Percolation-like behavior of some optimal coalition formation models

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The ground-state of an infinite-range Potts glass-type model with $\pm J$ bonds and unrestricted number of states is used to investigate coalition formation. As a function of the $q$ probability of $+J$ bonds in the system it is found that the $r$ relative size of the largest cluster (a cluster being the group of elements in the same state) shows a percolation like behavior. By a simple renormalization approach and several optimization methods we investigate the $r(q)$ curves for finite systems sizes. Non-trivial consequences for social percolation problems are discussed.

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I. INTRODUCTION

The Potts glass was originally introduced for studying various non-magnetic random orientational and structural glasses, which do not possess reflection or rotational symmetries. Apart of the specific solid-state and statistical physics applications, the infinite-range (or mean-field) version of the model recently received renewed interest from the viewpoint of coalition formation phenomenon in sociological systems. From this perspective the primary interest is in the ground-state of the infinite-range (or mean-field) $p$-state Potts glass.

The infinite-range $p$-state Potts glass is usually defined by the Hamiltonian:

$$H = -p \sum_{i<j} J_{ij} \delta_{\sigma(i)\sigma(j)}$$  

The $\delta(i)$ Potts states can take the values $0, 1, 2, \ldots, p - 1$. The sum is extended over all $N(N - 1)/2$ pairs in the lattice, $\delta_{\text{m}} = 1$ if $m = n$ and $\delta_{\text{m}} = 0$ otherwise. The $J_{ij}$ bonds are randomly distributed quenched variables with $J_0/N$ mean, and the variance is presumed to scale as $N^{-1}$. The system has a non-trivial frustration and computing the thermodynamic parameters is a complex task. The above model has been extensively studied by many authors through different methods. Within the replica theory a self-consistent description of the low-temperature glassy phase was obtained. For $p > 2$ and low enough temperatures it was found that the infinite-range Potts glass is finally always ferromagnetic. Here we consider a special case of the infinite-range Potts glass, which can be useful in understanding some universalities for coalition formation phenomena in sociological systems.

The main difference between the original Potts glass and the model studied here is that we consider an important class of the $J_{ij}$ bonds, where the variance scales as $N^{-2}$. Also, we consider unrestricted number of $p$ Potts states ($p = N$), and restrict the study on the ground-state ($T = 0$). In the $N \to \infty$ limit an interesting percolation-like transition is then revealed which is studied by different approximations for finite system sizes.

II. THE MODEL

In order to describe the process of aggregation or coalition-formation phenomena in politics, economics or sociological systems we introduce a model similar to the original Potts glass model. In such a system given a set of $N$ actors (in our case the Potts variables) we define an associated distribution of bilateral propensities towards either cooperation or conflict. The actors might be countries which ally into international coalitions, companies that adopt common standards and strategies, parties that make alliances, individuals which form different interest groups, and so on. The propensities will define the $Z_{ij}$ interactions between the actors. The $Z_{ij}$ bond is positive if there is a tendency towards cooperation and negative if it is a conflicting relation between actor $i$ and $j$. For simplicity reasons let us assume first that the $Z_{ij}$ links are symmetric ($Z_{ij} = Z_{ji}$), however later the case without this assumption is also considered. In addition to this, each actor has an associated distribution of bilateral propensities towards either cooperation or conflict. This non-trivial optimization problem can be mathematically formulated in the formalism of a zero-temperature Potts glass type model. To prove this, we define a cost-function, $K$, (a kind of energy of the system) that is increasing with $S_i S_j / |Z_{ij}|$ whenever two conflicting actors ($i$ and $j$) are in the same coalition or two actors which have a tendency towards collaboration.
are in different coalition. The cost-function is zero, when no propensity is in conflict with the formed coalitions. The number of possible coalitions is unrestricted (maximal possible number is \( N \)), and we denote the coalition in which actor \( i \) is by \( \sigma(i) \). The cost function then writes as:

\[
K = -\sum_{i<j} \delta_{\sigma(i)\sigma(j)} Z_{ij} S_i S_j + \frac{1}{2} \sum_{i<j} (Z_{ij} S_i S_j + | Z_{ij} S_i S_j |),
\]

(2)

It is immediate to realize that for a given distribution of the \( Z_{ij} \) interactions and \( S_i \) weight-factors the second term in equation (2) is constant (independent of the formed coalitions). Minimizing the \( K \) cost function is equivalent with finding the ground-state of the Hamiltonian with \( p = N \). Instead of \( S_i S_j Z_{ij} \) we now introduce the \( J_{ij} N \) notation. If \( Z_{ij} \) and \( S_i \) are independent of \( N \) we have that \( < J_{ij} > \) scales as \( N^{-1} \), and we introduce the notation: \( J_0 = N \cdot < J_{ij} > \). We consider now a somehow trivial but practically important and general case, when the variance of \( J_{ij} \) scales as \( N^{-2} \). (As an immediate example for this scaling is the simple case when \( S_i = S_j = 1 \) and \( Z_{ij} = 1 \) with a probability \( q \) and \(-1 \) with a probability \( 1-q \).) For this choice, the \( N \to \infty \) thermodynamic limit becomes simple, since the disorder in the system scales out. The infinite-range Potts glass becomes thus equivalent with a simple mean-field Potts-model, with \( J_0 \) interactions between the elements. While for \( J_0 > 0 \) the system has minimal cost function when all elements are in the same coalition, for \( J_0 < 0 \) in the ground-state each element has to be in a different coalition. As a function of \( J_0 \) a transition is thus expected. This transition resembles the one obtained in percolation or random graph models. Since the temperature has no role in this phenomenon, we call it geometrical phase transition. In the present paper we study this geometrical phase transition for finite \( N \) values and simple \( J_{ij} \) distributions. The finite \( N \) limit is however not as simple as the thermodynamic limit. Frustration effects are important and finding the ground-state is a complex NP hard optimization problem. (It is believed that for large \( N \) the number of steps necessary for an algorithm to find an exact optimum must, in general, grow faster than any polynomial in \( N \).) Several methods were used to investigate finite-size behavior in the expected transition. First a simple renormalization approach was considered. For small systems (up to \( N = 10 \)) an exact enumeration was then used. For larger systems (up to \( N = 60 \)) Monte-Carlo type simulated annealing and the recently proposed extreme optimization was applied.

The order parameter considered by us is the \( r \) relative size of the largest cluster. In the thermodynamic limit \( r \) has the right behavior, for \( J_0 < 0 \) we get \( r = 0 \), and for \( J_0 > 0 \) we obtain \( r = 1 \). More precisely, \( r \) is computed as

\[
r(J_0) = \max_{\sigma(i)} \left\{ \frac{C_x(i, J_0)}{N} \right\} > x,
\]

where \( C_x(i, J_0) \) stands for the number of elements in state \( i \) for an \( x \) realization of the \( J_{ij} \) distribution, when \( < J_{ij} >= J_0/N \). Since the ground-state might be degenerated (i.e. many possible configurations with the same minimal energy might exist) we make an average over all these states, denoted in (3) by the over-line. \( < .... >_x \) refers then for an ensemble average over \( J_{ij} \).

We focus now on the simplest model in which we expect this transition, i.e. when \( J_{ij} \) is a two valued quenched random variable, \( J_{ij} = 1/N \) with probability \( q \) and \(-1/N \) with probability \( 1-q \). We refer then for an ensemble average over \( J_{ij} \).

In the view of our previous arguments we expect that in the \( N \to \infty \) limit the \( r(q) \) curves will indicate a geometrical phase-transition at \( q = 1/2 \).

### III. RENORMALIZATION APPROACH

Our elementary renormalization approach estimates in a mean-field manner the new relative size of the largest state, whenever the system size is doubled. We start from a system composed by only two elements (step 1). In the ground-state, the probability to have these two elements in the same Potts state is \( q_1 = q \). The relative size of the largest cluster is then \( r_1 = q_1 + (1 - q_1)/2 \), since the largest cluster will be the total system with probability \( q_1 \), and the original half with probability \( 1 - q_1 \). In step 2 we now double the system size by linking through all possible \( J_{ij} \) connections two previous configurations (A and B) with maximal relative size \( r_1 \), each of them having two elements. Then, we reduce the four \( J_{ij} \) connections between the elements of A and B to a single one, and transform the system into a configuration similar to the one from step 1. This procedure is summarized in Fig. 1.
The new link will be positive (+1) with probability 

\[ q_2 = q_1^4 + 4q_1^3(1 - q_1) + 3q_1^2(1 - q_1)^2, \]

and the new relative size of the largest state is 

\[ r_2 = q_2 + (1 - q_2)r_1/2. \]

The factor 3 from the last term in the expression of \( q_2 \) results by considering the new link positive with 1/2 probability, whenever there are two positive and two negative links (6 possible realizations in total). This doubling procedure is then recursively repeated, leading to the simple renormalization equations:

\[ q_{k+1} = q_k^4 + 4q_k^3(1 - q_k) + 3q_k^2(1 - q_k)^2, \]  \hspace{1cm} (6)

\[ r_{k+1} = q_{k+1} + (1 - q_{k+1})r_k/2. \]  \hspace{1cm} (7)

The size of the system after \( k \) steps is \( N = 2^k \).

On the \([0, 1]\) interval, iteration (6) has two stable fixpoints: 0 and 1. There is also an unstable fixpoint \( q = 1/2 \). Starting the iteration from \( q \in [0, 1/2] \) we get \( \lim_{k \to \infty} q_k = 0 \) and \( \lim_{k \to \infty} r_k = 0 \). Choosing \( q \in (1/2, 1] \) we get \( \lim_{k \to \infty} q_k = 1 \) and \( \lim_{k \to \infty} r_k = 1 \). These results suggest that in an infinite system we have two distinct phases separated by \( q_0 = 1/2 \), as expected. In phase I the \( r \) order parameter converges to 0, and in phase II \( r \) converges to 1. We get thus the expected percolation-like transition as a function of \( q \).

Using equations (6,7) we can also easily plot the \( r(q) \) curves for different system sizes. Results in this sense are presented in Fig. 2. These results support our previous arguments.

### IV. EXACT ENUMERATION

For small system sizes (\( N \leq 10 \)) exact enumeration is possible. This means that one can computationally map the whole phase-space (all \( \sigma(i) \) realizations) for a generated \( J_{ij} \) configuration and determine the minimum energy states. Moreover, for \( N \leq 7 \) it was also possible to map all \( J_{ij} \) configurations as well, our results up to \( N = 7 \) are thus exact. In the \( 7 < N \leq 10 \) interval, although the minimum energy states are exactly found, due to greatly increased computational time and memory needed it was possible to generate only a reasonable ensemble average for \( J_{ij} \) (5000 configurations). Results are plotted on Fig. 3.

We performed this exact enumeration with two purposes. First, we checked the trends of the \( r(q) \) curves as a function of increasing system size. Secondly, these results offer a good "standard" for our less rigorous Monte-Carlo type optimization methods, used for larger system sizes. As the results in Fig. 3 shows the \( r(q) \) curves have a similar trend as those suggested by our renormalization approach, i.e. as the system size increases we find increasing slopes for \( r(q) \) around a nontrivial \( q \) value.
FIG. 4. Average degeneration level of the ground-state (a.) and difference between the maximal and minimal \( r \) value for different coalition structures in the ground state (b.), both as a function of the \( q \) probability of + interactions between the elements.

By exact enumeration we have also studied the degeneration level of the ground state. For a given \( J_{ij} \) bond-configuration, many different coalition structure can have the same ground-state energy. We can define thus a \( w \) degeneration level for each ground-state, and for a fixed \( q \) value we can calculate the \( \langle w \rangle \) ensemble average over all bond configurations. Different coalition structures in the ground-state might be characterized by different \( r \) values, as well. For a given bond configuration the difference between the maximal \( r \) value (\( r_{\text{max}} \)) and the minimal one (\( r_{\text{min}} \)) will characterize the maximal possible deviation in the order parameter. An ensemble average over this quantity (\( \langle r_{\text{max}} - r_{\text{min}} \rangle \)) will give information about the differences which are possible to get in the \( r \) order parameter, while choosing another equally optimal coalition structure. The values of \( \langle w \rangle \) and \( \langle r_{\text{max}} - r_{\text{min}} \rangle \) have been calculated as a function of the \( q \) parameter. The obtained results are presented in Fig. 4 according to their fitness. Considering the \( P(k) \sim k^{-\tau} \) probability distribution over the rank, \( k \), we then select an element for which the state will be changed. For this first step we found the optimal value of \( \tau = 0.25 \). In the second step we decide the new state of the chosen element by a similar procedure. For this second step the optimal value of \( \tau \) proved to be 4.

Simulated annealing and extreme optimization gave identical and practically indistinguishable results. Therefore in Fig. 5 we plot only the simulated annealing results. The shape of the

\[
\Delta r(q) = \sqrt{\langle r^2(q) \rangle_x - \langle r(q) \rangle_x^2}
\]

standard deviation was also computed (Fig. 4b), suggesting a non-trivial peak. In Fig. 5 the curves for \( N = 10, 20, 30 \) and 40 were obtained with an ensemble average of 5000 realizations, and the results for \( N = 60 \) with a statistics of 1000 realizations. For \( N = 10 \) the Monte-Carlo type results are in perfect agreement with the ones from exact enumerations (Fig. 5a), giving confidence in the used stochastic simulation methods.

V. MONTE CARLO TYPE OPTIMIZATION

Monte-Carlo type optimizations were used for computing the ground-state of larger systems. We considered both the classical simulated annealing [17] and the recently proposed extreme optimization method [18]. Both approaches are rather time-consuming and the necessary computational time increases sharply with system size. Our computational resources allowed to study systems with sizes up to \( N = 60 \).

Simulated annealing has been implemented in the standard fashion [17]. For the extreme optimization method we generalized the originally proposed method [18] by considering a two-step algorithm. In the first step we performed the usual optimization after the energies of the elements. As suggested in [18] we assigned a given fitness to each Potts element and ranked all the variables according to their fitness. Considering the \( P(k) \sim k^{-\tau} \) probability distribution over the rank, \( k \), we then select an element for which the state will be changed. For this first step we found the optimal value of \( \tau = 0.25 \). In the second step we decide the new state of the chosen element by a similar procedure. For this second step the optimal value of \( \tau \) proved to be 4.

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Our estimates suggest that extreme optimization was
faster by a factor of at least two, in comparison with simulated annealing. However, we found that extreme optimization is also strongly affected by the increasing system size, and for $N > 60$ we couldn’t get any good statistics in reasonable computational time.

The results plotted in Fig. 5 support the expected geometrical phase transition in the system. As the system size increases the $r(q)$ curves show a more and more sharper trend in the vicinity of $q = 1/2$. Also, the $\Delta r(q)$ standard deviation exhibits a non-trivial peak, which gets sharper and closer to $q = 1/2$ as the system size increases. By extrapolating the obtained results as a function of $N$ for $q = 0.1$, $q = 0.3$ and $q = 0.7$, one can show that $r \to 0$ as a power-law for $q = 0.1$ and $q = 0.3$, and $r \to 1$ for $q = 0.7$ (Fig. 6). This proves the existence of the presumed phases.

Dropping the symmetry requirement for $J_{ij}$ introduces an extra frustration in the system. While for symmetric $J_{ij}$ only subsets with more than two elements can be frustrated, in the asymmetric case subsets of two elements can become already frustrated. It is interesting to note however, that the nature of the observed transition is not affected by dropping this symmetry requirement and again, the same geometrical phase transition should appear in $q_c = 1/2$. Up to $N = 10$ we computed the $r(q)$ curves by exact enumerations and for $N = 20$ and 30 we used the extreme optimization method. No important deviations from the symmetric case were found.

VI. A MORE GENERAL CASE

Next, we briefly present our results for a more general case, where the $S_i$ factors are also randomly distributed.

![FIG. 6.](image)

FIG. 6. Finite-size scaling (log-log plots) for (a.) $q = 0.1$ and $q = 0.3$ (b.) $q = 0.7$. The best-fit lines for (a.) have slopes of $-0.714$ and $-0.4928$, respectively.

VII. DISCUSSIONS

The observed geometrical phase transition is interesting also from the viewpoint of the much discussed social percolation [19], where the emergence of a giant cluster is observed in many social phenomena. Our simple model suggests that large sociological systems can show tendencies to percolation-like behavior due to coalition formation phenomena. If a globally coupled large system has more propensities pointing towards collaboration than conflict, usually a single coalition satisfies optimally the apparently conflicting interactions. Contrary, when there are more conflicting propensities than collaborative ones, the society will fragment in large number of coalitions, and each element will isolate itself from the others. As expected, this percolation-like behavior is rather smooth for small systems sizes. The observed percolation-like behavior is also quite stable relative to the choice of the $Z_{ij}$ propensities and $S_i$ weight-factors.
It is also important to mention that according to the considered model the most unpredictable societies are the "equilibrated" ones, where the number of positive and negative links are roughly the same. From our numerical results one can see that in this case $\Delta r$ is big, and the value of $r$ is changing strongly with small variations of $q$. First, this means that the system is very sensitive to the explicit realization of the $J_{ij}$ values. Secondly, as seen in Fig. 4 in this region many equilibrium configurations with different $r$ values might co-exist, all of them having the same minimal $K$ value (degeneracy of the ground-state might be high). Third, a small difference in the measured $q$ value can result in large differences for the expected $r$ values. In these "equilibrated" societies statistical methods are useless for predicting the optimal clusterization. Specific analyses of the concrete situation is thus the only acceptable prediction method.

The fact that in the ground-state many equally-optimum configurations with quite different maximal cluster sizes are possible might also lead to interesting implications. It might well be possible the existence of some "mixed" states, where the system behavior can be described not from a clear coalition structure, but rather from a superposition of many coalition structures. The model considered by us is of course a very simple one, capturing only a few parameters that are important in understanding social coalition formation. In our model we have also neglected the dynamics of the system, and presumed that the system will clusterize in one of the optimal configurations. The system is however frustrated, and many configurations with local minimum exist. During its dynamics, the system might get trapped in a local minimum, and the formed coalitions might be the one corresponding to this case, rather than the global optimum case. The model considered by us and our results are usable thus only for statistically predicting the optimal clusterization, and not for understanding the coalitions that are formed in reality.

In conclusion, in the present study we presented evidences for a geometrical phase transition in the ground-state of an infinite-range Potts glass where the standard deviation of the bonds scale as $N^{-2}$. For finite system sizes three different methods were used to approach this NP hard optimization problem, all of them supporting the percolation-like behavior of the largest cluster size as a function of the positive links in the system. The model considered by us might be useful in understanding some social percolation phenomena in large sociological systems.

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