The Elliptic Algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$ and the Deformation of $W_N$ Algebra

Takeo KOJIMA * and Hitoshi KONNO **,†

* Department of Mathematics, College of Science and Technology, Nihon University, Chiyoda-ku, Tokyo 101-0062, Japan.
E-mail:kojima@math.cst.nihon-u.ac.jp

** Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739-8521, Japan.
E-mail:konno@mis.hiroshima-u.ac.jp

† Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK.

Abstract

After reviewing the recent results on the Drinfeld realization of the face type elliptic quantum group $B_{q,\lambda}(\hat{\mathfrak{sl}}_N)$ by the elliptic algebra $U_{q,p}(\hat{\mathfrak{sl}}_N)$, we investigate a fusion of the vertex operators of $U_{q,p}(\hat{\mathfrak{sl}}_N)$. The basic generating functions $\Lambda_j(z)$ ($1 \leq j \leq N - 1$) of the deformed $W_N$ algebra are derived explicitly.
1 Introduction

In recent papers [1, 2, 3], we showed that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ provides the Drinfeld realization of the face type elliptic quantum group $B_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ [4] tensored by a Heisenberg algebra. Based on this fact, we defined the $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ counterparts of the intertwining operators of the $B_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ modules and obtained their free field realization in the level one representation. The resultant vertex operators, called the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$, are identified with the vertex operators of the $\widehat{\mathfrak{sl}}_N$ type RSOS model in the algebraic analysis formulation [5]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ with $\mathfrak{g}$ being an affine Lie algebra provides the Drinfeld realization for the elliptic quantum group $B_{q,\lambda}(\mathfrak{g})$ and enables us to perform an algebraic analysis of the $\mathfrak{g}$ type RSOS model.

On the other hand, the $\widehat{\mathfrak{sl}}_N$ RSOS model is known as an off-critical deformation of the $W_N$ minimal model [6]. In this relation, it is remarkable that the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ in the $c = 1$ representation coincides with the algebra of the screening currents of the deformed $W_N$ algebra [7, 8, 9]. In general, we expect that the elliptic algebra $U_{q,p}(\mathfrak{g})$ provides an algebra of screening currents of the deformation of the coset CFT associated with $(\mathfrak{g})_c \oplus (\mathfrak{g})_{r-c-2}/(\mathfrak{g})_{r-2}[1, 2]$, which corresponds to the $c \times c$ fusion RSOS model of type $\mathfrak{g}$.

The purpose of this paper is to continue to discuss an explicit relation among the elliptic algebra $U_{q,p}(\mathfrak{g})$, the $\mathfrak{g}$ type RSOS model and the deformation of $W(\hat{\mathfrak{g}})$ algebra in the case $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$. We here investigate a fusion of the type II vertex operator of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and its dual, and show that the generating functions of the deformed $W_N$ algebra can be extracted from it. The idea of fusion of the vertex operators was used in [10, 11] to derive the generating function of the deformed Virasoro algebra (corresponding to the $\phi_{1,3}$ perturbation) from the ABF model in regime III, in [12] for the deformed $W_N$ algebra with the central charge $c_N = (N - 1) \left(1 - N(N+1) \frac{N_1(N+1) + N(N+1)}{r(r-1)}\right)$ at special point $r = N + 2$ (the $\mathbb{Z}_N$ parafermion point) from the ABF model in regime II, and in [13] for the deformed Virasoro algebra (corresponding to the $\phi_{1,2}$ perturbation) from the dilute $A_L$ model.

This article is organised as follows. In the next section, we briefly review the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ as the Drinfeld realization of the elliptic quantum group $B_{q,\lambda}(\widehat{\mathfrak{sl}}_N)$ according to [3]. In section 3, we give a summary of the results on the free field realization of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$. In section 4, we discuss a fusion of the vertex operators of $U_{q,p}(\widehat{\mathfrak{sl}}_N)$ and derive the basic generators of the deformed $W_N$ algebra.

Through this paper, we use the following symbols. $p = q^{2r}$, $p^* = pq^{-2c} = q^{2r^*}$ (with $r^* = 2$).
We also use the Jacobi theta functions
\[ \theta_p(z) = (z, p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty, \]
\[ \{z\} = (z; p, q^2)_\infty, \quad \{z\}^* = \{z\}_{p\to p^*}, \]
\[ (z; t_1, \cdots, t_k)_\infty = \prod_{n_1, \cdots, n_k \geq 0} (1 - z t_1^{n_1} \cdots t_k^{n_k}). \]

We take the normalization of the theta function to be
\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \]
\[ \Theta_p(z) = (z, p)_\infty (pz^{-1}; p)_\infty (p; p)_\infty, \]
\[ \{z\} = (z; p, q^2)_\infty, \quad \{z\}^* = \{z\}_{p\to p^*}, \]
\[ (z; t_1, \cdots, t_k)_\infty = \prod_{n_1, \cdots, n_k \geq 0} (1 - z t_1^{n_1} \cdots t_k^{n_k}). \]

We also use the Jacobi theta functions
\[ [v] = q^{\frac{v^2}{2} - v} \Theta_p(q^{2v}) (p; p)_\infty^3, \quad [v]^* = q^{\frac{v^2}{2} - v} \Theta_p(q^{2v^*}) (p^*; p^*_\infty)^3, \]
which satisfy \([-v] = [-v]\) and the quasi-periodicity property
\[ [v + r] = [v], \quad [v + r\tau] = e^{-\pi i \tau} \frac{2\pi i v}{[v]^*} [v]. \]

We take the normalization of the theta function to be
\[ \oint_{C_0} \frac{dz}{2\pi iz} [v] = 1, \quad \oint_{C_0} \frac{dz}{2\pi iz} [v]^* = \frac{[v]}{[v]^*} \bigg|_{v \to 0}, \]
where \(C_0\) is a simple closed curve in the \(v\)-plane encircling \(v = 0\) anticlockwise.

2 The Elliptic Algebra \(U_{q,p}(\hat{sl}_N)\)

2.1 Definition

**Definition 2.1 (Elliptic algebra \(U_{q,p}(\hat{sl}_N)\))** We define the elliptic algebra \(U_{q,p}(\hat{sl}_N)\) to be the associative algebra of the currents \(E_j(v), F_j(v) (1 \leq j \leq N - 1)\) and \(K_j(v) (1 \leq j \leq N)\) satisfying the following relations.

\[
E_i(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]^*}{[v_1 - v_2 - \frac{A_{ij}}{2}]^*} E_j(v_2) E_i(v_1), \tag{2.1}
\]
\[
F_i(v_1)F_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]}{[v_1 - v_2 + \frac{A_{ij}}{2}]} F_j(v_2) F_i(v_1), \tag{2.2}
\]
\[
[E_i(v_1), F_j(v_2)] = \frac{\delta_{ij}}{q - q^{-1}} \left( \delta(q^{-c} z_1/z_2) H^+_j \left( v_2 + \frac{c}{4} \right) - \delta(q^c z_1/z_2) H^-_j \left( v_2 - \frac{c}{4} \right) \right), \tag{2.3}
\]
\[
H^+_j \left( v + \frac{1}{2} \left( r - \frac{c}{2} \right) \right) = \kappa K_j \left( v + \frac{N - j}{2} \right) K_{j+1} \left( v + \frac{N - j}{2} \right)^{-1}, \tag{2.4}
\]
\[ K_j(v_1)K_j(v_2) = \rho(v_1 - v_2)K_j(v_2)K_j(v_1), \quad (2.5) \]
\[ K_{j_1}(v_1)K_{j_2}(v_2) = \rho(v_1 - v_2) \frac{[v_1 - v_2 - 1][v_1 - v_2]}{[v_1 - v_2 - 1][v_1 - v_2 + 1]} K_{j_2}(v_2)K_{j_1}(v_1) \]
\[ (1 \leq j_1 < j_2 \leq N), \quad (2.6) \]
\[ K_j(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2}]}{[v_1 - v_2 + \frac{j+r-N}{2} - 1]} E_j(v_2)K_j(v_1), \quad (2.7) \]
\[ K_{j+1}(v_1)E_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} + 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]} E_j(v_2)K_{j+1}(v_1), \quad (2.8) \]
\[ K_{j_1}(v_1)E_{j_2}(v_2) = E_{j_2}(v_2)K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1), \quad (2.9) \]
\[ K_j(v_1)F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} - 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]} F_j(v_2)K_j(v_1), \quad (2.10) \]
\[ K_{j+1}(v_1)F_j(v_2) = \frac{[v_1 - v_2 + \frac{j+r-N}{2} + 1]}{[v_1 - v_2 + \frac{j+r-N}{2}]} F_j(v_2)K_{j+1}(v_1), \quad (2.11) \]
\[ K_{j_1}(v_1)E_{j_2}(v_2) = F_{j_2}(v_2)K_{j_1}(v_1) \quad (j_1 \neq j_2, j_2 + 1), \quad (2.12) \]
\[ z_1 \frac{1}{(p^*q^2z_1/z; p^*)} \left\{ (z_2/z) \frac{1}{(p^*q^{-1}z_2/z; p^*)} E_i(v_1)E_i(v_2)E_j(v) + (z_1 \leftrightarrow z_2) = 0 \right\} \]
\[ \quad - [2q](p^*q^{-1}z_2/z; p^*) E_i(v_1)E_j(v)E_i(v_2) \]
\[ + (z_1/z_2) \frac{1}{(p^*q^{-1}z_1/z; p^*)} E_j(v)E_i(v_1)E_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0 \quad (|i - j| = 1). \quad (2.13) \]
\[ \frac{1}{(pq^{-2}z_2/z; p)} \left\{ (z_2/z) \frac{1}{(pq^{-1}z_2/z; p)} F_i(v_1)F_i(v_2)F_j(v) \right\} - \frac{1}{(pq^{-2}z_2/z; p)} F_i(v_1)F_j(v)F_i(v_2) \]
\[ + (z_2/z_1) \frac{1}{(pq^{-1}z_1/z; p)} F_j(v)F_i(v_1)F_i(v_2) \right\} + (z_1 \leftrightarrow z_2) = 0 \quad (|i - j| = 1). \quad (2.14) \]

Here \( A = (A_{jk}) \) is the Cartan matrix of \( \mathfrak{sl}_N \). The constant \( \kappa \) and the functions \( \rho(v) \) are given by
\[ \kappa = \frac{(p^*; p^*)}{(p^*q^2; p^*)}, \quad (2.15) \]
\[ \rho(v) = \frac{\rho^+(v)}{\rho^+(v)}, \quad (2.16) \]
\[ \rho^+(v) = q^N z^{-N} \frac{\{pq^2z\} \{pq^{2N-2}z\} \{1/z\} \{q^{2N}/z\}}{\{pz\} \{pq^{2N}z\} \{q^2/z\} \{q^{2N-2}/z\}}, \quad \rho^+(v) = \rho^+(v)|_{r \rightarrow r^*}. \quad (2.17) \]
2.2 Realization of $U_{q,p}(\mathfrak{sl}_N)$

The elliptic algebra $U_{q,p}(\mathfrak{sl}_N)$ can be realized by using the Drinfeld generators of $U_q(\mathfrak{sl}_N)$ and a Heisenberg algebra. Let $h_i$, $a_i^\dagger$, $x_{i,n}^\pm$ ($1 \leq i \leq N - 1 : m \in \mathbb{Z}_{\neq 0}, n \in \mathbb{Z}$), $c$, $d$ be the standard Drinfeld generators of $U_q(\mathfrak{sl}_N)[14, 3]$. Their generating functions $x_i^\pm(z)$, $\psi_i(z)$, $\varphi_i(z)$ are called the Drinfeld currents.

$$x_i^\pm(z) = \sum_{n \in \mathbb{Z}} x_{i,n}^\pm z^{-n}, \quad (2.18)$$

$$\psi_i(q^{\frac{r}{2}}z) = q^{h_i} \exp \left( (q - q^{-1}) \sum_{m > 0} a_{i,m} z^{-m} \right), \quad (2.19)$$

$$\varphi_i(q^{-\frac{r}{2}}z) = q^{-h_i} \exp \left( -(q - q^{-1}) \sum_{m > 0} a_{i,-m} z^m \right) \quad (1 \leq i \leq N - 1). \quad (2.20)$$

**Definition 2.1** We define “dressed” currents $e_i(z,p)$, $f_i(z,p)$, $\psi_i^\pm(z,p)$, ($1 \leq i \leq N - 1$) by

$$e_i(z,p) = u_i^+(z,p)x_i^+(z), \quad (2.21)$$

$$f_i(z,p) = x_i^-(z)u_i^-(z,p), \quad (2.22)$$

$$\psi_i^+(z,p) = u_i^+(q^{\frac{r}{2}}z,p)\psi_i(z)u_i^-(q^{-\frac{r}{2}}z,p), \quad (2.23)$$

$$\psi_i^-(z,p) = u_i^+(q^{-\frac{r}{2}}z,p)\varphi_i(z)u_i^-(q^{\frac{r}{2}}z,p), \quad (2.24)$$

where

$$u_i^+(z,p) = \exp \left( \sum_{m > 0} \frac{1}{[r^m]_q} a_{i,+m} (q^r z)^m \right), \quad (2.25)$$

$$u_i^-(z,p) = \exp \left( -\sum_{m > 0} \frac{1}{[r^m]_q} a_{i,-m} (q^{-r} z)^{-m} \right). \quad (2.26)$$

Setting $b_{j,m} = \frac{[r^m]_q}{[r^m]_q} a_{j,m}$ (for $m > 0$), $q^{\pm m} a_{j,m}$ (for $m < 0$), we introduce new generators, $B_m^j (1 \leq j \leq N; m \in \mathbb{Z})$, by

$$-B_m^j + B_m^{j+1} = \frac{m}{[m]_q} b_{j,m} q^{(N-j)m}, \quad \sum_{j=1}^N q^{2jm} B_m^j = 0. \quad (2.27)$$

From this and the commutation relation of the Drinfeld generators $a_{j,m}$, we derive the following commutation relations.

$$[B_m^j, B_{m'}^k] = \frac{[r^m]_q [c m]_q}{[r^m]_q [m]_q [N m]_q} \times \begin{cases} (N-1)_m [m]_q & (j = k) \\ -q^{-m} \text{sgn}(j-k) [m]_q & (j \neq k) \end{cases}, \quad (2.28)$$
for \(m, m' \in \mathbb{Z}_{\neq 0}, \ 1 \leq j, k \leq N\). Then defining new currents \(k_j(z, p)\) (\(1 \leq j \leq N\)) by

\[
k_j(z, p) =: \exp \left( \sum_{m \neq 0} \frac{[m]_q}{m[r^*_m]_q} B^j_m z^{-m} \right),
\]

we obtain the following decomposition.

\[
\psi_j^\pm(q^{r_z}(z^z)z, p) = \kappa q^{\pm h_j} k_j(q^{-j+1}z, p)k_{j+1}(q^{-j+1}z, p)^{-1}.
\]

On the other hand, let \(\epsilon_j\) (\(1 \leq j \leq N\)) be the orthonormal basis in \(\mathbb{R}^N\) with the inner product \(\langle \epsilon_j, \epsilon_k \rangle = \delta_{j,k}\). Setting \(\bar{\epsilon}_j = \epsilon_j - \epsilon, \ \epsilon = \frac{1}{N} \sum_{j=1}^N \epsilon_j\), we have the weight lattice \(P\) of type \(A^{(1)}_{N-1}\); \(P = \oplus_{j=1}^N \mathbb{Z} \bar{\epsilon}_j\). Then, for example, the simple roots \(\alpha_j\) (\(1 \leq j \leq N - 1\)) of \(\mathfrak{sl}_N\) are given by \(\alpha_j = -\bar{\epsilon}_j + \bar{\epsilon}_{j+1}\). Let us introduce operators \(h_\alpha, \beta (\alpha, \beta \in P)\) by

\[
[h_\epsilon_j, \bar{\epsilon}_k] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \quad [h_\epsilon_j, h_\epsilon_k] = 0 = [\bar{\epsilon}_j, \bar{\epsilon}_k],
\]

\(
h_\alpha = \sum_j n_j h_\epsilon_j \quad \text{for} \quad \alpha = \sum_j n_j \bar{\epsilon}_j \quad \text{and} \quad h_0 = 0.
\)

Note that \(\langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle = \delta_{j,k} - \frac{1}{N}\) and \([h_\alpha_j, \alpha_k] = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} = \delta_{j,k}\). We hence identify \(h_\alpha_j = -h_\epsilon_j + h_{\bar{\epsilon}_{j+1}}\) with \(h_j\) in the Drinfeld generators of \(U_q(\hat{\mathfrak{sl}}_N)\).

**Definition 2.2** We define the (centrally extended) Heisenberg algebra \(\mathbb{C}\{\hat{H}\}\) as an associative algebra generated by \(P_\epsilon, Q_\epsilon\) (\(1 \leq j \leq N\)) and \(\eta_j\) (\(1 \leq j \leq N - 1\)) with the relations

\[
[P_\epsilon_j, Q_\epsilon_k] = \langle \bar{\epsilon}_j, \bar{\epsilon}_k \rangle, \quad [P_\epsilon_j, P_\epsilon_k] = 0, \quad [Q_\epsilon_j, Q_\epsilon_k] = \left( \frac{1}{r} - \frac{1}{r^*_k} \right) \text{sgn}(j - k) \log q,
\]

\[
[Q_\epsilon_j, \eta_k] = \frac{1}{r} \text{sgn}(j - k) \log q,
\]

\[
[\eta_j, \eta_k] = \frac{1}{r} \text{sgn}(j - k) \log q,
\]

\[
[P_\epsilon_j, \eta_k] = 0, \quad \sum_{j=1}^N \eta_j = 0,
\]

\[
[\eta_j, \alpha] = [P_\epsilon_j, U_q(\hat{\mathfrak{sl}}_N)] = [Q_\epsilon_j, U_q(\hat{\mathfrak{sl}}_N)] = [\eta_j, U_q(\hat{\mathfrak{sl}}_N)] = 0.
\]

**Definition 2.3** We define the currents \(E_j(v), F_j(v), H_j^\pm(v)\) (\(1 \leq j \leq N - 1\)) and \(K_j(v)\) (\(1 \leq j \leq N\)) by

\[
E_j(v) = e_j(z, p)e^{\alpha_j}e^{-Q_{\alpha_j}(q^{-j+1}z, p)}^{-\frac{r_{\alpha_j}}{r^{z_j}}},
\]

\[
F_j(v) = f_j(z, p)e^{-\alpha_j}e^{-Q_{-\alpha_j}(q^{-j+1}z, p)}^{-\frac{r_{-\alpha_j}}{r^{-z_j}}} h_j,
\]

\[
H_j^\pm(v) = \psi_j^\pm(z, p)q^{\pm h_j}e^{-Q_{\alpha_j}(q^{-j+N\pm(1-r^z)z, p})^{1/2} + \frac{1}{r\alpha_j} (P_{\alpha_j}^{-1} + \frac{1}{r\alpha_j}) h_j},
\]

\[
K_j(v) = k_j(z, p)e^{Q_{\alpha_j}(q^{1/2}z)} z^{-\frac{1}{r\alpha_j} (P_{\alpha_j}^{-1} + \frac{1}{r\alpha_j}) h_j},
\]
where \( z = q^{2v} \) and \( \tilde{\alpha}_j = -\eta_j + \eta_{j+1} \).

Then it is easy to show that \( E_j(v), F_j(v), H_j^\pm(v) \) and \( K_j(v) \) satisfy the defining relations of the elliptic algebra \( U_{q,p}(\hat{\mathfrak{sl}}_N) \).

### 2.3 RLL relation

We next discuss a relation between two elliptic algebras \( U_{q,p}(\hat{\mathfrak{sl}}_N) \) and \( B_{q,\lambda}(\hat{\mathfrak{sl}}_N) \). We construct a \( L \)-operator by using the half currents and show that it satisfies the dynamical RLL-relation which characterizes \( B_{q,\lambda}(\hat{\mathfrak{sl}}_N) \). We use the following abbreviations

\[
P_{j,l} = -P_{\xi_j} + P_{\xi_l} = P_{\alpha_j} + P_{\alpha_{j+1}} + \cdots + P_{\alpha_{l-1}} \quad (2.42)
\]

\[
h_{j,l} = -h_{\xi_j} + h_{\xi_l} = h_j + h_{j+1} + \cdots + h_{l-1} \quad (2.43)
\]

for \( j < l \). From the definition of \( C\{\hat{\mathcal{H}}\} \) and (2.38)-(2.41), we have

\[
[K_j(v), P_{k,l}] = (\delta_{j,k} - \delta_{j,l})K_j(v) = [K_j(v), P_{k,l} + h_{k,l}], \quad (2.44)
\]

\[
[E_j(v), P_{k,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})E_j(v), \quad (2.45)
\]

\[
[F_j(v), P_{j,l} + h_{j,l}] = (\delta_{j,k} + \delta_{j+1,l} - \delta_{j,l} - \delta_{j+1,k})F_j(v), \quad (2.46)
\]

\[
[F_j(v), P_{k,l}] = 0 = [E_j(v), P_{k,l} + h_{k,l}], \quad (2.47)
\]

**Definition 2.2** We define the half currents \( F_{j,l}^+(v), E_{l,j}^+(v), (1 \leq j < l \leq N) \) and \( K_j^+(v) \) \((1 \leq j \leq N)\) by

\[
K_j^+(v) = K_j \left( v + \frac{r+1}{2} \right) \quad (1 \leq j \leq N), \quad (2.48)
\]

\[
F_{j,l}^+(v) = a_{j,l} \oint_{C(j,l)} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} F_{l-1}(v_{l-1})F_{l-2}(v_{l-2}) \cdots F_j(v_j) \times [v - v_{l-1} + P_{j,l} + h_{j,l} + \frac{l-N}{2} - 1][1] \times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m + P_{j,m+1} + h_{j,m+1} - \frac{1}{2}]^*[1]}{[v_{m+1} - v_m + \frac{1}{2}]^*[P_{j,m+1} + h_{j,m+1}]}, \quad (2.49)
\]

\[
E_{l,j}^+(v) = a_{j,l}^* \oint_{C^*(j,l)} \prod_{m=j}^{l-1} \frac{dw_m}{2\pi i w_m} E_j(v_j)E_{j+1}(v_{j+1}) \cdots E_{l-1}(v_{l-1}) \times [v - v_{l-1} - P_{j,l} + \frac{l-N}{2} + \frac{\xi}{2} + 1]^*[1]^* \times \prod_{m=j}^{l-2} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m + \frac{1}{2}]^*[P_{j,m+1} - 1]^*}, \quad (2.50)
\]
Here $w_m = q^{2vm}$ and the integration contour $C(j, l)$ and $C^*(j, l)$ are given by

\[ C(j, l) : |pq^{-N}z| < |w_{l-1}| < |q^{-N}z|, \]
\[ |pqw_{m+1}| < |w_m| < |qw_{m+1}|, \quad (2.51) \]
\[ C^*(j, l) : |p^*q^{-N+c}z| < |w_{l-1}| < |q^{-N+c}z|, \]
\[ |p^*qw_{m+1}| < |w_m| < |qw_{m+1}|, \quad (2.52) \]

where $m = j, j + 1, \ldots, l - 2$. The constants $a_{j,l}$ and $a_{j,l}^*$ are chosen to satisfy

\[ \frac{\kappa a_j a_{j,l}^*[1]}{q - q^{-1}} = 1. \quad (2.53) \]

### 2.4 $L$-operator

**Definition 2.3** By using the half currents, we define the $L$-operator $\hat{L}^+(v) \in \text{End}(\mathbb{C}^N) \otimes U_{q,p}(\hat{sl}_N)$ as follows.

\[
\hat{L}^+(z) = \left( \begin{array}{cccc}
1 & F_{1,2}^+(v) & F_{1,3}^+(v) & \cdots & F_{1,N}^+(v) \\
0 & 1 & F_{2,3}^+(v) & \cdots & F_{2,N}^+(v) \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 & F_{N-1,N}^+(v) \\
0 & \cdots & \cdots & 0 & 1
\end{array} \right) \left( \begin{array}{cccc}
K_1^+(v) & 0 & \cdots & 0 \\
0 & K_2^+(v) & \vdots \\
\vdots & \vdots & \ddots & 0 \\
0 & \cdots & 0 & K_N^+(v)
\end{array} \right)
\]

Then a direct comparison with the relations of the half currents leads us to the following conjecture.

**Conjecture 2.4** The $L$-operator $\hat{L}^+(v)$ satisfies the following RLL = LLR* relation.

\[
R^{+(12)}(u_1 - u_2, P + h)\hat{L}^+(1)(z_1)\hat{L}^+(2)(z_2) = \hat{L}^+(2)(z_2)\hat{L}^+(1)(z_1)R^{+(12)}(u_1 - u_2, P). \quad (2.55)
\]

Here $z_i = q^{2u_i}$ $(i = 1, 2)$. The $R$-matrix $R^+(v, P)$ is the image of the universal $R$-matrix $R(r, \{s_j\})$ of $B_{q, P}(\hat{sl}_N)$ in the evaluation representation $(\pi_{V_+} \otimes \pi_{V_+})$, $V \cong \mathbb{C}^N$, given by

\[
R^+(v, s) = \rho^+(v)\bar{R}(v, s), \quad (2.56)
\]
\[ R(v, s) = \sum_{j=1}^{N} E_{jj} \otimes E_{jj} + \sum_{1 \leq j < l \leq N} (b(v, s_{j,l})E_{jj} \otimes E_{ll} + b(v)E_{ll} \otimes E_{jj}) \\
+ \sum_{1 \leq j < l \leq N} (c(v, s_{j,l})E_{ij} \otimes E_{lj} + c(v, s_{j,l})E_{lj} \otimes E_{ji}), \quad (2.57) \]

where \( s_{j,l} = \sum_{m=j}^{l-1} s_j \) \((1 \leq j < l \leq N)\) and

\[ b(u, s) = \frac{[s + 1][s - 1][u]}{[s]^{2}[u + 1]}, \quad \tilde{b}(u) = \frac{[u]}{[u + 1]}, \quad (2.58) \]

\[ c(u, s) = \frac{[1][s + u]}{[s][u + 1]}, \quad \tilde{c}(u, s) = \frac{[1][s - u]}{[s][u + 1]} \quad (2.59) \]

And \( R^{+*}(v, s) = R^{+}(v, s)|_{r \to r^*} \). Up to a gauge transformation, \( R^{+}(v, P) \) coincides with the Boltzmann weight of the \( \widehat{\mathfrak{sl}}_N \) RSOS model[6].

The \( c = 1 \) case, the statement was proved by using the free field realization [3].

Now let us define the modified \( L \)-operator \( L^{+}(v, P) \) by

\[ L + (z, P) = \tilde{L} + (z) \begin{pmatrix} e^{-Q_{\ell_1}} & 0 & \cdots & 0 \\ 0 & e^{-Q_{\ell_2}} & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & e^{-Q_{\ell_N}} \end{pmatrix} = \tilde{L}^{+}(z) \exp \left\{ \sum_{m=1}^{N} h_{\ell_m}^{(1)} Q_{\ell_m} \right\} \quad (2.60) \]

Here \( h_{\ell_j}^{(1)} = h_{\ell_j} \otimes 1 \), \( h_{\ell_m} = -E_{mm} \) \((a \ N \times \ N \ matrix \ unit)\). We then show that the modified \( L \)-operator depends on neither \( Q_{\ell_j} \) nor \( \eta_j \) and satisfies the dynamical \( RLL \) relation of \( B_{q,\lambda}(\widehat{\mathfrak{sl}}_N)[4] \).

**Corollary 2.5**

\[ R^{+(12)}(v, P + h)L^{+(1)}(z_1, P)L^{+(2)}(z_2, P + h^{(1)}) = L^{+(2)}(z_2, P)L^{+(1)}(z_1, P + h^{(2)})R^{+(12)}(v, P), \quad (2.61) \]

where \( u = u_1 - u_2 \).

Hence, we regard the elliptic currents \( E_j(v) \), \( F_j(v) \) \((1 \leq j \leq N - 1)\) and \( K_j(v) \) \((1 \leq j \leq N)\) in \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \) as the Drinfeld realization of the elliptic algebra \( B_{q,\lambda}(\widehat{\mathfrak{sl}}_N) \) tensored by the Heisenberg algebra.

\[ U_{q,p}(\widehat{\mathfrak{sl}}_N) = B_{q,\lambda}(\widehat{\mathfrak{sl}}_N) \otimes_{\mathbb{C}(P_{r_1}, P_{r_2}, \ldots, P_{r_N})} \mathbb{C}\{\hat{H}\}. \quad (2.62) \]

### 3 Vertex Operators of \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \)

We here summarize a construction of the type II vertex operator of \( U_{q,p}(\widehat{\mathfrak{sl}}_N) \) and its dual vertex operator.
3.1 Definition

Let us first define an extension of the \( U_q(\mathfrak{sl}_N) (\cong \mathcal{B}_{q,\lambda}(\mathfrak{sl}_N)) \) modules by

\[
\hat{\mathcal{F}} = \bigoplus_{\mu_1, \ldots, \mu_N \in \mathbb{Z}} \mathcal{F} \otimes e^{\mu_1 Q_{e_1} + \cdots + \mu_{N-1} Q_{e_{N-1}}}.
\]

Let \( \Psi^*_W(z, P) \) be the type II intertwining operator of \( \mathcal{B}_{q,\lambda}(\mathfrak{sl}_N) \) [4]. We define the type II vertex operator \( \tilde{\Psi}^*_W(z) \) of \( U_{q,p}(\mathfrak{sl}_N) \) as the following extension.

\[
\tilde{\Psi}^*_W(z) = \Psi^*_W(z, P) \exp \left\{ \sum_{j=1}^{N} h^{(1)}_j Q_{e_j} \right\} : W_z \otimes \hat{\mathcal{F}} \rightarrow \hat{\mathcal{F}}'.
\]  

(3.1)

From the intertwining relation of the \( \mathcal{B}_{q,\lambda}(\mathfrak{sl}_N) \) intertwining operators, we derive the following relation for the new operator \( \tilde{\Psi}^*_W(z) \).

\[
\tilde{L}^{+(1)}_{V}(z_1) \tilde{\Psi}^{(2)}_W(z_2) = \tilde{\Psi}^{(2)}_W(z_2) \tilde{L}^{+(1)}_{V}(z_1) R_{V,W}^{+(12)}(u_1 - u_2, P - h^{(1)} - h^{(2)}).
\]

(3.2)

Let us consider the vector representation \( V = W = \mathbb{C}^N \) of \( \mathcal{B}_{q,\lambda}(\mathfrak{sl}_N) \). We denote a basis of \( V \) by \( \{v_m\}_{m=1}^N \). In this representation, the \( R \)-matrix \( R_{V,V}^{+(12)}(v, P) \) is given by \( R^+(v, P) \) in (2.56) and the \( L \)-operator \( \tilde{L}^+_{V}(z) \) by \( \tilde{L}^+(z) \) in (2.54). We define the components of the vertex operators by

\[
\tilde{\Psi}^*_V(q^{-c-1}z)(v_m \otimes \cdot) = \Psi^*_m(z),
\]

(3.3)

and the matrix elements of the \( L \)-operator \( \tilde{L}^+(z) \) by

\[
\tilde{L}^+(z)v_j = \sum_{1 \leq m \leq N} v_m L^+(z)_{mj}.
\]

(3.4)

3.2 Free field realizations

We here construct a free field realization of the vertex operators fixing \( c = 1 \). Let \( \alpha_j \) be the simple root operator. We make the standard central extension \( [\alpha_j, \alpha_k] = \pi i A_{jk} \) and set \( \tilde{\alpha}_j = \alpha_j + \alpha_j \), where \( \tilde{\alpha}_j \) is an element of the Heisenberg algebra \( \mathbb{C}\{\hat{H}\} \). Then we have

Proposition 3.1  The currents \( E_j(v) \) and \( F_j(v) \) given by

\[
E_j(v) = \exp \left( - \sum_{m \neq 0} \frac{[rm]_q}{m[r^*m]_q} (-B_m^j + B_{m+1}^j)(q^{N-j}z)^{-m} \right) : e^{\tilde{\alpha}_j z^{h_j}} e^{-Q_{\alpha_j}(q^{-j+N}z)} \frac{P_{\alpha_j} - 1}{\pi} \right],
\]

(3.5)

\[
F_j(v) = \exp \left( \sum_{m \neq 0} \frac{1}{m} (-B_m^j + B_{m+1}^j)(q^{N-j}z)^{-m} \right) : e^{-\tilde{\alpha}_j z^{h_j}} (q^{-j+N}z)^{\frac{P_{\alpha_j} - 1}{\pi} + h_j} \right],
\]

(3.6)

together with \( H_j^\pm(v), K_j(v) \) given in (2.40)-(2.41) satisfy the commutation relations in Definition 2.1 for level \( c = 1 \).
Theorem 3.2 The highest components of the type II vertex operator $\Psi^*_N(z)$ is realized in terms of a free field by

$$
\Psi^*_N(z) = \exp \left( \sum_{m \neq 0} \frac{[rm]}{m[r^*m]} B_m z^{-m} \right) : e^{-\hat{L}_{N-1} z^{-h_N}} e^{Q_N z^{1/2} P_N z^{(1+1)N-1} N^{-1}}, 
$$

where $\hat{L}_{N-1} = \frac{1}{N} (\hat{a}_1 + 2 \hat{a}_2 + \cdots + (N-1) \hat{a}_{N-1})$. The other components of the type II vertex $\Psi^*_j(z)$ ($1 \leq j \leq N$) are given by

$$
\Psi^*_j(z) = a^*_{j,N} \int_{C^*} \prod_{m=j}^{N-1} \frac{dw_m}{2\pi i w_m} \Psi^*_N(v) E_{N-1}(v_{N-1}) \cdots E_j(v_j) 
\times \prod_{m=j}^{N-1} \frac{[v_{m+1} - v_m - P_{j,m+1} + \frac{1}{2}]^*[1]^*}{[v_{m+1} - v_m - \frac{1}{2}]^*[P_{j,m+1} - 1]^*}. 
$$

Here $v_N = v$. The integration contour $C^*$ is specified as follows. For the integration contour for $w_m$ ($j \leq m \leq N - 1$), the poles at $w_m = p^* q^{-1} w_m + 1$ ($n = 0, 1, \ldots$) are inside, whereas the poles at $w_m = p^{* - n} q w_m + 1$ ($n = 0, 1, \ldots$) are outside.

Theorem 3.3 The free field realizations of the type-II vertex operator $\Psi^*_N(z)$ satisfies the following commutation relation.

$$
\Psi^*_j(z_1) \Psi^*_j(z_2) = \sum_{j_1, j_2 = 1}^{N} \Psi^*_j(z_2) \Psi^*_j(z_1) R^*_j j_2 (u_1 - u_2, P) 
$$

Here we set $R^*(v, P) = \mu^*(v) \hat{R}^*(v, P)$ with

$$
\mu^*(v) = z^{(\frac{1}{N} - 1) \frac{N-1}{N}} \frac{\{p q^{2N-2} z^2\}^* \{q^{2N} z^2\}^* \{p/z\}^* \{q^{2N}/z\}^* \{q^{2N}/z\}^*}{\{p/z\}^* \{q^{2N} z^2\}^* \{p q^{2N-2} z^2\}^* \{q^{2N}/z\}^*}. 
$$

3.3 The dual vertex operator

The dual of the type II vertex operator of $U_{q,p}(\hat{s}(N))$ is an operator satisfying

$$
\Psi(z) : \hat{F} \rightarrow V_z \otimes \hat{F}'. 
$$

We define its components in the vector representation by

$$
\Psi(z) = \sum_{j=1}^{N} v_j \otimes \Psi_j(z). 
$$
The following inversion relations hold.

\[
\Psi_j(z)\Psi_k(z') = \delta_{jk} \frac{g_N}{1 - q^{-1} q^z \frac{z - 1}{z}} + \cdots,
\]

\[
g_N = \sqrt{-1} \frac{g_N^{N+1}}{q^{z^z}} \frac{N^2 - 1}{2} \frac{(p^* q^2 z^p)_{\infty}}{(p^* z^p)_{\infty}} (q^{2N-2} q^2 p^* z^p)_{\infty}.
\]

as \( z' \to q z N \), as well as

\[
\sum_{j=1}^{N} \Psi_j(z)\Psi_j(z') = \frac{g_N' z^{1-N}}{1 - q^{-1} q^z \frac{z - 1}{z}} + \cdots,
\]

\[
\sum_{j=1}^{N} \Psi_j(z)\Psi_j(z') = \frac{g_N' z^{1-N}}{1 - q^{-1} q^z \frac{z - 1}{z}} + \cdots,
\]

where

\[
g_N' = \sqrt{-1} \frac{g_N'^{N+1}}{q^{z^z}} \frac{N^2 - 1}{2} \frac{(p^* q^2 z^p)_{\infty}}{(p^* z^p)_{\infty}} (q^{2N-2} q^2 p^* z^p)_{\infty}.
\]

as \( z' \to q z^{-N} \). The free field realizaton is given as follows.

\[
\Psi_j(z) = \oint_C \prod_{m=1}^{j-1} \frac{dz_m}{2\pi i z_m} \Psi_1(z) E_1(v_1) \cdots E_{j-1}(v_{j-1}) \]

\[
\times \prod_{m=1}^{j-1} \frac{1}{v_m - v_m - \frac{1}{2}} \sum_{n=1}^{N} \frac{[\alpha_m]}{[\alpha_m]} \frac{[\alpha_m]}{[\alpha_m]}
\]

where \( v = v_0 \) and

\[
\Psi_1(z) = : \exp \left( \sum_{m=0}^{[\alpha_m]} \frac{[\alpha_m]}{[\alpha_m]} B_m(q^N z)^{-m} \right) e^{-Q \zeta_1 (q^N z)^{-\frac{1}{2}} + \frac{\zeta_1}{2}}.
\]

with \( \tilde{\alpha}_1 = \frac{1}{2} ((N-1) \tilde{\alpha}_1 + (N-2) \tilde{\alpha}_2 + \cdots + \tilde{\alpha}_N) \). The integration contour \( C \) is specified by the condition : for the contour for \( w_m (1 \leq m \leq j-1) \), the poles at \( w_m = q^{-1} w_{m-1} p^m \) (\( n = 0, 1, 2, \ldots \)) are inside, whereas the poles at \( w_m = q w_{m-1} p^m \) (\( n = 0, 1, 2, \ldots \)) are outside.

Remark The free field realizations of the vertex operators in Theorem 3.2 and of the dual vertex operators are essentially the same as those of the \( \hat{a}_N \) RSOS model obtained in [15, 16].

4 Fusion of the Vertex Operators

We now consider the fusion of the type II vertex operator \( \Psi_1^*(z_2) \) and its dual \( \Psi_1(z_1) \). Namely, we consider a product \( \Psi_1(z_1)\Psi_1^*(z_2) \) and investigate the limits to the fusion points \( z_1 = q^{-N} p^m z_2 \) (\( n = 0, 1, 2, \ldots, N \)), where the contour in (3.8) for \( w_1 \) gets pinches.

For example, let us consider the case \( n = 1 \). If we take residues for the poles \( w_{N-1} = q^{-1} z_2, w_{j-1} = q^{-1} w_j \) (\( j = N - 1, N - 2, \ldots, 3 \)), the limit \( z_1 \to q^{-N} p^* z_2 \) causes pinches in the
consideration leads us to the following results. As \( z_1 \to q^{-N}p^*z_2 \),

\[
\Psi_1(z_1)\Psi_1^*(z_2) = \frac{z_1^{-N}}{1 - q^{-N}p^*z_2 z_1} \left\{ C_n \tilde{T}_n(q^{(n-1)r^*}z_2) + \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq N} C_{j_1j_2\ldots j_n} \cdot \Lambda_{j_1}(z_2q^{(2n-3)r^*}) \Lambda_{j_2}(z_2q^{(2n-3)r^*}) \cdots \Lambda_{j_n}(z_2q^{(2n-3)r^*}) \right\} + \cdots.
\]

(4.1)

Here

\[
\tilde{T}_n(z) = \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq N} \Lambda_{j_1}(z_2q^{(n-1)r^*}) \Lambda_{j_2}(z_2q^{(n-3)r^*}) \cdots \Lambda_{j_n}(z_2q^{-(n-1)r^*}),
\]

(4.2)

\[
\Lambda_j(z) = \exp \left( \sum_{m \neq 0} \frac{q^r_m - q^{-r_m}}{m} B_m z^{-m} \right) : q^{-2P_j}p^*h_j q^{\frac{2(1-N)}{N}} p^* z^{-j},
\]

(4.3)

\[
C_n = \sqrt{\frac{1}{N}} q^{\frac{n-1}{2} + \frac{n^2}{2}} \left( \frac{(p^* q; p^*)_{\infty}}{(p^*; p^*)_{\infty}} \right)^n \left( \frac{1 - pq^{-N}}{1 - q^{-N}} \right)^n \times \left( \frac{pq^{2N} p^*; q^2N, p^*}_{(q^{2N} p^*; q^2N, p^*)_{\infty}} \right),
\]

(4.4)

In (4.1), \( \sum' \) denotes the sum over the complementary set to \( 1 \leq j_1 < j_2 < \ldots < j_n \leq N \). \( C_{j_1j_2\ldots j_n} \) are constants not important here.

The basic operators \( \Lambda_j(z) \) (\( 1 \leq j \leq N - 1 \)) coincides with those in the deformed \( W_N \) algebra[7, 8]. The expressions for \( \tilde{T}_n \) (\( 1 \leq n \leq N \)) are almost same as those of the generating “currents” of the deformed \( W_N \) algebra, but the unit of the \( q \)-shift in the arguments in \( \Lambda_j(z) \) is different. In an identification of the parameters \( p_W = q^{-2} \), \( q_W = p = q^{2r} \), where \( p_W \) and \( q_W \) are \( p \) and \( q \) in [7, 8], respectively, the unit of the \( q \)-shift in [7, 8] is given by \( p_W \), whereas it is \( p^* = q^{2(r-1)} \) in our \( \tilde{T}_n(z) \). As a consequence, we have

\[
\tilde{T}_N(z) = : \Lambda_1(z_2q^{(N-1)r^*}) \Lambda_2(z_2q^{(N-3)r^*}) \cdots \Lambda_N(z_2q^{-(N-1)r^*}) : \neq 1.
\]

(4.5)

Therefore, our deformed \( W \) algebra generated by \( \tilde{T}_n \) (\( 1 \leq n \leq N \)) is \( \mathfrak{g}_l \) type instead of \( \mathfrak{s}_l \) type.
On the other hand, since the type II vertex operator $\Psi^*(z)$ and its dual $\Psi(z)$ are the creation operators of the physical excited particle and anti-particle, it is natural to identify the operators $\tilde{T}_n(z)$ ($1 \leq n \leq N$) with the creation operator of their bound states. The $S$-matrix of the bound state particles are calculated as follows.

$$\tilde{T}_n(z)\tilde{T}_m(w) = S_{n,m}(w/z) \tilde{T}_m(w)\tilde{T}_n(z),$$

(4.6)

$$S_{n,m}(z) = \prod_{k=1}^{n} \prod_{l=1}^{m} \phi_N \left( zq^{r^*(n-m+2(l-k))} \right),$$

(4.7)

$$\phi_N(z) = \frac{\Theta_{q^{-2}}(q^2z)\Theta_{q^2N}(p^{*}z)\Theta_{q^{2N}}(p^{*-1}q^{-2}z)}{\Theta_{q^2}(q^{-2}z)\Theta_{q^{2N}}(p^{*-1}z)\Theta_{q^{2N}}(p^{*}q^2z)}.$$  

(4.8)

Again, this $S$-matrix is different from the one obtained by Feigin and Frenkel (sec7.2 in [7]) only by the choice of the unit of the $q$-shift.

The scaling limit of the $\widehat{sl}_N$ RSOS model is expected to be the RSOS restriction of the affine Toda field theory with imaginary coupling constant. It is interesting to compare the scaling limit of our $S$-matrices, $R^*(v, P)$ for the excited particle and $S_{n,m}(z)$ for the bound states, with the bootstrap results [17, 18].

Acknowledgements We would like to thank Patrick Dorey, Tetsuji Miwa and Robert Weston for discussion. We also thank the organizers of RAQIS’03, Daniel Arnaudon, Jean Avan, Luc Frappat, Éric Ragoucy and Paul Sorba, for their kind invitation to the conference and their hospitality. H.K. is also grateful to DAMTP, University of Cambridge, for hospitality. This work is supported by the JSPS/Royal Society fellowship and the Grant-in-Aid for Young Scientists (B) (14740107) from the Ministry of Education, Japan.

References

[1] Konno, H.: An elliptic algebra $U_{q,p}(\widehat{sl}_2)$ and the fusion RSOS model, Commun.Math.Phys. 195, 373-403 (1998).

[2] Jimbo, M., Konno, H., Odake, S. and Shiraishi, J.: Elliptic Algebra $U_{q,p}(\widehat{sl}_2)$: Drinfeld Currents and Vertex Operators, Commun.Math.Phys.199, 605-647 (1999).

[3] Kojima, T and Konno, H.: The Elliptic Algebra $U_{q,p}(\widehat{sl}_N)$ and the Drinfeld Realization of the Elliptic Quantum Group $B_{q,\lambda}(\widehat{sl}_N)$, to appear in Commun.Math.Phys.

[4] Jimbo, M., Konno, H., Odake, S. and Shiraishi, J.: Quasi-Hopf Twistors for Elliptic Quantum Groups, Transformation Groups 4, 303-327 (1999).
[5] Jimbo, M. and Miwa, T. Algebraic Analysis of Solvable Lattice Models. CBMS Regional Conference Series in Mathematics vol. 85, AMS (1994).

[6] Jimbo, M., Miwa, T. and Okado, M.: Solvable Lattice Models Whose States are Dominant Integral Weights of $A_{n-1}^{(1)}$, Lett.Math.Phys. 14, 123-131 (1987).

[7] Feigin, B. and Frenkel, E.: Quantum $W$-Algebras and Elliptic Algebras, Commun. Math.Phys. 178 653–678 (1996).

[8] Awata, H., Kubo, H., Odake, S. and Shiraishi, J.: Quantum $W_N$ Algebras and Macdonald Polynomials , Commun. Math.Phys. 179 401–416 (1996).

[9] Frenkel, E. and Reshetikhin, N.: Deformation of $W$-Algebras Associated to Simple Lie Algebras, Commun. Math.Phys. 178 653–678 (1996).

[10] Jimbo, M., Konno, H., and Miwa, T.: Massless XXZ model and the Degeneration of the Elliptic Algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}}_2)$, Math.Phys.Studies 20, 117–138 (1997).

[11] Jimbo, M., and Shiraishi, J.: A Coset-type Construction for the Deformed Virasoro Algebra, Lett.Math.Phys.44, 349–352 (1998).

[12] Jimbo, M., Konno, H., Odake, S., Shiraishi, J. and Pugai, Y.: Free Field Construction for the ABF Model in Regime II, J.Stat.Phys.102, 883–921 (2001).

[13] Hara, Y., Jimbo, M., Konno, H., Odake, S., and Shiraishi, J.: Free Field Approach to the Dilute $A_L$ Models, J.Math.Phys.40, 3791–3826 (1999).

[14] Drinfeld, V.G.: A new realization of Yangians and quantized affine algebras, Soviet.Math.Dokl. 36, 212-216 (1988).

[15] Asai, Y., Jimbo, M., Miwa, T. and Pugai, Y.: Bosonization of vertex operators for the $A_{n-1}^{(1)}$ face model, J.Phys.A29, 6595-6616 (1996).

[16] Furutsu, H., Kojima, T. and Quano, Y.-H.: Type-II vertex operators for the $A_{n-1}^{(1)}$ face model, Int.J.Mod.Phys.15, 1533-1556 (2000).

[17] Johnson, P.R, : Exact Quantum $S$-matrices for Solitons in Simply-laced Affine Toda Field Theories, Nucl.Phys.B496, 505-550 (1997).

[18] Gandenberger, G.: Trigonometric $S$-matrices, Affine Toda Solitons and Supersymmetry, Int.J.Mod.Phys.A 13, 4553-4590 (1997).