Introduction to White Noise, Hida-Malliavin Calculus and Applications

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Abstract

This paper is based on lectures given by one of us (N.Agram) at the CIMPA Research School on Stochastic Analysis and Applications at Saida, Algeria, 1-9 March 2019. The purpose of the paper is threefold:

• We first give a short and simple survey of the Hida white noise calculus, and in this context we introduce the Hida-Malliavin derivative as a stochastic gradient with values in the Hida stochastic distribution space $(S)^*$. We show that this Hida-Malliavin derivative defined on $L^2(\mathcal{F}_T, P)$ is a natural extension of the classical Malliavin derivative defined on the subspace $\mathcal{D}_{1,2}$ of $L^2(P)$.

• Second, the Hida-Malliavin calculus allows us to prove new results under weaker assumptions than could be obtained by the classical theory. In particular, we prove the following:
  (i) A general integration by parts formula and duality theorem for Skorohod integrals,
  (ii) a generalised fundamental theorem of stochastic calculus, and
  (iii) a general Clark-Ocone theorem, valid for all $F \in L^2(\mathcal{F}_T, P)$.

Thirdly, we present applications of the above theory. For example, we discuss the following:

• A general representation theorem for backward stochastic differential equations with jumps, in terms of Hida-Malliavin derivatives,

• a general stochastic maximum principle for optimal control,

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• backward stochastic Volterra integral equations,
• optimal control of stochastic Volterra integral equations and other stochastic systems.

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## 1 White noise theory and Hida-Malliavin calculus with applications - and without tears ...

The purpose of these lectures is to give a short and easy introduction to the Hida white noise theory and the associated Hida-Malliavin calculus. This theory is important for many applications and we believe it deserves to be better known. The problem has been that most of the literature in this area has been too general and formidable for the reader who just wants to know enough of the basic features of the theory in order to be able to apply it to his or her area of research. These lectures aim to fill that gap in the literature. Moreover, they intend to convince the reader that there are indeed a lot of applications of this theory, and to explain where and how.

This is basically a survey paper. Most of the text in these lectures are taken from other published sources referred to in the reference list. However, some of the proofs are new and shorter than the originals.

The stochastic calculus of variations, now also know as Malliavin calculus, was introduced by P. Malliavin [26] as a tool for studying the smoothness of densities of solutions of stochastic differential equations. Subsequently other applications of this theory was found. In [28] Ocone used Malliavin calculus to prove an explicit representation theorem for Brownian motion functionals and in a subsequent paper [29] Karatzas and Ocone applied this to study portfolio problems in finance.

The original presentation of Malliavin was quite complicated, but subsequently simpler constructions of this theory have been found. See e.g. [13] and the references therein. In particular, we think that the use of white noise theory makes the theory of Malliavin calculus (in this context also known as Hida-Malliavin calculus) quite natural within the context of directional derivatives and Fréchet derivatives on the space $S'$ of tempered distributions, both in the Brownian motion case and in the case of Poisson random measure. See definitions below.

A major advantage of presenting the Malliavin calculus in the context of white noise theory is that the corresponding Hida-Malliavin derivative can be extended from the subspace $D_{1,2}$ to all of $L^2(P)$. This enables us to prove stronger results compared with what would be
possible in the classical setting. In particular, we will prove
(i) A general integration by parts formula and duality theorem for Skorohod integrals,
(ii) a generalised fundamental theorem of stochastic calculus,
(iii) a general Clark-Ocone theorem, valid for all $F \in L^2(F_T, P)$,
(iv) a general representation for solutions of backward stochastic differential equations (BS-
DEs) with jumps, in terms of Hida-Malliavin derivatives, and (v) applications to stochastic
control.

2 White noise theory for Brownian motion

In this section we give a short introduction to the Hida white noise calculus. A general
reference for this section is [16]. See also [15].

2.1 The white noise probability space

We start with the construction of the white noise probability space. Let $S = S(\mathbb{R}^d)$ be
the Schwartz space of rapidly decreasing smooth $C^\infty(\mathbb{R}^d)$ real functions on $\mathbb{R}^d$. The space
$S = S(\mathbb{R}^d)$ is a Fréchet space with respect to the family of seminorms:

$$
\|f\|_{K,\alpha} := \sup_{x \in \mathbb{R}^d} \{(1 + |x|^K)|\partial^\alpha f(x)|\},
$$

where $K = 0, 1, ..., \alpha = (\alpha_1, ..., \alpha_d)$ is a multi-index with $\alpha_j = 0, 1, ... (j = 1, ..., d)$ and

$$
\partial^\alpha f := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f
$$

for $|\alpha| = \alpha_1 + ... + \alpha_d$.

Let $S' = S'(\mathbb{R}^d)$ be its dual, called the space of tempered distributions. Let $B$ denote the family of all Borel subsets of $S'(\mathbb{R}^d)$ equipped with the weak* topology. If $\omega \in S'$ and $\phi \in S$ we let

$$
\omega(\phi) = \langle \omega, \phi \rangle
$$

denote the action of $\omega$ on $\phi$. For example, if $\omega = m$ is a measure on $\mathbb{R}^d$ then

$$
\langle \omega, \phi \rangle = \int_{\mathbb{R}^d} \phi(x) dm(x),
$$

and, in particular, if this measure $m$ is concentrated on $x_0 \in \mathbb{R}^d$, then

$$
\langle \omega, \phi \rangle = \phi(x_0)
$$

is the evaluation of $\phi$ at $x_0 \in \mathbb{R}^d$.

Other examples include

$$
\langle \omega, \phi \rangle = \phi'(x_1).
$$
i.e. \( \omega \) takes the derivative of \( \phi \) at a point \( x_1 \).

Or, more generally,

\[
\langle \omega, \phi \rangle = \phi^{(k)}(x_k),
\]

i.e. \( \omega \) takes the \( k \)'th derivative at the point \( x_k \), or linear combinations of the above.

From now on we consider only the 1-dimensional case, i.e. \( d = 1 \). For a multidimensional presentation see [16]. We fix the sample space to be \( \Omega = S'(\mathbb{R}) = S' \) and \( \mathcal{F} = \mathcal{B} \). In the following we will use the Bochner-Minlos-Sazonov theorem (see e.g. [16]), which in our setting states the following:

**Theorem 2.1 (Bochner-Minlos-Sazonov).** Let \( g : S \mapsto \mathbb{R} \) be given. Then there exists a probability measure \( \mu \) on \( \Omega = S'(\mathbb{R}) \) such that

\[
\mathbb{E}_\mu[e^{i\langle \omega, \phi \rangle}] := \int_{\Omega} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = g(\phi); \quad \text{for all } \phi \in S
\]

(2.2)

if and only if the function \( g \) satisfies the following 3 conditions:

(i) \( g(0) = 1 \)

(ii) \( g \) is continuous in the Fréchet topology on \( S \)

(iii) \( g \) is positive definite, i.e.

\[
\sum_{j, \ell} z_j \bar{z}_\ell g(\phi_j - \phi_\ell) \geq 0 \quad \text{for all } z_j \in \mathbb{C}, \phi_j \in S; \quad j = 1, 2, \ldots, n,
\]

(2.3)

where \( \mathbb{C} \) denotes the set of complex numbers.

In particular, if we choose

\[
g(\phi) = e^{-\frac{1}{2}||\phi||^2}; \quad \phi \in S,
\]

(2.4)

we can check that \( g \) satisfies the conditions (i) - (iii) in the above theorem, and hence we get that there exists a probability measure \( P \) on \( \Omega \) such that

\[
\mathbb{E}[e^{i\langle \omega, \phi \rangle}] := \int_{\Omega} e^{i\langle \omega, \phi \rangle} P(d\omega) = e^{-\frac{1}{2}||\phi||^2}, \quad \phi \in S,
\]

(2.5)

where

\[
||\phi||^2 = ||\phi||^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\phi(x)|^2 dx.
\]

The measure \( P \) is called the white noise probability measure and \( (\Omega, \mathcal{F}, P) = (S', \mathcal{B}, P) \) is called the white noise probability space.
Definition 2.2 The (smoothed) white noise process is the measurable map
\[ w : S \times S' \to \mathbb{R} \]
given by
\[ w(\phi, \omega) = w_\phi(\omega) = \langle \omega, \phi \rangle, \quad \phi \in S, \, \omega \in S'. \] (2.6)

From \( w_\phi \) we can construct a Brownian motion process \( B(t), t \in \mathbb{R} \), as follows:

Step 1 First we verify that the isometry
\[ \mathbb{E}[w_\phi^2] = \|\phi\|^2, \quad \phi \in S, \] (2.7)
holds true where, according to our notation, the left-hand side is
\[ \mathbb{E}[w_\phi^2] = \int_{S'} \langle \omega, \phi \rangle^2 P(d\omega). \]

Step 2 Next we use Step 1 to define the value \( \langle \omega, \psi \rangle \) for arbitrary \( \psi \in L^2(\mathbb{R}) \), as \( \langle \omega, \psi \rangle := \lim_{n \to \infty} \langle \omega, \phi_n \rangle \), where \( \phi_n \in S, n \in \mathbb{N} = \{1, 2, \ldots\} \), and \( \phi_n \to \psi \) in \( L^2(\mathbb{R}) \).

By Step 1 it follows that this definition does not depend on the choice of the approximating sequence \( \{\phi_n\}_{n \in \mathbb{N}} \).

Step 3 Using Step 2 we can define
\[ \tilde{B}(t, \omega) := \langle \omega, \chi_{[0,t]} \rangle, \quad t \in \mathbb{R}, \]
by choosing
\[ \psi(s) = \chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0, t) \text{ (or } s \in [t, 0), \text{ if } t < 0) \\ 0 & \text{otherwise} \end{cases} \]
which belongs to \( L^2(\mathbb{R}) \) for all \( t \in \mathbb{R} \).

Step 4 By the Kolmogorov continuity theorem (see [24]), we obtain that \( \tilde{B}(t), t \in \mathbb{R} \), has a continuous version \( B(t), t \in \mathbb{R} \), i.e. for all \( t \) we have \( P\{\tilde{B}(t) = B(t)\} = 1 \). This continuous process \( B(t), t \in \mathbb{R} \), is a Brownian motion (Wiener process).

Remark 2.3 Note that for each \( t \) we are defining \( \tilde{B}(t) = \tilde{B}(t, \omega) = \langle \omega, \chi_{[0,t]} \rangle \) by using a sequence of functions \( \phi^{(t)}_n \in S(\mathbb{R}) \) converging to \( \chi_{[0,t]} \) in \( L^2(\mathbb{R}) \). Hence \( \tilde{B}(t) \) is only defined almost everywhere on \( \Omega \), where the exceptional set of measure zero depends on \( t \). Since there are uncountably many \( t \in [0, \infty) \) there is no common set \( \Omega_0 \) of measure 0 in \( \Omega \) such that \( \tilde{B}(t, \omega) = \langle \omega, \chi_{[0,t]} \rangle \) is defined for all \( t \in [0, \infty) \) and for all \( \omega \in \Omega \setminus \Omega_0 \). Therefore we cannot prove the continuity of \( t \mapsto \tilde{B}(t, \omega) \) by arguing \( \omega \)-wise. But we can use the Kolmogorov continuity theorem to conclude that \( \tilde{B}(t, \omega) \) has a continuous version.
Note that when the Brownian motion process \( B(t, \omega), t \in \mathbb{R}, \omega \in \Omega \) with \( \Omega := \mathcal{S}'(\mathbb{R}) \) is constructed this way, then each \( \omega \in \Omega = \mathcal{S}'(\mathbb{R}) \) is a tempered distribution. Hence \( \Omega \) is a Fréchet space, i.e. a topological vector space with a topology given by a family of seminorms. This gives us a topological structure on \( \Omega \) which we will use frequently in the following.

From the above Step 2 it follows that the smoothed white noise \( w_\phi \) can be extended to all (deterministic) \( \phi \in L^2(\mathbb{R}) \) and that the relation between smoothed white noise \( w_\phi \) and the Brownian motion process \( B(t), t \in \mathbb{R} \), is

\[
w_\phi(\omega) = \int_{\mathbb{R}} \phi(t) dB(t, \omega), \quad \omega \in \Omega, \quad \phi \in L^2(\mathbb{R}),
\]

where the integral on the right-hand side is the Wiener-Itô integral. Note that the isometry (2.7) is then the classical Itô isometry.

### 2.2 The Wiener-Itô chaos expansion

We now present an orthogonal expansion of the space \( L^2(\mathcal{F}, P) \). This presentation will be useful for the extended Hida-Malliavin calculus we come to later.

The Hermite polynomials \( h_n(x) \) are defined by

\[
h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}) , \quad n = 0, 1, 2, \ldots
\]

The first Hermite polynomials are

\[
h_0(x) = 1, h_1(x) = x, h_2(x) = x^2 - 1, h_3(x) = x^3 - 3x, h_4(x) = x^4 - 6x^2 + 3, h_5(x) = x^5 - 10x^3 + 15x, \ldots
\]

Some useful properties of the Hermite polynomials are

- \( h'_n(x) = nh_{n-1}(x); \quad n = 1, 2, \ldots \)
- \( h_{n+1}(x) - 2xh_n(x) + 2nh_{n-1}(x) = 0; \quad n = 1, 2, \ldots \)

Let \( e_k \) be the \( k \)’th Hermite function defined by

\[
e_k(x) := \pi^{-\frac{1}{4}}((k - 1)!)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} h_{k-1}(\sqrt{2}x), \quad k = 1, 2, \ldots
\]

Then \( \{e_k\}_{k \geq 1} \) constitutes an orthonormal basis for \( L^2(\mathbb{R}) \) and \( e_k \in \mathcal{S}(\mathbb{R}) \) for all \( k \).

Define

\[
\theta_k(\omega) := \langle \omega, e_k \rangle = w_{e_k}(\omega) = \int_{\mathbb{R}} e_k(x) dB(x, \omega), \quad \omega \in \Omega.
\]
Definition 2.4 Let \( J \) denote the set of all finite multi-indices \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \), \( m = 1, 2, \ldots \), of non-negative integers \( \alpha_i \). If \( \alpha = (\alpha_1, \cdots, \alpha_m) \in J \), \( \alpha \neq 0 \), we put

\[
H_\alpha(\omega) := \prod_{j=1}^{m} h_{\alpha_j}(\theta_j(\omega)) = h_{\alpha_1}(\theta_1)h_{\alpha_2}(\theta_2)\cdots h_{\alpha_m}(\theta_m), \quad \omega \in \Omega.
\] (2.11)

We set \( H_0 := 1 \). Hereafter we put

\[
e^{(k)} = (0, 0, \ldots, 1, 0, \ldots, 0) \quad (2.12)
\]

with 1 on \( k \)'th position. For example, we have

\[
H_{e^{(k)}}(\omega) = h_1(\theta_k(\omega)) = \theta_k = \langle \omega, e_k \rangle,
\]

and if \( \alpha = (3, 0, 2) \), then

\[
H_{(3,0,2)} = h_3(\theta_1)h_0(\theta_2)h_2(\theta_3) = (\theta_1^3 - 3\theta_1)(\theta_3^2 - 1).
\]

We have the following fundamental result:

**Theorem 2.5 The Wiener-Itô chaos expansion theorem.** The family \( \{H_\alpha\}_{\alpha \in J} \) constitutes an orthogonal basis of \( L^2(\mathbb{P}) \). More precisely, for all \( \mathcal{F} \)-measurable \( X \in L^2(\mathbb{P}) \) there exist (uniquely determined) numbers \( c_\alpha \in \mathbb{R} \) such that

\[
X = \sum_{\alpha \in J} c_\alpha H_\alpha \quad \in L^2(\mathbb{P}).
\] (2.13)

Moreover, we have the isometry

\[
\|X\|_{L^2(\mathbb{P})}^2 = \sum_{\alpha \in J} \alpha! c_\alpha^2,
\] (2.14)

where \( \alpha! = \alpha_1! \alpha_2! \cdots \alpha_m! \) for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \).

**Example 2.6** To find the chaos expansion of the Brownian motion process \( B(t) \) at time \( t \), we proceed as follows:

\[
B(t) = \int_{\mathbb{R}} \chi_{[0,t]}(s)dB(s) = \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\chi_{[0,t]}, e_k)_{L^2(\mathbb{R})} e_k(s)dB(s)
\]

\[
= \sum_{k=1}^{\infty} \left( \int_0^t e_k(y)dy \right) \int_{\mathbb{R}} e_k(s)dB(s) = \sum_{k=1}^{\infty} \left( \int_0^t e_k(y)dy \right) H_{e^{(k)}}. \] (2.15)
2.3 The Hida stochastic test and distribution spaces

Analogous to the test functions $S(\mathbb{R})$ and the tempered distributions $S'(\mathbb{R})$ on the real line $\mathbb{R}$, there is a useful space of (Hida) stochastic test functions $(S)$ and a space of (Hida) stochastic distributions $(S)^*$ on the white noise probability space. In the following we will use the notation

\[(2N)^\alpha = \prod_{j=1}^{m} (2j)^{\alpha_j} = (2 \cdot 1)^{\alpha_1} (2 \cdot 2)^{\alpha_2} (2 \cdot 3)^{\alpha_3} \cdots (2m)^{\alpha_m}, \quad \text{for} \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathcal{J}. \quad (2.16)\]

**Definition 2.7** (Hida stochastic test function spaces $(S)_k, (S)$) Let $k \in \mathbb{R}$. We say that $f = \sum_{\alpha \in \mathcal{J}} a_{\alpha} H_{\alpha} \in L^2(P)$ belongs to the Hida test function Hilbert space $(S)_k$ if

\[\|f\|_k^2 := \sum_{\alpha \in \mathcal{J}} \alpha! a_{\alpha}^2 (2N)^{\alpha k} < \infty. \quad (2.17)\]

We define the Hida stochastic test function space $(S)$ as the space

\[(S) = \bigcap_{k \in \mathbb{R}} (S)_k\]

equipped with the projective topology, i.e.

$f_n \rightarrow f, n \rightarrow \infty, \text{ in } (S)$ if and only if $\|f_n - f\|_k \rightarrow 0, n \rightarrow \infty, \text{ for all } k$.

We illustrate this concept with an example:

**Example 2.8** The smoothed white noise $w_\phi$ belongs to $(S)$ if $\phi \in S(\mathbb{R})$.

In fact, if $\phi = \sum_{j=1}^{\infty} c_j e_j$ we have

\[w_\phi = \sum_{j=1}^{\infty} c_j H_{e_j}. \quad (2.18)\]

Therefore, using (2.17) we can see that $w_\phi \in (S)$ if and only if

\[\sum_{j=1}^{\infty} c_j^2 (2j)^k < \infty \]

for all $k$, which holds because $\phi \in S(\mathbb{R})$. See e.g. [37].

**Definition 2.9** (Hida stochastic distribution spaces $(S)_q, (S)^*$)
• Let $q \in \mathbb{R}$. We say that the formal sum $F = \sum_{\alpha \in J} b_{\alpha} H_{\alpha}$ belongs to the Hida distribution Hilbert space $(S)_{-q}$ if

$$
\|F\|_{-q}^2 := \sum_{\alpha \in J} \alpha! c_{\alpha}^2 (2N)^{-\alpha q} < \infty.
$$

(2.19)

We define the Hida stochastic distribution space $(S)^*$ as the space

$$(S)^* = \bigcup_{q \in \mathbb{R}} (S)_{-q}$$

equipped with the inductive topology, i.e. $F_n \to F$, $n \to \infty$, in $(S)^*$ if and only if there exists $q$ such that $\|F_n - F\|_{-q} \to 0$, $n \to \infty$.

• If $F = \sum_{\alpha \in J} b_{\alpha} H_{\alpha} \in (S)^*$, we define the generalized expectation $\mathbb{E}[F]$ of $F$ by

$$
\mathbb{E}[F] = b_0.
$$

(2.20)

(Note that if $F \in L^2(P)$ then the generalised expectation coincides with the usual expectation, since $\mathbb{E}[H_{\alpha}] = 0$ for all $\alpha \neq 0$).

Note that $(S)^*$ can be regarded as the dual of $(S)$: Namely, the action of $F = \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)^*$ on $f = \sum_{\alpha} a_{\alpha} H_{\alpha} \in (S)$, where $b_{\alpha}, a_{\alpha} \in \mathbb{R}$, is given by

$$
\langle F, f \rangle = \sum_{\alpha} \alpha! a_{\alpha} b_{\alpha}.
$$

We have the inclusions

$$(S) \subset (S)_k \subset L^2(P) \subset (S)_{-q} \subset (S)^*, \quad \text{for all} \quad k, q.
$$

**Example 2.10** The singular (also called pointwise) white noise $\dot{B}(t)$, $t \in \mathbb{R}$, is defined as follows:

$$
\dot{B}(t) := \sum_{k=1}^{\infty} e_k(t) H_{(k)}.
$$

(2.21)

We can verify that $\dot{B}(t) \in (S)^*$ for all $t$, as follows:

$$
\|\dot{B}(t)\|_{-q}^2 = \sum_{k=0}^{\infty} e_k^2(t) e^{(k)!} (2N)^{e^{(k)}}(2k)^{-q} = \sum_{k=0}^{\infty} e_k^2(t) (2k)^{-q} < \infty, \quad q \geq 2,
$$

because

$$
\sup_{t \in \mathbb{R}} |e_k(t)| = O(k^{-1/12}).
$$
Similarly we see that by (2.6) we get
\[
\frac{d}{dt}B(t) = \frac{d}{dt} \sum_{k=1}^{\infty} \left( \int_0^t e_k(y) dy \right) H_{\epsilon(k)} = \dot{B}(t),
\]
where the derivative is taken in \((S)^*\).

Thus we see that although the time derivative \(\frac{d}{dt}B(t)\) does not exist in the classical sense, it does exist as an element of \((S)^*\), and this derivative is the singular white noise \(\dot{B}(t)\).

The following useful characterisation of the Hida spaces is due to Zhang [43]:

**Theorem 2.11**

(i) The Hida stochastic test function space \((S)\) consists of those
\[
F = \sum_{\alpha \in \mathcal{J}} c_{\alpha} H_{\alpha} \in L^2(P)
\]
such that
\[
\sup_{\alpha} \left\{ c_{\alpha}^2 \alpha!(2N)^{k\alpha} \right\} < \infty \text{ for all } k \in \mathbb{N}. \tag{2.23}
\]

(ii) The Hida stochastic distribution space \((S)^*\) consists of those formal expansions
\[
F = \sum_{\beta \in \mathcal{J}} b_{\beta} H_{\beta}
\]
such that
\[
\sup_{\beta} \left\{ c_{\beta}^2 \beta!(2N)^{-q\beta} \right\} < \infty \text{ for some } q \in \mathbb{N}. \tag{2.24}
\]

We will also need the following result:

**Theorem 2.12**
\[
\sum_{\alpha \in \mathcal{J}} (2N)^{-\alpha q} < \infty \text{ if and only if } q > 1. \tag{2.25}
\]

### 3 The Wick product

In addition to a canonical vector space structure, the spaces \((S)\) and \((S)^*\) also have a natural multiplication given by the **Wick product**.

The main reference for this section is [16].

**Definition 3.1** If \(X = \sum_{\alpha} a_{\alpha} H_{\alpha} \in (S)^*, Y = \sum_{\beta} b_{\beta} H_{\beta} \in (S)^*\) then the Wick product \(X \diamond Y\) of \(X\) and \(Y\) is defined by
\[
X \diamond Y := \sum_{\alpha, \beta} a_{\alpha} b_{\beta} H_{\alpha+\beta} = \sum_{\gamma} \left( \sum_{\alpha+\beta=\gamma} a_{\alpha} b_{\beta} \right) H_{\gamma}. \tag{3.1}
\]

Using (2.19) and (2.17) one can now verify the following:
\[
X, Y \in (S)^* \Rightarrow X \diamond Y \in (S)^*. \tag{3.2}
\]
\[
X, Y \in (S) \Rightarrow X \diamond Y \in (S). \tag{3.3}
\]

See [16] Lemma 2.4.4. Note, however, that \(X, Y \in L^2(P) \nRightarrow X \diamond Y \in L^2(P)\) in general. See [16] Example 2.4.8.
Example 3.2  

(i) The Wick square of the singular white noise is

\[
(W)^2(t) = \sum_{k,m=1}^{\infty} e_k(t)e_m(t)H_{\epsilon(k)+\epsilon(m)}.
\]

One can show that

\[
(W)^2(t) \in (S)^*, \quad t \in \mathbb{R}.
\]  

(ii) The Wick square of the smoothed white noise is

\[
(w_\phi)^2 = \sum_{k,m=1}^{\infty} c_k c_m H_{\epsilon(k)+\epsilon(m)} \quad \text{if} \quad \phi = \sum_{k=1}^{\infty} c_k e_k \in L^2(\mathbb{R})
\]

Since

\[
H_{\epsilon(k)+\epsilon(m)} = \begin{cases} 
H_{\epsilon(k)} \cdot H_{\epsilon(m)} & \text{if } k \neq m \\
H_{\epsilon(k)}^2 - 1 & \text{if } k = m
\end{cases}
\]

we see that

\[
(w_\phi)^2 = w_\phi^2 - \sum_{k=1}^{\infty} c_k^2 = w_\phi^2 - \|\phi\|^2.
\]  

Note, in particular, that \((w_\phi)^2\) is not positive. In fact, \(\mathbb{E}[(w_\phi)^2] = 0\) by (2.21) and the fact that \(\mathbb{E}[H_\alpha] = 0\) for \(\alpha \neq 0\) (see Theorem 2.3).

Before proceeding further, we list some reasons that the Wick product is natural to use in stochastic calculus:

a) First, note that if (at least) one of the factors \(X, Y\) is deterministic, then

\[
X \diamond Y = X \cdot Y
\]

Therefore the two types of products, the Wick product and the ordinary (\(\omega\)-pointwise) product, coincide in the deterministic calculus. So when one extends a deterministic model to a stochastic model by introducing noise, it is not obvious which interpretation to choose for the products involved. The choice should be based on additional modelling and mathematical considerations.

b) The Wick product is the only product which is defined for singular white noise \(B\). Pointwise product \(X \cdot Y\) does not make sense in \((S)^*\)!

c) The Wick product has been used for 50 years already in quantum physics as a renormalization procedure.
d) There is a fundamental relation between Itô/Skorohod integrals and Wick products, given by
\[ \int_R Y(t)\delta B(t) = \int_R Y(t) \odot B(t)dt \] (3.6)
Here the integral on the right is interpreted as a Bochner integral with values in \((\mathcal{S})^*\). See Theorem 3.7 below.

e) A big class of strong solutions to stochastic differential equations can be explicitly solved by using the Wick product. See [25].

3.1 Some basic properties of the Wick product

We list below some useful properties of the Wick product. Some are easy to prove, others harder. For complete proofs see [16].

The Wick product is a binary operation on \((\mathcal{S})^*\), i.e. for all \(X, Y \in (\mathcal{S})^*\) we have \(X \odot Y \in (\mathcal{S})^*\). Moreover, for arbitrary \(X, Y, Z \in (\mathcal{S})^*\) we have
\[ X \odot Y = Y \odot X \quad \text{(commutative law),} \]
\[ X \odot (Y \odot Z) = (X \odot Y) \odot Z \quad \text{(associative law),} \]
\[ X \odot (Y + Z) = (X \odot Y) + (X \odot Z) \quad \text{(distributive law).} \] (3.7, 3.8, 3.9)

In view of the above we can define the Wick powers
\[ X^{\odot n} = X \odot X \odot \cdots \odot X \quad \text{(n times) for } X \in (\mathcal{S})^*, \quad n = 1, 2, \ldots. \]

We put \(X^{\odot 0} = 1\). Similarly, the Wick exponential of \(X \in (\mathcal{S})^*\) is defined by
\[ \exp^{\odot} X = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\odot n}, \]
if convergent in \((\mathcal{S})^*\). Thus the Wick algebra obeys the same rules as the ordinary algebra. For example,
\[ (X + Y)^{\odot 2} = X^{\odot 2} + 2X \odot Y + Y^{\odot 2} \] (3.11)
(no Itô formula!) and
\[ \exp^{\odot}(X + Y) = \exp^{\odot}(X) \odot \exp^{\odot}(Y). \] (3.12)

Note, however, that combinations of ordinary products and Wick products require caution. For example, in general we have
\[ X \cdot (Y \odot Z) \neq (X \cdot Y) \odot Z. \]

Note that since \(E[H_\alpha] = 0\) for all \(\alpha \neq 0\), we have that if \(X = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha \in L^2(P)\), then
\[ E[X] = c_\alpha. \]
From this we deduce the remarkable property of the Wick product:

$$\mathbb{E}[X \diamond Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y],$$  \hspace{1cm} (3.13)

whenever $X, Y$ and $X \diamond Y$ are $P$-integrable. Note that it is not required that $X$ and $Y$ are independent!

By induction it follows from (3.13) that

$$\mathbb{E}[\exp \diamond X] = \exp \mathbb{E}[X].$$  \hspace{1cm} (3.14)

From Example 3.2 (ii) we deduce that

$$w_{\phi} \diamond w_{\psi} = w_{\phi} \cdot w_{\psi} - \frac{1}{2} \int_{\mathbb{R}} \phi(t)\psi(t)dt, \ \phi, \psi \in L^2(\mathbb{R}).$$

In particular,

$$B^{\diamond 2}(t) = B^2(t) - t, \quad t \geq 0.$$  \hspace{1cm} (3.15)

Moreover, if $\text{supp } \phi \cap \text{supp } \psi = \emptyset$, then

$$w_{\phi} \diamond w_{\psi} = w_{\phi} \cdot w_{\psi}.$$  \hspace{1cm} (3.16)

Hence if $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$, then

$$(B(t_4) - B(t_3)) \diamond (B(t_2) - B(t_1)) = (B(t_4) - B(t_3)) \cdot (B(t_2) - B(t_1)).$$  \hspace{1cm} (3.17)

More generally, it can be proved that if $F$ is $\mathcal{F}_t$-measurable and $h > 0$, then

$$F \diamond (B(t + h) - B(t)) = F \cdot (B(t + h) - B(t)).$$  \hspace{1cm} (3.18)

For a proof see e.g. [16, Exercise 2.22].

### 3.2 Wick product and Hermite polynomials

There is a striking connection between Wick powers and Hermite polynomials $h_n; n = 0, 1, 2, \ldots$, as follows:

**Theorem 3.3** (a) Choose $\varphi \in L^2(\mathbb{R}).$ Then

$$h_n(w_{\varphi}) = ||\varphi||^{-n}w_{\varphi}^{\diamond n}.$$  \hspace{1cm} (3.19)

where $||\varphi|| = ||\varphi||_{L^2(\mathbb{R})}$.

(b) In particular, let $\theta_k = \int_{\mathbb{R}} c_k(s)dB(s); k = 1, 2, \ldots$. Then

$$h_n(\theta_k) = \theta_k^{\diamond n} : \quad n = 0, 1, \ldots$$  \hspace{1cm} (3.20)
Proof. (a). The generating function of $h_n$ is given by

$$\exp(tx - \frac{1}{2}t^2) = \sum_{n=0}^{\infty} h_n(x) \frac{t^n}{n!}. \quad (3.21)$$

On the other hand we know that

$$\exp\left(t \frac{w_\varphi}{||\varphi||} - \frac{1}{2}t^2\right) = \exp^\diamond\left(t \frac{w_\varphi}{||\varphi||}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{w_\varphi}{||\varphi||}\right)^n. \quad (3.22)$$

Substituting $x = \frac{w_\varphi}{||\varphi||}$ in (3.21) and comparing the terms with equal power of $t$ in (3.22) we get that

$$h_n\left(\frac{w_\varphi}{||\varphi||}\right) = \left(\frac{w_\varphi}{||\varphi||}\right)^n; \quad \text{for all } n. \quad (b)$$

This follows from (a) by using $\varphi = e_k$, using that $||e_k|| = 1$. \qed

Applying this result to the basis elements $H_\alpha$ defined in (2.11) and using that Wick products and ordinary products are the same for independent variables, we get

**Theorem 3.4**

$$H_\alpha = \theta_1^{\alpha_1} \theta_2^{\alpha_2} \cdots \theta_1^{\alpha_1} \diamond \theta_2^{\alpha_2} \diamond \cdots \quad (3.23)$$

3.3 Wick products and Skorohod integration

We now prove the fundamental relation (3.6) between Wick products and Skorohod integration.

**Definition 3.5** A function $Y : \mathbb{R} \rightarrow (\mathcal{S})^*$ (also called an $(\mathcal{S})^*$-valued process) is $(\mathcal{S})^*$-integrable if

$$(Y(t), f) \in L^1(\mathbb{R}), \quad \text{for all } f \in (\mathcal{S}).$$

Then the $(\mathcal{S})^*$-integral of $Y$, denoted by $\int_\mathbb{R} Y(t) dt$, is the (unique) element in $(\mathcal{S})^*$ such that

$$\left\langle \int_\mathbb{R} Y(t) dt, f \right\rangle = \int_\mathbb{R} \langle Y(t), f \rangle dt, \quad \text{for all } f \in (\mathcal{S}). \quad (3.24)$$

**Remark 3.6** The fact that (3.24) does indeed define $\int_\mathbb{R} Y(t) dt$ as an element of $(\mathcal{S})^*$ is a consequence of [13, Proposition 8.1].

Note that if $Y$ is $(\mathcal{S})^*$-integrable, then so is $Y\chi_{(a,b)}$, for all $a, b \in \mathbb{R}$, and we put

$$\int_a^b Y(t) dt := \int_\mathbb{R} Y(t)\chi_{(a,b)}(t) dt.$$
Theorem 3.7 ([3], Theorem 5.20)
Assume that \( \varphi(t) \) is \( \mathbb{F} \)-adapted and \( \mathbb{E}[\int_{\mathbb{R}} \varphi(t)^2 dt] < \infty \). Then \( \varphi(t) \diamond \dot{B}(t) \) is integrable in \( \mathcal{S}^* \) and
\[
\int_{\mathbb{R}} \varphi(t) dB(t) = \int_{\mathbb{R}} \varphi(t) \dot{B}(t) dt. \tag{3.25}
\]
In particular, note that if \( Y(t) = \sum_{i=1}^{n} c_i \chi_{(t_i, t_{i+1}]}(t), \ t \in \mathbb{R}, \) with \( c_i \in (\mathcal{S})^* \) for \( i = 1, \ldots, n \) and \( t_1 < \ldots < t_n \). Then we have
\[
\int_{\mathbb{R}} Y(t) \dot{B}(t) dt = \sum_{i=1}^{n} c_i \diamond (B(t_{i+1}) - B(t_i)).
\]

In view of the above theorem, the following terminology is natural.

Definition 3.8 Suppose \( Y \) is an \( (\mathcal{S})^* \)-valued process such that
\[
\int_{\mathbb{R}} Y(t) \dot{B}(t) dt \in (\mathcal{S})^*,
\]
then we call this integral the generalised Skorohod integral of \( Y \).

Combining the properties above with the fundamental relation \( (3.6) \) for Skorohod integration, we get a powerful calculation technique for stochastic integration. First of all, note that, by \( (3.6) \),
\[
\int_{0}^{t} \dot{B}(s) ds = B(t); \ t \in [0, T]. \tag{3.26}
\]
From this we deduce that
\[
\frac{d}{dt} B(t) \text{ exists in } (\mathcal{S})^* \text{ and } \frac{d}{dt} B(t) = \dot{B}(t); \ t \in [0, T]. \tag{3.27}
\]
Moreover, using \( (3.8) \) we get
\[
\int_{0}^{T} X \diamond Y(t) \dot{B}(t) dt = X \diamond \int_{0}^{T} Y(t) \dot{B}(t) dt, \tag{3.28}
\]
if \( X \) does not depend on \( t \). Compare this with the fact that for Skorohod integrals we generally have
\[
\int_{0}^{T} X \cdot Y(t) \delta B(t) \neq X \cdot \int_{0}^{T} Y(t) \delta B(t), \tag{3.29}
\]
even if \( X \) does not depend on \( t \).
Example 3.9 To illustrate the use of Wick calculus, let us consider the following:

\[
\int_{0}^{T} B(t) [B(T) - B(t)] \delta B(t) = \int_{0}^{T} B(t) \cdot (B(T) - B(t)) \circ \dot{B}(t) dt
\]

\[
= \int_{0}^{T} B(t) \circ B(T) \circ \dot{B}(t) dt - \int_{0}^{T} B^{\circ 2}(t) \circ \dot{B}(t) dt
\]

\[
= B(T) \circ \int_{0}^{T} B(t) \circ \dot{B}(t) dt - \frac{1}{3} B^{\circ 3}(T)
\]

\[
= \frac{1}{6} B^{\circ 3}(T) = \frac{1}{6} [B^{3}(T) - 3TB(T)],
\]

where we have correspondingly used (3.17), (3.9), (3.28) and Theorem 3.3, keeping in mind that \( ||\chi_{[0,T]}(\cdot)||_{L^2(\mathbb{R})} = T^{3/2} \).

We proceed to establish some useful properties of generalised Skorohod integrals.

Lemma 3.10 Suppose \( f \in \mathcal{S} \) and \( G(t) \in \mathcal{S}^{\circ -q} \) for all \( t \in \mathbb{R} \), for some \( q \in \mathbb{N} \). Put

\[
\hat{q} = q + \frac{1}{\log 2}.
\]

Then

\[
\int_{\mathbb{R}} |\langle G(t) \circ \dot{B}(t), f \rangle| dt \leq ||f||_{\hat{q}} \left( \int_{\mathbb{R}} ||G(t)||_{-q}^2 dt \right)^{1/2}.
\]

Proof. Suppose \( G(t) = \sum_{\alpha \in J} a_\alpha(t) H_\alpha \), \( f = \sum_{\beta \in J} b_\beta H_\beta \). Then

\[
\langle G(t) \circ \dot{B}(t), f \rangle = \left( \sum_{\alpha,k} a_\alpha(t) e_k(t) H_{\alpha+\epsilon(k)} \sum_{\beta \in J} b_\beta H_\beta \right)
\]

\[
= \sum_{\alpha,k} a_\alpha(t) e_k(t) b_{\alpha+\epsilon(k)} (\alpha + \epsilon(k))!.
\]
Hence
\[
\left| \int_{\mathbb{R}} \left\langle G(t) \odot \dot{B}(t), f \right\rangle dt \right| \leq \sum_{\alpha,k} |b_{\alpha+\epsilon(k)}| \alpha! (\alpha_k + 1) \int_{\mathbb{R}} |a_\alpha(t) e_k(t)| dt \\
\leq \sum_{\alpha,k} |b_{\alpha+\epsilon(k)}| \alpha! (\alpha_k + 1) \left( \int_{\mathbb{R}} a_\alpha^2(t) dt \right)^{1/2} \\
\leq \left( \sum_{\alpha,k} b_{\alpha+\epsilon(k)}^2 \right)^{1/2} \alpha! (\alpha_k + 1) \left( \int_{\mathbb{R}} a_\alpha^2(t) dt \right)^{1/2} \\
\leq \|f\|_q \left( \sum_{\alpha,k} \left( \int_{\mathbb{R}} a_\alpha^2(t) dt \right) \alpha! (\alpha_k + 1) (2N)^{-\frac{\alpha_k}{\log 2} (2N)^{-\frac{q}{2}} (2N)^{-q}} \right)^{1/2} \\
\leq \|f\|_q \left( \int_{\mathbb{R}} \|G(t)\|_{-q}^2 dt \right)^{1/2}.
\]

\[\square\]

Using this result we obtain the following:

**Theorem 3.11**

(i) Suppose \(G : \mathbb{R} \rightarrow (S)_{-q}\) satisfies

\[
\int_{\mathbb{R}} \|G(t)\|_{-q}^2 dt < \infty, \quad \text{for some } q \in \mathbb{N}.
\]

Then

\[
\int_{\mathbb{R}} G(t) \odot \dot{B}(t) dt \quad \text{exists in } (S)^*.
\]

(ii) Suppose \(F(t), F_n(t), n = 1, 2, \ldots, \) are elements of \((S)_{-q}\) for all \(t \in \mathbb{R}\) and

\[
\int_{\mathbb{R}} \|F_n(t) - F(t)\|_{-q}^2 dt \rightarrow 0, \quad n \rightarrow \infty.
\]

Then

\[
\int_{\mathbb{R}} F_n(t) \odot \dot{B}(t) dt \rightarrow \int_{\mathbb{R}} F(t) \odot \dot{B}(t) dt, \quad n \rightarrow \infty,
\]

in the weak*-topology on \((S)^*\).

**Proof.**

(i) The proof follows from Lemma 3.10 and Definition 3.5.

(ii) By Lemma 3.10 we have

\[
\left| \left\langle \int_{\mathbb{R}} (F_n(t) - F(t)) \odot \dot{B}(t) dt, f \right\rangle \right| \leq \int_{\mathbb{R}} \left| \left\langle (F_n(t) - F(t)) \odot \dot{B}(t), f \right\rangle \right| dt \\
\leq \|f\|_q \int_{\mathbb{R}} \|F_n(t) - F(t)\|_{-q}^2 dt \rightarrow 0, \quad n \rightarrow \infty.
\]

\[\square\]
4 The Hida-Malliavin calculus

As in previous sections we assume that the Brownian motion $B(t), t \in \mathbb{R}$, is constructed on the space $(\Omega, \mathcal{B}, P)$ with $\Omega = S'(\mathbb{R})$. Note that any $\gamma \in L^2(\mathbb{R})$ can be regarded as an element of $\Omega = S'(\mathbb{R})$ by the action

$$\langle \gamma, \phi \rangle = \int_{\mathbb{R}} \gamma(t)\phi(t)dt; \quad \phi \in \mathcal{S}(\mathbb{R}).$$

4.1 The Hida-Malliavin derivative

We are now ready to define the Hida-Malliavin derivative. A general reference for this section is [13].

Definition 4.1 (i) Let $F \in L^2(P)$ and let $\gamma \in L^2(\mathbb{R})$ be deterministic. Then the directional derivative of $F$ in $(S)^*$ (respectively, in $L^2(P)$) in the direction $\gamma$ is defined by

$$D_\gamma F(\omega) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ F(\omega + \varepsilon \gamma) - F(\omega) \right]$$

whenever the limit exists in $(S)^*$ (respectively, in $L^2(P)$).

(ii) Suppose there exists a function $\psi : \mathbb{R} \mapsto (S)^*$ (respectively, $\psi : \mathbb{R} \mapsto L^2(P)$) such that

$$\int_{\mathbb{R}} \psi(t)\gamma(t)dt \quad \text{exists in } (S)^* \quad \text{respectively, in } L^2(P) \quad \text{and} \quad D_\gamma F = \int_{\mathbb{R}} \psi(t)\gamma(t)dt, \quad \text{for all } \gamma \in L^2(\mathbb{R}).$$

Then we say that $F$ is Hida-Malliavin differentiable in $(S)^*$ (respectively, in $L^2(P)$) and we write

$$\psi(t) = D_t F, \quad t \in \mathbb{R}.$$

We call $D_t F$ the Hida-Malliavin derivative at $t$ in $(S)^*$ (respectively, in $L^2(P)$) or the stochastic gradient of $F$ at $t$.

Example 4.2 (i) Suppose $F(\omega) = \langle \omega, f \rangle = \int_{\mathbb{R}} f(t)dB(t)$, $f \in L^2(\mathbb{R})$. Then

$$D_\gamma F = \frac{1}{\varepsilon} \left[ \langle \omega + \varepsilon \gamma, f \rangle - \langle \omega, f \rangle \right] = \langle \gamma, f \rangle = \int_{\mathbb{R}} f(t)\gamma(t)dt.$$

Therefore $F$ is Hida-Malliavin differentiable and

$$D_t \left( \int_{\mathbb{R}} f(t)dB(t) \right) = f(t), \quad t - a.a.$$
(ii) Let $F \in L^2(P)$ be Hida-Malliavin differentiable in $L^2(P)$ for a.a. $t$. Suppose that $\varphi \in C^1(\mathbb{R})$ and $\varphi'(F)D_tF \in L^2(P \times \lambda)$. Then if $\gamma \in L^2(\mathbb{R})$ we have

$$D_{\gamma}(\varphi(F)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \varphi(F(\omega + \varepsilon \gamma)) - \varphi(F(\omega)) \right] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \varphi(F(\omega)) + \varepsilon D_{\gamma}F - \varphi(F(\omega)) \right] = \frac{1}{\varepsilon} \varphi'(F(\omega)) \varepsilon D_{\gamma}F = \varphi'(F)D_{\gamma}F = \int_{\mathbb{R}} \varphi'(F)D_tF \gamma(t) dt.$$ 

This proves that $\varphi(F)$ is also Hida-Malliavin differentiable and we have the chain rule

$$D_t(\varphi(F)) = \varphi'(F)D_tF. \quad (4.3)$$

More generally, the same proof gives the following extension:

**Theorem 4.3 (Chain rule)** Let $F_1, \ldots, F_m \in L^2(P)$ be Hida-Malliavin differentiable in $L^2(P)$. Suppose that $\varphi \in C^1(\mathbb{R}^m)$, $D_tF_i \in L^2(P)$, for all $t \in \mathbb{R}$, and $\frac{\partial \varphi}{\partial x_i}(F)D_tF_i \in L^2(\lambda \times P)$ for $i = 1, \ldots, m$, where $F = (F_1, \ldots, F_m)$. Then $\varphi(F)$ is Hida-Malliavin differentiable and

$$D_tF(\varphi(F)) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F)D_tF_i. \quad (4.4)$$

### 4.2 The general Hida-Malliavin derivative

It is useful to note how the Hida-Malliavin derivative can be expressed in terms of the Wiener-Itô chaos expansion (see Theorem 2.5). To this aim observe that from the chain rule (4.3) we have

$$D^1H = \sum_{k=1}^m \prod_{j \neq k} h_{\alpha_j}(\theta_j)\alpha_k h_{\alpha_k-1}(\theta_k)e_k(t) = \sum_{k=1}^m \alpha_k e_k(t)H_{\alpha-\epsilon(k)}. \quad (4.5)$$

In view of this, the following definition is natural:

**Definition 4.4 The general Hida-Malliavin derivative.**

If $F = \sum_{\alpha \in J} c_{\alpha}H_{\alpha} \in (S)^*$ we define the Hida-Malliavin derivative $D_tF$ of $F$ at $t$ in $(S)^*$ by the following expansion:

$$D_tF = \sum_{\alpha \in J} \sum_{k=1}^\infty c_{\alpha} \alpha_k e_k(t)H_{\alpha-\epsilon(k)}, \quad (4.6)$$

whenever this sum converges in $(S)^*$. We shall denote $\text{Dom}(D_t)$ the set of all $F \in (S)^*$ for which the above series converges in $(S)^*$ for all $t$. 

19
The following result gives that in fact $\text{Dom}(D_t) = (\mathcal{S})^*$:

**Theorem 4.5**  
(i) If $F \in (\mathcal{S})$ then $D_tF \in (\mathcal{S})$ for all $t$.

(ii) If $F \in (\mathcal{S})^*$ then $D_tF \in (\mathcal{S})^*$ for all $t$.

**Proof.** (ii) We prove only the second part; the proof of the first part being similar: Suppose $F = \sum_{\alpha \in J} c_{\alpha} H_{\alpha} \in (\mathcal{S})^*$. Then we know by (2.24) that there exists $q_0 \in \mathbb{N}$ such that

$$\alpha! c_{\alpha}^2 (2N)^{-q_0} \leq 1 \text{ for all } \alpha. \tag{4.7}$$

We have to prove that

$$D_tF = \sum_{\alpha \in J} \sum_{k=1}^{\infty} c_{\alpha} \alpha_k e_k(t) H_{\alpha - \epsilon(k)} \in (\mathcal{S})^*.$$

To this end, it suffices to prove that there exists $q_1 \in \mathbb{N}$ such that

$$(\alpha - \epsilon(k))! c_{\alpha}^2 \alpha_k^2 e_k^2(t)(2N)^{-(q_0+q_1)(\alpha - \epsilon(k))} \leq 1. \tag{4.8}$$

Since $\{e_k(t)\}$ is a bounded family we get by (4.7) that

$$\begin{align*}
(\alpha - \epsilon(k))! c_{\alpha}^2 \alpha_k^2 e_k^2(t)(2N)^{-(q_0+q_1)(\alpha - \epsilon(k))} & \leq C_1 (\alpha - \epsilon(k))! c_{\alpha}^2 \alpha_k^2 (2N)^{-(q_0+q_1)(\alpha - \epsilon(k))} \\
& = C_1 \alpha! \frac{\alpha_k - 1}{\alpha_k} c_{\alpha}^2 \alpha_k^2 (2N)^{-q_0(\alpha - \epsilon(k))} (2N)^{-q_1(\alpha - \epsilon(k))} \\
& = C_1 \alpha! c_{\alpha}^2 (2N)^{-q_0(\alpha - \epsilon(k))} (\alpha_k - 1) (\alpha_k) (2N)^{-q_1(\alpha - \epsilon(k))} \\
& \leq C_1 \alpha! c_{\alpha}^2 (2N)^{-q_0(\alpha - \epsilon(k))} (\alpha_k - 1) (2(k - 1))^{-q_1(\alpha_k - 1)} \\
& \leq C_1 \alpha! c_{\alpha}^2 (\alpha_k - 1) 2^{-q_1(\alpha_k - 1)} [\alpha_k (2k - q_1(\alpha_k - 1)] 1_{\alpha_k > 1} \\
& \leq 1,
\end{align*}$$

if $k > 1$ and $q_1$ is large enough.

We end this section by stating some crucial properties of the Hida-Malliavin derivative:

### 4.3 The fundamental theorem of stochastic calculus for $B(\cdot)$

**Theorem 4.6** *(Fundamental theorem)* Suppose that $\varphi \in L^2(\lambda \times P)$ is $\mathbb{F}$-adapted. Then

$$\int_{\mathbb{R}} \varphi(s) \diamond B(s) ds \in L^2(P), \tag{4.9}$$

and for all $t > 0$ we have
\[ D_t \left( \int_{\mathbb{R}} \varphi(s) \diamond \mathring{B}(s) ds \right) = \int_{\mathbb{R}} D_t \varphi(s) \diamond \mathring{B}(s) ds + \varphi(t) \quad (4.10) \]
\[= \int_{t}^{\infty} D_t \varphi(s) \diamond \mathring{B}(s) ds + \varphi(t). \quad (4.11) \]

Proof. The first statement (4.9) follows from Theorem 3.7. Recall that
\[ \mathring{B}(s) = \sum_{j} e_j(s) H_{\epsilon(j)}. \quad (4.12) \]
Hence, if we assume that \( \varphi \) has the expansion
\[ \varphi(s) = \sum_{\beta} c_{\beta}(s) H_{\beta}, \quad (4.13) \]
we get
\[ D_t \left( \int_{\mathbb{R}} \varphi(s) \diamond \mathring{B}(s) ds \right) = D_t \left( \int_{\mathbb{R}} \left( \sum_{\beta} c_{\beta}(s) H_{\beta} \right) \diamond \left( \sum_{j} e_j(s) H_{\epsilon(j)} \right) ds \right) \]
\[= D_t \left( \int_{\mathbb{R}} \sum_{\beta,j} c_{\beta}(s) e_j(s) H_{\beta+\epsilon(j)} ds \right) \]
\[= \int_{\mathbb{R}} \sum_{\beta,j,k} c_{\beta}(s) e_j(s) e_k(t) \beta_k H_{\beta+\epsilon(j)-\epsilon(k)} ds \]
\[= \int_{\mathbb{R}} \left( \sum_{\beta,k} c_{\beta}(s) e_k(t) \beta_k H_{\beta-\epsilon(k)} \right) \diamond \left( \sum_{j} e_j(s) H_{\epsilon(j)} \right) ds + \sum_{\beta,k} (c_{\beta}, e_k)_{L^2(\mathbb{R})} e_k(t) H_{\beta} \]
\[= \int_{\mathbb{R}} D_t \left( \sum_{\beta} c_{\beta}(s) H_{\beta} \right) \diamond \mathring{B}(s) ds + \sum_{\beta} c_{\beta}(t) H_{\beta} \]
\[= \int_{\mathbb{R}} D_t \varphi(s) \diamond \mathring{B}(s) ds + \varphi(t). \quad (4.14) \]
This proves \( (4.10) \). \( \square \)

### 4.4 A generalised Clark-Ocone theorem

We can apply Theorem 4.6 to prove the following, which was first obtained in [1] (with a different proof). See also [13]:

\[ \text{This proves (4.10).} \]
Theorem 4.7 *(Generalised Clark-Ocone theorem)* Let $F \in L^2(\mathcal{F}_T, P)$. Then $D_t F \in (\mathcal{S})^*$ for all $t$, $\mathbb{E}[D_t F|\mathcal{F}_t] \in L^2(\lambda \times P)$ and

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F|\mathcal{F}_t] dB(t) \quad (4.15)$$

Proof. By the Itô representation theorem there exists a unique $\mathbb{F}$-adapted process $\varphi \in L^2(\lambda \times P)$ such that

$$F = \mathbb{E}[F] + \int_0^T \varphi(s) dB(s). \quad (4.16)$$

Taking the Hida-Malliavin derivative and conditional expectation of both sides and applying the fundamental theorem (Theorem 4.6) we get

$$\mathbb{E}[D_t F|\mathcal{F}_t] = \mathbb{E}[D_t(\int_0^T \varphi(s) dB(s))|\mathcal{F}_t] = \mathbb{E}[\int_0^T D_t \varphi(s) dB(s) + \varphi(t)|\mathcal{F}_t]$$

$$= \mathbb{E}[\int_t^T D_t \varphi(s) dB(s) + \varphi(t)|\mathcal{F}_t] = \varphi(t), \quad (4.17)$$

since $D_t \varphi(s) = 0$ for all $s < t$. $\square$

### 4.5 Integration by parts

**Lemma 4.8** Suppose $g \in L^2(\mathbb{R})$ and $F \in \mathbb{D}_{1,2}$. Then

$$F \circ \int_\mathbb{R} g(t) dB(t) = F \int_\mathbb{R} g(t) dB(t) - \int_\mathbb{R} g(t) D_t F dt. \quad (4.18)$$

Proof. To ease the notation, let $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R})}$ and $(\cdot, \cdot) = (\cdot, \cdot)_{L^2(\mathbb{R})}$. For $y \in \mathbb{R}$ we define

$$G_y := \exp \left\{ y \int_\mathbb{R} g(t) dB(t) \right\} = \exp \left\{ y \int_\mathbb{R} g(t) dB(t) - \frac{1}{2} y^2 \| g \|^2 \right\}.$$

Choose $F = \exp \left\{ \int_\mathbb{R} f(t) dB(t) \right\} = \exp \left\{ \int_\mathbb{R} f(t) dB(t) - \frac{1}{2} \| f \|^2 \right\}$, where $f \in L^2(\mathbb{R})$. Then

$$F \circ G_y = \exp \circ \left\{ \int_\mathbb{R} f(t) dB(t) \right\} \circ \exp \left\{ y \int_\mathbb{R} g(t) dB(t) \right\}$$

$$= \exp \left\{ y \int_\mathbb{R} g(t) dB(t) \right\}$$

$$= \exp \left\{ \int_\mathbb{R} [f(t) + yg(t)] dB(t) - \frac{1}{2} f + yg \right\}$$

$$= \exp \left\{ \int_\mathbb{R} f(t) dB(t) \right\} \exp \left\{ \int_\mathbb{R} yg(t) dB(t) \right\} \exp \left\{ y(f, g) \right\}$$

$$= FG_y \exp \left\{ -y(f, g) \right\}.$$
Now differentiating with respect to $y$, we get
\[
\frac{d}{dy} (F \diamond G_y) = F \diamond (G_y \diamond \int_R g(t)dB(t)) \tag{4.19}
\]
and
\[
\frac{d}{dy} (FG_y \exp \{ -y(f,g) \}) \\
= FG_y \left[ \int_R g(t)dB(t) \exp \{ -y(f,g) \} - (f,g) \exp \{ -y(f,g) \} \right]. \tag{4.20}
\]
Comparing (4.19) and (4.20) we get
\[
F \diamond (G_y \diamond \int_R g(t)dB(t)) = FG_y \exp \{ -y(f,g) \} \left[ \int_R g(t)dB(t) - (f,g) \right].
\]
In particular, putting $y = 0$ we get
\[
F \diamond \int_R g(t)dB(t) = F \int_R g(t)dB(t) - F \int_R f(t)g(t)dt \\
= F \int_R g(t)dB(t) - \int_R g(t)D_tFdt. \tag{4.21}
\]
This proves the result if $F = \exp \circ \{ \int_R f(t)dB(t) \}$ for some $f \in L^2(\mathbb{R})$. Since linear combinations of such $F$’s are dense in $D_{1,2}$, the result follows by an approximation argument. \(\square\)

**Example 4.9** Choose $F = \int_R f(t)dB(t)$ with $f \in L^2(\mathbb{R})$. Then (4.18) gives
\[
\left( \int_R f(t)dB(t) \right) \circ \left( \int_R g(t)dB(t) \right) = \left( \int_R f(t)dB(t) \right) \left( \int_R g(t)dB(t) \right) - \int_R f(t)g(t)dt, \tag{4.21}
\]
which is in agreement with Theorem 3.3.

**Remark 4.10** A general formula for the relation between Wick products and ordinary products can be found in [17].

**Theorem 4.11** (Integration by parts) [13].
Suppose $Fu(t), 0 \leq t \leq T$, is Skorohod integrable, with $F \in L^2(\mathcal{F}_T, P)$. Then $Fu(t), 0 \leq t \leq T$, is Skorohod integrable and
\[
\int_0^T Fu(t)\delta B(t) = F\int_0^T u(t)\delta B(t) - \int_0^T u(t)D_tFdt. \tag{4.22}
\]
Proof. First assume that \( u(t), 0 \leq t \leq T \), is a simple function, i.e. it can be written as a finite linear combination of the form \( u(t) = \sum_i a_i \chi_{[t_i, t_{i+1})} (t), 0 \leq t \leq T \), where \( a_i \in \mathbb{D}_{1,2} \) for all \( i \). Then by applying Lemma \ref{lem:4.8} twice we get

\[
\int_0^T F u(t) \delta B(t) = \sum_i (Fa_i) \circ \Delta B(t_i)
\]

\[
= \sum_i Fa_i \Delta B(t_i) - \sum_i \int_{t_i}^{t_{i+1}} D_t (Fa_i) dt
\]

\[
= F \sum_i a_i \Delta B(t_i) - \sum_i \int_{t_i}^{t_{i+1}} D_t (Fa_i) dt
\]

\[
= F \left( \sum_i a_i \circ \Delta B(t_i) + \sum_i \int_{t_i}^{t_{i+1}} D_t a_i dt \right) - \sum_i \int_{t_i}^{t_{i+1}} D_t (Fa_i) dt
\]

\[
= F \int_0^T u(t) \delta B(t) - \int_0^T u(t) D_t F dt.
\]

Now approximate the general \( u \) by a sequence \( u_m \) of simple functions in \( \text{Dom}(\delta) \subseteq L^2(P \times \lambda) \) converging to \( u \) in \( L^2(P \times \lambda) \). We omit the details.

\[\square\]

### 4.6 The duality formula

The following useful result is a consequence of the generalised Clark-Ocone theorem:

**Theorem 4.12 (Generalised duality formula).**

Let \( F \in L^2(\mathcal{F}_T, P) \) and let \( u(t) = u(t, \omega) \in L^2(\lambda \times P) \) be \( \mathbb{F} \)-adapted. Then \( \mathbb{E}[D_tF|\mathcal{F}_t] \in L^2(\lambda \times P) \) and

\[
\mathbb{E} \left[ F \int_0^T u(t) dB(t) \right] = \mathbb{E} \left[ \int_0^T u(t) \mathbb{E}[D_tF|\mathcal{F}_t] dt \right].
\]

**Proof.** By the generalised Clark-Ocone theorem and the Itô isometry we have

\[
\mathbb{E} \left[ F \int_0^T u(t) dB(t) \right] = \mathbb{E} \left[ \mathbb{E}[F] + \int_0^T \mathbb{E}[D_tF|\mathcal{F}_t] dB(t) \int_0^T u(t) dB(t) \right]
\]

\[
= \mathbb{E} \left[ \int_0^T u(t) \mathbb{E}[D_tF|\mathcal{F}_t] dt \right].
\]

\[\square\]
4.7 Connection to the classical Malliavin derivative

In this section, we recall the basic definition and properties of the classical Malliavin calculus for Brownian motion. A general reference for this presentation is the book [13]. See also [22], [27] and [38].

A natural starting point is the classical Wiener-Itô chaos expansion theorem, which states that any \( F \in L^2(F_T, P) \) can be written

\[
F = \sum_{n=0}^{\infty} I_n(f_n) \tag{4.24}
\]

for a unique sequence of symmetric deterministic functions \( f_n \in L^2(\lambda^n) \), where \( \lambda \) is Lebesgue measure on \([0, T]\) and

\[
I_n(f_n) = n! \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n)dB(t_1)dB(t_2)\cdots dB(t_n) \tag{4.25}
\]

(the \( n \)-times iterated integral of \( f_n \) with respect to \( B(\cdot) \)) for \( n = 1, 2, \ldots \) and \( I_0(f_0) = f_0 \) when \( f_0 \) is a constant.

Moreover, we have the isometry

\[
\mathbb{E}[F^2] = \|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n!\|f_n\|^2_{L^2(\lambda^n)} =: \|F\|^2_{D(B)} \tag{4.26}
\]

**Definition 4.13 (Classical Malliavin derivative \( \tilde{D}_t \) with respect to \( B(\cdot) \))**

Let \( \mathbb{D}^{(B)}_{1,2} = \mathbb{D}_{1,2} \) be the space of all \( F \in L^2(F_T, P) \) such that its chaos expansion (2.1) satisfies

\[
\|F\|^2_{\mathbb{D}^{(B)}_{1,2}} := \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2(\lambda^n)} < \infty. \tag{4.27}
\]

For \( F \in \mathbb{D}^{(B)}_{1,2} \) and \( t \in [0, T] \), we define the classical Malliavin derivative \( \tilde{D}_t F \) of \( F \) at \( t \) (with respect to \( B(\cdot) \)), by

\[
\tilde{D}_t F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)), \tag{4.28}
\]

where the notation \( I_{n-1}(f_n(\cdot, t)) \) means that we apply the \((n-1)\)-times iterated integral to the first \( n-1 \) variables \( t_1, \ldots, t_{n-1} \) of \( f_n(t_1, t_2, \ldots, t_n) \) and keep the last variable \( t_n = t \) as a parameter.

Using the classical Itô isometry repeatedly, we can prove the following important isometry:

\[
\mathbb{E} \left[ \int_0^T (\tilde{D}_t F)^2 dt \right] = \sum_{n=1}^{\infty} nn!\|f_n\|^2_{L^2(\lambda^n)} =: \|F\|^2_{\mathbb{D}^{(B)}_{1,2}}. \tag{4.29}
\]

In particular, this proves that the function \((t, \omega) \to D_tF(\omega)\) belongs to \( L^2(\lambda \times P) \) if \( F \in \mathbb{D}_{1,2} \).
Example 4.14 If \( F = \int_0^T f(t) dB(t) \) with \( f \in L^2(\lambda) \) deterministic, then
\[
\tilde{D}_t F = f(t) \text{ for } a.a. t \in [0, T].
\]

We now proceed to compare the classical Malliavin derivative \( \tilde{D}_t \) with the Hida-Malliavin derivative \( D_t \):

The following crucial connection between Hermite polynomials \( h_n(x) \) and iterated Wiener-Itô integrals \( I_n \) was proved by Itô \[22\]:

**Theorem 4.15** Let \( g \in L^2(\lambda) \) be deterministic. Then
\[
I_n(g^\otimes n) = \|g\|^n h_n \left( \int_0^T g(t) dB(t) \right),
\]
where \( \|g\| = \|g\|_{L^2(\lambda)} \) and \( \otimes \) denotes the tensor product, i.e.
\[
g^\otimes n(t_1, t_2, ..., t_n) := g(t_1)g(t_2)...g(t_n).
\]

Combining this result with the chain rule for the Hida-Malliavin derivative and the properties of the Hermite polynomials, we obtain that if \( f_n = e_k^\otimes n \in \tilde{L}^2(\mathbb{R}^n) \) then
\[
D_t(I_n(f_n)) = D_t(h_n(\theta_k)) = h'_n(\theta_k)e_k(t) = nh_{n-1}(\theta_k)e_k(t)
\]
\[
= nI_{n-1}(f_n(\cdot, t)) = \tilde{D}_t(I_n(f_n)).
\]

Since any symmetric function \( f \) on \([0, T]^n\) can be written as a linear combination of tensor products of \( e_k \)'s we have proved the following:

**Theorem 4.16** Let \( F \in D_{1,2} \). Then the classical Malliavin derivative \( \tilde{D}_t F \) of \( F \) coincides with the Hida-Malliavin derivative \( D_t F \) of \( F \).

We conclude that the Hida-Malliavin derivative defined above on \( L^2(P) \) is an extension of the Malliavin derivative defined on the space \( D_{1,2} \). Therefore we can from now on without ambiguity use the notation \( D_t \) both for the Hida-Malliavin derivative and the classical Malliavin derivative.

5. **White noise theory for Lévy processes and Poisson random measures**

The construction we did in Section 2 of the white noise probability space for Brownian motion can be modified to apply to other processes. For example, we obtain a white noise theory for Lévy processes if we proceed as follows (see [13] for details):
Definition 5.1 Let $\nu$ be a measure on $\mathbb{R}_0$ such that
\[ \int_{\mathbb{R}} \zeta^2 \nu(d\zeta) < \infty. \] (5.1)

Define
\[ h(\varphi) = \exp(\int_{\mathbb{R}} \Psi(\varphi(x)) dx); \quad \varphi \in (S), \] (5.2)
where
\[ \Psi(w) = \int_{\mathbb{R}} (e^{iw\zeta} - 1 - iw\zeta) \nu(d\zeta); \quad w \in \mathbb{R}, \quad i = \sqrt{-1}. \] (5.3)

Then $h$ satisfies the conditions (i) - (iii) of the Bochner - Minlos- Sazonov theorem of Section 2. Therefore there exists a probability measure $Q$ on $\Omega = S'(\mathbb{R})$ such that
\[ \mathbb{E}_Q[e^{i\langle \omega, \varphi \rangle}] := \int_{\Omega} e^{i\langle \omega, \varphi \rangle} dQ(\omega) = h(\varphi); \quad \varphi \in (S). \] (5.4)

The triple $(\Omega, \mathcal{F}, Q)$ is called the (pure jump) Lévy white noise probability space.

One can now easily verify the following

1. $\mathbb{E}_Q[\langle \cdot, \varphi \rangle] = 0; \quad \varphi \in (S)$
2. $\mathbb{E}_Q[\langle \cdot, \varphi \rangle^2] = K \int_{\mathbb{R}} \varphi^2(y) dy; \quad \varphi \in (S)$, where $K = \int_{\mathbb{R}} \zeta^2 \nu(d\zeta)$.

As in Section 2 we use an approximation argument to define
\[ \tilde{\eta}(t) = \tilde{\eta}(t, \omega) = \langle \omega, \chi_{[0,t]} \rangle; \quad a.a. (t, \omega) \in [0, \infty) \times \Omega. \] (5.5)

Then the following holds:

Theorem 5.2 The stochastic process $\tilde{\eta}(t)$ has a càdlàg version. This version $\eta(t)$ is a pure jump Lévy process with Lévy measure $\nu$.

We can now proceed as in Section 2 and develop a white noise theory and Hida-Malliavin calculus with respect to the process $\eta$. We refer to [13] for more information.

6 Exercises

1. Prove that
\[ \tilde{B}(t) := \langle \omega, \chi_{[0,t]} \rangle, \quad t \in \mathbb{R}, \]
where
\[ \chi_{[0,t]} = \begin{cases} 1 & \text{if } s \in [0, t) \text{ (or } s \in [t, 0), \text{ if } t < 0), \\ 0 & \text{otherwise,} \end{cases} \]
has the following properties:
1a) $\mathbb{E}[\tilde{B}(t)] = 0; \quad t \geq 0,$
1b) $\mathbb{E}[\tilde{B}(t)^2] = t; \quad t \geq 0,$
1c) $\mathbb{E}[\tilde{B}(t)^4] = 3t^2; \quad t \geq 0.$
2. Prove that $L^2(\mathcal{F}_T, P) \subseteq (\mathcal{S})^*$.

3. Prove that the functional
   \[ g(\phi) = e^{-\frac{1}{2}||\phi||^2}; \quad \phi \in L^2(\mathbb{R}), \]
   is positive definite. (See (2.3).)

4. Define $F(t) = \mathbb{E}[\exp(\int_0^T f(s)dB(s))|\mathcal{F}_t]$, with $f \in L^2([0, T])$ deterministic. Prove that \[ D_t F = F f(t); \quad t \in [0, T] \]

5. Let $G = \int_0^T \int_{\mathbb{R}} \gamma(s, \zeta) \tilde{N}(ds, d\zeta)$ and $\varphi(G) = \exp(\int_0^T \int_{\mathbb{R}} \gamma(s, \zeta) \tilde{N}(ds, d\zeta))$. Prove that \[ D_{t, \zeta} \varphi(G) = \exp(G)[\exp(\gamma(t, \zeta)) - 1]. \]

6. Put $\varphi(G) = \exp(\int_0^T \int_{\mathbb{R}} \ln(1 + \gamma(s, \zeta)) \tilde{N}(ds, d\zeta))$. Prove that \[ D_{t, z} \varphi(G) = \varphi(G)\gamma(t, z). \]

7. Write down the expansion
   \[ F = \sum_{\alpha \in J} c_\alpha H_\alpha \]
   for the following random variables and use Definition 4.4 to find $D_t F$:
   - 7a. $F = B(t_0)$ for some $t_0 \in [0, T]$.
   - 7b. $F = \int_0^T f(s)dB(s)$ for some deterministic $f \in L^2(\lambda)$.
   - 7c. $F = \dot{B}(t_0)$ for some $t_0 \in [0, T]$.

7 Applications

7.1 Backward stochastic differential equations

Backward sde’s (bsde’s) were first introduced in their linear form by Bismut [11] in connection with a stochastic version of the Pontryagin maximum principle. Subsequently, this theory was extended by Pardoux and Peng [32] to the nonlinear case. The first work applying bsde to finance was the paper by El Karoui et al [14] where they studied several applications to option pricing and recursive utilities. All the above mentioned works are in the Brownian motion framework (continuous case). The discontinuous case is more involved. Tang and Li [39] proved an existence and uniqueness result in the case of a natural filtration associated with a Brownian motion and a Poisson random measure. Barles et al [8] proved a comparison theorem for such equations and later Royer [36] extended comparison theorem under weaker assumptions. We define the following spaces for the solution to live:
We observe that \( \tilde{V} \) filtration generated by \( \tilde{\{ \text{condition} \} \} \)
condition for forward SDE, has a unique continuous

Here we suppose for simplicity that the coefficients
Let us develop the basic idea which is behind BSDEs. For this let us consider a SDE driven
by the Brownian motion \( B \), which is of the following form
\[ dY(t) = f(Y(t))dt + \sigma(Y(t))dB(t), \quad t \in [0, T]. \]

Here we suppose for simplicity that the coefficients \( f, \sigma : \mathbb{R} \to \mathbb{R} \) are Lipschitz. Then it is
a classical result that, if \( Y(0) = \xi \in \mathbb{R} \) is an imposed initial condition, the SDE, also called
forward SDE, has a unique continuous \( \mathcal{F} \)-adapted solution \( Y \). But what does happen, if we
replace now the initial condition by a terminal one? As long as the condition \( \xi \) remains still
deterministic real value, we make a time inversion and write the equation with terminal
condition for \( V(t) := Y(T-t) \), \( \tilde{B}(t) := B(T) - B(T-t), \) \( t \in [0, T] \), as an SDE with initial condition:
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
  dV(t) &= -f(V(t))dt - \sigma(V(t))d\tilde{B}(t), \quad t \in [0, T], \\
  V(0) &= \xi.
\end{array}
\right.
\end{align*}
\]
We observe that \( \tilde{B} = (\tilde{B}(t))_{t \in [0,T]} \) is a Brownian motion, and due to the Lipschitz condition on
the coefficients, there is a unique continuous solution \( V \) adapted with respect to the
filtration generated by \( \tilde{B} \). This means that, \( Y(t) = V(T-t) \) is \( \tilde{\mathcal{F}}_t = \sigma(\{ B(T) - B(s), s \in [t, T]\}) \)-measurable, for all \( t \in [0, T] \) (\( \sigma \)-fields are considered as completed), and, for \( t_n^i = T - t + t_n^i, \) \( 0 \leq i \leq n \), the stochastic integral of the SDE for the process \( Y \) can be
described by
\[
\int_0^t \sigma(Y(s))dB(s) := \int_{T-t}^T \sigma(V(s))d\tilde{B}(s)
= L^2 - \lim_{n \to +\infty} \sum_{i=0}^{n-1} \sigma(Y(t^i_n)) (\tilde{B}(t^i_{n+1}) - \tilde{B}(t^i_n))
= L^2 - \lim_{n \to +\infty} \sum_{i=0}^{n-1} \sigma(Y(T - t^i_n)) (B(T - t^i_n) - B(T - t^i_{i+1}))
= L^2 - \lim_{n \to +\infty} \sum_{i=0}^{n-1} \sigma(Y(t - t^i_n)) (B(t - t^i_n) - B(t - t^i_{i+1}))
= L^2 - \lim_{n \to +\infty} \sum_{i=1}^{n} \sigma(Y(t^{i+1}_n)) (B(t^{i+1}_n) - B(t^i_n)),
\]
i.e., we have to do with the so-called Itô backward integral.

But how about a terminal condition \( Y(T) = \xi \) with \( \xi \in L^2(\mathcal{F}_T, P) \)? We see that in this case a time inversion of our SDE for \( Y \) leads to a forward equation for \( V(t) = Y(T-t) \) with an anticipating initial condition \( V(0) = \xi \). But we are not interested in studying SDEs with anticipation.

In order to understand better what to do, let us first consider the special case where \( f(y) = ay \), with \( a \in \mathbb{R} \) is a real constant, i.e., we have the SDE

\[
\begin{align*}
\left\{ \begin{array}{l}
dY(t) = aY(t)dt, \ t \in [0, T], \\
Y(T) = \xi \in L^2(\mathcal{F}_T, P).
\end{array} \right.
\]

The unique solution of this equation is the process \( Y(t) = \xi \exp\{-a(T-t)\}, \ t \in [0, T] \).

This process is, obviously, not adapted to the Brownian filtration \( \mathbb{F} \). Being interested in adapted solutions we replace this process \( Y(t), \ t \in [0, T] \), by its best approximation in \( L^2 \) by a process which is adapted with respect to the filtration \( \mathbb{F} \), i.e., we consider the process \( U(t) = \exp\{-a(T-t)\} \mathbb{E}[\xi|\mathcal{F}_t], \ t \in [0, T] \), which is just the optional projection of the process \( Y \).

In the special case, where \( a = 0 \), we see that \( U(t) = \mathbb{E}[\xi|\mathcal{F}_t], \ t \in [0, T] \), is just the martingale generated by \( U(T) = Y(T) = \xi \), and from the martingale representation property in a Brownian setting we have the existence and the uniqueness of a square integrable \( \mathbb{F} \)-adapted process \( Z \) such that

\[
\xi = \mathbb{E}[\xi] + \int_0^T Z(t)dB(t), \ \text{P-a.s.,}
\]

i.e.,

\[
U(t) = \mathbb{E}[\xi|\mathcal{F}_t] = \mathbb{E}[\xi] + \int_0^t Z(s)dB(s), \ t \in [0, T],
\]

which shows that \( dU(t) = Z(t)dB(t), \ t \in [0, T], \ U(T) = \xi \). This indicates that we have to reinterpret our equation for \( Y \) in the following sense:

\[
\left\{ \begin{array}{l}
dY(t) = (f(Y(t))dt + \sigma(Y(t))dB(t)) + V(t)dB(t) \ t \in [0, T], \\
Y(T) = \xi (\in L^2(\mathcal{F}_T, P)),
\end{array} \right.
\]

where the solution we have to look for is a couple of square integrable \( \mathbb{F} \)-adapted processes \((Y, V)\), where \( V \) has its origin from the martingale representation property. However, having \((Y, V)\), we can define the process \( Z = (Z(t))_{t \in [0, T]} \) by putting \( Z(t) := V(t) + \sigma(Y(t)), \ t \in [0, T] \) (observe that, knowing \((Y, Z)\) we can compute \( V(t) = Z(t) - \sigma(Y(t)) \)), and this leads to the SDE,

\[
\left\{ \begin{array}{l}
dY(t) = f(Y(t))dt + Z(t)dB(t) \ t \in [0, T], \\
Y(T) = \xi.
\end{array} \right.
\]

Finally, in order to have the backward SDE in the general form studied by Pardoux and Peng in 1990, we replace the Lipschitz function \( f : \mathbb{R} \to \mathbb{R} \) by a more general, adapted coefficient \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) which is now allowed to depend also on \((\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R} \), but also on the solution component \( Z(t) \), and in order to be coherent with the notation which
backward stochastic differential equation (BSDE) introduced and studied by Pardoux and Peng in their pioneering work of 1990 (see [32]):

\[
\begin{align*}
\left\{ \begin{array}{l}
dY(t) = -f(t,Y(t),Z(t))dt + Z(t)dB(t), \quad t \in [0,T], \\
Y(T) = \xi \in L^2(F_T, P).
\end{array} \right.
\]

**Theorem 7.1** [32] Let \( \xi \in L^2(F_T, P) \) and \( f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) a jointly measurable mapping satisfying the following assumptions:

i) \( f(\cdot, \cdot, 0, 0) \in L^2 \),

ii) \( f(\omega, t, \cdot, \cdot) \) is uniformly Lipschitz, \( dtP(\omega) \)-a.e., i.e., there is some constant \( C \in \mathbb{R} \) such that, \( dtP(\omega) \)-a.e., for all \( y, y', z, z' \in \mathbb{R} \),

\[
|f(t, \omega, y, z) - f(t, \omega, y', z')| \leq C(|y - y'| + |z - z'|).
\]

Then the BSDE

\[
\begin{align*}
\left\{ \begin{array}{l}
dY(t) = -f(t,Y(t),Z(t))dt + Z(t)dB(t), \quad t \in [0,T], \\
Y(T) = \xi.
\end{array} \right.
\]

possesses a unique solution \((Y, Z) \in S^2 \times L^2\).

**Remark 7.2** So far we have been dealing with BSDEs driven by Brownian motion \( B(\cdot) \) only. We now turn to the more general case, with BSDEs driven by both Brownian motion and an independent compensated Poisson random measure \( \tilde{N} \). As explained in Theorem 5.2, we can construct both \( B \) and \( \tilde{N} \) on the same space \( \Omega = S'(\mathbb{R}) \). We refer to [37] for more information on SDEs and BSDEs driven by Brownian motion and Poisson random measure and optimal control of such equations.

### 7.1.1 Representation of solutions of BSDE

The following result is new:

**Theorem 7.3** Suppose that \( f, p, q \) and \( r \) are given càdlàg adapted processes in \( L^2(\lambda \times P), L^2(\lambda \times P), L^2(\lambda \times P) \) and \( L^2(\lambda \times \nu \times P) \) respectively, and they satisfy a BSDE of the form

\[
\begin{align*}
\left\{ \begin{array}{l}
dp(t) = f(t)p(t^+) + q(t)dB(t) + \int_{\mathbb{R}_0} r(t, \zeta)\tilde{N}(dt, d\zeta); 0 \leq t \leq T, \\
p(T) = F \in L^2(F_T, P).
\end{array} \right.
\]

Then for a.a. \( t \) and \( \zeta \) the following holds:

\[
q(t) = D_tp(t^+) := \lim_{\varepsilon \to 0^+} D_\epsilon p(t + \varepsilon) \text{ (limit in } (S)^*),
\]

\[
q(t) = \mathbb{E}[D_\epsilon p(t^+)|\mathcal{F}_t] := \lim_{\varepsilon \to 0^+} \mathbb{E}[D_\epsilon p(t + \varepsilon)|\mathcal{F}_t] \text{ (limit in } L^2(P)),
\]

and

\[
r(t, \zeta) = D_{t, \zeta}p(t^+) := \lim_{\varepsilon \to 0^+} D_{t, \epsilon, \zeta} p(t + \varepsilon) \text{ (limit in } (S)^*),
\]

\[
r(t, \zeta) = \mathbb{E}[D_{t, \epsilon, \zeta} p(t^+)|\mathcal{F}_t] := \lim_{\varepsilon \to 0^+} \mathbb{E}[D_{t, \epsilon, \zeta} p(t + \varepsilon)|\mathcal{F}_t] \text{ (limit in } L^2(P)).
\]
Proof. Using white noise calculus and the Wick product representation of the stochastic integrals, we can write the BSDE (7.1) as a forward SDE

\[ p(t) = p_0 + \int_0^t f(u)du + \int_0^t q(u) \circ \dot{B}(u)du + \int_0^t \int_{\mathbb{R}_0} r(u, \zeta) \circ \dot{N}(u, d\zeta)du. \]  

for some initial value \( p(0) = p_0 \) (constant).

This implies that for all \( s < t \) we have, as equations in \( (\mathcal{S})^* \),

\[
D_t p(t + \varepsilon) = \int_t^{t+\varepsilon} D_t f(u)du + \int_t^{t+\varepsilon} D_t q(u) \circ \dot{B}(u)du + q(t)
\]

and

\[
D_t p(t + \varepsilon) = \int_t^{t+\varepsilon} D_t f(u)du + \int_t^{t+\varepsilon} \int_{\mathbb{R}_0} D_t r(u, \zeta) \circ \dot{N}(u, \zeta)du + r(t, \zeta)
\]

Taking the limit in \( (\mathcal{S})^* \) as \( \varepsilon \to 0^+ \) we get (7.2) and (7.4).

Taking the conditional expectation and then the limit as \( \varepsilon \) goes to \( 0^+ \), we get (7.3) and (7.5).

\[ \square \]

### 7.1.2 Closed formula for mean-field BSDE

We shall find the closed formula corresponding to the linear mean-field BSDE of the form

\[
\begin{align*}
    dY(t) &= -[\alpha_1(t)Y(t) + \beta_1(t)Z(t) + \int_{\mathbb{R}_0} \eta_1(t, \zeta)K(t, \zeta)\nu(d\zeta) + \alpha_2(t)\mathbb{E}[Y(t)]] \\
    &\quad + \beta_2(t)\mathbb{E}[Z(t)] + \int_{\mathbb{R}_0} \eta_2(t, \zeta)\mathbb{E}[K(t, \zeta)]\nu(d\zeta) + \gamma(t)]dt \\
    &\quad + Z(t)dB(t) + \int_{\mathbb{R}_0} K(t, \zeta)\dot{N}(dt, d\zeta), \quad t \in [0, T],
\end{align*}
\]

(7.7)

where the coefficients \( \alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \eta_1(t, \cdot), \eta_2(t, \cdot) \) are given deterministic functions; \( \gamma(t) \) is a given \( \mathbb{F} \)-adapted process and \( \xi \in L^2(\Omega, \mathcal{F}_T) \) is a given \( \mathcal{F}_T \) measurable random variable. Applying the closed formula for linear BSDE with jumps, the above linear mean-field bsde (7.7) can be written as follows.

\[
Y(t) = \mathbb{E}[(\xi \Gamma(t, T) + \int_t^T \Gamma(t, s)\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)] \\
+ \int_{\mathbb{R}_0} \eta_2(s, \zeta)\mathbb{E}[K(s, \zeta)]\nu(d\zeta) + \gamma(s)]ds)|\mathcal{F}_t], \quad t \in [0, T],
\]

(7.8)

where \( \Gamma(t, s) \) is the solution of the following linear sde

\[
\begin{align*}
    d\Gamma(t, s) &= \Gamma(t, s^-)[\alpha_1(t)dt + \beta_1(t)dB(t) + \int_{\mathbb{R}_0} \eta_1(t, \zeta)\dot{N}(dt, d\zeta)], \quad s \in [t, T], \\
    \Gamma(t, t) &= 1.
\end{align*}
\]

(7.9)

Since we are in one dimension, Equation (7.9) can be solved explicitly and the solution is given by

\[
\begin{align*}
    \Gamma(t, s) &= \exp\{\int_t^s \beta_1(r)dB(r) + \int_t^s \alpha_1(r) - \frac{1}{2}(\beta_1(r))^2)dr \\
    &\quad + \int_t^s \int_{\mathbb{R}_0} [\ln(1 + \eta_1(r, \zeta)) - \eta_1(r, \zeta)]\nu(d\zeta)dr \\
    &\quad + \int_t^s \int_{\mathbb{R}_0} [\ln(1 + \eta_1(r, \zeta))\dot{N}(dr, d\zeta)]\}.
\end{align*}
\]

(7.10)
Taking the expectation, we have
\[ \mathbb{E} \Gamma(t, s) = \exp \left\{ \int_t^s \alpha_1(r)dr \right\}. \] (7.11)

To solve (7.8) we take the expectation on both sides of (7.8). Denoting \( \overline{Y}(t) := \mathbb{E}[Y(t)], \) \( Z(t) := \mathbb{E}[Z(t)], \) and \( \overline{K}(t, \zeta) := \mathbb{E}[K(t, \zeta)], \) we obtain
\[ \overline{Y}(t) = \mathbb{E}[\xi \Gamma(t, T) + \int_t^T \Gamma(t, s)\{\alpha_2(s)\overline{Y}(s) + \beta_2(s)Z(s)\} + \int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds, \quad t \in [0, T]. \] (7.12)

To find equations for \( Z(t) \) and \( \overline{K}(t, \zeta) \) we write the original equation (7.7) as a forward one:
\[ Y(t) = Y(0) + \int_0^t \{\alpha_1(s)Y(s) + \alpha_2(s)\overline{Y}(s) + \beta_1(s)Z(s) + \beta_2(s)\overline{Z}(s) \]
\[ + \int_{\mathbb{R}_0} (\eta_1(s, \zeta)K(s, \zeta) + \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds \]
\[ + \int_0^t Z(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} K(s, \zeta)\tilde{N}(ds, d\zeta), \quad t \in [0, T], \]
for some deterministic initial value \( Y(0). \) Then using the fundamental theorem of stochastic calculus (Theorem 4.6), we compute the Hida-Malliavin derivative of \( Y(t) \) for all \( r < t \) as follows:
\[ D_rY(t) = \int_r^t D_r\{\alpha_1(s)Y(s) + \alpha_2(s)\overline{Y}(s) + \beta_1(s)Z(s) + \beta_2(s)\overline{Z}(s) \]
\[ + \int_{\mathbb{R}_0} (\eta_1(s, \zeta)K(s, \zeta) + \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds \]
\[ + \int_r^t D_rZ(s)dB(s) + Z(r). \]

Letting \( r \to t- \), we get that \( Z(t) = D_tY(t). \) Thus, to find \( Z(t) \) we only need to compute \( D_tY(t). \) We shall use the expression (7.8) for \( Y(t) \) and the identity
\[ D_t\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[D_tF|\mathcal{F}_t]. \]

We also notice that \( D_t\Gamma(t, T) = \Gamma(t, T)\beta_1(t). \) Then
\[ Z(t) = \mathbb{E}[\{D_t\xi \Gamma(t, T) + \xi \Gamma(t, T)\beta_1(t) + \int_t^T \Gamma(t, s)\beta_1(t)\{\alpha_2(s)\overline{Y}(s) \]
\[ + \beta_2(s)\overline{Z}(s) + \int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds \}. \]

Taking the expectation, we have
\[ \overline{Z}(t) = \mathbb{E}[D_t\xi \Gamma(t, T) + \beta_1(t)\mathbb{E}(\xi \Gamma(t, T)) + \int_t^T \mathbb{E}(\Gamma(t, s))\beta_1(t)\{\alpha_2(s)\overline{Y}(s) \]
\[ + \beta_2(s)\overline{Z}(s) + \int_{\mathbb{R}_0} \eta_2(s, \zeta)\overline{K}(s, \zeta)\nu(d\zeta) + \gamma(s)]ds]. \] (7.13)
Similarly, we have \( K(t, \zeta) = D_{t, \zeta} Y(t) \) which yields

\[
K(t, \zeta) = \mathbb{E}[D_{t, \zeta} \xi \Gamma(t, T) + \xi \Gamma(t, T) \eta_1(t, \zeta) + \int_t^T \Gamma(t, s) \eta_1(t, \zeta) \{ \alpha_2(s) \overline{Y}(s) + \beta_2(s) \overline{Z}(s) + \int_{\mathbb{R}_0} \eta_2(s, \zeta) \overline{K}(s, \zeta) \nu(d\zeta) + \gamma(s) \} ds] \mathcal{F}_t.
\]

Taking the expectation yields

\[
\overline{K}(t, \zeta) = \mathbb{E}[D_{t, \zeta} \xi \Gamma(t, T) + \xi \Gamma(t, T) \eta_1(t, \zeta) + \int_t^T \Gamma(t, s) \eta_1(t, \zeta) \{ \alpha_2(s) \overline{Y}(s) + \beta_2(s) \overline{Z}(s) + \int_{\mathbb{R}_0} \eta_2(s, \zeta) \overline{K}(s, \zeta) \nu(d\zeta) + \gamma(s) \} ds].
\]

Equations (7.12), (7.13) and (7.14) can be used to obtain \( \bar{Y}, \bar{Z}, \bar{K} \). In fact, we let

\[
V(t) = \begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t, \zeta) \end{pmatrix} = \begin{pmatrix} \overline{Y}(t) \\ \overline{Z}(t) \\ \overline{K}(t, \zeta) \end{pmatrix} \in L^2 \times L^2 \times H^2_\psi,
\]

and

\[
A(t, s, \zeta) = (A_{ij}(t, s, \zeta))_{1 \leq i,j \leq 3} \quad (7.15)
\]

\[
= \begin{pmatrix} \exp\{\int_t^s \alpha_1(r) dr\} \alpha_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \beta_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \eta_2(s, \zeta) \\ \exp\{\int_t^s \alpha_1(r) dr\} \beta_1(t) \alpha_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \beta_1(t) \beta_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \beta_1(t) \eta_2(s, \zeta) \\ \exp\{\int_t^s \alpha_1(r) dr\} \eta_1(t, \zeta) \alpha_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \eta_1(t, \zeta) \beta_2(s) & \exp\{\int_t^s \alpha_1(r) dr\} \eta_1(t, \zeta) \eta_2(s, \zeta) \end{pmatrix}.
\]

Define a mapping \( A = A^T \) from \( V = (V_1, V_2, V_3)^T \in L^2 \times L^2 \times H^2_\psi \) to itself by

\[
(AV)_i(t, \zeta) = \sum_{j=1}^2 \int_t^T A_{ij}(t, s)V_j(s)ds + \int_{\mathbb{R}_0} A_{i3}(t, s, \zeta)V_3(s, \zeta)\nu(d\zeta)ds. \quad (7.16)
\]

Then (7.12), (7.13) and (7.14) can be written as

\[
V = F + AV , \quad (7.17)
\]

where

\[
F(t, \zeta) = \begin{pmatrix} \mathbb{E}(\xi \Gamma(t, T)) + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_{t, \xi} \xi \Gamma(t, T) + \xi \Gamma(t, T)] + \int_t^T \gamma(s)ds \\ \mathbb{E}[D_{t, \xi} \xi \Gamma(t, T) + \xi \Gamma(t, T) \eta_1(t, \zeta)] + \int_t^T \gamma(s)ds \end{pmatrix}. \quad (7.18)
\]

Note that the operator norm of \( A \), \( ||A|| \), is less than 1 if \( t \) is close enough to \( T \). Therefore there exists \( \delta > 0 \) such that \( ||A|| < 1 \) if we restrict the operator to the interval \([T - \delta, T]\) for
some $\delta > 0$ small enough. In this case the linear equation equation (7.17) can now be solved easily as follows:

$$(I - A)V = F,$$

or

$$V = (I - A)^{-1}F = \sum_{n=0}^{\infty} A^n F; \quad t \in [T - \delta, T]. \quad (7.19)$$

Next, using $V(T - \delta)$ as the terminal value of the corresponding BSDE in the interval $[T - 2\delta, T - \delta]$ and repeating the argument above, we find that there exists a solution $V$ of the BSDE in this interval, given by the equation

$$V(t, \zeta) = V(T - \delta, \zeta) + A^{T-\delta}(t, \cdot, \zeta)V(\cdot); \quad T - 2\delta \leq t \leq T - \delta. \quad (7.20)$$

Proceeding by induction we end up with a solution on the whole interval $[0, T]$. We summarise this as follows:

**Theorem 7.4 (Closed formula [3])** Assume that $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t), \eta_1(t, \cdot), \eta_2(t, \cdot)$ are given bounded deterministic functions and that $\gamma(t)$ is $\mathbb{F}$-adapted and $\xi \in L^2(\Omega, \mathcal{F}_T)$. Then the component $Y(t)$ of the solution of the linear mean-field BSDE (7.7) can be written on its closed formula as follows

$$Y(t) = \mathbb{E}[(\xi \Gamma(t, T) + \int_t^T \Gamma(t, s)\{((\alpha_2(s), \beta_2(s), \eta_2(s, \zeta))V(s) + \gamma(s)\}ds)\mathcal{F}_t], t \in [0, T], \mathbb{P}\text{-a.s.},$$

where

$$\Gamma(t, s) = \exp\{\int_t^s \beta_1(r)dB(r) + \int_t^s (\alpha_1(r) - \frac{1}{2}(\beta_1(r))^2)dr$$

$$+ \int_t^s \int_{\mathbb{R}_0} (\ln(1 + \eta_1(r, \zeta)) - \eta_1(r, \zeta))\nu(d\zeta)dr$$

$$+ \int_t^s \int_{\mathbb{R}_0} (\ln(1 + \eta_1(r, \zeta))\tilde{N}(dr, d\zeta))\}. \quad (7.21)$$

and, inductively,

$$V(t, \zeta) = V(T - k\delta, \zeta) + A^{T-k\delta}(t, \cdot, \zeta)V(\cdot); \quad T - (k+1)\delta \leq t \leq T - k\delta; \quad k = 0, 1, 2, \ldots \quad (7.22)$$

Or, equivalently,

$$V(t, \zeta) = (A^{T-k\delta}(t, \cdot, \zeta))^n V(T - k\delta, \cdot); \quad T - (k+1)\delta \leq t \leq T - k\delta; \quad k = 0, 1, 2, \ldots$$

where $A^S; S > 0$ is given by (7.15) and $V(T, \zeta) = F$.

### 7.2 Stochastic maximum principles via Hida-Malliavin calculus

Recall the two main methods of optimal control of systems described by Itô- Lévy processes:

- **Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation**
  This method was introduced by R. Bellman in the 1950’s, first in the deterministic case.
• The stochastic maximum principle

This method was established at around the same time by L. Pontryagin and his group in the deterministic case.

The maximum principle was extended to the stochastic case by J.-M. Bismut for the linear-quadratic case and for Brownian motion driven SDE’s (1976) and subsequently further developed by A. Bensoussan, S. Peng, E. Pardoux and others (still for Brownian motion driven SDE’s, 1980 -1990), and then by S. Rong and N.-C. Framstad, A. Sulem & B. Øksendal (for jump diffusions, 1990 -).

Dynamic programming is efficient when applicable, but it requires that the system is Markovian. The maximum principle has the advantage that it also applies to non-Markovian SDE’s, but the drawback is the corresponding complicated BSDE for the adjoint processes.

The state of our system $X^u(t) = X(t)$ satisfies the following SDE

$$
\begin{align*}
\left\{ 
\begin{array}{l}
  dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\
  X(0) = x_0 \in \mathbb{R} \text{ (constant)},
\end{array}
\right.
\end{align*}
$$

(7.23)

where $b(t, x, u) = b(t, x, u, \omega) : [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}$, $\sigma(t, x, u) = \sigma(t, x, u, \omega) : [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}$ and $\gamma(t, x, u, \zeta) = \gamma(t, x, u, \zeta) : [0, T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \rightarrow \mathbb{R}$.

From now on we fix an open convex set $U$ such that $V \subset U$ and we assume that $b$, $\sigma$ and $\gamma$ are continuously differentiable and admits uniformly bounded partial derivatives in $U$ with respect to $x$ and $u$.

Moreover, we assume that the coefficients $b$, $\sigma$ and $\gamma$ are $\mathbb{F}$-adapted, and uniformly Lipschitz continuous with respect to $x$, in the sense that there is a constant $C$ such that, for all $t \in [0, T], u \in V, \zeta \in \mathbb{R}_0, x, x' \in \mathbb{R}$ we have

$$
\begin{align*}
& |b(t, x, u) - b(t, x', u)|^2 + |\sigma(t, x, u) - \sigma(t, x', u)|^2 \\
& + \int_{\mathbb{R}_0} |\gamma(t, x, u, \zeta) - \gamma(t, x', u, \zeta)|^2 \nu(d\zeta) \leq C|x - x'|^2, \text{ a.s.}
\end{align*}
$$

Under this assumption, there is a unique solution $X \in \mathcal{S}^2$ to the equation (7.23), such that

$$
X(t) = x_0 + \int_0^t b(s, X(s), u(s))ds + \int_0^t \sigma(s, X(s), u(s))dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X(s), u(s), \zeta)\tilde{N}(ds, d\zeta); 0 \leq t \leq T.
$$

For a given set $A$ of admissible controls, the performance functional has the form

$$
J(u) = \mathbb{E}[\int_0^T f(t, X(t), u(t))dt + g(X(T))], \quad u \in A,
$$

(7.24)

with given functions $f : [0, T] \times \mathbb{R} \times U \times \Omega \rightarrow \mathbb{R}$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, assumed to be $\mathbb{F}$-adapted and $\mathcal{F}_T$-measurable, respectively, and continuously differentiable with respect to $x$ and $u$ with bounded partial derivatives in $U$.

Suppose that $\hat{u}$ is an optimal control. Fix $\tau \in [0, T), 0 < \epsilon < T - \tau$ and a bounded $\mathcal{F}_\tau$-measurable $v$ and define the spike perturbed $u^\epsilon$ of the optimal control $\hat{u}$ by
\[
    u^\varepsilon(t) = \begin{cases} 
        \hat{u}(t); & t \in [0, \tau) \cup (\tau + \varepsilon, T], \\
        v; & t \in [\tau, \tau + \varepsilon].
    \end{cases}
\]

Let \( X^\varepsilon(t) := X^{u^\varepsilon}(t) \) and \( \hat{X}(t) := X^{\hat{u}}(t) \) be the solutions of (7.23) corresponding to \( u = u^\varepsilon \) and \( u = \hat{u} \), respectively.

Define

\[ Z^\varepsilon(t) := X^\varepsilon(t) - \hat{X}(t); \ t \in [0, T]. \tag{7.25} \]

Then by the mean value theorem \(^1\) we can write

\[ b'(t) - \hat{b}(t) = \frac{\partial b}{\partial x}(t)Z^\varepsilon(t) + \frac{\partial b}{\partial u}(t)(u^\varepsilon(t) - \hat{u}(t)), \]

where

\[ b'(t) = b(t, X^\varepsilon(t), u^\varepsilon(t)), \hat{b}(t) = b(t, \hat{X}(t), \hat{u}(t)), \]

and

\[ \frac{\partial \hat{b}}{\partial x}(t) = \frac{\partial b}{\partial x}(t, x, u)_{x=\hat{x}(t), u=\hat{u}(t)}; \]

and

\[ \frac{\partial \hat{b}}{\partial u}(t) = \frac{\partial b}{\partial u}(t, x, u)_{x=\hat{x}(t), u=\hat{u}(t)}. \]

Here \((\hat{u}(t), \hat{X}(t))\) is a point on the straight line between \((\hat{u}(t), \hat{X}(t))\) and \((u^\varepsilon(t), X^\varepsilon(t))\). With a similar notation for \( \sigma \) and \( \gamma \), we get

\[
    Z^\varepsilon(t) = \int^t_\tau \left\{ \frac{\partial b}{\partial x}(s)Z^\varepsilon(s) + \frac{\partial b}{\partial u}(s)(u^\varepsilon(s) - \hat{u}(s)) \right\} ds + \int^t_\tau \left\{ \frac{\partial \sigma}{\partial x}(s)Z^\varepsilon(s) + \frac{\partial \sigma}{\partial u}(s)(u^\varepsilon(s) - \hat{u}(s)) \right\} dB(s)
    + \int^t_\tau \int_{\mathbb{R}_0} \left\{ \frac{\partial \gamma}{\partial x}(s, \zeta)Z^\varepsilon(s) + \frac{\partial \gamma}{\partial u}(s, \zeta)(u^\varepsilon(s) - \hat{u}(s)) \right\} dN(ds, d\zeta); \tau \leq t \leq \tau + \varepsilon, \tag{7.27}
\]

and

\[
    Z^\varepsilon(t) = \int^t_{\tau + \varepsilon} \frac{\partial b}{\partial x}(s)Z^\varepsilon(s) ds + \int^t_{\tau + \varepsilon} \frac{\partial \sigma}{\partial x}(s)Z^\varepsilon(s) dB(s)
    + \int^t_{\tau + \varepsilon} \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(s, \zeta)(s)Z^\varepsilon(s) N(ds, d\zeta); \tau + \varepsilon \leq t \leq T. \tag{7.28}
\]

On other words,

\[
    \begin{cases}
        dZ^\varepsilon(t) &= \left\{ \frac{\partial b}{\partial x}(t)Z^\varepsilon(t) + \frac{\partial b}{\partial u}(t)(v - \hat{u}(t)) \right\} dt + \left\{ \frac{\partial \sigma}{\partial x}(t)Z^\varepsilon(t) + \frac{\partial \sigma}{\partial u}(t)(v - \hat{u}(t)) \right\} dB(t) \\
        &+ \int_{\mathbb{R}_0} \left\{ \frac{\partial \gamma}{\partial x}(t, \zeta)Z^\varepsilon(t) + \frac{\partial \gamma}{\partial u}(t, \zeta)(v - \hat{u}(t)) \right\} dN(dt, d\zeta); \tau \leq t \leq \tau + \varepsilon,
    \end{cases} \tag{7.28}
\]

\(^1\)Recall that if a function \( f \) is continuously differentiable on an open convex set \( U \subset \mathbb{R}^n \) and continuous on the closure \( \bar{U} \), then for all \( x, y \in \bar{U} \) there exists a point \( \tilde{x} \) on the straight line connecting \( x \) and \( y \) such that

\[
    f(y) - f(x) = f'(\tilde{x})(y - x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\tilde{x})(y_i - x_i) \tag{7.26}
\]
\[ dZ^\epsilon(t) = \frac{\partial H}{\partial x}(t)Z^\epsilon(t)dt + \frac{\partial^2 H}{\partial x^2}(t)Z^\epsilon(t)dB(t) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, \zeta)Z^\epsilon(t)d\tilde{N}(dt, d\zeta); \tau + \epsilon \leq t \leq T. \] (7.29)

**Remark 7.5**

1. Note that since the process \( \eta(t) := \int_{t_0}^t \int_{\mathbb{R}_0} \zeta \tilde{N}(ds, d\zeta); t \geq 0 \) is a Lévy process, we know that for every given (deterministic) time \( t \geq 0 \) the probability that \( \eta \) jumps at \( t \) is 0. Hence, for each \( t \), the probability that \( X \) makes jump at \( t \) is also 0. Therefore we have \( Z^\epsilon(\tau) = 0 \) a.s.

2. We remark that the equations (7.28) − (7.29) are linear SDE and then by our assumptions on the coefficients, they admit a unique solution.

Let \( \mathcal{R} \) denote the set of (Borel) measurable functions \( r : \mathbb{R}_0 \to \mathbb{R} \) and define the Hamiltonian

\[ H(t, x, y, u, p, q, r) := H(t, x, y, u, p, q) = f(t, x, u) + b(t, x, u)p + \sigma(t, x, u)q + \int_{\mathbb{R}_0} \gamma(t, x, y, u) r(t, \zeta)\nu(d\zeta). \] (7.30)

Let \( (p^\epsilon, q^\epsilon, r^\epsilon) \in S^2 \times L^2 \times L^2 \) be the solution of the following associated adjoint BSDE:

\[
\begin{aligned}
    dp^\epsilon(t) &= -\frac{\partial H}{\partial x}(t, x, y, u, p^\epsilon, q^\epsilon)dt + \frac{\partial \gamma}{\partial x}(t, x, u)q^\epsilon(t)dB(t) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, x, u) r^\epsilon(t, \zeta)d\tilde{N}(dt, d\zeta); t \in [0, T], \\
    p^\epsilon(T) &= \frac{\partial \tilde{g}}{\partial x}(\tilde{X}(T)),
\end{aligned}
\] (7.31)

where

\[ \frac{\partial H}{\partial x}(t, x, y, u, p^\epsilon, q^\epsilon, r^\epsilon) = \frac{\partial f}{\partial x}(t, x, u) + \frac{\partial b}{\partial x}(t, x, u)p^\epsilon(t) + \frac{\partial \sigma}{\partial x}(t, x, u)q^\epsilon(t) + \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, x, u) r^\epsilon(t, \zeta)\nu(d\zeta). \]

**Lemma 7.6** [4] The following holds,

\[ Z^\epsilon(t) \to 0 \text{ as } \epsilon \to 0^+; \text{ for all } t \in [\tau, T]. \] (7.32)

\[ (p^\epsilon, q^\epsilon, r^\epsilon) \to (\hat{p}, \hat{q}, \hat{r}) \text{ when } \epsilon \to 0^+, \] (7.33)

where \( (\hat{p}, \hat{q}, \hat{r}) \) is the solution of the BSDE

\[
\begin{aligned}
    d\hat{p}(t) &= -\frac{\partial \tilde{H}}{\partial x}(t, \tilde{X}(t))dt + \frac{\partial \tilde{g}}{\partial x}(t, \tilde{X}(t))dB(t) + \int_{\mathbb{R}_0} \frac{\partial \tilde{\gamma}}{\partial x}(t, \tilde{X}(t)) d\tilde{N}(dt, d\zeta); t \in [0, T], \\
    \hat{p}(T) &= \frac{\partial \tilde{g}}{\partial x}(\tilde{X}(T)).
\end{aligned}
\]
Proof. By the Itô formula, we see that the solutions of the equations (7.28) − (7.29), are

\[
Z^t(t) = Z^t(t + \epsilon) \exp\left( \int_{t+\epsilon}^t \left\{ \frac{\partial \tilde{V}}{\partial t}(s) - \frac{1}{2} \left( \frac{\partial^2 \tilde{V}}{\partial x^2}(s) \right)^2 + \int_{\mathbb{R}_0} [\log(1 + \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta)) - \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta)] \nu(d\zeta) \right\} ds \\
+ \int_{t+\epsilon}^t \frac{\partial \tilde{V}}{\partial x}(s) dB(s) + \int_{t+\epsilon}^t \int_{\mathbb{R}_0} [\log(1 + \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta))] \tilde{N}(ds, d\zeta) \right) dt, \quad \tau + \epsilon \leq t \leq T.
\]

and

\[
Z^t(t) = \mathcal{Y}(t)^{-1} \left[ \int_0^t \mathcal{Y}(s) \left( \frac{\partial \tilde{V}}{\partial t}(s) (u^t(s) - \hat{u}(s)) \right) ds \\
+ \int_{\mathbb{R}_0} \left( \frac{1}{1 + \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta)} - 1 \right) \frac{\partial \tilde{V}}{\partial x}(s, \zeta) (v - \hat{u}(s)) \nu(d\zeta) ds + \int_0^t \mathcal{Y}(s) \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta) dB(s) \\
+ \int_0^t \int_{\mathbb{R}_0} \mathcal{Y}(s) \left( \frac{\partial^2 \tilde{V}}{\partial x^2}(s, \zeta) \left( \frac{v - \hat{u}(s)}{\partial u^t(s)} \right) - 1 \right) \tilde{N}(ds, d\zeta) \right] \right]_{\tau \leq t \leq \tau + \epsilon},
\]

where

\[
\begin{aligned}
d\mathcal{Y}(t) &= \mathcal{Y}(t^-) \left[ - \frac{\partial \tilde{V}}{\partial x}(t) + \left( \frac{\partial \tilde{V}}{\partial x}(t) (u^t(t) - \hat{u}(t)) \right)^2 \\
&+ \int_{\mathbb{R}_0} \left( \frac{1}{1 + \frac{\partial^2 \tilde{V}}{\partial x^2}(t, \zeta)} - 1 \right) \frac{\partial \tilde{V}}{\partial x}(t, \zeta) \nu(d\zeta) dt - \frac{\partial \tilde{V}}{\partial x}(t) dB(t) \\
&+ \int_{\mathbb{R}_0} \left( \frac{1}{1 + \frac{\partial^2 \tilde{V}}{\partial x^2}(t, \zeta)} - 1 \right) \tilde{N}(dt, d\zeta) \right]_{\tau \leq t \leq \tau + \epsilon},
\end{aligned}
\]

\[
\mathcal{Y}(0) = 1.
\]

From (7.35) we see that \(Z^t(\tau + \epsilon) \to 0\) as \(\epsilon \to 0^+\), and then from (7.34) we deduce that \(Z^t(t) \to 0\) as \(\epsilon \to 0^+\), for all \(t\).

The BSDE (7.31) is linear, and we can write the solution explicitly as follows:

\[
p^t(t) = E[\Gamma(T) \frac{\partial \tilde{X}}{\partial x}(T) \mathcal{Y}(T) + \int_t^T \Gamma(s) \frac{\partial \tilde{X}}{\partial x}(s) ds | \mathcal{F}_t]; \quad t \in [0, T],
\]

where \(\Gamma(t) \in \mathcal{S}^2\) is the solution of the linear SDE

\[
\begin{aligned}
d\Gamma(t) &= \Gamma(t^-) \left[ \frac{\partial \tilde{V}}{\partial x}(t) dt + \frac{\partial \tilde{V}}{\partial x}(t) dB(t) + \int_{\mathbb{R}_0} \frac{\partial \tilde{V}}{\partial x}(t, \zeta) \tilde{N}(dt, d\zeta) \right] \right]_{t \in [0, T],
\end{aligned}
\]

\[
\Gamma(0) = 1.
\]

From this, we deduce that \(p^t(t) \to \hat{p}(t), q^t(t) \to \hat{q}(t)\) and \(r^t(t, \zeta) \to \hat{r}(t, \zeta)\) as \(\epsilon \to 0^+.\]

We now state and prove the main result of this part.

**Theorem 7.7 (Necessary maximum principle [4])** Suppose \(\hat{u} \in \mathcal{A}\) is maximizing the performance (7.24). Then for all \(t \in [0, T]\) and all bounded \(\mathcal{F}_t\)-measurable \(v \in V\), we have

\[
\frac{\partial H}{\partial t}(t, \hat{X}(t), \hat{u}(t), v - \hat{u}(t)) \leq 0.
\]
Proof. Consider

\[ J(u') - J(\hat{u}) = I_1 + I_2, \quad (7.37) \]

where

\[ I_1 = \mathbb{E}\left[ \int_{\tau}^{T} \{ f(t, X'(t), u'(t)) - f(t, \hat{X}(t), \hat{u}(t)) \} dt \right], \quad (7.38) \]

and

\[ I_2 = \mathbb{E}[g(X'(T)) - g(\hat{X}(T))]. \quad (7.39) \]

By the mean value theorem, we can write

\[ I_1 = \mathbb{E}\left[ \int_{\tau}^{\tau + \epsilon} \left\{ \frac{\partial f}{\partial x}(t)Z'(t) + \frac{\partial f}{\partial u}(t)(u'(t) - \hat{u}(t)) \right\} dt + \int_{\tau + \epsilon}^{T} \frac{\partial f}{\partial x}(t)Z'(t) dt \right], \quad (7.40) \]

and, applying the Itô formula to \( p^f(t)Z'(t) \) and by (7.29), we have

\[ I_2 = \mathbb{E}\left[ \frac{\partial p^f}{\partial x}(\hat{X}(T))Z'(T) \right] = \mathbb{E}[p^f(T)Z'(T)] \]

\[ = \mathbb{E}[p^f(\tau + \epsilon)Z'(\tau + \epsilon)] \]

\[ + \mathbb{E}\left[ \int_{\tau + \epsilon}^{T} p^f(t) dZ'(t) + \int_{\tau + \epsilon}^{T} Z'(t) dp^f(t) + \int_{\tau + \epsilon}^{T} d\langle p^f, Z' \rangle(t) \right] \]

\[ = \mathbb{E}\left[ p^f(t + \epsilon)\left( \int_{\tau}^{\tau + \epsilon} \left\{ \frac{\partial \hat{p}}{\partial x}(t)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t)(u'(t) - \hat{u}(t)) \right\} dt \right. \]

\[ + \int_{\tau}^{\tau + \epsilon} \left\{ \frac{\partial \hat{p}}{\partial x}(t)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t)(u'(t) - \hat{u}(t)) \right\} dB(t) \]

\[ + \int_{\tau}^{\tau + \epsilon} \int_{R_0} \left\{ \frac{\partial \hat{p}}{\partial x}(t, \zeta)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t, \zeta)(u'(t) - \hat{u}(t)) \right\} N(dt, d\zeta) \]

\[ + \mathbb{E}\left[ \int_{\tau + \epsilon}^{T} \left\{ p^f(t)\frac{\partial \hat{p}}{\partial x}(t)Z'(t) - \frac{\partial \hat{p}}{\partial u}(t)Z'(t) + q^f(t)\frac{\partial \hat{p}}{\partial x}(t)Z'(t) \right. \]

\[ + \int_{R_0} \nu(t, \zeta)\frac{\partial \hat{p}}{\partial x}(t, \zeta)Z'(t) \nu(d\zeta) \} dt \right] \quad (7.41) \]

Using the generalized duality formula, we get

\[ I_2 = \mathbb{E}\left[ \int_{\tau}^{\tau + \epsilon} p^f(\tau + \epsilon) \left( \frac{\partial \hat{p}}{\partial x}(t)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t)(u'(t) - \hat{u}(t)) \right) \right. \]

\[ + \mathbb{E}[D_t p^f(\tau + \epsilon)|\mathcal{F}_\tau] \left( \frac{\partial \hat{p}}{\partial x}(t)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t)(u'(t) - \hat{u}(t)) \right) \]

\[ + \int_{R_0} \mathbb{E}[D_{t, \zeta} p^f(\tau + \epsilon)|\mathcal{F}_\tau] \left\{ \frac{\partial \hat{p}}{\partial x}(t, \zeta)Z'(t) + \frac{\partial \hat{p}}{\partial u}(t, \zeta)(u'(t) - \hat{u}(t)) \right\} \nu(d\zeta) \} dt \]

\[ - \mathbb{E}\left[ \int_{\tau + \epsilon}^{T} \frac{\partial \hat{p}}{\partial x}(t)Z'(t) dt \right] \quad (7.42) \]
where by the definition of $H$ (7.30)

$$
\frac{\partial \hat{f}}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p'(t) - \frac{\partial \sigma}{\partial x}(t)q'(t) - \int_{\mathbb{R}_0} \frac{\partial \nu}{\partial x}(t, \zeta) r'(t, \zeta) \nu(d\zeta).
$$

Summing (7.40) and (7.42), we obtain

$$
I_1 + I_2 = \mathbb{E}\left[ \int_{\tau}^{T+} \left( \frac{\partial \hat{f}}{\partial x}(t) + p'(\tau + \epsilon) \frac{\partial b}{\partial x}(t) + \mathbb{E}[D_t p'(\tau + \epsilon)|\mathcal{F}_t] \frac{\partial \sigma}{\partial x}(t) \right) dt \right] + \mathbb{E}\left[ \int_{\tau}^{T+} \left( \frac{\partial \hat{f}}{\partial x}(t) + p'(\tau + \epsilon) \frac{\partial b}{\partial x}(t) + \mathbb{E}[D_t p'(\tau + \epsilon)|\mathcal{F}_t] \frac{\partial \sigma}{\partial x}(t) \right) dt \right] \int_{\mathbb{R}_0} \frac{\partial \nu}{\partial x}(t, \zeta) r'(t, \zeta) \nu(d\zeta) \right] (u'(t) - \hat{u}(t)) dt.
$$

By the estimate of $Z'$ (7.32), we get

$$
\lim_{\epsilon \to 0^+} X'(t) = \hat{X}(t); \text{ for all } t \in [\tau, T],
$$

and by (7.33) we have

$$
p'(t) \to \hat{p}(t), q'(t) \to \hat{q}(t) \text{ and } r'(t, \zeta) \to \hat{r}(t, \zeta) \text{ when } \epsilon \to 0^+, \tag{7.45}
$$

where $(\hat{p}, \hat{q}, \hat{r})$ solves the BSDE

$$
\begin{align*}
\left\{ \begin{array}{l}
d\hat{p}(t) = -\frac{\partial H}{\partial x}(t) dt + \hat{q}(t) dB(t) + \int_{\mathbb{R}_0} \hat{r}(t, \zeta) \hat{N}(dt, d\zeta); \tau \leq t \leq T, \\
\hat{p}(T) = \frac{\partial u}{\partial x}(\hat{X}(T)).
\end{array} \right.
\end{align*}
$$

Using the above and the assumption that $\hat{u}$ is optimal, we get

$$
0 \geq \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} (J(u') - J(\hat{u}))
\begin{align*}
&= \mathbb{E}\left[ \frac{\partial f}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau)) + \hat{p}(\tau) \frac{\partial b}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau)) + \mathbb{E}[D_{\tau} \hat{p}(\tau^+)|\mathcal{F}_t] \frac{\partial \sigma}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau)) + \int_{\mathbb{R}_0} \mathbb{E}[D_{\tau, \zeta} \hat{p}(\tau^+)|\mathcal{F}_t] \frac{\partial \nu}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau), \zeta) \nu(d\zeta) \right] (v - \hat{u}(\tau)) \\
&+ \int_{\mathbb{R}_0} \mathbb{E}[D_{\tau} \hat{p}(\tau^+)|\mathcal{F}_t] \frac{\partial \nu}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau), \zeta) \nu(d\zeta) \right] (v - \hat{u}(\tau)),
\end{align*}
$$

where, by Theorem (7.3)

$$
\begin{align*}
\mathbb{E}[D_{\tau} \hat{p}(\tau^+)|\mathcal{F}_t] &= \lim_{\epsilon \to 0^+} \mathbb{E}[D_{\tau} \hat{p}(\tau + \epsilon)|\mathcal{F}_t] = \hat{q}(\tau), \\
\mathbb{E}[D_{\tau, \zeta} \hat{p}(\tau^+)|\mathcal{F}_t] &= \lim_{\epsilon \to 0^+} \mathbb{E}[D_{\tau, \zeta} \hat{p}(\tau + \epsilon)|\mathcal{F}_t] = \hat{r}(\tau, \zeta).
\end{align*}
$$

Hence

$$
\mathbb{E}\left[ \frac{\partial H}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau))(v - \hat{u}(\tau)) \right] \leq 0.
$$

Since this holds for all bounded $\mathcal{F}_\tau$-measurable $v$, we conclude that

$$
\frac{\partial H}{\partial u}(\tau, \hat{X}(\tau), \hat{u}(\tau))(v - \hat{u}(\tau)) \leq 0 \text{ for all } v.
$$

\qed
7.2.1 Example [4]

We now illustrate Theorem 7.7 by applying it to a linear-quadratic stochastic control problem with a constraint, as follows:

Consider a controlled SDE of the form

\[
\begin{aligned}
    dX(t) &= u(t)dt + \sigma dB(t) + \int_{\mathbb{R}_0} \gamma(\zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\
    X(0) &= x_0 \in \mathbb{R}.
\end{aligned}
\]

Here \( u \in \mathcal{A} \) is our control process (see below) and \( \sigma \) and \( \gamma \) is a given constant in \( \mathbb{R} \) and function from \( \mathbb{R}_0 \) into \( \mathbb{R} \), respectively, with

\[
\int_{\mathbb{R}_0} \gamma^2(\zeta) \nu(d\zeta) < \infty.
\]

We want to control this system in such a way that we minimize its value at the terminal time \( T \) with a minimal average use of energy, measured by the integral \( \mathbb{E}[\int_0^T u^2(t)dt] \) and we are only allowed to use nonnegative controls. Thus we consider the following constrained optimal control problem:

**Problem 7.8** Find \( \hat{u} \in \mathcal{A} \) (the set of admissible controls) such that

\[
J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u),
\]

where

\[
J(u) = \mathbb{E}\left[ -\frac{1}{2} X^2(T) - \frac{1}{2} \int_0^T u^2(t)dt \right],
\]

and \( \mathcal{A} \) is the set of predictable processes \( u \) such that \( u(t) \geq 0 \) for all \( t \in [0, T] \) and

\[
\mathbb{E}\left[ \int_0^T u^2(t)dt \right] < \infty.
\]

Thus in this case the set \( V \) of admissible control values is given by \( V = [0, \infty) \) and we can use \( U = V \). The Hamiltonian is given by

\[
H(t, x, u, p, q, r) = -\frac{1}{2} u^2 + up + \sigma q + \int_{\mathbb{R}_0} \gamma(\zeta) r(\zeta) \nu(d\zeta),
\]

the adjoint BSDE for the optimal adjoint variables \( \hat{p}, \hat{q}, \hat{r} \) is given by

\[
\begin{aligned}
    d\hat{p}(t) &= \hat{q}(t) dB(t) + \int_{\mathbb{R}_0} \hat{r}(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\
    \hat{p}(T) &= -\hat{X}(T).
\end{aligned}
\]

Hence

\[
\hat{p}(t) = -\mathbb{E}[\hat{X}(T)|\mathcal{F}_t].
\]
Theorem 7.7 states that if \( \hat{u} \) is optimal, then
\[
(-\hat{u}(t) + \hat{p}(t))(v - \hat{u}(t)) \leq 0; \quad \text{for all } v \geq 0.
\]
From this we deduce that
\[
\begin{cases}
(i) \text{ if } \hat{u}(t) = 0, \quad \text{then } \hat{u}(t) \geq \hat{p}(t), \\
(ii) \text{ if } \hat{u}(t) > 0, \quad \text{then } \hat{u}(t) = \hat{p}(t).
\end{cases}
\]
Thus we see that we always have \( \hat{u}(t) \geq \max\{\hat{p}(t), 0\} \). We claim that in fact we have equality, i.e. that
\[
\hat{u}(t) = \max\{\hat{p}(t), 0\}.
\]
To see this, suppose the opposite, namely that \( \hat{u}(t) > \max\{\hat{p}(t), 0\} \).

Then in particular \( \hat{u}(t) > 0 \), which by (ii) above implies that \( \hat{u}(t) = \hat{p}(t) \), a contradiction.

We summarize what we have proved as follows:

Theorem 7.9 \cite{4} Suppose there is an optimal control \( \hat{u} \in A \) for Problem 7.8. Then
\[
\hat{u}(t) = \max\{\hat{p}(t), 0\} = \max\{-\mathbb{E}[\hat{X}(T)|\mathcal{F}_t], 0\},
\]
where \((\hat{p}, \hat{X})\) is the solution of the coupled forward-backward SDE system given by
\[
\begin{cases}
\begin{aligned}
d\hat{X}(t) &= \max\{\hat{p}(t), 0\}dt + \sigma dB(t) + \int_{\mathbb{R}_0} \gamma(\zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\
\hat{X}(0) &= x_0 \in \mathbb{R},
\end{aligned}
\end{cases}
\]
\[
\begin{cases}
\begin{aligned}
d\hat{p}(t) &= \hat{q}(t)dB(t) + \int_{\mathbb{R}_0} \hat{r}(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\
\hat{p}(T) &= -\hat{X}(T).
\end{aligned}
\end{cases}
\]

Remark 7.10 For comparison, in the case when there are no constraints on the control \( u \), we get from the well-known solution of the classical linear-quadratic control problem (see e.g. Øksendal \cite{28}, Example 11.2.4) that the optimal control \( u^* \) is given in feedback form by
\[
u^*(t) = -\frac{\hat{X}(t)}{T + 1 - t}; \quad t \in [0, T].
\]

EXERCISE

1. Let \( X(t) \) satisfy the equation
\[
\begin{cases}
\begin{aligned}
dX(t) &= (b_0(t) + b_1(t)X(t))dt + (\sigma_0(t) + \sigma_1(t)X(t))dB(t) \\
& \quad + \int_{\mathbb{R}_0} (\gamma_0(t, \zeta) + \gamma_1(t, \zeta) X(t))\tilde{N}(dt, d\zeta); \quad t \in [0, T], \\
X(0) &= x_0,
\end{aligned}
\end{cases}
\]

43
for given $\mathbb{F}$-predictable processes $b_0(n(t), b_1(t), \sigma_0(t), \sigma_1(t), \gamma_0(t, \zeta), \gamma_1(t, \zeta)$ with $\gamma_i(t, \zeta) \geq -1$ for $i = 0, 1$.

Suppose

$$\Upsilon(t) = \exp \left[ \int_0^t (-b_1(s) + \frac{1}{2} \sigma_1^2(s)) - \int_{\mathbb{R}_0} \{ \log(1 + \gamma_1(s, \zeta)) - \gamma_1(s, \zeta) \} \nu(d\zeta) ds \right.$$  
$$- \int_0^t \sigma_1(s) dB(s) + \int_0^t \int_{\mathbb{R}_0} \log(1 + \gamma_1(s, \zeta)) \tilde{N}(ds, d\zeta) \right]; t \in [0, T].$$

Then the unique solution $X(t)$ is given by

$$X(t) = \Upsilon(t)^{-1} \left[ x_0 + \int_0^t \Upsilon(s)(b_0(s) + \int_{\mathbb{R}_0} \frac{1}{1+\gamma_1(s, \zeta)} - 1) \gamma_0(s, \zeta) \nu(d\zeta) ds \right.$$  
$$+ \int_0^t \Upsilon(s) \sigma_0(s) dB(s) + \int_0^t \int_{\mathbb{R}_0} \Upsilon(s)(\frac{\gamma_0(s, \zeta)}{1+\gamma_1(s, \zeta)}) \tilde{N}(ds, d\zeta) \right]; t \in [0, T].$$

2. Suppose that $Y(t)$ satisfies the linear BSDE

$$\begin{cases} 
  dY(t) = -[\alpha(t)Y(t) + \beta(t)Z(t) + \int_{\mathbb{R}_0} \eta(t, \zeta) K(t, \zeta) \nu(d\zeta) + \gamma(t)] dt \\
  + Z(t) dB(t) + \int_{\mathbb{R}_0} K(t, \zeta) \tilde{N}(dt, d\zeta), \\
  Y(t) = \xi.
\end{cases}$$

Prove that the component of the solution $Y(t)$ can be written on its closed formula as

$$Y(t) = \mathbb{E} \left[ (\xi \Gamma(t, T) + \int_t^T \Gamma(t, s) \gamma(s) ds) | \mathcal{F}_t \right], \quad t \in [0, T],$$

where $\Gamma(t, s)$ is the solution of the following linear sde

$$\begin{cases} 
  d\Gamma(t, s) = \Gamma(t, s^{-})[\alpha(t) dt + \beta(t) dB(t) + \int_{\mathbb{R}_0} \eta(t, \zeta) \tilde{N}(dt, d\zeta)], \quad s \in [t, T], \\
  \Gamma(t, t) = 1.
\end{cases}$$

### 7.3 Stochastic Volterra integral equations (SVIEs)

In the following we put $\triangle := \{(t, s) \in [0, T]^2 : t \leq s \}$. We define the following spaces:

- $L^2_y$ consists of the $\mathbb{F}$-adapted càdlàg processes $Y : [0, T] \times \Omega \to \mathbb{R}$ equipped with the norm
  $$\| Y \|^2_{L^2_y} := \mathbb{E} \left[ \int_0^T |Y(t)|^2 dt \right] < \infty.$$

- $L^2_z$ consists of the $\mathbb{F}$-predictable processes
  $$Z : \triangle \times \Omega \to \mathbb{R},$$
such that $\mathbb{E}\left[\int_0^T \int_t^T |Z(t, s)|^2 dsdt\right] < \infty$ with $s \mapsto Z(t, s)$ being $\mathbb{F}$-predictable on $[t, T]$.

We equip $L^2_z$ with the norm

$$\| Z \|^2_{L^2_z} := \mathbb{E}\left[\int_0^T \int_t^T |Z(t, s)|^2 dsdt\right].$$

- $L^2_\nu$ consists of all Borel functions $K : \mathbb{R}_0 \to \mathbb{R}$, such that

$$\| K \|^2_{L^2_\nu} := \int_{\mathbb{R}_0} K(t, s, \zeta)^2 \nu(d\zeta) < \infty.$$

- $H^2_\nu$ consists of $\mathbb{F}$-predictable processes $K : \Delta \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, such that

$$\mathbb{E}\left[\int_0^T \int_t^T \int_{\mathbb{R}_0} |K(t, s, \zeta)|^2 \nu(d\zeta) dsdt\right] < \infty$$

and $s \mapsto K(t, s, \cdot)$ being $\mathbb{F}$-predictable on $[t, T]$. We equip $H^2_\nu$ with the norm

$$\| K \|^2_{H^2_\nu} := \mathbb{E}\left[\int_0^T \int_t^T \int_{\mathbb{R}_0} |K(t, s, \zeta)|^2 \nu(d\zeta) dsdt\right].$$

- Let $L^2_{\mathcal{F}_t}[0, T]$ be the space of all processes $\psi : [0, T] \times \Omega \to \mathbb{R}$ and $\psi$ is $\mathcal{F}_t$-measurable for all $t \in [0, T]$, such that

$$\| \psi \|^2_{L^2_{\mathcal{F}_t}[0, T]} = \mathbb{E}\left[\int_0^T |\psi(t)|^2 dt\right] < \infty.$$

- $L^2_F[0, T]$ is the space of all $\psi \in L^2_{\mathcal{F}_t}[0, T]$ that are $\mathbb{F}$-adapted.

Let us start by motivating what is a forward SVIE and then we will go to the BSVIE.

### 7.3.1 A motivating example

Stochastic Volterra integral equations (SVIEs) are a special type of integral equations. They represent interesting models for stochastic dynamics with memory, with applications to e.g.

- engineering,
- biology (e.g. population dynamics) and
- finance.

Moreover, they are useful tools for studying
• fractional Brownian motion,
• stochastic differential equations with delay and
• stochastic partial differential equations.

For example, let \( X^u(t) = X(t) \) be a given cash flow, modelled by the following stochastic Volterra integral equation:

\[
X(t) = x_0 + \int_0^t [b_0(t, s)X(s) - u(s)]ds + \int_0^t \sigma_0(s)X(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma_0(s, \zeta)X(s)\tilde{N}(ds, d\zeta); \quad t \geq 0,
\] (7.48)

or, in differential form,

\[
\begin{cases}
  dX(t) = [b_0(t, t)X(t) - u(t)]dt + \sigma_0(t)X(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t, \zeta)X(t)\tilde{N}(dt, d\zeta) + (\int_0^t \frac{\partial b_0}{\partial t}(t, s)X(s)ds)dt; \\
  X(0) = x_0.
\end{cases}
\] (7.49)

We see that the dynamics of \( X(t) \) contains a history (or memory) term represented by the \( ds \)-integral.

We assume that \( b_0(t, s), \sigma_0(s) \) and \( \gamma_0(s, \zeta) \) are given deterministic functions of \( t, s \), and \( \zeta \), with values in \( \mathbb{R} \), and that \( b_0(t, s) \) is continuously differentiable with respect to \( t \) for each \( s \). For simplicity we assume that these functions are bounded, and we assume that there exists \( \varepsilon > 0 \) such that \( \gamma_0(s, \zeta) \geq -1 + \varepsilon \) for all \( s, \zeta \) and the initial value \( x_0 \in \mathbb{R} \). Let \( A \) denote the set of admissible controls \( u \). We want to solve the following maximisation problem:

**Problem 7.11** Find \( \hat{u} \in A \), such that

\[
\sup_u J(u) = J(\hat{u}),
\] (7.50)

where

\[
J(u) = \mathbb{E}[\theta X(T) + \int_0^T \log(u(t))dt],
\] (7.51)

\( \theta = \theta(\omega) \) being a given \( \mathcal{F}_T \)-measurable random variable.

We will return to this example after some general theory on optimal control of SVIEs.

### 7.3.2 Backward stochastic Volterra integral equations (BSVIEs)

Recall that the BSDE \((Y, Z, K)\)

\[
-dY(t) = F(t, Y(t), Z(t), K(t, \cdot))dt - Z(t)dB(t) - \int_{\mathbb{R}_0} K(t, \zeta)\tilde{N}(dt, d\zeta), \quad Y(T) = \zeta,
\]

is equivalent to
\[ Y(t) = \zeta + \int_t^T F(s, Y(s), Z(s), K(s, \cdot)) ds - \int_t^T Z(s) dB(s) \quad (7.52) \]

\[ - \int_t^T \int_{\mathbb{R}_0} K(s, \zeta) \tilde{N}(ds, d\zeta). \]

The corresponding BSVIE has the form
\[ Y(t) = \xi(t) + \int_t^T F(t, s, Y(s), Z(t, s), K(t, s, \cdot)) ds \]
\[ - \int_t^T Z(t, s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K(t, s, \zeta) \tilde{N}(ds, d\zeta). \quad (7.53) \]

### 7.3.3 Representation of solutions of BSVIE

**Theorem 7.12 [6]** Suppose that \( F, Y, Z \) and \( K \) are given càdlàg adapted processes which satisfy a BSVIE of the form
\[ Y(t) = \xi(t) + \int_t^T F(t, s, Y(s), Z(t, s), K(t, s, \cdot)) ds \]
\[ - \int_t^T Z(t, s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K(t, s, \zeta) \tilde{N}(ds, d\zeta). \]

Then for a.a. \( t \) and \( \zeta \) the following holds:
\[ Z(t, s) = D_t Y(t^+) := \lim_{\varepsilon \to 0^+} D_t Y(t + \varepsilon) \ (\text{limit in } (S)^*), \]
\[ Z(t, s) = \mathbb{E}[D_t Y(t^+)|\mathcal{F}_t] := \lim_{\varepsilon \to 0^+} \mathbb{E}[D_t Y(t + \varepsilon)|\mathcal{F}_t] \ (\text{limit in } L^2(P)), \]

and
\[ K(t, s, \zeta) = D_{t, \zeta} Y(t^+) := \lim_{\varepsilon \to 0^+} D_{t, \zeta} Y(t + \varepsilon) \ (\text{limit in } (S)^*), \]
\[ K(t, s, \zeta) = \mathbb{E}[D_{t, \zeta} Y(t^+)|\mathcal{F}_t] := \lim_{\varepsilon \to 0^+} \mathbb{E}[D_{t, \zeta} Y(t + \varepsilon)|\mathcal{F}_t] \ (\text{limit in } L^2(P)). \]

### 7.3.4 Closed formula for linear BSVIE

Consider now the linear form. Let \((\Phi(t, s), 0 \leq t < s \leq T)\) and \((\xi(s), \beta(s, \zeta); 0 \leq s \leq T, \zeta \in \mathbb{R}_0)\) be given (deterministic) measurable functions of \( t, s, \) and \( \zeta, \) with values in \( \mathbb{R}_0. \) For simplicity we assume that these functions are bounded, and we assume that there exists \( \varepsilon > 0 \) such that \( \beta(s, \zeta) \geq -1 + \varepsilon \) for all \( s, \zeta. \) We consider the following linear backward stochastic Volterra integral equations in the unknown process triplet \((Y(t), Z(t, s), K(t, s, \zeta)):\)
\[ Y(t) = F(t) + \int_t^T \left[ \Phi(t, s) Y(s) + \xi(s) Z(t, s) + \int_{\mathbb{R}_0} \beta(s, \zeta) K(t, s, \zeta) \nu(d\zeta) \right] ds \]
\[ - \int_t^T Z(t, s) dB(s) - \int_t^T \int_{\mathbb{R}_0} K(t, s, \zeta) \tilde{N}(ds, d\zeta), \quad (7.54) \]

where \( 0 \leq t \leq T \) and \( \tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta) dt \) is the compensated Poisson random measure.
To this end, we define the probability measure $Q$ by

$$dQ = M(T)dP \text{ on } \mathcal{F}_T,$$

where

$$M(t) := \exp \left( \int_0^t \xi(s)dB(s) - \frac{1}{2} \int_0^t \xi^2(s)ds + \int_0^t \int_{\mathbb{R}_0} \ln(1 + \beta(s, \zeta))\tilde{N}(ds, d\zeta) 
+ \int_0^t \int_{\mathbb{R}_0} \{\ln(1 + \beta(s, \zeta)) - \beta(s, \zeta)\}\nu(d\zeta)ds \right); \quad 0 \leq t \leq T. \quad (7.56)$$

Then under the new probability measure $Q$ the process

$$B_Q(t) := B(t) - \int_0^t \xi(s)ds, \quad 0 \leq t \leq T. \quad (7.57)$$

is a Brownian motion, and the random measure

$$\tilde{N}_Q(dt, d\zeta) := \tilde{N}(dt, d\zeta) - \beta(t, \zeta)\nu(d\zeta)dt \quad (7.58)$$

is the $Q$-compensated Poisson random measure of $N(\cdot, \cdot)$, in the sense that the process

$$\tilde{N}_\gamma(t) := \int_0^t \int_{\mathbb{R}_0} \gamma(s, \zeta)\tilde{N}_Q(ds, d\zeta)$$

is a local $Q$-martingale, for all predictable processes $\gamma(t, \zeta)$ such that

$$\int_0^T \int_{\mathbb{R}_0} \gamma^2(t, \zeta)\beta^2(t, \zeta)\nu(d\zeta)dt < \infty. \quad (7.59)$$

We also introduce, for $0 \leq t \leq r \leq T$,

$$\Phi^{(1)}(t, r) = \Phi(t, r), \quad \Phi^{(2)}(t, r) = \int_t^r \Phi(t, s)\Phi(s, r)ds,$$

and inductively

$$\Phi^{(n)}(t, r) = \int_t^r \Phi^{(n-1)}(t, s)\Phi(s, r)ds, \quad n = 3, 4, \cdots. \quad (7.60)$$

**Remark 7.13** Note that if $|\Phi(t, r)| \leq C$ (constant) for all $t, r$, then by induction

$$|\Phi^{(n)}(t, r)| \leq \frac{C^m T^n}{n!}$$

for all $t, r, n$. Hence,

$$\sum_{n=1}^{\infty} |\Phi^{(n)}(t, r)| < \infty$$

for all $t, r$. 48
Theorem 7.14 [18] \[ \Psi(t, r) = \sum_{n=1}^{\infty} \Phi^{(n)}(t, r). \] (7.61)

Then we have the following explicit form of the solution triplet:

(i) The \( Y \) component of the solution triplet is given by

\[
Y(t) = \mathbb{E}_Q \left[ F(t) \left| \mathcal{F}_t \right. \right] + \int_t^T \Psi(t, r) \mathbb{E}_Q \left[ F(r) \left| \mathcal{F}_t \right. \right] dr
= \mathbb{E}_Q \left[ F(t) + \int_t^T \Psi(t, r) F(r) dr \left| \mathcal{F}_t \right. \right].
\] (7.62)

(ii) The \( Z \) and \( K \) components of the solution triplet are given by the following:

Define

\[ U(t) = F(t) + \int_t^T \Phi(t, r) Y(r) dr - Y(t); \quad 0 \leq t \leq T. \] (7.63)

Then \( Z(t, s) \) and \( K(t, s, \zeta) \) can be expressed by the Hida-Malliavin derivatives \( D_s \) and \( D_{s, \zeta} \) with respect to \( B \) and \( N \), respectively, as follows:

\[
Z(t, s) = \mathbb{E}_Q [D_s U(t) - U(t) \int_s^T D_s \xi(r) dB_Q(r) | \mathcal{F}_s]; \quad 0 \leq t \leq s \leq T
\] (7.64)

and

\[
K(t, s, \zeta) = \mathbb{E}_Q [U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_{s, \zeta} U(t) | \mathcal{F}_s]; \quad 0 \leq t \leq s \leq T,
\] (7.65)

where

\[
\tilde{H}_s = \exp \left[ \int_0^s \int_{\mathbb{R}_0} [D_{s, x} \beta(r, x) + \log(1 - \frac{D_{s, x, \beta}(r, x)}{1 - \beta(r, x)}) (1 - \beta(r, x)) \nu(dx) dr
+ \int_0^s \int_{\mathbb{R}_0} \log(1 - \frac{D_{s, x, \beta}(r, x)}{1 - \beta(r, x)}) \tilde{N}_Q(dr, dx) \right].
\] (7.66)

Proof. With the processes \( B_Q \) and \( \tilde{N}_Q \) defined in (7.57)-(7.58) we can eliminate the unknowns \( Z(t, s) \) and \( K(t, s, \zeta) \) inside the first integral in (7.54). More precisely, we can rewrite equation (7.54) as

\[ Y(t) = F(t) + \int_t^T \Phi(t, s) Y(s) ds - \int_t^T \int_{\mathbb{R}_0} K(t, s, \zeta) \tilde{N}_Q(ds, d\zeta), \] (7.67)

where \( 0 \leq t \leq T \). Taking the conditional \( Q \)-expectation on \( \mathcal{F}_t \), we get

\[
Y(t) = \mathbb{E}_Q \left[ F(t) + \int_t^T \Phi(t, s) Y(s) ds \left| \mathcal{F}_t \right. \right]
= \tilde{F}(t, t) + \int_t^T \Phi(t, s) \mathbb{E}_Q \left[ Y(s) \left| \mathcal{F}_t \right. \right] ds, \quad 0 \leq t \leq T.
\] (7.68)
Here, and in what follows, we denote
\[
\tilde{F}(t, s) = \mathbb{E}_Q \left[ F(t) \big| \mathcal{F}_s \right].
\]
(7.69)

Fix \( r \in [0, t] \). Taking the conditional \( Q \)-expectation on \( \mathcal{F}_r \) of (7.68), we get
\[
\mathbb{E}_Q[Y(t) \big| \mathcal{F}_r] = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \mathbb{E}_Q \left[ Y(s) \big| \mathcal{F}_r \right] ds,
\]
\( r \leq t \leq T \)

Denote
\[
\tilde{Y}(s) = \mathbb{E}_Q \left[ Y(s) \big| \mathcal{F}_r \right], \quad r \leq s \leq T.
\]
Then the above equation can be written as
\[
\tilde{Y}(t) = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \tilde{Y}(s) ds, \quad r \leq t \leq T.
\]

Substituting \( \tilde{Y}(s) = \tilde{F}(s, r) + \int_s^T \Phi(s, u) \tilde{Y}(u) du \) in the above equation, we obtain
\[
\tilde{Y}(t) = \tilde{F}(t, r) + \int_t^T \Phi(t, s) \tilde{Y}(s) ds + \int_t^T \Phi(t, u) \tilde{Y}(u) du, \quad r \leq t \leq T,
\]
By repeatedly using the above argument, we get
\[
\tilde{Y}(t) = \tilde{F}(t, r) + \sum_{n=1}^\infty \int_t^T \Phi^{(n)}(t, u) \tilde{F}(u, r) du
\]
\[
= \tilde{F}(t, r) + \int_t^T \Psi(t, u) \tilde{F}(u, r) du,
\]
(7.70)

where \( \Psi \) is defined by (7.61). Now substituting \( \mathbb{E}_Q(Y(s) \big| \mathcal{F}_t) = \tilde{Y}(s) \) (with \( r = t \) into (7.68) we obtain part (i) of the theorem. It remains to prove (7.64)-(7.65). By (7.67) we have
\[
U(t) = \int_t^T Z(t, s) dB_Q(s) + \int_t^T \int_{\mathbb{R}_0} K(t, s, \zeta) \tilde{N}_Q(ds, d\zeta); \quad 0 \leq t \leq s \leq T.
\]
(7.71)

Note that by the Clark-Ocone formula under change of measure (see [21]), extended to \( L^2(\mathcal{F}_T, P) \) as in [1], we get
\[
Z(t, s) = \mathbb{E}_Q[D_s U(t) - U(t) \int_s^T D_s \xi(r) dB_Q(r) \big| \mathcal{F}_s]; \quad t \leq s \leq T
\]
(7.72)

and
\[
K(t, s, \zeta) = \mathbb{E}_Q[U(t)(\tilde{H}_s - 1) + \tilde{H}_s D_s \xi U(t) \big| \mathcal{F}_s]; \quad t \leq s \leq T
\]
(7.73)
where
\[ \hat{H}_s = \exp \left[ \int_0^s \int_{\mathbb{R}_0} [D_{s,x} \beta(r, x) + \log \left( 1 - \frac{D_{s,x} \beta(r, x)}{1 - \beta(r, x)} \right)] \nu(dx)dr \\
+ \int_0^s \int_{\mathbb{R}_0} \log(1 - \frac{D_{s,x} \beta(r, x)}{1 - \beta(r, x)}) N_Q(dr, dx) \right]. \] (7.74)

as claimed. \(\square\)

We illustrate our result by a specific example:

**Example 7.15** \[18\] Let \( \Phi(t, r) = \rho(r-t) \) for some bounded function \( \rho \) defined on the positive half line and let \( \mathcal{L} \rho(s) = \int_0^\infty e^{-st} \rho(t) dt \) be the Laplace transform of \( \rho \). Then \( \Phi^n(t, r) = \rho_n(r-t) \), where \( \rho_n = \rho * \cdots * \rho \) is the n fold convolution of \( \rho \). The Laplace transform \( \mathcal{L} \rho_n(s) = (\mathcal{L} \rho(s))^n \). Thus if \( \Phi(t, r) = \rho(r-t) \), then

\[ \Psi(t, r) = \bar{\Psi}(r-t), \]

where the Laplace transform of \( \bar{\Psi} \) is

\[ \mathcal{L} \bar{\Psi}(s) = \sum_{n=1}^{\infty} (\mathcal{L} \rho(s))^n = \frac{\mathcal{L} \rho(s)}{1 - \mathcal{L} \rho(s)}. \]

In particular if \( \rho(x) = e^{-x}, x > 0 \), then \( \mathcal{L} \rho(s) = \frac{1}{1+s} \), which implies that \( \mathcal{L} \bar{\Psi}(s) = \frac{1}{s} \). Thus \( \bar{\Psi}(x) = 1 \).

### 7.3.5 Smoothness of the solution triplet

It is of interest to study when the solution components \( Z(t, s), K(t, s, \zeta) \) are smooth (\( C^1 \)) with respect to \( t \). Such smoothness properties are important in the study of optimal control. It is also important in the numerical solutions. Using the explicit form of the solution triplet, we can give sufficient conditions for such smoothness in the linear case.

**Theorem 7.16** \[18\] Assume that \( \xi, \beta \) are deterministic and that \( F(t) \) and \( \Phi(t, s) \) are \( C^1 \) with respect to \( t \) satisfying

\[ \mathbb{E}_Q \left[ \int_0^T \left\{ \int_t^T \left\{ F^2(t) + \Phi^2(t, s) + \left( \frac{dF(t)}{dt} \right)^2 + \left( \frac{\partial \Phi(t, s)}{\partial t} \right)^2 \right\} ds \right\} dt \right] < \infty. \] (7.75)

Then, for \( t < s \leq T \),

\[ Z(t, s) = \mathbb{E}_Q[D_s F(t) + \int_t^s \Phi(t, r) D_s Y(r) dr | \mathcal{F}_s], \] (7.76)

\[ K(t, s, \zeta) = \mathbb{E}_Q[D_{s, \zeta} F(t) + \int_t^s \Phi(t, r) D_{s, \zeta} Y(r) dr | \mathcal{F}_s]. \] (7.77)
In particular, we have
\[
\mathbb{E}_Q \left[ \int_0^T \left\{ \int_t^T \left( \frac{\partial Z}{\partial t}(t,s) \right)^2 ds \right\} dt + \int_0^T \left\{ \int_t^T \int_{\mathbb{R}_0} \left( \frac{\partial K}{\partial t}(t,s,\zeta) \right)^2 \nu(d\zeta) ds \right\} dt \right] < \infty.
\]
(7.78)

**Proof.** Since \(Y(t)\) is \(\mathcal{F}_t\)-measurable, we get that \(D_sY(t) = D_{s\zeta}Y(t) = 0\) for all \(s > t\). Hence by (7.63)
\[
\mathbb{E}_Q[D_sU(t)|\mathcal{F}_s] = \mathbb{E}_Q[D_sF(t) + \int_t^T \Phi(t,r)D_sY(r)dr|\mathcal{F}_s]
\]
(7.79) and
\[
\mathbb{E}_Q[D_{s,\zeta}U(t)|\mathcal{F}_s] = \mathbb{E}_Q[D_{s,\zeta}F(t) + \int_t^T \Phi(t,r)D_{s,\zeta}Y(r)dr|\mathcal{F}_s].
\]
(7.80)
Then the result follows from (7.64) and (7.65). □

### 7.4 Stochastic maximum principles for SVIEs

In this section, we study stochastic maximum principles of stochastic Volterra integral systems under partial information, i.e., the information available to the controller is given by a sub-filtration \(\mathcal{G} = \{\mathcal{G}_t\}_{t \geq 0}\) such that \(\mathcal{G}_t \subseteq \mathcal{F}_t\) for all \(t \geq 0\). The set \(U \subset \mathbb{R}\) is assumed to be convex. The set of admissible controls, i.e. the strategies available to the controller is given by a subset \(A_{\mathcal{G}}\) of the càdlàg, \(U\)-valued and \(\mathcal{G}\)-adapted processes.

The state of our system \(X^u(t) = X(t)\) satisfies the following SVIE
\[
X(t) = \xi(t) + \int_0^t b(t,s,X(s),u(s))ds + \int_0^t \sigma(t,s,X(s),u(s))dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(t,s,X(s),u(s),\zeta)N(ds,d\zeta); t \in [0,T],
\]
(7.81)

where \(b(t,s,x,u) = b(t,s,x,u,\omega) : [0,T]^2 \times \mathbb{R} \times U \times \Omega \to \mathbb{R}\), \(\sigma(t,s,x,u) = \sigma(t,s,x,u,\omega) : [0,T]^2 \times \mathbb{R} \times U \times \Omega \to \mathbb{R}\) and \(\gamma(t,s,x,u,\zeta) = \gamma(t,s,x,u,\zeta,\omega) : [0,T]^2 \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \to \mathbb{R}\).

The **performance functional** has the form
\[
J(u) = \mathbb{E} \left[ \int_0^T f(t,X(t),u(t))dt + g(X(T)) \right], \quad u \in A_{\mathcal{G}},
\]
(7.82)

with given functions \(f(t,x,u) = f(t,x,u,\omega) : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}\) and \(g(x) = g(x,\omega) : \mathbb{R} \times \Omega \to \mathbb{R}\).

We impose the following assumption:

**Assumption A1**

The processes \(b, \sigma, \gamma\) are \(\mathcal{F}_s\)-adapted for all \(s \leq t\), and twice continuously differentiable \((C^2)\) with respect to \(t\), \(x\) and continuously differentiable\((C^1)\) with respect to \(u\) for each \(s\). The driver \(g\) is assumed to be \(\mathcal{F}_T\)-measurable and \(C^1\) in \(x\). Moreover, all the partial derivatives are supposed to be bounded.
Note that the performance functional (7.82) is not of Volterra type.

Define the Hamiltonian functional associated to our control problem (7.81) and (7.82), as

\[ H(t, x, v, p, q, r(\cdot)) := H^0(t, x, v, p, q, r(\cdot)) + H^1(t, x, v, p, q, r(\cdot)), \]  

(7.83)

where

\[ H^0 : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times L^2_\nu \rightarrow \mathbb{R} \]

and

\[ H^1 : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times L^2_\nu \rightarrow \mathbb{R} \]

by

\[ H^0(t, x, v, p, q, r(\cdot)) := f(t, x, v) + p(t)b(t, t, x, v) + q(t, t)\sigma(t, t, x, v) \]

\[ + \int_{\mathbb{R}^0} r(t, t, \zeta)\gamma(t, t, x, v, \zeta)\nu(d\zeta), \]

(7.84)

\[ H^1(t, x, v, p, q, r(\cdot)) := \int_t^T p(s)\frac{\partial H}{\partial x}(s, t, x, v)ds + \int_t^T q(s, t)\frac{\partial x}{\partial s}(s, t, x, v)ds \]

\[ + \int_t^T \int_{\mathbb{R}^0} r(s, t, \zeta)\frac{\partial x}{\partial s}(s, t, x, v, \zeta)\nu(d\zeta)ds. \]

We may regard \( x, p, q, r = r(\cdot) \) as generic values for the processes \( X(\cdot), p(\cdot), q(\cdot), r(\cdot) \), respectively.

The BSVIE for the adjoint processes \( p(t), q(t, s), r(t, s, \cdot) \) is defined by

\[ p(t) = \frac{\partial g}{\partial x}(X(T)) + \int_t^T \frac{\partial H}{\partial x}(s, t, x, v)ds - \int_t^T q(t, s)dB(s) \]

\[ - \int_t^T \int_{\mathbb{R}^0} r(t, s, \zeta)\mathbb{N}(ds, d\zeta); t \in [0, T], \]

(7.85)

where we have used the simplified notation

\[ \frac{\partial H}{\partial x}(t) = \frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t, t), r(t, t, \cdot)). \]

**Remark 7.17** Using the definition of \( \mathcal{H} \) and the Fubini theorem, we see that the driver in
the BSVIE (7.85) can be explicitly written
\[
\int_t^T \frac{\partial H}{\partial x}(s)ds = \int_t^T \left\{ \frac{\partial f}{\partial x}(s, x, v) + p(s) \frac{\partial b}{\partial x}(s, s, x, v) + \int_s^T p(z) \frac{\partial^2 b}{\partial z \partial x}(z, t, x, v)dz \right. \\
+ q(s, s) \frac{\partial \sigma}{\partial x}(s, s, x, v) + \int_s^T q(z, t) \frac{\partial^2 \sigma}{\partial z \partial x}(z, t, x, v)dz \\
+ \int_{\mathbb{R}_0} r(s, s, \zeta) \frac{\partial \gamma}{\partial x}(s, s, x, v, \zeta) \nu(d\zeta) + \int_s^T \int_{\mathbb{R}_0} r(z, t, \zeta) \frac{\partial^2 \gamma}{\partial z \partial x}(z, t, x, v, \zeta) \nu(d\zeta)dz \right\}ds.
\]
From this it follows by Theorem 3.1 in Agram et al [6], that we have existence and uniqueness
of the solution of equation (7.85).

From now on we also make the following assumption:

Remark 7.18 Note that from equation (7.81), we get the following equivalent formulation, for each
\[
d X(t) = \xi'(t)dt + b(t, t, X(t), u(t)) dt + (\int_0^t \frac{\partial h}{\partial t}(t, s, X(s), u(s)) ds)dt \\
+ \sigma(t, t, X(t), u(t)) dB(t) + (\int_0^t \frac{\partial \sigma}{\partial t}(t, s, X(s), u(s)) dB(s))dt \\
+ \int_{\mathbb{R}_0} \gamma(t, t, X(t), u(t), \zeta) \tilde{N}(dt, d\zeta) + (\int_0^t \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial t}(t, s, X(s), u(s), \zeta) \tilde{N}(ds, d\zeta))dt,
\]
and from equation (7.85) under assumption A2, we have the following differential form
\[
\begin{cases}
dp(t) = -\left[ \frac{\partial h}{\partial x}(t) + \int_t^T \frac{\partial h}{\partial t}(t, s)dB(s) + \int_t^T \int_{\mathbb{R}_0} \frac{\partial \gamma}{\partial x}(t, s, \zeta) \tilde{N}(ds, d\zeta) \right] dt \\
+ q(t, t) dB(t) + \int_{\mathbb{R}_0} r(t, t, \zeta) \tilde{N}(dt, d\zeta), \\
p(T) = \frac{\partial h}{\partial x}(X(T)).
\end{cases}
\]

Remark 7.18 Assumption A2 is verified in a subclass of linear BSVIE with jumps, as we will see in section 5. For more details, we refer to Hu and Øksendal [18].

7.4.1 A sufficient maximum principle

We now state and prove a sufficient version of the maximum principle approach (a verification theorem).

Theorem 7.19 (Sufficient maximum principle [6]) Let \( \hat{u} \in \mathcal{A}_G \), with corresponding solutions \( \hat{X}(t), (\hat{p}(t), \hat{q}(t, s), \hat{r}(t, s, \cdot)) \) of (7.81) and (7.85) respectively. Assume that
• The functions
\[ x \mapsto g(x), \]
and
\[ (x, u) \mapsto \mathcal{H}(t, x, u, \hat{p}, \hat{q}, \hat{r}(\cdot)) \]
are concave.

• (The maximum condition)
\[
\sup_{v \in U} \mathbb{E}[\mathcal{H}(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t, t), \hat{r}(t, t, \cdot)) | \mathcal{G}_t] \\
= \mathbb{E}[\mathcal{H}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t, t), \hat{r}(t, t, \cdot)) | \mathcal{G}_t] \ \forall t \text{ P-a.s.} \tag{7.89}
\]

Then \( \hat{u} \) is an optimal control for our problem.

Proof. By considering a sequence of stopping times converging upwards to \( T \), we see that we may assume that all the \( dB \)- and \( \tilde{N} \)- integrals in the following are martingales and hence have expectation 0.

Choose \( u \in \mathcal{A}_C \), we want to prove that \( J(u) \leq J(\hat{u}) \).

By the definition of the cost functional \( (7.82) \), we have
\[
J(u) - J(\hat{u}) = I_1 + I_2, \tag{7.90}
\]
where we have used the shorthand notations
\[
I_1 = \mathbb{E} \left[ \int_0^T \tilde{f}(t) \, dt \right], \quad I_2 = \mathbb{E} [\tilde{g}(T)],
\]
and
\[
\tilde{f}(t) = f(t) - \hat{f}(t),
\]
with
\[
f(t) = f(t, X(t), u(t)), \quad \hat{f}(t) = f(t, \hat{X}(t), \hat{u}(t)),
\]
and similarly for \( b(t, t) = b(t, t, X(t), u(t)) \), and the other coefficients.

By the definition of the Hamiltonian \( (7.84) \), we get
\[
I_1 = \mathbb{E} \left[ \int_0^T \{ \tilde{H}(t) - \hat{H}(t) \hat{b}(t, t) - \hat{q}(t, t) \hat{\sigma}(t, t) - \int_{\mathbb{R}_0} \hat{r}(t, t, \zeta) \hat{\gamma}(t, t, \zeta) \nu(d\zeta) \} \, dt \right], \tag{7.91}
\]
where \( \tilde{H}(t) = H(t) - \tilde{H}(t) \) with
\[
H(t) = H(t, X(t), u(t), \hat{p}(t), \hat{q}(t, t), \hat{r}(t, t, \cdot)), \quad \tilde{H}(t) = \tilde{H}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t, t), \hat{r}(t, t, \cdot)).
\]
By the concavity of $g$ and the terminal value of the BSVIE \((7.85)\), we obtain

$$I_2 \leq \mathbb{E}[\frac{\partial \hat{b}}{\partial x}(T)\hat{X}(T)] = \mathbb{E}[\hat{p}(T)\hat{X}(T)].$$

Applying the Itô formula to $\hat{p}(t)\hat{X}(t)$, we get

$$I_2 \leq \mathbb{E}[\hat{p}(T)\hat{X}(T)]$$

$$= \mathbb{E}\left[\int_0^T \hat{p}(t)\{\hat{b}(t, t) + \int_0^t \frac{\partial \hat{b}}{\partial t}(t, s)ds + \int_0^t \frac{\partial \hat{\sigma}}{\partial t}(t, s)dB(s)\right]$$

$$+ \int_0^T \int_{\mathbb{R}_0} \frac{\partial \hat{\nu}}{\partial t}(t, s, \zeta)\tilde{N}(ds, d\zeta)dt + \int_0^T \hat{X}(t)\{-\frac{\partial \hat{\nu}}{\partial x}(t) + \int_t^T \frac{\partial \hat{\nu}}{\partial t}(t, s)dB(s)\}$$

$$+ \int_t^T \int_{\mathbb{R}_0} \frac{\partial \hat{\nu}}{\partial t}(t, s, \zeta)\tilde{N}(ds, d\zeta)dt + \int_0^T \hat{q}(t, t)\tilde{\sigma}(t, t)dt + \int_0^T \int_{\mathbb{R}_0} \tilde{r}(t, t, \zeta)\tilde{\gamma}(t, t, \zeta)\nu(d\zeta)dt\]$$

(7.92)

By the Fubini theorem, we get

$$\int_0^T \hat{p}(t)\left(\int_0^t \frac{\partial \hat{b}}{\partial t}(t, s)ds\right)dt = \int_0^T \left(\int_t^T \hat{p}(t)\frac{\partial \hat{b}}{\partial t}(t, s)dt\right)ds = \int_0^T \left(\int_t^T \hat{p}(s)\frac{\partial \hat{b}}{\partial t}(s, t)ds\right)dt.$$  

(7.93)

The generalized duality formula for the Brownian motion, yields

$$\mathbb{E}\left[\int_0^T \hat{p}(t)\left(\int_0^t \frac{\partial \hat{\sigma}}{\partial t}(t, s)dB(s)\right)dt\right] = \int_0^T \mathbb{E}\left[\int_0^t \hat{p}(t)\frac{\partial \hat{\sigma}}{\partial t}(t, s)dB(s)\right]dt$$

$$= \int_0^T \mathbb{E}\left[\int_0^t \mathbb{E}[D_s\hat{p}(t)|\mathcal{F}_s]\frac{\partial \hat{\sigma}}{\partial t}(t, s)ds\right]dt.$$

Fubini’s theorem gives

$$\mathbb{E}\left[\int_0^T \hat{p}(t)\left(\int_0^t \frac{\partial \hat{\nu}}{\partial t}(t, s)dB(s)\right)dt\right] = \int_0^T \mathbb{E}\left[\int_s^T \mathbb{E}[D_s\hat{p}(t)|\mathcal{F}_s]\frac{\partial \hat{\nu}}{\partial t}(t, s)dt\right]ds$$

$$= \mathbb{E}\left[\int_0^T \int_t^T \mathbb{E}[D_t\hat{p}(s)|\mathcal{F}_t]\frac{\partial \hat{\nu}}{\partial s}(s, t)dtds\right],$$

and by equality \((7.3)\), we end up with

$$\mathbb{E}\left[\int_0^T \hat{p}(t)\left(\int_0^t \frac{\partial \hat{\nu}}{\partial t}(t, s)dB(s)\right)dt\right] = \mathbb{E}\left[\int_0^T \int_t^T \hat{q}(s, t)\frac{\partial \hat{\nu}}{\partial s}(s, t)dtds\right].$$  

(7.94)

Doing similar considerations as for the Brownian setting for the jumps, such as the Fubini theorem, the generalized duality formula for jumps, we obtain
\[
\begin{align*}
\mathbb{E}\left[ \int_0^T \left( \int_0^t \hat{p}(t) \frac{\partial \phi}{\partial t}(t, s, \zeta) \tilde{N}(ds, d\zeta) \right) dt \right] &= \int_0^T \mathbb{E}\left[ \int_0^t \hat{p}(t) \frac{\partial \phi}{\partial t}(t, s, \zeta) \tilde{N}(ds, d\zeta) \right] dt \\
&= \int_0^T \mathbb{E}\left[ \int_0^t \int_0^T \mathbb{E}\left[ D_{s, \zeta} \hat{p}(t) | \mathcal{F}_s \right] \frac{\partial \phi}{\partial t}(t, s, \zeta) \nu(d\zeta) ds \right] dt \\
&= \mathbb{E}\left[ \int_0^T \int_t^T \int_0^T \mathbb{E}\left[ D_{t, \zeta} \hat{p}(s) | \mathcal{F}_t \right] \frac{\partial \phi}{\partial t}(s, t, \zeta) \nu(d\zeta) ds dt \right] \\
&= \mathbb{E}\left[ \int_0^T \int_t^T \int_0^T \hat{r}(s, t, \zeta) \frac{\partial \phi}{\partial \zeta}(s, t, \zeta) \nu(d\zeta) ds dt \right].
\end{align*}
\]

(7.95)

Substituting (7.93), (7.94) and (7.95) combined with (7.83) in (7.90), yields

\[
J(u) - J(\hat{u}) \leq \mathbb{E}\left[ \int_0^T \left\{ \mathcal{H}(t) - \hat{\mathcal{H}}(t) - \frac{\partial \hat{\mathcal{H}}}{\partial \zeta}(t) \tilde{X}(t) \right\} dt \right].
\]

By the concavity of \( \mathcal{H} \), we have

\[
\mathcal{H}(t) - \hat{\mathcal{H}}(t) \leq \frac{\partial \hat{\mathcal{H}}}{\partial \zeta}(t) \tilde{X}(t) + \frac{\partial \hat{\mathcal{H}}}{\partial \zeta}(t) \tilde{u}(t).
\]

Hence, since \( u = \hat{u} \) is \( \mathcal{G} \)-adapted and maximizes the conditional Hamiltonian,

\[
\begin{align*}
J(u) - J(\hat{u}) &\leq \mathbb{E}\left[ \int_0^T \frac{\partial \mathcal{H}}{\partial \zeta}(t)(u(t) - \hat{u}(t)) dt \right] \\
&= \mathbb{E}\left[ \int_0^T \mathbb{E}\left[ \frac{\partial \mathcal{H}}{\partial \zeta}(t) | \mathcal{G}_t \right] (u(t) - \hat{u}(t)) dt \right] \leq 0,
\end{align*}
\]

(7.96)

which means that \( \hat{u} \) is an optimal control. □

### 7.4.2 A necessary maximum principle

Suppose that a control \( u \in \mathcal{A}_G \) is optimal and that \( \beta \in \mathcal{A}_G \). If the function \( \lambda \mapsto J(u + \lambda \beta) \) is well-defined and differentiable on a neighbourhood of 0, then

\[
\frac{d}{d\lambda} J(u + \lambda \beta) \big|_{\lambda=0} = 0.
\]

Under a set of suitable assumptions on the coefficients, we will show that

\[
\frac{d}{d\lambda} J(u + \lambda \beta) \big|_{\lambda=0} = 0
\]

57
is equivalent to
\[ \mathbb{E}[\frac{\partial H}{\partial u}(t) \mid \mathcal{G}_t] = 0 \quad P - \text{a.s. for each } t \in [0, T]. \]

The details are as follows:
For each given \( t \in [0, T] \), let \( \eta = \eta(t) \) be a bounded \( \mathcal{G}_t \)-measurable random variable, let \( h \in [T - t, T] \) and define
\[ \beta(s) := \eta 1_{[t, t+h]}(s); s \in [0, T]. \] (7.97)

Assume that
\[ u + \lambda \beta \in \mathcal{A}_G, \] (7.98)
for all \( \beta \) and all \( u \in \mathcal{A}_G \), and all non-zero \( \lambda \) sufficiently small. Assume that the derivative process \( Y(t) \), defined by
\[ Y(t) = \frac{d}{d\lambda} X(u + \lambda \beta(t))|_{\lambda=0}, \] (7.99)
exists.
Then we see that
\[
Y(t) = \int_0^t \left( \frac{\partial b}{\partial x}(t,s)Y(s) + \frac{\partial b}{\partial u}(t,s)\beta(s) \right) ds \\
+ \int_0^t \left( \frac{\partial \sigma}{\partial x}(t,s)Y(s) + \frac{\partial \sigma}{\partial u}(t,s)\beta(s) \right) dB(s) \\
+ \int_0^t \int_{\mathbb{R}_0} \left( \frac{\partial \gamma}{\partial x}(t,s,\zeta)Y(s) + \frac{\partial \gamma}{\partial u}(t,s,\zeta)\beta(s) \right) \tilde{N}(ds,d\zeta),
\]
and hence
\[
dY(t) = \left[ \frac{\partial b}{\partial t}(t,t)Y(t) + \frac{\partial b}{\partial u}(t,t)\beta(t) + \int_0^t \left( \frac{\partial^2 b}{\partial t \partial x}(t,s)Y(s) + \frac{\partial^2 b}{\partial t \partial u}(t,s)\beta(s) \right) ds \\
+ \int_0^t \left( \frac{\partial^2 \sigma}{\partial t \partial x}(t,s)Y(s) + \frac{\partial^2 \sigma}{\partial t \partial u}(t,s)\beta(s) \right) dB(s) \\
+ \int_0^t \int_{\mathbb{R}_0} \left( \frac{\partial^2 \gamma}{\partial t \partial x}(t,s,\zeta)Y(s) + \frac{\partial^2 \gamma}{\partial t \partial u}(t,s,\zeta)\beta(s) \right) \tilde{N}(ds,d\zeta) \right] dt \\
+ \left( \frac{\partial b}{\partial t}(t,t)Y(t) + \frac{\partial b}{\partial u}(t,t)\beta(t) \right) dB(t) \\
+ \int_{\mathbb{R}_0} \left( \frac{\partial \gamma}{\partial x}(t,t,\zeta)Y(t) + \frac{\partial \gamma}{\partial u}(t,t,\zeta)\beta(t) \right) \tilde{N}(dt,d\zeta). \] (7.100)

We are now ready to formulate the result:

**Theorem 7.20 (Necessary maximum principle [6])** Suppose that \( \hat{u} \in \mathcal{A}_G \) is such that, for all \( \beta \) as in (7.97),
\[ \frac{d}{d\lambda} J(\hat{u} + \lambda \beta)|_{\lambda=0} = 0 \] (7.101)
and the corresponding solution \( \hat{X}(t) \), \((\hat{p}(t), \hat{q}(t), \hat{r}(t, t, \cdot))\) of \((7.81)\) and \((7.85)\) exists. Then,

\[
\mathbb{E}[\frac{\partial \mathcal{H}}{\partial u}(t)|\mathcal{G}_t]|_{u=\hat{u}(t)} = 0. \tag{7.102}
\]

Conversely, if \((7.102)\) holds, then \((7.101)\) holds.

Proof. By considering a suitable increasing family of stopping times converging to \(T\), we may assume that all the local martingales \((dB-\text{ and } \hat{N}\text{- integrals})\) appearing in the proof below are martingales. For simplicity of notation we drop the “hat” everywhere and write \(u\) in stead of \(\hat{u}\), \(X\) in stead of \(\hat{X}\) etc in the following. Consider

\[
\frac{d}{dx}J(u + \lambda \beta)|_{\lambda=0} = \mathbb{E}\left[ \int_0^T \left\{ \frac{\partial T}{\partial x}(t)Y(t) + \frac{\partial T}{\partial u}(t)\beta(t) \right\} dt + \frac{\partial T}{\partial x}(X(T))Y(T) \right]. \tag{7.103}
\]

Applying the Itô formula, we get

\[
\mathbb{E}\left[ \frac{\partial \mathcal{H}}{\partial x}(X(T))Y(T) \right] = \mathbb{E}[p(T)Y(T)]
\]

\[
= \mathbb{E}\left[ \int_0^T p(t)\left( \int_0^t \frac{\partial p}{\partial x}(s,t)Y(s) + \frac{\partial p}{\partial u}(s,t)\beta(s)ds \right) dt \right]
\]

\[
+ \int_0^T p(t)\left\{ \int_0^t \left\{ \int_0^s \frac{\partial^2 p}{\partial x^2}(r,s,t)Y(s) + \frac{\partial^2 p}{\partial x \partial u}(r,s,t)\beta(s)ds \right\} dr \right\} dt
\]

\[
- \int_0^T Y(t)\frac{\partial T}{\partial x}(t)dt + \int_0^T q(t,s)(\frac{\partial p}{\partial x}(t,s)Y(t) + \frac{\partial p}{\partial u}(t,s)\beta(t))dt
\]

\[
+ \int_0^T \int_0^s r(t,s,\zeta)(\frac{\partial^2 p}{\partial x^2}(t,s,\zeta)Y(t) + \frac{\partial^2 p}{\partial x \partial u}(t,s,\zeta)\beta(t))\nu(d\zeta)dt.
\]

From \((7.94)\) and \((7.95)\), we have

\[
\mathbb{E}\left[ p(T)Y(T) \right] = \mathbb{E}\left[ \int_0^T \left\{ \frac{\partial p}{\partial x}(t) + \int_t^T \left( \frac{\partial^2 p}{\partial s \partial x}(s,t)p(s) + \frac{\partial^2 p}{\partial s \partial u}(s,t)q(s,t) \right) ds \right\} \right] dt
\]

\[
+ \int_0^T \int_0^s \frac{\partial^2 p}{\partial x^2}(s,t,\zeta)\nu(d\zeta)Y(t) dt
\]

\[
+ \int_0^T \int_0^s \frac{\partial^2 p}{\partial x \partial u}(s,t,\zeta)\beta(t)\nu(d\zeta)Y(t) dt
\]

\[
+ \int_0^T \int_0^s \left( \frac{\partial p}{\partial x}(t,s,\zeta)Y(t) + \frac{\partial p}{\partial u}(t,s,\zeta)\beta(t) \right) r(t,s,\zeta)\nu(d\zeta)Y(t) dt
\]

\[
+ \int_0^T \int_0^s \left( \frac{\partial^2 p}{\partial x^2}(t,s,\zeta)Y(t) + \frac{\partial^2 p}{\partial x \partial u}(t,s,\zeta)\beta(t) \right) r(t,s,\zeta)\nu(d\zeta)Y(t) dt
\]

Using the definition of \( \mathcal{H} \) in \((7.83)\) and the definition of \( \beta \), we obtain

\[
\frac{d}{dx}J(u + \lambda \beta)|_{\lambda=0} = \mathbb{E}\left[ \int_0^T \frac{\partial \mathcal{H}}{\partial u}(s)\beta(s)ds \right] = \mathbb{E}\left[ \int_t^{t+h} \frac{\partial \mathcal{H}}{\partial u}(s)ds \right]. \tag{7.104}
\]

Now suppose that

\[
\frac{d}{dx}J(u + \lambda \beta)|_{\lambda=0} = 0. \tag{7.105}
\]

Differentiating the right-hand side of \((7.104)\) at \(h = 0\), we get

\[
\mathbb{E}[\frac{\partial \mathcal{H}}{\partial u}(t)\eta] = 0.
\]

59
Since this holds for all bounded $\mathcal{G}_t$-measurable $\eta$, we have
\[
\mathbb{E}[\frac{\partial H}{\partial u}(t)|\mathcal{G}_t] = 0. \quad (7.106)
\]
Conversely, if we assume that (7.106) holds, then we obtain (7.105) by reversing the argument
we used to obtain (7.104).

$\square$

**EXERCISE**

Let $X^u(t) = X(t)$ be a given cash flow, modelled by the following stochastic Volterra equation:
\[
X(t) = x_0 + \int_0^t [b_0(t,s)X(s) - u(s)] ds + \int_0^t \sigma_0(s)X(s) dB(s)
+ \int_0^t \int_{R_0} \gamma_0(s,\zeta) X(s) \tilde{N}(ds, d\zeta); \quad t \geq 0,
\]
(7.107)
or, in differential form,
\[
\begin{cases}
\quad dX(t) = [b_0(t,t)X(t) - u(t)] dt + \sigma_0(t)X(t) dB(t) \\
+ \int_{R_0} \gamma_0(t,\zeta) X(t) \tilde{N}(dt, d\zeta) + \left[ \int_0^t \frac{\partial b_0}{\partial t}(t,s)X(s)ds \right] dt; \quad t \geq 0.
\end{cases}
\]
(7.108)

We see that the dynamics of $X(t)$ contains a history or memory term represented by the $ds$-integral.
We assume that $b_0(t,s)$, $\sigma_0(s)$ and $\gamma_0(s,\zeta)$ are given deterministic functions of $t$, $s$, and $\zeta$, with values in $\mathbb{R}$, and that $b_0(t,s)$ is continuously differentiable with respect to $t$ for each $s$.
For simplicity we assume that these functions are bounded, and we assume that there exists $\varepsilon > 0$ such that $\gamma_0(s,\zeta) \geq -1 + \varepsilon$ for all $s, \zeta$ and the initial value $x_0 \in \mathbb{R}$.
We want to solve the following maximisation problem:
Find $\hat{u} \in \mathcal{A}_G$, such that
\[
\sup_u J(u) = J(\hat{u}), \quad (7.109)
\]
where
\[
J(u) = \mathbb{E} \left[ \theta X(T) + \int_0^T \log(u(t)) dt \right]. \quad (7.110)
\]
Here $\theta = \theta(\omega)$ is a given $\mathcal{F}_T$-measurable random variable.

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