Tighter generalized monogamy and polygamy relations for multiqubit systems

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We present a different kind of monogamy and polygamy relations based on concurrence and concurrence of assistance for multiqubit systems. By relabeling the subsystems associated with different weights, a smaller upper bound of the $\alpha$th (0 $\leq$ $\alpha$ $\leq$ 2) power of concurrence for multiqubit states is obtained. We also present tighter monogamy relations satisfied by the $\alpha$th (0 $\leq$ $\alpha$ $\leq$ 2) power of concurrence for $N$-qubit pure states under the partition $AB$ and $C_1 \cdots C_{N-2}$, as well as under the partition $ABC_1$ and $C_2 \cdots C_{N-2}$. These inequalities give rise to the restrictions on entanglement distribution and the trade off of entanglement among the subsystems. Similar results are also derived for negativity.

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INTRODUCTION

Quantum entanglement is an essential feature of quantum mechanics, which distinguishes the quantum from the classical world. One of the fundamental differences between classical and quantum correlations lies on the sharability among the subsystems. Different from the classical correlation, quantum correlation cannot be freely shared. The monogamy relations give rise to the restrictions on the distribution of entanglement in the multipartite setting. It is not possible to prepare three qubits in a way that any two qubits are maximally entangled. The monogamy relation was first quantified by Coffman, Kundu, and Wootters (CKW) for three qubits, $E_{A|BC} \geq E_{AB} + E_{AC}$, where $E_{A|BC}$ denotes the entanglement between systems $A$ and $BC$. The CKW inequality shows that the more entanglement shared between two qubits $A$ and $B$, the less entanglement between the qubits $A$ and $C$. CKW inequality was generalized to multiqubit systems and also studied intensively in more general settings and

Using concurrence of assistance as the measure of distributed entanglement, the polygamy of entanglement provides a lower bound for the distribution of bipartite entanglement in a multipartite system. Polygamy of entanglement is characterized by the polygamy inequality, $E_{aABC} \leq E_{aAB} + E_{aAC}$ for a tripartite quantum state $\rho_{ABC}$, where $E_{aA|BC}$ is the assisted entanglement between $A$ and $BC$. Polygamy of entanglement was generalized to multiqubit systems and arbitrary dimensional multipartite states. The authors have given the monogamy and polygamy relations with any qubits as the focus ones for multiqubit states. Furthermore, the case of the $\alpha$th (0 $\leq$ $\alpha$ $\leq$ 2) power of concurrence for $N$-qubit pure states under any partition was studied in.

In this paper, we study the general monogamy inequalities with qubits $AB$ as the focus qubits, satisfied by the concurrence and the concurrence of assistance (COA). A smaller (tighter) upper bound for the $\alpha$th (0 $\leq$ $\alpha$ $\leq$ 2) power of concurrence for multiqubit states is obtained. Then we establish the tighter monogamy relations of the $\alpha$th (0 $\leq$ $\alpha$ $\leq$ 2) power of concurrence in $N$-qubit pure states under the partition $AB$ and $C_1 \cdots C_{N-2}$, as well as under the partition $ABC_1$ and $C_2 \cdots C_{N-2}$. Based on the relations between negativity and concurrence, we also obtain similar results for negativity. Detailed examples are presented.

TIGHTER GENERALIZED MONOGAMY AND POLYGAMY RELATIONS OF CONCURRENCE

Let $H_X$ denote the finite dimensional vector space associated with qubit $X$. For a bipartite pure state $|\psi\rangle_{AB}$ in vector space $H_A \otimes H_B$, the concurrence is given by

$$C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}, \quad (1)$$

where $\rho_A$ is the reduced density matrix by tracing over the subsystem $B$, $\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$. The concurrence for a bipartite mixed state $\rho_{AB}$ is defined by the convex roof

$$C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

where the minimum is taken over all possible decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, with $p_i \geq 0$, $\sum_i p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$.

For a tripartite state $|\psi\rangle_{ABC}$, the concurrence of assistance (COA) is defined by

$$C_a(|\psi\rangle_{ABC}) = C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),$$

for all possible ensemble realizations of $\rho_{AB} = \text{Tr}_C(|\psi\rangle_{ABC}\langle\psi|) = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. When $\rho_{AB}$ is a pure state, one has $C(|\psi\rangle_{AB}) = C_a(\rho_{AB})$. 
For an $N$-qubit state $|\psi\rangle_{AB_1\cdots B_N} \in H_A \otimes H_{B_1} \otimes \cdots \otimes H_{B_N}$, the concurrence $C(|\psi\rangle_{AB_1\cdots B_N})$ of the state $|\psi\rangle_{A|B_1\cdots B_N}$ viewed as a bipartite partition $A$ and $B_1B_2\cdots B_N$, satisfies the monogamy inequality [20],

$$C^2(\rho_{AB_1B_2\cdots B_N}) \geq C^2(\rho_{AB_1}) + C^2(\rho_{AB_2}) + \cdots + C^2(\rho_{AB_{N-1}}),$$

(2)

where $C(\rho_{AB_i})$ is the concurrence of $\rho_{AB_i} = \text{Tr}_{B_1\cdots B_{i-1}B_{i+1}\cdots B_N}(\rho)$.

The dual inequality satisfied by COA for $N$-qubit states has the form [27],

$$C^2(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq \sum_{i=1}^{N-1} C^2_a(\rho_{AB_i}).$$

(3)

Furthermore, the authors in [3] presented a generalized monogamy relation for $\alpha \geq 2$, $C^\alpha(\rho_{AB_1B_2\cdots B_N}) \geq C^\alpha(\rho_{AB_1}) + C^\alpha(\rho_{AB_2}) + \cdots + C^\alpha(\rho_{AB_{N-1}})$. The dual inequality is given in [21] for $0 \leq \alpha \leq 2$,

$$C^\alpha(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq C^\alpha_\alpha(\rho_{AB_1}) + \frac{\alpha}{2} C^\alpha_\alpha(\rho_{AB_2}) + \cdots + \left(\frac{\alpha}{2}\right)^{N-2} C^\alpha_\alpha(\rho_{AB_{N-1}}).$$

(4)

In this paper, we first give a tighter upper bound satisfied by the $\alpha$th power of COA for $N$-qubit states. Then we present monogamy and polygamy relations for $N$-qubit states in terms of the $\alpha$th power of COA, which are tighter than the existing ones.

The concurrence [1] is related to the linear entropy $T(\rho)$ of a state $\rho$, $T(\rho) = 1 - \text{Tr}(\rho^2)$ [28]. For a bipartite state $\rho_{AB}$, $T(\rho)$ has the property [29],

$$|T(\rho_A) - T(\rho_B)| \leq T(\rho_{AB}) \leq T(\rho_A) + T(\rho_B).$$

(5)

For convenience, we rewrite (3) as follows,

$$C^2(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq \sum_{i=1}^{N-1} C^2_a(\rho_{AM_i}),$$

(6)

where $C^2_a(\rho_{AM_i}) = \sum_{j=M_{i-1}+1}^{M_i} C^2_a(\rho_{AB_j})$ with $M_0 = 0$, $\sum_{i=1}^{k} M_i = N - 1$, $1 \leq k \leq N - 1$. The summation on the right hand side of (6) has been separated into $k$ parts. There is always a choice of $M_i$, such that the above relations is true.

**Theorem 1.** For any $N$-qubit pure state $|\psi\rangle_{AB_1B_2\cdots B_N}$, we have

$$C^\alpha(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq C^\alpha_a(\rho_{AM_1}) + h C^\alpha_a(\rho_{AM_2}) + \cdots + h^{k-1} C^\alpha_a(\rho_{AM_k}),$$

(7)

for all $0 \leq \alpha \leq 2$, where $h = 2^{\frac{\alpha}{2}} - 1$.

**Proof.** Without loss of generality, we can always assume that $C^2_a(\rho_{AM_i}) \geq \sum_{j=M_{i-1}+1}^{M_i} C^2_a(\rho_{AM_j})$, $1 \leq t \leq k - 1$, $2 \leq k \leq N - 1$, by reordering $M_1, M_2, \cdots, M_k$ and/or relabeling the subsystems in need. Form the result in [27], we have

$$C^\alpha(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq \left( C^2_a(\rho_{AM_1}) + \sum_{i=2}^{k} C^2_a(\rho_{AM_i}) \right)^{\frac{\alpha}{2}}$$

$$= C^\alpha_a(\rho_{AM_1}) \left( 1 + \sum_{i=2}^{k} \frac{C^2_a(\rho_{AM_i})}{C^2_a(\rho_{AM_1})} \right)^{\frac{\alpha}{2}}$$

$$\leq C^\alpha_a(\rho_{AM_1}) \left[ 1 + h \left( \sum_{i=2}^{k} C^2_a(\rho_{AM_i}) \right) \right]^{\frac{\alpha}{2}}$$

$$= C^\alpha_a(\rho_{AM_1}) + h \left( \sum_{i=2}^{k} C^2_a(\rho_{AM_i}) \right)^{\frac{\alpha}{2}}$$

$$\leq \cdots \leq \sum_{i=1}^{k} h^{i-1} C^\alpha_a(\rho_{AM_i}),$$

(8)

where the first inequality is due to [20]. By using the fact that [30], for any real numbers $x$ and $t$ such that $0 \leq t \leq 1$ and $0 \leq x \leq 1$, $(1 + t)^x \leq 1 + (2^x - 1)t^x$, we get the second inequality.

Theorem 1 gives a tighter polygamy relation of the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence for $N$-qubit pure state $|\psi\rangle_{A|B_1B_2\cdots B_N}$ based on the COA. For the case of $k = N - 1$, we have the following result,

$$C^\alpha(|\psi\rangle_{A|B_1B_2\cdots B_N}) \leq C^\alpha_a(\rho_{AB_1}) + h C^\alpha_a(\rho_{AB_2}) + \cdots + h^{N-2} C^\alpha_a(\rho_{AB_{N-1}}).$$

(9)

Specially, for $\alpha = 2$, inequality [7] or [9] reduces to the result [27]. For $0 < \alpha < 2$, inequality [7] or [9] is tighter than the result [31] in [21].

**Example 1.** Let us consider the three-qubit state $|\psi\rangle$ in the generalized Schmidt decomposition form [32, 33],

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\psi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

where $\lambda_i \geq 0$, $i = 0, \cdots, 4$ and $\sum_{i=0}^{4} \lambda_i^2 = 1$. We have $C(\rho_{AB|C}) = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2 + \lambda_5^2}$, $C(\rho_{AC|B}) = 2\lambda_0\lambda_3$, $C(\rho_{AB}) = 2\lambda_0\lambda_5$, $C(\rho_{AC}) = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_4^2}$, $C(\rho_{BC}) = 2\lambda_0\sqrt{\lambda_3^2 + \lambda_5^2}$. Set $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \sqrt{\frac{\lambda_5}{5}}$. One gets $C^\alpha(\rho_{AB|C}) = (\frac{2\lambda_0}{5})^\alpha$, $C^\alpha(\rho_{AB}) + \frac{\alpha}{2} C^\alpha(\rho_{AC}) = (1 + \frac{\alpha}{2}) (\frac{2\lambda_0}{5})^\alpha$, $C^\alpha(\rho_{AC}) + h C^\alpha(\rho_{BC}) = 2^\frac{\alpha}{2} (\frac{2\lambda_0}{5})^\alpha$. One can see that our result is better than that of [4] for $0 < \alpha < 2$, see Fig. 1.

In the following, by using the conclusion of Theorem 1 and Lemma 2, we present some monogamy-type inequal-
Theorem 1. For arbitrary two real numbers $x$ and $y$ such that $x \geq y \geq 0$, we have $(x-y)^2 \geq x^2 - y^2$ and $(x+y)^2 \leq x^2 + y^2$ for $0 \leq \alpha \leq 1$.

[Proof]. Without loss of generality, there always exists a proper ordering of the subsystems $M_i, M_{i+1}, \ldots, M_k$ $(i = 1, 2)$ such that $C^2_a(\rho_{AM_i}) \geq \sum_{i=1}^{k_1} C^2_a(\rho_{AM_l})$ and $C^2_a(\rho_{BM_k}) \geq \sum_{i=k_2+1}^{N} C^2_a(\rho_{BM_l})$, $1 \leq t \leq k_1 - 1$ $(i = 1, 2)$, $2 \leq k_1, k_2 \leq N - 1$.

For $N$-qubit pure state $\rho_{ABC1 \cdots C_{N-2}}$, if $C(\rho_{A|BC1 \cdots C_{N-2}}) \geq C(\rho_{B|AC1 \cdots C_{N-2}})$, one has

$$C^\alpha(\rho_{AB|C1 \cdots C_{N-2}})$$

$$\geq \max \left\{ h \sum_{i=1}^{k_1-1} C^\alpha(\rho_{AM_i}) + C^\alpha(\rho_{AM_{k_1}}) - J_B, \right.$$

$$\left. h \sum_{i=1}^{k_2-1} C^\alpha(\rho_{BM_i}) + C^\alpha(\rho_{BM_{k_2}}) - J_A \right\},$$

for $0 \leq \alpha \leq 2$, $N \geq 4$, where $C^2_a(\rho_{AM_i})$ is defined in (6) and $J_A = \sum_{i=1}^{k_1} h^{i-1} C^\alpha_a(\rho_{AM_i}), J_B = \sum_{i=1}^{k_2} h^{i-1} C^\alpha_a(\rho_{BM_i})$. $h = 2^\alpha - 1, k_1, k_2$ are defined similar to inequality (4).

[Proof]. Without loss of generality, there always exists a proper ordering of the subsystems $M_i, M_{i+1}, \ldots, M_k$ $(i = 1, 2)$ such that $C^2_a(\rho_{AM_i}) \geq \sum_{i=1}^{k_1} C^2_a(\rho_{AM_l})$ and $C^2_a(\rho_{BM_k}) \geq \sum_{i=k_2+1}^{N} C^2_a(\rho_{BM_l})$, $1 \leq t \leq k_1 - 1$ $(i = 1, 2)$, $2 \leq k_1, k_2 \leq N - 1$.

For $N$-qubit pure state $\rho_{ABC1 \cdots C_{N-2}}$, if $C(\rho_{A|BC1 \cdots C_{N-2}}) \geq C(\rho_{B|AC1 \cdots C_{N-2}})$, one has

$$C^\alpha(\rho_{AB|C1 \cdots C_{N-2}})$$

$$\geq \max \left\{ h \sum_{i=1}^{k_1-1} C^\alpha(\rho_{AM_i}) + C^\alpha(\rho_{AM_{k_1}}) - J_B, \right.$$

$$\left. h \sum_{i=1}^{k_2-1} C^\alpha(\rho_{BM_i}) + C^\alpha(\rho_{BM_{k_2}}) - J_A \right\},$$

where the first inequality is due to the left inequality in (2). From Lemma, one gets the second inequality. Using the inequality $(1 + t)^x \geq 1 + (2^x - 1)t^x$, $t \geq 1, 0 \leq x \leq 1$, we get the third inequality. The last inequality is due to Theorem 1.

If $C(\rho_{A|BC1 \cdots C_{N-2}}) \leq C(\rho_{B|AC1 \cdots C_{N-2}})$, similar to the above derivation, we can obtain another inequality in Theorem 2. □

Theorem 2 shows that the entanglement contained in the pure states $\rho_{ABC1 \cdots C_{N-2}}$ is related to the sum of entanglement between bipartitions of the system. The lower bound in inequalities (6) is easily calculable. As an example, let us consider the four-qubit pure state $|\psi\rangle_{ABCD} = \frac{1}{\sqrt{2}}(|0000\rangle + |1001\rangle)$. We have $C(\rho_{AB}) = C(\rho_{AC}) = 0$, $C(\rho_{AB}) = 1$, and $C(\rho_{BC}) = C(\rho_{BD}) = 0$. Therefore, $C(\psi)_{ABCD} \geq 2^\alpha - 1, 0 \leq \alpha \leq 2$. Namely, the state $|\psi\rangle_{ABCD}$ saturates the inequality (10) for $\alpha = 2$.

Similar to the proof of Theorem 2, from (2) we can derive another upper bound of the $\alpha$th power of concurrence as follows.

[Theorem 3]. For any $N$-qubit state $|\psi\rangle_{ABC1 \cdots C_{N-2}}$, we have

$$C^\alpha(\rho_{AB|C1 \cdots C_{N-2}})$$

$$\geq \max \left\{ \left( \sum_{i=1}^{N-2} C^2(\rho_{AC_i}) + C^2(\rho_{AB}) \right)^{\frac{\alpha}{2}} - J_B, \right.$$

$$\left. \left( \sum_{i=1}^{N-2} C^2(\rho_{BC_i}) + C^2(\rho_{AB}) \right)^{\frac{\alpha}{2}} - J_A \right\},$$

for $0 \leq \alpha \leq 2$, $N \geq 4$, where $C^2_a(\rho_{AM_i})$ is defined in (6) and $J_A = \sum_{i=1}^{k_1} h^{i-1} C^\alpha_a(\rho_{AM_i}), J_B = \sum_{i=1}^{k_2} h^{i-1} C^\alpha_a(\rho_{BM_i})$. $h = 2^\alpha - 1, k_1, k_2$ are defined similar to inequality (4).
FIG. 2: \( Y \) stands for the differences between the left and right of the generalized monogamy inequalities: solid (red) line for (11); dashed (blue) line for (9) in [21].

**Example 2.** Let us consider the 4-qubit generalized W-class state,

\[
|W\rangle_{ABC:C_2} = \lambda_1|1000\rangle + \lambda_2|0100\rangle + \lambda_3|0010\rangle + \lambda_4|0001\rangle,
\]

where \( \sum_{i=1}^{4} \lambda_i^2 = 1 \). We have \( C(|W\rangle_{ABC:C_2}) = 2\sqrt{(\lambda_1^2 + \lambda_2^2)(\lambda_3^2 + \lambda_4^2)} \), \( C(\rho_{AB}) = C_4(\rho_{AB}) = 2\lambda_1\lambda_2 \), \( C(\rho_{AC_1}) = C_6(\rho_{AC_1}) = 2\lambda_1\lambda_3 \), \( C(\rho_{AC_2}) = C_4(\rho_{AC_2}) = 2\lambda_1\lambda_4 \), taking \( \lambda_1 = \frac{\lambda}{\sqrt{4}} \), \( \lambda_2 = \frac{\lambda}{\sqrt{4}} \), \( \lambda_3 = \frac{\lambda}{\sqrt{4}} \) and \( \lambda_4 = \frac{\lambda}{4} \), we get \( J_A = J_B = \left(\frac{3}{4}\right)^{\alpha} + h\left(\frac{\sqrt{\alpha}}{2}\right)^{\alpha} + h^2\left(\frac{\alpha}{4}\right)^{\alpha} \). Set \( y_1 = C^\alpha(|W\rangle_{ABC:C_2}) + C^2(\rho_{AC_1}) + C^2(\rho_{AC_2}) \) to be the difference between the left and right side of (11). We have

\[
y_1 = \left(\frac{\sqrt{\alpha}}{8}\right)^{\alpha} - \left(\frac{\sqrt{\alpha}}{8}\right)^{\alpha} + \left(\frac{\alpha}{4}\right)^{\alpha} + h\left(\frac{\sqrt{\alpha}}{2}\right)^{\alpha} + h^2\left(\frac{\alpha}{4}\right)^{\alpha}.
\]

From the inequality (9) in [21], such difference is given by

\[
y_2 = \left(\frac{\sqrt{\alpha}}{8}\right)^{\alpha} - \left(\frac{\sqrt{\alpha}}{8}\right)^{\alpha} + \left(\frac{\alpha}{4}\right)^{\alpha} + h\left(\frac{\sqrt{\alpha}}{2}\right)^{\alpha} + h^2\left(\frac{\alpha}{4}\right)^{\alpha}.
\]

From Fig. 2 we can see that the difference between the left and right of the generalized monogamy inequality (11) is smaller than that of the result from [21].

Different from the usual monogamy inequalities under the partition \( A \) and \( B_1 \cdots B_{N-2} \) [29], Theorem 2 and Theorem 3 give monogamy relations under the partition \( AB \) and \( C_1 \cdots C_{N-2} \), which present finer weighted characterizations of the entanglement distributions among the subsystems, as illustrated in Example 2. Moreover, the result in Ref. [21] is a special case of Theorem 3 for \( \alpha = 2 \).

Theorem 2 and Theorem 3 give rise to monogamy-type lower bound of \( C(\psi_{AB|C_1 \cdots C_{N-2}}) \). According to the subadditivity of the linear entropy, we also have the following conclusion:

**Theorem 4.** For any \( 2 \otimes 2 \otimes \cdots \otimes 2 \) pure state \( |\psi\rangle_{ABC \cdots C_{N-2}} \), we have

\[
C^\alpha(|\psi\rangle_{AB|C_1 \cdots C_{N-2}}) \leq J_A + J_B
\]

for \( 0 \leq \alpha \leq 2 \), \( N \geq 4 \), where \( J_A \) and \( J_B \) are defined similarly as in Theorem 2.

**Proof.** Without loss of generality, there always exists a proper ordering of the subsystems such that \( C^2(\rho_{AB}) \geq \sum_{t=t+1}^{N} C^2(\rho_{AB}) \) and \( C^2(\rho_{AC_1}) \geq \sum_{t=t+1}^{N} C^2(\rho_{AC_1}) \) for any \( 1 \leq t, t_1 \leq k-1 \) and \( 2 \leq k, k_2 \leq N-1 \). For qubit state \( |\psi\rangle_{ABC_1 \cdots C_{N-2}} \), one has

\[
C^\alpha(|\psi\rangle_{AB|C_1 \cdots C_{N-2}}) \leq J_A + J_B
\]

where the first inequality is due to the right inequality in (13). The second inequality is due to Lemma. Using the Theorem 1, one gets the last inequality. □

Let us consider the following four-qubit pure state, \( |\psi\rangle_{ABCD} = \frac{1}{\sqrt{3}}(|0000\rangle + |0010\rangle + |1011\rangle) \). Then from the result in [21], one gets \( C^\alpha(|\psi\rangle_{AB|CD}) \leq (\frac{2\sqrt{\alpha}}{8})^\alpha + h(\frac{\alpha}{4})^\alpha \). While from our Theorem 4, we have \( C^\alpha(|\psi\rangle_{AB|CD}) \leq (\frac{2\sqrt{\alpha}}{8})^\alpha + h(\frac{\alpha}{4})^\alpha \) for any \( 0 \leq \alpha \leq 2 \), where \( h = 2^{\frac{\alpha}{2}} - 1 \), see Fig. 3.

Now we generalize our results to the concurrence \( C_{ABC_1|C_2 \cdots C_{N-1}}(|\psi\rangle) \) under the partition \( ABC_1 \) and \( C_2 \cdots C_{N-2} (N \geq 6) \) for pure state \( |\psi\rangle_{ABC_1 \cdots C_{N-2}} \). Similar to Theorem 2, Theorem 3 and Theorem 4, we obtain the following corollaries:

**Corollary 1.** For any \( N \)-qubit pure state \( |\psi\rangle \)
where the first inequality is due to $k - C$ and the right inequalities of (5) and (7). Analogously, by or $(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2})$

$$C^{\alpha}(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2})$$

$$\geq \max \left\{ h \sum_{i=1}^{K} C^{\alpha}(\rho_{AM_{i}}) + C^{\alpha}(\rho_{AM_{K+1}} - J_{B},$$

$$h \sum_{i=1}^{K} C^{\alpha}(\rho_{BM_{i}}) + C^{\alpha}(\rho_{BM_{K+1}} - J_{A}) - J_{C_{1}} \right\}$$

or

$$C^{\alpha}(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2})$$

$$\geq \max \left\{ \left( \sum_{i=1}^{N-2} C^{2}(\rho_{AC_{i}}) + C^{2}(\rho_{AB}) \right)^{\frac{\alpha}{2}} - J_{B},$$

$$\left( \sum_{i=1}^{N-2} C^{2}(\rho_{BC_{i}}) + C^{2}(\rho_{AB}) \right)^{\frac{\alpha}{2}} - J_{A} \right\} - J_{C_{1}} \right\}$$

where $J_{A}$, $J_{B}$ are defined as in Theorem 2, $J_{C_{1}} = \sum_{i=1}^{K} h^{-1} C^{\alpha}(\rho_{C_{1}M_{i}}), h = 2^{\frac{1}{4}} - 1, 2 \leq m \leq N - 3, N \geq 6$.

[Proof.] For any $N$-qubit pure state $|\psi_{ABC}^{C} | C_{2} \cdots C_{N-2}$, if $C(|\psi_{ABC}^{C} | C_{2} \cdots C_{N-2}) \geq C(|\psi_{ABC}^{C} | C_{2} \cdots C_{N-2})$, we have

$$C^{\alpha}(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2})$$

$$= (2T(\rho_{ABC}^{C}))^{\frac{\alpha}{2}}$$

$$\geq 2T(\rho_{AB}) - 2T(\rho_{C_{1}})^{\frac{\alpha}{2}}$$

$$= C^{2}(\rho_{AC_{1}}) - C^{2}(\rho_{C_{1}AC_{2} \cdots C_{N-2}})$$

$$\geq C^{\alpha}(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2}) - C^{\alpha}(\psi_{ABC}^{C} | C_{2} \cdots C_{N-2}),$$

from the definition of negativity. Given a bipartite state $\rho_{AB}$ in $H_{A} \otimes H_{B}$, the negativity is defined by $\hat{\rho}_{AB}$,

$$N(\rho_{AB}) = ||\rho_{AB}^{T}||_{1} - 1,$$

where $\rho_{AB}^{T}$ is the partially transposed matrix of $\rho_{AB}$ with respect to the subsystem $A$, $||X||$ denotes the trace norm of $X$, i.e $||X|| = Tr|X|$. Negativity is a computable measure of entanglement, and is a convex function of $\rho_{AB}$. It vanishes if and only if $\rho_{AB}$ is separable for the 2 $\otimes$ 2 and 2 $\otimes$ 3 systems. For the purposes of discussion, we use the following definition of negativity: $N(\rho_{AB}) = ||\rho_{AB}^{T}||_{1} - 1$.

For any bipartite pure state $|\psi_{AB}^{C} |$ in a $d \otimes d$ quantum system with Schmidt rank $d$, $|\psi_{AB}^{C} | = \sum_{i=1}^{d} \sqrt{\lambda_{i}} |ii\rangle$, one has

$$N(|\psi_{AB}^{C} |) = 2 \sum_{i<j} \sqrt{\lambda_{i} \lambda_{j}},$$

from the definition of concurrence, we have

$$C(|\psi_{AB}^{C} |) = 2 \sum_{i<j} \sqrt{\lambda_{i} \lambda_{j}}.$$

Combining (18) with (19), one obtains

$$N(|\psi_{AB}^{C} |) \geq C(|\psi_{AB}^{C} |).$$

For any bipartite pure state $|\psi_{AB}^{C} |$ with Schmidt rank $2$, one has $N(|\psi_{AB}^{C} |) = C(|\psi_{AB}^{C} |)$ from (18) and (19). For a mixed state $\rho_{AB}$, the convex-roof extended negativity (CREN) is defined by

$$N_{c}(\rho_{AB}) = \min \sum_{i} p_{i} N(|\psi_{i}^{C} |).$$
for 0 ≤ α ≤ 2, N ≥ 4, where $J_A' = \sum_{i=1}^{k_1} h^{i-1} N_\alpha^a(\rho_{AM_i})$, $J_B' = \sum_{i=1}^{k_2} h^{i-1} N_\alpha^a(\rho_{BM_i})$, $h = 2^{2\sqrt{2}} - 1$.

**[Proof]**. Without loss of generality, there always exists a proper ordering of the subsystems such that $N_\alpha^a(\rho_{AM_i}) \geq \sum_{i=t_i+1}^{k_2} N_\alpha^a(\rho_{BM_i})$ and $N_\alpha^a(\rho_{BM_i}) \geq \sum_{i=t_i+1}^{k_2} N_\alpha^a(\rho_{BM_i})$, $1 \leq t_i \leq k_i - 1$ ($i = 1, 2$), $2 \leq k_1, k_2 \leq N - 1$. For pure state $|\psi\rangle_{ABC_1\cdots C_{N-2}}$, we have

\[
N_\alpha^a(|\psi\rangle_{AB(C_1\cdots C_{N-2})}) \geq \max \left\{ \sum_{i=1}^{k_1} N_\alpha^c(\rho_{AM_i}) + N_\alpha^c(\rho_{AM_{k_1}}) - J_B, \sum_{i=1}^{k_2} N_\alpha^c(\rho_{BM_i}) + N_\alpha^c(\rho_{BM_{k_2}}) - J_A' \right\},
\]

where the first inequality is due to [20], the second inequality is from Theorem 2, the equality is based on [21] and [22].

**[Theorem 7]**. For any qubit state $|\psi\rangle_{ABC_1\cdots C_{N-2}}$, we have

\[
N_\alpha^a(|\psi\rangle_{AB(C_1\cdots C_{N-2})}) \geq \max \left\{ \sum_{i=1}^{N-2} N_\alpha^c(\rho_{AC_i}) + N_\alpha^c(\rho_{AB}) \right\}^{\frac{2}{\sqrt{2}}} - J_B',
\]

for 0 < α ≤ 2, N ≥ 4, where $J_A' = \sum_{i=1}^{k_1} h^{i-1} N_\alpha^a(\rho_{AM_i})$, $J_B' = \sum_{i=1}^{k_2} h^{i-1} N_\alpha^a(\rho_{BM_i})$, $h = 2^{2\sqrt{2}} - 1$.

For N-qubit pure state $|\psi\rangle_{ABC_1B_2\cdots B_{N-1}}$, based on the result in [39, 40], one has $N(|\psi\rangle_{AB(C_1\cdots C_{N-2})}) \leq \sqrt{\frac{r(r-1)}{2}} C(|\psi\rangle_{AB(C_1\cdots C_{N-2})})$, where $r$ is the Schmidt rank of the pure state $|\psi\rangle_{ABC_1\cdots C_{N-2}}$. From Theorem 4, we can obtain the upper bound of negativity under the partition $AB$ and $C_1\cdots C_{N-2}$.

**[Theorem 8]**. For any qubit state $|\psi\rangle_{ABC_1\cdots C_{N-2}}$, we have

\[
N_\alpha^a(|\psi\rangle_{AB(C_1\cdots C_{N-2})}) \leq \left( \frac{r(r-1)}{2} \right)^{\frac{2}{\sqrt{2}}} (J_A' + J_B'),
\]

for 0 ≤ α ≤ 2, where $J_A'$, $J_B'$ are given in Theorem 6.

**CONCLUSION**

Entanglement monogamy and polygamy relations are fundamental properties of multipartite entan-
gled states. We have presented tighter monogamy relations of the $\alpha$th power of concurrence for $N$-qubit systems by showing the relations among $C(|\psi\rangle_{AB}|C_1\cdots C_{N-2}), C(\rho_{AB}), C(\rho_{AC}), C(\rho_{BC}), C_{a}(\rho_{AC}),$ and $C_{a}(\rho_{BC}), 1 \leq i \leq N−2$, which give rise to the larger lower bounds and smaller upper bounds on the entanglement sharing among the partitions. The monogamy relations based on concurrence and COA have been investigated. We have obtained the smaller upper bound of the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence based on COA. We then have derived the tighter monogamy and polygamy relations satisfied by the $\alpha$th ($0 \leq \alpha \leq 2$) power of concurrence in $N$-qubit pure states under the partition $AB$ and $C_1\cdots C_{N-2}$, as well as under the partition $ABC_1$ and $C_1\cdots C_{N-2}$. These relations also give rise to a kind of trade-off relationship restricted by the lower and upper bounds of concurrences. Based on the relations between negativity and concurrence, we have also obtained the similar results for CRENOA. These results may be generalized to monogamy and polygamy relations under arbitrary partitions $C_{AB\hat{C}_1\hat{C}_2\cdots\hat{C}_{i−1}|C_{i+1}\cdots C_{N−2}}, 2 \leq i \leq N−2$. Our approach may be also used for the investigation of entanglement distribution based on other measures of quantum correlations.

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