ON ARITHMETIC OF ONE CLASS OF PLANE MAPS

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ABSTRACT. We study bipartite maps on the plane with one infinite face and one face of perimeter 2. At first we consider the problem of their enumeration and then the connection between the combinatorial structure of a map and the degree of its definition field. The second problem is considered when the number of edges is $p + 1$, where $p$ is prime.

1. INTRODUCTION

We will study connected maps in the plane with two faces — one infinite, another — of perimeter 2. Such maps can be obtained from a plane trees by doubling one edge. For example,

$$
\begin{array}{c}
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r}
\end{array}
\Rightarrow
\begin{array}{c}
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r}
\end{array}
$$

All our maps will be bipartite: vertices are white and black and adjacent vertices have different colors. The map above generates two bipartite maps:

$$
\begin{array}{c}
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r}
\end{array}
\text{ and }
\begin{array}{c}
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r} \\
\text{r} \quad \text{r}
\end{array}
$$

In what follows such maps will be called $2^1$-maps, i.e. maps with one face of perimeter 2 (and one infinite face).

Let $D$ be a $2^1$-map and $k_1, k_2, \ldots (l_1, l_2, \ldots)$ be numbers of its white (black) vertices of degrees 1, 2, and so on. The passport of the map $D$ is an expression

$$
a_1^{k_1}a_2^{k_2} \cdots b_1^{l_1}b_2^{l_2} \cdots ,
$$

where $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are formal variables. Thus, maps above have passports

$$
a_1a_3^2b_1^2b_2b_3 \text{ and } a_1^2a_2a_3b_1b_3^2.
$$

In Section 2 we will construct a generating function for $2^1$-maps in terms of passports.

Beginning from Section 3 we will assume that $2^1$-map $D$ has $p + 1$ edges, where $p$ is prime.

Remark 1.1. We follow the work [6], where arithmetic of plane bipartite trees with prime number of edges is studied.
Let $\beta$ be a Belyi function for $D, K$ — its big field of definition ($K$ contains coordinates of all vertices), $\rho$ — a prime divisor in $K$, that divides $p, v_\rho$ — the corresponding valuation and $I \subset K$ — the ideal of elements with positive $v_\rho$ valuation. Function $\beta$ is called a normalized model for $2^1$-map $D$, if

- some white vertex of degree $> 1$ is at 0 and some black vertex of degree $> 1$ is at 1;
- coordinates of all white vertices, except, maybe, one of degree 1, belong to $I$;
- coordinates of all black vertices, except, maybe, one of degree 1, belong to the set $1 + I$.

If besides the value of $\beta$ in white vertices is 0 and in black — 1, then $\beta$ will be called a totally normalized model.

**Remark 1.2.** In work [6] a Shabat $p$ polynomial for a tree $T$ is called normalized, if

- some white vertex is in zero and some black — in 1;
- coordinates of all white vertices belong to $I$;
- coordinates of all black vertices belong to the set $1 + I$.

In Section 3 the existence theorem (Theorem 3.1) is proved: each $2^1$-map with $p + 1$ edges has a normalized model.

Arithmetic of normalized model is investigated in Section 5. Let a $2^1$-map $D$ has $n + 1$ white vertices $v_0, \ldots, v_n$ with coordinates $x_0, \ldots, x_n$ and $m + 1$ black vertices $u_0, \ldots, u_m$ with coordinates $y_0, \ldots, y_m$. Let $e = v_\rho(p)$ be the ramification index. The following statement holds (Theorem 5.1): let $\beta$ be a normalized model, then there are three cases.

- Coordinate $x_s$ of one white vertex $v_s$ of degree 1 does not belong to $I$. In this case $v_\rho(x_i) = e/(n - 1), i \neq s$, and $v_\rho(x_i - x_j) = e/(n - 1), i, j \neq s$.
- $v_\rho(x_i) = e/(n + 1)$ for all $i$ and $v_\rho(x_i - x_j) = e/(n + 1)$ for all $i \neq j$.
- $v_\rho(x_i) = l > 0$ for all white vertices, except one $v_s$ of degree 1. Here $v_\rho(x_s) = k > 0, k < l, l = (e - 2k)/(n - 1)$, and $v_\rho(x_i - x_j) = l$ for $i, j \neq s$.

An analogous result is valid for black vertices.

**Remark 1.3.** In work [6] the result is as follows: if a tree $T$ with prime number of edges has $n + 1$ white vertices and $p$ is its normalized polynomial, then

- $v_\rho(x_i) = e/n$ for all $i$;
- $v_\rho(x_i - x_j) = e/n$ for all $i, j, i \neq j$.

**Corollary.** If vertices $v_i$ and $v_j$ both have degree $> 1$ then

$$v_\rho(x_i - x_j) = e/(n - 1) \text{ or } e/(n + 1) \text{ or } (e - 2k)/(n - 1) > k.$$  

In Section 6 (Theorem 6.1) we prove the existence and the uniqueness of the canonical model, i.e. Belyi function $\beta$ with the following properties:

- the value of $\beta$ in white vertices is 0 and in black — 1;
- the sum of coordinates of white vertices of degree $> 1$ is 0;
- the sum of coordinates of black vertices of degree $> 1$ is 1.

**Remark 1.4.** In [6] the canonical model is such Shabat polynomial that: the sum of coordinates of all white vertices is 0 and the sum of coordinates of all black vertices is 1.
We obtain the canonical model from a normalized model with the use of some coordinate change that preserves the big definition field and the following property: if vertices $v_i$ and $v_j$ both have degree $> 1$ then

$$v_\rho(x_i - x_j) = e/(n-1) \text{ or } e/(n+1) \text{ or } (e - 2k)/(n-1) > k.$$ 

The field of definition $L \subset K$ of a map $D$ coincides with the field of definition of its canonical model. Let $\tau$ be a prime divisor in $L$ that divide $p$ and is divided by $\rho$, $v_\tau$ be the corresponding valuation in $L$ and $e_\tau = v_\tau(p)$ — ramification index.

Let $k$ be the maximal degree of white vertices of map $D$ and $s_i, i = 1, \ldots, k$, — the number of white vertices of degree $i$. Let

$$s = \gcd(s_i, 2 \leq i < j \leq k, s_i(s_i - 1), 2 \leq i \leq k).$$

In our case (number of edges is $p + 1$) we can estimate (Theorem 7.1) the degree of field $L$:

- either $se_\tau/(n-1)$ is integer;
- or $se_\tau/(n+1)$ is integer;
- or $s(e_\tau - 2k)/(n-1)$ is integer and $> k$.

Remark 1.5. In [6]

$$s = \gcd(s_is_j, 1 \leq i < j \leq k, s_i(s_i - 1), 1 \leq i \leq k)$$

and $se_\tau/n$ is integer. Thus, Zapponi obtained better estimations on degree of definition field of a plane tree with prime number of edges.

Remark 1.6. We will need the following result from algebraic number theory (see [1]). Let $K$ be an algebraic field of degree $n$, $p$ — a prime number, $\rho_1, \ldots, \rho_m$ — all prime divisors in $K$, that divide $p$, i.e.

$$p = c \cdot \rho_1^{e_1} \cdot \ldots \cdot \rho_m^{e_m},$$

where $c$ is a unit and $e_i$ — ramification indices. Then

$$n = \sum_{i=1}^{m} c_i n_i,$$ (1.1)

where $n_i$ — inertia indices.

The case of passport $a_1^4a_2^3b_1b_2$ is considered in Example 7.1. We have five maps with this passport. Here $n + 1 = 6$ and $s = 2$. Thus, either $e_\tau$ is even or $e_\tau$ is divisible by 3. As degree of definition field is $\leq 5$, we have two possibilities: either we have two Galois orbits of cardinality 2 and 3, or number 7 is the product of two prime divisors $7 = c\rho_1^2\rho_2^1$ (c is a unit). Actually, the first possibility holds.

2. Enumeration of $2^1$ maps

A $2^1$-map can be obtained by joining a white vertex of some tree $T_1$ with black vertex of another tree $T_2$ by a double edge:

At first we will explain how plane bipartite trees were enumerated in the work [5].
Let $M$ be the set of all plane bipartite trees with the given passport $P = a_1^{k_1}a_2^{k_2} \cdots b_1^{l_1}b_2^{l_2} \cdots$. Here $n = k_1 + k_2 + \cdots$ and $m = l_1 + l_2 + \cdots$ are numbers of white and black vertices, respectively. Let $\#\text{Aut}(T)$ be the order of the group of automorphisms of plane bipartite tree $T$, then

$$\sum_{T \in M} \frac{1}{\#\text{Aut}(T)} = \frac{(n-1)! (m-1)!}{\prod_i (k_i)! \prod_j (l_j)!}.$$ 

Example 2.1. There are two trees with passport $P = a_2^2 a_4 b_2^2 b_2^2$:

```
\[ \bullet \rightarrow \bullet \quad \text{and} \quad \bullet \rightarrow \bullet \rightarrow \bullet \]
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The group of automorphisms of the left tree has order 2, the group of automorphisms of the right tree is trivial. Thus

$$1 + \frac{1}{2} = \frac{3}{2} = \frac{2! \cdot 3!}{2! \cdot 2! \cdot 2!}.$$ 

Let $K$ be the set of infinite sequences of nonnegative integers $k = (k_1, k_2, \ldots)$ such that only finite number of $k_i$ are positive. Let $|k| = \sum_{i=1}^{\infty} k_i$, $||k|| = \sum_{i=1}^{\infty} i \cdot k_i$, $k! = \prod_{i=1}^{\infty} (k_i)!$ and $a^k = \prod_{i=1}^{\infty} a_i^{k_i}$, where $a_i$ are formal variables. Let us consider series

$$A(a, t, x) = \sum_{k \in K} \frac{(|k| - 1)! a^k}{k!} \cdot t^{||k||} x^{||k||} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot (t \cdot x \cdot a_1 + t \cdot x^2 \cdot a_2 + t \cdot x^3 \cdot a_3 + \ldots)^n$$

and analogously defined series

$$B(b, t, y) = \sum_{l \in K} \frac{(||l|| - 1)! b^l}{l!} \cdot t^{||l||} y^{||l||} = \sum_{m=1}^{\infty} \frac{1}{m} \cdot (t \cdot y \cdot b_1 + t \cdot y^2 \cdot b_2 + t \cdot y^3 \cdot b_3 + \ldots)^m.$$ 

Now we can construct the generating function $T(a, b, t, x)$ for the set of plane bipartite trees: from the product $A(a, t, x) \cdot B(b, t, y)$ we delete those terms, where $\deg(x) \neq \deg(y)$ or $\deg(x) \neq \deg(t) - 1$ and add terms $a_0$ and $b_0$ that correspond to trees with one vertex (white or black).

Let $D_a$ and $D_b$ be differential operators:

$$D_a = a_2 x \frac{\partial}{\partial a_0} + a_3 x \frac{\partial}{\partial a_1} + 2a_4 x \frac{\partial}{\partial a_2} + 3a_5 x \frac{\partial}{\partial a_3} + \ldots$$

and

$$D_b = b_2 x \frac{\partial}{\partial b_0} + b_3 x \frac{\partial}{\partial b_1} + 2b_4 x \frac{\partial}{\partial b_2} + 3b_5 x \frac{\partial}{\partial b_3} + \ldots$$

Theorem 2.1. Let $M(a, b, t, x)$ be the generating function of $2^1$-maps, then

$$M(a, b, t, x) = D_a(T) \cdot D_b(T).$$
Proof. We connect a white vertex \( v \), \( \text{deg}(v) = k \), of one tree with a black vertex \( u \), \( \text{deg}(u) = l \) of another by the double edge. There are \( kl \) ways to draw this double edge. This procedure increases the degree of \( v \) by 2 and the degree of \( u \) by two. It remains to note that the group of automorphisms of a \( 2^1 \)-map is trivial. 

Example 2.2. Terms of \( M(a, b, t, x) \) with \( \text{deg}(x) = \text{deg}(t) = 5 \) are as follows:

\[
(a_3b_2b_1^3 + a_2a_1b_5 + a_4a_1b_3b_1^2 + a_3a_2b_4b_1 + 2a_4a_1b_5b_1 + \\
+ 2a_2a_1b_4b_1 + 2a_3a_2b_3b_1^2 + 2a_3a_1b_3b_2 + a_3a_2b_2b_1 + a_2a_1b_3b_2)x^5t^5
\]

We see that there are fourteen \( 2^1 \)-maps with 5 edges. Here they are:

\[
\begin{align*}
& a_3b_2b_1^3, & a_4a_1b_3b_1^2, & a_2a_1b_5, & a_1a_4b_2b_3, & a_1a_3b_1b_4, \\
& a_1a_4b_1b_2^3, & a_1a_4b_1b_2^2, & a_1b_4b_1^2, & a_1a_3b_1b_4, \\
& a_2a_3b_1b_3, & a_2a_3b_2b_3, & a_2a_3b_2b_3, & a_1a_3b_2b_3. \\
\end{align*}
\]

3. Normalized Model

We will study \( 2^1 \)-maps with \( p + 1 \) edges, where \( p \) is a prime number. Let \( D \) be such map with \( n + 1 \) white vertices and \( m + 1 \) black (here \( n + m = p - 1 \)). Let \( \beta \) be a Belyi function for \( D \) such, that some white vertex of degree \( > 1 \) is in 0 and some black vertex in 1. Let \( v_0, v_1, \ldots, v_n \) be white vertices of degrees \( k_0, k_1, \ldots, k_n \), respectively, and let \( x_0, x_1, \ldots, x_n \) be their coordinates. Analogously, let \( u_0, u_1, \ldots, u_m \) be black vertices of degrees \( l_0, l_1, \ldots, l_m \), respectively, and let \( y_0, y_1, \ldots, y_m \) be their coordinates. Function \( \beta \) has a pole at point \( c \) inside the face of perimeter 2. We can assume that

\[
\beta = \prod_{i=0}^{n} (z - x_i)^{k_i} / (z - c).
\]
If $\beta(y_j) = r$, $j = 0, \ldots, m$, then
$$\beta - r = \frac{\prod_{j=0}^{m}(z - y_j)^{l_j}}{z - c}.$$ Let $K$ be the big field of definition of function $\beta$, i.e. $K$ contains coordinates of all vertices (and numbers $c$ and $r$). Let $\rho$ be a prime divisor in $K$ that divides $p$, $v_\rho$ be the corresponding valuation and $e$ — ramification index of $p$ with respect to $\rho$. Let $O$ be the ring of $\rho$-integral elements, i.e.
$$O = \{k \in K \mid v_\rho(k) \geq 0\}$$ and let $I = \{k \in K \mid v_\rho(k) > 0\}$ be the maximal ideal in $O$.
If
$$w = \min_{0 \leq j \leq m} v_\rho(y_j) \text{ and } v_\rho(y_i) = w,$$ then $w \leq 0$, because some black vertex is at 1.

Now let us make a change of variables: all coordinates we divide by $y_i$. We preserve the notation, but will remember that coordinates of all black vertices are $\rho$-integral (and some black vertex is at 1).

In equation
$$\prod_{i=0}^{n}(z - x_i)^{k_i} - r(z - c) = \prod_{j=0}^{m}(z - y_j)^{l_j}$$ (3.1)
the right-hand side is $\rho$-integral, hence, the left-hand side is also $\rho$-integral. But some white vertex of degree $> 1$ is at 0, so
$$\prod_{i=0}^{n}(z - x_i)^{k_i} = z^{p+1} + a_1z^p + a_2z^{p-1} + \ldots + a_{p-1}z^2$$ and coefficients $a_1, \ldots, a_{p-1}$ are $\rho$-integral. Hence, coordinates $x_0, \ldots, x_n$ are also $\rho$-integral.

Let us consider the numerator of derivative $\beta'$:
$$(p + 1)z^p + pa_1z^{p-1} + (p - 1)a_2z^{p-2} + \ldots + 2a_{p-1}z)(z - c) - (z^{p+1} + a_1z^p + a_2z^{p-1} + \ldots + a_{p-1}z^2) =$$
$$= p \prod_{i=0}^{n}(z - a_i)^{k_i - 1} \prod_{j=0}^{m}(z - y_j)^{l_j - 1}.$$ Coefficients of the polynomial in the right-hand side of (3.2) belong to $I$, hence, the same is true about the left side of (3.2). So, we have the following relations:
$$-(p + 1)c + (p - 1) a_1 \in I$$
$$-p c a_1 + (p - 2) a_2 \in I$$
$$-(p - 1) c a_2 + (p - 3) a_3 \in I$$
$$\ldots \ldots \ldots$$
$$-3c a_{p-2} + a_{p-1} \in I$$ (3.3)
The first relation gives us the $\rho$-integrality of $c$. The second — that $a_2 \in I$. The third — that $a_3 \in I$, and so on. We see that all coefficients of the polynomial $z^{p+1} + a_1z^p + \ldots + a_{p-1}z^2$ except, maybe $a_1$, belong to $I$. It means that coordinates of all white vertices except, maybe one of degree 1, belong to $I$. 

Now let us consider black vertices.

**Proposition 3.1.** There exists a black vertex of degree > 1 such, that its coordinate does not belong to I.

The proof of this proposition will be given in Section 4.

The coordinate $y_s$ of some black vertex $u_s$ of degree > 1 does not belong to ideal $I$, i.e. $v_p(y_s) = 0$. Let us make a change of variable: all coordinates we divide by $y_s$. As before some white vertex of degree > 1 is at 0 and coordinates of all white vertices except, maybe one of degree 1, belong to $I$. But now some black vertex of degree > 1 is at 1.

Now the change $z := 1 - z$ and $\beta := \beta - r$ allows one to apply the above reasoning to black vertices also.

**Definition 3.1.** Belyi function $\beta$ of some $2^1$-map $D$ with $p + 1$ edges is called its *normalized model* (with respect to a prime divisor $p$), if:

- coordinates of all vertices (black and white) and coordinate of the pole are $\rho$-integral
- some white vertex of degree > 1 is at 0 and some black vertex of degree > 1 is at 1;
- coordinates of all white vertices except, maybe one of degree 1, belong to the ideal $I$;
- coordinates of all black vertices except, maybe one of degree 1, belong to the set $\{1 + I\}$.

$\beta$ is called *completely normalized*, if its critical values are 0 (value in white vertices) and 1 (value in black vertices).

**Theorem 3.1.** For each $2^1$-map with $p + 1$ edges there exists a normalized model.

4. The proof of Proposition 3.1.

*Proof.* Let us assume that there exists a $2^1$-map $D$ and its Belyi function $\beta$ such, that

- coordinates of all vertices are $\rho$-integral and the same is true for $c$;
- a white vertex of degree > 1 is at 0 and coordinates of all other white vertices (except, maybe one of degree 1) belong to ideal $I$;
- some black vertex is at 1 and coordinates of all black vertices of degree > 1 belong to $I$.

As coefficients of polynomial

$$\prod_{j=0}^{m} (z - y_j)^{l_j} = z^{p+1} + b_1 z^p + \ldots + b_{p-1} z^2 + b_p z + b_{p+1}$$

are equal to the corresponding coefficients $a_i$ of polynomial $\prod_i (z - x_i)^{k_i}$ for $i < p$, then $b_i \in I$ for $i > 1$ and coordinates of all black vertices (except one that is at 1) belong to $I$.

As $a_1 = b_1$, there exists a white vertex of degree 1 whose coordinate belongs to the set $\{1 + I\}$. Let $y_0 = 1$, $y_j \in I$ for $j > 0$, $x_0 - 1 \in I$ and $x_i \in I$ for $i > 0$. As $c + a_1 \in I$ (see (3.3)), then $c - 1 \in I$. 


Let

\[ k = \min\{\min_{i>0} v_p(x_i), \min_{j>0} v_p(y_j)\} > 0. \]

We can assume that \( v_p(x_s) = k, s > 0 \). Let us consider the function

\[ \beta_1 = x_s^{-p} \beta(x_s z) = x_s^{-p} \cdot \frac{\prod_{i=0}^{n}(x_s z - x_i)^{k_i}}{x_s z - c} = \frac{(x_s z - x_0) \prod_{i>0}(z - \bar{x}_i)^{k_i}}{x_s z - c}, \]

where \( \bar{x}_i = x_i / x_s \). Analogously, let \( \bar{y}_j = y_j / x_s \).

Let us find the derivative \( \beta_1' \):

\[
\begin{align*}
\beta_1' &= x_s^{-p} x_s' \beta'(x_s z) = x_s^{-p} x_s \frac{p \prod_{i=0}^{n}(x_s z - x_i)^{k_i-1} \prod_{j=0}^{m}(x_s z - y_j)^{l_j-1}}{(x_s z - c)^2} \\
&= x_s^2 p \prod_{i=0}^{n}(z - \bar{x}_i)^{k_i-1} \prod_{j=0}^{m}(z - \bar{y}_j)^{l_j-1} / (x_s z - c)^2
\end{align*}
\]

(because \( n + m = p - 1 \)). We see that all coefficients of the numerator belong to \( I \).

On the other hand, this numerator is equal to

\[ \left( (x_s z - x_0) \prod_{i=1}^{n} (z - \bar{x}_i)^{k_i} \right)' (x_s z - c) - x_s (x_s z - x_0) \prod_{i=1}^{n} (z - \bar{x}_i)^{k_i}. \quad (4.1) \]

As \( x_0, \bar{x}_s \) and \( c \) do not belong to \( I \), but \( x_s \in I \), then not all coefficients of polynomial (4.1) belong to the ideal \( I \). We have a contradiction. \( \square \)

5. **Arithmetic of normalized model.**

Let \( \beta \) be a normalized model.

5.1. **The first case: the coordinate of some white vertex of degree 1 does not belong to \( I \).** Then \( c \notin I \) and coordinate of some black vertex of degree 1 does not belong to the set \{1 + \( I \)\}. Let us assume that \( x_0 \notin I \) and \( y_0 - 1 \notin I \).

On one hand

\[
\text{numerator}(\beta') = \prod_{i=0}^{n} (z - x_i)^{k_i} \left( \sum_{i=0}^{n} \frac{k_i}{z - x_i} \right) (z - c) - \prod_{i=0}^{n} (z - x_i)^{k_i}. 
\]

On the other hand

\[
\text{numerator}(\beta') = p \prod_{i=0}^{n} (z - x_i)^{k_i-1} \prod_{j=0}^{m} (z - y_j)^{l_j-1}. 
\]

Thus,

\[
\prod_{i=0}^{n} (z - x_i) \cdot \left( \sum_{i=0}^{n} \frac{k_i}{z - x_i} \right) (z - c) - \prod_{i=0}^{n} (z - x_i) = p \prod_{j=0}^{m} (z - y_j)^{l_j-1}. \quad (5.1)
\]

Let the vertex \( v_1 \) be at zero, i.e. \( x_1 = 0 \). Then

\[
k_1 x_0 \prod_{i>1} x_i = \pm p \prod_{j=0}^{m} y_j^{l_j-1} \Rightarrow \sum_{i=2}^{n} v_p(x_i) = e \Rightarrow (n - 1) k \leq e, \quad (5.2)
\]

where \( k = \min_{i>1} v_p(x_i) \).
Now, as in previous section, we will work with the function
\[ \beta_1 = x_s^{-p} \beta(x_s z) = x_s^{-p} \frac{(x_s z - x_0) \prod_{i=1}^{n} (x_s z - x_i)^{k_i}}{x_s z - c} = (x_s z - x_0) \frac{\prod_{i=1}^{n} (z - \tilde{x}_i)^{k_i}}{x_s z - c}, \]
where \( v_\rho(x_s) = k \) and \( \tilde{x}_i = x_i/x_s \).

As in the previous section we deduce, that not all coefficients of the numerator of derivative
\[ \beta'_1(z) = x_s^{1-p} \beta'(x_s z) = x_s^{1-p} \frac{p \prod_{i=1}^{n} (x_s z - x_i)^{k_i-1} \prod_{j=0}^{m} (x_s z - y_j)^{l_j-1}}{(x_s z - c)^2} = x_s^{1-n} \frac{p \prod_{i=1}^{n} (z - \tilde{x}_i)^{k_i-1} \prod_{j=0}^{m} (x_s z - y_j)^{l_j-1}}{(x_s z - c)^2} \]
belong to \( I \). Thus, \((n-1)k \geq e\). Taking (5.2) into account we have that
\[ v_\rho(x_i) = \frac{e}{n-1} \text{ for } i > 0 \text{ and } x_i \neq 0. \] (5.3)
The substitution \( x_j \) instead of \( z \) in (5.1), where \( j > 0, x_j \neq 0 \), and subsequent computing valuations gives the equality
\[ \sum_{i>0,i \neq j} v_\rho(x_i - x_j) = e. \] (5.4)

5.2. The second case: all \( x_i \in I \). In this case \( c \in I \) and coordinates of all black vertices belong to the set \( \{1+i\} \), except one of degree 1 — coordinate of this vertex belong to \( I \). Let \( k = \min v_\rho(x_i) \) and let
\[ J = \{i, 0 \leq i \leq n \mid v_\rho(x_i) = k\}. \]

5.2.1. Either the cardinality of \( J \) is \( > 1 \), or \( J = \{l\} \) and degree of \( v_l \) is \( > 1 \). Let \( x_0 = 0 \). After the substitution \( z = 0 \) in (5.1) and computing valuations we get the equality
\[ \sum_{i=1}^{n} v_\rho(x_i) + v_\rho(c) = e. \] (5.5)
Thus, \( v_\rho(c) < e \) and \( v_\rho(x_i) < e \) for \( i = 1, \ldots, n \). As
\[ v_\rho(-(p+1)c + (p-1)a_1) \geq e \]
(see (3.2)), then \( v_\rho(c) \geq k \) and from (5.5) we get the relation \((n+1)k \leq e\).

We will work with function
\[ \beta_1 = x_s^{-p} \beta(x_s z) = \frac{\prod_{i=1}^{n} (z - \tilde{x}_i)^{k_i}}{z - c}, \]
where, as above, \( v_\rho(x_s) = k, \tilde{x}_i = x_i/x_s \) and \( c = c/x_s \). Let
\[ \prod_{i=0}^{n} (z - \tilde{x}_i)^{k_i} = z^{p+1} + \tilde{a}_1 z^p + \tilde{a}_2 z^{p-1} + \ldots \]
Theorem 5.1. Let us summarize.

There exists $j > 1$ such, that $v_p(\tilde{a}_j) = 0$, but $v_p(\tilde{a}_i) > 0$ for $i > j$. Now

\[
\text{numerator}(\beta'_j) = \ldots + \left[-(p-j+2)\tilde{a}_{j-1}c + (p-j)\tilde{a}_j\right] z^{p-j+1} + \left[-(p-j+1)\tilde{a}_j c + (p-j-1)\tilde{a}_{j+1}\right] z^{p-j} \ldots
\]

If $c \in I$, then coefficient at $z^{p-j+1}$ does not belong to $I$. If $c \notin I$, then coefficient at $z^{p-j}$ does not belong to $I$.

On the other hand

\[
\beta'_1 = x_s^{-(n+1)} \frac{p \prod_{i=0}^{n}(z - \tilde{x}_i)k_i^{-1} \prod_{j=0}^{m}(x_s z - y_j)l_j^{-1}}{(z-c)^2}.
\]

As not all coefficients of numerator belong to $I$, then $e \leq (n+1)k$. Hence,

\[
v_p(x_1) = \ldots = v_p(x_n) = v_p(c) = \frac{e}{n+1} \quad \text{and} \quad v_p(x_i - x_j) = \frac{e}{n+1} \quad \text{for} \quad i \neq j. \quad (5.6)
\]

5.2.2. The degree of $v_0$ is 1, $v_p(x_0) = k$ and $v_p(x_i) > k$, if $i > 0$. Then $v_p(c) = k$.

Let $\min_{i>0} v_p(x_i) = l > k$. As

\[
\sum_{x_i \neq 0} v_p(x_i) + v_p(c) = e, \quad \text{then} \quad 2k + (n-1)l \leq e.
\]

The same computations with the function $\beta_1 = x_s^{-p} \beta(x_s z)$, where $v_p(x_s) = l$, give us the relation $2k + (n-1)l \geq e$, i.e.

\[
v_p(x_i) = \frac{e - 2k}{n-1} \quad \text{for} \quad i > 0 \quad \text{and} \quad x_i \neq 0 \quad \text{and} \quad v_p(x_i - x_j) = \frac{e - 2k}{n-1} \quad \text{for} \quad i, j > 0. \quad (5.7)
\]

Let us summarize.

**Theorem 5.1.** Let $\beta$ be a normalized model of some $21^\text{-map}$ with $p + 1$ edges, $n + 1$ white vertices and $m + 1$ black ones. Let $\rho$ be a prime divisor in the big definition field $K$, that divides $p$, $\nu_\rho$ be the corresponding valuation and

\[ I = \{ k \in K \mid v_\rho(k) > 0 \} \subset K. \]

Let $e = v_\rho(p)$ be the ramification index. There are three options.

- $v_\rho(x_i) = e/(n-1)$ for all white vertices, except one $v_s$ of degree 1 and $x_s \notin I$. Moreover, $v_\rho(x_s - x_j) = e/(n-1)$ for $i, j \neq s$.
- $v_\rho(x_i) = e/(n+1)$ for all $i$, $v_\rho(x_i - x_j) = e/(n+1)$ for $i, j$.
- $x_i \in I$ for all $I$ and $v_\rho(x_i) = l$ for all white vertices, except one $v_s$ of degree 1 for which $v_\rho(x_s) = k < l$. Moreover $l = (e - 2k)/(n-1)$ and $v_\rho(x_i - x_j) = l$ for all $i, j \neq s$.

**Remark 5.1.** Analogous results are valid for black vertices.
6. Canonical model

Definition 6.1. A Belyi function $\beta$ of some 2^{1}\text{-map} is called its canonical model if

- the value of $\beta$ is 0 in white vertices and 1 — in black;
- sum of coordinates of white vertices of degree $> 1$ is 0;
- sum of coordinates of black vertices of degree $> 1$ is 1.

Theorem 6.1. For each 2^{1}\text{-map} with $p + 1$ edges there exists the unique canonical model.

Proof. Let $\beta$ be a normalized model of our map with the big definition field $K$, $X$ be the sum of coordinates of white vertices of degree $> 1$ and $Y$ be the sum of black vertices of degree $> 1$. Then $X \in I$, but $Y \notin I$. Indeed, otherwise the number of black vertices of degree $> 1$ is $p$ and the number of edges is not less than $2p$.

For a new variable $z' = az + b$ these sums are $X' = aX + bk$ and $Y' = aY + bl$, where $k$ is the number of white vertices of degree $> 1$ and $l$ is the corresponding number of black vertices. We have to find numbers $a$ and $b$ such, that

\[
\begin{cases}
  aX + bk = 0 \\
  aY + bl = 1
\end{cases}
\]

As $Xl - Yk$ — the determinant of the system, is $\notin I$ (and, thus, $\neq 0$), then the system has the unique solution and

\[
a = -\frac{k}{Xl - Yk}, \quad b = \frac{X}{Xl - Yk}.
\]

We see, that $a$ is $\rho$-integral, but $a \notin I$. Let $\bar{x}_i = ax_i + b, \ i = 0, \ldots, n, \bar{y}_j = ay_j + b, \ j = 0, \ldots, m$, and $\bar{c} = ac + b$. As $\bar{x}_i - \bar{x}_j = a(x_i - x_j)$, then

\[
v_p(\bar{x}_i - \bar{x}_j) = v_p(x_i - x_j).
\]

Let $L$ be the field of definition of polynomials

\[
P = \prod_{i=0}^{n}(z - \bar{x}_i)^{k_i} \text{ and } Q = \prod_{j=0}^{m}(z - \bar{y}_j)^{l_j}.
\]

As $P - r(z - \bar{c}) = Q$, then $r \in L$ and $\bar{c} \in L$. Hence, the function $\bar{\beta} = P/r(z - \bar{c})$ will be the canonical model. \hfill \square

Remark 6.1. The uniqueness of canonical model means that its field of definition coincides with the definition field of the map

7. Geometrical ramification

Let $D$ be a 2^{1}\text{-map}, $k$ — the maximal degree of its white vertices and $s_i, i = 2, \ldots, k$, — the number of white vertices of degree $i$. Let

\[
s = \gcd(s_is_j, 2 \leq i < j \leq k, s_i(s_i - 1), 2 \leq i \leq k).
\]

There exists integers $c_i$ and $c_{ij}$ such, that

\[
S = \sum_{i} c_is_i(s_i - 1) + \sum_{i<j} c_{ij}s_is_j.
\]
Let $\beta$ be the canonical model of a map $D$, $v_0,\ldots,v_n$ — white vertices, $x_0,\ldots,x_n$ — their coordinates and $d_0,\ldots,d_n$ — their degrees. Function $\beta$ is defined over the field $L$. Elements 

$$t_i = \prod_{i_1 \neq i_2, \ d_{i_1} = d_{i_2} = i} (x_{i_1} - x_{i_2}), \ i > 1,$$

and 

$$t_{ij} = \prod_{i_1, i_2, d_{i_1} = i, d_{i_2} = j} (x_{i_1} - x_{i_2}), \ 1 < i < j$$

belong to $L$.

Let $\tau$ be a prime divisor of the field $L$ that divides $p$ and is divided by $\rho$, $v_\tau$ be the corresponding valuation and $e_\tau$ be the ramification index in the field $L$: $e_\tau = v_\tau(p)$. For each $u \in L$ we have that 

$$v_\rho(u) = v_\rho(\tau)v_\tau(u), \ \text{thus,} \ e = v_\rho(\tau)e_\tau$$

(7.1)

Let us consider the element 

$$t = \prod_i t_i^{e_i} \prod_{i < j} t_{ij}^{e_{ij}} \in L.$$

According to Theorem 5.1, 

$$v_\rho(t) = \begin{cases} 
\text{either } se/(n - 1), \\
\text{or } se/(n + 1), \\
\text{or } s(e - 2k)/(n - 1) > k.
\end{cases}$$

Thus, we have the following statement.

**Theorem 7.1.**  

- Either $se_\tau/(n - 1)$ is integer;  
- or $se_\tau/(n + 1)$ is integer;  
- or $s(e - 2k)/(n - 1)$ is integer and $> k$.

The same can be said about black vertices.

**Example 7.1.** Let us consider $2^1$-maps with passport $a_1^4a_2^3b_1b_7$. The number of edges is $8 = 7 + 1$. There are 5 such maps:

```
 to the right side of the map we must add a "tail" .
```

Here $n + 1 = 6$ and $s = 2$, hence, either $e_\tau$ is even, or $e_\tau$ is divisible by 3. Thus (see (1.1)), either we have two Galois orbits (one of cardinality 2 and another of cardinality 3), or 7 is a product of two prime divisors $\rho_1$ and $\rho_2$ with ramifications 2 and 3, respectively, i.e. $7 = \rho_1^2\rho_2^3$. (The first case is realized.)

**Example 7.2.** Let us consider $2^1$-maps with passport $a_1^3a_2a_3b_1^2b_6$. The number of edges is $8 = 7 + 1$. There are 8 such maps:

```
``
Here $n + 1 = 5$ and $s = 1$, hence, either $e_\tau$ is divisible by 3, or — by 5. Thus (see (1.1)), either we have two Galois orbits (one of cardinality 3 and another of cardinality 5), or $7$ is a product of two prime divisors $\rho_1$ and $\rho_2$ with ramifications 3 and 5, respectively, i.e. $7 = \rho_1^3 \rho_2^5$. (The second case is realized.)

In normalized model, where white vertex of degree 3 is at zero, black vertex of degree 6 is at 1 and white vertex of degree 2 is at point $a$ we have the following equation on $a$:

$$15a^8 - 7 \cdot 24a^7 + 7 \cdot 121a^6 - 7 \cdot 360a^5 + 7^2 \cdot 99a^4 - 7 \cdot 880a^3 + 7^2 \cdot 102a^2 - 7^2 \cdot 48a + 7^2 = 0.$$ 

Let $x = v_\rho(a)$, then in this expression there are 3 terms that may have the minimal valuation: $15a^8$ with valuation $8x$, $7 \cdot 880a^3$ with valuation $3x + e$ and 49 with valuation $2e$. At least two terms must have the minimal valuation.

- $8x = 3x + e$ minimal $\Rightarrow x = e/5$, $2e > 3x + e$, $e = 5$.
- $8x = 2e$ minimal $\Rightarrow x = e/4$, $3x + e < 2e$, contradiction.
- $3x + e = 2e$ minimal $\Rightarrow x = e/3$, $8x > 2e$, $e = 3$.

Thus, it again follows that $7 = \rho_1^3 \rho_2^5$.

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**Supplement**

The difference between normalized and canonical models can be illustrated on example of their Julia sets (see [2] or [4]). Let us consider two maps

with passport $a_2^2a_4b_1b_2b_3$.

In normalized model white vertex of degree 4 is at 0, black vertex of degree 3 is at 1 and 0 and 1 are critical values. Then here are two attracting fixed points — 0 and 1. In canonical model there is only one attracting fixed point — 0. Let $O$ be the union of small neighborhoods of attracting fixed points and a neighborhood of
infinity. Points that come into $O$ in $\leq 5$ steps are white, in 6 or 7 steps — yellow, in 8 or 9 steps — red. Other points (among them points of Julia set) are blue.

The first map

![Normalized model](image1)

![Canonical model](image2)

The second map

![Normalized model](image3)

![Canonical model](image4)

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