nabla u_h| \, dx - \int_{\Omega} |\nabla u_h|_\varepsilon \, dx
\]
\[
\geq -\varepsilon |\Omega|,
\]
which is the first inequality. Using \( \Pi_K u_{h,\varepsilon,\delta} = s_{h,\varepsilon,\delta} \) and the arguments in the proof of Proposition 4 we obtain
\[
E_{h}^{seg}(u_h) - E_{h,\varepsilon,\delta}^{seg}(u_{h,\varepsilon,\delta}, s_{h,\varepsilon,\delta})
\]
\[
= \int_{\Omega} |\nabla u_h| \, dx + \alpha \int_{\Omega} f u_h \, dx
\]
\[
- \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \, dx - \alpha \int_{\Omega} f u_{h,\varepsilon,\delta} \, dx - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2
\]
\[
\leq \int_{\Omega} |\nabla u_h| \, dx + \alpha \int_{\Omega} f u_h \, dx
\]
\[
- \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \, dx - \alpha \int_{\Omega} f u_{h,\varepsilon,\delta} \, dx - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2
\]
\[
\leq \int_{\Omega} |\nabla u_h| \, dx + \alpha \int_{\Omega} f u_h \, dx
\]
\[
- \int_{\Omega} |\nabla u_{h,\varepsilon,\delta}| \, dx - \alpha \int_{\Omega} f u_{h,\varepsilon,\delta} \, dx - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2
\]
\[
\leq \varepsilon^2 \frac{|\Omega|}{h^2} + \frac{1}{8\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2 - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2
\]
\[
+ \alpha \|f\| \|u_{h,\varepsilon,\delta} - s_{h,\varepsilon,\delta}\|
\]
\[
\leq \varepsilon^2 \frac{|\Omega|}{h^2} + \frac{1}{8\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2 - \frac{1}{2\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2
\]
\[
+ \frac{1}{8\delta} \|s_{h,\varepsilon,\delta} - u_{h,\varepsilon,\delta}\|_h^2 + 2\alpha^2 \|f\|^2.
\]
This proves the assertion. \(\square\)

Remark 11. In order to recover the convergence rate from Lemma 6.1 we choose \(\varepsilon = h^{1/2}/|\Omega|\) and \(\delta = h^{5/2}/(\alpha\|f\|)\), if \(d = 2\), and \(\varepsilon = h^{1/3}/|\Omega|\), \(\delta = h^{7/3}/(\alpha\|f\|)\), if \(d = 3\).

Remark 12. Again, if we use a triangulation consisting of right triangles or tetrahedra with mutually orthogonal edges, we can relax the condition on \(\delta\) as described in Remark 7 and may choose \(\delta = h^{1/2}/(\alpha\|f\|)\).

7. Numerical experiments.

7.1. Constrained ROF. We investigate two examples which differ only in the choice of the parameter \(\alpha > 0\). In both cases the algorithms are initialized as follows: we set \(p_h^0 = \lambda_h^0 = 0, \eta_h^0 = 0, s_h^0 = (\chi + \psi)/2\) and \(\sigma_3 = h^{-3/2}\) (see 3) in Algorithm 3. We start Algorithm 2 with \(\varepsilon = h, \tau = 1, u_h^0 = (\chi + \psi)/2, \)
\(p_h^0 = \varepsilon^{1/2}\) and \(s_h^0 = u_h^0, \) and Algorithm 4 with \(\varepsilon = h, \tau = 1, u_h^0 = (\chi + \psi)/2, \)
\(p_h^0 = \varepsilon^{1/2}, s_h^0 = u_h^0\) and \(\eta_h^0 = 0\). To compare the accuracy of the algorithms we use precomputed approximate minimizers \(\tilde{u}_h\) of \(E\) generated by Algorithm 3 with \(\sigma_2 = \alpha, (\tilde{u}_{h,\varepsilon,\delta}, \tilde{s}_{h,\varepsilon,\delta})\) of \(E_{h,\varepsilon,\delta}\) generated by Algorithm 2 with both \(\delta = h/\alpha\) and
\[ \delta = h \text{, and } \tilde{u}_{h,\varepsilon} \text{ of } E_\varepsilon \text{ generated by Algorithm 4 with } \sigma = \alpha \text{ after } 10^4 \text{ iterations.} \]

The stopping criteria have been chosen as follows:

- **Split-split method:**
  \[ \|\tilde{u}_h - u^j_h\| \leq 10^{-3} \] (5)

- **Heron-penalty method:**
  \[ \max\{\|\tilde{u}_{h,\varepsilon,\delta} - u^j_h\|, \|\tilde{s}_{h,\varepsilon,\delta} - s^j_h\|\} \leq 10^{-3} \] (6)

- **Heron-split method:**
  \[ \|\tilde{u}_{h,\varepsilon} - u^j_h\| \leq 10^{-3} \] (7)

These stopping criteria have been chosen for better comparison of the methods. In general, the stopping criteria mentioned in the descriptions of the algorithms should be used.

**Example 1.** We let \( d = 2, \Omega = (0, 1)^2, \chi \equiv 1/5, \psi \equiv 1/2 \) and \( \alpha = 50 \). Moreover, given a uniform triangulation \( T_5 \) of \( \Omega \) consisting of halved squares with mesh size \( h = \sqrt{2}/2^5 \), we let \( \tilde{g} \in S^1(T_5) \) to be the continuous, piecewise affine approximation of the characteristic function \( \chi_{B_{1/5}(x_\Omega)} \) of the closed ball with radius \( 1/5 \) around the center \( x_\Omega \) of \( \Omega \), i.e., \( \tilde{g} \) is defined by

\[
\tilde{g}(z) := \begin{cases} 
1, & \text{if } z \in B_{1/5}(x_\Omega) \\
0, & \text{else}
\end{cases}
\]

for \( z \in N_5 \). We then set \( g := \tilde{g} + \xi_5 \) where \( \xi_5 \in S^1(T_5) \) whose coefficient vector is a sample of a random variable uniformly distributed in the interval \([-1/2, 1/2]\).

|          | Split-split | Heron-penalty | Heron-split |
|----------|-------------|--------------|-------------|
| \( \sigma_2 \) | 1           | \( 1/h \)    | \( h/\alpha \) |
| \( \delta \) | 344         | 39           | 37          |
| \( \sigma \)  | 89          | 42           | 101         |
| \( \alpha \) | 289         | 79           | 79          |
| \( h/\alpha \) | 143         | 55           | 57          |
| \( \beta \)  | 283         | 77           | 78          |
| \( h/\alpha \) | 211         | 69           | 94          |
| \( \gamma \) | 287         | 115          | 120         |
| \( h/\alpha \) | 426         | 101          | 151         |

Table 1. Iteration numbers with (5), (6), (7) for Example 1 and \( \sigma_1 = h^{-3/2} \) for the split-split method and \( \tau = 1 \) for the Heron-penalty and the Heron-split method.

We first of all remark that for the chosen \( \Omega, g \) and \( \alpha \) the inclusion of the obstacle functions \( \chi \equiv 1/5 \) and \( \psi \equiv 1/2 \) is not redundant. The solution for the classical ROF problem with these model parameters attains its maximal and minimal value which are greater than 3/4 and smaller than 1/10, respectively. Moreover, the scaling of the penalization parameter as \( \delta = O(h) \) is justified by Remark 4 since the considered obstacles are globally constant and we use a triangulation consisting of halved squares. Particularly, Assumption 3 is satisfied. In Table 1, we displayed the iteration numbers needed by the algorithms to satisfy the stopping criteria (5), (6), and (7). Since the optimal choice of the step sizes \( \sigma_2 \) and \( \sigma \) for the split-split method and the Heron-split method, respectively, are not clear, we ran the algorithms for three different choices of step sizes. As one can observe the choice \( \sigma_2 = \alpha \) and \( \sigma = \alpha \), respectively, lead to the smallest iteration numbers which is due to the balance of
fit-to-data and fit-to-constraint, i.e., when \( \sigma_2 = \sigma = \alpha \) is chosen the split-split algorithm and the Heron-split method try to recover the datum \( g \) and to satisfy the bilateral constraint simultaneously. Note that the iteration numbers for \( \sigma_2 = 1 \) and \( \sigma = 1 \) for the split-split method and the Heron-split method, respectively, are astonishingly stable for different mesh sizes. However, this behavior cannot be inferred from the stability estimate given in Proposition 3. Regarding the Heron-penalty method, we ran the algorithm for the choices \( \delta = h/\alpha \) and \( \delta = h \). Obviously, due to a worse conditioning, the iteration numbers for the choice \( \delta = h/\alpha \) are much larger than for \( \delta = h \). However, for \( \delta = h \) the iterates \( u^j_h \) generated by the Heron-penalty method stay far from the minimizer \( u_h \) of \( E \) and the distance corresponds to the magnitude of \( \alpha \) as predicted by Proposition 4. This effect is also illustrated in Figure 1, where the \( L^2 \)-error between the approximate minimizer \( \tilde{u}_h \) of \( E \) generated by the split-split method and the iterates \( u^j_h \) and \( s^j_h \), respectively, generated by the Heron-penalty method for both \( \delta = h/\alpha \) and \( \delta = h \) is plotted against the number of iterations. For \( \delta = h/\alpha \) we see that the final iterates \( u^N_h \) and \( s^N_h \) serve as good approximations of \( \tilde{u}_h \) and the approximation error is of order \( h \). Yet, for \( \delta = h \), the iterates \( u^j_h \) do not have practical approximation properties and one would have to decrease the mesh size significantly to obtain a similar accuracy as for \( \delta = h/\alpha \). Still, one can also observe that the different choices of \( \delta \) do not considerably affect the approximation properties of the iterates \( s^j_h \), i.e., one may set

**Figure 1.** \( L^2 \)-error between approximate minimizer \( \tilde{u}_h \) of \( E \) (generated by the split-split method) and iterates \( u^j_h \), \( s^j_h \) of the Heron-penalty method, \( h = \sqrt{2} 2^{-6}, \varepsilon = h \), for Example 1. Note that the iterates \( u^j_h \) serve as good approximations of \( \tilde{u}_h \) when \( \delta = h/\alpha \).
\[ \delta = h \] and choose the final iterate \( s_N \) as an approximation of \( \tilde{u}_h \). Yet, this is not justified theoretically by Proposition 4. Furthermore, the behavior of the active set \( \{ z \in \mathcal{N}_h : u_j^h(z) \leq \chi(z) \text{ or } u_j^h(z) \geq \psi(z) \} \) is encoded in the error \( \| \tilde{u}_h - u_j^h \| \) since \( \tilde{u}_h \approx u_h \in K \). The error curves for \( \| \tilde{u}_h - u_j^h \| \) and \( \| \tilde{u}_h - s_j^h \| \) for the split-split method are similar to those shown in Figure 1 for \( \delta = h/\alpha \) the only difference being that we have \( \| \tilde{u}_h - u_j^h \|, \| \tilde{u}_h - s_j^h \| \to 0 \) as \( j \to \infty \) in contrast to the Heron-penalty method.

**Example 2.** The setting is as in Example 1 except for the parameter \( \alpha \), which is set to \( \alpha = 500 \) here.

In Table 2 the iteration numbers for Example 2 are displayed. We see that the effect of higher iteration numbers for the split-split method and the Heron-split method for the choice \( \sigma_2 = \sigma = 1 \) and for the choice \( \delta = h/\alpha \) in the Heron-penalty method are now even more pronounced due to the choice of a large \( \alpha \). Moreover, Figure 2 shows once again that the iterates \( u_j^h \) generated by the Heron-penalty method for \( \delta = h \) may not be used as a good approximation of \( \tilde{u}_h \) since the constant \( \alpha \) affects the \( L^2 \)-distance between \( u_j^h \) and \( \tilde{u}_h \) while the iterates \( s_j^h \) seem to define accurate approximations of \( \tilde{u}_h \) almost independently of the choice of \( \delta \). Again, the iteration numbers for \( \sigma_2 = 1 \) and \( \sigma = 1 \) appear to be stable.

The overall conclusion of the experiments is that the split-split method with \( \sigma_2 = \alpha \), Heron-penalty method with \( \delta = h \) and the Heron-split method with \( \sigma = \alpha \) lead to the smallest iteration numbers and have a similar performance. An advantage of the Heron-penalty method is that the choice of parameters is clear while the choice of the step sizes \( \sigma_2 \) and \( \sigma \) in the split-split method and the Heron-split method, respectively, remain unclear in general. However, the penalization parameter \( \delta \) in the Heron-penalty method has to scale as \( h/\alpha \) (or at least as \( h^{1/2}/\alpha \) to recover the convergence rate as stated in Remark 7) for the iterates \( u_j^h \) to be close to \( u_h \).

|       | Split-split | Heron-penalty | Heron-split |
|-------|-------------|---------------|-------------|
|       | \( \sigma_2 \) | \( \delta \) | \( \sigma \) |
| \( h \) | \( 1/\alpha \) | \( 1/h \) | \( h/\alpha \) | \( h \) | \( 1 \) | \( \alpha \) | \( 1/h \) |
| \( \sqrt{2}/2^6 \) | 2745 | 10 | 124 | 74 | 24 | 4592 | 15 | 189 |
| \( \sqrt{2}/2^6 \) | 2808 | 15 | 65 | 156 | 27 | 3586 | 21 | 72 |
| \( \sqrt{2}/2^7 \) | 2847 | 23 | 35 | 298 | 35 | 3133 | 25 | 42 |
| \( \sqrt{2}/2^8 \) | – | 29 | 30 | 572 | 43 | – | 33 | 37 |

Table 2. Iteration numbers with \( 5 \cdot 10^3, 10^3, 2^7 \) for Example 2 and \( \sigma_1 = h^{-3/2} \) for the split-split method and \( \tau = 1 \) for the Heron-penalty and the Heron-split method.

### 7.2. Piecewise constant segmentation.

In this section we report numerical experiments for the binary image segmentation model presented in Section 6. We consider the two-phase image segmentation of the image “cameraman”. To this extent, we let \( g \) be the continuous piecewise affine function with coefficient vector whose entries consist of the (scaled) gray-values of the image ranging from 0 to 1. We further let \( \alpha = 1500 \), \( \chi \equiv 0 \), \( \psi \equiv 1 \), and consider the following algorithm of [13] which involves the solution of a constrained TV-minimization problem.

Figure 2. $L^2$-error between approximate minimizer $\tilde{u}_h$ of $E$ (generated by the split-split method) and iterates $u_j^h$, $s_j^h$ of the Heron-penalty method, $h = \sqrt{2}^{-6}$, $\varepsilon = h$, for Example 2. Again, the iterates $u_j^h$ approximate $\tilde{u}_h$ properly when $\delta = h/\alpha$.

Algorithm 5. Choose $c_1^0, c_2^0 \in \mathbb{R}$ and $\gamma \in [0, 1]$. Set $k = 0$.
(1) Compute for $f = (c_1^k - g)^2 - (c_2^k - g)^2$ an approximate minimizer $\tilde{u}_h^{k+1} \in S^1(T_h)$ of the functional

$$E^{seg}(v_h) = \int_{\Omega} |\nabla v_h| + \alpha \int_{\Omega} f v_h \, dx + I_K(v_h)$$

using either the split-split method or the Heron-penalty method or the Heron-split method.
(2) Define $u_h^{k+1}$ by

$$u_h^{k+1}(z) = \begin{cases} 1, & \text{if } \tilde{u}_h^{k+1}(z) \geq \gamma, \\ 0, & \text{if } \tilde{u}_h^{k+1}(z) < \gamma, \end{cases}$$

for $z \in N_h$. Set

$$\Sigma^{k+1} = \bigcup \{ T \in T_h : u_h^{k+1} \mid_T \equiv 1 \}.$$

(3) Set

$$c_1^{k+1} = \frac{1}{|\Sigma^{k+1}|} \int_{\Sigma^{k+1}} g \, dx \quad \text{and} \quad c_2^{k+1} = \frac{1}{|\Omega \setminus \Sigma^{k+1}|} \int_{\Omega \setminus \Sigma^{k+1}} g \, dx.$$

(4) Stop if

$$\frac{|MS(\Sigma^{k+1}, c_1^{k+1}, c_2^{k+1}) - MS(\Sigma^k, c_1^k, c_2^k)|}{|MS(\Sigma^{k+1}, c_1^{k+1}, c_2^{k+1})|} \leq \varepsilon_{\text{stop}}.$$
Otherwise, increase \( k \to k + 1 \) and continue with (1).

Using a uniform triangulation consisting of halved squares with mesh size \( h \sim 2^{-8} \) we chose \( c_1^0 = 1 \) and \( c_2^0 = 0 \) and set the thresholding parameter to \( \gamma = 1/2 \). We set \( \varepsilon_{\text{stop}} = 10^{-3} \) in step (4). In step (2), we initialize the algorithms as in the preceding subsection and used \( \varepsilon = h^{1/2} \) for the Heron-penalty and the Heron-split method. We set \( \delta = h^{1/2} \) in the Heron-penalty method and used \( \sigma_2 = \sigma = \alpha\|f\| \) in the split-split and the Heron-split method and \( \tau = 1 \) in the Heron-penalty and the Heron-split method. The stopping criteria in step (2) were chosen as follows:

- **Split-split method:**
  \[
  \left( \|\lambda_h^{j+1} - \lambda_h^j\|_w^2 + h^{-3}\|p_h^{j+1} - p_h^j\|_w^2 \right)^{1/2} \leq 10^{-3} \quad \text{and} \quad \left( \frac{1}{\sigma_2^2}\|\eta_h^{j+1} - \eta_h^j\|_h^2 + \|s_h^{j+1} - s_h^j\|_h^2 \right)^{1/2} \leq 10^{-3}
  \]
  \[ \tag{8} \]

- **Heron-penalty method:**
  \[
  \left( \|d_hu_h^{j+1}\|_h^2 + \|dp_h^{j+1}\|_h^2 + \|ds_h^{j+1}\|_h^2 \right)^{1/2} \leq 10^{-3}
  \]
  \[ \tag{9} \]

- **Heron-split method:**
  \[
  \left( \|d_hu_h^{j+1}\|_h^2 + \|dp_h^{j+1}\|_h^2 \right)^{1/2} \leq 10^{-3} \quad \text{and} \quad \left( \frac{1}{\sigma_2^2}\|\eta_h^{j+1} - \eta_h^j\|_h^2 + \|s_h^{j+1} - s_h^j\|_h^2 \right)^{1/2} \leq 10^{-3}
  \]
  \[ \tag{10} \]

Note that by multiplying the first inequality in (8) by \( h^{1/2} \) the stopping criterion corresponds to \( \varepsilon_{\text{stop}}^{(1)} = h^{3/2}10^{-3} \) in Algorithm 4. We incorporated an \( h \)-dependent factor in \( \varepsilon_{\text{stop}}^{(1)} \) so that the gradient of \( u_h \) is sufficiently well approximated.

Figure 3 shows the original image and the outputs of Algorithm 5 using the split-split method, the Heron-penalty method and the Heron-split method in step (2), respectively, where we used the iterates \( s_h^j \) as approximations of a minimizer in step (2) in the Heron-penalty method. The three white horizontal lines in the outputs are due to image conversion. One should take into account that a uniform triangulation has been used based on the resolution of the original image (256×256) which makes the images seem to be slightly distorted. One can observe that all outputs are visually almost the same. The output of the split-split method and the Heron-split method visually do not differ at all, whereas the image generated by the Heron-penalty method has two black dots which do not appear in the other two images. These dots are not contained in the output of the Heron-penalty method when using \( \delta = \frac{h^{1/2}}{\|f\|} \). The final values of \( c_1 \) and \( c_2 \) were \( c_1 = 0.5946 \) and \( c_2 = 0.1144 \) for both the split-split method and the Heron-split method and \( c_1 = 0.5951 \), \( c_2 = 0.1147 \) for the Heron-penalty method. The final values of the energy \( MS(\Sigma, c_1, c_2) \) were 25.3745, 25.3863 and 25.3711 for the split-split method, the Heron-penalty method and the Heron-split method, respectively. The split-split method needed 5000 iterations in total for termination (6 outer iterations), while the Heron-penalty method needed 1382 iterations in total (7 outer iterations) and the Heron-split method needed 857 iterations in total (6 outer iterations). The Heron-penalty method with \( \delta = \frac{h^{1/2}}{\|f\|} \) needed 1675 iterations in total (6 outer iterations). Using \( u_h^k \) generated by the Heron-penalty method as approximations for the minimizer in step (2) led to the same results as for the case using \( s_h^k \).
8. Conclusion. We discussed a finite element discretization of a prototypical bilaterally constrained total variation minimization problem, proved convergence as $h \to 0$ under a density assumption of $C^\infty(\Omega) \cap K$ in $BV(\Omega) \cap K$, provided an experimental comparison between three approaches for the numerical solution of the discretized problem which differ in the numerical treatment of the nondifferentiable $BV$-seminorm and the nondifferentiable indicator functional encoding the constraint and addressed their stability. The numerical experiments reveal that the performance of the Heron-penalty method is negatively affected by the penalization parameter $\delta$ which determines how well $u_{h, \varepsilon, \delta}$ approximates the discrete minimizer $u_h$. However, we could have obtained lower iteration numbers if the penalization parameter had been chosen as $\delta = \mathcal{O}(h^{1/2})$ instead of $\delta = \mathcal{O}(h)$. On the other hand, the split-split method avoids additional regularization of the nondifferentiable terms and yields low iteration numbers if the step sizes are chosen properly. The Heron-split method also yields low iteration numbers for appropriate step sizes, however, it lacks a stability estimate. Therefore, in our opinion, the split-split method seems to be the most appropriate out of these three algorithms.
for bilaterally constrained total variation minimization. Still, the choice of a proper step size in splitting methods is crucial and remains unclear. Here, it might be interesting to consider an adaptive step size adjustment to improve the convergence rate of splitting methods. The Heron-penalty method is also stable independent of the choice of the step size. The (optimal) scaling of the regularization and penalization parameters is clear but in most cases restrictive with regard to the penalization parameter \( \delta \) which has to be chosen as \( \delta = O(h^{5/2}) \) for \( d = 2 \) and \( \delta = O(h^{7/3}) \) for \( d = 3 \). It will be interesting to investigate if this restrictive condition on the penalization parameter for the case of non-constant obstacles or triangulations with triangles which are not right-angled is really necessary. These topics as well as the application to total variation regularized damage evolution models is left for future research.

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E-mail address: bartels@mathematik.uni-freiburg.de
E-mail address: marijo.milicevic@mathematik.uni-freiburg.de