Multiscale decompositions of Hardy spaces

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1 Introduction

We would like to elaborate on a program of analysis pursued by Alex Grossmann and his collaborators on the analytic utilization of the phase of Hardy functions, as a multiscale signal processing tool.

An inspiration at the origin of ”wavelet” analysis (when Grossmann, Morlet, Meyer and collaborators were interacting and exploring versions of multiscale representations) was provided, by the analysis of holomorphic signals, for which, the images of the phase of Cauchy wavelets were remarkable in their ability to reveal intricate singularities or dynamic structures, such as instantaneous frequency jumps, in musical recordings. This work which was pursued by Grossmann, Kronland Martinet et al [11] exploiting phase and amplitude variability of holomorphic signals was challenged by computational complexity as well as by the lack of simple, efficient, mathematical processing, and generalizations to higher dimensional signals. It was mostly bypassed by the orthogonal wavelet transforms. We aim to show that these ideas are powerful nonlinear subtle tools.

Our goal here is to follow their seminal work and introduce recent developments in nonlinear analysis. In particular we will sketch methods extending conventional Fourier analysis, exploiting both phase and amplitudes of holomorphic functions.

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The miracles of nonlinear complex analysis, such as factorization and composition of functions lead to new versions of holomorphic wavelets, and relate them to multiscale dynamical systems.

Our story interlaces the role of the phase of signals with their analytic/geometric properties. The Blaschke factors are a key ingredient, in building analytic tools, starting with the Malmquist Takenaka orthonormal bases of the Hardy space $H^2(T)$, continuing with "best" adapted bases obtained through phase unwinding, and concluding with relations to composition of Blaschke products and their dynamics (on the disc, and on invariant subspaces of $H^2(T)$). Specifically we discuss multiscale orthonormal holomorphic wavelet bases, related to Grossmann’s and Morlet’s program [8], and associated generalized scaled holomorphic orthogonal bases, to dynamical systems, obtained by composing Blaschke factors.

We also, remark, that the phase of a Blaschke product is a one layer neural net with (arctan as an activation sigmoid) and that the composition is a "Deep Neural Net" whose depth is the number of compositions, our results provide a wealth of related libraries of orthogonal bases .

We sketch these ideas in various "vignette" subsections and refer for more details on analytic methods [2], related to the Blaschke based nonlinear phase unwinding decompositions [3][4][14], we also consider orthogonal decompositions of invariant subspaces of Hardy spaces. In particular we constructed a multiscale decomposition, described below, of the Hardy space of the upper half-plane.

Such a decomposition can be carried in the unit disk by conformal mapping. A somewhat different multiscale decomposition of the space $H^2(T)$ has been constructed by using Malmquist-Takenaka bases associated with Blaschke products whose zeros are $(1 - 2^{-n})e^{2\pi i j/2^n}$ where $n \geq 1$ and $0 \leq j < 2^n$. Here we provide a variety of multiscale decompositions by considering iterations of Blaschke products.

2 Preliminaries and notation

For $p \geq 1$, $H^p(T)$ stands for the space of analytic functions $f$ on the unit disk $\mathbb{D}$ such that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < +\infty.$$  

Such functions have boundary values almost everywhere, and the Hardy space $H^p(T)$ can be identified with the set of $L^p$ functions on the torus $T = \partial \mathbb{D}$ whose Fourier coefficients of negative order vanish.

A subspace of $H^p(T)$ is invariant if it is invariant under multiplication by $e^{i\theta}$ (or by $z$), depending whether these functions are considered as functions on $T$ or $\mathbb{D}$. An inner function is a bounded analytic function on the unit disk whose boundary values have modulus 1 almost everywhere. It is
known that the invariant subspaces are of the form \( uH^p(T) \) where \( u \) is an inner function \([9, 10]\). The inner function \( u \) is determined by the invariant subspace up to multiplication by a constant of modulus 1.

If \( f \) and \( g \) are two functions on \( T \) (in \( L^p \) and \( L^p/(p-1) \) for some \( p \in [1, +\infty) \)), let
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \, d\theta.
\]

Let \( \mathcal{H} \) be the operator of orthogonal projection of \( L^2(T) \) onto \( H^2(T) \). It results from the properties of the Hilbert transform that this operator extends as a bounded operator from \( L^p(T) \) to \( H^p(T) \) for \( 1 < p < +\infty \).

If \( u \) is an inner function, let \( \chi_u \) be the operator of multiplication by \( u \) (which is an isometry of all the \( L^p \)). Then the operator \( \mathcal{H}_u = \chi_u \mathcal{H} \chi_u^{-1} \) is the operator of orthogonal projection of \( L^2 \) onto \( uH^2(T) \). It results that this operator extends as a bounded operator from \( L^p(T) \) to \( H^p(T) \) for all \( p \in (1, +\infty) \) with a norm independent of \( u \). In other terms, for all \( p > 1 \), there exists \( C_p \) such that, for all \( u \) and all \( f \in L^p(T) \),
\[
\|\mathcal{H}_u f\|_p \leq C_p \|f\|_p. \tag{1}
\]

There is a parallel theory for analytic functions on the upper half plane \( \mathbb{H} = \{x + iy : y > 0\} \). The space of analytic functions \( f \) on \( \mathbb{H} \) such that
\[
\sup_{y > 0} \|f(\cdot + iy)\|_{L^p(\mathbb{R})} < +\infty
\]
is denoted by \( H^p(\mathbb{R}) \). These functions have boundary values in \( L^p(\mathbb{R}) \) when \( p \geq 1 \). The space \( H^p(\mathbb{R}) \) is identified to the space of \( L^p \) functions whose Fourier transform vanishes on the negative half line \( (-\infty, 0) \).

A subspace of \( H^2(\mathbb{R}) \) is said to be invariant if it is stable by multiplication by the functions \( e^{\pm i\pi \xi x} \) for all \( \xi > 0 \). As previously, the invariant subspaces are of the form \( uH^2 \) where \( u \) is an inner function, i.e., a bounded analytic function on \( \mathbb{H} \) whose boundary values are of modulus 1 almost everywhere.

As previously, the operators of orthogonal projections on invariant subspaces extend, for any \( p \in (1, +\infty) \), as continuous operators on \( H^p(\mathbb{R}) \) with a uniform bound for their norms.

### 3 Malmquist-Takenaka bases on the torus

**Lemma 1** Let \( a \) be a complex number of modulus less than 1 and \( u \) be an inner function. Then \((z - a)uH^2\) has codimension 1 in \( uH^2 \) and \( \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}z} u \) is a unit vector in the orthogonal complement of \((z - a)uH^2\) in \( uH^2 \).
Proof Since $f \mapsto uf$ is an isometry of $H^2$ onto $uH^2$, it is enough to consider the case $u = 1$. One has
\[
\langle (z - a)f(z), \frac{\sqrt{1 - |a|^2}}{1 - \overline{a}z} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\theta} - a)f(e^{i\theta}) \frac{\sqrt{1 - |a|^2}}{1 - ae^{-i\theta}} d\theta
\]
\[
= \frac{\sqrt{1 - |a|^2}}{2\pi} \int_{-\pi}^{\pi} e^{i\theta} f(e^{i\theta}) d\theta = 0.
\]
Also, if $f$ is orthogonal to $(1 - \overline{a}z)^{-1}$ one has
\[
0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) d\theta = \frac{1}{2\pi} \oint \frac{f(z)}{z - a} dz = f(a),
\]
so $f \in (z - a)H^2$.

Now let $(a_n)_{n>0}$ be a sequence of complex numbers of modulus less than 1. For $n \geq 0$, let
\[
B_n(z) = \prod_{0 \leq j < n} \frac{z - a_j}{1 - \overline{a}_j z} \quad \text{and} \quad \phi_n(z) = B_n(z) \frac{\sqrt{1 - |a_n|^2}}{1 - \overline{a}_n z}.
\]
It results from Lemma 1 that, if
\[
\sum_{n \geq 1} (1 - |a_j|^2) = +\infty,
\]
the functions $\phi_n$ form an orthonormal basis of $H^2$.

If $\sum_{n \geq 1} (1 - |a_j|^2) < +\infty$ the functions $\phi_n$ form an orthonormal basis of $H^2 \ominus BH^2$, where $B$ is the convergent Blaschke product
\[
B(z) = \prod_{j>0} \frac{\overline{a}_j}{|a_j|} \frac{z - a_j}{1 - \overline{a}_j z}.
\]

Consider a sequence $(B_m)_{m \geq 1}$:
\[
B_m(z) = \prod_{j>0} \frac{\overline{a}_{m,j}}{|a_{m,j}|} \frac{z - a_{m,j}}{1 - \overline{a}_{m,j} z}
\]
of convergent Blaschke products such that $\sum_{m,j>0} (1 - |a_{m,j}|^2) = +\infty$.

Let $B_0 = 1$ and, for $n \geq 0$ and $m \geq 1$,
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\[ B_{m,n}(z) = \prod_{0 \leq j < n} \frac{z - a_{m,j}}{1 - \overline{a}_{m,j} z} \]

\[ \phi_{m,n}(z) = B_{m-1}(z) B_{m,n}(z) \frac{\sqrt{1 - |a_{m,n}|^2}}{1 - \overline{a}_{m,n} z}. \]

Then, \((\phi_{m,n})_{n \geq 1}\) is an orthonormal basis of \(B_{m-1} H^2 \ominus B_{m-1} B m H^2\), and \((\phi_{m,n})_{m \geq 1, n \geq 1}\) is an orthonormal basis of \(H^2\).

The bases so obtained are the Malmquist-Takenaka bases [23].

4 The upper half plane

We present some prior results [2], without proof. In this section one simply writes \(H^2\) instead of \(H^2(\mathbb{R})\).

4.1 Malmquist-Takenaka bases

Let \((a_j)_{1 \leq j}\) be a sequence (finite or not) of complex numbers with positive imaginary parts and such that

\[ \sum_{j \geq 0} \frac{\Re a_j}{1 + |a_j|^2} < +\infty. \] (2)

The corresponding Blaschke product is

\[ B(x) = \prod_{j \geq 0} \frac{|1 + a_j^2|}{1 + a_j^2} \frac{x - a_j}{x - \overline{a}_j}, \]

where, 0/0, which appears if \(a_j = i\), should be understood as 1. The factors \(\frac{|1 + a_j^2|}{1 + a_j^2}\) insure the convergence of this product when there are infinitely many zeroes. But, in some situations, it is more convenient to use other convergence factors as we shall see below.

Whether the series (2) is convergent or not, one defines (for \(n \geq 0\)) the functions

\[ \phi_n(x) = \frac{1}{\sqrt{\pi}} \left( \prod_{0 \leq j < n} \frac{x - a_j}{x - \overline{a}_j} \right) \frac{1}{x - \overline{a}_n}. \]

Then these functions form a orthonormal system in \(H^2\). If the series (2) diverges, it is a basis of \(H^2\), otherwise it is a basis of the orthogonal complement of \(B H^2\) in \(H^2\).
4.2 A multiscale Wavelet decomposition

The infinite products

\[ G_n(x) = \prod_{j \leq n} j - i \ x - j - i \quad \text{and} \quad G(x) = \prod_{j \in \mathbb{Z}} j - i \ x - j + i \]

can be expressed in terms of known functions:

\[ G_n(x) = \Gamma(i - n) \ \Gamma(x - n + i) \ \Gamma(x - n - i) \ \Gamma(1 - n) \quad \text{and} \quad G(x) = \frac{\sin \pi(i - x)}{\sin \pi(i + x)}. \]

4.3 An orthonormal system

Consider the function

\[ \phi(x) = \frac{\Gamma(x - 1 + i)}{\sqrt{\pi} \Gamma(x - i)}. \]

It is easily checked that

\[ \phi(x - n) = \frac{\Gamma(i - n)}{\Gamma(-i - n)} \frac{G_n(x)}{\sqrt{\pi} (x - (n + 1) + i)}. \]

Set \( \phi_n(x) = \phi(x - n) \). For fixed \( m \), the functions \( \phi_n/G_m \), for \( n \geq m \), form a Malmquist-Takenaka basis of \((G/G_m)H^2\). In other terms, the functions \( \phi_n \), for \( n \geq m \), form an orthonormal basis of \( G_mH^2 \ominus GH^2 \). This means that the functions \( \phi_n \) (for \( n \in \mathbb{Z} \)) form a Malmquist-Takenaka basis of the orthogonal complement of \( GH^2 \) in \( H^2 \).

4.3.1 Multiscale decomposition

As \(|1 - G(2^n x)| \leq C 2^n\) all the products

\[ B_n(x) = \prod_{j < n} G(2^j x) \quad \tag{3} \]

are convergent and \( \lim_{n \to \infty} B_n = 1 \) uniformly.

Let \( B = B_0 \). Obviously, \( B_n(x) = B(2^n x) \).

Consider the following subspaces of \( H^2 \):

\[ E_n = B_n H^2. \]
This is a decreasing sequence. The space $E_{+\infty} = \bigcap_{n \in \mathbb{Z}} E_n$ is equal to $\{0\}$ since a function orthogonal to this space would have too many zeros, and the space $E_{-\infty} = \text{closure of } \bigcup_{n \in \mathbb{Z}} E_n$ is equal to $H^2$ since $B_n$ converges uniformly to 1 when $n$ goes to $-\infty$.

For all $n$ and $j$, let
\[
\phi_{n,j}(x) = 2^{n/2}\phi(2^n x - j)B(2^n x).
\]

Then, for all $n$, $(\phi_{n,j})_{j \in \mathbb{Z}}$ is an orthonormal basis of $E_n \ominus E_{n+1}$. At last $(\phi_{n,j})_{n,j \in \mathbb{Z}}$ is an orthonormal basis of $H^2$.

5 Adapted MT bases, ”phase unwinding”

Our goal is to find a ”best” adapted Malmquist Takenaka basis to analyze a given function the idea is to peel off the oscillation of a function by dividing by its Blaschke product, this procedure is iterated to yield an expansion in an orthogonal collection of functions or Blaschke products which of course are naturally embedded in a MT basis, once the zeroes are ordered.

5.1 The unwinding series.

There is a natural way to iterate the Blaschke factorization, it is inspired by the power series expansion of a holomorphic function. If $G$ has no zeroes inside $\mathbb{D}$, its Blaschke factorization is the trivial one $G = 1 \cdot G$, however, the function $G(z) - G(0)$ certainly has at least one root inside the unit disk $\mathbb{D}$ and will therefore yield some nontrivial Blaschke factorization $G(z) - G(0) = B_1G_1$.

We write
\[
F = B \cdot G \\
= B \cdot (G(0) + (G(z) - G(0))) \\
= B \cdot (G(0) + B_1 G_1) \\
= G(0)B + BB_1 G_1.
\]

An iterative application gives rise to the unwinding series
\[
F = a_1 B_1 + a_2 B_1 B_2 + a_3 B_1 B_2 B_3 + a_4 B_1 B_2 B_3 B_4 + \ldots
\]

This orthogonal expansion first appeared in the PhD thesis of Michel Na- hon [14] and independently by T. Qian in [13, 21]. Given a general function $F$ it is not numerically feasible to actually compute the roots of the function;
a crucial insight in [14] is that this is not necessary – one can numerically obtain the Blaschke product in a stable way by using a method that was first mentioned in a paper of Guido and Mary Weiss [24] and has been investigated with respect to stability by Nahon [14]. Using the boundedness of the Hilbert transform one can prove easily convergence in $L^p, 1 < p < \infty$.

5.2 The fast algorithm of Guido and Mary Weiss [24]

Our starting point is the theorem that any Hardy function can be decomposed as

$$F = B \cdot G,$$

where $B$ is a Blaschke product, that is a function of the form

$$B(z) = z^m \prod_{i \in I} \frac{a_i}{|a_i|} \frac{z - a_i}{1 - \overline{a_i}z},$$

where $m \in \mathbb{N}_0$ and $a_1, a_2, \ldots \in \mathbb{D}$ are zeroes inside the unit disk $\mathbb{D}$ and $G$ has no roots in $\mathbb{D}$. For $|z| = 1$ we have $|B(z)| = 1$ which motivates the analogy

$$B \sim \text{frequency and } G \sim \text{amplitude}$$

for the function restricted to the boundary. However, the function $G$ need not be constant: it can be any function that never vanishes inside the unit disk. If $F$ has roots inside the unit disk, then the Blaschke factorization $F = B \cdot G$ is going to be nontrivial (meaning $B \not\equiv 1$ and $G \not\equiv F$). $G$ should be 'simpler' than $F$ because the winding number around the origin decreases.

In fact since $|F| = |G|$ and $\ln(G)$ is analytic in the disk we have formally

$$G = \exp(\ln |F| + i(\ln |F|)^\circ) = \exp(\mathcal{H}(\ln |F|))$$

and $B = F/G$.

$G$ can be computed easily using the FFT [14].

6 Iteration of Blaschke products.

We are interested in the case we iterate finite Blaschke products:

$$B(z) = e^{i\theta} z^\nu \prod_{j=1}^\mu \frac{z + a_j}{1 + \overline{a_j}z},$$

where $\mu$ and $\nu$ are nonnegative integers and the $a_j$ are complex numbers of modulus less than 1.
It is well known that $\mathbb{T}$ and $\mathbb{D}$ are globally invariant under $B$, as well as the complement of $\overline{\mathbb{D}}$ in the Riemann sphere.

We have
\[ e^{-i\varphi}B(e^{i\varphi}z) = e^{i(\theta + (\nu + \mu - 1)\varphi)}z^{\nu}b_1 \prod_{j=1}^{\mu} \frac{z + b_j}{1 + \overline{b_j}z}, \]
where $b_j = e^{-i\varphi}a_j$. This means that for iteration purpose we may assume $\theta = 0$.

When $\nu \geq 1$, it results from the Schwarz lemma that 0 is the unique fixed point of $B$ in $\mathbb{D}$ and that this fixed point is attracting. The basin of attraction of 0 contains a disk centered at 0. The sequence of iterates $B_n$ is a normal family, since it converges towards 0 on some neighborhood of 0, it has a unique limit point. This means that $B_n$ converges to 0 uniformly on any compact subset of $\mathbb{D}$.

Now, suppose $\nu = 0$. We have
\[ B(z)\overline{B(1/z)} = 1, \]
so, if $\alpha$ is a fixed point of $B$, so is $1/\alpha$.

we have two possibilities:

1. There exists a fixed point $\alpha \in \mathbb{D}$ of $B$. Then if $\phi(z) = \frac{z + \alpha}{1 + \overline{\alpha}z}$, then 0 is a fixed point of the Blaschke product $\phi^{-1} \circ B \circ \phi$. It results from the preceding discussion that $\alpha$ is the unique fixed point of $B$ in $\mathbb{D}$, that it is attracting, and that the sequence of iterates of $B$ converges to the constant $\alpha$.
2. All the fixed points lie in $\mathbb{T}$. Then according to the Denjoy-Wolff theorem [1], again the sequence of iterates of $B$ converges to some constant.

Moreover, the fixed points of $B$ are the roots of an equation of degree $\mu + 1$, so if $B$ has a fixed point in $\mathbb{D}$, it has $\mu - 1$ fixed points in $\mathbb{T}$ (none of them being attracting). In this case, the dynamics on $\mathbb{T}$ is that of a cookie cutter.

When $B_1$ and $B_2$ are finite Blaschke products with $n_1$ and $n_2$ zeros, then $B_2 \circ B_1$ is a finite Blaschke products with $n_1 n_2$ zeros (of course zeros are counted with their multiplicities). This is obvious by considering the variation of arguments on the boundary of the disk.

Let $B$ be a finite Blaschke product having at least two zeros. Then, one may consider the dynamical system which it generates. Let $B_0(z) = z$, and $B_{n+1} = B_n \circ B$.

**Lemma 2** Consider a Blaschke product $F$ of the form $F(z) = zB(z)$, where $B$ is nonconstant. Then there exists a sequence $0, a_1, a_2, \ldots, a_j, \ldots$ of complex numbers in the unit disk and an increasing sequence $(\nu_j)_{j \geq 1}$ of positive integers such that $a_1, a_2, \ldots, a_{\nu_j}$ are the zeros, counted according to their multiplicity, of $F_n$. Moreover $\sum_{j \geq 1} (1 - |a_j|) = +\infty$. □

**Proof** A simple recursion shows
This proves the existence of the sequences \((a_j)\) and \((\nu_j)\). In fact, if \(B\) has \(k-1\) zeros, \(\nu_n = k^n\).

If \(B(0) = 0\), then the multiplicity of 0 in \(F_n\) increases with \(n\), so
\[
\sum_{j \geq 1} (1 - |a_j|) = +\infty.
\]

Otherwise, let us compute the derivative \(F'_n(0)\). On the one hand, due to (4) and \(F(0) = 0\), we have \(F'_n(0) = B(0)\). Therefore \(\lim_{n \to \infty} F'_n(0) = 0\). On the other hand,
\[
F'_n(0) = \lim_{z \to 0} \frac{F_n(z)}{z}, \quad \text{so} \quad |F'_n(0)| = \prod_{j=1}^{\nu_n} |a_j|.
\]

Thus \(\prod_{j \geq 1} |a_j| = 0\). This ends the proof.

**Corollary 1** Let \(F\) be a finite Blaschke product with a fixed point \(\alpha\) inside the unit disk. Then there exits a sequence \(\alpha, a_1, a_2, \ldots, a_j, \ldots\) of complex numbers in the unit disk and an increasing sequence \((\nu_j)_{j \geq 1}\) of positive integers such that \(a_1, a_2, \ldots, a_{\nu_n}\) are the zeros, counted according to their multiplicity, of \(F_n\). Moreover \(\sum_{j \geq 1} (1 - |a_j|) = +\infty\). □

**Proof** Just consider \(\varphi_\alpha \circ F \circ \varphi_\alpha\), with \(\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}\), and use Lemma 2.

When \(F = zB\), if \(|w| < 1\) and \(F(z) = w\), we have \(|w| = |z||B(z)| < |z|\). A refinement of this observation leads in some circumstances to an estimate of the speed of accumulation of the zeros to the boundary. Here is an example.

**Lemma 3** Let \(k\) be a positive integer and \(a \in \mathbb{D}, a \neq 0\). Let \(F(z) = z^k - a^k\). Then there exist two increasing and concave functions \(g\) and \(h\) on \([0, 1]\) such that \(g(1) = h(1) = 1, g(0) = 0, g(t) < h(t)\) if \(t < 1\), and such that, if \(F(z) = w\) with \(|w| \geq |a|\), we have \(g(|w|) < |z| < h(|w|)\). □

**Proof** One can check the following equalities, valid for \(0 < \rho, r < 1\) and any real \(\varphi\):
\[
\frac{r + \rho}{1 + r \rho} = \inf_\theta \left| \frac{r e^{i\theta} - \rho e^{i\varphi}}{1 - r \rho e^{i(\theta - \varphi)}} \right| < \left( \sup_\theta \left| \frac{r e^{i\theta} - \rho e^{i\varphi}}{1 - r \rho e^{i(\theta - \varphi)}} \right| \right) = \frac{|r - \rho|}{1 - r \rho}.
\]

Let \(\rho = |a|\). Suppose \(F(z) = w\), with \(|w| = t \geq \rho\), and \(|z| = r\). As already observed \(|z| > |w|\). Then the preceding equalities give
$$\frac{r^k + \rho^k}{1 + r^k \rho^k} \leq t \leq \frac{r^k - \rho^k}{1 - r^k \rho^k}.$$ 

Let $\varphi(r) = \frac{r^k - \rho^k}{r^k + \rho^k}$ and $\psi(r) = \frac{r^k + \rho^k}{r^k - \rho^k}$. It is routine to check that $\varphi$ and $\psi$ is increasing and convex, $\psi$ from $[0, 1]$ onto $[0, 1]$ and $\varphi$ from $[\rho, 1]$ onto $[0, 1]$. Then the inverse functions $g$ and $h$ of $\psi$ and $\varphi$ will do.

**Proposition 1** Keep the notation and hypotheses of Lemma 3. Then, for all $w$ such that $F_{n+1}(w)/F_n(0) = 0$, we have $g(|a|) < |w| < h(|a|)$. □

**Proof** The zeros of $F_{n+1}/F_n$ are inverse image by $F$ of the zeros of $F_n/F_{n-1}$. Also the zeros of $F/F_0$ are $e^{2i\pi k/a}$. Knowing that, if $z \neq 0$ we have $|F(z)| < |z|$, we conclude that all the zeros (except 0!) have a modulus larger than or equal to $|a|$.

Lemma 3 shows that for all zero $w$ of $F_2/F_1$ we have $g(|a|) \leq |w| \leq h(|a|)$. Then we can perform a recursion. Assuming that all zeros of $F_n/F_{n-1}$ have an absolute value between $g_{n-1}(|a|)$ and $h_{n-1}(|a|)$, Lemma 3 gives that the absolute value of the zeros of $F_{n+1}/F_n$ all lie between $g_n(|a|)$ and $h_n(|a|)$.

It is worth noticing that 1 is an attracting fixed point of both $g$ and $h$. Besides we have

$$g'(1) = \frac{1}{1 + k \frac{1 - |a|^k}{1 + |a|^k}} \quad \text{and} \quad h'(1) = \frac{1}{1 + k \frac{1 + |a|^k}{1 - |a|^k}}.$$ 

**Fig. 1** The functions $h$ and $g$, with $a = 0.5$ and $k = 1, 3, 5$. 

6.1 Multiscale decomposition

Each Blaschke product $B$ defines invariant subspaces of $H^p$. The projection on this space is given by the kernel $\frac{B(z)\overline{B(w)}}{z-w}$. This projection is continuous for $1 < p < +\infty$.

Let $F$ be a Blaschke product of degree at least 2 with a fixed point inside the unit disk. Its iterates define a hierarchy of nested invariant subspaces $E_n = F_n H^2$.

Due to Corollary 1, $\bigcap_{n \geq 1} E_n = \{0\}$.

The Takenaka construction provides orthonormal bases of $E_n \ominus E_{n+1}$. But this is not canonical as it depends on an ordering of the zeros of $F_{n+1}/F_n$.

Figure 2 shows 1st, 3rd, and 5th iterates of $F(z) = z(z - 2^{-1})/(1 - 2^{-1}z)$. Figure 3 displays the same for $F(z) = z(z - 2^{-2})/(1 - 2^{-2}z)$. The upper pictures display the phases modulo $2\pi$ (values in the interval $(-\pi, \pi]$) of these Blaschke products while the lower pictures display minus the logarithms of their absolute value. The coordinates $(x, y)$ correspond to the point $e^{-y+ix}$. On these figures it is easy to locate the zeros, specially by looking at the phase which then has an abrupt jump.
Fig. 3 The argument and the absolute value of $F(z)$, $F^{(3)}(z)$, and $F^{(5)}(z)$, with $F(z) = \frac{z^{2} - 2^{-2}}{1 - 2^{-2}z^{2}}$ and $z = \exp(-y + ix)$.

7 Remarks on Iteration of Blaschke products as a
"Deep Neural Net"

In the upper half plane let $(a_{j})_{1\leq j}$ be a finite sequence of complex numbers with positive imaginary parts.

The corresponding Blaschke product on the line is

$$B(x) = \prod_{j \geq 0} \frac{(x - a_{j})/(x - \bar{a}_{j})}{(x - a_{j})/(x - \bar{a}_{j})},$$

We can write $B(x) = \exp(i\theta(x)).$ where

$$\theta(x) = \sum_{j \geq 0} \sigma\left([x - \alpha_{j}] / (\beta_{j})\right)$$

where $a_{j} = \alpha_{j} + i\beta_{j}$ and $\sigma = \arctan(x) + \pi / 2$ is a sigmoid.

This is precisely the form of a single layer in a Neural Net, each unit has a weight and bias determined by $a_{j}$. We obtain the various layers of a deep net through the composition of each layer with a preceding layer. In our preceding examples we took a single short layer given by a simple Blaschke term with two zeroes in the first layer that we iterated to obtain an orthonormal Malmquist Takanaka basis (we could have composed different elementary products at each layer), demonstrating the versatility of the method to generate highly complex functional representations.
As an example let $F(z)$ be mapped from $G$ (4.2) in the section on wavelet construction.

$$F(z) = G(w) = \frac{\sin(\pi(i - w))}{\sin(\pi(i + w))}$$

with $w = \frac{i(1-z)}{(1+z)}$

we can view the phase of $F$ as a neural layer which when composed with itself results in a Phase which is a two layer neural net represented graphically in fig 4.

Where each end of a color droplet corresponds to one zero or unit of the two layer net.

Fig. 4 Two iterations of $F(z)$
We conclude by referring to Daubechies et al. [5] for a description of a similar iteration for piecewise affine functions in which simple affine functions play the role of a Blaschke product.

Appendix: An example

In this section \( a \in \mathbb{D} \) and \( B(z) = \left( \frac{z + a}{1 + \pi z} \right)^2 \).

**Proposition 2** 1. If \( 27 |a|^4 - 18 |a|^2 + 8 \Re a < 1 \), then \( B \) has a unique fixed point inside \( \mathbb{D} \) and a unique fixed point in \( \mathbb{T} \). Moreover \( B_n \) converges, uniformly on any compact subset of \( \mathbb{D} \), towards the attracting fixed point.

2. If \( 27 |a|^4 - 18 |a|^2 + 8 \Re a \geq 1 \), \( B \) has no fixed point in \( \mathbb{D} \); it has three fixed points in \( \mathbb{T} \) and only one of them is attracting. Moreover \( B_n \) converges, uniformly on any compact subset of \( \mathbb{D} \), towards the attracting fixed point. \( \square \)

**Proof** The fixed points of \( B \) are the roots of the equation

\[
\pi^2 z^3 + (2\pi - 1)z^2 - (2a - 1)z - a^2 = 0. \tag{5}
\]

If we write \( z = x + iy \) and \( a = t + iu \), with \( x, y, t, \) and \( u \) real, this equation becomes

\[
-t^2 x^3 + 3t^2 xy^2 - 6txy^3 + 2tuy^3 + u^2 x^3 - 3u^2 xy^2 - 2tx^2 + 2ty^2 - 4uxy + t^2 + 2tx - u^2 - 2uy + x^2 - y^2 - x \\
+ i(-3t^2 x^2 y + t^2 y^3 + 2txy^3 - 6txy^3 + 3u^2 x^2 y - u^2 y^3 - 4txy + 2ux^2 - 2uy^2 + 2tu + 2ty + 2ux + 2xy - y) = 0. \tag{6}
\]

So \( x \) and \( y \) are the real solutions to a system of two polynomial equations. If we look for the fixed points in \( \mathbb{T} \) we have to add the equation \( x^2 + y^2 = 1 \). By substituting \( 1 - x^2 \) to \( y^2 \) in the real and imaginary parts of the previous equation we get the following two equations

\[
(2x - 1)(2t^2 x - 2u^2 x + t^2 - u^2 + 2t - 1)y \\
= -2u(x + 1)(2x - 1)(2tx - t + 1) \tag{6}
\]

and

\[
2u(2x + 1)(2tx - t + 1)y \\
= +(2x + 1)(x - 1)(2t^2 x - 2u^2 x + t^2 - u^2 + 2t - 1). \tag{7}
\]
By eliminating \( y \) between Equations \( 5 \) and \( 7 \) we get the polynomial

\[
P = 4(t^2 + u^2)^2 x^3 + (8t^4 + 8tu^2 - 4t^2 + 4u^2)x^2
\]
\[
+ (-3t^4 - 6t^2 u^2 - 3u^4 - 4t^3 + 12tu^2 + 6u^2 + 2u^2 - 4t + 1)x
\]
\[
- (t^2 + 2tu - u^2 + 2t - 2u - 1)(t^2 - 2tu - u^2 + 2t + 2u - 1).
\]

If \( x_0 \) is a real root of \( P \), then \( y_0 \) computed by substituting \( x_0 \) to \( x \) in Equation \( 6 \) or in Equation \( 7 \) is real and satisfies \( x_0^2 + y_0^2 = 1 \). So all depends on the sign of \( Q \).

The real roots of \( P \) are the real parts of the fixed points of \( B \) on the circle \( T \). So, the number of fixed points of modulus 1, depends on the sign of the discriminant of \( P \) viewed as a polynomial in \( x \). This discriminant is

\[-(t^2 + u^2 - 1)^2(27(t^2 + u^2)^2 - 18(t^2 + u^2) + 8t - 1).\]

So all depends on the sign of \( Q = 27(t^2 + u^2)^2 - 18(t^2 + u^2) + 8t - 1 \), i.e., the sign of \( 27|a|^4 - 18|a|^2 + 8\Re a - 1 \). This accounts for the first assertion and part of the second one.

Now we give an algebraic proof that, according to the Denjoy-Wolff theorem, when \( Q > 0 \) there is one fixed point where \( |B'| < 1 \).

Let \( P_1 \) be the polynomial in \( z \) obtained by replacing \( a \) by \( t + iu \) in the left-hand side of Equation \( 5 \). The resultant of \( P_1 \) and \( 1 - wB'(z) \) considered as polynomials in \( z \), is \((iu - t)^3(t^2 + u^2 - 1)^4R(w)\), where

\[
R(w) = 8(t^2 + u^2)(t^2 + u^2 - 1)w^3 + (12t^4 + 24t^2 u^2 + 12u^4 - 4t^2 - 4u^2 + 8t)w^2
\]
\[
+ 2(3t^2 + 3u^2 + 1)(t^2 + u^2 - 1)w + (t^2 + u^2 - 1)^2.
\]

The roots of \( R \), considered as a polynomial in \( w \), are the inverses of the derivative of \( B \) evaluated at the fixed points of \( B \).

Let \( R_1(w) = R(1 + w) \). We have

\[
R_1(w) = 8(t^2 + u^2)(t^2 + u^2 - 1)w^3
\]
\[
+ 4\left( 9(t^2 + u^2)^2 - 7(t^2 + u^2) + 2t \right)w^2 + 2Qw + Q.
\]

We see that the coefficients of \( R_1 \) present one variation of sign. This means that this polynomial has a positive root. Therefore, due to Descartes’ rule, \( R \) has a root larger than one. This means that there is a fixed point where the derivative of \( B \) is in the interval \((0, 1)\).

The following figure shows the regions corresponding to the previous discussion.

The equation of the red curve (a cardioid) is \( 27(t^2 + u^2)^2 - 18(t^2 + u^2) + 8t - 1 = 0 \).
When \((t,u)\) lies between the cardioid and the circle, all the fixed points have modulus 1, when it lies inside, there is a fixed point inside \(\mathbb{D}\).

When the attracting fixed point is on the boundary of the disk, \(|B_n(0)|\) converges exponentially fast towards 1. Therefore, if \((z_j)_{j \geq 0}\) is the sequence of the zeroes (counted according to their multiplicities) of all the iterates of \(B\), one has

\[
\sum_{j \geq 0} (1 - |z_j|) < +\infty.
\]

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