Unconventional band structure for a periodically gated surface of a three-dimensional topological insulator

Puja Mondal and Sankalpa Ghosh

Department of Physics, Indian Institute of Technology Delhi, New Delhi-110016, India

E-mail: sankalpa@physics.itd.ac.in

Received 13 March 2015
Accepted for publication 28 October 2015
Published 24 November 2015

Abstract

The surface states of the three-dimensional (3D) topological insulators are described by a two-dimensional (2D) massless dirac equation. A gate-voltage-induced one-dimensional potential barrier on such surfaces creates a discrete bound state in the forbidden region outside the dirac cone. Even for a single barrier it is shown that such a bound state can create an electrostatic analogue of Shubnikov de Haas oscillation which can be experimentally observed for relatively smaller size samples. However, when these surface states are exposed to a periodic arrangement of such gate-voltage-induced potential barriers, the band structure of the same was significantly modified. This is expected to significantly alter the properties of the macroscopic system. We also suggest that, within suitable limits, the system may offer ways to control electron spin electrostatically, which may be practically useful.

Keywords: massless Dirac fermions, surface states of topological insulators, bound states

Online supplementary data available from stacks.iop.org/JPhysCM/27/495301/mmedia

(Some figures may appear in colour only in the online journal)
gated barrier region for macroscopic samples, such conductance oscillation is hard to observe. To observe the effect of such bound states in the macroscopic sample we therefore consider a periodic array of such barriers on the surface of 3DTI in this paper. We show that the resulting band structure is unique for such MDFs and consists of two distinct parts, one inside the Dirac cone formed out of continuum scattering states and the other outside the Dirac cone originating from the bound states. When we consider the potential barrier in the $\delta$-function limit, the corresponding bound states are one-dimensional helical states with spin and momentum locked. In a periodic set-up of such $\delta$-function potential bands formed by helical states may provide methods to control spin through electrostatic means.

The effective Hamiltonian describing surface states of 3DTI can be written as

$$H_{\text{tot}} = v_F(\sigma \cdot p) + \frac{\lambda}{2} \sigma_3 (k^3_x + k^3_y)$$

(1)

Here $v_F$ and $\lambda$ are the Fermi velocity and wrapping parameter. $\sigma = \sigma_i \hat{\sigma} + \sigma_j \hat{j}$ is the Pauli matrix vector describing the real spin of the electron and $k_x = k_x \pm i k_y$. The first term in the Hamiltonian corresponds to that for two-dimensional MDFs giving circular-shaped energy contours for ungated 3DTI surface states centered around the Dirac point and dominates up to a certain energy value (e.g. in the case of Bi$_2$Te$_3$ it is up to 150 meV and in case of Bi$_2$Se$_3$ it is up to 100 meV). The contribution to the Hamiltonian due to the hexagonal wrapping (HW) effect is given by the second term $\frac{\lambda}{2} \sigma_3 (k^3_x + k^3_y)$ which becomes more effective as one moves away from the Dirac point [17, 18] and shows deviation from the circular energy plot. Therefore in the vicinity of the Dirac point within the above mentioned energy range the surface states of 3DTI have same Dirac-like Hamiltonian as Graphene, namely

$$H = v_F(\sigma \cdot p)$$

(2)

but without valley degeneracy like the latter.

The other approximation included in the Hamiltonian (2) is that the anisotropy in the Fermi velocity is ignored [17]. This is again valid in the close vicinity of Dirac points. We also consider here a single Fermi crossing for the surface states as opposed to greater numbers (odd number higher than one) of fermi crossings of the surface states [6]. Thus we model the surface states of the 3DTI with the the Hamiltonian (2) assuming that the height of the potential barrier (3) is within the stipulated limit satisfying the above mentioned conditions.

It may be also noted that such surface states can alternatively be described by the effective Hamiltonian $H' = (v_F \sigma \times k)_z$ [17] which can be obtained from (2) through a unitary transformation. We consider such surface states in a scalar potential barrier (figure 1(a))

$$V(x) = \begin{cases} V_0 & \text{if } |x| < d/2 \\ 0 & \text{if } |x| > d/2 \end{cases}$$

(3)

which only varies along the $x$-direction. We chose the height of the potential barrier to be less than the bulk gap of a 3DTI so that it does not create bulk excitation in the system. As known from the experimental work, the 3DTI has large bulk band gaps of the order of 0.3 eV for Bi$_2$Se$_3$ [19] and 0.15 eV for Bi$_2$Te$_3$ [20]. It may be also noted that such potential respects the time reversal symmetry of these surface states.

The model Hamiltonian (2) describes massless dirac fermions with zero chemical potential which is strictly valid only at (or in the immediate neighborhood of) the dirac point.

Writing the stationary solutions of the Schrödinger equation with energy $E$ as $\psi(x, y) = \psi(x) e^{i q_y y} e^{-i d x}$ for a given $V_0$, $\epsilon = \frac{E}{h v_F}$, the $x$-component of the wave vector is given by

$$q_x = \sqrt{\epsilon^2 - q_y^2}, |x| > \frac{d}{2}$$

$$q_x = \sqrt{\left(\frac{E - V_0}{h v_F}\right)^2 - q_y^2}, |x| < \frac{d}{2}$$

(4)

We define $\kappa = i q_x/c$ and $\alpha = \tanh{\left(\frac{\epsilon}{W}\right)}$, $q_x = \sqrt{-q_y^2 + (v_F d - \epsilon)^2}$.

Figure 1. (a) Schematic of the potential barrier and the bound state wavefunction (red) on the surface of a 3DTI. (b) Schematic figure of periodic arrangement of such potential barrier. Here $V = V_0$ in both figures.
exist if $\varepsilon < |q_y| < |v_yd - \varepsilon|$. This condition can only be satisfied with linear dispersion for the MDF. Such bound states cannot be created in the case of 2D NREG with quadratic dispersion. The wave functions for such solutions are:

$$
\psi(x) = \begin{cases} 
Ae^{x(x+d/2)} \left( \exp \left( i \left( \frac{\pi}{2} + i\alpha \right) \right) \right) & x < -d/2 \\
Be^{-x(x-d/2)} \left( \exp \left( i \left( \frac{\pi}{2} - i\alpha \right) \right) \right) & x > d/2 
\end{cases}
$$

(5)

$$
\psi(x) = C e^{i\alpha x} \left( \frac{1}{\varepsilon} \right) + De^{-i\alpha x} \left( \frac{1}{-\varepsilon} \right) \cdot |x| < d/2
$$

(6)

A schematic profile of such bound state wave functions is given in figure 1(a). The continuity of the wave function at $x = \pm d/2$ determines $A$, $B$, $C$, $D$ whose nontrivial solutions gives the quantization condition

$\tan \left( (\varepsilon - v_y)^2 - q_y^2 \right) + \frac{\sqrt{q_y^2 - \varepsilon^2}}{q_y^2 - (v_y - \varepsilon)} = 0.$

(7)

Here $q_y = q_yd$ and $\varepsilon = \varepsilon d$ are dimensionless. Equation (7) can be solved numerically to yield the bound states solutions, bounded between two parallel lines from $\varepsilon = \pm |q_y|$ to $\varepsilon = v_y \pm |q_y|$ (figure 2(a)). These bound state solutions have real energy and exist outside the dirac cone. The situation is contrasted with the bound state formation for massless dirac fermions inside a potential well (see figure 2(b)) (for details see footnote1). The quantized energy values for a given value of the gate voltage is given by

$$
\varepsilon = -\pi n v_y^2.
$$

Figure 2. (a) Bound states (blue) for potential barrier outside the dirac cone (black lines) in between $\varepsilon = \pm |q_y|$ (black) to $\varepsilon = v_y \pm |q_y|$ (magenta) for a potential barrier (b) For comparison bound states inside the dirac cone (black line) for the potential well problem for MDF (see the discussion in footnote2 section I) are plotted. (c) Relative DOS due to bound states as a function of $v_y$. In the inset similar DOS for an NREG in a uniform magnetic field $B^z \omega_c = \frac{eB}{mc}$ is shown (d) Conductance oscillation due to bound states as a function of $v_y$.

1 See the supplementary material for the discussion on difference between bound states in potential well and potential barrier in section I for the discussion of Green’s function in section II, for the discussion of details in the calculation of conductance of MDF, $\sigma_N^C$, in section IIIA and for some details of the calculation of conductance $\sigma_N^B$, due to bound states in section III B (stacks.iop.org/JPhysCM/27/495301/mmedia).

2 See footnote 1.
Such tunable gate-voltage, profoundly affects the DOS and consequently other properties due to the discrete energy spectrum. Here $v_F = 5 \times 10^6$ m s$^{-1}$ of Bi$_2$S$_3$ for our calculation [21]. For the continuum states of MDF on the surface of a 3DTI obeying $\epsilon(\mathbf{q}) = \mathbf{q}$, DOS $\rho$ is

$$\rho_b(\varepsilon) = \frac{2L_xL_y}{4\pi^2} \int d^2 q (E - E(\mathbf{q})) = \rho_0 |\varepsilon|$$

(8)

with $\rho_0 = \frac{L_xL_y}{2\pi \hbar v_F}$ is the sample length along $x, y$. The contribution to the DOS due to the discrete bound states ($\rho_b$) can be calculated from equation (7) as

$$\rho_b(\varepsilon) = 2\rho_0 \frac{d}{L_x} \sum_n \left| \frac{\partial \tilde{q}_n}{\partial \varepsilon} \right| \delta(\tilde{q}_n - \varepsilon)$$

(9)

where

$$\frac{\partial \tilde{q}_n}{\partial \varepsilon} = \tilde{q}_n - \varepsilon + \frac{\varepsilon - \tilde{q}_n^2}{\varepsilon - \tilde{q}_n^2 - \tilde{q}_n^2 \varepsilon - \varepsilon^2}$$

(10)

$\rho_b$ scales with $d$ and its variation with $\varepsilon$ is plotted in figure 2(c). When for a given energy the condition $\varepsilon = \varepsilon_n$ is satisfied for a given $v_F$, a jump occurs in DOS as expected from equation (9). From figure 2(c) one finds the behavior of $\rho_b$ as a function of the gate voltage is very similar to that of the DOS of an NREG in the presence of a magnetic field. To show how such DOS influences the transport, we calculate the conductance in presence of such bound states.

Since the DOS receives contributions both from the free massless dirac fermions as well as the bound states, either of these states can contribute to the conductivity tensor. The conductance of free 2D MDF was already studied [22–24]. Briefly, in terms of energy eigen states, the expression for the frequency ($\omega$) dependent conductivity tensor at finite temperature ($T$)

$$\sigma_{\nu\nu}(\omega, \beta) = \frac{i}{\omega} \int d\varepsilon \int \frac{d\mathbf{r}d\mathbf{r}'}{L_xL_y} \sum_{m,n} \langle m| \hat{j}_n(\mathbf{r})n \rangle \langle \hat{j}_m(\mathbf{r}')|m \rangle\times \delta(\varepsilon - \varepsilon_n) \left\{ f(\varepsilon) - f(\varepsilon_n) \right\}$$

(11)

where $f(\varepsilon) = \frac{1}{\exp(\hbar \omega + \beta\varepsilon) + 1}$ is the Fermi–Dirac distribution at temperature $T = k_B(\beta)^{-1}$ and zero chemical potential. By taking $L_{x,y} \rightarrow \infty$, then $\hbar \beta \rightarrow 0$ and $\omega \rightarrow 0$ in (11) expression for dc conductivity can be obtained.

In this paper, our main purpose is to see change in the conductivity due to the presence of bound states. Therefore the contribution to the conductivity by the scattering electrons is used as an overall scale factor for such bound-gate-induced conductivity calculated up to the leading order. To this purpose, we have only considered the first term in Kubo formula in the linear response regime to calculate conductivity for continuum and bound states without considering any vertex correction. Scattering of surface electrons are intrinsically anisotropic because of the fact that surface states $|\psi(k)\rangle$ and $|\psi(-k)\rangle$ form Kramers pairs and they are orthogonal. In this case, transport of surface electrons is not equal to scattering time of ordinary electrons in the presence of disorder. Transport time is equal to twice the scattering time of conventional electrons for scalar isotropic disorder i.e. $\tau_s = 2\tau_c$.

The next leading correction to the conductivity calculated here is the vertex correction. This will add another term to the classical conductivity which will be from corrections due to ladder diagram or diffusion. In this case re-normalized vertex current is proportional to the bare current. The contribution of diffusion to the conductivity will be the same order of bare current. Corresponding calculations for surface states of 3DTI were performed in the literature [25]. Contributions to the conductivity from diffusion is of the form

$$\sigma_s = \frac{\hbar}{2\pi} \text{tr} [J_s \Gamma^{(d)} J_s]$$

where $\Gamma^{(d)}$ and $J_s$ are diffusion structure factor and re-normalized vertex current. Here we have not included such contributions. Within the above mentioned approximations the expression of zero temperature dc conductance for MDF due to the continuum of the scattering states is obtained as (details in footnote$^3$)

$$\sigma_{yy}(\omega \rightarrow 0) = \frac{e^2L_x\pi}{L_y \hbar} \left\{ \frac{\epsilon \varepsilon_n}{\hbar} + \frac{1}{\pi} \left( 1 - \frac{\epsilon \varepsilon_n}{\hbar} \tan^{-1}(\hbar\epsilon \varepsilon_n) \right) \right\}$$

(12)

where $\varepsilon_n = 2\tau_s$ is the transport time of surface electrons in the presence of disorder. Similarly, the expression of the conductance in the $\omega \rightarrow 0$ and $T \rightarrow 0$ limit due to the bound states can be calculated as (details in footnote$^4$)

$$\sigma_{yy}^b = \frac{4e^2d}{\pi L_x \hbar} \sum_n \left| \frac{\partial \tilde{q}_n}{\partial \varepsilon} \right| \chi(\varepsilon_n) \eta$$

(13)

where $\chi(\varepsilon) = |\int_0^{\infty} \psi^{\dagger}(x)\psi(x) dx|^2$. The ratio of the free particle and bound state contribution to $\sigma_{yy}$ (we drop common $\chi$),

$$\frac{\sigma_{yy}^b}{\sigma_{yy}} = \frac{4d}{\pi L_x \epsilon_F} \left| \frac{\partial \tilde{q}_n}{\partial \varepsilon} \right| \chi(\epsilon_F)$$

(14)

which oscillates with the changing $v_F$. This is plotted in figure 2(d). As expected, this oscillation is similar to the SdH oscillation in the presence of a magnetic field due to the discrete nature of the bound states. As the current oscillation suggests, a gaped surface of the 3DTI can therefore be used for switching purposes [26]. However, unlike in the case of SdH oscillation, here $\sigma_s$ scales with the $d_{\beta}$, the relative width of the barrier. Therefore, whereas for a mesoscopic sized sample, such single-barrier-induced oscillation may be observed [27], in a macroscopic sample such effects will vanish.

Even though local measurement such as local DOS [28, 29] can detect such bound state formations by single barrier, creating a global effect on macroscopic samples will be more desirable for application. An obvious way to achieve this is to tile the surface with a periodic array of such gate-voltage-induced potential barriers. Such tunable, superlattice structures of periodic potential have recently been realized.

$^3$ See footnote 1.

$^4$ See footnote 1.
For massless Dirac fermions in the case of Graphene [30, 31] and recently in the case of Topological insulators [32]. Very recently, persistent optical gating of Topological Insulators has been achieved through which such gated structures [33] can also be achieved. We consider a periodic array of the potential barrier (see figure 1(b)) given as

$$V(x) = \sum_{n} V_{0}\Theta\left(x + \frac{d}{2} - nL\right)\Theta\left(nL + \frac{d}{2} - x\right), n \in I$$

where $\Theta(x)$ is the Heaviside step function. Here $L$ is unit cell size, $b = L - d$ is the inter-barrier separation. In the $n-1$th unit cell the wave functions in the region $I$ and $II$ are respectively given by

$$\psi_I(x) = C_n e^{iq_{I}(x-nL)}\left(\frac{1}{\mp e^{ib}}\right) + D_n e^{-iq_{I}(x-nL)}\left(\frac{1}{\mp e^{-ib}}\right)$$

$$\psi_{II}(x) = A_n e^{iq_{II}(x-nL)}\left(\frac{1}{\pm e^{ib}}\right) + B_n e^{-iq_{II}(x-nL)}\left(\frac{1}{\pm e^{-ib}}\right)$$

The wave function in the $n$th cell is given by

$$\psi(x) = C_n e^{iq_{I}(x-nL)}\left(\frac{1}{\pm e^{ib}}\right) + D_n e^{-iq_{I}(x-nL)}\left(\frac{1}{\mp e^{-ib}}\right)$$

Matching the boundary conditions in the interfaces $x = (n-1)L$ and $x = nL$, we get (for details of the method see [34, 35])

$$\begin{pmatrix} C_{n-1} \\ D_{n-1} \end{pmatrix} = M \begin{pmatrix} C_n \\ D_n \end{pmatrix}$$

where $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is a unimodular transfer matrix that connects equivalent unit cells. However, in a periodic potential the Bloch equation demands

$$\left(\begin{array}{c} C_n \\ D_n \end{array}\right) = e^{ikL} \left(\begin{array}{c} C_{n-1} \\ D_{n-1} \end{array}\right)$$

Equating these two relations one gets the eigenvalue condition as $\det[M_{\text{total}} - \lambda I] = 0$ whose solution gives the Bloch vector as

$$\lambda_1 + \lambda_2 = e^{-iKL} + e^{iKL} \Rightarrow K = \frac{1}{L} \cos^{-1}\left[\frac{1}{2} \text{Tr}(M_{ij})\right]$$

Equation (18) when explicitly written in terms of the matrix element takes the usual Kronig–Penny form ($k_i = i\kappa$)
The dispersion $E = \pm q y d$ due to the bound states (magenta in figure 3), separated by states (blue in figure 3) and the other region outside the Dirac cone due to the presence of scattering $v_g$ superposed. Color of legends same as the color of bands. For two $\delta$-function barrier $v_g$ modes for four different values of $|q y|$, with corresponding $E$ $\neq 0$, so do the band properties such as band gap, band position etc. Because of the discreteness of the resulting band structure over a wide range of barrier strength, it is expected that the DOS in the presence of such periodic potential will oscillate in a similar manner like $\rho_b$ in equation (9). This will in turn affect various properties of a system.

It may be noted that in the band structure depicted in figure 3 a special situation arose when $E = V$ within the Dirac cone. This is because at that particular point the solution of the Dirac equation is different. Such points represent zero mode solutions, which have been discussed in a number of works [36, 37] earlier. Briefly, at $E = V$, the Dirac equation has the form

$$[\partial_{q y}^2 - k^2] \psi_{1,2}(x) = 0$$

The solution of equation (21) will have the form

$$\psi(x) = C e^{i k x} + D e^{-i k x}, \quad |x| < d/2$$

We have obtained a transcendental equation for the $E = V$ point by using the same transfer matrix method for scattering state solutions within the Dirac cone, namely

$$\cos KL = \cos(k_b) \cos(q_y d) + \sinh(k_b) \sin(q_y d)$$

$$\times \left[\tan \theta \cot \alpha - \frac{1}{\cos \theta \sinh \alpha}\right], \quad \varepsilon < \bar{q}_y$$

The band structures corresponding to equations (19) and (20) belong to two distinct region of bands in the $E - q y$ plane, one within the Dirac cone due to the presence of scattering states (blue in figure 3) and the other region outside the Dirac due to the bound states (magenta in figure 3), separated by $\varepsilon = \bar{q}_y$. Such band structure is unique to the MDF because of the formation of bound states in a potential barrier and constitutes one of the most important results of this paper. The band structure that is formed within the Dirac cone can again be analyzed to extract information for a number of interesting properties such as additional Dirac points [36–38], miniband formation [39, 40] which was already been studied for Dirac fermions in other contexts.

Here we only explain bands in the region $\varepsilon < \bar{q}_y$ using the tight-binding approximation. These bands arise in a similar way like in a generic tight-binding model due to the lifting of degeneracy of the bound states formed in each barrier by hopping amplitude. A typical example is that of the Landau bands in a Hofstadter butterfly [41] where the degeneracy of Landau levels is lifted by the introduction of a lattice potential. However, now they co-exist with the bands formed out of scattering states within the Dirac cone which set them apart from the Hofstadter problem. When the barrier separation ($b$) relative to the barrier width ($d$) is increased, the hopping amplitude is decreased. This leads to the shrinking of the band width and can be clearly seen by comparing the band structure in the left and right column of the figure 3. Since the number of bound states and their position changes with $V_0$, so do the band properties such as band gap, band position etc. Because of the discreteness of the resulting band structure over a wide range of barrier strength, it is expected that the DOS in the presence of such periodic potential will oscillate in a similar manner like $\rho_b$ in equation (9). This will in turn affect various properties of a system.

To explore further the non-trivial effects due to the bound state formation we consider the limit $d \rightarrow 0$ and $V_0 \rightarrow \infty$ such that $Z = V_0 d$ constant. Substitution of this in (4) gives $q_y d = \frac{Z}{\hbar v_F} = v y$ such that equation (7) gives $\tan(v y) = \frac{Z}{v y} = \frac{\varepsilon - E/\hbar}{v_F}$. The dispersion relation of the corresponding states are $\varepsilon = \pm q_y \cos(v y)$. Substitution of these results in equations (19) and (20) gives

$$\cos KL = \cos(k_b) \cos(q_y d) + \sinh(k_b) \sin(q_y d)$$

$$\times \left[\tan \theta \cot \alpha - \frac{1}{\cos \theta \sinh \alpha}\right], \quad \varepsilon < \bar{q}_y$$

$$\cos Kb = \cos(k_b) \cos(q_y d) + \sinh(k_b) \sin(q_y d) \left(\frac{\varepsilon}{k_b}\right), \quad \varepsilon < \bar{q}_y$$

$$\cos Kb = \cos(k_b) \cos(q_y d) + \sinh(k_b) \sin(q_y d) \left(\frac{\varepsilon}{k_b}\right), \quad \varepsilon > \bar{q}_y$$

The band structure generated by the bound states in this limit given by equation (25) are plotted in figure 4. Each band corresponds to a given value of $Z$. It is known [11, 29] that in this limit the bound states corresponding to the helical edge modes on the surface of a 3DTI at the interface of each potential barrier forming Tomonaga–Luttinger states [42]. For such states the momentum is locked with the spin whose sign (up/down) is determined by the Z. By changing electrostatic potential $V_0$ and thereby Z, one can flip the spin of such helical states. The bands shown in figure 4 by such helical modes can therefore play a very important role in spintronics [43].
out of bound states that exist outside the Dirac cone. They can create experimentally observable effects and in suitable limits may lead to the possibility of interesting applications. We thank K Sengupta and D Kumar for helpful discussion. PM is supported by a UGC fellowship and SG is partially supported by a UGC grant under UGC-UKIERI thematic partnership.

References

[1] Kane C L and Mele E J 2005 Phys. Rev. Lett. 95 146802
[2] Benevige B A, Hughes T L and Zhang S C 2006 Science 314 1757
[3] König M et al 2007 Science 318 766
[4] Moore J E and Balents L 2007 Phys. Rev. B 75 121306
[5] Fu L, Kane C L and Mele E J 2007 Phys. Rev. Lett. 98 106803
[6] Hsieh D et al 2008 Nature 452 970
[7] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[8] Qi X L and Zhang S C 2010 Phys. Rev. Lett. 105 236802
[9] Kuroda K et al 2010 Phys. Rev. Lett. 105 076802
[10] Xia Y et al 2009 Nat. Phys. 5 398
[11] Chen Y L et al 2009 Science 325 175
[12] Zhang H, Liu C X, Qi X L, Dai X, Fang Z and Zhang S C 2009 Nat. Phys. 5 438
[13] Ryu S, Murdy C, Furasaki A and Ludwig A W W 2007 Phys. Rev. B 75 205344
[14] Ludwig A A W, Fischer M P A, Shankar R and Grinstein G 1994 Phys. Rev. B 50 7526
[15] Shanz H, Liu C X, Qi X L, Dai X, Fang Z and Zhang S C 2009 Nat. Phys. 5 438
[16] Pereira J M, Minar V , Peeters F M and Vasilopoulos P 2006 Phys. Rev. B 74 045424
[17] Matulis A and Peeters F M 2008 Phys. Rev. B 77 115423
[18] Nguyen H C, Tien Hoang M and Lien Nguyen V 2009 Phys. Rev. B 79 035411
[19] Solyom J 2009 Fundamentals of the Physics of Solids vol 2 (Berlin: Springer) chapter 22
[20] Fu L 2009 Phys. Rev. Lett. 103 266801
[21] Yeats A L et al 2015 Sci. Adv. 1 E1500640
[22] Hofstadter D R 1976 Phys. Rev. B 14 2239
[23] Ghosh S and Sharma M 2009 J. Phys.: Condens. Matter 21 292204
[24] Park C H et al 2008 Phys. Rev. Lett. 101 126804
[25] Giamarchi T 2004 Quantum Physics in One Dimension (Oxford: Claredon Press)
[26] Bandyopadhyay S and Calhuy M 2008 Introduction to Spintronics (New York: CRC Press)