DENSITY FUNCTIONS FOR QUICKQUANT AND QUICKVAL

JAMES ALLEN FILL AND WEI-CHUN HUNG

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Abstract. We prove that, for every $0 \leq t \leq 1$, the limiting distribution of the scale-normalized number of key comparisons used by the celebrated algorithm QuickQuant to find the $t$th quantile in a randomly ordered list has a Lipschitz continuous density function $f_t$ that is bounded above by 10. Furthermore, this density $f_t(x)$ is positive for every $x > \min\{t, 1 - t\}$ and, uniformly in $t$, enjoys superexponential decay in the right tail. We also prove that the survival function $1 - F_t(x) = \int_x^\infty f_t(y) \, dy$ and the density function $f_t(x)$ both have the right tail asymptotics $\exp[-x \ln x - x \ln \ln x + O(x)]$. We use the right-tail asymptotics to bound large deviations for the scale-normalized number of key comparisons used by QuickQuant.

1. Introduction and summary

1.1. Introduction. QuickQuant is closely related to an algorithm called QuickSelect, which in turn can be viewed as a one-sided analogue of QuickSort. In brief, QuickSelect$(n, m)$ is an algorithm designed to find a number of rank $m$ in an unsorted list of size $n$. It works by recursively applying the same partitioning step as QuickSort to the sublist that contains the item of rank $m$ until the pivot we pick has the desired rank or the size of the sublist to be explored has size one. Let $C_{n,m}$ denote the number of comparisons needed by QuickSelect$(n, m)$, and note that $C_{n,m}$ and $C_{n,n+1-m}$ have the same distribution, by symmetry. Knuth [20] finds the formula

$$\mathbb{E} C_{n,m} = 2 \left[ (n+1)H_n - (n+3-m)H_{n+1-m} - (m+2)H_m + (n+3) \right]$$

(1.1)

for the expectation.

The algorithm QuickQuant$(n, t)$ refers to QuickSelect$(n, m_n)$ such that the ratio $m_n/n$ converges to a specified value $t \in [0, 1]$ as $n \to \infty$. It is easy to see that (1.1) tells us about the limiting behavior of the expected number of comparisons after standardizing:

$$\lim_{n \to \infty} \mathbb{E}[n^{-1}C_{n,m_n}] = 2 + 2H(t),$$

(1.2)

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where $H(x) := -x \ln x - (1 - x) \ln(1 - x)$ with $0 \ln 0 := 0$.

We follow the set-up and notation of Fill and Nakama [12], who use an infinite sequence $(U_i)_{i \geq 1}$ of independent Uniform$(0,1)$-distributed random variables to couple the number of key comparisons $C_{n,m}$ for all $n$. Let $L_0(n) := 0$ and $R_0(n) := 1$. For $k \geq 1$, inductively define

$$
\tau_k(n) := \inf\{i : L_{k-1}(n) < U_i < R_{k-1}(n)\},
$$

and let $r_k(n)$ be the rank of the pivot $U_{\tau_k(n)}$ in the set $\{U_1, \ldots, U_n\}$ if $\tau_k(n) < \infty$ and be $m_n$ otherwise. [Recall that the infimum of the empty set is $\infty$; hence $\tau_k(n) = \infty$ if and only if $L_{k-1}(n) = R_{k-1}(n)$.] Also, inductively define

$$
L_k(n) := \mathbb{1}(r_k(n) \leq m_n) U_{\tau_k(n)} + \mathbb{1}(r_k(n) > m_n) L_{k-1}(n),
$$

$$
R_k(n) := \mathbb{1}(r_k(n) \geq m_n) U_{\tau_k(n)} + \mathbb{1}(r_k(n) < m_n) R_{k-1}(n),
$$

if $\tau_k(n) < \infty$, but

$$(L_k(n), R_k(n)) := (L_{k-1}(n), R_{k-1}(n)),$$

if $\tau_k(n) = \infty$. The number of comparisons at the $k$th step is then

$$S_{n,k} := \sum_{i: \tau_k < i \leq n} \mathbb{1}(L_{k-1}(n) < U_i < R_{k-1}(n)),$$

and the normalized total number of comparisons equals

$$n^{-1}C_{n,m} := n^{-1} \sum_{k \geq 1} S_{n,k}. \tag{1.5}$$

Mahmoud, Modarres and Smythe [23] studied QuickSelect in the case that the rank $m$ is taken to be a random variable $M_n$ uniformly distributed on $\{1, \ldots, n\}$ and assumed to be independent of the numbers in the list. They used the Wasserstein metric to prove that $Z_n := n^{-1}C_{n,M_n} \overset{\mathcal{L}}{\rightarrow} Y$ as $n \rightarrow \infty$ and identified the distribution of $Y$. In particular, they proved that $Y$ has an absolutely continuous distribution function. Gr"ubel and R"osler [15] treated all the quantiles $t$ simultaneously by letting $m_n := m_n(t)$. Specifically, they considered the normalized process $X_n$ defined by

$$X_n(t) := n^{-1}C_{n,|nt|+1} \text{ for } 0 \leq t < 1, \quad X_n(t) := n^{-1}C_{n,n} \text{ for } t = 1. \tag{1.6}$$

Working in the Skorohod topology (see Billingsley [1, Chapter 3]), they proved that this process has a limiting distribution as $n \rightarrow \infty$, and the value of the limiting process at argument $t$ is the sum of the lengths of all the intervals encountered in all the steps of searching for population quantile $t$.

We can use the same sequence $(U_i)_{i \geq 1}$ of Uniform$(0,1)$ random variables to express the limiting stochastic process. For $t \in [0,1]$, let $L_0(t) := 0$ and $R_0(t) := 1$, and let $\tau_0(t) := 0$. For $t \in [0,1]$ and $k \geq 1$, inductively define

$$\tau_k(t) := \inf\{i > \tau_{k-1}(t) : L_{k-1}(t) \leq U_i \leq R_{k-1}(t)\}, \tag{1.7}$$

$$L_k(t) := \mathbb{1}(U_{\tau_k(t)} \leq t) U_{\tau_k(t)} + \mathbb{1}(U_{\tau_k(t)} > t) L_{k-1}(t), \tag{1.8}$$

$$R_k(t) := \mathbb{1}(U_{\tau_k(t)} \leq t) R_{k-1}(t) + \mathbb{1}(U_{\tau_k(t)} > t) U_{\tau_k(t)}, \tag{1.9}$$
It is not difficult to see that
\[ P(\tau_k(t) < \infty \text{ and } 0 \leq L_k(t) \leq t \leq R_k(t) \leq 1 \text{ for all } 0 \leq t \leq 1 \text{ and } k \geq 0) = 1 \]
and that for each fixed \( t \in (0, 1) \) we have
\[ P(L_k(t) < t < R_k(t) \text{ for all } k \geq 0) = 1. \]
The limiting process \( Z \) can be expressed as
\[ Z(t) := \sum_{k=0}^{\infty} [R_k(t) - L_k(t)] = 1 + \sum_{k=1}^{\infty} [R_k(t) - L_k(t)]; \quad (1.10) \]
it is not hard to see that
\[ P(1 < Z(t) < \infty \text{ for all } 0 \leq t \leq 1) = 1. \]
Note also that the processes \( Z \) and \( (Z(1 - t))_{t \in [0, 1]} \) have the same finite-dimensional distributions. Gr"{u}bel and R"{o}sler [15, Theorem 8] proved that we can replace the subscript \( \lfloor nt \rfloor + 1 \) in (1.6) by any \( m_n(t) \) with \( 0 \leq m_n(t) \leq n \) such that \( m_n(t)/n \to t \) as \( n \to \infty \), and then the normalized random variables \( n^{-1}C_{n,m_n} \) converge (univariately) to the limiting random variable \( Z(t) \) for each \( t \in [0, 1] \).

Among the univariate distributions of \( Z(t) \) for \( t \in [0, 1] \), only the common distribution of \( Z(0) \) and \( Z(1) \) is known at all explicitly. As established by Hwang and Tsai [17], this distribution is the Dickman distribution; see their paper for a wealth of information about the distribution. The overarching goal of this paper is to establish basic properties of the distributions of \( Z(t) \) for other values of \( t \).

Kodaj and Móri [21] proved the (univariate) convergence of (1.5) to \( Z(t) \) in the Wasserstein metric. Using the coupling technique and induction, they proved that (1.5) is stochastically smaller than its continuous counterpart (1.10). Combining this fact with knowledge of their expectations (see (1.1) and [21, Lemma 2.2]), they proved that (1.5) converges to (1.10) in the Wasserstein metric and thus in distribution.

Gr"{u}bel [14] connected \( \text{QuickSelect}(n, m_n) \) to a Markov chain to identify the limiting process. For each fixed \( n \geq 1 \), he considered the Markov chain \( (Y_{m,n}^{(n)})_{m\geq 0} \) on the state space \( I_n := \{(i, j) : 1 \leq j \leq i \leq n\} \) with \( Y_0^{(n)} := (n, m_n) \). Transition probabilities of \( Y^{(n)} \) from the state \((i, j)\) are determined by the partition step of \( \text{QuickSelect}(i, j) \) as follows. If \( Y_m^{(n)} = (i, j) \), then \( Y_{m+1}^{(n)} \) is selected uniformly at random from the set
\[ \{(i - k, j - k) : k = 1, \ldots, j - 1\} \cup \{(1, 1)\} \cup \{(i - k, j) : k = 1, \ldots, i - j\}; \]
in particular, \( (1, 1) \) is an absorbing state for \( Y^{(n)} \). If we write \( Y_m^{(n)} = (S_m^{(n)}, Q_m^{(n)}) \), then we know
\[ n^{-1}C_{n,m_n} \overset{\mathcal{L}}{=} n^{-1} \sum_{m \geq 0} (S_m^{(n)} - 1). \quad (1.11) \]
Grübel [14] constructed another Markov chain $Y = (Y_m) = ((S_m, Q_m))$, which is a continuous-value counterpart of the process $Y^{(n)}$, and he proved that for all $m \geq 0$, the random vector $Y_m^{(n)}$ converges to $Y_m$ almost surely. Using the dominated convergence theorem, he proved that the random variables $n^{-1} \sum_{m=0}^{\infty} (S_m^{(n)} - 1)$ converge almost surely to $\sum_{m=0}^{\infty} S_m$; the limiting random variable here is exactly $Z(t)$ of (1.10). Combining with (2.14), he concluded that $n^{-1}C_{n,m}$ converges in distribution to (1.10). As previously mentioned, Hwang and Tsai [17] identified the limiting distribution of (1.5) when $m = o(n)$ as the Dickman distribution.

Fill and Nakama [12] studied the limiting distribution of the cost of using QuickSelect for a variety of cost functions. In particular, when there is simply unit cost of comparing any two keys, then their work reduces to study of the number of key comparisons, to which we limit our focus here. They proved $L^p$-convergence of (1.5) for QuickQuant$(n, t)$ to (1.10) for $1 \leq p < \infty$ by first studying the distribution of the number of key comparisons needed for another algorithm called QuickVal, and then comparing the two algorithms. The algorithm QuickVal$(n, t)$ finds the rank of the population $t$-quantile in the sample, while its cousin QuickQuant$(n, t)$ looks for the sample $t$-quantile. Intuitively, when the sample size is large, we expect the rank of the population $t$-quantile to be close to $nt$. Therefore, the two algorithms should behave similarly when $n$ is large. Given a set of keys $\{U_1, \ldots, U_n\}$, where $U_i$ are i.i.d. Uniform$(0, 1)$ random variables, one can regard the operation of QuickVal$(n, t)$ as that of finding the rank of the value $t$ in the augmented set $\{U_1, \ldots, U_n, t\}$. It works by first selecting a pivot uniformly at random from the set of keys $\{U_1, \ldots, U_n\}$ and then using the pivot to partition the augmented set (we don’t count the comparison of the pivot with $t$). We then recursively do the same partitioning step on the subset that contains $t$ until the set of the keys on which the algorithm operates reduces to the singleton $\{t\}$. For QuickVal$(n, t)$ with the definitions (1.7)–(1.9) and with

$$S_{n,k}(t) := \sum_{\tau_k(t) < i \leq n} \mathbb{1}(L_{k-1}(t) < U_i < R_{k-1}(t)),$$

Fill and Nakama [12] showed that $n^{-1} \sum_{k \geq 1} S_{n,k}(t)$ converges (for fixed $t$) almost surely and also in $L^p$ for $1 \leq p < \infty$ to (1.10). They then used these facts to prove the $L^p$ convergence (for fixed $t$) of (1.5) to (1.10) for QuickQuant$(n, t)$ for $1 \leq p < \infty$.

Fill and Matterer [11] treated distributional convergence for the worst-case cost of Find for a variety of cost functions. Suppose, for example, that we continue, as at the start of this section, to assign unit cost to the comparison of any two keys, so that $C_{n,m}$ is the total cost for QuickSelect$(n, m)$. Then (for a list of length $n$) the cost of worst-case Find is $\max_{1 \leq m \leq n} C_{n,m}$, and its distribution depends on the joint distribution of $C_{n,m}$ for varying $m$.

We shall not be concerned here with worst-case Find, but we wish to review
the approach and some of the results of [11], since there is relevance of their work to QuickQuant\((n, t)\) for fixed \(t\).

Fill and Matterer [11] considered tree-indexed processes closely related to the operation of the QuickSelect algorithm, as we now describe. For each node in a given rooted ordered binary tree, let \(\theta\) denote the binary sequence (or string) representing the path from the root to this node, where 0 corresponds to taking the left child and 1 to taking the right. The value of \(\theta\) for the root is thus the empty string, denoted \(\varepsilon\). Define \(L_\varepsilon := 0\), \(R_\varepsilon := 1\), and \(\tau_\varepsilon := 1\). Given a sequence of i.i.d. Uniform(0, 1) random variables \(U_1, U_2, \ldots\), recursively define

\[
\tau_\theta := \inf\{i : L_\theta < U_i < R_\theta\},
\]

\[
L_{\theta 0} := L_\theta, \quad L_{\theta 1} := U_{\tau_\theta},
\]

\[
R_{\theta 0} := U_{\tau_\theta}, \quad R_{\theta 1} := R_\theta.
\]

Here the concatenated string \(\theta 0\) corresponds to the left child of the node with string \(\theta\), while \(\theta 1\) corresponds to the right child. Observe that, when inserting a key \(U_i\) arriving at time \(i > \tau_\theta\) into the binary tree, this key is compared with the “pivot” \(U_{\tau_\theta}\) if and only if \(U_i \in (L_\theta, R_\theta)\). For \(n\) insertions, the total cost of comparing keys with pivot \(U_{\tau_\theta}\) is therefore

\[
S_n,\theta := \sum_{\tau_\theta < i \leq n} 1(L_\theta < U_i < R_\theta).
\]

We define a binary-tree-indexed stochastic process \(S_n = (S_n,\theta)_{\theta \in \Theta}\), where \(\Theta\) is the collection of all finite-length binary sequences.

For each \(1 \leq p \leq \infty\), Fill and Matterer [11, Definition 3.10 and Proposition 3.11] defined a Banach space \(B^{(p)}\) of binary-tree-indexed stochastic processes that corresponds in a natural way to the Banach space \(L^p\) for random variables. Let \(I_\theta := R_\theta - L_\theta\) and consider the process \(I = (I_\theta)_{\theta \in \Theta}\). Fill and Matterer [11, Theorem 4.1 with \(\beta \equiv 1\)] proved the convergence of the processes \(n^{-1}S_n\) to \(I\) in the Banach space \(B^{(p)}\) for each \(2 \leq p < \infty\).

For the simplest application in [11], namely, to QuickVal\((n, t)\) with \(t\) fixed, let \(\gamma(t)\) be the infinite path from the root to the key having value \(t\) in the (almost surely) complete binary search tree formed by successive insertions of \(U_1, U_2, \ldots\) into an initially empty tree. The total cost (call it \(V_n\)) of QuickVal\((n, t)\) can then be computed by summing the cost of comparisons with each (pivot-)node along the path, that is,

\[
V_n := \sum_{\theta \in \gamma(t)} S_n,\theta.
\]

Using their tree-process convergence theorem described in our preceding paragraph, Fill and Matterer [11, Proposition 6.1 with \(\beta \equiv 1\)] established \(L^p\)-convergence, for each \(0 < p < \infty\), of \(n^{-1}V_n\) to \(I_{\gamma(t)}\) as \(n \to \infty\), where \(I_{\gamma(t)} := \sum_{\theta \in \gamma(t)} I_\theta\). Moreover ([11, Theorem 6.3 with \(\beta \equiv 1\)], they also
proved $L^p$-convergence of $n^{-1}Q_n$ to the same limit, again for every $0 < p < \infty$, where $Q_n$ denotes the cost of QuickQuant($n, t$).

Throughout this paper, we will use the standard notations $1(A)$ to denote the indicator function of the event $A$ and $\mathbb{E}[f; A]$ for $\mathbb{E}[f1(A)]$.

1.2. Summary. In Section 2, by construction we establish the existence of densities $f_t$ for the random variables $Z(t)$ defined in (1.10). In Section 3 we prove that these densities are uniformly bounded and in Section 4 that they are uniformly continuous. As shown in Section 5, the densities satisfy a certain integral equation for $0 < t < 1$. The right-tail behavior of the density functions is examined in Section 6, and the left-tail behavior in Section 8. In Section 7 we prove that $f_t(x)$ is positive if and only if $x > \min\{t, 1 - t\}$, and we improve the result of Section 4 by showing that $f_t(x)$ is Lipschitz continuous in $x$ for fixed $t$ and jointly continuous in $(t, x)$. Sections 9–10 are devoted to sharp logarithmic asymptotics for the right tail of $f_t$, and Section 11 uses the results of those two sections to treat right-tail large deviation behavior of QuickQuant asymptotics for the right tail of $f_t$, and for large but finite $n$.

2. Existence (and construction) of density functions

In this section, we prove that $Z \equiv Z(t)$ defined in (1.10) for fixed $0 \leq t \leq 1$ has a density. For notational simplification, we let $L_k \equiv L_k(t)$ and $R_k \equiv R_k(t)$. Let $J \equiv J(t) := Z(t) - 1 = \sum_{k=1}^{\infty} \Delta_k$ with $\Delta_k \equiv \Delta_k(t) := R_k(t) - L_k(t)$. We use convolution notation as in Section V.4 of Feller [7]. The following lemma is well known and can be found, for example, in Feller [7, Theorem V.4.4] or Durrett [6, Theorem 2.1.11].

**Lemma 2.1.** If $X$ and $Y$ are independent random variables with respective distribution functions $F$ and $G$, then $Z = X + Y$ has the distribution function $F \ast G$. If, in addition, $X$ has a density $f$ (with respect to Lebesgue measure), then $Z$ has a density $f \ast G$.

Let $X = \Delta_1 + \Delta_2$ and $Y = \sum_{k=3}^{\infty} \Delta_k$. For the remainder of this paragraph, we suppose $0 < t < 1$. If we condition on $(L_3, R_3) = (l_3, r_3)$ for some $0 \leq l_3 < t < r_3 \leq 1$ with $(l_3, r_3) \neq (0, 1)$, we then have

$$Y = (r_3 - l_3) \sum_{k=3}^{\infty} \frac{R_k - L_k}{r_3 - l_3} = (r_3 - l_3) \sum_{k=3}^{\infty} (R_k' - L_k'),$$

(2.1)

where we set $L_k' = (L_k - l_3)/(r_3 - l_3)$ and $R_k' = (R_k - l_3)/(r_3 - l_3)$ for $k \geq 3$. Observe that, by definitions (1.7)–(1.9), the stochastic process $((L_k', R_k'))_{k \geq 3}$, conditionally given $(L_3, R_3) = (l_3, r_3)$, has the same distribution as the (unconditional) stochastic process of intervals $((L_k, R_k))_{k \geq 0}$ encountered in all the steps of searching for population quantile $(t - l_3)/(r_3 - l_3)$ (rather than $t$) by QuickQuant. Note also that (again conditionally) the stochastic processes $((L_k, R_k))_{0 \leq k \leq 2}$ and $((L_k', R_k'))_{k \geq 3}$ are independent. Thus (again conditionally) $Y/(r_3 - l_3)$ has the same distribution as the (unconditional) random variable $Z ((t - l_3)/(r_3 - l_3))$ and is independent of $X$. We will
prove later (Lemmas 2.3–2.4) that, conditionally given \((L_3, R_3) = (l_3, r_3)\), the random variable \(X\) has a density. Let
\[
f_{l_3, r_3}(x) := \mathbb{P}(X \in dx \mid (L_3, R_3) = (l_3, r_3))/dx
\]
be such a conditional density. We can then use Lemma 2.1 to conclude that
\(J = X + Y\) has a conditional density
\[
h_{l_3, r_3}(x) := \mathbb{P}(J \in dx \mid (L_3, R_3) = (l_3, r_3))/dx.
\]
By mixing \(h_{l_3, r_3}(x)\) for all possible values of \(l_3, r_3\), we will obtain an unconditional density function for \(J\), as summarized in the following theorem.

**Theorem 2.2.** For each \(0 \leq t \leq 1\), the random variable \(J(t) = Z(t) - 1\) has a density
\[
f_t(x) := \int \mathbb{P}((L_3, R_3) \in d(l_3, r_3)) \cdot h_{l_3, r_3}(x), \quad (2.2)
\]
and hence the random variable \(Z(t)\) has density \(f_t(x - 1)\).

Now, as promised, we prove that, conditionally given \((L_3, R_3) = (l_3, r_3)\), the random variable \(X\) has a density \(f_{l_3, r_3}\). We begin with the case \(0 < t < 1\).

**Lemma 2.3.** Let \(0 \leq l_3 < t < r_3 \leq 1\) with \((l_3, r_3) \neq (0, 1)\). Conditionally given \((L_3, R_3) = (l_3, r_3)\), the random variable \(X = \Delta_1 + \Delta_2\) has a right continuous density \(f_{l_3, r_3}\).

**Proof.** We consider three cases based on the values of \((l_3, r_3)\).

**Case 1:** \(l_3 = 0\) and \(r_3 < 1\). Since \(L_k\) is nondecreasing in \(k\), from \(L_3 = 0\) it follows that \(L_1 = L_2 = 0\). The unconditional joint distribution of \((L_1, R_1, L_2, R_2, L_3, R_3)\) satisfies
\[
\mathbb{P}(L_1 = 0, R_1 \in dr_1, L_2 = 0, R_2 \in dr_2, L_3 = 0, R_3 \in dr_3) = \mathbb{1}(t < r_3 < r_2 < r_1 < 1) dr_1 \frac{dr_2 dr_3}{r_1 r_2} \quad (2.3)
\]
and hence
\[
\mathbb{P}(L_3 = 0, R_3 \in dr_3) = \mathbb{1}(t < r_3 < 1) dr_3 \int_{r_2=r_3}^1 \frac{dr_2}{r_2} \int_{r_1=r_2}^1 \frac{dr_1}{r_1}

= \mathbb{1}(t < r_3 < 1) dr_3 \int_{r_2=r_3}^1 \frac{dr_2}{r_2} (- \ln r_2)

= \frac{1}{2} (\ln r_3)^2 \mathbb{1}(t < r_3 < 1) dr_3. \quad (2.4)
\]
Dividing (2.3) by (2.4), we find
\[
\mathbb{P}(L_1 = 0, R_1 \in dr_1, L_2 = 0, R_2 \in dr_2 \mid L_3 = 0, R_3 = r_3)

= \frac{2 r_1^{-1} r_2^{-1} dr_1 dr_2}{(\ln r_3)^2} \mathbb{1}(t < r_3 < r_2 < r_1 < 1). \quad (2.5)
\]
Thus for $x \in (2r_3, 2)$, we find

$$f_{0,r_3}(x) = \mathbb{P}(X \in dx \mid L_3 = 0, R_3 = r_3)/dx$$

$$= \int_{r_2}^{x/2} \mathbb{P}(R_2 \in dr_2, R_1 \in dx - r_2 \mid L_3 = 0, R_3 = r_3)/dx$$

$$= \frac{2}{(\ln r_3)^2} \int_{r_2=r_3\vee(x-1)}^{x/2} (x - r_2)^{-1} r_2^{-1} dr_2$$

$$= \frac{2}{(\ln \frac{1}{r_3})^2} \frac{1}{x} \left[ \ln \left( \frac{x - r_3}{r_3} \right) \mathbb{1}(2r_3 \leq x < 1 + r_3) 

+ \ln \left( \frac{1}{x - 1} \right) \mathbb{1}(1 + r_3 \leq x < 2) \right]; \quad (2.6)$$

we set $f_{0,r_3}(x) = 0$ for $x \notin (2r_3, 2)$.

*Case 2*: $l_3 > 0$ and $r_3 = 1$. This condition implies that $R_1 = R_2 = 1$. Invoking symmetry, we can skip the derivation and immediately write

$$f_{l_3,1}(x) = \frac{2}{(\ln \frac{1}{l_3})^2} \frac{1}{x} \left[ \ln \left( \frac{x - 1 + l_3}{1 - l_3} \right) \mathbb{1}(2 - 2l_3 \leq x < 2 - l_3) 

+ \ln \frac{1}{x - 1} \mathbb{1}(2 - l_3 \leq x < 2) \right] \quad (2.7)$$

for $x \in (2 - 2l_3, 2)$; we set $f_{l_3,1}(x) = 0$ for $x \notin (2 - 2l_3, 2)$.

*Case 3*: $0 < l_3 < t < r_3 < 1$. There are six possible scenarios for the random vector $(L_1, R_1, L_2, R_2, L_3, R_3)$, and to help us discuss the cases, we consider values $l_2, r_2$ satisfying $0 < l_2 < l_3 < t < r_3 < r_2 < 1$.

(a) $L_1 = l_2, L_2 = L_3 = l_3$ and $R_1 = R_2 = 1, R_3 = r_3$.

In this subcase, we consider the event that the first pivot we choose locates between 0 and $l_3$, the second pivot has value $l_3$, and the third pivot has value $r_3$. Denote this event by $E_{llr}$ (with $llr$ indicating that we shrink the search intervals by moving the lefthand, lefthand, and then righthand endpoints). We have

$$\mathbb{P}(L_1 \in dl_2, R_1 = 1, L_2 \in dl_3, R_2 = 1, L_3 \in dl_3, R_3 \in dr_3)$$

$$= \mathbb{1}(0 < l_2 < l_3 < t < r_3 < 1) dl_2 \frac{dl_3}{1 - l_2} \frac{dr_3}{1 - l_3}. \quad (2.8)$$

Integrating over all possible values of $l_2$, we get

$$\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_{llr})$$

$$= \mathbb{1}(0 < l_3 < t < r_3 < 1) \frac{1}{1 - l_3} \ln \left( \frac{1}{1 - l_3} \right) dl_3 dr_3.$$
(b) \( L_1 = L_2 = 0, L_3 = l_3 \) and \( R_1 = r_2, R_2 = R_3 = r_3 \).

In this and all subsequence subcases, we use notation like that in subcase (a). In this subcase, we invoke symmetry in comparison with subcase (a). The results are

\[
\mathbb{P}(L_1 = 0, R_1 \in dr_2, L_2 = 0, L_3 \in dl_3, R_2 = R_3 \in dr_3) = 1(0 < l_3 < t < r_3 < r_2 < 1) dl_2 \frac{dr_2}{r_2} dl_3 \frac{dr_3}{r_3} \tag{2.9}
\]

and

\[
\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_r(t)) = 1(0 < l_3 < t < r_3 < 1) \frac{1}{r_3} \ln \left( \frac{1}{r_3} \right) dl_3 dr_3.
\]

(c) \( L_1 = L_2 = l_2, L_3 = l_3 \) and \( R_1 = 1, R_2 = R_3 = r_3 \).

In this subcase we have

\[
\mathbb{P}(R_1 = 1, L_1 = L_2 \in dl_2, L_3 \in dl_3, R_2 = R_3 \in dr_3) = 1(0 < l_2 < l_3 < t < r_3 < 1) dl_2 \frac{dr_2}{r_2} dl_3 \frac{dr_3}{r_3} \tag{2.10}
\]

Integrating over the possible values of \( l_2 \), we find

\[
\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_r(t))/(dl_3 dr_3)
\]

\[
= 1(0 < l_3 < t < r_3 < 1) \frac{1}{1 - r_3} \left[ \ln \left( \frac{1}{r_3 - l_3} \right) - \ln \left( \frac{1}{r_3} \right) - \ln \left( \frac{1}{1 - l_3} \right) \right]. \tag{2.11}
\]

(d) \( L_1 = 0, L_2 = L_3 = l_3 \) and \( R_1 = R_2 = r_2, R_3 = r_3 \).

In this subcase, by symmetry with subcase (c), we have

\[
\mathbb{P}(L_1 = 0, L_2 = L_3 \in dl_3, R_1 = R_2 \in dr_2, R_3 \in dr_3) = 1(0 < l_3 < t < r_3 < r_2 < 1) dl_2 \frac{dr_2}{r_2} dr_3 \frac{dr_3}{r_3 - l_3} \tag{2.11}
\]

and

\[
\mathbb{P}(L_3 \in dl_3, R_3 \in dr_3, E_r(t))/(dl_3 dr_3)
\]

\[
= 1(0 < l_3 < t < r_3 < 1) \frac{1}{l_3} \left[ \ln \left( \frac{1}{r_3 - l_3} \right) - \ln \left( \frac{1}{r_3} \right) - \ln \left( \frac{1}{1 - l_3} \right) \right]. \tag{2.12}
\]

(e) \( L_1 = 0, L_2 = l_2, L_3 = l_3 \) and \( R_1 = R_2 = R_3 = r_3 \).

In this subcase we have

\[
\mathbb{P}(L_1 = 0, L_2 \in dl_2, L_3 \in dl_3, R_1 = R_2 = R_3 \in dr_3)
\]

\[
= 1(0 < l_2 < l_3 < t < r_3 < 1) dl_2 \frac{dr_2}{r_2} dl_3 \frac{dr_3}{r_3 - l_2}. \tag{2.12}
\]
Integrating over the possible values of $l_2$, we have
\[
\mathbb{P}(L_3 \in d l_3, R_3 \in d r_3, E_{irl})
\]
\[
= \mathbbm{1}(0 < l_3 < t < r_3 < 1) \frac{1}{r_3} \left[ \ln \left( \frac{1}{r_3 - l_3} \right) - \ln \left( \frac{1}{r_3} \right) \right] d l_3 \, d r_3.
\]

(f) $L_1 = L_2 = L_3 = l_3$ and $R_1 = 1, R_2 = r_2, R_3 = r_3$.

In this final subcase, by symmetry with subcase (e), we have
\[
\mathbb{P}(L_1 = L_2 = L_3 \in d l_3, R_1 = 1, R_2 \in d r_2, R_3 \in d r_3)
\]
\[
= \mathbbm{1}(0 < l_3 < t < r_3 < r_2 < 1) d l_3 \left( \frac{r_2 - l_3}{1 - l_3} \right) \frac{d r_2}{d r_3} \frac{d r_3}{d l_3} = \mathbbm{1}(0 < l_3 < t < r_3 < r_2 < 1) d l_3 \frac{d r_2}{d r_3} \frac{d r_3}{d l_3} \tag{2.13}
\]
and
\[
\mathbb{P}(L_3 \in d l_3, R_3 \in d r_3, E_{irr})/(d l_3 \, d r_3)
\]
\[
= \mathbbm{1}(0 < l_3 < t < r_3 < 1) \frac{1}{1 - l_3} \left[ \frac{1}{r_3 - l_3} - \frac{1}{1 - l_3} \right].
\]

Summing results from the six subcases, we conclude in Case 3 that
\[
\mathbb{P}(L_3 \in d l_3, R_3 \in d r_3) = \mathbbm{1}(0 < l_3 < t < r_3 < 1) g(l_3, r_3) d l_3 \, d r_3, \tag{2.14}
\]
where
\[
g(l_3, r_3) := \left[ \frac{1}{l_3(1 - l_3)} + \frac{1}{r_3(1 - r_3)} \right] \ln \left( \frac{1}{r_3 - l_3} \right) - \left( \frac{1}{l_3} + \frac{1}{1 - r_3} \right) \left[ \ln \left( \frac{1}{r_3} \right) + \ln \left( \frac{1}{1 - l_3} \right) \right]. \tag{2.15}
\]

The conditional joint distribution of $(L_1, R_1, L_2, R_2)$ given $(L_3, R_3) = (l_3, r_3)$ can be derived by dividing each of (2.8)–(2.13) by (2.14), and we can then compute $f_{l_3, r_3}$ from these conditional distributions. Let us write
\[
f_{l_3, r_3}(x) = \mathbbm{1}(0 < l_3 < t < r_3 < 1) \frac{1}{g(l_3, r_3)} \sum_{i=1}^{6} f_{l_3, r_3}^{(i)}(x), \tag{2.16}
\]
where $f_{l_3, r_3}^{(i)}(x) d l_3 \, d r_3 \, d x$ is the contribution to
\[
\mathbb{P}(L_3 \in d l_3, R_3 \in d r_3, X \in d x)
\]
 arising from the $i$th subcase of the six.

In subcase (a) we know that $X = R_1 - L_1 + R_2 - L_2 = 2 - l_2 - l_3$. Changing variables from $l_2$ to $x$, from (2.8) we find
\[
f_{l_3, r_3}^{(1)}(x) = \mathbbm{1}(2 - 2l_3 \leq x < 2 - l_3) \frac{1}{1 - l_3} \frac{1}{x - 1 + l_3}.
\]

In subcase (b) we know that $X = r_2 + r_3$. Changing variables from $r_2$ to $x$, from (2.9) we find
\[
f_{l_3, r_3}^{(2)}(x) = \mathbbm{1}(2r_3 \leq x < 1 + r_3) \frac{1}{r_3} \frac{1}{x - r_3}.
\]
In subcase (c), we know that \( X = 1 - 2t_2 + r_3 \). Changing variables from \( l_2 \) to \( x \), from (2.10) we find
\[
f_{l_3,r_3}^{(3)}(x) = 1(1 + r_3 - 2r_3 \leq x < 1 + r_3) \frac{1}{x + 1 - r_3} \frac{2}{x + r_3 - 1}.
\]

In subcase (d), we know that \( X = 2r_2 - l_3 \). Changing variables from \( r_2 \) to \( x \), from (2.11) we find
\[
f_{l_3,r_3}^{(4)}(x) = 1(2r_3 - l_3 \leq x < 2 - l_3) \frac{1}{x + r_3 - l_3}.
\]

In subcase (e), we know that \( X = 2r_3 - l_2 \). Changing variables from \( l_2 \) to \( x \), from (2.12) we find
\[
f_{l_3,r_3}^{(5)}(x) = 1(2r_3 - l_3 \leq x < 2r_3) \frac{1}{r_3} \frac{1}{x - r_3}.
\]

Finally, in subcase (f), we know that \( X = 1 + r_2 - l_3 \). Changing variables from \( r_2 \) to \( x \), from (2.13) we find
\[
f_{l_3,r_3}^{(6)}(x) = 1(r_3 - 2l_3 + 1 \leq x < 2 - 2l_3) \frac{1}{1 - l_3} \frac{1}{x - 1 + l_3}.
\]

The density functions \( f_{0,r_3} \) and \( f_{1,1} \) we have found in Cases 1 and 2 are continuous. We have chosen to make the functions \( f_{l_3,r_3}^{(i)} \) (for \( i = 1, \ldots, 6 \)) right continuous in Case 3. Thus the density \( f_{l_3,r_3} \) we have determined at (2.16) in Case 3 is right continuous.

Our next lemma handles the cases \( t = 0 \) and \( t = 1 \) that were excluded from Lemma 2.3, and its proof is the same as for Cases 1 and 2 in the proof of Lemma 2.3.

**Lemma 2.4.** (a) Suppose \( t = 0 \). Let \( 0 < r_3 < 1 \). Conditionally given \((L_3, R_3) = (0, r_3)\), the random variable \( X = \Delta_1 + \Delta_2 \) has the right continuous density \( f_{0,r_3} \) specified in the sentence containing (2.6).

(b) Suppose \( t = 1 \). Let \( 0 < l_3 < 1 \). Conditionally given \((L_3, R_3) = (l_3, 1)\), the random variable \( X = \Delta_1 + \Delta_2 \) has the right continuous density \( f_{1,1} \) specified in the sentence containing (2.7).

We need to check the trivariate measurability of the function \( f_t(l_3, r_3, x) := f_{l_3,r_3}(x) \) before diving into the derivation of the density function of \( J \). Given a topological space \( S \), let \( \mathcal{B}(S) \) denote its Borel \( \sigma \)-field, that is, the \( \sigma \)-field generated by the open sets of \( S \). Also, given \( 0 < t < 1 \), let
\[
S_t := \{(l_3, r_3) \neq (0, 1) : 0 \leq l_3 < t < r_3 \leq 1\}.
\]

**Lemma 2.5.** (a) For \( 0 < t < 1 \), the conditional density function \( f_t(l_3, r_3, x) \), formed to be a right continuous function of \( x \), is measurable with respect to \( \mathcal{B}(S_t \times \mathbb{R}) \).

(b) For \( t = 0 \), the conditional density function \( f_0(r_3, x) := f_{0,r_3}(x) \), continuous in \( x \), is measurable \( \mathcal{B}((0, 1) \times \mathbb{R}) \).

(c) For \( t = 1 \), the conditional density function \( f_1(l_3, x) := f_{l_3,1}(x) \), continuous in \( x \), is measurable \( \mathcal{B}((0, 1) \times \mathbb{R}) \).
We introduce (the special case of real-valued $f$ of) a lemma taken from Gowrisankaran [13, Theorem 3] giving a sufficient condition for the measurability of certain functions $f$ defined on product spaces. The lemma will help us prove Lemma 2.5.

**Lemma 2.6** (Gowrisankaran [13]). Let $(\Omega, \mathcal{F})$ be a measurable space. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. Suppose that the section mapping $f(\cdot, y)$ is $\mathcal{F}$-measurable for each $y \in \mathbb{R}$ and that the section mapping $f(\omega, \cdot)$ is either right continuous for each $\omega \in \Omega$ or left continuous for each $\omega \in \Omega$. Then $f$ is measurable with respect to the product $\sigma$-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

**Proof of Lemma 2.5.** We prove (b), then (c), and finally (a).

(b) Recall the expression (2.6) for $f_0(r_3, x)$ [for $0 < r_3 < 1$ and $x \in (2r_3, 2)$]. We apply Lemma 2.6 with $(\Omega, \mathcal{F}) = ((0, 1), \mathcal{B}((0, 1)))$. The right continuity of $f_0(r_3, \cdot)$ has already been established in Lemma 2.4(a). On the other hand, when we fix $x$ and treat $f(0, r_3, x)$ as a function of $r_3$, the conditional density function can be separated into the following cases:

- If $x \leq 0$ or $x \geq 2$, then $f_0(r_3, x) \equiv 0$.
- If $0 < x < 2$, then from (2.6) we see that $f_0(r_3, x)$ is piecewise continuous (with a finite number of measurable domain-intervals), and hence measurable, in $r_3$.

Since the product $\sigma$-field $\mathcal{B}((0, 1)) \otimes \mathcal{B}(\mathbb{R})$ equals $\mathcal{B}((0, 1) \times \mathbb{R})$, the desired result follows.

(c) Assertion (c) can be proved by a similar argument or by invoking symmetry.

(a) We apply Lemma 2.6 with $(\Omega, \mathcal{F}) = (S_t, \mathcal{B}(S_t))$. The right continuity of $f(l_3, r_3, \cdot)$ has already been established in Lemma 2.3, so it suffices to show for each $x \in \mathbb{R}$ that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in S_t$. For this it is clearly sufficient to show that $f(0, r_3, x)$ is measurable in $r_3 \in (t, 1)$, that $f(l_3, 1, x)$ is measurable in $l_3 \in (0, t)$, and that $f(l_3, r_3, x)$ is measurable in $(l_3, r_3) \in (0, t) \times (t, 1)$. All three of these assertions follow from the fact that piecewise continuous functions (with a finite number of measurable domain-pieces) are measurable; in particular, for the third assertion, note that the function $g$ defined at (2.15) is continuous in $(l_3, r_3) \in (0, t) \times (t, 1)$ and that each of the six expressions $f_{l_3, r_3}^{(i)}(x)$ appearing in (2.16) is piecewise continuous (with a finite number of measurable domain-pieces) in these values of $(l_3, r_3)$ for each fixed $x \in \mathbb{R}$.

This complete the proof. \qed

As explained at the outset of this section, a conditional density $h_{l_3, r_3}(\cdot)$ for $J(t)$ given $(L_3, R_3) = (l_3, r_3)$ can now be formed by convolving the conditional density function of $X$, namely $f_{l_3, r_3}(\cdot)$, with the conditional distribution function of $Y$. That is, we can write

$$h_{l_3, r_3}(x) = \int f_{l_3, r_3}(x - y) \, \mathbb{P}(Y \in dy | (L_3, R_3) = (l_3, r_3)). \quad (2.17)$$
We now prove in the next two lemmas that the joint measurability of $f_{l_3,r_3}(x)$ with respect to $(l_3, r_3, x)$ ensures the same for $h_{l_3,r_3}(x)$.

**Lemma 2.7.** Let $(\Omega, \mathcal{F})$ be a measurable space. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function measurable with respect to the product $\sigma$-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. Let $V$ and $Y$ be two measurable functions defined on a common probability space and taking values in $\Omega$ and $\mathbb{R}$, respectively. Then a regular conditional probability distribution $P(Y \in dy \mid V)$ for $Y$ given $V$ exists, and the function $T g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$T g(v, x) := \int g(v, x - y) P(Y \in dy \mid V = v)$$

is measurable with respect to the product $\sigma$-field $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$.

**Proof.** This is standard. For completeness, we provide a proof making use of Kallenberg [19, Lemma 3.2(i)]. First, since $Y$ is a real-valued random variable, by Billingsley [2, Theorem 33.3] or Durrett [6, Theorem 4.1.18] or Kallenberg [19, Theorem 8.5] there exists a regular conditional probability distribution for $Y$ given $V$; this is a probability kernel from $\Omega$ to $\mathbb{R}$ and can trivially be regarded as a kernel from $\Omega \times \mathbb{R}$ to $\mathbb{R}$. Let $S := \Omega \times \mathbb{R}$, $T := \mathbb{R}$, $\mu_{v,x}(dy) := P(Y \in dy \mid V = v)$, and $f((v, x), y) := g(v, x - y)$. The conclusion of our lemma is then an immediate consequence of the first assertion in the aforementioned Kallenberg lemma. $\square$

We can now handle the measurability of $(l_3, r_3, x) \mapsto h_{l_3,r_3}(x)$.

**Lemma 2.8.**

(a) For $0 < t < 1$, the mapping $(l_3, r_3, x) \mapsto h_{l_3,r_3}(x)$ is $\mathcal{B}(S_t \times \mathbb{R})$ measurable.

(b) For $t = 0$, the mapping $(r_3, x) \mapsto h_{0,r_3}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.

(c) For $t = 1$, the mapping $(l_3, x) \mapsto h_{l_3,1}(x)$ is $\mathcal{B}((0, 1) \times \mathbb{R})$ measurable.

**Proof.** We prove (a); the claims (b) and (c) are proved similarly. Choosing $\Omega = S_t$ and $g(l_3, r_3, x) = f_{l_3,r_3}(x)$ with $(l_3, r_3) \in \Omega$ in Lemma 2.7, we conclude that

$$(l_3, r_3, x) \mapsto h_{l_3,r_3}(x) = T g(l_3, r_3, x)$$

is $\mathcal{B}(S_t \times \mathbb{R})$ measurable. $\square$

Recall the definition of $f_t(x)$ at (2.2). It then follows from Lemma 2.8 that $f_t(x)$ is well defined and measurable with respect to $x \in \mathbb{R}$ for fixed $0 \leq t \leq 1$. This completes the proof of Theorem 2.2.

3. **Uniform boundedness of the density functions**

In this section, we prove that the functions $f_t$ are uniformly bounded for $0 \leq t \leq 1$.

**Theorem 3.1.** The densities $f_t$ are uniformly bounded by 10 for $0 \leq t \leq 1$. 

The proof of Theorem 3.1 is our later Lemmas 3.2 and 3.6. In particular, the numerical value 10 comes from the bound in the last line of the proof of Lemma 3.2 plus two times the bound in the last sentence of the proof of Lemma 3.6. A bound on the function $f_t$ is established by first finding a bound on the conditional density function $f_{l_3,r_3}$. Observe that the expressions in the proof of Lemma 2.3 for $f_{l_3,r_3}^{(i)}(x)$ (for $i = 1, \ldots, 6$) in Case 3 all involve indicators of intervals. The six endpoints of these intervals are

$$2r_3 - l_3, 2r_3, 1 + r_3 - 2l_3, 1 + r_3, 2 - 2l_3,$$

with $0 < l_3 < t < r_3 < 1$. The relative order of these six endpoints is determined once we know the value of $\rho = \rho(l_3, r_3) := l_3/(1 - r_3)$. Indeed:

- When $\rho \in (0, 1/2)$, the order is
  $$2r_3 - l_3 < 2r_3 < 1 + r_3 - 2l_3 < 1 + r_3 < 2 - 2l_3 < 2 - l_3.$$

- When $\rho \in (1/2, 1)$, the order is
  $$2r_3 - l_3 < 1 + r_3 - 2l_3 < 2r_3 < 2 - 2l_3 < 1 + r_3 < 2 - l_3.$$

- When $\rho \in (1, 2)$, the order is
  $$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2r_3 < 2 - l_3 < 1 + r_3.$$

- When $\rho \in (2, \infty)$, the order is
  $$1 + r_3 - 2l_3 < 2r_3 - l_3 < 2 - 2l_3 < 2 - l_3 < 2r_3 < 1 + r_3.$$

When $\rho = 0$ (i.e., in Case 1 in the proof of Lemma 2.3: $l_3 = 0 < r_3 < 1$), the function $f_{l_3,r_3}$ is given by $f_{0,r_3}$ at (2.6). When $\rho = \infty$ (i.e., in Case 2 in the proof of Lemma 2.3: $0 < l_3 < r_3 = 1$), the function $f_{l_3,r_3}$ is given by $f_{l_3,1}$ at (2.7). The result (2.16) for Case 3 in the proof of Lemma 2.3 can be reorganized as follows, where we define the following functions to simplify notation:

$$m_1(x, l_3, r_3) := \frac{1}{r_3(x - r_3)} + \frac{2}{(x + l_3)(x - l_3)},$$

$$m_2(x, l_3, r_3) := \frac{1}{(1 - l_3)(x - 1 + l_3)} + \frac{2}{(x + l_3)(x - l_3)},$$

$$m_3(x, l_3, r_3) := \frac{2}{(x + 1 - r_3)(x + r_3 - 1)} + \frac{1}{(1 - l_3)(x + l_3 - 1)},$$

$$m_4(x, l_3, r_3) := \frac{1}{r_3(x - r_3)} + \frac{2}{(x + 1 - r_3)(x + r_3 - 1)}.$$

When $\rho \in (0, 1)$, the conditional density $f_{l_3,r_3}$ satisfies

$$f_{l_3,r_3}(x) g(l_3, r_3) = \mathbb{1}(2r_3 - l_3 \leq x < 1 + r_3 - 2l_3) m_1(x, l_3, r_3) + \mathbb{1}(1 + r_3 - 2l_3 \leq x < 1 + r_3) [m_2(x, l_3, r_3) + m_4(x, l_3, r_3)] + \mathbb{1}(1 + r_3 \leq x < 2 - l_3) m_2(x, l_3, r_3).$$

Lastly, when $\rho \in (1, \infty)$, the conditional density $f_{l_3,r_3}$ satisfies

$$f_{l_3,r_3}(x) g(l_3, r_3) = \mathbb{1}(1 + r_3 - 2l_3 \leq x < 2r_3 - l_3) m_3(x, l_3, r_3)$$

(3.2)
that $1 + r_3 < x < 2 - l_3$ \[ m_2(x, l_3, r_3) + m_4(x, l_3, r_3) \]
+ $1(2 - l_3 < x < 1 + r_3) m_4(x, l_3, r_3)$.

Recall the definition of $f_\ell(x)$ at (2.2). For any $x \in \mathbb{R}$ we can decompose $f_\ell(x)$ into three contributions:

$$f_\ell(x) = \mathbb{E} h_{L_3, R_3}(x) = \mathbb{E}[h_{L_3, R_3}(x); \rho(L_3, R_3) = 0] + \mathbb{E}[h_{L_3, R_3}(x); \rho(L_3, R_3) = \infty]
+ \mathbb{E}[h_{L_3, R_3}(x); 0 < \rho(L_3, R_3) < \infty].$$

We first consider the contribution from the case $0 < \rho(L_3, R_3) < \infty$ for any $0 < \ell < 1$, noting that this case doesn’t contribute to $f_\ell(x)$ when $t = 0$ or $t = 1$. Define

$$b(l_3, r_3) = \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[ \frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right]. \tag{3.3}$$

**Lemma 3.2.** The contribution to the density function $f_\ell$ from the case $0 < \rho(L_3, R_3) < \infty$ is uniformly bounded for $0 < \ell < 1$. More precisely, given $0 < \ell < 1$ and $0 < l_3 < t < r_3 < 1$, we have

$$f_{l_3, r_3}(x) \leq b(l_3, r_3) \quad \text{and} \quad h_{l_3, r_3}(x) \leq b(l_3, r_3) \quad \text{for all} \ x \in \mathbb{R}; \tag{3.4}$$

and, moreover, $\mathbb{E}[h_{L_3, R_3}(x); 0 < \rho(L_3, R_3) < \infty]$ is uniformly bounded for $0 < \ell < 1$.

**Proof.** For (3.4), we need only establish the bound on $f_\ell$.

We start to bound (3.1) for $0 < \rho < 1$. The function $m_1$ is a decreasing function of $x$ and thus reaches its maximum in (3.1) when $x = 2r_3 - l_3$:

$$m_1(x, l_3, r_3) \leq m_1(2r_3 - l_3, l_3, r_3) = \frac{3}{2} \frac{1}{r_3(r_3 - l_3)}.$$ 

The function $m_2 + m_4$ is also a decreasing function of $x$, and the maximum in (3.1) occurs at $x = 1 + r_3 - 2l_3$. Plug in this $x$-value and use the fact that $1 - l_3 > r_3$ when $\rho < 1$ to obtain

$$(m_2 + m_4)(x, l_3, r_3) \leq \frac{3}{2} \frac{1}{r_3(r_3 - l_3)} + \frac{3}{2} \frac{1}{(1 - l_3)(r_3 - l_3)}.$$ 

Finally, the function $m_2$ is again a decreasing function of $x$, and the maximum in (3.1) occurs at $x = 1 + r_3$. Plug in this $x$-value and use the facts that $1 + r_3 - l_3 < 2r_3$ and $1 + r_3 + l_3 > r_3 - l_3$ to conclude

$$m_2(x, l_3, r_3) \leq \frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)}.$$ 

By the above three inequalities, we summarize that for $0 < \rho < 1$ we have for all $x$ the inequality

$$f_{l_3, r_3}(x) \leq \frac{1}{g(l_3, r_3)} \frac{3}{2} \left[ \frac{1}{r_3(r_3 - l_3)} + \frac{1}{(1 - l_3)(r_3 - l_3)} \right].$$

The method to upper-bound (3.2) is similar to that for (3.1), or one can again invoke symmetry, and we skip the proof here.
For the expectation of $b(L_3, R_3)$, we see immediately that
\[
\mathbb{E}[b(L_3, R_3); 0 < \rho(L_3, R_3) < \infty] = \frac{3}{2} \int_0^t \int_t^1 \left[ \frac{1}{r(r-l)} + \frac{1}{(1-l)(r-l)} \right] \, dr \, dl = \frac{\pi^2}{4} + \frac{3}{2} (\ln t)(\ln(1-t)) \leq \frac{\pi^2}{4} + \frac{3}{2} (\ln 2)^2.
\]

For the cases $\rho(L_3, R_3) = 0$ and $\rho(L_3, R_3) = \infty$, we cannot find a constant bound $b(l_3, r_3)$ on the function $f_{l_3,r_3}$ such that the corresponding contributions to $\mathbb{E}[b(L_3, R_3)]$ are bounded for $0 \leq t \leq 1$. Indeed, although we shall omit the proof since it would take us too far afield, there exists no such bound $b(l_3, r_3)$.

Instead, to prove the uniform boundedness of the contributions in these two cases, we take a different approach. The following easily-proved lemma comes from Gr"ubel and R"osler [15, proof of Theorem 9].

**Lemma 3.3.** Consider a sequence of independent random variables $V_1, V_2, \ldots$, each uniformly distributed on $(1/2, 1)$, and let
\[
V := 1 + \sum_{n=1}^{\infty} \prod_{k=1}^n V_k.
\]

Then the random variables $Z(t)$, $0 \leq t \leq 1$, defined at (1.10) are all stochastically dominated by $V$. Furthermore, $\mathbb{E}V = 4$; and $V$ has everywhere finite moment generating function $m$ and therefore superexponential decay in the right tail, in the sense that for any $\theta \in (0, \infty)$ we have $P(V \geq x) = o(e^{-\theta x})$ as $x \to \infty$.

The following lemma pairs the stochastic upper bound $V$ on $Z(t)$ with a stochastic lower bound. These stochastic bounds will be useful in later sections. Recall that the Dickman distribution with support $[1, \infty)$ is the distribution of $Z(0)$.

**Lemma 3.4.** Let $D$ be a random variable following the Dickman distribution with support $[1, \infty)$. Then for all $0 \leq t \leq 1$ we have $D \leq Z(t) \leq V$ stochastically.

**Proof.** Recall that $\Delta_1(t) = R_1(t) - L_1(t)$. We first use a coupling argument to show that $\Delta_1(t)$ is stochastically increasing for $0 \leq t \leq 1/2$. Let $U = U_1 \sim \text{uniform}(0, 1)$ be the first key in the construction of $Z$ as described in (1.7)–(1.10). Let $0 \leq t_1 < t_2 \leq 1/2$. It is easy to see that $\Delta_1(t_1) = \Delta_1(t_2)$ unless $t_1 < U < t_2$, in which case $\Delta_1(t_1) = U < t_2 \leq 1/2 < 1 - U = \Delta_1(t_2)$.

Let $V_1 \sim \text{uniform}(1/2, 1)$, as in Lemma 3.3, and let $0 \leq t \leq 1/2$. Since $\Delta_1(0) \overset{d}{=} U$ and $\Delta_1(1/2) \overset{d}{=} V_1$, we immediately have $U \leq \Delta_1(t) \leq V_1$ stochastically. This implies by a simple induction argument on $k$ involving the conditional distribution of $\Delta_k(t)$ given $\Delta_{k-1}(t)$ that $D \leq Z(t) \leq V$ stochastically. \qed
Remark 3.5. (a) Note that we do not claim that \( Z(t) \) is stochastically increasing in \( t \in [0, 1/2] \). Indeed, other than the stochastic ordering \( D = Z(0) \leq Z(t) \), we do not know whether any stochastic ordering relations hold among the random variables \( Z(t) \).

(b) The random variable \( V \) can be interpreted as a sort of “limiting greedy (or ‘on-line’) worst-case QuickQuant normalized key-comparisons cost”. Indeed, if upon each random bisection of the search interval one always chooses the half of greater length and sums the lengths to get \( V^{(n)} \), then the limiting distribution of \( V^{(n)}/n \) is that of \( V \).

Lemma 3.6. The contributions to the density function \( f_t \) from the cases \( \rho(L_3, R_3) = 0 \) and \( \rho(L_3, R_3) = \infty \) are uniformly bounded for \( 0 \leq t \leq 1 \).

Proof. Because the Dickman density is bounded above by \( e^{-\gamma} \), we need only consider \( 0 < t < 1 \). The case \( \rho(L_3(t), R_3(t)) = 0 \) corresponds to \( L_3(t) = 3 \), while the case \( \rho(L_3(t), R_3(t)) = \infty \) corresponds to \( R_3(t) = 1 \). By symmetry, the contribution from \( R_3(t) = 1 \) is the same as the contribution from \( L_3(1-t) = 0 \), so we need only show that the contribution from \( L_3(t) = 0 \) is uniformly bounded. We will do this by showing that the larger contribution from \( L_2(t) = 0 \) is uniformly bounded.

By conditioning on the value of \( R_2(t) \), the contribution from \( L_2(t) = 0 \) is

\[
c_t(x) := \mathbb{P}(L_2(t) = 0, J(t) \in dx) / dx
\]

\[
= \int_{t \leq t} \int_{z > 1} 1(r \leq x - rz < 1) (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) dr. \tag{3.6}
\]

If \( z > 1 \) and \( r \leq x - rz \), then \((x/r) - 1 \geq z > 1 \) and so \( r < x/2 \). Therefore we find

\[
c_t(x) \leq \int_{t} \int_{\max(1, (x-1)/r) < z \leq (x/r) - 1} (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) dr.
\]

The integrand (including the implicit indicator function) in the inner integral is an increasing function of \( z \) over the interval \((-\infty, (x/r) - 1]\), with value \( r^{-1} \) at the upper endpoint of the interval. We can thus extend it to a nondecreasing function \( \phi \equiv \phi_{x, r} \) with domain \( \mathbb{R} \) by setting \( \phi(z) = r^{-1} \) for \( z > (x/r) - 1 \). It then follows that

\[
c_t(x) \leq \int_{0}^{\min(1, x/2)} \int_{\max(1, (x-1)/r) < z \leq (x/r) - 1} (x - rz)^{-1} \mathbb{P}(V \in dz) \bigg[ \mathbb{P}(V > \frac{x}{r} - 1) dr 
+ r^{-1} \mathbb{P}(V > \frac{x}{r} - 1) \bigg]
+ \int_{0}^{\min(1, x/2)} \int_{\max(1, (x-1)/r) < z \leq (x/r) - 1} (x - rz)^{-1} \mathbb{P}(V \in dz) dr
+ \int_{0}^{x/2} r^{-1} \mathbb{P}(V > \frac{x}{r} - 1) dr. \tag{3.7}
\]
By the change of variables $v = (x/r) - 1$, the second term in (3.7) equals
\[
\int_1^\infty (v + 1)^{-1} \mathbb{P}(V > v) \, dv \leq \frac{1}{2} \int_0^\infty \mathbb{P}(V > v) \, dv = \frac{1}{2} \mathbb{E} V = 2.
\]
Comparing the integrals $c_t(x)$ at (3.6) and the first term in (3.7), we see that the only constraint that has been discarded is $r > t$. We therefore see by the same argument that produces (3.6) that the first term in (3.7) is the value of the density for $W := U_1(1 + U_2 V)$ at $x$, where $U_1$, $U_2$, and $V$ are independent and $U_1$ and $U_2$ are uniformly distributed on $(0, 1)$. Thus to obtain the desired uniform boundedness of $f_t$ we need only show that $W$ has a bounded density. For that, it suffices to observe that the conditional density of $W$ given $U_2$ and $V$ is bounded above by 1 (for any values of $U_2$ and $V$), and so the unconditional density is bounded by 1. We conclude that $c_t(x) \leq 3$, and this completes the proof.

**Remark 3.7.** Based on simulation results, we conjecture that the density functions $f_t$ are uniformly bounded by $e^{-\gamma}$ (the sup-norm of the right-continuous Dickman density $f_0$) for $0 \leq t \leq 1$.

### 4. Uniform Continuity of the Density Function $f_t$

From the previous section, we know that for $0 < t < 1$ in the case $0 < l < t < r < 1$ (i.e., the case $0 < \rho < \infty$) the function $f_{l,r}$ is càdlàg (that is, a right continuous function with left limits) and bounded above by $b(l, r)$, where the corresponding contribution $\mathbb{E}[b(L_3, R_3) ; 0 < \rho(L_3, R_3) < \infty]$ is finite. Applying the dominated convergence theorem, we conclude that the contribution to $f_t$ from this case is also càdlàg.

For the cases $0 = l < t < r < 1$ ($\rho = 0$) and $0 < l < t < r = 1$ ($\rho = \infty$), the functions $f_{0,r}$ and $f_{l,1}$ are both continuous on the real line. In this section, we will build bounds $b_1(l, r)$ (note that these bounds depend on $t$) for these two cases (Lemma 4.1 for $\rho = 0$ and Lemma 4.2 for $\rho = \infty$) in similar fashion as for Lemma 3.6 such that both $\mathbb{E}[b_1(L_3, R_3) ; \rho(L_3, R_3) = 0]$ and $\mathbb{E}[b_1(L_3, R_3) ; \rho(L_3, R_3) = \infty]$ are finite for any $0 < t < 1$. Given these bounds, we can apply the dominated convergence theorem to conclude that the density $f_t$ is càdlàg. Later, this result will be sharpened substantially in Theorem 4.4.

Let $\alpha \approx 3.59112$ be the unique real solution of $1 + x - x \ln x = 0$ and let $\beta := 1/\alpha \approx 0.27846$. Define
\[
b_1(r) := \frac{2}{\ln r - 1} \frac{1}{1 + r} \quad \text{and} \quad b_2(r) := \frac{2}{(\ln r - 1)^2} \frac{1}{r \beta}.
\]

**Lemma 4.1.** Suppose $\rho = 0$, i.e., $0 = l_3 < t < r_3 < 1$. If $t \geq \beta$, then the optimal constant upper bound on $f_{l_3,r_3}$ is
\[
b_t(l_3, r_3) = b_1(r_3),
\]
with corresponding contribution
\[ \mathbb{E}[b_t(L_3(t), R_3(t)); \rho(L_3(t), R_3(t)) = 0] = \int_{t}^{1} \frac{\ln r^{-1}}{1 + r} \, dr \leq \int_{\beta}^{1} \frac{\ln r^{-1}}{1 + r} \, dr < \frac{1}{4} \]

to \( \mathbb{E} b_t(L_3(t), R_3(t)) \). If \( t < \beta \), then the optimal constant upper bound on \( f_{l_3, r_3} \) is the continuous function
\[ b_t(l_3, r_3) = b_1(r_3) \mathbb{I}(\beta \leq r_3 < 1) + b_2(r_3) \mathbb{I}(t < r_3 < \beta) \]
of \( r_3 \in (t, 1] \), with corresponding contribution
\[ \mathbb{E}[b_t (L_3(t), R_3(t)) ; \rho(L_3(t), R_3(t)) = 0] = \int_{\beta}^{1} \frac{\ln r^{-1}}{1 + r} \, dr + \beta(\ln \beta - \ln t) \]
\[ < \frac{1}{4} + \beta(\ln \beta - \ln t). \]

**Lemma 4.2.** Suppose \( \rho = \infty \), i.e., \( 0 < l_3 < t < r_3 = 1 \). If \( t \leq 1 - \beta \), then the optimal constant upper bound on \( f_{l_3, r_3} \) is
\[ b_t(l_3, r_3) = b_1(1 - l_3), \]

with corresponding contribution
\[ \mathbb{E}[b_t (L_3(t), R_3(t)) ; \rho(L_3(t), R_3(t)) = 0] = \int_{1-t}^{1} \frac{\ln r^{-1}}{1 + r} \, dr \leq \int_{\beta}^{1} \frac{\ln r^{-1}}{1 + r} \, dr < \frac{1}{4}. \]

If \( t > 1 - \beta \), then the optimal constant upper bound on \( f_{l_3, r_3} \) is the continuous function
\[ b_t(l_3, r_3) = b_1(1 - l_3) \mathbb{I}(0 < l_3 \leq 1 - \beta) + b_2(1 - l_3) \mathbb{I}(1 - \beta < l_3 < t) \]
of \( l_3 \in [0, t) \), with corresponding contribution
\[ \mathbb{E}[b_t (L_3(t), R_3(t)) ; \rho(L_3(t), R_3(t)) = 0] = \int_{\beta}^{1} \frac{\ln r^{-1}}{1 + r} \, dr + \beta[\ln \beta - \ln(1 - r)] \]
\[ < \frac{1}{4} + \beta[\ln \beta - \ln(1 - t)]. \]

We prove Lemma 4.1 here, and Lemma 4.2 follows similarly or by symmetry.

**Proof of Lemma 4.1.** When \( \rho = 0 \), we have \( l_3 = 0 \), and the conditional density function is \( f_{0, r_3} \) in (2.6). The expression in square brackets at (2.6) is continuous and unimodal in \( x \), with maximum value at \( x = 1 + r_3 \). Because the factor \( 1/x \) is decreasing, it follows that the maximum value of \( f_{0, r_3}(x) \) is the maximum of
\[ \frac{2}{(\ln r_3^{-1})^2} x \ln \left( \frac{x - r_3}{r_3} \right) \]
over \( x \in [2r_3, 1 + r_3] \), i.e., the maximum of
\[ \frac{2}{r_3 (\ln r_3^{-1})^2} \frac{\ln y}{1 + y} \]
over \( y \in [1, 1/r_3] \). A simple calculation shows that the displayed expression is strictly increasing for \( y \in [1, \alpha] \) and strictly decreasing for \( y \in [\alpha, \infty) \). Thus the maximum for \( y \in [1, 1/r_3] \) is achieved at \( y = \alpha \) if \( \alpha \leq 1/r_3 \) and at \( y = 1/r_3 \) if \( \alpha \geq 1/r_3 \). Equivalently, \( f_{0,r_3}(x) \) is maximized at \( x = r_3(\alpha + 1) \) if \( r_3 \leq \beta \) and at \( x = 1 + r_3 \) if \( r_3 \geq \beta \). The claims about the optimal constant upper bound on \( f_{l_3,r_3} \) and the contribution to \( \mathbb{E} b_t(L_3(t), R_3(t)) \) now follow readily.

**Remark 4.3.** If we are not concerned about finding the best possible upper bound, then for the case \( \rho = 0 \) we can choose \( b_l(l, r) := b_2(r) \); for the case \( \rho = \infty \), we can choose \( b_l(l, r) := b_2(1-l) \). These two bounds still get us the desired finiteness of the contributions to \( \mathbb{E} b_t(L_3, R_3) \) for any \( 0 < t < 1 \).

**Theorem 4.4.** For \( 0 < t < 1 \), the density function \( f_t : \mathbb{R} \to [0, \infty) \) is uniformly continuous.

**Proof.** Fix \( 0 < t < 1 \). By the dominated convergence theorem, the contributions to \( f_t(x) \) from \( 0 = l < t < r < 1 \) and \( 0 < l < t < r = 1 \), namely, the functions

\[
c_0(x) := \int_{r,y} f_{0,r}(x-y) \mathbb{P}(L_3(t) = 0, R_3(t) \in dr, Y \in dy)
\]

and

\[
c_1(x) := \int_{l,y} f_{1,l}(x-y) \mathbb{P}(L_3(t) \in dl, R_3(t) = 1, Y \in dy),
\]

are continuous for \( x \in \mathbb{R} \). Further, according to (2.14) and (2.16), the contribution from \( 0 < l < t < r < 1 \) is \( \sum_{i=1}^6 c^{(i)}(x) \), where we define

\[
c^{(i)}(x) := \int_{l,r,y} f^{(i)}_{l,r}(x-y) \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l, r)) \, dl \, dr.
\]

It is easy to see that all the functions \( c_0, c_1, \) and \( c^{(i)} \) for \( i = 1, \ldots, 6 \) vanish for arguments \( x \leq 0 \). To prove the uniform continuity of \( f_t(x) \) for \( x \in \mathbb{R} \), it thus suffices to show that each of the six functions \( c^{(i)} \) for \( i = 1, \ldots, 6 \) is continuous on the real line and that each of the eight functions \( c_0, c_1, \) and \( c^{(i)} \) for \( i = 1, \ldots, 6 \) vanishes in the limit as argument \( x \to \infty \).

Fix \( i \in \{1, \ldots, 6\} \). Continuity of \( c^{(i)} \) holds since \( f^{(i)}_{l,r} \) is bounded by \( b(l, r) \) defined at (3.3) and is continuous except at the boundary of its support. To illustrate, consider, for example, \( i = 3 \). For each fixed \( 0 < l < t < r < 1 \) and \( x \in \mathbb{R} \), we have \( f^{(3)}_{l,r}(x+h-y) \to f^{(3)}_{l,r}(x-y) \) as \( h \to 0 \) for all but two exceptional values of \( y \), namely, \( y = x - (1 + r - 2l) \) and \( y = x - (1 - r) \). From the discussion following (2.1) and from Theorem 2.2, we know that the conditional law of \( Y \) given \( (L_3(t), R_3(t)) = (l, r) \) has a density with respect to Lebesgue measure, and hence the set of two exceptional points has zero measure under this law. We conclude from the dominated convergence theorem that

\[
\int_y f^{(i)}_{l,r}(x-y) \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l, r))
\]
is continuous in $x \in \mathbb{R}$. It now follows by another application of the dominated convergence theorem that $c^{(i)}$ is continuous on the real line.

Since the eight functions $f_{0,r}$, $f_{1,1}$, and $f_{i,r}$ for $i = 1,\ldots,6$ all vanish for all sufficiently large arguments, another application of the dominated convergence theorem shows that $c_0(x), c_1(x)$, and $c^{(i)}(x)$ for $i = 1,\ldots,6$ all vanish in the limit as $x \to \infty$. This completes the proof. \hfill \Box

**Remark 4.5.** For any $0 < t < 1$, by the fact that $J(t) \geq R_1(t) - L_1(t) \geq \min(t,1-t)$,

we have $\mathbb{P}(J(t) < \min(t,1-t)) = 0$ and thus $f_t(\min(t,(1-t))) = 0$ by Theorem 4.4. This is a somewhat surprising result since we know that the right-continuous Dickman density $f_0$ satisfies $f_0(0) = e^{-\gamma} > 0$.

**Remark 4.6.** Since $f_0$ is both (uniformly) continuous on and piecewise differentiable on $(0,\infty)$, it might be natural to conjecture that the densities $f_t$ for $0 < t < 1$ are also piecewise differentiable.

Later, in Theorem 7.4, we prove that the densities $f_t$ are Lipschitz continuous, which implies that each of them is almost everywhere differentiable.

### 5. Integral Equation for the Density Functions

In this section we prove that for $0 \leq t < 1$ and $x \in \mathbb{R}$, the density function $f_t(x)$ is jointly Borel measurable in $(t,x)$. By symmetry, we can conclude that $f_t(x)$ is jointly Borel measurable in $(t,x)$ for $0 \leq t \leq 1$. We then use this result to establish an integral equation for the densities.

Let $F_t$ denote the distribution function for $J(t)$. Because $F_t$ is right continuous, it is Borel measurable (for each $t$).

**Lemma 5.1.** For each positive integer $n$, the mapping

$$(t,x) \mapsto F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \quad (0 \leq t < 1, \ x \in \mathbb{R})$$

is Borel measurable.

**Proof.** Note that

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) = \sum_{j=1}^{n} \mathbb{1}_{\left(\frac{j-1}{n} \leq t < \frac{j}{n}\right)} F_{\frac{j}{n}}(x).$$

Each term is the product of a Borel measurable function of $t$ and a Borel measurable function of $x$ and so is a Borel measurable of $(t,x)$. The same is then true of the sum. \hfill \Box

**Lemma 5.2.** For each $0 \leq t < 1$ and $x \in \mathbb{R}$, as $n \to \infty$ we have

$$F_{\frac{\lfloor nt \rfloor + 1}{n}}(x) \to F_t(x).$$
Proof. We reference Grübel and Rösler [15], who construct a process \( J = (J(t))_{0 \leq t \leq 1} \) with \( J(t) \) having distribution function \( F_t \) for each \( t \) and with right continuous sample paths. It follows (for each \( t \in [0,1) \)) that \( F_u \) converges weakly to \( F_t \) as \( u \downarrow t \). But we know that \( F_t \) is a continuous (and even continuously differentiable) function, so for each \( x \in \mathbb{R} \) we have \( F_u(x) \to F_t(x) \) as \( u \downarrow t \). The result follows. \( \square \)

Proposition 5.3. The mapping

\[ (t, x) \mapsto F_t(x) \quad (0 \leq t < 1, \ x \in \mathbb{R}) \]

is Borel measurable.

Proof. According to Lemmas 5.1–5.2, this mapping is the pointwise limit as \( n \to \infty \) of the Borel measurable mappings in Lemma 5.2. \( \square \)

Let \( f_t \) denote the continuous density for \( F_t \), as in Theorem 4.4.

Theorem 5.4. The mapping

\[ (t, x) \mapsto f_t(x) \quad (0 \leq t < 1, \ x \in \mathbb{R}) \]

is Borel measurable.

Proof. By the fundamental theorem of integral calculus, \( f_t = F_t' \). The mapping in question is thus the (sequential) limit of difference quotients that are Borel measurable by Proposition 5.3 and hence is Borel measurable. \( \square \)

Now we are ready to derive integral equations. We start with an integral equation for the distribution functions \( F_t \).

Proposition 5.5. The distribution functions \( (F_t) \) satisfy the following integral equation for \( 0 \leq t \leq 1 \) and \( x \in \mathbb{R} \):

\[
F_t(x) = \int_{l \in (0, t)} F_{t-1} \left( \frac{x}{1-l} - 1 \right) \, dl + \int_{r \in (t, 1)} F_{t} \left( \frac{x}{r} - 1 \right) \, dr. \tag{5.1}
\]

Proof. This follows by conditioning on the value of \((L_1(t), R_1(t))\). Observe that each of the two integrands is (by Proposition 5.3 for \( t \notin \{0,1\} \) and by right continuity of \( F_0 \) and \( F_1 \) for \( t \in \{0,1\} \)) indeed [for fixed \((t, x)\)] a Borel measurable function of the integrating variable. \( \square \)

Remark 5.6. It follows from (i) the changes of variables from \( l \) to \( v = (t-l)/(1-l) \) in the first integral in (5.1) and from \( r \) to \( v = t/r \) in the second integral, (ii) the joint continuity of \( f_t(x) \) in \((t, x)\) established later in Corollary 7.11, and (iii) Leibniz’s formula that \( F_t(x) \) is differentiable with respect to \( t \in (0,1) \) for each fixed \( x \in \mathbb{R} \).

Integral equation (5.1) for the distribution functions \( F_t \) immediately leads us to an integral equation for the density functions \( f_t \).
**Proposition 5.8.** The continuous density functions ($f_t$) satisfy the following integral equation for $0 < t < 1$ and $x \in \mathbb{R}$:
\[
f_t(x) = \int_{l \in (0,t)} (1-l)^{-1} f_{\frac{x}{1-l}} \left( \frac{x}{1-l} - 1 \right) \, dl + \int_{r \in (t,1)} r^{-1} f_{\frac{x}{r}} \left( \frac{x}{1-r} - 1 \right) \, dr.
\]

**Proof.** Fix $t \in (0,1)$. Differentiate (5.1) with respect to $x$. It is easily proved by an argument applying the dominated convergence theorem to difference quotients and the mean value theorem that we can differentiate under the integral signs in (5.1) provided that
\[
\int_{l \in (0,t)} (1-l)^{-1} f_{\frac{x}{1-l}} \, dl + \int_{r \in (t,1)} r^{-1} f_{\frac{x}{r}} \, dr
\]
is finite, where $f_t^*$ denotes any upper bound on $f_t(x)$ as $x$ varies over $\mathbb{R}$. By Theorem 3.1 we can simply choose $f_t^* = 10$. Then (5.2) equals 10 times
\[-\ln(1-t) - \ln t,
\]
which is finite. \qed

In the next proposition, we provide an integral equation based on the formula for $f_t$ in (2.2); this integral equation will be useful in the next section. Recall that $Y(t) = \sum_{k=3}^{\infty} \Delta_k(t)$. Using (2.1), the conditional distribution of $Y(t)/(t_3 - l_3)$ given $(L_3, R_3) = (l_3, r_3)$ is the (unconditional) distribution of $Z(\frac{l_3}{l_3 - r_3}) = 1 + J(\frac{l_3}{l_3 - r_3})$. Apply Theorem 2.2 on $Z(\frac{l_3}{l_3 - r_3})$ leads us to an integral equation for the density function of $J(t)$.

**Proposition 5.7.** The continuous density functions $f_t$ for the random variables $J(t) = Z(t) - 1$ satisfy the integral equation
\[
f_t(x) = \int P((L_3(t), R_3(t)) \in d(l_3, r_3)) \cdot h_t(x \mid l_3, r_3)
\]
for $x \geq 0$, where
\[
h_t(x \mid l_3, r_3) = \int f_{l_3, r_3}(x - y) (r_3 - l_3)^{-1} f_{\frac{x-y}{r_3-l_3}} \left( \frac{y}{r_3-l_3} - 1 \right) \, dy.
\]

6. **Right-tail behavior of the density function**

In this section we will prove, uniformly for $0 < t < 1$, that the continuous density functions $f_t$ enjoy the same superexponential decay bound as Gribel and Rösler [15, Theorem 9] proved for the survival functions $1 - F_t$. By a separate and easier argument, one could include the cases $t = 0,1$. Let $m_t$ denote the moment generating function of $Z(t)$ and recall that $m$ denotes the moment generating function of $V$ at (3.5). By Lemma 3.3, the random variables $Z(t)$, $0 \leq t \leq 1$, are stochastically dominated by $V$. As a consequence, if $\theta \geq 0$, then
\[
m_t(\theta) \leq m(\theta) < \infty
\]
for every $t \in (0,1)$. 
Theorem 6.1. Uniformly for \( t \in (0,1) \), the continuous \texttt{QuickQuant} density functions \( f_t(x) \) enjoy superexponential decay when \( x \) is large. More precisely, for any \( \theta > 0 \) we have

\[
f_t(x) < 4\theta^{-1}e^{2\theta}m(\theta)e^{-\theta x}
\]

for \( x \geq 3 \), where \( m \) is the moment generating function of the random variable \( V \) at (3.5).

Proof. Our starting point is the following equation from the discussion preceding Proposition 5.8:

\[
f_t(x) = \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) \int f_{l,r}(x - y) \mathbb{P}\left((r - l)Z\left(\frac{t - l}{r - l}\right) \in dy\right)
\]

By Lemma 3.3, for any \( \theta \in \mathbb{R} \) we can obtain a probability measure \( \mu_{t,\theta}(dz) := m_t(\theta)^{-1}e^{\theta z} \mathbb{P}(Z(t) \in dz) \) by exponential tilting. Since \( m_t(\theta) \leq m(\theta) < \infty \) for every \( \theta \geq 0 \) and \( t \in (0,1) \), we can rewrite and bound (6.1) as follows (for any \( \theta \geq 0 \)):

\[
f_t(x) \leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) \int e^{-\theta z} f_{l,r}(x - (r - l)z) \mu_{t,\theta}(dz)
\]

Recall that \( f_{l,r}(x) \) is bounded above by \( b_t(l, r) \) (Lemmas 3.2 and 4.1–4.2) and vanishes for \( x \geq 2 \). Therefore, if \( \theta \geq 0 \) then

\[
f_t(x) \leq m(\theta) \int_{l,r} \mathbb{P}((L_3, R_3) \in (dl, dr)) b_t(l, r) \int e^{-\theta z} \mu_{t,\theta}(dz)
\]

Suppose \( x \geq 3 \) and \( \theta > 0 \). We now consider in turn the contribution to (6.2) for \( l = 0 \), for \( r = 1 \), and for \( 0 < l < t < r < 1 \). For \( l = 0 \), the contribution is \( m(\theta) \) times the following:

\[
\int_0^1 r^{-1} \beta \exp[-\theta r^{-1}(x - 2)] dr 
\]

\[
\leq \beta \int_0^1 r^{-2} \exp[-\theta r^{-1}(x - 2)] dr 
\]

\[
= \beta \left[\theta(x - 2)^{-1} \exp[-\theta(x - 2)]\right] \leq \beta \theta^{-1}e^{2\theta}e^{-\theta x}.
\]
Similarly (or symmetrically), the contribution for \( r = 1 \) is bounded by the same \( 3\theta^{-1}e^{2\theta} m(\theta)e^{-\theta x} \). For \( 0 < l < t < r < 1 \), by symmetry we may without loss of generality suppose that \( 0 < t \leq 1/2 \), and then the contribution is \( \frac{3}{2}m(\theta) \) times the following:

\[
\int_0^t \int_{t-l}^{t-l} \left[ \frac{1}{r(r-l)} + \frac{1}{(1-l)(r-l)} \right] \exp \left( -\frac{\theta x}{r-l} \right) \, dr \, dl \\
= \int_0^t \int_{t-l}^{1-l} \left[ \frac{1}{(s+l)s} + \frac{1}{(1-l)s} \right] \exp[-\theta s^{-1}(x-2)] \, ds \, dl \\
= \int_0^t \int_{t-l}^{1-l} (1-l)^{-1}(s+l)^{-1}(1+s)s^{-1} \exp[-\theta s^{-1}(x-2)] \, ds \, dl \\
\leq 4 \int_0^t \int_{t-l}^{1-l} s^{-2} \exp[-\theta s^{-1}(x-2)] \, ds \, dl \\
\leq 4 \int_0^{1/2} \int_0^1 s^{-2} \exp[-\theta s^{-1}(x-2)] \, ds \, dl \\
= 2[\theta(x-2)]^{-1} \exp[-\theta(x-2)] \leq 2\theta^{-1}e^{2\theta}e^{-\theta x}.
\]

Summing all the contributions, we find

\[ f_t(x) \leq (3 + 2\beta)\theta^{-1}e^{2\theta} m(\theta)e^{-\theta x} < 4\theta^{-1}e^{2\theta} m(\theta)e^{-\theta x}, \quad (6.3) \]

for any \( 0 < t < 1 \), \( x \geq 3 \), and \( \theta > 0 \), demonstrating the uniform superexponential decay. \(\Box\)

**Remark 6.2.** Since \( f_t \) is uniformly bounded by 10 by Theorem 3.1, for any \( \theta > 0 \), by choosing the coefficient \( C_\theta := \max\{10e^{3\theta}, 4\theta^{-1}e^{2\theta} m(\theta)\} \), we can extend the superexponential bound on \( f_t(x) \) in Theorem 6.1 for \( x \geq 3 \) to \( x \in \mathbb{R} \) as

\[ f_t(x) \leq C_\theta e^{-\theta x} \quad \text{for } x \in \mathbb{R} \text{ and } 0 < t < 1. \quad (6.4) \]

Note that this bound is not informative for \( x \leq \min\{t, 1-t\} \) since we know \( f_t(x) = 0 \) for such \( x \) by Theorem 4.4, but it will simplify our proof of Theorem 7.4.

### 7. Positivity and Lipschitz Continuity of the Continuous Density Functions

In this section we establish several properties of the continuous density function \( f_t \). We prove that \( f_t(x) \) is positive for every \( x > \min\{t, 1-t\} \) (Theorem 7.1), Lipschitz continuous for \( x \in \mathbb{R} \) (Theorem 7.4), and jointly continuous for \( (t, x) \in (0,1) \times \mathbb{R} \) (Corollary 7.11).

#### 7.1. Positivity

**Theorem 7.1.** For any \( 0 < t < 1 \), the continuous density \( f_t \) satisfies

\[ f_t(x) > 0 \text{ if and only if } x > \min\{t, 1-t\}. \]
We already know that $f_t(x) = 0$ if $x \leq \min\{t, 1 - t\}$, so we need only prove the “if” assertion. Our starting point for the proof is the following lemma. Recall from Chung [4, Exercise 1.6] that a point $x$ is said to belong to the support of a distribution function $F$ if for every $\epsilon > 0$ we have

$$F(x + \epsilon) - F(x - \epsilon) > 0.$$ (7.1)

Note that to prove that $x$ is in the support of $F$ we may choose any $\epsilon_0(x) > 0$ and establish (7.1) for all $\epsilon \in (0, \epsilon_0(x))$.

**Lemma 7.2.** For any $0 < t < 1$, the support of $F_t$ is $[\min\{t, 1 - t\}, \infty)$.

**Proof.** Clearly the support of $F_t$ is contained in $[\min\{t, 1 - t\}, \infty)$, so we need only establish the reverse containment. Since $F_t = F_{1-t}$ by symmetry, we may fix $t \leq 1/2$. Also fixing $x \geq t$, write

$$x = t + K + b$$

where $K \geq 0$ is an integer and $b \in [0, 1)$. We will show that $x$ belongs to the support of $F_t$. Let

$$A := \bigcap_{k=1}^{K} \{1 - k\epsilon < R_k < 1 - (k - 1)\epsilon\}.$$ (7.2)

We break our analysis into four cases: (i) $t < b < 1$, (ii) $b = t$, (iii) $0 < b < t$, and (iv) $b = 0$.

(i) $t < b < 1$. Let

$$B := \{b < R_{K+1} < b + \epsilon\} \bigcap \{t < R_{K+2} < t + \epsilon\} \bigcap \{t - \epsilon < L_{K+3} < t\}$$

and

$$C := \left\{0 \leq \sum_{k=K+4}^{\infty} \Delta_k < 6\epsilon\right\}.$$ (7.3)

Upon observing that for $\delta_1, \delta_2 \in (0, \epsilon)$ we have by use of Markov’s inequality that

$$\mathbb{P}(C \mid (L_{K+3}, R_{K+3}) = (t - \delta_1, t + \delta_2))$$

$$= \mathbb{P}\left((\delta_1 + \delta_2)J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) < 6\epsilon\right) \geq \mathbb{P}\left(J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) < 3\right)$$

$$\geq 1 - \frac{1}{3} \mathbb{E}J\left(\frac{\delta_1}{\delta_1 + \delta_2}\right) \geq 1 - \frac{1}{3} \left[1 + 2H\left(\frac{1}{2}\right)\right] > 0.2 > 0.$$ (7.4)

We then see that $\mathbb{P}(A \cap B \cap C) > 0$ for all sufficiently small $\epsilon$. But if the event $A \cap B \cap C$ is realized, then

$$J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + b + t = x - \left(\frac{K + 1}{2}\right)\epsilon$$
and

\[ J(t) < \sum_{k=1}^{K} \left[ 1 - (k - 1)\epsilon \right] + (b + \epsilon) + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 10\epsilon. \]

We conclude that \( x \) is in the support of \( F_t \).

(ii) \( b = t \). Let

\[ B := \{ t < R_{K+2} < R_{K+1} < t + \epsilon \} \bigcap \{ t - \epsilon < L_{K+3} < t \} \]

and define \( C \) by (7.2). We then see that \( \mathbb{P}(A \cap B \cap C) > 0 \) for all sufficiently small \( \epsilon \). But if the event \( A \cap B \cap C \) is realized, then

\[ J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t + t = x - \left( \frac{K + 1}{2} \right) \epsilon \]

and

\[ J(t) < \sum_{k=1}^{K} \left[ 1 - (k - 1)\epsilon \right] + (b + \epsilon) + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 10\epsilon. \]

We conclude that \( x \) is in the support of \( F_t \).

(iii) \( 0 < b < t \). Let

\[ B := \{ t < R_{K+1} < t + \epsilon \} \bigcap \{ t - b - \epsilon < L_{K+2} < t - b \} \bigcap \{ t - \epsilon < L_{K+3} < t \} \]

and define \( C \) by (7.2). We then see that \( \mathbb{P}(A \cap B \cap C) > 0 \) for all sufficiently small \( \epsilon \). But if the event \( A \cap B \cap C \) is realized, then

\[ J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t + b = x - \left( \frac{K + 1}{2} \right) \epsilon \]

and

\[ J(t) < \sum_{k=1}^{K} \left[ 1 - (k - 1)\epsilon \right] + (t + \epsilon) + (b + 2\epsilon) + 2\epsilon + 6\epsilon \leq x + 11\epsilon. \]

We conclude that \( x \) is in the support of \( F_t \).

(iv) \( b = 0 \). Let

\[ B := \{ t < R_{K+1} < t + \epsilon \} \bigcap \{ t - \epsilon < L_{K+2} < t \} \]

and define \( C \) by (7.2), but with \( K + 4 \) there changed to \( K + 3 \). We then see that \( \mathbb{P}(A \cap B \cap C) > 0 \) for all sufficiently small \( \epsilon \). But if the event \( A \cap B \cap C \) is realized, then

\[ J(t) > \sum_{k=1}^{K} (1 - k\epsilon) + t = x - \left( \frac{K + 1}{2} \right) \epsilon \]
and
\[ J(t) < \sum_{k=1}^{K} [1 - (k - 1)\epsilon] + (t + \epsilon) + 2\epsilon + 6\epsilon \leq x + 9\epsilon. \]

We conclude that \( x \) is in the support of \( F_t \).

We next use (3.6) together with Lemma 7.2 to establish Theorem 7.1 in a special case.

**Lemma 7.3.** For any \( 0 < t < 1 \), the continuous density \( f_t \) satisfies
\[ f_t(x) > 0 \quad \text{for all} \quad x > 2 \min\{t, 1 - t\}. \]

**Proof.** We may fix \( t \leq 1/2 \) and \( x > 2t \) and prove \( f_t(x) > 0 \). To do this, we first note from (3.6) that
\[ f_t(x) \geq c_t(x) = \mathbb{P}(L_2(t) = 0, J(t) \in dx) / dx \]
\[ = \int_{r \in (t, 1)} \int (1 - l)(r - rz < 1) (x - rz)^{-1} \mathbb{P}(Z(t/r) \in dz) \, dr \]
\[ \geq \int_{r \in (t, 1)} \int_{(x-1)/r}^{(x/r)-1} (x/r - 1) \mathbb{P}(Z(t/r) \in dz) \, dr \]
\[ \geq \int_{r \in (t, 1)} \mathbb{P}\left( x/r - 1 < Z(t/r) < x/r - 1 \right) \, dr. \]

According to Lemma 7.2, for the integrand in this last integral to be positive, it is necessary and sufficient that \( (x/r - 1)(x/r - 1)/r < 1 \) (equivalently, \( r < 1 \)) and
\[ \frac{x}{r} - 1 > 1 + \min\{\frac{t}{r}, 1 - \frac{t}{r}\} \]
for which it is sufficient that \( r < (x + t)/3 \). Thus
\[ f_t(x) \geq \int_{r \in (t, \min\{(x+t)/3, 1\})} \mathbb{P}\left( x/r - 1 < Z(t/r) < x/r - 1 \right) \, dr > 0 \]
because (recalling \( x > 2t \)) the integrand here is positive over the nondegenerate interval of integration.

Finally, we use a different contribution to \( f_t(x) \) together with Lemma 7.3 to establish Theorem 7.1.

**Proof of Theorem 7.1.** We may fix \( t \geq 1/2 \) and \( x > 1 - t \) and prove \( f_t(x) > 0 \). To do this, we first note that
\[ f_t(x) \geq \int_{l \in (0, t)} \int_{r \in (t, 1)} \mathbb{P}(L_1(t) = L_2(t) \in dl, R_2(t) \in dr) \]
\[ \left[ \mathbb{P}\left( (r - l)J\left( \frac{t - l}{r - l}\right) \in dx - \left[ (1 - l) + (r - l) \right] \right) / dx \right] \]
\[ = \int_{l \in (0, t)} \int_{r \in (t, 1)} (1 - l)^{-1}(r - l)^{-1} f_t\left( \frac{x - (1 + r - 2l)}{r - l} \right) \, dr \, dl. \]
According to Lemma 7.3, for the integrand in this double integral to be positive, it is sufficient that
\[
\frac{x - (1 + r - 2l)}{r - l} > 2 \min \left\{ \frac{t - l}{r - l}, \frac{r - t}{r - l} \right\},
\]
or, equivalently,
\[
x > \min \{1 + 2t + r - 4l, 1 - 2t + 3r - 2l\}.
\]
This strict inequality is true (because \(x > 1 - t\) when \(l = t\) and \(r = t\) and so, for sufficiently small \(\epsilon > 0\) is true for \(l \in (t - \epsilon, t)\) and \(r \in (t, t + \epsilon)\). Thus
\[
f_t(x) \geq \int_{t \in (t - \epsilon, t)} \int_{r \in (t, t + \epsilon)} (1 - l)^{-1} (r - l)^{-1} f_{l \rightarrow l} \left( \frac{x - (1 + r - 2l)}{r - l} \right) \, dr \, dl > 0
\]
because the integrand here is positive over the fully two-dimensional rectangular region of integration. □

7.2. Lipschitz continuity. We now prove that, for each \(0 < t < 1\), the density function \(f_t\) is Lipschitz continuous, which is a result stronger than Theorem 4.4. 

**Theorem 7.4.** For each \(0 < t < 1\), the density function \(f_t\) is Lipschitz continuous.

That is, there exists a constant \(\Lambda_t \in (0, \infty)\) such that for any \(x, z \in \mathbb{R}\), we have \(|f_t(z) - f_t(x)| \leq \Lambda_t |z - x|\). The proof of Theorem 7.4 will reveal that one can take \(\Lambda_t = \Lambda [t^{-1} \ln t][(1 - t)^{-1} \ln(1 - t)]\) for some constant \(\Lambda < \infty\). Thus the densities \(f_t\) are in fact uniformly Lipschitz continuous for \(t\) in any compact subinterval of \((0, 1)\).

We break the proof of Theorem 7.4 into two lemmas. Lemma 7.5 deals with the contribution to \(f_t\) from the disjoint-union event \(\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}\) while Lemma 7.6 deals with the contribution from the event \(\{0 < L_3(t) < t < R_3(t) < 1\}\).

**Lemma 7.5.** For each \(0 < t < 1\), the contribution to \(f_t\) from the event \(\{0 = L_3(t) < t < R_3(t) < 1\} \cup \{0 < L_3(t) < t < R_3(t) = 1\}\) is Lipschitz continuous.

**Proof.** Fix \(0 < t < 1\). By symmetry, we need only consider the contribution to \(f_t(x)\) from the event \(\{0 = L_3(t) < t < R_3(t) < 1\}\). Recall that this contribution is
\[
c_0(x) := \frac{1}{2} \int_{r,y} (\ln r)^2 f_{0,r}(x - y) \, \mathbb{P}(Y \in dy \mid L_3(t) = 0, R_3(t) = r) \, dr,
\]
and that the conditional probability in the integrand can be written as
\[
\mathbb{P}(Y \in dy \mid L_3(t) = 0, R_3(t) = r) = \frac{1}{r} f_r \left( \frac{y}{r} - 1 \right) \, dy.
\]
Let \(z, x \in \mathbb{R}\) with \(z > x\) and fixed \(r \in (t, 1)\). Writing
\[
d_r(x, z, y) := \frac{1}{2} (\ln r)^2 \left| f_{0,r}(z - y) - f_{0,r}(x - y) \right|,
\]
we are interested in bounding the absolute difference
\[ |c_0(z) - c_0(x)| \leq \int_{r,y} |d_r(x, z, y)| \frac{1}{r} f_{\frac{y}{r}} \left( \frac{y}{r} - 1 \right) dy. \]

**Case 1.** \( z - x \leq 1 - r \). We bound \( d_r(x, z, y) \) for \( y \) in each of the seven subintervals of the real line determined by the six partition points
\[ x - 2 < z - 2 \leq x - (1 + r) < z - (1 + r) \leq x - 2r < z - 2r, \]
and then the contribution to our bound on \( |c_0(z) - c_0(x)| \) from all \( y \) in that subinterval (and all \( r \) satisfying the restriction of Case 1). For the two subcases \( y \leq x - 2 \) and \( y > z - 2r \), we have \( d_r(x, z, y) = 0 \). We bound the five nontrivial subcases as follows.

**Subcase 1(a).** \( x - 2 < y \leq z - 2 \). We have
\[ |d_r(x, z, y)| = \left| \frac{1}{x - y} \right| \ln \left( \frac{1}{x - y - 1} \right) \leq \frac{1}{1 + r} \ln \frac{1}{r}, \]
and the contribution to \( |c_0(z) - c_0(x)| \) is bounded by
\[ \int_{r=t}^{1} \frac{1}{r(1 + r)} \left( \ln \frac{1}{r} \right) \int_{y=x-2}^{z-2} f_{\frac{y}{r}} \left( \frac{y}{r} - 1 \right) dy dr \leq 10(z - x) \frac{1 - t}{t(1 + t)} \ln \frac{1}{t} \]
since \( f_{\frac{y}{r}} \) is bounded by 10.

**Subcase 1(b).** \( z - 2 < y \leq x - (1 + r) \). We have
\begin{align*}
  d_r(x, z, y) &= \frac{1}{z - y} \ln \left( \frac{1}{z - y - 1} \right) - \frac{1}{x - y} \ln \left( \frac{1}{x - y - 1} \right) \\
  &= \frac{1}{z - y} \left[ \ln \left( \frac{1}{z - y - 1} \right) - \ln \left( \frac{1}{x - y - 1} \right) \right] \\
  &\quad + \left( \frac{1}{z - y} - \frac{1}{x - y} \right) \ln \left( \frac{1}{x - y - 1} \right).
\end{align*}

Observe that \( z - y > x - y > 1 + r \) and that the function \( \ln[1/(x - 1)] \) is differentiable for \( x > 1 \). We then use the mean value theorem to obtain
\[ |d_r(x, z, y)| \leq \frac{1}{1 + r} \left| \ln \left( \frac{1}{z - y - 1} \right) - \ln \left( \frac{1}{x - y - 1} \right) \right| + \frac{(z - x)}{(1 + r)^2} \ln \frac{1}{r} \]
\[ \leq (z - x) \left[ \frac{1}{r(1 + r)} + \frac{1}{(1 + r)^2} \ln \frac{1}{r} \right]. \]
The contribution to \( |c_0(z) - c_0(x)| \) is then bounded by
\[ (z - x) \int_{r=t}^{1} \left[ \frac{1}{r(1 + r)} + \frac{1}{(1 + r)^2} \ln \frac{1}{r} \right] dr \leq (z - x) \frac{1 - t}{1 + t} \left( \frac{1}{t} + \frac{1}{1 + t} \ln \frac{1}{t} \right). \]

**Subcase 1(c).** \( x - (1 + r) < y \leq z - (1 + r) \). We have
\[ d_r(x, z, y) = \frac{1}{z - y} \ln \left( \frac{1}{z - y - 1} \right) - \frac{1}{x - y} \ln \left( \frac{x - y - r}{r} \right) \]
\[ \leq (z - x) \left[ \frac{1}{r(1 + r)} + \frac{1}{(1 + r)^2} \ln \frac{1}{r} \right]. \]
Using the inequality \( z - y \geq 1 + r \) and \( x - y > 2r \), we have

\[
|d_r(x, z, y)| = \frac{1}{1 + r} \left| \ln \left( \frac{1}{z - y - 1} \right) - \ln \left( \frac{x - y - r}{r} \right) \right|
+ \frac{z - x}{2r(1 + r)} \ln \frac{1}{r}.
\]

We can bound the absolute-value term here by

\[
\left| \ln \frac{1}{z - y - 1} - \ln \frac{1}{(1 + r) - 1} \right|
+ \left| \ln \frac{(1 + r) - r}{r} - \ln \frac{x - y - r}{r} \right|
\leq \frac{1}{r} [z - y - (1 + r)] + \frac{1}{r} [(1 + r) - (x - y)] = (z - x) \frac{1}{r},
\]

where the above inequality comes from two applications of the mean value theorem. The contribution to \(|c_0(z) - c_0(x)|\) is then bounded by

\[
(z - x) \int_{r=t}^{1} \left( \frac{1}{r(1 + r)} + \frac{1}{2r(1 + r)} \ln \frac{1}{r} \right) \, dr \leq (z - x) \frac{1 - t}{t(1 + t)} \left( 1 + \frac{1}{2} \ln \frac{1}{t} \right).
\]

**Subcase 1(d).** \( z - (1 + r) < y \leq x - 2r \). We have

\[
d_r(x, z, y) = \frac{1}{z - y} \ln \left( \frac{z - y - r}{r} \right) - \frac{1}{x - y} \ln \left( \frac{x - y - r}{r} \right)
= \frac{1}{z - y} \left[ \ln \left( \frac{z - y - r}{r} \right) - \ln \left( \frac{x - y - r}{r} \right) \right]
+ \left( \frac{1}{z - y} - \frac{1}{x - y} \right) \ln \left( \frac{x - y - r}{r} \right).
\]

Using the inequality \( z - y > x - y \geq 2r \), we obtain

\[
|d_r(x, z, y)| \leq \frac{1}{2r} [\ln(z - y - r) - \ln(x - y - r)] + \frac{z - x}{(2r)^2} \ln \frac{1}{r}
\leq (z - x) \left[ \frac{1}{2r} + \frac{1}{(2r)^2} \ln \frac{1}{r} \right]
\]

by the differentiability of \( \ln(x - r) \) for \( x > r \) and the mean value theorem. The contribution to \(|c_0(z) - c_0(x)|\) is then bounded by

\[
(z - x) \int_{r=t}^{1} \left( \frac{1}{2r} + \frac{1}{(2r)^2} \ln \frac{1}{r} \right) \, dr = (z - x) \frac{(1 - t) + \ln(1/t)}{4t}.
\]

**Subcase 1(e).** \( x - 2r < y \leq z - 2r \). Using the inequality \( 2r \leq z - y < 1 + r \), we have

\[
|d_r(x, z, y)| = \frac{1}{z - y} \ln \left( \frac{z - y - r}{r} \right) \leq \frac{1}{2r} \ln \frac{1}{r},
\]
and the contribution to $|c_0(z) - c_0(x)|$ is then bounded by
\[ \int_{r=t}^1 \frac{1}{2r^2} \left( \ln \frac{1}{r} \right) \int_{y=x-2r}^{z-2r} f_{l,r} \left( \frac{y}{r} - 1 \right) \, dy \, dr \leq 10(z - x) \frac{1-t}{2t^2} \ln \frac{1}{t}. \]

This completes the proof for Case 1.

**Case 2.** $z - x > 1 - r$. We directly bound
\[ |d_r(x, z, y)| \leq \frac{1}{2} (\ln r)^2 [f_{0,r}(z - y) + f_{0,r}(x - y)]. \]

If $z - x \leq 1 - t$, use the bound in Remark 4.3; we can then bound the contribution to $|c_0(z) - c_0(x)|$ by
\[ \int_{r=1-(z-x)}^1 \frac{2\beta}{r} \, dr \leq (z - x) \frac{2\beta}{t}. \]

On the other hand, if $z - x > 1 - t$, then we can bound the contribution to $|c_0(z) - c_0(x)|$ by
\[ \frac{z - x}{1-t} \int_{r=t}^1 \frac{2\beta}{r} \, dr \leq (z - x) \frac{2\beta}{t}. \]

This completes the proof for Case 2. We conclude that $c_0$ is a Lipschitz continuous function; note that the Lipschitz constant we have obtained depends on $t$. \(\square\)

**Lemma 7.6.** For each $0 < t < 1$, the contribution to $f_t$ from the event \( \{0 < L_3(t) < t < R_3(t) < 1\} \) is Lipschitz continuous.

**Proof.** Fix $0 < t < 1$. According to (2.14) and (2.16), the contribution from the event in question to $f_t(x)$ is $\sum_{i=1}^6 c^{(i)}(x)$, where we define
\[ c^{(i)}(x) := \int_{l,r,y} f^{(i)}_{l,r} (x - y) \, \mathbb{P}(Y \in dy \mid (L_3(t), R_3(t)) = (l,r)) \, dl \, dr. \]

We show here that $c^{(3)}$ is Lipschitz continuous, and the claims that the other contributions $c^{(i)}$ are Lipschitz continuous are proved similarly.

Let $x, z \in \mathbb{R}$ with $z > x$ and consider $(l, r)$ satisfying $0 < l < t < r < 1$. Define
\[ d_{l,r}(x, z, y) := f^{(3)}_{l,r} (z - y) - f^{(3)}_{l,r} (x - y) \]
and reformulate
\[ f^{(3)}_{l,r} (x) = 1(1 + r - 2l \leq x < 1 + r) \frac{1}{x} \left( \frac{1}{x + 1 - r} + \frac{1}{x + r - 1} \right) \]
from the expression for $f^{(3)}_{l,r}(x)$ found in Section 2. We are interested in bounding the quantity
\[ |c^{(3)}(z) - c^{(3)}(x)| \leq \int_{l,r,y} |d_{l,r}(x, z, y)| \, \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l,r)) \, dl \, dr, \]

(7.4)
where the conditional probability can also be written in density terms as
\[ P(Y \in dy \mid (L_3, R_3) = (l, r)) = \frac{1}{r - l} f_{l, r} \left( \frac{y}{r - l} - 1 \right) dy. \]

Just as we did for Lemma 7.5, we break the proof into consideration of two cases.

Case 1. \( z - x < 2l \). As in the proof for Case 1 of Lemma 7.5, we bound \( d_{l,r}(x, z, y) \) for \( y \) in each of the five subintervals of the real line determined by the four partition points
\[
x - (1 + r) < z - (1 + r) < x - (1 + r - 2l) < z - (1 + r - 2l).
\]

For the two subcases \( y \leq x - (1 + r) \) and \( y > z - (1 + r - 2l) \), we have \( d_{r}(x, z, y) = 0 \). We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

Subcase 1(a). \( z - (1 + r) < y \leq x - (1 + r - 2l) \). We have
\[
d_{l,r}(x, z, y) = \frac{1}{z - y} \left( \frac{1}{z - y + 1 - r} - \frac{1}{z - y + r - 1} \right) - \frac{1}{x - y} \left( \frac{1}{x - y + 1 - r} - \frac{1}{x - y + r - 1} \right) + \frac{1}{z - y} \left( \frac{1}{z - y + 1 - r} - \frac{1}{z - y + r - 1} \right) \frac{1}{x - y + 1 - r} - \frac{1}{z - y} \left( \frac{1}{z - y + r - 1} - \frac{1}{x - y + r - 1} \right) - \frac{1}{z - y} \left( \frac{1}{z - y + r - 1} - \frac{1}{x - y + r - 1} \right) \frac{1}{x - y + r - 1}.
\]

Using the inequality \( z - y > x - y \geq 1 + r - 2l \), we obtain
\[
|d_{l,r}(x, z, y)| \leq \frac{1}{1 + r - 2l} \left( \frac{z - x}{(2 - 2l)^2} + \frac{z - x}{(1 + r - 2l)^2} \right) + \frac{1}{1 + r - 2l} \left( \frac{z - x}{(z - y + r - 1)(x - y + r - 1)} \right) + \frac{1}{1 + r - 2l} \left( \frac{z - x}{2(r - l)} \right).
\]

Except for the third term, it is easy to see (by direct computation) that the corresponding contribution to the bound (7.4) on \( |c^{(3)}(z) - c^{(3)}(x)| \) is bounded by a constant (depending on \( t \)) times \( z - x \). So we now focus on bounding the contribution from the third term. Note that since \( 1 + r - 2l > 1 - t > 0 \), we need only bound
\[
\int_{l, r, y} \left( \frac{1}{(z - y + r - 1)(x - y + r - 1)} \right) \P(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr,
\]

by a constant (which is allowed to depend on \( t \), but our constant will not).
We first focus on the integral in (7.5) with respect to \( y \) and write it, using a change of variables, as
\[
\int_{y \in I} d^{*}_{l,r}(x, z, y) f_{\frac{t-l}{r-l}}(y) \, dy, \tag{7.6}
\]
with
\[
d^{*}_{l,r}(x, z, y) = \frac{1}{[z - (r - l)(y + 1) + r - 1][x - (r - l)(y + 1) + r - 1]}
\]
and \( I := \{ y : y \leq \frac{x - (1 + r - 2l)}{r - l} - 1 \} \). Because the support of the density \( f_{\frac{t-l}{r-l}} \) is contained in the nonnegative real line, the integral (7.6) vanishes unless the right endpoint of the interval \( I \) is positive, which is true if and only if
\[
r < \frac{x - 1 + 3l}{2}.
\]
So we see that the integral of (7.6) over \( r \in (t, 1) \) vanishes unless this upper bound on \( r \) is larger than \( t \), which is true if and only if
\[
l > \frac{1 - x + 2t}{3}. \tag{7.7}
\]
But then the integral of (7.6) over \( \{ (l, r) : 0 < l < t < r < 1 \} \) vanishes unless this lower bound on \( l \) is smaller than \( t \), which is true if and only if \( x > 1 - t \); we conclude that for \( x \leq 1 - t \), that integral vanishes.

So we may now suppose \( x > 1 - t \), and we have seen that the integral of (7.6) over \( \{ (l, r) : 0 < l < t < r < 1 \} \) is bounded above by its integral over the region
\[
R := \left\{ (l, r) : \frac{1 - x + 2t}{3} \vee 0 < l < t < r < 1 \land \frac{x - 1 + 3l}{2} \right\}.
\]
Observe that on \( R \) we have
\[
\frac{x - (1 + r - 2l)}{r - l} - 1 = \frac{x - 1 + l}{r - l} - 2 > \frac{2x + t - 1}{3} - 2 > \frac{1x + t - 1}{2} - 2.
\]
Define
\[
B := \left\{ (l, r) : \frac{x + t - 1}{2(r - l)} > 2 \right\}.
\]
We now split our discussion of the contribution to the integral of (7.6) over \( (l, r) \in R \) into two terms, corresponding to (i) \( R \cap B^c \) and (ii) \( R \cap B \).

Term (i). \( R \cap B^c \). Using (7.8), we can bound (7.6) by extending the range of integration from \( I \) to
\[
I^* := \left\{ y : \frac{x + t - 1}{2(r - l)} - 2 < y \leq \frac{x - (1 + r - 2l)}{r - l} - 1 \right\}.
\]
Making use of the inequality (6.4), the integral (7.6) is bounded, for any \( \theta > 0 \), by
\[
\int_{y \in I^*} \frac{1}{4(r-l)^2} C_\theta e^{-\theta y} \, dy \leq \frac{C_\theta}{4\theta(r-l)^2} \exp \left[ -\frac{x + t - 1}{2(r-l)} \theta + 2\theta \right].
\]
The integral over \((l, r) \in R \cap B^c\) of (7.6) is therefore bounded by
\[
\frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^t \int_{r=t}^{(x-1+3l)/2} \frac{1}{(r-l)^2} \exp \left[ -\frac{x + t - 1}{2(r-l)} \theta \right] \, dr \, dl
\]
\[
= \frac{C_\theta}{4\theta} e^{2\theta} \int_{l=(1-x+2t)/3}^t \int_{s=t}^{(x-1+l)/3} \frac{1}{s^2} \exp \left( -\frac{x + t - 1}{2s} \theta s^{-1} \right) \, ds \, dl
\]
\[
\leq \frac{C_\theta}{2\theta^2} e^{2\theta} \frac{1}{x + t - 1} e^{-\theta} \left( t - \frac{1 - x + 2t}{3} \right) = \frac{C_\theta}{60^2} e^{\theta} < \infty. \tag{7.9}
\]

**Term (ii).** \( R \cap B \). We can bound (7.6) by the sum of the integrals of the same integrand over the intervals \( I^* \) and
\[
I' := \left\{ y : 0 < y \leq \frac{x + t - 1}{2(r-l)} - 2 \right\}.
\]
The bound for the integral over \( I^* \) is the same as the bound for the \( R \cap B^c \) term. To bound the integral over \( I' \), we first observe that
\[
d_{l,r}^*(x, z, y) \leq \frac{1}{\frac{1}{2}(x - t - 1) + 2r - l} \leq \frac{4}{(x + t - 1)^2},
\]
where the last inequality holds because \( l < t < r \). The contribution to (7.5) can be bounded by integrating \( 4/(x+t-1)^2 \) with respect to \((l, r) \in R \cap B \). We then extend this region of integration to \( R \), and thus bound the contribution by
\[
\frac{4}{(x + t - 1)^2} \int_{l=(x+1+r)/3}^t \left( \frac{x - 1 + 3l}{2} - t \right) \, dl \leq \frac{2}{(x + t - 1)} \left( t - \frac{2t + 1 - x}{3} \right)
\]
\[
= 2/3.
\]
This completes the proof for Subcase 1(a).

**Subcase 1(b).** \( x - (1 + r - 2l) < y \leq z - (1 + r - 2l) \). First note that in this subcase we have \( f^{(3)}(x - y) = 0 \). We proceed in similar fashion as for Subcase 1(a), this time setting
\[
I := \left\{ y : \frac{x - 1 + l}{r - l} - 2 < y \leq \frac{z - 1 + l}{r - l} - 2 \right\}.
\]
Again using a linear change of variables, the integral (with respect to \( y \) only, in this subcase) appearing on the right in (7.4) in this subcase can be written
as
\[ \int_{y \in I} d_{l,r}^*(z, y) f_{l,r}(y) \, dy \quad (7.10) \]
where now
\[ d_{l,r}^*(z, y) = \frac{1}{z - (r - l)(y + 1) + 1 - r} \times \frac{2}{z - (r - l)(y + 1) + r - 1}. \]
Note that, unlike its analogue in Subcase 1(a), here \( d_{l,r}^*(z, y) \) does not possess an explicit factor \( z - x \).

By the same discussion as in Subcase 1(a), we are interested in the integral of (7.10) with respect to \( (l, r) \in R \), where this time \( R := \left\{ (l, r) : 1 - z + 2t/3 \leq l < t < r < 1 \land \frac{z - 1 + 3l}{2} \right\} \).

and we may suppose that \( z > 1 - t \).

Observe that on \( R \) we have
\[ \frac{z - 1 + l}{r - l} - 2 > \frac{2z + t - 1}{3} - 2 > \frac{z + t - 1}{2} - 2. \]
Following a line of attack similar to that for Subcase 1(a), we define
\[ W := \left\{ (l, r) : \frac{x - 1 + l}{r - l} - 2 > \frac{z + t - 1}{2} - 2 \right\} \]
and split our discussion of the integral of (7.10) over \( (l, r) \in R \) into two terms, corresponding to (i) \( R \cap W^c \) and (ii) \( R \cap W \).

**Term (i).** \( R \cap W \). We bound (7.10) by using the inequality (6.4) (for any \( \theta > 0 \)) and obtain
\[ \int_{y \in I} \frac{1}{2 - 2t} \frac{1}{r - l} C_\theta \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l)} - 2 \right) \right] \, dy \leq \frac{1}{2} \frac{1}{1 - t(r - l)^2} C_\theta e^{2\theta} \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l)} \right) \right] (z - x). \]

Integrate this terms with respect to \( (l, r) \in R \cap W \), we get no more than
\[ \frac{1}{2} \frac{(z - x)}{1 - t} C_\theta e^{2\theta} \int_{l=(1-z+2t)/3}^{t} \int_{r=t}^{(z-1+3l)/2} \frac{1}{(r - l)^2} \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l)} \right) \right] \, dr \, dl, \]
which [consult (7.9)] is bounded by \( (z - x) \) times a constant depending only on \( t \) and \( \theta \).

**Term (ii).** \( R \cap W^c \). We partition the interval \( I \) of \( y \)-integration into the two subintervals
\[ I^* := \left\{ y : \frac{z + t - 1}{2(r - l)} - 2 < y \leq \frac{z - 1 + l}{r - l} - 2 \right\} \]
and
\[ I' := \left\{ y : \frac{x - 1 + l}{r - l} - 2 < y \leq \frac{z + t - 1}{2(r - l)} - 2 \right\}. \]
Observe that the length of each of the intervals $I^*$ and $I'$ is no more than the length of $I$, which is $(z - x)/(r - l)$. We can bound the integral over $y \in I^*$ and $(l, r) \in R \cap W^c$ just as we did for Term (i). For the integral over $y \in I'$ and $(l, r) \in R \cap W^c$, observe the following inequality:

$$d_{l,r}(z, y) \leq \frac{1}{2 - 2t} \frac{2}{(z + t - 1) + 2r - l - t}.$$  

Using the constant bound in Theorem 3.1, the integral of $d_{l,r}(z, y) f_{z, y}^{-1}(y)$ with respect to $y \in I'$ and $(l, r) \in R \cap W^c$ is bounded above by

$$10 \frac{(z - x)}{1 - t} \int_{l = (1 - z + 2t) / 3}^{t} \left( \frac{1}{r - l} - \frac{2}{2r - l - t + \frac{z + t - 1}{2}} \right) \frac{1}{l - t + \frac{z + t - 1}{2}} \, dl \, dr.$$  

(7.11)

Write the integrand here in the form

$$\frac{1}{r - l} \frac{1}{2t - 2r - l - t} = \left( \frac{1}{r - l} - \frac{2}{2r - l - t + \frac{z + t - 1}{2}} \right) \frac{1}{l - t + \frac{z + t - 1}{2}},$$

and observe that $l - t + \frac{z + t - 1}{2} > \frac{z + t - 1}{6} > 0$. Hence we can bound (7.11) by

$$10 \frac{(z - x)}{1 - t} \int_{l = (1 - z + 2t) / 3}^{t} \frac{1}{l - t + \frac{z + t - 1}{2}} \left[ \ln \frac{z - 1 + l}{2} - \ln \left( t - l \right) \right] \, dl$$

$$\leq 10 \frac{(z - x)}{1 - t} \frac{6}{z + t - 1} \left[ \frac{z + t - 1}{3} \ln \frac{z + t - 1}{2} - \int_{l = 1 - \frac{z + t - 1}{2}}^{t} \ln \left( t - l \right) \, dl \right]$$

$$= 20 \frac{(z - x)}{1 - t} \left( \ln \frac{z + t - 1}{3} - \ln \left( \frac{z + t - 1}{2} \right) + 1 \right)$$

$$= 20 \left( 1 + \ln \frac{3}{2} \right) \frac{(z - x)}{1 - t}.$$  

This completes the proof for Subcase 1(b).

**Subcase 1(c).** $x - (1 + r) < y \leq z - (1 + r)$. In this case, the contribution from $f^{(3)}(z - y)$ vanishes. Without loss of generality we may suppose $z - x < t$, otherwise we can insert a factor $(z - x)/t$ in our upper bound, and the desired upper bound follows from the fact that the densities $f_r$ are all bounded by 10.

Observe that the integrand $|d_{l,r}(x, z, y)|$ in the bound (7.4) is

$$\frac{1}{x - y + 1 - r} \leq \frac{2}{x - z + 2} \leq \frac{2}{2 - r} \leq \frac{2}{t(2 - t)}.$$  

Integrate this constant bound directly with respect to

$$P(Y \in dY | (L_3, R_3) = (l, r)) \, dl \, dr$$

on the region $x - (1 + r) < y \leq z - (1 + r)$ and $0 < l < t < r < 1$ and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by $(z - x)$ times a constant that depends on $t$. This completes the proof for Subcase 1(c) and also for Case 1.
Case 2. \( z - x \geq 2l \). In this case we simply use
\[
|d_{l,r}(x, z, y)| \leq f^{(3)}(z - y) + f^{(3)}(x - y),
\]
and show that each of the two terms on the right contributes at most a constant (depending on \( t \)) times \((z - x)\) to the bound in (7.4). Accordingly, let \( w \) be either \( x \) or \( z \). We are interested in bounding
\[
\int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{w-y+1-r} \frac{2}{w-y+r-1} \mu(dy, dr, dl) \quad (7.12)
\]
with \( \mu(dy, dr, dl) := P(Y \in dy \mid (L_3, R_3) = (l, r)) \, dr \, dl \). We bound the integrand as follows:
\[
\frac{1}{w-y+1-r} \frac{2}{w-y+r-1} \leq \frac{1}{2-2l} \frac{2}{2r-2l} \leq \frac{1}{2} \frac{1}{1-t} \frac{1}{r-l}.
\]
We first suppose \( z - x < t \) and bound (7.12) by
\[
\frac{1}{2} \frac{1}{1-t} \int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{r-l} \, dr \, dl \leq \frac{1}{2} \frac{1}{1-t} \int_{l=0}^{t} [-\ln(t-l)] \, dl \leq \frac{1}{2} \frac{1}{1-t} \left[-\ln \left(t - \frac{z-x}{2} \right) \right] \frac{z-x}{2} \leq (z-x) \frac{\ln(2/t)}{4(1-t)}.
\]
If instead \( z - x \geq t \), we bound (7.12) by
\[
\frac{1}{2} \frac{z-x}{t} \int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{1-l} \frac{1}{r-l} \, dr \, dl \leq (z-x) \frac{\pi^2}{12t}.
\]
This completes the proof for Case 2 and thus the proof of Lipschitz continuity of \( c^{(3)} \).

We immediately get the following corollary from the proof of Theorem 7.4.

**Corollary 7.7.** For any \( 0 < \eta < 1/2 \), the uniform continuous family \( \{f_t : t \in [\eta, 1-\eta]\} \) is a uniformly equicontinuous family.

**Proof.** We observe from the proof of Theorem 7.4 that for any \( 0 < \eta < 1/2 \), the Lipschitz constants \( L_t \) in Theorem 7.4 are bounded for \( t \in [\eta, 1-\eta] \) by some universal constant \( C < \infty \). The result follows.

### 7.3. Joint continuity.

As noted in the proof of Lemma 5.2, we reference Grübel and Rösler [15] to conclude that for each \( t \in [0, 1) \), the distribution functions \( F_u \) converge weakly to \( F_t \) as \( u \downarrow t \). It follows by symmetry that the convergence also holds for each \( t \in (0, 1] \) as \( u \uparrow t \). We now deduce the convergence from \( f_u \) to \( f_t \) for each \( t \in (0, 1) \) as \( u \to t \), according to the following lemma.

**Lemma 7.8.** For each \( 0 < t < 1 \) we have \( f_u \to f_t \) uniformly as \( u \to t \).
Proof. We fix $0 < t \leq 1/2$ and choose $0 < \eta < t$. By the weak convergence of $F_u$ to $F_t$ as $u \to t$, the uniform boundedness of the density functions (Theorem 3.1), the fact that $f_l(x) \to 0$ as $x \to \pm \infty$, and the uniform equicontinuity of the family $\{f_u : u \in [\eta, 1-\eta]\}$ (Corollary 7.7), we conclude from Boos [3, Lemma 1] (a converse to Scheffé's theorem) that $f_u \to f_l$ uniformly as $u \to t$.

Remark 7.9. The uniform equicontinuity in Corollary 7.7 does not hold for the family $\{f_t : t \in (0, 1)\}$. Here is a proof. For the sake of contradiction, suppose to the contrary. We symmetrize $f_l(x)$ at $x = 0$ for every $0 \leq t \leq 1$ to create another family of continuous densities $g_t$; that is, consider $g_t(x) := [f_l(x) + f_l(-x)]/2$. Observe that the supposed uniform equicontinuity of the functions $f_t$ for $t \in (0, 1)$ extends to the functions $g_t$. Now suppose (for each $t \in [0, 1]$) that $W(t)$ is a random variable with density $g_t$. By a simple calculation we have $W(t) \Rightarrow W(0)$, and it follows by Boos [3, Lemma 1] that $g_t(x) \to g_0(x)$ uniformly in $x$. This contradicts to the fact that $g_t(0) = 0$ for all $t \in (0, 1)$ but $g_0(0) = e^{-\gamma}$.

Remark 7.10. Since $(F_l)_{t \in [0,1]}$ is weakly continuous in $t$ and $F_l$ is atomless for $0 \leq t \leq 1$, it follows from a theorem of Pólya ([4, Exercise 4.3.4]) that $(F_l)_{t \in [0,1]}$ is continuous in the sup-norm metric, i.e., that $(J(t))$ or $(Z(t))$ is continuous in the Kolmogorov–Smirnov metric on distributions.

Corollary 7.11. The density $f_l(x)$ is jointly continuous in $(t, x) \in (0, 1) \times \mathbb{R}$.

Proof. As $(t', x') \to (t, x) \in (0, 1) \times \mathbb{R}$, we have

$$
\limsup |f_{t'}(x') - f_t(x)| \leq \limsup |f_{t'}(x') - f_l(x')| + \limsup |f_l(x') - f_t(x)| \\
\leq \limsup \|f_{t'} - f_l\|_\infty + \delta_t(|x' - x|) \\
= 0
$$

where the sup-norm $\|f_{t'} - f_l\|_\infty$ tends to $0$ as $t' \to t$ by Lemma 7.8 and the modulus of uniform continuity $\delta_t$ of the function $f_t$ tends to $0$ as $x' \to x$ by Theorem 4.4.

Remark 7.12. The positivity of $f_l(x)$ for each $0 < t < 1$ and $x > \min \{t, 1-t\}$ in Theorem 7.1 can be proved alternatively by using the integral equation Proposition 5.7 and the joint continuity result of Corollary 7.11. Here is the proof.

Fix (for now) $t_0, t_1, t_2 \in (0, 1)$ with $t_1 > t_0 > t_2$. We will show that $f_{t_0}(x) > 0$ for all $x > t_0$, using $t_1$ and $t_2$ in auxiliary fashion. Since this is true for arbitrarily chosen $t_0$, invoking symmetry $(f_t \equiv f_{1-t})$ then completes the proof.

We certainly know that $f_{t_0}(y_0) > 0$ for some $y_0 > t_0$; choose and fix such a $y_0$. Use Proposition 5.7 to represent the density $f_{t_1}(x)$. We observe that the integrand of the integral with respect to $l$ is positive at $l = l_1 = (t_1 - t_0)/(1 - t_0)$ and $x = y_1 = (1 - l_1)(y_0 + 1)$. From Corollary 7.11 we
conclude that the integrand is positive in a neighborhood of $t_1$ and thus $f_{t_1}(y_1) > 0$.

Further, use Proposition 5.7 to represent the density $f_{t_0}(x)$. We observe that the integrand of the integral with respect to $r$ is positive at $r = r_2 = \frac{t_0}{t_1}$ and $x = y_2 = r_2(y_1 + 1)$. From $f_{t_1}(y_1) > 0$ and Corollary 7.11 we conclude that $f_{t_0}(y_2) > 0$.

Now letting $y_2 = y_0 + \epsilon_1$, we have

$$
\epsilon_1 = \left( \frac{t_0}{t_1} \frac{1 - t_1}{1 - t_0} - 1 \right) y_0 + \frac{t_0}{t_1} \left( 1 + \frac{1 - t_1}{1 - t_0} \right)
$$

Observe that as $t_1 \downarrow t_0$ we have $\epsilon_1 \to 2$, while as $t_1 \uparrow 1$ we have $\epsilon_1 \downarrow -y_0 + t_0 < 0$. Thus, given $\delta \in (0, 2 - t_0 + y_0)$ it is possible to choose $t_1 \in (t_0, 1)$ such that $\epsilon_1 = -y_0 + t_0 + \delta$, i.e., $y_2 = t_0 + \delta$. We conclude that $f_{t_0}(x)$ is positive for every $x > t_0$, as desired.

8. Left-tail behavior of the density function

We consider the densities $f_t$ with $t \in (0, 1)$; since $f_t \equiv f_{1-t}$ by symmetry, we may without loss of generality suppose $t \in (0, 1/2]$. As previously noted (recall Theorems 7.1 and 4.4), $f_t(x) = 0$ for all $x \leq t$ and $f_t(x) > 0$ for all $x > t$. In this section we consider the left-tail behavior of $f_t$, by which we mean the behavior of $f_t(x)$ as $x \downarrow t$.

As a warm-up, we first show that $f_t$ has a positive right-hand derivative at $t$ that is large when $t$ is small.

Lemma 8.1.

(a) Fix $t \in (0, 1/2)$. Then the density function $f_t$ has right-hand derivative $f'_t(t)$ at $t$ equal to $c_1/t$, where

$$
c_1 := \int_0^1 E[2 - w + J(w)]^{-2} \, dw \in (0.0879, 0.3750).
$$

(b) Fix $t = 1/2$. Then the density function $f_t$ has right-hand derivative $f'_t(t)$ at $t$ equal to $2c_1/t = 4c_1$.

Proof. (a) We begin with two key observations. First, if $L_1(t) > 0$, then $J(t) > 1 - t$. Second, if $1 > R_1(t) > R_2(t)$, then $J(t) > 2t$. It follows that if $0 < z < \min\{1 - 2t, t\}$, then, with $Y \equiv Y(t)$ as defined at (2.1),

$$
f_t(t + z) \, dz = \mathbb{P}(J(t) - t \in dz) = \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz)
$$

$$
= \int_{y > x > 0; x + y < z, x < 1 - t, y - x < t} \mathbb{P}(R_1(t) - t \in dx, t + x - L_2(t) \in dy, Y(t) \in dz - x - y)
$$

$$
= \int_{y > x > 0; y + x < z, x < 1 - t, y - x < t} dx \frac{dy}{t + x} \mathbb{P} \left( y J \left( \frac{y - x}{y} \right) \in dz - x - y \right)
$$
\[
\int_{y>x>0: x+y<z, x<1-t, y-x<t} dx \frac{dy}{t+x} y^{-1} f_{1-\frac{z}{y}} \left( \frac{z-x-y}{y} \right) dz.
\]

Now make the changes of variables from \(x\) to \(u = x/z\) and from \(y\) to \(v = y/z\). We then find
\[
f_t(t+z) = z \int_{v>u>0: u+v<1} (t+uz)^{-1} v^{-1} f_{1-\frac{u+v}{v}} \left( \frac{1-u-v}{v} \right) du dv
\]
\[
= z \int_{v>u>0: u+v<1} (t+uz)^{-1} v^{-1} f_{1-\frac{u+v}{v}} \left( \frac{1-u-v}{v} \right) du dv,
\]
where the second equality follows because \((1-t)/z > (1-t)/z + (1-2t)/z > 1\) and \(t/z > 1\) by assumption. Thus, as desired,
\[
f_t(t+z) \sim \frac{c_1 z}{t}
\]
as \(z \downarrow 0\) by the dominated convergence theorem, if we can show that
\[
\tilde{c} := \int_{v>u>0: u+v<1} v^{-1} f_{1-\frac{u+v}{v}} \left( \frac{1-u-v}{v} \right) du dv
\]
equals \(c_1\). For that, make another change of variables from \(u\) to \(w = u/v\); then we find
\[
\tilde{c} = \int_0^1 \int_0^{(1+w)^{-1}} f_{1-w} (v^{-1} - (1+w)) dv dw
\]
\[
= \int_0^1 \int_0^{(2-w)^{-1}} f_w (v^{-1} + w - 2) dv dw.
\]
Make one last change of variables, from \(v\) to \(s = v^{-1} + w - 2\), to conclude
\[
\tilde{c} = \int_0^1 \int_0^{\infty} (2-w+s)^{-2} f_w(s) ds dw = c_1,
\]
as claimed.

To obtain the claimed upper bound on \(c_1\), we note, using the facts that \(J(w)\) and \(J(1-w)\) have the same distribution and that \(J(w) > w\) for \(w \in (0, 1/2)\), that
\[
c_1 = \int_0^{1/2} \mathbb{E}[2-w+J(w)]^{-2} dw + \int_{1/2}^1 \mathbb{E}[2-w+J(w)]^{-2} dw
\]
\[
= \int_0^{1/2} \mathbb{E}[2-w+J(w)]^{-2} dw + \int_{1/2}^{1/2} \mathbb{E}[1+w+J(w)]^{-2} dw
\]
\[
< \int_0^{1/2} \frac{1}{w} dw + \int_0^{1/2} (1+2w)^{-2} dw = \frac{1}{2} + \frac{1}{3} = \frac{3}{8} = 0.3750.
\]

To obtain the claimed lower bound on \(c_1\), we combine Jensen’s inequality with the known fact [cf. (1.2)] that \(\mathbb{E}J(w) = 1 + 2H(w)\) with \(H(w) = \)
\( -w \ln w - (1 - w) \ln(1 - w): \)
\[
c_1 = \int_0^1 \mathbb{E}[2 - w + J(w)]^{-2} \, dw \\
\geq \int_0^1 (\mathbb{E}[2 - w + J(w)])^{-2} \, dw = \int_0^1 (3 - w + 2H(w))^{-2} \, dw > 0.0879.
\]

(b) By an argument similar to that at the start of the proof of (a), if \(0 < z < 1/2\), then, using symmetry at the third equality,
\[
f_t(t + z) \, dz = \mathbb{P}(J(t) - t \in dz) \\
= \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz) + \mathbb{P}(L_1(t) > 0, R_2(t) < 1, J(t) - t \in dz) \\
= 2 \mathbb{P}(R_1(t) < 1, L_2(t) > 0, J(t) - t \in dz) \\
\sim \frac{2c_1 z}{t} = 4c_1 z.
\]

Here the asymptotic equivalence is as \(z \downarrow 0\) and follows by the same argument as used for (a).

We are now prepared for our main result about the left-tail behavior of \(f_t\).

**Theorem 8.2.**

(a) Fix \(t \in (0, 1/2)\). Then \(f_t(t + tz)\) has the uniformly absolutely convergent power series expansion
\[
f_t(t + tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k
\]
for \(z \in [0, \min\{t^{-1} - 2, 1\})\), where for \(k \geq 1\) the coefficients
\[
c_k := \int_0^1 (1 - w)^{k-1} \mathbb{E}[2 - w + J(w)]^{-(k+1)} \, dw,
\]
not depending on \(t\), are strictly positive, have the property that \(2^k c_k\) is strictly decreasing in \(k\), and satisfy
\[
0 < (0.0007)2^{-(k+1)}(k + 1)^{-2} < c_k < 2^{-(k+1)}k^{-1}(1 + 2^{-k}) < 0.375 < \infty.
\]
[In particular, \(2^k c_k\) is both \(O(k^{-1})\) and \(\Omega(k^{-2})\).]

(b) Fix \(t = 1/2\). Then \(f_t(t + tz)\) has the uniformly absolutely convergent power series expansion
\[
f_t(t + tz) = 2 \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k
\]
for \(z \in [0, 1)\).
Proof. (a) As shown in the proof of Lemma 8.1, for \( z \in [0, \min\{t^{-1} - 2, 1\} \) we have

\[
 f_t(t + tz) = z \int_{v > u > 0; u + v < 1} (1 + uz)^{-1} v^{-1} f_{1 - \frac{u}{v}} \left( \frac{1 - u - v}{v} \right) \, du \, dv.
\]  

(8.1)

Note that the expression on the right here doesn’t depend on \( t \). Further, since \( z \leq 1 \) and \( 0 < u < 1/2 \) in the range of integration,

\[
 \frac{1}{2} \int_{v > u > 0; u + v < 1} (1 - uz)^{-1} v^{-1} f_{1 - \frac{u}{v}} \left( \frac{1 - u - v}{v} \right) \, du \, dv
\]

\[
 < \int_{v > u > 0; u + v < 1} v^{-1} f_{1 - \frac{u}{v}} \left( \frac{1 - u - v}{v} \right) \, du \, dv
\]

\[
 = \hat{c} = c_1 < 3/8 < \infty,
\]

with \( \hat{c} \) and \( c_1 \) as in the proof of Lemma 8.1. It follows that \( f_t(t + tz) \) has the uniformly absolutely convergent power series expansion

\[
 f_t(t + tz) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k
\]

for \( z \in [0, \min\{t^{-1} - 2, 1\} \), where for \( k \geq 1 \) we have

\[
 c_k = 2 \times 2^{-k} \int_{v > u > 0; u + v < 1} (2u)^{k-1} v^{-1} f_{1 - \frac{u}{v}} \left( \frac{1 - u - v}{v} \right) \, du \, dv
\]

\[
 = \int_0^1 (1 - w)^{k-1} \mathbb{E}[2 - w + J(w)]^{-k+1} \, dw;
\]

the second equality follows just as for \( c = c_1 \) in the proof of Lemma 8.1. From the first equality it is clear that these coefficients have the property that \( 2^k c_k \) is strictly decreasing in \( k \).

To obtain the claimed upper bound on \( c_k \), proceed just as in the proof of Lemma 8.1 to obtain

\[
 c_k < 2^{-(k+1)} \int_0^{1/2} (1 - w)^{k-1} \, dw + \int_0^{1/2} w^{k-1} (1 + 2w)^{-k-1} \, dw
\]

\[
 = 2^{-(k+1)} k^{-1} (1 - 2^{-k}) + k^{-1} 2^{-(k+1)} k^{-1} (1 + 2^{-k})
\]

The claimed lower bound on \( c_k \) follows from Lemma 8.1 for \( k = 1 \) but for \( k \geq 2 \) requires more work. We begin by establishing a lower bound on \( \mathbb{P}(J(w) \leq 2w) \) for \( w \leq 1/3 \), using what we have already proved:

\[
 \mathbb{P}(J(w) \leq 2w) = \int_0^w f_w(w + x) \, dx = w \int_0^1 f_w(w + wz) \, dz
\]

\[
 \geq w \int_0^1 (c_1 z - c_2 z^2) \, dz = w (\frac{1}{2} c_1 - \frac{1}{3} c_2)
\]

\[
 > [0.04395 - (1/3)(1/8)(1/2)(5/4)]w > 0.0179 w.
\]
Thus $c_k$ is at least $0.0179 \cdot 2^{-(k+1)}$ times the following expression:

$$2^{k+1} \int_0^{1/3} w(1 - w)^{k-1}(2 + w)^{-k-1} \, dw$$

$$\geq \int_0^{1/3} w \exp[-2(k+1)w] \exp[-(k+1)w/2] \, dw$$

$$\geq \int_0^{1/(k+1)} w \exp[-5(k+1)w/2] \, dw$$

$$\geq e^{-5/2} \int_0^{1/(k+1)} w \, dw = 1 - 2e^{-5/2(k+1)^2}.$$

(b) The claim of part (b) is clear from the proof of Lemma 8.1.

Corollary 8.3.

(a) Fix $t \in (0, 1/2)$. Then, for all $x \in (t, \min\{1-t, 2t\})$, the density $f_t(x)$ is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.

(b) Fix $t = 1/2$. Then for all $x \in [1/2, 1)$, the density $f_{1/2}(x)$ is infinitely differentiable, strictly increasing, strictly concave, and strictly log-concave.

Proof. Once again it is clear that we need only prove (a). The result is actually a corollary to (8.1), rather than to Theorem 8.2. It is easy to justify repeated differentiation with respect to $z$ under the double integral of (8.1). In particular, for $z \in (0, \min\{t^{-1} - 2, 1\})$ we have

$$tf'_t(tz) = \int\int_{v > u > 0: u + v < 1} (1 + uz)^{-1} v^{-1} f_{1-\frac{t}{2}} \left(1 - \frac{u - v}{v}\right) \, du \, dv$$

$$- z \int\int_{v > u > 0: u + v < 1} u (1 + uz)^{-2} v^{-1} f_{1-\frac{t}{2}} \left(1 - \frac{u - v}{v}\right) \, du \, dv$$

$$= \int\int_{v > u > 0: u + v < 1} (1 + uz)^{-2} v^{-1} f_{1-\frac{t}{2}} \left(1 - \frac{u - v}{v}\right) \, du \, dv > 0$$

and

$$t^2 f''_t(tz) = -2 \int\int_{v > u > 0: u + v < 1} u (1 + uz)^{-3} v^{-1} f_{1-\frac{t}{2}} \left(1 - \frac{u - v}{v}\right) \, du \, dv$$

$$< 0.$$

Strict log-concavity of the positive function $f_t$ follows immediately from strict concavity.

Remark 8.4. (a) By extending the computations of the first and second derivatives of $f_t$ in the proof of Corollary 8.3 to higher-order derivatives, it is easy to see that $f_t(x)$ is real-analytic for $x$ in the intervals as specified in Corollary 8.3(a)–(b). For the definition of real analytic function, see Krantz and Parks [22, Definition 1.1.5].
(b) It may be that, like the Dickman density \( f_0 \), the densities \( f_t \) with \( 0 < t < 1 \) are log-concave everywhere and hence strongly unimodal. Even if this is false, we conjecture that the densities \( f_t \) are all unimodal.

9. Improved right-tail asymptotic upper bound

In this section, we will prove that for \( 0 < t < 1 \) and \( x > 4 \), the continuous density function \( f_t \) satisfies

\[
f_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)]
\]

uniformly in \( t \). We first bound the moment generating function of the random variable \( V \) treated in Lemma 3.3.

**Lemma 9.1.** Denote the moment generating function of \( V \) by \( m \). Then for every \( \epsilon > 0 \) there exists a constant \( a \equiv a(\epsilon) > 0 \) such that for all \( \theta > 0 \) we have

\[
m(\theta) \leq \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta]. \tag{9.1}
\]

**Proof.** The idea of the proof comes from Janson [18, Lemma 6.1]. Observe that the random variable \( V \) satisfies the following distributional identity

\[
V \overset{\text{d}}{=} 1 + V_1 \cdot V
\]

where \( V_1 \sim \text{Uniform}(1/2, 1) \) is independent of \( V \). It follows by conditioning on \( V_1 \) that the moment generating function \( m \) satisfies

\[
m(\theta) = 2e^\theta \int_{v=1/2}^{1} m(\theta v) \, dv = 2e^\theta \int_{u=0}^{1/2} m(\theta(1-u)) \, du. \tag{9.2}
\]

Since \( m \) is continuous and \( m(0) = 1 \), there exists a \( \theta_1 > 0 \) such that the inequality (9.1) holds (for any constant \( a > 0 \)) for \( \theta \in [0, \theta_1] \). Choose and fix \( \theta_2 > \max\{\theta_1, 5\} \) and choose \( a \in [1, \infty) \) large enough such that the inequality (9.1) holds for \( \theta \in [\theta_1, \theta_2] \).

We now suppose for the sake of contradiction that (9.1) fails at some \( \theta > \theta_2 \). Define \( T := \inf\{\theta > \theta_2 : (9.1) \text{ fails}\} \); then by continuity we have \( m(T) = \exp[(2 + \epsilon)T^{-1}e^T + aT] \).

Since \( m(\theta u) \geq 1 \) for any \( \theta > 0 \) and \( 0 < u < 1/2 \), we can conclude from (9.2) that \( m \) satisfies

\[
m(\theta) \leq 2e^\theta \int_{u=0}^{1/2} m(\theta u) m(\theta(1-u)) \, du \tag{9.3}
\]

for every \( \theta > 0 \), including for \( \theta = T \). The proof is now completed effortlessly by applying exactly the same argument as for the limiting QuickSort moment generating function in Fill and Hung [9, proof of Lemma 2.1]; indeed, using only (9.3) they prove that when \( \theta = T \) the right-hand side of (9.3) is strictly smaller than \( m(T) \), which is the desired contradiction. \( \square \)
Thus, for $\epsilon > 0$ and $\theta > 0$, the moment generating functions $m_t$ all satisfy

$$m_t(\theta) \leq m(\theta) \leq \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta]. \quad (9.4)$$

We now deduce a uniform right-tail upper bound on the survival functions $1 - F_t$ for $0 < t < 1$.

**Theorem 9.2.** Uniformly in $0 < t < 1$, for $x > 1$ the distribution function $F_t$ for $J(t)$ satisfies

$$1 - F_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

**Proof.** The proof is essentially the same as for Fill and Hung [9, proof of Proposition 1.1], but for completeness we sketch the simple proof here. Fix $\epsilon > 0$. For any $\theta > 0$ we have the Chernoff bound

$$1 - F_t(x) = \mathbb{P}(J(t) > x) \leq \mathbb{P}(Z(t) > x) \leq e^{-\theta x}m_t(\theta) \leq e^{-\theta x} \exp[(2 + \epsilon)\theta^{-1}e^\theta + a\theta]$$

by (9.4). Letting $\theta = \ln[(2 + \epsilon)^{-1}x \ln x]$, and then $\epsilon \downarrow 0$ we get the desired upper bound—in fact, with the following improvement we will not find useful in the sequel:

$$1 - F_t(x) \leq \exp[-x \ln x - x \ln \ln x + (1 + \ln 2)2 + o(x)]. \quad \Box$$

The continuous density function $f_t(x)$ enjoys the same uniform asymptotic bound for $0 < t < 1$ and $x > 4$.

**Theorem 9.3.** Uniformly in $0 < t < 1$, for $x > 4$ the continuous density function $f_t$ satisfies

$$f_t(x) \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

**Proof.** Fix $0 < t < 1$ and let $x > 4$. We first use the integral equation in Proposition 5.8, namely,

$$f_t(x) = \int \mathbb{P}((L_3(t), R_3(t)) \in d(l, r)) \cdot h_t(x | l, r),$$

for $x \geq 0$, where, by a change of variables,

$$h_t(x | l, r) = \int f_{l,r}((r - l)(y - 1)) f_{r-l}\left(\frac{x}{r-l} - y\right) dy; \quad (9.5)$$

we consider the contribution to $f_t(x)$ from values $(l, r)$ satisfying $0 < l < t < r < 1$. Recall that the conditional density $f_{l,r}(z)$ vanishes if $z \geq 2$. Thus the only nonzero contribution to (9.5) is from values of $y$ satisfying

$$y \leq \frac{2}{r-l} + 1.$$

If this inequality holds, then the argument for the factor $f_{(t-l)/(r-l)}$ satisfies

$$\frac{x}{r-l} - y \geq \frac{x - 2}{r-l} - 1 \geq x - 3.$$
Using $b(l, r)$ of Lemma 3.2 and (3.3) to bound the $f_{t,l,r}$ factor, we obtain

$$h_t(x | l, r) \leq b(l, r) \left( 1 - F_{t-l}(x - 3) \right).$$

By Theorem 9.2 and the last display in the proof of Lemma 3.2, the contribution in question is thus bounded by $\exp[-x \ln x - x \ln \ln x + O(x)]$, uniformly in $t$, for $x > 4$.

For the contribution to $f_t(x)$ corresponding to the cases $0 = L_3(t) < t < R_3(t)$ and $0 < L_3(t) < t < R_3(t) = 1$, we use the same idea as in the proof of Lemma 3.6. By symmetry, we need only consider the first of these two cases. Recall from the proof of Lemma 3.6 that the contribution in question is bounded by the sum of $f_{W}(x)$, which is the density of $W = U_1(1 + U_2 V)$ evaluated at $x$ [where $U_1$, $U_2$, and $V$ are independent, $U_1$ and $U_2$ are uniformly distributed on $(0,1)$, and $V$ is as in Lemma 3.3], and the integral

$$\int_{r=0}^{1} r^{-1} \mathbb{P} \left( V > \frac{x}{r} - 1 \right) \, dr = \int_{v=x-1}^{\infty} (v + 1)^{-1} \mathbb{P}(V > v) \, dv \leq x^{-1} \int_{v=x-1}^{\infty} \mathbb{P}(V > v) \, dv \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

The last inequality here is obtained by applying a Chernoff bound and Lemma 9.1 to the integrand and integrating; we omit the straightforward details. To bound the density of $W$ at $x$, observe that by conditioning on the values of $U_2$ and $V$, we have

$$f_{W}(x) = \int_{u,v} (1 + uv)^{-1} \mathbb{1}(0 \leq x \leq 1 + uv) \, \mathbb{P}(U_2 \in du, V \in dv)$$

$$= \int_{u=0}^{1} \int_{v=(x-1)/u}^{\infty} (1 + uv)^{-1} \, \mathbb{P}(V \in dv) \, du \leq x^{-1} \int_{u=0}^{1} \mathbb{P} \left( V > \frac{x - 1}{u} \right) \, du \leq x^{-1} \mathbb{P}(V > x - 1) \leq \exp[-x \ln x - x \ln \ln x + O(x)].$$

This completes the proof. 

10. Matching right-tail asymptotic lower bound

In this section we will prove for each fixed $t \in (0,1)$ that the continuous density function $f_t$ satisfies

$$f_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)]$$

as $x \to \infty$, matching the upper bound of Theorem 9.2 to two logarithmic asymptotic terms, with remainder of the same order of magnitude. While we are able to get a similarly matching lower bound to Theorem 9.3 for the survival
function $1 - F_t$ that is uniform in $t$, we are unable to prove uniformity in $t$
for the density lower bound.
We begin with consideration of the survival function.

**Theorem 10.1.** Uniformly in $0 < t < 1$, the distribution function $F_t$ for $J(t)$ satisfies

$$1 - F_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)].$$

**Proof.** With $D$ denoting a random variable having the Dickman distribution
with support $[1, \infty)$, for any $0 < t < 1$ we have from Lemma 3.4 that

$$1 - F_t(x) = \mathbb{P}(J(t) > x) = \mathbb{P}(Z(t) > x + 1) \geq \mathbb{P}(D > x + 1)$$

$$= \exp[-x \ln x - x \ln \ln x + O(x)] \text{ as } x \to \infty.$$  

The asymptotic lower bound here follows by substitution of $x + 1$ for $u$ in
equation (1.6) (for the unnormalized Dickman function) of Xuan [24], who
credits earlier work of de Bruijn [5] and of Hua [16].  \hfill \Box

Now we turn our attention to the densities.

**Theorem 10.2.** For each fixed $t \in (0, 1)$ we have

$$f_t(x) \geq \exp[-x \ln x - x \ln \ln x + O(x)] \text{ as } x \to \infty.$$  

**Proof.** From the calculations at the beginning of the proof of Lemma 8.1,
for all $z > 0$ we have

$$f_t(t + z) \geq z \int_{v > u \geq 0; u + v < 1, u < (1 - t)/z, v - u < t/z} (t + uz)^{-1} v^{-1} f_{1 - \frac{u}{v}} \left(\frac{1 - u - v}{v}\right) \, du \, dv.$$  

Thus, changing variables from $u$ to $w = 1 - (u/v)$, we have

$$f_t(t + tz) \geq z \int_0^1 \int_0^\infty \Upsilon(t, z, w) \left[1 + v(1 - w)z\right]^{-1} f_w\left(v^{-1} + w - 2\right) \, dv \, dw,$$

where $\Upsilon(t, z, w) := \min\{2(1 - w)^{-1}, (1 - t)(tz)^{-1}(1 - w)^{-1}, z^{-1}w^{-1}\}$. Now let

$$\Lambda(t, z, w) := \max\{0, t(1 - t)^{-1}z(1 - w) + w - 2, zw + w - 2\}$$

and change variables from $v$ to $s = v^{-1} + w - 2$ to find

$$f_t(t + tz)$$

$$\geq z \int_0^1 \int_0^{\Lambda(t, z, w)} [1 + (2 - w + s)^{-1}(1 - w)z]^{-1} (2 - w + s)^{-2} f_w(s) \, ds \, dw.$$  

Observe that if $\delta > 0$ and $t \leq w \leq (1 + \delta)t \leq 1$, then

$$\Lambda(t, z, w) < (1 + \delta)tz$$

and so

$$f_t(t + tz)$$

$$\geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{\infty} [1 + (2 - w + s)^{-1}(1 - w)z]^{-1} (2 - w + s)^{-2} f_w(s) \, ds \, dw.$$
If \( \delta \leq 1 \), it follows that
\[
f_t(t + tz) \\
\geq z \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} [1 + (2 - w + s)^{-1}(1 - w)]^{-1} (2 - w + s)^{-2} f_w(s) \, ds \, dw \\
\geq \frac{z}{(2 + 2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \frac{1}{1 + (2 - w + (1 + \delta)tz)^{-1}(1 - w)} f_w(s) \, ds \, dw \\
\geq \frac{z}{(2 + 2tz)^2} \int_t^{(1+\delta)t} \int_{(1+\delta)tz}^{2tz} \left[1 + \frac{1 - w}{(1 + \delta)tz}\right]^{-1} f_w(s) \, ds \, dw \\
\geq \frac{tz}{(2 + 2tz)^2} \int_t^{(1+\delta)t} f_w(s) \, ds \, dw \\
= \frac{tz}{(2 + 2tz)^2} \int_t^{(1+\delta)t} \left[ \mathbb{P}(J(w) > (1 + \delta)tz) - \mathbb{P}(J(w) > 2tz) \right] \, dw.
\]

Recall that \( D \) defined in Lemma 3.4 is a random variable having the Dickman distribution on \([1, \infty)\) and that \( V \) is defined in (3.5). By Lemma 3.4, we have \( D - 1 \leq J(w) \leq V - 1 \) stochastically, and thus we can further lower-bound the density function as follows:
\[
f_t(t + tz) \geq \frac{tz}{(2 + 2tz)^2} \int_t^{(1+\delta)t} \left[ \mathbb{P}(D - 1 > (1 + \delta)tz) - \mathbb{P}(V > 2tz) \right] \, dw
\]
\[
= \delta t \frac{tz}{(2 + 2tz)^2} \left[ \mathbb{P}(D - 1 > (1 + \delta)z) - \mathbb{P}(V > 2z) \right].
\]
That is, if \( 0 < \delta \leq \min\{1, t^{-1} - 1\} \), then for every \( z > 0 \) we have
\[
f_t(t + z) \geq \frac{\delta t}{(2 + 2z)^2} \left[ \mathbb{P}(D - 1 > (1 + \delta)z) - \mathbb{P}(V > 2z) \right].
\]

If \( z \geq \max\{1, t/(1 - t)\} \), then we can choose \( \delta \equiv \delta_z = z^{-1} \) and conclude
\[
f_t(t + z) \geq t(2 + 2z)^{-2} \left[ \mathbb{P}(D - 1 > z + 1) - \mathbb{P}(V > 2z) \right].
\]
Moreover, as \( z \to \infty \), we have
\[
(2 + 2z)^{-2} \left[ \mathbb{P}(D - 1 > z + 1) - \mathbb{P}(V > 2z) \right] = \exp[-z \ln z - z \ln \ln z + O(z)].
\]
The stated result follows readily.  \( \square \)

**Remark 10.3.** The proof of Theorem 10.1 reveals that the result in fact holds uniformly for \( t \) in any closed subinterval of \((0, 1)\). In fact, the proof shows that the result follows uniformly in \( t \in (0, 1) \) and \( x \to \infty \) satisfying \( x = \Omega(\ln[1/ \min\{t, 1 - t\}]) \).
11. Right-tail large deviation behavior of QuickQuant\( (n,t) \)

In this section, we investigate the right-tail large deviation behavior of QuickQuant\( (n,t) \), that is, of QuickSelect\( (n,m_n(t)) \). Throughout this section, for each fixed \( 0 \leq t \leq 1 \) we consider any sequence \( 1 \leq m_n(t) \leq n \) such that \( m_n(t)/n \to t \) as \( n \to \infty \). We abbreviate the normalized number of key comparisons of QuickSelect\( (n,m_n(t)) \) discussed in Section 1 as \( C_n(t) := \frac{n-1}{m_n(t)} \).

Kodaj and Mőri [21, Corollary 3.1] bound the convergence rate of \( C_n(t) \) to its limit \( Z(t) \) in the Wasserstein \( d_1 \)-metric, showing that the distance is \( O(\delta_{n,t}^{-1} \log(\delta_{n,t}^{-1})) \), where \( \delta_{n,t} = |n^{-1}m_n(t) - t| + n^{-1} \). Using their result, we bound the convergence rate in Kolmogorov–Smirnov distance in the following lemma.

**Lemma 11.1.** Let \( d_{KS}(\cdot,\cdot) \) be Kolmogorov–Smirnov (KS) distance. Then

\[
d_{KS}(C_n(t), Z(t)) = \exp \left[ -\frac{1}{2} \ln \frac{1}{\delta_{n,t}} + \frac{1}{2} \ln \ln \frac{1}{\delta_{n,t}} + O(1) \right].
\]

\[(11.1)\]

**Proof.** The lemma is an immediate consequence of Fill and Janson [10, Lemma 5.1], since the random variable \( Z(t) \) has a density function bounded by 10, according to Theorem 3.1. Indeed, by that result we have

\[
d_{KS}(C_n(t), Z(t)) \leq 2^{1/2} [10 d_1(C_n(t), Z(t))]^{1/2} = O([\delta_{n,t} \log(\delta_{n,t}^{-1})]^{1/2}).
\]

\( \square \)

Using the right-tail asymptotic bounds on the limiting QuickQuant\( (t) \) distribution function \( F_t \) in Theorems 9.2 and 10.1 (which extend to \( t \in \{0,1\} \) by known results about the Dickman distribution), we can now derive the right-tail large-deviation behavior of \( C_n(t) \).

**Theorem 11.2.** Fix \( t \in [0,1] \) and abbreviate \( \delta_{n,t} \) as \( \delta_n \). Let \( (\omega_n) \) be any sequence diverging to \( +\infty \) as \( n \to \infty \) and let \( c > 1 \). For integer \( n \geq 3 \), consider the interval

\[ I_n := \left[ c, \frac{1}{2 \ln \ln \delta_n^{-1}} \left( 1 - \frac{\omega_n}{\ln \ln \delta_n^{-1}} \right) \right]. \]

(a) Uniformly for \( x \in I_n \) we have

\[
P(C_n(t) > x) = (1 + o(1))P(Z(t) > x) \quad \text{as} \quad n \to \infty.
\]

\[(11.2)\]

(b) If \( x_n \in I_n \) for all large \( n \), then

\[
P(C_n(t) > x_n) = \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)].
\]

\[(11.3)\]

**Proof.** The proof is similar to that of Fill and Hung [9, Theorem 3.3] or its improvement in [8, Theorem 3.3]. We prove part (a) first. By Lemma 11.1, it suffices to show that

\[
\exp \left[ -\frac{1}{2} \ln \frac{1}{\delta_n} + \frac{1}{2} \ln \ln \frac{1}{\delta_n} + O(1) \right] \leq o(P(Z(t) > x_n))
\]
with $x_n \equiv \frac{1}{2} \ln \frac{\delta_n \ln \delta_n}{\ln \ln \delta_n} \left(1 - \frac{\omega_n}{\ln \ln \delta_n}\right)$ and $\omega_n = o(\ln \ln \delta_n^{-1})$. Since, by Theorem 10.1, we have

$$P(Z(t) > x_n) \geq \exp[-x_n \ln x_n - x_n \ln \ln x_n + O(x_n)],$$

it suffice to show that for any constant $C < \infty$ we have

$$-\frac{1}{2} \ln \frac{1}{\delta_n} + \frac{1}{2} \ln \ln \frac{1}{\delta_n} + C + x_n \ln x_n + x_n \ln \ln x_n + Cx_n \to -\infty. \quad (11.4)$$

This is routine and similar to what is done in [9, proof of Theorem 3.3]. This completes the proof of part (a).

Part (b) is immediate from part (a) and Theorems 9.2 and 10.1. □

Remark 11.3. Consider the particular choice $m_n(t) = \lfloor nt \rfloor + 1$ (for $t \in [0, 1]$, with $m_n(1) = n$) of the sequences $(m_n(t))$. That is, suppose that $C_n(t) = X_n(t)$ as defined in (1.6). In this case, large-deviation upper bounds based on tail estimates of the limiting $F_t$ have broader applicability than as described in Theorem 11.2 and are easier to derive, too. The reason is that, by Kodaj and Móri [21, Lemma 2.4], the random variable $X_n(t)$ is stochastically dominated by its continuous counterpart $Z(t)$. Then, by Theorem 10.1, uniformly in $t \in [0, 1]$, we have

$$P(X_n(t) > x) \leq P(Z(t) > x) \leq \exp[-x \ln x - x \ln \ln x + O(x)] \quad (11.5)$$

for $x > 1$; there is no restriction at all on how large $x$ can be in terms of $n$ or $t$.

Here is an example of a very large value of $x$ for which the tail probability is nonzero and the aforementioned bound still matches logarithmic asymptotics to lead order of magnitude, albeit not to lead-order term. The largest possible value for the number $C_{n,m}$ of comparisons needed by $\text{QuickSelect}(n,m)$ is $\binom{n}{2}$, corresponding in the natural coupling to any permutation of the $n$ keys for which the $m-1$ keys smaller than the target key appear in increasing order, the $n-m$ keys larger than the target key appear in decreasing order, and the target key appears last; thus

$$P\left(C_{n,m} = \binom{n}{2}\right) = \frac{1}{n!} \binom{n-1}{m-1},$$

which lies between $1/n!$ and $(\frac{n-1}{(n-1)!})/n! \sim 2^{-n-1/2}/(n! \sqrt{n})$. We conclude that for $x_n = (n-1)/2$ we have, uniformly in $t \in [0, 1]$, that

$$P(X_n(t) \geq x_n) = P(X_n(t) = x_n) = \exp[-2x_n \ln x_n + O(x_n)].$$

The bound (11.5) on $P(X_n(t) > x)$ is in fact also (by the same proof) a bound on the larger probability $P(X_n(t) \geq x)$, and in this case implies

$$P(X_n(t) \geq x_n) = \exp[-x_n \ln x_n + O(x_n \log \log x_n)].$$

The bound (11.5) is thus loose only by an asymptotic factor of 2 in the logarithm of the tail probability.
Remark 11.4. (a) We can use another result of Kodaj and Móri, namely, [21, Lemma 3.2], in similar fashion to quantify the Kolmogorov–Smirnov continuity of the process \(Z\) discussed in Remark 7.10. Let \(0 \leq t < u \leq 1/2\) and \(\delta = u - t\). Then the lemma asserts
\[
d_1(Z(t), Z(u)) < 4\delta(1 + 2\log\delta^{-1}).
\]
It follows using Fill and Janson [10, Lemma 5.1] that
\[
d_{KS}(Z(t), Z(u)) \leq O((\delta \log\delta^{-1})^{1/2}) = \exp \left[ -\frac{1}{2} \ln\delta^{-1} + \frac{1}{2} \ln\ln\delta^{-1} + O(1) \right],
\]
uniformly for \(|u - t| \leq \delta\), as \(\delta \downarrow 0\). We thus have uniform Kolmogorov–Smirnov continuity of \(Z\).

(b) Kodaj and Móri [21] did not consider a lower bound on either of the distances in (a), but we can rather easily obtain a lower bound on the KS distance that is of order \(\delta^2\) uniformly for \(t\) and \(u\) satisfying \(0 < t < t + \delta = u \leq \min\{1/2, 2t\}\).

Indeed, for such \(t\) and \(u\) we have \(P(J(u) \leq u) = 0\) and, by Theorem 8.2 (since \(t \leq u \leq 1/2 \leq \min\{1 - t, 2t\}\), as required by the hypotheses of the theorem) and in the notation of that theorem,
\[
\begin{align*}
P(J(t) \leq u) &= \int_{t}^{u} f_t(x) \, dx = t \int_{0}^{(u/t) - 1} \sum_{k=1}^{\infty} (-1)^{k-1} c_k z^k \, dz \\
&\geq t \int_{0}^{(u/t) - 1} (c_1 z - c_2 z^2) \, dz \\
&= t \left[ \frac{1}{2} c_1 \left( \frac{u}{t} - 1 \right)^2 - \frac{1}{3} c_2 \left( \frac{u}{t} - 1 \right)^3 \right] \\
&\geq \frac{1}{3} c_1 t \left( \frac{u}{t} - 1 \right)^2 > \frac{1}{150} (u - t)^2 = \frac{1}{150} \delta^2,
\end{align*}
\]
where the penultimate inequality holds because \(\frac{u}{t} - 1 = \frac{\delta}{t} < 1\) and \(0 < c_2 \leq \frac{1}{2} c_1\).

(c) The lower bound in (b) can be improved to order \(\delta\) when \(t = 0\). Then for every \(u \in [0, 1]\) we have \(P(J(0) \leq u) = e^{-\gamma} u\), and so for \(u \in [0, 1/2]\) we have
\[
d_{KS}(Z(0), Z(u)) \geq e^{-\gamma} u.
\]

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Appendix A. Proof of Lemma 7.6

In the proof of Lemma 7.6, we show only that the function \(c^{(3)}\) is Lipschitz continuous. In this Appendix, we show by similar arguments that \(c^{(i)}\) is Lipschitz continuous for \(i = 1, 2, 4, 5, 6\). Let \(x, z \in \mathbb{R}\) with \(z > x\) and consider \((l, r)\) satisfying \(0 < l < t < r < 1\).

Lemma A.1. The function \(c^{(1)}\) is Lipschitz continuous.
Proof. Redefine
\[ d_{l,r}(x, z, y) := f_{l,r}^{(1)}(z - y) - f_{l,r}^{(1)}(x - y) \]
and recall that
\[ f_{l,r}^{(1)}(x) = \begin{cases} 1 & (2 - 2l) \leq x < 2 - l \\ \frac{1}{1 - l} & x - 1 \leq l. \end{cases} \]
We are interested in bounding the quantity
\[ |c^{(1)}(z) - c^{(1)}(x)| \leq \int_{l,r,y} |d_{l,r}(x, z, y)| \mathbb{P}(Y \in dy | (L_3, R_3) = (l, r)) \, dl \, dr. \tag{A.1} \]

Just as we did for Lemma 7.6, we break the proof into consideration of two cases.

**Case 1.** \( z - x < l \). We bound \( d_{l,r}(x, z, y) \) for \( y \) in each of the five subintervals of the real line determined by the four partition points
\[ x - (2 - l) < z - (2 - l) < x - (2 - 2l) < z - (2 - 2l). \]
For the two subcases \( y \leq x - (2 - l) \) and \( y > z - (2 - 2l) \), we have \( d_{r}(x, z, y) = 0 \). We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

**Subcase 1(a).** \( z - (2 - 2l) < y \leq (2 - 2l) \). We have
\[ d_{l,r}(x, z, y) = \frac{1}{1 - l} \left( \frac{1}{z - y - 1 + l} - \frac{1}{x - y - 1 + l} \right) \]
\[ = -\frac{1}{1 - l} \frac{z - x}{(z - y - 1 + l)(x - y - 1 + l)} . \]
Using the inequality \( z - y > x - y \geq 2 - 2l \), we obtain
\[ |d_{l,r}(x, z, y)| \leq \frac{1}{1 - l} \frac{|z - x|}{(1 - l)^2} . \]
Using inequality (A.1), we conclude that the contribution from this subcase is bounded above by \( \frac{l(2 - l)}{2(l - 1)^2} |z - x| \).

**Subcase 1(b).** \( (2 - l) < y \leq z - (2 - l) \). First note that in this subcase we have \( f^{(1)}(z - y) = 0 \). Using the fact that \( y \leq z - (2 - l) < x - (2 - 2l) \), we have \( x - y \geq 2 - 2l \) and
\[ |d_{l,r}(x, z, y)| \leq \frac{1}{(1 - l)^2} \frac{1}{x - y - 1 + l} \leq \frac{1}{(1 - l)^2} . \]
By the constant bound on \( f_t \) in Theorem 3.1, the contribution to \( |c^{(1)}(z) - c^{(1)}(x)| \) is bounded above by
\[ \int_{t=0}^{1} \int_{r=t}^{1} \int_{y=x-(2-l)}^{z-(2-l)} \frac{1}{1 - l} \frac{1}{r - l} f_{l,r}^{(1)} \left( \frac{y}{r - l} - 1 \right) \, dy \, dr \, dl \]
\[ \leq 10(z - x) \int_{t=0}^{1} \int_{r=t}^{1} \frac{1}{(1 - l)^2} \frac{1}{r - l} \, dr \, dl . \]
\[ (z - x) \frac{10 t (1 - \ln t)}{1 - t}. \]

**Subcase 1(c).** \( x - (2 - 2l) < y \leq z - (2 - 2l) \). In this case, the contribution from \( f^{(1)}(x - y) \) vanishes. We have

\[ |d_{t,r}(x,y)| \leq \frac{1}{1 - l} \frac{1}{z - y - 1 + l} \leq \frac{1}{(1 - l)^2}. \]

The corresponding contribution to \( |c^{(1)}(z) - c^{(1)}(x)| \) can be bounded just as in Subcase 1(b).

This completes the proof for Case 1.

**Case 2.** \( z - x \geq l \). In this case we simply use

\[ |d_{t,r}(x,y)| \leq f^{(1)}(z - y) + f^{(1)}(x - y), \]

and show that each of the two terms on the right contributes at most a constant \((\text{depending on } t)\) times \((z - x)\) to the bound in (A.1). Accordingly, let \( w \) be either \( x \) or \( z \). We are interested in bounding

\[ \int_{l=0}^{(z-x)\wedge t} \int_{r=t}^{1} \int_{y=(w-(2-2l))}^{1} \frac{1}{1 - l} \frac{1}{\mu(\text{dy}, \text{dr}, \text{dl})} \] (A.2)

with \( \mu(\text{dy}, \text{dr}, \text{dl}) := P(Y \in \text{dy} \mid (L_3, R_3) = (l, r)) \text{dr \text{dl}} \). We bound the integrand as follows:

\[ \frac{1}{1 - l} \frac{1}{\mu(\text{dy}, \text{dr}, \text{dl})} \leq \frac{1}{(1 - l)^2}. \]

We first suppose \( z - x < t \) and bound (A.2) by

\[ \int_{l=0}^{z-x} \int_{r=t}^{1} \frac{1}{(1 - l)^2} \text{dr \text{dl}} \leq \frac{1}{1 - t} \int_{l=0}^{z-x} \text{dl} = \frac{z - x}{1 - t}. \]

If instead \( z - x \geq t \), we bound (A.2) by

\[ \frac{z - x}{t} \int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{(1 - l)^2} \text{dr \text{dl}} = z - x. \]

This completes the proof for Case 2 and thus the proof of Lipschitz continuity of \( c^{(1)} \).

\[ \square \]

**Lemma A.2.** The function \( c^{(2)} \) is Lipschitz continuous.

**Proof.** The Lipschitz continuity of \( c^{(2)} \) follows immediately from \( c^{(1)} \) because

\[ f^{(2)}_{l,r}(x) = \mathbb{1}(2r \leq x < 1 + r) \frac{1}{r x - r}, \]

and we can replace \( l \) in the proof of \( c^{(1)} \) by \( 1 - r \) to see that if \( \Lambda^{(1)}(t) \) is a bound on the Lipschitz constant for \( c^{(1)} \), then \( \Lambda^{(1)}(1 - t) \) is a bound on the Lipschitz constant for \( c^{(2)} \).

\[ \square \]

**Lemma A.3.** The function \( c^{(4)} \) is Lipschitz continuous.
Proof. Redefine
\[ d_{l,r}(x, z, y) := f^{(4)}_{l,r}(z - y) - f^{(4)}_{l,r}(x - y) \]
and reformulate
\[ f^{(4)}_{l,r}(x) = 1(2r - l < x < 2 - l) \left( \frac{1}{x} \left( \frac{1}{x + l} + \frac{1}{x - l} \right) \right) \]
from the expression for \( f^{(4)}_{l,r}(x) \) found in Section 2. We are interested in bounding the quantity
\[ |c^{(4)}(z) - c^{(4)}(x)| \leq \int_{l,r,y} |d_{l,r}(x, z, y)| \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr, \]
where the conditional probability can also be written in density terms as
\[ \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) = \frac{1}{r - l} f_{l-r} \left( \frac{y}{r - l} - 1 \right) \, dy. \]
Just as we did for Lemma 7.6, we break the proof into consideration of two cases.

Case 1. \( z - x < 2(1 - r) \). As in the proof for Case 1 of Lemma 7.6, we bound \( d_{l,r}(x, z, y) \) for \( y \) in each of the five subintervals of the real line determined by the four partition points
\[ x - (2 - l) < z - (2 - l) < x - (2r - l) < z - (2r - l). \]
For the two subcases \( y \leq x - (2 - l) \) and \( y > z - (2r - l) \), we have \( d_{l,r}(x, z, y) = 0 \). We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

Subcase 1(a). \( z - (2 - l) < y \leq x - (2r - l) \). We have
\[
\begin{align*}
|d_{l,r}(x, z, y)| &\leq \frac{1}{z - y} \left( \frac{1}{z - y + l} + \frac{1}{z - y - l} \right) - \frac{1}{x - y} \left( \frac{1}{x - y + l} + \frac{1}{x - y - l} \right) \\
&= \frac{1}{z - y} \left( \frac{1}{z - y + l} - \frac{1}{x - y + l} \right) + \frac{1}{z - y} \left( \frac{1}{z - y - l} - \frac{1}{x - y - l} \right) + \frac{1}{z - y} \left( \frac{1}{z - y - l} - \frac{1}{x - y - l} \right).
\end{align*}
\]
Using the inequality \( z - y > x - y \geq 2r - l \), we obtain
\[
|d_{l,r}(x, z, y)| \leq \frac{1}{2r - l} \left( \frac{z - x}{(2r - l)^2} \right) + \frac{z - x}{2r - l} \frac{1}{(z - y - l)(x - y - l)} + \frac{z - x}{(2r - l)^2} \frac{1}{2r - l}.
\]
Except for the third term, it is easy to see (by direct computation) that the corresponding contribution to the bound (A.3) on \( |c^{(4)}(z) - c^{(4)}(x)| \) is
bounded by a constant (depending on $t$) times $z - x$. So we now focus on bounding the contribution from the third term. Note that since $2r - l > t > 0$, we need only bound
\[
\int_{l, r, y} \frac{1}{(z - y - l)(x - y - l)} \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr \tag{A.4}
\]
by a constant (which is allowed to depend on $t$, but our constant will not).

We first focus on the integral in (A.4) with respect to $y$ and write it, using a change of variables, as
\[
\int_{y \in I} \frac{d^*_l, r(x, y)}{f_{l-t}(y)} \, dy, \tag{A.5}
\]
with
\[
d^*_l, r(x, y) = \frac{1}{[z - (r - l)(y + 1) - l][x - (r - l)(y + 1) - l]}
\]
and $I := \left\{ y : \frac{x - (2r - l)}{r - l} - 1 < y \leq \frac{x - (2r - l)}{r - l} - 1 \right\}$. Because the support of the density $f_{l-t}$ is contained in the nonnegative real line, the integral (A.5) vanishes unless the right endpoint of the interval $I$ is positive, which is true if and only if
\[
l > \frac{3r - x}{2}.
\]
So we see that the integral of (A.5) over $l \in (0, t)$ vanishes unless this lower bound on $l$ is smaller than $t$, which is true if and only if
\[
r < \frac{2t + x}{3}. \tag{A.6}
\]
But then the integral of (A.5) over $\{(l, r) : 0 < l < t < r < 1\}$ vanishes unless this upper bound on $r$ is greater than $t$, which is true if and only if $x > t$; we conclude that for $x \leq t$, that integral vanishes.

So we may now suppose $x > t$, and we have seen that the integral of (A.5) over $\{(l, r) : 0 < l < t < r < 1\}$ is equal to its integral over the region
\[
R := \left\{ (l, r) : \frac{3r - x}{2} \vee 0 < l < t < r < 1 \land \frac{2t + x}{3} \right\}.
\]
Observe that on $R$ we have
\[
\frac{x - (2r - l)}{r - l} - 1 = \frac{x - l}{r - t} - 3 > \frac{x - t}{r - l} - 3 > \frac{1}{2} \frac{x - t}{2(r - l)} - 3. \tag{A.7}
\]
Define
\[
B := \left\{ (l, r) : \frac{x - t}{2(r - l)} - 3 > 0 \right\}.
\]
We now split our discussion of the contribution to the integral of (A.5) over $(l, r) \in R$ into two terms, corresponding to (i) $R \cap B^c$ and (ii) $R \cap B$. 

Term (i). $R \cap B^c$. Since the support of the density $f_{L_{c+L}}$ is contained in the nonnegative real line, the value of (A.5) remains unchanged if we integrate $y$ over the positive part of $I$ instead. Let

$$I^* := \left\{ y : \frac{x - t}{2(r - l)} - 3 < y \leq \frac{x - (2r - l)}{r - l} - 1 \right\}.$$  

Since the left endpoint of $I^*$ is nonpositive, we can bound (A.5) by extending the range of integration from $I$ to $I^*$. Making use of the inequality (6.4), the integral (A.5) is bounded, for any $\theta > 0$, by

$$\int_{y \in I^*} \frac{1}{4(r - l)^2} C_\theta e^{-\theta y} dy \leq \frac{C_\theta}{4\theta(r - l)^2} \exp \left[-\frac{x - t}{2(r - l)} \theta + 3\theta\right].$$  

The integral over $(l, r) \in R \cap B^c$ of (A.5) is therefore bounded by

$$\frac{C_\theta}{4\theta^3} 3^\theta \int_{r = t}^{(2t + x)/3} \int_{r = (3r - x)/2}^{t} \frac{1}{(r - l)^2} \exp \left[-\frac{x - t}{2(r - l)} \theta\right] dl \, dr$$

$$= \frac{C_\theta}{4\theta^3} e^{3\theta} \int_{r = t}^{(2t + x)/3} \int_{s = r - t}^{(x - r)/2} \frac{1}{s^2} \exp \left(-\frac{x - t}{2} \theta s^{-1}\right) ds \, dr$$

$$\leq \frac{C_\theta}{4\theta^3} e^{3\theta} \left(\frac{2}{\theta(x - t)}\right) \int_{r = t}^{(2t + x)/3} \exp \left(-\frac{x - t}{2} \frac{2}{x - r}\right) dr$$

$$\leq \frac{C_\theta}{2\theta^2} e^{3\theta} \frac{1}{x - t} e^{-\theta} \left(\frac{2t + x}{3} - t\right) = \frac{C_\theta}{6\theta^2} e^{2\theta} < \infty. \quad (A.8)$$

Term (ii). $R \cap B$. We can bound (A.5) by the sum of the integrals of the same integrand over the intervals $I^*$ and

$$I^* := \left\{ y : 0 < y \leq \frac{x - t}{2(r - l)} - 3 \right\}.$$  

The bound for the integral over $I^*$ is the same as the bound for the $R \cap B^c$ term. To bound the integral over $I^*$, we first observe that

$$d_{L_{c+L}}(x, z, y) \leq \frac{1}{\frac{1}{2}(x + t) + 2r - 3l} \leq \frac{4}{(x - t)^2},$$

where the last inequality holds because $l < t < r$. The contribution to (A.4) can be bounded by integrating $4/(x - t)^2$ with respect to $(l, r) \in R \cap B$. We then extend this region of integration to $R$, where $(3r - x)/2$ is at most the lower bound on $l$ and $(2t + x)/3$ is at least the upper bound on $r$. Thus we bound the contribution by

$$\frac{4}{(x - t)^2} \int_{r = t}^{(2t + x)/3} \left(t - \frac{3r - x}{2}\right) \, dr = \frac{1}{3}.$$  

This completes the proof for Subcase 1(a).
Subcase 1(b). $x - (2r - l) < y \leq z - (2r - l)$. First note that in this subcase we have $f^{(4)}(x - y) = 0$. We proceed in similar fashion as for Subcase 1(a), this time setting

$$I := \left\{ y : \frac{x - l}{r - l} - 3 < y \leq \frac{z - l}{r - l} - 3 \right\}.$$

Again using a linear change of variables and the original expression for $f^{(4)}$ defined in (2.16), the integral (with respect to $y$ only, in this subcase) appearing on the right in (A.3) in this subcase can be bounded by

$$\int_{y \in I} d_{l,r}^{*}(z, y) f_{\frac{t}{r-l}}(y) dy \quad (A.9)$$

where now

$$d_{l,r}^{*}(z, y) = \frac{1}{z - (r - l)(y + 1) + l} \times \frac{2}{z - (r - l)(y + 1) - l}.$$

Note that, unlike its analogue in Subcase 1(a), here $d_{l,r}^{*}(z, y)$ does not possess an explicit factor $z - x$.

By the same discussion as in Subcase 1(a), we are interested in the integral of (A.9) with respect to $(l, r) \in R$, where this time

$$R := \left\{ (l, r) : \frac{3r - z}{2} \vee 0 < l < t < r < 1 \wedge 2t + z \right\}.$$

and we may suppose that $z > t$.

Observe that on $R$ we have

$$\frac{z - l}{r - l} - 3 > \frac{z - t}{r - l} - 3 > \frac{1}{2} \frac{z - t}{2r - l} - 3.$$

Following a line of attack similar to that for Subcase 1(a), we define

$$W := \left\{ (l, r) : \frac{x - l}{r - l} - 3 > \frac{z - t}{2(r - l)} - 3 \right\}$$

and split our discussion of the integral of (A.9) over $(l, r) \in R$ into two terms, corresponding to (i) $R \cap W^c$ and (ii) $R \cap W$.

**Term (i).** $R \cap W$. We bound (A.9) by using the inequality (6.4) (for any $\theta > 0$) and obtain

$$\int_{y \in I} \frac{1}{r} \frac{1}{2(r-l)} C_{\theta} \exp \left[ -\theta \left( \frac{z - t}{2(r - l)} - 3 \right) \right] dy \quad (A.9)$$

$$\leq \frac{11}{2} \frac{1}{t} \frac{1}{(r-l)^2} C_{\theta} e^{3\theta} \exp \left[ -\theta \left( \frac{z - t}{2(r - l)} \right) \right] (z - x).$$

Integrating this bound with respect to $(l, r) \in R \cap W$, we get no more than

$$(z - x) \frac{11}{2} \frac{1}{t} C_{\theta} e^{3\theta} \int_{r=t}^{(2t+z)/3} \int_{l=(3r-z)/2}^{t} \frac{1}{(r-l)^2} \exp \left[ -\theta \left( \frac{z - t}{2(r - l)} \right) \right] dl \, dr,$$

which [compare (7.9)] is bounded by $(z - x)$ times a constant depending only on $t$ and $\theta$.
Term (ii). $R \cap W^c$. We partition the interval $I$ of $y$-integration into the two subintervals

$$I^* := \left\{ y : \frac{z-t}{2(r-l)} - 3 < y \leq \frac{z-l}{r-l} - 3 \right\}$$

and

$$I' := \left\{ y : \frac{x-l}{r-l} - 3 < y \leq \frac{z-t}{2(r-l)} - 3 \right\}.$$

Observe that the length of each of the intervals $I^*$ and $I'$ is no more than the length of $I$, which is $(z-x)/(r-l)$. We can bound the integral over $y \in I^*$ and $(l, r) \in R \cap W^c$ just as we did for Term (i). For the integral over $y \in I'$ and $(l, r) \in R \cap W^c$, observe the following inequality:

$$d_{l,r}^*(z, y) \leq \frac{2}{t(z+t)+2r-l} \times \frac{1}{t(z+t)+2r-3l} \leq \frac{4}{3t} \frac{2}{z-t}.$$

Using the constant bound in Theorem 3.1, the integral of $d_{l,r}^*(z, y)f_{x-l}^*(y)$ with respect to $y \in I'$ and $(l, r) \in R \cap W^c$ is bounded above by

$$80 \frac{z-x}{3t} \frac{1}{z-t} \int_{r=l}^{(2t+z)/3} \int_{r=(3r-z)/2}^t \frac{1}{r-l} \, dl \, dr \leq 80 \frac{z-x}{3t} (\ln 3 - \ln 2).$$

This completes the proof for Subcase 1(b).

Subcase 1(c). $x - (2 - l) < y \leq z - (2 - l)$. In this case, the contribution from $f^{(4)}(z - y)$ vanishes. Without loss of generality we may suppose $z - x < 1 - t$, because otherwise we can insert a factor $(z-x)/(1-t)$ in our upper bound, and the desired upper bound follows from the fact that the densities $f_r$ are all bounded by 10. Observe that the integrand $|d_{l,r}(x, z, y)|$ in the bound (A.3) is

$$\frac{2}{x-y+l} \times \frac{1}{x-y-l} \leq \frac{1}{x-z+2} \times \frac{2}{x-z+2-2l} \leq \frac{2}{1+t} \times \frac{2}{1+t-2l} \leq \frac{2}{(1+t)(1-t)}.$$

Integrate this constant bound directly with respect to

$$P(Y \in dy|(L_3, R_3) = (l, r)) \, dl \, dr$$

on the region $x - (2 - l) < y \leq z - (2 - l)$ and $0 < l < t < r < 1$ and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by $(z-x)$ times a constant that depends on $t$. This completes the proof for Subcase 1(c) and also for Case 1.

Case 2. $z - x \geq 2(1 - r)$. In this case we simply use

$$|d_{l,r}(x, z, y)| \leq f^{(4)}(z - y) + f^{(4)}(x - y),$$

and show that each of the two terms on the right contributes at most a constant (depending on $t$) times $(z-x)$ to the bound in (A.3). Accordingly,
let \( w \) be either \( x \) or \( z \). We are interested in bounding

\[
\int_{r=\left[1-\frac{1}{2}(z-x)\right]}^{t} \int_{y=\left(r-w-(2t-l)\right)}^{1} \frac{2}{w-y+l} \frac{1}{w-y-l} \mu(dy, dl, dr) \tag{A.10}
\]

with \( \mu(dy, dl, dr) := P(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr \). We bound the integrand as follows:

\[
\frac{2}{w-y+l} \frac{1}{w-y-l} \leq \frac{1}{r} \frac{1}{2r-2l} \leq \frac{1}{2t} \frac{1}{r-t}.
\]

We first suppose \( z - x < 1 - t \) and bound (A.10) by

\[
\frac{1}{2t} \int_{r=1-\frac{1}{2}(z-x)}^{t} \int_{l=0}^{1} \frac{1}{r} dl \, dr \leq \frac{1}{2t} \int_{r=1-\frac{1}{2}(z-x)}^{t} \left[-\ln(r-t)\right] dr
\]

\[
\leq \frac{1}{2t} \left[-\ln\left(1 - \frac{z - x}{2} - t\right)\right] \frac{z - x}{2}
\]

\[
\leq \frac{(z - x) \ln(2/(1-t))}{4t}.
\]

If instead \( z - x \geq 1 - t \), we bound (A.10) by

\[
\frac{1}{2t} \int_{l=0}^{r=t} \int_{y=1-\frac{1}{2}(z-x)}^{1} \frac{1}{r} \, dl \, dr \leq \frac{1}{2t} \int_{l=0}^{t} \left[-\ln(t-l)\right] dl
\]

\[
\leq \frac{1}{2(1-t)} \frac{1 - \ln t}{2(1-t)}.
\]

This completes the proof for Case 2 and thus the proof of Lipschitz continuity of \( c^{(4)} \). \( \square \)

**Lemma A.4.** The function \( c^{(5)} \) is Lipschitz continuous.

**Proof.** Redefine

\[ d_{l,r}(x, z, y) := f_{l,r}^{(5)}(z - y) - f_{l,r}^{(5)}(x - y) \]

and recall

\[ f_{l,r}^{(5)}(x) = \mathbb{1}(2r - l \leq x < 2r) \frac{1}{r} \frac{1}{x-r}. \]

We are interested in bounding the quantity

\[
|c^{(5)}(z) - c^{(5)}(x)| \leq \int_{l,r,y} \left| d_{l,r}(x, z, y) \right| P(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr,
\]

where the conditional probability can also be written in density terms as

\[
P(Y \in dy \mid (L_3, R_3) = (l, r)) = \frac{1}{r-l} f_{l,r}^{(4)} \left( \frac{y}{r-l} - 1 \right) dy.
\]

Just as we did for Lemma 7.6, we break the proof into consideration of two cases.
Case 1.  $z - x < l$. As in the proof for Case 1 of Lemma 7.6, we bound $d_{l,r}(x, z, y)$ for $y$ in each of the five subintervals of the real line determined by the four partition points

$$x - 2r < z - 2r < x - (2r - l) < z - (2r - l).$$

For the two subcases $y \leq x - 2r$ and $y > z - (2r - l)$, we have $d_{r}(x, z, y) = 0$. We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

Subcase 1(a). $z - 2r < y \leq x - (2r - l)$. We have

$$d_{l,r}(x, z, y) = \frac{1}{r} \left( \frac{1}{z - y - r} - \frac{1}{x - y - r} \right).$$

Note that since $r > t$, we need only bound

$$\int_{l,r,y} \frac{1}{(z - y - r)(x - y - r)} \mathbb{P}(Y \in dy \mid (L_{3}, R_{3}) = (l, r)) \, dl \, dr \quad (A.12)$$

by a constant (which is allowed to depend on $t$, but our constant will not).

We first focus on the integral in (A.12) with respect to $y$ and write it, using a change of variables, as

$$\int_{y \in I} d_{l,r}^{*}(x, z, y) f_{\frac{x-l}{r}}(y) \, dy, \quad (A.13)$$

with

$$d_{l,r}^{*}(x, z, y) = \frac{1}{[z - (r - l)(y + 1) - 1][x - (r - l)(y + 1) - 1]}$$

and $I := \left\{ y : \frac{z - (2r)}{r - l} - 1 < y \leq \frac{x - (2r - l)}{r - l} - 1 \right\}$. Because the support of the density $f_{\frac{x-l}{r}}$ is contained in the nonnegative real line, the integral (A.13) vanishes unless the right endpoint of the interval $I$ is positive, which is true if and only if

$$l > \frac{3r - x}{2}.$$

So we see that the integral of (A.13) over $l \in (0, t)$ vanishes unless this lower bound on $l$ is smaller than $t$, which is true if and only if

$$r < \frac{2t + x}{3}. \quad (A.14)$$

But then the integral of (A.13) over $\{(l, r) : 0 < l < t < r < 1\}$ vanishes unless this upper bound on $r$ is greater than $t$, which is true if and only if $x > t$; we conclude that for $x \leq t$, that integral vanishes.

So we may now suppose $x > t$, and we have seen that the integral of (A.13) over $\{(l, r) : 0 < l < t < r < 1\}$ is bounded above by its integral over the
the integral (A.13) is bounded, for any \( \theta > 1 \) over \((l, r)\) in the range of integration from \(l\) to \(r\). We now split our discussion of the contribution to the integral of (A.13) over \((l, r)\) in R into two terms, corresponding to (i) \( R \cap B^c \) and (ii) \( R \cap B \).

**Term (i).** \( R \cap B^c \). Since the support of the density \( f_{\theta} \) is contained in the nonnegative real line, the value of (A.13) remains unchanged if we integrate \( y \) over the positive part of \( y \) instead. Let

\[
I^* := \left\{ y : \frac{x-t}{2(r-l)} - 3 < y \leq \frac{x-(2r-l)}{r-l} - 1 \right\}.
\]

Since the left endpoint of \( I^* \) is nonpositive, we can bound (A.13) by extending the range of integration from \( I \) to \( I^* \). Making use of the inequality (6.4), the integral (A.13) is bounded, for any \( \theta > 0 \), by

\[
\int_{y \in I^*} \frac{1}{(r-l)^2} C_\theta e^{-\theta y} \, dy \leq \frac{C_\theta}{\theta(r-l)^2} \exp \left[ -\frac{x-t}{2(r-l)} \theta + 3\theta \right].
\]

The integral over \((l, r) \in R \cap B^c \) of (A.13) is therefore bounded by

\[
\frac{C_\theta}{\theta} e^{3\theta} \int_{r=t}^{(2t+x)/3} \int_{l=(3r-x)/2}^{t} \frac{1}{(r-l)^2} \exp \left[ -\frac{x-t}{2(r-l)} \theta \right] \, dl \, dr
= \frac{C_\theta}{\theta} e^{3\theta} \int_{r=t}^{(2t+x)/3} \int_{s=r-t}^{(x-r)/2} \frac{1}{s^2} \exp \left( -\frac{x-t}{2} \frac{2}{x-r} \theta s^{-1} \right) \, ds \, dr
\leq \frac{C_\theta}{\theta} e^{3\theta} \frac{2}{\theta(x-t)} \int_{r=t}^{(2t+x)/3} \exp \left( -\frac{x-t}{2} \frac{2}{x-r} \right) \, dr
\leq \frac{C_\theta}{\theta^2} e^{3\theta} \frac{2}{x-t} e^{-\theta} \left( \frac{2t+x}{3} - t \right) = \frac{2C_\theta}{3\theta^2} e^{2\theta} < \infty. \tag{A.16}
\]

**Term (ii).** \( R \cap B \). We can bound (A.13) by the sum of the integrals of the same integrand over the intervals \( I^* \) and

\[
I' := \left\{ y : 0 < y \leq \frac{x-t}{2(r-l)} - 3 \right\}.
\]

The bound for the integral over \( I^* \) is the same as the bound for the \( R \cap B^c \) term. To bound the integral over \( I' \), we first observe that

\[
d_{t,r}^*(x, z, y) \leq \frac{1}{\left[ \frac{1}{2}(x+t) + r - 2t \right]^2} \leq \frac{4}{(x-t)^2}.
\]
where the last inequality holds because $l < t < r$. The contribution to (A.12) can be bounded by integrating $4/(x-t)^2$ with respect to $(l, r) \in R \cap B$. We then extend this region of integration to $R$, where $(3r-x)/2$ is at most the lower bound on $l$ and $(2t + x)/3$ is at least the upper bound on $r$. Thus we bound the contribution by
\[
\frac{4}{(x-t)^2} \int_{r=t}^{(2t+x)/3} \left( \frac{t - 3r - x}{2} \right) \, dr = \frac{1}{3}.
\]
This completes the proof for Subcase 1(a).

Subcase 1(b). $x - (2r - l) < y \leq z - (2r - l)$. First note that in this subcase we have $f(5)(x - y) = 0$. We proceed in similar fashion as for Subcase 1(a), this time setting
\[I := \left\{ y : \frac{x-l}{r-l} - 3 < y \leq \frac{z-l}{r-l} - 3 \right\}.
\]
Again using a linear change of variables, the integral (with respect to $y$ only, in this subcase) appearing on the right in (A.11) in this subcase can be bounded by
\[
\int_{y \in I} d_{l,r}^*(z, y) f_{\frac{1}{3}}(y) \, dy \tag{A.17}
\]
where now
\[d_{l,r}^*(z, y) = \frac{1}{r} \times \frac{1}{z - (r-l)(y+1) - r}.
\]
Note that, unlike its analogue in Subcase 1(a), here $d_{l,r}^*(z, y)$ does not possess an explicit factor $z - x$.

By the same discussion as in Subcase 1(a), we are interested in the integral of (A.17) with respect to $(l, r) \in R$, where this time
\[R := \left\{ (l, r) : \frac{3r - z}{2} \lor 0 < l < t < r < 1 \land \frac{2t + z}{3} \right\}.
\]
and we may suppose that $z > t$.

Observe that on $R$ we have
\[
\frac{z-l}{r-l} - 3 > \frac{z-t}{r-l} - 3 > \frac{1}{2} \frac{z-t}{2(r-l)} - 3.
\]
Following a line of attack similar to that for Subcase 1(a), we define
\[W := \left\{ (l, r) : \frac{x-l}{r-l} - 3 > \frac{z-t}{2(r-l)} - 3 \right\}
\]
and split our discussion of the integral of (A.9) over $(l, r) \in R$ into two terms, corresponding to (i) $R \cap W^c$ and (ii) $R \cap W$.

Term (i). $R \cap W$. We bound (A.17) by using the inequality (6.4) (for any $\theta > 0$) and obtain
\[
\int_{y \in I} \frac{1}{r} \frac{1}{(r-l)} C_\theta \exp \left[ -\theta \left( \frac{z-t}{2(r-l)} - 3 \right) \right] \, dy
\]
\[ \leq \frac{1}{t} \frac{1}{(r - l)^2} C_\theta e^{3\theta} \exp \left[ -\theta \left( \frac{z - t}{2(r - l)} \right) \right] (z - x). \]

Integrating this bound with respect to \((l, r) \in R \cap W\), we get no more than
\[ (z - x) \frac{1}{t} C_\theta e^{3\theta} \int_{r=t}^{(2t+z)/3} \int_{l=(3r-z)/2}^{t} \frac{1}{(r - l)^2} \exp \left[ -\frac{z - t}{2(r - l)} \theta \right] \, dl \, dr, \]
which [compare (7.9)] is bounded by \((z - x)\) times a constant depending only on \(t\) and \(\theta\).

**Term (ii).** \(R \cap W^c\). We partition the interval \(I\) of \(y\)-integration into the two subintervals
\[ I^* := \left\{ y : \frac{z - t}{2(r - l)} - 3 < y \leq \frac{z - l}{r - l} - 3 \right\} \]
and
\[ I' := \left\{ y : \frac{x - l}{r - l} - 3 < y \leq \frac{z - t}{2(r - l)} - 3 \right\}. \]

Observe that the length of each of the intervals \(I^*\) and \(I'\) is no more than the length of \(I\), which is \((z - x)/(r - l)\). We can bound the integral over \(y \in I^*\) and \((l, r) \in R \cap W^c\) just as we did for Term (i). For the integral over \(y \in I'\) and \((l, r) \in R \cap W^c\), observe the following inequality:
\[ d_{l,r}(z, y) \leq \frac{1}{r} \frac{1}{2(z + t) + r - 2l} \leq \frac{1}{t} \frac{2}{z - t}. \]

Using the constant bound in Theorem 3.1, the integral of \(d_{l,r}(z, y)f_{x,y}(y)\) with respect to \(y \in I'\) and \((l, r) \in R \cap W^c\) is bounded above by
\[
\begin{align*}
20 \frac{z - x}{t} \frac{1}{z - t} & \int_{r=t}^{(2t+z)/3} \int_{l=(3r-z)/2}^{t} \frac{1}{(r - l)^2} \, dl \, dr \\
& = 20 \frac{z - x}{t} \frac{1}{z - t} \int_{r=t}^{(2t+z)/3} \left[ -\ln(r - t) + \ln \left( \frac{1}{2} (z - r) \right) \right] \, dr \\
& = 20 \frac{z - x}{t} \frac{1}{z - t} \frac{z - t}{3} \left[ 1 - \ln \left( \frac{z - t}{3} \right) + \ln(z - t) - \left( 1 + \ln \left( \frac{8}{9} \right) \right) \right] \\
& = 20 \frac{z - x}{t} \ln(3 - \ln 2).
\end{align*}
\]

This completes the proof for Subcase 1(b).

**Subcase 1(c).** \(x - 2r < y \leq z - 2r\). In this case, the contribution from \(f^{(5)}(z - y)\) vanishes. Without loss of generality we may suppose \(z - x < t/2\), otherwise we can insert a factor \(2(z - x)/t\) in our upper bound, and the desired upper bound follows from the fact that the densities \(f_z\) are all bounded by 10. Observe that the integrand \(|d_{l,r}(x, z, y)|\) in the bound (A.11) is
\[ \frac{1}{r} \times \frac{1}{x - y + r} \leq \frac{1}{r} \times \frac{1}{x - z + r} \leq \frac{1}{t} \times \frac{1}{r - (t/2)} \leq \frac{2}{t^2}. \]
Integrate this constant bound directly with respect to  
\[ P(Y \in \text{dy} | (L_3, R_3) = (l, r)) \, dr \, dl \]
on the region \( x - 2r < y \leq z - 2r \) and \( 0 < l < t < r < 1 \) and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by \((z - x)\) times a constant that depends on \(t\). This completes the proof for Subcase 1(c) and also for Case 1.

**Case 2.** \( z - x \geq l \). In this case we simply use  
\[ |d_{l,r}(x, z, y)| \leq f^{(5)}(z - y) + f^{(5)}(x - y), \]
and show that each of the two terms on the right contributes at most a constant (depending on \(t\)) times \((z - x)\) to the bound in (A.11). Accordingly, let \(w\) be either \(x\) or \(z\). We are interested in bounding  
\[
\int_{l=0}^{(z-x)/t} \int_{r=t}^{1} \int_{y=w-2r}^{w-(2r-l)} \frac{1}{r} \frac{1}{w-y-r} \mu(\text{dy}, \text{dr}, \text{dl}) \quad (A.18)
\]
with \(\mu(\text{dy}, \text{dr}, \text{dl}) := P(Y \in \text{dy} \mid (L_3, R_3) = (l, r)) \, dr \, dl\). We bound the integrand as follows:
\[
\frac{1}{r} \frac{1}{w-y-r} \leq \frac{1}{r} \frac{1}{r-l} \leq \frac{1}{t} \frac{1}{r-l}.
\]
We first suppose \(z - x < t/2\) and bound (A.18) by  
\[
\frac{1}{t} \int_{l=0}^{(z-x)/t} \int_{r=t}^{1} \frac{1}{r-l} \, dr \, dl \leq \frac{1}{t} \int_{l=0}^{(z-x)/t} \left[ -\ln(t-l) \right] \, dl 
\leq \frac{1}{t} \left[ -\ln \left( t - (z - x) \right) \right] (z - x) 
\leq (z - x) \frac{\ln(2/t)}{t}.
\]
If instead \(z - x \geq t/2\), we bound (A.18) by  
\[
\frac{z-x}{t/2} \int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{r-l} \, dr \, dl \leq \frac{2(z-x)}{t^2} \int_{l=0}^{t} \left[ -\ln(t-l) \right] \, dl 
\leq 2(z - x) \frac{1 - \ln t}{t}.
\]
This completes the proof for Case 2 and thus the proof of Lipschitz continuity of \(c^{(5)}\). \(\square\)

**Lemma A.5.** The function \(c^{(6)}\) is Lipschitz continuous.

**Proof.** Redefine  
\[ d_{l,r}(x, z, y) := f^{(6)}_{l,r}(z - y) - f^{(6)}_{l,r}(x - y) \]
and recall  
\[ f^{(6)}_{l,r}(x) = \mathbb{1}(1 + r - 2l \leq x < 2 - 2l) \frac{1}{1 - x - l + 1}. \]
We are interested in bounding the quantity
\[ |c^{(6)}(z) - c^{(6)}(x)| \leq \int_{l, r, y} |d_{l, r}(x, z, y)| \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr, \]
where the conditional probability can also be written in density terms as
\[ \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) = \frac{1}{r - l} f_{r-l} \left( \frac{y}{r - l} - 1 \right) \, dy. \]

Just as we did for Lemma 7.5, we break the proof into consideration of two cases.

Case 1. \( z - x < 1 - r \). As in the proof for Case 1 of Lemma 7.6, we bound \( d_{l, r}(x, z, y) \) for \( y \) in each of the five subintervals of the real line determined by the four partition points
\[ x - (2 - 2l) < z - (2 - 2l) < x - (1 + r - 2l) < z - (1 + r - 2l). \]

For the two subcases \( y \leq x - (2 - 2l) \) and \( y > z - (1 + r - 2l) \), we have \( d_{r}(x, z, y) = 0 \). We bound the three nontrivial subcases (listed in order of convenience of exposition, not in natural order) as follows.

Subcase 1(a). \( z - (2 - 2l) < y \leq x - (1 + r - 2l) \). We have
\[ d_{l, r}(x, z, y) = \frac{1}{1 - l} \left( \frac{1}{z - y + 1 + l} - \frac{1}{x - y + 1 + l} \right) \]
\[ = \frac{z - x}{1 - l} \frac{1}{(z - y - 1 + l)(x - y - 1 + l)}. \]

Note that since \( 1 - l > 1 - t > 0 \), we need only bound
\[ \int_{l, r, y} \frac{1}{(z - y - 1 + l)(x - y - 1 + l)} \mathbb{P}(Y \in dy \mid (L_3, R_3) = (l, r)) \, dl \, dr \]
by a constant (which is allowed to depend on \( t \), but our constant will not).

We first focus on the integral in (A.20) with respect to \( y \) and write it, using a change of variables, as
\[ \int_{y \in I} d^*_{l, r}(x, z, y) f_{r-l}(y) \, dy, \]
with
\[ d^*_{l, r}(x, z, y) = \frac{1}{[z - (r - l)(y + 1) - 1 + l][x - (r - l)(y + 1) - 1 + l]} \]
and \( I := \left\{ y : \frac{z - (2 - 2l)}{r - l} - 1 < y \leq \frac{x - (1 + r - 2l)}{r - l} - 1 \right\} \). Because the support of the density \( f_{r-l} \) is contained in the nonnegative real line, the integral (A.21) vanishes unless the right endpoint of the interval \( I \) is positive, which is true if and only if
\[ r < x - 1 + 3l \]
The integral (A.21) is bounded, for any \( \theta > 1 \) in the range of integration from \( l \) to \( r \).

We now split our discussion of the contribution to the integral of (A.21) into two terms, corresponding to (i) \( R \cap B^c \) and (ii) \( R \cap B \).

**Term (i).** \( R \cap B^c \). Since the support of the density \( f_{\frac{1}{\theta}} \) is contained in the nonnegative real line, the value of (A.21) remains unchanged if we integrate \( y \) over the positive part of \( I \) instead. Let

\[
I^* := \left\{ y : \frac{x + t - 1}{2(r - l)} - 2 < y \leq \frac{x - (1 + r - 2l)}{r - l} - 1 \right\}.
\]

Since the left endpoint of \( I^* \) is nonpositive, we can bound (A.21) by extending the range of integration from \( I \) to \( I^* \). Making use of the inequality (6.4), the integral (A.21) is bounded, for any \( \theta > 0 \), by

\[
\int_{y \in I^*} \frac{1}{(r - l)^2} C_\theta e^{-\theta y} dy \leq \frac{C_\theta}{\theta (r - l)^2} \exp \left[ -\frac{x + t - 1}{2(r - l)} \theta + 2\theta \right].
\]

The integral over \((l, r) \in R \cap B^c\) of (A.21) is therefore bounded by

\[
\frac{C_\theta}{\theta} e^{2\theta} \int_{l=1-x+2t/3}^{l+(x-1+3t)/2} \frac{1}{(r - l)^2} \exp \left[ -\frac{x + t - 1}{2(r - l)} \theta \right] dr dl
\]

\[
= \frac{C_\theta}{\theta} e^{2\theta} \int_{l=1-x+2t/3}^{l+(x-1+3t)/2} \frac{1}{s^2} \exp \left( -\frac{x + t - 1}{2s} \theta s^{-1} \right) ds dl
\]

\[
\leq \frac{C_\theta}{\theta} e^{2\theta} \int_{l=1-x+2t/3}^{l+(x-1+3t)/2} \exp \left( -\frac{x + t - 1}{2} \theta \frac{2}{x - 1 + l} \right) dl
\]

\[
\leq 2 \frac{C_\theta}{\theta^2} e^{2\theta} \frac{1}{x + t - 1} e^{-\theta} \left( t - \frac{1 - x + 2t}{3} \right) = \frac{2C_\theta}{3\theta^2} e^{\theta} < \infty. \tag{A.24}
\]
Term (ii). $R \cap B$. We can bound (A.21) by the sum of the integrals of the same integrand over the intervals $I^*$ and

$$I^* := \left\{ y : 0 < y \leq \frac{x + t - 1}{2(r - l)} - 2 \right\}.$$

The bound for the integral over $I^*$ is the same as the bound for the $R \cap B^c$ term. To bound the integral over $I'$, we first observe that

$$d^*_{l,r}(x, z, y) = \frac{1}{f(x - y)} \leq \frac{4}{(x + t - 1)^2},$$

where the last inequality holds because $r > t$. The contribution to (A.20) can be bounded by integrating $4/(x + t - 1)^2$ with respect to $(l, r) \in R \cap B$. We then extend this region of integration to $R$, where $(2t + 1 - x)/3$ is at most the lower bound on $l$ and $(x - 1 + 3l)/2$ is at least the upper bound on $r$. Thus we bound the contribution by

$$\frac{4}{(x + t - 1)^2} \int_{l=r+1}^{t} x^2 \left( \frac{x - 1 + 3l}{2} - t \right) \, dl = \frac{1}{3}.$$

This completes the proof for Subcase 1(a).

Subcase 1(b). $x - (1 + r - 2l) < y \leq z - (1 + r - 2l)$. First note that in this subcase we have $f^{(6)}(x - y) = 0$. We proceed in similar fashion as for Subcase 1(a), this time setting

$$I := \left\{ y : \frac{x - 1 + l}{r - l} - 2 < y \leq \frac{z - 1 + l}{r - l} - 2 \right\}.$$

Again using a linear change of variables, the integral (with respect to $y$ only, in this subcase) appearing on the right in (A.19) in this subcase can be written as

$$\int_{y \in I} d^*_{l,r}(z, y) f_{\frac{r-l}{r-l}}(y) \, dy \quad \text{(A.25)}$$

where now

$$d^*_{l,r}(z, y) = \frac{1}{1 - l} \times \frac{1}{z - (r - l)(y + 1) - 1 + l}.$$

Note that, unlike its analogue in Subcase 1(a), here $d^*_{l,r}(z, y)$ does not possess an explicit factor $z - x$.

By the same discussion as in Subcase 1(a), we are interested in the integral of (A.25) with respect to $(l, r) \in R$, where this time

$$R := \left\{ (l, r) : \frac{1 - z + 2t}{3} \lor 0 < l < t < r < 1 \land \frac{z - 1 + 3l}{2} \right\}.$$

and we may suppose that $z > 1 - t$.

Observe that on $R$ we have

$$\frac{z - 1 + l}{r - l} - 2 > \frac{2}{3} \frac{z + t - 1}{r - l} - 2 > \frac{1}{2} \frac{z + t - 1}{r - l} - 2.$$
Following a line of attack similar to that for Subcase 1(a), we define

$$W := \left\{ (l, r) : \frac{x - 1 + l}{r - l} - 2 > \frac{z + t - 1}{2(r - l) - 2} \right\}$$

and split our discussion of the integral of (A.25) over \((l, r) \in R\) into two terms, corresponding to (i) \(R \cap W^c\) and (ii) \(R \cap W\).

**Term (i).** \(R \cap W\). We bound (A.25) by using the inequality (6.4) (for any \(\theta > 0\)) and obtain

$$\int_{y \in I} \frac{1}{1 - l} \frac{1}{1 - r} C_\theta \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l) - 2} \right) \right] dy \leq \frac{1}{1 - t} \frac{1}{1 - (r - l)^2} C_\theta e^{2\theta} \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l)} \right) \right] (z - x).$$

Integrating this bound with respect to \((l, r) \in R \cap W\), we get no more than

$$(z - x) \frac{1}{1 - t} C_\theta e^{2\theta} \int_{l}^{t} \int_{r=t}^{(z-1+3\theta)/2} \frac{1}{(r - l)^2} \exp \left[ -\theta \left( \frac{z + t - 1}{2(r - l)} \right) \right] dx,$$

which [see (7.9)] is bounded by \((z - x)\) times a constant depending only on \(t\) and \(\theta\).

**Term (ii).** \(R \cap W^c\). We partition the interval \(I\) of \(y\)-integration into the two subintervals

$$I^* := \left\{ y : \frac{z + t - 1}{2(r - l)} - 2 < y \leq \frac{z - 1 + l}{r - l} - 2 \right\}$$

and

$$I' := \left\{ y : \frac{x - 1 + l}{r - l} - 2 < y \leq \frac{z + t - 1}{2(r - l)} - 2 \right\}.$$

Observe that the length of each of the intervals \(I^*\) and \(I'\) is no more than the length of \(I\), which is \((z - x)/(r - l)\). We can bound the integral over \(y \in I^*\) and \((l, r) \notin R \cap W^c\) just as we did for Term (i). For the integral over \(y \in I'\) and \((l, r) \in R \cap W^c\), observe the following inequality:

$$d_{l,r}^y(z, y) \leq \frac{1}{1 - t} \times \frac{1}{2(z - t - 1) + r}.$$

Using the constant bound in Theorem 3.1, the integral of \(d_{l,r}^y(z, y)f_{I-l}^y(y)\) with respect to \(y \in I'\) and \((l, r) \in R \cap W^c\) is bounded above by

$$10 \frac{z - x}{1 - t} \int_{l=(1-z+2t)/3}^{t} \int_{r=t}^{(z-1+3\theta)/2} \frac{1}{r - l} \frac{1}{1/2(z - t - 1) + r} dr dl. \quad (A.26)$$

Write the integrand here in the form

$$\frac{1}{r - l} \frac{1}{\left( \frac{z - t - 1}{2} + r \right)} = \left( \frac{1}{r - l} - \frac{1}{r + \frac{z - t - 1}{2}} \right) \left( \frac{1}{l + \frac{z - t - 1}{2}} \right)$$
and observe that \( l + \frac{z-t-1}{2} > \frac{z+t-1}{6} > 0 \). Hence we can bound (A.26) by

\[
10\frac{z-x}{1-t} \int_{l=(1-z+2t)/3}^{t} \left[ \ln \frac{z+2l-1}{2} - \ln(t-l) \right] dl \\
\leq 10\frac{z-x}{1-t} \int_{l=\frac{z-t-1}{2}}^{t} \left[ \ln \frac{z+2l-1}{2} - \ln(t-l) \right] dl \\
= 20\frac{z-x}{1-t} \left[ \ln \frac{z+2l-1}{2} - \ln \left( \frac{z+2l-1}{3} + 1 \right) \right] \\
= 20 \left( 1 + \ln 3 - \ln 2 \right) \frac{(z-x)}{1-t}.
\]

This completes the proof for Subcase 1(b).

**Subcase 1(c).** \( x-(2-2l) < y \leq z-(2-2l) \). In this case, the contribution from \( f^{(0)}(z-y) \) vanishes. Without loss of generality we may suppose \( z-x < (1-t)/2 \), otherwise we can insert a factor \( 2(z-x)/(1-t) \) in our upper bound, and the desired upper bound follows from the fact that the densities \( f_r \) are all bounded by 10. Observe that the integrand \( |d_{l,r}(x,z,y)| \) in the bound (A.19) is

\[
\frac{1}{1-l} x - y - 1 + l \leq \frac{1}{1-l} x - z + 1 - l \leq \frac{2}{1-t} \left( 1 + t - 2l \right) \leq \frac{2}{(1-t)^2}.
\]

Integrate this constant bound directly with respect to

\[
P(Y \in dy | (L_3, R_3) = (l,r)) dl dr
\]

on the region \( x-(2-2l) < y \leq z-(2-2l) \) and \( 0 < l < t < r < 1 \) and use the fact that the density is bounded by 10; we conclude that this contribution is bounded by \((z-x)\) times a constant that depends on \( t \). This completes the proof for Subcase 1(c) and also for Case 1.

**Case 2.** \( z-x \geq 1-r \). In this case we simply use

\[
|d_{l,r}(x,z,y)| \leq f^{(0)}(z-y) + f^{(0)}(x-y),
\]

and show that each of the two terms on the right contributes at most a constant (depending on \( t \)) times \((z-x)\) to the bound in (A.19). Accordingly, let \( w \) be either \( x \) or \( z \). We are interested in bounding

\[
\int_{r=1-(z-x)}^{1} \int_{l=0}^{t} \int_{y=w-(2-2l)}^{w-(1+r-2l)} \frac{1}{1-l} \frac{2}{w-y-1+l} \mu(dy,dl,dr) \tag{A.27}
\]

with \( \mu(dy,dl,dr) := P(Y \in dy | (L_3, R_3) = (l,r)) dl dr \). We bound the integrand as follows:

\[
\frac{1}{1-l} \frac{2}{w-y-1+l} \leq \frac{1}{1-l} \frac{2}{r-l} \leq \frac{1}{1-t} \frac{2}{r-l}.
\]

We first suppose \( z-x < (1-t)/2 \) and bound (A.27) by

\[
\frac{2}{1-t} \int_{r=1-(z-x)}^{1} \int_{l=0}^{t} \frac{1}{r-l} dl dr \leq \frac{2}{1-t} \int_{r=1-(z-x)}^{1} [-\ln(r-t)] dr
\]
\[
\leq \frac{2}{1-t} \left[ -\ln(1-(z-x)-t) \right](z-x) \\
\leq 2(z-x)\frac{\ln[2/(1-t)]}{(1-t)}.
\]

If instead \( z-x \geq (1-t)/2 \), we bound (A.27) by
\[
\frac{4(z-x)}{1-t} \int_{l=0}^{t} \int_{r=t}^{1} \frac{1}{1-t} \frac{1}{r} dr dl \leq (z-x)\frac{2\pi^2}{3(1-t)}.
\]

This completes the proof for Case 2 and thus the proof of Lipschitz continuity of \( c^{(6)} \).

\[ \square \]

Lemmas A.1–A.5, together with Lemmas 7.5–7.6, complete the proof of Theorem 7.4.

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