Lie bialgebra structures on the $W$-algebra $W(2, 2)$

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Abstract. Verma modules over the $W$-algebra $W(2, 2)$ were considered by Zhang and Dong, while the Harish-Chandra modules and irreducible weight modules over the same algebra were classified by Liu and Zhu etc. In the present paper we shall investigate the Lie bialgebra structures on the referred algebra, which are shown to be triangular coboundary.

Key words: Lie bialgebras, Yang-Baxter equation, $W$ algebra $W(2, 2)$.

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§1. Introduction

It is well known that the notion of Lie bialgebras was originally introduced by Drinfeld in 1983 (cf. [1]) during the search for the solutions of the Yang-Baxter quantum equation. During the recent years, there have appeared several papers on Lie bialgebras (e.g., [5] and [9–16]). Witt and Virasoro type Lie bialgebras were introduced in [15], of which type Lie bialgebras were further classified in [12], while the generalized case was considered in [14]. Lie bialgebra structure on generalized Virasoro-like and Block Lie algebras were investigated in [16] and [5] respectively.

In this paper we shall investigate Lie bialgebra structures on the $W$ algebra $W(2, 2)$ introduced in [17], denoted by $W$ here, which have been quantized in [6] by the authors using the method introduced in [3] and generalized in [4]. This algebra is an infinite-dimensional Lie algebra with a $\mathbb{C}$-basis $\{L_n, W_n, c | n \in \mathbb{Z}\}$ and the following Lie brackets (other components vanishing):

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c,$$  \hspace{0.5cm} (1.1)

$$[L_m, W_n] = (m - n)W_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c.$$  \hspace{0.5cm} (1.2)

The verma modules on $W$ were investigated in [17]. Later all irreducible weight modules with finite dimensional weight spaces and all indecomposable modules with less than one dimensional weight space on $W$ were classified in [8]. Meanwhile, irreducible weight modules possessing at least one nontrivial finite-dimensional weight space were also classified in [7].

Let us recall the definitions related to Lie bialgebras. For convenience, we introduce them being assort to the notation $L$. Let $L$ be any vector space over the complex field $\mathbb{C}$

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of characteristic zero. Denote by $\xi$ the cyclic map of $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$ cyclically permuting the coordinates, namely, $\xi(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_3 \otimes x_1$ for $x_1, x_2, x_3 \in \mathcal{L}$, and by $\tau$ the twist map of $\mathcal{L} \otimes \mathcal{L}$, i.e., $\tau(x \otimes y) = y \otimes x$ for $x, y \in \mathcal{L}$.

First one need to reformulate the definitions of a Lie algebra and Lie coalgebra as follows. A Lie algebra is a pair $(\mathcal{L}, \delta)$ of a vector space $\mathcal{L}$ and a linear map $\delta : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L}$ (the bracket of $\mathcal{L}$) satisfying the conditions:

$$\text{Ker}(1 - \tau) \subset \text{Ker} \delta,$$

$$\delta \cdot (1 \otimes \delta) \cdot (1 + \xi + \xi^2) = 0 : \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L},$$

A Lie coalgebra is a pair $(\mathcal{L}, \Delta)$ of a vector space $\mathcal{L}$ and a linear map $\Delta : \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{L}$ (the cobracket of $\mathcal{L}$) satisfying the conditions:

$$\text{Im} \Delta \subset \text{Im}(1 - \tau),$$

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta = 0 : \mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L},$$

For a Lie algebra $\mathcal{L}$, we always use $[x, y] = \delta(x, y)$ to denote its Lie bracket and use the symbol "·" to stand for the diagonal adjoint action

$$x \cdot (\sum_i a_i \otimes b_i) = \sum_i ([x, a_i] \otimes b_i + a_i \otimes [x, b_i]) \quad \text{for} \quad x, a_i, b_i \in \mathcal{L}.$$  

Definition 1.1. A Lie bialgebra is a triple $(\mathcal{L}, \delta, \Delta)$ satisfying the conditions:

$(\mathcal{L}, \delta)$ is a Lie algebra, $(\mathcal{L}, \Delta)$ is a Lie coalgebra,

$$\Delta \delta(x, y) = x \cdot \Delta y - y \cdot \Delta x \quad \text{for} \quad x, y \in \mathcal{L} \quad \text{(compatibility condition)}.$$  

Denote by $\mathcal{U}$ the universal enveloping algebra of $\mathcal{L}$ and by $1$ the identity element of $\mathcal{U}$. For any $r = \sum_i a_i \otimes b_i \in \mathcal{L} \otimes \mathcal{L}$, define $r^{ij}, c(r), i, j = 1, 2, 3$ to be elements of $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ by (where the bracket in (1.10) is the commutator):

$$r^{12} = \sum_i a_i \otimes b_i \otimes 1, \quad r^{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad r^{23} = \sum_i 1 \otimes a_i \otimes b_i,$$

$$c(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

Definition 1.2. (1) A coboundary Lie bialgebra is a 4-tuple $(\mathcal{L}, \delta, \Delta, r)$, where $(\mathcal{L}, \delta, \Delta)$ is a Lie bialgebra and $r \in \text{Im}(1 - \tau) \subset \mathcal{L} \otimes \mathcal{L}$ such that $\Delta = \Delta_r$ is a coboundary of $r$, where $\Delta_r$ is defined by

$$\Delta_r(x) = x \cdot r \quad \text{for} \quad x \in \mathcal{L}.$$  

(2) A coboundary Lie bialgebra $(\mathcal{L}, \delta, \Delta, r)$ is called triangular if it satisfies the following classical Yang-Baxter Equation (CYBE):

$$c(r) = 0.$$
The main result of this paper can be formulated as follows.

**Theorem 1.3.** Every Lie bialgebra structure on \( \mathcal{W} \) is triangular coboundary.

Throughout the paper, we denote by \( \mathbb{Z}^* \) the set of all nonzero integers, \( \mathbb{Z}_+ \) the set of all nonnegative integers and \( \mathbb{C}^* \) the set of all nonzero complex numbers.

§2. Proof of the main results

The following result can be found in [1, 2, 12].

**Lemma 2.1.** Let \( \mathcal{L} \) be a Lie algebra and \( r \in \text{Im}(1 - \tau) \subset \mathcal{L} \otimes \mathcal{L} \).

1. The triple \((\mathcal{L}, [\cdot, \cdot], \Delta_r)\) is a Lie bialgebra if and only if \( r \) satisfies CYBE (1.12).
2. We have
\[
(1 + \xi + \xi^2) \cdot (1 \otimes \Delta) \cdot \Delta(x) = x \cdot c(r) \text{ for all } x \in \mathcal{L}.
\]

**Lemma 2.2.** Let \( \mathcal{W}^n = \mathcal{W} \otimes \cdots \otimes \mathcal{W} \) be the tensor product of \( n \) copies of \( \mathcal{W} \), and regard \( \mathcal{W}^n \) as a \( \mathcal{W} \)-module under the adjoint diagonal action of \( \mathcal{W} \). Suppose \( r \in \mathcal{W}^n \) satisfying \( x \cdot r = 0 \) for all \( x \in \mathcal{W} \). Then \( r \in \mathbb{C}c^n \).

*Proof.* It can be proved directly by using the similar arguments as those presented in the proof of Lemma 2.2 of [16]. \( \Box \)

An element \( r \in \text{Im}(1 - \tau) \subset \mathcal{W} \otimes \mathcal{W} \) is said to satisfy the *modified Yang-Baxter equation* (MYBE) if
\[
x \cdot c(r) = 0 \text{ for all } x \in \mathcal{W}.
\]

As a conclusion of Lemma 2.2 one immediately obtains

**Corollary 2.3.** An element \( r \in \text{Im}(1 - \tau) \subset \mathcal{W} \otimes \mathcal{W} \) satisfies CYBE (1.12) if and only if it satisfies MYBE (2.2).

Regard \( \mathcal{V} = \mathcal{W} \otimes \mathcal{W} \) as a \( \mathcal{W} \)-module under the adjoint diagonal action. Denote by \( \text{Der}(\mathcal{W}, \mathcal{V}) \) the set of *derivations* \( D : \mathcal{W} \rightarrow \mathcal{V} \), namely, \( D \) is a linear map satisfying
\[
D([x, y]) = x \cdot D(y) - y \cdot D(x) \text{ for } x, y \in \mathcal{W},
\]
and \( \text{Inn}(\mathcal{W}, \mathcal{V}) \) the set consisting of the derivations \( v_{\text{inn}}, v \in \mathcal{V} \), where \( v_{\text{inn}} \) is the *inner derivation* defined by
\[
v_{\text{inn}} : x \mapsto x \cdot v \text{ for } x \in \mathcal{W}.
\]

Then it is well known that
\[
H^1(\mathcal{W}, \mathcal{V}) \cong \text{Der}(\mathcal{W}, \mathcal{V})/\text{Inn}(\mathcal{W}, \mathcal{V}),
\]
where \( H^1(\mathcal{W}, \mathcal{V}) \) the *first cohomology group* of the Lie algebra \( \mathcal{W} \) with coefficients in the \( \mathcal{W} \)-module \( \mathcal{V} \).
Proposition 2.4. $\text{Der}(\mathcal{W}, \mathcal{V}) = \text{Inn}(\mathcal{W}, \mathcal{V})$, equivalently, $H^1(\mathcal{W}, \mathcal{V}) = 0$.

Proof. Note that $\mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n$ and $\mathcal{V} = \mathcal{W} \otimes \mathcal{W} = \bigoplus_{n \in \mathbb{Z}} \mathcal{V}_n$ are $\mathbb{Z}$-graded with

$$\mathcal{W}_n = \text{Span}\{ L_n, W_n | n \in \mathbb{Z} \} \oplus \delta_{n,0} \mathcal{C}$$ and $\mathcal{V}_n = \sum_{p,q \in \mathbb{Z}, p+q=n} \mathcal{W}_p \otimes \mathcal{W}_q$ for $n \in \mathbb{Z}$. \hfill (2.6)

A derivation $D \in \text{Der}(\mathcal{W}, \mathcal{V})$ is homogeneous of degree $\alpha \in \mathbb{Z}$ if $D(\mathcal{W}_n) \subset \mathcal{W}_{\alpha+n}$ for all $n \in \mathbb{Z}$. Denote

$$\text{Der}(\mathcal{W}, \mathcal{V})_\alpha = \{ D \in \text{Der}(\mathcal{W}, \mathcal{V}) | \deg D = \alpha \} \quad \text{for} \quad \alpha \in \mathbb{Z}.$$

Let $D \in \text{Der}(\mathcal{W}, \mathcal{V})$. For any $\alpha \in \mathbb{Z}$, we define the linear map $D_\alpha : \mathcal{W} \to \mathcal{V}$ as follows: For any $\mu \in \mathcal{W}$ with $n \in \mathbb{Z}$, write $D(\mu) = \sum_{p \in \mathbb{Z}} \mu_p$ with $\mu_p \in \mathcal{V}_p$, then we set $D_\alpha(\mu) = \mu_{n+\alpha}$. Obviously, $D_\alpha \in \text{Der}(\mathcal{W}, \mathcal{V})_\alpha$ and we have

$$D = \sum_{\alpha \in \mathbb{Z}} D_\alpha,$$ \hfill (2.7)

which holds in the sense that for every $u \in \mathcal{W}$, only finitely many $D_\alpha(u) \neq 0$, and $D(u) = \sum_{\alpha \in \mathbb{Z}} D_\alpha(u)$ (we call such a sum in \hfill (2.7) summable).

We shall prove this proposition by several claims.

Claim 1. If $\alpha \in \mathbb{Z}^*$, then $D_\alpha \in \text{Inn}(\mathcal{W}, \mathcal{V})$.

For $\alpha \neq 0$, denote $\gamma = \alpha^{-1} D_\alpha(L_0) \in \mathcal{V}_\alpha$. Then for any $x_n \in \mathcal{W}_n$, applying $D_\alpha$ to $[L_0, x_n] = -nx_n$, using $D_\alpha(x_n) \in \mathcal{V}_{n+\alpha}$, we obtain

$$-(\alpha + n)D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = L_0 \cdot D_\alpha(x_n) - x_n \cdot D_\alpha(L_0) = -nD_\alpha(x_n),$$ \hfill (2.8)

i.e., $D_\alpha(x_n) = \gamma_{\text{inn}}(x_n)$. Thus $D_\alpha = \gamma_{\text{inn}}$ is inner.

For convenience, we always use “$\equiv$” to denote equal modulo $\mathbb{C}(c \otimes c)$ in the following.

Claim 2. $D_0(L_0) \equiv D_0(c) \equiv 0$.

For any $n \in \mathbb{Z}$ and $x_n \in \mathcal{W}_n$, applying $D_0$ to $[L_0, x_n] = -nx_n$ and $[x_n, c] = 0$ respectively, one has $x_n \cdot D_0(L_0) = x_n \cdot c = 0$. Thus by Lemma \hfill (2.2) $D_0(L_0) \equiv D_0(c) \equiv 0$.

Claim 3. Replacing $D_0$ by $D_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $D_0(\mathcal{W}) \equiv 0$.

For any $m \in \mathbb{Z}^*$, $n \in \mathbb{Z}$, one can write $D_0(L_m)$ and $D_0(W_n)$ as follows

$$D_0(L_m) \equiv \sum_{p \in \mathbb{Z}} a_{m,p} L_p \otimes L_{m-p} + \sum_{p \in \mathbb{Z}} b_{m,p} L_p \otimes W_{m-p} + a_m L_m \otimes c + b_m c \otimes L_m$$
$$+ \sum_{p \in \mathbb{Z}} c_{m,p} W_p \otimes L_{m-p} + \sum_{p \in \mathbb{Z}} d_{m,p} W_p \otimes W_{m-p} + c_m W_m \otimes c + d_m c \otimes W_m,$$ \hfill (2.9)

$$D_0(W_n) \equiv \sum_{p \in \mathbb{Z}} e_{n,p} L_p \otimes L_{n-p} + \sum_{p \in \mathbb{Z}} f_{n,p} L_p \otimes W_{n-p} + e_n L_n \otimes c + f_n c \otimes L_n$$
$$+ \sum_{p \in \mathbb{Z}} g_{n,p} W_p \otimes L_{n-p} + \sum_{p \in \mathbb{Z}} h_{n,p} W_p \otimes W_{n-p} + g_n W_n \otimes c + h_n c \otimes W_n,$$ \hfill (2.10)
where all the coefficients of the tensor products are complex numbers, and the sums are all finite. For any $p \in \mathbb{Z}$, the following identities hold:

\[
L_1 \cdot (L_0 \otimes c) = L_1 \otimes c, \quad L_1 \cdot (c \otimes L_0) = c \otimes L_1,
\]

\[
L_1 \cdot (W_0 \otimes c) = W_1 \otimes c, \quad L_1 \cdot (c \otimes W_0) = c \otimes W_1,
\]

\[
L_1 \cdot (L_p \otimes L_{-p}) = (1 - p)L_{p+1} \otimes L_{-p} + (1 + p)L_p \otimes L_{1-p},
\]

\[
L_1 \cdot (L_p \otimes W_{-p}) = (1 - p)L_{p+1} \otimes W_{-p} + (1 + p)L_p \otimes W_{1-p},
\]

\[
L_1 \cdot (W_p \otimes L_{-p}) = (1 - p)W_{p+1} \otimes L_{-p} + (1 + p)W_p \otimes L_{1-p},
\]

\[
L_1 \cdot (W_p \otimes W_{-p}) = (1 - p)W_{p+1} \otimes W_{-p} + (1 + p)W_p \otimes W_{1-p}.
\]

Denote

\[
M_1 = \max\{|p| \mid a_{1,p} \neq 0\}, \quad M_2 = \max\{|p| \mid b_{1,p} \neq 0\},
\]

\[
M_3 = \max\{|p| \mid c_{1,p} \neq 0\}, \quad M_4 = \max\{|p| \mid d_{1,p} \neq 0\}.
\]

Using the induction on $\sum_{i=1}^4 M_i$ in the above identities, and replacing $D_0$ by $D_0 - u_{\text{inn}}$, where $u$ is a combination of some $L_p \otimes L_{-p}$, $L_p \otimes W_{-p}$, $W_p \otimes L_{-p}$, $W_p \otimes W_{-p}$, $L_1 \otimes c$, $c \otimes L_1$, $W_p \otimes c$ and $c \otimes W_1$, one can safely suppose

\[
a_1 = b_1 = c_1 = d_1 = 0,
\]

\[
a_{1,p} = b_{1,p} = c_{1,p} = d_{1,p} = 0 \quad \text{if} \quad p \neq -1, 2.
\]

Thus the expression of $D_0(L_1)$ can be simplified as (recalling Claim 2)

\[
D_0(L_1) \equiv a_{1,-1}L_{-1} \otimes L_2 + a_{1,2}L_2 \otimes L_{-1} + b_{1,-1}L_{-1} \otimes W_2 + b_{1,2}L_2 \otimes W_{-1}
\]

\[
+ c_{1,-1}W_{-1} \otimes L_2 + c_{1,2}W_2 \otimes L_{-1} + d_{1,-1}W_{-1} \otimes W_2 + d_{1,2}W_2 \otimes W_{-1}. \quad (2.11)
\]

Applying $D_0$ to $[L_{-1}, L_1] = 2L_0$, under modulo $\mathbb{C}(c \otimes c)$, we obtain

\[
\sum_{p \in \mathbb{Z}} ((2 - p)a_{-1,p-1} + (2 + p)a_{-1,p})L_p \otimes L_{-p} + 3a_{1,-1}L_{-1} \otimes L_1 + 3a_{1,2}L_1 \otimes L_{-1}
\]

\[
+ \sum_{p \in \mathbb{Z}} ((2 - p)b_{-1,p-1} + (2 + p)b_{-1,p})L_p \otimes W_{-p} + 3b_{1,-1}L_{-1} \otimes W_1 + 3b_{1,2}L_1 \otimes W_{-1}
\]

\[
+ \sum_{p \in \mathbb{Z}} ((2 - p)c_{-1,p-1} + (2 + p)c_{-1,p})W_p \otimes L_{-p} + 3c_{1,-1}W_{-1} \otimes L_1 + 3c_{1,2}W_1 \otimes L_{-1}
\]

\[
+ \sum_{p \in \mathbb{Z}} ((2 - p)d_{-1,p-1} + (2 + p)d_{-1,p})W_p \otimes W_{-p} + 3d_{1,-1}W_{-1} \otimes W_1 + 3d_{1,2}W_1 \otimes W_{-1}
\]

\[
+ 2a_{-1}L_0 \otimes c + 2b_{-1}c \otimes L_0 + 2c_{-1}W_0 \otimes c + 2d_{-1}c \otimes W_0 = 0.
\]

For any $p \in \mathbb{Z}$, comparing the coefficients of $L_p \otimes L_{-p}$, $L_p \otimes W_{-p}$, $W_p \otimes L_{-p}$, $W_p \otimes W_{-p}$,
\[ L_0 \otimes c, c \otimes L_0, W_0 \otimes c \text{ and } c \otimes W_0 \text{ respectively in the above equation, on has} \]
\[ a_{-1} = b_{-1} = c_{-1} = d_{-1} = 0, \]
\[ 3a_{1,2} + a_{-1,0} + 3a_{-1,1} = 3b_{1,2} + b_{-1,0} + 3b_{-1,1} = 0, \]
\[ 3c_{1,2} + c_{-1,0} + 3c_{-1,1} = 3d_{1,2} + d_{-1,0} + 3d_{-1,1} = 0, \]
\[ 3a_{1,-1} + 3a_{-1,2} + a_{-1,-1} = 3b_{1,-1} + 3b_{-1,2} + b_{-1,-1} = 0, \]
\[ 3c_{1,-1} + 3c_{-1,2} + c_{-1,-1} = 3d_{1,-1} + 3d_{-1,2} + d_{-1,-1} = 0, \]
\[ (p-2)a_{-1,p-1} - (p+2)a_{-1,p} = (p-2)b_{-1,p-1} - (p+2)b_{-1,p} = 0, \quad p \neq \pm 1, \]
\[ (p-2)c_{-1,p-1} - (p+2)c_{-1,p} = (p-2)d_{-1,p-1} - (p+2)d_{-1,p} = 0, \quad p \neq \pm 1, \]

which give the following identities:
\[ 3a_{1,2} + a_{-1,0} + 3a_{-1,1} = 3b_{1,2} + b_{-1,0} + 3b_{-1,1} = 0, \]
\[ 3c_{1,2} + c_{-1,0} + 3c_{-1,1} = 3d_{1,2} + d_{-1,0} + 3d_{-1,1} = 0, \]
\[ 3a_{1,-1} + 3a_{-1,2} - a_{-1,0} = 3b_{1,-1} + 3b_{-1,2} - b_{-1,0} = 0, \]
\[ 3c_{1,-1} + 3c_{-1,2} - c_{-1,0} = 3d_{1,-1} + 3d_{-1,2} - d_{-1,0} = 0, \]
\[ a_{-1,p} = b_{-1,p} = c_{-1,p} = d_{-1,p} = 0, \quad \forall \ p \in \mathbb{Z}, \ p \neq -2, -1, 0, 1, \]
\[ a_{-1,-1} + a_{-1,0} = b_{-1,-1} + b_{-1,0} = c_{-1,-1} + c_{-1,0} = d_{-1,-1} + d_{-1,0} = 0. \]

Thus \( D_0(L_{-1}) \) and \( D_0(L_1) \) can respectively be rewritten as
\[
D_0(L_{-1}) \equiv a_{-1,-2}L_{-2} \otimes L_1 - a_{-1,0}L_{-1} \otimes L_0 + a_{-1,0}L_0 \otimes L_{-1} + a_{-1,1}L_1 \otimes L_{-2}
+b_{-1,-2}L_{-2} \otimes W_1 - b_{-1,0}L_{-1} \otimes W_0 + b_{-1,0}L_0 \otimes W_{-1} + b_{-1,1}L_1 \otimes W_{-2}
+c_{-1,-2}W_{-2} \otimes L_1 - c_{-1,0}W_{-1} \otimes L_0 + c_{-1,0}W_0 \otimes L_{-1} + c_{-1,1}W_1 \otimes L_{-2}
+d_{-1,-2}W_{-2} \otimes W_1 - d_{-1,0}W_{-1} \otimes W_0 + d_{-1,0}W_0 \otimes W_{-1} + d_{-1,1}W_1 \otimes W_{-2},
\]
\[
D_0(L_1) \equiv \left( \frac{a_{-1,0}}{3} - a_{-1,-2} \right)L_{-1} \otimes L_2 - \left( \frac{a_{-1,0}}{3} \right)L_2 \otimes L_{-1}
+ \left( \frac{b_{-1,0}}{3} - b_{-1,-2} \right)L_{-1} \otimes W_2 - \left( \frac{b_{-1,0}}{3} \right)L_2 \otimes W_{-1}
+ \left( \frac{c_{-1,0}}{3} - c_{-1,-2} \right)W_{-1} \otimes L_2 - \left( \frac{c_{-1,0}}{3} \right)W_2 \otimes L_{-1}
+ \left( \frac{d_{-1,0}}{3} - d_{-1,-2} \right)W_{-1} \otimes W_2 - \left( \frac{d_{-1,0}}{3} \right)W_2 \otimes W_{-1}.
\]

Applying \( D_0 \) to \([ L_2, L_{-1} ] = 3L_1 \), under modulo \( \mathbb{C}(c \otimes c) \), we obtain
\[ 2a_{-1,0}L_2 \otimes L_{-1} + 3a_{-1,0}L_0 \otimes L_1 + a_{-1,1}L_3 \otimes L_{-2} + 4a_{-1,1}L_1 \otimes L_0
+ 4a_{-1,-2}L_0 \otimes L_1 + a_{-1,-2}L_{-2} \otimes L_3 - 3a_{-1,0}L_1 \otimes L_0 - 2a_{-1,0}L_{-1} \otimes L_2
+ 2b_{-1,0}L_2 \otimes W_{-1} + 3b_{-1,0}L_0 \otimes W_1 + b_{-1,1}L_3 \otimes W_{-2} + 4b_{-1,1}L_1 \otimes W_0
+ 4b_{-1,-2}L_0 \otimes W_1 + b_{-1,-2}L_{-2} \otimes W_3 - 3b_{-1,0}L_1 \otimes W_0 - 2b_{-1,-1}L_{-1} \otimes W_2\]
\[+ 2c_{-1,0}W_2 \otimes L_{-1} + 3c_{-1,0}W_0 \otimes L_1 + c_{-1,1}W_3 \otimes L_{-2} + 4c_{-1,1}W_1 \otimes L_0 \]
\[+ 4d_{-1,-2}W_0 \otimes W_1 + d_{-1,-2}W_2 \otimes W_3 - 3d_{-1,0}W_1 \otimes W_0 - 2d_{-1,0}W_{-1} \otimes W_2 \]
\[+ 4c_{-1,-2}W_0 \otimes L_1 + c_{-1,-2}W_2 \otimes L_3 - 3c_{-1,0}W_1 \otimes L_0 - 2c_{-1,0}W_{-1} \otimes L_2 \]
\[+ 2d_{-1,0}W_2 \otimes W_{-1} + 3d_{-1,0}W_0 \otimes W_1 + d_{-1,1}W_3 \otimes W_{-2} + 4d_{-1,1}W_1 \otimes W_0 \]
\[- (a_{-1,0} - 3a_{-1,2})L_1 \otimes L_2 + (a_{-1,0} + 3a_{-1,1})L_2 \otimes L_1 - (b_{-1,0} - 3b_{-1,2})L_{-1} \otimes L_2 \]
\[+ (b_{-1,0} + 3b_{-1,1})L_2 \otimes W_{-1} - (c_{-1,0} - 3c_{-1,2})W_{-1} \otimes L_2 + (c_{-1,0} + 3c_{-1,1})W_2 \otimes L_{-1} \]
\[- (d_{-1,0} - 3d_{-1,2})W_{-1} \otimes W_2 + (d_{-1,0} + 3d_{-1,1})W_2 \otimes W_{-1} + \sum_{p \in \mathbb{Z}} (1 + p)a_{2,p}L_{p-1} \otimes L_{2-p} \]
\[+ \sum_{p \in \mathbb{Z}} (3 - p)a_{2,p}L_p \otimes L_{1-p} + \sum_{p \in \mathbb{Z}} (1 + p)b_{2,p}W_{p-1} \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)b_{2,p}L_p \otimes W_{1-p} \]
\[+ \sum_{p \in \mathbb{Z}} (1 + p)c_{2,p}W_{p-1} \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)c_{2,p}W_p \otimes L_{1-p} + \sum_{p \in \mathbb{Z}} (1 + p)d_{2,p}W_{p-1} \otimes W_{2-p} \]
\[+ \sum_{p \in \mathbb{Z}} (3 - p)d_{2,p}W_p \otimes W_{1-p} + 3a_2L_1 \otimes c + 3b_2c \otimes L_1 + 3c_2W_1 \otimes c + 3d_2c \otimes W_1 = 0. \]

For any \( p \in \mathbb{Z} \), comparing the coefficients of \( L_1 \otimes c \), \( c \otimes L_1 \), \( W_1 \otimes c \), \( c \otimes W_1 \), \( L_p \otimes L_{1-p} \), \( L_p \otimes W_{1-p} \), \( W_p \otimes L_{1-p} \) and \( W_p \otimes W_{1-p} \) respectively in the above equation, we firstly obtain
\[a_2L_1 \otimes c = b_2c \otimes L_1 = c_2W_1 \otimes c = d_2c \otimes W_1 = 0,\]
\[\sum_{p \in \mathbb{Z}} (1 + p)a_{2,p}L_{p-1} \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)a_{2,p}L_p \otimes L_{1-p} \]
\[+ a_{-1,-2}L_{-2} \otimes L_3 + 3(a_{-1,-2} - a_{-1,0})L_{-1} \otimes L_2 + (4a_{-1,-2} + 3a_{-1,0})L_0 \otimes L_1 \]
\[+ (4a_{-1,1} - 3a_{-1,0})L_1 \otimes L_0 + 3(a_{-1,0} + a_{-1,1})L_2 \otimes L_{-1} + a_{-1,1}L_3 \otimes L_{-2} = 0,\]
\[\sum_{p \in \mathbb{Z}} (1 + p)b_{2,p}L_{p-1} \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)b_{2,p}L_p \otimes W_{1-p} \]
\[+ b_{-1,-2}L_{-2} \otimes W_3 + 3(b_{-1,-2} - b_{-1,0})L_{-1} \otimes W_2 + (4b_{-1,-2} + 3b_{-1,0})L_0 \otimes W_1 \]
\[+ (4b_{-1,1} - 3b_{-1,0})L_1 \otimes W_0 + 3(b_{-1,0} + b_{-1,1})L_2 \otimes W_{-1} + b_{-1,1}L_3 \otimes W_{-2} = 0,\]
\[\sum_{p \in \mathbb{Z}} (1 + p)c_{2,p}W_{p-1} \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)c_{2,p}W_p \otimes L_{1-p} \]
\[+ c_{-1,-2}W_{-2} \otimes L_3 + 3(c_{-1,-2} - c_{-1,0})W_{-1} \otimes L_2 + (4c_{-1,-2} + 3c_{-1,0})W_0 \otimes L_1 \]
\[+ (4c_{-1,1} - 3c_{-1,0})W_1 \otimes L_0 + 3(c_{-1,0} + c_{-1,1})W_2 \otimes L_{-1} + c_{-1,1}W_3 \otimes L_{-2} = 0,\]
\[\sum_{p \in \mathbb{Z}} (1 + p)a_{2,p}W_{p-1} \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} (3 - p)a_{2,p}W_p \otimes W_{1-p} \]
\[+ a_{-1,-2}W_{-2} \otimes W_3 + 3(a_{-1,-2} - a_{-1,0})W_{-1} \otimes W_2 + (4a_{-1,-2} + 3a_{-1,0})W_0 \otimes W_1 \]
\[+ (4a_{-1,1} - 3a_{-1,0})W_1 \otimes W_0 + 3(a_{-1,0} + a_{-1,1})W_2 \otimes W_{-1} + a_{-1,1}W_3 \otimes W_{-2} = 0,\]

Then for any \( p \in \mathbb{Z} \), \( p \neq -2, -1, 0, 1, 2, 3 \), one has
\[a_2 = b_2 = c_2 = d_2 = 0,\]
a_{-1,-2} + 5a_{2,-2} = a_{-1,1} + 5a_{2,1} = (p + 2)a_{2,p+1} - (p - 3)a_{2,p} = 0,
4a_{-1,1} - 3a_{1,0} + 3a_{2,2} + 2a_{2,1} = 3(a_{-1,0} + a_{1,1}) + 4a_{2,3} + a_{2,2} = 0,
3a_{-1,-2} - 3a_{-1,0} + a_{2,0} + 4a_{2,-1} = 4a_{-1,-2} + 3a_{-1,0} + 2a_{2,1} + 3a_{2,0} = 0,
b_{-1,-2} + 5b_{2,-2} = b_{-1,1} + 5b_{2,1} = (p + 2)b_{2,p+1} - (p - 3)b_{2,p} = 0,
4b_{-1,1} - 3b_{1,0} + 3b_{2,2} + 2b_{2,1} = 3(b_{-1,0} + b_{1,1}) + 4b_{2,3} + b_{2,2} = 0,
3b_{-1,-2} - 3b_{-1,0} + b_{2,0} + 4b_{2,-1} = 4b_{-1,-2} + 3b_{-1,0} + 2b_{2,1} + 3b_{2,0} = 0,
c_{-1,-2} + 5c_{2,-2} = c_{-1,1} + 5c_{2,1} = (p + 2)c_{2,p+1} - (p - 3)c_{2,p} = 0,
4c_{-1,1} - 3c_{1,0} + 3c_{2,2} + 2c_{2,1} = 3(c_{-1,0} + c_{1,1}) + 4c_{2,3} + c_{2,2} = 0,
3c_{-1,-2} - 3c_{-1,0} + c_{2,0} + 4c_{2,-1} = 4c_{-1,-2} + 3c_{-1,0} + 2c_{2,1} + 3c_{2,0} = 0,
d_{-1,-2} + 5d_{2,-2} = d_{-1,1} + 5d_{2,1} = (p + 2)d_{2,p+1} - (p - 3)d_{2,p} = 0,
4d_{-1,1} - 3d_{1,0} + 3d_{2,2} + 2d_{2,1} = 3(d_{-1,0} + d_{1,1}) + 4d_{2,3} + d_{2,2} = 0,
3d_{-1,-2} - 3d_{-1,0} + d_{2,0} + 4d_{2,-1} = 4d_{-1,-2} + 3d_{-1,0} + 2d_{2,1} + 3d_{2,0} = 0,

which together give the following identities:

\[ a_{-1,-2} = -5a_{2,-2} = a_{2,p} = 0, \quad a_{2,-1} = \frac{1}{4}(3a_{-1,0} - a_{2,0}), \quad a_{2,1} = -\frac{3}{2}(a_{2,0} + a_{-1,0}), \]
\[ a_{2,4} = -\frac{1}{5}a_{-1,1}, \quad a_{2,2} = a_{2,0} + 2a_{-1,0} - \frac{4}{3}a_{-1,1}, \quad a_{2,3} = -\frac{1}{4}a_{2,0} - \frac{5}{4}a_{-1,0} - \frac{5}{12}a_{-1,1}, \]
\[ b_{-1,-2} = -5b_{2,-2} = b_{2,p} = 0, \quad b_{2,-1} = \frac{1}{4}(3b_{-1,0} - b_{2,0}), \quad b_{2,1} = -\frac{3}{2}(b_{2,0} + b_{-1,0}), \]
\[ b_{2,4} = -\frac{1}{5}b_{-1,1}, \quad b_{2,2} = b_{2,0} + 2b_{-1,0} - \frac{4}{3}b_{-1,1}, \quad b_{2,3} = -\frac{1}{4}b_{2,0} - \frac{5}{4}b_{-1,0} - \frac{5}{12}b_{-1,1}, \]
\[ c_{-1,-2} = -5c_{2,-2} = c_{2,p} = 0, \quad c_{2,-1} = \frac{1}{4}(3c_{-1,0} - c_{2,0}), \quad c_{2,1} = -\frac{3}{2}(c_{2,0} + c_{-1,0}), \]
\[ c_{2,4} = -\frac{1}{5}c_{-1,1}, \quad c_{2,2} = c_{2,0} + 2c_{-1,0} - \frac{4}{3}c_{-1,1}, \quad c_{2,3} = -\frac{1}{4}c_{2,0} - \frac{5}{4}c_{-1,0} - \frac{5}{12}c_{-1,1}, \]
\[ d_{-1,-2} = -5d_{2,-2} = d_{2,p} = 0, \quad d_{2,-1} = \frac{1}{4}(3d_{-1,0} - d_{2,0}), \quad d_{2,1} = -\frac{3}{2}(d_{2,0} + d_{-1,0}), \]
\[ d_{2,4} = -\frac{1}{5}d_{-1,1}, \quad d_{2,2} = d_{2,0} + 2d_{-1,0} - \frac{4}{3}d_{-1,1}, \quad d_{2,3} = -\frac{1}{4}d_{2,0} - \frac{5}{4}d_{-1,0} - \frac{5}{12}d_{-1,1}, \]

for any \( p \in \mathbb{Z}, \ p \neq -1, 0, 1, 2, 3 \). Then one can rewrite \( D_0(L_{-1}) \), \( D_0(L_1) \) and \( D_0(L_2) \) respectively as

\[
D_0(L_{-1}) \equiv a_{-1,0}(L_0 \otimes L_{-1} - L_{-1} \otimes L_0) + b_{-1,0}(L_0 \otimes W_{-1} - L_{-1} \otimes W_0) + c_{-1,0}(W_0 \otimes L_{-1} - W_{-1} \otimes L_0) + d_{-1,0}(W_0 \otimes W_{-1} - W_{-1} \otimes W_0),
\]
\[
D_0(L_1) \equiv \frac{a_{-1,0}}{3}(L_{-1} \otimes L_2 - L_2 \otimes L_{-1}) + \frac{b_{-1,0}}{3}(L_{-1} \otimes W_2 - L_2 \otimes W_{-1})
+ \frac{c_{-1,0}}{3}(W_{-1} \otimes L_2 - W_2 \otimes L_{-1}) + \frac{d_{-1,0}}{3}(W_{-1} \otimes W_2 - W_2 \otimes W_{-1}),
\]

\[
D_0(L_2) \equiv \frac{1}{4}(3a_{-1,0} - a_{2,0})L_{-1} \otimes L_3 + a_{2,0}L_0 \otimes L_2 - \frac{3}{2}(a_{2,0} + a_{-1,0})L_1 \otimes L_1
+ (a_{2,0} + 2a_{-1,0})L_2 \otimes L_0 - \frac{1}{4}(a_{2,0} + 5a_{-1,0})L_3 \otimes L_{-1}
+ \frac{1}{4}(3b_{-1,0} - b_{2,0})L_{-1} \otimes W_3 + b_{2,0}L_0 \otimes W_2 - \frac{3}{2}(b_{2,0} + b_{-1,0})L_1 \otimes W_1
+ (b_{2,0} + 2b_{-1,0})L_2 \otimes W_0 - \frac{1}{4}(b_{2,0} + 5b_{-1,0})L_3 \otimes W_{-1}
+ \frac{1}{4}(3c_{-1,0} - c_{2,0})W_{-1} \otimes L_3 + c_{2,0}W_0 \otimes L_2 - \frac{3}{2}(c_{2,0} + c_{-1,0})W_1 \otimes L_1
+ (c_{2,0} + 2c_{-1,0})W_2 \otimes L_0 - \frac{1}{4}(c_{2,0} + 5c_{-1,0})W_3 \otimes L_{-1}
+ \frac{1}{4}(3d_{-1,0} - d_{2,0})W_{-1} \otimes W_3 + d_{2,0}W_0 \otimes W_2 - \frac{3}{2}(d_{2,0} + d_{-1,0})W_1 \otimes W_1
+ (d_{2,0} + 2d_{-1,0})W_2 \otimes W_0 - \frac{1}{4}(d_{2,0} + 5d_{-1,0})W_3 \otimes W_{-1}.
\]

Applying \(D_0\) to \([L_1, L_{-2}] = 3L_{-1}\), under modulo \(\mathbb{C}(c \otimes c)\), we obtain

\[
\sum_{p \in \mathbb{Z}} 3(1 - p)a_{-2,p}L_{p+1} \otimes L_{-2-p} + 3 \sum_{p \in \mathbb{Z}} (3 + p)a_{-2,p}L_p \otimes L_{-1-p}
+ 3 \sum_{p \in \mathbb{Z}} (1 - p)b_{-2,p}L_{1+p} \otimes W_{-2-p} + 3 \sum_{p \in \mathbb{Z}} (3 + p)b_{-2,p}L_p \otimes W_{-1-p}
+ 3 \sum_{p \in \mathbb{Z}} (1 - p)c_{-2,p}W_{1+p} \otimes L_{-2-p} + 3 \sum_{p \in \mathbb{Z}} (3 + p)c_{-2,p}W_p \otimes L_{-1-p}
+ 3 \sum_{p \in \mathbb{Z}} (1 - p)d_{-2,p}W_{1+p} \otimes W_{-2-p} + 3 \sum_{p \in \mathbb{Z}} (3 + p)d_{-2,p}W_p \otimes W_{-1-p}
+ 9a_{-2}L_{-1} \otimes c + 9b_{-2}c \otimes L_{-1} + 9c_{-2}W_{-1} \otimes c + 9d_{-2}c \otimes W_{-1}
- 13a_{-1,0}L_0 \otimes L_{-1} - a_{1,0}L_2 \otimes L_{-3} + a_{-1,0}L_3 \otimes L_2 + 13a_{-1,0}L_{-1} \otimes L_0
- 13b_{-1,0}L_0 \otimes W_{-1} - b_{1,0}L_2 \otimes W_{-3} + b_{-1,0}L_3 \otimes W_2 + 13b_{-1,0}L_{-1} \otimes W_0
- 13c_{-1,0}W_0 \otimes L_{-1} - c_{1,0}W_2 \otimes L_{-3} + c_{-1,0}W_3 \otimes L_2 + 13c_{-1,0}W_{-1} \otimes L_0
- 13d_{-1,0}W_0 \otimes W_{-1} - d_{1,0}W_2 \otimes W_{-3} + d_{-1,0}W_3 \otimes W_2 + 13d_{-1,0}W_{-1} \otimes W_0
- 9a_{-1,0}L_0 \otimes L_{-1} + 9a_{-1,0}L_{-1} \otimes L_0 - 9b_{-1,0}L_0 \otimes W_{-1} + 9b_{-1,0}L_{-1} \otimes W_0
- 9c_{-1,0}W_0 \otimes L_{-1} + 9c_{-1,0}W_{-1} \otimes L_0 - 9d_{-1,0}W_0 \otimes W_{-1} + 9d_{-1,0}W_{-1} \otimes W_0 = 0.
\]

For any \(p \in \mathbb{Z}\), comparing the coefficients of \(L_p \otimes L_{-1-p}\), \(L_p \otimes W_{-1-p}\), \(W_p \otimes L_{-1-p}\), \(W_p \otimes W_{-1-p}\)
\( L_{-1} \otimes c, c \otimes L_{-1}, \ W_{-1} \otimes c \text{ and } c \otimes W_{-1} \) respectively in the above equation, one has
\[
(p - 2)a_{-2,p - 1} - (p + 3)a_{-2,p} = 0,
\]
\[
a_{-2} = b_{-2} = c_{-2} = d_{-2} = 9a_{-2,-2} + 6a_{-2,-1} + 22a_{-1,0} = 0,
\]
\[
6a_{-2,-1} + 9a_{-2,0} - 22a_{-1,0} = 15a_{-2,2} - a_{-1,0} = 15a_{-2,-4} + a_{-1,0} = 0,
\]
\[
(p - 2)b_{-2,p - 1} - (p + 3)b_{-2,p} = 9b_{-2,-2} + 6b_{-2,-1} + 22b_{-1,0} = 0,
\]
\[
6b_{-2,-1} + 9b_{-2,0} - 22b_{-1,0} = 15b_{-2,2} - b_{-1,0} = 15b_{-2,-4} + b_{-1,0} = 0,
\]
\[
(p - 2)c_{-2,p - 1} - (p + 3)c_{-2,p} = 9c_{-2,-2} + 6c_{-2,-1} + 22c_{-1,0} = 0,
\]
\[
6c_{-2,-1} + 9c_{-2,0} - 22c_{-1,0} = 15c_{-2,2} - c_{-1,0} = 15c_{-2,-4} + c_{-1,0} = 0,
\]
\[
(p - 2)d_{-2,p - 1} - (p + 3)d_{-2,p} = 9d_{-2,-2} + 6d_{-2,-1} + 22d_{-1,0} = 0,
\]
\[
6d_{-2,-1} + 9d_{-2,0} - 22d_{-1,0} = 15d_{-2,2} - d_{-1,0} = 15d_{-2,-4} + d_{-1,0} = 0,
\]
for any \( p \in \mathbb{Z}, \neq -3, -1, 0, 2 \), which together force
\[
a_{-1,0} = a_{-2,p} = b_{-2,p} = c_{-2,p} = d_{-2,p} = 0 \ \forall \ p \in \mathbb{Z}, \ p \neq 1,
\]
Thus \( D_0(L_{-1}), D_0(L_1), D_0(L_{-1}) \) and \( D_0(L_1) \) can respectively be rewritten as
\[
D_0(L_{-1}) \equiv D_0(L_1) \equiv 0,
\]
\[
D_0(L_{-2}) \equiv a_{-2,1}L_{1} \otimes L_{-3} + b_{-2,1}L_{1} \otimes W_{-3} + c_{-2,1}W_{1} \otimes L_{-3} + d_{-2,1}W_{1} \otimes W_{-3},
\]
\[
D_0(L_2) \equiv \frac{1}{4}a_{2,0}L_{-1} \otimes L_{3} + a_{2,0}L_{0} \otimes L_{2} - \frac{3}{2}a_{2,0}L_{1} \otimes L_{1} + a_{2,0}L_{2} \otimes L_{0}
\]
\[
- \frac{1}{4}a_{2,0}L_{3} \otimes L_{1} - \frac{1}{4}b_{2,0}L_{-1} \otimes W_{3} + b_{2,0}L_{0} \otimes W_{2} - \frac{3}{2}b_{2,0}L_{1} \otimes W_{1}
\]
\[
+ b_{2,0}L_{2} \otimes W_{0} - \frac{1}{4}b_{2,0}L_{3} \otimes W_{-1} - \frac{1}{4}c_{2,0}W_{1} \otimes L_{3} + c_{2,0}W_{0} \otimes L_{2}
\]
\[
- \frac{3}{2}c_{2,0}W_{1} \otimes L_{1} + c_{2,0}W_{2} \otimes L_{0} - \frac{1}{4}c_{2,0}W_{3} \otimes L_{-1} - \frac{1}{4}d_{2,0}W_{-1} \otimes W_{3}
\]
\[
+ d_{2,0}W_{0} \otimes W_{2} - \frac{3}{2}d_{2,0}W_{1} \otimes W_{1} + d_{2,0}W_{2} \otimes W_{0} - \frac{1}{4}d_{2,0}W_{3} \otimes W_{-1}.
\]
Applying \( D_0 \) to \( [L_1, L_{-2}] = 3L_{-1} \), under modulo \( \mathbb{C}(c \otimes c) \), we obtain
\[
a_{2,0}L_{-3} \otimes L_{3} - 8a_{2,0}L_{-2} \otimes L_{2} + 39a_{2,0}L_{-1} \otimes L_{1} - 32a_{2,0}L_{0} \otimes L_{0}
\]
\[
+(27a_{2,0} - 20a_{2,-1})L_{1} \otimes L_{-1} - 8a_{2,0}L_{2} \otimes L_{-2} + (a_{2,0} - 4a_{2,-1})L_{3} \otimes L_{-3}
\]
\[
b_{2,0}L_{-3} \otimes W_{3} - 8b_{2,0}L_{-2} \otimes W_{2} + 39b_{2,0}L_{-1} \otimes W_{1} - 32b_{2,0}L_{0} \otimes W_{0}
\]
\[
+(27b_{2,0} - 20b_{2,-1})L_{1} \otimes W_{-1} - 8b_{2,0}L_{2} \otimes W_{-2} + (b_{2,0} - 4b_{2,-1})L_{3} \otimes W_{-3}
\]
\[
c_{2,0}W_{-3} \otimes L_{3} - 8c_{2,0}W_{-2} \otimes L_{2} + 39c_{2,0}W_{-1} \otimes L_{1} - 32c_{2,0}W_{0} \otimes L_{0}
\]
\[
+(27c_{2,0} - 20c_{2,-1})W_{1} \otimes W_{-1} - 8c_{2,0}W_{2} \otimes W_{-2} + (c_{2,0} - 4c_{2,-1})W_{3} \otimes W_{-3}
\]
\[
d_{2,0}W_{-3} \otimes W_{3} - 8d_{2,0}W_{-2} \otimes W_{2} + 39d_{2,0}W_{-1} \otimes W_{1} - 32d_{2,0}W_{0} \otimes W_{0}
\]
\[
+(27d_{2,0} - 20d_{2,-1})W_{1} \otimes W_{-1} - 8d_{2,0}W_{2} \otimes W_{-2} + (d_{2,0} - 4d_{2,-1})W_{3} \otimes W_{-3} = 0.
\]
which together force (comparing the coefficients of the tensor products)

\[ a_{2,0} = b_{2,0} = c_{2,0} = d_{2,0} = a_{-2,1} = b_{-2,1} = c_{-2,1} = d_{-2,1} = 0. \]

Thus we can deduce

\[ D_0(L_{-2}) \equiv D_0(L_2) \equiv 0. \] (2.13)

Since the Virasoro subalgebra of \( \mathcal{W} \), denoted by \( \mathcal{V}ir := \text{Span}\{L_n \mid n \in \mathbb{Z}\} \) can be generated by the set \( \{L_{-2}, L_{-1}, L_1, L_2\} \), then by (2.12) and (2.13), one has

\[ D_0(L_n) \equiv 0, \quad \forall n \in \mathbb{Z}. \] (2.14)

Applying \( D_0 \) to \([L_0, [L_0, W_2]] = 2W_2\) and using (2.14), one has

\[ L_0 \cdot L_0 \cdot D_0(W_2) = 2D_0(W_2). \]

Then using (2.10), we obtain

\[
\begin{align*}
&\sum_{p \in \mathbb{Z}} 2e_{2,p}L_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} 2f_{2,p}L_p \otimes W_{2-p} + 2e_{2}L_2 \otimes c + 2f_{2}c \otimes L_2 \\
&+ \sum_{p \in \mathbb{Z}} 2g_{2,p}W_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} 2h_{2,p}W_p \otimes W_{2-p} + 2g_{2}W_2 \otimes c + 2h_{2}c \otimes W_2 \\
&= \sum_{p \in \mathbb{Z}} p^2e_{2,p}L_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)e_{2,p}L_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)e_{2,p}L_p \otimes L_{2-p} \\
&+ \sum_{p \in \mathbb{Z}} (p-2)^2e_{2,p}L_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p^2f_{2,p}L_p \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)f_{2,p}L_p \otimes W_{2-p} \\
&+ \sum_{p \in \mathbb{Z}} p(2-p)f_{2,p}L_p \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} ((p-2)^2f_{2,p}L_p \otimes W_{2-p} + 4e_nL_2 \otimes c + 4f_{2}c \otimes L_2 \\
&+ \sum_{p \in \mathbb{Z}} p^2g_{2,p}W_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)g_{2,p}W_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)g_{2,p}W_p \otimes L_{2-p} \\
&+ \sum_{p \in \mathbb{Z}} (p-2)^2g_{2,p}W_p \otimes L_{2-p} + \sum_{p \in \mathbb{Z}} p^2h_{2,p}W_p \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} p(2-p)h_{2,p}W_p \otimes W_{2-p} \\
&+ \sum_{p \in \mathbb{Z}} p(2-p)h_{2,p}W_p \otimes W_{2-p} + \sum_{p \in \mathbb{Z}} (p-2)^2h_{2,p}W_p \otimes W_{2-p} + 4g_{2}W_2 \otimes c + 4h_{2}c \otimes W_2.
\end{align*}
\]

Comparing the coefficients of the tensor products, one immediately has

\[ e_{2,p} = f_{2,p} = g_{2,p} = h_{2,p} = e_2 = f_2 = g_2 = h_2 = 0, \quad \forall p \in \mathbb{Z}, \]

which implies

\[ D_0(W_2) \equiv 0. \] (2.15)

According to the fact that the algebra \( \mathcal{W} \) is generated by the set \( \{L_{-2}, L_1, L_2, W_2\} \), using (2.14) and (2.15), we obtain

\[ D_0(\mathcal{W}) \equiv 0. \] (2.16)

This proves Claim 3. □
Claim 4. $D_0 = 0$.

By Claims 1, 2 and 3 we have $D_0(W) \subseteq \mathbb{C}(c \otimes c)$. Since $[W, W] = W$, we obtain $D_0(W) \subseteq W \cdot (D_0(W)) = 0$. Then Claim 4 follows. \hfill \Box

Claim 5. For every $D \in \text{Der}(W, V)$, (2.7) is a finite sum.

By the above claims, we can suppose $D_n = (v_n)_{\text{im}}$ for some $v_n \in \mathcal{V}_n$ and $n \in \mathbb{Z}$. If $\mathbb{Z}' = \{ n \in \mathbb{Z}^* | v_n \neq 0 \}$ is an infinite set, we obtain $D(L_0) = \sum_{n \in \mathbb{Z}'/\cup(0)} L_0 \cdot v_n = - \sum_{n \in \mathbb{Z}'} n v_n$ is an infinite sum, which is not an element in $\mathcal{V}$, contradicting with the fact that $D$ is a derivation from $W$ to $V$. This proves Claim 5 and the proposition. \hfill \Box

Lemma 2.5. Suppose $v \in \mathcal{V}$ such that $x \cdot v \in \text{Im}(1 - \tau)$ for all $x \in W$. Then $v \in \text{Im}(1 - \tau)$.

Proof. First note that $W \cdot \text{Im}(1 - \tau) \subset \text{Im}(1 - \tau)$. We shall prove that after a number of steps in each of which $v$ is replaced by $v - u$ for some $u \in \text{Im}(1 - \tau)$, the zero element is obtained and thus proving that $v \in \text{Im}(1 - \tau)$. Write $v = \sum_{n \in \mathbb{Z}} v_n$. Obviously,

$$v \in \text{Im}(1 - \tau) \iff v_n \in \text{Im}(1 - \tau) \text{ for all } n \in \mathbb{Z}. \quad (2.17)$$

Then $\sum_{n \in \mathbb{Z}} n v_n = L_0 \cdot v \in \text{Im}(1 - \tau)$. By (2.17), $n v_n \in \text{Im}(1 - \tau)$, in particular, $v_n \in \text{Im}(1 - \tau)$ if $n \neq 0$. Thus by replacing $v$ by $v - \sum_{n \in \mathbb{Z}} v_n$, we can suppose $v = v_0 \in \mathcal{V}_0$. Now we can write

$$v = \sum_{p \in \mathbb{Z}} a_p L_p \otimes L_{-p} + \sum_{p \in \mathbb{Z}} b_p L_p \otimes W_{-p} + d_0 L_0 \otimes c + b'_0 c \otimes L_0$$

$$+ \sum_{p \in \mathbb{Z}} c_p W_p \otimes L_{-p} + \sum_{p \in \mathbb{Z}} d_p W_p \otimes W_{-p} + c'_0 W_0 \otimes c + d'_0 c \otimes W_0,$$

where all the coefficients of the tensor products are complex numbers and the sums are all finite. Fix the normal total order on $\mathbb{Z}$ compatible with its additive group structure. Since the elements of the form $u_{1,p} := L_p \otimes L_{-p} - L_p \otimes L_p$, $u_{2,p} := L_p \otimes W_{-p} - W_{-p} \otimes L_p$, $u_{3,p} := W_p \otimes W_{-p} - W_p \otimes W_p$, $u_1 := L_0 \otimes c \otimes c \otimes L_0$ and $u_2 := W_0 \otimes c \otimes c \otimes W_0$ are all in $\text{Im}(1 - \tau)$, replacing $v$ by $v - u$, where $u$ is a combination of some $u_{1,p}$, $u_{2,p}$, $u_{3,p}$, $u_1$ and $u_2$, we can suppose

$$b'_0 = c_p = d'_0 = 0, \ \forall \ p \in \mathbb{Z}, \quad (2.18)$$

$$a_p \neq 0 \implies p > 0 \text{ or } p = 0, \quad (2.19)$$

$$d_p \neq 0 \implies p > 0 \text{ or } p = 0. \quad (2.20)$$

Then the $v$ can be rewritten as

$$v = \sum_{p \in \mathbb{Z}^+} a_p L_p \otimes L_{-p} + \sum_{p \in \mathbb{Z}} b_p L_p \otimes W_{-p} + \sum_{p \in \mathbb{Z}^+} d_p W_p \otimes W_{-p} + d'_0 L_0 \otimes c + c'_0 W_0 \otimes c. \quad (2.21)$$
First assume that $a_p \neq 0$ for some $p > 0$. Choose $q > 0$ such that $q \neq p$. Then we see that the term $L_{p+q} \otimes L_{-p}$ appears in $L_q \cdot v$, but (2.19) implies that the term $L_{-p} \otimes L_{p+q}$ does not appear in $L_q \cdot v$, a contradiction with the fact that $L_q \cdot v \in \text{Im}(1-\tau)$. Then one further can suppose $a_p = 0$, \(\forall p \in \mathbb{Z}^*\). Similarly, one also can suppose $d_p = 0$, \(\forall p \in \mathbb{Z}^*\). Therefore, the identity (2.21) becomes

$$v = a_0L_0 \otimes L_0 + d_0W_0 \otimes W_0 + a'_0L_0 \otimes c + c'_0W_0 \otimes c + \sum_{p \in \mathbb{Z}} b_pL_p \otimes W_{-p}. \quad (2.22)$$

Finally, we mainly use the fact $\text{Im}(1-\tau) \subset \text{Ker}(1+\tau)$ and our suppose $W \cdot v \subset \text{Im}(1-\tau)$ to deduce $a_0 = d_0 = a'_0 = c'_0 = b_p = 0$, \(\forall p \in \mathbb{Z}\). One has the computation

$$0 = (1 + \tau)L_1 \cdot v = 2a_0(L_1 \otimes L_0 + L_0 \otimes L_1) + 2d_0(W_1 \otimes W_0 + W_0 \otimes W_1) + a'_0(L_1 \otimes c + c \otimes L_1)$$

$$+ \sum_{c \in \mathbb{Z}} ((2-p)b_{p-1} + (1+p)b_p) L_p \otimes W_{1-p} + \sum_{p \in \mathbb{Z}} ((2-p)b_{p-1} + (1+p)b_p) W_{1-p} \otimes L_p$$

$$+ c'_0(W_1 \otimes c + c \otimes W_1).$$

Then noticing the set \(\{p \mid b_p \neq 0\}\) of finite rank and comparing the coefficients of the tensor products, one immediately gets

$$b_0 = -2b_{-1} = -2b_1,$$

$$a_0 = d_0 = a'_0 = c'_0 = b_p = 0, \quad \forall p \in \mathbb{Z}, p \neq 0, \pm 1.$$

Then (2.22) can be rewritten as

$$v = b_1(L_{-1} \otimes W_1 - 2L_0 \otimes W_0 + L_1 \otimes W_{-1}). \quad (2.23)$$

observing the computation

$$0 = (1 + \tau)L_2 \cdot v = b_1(1 + \tau)L_2 \cdot (L_{-1} \otimes W_1 - 2L_0 \otimes W_0 + L_1 \otimes W_{-1})$$

$$= b_1(1 + \tau)(6L_1 \otimes W_1 + L_{-1} \otimes W_3 - 4L_2 \otimes W_0 - 4L_0 \otimes W_2 + L_3 \otimes W_{-1}),$$

which forces $b_1 = 0$, then $b_0 = -2b_{-1} = -2b_1 = 0$. Then the lemma follows. \(\square\)

**Proof of Theorem 7.3.** Let $(W, [\cdot, \cdot], \Delta)$ be a Lie bialgebra structure on $W$. By (1.9), (2.3) and Proposition 2.4, $\Delta = \Delta_r$ is defined by (1.11) for some $r \in W \otimes W$. By (1.5), $\text{Im} \Delta \subset \text{Im}(1-\tau)$. Thus by Lemma 2.5, $r \in \text{Im}(1-\tau)$. Then (1.6), (2.1) and Corollary 2.3 show that $c(r) = 0$. Thus Definition 1.2 says that $(W, [\cdot, \cdot], \Delta)$ is a triangular coboundary Lie bialgebra. \(\square\)
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