EQUIANGULAR LINES IN EUCLIDEAN SPACES

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Abstract. We obtain several new results contributing to the theory of real equiangular line systems. Among other things, we present a new general lower bound on the maximum number of equiangular lines in $d$ dimensional Euclidean space; we describe the two-graphs on 12 vertices; and we investigate Seidel matrices with exactly three distinct eigenvalues. As a result, we improve on two long-standing upper bounds regarding the maximum number of equiangular lines in dimensions $d = 14$, and $d = 16$. Additionally, we prove the nonexistence of certain regular graphs with four eigenvalues, and correct some tables from the literature.

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1. Introduction

Let $d \geq 1$ be an integer and let $\mathbb{R}^d$ denote the Euclidean $d$-dimensional space equipped with the usual inner product $\langle \cdot, \cdot \rangle$. A set of $n \geq 1$ lines, represented by the unit vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$, is called equiangular if there exists a constant $\alpha \geq 0$, such that $\langle v_i, v_j \rangle = \pm \alpha$ for all $1 \leq i < j \leq n$. This constant is referred to as the common angle between the lines. If $\alpha = 0$ then the line system is just a subset of an orthonormal basis; if $n \leq d$ then it is easy to construct equiangular line systems for all $0 \leq \alpha \leq 1$. Therefore one excludes these trivial cases by assuming that $n > d$ and consequently $\alpha > 0$.

Equiangular lines were introduced first by Haantjes [16] in 1948 and then investigated in detail by Van Lint and Seidel in a seminal paper [23] and were subject of active research during the 1970s [5], [7], [32], [42]. Recently there has been a renewed interest in some special, highly structured complex equiangular line systems [14], called tight frames, arising from applications in signal processing [12], [19] and quantum tomography [11], [34].

The fundamental problem in this area is the determination of the maximum number of equiangular lines $N(d)$ in $\mathbb{R}^d$. In the following table we display lower and upper bounds on $N(d)$ for the first few values of $d$ (cf. [22]).

| $d$ | 2  | 3  | 4  | 5  | 6  | 7-13 | 14  | 15  | 16  | 17  | 18  | 19  | 20  | 21  | 22  | 23-41 |
|-----|----|----|----|----|----|------|----|----|----|----|----|----|----|----|------|
| $N(d)$ | 3  | 6  | 6  | 10 | 16 | 28   | 28-29 | 36  | 40-41 | 48-50 | 48-61 | 72-76 | 90-96 | 126 | 176 | 276  |
| $1/\alpha$ | 2 $\sqrt{5}$ | 3 $\sqrt{5}$ | 3 | 3 | 3 | 3,5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |

Table 1. The maximum number of equiangular lines for $d \leq 41$. 

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We remark that there exist several incorrectly revised tables in the current literature (e.g. the one in [35, Section 3.3]) which might suggest to the uninitiated that \( N(d) \) is known for small \( d \). This is, however, not the case as \( d = 14 \) is still undecided. Table 1 shows that, despite a considerable amount of research in the past 40 years, determining \( N(d) \) even for relatively small values of \( d \) is still out of reach. The methods used to obtain configurations with the above indicated number of lines are discussed in detail throughout the scattered literature [22], [35], [41], [42] and [43], while the upper bounds will be mentioned later. The reader might wish to jump ahead to Section 5.3 where the new results regarding dimensions \( d = 14 \) and \( d = 16 \) are presented.

**Remark 1.1.** Seidel seems to claim in [35, Section 3.3] that the lower bounds indicated above cannot be improved unless \( d = 19 \) or 20. This might be true, but it is unclear whether or not his statement follows implicitly from the cited literature.

The Gram matrix of the equiangular line system \([G]_{i,j} := \langle v_i, v_j \rangle, 1 \leq i, j \leq n\), is of fundamental interest. It contains all the relevant parameters of the line system and thus study via matrix theoretical and linear algebraic tools is possible. It is, however, more convenient to consider the Seidel matrix \( S := (G - I) / \alpha \) instead, which is a symmetric matrix with zero diagonal and \( \pm 1 \) entries otherwise. The multiplicity of the smallest eigenvalue \( \lambda_0 \) of \( S \) describes the smallest possible dimension \( d \) where the line system fits in with common angle \( \alpha = -1 / \lambda_0 \). Seidel matrices are the central objects of this work.

The outline of this paper is as follows. In Section 2 we provide an overview of the subject and present both classical and recent results on fundamental properties of equiangular lines. The main result of this part is an improved quadratic lower bound on the maximum number of equiangular lines (cf. [9]). In Section 3 we prove new modular identities involving the determinant and the permanent of Seidel matrices. In Section 4 we report on our computational results: we present a full classification of Seidel matrices up to order 12 (cf. [5], [28], [37]). In Section 5 we investigate Seidel matrices with three distinct eigenvalues in detail. In Section 6 we discuss equiangular line systems with common angle 1/5. During the course of the paper we extend (and occasionally correct) several tables from the literature.

## 2. Large sets of equiangular lines

In this section we recall some fundamental structural results on equiangular lines, and discuss several lower and upper bounds on \( N(d) \). The main contribution here is Theorem 2.7 which improves upon an earlier construction of de Caen [9] and describes a new general lower bound for \( N(d) \). The first result is the absolute bound.

**Theorem 2.1 (Gerzon [22]).** The maximum number of equiangular lines in \( \mathbb{R}^d \) is bounded by

\[
(1) \quad \frac{d(d + 1)}{2} \leq N(d),
\]

and if equality holds then \( d = 2, 3, \) or \( d + 2 \) is the square of an odd integer.

Equality indeed holds for \( d = 2, 3, 7, \) and 23. For long it was conjectured that the upper bound \( (1) \) is attained whenever \( d + 2 \) is an odd square, until Makhnev, in a surprising breakthrough paper [25], proved that \( N(47) < 24 \cdot 47 \). Additional counterexamples were found subsequently by a different method [2]; and recently,
by employing semidefinite programming, the upper bound \( N(d) \) for \( N(d) \) was considerably improved for various dimensions \( d \leq 136 \) \[3\]. It is worthwhile to note that these are three fundamentally different approaches.

It turns out that the common angle \( \alpha \) is heavily restricted in case \( n > 2d \).

**Theorem 2.2** (Neumann \[22\]). Assume that there exists \( n > 2d \) equiangular lines in \( \mathbb{R}^d \) with common angle \( \alpha \). Then \( 1/\alpha \) is an odd integer.

Theorem 2.2 is particularly useful for deciding the maximum number of equiangular lines in \( \mathbb{R}^d \) for small \( d \). The result above can be combined with the following improved upper bound, called the relative bound, which takes into consideration the common angle under some assumptions on the ambient dimension \( d \).

**Proposition 2.3** (see \[23\]). Assume that there exist \( n \) equiangular lines in \( \mathbb{R}^d \) with common angle \( \alpha \leq 1/\sqrt{d} + 2 \). Then

\[
n \leq \frac{d(1-\alpha^2)}{1-d\alpha^2},
\]

and equality holds if and only if the corresponding Seidel matrix has exactly two distinct eigenvalues.

**Remark 2.4.** The relative bound, described by Proposition 2.3, does not exceed the absolute bound, given by Theorem 2.1.

We provide a slight generalization of Proposition 2.3 in Section 5 (see Theorem 5.6), and investigate the case of “almost” equality in Theorem 5.24.

Now we turn to the discussion of lower bounds. It is easy to show that \( N(d) \geq cd\sqrt{d} \) for some \( c > 0 \) (see \[22\]), but for many years the asymptotic behavior of \( N(d) \) was unknown (see \[35\], p. 884}). This was determined a while ago (see \[39\], p. 80]) based on a construction by de Caen who used a family of three-class association schemes and exhibited the first quadratic lower bound on real equiangular lines in specific dimensions \[9\]. We analyze de Caen’s construction and improve upon the number of equiangular lines arising from it by a factor of 2. First we need the following technical result.

**Proposition 2.5.** For each \( i \geq 3 \) and \( m = 4^i \) there exists an equiangular set of

(a) \( n = m(m/2+1) \) lines in dimension \( d = m + m/2 + 1 \); and
(b) \( n = m(m/2+1)-1 \) lines in dimension \( d = m + m/2 \); and
(c) \( n = mj \) lines in dimension \( d = m + j - 1 \) for every \( m/32 + 1 \leq j \leq m/2 \)

with common angle \( \alpha = 1/(2^i + 1) \).

The proof is based on the existence of a complete set of \( m/2+1 \) pairwise mutually unbiased bases in \( \mathbb{R}^m \), whenever \( m = 4^i \) for some \( i \geq 1 \). For a gentle introduction see \[27\]; for more details see Kantor’s work (but be aware of a typo in the relevant example in \[20\], p. 157]). We remark here that the case \( j = m/2 \) in part (c) is exactly de Caen’s construction \[9\].

**Proof.** Let \( i \geq 3 \) and consider a complete set of real mutually unbiased bases in \( \mathbb{R}^m \), where \( m = 4^i \), i.e. \( m/2 + 1 \) orthonormal bases with the additional property that \( \langle e, f \rangle = \pm 1/\sqrt{m} \) whenever \( e \) and \( f \) are column vectors from distinct bases. Embed these vectors into \( \mathbb{R}^{m+m/2+1} \) in the natural way by adjoining \( m/2 + 1 \) zero coordinates to all of them. Let \( a = 1/\sqrt{m+1} \) and \( b = a/\sqrt{m} \). If a vector \( v \in \mathbb{R}^{m+m/2+1} \) is a member of the \( j \)th orthonormal basis \( (j = 1, \ldots, m/2 + 1)\),
then consider the scaled vector $bv$ and replace its $(m + j)$th coordinate with $a$. These new vectors form the desired configuration of equiangular lines described in part (a). To see part (b), observe that without loss of generality one of the $m + 1$ orthonormal bases can be the standard basis of $\mathbb{R}^m$. By the unbiasedness condition and a trivial column-normalization it follows that the vectors coming from all the $m$ other basis must have the same first coordinate. Therefore, discarding the column vector from the standard basis whose first coordinate is 1 leads, after repeating the same construction described in part (a), to a linearly dependent set of (row) vectors forming an equiangular line system in $\mathbb{R}^{m + m/2}$. Part (c) follows immediately by using only $j$ ($m/32 + 1 \leq j \leq m/2$) mutually unbiased bases in the proof of part (a).

Remark 2.6. To maximize the number of equiangular lines in a given dimension $d$ the choice of the angle $\alpha$ is crucial. Indeed, in $\mathbb{R}^{95}$ we have $n \leq 438$ for $\alpha \leq 1/11$ by Proposition 2.3. However, for $\alpha = 1/9$, for which the previously mentioned bound does not apply, we can construct 2048 lines via part (c) of Proposition 2.5. On the other hand, for $\alpha = 1/3$ the maximum number of equiangular lines is just 188 (see [22]), and hence using angles that are too small or too large are both unsatisfactory in general.

The following is the main result of this section.

**Theorem 2.7.** Let $d \geq 25$ and let $m = 4^i$, $i \geq 2$ be the unique integer number for which $3m/2 + 1 \leq d \leq 6m$. Then

\[
N(d) \geq \begin{cases} 
  m(m/2 + 1) & 3m/2 + 1 \leq d \leq 33m/8 - 1; \\
  4m(d - 4m + 1) & 33m/8 \leq d \leq 6m - 1; \\
  4m(2m + 1) - 1 & d = 6m.
\end{cases}
\]

**Proof.** The theorem is an immediate consequence of Proposition 2.5. The first case follows from part (a) for $i \geq 3$, and since there exists a complete set of mutually unbiased bases in $\mathbb{R}^{16}$, it follows for $i = 2$ as well. The second case follows from part (c) by replacing $m$ with $4m$. Indeed, by using $j$ mutually unbiased bases of size $4m (m/8 + 1 \leq j \leq 2m)$ we have $4mj \geq 4m(m/8 + 1) > m(m/2 + 1)$ lines in dimension $d = 4m + j - 1 \geq 33m/8$. The third case follows from part (b) by replacing $m$ with $4m$.

We note the following general lower bound.

**Corollary 2.8 (cf. [9] and [39, p. 80]).** For every $d \geq 2$ we have

\[
N(d) \geq \left\lceil \frac{1}{1089} \left( 32d^2 + 328d + 296 \right) \right\rceil.
\]

This is an improvement upon an implicit result of de Caen by a factor of 2, see [39, p. 79]. It was obtained in two steps: first, by observing that de Caen used $m/2$ mutually unbiased bases only; second, by removing a suitable set of lines in order to reduce the dimension.

**Proof.** Let $f(d) := (32d^2 + 328d + 296)/1089$. For $2 \leq d \leq 22$ the statement is trivial since $f(d) < d$. For $23 \leq d \leq 24$ we have $f(d) < 276$ and hence the 276 lines in $\mathbb{R}^{24}$ (see Table 1) satisfy the claim. Therefore we can assume that $d \geq 25$ and so invoke Theorem 2.7. We readily see that $f$ is the quadratic curve fit to the set
of points \( \{(33m/8 - 1, m(m/2 + 1)) : m = 4^i, i \geq 2\} \subset \mathbb{R} \times \mathbb{R} \), and by noting that 
\( N(33m/8) \geq 4m(m/8 + 1) > f(33m/8) \) for \( m = 4^i, i \geq 2 \), the convexity of \( f \) yields the desired result. \( \square \)

It would be interesting to see whether the constant \( 32/1089 \) can be further improved and pushed closer to \( 1/2 \) (cf. Theorem 2.1). This, however, might require some further new ideas.

3. Some structural results on Seidel matrices

The switching class of a Seidel matrix \( S \) is the set of all Seidel matrices \( PDSDP^T \) where \( P \) is a permutation matrix and \( D \) is a \( \pm 1 \) diagonal matrix. Two Seidel matrices, \( S_1 \) and \( S_2 \), are called switching equivalent, if \( S_2 = PDS_1DP^T \) holds for some \( P \) and \( D \). This equivalence captures the symmetries of the equiangular line systems. Indeed, the described operations correspond to relabeling the spanning unit vectors and replacing some of them with their negatives. It is customary to associate an ambient graph \( \Gamma \) to a Seidel matrix \( S \), defined via its adjacency matrix \( A := (J - S - I)/2 \), where \( J \) is the matrix with all entries 1, and think of \( S \) as a signed complete graph. In this terminology, by switching with respect to a vertex \( v \) in \( \Gamma \), we obtain the graph \( \Gamma' \) in which the neighbors of \( v \) in \( \Gamma' \) are its nonneighbors in \( \Gamma \) and vice versa.

In this section we state some new structural results on Seidel matrices. To motivate our efforts we begin this part with the following trivial warm-up result.

**Lemma 3.1.** Let \( \Gamma \) be a connected \( k \)-regular graph on \( n \) vertices with \( e \) edges and let \( A \) be its adjacency matrix. Then \( \det (J - 2A - I) \equiv (-1)^n(1 - n) + 4en \) (mod 8).

In particular, if the switching class of a Seidel matrix contains a connected regular graph then we know its determinant modulo 8.

**Proof.** The proof follows from a standard spectral analysis. Let \( k \) and \( \lambda_2, \lambda_3, \ldots, \lambda_n \) be the eigenvalues of \( A \). Note that \( \text{tr}A = 0 \) and \( \text{tr}A^2 = nk = 2e \). The eigenvalues of \( S := J - 2A - I \) are \( n - 2k - 1 \) and \(-2\lambda_i - 1 \) for \( 2 \leq i \leq n \). We find

\[
\det S = (n - 2k - 1) \prod_{i=2}^{n} (-2\lambda_i - 1)
\equiv (-1)^n (2k + 1 - n) \left( 1 + 2 \sum_{i=2}^{n} \lambda_i + 4 \sum_{2 \leq i < j \leq n} \lambda_i \lambda_j \right)
\equiv (-1)^n (2k + 1 - n)(1 - 2k + 4k^2 - 4e) \equiv (-1)^n(1 - n) + 4en \pmod{8}. \square
\]

We generalize Lemma 3.1 and prove it for arbitrary graphs (cf. Theorem 3.4), along with a related result on permanents. Let us recall that a derangement is a permutation without any fixed points. It is clear from the definition that the number of derangements on \( n \) elements, \( \delta(n) \), is exactly \( \text{per}(J - I) \). By expanding this permanent along the first row we find that \( \delta(n) = (n-1)\delta(n-1) + \delta(n-2) \) for \( n \geq 3 \). Combining this with the initial values \( \delta(1) = 0 \) and \( \delta(2) = 1 \) and using induction on \( n \) we obtain for \( n \geq 2 \) that

\[
\delta(n) = n\delta(n-1) + (-1)^n.
\]

\[ (2) \]
Remark 3.2. It is possible to determine $\delta(n) \pmod{8}$ based on equation (2). An easy inductive argument shows that $\delta(n) \equiv 1 \pmod{8}$ for $n$ even, and $\delta(n) \equiv n - 1 \pmod{8}$ for $n$ odd. In particular $\delta(n) \equiv n - 1 \pmod{2}$.

Lemma 3.3. Let $[S]_{i,j} = s_{i,j}$ be a Seidel matrix of order $n \geq 2$. Let $S'$ be obtained from $S$ by changing $s_{12} = s_{21}$ to their negative. Then, we have $\det S' \equiv \det S + 4n \pmod{8}$.

We write $\mathfrak{S}_n$ to denote the symmetric group on $n$ elements.

Proof. For $n = 2$ the result is immediate, therefore we can assume that $n \geq 3$. Let us define $R_n := \{\sigma \in \mathfrak{S}_n : \sigma(1) = 2, \sigma(2) \neq 1, \sigma(i) \neq i \text{ for } 3 \leq i \leq n\}$ and note that its cardinality is $\delta(n - 1)$. Since the inversion number obeys $I(\sigma) = I(\sigma^{-1})$ we readily find that

$$\det S - \det S' = 4 \sum_{\sigma \in R_n} (-1)^{I(\sigma)} \prod_{i=1}^{n} s_{i,\sigma(i)},$$

and hence, since we are working modulo 8, it is enough to determine the parity of the above sum, or, since all terms are $\pm 1$, the parity of $\delta(n - 1)$. By Remark 3.2 this is exactly the same as the parity of $n$. $\square$

The following is the main contribution of this section.

Theorem 3.4 (cf. [17, p. 659]). Let $\Gamma$ be a graph on $n \geq 2$ vertices with $e$ edges and let $A$ be its adjacency matrix. Then $\det(J - 2A - I) \equiv (-1)^n(1 - n) + 4en \pmod{8}$.

Proof. Let $S = J - I$ and observe that $S' = J - 2A - I$ can be obtained from $S$ by changing the sign of $2e$ off-diagonal entries. Since $\det S = (-1)^{n-1}(n-1)$, repeated application of Lemma 3.3 yields the desired result. $\square$

We will use Theorem 3.4 in the following qualitative form.

Corollary 3.5. Let $S$ be a Seidel matrix of order $n$. Then $\det S \equiv 1 - n \pmod{4}.$

Proof. If $n = 1$ then $\det S = 0$. Otherwise the statement follows from Theorem 3.4 $\square$

We point out the following analogous property.

Theorem 3.6. Let $\Gamma$ be a graph on $n \geq 2$ vertices with $e$ edges and let $A$ be its adjacency matrix. Then $\per(J - 2A - I) \equiv (-1)^n + n(1 - (-1)^n)/2 + 4en \pmod{8}$.

Proof. Follows along the same lines as the proof of Lemma 3.3 and Theorem 3.4 mutatis mutandis. Since $\per(J - I) = \delta(n)$, we find that $\per(J - 2A - I) \equiv \delta(n) + 4en \pmod{8}$. The result follows from Remark 3.2 $\square$

As an application, we prove the following result. Recall that a Seidel matrix $S$ is self-complementary, if $S$ and $-S$ are switching equivalent.

Proposition 3.7. Let $S$ be a Seidel matrix of order $n \equiv 3 \pmod{4}$. Then $\det(-S) \equiv \det S + 4 \pmod{8}$. In particular, $S$ cannot be self-complementary.

Proof. From Theorem 3.4 it follows that $\det S \equiv \pm 2 \pmod{8}$ and hence $\det(-S) = -\det S \equiv \mp 2 \equiv \pm 2 + 4 \equiv \det S + 4 \pmod{8}$. $\square$

We do not know any applications of Theorem 3.4 but we will use Corollary 3.5 several times during the following sections. Understanding the determinant modulo further values might lead to some non-existence results on Seidel matrices with prescribed spectrum.
In this section we report on some computational results on Seidel matrices. Prior to this work one representative from each of the switching classes was available up to order $n \leq 11$ [5], [28], [37]. The aim of this section is to present a complete classification of Seidel matrices up to order $n \leq 12$. We achieve this result via two ways: first, we present a list of Seidel matrices of order $n = 12$ thus complementing the earlier results; second, and perhaps more importantly, we provide an efficient way to determine the equivalence of Seidel matrices of these orders via various invariants.

Interestingly, the number of switching classes are known explicitly, due to the following spectacular equicardinality result. Recall that an Euler graph is a (not necessarily connected) graph, all of whose vertices are of even degree.

**Theorem 4.1** (Mallows–Sloane [26]). The number of switching classes of Seidel matrices of order $n$ and the number of Euler graphs on $n$ vertices are equal in number.

Since the number of Euler graphs are known due to explicit formulae [24], [33], discussed in detail in [7], [26], once enough inequivalent switching classes are identified, they constitute a full classification. It is worthwhile to note that while, for $n$ odd, each switching class contains a unique Euler graph [18], [36], this property fails to hold for $n$ even [26].

Recall that a complete invariant $\varphi_n$ is a surjective class function mapping Seidel matrices of order $n$ into their switching classes. Thus two Seidel matrices $S_1$ and $S_2$ of order $n$ are equivalent if and only if $\varphi_n(S_1) = \varphi_n(S_2)$. A complete invariant provides a convenient way to determine equivalence of Seidel matrices. Therefore the most time-consuming part of the classification, which is equivalence rejection, can be avoided by employing suitable chosen invariants. The choice of the invariants introduced in this section was motivated by the graph reconstruction conjecture (cf. [29]).

Let $\det(xI - S) = x^n + \sum_{i=0}^{n-2} a_i x^i$ be the characteristic polynomial of a Seidel matrix of order $n$. It is easy to see that the truncated coefficient list, denoted by

$$\chi_n(S) := [a_0, a_1, \ldots, a_{n-2}]$$

is a complete invariant for $n \leq 7$ (see [3]). One slight inconvenience is the data structure of $\chi_n$: comparing and storing vectors is infeasible for higher $n$ due to the large number of inequivalent matrices. Instead, we map various vector valued invariants into the cyclic group $\mathbb{Z}/m\mathbb{Z}$, which we identify with \{0, 1, 2, ..., $m-1$\}. We invite the reader to verify the following statements.

**Lemma 4.2.** Let $S$ be a Seidel matrix of order $n$. If $n \leq 6$ then $\chi_n(S)$, defined in (3), is a complete invariant. For $n \in \{7, 8, 9, 10\}$ the following functions $\varphi_n : \mathcal{M}_n(\mathbb{Z}) \to \mathbb{Z}/m\mathbb{Z}$

$$\varphi_7(S) := \prod_{a_i \in \chi_7(S)} a_i \pmod{409}, \quad \varphi_8(S) := \prod_{S'} \varphi_7(S') \pmod{7507},$$

$$\varphi_9(S) := \prod_{S'} \varphi_8(S') \pmod{268921}, \quad \varphi_{10}(S) := \prod_{S'} \varphi_9(S') \pmod{45131767},$$

where $S'$ runs through all $(n-1) \times (n-1)$ principal minors, are complete invariants.
Proof. Verification by computers is straightforward based on the full classification of Seidel matrices up to order \( n \leq 10 \) \([5], [37]\). □

The invariants \( \varphi_8, \varphi_9 \) and \( \varphi_{10} \) are recursively defined, and their computation becomes less and less efficient. Therefore we seek for an improved invariant for \( n = 11 \), one which can be computed faster, as follows. Let \( S \) be a Seidel matrix of order 11, and let us define the following functions:

\[
\psi(S) := \{(v_i, m_i) : v_i \text{ is a } 9 \times 9 \text{ principal minor of } S \text{ of multiplicity } m_i\},
\]

\[
\varphi_{11}(S) := \det S \prod_{(v_i, m_i) \in \psi(S)} (v_i + 1)(m_i + 1) \quad \text{mod } 97124414801.
\]

(4)

We have the following.

**Proposition 4.3.** The Seidel matrices \( S_1 \) and \( S_2 \) of order 11 are switching equivalent if and only if \( \varphi_{11}(S_1) = \varphi_{11}(S_2) \), where \( \varphi_{11} \) is defined in (4).

**Proof.** This can be verified by using the classification of Euler graphs on 11 vertices \([28]\). □

The main achievement of this section is the following result.

**Theorem 4.4.** The Seidel matrices \( S_1 \) and \( S_2 \) of order 12 are switching equivalent if and only if \( \varphi_{12}(S_1) = \varphi_{12}(S_2) \), where \( \varphi_{11} \) is defined in (4) and \( \varphi_{12}(S) \) is the following multiset:

\[
\varphi_{12}(S) = \{\varphi_{11}(S') : S' \text{ is a } 11 \times 11 \text{ principal submatrix of } S\}.
\]

**Proof.** From Theorem 4.1 we know that there exists exactly 87723296 switching classes of order \( n = 12 \) (see an explicit table in \([26]\)). We have generated this number of matrices with distinct invariant \( \varphi_{12} \), which are available as a supplement on the web page \([40]\). □

As an application, we recall that if \( S \) is a Seidel matrix of order \( n \) with eigenvalues \( \lambda_i, 1 \leq i \leq n \), then \( S(S) := \sum_{i=1}^{n} |\lambda_i| \) is the energy of \( S \). Haemers conjectures, and R. Swinkels verifies for \( n \leq 10 \), that \( S(S) \geq 2(n - 1) \) for all \( S \) (see \([13], [17], [21]\)). We contribute some additional empirical evidence.

**Corollary 4.5.** Let \( S \) be a Seidel matrix of order \( n \leq 12 \). Then \( S(S) \geq 2(n - 1) \), with equality if and only if \( S = \pm(J_n - I_n) \), up to switching equivalence.

**Proof.** Verification by computers. □

Recall that the automorphism group of a Seidel matrix \( S \), denoted by \( \text{Aut}(S) \), is formed by the pairs \((P, D)\) where \( P \) is a permutation matrix, \( D \) is a diagonal \( \pm 1 \) matrix, such that \( S = PDSDP^T \) hold, and where we do not make a distinction between the group elements \((P, D)\) and \((P, -D)\). Having classified all Seidel matrices of order 12, we have enumerated those with interesting automorphism group. One of the (many) fascinating properties of Seidel matrices is that their automorphism group can be larger than the automorphism group of any of the ambient graphs contained in their switching class. This property is captured by Cameron’s first cohomology invariant \( \gamma \), which is nonzero if and only if \( |\text{Aut}(S)| > \max\{|\text{Aut}(\Gamma)| : \Gamma \text{ is an ambient graph in the switching class of } S\} \). Since almost all two-graphs satisfy \( \gamma = 0 \) \([7]\), those for which this does not hold is of exceptional interest. The number of such Seidel matrices up to order \( n \leq 12 \) are presented in
the concluding summarizing table (cf. [5], [7], but note that both papers incorrectly report the number of $8 \times 8$ Seidel matrices with $\gamma \neq 0$).

| $n$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| total | 1 | 1 | 2 | 3 | 7 | 16 | 54 | 243 | 2038 | 33120 | 87723296 |
| $\gamma \neq 0$ | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 21 | 0 | 392 | 0 | 15274 |
| self-compl. | 1 | 1 | 0 | 1 | 1 | 4 | 0 | 19 | 10 | 320 | 0 | 25112 |
| $\lambda_{\min} = -5$ | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 8 | 33 | 306 | 6727 | 219754 |
| invariant | det | det | det | $\chi_4$ | $\chi_5$ | $\chi_6$ | $\varphi_7$ | $\varphi_8$ | $\varphi_9$ | $\varphi_{10}$ | $\varphi_{11}$ | $\varphi_{12}$ |

Table 2. Summary of Seidel matrices of order $n \leq 12$.

Note that there do not exist self-complementary Seidel matrices of order $n \equiv 3 \pmod{4}$ by Proposition 3.7. Also $\gamma \neq 0$ can hold only if $n$ is even [7].

Remark 4.6. We remark here that instead of considering Seidel matrices, one might want to construct the frame vectors $v_i$, $1 \leq i \leq n$ directly. It is not too hard to see that up to change of basis all coordinates of the frame vectors are the signed square root of some rational number. This kind of representation is particularly useful for extending a linearly independent set of equiangular lines.

5. TWO-GRAPHS WITH THREE EIGENVALUES

Two-graphs were introduced as a tool for studying doubly transitive groups [33]. We do not recall the concept here rigorously, but rather remark that two-graphs form a class of three uniform hypergraphs and are in one-to-one correspondence with switching classes of Seidel matrices (see [35, p. 881]). A two-graph is said to be regular if the corresponding Seidel matrix has two distinct eigenvalues only. These objects are extremely useful as they correspond to the equality case in the relative bound (see Proposition 2.3). However, regular two-graphs do not always exist, and we seek for a relaxed concept leading to large equiangular line systems. We observe that there exist various large sets of equiangular line systems, whose corresponding Seidel matrices have exactly three distinct eigenvalues (see Examples 5.12, 5.20, 5.21, and 5.22) and therefore, in this section, we begin to investigate them in detail. The analogous question for graph adjacency matrices was considered in a series of papers [5], [10], [31].

5.1. Existence and structure. In this section we completely settle the existence of Seidel matrices with exactly three distinct eigenvalues (see Theorem 5.10). We also show in Theorem 5.2 that every such Seidel matrix is switching equivalent to one having an Euler graph as an ambient graph.

We begin by recalling a fundamental tool, crucial to this section, called interlacing.

Lemma 5.1 (see [4]). Let

$$
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix}
$$

be a real symmetric matrix of order $n$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ where $A$ is of order $m$ with eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$. Then we have $\lambda_i \leq \mu_i \leq \lambda_{n-m+i}$ for all $1 \leq i \leq m$. 
In what follows we will be interested in Seidel matrices of order $n$, having spectrum \{$(\lambda)^{n-d}, [\mu]^m, [\nu]^{d-m}$\}, where $2 \leq d \leq n-1$ and $1 \leq m \leq d-1$. If $\lambda$ happens to be the smallest eigenvalue then such a matrix corresponds to $n$ equiangular lines in $\mathbb{R}^d$. We spell out the standard equations which we will frequently employ during the course of this section:

\begin{align}
\text{(5)} & \quad \text{tr}S = (n-d)\lambda + m\mu + (d-m)\nu = 0, \\
\text{(6)} & \quad \text{tr}S^2 = (n-d)\lambda^2 + m\mu^2 + (d-m)\nu^2 = n(n-1).
\end{align}

Since $S$ is real and symmetric, its minimal polynomial is $(x-\lambda)(x-\mu)(x-\nu)$, we obtain

\begin{equation}
S^3 = (\lambda + \mu + \nu)S^2 - (\lambda \mu + \mu \nu + \nu \lambda)S + \lambda \mu \nu I.
\end{equation}

We get another formula by cubing $S = (J - 2A - I)^3$, namely

\begin{align}
\text{(8)} & \quad S^3 = (n^2 - 3n + 3)J + 4(A^2J + JA^2) + 4AJA - 2JAJ + 2(3 - n)(AJ + JA) \\
& \quad \quad - 8A^3 - 12A^2 - 6A - I.
\end{align}

We start with a structural result, interesting in its own.

**Theorem 5.2.** Let $S$ be a Seidel matrix with exactly three distinct eigenvalues. Then $S$ is switching equivalent to a Seidel matrix $S'$, whose ambient graph is an Euler graph.

**Proof.** Assume that $S$ is of order $n$. For $n$ odd, the statement follows from a general result due to Seidel \cite{17}, \cite{20}. Suppose that $n$ is even, and that $S$ is a Seidel matrix whose ambient graph $\Gamma$ has $e$ edges and has a vertex $v_1$ with vertex degree $d(v_1) = 0$. From equation (7) we observe that for all $1 \leq i \leq n$ we have

\begin{equation}
[S^3]_{ii} = [S^3]_{11}.
\end{equation}

From equation (8) it follows, by using $JAJ = 2eJ$, that

\begin{equation}
[S^3]_{ii} = n^2 - 3n + 2 - 4e + 4(A^2J + JA^2)_{ii} - 8[A^3]_{ii} + 4d^2(v_i) - 4nd(v_i).
\end{equation}

Plug in $i = 1$ to conclude that $[S^3]_{11} = n^2 - 3n + 2 - 4e$. Therefore, by combining equations (9) and (10), it follows that $4(A^2J + JA^2)_{ii} - 8[A^3]_{ii} + 4d^2(v_i) - 4nd(v_i) = 0$ for all $1 \leq i \leq n$. Since $n$ is even so is $d^2(v_i)$ for all $1 \leq i \leq n$, and hence $\Gamma$ is an Euler graph as claimed. \hfill \square

The following is a consequence of Theorem 5.2.

**Corollary 5.3.** Let $S$ be a Seidel matrix of order $n$ with exactly three distinct eigenvalues $\lambda$, $\mu$, and $\nu$. Then

\begin{enumerate}
\item[(a)] $(n-1)(\lambda + \mu + \nu) + \lambda \mu \nu \equiv n^2 + n + 2 \pmod{4}$;
\item[(b)] $(n-2)(\lambda + \mu + \nu) + \lambda \mu + \mu \nu + \nu \lambda \equiv n^2 + n + 1 \pmod{4}$;
\item[(c)] $\lambda \mu + \mu \nu + \nu \lambda \equiv 1 \pmod{2}$.
\end{enumerate}

**Proof.** We can assume, by Theorem 5.2 that $S = J - 2A - I$, where $A$ is the adjacency matrix of some Euler graph $\Gamma$. Therefore $AJ + JA \equiv 0 \pmod{2}$. Consider equation (8) modulo 4 and note that $JAJ = 2eJ$, where $e$ is the number of edges in $\Gamma$. We have $S^3 \equiv 2A - I + (n^2 + n - 1)J \pmod{4}$. Comparing this with equation (7) and replacing $S^2$ by $(J - 2A - I)^2$ yields

\begin{align*}
(\lambda + \mu + \nu)((n-2)J + I) & \quad - (\lambda \mu + \mu \nu + \nu \lambda)(J - 2A - I) + \lambda \mu \nu I \\
& \equiv 2A - I + (n^2 + n - 1)J \pmod{4}.
\end{align*}
Consider the \((i,j)\)th entry of this congruence. By taking \(i = j\) we obtain part (a). By taking two distinct off-diagonal coordinates \(i \neq j\) and \(k \neq \ell\), for which \(a_{ij} = 0\) and \(a_{k\ell} = 1\), respectively, we obtain a system of two congruences, being equivalent to part (b) and (c).

Next we prove that if \(S\) is a Seidel matrix with exactly three distinct eigenvalues, then the multiplicities of these cannot be all equal.

**Proposition 5.4.** Let \(S\) be a Seidel matrix of order \(n\) with spectrum \([\lambda]^{n-d}, [\mu]^m, [\nu]^{d-m}\) for some \(2 \leq d \leq n-1\) and \(1 \leq m \leq d-1\). Then the case \(m = n-d = d-m\) is impossible.

**Proof.** Suppose the contrary, i.e., there exists a Seidel matrix where \(m = n-d = d-m = n/3\). It follows that \(n \equiv 0 \pmod{3}\). Using equations (5) and (6) we find that \(\lambda + \mu + \nu = 0\) and \(\lambda \mu + \mu \nu + \nu \lambda = -3(n-1)/2\). In particular, \(n\) is odd. Plugging these into part (a) of Corollary 5.3 we find that \(n \equiv 3 \pmod{4}\). Now we can put \(n = 12M + 3\) and use Corollary 5.5 to infer that \(\det S = (\lambda \mu \nu)^{4M+1} \equiv 1 - n \equiv 2 \pmod{4}\). This forces \(\lambda \mu \nu\) to be even and \(M = 0\), and since there is no such \(3 \times 3\) Seidel matrix, we have a contradiction.

We will see further applications of Corollary 5.3 later in Remark 5.7.

**Corollary 5.5.** Every Seidel matrix with exactly three distinct eigenvalues has an integer eigenvalue.

**Proof.** Follows immediately from Proposition 5.4.

The following is a generalization of Proposition 2.3.

**Theorem 5.6.** Let \(d \geq 1\) and let \(S\) be a Seidel matrix of order \(n \geq 2\) with smallest eigenvalue \(\lambda_0\) of multiplicity \(n-d \geq 1\). Assume that \(S\) has another eigenvalue \(\mu\) of multiplicity \(m \leq d\). Then

\[
(11) \quad \left| \frac{\lambda_0(n-d)}{d} + \frac{\mu(n-d)}{d} \right| \leq \sqrt{n(d(n-1) - \lambda_0^2(n-d))} \cdot \sqrt{\frac{d-m}{m}}.
\]

Equality holds if and only if \(S\) has at most three distinct eigenvalues.

The proof is standard, and follows the same spectral analysis that appears in [23].

**Proof.** First note that the right hand side is well-defined by [23] Lemma 6.1]. Let us denote the remaining eigenvalues of \(S\) by \(\lambda_1, \lambda_2, \ldots, \lambda_{d-m}\). Then, by using that \(\text{tr} S = 0\) and \(\text{tr} S^2 = n(n-1)\), we find, after an application of the Cauchy–Schwartz inequality, that

\[
((n-d)\lambda_0 + m\mu)^2 = \left( \sum_{i=1}^{d-m} \lambda_i \right)^2 \leq (d-m) \sum_{i=1}^{d-m} \lambda_i^2 \leq (d-m) (n(n-1) - (n-d)\lambda_0^2 - m\mu^2).
\]

The result follows after some algebraic manipulations.
If $S$ is a Seidel matrix with exactly three distinct eigenvalues, then Theorem 5.6 boils down to an algebraic identity, which in turn can be used to tabulate feasible parameter sets of such Seidel matrices. Some interesting parameter sets are shown in Appendix A.

**Remark 5.7.** Even though they satisfy the conditions of Theorem 5.6 and the necessary condition described by Corollary 5.5, the following two spectra $\{[-7]^1, [-1]^3, [2]^5\}$, and $\{[-\sqrt{10}]^1, [0]^3, [\sqrt{10}]^1\}$, fail to meet part (a) and part (c) of Corollary 5.5 respectively.

Now we turn to a description of the case when two of the three multiplicities are equal.

**Proposition 5.8.** Let $S$ be a Seidel matrix of order $n$, let $(n+1)/2 \leq d \leq n-1$, and suppose that $S$ has smallest eigenvalue $\lambda_0$ of multiplicity $n-d$. Assume further that $S$ has two additional eigenvalues $\mu, \nu \in \mathbb{Z}$ of multiplicity $2d-n$ and $n-d$, respectively. Then

$$\lambda_0, \nu = \frac{(n-2d)\mu + \sqrt{n(2(n-1)(n-d) + (n-2d)\mu^2)}}{2(n-d)},$$

and hence necessarily

$$d\mu \equiv 0 \pmod{n-d}, \quad \text{and} \quad n(n-1)/2 + (1 + (-1)^\mu)d\mu/4 \equiv 0 \pmod{n-d}.$$ 

Note that Proposition 5.8 covers all cases where two out of the three eigenvalue multiplicities are equal, up to taking the negative of the Seidel matrix if it is necessary.

**Proof.** The values of $\lambda_0$ and $\nu$ follow from Theorem 5.6 while the necessary conditions follow from the fact that $\lambda_0 + \nu$ and $\lambda_0\nu$ are both algebraic integers. Indeed, from the sum condition we readily see that there exists an integer $M$ such that $d\mu = M(n-d)$. From the product condition we observe that $-2(n-d)\lambda_0\nu = n(n-1) + d\mu(\mu - M)$, and hence

$$n(n-1) + d\mu(\mu - 1) = n(n-1) + d\mu(\mu - 2) + d\mu \equiv M(M-1)(n-d) \pmod{2(n-d)}.$$ 

For $\mu$ odd we have $\mu - 1$ even and the statement follows from the left hand side. Otherwise, if $\mu$ is even the statement follows from the middle part.

An infinite family of examples where the multiplicity of two eigenvalues agree can be obtained from the symmetric Paley matrices of prime power order $n \equiv 1 \pmod{4}$; the spectrum of these matrices is $\{[-\sqrt{n}]^{(n-1)/2} [0]^1, [\sqrt{n}]^{(n-1)/2}\}$. There exists an additional example of order $n = 12 \equiv 0 \pmod{4}$ with spectrum $\{[-1-2\sqrt{5}]^3, [1]^6, [-1+2\sqrt{5}]^3\}$, coming from the icosahedron. Note that in all of the previous examples the integer eigenvalue is neither the smallest, nor the largest one. This is not the case in general, as there exists a Seidel matrix of order $n = 10 \equiv 2 \pmod{4}$ with spectrum $\{[-3]^1, [2 - \sqrt{5}]^3, [2 + \sqrt{5}]^3\}$. Further examples with quadratic irrational eigenvalues come from Proposition 5.14 but none of them is of order $n \equiv 3 \pmod{4}$, as we show next.

**Lemma 5.9.** Let $S$ be a Seidel matrix of order $n$ with exactly three distinct eigenvalues such that two of them have the same multiplicity. Then $n \not\equiv 3 \pmod{4}$. 

Let $\Gamma$ be a connected $k$-regular graph on $n$ vertices with adjacency matrix $A$. Assume that $A$ has exactly three distinct eigenvalues. Then $S := J - 2A - I$ has at most three distinct eigenvalues.

Proof. If the eigenvalues of $A$ are $k$, $\lambda$, and $\mu$, then the eigenvalues of $S$ are $n - 2k - 1$, $-2\lambda - 1$, and $-2\mu - 1$, respectively. □

Example 5.12. Let $\Gamma$ be a strongly regular graph with spectrum $\{2\}^{24}, [-4]^{15}, [12]^4$ (see [38]). Then from Lemma 5.11 we obtain a Seidel matrix with spectrum $\{[-5]^{24}, [7]^{15}, [15]^4\}$. The corresponding 40 lines form the largest known equiangular line system in $\mathbb{R}^{16}$.

Connected regular graphs with exactly four distinct eigenvalues [11] are also useful.

Lemma 5.13. Let $\Gamma$ be a connected $k$-regular graph on $n$ vertices with adjacency matrix $A$. Assume that $A$ has exactly four distinct eigenvalues. Then $S := J - 2A - I$ has at least three distinct eigenvalues.
Proof. If the eigenvalues of $A$ are $k$, $\lambda$, $\mu$, and $\nu$, then the eigenvalues of $S$ are $n - 2k - 1$, $-2\lambda - 1$, $-2\mu - 1$, and $-2\nu - 1$, respectively. □

The next result is a far reaching generalization of \[ \text{Construction 6.2}. \]

**Proposition 5.14.** Let $S$ be a Seidel matrix with spectrum $\{[\lambda_0]^{n-c}, [\lambda_1]^d\}$. Then for all $b \geq 2$ there exists a Seidel matrix with spectrum $\{[1 - (1 - \lambda_0)b]^{a-c}, [1]^{a(b-1)}, \times\}$.

Proof. The Seidel matrix $J_b \otimes (S - I_a) + I_{ab}$ of order $ab$ has the desired spectrum. □

The following, besides being a folklore technical result, serves as an other example of Seidel matrices with three distinct eigenvalues.

**Lemma 5.15.** Let $S$ be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ for some $1 \leq d \leq n - 1$. Let $S'$ be a principal $(n - 1) \times (n - 1)$ submatrix of $S$. Then the spectrum of $S'$ is $\{[\lambda_0]^{n-d-1}, [\lambda_1]^{d-1}, [\lambda_0 + \lambda_1]^{1}\}$.

Proof. By interlacing (see Lemma 5.1) $S'$ has spectrum $\{[\lambda_0]^{n-d-1}, [\lambda_1]^{d-1}, [x]^1\}$ for some $\lambda_0 \leq x \leq \lambda_1$. Since $\text{tr}S = \text{tr}S' = 0$, it follows that $x = \lambda_0 + \lambda_1$, as claimed. □

**Proposition 5.16.** Let $S$ be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ for some $1 \leq d \leq n - 1$ and assume that for some $1 \leq c \leq d$ it admits the block partition

$$S = \begin{bmatrix} J_c - I_c & * \\ * & S' \end{bmatrix}.$$

Then $\text{tr}((S')^3) = (n - 3c)(n - 1)(\lambda_0 + \lambda_1) + c(c-1)(3\lambda_0 + 3\lambda_1 - c + 2)$.

Proof. For $c = 1$ the spectrum is known by Lemma 5.15 while for $c = 2$ it can be determined from the standard equations \[ \text{Lemma 5.15} \] and \[ \text{Lemma 5.15} \], and hence the claim follows in these cases. Therefore we assume that $c \geq 3$ and compare $\text{tr}((S')^3)$ with $\text{tr}(S^3)$ to find:

$$\text{tr}(S^3) = 3 \sum_{M \in \mathcal{M}} \det M$$

$$= \text{tr}((S')^3) + 3 \sum_{M \in \mathcal{M}_1} \det M + 3 \sum_{M \in \mathcal{M}_2} \det M + 3 \sum_{M \in \mathcal{M}_3} \det M,$$

where $\mathcal{M}$ is the subset of all $3 \times 3$ principal minors of $S$, while $\mathcal{M}_i$, $i = 1, 2, 3$ are those subsets of $\mathcal{M}$ whose members intersect the first $c$ rows of $S$ in exactly $i$ places. We calculate these sums separately.

The last term is straightforward, as we have $\binom{c}{3}$ minors within the first $c \times c$ submatrix of $S$, each contributing the value 2 to the sum:

$$\sum_{M \in \mathcal{M}_3} \det M = 2 \binom{c}{3}.$$

The value of the sum over $\mathcal{M}_2$ can be obtained by evaluating the inner products of the rows of $S$, $\langle S_i, S_j \rangle$, via two different ways. First, since $S$ has two eigenvalues only, it satisfies

$$S^2 = (\lambda_0 + \lambda_1)S + (n - 1)I.$$


and consequently for every 1 ≤ i < j ≤ c we have \( \langle S_i, S_j \rangle = [S^2]_{ij} = \lambda_0 + \lambda_1 \). Second, we count the number of vertical pairs \([1, 1]^T, [1, -1]^T, [-1, 1]^T\) within the \(i\)th and \(j\)th rows of \(S\), which we denote by \(w_{i,j}, x_{i,j}, y_{i,j}, z_{i,j}\). It follows that \(\langle S_i, S_j \rangle = w_{i,j} - x_{i,j} - y_{i,j} + z_{i,j}\). By combining these two equations for the inner products we find that the quantity \(w_{i,j} - x_{i,j} - y_{i,j} + z_{i,j}\) does not depend on the subscripts, and hence

\[
\sum_{M \in M_2} \det M = 2 \sum_{i<j}^{c} (w_{i,j} - x_{i,j} - y_{i,j} + z_{i,j} - (c - 2)) = c(c-1)(\lambda_0 + \lambda_1 - c + 2).
\]

The value of the final sum can be obtained by counting the (sum of the) number of signed closed 3-walks originating from the first \(c\) vertices of \(S\) in two ways. Firstly, by counting, double counting, and triple counting, we have

\[
\sum_{i=1}^{c} [S^3]_{ii} = \sum_{i=1}^{c} \sum_{k,l} s_{ik}s_{kl}s_{li} = \sum_{i=1}^{c} \sum_{k,l} s_{kl}s_{li} = \sum_{M \in M_1} \det M + 2 \sum_{M \in M_2} \det M + 3 \sum_{M \in M_3} \det M.
\]

Secondly, we know from (14) that \([S^3]_{ii} = (n-1)(\lambda_0 + \lambda_1)\) for all \(1 \leq i < j \leq n\) and hence

\[
\sum_{i=1}^{c} [S^3]_{ii} = c(n-1)(\lambda_0 + \lambda_1).
\]

Comparing these two we find that

\[
\sum_{M \in M_1} \det M = c(n-1)(\lambda_0 + \lambda_1) - 2 \sum_{M \in M_2} \det M - 3 \sum_{M \in M_3} \det M,
\]

and therefore combining (13), (15), and (16) with (12) yields

\[
\text{tr} \left((S')^3\right) = \text{tr}(S^3) - 3c(n-1)(\lambda_0 + \lambda_1) + 3 \sum_{M \in M_2} \det M + 6 \sum_{M \in M_3} \det M
\]

\[
= (n-3c)(n-1)(\lambda_0 + \lambda_1) + c(c-1)(3\lambda_0 + 3\lambda_1 - c + 2). \quad \Box
\]

Now we can generalize Lemma 5.15 as follows.

**Proposition 5.17.** Let \(S\) be a Seidel matrix of order \(n \geq 2\) with spectrum \(\{\lambda_0\}^{n-d}, [\lambda_1]^d\) for some \(1 \leq d \leq n - 1\) and assume that for some \(1 \leq c \leq \min\{d, n - d\}\) it admits the block partition

\[
S = \begin{bmatrix}
J_c & -I_c & \ast \\
\ast & S' \ast 
\end{bmatrix}.
\]

Then the spectrum of \(S'\) is \(\{\lambda_0\}^{n-d-c}, [\lambda_1]^{d-c}, [\lambda_0 + \lambda_1 + 1 - c]^1, [\lambda_0 + \lambda_1 + 1]^{c-1}\}.

**Remark 5.18.** If \(J_c - I_c\) is a principal submatrix of a Seidel matrix \(S\) of order \(n \geq 2\) with largest eigenvalue \(\lambda_1\), then \(c \leq \min\{d, n - d\}\) unless \(S\) is switching equivalent to \(J - I\). Another bound, \(c \leq \lambda_1 + 1\) follows from interlacing.

**Proof.** We prove by induction on \(c\). For \(c = 1\) the claim follows from Lemma 5.15 and, for \(c = 2\), it is easy to determine the spectrum from the standard equations (5) and (6). Therefore we assume that \(c \geq 3\). By removing the first \(c - 1\) rows and
columns from $S$ we obtain, by our induction hypothesis, a Seidel matrix whose spectrum is the following: $\{[\lambda_0]^{n-d-c+1}, [\lambda_1]^d, [\lambda_0 + \lambda_1 + 2 - c], [\lambda_0 + \lambda_1 + 1]^{c-2}\}$, where all the multiplicities are at least 1. We now remove the $c$th row and column from $S$ as well and use interlacing to find that the spectrum of $S'$ reads $\{[\lambda_0]^{n-d-c}, [\lambda_1]^d, [\lambda_0 + \lambda_1 + 1]^{c-3}, [x]^2, [y]^2, [z]^2\}$ for some real numbers $x, y$, and $z$. By the standard equations and Proposition 5.16 we find that
\[
\text{tr}(S') = (n - d - c)\lambda_0 + (d - c)\lambda_1 + (c - 3)(\lambda_0 + \lambda_1 + 1) + x + y + z = 0,
\]
\[
\text{tr}((S')^2) = (n - d - c)\lambda_0^2 + (d - c)\lambda_1^2 + (c - 3)(\lambda_0 + \lambda_1 + 1)^2 + x^2 + y^2 + z^2
\]
\[
= (n - c)(n - c - 1),
\]
\[
\text{tr}((S')^3) = (n - d - c)\lambda_0^3 + (d - c)\lambda_1^3 + (c - 3)(\lambda_0 + \lambda_1 + 1)^3 + x^3 + y^3 + z^3
\]
\[
= (n - 3c)(n - 1)(\lambda_0 + \lambda_1) + (c - 1)(3\lambda_0 + 3\lambda_1 - c - 2).
\]

Now by noting that $\lambda_1 = -(n - d)\lambda_0/d$ and $\lambda_0^2 = d(n - 1)/(n - d)$ it is immediate to check that $\{x, y, z\} = \{\lambda_0 + \lambda_1 + 1 - c, \lambda_0 + \lambda_1 + 1, \lambda_0 + \lambda_1 + 1\}$ (in any order) is a solution. Since the elementary symmetric polynomials $x + y + z$, $xy + yz + zx$ and $xyz$ can be expressed using the equations above, it follows that this is the only solution.

The following is the main result of this section.

**Theorem 5.19.** Let $S$ be a Seidel matrix of order $n \geq 2$ with spectrum $\{[\lambda_0]^{n-d}, [\lambda_1]^d\}$ with $\lambda_1 \leq \min\{d - 1, n - d - 1\}$ and assume that it admits the block partition $S = \begin{bmatrix} J_{\lambda_1+1} & I_{\lambda_1+1} & \ast \\ \ast & \ast & S' \end{bmatrix}$.

Then the spectrum of $S'$ is $\{[\lambda_0]^{n-d-\lambda_1}, [\lambda_1]^{d-\lambda_1-1}, [\lambda_0 + \lambda_1 + 1]^{\lambda_1}\}$.

**Proof:** Setting $c = \lambda_1 + 1$ in Proposition 5.17 and observing that in this case $\lambda_0 + \lambda_1 + 1 - c = \lambda_0$ we obtain the desired result. \(\square\)

Note that Theorem 5.19 describes a method to obtain equiangular lines in a smaller dimensional space by removing a maximal independent set (see Remark 5.18) from the underlying graph. One consequence is the following result of Tremain.

**Example 5.20** (cf. [13]). We construct 28 equiangular lines in $\mathbb{R}^{14}$ with common angle $1/5$ from a regular two-graph on 36 vertices. Consider the $35 \times 35$ matrix $S$ in [6, p. 75]. Let $S'$ be constructed from $-S$ by extending it with a 36th row and column with all entries equal to 1 except for $S'_{36,36} = 0$. Upon removing rows $\{1, 6, 11, 17, 20, 33, 34, 36\}$ along with the same set of columns from $S'$ we find, by Theorem 5.12, that the spectrum of the resulting Seidel matrix of order 28 is $\{[-5]^{14}, [3^7], [7^7]\}$. Additional inequivalent examples might be obtained from other regular two-graphs on 36 vertices; these have been classified in [30].

Next we discuss some relevant, sporadic examples of equiangular line systems.

**Example 5.21** (see [22]). Let $B := \{4^i + j, 7 \cdot 4^i + j, 11 \cdot 4^i + j\} \pmod{19} : 0 \leq i \leq 2, 0 \leq j \leq 18$ be the set of blocks of the Netto triple system on the ground set $X = \{1, 2, \ldots, 19\}$; let $e_i$ denote the standard basis vectors in $\mathbb{R}^{19}$ for $1 \leq i \leq 19$ and for a subset $T \subseteq X$ let us denote $e_T := \sum_{i \in T} e_i$. The vectors $v_B := (6e_B + e_1 - e_X)/\sqrt{90}$ for which $1 \notin B \in B$ are all orthogonal to both $e_1$ and $e_X$ and form an equiangular
line system of 48 lines in \( \mathbb{R}^{17} \). Moreover, the corresponding Seidel matrix has spectrum \{[-5]^{31}, [7]^{8}, [11]^{9} \}.

The known maximal set of equiangular lines in dimensions 19-23 all come from the Witt-design. The examples in dimension 21, 22, and 23 are regular two-graphs; Taylor’s example in dimension 20 has four distinct eigenvalues \([42]\); while the following construction, discovered by Asche, leads to a Seidel matrix with three distinct eigenvalues in dimension 19.

**Example 5.22** (see \([42]\) p. 124)). Let \( B \) be the set of 759 blocks of the Witt-design, the “octads”, defined on the ground set \( X = \{ 1, 2, \dots, 24 \} \), let \( e_i \) denote the standard basis vectors in \( \mathbb{R}^{24} \) for \( 1 \leq i \leq 24 \) and for a subset \( T \subseteq X \) let us denote \( e_T := \sum_{i \in T} e_i \). Let \( B_1, B_2 \in B \) such that \( 1 \notin B_1, B_2 \) and \( B_1 \cap B_2 = \{ 2, 3 \} \). The vectors \( v_B := (4e_B - 4e_1 - e_X) / \sqrt{80} \) for which \( 1 \in B \in B \) are all orthogonal to \( 4e_1 + e_X \). Those, which in addition are orthogonal to all of \( e_1 - e_2, e_1 - e_3, v_{B_1}, \) and \( v_{B_2} \), form an equiangular line system of 72 lines in \( \mathbb{R}^{19} \). Moreover, the corresponding Seidel matrix has spectrum \{[-5]^{33}, [13]^{16}, [19]^{3} \}.

The following observation nicely connects the existence of some hypothetical strongly regular graphs on 76 and 96 vertices, corresponding to some of the upper bounds in Table 1.

**Remark 5.23.** One can see the following by repeated application of Theorem 5.19. Assume that there exists a Seidel matrix of order 96 with spectrum \{[-5]^{76}, [19]^{20} \} containing a \( J_{20} - I_{20} \) principal minor (see Remark 5.18). Then, there exists a Seidel matrix of order 76 with spectrum \{[-5]^{57}, [15]^{19} \}. If, in addition, this latter contains a \( J_{16} - I_{16} \) principal minor, then there further exists a Seidel matrix of order 60 with spectrum \{[-5]^{42}, [11]^{15}, [15]^{3} \}. These examples would improve upon the best known lower bounds on the number of equiangular lines in dimensions 20, 19, and 18. See \([14]\) for regular two-graphs with maximal cliques.

In Table 3 we display the number of inequivalent Seidel matrices with exactly three distinct eigenvalues up to order \( n \leq 12 \) (cf. Theorem 5.10 and with the results of Section 4).

| \( n \) | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|----|----|----|
| \( \Delta \) | 0 | 0 | 1 | 2 | 0 | 2 | 3 | 4 | 0 | 10 |

**Table 3.** The number of Seidel matrices of order \( n \) with exactly three distinct eigenvalues, up to switching equivalence.

### 5.3. A nonexistence result.

In this section we prove, under some assumptions, that if the number of equiangular lines is “very close” to the relative bound, described in Proposition 2.3, then such a configuration must correspond to a Seidel matrix with at most three distinct eigenvalues.

**Theorem 5.24.** Let \( d \geq 3 \) and assume that there exists \( n \) equiangular lines in \( \mathbb{R}^{d} \), with corresponding Seidel matrix \( S \) having eigenvalues \( \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_d \). Assume further that \( \lambda_0 > d \), \( \Delta := d^2 - d \lambda_0^2 n - dn^2 + dn + \lambda_0^3 n^2 \) is a square of an integer, and for some sign choice \( s := (-\lambda_0 (n - d) \pm \sqrt{\Delta}) / d \) is an integer. If \( s \neq \lambda_i \) for all \( 1 \leq i \leq d \), then \( S \) has at most three distinct eigenvalues.
Proof. We have the following chain of (in)equalities:

\[ d = n(n - 1) + s^2 n - (\lambda_0 - s)^2(n - d) = \sum_{i=1}^{d} (\lambda_i - s)^2 \geq d \prod_{i=1}^{d} (\lambda_i - s)^{2/d} \geq d, \]

where the leftmost equality follows from the assumptions; the middle equality follows from evaluating \( \text{tr}((S - sI)^2) \); the next relation is the inequality of arithmetic and geometric means; and the rightmost inequality follows from the fact that the set \( \{ \lambda_i - s : i = 1, \ldots, d \} \) consists of nonzero algebraic integers closed under algebraic conjugation. In particular, we have equality everywhere, and therefore \( \lambda_i = s \pm 1 \) for all \( i = 1, \ldots, d \).

\[ \blacksquare \]

Remark 5.25. We remark that if the conditions of Theorem 5.24 are met, and \( d \leq \lambda_0^2 - 2 \), then \( n = \left( \frac{d(\lambda_0^2 - 1)}{\lambda_0^2 - d} \right) \). Indeed, the implicit bound \( \Delta \geq 0 \) in Theorem 5.24 and the relative bound given by Proposition 2.3 together lead to

\[ L := \frac{d(\lambda_0^2 - 1) + d\sqrt{(\lambda_0^2 - 3)^2 + 4d - 8}}{2(\lambda_0^2 - d)} \leq n \leq \frac{d(\lambda_0^2 - 1)}{\lambda_0^2 - d} =: R. \]

Since \( R - L = 2d/(\lambda_0^2 - 1 + \sqrt{(\lambda_0^2 - 3)^2 + 4d - 8}) < 1 \), the claim follows. In particular, \( n \) is as close to the relative bound as it can be.

It follows from Theorem 5.24 that 30 lines in \( \mathbb{R}^{14} \) must correspond to a Seidel matrix with spectrum \( \{ -5, 16, 5, 9, 7 \} \). Similarly, 42 lines in \( \mathbb{R}^{16} \) must come from a Seidel matrix with spectrum \( \{ -5, 26, 7, 9 \} \). We prove that these Seidel matrices do not exist.

Lemma 5.26. Let \( M \) be a positive semi-definite \( \{ 0, \pm 1 \} \) matrix of order \( n \) with constant diagonal entries 1. Then there exist positive integers \( c, k_1, \ldots, k_c \), such that \( M \) is switching equivalent to the block diagonal matrix \( \text{diag}[J_{k_1}, \ldots, J_{k_c}] \).

Proof. The proof goes by induction on \( n \). For \( n = 1 \) there is nothing to prove, so we can consider an \((n + 1) \times (n + 1)\) matrix \( M' \). We can assume, by the inductive hypothesis, that its leading principal \( n \times n \) submatrix \( M \) is already in the desired form, namely \( M = \text{diag}[J_{k_1}, \ldots, J_{k_c}] \) for some positive integers \( c, k_1, \ldots, k_c \). Since \( M' \) is positive semi-definite, it follows that it cannot have any principal submatrices, switching equivalent to either

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & 1
\end{bmatrix}.
\]

Therefore either \( M' \) is switching equivalent to \( \text{diag}[J_{k_1}, \ldots, J_{k_c}, J_1] \) or, up to re-ordering the blocks, \( M' \) is switching equivalent to \( \text{diag}[J_{k_1}, \ldots, J_{k_c+1}] \).

\[ \blacksquare \]

Theorem 5.27. Let \( S \) be a Seidel matrix with spectrum \( \{ [\lambda]^n, [\mu]^b, [\nu]^c \} \) of order \( n \equiv 2 \pmod{4} \). Assume that \( \lambda + \mu \equiv 0 \pmod{4} \), and \( |n - 1 + \lambda \mu| = 4 \). Then \( |(\nu - \lambda)(\nu - \mu)|/4 = n/c \in \mathbb{Z} \), and \( |\nu| \leq n/c - 1 \).

Proof. Let \( \varphi := (\nu - \lambda)(\nu - \mu) \) and let \( \sigma := \text{sgn} \varphi \). Consider the positive semi-definite matrix \( M := \sigma(S - \lambda I)(S - \mu I) \) with spectrum \( \{ [0]^n + b, [\varphi]^c \} \). Let \( A \) be the adjacency matrix of the ambient graph corresponding to \( S \). By Theorem 5.2 it follows that \( AJ + JA \equiv 0 \pmod{2} \) and hence \( S^2 - I = (n - 2)J - 2(AJ + JA) + 4(A^2 + A) \equiv 0 \pmod{4} \). Therefore the off-diagonal entries of \( M \) are all divisible by...
4, while the diagonal entries of $M$ are all equal to $4 = \sigma(n - 1 + \lambda\mu) \geq 0$. Since every $2 \times 2$ submatrix of $M$ must be positive semi-definite, it follows that every off-diagonal entry $m$ satisfies $|m| \leq |n - 1 + \lambda\mu| = 4$. Therefore $m \in \{0, \pm 4\}$, and hence $M$ is a $\{0, \pm 4\}$-matrix, and $|q|$ is divisible by 4. By Lemma 5.26 we may assume that $M = 4\text{diag}[J_{k_1}, \ldots, J_{k_c}]$ for some positive integers $k_1, \ldots, k_c$. Since each block must have an eigenvalue $|q|/4$, it follows that $k_i = |q|/4$ for all $i$, and thus $c|q|/4 = n$. Therefore $M = 4J_{n/c} \otimes I_c$. On the one hand, we find that

$$[\sigma^3]_{11} = [(\sigma M + (\lambda + \mu)S - \lambda\mu I)S]_{11} = [(4\sigma J_{n/c} \otimes I_c)S]_{11} + (\lambda + \mu)(n - 1).$$

On the other hand, from equation (7) we have $[\sigma^3]_{11} = (\lambda + \mu + \nu)(n - 1) + \lambda\mu\nu$. Therefore, we have $|\nu(n - 1 + \lambda\mu)| = 4|[J_{n/c} \otimes I_c]S]_{11}| \leq 4(n/c - 1)$, and the result follows.

**Corollary 5.28** (cf. [11 Appendix A]). There do not exist regular graphs with spectrum $\{[11]^{\dagger}, [2]^{16}, [-3]^{9}, [-4]^4\}$ or $\{[12]^{\dagger}, [2]^{16}, [-3]^{9}, [-4]^8\}$.

**Proof.** Either of these graphs would lead, via Lemma 5.13 to a Seidel matrix with spectrum $\{[5]^{16}, [5]^9, [7]^5\}$, which does not exist by Theorem 5.27.

In [11] the authors used a computer search to claim the nonexistence of regular graphs with spectrum $\{[11]^{\dagger}, [2]^{16}, [-3]^{9}, [-4]^4\}$. However, the case $\{[12]^{\dagger}, [2]^{16}, [-3]^{9}, [-4]^8\}$ was left open.

By combining Theorems 5.24 and 5.27 we obtain the main result of this subsection.

**Corollary 5.29.** The maximum number of equiangular lines in $\mathbb{R}^{14}$ is at most 29. The maximum number of equiangular lines in $\mathbb{R}^{16}$ is at most 41.

**Proof.** In both cases $s$, appearing in Theorem 5.24 is even, and therefore it cannot be an eigenvalue of $S$ by Corollary 5.5.

6. Equiangular lines with common angle 1/5

Maximal equiangular line systems with common angle $\alpha = 1/3$ are completely understood: it is easy to see that the corresponding Seidel matrices must contain an $I_4 - J_4$ principal submatrix [22], and consequently one cannot have more than $2(d - 1)$ such equiangular lines in $\mathbb{R}^d$ for $d \geq 15$ (see also [16] and [23]). In this section we discuss results regarding $N_5(d)$, the maximum number of equiangular lines with common angle 1/5.

**Theorem 6.1** (see [22]). Let $S$ be a Seidel matrix of order $n$ with smallest eigenvalue $-5$ of multiplicity $n - d$, containing an $I_6 - J_6$ principal submatrix. Then $n \leq 276$ for $43 \leq d \leq 185$, and $n \leq \lfloor 3(d - 1)/2 \rfloor$ for $d \geq 186$.

**Remark 6.2.** One out of the four Seidel matrices of order 12 corresponding to maximal equiangular line systems in $\mathbb{R}^9$ with common angle 1/5 does not contain (up to switching) any $I_6 - J_6$ principal submatrix.

Theorem 6.1 was subsequently improved by Neumaier, who determined the maximum number of equiangular lines with common angle 1/5 in $\mathbb{R}^d$ for large $d$ [32]. It turns out that the Seidel matrix of all such line systems is switching equivalent to one whose ambient graph has largest eigenvalue at most 2. Such graphs are called Dynkin graphs [22].
Theorem 6.3 (Neumaier [32]). There exists a positive integer $V$, for which if $S$ is a Seidel matrix of order $n \geq V$ with smallest eigenvalue $-5$, then $S$ is switching equivalent to some Seidel matrix $S'$ such that its ambient graph $\Gamma'$ is a Dynkin graph.

Remark 6.4. Neumaier claims, without proof, that $2486 \leq V \leq 45374$ is true [32].

The case $t = 2$ of the following technical result is the one we are interested in.

Lemma 6.5. Let $\Gamma$ be a graph with adjacency matrix $A$ having largest eigenvalue $t$ of multiplicity $m$ and let $S := J - 2A - I$ be the corresponding Seidel matrix. Then $S$ has smallest eigenvalue $\lambda_0 \geq -(2t + 1)$ with equality if and only if $m \geq 2$; in this case $\lambda_0$ has multiplicity $m - 1$.

Proof. We have $x^TSx = x^Tx - 2x^TAx - x^T Ix \geq -(2t + 1)$ for any unit vector $x$ and hence $\lambda_0 \geq -(2t + 1)$, as claimed. Moreover, equality holds if and only if $\dim(\ker J \cap \ker (A - tI)) \geq 1$. By Perron–Frobenius theory, $A$ has a nonnegative $t$-eigenvector, which cannot be a 0-eigenvector of $J$, and consequently $\dim(\ker J + \ker (A - tI)) = n$. Therefore

$$\dim(\ker (S - (2t + 1)I)) = \dim(\ker J + \ker (A - tI)) = \dim(\ker J) + \dim(\ker (A - tI)) - \dim(\ker J + \ker (A - tI)) = n - 1 + m - n = m - 1.$$ 

The following result was announced, without a proof, in [32].

Corollary 6.6. Let $d \geq \lceil (2V + 5)/3 \rceil$, where $V$ is the number in Theorem 6.3. Then $N_5(d) = \lceil 3(d - 1)/2 \rceil$.

Proof. First we argue that $N_5(d) \geq \lfloor 3(d - 1)/2 \rfloor$ for $d \geq 5$. For $d$ odd this immediately follows by setting $a = (d - 1)/2$, $S = J_a - I_a$, and $b = 3$ in Proposition 6.4. For $d$ even consider the graph $\Gamma$ on $n = 3(d - 2)/2 + 1 = \lfloor 3(d - 1)/2 \rfloor$ vertices formed by $(d - 2)/2$ disjoint triangles and an additional isolated node. By Lemma 6.5 the smallest eigenvalue of the corresponding Seidel matrix $S$ is $-5$ of multiplicity $(d - 2)/2 - 1$ and hence we have the desired configuration. Note that this construction is only interesting for $d \geq 186$.

Secondly we show that $N_5(d) \leq \lfloor 3(d - 1)/2 \rfloor$ for $d \geq \lceil (2V + 5)/3 \rceil$. Suppose that $S$ is a Seidel matrix of the largest set of equiangular lines in $\mathbb{R}^d$ with common angle $1/5$. We estimate $n - d$, which is the multiplicity of the $-5$ eigenvalue of $S$ through the ambient graph $\Gamma$ as follows. From the first part and from the assumption on $d$ we find that $n \geq \lfloor 3(d - 1)/2 \rfloor \geq V$. Therefore Theorem 6.3 applies and we may assume that the ambient graph $\Gamma$ with adjacency matrix $A = (J - S - I)/2$ is a Dynkin graph. By applying Lemma 6.5 it follows that $A$ has largest eigenvalue at least 2 of multiplicity $m = n - d + 1$. Furthermore, since $m \leq \lfloor n/3 \rfloor$, we have $n - d \leq \lfloor n/3 \rfloor - 1$ and we obtain $n \leq \lfloor 3(d - 1)/2 \rfloor$ as claimed.

We conclude this section with an updated table containing bounds on $N_5(d)$.

**Theorem 6.7.** Bounds for the maximum number of equiangular lines in $\mathbb{R}^d$ with common angle $\alpha = 1/5$ is given in Table 7 below.

| $d$ | $2-4$ | $5$ | $6$ | $7$ | $8$ | $9$ | $10$ | $11$ | $12$ | $13$ | $14$ | $15$ | $16$ | $17$ | $18$ | $19$ | $20$ |
|-----|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $N_5(d)$ | $d$ | $6$ | $7$ | $9$ | $10$ | $12$ | $16$ | $18$ | $20-22$ | $26$ | $28-29$ | $36$ | $40-41$ | $48-50$ | $48-61$ | $72-76$ | $90-96$ |
\[
\begin{array}{cccccccc}
d & 21 & 22 & 23-60 & 61-136 & 137-185 & 186-\left[(2V + 5)/3\right] - 1 & \left[(2V + 5)/3\right] - 1 \\
N_5(d) & 126 & 176 & 276 & 276-B(d) & 276-d(d + 1)/2 & [3(d - 1)/2]-V & [3(d - 1)/2] \\
\end{array}
\]

Table 4. Bounds for the maximum number of equiangular lines with \(\alpha = 1/5\).

**Proof.** We compare upper bounds given by Proposition 2.3 with lower bounds arising from direct constructions. For \(d \leq 8\) we have \(n \leq 12\) and we infer the result from the full classification of Seidel matrices (see Section 4). For \(d = 9\) we have \(n \leq 13\), but \(n = 13\) lines could have only been obtained by extending one of the four Seidel matrices of order 12 with smallest eigenvalue \(-5\) of multiplicity 3. This is shown to be impossible by a simple computer search. For \(d = 10\) there is equality in the relative bound and 16 lines can be obtained from the symmetric Hadamard matrix \(H = (J_4 - 2I_4) \otimes (J_4 - 2I_4)\) after removing its diagonal. The Seidel matrix \(H - I_{16}\) can be extended, via an easy computer search, with further two lines to obtain 18 equiangular lines in \(\mathbb{R}^{11}\) (see Remark 4.6). For \(d = 13\) we have four inequivalent conference graphs \([30\), \(37\), and an application of Theorem 5.19 shows the existence of 20 lines in \(\mathbb{R}^{12}\). Finally, cases \(14 \leq d \leq 23\) agree with the values of Table 4; the bounds \(B(d)\), regarding cases \(24 \leq d \leq 136\) are discussed in a forthcoming paper \([3]\); and the remaining values follow from Corollary 6.6.

Finally, we remark that it would be nice to see a combinatorial interpretation of Seidel matrices with three distinct eigenvalues. Such new perspective might shed some light on the existence of the hypothetical Seidel matrices highlighted in the appendix. This will hopefully lead to further improvements upon the best known bounds on the number of equiangular lines in small dimensions.

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Appendix A. A supplementary table

Here we display a list of feasible spectra for Seidel matrices whose existence would lead to an attainment or improvement upon the known number of equiangular lines in dimensions 14 and 16-20. The table was compiled using Theorem 5.6, Corollary [5.5] and Corollary [5.3].

| n  | d  | λ    | µ    | ν    | Existence | Remark       |
|----|----|------|------|------|-----------|--------------|
| 28 | 14 | $-5^1$ | $[3]^7$ | $[7]^7$ | Y         | Example 5.20 |
| 30 | 14 | $-5^1$ | $[5]^9$ | $[7]^7$ | N         | Theorem 5.27 |
| 40 | 16 | $-5^1$ | $[5]^6$ | $[9]^10$ | ?         |              |
| 40 | 16 | $-5^1$ | $[7]^5$ | $[15]^1$ | Y         | Example 5.12 |
| 42 | 16 | $-5^1$ | $[7]^7$ | $[9]^9$ | N         | Theorem 5.27 |
| 48 | 17 | $-5^1$ | $[7]^8$ | $[11]^9$ | Y         | Example 5.21 |
| 49 | 17 | $-5^1$ | $[7]^8$ | $[16]^1$ | ?         |              |
| 48 | 18 | $-5^1$ | $[3]^9$ | $[11]^12$ | ?        |              |
| 48 | 18 | $-5^1$ | $[7]^7$ | $[19]^2$ | ?         |              |
| 54 | 18 | $-5^1$ | $[7]^9$ | $[16]^1$ | ?         |              |
| 60 | 18 | $-5^1$ | $[11]^5$ | $[15]^3$ | ?         | Remark 5.23  |
| 72 | 19 | $-5^1$ | $[11]^6$ | $[19]^3$ | Y         | Example 5.22 |
| 75 | 19 | $-5^1$ | $[10]^4$ | $[15]^8$ | ?         | Lemma 5.15   |
| 90 | 20 | $-5^1$ | $[13]^5$ | $[19]^5$ | ?         |              |
| 95 | 20 | $-5^1$ | $[14]^1$ | $[19]^9$ | ?         | Lemma 5.15   |

Table 5. Feasible parameters of Seidel matrices with exactly three distinct eigenvalues.

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