Relativistic diffusive motion in thermal 
electromagnetic fields

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Abstract

We discuss relativistic dynamics in a random electromagnetic field 
which can be considered as a high temperature limit of the quantum 
electromagnetic field in a heat bath (cavity) moving with a uniform velocity \( w \).
We derive a diffusion approximation for particle’s dynamics generalizing 
the diffusion of Schay and Dudley. It is shown that Jüttner distribution 
is the equilibrium state of the diffusion.

1 Introduction

The dynamics of a test (tracer) particle in a medium of some other particles 
is often approximated by diffusion [1]. It is an old result [2][3] that the Vlasov 
equation describing a tracer particle moving in a random electromagnetic field 
produced by a chaotic motion of other particles can be approximated by a 
diffusion equation. The diffusion approximation finds applications in heavy 
ion collisions [4][5], plasma physics and in astrophysics [2][3][6][7]. A detailed 
discussion of the diffusion approximation can be found in [8][9] [10][11] [12].

The mathematical theory of a relativistic diffusion is not so well developed 
as the non-relativistic one (see the reviews in [13][14]). A relativistic diffusion 
in the configuration space does not exist [15][16]. If the diffusing particle has a 
fixed mass then an analog of the Kramers diffusion (on the phase space) without 
friction is uniquely determined. The generator of the diffusion is defined by the 
Laplace-Beltrami operator on the mass shell as has been shown by Schay [17] 
and Dudley [18]. In our earlier papers [19][20]we have studied a friction term in 
the framework of the relativistic Brownian motion of Schay, Dudley, Franchi and 
Le Jan [21]. The friction term has been defined as a drift uniquely determined by 
the equilibrium state and by the requirement of the reversibility of the process 
(in physical terms by the detailed balance). Such a drift term has been obtained
by other means in [22][23]. Without a friction the relativistic Brownian motion is unphysical, e.g., the particle energy grows exponentially fast [20].

Following a development of the theory of the non-relativistic Brownian motion we may expect that relativistic deterministic motion, which is chaotic in certain sense, can also be approximated by a relativistic diffusion. In our earlier paper [24] we have discussed the dynamics of a relativistic particle in a Poincare invariant random electromagnetic field. We have shown that the relativistic diffusion of Schay and Dudley can be considered as an approximation of the proper time dynamics in this field. However, such a Poincare invariant electromagnetic field is rather unphysical. In this paper we discuss particle’s dynamics in a thermal (random) electromagnetic field with a distinguished frame of reference. We can think of the Universe filled with the CMB radiation as an example of a Poincare covariant system with a heat bath [25]. When describing a random system it is more appropriate to apply a statistical (Liouville) description. The evolution in the proper time is determined by the expectation value of an exponential of the Liouville operator. The expectation value is defined by the probability distribution of the electromagnetic field. The two-point function is expressed by its Fourier transform $\tilde{G}$. We do not need to specify the thermal distribution $\tilde{G}$ (it can be the Planck distribution). We consider an observer moving in an inertial frame with the velocity $w$ (this is equivalent to moving the heat bath and keeping the observer at rest). For such an observer the thermal distribution is a function of the momentum $k$ of the electromagnetic modes and $w$ [25][26]. We restrict ourselves to the expansion of the exponential of the Liouville operator till the second order in the electromagnetic field strength (the square of the Liouville operator). Assuming that the correlation functions of the electromagnetic modes (Fourier components) are decaying fast for the time larger than the correlation time $\tau_c$ we can apply the Markov approximation treating the particle’s subsequent scatterings upon electromagnetic modes as independent. According to Kubo’s theory [8]-[9] the second order differential operator arising in the small time expansion is the generator of the diffusion defining the evolution of the system also at times large in comparison to the correlation time $\tau_c$. We show that the form of this diffusion is determined uniquely by the assumptions of the Lorentz invariance of $\tilde{G}$. This assumption allows to express the expectation value of the square of the Liouville operator unequivocally up to two constants. It follows that the generator of the diffusion depends only on the momenta and on the frame $w$. The diffusion equation in the proper time is a generalization of the diffusion of Schay [17] and Dudley [18] (corresponding to zero temperature). The particle mass is preserved by the dynamics. It enters in an additive way in the model. We can sum over the mass when computing expectation values of observables, e.g., the energy-momentum or entropy. We think that keeping the mass as a variable (we can fix it at the end if necessary) can be a useful tool in some models of diffusion. In quantum physics [27] the Wigner function is an analog of the classical phase space distribution. The square of four-momentum is off the mass shell in the Wigner
function description [27] (a matter with a continuous mass spectrum has been suggested recently in [28]). In an application of the diffusion equation to cosmology [29][30] the mass of a galaxy can be considered as a parameter varying continuously. The diffusion equation, which we have obtained in a model of an interaction with the electromagnetic thermal radiation, could have a broader application to any diffusive dynamics in an environment characterized by a distinguished reference frame (the heat bath). The diffusion in the proper time originated from the dynamics in a random field. In a random system the meaning of the proper time is loosing its physical meaning. We should express the evolution of states and observables (e.g., the energy-momentum) in the coordinate (laboratory) time. We obtain a transport equation in the laboratory time from the requirement that probability density should not depend on the proper time. There is a certain similarity between our transport equation and the one of refs.[31][32][33]. These authors describe a diffusion of a particle in a fluid. The fluid supplies a distinguished reference frame. Our form of the equation and the constants appearing in it result from a concrete physical model which has not been considered in refs.[31]-[33].

The plan of the paper is the following. In secs.2-3 we repeat the formulation [24] of the kinetic equation in an electromagnetic field in order to fix the notation. In sec.4 the two-point function of a random electromagnetic field depending additionally on the four-velocity \( w^\mu \) is discussed. The exponential of the Liouville operator is expanded till the second order in the proper time. Its expectation value is calculated. In sec.5 we take the diffusion generator as calculated in previous sections as a basis of the relativistic diffusion theory. We show that Jüttner distribution is the equilibrium state of the diffusion.

2 Kinetic equation of particles in an electromagnetic field

The dynamics of a particle in an electromagnetic field is described by the equations [34]

\[
\sqrt{p^2} \frac{dx^\mu}{d\tau} = p^\mu, \tag{1}
\]

\[
\sqrt{p^2} \frac{dp_\mu}{d\tau} = \mathcal{F}_{\mu\nu} p^\nu, \tag{2}
\]

where \( \mu = 0, 1, 2, 3 \), \( x \) denotes the coordinate and \( p \) the four-momentum. \( \mathcal{F} \) is an antisymmetric field strength of the electromagnetic field. We could express the mass as \( \sqrt{p^2} \) without changing the meaning of eqs.(1)-(2) because it follows from eq.(2) that \( p^2 = \eta_{\mu\nu} p^\mu p^\nu = const \) as a consequence of the antisymmetry of \( \mathcal{F}_{\mu\nu} \) and

\[
\eta_{\mu\nu} p_\mu \frac{d}{d\tau} p_\nu = 0 \tag{3}
\]
with $\eta_{\mu\nu} = (1, -1, -1, -1)$. From eqs.(1)-(2) it follows that $\tau$ is proportional to the proper time as

$$d\tau^2 = dx^\mu dx_\mu = c^2 dt^2 (1 - c^{-2} \left(\frac{dx}{dt}\right)^2),$$

where $x^0 = ct$ (we set $c = 1$ from now on for convenience). We can eliminate $\tau$ from eqs.(1)-(2) in favor of $x^0$ if we wish.

In the kinetic theory of classical particles we consider a distribution $\mathcal{K}_\tau$ of trajectories $(x(\tau, y, q), p(\tau, y, q))$ in the phase space starting from $(y, q)$

$$\mathcal{K}_\tau(x, p; y, q) = \delta(x - x(\tau, y, q))\delta(p - p(\tau, y, q)).$$

(4)

We can consider more general distributions of trajectories $\Phi_\tau$ by spreading the initial points in the phase space with a certain probability distribution $\Phi$

$$\Phi_\tau(x, p) = \int \mathcal{K}_\tau(x, p; y, q) \Phi(y, q) dy dq.$$  

(5)

$\mathcal{K}_\tau$ as well as $\Phi_\tau$ satisfy the differential equation

$$\frac{d}{d\tau} \Phi_\tau \equiv G^+ \Phi_\tau = -\frac{1}{\sqrt{p^2}}(p^\mu \frac{\partial \Phi_\tau}{\partial x^\mu} + \mathcal{F}^{\mu\nu} p_\nu \frac{\partial \Phi_\tau}{\partial p^\mu}).$$

(6)

$\mathcal{K}_\tau$ with the initial condition $\delta(x - y)\delta(p - q)$ and $\Phi_\tau(x, p)$ with the initial condition $\Phi(x, p)$ . We defined here a first order operator $G^+$ which will be discussed later in more detail. In eq.(6) we assumed that $p_0$ is an independent variable. Working with the four-momentum off the mass shell allows to preserve explicit Lorentz invariance. A formulation with an independent $p_0$ may have applications to quantum phase space methods based on the Wigner function [27][35][36]. If $p$ is on the mass shell then we skip $p_0$ derivative in eq.(6).

The density of trajectories $\Omega$ in the laboratory time $x^0$ can be expressed by $\Phi_\tau$ [27]

$$\Omega(x, p) = \int d\tau \Phi_\tau(x, p).$$

(7)

It satisfies the transport equation

$$G^+ \Omega = 0.$$  

(8)

In the Liouville approach to the classical statistical mechanics we consider functions $W$ on the phase space as observables. We may define the expectation value $\Phi_\tau(W)$ of $W$ in the state $\Phi_\tau$ as

$$\Phi_\tau(W) = (\Phi_\tau, W) = \int dx dp \Phi_\tau(x, p) W(x, p) \equiv (\Phi, W_\tau) = \int dy dq \Phi(y, q) W(x(\tau, y, q), p(\tau, y, q)),$$

(9)
The semigroup \( (T_\tau W)(y, q) \equiv W_\tau(y, q) = W(x(\tau, y, q), p(\tau, y, q)) \) is unitary in the scalar product (9). We have from the definition of \( W_\tau \) in eq.(9)

\[
\frac{d}{d\tau} (T_\tau W)(y, q) = (T_\tau G W)(y, q) = (G T_\tau W)(y, q) = G W_\tau,
\]

where the generator \( G \) of the semigroup \( T_\tau = \exp(\tau G) \) is the adjoint of \( G^\dagger \) of eq.(6) in the Hilbert space \( L^2(dydq) \) (by \( dy \) or \( dq \) we denote an integral over \( R^4 \); vectors from \( R^3 \) will be denoted by a boldface letter). In the derivation of eq.(10) we apply the semigroup law \( T_{\tau+s} = T_\tau T_s \) in the form \( W(x(s, x_\tau, p_\tau), p(s, x_\tau, p_\tau)) = W(x(\tau, x_s, p_s), p(\tau, x_s, p_s)) \) (where \( (x_\tau, p_\tau) \) is the shorthand notation for the trajectory of eq.(4)) resulting from the composition of trajectories. Then, the equality \( T_\tau G = G T_\tau \) in eq.(10) follows by differentiation. The requirement of the \( \tau \)-independence of the probability distribution \( \frac{d}{d\tau} \Phi_\tau = 0 \) gives the kinetic equation in the laboratory time \( t \). This is the same equation as the one which can be derived by an elimination of \( \tau \) in favor of \( t \) in the evolution equations (1)-(2).

### 3 Random dynamics

In this section we repeat some definitions used in our earlier paper [24] in order to fix the notation. We write the formula (10) for an evolution of a function of trajectories in a general form as

\[
\frac{d}{d\tau} W = (X + Y)W,
\]

where

\[
X = \frac{1}{\sqrt{p^2}} p^\mu \partial_\mu
\]

and

\[
Y = \frac{1}{\sqrt{p^2}} F^{\mu\nu} p_\nu \partial_\mu.
\]

From now on derivatives over space-time coordinates will have an index \( x \) and the derivatives without an index mean the derivatives over momenta. Let

\[
Y(\tau) = \exp(-\tau X)Y \exp(\tau X) = \frac{1}{\sqrt{p^2}} F^{\mu\nu}(x - \frac{1}{\sqrt{p^2}} \tau p_\nu)(\partial_\mu + \frac{1}{\sqrt{p^2}} \tau \partial_\nu^\mu).
\]

Then, the solution of eq.(11) can be expressed in the form (an analog of the interaction picture in quantum mechanics)

\[
W_\tau = \exp(\tau X)W_\tau^I,
\]
where
\[ \partial_\tau W^\tau_I = Y(\tau)W^\tau_I. \] (16)

We assume that the electromagnetic field is random. We consider an expectation value of $W_I$ (we could equivalently discuss $\Phi_I$ with the same result). In general, we have the cumulant expansion for the expectation value (the expectation value over the electromagnetic field $F$ is denoted by $\langle . \rangle$, the initial condition $W$ is assumed to be independent of $F$)

\[ \langle W_I^\tau \rangle = \exp \left( \int_0^\tau ds \langle Y(s) \rangle W_I^\tau + \frac{1}{4} \int_0^\tau ds \int_0^s ds' \langle (A(s)A(s') + A(s')A(s)) \rangle + \ldots \right) W, \] (17)

here
\[ A(s) = Y(s) - \langle Y(s) \rangle \]

If $[Y(s), Y(s')] = 0$ and $Y$ is a linear function of Gaussian variables then eq.(17) is exact (with no higher order terms). The expansion of the dynamics (11) till the second order term reads

\[ \langle W_I^\tau \rangle = W + \int_0^\tau ds \langle Y(s) \rangle W + \frac{1}{4} \langle \int_0^\tau ds A(s) \rangle^2 W + \frac{1}{2} \langle \int_0^\tau ds A(s) \rangle W + \ldots \] (18)

The approach of Kubo [8]-[9] approximates the random Liouville operator in the exponential (17) by the $\tau^2$ term in the expansion (18)(we discuss this approximation in the next section).

4 Diffusion in random electromagnetic fields

In ref.[24] we have calculated the generator of the diffusion in a Poincare invariant random electromagnetic field. We have assumed that the correlation functions of the electromagnetic field depend only on the space-time coordinates. In this paper we discuss the correlation functions which may depend also on a time-like vector $w^\mu$ (without loss of generality we may assume $w^\mu w_\mu = 1$; as will be seen from the computations the vector $w^\mu$ plays also a role of an ultraviolet cutoff). We have in mind a model of a particle moving in the electromagnetic field of the black body radiation at finite temperature $\beta^{-1}$ (the heat bath). The Lorentz invariance of the particle system is lost if we keep the heat bath at rest. We can express the Lorentz invariance of the whole system (a particle interacting with the electromagnetic field and the environment) considering a heat bath moving with a velocity $w^\mu$. In quantum field theory the correlation functions of the quantum electromagnetic field are usually calculated in the rest frame of the heat bath [37](for explicitly Lorentz covariant calculations see [38]). Correlation functions of quantum fields are not symmetric (quantum fields do not commute). Hence, they do not describe a random field. However, in the high temperature limit the non-symmetric part can be neglected [37]. In this limit the correlations can be represented as correlation functions of random fields. We split
\[ F_{\mu\nu} = \langle F_{\mu\nu} \rangle + F_{\mu\nu}, \]  

where \( F_{\mu\nu} \) has zero mean value. We assume that an average \( \langle \cdot \rangle \) over \( \mathcal{F} \) is defined which preserves the Poincaré symmetry. This means that the two-point function defined by

\[ \langle F_{\mu\nu}(x)F_{\sigma\rho}(x') \rangle = G_{\mu\nu;\sigma\rho}(x - x') \]  

is a tensor. \( G_{\mu\nu;\sigma\rho} \) is symmetric under the exchange of indices \( (\mu\nu; x) \) and \( (\sigma\rho; x') \) and antisymmetric under the exchange \( \mu \rightarrow \nu \) and \( \sigma \rightarrow \rho \). We impose the Bianchi identities (as \( \mathcal{F} \) is expressed by a potential)

\[ \partial_\sigma F_{\mu\nu,\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0. \]  

In terms of the two-point function

\[ \partial_\sigma \epsilon^{\alpha\beta\mu\nu} G_{\mu\nu;\sigma\rho} = 0. \]  

For Fourier transforms eq.(20) reads

\[ \langle \tilde{F}_{\mu\nu}(k)\tilde{F}_{\sigma\rho}(k') \rangle = \tilde{G}_{\mu\nu;\sigma\rho}(k)\delta(k - k'), \]  

where \( \tilde{G}_{\mu\nu;\sigma\rho}(k) \) is a tensor which must be constructed from the vectors \( k_\mu, w_\mu \) and the fundamental four-dimensional tensors \( \eta_{\mu\nu} \) and \( \epsilon_{\mu\nu\rho\sigma} \). Hence, in general we could have with some scalar functions \( a_\nu \) (higher order tensors in \( k \) and \( w \) would not satisfy the antisymmetry requirements mentioned below eq.(20))

\[ \tilde{G}_{\mu\nu;\sigma\rho}(k) = a_1(\eta_{\mu\sigma}k_\rho - \eta_{\rho\sigma}k_\mu) + a_2(\eta_{\mu\sigma}w_\rho - \eta_{\rho\sigma}w_\mu) + a_3(\eta_{\mu\sigma}w_\rho - \eta_{\rho\sigma}w_\mu). \]

However, the Bianchi identities (22) lead to \( a_2 = a_3 = 0 \). The reason is that in eq.(22) (in the \( x \)-representation) we need in addition to \( \partial_\sigma \epsilon^{\alpha\beta\mu\nu} \) one more derivative with an index \( \nu \) or \( \mu \) in order to give zero on the rhs. There are terms multiplied by \( a_2 \) without such a derivative, hence \( a_2 = 0 \). It is clear that the \( a_3 \) terms cannot satisfy (22) because they are not symmetric in \( (\mu, \nu) \). Hence, \( a_3 = 0 \). The requirement of positivity of the probability measure defining the expectation value (20) leads to \( a_0 = 0 \) (as discussed in [24]); in the case of a quantum free electromagnetic field at finite temperature [37] we obtain also the representation (24) where only \( a_1 \neq 0 \). Summarizing, the two-point function of a random electromagnetic field satisfying the Bianchi identities must have the form

\[ G_{\mu\nu;\sigma\rho}(x, x') = \int dk \tilde{G}_{\mu\nu;\sigma\rho}(k) \exp(ik(x - x')) \]

\[ = \int dk \tilde{G}(k) \exp(ik(x - x'))(\eta_{\mu\sigma}k_\rho - \eta_{\rho\sigma}k_\mu) \]

\[ = \int dk \tilde{G}(k) \exp(ik(x - x'))(\eta_{\mu\sigma}k_\rho - \eta_{\rho\sigma}k_\mu), \]

Hence,

\[ \langle F_{\mu\nu}(x)F_{\sigma\rho}(x') \rangle_\beta = -D_{\mu\nu;\sigma\rho}G(x - x'), \]  

where \( D_{\mu\nu;\sigma\rho} \) is a tensor which must be constructed from the vectors \( k_\mu, w_\mu \) and the fundamental four-dimensional tensors \( \eta_{\mu\nu} \) and \( \epsilon_{\mu\nu\rho\sigma} \). Hence, in general we could have with some scalar functions \( a_\nu \) (higher order tensors in \( k \) and \( w \) would not satisfy the antisymmetry requirements mentioned below eq.(20))

\[ D_{\mu\nu;\sigma\rho}(k) = a_1(\eta_{\mu\sigma}k_\rho - \eta_{\rho\sigma}k_\mu) + a_2(\eta_{\mu\sigma}w_\rho - \eta_{\rho\sigma}w_\mu) + a_3(\eta_{\mu\sigma}w_\rho - \eta_{\rho\sigma}w_\mu). \]
where
\[ D_{\mu\nu;\sigma\rho} = -\eta_{\mu\sigma} \partial^\sigma_x \partial^\rho_x + \eta_{\mu\rho} \partial^\rho_x \partial^\sigma_x - \eta_{\nu\rho} \partial^\rho_x \partial^\nu_x + \eta_{\nu\sigma} \partial^\sigma_x \partial^\mu_x. \] (27)

The two-point function is positive definite if and only if \( \tilde{G}(k) \) in eq.(25) satisfies the condition
\[ \tilde{G}(k) \geq 0 \] (28)
and \( \tilde{G}(k) = 0 \) if \( k^2 < 0 \) (as we have shown in [24]).

It follows from eq.(25) that
\[ (D_{\mu\nu;\sigma\rho}G)(x) = \eta_{\mu\sigma} T_{\nu\rho}(x) - \eta_{\mu\rho} T_{\nu\sigma}(x) + \eta_{\nu\rho} T_{\mu\sigma}(x) - \eta_{\nu\sigma} T_{\mu\rho}(x), \] (29)
where
\[ T_{\mu\nu}(x) = -\partial^\mu_x \partial^\nu_x G(x) = \int dk \tilde{G} k_{\mu} k_{\nu} \exp(ikx). \] (30)

We shall need solely
\[ T_{\mu\nu}(0) = \int dk \tilde{G} k_{\mu} k_{\nu} = \epsilon w_{\nu} - \pi_\epsilon (\eta_{\mu\nu} - w_{\mu} w_{\nu}) \] (31)

The decomposition of the rhs of eq.(31) follows from the Lorentz invariance. If \( \tilde{G} \) has the meaning of the distribution of electromagnetic modes (photons) then \( \epsilon \) is the energy density and \( \pi_\epsilon \) the pressure. The constants in eq.(31) can be expressed by \( \tilde{G} \)
\[ \epsilon = \int dk \tilde{G}(k)(kw)^2, \]
\[ \pi_\epsilon = \frac{1}{3} \int dk \tilde{G}(k)((kw)^2 - k^2). \]

If \( \tilde{G}(k) \) is vanishing for \( k^2 < 0 \) (as is the case for a probability distribution of electromagnetic modes) then from eq.(31) we can derive the inequality
\[ \epsilon \geq 3 \pi_\epsilon \geq 0. \] (32)

We could express the expectation value \( \langle A(s)A(s') \rangle \), needed for a calculation of the diffusion, by \( \partial_{\mu} \partial_{\nu} G\left( \frac{1}{\sqrt{p^2}}(s - s')p \right) \) of eq.(30) calculated at the point \((s - s')p \) (it comes from the correlation functions of \( F^{\mu\nu}(x - \frac{1}{\sqrt{p^2}}ps, p) \) of eq.(14) calculated at \( s \) and \( s' \) with the covariance (26)). This function has more involved tensor decomposition than \( T^{\mu\nu} \) of eq.(31) because there are still some tensors constructed from \( p_{\mu} \) and \( w_{\mu} \) on the rhs of eq.(30). However, these terms will be of higher order than \( \tau^2 \). In the limit of small \( \tau \) we calculate
\[ A_w = \lim_{\tau \to 0} \frac{1}{\tau^2} \left( \int_0^\tau A(s) \, ds \right)^2 = \lim_{\tau \to 0} \frac{1}{\tau^2} \int_0^\tau ds \int_0^\tau ds' \langle A(s)A(s') + A(s')A(s) \rangle \]
\[ = \left( -\eta^{\mu\sigma} T^{\nu\rho}(0) + \eta^{\mu\rho} T^{\nu\sigma}(0) - \eta^{\nu\rho} T^{\mu\sigma}(0) + \eta^{\nu\sigma} T^{\mu\rho}(0) \right) \frac{1}{\sqrt{p^2}} p_{\rho} \partial_{\sigma} - \frac{1}{\sqrt{p^2}} p_{\sigma} \partial_{\rho}. \] (33)
We dropped the $s\partial^s$ term from eq.(14) because it does not contribute to the limit of small $\tau$. If we apply the formula (31) then we obtain

$$A_w \equiv 2\pi \eta \eta_{\mu\sigma} \partial_{\mu} \eta_{\sigma
u} - 6p^2 \eta_{\mu\nu} \partial_{\mu} \partial_{\nu}$$

$$- (\epsilon + \pi_\sigma) p^2 (p^\mu p^\nu + w^\mu p^\nu) \partial_{\mu} \partial_{\nu} + p^2 w^\mu w^\nu \partial_{\mu} \partial_{\nu}$$

$$+ (\epsilon + \pi_\sigma) p^{-2}(p^\mu \partial_{\mu} \eta_{\sigma\rho} + 2w^\mu w^\sigma \partial_{\mu})$$

$$\equiv \partial_{\mu} \alpha_{\mu\nu} \partial_{\nu},$$

(34)

where

$$\alpha_{\mu\nu} = 2\pi \eta \eta_{\mu\sigma} p^2 - (\epsilon + \pi_\sigma)(w^\mu p^\nu + p^\mu w^\nu) = P_{\mu\sigma} C_{\sigma\rho} P_{\rho\nu}$$

(35)

with

$$p^2 P_{\mu\sigma} = \eta_{\mu\sigma} p^2 - p^\mu p^\sigma$$

(36)

and

$$C_{\sigma\rho} = 2\pi \eta \eta_{\sigma\rho} - (\epsilon + \pi_\sigma)(w^\sigma p^\rho + w^\rho p^\sigma).$$

(37)

It should be pointed out that without the velocity $w$ the limit in eq.(33) would not exist. In fact, $T_{\mu\nu}(0)$ is infinite if $\tilde{G}$ depends only on $k^2$. A damping factor in $\tilde{G}$ is necessary if the integral (31) is to exist (such a damping factor is present in the Planck distribution (47)).

The approach of Kubo [8]-[9] approximates the random Liouville operator on the rhs of eq.(18) by the $\tau^2$ term. Kubo shows [8]-[9] that the kinematic $\tau^2$ behaviour in random dynamics at times short in comparison to the correlation time $\tau_c$ (of the electromagnetic modes) goes into the diffusive $\tau$ behaviour at times large in comparison to the correlation time. Kubo’s argument can be considered as a Markov approximation (no memory of successive steps) to the random dynamics.

Another interpretation of the diffusion approximation (which can be made rigorous [11]) interprets the small $\tau$ expansion as a weak field expansion. The weak field can be rescaled to a large time. The rescaling can be interpreted as a transformation of a microscopic behaviour of a particle system into an evolution at a macroscopic time scale. In more detail, eq.(2) is rewritten as

$$\frac{dp^\mu}{d\tau} = \delta^2 \frac{1}{\sqrt{p^2}} F^\mu\nu p^\nu.$$  

Then, rescaling the time and an expansion in $\delta$ lead to the diffusion with the diffusion tensor

$$\tilde{\alpha}_{\mu\sigma} = \int_0^\infty ds (F^{\mu\nu}(x) F^{\rho\sigma}(x - s \frac{1}{\sqrt{p^2}})) p_\nu p_\rho.$$  

(38)

The integral (38) is proportional to the correlation time $\tau_c$. As discussed in [8]-[9] even if the correlations of the random fields are not decaying exponentially then
nevertheless the behaviour of the particle dynamics in random fields described by the matrix $\alpha$ for times small in comparison to the correlation time determines the behaviour for times large in comparison with the correlation time. The large time behaviour is diffusive with the diffusion matrix $\tilde{\alpha}$ related to $\alpha$ in eq.(34) by

$$\tilde{\alpha}^{\mu\sigma} = \tau_c \alpha^{\mu\sigma}. \quad (39)$$

The Kubo’s formulas (the analogs of eqs.(38)-(39)) are widely applied in the transport theory of random systems.

As will be discussed in sec.5 in general the diffusion generated by $A_\nu$ (34) has no equilibrium. We need a friction term in order to stabilize the behaviour at large time. We would like to point out that if $\langle \partial^\nu x \sigma F^\mu\nu \rangle \neq 0$ then in the expansion (18) there is a drift term which can lead to a friction. Let us consider in eq.(18) the term which is of the first order in the field. Expanding $F(x - s \frac{1}{\sqrt{p^2}} p)$ in $s$ we obtain

$$K = \int_0^\tau ds \langle Y(s) \rangle = \tau \langle F^\mu\nu(x) \rangle \frac{1}{\sqrt{p^2}} p^\nu \partial^\mu - \tau^2 \langle \partial^\nu x \sigma F^\mu\nu \rangle p^{-2} p^\sigma p^\nu \partial^\mu + o(\tau^2). \quad (40)$$

Poincare invariance implies that

$$\langle F^\mu\nu \rangle = 0 \quad (41)$$

because no antisymmetric tensor can be built from $\epsilon^{\mu\nu\alpha\beta}$, $\eta^{\mu\nu}$ and $w^\mu$. Assuming that the second term on the rhs of eq.(40) is non-zero we have from the Poincare invariance

$$\langle \partial^\nu x \sigma F^\mu\nu \rangle = -\frac{r}{2} (\eta_{\sigma\nu} w^\mu - \eta_{\sigma\mu} w^\nu) \quad (42)$$

with a certain constant $r$. Hence,

$$K = rw p p^{-2} p^\mu \partial^\mu - rw^\mu \partial^\mu = -r P^{\mu\nu} w_\nu \partial^\mu. \quad (43)$$

with $P^{\mu\nu} = \eta^{\mu\nu} - p^{-2} p^\mu p^\nu$. We can get an interpretation of eq.(42) if we calculate an electric current of the electromagnetic field

$$J_\nu = \partial^\mu F^\mu\nu. \quad (44)$$

From eq.(42)

$$\langle J^\mu \rangle = -\frac{3r}{2} w^\mu. \quad (45)$$

The current (45) results from a charge $-\frac{3r}{2}$ moving with the velocity $w^\mu$. We can obtain such a current in a finite temperature quantum field theory of interacting electromagnetic and complex scalar fields $\phi$ [37]. The density matrix is

$$\rho_\beta = \exp(-\beta P^{\nu} w_\nu), \quad (46)$$
where $\beta^{-1}$ is the temperature, $P_\mu$ is the four-momentum of the quantum fields and $w^\mu$ describes the moving frame. Then, calculating the expectation value (of the electric current) we obtain in the lowest order of the expansion in the coupling constant

$$\langle J_\mu \rangle_\beta = -i \text{Tr} \left( \exp(-\beta P^\nu w_\nu) (\overline{\phi} \phi - \phi \overline{\phi}) \right)$$

$$= -\int dk k_\mu^{-1} k_\mu (\exp(\beta k w) - 1)^{-1} = -\frac{\beta}{2} w_\mu$$

with a certain constant $r$. Here, $\langle \cdot \rangle_\beta$ denotes an expectation value (defined by the trace) with respect to the state (46).

5 Relativistic diffusion at finite temperature

We define the diffusion generator by the lowest order ($\tau^2$) term in the expansion (18) in proper time (including the friction term coming from a constant current as discussed at the end of sec.4). We incorporate Kubo’s argument that this operator multiplied by the correlation time generates a diffusion properly approximating the random dynamics. In this way we are led to the following diffusion equation for particles with a continuous mass spectrum (by a certain abuse of notation we denote by $W$ the expression which has the meaning of $\langle W \rangle$ in random dynamics of sec.3)

$$\frac{1}{\tau_c} \partial_\tau W \equiv G^w W = \frac{1}{\tau_c} p^\rho \partial_\rho W + \partial_\mu \alpha^{\mu\nu} \partial_\nu W - r P^\mu w_\mu \partial_\mu W,$$  (48)

where $r$ is determined by the expectation value (42).

The diffusion in the proper time originates from the relativistic dynamics in the proper time. Then, the proper time has a well-defined meaning. This is the time measured in the frame moving with the particle. However, if the particle motion is random then this frame is random as well. Such a random frame is losing a physical meaning. For this reason we should express the kinetic transport in terms of the coordinate (deterministic, laboratory) time. The probability distribution evolves according to the adjoint equation (6). The independence of the proper time parametrization ($\frac{d}{d\tau} \Phi_\tau = 0$) is equivalent to the replacement of the proper time by the physical time $t$ in the kinetic equation (8). We extend this requirement to the diffusion (48). In this way we obtain the transport equation

$$G^+_w \Omega = -\frac{1}{\tau_c} p^\rho \partial_\rho \Omega + A_w \Omega + r \partial_\mu P^{\mu\nu} w_\nu \Omega = 0,$$  (49)

where the adjoint of $G_w$ is in $L^2(dx dp)$.

Explicitly,

$$\frac{1}{\tau_c} p^\rho \partial_\rho \Omega = \partial_\mu (\alpha^{\mu\rho} \partial_\rho + r P^{\mu\rho} w_\rho) \Omega \equiv \partial_\mu \left( 2 \pi \tau_c P^\rho \partial_\rho + r P^{\mu\rho} w_\rho \ight. \left. - (\epsilon + \pi_c) p^{-2} (w^\mu p^\rho - (wp)^2 \partial^\mu - (wp)(w^\mu p^\rho + w^\rho p^\mu) \partial_\rho + p^2 w^\mu w^\rho \partial_\rho) \right) \Omega.$$  (50)
If the condition (32) is satisfied then $\partial_\mu \alpha^{\mu\nu} \partial_\nu$ is a non-positive operator as an expectation value (33) of the square of a differential operator. It follows (see e.g.,[39]) that the momentum dependent coefficients $\alpha^{\mu\nu}$ satisfy the positivity condition

$$a_\mu a_\nu \alpha^{\mu\nu} \geq 0.$$  \hspace{1cm} (51)

However, we can prove that the inequality (51) is satisfied under the condition

$$\epsilon \geq \pi_\epsilon \geq 0$$  \hspace{1cm} (52)

weaker than (32). We can use the Lorentz invariance of the inequality (51) in order to choose a proper Lorentz frame. So, if $a^2 > 0$ we choose a frame such that $a = (a_0, \mathbf{0})$. In this frame (from eq.(35)) we have

$$a_\mu a_\nu \alpha^{\mu\nu} = a_0^2 p^{-2}((\epsilon - \pi_\epsilon) p^2 + (\epsilon + \pi_\epsilon)(p^2 w^2 - (p w)^2),$$  \hspace{1cm} (53)

where the normalization $w^2 = w_0^2 - w^2 = 1$ has been used. It follows from the Schwarz inequality that (53) is non-negative if the condition (52) is satisfied. If $a^2 \leq 0$ then we choose the Lorentz frame such that $w = (1, \mathbf{0})$. In this frame

$$a_\mu a_\nu \alpha^{\mu\nu} = (\epsilon - \pi_\epsilon) p^{-2}(a_0^2 p^2 + p_0^2 a^2 - 2 p_0 a_0) + 2 \pi_\epsilon p^{-2}(p^2 a^2 - (p a)^2).$$  \hspace{1cm} (54)

(54) is non-negative under the condition (52) again by Schwarz inequality. It follows from eqs.(53)-(54) that the conditions (52) are sufficient and necessary for a definition of the diffusions (48) and (50) (the condition (51) is necessary for any definition of a diffusion because the diffusion matrix $\alpha^{\mu\nu}$ should be equal to the correlation function of particle’s diffusing paths which is defined by a positive probability measure).

We find the equilibrium solution $\Omega_E$ from the requirement

$$(\alpha^{\mu\nu} \partial_\nu + r P^{\mu\nu} a_\nu) \Omega_E = 0.$$  \hspace{1cm} (55)

There is a solution (which is the Jüttner equilibrium [40]), if

$$r = \beta (\epsilon - \pi_\epsilon).$$  \hspace{1cm} (56)

Then,

$$\Omega_E = \exp(-\beta w p - \gamma p^2),$$  \hspace{1cm} (57)

where $\gamma$ is an arbitrary constant. Eq.(56) can be satisfied for $r = 0$ if $\pi_\epsilon = \epsilon$ which is still in agreement with eq.(52). However, the condition $\pi_\epsilon = \epsilon$ is in conflict with the positivity of the measure determining the correlation functions of the electromagnetic field (see eq.(32)).

If the equilibrium $\Omega_E$ is known then the form of the friction can be determined from the condition of the detailed balance (see an interpretation in terms of transition rates in [41] and as Kolmogorov’s reversibility condition in [42])

$$\int d\mathbf{p} \Omega_E \Phi, W = \int d\mathbf{p} \Omega_E \Phi W_\tau.$$  \hspace{1cm} (58)
This condition is equivalent to the requirement that the generator of the diffusion is of the form
\[ G_w = \frac{1}{\tau_c} p^\mu \partial_\mu + \Omega^{-1} \partial_\mu \alpha^{\mu\nu} \Omega_\nu \partial_\nu \] (59)

From eq.(59) it follows that the friction term of a reversible diffusion must have the form
\[ \alpha^{\mu\nu} (\Omega^{-1}_E \partial_\mu \Omega_\nu) \partial_\nu. \] (60)

Comparing eq.(60) with eq.(48) we can see that the diffusion (48) satisfies the detailed balance condition (58) if \( r \) is determined by eq.(56).

6 Discussion and summary

We have discussed Liouville description of a particle in an electromagnetic field coming from a thermal radiation. In such a system there is a distinguished reference frame associated with the heat bath. The Poincare invariance of the whole system (particle plus the heat bath) can be expressed as a covariance with respect to the transformations of the reference frame of an observer. In such a description the thermal field becomes a function of the space-time point and the reference frame. We have shown that the lowest order term in the expansion of the Liouville evolution in powers of the random electromagnetic field is described by a diffusion operator which is a generalization of the relativistic diffusion of Schay and Dudley. We developed a diffusion theory assuming (after Kubo) that the generator of the short time dynamics after multiplication by the environmental correlation time determines the diffusion at long time. We treat the square of the momentum as a continuous variable. The off shell formulation allows to preserve explicit Lorentz covariance. It may have applications to quantum field theory (heavy ion collisions) and cosmology (diffusion of galactic matter). It seems that the diffusion equation derived from an interaction with a random electromagnetic field could also be inferred on the basis of general assumptions of the preservation of the square of the momentum and the Poincare invariance of a system described by the phase point \((x, p)\) and a fixed vector \( w \) which can be interpreted as the velocity of a frame in which the particle is studied. Demanding that similarly as in the case of the deterministic dynamics the probability distribution does not depend on the proper time we obtain a diffusive transport equation in the laboratory time. A similar transport equation (called a relativistic Ornstein-Uhlenbeck process) has been discussed in refs.[31]-[33] without relating it to the physical model of an interaction with an electromagnetic field. The authors in [31]-[33] investigate diffusion of particles in a fluid which supplies a preferred reference frame. However, their diffusion does not coincide with the one defined in this paper. Owing to the dependence of \( \alpha^{\mu\nu} \) in eq.(50) on the momenta there is no choice of parameters and no choice of the frame which enables to reduce the diffusion (50) to the Ornstein-Uhlenbeck diffusion (the Ornstein-Uhlenbeck diffusion appears only in...
the non-relativistic limit). It is reassuring that a mathematical construction of the diffusion, discovered first by Schay [17] and Dudley [18] and generalized here to realistic systems at finite temperature, is the one which may be realized in nature (in particular, in a particle’s interaction with a thermal radiation).

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