DELTA SHOCK WAVE FORMATION IN THE CASE OF TRIANGULAR HYPERBOLIC SYSTEM OF CONSERVATION LAWS

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Abstract. We describe $\delta$ shock wave arising from continuous initial data in the case of triangular conservation law system arising from "generalized pressureless gas dynamics model" (e.g. [12]). We use the weak asymptotic method [8, 9].

In this paper we investigate formation of $\delta$-shock wave in the case of triangular system of conservation laws:

\[(1) \quad u_t + (f(u))_x = 0, \]
\[(2) \quad v_t + (vg(u))_x = 0, \]

with continuous initial data

\[(3) \quad u|_{t=0} = \hat{u}(x) = \begin{cases} 
U_1, & x < a_2, \\
U_0, & a_1 < x \leq a_2, \\
u_0(x), & a_1 \leq x \leq a_2, 
\end{cases} \]
\[(4) \quad v|_{t=0} = \hat{v}(x) = \begin{cases} 
V_1, & x < a_2, \\
V_0, & a_1 < x \leq a_2, \\
v_0(x), & a_1 \leq x \leq a_2, 
\end{cases} \]

where $u_0$ and $v_0$ are continuous functions defined on $[a_2, a_1]$ such that $v_0$ is bounded and $u_0$ satisfies:

\[f'(u_0(x)) = -Kx + b, \quad x \in [a_2, a_1],\]

and $K$ and $b$ are constants determined from the continuity conditions:

\[f'(U_1) = -Ka_2 + b, \quad f'(U_0) = -Ka_1 + b\]

i.e.

\[K = \frac{f'(U_1) - f'(U_0)}{a_1 - a_2}, \quad b = \frac{f'(U_1)a_1 - f'(U_0)a_2}{a_1 - a_2}.\]

For the functions $f$ and $g$ we assume:

\[f \in C^2([U_0, U_1]), \quad g \in C^1([U_0, U_1]),\]
\[f'' > 0 \text{ on } [U_0, U_1],\]
\[g' - f'' \geq 0 \text{ on } [U_0, U_1],\]
\[\exists \bar{U} \in (U_0, U_1) \text{ such that } g(\bar{U}) = f'(\bar{U}).\]

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As we will see in the next section, such conditions provide appearance of admissible \( \delta \) shock wave.

The main result of the paper is construction of formulas which smoothly and globally in \( t \in \mathbb{R} \) represents approximate solution to the problem. We have chosen initial data so that such formulas correspond to the process of delta shock wave formation from continuous initial data.

It is well known that if the solution to equation (1) is discontinuous function then unknown function \( v \) contained in (2) can contain \( \delta \) distribution. Such form of the function \( v \) is natural from the viewpoint of applications.

The basic difficulty lies in giving the sense to the solution containing \( \delta \) distribution. Namely, the second equation of the system contains nonlinearity generally implying the problem of defining the product of \( \delta \) distribution with Heaviside function.

Of course, this problem appears only if we directly substitute functions containing mentioned singularities into the equation. In that case, the product of \( \delta \) and Heaviside function is defined using measure theory \[11, 12, 22, 19\]. Also, one can use regularization of \( \delta \) and Heaviside distribution and define the product as the weak limit of the product of the approximations \[16, 18\]. Finally, recently it has been proposed \[7\] to define solutions of \( \delta \) type (i.e. solutions containing \( \delta \) distributions) trough appropriate integral equalities as done in the case of \( L^\infty \cap L^1 \) weak solutions, thus avoiding problem of multiplication of singularities (see also \[13\]).

According to all said above, we see that the problem of propagation of already formed \( \delta \) shocks has been explored rather thoroughly.

But, the problem of \( \delta \) shock formation is much less studied. Even if it is studied, it has been always done for Riemann problem and always using vanishing viscosity approach. One of the first result of finding global smooth approximation to the problem of type (1)-(4) can be found in \[15\] for the case \( f(u) = u^2/2 \) and \( g(u) = u \).

There, approximate solution \( u_\varepsilon \) to the first equation of the system is found in explicit form by using vanishing viscosity approximation and Hopf-Cole transformation. Then, substituting \( u_\varepsilon \) in the place of \( u \) in equation (2), the equation becomes linear equation in \( v \) and it is solved by the method of characteristics.

General situation using the same approach (vanishing viscosity but with the vanishing term of the form \( \varepsilon t(u, v)_{xx} \)) was considered in \[14\]. There, for the Riemann initial data author proves that system (1,2) admits approximate solutions converging to \( \delta \) type distribution. Author obtains the result by using various apriori estimates obtained from the equations with vanishing viscosity.

Here, we consider more general initial conditions (continuous initial conditions), and also give explicit formula for approximate solution to the problem. It is very important when generalizing result to multidimensional situation since it provides us to describe analytically geometric singularities \[4\].

Other possible approach in general situation is to use Oleinik-Lax formula which gives the solution of the first equation \[11\]. Still, in this case one has to add additional assumptions concerning behavior of the solution in the moment of bifurcation.

Assumptions on the weak continuity of the solution are not enough \[6\], and additional assumptions on boundedness are not quite justified for \( \delta \) shock wave type solutions.
In this paper, we will use the weak asymptotic method which appeared to be rather effective in lots of situations involving nonlinear waves formation and interaction \cite{5, 7, 8, 9}.

We give basic definitions of the weak asymptotic method.

**Definition 1.** By $\mathcal{O}_D(\varepsilon^\alpha) \subset \mathcal{D}'(\mathbb{R})$, $\alpha \in \mathbb{R}$, we denote the family of distributions depending on $\varepsilon \in (0, 1)$ and $t \in \mathbb{R}^+$ such that for any test function $\eta(x) \in C^1_0(\mathbb{R})$, the estimate

$$\langle \mathcal{O}_D(\varepsilon^\alpha), \eta(x) \rangle = O(\varepsilon^\alpha), \quad \varepsilon \to 0,$$

holds, where the estimate on the right-hand side is understood in the usual sense and locally uniform in $t$, i.e., $|O(\varepsilon^\alpha)| \leq C_T \varepsilon^\alpha$ for $t \in [0, T]$.

**Definition 2.** The family of pairs of functions $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(x, t), v_\varepsilon(x, t)), \varepsilon > 0$, is called a weak asymptotic solution of problem (1)-(4) if

$$u_\varepsilon t + (f(u_\varepsilon))_x = \mathcal{O}_D(\varepsilon),$$

$$v_\varepsilon t + (v_\varepsilon g(u_\varepsilon))_x = \mathcal{O}_D(\varepsilon),$$

$$u_\varepsilon \bigg|_{t=0} - \hat{u} = \mathcal{O}_D(\varepsilon), \quad v_\varepsilon \bigg|_{t=0} - \hat{v} = \mathcal{O}_D(\varepsilon), \quad \varepsilon \to 0.$$

According to the previous definitions, an approximation constructed by the means of the vanishing viscosity is indeed weak asymptotic solution to equation (1). In the case of the quadratic nonlinearity (i.e. when $f(u) = u^2$) weak asymptotic solution is constructed in \cite{3}, and in the case when $f$ is arbitrary convex function in \cite{9}. In both of the latter papers Cauchy problems with special initial data of type (3) are considered.

Such initial data has property that the solution of corresponding Cauchy problem can always be represented as linear combination of the Heaviside functions (see \cite{9} and Theorem 6 below). This, in turn, allows us to use the techniques of the weak asymptotic method \cite{5, 8}.

We stress that in \cite{3, 9}, in the case of gradient catastrophe the solution is approximated by the smooth function which describes interaction of nonlinear waves.

Still, in the case of general piecewise monotone continuous initial data the problem can be solved by the same technique as in the case of initial data (3). We describe briefly the procedure \cite{10}.

On the first step, we replace given (arbitrary piecewise monotone continuous) initial data by the new initial data which differs from the original one only in the neighborhood $(x_0 - \varepsilon^{1/2}, x_0 + \varepsilon^{1/2})$ of the point $x_0$ from which the shock wave will appear the first (i.e. along which we will have the first gradient catastrophe). In the interval $(x_0 - \varepsilon^{1/2}, x_0 + \varepsilon^{1/2})$ the new initial data will be of the type (3) (where $a_2 = x_0 - \varepsilon^{1/2}$ and $a_1 = x_0 + \varepsilon^{1/2}$). The discrepancy caused by such replacement is of order $O(\varepsilon^{1/2})$ in $L^1(\mathbb{R})$ sense.

With new initial data we can construct the solution using formulas from \cite{9} or Theorem 6 below.

In this paper, we give new formulas for the construction of the weak asymptotic solution to problem (1)-(3) (which are simpler then ones appearing in \cite{9}). Then, after obtaining the weak asymptotic solution to (1)-(3) we substitute it in equation (2) and solve problem (2), (4) using the method of characteristics. We will explain this more closely.
The system we consider is triangular system. The first equation is scalar conservation law with convex nonlinearity. The problem of shock wave formation for this equation is solved in [9]. It was done through the concept of ‘new characteristics’. In a matter of fact, generalized characteristics (see [2], Definition 10.2.1) are approximated by ‘new characteristics’. But, unlike generalized characteristics which are nonsmooth and intersecting curves, the ‘new characteristics’ are globally defined smooth nonintersecting curves along which solution remains constant. Using this concept, we obtain approximating solution (in the sense of Definition 2) to equation (1). This solution is smooth and we can substitute it into equation (2) and differentiate according to Leibnitz rule the product $v_\v g(u_\v)$ appearing there. Then, we can solve obtained equation using standard method of characteristics.

According to the weak asymptotic method and the procedure previously described, we replace problem (1-4) by the family of problems

\begin{align*}
(5) & \quad u_\v_t + (f(u_\v))_x = \mathcal{O}_D'(\v), \\
(6) & \quad v_\v_t + (v_\v g(u_\v))_x = 0, \quad \v > 0,
\end{align*}

\begin{align*}
(7) & \quad u_\v|_{t=0} = \hat{u}(x) + \mathcal{O}_D'(\v) = \begin{cases} U_1, & x < a_2, \\
U_0, & a_2 \leq x \end{cases}, \\
(8) & \quad v_\v|_{t=0} = \hat{v}(x) + \mathcal{O}_D'(\v) = \begin{cases} V_1, & x < a_2, \\
V_0, & a_2 \leq x \end{cases},
\end{align*}

where $\mathcal{O}_D'$ will be precised in Theorem 6.

We expose the plan of the paper in more details.

In Section 1 we recall necessary conditions for appearance of admissible $\delta$ shock wave for system (1), (2). Then, we quote result in the framework of the weak asymptotic method that we shall need.

In Section 2 we construct the weak asymptotic solution to problem (1), (3).

In Section 3 we construct the weak asymptotic solution to problem (2), (4).

Finally, in Section 4 we find weak limit of the constructed weak asymptotic solution to problem (1-4).

1. Conditions for $\delta$ shock wave appearance and some weak asymptotic formulas

Consider system (1), (2) with Riemann initial data:

\begin{align*}
(9) & \quad u|_{t=0} = \begin{cases} U_1, & x < 0, \\
U_r, & x \geq 0 \end{cases}, \\
(10) & \quad v|_{t=0} = \begin{cases} V_1, & x < 0, \\
V_r, & x \geq 0 \end{cases}.
\end{align*}

Since the aim of the paper is to describe formation of $\delta$ shock waves, we want to determine sufficient condition on $f$ and $g$ which provides $\delta$-shock wave formation from initial data (9), (10). In other words, we want to determine conditions on $f$ and $g$ such that Riemann problem (1), (2), (9), (10) admits solution of the type:
The solution is understood in the sense of Definition 1 from paper [5].

As the admissibility conditions for δ shocks we shall use overcompressive conditions (as in [14, 16, 17, 21]):

\[ \lambda_i(U_r, V_r) \leq c \leq \lambda_i(U_l, V_l), \quad i = 1, 2, \]

where \( \lambda_i, i = 1, 2, \) are eigenvalues of system (2), i.e.

\[ \lambda_1(u, v) = f'(u), \quad \lambda_2(u, v) = g(u). \]

From (13) and expressions for \( \lambda_i, i = 1, 2 \) we have:

\[ f'(U_r) \leq c \leq f'(U_l) \]

\[ g(U_r) \leq c \leq g(U_l). \]

The following conditions were used in [14]:

\[ g' > 0, \quad f'' > 0, \quad f' < g. \]

Still, such conditions will not necessarily give δ shock even if the classical solution \( u \) to (1), (3) blows up after certain time. Since in this paper we are interested only on the δ shock appearance phenomenon, we shall need more restrictive conditions. The conditions which we shall derive below ensures δ shock wave appearance if the classical solution to (1), (3) blows up. We stress that δ shock wave can arise also in the case of less restrictive conditions on \( f \) and \( g \) but in the special case of initial data.

We proceed with deriving of necessary conditions. Initial assumption is convexity of the function \( f \), i.e. \( f'' > 0 \). We have to find conditions on \( g \) such that (14) is satisfied. The following condition obviously implies (14):

\[ g(U_r) \leq f'(U_r) \leq c \leq f'(U_l) \leq g(U_l). \]

Since \( f' \) is increasing it is clear that it has to be \( U_r > U_l \). If we assume that \( F = g - f' \) is increasing in the interval \([U_r, U_l]\) and that \( F \) attains zero in that interval, obviously (15) will be satisfied (since \( F \) changes sign on \([U_r, U_l]\)). We can collect previous considerations in the following theorem:

**Theorem 3.** Assume that the functions \( f, g \in C^2(\mathbb{R}) \) satisfy

1. \( f'' > 0 \) on \( \mathbb{R} \)
2. \( g' - f'' > 0 \) on \([U_r, U_l]\),
3. \( \exists \hat{U} \in (U_r, U_l) \) such that \( g(\hat{U}) = f'(\hat{U}) \).

If \( \hat{U} \in [U_r, U_l] \) then Riemann problem (1), (2), (13), (10) admits δ type solution of the form (12).

Next, we give very important theorem in the framework of the weak asymptotic method (sometimes called nonlinear superposition law):
Theorem 4. \[9\] Let \( \theta_{i\varepsilon}(x) = \omega_i(x/\varepsilon), \ i = 1, 2, \) where \( \lim_{z \to +\infty} \omega_i(z) = 1, \ \lim_{z \to -\infty} \omega_i(z) = 0 \) and \( \frac{d\omega_i(z)}{dz} \in \mathcal{S}(\mathbb{R}) \) where \( \mathcal{S}(\mathbb{R}) \) is the Schwartz space of rapidly decreasing functions. For the bounded functions \( a, b, c \) depending on \( (x, t) \in \mathbb{R}^+ \times \mathbb{R} \) we have

\[
\begin{align*}
0 & \leq a \leq b \leq c, \\
\frac{d}{dt}(a + b\theta_1(\varphi_1 - x) + c\theta_2(\varphi_2 - x)) = \\
&= f(a) + \theta_1(\varphi_1 - x) \left( f(a + b + c)B_1 + f(a + b)B_2 - f(a + c)B_1 - f(a)B_2 \right) + \\
&\quad + \theta_2(\varphi_2 - x) \left( f(a + b + c)B_2 - f(a + b)B_2 + f(a + c)B_1 - f(a)B_1 \right) + O_D(\varepsilon),
\end{align*}
\]

where for \( \rho \in \mathbb{R} \) we have

\[
B_1(\rho) = \int \omega_1(z) \omega_2(z + \rho) dz \quad \text{and} \quad B_2(\rho) = \int \omega_2(z) \omega_1(z - \rho) dz,
\]

and

\( B_1(\rho) + B_2(\rho) = 1. \)

2. Weak asymptotic solution to (11), (13)

First, we determine the function \( u_\varepsilon \). In the sequel we use the following notation (as usual \( x \in \mathbb{R}, \ t \in \mathbb{R}^+ \)):

\[
\begin{align*}
u_1 &= u_1(x, t, \varepsilon), \quad B_1 = B_i(\rho), \quad \varphi_i = \varphi_i(t, \varepsilon), \\
\theta_{i\varepsilon} &= \theta_{i\varepsilon}(\varphi_i - x) = \omega_i(\frac{\varphi_i - x}{\varepsilon}), \\
\delta_{i\varepsilon} &= -\frac{d}{dx} \theta_{i\varepsilon}(\varphi_i - x) = -\frac{d}{dx} \omega_i(\frac{\varphi_i - x}{\varepsilon}), \ i = 1, 2, \\
\tau &= \frac{f'(U) t + a_2 - f'(u_0 t)}{\varepsilon} = \psi_0(t), \\
t^* &= \frac{a_1 - a_2}{f'(U_1) - f'(U_0)}, \\
x^* &= f'(U_1) t^* + a_2 = f'(U_0) t^* + a_1 = \frac{f'(U_0) a_1 - f'(U_0) a_2}{f'(U_1) - f'(U_0)}.
\end{align*}
\]

The function \( \tau \) is so-called 'fast variable'. It is equal to difference of standard characteristics of equation (9) emanating from \( a_2 \) and \( a_1 \), respectively. When we are in the domain of existence of classical solution to (11), (13) we have \( \tau \to -\infty \), while when we are in the domain where solution to (11) (3) is discontinuous (i.e. in the form of the shock wave) we have \( \tau \to \infty \).

The point \((t^*, x^*)\) is the point of blow up of the classical solution to (11), (3).

Also notice that \( w - \lim_{x \to 0} \theta_{i\varepsilon} = \theta(\varphi_i - x) \) and \( w - \lim_{x \to 0} \delta_{i\varepsilon} = \delta(\varphi_i - x) \) for Heaviside function \( \theta \) and Dirac distribution \( \delta \).

First, we will describe the shock wave formation process for problem (11), (13).

We explain the procedure we shall use before we formulate the theorem.

It is well known problem (11), (13) will have classical solution up to the moment \( t = t^* \). The choice of our initial data is such that in the moment of blow up of the classical solution the shock wave will be formed and it will not change its shape for any \( t > t^* \). This is because all the characteristics emanating from \([a_2, a_1]\) intersect in one point \((t^*, x^*)\).

So, for \( t > t^* \) we have to pass to the weak solution concept. In other words, in the moment \( t = t^* \) we stop the time and solve Riemann problem for equation (11).
Our aim is to find global in time approximate solution to (1), (3) which is at least continuous. To do this we have to avoid intersection of characteristics.

Natural idea is to smear the discontinuity line, i.e. to take ε neighborhood of the discontinuity line and to dispose characteristics in that neighborhood in a way that they do not intersect and as ε → 0 all of them lump together into the discontinuity line. Of course, this will not be the standard characteristics for problem (1), (3). Nevertheless, along them approximate solution to our problem will remain constant. Such lines we call ‘new characteristics’.

Another question that arises here is how to distribute ‘new characteristics’ in the ε neighborhood of the discontinuity line. The obvious way to accomplish this is to distribute the ‘new characteristics’ uniformly in the mentioned area, i.e. in a way that every of them is parallel to the discontinuity line.

We use Theorem ?? and ‘switch’ functions $B_i$, $i = 1, 2$, appearing there. Denote by $\phi_i$, $i = 1, 2$, the new characteristics emanating from the points $a_i$, $i = 1, 2$, respectively. They are given by the following Cauchy problems:

$$
\begin{align*}
\frac{d}{dt}\phi_1(t, \varepsilon) &= (B_2(\rho) - B_1(\rho)) f'(U_1) + cB_1(\rho), \quad \phi_1(0, \varepsilon) = a_1 + A\varepsilon\frac{a_1 + a_2}{2}, \\
\frac{d}{dt}\phi_2(t, \varepsilon) &= (B_2(\rho) - B_1(\rho)) f'(U_0) + cB_1(\rho), \quad \phi_2(0, \varepsilon) = a_2 - A\varepsilon\frac{a_1 + a_2}{2},
\end{align*}
$$

for large enough constant $A$. As we shall see later, it will be necessary to extend a little bit the interval $[a_2, a_1]$. Therefore, we have $A\varepsilon\frac{a_1 + a_2}{2}$ accompanying initial data in (18). Also, in (18) we define:

$$
\rho = \frac{\phi_2(t, \varepsilon) - \phi_1(t, \varepsilon)}{\varepsilon}.
$$

According to what we said above, we expect that for every $t > t^*$ it should be (since new characteristics should be ‘close’ one to another for $t > t^*$, i.e. equal as $\varepsilon \to 0$):

$$
\phi_1(t, \varepsilon) - \phi_2(t, \varepsilon) = O(\varepsilon), \quad t > t^*,$$

and (since new characteristics should be ‘close’ to the discontinuity line):

$$
\frac{d}{dt}\varphi(t, \varepsilon) - \frac{f(U_1) - f(U_0)}{U_1 - U_0} = O(\varepsilon), \quad t > t^*.
$$

More precisely, we expect that for $t > t^*$ the expression $B_2(\rho) - B_1(\rho)$ to be close to zero thus eliminating nonlinearity $f'$ appearing in the equation of new characteristics (18). This means that, according to Theorem ?? we have $B_2(\rho) + B_1(\rho) = 1$, $B_1$ and $B_2$ are close to $1/2$, and this implies that $c$ from (18) should be close to $2\frac{f(U_1) - f(U_0)}{U_1 - U_0}$ (Rankine-Hugoniot conditions). Actually, here we use the following simple observation.

Once the shock wave is formed, it continuous to move according to Rankine-Hugoniot conditions and it does not change its shape along entire time axis. Therefore, the linear equation:

$$
\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad c = 2\frac{f(U_1) - f(U_0)}{U_1 - U_0}.
$$
and equation (1) with the same initial condition:

\[ u|_{t=0} = \begin{cases} U_1, & x < 0, \\ U_0, & x \geq 0, \end{cases} \]

will have the same solutions. Clearly, it is much easier to solve linear equation (20) than nonlinear equation (1). Still, the question is how to pass from nonlinear equation (1) to linear equation (20) in the domains where they give the same solution (in the case of our initial data it will be after the shock wave formation). We explain briefly how we do it.

Define ‘new characteristics’ as the solutions to the following Cauchy problem:

\[
\begin{align*}
\dot{x} &= (B_2 - B_1)f'(u_1) + cB_1, \\
\dot{u}_1 &= 0, \quad u_1(0) = u_0(x_0), \quad x(0) = x_0 + \varepsilon A(x_0 - \frac{a_1 + a_2}{2}), \quad x_0 \in [a_2, a_1].
\end{align*}
\]

(21)

Thus, \( \varphi_i(t, \varepsilon) = x(a_i, t, \varepsilon), \ i = 1, 2, \) where \( x \) is the solution to (21). We will show later that it is possible to choose the constant \( A \) so that for \( x_0 \in [a_2, a_1] \) every \( t > 0 \) we have

\[
\frac{\partial x}{\partial x_0} > 0.
\]

This means that ‘new characteristics’ indeed do not intersect which in turn means that there exists the solution \( x_0 \) of the implicit equation:

\[ x(x_0, t, \varepsilon) = x. \]

(22)

Bearing in mind that \( B_1 \sim B_2 \) after the interaction we see that, using the new characteristics, we have smoothly passed from the characteristics of equation (1) to the characteristics of equation (21), i.e. form equation (1) to equation (20).

We formalize the previous considerations in Theorem 5. The theorem is analogue to the main result from [9]. The problem which we consider here, i.e. problem (1), (3) can be solved in more elegant manner (see Theorem 7 below). Still, approach used in Theorem 5 can be used on the case of arbitrary piecewise monotone initial data. Also, Theorem 5 represents motivation for Theorem 7.

**Theorem 5.** The weak asymptotic solution of problem (1), (3) has the form:

\[
(23) \quad u_\varepsilon(x, t) = U_0 + (u_1(x, t, \varepsilon) - U_0) \omega_1\left(\frac{\varphi_1(t, \varepsilon) - x}{\varepsilon}\right) + (U_1 - u_1(x, t, \varepsilon)) \omega_2\left(\frac{\varphi_2(t, \varepsilon) - x}{\varepsilon}\right),
\]

where \( \omega_i \) satisfies the conditions from Theorem 5 and:

\[
\omega_1(z) = 1 \quad \text{for} \quad z > 0 \quad \text{and} \quad \omega_2(z) = 0 \quad \text{for} \quad z < 0.
\]

The functions \( \varphi_i(t, \varepsilon), i = 1, 2, \) are given by (18) and the function \( \rho \) is given by (19).

The function \( u_1(x, t, \varepsilon) \) is given by

\[
u_1(x, t, \varepsilon) = u_0(x_0(x, t, \varepsilon))\]

where \( x_0 \) is the inverse function to the function \( x = x(x_0, t, \varepsilon), t > 0, \varepsilon > 0, \) of ‘new characteristics’ defined through Cauchy problem (21).
In the sequel we write only
\( \theta \)
After using (16) and Leibnitz rule, and collecting terms multiplying \( \theta \) intersect. It appears that it is much easier to accomplish this if we perturb initial
The system of characteristics for this problem reads:
\[
\begin{aligned}
\left(U_0 + (u_1(x,t,\varepsilon) - U_0)\right)\omega_1(\frac{\varphi_1(t,\varepsilon) - x}{\varepsilon}) + (U_1 - u_1(x,t,\varepsilon))\omega_2(\frac{\varphi_2(t,\varepsilon) - x}{\varepsilon}) + (f(U_0 + (u_1(x,t,\varepsilon) - U_0)\omega_1(\frac{\varphi_1(t,\varepsilon) - x}{\varepsilon}) + (U_1 - u_1(x,t,\varepsilon))\omega_2(\frac{\varphi_2(t,\varepsilon) - x}{\varepsilon}))
\right)
= O_D(\varepsilon)
\end{aligned}
\]
After using (16) and Leibnitz rule, and collecting terms multiplying \( \theta \)
\( \theta_2 = \omega_2(\frac{\varphi_2(t,\varepsilon) - x}{\varepsilon}) \), \( \delta_1 = \left(\omega_1(\frac{\varphi_1(t,\varepsilon) - x}{\varepsilon})\right)' \) and \( \delta_2 = \left(\omega_2(\frac{\varphi_2(t,\varepsilon) - x}{\varepsilon})\right)' \) we have:
\[
\begin{aligned}
\left[ \frac{\partial u_1}{\partial t} + B_2(\rho)f'(u_1)\frac{\partial}{\partial x} + B_1(\rho)f'(U_1 + U_0 - u_1)\frac{\partial u_1}{\partial x} \right] \, \theta_1 + \\
\left[ - \frac{\partial u_1}{\partial t} + B_2(\rho)f'(u_1)\frac{\partial}{\partial x} - B_1(\rho)f'(U_1 + U_0 - u_1)\frac{\partial u_1}{\partial x} \right] \, \theta_2 + \\
((u_1 - U_0)\varphi_1 - B_2(\rho)(f(u_1) - f(U_0)) - B_1(\rho)(f(U_1) - f(U_1 + U_0 - u_1))) \, \delta_1 + \\
((U_1 - u_1)\varphi_1 - B_2(\rho)(f(u_1) - f(u_1)) - B_1(\rho)(f(U_1 + U_0 - u_1) - f(U_0))) \, \delta_2 = O_D(\varepsilon).
\end{aligned}
\]
In the sequel we write only \( B_i \) instead of \( B_i(\rho), i = 1, 2 \).
We rearange this expression using the following simple formula \( C\theta_1 + D\theta_2 = (C + D)\theta_2 + C(\theta_1 - \theta_2) \):
\[
\begin{aligned}
\left( \frac{\partial u_1}{\partial t} + [(B_2 - B_1)f'(u_1)]\frac{\partial u_1}{\partial x} \right) \, (\theta_1 - \theta_2) + \\
B_1\left[ \frac{d}{dx} (f(U + u_0^0 - u_1) + f(u_1)) \right] \, (\theta_1 - \theta_2) + \\
((u_1 - U_0)\varphi_1 - B_2(\rho)(f(u_1) - f(U_0)) - B_1(\rho)(f(U_1) - f(U_1 + U_0 - u_1))) \, \delta_1 + \\
((U_1 - u_1)\varphi_1 - B_2(\rho)(f(u_1) - f(u_1)) - B_1(\rho)(f(U_1 + U_0 - u_1) - f(U_0))) \, \delta_2 = O_D(\varepsilon).
\end{aligned}
\]
For an unknown constant \( c \) we add and subtract the term \( cB_1 \frac{\partial u_1}{\partial x} \) in the coefficient multiplying \( (\theta_1 - \theta_2) \) and then we rewrite the last expression in the following form:
\[
\begin{aligned}
(\frac{\partial u_1}{\partial t} + [(B_2 - B_1)f'(u_1) + cB_1]\frac{\partial u_1}{\partial x}) \, (\theta_1 - \theta_2) + \\
B_1\left[ \frac{d}{dx} (f(U + u_0^0 - u_1) + f(u_1) - cu_1)) \right] \, (\theta_1 - \theta_2) + \\
((u_1 - U_0)\varphi_1 - B_2(\rho)(f(u_1) - f(U_0)) - B_1(\rho)(f(U_1) - f(U_1 + U_0 - u_1))) \, \delta_1 + \\
((U_1 - u_1)\varphi_1 - B_2(\rho)(f(u_1) - f(u_1)) - B_1(\rho)(f(U_1 + U_0 - u_1) - f(U_0))) \, \delta_2 = O_D(\varepsilon).
\end{aligned}
\]
We put
\[
\begin{aligned}
\frac{\partial u_1}{\partial t} + [(B_2 - B_1)f'(u_1) + cB_1]\frac{\partial u_1}{\partial x} = 0, \quad u_1(x,0,\varepsilon) = u_0(x), \quad x \in [a_2, a_1].
\end{aligned}
\]
The system of characteristics for this problem reads:
\[
\begin{aligned}
\dot{x} &= (B_2 - B_1)f'(u_1) + cB_1, \\
\dot{u_1} &= 0, \quad u_1(0) = u_0(x_0), \quad x(0) = x_0 \in [a_2, a_1].
\end{aligned}
\]
The aim is to prove that characteristics defined by the previous system do not intersect. It appears that it is much easier to accomplish this if we perturb initial
data for \( x \) in the previous system for a parameter of order \( \varepsilon \). More precisely, instead of (25) we shall consider system (21) (the same is done in [9]).

It is clear that such perturbation changes the solution of (21) for \( O_D(\varepsilon) \) since initial condition in (21) is continuous.

We pass to the proof that the characteristics given by (21) do not intersect. From the second equation in (21) it follows \( u_1 \equiv u_0(x_0) \). We substitute this into the first equation of (21) and use \( f'(u_0(x_0)) = -Kx_0 + b, x_0 \in [a_2, a_1] \). We have:

\[
\dot{x} = (B_2 - B_1)(-Kx_0 + b) + cB_1, \quad x(0) = x_0 + \varepsilon A \left( x_0 - \frac{a_1 + a_2}{2} \right).
\]

Out of the segment \([a_2 - \varepsilon A^{\frac{a_1+a_2}{2}}, a_1 + \varepsilon A^{\frac{a_1+a_2}{2}}]\) initial function is constant and we define the solution \( u_1 \) of problem (21) to be equal to \( U_1 \) on the left-hand side of the characteristic emanating from \( a_2 - A^{\frac{a_1+a_2}{2}} \) and to be equal to \( U_0 \) on the right-hand side of the characteristic emanating from \( a_1 + A^{\frac{a_1+a_2}{2}} \).

For the functions \( \varphi_1, \varphi_2 \) as the characteristics emanating from \( a_1 + A^{\frac{a_1+a_2}{2}} \) and \( a_2 - A^{\frac{a_1+a_2}{2}} \) respectively, we have (compare to (18))

\[
\begin{align*}
\varphi_{11} &= (B_2 - B_1)(-K_1 + b) + cB_1, \\
\varphi_{21} &= (B_2 - B_1)(-K_2 + b) + cB_1.
\end{align*}
\]

Now, we show how to effectively determine \( \rho \) given by (19). We apply standard procedure (see [3, 5, 9]). Subtracting (27) from (25) we get:

\[(\varphi_2 - \varphi_1)_t = \varepsilon \left( \frac{\varphi_2 - \varphi_1}{\varepsilon} \right)_t = \varepsilon \rho_t = (B_2 - B_1)\psi_0(t).\]

Then, passing from the ”slow” variable \( t \) to the ”fast” variable \( \tau \) we obtain (we also use \( B_2 + B_1 = 1 \)):

\[
\rho_\tau = 1 - 2B_1(\rho), \quad \rho \bigg|_{\tau \to -\infty} = 1.
\]

We explain the condition \( \lim_{\tau \to -\infty} \frac{\rho}{\tau} = 1 \). We have from (29) and (28)

\[
\rho \bigg|_{\tau = t} = \frac{\int_0^t 2(U - u_0)^2(B_2 - B_1)dt'}{2(u - u_0)^2t + a_2 - a_1}.
\]

Putting \( t = 0 \) in the previous relation we see that

\[
\rho \bigg|_{t = 0} = 1.
\]

When we let \( \varepsilon \to 0 \) when \( t = 0 \) we have \( \tau \to -\infty \). Therefore, from (30) it follows

\[
\rho \bigg|_{\tau \to -\infty} = 1.
\]

This relation practically means that new characteristics emanating from \( a_i, i = 1, 2 \), coincides at least in the initial moment with standard characteristics up to some small parameter \( \varepsilon \). Still, since \( \tau \to -\infty \) for every \( t < t^* \) (which means \( B_1 \to 0 \)) we see from (27) and (25) that new characteristics coincides with standard ones for every \( t < t^* \) up to some small parameter \( \varepsilon \).

Next, we analyze (29). From the standard theory of ODE we see that \( \rho \to \rho_0 \) as \( \tau \to +\infty \) where \( \rho_0 \) is constant such that \( B_1(\rho_0) = B_2(\rho_0) = 1/2 \). That means that after the interaction, i.e. for \( t > t^* \), we have

\[
\rho = \frac{\varphi_1 - \varphi_2}{\varepsilon} = \rho_0 + O(\varepsilon) \quad \Rightarrow \quad \varphi_1 = \varphi_2 + O(\varepsilon), \quad \varepsilon \to 0,
\]
or, after letting \( \varepsilon \to 0 \), for \( t > t^* \) we have shock wave concentrated at (see text in front of theorem for notations):

\[
(31) \quad \varphi(t) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(t, \varepsilon) = \frac{c}{2}(t-t^*) + x^*.
\]

Now, we can prove global solvability of Cauchy problem (21).

Problem (21) is globally solvable if characteristics emanating from the interval \([a_2 - \varepsilon A \frac{B_2 + B_1}{2}, a_1 + \varepsilon A \frac{B_2 - B_1}{2}]\) do not intersect. To prove that we will use the inverse function theorem. We will prove that for every \( t \) we have \( \frac{\partial x}{\partial x_0} > 0 \) which means that for every \( x = x(x_0, t, \varepsilon) \), \( x_0 \in [a_2, a_1] \), we have unique \( x_0 = x_0(x, t, \varepsilon) \) and we can write \( u_1(x(x_0, t, \varepsilon), t) = u_0(x_0(x, t, \varepsilon)) \).

Differentiating (26) in \( x_0 \) and integrating from 0 to \( t \) we obtain (we remind \( B_2 + B_1 = 1 \)):

\[
(32) \quad \frac{\partial x}{\partial x_0} = 1 + \varepsilon A - K \int_0^t (B_2 - B_1)dt' = 1 + \varepsilon - K \int_0^t (1 - 2B_1)dt'.
\]

For \( t \in [0, t^*] \) we have (notice that \( 1 - Kt^* = 0 \)):

\[
\frac{\partial x}{\partial x_0} = 1 + \varepsilon A - K \int_0^t dt + K \int_0^t 2B_1 dt \geq 1 + \varepsilon A - K \int_0^t dt + K \int_0^t 2B_1 dt = \varepsilon A + K \int_0^t 2B_1 dt > 0.
\]

So, everything is correct for \( t \leq t^* \).

To see what is happening for \( t > t^* \), initially we estimate \( 1 - 2B_1(\rho) \) when \( \tau \to \infty \). From equation (29) we have (we use Taylor expansion of \( B_1 \) around the point \( \rho = \rho_0 \)):

\[
\rho_\tau = 1 - 2B_1(\rho) = -2(\rho - \rho_0)B_1'(\tilde{\rho}),
\]

for some \( \tilde{\rho} \) belonging to the interval with ends in \( \rho \) and \( \rho_0 \). From here we see:

\[
\rho - \rho_0 = (\rho(\tau_0) - \rho_0) \exp(\int_{\tau_0}^\tau -2B_1'((\tilde{\rho})d\tau') = (\rho(\tau_0) - \rho_0) \exp((\tau_0 - \tau)2B_1'(\tilde{\rho})),
\]

for some fixed \( \rho_0 \in \mathbb{R} \) and \( \tilde{\rho} \in (\rho(\tau_0), \rho(\tau)) \in [\rho(\tau_0), \rho_0] \). We remind that \( B_1'(\tilde{\rho}) \geq c > 0 \), for some constant \( c \), since \( B_1 \) is increasing function and \( \tilde{\rho} \) belongs to the compact interval \([\rho(\tau_0), \rho_0] \). Letting \( \tau \to \infty \) we conclude that for any \( N \in \mathbb{N} \)

\[
\rho - \rho_0 = \mathcal{O}(1/\tau^N), \quad \tau \to \infty.
\]

From here we have \( \rho_\tau = \mathcal{O}(1/\tau^N), \quad \tau \to \infty \), since:

\[
\lim_{\tau \to \infty} \frac{\rho_\tau}{\rho - \rho_0} = \lim_{\tau \to \infty} \frac{1 - 2B_1(\rho)}{\rho - \rho_0} = \lim_{\tau \to \infty} -2B_1'(\rho) = -2B_1'(\rho_0) = \text{const.} < 0.
\]

This, in turn, means that for every \( N \in \mathbb{N} \) and \( t > t^* \) we have

\[
(33) \quad 1 - 2B_1(\rho) = \rho_\tau = \mathcal{O}(\tau^{-N}) = \mathcal{O}(\varepsilon^N), \quad \varepsilon \to \infty,
\]

since for fixed \( t > t^* \) we have \( \tau = \frac{\Theta_0(t)}{\varepsilon} \to \infty \) as \( \varepsilon \to 0 \).
Now we can prove that \( \frac{\partial x}{\partial x_0} \) for \( t > t^* \). We have

\[
(34) \quad \frac{\partial x}{\partial x_0} = 1 + \varepsilon A - 2K \int_0^{t^*} (1 - 2B_1)dt' = 
1 + \varepsilon A - 2K \int_0^{t^*} (1 - 2B_1)dt' - 2K \int_{t^*}^t (1 - 2B_1)dt' = 
\varepsilon A + 4 \int_0^{t^*} B_1 dt' - 2K \int_{t^*}^t (1 - 2B_1)dt' > \varepsilon A - 2K \int_{t^*}^t (1 - 2B_1)dt'.
\]

Recall that

\[
B_1 = B_1(\rho(\tau)) = B_1(\rho\left(\frac{\psi_0(t)}{\varepsilon}\right)).
\]

Consider the last term in expression (34):

\[
2K \int_{t^*}^t (1 - 2B_1)dt' = 2K \int_{t^*}^t (1 - 2B_1(\rho\left(\frac{\psi_0(t')}{\varepsilon}\right)))dt' = 
\left( \frac{\psi_0(t')}{\varepsilon} = z \implies (u - u_0^\varepsilon)dt' = \varepsilon dz; \right) 
\left( t^* < t' < t \implies 0 < z < \frac{\psi_0(t)}{\varepsilon} \right) = 
2K\varepsilon \int_0^{\frac{\psi_0(t)}{\varepsilon}} (1 - 2B_1(\rho(z)))dz < \varepsilon 2KC,
\]

where

\[
C = \int_0^{\infty} (1 - 2B_1(\rho(z)))dz < \infty,
\]

since from (33) we know \( 1 - 2B_1(\rho(z)) = O(z^{-N}), z \to \infty \) and \( N \in \mathbb{N} \) arbitrary.

Therefore, for \( A \) large enough (more precisely for \( A > C \)) we have \( \frac{\partial x}{\partial x_0} > 0 \) what we wanted to prove.

Next step is to obtain the constant \( c \). We multiply (24) by \( \eta \in \mathcal{C}_0^1(\mathbb{R}) \), integrate over \( \mathbb{R} \) with respect to \( x \) and use (24) (so, we remove the first term in (24)):

\[
\int B_1 \left[ \frac{d}{dx}(f(U_1 + U_0 - u_1) + f(u_1) - cu_1)](\theta_{1\varepsilon} - \theta_{2\varepsilon})\eta(x)dx + 
((u_1 - U_0)\varphi_{1\varepsilon} - B_2(f(u_1) - f(U_0)) - B_1(f(U_1) - f(U_1 + U_0 - u_1)))\delta_{1\varepsilon} + 
((U_1 - u_1)\varphi_{2\varepsilon} + B_2(f(u_1) - f(U_1)) + B_1(f(U_0) - f(U_1 + U_0 - u_1)))\delta_{2\varepsilon} = O(\varepsilon).
\]

We apply partial integration on the first integral in the previous expression to obtain:

\[
(35) \quad \int B_1[f(U_1 + U_0 - u_1) + f(u_1) - cu_1)](\theta_{1\varepsilon} - \theta_{2\varepsilon})\eta'(x)dx + 
\int ((u_1 - U_0)\varphi_{1\varepsilon} - B_2(f(u_1) - f(U_0)) + B_1(f(u_1) + f(U_0) - cu_1))\eta(x)\delta_{1\varepsilon}dx + 
\int ((U_1 - u_1)\varphi_{2\varepsilon} + B_2(f(u_1) - f(U_1)) - B_1(f(u_1) + f(U_1) - cu_1))\eta(x)\delta_{2\varepsilon}dx = O(\varepsilon).
\]

Now we use the choice of regularizations \( \omega_i, i = 1, 2 \). Since,

\[
\delta_{1\varepsilon} = \frac{1}{\varepsilon} \omega_1'(\frac{\varphi_1 - x}{\varepsilon}) = 0 \text{ for } x < \varphi_1
\]

\[
\delta_{2\varepsilon} = \frac{1}{\varepsilon} \omega_2'(\frac{\varphi_2 - x}{\varepsilon}) = 0 \text{ for } x > \varphi_2
\]
we have from (35):

\[\varepsilon \rho B_1 \int [f(U_1 + U_0 - u_1) + f(u_1) - cu_1] \frac{\theta_{1x} - \theta_{2x}}{\varphi_2 - \varphi_1} \eta'(x) dx +\]

\[\int_{\varphi_1}^{\varphi_2} ((u_1 - U_0) \varphi_{1x} - B_2 (f(u_1) - f(U_0)) + B_1 (f(u_1) + f(U_0) - cu_1)) \eta(x) \delta_{1x} dx +\]

\[\int_{-\infty}^{\varphi_2} ((U_1 - u_1) \varphi_{2x} + B_2 (f(u_1) - f(U_1)) - B_1 (f(u_1) + f(U_1) - cu_1)) \eta(x) \delta_{2x} dx =\]

(now, we use \(u_1 \equiv U_1\) for \(x > \varphi_1\) and \(u_1 \equiv U_0\) for \(x < \varphi_2\))

\[\varepsilon \rho B_1 \int [f(U_1 + U_0 - u_1) + f(u_1) - cu_1] \frac{\theta_{1x} - \theta_{2x}}{\varphi_2 - \varphi_1} \eta'(x) dx +\]

\[\int_{\varphi_1}^{\infty} B_1 (2f(U_0) - cU_0) \eta(x) \delta_{1x} dx -\]

\[\int_{-\infty}^{\varphi_2} B_1 (-2f(U_1) + cU_1) \delta_{2x} \eta(x) dx = O(\varepsilon).\]

To continue, notice that we have \(|\rho B_1| < \infty\) for every \(\tau \in \mathbb{R}\). Namely,

\[|\rho B_1(\rho)| \to 0 \quad \text{as} \quad \tau \to -\infty \quad \text{since in that case} \quad B_1(\rho(\tau)) \sim B_1(\tau) \sim \frac{1}{\tau^N} \sim \frac{1}{\rho^N},\]

\[|\rho B_1(\rho)| \to \rho_0 B_1(\rho_0) \quad \text{as} \quad \tau \to \infty \quad \text{since in that case} \quad \rho \to \rho_0.\]

This fact, together with the fact that \(\delta_{1x} \to \delta(\varphi_1 - x)\), reduces expression (36) to:

\[B_1 (2f(U_0) - cU_0) \eta(\varphi_1) - B_1 (-2f(U_1) + cU_1) \eta(\varphi_2) = O(\varepsilon).\]

Rewrite this expression in the following manner:

\[B_1 (2f(U_0) - f(U_1)) - c(U_0 - U_1) \eta(\varphi_1) + B_1 (-2f(U_1) + cU_1) (\eta(\varphi_2) - \eta(\varphi_1)) =\]

\[B_1 (2f(U_0) - f(U_1)) - c(U_0 - U_1) \eta(\varphi_1) +\]

\[\varepsilon \rho B_1(\rho) (-2f(U_1) + cU_1) \frac{\eta(\varphi_2) - \eta(\varphi_1)}{\varphi_2 - \varphi_1} \equiv \quad \text{(37)}\]

\[B_1 (2f(U_0) - f(U_1)) - c(U_0 - U_1) \eta(\varphi_1) = O(\varepsilon).\]

From here, we see that the last relation is satisfied for

\[c = \frac{2f(U_1) - f(U_0)}{U_1 - U_0}.\]

The theorem is proved. \(\square\)

**Remark 6.** In the case such as our, when \(U_0\) and \(U_1\) are constants, is possible to replace formula (23) by

\[u_\varepsilon(x, t) = \hat{u}(x_0(x, t, \varepsilon)),\]

where, as before, the function \(x_0\) is the solution to implicit equation (22) and \(\hat{u}\) are initial data \((3)\).

The proof of this fact obviously follows after comparing the trajectories. We give precise formulation in the next theorem. We leave it without proof since it is completely analogical to the proof of the previous theorem.
The difference between the previous and the next theorem is in the form of characteristics along which we solve our problem. In the previous theorem, for fixed \( \varepsilon \), the weak asymptotic solution \( u_\varepsilon \) to (1), (3) was generator of continuous semigroup of transformations (since characteristics intersect along \( x = \varphi_i \)) and in the following theorem the weak asymptotic solution \( u_\varepsilon \) to (1), (3) forms continuous group of transformation since appropriate characteristics do not intersect. Still, approach from the next theorem can be used only in the case of special initial data.

**Theorem 7.** The weak asymptotic solution \( u_\varepsilon, \varepsilon > 0 \), to Cauchy problem

\[
\frac{\partial u}{\partial t} + (f(u))_x = 0, \quad u|_{t=0} = \hat{u}(x),
\]

is given by

\[
u_\varepsilon(x,t) = \hat{u}(x_0(x,t,\varepsilon)),
\]

where \( x_0 \) is inverse function to the function \( x = x(x_0,t,\varepsilon) \), \( t > 0, \varepsilon > 0 \), of 'new characteristics' defined through the Cauchy problem:

\[
\dot{x} = f'(u_\varepsilon)(B_2(\rho) - B_1(\rho)) + cB_1(\rho), \quad x(0) = x_0 + \varepsilon A \left( x_0 - \frac{a_1 + a_2}{\varepsilon} \right),
\]

where \( A \) is large enough, the functions \( B_1 \) and \( B_2 \) are defined in Theorem 7, constant \( c \) is given in (20) and \( \rho = \rho(\psi_0(t)/\varepsilon) \) is the solution of Cauchy problem (19).

The following corollary is obvious. It claims that the weak asymptotic solution defined in arbitrary of the previous theorems tends to the shock wave with the states \( U_1 \) on the left and \( U_0 \) on the right (see (31)).

**Corollary 8.** With the notations from the previous theorems, for \( t > t^* \) the weak asymptotic solution \( u_\varepsilon \) to problem (1), (3) we have for every fixed \( t > 0 \):

\[
u_\varepsilon(x,t) \rightarrow \begin{cases} U_1, & x < \frac{c}{2}(t - t^*) + x^*, \\ U_0, & x > \frac{c}{2}(t - t^*) + x^*, \end{cases}
\]

where \( \rightarrow \) means convergence in the weak sense with respect to the real variable.

3. **The Weak Asymptotic Solution to (2), (4)**

At the beginning of the section, we explain some general moments.

The plan is to substitute smooth function \( u_\varepsilon \) given by (23) into (2). Thus, we obtain equation (6). Augmented by initial data (4), this linear partial differential equation of the first order has global differentiable solution.

But, as \( \varepsilon \rightarrow 0 \) we can have discontinuities in \( v = \lim_{\varepsilon \rightarrow 0} u_\varepsilon \) not only on the line on which the shock wave of standard admissible weak solution \( u \) of (1), (3) is supported, and which appears for \( t > t^* \). Also, for \( t < t^* \) discontinuities can appear along standard characteristics for problem (1), (3) emanating from the points \( a_2 \) and \( a_1 \). Discontinuity arises due to non-smoothness of initial data \( \psi_0 \) in the points \( a_2 \) and \( a_1 \) (see Example 13).

The situation is different if instead of (23) we put in (2) in the place of of \( u \) the function \( u_\varepsilon \) given by (41) in Theorem 7. The function \( u_\varepsilon \) is not smooth since we do not use regularizations of Heaviside function to smooth weak discontinuities.
appearing in the initial data. Accordingly, for $t < t^*$ it admits the same behavior as the standard admissible weak solution $u$ of problem (1), (3) (it can produce discontinuities in $v_\varepsilon$ even for $t < t^*$).

Therefore, for the sake of consistency (we have the same situation for $\varepsilon > 0$ and as $\varepsilon \to 0$), we will use the function given by (41) in the place of $u$ appearing in (2).

Weak asymptotic solution to Cauchy problem (2), (4) we will solve separately in five areas of $(x, t)$ plane in which the solution is certainly smooth. Then, we will connect solutions in those domains and prove that the function formed in that way represents weak solution to our problem.

In order to single out those domains we substitute (41) into (2) and, formally, use Leibnitz rule for derivative of product:

$$v_\varepsilon t + g(u_\varepsilon)v_\varepsilon x = -(g(u_\varepsilon))_x v_\varepsilon.$$  

(43)

Remark 9. We say "formally" since the function $v_\varepsilon$ can have discontinuities, at least for $\varepsilon = 0$. We repeat that this is caused by non-smoothness of the new characteristics if we use (41) as well as standard characteristics in $x_0 = a_1$ and $x_0 = a_2$.

The system of characteristics corresponding to (43), (4) is:

$$\dot{X} = g(u_\varepsilon), \quad X(0) = x_0,$$

$$\dot{v}_\varepsilon = -v_\varepsilon (g(u_\varepsilon))_x, \quad v_\varepsilon(0) = \hat{v}(x_0).$$

(44)

We prove global resoluteness of this ODE system for $x_0 \in [a_2 - \varepsilon a_2 + a_1, a_1 + \varepsilon A + a_1]$. According to the inverse function theorem it is enough to prove that along entire temporal axis we have

$$\frac{\partial X}{\partial x_0} > 0.$$

Denote by $J = \frac{\partial x}{\partial x_0}$ where $x = x(x_0, t, \varepsilon)$ is the function defined by Cauchy problem (26). We have proved in the previous theorem that $J > 0$ for every $t > 0$. Recall that

$$u_\varepsilon(x, t) = u_0(\hat{x}_0(x, t, \varepsilon)),$$

where $\hat{x}_0$ is inverse function to the function $x$ defined trough (26). From (41) we have (we write below $g'(u_0) = g'(u_0(\hat{x}_0(x, t, \varepsilon)))$)

$$\frac{d}{dt} \frac{\partial X}{\partial x_0} = g'(u_0)u'_0 \frac{\partial \hat{x}_0}{\partial X} \frac{\partial X}{\partial x_0} = g'(u_0)u'_0 J^{-1} \frac{\partial X}{\partial x_0}, \quad \frac{\partial X}{\partial x_0}|_{t=0} = 1.$$

After integrating this differential equation with respect to the unknown function $\frac{\partial X}{\partial x_0}$ we obtain:

$$\frac{\partial X}{\partial x_0} = \exp\left(\int_0^t g'(u_0)u'_0 J^{-1} dt\right) > 0, \quad t > 0,$$

which implies existence of inverse function $x_0 = x_0(X, t, \varepsilon)$ along entire temporal axis, which, in turn, implies global resoluteness of problem (41).

Denote by $\varphi^*_i, i = 1, 2$, solutions of the following Cauchy problems:

$$\dot{X} = g(u_\varepsilon),$$

$$X(0) = a_i, \quad i = 1, 2.$$

Now, we can introduce domains in which we will separately solve Cauchy problem (2), (4).
Remark 10. In the domains to be introduced, it is equivalent to say Cauchy problem (43), (4) instead of (2), (4) since in those domains problem (2), (4) has smooth solution.

We set
\[ D_1 = \{(x,t) | x < \phi_2\}, \quad D_2 = \{(x,t) | x > \phi_1\}, \]
\[ D_3 = \{(x,t) | \phi_2 < x < \phi_2^*\}, \quad D_4 = \{(x,t) | \phi_1^* < x < \phi_1\}, \]
\[ D_5 = \{(x,t) | \phi_2^* < x < \phi_1^*\} \]

On the beginning, we prove that those domains are disjunct. Accordingly, we inspect relations between the functions \( \phi_i \) and \( \phi_i^* \), \( i = 1, 2 \). We have to prove the following fact for every \( t \in \mathbb{R}^+ \):
\[ \phi_2(t, \varepsilon) \leq \phi_2^*(t, \varepsilon) < \phi_1^*(t, \varepsilon) \leq \phi_1(t, \varepsilon), \]
(45)

First, we prove that \( \phi_2 \leq \phi_2^* \).

In the moment \( t = 0 \) we have
\[ (\phi_2)' = f(U_1)(B_2 - B_1) + cB_1 \quad \text{and} \quad (\phi_2^*)' = g(U_1), \]
(47)
and (see (15))
\[ g(U_1) > f'(U_1) > f'(U_1)(B_2 - B_1) + cB_1, \]
since \( f'(U_1) > c/2 \). Using well known theorem from ODE-s ("who goes slower does not reach further", (1)) from (17) we see that in some neighborhood of \( t = 0 \) we have \( \phi_2 < \phi_2^* \). Assume now that \( t_0 \) is the smallest \( t > 0 \) such that \( \phi_2 = \phi_2^* \). In this case we have the same situation as in the moment \( t = 0 \), i.e. there exists neighborhood \((t_0, t_0 + \delta)\) such that \( \phi_2 < \phi_2^* \) in \((t_0, t_0 + \delta)\). Continuing like this we see that we indeed have (46).

In the completely same manner we prove that
\[ \phi_1^* \leq \phi_1, \]
(48)
It is remained to prove that:
\[ \phi_2^* < \phi_1^*. \]
(49)
This directly follows from the fact that characteristics of problem (44) do not intersect. That means that relation between two characteristics remains the same along entire time axis. Therefore,
\[ \phi_2^* = X(a_2, t, \varepsilon) < X(a_1, t, \varepsilon) = \phi_1^*, \]
since \( a_2 < a_1 \). This proves (49).

Collecting (16), (18) and (19) we obtain (15). Very important implication of relation (15) and the fact that for \( t > t^* \)
\[ \lim_{\varepsilon \rightarrow 0} \phi_i(t, \varepsilon) = \frac{c}{2}(t - t^*) + x^*, \quad i = 1, 2 \]
is the following. For \( t > t^* \) we have:
\[ \lim_{\varepsilon \rightarrow 0} \phi_i^*(t, \varepsilon) = \frac{c}{2}(t - t^*) + x^*, \quad i = 1, 2. \]
(50)
We remind that the constants \( t^* \) and \( x^* \) are introduced in front of Theorem [5].

Next, we solve problem (43), (4) separately in domains \( D_i, i = 1, ..., 5 \).
In domains $D_1$ and $D_2$ we have $u_\varepsilon \equiv \text{const.}$ and therefore the characteristics corresponding to $v_\varepsilon$ there are straight lines. More precisely, we have:

\[
v_\varepsilon(x, t) \equiv V_1, \quad (x, t) \in D_1,
v_\varepsilon(x, t) \equiv V_0, \quad (x, t) \in D_2.
\]

Another two domains are:

\[
D_3 = \{(x, t) | \varphi_2 < x < \varphi_2^*\}, \quad D_4 = \{(x, t) | \varphi_1^* < x < \varphi_1\}.
\]

In those domains we solve the following Cauchy problems:

\[
v_\varepsilon t + g(u_\varepsilon)v_\varepsilon x = -(g(u_\varepsilon)_x v_\varepsilon,\]
\[
v_\varepsilon|_{x=\varphi_1} = V_0 \quad \text{(initial data for the first Cauchy problem)},
v_\varepsilon|_{x=\varphi_2} = V_1 \quad \text{(initial data for the second Cauchy problem)}.
\]

We use standard method of characteristics. Note that in this case characteristics emanate from the lines $x = \varphi_i$, $i = 1, 2$, and not from $x$ axis as usual The system of characteristics for ones emanating from the line $\varphi_1$ has the form:

\[
\dot{X} = g(u_\varepsilon),
\dot{v}_\varepsilon = -v_\varepsilon(g(u_\varepsilon))_x,
X(t_0) = \varphi_1(t_0) = x_0, \quad v_\varepsilon(t_0) = V_0.
\]

and for the characteristics emanating from the line $\varphi_2$ has the form:

\[
\dot{X} = g(u_\varepsilon),
\dot{v}_\varepsilon = -v_\varepsilon(g(u_\varepsilon))_x,
X(t_0) = \varphi_2(t_0) = x_0, \quad v_\varepsilon(t_0) = V_1.
\]

Global solvability of this system can be proved in the same way as one for the system (44).

Next step is to solve second equation of (51) (or analogically of (51)). We have for problem (51):

\[
\dot{v}_\varepsilon = -(g(u_\varepsilon))_x v_\varepsilon \implies \]
\[
v_\varepsilon = V_0 \exp\left(-\int_0^t (g(u_\varepsilon))_x dt'\right) \implies \]
\[
v_\varepsilon = V_0 \exp\left(-\int_0^t \frac{dX}{dt} dt'\right) \implies \]
\[
v_\varepsilon = V_0 \exp\left(-\int_0^t \frac{\partial}{\partial x_0} \frac{dX}{dt} dt'\right) \implies \]
\[
v_\varepsilon = V_0 \exp\left(-\int_0^t \frac{\partial}{\partial X} \frac{\partial x_0}{\partial X} dt'\right) \implies \]
\[
v_\varepsilon = \frac{V_0}{\frac{\partial X}{\partial x_0}}.
\]

and, similarly, for (51):

\[
v_\varepsilon = \frac{V_1}{\frac{\partial X}{\partial x_0}}.
\]
Previous implies:

\[ v_x(x,t) = V_1 \frac{\partial x_{03}}{\partial x}(x,t,\varepsilon), \quad (x,t) \in D_3, \]

\[ v_x(x,t) = V_0 \frac{\partial x_{04}}{\partial x}(x,t,\varepsilon), \quad (x,t) \in D_4, \]

where \( x_{03} = x_0(X,t,\varepsilon) = \varphi_1(t_0) \) and \( x_{04} = x_0(X,t,\varepsilon) = \varphi_2(t_0) \) (for appropriate \( t_0, i = 1, 2 \), depending on \( (X,t) \)) are inverse functions to the function \( X \) determined by (51) and (52), respectively.

Finally, we solve problem (43), (4) in the domain:

\[ D_5 = \{ (x,t) \mid \varphi_2^* < x < \varphi_1^* \}. \]

We apply similar procedure as in the previous case. The solution in this domain is:

\[ v_x(x,t) = v_0(x_0(x,t,\varepsilon)) \frac{\partial x_{05}}{\partial x}(x,t,\varepsilon), \]

where \( x_{05} = x_0(X,t,\varepsilon) \) is inverse function to the function \( x \) determined by (54) (\( x_0 \) restricted on \([a_2 - \varepsilon A^{a_2 + a_1}, a_1 + \varepsilon A^{a_2 + a_1}]\)).

Thus, we have proved the following theorem:

**Theorem 11.** The weak asymptotic solution to problem (43), (4) is given by the formula:

\[
\begin{cases}
V_0, & (x,t) \in D_1, \\
V_0 \frac{\partial x_{03}}{\partial x}(x,t,\varepsilon), & (x,t) \in D_3, \\
v_0(x_0(x,t,\varepsilon)) \frac{\partial x_{05}}{\partial x}(x,t,\varepsilon), & (x,t) \in D_5, \\
V_1 \frac{\partial x_{04}}{\partial x}(x,t,\varepsilon), & (x,t) \in D_4, \\
V_1, & (x,t) \in D_2.
\end{cases}
\]

Let us analyze the function \( v_x \) more closely. We have to inspect its behavior on the boundaries of the domains \( D_i, i = 1, 2, 3, 4, 5 \), since in the domains we know that equation (43) is satisfied.

On the lines \( D_1 \cap D_3 \) and \( D_2 \cap D_4 \) the function is continuous and therefore, in the domain \( D_1 \cup D_2 \cup D_3 \cup D_4 \) the function \( v_x \) represents admissible weak solution to (43).

On the lines \( D_3 \cap D_5 = \varphi_2^* \) and \( D_4 \cap D_5 = \varphi_1^* \) the function \( v_x \) can have discontinuities (see Remark 9 and Example 13). Therefore, we have to check if Rankine-Hugoniot conditions are satisfied on \( \varphi_i^*, i = 1, 2 \) for equation (43) given in divergent form:

\( \varepsilon \sum_{i=1}^{2} \left( \varphi_i^* \right)_t + \left( v_x g(u_x) \right)_x = 0. \)

Accordingly, we have to check:

\( \left( \frac{\partial \varphi_i^*}{\partial t} \right)_t = \left. \frac{g(u_x)}{v_x} \right|_{x=\varphi_i^*}. \)

Since \( u_x \) is continuous function from here it follows:

\( \left( \varphi_i^* \right)_t = \left. g(u_x) \right|_{x=\varphi_i^*}, \)

which is exactly the definition of the function \( \varphi_i^*, i = 1, 2 \).

This concludes the proof that the function \( v_x \) represents weak solution to problem (54), (4).
4. Weak Limit of the Solution

It remains to inspect the weak limit of the weak asymptotic solution \((u_\varepsilon, v_\varepsilon)\) of problem (2), (4) for \(t > t^*\) (since for \(t < t^*\) we have classical solution of the considered problem).

We have already known from Corollary 9 that for \(t \geq t^*\) we have:

\[
(55) \quad w - u_\varepsilon(x, t) \to \begin{cases} 
U_1, & x < \frac{c}{2}(t - t^*) + x^*, \\
U_0, & x \geq \frac{c}{2}(t - t^*) + x^*.
\end{cases}
\]

So, we have to inspect weak limit of the weak asymptotic solution \((u_\varepsilon, v_\varepsilon)\) = \((x, t, \varepsilon)\) of problem (2), (4) for \(t > t^*\).

**Theorem 12.** For every fixed \(t > t^*\) the function \(v_\varepsilon\) given by (53) satisfies as \(\varepsilon \to 0\)

\[
(56) \quad v_\varepsilon(x, t) \to \begin{cases} 
V_1, & x < \frac{c}{2}(t - t^*) + x^*, \\
V_0, & x \geq \frac{c}{2}(t - t^*) + x^*.
\end{cases} + \left[V_1(a_2 + g(U_1)t - \frac{c}{2}(t - t^*) - x^*) + V_0(\frac{c}{2}(t - t^*) + x^* - a_1 - g(U_0)t) + \int_{a_2}^{t^1} v_0(x_0)dx_0\right] \delta(x - \frac{c}{2}(t - t^*) - x^*),
\]

where \(\to\) means convergence in the weak sense with respect to the real variable.

**Proof:** To begin, note that we can write function \(v_\varepsilon\) from (53) in the following manner:

\[v_\varepsilon(x, t) = \hat{v}(x_0(x, t, \varepsilon)) \frac{\partial x_0}{\partial x}(x, t, \varepsilon),\]

where

\[
(57) \quad \begin{cases} 
x - g(U_1)t, & (x, t) \in \bar{D}_1, \\
x_{03}^{-1}(x, t, \varepsilon) - g(U_1)\varphi_2^{-1}(x_{03}^{-1}(x, t, \varepsilon)), & (x, t) \in \bar{D}_3, \\
\text{(here first we go by } x_{03}^{-1} \text{ to the line } \varphi_2 \text{ so that } x_{03}^{-1}(x, t, \varepsilon) = \varphi_2(t_0) \\
\text{and then proceed to the line } t = 0 \text{ along the straight line } x - g(U_1)t),
\end{cases}
\]

\[
\begin{cases} 
x_{05}^{-1}(x, t, \varepsilon), & (x, t) \in \bar{D}_5, \\
x_{04}^{-1}(x, t, \varepsilon) - g(U_0)\varphi_1^{-1}(x_{04}^{-1}(x, t, \varepsilon)), & (x, t) \in \bar{D}_4, \\
\text{(here first we go by } x_{04}^{-1} \text{ to the line } \varphi_1 \text{ so that } x_{04}^{-1}(x, t, \varepsilon) = \varphi_1(t_0) \\
\text{and then proceed to the line } t = 0 \text{ along the straight line } x - g(U_0)t)
\end{cases}.
\]

and

\[
\frac{\partial x_0}{\partial x}(x, t, \varepsilon) = \begin{cases} 
1, & (x, t) \in \bar{D}_1, \\
\frac{\partial x_0}{\partial x}, & (x, t) \in \bar{D}_3, \\
\frac{\partial x_0}{\partial x}, & (x, t) \in \bar{D}_5, \\
\frac{\partial x_0}{\partial x}, & (x, t) \in \bar{D}_4, \\
1, & (x, t) \in \bar{D}_2.
\end{cases}
\]
We take $\eta \in C^1_0(\mathbb{R})$ and write using (53):

$$\int v_\varepsilon(x, t)\eta(x)dx = \int_{-\infty}^{\varphi_2 - \varepsilon} v_\varepsilon(x, t)\eta(x)dx + \int_{\varphi_2 - \varepsilon}^{\varphi_1} v_\varepsilon(x, t)\eta(x)dx + \int_{\varphi_1 + \varepsilon}^{\infty} v_\varepsilon(x, t)\eta(x)dx + \int_{\infty}^{\varphi_1 + \varepsilon} v_\varepsilon(x, t)\eta(x)dx + \int_{\varphi_2 - \varepsilon}^{\varphi_1} V_1\eta(x)dx + \int_{\varphi_2 - \varepsilon}^{\varphi_1} V_1\frac{\partial x_0}{\partial x}\eta(x)dx + \int_{\varphi_1 + \varepsilon}^{\varphi_2} \int_{-\infty}^{\varepsilon} V_0\eta(x)dx + \int_{\varepsilon}^{\varphi_1 + \varepsilon} V_0\eta(x)dx.$$ 

Here, we have written $\varphi_i \pm \varepsilon$ in order to avoid possible $\varphi_i = \varphi^*_i$.

Then, we use the change of variables $x = X(x_0, t, \varepsilon)$ where $X$ is inverse function of the function $x_0 = x_0(X, t, \varepsilon)$ given by (57). We have:

$$\int v_\varepsilon(x, t)\eta(x)dx = \int_{-\infty}^{\varphi_2 - \varepsilon} v_\varepsilon(x, t)\eta(x)dx + \int_{\varphi_2 - \varepsilon}^{\varphi_1} V_1\eta(x_0(x_0, t, \varepsilon))dx_0 + \int_{\varphi_1 + \varepsilon}^{\infty} V_0\eta(x_0(x_0, t, \varepsilon))dx_0 + \int_{\varphi_1 + \varepsilon}^{\varphi_2} V_0\frac{\partial x_0}{\partial x}\eta(x_0(x_0, t, \varepsilon))dx_0 + \int_{\varphi_1 + \varepsilon}^{\varphi_2} V_0\eta(x_0(x_0, t, \varepsilon))dx_0,$$

and we remind that:

$$x_0(\varphi_1, t, \varepsilon) = \varphi_1 + \varepsilon - g(U_0)t, \quad x_0(\varphi_2, t, \varepsilon) = \varphi_2 - \varepsilon - g(U_1)t,$$

and for $t > t^*$ we have (see (27) and (28)):

$$x_0(\varphi_1 + \varepsilon, t, \varepsilon) \rightarrow \frac{c}{2}(t - t^*) + x^* - g(U_0)t, \quad \varepsilon \rightarrow 0, \quad x_0(\varphi_2 - \varepsilon, t, \varepsilon) \rightarrow \frac{c}{2}(t - t^*) + x^* - g(U_1)t, \quad \varepsilon \rightarrow 0.$$

Accordingly, for $t > t^*$ after letting $\varepsilon \rightarrow 0$ we have from (58) exactly (56).

This concludes the theorem. $\square$

It remains to give a comment concerning admissibility of the solution. Actually, it follows from assumptions on $f'$ and $g$ quoted in Theorem 3 providing:

$$g(U_1) < f'(U_1), \quad f'(U_0) < g(U_0).$$

Admissibility of the shock wave appearing in the solution to problem (1), (3) implies

$$f'(U_1) < c/2 < f'(U_0)$$

which together with (59) implies:

$$g(U_1) < c/2 < g(U_0),$$

which proves overcompressibility of the shock and $\delta$ shock wave appearing in (55) and (56).
Example 13. In this example we look for the classical solution to the following problem:

\[ u_t + \left( \frac{1}{2}u^2 \right)_x = 0 \]
\[ v_t + (2uv)_x = 0 \]

for initial functions given by:

\[ u \big|_{t=0} = \begin{cases} 1, & x \leq -1 \\ -x, & -1 \leq x \leq 1 \\ -1, & 1 \leq x \end{cases} \]
\[ v \big|_{t=0} = \begin{cases} 1, & |x| \geq 1 \\ x^{2/3}, & -1 \leq x \leq 1. \end{cases} \]

Using previous notation on this case, we have:

\[ a_1 = 1, \quad a_2 = -1, \quad U = 1, \quad u_0 = -1 \]
\[ b = 0, \quad K = 1. \]

Also, for the inverse function \( x_0 = x_0(x,t) \) we have:

\[ x_0 = \begin{cases} x - 2t, & x \leq t - 1, \\ -2 - \frac{(t-1)^2}{x}, & t - 1 \leq x \leq -(t-1)^2, \\ \frac{x}{(1-t)^2}, & -(t-1)^2 \leq x \leq (t-1)^2, \\ 2 - \frac{(t-1)^2}{x}, & (t-1)^2 \leq x \leq 1 - t, \\ x + 2t, & 1 - t \leq x. \end{cases} \]

From here we can immediately compute \( \frac{\partial x}{\partial x_0} \) and then

\[ \frac{\partial x}{\partial x_0} = \frac{1}{\frac{\partial x_0}{\partial x}}. \]

The solution of the problem is:

\[ u(x,t) = \begin{cases} 1, & x > t - 1, \\ \frac{x}{t-1}, & t - 1 \leq x < -(t-1), \\ -1, & -(t-1) \leq x \leq -t + 1. \end{cases} \]

\[ v(x,t) = \begin{cases} 1, & x < t - 1, \\ \frac{(t-1)^2}{x^{2/3}}, & t - 1 \leq x < -(t-1)^2, \\ \frac{(t-1)^2}{x^{2/3}}, & -(t-1)^2 \leq x < (t-1)^2, \\ 1, & (t-1)^2 \leq x < -t + 1, \end{cases} \]

Notice that the function \( v \) has discontinuities on the following lines:

\[ x = (t-1)^2, \quad x = -(t-1)^2. \]

and that on that line Rankine-Hugoniot conditions are satisfied.

Notice, also, that if instead of \( x^{2/3} \) in the initial data for \( v \) we would put \( x^{2n} \) the solution of our problem would be continuous for \( t < t^* \).
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