On a Family of Random Noble Means Substitutions

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Abstract

In 1989, Godrèche and Luck [1] introduced the concept of local mixtures of primitive substitution rules along the example of the well-known Fibonacci substitution and foreshadowed heuristic results on the topological entropy and the spectral type of the diffraction measure of associated point sets. In this contribution, we present a generalisation of this concept by regarding the so-called ‘noble means families’, each consisting of finitely many primitive substitution rules that individually all define the same two-sided discrete dynamical hull. We report about results in the randomised case on topological entropy, ergodicity of the two-sided discrete hull, and the spectral type of the diffraction measure of related point sets.
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1 The Setting

Consider the binary alphabet $A_2 = \{a, b\}$. For an arbitrary but fixed integer $m \geq 1$ and $0 \leq i \leq m$, we define the noble means substitution (NMS) rule $\zeta_{m,i}: A_2 \to A_2^*$ by

$$\zeta_{m,i}: \begin{cases} a \mapsto a^i ba^{m-i}, & \text{where } M_m := \begin{pmatrix} m & 1 \\ 1 & 0 \end{pmatrix} \\ b \mapsto a, \end{cases}$$

is its (unimodular) substitution matrix. The family

$$N_m := \{\zeta_{m,i} \mid m \in \mathbb{N}, 0 \leq i \leq m\}$$

is called a noble means family and each of its members is a primitive Pisot substitution with inflation multiplier $\lambda_m := (m + \sqrt{m^2 + 4})/2$ and algebraic conjugate $\lambda_m^* := (m - \sqrt{m^2 + 4})/2$. The two-sided discrete (symbolic) hull $X_{m,i}$ of $\zeta_{m,i}$ is defined as the orbit closure of a fixed point in the local topology. Each $X_{m,i}$ is reflection symmetric and aperiodic in the sense that it does not contain any periodic element. For fixed $m \in \mathbb{N}$, one observes that the $\zeta_{m,i}$ are pairwise conjugate and therefore all individual $X_{m,i}$ coincide; see [2] Ch. 4 for background.

Now, we fix $m \in \mathbb{N}$ and a (strictly positive) probability vector $p_m = (p_0, \ldots, p_m)$ and define a random substitution $\zeta_m$ on $A_2$ by

$$\zeta_m: \begin{cases} a \mapsto \zeta_{m,0}(a) \text{ with probability } p_0, \\ \vdots \\ b \mapsto \zeta_{m,m}(a) \text{ with probability } p_m, \end{cases}$$

We refer to $\zeta_m$ as a random noble means substitution (RNMS). Both the substitution matrix and the inflation multiplier are the same as in the NMS case. We aim at the local mixture of all members of $N_m$, which means that we independently apply $\zeta_m$ to each letter of some word $w \in A_2^\ast$. In this case, the two-sided discrete stochastic hull $X_m$ is defined as the smallest closed and shift-invariant subset of $A_2^\ast$ with the property that $X_m \subset X_m$, where

$$X_m := \{w \in A_2^\ast \mid w \text{ is an accumulation point of } (\zeta_m^k(a)a)_{k \in \mathbb{N}}\}.$$ 

Both $X_{m,i}$ and $X_m$ are completely characterised by the legal subwords. Here, a word $w \in A_2^\ast$ is $\zeta_m$-legal if there is a $k \in \mathbb{N}$ such that $w$ is a subword of at least one realisation of the random variable $\zeta_m^k(b)$. The set of $\zeta_m$-legal words of length $\ell$ is henceforth denoted by $D_{m,\ell}$. One can show that $X_{m,1} \subseteq X_m$ by considering the subword $bb$ and that the system $(X_m, S)$, where $S$ denotes the shift, is topologically transitive but not minimal. Note that $X_m$ is invariant under alterations of $p_m$ as long as $p_m$ is strictly positive.

2 Topological entropy

For $m \in \mathbb{N}$ and $n \geq 3$, the set of exact RNMS words is given by

$$G_{m,n} := \bigcup_{i=0}^{m} \prod_{j=0}^{m} G_{m,n-1-\delta_{ij}},$$

where $G_{m,1} := \{b\}$, $G_{m,2} := \{a\}$ and $\delta_{ij}$ denotes the Kroenecker function. The product in Eq. (1) is understood via concatenation of words. Now, assume that $p_m$ is strictly positive. The complexity function

$$C_m : \mathbb{N} \to \mathbb{N}, \quad \ell \mapsto |D_{m,\ell}|$$
Theorem 1 \((2)\)

word \(m\) cylinder sets \(Z\) shift-invariant probability measure.

We define a \((\mathbb{R}^N)\) case. One obtains a random substitution rule

\[ \begin{align*}
\lambda_m &= \lim_{n \to \infty} \log(C_m(\ell_{m,n})) = \lim_{n \to \infty} \log(|G_{m,n}|) \\
&= \lambda_m \frac{1}{\ell_{m,n}} \sum_{i=1}^{\infty} \log(m(i-1)+1) \lambda_i^m > 0,
\end{align*} \]

where \(\ell_{m,n}\) is the length of any word \(w \in G_{m,n}\). The numerical values of \(\mathcal{H}_m\) for \(1 \leq m \leq 4\) are shown in Table 1.

Table 1: Numerical values of \(\mathcal{H}_m\) for \(1 \leq m \leq 4\).

| \(m\) | \(\mathcal{H}_m\) |
|------|---------|
| 1    | 0.44439 |
| 2    | 0.40855 |
| 3    | 0.37139 |
| 4    | 0.33862 |

One can prove that \(\mathcal{H}_m > \mathcal{H}_{m+1}\) for all \(m \in \mathbb{N}\) and \(\mathcal{H}_m \to_{m \to \infty} 0\).

3 Ergodicity of \((\mathbb{X}_m, S)\)

The known concept of the induced substitution [5] Ch. 5] that acts on the alphabet of legal subwords of a fixed length can be generalised to the stochastic setting of the RNMS case. One obtains a random substitution rule \((\zeta_m)_{\tau}: \mathcal{D}_{m,n} \to \mathcal{P}_{m,n}\) and one can prove that the induced substitution matrix \(M_{m,n}\) is a primitive matrix which enables the application of Perron–Frobenius (PF) theory.

For fixed \(m, n \in \mathbb{N}\), let \(w \in \mathcal{D}_{m,n}\) be a \(\zeta_m\)-legal word. We define a shift-invariant probability measure \(\mu_m\) on the cylinder sets \(Z_k(w) = \{v \in \mathbb{X}_m | v_{[k,k+\ell-1]} = w\}\) for any \(k \in \mathbb{Z}\) by

\[ \mu_m(Z_k(w)) = R_{m,n}(w), \] \(2\)

where \(R_{m,n}(w)\) is the entry of the (statistically normalised) right PF eigenvector of \(M_{m,n}\) according to the word \(w\).

Theorem 1 \([3, 6]\). Let \(\mathbb{X}_m \subset A_{\mathcal{F}}^\mathcal{F}\) be the hull of the random noble means substitution for \(m \in \mathbb{N}\) and \(\mu_m\) the shift-invariant probability measure of Eq. (2) on \(\mathbb{X}_m\). For any \(f \in L^1(\mathbb{X}_m, \mu_m)\) and for an arbitrary but fixed \(s \in \mathbb{Z}\),

\[ \lim_{N \to \infty} \frac{1}{N} \sum_{i=s}^{N-1} f(S^i x) = \int_{\mathbb{X}_m} f \, d\mu_m \]

holds for \(\mu_m\)-almost every \(x \in \mathbb{X}_m\).

Theorem 4 \([4]\) implies that \(\mu_m\) is ergodic. The proof can be accomplished via an application of Etemadi’s formulation of the strong law of large numbers \([7]\) and a suitable reorganisation of the summation over the characteristic function of some cylinder set.

4 Cut and project

The geometric realisations \(\Lambda_{m,i}\) of fixed points of each noble means substitution can be derived as regular model sets within the cut and project scheme \((\mathbb{R}, \mathbb{R}, \mathcal{L}_m)\), where \(\mathcal{L}_m := \{(x, x') | x \in Z[\lambda_m]\}\); see Figure 4. Here, the letters \(a \) and \(b \) are identified with closed intervals of length \(\lambda_m\) and \(1\), respectively, and the left endpoints are chosen as control points. The windows \(W_{m,i}\) for \(\Lambda_{m,i}\), in the generic cases \(0 < i < m\), are

\[ W_{m,i} := i \tau_m + [\lambda_m', 1] \text{ with } \tau_m := -\frac{1}{m}(\lambda_m' + 1), \]

while in the singular cases \(i = 0\) and \(i = m\), one finds

\[ W_{m,0}^{(a)} := [\lambda_m', 1], \quad W_{m,0}^{(b)} := [\lambda_m', 1], \]

\[ W_{m,m}^{(a)} := [-1, -\lambda_m'], \quad W_{m,m}^{(b)} := [-1, -\lambda_m'], \]

distinguished according to the legal two-letter seeds. Now, one can prove that each member of the continuous RNMS hull \(\mathbb{Y}_m\) is a relatively dense subset of an element of the LI class of the model set \(\Theta(W_m)\), within the cut and project scheme of Figure 4 with window \(W_m = [\lambda_m', 1, 1 - \lambda_m']\) and therefore a Meyer set by [8 Thm. 9.1]. The volume of the interval \(W_m\) is minimal with this property, and it strictly contains \(\bigcup_{i=0}^{m} W_{m,i}\); see Figure 2 for an illustration in the case of \(m = 2\).

Consequently, each geometric realisation of a random noble means word is a naturally arising instance of a Meyer set with entropy.

5 Diffraction

The diffraction of the NMS cases is well understood due to their characterisation as regular model sets [2 Ch. 9].
is given by the Fourier transform of $\gamma$ averaged convolution by balls. The diffraction measure $R$ where $\Lambda$ finally lead to an explicit expression for the measure of a continuous part in the diffraction spectrum in the RNMS case.

Because of Theorem 1, the suspension $\Gamma$ means set $\Lambda \in \mathbb{Y}_m$. Now, let $\delta_\Lambda := \sum_{x \in \Lambda} \delta_x$ be the Dirac comb for a random noble means set $\Lambda \in \mathbb{Y}_m$. One can compute the autocorrelation of $\delta_\Lambda$ to be $\nu_m$-almost surely given by

$$\gamma := \lim_{R \to \infty} \frac{\delta_\Lambda * \delta_{\Lambda}}{\text{vol}(B_R)} = \mathbb{E}(\delta_\Lambda \circ \delta_\Lambda),$$

where $\Lambda_R := \Lambda \cap B_R(0)$ and $\circ$ denotes the volume-averaged convolution by balls. The diffraction measure is given by the Fourier transform of $\gamma$ and reads

$$\hat{\gamma} = \lim_{R \to \infty} \frac{1}{\text{vol}(B_R)}|\mathbb{E}(\delta_{\Lambda_R})|^2 + \lim_{R \to \infty} \frac{1}{\text{vol}(B_R)} \mathbb{V}(\delta_{\Lambda_R}),$$

where $\mathbb{E}$ and $\mathbb{V}$ refer to mean and variance with respect to the measure $\nu_m$. Now, the following two key properties finally lead to an explicit expression for $\hat{\gamma}$.

- It is enough to study $\hat{\gamma}$ on the basis of exact RNMS words, as defined in Eq. (1), because $\zeta_m$-legality of a word $w \in A_2^*$ means that $w$ is a subword of a word in $\mathcal{G}_{m,n}$ for a suitably chosen $n \in \mathbb{N}$.

- It is not difficult to prove that

$$\mathcal{G}_{m,n} = \{w \in A_2^* \mid w = \zeta_m^{n-1}(b)\}$$

and even more that the two stochastic processes, based on the substitution rule and the concatenation rule, are equal. Note that the equality in Eq. (3) means that there is at least one realisation of the random variable $\zeta_m^{n-1}(b)$ that equals $w$.

For convenience, we restrict to $m = 1$ in the following and define for $n \geq 2$ the complex-valued random variable $X_n(k)$ by

$$X_n(k) := \begin{cases} X_{n-2}(k) + e^{-2\pi ik\lambda_1} X_{n-1}(k), & \langle p_0 \rangle, \\ X_{n-1}(k) + e^{-2\pi ik\lambda_1} X_{n-2}(k), & \langle p_1 \rangle, \end{cases}$$

with $X_0(k) = e^{-2\pi ik}$ and $X_1(k) = e^{-2\pi ik\lambda_1}$. Here, $X_n(k)$ corresponds to exact RNMS words in $\mathcal{G}_{1,n+1}$. Therefore, we consider averaging over the sequence $L_n = \lambda_1^n$ and find the following result.

**Proposition 2** ([3]). For any $n \in \mathbb{N}$, consider the function $\phi_n: \mathbb{R} \to \mathbb{R}_+$, defined by

$$\phi_n(k) := \frac{1}{L_n} \mathbb{E}(X_n(k)).$$

The sequence $\langle \phi_n \rangle_{n \in \mathbb{N}}$ converges uniformly to the continuous function $\phi: \mathbb{R} \to \mathbb{R}_+$, with

$$\phi(k) := \frac{2p_0 p_1 \lambda_1}{\sqrt{5}} \sum_{i=2}^{\infty} \lambda_1^{-i} \Psi_i(k). \quad (4)$$

Here, $\Psi_n: \mathbb{R} \to \mathbb{R}_+$ is a bounded and smooth function that monotonically decreases in $n$, defined by

$$\Psi_n(k) := \frac{1}{2} \left( (1 - e_n^{-2}) E_{n-1} - (1 - e_n^{-1}) E_{n-2} \right)^2,$$

where $e_n := e^{-2\pi i \lambda_1}$ and $E_n := \mathbb{E}(X_n(k))$. This fixes the absolutely continuous part of $\hat{\gamma}$. The pure point part can be computed via the recursion relation

$$E_n = (p_1 + p_0 e_n^{-2}) E_{n-1} + (p_0 + p_1 e_n^{-1}) E_{n-2}, \quad (5)$$

where $E_0 := e^{-2\pi i \lambda_1}$ and $E_1 := e^{-2\pi i \lambda_1}$. This yields

$$\hat{\gamma}([k]) = \lim_{n \to \infty} \frac{1}{L_n^2} \mathbb{E}(X_n(k))^2$$

and an approximation of $\hat{\gamma}_{pp}$ and $\hat{\gamma}_{ac}$ is illustrated in Figure 3 together with a sketch of the full diffraction, based on the recursion of Eq. (5) with $n = 6$, in Figure 4. Considering the Lebesgue decomposition $\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc}$, we find that

$$\hat{\gamma} = \hat{\gamma}_{pp} + \phi(k) \lambda,$$

where $\lambda$ denotes the Lebesgue measure and $\phi$ the density function of Eq. (4).
As \( \xi \) where the matrices \( A \) work is supported by the German Research Foundation
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Figure 5. Distribution of control points generated by \( a \) (dark) and \( b \) (light) in the internal space in the case of \( p_1 = (1/2, 1/2) \). The plot is generated by a lift of \( \xi^{12}(b) \) (i.e. 2178909 points) to the internal space.

It is possible to compute the pure point part from the recursion relation in Eq. (5). Another interesting approach comes from the theory of iterated function systems and inflation-invariant measures. Here, one finds that \( \sum_{k \in \mathcal{L}_1} |\hat{\eta}_a(k) - \hat{\eta}_b(k)|^2 \delta_k \),
where \( \mathcal{L}_1 \) is the Fourier module. In the following, we write \( \xi = \lambda_1 \). The invariant measures \( \hat{\eta}_a, \hat{\eta}_b \) can be approximated via the recursion relation
\[
\left( \begin{array}{c} \hat{\eta}_a(k) \\ \hat{\eta}_b(k) \end{array} \right) = \left[ e^{-2\pi i k \xi} \right] \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \left( \begin{array}{c} \hat{\eta}_a(0) \\ \hat{\eta}_b(0) \end{array} \right),
\]
where the matrices \( A(k) \) and \( B(k) \) are given by
\[
\begin{pmatrix} e^{-2\pi i k \xi} & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ e^{-2\pi i k \xi} & 1 \end{pmatrix}.
\]
As \( \xi \to 0 \) for \( n \to \infty \), an appropriate choice of the eigenvector \( (\hat{\eta}_a(0), \hat{\eta}_b(0))^T \) for the equation
\[
\left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right) \begin{pmatrix} \hat{\eta}_a(0) \\ \hat{\eta}_b(0) \end{pmatrix} = \lambda_1 \begin{pmatrix} \hat{\eta}_a(0) \\ \hat{\eta}_b(0) \end{pmatrix}
\]
fixed the recursion. Since \( \hat{\eta}_a(0) + \hat{\eta}_b(0) \) must be the point density of some random golden means set, which always is \( \lambda_1/\sqrt{5} \), one finds \( \hat{\eta}_a(0) = 1/\sqrt{5} \) and \( \hat{\eta}_b(0) = (\lambda_1 - 1)/\sqrt{5} \). The distribution of control points in the internal space distinguished for \( a \) and \( b \), respectively, is illustrated in Figure 5.

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