The objective past of a quantum universe—Part 1: Redundant records of consistent histories

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We extend the framework of quantum Darwinism onto the consistent histories approach to quantum mechanics. We use the orthogonality of records on subsystems, and Finkelstein’s concept of partial-trace consistency, to define the redundancy of a history. The requirement of redundant records is proposed to reduce the size of the so-called set-selection problem.

I. INTRODUCTION

“Into what mixture does the wavepacket collapse?” \cite{1}. This is the preferred basis problem in quantum mechanics, which launched the detailed study of decoherence \cite{2}. The theory of decoherence is the foundation of the modern understanding of the quantum-classical transition \cite{3-5}.

The preferred basis problem has been solved exactly in the case of so-called pure decoherence \cite{6} between a system and an environment by identifying the pointer basis \cite{7}, whose origin can be traced back to the interaction Hamiltonian between a quantum system $S$ and its environment $\mathcal{E}$ \cite{1,7,8}. An approximate pointer basis exists for many other situations (e.g., \cite{9,12}).

The consistent (or decoherent) histories framework \cite{14,17} is a mathematical formalism for applying quantum mechanics to completely closed systems, up to and including the whole universe. It has been argued that quantum mechanics within this framework would be a fully satisfactory physical theory—avoiding the quantum measurement problem—only if it were supplemented with an unambiguous mechanism for identifying a preferred set of histories corresponding (at the least) to the perceptions of observers \cite{18-23} (although see Refs. \cite{16,24,25}). In other words, “What are the branches in the wavefunction of the universe?” This is the set selection problem, the global analog to the preferred basis problem. The deep relationship between the two can be seen by noting that the interaction between a quantum system and a measuring apparatus selects the pointer basis, and the measurement results (which obey the classical rules of probability) are then amplified by decoherence to macroscopic scales.

It is clear that such a set of histories must satisfy the mathematical requirement of consistency. But the set selection problem still looms large as almost all consistent sets bear no resemblance to the classical reality we perceive \cite{20,28}. Classical reasoning can only be done relative to a single consistent set; simultaneous reasoning from different sets leads to contradictions \cite{18,20,29,30}. A preferred set would allow one to unambiguously compute probabilities\cite{4} for all observations from first principles, that is, from (1) a wavefunction of the universe and (2) a Hamiltonian describing the fundamental interactions.

To agree with our expectations, a preferred set would describe macroscopic systems as course-grained variables approximately obeying classical equations of motion, thereby constituting a “quasiclassical domain” \cite{11,19,20,29,35}. Various principles for its identification have been explored, both within the consistent histories formalism \cite{12,22,28,35,41} and outside of it \cite{42,45}. None have gathered broad support.

It has long been recognized that records may play a key role in identifying the quasiclassical domain \cite{8,11,35,46-48}. In the case of a pure global state, Gell-Mann and Hartle pointed out that there must exist a formal record of consistent histories \cite{11}. (Here “global” refers to everything: both the system and its environment, which can encompass the universe.) But this argument guarantees only the existence of one record. Furthermore, without a stronger principle for identifying a preferred set (which they explore), this record will generally correspond to a highly complicated observable involving the entire global state, completely inaccessible to a realistic observer. And when the global state is mixed, there need be no record at all.

Establishing realistic records will require something more. It has been noted that the decoherence of a quantum system is connected to the production of records about that system \cite{1,8,47}, but that the ability of many observers to infer the state of the system by sampling an intermediate environment is predicated on properties of a global state which do not follow merely from decoherence (or consistency). Quantum Darwinism \cite{49,50} is a paradigm for describing and quantifying what distinguishes such states awash in the enormous sea of Hilbert space. It has been successfully applied to several prototypical models of decoherence \cite{51,52} (with redundant...
records appearing generically \[6, 59, 60\]), but models with non-trivial histories have not yet been investigated. (By “non-trivial” histories, we mean those featuring multiple events at different times, each with multiple outcomes.)

Almost two decades ago, J. Finkelstein introduced the idea of partial-trace consistency (or partial-trace decoherence) to identify when the consistency of a set of histories of a system is due to that system’s interaction with an environment \[61\]. (See also \[27, 46\].) Partial-trace consistency allows us to say that records exist—not just somewhere—but in the environment. This provides a key link between consistency, decoherence, and localized records.

As we will show, partial-trace consistency (which we slightly generalize) allows us to specify which parts of the environment are responsible for consistency. For realistic systems that are decohered by massive environments, we expect the consistency to be highly redundant in the following sense. The environment will have many small parts that are each individually sufficient to decohere the system. Likewise, we expect records of the history to be widely available in many small parts of the environment. (Indeed, observers need only capture a fraction of the photons in a room to infer the state of macroscopic degrees of freedom.)

With this in hand, we can rigorously define the idea that a history is recorded redundantly in the environment. This allows us to establish the objectivity of a set of histories, in the sense that many independent observers can all agree on the history by accessing only small, disjoint subsets of the environment. Given that the quasiscalar historical correspondences to our everyday macroscopic observations are known to be objective in this sense, and that most consistent histories will not turn out to be objective, this gives a new tool for reducing the size of the set selection problem and confirming that our observations of persistent classicality are compatible with a fundamentally quantum universe.

Sections \[II\] and \[III\] review decoherence and the consistent histories framework. Readers familiar with these ideas can skip those sections after noting our choice of notation. We describe partial-trace consistency in Section \[IV\] expanding on the results of Ref. \[61\]. In particular, we prove an identity linking the fidelity of two conditional partial-trace decoherence function. The connection to quantum Darwinism is completed in a forthcoming companion paper \[62\]. See also Ref. \[63\] for a modest uniqueness theorem suggesting that redundant records may be a surprisingly powerful set-selection criterion.

We work in the Schrödinger picture so that, although not written explicitly, states like \(\psi_a\) and \(\rho^E\) have implicit dependence on \(t\). Time-evolved projectors will be indicated with explicit time dependence, i.e. \(P(t_m)\). Initial states will be tagged with a null superscript (e.g. \(\rho^0, |\psi^0\rangle\)) and will be considered fixed, as will certain basis states (e.g. \(|S_a\rangle, |\rangle\)). Decoherence factors \((\Gamma_{ab})\) and information theoretic quantities (e.g. \(I_{S_aE}, H_E\)) have implicit \(t\) dependence. We take \(\hbar = 1\). Symbols like \(S\) will be used to both label a quantum system and also to denote its associated Hilbert space.

\section{II. DECOHERENCE}

Observers and other macroscopic quantum system do not exist in isolation, but are virtually always immersed in various large environmental baths \[3–5, 64\]. Below we briefly review some basic decoherence concepts.

\subsection{A. Pure decoherence}

Suppose a system \(S\) begins in an arbitrary pure state
\[|\psi^{S,0}\rangle = \sum_a c_a |S_a\rangle\]  
(1)
in a fixed basis \(|S_a\rangle\) and it is coupled to an environment \(E\) initially in some state \(\rho^{E,0}\). \(S\) is said to undergo pure decoherence \[3, 64\] by a (monolithic) environment when the evolution takes the form of a controlled unitary
\[e^{-it\hat{H}} = \sum_a |S_a\rangle \langle S_a| \otimes U_a,\]  
(2)
where the \(U_a\) are arbitrary unitaries governing the evolution of \(E\) conditional on the state \(|S_a\rangle\) of \(S\). In this case, the set \(\{|S_a\rangle\}\) forms an unambiguous pointer basis for the system \(S\) \[1\]. The global state is then
\[\rho = \sum_{a,b} c_a c_b^* |S_a\rangle \langle S_b| \otimes U_a \rho^{E,0} U_b^\dagger\]  
(3)
and the density matrix of \(S\) is
\[\rho^S = \sum_{a,b} \Gamma_{ab} c_a c_b^* |S_a\rangle \langle S_b|\]  
(4)
where the decoherence factors are
\[\Gamma_{ab} = \text{Tr}[U_a \rho^{E,0} U_b^\dagger] = \Gamma_{ba}^*\]  
(5)
When \(\Gamma_{ab}\) vanishes for all \(a \neq b\), we say that the system has been fully decohered by the environment; it takes the diagonal form
\[\rho^S = \sum_a |c_a|^2 |S_a\rangle \langle S_a|\]  
(6)

\subsection{B. Records}

Given that the von Neumann measurement scheme is the prototypical example of pure decoherence, one might expect that pure decoherence necessarily leaves an imprint of the \(|S_a\rangle\) in \(E\). In fact, for a general mixed initial
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E tells one nothing about the pointer basis. More generally, point in the same direction. When is not accessible in just.

We can produce pure decoherence using the CNOT gate

\[ U_{\text{CNOT}}^{\mathcal{E}} = |0\rangle_S |0\rangle_R \otimes I^\mathcal{E} + |1\rangle_S |1\rangle_R \otimes (|\uparrow\rangle_\mathcal{E} |\downarrow\rangle_\mathcal{R} + |\downarrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R}) \]

which flips the state of when is in the state |\uparrow\rangle_S. This produces

\[ U_{\text{CNOT}}^{\mathcal{E}}|\psi\rangle_S |E\rangle_\mathcal{E}R = |0\rangle_S \left[ |\uparrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R} + |\downarrow\rangle_\mathcal{E} |\downarrow\rangle_\mathcal{R} \right] \]

\[ + |1\rangle_S \left[ |\downarrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R} + |\uparrow\rangle_\mathcal{E} |\downarrow\rangle_\mathcal{R} \right]. \]

There is no record in about the pointer basis \{\uparrow\rangle_S, |\downarrow\rangle_S\} even though this is the basis in which is decohered. The distinction between |\uparrow\rangle_S and |\downarrow\rangle_S has been recorded in the expanded environment \mathcal{E} \otimes \mathcal{R}, but this information is not accessible in just \mathcal{E}. Rather, it is encoded in the correlation between \mathcal{E} and \mathcal{R}: When is up, \mathcal{E} and \mathcal{R} point in the same direction. When \mathcal{S} is down, they point in opposite directions. But looking at \mathcal{E} (or just \mathcal{R}) tells one nothing about the pointer basis. More generally, when the initial state of \mathcal{E} is partially mixed, the ability of \mathcal{E} to learn information about \mathcal{S} can be analyzed using methods from the theory of communication by treating \mathcal{E} as a noisy channel.

For an arbitrary pointer basis \{|S_a\rangle\}, the crucial difference between mere decoherence (of \{|S_a\rangle\} by \mathcal{E}) and the existence of records (about \{|S_a\rangle\} in \mathcal{E}) is encoded in the distinguishability of the conditional states of the environment \rho_{\mathcal{E}|S} = U_{ab} \rho_{\mathcal{E}|S} U_{ab}^\dagger. For an observer to be able to clearly discriminate between the various pointer states by making a measurement on \mathcal{E}, the conditional states \rho_{\mathcal{E}|S} must be mutually orthogonal. This automatically implies that decoherence factors \Gamma_{ab} vanish for \(a \neq b\) (records imply decoherence) but the converse is not true, as the above example above shows.

We can get a final state with the same form as \[10\] using a pure global state by allowing inter-environmental interactions. (Remember that purity of the global state is an invariant property for a closed quantum universe.) For instance, starting in the product initial state \(|\psi\rangle_S |\uparrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R} \) and applying a CNOT gate \(U_{\text{CNOT}}^{SR} \) to \mathcal{R} (conditional on \mathcal{S}) just like \[9\] yields the GHZ state:

\[ U_{\text{CNOT}}^{SR} U_{\text{CNOT}}^{SE} |\psi\rangle_S |\uparrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R} = |0\rangle_S |\uparrow\rangle_\mathcal{E} |\uparrow\rangle_\mathcal{R} + |1\rangle_S |\downarrow\rangle_\mathcal{E} |\downarrow\rangle_\mathcal{R}. \]

This is pure decoherence with a product global state, and it yields redundant records (which was unavoidable \[6\]). At this point, both \mathcal{E} and \mathcal{R} have a complete record of \mathcal{S}. But now one can see that a local interaction between \mathcal{E} and \mathcal{R} can recover \[10\].

C. Fragmentation

The location and accessibility of records will be of paramount importance. For this reason, we will be interested in fragments of the environment formed by proper subsets of its subsystems, as depicted in Figure 1. In this case, pure decoherence is of the form

\[ e^{-i t H} = \sum_a |S_a\rangle \langle S_a| \otimes \bigotimes_{i=1}^N U_{E_i}^a, \]

where the environment is composed of \(N\) parts, \(E = \bigotimes_{i=1}^N E_i\), that is initially uncorrelated, \( \rho^{SE} = \rho^{S|0} \otimes \bigotimes_{i=1}^N \rho^{E_i|0} \). A fragment is \( F = \bigotimes_{i \in F} E_i \), where \( F \subset \{1, \ldots, N\} \). We also define the complement fragment \( \overline{F} = \bigotimes_{i \notin F} E_i \) so \( E = \mathcal{E} \otimes \overline{F} \). This allows us to isolate a fragment’s contribution to the decoherence factor:

\[ \Gamma_{ab} = \Gamma_{ab}^F \Gamma_{ab}^{\overline{F}} \]

where

\[ \Gamma_{ab}^{\overline{F}} = \text{Tr} \left[ U_a^F \rho^{\overline{F}|0} U_b^{\overline{F}} \right] = \Gamma_{ba}^F. \]

Equation \[11\] features two records of \(S\), one in \(E\) and one in \(\mathcal{R}\). But, as discussed, we expect real-life quantum systems to have many records, taking the form of branching states \[58\] \[65\]:

\[ |\psi\rangle \approx \sum_a |S_a\rangle |E_1^{(a)}\rangle |E_2^{(a)}\rangle |F_3^{(a)}\rangle \ldots. \]

These are approximate GHZ states, with each term in the sum corresponding to a “branch” in the global wavefunction. We don’t necessarily expect this idealized branching
structure on the microscopic level. (Individual particle may be entangled within a branch, e.g. in spin singlet states.) But, for macroscopic systems, large fragments \( \mathcal{F}_t \) will often take the above approximate form which exhibits redundancy of records.

### III. CONSISTENT HISTORIES

In this section, the consistent histories framework [14] is briefly summarized. See [15] [17] and references therein for a broad review.

#### A. The consistent histories framework

At a fixed time \( t \), alternative outcomes within a closed quantum system are represented by a complete set of orthogonal projection operators \( \{ P_\alpha \} \), satisfying

\[
\sum_\alpha P_\alpha = 1, \tag{15}
\]

\[
P_a P_b = \delta_{ab} P_a, \tag{16}
\]

\[
P_\alpha^\dagger = P_\alpha. \tag{17}
\]

The probabilities of these outcomes for a given state \( \rho \) are given by

\[
p_\alpha = \text{Tr}[P_\alpha \rho P_\alpha^\dagger] = \text{Tr}[P_\alpha \rho]. \tag{18}
\]

In the case of a pure global state \( \rho = |\psi\rangle \langle \psi| \), the probability is just the squared norm of the conditional state \( |\psi_\alpha\rangle = P_\alpha |\psi\rangle \).

A **history** is a conjunction of several outcomes at different times \( t_m \), and it is represented by a class operator[^2] of the form

\[
C_\alpha = P_{a_M}^{(M)}(t_M)...P_{a_1}^{(1)}(t_1), \tag{19}
\]

where \( \alpha = (a_1,...,a_M) \). Here,

\[
P_{a_m}^{(m)}(t_m) = U_{t_m,t_{m+1}}^\dagger P_{a_m} U_{t_m,t_{m+1}} \tag{20}
\]

are the time-evolved projectors with implicit \( t \) dependence and \( U_{t',t''} = e^{-i(t''-t')H} \) is the unitary which evolves from \( t' \) to \( t'' \). The set \( \{ P_{a_m}^{(m)} \} \) is a complete set of projectors, indexed by \( a_m \), for each fixed value of \( m \). The class operators obey

\[
\sum_\alpha C_\alpha = I. \tag{21}
\]

One would like to assign probabilities to histories according to

\[
p_\alpha = \text{Tr}[C_\alpha \rho C_\alpha^\dagger]. \tag{22}
\]

In the case of a pure state \( |\psi\rangle \), this would again be just the squared norm of the conditional state \( |\psi_\alpha\rangle = C_\alpha |\psi\rangle \). Although such numbers necessarily are real, positive, and sum to unity, they do not obey the probability sum rules. That is, it is not generally true that the probability [as defined by (22)] that either of two mutually exclusive histories is realized equals the sum of the probabilities for the two individually:

\[
p_{\alpha \lor \beta} = p_\alpha + p_\beta, \tag{23}
\]

where \( C_{\alpha \lor \beta} = C_\alpha + C_\beta \) is a coarse-graining of \( C_\alpha \) and \( C_\beta \). \( C_\alpha \) and \( C_\beta \) are fine-grainings of \( C_{\alpha \lor \beta} \), and the operations of coarse- and fine-graining of histories form a partial order [11]. The failure of this probability sum rule to hold is a manifestation of quantum interference, which prevents the indiscriminate application of classical logic to quantum systems.

We therefore require that a set of histories obey the condition

\[
\mathcal{D}(\alpha,\beta) = 0, \quad \alpha \neq \beta \tag{24}
\]

where

\[
\mathcal{D}(\alpha,\beta) = \text{Tr}[C_\alpha \rho C_\beta^\dagger] \tag{25}
\]

is the decoherence functional. If so, we say that the histories are consistent[^3].

[^2]: The term “class” for these operators reflects the fact that, in general, the projectors composing them will be multi-dimensional. Therefore, the class operators select many fine-grained histories, e.g. Feynman paths.

[^3]: There are reasons to consider other conditions [23] [64] [68] besides...
(Equation 24 has often been called medium decoherence [39]. We resist this practice and reserve “decoherence” for the physical process predicated on a system-environment distinction as discussed in Section III C.)

Importantly, the decoherence functional does not have time dependence and takes the same form in the Heisenberg picture. Consistency is a (non-dynamical) property of a state $\rho$ and a set of histories $\{C_\alpha\}$.

We can also define for any subsystem $A$ (where $\mathcal{H} = A \otimes B$) the normalized state of $A$ conditional on $\alpha$:

$$\rho^A_\alpha = \frac{\text{Tr}_B[C_\alpha \rho C^\dagger_\alpha]}{\text{Tr}[C_\alpha \rho C^\dagger_\alpha]} = \frac{\text{Tr}_B[C_\alpha \rho C^\dagger_\alpha]}{p_\alpha}.$$  \hfill (26)

In particular, the conditional global states are $\rho_\alpha = C_\alpha \rho C^\dagger_\alpha / p_\alpha$.

### B. Records

The consistency condition (24) is the mixed-state generalization of the natural requirement that the conditional pure states do not overlap, $\langle \psi_\alpha | \psi_\beta \rangle = 0$, and it ensures that (23) holds. For a pure state, the branches of a consistent set of histories will form the tree structure in Figure 2 and there will necessarily exist records, i.e. a complete set of orthogonal projectors $R_\alpha$ such that $R_\alpha |\psi\rangle = C_\alpha |\psi\rangle = |\psi_\alpha\rangle$ [11]. (The class operators $C_\alpha$ are generally not projectors.)

In the case of a mixed state, the existence of records is defined as [11]

$$R_\alpha \rho = C_\alpha \rho.$$  \hfill (27)

This is equivalent to saying that the supports of the conditional states $\rho_\alpha$ lie in orthogonal subspaces. It is a sufficient but not necessary condition for (24) to hold. (Equation (27) was once called “strong decoherence” [11], although that term was later re-appropriated [28] for a modified version of (24).)

Records are important because in principle they allow observers within the Universe to become correlated with the history by making a projective measurement. However, these need not be feasible measurements because the records may not be confined to any particular subsystem; rather, they will usually be encoded across the entire global state. Gell-Mann and Hartle emphasize this by calling (27) generalized records. Naively, it might lead us think that deducing the path taken by a dust grain could require measuring an observable that involves all the photons in the room. The potential inaccessibility of such records is a consequence of the fact that only consistency of the set of histories has been assumed. In contrast, we aim in this article to describe why reasonably accessible and redundant records of histories should exist in real-life quantum systems.

### C. Pure decoherence and approximate consistency

The decoherence functional satisfies [71]

$$|\mathcal{D}(\alpha, \beta)|^2 = |\mathcal{D}(\alpha, \alpha)\mathcal{D}(\beta, \beta)|$$  \hfill (28)

with the probability sum rules obeyed if $|\mathcal{D}(\alpha, \beta)|^2 = 0$ for $\alpha \neq \beta$. For realistic quantum systems with simple, physically meaningful projectors, a set of histories will almost never be precisely consistent. But suppose that the off-diagonal terms of the decoherence functional are small, to some order $\epsilon$, compared to the on-diagonal terms:

$$|\mathcal{D}(\alpha, \beta)|^2 < \epsilon |\mathcal{D}(\alpha, \alpha)\mathcal{D}(\beta, \beta)|, \quad \alpha \neq \beta.$$  \hfill (29)

Then one can show that the probability sum rules are valid to the same order $\epsilon$ for almost every possible coarse-graining [71].

So define, for any two different histories $\alpha$ and $\beta$, a consistency factor

$$\mathcal{C}_{\alpha, \beta} = \frac{|\mathcal{D}(\alpha, \beta)|}{\sqrt{|\mathcal{D}(\alpha, \alpha)| |\mathcal{D}(\beta, \beta)|}} = |\mathcal{D}(\alpha, \beta)| / \sqrt{p_\alpha p_\beta}.$$  \hfill (30)

We say that the histories are approximately consistent to order $\epsilon$ when $|\mathcal{C}_{\alpha, \beta}| < \epsilon$ for $\alpha \neq \beta$ [71, 72].

Now consider the case of pure decoherence discussed in Section II A. We choose by hand the simple set of histories

$$\{C_\alpha = P_a = P_a^S \otimes I^E\},$$  \hfill (31)

where the projectors act non-trivially only on $S$. Such histories of a single quantum subsystem are reasonably called trajectories of the subsystem. The consistency factor (30) is just the decoherence factor [5] up to a phase:

$$\mathcal{C}_{\alpha, \beta} = \frac{\mathcal{D}(\alpha, \beta)}{\sqrt{p_\alpha p_\beta}} = \frac{\text{Tr} [P_a |\psi\rangle \langle \psi | P_b]}{\sqrt{\text{Tr} [P_a |\psi\rangle \langle \psi | P_a] \text{Tr} [P_b |\psi\rangle \langle \psi | P_b]}}$$  \hfill (32)

$$= \frac{c_a c_b^*}{|c_a c_b|^2} \text{Tr} [U_a^E U_b^E]$$

$$= \frac{c_a c_b^*}{|c_a c_b|^2} \Gamma_{\alpha, \beta}.$$

(This phase can be absorbed into the states $|S_a\rangle$.) There are $N(N - 1)/2$ (fixed) factors measuring consistency for a set of $N$ histories, which we can exactly identify with the $N(N - 1)/2$ decoherence factors in the case of pure
decoherence. Very loosely, pure decoherence is to states as consistency is to histories.

That said, we emphasize that decoherence is a dynamical physical process predicated on a distinction between system and environment, whereas consistency is a static property of a set of histories, a Hamiltonian, and an initial state. For a given decohering quantum system, there is generally a preferred basis of pointer states [18]. In contrast, the mere requirement of consistency does not distinguish a preferred set of histories which describe classical behavior from any of the many sets with no physical interpretation. For an arbitrary set of histories, consistency factors will be defined but there may be no system-environment decomposition, and hence no pointer basis or decoherence factors. Therefore, we will retain the semantic distinction between consistency factors and decoherence factor, using the latter only when there is at least an approximate pointer basis (such as Gaussian wavepackets [9] or hydrodynamical variables [11][13]). For more discussion of the connection between the consistency of histories and the physical process of decoherence, see Refs. [26] [27] [46] [61] [73].

IV. PARTIAL-TRACE CONSISTENCY

We now define the concept of partial-trace consistency [61], which bridges the gap between decoherence and consistent histories. Given a decomposition of Hilbert space into two parts, \( \mathcal{H} = \mathcal{A} \otimes \mathcal{B} \), we define a partial-trace decoherence functional by tracing over only \( \mathcal{B} \):

\[
\mathcal{D}_\mathcal{B}(\alpha, \beta) = \text{Tr}_\mathcal{B} \left[ C_\alpha \rho C_\beta^\dagger \right]
\]

\[
= \text{Tr}_\mathcal{B} \left[ U_{t_M+1} P_{\alpha M} (t_M) U_{t_M+1}^\dagger \ldots P_{\alpha_1} (t_1) U_{t_1} \right] \rho U_{t_1}^\dagger \rho U_{t_1} \ldots P_{\alpha_1} (t_1) U_{t_1}^\dagger (33)
\]

Note that \( \mathcal{D}_\mathcal{B} \) is an operator acting on \( \mathcal{A} \) and has implicit \( t \) dependence, unlike \( \mathcal{D} \). (As we shall see, this reflects the fact that records exist at certain times in certain places.) In contrast to Ref. [61] (see equation (1) therein), we have retained the traditional definition of the class operators [19] using time-evolved projectors. This allows us to consider a partial-trace decoherence functional evaluated at any time \( t \) rather than just at the final time step \( t_M \).

We define partial-trace consistency with respect to \( \mathcal{B} \) at some time \( t \), or \( \mathcal{B} \)-consistency, as

\[
\mathcal{D}_\mathcal{B}(\alpha, \beta) = 0.
\]

The partial-trace decoherence functional obeys the following basic operator relations:

\[
\text{Tr}_\mathcal{A} \mathcal{D}_\mathcal{B}(\alpha, \beta) = \mathcal{D}(\alpha, \beta),
\]

\[
\mathcal{D}_\mathcal{B}(\beta, \alpha) = \mathcal{D}_\mathcal{B}^\dagger(\alpha, \beta),
\]

\[
\sum_{\alpha, \beta} \mathcal{D}_\mathcal{B}(\alpha, \beta) = \text{Tr}_\mathcal{B} \rho = \rho^\mathcal{A}.
\]

When originally introduced, the partial-trace decoherence functional was taken with respect to the environment \( (\mathcal{A} \rightarrow \mathcal{S}, \mathcal{B} \rightarrow \mathcal{E}) \). For our purposes, we will assume that the environment can be broken into a fragment and its complement, \( \mathcal{E} = \mathcal{F} \otimes \mathcal{F} \), and study \( \mathcal{F} \)-consistency.

A. Properties

We now collect some known properties of the partial-trace decoherence functional. We fill in those proofs which were originally absent from Ref. [61] and make the extension from \( \mathcal{E} \)-consistency to \( \mathcal{F} \)-consistency where appropriate. We will occasionally be sloppy with the order of the tensor product decomposition notation by interchanging \( \mathcal{S} \otimes \mathcal{E} \otimes \mathcal{F} \) and \( \mathcal{S} \otimes \mathcal{F} \otimes \mathcal{F} \) where the meaning is clear.

Inheritance of consistency:

Any histories which are \( \mathcal{F} \)-consistent are automatically \( \mathcal{E} \)-consistent and therefore consistent.

\[
\mathcal{D}_\mathcal{E}(\alpha, \beta) = 0 \Rightarrow \mathcal{D}_\mathcal{F}(\alpha, \beta) = 0
\]

This follows immediately from taking the trace.

Consistency implies diagonalization:

For trajectories of the system \( \mathcal{S} \) (i.e. histories constructed of projectors of the form \( \mathcal{P}^\mathcal{S} \otimes I^\mathcal{E} \)), \( \mathcal{E} \)-consistency (and hence \( \mathcal{F} \)-consistency) at the final time step \( t = t_M \) implies the density matrix of the system \( \rho^\mathcal{S} \) is block diagonal on subspaces associated with those projectors.

To show this, sum \( \mathcal{D}_\mathcal{E}(\alpha, \beta) \) over all projectors for \( \alpha \) and \( \beta \) except at the final time. If \( a_M \neq b_M \), then \( \alpha \neq \beta \) and

\[
0 = \sum_{a_1} \ldots \sum_{a_{M-1}} \sum_{b_1} \ldots \sum_{b_{M-1}} \sum \left[ \mathcal{D}_\mathcal{E}(\alpha, \beta) |_{a_M} \right]
\]

\[
= \text{Tr}_\mathcal{E} \left[ \left( \mathcal{P}^\mathcal{S} \otimes I^\mathcal{E} \right) \left( U_{t_M+1} P_{a_M} (t_M) U_{t_M+1}^\dagger \ldots P_{a_1} (t_1) U_{t_1} \right) \rho U_{t_1} \ldots P_{a_1} (t_1) U_{t_1}^\dagger \right]
\]

\[
= \text{Tr}_\mathcal{E} \left[ \left( \mathcal{P}^\mathcal{S} \otimes I^\mathcal{E} \right) \rho |_{a_M} \right] \left( \mathcal{P}^\mathcal{S} \otimes I^\mathcal{E} \right)
\]

\[
= \mathcal{P}^\mathcal{S} \rho |_{a_M} \mathcal{P}^\mathcal{S}.
\]

Effective wavefunction collapse:

\( \mathcal{E} \)-consistency implies that the conditional states of the system, \( \rho^\mathcal{S} = \text{Tr}_\mathcal{E} \left[ C_\alpha \rho C_\beta^\dagger |_{a_M} \right] \), obey

\[
p_{a_M \beta} = p_{a_M \alpha} + p_{a_M \beta} \mathcal{F}.
\]

This is a generalization of the probability sum rule, [23]. To prove (40), recall that \( C_{a_M \beta} = C_{a_M} + C_{\beta} \) so that

\[
\mathcal{D}_\mathcal{E}(\alpha \vee \beta, \alpha \vee \beta) = \mathcal{D}_\mathcal{E}(\alpha, \alpha) + \mathcal{D}_\mathcal{E}(\alpha, \beta)
\]

\[
\mathcal{D}_\mathcal{E}(\beta, \alpha) + \mathcal{D}_\mathcal{E}(\beta, \beta) \]

\[
= \mathcal{D}_\mathcal{E}(\alpha, \alpha) + \mathcal{D}_\mathcal{E}(\beta, \beta),
\]

(41)
where the second line is due to $\mathcal{E}$-consistency. We get (40) from the definition of the conditional state and the fact that $p_{\alpha'\beta'} = p_{\alpha} + p_{\beta}$ by the consistency of the histories.

**Extension of histories:**

Insofar as the fragment decouples from the system and the rest of the environment—thus precluding recoherence—any two particular histories $\alpha$ and $\beta$ of the system that are $\mathcal{F}$-consistent will remain so when extended into the future with additional histories $\alpha'$ and $\beta'$. More precisely, suppose $\mathcal{D}_\mathcal{F}(\alpha, \beta) = \text{Tr}_\mathcal{F}[C_{\alpha}\rho C_{\beta}^\dagger] = 0$ for histories up to some fixed time $t \geq t_M$, and the Hamiltonian for times after $t$ takes the form $H' = H'_{\mathcal{F}} \otimes I^\mathcal{F} + I^\mathcal{F} \otimes H^\mathcal{F}$. Then the history $\alpha$ of the system extended to some later time $t' \geq t$ is of the form $C_{\alpha'\alpha'} = C_{\alpha'}(U_{t't}^\mathcal{F} \otimes U_{t't'}^\mathcal{F})C_{\alpha}$ where

$$C_{\alpha'} = C_{\alpha'}^\mathcal{F} \otimes I^\mathcal{F},$$

$$C_{\alpha'}^\mathcal{F} = P_{\alpha_m'}^{(M+m')}(t_{M+m'}) \cdots P_{\alpha_{M+1}'}(t_{M+1}),$$

$$P_{\alpha_{M+m}'}(t_{M+m'}) = U_{t't, t_{M+m'}} P_{\alpha_{M+m}'}U_{t't, t_{M+m'}}.$$  

A similar extension is made for the history $\beta$, and the two extended histories can be seen to be $\mathcal{F}$-consistent at time $t'$:

$$\mathcal{D}_\mathcal{F}(\alpha \wedge \alpha', \beta \wedge \beta') = \text{Tr}_\mathcal{F}[C_{\alpha'\alpha'}\rho C_{\beta\beta'}] = C_{\alpha'}^\mathcal{F}U_{t't}^\mathcal{F}\text{Tr}_\mathcal{F}[C_{\alpha}\rho C_{\beta}^\dagger]U_{t't'}^\mathcal{F}C_{\beta'}^\dagger = 0$$

This ability of small decoupled fragments to guarantee partial-trace consistency was discussed earlier in Ref. [25].

As emphasized by Finkelstein, these properties are evidence that $\mathcal{E}$-consistency is the natural requirement for saying that a set of histories are consistent because of the physical process of decoherence, i.e. that the histories truly decohere rather than merely attain consistency. The inherent time dependence of the partial-trace decoherence functional emphasizes the important distinction between the physical process of decoherence (which occurs between two quantum systems, such as $\mathcal{S}$ and $\mathcal{F}$, during an interval of time) and the mathematical condition of consistency (which applies timeless to a set of histories, and which does not require a decomposition of the Hilbert space into subsystems). Finkelstein's concept of partial-trace consistency elegantly links these two concepts.

**B. Partial-trace consistency factor**

For an arbitrary decomposition $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$, the partial-trace decoherence functional has matrix elements bounded as [61]

$$|\langle A_i|\mathcal{D}_\mathcal{B}(\alpha, \beta)|A_j \rangle|^2 \leq \langle A_i|\mathcal{D}_\mathcal{B}(\alpha, \alpha)|A_i \rangle \langle A_j|\mathcal{D}_\mathcal{B}(\beta, \beta)|A_j \rangle$$  

(46)

for any basis $|A_i \rangle$ of $\mathcal{A}$. This suggests that the proper way to generalize the approximate consistency condition for the partial-trace decoherence functional is to require that the left-hand side of (46) is much smaller than the right-hand side for all $i$ and $j$ [61]. More specifically, define a partial-trace consistency factor

$$\mathcal{E}_\alpha^\mathcal{B} = \frac{|\langle A_i|\mathcal{D}_\mathcal{B}(\alpha, \beta)|A_j \rangle|^2}{\sqrt{\text{Tr}_\mathcal{B}[C_{\alpha}\rho C_{\beta}^\dagger]\text{Tr}_\mathcal{B}[C_{\alpha}\rho C_{\beta}^\dagger]}} \leq \sqrt{\text{Tr}_\mathcal{B}[C_{\alpha}\rho C_{\beta}^\dagger]}.  \quad (47)$$

We now relate the partial-trace consistency factor to the decoherence factor, and to the fidelity between conditional states.

**Decoherence factor and consistency factor:**

In the special case of pure decoherence, [22] and [31], with orthonormal pointer states $|S_{\alpha} \rangle$ that record the history, the partial-trace consistency factor for the environment $\mathcal{E}$ is

$$\mathcal{E}_\alpha^\mathcal{E} = \mathcal{E}_\alpha^\mathcal{B} = \mathcal{E}_\alpha^\mathcal{B} = \Gamma_{\alpha\beta}|S_{\alpha}\rangle(S_{\beta}\rangle.$$  

(48)

When (as considered in Section II C) there are no intra-environmental interactions and there is a product initial state, we can more generally write

$$\mathcal{E}_\alpha^\mathcal{E} = \mathcal{E}_\alpha^\mathcal{B} = \mathcal{E}_\alpha^\mathcal{B} = \mathcal{E}_\alpha^\mathcal{B} = \Gamma_{\alpha\beta}|S_{\alpha}\rangle(S_{\beta}\rangle \otimes U_{\alpha}^\mathcal{F}p^\mathcal{F}U_{\beta}^\mathcal{F}\dagger.$$  

(49)

The operator $U_{\alpha}^\mathcal{F}p^\mathcal{F}U_{\beta}^\mathcal{F}\dagger$ is not generally a density matrix, but for any unitarily invariant norm $\| \cdot \|$ (like the trace norm; see below) we have $\| \mathcal{E}_\alpha^\mathcal{B} \| = \mathcal{E}_\alpha^\mathcal{B} \| p^\mathcal{F} \|$, where $\| p^\mathcal{F} \|$ is fixed by initial conditions. So for pure decoherence without intra-environmental interactions, the decoherence factor $\Gamma_{\alpha\beta}^\mathcal{F}$ controls the norm of the consistency factor $\mathcal{E}_\alpha^\mathcal{B}$. One vanishes if and only if the other does.

**Fidelity and consistency factor:** When the global state is pure, $\rho = |\psi\rangle\langle\psi|$, the trace norm of the partial-trace consistency factor equals the fidelity between the relevant conditional states:

$$F(\rho_\alpha^\mathcal{A}, \rho_\beta^\mathcal{A}) = \| \mathcal{E}_\alpha^\mathcal{B} \|_1,$$  

(50)

where $\| W \|_1 = \text{Tr}|W|^2 = \text{Tr}\sqrt{W\dagger W}$ is the trace norm of an operator $W$ and $F(\sigma_1, \sigma_2) = \| \sqrt{\sigma_1} \sqrt{\sigma_2} \|_1$ is
the fidelity. (Note that we use the un-squared convention; some authors refer to \( F^2 \) as the fidelity.) To show this, let the Schmidt decomposition of the branches \( |\psi_a\rangle \) be
\[
|\psi_a\rangle = p_a \sum_r \sqrt{d_r^{(a)}} |A_r^{(a)}\rangle |B_r^{(a)}\rangle
\] (51)
where, for each \( a \), the \( d_r^{(a)} \) are positive coefficients and the \( |A_r^{(a)}\rangle \) and \( |B_r^{(a)}\rangle \) form orthonormal bases for \( A \) and \( B \). Then the partial-trace consistency factors are
\[
\mathcal{C}_{\alpha,\beta} = \sum_{r,s} \sqrt{d_r^{(a)} d_s^{(b)}} \langle A_r^{(a)}|A_s^{(b)}\rangle \langle B_r^{(a)}|B_s^{(a)}\rangle.
\] (52)

On the other hand, the conditional states of \( A \) are
\[
\rho_{\alpha}^A = \frac{\text{Tr}_B[|\psi_a\rangle\langle\psi_a|]}{p_a} = \sum_r d_r^{(a)} |A_r^{(a)}\rangle \langle A_r^{(a)}|.
\] (53)

Direct computation then shows that
\[
F(\rho_{\alpha}^A, \rho_{\beta}^A) = \text{Tr} \sqrt{M} = \| \mathcal{C}_{\alpha,\beta} \|_1
\] (54)
where \( M \) is a matrix with elements
\[
M_{r,r'} = \sum_s \langle A_s^{(r)}|A_s^{(r')}\rangle \langle A_s^{(r)}|A_s^{(r')}\rangle d_s^{(r)} \sqrt{d_s^{(r)} d_s^{(r')}}.
\] (55)

C. Records

We now discuss the relationship between records and consistency.

**Conditions for records:**

The following statements are equivalent conditions for saying a record of the history \( \alpha \) exists in \( A \).

1. The conditional states \( \rho_{\alpha}^A \) are orthogonal. That is, their respective supports are mutually orthogonal subspaces.
2. The fidelities of the conditional states obey \( F(\rho_{\alpha}^A, \rho_{\beta}^A) = \delta_{\alpha,\beta} \).
3. There are orthogonal projectors \( R_\alpha = R_\alpha^A \otimes I_B \) acting non-trivially only on \( A \) which pick out the conditional states \( \rho_\alpha \):
\[
(R_\alpha^A \otimes I_B) \rho = C_\alpha \rho
\] (56)

This implies \( R_\alpha \rho R_\alpha = C_\alpha \rho C_\alpha^\dagger = p_\alpha \rho_\alpha \) and \( R_\alpha^A \rho R_\alpha = \rho R_\alpha^A \rho R_\alpha = p_\alpha \rho_\alpha^A \).

This means that a observer could realistically determine the history \( \alpha \) by making only a local measurement on \( A \).

That (1) \( \iff \) (2) is a basic property of the quantum fidelity \( \mathcal{F} \). We get (1) \( \iff \) (3) by defining \( R_\alpha^A \) to project onto the support of \( p_\alpha^A(T) \).

**Records imply consistency:**

If there is a record of the history in \( A \), then the histories are \( A \)-consistent.

This can be shown by first unraveling the global state, \( \rho = \sum \lambda_z |z\rangle \langle z| \) (where \( \lambda_z > 0 \), and considering the Schmidt decomposition of \( C_\alpha |z\rangle \) over the \( A-B \) decomposition:
\[
C_\alpha |z\rangle = \sum_j c_{j,z}^{(a)} |A_{j,z}^{(a)}\rangle |B_{j,z}^{(a)}\rangle.
\] (57)

Expanding the condition for orthogonality of the conditional states \( (\text{Tr}_A[\rho_\alpha^A \rho_\beta^A] = 0 \) when \( \alpha \neq \beta \)) with \( \text{(57)} \) shows that \( \langle A_{j,z}^{(a)}|A_{j',z'}^{(a)}\rangle = 0 \) for all \( j, j', z, z' \) whenever \( \alpha \neq \beta \). \( A \)-consistency then follows by evaluating \( \mathcal{D}_A(\alpha, \beta) \) with \( \text{(57)} \).

**Consistency with purity implies records:**

If the global state is pure, \( \rho = |\psi\rangle \langle \psi| \), then \( A \)-consistency implies there is a record of the history in \( A \). This follows from \( \text{(50)} \).

In short: the existence of a local record is a sufficient condition for partial-trace consistency. When the global state is pure, it is a necessary condition. This is a “historical generalization” of the previously known relationship between generalized records and consistency (in the consistent histories setting) and between records and decoherence (in the quantum Darwinism setting).

D. Records correlated in time

Consider an \( \mathcal{F} \)-consistent set of histories \( \{\alpha\} \) of the system \( S \) with a pure global state \( \rho = |\psi\rangle \langle \psi| \), so that the tree structure in Figure 1 is realized. By the last property in the previous section, we know that there exist branches \( |\psi_a\rangle \in S \otimes \mathcal{F} \otimes \mathcal{F}_\alpha \), where the \( \mathcal{F}_\alpha \) are mutually orthogonal subspaces of \( \mathcal{F} \). Now, let \( \{\hat{\alpha}\} \) be a coarse-grained set of histories obtained by summing over the projectors at the final time step:
\[
C_{\hat{\alpha}} = \sum_{a_M} C_{\alpha}
\]
\[
= \sum_{a_M} P_{a_M}(t_M) P_{a_{M-1}}(t_{M-1}) \ldots P_{a_1}(t_1)
\] (58)
\[
= P_{a_M}(t_M) \ldots P_{a_1}(t_1).
\]

Then
\[
|\psi_{\hat{\alpha}}\rangle = \sum_{a_M} |\psi_a\rangle \in S \otimes \mathcal{F} \otimes \mathcal{F}_{\hat{\alpha}}
\] (59)
where the \( \mathcal{F}_{\hat{\alpha}} = \bigoplus_{a_M} \mathcal{F}_\alpha \) are direct sums of orthogonal subspaces \( \mathcal{F}_\alpha \). The \( \mathcal{F}_{\hat{\alpha}} \) are therefore also orthogonal.

By repeatedly summing over the final set of projectors, we can continue this process and show that a branch structure on the Hilbert space of \( \mathcal{F} \) can be found which is equivalent to the branch structure of the \( |\psi_{\hat{\alpha}}\rangle \) (seen in
More precisely, it follows that for all $\tilde{m}$ (with $1 \leq \tilde{m} \leq \tilde{M}$) we can decompose

$$\mathcal{F}(a_1, \ldots, a_{m-1}) = \bigoplus_{a_m} \mathcal{F}(a_1, \ldots, a_{m-1}, a_m)$$

(60)

into mutually orthogonal subspaces where

$$|\psi(a_1, \ldots, a_{m-1})\rangle \in \mathcal{S} \otimes \mathcal{F} \otimes \mathcal{F}(a_1, \ldots, a_{m-1}).$$

(61)

If we allow for a non-interacting auxiliary environment $\mathcal{R}$ to be appended to the fragment $\mathcal{F}$, then there exists a decomposition of the supplemented fragment $\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{R}$ into subsystems

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}(1) \otimes \cdots \otimes \tilde{\mathcal{F}}(M)$$

(62)

and each subsystem into subspaces

$$\tilde{\mathcal{F}}(m) = \bigoplus_{a_m} \tilde{\mathcal{F}}(a_m)$$

(63)

(for each $m$) such that

$$|\psi(a_1, \ldots, a_M)\rangle \in \mathcal{S} \otimes \mathcal{F} \otimes \mathcal{F}(a_1, \ldots, a_M)$$

$$\subset \mathcal{S} \otimes \tilde{\mathcal{F}}(1) \otimes \cdots \otimes \tilde{\mathcal{F}}(M).$$

(64)

In other words, an observer can infer the state of $\mathcal{S}$ at time $t_m$ (more precisely, which of the projectors $\mathcal{P}_m$ it satisfied) by making a measurement of a subsystem $\tilde{\mathcal{F}}(m)$ at later time $t \geq t_M > t_m$.

Of course, there is no need for the decomposition in (62) to have anything to do with a natural separation of the environment into parts accessible to a realistic observer, so such a measurement may be completely infeasible. But in forthcoming paper [75], we show that collisional decoherence by photons provides a natural example where this decomposition is physically meaningful and exploited every day.

V. DISCUSSION

In the case of a pure global state, the consistency of a history is a global property representing orthogonality of the branches. But partial-trace consistency means the stronger fact that the history is consistent due to orthogonality in the particular subsystem traced out. Therefore, partial-trace consistency mathematically captures the intuitive idea that a history has been decohered by a particular subsystem. When a history is partial-trace consistent with respect to many systems, we can say it is redundantly decohered.

In the case of a mixed global state, decoherence does not imply records. This arose in the contexts of decoherence [8], quantum Darwinism [54, 57], and consistent histories [11, 28]. Using the definition of redundancy in the consistent histories framework we have introduced in this paper, we can combine these observations in a single stronger principle: The redundant consistency of a history is a necessary condition for the existence of redundant records (for any degree of redundancy $R$), and it is sufficient condition when the global state is pure.

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The inclusion of $\mathcal{R}$ ensures that there will be sufficient dimensionality in $\tilde{\mathcal{F}}$ to fit the entire tensor product (62). This might not otherwise be true if some histories have zero probability.

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