INARIANT CONNECTIONS WITH SKEW-TORSION AND ∇-EINSTEIN NATURALLY REDUCTIVE MANIFOLDS

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Abstract. For a compact connected Lie group $G$ we study the class of bi-invariant affine connections whose geodesics through $e \in G$ are the 1-parameter subgroups. We show that the bi-invariant affine connections which induce derivations on the corresponding Lie algebra $\mathfrak{g}$ coincide with the bi-invariant metric connections. In the sequel, we focus on the geometry of a naturally reductive space $(M = G/K, g)$ endowed with a family of $G$-invariant connections $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ whose torsion is a multiple of the torsion of the canonical connection $\nabla^c$, i.e. $T^\alpha = \alpha \cdot T^c$. For the spheres $S^0$ and $S^7$ we prove that the space of $G_2$ (resp. Spin(7))-invariant affine or metric connections consists of the family $\nabla^0$. In the compact case we examine the flatness condition $R^0 \equiv 0$ and we state a refinement of the classical Cartan-Schouten theorem. The “constancy” of the induced Ricci tensor $\text{Ric}^c$ is also described. We prove that any compact isotropy irreducible naturally reductive Riemannian manifold, which is not a symmetric space of Type I, carries at least two $\nabla^c$-Einstein structures with skew-torsion, namely those which occur for $\alpha = \pm 1$. A generalization of this result is given also for a class of compact normal homogeneous spaces $M = G/K$ with two isotropy summands. We introduce a new 2-parameter family of $G$-invariant connections on $M = G/K$, namely $\nabla^{s,t}$ with $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$; for the Killing metric $t = 1/2$ skew-torsion appears and we examine the $\nabla^{s,1/2}$-Einstein condition. We show that $M$ is normal Einstein, if and only if, $M$ is a $\nabla^{s,1/2}$-Einstein manifold with skew-torsion for one of the values $s = 0, 2$. In this way we provide a series of new examples of manifolds admitting these structures.

1. Introduction

Given a Riemannian manifold $(M^n, g)$, metric connections whose torsion is a 3-form are geometrically the connections which have the same geodesics as the Levi-Civita connection. These connections play a crucial role in the theory of non-integrable geometries and they admit physical applications in type II string theory [FG1, A2]. A very remarkable example is the so-called canonical connection (for some $\alpha$, cubic Dirac operator the so-called $G$-invariant affine or metric connections consists of the family $\nabla^0$. In the compact case we examine the flatness condition $R^0 \equiv 0$ and we state a refinement of the classical Cartan-Schouten theorem. The “constancy” of the induced Ricci tensor $\text{Ric}^c$ is also described. We prove that any compact isotropy irreducible naturally reductive Riemannian manifold, which is not a symmetric space of Type I, carries at least two $\nabla^c$-Einstein structures with skew-torsion, namely those which occur for $\alpha = \pm 1$. A generalization of this result is given also for a class of compact normal homogeneous spaces $M = G/K$ with two isotropy summands. We introduce a new 2-parameter family of $G$-invariant connections on $M = G/K$, namely $\nabla^{s,t}$ with $s \in \mathbb{R}$ and $t \in \mathbb{R}_+$; for the Killing metric $t = 1/2$ skew-torsion appears and we examine the $\nabla^{s,1/2}$-Einstein condition. We show that $M$ is normal Einstein, if and only if, $M$ is a $\nabla^{s,1/2}$-Einstein manifold with skew-torsion for one of the values $s = 0, 2$. In this way we provide a series of new examples of manifolds admitting these structures.

Theorem 1.1. Given a compact connected Lie group $G$ with a bi-invariant metric $\rho$, the class of bi-invariant affine connections which induce derivations $\Lambda : \mathfrak{g} \to \text{Der}(\mathfrak{g})$ on the corresponding Lie algebra $\mathfrak{g}$ coincides with the class of bi-invariant metric connections on $G$ with respect to $\rho$.

Next we treat naturally reductive Riemannian manifolds $(M = G/K, g)$ endowed with a $G$-invariant metric connection with torsion that is a multiple of the torsion $T^c$ of the canonical connection $\nabla^c$, say $T^\alpha = \alpha \cdot T^c$ for some $\alpha \in \mathbb{R}$. Connections of this kind are of particular interest, for example in the spin case they induce the so-called cubic Dirac operator [A1]. From now on let us denote this family by $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$. Obviously, it joins the canonical connection ($\alpha = 1$) with the Riemannian connection ($\alpha = 0$) and the associated 3-tensor
$T^α(X,Y,Z) = -α([X,Y]_m,Z)$ is a 3-form on $m = T_α M$. Hence $∇^α$ is a $G$-invariant connection with skew-torsion for any $α \in \mathbb{R}\setminus\{0\}$. For an irreducible symmetric space $(M = G/K,g)$ of Type I one can show that the space of $G$-invariant metric connections consists only of the canonical connection $∇^c ≡ ∇^g$. This observation follows by the classification of $G$-invariant affine connections on $M = G/K$ [2 Thm. 2.1] and an important remark about their metric property [AFH Rem. 3.2]. In fact, it fails if one drops the “metric” condition, i.e., there exist symmetric spaces of Type I carrying invariant affine connections different than the canonical connection [L2 Thm. 2.1]. Here we primarily focus on symmetric spaces which can be (re)presented as cosets of distinct Lie groups, e.g. the spheres $S^6$ and $S^7$. We show that

**Theorem 1.2.** The space of $G_2$-invariant affine (or metric) connections on the sphere $S^6 = G_2 / SU(3)$ is 1-dimensional; it consists of the family $\{∇^α : α \in \mathbb{R}\}$ described above. The same is true for the space of $Spin(7)$-invariant affine (or metric) connections on the 7-sphere $S^7 = Spin(7)/G_2$.

Notice that the unique $SO(7)$- (resp. $SO(8)$)-invariant affine (or metric) connection on the symmetric space $S^6 = SO(7)/SO(6)$ (resp. $S^7 = SO(8)/SO(7)$) is the canonical connection [L2]. The 1-dimensional family in Theorem [L2] occurs since the cosets $G_2 / SU(3)$ and $Spin(7)/G_2$, although diffeomorphic to a symmetric space (namely a sphere), do not provide us with symmetric pairs (see [Bes 7.107 Tab. 6]).

In Section 4 we focus on the geometric properties of a naturally reductive space $(M = G/K,g)$ with respect to the family $∇^α$. We examine the flatness condition, i.e. the equation $R^α = 0$, where $R^α$ stands for the associated curvature tensor; in the spirit of [D'AN Prop. 3.7 (d)] we explain why this equation implies the $∇^α$-parallelism of the torsion $T^α$ (see Proposition 4.2). Then we prove that

**Theorem 1.3.** Let $(M^n = G/K,g)$ be a compact naturally reductive Riemannian manifold (irreducible as Riemannian manifold) admitting a transitive and effective action of a compact Lie group $G$. Assume that $M = G/K$ is flat with respect to a family of $G$-invariant metric connections whose torsion $T^α$ is such that $0 ≠ T^α = α \cdot T^c$, for any $α \in \mathbb{R}\setminus\{0\}$. Then, $M ≅ G$ is isometric to a compact Lie group endowed with a bi-invariant metric and one of the $±1$-connections, i.e. $α = ±1$. In particular, $G$ is simple.

This result is a refinement of the classical theorem of Cartan-Schouten which states that the unique Riemannian manifolds carrying flat metric connections with skew-torsion are the compact (simple) Lie groups and the 7-sphere $S^7 = Spin(7)/G_2$ [CS2 [D'AN]. Nowadays, this theorem has been proved in terms of holonomy algebras associated to metric connections with skew-torsion (see [N Thm. 1.4], [AF2 Thm. 2.2] and [R1 Thm. 4.5]). Our approach however avoids this machinery and our arguments rely on the property $T^α = α \cdot T^c$ and the special homogeneous structure that $M = G/K$ carries (Kostant’s theorem). The reason that $Spin(7)/G_2$ does not appear in Theorem [L2] is due to our invariant-torsion scenario which does not allow non-invariant connections; in contrast the connection which induces an absolute parallelism in $S^7$, although metric and with skew-torsion, is not invariant (see [AF2] and Remark [R3]).

A natural subsequent step after the examination of the flatness condition for the family $\{∇^α : α \in \mathbb{R}\}$ is the investigation of the “constancy” of the corresponding Ricci tensor $Ric^α$. This takes place in Section 5 where we describe $∇^α$-Einstein structures with skew-torsion on compact naturally reductive spaces $(M = G/K,g)$. Given a Riemannian manifold $(M^n,g)$ $(n ≥ 3)$ equipped with a metric connection $∇$ with non-trivial skew-torsion $T$, a $∇$-Einstein structure with skew-torsion, or in short a $∇$-Einstein structure, is a mathematical generalization of the Riemannian Einstein condition given by a tuple $(M^n,g,∇,T)$ as above, satisfying the equation $Ric = (Scal \cdot g)/n$. Here, $Ric_S$ denotes the symmetric part of the Ricci tensor associated to $∇$ and Scal is the corresponding scalar curvature. Solutions of this equation naturally appear in the theory of non-integrable geometries. Their variational approach and their interaction with the so-called characteristic connection (parallel skew-torsion) has been recently described in [AFc]. In this note we explain the $∇^α$-Einstein condition on compact naturally reductive Riemannian manifolds (or normal homogeneous spaces) in terms of Casimir elements. We begin with the isotropy irreducible case, where one has to exclude the symmetric spaces of Type I (because $∇^α ≡ ∇^c ≡ ∇^g$, see Proposition 1.2). We prove the following result.

**Theorem 1.4.** Let $(M = G/K,g)$ be a compact isotropy irreducible naturally reductive manifold $(M^n = G/K,g)$, which is not a symmetric space of Type I. Then, $(M = G/K,g)$ becomes a $∇^α$-Einstein manifold with skew-torsion with respect to the family $\{∇^α : α \in \mathbb{R}\setminus\{0\}\}$, at least for the values $α = ±1$.

Therefore, any compact isotropy irreducible non-symmetric normal homogeneous Riemannian manifold $(M^n = G/K,g)$ is a $∇^1$-Einstein manifold. Notice that the connections $∇^c = ∇^1$ and $∇^{-1}$ have some common geometric properties, e.g. their Ricci tensors are identical. However they have a crucial difference: $∇^1T^1 = 0 ≠ ∇^{-1}T^{-1}$. This is a remarkable result: until now most of the known $∇$-Einstein structures (e.g. nearly Kähler manifolds, Sasakian manifolds, nearly parallel $G_2$-manifolds and other) are related to a metric
connection with parallel skew-torsion (e.g. the characteristic connection). Here we prove that a large class of compact homogeneous spaces (see [Woo1, Bes]), naturally admits $\nabla$-Einstein structures with both parallel and non-parallel skew-torsion $T \neq 0$.

In the final section we extend this result to the non-isotropy irreducible case. We examine compact connected homogeneous Riemannian manifolds $M = G/K$ of a compact connected semi-simple Lie group $G$, whose isotropy representation decomposes into two (non-trivial) irreducible and inequivalent $K$-submodules satisfying

\[(1.1) \ g = \mathfrak{t} \oplus m_1 \oplus m_2, \quad [\mathfrak{t}, m_1] \subset m_1, \quad [\mathfrak{t}, m_2] \subset m_2, \quad [m_1, m_1] \subset \mathfrak{t} \oplus m_2, \quad [m_1, m_2] \subset m_1, \quad [m_2, m_2] \subset \mathfrak{t}.\]

Such cosets occur often as fibrations over symmetric spaces (see Remark 4.2). Well-known examples are: connected semi-simple Lie groups, flag manifolds with two isotropy summands, odd-dimensional spheres, 3- and 4-symmetric spaces (see [K1, K2, AC1, AC2] and the references therein).

As a first step, we characterize the invariant metric connections of $(M = G/K, m_1 \oplus m_2, g_1)$ which have skew-torsion; they exist only for the Killing metric $t = 1/2$, under the further assumption that the associated Nomizu map $\Lambda_m : m \to so(m)$ satisfies $\Lambda_m(X)X = 0$, for any $X \in m$ (see Theorem 7.2). Notice that this is not such a surprising result; we already know several cosets $M = G/K$ satisfying the characteristic connection. Such an example is the 6-dimensional homogeneous nearly Kähler manifold $CP^3 = SO(5)/U(2) = Sp(2)/Sp(1) \times U(1)$ endowed with the so-called Chern connection, which coincides with the canonical connection $\nabla^c$ and thus trivially satisfies the condition $\Lambda(X)X = 0$ (see [A2, Lem. 2.2] and for the uniqueness of the characteristic connection see [AFH, Thm. 2.1]).

If we relax the irreducible assumption of the isotropy components, another well-known example is the full flag manifold $F_{1,2} = SU(3)/T_{\text{max}}$ (see [BF 6]). Based on this characterization, we introduce a new 2-parameter family of $G$-invariant metric connections on $M = G/K$, given by

\[
\{ \nabla^{s,t}_X Y = \nabla^X Y + s \Lambda^t(X)Y = \nabla^X Y + s \Lambda^{g_1}(X)Y : s \in \mathbb{R}, t \in \mathbb{R}_+ \},
\]

where $\Lambda^{g_1} \equiv \Lambda^t$ denotes the Nomizu map of the Levi-Civita connection $\nabla^t(= \nabla^{g_1})$ on $M = G/K$. The family $\nabla^{s,t}$ joins the connections $\nabla^t$ and $\nabla^c$ and for the Killing metric it gives rise to 1-parameter family of $G$-invariant metric connections with skew-torsion, namely $\{ \nabla^{s,t} : s \in \mathbb{R} \}$. We describe the basic geometric features of $(M = G/K, g_1)$ with respect to $\nabla^{s,t}$ and next we examine the $\nabla^{s,t}$-Einstein condition in terms of the Casimir eigenvalues $\text{Cas}_1$ and $\text{Cas}_2$. We obtain the following correspondence.

**Theorem 1.5.** Let $(M = G/K, m_1 \oplus m_2, g_{1/2})$ be a compact homogeneous Riemannian manifold with two isotropy summands satisfying (1.1). Then, $M = G/K$ is a $\nabla^t$-Einstein manifold with skew-torsion $0 \neq T^s \in \Lambda^3(m)$ for the values $s = 0$ or $s = 2$, if and only if, the Killing metric $g_B \equiv g_{1/2}$ is a $G$-invariant Einstein metric, i.e. $\text{Cas}_1 = \text{Cas}_2$. In particular, if $g_B$ is an Einstein metric then the Ricci tensors of $\nabla^0$ and $\nabla^2$ are identical.

In this way, we verify the existence of a new $\nabla$-Einstein structure with skew-torsion on $CP^3$.

**Theorem 1.6.** The homogeneous nearly Kähler manifold $(M = G/K = CP^3, g_{1/2})$ admits exactly two $\nabla^{s,t}$-Einstein structures with skew-torsion. The first one is the well-known $\nabla^{0,t} \equiv \nabla^c$-Einstein structure with parallel skew-torsion (Gray’s theorem, see [G3, A2, AFc]). The second occurs for $s = 2$ and this is such that $\nabla^2 \oplus T^2 \neq 0$.

Aside this example, we use results of author’s Phd thesis [AC2, AC1] to construct new classes of homogeneous spaces carrying $\nabla^t$-Einstein structures with skew-torsion, for the values $s = 0, 2$. These are flag manifolds with two isotropy summands and they are the first known examples of non-isotropy irreducible homogeneous Riemannian manifolds, admitting $\nabla$-Einstein structures with skew-torsion.

2. **Homogeneous Riemannian manifolds and invariant connections**

Consider a connected homogeneous Riemannian manifold $(M = G/K, g)$, where $G \subset I(M)$ is a closed subgroup of the isometry group and $K$ is the isotropy subgroup at a fixed point $o = eK \in M$. Assume for simplicity that the transitive $G$-action is effective and that $K$ is connected. Because $K$ is compact one can always fix an $\text{Ad}(K)$-invariant complement $\mathfrak{m}$ of $\mathfrak{t}$ in $\mathfrak{g}$ such that $g = \mathfrak{t} \oplus \mathfrak{m}$ and $[\mathfrak{t}, \mathfrak{m}] \subset \mathfrak{m}$. Since the natural projection $\pi : G \to M = G/K$ is a submersion we identify $\ker(d\pi)_e = \mathfrak{t}$ and we obtain a canonical isomorphism $\mathfrak{m} \ni X \mapsto X_o := \frac{d}{dt} \exp(tX)|_{t=0} \in T_oM$ between $\mathfrak{m}$ and the tangent space $T_oM$. Hence we identify the isotropy representation $\chi : K \to SO(\mathfrak{m}) \subset \text{Aut}(\mathfrak{m})$ of $K$ with the restriction of the adjoint representation $\text{Ad}|_{\mathfrak{m}}$ on $\mathfrak{m}$. As usual, we pull back the Riemannian metric $g := \langle \cdot , \cdot \rangle_o$ on $T_oM$ to an $\text{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$. Let $T(G/K) = G \times K$ $\mathfrak{m}$ (resp. $F(G/K) = G \times K \text{GL}(\mathfrak{m})$).
be the tangent (resp. frame) bundle of \( M = G/K \). By a theorem of H. C. Wang \([W]\) it is known that a linear \( G \)-invariant connection \( \nabla : \Gamma(TM) \to \Gamma(T^*M \otimes TM) \) is described by a \( \mathbb{R} \)-linear map \( \Lambda_m : m \to \mathfrak{g}(m) \) which is equivariant under the isotropy representation, i.e. \( \Lambda_m(\text{Ad}(k)X) = \text{Ad}(k)\Lambda_m(X) \text{Ad}(k)^{-1} \) for all \( X \in m \) and \( k \in K \). The identification \( \Lambda_m(X)Y = \eta(Y, X) \) shows that the space of \( G \)-invariant affine connections can be equivalently described as the set of all \( \text{Ad}(K) \)-equivariant bilinear maps \( \eta : m \times m \to m \), i.e. \( \eta(\text{Ad}(k)X, \text{Ad}(k)Y) = \text{Ad}(k)\eta(X, Y) \) for any \( X, Y \in m \) and \( k \in K \). Hence one can establish the identification (see \([CS, L2]\))

\[
\mathcal{A}ff_G(F(G/K)) \cong \text{Hom}_K^B(m \otimes m, m),
\]

where in general \( \mathcal{A}ff_G(P) \) denotes the space of \( G \)-invariant affine connections on a homogeneous principal bundle \( P \to G/K \) over \( M = G/K \) and \( \text{Hom}_K^B(m \otimes m, m) \) is the space of \( K \)-intertwining maps \( m \otimes m \to m \). The linear map \( \Lambda_m \) is commonly referred to us as the Nomizu map \( \text{Nomizu map} \) (for details see \([AVL, KN]\)) and it nicely describes the properties of \( \nabla \). For example, \( \nabla \) is metric, i.e. \( \Lambda_m(X) \) lies in \( \mathfrak{so}(m) \) for any \( X \in m \) if and only if \( \langle \Lambda_m(X)Y, Z \rangle + \langle Y, \Lambda_m(X)Z \rangle = 0 \) for any \( X, Y, Z \in m \). Furthermore, its torsion and curvature are given by

\[
\begin{align*}
T(X, Y)_o & = \Lambda_m(X)Y - \Lambda_m(Y)X - [X, Y]_m \\
R(X, Y)_o & = [\Lambda_m(X), \Lambda_m(Y)] - \Lambda_m([X, Y]_m) - \text{ad}([X, Y]_t)
\end{align*}
\]

Viewing the torsion as a 3-tensor \( T(X, Y, Z) := (T(X, Y), Z) \) we will call \( T \) the torsion form if and only if it is skew-symmetric in \( Y \) and \( Z \) (and hence totally skew-symmetric).

Recall now that the Nomizu map \( \Lambda_m : m \to \mathfrak{so}(m) \) associated to the so-called canonical connection \( \nabla^c \) on \( M = G/K \) is the zero map \( \Lambda_m \equiv 0 \), i.e. \( \Lambda_m(X) = 0 \), for any \( X \in m \) \([AVL, KN]\). The canonical connection is induced by the principal \( K \)-bundle \( G \to G/K \) and depends on the choice of \( m \). It is well-known that it has parallel torsion \( T^c(X, Y) = -[X, Y]_m \) and curvature \( R^c(X, Y) = -\text{ad}([X, Y]_t) \) (Ambrose-Singer theorem), in particular any \( G \)-invariant tensor field on \( M = G/K \) is \( \nabla^c \)-parallel.

**Definition 2.1.** A homogeneous Riemannian manifold \( (M = G/K, g) \) is called naturally reductive if and only if the torsion of the canonical connection \( \nabla^c \) is a 3-form on \( m \), i.e. \( T^c \in \Lambda^3(m) \).

The notion of natural reductivity is equivalent to the geometric property that for each vector \( X \in m \) the orbit \( \gamma(t) := \exp(tX)g \) is a geodesic on \( M \), which means the Riemannian geodesics coincide with the \( \nabla^c \)-geodesics. Riemannian symmetric spaces and isotropy irreducible homogeneous Riemannian manifolds are the most typical examples of naturally reductive spaces, but there are much more (see for example \([A2]\)). An important result of B. Kostant \([Ko]\) states that if \( (M = G/K, g) \) is naturally reductive with respect to an effective action of \( G \), then \( \mathfrak{g} := m + [m, m] \) is an ideal of \( \mathfrak{g} \) and the corresponding Lie group \( \tilde{G} \subset G \) acts transitively on \( M \). Moreover, there exists a unique \( \text{Ad}(\tilde{G}) \)-invariant symmetric non-degenerate bilinear form \( Q \) on \( \mathfrak{g} \), not necessarily positive definite, such that \( Q(t\mathfrak{g}, \mathfrak{m}) = 0 \) and \( Q_{\mathfrak{m}} = \langle \cdot, \cdot \rangle \). Conversely, if \( G \) is connected, then any \( \text{Ad}(G) \)-invariant symmetric and non-degenerate bilinear form \( Q \) on \( \mathfrak{g} \), which is non-degenerate on \( \mathfrak{t} \) and positive definite on \( m = \mathfrak{t}^\perp \), induces a naturally reductive metric on \( M \) given by \( g_Q = Q_{\mathfrak{m}} \). In this case one has \( \tilde{G} = G \) and \( \mathfrak{g} = \mathfrak{g} = m + [m, m] \); hence the commutator \([m, m] \) generates the isotropy subalgebra \( \mathfrak{t} \) as a vector space, i.e. \( \mathfrak{t} = \text{span}\{[X, Y]_t : X, Y \in m\} \). A special class of naturally reductive manifolds \( M = G/K \) consists of the so-called normal homogeneous Riemannian manifolds; here there is an \( \text{Ad}(G) \)-invariant inner product \( Q \) on \( \mathfrak{g} \) such that \( Q(\mathfrak{t}, \mathfrak{m}) = 0 \), i.e. \( \mathfrak{t} = \mathfrak{t}^\perp \) and \( Q_{\mathfrak{m}} = \langle \cdot, \cdot \rangle \). Thus, a normal metric is defined by a positive definite bilinear form \( Q \). If \( Q = B \), where \( B \) denotes the negative of the Killing form of \( \mathfrak{g} \), then the normal metric is the so-called Killing metric; this is the case if the Lie group \( G \) is compact and semi-simple.

**Lemma 2.2.** \([A1]\) A \( G \)-invariant metric connection \( \nabla \) on a naturally reductive Riemannian manifold \((M = G/K, g)\) has (totally) skew-symmetric torsion \( T \in \Lambda^3(m) \) if and only if the associated Nomizu map \( \Lambda_m : m \to \mathfrak{so}(m) \) satisfies \( \Lambda_m(Z)Z = 0 \), for any \( Z \in m \).

3. **Bi-invariant affine connections on Lie groups and derivations**

3.1. **Bi-invariant connections.** A compact connected Lie group \( M = G \) with a bi-invariant metric \( \rho \) can be viewed as a symmetric space of the form \((G \times G)/\Delta G\). The Cartan decomposition is given by \( \mathfrak{g} \oplus \mathfrak{g} = \Delta \mathfrak{g} \oplus \mathfrak{p} \), where both \( \Delta \mathfrak{g} := \{(X, X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \) and \( \mathfrak{p} := \{(X, -X) \in \mathfrak{g} \oplus \mathfrak{g} : X \in \mathfrak{g}\} \) are isomorphic to \( \mathfrak{g} \), as \( G \)-modules. The isotropy representation is the adjoint representation of \( G \), i.e. \( \chi(g)(X, -X) := (\text{Ad}(g)X, -\text{Ad}(g)X) \); so \( G \) is isotropy irreducible if and only if \( G \) is simple. In this note we are interested in bi-invariant connections on \( G \), i.e. \((G \times G)\)-invariant affine connections. Such a connection, say \( \nabla^\rho \), is completely described by a bilinear map \( \eta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \) such that \( \eta(\text{Ad}(g)X, \text{Ad}(g)Y) = \text{Ad}(g)\eta(X, Y) \), for any \( g \in G \) and \( X, Y \in \mathfrak{g} \) \([L]\). The associated Nomizu map \( \Lambda^\rho : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is given by \( \Lambda^\rho(X)Y := \eta(X, Y) \) and the equivariant condition is expressed by \( \Lambda^\rho(\text{Ad}(g)X) = \text{Ad}(g)\Lambda^\rho(X) \text{Ad}(g)^{-1} \) with \( g \in G \) and \( X \in \mathfrak{g} \).
For a (compact) simple Lie group $G$ there exists a 1-dimensional family of canonical connections (see for example [OR] Remark 6.1) or [AFH, p. 18]). This family can be viewed as a line in the space of bi-invariant affine connections of $G$ which joins the Levi-Civita connection with the well-known $\pm 1$-connections of Cartan-Schouten. With the aim to set up notation, let us present a short proof of this fact.

**Theorem 3.1.** On a compact simple Lie group $G \cong (G \times G)/\Delta G$ with a bi-invariant metric $\rho$ there exists a 1-dimensional family of bi-invariant canonical connections, which is given by $\nabla^\alpha_X Y = \eta^\alpha(X,Y) = \frac{1-\alpha^2}{2}[X,Y]/4$ (\(\alpha \in \mathbb{R}\)) (up to scalar and sign). The associated curvature has the form $R^\alpha(X,Y)Z = (1-\alpha^2)[Z,[X,Y]]/4$ for any $X,Y,Z \in \mathfrak{g}$; thus for $\alpha = \pm 1$ the Lie group $(G,\rho)$ endowed with one of the connections $\nabla^{\pm 1}$ becomes flat, i.e. $R^{\pm 1} \equiv 0$. Moreover, the associated torsion $T^\alpha$ is $\nabla^\alpha$-parallel for all $\alpha \in \mathbb{R}$ by the Jacobi identity.

**Proof.** Consider the decomposition $\mathfrak{g} \oplus \mathfrak{g} = \Delta \mathfrak{g} \oplus \mathfrak{p}_\alpha$, with $\mathfrak{p}_\alpha := \{ \mathfrak{X}_\alpha := \{ \frac{\alpha + 1}{4}[X,Y] \} \in \mathfrak{g} : X \in \mathfrak{g} \} \cong \mathfrak{g}$, for some $\alpha \in \mathbb{R}$. This is reductive decomposition and for any $X, \alpha Y \in \mathfrak{p}_\alpha$ the commutator $[X, \alpha Y] = \frac{(\alpha + 1)^2}{4}[X,Y], \{ \frac{(\alpha - 1)^2}{4}[X,Y] \} \in \mathfrak{g}$ can be expressed by $[X, \alpha Y] = [X, \alpha Y] \Delta \mathfrak{g} + [X, \alpha Y] \mathfrak{p}_\alpha$, with

$$[X, \alpha Y] \Delta \mathfrak{g} = \frac{1-\alpha^2}{4}[X,Y], \text{ and } [X, \alpha Y] \mathfrak{p}_\alpha = \alpha\left(\frac{\alpha + 1}{2}[X,Y], \frac{\alpha - 1}{2}[X,Y]\right),$$

respectively. Hence, the torsion of the induced connection is given by $T^\alpha(X,Y) := -[X, \alpha Y] \mathfrak{p}_\alpha = -[X,Y], \text{ for any } X, Y \in \mathfrak{g}$. Let now $(\cdot, \cdot)$ be the $\text{Ad}(G)$-invariant inner product on $\mathfrak{g}$ corresponding to $\rho$ (this is a multiple of the negative of the Killing form of $G$). Because the associated 3-tensor $T^\alpha(X,Y,Z) = \{T^\alpha(X,Y), Z\}$ is a 3-from, $\nabla^\alpha$ is metric (see [AL2] §2.1) and we compute that $\eta^\alpha(X,Y) = \frac{1-\alpha^2}{2}[X,Y]$. In fact, by applying (6.11) we get $\langle \nabla^\alpha_3 X, Y \rangle = \langle \nabla^\alpha_3 Y, Z \rangle + \frac{1}{2}T^\alpha(X,Y,Z) = \{\eta^\alpha(X,Y), Z\}$ for all $X,Y,Z \in \mathfrak{g}$, or in other words $\nabla^\alpha_3 X = \eta^\alpha(\mathfrak{g})$. For $\alpha = 0$ one gets the Levi-Civita connection $\nabla^0 = \nabla^\rho$ and the usual Cartan decomposition, i.e. $\mathfrak{p}_0 \cong p$. The values $\alpha = \pm 1$ define the flat $\pm 1$-connections of Cartan and Schouten (see [CS1, KN]). Indeed,

$$R^\alpha(X,Y)Z = \eta^\alpha(X, \eta^\alpha(Y,Z)) - \eta^\alpha(Y, \eta^\alpha(X,Z)) - \eta^\alpha([X,Y], Z) = \frac{(1-\alpha^2)^2}{4}[X, [Y,Z]] + [Y, [Z,X]] + \frac{1-\alpha^2}{2}[Z, [X,Y]] = \frac{1-\alpha^2}{4}[Z, [X,Y]].$$

We recall now the classification of metric bi-invariant connections on a compact Lie group $G$ by [AFH]. For the sake of completeness, and since we will use this result, we explain the main idea of the proof (adapted in our notation). This is essentially based on the classification of bi-invariant affine connections given in [LI].

**Theorem 3.2.** ([AFH, p. 18]) Let $G$ be a compact connected Lie group with a bi-invariant metric $\rho$ and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ be a decomposition of the corresponding Lie algebra $\mathfrak{g}$ into its centre $\mathfrak{g}_0 \equiv Z(\mathfrak{g})$ and simple ideals $\mathfrak{g}_i$, $1 \leq i \leq r$. Then, a bi-invariant metric connection on $G$ is given by (up to scalar and sign)

$$\nabla^\alpha_X Y := \eta^\alpha(X, Y) = \sum_{i=1}^r (1-\alpha_i)/2 \cdot [X,Y]_{\mathfrak{g}_i}, \quad \alpha := (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r,$$

(3.1)

The torsion and the curvature of this $r$-parameter family are given by $T^\alpha(X,Y) = -\sum_{i=1}^r \alpha_i \cdot [X,Y]_{\mathfrak{g}_i}$, and $R^\alpha(X,Y)Z = \sum_{i=1}^r ((1-\alpha_i)/4) \cdot [Z, [X,Y]_{\mathfrak{g}_i}]_{\mathfrak{g}_i}$, respectively.

**Proof.** Consider first a bilinear $\text{Ad}(G)$-equivariant map $\eta : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ corresponding to a bi-invariant metric connection $\nabla$ on $G$. Since $\nabla$ is metric with respect to $\rho$, $\eta$ is skew-symmetric with respect to the induced $\text{Ad}(G)$-invariant inner product $(\cdot, \cdot)$, i.e. $\eta_X := \Lambda(X) \in \text{so}(\mathfrak{g})$, for any $X \in \mathfrak{g}$. Thus, as a first step we obtain the following correspondence (see also Lemma 2.2).

A bi-invariant metric connection $\nabla$ has skew-torsion $T \in \Lambda^3(\mathfrak{g}) \iff \eta(\lambda(\mathfrak{X}), \mathfrak{X}) = \lambda(\mathfrak{X})X = 0 \quad \forall \mathfrak{X} \in \mathfrak{g}$, and this corrects a small error in [AFH, Lemma 3.1], where the condition $\lambda(\mathfrak{X})X = 0$ is missing from the first two statements. Obviously, a connection induced by the adjoint representation of $\mathfrak{g}$ verifies this “skew-torsion” condition. Hence, the most interesting part of the proof is that of uniqueness. Let us break the argument up into two steps.

**1st Step:** We begin with the additional assumption that $G$ is simple. By Theorem 3.1 we know that the bilinear map $\lambda : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\lambda(X,Y) = (1-\alpha)[X,Y]/2$ defines a 1-dimensional family of bi-invariant metric connections on $G$ with torsion $T^\alpha(X,Y) = -\alpha[X,Y]$, for any $X,Y \in \mathfrak{g}$. We need now to show that this is the unique family (up to constant and sign). The space of bi-invariant affine connections on $G$ is isomorphic to the space $\text{Hom}^3(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ of all $G$-intertwining maps $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. Since $\mathfrak{g}$ is irreducible (and of real type), it is sufficient to compute the multiplicity of $\mathfrak{g}$ inside $\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^2 \oplus \Lambda^2(\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ always lies in $\Lambda^2(\mathfrak{g})$ since the map $\Lambda^2(\mathfrak{g}) \rightarrow \mathfrak{g}$ is surjective. Moreover, in [LI] H. T. Laquer confirms that the
multiplicity of \( g \) in \( \Lambda^2(g) \) is one for any compact simple Lie group and only for \( SU(n) \ (n \geq 3) \) there is a new copy of \( g \) inside \( S^2(g) \) with the same multiplicity. Thus for \( G \) simple, different from \( SU(n) \), the unique family of bi-invariant affine connections is determined by the Lie bracket, i.e. the bilinear map \( \lambda \). For \( SU(n) \) the “exceptional” family is induced by the symmetric bilinear map \( \eta^{\text{exc}}(X, Y) = iXY + YX - (2/n)\text{tr}(XY) \cdot I \), where \( I \) is the \( n \times n \) identity matrix. However, the induced affine connection is not metric with respect to a bi-invariant metric, e.g. the negative of the Killing form \([AFH]\). This proves the claim for \( G \) simple.

2nd Step: Let us drop now the latter condition and explain the more general case of a compact Lie group \( G \). Consider the decomposition of the corresponding Lie algebra \( g = T_eG \) into its centre and simple ideals \( g = g_0 \oplus g_1 \oplus \cdots \oplus g_r \) and write \( X = x_0 + x_1 + \cdots + x_r \). Then \( [X, Y] = [X, Y]_{g_0} + \cdots + [X, Y]_{g_r} \), where \( [X, Y]_{g_i} := [x_i, y_i] \). For any simple ideal \( g_i \) one can apply the method described in the first step, by using the bi-invariant connection \( \nabla^{g_i} \) induced by the bilinear map \( \eta^{g_i}(X, Y) = ((1 - \alpha_i)/2) \cdot [X, Y]_{g_i} \) for some \( \alpha_i \in \mathbb{R} \). Obviously, \( \nabla^{g_i} \) is metric with respect to the restriction \( \langle \ , \ \rangle_{g_i} = x_i \rho |_{g_i} \), where \( x_i \) are real positive numbers for any \( i = 1, \ldots, r \). Consider now some scalar product \( b \) on the centre \( g_0 \) and notice that \( \eta^{g_0} \equiv 0 \). The \( \text{Ad}(G) \)-invariant scalar product \( \langle \ , \ \rangle \) can be expressed by (up to scalar)

\[
\langle \ , \ \rangle = b|_{g_0} + x_1 \rho |_{g_1} + \cdots + x_r \rho |_{g_r}, \quad x_i \in \mathbb{R}_+.
\]

Hence, it is not difficult to see that the map defined by \( \eta^{\alpha}(X, Y) := \sum_{i=1}^{r} \eta^{g_i}(X, Y) \) with \( \alpha := (\alpha_1, \ldots, \alpha_r) \), induces a family of bi-invariant connections on \( G \) which are metric with respect to \( \langle \ , \ \rangle \). The associated torsion is given by \( T^\alpha(X, Y) = -\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \cdot [X, Y]_{g_j} \), and the induced 3-tensor is obviously a 3-form on \( g \) (since \( \langle \ , \ \rangle \) is also \( \text{ad}(g) \)-invariant). On the other hand, by \([L1]\) it is known that besides \( SU(n) \), only for \( U(n) \ (n \geq 2) \) one can construct affine bi-invariant connections corresponding to \( \text{Ad}(U(n)) \)-equivariant bilinear maps different from the Lie bracket (for details see \([L1]\) Thm. 9.1 and Theorem \([AFH]\) Thm 3.1). However, as for \( SU(n) \), in \([AFH]\) Thm 9.1 we conclude that for an arbitrary compact Lie group \( G \) a bi-invariant metric connection necessarily corresponds to a copy of \( g \) inside \( \Lambda^2(g) \), and this is given by \([3.1]\) (up to scalar and sign). The assertion for the curvature follows easily: \( R^\alpha = \sum_{i=1}^{r} R^{g_i} \), where \( R^{g_i} \) is the curvature tensor of \( g_i \) for \( i = 1, \ldots, r \).

**Corollary 3.3.** Any bi-invariant metric connection \( \nabla \) on a compact connected Lie group \( G \) endowed with a bi-invariant metric, has (totally) skew-symmetric torsion \( T \in \Lambda^2(g) \).

3.2. A certain class of bi-invariant affine connections. After Corollary 3.3 on can naturally pose the following question.

**Question 3.4.** Let \( G \) be a compact connected Lie group with Lie algebra \( g \). Given an arbitrary bi-invariant affine connection \( \nabla \) whose Nomizu map \( \Lambda : g \to \text{End}(g) \) satisfies the “skew-torsion” condition, namely

\[
\Lambda(X)X = 0, \quad \forall X \in g,
\]

is it true that \( \nabla \) is metric with respect to the bi-invariant metric? In other words, are the conditions \( \Lambda(X)X = 0 \) and \( \Lambda(X) \in \text{so}(g) \equiv \text{equivariant for any bi-invariant affine connection on } G? \)

A bi-invariant connection satisfying \([3.2]\) has as geodesics orbits of the one-parameter subgroups of \( G \) (in the simple case the same geodesics with the 1-parameter family of canonical connections on \( G \), see \([KN]\) Prop. 2.9, Ch. X). Hence, as we explained before, if \( \nabla \) is metric with respect to a bi-invariant metric on \( G \) then its torsion must be a 3-form on \( g \). However, the “converse” is not true, that is: the previous question admits a negative answer with counterexamples appearing for \( U(n) \) (see \([L1]\) \([AFH]\) and for more details the proof of Theorem 3.11). The goal of this paragraph is to specify exactly the subclass of bi-invariant affine connections on \( G \) which satisfy \([3.2]\) and provide us with a positive answer to Question 3.4. Hence, we ask:

**Question 3.5.** Which further conditions do we have to impose on the Nomizu map \( \Lambda : g \to \text{End}(g) \) of a bi-invariant affine connection on \( G \) satisfying \([3.2]\) in order to be metric with respect to a bi-invariant metric? In other words, which subclass of bi-invariant affine connections satisfying \([3.2]\) can be identified with the class of bi-invariant metric connections on \( G \)?

Our answer relates the flat connections of this type, which coincide with the \( \pm 1 \)-connections discussed in Theorem 3.1. We should emphasize once more that here we drop the condition that \( G \) is simple.

**Lemma 3.6.** Let \( G \) be a compact connected Lie group and let \( \nabla \) be a bi-invariant affine connection with \( \Lambda(X)X = 0 \), for any \( X \in g \). Then the following are equivalent:

(a) \( \nabla \) is flat \( R \equiv 0 \), i.e. \( \Lambda(X) : g \to g \) is a representation for any \( X \in g \).

(b) \( \Lambda(X) = \text{ad}(X) \), or \( \Lambda(X) = 0 \) for any \( X \in g \), and these are the unique bi-invariant linear connections which satisfy (a).
Proof. By definition, \( R(X, Y) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]) \) and thus \( \nabla \) is flat if and only if \( \Lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is a representation (for example, the Riemannian connection does not induce a representation). Assume that \( R \equiv 0 \), i.e. \( \Lambda(X)\Lambda(Y)Z - \Lambda(Y)\Lambda(X)Z - \Lambda([X, Y])Z = 0 \), for any \( X, Y, Z \in \mathfrak{g} \). By polarization it follows that (3.2) is equivalent to \( \Lambda(Y)\Lambda(X)Y + \Lambda(Y)[X, Y]X = 0 \) for any \( X, Y \in \mathfrak{g} \). Thus \( \Lambda(Y)[X, Y] = -\Lambda(Y)(\Lambda(X)Y - [X, Y]) \) and by setting \( Y = Z \) in the equation \( R \equiv 0 \), we get that

\[
0 = -\Lambda(Y)\Lambda(X)Y - \Lambda([X, Y])X = -\Lambda(Y)\Lambda(X)Y + \Lambda(Y)[X, Y]X = -\Lambda(Y)(\Lambda(X)Y - [X, Y]).
\]

Therefore \( \Lambda(X) = \text{ad}(X) \) or \( \Lambda(X) = 0 \), for any \( X \in \mathfrak{g} \). The converse is trivial. \( \square \)

From now on we will denote the special connections presented in Lemma 3.6 (b) by \( \nabla^+ \) and \( \nabla^- \); respectively; their torsion is given by \( T^\pm(X, Y) = \pm[X, Y] \). Both can be viewed as special members of these bi-invariant linear connections on \( G \), whose Nomizu map induces derivations on the corresponding Lie algebra \( \mathfrak{g} \) (for \( \nabla^- \) trivially). In the sequel we show that this is the desired condition that answers Question 3.5. First we propose a formula which allows us to characterize the Ad(\( g \))-equivariant derivations on \( \mathfrak{g} \) in terms of the curvature and the covariant derivative of the torsion of a bi-invariant connection satisfying (3.2).

**Proposition 3.7.** Let \( G \) be a compact connected Lie group and let \( \nabla \) be a bi-invariant affine connection whose Nomizu map \( \Lambda : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) satisfies (3.2). Then, \( \Lambda : \mathfrak{g} \to \text{Der}(\mathfrak{g}) \subset \text{End}(\mathfrak{g}) \) is a derivation of \( \mathfrak{g} \) if and only if the curvature \( R \) and the covariant derivative of the torsion \( T \) of \( \nabla \) satisfy the following relation:

\[
(\nabla T)(X, Y) = 2(R(Z, X)Y - \Lambda(Y)([Z, X] - \Lambda(Z)X)), \quad \forall X, Y, Z \in \mathfrak{g}.
\]

Proof. The proof is direct. Crucial is our assumption \( \Lambda(X)X = 0 \) and hence we mention that for bi-invariant connections without this property our claim fails. For simplicity set \( D(Z, X, Y) := \Lambda(Z)[X, Y] - \Lambda([Z, X], Y] - \lambda\), and notice that the endomorphism \( \Lambda : \mathfrak{g} \to \mathfrak{g} \) is a derivation if and only if \( D(Z, X, Y) = 0 \), for any \( X, Y, Z \in \mathfrak{g} \). Now, for any \( Z \in \mathfrak{g} \) we view the covariant derivative of the torsion \( T(X, Y) = 2\Lambda(X)Y - [X, Y] \) as a bilinear map \( \nabla T : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} \). Then, because \( \Lambda(\Lambda(Z)X)Y = -\Lambda(Y)\Lambda(Z)X \) and \( \Lambda([X, Y])Y = -\Lambda(Y)[X, Y] \) for any \( X, Y, Z \in \mathfrak{g} \), we obtain that

\[
(\nabla T)(X, Y) = \Lambda(Z)T(X, Y) - T(\Lambda(Z)X, Y) - T(X, \Lambda(Z)Y)
= 2\Lambda(Z)\Lambda(X)Y - \Lambda([Z, X], Y] - 2\Lambda\Lambda(X)X)Y + \Lambda([X, Y] - 2\Lambda(\Lambda(Z)X)Y + [X, \Lambda(Z)Y]
= 2\Lambda(\Lambda(Z)X)Y + 2\Lambda(\Lambda(Z)X) - 2\Lambda(X)\Lambda(Z)Y - D(Z, X, Y)
\]

where in (†) we insert the formula \( R(Z, X)Y = \Lambda(\Lambda(Z)X)Y - \Lambda(Y)\Lambda(Z)X \) and \( \Lambda([X, Y])Y = -\Lambda(Y)[X, Y] \). \( \square \)

A simple but important consequence of Proposition 3.7 is the following.

**Corollary 3.8.** Let \( G \) be a compact connected Lie group endowed with a bi-invariant affine connection \( \nabla \) whose Nomizu map \( \Lambda : \mathfrak{g} \to \text{Der}(\mathfrak{g}) \subset \text{End}(\mathfrak{g}) \) is a derivation and satisfies (3.2). If \( \nabla \) is flat, i.e. \( R \equiv 0 \), or \( R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X) \), then the corresponding torsion \( T \) is \( \nabla \)-parallel.

Proof. If \( R \equiv 0 \), then equation (3.2) reduces to \( (\nabla T)(X, Y) = -2\Lambda(Y)([Z, X] - \Lambda(Z)X) \). Parallel, Lemma 3.6 ensures that \( \Lambda = \Lambda^\pm \) and then \( -2\Lambda(Y)([Z, X] - \Lambda(Z)X) = 0 \). Thus \( \nabla T = 0 \), i.e. \( \nabla \times T^\pm = 0 \). For example, this is the case if \( G \) is semi-simple, since then any derivation is inner (however notice that in the compact case this argument fails, see Proposition 3.10). Now, if \( R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X) \) then it is immediate from (3.3) that \( \nabla T \equiv 0 \). An alternative way that avoids (3.3) but includes a few more computations occurs due to the following observation. For a bi-invariant affine connection on \( G \) satisfying our assumptions, it is not difficult to prove that the equation \( R(Z, X)Y = \Lambda(Y)([Z, X] - \Lambda(Z)X) \) is equivalent to \( \Lambda([Z, X], Y] = -\lambda(Z)Y \) (as an endomorphism of \( \mathfrak{g} \)), or in other words \( \Lambda(Z)YX = -\lambda(X)\lambda(Z)Y \) for any \( X, Y, Z \in \mathfrak{g} \). By using this relation and the properties of \( \Lambda \), a straightforward computation shows that \( (\nabla T)(X, Y) = 0 \) for any \( X, Y, Z \in \mathfrak{g} \). \( \square \)

By combining this with Lemma 3.6 we conclude that

**Corollary 3.9.** On a compact connected Lie group \( G \) there exist exactly two bi-invariant affine connections satisfying (3.3), which are flat and have parallel torsion. These are the connections \( \nabla^\pm \) described in Lemma 3.6 and they coincide with the ±1-connections of Cartan-Schouten.

**Proposition 3.10.** Let \( \mathfrak{g} \) be a reductive Lie algebra. Then, any derivation \( D : \mathfrak{g} \to \text{Der}(\mathfrak{g}) \) is given by \( D(X) = \phi(Z) \oplus \text{ad}(X) \) for some linear map \( \phi : \mathfrak{g}_0 \to \text{End}(\mathfrak{g}_0) \) in the centre \( \mathfrak{g}_0 \). In fact \( H^1(\mathfrak{g}, \mathfrak{g}) \cong \text{End}(\mathfrak{g}_0) \).
Proof. Consider the decomposition of \( g \) into its centre and semi-simple part, i.e., \( g = g_0 \oplus [g,g] = g_0 \oplus g_{ss} \) and express any \( X \in g \) in a unique way by \( X = Z + X_s \), where \( Z \in g_0 \) and \( X_s \in g_{ss} \). Then define \( D : g \rightarrow \text{End}(g) \) by \( D(X) := \phi(Z) + \text{ad}(X_s) = (\phi(Z) + \text{ad}(X_s)) \) for some linear map \( \phi : g_0 \rightarrow \text{End}(g_0) \). Obviously, this is a derivation of \( g \) and in order to prove our claim it is sufficient to show that for any \( Z \in g_0 \) and \( X_s \in g_{ss} \) derivations of the form \( D_1(Z) : g_0 \rightarrow g_{ss} \) and \( D_2(X_s) : g_{ss} \rightarrow g_0 \) are necessarily trivial. This follows easily for \( D_1 \), because the centre of any Lie algebra is a characteristic ideal, i.e., remains invariant under derivations. Consider now some \( \alpha, \beta \in g \) with \( [\alpha, \beta] \in g_{ss} \) and assume that for any \( X_s \in g_{ss} \), the linear map \( D_2(X_s) : g_{ss} \rightarrow g_0 \) is a non-trivial derivation. Then, \( D_2(X_s) \) acts on \( [\alpha, \beta] \) as an inner derivation, i.e., \( D_2(X_s)[\alpha, \beta] = \text{ad}_{X_s}[\alpha, \beta] = [X_s, [\alpha, \beta]] \). On the other hand, we have that \( D_2(X_s)[\alpha, \beta] = D_2(X_s)\alpha \beta + [\alpha, D_2(X_s)\beta] = 0 \), since \( D_2(X_s)\alpha, D_2(X_s)\beta \in g_0 \). Because \( X_s, [\alpha, \beta] \in g_{ss} \), this gives a contradiction. In this way we conclude that the spaces \( \text{Der}(g_0, g_{ss}) \) and \( \text{Der}(g_0, g_0) \) must be trivial and for the Lie algebra \( \text{Der}(g) \) we get the direct sum decomposition

\[
\text{Der}(g) = \text{Der}(g_0) \oplus \text{Der}(g_{ss}) = \text{End}(g_0) \oplus \text{ad}(g) = \text{Out}(g) \oplus \mathcal{I}(g),
\]

where \( \mathcal{I}(g) \cong g \setminus g_0 = \{\text{ad}(X) : X \in g\} = \text{ad}(g) \) denotes the space of all inner derivations and \( \text{Out}(g) \) is the quotient algebra of outer derivations \( \text{Out}(g) \cong \text{Der}(g) / \mathcal{I}(g) \). The Lie subalgebra \( \mathcal{I}(g) \) is an ideal in \( \text{Der}(g) \), the so-called adjoint algebra of \( g \). On the other hand, the algebra \( \text{Out}(g) \) coincides with the first cohomology \( H^1(\mathfrak{g}, g) \) of \( g \) acting on itself by the adjoint representation, see [GOV, p. 57]. Hence \( H^1(\mathfrak{g}, g) \cong \text{End}(g_0) \). 

We conclude that given a compact Lie group \( G \) and an arbitrary derivation \( D : g \rightarrow \text{Der}(g) \), the relation \( D(X)X = 0 \) is not necessarily true for any \( X \in g \). Next we will show that if \( D : g \rightarrow \text{Der}(g) \) is an \( \text{Ad}(G) \)-equivariant derivation with \( D(X)X = 0 \) for any \( X \in g \), then the map \( \phi \) appearing in the splitting \( D = \phi \oplus \text{ad} \) must be trivial \( \phi = 0 \), i.e., \( D \) oughts to be an inner derivation. Although for non-central elements \( g \notin Z(G) \) one can prove this results easily, for central elements the equivariance condition does not provide any further information and a proof of the claim seems difficult. We overpass this problem by using Proposition 3.7.

**Theorem 3.11.** Let \( G \) be a compact connected Lie group and let \( D : g \rightarrow \text{Der}(g) \) be a derivation of \( g = T_eG \). Assume that \( D(\text{Ad}(g)X) = \text{Ad}(g)D(X)\text{Ad}(g)^{-1} \) for all \( g \in G, X \in g \) and that \( D(X)X = 0 \) for any \( X \in g \). Then \( D = \text{ad} \) is an inner derivation.

**Proof.** 1st way: By Proposition 3.10 write \( D = \phi \oplus \text{ad} \) for some linear map \( \phi : g_0 \rightarrow \text{End}(g_0) \). Because \( \text{Ad}(G)g_0 = g_0 \) and the adjoint representation of \( g \) is \( \text{Ad}(G) \)-equivariant, it turns out that \( D \) has the same property if and only if \( \phi(\text{Ad}(g)Z) = \phi(Z)(\equiv \text{Ad}(g)\phi(Z)\text{Ad}(g)^{-1}) \), for any \( g \in G \) and \( Z \in g_0 \) (where we view \( \text{Ad}(g)\phi(Z)\text{Ad}(g)^{-1} \) as an endomorphism \( g_0 \rightarrow g_0 \)). In addition, the condition \( D(X)X = 0 \) for any \( X \in g \) is equivalent to \( \phi(Z)Z = 0 \) for any \( Z \in g_0 \). Now, it is sufficient to prove that \( \phi = 0 \). Assume in contrast that \( \phi(Y) \neq 0 \) for some \( Y \in g \). We view the centre \( Z(G) \) of \( G \) as a compact Lie group itself and we identify \( T_e(Z(G)) = g_0 \) (the centre \( Z(G) \) is closed subgroup of \( G \)). Because for any \( Z \in g_0 \) the endomorphism \( \phi(Z) : g_0 \rightarrow g_0 \) is (trivially) a derivation which satisfies the properties of Proposition 3.7, the associated bi-invariant affine connection on \( Z(G) \) satisfies \( \sum \) for any \( X, Y, Z \in g_0 \). Let us denote this connection by \( \nabla^\phi \). Obviously \( R_{^\phi} = 0 \) (since \( g_0 \) is abelian) and \( \nabla^\phi(Z, Z') = \phi(Z)Z' - \phi(Z')Z \) for any \( Z, Z' \in g_0 \). An easy computation also shows that

\[
(\nabla^\phi_2 T^\phi)(X, Y) = \phi(Z)(T^\phi(X, Y)) - T^\phi(\phi(Z)X, Y) - T^\phi(X, \phi(Z)Y) = \phi(\phi(Z)Y)X - \phi(\phi(Z)X)Y,
\]

since for example \( \phi(Z)\phi(X) = \phi(X)\phi(Z) \) for any \( Z, X \in g_0 \). Finally, the relation \( \sum \) takes the form

\[
\phi(\phi(Z)Y)X - \phi(\phi(Z)X)Y = 2\phi(Y)\phi(\phi(Z)X) \quad \forall X, Y, Z \in g_0.
\]

Now, for \( X = Z \) it reduces to \( \phi(\phi(X)Y)X = 0 \), for any \( X, Y \in g_0 \). Because the identity \( \phi(Z)Z = 0 \) is equivalent to \( \phi(X)Y + \phi(Y)X = 0 \) for any \( X, Y \in g_0 \), we can write \( \phi(Y)X \) and \( Y = 0 \) for any \( Y \in g_0 \), which gives rise to a contradiction. Thus \( \phi = 0 \) and \( D = \text{ad} \).

2nd way: Proposition 3.7 characterizes the \( \text{Ad}(G) \)-equivariant derivations \( D : g \rightarrow \text{Der}(g) \) on the Lie algebra \( g \) of a compact connected Lie group \( G \) satisfying the condition \( D(X)X = 0 \) for any \( X \in g \). Such derivations correspond to bi-invariant affine connections of \( G \), whose Nomizu map \( \Lambda : g \rightarrow \text{gl}(g) \) satisfies the relations \( \sum \) and \( \sum \). For a compact simple Lie group, except \( G = SU(n) \), the bi-invariant affine connections are described (up to scalar and sign) by the 1-parameter family \( \Lambda^\alpha : g \rightarrow \text{End}(g) \) with \( \Lambda^\alpha := ((1 - \alpha)/2) \cdot \text{ad} \), for some \( \alpha \in \mathbb{R} \), which is obviously an \( \text{Ad}(G) \)-equivariant derivation. For the general compact case, by adopting the notation of Theorem 3.2 it is easy to see that for some \( \alpha_i \in \mathbb{R} \) the expression \( D(X) := \sum_{i=1}^n \Lambda_i^\alpha(X) = \sum_{i=1}^n ((1 - \alpha_i)/2) \cdot \text{ad}(X) | \| g \) is a derivation on \( g \), which turns out to be inner (by linearity of \( \text{ad} \)). In order to prove our claim, there remains to exclude the exotic connections of \( SU(n) \) and \( U(n) \). Indeed, a routine computation shows that these are not derivations, in particular: the unique bi-invariant linear connections...
of a compact connected Lie group which induce derivations on the corresponding Lie algebra are induced by
the Lie bracket. For example, for SU(n) the bilinear map \( \eta_{\text{trace}} \) described in Theorem 3.2 does not induce a
derivation nor does it satisfy (3.2). The special families for \( U(n) \) are more complicated. For \( n = 2 \),
aside the skew-symmetric map induced by the Lie bracket, the new families of symmetric bilinear \( \text{Ad}(U(n)) \)-
equivariant maps span a 3-dimensional space \([L1]\). However, neither these are derivations and the condition
\( \eta(X,X) = 0 \) also fails. The same is true for \( n \geq 3 \); there is a 3-dimensional space generated by symmetric
bilinear maps \( \eta_i : u(n) \times u(n) \to u(n) \) which do not induce derivations, namely \( \eta_1(X,Y) := i(X \cdot Y + Y \cdot X) \),
\( \eta_2(X,Y) := \text{tr}(X \cdot Y) \cdot iI \), and \( \eta_3(X,Y) := \text{tr}(X) \text{tr}(Y) \cdot iI \), but also the skew-symmetric map \( \mu(X,Y) = i(\text{tr}(X) Y - \text{tr}(Y) X) \) (see \([L1]\) Thm. 10.1) or \([AP1]\) Thm. 3.1). Because \( \mu(X,X) = 0 \) for any \( X \in u(n) \), \( \mu \) is
at least a candidate of Proposition 3.7. However a quick check implies that neither this is a derivation. Now,
although a linear combination \( \eta_i(X,Y) := c_1\eta_1(X,Y) + c_2\eta_2(X,Y) + c_3\eta_3(X,Y) + c_4\mu(X,Y) \) gives rise to an
\( \text{Ad}(U(n)) \)-equivariant bilinear map on \( g \), the condition \( \eta_i(X,X) = 0 \) for any \( X \in u(n) \) is true, if and only
if, \( c_1 = c_2 = c_3 = 0 \). Because on an arbitrary compact Lie group \( G \) these connections exhaust all possible
bi-invariant affine connections \([L1]\) Thm. 9.1, the proof is complete.

Based on Theorem 3.11 we are now in the position to present the following result and proceed to the proof of
Theorem 1.1.

Corollary 3.12. Let \( G \) be a compact connected Lie group with a bi-invariant metric \( \rho \) and let \( \nabla \) be a bi-invariant
affine connection corresponding to a linear \( \text{Ad}(G) \)-equivariant map \( \Lambda : g \to g(g) \) satisfying (3.2). Then \( \Lambda \) is a derivation if and only if \( \nabla \) is metric with respect to \( \rho \).

Proof. We need only to prove the sufficient condition since the converse is obvious due to Theorem 3.2.
Recall that the Killing form of a Lie algebra \( g \) (which here we denote by \( B = B_g \)) satisfies the relation
\( B(AX,AY) = B(X,Y) \), for any automorphism \( A : g \to g \), see [GOV] p. 13. If \( \Lambda(X) \in \text{Der}(g) \), then
\( \exp(t\Lambda(X)) \in \text{Aut}(g) \). Thus, the derivative of the relation \( B(AX,AY) = B(X,Y) \) at \( t = 0 \) for \( A = \exp(t\Lambda(X)) \) implies
\( B(\Lambda(X)Y,Z) + B(Y,\Lambda(X)Z) = 0 \) for any \( X,Y,Z \in g \). If the Lie group \( G \) is simple, then any \( \text{Ad}(G) \)-invariant inner product is a multiple of \( -B \), so \( \Lambda(X) \in so(g) \). If \( G \) is compact, then we express the \( \text{Ad}(G) \)-invariant inner product associated to \( \rho \) by \( ( , ) = b|_{g_0} - \sum_{i=1}^r c_i \cdot B|_{g_i} \) for some \( c_i > 0 \) (see Theorem 3.2); this is true because \( \rho|_{g_i} \) is multiple of \( -B_i \) where \( B_i \equiv B_{g_i} = B|_{g_i} \), for any \( i = 1,\ldots,r \). As we explained above, for any simple ideal \( g_i \) \( (1 \leq i \leq r) \) (inner) derivations become metric with respect to \( B|_{g_i} \).
For the centre \( g_0 \) not all the derivations are necessarily metric with respect to the scalar product \( b \). However,
\( \Lambda : g \to \text{Der}(g) \) is an \( \text{Ad}(G) \)-equivariant derivation with \( \Lambda(X)X = 0 \) and Theorem 3.11 guarantees that this
is inner. Hence the centre has no contribution and we finally obtain \( \Lambda(X) \in so(g) \) for any \( X \in g \).

Proof of Theorem 1.1. By Corollary 3.12 Theorem 1.1 is valid if one can drop the condition \( \Lambda(X)X = 0 \). In
fact, this is the case because a linear combination of the exotic connections on \( U(n) \) \( (n \geq 2) \) fails to induce a
non-trivial derivation (nor satisfies (3.2) as we explained in the proof of Theorem 3.11). The same time, the
endomorphism \( \Lambda(X) := \sum_{i=1}^r (1 - \alpha_i)/2 \cdot \text{ad}(X)|_{g_i} \) is an equivariant derivation which trivially verifies the
condition \( \Lambda(X)X = 0 \), for any \( X \in g \). This proves our assertion.

4. INvariant metric connections with skew-torsion on Naturally REductive spaces

Let \( (M = G/K, g) \) be a connected naturally reductive Riemannian manifold. Next we study \( G \)-invariant
metric connections whose torsion is proportional to the torsion of the canonical connection \( \nabla^c \). Let \( g = t \oplus m \)
be a reductive decomposition. We write \( \Lambda^g : m \to so(m) \) for the Nomizu map of the Levi-Civita connection
\( \nabla^g \) on \( (M = G/K, g) \) and \( \nabla^c \) for the canonical connection associated to \( m \). Recall that the \( \text{Ad}(K) \)-invariant
inner product \( ( , ) \) that \( g \) induces on \( m \) is such that \( (X,Y)_m) + (Y,[X,Z]) = 0 \), for any \( X,Y,Z \in m \).

Proposition 4.1. (a) For any \( \alpha \in \mathbb{R} \) there is a bijective correspondence between linear \( \text{Ad}(K) \)-equivariant
maps \( \Lambda^\alpha : m \to so(m) \), defined by
\[
(4.1) \quad \Lambda^\alpha(X)Y = \frac{1 - \alpha}{2}[X,Y]_m + (1 - \alpha)\Lambda^g(X)Y, \quad \forall X, Y \in m,
\]
and \( G \)-invariant metric connections \( \nabla^\alpha \) on \( T(G/K) \) with (totally) skew-symmetric torsion \( T^\alpha \in \Lambda^3(m) \) such that
\( T^\alpha = \alpha \cdot T^c \), where \( T^c \) is the torsion of the canonical connection \( \nabla^c \).

(b) If the Lie group \( G \) is compact and simple, then \( M = G/K \) is normal homogeneous and the family
\( \{ \nabla^\alpha : \alpha \in \mathbb{R} \} \) is naturally induced by a bi-invariant metric connection of \( G \).

Proof. (a) The direct statement is well-known [A1]. The converse is also very easy. Because \( (M = G/K, g) \) is
naturally reductive with respect to \( G \), \( \nabla^\alpha \) is a \( G \)-invariant metric connection with skew-torsion \( T^\alpha \in \Lambda^3(m) \) if
and only if the corresponding Nomizu map, say $\Lambda_m : m \to so(m)$, is such that $\Lambda_m(X)Y + \Lambda_m(Y)X = 0$ for any $X, Y \in m$, see Lemma $\ref{lemma}$ Because $T^n = \alpha \cdot T^c$ we also obtain that $\Lambda_m(X)Y - \Lambda_m(Y)X = (1 - \alpha)[X, Y]_m$, for any $X, Y \in m$ and now our claim follows.

(b) By $\cite{[L]_{2}}$ Thm. 6.1 it is known that given a homogenous space $M = G/K$ with a reductive decomposition $g = \mathfrak{t} \oplus \mathfrak{m}$, then there is a natural mapping

$$Aff_{G \times G}(F(G)) \to Aff_G(F(G/K)), \quad \eta \to \pi_\ast \eta, \quad \text{with } (\pi_\ast \eta)(X, Y) := \eta(X, Y)_m, \quad \forall X, Y \in m.$$ 

Here, $\eta$ is a bi-invariant linear connection on $G$, $\pi_\ast \equiv d\pi_\ast : g \to m$ is the differential of $\pi$ at $e$ and $\eta(X, Y)_m$ is the $m$-part of $\eta(X, Y)$, i.e. $\eta(X, Y) = \eta(X, Y)_m + \eta(X, Y)_k$. If $\eta$ is a bi-invariant metric connection on $G$, then the induced $G$-invariant connection on $M = G/K$ will be metric, since the inner product on $m$ is the restriction of an $\text{Ad}(G)$-invariant inner product of $g$. In our case, and since $g$ has been assumed to be naturally reductive, $\eta$ oughts to induce a $G$-invariant metric connection with skew-torsion. For $G$ compact and simple any bi-invariant metric connection is given by the map $\eta^\alpha(X, Y) = ((1 - \alpha)/2)[X, Y]$ for some $\alpha \in \mathbb{R}$ (up to scalar and sign). Consider the composition $\pi_\ast \eta^\alpha : g \times g \to m$ and write $X = X_m + X_k$. Then, by restricting $\pi_\ast \eta^\alpha$ on $m \times m$ we obtain a well-defined bilinear map $\lambda^\alpha : m \times m \to m$ with $\lambda^\alpha(X_m, Y_m) = (\pi_\ast \eta^\alpha)(X_m, Y_m) := \eta^\alpha(X_m, Y_m) = ((1 - \alpha)/2)[X_m, Y_m]_m$. This is an $\text{Ad}(K)$-equivariant map satisfying $\langle \lambda^\alpha(X_m, Y_m), Z_m \rangle + (Y_m, \lambda^\alpha(X_m, Z_m)) = 0$. The associated Nomizu map $\Lambda^\alpha : m \to so(m)$ is obviously the family discussed in (a).

We describe now the case of a symmetric space of Type I.

**Theorem 4.2.** Let $(M = G/K, g)$ be an (irreducible) Riemannian symmetric space of Type I. Then

(a) If $\nabla$ is a $G$-invariant metric connection with torsion a multiple of the torsion of the canonical connection, then necessarily $\nabla \equiv \nabla^c \equiv \nabla^g$. In other words, the unique $G$-invariant torsion-free metric connection on $M = G/K$ is the canonical connection which coincides with the Riemannian connection.

(b) The space of $G$-invariant metric connections consists of just the canonical connection $\nabla^c \equiv \nabla^g$.

**Proof.** (a) By definition, an irreducible Riemannian symmetric space $M = G/K$ of Type I is compact and simply connected and the Lie group $G$ is compact and simple, in fact $G = \text{Iso}_0(M)$ $\cite{[Bes]}$ (7.82)]. Hence, one can apply the method described in part (b) of Proposition 4.1. However, this never produces invariant connections different than the canonical connection. This is true because $(G, K)$ is a symmetric pair and hence $g = \mathfrak{t} \oplus \mathfrak{m}$ is the Cartan decomposition, i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}$. Indeed, let $\Lambda : m \to so(m)$ be the Nomizu map associated to $\nabla$. Then, by assumption $T = \alpha \cdot T^c = 0$ for any $\alpha \in \mathbb{R}$ and $\Lambda(X)Y = ((1 - \alpha)/2)[X, Y]_m = 0$. Then $\nabla^g X Y = \nabla^c X Y + \Lambda(X)Y = \nabla^g X Y \equiv \nabla^g X Y$.

(b) Because $M = G/K$ is irreducible (as Riemannian manifold) the isotropy representation is (strongly) isotropy irreducible (since for example the isotropy representation coincides with the holonomy representation). Therefore, for the classification of $G$-invariant affine connections one has to compute the multiplicity of $m$ inside the $K$-module $m \otimes m = S^2(m) \oplus \Lambda^2(m)$. This was done in $\cite{[L]_{2}}$, where it was proved that beyond the canonical connection, only the symmetric spaces $SU(n)/SO(n), SU(2n)/Sp(n)$ ($n \geq 3$) and $E_6/F_4$ admit a new 1-dimensional family of invariant affine connections. For the first two cases this family appears by applying Proposition 4.1 (b), for the exceptional connection $\eta^{exc}$ that $SU(n)$ admits. However, we already know that $\eta^{exc}$ is not metric, so it fails to induce $SU(n)$-invariant metric connections on the associated symmetric space. The same is true for $E_6/F_4$: although $S^2(m)$ includes a copy of $m$, the induced $E_6$-invariant connection does not preserve the Killing metric (and thus any $E_6$-invariant metric, see $\cite{[APH]}$ Rem. 3.2)).

**Remark 4.3.** According to $\cite{[OR]}$ Thm. 1.2, given a compact naturally reductive Riemannian manifold $(M = G/K, g)$ (locally irreducible) the canonical connection is unique under the assumption that $M$ is not isometric to a real, a projective space, or a compact simple Lie group with a bi-invariant metric. Viewing the sphere $S^n$ as a compact quotient $M = G/K$ this anomaly appears since $G$ is not necessarily equal to the full isometry group $\text{Iso}(M)$ of its connected component $\text{Iso}_0(M)$, in contrast to an isotropic irreducible symmetric space of Type I. Actually, let $(M = G/K, \mathcal{B})$ be an (effective) simply connected normal homogeneous manifold with $G$ compact connected and simple and assume that the isotropy representation is (strongly) irreducible. Then $G = \text{Iso}_0(M)$, unless $G = G_2/SU(3) = S^6$ or $M = Spin(7)/G_2 = S^7$ where $\text{Iso}_0(M, g_B) = SO(7), SO(8)$, respectively (see $\cite{[Wo]}$ Thm. 17.1 or $\cite{[WZ]}$ p. 623)). It is well-known that there are more spheres that can be represented as quotients of distinct Lie groups. This result goes back to the theory of enlargements of transition actions developed by A. L. Oniščik (1966) who classified all simple compact Lie algebras $g$ with Lie subalgebras $\mathfrak{k}_1, \mathfrak{k}_2$ such that $g = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ (see for example $\cite{[KI]}$ $\cite{[KS]}$). If $G$ is the compact simply connected Lie group corresponding to $g$ and $K_1, K_2 \subset G$ are the Lie subgroups associated to the Lie algebras $\mathfrak{k}_1, \mathfrak{k}_2$, then $g = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ if and only if $K_1$ acts transitively on $G/K_2$. Hence, in the Lie group level we
have the identifications $G/K_1 = K_2/(K_1 \cap K_2)$ (and $G/K_2 = K_1/(K_1 \cap K_2)$). Oniščič’s list (for symmetric cosets) contains several spheres. Let us present them.

| $G/K_1$ | $K_2/K_1 \cap K_2$ | $G/K_1$ | $K_2/K_1 \cap K_2$ |
|----------|-------------------|----------|-------------------|
| $S^{4n-1}$ | $SO(4n)/SO(4n-1)$ | $Sp(n)/Sp(n-1)$ | $S^n$ | $SO(7)/SO(6)$ | $G_2/\text{SU}(3)$ |
| $S^{4n-1}$ | $SO(4n)/SO(4n-1)$ | $Sp(n)U(1)/Sp(n-1)U(1)$ | $S^7$ | $SO(8)/SO(7)$ | $\text{Spin}(7)/G_2$ |
| $S^{4n-1}$ | $SO(4n)/SO(4n-1)$ | $Sp(n)Sp(1)/Sp(n-1)Sp(1)$ | $S^{15}$ | $SO(16)/SO(15)$ | $\text{Spin}(9)/\text{Spin}(7)$ |
| $S^{2n-1}$ | $SO(2n)/SO(2n-1)$ | $U(n)/U(n-1)$ | $S^{2n-1}$ | $SO(2n)/SO(2n-1)$ | $SU(n)/SU(n-1)$ |

In this table, although any $M = G/K_1$ is an isotropy irreducible symmetric space, only the presentations of $S^6$ and $S^7$ are (strongly) isotropy irreducible. Another fact that deserves our attention is that although the cosets $K_2/(K_1 \cap K_2)$ are diffeomorphic to a Riemannian symmetric space, namely a sphere, the pairs $(K_2, K_1 \cap K_2)$ are not necessarily symmetric. For example

$$S^7 = G/K = SO(8)/SO(7) \cong U(4)/U(3) \cong Sp(2)/Sp(1) \cong Sp(2)U(1)/Sp(1)U(1) \cong \text{Spin}(7)/G_2,$$

but taking a reductive decomposition $g = \mathfrak{t} \oplus \mathfrak{m}$ the relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{t}$ just holds for the first presentation. The spheres $S^6 = G_2/\text{SU}(3)$ and $S^{15} = \text{Spin}(9)/\text{Spin}(7)$ are also remarkable, with the 15-dimensional sphere being a homogeneous space with two isotropy summands satisfying (1.1). Due to this observation and since $K_2 \subset G$ and $M = G/K_1 = K_2/(K_1 \cap K_2)$, one may expect more $K_2$-invariant affine connections on $M$ than $G$-invariant connections. Let us examine this interesting problem for the irreducible cosets $K_2/(K_1 \cap K_2)$ appearing above and prove Theorem 1.2. A detailed study of $G$-invariant affine connections on compact (non-symmetric) strongly isotropy irreducible homogenous Riemannian manifolds $M = G/K$ will be treated in a forthcoming paper.

**Proof of Theorem 1.2.** We mention that viewing the spheres $S^6$ and $S^7$ as symmetric spaces, [22] Thm. 2.1 states that both the spaces $Af_{SO(7)}(F(SO(7)/SO(6)))$ and $Af_{SO(8)}(F(SO(8)/SO(7)))$ consist of just a point, corresponding to the canonical connection (induced by the associated Cartan decomposition), see also the proof of Theorem 1.2. Because the non-symmetric presentations of $S^6$ and $S^7$ are still (strongly) isotropy irreducible, the dimensions of the spaces $Af_{G_2}(F(G_2/\text{SU}(3)))$ and $Af_{Spin(7)}(F(Spin(7)/G_2))$ coincide with the multiplicity of the corresponding isotropy representation $\mathfrak{m}$ inside $\mathfrak{m} \otimes \mathfrak{m} = \Lambda^2(\mathfrak{m}) \oplus \text{Sym}^2(\mathfrak{m})$. However, we need now to view $\mathfrak{m}$ as a $\text{SU}(3)$- (resp. $G_2$-) module. Consider first the 7-sphere $S^7 \subset \mathbb{R}^8$ and identify $\mathbb{R}^8 \cong \mathbb{O}$, where $\mathbb{O}$ are the Cayley numbers. We view $G_2 \cong \text{Aut}(\mathbb{O})$ as a subgroup of $Spin(7) \subset Cl(\mathbb{R}^7)$ preserving the spinor $\psi_0 = (1,0,\ldots,0)^t \in \Delta_7$, where $\Delta_7 := \mathbb{R}^8$ is the 8-dimensional spin representation of Spin(7). Because Spin(7) acts transitively on $S^7$ we get the diffeomorphism $S^7 \cong Spin(7)/G_2$, see [FKMS]. As usual, we write $V^{a,b}$ for the irreducible representation of $G_2$ corresponding to highest weight $(a,b)$, where both $a, b$ are non-negative integers; for example $V^{0,0} \cong \mathbb{R}$ is the trivial representation, $\phi_7 := V^{1,0} \cong \mathbb{R}^7$ is the standard representation of $G_2$ and $V^{0,1} \cong \phi_2$ is its adjoint representation. Let now $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$ be a reductive decomposition. The isotropy representation $\mathfrak{m}$ coincides with the standard representation $\mathfrak{m} \cong \phi_7 = \{X,\omega : X \in \mathbb{R}^7\} \cong \mathbb{R}^7$, where $\omega$ states for the (generic) 3-form on $\mathbb{R}^7$ preserved by $G_2$, see [FKMS]. For the (real) $G_2$-modules $\Lambda^2(\mathfrak{m})$ and $\text{Sym}^2(\mathfrak{m})$ we get the decompositions (we use the Lie software package):

$$\Lambda^2(\mathfrak{m}) \cong \text{so}(7) = V^{0,1} \oplus V^{1,0} = \mathfrak{g}_2 \oplus \phi_7 = \mathfrak{g}_2 \oplus \mathfrak{m}, \quad \text{Sym}^2(\mathfrak{m}) = V^{2,0} \oplus \mathbb{R},$$

where $V^{2,0} \cong \mathbb{R}^7$ with dim $V^{2,0} = 27$. Thus, there is only one copy of $\mathfrak{m}$ inside the $G_2$-module $\mathfrak{m} \oplus \mathfrak{m}$ which belongs to $\Lambda^2(\mathfrak{m})$. In other words, there is skew-symmetric bilinear $Ad(G_2)$-equivariant map $\eta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ which induces a 1-dimensional family of Spin(7)-invariant affine connections on $S^7$. Indeed, because $\mathfrak{m}$ is irreducible, Schur’s lemma tells us that $\eta$ must be a multiple of the Lie bracket, say $\eta(X,Y) = \frac{1}{(1-\alpha)} [X,Y]_m$ for some $\alpha \in \mathbb{R}$, with $X, Y \in \mathfrak{m}$. This induces the family $\{V^{a,b} : \alpha \in \mathbb{R}\}$ discussed in Proposition 1.1.

Let us treat now the 6-sphere. Recall that $G_2$ preserves the imaginary octonions $\text{Im}(\mathbb{O}) \cong \mathbb{R}^7$ and acts transitively on $S^6 \subset \text{Im}(\mathbb{O})$ with stabilizer diffeomorphic to SU(3), i.e. $S^6 \cong G_2/\text{SU}(3)$, see [11] Lem. 5.1. The weights of SU(3), similarly with $G_2$, are given by pairs of non-negative integers $(a, b)$ and irreducible SU(3)-representations will be labeled again by $V^{a,b}$. In particular, it is dim $V^{a,b} = \frac{1}{2}(a+1)(b+1)(a+b+2)$ and $V^{b,a} = \overline{V^{a,b}}$. Obviously, $V^{1,0} \cong C^3 := \mu_3$ is the standard (complex) representation of SU(3), $V^{0,1} \cong C \cong \mu_3^c$ is its conjugate and $V^{1,1} \cong su(3)^C$ is the complexified adjoint representation. Let $\mathfrak{g}_3 = su(3) \oplus \mathfrak{m}$ be a reductive decomposition. It follows that $\mathfrak{m} = [\mu_3, \mathfrak{m}]$, where for a complex representation $V$ we denote by $[V]_r$ the underlying real representation (whose real dimension is twice the complex dimension of $V$). Thus, it is more convenient to use the complexified isotropy representation, which splits now into two conjugate
(inequivalent) submodules: \( m \otimes_{\mathbb{R}} C = \mu_3 \oplus \overline{\mu}_3 = C^3 \oplus \overline{C^3} \). Then, for the SU(3)-module \( \Lambda^2(m) \otimes_{\mathbb{R}} C \) we get
\[
\Lambda^2(m) \otimes_{\mathbb{R}} C = \Lambda^2(m^c) = \Lambda^2(C^3) \oplus \Lambda^2(C^3) \oplus (C^3 \otimes \overline{C^3}) = (V^{1,0} \oplus V^{0,1}) \oplus V^{1,1} \oplus C,
\]
since \( C \cong V^{0,0} \), \( \Lambda^2(C^3) \cong \Lambda^2(\mu_3) \cong V^{1,0} = \mu_3 \) (see also [K2], p. 125)) and \( \Lambda^2(C^3) \cong \Lambda^2(C^3) \cong V^{1,0} \cong V^{0,1} = \overline{\mu}_3 \). Hence we finally conclude that
\[
\Lambda^2(m) = [V^{1,0}]_R \oplus \mathfrak{su}(3) \oplus R = [\mu_3]_R \oplus \mathfrak{su}(3) \oplus R = m \oplus \mathfrak{su}(3) \oplus R.
\]

The trivial summand \( R \) can be interpreted in terms of the natural SU(3)-structure that \( S^6 \) admits. This is of Gray-Hervella type \( \mathcal{W}_1 \), hence \( S^6 \) carries a (homogeneous) nearly Kähler structure which may be specified by an \( \mathfrak{su}(3) \)-connection and a complex structure \( \phi \) of \( \mathfrak{su}(3) \)-invariant affine connections on \( S^6 \). Under the action of \( \text{SU}(3) \), we also get
\[
\text{Sym}^2(m) \otimes_{\mathbb{R}} C = \text{Sym}^2(C^3) = \text{Sym}^2(C^3) \oplus \text{Sym}^2(C^3) \oplus (C^3 \otimes \overline{C^3})
\]
with \( V^{2,0} \cong \text{Sym}^2(C^3) \). Consequently, the decomposition of \( \text{Sym}^2(m) \) into irreducible \( \text{SU}(3) \)-submodules is given by \( \text{Sym}^2(m) = [V^{2,0}]_R \oplus \mathfrak{su}(3) \oplus R \). This proves our claim for the affine case. Now, the assertion about the metric property occurs very easily: by Schur’s lemma any \( \mathfrak{g}_2 \)-invariant metric on \( S^6 = G_2 / \text{SU}(3) \) must be a multiple of the negative of the Killing form of \( G_2 \), restricted on \( m \); hence the family \( \{ \nabla^\alpha : \alpha \in \mathbb{R} \} \) is necessarily metric. Similarly for \( S^7 = \text{Spin}(7) / G_2 \).

**Remark 4.4.** The embedding of \( S^7 \) inside the spin representation \( \Delta_T \cong \mathbb{R}^8 \) induces on the associated tangent bundle \( TS^7 \) an affine connection \( \nabla^{\text{flat}} \) which is metric and has (non-parallel) skew-torsion \( T^{\text{flat}} \neq 0 \) [AF2]. The 7-sphere endowed with this connection and a Riemannian metric of constant sectional curvature becomes flat, and together with the compact (simple) Lie groups endowed with a bi-invariant metric and one of the \( \pm 1 \)-connections, exhaust all Riemannian manifolds carrying a flat metric connection with non-trivial skew-torsion (Cartan-Schouten theorem). Viewing the sphere \( S^7 = \text{Spin}(7) / G_2 \) as a \( G_2 \)-manifold, I. Agricola and Th. Friedrich [AF2] pp. 7–9 described this connection as a \( G_2 \) connection whose torsion \( T^{\text{flat}} \) doesn’t have constant coefficients. Hence, \( \nabla^{\text{flat}} \) is not an invariant connection and this is the reason that it does not appear in Theorem [12] (for example, the difference \( D := \nabla^{\text{flat}} - \nabla^c \) is not an \( \text{Ad}(G_2) \)-invariant tensor and hence given a reducible decomposition \( \text{spin}(7) = \mathfrak{g}_2 \oplus m \), the relation \( T^{\text{flat}} = \alpha \cdot T^c \) fails for any \( \alpha \in \mathbb{R} \)).

5. \( \nabla^\alpha \)-flat naturally reductive Riemannian manifolds

We describe now the case that the family \( \{ \nabla^\alpha : \alpha \in \mathbb{R} \setminus \{0\} \} \) is flat and we provide a proof of Theorem [13].

**Assumption 5.1.** In this section we assume that \( (M = G/K, g) \) is a compact naturally reductive manifold, irreducible as Riemannian manifold, endowed with an effective and transitive action of a compact connected Lie group \( G \) and hence a reducible decomposition \( \mathfrak{g} = \mathfrak{t} \oplus m \) such that \( \mathfrak{g} = \mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \) in terms of Kostant’s theorem. The irreducible condition is not necessary, but it is sufficient when we will examine the flat case.

**Remark 5.2.** Given a homogeneous space \( M = G/K \) with a reducible decomposition \( \mathfrak{g} = \mathfrak{t} \oplus m \) the space \( \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] \) is identified with the Lie algebra \( \text{tr}(\nabla^c) \) of the transvection group \( \text{Tr}(\nabla^c) \) of the canonical connection associated to \( m \) [R2] Rem. 4.1]. The group \( \text{Tr}(\nabla^c) \) is a connected and normal subgroup of \( \text{Aff}_0(\nabla^c) \) (the connected component of the affine group of \( \nabla^c \)), which consists of all \( \nabla^c \)-affine transformations that preserve any \( \nabla^c \)-holonomy subbundle of the orthonormal frame bundle. In general \( \text{Tr}(\nabla^c) \subseteq G \). However, if \( M = G/K \) is a compact normal homogeneous space, then \( G = \text{Tr}(\nabla^c) \) [R2 Prop. 4.2] and hence any such (irreducible) space gives rise to a naturally reductive manifold satisfying the conditions of Assumption 5.1. In fact, our assumption says that \( M = G/K \) is a compact naturally reductive manifold with respect to the decomposition \( \mathfrak{g} = \mathfrak{t} \oplus m \), where \( G = \text{Tr}(\nabla^c) \) is the group of transvections of the canonical connection \( \nabla^c \) associated to \( m \).

Let \( \text{Jac}_m : m \times m \times m \to m \) be the trilinear map defined by \( \text{Jac}_m(X, Y, Z) := \mathcal{S}^{X, Y, Z}[X, [Y, Z]_m]_m \), where \( \mathcal{S} \) denotes the cyclic sum over the vectors \( X, Y, Z \in m \). For a Riemannian symmetric space \( (M = G/K, g) \) it is \( \text{Jac}_m \equiv 0 \) identically, since \( [m, m] \subset \mathfrak{t} \). By using Theorem [12] we can prove that the vanishing of this map on a naturally reductive space \( M = G/K \), different than a symmetric space of Type I, implies the isometry of \( M \) with a compact Lie group endowed with a bi-invariant metric.
Lemma 5.3. Let \((M^n = G/K, g)\) be a compact naturally reductive Riemannian manifold as in our assumption, endowed with a \(G\)-invariant metric connection \(\nabla^\alpha\) whose torsion is such that \(T^\alpha = \alpha \cdot T^e\) for some \(\alpha \in \mathbb{R} \setminus \{0, 1\}\). Then the following conditions are equivalent

\[
\text{Jac}_m \equiv 0 \iff [m, m] \subset m \iff M \cong G\text{ endowed with a bi-invariant metric.}
\]

Proof. According to Proposition 4.1, we write \(\nabla^\alpha Y = \nabla_Y X + \Lambda^\alpha(Y)X\), where \(\Lambda^\alpha(Y)X = \frac{1}{2\alpha}[X, Y]_m\) with \(\alpha \neq 0, 1\). Since \(\alpha \neq 1\), \(\nabla^\alpha\) cannot be the canonical connection associated to \(m\) and by Theorem 4.2, \(M = G/K\) cannot be a symmetric space of Type I. First we prove that the condition \(\text{Jac}_m \equiv 0\) implies that \([m, m] \subset m\). In contrast, assume that there exist some \(X, Y \in m\) such that \([X, Y] \notin m\). Then, their Lie bracket \([X, Y]\) has no \(m\)-part, i.e. \([X, Y] = [X, Y]_f\) and the equation \(\text{Jac}_m(X, Y, Z) = 0\) takes the form \([X, [Y, Z]]_m + [Y, [Z, X]]_m = 0\). By computing now the curvature \(R^\alpha\) of \(\nabla^\alpha\), it is not difficult to see that

\[
R^\alpha(X, Y)Z = \frac{(1-\alpha)^2}{4}([X, [Y, Z]]_m + [Y, [Z, X]]_m) - \frac{(1-\alpha)}{2}([X, Y]_m, Z)_m - ([X, Y]_f, Z)_m.
\]

Hence, \(R^\alpha\) is identical with the curvature associated to the canonical connection and since \(\alpha \neq 1\) we obtain a contradiction. Conversely, assume that \([m, m] \subset m\). Then, because the commutator \([m, m]\) spans all of \(\mathfrak{t}\) (by assumption), \(\mathfrak{t}\) must be necessarily trivial, i.e. \(K = \{e\}\) and \(M^n \cong G\). Hence \(g = m\) and by the Jacobi identity we see that \(\text{Jac}_m \equiv 0\). \(\square\)

Theorem 5.4. Let \((M^n = G/K, g)\) be a compact naturally reductive Riemannian manifold as in our assumption, endowed with the family of \(G\)-invariant metric connections \(\{\nabla^\alpha : \alpha \in \mathbb{R}\}\) described in Proposition 4.7. If \(\nabla^\alpha T^\alpha = 0\) for any \(\alpha \neq 0, 1\) then \(M^n \cong G\) is isometric to a compact Lie group endowed with a bi-invariant metric.

Proof. By [A2] p. 61 we know that for any \(\alpha \in \mathbb{R} \setminus \{0\}\) the covariant derivative of \(T^\alpha\) is given by \((\nabla^\alpha T^\alpha)(X, Y) = (\alpha(\alpha - 1)/2)\text{Jac}(X, Y, Z)\). Because \(\alpha \neq 0, 1\), it follows that the equation \(\nabla^\alpha T^\alpha = 0\) is equivalent to \(\text{Jac}(X, Y, Z) = 0\) for any \(X, Y, Z \in m\) and then the isometry \(M^n \cong G\) was explained in Lemma 5.3. \(\square\)

Let us investigate the case when the family \(\{\nabla^\alpha : \alpha \in \mathbb{R}, \alpha \neq 0\}\) is flat, i.e. \(R^\alpha \equiv 0\). First, in the spirit of [DAN Prop. 3.7, (d)] (see also [AF2] p. 4), we show the following: On a naturally reductive Riemannian manifold \((M = G/K, g)\) a flat geometry with respect to a \(G\)-invariant metric connection \(\nabla\) with torsion \(0 \neq T = \alpha \cdot T^e\) is necessarily related to a “canonical” connection in the sense that \(\nabla T = 0\). Notice that we already know that the flatness implies the parallelism of the torsion when \(M^n \cong G\) is a compact Lie group endowed with a bi-invariant metric connection (combine Corollaries 3.8 and 3.12).

Proposition 5.5. ([DAN]) On a compact naturally reductive Riemannian manifold \((M = G/K, g)\) as in our assumption, the flatness condition of the family \(\{\nabla^\alpha : \alpha \in \mathbb{R} \setminus \{0\}\}\) implies the parallelism of the associated torsion form \(0 \neq T^\alpha \in \Lambda^3(m)\), i.e. \(\nabla^\alpha T^\alpha = 0\) for any \(\alpha \in \mathbb{R} \setminus \{0\}\).

Proof. We identify \(m \cong T_o M\), and we write \(X^\ast \in \mathcal{X}(M)\) for the Killing vector field on \(M\) induced by some \(X \in T_o M \subset g\). Recall that \([X^\ast, Y^\ast]_o = -[X, Y]_m = -[X, Y]^\ast_o\) (see [WZ p. 566]). Since \(M\) is \(\nabla^\alpha\)-flat for any \(\alpha \neq 0\), there is an orthonormal frame of Killing vector fields around \(o = eK\), say \(\{X^\ast_1, \ldots, X^\ast_n\} = \{X_1, \ldots, X_n\} : X_i \in m\), with respect to which the Christoffel symbols of \(\nabla^\alpha\) vanish, i.e. \((\nabla^\alpha_{X^\ast_i} X^\ast_j)_o = (\nabla^g_{X^\ast_i} X^\ast_j)_o = 0\). We view the torsion \(T^\alpha\) as a \((2, 1)\)-tensor field and we identify its evaluation at \(o \in M\) by a vector in \(m\), i.e. \(T^\alpha(X^\ast_i, X^\ast_k)_o \in m\). Then we immediately get

\[
T^\alpha(X^\ast_i, X^\ast_k)_o = -[X^\ast_i, X^\ast_k]_o = [X_i, X_k]_m = [X_i, X_k]^\ast_o.
\]

Since \(\nabla^g\), \(X^\ast = 0\) we also have \((\nabla^g_{X^\ast_i} X^\ast_j)_o = -\nabla^g_{X^\ast_i} X^\ast_j)\) and by \(\nabla^\alpha = \nabla^g + T^\alpha\) we take \(2(\nabla^g_{X^\ast_i} X^\ast_j)_o = -T^\alpha(X^\ast_i, X^\ast_j)_o = [X^\ast_i, X^\ast_j]_o = -[X_i, X_j]_m\). Finally notice that

\[
(\nabla^g_{X^\ast_i} X^\ast_j)_o - (\nabla^g_{X^\ast_i} X^\ast_j)_o = [X^\ast_i, [X^\ast_j, X^\ast_k]]_o = -[X^\ast_k, [X^\ast_i, X^\ast_j]]_o = -[X^\ast_k, [X_i, X_j]]_m = [X^\ast_k, [X_i, X_j]]_m\] \[\frac{(1-\alpha)^2}{4}([X, [Y, Z]]_m + [Y, [Z, X]]_m) - \frac{(1-\alpha)}{2}([X, Y]_m, Z)_m - ([X, Y]_f, Z)_m.
\]

where the second identity translates the Killing condition for the vector field \(X^\ast_k\). By using these relations and the expression of the product \((X^\ast_k)_o \cdot [X^\ast_i, X^\ast_j]_o, (X^\ast_i)_o \cdot (X^\ast_i)_o\) in terms of the Levi-Civita connection, it is not difficult to prove that

\[
([X^\ast_i, X^\ast_j]_o \cdot [X^\ast_k, X^\ast_l]_o) + ([X^\ast_k, X^\ast_i]_o \cdot [X^\ast_j, X^\ast_l]_o) = (X^\ast_k)_o \cdot ([X^\ast_i, X^\ast_j]_o \cdot (X^\ast_l)_o).
\]
In fact, because \( \langle \cdot, \cdot \rangle \) is naturally reductive, the left-hand side of this identity vanishes and it finally reduces to the simple expression \( X_k(\langle X_i, X_j \rangle_m, X_i) = 0 \). Hence, the function \( \langle X_i, X_j \rangle_m, X_i \) must be constant which is equivalent to say that \( \nabla^a T^a = 0 \). Indeed, we will show that (see [D'A-N] p. 405 or [AF2] p. 4) for a general Riemannian manifold endowed with a flat metric connection with skew-torsion

\[
\langle X_k \rangle_o(\langle X_i, X_j \rangle_o, (X_i)_o) = -(\nabla^a X^a)(X_i, X_j, X_i)_o.
\]

At a first step, for the left-hand side it is

\[
\langle X_k \rangle_o(\langle X_i, X_j \rangle_o, (X_i)_o) = \langle (\nabla^a X^a)(X_i, X_j, X_i)_o \rangle_o + \langle (X_i, X_j)_o, (\nabla^a X^a)_o \rangle_o
\]

\[
= -(\langle (\nabla^a X^a)(X_i, X_j)_o \rangle_o, \langle X_i, X_j \rangle_o) + \frac{1}{2}\langle [X_i, X_j]_o, [X_k, X_i]_o \rangle_o
\]

\[
= -(\nabla^a T^a(X_i, X_j), X_i) + \frac{1}{2}\langle [X_i, X_j]_m, [X_k, X_i]_m \rangle.
\]

Because \( \tilde{T}_{jk} = 0 \) the expression of the exterior derivative \( \nabla^a T^a \) is also simplified. In particular

\[
\langle X_k \rangle_o(\langle X_i, X_j \rangle_o, (X_i)_o) = \langle (\nabla^a X^a)(X_i, X_j, X_i)_o \rangle_o = \langle (\nabla^a X^a)(X_i, X_j)_o \rangle_o + \frac{1}{2}\langle [X_i, X_j]_o, (X_i)_o \rangle_o
\]

\[
= \langle (\nabla^a X^a)(X_i, X_j)_o \rangle_o + \frac{1}{2}\langle [X_i, X_j]_o, (X_i)_o \rangle_o
\]

\[
= \langle (\nabla^a X^a)(X_i, X_j) + \frac{1}{2}[X_k, [X_i, X_j]_m, X_i]
\]

\[
= \langle (\nabla^a X^a)(X_i, X_j) - \frac{1}{2}[X_i, X_j]_m, [X_k, X_i]_m \rangle
\]

\[
= -(X_k)_o(\langle X_i, X_j \rangle_o, (X_i)_o),
\]

and this finishes the proof. \( \square \)

**Proof of Theorem 1.3.** Assume that \( R^a(X, Y)Z = 0 \) for any \( X, Y, Z \in \mathfrak{m} \). Then, Proposition 5.3 ensures that \( \nabla^a T^a = 0 \) for any \( a \neq 0 \). Recall that \( \langle \nabla^a T^a \rangle(X, Y) = \frac{\alpha(a-1)}{2} \text{Jac}_m(X, Y, Z) \). Hence, since \( \alpha \neq 0 \) by assumption, there are two possibilities: either \( \alpha = 1 \), i.e. \( \nabla^1 = \nabla^0 \) is the canonical connection associated to reductive decomposition \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m} \), or \( \text{Jac}_m = 0 \). If \( M = G/K \) is a symmetric space of Type I by (a) or (b) of Theorem 1.2 it must be necessarily \( \alpha = 1 \), in particular \( \nabla^a \) is torsion free. Because \( T^a \neq 0 \) by assumption, it follows that \( M \) cannot be isometric with an irreducible symmetric space of Type I. If \( \alpha \neq 1 \) and \( \text{Jac}_m \equiv 0 \), then Lemma 5.3 implies the desired isometry \( M^n \cong G \). Now, for \( \alpha = 1 \), by [5.1] we get \( R^a(X, Y)Z = -\frac{1}{2}[X, Y]_m, Z \) for any \( X, Y, Z \in \mathfrak{m} \). Because \( G \) acts affinely on \( M = G/K \), the isotropy representation is faithful [Bec] and hence the condition \( R^3 = 0 \) implies that \( [X, Y]_m = 0 \), for any \( X, Y \in \mathfrak{m} \). Since \( g = m + [m, m] \) we finally conclude that \( M^n \cong G \) endowed with the connection \( \nabla^p \). The value \( \alpha = -1 \) and the connection \( \nabla^p \) appears more naturally by combining (5.2) and the relation \( T^a = \alpha \cdot T^a \); because \( M \) is \( \nabla^a \)-flat it follows that

\[
T^a(X_i, X_j) = [X_i, X_j]_m \iff -\alpha[X_i, X_j]_m = [X_i, X_j]_m \iff \alpha = -1.
\]

Because the condition \( R^2 = 0 \) implies that \( \nabla^1 T^1 = 0 \), we conclude that \( \text{Jac}_m(X, Y, Z) = 0 \) for any \( X, Y, Z \in \mathfrak{m} \) and hence similarly it follows that \( M^n \cong G \). Now, since \( M \) is irreducible and isometric to a compact Lie group with a bi-invariant metric, i.e. a symmetric space, we conclude that \( M \) must be simple.

**Remark 5.6.** The relation (5.1) can be rephrased as follows:

\[
R^a(X, Y)Z = \frac{(1-\alpha)^2}{4} \text{Jac}_m(X, Y, Z) + \frac{1-\alpha^2}{4}[Z, [X, Y]_m]_m - [[X, Y]_m, Z]
\]

\[
= R^a(X, Y)Z + \frac{\alpha(\alpha-2)}{4} \text{Jac}_m(X, Y, Z) - \frac{\alpha^2}{4}[Z, [X, Y]_m]_m.
\]

Hence, if \( R^a \equiv 0 \) it follows that \( R^a(X, Y)Z = \frac{\alpha(a-1)}{4} \text{Jac}_m(X, Y, Z) + \frac{\alpha^2}{4}[Z, [X, Y]_m]_m \). Now, for this case Theorem 1.3 also states that \( \alpha = \pm 1 \) and \( M^n \cong G \). Thus it is \( \text{Jac}_m \equiv 0 \) and we have proved that

\[
R^a(X, Y)Z = -\frac{1}{4}[X, Y]_m, Z \implies X, Y, Z \in \mathfrak{m}.
\]

As a simple consequence of the natural reductivity, one can show now that each \( R^a(X_i, X_j)_k \) is a Killing vector field, where \( \{X_1, \ldots, X_n\} \) is the parallel Killing frame. This is the interpretation of [D'A-N] Lem. 3.4 or [AF2] Prop. 2.3 for a naturally reductive manifold.
6. $\nabla^\alpha$-Einstein naturally reductive manifolds with skew-torsion

Let us discuss now the “constancy” of the Ricci tensor $\text{Ric}^\alpha$ associated to the family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ and describe compact naturally reductive manifolds that admit $\nabla^\alpha$-Einstein structures with skew-torsion.

6.1. Curvature of connections with skew-torsion and $\nabla$-Einstein manifolds. We need to recall identities of the Ricci tensor and the scalar curvature of a metric connection $\nabla$ with skew-symmetric torsion $0 \neq T \in \Lambda^2(M^n)$. We limit ourselves only in a few details and for a general picture we refer to [FKS, AF1, A2].

As usual, we write

$$g(\nabla_X Y, Z) = g(\nabla^S_X Y, Z) + \frac{1}{2}T(X, Y, Z),$$

where $\nabla^S$ denotes the Levi-Civita connection of the fixed Riemannian manifold $(M^n, g)$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of $M$. In terms of the co-differential $\delta T$ and the normalized length $\|T\|^2 := (1/6)\sum_{i,j} g(T(e_i, e_j), T(e_i, e_j))$ of $T$, one has the following formulas:

$$\text{Ric}(X, Y) = \text{Ric}^S(X, Y) - \frac{1}{4} \sum_{i=1}^n g(T(e_i, X), T(e_i, Y)) - \frac{1}{2}(\delta^S T)(X, Y), \quad \text{Scal} = \text{Scal}^S - \frac{3}{2}\|T\|^2.$$

The co-differential (with respect to $\nabla$) of a $n$-form $\omega$ on $M$ is given by $\delta \omega := -\sum_i e_i \lrcorner \omega$. For the torsion 3-form it holds that $\delta \nabla T = \delta^S T$. Notice also that the Ricci tensor of $\nabla$ is not necessarily symmetric; it decomposes into a symmetric and antisymmetric part $\text{Ric} = \text{Ric} + \text{Ric}_A$, which are given respectively by

$$\text{Ric}_S(X, Y) := \text{Ric}^S(X, Y) - \frac{1}{4} \sum_{i=1}^n g(T(e_i, X), T(e_i, Y)), \quad \text{Ric}_A(X, Y) := -\frac{1}{2}(\delta^S T)(X, Y).$$

Now, similarly to compact Einstein manifolds, $\nabla$-Einstein manifolds with skew-torsion admit a variational approach based on the functional $(g, T) \mapsto \int_M \left(\text{Scal} - 2\Lambda\right) d\text{vol}_g$, where $\Lambda$ is a cosmological constant. In particular, by [AF2, Thm. 2.1] it is known that critical points of this functional are pairs $(g, T)$ as above, satisfying the equation

$$-\text{Ric}_S + \frac{1}{2}\text{Scal} \cdot g - \Lambda \cdot g = 0.$$

For this reason, one has the following formal definition:

**Definition 6.1.** We call a 4-tuple $(M^n, g, \nabla, T)$ a $\nabla$-Einstein manifold with skew-torsion $T$, or in short, a $\nabla$-Einstein manifold, if the symmetric part of the Ricci tensor of $\nabla$ satisfies the equation $\text{Ric}_S = \frac{\text{Scal}}{n} g$.

Notice that in contrast to the Riemannian case, for a $\nabla$-Einstein manifold the scalar curvature is not necessarily constant; counterexamples appear in [AF5]. In fact, if the torsion $T$ is $\nabla$-parallel then $\delta^\nabla T = 0$ [FKS], and the Ricci tensor becomes symmetric $\text{Ric} = \text{Ric}_S$. If in addition it holds that $\delta\text{Ric}^S = 0$, then the scalar curvature is constant as the classical case of an Einstein manifold. This is the case for any $\nabla$-Einstein manifold $(M, g, \nabla, T)$ with parallel skew-torsion [AF2, Prop.2.7].

**Remark 6.2.** Several examples of $\nabla$-Einstein manifolds with skew-torsion are known and most of them are related to some special metric (and the characteristic connection). This phenomenon ensures the importance of $\nabla$-Einstein structures. For instance, such are the 6-dimensional homogeneous nearly Kähler manifolds $S^6, S^3 \times S^3, \mathbb{C}P^3$ and $F_4$, endowed with the Chern connection [G]. Nearly parallel $G_2$-manifolds in dimension 7, e.g. the Aloff-Wallach spaces $\text{SU}(3)/\mathbb{S}^1$ and the 7-sphere $S^7$, are also $\nabla$-Einstein [FKMS, FLS]. Compact Lie groups which are Einstein as Riemannian manifolds are also $\nabla$-Einstein manifolds. The converse is it also true (see [AF2] and Example [G.7]). It worthy to mention that most of the known $\nabla$-Einstein structures are characterized by the fact that their torsion is $\nabla$-parallel. However, next we will show that one can construct several infinite families of compact homogeneous $\nabla$-Einstein manifolds $(M = G/K, g, \nabla, T)$, not necessarily isotropy irreducible, with both parallel and non-parallel skew-torsion $T \neq 0$.

6.2. The $\nabla^\alpha$-Einstein condition on naturally reductive spaces. Let $(M^n = G/K, g)$ be a connected (not necessarily compact) naturally reductive Riemannian manifold with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. In terms of Section 4 (and similarly with Section 5) we also assume that $\mathfrak{g} = \tilde{\mathfrak{g}} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ and we will use the $\text{Ad}(G)$-invariant extension $Q$ of the (naturally reductive) inner product $\langle . , . \rangle$. We shall denote by $Q_\mathfrak{k}$ the restriction of $Q$ on $\mathfrak{k}$, i.e. $Q_\mathfrak{k}(X, Y) = Q(X_\mathfrak{k}, Y_\mathfrak{k})$ where $X_\mathfrak{k}$ is the $\mathfrak{k}$-component of $X \in \mathfrak{g}$. 


Let us consider the family of $G$-invariant metric connections $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ described in Proposition 4.1. Because $T^\alpha(X,Y,Z) := (T^\alpha(X,Y), Z) = -\alpha([X,Y]_m, Z)$ is totally skew-symmetric, we write
\[
(\nabla^\alpha_X Y, Z) = (\nabla^\alpha_X Y, Z) + \frac{1}{2} T^\alpha(X,Y,Z), \quad \forall \ X, Y, Z \in m.
\]

Associated to $Q_\mathfrak{f}$ and the isotropy representation $\chi_* : \mathfrak{f} \to \mathfrak{so}(m)$, one has the Casimir element which is the linear operator $C_\chi := C_\chi, Q_\mathfrak{f} : m \to m$ defined by
\[
C_\chi := -\sum_{\ell \in \dim \mathfrak{f}} \chi_*(k_\ell) \circ \chi_*(k_\ell'),
\]
where $\{k_\ell, k'_\ell\}$ are dual bases of $\mathfrak{f}$ with respect to $Q_\mathfrak{f}$. Because $Q_\mathfrak{f}$ is not necessarily positive definite, the eigenvalues of $C_\chi$ can be of either sign. If $\chi$ is an irreducible representation, then $C_\chi$ is a scalar operator, i.e. $C_\chi = \text{Cas} \cdot \text{Id}_m$ for some constant $\text{Cas} \in \mathbb{R}$. Hence, if $\{Z_1, \ldots, Z_n\}$ is an $(\ ,\ )$-orthonormal basis of $m$ we get
\[
\langle C_\chi Z_i, Z_i \rangle = \langle \text{Cas} \cdot Z_i, Z_i \rangle = \text{Cas} \cdot \langle Z_i, Z_i \rangle = \text{Cas}.
\]

Consider also the symmetric bilinear map $A$ on $m$, given by $A(X,Y) = \langle C_\chi X, Y \rangle$ for any $X, Y \in m$, and let us denote by $B$ the negative of the restriction of the Killing form of $\mathfrak{g}$ on $m$. Then, the following relations are true (see for example [WZ, Bes, A1])
\[
A(X,Y) = \sum_j Q_\mathfrak{f}([X,Z_j],[Y,Z_j]), \quad B(X,Y) = \sum_i ([X,Z_i]_m, [Y,Z_i]_m) + 2A(X,Y).
\]

So, if $m$ is irreducible it follows that
\[
\text{Cas} = \langle C_\chi Z_i, Z_i \rangle = A(Z_i, Z_i) = \sum_j Q_\mathfrak{f}([Z_i, Z_j], [Z_i, Z_j]).
\]

Now we are able to present the following nice formulas.

**Lemma 6.3.** The Ricci curvature of the naturally reductive Riemannian manifold $(M = G/K, g)$ endowed with the family $\{\nabla^\alpha : \alpha \in \mathbb{R}\}$ of $G$-invariant metric connections presented in Proposition 4.1 is given by
\[
\text{Ric}^\alpha(X,Y) = \frac{1 - \alpha^2}{4} \sum_{i=1}^n ([X,Z_i]_m, [Y,Z_i]_m) + A(X,Y) = \frac{1 - \alpha^2}{4} B(X,Y) + \frac{1 + \alpha^2}{2} A(X,Y).
\]

The corresponding scalar curvature $\text{Scal}^\alpha : M \to \mathbb{R}$ has the form
\[
\text{Scal}^\alpha = \frac{1 - \alpha^2}{4} \sum_{i,j=1}^n \|Z_i, Z_j\|_m^2 \|Z_i, Z_j\|_m^2 + \sum_{i,j} Q_\mathfrak{f}([Z_i, Z_j], [Z_i, Z_j]) = \frac{1 - \alpha^2}{4} \sum_{i,j=1}^n \|Z_i, Z_j\|_m^2 + \sum_i A(Z_i, Z_i),
\]
where $\|Z\| := \sqrt{\langle Z, Z \rangle}$ is the norm of a vector $Z \in m$ with respect to the $\text{Ad}(K)$-invariant inner product $\langle \ , \ \rangle$.

For a moment, notice that
\[
\text{Ric}^g(X,Y) \equiv \text{Ric}^0(X,Y) = \frac{1}{4} \sum_{i=1}^n ([X,Z_i]_m, [Y,Z_i]_m) + A(X,Y) = \frac{1}{4} B(X,Y) + \frac{1}{2} A(X,Y),
\]
which is the classical formula of the Riemannian Ricci tensor of a naturally reductive homogeneous Riemannian manifold $(M = G/K, g)$ with $\mathfrak{g} = \mathfrak{k}$, see [WZ] Prop. 1.9, pp. 569–570 or [Bes] (7.89b). Now, it is also known that $\delta^\alpha T^\alpha = 0$ for any $\alpha \in \mathbb{R}$ (see [A1, A2] or Lemma 7.11 for a similar procedure). Therefore,

**Corollary 6.4.** The Ricci tensor $\text{Ric}^\alpha$ associated to the family $\nabla^\alpha$ is symmetric for any $\alpha \in \mathbb{R}$, i.e. $\text{Ric}^\alpha = \text{Ric}^\alpha_S = \text{Ric}^0 - \frac{1}{2} S^\alpha$, where $S^\alpha$ is the symmetric tensor defined by $S^\alpha(X,Y) = \sum_i (T^\alpha(Z_i, X), T^\alpha(Z_i, Y))$. In full details
\[
S^\alpha(X,Y) = \alpha^2 \sum_i ([X,Z_i]_m, [Y,Z_i]_m) = \alpha^2 \left\{ B(X,Y) - 2A(X,Y) \right\}.
\]

**Remark 6.5.** We observe that the connections given for $\alpha = \pm 1$ have some common geometric features. For example, they have identical Ricci tensor $\text{Ric}^{\pm 1}(X,Y) = A(X,Y)$ and symmetric tensor $S^{\pm 1}(X,Y) = B(X,Y) - 2A(X,Y)$. Notice however that $\nabla^{-1} T^{-1} = 0$ does not imply $\nabla^{-1} T^{-1}$. Let us exam now the $\nabla^\alpha$-Einstein condition on $(M^n = G/K, g)$, i.e. the equation
\[
(6.2) \quad \text{Ric}^0(X,Y) - \frac{1}{4} S^0(X,Y) = \frac{\text{Scal}}{n}(X,Y) \iff \frac{(1 - \alpha^2)}{4} B(X,Y) + \frac{1 + \alpha^2}{2} A(X,Y) = \frac{\text{Scal}}{n}(X,Y).
\]
Invariant Connections with Skew-Torsion and ∇-Einstein Naturally Reductive Manifolds

Remark 6.6. If the naturally reductive manifold \((M^n = G/K, g)\) \((n \geq 3)\) is Einstein with Einstein constant \(c\), then \(M\) is \(\nabla^\alpha\)-Einstein with skew-torsion if and only if the symmetric tensor \(S^\alpha\) is such that

\[ S^\alpha(X, Y) = \frac{4(nc - \text{Scal})}{n}(X, Y), \quad \forall \, X, Y \in \mathfrak{m}. \]

Conversely, if \(M\) is \(\nabla^\alpha\)-Einstein and the symmetric tensor \(S^\alpha\) satisfies the given relation for some \(c \in \mathbb{R}\), then \(M\) is Einstein, i.e. \(\text{Ric}^g = cg\). Therefore, for 4-tuples \((M^n, g, \nabla, T)\) which have symmetric tensor \(S\) “proportional” to the metric tensor, the notions of \(\nabla\)-Einstein and Einstein manifolds become equivalent. Here, by the term proportional we mean that there is a smooth function on \(M\) such that \(S = f(x)g\) for any \(x \in M\). This function depends on the scalar curvature of \(M\) with respect to \(\nabla\), so it is not necessarily constant. However, if the Einstein manifold \((M, g, \nabla, T)\) is such that \(\nabla T = 0\), then the Riemannian scalar curvature is constant \(\text{Scal}^g = \text{tr} \text{Ric}^g = c \cdot \text{tr}(g) = c \cdot n\) and since \(\nabla T = 0\) the same is true for the length of \(T\). Then, the formula \(\text{Scal} = \text{Scal}^g - \frac{4}{3n}||T||^2\) implies that \(f : M \to \mathbb{R}\) is constant. Usually, 3-forms \(T\) with associated symmetric tensor \(S\) being a multiple of the metric tensor, are called of \(Einstein\) type, see [AFg] for details.

Example 6.7. (see also [AFc Section 2.3.2.]) An example that related with Remark 6.6 is a compact connected Lie group \(M^n \cong G\) with a bi-invariant metric \(g\); then the \(\nabla\)-Einstein condition is equivalent to the classical Einstein condition. Indeed, in this case \(A\) is identically equal to zero, thus the Ricci tensor associated to \(\nabla^\alpha\) is proportional to Riemannian Ricci curvature. For simplicity, let us use the family \(\nabla^\alpha_{\lambda} Y = \eta^\alpha(X, Y) := ((1 - \alpha)/2)[X, Y]\). We compute that

\[ S^\alpha(X, Y) = \alpha^2([X, Z_i], [Y, Z_i]), \quad \text{Ric}^\alpha(X, Y) = \frac{1 - \alpha^2}{4} \sum \langle [X, Z_i], [Y, Z_i] \rangle, \]

or in other words \(\text{Ric}^\alpha = (1 - \alpha^2) \text{Ric}^g\), where \(\text{Ric}^g \equiv \text{Ric}^g\) is the Riemannian Ricci curvature. Moreover, it is \(\text{Scal}^\alpha = \frac{1 - \alpha^2}{n} \sum \langle [Z_i, Z_j] \rangle\) and \(||T^\alpha||^2 = \frac{\alpha^2}{n^2} \sum \langle [Z_i, Z_j] \rangle\). Hence, if \(g\) is a bi-invariant Einstein metric with Einstein constant \(c\), then \(G\) is \(\nabla^\alpha\)-Einstein with (constant) scalar curvature \(\text{Scal}^\alpha = n(1 - \alpha^2)c\) and torsion \(T^\alpha\), such that \(6||T^\alpha||^4 = 4n\alpha^2\). Conversely, if \(M^n = G\) is \(\nabla^\alpha\)-Einstein for \(\alpha \neq \pm 1\), then its scalar curvature \(\text{Scal}^\alpha\) is constant (since \(\nabla T^\alpha = 0\) for any \(\alpha \in \mathbb{R}\)) and the bi-invariant metric \(g\) is Einstein with Einstein constant \(c = \frac{\text{Scal}^\alpha}{n(1 - \alpha^2)}\). For \(\alpha = \pm 1\) we know that \(G\) becomes \(\nabla^\pm\)-flat, in particular it is \(\text{Ric}^\pm\)-flat, i.e. \(\text{Ric}^\pm = 0\) and thus it is trivially a \(\nabla^\pm\)-Einstein manifold [AFg]. When \(G\) is simple the Killing metric \(g_B = -B\) is always Einstein with \(c = 1/4\) [WZ]; hence any compact simple Lie group \(G\) is a \(\nabla^\alpha\)-Einstein manifold with constant scalar curvature \(\text{Scal}^\alpha = n(1 - \alpha^2)/4\) and skew-torsion \(T^\alpha\) such that \(||T^\alpha||^2 = n\alpha^2/6\). In fact, in the simple case the \(\nabla^\pm\)-Einstein structures occur also by applying Theorem [1.4].

We treat now the compact isotropy irreducible case with goal to describe specific solutions of the \(\nabla^\alpha\)-Einstein condition and prove Theorem [1.4]. Let us mention once more that Proposition [1.2] forces us to exclude the (trivial) symmetric spaces of Type I.

Proof of Theorem [1.4] By assumption, the \(K\)-representation \(\mathfrak{m}\) is irreducible and Schur’s lemma ensures that \(\langle \cdot, \cdot \rangle\) is the unique \(G\)-invariant (Einstein) metric on \(M\). For the same reason, the Casimir element \(C_X \equiv C_X, \mathfrak{g}_K : \mathfrak{m} \to \mathfrak{m}\) is such that \(C_X = \text{Cas} \cdot \text{Id}_\mathfrak{m}\) with \(\text{Cas} = A(Z_i, Z_i)\) (not necessarily positive), where \(A(X, Y) = \langle C_X, Y \rangle\). Now, by Lemma [6.4] we see that the \(\nabla^\alpha\)-Einstein condition [6.2] takes the form:

\[ \frac{1 - \alpha^2}{4} B(X, Y) + \frac{1 + \alpha^2}{2} A(X, Y) = \left\{ \frac{(1 - \alpha^2)}{n} \sum \langle [Z_i, Z_j] \rangle \right\} \left\{ \frac{4n}{\text{Scal}} \sum A(Z_i, Z_i) \right\} (X, Y). \]

Looking for \(\nabla^\alpha\)-Einstein structures with skew-torsion for the values \(\alpha = \pm 1\), this formula reduces to

\[ A(X, Y) = \frac{\sum A(Z_i, Z_i)}{n} (X, Y) = \frac{\text{Cas}}{n} (X, Y) = \langle C_X, X \rangle = \langle C_X, Y \rangle, \]

which is an identity, by the definition of \(A\). This proves our claim.

Corollary 6.8. Any compact non-symmetric isotropy irreducible normal homogeneous Riemannian manifold is a \(\nabla^\alpha\)-Einstein manifold with skew-torsion for the values \(\alpha = \pm 1\).

Remark 6.9. A non-compact isotropy irreducible homogeneous Riemannian manifold is necessarily symmetric, see [Bus] Prop. 7.46. In this case, the form of the family \(\nabla^\alpha\) and the classification of non-compact irreducible symmetric spaces, restricts the \(\nabla^\alpha\)-Einstein condition to be valid only for non-compact simple Lie groups, e.g. \(\text{Sl}(2, \mathbb{C})\). The interesting non-compact case is beyond the scope of this paper and it will be treated separately.
Remark 6.10. Let \((M = G/K, g_B)\) be a compact normal homogeneous Riemannian manifold and assume that \(\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r\) is a decomposition of \(\mathfrak{m} = T_0M\) into \(r\) irreducible \(K\)-submodules with \(\mathfrak{m}_i \neq \mathfrak{m}_j\) for any \(i \neq j\). Let \(\Lambda^0 : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})\) be the Nomizu map associated to the Levi-Civita connection on \(M\). Then, the linear function \(\Lambda^0 \cdot \Lambda^0 \) induces a family of \(G\)-invariant metric connections with skew-torsion \(T^\alpha = \alpha \cdot T^c\). Thus, Theorem 3.14 can be appropriately modified under the assumption that the Killing metric \(g_B\) is Einstein (which similarly is given terms of the Casimir constants \(C_1, \ldots, C_r\) associated to irreducible submodules, see [WZ]). However, since the explicit expression of \(\Lambda^0 \) depends on the decomposition of \(\mathfrak{m}\), in general, this is all that one can say for \(\nabla^\alpha\)-Einstein structures. Next we will describe this interesting situation for a certain class of normal homogeneous Riemannian manifolds with two isotropy summands.

7. A CLASS OF COMPACT HOMOGENEOUS RIEMANNIAN MANIFOLDS WITH TWO ISOTROPY SUMMANDS

7.1. Homogeneous spaces with two isotropy summands. We treat now compact connected homogeneous Riemannian manifolds \((M = G/K, g)\) whose isotropy representation \(\chi : K \to \mathfrak{so}(\mathfrak{m})\) decomposes into two (non-trivial) inequivalent and irreducible \(K\)-submodules satisfying \([1,1]\). Without loss of generality we assume that the compact Lie group \(G\) is connected and semi-simple and that \(K\) is connected, see [Bes, WZ]. We consider the 1-parameter family of \(G\)-invariant Riemannian metrics on \(M = G/K\), given by

\[
 g_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2t \cdot B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}, \quad t \in \mathbb{R},
\]

where \(B\) denotes the negative of the Killing form of \(g\). It follows that any \(G\)-invariant Riemannian metric on \(M = G/K\) is a multiple of \(g_t\). Notice that the value \(t = 1/2\) defines the Killing metric \(g_{1/2} = g_B = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + B|_{\mathfrak{m}_2 \times \mathfrak{m}_2}\); hence \((M = G/K, g_{1/2})\) is a normal homogeneous Riemannian manifold.

Remark 7.1. There is a natural construction that gives rise to compact homogeneous spaces with two isotropy summands satisfying \([1,1]\). Consider a semi-simple Lie algebra \(\mathfrak{g}\) and assume that the pairs \((\mathfrak{g}, \mathfrak{t} \oplus \mathfrak{m}_2)\) and \((\mathfrak{t} \oplus \mathfrak{m}_2, \mathfrak{t})\) are orthogonal symmetric pairs such that \(\mathfrak{m}_1\) be an orthogonal complement of \(\mathfrak{t} \oplus \mathfrak{m}_2\) in \(\mathfrak{g}\), with respect to the Killing form of \(\mathfrak{g}\). Then, by setting \(\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2\) one can easily verify the inclusions given by \([1,1]\). In this way we obtain a Riemannian submersion \(G/K \to U\) where \(U\) is the connected Lie group generated by the Lie algebra \(\mathfrak{u} = \mathfrak{t} \oplus \mathfrak{m}_2\), and both (effective) quotients \(U/K\) and \(G/U\) are symmetric spaces. If \(\mathfrak{p}\) denotes an orthogonal complement of \(\mathfrak{u}\) in \(\mathfrak{g}\), i.e. \(\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}\), then we may identify \(\mathfrak{p} = T_0G/U = \mathfrak{m}_1\) and \(\mathfrak{m} = T_0G/K = \mathfrak{m}_1 \oplus \mathfrak{m}_2\). Such fibrations occur for example when \(M = G/K\) is a (generalized) flag manifold with two isotropy summands, and then the (effective) fiber \(U/K\) is a compact Hermitian symmetric space.

We want now to characterize the \(G\)-invariant metric connections of \(M = G/K\) which have totally antisymmetric torsion. It turns out that this is possible only for the Killing metric and under further assumptions.

Theorem 7.2. Let \(\nabla\) be a \(G\)-invariant metric connection of the homogeneous Riemannian manifold \((M = G/K, g_t, g_t)\) with non-trivial torsion \(T \neq 0\). Then, \(T\) is totally skew-symmetric if and only if \(t = 1/2\) and \(\Lambda(X)X = 0\) for any \(X, Y \in \mathfrak{m}\), where \(\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})\) denotes the associated Nomizu map.

Proof. The tensor field \(T(X, Y, Z) = g_t(T(X, Y), Z)\) which occurs from \(T\) by contraction with \(g_t\) is already skew-symmetric with respect to \(X, Y\). Set \(T(X, Y, Z) := T(X, Y, Z) + T(X, Z, Y)\). Then, the condition \(T \in \Lambda^3(\mathfrak{m})\) is equivalent to say that \(T(X, Y, Z) = 0\), for any \(X, Y, Z \in \mathfrak{m}\). An easy computation shows that

\[
 T(X, Y, Z) = g_t(\Lambda(X)Y, Z) - g_t(\Lambda(Y)X, Z) - g_t([X, Y]|_{\mathfrak{m}}\cdot Z) + g_t(\Lambda(X)Z, Y) - g_t(\Lambda(Z)X, Y) - g_t([X, Z]|_{\mathfrak{m}}\cdot Y),
\]

for any \(X, Y, Z \in \mathfrak{m}\). Since \(\Lambda(X) \in \mathfrak{so}(\mathfrak{m})\) for all \(X \in \mathfrak{m}\), we see that the terms \((\alpha'), (\beta')\) and \((\gamma')\) cancel one another: \(g_t(\Lambda(X)Y, Z) + g_t(\Lambda(Y)Z, Y) - g_t(\Lambda(X)Z, Y) + g_t(\Lambda(Z)X, Y) = 0\). For the same reason it is

\[
 -(\beta') - (\gamma') = -g_t(\Lambda(Y)X, Z) - g_t(\Lambda(Z)X, Y) = g_t(\Lambda(Y)Z, X) + g_t(\Lambda(Z)Y, X) = g_t(\Lambda(Y)Z + \Lambda(Z)Y, X).\]

Hence, the equation \(T(X, Y, Z) = 0\) becomes equivalent to

\[
 g_t(\Lambda(Y)Z + \Lambda(Z)Y, X) - g_t([X, Y]|_{\mathfrak{m}}\cdot Z) - g_t([X, Z]|_{\mathfrak{m}}\cdot Y) = 0, \quad \forall X, Y, Z \in \mathfrak{m}.
\]

By using this formula one needs to examine each possible case separately. Consider for example some non-zero vectors \(X, Y \in \mathfrak{m}_1\) and \(Z \in \mathfrak{m}_2\). Then

\[
 -(\beta) - (\gamma) = g_t([Y, X]|_{\mathfrak{m}_2}\cdot Z) + g_t([Z, X]|_{\mathfrak{m}_2}, Y) = 2tB([Y, X]|_{\mathfrak{m}_2}, Z) + B([Z, X]|_{\mathfrak{m}_2}, Y).
\]

\[
 = -2tB([Y, Z], Z) - B([Z, Y], X) = -(2t - 1)B([Y, Z], X) = -g_t((2t - 1)[Y, Z], X).
\]
Hence the skew-symmetry of $T$ reduces to the equation $g_t(A(Y)Z + A(Z)Y - (2t - 1)[Y, Z], X) = 0$, for any $X \in m_1, Y \in m_1$ and $Z \in m_2$, which means that $A(Y)Z + A(Z)Y = (2t - 1)[Y, Z]$, for all $Y \in m_1, Z \in m_2$. Let now $X \in m_2, Y \in m_1$ and $Z \in m_2$. Then we get $A(Y)Z + A(Z)Y = 0$, for any $Y \in m_1, Z \in m_2$, and this gives rise to the following system of equations

$$\left\{ A(Y)Z + A(Z)Y = (2t - 1)[Y, Z], \quad A(Y)Z + A(Z)Y = 0, \quad \forall Y \in m_1, Z \in m_2 \right\}.$$ 

Thus, it must be $t = 1/2$ and $A(Y)Z + A(Z)Y = 0$, for all $Y \in m_1, Z \in m_2$. One can obtain the same result by comparing the cases $X \in m_1, Y \in m_2, Z \in m_1$ and $X, Y, Z \in m_1$, respectively, i.e. $t = 1/2$ and $A(Y)Z + A(Z)Y = 0$, for any $Y \in m_2, Z \in m_1$. Similar are treated the other cases. The converse direction follows by Lemma 2.2 since for $t = 1/2$ we obtain the Killing metric $g_{1/2}$ which is naturally reductive.

**Proposition 7.3.** (BFGK Lem. 10, p. 141) The Nomizu map $\Lambda_t : m \to so(m)$ associated to the Levi-Civita connection $\nabla^{\xi}(\equiv \nabla^{g_t})$ $(t > 0)$ of the homogeneous Riemannian manifold $(M = G/K, m_1 \oplus m_2, g_t)$ is defined by the following relations

$$\begin{align*}
\Lambda_t(m_1)m_1 &= (1/2)[m_1, m_1]m_2, \\
\Lambda_t(m_2)m_1 &= (1 - t)[m_2, m_1], \\
\Lambda_t(m_2)m_2 &= t[m_1, m_2], \\
\Lambda_t(m_2)m_2 &= (1 - t)[m_2, m_1].
\end{align*}$$

**Proof.** By the definition of the Riemannian connection (see [KN]) we obtain that $g_t(A_t(X)Y, Z) = (1/2)\{g_t([X, Y]m_1, Z) + g_t([X, Z]m_1, Y) + g_t([Y, Z]m_1, X)\}$, for any $X, Y, Z \in m$, and the result easily follows. For a slightly different argument, see [BFGK].

**7.2. A new family of G-invariant metric connections with skew-torsion.** Let us introduce now a new family of G-invariant metric connections on $M = G/K$ for which the Killing metric has totally skew-symmetric torsion. We use the Levi-Civita connection $\nabla^t$ corresponding to the linear map $A_t : m \to so(m)$ and for a new parameter $s \in \mathbb{R}$ we consider the map $A^m_{s,t} = \{A_{s,t} : m \to so(m)\}$, which is given by $A_{s,t}(X) := s \cdot A_t(X)$, i.e.

$$\begin{align*}
A_{s,t}(m_1)m_1 &= (s/2)[m_1, m_1]m_2, \\
A_{s,t}(m_2)m_1 &= s(1 - t)[m_2, m_1], \\
A_{s,t}(m_1)m_2 &= st[m_1, m_2], \\
A_{s,t}(m_2)m_2 &= s(1 - t)[m_2, m_1].
\end{align*}$$

(7.2)

Obviously, $A_{s,t}$ is an Ad$(K)$-equivariant linear map such that $A_{s,t}(X) \in so(m)$ for any $X \in m$. Thus, it induces a 2-parameter family of G-invariant metric connections $\{\nabla^{s,t} : s \in \mathbb{R}, t \in \mathbb{R}_+\}$ defined as follows:

$$\begin{align*}
\nabla^{s,t}X &:= \nabla^0X + A_{s,t}(X)Y = \nabla^0X + s \cdot A_t(Y), \\
\nabla^{s,t}Y &:= \nabla^0Y + A_{s,t}(X)Y = \nabla^0Y + s \cdot A_t(Y), \\
\nabla^{s,t}Z &:= \nabla^0Z + A_{s,t}(X)Y = \nabla^0Z + s \cdot A_t(Y).
\end{align*}$$

(7.3)

The family $\nabla^{s,t}$ can be thought of as a line in the space of invariant (affine) metric connections of $M = G/K$ which joins the canonical connection $\nabla^0 \equiv \nabla^c (s = 0)$ and the torsion-free Riemannian connection $\nabla^t \equiv \nabla^{1,t} (s = 1)$. By applying (2.1) we also obtain that

**Corollary 7.4.** The torsion $T^{s,t}$ of the G-invariant metric connection $\nabla^{s,t}$ is given as follows:

$$\begin{align*}
T^{s,t}(m_1, m_1) &= (s - 1)[m_1, m_1]m_2, \\
T^{s,t}(m_2, m_1) &= (s - 1)[m_2, m_1], \\
T^{s,t}(m_1, m_2) &= (s - 1)[m_1, m_2], \\
T^{s,t}(m_2, m_2) &= 0.
\end{align*}$$

(7.4)

Notice that similarly to $A_{s,t}$ (or $A_t$), the torsion $T^{s,t}$ has zero pure $m_2$-part since $[m_2, m_2] \subseteq \mathfrak{k}$. Consequently, the associated 3-tensor $T^{s,t}(X, Y, Z) := g_t(T^{s,t}(X, Y), Z)$ is such that

$$\begin{align*}
T^{s,t}(m_1, m_1, m_2) &= 2t(s - 1)B([m_1, m_1]m_2, m_2), \\
T^{s,t}(m_2, m_1, m_1) &= (s - 1)B([m_2, m_1], m_1), \\
T^{s,t}(m_1, m_2, m_2) &= (s - 1)B([m_1, m_2], m_1), \\
T^{s,t}(m_2, m_2, m_2) &= 0,
\end{align*}$$

and all the other combinations are zero.

In general, $T^{s,t}(X, Y, Z)$ is not a 3-form; Theorem 7.2 states that this is possible only for $t = 1/2$ and under the further assumption $A_{s,t}(X)X = 0$ for any $X \in m$. By writing $m \ni X = \mathfrak{v} + Z$ with $V \in m_1, Z \in m_2$ and $0 \neq [V, Z] \in m_1$, it is easy to see that the latter condition is equivalent to $s(2t - 1)[V, Z] = 0$. Therefore, for $t = 1/2$ it is always satisfied, i.e. $A_{s,1/2}(X)X = 0$ for any $X \in m$, as required. In fact, by 7.2 and for non-zero vectors $X \in m_1, Y \in m_1$ and $Z \in m_2$ we see that

$$T^{s,t}(X, Y, Z) + T^{s,t}(X, Z, Y) = 0 \iff (s - 1)(2t - 1)B([X, Y]m_2, Z) = 0.$$ 

Consequently, there are two possibilities: $s = 1$ or $t = 1/2$. The first fails, since it corresponds to the Riemannian connection. Thus, the only possible value is $t = 1/2$ which corresponds to the Killing metric. Similarly are treated the other cases. Hence, we have proved the following correspondence.

**Proposition 7.5.** For any $s \neq 1$ it holds that $0 \neq T^{s,t}(X, Y, Z) \in \Lambda^3(m)$ for any $X, Y, Z \in m \iff t = 1/2$. 

7.3. The Ricci tensor and the scalar curvature associated to the family $\nabla^s,t$. We want now to describe the Ricci tensor and the scalar curvature of $(M = G/K, m_1 \oplus m_2, g_t, \nabla^s,t)$. The Ricci tensor is given by Ric$^{s,t}(X, Y) = \sum_{i=1}^{n} g_t(R^{s,t}(X, Z_i)Z_i, Y)$, where $R^{s,t}(X, Y)$ is the associated curvature tensor and $\{Z_1, \ldots, Z_n\}$ denotes a $g_t$-orthonormal basis of $m = T_cG/K$. By $B_t := B|_{t \times t}$ we denote the restriction of $B$ on the Lie subalgebra $\mathfrak{k}$. Now we need the following useful lemma.

**Lemma 7.6.** Consider the homogeneous Riemannian manifold $(M = G/K, m_1 \oplus m_2, g_t, \nabla^s,t)$.

a) Let $X \in m_1$ and $Y \in m_1$. Then

$$g_t(R^{s,t}(X, Z)Z, Y) = \frac{(s^2t - 2s + 2st)}{2} B([X, Z]_{m_2}, Z), \quad g_t(R^{s,t}(X, Z)Z, Y) - B_t([X, Z], [Z, Y]), \quad \text{if } Z \in m_1,$$

$$g_t(R^{s,t}(X, Z)Z, Y) = st(s - st - 1) B([[X, Z], [Z, Y])], \quad \text{if } Z \in m_2. $$

b) Let $X \in m_1$ and $Y \in m_2$, or $X \in m_2$ and $Y \in m_1$. Then $g_t(R^{s,t}(X, Z)Z, Y) = 0$, for any $Z \in m$.

c) Let $X \in m_2$ and $Y \in m_2$. Then

$$g_t(R^{s,t}(X, Z)Z, Y) = st(s - st - 1) B([[X, Z], [Z, Y])], \quad \text{if } Z \in m_1,$$

$$g_t(R^{s,t}(X, Z)Z, Y) = -2tB_t([X, Z], [Z, Y]), \quad \text{if } Z \in m_2.$$

**Proof.** By (2.1) it follows that

$$g_t(R^{s,t}(X, Z)Z, Y) = g_t(-\Delta_{s,t}(Z)\Lambda_{s,t}(X)Z, Y) + g_t(-\Delta_{s,t}([X, Z]_m)Z, Y) + g_t(-\text{ad}([X, Z]_t)Z, Y).$$

Then, the given formulas are an immediate application of this identity. Let us describe this first case in details. Assume that $0 \neq X, Y \in m_1$. Then, by using (2.1) and (2.2) we compute

$$g_t(R^{s,t}(X, Z)Z, Y) = g_t\left(-\frac{s^2t}{2}[X, Z]_{m_2}, Y\right) + g_t(-s(1 - t)[[X, Z]_{m_2}, Z], Y) + g_t(-[[X, Z]_t, Z], Y)$$

$$= g_t\left(\frac{s^2t - 2s(1 - t)}{2}[X, Z]_{m_2}, Z - [[X, Z]_t, Z], Y\right)$$

$$= \frac{s^2t - 2s(1 - t)}{2} B([X, Z]_{m_2}, Z), \quad \text{if } Z \in m_1,$$

and the claim easily follows, since $B([[X, Z]_t, Z], Y) = B_t([[X, Z], [Z, Y])].$ Assume now that $0 \neq X, Y \in m_1$ and $0 \neq Z \in m_2$. Then we always have $[X, Z] \in m_1$, so $\text{ad}([X, Z]_t)Z = 0$. Thus

$$g_t(R^{s,t}(X, Z)Z, Y) = g_t(-s^2t(1 - t)[X, Z]_t, Y) + g_t(-st[X, Z]_t, Y)$$

$$= g_t\left(s^2t(1 - t) - st\right)[[X, Z], Y] = st(s - st - 1) B([[X, Z], [Z, Y]),$$

which is the formula stated above. The other cases are treated similarly. \hfill \square

From now on set $d_i := \dim m_i \quad (i = 1, 2)$ and fix a $B$-orthonormal basis of $m$ adapted to the decomposition $m = m_1 \oplus m_2$, say $m_1 = \text{span}_R\{X_i\}_{i=1}^{d_1}$, $m_2 = \text{span}_R\{Y_k\}_{k=1}^{d_2}$, with $B(X_i, Y_k) = 0$ and $B(X_i, X_j) = \delta_{i,j}$, $B(Y_k, Y_l) = \delta_{k,l}$, respectively. The associated $g_t$-orthonormal bases are of the form

$$m_1 = \text{span}_R\{V_i := X_i : 1 \leq i \leq d_1\}, \quad m_2 = \text{span}_R\{W_k := Y_k/\sqrt{2t} : 1 \leq k \leq d_2\}.$$

It is useful to express the splitting $m = m_1 \oplus m_2$ by $\chi_\xi = \chi_1^\xi \oplus \chi_2^\xi$, where the sub-representations $\chi_\xi : \mathfrak{k} \to \mathfrak{so}(m_i)$ are given by $\chi_\xi(Y) := \text{ad}(Y)|_{m_i}$, for any $Y \in \mathfrak{k}$. Then, for the Casimir element $C_{X_i, Y_j} : m \to m$ we write

$$C_{X_i} = C_{X_i} \oplus C_{X_i}^*, \quad \text{with } C_{X_i} : m_i \to m_i \quad (i = 1, 2),$$

defined by $C_{X_i} = -\sum_{\alpha=1}^{\dim \mathfrak{k}} \chi_\alpha^i(k_\alpha) \circ \chi_\alpha^i(k_\alpha)$ and $C_{X_i}^* = -\sum_{\alpha=1}^{\dim \mathfrak{k}} \chi_\alpha^i(k_\alpha) \circ \chi_\alpha^i(k_\alpha)^\vee$, respectively. Here, as usual $\{k_\alpha, k_\alpha^\vee\}$ are dual bases of $\mathfrak{k}$ with respect to $B_{\mathfrak{k}}$. Because $B$ is the Killing metric, it is necessarily $C_{X_i} = \text{Cas}_i \cdot I_{m_i}$, with $\text{Cas}_i = B(\lambda_i, \lambda_i + \delta) > 0$, where $\lambda_i$ is the dominant weight of the $K$-module $m_i$ and $\delta$ denotes the half of the sum of positive roots of $\mathfrak{k} \otimes \mathbb{C}$. In other words (see [WZ] or [B85, p. 197])

$$\text{Cas}_1 = B(C_{X_1}X_j, X_j) = \sum_{i=1}^{d_1} B_t([X_j, X_i], [X_j, X_i]) = A_1(X_j, X_j),$$

$$\text{Cas}_2 = B(C_{X_2}Y_i, Y_i) = \sum_{k=1}^{d_2} B_t([Y_i, Y_k], [Y_i, Y_k]) = A_2(Y_i, Y_i),$$

where $A_1$ and $A_2$ are the scalar curvature associated to the family $\nabla^s,t$. We want now to describe the Ricci tensor and the scalar curvature of $(M = G/K, m_1 \oplus m_2, g_t, \nabla^s,t)$. The Ricci tensor is given by Ric$^{s,t}(X, Y) = \sum_{i=1}^{n} g_t(R^{s,t}(X, Z_i)Z_i, Y)$, where $R^{s,t}(X, Y)$ is the associated curvature tensor and $\{Z_1, \ldots, Z_n\}$ denotes a $g_t$-orthonormal basis of $m = T_cG/K$. By $B_t := B|_{t \times t}$ we denote the restriction of $B$ on the Lie subalgebra $\mathfrak{k}$. Now we need the following useful lemma.

**Lemma 7.6.** Consider the homogeneous Riemannian manifold $(M = G/K, m_1 \oplus m_2, g_t, \nabla^s,t)$.

a) Let $X \in m_1$ and $Y \in m_1$. Then

$$g_t(R^{s,t}(X, Z)Z, Y) = \frac{(s^2t - 2s + 2st)}{2} B([X, Z]_{m_2}, Z), \quad g_t(R^{s,t}(X, Z)Z, Y) - B_t([X, Z], [Z, Y]), \quad \text{if } Z \in m_1,$$

$$g_t(R^{s,t}(X, Z)Z, Y) = st(s - st - 1) B([[X, Z], [Z, Y])], \quad \text{if } Z \in m_2. $$

b) Let $X \in m_1$ and $Y \in m_2$, or $X \in m_2$ and $Y \in m_1$. Then $g_t(R^{s,t}(X, Z)Z, Y) = 0$, for any $Z \in m$.

c) Let $X \in m_2$ and $Y \in m_2$. Then

$$g_t(R^{s,t}(X, Z)Z, Y) = st(s - st - 1) B([[X, Z], [Z, Y])], \quad \text{if } Z \in m_1,$$

$$g_t(R^{s,t}(X, Z)Z, Y) = -2tB_t([X, Z], [Z, Y]), \quad \text{if } Z \in m_2.$$
where similarly with paragraph 6.2 we define symmetric bilinear maps $A_i$ on $m_i$ $(i = 1, 2)$ by

$$A_1(X, Y) := B(C_{X_1}, X, Y) = \sum_{i=1}^{d_1} B_t([X, X_i], [Y, X_i]), \quad X, Y \in m_1,$$

$$A_2(X, Y) := B(C_{X_2}, X, Y) = \sum_{k=1}^{d_2} B_t([X, Y_k], [Y, Y_k]), \quad X, Y \in m_2.$$

Theorem 7.7. The Ricci tensor $\text{Ric}^{s,t}$ of the homogeneous Riemannian manifold $(M = G/K, m_1 \oplus m_2, g_t)$ endowed with the family of $G$-invariant metric connections $\{\nabla^{s,t} : s \in \mathbb{R}, t \in \mathbb{R}^+\}$, is expressed as follows:

(a) Let $X, Y \in m_1$. Then

$$\text{Ric}^{s,t}(X, Y) = \sum_{i=1}^{d_1} \frac{(s^2t - 2s + 2st)}{2} B([[X, X_i], X_i], Y) + \sum_{k=1}^{d_2} \frac{(s^2t - s^2 - s)}{2} B([[X, Y_k], Y_k], Y) + A_1(X, Y).$$

(b) Let $X \in m_1$, $Y \in m_2$, or $X \in m_2$, $Y \in m_1$. Then $\text{Ric}^{s,t}(X, Y) = 0$.

(c) Let $X, Y \in m_2$. Then

$$\text{Ric}^{s,t}(X, Y) = \sum_{i=1}^{d_1} (s^2t - s^2t^2 - st) B([[X, X_i], X_i], m_2, Y) + A_2(X, Y).$$

Proof. Given some $g_t$-orthonormal vectors $V_i \in m_1, W_k \in m_2$, it is $R^{s,t}(X, V_i)V_i = R^{s,t}(X, X_i)X_i$ and $R^{s,t}(X, W_k)W_k = (1/2t)R^{s,t}(X, Y_k)Y_k$, respectively. Thus, for any $X, Y \in m$ we compute $\text{Ric}^{s,t}(X, Y) = \sum_{i=1}^{d_1} g_t(R^{s,t}(X, X_i)X_i, Y) + (1/2t) \sum_{k=1}^{d_2} g_t(R^{s,t}(X, Y_k)Y_k, Y)$ and the result occurs by Lemma 7.6.

Corollary 7.8. The scalar curvature $\text{Scal}^{s,t}$ of $(M = G/K, m_1 \oplus m_2, g_t, \nabla^{s,t})$ is the function on $M$ given by

$$\text{Scal}^{s,t} = -\frac{(s^2t - 2s + 2st)}{2} \sum_{i,j} \|[X, X_i]_{m_2}\|^2 - \sum_{i,k} (s^2t - s^2t^2 - s)\|[X, Y_k]\|^2$$

$$+ \sum_{i} \frac{1}{t} A_1(X_i, X_i) + \frac{1}{2t} \sum_{k} A_2(Y_k, Y_k).$$

7.4. The homogeneous Einstein equation. Before the description of the $\nabla^{s,t}$-Einstein condition on $(M = G/K, m_1 \oplus m_2, g_t)$, let us shortly illustrate the traditional homogeneous Einstein equation $\text{Ric}^{s,t} = cg_t$ (where $c \in \mathbb{R}^+$ is the Einstein constant). We need the components $r_1 = \text{Ric}^{1,t}(V_j, V_j)$ and $r_2 = \text{Ric}^{1,t}(W_i, W_i)$ of the Riemannian Ricci tensor, for some $g_t$-orthonormal vectors $V_j \in m_1$ and $W_i \in m_2$, respectively. Because $m_1 \not\cong m_2$ as $K$-representations, it is $\text{Ric}^{1,t}(m_1, m_2) = 0$ and all homogeneous Einstein metrics, if existent, appear as real positive solutions of the equation $r_1 - r_2 = 0$. As a first step, by Theorem 7.7 we obtain that

Corollary 7.9. Let $\{X_i\}^d_{i=1}$ and $\{Y_k\}^d_{k=1}$ be the $B$-orthonormal bases of $m_1$ and $m_2$, respectively. Then, it holds that $\text{Ric}^{s,t}(X_i, Y_k) = 0 = \text{Ric}(Y_k, X_i)$, and

$$\text{Ric}^{s,t}(X_j, X_j) = -\frac{(s^2t - 2s + 2st)}{2} \sum_{i=1}^{d_1} \|[X, X_i]_{m_2}\|^2 - \frac{(s^2t - 2s^2 - s)}{2} \sum_{k=1}^{d_2} \|[X, Y_k]\|^2 + \text{Cas}_1,$$

$$\text{Ric}^{s,t}(Y_i, Y_i) = -st(s - st - 1) \sum_{i=1}^{d_1} \|[Y, X_i]\|^2 + \text{Cas}_2.$$

The Riemannian Ricci tensor occurs for $s = 1$, and then we have $r_1 = \text{Ric}^{1,t}(V_j, V_j) = \text{Ric}^{1,t}(X_j, X_j)$ and $r_2 = \text{Ric}^{1,t}(W_i, W_i) = (1/2t)\text{Ric}^{1,t}(Y_i, Y_i)$, respectively. Thus, and since $t \not= 0$, it is not difficult to see that the homogeneous Einstein equation is given by the quadratic equation $\alpha \cdot t^2 + \beta \cdot t + \gamma = 0$, where

$$\alpha = -3 \sum_i \|[X, X_i]_{m_2}\|^2 + \sum_k \|[X, Y_k]\|^2 - \sum_i \|[Y, X_i]\|^2,$$

$$\beta = 2 \left( \sum_i \|[X, X_i]_{m_2}\|^2 + \text{Cas}_1 \right), \quad \gamma = -\text{Cas}_2.$$

Although the quantities $\beta$ and $\gamma$ have known sign, the sign of $\alpha$ depends in general on the underlying manifold $M = G/K$. We conclude that the number of invariant Einstein metrics $\mathcal{E}(M)$ on $M = G/K$ must be such that $0 \leq \mathcal{E}(M) \leq 2$, see also [Bes, 9.99] and [DK]. If $M = G/K$ is a flag manifold with $m = m_1 \oplus m_2$, homogeneous Einstein metrics have been classified by the author in terms of variational analysis as a part of
his Ph.D thesis [AC2]; any such manifold admits exactly two invariant Einstein metrics, one of them being Kähler. A special member here is the complex projective space \( \mathbb{C}P^{2\ell - 1} = \text{Sp}(\ell)/\text{Sp}(\ell - 1) \times U(1) \) and the classification of invariant Einstein metrics for this coset was for first time described by W. Ziller [Z].

**Example 7.10.** Consider the complex projective space \( \mathbb{C}P^3 = \text{Sp}(2)/\text{Sp}(1) \times U(1) \cong SO(5)/U(2) \). We will use the second presentation and we refer to [BFGK] for a description associated to the first one. For the Lie algebra \( \mathfrak{so}(5) \) we fix a reductive decomposition related to the twistor fibration of \( \mathbb{C}P^3 \) over the 4-sphere \( S^4 \) (see [BFGK]). We shall denote by \( D_{ij} \) the \( (n \times n) \)-matrix having 1 in the \((i, j)\)-entry and zeros elsewhere and we set \( E_{ij} = -D_{ij} + D_{ji} \), with \( 1 \leq i \neq j \leq n \). The matrices \( \{ E_{ij} : i < j \} \) form an orthonormal basis of \( \mathfrak{so}(n) \) with respect to the scalar product \( B' = -(1/2) \text{tr} AB \) (which is such that \( B_{SO(n)} = 2(n - 2)B' \)). By definition, it is \( \mathfrak{so}(5) = \text{span}_\mathbb{R}\{E_{1,2}, \ldots, E_{4,5}\} \) and we set

\[
\mathfrak{e} = \mathfrak{u}(2) = \text{span}_\mathbb{R}\{k_1 := E_{1,2}, k_2 := E_{3,4}, k_3 := (E_{1,3} - E_{2,4})/\sqrt{2}, k_4 := (E_{1,4} + E_{2,3})/\sqrt{2}\} \cong \mathfrak{u}(2).
\]

Notice that \( B'(k_1, k_2) = \delta_{i,j} \), for any \( 1 \leq i, j \leq 4 \). Let \( \mathfrak{m} \) be the invariant \( B' \)-orthogonal complement of \( \mathfrak{u}(2) \) into \( \mathfrak{so}(5) \); an orthonormal basis with respect to \( B' \) is given by the vectors \( e_1 := E_{1,5}, e_2 := E_{2,5}, e_3 := E_{3,5}, e_4 := E_{4,5}, e_5 := (E_{1,3} + E_{2,4})/\sqrt{2} \), and \( e_6 := (E_{1,4} - E_{2,3})/\sqrt{2} \). We set

\[
\mathfrak{m}_1 := \text{span}_\mathbb{R}\{e_1, e_2, e_3, e_4\}, \quad \mathfrak{m}_2 := \text{span}_\mathbb{R}\{e_5, e_6\},
\]

such that \( \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \), and then the inclusions given by (7.1) are true. Up to scaling, any invariant Riemannian metric on \( \mathbb{C}P^3 \) is given by \( g_\alpha = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2B'|_{\mathfrak{m}_2 \times \mathfrak{m}_2} \). For \( X_i = e_1 \) and \( Y_i = e_5 \), the coefficients \( \alpha, \beta, \gamma \) of Einstein equation are given respectively by \( \alpha = -3\sum_{i=1}^4 \|e_1, e_1\|_{m_2}^2 + \sum_{i=5}^6 \|e_1, e_k\|_{m_2}^2 - \sum_{i=1}^4 \|e_5, e_i\|_{m_2}^2 \), \( \beta = 2\left(\sum_{i=1}^4 \|e_1, e_1\|_{m_2}^2 + B'(C_{X_i} e_1, e_1)\right) \) and \( \gamma = -B'(C_{X_5} e_5, e_5) \). We need the Lie brackets \( \text{ad}_{e_i} \), \( e_j = [e_i, e_j] \) \( (1 \leq i, j \leq 6) \) of the base vectors, which we present in the following table.

\[
\begin{array}{cccccc}
\text{ad}_{1,2} & \text{ad}_{2,3} & \text{ad}_{3,4} & \text{ad}_{4,5} & \text{ad}_{5,6} \\
 e_1 & 0 & -k_1 & -E_{1,3} & -(\sqrt{2}/2)e_3 & (\sqrt{2}/2)e_4 \\
 e_2 & k_1 & 0 & -E_{1,3} & -(\sqrt{2}/2)e_4 & -(\sqrt{2}/2)e_3 \\
 e_3 & E_{1,3} & E_{2,3} & 0 & -k_2 & (\sqrt{2}/2)e_1 \\
 e_4 & E_{1,4} & E_{2,4} & k_2 & 0 & (\sqrt{2}/2)e_2 \\
 e_5 & -(\sqrt{2}/2)e_3 & -(\sqrt{2}/2)e_4 & (\sqrt{2}/2)e_1 & (\sqrt{2}/2)e_2 & 0 \\
 e_6 & -(\sqrt{2}/2)e_4 & (\sqrt{2}/2)e_3 & -(\sqrt{2}/2)e_2 & (\sqrt{2}/2)e_1 & -k_1 - k_2
\end{array}
\]

Notice that

\[
E_{1,3}|_{m_2} = (\sqrt{2}/2)e_5, \quad E_{1,4}|_{m_2} = (\sqrt{2}/2)e_6, \quad E_{2,3}|_{m_2} = -(\sqrt{2}/2)e_6, \quad E_{2,4}|_{m_2} = (\sqrt{2}/2)e_5, \quad E_{3,4}|_{m_2} = (\sqrt{2}/2)e_3, \quad E_{3,5}|_{m_2} = -(\sqrt{2}/2)e_3.
\]

Hence we compute \( \gamma = B'(C_{X_5} e_5, e_5) = -2 = B'(C_{X_5} e_1, e_1) \), \( \alpha = -4 \), and \( \beta = 6 \). Therefore on \( \mathbb{C}P^3 \) the Einstein equation \( \alpha \cdot t^2 + \beta \cdot t + \gamma = 0 \) has two positive solutions, namely \( t = 1 \) and \( t = 1/2 \). The first value defines the Kähler-Einstein metric \( g_1 = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + 2B'|_{\mathfrak{m}_2 \times \mathfrak{m}_2} \) and the second one corresponds to the Killing metric \( g_{1/2} = B'|_{\mathfrak{m}_1 \times \mathfrak{m}_1} + B'|_{\mathfrak{m}_2 \times \mathfrak{m}_2} \) which is a homogeneous Einstein metric for \( \mathbb{C}P^3 \), see also [Z] [WZ] [BFGK] [AC2].

### 7.5. \( \nabla^s \frac{s}{2} \)-Einstein Structures.

Proposition 7.3 ensures that for \( s \neq 1 \) and \( t = 1/2 \) the family \( \{ \nabla^s \frac{s}{2} : s \in \mathbb{R} \} \) has non-trivial skew-symmetric torsion \( T^s \frac{s}{2} \in \Lambda^3(m) \). Thus, if \( \nabla^{g_{1/2}} \equiv \nabla^1 \frac{1}{2} \) denotes the Levi-Civita connection on the normal homogeneous Riemannian manifold \( (M = G/K, \mathfrak{m}_1 \oplus \mathfrak{m}_2, g_{1/2}) \), one can write

\[
B(\nabla^s_X Y, Z) = B(\nabla^{g_{1/2}}_X Y, Z) + \frac{1}{2} T^s(X, Y, Z)
\]

This is a connection with skew-torsion of the form \( (1.1) \), which of course is identified with the family \( \nabla^s \frac{s}{2} \) defined by (7.3) for \( t = 1/2 \). Hence, for the Killing metric \( g_{1/2} \equiv B \) it makes sense to examine the existence of a \( \nabla^s \frac{s}{2} \)-Einstein structure with skew-torsion. Because the value \( t = 1/2 \) will be fixed from now on, for simplicity we will write \( \nabla^s \frac{s}{2} \equiv \nabla^s, T^s \frac{s}{2} \equiv T^s \), e.t.c. Let us study first some structural properties of the torsion form \( T^s \in \Lambda^3(m) \). In order to unify our computations it is useful to introduce the following maps

\[
\begin{align*}
\text{Jac}_m : \mathfrak{m}_1 \times \mathfrak{m}_1 \times \mathfrak{m}_1 \rightarrow \mathfrak{m}_1, \quad \text{Jac}_m(X_j, X_r, X_s) & := \mathfrak{g}_{j, r, s}[X_j, [X_r, X_s]]_{m_2}, \\
\text{Jac}_m : \mathfrak{m}_1 \times \mathfrak{m}_1 \times \mathfrak{m}_2 \rightarrow \mathfrak{m}_2, \quad \text{Jac}_m(X_i, X_j, X_k) & := [X_i, [X_j, X_k]]_{m_2} + [X_j, X_k]_{m_2} + [X_k, X_i]_{m_1}, \\
\text{Jac}_m : \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_2 \rightarrow \mathfrak{m}_1, \quad \text{Jac}_m(X_i, Y_k, Y_l) & := \mathfrak{g}_{i, k, l}[X_i, [Y_k, Y_l]], \\
\end{align*}
\]
Here, the vectors \( \{X_i\}_{i=1}^{d_1} \) and \( \{Y_k\}_{k=1}^{d_2} \) stand for the fixed \( B \)-orthonormal bases. Although these trilinear maps are different each other, we use the same notation since in any case their definition is obvious by (1.1). It is easy to see that the mixed Jacobians \( \text{Jac}_m(X_i, X_j, Y_k) \) and \( \text{Jac}_m(X_i, Y_k, Y_l) \) are identically equal to zero. Indeed, by viewing the vectors \( X_i, Y_j, Y_l \in m \) as left-invariant vector fields, we see that \( \text{Jac}_m(Y_k, Y_j, Y_l) = 0 \); because \( B(\text{Jac}_m(X_i, X_j, Y_k), Y_l) = B(\text{Jac}_m(X_l, Y_j, Y_l), X_j) \) it is also \( \text{Jac}_m(X_i, Y_j, Y_k) = 0 \). Without assuming any notation compatible with tensors we shall use also the conventions \( \nabla^s_{\alpha\beta\gamma} := \nabla^s_{\alpha\beta\gamma}(A, B) \) and

\[
\nabla^s_{\alpha\beta\gamma} := (\nabla^s_{\alpha\beta\gamma}(A, B, D) := g_{1/2}(\nabla^s_{\alpha\beta\gamma}(A, B, D) = B(\nabla^s_{\alpha\beta\gamma}(A, B, D)),
\]

for some vectors \( A, B \in m, \beta, C \in m, D \in m \) with \( 1 \leq \alpha, \beta, \gamma, \delta \leq 2 \).

**Lemma 7.11.** The covariant derivative \( \nabla^s T^* : m \times m \to m \) of the torsion form \( T^{*,^{1/2}} \equiv T^* \) is given by

\[
\nabla^s_{111} := (\nabla^s_{X_i, T^*})(X_i, X_j) = \frac{s(s-1)}{2} \text{Jac}_m(X_i, X_j, X_l),
\]

\[
\nabla^s_{112} := (\nabla^s_{Y_k, T^*})(X_i, X_j) = -\frac{s(s-1)}{2}[Y_k, [X_i, X_j]],
\]

\[
\nabla^s_{221} := (\nabla^s_{X_i, T^*})(Y_k, Y_l) = -\frac{s(s-1)}{2}[X_i, [Y_k, Y_l]],
\]

with \( \nabla^s_{112} = -\nabla^s_{121} = \nabla^s_{211}, \nabla^s_{221} = -\nabla^s_{212} = \nabla^s_{122} \) and all the other combinations are zero. On the other hand, the co-differential \( \delta^s T^* \) vanishes for any \( s \in \mathbb{R} \).

**Proof.** For any \( X, Y, Z \in m \) it is \( \nabla^s_{Z^*}(X, Y) := \nabla^s_{Z^*}(T^*(X, Y)) = T^*(\nabla^s_{Z, X}, Y) \) and we obtain the results by computing these three terms, say \((\alpha), (\beta), (\gamma), \) in any case separately. Let us present the first two cases and the other are treated similarly. We use the expression \( \nabla^s_{Z^*} = \nabla^s_{Z^*} + \Lambda_s(Z)(Z) \) for any \( Z, Z' \in m \), where the Nomizu map \( \Lambda_s : m \to \mathfrak{so}(m) \) is determined by \( (2.1) \) for \( t = 1/2 \). Let \( X, Y, Z \in m \) be three non-zero vectors. Then

\[
\begin{align*}
(\alpha) &= \nabla^s_{Z^*}(T^*(X, Y)) = \nabla^s_{Z^*}(T^*(X, Y)) + \frac{s(s-1)}{2}[Z, [X, Y]]_m, \\
(\beta) &= -T^*(\nabla^s_{Z^*}X, Y) - T^*(\Lambda_s(Z)X, Y) = -T^*(\nabla^s_{Z^*}X, Y) - \frac{s(s-1)}{2}[Z, [X, Y]]_m, \\
(\gamma) &= -T^*(X, \nabla^s_{Z^*}Y) - T^*(X, \Lambda_s(Z)Y) = -T^*(X, \nabla^s_{Z^*}Y) - \frac{s(s-1)}{2}[X, [Y, Z]]_m.
\end{align*}
\]

Since \( T^*(X, Y) = -(s-1)T^*(X, Y) \) it follows that

\[
\nabla^s_{Z^*}(T^*(X, Y)) - T^*(\nabla^s_{Z^*}X, Y) - T^*(X, \nabla^s_{Z^*}Y) = -(s-1)(\nabla^s_{Z^*}T^*)/(X, Y),
\]

and then our formula comes true due to the Ambrose-Singer theorem: \( \nabla^s T^* = 0 \). Assume now that \( X, Y \in m_1 \) and \( Z \in m_2 \). In this case, the sum of \((\alpha), (\beta)\) and \((\gamma)\) equals to \( \nabla^s_{Z^*}(X, Y) = \frac{s(s-1)}{2}[X, [Y, Z]]_m \) and the result occurs by the vanishing of \( \text{Jac}_m(X_i, X_j, Y_k) \). Notice that the relations \( \nabla^s_{112} = -\nabla^s_{121} = \nabla^s_{211}, \nabla^s_{221} = -\nabla^s_{212} = \nabla^s_{122} \), mean that \( \nabla^s_{Z^*}X_i, X_j = -\nabla^s_{Z^*}X_l, Y_k = \overrightarrow{\nabla^s_{Z^*}X_i, Y_k} = (\nabla^s_{Z^*}T^*)(Y_k, X_l) \) and \( \nabla^s_{Z^*}(Y_k, Y_l) = -\nabla^s_{Z^*}(Y_l, Y_k) = (\nabla^s_{Z^*}T^*)(X_l, Y_k), \) respectively. Now, the co-differential of \( T^* \) is given by \( \delta^s T^* = -\sum_{i=1}^n Z_i \nabla^s_{Z^*}T^* \), where \( \{Z_1, \ldots, Z_n\} \) is a \( g_{1/2} \)-orthonormal basis of \( m \). Hence one needs the non-zero contractions \( \nabla^s_{\alpha\beta\gamma} \) with \( 1 \leq \alpha, \beta, \gamma, \delta \leq 2 \):

\[
\begin{align*}

abla^s_{111} := B(\nabla^s_{X_i, T^*})(X_i, X_j, X_k) = \frac{s(s-1)}{2} B(\text{Jac}_m(X_i, X_j, X_k), X_l),
\end{align*}
\]

\[
\begin{align*}

abla^s_{112} := B(\nabla^s_{Y_k, T^*})(X_i, X_j, Y_l) = \frac{s(s-1)}{2} B([X_i, X_j], [Y_k, Y_l]),
\end{align*}
\]

\[
\begin{align*}

abla^s_{221} := B(\nabla^s_{X_i, T^*})(Y_k, Y_l, Y_m) = \frac{s(s-1)}{2} B([X_i, Y_k], [Y_l, Y_m]).
\end{align*}
\]

In fact, one can easily see that \( \nabla^s_{112} = -\nabla^s_{121} = \nabla^s_{211}, \nabla^s_{221} = -\nabla^s_{212} = \nabla^s_{122} \) and by an easy computation we obtain that \( \delta^s T^* = 0 (= \delta T^*) \).

The vanishing of the co-differential \( \delta^s T^* \) ensures that the Ricci tensor \( \text{Ric}^s \) is symmetric. In more details

**Proposition 7.12.** Let \( \{X_i\}_{i=1}^{d_1} \) and \( \{Y_k\}_{k=1}^{d_2} \) be the \( B \)-orthonormal bases of \( m_1 \) and \( m_2 \), respectively. Then, the Ricci tensor associated to the 1-parameter family \( \{\nabla^s \equiv \nabla^{s,1/2} : s \in \mathbb{R}\} \) satisfies the following relations:

\[
\text{Ric}^s(X_j, X_j) = \text{Ric}^{s,1/2}(X_j, X_j) - \frac{1}{4} S^s(X_j, X_j), \quad \text{Ric}^s(Y_l, Y_l) = \text{Ric}^{s,1/2}(Y_l, Y_l) - \frac{1}{4} S^s(Y_l, Y_l),
\]

where the non-zero parts of the symmetric tensor \( S^s \) are of the form

\[
S^s(X_j, X_j) = (s-1)^2 \left\{ \sum_i \|X_j, X_i\|_m^2 \right\} + \sum_k \|X_j, Y_k\|_m^2 \right\}, \quad S^s(Y_{l}, Y_{l}) = (s-1)^2 \sum_i \|Y_{l}, X_i\|_m^2.
\]
Proof. By Corollary 7.5, the Ricci tensor $Ric^s \equiv Ric^{s,2}$ with respect to $\nabla^s$ has the form
\[
Ric^s(X_j, X_j) = -\frac{(s^2 - 2s)}{4} \left\{ \sum_{i=1}^{d_1} \|[X_j, X_i]_m^2\|^2 + \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 \right\} + Cas_1,
\]
\[
Ric^s(Y_i, Y_i) = -\frac{(s^2 - 2s)}{4} \sum_{i=1}^{d_1} \|[Y_i, X_i]\|^2 + Cas_2.
\]
By using now the Riemannian Ricci tensor $Ric^{1,2} \equiv Ric^1 \equiv Ric^{g_{1/2}}$ and the definition of the symmetric tensor $S^s$, one can obtain the given expressions. For example, let $0 \neq X, Y \in m_1$. Then
\[
S^s(X, Y) := \sum_{j=1}^{d_1} B(T^s(X_j, X), T^s(X_j, Y)) + \sum_{j=1}^{d_2} B(T^s(Y_i, X), T^s(Y_i, Y)) = (s - 1)^2 \left\{ \sum_j B([X_j, X]_m^2, [X_j, X]_m^2) + \sum_l B([Y_l, X], [Y_l, Y]) \right\}
\]
Similarly, for $X, Y \in m_2$ we get $S^s(X, Y) = \sum_j B(T^s(X_j, X), T^s(X_j, X)) = (s - 1)^2 \sum_j B([X_j, X], [X_j, Y])$, and finally for $X \in m_1, Y \in m_2$ it is $S^s(X, Y) = 0 = S^s(Y, X)$. □

Now we are ready to proceed with the proof of Theorem 1.5.

Proof of Theorem 1.5. A $\nabla^s$-Einstein structure on $(M = G/K, m_1 \oplus m_2, g_{1/2})$ is given as a solution (with respect to $s$) of the following system:
\[
\begin{align*}
\{ \text{Ric}^s(X_j, X_j) &= \frac{\text{Scal}^s}{n} B(X_j, X_j) = \text{Scal}^d, \quad \text{Ric}^s(Y_i, Y_i) = \frac{\text{Scal}^s}{n} B(Y_i, Y_i) = \text{Scal}^d \},
\end{align*}
\]
This is equivalent to the equation $Ric^s(X_j, X_j) - Ric^s(Y_i, Y_i) = 0$, which may expressed by
\[
(s^2 - 2s) \left\{ \sum_{i=1}^{d_1} \|[X_j, X_i]_m^2\|^2 + \sum_{k=1}^{d_2} \|[X_j, Y_k]\|^2 - \sum_{i=1}^{d_1} \|[Y_i, X_i]\|^2 \right\} = 4(\text{Cas}_1 - \text{Cas}_2).
\]
Since $m = m_1 \oplus m_2$ is a $B$-orthogonal decomposition of $m = T\theta G/K$ and both $m_1, m_2$ have been assumed to be irreducible and non-equivalent, by [WZ] Thm. 1.11 it follows that $M = G/K$ is a normal homogeneous Einstein manifold, if and only if, the Casimir eigenvalues coincide, i.e., $\text{Cas}_1 = \text{Cas}_2$. Assume first that $M = G/K$ admits a $\nabla^s$-Einstein structure for $s = 0$ or $s = 2$. Then the left-hand side of (7.5) vanishes and hence $(M = G/K, m_1 \oplus m_2, g_{1/2})$ must be a normal homogeneous Einstein manifold. Conversely, let us assume that $g_{1/2} \equiv g_{0}$ is a $G$-invariant Einstein metric on $M$. Then, the right-hand side of (7.5) vanishes and the values $s = 0$ and $s = 2$ trivially satisfy the reduced equation. The last assertion is easy. □

Remark 7.13. Recall that $T^s(X, Y) = (s - 1)[X, Y]_m$, hence the torsion of the connections $\nabla^0$ and $\nabla^2$ are such that $T^0 = -T^2$. Although $\nabla^0 T^0 = 0$ (Ambrose-Singer theorem), Lemma 7.14 ensures that $\nabla^2 T^2 \neq 0$ and thus the value $s = 2$ gives rise to a $\nabla^s$-Einstein structure with non-parallel skew-torsion (notice the similarity with the isotropy irreducible case).

Proof of Theorem 1.6. By Example 7.10 it follows that $\mathbb{C}P^3$ is a normal homogeneous Einstein manifold with two isotropy summands, i.e., $\text{Cas}_1 = \text{Cas}_2$ (see also [Z], [WZ], [BF], [AC2]). Therefore, Theorem 1.5 states that for $s = 0, 2$, $\mathbb{C}P^3$ is a $\nabla^s$-Einstein manifold with skew-torsion $T^0(X, Y) = -[X, Y]_m$ and $T^2(X, Y) = [X, Y]_m$, respectively. The uniqueness of these structures is obvious due to the $\nabla^s$-Einstein equation (7.5). The existence of the $\nabla^0$-Einstein structure where $\nabla^0 = \nabla^c$ is the canonical connection, is a well-known result related with the homogeneous nearly-Kähler structure that $\mathbb{C}P^3$ admits and the identification of the Chern connection with $\nabla^c$ (see [G], [A2], [AFc]). On the other hand, the $\nabla^s$-Einstein structure with skew-torsion corresponding to the value $s = 2$ is new. □

7.6. Examples. Let us present now a series of manifolds that Theorem 1.5 can be applied. We focus on flag manifolds and we prove that several of them carry $\nabla^s$-Einstein structures for $s = 0, 2$. Let us fix a compact simple Lie group $G$ and let $M = G/K$ be a flag manifold with two isotropy summands, say $m = m_1 \oplus m_2$. Such spaces have been classified in terms of painted Dynkin diagrams in [AC1]. Since both $m_1$ and $m_2$ are irreducible and inequivalent, any $G$-invariant Riemannian metric on $M = G/K$ is given by $\langle X, X \rangle$ (up to scalar).
In [AC2 Thm 1.1] it was shown that $M$ admits precisely two $G$-invariant Einstein metrics; one of them is Kähler and appears for $t = 1$, and the non-Kähler is given by $t = \frac{2d_1 + d_2}{d_1 + 2d_2}$, i.e.

$$g_{\frac{2d_1 + d_2}{d_1 + 2d_2}} = B|_{m_1 \times m_1} + \frac{4d_2}{d_1 + 2d_2} B|_{m_2 \times m_2}.$$

**Corollary 7.14.** Let $G$ be a compact connected simple Lie group. A generalized flag manifold $M = G/K$ whose isotropy representation is such that $m = m_1 \oplus m_2$, is a normal homogeneous Einstein manifold if and only if $d_1 = 2d_2$, where $d_i = \text{dim } m_i$ for $i = 1, 2$.

A quick check of the dimensions of the isotropy summands implies that there no exceptional flag manifolds, with $m = m_1 \oplus m_2$, for which the Killing metric can be a $G$-invariant Einstein metric (see [AC2 p. 245]). However, several examples appear for adjoint orbits corresponding to the classical Lie groups $B_\ell = \text{SO}(2\ell + 1)$, $C_\ell = \text{Sp}(\ell)$, or $D_\ell = \text{SO}(2\ell)$.

| Classical flag manifolds $M = G/K$ with $m = m_1 \oplus m_2$ | Conditions |
|------------------------------------------------------------|-------------|
| $B(\ell, p) := \text{SO}(2\ell + 1)/U(p) \times \text{SO}(2(\ell - p) + 1)$ | $2 \leq p \leq \ell$, $\ell \geq 2$ |
| $C(\ell, p) := \text{Sp}(\ell)/U(p) \times \text{Sp}(\ell - p)$ | $1 \leq p \leq \ell - 1$, $\ell \geq 2$ |
| $D(\ell, p) := \text{SO}(2\ell)/U(p) \times \text{SO}(2(\ell - p))$ | $2 \leq p \leq \ell - 2$, $\ell \geq 4$ |

**Example 7.15.** For the family $B(\ell, p)$ we compute $d_1 = 4p(\ell - p) + 2p$ and $d_2 = p(p - 1)$. According to Corollary 7.14, the Killing metric $g_B$ is Einstein if and only if $p = 2(\ell + 1)/3 \in \mathbb{Z}_+$. Hence we conclude that the manifold $B(\ell, 2(\ell + 1)/3) = \text{SO}(2\ell + 1)/U(2(\ell + 1)/3) \times \text{SO}(2(\ell - 2)/3 + 1)$, with $\ell = 2 + 3k$ and $k = 0, 1, 2, 3, \ldots$, is a $\nabla^s$-Einstein manifold for the values $s = 0, 2$. Let us list the first examples:

| $\ell$ | $p = 2(\ell + 1)/3 : p \in \mathbb{Z}_+$ | $(M = G/K, m_1 \oplus m_2, g_B)$ | $\nabla^s$-Einstein for $s = 0, 2$ |
|--------|---------------------------------|-------------------------------|----------------------------------|
| $\ell = 2$ | $p = 2$ | $\mathbb{C}P^3 = \text{SO}(5)/U(2)$ | $\checkmark$ |
| $\ell = 5$ | $p = 4$ | $\text{SO}(11)/U(4) \times \text{SO}(3)$ | $\checkmark$ |
| $\ell = 8$ | $p = 6$ | $\text{SO}(17)/U(6) \times \text{SO}(5)$ | $\checkmark$ |
| : | : | : | : |

**Example 7.16.** For the space $C(\ell, p)$ it is $d_1 = 4p(\ell - p)$ and $d_2 = p(p + 1)$ and the condition $d_1 = 2d_2$ takes the form $p = (2\ell - 1)/3 \in \mathbb{Z}_+$. Thus the family $C(\ell, (2\ell - 1)/3) = \text{Sp}(\ell)/U((2\ell - 1)/3) \times \text{Sp}((\ell + 1)/3)$, with $\ell = 2 + 3k$ and $k = 0, 1, 2, 3, \ldots$, is a normal homogeneous Einstein manifold. Moreover, for $s = 0, 2$ it becomes a $\nabla^s$-Einstein manifold with skew-torsion.

| $\ell$ | $p = (2\ell - 1)/3 : p \in \mathbb{Z}_+$ | $(M = G/K, m_1 \oplus m_2, g_B)$ | $\nabla^s$-Einstein for $s = 0, 2$ |
|--------|---------------------------------|-------------------------------|----------------------------------|
| $\ell = 2$ | $p = 1$ | $\mathbb{C}P^3 = \text{Sp}(2)/U(1) \times \text{Sp}(1)$ | $\checkmark$ |
| $\ell = 5$ | $p = 3$ | $\text{Sp}(5)/U(3) \times \text{Sp}(2)$ | $\checkmark$ |
| $\ell = 8$ | $p = 5$ | $\text{Sp}(8)/U(5) \times \text{Sp}(3)$ | $\checkmark$ |
| : | : | : | : |

**Example 7.17.** For the flag manifold $D(\ell, p)$ it is $d_1 = 4p(\ell - p)$ and $d_2 = p(p - 1)$. Hence $D(\ell, p)$ is a normal homogeneous Einstein manifold if and only if $p = (2\ell + 1)/3 \in \mathbb{Z}_+$. It follows that the family $D(\ell, (2\ell + 1)/3) = \text{SO}(2\ell)/U((2\ell + 1)/3) \times \text{SO}(2(\ell - 1)/3)$, with $\ell = 2 + 3k$ and $k = 0, 1, 2, 3, \ldots$, admits $\nabla^s$-Einstein structures with skew-torsion for the values $s = 0, 2$.

| $\ell$ | $p = (2\ell + 1)/3 : p \in \mathbb{Z}_+$ | $(M = G/K, m_1 \oplus m_2, g_B)$ | $\nabla^s$-Einstein for $s = 0, 2$ |
|--------|---------------------------------|-------------------------------|----------------------------------|
| $\ell = 4$ | $p = 3$ | $\text{SO}(8)/U(3) \times \text{SO}(2)$ | $\checkmark$ |
| $\ell = 7$ | $p = 5$ | $\text{SO}(14)/U(5) \times \text{SO}(4)$ | $\checkmark$ |
| $\ell = 10$ | $p = 7$ | $\text{SO}(20)/U(7) \times \text{SO}(6)$ | $\checkmark$ |
| : | : | : | : |

**Acknowledgements:** This research has been fully supported by Masaryk University under the Grant Agency of Czech Republic, project no.14-2464P. It is my sincere pleasure to thank I. Agricola and D. V. Alekseevsky for their enlightening conversations and comments, as well as Y. Sakane for his constant interaction and support. I also acknowledge A. Arvanitoyeorgos and S. Chiossi for reading earlier versions of this paper and for their valuable remarks.
REFERENCES

[A1] I. Agricola: Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory, Comm. Math. Phys., 232, no. 3, (2003), 535–563.

[A2] I. Agricola: The Srní lectures on non-integrable geometries with torsion, Arch. Math. 42 (2006), 5–84. With an appendix by M. Kassuba.

[AF] I. Agricola and A. C. Ferreira: Einstein manifolds with skew torsion, Oxford Quart. J. Math. (2013) (doi:10.1093/qmath/hat050).

[AF1] I. Agricola and Th. Friedrich: On the holonomy of connections with skew-symmetric torsion, Math. Ann. 328 (2004), 711–748.

[AF2] I. Agricola and Th. Friedrich: A note on flat metric connections with antisymmetric torsion, Diff. Geom. Appl. 28 (2010), 480–487.

[AFH] I. Agricola, Th. Friedrich and J. Hâll: Sp(3)-structures on 14-dimensional manifolds, J. Geom. Phys. 69 (2013), 12–30.

[AVL] D. V. Alekseevsky, A. M. Vinogradov and V. V. Lychagin: Geometry I - Basic Ideas and Concepts of Differential Geometry, Encyclopaedia of Mathematical Sciences, Vol. 28, Springer-Verlag, Berlin, 1991.

[AC1] A. Arvanitoyeorgos and I. Chrysikos: Motion of charged particles and homogeneous geodesics in Kähler C-spaces with two isotropy summands, Tokyo J. Math. 32 (2), (2009), 487–500.

[AC2] A. Arvanitoyeorgos and I. Chrysikos: Invariant Einstein metrics on generalized flag manifolds with two isotropy summands, J. Austral. Math. Soc. 90 (02), (2011), 237–251.

[BFGK] H. Baum, Th. Friedrich, R. Grunewald and I. Kath: Twistors and Killing spinors on Riemannian manifolds, etc.: B. G. Teubner Verlagsgesellschaft, 1991.

[But] J-B. Butruille: Homogeneous nearly Kahler manifolds. Handbook of pseudo-Riemannian geometry and supersymmetry, 399–423, IRMA Lect. Math. Theor. Phys., 16, Eur. Math. Soc., Zürich, 2010.

[CS] A. Charbon and J. Slovák: Parabolic geometries I: Background and general theory, Mathematical Surveys and Monographs, 154., A.M.S., RI, 2009.

[DK] W. Dickinson and M. Kerr: The geometry of compact homogeneous spaces with two isotropy summands, Ann. Glob. Anal. Geom. 34, (2008), 329–350.

[FKMS] Th. Friedrich and S. Ivanov: Parallel spinors and connections with skew-symmetric torsion in string theory, Asian J. Math. 6 (2002), no. 2, 303–335.

[GOV] V. V. Gortzevich, A. L. Onishchik, and E. B. Vinberg: Lie Groups and Lie Algebras III, Encyclopaedia of Mathematical Sciences,Vol. 20, Springer-Verlag, Berlin, 1993.

[G] A. Gray: The structure of nearly Kähler manifolds, Math. Ann. 223 (1976), 233–248.

[K1] M. Kerr: Some new homogeneous Einstein metrics on symplectic spaces, Trans. Amer. Math. Soc. 348, (1996), 153–171.

[K2] M. Kerr: New examples of homogeneous Einstein metrics, Michigan J. Math., 45, (1998), 115–134.

[KS] M. Kerin and K. Shankar: Riemannian submersions from simple, compact Lie groups, Münster J. of Math. 5 (2012), 25–49.

[KN] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry Vol II, Wiley - Interscience, New York, 1969.

[Ko] B. Kostant: On differential geometry and homogeneous spaces, II, Proc. N. A. S. U. S. A. 42, (1956), 354–357.

[L1] H. T. Laquer: Invariant affine connections on Lie groups, Trans. Amer. Math. Soc., 331, (2), (1992), 511–511.

[L2] H. T. Laquer: Invariant affine connections on symmetric spaces, Proc. Am. Math. Soc., 115, (2), (1992), 447–454.

[MS] A. Moroianu and U. Semmelmann: The Hermitian Laplace operator on nearly Kähler manifolds, Commun. Math. Phys. 294, (2010), 251–272.

[N] P-A. Nagy: Skew-symmetric prolongations of Lie algebras and applications, Journal of Lie Theory, 23 (2013), (1), 1–33.

[OR] C. Olmos and S. Reggiani: The skew-torsion holonomy theorem and naturally reductive spaces, J. Reine Angew. Math. 664 (2012), 29–53.

[R1] S. Reggiani: A Berger-type theorem for metric connections with skew-symmetric torsion, J. Geom. Phys., 65 (2013), 26–34

[R2] S. Reggiani: On the affine group of a normal homogeneous manifold, Ann. Glob. Anal. Geom. 37, (2010), 351–359.

[W] H. C. Wang: On invariant connections over a principal fibre bundle, Nagoya Math. J. 13, (1958), 1–19.

[WZ] M. Y. Wang and W. Ziller: On normal homogeneous Einstein manifolds, Ann. Scient. Éc. Norm. Sup. 18, (4), (1985), 569–633.

[Wol] J. A. Wolf: The geometry and the structure of isotropy irreducible homogeneous spaces, Acta. Math., 120 (1968) 59–148; correction: Acta Math., 152, (1984), 141–142.

[Z] W. Ziller: Homogeneous Einstein metrics on spheres and projective spaces, Math. Ann. 259, (1982) 351–358.

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