Nonexistence of Finite-dimensional Quantizations of a Noncompact Symplectic Manifold

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Abstract We prove that there is no faithful finite-dimensional representation by skew-hermitian matrices of a “basic algebra of observables” $\mathcal{B}$ on a noncompact symplectic manifold $M$. Consequently there exists no finite-dimensional quantization of any Lie subalgebra of the Poisson algebra $C^\infty(M)$ containing $\mathcal{B}$. 
1. Introduction

Let $M$ be a connected noncompact symplectic manifold. On physical grounds one expects a quantization of $M$, if it exists, to be infinite-dimensional. This is what we rigorously prove here, in the framework of the paper [GGT]. Our precise hypotheses are spelled out below.

A key ingredient in the quantization process is the choice of a basic set of observables in the Poisson algebra $C^\infty(M)$. This is a finite-dimensional linear subspace $B$ of $C^\infty(M)$ such that

- (B1) (Completeness) the Hamiltonian vector fields $X_f$, $f \in B$, are complete,
- (B2) (Transitivity) $\{X_f \mid f \in B\}$ spans $TM$, and
- (B3) (Minimality) $B$ is minimal with respect to these conditions.

In addition to these conditions we assume in this paper that $B$ forms a Lie algebra under the Poisson bracket. We then refer to $B$ as a basic algebra. (Note also that unlike in [GGT], we do not require here that $1 \in B$.)

Now fix a basic algebra $B$, and let $O$ be any Lie subalgebra of $C^\infty(M)$ containing $1$ and $B$. Then by a finite-dimensional quantization of the pair $(O, B)$ we mean a Lie representation $Q$ of $O$ by skew-hermitian matrices on $\mathbb{C}^n$ such that

- (Q1) $Q(1) = I$,
- (Q2) $Q(B)$ is irreducible, and
- (Q3) $Q$ is faithful on $B$.

We refer the reader to [GGT] for a detailed discussion of these matters. We remark that in the infinite-dimensional case there are additional conditions which must be imposed upon a quantization. We also elaborate briefly on (Q3). Although faithfulness is not usually assumed in the definition of a quantization, it seems to us a reasonable requirement in that a classical observable can hardly be regarded as “basic” in a physical sense if it is in the kernel of a quantization map. In this case, it cannot be obtained in any classical limit from the quantum theory.
2. The Obstruction

Given the definitions above, we state our result:

Theorem. Let $M$ be a noncompact symplectic manifold, $\mathcal{B}$ a basic algebra on $M$, and $\mathcal{O}$ any Lie subalgebra of $C^\infty(M)$ containing $\mathcal{B}$. Then there is no finite-dimensional quantization of $(\mathcal{O}, \mathcal{B})$.

As the proof will show, we do not need conditions (Q1) or (Q2) to obtain the theorem. Moreover, the subalgebra $\mathcal{O}$ is irrelevant since the proof depends only on the Lie theoretical properties of the basic algebra $\mathcal{B}$ and its action on $M$.

Proof: We argue by contradiction. Suppose there exists a finite-dimensional quantization $Q$ of the basic algebra $\mathcal{B}$. Since $Q(\mathcal{B})$ consists of skew-hermitian matrices, it is completely reducible. Since $Q$ is faithful, one deduces from [V, Thm 3.16.3] that $\mathcal{B}$ is reductive, i.e. $\mathcal{B} = s \oplus \mathfrak{z}$ where $s$ is semisimple and $\mathfrak{z}$ is the center of $\mathcal{B}$.

We show that $\mathfrak{z} = \{0\}$. Indeed, by the transitivity condition (B2), the elements of $\mathfrak{z}$ must be constant but, if these are nonzero, then $s$ alone would serve as a basic algebra, contradicting the minimality condition (B3). Thus $\mathfrak{z} = \{0\}$ and $\mathcal{B} = s$ is semisimple.

Let $B$ be the connected, simply connected Lie group with Lie algebra $\mathcal{B}$. We show that $B$ is noncompact. Let $\mathfrak{g}$ be the Lie algebra $\{X_f \mid f \in \mathcal{B}\}$. By (B1) the vector fields in $\mathfrak{g}$ are complete and so by [V, Thm. 2.16.13] this infinitesimal action of $\mathfrak{g}$ can be integrated to an action of the connected, simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Condition (B2) implies that this action is locally transitive and thus globally transitive as $M$ is connected. Thus the noncompact manifold $M$ is a homogeneous space for $G$, and so $G$ must be noncompact as well. Now $\mathcal{B}$ is isomorphic either to $\mathfrak{g}$ or to a central extension of $\mathfrak{g}$ by constants. Since $\mathcal{B}$ is semisimple, the latter alternative is impossible. Hence $B$ is isomorphic to $G$ and so is noncompact.

Now consider a unitary representation $\pi$ of $B$ on $\mathbb{C}^n$. Decompose $B$ into a product $B_1 \times \cdots \times B_K$ of simple groups. Then (at least) one of these, say $B_1$, must be
noncompact. But it is well-known that a connected, simple, noncompact Lie group has no nontrivial unitary representations [BR, Thm. 8.1.2]. Thus \( \pi|B_1 \) is trivial, i.e. \( \pi(b) = I \) for all \( b \in B \). Since every finite-dimensional quantization \( Q \) of \( B \) is a derived representation of some unitary representation \( \pi \) of \( B \), it follows that \( Q|B_1 = 0 \), and so \( Q \) cannot be faithful.

3. Discussion

This theorem is complementary to a recent result of [GGG] which states that there are no nontrivial quantizations (finite-dimensional or otherwise) of \( (P(B), B) \) on a compact symplectic manifold \( M \), where \( P(B) \) is the Poisson algebra of polynomials generated by the basic algebra \( B \). The proof of that result leaned heavily on the algebraic structure of \( P(B) \); indeed, when \( M \) is compact, it turns out that \( B \) must be compact semisimple, and such algebras do have faithful finite-dimensional representations by skew-hermitian matrices. Thus in the compact case, the obstruction to the existence of a quantization is Poisson, rather than Lie theoretical. Combining [GGG] with the present theorem, we can now assert, roughly speaking, that no symplectic manifold with a basic algebra has a finite-dimensional quantization.

We hope to address the issue of whether there are obstructions in general to infinite-dimensional quantizations of noncompact symplectic manifolds in future work. Certainly such obstructions exist in specific examples, such as \( \mathbb{R}^{2n} \) [GGT] and \( T^\ast S^1 \) [GG]. This appears to be a difficult problem, however.

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