The wave equation on Schwarzschild-de Sitter spacetimes

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Abstract

We consider solutions to the linear wave equation $\Box_g \phi = 0$ on a non-extremal maximally extended Schwarzschild-de Sitter spacetime arising from arbitrary smooth initial data prescribed on an arbitrary Cauchy hypersurface. (In particular, no symmetry is assumed on initial data, and the support of the solutions may contain the sphere of bifurcation of the black/white hole horizons and the cosmological horizons.) We prove that in the region bounded by a set of black/white hole horizons and cosmological horizons, solutions $\phi$ converge pointwise to a constant faster than any given polynomial rate, where the decay is measured with respect to natural future-directed advanced and retarded time coordinates. We also give such uniform decay bounds for the energy associated to the Killing field as well as for the energy measured by local observers crossing the event horizon. The results in particular include decay rates along the horizons themselves. Finally, we discuss the relation of these results to previous heuristic analysis of Price and Brady et al.

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1 Introduction

The introduction of a positive cosmological constant in the Einstein equations of general relativity gives rise to a wide variety of new interesting solution space-times, in particular, spacetimes containing both “black hole” and “cosmological” regions. As in the case of black-hole spacetimes with vanishing cosmological constant, the stability of these spacetimes as solutions to the Einstein equations is a fundamental open problem of gravitational physics. Yet even the simplest questions concerning the behaviour of linear waves on such spacetime backgrounds today remain unanswered. In this paper, we initiate in the above context the mathematical study of decay for solutions to the linear wave equation.

The simplest family of black-hole spacetimes with positive cosmological constant is the so-called Schwarzschild-de Sitter family. If the cosmological constant $\Lambda > 0$ is considered fixed, this is a 1-parameter family of solutions $(\mathcal{M}, g)$ to the Einstein vacuum equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu},$$

(1)
with parameter $M$, called the *mass*. We shall consider only the non-extremal black-hole case, corresponding to parameter values

$$0 < M < \frac{1}{3\sqrt{\Lambda}}$$

(2)

As with the Schwarzschild family, the first manifestation of the Schwarzschild-de Sitter family of solutions was an expression for the metric in local coordinates, in this case first published in 1918 by Kottler [17], and independently by Weyl [20], in the form

$$-\left(1 - \frac{2}{r}M - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{2}{r}M - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2 d\sigma_{S^2}. \quad (3)$$

Here $d\sigma_{S^2}$ denotes the standard metric on the unit 2-sphere. The global structure of maximal spherically symmetric vacuum extensions of such metrics was only understood much later [8, 18, 13] based on the methods of formal Penrose diagrams introduced by B. Carter. In fact, maximally extended spherically symmetric vacuum spacetimes $(\mathcal{M}, g)$ with various different topologies can be constructed, all of which equally well merit the name “Schwarzschild-de Sitter with parameter $M$ and cosmological constant $\Lambda$”. Such solutions $(\mathcal{M}, g)$ all share the property that the universal cover $\tilde{Q}$ of the 2-dimensional Lorentzian quotient $Q = \mathcal{M}/SO(3)$ consists of an infinite chain of regions as depicted in the Penrose diagram below:

The results of this paper do not depend on the topology, and for definiteness, one may assume in what follows that the name “Schwarzschild de-Sitter” and the notation $(\mathcal{M}, g)$ refer to the spacetime with quotient precisely the universal cover depicted above.

It is then the wave equation

$$\Box_g \phi = 0$$

(4)

on this background $(\mathcal{M}, g)$ whose mathematical study we wish to initiate here. There is already a rich body of heuristic work on this problem in the physics literature. (See Section 1.4 below for a discussion.) The motivation for the study of (4) in the present context is multifold. In particular, as in the case of vanishing cosmological constant, studied in our previous [12], we believe that proving bounds on decay rates for solutions to (4) is a first step to a mathematical
understanding of non-linear stability problems for spacetimes containing black holes, that is to say, to the problem of stability in the context of the dynamics of (1). For a more detailed discussion, we refer the reader to the introductory remarks of [12].

1.1 The initial value problem for the wave equation

We are interested in solutions of (4) arising from suitably regular initial data prescribed on a Cauchy surface $\Sigma$ of $M$. For future applications to non-linear stability problems, it is crucial that all assumptions have a natural geometric interpretation independent of special coordinate systems. Moreover, our primary concern in this paper is the region $D$ bounded by a set of black/white hole horizons $H^+ \cup H^-$ and cosmological horizons $\overline{H}^+ \cup \overline{H}^-$:

$$D = \text{clos} \left( J^-(H^+ \cup \overline{H}^+) \cap J^+(H^- \cup \overline{H}^-) \right)$$

as depicted below.

By causality, the global behaviour of $\phi$ in $D$ can be understood independently of the behaviour near $r = 0$ and $r = \infty$. The behaviour in say $D \cap J^+(\Sigma)$ is completely determined by the behaviour of appropriate initial data on $\Sigma \cap J^-(D)$.\(^2\) We review briefly in the next paragraph the solvability and domain of dependence property for the initial value problem for (4).

Let $\Sigma \subset M$ be a smooth Cauchy surface and let $n^\mu$ denote the future-directed unit normal of $\Sigma$. For $s \geq 1$, let $\varphi$ be an $H^s_{\text{loc}}(\Sigma)$ function and $\varphi : \Sigma \to \mathbb{R}$ an $H^{s-1}_{\text{loc}}(\Sigma)$ function. Then there exists a unique global solution $\phi : M \to \mathbb{R}$ of

$$\Box_g \phi = 0$$

such that for all smooth spacelike hypersurfaces $\Sigma$ with future directed unit normal $n$, $\varphi|_{\Sigma'} \in H^s_{\text{loc}}$, $(\hat{n} \varphi)|_{\Sigma'} \in H^{s-1}_{\text{loc}}$, and $\phi|_{\Sigma} = \varphi$, $n\phi|_{\Sigma} = \varphi$. Moreover, if $K \subset M$ is closed and $\phi_1, \phi_2$ are two such solutions corresponding to data $(\varphi_1, \varphi_1), (\varphi_2, \varphi_2)$ such that $\varphi_1|_{\Sigma \cap K} = \varphi_2|_{\Sigma \cap K}, \varphi_1|_{\Sigma \cap K} = \varphi_2|_{\Sigma \cap K}$, then $\phi_1 = \phi_2$ on $M \setminus (J^+(\Sigma \setminus K) \cup J^-(\Sigma \setminus K))$. In particular, setting $K = J^-(D)$, we obtain that $\phi_1 = \phi_2$ on $J^+(\Sigma) \cap D$.

\(^1\)We employ in this paper the standard notation of Lorentzian geometry (e.g. $J^+$, $J^-$, etc.), and Penrose diagrams. See [13].

\(^2\)Note, as depicted, that $\Sigma \cap J^-(D)$ is not necessarily $\Sigma \cap D$. 
1.2 The main theorem

1.2.1 Norms on initial data

Since our results will be quantitative, we need to introduce relevant norms on the compact manifold with boundary \( \Sigma \cap J^- (\mathcal{D}) \). Let \( \| \cdot \| \) denote the Riemannian \( L^2 \) norm on \( \Sigma \cap J^- (\mathcal{D}) \). This induces a norm on sections of the tangent bundle, a norm we will denote also by \( \| \cdot \| \). If \( \varphi \in H^1_{\text{loc}}(\Sigma) \), \( \dot{\varphi} \in L^2_{\text{loc}}(\Sigma) \), then let us denote by \( \phi \) the unique solution of \( \Box g \phi = 0 \) corresponding to initial data \((\varphi, \dot{\varphi})\).

Let us define now for all real \( s \geq 0 \) the quantity

\[
E_s(\varphi, \dot{\varphi}) = \| \nabla_{\Sigma} \varphi \|^2 + \| \dot{\varphi} \|^2 + \sum_{\ell \geq 1} r^{2s} \ell^{2s} \| \nabla_{\Sigma} \phi_{\ell} \|^2 + r^{2s} \ell^{2s} \| n \phi_{\ell} \|^2,
\]

where \( \phi_{\ell} \) denotes the projection of \( \phi \) to the \( \ell \)th eigenspace of \( \Delta \), i.e. the \( \ell \)th spherical harmonic of \( \phi \). The function \( r \) is discussed in Section 2. If \( \Sigma \) itself is spherically symmetric, then we may replace \( \phi_{\ell} \), \( n \phi_{\ell} \) by \( \varphi_{\ell} \), \( \varphi_{\ell} \), and the above expression is a sum of integrals on initial data. For general \( \Sigma \), a sufficient condition for the finiteness of \( \Box g \phi = 0 \) is that \( \varphi \in H^{s+1}_{\text{loc}}(\Sigma) \), \( \dot{\varphi} \in H^s_{\text{loc}}(\Sigma) \).

In the case \( m \geq 0 \) an integer, we can characterize \( E_m \) geometrically as follows. Let \( \Omega_i \), \( i = 1, \ldots, 3 \) denote a basis of Killing fields generating the Lie algebra \( \mathfrak{so}(3) \) associated to the spherical symmetry of \((\mathcal{M}, g)\). We call \( \Omega_i \) angular momentum operators. It easily follows that

\[
E_m(\varphi, \dot{\varphi}) \sim \sum_{p_1, \ldots, p_{m-1}=0,1} \sum_{1 \leq i_1, \ldots, i_{m-1} \leq 3} \| \nabla_{\Sigma} (\Omega^{p_1}_{i_1} \cdots \Omega^{p_{m-1}}_{i_{m-1}} \varphi) \|^2 + \| (\Omega^{p_1}_{i_1} \cdots \Omega^{p_{m-1}}_{i_{m-1}} \varphi) \|^2.
\]

Again, if \( \Sigma \) itself is spherically symmetric, we may replace \( \phi \) with \( \varphi \) in the first term, and remove the \( n \) from the second, replacing \( \phi \) with \( \dot{\varphi} \).

1.2.2 First statement of the theorem

The main result of this paper is contained in the following

**Theorem 1.1.** Let \((\mathcal{M}, g)\) denote the Schwarzschild-de Sitter spacetime with parameter \( M \) and cosmological constant \( \Lambda \) satisfying \( \Box g \) and let \( \Sigma \) be a Cauchy surface for \( \mathcal{M} \). Let \( \mathcal{D} \subset \mathcal{M} \) denote a region as defined in \( \Box g \) and let \( s \geq 0 \).

Then, there exist constants \( C_s \) depending only on \( s \), \( M \), \( \Lambda \), and the geometry of \( \Sigma \cap J^- (\mathcal{D}) \) such that for all solutions \( \phi \) of the wave equation \( \Box g \phi = 0 \) on \( \mathcal{M} \) such that \( E_s(\varphi, \dot{\varphi}) \) is finite, where \( \varphi \equiv \phi|_{\Sigma} \), \( \dot{\varphi} \equiv n\phi|_{\Sigma} \), and for all achronal hypersurfaces \( \Sigma' \subset \mathcal{D} \cap J^+(\Sigma') \), the bound

\[
\int_{\Sigma'} T_{\mu \nu}(\phi) T^{\mu \nu} \leq C_s \ E_s(\varphi, \dot{\varphi})(v_{+}(\Sigma')^{-s} + u_{+}(\Sigma')^{-s})
\]

holds, where \( u \) and \( v \) denote fixed Eddington-Finkelstein advanced and retarded coordinates, \( u_{+} = \max\{ u, 1 \} \), \( u_{+}(\Sigma) = \inf_{x \in \Sigma} u_{+}(x) \), etc, \( T^{\mu \nu} \) denotes the

\[\text{See Section 2. Although these coordinates are only defined in } \mathcal{D} \text{, the statements } (7), (8) \text{ can be interpreted in all of } \mathcal{D} \text{ in view of conventions [10 - 22].} \]
Killing field coinciding in the interior of $\mathcal{D}$ with $\frac{\partial}{\partial t}$, $T_{\mu\nu}(\phi)$ denotes the standard energy-momentum tensor, and $\nu^\mu$ is the future-directed unit normal wherever $\Sigma'$ is spacelike, in which case the integral is taken with measure the induced volume form $\mathcal{D}$. 

In addition, (7) holds if $T^\mu$ is replaced by the vector field $N^\mu$ defined in Section 6. If $s > 1$, then the pointwise bound

$$|\phi - \phi| \leq C_s \mathbf{E}_s^\frac{1}{4}(\varphi, \dot{\varphi}) \left( \frac{v_+^{-s}}{v_+^{1-s}} + u_+^{1-s} \right)$$

holds in $J^+(\Sigma) \cap \mathcal{D}$, where $\phi$ is a constant satisfying

$$|\phi| \leq \sup_{x \in \Sigma} |\phi(x)| + C_0 \mathbf{E}_0^\frac{1}{4}(\varphi, \dot{\varphi}).$$

In particular, the theorem applies to arbitrary smooth initial data $\varphi \in C^\infty(\Sigma)$, $\dot{\varphi} \in C^\infty(\Sigma)$, where $s$ can be taken arbitrarily large. There are no unphysical assumptions regarding vanishing of $\phi$ at the sphere of bifurcation of the horizons, i.e. at the sets $\mathcal{H}^+ \cap \mathcal{H}^-$ and $\overline{\mathcal{H}^+} \cap \overline{\mathcal{H}^-}$. The decay rates (8), (9) are uniform, i.e. they hold up to and including the horizons, setting $u_+ = \infty$ or $v_+ = \infty$. In particular, $\Sigma'$ in (9) can be taken (as depicted below) to contain subsets of $\mathcal{H}^+$ and/or $\overline{\mathcal{H}^+}$.

### 1.2.3 Comparison with the Schwarzschild case

The statement of Theorem 1.1 should be compared with the results of our previous [10, 12] concerning the wave equation on a Schwarzschild exterior. Recall that in the region $r > 2M$, the Schwarzschild metric is given by the expression $3$ for $\Lambda = 0$, $M > 0$. The Penrose diagramme of the closure of this

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$^4$A correct interpretation of $n^\mu$ and the measure of integration for general achronal $\Sigma'$ can be derived by a limiting procedure.
region in the maximally extended Schwarzschild spacetime is given below:

\[
H^+ + H^- + I^+ + I^- \]

In [10], an analogue of (7) is proven for all \( s < 6 \) and spherically symmetric initial data. Modulo an \( \epsilon \), this result is expected to be sharp, as it not expected to be true for \( s > 6 \), in view of heuristic arguments due to Price [19].

In [12], an analogue of (7) is proven for \( s = 2 \) for arbitrary, not necessarily spherically symmetric, initial data.

In view of the fact that solutions of the wave equation vanish on \( I^+ \), the results of [12] allow one to obtain the uniform pointwise decay rate

\[
|\phi| \leq C v^{-1} + \sum_i \phi_i
\]

As a uniform decay bound in \( v \), this decay rate is in fact sharp.

1.2.4 Second statement of the theorem

The loss of angular derivatives in the result of Theorem 1.1 can be more precisely quantified by decomposing \( \phi \) into spherical harmonics. Each spherical harmonic \( \phi_\ell \) decays at least exponentially, but the bound on the exponential rate obtained here decreases inverse quadratically in the spherical harmonic number. We have the following

**Theorem 1.2.** Let \( (M, g), \Sigma, D \) be as in Theorem 1.1. Then there exists a constant \( c \) depending only on \( M \) and \( \Lambda \), and \( C_0 \) depending only on \( M, \Lambda, \) and the geometry of \( \Sigma \cap J^- (D) \), such that for all \( \phi_\ell \) solutions of the wave equation on \( M \) with spherical harmonic number \( \ell \) with

\[
E_0(\phi_\ell, \dot{\phi}_\ell) = \|\nabla \phi_\ell\|^2 + \|\dot{\phi}_\ell\|^2 < \infty
\]

and all achronal hypersurfaces \( \Sigma' \subset J^+ (\Sigma) \cap D \), the bound

\[
\int_{\Sigma'} T_{\mu\nu}(\phi_\ell) T^{\mu\nu} \leq C_0 E_0(\phi_\ell, \dot{\phi}_\ell) \left( e^{-2cv+ (\Sigma')/\ell^2} + e^{-2cu_+ (\Sigma')/\ell^2} \right)
\]

holds for all \( \ell \geq 0 \), and, again as before, also with \( T^\mu \) above replaced with \( N^\mu \) defined in Section 6. In addition, the pointwise bounds

\[
|\phi_\ell(u, v)| \leq C_0 E_0(\phi_\ell, \dot{\phi}_\ell) (e^{-cv_+/\ell^2} + e^{-cu_+/\ell^2})
\]

for \( \ell \geq 1 \), and

\[
|\phi_0(u, v) - \bar{\phi}| \leq C_0 E_0(\phi_0, \dot{\phi}_0) (e^{-cv_+/\ell^2} + e^{-cu_+/\ell^2}),
\]
for $\ell = 0$, hold in $J^+(\Sigma) \cap D$, where $\phi$ is a constant satisfying
\[ |\phi| \leq \inf_{x \in \Sigma} |\phi_0(x)| + C_0 E_0^2(\phi_0, \dot{\phi}_0). \]

The above theorem can easily be seen to imply Theorem 1.1.

1.3 Overview of the proof

In this paper, we insist on a framework of proof that in principle may have relevance to the non-linear stability problem, that is to say, the problem of the dynamics of (1) starting from initial data close to those induced on a Cauchy hypersurface $\Sigma$ of Schwarzschild-de Sitter. This leads us to try to exploit compatible currents. In this section, we will describe this general approach, and the natural relation of the currents we will define with various geometric and analytical aspects of the problem at hand.

1.3.1 Vector fields and compatible currents

For quasilinear hyperbolic systems (like (1)) in $3 + 1$ dimensions, all known techniques for studying the global dynamics are based on $L^2$ estimates. In the Lagrangian case, the origin of such estimates can be understood geometrically in terms of compatible currents (see Christodoulou [9]). These are 1-forms $J_\mu$ such that at each point $x \in M$, both $J_\mu$ and the divergence $K = \nabla_\mu J^\mu$ depend only on the 1-jet of $\phi$. An important class of these are the currents $J_\mu^V$ obtained by contracting the energy momentum tensor $T_{\mu\nu}$ with an arbitrary vector field $V^\mu$. See Section 3 for a discussion in the context of the linear wave equations $\square_\gamma \phi = 0$ studied here. All estimates in this paper are obtained by exploiting the integral identities
\[ \int_{\mathcal{R}} K = \int_{\partial \mathcal{R}} J_\mu n^\mu \quad (10) \]
corresponding to compatible currents of the form $J_\mu^V$ and straightforward modifications thereof $J_\mu = J_\mu^V + \cdots$, where the vector fields $V$ are directly related to the geometry of the problem, and the region $\mathcal{R}$ is suitably chosen.

1.3.2 The photon sphere and the currents $J_\mu^X$

The timelike hypersurface $r = 3M$ is known as the photon sphere. This has the ominous property of being spanned by null geodesics. If additional regularity is not imposed, then it is clear by a geometric optics approximation that solutions of the wave equation can concentrate their energy along such geodesics for arbitrary long times, and one can thus not achieve a quantitative bound for the rate of decay in terms of initial energy alone. In particular, (7) cannot hold for $s > 0$ if $E_s$ is replaced by $E_0$.

It is truly remarkable that this obstruction arising from geometrical optics is captured, and quantified, by a current $J_\mu$ associated to a vector field $V$.
of the form $f(r^*)\partial_{r^*}$ for a well-chosen function $f$. The story is not entirely straightforward, however. The desired current is in fact not precisely of the form $J^Y_\mu$, but a modification thereof, to be denoted $J^{X,3}_\mu$, which is associated in a well defined way to a collection of vector fields $X_\ell = f_\ell(r^*)\partial_{r^*}$. The current is defined by summing over currents $J^{X,3}_{\mu}$ which act on individual spherical harmonics $\phi_\ell$.

The current $J^{X,3}_{Y}$ yields a nonnegative $K^{X,3}_Y$, modulo an error term supported near the horizons. In a first approximation, we may pretend that in fact $K^{X,3}_Y \geq 0$, but degenerates (in regular coordinates) near the horizon. The identity (10) can then be used as an estimate for its left hand side, in view of the fact that its right hand side will in fact be bounded by the flux of $J_\mu^T$ for the Killing field $T$, which is conserved. The role of the photon sphere will be exemplified by the degeneration at $r = 3M$ of the quantity controlled by this spacetime integral.

In order to obtain decay results from the above, one would have to gain information about the quantity estimating the boundary terms—namely $\int_{\partial \cal R} J_\mu^T n^\mu$, from the control of spacetime integral. The difficulty for this is that the spacetime integral estimates one obtains degenerate at the photon sphere $r = 3M$ and at the horizons. This does not allow one to control $J_\mu^T n^\mu$ there.

The problem at the photon sphere is cured by applying the estimate also to angular derivatives. It is here that the argument “loses” an angular derivative. It is this loss that leads to the form of decay proven in (10).

The problem on the horizon, on the other hand, turns out to be illusionary. The horizon is in fact a very favourable place for estimating the solution! For this, we will need to consider the “local observer” vector fields $Y, \overline{Y}$, to be described in the next section.

### 1.3.3 The red-shift effect and the currents $J^Y_\mu$ and $J^{\overline{Y}}_\mu$

The heuristic mechanism ensuring decay near the horizons has been understood for many years, and is known as the red shift effect. This is typically described in the language of geometric optics. If two observers $A$ and $B$ cross the event horizon at advanced times $v_A < v_B$, and $A$ sends a signal to $B$ at a certain frequency, as he ($A$) measures it, then the frequency at which $B$ receives it is exponentially damped in the quantity $v_B - v_A$.

It turns out that this exponential damping property can be captured by the integral identities (10) corresponding to the currents $J^Y_\mu$ and $J^{\overline{Y}}_\mu$ associated to vector fields $Y, \overline{Y}$, defined in Section 8. These vector fields are supported near the horizons $\cal H^+, \overline{\cal H}^+$, respectively. The estimates (10) corresponding to the currents $J^Y_\mu, J^{\overline{Y}}_\mu$ fulfill the double role of (a) correcting for the error region where $K^{X,3}_Y < 0$ by dominating this term near the horizon by $K^Y + K^{\overline{Y}}$ and (b) controlling the spacetime integrated energy measured by local observers near the horizon. The choice of $Y, \overline{Y}$ is delicate, because there is an “error region” where $K^Y + K^{\overline{Y}} < 0$, which must be controlled with the help of the currents of

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5This insight, in the case of the wave equation on the Schwarzschild solution, is originally due to Blue and Soffer [2]. See, however, [3].
the previous section. The use of the currents corresponding to \( X, Y \) and \( \overline{Y} \) are thus strongly coupled.

1.3.4 Comparison with the Schwarzschild case

To see the above arguments in context, the reader may wish to compare with our previous \cite{12}, where versions of the currents \( J_{\mu}^X, J_{\mu}^Y \) are also employed. The relation of our arguments with the physical mechanisms at play are in fact much clearer in the present paper, than in \cite{12}. This is due on the one hand to the absence here of the Morawetz-type vector field (denoted \( K \) in \cite{12}), and, on the other hand, to the relative simplicity here in the construction of the current \( J_{\mu}^X \). We give here some comments on these points.

The Morawetz vector field employed in \cite{12} is a highly unnatural quantity at the horizon from the geometric point of view. On the other hand, in view of its weights, it somewhat magically captures a polynomial (as opposed to the proper exponential) version of the red shift. The pointwise decay rates achieved via \( K \) at the horizon are worse than the decay rates away from the horizon, but sufficient if one is only interested in the behaviour of the solution away from the horizon. (See also \cite{4}.) In our \cite{12}, uniform decay rates up to the horizon were indeed obtained with the help of \( J_{\mu}^Y \). But these estimates could be obtained \textit{a posteriori}. From the point of view of the non-linear stability problem, this decoupling appears to be an exceptional feature. It is in this sense that the scheme proposed in the present paper is perhaps more naturally connected to the geometry of general black holes.

The second point to be made here concerns the construction of \( J_{\mu}^X \). In \cite{12}, positivity of the analogue of what we denote here \( K^{X,3} \) relied on an unmotivated recentring and rescaling of the derivatives of the functions \( f_l \) which obscured perhaps the fundamental connection with the photon sphere. Here, this connection appears much more clear. Of course, this is at the expense of having to bound \( -K^{X,3} \) from \( K^Y \) and \( K^{\overline{Y}} \). This should in no way be thought of as a disadvantage. The red-shift effect has a lot to offer. It should be used and not obscured.

1.4 Discussion

As noted above, the study of the asymptotic behaviour of solutions to \( \Box_g \phi = 0 \) on both Schwarzschild and Schwarzschild-de Sitter backgrounds has a long tradition in the physics literature. In the Schwarzschild case, the pioneering heuristic study is due to Price \cite{19}. See also \cite{11}. For the Schwarzschild-de Sitter case, there is numerical work of Brady et al \cite{6}, the subsequent \cite{7}, and references therein.

The above studies are based entirely upon decomposition of \( \phi \) into spherical harmonics. The results of these heuristics or numerics are typically presented in terms of the asymptotic behaviour of the tail:

\[
\phi_{\ell}(r,t) \sim t^{-2\ell-3}, \quad \phi_{\ell}(u,v) \sim u^{-2\ell-3}, \quad r\phi_{\ell}(u,v) \sim u^{-2\ell-2} \tag{11}
\]
for Schwarzschild, where $2M < r < \infty$ is fixed in the first formula, $u \geq v$ in the second, and $v \geq 2u$ in the third, and
\[
\phi_\ell(r, t) \sim e^{-c_\ell t}, \quad \phi_\ell(u, v) \sim e^{-c_\ell t}, \quad \phi_\ell(u, v) \sim e^{-c_\ell u}
\]
for Schwarzschild-de Sitter and $\ell \geq 1$, where $r_b < r < r_c$ is fixed in the first formula, and $u \geq v$ in the second, and $v \geq u$ in the third.

At first glance, statements (12) may appear stronger than what is actually proven in Theorem 1.2. As quantitative statements of decay, however, statements (11) and (12) are in fact much weaker than what has now been mathematically proven, here and in [12]. For, rewriting, in particular, the first formula of (12) as
\[
|\phi_\ell(r, t)| \leq C_\ell(r)e^{-c_\ell t},
\]
then there is no indication as to what $C_\ell(r)$ depends on, indeed, if there is any bound on $C_\ell$ provided by some norm of initial data, and if so, what is the behaviour as $\ell \to \infty$. This does not concern a mathematical pathology, but is intimately connected with the physical effect caused by the photon sphere. Indeed, a geometric optics approximation shows easily that if (13) is to hold and if $C_\ell$ is to depend, say, on the initial energy of the spherical harmonic, then $C_\ell \to \infty$ as $\ell \to \infty$. It is the rate of this divergence that would then determine the decay rate (if any) for $\phi$.

If one is interested in quantitative statements of decay, a statement like (12) provides no more information than the statement
\[
\lim_{(u, v) \to (\infty, \infty)} \phi_\ell(u, v) = 0.
\]
It is worth noting that the above statement at the level of individual spherical harmonics, together with the (uniform) boundedness result
\[
|\phi| \leq C \sup R \{\phi\} + C E_1^s(\phi, \dot{\phi}),
\]
can indeed be used to show, for fixed $r$, the statement
\[
\lim_{(u, v) \to (\infty, \infty)} (\phi - \phi_0)(u, v) = 0,
\]
for the total $\phi$. This can be termed the statement of (uniform) decay without a rate.

Thus it is truly only (16), and not the results of [12] or Theorem 1.1, that can be said to be suggested by heuristic and numerical studies.

Results like (16), and not the results of [12] or Theorem 1.1, that can be said to be suggested by heuristic and numerical studies.

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6The result [15] was shown for Schwarzschild in fundamental work of Kay and Wald [16]. Our [12] gives an alternative proof not relying on the discrete symmetries of the maximal development. For Schwarzschild-de Sitter, the statement [16] of course follows from Theorem 1.1 for any $s > 1$. We have not found another statement of this in the literature.

7Sometimes, even the statement \( \forall r_b < r < r_c, \lim_{t \to \infty} \phi(r, t) = 0 \) is termed “linear stability”. Such a result does not even imply [15]. It is in fact entirely consistent with the statement sup\( \phi \in \ell \in [0, \infty) \mid \phi(r, t) \mid = \infty \).
the sharp decay result, it would in fact suggest instability for Schwarzschild or Schwarzschild-de Sitter once one passes to the next order in perturbation theory. At very least, it would exclude all known techniques for proving non-linear stability for supercritical non-linear wave equations like (1). It is only quantitative uniform decay bounds with decay rate sufficiently fast, such that moreover the bound depends only on a suitable norm of initial data, which indeed can the thought of as suggestive of non-linear stability. One should thus be careful in associating the heuristic and numerical tradition exemplified in [19] with the conjecture that black holes are stable.

1.5 Note added

While the final version of this manuscript was being prepared, an interesting preprint [5] appeared addressing a special case of the problem under consideration here with the methods of time-independent scattering theory. The special case where $\phi$ is not supported at $\mathcal{H}^+ \cap \mathcal{H}^-$ and $\mathcal{H}^+ \cap \mathcal{H}^-$ is considered and quantitative exponential decay bounds are proven for $\int_{\Sigma} T_{\mu\nu} T^{\mu\nu}$ in the coordinate $t$, where one must restrict to $\Sigma' = \{t\} \times [r_0, R_0]$, for $r_b < r_0$, $R_0 < r_c$. The bounds lose only an $\epsilon$ of an angular derivative, but depend on $r_0$, $R_0$, and the initial support of $\phi$ in an unspecified way. The work [5] depends in an essential way on a previous detailed analysis of Sá Barreto and Zworski [21] concerning resonances of an associated elliptic problem.

For the special case of the data considered in [5], given that result, then the estimates of the present paper, in particular, those provided by the currents $J^Y_\mu$, $J^\Sigma_\mu$, can be applied a posteriori to obtain uniform (i.e. holding up to the horizons) exponential decay bounds.

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2 The Schwarzschild de-Sitter metric in coordinates

We refer the reader to the references [8, 13, 18] for detailed discussions of the geometry of Schwarzschild-de Sitter.

2.1 Schwarzschild coordinates $(r, t)$

We recall that so-called Schwarzschild coordinates $(r, t)$ map $D^o$ onto $(r_b, r_c) \times (-\infty, \infty)$, in which the metric takes the form (3). Let the choice of the $t$ coordinate be fixed. Here $0 < r_b < r_c$ denote the two positive roots of the
The function \( r \) can be given a geometric interpretation
\[
r(p) = \sqrt{\text{Area}(\hat{\pi}^{-1}(\hat{\pi}(p)))}/4\pi,
\]
where here \( \hat{\pi} : \mathcal{M} \rightarrow \mathcal{Q} \) is the natural projection. Thus \( r \) can be defined as a smooth function on all of \( \mathcal{M} \). It is known as the \textit{area-radius function}. This function also clearly descends to \( \mathcal{Q} \).

The \((r, t)\) coordinates degenerate along the horizons \( \mathcal{H}^+ \cup \mathcal{H}^- \) and \( \overline{\mathcal{H}}^+ \cup \overline{\mathcal{H}}^- \), on which \( r = r_b, r = r_c \), respectively.

It is immediate from the explicit form of the metric that the vector field \( \partial_t \) is Killing in \( \mathcal{D}^\circ \). This extends to a globally defined Killing field \( T \) on \( (\mathcal{M}, g) \), which is null along \( \mathcal{H}^+ \cup \mathcal{H}^- \) and \( \overline{\mathcal{H}}^+ \cup \overline{\mathcal{H}}^- \), and vanishes along \( \mathcal{H}^+ \cap \mathcal{H}^- \) and \( \overline{\mathcal{H}}^+ \cap \overline{\mathcal{H}}^- \).

### 2.2 Regge-Wheeler coordinates \((r^*, t)\)

We now proceed to define two related coordinate systems on \( \mathcal{D}^\circ \). Let us denote the unique negative root of (17) as \( r^- \), and let us set
\[
\kappa_b = \frac{d}{dr} \left( 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \right) \bigg|_{r=r_b},
\]
and similarly \( \kappa_c, \kappa_- \). We now set
\[
r^* = -\frac{1}{2\kappa_c} \log \left| \frac{r}{r_c} - 1 \right| + \frac{1}{2\kappa_b} \log \left| \frac{r}{r_b} - 1 \right|
+ \frac{1}{2\kappa_-} \log \left| \frac{r}{r_-} - 1 \right| - C^*
\]
where \( C^* \) is a constant we may choose arbitrarily. For convenience, let us choose \( C^* \) so that \( r^* = 0 \) when \( r = 3M \), the so-called \textit{photon sphere}. We call the coordinates \((r^*, t)\) so-defined \textit{Regge-Wheeler coordinates}.

### 2.3 Eddington-Finkelstein coordinates \((u, v)\)

From Regge-Wheeler coordinates \((r^*, t)\), we can define now \textit{retarded} and \textit{advanced} Eddington-Finkelstein coordinates \( u \) and \( v \), respectively, by
\[
t = v + u
\]
and
\[
r^* = v - u.
\]

These coordinates turn out to be null: Setting \( \mu = \frac{2M}{r} + \frac{1}{3} \Lambda r^2 \), the metric takes the form
\[
-4(1 - \mu)du dv + r^2 d\varphi^2,
\]
We shall move freely between the two coordinate systems \((r^*, t)\) and \((u, v)\) in this paper. Note that in either, region \(\mathcal{D}^o\) is covered by \((-\infty, \infty) \times (-\infty, \infty)\).

By appropriately rescaling \(u\) and \(v\) to have finite range, one can construct coordinates which are in fact regular on \(\mathcal{H}^\pm\) and \(\tilde{\mathcal{H}}^\pm\). By a slight abuse of language, one can parametrize the future and past horizons in our present \((u, v)\) coordinate systems as

\[
\mathcal{H}^+ = \{(\infty, v)_{v \in (-\infty, \infty)} \},
\]

\[
\mathcal{H}^- = \{(u, -\infty)_{u \in (-\infty, \infty)} \},
\]

\[
\mathcal{H}_+^\pm = \{(u, \infty)_{u \in (-\infty, \infty)} \},
\]

\[
\mathcal{H}_-^\pm = \{(-\infty, v)_{v \in (-\infty, \infty)} \}.
\]

Under these conventions, the statements of Theorem 1.1 can be applied up to the boundary of \(\mathcal{D}\).

### 2.4 Useful formulae

Finally, we collect various formulas for future reference:

\[
\mu = \frac{2M}{r} + \frac{1}{3} \Lambda r^2,
\]

\[
g_{uv} = (g^{uv})^{-1} = -2(1 - \mu),
\]

\[
\partial_v r = (1 - \mu), \quad \partial_u r = -(1 - \mu)
\]

\[
dt = dv + du, d\tau^* = dv - du,
\]

\[
T = \frac{\partial}{\partial t} = \frac{1}{2} \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial u} \right),
\]

\[
\frac{\partial}{\partial r^*} = \frac{1}{2} \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right),
\]

\[
dVol_M = 2r^2(1 - \mu) du dv dA_{S^2},
\]

\[
dVol_{t=\text{const}} = r^2 \sqrt{1 - \mu} \, dr^* dA_{S^2},
\]

\[
\Box \psi = \nabla^\alpha \nabla_\alpha \psi = -(1 - \mu)^{-1} \left( \partial_t^2 \psi - r^{-2} \partial_r \left( r^2 \partial_r \psi \right) \right) + \nabla^A \nabla_A \psi.
\]

Here \(\nabla\) denotes the induced covariant derivative on the group orbit spheres.
As discussed in the introduction, the results of this paper will rely on $L^2$-based estimates. Such estimates arise naturally in view of the Lagrangian structure of the wave equation. We review briefly here.

Let $\phi$ be a solution of $\Box_\phi \phi = 0$. In general coordinates, the energy-momentum tensor $T_{\alpha\beta}$ for $\phi$ is defined by the expression

$$T_{\alpha\beta}(\phi) = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi.$$ 

The tensor $T_{\alpha\beta}$ is symmetric and divergence-free, i.e. we have

$$\nabla^\alpha T_{\alpha\beta} = 0.$$ 

(23)

For the null coordinate system $u, v, x^A, x^B$ we have defined, where $x^A, x^B$ denote coordinates on $S^2$, we compute the components

$$T_{uu} = (\partial_u \phi)^2,$$

$$T_{vv} = (\partial_v \phi)^2,$$

$$T_{uv} = \left( - \frac{1}{2} g_{uv} |\nabla/\phi|^2 - (1-\mu) \right) + \left( \partial_u V_v (1-\mu) - \frac{1}{2} |\nabla/\phi|^2 - \phi^a \phi_a \right).$$

Here the notation $|\nabla/\phi|^2 = g^{AB} \partial_A \phi \partial_B \phi = r^{-2} |d\psi|^2$. Note moreover that $|\nabla/\psi|^2 = r^{-2} \sum_{i=1}^3 |\Omega_i \psi|^2$.

Let $V^\alpha$ denote an arbitrary vector field. Let $\pi^\alpha_{V\beta}$ denote the deformation tensor of $V$, i.e.,

$$\pi^\alpha_{V\beta} = \frac{1}{2} (\nabla^\alpha V^\beta + \nabla^\beta V^\alpha).$$

(24)

In local coordinates we have the following expression:

$$T_{\alpha\beta}(\phi) \pi^\alpha_{V\beta} = \frac{1}{4(1-\mu)} \left( (\partial_u \phi)^2 \partial_v (V_v (1-\mu)^{-1}) + (\partial_v \phi)^2 \partial_u (V_u (1-\mu)^{-1}) \right) + |\nabla/\phi|^2 (\partial_u V_v + \partial_v V_u) - \frac{1}{2} (V_u - V_v)(|\nabla/\phi|^2 - \phi^a \phi_a).$$

Set

$$J^\alpha_{V\alpha} = T_{\alpha\beta}(\phi).$$

(25)

The relations (23) and (24) give

$$K^V = \nabla^\alpha J^\alpha_{V\alpha} = T_{\alpha\beta}(\phi) \pi^\alpha_{V\beta},$$

and the divergence theorem applied to an arbitrary region $\mathcal{R}$ gives (10).

Identity (10) is particularly useful when the vector field $V$ is Killing, for instance the vector field $T$ defined previously. For then, $K^T = 0$ and one obtains a conservation law for the boundary integrals. Moreover, when $\partial \mathcal{R}$
corresponds to two homologous timelike hypersurfaces, the integrands \( J_{\mu}^T(\phi)n^\mu \) on the right hand side of (20) when properly oriented are positive semi-definite in the derivatives of \( \phi \).

Were the vector field \( T \) timelike in all of \( D \), then by applying (10) to \( \phi, \Omega_{\rho\phi} \), etc., one could show the uniform boundedness of all derivatives of \( \phi \). Since \( T \) becomes null on \( \mathcal{H}^+ \cup \overline{\mathcal{H}}^+ \), the integrand does not control all quantities on the horizon. It is for this reason that even proving uniform boundedness for solutions of \( \Box_\phi = 0 \) on \( D \) is non-trivial. (See [16].)

For \( \phi \) a solution to \( 2g \phi = 0 \), the 1-form \( J^V_{\mu}(\phi) \) defined above has the property that both it and its divergence \( \nabla_{\mu}J^V_{\mu} \) depend only on the 1-jet of \( \phi \). Following Christodoulou [9], we shall call one-forms \( J_{\mu} \) and their divergences \( K_{\mu} = \nabla_{\mu}J_{\mu} \) with the aforementioned property (thought of as form-valued and scalar valued maps on the bundle of 1-jets, respectively) compatible currents.

4 Constants and cutoffs

4.1 The special values \( r_i, R_i \)

In the course of this proof we shall require special values \( r_i, R_i, i = 0, 1, 2 \), satisfying
\[
-\infty < \frac{1}{2} r_0^* < 2 r_1^* < \frac{1}{2} r_1^* < 2 r_2^* < 0 < 2 R_2^* < \frac{1}{2} R_1^* < 2 R_1^* < \frac{1}{2} R_0 < \infty.
\]

Eventually, specific choices of these constants will be made, and these choices will depend only on \( M, \Lambda \). Constants \( r_2, R_2 \) are in fact only constrained by Lemma 7.3.1. Constants \( r_1, R_1 \) are constrained by the necessity of satisfying Proposition 8.3.1 of Section 8.3.

To choose \( r_0, R_0 \), on the other hand, is more subtle, as we will have to keep track of a certain competition of constants as \( r_0, R_0 \) vary. We adopt, thus, the convention described in the next section.

4.2 Dependence of constants \( C, E, \) and \( \epsilon \) on \( r_i, R_i \)

In all formulas that follow in this paper, constants which can be chosen independently of \( r_i^*, R_0^* \) shall be denoted by \( C \). Constants \( C \) will thus depend on \( M, \Lambda, r_i, R_i \), for \( i = 1, 2 \), and, after \( r_i, R_i \) have been chosen, will depend only on \( M, \Lambda \).

Constants which depend on \( M, \Lambda, r_0^* \) and \( R_0^* \) and tend to 0 as \( r_0^* \to -\infty, R_0^* \to \infty \) will be denoted by \( \epsilon \). Finally, all other constants depending on \( M, \Lambda, r_0^* \) and \( R_0^* \), will be denoted by \( E \). Constants denoted by \( E \) in principle diverge as \( r_0^* \to \infty, R_0^* \to \infty \).

We will also use the convention \( A \approx B \) whenever \( C^{-1}A \leq B \leq CA \) with a constant \( C \) understood as above.

In view of our above conventions, note finally the obvious algebra of constants: \( C \pm C = C, \epsilon C = \epsilon, CE = E, \epsilon E = E \), etc.
4.3 Cutoffs

Associated to these special values of \( r \), we will define a number of cutoff functions. It is convenient to introduce also the notation

\[
r_{-1}^* = 4r_0^*, \quad R_{-1}^* = 4R_0^*.
\]

4.3.1 The cutoffs \( \eta_i \)

Let \( \eta : [0, \infty) \to \mathbb{R} \) be a nonnegative smooth cutoff function which is equal to 1 in \([0, 1]\) and 0 outside \([0, 2]\).

For \( i = -1, 0, 1, 2 \), define

\[
\eta_i(r^*) = \begin{cases} 
\eta(r^*/r_i^*) & \text{for } r^* \leq 0 \\
\eta(r^*/R_i^*) & \text{for } r^* \geq 0.
\end{cases}
\]

Clearly \( \eta_i \) has the property that \( \eta_i = 1 \) in \([r_i^*, R_i^*]\) and \( \eta_i = 0 \) in \((-\infty, \frac{1}{2}r_{i-1}^*] \cup (\frac{1}{2}R_{i-1}^*, \infty)\). Moreover, \( \sup \eta_i' \to 0 \) as \( r_i^* \to -\infty \), \( R_i^* \to \infty \).

4.3.2 The cutoffs \( \chi_i \) and \( \overline{\chi}_i \)

Now let \( \chi : (-\infty, \infty) \to \mathbb{R} \), \( \overline{\chi} : (\infty, \infty) \to \mathbb{R} \) be nonnegative smooth cutoff functions such that \( \chi \) is 1 in \((-\infty, -1]\) and 0 in \([-\frac{1}{2}, \infty)\), and \( \overline{\chi} \) is 1 in \([1, \infty)\) and 0 in \((-\infty, \frac{1}{2}]\).

Now for \( i = -1, 0, 1 \) define

\[
\chi_i = \chi(-r^*/r_i^*) \\
\overline{\chi}_i = \overline{\chi}(r^*/R_i^*).
\]

Clearly, \( \chi_i \) has the property that \( \chi_i = 1 \) in \((-\infty, r_i^*]\), and 0 in \([2r_{i+1}^*, \infty)\). Similarly \( \overline{\chi}_i \) has the property that \( \overline{\chi}_i = 1 \) in \([R_i^*, \infty)\), and 0 in \((-\infty, 2R_{i+1}^*)\).

5 The hypersurfaces \( \Sigma_t \) and the region \( \mathcal{R}(t_1, t_2) \)

Let \( r_1, R_1 \) be the special values announced in Section 4.1. For all \( t \), define \( \Sigma_t = \{t\} \times [r_1, R_1] \cup \{(t - R_1^*)/2 \times [(t + R_1^*)/2, \infty] \cup [(t - r_1^*)/2, \infty] \times \{(t + r_1^*)/2\} \)

and for any \( t_2 > t_1 \), define \( \mathcal{R}(t_1, t_2) = J^+(\Sigma_{t_1}) \cap J^- (\Sigma_{t_2}) \).
The diagram below may be helpful:

We shall repeatedly apply the identity (10) in the region \( \mathcal{R}(t_1, t_2) \). We record below its explicit form:

\[
0 = \int_{\mathcal{R}(t_1, t_2)} K(\phi) + \int_{\Sigma t_2} J_\mu(\phi)n^\mu - \int_{\Sigma t_1} J_\mu(\phi)n^\mu + \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J_\mu(\phi)n^\mu + \int_{\mathcal{H}^- \cap \{u_1 \leq u \leq v_2\}} J_\mu(\phi)n^\mu.
\] (26)

Here, \( n = (1 - \mu)^{-\frac{3}{2}}T \) whenever \((u, v)\) belongs to the space-like portion of \( \Sigma_t \), and the measure of integration, call it \( dm \), is there understood to be the induced volume form. Let us moreover define \( n = \frac{\partial}{\partial u} \) and \( n = \frac{\partial}{\partial v} \) whenever \((u, v)\) belongs to the \( v = \text{const} \) and \( u = \text{const} \) portions of \( \Sigma_t \) respectively. With this choice, the measure of integration \( dm \) in the respective null segments is understood to be given by

\[
dm_{v=\text{const}} = r^2 \, dA_{g^2} \, du, \quad dm_{u=\text{const}} = r^2 \, dA_{g^2} \, dv.
\]

All integrals over \( \Sigma_t \) that appear in the sections that follow are understood to be with respect to the measure \( dm \) defined above. The integral over \( \mathcal{R}(t_1, t_2) \) is to be understood to be with respect to the spacetime volume form. See Section 2.4.

6 The main estimates

In this section we will give a geometric statement of the main estimates.
6.1 The vectorfields $N$, $\tilde{N}$ and $P$

Recall the cutoffs functions $\eta_i$, $\chi_1$, $\bar{\chi}_1$, from Section 4.3. Define the vector fields

$$N \doteq \frac{\chi_1}{1 - \mu} \frac{\partial}{\partial u} + \bar{\chi}_1 \frac{\partial}{\partial v} + T,$$

$$P_i \doteq \eta_i (1 - \mu)^{-1/2} \frac{\partial}{\partial v^i},$$

$$\tilde{N} \doteq (r - 3M)^2 N. \quad \text{(29)}$$

For convenience, let us denote $P_2$ by $P$.

The above vector fields will provide the fundamental directions in which the energy momentum tensor $T^{\mu\nu}$ is to be contracted and/or $\phi$ is to be differentiated in the definition of the fundamental quantities appearing in the main estimates. See Section 6.2 below.

The coordinate-dependent definitions given above notwithstanding, the important features of these vector fields can be understood geometrically. All three are invariant with respect to the action of $\Psi_t$, the one-parameter group of differentiable maps $\mathcal{D} \to \mathcal{D}$ generated by the Killing field $T$. $N$ is future-directed timelike on $\Sigma_t$, $\tilde{N}$ is future-directed timelike everywhere except $r = 3M$, where it vanishes quadratically, and $P$ is supported away from the horizon and orthogonal to $T$.

6.2 The quantities $Z^{\tilde{N},P}$, $Z^N$ and $Q$

Let $T^{\mu\nu}$ be the energy-momentum tensor defined in Section 3. Define the quantities

$$Z^{\tilde{N},P}_{\phi} (t) \doteq \int_{\Sigma_t} \left( T^{\mu\nu}(\phi) \tilde{N}^{\mu} n^{\nu} + (P\phi)^2 \right),$$

$$Z^N_{\phi} (t) \doteq \int_{\Sigma_t} T^{\mu\nu}(\phi) N^{\mu} n^{\nu},$$

$$Q_{\phi}(t_1, t_2) \doteq \int_{t_1}^{t_2} Z^{\tilde{N},P}_{\phi} (t) \, dt.$$

The quantity $Q_{\phi}$ is equivalent to the spacetime integral of the density $q(\phi)$ defined by

$$q(\phi) \doteq \left( T^{\mu\nu}(\phi) \tilde{N}^{\mu} \frac{n^{\nu}}{1 - \mu} + (P\phi)^2 \right),$$

in the sense of the formula

$$Q_{\phi}(t_1, t_2) \approx \int_{\mathcal{R}(t_1, t_2)} q(\phi),$$

understood with the conventions of Section 4.2. We will make use of this equivalence often in what follows. Recall also that the spacetime integral on the right
hand side of (33) is to be understood with respect to the volume form. See Section 2.4.

Note that the quantity $\mathbf{Z}_N^\phi$ has integrand positive definite in $d\phi$. It is in fact precisely the flux through $\Sigma_t$ of the current $J_\mu^N(\phi)$. The quantity $\bar{\mathbf{Z}}_{\tilde{N},P}^\phi$ differs from $\mathbf{Z}_N^\phi$ in that control of the angular and $t$-derivatives degenerates quadratically at $r = 3M$. Similarly, the integrand of $Q_{\phi}(t_1, t_2)$ also degenerates at $r = 3M$. This hypersurface $r = 3M$ is the so-called photon sphere discussed already in the introduction.

6.3 Statement of the estimates

The main estimates of this paper are contained in the following

**Theorem 6.1.** There exists a constant $C$ depending only on $M, \Lambda$ such that for all $t_2 > t_1$ and all sufficiently regular solutions $\phi$ of $\Box \phi = 0$ in $\mathcal{R}(t_1, t_2)$ we have

$$Q_{\phi}(t_1, t_2) \leq C \mathbf{Z}^N_{\phi}(t_1),$$

$$\mathbf{Z}_N^\phi(t_2) \leq C \mathbf{Z}_{\tilde{N},P}^\phi(t_2) + C(t_2 - t_1)^{-1}\left(\mathbf{Z}_N^\phi(t_1) + Q_{\phi}(t_1, t_2) + \sum_{i=1}^3 Q_{\Omega_i}(t_1, t_2)\right),$$

More generally than (36), if $\Sigma' \subset \mathcal{R}(t_1, t_2)$ is achronal then

$$\int_{\Sigma'} T_{\mu\nu}(\phi) N^\mu n^\nu \leq C(\mathbf{Z}_N^\phi(t_1) + Q_{\phi}(t_1, t_2)).$$

These estimates will be used in Section 11 to prove Theorems 1.1 and 1.2.

As described in the introduction, the proof of Theorem 6.1 will be accomplished in Section 10 with the help of so called energy currents $J_\mu$ associated to the vector fields $X_\ell, Y, \overline{Y}$ and $\Theta$. We turn in the next sections to the definition of these currents.

6.4 Discussion

Were it $\mathbf{Z}_{\tilde{N},P}^\phi$ on the right hand side of (34), or alternatively, were $Q$ defined as the time-integral of $\mathbf{Z}^N_{\phi}$, then inequality (34) would immediately lead to exponential decay in $t$ for $Q(t, t_*)$ (cf. Lemma 11.1.1).

The appearance of $Q_{\Omega_i}$ on the right hand side of (34) signifies that the estimates “lose” an angular derivative. At the level of any fixed spherical harmonic $\phi_\ell$, estimates (34) and (35) lead immediately to exponential decay for $Q_{\phi_{\ell}}(t, t_*)$ and $\mathbf{Z}_{\phi_{\ell}}^N$. The nature of the loss of angular derivative in (35) means that for the total $Q_{\phi}(t, t_*)$ and $\mathbf{Z}_{\phi}^N$, one can only obtain polynomial decay in $t$, where the bound on the decay rate exponent is linear in the angular derivatives lost. Exponential decay for $\phi$ would be retrieved if the “loss in angular derivatives” in estimate (34) were logarithmic. See the dependence in $\ell$ in Lemma 11.1.1.
7 The $J^X$ family of currents

We define in this section a family of currents, all loosely based on vector fields parallel to $\frac{\partial}{\partial r^*}$. The role of these currents in capturing the role of the “photonsphere” has already been discussed in the introduction.

7.1 Template currents $J^V_{\mu i}$ for an arbitrary $V = f \frac{\partial}{\partial r^*}$

Let $f$ be a function of $r^*$ and consider a vector field

$$V = f(r^*) \frac{\partial}{\partial r^*}.$$  \hspace{1cm} (38)

Define the currents

$$J^V_{\mu 0}(\phi) = T_{\mu \nu}(\phi) V^\nu,$$

$$J^V_{\mu 1}(\phi) = T_{\mu \nu}(\phi) V^\nu + \frac{1}{4} \left( f' + 2 \frac{1 - \mu}{r} f \right) \partial_{\mu}(\phi)^2 - \frac{1}{4} \partial_{\mu} \left( f' + 2 \frac{1 - \mu}{r} f \right) \phi^2,$$

$$J^V_{\mu 2}(\phi) = T_{\mu \nu}(\phi) V^\nu + \frac{1}{4} \left( f' + 2 \frac{1 - \mu}{r} f \right) \partial_{\mu}(\phi)^2 - \frac{1}{4} \partial_{\mu} \left( f' + 2 \frac{1 - \mu}{r} f \right) \phi^2 - \frac{1}{2} \frac{f'}{r} V_{\mu} \phi^2,$$

$$J^V_{\mu 3}(\phi) = T_{\mu \nu}(\phi) V^\nu + \frac{1}{4} \left( f' + 2 \frac{1 - \mu}{r} f \right) \partial_{\mu}(\phi)^2 - \frac{1}{4} \partial_{\mu} \left( f' + 2 \frac{1 - \mu}{r} f \right) \phi^2 - \frac{1}{2} \frac{f'}{r} V_{\mu} \phi^2 - \frac{1}{2} \frac{1}{3M/r} f \phi \nabla_{\mu} \phi,$$

$$J^V_{\mu 4}(\phi) = T_{\mu \nu}(\phi) V^\nu + \frac{1}{4} f' \partial_{\mu}(\phi)^2 - \frac{1}{4} \partial_{\mu} f' \phi^2,$$

and the divergences

$$K^V_{\mu i} = \nabla^\mu J^V_{\mu i}.$$

Note the identities

$$\frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} = \frac{r - 3M}{r^2},$$

$$\frac{1}{2r} \left( \frac{\mu'}{1 - \mu} - \frac{\mu'}{r} \right) = \frac{M}{r^4} \left( 3 - \frac{8M}{r} \right) + \frac{M\Lambda}{3r^2} - \frac{2\Lambda^2 r}{9}.$$  \hspace{1cm} (39)

We compute

$$K^V_{\mu 0}(\phi) = \frac{f' \partial_{\mu} \phi^2}{1 - \mu} + |\nabla \phi|^2 \left( \frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} \right) f - \frac{1}{4} \left( 2f' + 4 \frac{1 - \mu}{r} f \right) \phi^2 \phi_\alpha.$$  \hspace{1cm} (40)
\[ K^{V,1}(\phi) = \frac{f'}{1 - \mu} (\partial_r \phi)^2 + |\nabla \phi|^2 \left( \frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} \right) f \\
- \frac{1}{4} \left( \Box \left( f' + 2 \frac{1 - \mu}{r} f \right) \right) \phi^2 \\
= \frac{f'}{1 - \mu} (\partial_r \phi)^2 + |\nabla \phi|^2 \left( \frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} \right) f \\
- \frac{1}{4} \left( \frac{1}{1 - \mu} f'' + \frac{4}{r} f'' - \frac{4\mu'}{r(1 - \mu)} f' + \frac{2}{(1 - \mu)r} \left( \frac{\mu' (1 - \mu)}{r} - \mu'' \right) f \right) \phi^2. \]

\[ K^{V,2}(\phi) = \frac{f'}{(1 - \mu)r^2} (\partial_r (r\phi))^2 + \frac{1 - 3M/r}{r} f |\nabla \phi|^2 - \frac{1}{4} \frac{1}{1 - \mu} f'' \phi^2 \\
+ f \left( \frac{M}{r^4} \left( 3 - \frac{8M}{r} \right) + \frac{MA^2r}{3r^2} - \frac{2A^2r^3}{9} \right) \phi^2, \]

\[ K^{V,3}(\phi) = \frac{f'}{(1 - \mu)r^2} (\partial_r (r\phi))^2 - \frac{1 - 3M/r}{r} f \phi \Delta \phi - \frac{1}{4} \frac{1}{1 - \mu} f'' \phi^2 \\
+ f \left( \frac{M}{r^4} \left( 3 - \frac{8M}{r} \right) + \frac{MA^2r}{3r^2} - \frac{2A^2r^3}{9} \right) \phi^2, \]

\[ K^{V,4}(\phi) = \frac{f'}{1 - \mu} (\partial_r \phi)^2 + |\nabla \phi|^2 \left( \frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} \right) f \\
- \frac{1 - \mu}{r} f \phi^2 - \frac{1}{4} \left( \Box f' \right) \phi^2 \\
= \frac{f'}{1 - \mu} (\partial_r \phi)^2 + |\nabla \phi|^2 \left( \frac{\mu'}{2(1 - \mu)} + \frac{1 - \mu}{r} \right) f \\
- \frac{1 - \mu}{r} f \phi^2 - \frac{1}{4} \left( \frac{1}{1 - \mu} f'' + \frac{2}{r} f'' \right) \phi^2. \]

The expression \( J^{V,3}_\mu \) is not a compatible current in the sense of Section 3, since \( K^{V,3}_\mu \) depends on the 2-jet of \( \phi \), but it can be treated as such when restricted to eigenfunctions of \( \Delta \).

### 7.2 Discussion

The relation of the photon sphere to currents based on vector fields \( V \) of the form \( f(r^*) \partial_r \phi \) is most clear upon examining the modified current \( J^{V,1}_\mu \) and noting the coefficient of \( |\nabla \phi|^2 \) in \( K^{V,1} \) vanishes precisely at \( r = 3M \) in view of (39). This indicates that if one is to have say \( K^{V,1} \geq 0 \), the function \( f \) must change sign at \( r = 3M \), and the control of the angular derivatives must degenerate at least quadratically.

The task of choosing a suitable \( f \) is simplified by passing to the further modified current \( J^{V,2} \) which effectively “borrows” positivity from the \( \partial_r \phi \) term.
Finally, one can take advantage of the further flexibility provided by choosing separately $V$ for each spherical harmonic, where now, after passing to the current $J_{\mu}^{X,3}$, the 0' th order terms are united with the angular derivative terms. In Section 7.3 we shall construct a current

$$J_{\mu}^{X,3}(\phi) = J_{\mu}^{X_0}(\phi_0) + \sum_{\ell} J_{\mu}^{X_{\ell},3}(\phi_{\ell}).$$

We do not in fact ensure that $K_{X,3} \geq 0$ everywhere, but rather, only in the region $r_0 \leq r \leq R_0$. The region near the horizon will be handled with the help of the spacetime integral terms controlled by the currents $J^Y$, $J^Y$ to be discussed in the next section.

Once an initial positive definite spacetime integral (albeit modulo an error) is constructed, other quantities can be controlled with the help of auxiliary currents. These are defined in Section 7.4. It is there where we also use the current template $J_{\mu}^{X,4}$.

### 7.3 The vector fields $X_{\ell}$

#### 7.3.1 The case $\ell = 0$

Define

$$f_0 = -r^{-2},$$

and set $X_0 = f_0 \partial_r$. Let $\phi_0$ denote the 0' th spherical harmonic of a solution $\Box g \phi = 0$ of the wave equation. Consider $J_{\mu}^{X_0,0}(\phi_0)$, $K_{X_0,0}(\phi_0)$ as defined above. We compute

$$K_{X_0,0}(\phi_0) = 2r^{-3} \left( \partial_r \phi_0 \right)^2 1 - \mu. \tag{41}$$

Note that

$$K_{X_0,0}(\phi_0) \geq 0.$$

#### 7.3.2 The case $\ell \geq 1$

Consider for now the expression $K^{V,3}$ for a general vector field $V$ of the form $f_0 \partial_r$, applied to a spherical harmonic $\phi_{\ell}$ with spherical harmonic number $\ell \geq 1$. We have

$$K^{V,3}(\phi_{\ell}) = \frac{f'}{(1 - \mu)r^2} \left( \partial_r (r \phi_{\ell}) \right)^2 - \frac{1}{4} \frac{1}{1 - \mu} f''^2 \phi_{\ell}^2$$

$$+ f \left( \ell(\ell + 1) \frac{1 - 3M/r}{r^3} + \frac{M}{r^4} \left( 3 - \frac{8M}{r} \right) + \frac{MA}{3r^2} - \frac{2A^2 r}{9} \right) \phi_{\ell}^2. \tag{42}$$

Define

$$h_{\ell}(r) = \ell(\ell + 1) \frac{1 - 3M/r}{r^3} + \frac{M}{r^4} \left( 3 - \frac{8M}{r} \right) + \frac{MA}{3r^2} - \frac{2A^2 r}{9}.$$ 

The following lemma is proven in Appendix A.
Lemma 7.3.1. For all $\ell \geq 1$, there exists a unique zero $r_{h\ell}$ of the function $h\ell(r)$ in $[r_b, r_c]$, and there exist constants $r^*_2 < 0$, $R^*_2 > 0$, depending only on $M$ and $\Lambda$, such that
\[ r_2 < r_{h\ell} < R_2. \]
Moreover, $\lim_{\ell \to \infty} r_{h\ell} \to 3M$.

Let $r^*_2$, $R^*_2$ now be fixed, chosen according to the above lemma.
Were the middle term on the right hand side of (42) absent, then we would have $K^{V,3}(\phi_\ell) \geq 0$ for any $f$ such that $f' \geq 0$, $f(r_{h\ell}) = 0$.

The middle term of (42) vanishes if $f''' = 0$, but, with the requirement that $f(r_{h\ell}) = 0$, in this case the function $f$ cannot be bounded. By suitably cutting off a function linear in $r^*$, we can ensure that $K^{V,3}(\phi_\ell) \geq 0$ in $r^*_0 \leq r \leq R^*_0$.

Let $\eta_0$ be the cutoff of Section 4.3.2. Define
\[ f_\ell(r^*) = \int_{r^*_\ell}^{r^*_0} \eta_0 \, dr^*, \]
and the vector field $X_\ell$ by
\[ X_\ell = \frac{1}{2} f_\ell \frac{\partial}{\partial v} - \frac{1}{2} f_\ell \frac{\partial}{\partial u} = f_\ell \frac{\partial}{\partial r^*}. \] (43)

We have
\[ K^{X_\ell,3}(\phi_\ell) \geq 0 \] (44)
in the region $r^*_0 \leq r \leq R^*_0$. Moreover, in this region we have in fact,
\[ \frac{(\partial_{r^*}(r\phi_\ell))^2}{1 - \mu} + (r - r_{h\ell})^2 \phi^2_\ell \leq CK^{X_\ell,3}(\phi_\ell). \] (45)
(Recall our conventions for constants $C$ from Section 4.2.)

7.3.3 The currents $J^{X,i}$
Finally, for $i = 1, 2, 3$ define the “total” currents
\[ J^{X,i}_\mu(\phi) = J^{X,0}_\mu + \sum_{\ell \geq 1} J^{X_\ell,i}_\mu(\phi_\ell), \]
and their divergences
\[ K^{X,i}_\mu(\phi_\ell) = \nabla^\mu J^{X,i}_\mu(\phi_\ell), \]
\[ K^{X,i}(\phi) = K^{X,0}(\phi_0) + \sum_{\ell \geq 1} K^{X_\ell,i}(\phi_\ell) = \nabla^\mu J^{X,i}_\mu(\phi). \]
7.3.4 Controlling the error

Besides obtaining nonnegativity for \( K^{X,3}(\phi) \) in the region \( r_0 \leq r \leq R_0 \), we need to understand the error in the region \( r \leq r_0 \) and \( r \geq R_0 \). It turns out that this error can be controlled by \( \epsilon \hat{q}(\phi) \), where \( \hat{q}(\phi) \) is a slightly stronger quantity than the energy density \( q(\phi) \).

Define the quantity

\[
\hat{q}(\phi) = \left( T_{\mu\nu}(\phi) \tilde{N}^\mu \frac{n^\nu}{1-\mu} + \frac{\eta-1(\chi_1 + \overline{\chi}_1)}{1-\mu} |r^*|^{-\delta-1} |\nabla \phi|^2 + \eta_1 (P\phi)^2 \right)
\]

(46)

Here \( \eta_1, \chi_1, \overline{\chi}_1 \) are the cut-off functions defined in Section 4.3. Note that \( \hat{q}(\phi) \approx \chi_1 (|\partial_u \phi|^2 + |\partial_v \phi|^2 + (\partial_r \phi)^2 + |\nabla \phi|^2) \), where \( q(\phi) \) is the density of the main quantity \( Q_\phi \) defined in (32).

The inequality replacing (44) which holds globally is given by Lemma 7.3.2.

The inequality

\[
\int_{S^2} K^{X,3}(\phi) \geq -\epsilon \int_{S^2} \hat{q}(\phi)
\]

(49)

holds on all spheres of symmetry.

Proof. It suffices to consider the regions \( r \leq r_0 \) and \( r \geq R_0 \). Relation (47) implies

\[
C \hat{q}(\phi) \geq \left( 1 + \frac{\eta-1(\chi_1 + \overline{\chi}_1)}{1-\mu} \right) |r^*|^{-\delta-1} |\nabla \phi|^2,
\]

in these two regions, while

\[
\int_{S^2} K^{X,3}(\phi) \geq \sum_{\ell \geq 1} \int_{S^2} K^{X,3}(\phi_\ell)
\]

\[
\geq -\sum_{\ell \geq 1} \int_{S^2} \frac{f_\ell'''}{1-\mu} \phi_\ell^2.
\]

The function \( f_\ell''' \) is supported in the region \([2r_0^*, r_0^*] \cup [R_0^*, 2R_0^*] \) and obeys the pointwise bound

\[
|f_\ell'''(r^*)| \leq C |r^*|^{-2},
\]

(50)

\(^8\)Recall the conventions regarding constants \( \epsilon \) in Section 4.2.
under our conventions for the constants denoted $C$. Note that $\eta_1(x_1 + \overline{x}_1) = 1$ in the support of $f''$. Therefore,

$$\int_{S^2} K^{X,3}(\phi) \geq -C \int_{S^2} \frac{1}{1-\mu} |r^*|^2 (\phi - \phi_0)^2$$

holds on all spheres of symmetry. We obtain \ref{inequality} with

$$\epsilon = C (|r_0^*|^{-1+\delta} + |R_0^*|^{-1+\delta}).$$

\[ \square \]

### 7.4 Auxiliary currents

We will also need several “auxiliary” currents.

#### 7.4.1 Auxiliary positive definite pointwise quantities

Let us first define, however, certain auxiliary positive definite quantities. The auxiliary currents $K$ will be seen to bound these quantities when integrated on spheres of symmetry in the region $r_0 \leq r \leq R_0$.

Define

$$q_1(\phi) \doteq \eta_1 q(\phi) = \eta_1 (T_{\mu\nu}(\phi) \tilde{N}^\mu n^\nu (1 - \mu)^{-1} + (P_2 \phi)^2)$$

$$q_1^a(\phi) \doteq \eta_1 (P_{-1} \phi)^2,$$

$$q_1^a(\phi) \doteq \eta_1 (\phi - \phi_0)^2,$$

$$q_1^b(\phi) \doteq \eta_1 (r - 3M)^2 |\nabla \phi|^2,$$

$$q_1^d(\phi) \doteq \eta_1 (r - 3M)^2 (T \phi)^2.$$

Here $\eta_1$ are the cut-off functions defined in Section 4.3. Note that

$$q_1(\phi) \approx C (q_1^a(\phi) + q_1^b(\phi) + q_1^d(\phi)),$$

$$q_{i+1}^x \leq q_i^x$$

for $x = \emptyset, a, a', b, d$, and that

$$r^{-2}(\ell + 1) \int_{S^2} (r - 3M)^2 q_1^a(\phi) r^2 dA_{S^2} = \int_{S^2} q_1^b(\phi) r^2 dA_{S^2}$$

for all spheres of symmetry.

The currents to be described in what follows are motivated by the problem of bounding the positive definite quantities whose latin superscripts they share.
7.4.2 The current $J^{X_n,3}$

Define

$$f_\ell^\alpha(r^\ast) \doteq -\frac{1}{6} \eta_2(r^\ast)(r^\ast - r_{h\ell})^3,$$

where $\eta_2$ is as defined in Section 4.3.2.

Note that $(f_\ell^\alpha)'' = -1$ on $[r_2, R_2]$, $f_\ell^\alpha(r_{h\ell}) = 0$, and $f_\ell^\alpha = 0$ for $r^\ast \leq 2r_2^2$.

Set $X_\ell^\alpha = f_\ell^\alpha \partial_r$, and define

$$J^{X_n,3}_{\mu}(\phi) = \sum_{\ell \geq 1} J^{X_n,3}_{\mu\ell}(\phi),$$

$$K^{X_n,3}(\phi) = \sum_{\ell \geq 1} K^{X_n,3}_{\ell}(\phi) = \nabla_{\mu} J^{X_n,3}_{\mu}(\phi).$$

The current $J^{X_n,3}(\phi)$ will allow us to bound the spacetime integrals of the quantities $q_0^0(\phi)$ and $q_0^\alpha(\phi)$. See Lemma 10.4.1. At this point, we can see from (42) and (43) that for each $\ell$, the pointwise bound

$$q_0^\alpha(\phi) + q_0^\alpha(\phi) + \ell(\ell + 1)(r - r_{h\ell})^2 q_0^\alpha(\phi) \leq CK^{X_n,3}(\phi) + CK^{X_n,3}_{\ell}(\phi)$$

holds in $r_0 \leq r \leq R_0$.

7.4.3 The currents $J^{X_n,2}$, $J^{X_n,0}$

Define

$$f_\ell^\beta(r^\ast) \doteq \eta_2(r^\ast)(r - 3M)$$

Set $X_\ell = f_\ell^\beta \partial_r$ and define as before the currents $J^{X_n,2}(\phi - \phi_0)$, $J^{X_n,0}(\phi)$ and $K^{X_n,2}(\phi - \phi_0)$, $K^{X_n,0}(\phi)$.

The current $J^{X_n,2}(\phi - \phi_0)$ will allow us to bound--in addition to the previous--the spacetime integral of the quantity $q_0^\alpha(\phi)$. See Lemma 10.4.2. At this point, we can deduce from (41), (42) and (43) the bound

$$\int_{S^2} (q_0^\alpha(\phi) + q_0^\alpha(\phi) + q_0^\alpha(\phi)) r^2 dA_{S^2}$$

$$\leq \int_{S^2} (CK^{X_n,3}(\phi) + CK^{X_n,3}(\phi) + CK^{X_n,2}(\phi - \phi_0)) r^2 dA_{S^2}$$

on each sphere of symmetry with $r^\ast \in [r_0^\ast, R_0^\ast]$.

The current $J^{X_n,0}(\phi)$, in conjunction also with $J^{X_n,0}(\phi_0)$ and $J^{X_n,4}(\phi - \phi_0)$ to be defined below, will be used to help bound the quantity $Z^N_\phi$ in Proposition 10.5.1. For now, note that from (10), (41), (42) and (43), the estimate

$$\int_{S^2} (q_0^\alpha(\phi) + q_0^\alpha(\phi) + q_0^\alpha(\phi))$$

$$- C(\eta_2(1 - \mu)(2(r - 3M)/r + 1) + \eta_2^2(r - 3M)) \partial_\mu \phi \partial_\mu \phi r^2 dA_{S^2},$$

$$\leq \int_{S^2} (CK^{X_n,3}(\phi) + CK^{X_n,3}(\phi) + CK^{X_n,2}(\phi - \phi_0) + CK^{X_n,0}(\phi)) r^2 dA_{S^2}$$
holds on each sphere of symmetry with \( r^* \in [r_0^*, R_0^*] \). Note that the left hand side of (54) \textit{a priori} does not have a sign, in view of the term containing \( \partial \mu \phi \partial \nu \phi \). After defining \( J^{X,0}(\phi_0) \) and \( J^{X,4}(\phi - \phi_0) \) below, we shall be able to improve \( (54) \) with \( (57) \).

### 7.4.4 The current \( J^{X,0} \)

Define

\[
f^c(r^*) = r^2.
\]

Set \( X^c = f^c \frac{\partial}{\partial r} \), and define as before \( J^{X,0}, K^{X,0} \).

We have that

\[
K^{X,0}(\phi_0) = 2r(\partial_t \phi_0)^2. \quad (55)
\]

### 7.4.5 The current \( J^{X,4} \)

Let us then finally define

\[
f^d(r^*) = \eta_1(r^*)(r - 3M)^3.
\]

Set \( X^d = f^d \frac{\partial}{\partial r} \) and define as before the currents \( J^{X,4} \) and \( K^{X,4} \).

The currents \( J^{X,0}(\phi_0) \) and \( J^{X,4}(\phi - \phi_0) \) will allow us to bound the space-time integral of the quantity \( q^d_1(\phi) \). See Lemma 10.4.3. For now, it follows from (40), (53) and (55) that on each sphere of symmetry with \( r^* \in [r_0^*, R_0^*] \), we have

\[
\int_{S^2} \left( q^a_0(\phi) + q^a_1(\phi) + q^0_1(\phi) + q^d_1(\phi) \right) r^2 dA_{S^2} \leq \int_{S^2} \left( CK^{X,3}(\phi) + CK^{X,3}(\phi) + CK^{X,2}(\phi - \phi_0) + CK^{X,0}(\phi_0) + CK^{X,4}(\phi - \phi_0) \right) r^2 dA_{S^2}. \quad (56)
\]

The currents \( J^{X,0}(\phi), J^{X,0}(\phi_0) \) and \( J^{X,4}(\phi - \phi_0) \) together allow us to estimate, on each sphere of symmetry for \( r^* \in [r_0^*, R_0^*] \), the quantity

\[
\int_{S^2} C\eta_2(2(r - 3M)/r + 1)((\partial_t \phi)^2 - |\nabla \phi|^2) r^2 dA_{S^2} \leq \int_{S^2} \left( CK^{X,3}(\phi) + CK^{X,3}(\phi) + CK^{X,2}(\phi - \phi_0) + CK^{X,0}(\phi) + CK^{X,0}(\phi_0) + CK^{X,4}(\phi - \phi_0) \right) r^2 dA_{S^2}. \quad (57)
\]

Again, the left hand side of (57) does not have a sign.
8 The currents $J^Y$ and $J^{\overline{Y}}$

In this section, we define the currents $J^Y_\mu$ and $J^{\overline{Y}}_\mu$, associated to two vector fields $Y$ and $\overline{Y}$, supported near the black hole and cosmological horizons, respectively. The role of these currents in capturing the “red-shift” effect has been discussed in the introduction.

8.1 The vector fields $Y$ and $\overline{Y}$

Let $0 < \delta < 1$ be a small number, and define functions $\alpha(r^*)$, $\beta(r^*)$ as follows. Recall the cutoff functions $\chi_1$, $\eta_{-1}$ from Section 4.3.2, and define

$$\alpha(r^*) = \chi_1(r^*)(2 - \mu + \eta_{-1}(r^*)|r^*|^{-\delta}), \quad (58)$$

$$\beta(r^*) = 2r_b^{-2}(2M/r_b^2 - 2\Lambda r_b/3)^{-1}\chi_1(r^*)(1 - \mu + \eta_{-1}(r^*)|r^*|^{-\delta}). \quad (59)$$

We have $\alpha \geq 0$, $\beta \geq 0$. To see the latter, note that

$$\frac{d}{dr}(1 - \mu)\bigg|_{r=r_b} = \left(\frac{2M}{r^2} - \frac{2}{3}\Lambda r\right)\bigg|_{r=r_b} > 0$$

since $r_ - < r_b < r_c$ are three non-degenerate roots of $(1 - \mu)$, which is non-negative on $[r_b, r_c]$. The above expression approaches 0 as $M$ and $\Lambda$ tend to the extremal values.

Similarly, define functions $\overline{\alpha}$, $\overline{\beta}$ as follows. Recall the cutoff $\overline{\chi}_1$ from Section 4.3.2, and define

$$\overline{\alpha}(r^*) = \overline{\chi}_1(r^*)(2 - \mu + \eta_{-1}|r^*|^{-\delta}), \quad (60)$$

$$\overline{\beta}(r^*) = 2r_c^{-2}(2\Lambda r_c/3 - 2M/r_c^2)^{-1}\overline{\chi}_1(r^*)(1 - \mu + \eta_{-1}|r^*|^{-\delta}). \quad (61)$$

Again, note that $\overline{\alpha} \geq 0$, $\overline{\beta} \geq 0$.

Set $Y$ to be the vector field

$$Y = \frac{\alpha(r^*)}{1 - \mu} \frac{\partial}{\partial u} + \frac{\beta(r^*)}{\partial v}$$

and $\overline{Y}$ to be

$$\overline{Y} = \frac{\overline{\alpha}(r^*)}{1 - \mu} \frac{\partial}{\partial u} + \frac{\overline{\beta}(r^*)}{\partial v}.$$

The application of cutoff $\eta_{-1}$ is in fact not essential for the arguments of the paper. The cutoff simply ensures that $Y$ and $\overline{Y}$ have smooth extensions beyond $\mathcal{H}^+$ and $\overline{\mathcal{H}}^+$. 
8.2 Definition of the currents

Define the currents

\[ J^Y_\mu (\phi) = T_{\mu \nu} Y^\nu (\phi), \]
\[ J^Y_{\mu} (\phi) = T_{\mu \nu} Y^\nu (\phi), \]

and set

\[ K^Y (\phi) = \nabla^\nu J^Y_\mu (\phi), \]
\[ K^Y (\phi) = \nabla^\nu J^Y_{\mu} (\phi). \]

We have

\[ K^Y (\phi) = \frac{\partial_u \phi}{2(1 - \mu)} \left( \frac{2M}{r^2} + \frac{2\Lambda r}{3} - \alpha' \right) + \frac{\partial_v \phi}{2(1 - \mu)} \beta' \]
\[ + \frac{1}{2} \| \nabla \phi \|^2 \left( \frac{\alpha'}{1 - \mu} - \frac{(\beta(1 - \mu))'}{1 - \mu} \right) - \frac{1}{r} \left( \frac{\alpha}{1 - \mu} - \beta \right) \partial_u \phi \partial_v \phi \]

and similarly

\[ K^Y_{\mu} (\phi) = \frac{\partial_v \phi}{2(1 - \mu)} \left( -\frac{2M}{r^2} - \frac{2\Lambda r}{3} + \alpha' \right) - \frac{\partial_u \phi}{2(1 - \mu)} \beta', \]
\[ - \frac{1}{2} \| \nabla \phi \|^2 \left( \frac{\alpha}{1 - \mu} - \frac{(\beta(1 - \mu))'}{1 - \mu} \right) + \frac{1}{r} \left( \frac{\alpha}{1 - \mu} - \beta \right) \partial_u \phi \partial_v \phi. \]

8.3 Discussion and the choice of \( r_1, R_1 \)

In this section we exhibit the “red-shift” property of the currents \( J^Y, J^Y_{\mu} \). This is essentially contained in Proposition 8.3.1 below. In the context of proving this proposition, we will choose the constants \( r_1, R_1 \).

Note that the polynomial powers in the definitions (58)–(59) would be unnecessary had they not been present in the definition (46). (Their presence in (46) is in turn necessitated by our application of (50).)

A slightly unpleasant feature of these polynomial decaying expressions is that, if left bare, they would lead to vector fields \( Y, Y_{\mu} \) which fail to be \( C^1 \) at \( \mathcal{H}^+, \mathcal{H}^{+} \), respectively. This would not in fact pose a problem for the analysis here. We prefer, however, to introduce a cutoff \( \eta_{1} \) in (58)–(59) to emphasize the geometric nature of all objects involved in the proof.

**Proposition 8.3.1.** For \( r_1^* \) sufficiently small and \( R_1^* \) sufficiently large, depending only on \( M, \Lambda \) we have

\[ CK^Y (\phi) \geq \bar{q}(\phi), \]
\[ CK^Y_{\mu} (\phi) \geq \bar{q}(\phi) \]

in \( r \leq r_1 \) and \( r \geq R_1 \), respectively.
Proof. We give the proof only for $K^Y$. The proof for $K^T$ is similar.

Recall the functions $\alpha$, $\beta$ from Section 8. Let us denote by

$$\gamma_b = 2r_b^{-2}(2M/r_b^2 - 2\Lambda r_b/3)^{-1}.$$  

In the region $r \leq r_1$, we have

$$\alpha = 2 - \mu + \eta_1(r^*)|r^*|^{-\delta}, \quad \beta = \gamma_b(1 - \mu + \eta_1(r^*)|r^*|^{-\delta}).$$

As a consequence, for all $r_1 < 0$,

$$\alpha' = \left(\frac{2M}{r^2} - \frac{2}{3} \Lambda r\right) (1 - \mu) + \eta_1'(r^*)|r^*|^{-\delta} + \delta \eta_1''(r^*),$$

$$\beta' = \gamma_b \left(\frac{2M}{r^2} - \frac{2}{3} \Lambda r\right) (1 - \mu) + \gamma_b \eta_1'(r^*)|r^*|^{-\delta} + \gamma_b \delta \eta_1''(r^*)^{-1-\delta}.$$  

Recall that

$$\left(\frac{2M}{r^2} - \frac{2}{3} \Lambda r\right) \bigg|_{r = r_b} > 0$$

and note in addition that $\eta_1' \geq 0$ in the region $r \leq r_1$.

Recall now the expression (62) for $K^Y$. In the region $r \leq r_1$, we compute

$$\frac{\alpha}{1 - \mu} - \frac{(\beta(1 - \mu))'}{1 - \mu} = (\mu + \gamma_b(1 - \mu) - \gamma_b \eta_1(r^*)|r^*|^{-\delta}) \left(\frac{2M}{r^2} - \frac{2}{3} \Lambda r\right)$$

$$+ ((1 - \mu)^{-1} - \gamma_b)(\eta_1'(r^*)|r^*|^{-\delta} + \delta \eta_1''(r^*)^{-1-\delta}),$$

$$\frac{1}{r} \left(\frac{\alpha}{1 - \mu} - \beta\right) = r^{-1}((1 - \mu)^{-1}(1 + \eta_1(r^*)|r^*|^{-\delta-1}) + 1$$

$$- \gamma_b(1 - \mu) - \gamma_b \eta_1(r^*)|r^*|^{-\delta}).$$

Note that

$$r^{-1}(1 - \mu)^{-1} \leq \frac{1}{4}(2M/r^2 - 2\Lambda r/3)(1 - \mu)^{-2} + r^{-2}(2M/r^2 - 2\Lambda r/3)^{-1}. \quad (64)$$

It follows that $r_1$ can be chosen sufficiently close to $r_b$, where the choice depends only on $M$, $\Lambda$, such that, in the region $r \leq r_1$, we have

$$CK^Y(\phi) \geq \left(\frac{\partial_n \phi}{1 - \mu}\right)^2 + \left(1 + \frac{\eta_1}{1 - \mu} \right) \left((\partial_n \phi)^2 + |\nabla \phi|^2\right). \quad (65)$$

In deriving (65), we have used the Cauchy-Schwarz inequality to bound the $\partial_n \phi \partial_n \phi$ term. (The fact that the constants work out is ensured by the limiting inequality (64). It is here that the presence of the constant factor $\gamma_b$ in the definition of $\beta$ is paramount.)
It now follows from (47) that, in the region $r \leq r_1$,

$$CK^Y(\phi) \geq \hat{q}(\phi),$$

as desired.

Note finally, that, in conformance with our conventions of Section 4.2, the constant $C$ above does not depend on $r_0, R_0$, despite the appearance of the cutoff $\eta_1$.

Henceforth, let $r_1, R_1$ be chosen so that the conclusion of the above proposition holds.

The special values $r_1, R_1, r_2, R_2$ now being fixed, constants denoted $C$ now depend only on $M, \Lambda$.

### 9 The current $J^\Theta$

Let $\zeta: [0, 1] \to [0, 1]$ be a cutoff function such that

$$\zeta(x) = 1, \quad x \geq 3/4,$$

$$\zeta(x) = 0, \quad x \leq 1/2.$$

Define

$$\zeta_{(t_1, t_2)}(\tau) = \zeta((\tau - t_1)/(t_1 - t_2)).$$

Let $\theta$ be the Heaviside step-function and let $\Theta$ to be the vector field

$$\Theta = \theta(r^* - r) \zeta_{(t_1, t_2)}(2v - r^*) \left[ (\partial_r \phi)^2 + \frac{1}{2} (1 - \mu) \|\nabla \phi\|^2 \right] + 2\theta(r^* - r^*) \theta(R^*_1 - r^*) \zeta_{(t_1, t_2)}(t) \left[ (\partial_r \phi)^2 + \frac{1}{2} (1 - \mu) \|\nabla \phi\|^2 \right]$$

and define as before the currents $J^\Theta$ and $K^\Theta$. Despite the appearance of the Heaviside function, $\Theta$ is a $C^{0, 1}$ vector field. It is of the form $\Theta = \xi T$, where $\xi$ is a spacetime cut-off function adapted to the $C^{0, 1}$ foliation $\Sigma_t$, constant on each leaf $\Sigma_t$. We have

$$K^\Theta(\phi) = T_{\mu\nu}(\phi)n_{\Theta}^{\mu\nu}$$

$$= -\frac{1}{2(1 - \mu)} \left[ (\theta(r^*_1 - r^*) \zeta_{(t_1, t_2)}(2v - r^*_1) \left[ (\partial_r \phi)^2 + \frac{1}{2} (1 - \mu) \|\nabla \phi\|^2 \right] \right.$$

$$+ 2\theta(r^* - r^*_1) \theta(R^*_1 - r^*_1) \zeta_{(t_1, t_2)}(t) \left[ (\partial_r \phi)^2 + (\partial_r \phi)^2 + (1 - \mu) \|\nabla \phi\|^2 \right]$$

$$+ \theta(r^* - R^*_1) \zeta_{(t_1, t_2)}(2u + R^*_1) \left( (\partial_r \phi)^2 + \frac{1}{2} (1 - \mu) \|\nabla \phi\|^2 \right).$$
10 Proof of the main estimates

10.1 Auxiliary integral quantities

Let us define the auxiliary quantities

\[ Q_{i,\phi}(t_1, t_2) = \int_{R(t_1, t_2)} q_{x_i}(\phi) \]

\[ \hat{Q}_{\phi}(t_1, t_2) = \int_{R(t_1, t_2)} \hat{q}(\phi) \]

\[ F^T_{\phi}(t_1, t_2) = \int_{I^+ \cap \{v_1 \leq v \leq v_2\}} J^T(\phi)n^\mu + \int_{H^+ \cap \{u_1 \leq u \leq u_2\}} J^T_\mu(\phi)n^\mu \]

\[ Z^T_{\phi}(t_i) = \int_{\Sigma_i} J^T_\mu n^\mu \]

where \( q_{x_i}(\phi) \) is as defined in Section 7.4.1, with \( x = \emptyset, a, b, \ldots \), and \( \hat{q}(\phi) \) is as defined in Section 7.3.4.

10.2 Preliminary inequalities

10.2.1 Auxiliary quantity inequalities

For the boundary terms, note first that on the support of \( \eta_1 \),

\[ n = \frac{1}{\sqrt{1 - \mu}} T. \]

On the other hand, in the expression (27), each term gives a nonnegative contribution to \( J^N_\mu n^\mu \), and we have on \( \Sigma_i \)

\[ J^N_\mu n^\mu \approx \chi^1 \left( \frac{\partial \phi}{1 - \mu} \right)^2 + \chi^1 \left( \frac{\partial \phi}{1 - \mu} \right)^2 + (\partial_t \phi)^2 + (\partial_r \phi)^2 + (1 - \mu)|\nabla \phi|^2. \] (66)

As a consequence,

\[ Z^{S,P}_{\phi}(t_i) \leq C Z^N_{\phi}(t_i), \] (67)

and

\[ Z^T_{\phi}(t_i) \leq C Z^N_{\phi}(t_i). \] (68)

10.2.2 Boundary term inequalities for currents \( J \)

In this section we address questions of size of the boundary terms generated by the currents \( J^X_i, J^X_i^a, J^X_i^b, J^X_i^c, J^X_i^d \) from Section 7 and \( J^Y, J^Y \) from Section 8.

Proposition 10.2.1. For \( J = J^{X,i}(\phi), J^{X,a,3}(\phi), J^{X,b,2}(\phi - \phi_0), J^{X,b,1}(\phi), J^{X,0}(\phi_0), J^{X,4}(\phi - \phi_0) \) we have

\[ \left| \int_{\Sigma_i} J_\mu n^\mu \right| \leq E Z^T_{\phi}(t_i), \]
\[
\int_{\mathcal{H} \cap \{v_1 \leq v \leq v_2\}} J_\mu n^\mu + \int_{\mathcal{H}^\nu \cap \{u_1 \leq u \leq u_2\}} J_\mu n^\mu \leq E \mathbf{F}^T_\phi (t_1, t_2).
\]

**Proof.** We shall here only prove the proposition in a representative case of the current

\[ J^{X,2}(\phi) = J^{X,0}(\phi_0) + \sum_{\ell \geq 1} J^{X,2}_\ell (\phi_\ell), \]

defined in Section 7.3.3.

Recall that

\[ Z^T T_\phi (t) = \int_{\Sigma_t} T_{\mu \nu} (\phi) T^\mu n^\nu. \]

Since on the space-like part of \( \Sigma_t \), we have \( n = (1 - \mu)^{-1/2} T \), it follows that there

\[ T_{\mu \nu} (\phi) T^\mu n^\nu = \frac{1}{\sqrt{1 - \mu}} T_{\mu \nu} (\phi) T^\mu T^\nu = \frac{1}{4 \sqrt{1 - \mu}} (T_{uu} (\phi) + 2 T_{uv} (\phi) + T_{vv} (\phi)), \]

since in addition \( T = 1/2 (\partial / \partial u + \partial / \partial v) \). On the other hand, since \( n = \partial / \partial u \) and \( n = \partial / \partial v \) on the null segments \( v = \text{const} \) and \( u = \text{const} \) of \( \Sigma_t \), respectively,

\[ T_{\mu \nu} (\phi) T^\mu n^\nu = T_{\mu \nu} (\phi) T^\mu \left( \frac{\partial}{\partial u} \right)^\nu = \frac{1}{2} (T_{uu} (\phi) + T_{uv} (\phi)), \]

\[ T_{\mu \nu} (\phi) T^\mu n^\nu = T_{\mu \nu} (\phi) T^\mu \left( \frac{\partial}{\partial v} \right)^\nu = \frac{1}{2} (T_{vv} (\phi) + T_{uv} (\phi)). \]

Since the space-like portion of \( \Sigma_t \) corresponds to \( r \)-values \( r_1 \leq r \leq R_1 \) we have that

\[ Z^T T_\phi (t) \approx \int_{\Sigma_t} (T_{uv} (\phi) + (1 - \chi_1) T_{uu} (\phi) + (1 - \chi_1) T_{vv} (\phi)). \]

Therefore,

\[ Z^T T_\phi (t) \approx \int_{\Sigma_t} \left( (1 - \chi_1) (\partial_r \phi)^2 + (1 - \chi_1) (\partial_u \phi)^2 + (1 - \mu) |\nabla \phi|^2 \right). \quad (69) \]

On the other hand,

\[ \mathbf{F}^T T_\phi (t_1, t_2) = \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J^T (\phi) n^\mu + \int_{\mathcal{H}^\nu \cap \{u_1 \leq u \leq u_2\}} J^T_\mu (\phi) n^\mu \]

\[ = \frac{1}{2} \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} (\partial_v \phi)^2 + \frac{1}{2} \int_{\mathcal{H}^\nu \cap \{u_1 \leq u \leq u_2\}} (\partial_u \phi)^2. \]

We compare now (69) with the expression for \( \int_{\Sigma_t} J^{X,2}_\mu n^\mu \). Start with the current

\[ J^{X,0}_\mu (\phi_0) = -\frac{1}{r^2} T_{\mu \nu} (\phi_0) \left( \frac{\partial}{\partial r^*} \right)^\mu \].
Since \( \partial / \partial r^* = 1/2 (\partial / \partial v - \partial / \partial u) \), we infer that on \( \Sigma_t \)
\[
\int_{\Sigma_t} |J_{\mu}^{X, \nu, 0}(\phi_0) n^\nu| \leq C \int_{\Sigma_t} ((1 - \chi_1)(\partial_u \phi_0)^2 + (1 - \chi_1)(\partial_v \phi_0)^2 + (1 - \mu)|\nabla \phi_0|^2)
\leq C Z^T_{\phi_0}(t).
\]

For \( \ell \geq 1 \), consider now
\[
J_{\mu}^{X, \tau, 2}(\phi_\ell) = -f_\ell T_{\mu \nu}(\phi_\ell) (\partial r^*)^\mu - \frac{1}{4} \left( f'_\ell + 2 \frac{1 - \mu}{r} f_\ell \right) \partial_\mu \phi_\ell^2 + \frac{1}{4} \partial_\mu \left( f'_\ell + 2 \frac{1 - \mu}{r} f_\ell \right) \phi_\ell^2.
\]

Recall that the functions \( f'_\ell(r^*) = 1 \) on the interval \([r_0^*, R_0^*]\) and vanish for \( r^* \leq 2r_0^* \) and \( r^* \geq 2R_0^* \). On the space-like portion of \( \Sigma_t \), i.e., for \( r \in [r_1, R_1] \), we have
\[
|J_{\mu}^{X, \tau, 2}(\phi_\ell) n^\mu| \leq C (T_{uv}(\phi_\ell) + T_{uu}(\phi_\ell) + T_{vv}(\phi_\ell) + \phi_\ell^2),
\]
while on the null segments of \( \Sigma_t \), we have
\[
|J_{\mu}^{X, \tau, 2}(\phi_\ell) \frac{\partial}{\partial u} \phi_\ell^2 | \leq E (T_{uv}(\phi_\ell) + T_{uu}(\phi_\ell) + (\partial_v \phi_\ell)^2 + (1 - \mu)\phi_\ell^2)
\]
for \( v = \text{const} \), and
\[
|J_{\mu}^{X, \tau, 2}(\phi_\ell) \frac{\partial}{\partial u} \phi_\ell^2 | \leq E (T_{uv}(\phi_\ell) + T_{vv}(\phi_\ell) + (\partial_u \phi_\ell)^2 + (1 - \mu)\phi_\ell^2)
\]
for \( u = \text{const} \). Therefore,
\[
\int_{\Sigma_t} |J_{\mu}^{X, \tau, 2}(\phi_\ell) n^\mu| \leq E \int_{\Sigma_t} (T_{uv}(\phi_\ell) + (1 - \chi_1) T_{uu}(\phi_\ell) + (1 - \chi_1) T_{uv}(\phi_\ell) + (1 - \mu)\phi_\ell^2)
\leq E \int_{\Sigma_t} ((1 - \chi_1)(\partial_u \phi_\ell)^2 + (1 - \chi_1)(\partial_v \phi_\ell)^2 + (1 - \mu)(|\nabla \phi_\ell|^2 + \phi_\ell^2)).
\]

Summing over \( \ell \) and using the identity
\[
\ell(\ell + 1) \int_{S^3} \frac{\phi_\ell^2}{r^2} r^2 dA_0^2 = \int_{S^2} |\nabla \phi_\ell|^2 r^2 dA_0^2,
\]
we obtain
\[
\int_{\Sigma_t} |J_{\mu}^{X, \tau, 2}(\phi) n^\mu| \leq E \int_{\Sigma_t} ((1 - \chi_1)(\partial_u \phi)^2 + (1 - \chi_1)(\partial_v \phi)^2 + (1 - \mu)|\nabla \phi|^2) \leq E Z^T_{\phi}(t),
\]
as desired. On the other hand,
\[
\int_{H^+ \cap \{ v_1 \leq v \leq v_2 \}} J_{\mu}^{X, \tau, 2}(\phi) n^\mu = \int_{H^+ \cap \{ v_1 \leq v \leq v_2 \}} \left( J_{\mu}^{X, \nu, 0}(\phi_0) + \sum_{\ell \geq 1} J_{\mu}^{X, \tau, 2}(\phi_\ell) \right) \left( \frac{\partial}{\partial v} \right)^\mu.
\]
Moreover, as desired. Similar arguments give the inequality on the horizon 

\[ \mathcal{H}^+ \]

Once again, we shall here consider only the current

\[ J^X_\mu(\phi_0) \left( \frac{\partial}{\partial v} \right)^\mu = -\frac{1}{r^2} T_{\mu
u}(\phi_0) \left( \frac{\partial}{\partial v} \right)^\mu = -\frac{1}{2r^2} (\partial_v \phi_0)^2, \]

\[ J^X_\mu(\phi_t) \left( \frac{\partial}{\partial v} \right)^\mu = -f_t T_{\mu
u}(\phi_t) (\partial_v r^+)^\mu - \frac{1}{4} (\frac{f_t}{r} + 2 \frac{1 - \mu}{r} f_t) \partial_\mu r^+ \]

\[ + \frac{1}{4} \partial_\mu (f_t^\prime + 2 \frac{1 - \mu}{r} f_t) r^+ \left( \frac{\partial}{\partial v} \right)^\mu = -f_t (\partial_v \phi_t)^2. \]

As a consequence, since \(|f_t(r^+)| \leq E|\),

\[ \left| \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J^X_\mu(\phi)n^\mu \right| \leq \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} \left( \frac{1}{2r^2} (\partial_v \phi_t)^2 + \sum_{t \geq 1} \left| f_t \right| (\partial_v \phi_t)^2 \right) \]

\[ \leq E \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} (\partial_v \phi)^2 \leq E F^T_\phi(t_1, t_2), \]

as desired. Similar arguments give the inequality on the horizon \(\mathcal{H}^+\), as well as the inequalities for the other currents of the statement of the proposition. \(\square\)

**Proposition 10.2.2.** For \( J = J^Y(\phi), J^\mathcal{F}(\phi) \), we have

\[ 0 \leq \int_{\Sigma_\tau} J_\mu n^\mu \leq C Z^S_\phi^{S, P}(t), \]

Moreover

\[ \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J^Y_\mu(\phi)n^\mu \geq 0, \]

\[ \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J^\mathcal{F}_\mu(\phi)n^\mu = 0, \]

\[ \int_{\mathcal{H}^+ \cap \{u_1 \leq u \leq u_2\}} J^Y_\mu(\phi)n^\mu = 0, \]

\[ \int_{\mathcal{H}^+ \cap \{u_1 \leq u \leq u_2\}} J^\mathcal{F}_\mu(\phi)n^\mu \geq 0. \]

**Proof.** Once again, we shall here consider only the current \( J^Y(\phi) \). The considerations for \( J^\mathcal{F} \) are practically identical.

By the construction in Section \( \Box \) the support of \( J^Y(\phi) \) is contained in the region \( r^+ \leq \frac{1}{2} r^* \). This immediately implies that \( J^Y(\phi)|_{\mathcal{H}^+} = 0 \). Moreover \( Y \) is a bounded future-directed time-like vector field in the region \( r^*_1 \leq r^+ < \frac{1}{2} r^*_1 \), which implies that there we have

\[ 0 \leq J^Y_\mu(\phi)n^\mu = T_{\mu
u}(\phi) Y^\nu n^\mu \]

\[ \leq CT_{\mu
u}(\phi) T^\nu n^\mu \]

\[ = C J^\mathcal{F}_\mu(\phi)n^\mu \leq C J^S_\mu(\phi)n^\mu. \]
Restricted to the support of $Y$, the remaining part of $\Sigma_t$ is contained in a null segment $v = \text{const}$. Thus, we have

$$J^Y_\mu (\phi) n^\mu = T_{\mu \nu} (\phi) Y^\nu \left( \frac{\partial}{\partial v} \right)^\mu = \frac{\alpha}{1 - \mu} T_{uu} + \beta T_{uv}$$

$$= \frac{\alpha}{1 - \mu} (\partial_u \phi)^2 + \beta (1 - \mu) |\nabla \phi|^2.$$ 

The functions $\alpha, \beta$ are non-negative and in the region $r \leq r_1$ are given by

$$\alpha = 2 - \mu + \eta - 1 (r^*) |r^*|^{-\delta}, \quad \beta = \gamma_b (1 - \mu + \eta - 1 (r^*) |r^*|^{-\delta}),$$

which implies that

$$0 \leq J^Y_\mu (\phi) n^\mu \leq C \left( \frac{(\partial_u \phi)^2}{1 - \mu} + (1 - \mu) |\nabla \phi|^2 \right).$$

Comparing this to the expression for $J^N_\mu (\phi) n^\mu$, given in (66), we see that

$$0 \leq J^Y_\mu (\phi) n^\mu \leq C J^N_\mu (\phi) n^\mu$$
on the null portion $v = \text{const}$ of $\Sigma_t$. Since on this portion we have $N = \tilde{N}$, we finally obtain the desired inequality

$$0 \leq \int_{\Sigma_t} J^Y_\mu (\phi) n^\mu \leq C Z \tilde{N} P (t).$$

On the other hand, on $\mathcal{H}^+$ we have $n = \frac{\partial}{\partial v}$ and

$$J^Y_\mu (\phi) n^\mu = 2 \frac{\alpha}{1 - \mu} T_{uv} + \beta T_{uv} = 2 |\nabla \phi|^2,$$

since $\beta = (1 - \mu) = 0$ and $T_{uv} = (1 - \mu) |\nabla \phi|^2$ on $\mathcal{H}^+$. Thus

$$\int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} J^Y_\mu (\phi) n^\mu = 2 \int_{\mathcal{H}^+ \cap \{v_1 \leq v \leq v_2\}} |\nabla \phi|^2.$$

\[\square\]

### 10.3 Applications of the integral identity for currents

In this section we exploit the divergence theorem for compatible currents to relate various integral quantities. From Proposition 10.2.1 Proposition 10.2.2 and identity (26), the following two propositions follow immediately:

**Proposition 10.3.1.** For $J = J^{X,0} (\phi), J^{X,1} (\phi), J^{X,2} (\phi), J^{X,3} (\phi), J^{X,4} (\phi - \phi_0), J^{X,5} (\phi - \phi_0), J^{X,6} (\phi - \phi_0)$ and $K = \nabla^\mu J_\mu$, we have

$$\left| \int_{\mathcal{R}(t_1, t_2)} K \right| \leq E (Z_\phi^\tau (t_1) + Z_\phi^\tau (t_2) + F_\phi^\tau (t_1, t_2)).$$
Proposition 10.3.2. We have
\[
\int_{\mathcal{R}(t_1,t_2)} K^Y(\phi) + \int_{\Sigma_{t_2}} J^Y_\mu(\phi) n^\mu + \int_{\mathcal{H}^+ \cap \{u_1 \leq v \leq u_2\}} J^Y_\mu(\phi) n^\mu \leq C Z^N_{\phi}(t_1), \tag{70}
\]
\[
\int_{\mathcal{R}(t_1,t_2)} K^Y(\bar{\phi}) + \int_{\Sigma_{t_2}} J^Y_\mu(\bar{\phi}) n^\mu + \int_{\mathcal{H}^+ \cap \{u_1 \leq v \leq u_2\}} J^Y_\mu(\bar{\phi}) n^\mu \leq C Z^N_{\phi}(t_1). \tag{71}
\]

Applying (26) with the energy current \(J^T\mu\) gives us the following proposition

Proposition 10.3.3.
\[
Z^T_{\phi}(t_2) + F^T_{\phi}(t_1,t_2) = Z^T_{\phi}(t_1) \leq C Z^N_{\phi}(t_1),
\]

Proof. The statement follows immediately from \(K^T = 0\) and (67), (68). \(\square\)

Applying (26) with the energy current \(J^T\mu + J^Y\mu + J^\bar{Y}\mu\), we obtain

Proposition 10.3.4. Let \(\Sigma' \subset \mathcal{R}_{t_1,t_2}\) be achronal. Then
\[
\int_{\Sigma'} J^N_\mu(\phi) n^\mu \leq Z^N_{\phi}(t_1) + C \int_{\mathcal{R}(t_1,t_2) \cap J^-(\Sigma')} -K^Y(\phi) - K^\bar{Y}(\phi)
\]

In particular,
\[
Z^N_{\phi}(t_2) \leq Z^N_{\phi}(t_1) + C \int_{\mathcal{R}(t_1,t_2)} -K^Y(\phi) - K^\bar{Y}(\phi).
\]

Proof. The vector field \(T + Y + \bar{Y}\) is timelike and one sees easily
\[
\left(J^T_\mu + J^Y_\mu + J^\bar{Y}_\mu\right) n^\mu \approx J^N_\mu n^\mu,
\]
while certainly \(K^T + K^Y + K^\bar{Y} = K^Y + K^\bar{Y}\). \(\square\)

We have an alternative bound on \(Z^N_{\phi}(t_2)\) as follows

Proposition 10.3.5.
\[
Z^N_{\phi}(t_2) \leq C Z^N_{\phi}(t_2) - C \int_{\mathcal{R}(t_1,t_2)} K^\Theta(\phi).
\]

Proof. Recall that by the construction given in Section 9 the vector field \(\Theta\) is future timelike, \(\Theta|_{\Sigma_{t_2}} = T\) and \(\Theta|_{\Sigma_1} = 0\). The result now follows from the statement
\[
Z^N_{\phi}(t_2) \leq C Z^N_{\phi}(t_2) + C \int_{\Sigma_{t_2}} J^\Theta_\mu(\phi) n^\mu
\]
and the divergence theorem. \(\square\)
10.4 Bounding $Q_\phi$ from $Z_\phi^N$

In this section we establish the first key part of Theorem 6.1, that is to say, statement (34). We begin with the following

**Proposition 10.4.1.**

$$Q_{1,\phi}(t_1,t_2) \leq E Z_\phi^N(t_1) + \epsilon \tilde{Q}_\phi(t_1,t_2)$$

**Proof.** The proposition follows from the three Lemmas below:

**Lemma 10.4.1.**

$$Q_{1,\phi}^a + Q_{1,\phi}^b \leq C \int_{\mathcal{R}(t_1,t_2)} K^{X,3}(\phi) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^a,3}(\phi) + \epsilon \tilde{Q}_\phi.$$ 

**Lemma 10.4.2.**

$$Q_{1,\phi}^c \leq C \int_{\mathcal{R}(t_1,t_2)} K^{X,3}(\phi) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^a,3}(\phi) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^b,2}(\phi - \phi_0) + \epsilon \tilde{Q}_\phi.$$ 

**Lemma 10.4.3.**

$$Q_{1,\phi}^d \leq C \int_{\mathcal{R}(t_1,t_2)} K^{X,3}(\phi) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^a,3}(\phi) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^b,2}(\phi - \phi_0) + C \int_{\mathcal{R}(t_1,t_2)} K^{X^c,4}(\phi - \phi_0) + \epsilon \tilde{Q}_\phi.$$ 

**Proof.** The statements of the above three Lemmas follow directly from (45), (52), (53) and (56), combined with the observation that, since the supports of the currents $J^{X,3}, J^{X^a,2}$ and $J^{X^c,4}$, as well as of the quantities $q_{X}^i(\phi)$, are contained in the region $\{2r_1^* \leq r^* \leq 2R_1^*\}$, since $K^{X^c,0}(\phi_0)$ is nonnegative, and since $K^{X,3}$ is nonnegative in $\{r_0^* \leq r^* \leq R_0^*\}$, it suffices to apply Lemma 7.3.2.

Proposition 10.4.1 now follows from Proposition 10.3.1 and Proposition 10.3.3.

**Proposition 10.4.2.**

$$\int_{\mathcal{R}(t_1,t_2)} K^Y(\phi) + K^\nabla(\phi) \leq C Z_\phi^{R_p}(t_1).$$ 

**Proof.** This follows immediately by adding (70), (71) of Proposition 10.3.2, in view also of Proposition 10.2.2.

**Proposition 10.4.3.**

$$\tilde{Q}_\phi(t_1,t_2) \leq C Q_{1,\phi}(t_1,t_2) + C \int_{\mathcal{R}(t_1,t_2)} K^Y(\phi) + K^\nabla(\phi)$$
Proof. The statement follows from the inequality

\[ \hat{q}(\phi) \leq Cq_1(\phi) + KY(\phi) + KY(\phi), \]

which is a direct consequence of (51), (47) and Proposition 8.3.1.

It now immediately follows from Proposition 10.4.1, Proposition 10.4.2 and Proposition 10.4.3 and our conventions regarding constants \( C, E \) and \( \epsilon \) that the following holds:

**Proposition 10.4.4.** If \( r^*_0 \) is chosen sufficiently small, and \( R^*_0 \) is chosen sufficiently large, depending only on \( M, \Lambda \), then

\[ \hat{Q}_\phi(t_1, t_2) \leq E Z^N_\phi(t_1). \]

Henceforth, let \( r_0, R_0 \) be so chosen so that the conclusion of the above proposition holds. In particular, in what follows we shall need only make use of constants \( C \) depending only on \( M, \Lambda \).

Finally, since by (47), (48)

\[ Q_\phi(t_1, t_2) \leq \hat{Q}_\phi(t_1, t_2), \]

the statement of Theorem 6.1 follows immediately.

### 10.5 Bounding \( Z^N_\phi \) from \( Z^N_\phi, Q_\phi \) and \( Q_{\Omega_i \phi} \)

Bound (35) of Theorem 6.1 is contained in the following

**Proposition 10.5.1.** For all \( t_1 < t_2 \)

\[ Z^N_\phi(t_2) \leq C Z^N_\phi(t_2) + (t_2 - t_1)^{-1} C \left( Z^N_\phi(t_1) + Q_\phi(t_1, t_2) + \sum_{i=1}^{3} Q_{\Omega_i \phi}(t_1, t_2) \right). \]

**Proof.** By Proposition 10.3.5

\[ Z^N_\phi(t_2) \leq C Z^N_\phi(t_2) - \int_{\mathcal{R}(t_1, t_2)} K^\Theta(\phi). \]

The current \( K^\Theta(\phi) \) was defined in Section 3. One easily sees

\[ -K^\Theta(\phi) \leq \frac{C}{t_2 - t_1} \left( x_1 \frac{(\partial_\phi \phi)^2}{1 - \mu} + x_1 \frac{(\partial_{\tau_\phi} \phi)^2}{1 - \mu} + |\nabla \phi|^2 + \eta_1 (\partial_t \phi)^2 + (\partial_\tau \phi)^2 \right). \]

Comparing this with (48), we infer that the statement of the proposition would follow from the estimate

\[ \int_{\mathcal{R}(t_1, t_2)} \eta_2 \left( (\partial_t \phi)^2 + |\nabla \phi|^2 \right) \leq C \left( Z^N_\phi(t_1) + Q_\phi(t_1, t_2) + \sum_{i=1}^{3} Q_{\Omega_i \phi}(t_1, t_2) \right). \]
From (48) we have that
\[
C_Q\phi(t_1, t_2) \geq \int_{\mathcal{R}(t_1, t_2)} (r - 3M)^2 |\nabla \phi|^2 + (\partial_r \phi)^2,
\]
\[
C_3 \sum_{i=1}^{3} Q_{\Omega_i, \phi}(t_1, t_2) \geq \int_{\mathcal{R}(t_1, t_2)} \eta_1 |\partial_r \nabla \phi|^2.
\]
A one-dimensional Poincaré inequality immediately implies that
\[
C_Q\phi(t_1, t_2) + \sum_{i=1}^{3} Q_{\Omega_i, \phi}(t_1, t_2) \geq \int_{\mathcal{R}(t_1, t_2)} \eta_2 |\nabla \phi|^2.
\]
It thus follows from (57) that
\[
\int_{\mathcal{R}(t_1, t_2)} (CK_{X, 3}(\phi) + CK_{X, a}(\phi) + CK_{X, b}(\phi - \phi_0) + CK_{X, b, 0}(\phi))
\]
\[
\geq \int_{\mathcal{R}(t_1, t_2)} \eta_2 (\partial_t \phi)^2 - C_Q\phi(t_1, t_2) + \sum_{i=1}^{3} Q_{\Omega_i, \phi}(t_1, t_2).
\]
The desired (72) now follows from Propositions 10.3.1 and 10.3.3. \(\square\)

10.6 Bounding \(Z^N_{\phi}(t_2)\) from \(Z^N_{\phi}(t_1)\) and \(Q_{\phi}\)

The final statements of Theorem 6.1 follows from Proposition 10.6.1.

Proposition 10.6.1. Let \(\Sigma' \subset \mathcal{R}(t_1, t_2)\) be achronal. Then
\[
\int_{\Sigma'} T_{\mu\nu}(\phi) N^\mu n^\nu \leq C \left( Z^N_{\phi}(t_1) + Q_{\phi}(t_1, t_2) \right). \tag{73}
\]
In particular,
\[
Z^N_{\phi}(t_2) \leq C \left( Z^N_{\phi}(t_1) + Q_{\phi}(t_1, t_2) \right). \tag{74}
\]

Proof. By Proposition 10.3.4
\[
\int_{\Sigma'} J^N_{\mu}(\phi) n^\mu \leq Z^N_{\phi}(t_1) + C \int_{\mathcal{R}(t_1, t_2) \cap J^-(\Sigma')} C^{-1} Y(\phi) - K^{-1}(\phi)
\]
Recall that the current \(K^{-1}(\phi)\) (respectively, \(K^{-1}(\phi)\)) is positive for \(r \leq r_1\) (respectively, \(r \geq R_1\)) and vanishes for \(r^* \geq \frac{1}{2} r_1^*\) (respectively, \(r^* \leq \frac{1}{2} R_1^*\)). Moreover, comparing (48) and (62) easily implies the bound
\[
-C^{-1} Y(\phi) - K^{-1}(\phi) \leq C q(\phi)
\]
for \(r_1^* \leq r^* \leq \frac{1}{2} r_1^*\), and \(\frac{1}{2} R_1^* \leq r^* \leq \frac{1}{2} R_1^*\). The result now follows immediately. \(\square\)
11 Proof of Theorem 1.1

11.1 Energy decay

**Proposition 11.1.1.** There exist constants $C, c$, depending only on $M, \Lambda$, such that for all $\phi_\ell$ solutions of $\Box_g \phi_\ell = 0$ in $J^+(\Sigma_0) \cap \mathcal{D}$ with spherical harmonic number $\ell$, then

$$Z_{\phi_\ell}^N(t) + Q_{\phi_\ell}(t, t^*) \leq C Z_{\phi_\ell}^N(0) e^{-2ct/\ell^2}$$

for all $t$ and all $t^* \geq t$.

**Proof.** This follows immediately from estimates (34) and (35), in view of the following lemma, proved in Appendix B, applied to the functions $f(t) = Z_{\phi_\ell}^N(t)$ and $h(t) = Z_{\phi_\ell}^N(t)$:

**Lemma 11.1.1.** Let $k, k_0$ be positive constants and let $g, h : [0, \infty) \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$ be nonnegative continuous functions satisfying

$$h(t_2) + \int_{t_2}^{t_3} f(\tau) d\tau \leq k \left( f(t_2) + (t_2 - t_1)^{-1} \left( h(t_1) + \ell^2 \int_{t_1}^{t_2} f(\tau) d\tau \right) \right)$$

for all $t_3 > t_2 > t_1 \geq 0$, and $\int_0^\infty f \leq k_0$. Then there exists a constants $c$ depending only on $k$, and a universal constant $C$ such that

$$h(t) + \int_{t}^{t^*} f(\tilde{t}) d\tilde{t} \leq C (\max\{h(0), k_0\}) e^{-ct/\ell^2}$$

for all $t$ and for all $t^* \geq t$. \hfill \qed

**Proposition 11.1.2.** There exist constants $C, c$, depending only on $M, \Lambda$, such that for all $\phi_\ell$ solutions of $\Box_g \phi_\ell = 0$ in $J^+(\Sigma_0) \cap \mathcal{D}$ with spherical harmonic number $\ell$ and for all achronal $\Sigma' \subset \mathcal{D} \cap J^+(\Sigma_0)$,

$$\int_{\Sigma'} T_{\mu\nu}(\phi_\ell) N^\mu N^\nu \leq C Z_{\phi_\ell}^N(0) \left( e^{-2cv_+(\Sigma')/\ell^2} + e^{-2cv_-(\Sigma')/\ell^2} \right),$$

in particular

$$Z_{\phi_\ell}^N(t) \leq C Z_{\phi_\ell}^N(0) e^{-2ct/\ell^2}.$$

**Proof.** This follows immediately from Propositions 11.1.1 and the inequality (36) of Theorem 6.1. \hfill \qed

The energy decay statements of Theorem 1.2 now follow from the following

**Proposition 11.1.3.** Let $\Sigma$ be a Cauchy surface for $\mathcal{M}$, and let $\phi, \dot{\phi}, E_0(\phi, \dot{\phi})$ be as in the statement of Theorem 1.1 or Theorem 1.2. Then, there exist constants $C, t_0 \geq 0$ depending only on $M, \Lambda$ and the geometry of $\Sigma \cap J^{-}(\mathcal{D})$, such that

$$\Sigma_{t_0} \subset J^+(\Sigma) \cap \mathcal{D},$$
and such that for all solutions \( \phi \) to the wave equation \( \Box_g \phi = 0 \) on \( J^+ (\Sigma) \cap J^- (D) \), the estimate

\[
Z^N_N (t_0) \leq C E_0 (\phi, \dot{\phi})
\]

holds.

**Proof.** This is completely standard. We give a sketch to emphasize here too the role of compatible currents based on vector fields! Extend say \( N_0 \) from \( \Sigma_0 \) to an arbitrary future-timelike vector field \( N \) in \( J^- (\Sigma_{t_0}) \cap J^+ (\Sigma) \), and consider an arbitrary spacelike foliation \( S_\tau \) of this region by manifolds with boundary, such that \( S_{-1} = \Sigma, S_0 = \Sigma_T \), and \( \partial S_\tau \subset J^- (\Sigma_T) \setminus I^- (\Sigma_T) \). Consider the analogue of (26) in \( J^- (S_\tau) \cap J^+ (S_{-1}) \). Set

\[
f(\tau) = \int_{S_\tau} N^\mu n_\mu.
\]

In view of the fact that \( N \) is future timelike, we obtain

\[
f(\tau) \leq f (-1) + \int_{J^- (S_\tau) \cap J^+ (S_{-1})} K.
\]

On the other hand, one easily sees that there exists a \( C \) depending on the geometry of our chosen foliation of the compact set \( J^- (\Sigma_T) \cap J^+ (\Sigma) \) such that for all \( \tau \in [-1, 0] \),

\[
\int_{J^- (S_\tau) \cap J^+ (S_{-1})} K \leq C \int_{-1}^{\tau} f (\check{\tau}) d\check{\tau}.
\]

It now follow that \( f (0) \leq e^C f (-1) \).

The result now follows by noting that

\[
f (-1) \leq C E_0 (\phi, \dot{\phi}), \quad Z^N_N (t_0) = f (0).
\]

**11.2 Pointwise decay**

The pointwise decay statements follow easily.

**A Proof of Lemma 7.3.1**

Consider the function

\[
A (r) \doteq r^3 h_\ell (r) = \ell (\ell + 1) \left( 1 - \frac{3M}{r} \right) + \frac{3M}{r} + \frac{\Lambda r^2}{3} - \frac{2\Lambda^2 r^4}{9}.
\]

Recall that

\[
\mu (r) = \frac{2M}{r} + \frac{\Lambda r^2}{3}, \quad \mu^2 = \frac{4M^2}{r^2} + \frac{4\Lambda M r}{3} + \frac{\Lambda^2 r^4}{9}.
\]
We may thus rewrite the function as
\[ A(r) = \ell(\ell + 1)\left(1 - \frac{3M}{r}\right) + \frac{3M}{r} + 3\Lambda Mr - 2\mu^2. \]

For \( \ell \geq 1 \), \( A \) is concave in \( r \):
\[
\frac{d^2 A}{dr^2} = -6M(\ell^2 + \ell - 1)r^{-3} - 4 \left( \frac{d\mu}{dr} \right)^2 - 4\mu \frac{d^2\mu}{dr^2} - 4\mu \frac{d^2\mu}{dr^2} r^{-2} = 4Mr^{-3} + \frac{2\Lambda}{3} > 0.
\]

The value of \( A \) at \( r_c \) is given by
\[ A(r_c) = (\ell(\ell + 1) - 2)\left(1 - \frac{3M}{r_c}\right) - \frac{3M}{r_c} + 3\Lambda Mr_c + 2 - 2\mu^2(r_c) \]
\[ = (\ell(\ell + 1) - 2)\left(1 - \frac{3M}{r_c}\right) - \frac{3M}{r_c} + 3\Lambda Mr_c, \]
where we have used \( \mu(r_c) = 1 \). The same formula evidently holds at \( r = r_b \) with \( r_c \) replacing everywhere.

Since \( r_c > 3M, (\ell(\ell + 1) \geq 2 \) and \( \Lambda r_c^2 = 3 - \frac{6M}{r_c} \), we have
\[ A(r_c) \geq \frac{3M}{r_c}(\Lambda r_c^2 - 1) = \frac{3M}{r_c} \left(2 - \frac{6M}{r_c}\right) > 0. \]

From \( r_b < 3 \), we obtain similarly
\[ A(r_b) \leq \frac{3M}{r_b} \left(2 - \frac{6M}{r_b}\right) < 0. \]

By concavity the function \( A(r) \) then has exactly one zero on the interval \((r_b, r_c)\), and thus, so does \( h_\ell(r) = r^{-3}A(r) \).

The second statement of the lemma is clear from the final, which in turn follows immediately from the form of the function \( h_\ell \).

**B  Proof of Lemma 11.1.1**

By replacing \( k \) with \( \max\{k, 1\} \), we may assume in what follows that \( k \geq 1 \). Let \( t_1, \ldots, t_i \) be a sequence with \( 18k(\ell^2 + 1) \geq t_i - t_{i-1} \geq 9k(\ell^2 + 1) \). We can choose
\[ t_{i+1} \in [t_i + 9k(\ell^2 + 1), t_i + 18k(\ell^2 + 1)] \]
such that by pigeonhole principle
\[ f(t_{i+1}) \leq k^{-1}(9\ell^2 + 9)^{-1} \int_{t_i+(18k\ell^2+18)}^{t_i+(18k\ell^2+18)} f(\tau) d\tau. \]

Assumptions of the Lemma then imply that
\[ h(t_i) + \int_{t_i}^{t_{i+1}} f(\tau) d\tau \leq kf(t_i) + \frac{1}{9}h(t_{i-1}) + \frac{1}{9} \int_{t_{i-1}}^{t_i} f(\tau) d\tau, \]
\[
\int_{t_i+(9\ell^2+9)}^{t_i+(18\ell^2+18)} f(\tau) d\tau \leq kf(t_i) + \frac{1}{9} h(t_{i-1}) + \frac{1}{9} \int_{t_{i-1}}^{t_i} f(\tau) d\tau.
\]

Therefore,
\[
f(t_{i+1}) \leq (9\ell^2 + 9)^{-1} \left( f(t_i) + \frac{1}{9k} h(t_{i-1}) + \frac{1}{9k} \int_{t_{i-1}}^{t_i} f(\tau) d\tau \right).
\]

Thus if
\[
f(t_i) \leq \bar{C} 2^{-i-1} \tag{76}
\]
\[h(t_{i-1}) + \int_{t_{i-1}}^{t_i} f(\tau) d\tau \leq \bar{C} k 2^{-i+1} \tag{77}
\]
then
\[
f(t_{i+1}) \leq \bar{C} (2^{-i-2} + 3^{-4} 2^{-i+2}) \leq \bar{C} 2^{-i-1}
\]
\[h(t_i) + \int_{t_i}^{t_{i+1}} f(\tau) d\tau \leq \bar{C} (k 2^{-i-1} + 3^{-2} k 2^{-i+1}) \leq \bar{C} k 2^{-i}.
\]

Now we have that \(76\), \(77\) indeed hold for \(i = 1\), where \(t_0 = 0\), and \(t_1\) is defined so as to satisfy \(75\), with \(\bar{C} = \max\{h(0), k_0\}\). This proves that for the sequence \(t_i\), defined above,
\[h(t_i) + \int_{t_i}^{t_{i+1}} f(\tau) d\tau \leq \bar{C} k 2^{-i} \leq \bar{C} k e^{-ct_i/k\ell^2}.
\]

Using the assumptions of the Lemma again we immediately obtain the desired statement.

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