A Van Benthem Theorem for Modal Team Semantics

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Abstract. We study the expressive power of variants of modal dependence logic, MDL, formulas of which are not evaluated in worlds but in sets of worlds, so called teams. The logic can then express that in a team, the value of a certain variable is functionally dependent on values of some other variables. The formulas of MDL, and all of its variants studied in the literature so far, are invariant under bisimulation when lifted from single worlds to sets of worlds in a natural way. In this paper we study the problem whether there is a logic that captures exactly the bisimulation invariant properties of Kripke structures and teams. Our main result shows that an extension of MDL, modal team logic MTL, extending MDL (or even only ML) by classical negation, is such a logic. We also give two further alternative characterizations for the bisimulation invariant properties in terms of extended modal inclusion logic with classical disjunction, and an extension of ML by so-called first-order definable generalized dependence atoms.

Keywords: modal logic, dependence logic, team semantics, expressivity, bisimulation, independence, inclusion, generalized dependence atom

1 Introduction

The concepts of dependence and independence are ubiquitous in many scientific disciplines such as experimental physics, social choice theory, computer science, and cryptography. Dependence logic $\mathcal{D}$ \cite{4} and its so-called team semantics have given rise to a new logical framework in which various notions of dependence and independence can be formalized and studied. Dependence logic extends first-order logic by dependence atoms

$$=(x_1, \ldots, x_{n-1}, x_n),$$

expressing that the value of the variable $x_n$ is functionally dependent on the values of $x_1, \ldots, x_{n-1}$. The formulas of dependence logic are evaluated over teams, i.e., sets of assignments, and not over single assignments as in first-order logic.
In [19] a modal variant of dependence logic MDL was introduced. In the modal framework teams are sets of worlds, and a dependence atom

\[(p_1, \ldots, p_{n-1}, p_n)\]

holds in a team \(T\) if there is a Boolean function that determines the value of the propositional variable \(p_n\) from those of \(p_1, \ldots, p_{n-1}\) in all worlds in \(T\). One of the fundamental properties of MDL (and of dependence logic) is that its formulas satisfy the so-called downwards closure property: if \(M, T \models \varphi\), and \(T' \subseteq T\), then \(M, T' \models \varphi\). Still, the modal framework is very different from the first-order one, e.g., dependence atoms between propositional variables can be eliminated with the help of the classical disjunction \(\lor\) [19]. On the other hand, it was recently shown that eliminating dependence atoms using disjunction causes an exponential blow-up in the formula size, that is, any formula of ML(\(\lor\)) logically equivalent to the atom in (2) is bound to have length exponential in \(n\) [11].

The central complexity theoretic questions regarding MDL have been solved in [17,14,5,15].

Extended modal dependence logic, EMDL, was introduced in [6]. This extension is defined simply by allowing ML formulas to appear instead of just propositions inside dependence atoms. EMDL can be seen as the first towards combining dependencies with temporal reasoning. EMDL is strictly more expressive than MDL but its formulas still have the downwards closure property. In fact, EMDL has recently been shown to be equivalent to the logic ML(\(\lor\)) [6,11].

In the first-order case, several interesting variants of the dependence atoms have been introduced and studied. The focus has been on independence atoms

\[(x_1, \ldots, x_\ell) \perp (y_1, \ldots, y_m)(z_1, \ldots, z_n),\]

and inclusion atoms

\[(x_1, \ldots, x_\ell) \subseteq (y_1, \ldots, y_\ell),\]

which were introduced in [10] and [7], respectively. The intuitive meaning of the independence atom is that the variables \(x_1, \ldots, x_\ell\) and \(z_1, \ldots, z_n\) are independent of each other for any fixed value of \(y_1, \ldots, y_m\), whereas the inclusion atom declares that all values of the tuple \((x_1, \ldots, x_\ell)\) appear also as values of \((y_1, \ldots, y_\ell)\). In [12] a modal variant, MIL, of independence logic was introduced. The logic MIL contains MDL as a proper sublogic, in particular, its formulas do not in general have the downwards closure property. In [12] it was also noted that all MIL formulas are invariant under bisimulation when this notion is lifted from single worlds to a relation between sets of words in a natural way. At the same time (independently) in [11] it was shown that EMDL and ML(\(\lor\)) can express exactly the bisimulation invariant properties of Kripke structures and teams that are downwards closed.

A famous theorem by Johan van Benthem [2,3] states that modal logic is exactly the fragment of first-order logic that is invariant under bisimulation. In this paper we study this question in the context of team semantics, and we ask if there is a logic that captures exactly the bisimulation invariant properties of
Kripke structures and teams. Our results show that such a logic is obtained by extending ML by classical negation $\sim$. We call this logic Modal Team Logic MTL (in analogy to (first order) team logic [18]). The classical negation turns out to be a very powerful connective in the modal team semantics setting. For example, we show that dependence atoms have a polynomial size definition in MTL, while—as already mentioned—in ML(\$\otimes\$) this is not the case [11].

We also study whether all bisimulation invariant properties can be captured by variants of EMDL. Our results show that both extended modal independence and inclusion logic EMIL and EMINCL fail to capture all bisimulation invariant properties. On the other hand, we show that EMINCL(\$\otimes\$) (EMINCL extended with classical disjunction) is in fact as expressive as MTL, which the analogously defined EMIL is weaker than MTL. Finally, we show that the extension ML\$^F_O\$ of ML by all first-order definable generalized dependence atoms (see [12]) gives rise to a logic that is as well equivalent to MTL.

2 Preliminaries

A Kripke model is a tuple $M = (W, R, \Pi)$ where $W$ is a nonempty set of worlds, $R \subseteq W \times W$, and $\Pi: P \to 2^W$, where $P$ is the set of propositional variables. A team of a model $M$ as above is simply a set $T \subseteq W$. The central basic concept underlying Väänänen’s modal dependence logic and all its variants is that modal formulas are evaluated not in a world but in a team, i.e., a set of worlds. This is made precise in the following definitions.

We first recall the usual syntax of modal logic ML:

$$\varphi ::= p | \neg p | (\varphi \land \varphi) | (\varphi \lor \varphi) | \lozenge \varphi | \square \varphi,$$

where $p$ is a propositional variable. Note that we consider only formulas in negation normal form, i.e., negation appears only in front of atoms. As will become clear from the definition of team semantics of ML, that we present next, $p$ and $\neg p$ are not dual formulas, consequently tertium non datur does not hold in the sense that it is possible that $M, T \models p$ and $M, T \models \neg p$ (however, we still have that $M, T \models p \lor \neg p$ for all models $M$ and teams $T$).

**Definition 2.1.** Let $M = (W, R, \pi)$ be a Kripke model, let $T \subseteq W$ be a team, and let $\varphi$ be an ML-formula. We define when $M, T \models \varphi$ holds inductively:

- If $\varphi = p$, then $M, T \models \varphi$ iff $T \subseteq \Pi(p)$,
- If $\varphi = \neg p$, then $M, T \models \varphi$ iff $T \cap \Pi(p) = \emptyset$,
- If $\varphi = \psi \lor \chi$ for some formulas $\psi$ and $\chi$, then $M, T \models \varphi$ iff $T = T_1 \cup T_2$ with $M, T_1 \models \psi$ and $M, T_2 \models \chi$,
- If $\varphi = \psi \land \chi$ for some formulas $\psi$ and $\chi$, then $M, T \models \varphi$ iff $M, T \models \psi$ and $M, T \models \chi$,
- If $\varphi = \lozenge \psi$ for some formula $\psi$, then $M, T \models \varphi$ iff there is some team $T'$ of $M$ such that $M, T' \models \psi$ and 1. for each $w \in T$, there is some $w' \in T'$ with $(w, w') \in R$, and 2. for each $w' \in T'$, there is some $w \in T$ with $(w, w') \in R$. 


If $\varphi = \square \psi$ for some formula $\psi$, then $M, T \models \psi$ iff $M, T' \models \psi$, where $T'$ is the set $\{w' \in M \mid (w, w') \in R \text{ for some } w \in T\}$.

We define MTL, modal team logic, to extend ML by a second type of negation, denoted by $\sim$ and interpreted just as classical negation. The syntax is formally given as follows:

$$\varphi ::= p \mid \neg p \mid \sim \varphi \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid \Diamond \varphi \mid \Box \varphi,$$

where $p$ is a propositional variable. The semantics of MTL is defined by extending Def. 2.1 by the following clause:

- If $\varphi = \sim \psi$ for some formula $\psi$, then $M, T \models \varphi$ iff $M, T \not\models \psi$.

We note that usually (see [16]), MTL also contains dependence atoms; however since these atoms can be expressed in MTL we omit them in the syntax (see Proposition 2.3 below). The classical disjunction $\lor$ is also readily expressed in MTL: $\varphi \lor \psi$ is logically equivalent to $\sim(\sim \varphi \land \sim \psi)$.

Analogously to the first-order setting, ML-formulas satisfy the following flatness property:

**Proposition 2.2.** [19] Let $M$ be a Kripke model and $T$ a team of $M$. Let $\varphi$ be a ML-formula. Then the following are equivalent:

1. $M, T \models \varphi$,
2. $M, \{w\} \models \varphi$ for each $w \in T$,
3. $M, w \models \varphi$ for each $w \in T$.

In [19] the connective $\neg$ is allowed to appear freely in MDL formulas. The well-known dualities from classical modal logic are also true for MDL formulas hence any ML-formula (even MDL) can be rewritten in such a way that $\neg$ only appears in front of propositional variables (without increasing the formula size). For a ML formula $\varphi$, we let $\varphi^{\text{dual}}$ denote the formula that is obtained by transforming $\neg \varphi$ to negation normal form. Now by Proposition 2.2 it follows that

$$M, T \models \varphi^{\text{dual}} \text{ iff } M, w \not\models \varphi \text{ for all } w \in T,$$

hence it follows that $M, T \models \sim \psi^{\text{dual}}$ if and only if $M, T \not\models \psi^{\text{dual}}$ if and only if there is some $w \in T$ with $M, w \models \psi$. We therefore often write $E \psi$ instead of $\sim \psi^{\text{dual}}$.

The next proposition shows that dependence atoms can be easily expressed in MTL.

**Proposition 2.3.** The dependence atom (2) can be expressed in MTL by a formula that has length polynomial in the length of $n$.

**Proof.** Note first that, analogously to the first-order case [1], (2) is logically equivalent with

$$\left( \bigwedge_{1 \leq i \leq n-1} = (p_i) \right) \rightarrow = (p_n),$$

...
where $\neg\circ$ is the so-called intuitionistic implication with the following semantics:

\[ M, T \models \varphi \rightarrow \psi \text{ iff for all } T' \subseteq T: \text{ if } M, T' \models \varphi \text{ then } M, T' \models \psi. \]

The connective $\neg\circ$ has a short logically equivalent definition in MTL (see [16]):

$\varphi \rightarrow \psi$ is equivalent to $(\neg\varphi \circ \psi) \circ \bot$, where $\circ$ is the dual of the splitjunction $\lor$, i.e., $\varphi \circ \psi$ is defined as $\sim(\sim\varphi \lor \sim\psi)$. Finally, $=(p_i)$ can be written as $p_i \oplus \neg p_i$, hence the claim follows. \hfill $\square$

The intuitionistic implication has been studied in modal team semantics context in [20].

We now introduce the central concept of bisimulation. Intuitively, two pointed models (i.e., pairs of models and worlds from the model) $(M_1, w_1)$ and $(M_2, w_2)$ are bisimilar, if they are indistinguishable from the point of view of modal logic. The notion of $k$-bisimilarity introduced below corresponds to indistinguishability by formulas with modal depth up to $k$: For a formula $\varphi$ in any of the logics considered in this paper, the modal depth of $\varphi$, denoted with $md(\varphi)$, is the maximal nesting degree of modal operators (i.e., $\Box$ and $\Diamond$) in $\varphi$.

**Definition 2.4.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models. We define inductively what it means for states $w_1 \in W_1$ and $w_2 \in W_2$ to be $k$-bisimilar, for some $k \in \mathbb{N}$, written as $(M_1, w_1) \equiv_k (M_2, w_2)$.

- $(M_1, w_1) \equiv_0 (M_2, w_2)$ holds if for each propositional variable $p$, we have that $M_1, w_1 \models p$ if and only if $M_2, w_2 \models p$.
- $(M_1, w_1) \equiv_{k+1} (M_2, w_2)$ holds if the following three conditions are satisfied:
  1. $(M_1, w_1) \equiv_0 (M_2, w_2)$,
  2. for each successor $w'_1$ of $w_1$ in $M_1$, there is a successor $w'_2$ of $w_2$ in $M_2$ such that $(M_1, w'_1) \equiv_k (M_2, w'_2)$ (forward condition),
  3. for each successor $w'_2$ of $w_2$ in $M_2$, there is a successor $w'_1$ of $w_1$ in $M_1$ such that $(M_1, w'_1) \equiv_k (M_2, w'_2)$ (backward condition).

Bisimulation is a characterization of the following semantic notion of equivalence:

**Definition 2.5.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, and let $w_1 \in W_1$, $w_2 \in W_2$. Then $(M_1, w_1)$ and $(M_2, w_2)$ are $k$-equivalent for some $k \in \mathbb{N}$, written $(M_1, w_1) \equiv_k (M_2, w_2)$ if for each modal formula $\varphi$ with $md(\varphi) \leq k$, we have that $M_1, w_1 \models \varphi$ if and only if $M_2, w_2 \models \varphi$.

Again, we simply write $w_1 \equiv_k w_2$ if the models $M_1$ and $M_2$ are clear from the context. As mentioned above, $k$-bisimulation and $k$-equivalent coincide. The following result is standard:

**Proposition 2.6.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, and let $w_1 \in W_1$, $w_2 \in W_2$. Then $(M_1, w_1) \equiv_k (M_2, w_2)$ if and only if $(M_1, w_1) \equiv_k (M_2, w_2)$. 

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For MTL and more generally logics with semantics defined on the team level, the above notion of bisimulation can be lifted to teams. The following definition is adapted from [12]:

**Definition 2.7.** Let $M_1 = (W_1, R_1, \pi_1)$ and $M_2 = (W_2, R_2, \pi_2)$ be Kripke models, let $T_1$ and $T_2$ be teams of $M_1$ and $M_2$. Then $(M_1, T_1)$ and $(M_2, T_2)$ are $k$-bisimilar, written as $M_1, T_1 \models_k M_2, T_2$ if the following holds:

- for each $w_1 \in T_1$, there is some $w_2 \in T_2$ such that $(M_1, w_1) \models_k (M_2, w_2)$,
- for each $w_2 \in T_2$, there is some $w_1 \in T_1$ such that $(M_1, w_1) \models_k (M_2, w_2)$.

To simplify notation, we often write $T_1 \models_k T_2$ if $M_1$ and $M_2$ are clear from the context. The following result is implied by the proof in [12]:

**Proposition 2.8.** If $(M_1, T_1) \models_k (M_2, T_2)$, then for each formula $\varphi \in \text{MTL}$ with $md(\varphi) \leq k$, we have that $M_1, T_1 \models \varphi$ if and only if $M_2, T_2 \models \varphi$.

The proof (a straight-forward adaptation of the one in [12]) will be given in the full version of this paper.

By the famous theorem of van Benthem, the expressive power of classical modal logic (i.e., without team semantics) can be characterized by bisimulations. In particular, for every pointed model $(M, w)$, there is a modal formula of modal depth $k$ that exactly characterizes $(M, w)$ up to $k$-bisimulation.

In the following, we restrict ourselves to a finite set of propositional variables.

**Theorem 2.9.** [4, Theorem 32] For each pointed Kripke model $(M, w)$ and each natural number $k$, there is a Hintikka formula $\phi_{M, w}^k \in \text{ML}$ with $md(\phi_{M, w}^k) = k$ such that for each pointed model $(M', w')$, the following are equivalent:

1. $M', w' \models \phi_{M, w}^k$,
2. $(M, w) \models_k (M', w')$.

Clearly, we can choose the Hintikka formulas such that $\phi_{M, w}^k$ is uniquely determined by the bisimilarity type of $(M, w)$, such that for $k$-bisimilar pointed models $(M_1, w_1)$ and $(M_2, w_2)$, the formulas $\phi_{M_1, w_1}^k$ and $\phi_{M_2, w_2}^k$ are identical.

It it clear that Theorem 2.9 does not hold for an infinite set of propositional symbols, since a finite formula can only specify the values of finally many variables.

### 3 Expressiveness of MTL

In this Section, we study the expressive power of MTL. As usual, we measure the expressive power of such a logic by the set of properties expressible in it. For this, we say that a team property is a set of pairs $(M, T)$ where $M$ is a Kripke model and $T$ a team of $M$. For an MTL-formula $\varphi$, we say that $\varphi$ expresses the property $\{(M, T) \mid M, T \models \varphi\}$.
Definition 3.1. Let $P$ be a team property. Then $P$ is invariant under $k$-bisimulation if for each pair of Kripke models $M_1$ and $M_2$ and teams $T_1$ and $T_2$ with $(M_1, T_1) \equiv_k (M_2, T_2)$ and $(M_1, T_1) \in P$, it follows that $(M_2, T_2) \in P$.

From Proposition 2.8 we know that every property expressed by an MTL-formula $\varphi$ with $md(\varphi) \leq k$ is invariant under $k$-bisimulation. The following is our main result on the expressive power of MTL, and shows that the converse of the above is true as well: MTL can express exactly the properties that are invariant under $k$-bisimulation, for some finite $k$.

Theorem 3.2. Let $P$ be a team property and $k \in \mathbb{N}$. Then the following are equivalent:

1. $P$ is invariant under $k$-bisimulation,
2. there is an MTL-formula $\varphi$ with $md(\varphi) = k$ that characterizes $P$.

The remainder of Section 3 is devoted to the proof of Theorem 3.2. We start with a natural characterization of the semantics of splitjunction—note that the following lemma is true for ML-formulas only: As an example, let $\psi$ be the dependence atom $=(p_1, \ldots, p_{n-1}, p_n)$. Then for every world $w$ of every model $M$, we have $M, \{w\} \models \psi$, but clearly, $\psi$ is not a tautology on the level of teams. In particular, the lemma fails for $S = \{\psi\}$ and any $M, T$ with $M, T \not\models \psi$.

Lemma 3.3. Let $S$ be a finite set of ML-formulas, let $M$ be a model and $T$ a team. Then $M, T \models \bigvee_{\varphi \in S} \varphi$ if and only if for each world $w \in T$, there is a formula $\varphi \in S$ with $M, w \models \varphi$.

Proof. Let $M$ be a model and $T$ a team. For $|S| = 1$, the claim follows from Proposition 2.2. Therefore assume that the lemma holds for $S' = \{\varphi_1, \ldots, \varphi_{n-1}\}$, and consider a set $S = S' \cup \{\varphi_n\}$. Then $\bigvee_{\varphi \in S} \varphi = \bigvee_{\varphi \in S'} \varphi \lor \varphi_n$. Therefore, $T = T_1 \cup T_2$ with

- $M, T_1 \models \bigvee_{\varphi \in S'} \varphi$;
- $M, T_2 \models \varphi_n$.

Due to the induction assumption, we know that for every world $w \in T_1$, there is a formula $\varphi \in S' \subseteq S$ with $M, w \models \varphi$. Due to Proposition 2.2, we know that each world $w \in T_2$ satisfies the formula $\varphi_n \in S$, therefore each world $w \in T$ satisfies some formula $\varphi \in S$ as claimed.

For the converse, assume that for each $w \in T$, there is some formula $\varphi \in S$ with $M, w \models \varphi$. Let $T_1 = \{w \in T \mid M, w \models \varphi_i \text{ for some } i \leq n - 1\}$, and let $T_2 = \{w \in T \mid M, w \models \varphi_n\}$. By the prerequisites we know that $T = T_1 \cup T_2$. Due to induction, we have that $M, T_1 \models \bigvee_{\varphi \in S'} \varphi$, and, due to Proposition 2.2 we know that $M, T_2 \models \varphi_n$. It therefore follows that $M, T \models \bigvee_{\varphi \in S'} \varphi \lor \varphi_n = \bigvee_{\varphi \in S} \varphi$ as claimed.

We now define the set of all Hintikka formulas that will appear in our later constructions. Informally, $\Phi^{=k}$ is the set of all Hintikka formulas characterizing models up to $k$-bisimilarity:
Definition 3.4. For $k \in \mathbb{N}$, the set $\Phi^\equiv_k$ is defined as
$$\Phi^\equiv_k = \{ \Phi^k_{M,w} \mid (M, w) \text{ is a pointed Kripke model} \}.$$  

An important observation is that $\Phi^\equiv_k$ is a finite set: This follows since above, we chose the representatives $\Phi^k_{M,w}$ to be identical for $k$-bisimilar models, and since there are only finitely many pointed models up to $k$-bisimulation. Since $\Phi^\equiv_k$ is finite, we can in the following freely use disjunctions over arbitrary subsets of $\Phi^\equiv_k$ and still obtain a finite formula.

Our next definition is used to characterize a team, again up to $k$-bisimulation. Since teams are sets of worlds, we use sets of formulas to characterize teams in the natural way, by choosing, for each world in the team, one formula that characterizes it.

**Definition 3.5.** For a model $M$ and a team $T$, let
$$\Phi^\equiv_k^{M,T} = \{ \varphi \in \Phi^\equiv_k \mid \text{there is some } w \in T \text{ with } (M, w) \models \varphi \}.$$  

Since $\Phi^\equiv_k^{M,T} \subseteq \Phi^\equiv_k$, it follows that $\Phi^\equiv_k^{M,T}$ is finite as well. In fact, it is easy to see that $|\Phi^\equiv_k^{M,T}|$ is exactly the number of $k$-bisimilarity types in $T$, i.e., the size of a maximal subset of $T$ containing only worlds such that the resulting pointed models are pairwise non-$k$-bisimilar.

We now combine the formulas from $\Phi^\equiv_k^{M,T}$ to be able to characterize $M$ and $T$ (up to $k$-bisimulation) in a single formula:

**Definition 3.6.** For a model $M$ with a team $T$, let
$$\varphi^\equiv_k^{M,T} = \left( \bigwedge_{\varphi \in \Phi^\equiv_k^{M,T}} E \varphi \right) \land \left( \bigvee_{\varphi \in \Phi^\equiv_k^{M,T}} \varphi \right).$$  

Intuitively, the formula $\varphi^\equiv_k^{M,T}$ expresses that in a model $M'$ and $T'$ with $M', T' \models \varphi^\equiv_k^{M,T}$, for each world $w \in T$ there must be some $w' \in T'$ such that $(M, w) \equiv^k_k (M', w')$, and conversely, for each $w' \in T'$, there must be some $w \in T$ with $(M, w) \equiv^k_k (M', w')$, which then implies that $(M, T)$ and $(M', T')$ are indeed bisimilar.

**Lemma 3.7.** Let $M$ be model and $T$ be a team. Then by definition and with Lemma 3.5, we can derive the following equivalences:

1. $\bigwedge_{\varphi \in \Phi^\equiv_k^{M,T}} E \varphi \Leftrightarrow \forall \varphi \in \Phi^\equiv_k^{M,T} \exists w \in T \text{ such that } M, w \models \varphi$ holds.
2. $\bigvee_{\varphi \in \Phi^\equiv_k^{M,T}} \varphi \Leftrightarrow \forall w \in T \exists \varphi \in \Phi^\equiv_k^{M,T} \text{ such that } M, w \models \varphi$ holds.

From the above, it follows that $\varphi^\equiv_k^{M,T}$ is a finite MTL-formula. Therefore together with Lemma 3.7, it holds that $\varphi^\equiv_k^{M,T}$ expresses $k$-bisimilarity with $(M, T)$:

**Lemma 3.8.** Let $M_1, M_2$ be Kripke models with teams $T_1, T_2$. Then the following are equivalent:

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\(- (M_1, T_1) \equiv_k (M_2, T_2)\)
\(- M_1, T_1 \models \varphi_{M, T}^k\).

Proof. First assume that \((M_1, T_1) \equiv_k (M_2, T_2)\). To see that \(M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M, T}^k} \mathbb{E}\varphi\), let \(\varphi \in \Phi_{M, T}^k\). By definition, \(\varphi\) is an ML-formula with \(md(\varphi) \leq k\) and there is some world \(w_2 \in T_2\) such that \(M_2, w_2 \models \varphi\). Due to the bisimulation condition, we know that there is a world \(w_1 \in T_1\) with \((M_1, w_1) \equiv_k (M_2, w_2)\). From Proposition 2.8 it follows that \(M_1, w_1 \models \varphi\). Since \(w_1 \in T_1\), it follows that \(M_1, T_1 \models \mathbb{E}\varphi\).

Since this is true for each \(\varphi \in \Phi_{M, T}^k\), it follows that \(M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M, T}^k} \mathbb{E}\varphi\).

It remains to show that \(M_1, T_1 \models \bigvee_{\varphi \in \Phi_{M, T}^k} \varphi\). Due to Lemma 3.3 it suffices to show that for each world \(w_1 \in T_1\), there is some formula \(\varphi \in \Phi_{M, T}^k\) with \(M_1, w_1 \models \varphi\). Therefore, let \(w_1 \in T_1\). Due to the bisimulation condition, we know that there is some \(w_2 \in T_2\) with \((M_1, w_1) \equiv_k (M_2, w_2)\). Due to the definition of \(\Phi_{M, T}^k\), we know that for the formula \(\varphi := \Phi_{M, w_2}^k\) we have that \(\varphi \in \Phi_{M, T}^k\) and \(M_2, w_2 \models \varphi\). Since \((M_1, w_1) \equiv_k (M_2, w_2)\), it follows that \(M_1, w_1 \models \varphi\), and hence for each \(w_1 \in T_1\) there is a formula \(\varphi\) as required.

For the converse, assume that \((M_1, T_1) \equiv (M_2, T_2)\). In particular, it follows that \(M_1, T_1 \models \bigvee_{\varphi \in \Phi_{M, T}^k} \varphi\) due to Lemma 3.3 it follows that for each \(w_1 \in T_1\), there is some formula \(\varphi \in \Phi_{M, T}^k\) with \(M_1, w_1 \models \varphi\). Due to the definition of \(\Phi_{M, T}^k\), we know that \(\varphi \in \Phi_{M, T}^k\), and that there is some \(w_2 \in T_2\) with \(M_2, w_2 \models \varphi\). From Theorem 2.9 it therefore follows that for each \(w_1 \in T_1\), there is some \(w_2 \in T_2\) with \((M_1, w_1) \equiv_k (M_2, w_2)\).

Let \(w_2\) be a world in \(T_2\), and let \(\varphi := \Phi_{M, w_2}^k\). It then follows that \(\varphi \in \Phi_{M, T}^k\). Since \(M_1, T_1 \models \bigwedge_{\varphi \in \Phi_{M, T}^k} \mathbb{E}\varphi\), it follows from Lemma 3.7 that there is some \(w_1 \in T_1\) with \(M_1, w_1 \models \varphi\). Due to the choice of \(\varphi\), Theorem 2.9 implies that \((M_1, w_1) \equiv_k (M_2, w_2)\) as required.

With the above results, we can now prove Theorem 3.2. For the converse, assume that \(P\) is invariant under \(k\)-bisimulation. We claim that the formula

\[
\varphi P := \bigoplus_{(M, T) \in P} \varphi_{M, T}^k
\]

expresses \(P\).

First note that \(\varphi\) is in fact the disjunction of only finitely many different formulas: Each \(\varphi_{M, T}^k\) is uniquely defined by a subset of the finite set \(\Phi_{M, T}^k\), therefore only finitely many different formulas appear in the disjunction. We now show that for each model \(M\) and team \(T\), we have that \((M, T) \in P\) if and only if \(M, T \models \varphi P\).

First assume that \((M, T) \in P\). Then the fact that \((M, w) \equiv_k (M, w)\) for each model \(M\), each world \(w\) and each number \(k\) and Lemma 3.3 imply that \(M, T \models \varphi_{M, T}^k\). Therefore, \(M, T \models \varphi P\).

For the converse, assume that \(M, T \models \varphi P\). Then there is some \((M', T') \in P\) with \(M, T \models \varphi_{M', T'}^k\). Due to Lemma 3.8 it follows that \((M, T) \equiv_k (M', T')\). Since \(P\) is invariant under \(k\)-bisimulation, it follows that \((M, T) \in P\) as required.
4 Extended Dependence Atoms

In the literature, three “dependence-like” atoms have been studied in detail, namely the dependence atom itself (see equation 1), the independence atom [10] and the inclusion atom [7]. In [19] and [12], the dependence and independence atoms were applied in the modal context to define Modal Dependence Logic (MDL) and Modal Independence Logic (MIL), respectively. In these logics, the dependence and independence atoms are only applied to propositional variables.

In [6], Ebbing et al. introduced Extended Modal Dependence Logic (EMDL), where this restriction is relaxed: In EMDL, the dependence atom may be applied to modal formulas (however, no nesting of the dependence operator is allowed in EMDL). EMDL was later shown to be equivalent in expressive power to modal logic with classical disjunction, and able to express all downwards-closed properties that are invariant under k-bisimulation for some natural number k [11].

Analogously to this treatment of the dependence operator in [6], in this section we consider the independence- and inclusion operators. Depending on whether we also allow classical negation or not, this gives four logics, namely MIL (as introduced in [12]), EMIL (Extended Modal Independence Logic), EMIL (EMIL extended with classical negation), EMINC (Extended Modal Inclusion Logic) and EMINC (EMINC extended with classical negation). We study the expressiveness of these logics, and show that while EMINC is as expressive as MTL, for each of the other three logics there is an MTL-expressible property that cannot be expressed in the logic. We also study ML with the addition of arbitrary first-order definable atoms, and show that the resulting logic—even without the addition of classical disjunction—is also equally expressible as MTL.

We note that adding classical negation to any of the logics discussed in this section clearly results in a logic that is equivalent in expressiveness to MTL.

4.1 Extended Modal Independence Logic (EMIL)

We first consider Extended Modal Independence Logic (EMIL). Syntactically, EMIL extends ML by the following: If $P$, $Q$, and $R$ are sets of ML-formulas, then $P \perp R Q$ is an EMIL-formula. The semantics of this extended independence atom are defined by lifting the definition for propositional variables given in [12] to ML-formulas as follows.

For a formula $\varphi$ and a team $T$, we write $\varphi(T)$ for the function defined as $\varphi(T) = 1$ if $M, T \models \varphi$, and $\varphi(T) = 0$ otherwise (the model $M$ will always be clear from the context). For a world $w$, we simply write $\varphi(w)$ instead of $\varphi(\{w\})$.

For a set of formulas $\overline{P}$ and worlds $w_1, w_2$, we write $w_1 \equiv_{\overline{P}} w_2$ if $\varphi(w_1) = \varphi(w_2)$ for each $\varphi \in \overline{P}$.

$$M, T \models \overline{P} \perp_{\overline{P}} Q \iff \forall w, w' \in T : w \equiv_{\overline{P}} w' \text{ implies } \exists w'' \in T : w'' \equiv_{\overline{P}} w \text{ and } w'' \equiv_{\overline{P}} w' \text{ and } w'' \equiv_{\overline{P}} w.$$
EMIL. Since EMDL is downwards-closed, and EMIL is not, it follows trivially that EMIL is strictly more expressive than EMDL. However, EMIL is strictly less expressive than MTL, and this even remains true when we also add classical disjunction to EMIL (we denote the resulting logic with EMIL\(^\oplus\)):

**Theorem 4.1.** There is a team property that is invariant under 0-bisimulation and which cannot be expressed in EMIL\(^\oplus\) and in EMIL.

**Proof.** Clearly it suffices to show the claim for EMIL\(^\oplus\). Let \(P\) be the property defined by \(\{(M,T) \mid M,T \models E_p\}\), i.e., the set of models \(M\) with teams \(T\) such that \(T\) contains at least one world satisfying the propositional variable \(p\). It is obvious (and also follows from Theorem 3.2) that \(P\) is invariant under 0-bisimulation.

Consider the model \(M\) with worlds \(w_1\) and \(w_2\), where \(p\) is true in \(w_1\) and false in \(w_2\), and the empty accessibility relation. Let \(T_1 = \{w_1, w_2\}\), and let \(T_2 = \{w_2\}\). Obviously, \((M,T_1) \in P\) and \((M,T_2) \not\in P\). We show that for every EMIL-formula \(\varphi\) with \(M,T_1 \models \varphi\), it also follows that \(M,T_2 \not\models \varphi\), which then in particular shows that there is no EMIL-formula expressing \(P\). Hence assume indirectly that \(M,T_1 \models \varphi\) and \(M,T_2 \not\models \varphi\). Without loss of generality, we can assume that \(\varphi\) does not contain any independence atoms: Trivially, \(M,T_2 \models \varphi_{P,Q,R}\) for any sets \(P, Q, R\) of formulas, since \(T_2\) is a singleton team. Since EMIL is monotone with respect to independence atoms (i.e., if we replace the independence atoms with variables \(x_1, \ldots, x_n\), then making a variable \(x_i\) true in some world never falsifies a formula that was true before), we can replace any occurrence of an independence atom in \(\varphi\) with \(\top\), and we still have that \(M,T_2 \not\models \varphi\) (as here, the independence atom is satisfied and therefore the replacement does not change whether \(\varphi\) is true) and \(M,T_1 \models \varphi\) (due to monotonicity). However, EMIL without independence atoms is simply ML (even in the presence of \(\oplus\), which in ML is equivalent to \(\lor\)), and thus downwards-closed. In particular, from \(M,T_1 \models \varphi\) it follows that \(M,T_2 \models \varphi\), which is a contradiction. \(\square\)

Since every property expressible in EMIL is invariant under bisimulation, it follows that MTL can express every EMIL-expressible property due to Theorem 3.2. Since Theorem 4.1 shows that there is a bisimulation-invariant (and hence, MTL-expressible) property that cannot be expressed in EMIL, we obtain the following:

**Corollary 4.2.** MTL is strictly more expressive than EMIL.

### 4.2 Extended Modal Inclusion Logic

Analogously to EMIL, we now define *Extended Modal Inclusion Logic*, EMINCL. EMINCL extends the syntax of ML with the following rule: If \(\varphi_1, \ldots, \varphi_n\) and \(\psi_1, \ldots, \psi_n\) are ML-formulas, then \((\varphi_1, \ldots, \varphi_n) \subseteq (\psi_1, \ldots, \psi_n)\) is an EMINCL-formula. The semantics of this inclusion atom are lifted from the first-order setting \([2]\) to the extended modal case:
Hence from the semantics of the inclusion atom, it is clear that
\( E \) to \((x, \ldots, \varphi_n) \subseteq (\psi_1, \ldots, \psi_n)\) if for every world \( w \in T \) there is a
world \( w' \in T \) such that \( \varphi_i(w) = \psi_i(w') \) for each \( i \in \{1, \ldots, n\} \).

**Theorem 4.3.** There is a 0-bisimulation invariant property that cannot be ex-
pressed with EMINCL. In particular, EMINCL is strictly less expressive than MTL.

**Proof.** Let \( P \) be the property

\[
\{(M, T) \mid \text{there is exactly one } i \in \{1, 2\} \text{ with } M, T \models E_p_i\}.
\]

Clearly (and also due to Theorem 3.2), \( P \) is invariant under 0-bisimulation.
Now, let \( M \) be a model with worlds \( w_1 \) and \( w_2 \) such that in \( w_i \), the variable \( p_i \)
is true and \( p_3-i \) is false. Let \( T_1 = \{w_1\} \), and \( T_2 = \{w_2\} \). Then, by construction,
\((M, T_1) \in P \) and \((M, T_2) \in P \), but \((M, T_1 \cup T_2) \notin P \). Now assume that \( \varphi \) is an
EMINCL-formula that expresses \( P \).

Then, in particular \( M, T_1 \models \varphi \), \( M, T_2 \models \varphi \) and \( M, (T_1 \cup T_2) \not\models \varphi \). However,
it easily follows that EMINCL is union-closed, i.e. if \( M, T_1 \models \varphi \) and \( M, T_2 \models \varphi \),
then also \( M, (T_1 \cup T_2) \models \varphi \) (see, e.g., [8], the property trivially transfers to the
modal setting). Therefore, we have a contradiction. \( \square \)

With \( \text{EMINCL}^\oplus \), we denote \( \text{EMINCL} \) extended with classical disjunction. It
turns out that \( \text{EMINCL}^\oplus \) is as powerful as MTL:

**Theorem 4.4.** Let \( P \) be a team property. Then the following are equivalent:

1. \( P \) is invariant under \( k \)-bisimulation
2. there is an \( \text{EMINCL}^\oplus \)-formula \( \varphi \) with \( \text{md}(\varphi) = k \) that characterizes \( P \).

In particular, \( \text{EMINCL}^\oplus \) is equally expressive as MTL and strictly more expressive
than EMINCL.

**Proof.** The direction from [2] to [11] follows by a straight-forward extension of the
proof in [12] that \( \text{ML} \) is invariant under bisimulation. For the converse, assume
that \( P \) is invariant under \( k \)-bisimulation. From the proof of Theorem 3.2 we
know that it suffices to construct an \( \text{EMINCL}^\oplus \)-formula \( \varphi \) that is equivalent to
the MTL-formula \( \bigodot_{(M, T) \in P} \varphi_{M,T}^\oplus \). Since the \( \bigodot \)-operator is available in \( \text{EMINCL}^\oplus \),
it suffices to show how to express the formula \( \varphi_{M,T}^\oplus \) for each model \( M \) and team
\( T \) as an \( \text{EMINCL}^\oplus \)-formula. Recall that

\[
\varphi_{M,T}^\oplus = \left( \bigwedge_{\varphi \in \Phi_{M,T}^\oplus} E \varphi \right) \land \left( \bigvee_{\varphi \in \Phi_{M,T}^\oplus} \varphi \right).
\]

The second conjunct already is an \( \text{EMINCL}^\oplus \)-formula, hence it suffices to show
how \( E \varphi \) can be expressed for an \( \text{ML} \)-formula \( \varphi \). As discussed earlier, \( M, T \models E \varphi \)
for an \( \text{ML} \)-formula \( \varphi \) if and only if there is a world \( w \in T \) with \( M, \{w\} \models \varphi \).
Hence from the semantics of the inclusion atom, it is clear that \( E \varphi \) is equivalent
to \((x \lor \neg \varphi) \subseteq (\varphi)\). This concludes the proof. \( \square \)
4.3 ML with arbitrary FO-definable dependence atoms

In this section we show that MTL, and the bisimulation invariant properties, can be captured as the extension of ML by all first-order definable generalized dependence atoms.

The notion of a generalized dependence atom in the modal context was introduced in [12]. A closely related notion was introduced and studied in the first-order context in [13]. The semantics of a generalized dependence atom $D$ is determined essentially by a property of teams. We do not present the full definitions here but refer to [12] for details.

In the following we are interested in generalized dependence atoms definable by first-order formulae, defined as follows: Suppose that $D$ is an atom of width $n$, that is, an atom that applies to $n$ propositional variables (for example the atom in (2)). We say that $D$ is FO-definable if there exists a FO-sentence $\phi$ over signature $\langle A_1, \ldots, A_n \rangle$ such that for all Kripke models $M = (W, R, \pi)$ and teams $T$, 

$$M, T \models D(p_1, \ldots, p_n) \iff A \models \phi,$$

where $A$ is the first-order structure with universe $T$ and relations $A_i$ for $1 \leq i \leq n$, where for all $w \in T$, $w \in A_i$ if and only if $p_i \in \pi(w)$.

In our “extended” setting the arguments to a generalized dependence atom $D(\varphi_1, \ldots, \varphi_n)$ can be arbitrary ML-formulas instead of propositional variables. It is easy to check that this generalization goes through without problems. In particular, in the definition of a FO-definable atom, the relation $A_i$ is now interpreted by the worlds of $T$ in which $\varphi_i$ is satisfied. We denote by $ML^{\text{FO}}$ the extension of ML by all FO-definable generalized dependence atoms $D$ with the extended interpretation, that is, $D$ can be applied to arbitrary ML-formulas.

**Theorem 4.5.** $ML^{\text{FO}}$ is equally expressive as MTL.

**Proof.** It easily follows from the proof of Theorem 6.8 in [12] that $ML^{\text{FO}}$ is invariant under bisimulation, hence $ML^{\text{FO}}$ is not more expressive than MTL. For the converse, let $P$ be a property that can be expressed in MTL. From Theorem 3.2 it follows that $P$ is invariant under $k$-bisimulation, and from the proof of Theorem 3.2 we know that it suffices to express the formula $\bigotimes_{(M,T) \in P} \varphi^k_{M,T}$ in $ML^{\text{FO}}$. We can do this with the following first-order-definable atom:

$$M, T \models D(\varphi_1, \ldots, \varphi_n, \varphi_1', \ldots, \varphi_n') \text{ if and only if there is some } k \in \{1, \ldots, n\} \text{ such that each } w \in T \text{ satisfies some } \varphi_i^k, \text{ and for each } j \in \{1, \ldots, n\}, \text{ there is some } w \in T \text{ that satisfies } \varphi_j^k.$$

$D$ can be FO-defined by replacing the exists/for all quantifiers on the indices with disjunctions/conjunctions:

$$\bigvee_{k \in \{1, \ldots, n\}} \left( \forall x (A_1^k(x) \lor \cdots \lor A_n^k(x)) \land \bigwedge_{j \in \{1, \ldots, n\}} (\exists x A_j^k(x)) \right)$$

Then, the atom $D$ applied to the formulas in $\varphi^k_{M,T}$ for all $(M, T) \in P$ gives a formula expressing $P$. 

\[
13
\]
5 Conclusion

We settled the question of the expressive power of modal logic with team semantics extended by various atoms defining team properties like dependence, inclusion, or independence. We proved that MTL, modal team logic, i.e., modal logic just extended by classical negation, gives the upper bound of all these logics:

- All these logics can be embedded into MTL, and MTL is equivalent to ML extended by arbitrary FO-definable generalized dependence atoms.
- MTL can express exactly those team properties that are invariant under bisimulation.

Overall, an interesting picture of the characterization of the expressiveness of modal logic in terms of bisimulations emerges: Let us say that “invariant under bounded bisimulation” means invariant under $k$-bisimulation for some finite $k$. Then we have the following hierarchy of logics:

- Due to van Benthem’s theorem [3], ML can exactly express all properties of pointed models that are invariant under bounded bisimulation.
- Due to [11], ML with team semantics and extended with classical disjunction $\otimes$ can exactly express all properties of teams that are invariant under bounded bisimulation and additionally downwards-closed.
- Our result shows that ML with team semantics and extended with classical negation $\sim$ can exactly express all properties of teams that are invariant under bounded bisimulation.

A number of open questions in the realm of modal logics with team semantics remain:

1. Note that, in the proof of Theorem 4.5, for each $k$, there is only a finite width of the $D$-operator above needed to express all properties that are invariant under $k$-bisimulation. However, the theorem leaves open the question whether there a “natural” atom $D$ or an atom with “restricted width” that gives the entire power of MTL—the results from Sections 4.1 and 4.2 and [6] show that the dependence atom, independence atom and inclusion atom do not suffice to give the expressive power of all of MTL unless (in the case of the inclusion atom) combined with classical disjunction.

2. Can we axiomatize MTL? Fan Yang has studied axiomatizability of the sublogic MID of MTL in her thesis [20].

3. While we mentioned a number of complexity results on modal dependence logic and some of its extensions, this issue remains unsettled for full MTL. In particular, what is the complexity of satisfiability and validity of MTL?

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