OPTIMAL CONTROL STRATEGY FOR AN AGE-STRUCTURED SIR ENDEMIC MODEL

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Abstract. In this article, we consider an age-structured SIR endemic model. The model is formulated from the available literature while adding some new assumptions. In order to control the infection, we consider vaccination as a control variable and a control problem is presented for further analysis. The method of weak derivatives and minimizing sequence argument are used for deriving necessary conditions and existence results. The desired criterion is achieved and sample simulations were presented which shows the effectiveness of the control.

1. Introduction. The chronological age is one of the key factors that distinguish individuals in a population and because of its importance it should be incorporated in epidemic and population models [15]. Various features and qualities such as death and birth rates are markedly different among various age groups. Mixing pattern in epidemic models are also age dependent, for example, children mix with other children of similar age and with the individuals of the age of their parents. Malaria infect all age groups, however, children are at great risk and exhibit highest death rate and symptoms [31]. Similarly, HIV could be seen in the individuals having age between twenty and forty. Disease models especially endemic model must include

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age-structure since with the passage of time the age profile of population is also changing [24].

There are several excellent introductory text books and papers in the literature on disease modeling that includes age-structured such as [16, 18, 2, 23, 19, 22, 14]. Most of these works are concerned about the stability analysis of the developed models, whereas, the controlling aspects of such diseases were given less attention. Due to the unpredictable nature, infectious diseases are the source of fear for humanity since the first age of human civilization. The key idea of modeling epidemics is to provide a rational basis for policy makers to control the spread of such disease. In order to control the impact of such outbreaks, researchers are trying to investigate an effective strategy by setting an optimal control problem [5, 9, 27, 29, 30].

Optimal control problem comprise of ordinary differential equations were studied by a number of researchers. Similarly, literature have some novel work on the control problems that include coupled equations of PDEs and ODEs. Recently, Rahman et al [28] developed a mixed giving up smoking model in which the authors beside the control strategies, studied the dynamical behavior of the model concerned. Khan and Zaman in [20] developed a SEIR model having age-since-infection in infectious classes and they set a control strategy using a single control variable. For a detail analysis of mixed control problems, the readers are advised to see the work of [32, 1, 6]. It is interesting to see that one can find a few papers on pure PDEs control problems such as [26, 8]. The authors in these articles derived some effective control strategies for PDE models, however, these models were spatial structured. For optimal control of age-structured model, Anita considered the optimal harvesting problem in just one equation [2]. For problem related to boundary control in epidemiology, readers are advised to see [4]. Interested readers in the optimal control of interacting species are referred to see [10, 7] and references cited therein.

In the current study, the researchers has considered an age-structured SIR endemic model somehow similar to that of Zaman and Khan [31]. In order to control the infection, usually there are two possibilities; treatment of infectives or vaccination of susceptible. We will consider vaccination of the susceptible individuals as a control variable in the model. Although the technique we are using is not a new idea, however, the application of this techniques to the model developed will provide novel insights to age-structured modeling.

The main objectives the present work are to improve our present knowledge regarding control theory in age-structured modeling and to clarify the techniques for setting an optimal control problem. Since, the model under discussion is from infinite dimensional space, therefore, it will help those readers who are dealing with control problems lying in such spaces. Moreover, we considered a general age-structured SIR model, thus by using this work, one can easily obtain the desired control criteria for many infectious diseases like HIV/AIDS, TB etc. In addition, this work is more cost effective as we are vaccinating susceptible of particular ages. Finally, this study will contribute to the relevant available literature as well as provide the readers with new and authentic information.

The organization of this work is as follows. Section 2 is about formulation of the proposed mathematical model and existence of a unique solution. In Section 3, we will set an objective functional subject to the developed model and will be presented for further analysis. The traditional techniques will be implemented for
obtaining the desired criterion. In Section 4, we will use finite difference method to obtain numerical solutions for supporting analytical results.

2. Formulation of the model. In order to formulate the model, we stratified the whole population into three main classes: the susceptible, the infective and the recovered classes. Let us denote these classes by \( s(t,a), i(t,a) \) and \( r(t,a) \) which represent the density of the respective classes at any time \( t \) and age \( a \). We set the maximum age \( A \) and maximum time \( T \), that is, the desired intervals of age and time are \((0, A)\) and \((0, T)\). Given a control \( u(t,a) \) that denotes the density of quantity of a vaccine given to an individual of age \( a \) at time \( t \), the associated state variables satisfy the following system:

\[
\begin{align*}
\frac{\partial s(t,a)}{\partial t} + \frac{\partial s(t,a)}{\partial a} &= B - \left( \hat{\lambda}(t,a)i(t,a) + \mu(a) + u(t,a) \right)s(t,a) , \\
\frac{\partial i(t,a)}{\partial t} + \frac{\partial i(t,a)}{\partial a} &= -\left( \gamma(a) + \mu(a) \right)i(t,a) + \hat{\lambda}(t,a)i(t,a)s(t,a) , \\
\frac{\partial r(t,a)}{\partial t} + \frac{\partial r(t,a)}{\partial a} &= -\mu(a)r(t,a) + \gamma(a)i(t,a) + u(t,a)s(t,a), \\
s(t,0) &= \int_{0}^{\infty} b(a) \left[ s(t,a) + i(t,a) + r(t,a) \right] da , \quad i(t,0) = r(t,0) = 0,
\end{align*}
\]

for \((t,a) \in \chi = (0, T) \times (0, A)\). Further, we denote the size of each class by \( S(t), I(t) \) and \( R(t) \), where, \( S(t) = \int_{0}^{\infty} s(t,a) da \), \( I(t) = \int_{0}^{\infty} i(t,a) da \) and \( R(t) = \int_{0}^{\infty} r(t,a) da \).

In model formulation, we assumed that the new recruitment into the population is only into the susceptible class at a constant rate \( B > 0 \). Further, we take the same natural death rate \( \mu(a) \) for all classes. The notion \( \hat{\lambda}(t,a) \) stand for mass action coefficient and thus the term \( \hat{\lambda}si \) assume the rate of infection of susceptible. The terms \( b(a) \) and \( \gamma(a) \) respectively represent the age-specific birth and recovery rate of total and infected population. The rate at which susceptible of age \( a \) at time \( t \) are effectively vaccinated depends on the amount of vaccine \( u(t,a) \) available for age \( a \) at time \( t \). The following are assumed to hold:

- **H1.** \( s_{0}(a) > 0, \; i_{0}(a) > 0, \; r_{0}(a) \geq 0, \) and \( s_{0}(a), \; i_{0}(a), \; r_{0}(a) \in L^{\infty}(0,A). \)
- **H2.** \( \mu(a), \; \gamma(a), \; b(a) \in L^{\infty}(0,A) \) and \( \hat{\lambda} \in L^{\infty}(\chi) \) are non-negative.

By using the method of characteristics, we can find solution of system (1) of the form

\[
s(t,a) = \begin{cases}
\quad s_{0}(a-t)e^{-\int_{0}^{t} \mu(a-t+\eta)d\eta} + \int_{0}^{t} e^{-\int_{\tau}^{t} \mu(a-t+\eta)d\eta} \left[ B - \left\{ \hat{\lambda}(\tau,a-t+\tau) \right\} \right] d\tau , \quad a > t, \\
\quad s(t-a,0)e^{-\int_{0}^{a} \mu(\eta)d\eta} + \int_{0}^{a} e^{-\int_{\eta}^{a} \mu(\eta)d\eta} \left[ B - \left\{ \hat{\lambda}(t-a+\tau) \right\} \right] d\tau , \quad a \leq t, \\
\quad \times i(t-a+\tau,\tau) + u(t-a+\tau,\tau) \right] s(t-a+\tau,\tau) \right) d\tau ,
\end{cases}
\]

\[
i(t,a)
\]
Upon integrating relations (2), (3) and (4), we get the following integral equations:

\[
S(t) = \int_{0}^{\infty} s(t,a)da = \int_{0}^{t} s(t,a)da + \int_{t}^{\infty} s(t,a)da,
\]

\[
I(t) = \int_{0}^{\infty} i(t,a)da = \int_{0}^{t} i(t,a)da + \int_{t}^{\infty} i(t,a)da,
\]

\[
R(t) = \int_{0}^{\infty} r(t,a)da = \int_{0}^{t} r(t,a)da + \int_{t}^{\infty} r(t,a)da,
\]

and

\[
S(t) = \int_{0}^{\infty} s(t,a)da = \int_{0}^{t} s(t,a)da + \int_{t}^{\infty} s(t,a)da,
\]
The desired functional space for the problem under consideration is
\[ Z = C \left( [0, T]; L^2(0, A) \right) \cap \left\{ u \in L^\infty(\chi) \mid 0 \leq u \leq u_{\text{max}} \text{ a.e. } (t, a) \in \chi \right\}, \] (8)
as it provides biologically justified \( L^\infty(\chi) \) bounds on \( s, i, \) and \( r \). Since, the force of infection is essentially bounded, measurable and positive function. Therefore, by using standard fixed point theorem argument of Grippenberg et al [12], it is easy to show that there exist unique nonnegative solutions of equations (5), (6) and (7).

3. **Optimal control strategy.** Optimal control theory is a powerful mathematical technique used for exploring control laws for dynamical and many other systems. In control theory applied to epidemiology, the main focus of health authorities and policy makers is to control the spread of various infectious diseases. In this regard, a variety of models have been developed [11, 21, 17] and optimal control problems were presented therein. The current study focuses on the control of those infectious diseases whose dynamics are changing with respect to age and time.

The proposed model (1) consist of three state variables \( s, i, r \) and a single control variable \( u(t, a) \) from the admissible control set. The admissible control variable is assumed to be from the set
\[ U = \left\{ u(t, a) \mid u : \chi \rightarrow [0, u_{\text{max}}], (t, a) \in \chi \right\}. \] (9)

Since, the control represent vaccination of susceptible individuals, therefore, \( u \) must be nonnegative and bounded by \( u_{\text{max}} \in R^+ \) due to the physical limitations.

Among the best control strategies, one has to minimize the number of infectives as well as the cost of vaccination and to maximize the susceptible and recovered population. Therefore, by considering the same strategy our objective functional will takes the form
\[ J(u) = \int_\chi \left[ A_1 i - B_1 s + \frac{B_2}{2} u^2 \right] dadt. \] (10)
subject to system (1). The notion \( A_1, B_1 \) and \( B_2 \) are constant weights and \( \frac{B_2}{2} u^2 \) represent the cost of vaccination. Our aim in this work is to minimize the objective functional. Using the continuous dependency of state variables on the control variable and minimizing sequence argument, we established the existence of an optimal control in the following theorem.

**Theorem 3.1.** There exists an optimal control variable \( u^* \in U \) such that \( J(u^*) = \min_{u \in U} J(u) \), subject to the control system (1).

**Proof.** Clearly, from the definition of the control variable \( u \) and from the existence of unique nonnegative solution of the model, we can say that the control variable \( u \) and the state variables \( s, i \) and \( r \) are uniformly bounded in \( \chi \). Therefore, \( \inf_{u \in U} J(u) > -\infty \) and there exist a minimizing sequence \( u^n \in U \) such that \( \lim_{n \rightarrow \infty} J(u^n) = \inf_{u \in U} J(u) \). Thus, by using the fact that the state variables continuously dependent on \( u \), we have \( (s^n, i^n, r^n) = (s, i, r)(u^n) \) and \( \left( ||s^n||, ||i^n||, ||r^n|| \right) \) is uniformly bounded for each \( n \). Hence, \( (s^n, i^n, r^n) \rightarrow (s, i, r)(u^*) \) where \( s^*, i^*, r^* \) are the associated states correspond to the optimal control \( u^* \).

Finally, the function \( \frac{B_2}{2} u^2 \) is a lower semi-continuous convex function in the objective functional, thus
\[ \int_\chi B_2 (u^*)^2 dadt \leq \lim_{n \rightarrow \infty} \inf \int_\chi \frac{B_2}{2} (u^n)^2 dadt. \] (11)
The property (11) gives us
\[
J(u^*) = \int_\chi [A_1 i^* - B_1 s^* + \frac{B_2}{2} (u^*)^2] \, da \, dt,
\]
\[
\leq \lim_{n \to \infty} \inf_{u \in U} \int_\chi [A_1 i^n - B_1 s^n + \frac{B_2}{2} (u^n)^2] \, da \, dt,
\]
\[
= \lim_{n \to \infty} \int_\chi [A_1 i^n - B_1 s^n + \frac{B_2}{2} (u^n)^2] \, da \, dt,
\]
\[
= \inf_{u \in U} J(u). \quad (12)
\]

From relation (12), we have \( J(u^*) \leq \inf_{u \in U} J(u) \). Thus, \( u^* \) is an optimal control which minimize the objective functional.

To obtain the necessary conditions for the control problem, first we introduce another control \( u' (t, a) = u(t, a) + \epsilon h(t, a) \), where \( 1 > \epsilon > 0 \) and \( h(t, a) \in L^\infty (\lambda) \) is called variation function. The variation function may be chosen such that \( u' \in U \). As state variable continuously depend on the control variable, thus, the introduction of new control will change the state variables to \( s^r = s(u'), i^r = i(u') \) and \( r^r = r(u') \).

More precisely, we can write \( (s', i', r') = (s', i, r)(u + \epsilon h) \). Corresponding to \( u^r \), system (1) will takes the form

\[
\frac{\partial s'(t, a)}{\partial t} + \frac{\partial s'(t, a)}{\partial a} = B - (\lambda(t, a)i'(t, a) + \mu(a) + u(t, a) + \epsilon h(t, a))s'(t, a), \quad (13a)
\]
\[
\frac{\partial i'(t, a)}{\partial t} + \frac{\partial i'(t, a)}{\partial a} = - (\gamma(a) + \mu(a))i'(t, a) + \lambda(t, a)i'(t, a)s'(t, a), \quad (13b)
\]
\[
\frac{\partial r'(t, a)}{\partial t} + \frac{\partial r'(t, a)}{\partial a} = - \mu(a)r'(t, a) + \lambda(t, a)i'(t, a) + (u(t, a) + \epsilon h(t, a))s'(t, a), \quad (13c)
\]
\[
s'(t, 0) = \int_0^\infty b(a)[s'(t, a) + i'(t, a) + r'(t, a)] \, da, \quad i'(0, a) = i_0(a), \quad r'(0, a) = r_0(a). \quad (13c)
\]

In order to find the weak derivatives of the state variables, we form the difference quotients \( \frac{s(u + \epsilon h) - s(u)}{\epsilon} \) and \( \frac{r(u + \epsilon h) - r(u)}{\epsilon} \) by subtract system (1) from (13). Further, we set the weak derivatives as \( \tilde{s}(t, a) = \lim_{\epsilon \to 0} \frac{s(u + \epsilon h) - s(u)}{\epsilon} \), \( \tilde{i}(t, a) = \lim_{\epsilon \to 0} \frac{i(u + \epsilon h) - i(u)}{\epsilon} \) and \( \tilde{r}(t, a) = \lim_{\epsilon \to 0} \frac{r(u + \epsilon h) - r(u)}{\epsilon} \) also known as sensitivities. The following linearized version of system (1) is satisfied by the sensitivities:

\[
\frac{\partial \tilde{s}(t, a)}{\partial t} + \frac{\partial \tilde{s}(t, a)}{\partial a} = - \lambda(t, a)(\tilde{s}(t, a)i(t, a) + s(t, a)\tilde{i}(t, a)) - (\mu(a) + u(t, a))\tilde{s}(t, a) - h(t, a)s(t, a), \quad (14a)
\]
\[
\frac{\partial \tilde{i}(t, a)}{\partial t} + \frac{\partial \tilde{i}(t, a)}{\partial a} = - (\gamma(a) + \mu(a))\tilde{i}(t, a) + \lambda(t, a)(\tilde{s}(t, a)i(t, a) + s(t, a)\tilde{i}(t, a)), \quad (14b)
\]
\[
\frac{\partial \tilde{r}(t, a)}{\partial t} + \frac{\partial \tilde{r}(t, a)}{\partial a} = - \mu(a)\tilde{r}(t, a) + \lambda(t, a)i(t, a) + (u(t, a) + s(t, a))\tilde{i}(t, a) + h(t, a)s(t, a), \quad (14c)
\]
\[ s(t, 0) = \int_0^\infty b(a) \left[ s(t, a) + \bar{r}(t, a) + \tilde{r}(t, a) \right] da, \quad \tilde{r}(t, 0) = 0, \quad (14d) \]

\[ \bar{s}(0, a) = 0, \quad \bar{\tilde{r}}(0, a) = 0. \quad (14c) \]

In the same way, we will differentiate the objective functional with respect to the control variable \( u \). In the direction of \( h \), the Gateaux derivative of \( J \) with respect to \( u \) is

\[ J'(u) = \lim_{\epsilon \to 0} \frac{J(u^{\epsilon}) - J(u)}{\epsilon} = \int \chi \left[ A_1 \bar{\tilde{r}}(t, a) - B_1 \bar{s}(t, a) + B_2 u(t, a) h(t, a) \right] d\chi, \quad (15) \]

where \( d\chi = d\alpha dt \).

In order to find the adjoint equations, we can write equation (14a) as

\[ \begin{align*}
0 &= \left\langle \left\{ \frac{\partial}{\partial t} \bar{s}(t, a) + \frac{\partial}{\partial a} \bar{s}(t, a) + (\mu(a) + u(t, a)) \bar{s}(t, a) + \bar{\lambda}(t,a) \bar{s}(t,a) \tilde{r}(t,a) + \lambda(t,a) \bar{s}(t,a) \tilde{i}(t,a) \right\} \right\rangle, \\
0 &= \left\langle \left\{ \bar{s}(t,a), -\frac{\partial}{\partial t} \bar{\lambda}(t,a) - \frac{\partial}{\partial a} \bar{\lambda}(t,a) + (\mu(a) + u(t,a)) \lambda(t,a) \right\} \right\rangle - \int \chi \left[ \bar{\lambda}(t,a) \bar{\tilde{i}}(t,a) + h(t,a) \right] s(t,a) \lambda(t,a) d\chi
\end{align*} \]

with the conditions

\[ \lambda_1(t, A) = 0, \quad \lambda_1(T, a) = 0, \]

where \( \bar{N}(t, a) = \bar{s}(t, a) + \bar{\tilde{i}}(t, a) + \bar{\tilde{r}}(t, a) \) and \( \langle \langle f, g \rangle \rangle = \int_0^T \int \infty f(t,a)g(t,a) dt \). In the same way, equation (14b) can be written as

\[ \begin{align*}
0 &= \left\langle \left\{ \frac{\partial}{\partial t} \bar{i}(t, a) + \frac{\partial}{\partial a} \bar{i}(t, a) + (\mu(a) + \gamma(a)) \bar{i}(t,a) - \bar{\lambda}(t,a) \bar{s}(t,a) \tilde{r}(t,a) \right\} \right\rangle, \\
0 &= \left\langle \left\{ \bar{i}(t,a), -\frac{\partial}{\partial t} \lambda_2(t,a) - \frac{\partial}{\partial a} \lambda_2(t,a) + (\mu(a) + \gamma(a)) \lambda_2(t,a) \right\} \right\rangle - \int \chi \bar{\lambda}(t,a) \bar{s}(t,a) \tilde{i}(t,a) \lambda_2(t,a) d\chi, \quad (17)
\end{align*} \]

with the conditions

\[ \lambda_2(t, A) = 0, \quad \lambda_2(T, a) = 0, \]

and equation (14c) will take the form

\[ \begin{align*}
0 &= \left\langle \left\{ \frac{\partial}{\partial t} \bar{r}(t, a) + \frac{\partial}{\partial a} \bar{r}(t, a) + \mu(a) \bar{r}(t,a) - u(t,a) \bar{s}(t,a) - h(t,a) s(t,a) \right\} \right\rangle, \\
0 &= \left\langle \left\{ \bar{r}(t,a), -\frac{\partial}{\partial t} \lambda_3(t,a) - \frac{\partial}{\partial a} \lambda_3(t,a) + (\mu(a) \lambda_3(t,a)) \right\} \right\rangle \\
- \int \chi \gamma(a) \bar{i}(t, a) \lambda_3(t, a) d\chi - \int \chi u(t,a) \bar{s}(t,a) \lambda_3(t,a) d\chi
\end{align*} \]
with the conditions

$$\lambda_3(t, A) = 0, \lambda_3(T, a) = 0.$$  

Next, we will combine equations (16), (17) and (18) along with the conditions and by rearranging the terms, we get the following adjoint system

$$\frac{\partial}{\partial t} \lambda_1(t, a) + \frac{\partial}{\partial a} \lambda_1(t, a) = \left( \mu(a) + u(t, a) \right) \lambda_1(t, a) + \dot{\lambda}(t, a) \lambda_1(t, a)i(t, a)$$

$$-b(a) \lambda_1(t, 0) - \dot{\lambda}(t, a) \lambda_2(t, a)i(t, a) - u(t, a) \lambda_3(t, a) - B_1, \quad (19a)$$

$$\frac{\partial}{\partial t} \lambda_2(t, a) + \frac{\partial}{\partial a} \lambda_2(t, a) = \left( \gamma(a) + \mu(a) \right) \lambda_2(t, a) - \dot{\lambda}(t, a) \lambda_2(t, a)s(t, a)$$

$$-\gamma(a) \lambda_3(t, a) - \dot{\lambda}(t, a) \lambda_1(t, a)s(t, a) - b(a) \lambda_1(t, 0) + A_1, \quad (19b)$$

$$\frac{\partial}{\partial t} \lambda_3(t, a) + \frac{\partial}{\partial a} \lambda_3(t, a) = \mu(a) \lambda_3(t, a) - b(a) \lambda_1(t, 0), \quad (19c)$$

with transversality conditions

$$\lambda_1(T, a) = 0, \lambda_2(T, a) = 0, \lambda_3(T, a) = 0,$$  

and the boundary conditions are

$$\lambda_1(t, A) = \lambda_2(t, A) = \lambda_3(t, A) = 0.$$  

**Theorem 3.2.** If $u^* \in U$ is an optimal control which minimizing (10) and $(s^*(t, a), i^*(t, a), r^*(t, a))$ and $(\lambda_1(t, a), \lambda_2(t, a), \lambda_3(t, a))$ are the corresponding state and adjoint variables, respectively, then

$$u^*(t, a) = \min \left\{ u_{\text{max}}, \max \left\{ 0, \frac{(\lambda_1(t, a) - \lambda_3(t, a))s^*(t, a)}{B_2} \right\} \right\}.$$  

**Proof.** By using system (19), we can write relation (15) as

$$J'(u) = \int_{\chi} \left[ A_1 i(t, a) - B_1 s(t, a) + B_2 u(t, a) h(t, a) \right] d\chi,$$

$$= \int_{\chi} i(t, a) \left[ \frac{\partial}{\partial t} \lambda_2(t, a) + \frac{\partial}{\partial a} \lambda_2(t, a) - (\gamma(a) + \mu(a)) \lambda_2(t, a) + \dot{\lambda}(t, a) \lambda_2(t, a)s(t, a) + \dot{\lambda}(t, a) \lambda_1(t, a) \lambda_3(t, a) + \gamma(a) \lambda_3(t, a) \right] d\chi - \int_{\chi} s(t, a) \left[ - \frac{\partial}{\partial t} \lambda_1(t, a) - \lambda_1(t, a) (\mu(a) + u(t, a)) \lambda_1(t, a) + \dot{\lambda}(t, a) \lambda_1(t, a)i(t, a) - b(a) \lambda_1(t, 0) - \dot{\lambda}(t, a) \lambda_2(t, a)i(t, a) - u(t, a) \lambda_3(t, a) \right] d\chi + \int_{\chi} B_2 u(t, a) h(t, a) d\chi$$

$$- \int_{\chi} i(t, a) \left[ - \frac{\partial}{\partial t} \lambda_3(t, a) - \frac{\partial}{\partial a} \lambda_3(t, a) + \mu(a) \lambda_3(t, a) - b(a) \lambda_1(t, 0) \right] d\chi.$$  

(23)
Now keeping in mind the fact that \(0 \leq J'(u)\) and considering equations (16), (17) and (18), we can write equation (23) as

\[
0 \leq \int h(t,a) \left[ s(t,a) \left( \lambda_3(t,a) - \lambda_1(t,a) \right) + B_2 u(t,a) \right] \, d\chi, \quad \text{for all } u(t,a) \in U.
\]

Thus on this set, in the case when \(h(t,a) \neq 0\), the rest of the integrand must be equal to zero. Hence

\[
u(t,a) = \frac{\left( \lambda_1(t,a) - \lambda_3(t,a) \right) s(t,a)}{B_2}.
\]

By taking into account the upper and lower bound of the control variable, we obtain

\[
u^*(t,a) = \min \left\{ u_{max}, \max \left\{ 0, \frac{\left( \lambda_1(t,a) - \lambda_3(t,a) \right) s^*(t,a)}{B_2} \right\} \right\}.
\]

Here the formula (24) represents the characterization of the optimal control.

4. Numerical setting. Since, majority of the nonlinear PDE models cannot be solved analytically or even directly by CAS (Computer Algebra System). Therefore, one must use some numerical method to obtain a discrete form of the model concerned. After discretization, the model may be coded in some programming language. Further, by using numerical methods we can find an approximate solution not exact one. Several methods are available in the literature for the numerical solution of PDEs like finite difference method, the method of lines, and the discretization method etc.

In this work, we will use the finite difference method for numerical simulation. Since, there is nonlinearities in the right-hand side of the model. Therefore, by evaluating the whole right-hand side at the next time to obtain an implicit method, the resulting equation for the next time level will become nonlinear and very hard to solve. In order to overcome this difficulty, we will partially linearize the nonlinear terms. The rest of the method is illustrated below:

First of all, we will discretize the domain

\[
D = \{ (t,a) : T_+ > t \geq 0, \quad 0 \leq a \leq A_+ \}.
\]
where $T_+$ is the maximum time and $A_+$ is the maximum age for simulation. With the same step size $\Delta a = \Delta t$, we will discretize the age and time direction, i.e., $a_i = i\Delta t$ and $t_n = n\Delta t$. With out any ambiguity, we may calculate the number of steps by using $M = \left\lfloor \frac{A_+}{\Delta a} \right\rfloor$ and $N = \left\lfloor \frac{T_+}{\Delta t} \right\rfloor$. This discretization will generate a square mesh in the domain.

Next, we replace the sum of partial derivatives by a difference quotient along the characteristic lines. In particular,

\[
\begin{align*}
    s_a(t, a) + s_t(t, a) &\approx \frac{s_{i+1}^{n+1} - s_i^n}{\Delta t}, \\
    i_a(t, a) + i_t(t, a) &\approx \frac{i_{i+1}^{n+1} - i_i^n}{\Delta t}, \\
    r_a(t, a) + r_t(t, a) &\approx \frac{r_{i+1}^{n+1} - r_i^n}{\Delta t}.
\end{align*}
\]

We evaluate the right-hand side at $a_{i+1}$ and $t_n$ and approximate the terms. As a first step, we have

\[
\begin{align*}
    \frac{s_{i+1}^{n+1} - s_i^n}{\Delta t} &= B - \left( \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} + \mu_{i+1} + \nu_{i+1}^n \right) s_{i+1}^{n+1}, \\
    \frac{i_{i+1}^{n+1} - i_i^n}{\Delta t} &= - \left( \gamma_{i+1} + \mu_{i+1} + \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} \right) i_{i+1}^{n+1} + \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} s_{i+1}^{n+1}, \\
    \frac{r_{i+1}^{n+1} - r_i^n}{\Delta t} &= -\mu_{i+1} r_{i+1}^{n+1} + \gamma_{i+1} i_{i+1}^{n+1} + \mu_{i+1} s_{i+1}^{n+1}.
\end{align*}
\]

Here we can easily solve the system for level $n + 1$, but the solution will be nonnegative, and we will approximate the continuous solution only for a very small step $\Delta t$. Thus, to make the method implicit, we replace

\[
\begin{align*}
    s_i^{n+1} &\approx \hat{s}_{i+1}^{n+1}, \\
    i_i^{n+1} &\approx i_{i+1}^{n+1}, \\
    r_i^{n+1} &\approx r_{i+1}^{n+1},
\end{align*}
\]

while keeping $\hat{\lambda}_{i+1}^n$ evaluated at level $n$. We obtain the following system approximating the differential equations:

\[
\begin{align*}
    \frac{s_{i+1}^{n+1} - s_i^n}{\Delta t} &= B - \left( \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} + \mu_{i+1} + \nu_{i+1}^n \right) s_{i+1}^{n+1}, \\
    \frac{i_{i+1}^{n+1} - i_i^n}{\Delta t} &= - \left( \gamma_{i+1} + \mu_{i+1} + \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} \right) i_{i+1}^{n+1} + \hat{\lambda}_{i+1}^n i_{i+1}^{n+1} s_{i+1}^{n+1}, \\
    \frac{r_{i+1}^{n+1} - r_i^n}{\Delta t} &= -\mu_{i+1} r_{i+1}^{n+1} + \gamma_{i+1} i_{i+1}^{n+1} + \mu_{i+1} s_{i+1}^{n+1}.
\end{align*}
\]

This system is linear in $s_{i+1}^{n+1}$, $i_{i+1}^{n+1}$, $r_{i+1}^{n+1}$ and can be easily solved. It is easy to verify that the solutions are nonnegative, and if all equation are added, the equation for the total population size can be obtained. At the same time, the method still possesses the good properties of the implicit schemes. The initial and boundary conditions can be similarly discretized:

\[
\begin{align*}
    s_0^{n+1} &= \sum_{j=1}^M b_j (s_j^{n+1} + r_j^{n+1} + r_j^{n+1}) \Delta t, & n = 0, \ldots, N - 1, \\
    i_0^{n+1} &= 0, \\
    r_0^{n+1} &= 0.
\end{align*}
\]
s_j^0 = s_0(a_j), \ i_j^0 = i_0(a_j), \ r_j^0 = r_0(a_j), \ j = 0, \cdots, M.

In conclusion, we would like to point out that this method is convergent, and the order of convergent is $O(\Delta t)$. Clearly, this method is acceptable as any numerical method is acceptable if it is convergent at least of order $O(\Delta t)$ [3, 25].

Using the same approach, we will discretize the adjoint equations. The discretized version of the model (25), adjoint system (19) and control variable (24) were coded in MATLAB and sample simulations were carried out. The values of the parameters for simulation are given in Table 1.

Table 1. Parameters values used in numerical simulation

| Parameters | Values            | References |
|------------|-------------------|------------|
| B          | 0.02              | Assumed    |
| $\lambda(t, a)$ | 0.03             | [26]       |
| $\mu(a)$  | 0.01(1 + sin((a - 20)\pi/40)) | Assumed    |
| $\gamma(a)$ | 0.2               | [13]       |
| $b(a)$     | \begin{cases} 0.2\sin^2((a - 15)\pi/30), & 15 < a < 40, \\ 0, & \text{otherwise}. \end{cases} | [1] |
| $u_{\text{max}}$ | 0.70             | [6]        |
| $A_1$      | 100               | [6]        |
| $B_1$      | 1                 | Assumed    |
| $B_2$      | 1000              | [6]        |

Moreover, we have assume the maximum age and time to be 60 years and 60 days, respectively. The constant step size of 0.08 is considered in time direction, whereas, the initial age distribution of susceptible, infectives and recovered are considered in line with model’s assumptions.

4.1. Simulation results. Since, in the present study our main focus was to set a control strategy for an age-structured SIR endemic model. In section 3, we have presented some analytical results, whereas in this section, we attempted to verify these results numerically. In the process of simulation, principle objectives was to identify the effectiveness of the control, i.e., vaccination and to know which age group should be vaccinated in order to control the infection with a minimum cost. Based on simulations, we have the following results:

Figure 1 shows the density of susceptible population of all ages in the case of no vaccination, whereas, Figure 2 represent the same population density with optimal vaccination strategy. It is clear from these two figures that compare to the scenario of no vaccination, the number of susceptible people have been reduced when the control is imposed. Upon implementation of vaccination, the peak value of susceptible density was reduced to 0.6 from 0.8. This means that before vaccination, more people was at risk to the infection which is markedly decreased after vaccination. Moreover, it is clear from these two figures that vaccination is more effective in long run for the higher age groups compare to lower age groups. In Figure 8 and 9, we picked some selective curves from $s(t, a)$ both with and without control which are clarifying our results more precisely. The blue, green and red curves in Figure 8 represents the solution profile of $s(t, a)$ at fixed ages 12, 28 and 52, respectively. Whereas, the plot 9 shows the solution profile of $s(t, a)$ at fixed time 8, 24 and 40 represented by blue, green and red curves, respectively.
Figure 1. The plot represents the density of susceptible population $s(t, a)$ without control.

Figure 2. The plot shows the behavior of the density of susceptible population $s(t, a)$ with control at time $t$ and age $a$. 
Figure 3. The plot represents the density of infected population $i(t, a)$ without control.

Figure 4. The plot shows the behavior of the density of infected population $i(t, a)$ with control at time $t$ and age $a$. 
Figure 5. The plot represent the density of recovered population $r(t, a)$ without control.

Figure 6. The plot shows the behavior of the density of recovered population $r(t, a)$ with control at time $t$ and age $a$. 
Figure 7. Behavior of the control variable with respect to time $t$ and age $a$.

Figure 8. The curves blue, green and red represents the solution profile of $s(t,a)$ at fixed ages 12, 28 and 52, respectively.
Figure 9. The plot shows the solution profile of $s(t,a)$ at fixed time 8, 24 and 40 represented by blue, green and red curves, respectively.

Figure 10. The curves blue, green and red represents the solution profile of $i(t,a)$ at fixed ages 12, 28 and 52, respectively.
Figure 11. The plot shows the solution profile of $i(t,a)$ at fixed time 8, 24 and 40 represented by blue, green and red curves, respectively.

Figure 12. The curves blue, green and red represents the solution profile of $r(t,a)$ at fixed ages 12, 28 and 52, respectively. The solid and dotted curves represent solution profile without and with control, respectively.
Figure 13. The plot shows the solution profile of $r(t,a)$ at fixed time 8, 24 and 40 represented by blue, green and red curves, respectively. Whereas, the solid and dotted curves represent solution profile without and with control, respectively.

Figure 14. The solid (blue), dotted (red) and dashed (green) represents sample curves of $u(t,a)$ at different ages 12, 28 and 52, respectively.
The control variable have a very small effect on infected class as well. Vaccination effect the elder infected population up to the mark, however, infected young and children are almost the same before and after vaccination. These facts are clear from Figures 3 and 4 along with Figures 10 and 11.

Similarly, we plotted the density of recovered population $r(t, a)$ both with and without control in Figures 5 and 6. In the case of no vaccination, the highest density of recovered people was 0.35 which increased to 0.5 when vaccine was supplemented. Furthermore, it is interesting to see that vaccination strongly effect the elder recovered population as suggested by Figures 12 and 13. Figure 7 represent the amount of vaccine available to a person of age $a$ at time $t$. Moreover, it is clear from Figures 14 and 15 that with increasing time and age direction, the amount of vaccine tends to increase.

5. Concluding remarks. In this article, we developed an age-structured SIR endemic model from the available literature. Since, the aim of the study was to control the infection, therefore, we considered vaccination as a control variable and a control problem was analyzed rigorously. For the derivation of necessary conditions, we used the weak derivatives and minimizing sequence argument. The desired criterion is achieved and sample simulations were carried out which shows the effectiveness of the control.

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