A directed network model for World-Wide Web

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In this paper, a directed network model for world-wide web is presented. The out-degree of the added nodes are supposed to be scale-free and its mean value is m. This model exhibits small-world effect, which means the corresponding networks are of very short average distance and highly large clustering coefficient. More interesting, the in-degree distribution obeys the power-law form with the exponent $\gamma = 2 + 1/m$, depending on the average out-degree. This finding is supported by the empirical data, which has not been emphasized by the previous studies on directed networks.

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I. INTRODUCTION

The last few years have burst a tremendous activity devoted to the characterization and understanding of complex network[1, 2, 3, 4]. Researchers described many real-world systems as complex networks with nodes representing individuals or organizations and edges mimicking the interaction among them. Commonly cited examples include technological networks, information networks, social networks and biological networks [4]. The results of many experiments and statistical analysis indicate that the networks in various fields have some common characteristics. They have small average distances like random graphs, large clustering coefficients like regular networks, and power-law degree distributions. The above characters are called the small-world effect[5] and scale-free property[6].

Motivated by the empirical studies on various real-life networks, some novel network models were proposed recently. The first successful attempt to generate networks with high clustering coefficient and small average distance is that of Watts and Strogatz (WS model) [5]. The WS model starts with a ring lattice with N nodes wherein every node is connected to its first 2m neighbors. The small-world effect emerges by randomly rewiring each edge of the lattice with probability p such that self-connections and duplicate edges are excluded. The rewiring edges are called long-range edges which connect nodes that otherwise may be part of different neighborhoods. Recently, some authors have demonstrated that the small-world effect can also be produced by using deterministic methods[7, 8, 9].

Another significant model capturing the scale-free property is proposed by Barabási and Albert (BA network)[6, 10]. Two special features, i.e., the growth and preferential attachment, is investigated in the BA networks for the free scaling of the Internet, WWW and scientific co-authorship networks, etc. These points to the fact that many real-world networks continuously grow by the way that new nodes added to the network, and would like to connect to the existing nodes with large number of neighbors.

While the BA model captures the basic mechanism which is responsible for the power-law distribution, it is still a minimal model with several limitations: it only predicts a fixed exponent in a power-law degree distribution, and the clustering coefficients of BA networks is very small and decrease with the increasing of network size, following approximately $C \sim \ln^{2}N/N[11]$. To further understand various microscopic evolution mechanisms and overcome the BA model’s discrepancies, there have been several promising attempts. For example, the aging effect on nodes’ charms leads the studies on the aging models[11, 12, 13, 14], the geometrical effect on the appearance probability of edges leads the studies on the networks in Euclidean space[15, 16, 17], and the self-similar effect on the existence of hierarchical structures leads the studies on the hierarchical models[18, 19, 20, 21, 22, 23].

One of the extensively studied networks is the World-Wide Web[24, 25, 26, 27], which can be treated as a directed network having power-law distributions for both in-degree and out-degree. In addition, it is a small-world networks. Since the knowledge of the evolution mechanism is very important for the better understanding of the dynamics built upon WWW, many theoretical models have been constructed previously[28, 29, 30]. However, these models haven’t considered the relationship between the in-degree distribution and the out-degree distribution.

In this paper, we propose a directed network model for the World-Wide Web. This model displays both scale-free and small-world properties, and its power-law exponent of out-degree distribution is determined by the average in-degree. Comparisons among the empirical data, analytic results and simulation results strongly suggest the present model a valid one. The rest of this paper is organized as follows: In section 2, the present model is
introduced. In section 3, the analyzes and simulations on network properties are shown, including the degree distribution, the average distance, and the clustering coefficient. Finally, in section 4, the main conclusion is drawn.

II. THE MODEL

Our model starts with a connected graph of $N_0$ nodes and $m_0$ edges. At each time step $i$, a new node $v_i$ is added and $2e_i$ existing nodes are chosen to be its neighbors. The choosing procedure involves two processes: preferential attachment[6] and neighboring attachment[31]. Firstly, in the preferential attachment process, $e_i$ nodes, denoted by the set $Q_i$, are selected with probability proportional to their in-degrees. And then, in the neighboring attachment process, for each node $x$ in $Q_i$, one of its neighbors is randomly selected to connect to $v_i$. Combine these two processes, there are in total $2e_i$ nodes having been chosen as the new one’s neighbors. In the whole evolution processes, the self-connections and duplicate edges are excluded.

It should be emphasized that, since the out-degree of the WWW network is not fixed but approximately obeying a power law, the number of newly added edges during one time step, $2e_i$, is not a constant but a random number also obeying a power-law. And the average out-degree $m$ is fixed, which significantly affects the in-degree distribution exponent, average distance and clustering coefficient of the whole network.

III. THE STATISTICAL CHARACTERISTICS

In this section, the scale-free small-world characteristics of the present model are shown.

A. The Scale-free Property

The probability that a newly appearing node connects to a previous node is simply proportional to the in-degree $k$ of the old vertex. Suppose the newly added node’s attraction is $A$, then the probability of attachment to the old vertices should be proportional to $k + A$, where $A$ is a constant and we set $A = 1$ for simplicity[32]. The probability that a new edge attaches to any of the vertices with degree $k$ is

$$\sum_k (k + 1)p_k = \frac{(k + 1)p_k}{m + 1}. \tag{1}$$

The mean out-degree of the newly added node is simply $m$, hence the mean number of new edges to vertices with current in-degree $k$ is $(k + 1)p_km/(m + 1)$. Denote $p_{k,n}$ the value of $p_k$ when the network size is $n$, then the change of $np_k$ is

$$\begin{cases}
(n + 1)p_{k,n+1} - np_{k,n} = m(kp_{k-1,n} - (k+1)p_k,n) / (m + 1) & k \geq 1 \\
(n + 1)p_{0,n+1} - np_{0,n} = 1 - p_{0,n}m / (m+1) & k = 0
\end{cases} \tag{2}$$

The stationary condition $p_{k,n+1} = p_{k,n} = p_k$ yields

$$p_k = \begin{cases}
[kp_{k-1} - (k + 1)p_k]m / (m + 1), & k \geq 1; \\
1 - p_0m / (m + 1), & k = 0
\end{cases} \tag{3}$$

Rearranging, one gets

$$p_k = \begin{cases}
\frac{k}{k + 1/m}p_{k-1}, & k \geq 1; \\
(m + 1) / (2m + 1), & k = 0.
\end{cases} \tag{4}$$

This yields

$$p_k = \frac{k(k-1)\cdots 1}{(k+2+1/m)\cdots (3+1/m)}p_0$$

$$= (1 + 1/m)B(k + 1, 2 + 1/m), \tag{5}$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ is Legendre’s beta function, which goes asymptotically as $a^{-b}$ for large $a$ and fix $b$, hence

$$p_k \sim k^{-(2+1/m)}. \tag{6}$$

This leads to $p_k \sim k^{-\gamma_i}$ with $\gamma_i = (2 + 1/m)$ for large $N$, where $\gamma_i$ is the exponent of the in-degree degree distribution.

In Fig. 1, the degree distributions for $m = 1, 2, 3, 4$ are shown. The simulation results agree with the analytic one very well and indicate that the exponents of the degree distribution have no relationship to the network size $N$.\n
FIG. 1: Degree distributions for different $N$ and $m$. In this figure, $p(k)$ denotes the probability that a randomly selected node is of in-degree $k$. When $m = 1$, the power-law exponent $\gamma$ of the density functions are $\gamma_{N=80000} = 2.95 \pm 0.06$ and $\gamma_{N=20000} = 2.97 \pm 0.04$. When $m = 2$, $\gamma_{N=80000} = 2.46 \pm 0.07$ and $\gamma_{N=20000} = 2.47 \pm 0.03$. When $m = 3$, $\gamma_{N=80000} = 2.29 \pm 0.08$ and $\gamma_{N=20000} = 2.31 \pm 0.03$. When $m = 4$, $\gamma_{N=80000} = 2.21 \pm 0.07$ and $\gamma_{N=20000} = 2.23 \pm 0.03$. The four dash lines of $m = 1, 2, 3, 4$ have slope -$3, -2, -1/2, -1/3$ and -$2, -1/4$ for comparison, respectively.
One of the significant empirical results on the in- and out-degree distributions is reported by Albert, Jeong and Barabási [33]. In this paper the crawl from Altavista was used. The appearance of the WWW from the point of view of Altavista is as following [3]:

- In May 1999 the Web consisted of $203 \times 10^6$ vertices and $1466 \times 10^6$ hyperlinks. The average in- and out-degree were $k_{in} = k_{out} = 7.22$.
- In October 1999 there were already $271 \times 10^6$ vertices and $2130 \times 10^6$ hyperlinks. The average in- and out-degree were $k_{in} = k_{out} = 7.85$.

The distributions were found to be of a power-law form with exponent $\gamma_i = 2.1$ and $\gamma_o = 2.7$, where $\gamma_o$ is the exponent of the out-degree distribution. When $k_{out} = 7.22$ and 7.85, one can obtained from $\gamma_i = 2 + 1/m$ that $\gamma_i = 2.138$ and 2.127 respectively, which is very close to 2.1, thus give a strong support to the validity of the present model.

**B. The Average Distance**

The average distance plays a significant role in measuring the transmission delay, thus is one of the most important parameters to measure the efficiency of communication network. Since the original conception of small-world effect is defined based on undirected networks, hereinafter we only consider the undirected version of our model, that is, the directed edge $E_{ij}$ from node $i$ to $j$ is considered to be an bidirectional edge between node $i$ and $j$. When the node is added to the network, each node of the network according to the time is marked. Denote $d(i, j)$ the distance between nodes $i$ and $j$, the average distance with network size $N$ is defined as

$$L(N) = \frac{2\sigma(N)}{N(N-1)}, \quad (7)$$

where the total distance is:

$$\sigma(N) = \sum_{1 \leq i < j \leq N} d(i, j). \quad (8)$$

Clearly, the distance between the existing nodes will not increase with the network size $N$, thus we have

$$\sigma(N+1) \leq \sigma(N) + \sum_{i=1}^{N} d(i, N+1). \quad (9)$$

Denote $y = \{y_1, y_2, \cdots, y_l\}$ as the node set that the $(N+1)$th node have connected. The distance $d(i, N+1)$ can be expressed as following

$$d(i, N+1) = \min\{d(i, y_j) | j = 1, 2, \cdots, l\} + 1. \quad (10)$$

Combining the results above, we have

$$\sigma(N+1) \leq \sigma(N) + (N-l) + \sum_{\Lambda} D(i, y), \quad (11)$$

where $\Lambda = \{1, 2, \cdots, N\} - \{y_1, y_2, \cdots, y_l\}$ is a node set with cardinality $N-l$. Consider the set $y$ as a single node, then the sum $\sum_{i\in\Lambda} d(i, y)$ can be treated as the distance from all the nodes in $\Lambda$ to $y$, thus the sum $\sum_{i\in\Lambda} d(i, y)$ can be expressed approximatively in terms of $L(N-l)$

$$\sum_{i\in\Lambda} d(i, y) \approx (N-l)L(N-l). \quad (12)$$

Because the average distance $L(N)$ increases monotonously with $N$, this yields

$$(N-l)L(N-l) = (N-l) \left( \frac{2\sigma(N-l)}{(N-l)(N-l-1)} \right) < \frac{2\sigma(N)}{N-l-1}. \quad (13)$$

Then we can obtain the inequality

$$\sigma(N+1) < \sigma(N) + (N-l) + \frac{2\sigma(N)}{N-l-1}. \quad (14)$$

Enlarge $\sigma(N)$, then the upper bound of the increasing tendency of $\sigma(N)$ will be obtained by the following equation.

$$\frac{d\sigma(N)}{dN} = N-l + \frac{2\sigma(N)}{N-l-1}. \quad (15)$$

This leads to the following solution:

$$\sigma(N) = (N-l-1)^2 \log(N-l-1) - (N-l-1) + C_1(N-l-1). \quad (16)$$

From Eq.16, we have that $\sigma(N) \sim N^2 L(N)$, thus $L(N) \sim \ln N$. Since Eq.14 is an inequality, the precise increasing tendency of the average distance $L(N)$ may be a little slower than $\ln N$. The simulation results are reported in figure 2.
which is in good accordance with both the analytic and empirical data. In figure 4, we report the simulation result about the relationship between \( C(k) \) and \( k \), which is in good accordance with both the analytic and empirical data. Consequently, we have

\[
C = \frac{2}{N} \sum_{i=1}^{N} \frac{k_i - e_i}{k_i(k_i - 1)} = \frac{2}{N} \sum_{i=1}^{N} \frac{k_{in}}{k_i(k_i - 1)},
\]

where \( k_{in} \) denotes the in-degree of the \( i \)th node. Because the average out-degree is \( m \), one can replace the out-degree of each node by \( m \). From Fig. 3, one can get that the degree distribution of the undirected network is \( p(k) \sim k^{-3} \), where \( k = k_{min}, k_{min} + 1, \ldots, k_{max} \). As an example, the clustering coefficient \( C \) when \( m = 1 \) can be rewritten as

\[
C = \frac{2}{N} \sum_{i=1}^{N} \frac{1}{k_i}.
\]

Since the degree distribution is \( p(k) = c_1 k^{-3} \), where \( k = 2, 3, \ldots, k_{max} \). The clustering coefficient \( C \) can be rewritten as

\[
C = \frac{k_{max}}{N} \frac{2}{k} \sum_{k=2}^{k_{max}} Np(k) = 2c_1 \sum_{k=2}^{k_{max}} k^{-4}.
\]

For sufficient large \( N \), \( k_{max} \gg 2 \). The parameter \( c_1 \) satisfies the normalization equation

\[
\sum_{k=2}^{k_{max}} p(k) dk = 1.
\]

It can be obtained that \( c_1 = 4.9491 \) and \( C = 2 \times 4.9491 \times \sum_{k=2}^{k_{max}} k^{-4} = 0.8149 \). The demonstration exhibits that most real-life networks have large clustering coefficients no matter how many nodes they have. From Fig. 5, one can get that as the average out-degree increases, the clustering coefficient decreases dramatically, which indicates that the clustering coefficient \( C \) is relevant to the average out-degree \( m \).

IV. CONCLUSION AND DISCUSSION

In summary, we have constructed a directed network model for World-Wide Web. The presented networks are
FIG. 5: The clustering coefficient vs the network size $N$ to different $m$ of the undirected versions of the present model. In this figure, when $m = 1, 2, 3, 4$, one can find that the clustering coefficient of the network is almost a constant 0.74, 0.28, 0.18 and 0.14, respectively. This indicates that the average clustering coefficient is relevant to the average out-degree $m$.

both of very large clustering coefficient and very small average distance. We argue that the degree distribution of many real-life directed networks may be fitted appropriately by two power-law distributions, i.e., in- and out-degree power-law distributions, such as the citation network, Internet network and World-Wide Web. Both the analytic and numerical studies indicate the exponent of the in-degree distribution of the presented networks can be well fitted by $2 + 1/m$, which has been observed in the empirical data. Although this model is simple and rough, it offers a good starting point to explain the existing empirical data and the relationship between the in- and out-degree distribution exponents.

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