Regular components of moduli spaces of stable maps

GAVRIL FARKAS

1 Introduction

The purpose of this note is to prove the existence of ‘nice’ components of the Hilbert scheme of curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus $g \geq 2$ and bidegree $(k,d)$. We can also phrase our result using the Kontsevich moduli space of stable maps to $\mathbb{P}^1 \times \mathbb{P}^r$. We work over an algebraically closed field of characteristic zero.

For a smooth projective variety $Y$ and a class $\beta \in H_2(Y, \mathbb{Z})$, one considers the moduli stack $\mathcal{M}_g(Y, \beta)$ of stable maps $f : C \to Y$, with $C$ a reduced connected nodal curve of genus $g$ and $f^*([C]) = \beta$ (see [FP] for the construction of these stacks). The open substack $\mathcal{M}_g(Y, \beta)$ of $\mathcal{M}_g(Y, \beta)$ parametrizes maps from smooth curves to $Y$. By $\mathcal{M}_g(Y, \beta)$ we denote the coarse moduli space corresponding to the stack $\mathcal{M}_g(Y, \beta)$ and similarly $\mathcal{M}_g$ is the moduli space corresponding to the stack $\mathcal{M}_g$ of stable curves of genus $g$. We denote by $\pi : \mathcal{M}_g(Y, \beta) \to \mathcal{M}_g$ the natural projection. The expected dimension of the stack $\mathcal{M}_g(Y, \beta)$ is

$$\chi(g, Y, \beta) = \dim(Y) (1 - g) + 3g - 3 - \beta \cdot K_Y.$$ 

Since in general the geometry of $\mathcal{M}_g(Y, \beta)$ is quite messy (e.g. existence of many components, some nonreduced and/or not of expected dimension), it is not obvious what the definition of a nice component of $\mathcal{M}_g(Y, \beta)$ should be. Following Sernesi [Se] we introduce the following terminology:

Definition. A component $V$ of $\mathcal{M}_g(Y, \beta)$ is said to be regular if it is generically smooth and of dimension $\chi(g, Y, \beta)$. We say that $V$ has the expected number of moduli if

$$\dim \pi(V) = \min(3g - 3, \chi(g, Y, \beta) - \dim \text{Aut}(Y)).$$

In this paper we only construct regular components of moduli spaces of stable maps. We study the stacks $\mathcal{M}_g(Y, \beta)$ when $Y = \mathbb{P}^1 \times \mathbb{P}^r$, $r \geq 3$ and $\beta = (k, d) \in H_2(\mathbb{P}^1 \times \mathbb{P}^r, \mathbb{Z})$. We denote by $\rho(g, r, d) = g - (r + 1)(g - d + r)$ the Brill-Noether number governing the existence of $g^r_d$’s on curves of genus $g$. Our main result is the following:

Theorem 1 Let $g, r, d$ and $k$ be positive integers with $r \geq 3$, $\rho(g, r, d) < 0$ and

$$(2 - \rho(g, r, d))r + 2 \leq k \leq (g + 2)/2.$$

Then there exists a regular component of the stack of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$. 

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We introduce the Brill-Noether locus \( M_{g,d}^r = \{ [C] \in M_g : C \text{ has a } g_d^r \} \), in the case \( \rho(g, r, d) < 0 \). The expected codimension of \( M_{g,d}^r \) inside \( M_g \) is \( -\rho(g, r, d) \). We view Theorem 1 as a tool in the study of the relative position of the loci \( M_{g,k}^1 \) and \( M_{g,d}^r \) when \( r \geq 3, \rho(g, 1, k) < 0 \) and \( \rho(g, r, d) < 0 \). The stack \( M_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \) comes naturally into play when looking at the intersection in \( M_g \) of the loci \( M_{g,k}^1 \) and \( M_{g,d}^r \). In such a setting, if \( V \) is a regular component of \( M_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \), then \( M_{g,k}^1 \) and \( M_{g,d}^r \) intersect properly along \( \pi(V) \). It is very plausible that one has a similar statement to Theorem 1 when \( \rho(g, r, d) \geq 0 \) and\( or \rho(g, 1, k) \geq 0 \), but from our perspective that seems of less interest because it would be essentially a statement about linear series on the general curve of genus \( g \) with no implications on the problem of understanding the geography of the Brill-Noether loci inside \( M_g \).

Regarding the problem of existence of regular components of \( \mathcal{M}_g(Y, \beta) \), so far the spaces \( \mathcal{M}_g(\mathbb{P}^r, d) \) have received the bulk of attention. When \( r = 1, 2 \) the problem boils down to the study of the Hurwitz scheme and of the Severi variety of plane curves which are known to be irreducible and regular. For \( r \geq 3 \) we have the following result of Sernesi (cf. [Se, p. 26]):

**Proposition 1.1** For all \( g, r, d \) such that \( d \geq r + 1 \) and

\[
-\frac{g}{r} + \frac{r + 1}{r} \leq \rho(g, r, d) < 0,
\]

there exists a regular component \( V \) of \( \mathcal{M}_g(\mathbb{P}^r, d) \) which has the expected number of moduli. A general point of \( V \) corresponds to an embedding \( C \hookrightarrow \mathbb{P}^r \) by a complete linear system (i.e. \( h^0(C, \mathcal{O}_C(1)) = r + 1 \)), the normal bundle \( N_C \) satisfies \( H^1(C, N_C) = 0 \) and the Petri map

\[
\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \to H^0(C, K_C)
\]

is surjective.

A. Lopez has obtained significant improvements on the range of \( g, r, d \) such that there exists a regular component of \( \mathcal{M}_g(\mathbb{P}^r, d) \): if \( h(r) = (4r^3 + 8r^2 - 9r + 3)/(r + 3) \), then for all \( g, r, d \) such that \( -(2 - 6/(r + 3))g + h(r) \leq \rho(g, r, d) < 0 \) there exists a regular component of \( \mathcal{M}_g(\mathbb{P}^r, d) \) with the expected number of moduli (cf. [Lo]).

When \( Y \) is a smooth surface, methods from [AC] can be employed to show that if \( V \) is a component of \( \mathcal{M}_g(Y, \beta) \) with \( \dim(V) \geq g + 1 \) and which contains a point \( \{ f : C \to Y \} \) with \( \deg(f) = 1 \) (i.e. \( f \) is generically injective), then \( V \) is regular. Here it is crucial that the normal sheaf \( N_f \) is of rank 1 as then the Clifford Theorem provides an easy criterion for the vanishing of \( H^1(C, N_f) \), which turns out to be a sufficient criterion for regularity (see Section 2).

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2 Deformations of maps and smoothings of space curves

We review some facts about deformations of maps and smoothings of reducible nodal curves in $\mathbb{P}^r$. Our references are [Ran] and [Se].

We start by describing the deformation theory of maps between complex algebraic varieties when the source is (possibly) singular and the target is smooth. Let $f : X \to Y$ be a morphism between complex projective varieties, with $Y$ being smooth. We denote by $\text{Def}(X, f, Y)$ the space of first-order deformations of the map $f$ when $X$ and $Y$ are not considered fixed. The space of first-order deformations of $X$ (resp. $Y$) is denoted by $\text{Def}(X)$ (resp. $\text{Def}(Y)$). We have the standard identification $\text{Def}(X) = \text{Ext}^1(\Omega_X, \mathcal{O}_X)$. The deformation space $\text{Def}(X, f, Y)$ fits in the following exact sequence:

$$\text{Hom}_{\mathcal{O}_Y}(f^*\Omega_Y, \mathcal{O}_X) \to \text{Def}(X, f, Y) \to \text{Def}(X) \oplus \text{Def}(Y) \to \text{Ext}^1_Y(\Omega_Y, \mathcal{O}_X).$$  (1)

The second arrow is given by the natural forgetful maps, the space $\text{Hom}_{\mathcal{O}_Y}(f^*\Omega_Y, \mathcal{O}_X) = H^0(X, f^*T_Y)$ parametrizes first-order deformations of $f : X \to Y$ when both $X$ and $Y$ are fixed, while for $A, B$, respectively $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, $\text{Ext}^1_Y(B, A)$ denotes the derived functor of $\text{Hom}_Y(B, A) = \text{Hom}_{\mathcal{O}_Y}(f^*B, A) = \text{Hom}_{\mathcal{O}_Y}(B, f_*A)$. Under reasonable assumptions (trivially satisfied when $f$ is a finite map between nodal curves) one has that $\text{Ext}^1_Y(\Omega_Y, \mathcal{O}_X) = \text{Ext}^1_Y(f^*\Omega_Y, \mathcal{O}_X)$. Using (1) it follows that when $X$ is smooth and irreducible and $Y$ is rigid (e.g. a product of projective spaces) $\text{Def}(X, f, Y) = H^0(X, N_f)$, where $N_f = \text{Coker}\{T_X \to f^*T_Y\}$ is the normal sheaf of the map $f$.

For a smooth variety $Y$, a class $\beta \in H_2(Y, \mathbb{Z})$ and a point $[f : C \to Y] \in \mathcal{M}_g(Y, \beta)$ we have that $T_{[f]}(\overline{\mathcal{M}}_g(Y, \beta)) = H^0(C, N_f)$. If moreover $\text{deg}(f) = 1$ and $H^1(C, N_f) = 0$, then every class in $H^0(C, N_f)$ is unobstructed, $f$ is an immersion (cf. [AC, Lemma 1.4]) and $\overline{\mathcal{M}}_g(Y, \beta)$ is smooth and of the expected dimension at the point $[f]$, that is, $[f]$ belongs to a regular component of $\overline{\mathcal{M}}_g(Y, \beta)$.

Let $C \subseteq \mathbb{P}^r$ be a stable curve of genus $g$ and degree $d$. If $\mathcal{I}_C$ is the ideal sheaf of $C$ we denote by $N_C := \text{Hom}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ the normal sheaf of $C$ in $\mathbb{P}^r$. Assume that $H^1(C, N_C) = 0$ and that $h^0(C, \mathcal{O}_C(1)) = r + 1$, that is, $C$ is embedded by a complete linear system. The differential of the map $\pi : \overline{\mathcal{M}}_g(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_g$ at the point $[C] \in \mathbb{P}^r$ is given by the natural map $H^0(C, N_C) \to \text{Ext}^1(\Omega_C, \mathcal{O}_C)$. If $\omega_C$ denotes the dualizing sheaf of $C$, then $\text{rk}(d\pi)|_{C \subseteq \mathbb{P}^r} = 3g - 3 - \dim \text{Ker}\mu_0(C)$, where

$$\mu_0(C) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, \omega_C(-1)) \to H^0(C, \omega_C)$$

is the Petri map. In particular $(d\pi)|_{C \subseteq \mathbb{P}^r}$ has rank $3g - 3 + \rho(g, r, d)$ if and only if $\mu_0(C)$ is surjective.

In the same setting, via the standard identification $T_{[C]}(\overline{\mathcal{M}}_g)^\vee = H^0(C, \omega_C \otimes \Omega_C)$, the annihilator $(\text{Im}(d\pi)|_{C \subseteq \mathbb{P}^r})^\perp \subseteq H^0(C, \omega_C \otimes \Omega_C)$ can be naturally identified with $\text{Im}(\mu_1(C))$, where

$$\mu_1(C) : \text{Ker}\mu_0(C) \to H^0(C, \Omega_C \otimes \omega_C)$$
is the Gaussian map obtained from taking the ‘derivative’ of $\mu_0(C)$ (cf. [CGGH, p. 163]).

In Section 3 we will smooth curves $X \subseteq \mathbb{P}^r$ which are unions of two smooth curves $C$ and $E$ meeting quasi-transversally (i.e. having distinct tangent lines) at a finite set $\Delta$. For such a curve one has the exact sequences (cf. [Se, p. 35])

$$0 \rightarrow \mathcal{O}_E(-\Delta) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0, \quad (2)$$

and

$$0 \rightarrow \Omega_E \rightarrow \omega_X \rightarrow \Omega_C(\Delta) \rightarrow 0. \quad (3)$$

Also in Section 3 we will use an inductive procedure to construct curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ with $H^1(C, N_C/\mathbb{P}^1 \times \mathbb{P}^r) = 0$. The induction step uses the following result (cf. [BE, Lemma 2.3]):

**Proposition 2.1** Let $C \subseteq \mathbb{P}^r$ be a smooth curve with $H^1(C, N_C) = 0$. We take $r + 2$ points $p_1, \ldots, p_{r+2} \in C$ in general linear position and a smooth rational curve $E \subseteq \mathbb{P}^r$ of degree $r$ which meets $C$ quasi-transversally at $p_1, \ldots, p_{r+2}$. Then $X = C \cup E$ is smoothable in $\mathbb{P}^r$ and $H^1(X, N_X) = 0$.

### 3 Existence of regular components of $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$

In this section we prove the existence of regular components of $\overline{\mathcal{M}}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$ in the case $k \geq r + 2, d \geq r \geq 3$, and $\rho(g, r, d) < 0$. We achieve this by constructing smooth curves $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of bidegree $(k, d)$ satisfying $H^1(C, N_C/\mathbb{P}^1 \times \mathbb{P}^r) = 0$.

Let us fix integers $g \geq 2, d \geq r \geq 3$ and $k \geq 2$, as well as a smooth curve $C$ of genus $g$ with maps $f_1 : C \to \mathbb{P}^1$, $f_2 : C \to \mathbb{P}^r$, such that $\deg(f_1) = k$, $\deg(f_2(C)) = d$ and $f_2$ is generically injective. Let us denote by $f : C \to \mathbb{P}^1 \times \mathbb{P}^r$ the product map. As usual we denote by $G^g_r(C)$ the scheme parametrizing $g^r$’s on $C$.

There is a commutative diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \rightarrow & T_C & \rightarrow & f^*(T_{\mathbb{P}^1 \times \mathbb{P}^r}) & \rightarrow & N_f & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & T_C \oplus T_C & \rightarrow & f_1^*(T_{\mathbb{P}^1}) \oplus f_2^*(T_{\mathbb{P}^r}) & \rightarrow & N_{f_1} \oplus N_{f_2} & \rightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

By taking cohomology in the last column, we see that the condition $H^1(C, N_f) = 0$ is equivalent to $H^1(C, N_{f_1}) = 0$ (automatic), $H^1(C, N_{f_2}) = 0$, and

$$\text{Im}\{\delta_1 : H^0(C, N_{f_1}) \to H^1(C, T_C)\} + \text{Im}\{\delta_2 : H^0(C, N_{f_2}) \to H^1(C, T_C)\} = H^1(C, T_C), \quad (4)$$
where \( \delta_1 \) and \( \delta_2 \) are coboundary maps. Condition (4) is equivalent (cf. Section 2) to

\[
(d\pi_1)_{|f_1} \left( T_{|f_1}(\mathcal{M}_g(\mathbb{P}^1, k)) \right) + (d\pi_2)_{|f_2} \left( T_{|f_2}(\mathcal{M}_g(\mathbb{P}^r, d)) \right) = T_{|C}(\mathcal{M}_g),
\]

where the projections \( \pi_1 : \mathcal{M}_g(\mathbb{P}^1, k) \to \mathcal{M}_g \) and \( \pi_2 : \mathcal{M}_g(\mathbb{P}^r, d) \to \mathcal{M}_g \) are the natural forgetful maps. Slightly abusing terminology, if \( C \) is a smooth curve and \((l_1, l_2) \in G_k^1(C) \times G_d^0(C)\) is a pair of base point free linear series on \( C \), we say that \((C, l_1, l_2)\) satisfies (5), if \((C, f_1, f_2)\) satisfies (5), where \( f_1 \) and \( f_2 \) are maps associated to \( l_1 \) and \( l_2 \).

Recall that a base point free pencil \( g_k^1 \) is said to be simple if the induced covering \( f : C \to \mathbb{P}^1 \) has a single ramification point \( x \) over each branch point and moreover \( e_2(f) = 2 \).

We prove the existence of regular components of \( \mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \) using the following inductive procedure:

**Proposition 3.1** Fix positive integers \( g, r, d \) and \( k \) with \( d \geq r \geq 3, k \geq r + 2 \) and \( \rho(g, r, d) < 0 \). Let us assume that \( C \subseteq \mathbb{P}^r \) is a smooth nondegenerate curve of degree \( d \) and genus \( g \), such that \( h^1(C, N_C) = 0, h^0(C, \mathcal{O}_C(1)) = r + 1 \) and the Petri map

\[
\mu_0(C) = \mu_0(C, \mathcal{O}_C(1)) : H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) \to H^0(C, K_C)
\]

is surjective. Assume furthermore that \( C \) possesses a simple base point free pencil \( g_k^1 \) say \( l \), such that \( |\mathcal{O}_C(1)|(-l) = \emptyset \) and \( (C, l, |\mathcal{O}_C(1)|) \) satisfies (5).

Then there exists a smooth nondegenerate curve \( Y \subseteq \mathbb{P}^r \) with \( g(Y) = g + r + 1 \),
\( \text{deg}(Y) = d + r \) and a simple base point free pencil \( l' \in G_k^1(Y) \), so that \( Y \) enjoys exactly the same properties: \( h^1(Y, N_Y) = 0, h^0(Y, \mathcal{O}_Y(1)) = r + 1 \), the Petri map \( \mu_0(Y) \) is surjective, \( |\mathcal{O}_Y(1)|(-l') = \emptyset \) and \( (Y, l', |\mathcal{O}_Y(1)|) \) satisfies (5).

**Proof.** We first construct a reducible \( k \)-gonal nodal curve \( X \subseteq \mathbb{P}^r \), with \( p_a(X) = g + r + 1 \), \( \deg(X) = d + r \), having all the required properties, then we prove that \( X \) can be smoothed in \( \mathbb{P}^r \) preserving all properties we want.

Let \( f_1 : C \to \mathbb{P}^1 \) be the degree \( k \) map corresponding to the pencil \( l \). The covering \( f_1 \) is simple hence the monodromy of \( f_1 \) is the full symmetric group. Then since \( |\mathcal{O}_C(1)|(-l) = \emptyset \), we have that for a general \( \lambda \in \mathbb{P}^1 \) the fibre \( f_1^{-1}(\lambda) = p_1 + \cdots + p_k \) consists of \( k \) distinct points in general linear position. Let \( \Delta = \{p_1, \ldots, p_{r+2}\} \) be a subset of \( f_1^{-1}(\lambda) \) and let \( E \subseteq \mathbb{P}^r \) be a rational normal curve (\( \deg(E) = r \)) passing through \( p_1, \ldots, p_{r+2} \). (Through any \( r + 3 \) points in general linear position in \( \mathbb{P}^r \), there passes a unique rational normal curve.) We set \( X := C \cup E \), with \( C \) and \( E \) meeting quasi-transversally at \( \Delta \). Of course \( p_a(X) = g + r + 1 \) and \( \deg(X) = d + r \). Note that \( \rho(g, r, d) = \rho(g + r + 1, r, d + r) \).

We first prove that \( [X] \in \mathcal{M}_{g+r+1,k}^1 \) (that is, \( X \) is a limit of smooth \( k \)-gonal curves), by constructing an admissible covering of degree \( k \) having as domain a curve \( X' \), stably equivalent to \( X \). Let \( X' := X \cup D_r+3 \cup \ldots \cup D_k \), where \( D_i \simeq \mathbb{P}^1 \) and \( D_i \cap X = \{p_i\} \), for \( i = r + 3, \ldots, k \). Take \( Y := (\mathbb{P}^1)_1 \cup \lambda (\mathbb{P}^1)_2 \) a union of two lines identified at \( \lambda \). We construct a degree \( k \) admissible covering \( f' : X' \to Y \) as follows: take \( f'_{|C} = f_1 : C \to \mathbb{P}^1 \)
(\mathbb{P}^1)_l, f'_{lE} = f_2 : E \to (\mathbb{P}^1)_l \text{ a map of degree } r + 2 \text{ sending the points } p_1, \ldots, p_{r+2} \text{ to } \lambda, \text{ and finally } f'_{D_1} : D_1 \simeq (\mathbb{P}^1)_2 \text{ isomorphisms sending } p_i \text{ to } \lambda. \text{ Clearly } f' \text{ is an admissible covering, so } X \text{ which is stably equivalent to } X' \text{ is a } k \text{-gonal curve.}

Let us consider now the space \( \overline{H}_{g+r+1,k} \) of Harris-Mumford admissible coverings of degree \( k \) (cf. [HM]) and denote by \( \pi_1 : \overline{H}_{g+r+1,k} \to \overline{M}_{g+r+1} \) the natural projection which sends a covering to the stable model of its source. We assume for simplicity that \( \text{Aut}(C') = \{ \text{Id}_C \} \) which implies that \( \text{Aut}(f') = \{ \text{Id}_{X'} \} \), so \( [f'] \) is a smooth point of \( \overline{H}_{g+r+1,k} \). In the case when \( C \) has nontrivial automorphisms the argument carries through without change if we replace the space of admissible coverings with the space of twisted covers of Abramovich, Corti and Vistoli (cf. [ACV]).

We compute the differential of the map \( \pi_1 \) at \([f]\). We have \( T_{[f]}(\overline{H}_{g+r+1,k}) = \text{Def}(X', f', Y) = \text{Def}(X, f, Y) \), where \( f = f'_{|X} : X \to Y \). The differential \( (d\pi_1)_{[f]} \) is the forgetful map \( \text{Def}(X, f, Y) \to \text{Def}(X) \) and from the sequence (2.1) we get that \( \text{Im}(d\pi_1)_{[f]} = u^{-1}_r(\text{Im} u_2) \), where \( u_1 : \text{Def}(X) \to \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X) \) and \( u_2 : \text{Def}(Y) \to \text{Ext}^1(f^*\Omega_Y, \mathcal{O}_X) \) are the dual maps of \( u^*_1 : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^1(X, \omega_X \otimes \Omega_X) \) and \( u^*_2 : H^0(X, \omega_X \otimes f^*\Omega_Y) \to H^0(Y, \omega_Y \otimes \Omega_Y) \). Here \( u^*_2 \) is induced by the trace map \( \text{tr} : f_*\omega_X \to \omega_Y \). Starting with the exact sequence on \( X \),

$$0 \to \text{Tors}(\omega_X \otimes \Omega_X) \to \omega_X \otimes \Omega_X \to \Omega^0_C(\Delta) \oplus \Omega^0_E(\Delta) \to 0,$$

we can write the following commutative diagram of sequences

$$
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) & \to & H^0(\omega_X \otimes f^*\Omega_Y) \\
(u^*_1)_{\text{tors}} & \downarrow & \uparrow u^*_2 \\
H^0(\text{Tors}(\omega_X \otimes \Omega_X)) & \to & H^0(\omega_X \otimes \Omega_X) \\
\end{array}
$$

where \( R_1 \) (resp. \( R_2 \)) is the ramification divisor of the map \( f_1 \) (resp. \( f_2 \)). Taking into account that \( H^0(E, 2K_E - R_2 + \Delta) = 0 \) and that \( H^0(Y, \omega_Y \otimes \Omega_Y) = H^0(\text{Tors}(\omega_Y \otimes \Omega_Y)) \), we obtain that

$$\text{Im}(d\pi_1)_{[f]} = (H^0(C, 2K_C - R_1 + \Delta) \oplus \text{Ker}(u^*_2)_{\text{tors}})^\perp,$$

where \( (u^*_2)_{\text{tors}} : H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \to H^0(\text{Tors}(\omega_Y \otimes \Omega_Y)) \) is the restriction of \( u^*_2 \). The space \( \text{Ker}(u^*_2)_{\text{tors}} \) is just a hyperplane in \( H^0(\text{Tors}(\omega_X \otimes f^*\Omega_Y)) \cong \mathbb{C}^{r+2} \).

**Intermezzo.** If we also assume that \( \rho(g, 1, k) < 0 \) and that \( [C] \) is a smooth point of \( \overline{M}^1_{g,k} \) (which happens precisely when \( \text{Aut}(C) = \{ \text{Id}_C \} \); \( C \) has exactly one \( g_1 \) and \( \dim \{2g_1 \} = 2 \)), then we can prove that the locus \( \overline{M}^1_{g+r+1,k} \) is smooth at \([X]\) as well. Indeed, since \( \Delta \in C_{r+2} \) was chosen generically in a fibre of the \( g_1 \) on \( C \), from Riemann-Roch we have that \( h^0(C, 2K_C - R_1 + \Delta) = g - 2k + 3 + r = \text{codim}(\overline{M}^1_{g+r+1,k}, \overline{M}_{g+r+1}) \).

The fibre over \([X]\) of the map \( \pi_1 : \overline{H}_{g+r+1,k} \to \overline{M}_{g+r+1} \) is identified with the space of degree \( r + 1 \) maps \( f_2 : E \to \mathbb{P}^1 \) such that \( f_2(p_1) = \ldots = f_2(p_{r+2}) = \lambda \), hence it is \( r + 1 \) dimensional. We compute the tangent cone
from the sequence (2) we have that

\[ \text{im}(\mu) = \text{im}(d\pi_1 z : z \in \pi_1^{-1}([X])) = H^0(C, 2K_C - R_1 + \Delta)^\perp, \]

which shows that \([X]\) is a smooth point of the locus \(\mathcal{M}_{g+r+1, k}^1\).

We compute now the differential

\[ (d\pi_2)|_X : T_{[X]}(\text{Hilb}_{d+r, g+r+1, r}) \to T_{[X]}(\mathcal{M}_{g+r+1}), \]

which is the same thing as the differential at the point \([X] \mapsto \mathbb{P}^r\) of the projection \(\pi_2 : \mathcal{M}_{g+r+1}(\mathbb{P}^r, d + r) \to \mathcal{M}_{g+r+1}\). We start by noticing that \(X\) is smoothable in \(\mathbb{P}^r\) and that \(H^1(X, N_X) = 0\) (apply Proposition 2.1). We also have that \(X\) is embedded in \(\mathbb{P}^r\) by a complete linear system, that is, \(h^0(X, \mathcal{O}_X(1)) = r + 1\). Indeed, on one hand, since \(X\) is nondegenerate, \(h^0(X, \mathcal{O}_X(1)) \geq h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) = r + 1\); on the other hand from the sequence (2) we have that \(h^0(X, \mathcal{O}_X(1)) \leq h^0(C, \mathcal{O}_C(1)) = r + 1\).

If \(X\) is embedded in \(\mathbb{P}^r\) by a complete linear system, we know (cf. Section 2) that

\[ \text{im}(d\pi_2|_X) = (\text{im} \mu_1(X))^\perp, \]

where \(\mu_1(X) : \ker \mu_0(X) \to H^0(X, \omega_X \otimes \Omega_X)\) is the ‘derivative’ of the Petri map \(\mu_0(X) : H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) \to H^0(X, \omega_X)\). We compute the kernel of \(\mu_0(X)\) and show that \(\mu_0(X)\) is surjective in a way that resembles the proof of Proposition 2.3 in [Se].

From the sequence (3) we obtain \(H^0(X, \omega_X) = H^0(C, K_C + \Delta)\), while from (2) we have that \(H^0(X, \mathcal{O}_X(1)) = H^0(E, \mathcal{O}_E(1))\) (keeping in mind that \(H^0(C, \mathcal{O}_C(1)(-\Delta)) = 0\), as \(p_1, \ldots, p_{r+2}\) are in general linear position). Finally, using (3) again, we have that \(H^0(X, \omega_X(-1)) = H^0(C, K_C(-1) + \Delta)\). Therefore we can write the following commutative diagram:

\[
\begin{array}{ccc}
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1)) & \xrightarrow{\mu_0(C)} & H^0(C, K_C) \\
\downarrow & & \downarrow \\
H^0(C, \mathcal{O}_C(1)) \otimes H^0(C, K_C(-1) + \Delta) & \rightarrow & H^0(C, K_C + \Delta) \\
\downarrow & = & \downarrow \\
H^0(X, \mathcal{O}_X(1)) \otimes H^0(X, \omega_X(-1)) & \xrightarrow{\mu_0(X)} & H^0(X, \omega_X) \\
\end{array}
\]

It follows that \(\ker \mu_0(C) \subseteq \ker \mu_0(X)\). By Corollary 1.6 from [CR], our assumptions \((\mu_0(C)\text{ surjective and } \operatorname{card}(\Delta) \geq 4)\) enable us to conclude that \(\mu_0(X)\) is surjective too. Then \(\ker \mu_0(C) = \ker \mu_0(X)\) for dimension reasons, hence also \(\text{im} \mu_1(X) = \text{im} \mu_1(C) \subseteq H^0(C, 2K_C) \subseteq H^0(X, \omega_X \otimes \Omega_X)\). We thus get that \(\text{im}(d\pi_2|_X) = (\text{im} \mu_1(X))^\perp = (\text{im} \mu_1(C))^\perp\).

The assumption that \((C, f_1, f_2)\) satisfies (5) can be rewritten by passing to duals as

\[ H^0(C, 2K_C - R_1)^\perp + (\text{im} \mu_1(C))^\perp = H^1(C, T_C) \iff H^0(C, 2K_C - R_1) \cap \text{im} \mu_1(C) = 0. \]
Then it follows that \( \text{Im} \mu_1(X) \cap (H^0(C, 2K_C - R_1 + \Delta) \oplus \text{Ker}((u^2_\gamma)_{\text{tors}})) = 0 \), which is the same thing as

\[
(d\pi_1)[f'] \left( T_{[f']}(\overline{H}_{g+r+1,k}) \right) + (d\pi_2)[X \to \mathbb{P}^r] \left( T_{[X \to \mathbb{P}^r]}(\overline{M}_{g+r+1}(\mathbb{P}^r, d + r)) \right) = \text{Ext}^1(\Omega_X, \mathcal{O}_X).
\]

This means that the images of \( \overline{H}_{g+r+1,k} \) under the map \( \pi_1 \) and of \( \overline{M}_{g+r+1}(\mathbb{P}^r, d + r) \) under the map \( \pi_2 \), meet transversally at the point \([X] \in \overline{M}_{g+r+1} \).

**Claim.** The curve \( X \) can be smoothed in such a way that the \( g_k^i \) and the very ample \( g_{d+r}^i \) are preserved (while (7) is an open condition on \( \overline{H}_{g+r+1,k} \times \overline{M}_{g+r+1}(\mathbb{P}^r, d + r) \)).

Indeed, the tangent directions that fail to smooth at least one node of \( X \) are those of \( \bigcup_{r=1}^{r+2} H^0(\text{Tors}_{\mu_i}(\omega_X \otimes \Omega_X)) \perp \), whereas the tangent directions that preserve both the \( g_k^i \) and the \( g_{d+r}^i \) are those in

\[
\left( (\text{Im} \mu_1(C) + H^0(C, 2K_C - R_1 + \Delta)) \oplus \text{Ker}((u^2_\gamma)_{\text{tors}}) \right) \perp.
\]

Since \( H^0(\text{Tors}_{\mu_i}(\omega_X \otimes \Omega_X)) \not\subseteq \text{Ker}((u^2_\gamma)_{\text{tors}}) \) for \( i = 1, \ldots, r + 2 \), by moving in a suitable direction in the tangent space at \([f']\) of \( \pi_1^{-1} \pi_2(\overline{M}_{g+r+1}(\mathbb{P}^r, d + r)) \), we finally obtain a smooth curve \( Y \subseteq \mathbb{P}^r \) with \( g(Y) = g + r + 1 \), \( \deg(Y) = d + r \) and satisfying all the required properties.

Using the previous result together with Proposition 1.1 we construct now regular components of \( \mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \).

**Theorem 1** Let \( g, r, d \) and \( k \) be positive integers such that \( r \geq 3 \), \( \rho(g, r, d) < 0 \) and

\[
(2 - \rho(g, r, d))r + 2 \leq k \leq (g + 2)/2.
\]

Then there exists a regular component of the stack of maps \( \mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d)) \).

**Proof.** All integer solutions \((g_0, d_0)\) of the equation \( \rho(g_0, r, d_0) = \rho(g, r, d) \) with \( g_0 \leq g \) and \( d_0 \leq d \), are of the form \( g_0 = g - a(r + 1) \) and \( d_0 = d - ar \) with \( a \geq 0 \). Using our numerical assumptions, by a routine check we find that there exists \( a > 0 \) such that \( g_0 = g - a(r + 1) > 0 \), \( d_0 = d - ar \geq r + 1 \), \( k \geq g_0 + 1 \) and

\[
\frac{-g_0}{r} + \frac{r + 1}{r} \leq \rho(g_0, r, d_0) < 0.
\]

By Proposition 1.1 there exists a smooth curve \( C_0 \subseteq \mathbb{P}^r \) of genus \( g_0 \) and degree \( d_0 \), with \( H^1(C_0, N_{C_0/\mathbb{P}^r}) = 0 \), \( h^0(C_0, \mathcal{O}_{C_0}(1)) = r + 1 \) and \( \mu_0(C_0) \) surjective. Moreover, since \( k \geq g_0 + 1 \), there exists an open dense subset \( U \subseteq \text{Pic}^k(C_0) \) such that for each \( L_1 \in U \) there exists a pencil \( l_1 = (L_1, V_1) \in G^k_1(C_0) \) with \( V_1 \in \text{Gr}(2, H^0(C_0, L)) \), such that \( l_1 \) is simple and base point free (cf. [Fu, Proposition 8.1]).

We denote by \( \pi_1 : \mathcal{M}_{g_0}(\mathbb{P}^1, k_0) \to \mathcal{M}_{g_0} \) the natural projection and by \( f_1 : C \to \mathbb{P}^1 \) the map corresponding to \( l_1 \). By Riemann-Roch we have \( H^1(C_0, L_1^{\otimes 2}) = 0 \), hence using
Applying semicontinuity, for a general element $C$

These curves will have rather few moduli ($r$

general $C$

Theorem 2

for which there is a regular component:

\[ \bullet \]

follows. Let $\mathcal{O}_{C_0}(1) \otimes L^\vee \subset W_{d_0-k}(C_0)$ for a general $L \in \text{Pic}^k(C_0)$. This is possible only for $d_0-k \geq g_0$, hence

\[ r+2 \leq k \leq d_0-g_0 < r, \text{ (because } \rho(g_0, r, d_0) = \rho(g, r, d) < 0), \]

a contradiction. Thus $(C_0, |\mathcal{O}_{C_0}(1)|, l_1)$ satisfies all conditions required by Proposition 3.1 which we can now apply a times to get a smooth curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^r$ of genus $g$ and bidegree $(k, d)$ such that $H^1(C, N_{C/\mathbb{P}^1 \times \mathbb{P}^r}) = 0$. The conclusion of Theorem 1 now follows.

In the special case $\rho(g, r, d) = -1$ we can extend the range of possible $g, r, d$ and $k$ for which there is a regular component:

**Theorem 2** Let $g, r, d, k$ be positive integers such that $r \geq 3$, $\rho(g, r, d) = -1$ and

\[ \frac{2r^2 + r + 1}{r - 1} \leq k \leq \frac{g + 2}{2}. \]

Then there exists a regular component of the stack of maps $\mathcal{M}_g(\mathbb{P}^1 \times \mathbb{P}^r, (k, d))$.

**Proof.** We find a solution $(g_0 = g-a(r+1), d_0 = d-ar)$ of the equation $\rho(g_0, r, d_0) = \rho(g, r, d)$ with $a \in \mathbb{Z}_{\geq 0}$ such that $d_0 \geq k + r$ and $\rho(g_0, 1, k) \geq r - 1$. Our numerical assumptions ensure that such an $a \geq 0$ exists. Note that in this case $k \leq g_0 + 1$, so we are not in the situation covered by Theorem 1.

It also follows that $-\frac{2r}{r} + \frac{r+1}{r} \leq -1 = \rho(g_0, r, d_0)$ and $d_0 \geq r + 1$, hence by Proposition 1.1 there exists an irreducible smooth open subset $U$ of $\mathcal{M}_g(\mathbb{P}^1, d_0)$ of the expected dimension, such that all points of $U$ correspond to embeddings of smooth curves $C \hookrightarrow \mathbb{P}^r$, with $h^1(C, N_C) = 0$, $h^0(C, \mathcal{O}_C(1)) = r + 1$ and $\mu_0(C)$ surjective.

Since we are in the case $\rho(g_0, r, d_0) = -1$, a combination of results by Eisenbud, Harris and Steffen gives that the Brill-Noether locus $M_{g_0, d_0}^r$ is an irreducible divisor in $M_{g_0}$ (see [St, Theorem 0.2]). It follows that the natural projection $\pi_2 : U \to M_{g_0, d_0}^r$ is dominant.

To apply Proposition 3.1 we now find a curve $[C_0] \in M_{g_0, d_0}^r$ having a complete base point free $g_0^1$ such that $2g_0^1$ is non-special. Then by semicontinuity we get that the general $[C] \in U$ also possesses a pencil $g_0^1$ with these properties. To find one particular such curve we proceed as follows: take $C_0$ a general $(r+1)$-gonal curve of genus $g_0$. These curves will have rather few moduli ($r+1 < \lceil (g+3)/2 \rceil$) but we still have that $[C_0] \in M_{g_0, d_0}^r$. Indeed, according to [CM] we can construct a $g_0^1 = [g_0^1 + F]$ on $C_0$, where $F$ is an effective divisor on $C_0$ with $h^0(C_0, F) = 1$. Since $k \leq g_0$, using Corollary 2.2.3 from [CKM] we find that $C_0$ also carries a complete base point free $g_0^1$, not composed with the $g_0^1$ computing $\text{gon}(C_0)$, and such that $2g_0^1$ is non-special. Since these are open conditions, they will hold generically along a component of $G_k^1(C_0)$. Applying semicontinuity, for a general element $[C] \in M_{g_0, d_0}^r$ (hence also for a general
element \([C] \in U\), the variety \(G^1_k(C)\) will contain a component \(A\) with general point \(l \in A\) being complete, base point free and with \(2l\) non-special.

We claim that for a general \(l \in A\) we have that \(|O_C(1)(-l)| = \emptyset\). Suppose not. Then if we denote by \(V^{r-1}_{d_0-k}(|O_C(1)|)\) the variety of effective divisors of degree \(d_0 - k\) on \(C\) imposing \(r - 1\) conditions on \(|O_C(1)|\), we obtain

\[
\dim V^{r-1}_{d_0-k}(|O_C(1)|) \geq \dim A \geq \rho(g, 1, k) \geq r - 1.
\]

Therefore \(C \subseteq \mathbb{P}^r\) has at least \(\infty^{r-1} (d_0 - k)\)-secant \((r - 2)\)-planes, hence also at least \(\infty^{r-1} r\)-secant \((r - 2)\)-planes (because \(d_0 - k \geq r\)). This last statement contradicts the Uniform Position Theorem (see [ACGH, p. 112]), hence the general point \([C] \in U\) enjoys all properties required to make Proposition 3.1 work. \(\blacksquare\)

**Remark.** From the proof of Theorem 2 the following question appears naturally: let us fix \(g, k\) such that \(g/2 + 1 \leq k \leq g\). One knows (cf. [ACGH]) that if \(l \in G^1_k(C)\) is a complete, base point free pencil then \(\dim T_l(G^1_k(C)) = \rho(g, 1, k) + h^1(C, 2l)\). Therefore if \(A\) is a component of \(G^1_k(C)\) such that \(\dim A = \rho(g, 1, k)\) and the general \(l \in A\) is base point free such that \(2l\) is special, then \(A\) is nonreduced. What is then the dimension of the locus

\[
V_{g,k} := \{[C] \in M_g : \text{ every component of } G^1_k(C) \text{ is nonreduced } \}.
\]

A result of Coppens (cf. [Co]) says that for a curve \(C\), if the scheme \(W^1_k(C)\) is reduced and of dimension \(\rho(g, 1, k)\), then the scheme \(W^1_{k+1}(C)\) is reduced too and of dimension \(\rho(g, 1, k+1)\). It would make then sense to determine \(\dim(V_{g,k})\) when \(\rho(g, 1, k) \in \{0, 1\}\) (depending on the parity of \(g\)). We suspect that \(V_{g,k}\) depends on very few moduli and if \(g\) is suitably large we expect that \(V_{g,k} = \emptyset\).

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University of Michigan, Department of Mathematics
East Hall, 525 East University, Ann Arbor, MI 48109-1109

e-mail: gfarkas@umich.edu