Discretized Gravity on the Hyperbolic Disk

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Abstract. We consider discretized gravity in six dimensions, where the two extra dimensions have been compactified on a hyperbolic disk of constant curvature. We analyze a fine-grained realization of lattice gravity on the hyperbolic disk at the level of an effective field theory for massive gravitons. It is shown that a nonzero curvature or warping in radial direction allows to obtain a strong coupling scale that becomes in the infrared regime larger than in discretized warped five-dimensional space. In particular, when approaching the boundary of the discretized warped hyperbolic disk, the local strong coupling scale can be as large as the local Planck scale.

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INTRODUCTION

Curved space-time manifolds admit to realize a number of extremely interesting ideas ranging from the Randall-Sundrum (RS) models [1] to the AdS/CFT correspondence [2] or moduli stabilization in string compactifications with fluxes [3]. A strong space-time curvature is also advantageous when formulating lattice gravity in the context of an effective field theory (EFT) [4, 5]. In 5D flat space, the ultraviolet (UV) strong coupling scale depends on the bulk volume, or infrared (IR) scale, via a so-called “UV/IR connection” that would forbid to take the large volume limit within a sensible EFT [5]. Recently, it has been shown that this UV/IR connection can be avoided for discretized gravity in five-dimensional (5D) warped space-time [6, 7] such that a large volume limit like in RS II becomes possible. However, the strong coupling scale is in discretized 5D warped space still everywhere smaller than the local Planck scale.

In this talk, we consider a six-dimensional (6D) lattice gravity construction, where the two extra dimensions are compactified on a discretized hyperbolic disk [8] (see also Ref. [9]). We study a fine-grained latticization of a hyperbolic disk that is warped along the radial direction and estimate there the local strong coupling scale. It turns out that the presence of the 6th dimension can yield the theory on the discretized warped hyperbolic disk more weakly coupled than in the corresponding 5D warped case. In particular, the local strong coupling scale can become on the boundary of the disk as large as the local Planck scale.

1 Talk presented at SUSY06, the 14th Conference on Supersymmetry and the Unification of Fundamental Interactions, UC Irvine, California, 12-17 June 2006
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Consider 6D general relativity compactified to four dimensions on an orbifold $K_2/Z_2$, where $K_2$ is a two-dimensional hyperbolic disk of constant negative curvature. The 6D coordinates $x^M$ are labeled by capital Roman indices $M = 0, 1, 2, 3, 5, 6$ while Greek indices $\mu = 0, 1, 2, 3$ denote the usual 4D coordinates $x^\mu$. The 6D Minkowski metric is $\eta_{MN} = \text{diag}(1,-1,-1,-1,-1,-1)$. A point on the hyperbolic disk $K_2$ is described by polar coordinates $(r, \phi)$, where $r = x^5$ and $\phi = x^6$ with $r \in [0, L]$ and $\phi \in [0, 2\pi)$. Here, $r$ is the geodesic distance of the point $(r, \phi)$ from the center, i.e., $L$ is the hyperbolic radius of the disk. The disk is warped along the $r$ direction and the 6D metric for the warped hyperbolic disk is given by the line element

$$
 ds^2 = e^{2\sigma(r)}g_{\mu\nu}(x^\mu, r, \phi)dx^\mu dx^\nu - dr^2 - v^{-2}\sinh^2(vr)d\phi^2, 
$$

where $1/v > 0$ is the curvature radius of the disk, $g_{\mu\nu}(x^\mu, r, \phi)$ is the 4D metric, and $\sigma(r) = -wr$, where $w$ is the curvature scale for the warping. The geometry of the orbifold $K_2/Z_2$ with the definition of the UV and IR branes is shown in Fig. 1. Expanding $g_{\mu\nu}$ in terms of small fluctuations around 4D Minkowski space as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the relevant 6D kinetic part of the graviton Lagrangian density is then, to quadratic order, found to be of the form

$$
 \mathcal{S}_{\text{lin}} = M_6^4 \int d^6x v^{-1}\sinh(vr) \left[ \frac{1}{4}e^{4\sigma(r)}(\partial_r h_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(\partial_r h_{\alpha\beta}) 
 + \frac{1}{4}e^{4\sigma(r)}v^2\sinh^{-2}(vr)(\partial_\phi h_{\mu\nu})(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta})(\partial_\phi h_{\alpha\beta}) \right],
$$

where $M_6$ is the 6D Planck scale, $h = h_{\mu\nu}$, $h_\nu = \partial^\nu h_{\mu\nu}$, and $h_{5M} = h_{6M} = 0$.

**DISCRETIZATION AND STRONG COUPLING**

Let us consider a fine-grained latticization of the hyperbolic disk $K_2$ which is of the type shown in Fig. 2 where we will throughout interpret the sites (drawn as circles) and
FIGURE 2. Fine-grained discretization of the hyperbolic disk for $k_{\text{max}} = 4$. The small solid circles represent the sites and are labeled by an index $i$. On each site $i$ lives a graviton field $g^i_{\mu\nu}$ and two neighboring sites $i$ and $j$ on adjacent circles are connected by a link $(i,j)$ (solid lines). The sites are distinguished by an index $i$ and each site $i$ is equipped with its own metric $g^i_{\mu\nu}$ that can be expanded around 4D flat space as $g^i_{\mu\nu} = \eta_{\mu\nu} + h^i_{\mu\nu}$. The sites lie on $k_{\text{max}}$ concentric circles (dashed lines) that are labeled as $k = 0, 1, 2, \ldots, k_{\text{max}}$ when going from the center ($k = 0$) to the boundary ($k = k_{\text{max}}$). Fig. 2 shows the special case $k_{\text{max}} = 4$. On the $k$th circle, we have $N_k = 2^k$ sites, and the total number of sites on the first $k$ circles is $N_{\text{total}} = 2^{k+1} - 1$.

The exponential growth of $N_k$, when moving outward on the disk, is a salient feature of our graph that is important for the strong coupling behavior near the IR branes on the boundary. Like in Ref. [6], we assume that the radial geodesic coordinates of the sites are integer multiples of a common proper radial lattice spacing $a \equiv 1/m_*$ and take the values $k \cdot a$, where $k = 0, 1, 2, \ldots, k_{\text{max}}$, i.e., the $k$th circle has a proper radius $k \cdot a$. For a given site $i$, we will denote by $r_i$ the proper radius of the concentric circle on which the site $i$ is located. To introduce the warping along the radial direction of the fine-grained latticized disk, we proceed exactly like in the discretized 5D RS model in Ref. [6] and replace in the linearized gravitational action in Eq. (2), e.g., the derivatives in the $r$ direction by $e^{-2wr_i} \partial_r h^i_{\mu\nu} \rightarrow \frac{1}{a} e^{-2wr_i} (h^i_{\mu\nu} - h^j_{\mu\nu})$, where it is understood that $i$ and $j$ are neighboring sites on two adjacent concentric circles that are connected by a link $(i,j)$.

We thus obtain for the discretized gravitational Lagrangian on the warped hyperbolic disk $\mathcal{L}_{\text{lin}} = M_4^2 \sum_i e^{-2wr_i} h^i_{\mu\nu} \Box h^i_{\mu\nu} + \mathcal{L}_{\text{FP}}$, where $\mathcal{L}_{\text{FP}}$ are Fierz-Pauli mass terms that are schematically given by

$$\mathcal{L}_{\text{FP}} = M_4^2 \sum_{(i,j)} m_*^2 e^{-4wr_i} (h^i_{\mu\nu} - h^j_{\mu\nu}) (\eta^i_{\mu\nu} \eta^i_{\alpha\beta} - \eta^i_{\mu\alpha} \eta^i_{\nu\beta}) (h^j_{\alpha\beta} - h^j_{\alpha\beta}), \quad (3)$$

where $i$ and $j$ are neighboring sites connected by a link $(i,j)$ and $M_4$ is the universal 4D Planck scale on the sites. Restoring like in Refs. [4,6] general coordinate invariances by adding Goldstone bosons to $\mathcal{L}_{\text{lin}}$, we find for the local strong coupling scale on the $k$th
\[ \Lambda_{\text{warp}}^{6D}(k) = \sqrt{\frac{w}{M_4}(\sqrt{N_k M_k m_4^4 e^{-4wka}})^{1/5}} = N_k^{1/10} \Lambda_{\text{warp}}^{5D}(k), \tag{4} \]

where \( M_k \equiv M_4 e^{-wka} \) is the local Planck scale on the \( k \)th circle and \( \Lambda_{\text{warp}}^{5D}(k) \) is the strong coupling scale in the corresponding 5D warped case. The important point is here the presence of the factor \( N_k \) that can render the model more weakly coupled than in 5D warped space when \( N_k \) becomes exponentially large. For \( w/M_4 \simeq 0.1 \), we see that having \( \Lambda_{\text{warp}}^{5D}(k) \) as large as \( M_k \) requires exponentially many sites \( N_k \) on the \( k \)th circle, which is in our model possible in the outer regions of \( K_2 \) at the IR branes. In 5D warped space, \( \Lambda_{\text{warp}}^{5D}(k) \) is always smaller than the local Planck scale. Choosing, instead, in our 6D model, \( e.g., m_\ast = M_4 \) and \( w = (0.1) \times M_4 \), the local strong coupling scale can become on the hyperbolic disk as large as the local Planck scale \( M_k \) for values \( k = O(10^2) \).

To summarize, we have seen for two discrete gravitational extra dimensions compactified on a warped hyperbolic disk that a high curvature or strong warping allows to avoid the UV/IR connection problem of lattice gravity in flat space. By going to six dimensions on the hyperbolic disk, it is possible to improve further the strong coupling behavior of the 5D flat or warped case and achieve on the IR branes a description of lattice gravity that is valid up to the local Planck scale. It would be interesting to relate our analysis, \( e.g., \) also to moduli stabilization in effective theories, to theories with spontaneously broken space-time symmetries \([10]\), and to studies on the implications of latticized extra dimensions for cosmology \([11]\).

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**REFERENCES**

1. L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370, [hep-ph/9905221](http://arxiv.org/abs/hep-ph/9905221); Phys. Rev. Lett. 83 (1999) 4690, [hep-th/9906064](http://arxiv.org/abs/hep-th/9906064).
2. J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113], [hep-th/9711200](http://arxiv.org/abs/hep-th/9711200); E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253.
3. For a review see, \( e.g., \) M. Grana, Phys. Rept. 423 (2006) 91, [hep-th/0509003](http://arxiv.org/abs/hep-th/0509003).
4. N. Arkani-Hamed, H. Georgi, and M.D. Schwartz, Annals Phys. 305 (2003) 96, [hep-th/0210184](http://arxiv.org/abs/hep-th/0210184).
5. N. Arkani-Hamed and M.D. Schwartz, Phys. Rev. D 69 (2004) 104001, [hep-th/0302110](http://arxiv.org/abs/hep-th/0302110).
6. L. Randall, M.D. Schwartz, and S. Thambyapillai, JHEP 0510 (2005) 110, [hep-th/0507102](http://arxiv.org/abs/hep-th/0507102).
7. J. Gallicchio and I. Yavin, JHEP 0306 (2006) 079, [hep-th/0507105](http://arxiv.org/abs/hep-th/0507105).
8. F. Bauer, T. Hallgren, and G. Seidl, [hep-th/0608176](http://arxiv.org/abs/hep-th/0608176).
9. F. Bauer, [hep-th/0610178](http://arxiv.org/abs/hep-th/0610178).
10. I. Kirsch, Phys. Rev. D 72 (2005) 024001, [hep-th/0503024](http://arxiv.org/abs/hep-th/0503024); N. Boulanger and I. Kirsch, Phys. Rev. D 73 (2006) 124023, [hep-th/0602225](http://arxiv.org/abs/hep-th/0602225).
11. T. Hallgren and T. Ohlsson, JCAP 0606 (2006) 014, [hep-ph/0510174](http://arxiv.org/abs/hep-ph/0510174); T. Hallgren, [hep-ph/0610367](http://arxiv.org/abs/hep-ph/0610367).