HODGE DECOMPOSITION AND THE SHAPLEY VALUE
OF A COOPERATIVE GAME

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ABSTRACT. We show that a cooperative game may be decomposed into
a sum of component games, one for each player, using the combinatorial
Hodge decomposition on a graph. This decomposition is shown to
satisfy certain efficiency, null-player, symmetry, and linearity properties.
Consequently, we obtain a new characterization of the classical Shapley
value as the value of the grand coalition in each player’s component game.
We also relate this decomposition to a least-squares problem involving
inessential games (in a similar spirit to previous work on least-squares and
minimum-norm solution concepts) and to the graph Laplacian. Finally,
we generalize this approach to games with weights and/or constraints on
coalition formation.

1. INTRODUCTION

In cooperative game theory, one of the central questions is that of fair
division: if players form a coalition to achieve a common goal, how should
they split the profits (or costs) of that achievement among themselves? (We
restrict our attention to transferable utility games, also called TU games,
whose total value may be freely divided and distributed among the players.)
Shapley [25] introduced one of the classical solution concepts to this problem,
now known as the Shapley value, which he proved to be the unique allocation
that satisfies certain axioms.

In this paper, we show that a cooperative game may be decomposed into
a sum of component games, one for each player, where these components are
uniquely defined in terms of the combinatorial Hodge decomposition on a
hypercube graph associated with the game. (That is, the value is apportioned
among the players for each possible coalition, not just the grand coalition
consisting of all players.) We prove that the Shapley value is precisely the
value of the grand coalition in each player’s component game.

This characterization of the game components and the Shapley value also
implies two equivalent characterizations: one in terms of the least-squares
solution to a linear problem, whose solution is exact if and only if the game is
inessential; the other in terms of the graph Laplacian. The first of these two
characterizations is related to the least-square and minimum-norm solution
concepts of Ruiz et al. [23] and Kultti and Salonen [21].

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Furthermore, since the combinatorial Hodge decomposition holds for arbitrary weighted graphs, this decomposition of cooperative games also generalizes to cases where edges of the hypercube graph are weighted or removed altogether. This may be seen as modeling variable willingness or unwillingness of players to join certain coalitions, as in some models of restricted cooperation. In the latter case, we compare the resulting solution concepts with other “Shapley values” for games with cooperation restrictions, such as the precedence constraints of Faigle and Kern [11] and the even more general digraph games of Khmelnitskaya et al. [18].

We note that the combinatorial Hodge decomposition has recently been used to decompose noncooperative games (Candogan et al. [2]) and has also been applied to other problems in economics, such as ranking of social preferences (Hirani et al. [12], Jiang et al. [17]). Here, we show that it can also lend insight to cooperative game theory.

2. Preliminaries

2.1. Cooperative games and the Shapley value. A cooperative game consists of a finite set $N$ of players and a function $v: 2^N \to \mathbb{R}$, which assigns a value $v(S)$ to each coalition $S \subseteq N$, such that $v(\emptyset) = 0$. Assuming that all players cooperate (forming the “grand coalition” $N$), the question of interest is how to split the value $v(N)$ among the players.

The Shapley value $\phi_i(v)$ allocated to player $i \in N$ is based entirely on the marginal value $v(S \cup \{i\}) - v(S)$ the player contributes when joining each coalition $S \subseteq N \setminus \{i\}$. It is uniquely defined according to the following theorem.

**Theorem 2.1** (Shapley [25]). There exists a unique allocation $v \mapsto \left( \phi_i(v) \right)_{i \in N}$ satisfying the following conditions:

(a) **Efficiency**: $\sum_{i \in N} \phi_i(v) = v(N)$.
(b) **Null-player property**: If $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, then $\phi_i(v) = 0$.
(c) **Symmetry**: If $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$, then $\phi_i(v) = \phi_j(v)$.
(d) **Linearity**: If $v, v'$ are two games with the same set of players $N$, then $\phi_i(\alpha v + \alpha' v') = \alpha \phi_i(v) + \alpha' \phi_i(v')$ for all $\alpha, \alpha' \in \mathbb{R}$.

Moreover, this allocation is given by the following explicit formula:

$$
\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - 1 - |S|)!}{|N|!} \left( v(S \cup \{i\}) - v(S) \right).
$$

The conditions (a)–(d) listed above are often called the Shapley axioms. Simply stated, they say that (a) the value obtained by the grand coalition is fully distributed among the players, (b) a player who contributes no marginal value to any coalition receives nothing, (c) equivalent players receive equal amounts, and (d) the allocation is linear in the game values.
The formula (1) has the following useful interpretation. Suppose the players form the grand coalition by joining, one-at-a-time, in the order defined by a permutation \( \sigma \) of \( N \). That is, player \( i \) joins immediately after the coalition \( S_{\sigma,i} = \{ j \in N : \sigma(j) < \sigma(i) \} \) has formed, contributing marginal value \( v(S_{\sigma,i} \cup \{ i \}) - v(S_{\sigma,i}) \). Then \( \phi_i(v) \) is the average marginal value contributed by player \( i \) over all \( |N|! \) permutations \( \sigma \), i.e.,
\[
\phi_i(v) = \frac{1}{|N|!} \sum_{\sigma} \left( v(S_{\sigma,i} \cup \{ i \}) - v(S_{\sigma,i}) \right).
\]
The equivalence of (1) and (2) is due to the fact that \( |S|!(|N| - 1 - |S|)! \) is precisely the number of permutations \( \sigma \) for which \( S = S_{\sigma,i} \), since there are \( |S|! \) ways to permute the preceding players and \( (|N| - 1 - |S|)! \) ways to permute the succeeding players.

For purposes of computation, of course, (1) is preferable to (2), since it contains \( 2^{|N|-1} \) terms rather than \( |N|! \) terms. Computing the Shapley value is \( \#P \)-complete (Deng and Papadimitriou [8]), although some recent work has explored polynomial algorithms for obtaining approximations to the Shapley value (Castro et al. [3, 4]).

**Example 2.2.** The “glove game” is a classic illustrative example of a cooperative game. Let \( N = \{1, 2, 3\} \), and suppose that player 1 has a left-hand glove, while players 2 and 3 each have a right-hand glove. The players wish to put together a pair of gloves, which can be sold for value 1, while unpaired gloves have no value. That is, \( v(S) = 1 \) if \( S \subset N \) contains both a left and a right glove (i.e., player 1 and at least one of players 2 or 3) and \( v(S) = 0 \) otherwise. The Shapley values for this game are
\[
\phi_1(v) = \frac{2}{3}, \quad \phi_2(v) = \phi_3(v) = \frac{1}{6}.
\]
This is perhaps easiest to interpret from the “average-over-permutations” perspective: player 1 contributes marginal value 0 when joining the coalition first (2 of 6 permutations) and marginal value 1 otherwise (4 of 6 permutations), so \( \phi_1(v) = \frac{2}{3} \). Efficiency and symmetry immediately give \( \phi_2(v) = \phi_3(v) = \frac{1}{6} \).

2.2. **Hodge theory: from differential forms to cochains.** Before discussing the Hodge decomposition on graphs, we first give some brief historical background on continuous and combinatorial Hodge theories.

The classical version of Hodge theory (Hodge [13, 14, 15, 16], Kodaira [19] equips the de Rham complex of differential forms \( (\Omega^\bullet(M), d^\bullet) \) on a manifold \( M \) with the \( L^2 \) inner product induced by a Riemannian metric on \( M \). One of the key tools is the Hodge decomposition, which states that any \( L^2 \) differential \( k \)-form may be orthogonally decomposed into the sum of an exact \( k \)-form (in the range of the exterior derivative \( d \)), a coexact \( k \)-form (in the range of \( d^\ast \)), and a harmonic \( k \)-form (in the kernel of both \( d \) and \( d^\ast \)). That is, if \( f \in L^2\Omega^k(M) \), then
\[
f = d\alpha + d^\ast \beta + \gamma,
\]
where \( \alpha \in L^2\Omega^{k-1}, \beta \in L^2\Omega^{k+1}, \gamma \in L^2\Omega^k(M), \) with \( d\gamma = 0 \) and \( d^*\gamma = 0. \) (This generalizes the Helmholtz decomposition of \( L^2 \) vector fields on \( \mathbb{R}^3 \) into divergence-free and curl-free components.) The Hodge decomposition is intimately related to the Laplace operator

\[
L = dd^* + d^*d
\]

on differential forms, since solutions to \( Lu = f \) give the Hodge decomposition \( f = dd^*u + d^*du \) (so in particular, \( f \) must have vanishing harmonic component in order for solutions to exist), while \( Lu = 0 \) for harmonic \( k \)-forms \( u. \) (See Schwarz [24] for more on the relationship between the Hodge decomposition and elliptic PDE theory.) Moreover, the space of harmonic \( k \)-forms is isomorphic to the \( k \)th de Rham cohomology space of \( M. \)

However, there is also a combinatorial version of Hodge theory on a finite simplicial complex \( K \) (Eckmann [10], Dodziuk [9]), which is related to simplicial cohomology rather than de Rham cohomology. Suppose we equip the complex of simplicial cochains \( (C^\bullet(K), d^\bullet) \), where \( d \) is the dual to the boundary operator \( \partial \) on simplicial chains, with the \( \ell^2 \) inner product over simplices of each degree. Then any \( k \)-cochain \( f \in C^k(K) \) may be decomposed into exact, coexact, and harmonic components—formally, just as in (3). One may also define a "discrete Laplace operator" \( L = dd^* + d^*d \) on cochains, whose properties are analogous to the Laplace operator on \( k \)-forms.

The simplest case of combinatorial Hodge theory is on an oriented graph \( G = (V,E) \), which we consider as a simplicial 1-complex\(^{1}\). Then \( C^0(G) \) and \( C^1(G) \) consist of real-valued functions on \( V \) and \( E \), respectively. If \( c \in C^1(G) \) and \( (a,b) \in E \) is an oriented edge, then on the reverse-oriented edge \((b,a)\) we define \( c(b,a) = -c(a,b) \). The operator \( d: C^0(G) \to C^1(G) \) is then defined by

\[
du(a,b) := u(b) - u(a). \]

With respect to the bases defined by \( V \) and \( E \), the matrix of \( d \) is precisely the transpose of the oriented incidence matrix of \( G \). If we equip \( C^1(G) \) and \( C^1(G) \) with the \( \ell^2 \) inner product (corresponding to counting measure on \( V \) and \( E \), respectively), then we have the complex

\[
0 \to \ell^2(V) \xrightarrow{d} \ell^2(E) \to 0. \]

Since the first and last arrows are trivial, the “harmonic” 0-cochains (resp., 1-cochains) are just those in the kernel of \( d \) (resp., \( d^* \)). Hence, the Hodge decompositions of \( \ell^2(V) \) and \( \ell^2(E) \) are

\[
\ell^2(V) = \mathcal{R}(d^*) \oplus \mathcal{N}(d), \quad \ell^2(E) = \mathcal{R}(d) \oplus \mathcal{N}(d^*),
\]

where \( \mathcal{R}(\cdot) \) and \( \mathcal{N}(\cdot) \) denote range and kernel (nullspace). In the general setting of infinite-dimensional Hilbert complexes, the Hodge decomposition is a consequence of Banach’s closed range theorem (Brüning and Lesch [1]). However, since \( \ell^2(V) \) and \( \ell^2(E) \) are finite dimensional, the combinatorial

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\(^{1}\)In graph theory, one may also consider the clique complex of a graph, where \( k \)-simplices correspond to \((k+1)\)-cliques. However, the hypercube graphs we will encounter in cooperative game theory contain only 1- and 2-cliques, i.e., vertices and edges, so they have no \( k \)-simplices for \( k > 1 \).
Hodge decomposition (4) is just the “fundamental theorem of linear algebra” (so-called by Strang [26]) applied to the linear map $d$.

The Laplace operator on 0-cochains is given by $L = d^*d : \ell^2(V) \to \ell^2(V)$. This is precisely the usual graph Laplacian encountered in, e.g., spectral graph theory (Chung [6]), usually expressed as $L = D - A$, where $D$ is the degree matrix and $A$ is the (unsigned) adjacency matrix of the graph $G$.

These expressions for $L$ are seen to be identical by observing that, for any vertex $a \in V$, we have

$$(d^*du)(a) = \sum_{b \sim a} du(b, a) = \sum_{b \sim a} (u(a) - u(b)) = \deg(a)u(a) - \sum_{b \sim a} u(b),$$

where $b \sim a$ denotes that $(a, b) \in E$ or $(b, a) \in E$. There is also another Laplace operator $dd^*: \ell^2(E) \to \ell^2(E)$ on 1-cochains, sometimes called the graph Helmholtzian (Jiang et al. [17]), but we omit further discussion of it since we will not encounter it in this paper.

3. Decomposition of cooperative games

3.1. Cooperative games and cochains on the hypercube graph. Given the set of players $N$, define the oriented graph $G = (V, E)$ by

$$V = 2^N, \quad E = \{(S, S \cup \{i\}) \in V \times V : S \subset N \setminus \{i\}, i \in N\}.$$  

This is precisely the $|N|$-dimensional hypercube graph, where each vertex corresponds to a coalition $S \subset N$, and where each edge corresponds to the addition of a single player $i \notin S$ to $S$, oriented in the direction of the inclusion $S \hookrightarrow S \cup \{i\}$.

With respect to this graph, a cooperative game is precisely a 0-cochain $v \in \ell^2(V)$ such that $v(\emptyset) = 0$. Furthermore, the 1-cochain $dv \in \ell^2(E)$ gives the marginal value on each edge, i.e., $dv(S, S \cup \{i\})$ is the marginal value contributed by player $i$ joining coalition $S \subset N \setminus \{i\}$. In order to talk about the marginal contributions of an individual player $i \in N$, ignoring those of the other players $j \neq i$, we introduce the following collection of “partial differentiation” operators.

Definition 3.1. For each $i \in N$, let $d_i : \ell^2(V) \to \ell^2(E)$ be the operator

$$d_iu(S, S \cup \{j\}) = \begin{cases}  
du(S, S \cup \{i\}) & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}$$

Therefore, $d_iv \in \ell^2(E)$ encodes the marginal value contributed by player $i$ to the game $v$. For any permutation $\sigma$ of $N$, which defines a path from $\emptyset$ to $N$, the marginal value contributed by player $i$ along this path is

$$\sum_{j \in N} d_i v(S_{\sigma,j}, S_{\sigma,j} \cup \{j\}) = dv(S_{\sigma,i}, S_{\sigma,i} \cup \{i\}) = v(S_{\sigma,i} \cup \{i\}) - v(S_{\sigma,i}),$$

which can be interpreted as a discrete “line integral” of $d_i v$ along the path.
3.2. Decomposition of inessential games. From Definition 3.1 we immediately see that $d = \sum_{i \in N} d_i$. However, in general, $\mathcal{R}(d_i) \not\subseteq \mathcal{R}(d)$. To see this, observe that for any permutation $\sigma$,

$$
\sum_{j \in N} du(S_{\sigma,j}, S_{\sigma,j} \cup \{j\}) = \sum_{j \in N} \left( u(S_{\sigma,j} \cup \{j\}) - u(S_{\sigma,j}) \right) = u(N) - u(\emptyset),
$$

since the sum telescopes. This value is the same for every permutation $\sigma$, which is a discrete analog of the fact that the line integral of a conservative vector field depends only in the endpoints, not the particular path chosen. Contrast this with the previous expression: we have already seen that $v(S_{\sigma,i} \cup \{i\}) - v(S_{\sigma,i})$ may be different, depending on the permutation $\sigma$, as in the glove game of Example 2.2. The question of when $d_i v \in \mathcal{R}(d)$ is related to the notion of inessential games, as we now show.

Definition 3.2. The game $v$ is inessential if $v(S) = \sum_{i \in S} v\{i\}$ for all $S \subset N$. That is, each coalition obtains the same value working together as its individual members would obtain working separately.

Proposition 3.3. The game $v$ is inessential if and only if $d_i v \in \mathcal{R}(d)$ for all $i \in N$.

Proof. If $d_i v \in \mathcal{R}(d)$, then from the calculation above, it follows that the marginal value $v(S \cup \{i\}) - v(S)$ is the same for all coalitions $S \subset N \setminus \{i\}$. Taking $S = \emptyset$, we see that this value is precisely $v\{i\}$. If this holds for all players $i \in N$, then we conclude that $v$ is inessential.

Conversely, suppose that $v$ is inessential, and define the game

$$
v_i(S) = \begin{cases} v\{i\} & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}
$$

It follows immediately that $d_i v = dv_i \in \mathcal{R}(d)$, which completes the proof. 

Therefore, inessential games have the decomposition $v = \sum_{i \in N} v_i$, where $v_i$ is the game described in the proof above. In the next section, we show how this decomposition generalizes to arbitrary games, and how the Shapley value naturally arises from the generalized decomposition.

3.3. Decomposition of arbitrary games and the Shapley value. For an arbitrary game $v$, we cannot hope to find games $v_i$ such that $d_i v = dv_i$ (unless $v$ is inessential, as shown in Proposition 3.3). However, the Hodge decomposition ensures that we can write $d_i v \in \ell^2(E)$ as the sum of some $dv_i \in \mathcal{R}(d)$ and an element of $\mathcal{N}(d^*)$. Moreover, since $G$ is connected, $\mathcal{N}(d) \cong \mathbb{R}$, so $v_i$ is uniquely determined by the condition $v_i(\emptyset) = 0$, i.e., that $v_i$ is itself a game.

Theorem 3.4. For each $i \in N$, let $v_i \in \ell^2(V)$ with $v_i(\emptyset)$ be the unique game such that $dv_i = Pd_i v$, where $P : \ell^2(E) \to \mathcal{R}(d)$ is the orthogonal projection onto $\mathcal{R}(d)$. Then the games $v_i$ satisfy the following:
Consequently, \( v_i = v \).

(b) If \( v(S \cup \{i\}) - v(S) = 0 \) for all \( S \subset N \setminus \{i\} \), then \( v_i = 0 \).

(c) If \( \sigma \) is a permutation of \( N \) and \( \sigma^*v \) is the game \( (\sigma^*v)(S) = v(\sigma(S)) \), then \( (\sigma^*v)_i = \sigma^*(v_{\sigma(i)}) \). In particular, if \( \sigma \) is the permutation swapping \( i \) and \( j \), and if \( \sigma^*v = v \), then \( v_i = \sigma^*(v_j) \).

(d) For any two games \( v, v' \) and \( \alpha, \alpha' \in \mathbb{R} \), \( (\alpha v + \alpha' v')_i = \alpha v_i + \alpha' v'_i \).

Consequently, \( v_i(N) = \phi_i(v) \) is the Shapley value for each player \( i \in N \).

Proof. First, since \( d = \sum_{i \in N} d_i \), we have

\[
\sum_{i \in N} v_i = \sum_{i \in N} d v_i = \sum_{i \in N} P d_i v = P \sum_{i \in N} d_i v = P d v = d v.
\]

Since \( G \) is connected, it follows that \( \sum_{i \in N} v_i \) and \( v \) differ by a constant. But this constant is \( v(\emptyset) - \sum_{i \in N} v_i(\emptyset) = 0 \), which proves (a).

Next, if \( v(S \cup \{i\}) - v(S) = 0 \) for all \( S \subset N \setminus \{i\} \), then \( d_i v = 0 \). It follows that \( d v_i = P d_i v = 0 \). Hence, \( v_i \) is constant, but since \( v_i(\emptyset) = 0 \), we conclude that \( v_i = 0 \), which proves (b).

Next, if \( \sigma \) is a permutation of \( N \), then direct calculation shows that \( d \sigma^* = \sigma^* d \) and \( d_i \sigma^* = \sigma^* d_{\sigma(i)} \). Furthermore, \( \sigma \) preserves counting measure and hence the \( l^2 \) inner product, so \( P \sigma^* = \sigma^* P \). Thus,

\[
d(\sigma^*v)_i = P d_i (\sigma^*v) = P \sigma^* (d_{\sigma(i)} v) = \sigma^* (P d_{\sigma(i)} v) = \sigma^* (d v_{\sigma(i)}) = d \sigma^* (v_{\sigma(i)}),
\]

so \( (\sigma^*v)_i \) and \( \sigma^* (v_{\sigma(i)}) \) differ by a constant. But this constant is \( (\sigma^*v)_i(\emptyset) - \sigma^* (v_{\sigma(i)})(\emptyset) = v_i(\emptyset) - v_{\sigma(i)}(\emptyset) = 0 \), which proves (c).

Next, since \( d, d_i, \) and \( P \) are all linear maps,

\[
d(\alpha v + \alpha' v')_i = P d_i (\alpha v + \alpha' v') = \alpha P d_i v + \alpha' P d_i v' \\
= \alpha d v_i + \alpha' d v'_i = d (\alpha v_i + \alpha' v'_i).
\]

Hence, the games \( (\alpha v + \alpha' v')_i \) and \( (\alpha v_i + \alpha' v'_i) \) differ by a constant—but just as above, this constant must be zero, which proves (d).

Finally, having shown (a)–(d), it follows that the allocation \( v \mapsto (v_i(N))_{i \in N} \) satisfies the Shapley axioms of Theorem 2.1. Indeed, condition (a) implies efficiency, (b) implies the null-player property, (c) implies symmetry, and (d) implies linearity. Hence, by the uniqueness property of the Shapley value, we must have that \( v_i(N) = \phi_i(v) \) for all \( i \in N \), which completes the proof. \( \square \)

Remark 3.5. An immediate corollary of Proposition 3.3 and Theorem 3.4 is the standard result that \( \phi_i(v) = v(\{i\}) \) for all \( i \in N \) whenever \( v \) is inessential.

Remark 3.6. Since \( P \) is orthogonal projection onto \( R(d) \), we may also view the game \( v_i \) as the least-squares solution to \( d v_i = d_i v \), which only has an exact solution when \( v \) is inessential. That is, we have

\[
v_i = \arg\min_{u \in l^2(V)} \| d u - d_i v \|_{l^2(E)}.
\]
This is similar in spirit to the work of Kultti and Salonen [21] on minimum-norm solution concepts, including the least-square values of Ruiz et al. [23]. Specifically, Kultti and Salonen [21] consider the projection of $v$ itself onto the subspace of inessential games in $\ell^2(V)$. By contrast, the projection in our approach is performed on $\ell^2(E)$.

**Remark 3.7.** The decomposition of Theorem 3.4 also gives a straightforward way to derive the Shapley formula (1). This formula is equivalent to the statement that, if $\chi(S,S \cup \{i\}) \in \ell^2(E)$ is the indicator function equal to 1 on $(S,S \cup \{i\})$ and 0 on all other edges, and if $u \in \ell^2(V)$ is the solution to $du = P\chi(S,S \cup \{i\})$ with $u(\emptyset) = 0$, then $u(N) = \frac{|S|!(|N|-1-|S|)!}{|N|!}$. (Here, $u$ can be seen as a kind of discrete Green’s function, in a sense similar to that of Chung and Yau [5].) To see this, consider the game

$$v(T) := \begin{cases} 1 & \text{if } |T| > |S|, \\ 0 & \text{if } |T| \leq |S|, \end{cases}$$

so that

$$dv = \sum_{|T|=|S|, j \notin T} \chi(T,T \cup \{j\}).$$

This sum contains $\binom{|N|}{|S|} (|N| - |S|) = \frac{|N|!}{|S|!(|N|-1-|S|)!}$ terms, so by symmetry, we must have

$$v(N) = \frac{|N|!}{|S|!(|N| - 1 - |S|)!} u(N).$$

Finally, since $v(N) = 1$, we obtain the claimed expression for $u(N)$.

**Example 3.8.** We now illustrate the decomposition of Theorem 3.4 by applying it to the glove game introduced in Example 2.2. Since $|N| = 3$, the graph $G$ consists of vertices and edges of the ordinary, three-dimensional cube.

Table 1 contains the values of $v$ and the component games $v_1$, $v_2$, and $v_3$. Several of the properties proved in Theorem 3.4 are immediately apparent. In particular, we have the decomposition $v = v_1 + v_2 + v_3$, while $v_1(N) = \frac{2}{3}$ and $v_2(N) = v_3(N) = \frac{1}{6}$ are the Shapley values previously obtained in Example 2.2. Furthermore, the symmetry of players 2 and 3 is evident in all three component games, not just $v_2$ and $v_3$. Indeed, if $\sigma$ is the permutation swapping players 2 and 3, then $v_1 = \sigma^* v_1$, $v_2 = \sigma^* v_3$, $v_3 = \sigma^* v_2$, which can be observed in Table 1. We also point out that, although $v \geq 0$, the component games $v_1, v_2, v_3$ may take negative values. Note also that $dv_i \neq dv$, corresponding to the fact that the glove game is not inessential.
3.4. Decomposition via the hypercube graph Laplacian. We now briefly show how the component games \( v_i \) may be computed using the graph Laplacian \( L = d^* d \), without having to explicitly compute the orthogonal projection operator \( P \). Denote \( L_i := d^* d_i \); this is in fact a weighted graph Laplacian, where the edge \( (S, S \cup \{j\}) \) has weight 1 if \( i = j \) and 0 otherwise. (We will say more about weighted graph Laplacians in Section 4.1.)

**Proposition 3.9.** For each \( i \in N \), the component game \( v_i \) of Theorem 3.4 is the unique solution to \( L v_i = L_i v \) such that \( v_i(\emptyset) = 0 \).

**Proof.** Since \( (dv_i - d_i v) \in N(d^*) \), we immediately have

\[
0 = d^* (dv_i - d_i v) = d^* dv_i - d^* d_i v = Lv_i - L_i v,
\]
so \( L v_i = L_i v \) as claimed. To show uniqueness, suppose that \( v'_i \) is another solution. Then \( L(v_i - v'_i) = 0 \), and since the hypercube graph is connected, we must have \( v_i - v'_i \) constant. But \( v_i(\emptyset) = v'_i(\emptyset) \), so it follows that \( v_i = v'_i \). \( \square \)

**Remark 3.10.** Equivalently, recall from Remark 3.6 that \( v_i \) may also be seen as the least-squares solution to \( dv_i = d_i v \). From this point of view, \( d^* dv_i = d^* d_i v \) is precisely the system of normal equations corresponding to this least-squares problem.

4. Weighted decompositions and restricted cooperation

4.1. Decomposition of cooperative games with weighted edges. Suppose that each edge \( (S, S \cup \{i\}) \in E \) of the hypercube graph is assigned a weight \( w(S, S \cup \{i\}) > 0 \). We define \( \ell^2_w(E) \) to be the space of 1-cochains.
equipped with the weighted $\ell^2$ inner product,
\[ \langle f, g \rangle_w := \sum_{(S,S \cup \{i\}) \in E} w(S, S \cup \{i\}) f(S, S \cup \{i\}) g(S, S \cup \{i\}). \]
The setting of Section 3 corresponds to the special case where $w = 1$ on every edge. (Equivalently, $w$ may be any positive constant, not necessarily 1.)

The weighted inner product affects the Hodge decomposition as follows. Although $d : \ell^2(V) \to \ell^2_w(E)$ is unchanged, $d^*_w : \ell^2_w(E) \to \ell^2(V)$ is now the adjoint with respect to the weighted inner product. We then have the combinatorial Hodge decomposition
\[ \ell^2_w(E) = \mathcal{R}(d) \oplus \mathcal{N}(d^*_w), \]
where the direct sum is $\ell^2_w$-orthogonal rather than $\ell^2$-orthogonal.

Denote by $P_w : \ell^2_w(E) \to \mathcal{R}(d)$ the $\ell^2_w$-orthogonal projection onto $\mathcal{R}(d)$. The decomposition of cooperative games in Theorem 3.4 may then be generalized as follows.

**Theorem 4.1.** For each $i \in N$, let $v_{i,w} \in \ell^2(V)$ with $v_{i,w}(\emptyset) = 0$ be the unique game such that $d v_{i,w} = P_w d_i v$. Then:

(a) $\sum_{i \in N} v_{i,w} = v$.

(b) If $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, then $v_{i,w} = 0$.

(c) If $\sigma$ is a permutation of $N$, then $(\sigma^* v)_{i,\sigma^* w} = \sigma^* (v_{\sigma(i),w})$. In particular, if $\sigma$ is the permutation swapping $i$ and $j$, and if $\sigma^* v = v$ and $\sigma^* w = w$, then $v_{i,w} = \sigma^*(v_{j,w})$.

(d) For any two games $v, v'$ and $\alpha, \alpha' \in \mathbb{R}$, $(\alpha v + \alpha' v')_{i,w} = \alpha v_{i,w} + \alpha' v'_{i,w}$.

**Proof.** The proofs of (a), (b), and (d) are just as in Theorem 3.4, since the weighted projection $P_w$ is still linear and equal to the identity on $\mathcal{R}(d)$.

For (c), we can no longer assume that a permutation preserves the $\ell^2_w$ inner product. However, we do have $\langle f, g \rangle_w = \langle \sigma^* f, \sigma^* g \rangle_{\sigma^* w}$, which implies $P_{\sigma^* w} \sigma^* = \sigma^* P_w$. Therefore,
\[ d(\sigma^* v)_{i,\sigma^* w} = P_{\sigma^* w} d_i (\sigma^* v) = P_{\sigma^* w} \sigma^* (d_{\sigma(i)} v) = \sigma^* (P_w d_{\sigma(i)} v) = \sigma^* (dv_{\sigma(i),w}) = d\sigma^*(v_{\sigma(i),w}), \]
and the rest of the argument proceeds as in the proof of Theorem 3.4.

**Corollary 4.2.** If $\sigma^* w = w$ for all permutations $\sigma$, then $\phi_i(v) = v_{i,w}(N)$.

**Proof.** If $w$ is invariant under permutations, then (a)–(d) imply that $v_{i,w}(N)$ satisfies the Shapley axioms, so it must be the Shapley value $\phi_i(v)$.

As in Remark 3.6, we may view the component $v_i$ as a weighted least-squares solution to $d v_i = d_i v$, in the sense that
\[ v_i = \arg \min_{u \in \ell^2(V)} \| d u - d_i v \|_{\ell^2_w(E)}. \]
We can also cast this in terms of the weighted graph Laplacians $L_w := d^* d$ and $L_{w_i} := d^*_i d_i = d^*_w d$, where the weight function $w_i$ is defined by

$$w_i(S, S \cup \{j\}) := \begin{cases} w(S, S \cup \{i\}) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The following generalization of Proposition 3.9 is stated without proof, since the proof is essentially identical. It can also be seen as an expression of the normal equations for the weighted least-squares problem.

**Proposition 4.3.** For each $i \in N$, the component game $v_i$ of Theorem 4.1 is the unique solution to $L_w v_i = L_{w_i} v_i$ such that $v_i(\emptyset) = 0$.

One consequence of Corollary 4.2 is that, although the Shapley value is unique, the decomposition $v = \sum_{i \in N} v_i$ of Theorem 3.4 generally is not. Indeed, any totally symmetric weight function $w$ will yield a decomposition satisfying the conditions of Theorem 3.4 and these will generally not agree with one another—except at $N$, where they all give the Shapley value. This is illustrated in the following example.

**Example 4.4.** In Example 3.8, we decomposed the glove game with respect to the $l^2$ inner product, corresponding to the constant weight function $w \equiv 1$. Suppose instead that we take $w(S, S \cup \{i\}) = |S| + 1$, which is totally symmetric but not constant. The resulting decomposition is shown in Table 2 and is distinct from that obtained in Example 3.8 and shown in Table 1. However, due to the symmetry of $w$, the Shapley values are again recovered as the value of the grand coalition in each component game. Note that the symmetry of players 2 and 3 is still apparent in the component games.

**Example 4.5.** Again, consider the glove game, but take the weight function to be $w(\emptyset, \{1\}) = \frac{1}{2}$ and $w = 1$ otherwise. This may be interpreted as player 1 being reluctant (but not totally unwilling) to be the first player to join the coalition. The resulting decomposition is shown in Table 3. Unlike in the previous examples, this $w$ is not totally symmetric, and consequently the values $v_1(N) = \frac{13}{17}$ and $v_2(N) = v_3(N) = \frac{2}{17}$ no longer agree with the Shapley values. Since player 1 is less willing to join the coalition first (i.e., to contribute zero marginal value), the payoff to player 1 is increased from $\frac{2}{3}$ to $\frac{13}{17}$ at the expense of players 2 and 3, the payoff to each of whom is reduced from $\frac{5}{6}$ to $\frac{2}{17}$. Note that the symmetry of players 2 and 3 is still maintained.

4.2. **Decomposition of games with restricted cooperation.** The framework discussed in the preceding sections, as in Shapley [25], assumes that every player $i \in N$ is willing to join every coalition $S \subset N$, so every such coalition may be feasibly formed en route to the grand coalition. In models of restricted cooperation, however, this is not the case. The precedence constraints of Faigle and Kern [11] impose a partial ordering on $N$, so that some players are constrained to join the coalition prior to others. Khmelnitskaya
Table 2. Decomposition of the three-player glove game as $v = v_1 + v_2 + v_3$, following Theorem 4.1, with weight function $w(S) = |S| + 1$. The Shapley values of $\frac{2}{3}$, $\frac{1}{6}$, $\frac{1}{6}$ appear in bold on the last line, since the weight function is totally symmetric, but the components are elsewhere distinct from those in Table 1.

| $S$    | $v$ | $v_1$ | $v_2$ | $v_3$ |
|--------|-----|-------|-------|-------|
| {}     | 0   | 0     | 0     | 0     |
| {1}    | 0   | $\frac{16}{31}$ | $-\frac{8}{31}$ | $-\frac{8}{31}$ |
| {2}    | 0   | $-\frac{8}{31}$ | $\frac{6}{31}$ | $\frac{2}{31}$ |
| {3}    | 0   | $-\frac{8}{31}$ | $\frac{2}{31}$ | $\frac{6}{31}$ |
| {1, 2} | 1   | $\frac{20}{31}$ | $\frac{21}{62}$ | $\frac{1}{62}$ |
| {1, 3} | 1   | $\frac{20}{31}$ | $\frac{1}{62}$ | $\frac{21}{62}$ |
| {2, 3} | 0   | $-\frac{9}{31}$ | $\frac{9}{62}$ | $\frac{62}{62}$ |
| {1, 2, 3} | 1 | $\frac{2}{3}$ | $\frac{1}{6} | \frac{1}{6} |

Table 3. Decomposition of the three-player glove game as $v = v_1 + v_2 + v_3$, following Theorem 4.1, with weight function $w(\emptyset, \{1\}) = \frac{1}{2}$ and $w = 1$ otherwise. The bold values $v_i(N)$ on the last line no longer correspond to the Shapley values, since $w$ is not totally symmetric.

| $S$    | $v$ | $v_1$ | $v_2$ | $v_3$ |
|--------|-----|-------|-------|-------|
| {}     | 0   | 0     | 0     | 0     |
| {1}    | 0   | $\frac{10}{17}$ | $-\frac{5}{17}$ | $-\frac{5}{17}$ |
| {2}    | 0   | $-\frac{5}{31}$ | $\frac{37}{272}$ | $\frac{3}{272}$ |
| {3}    | 0   | $-\frac{5}{31}$ | $\frac{3}{272}$ | $\frac{37}{272}$ |
| {1, 2} | 1   | $\frac{25}{31}$ | $\frac{87}{272}$ | $-\frac{15}{272}$ |
| {1, 3} | 1   | $\frac{25}{31}$ | $-\frac{15}{272}$ | $\frac{87}{272}$ |
| {2, 3} | 0   | $-\frac{3}{17}$ | $\frac{3}{272}$ | $\frac{3}{272}$ |
| {1, 2, 3} | 1 | $\frac{13}{17}$ | $\frac{2}{272}$ | $\frac{2}{17}$ |

et al. [18] have recently generalized this to so-called digraph games, where precedence is determined by a diagraph on $N$ that (unlike the Faigle and Kern [11]) may contain cycles; a player $i$ may be required to precede another
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Table 4. Decomposition of the three-player glove game as $v = v_1 + v_2 + v_3$, where $\{1\}$ and its incident edges are removed from the hypercube graph. This causes players 2 and 3 to become null players, so player 1 receives all the value, and the game becomes inessential.

| $S$       | $v$ | $v_1$ | $v_2$ | $v_3$ |
|-----------|-----|-------|-------|-------|
| $\emptyset$ | 0   | 0     | 0     | 0     |
| $\{2\}$   | 0   | 0     | 0     | 0     |
| $\{3\}$   | 0   | 0     | 0     | 0     |
| $\{1,2\}$ | 1   | 1     | 0     | 0     |
| $\{1,3\}$ | 1   | 1     | 0     | 0     |
| $\{2,3\}$ | 0   | 0     | 0     | 0     |
| $\{1,2,3\}$ | 1  | 1       | 0     | 0     |

player $j$ in some coalitions but not others. (For another recent model of restricted cooperation, see Koshevoy et al. [20].)

The constraints above all correspond to situations where a player $i$ is forbidden to join a coalition $S \subset N \setminus \{i\}$. In this case, we say that the edge $(S, S \cup \{i\})$ is infeasible, and we remove it from the hypercube graph. If we continue in this manner, removing all edges and vertices that are incompatible with the constraints, then we arrive at a graph $G = (V, E)$ which is a subgraph of the hypercube graph. Here, $V$ contains the so-called feasible coalitions that are compatible with the constraints.

Assume that $G$ is connected and that $\emptyset, N \in V$, so that a coalition is feasible if and only if it can be formed starting from $\emptyset$, and the grand coalition is feasible. Since the Hodge decomposition may be defined on any graph—in particular, on the subgraph $G$ of the hypercube graph—we again obtain a decomposition $v = \sum_{i \in N} v_i$, defined by $dv_i = Pd_i v$ with $v_i(\emptyset) = 0$ for $i \in N$. Since $G$ is connected, we again have $N(d) \cong \mathbb{R}$, so the decomposition is unique; moreover, it satisfies conditions (a)–(d) of Theorem 4.1 if we interpret the missing edges as having weight zero.

Example 4.6. In the glove game, suppose that player 1 refuses to join the coalition first, so that $\{1\}$ and all its incident edges are removed from the graph. The resulting decomposition is shown in Table 4. Note that $v_1 = v$ and $v_2 = v_3 = 0$, so that player 1 captures all of the value of the game. The reason for this is that, by removing the only edges on which players 2 and 3 contribute marginal value, the constraints have turned players 2 and 3 into null players. Observe also that $dv_i = d_i v$ for all $i \in N$, so the constraints have effectively made the game inessential.
Table 5. Decomposition of the three-player glove game as $v = v_1 + v_2 + v_3$, where $\{2\}$ and its incident edges are removed from the hypercube graph.

| $S$ | $v$ | $v_1$ | $v_2$ | $v_3$ |
|-----|-----|-------|-------|-------|
| $\emptyset$ | 0   | 0     | 0     | 0     |
| $\{1\}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{1}{5}$ |
| $\{3\}$ | 0   | $-\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{1}{5}$ |
| $\{1, 2\}$ | 1   | $\frac{2}{5}$ | $\frac{3}{5}$ | 0     |
| $\{1, 3\}$ | 1   | $\frac{1}{2}$ | $\frac{1}{10}$ | $\frac{2}{5}$ |
| $\{2, 3\}$ | 0   | $-\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ |
| $\{1, 2, 3\}$ | 1   | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{5}$ |

**Example 4.7.** On the other hand, suppose that player $\{2\}$ refuses to join the coalition first, so that $\{2\}$ and all its incident edges are removed from the graph. The resulting decomposition is shown in Table 5. Unlike in the previous example, all three players still contribute marginal value on some of the remaining feasible edges. However, removing an edge on which player 2 contributes zero marginal value causes the payoff to player 2 to increase from $\frac{1}{6}$ to $\frac{3}{10}$. Interestingly, player 3 also receives a slightly increased payoff, from $\frac{1}{6}$ to $\frac{7}{10}$, since player 3 contributes zero marginal value to any coalition that already contains player 2, and one such coalition has been removed from consideration. Both players 2 and 3 benefit at the expense of player 1, whose payoff is decreased from $\frac{2}{3}$ to $\frac{1}{2}$.

**Remark 4.8.** The values obtained in Example 4.7 are different from the Shapley values with precedence constraints in Faigle and Kern [11], Khmelevskaya et al. [18], which are $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. These take the approach of averaging over feasible permutations, a generalization of [2]. In this case, there are four feasible permutations—$(1, 2, 3)$, $(1, 3, 2)$, $(3, 1, 2)$, and $(3, 2, 1)$—and player 1 contributes marginal value 1 in two of these, while players 2 and 3 each contribute marginal value 1 in one permutation. This illustrates that, unlike the case in Section 3, these solution concepts are generally distinct.

(We also note that, as pointed out in Section 2.1, it is computationally undesirable to average over permutations, since this is factorial in $|N|$, whereas solving a $|V| \times |V|$ linear system is only exponential in $|N|$.)

Finally, we note that we may also consider *weighted* subgraphs of the hypercube graph, combining the approach above with that of Section 4.1 in the obvious way. The next example illustrates that if we weight the
edges according to the degrees of the incident vertices—which are generally non-constant once edges have been removed—we may obtain different decompositions of the game than if we used a constant edge weight.

**Example 4.9.** Consider again the restricted glove game of Example 4.7, where player 2 refuses to join the coalition first, so that \( \{2\} \) and its incident edges are removed from the hypercube graph. Instead of taking the edge weights \( w \equiv 1 \), suppose we take \( w(a, b) = \text{deg}(a)\text{deg}(b) \). This weight function is related to a graph Laplacian with vertex weights considered by Lovász \[22\], as observed by Chung and Langlands \[7\].

The resulting decomposition is shown in Table 6. In this case, the players receive the payoffs \( (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \), which is distinct from the constant-weight payoffs \( (\frac{1}{2}, \frac{3}{10}, \frac{1}{5}) \) of Example 4.7. (This also happens to agree with the Shapley values with precedence constraints in Remark 4.8, although this need not be the case for an arbitrary game.) Note that, although \( v_2(N) = v_3(N) \), the component games \( v_2 \) and \( v_3 \) display asymmetry elsewhere: e.g., \( v_2(\{2, 3\}) \neq v_3(\{2, 3\}) \). This is due to the asymmetry between players 2 and 3 in the graph.

### 5. Conclusion

We have used the combinatorial Hodge decomposition on a hypercube graph to show that any cooperative game may be decomposed into a sum of component games, one component for each player, so that this decomposition satisfies appropriate efficiency, null-player, symmetry, and linearity properties. This yields a new characterization of the Shapley value as the value of the grand coalition in each player’s component game.

We have also shown that this game decomposition may be understood in terms of the least-squares solution to a linear problem, where the solution is
exact if and only if the game is inessential. In this sense, our decomposition may be considered as an edge-based (rather than vertex-based) variant of the least-squares and minimum-norm solution concepts of Ruiz et al. [23] and Kultti and Salonen [21]. The normal equations for this linear problem yield another, equivalent characterization of the game decomposition in terms of the well-studied graph Laplacian.

Finally, we have shown how this decomposition may be generalized, in a natural way, using the combinatorial Hodge decomposition for weighted graphs and subgraphs of the hypercube graph. These generalized decompositions preserve the efficiency, null-player, symmetry (in an appropriate sense, modulo the symmetry of the weights and the subgraph), and linearity properties obtained earlier. This yields a family of decompositions, and corresponding solution concepts, for problems where players exhibit variable willingness or unwillingness to join certain coalitions, and we have compared and contrasted these solution concepts with those of Faigle and Kern [11], Khmelnitskaya et al. [18] for certain models of restricted cooperation.

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