Born’s group and Generalized isometries

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Abstract

We define the Born group as the group of transformations that leave invariant the line element of Minkowski’s spacetime written in terms of Fermi coordinates of a Born congruence. This group depends on three arbitrary functions of a single argument. We construct implicitly the finite transformations of this group and explicitly the corresponding infinitesimal ones. Our analysis of this group brings out the new concept of Generalized group of isometries. The limiting cases of such groups being, at one end, the Groups of isometries of a spacetime metric and, at the other end, the Group of diffeomorphisms of any spacetime manifold. We mention two examples of potentially interesting generalizations of the Born congruences.

1 Fermi, Born and Killing congruences

Fermi-Walker transport, ([6], [7], [8]). Let $L$ be a worldline and $\pi^\alpha$ be its tangent unit vector field. Let $\underline{\pi}^\alpha$ be a vector field defined along $L$. By definition this vector field is Fermi-Walker transported if it is a solution of the differential equation:

$$\frac{D\underline{\pi}^\alpha}{d\tau} \equiv \frac{\nabla \underline{\pi}^\alpha}{d\tau} + \underline{\pi}^\alpha \frac{\nabla \underline{\pi}^\eta}{d\tau} + \frac{1}{2} \underline{\Omega}^\alpha_{\rho}\underline{\pi}^\rho = 0,$$

(1)

where $\underline{\Omega}^\alpha_{\rho\eta}$ is a skewsymmetric 2-rank tensor orthogonal to $\underline{\pi}^\alpha$:

$$\underline{\Pi}^\alpha_{\rho\eta} = - \underline{\Omega}^\alpha_{\rho\eta}, \quad \underline{\Pi}^\alpha_{\rho\eta} \underline{\pi}^\eta = 0$$

defined along $L$, and where:

$$\frac{\nabla \underline{\pi}^\alpha}{d\tau} \equiv \frac{d\underline{\pi}^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} \underline{\pi}^\beta \underline{\pi}^\gamma, \quad \underline{\pi}^\alpha \equiv \frac{\nabla \underline{\pi}^\alpha}{d\tau} \equiv \frac{d\underline{\pi}^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} \underline{\pi}^\beta \underline{\pi}^\gamma,$$

$\tau$ being the proper time measured along $L$ with arbitrary origin, and $\Gamma^\alpha_{\beta\gamma}$ being the restriction of the Christoffel symbols to $L$.

It follows from this definition that the tangent unit vector field $\pi^\alpha$ is a solution of Eq. (1). Therefore it can be thought of as being Fermi-Walker propagated along $L$. Another important property is that the scalar product of any two Fermi-Walker propagated vector fields is constant along $L$.

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Fermi coordinates, [6,12,13]. Let us consider a worldline \( L \) and let \( \mathbf{e}_0 \) be three vectors defined at some event \( O \) on \( L \) such that:

\[
\mathbf{e}_0^i \mathbf{e}_0^j = \delta_{ij}, \quad \mathbf{e}_0^i \mathbf{e}_\alpha \mathbf{e}_\alpha = 0,
\]

\( \mathbf{e}_0^i \) being the unit vector tangent to \( L \) at \( O \). Fermi-Walker propagating each of these three spacelike vectors along \( L \) we obtain at each event of \( L \) an orthonormal frame of reference adapted to \( L \):

\[
\mathbf{e}_i^i \mathbf{e}_j^j = \delta_{ij}, \quad \mathbf{u}_\alpha \mathbf{e}_\alpha = 0.
\]

We shall say that \( T \) is the Normal tube of \( L \) if for each event \( E \) of \( T \) there exists one and only one spacelike geodesic passing through \( E \) and being orthogonal to \( L \). We shall note this geodesic \( G(E_0, E) \), \( E_0 \) being its intersection with \( L \).

By definition the Fermi coordinates with Baseline \( L \) of any event \( E \) of \( T \) are:

\[
z^0 = \tau, \quad z^i = n'^i \mathbf{r},
\]

where \( \tau \) is the proper-time interval measured along \( L \) from \( O \) to \( E_0 \), \( n^\alpha \) is the unit tangent vector to \( G(E_0, E) \) at \( E_0 \), and \( r \) is the geodesic distance from \( E_0 \) to \( E \).

Fermi congruences. Given a worldline \( L \), a Fermi congruence \( C(L) \) with Baseline \( L \) is the congruence with parametric equations:

\[
z^0 = \tau, \quad z^i = c^i
\]

\( z^\alpha \) being a system of Fermi coordinates with Baseline \( L \) and \( c^i \) three arbitrary constants.

Equivalently a Fermi congruence \( C(L) \) with Baseline \( L \) can be defined as a congruence such that the orthogonal geodesic distance \( G(E_0, E) \) remains constant when the event \( E \) is displaced along the worldline of the congruence passing through \( E \).

Let \( C \) be a timelike congruence defined on some world tube \( T \), and \( u^\alpha \) be its unit tangent vector. There are three basic geometrical objects that can be associated to it besides its Projector:

\[
\hat{g}_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta.
\]

The Curvature:

\[
b^\alpha = u^\rho \nabla^\rho u^\alpha, \quad u^\alpha b_\alpha = 0.
\]

The Deformation rate of the congruence, which is the symmetric 2-rank tensor orthogonal to \( u^\alpha \):

\[
\Sigma_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta + \hat{\nabla}_\beta u_\alpha, \quad \Sigma_{\alpha\beta} u^\alpha = 0,
\]

where:

\[
\hat{\nabla}_\alpha u_\beta \equiv \hat{g}^\gamma_{\alpha \beta} \nabla_\gamma u_\sigma.
\]

And the Rotation rate of the congruence, which is the skewsymmetric 2-rank tensor orthogonal to \( u^\alpha \):

\[
\Omega_{\alpha\beta} = \hat{\nabla}_\alpha u_\beta - \hat{\nabla}_\beta u_\alpha, \quad \Omega_{\alpha\beta} u^\alpha = 0.
\]
**Chorodesics.** We shall say that a curve $x^\alpha = x^\alpha(\lambda)$ is a chorodesic of a timelike congruence $C$ if it is a solution of the following system of differential equations:

$$\frac{dx^\alpha}{d\lambda} = p^\alpha \quad \frac{dp^\alpha}{d\lambda} + \Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu = \frac{1}{2} u^\alpha \Sigma_{\lambda\mu} p^\lambda p^\mu,$$

where $u^\alpha$ is the tangent vector to $C$ and $\Sigma_{\lambda\mu}$ is its Deformation rate. If $\Sigma_{\lambda\mu} = 0$ then the chorodesics of $C$ and the geodesics of the spacetime coincide.

**Lemma 1**: If a spacelike chorodesic is orthogonal to a worldline of a congruence $C$ then it is orthogonal to all the worldlines of the congruence $C$ that it crosses.

In fact from Eqs. (6) it follows that:

$$\frac{d}{d\lambda}(u_\alpha p^\alpha) = p^\alpha p^\beta \left( \nabla_\alpha u_\beta - \frac{1}{2} \Sigma_{\alpha\beta} \right);$$

or, using the definition (4) of $\Sigma_{\alpha\beta}$:

$$\frac{d}{d\lambda}(u_\alpha p^\alpha) = (u_\alpha p^\alpha)(b_\beta p^\beta).$$

Therefore if $u_\alpha p^\alpha$ is zero at one event it will remain zero all along the chorodesic passing through this event and having $p^\alpha$ as tangent.

**Lemma 2**: If $C$ is a Fermi congruence with Baseline $L$ then the restriction of the Deformation rate tensor on $L$ is zero: $\Sigma_{\alpha\beta} = 0$.

We shall say that $C$ is an *Homogeneous Fermi congruence* on $T$ if each worldline of $C$ can be viewed as a Baseline of $C$.

**Born congruences, (3,10,13,15)**. Born congruences were initially defined as those congruences for which the infinitesimal orthogonal distance between neighboring worldlines was constant along the worldlines. It follows from this definition that the deformation rate tensor of a Born Congruence is zero:

$$\Sigma_{\alpha\beta} = 0. \quad (7)$$

**Lemma 3**: If a spacelike geodesic is orthogonal to a worldline of a Born congruence then it is orthogonal to all the worldlines of the congruence that it crosses. Reciprocally, if $C$ is a congruence such that if a geodesic is orthogonal to a worldline of $C$ then it is orthogonal to all the worldlines of $C$ that it crosses, then $C$ is a Born congruence.

The first part of this Lemma is a corollary of lemma 1.

**Theorem 1**: If $L_0$ and $L_1$ are two worldlines of a Born congruence then the length of any geodesic orthogonal arc $G(E_0, E_1)$ intercepted by these two worldlines remains constant when $E_0$ and $E_1$ move accordingly along the two worldlines.

This follows immediately from lemma 1.

**Theorem 2**: Every Homogeneous Fermi congruence is a Born congruence, and reciprocally every Born congruence is an Homogeneous Fermi congruence.

The first part of this Theorem follows immediately from the Lemma 1. The second part can be proved as follows: let $L_0$ and $L_1$ be two worldlines of a Born congruence. From the Lemma 1 it follows that the family of geodesics which intersect both worldlines and are orthogonal to $L_0$ are also orthogonal to $L_1$. 

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The length of any geodesic arc \( G(E_0, E_1) \) intercepted by these two worldlines will be given by the integral:

\[
L = \int_{E_0}^{E_1} \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta},
\]
calculated along \( G(E_0, E_1) \). The variation of this integral when \( E_0 \) and \( E_1 \) move along \( L_0 \) and \( L_1 \) is:

\[
\delta L = g_{\alpha\beta}(x\rho_1) \delta x^\alpha n_\alpha - g_{\alpha\beta}(x\rho_0) \delta x^\alpha n_\alpha
\]

\( n_\alpha \) and \( n_\alpha \) being the unit tangent vectors of \( G(E_0, E_1) \) at the events \( E_0 \) and \( E_1 \). Since at both ends \( \delta x^\alpha \) and \( n_\alpha \) are orthogonal, it follows that \( \delta L = 0 \) which proves that the geodesic arc length of \( G(E_0, E_1) \) remains constant. Since \( L_0 \) and \( L_1 \) are arbitrary it follows that the congruence is an homogeneous Fermi congruence.

**Killing congruences.** (2) A Killing vector field \( \xi^\alpha \) is a generator of an infinitesimal symmetry of spacetime. It satisfies the Killing equations:

\[
\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0
\]

A Killing congruence \( K \) is a congruence tangent to a Killing vector field. If \( K \) is timelike and \( u^\alpha \) is the corresponding unit tangent vector field to \( K \) then it can be characterized by the following equations:

\[
\Sigma_{\alpha\beta} = 0, \quad \partial_\alpha b_\beta - \partial_\beta b_\alpha = 0
\]

It follows from this last characterization that any Killing congruence is a Born congruence. We shall call Pure Born those Born congruences which are not Killing congruences, i.e., those for which \( b_\alpha \) is not a gradient.

**Theorem 3.** (4,5): In Minkowski spacetime, \( M_4 \), any Born congruence \( C \) is either a Killing congruence or an irrotational congruence. More precisely: if it is a Killing congruence then the tangent unit vector field is colinear to one of the generators of a Poincaré transformation. If it is a Pure Born congruence then it is the irrotational Fermi congruence with Baseline any worldline of the congruence.

From the last part of this theorem it follows that if \( x^\alpha \) is a Galilean system of coordinates of \( M_4 \):

\[
ds^2 = -dt^2 + \delta_{ij} dx^i dx^j,
\]
and \( \bar{y}^\alpha(\tau) \) are the parametric equations of a timelike worldline \( L \) then, using Fermi coordinates \((\tau, z^i)\) with Baseline \( L \), the parametric equations of the unique Pure Born congruence containing \( L \) are:

\[
x^\alpha = \bar{e}_i^\alpha(\tau) z^i + \bar{y}^\alpha(\tau),
\]
where \( \bar{e}_i^\alpha(\tau) \) is a system of orthonormal vectors Fermi-Walker propagated along \( L \).

Differentiating these equations and substituting in (8) the line element of \( M_4 \) becomes:

\[
ds^2 = -[1 + a_i(\tau) z^i]^2 d\tau^2 + \delta_{ij} dz^i dz^j,
\]
where \( a_i = \bar{e}_i^\alpha b_\alpha \).
2 Poincaré and Born’s group

Let us consider Minkowski’s spacetime $M_4$, a system of Galilean coordinates, and the Poincaré group:

$$x'^\alpha = L^\alpha_\beta (x^\beta, \Lambda_I) \equiv L^\alpha_\beta (x^\beta - \Lambda^\beta)$$

where $L^\alpha_\beta$ are Lorentz matrices. The line element (8) is invariant under this group which means that in terms of the new Galilean coordinates $x'^\alpha$ the same line element becomes:

$$ds^2 = -dt'^2 + \delta_{ij} dx'^i dx'^j.$$  \hspace{1cm} (11)

Let us assume that we perform the coordinate transformation (9), which we write for short as:

$$z'^\alpha = F^\alpha (x^\rho), \quad x^\rho = \bar{F}^\rho (z'^\alpha),$$

leading to Minkowski’s line element (10), and let us consider the family of composite transformations:

$$z'^\alpha = F^\alpha \{ L^\beta [\bar{F}^\gamma (z'^\delta)] \}$$

By construction this family of transformations is still a group. More precisely it is the realization of the Poincaré group which leaves invariant the line element of Minkowski spacetime written as in (10). This meaning that in terms of the new coordinates we shall have:

$$ds^2 = -[1 + a^k_\tau (\tau')]^{2} d\tau'^2 + \delta_{ij} dz'^i dz'^j.$$ \hspace{1cm} (12)

Notice that the functions $a^k_\tau$ are the same in expressions (10) and (12).

We define the Fermi realization of the Born group as the family of transformations:

$$z'^\alpha = B'^\alpha (z^\rho), \quad z^\rho = B^\rho (z'^\alpha)$$

which leave invariant the form of the line element (10):

$$ds^2 = -[1 + a'_k (\tau')]^{2} d\tau'^2 + \delta_{ij} dz'^i dz'^j$$

without necessarily leaving invariant the functions $a^k$. That is to say: in general $a'_k \neq a_k$.

By construction the Born group contains as subgroup the Poincaré group. It contains also an isotropy subgroup which leaves invariant the Born congruence without leaving invariant the Baseline on which the Fermi coordinates are based. This subgroup which depends on three parameters $\lambda^i$ is:

$$z'^i = z^i - \lambda^i \quad \tau' = \tau + \nu_k (\tau) \lambda^k$$

where:

$$\nu_k (\tau) = \int_{0}^{\tau} a_k (u) du$$

Under a transformation of this subgroup the functions $a_k$ become:

$$a'_k (u, \lambda^i) = \frac{a_k (u)}{1 + a_j (u) \lambda^j}$$
This subgroup is abelian and translates the Baseline from a worldline of the congruence to another. It does not have to be confused with the translation subgroup of the Poincaré subgroup which leaves invariant the functions \( a_k \).

**Infinitesimal transformations.** Let \( L \) and \( L' \) be two neighbouring worldlines in Minkowski’s spacetime

\[
\vec{y}^\alpha(\tau') = \vec{y}^\alpha(\tau') + \delta\vec{y}^\alpha(\tau')
\]

Let us assume that the origins of the proper times are close to each other and that at these origins two adapted orthonormal frames of reference have been defined and are such that:

\[
e_i^\alpha (0) = e_i^\alpha (0) + \delta \Omega_j^i (0) e_j^\alpha (0)
\]

Let us finally assume that the curvatures of both worldlines do not differ much when considered at the same value of their arguments:

\[
a_i' = a_i + \zeta_i
\]

If \((\tau, z^i)\) and \((\tau', z'^i)\) are the corresponding Fermi coordinates with Baselines \( L \) and \( L' \) then we have:

\[
\tau' = \tau + \xi^0 (z^\alpha), \quad z'^i = z^i + \xi^i (z^\alpha)
\]

where:

\[
\xi^0 = [1 + a_i z^i]^{-1} (\delta \vec{y}^0 + z^i \delta \Omega_i^i), \quad \xi^i = -(\delta \vec{y}^i + z^i \delta \Omega_i^j),
\]

These transformations are the infinitesimal Born transformations corresponding to its Fermi realization, \( \xi^\alpha \) being the components of the appropriate generalization of a Killing vector.

**Generalized Killing equations.** A straightforward calculation shows that the generalized Killing vector satisfies the following generalized Killing equations:

\[
\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \frac{\partial g_{\alpha \beta}}{\partial a_i} \zeta_i = 0
\]

The Born group defined in this section has a very precise physical meaning, since it is in fact the special relativistic generalization of the Irrotational rigid motion group of classical mechanics:

\[
t' = t \quad z'^i = z^i + \xi^i (t)
\]

to which it contracts when the speed of light in vacuum is assumed to be infinity.

3 **Generalized Isometries**

Let \( g_{\alpha \beta} (x^\rho) \) be the components of a spacetime metric and let us assume that we can find some functions \( \bar{g}_{\alpha \beta} (x^\rho, \lambda_a) \), and some functions \( f_a (x^\rho) \) belonging to some set \( S \) such that we have:
\[ g_{\alpha\beta}(x^\rho) = \bar{g}_{\alpha\beta}(x^\rho, f_a(x^\sigma)) \]  
(13)

We shall then say that the \( \bar{g}_{\alpha\beta} \) are a Compound description of the \( g_{\alpha\beta} \) with parameter set \( S \). Any metric possesses two limiting Compound descriptions: The Elementary description for which the set \( S \) is the empty set and therefore \( \bar{g}_{\alpha\beta}(x^\rho) = g_{\alpha\beta}(x^\rho) \). And the Trivial description for which \( \bar{g}_{\alpha\beta} = g_{\alpha\beta}(f_a) \) with \( a = 1 \cdots 10 \), and \( f_a \) being functions of the four coordinates.

**Definition:** If:

\[ x'^\alpha = \varphi'^\alpha(x'^\rho), \quad x^\alpha = \varphi^\alpha(x'^\rho) \]  
(14)

is a transformation of the spacetime and there exist functions \( f_a \) such that:

\[ g'_{\alpha\beta}(x'^\gamma) = g_{\alpha\beta}[x'^\gamma, f'_a(x'^\gamma)] \]

where:

\[ g'_{\alpha\beta}(x'^\gamma) = \partial\varphi^\mu/\partial x'^\alpha \partial\varphi^\nu/\partial x'^\beta [\varphi^\delta(x'^\gamma)] g_{\mu\nu}[\varphi^\delta(x'^\gamma), f_a(\varphi^\rho(x'^\gamma))] \]

then we shall say that the transformations (14) define a Generalized isometry of the spacetime metric \( g_{\alpha\beta} \).

**Generalized Killing equations.** From this definition it follows that if:

\[ x'^\rho = x^\rho + \xi^\rho(x^\alpha), \quad f'_a = f_a + \zeta_a(x^\alpha) \]

are the transformations of an infinitesimal generalized isometry, then the \( \xi^\rho \) satisfy a system of differential equations:

\[ \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + \partial g_{\alpha\beta}/\partial f_a \zeta_a = 0 \]

which are a generalization of the Killing equations.

Ordinary isometries of a spacetime, when they exist, can be considered as a limiting case of a generalized isometry. They can be indeed considered as isometries of the Elementary description. The opposite limiting case is the group of diffeomorphisms of a 4-dimensional spacetime, since it can be considered as a Generalized isometry of the Trivial representation of any metric.

The Born group that we considered in Section 2 is an intermediate generalized isometry group. The parameter set being in this case the triads of functions of one single argument.

4 Generalizations of the Born congruences

The most obvious generalization of the Born group as it was defined in Section 2 is to consider any spacetime \( V_4 \) for which the Born equations have at least one solution and define the Born group \( B(V_4) \) as the group which leaves invariant the metric of \( V_4 \) written in terms of the Fermi coordinates with Baseline any worldline of any of the Born congruences.

We mention below two potentially interesting generalizations of the Born congruences of any spacetime. Each of these generalizations would lead to a corresponding generalization of the Born group in the form of a Generalized isometry of the spacetime on which they are considered. The parameter set of
the corresponding Compound description of the metric depending of course on
the number and type of the arbitrary functions contained in the general solution
of the conditions defining these generalizations.

The first example are Cattaneo’s affinities, which are particularly interesting
from the geometrical point of view. They are defined as those congruences for
which one has:

\[ \hat{\nabla}_\gamma \Sigma_{\alpha\beta} + b_\gamma \Sigma_{\alpha\beta} = 0, \quad \hat{\nabla}_\gamma \hat{\Sigma}_{\alpha\beta} \equiv \hat{g}^\lambda_\gamma \hat{g}^{\alpha\beta} \nabla_\lambda \Sigma_{\rho\sigma}. \]

As it is well known the Born conditions (7) express that the metric of the space-
time \( g_{\alpha\beta} \) can be projected onto a metric \( \hat{g}_{ij} \) of the quotient manifold \( V_3 = V_4 / \mathcal{R}, \)
\( \mathcal{R} \) being the equivalence relation defined by the corresponding Born congruence.
The equations above express a more general condition. Namely that the linear
connection \( \Gamma^\alpha_{\beta\gamma} \) can be projected onto a linear connection of \( V_3. \) This connection
is the restriction to these particular congruences of a more general object first
defined by Cattaneo[9].

The second example is in our opinion interesting both from the geometrical
and the physical point of view for reasons which will be discussed elsewhere.
We shall say that a timelike congruence is quo-harmonic if the tangent vector
field \( u^\alpha \) is such that there exist three independent functions \( f_i \)
solutions of the following equations:

\[ u^\rho \partial_\rho f^i = 0, \quad i = 1, 2, 3 \]

\[ \hat{\Delta} f^i - (3 - \text{dim}) b^\rho \partial_\rho f^i = 0 \iff \Delta f^i - (4 - \text{dim}) b^\rho \partial_\rho f^i = 0, \]

where:

\[ \Delta \equiv g^{\alpha\beta} \nabla_\alpha \partial_\beta, \quad \hat{\Delta} \equiv \hat{g}^{\alpha\beta} \nabla_\alpha \partial_\beta \]

and where \( \text{dim} = 3. \) We call these congruences quo-harmonic to distinguish
them from the Harmonic congruences which are defined as those congruences
for which there exist three independent functions \( f^i \) solutions of equations (15)
and (16) with \( \text{dim} = 4. \) Harmonic congruences were extensively discussed in
ref.[15] where it was proved in particular that irrotational Born congruences are
never harmonic. This is the reason why harmonic congruences do not qualify
as appropriate generalizations of the Born congruences. On the contrary, every
Born congruence is quo-harmonic. Indeed, using any system of adapted coordi-
nates, i.e. any system of coordinates for which \( u^\alpha = 0, \) eqs. (15) and (16)
become:

\[ \partial_b f^i = 0, \quad \hat{\Delta} f^i = 0, \]

where \( \hat{\Delta} \) is here the Laplacian of the quotient metric \( \hat{g}_{ij}(x^k). \) The above equa-
tions always have local triads of independent solutions and this proves our as-
sertion.

References

[1] Ll. Bel in Relativity in General, Eds. J. Diaz, M. Lorente, Editions
Frontières, 47 (1994)
[2] Killing, W., Journal für die reine und angew. Math.(Crelle), 109, pp. 121-186 (1892)

[3] Born, M., Ann. Physik (4), 30, p. 1 (1909)

[4] Herglotz, G., Ann. Physik (4), 31, p. 393 (1910)

[5] Noether, F., Ann. Physik (4), 31, p. 919 (1910)

[6] Fermi, E., R.C. Acc. Naz. Lincei, 31, 21-23, 51-52 (1922)

[7] Walker, A.G., Proc. Roy. Soc. Edinb., 52, p. 345 (1932)

[8] Mast, C.B. and Strathdee, J., Proc. Royal Soc., A, 252, pp. 476-487 (1959)

[9] Cattaneo, C., Ann. di Mat. pura ed appl., S, IV, T. XLVIII, p. 361 (1959)

[10] Synge J.L., Relativity: The General Theory, North Holland (1960). And references therein.

[11] Møller C., The theory of Relativity, Oxford University Press (1962)

[12] Manasse, F.K., and Misner, C.W., J. Math. Phys., 4, p. 735 (1963)

[13] Synge J.L., Relativity: The Special Theory, North Holland (1965). And references therein.

[14] Ni, W.T. and Zimmermann, M., Phys. Rev. D, 17, p. 1473 (1978)

[15] Bel, Ll. and Coll, B., Gen. Rel. Grav., 26, 6, p. 613 (1993)