AN APPROXIMATE VERSION OF THE STRONG NINE DRAGON TREE CONJECTURE

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Abstract. The Strong Nine Dragon Tree Conjecture asserts that for any integers $k$ and $d$ any graph with fractional arboricity at most $k + \frac{d}{k + 1}$ decomposes into $k + 1$ forests, such that for at least one of the forests, every connected component contains at most $d$ edges. We prove this conjecture when $d \leq k + 1$.

We also prove an approximate version of this conjecture, that is, we prove that for any positive integers $k$ and $d$, any graph with fractional arboricity at most $k + \frac{d}{k + 1}$ decomposes into $k + 1$ forests, such that one for at least one of the forests, every connected component contains at most $d + \frac{d(k + 2)}{k + 1} - k$ edges.

1. Introduction

Throughout this paper, all graphs are finite and may contain multiple edges, but have no loops. All undefined graph theory terminology can be found in any standard textbook, for example [1]. For a graph $G$, $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set. We will let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For any graph $G$, we say a decomposition of $G$ is a set of subgraphs of $G$ such that the union of their edges sets is $E(G)$. We will say the fractional arboricity of $G$ is

$$\Gamma_f(G) := \max_{H \subseteq G, v(H) > 1} \frac{e(H)}{v(H) - 1}.$$ 

The goal of this paper is to generalize the celebrated Nash-Williams Theorem, which characterizes when a graph decomposes into $k$ forests.

**Theorem 1.1** (Nash-Williams Theorem [9]). A graph $G$ decomposes into $k$ forests if and only if $\Gamma_f(G) \leq k$.

Suppose we have a graph where $\Gamma_f(G) = k - 1 + \varepsilon$ for some small $\varepsilon > 0$. Then Nash-Williams Theorem says that $G$ decomposes into $k$ forests, and you cannot decompose $G$ into $k - 1$ forests. Intuitively, the fractional arboricity is only minutely over $k$, so you only barely need $k$ forests. Hence, you might hope that you can say more than just that the graph decomposes into $k$ forests. One way to say more would be to restrict the types of forests that can appear in at least one of the forests in the forest decomposition. For example, you might hope that if $\varepsilon$ was sufficiently small, one of the forests can be assumed to be a matching. Jiang and Yang proved the Nine Dragon Tree Theorem which not only confirms that if $\varepsilon$ is small enough you can assume one of the forests is a matching, but proves a general bound about the maximum degree of one of the forests.

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Theorem 1.2 (Nine Dragon Tree Theorem \[6\]). Let \( k \) and \( d \) be positive integers. Let \( G \) be a graph with fractional arboricity at most \( k + \frac{d}{k + d + 1} \). Then \( G \) decomposes into \( k + 1 \) forests, such that one of the forests has maximum degree \( d \).

It was shown by Montassier, Ossona de Mendez, Raspaund and Zhu in \[8\] that the fractional arboricity bound in the Nine Dragon Tree Theorem is best possible. Despite this, they posed a significant strengthening of the Nine Dragon Tree Theorem, which aptly is called the Strong Nine Dragon Tree Conjecture.

Conjecture 1.3 (Strong Nine Dragon Tree Conjecture). Let \( k \) and \( d \) be positive integers. Let \( G \) be a graph with fractional arboricity at most \( k + \frac{d}{k + d + 1} \). Then \( G \) decomposes into \( k + 1 \) forests, such that for one of the forests, each connected component has at most \( d \) edges.

While plenty of progress was made towards the Nine Dragon Tree Theorem before its eventual proof by Jiang and Yang (A survey of results can be found in version one of \[2\]), very little progress has been made towards the Strong Nine Dragon Tree Conjecture.

Of course, the Nine Dragon Tree Theorem implies the Strong Nine Dragon Tree Conjecture when \( d = 1 \) (we note this case was first shown by Yang in \[10\]). Other than this, the Strong Nine Dragon Tree Conjecture is known to hold when \( k = 1 \) and \( d = 2 \), which was proven by Kim, Kostochka, West, Wu and Zhu in \[7\] (they in fact give a slightly better fractional arboricity bound).

The main contribution of this paper is to prove the Strong Nine Dragon Tree Conjecture when \( d \leq k + 1 \), and more generally, prove an approximate version for all \( k \) and \( d \).

Theorem 1.4. Fix integers \( k \) and \( d \). If \( d \leq k + 1 \), let \( c(d, k) = 0 \). Otherwise, let

\[
c(d, k) = \frac{d(k(\lfloor \frac{d}{k + 1} + 2 \rfloor)\lceil \frac{d}{k + 1} + 2 \rceil - k)}{k + 1}.
\]

Any graph with fractional arboricity at most \( k + \frac{d}{k + d + 1} \) decomposes into \( k + 1 \) forests, such that for at least one of the forests, every connected component contains at most \( d + c(d, k) \) edges.

This generalizes all of the previous known results on the Strong Nine Dragon Tree Conjecture. The proof of Theorem 1.4 is self contained, and uses a significantly different approach than the \( k = 1, d = 2 \) proof in \[7\] and also the \( d = 1 \) proof in \[10\].

To understand the approach of this paper, it is instructive to look at pseudoforests. Recall a pseudoforest is a graph where each connected component contains at most one cycle. Hakimi’s Theorem gives a decomposition theorem for pseudoforests which looks similar to Nash-Williams Theorem. Recall, the maximum average degree of a graph \( G \) is

\[
\text{mad}(G) = \max_{H \subseteq G} \frac{2e(H)}{v(H)}.
\]

Theorem 1.5 (Hakimi’s Theorem \[5\]). A graph \( G \) admits an orientation where each vertex has indegree at most \( k \) if and only if \( \text{mad}(G) \leq 2k \).

Observe that a graph \( G \) is a pseudoforest if and only if \( G \) admits an orientation where each vertex has indegree at most one. Hence, we can rephrase Hakimi’s Theorem as

Theorem 1.6 (Hakimi’s Theorem rephrased). A graph decomposes into \( k \) pseudoforests if and only if \( \text{mad}(G) \leq 2k \).
As it is very indicative of our approach, we give a short proof of Hakimi’s Theorem.

**Proof.** Suppose towards a contradiction that $G$ does not admit an orientation where every vertex has indegree at most $k$. Consider an orientation of $G$ where the number
\[
\rho := \sum_{v \in V(G)} \max\{\text{indegree}(v) - k, 0\},
\]
is minimized.

Consider any vertex $v$ with indegree at least $k + 1$. Let $H$ be the subgraph induced by the set of vertices with a directed path to $v$. Observe that if there is a directed path from a vertex $x$ to $v$, where $x$ has indegree at most $k - 1$, then reversing the orientation on each edge in the path, we obtain an orientation with a smaller value for $\rho$. Hence every vertex with a directed path to $v$ has indegree at least $k$. Further for all $u \in V(H)$, every vertex with an arc into $u$ is in $V(H)$. In particular, all of the in-edges into $u$ lie in $E(H)$. But now
\[
\frac{2e(H)}{v(H)} > \frac{2k v(H)}{v(H)} = 2k,
\]
a contradiction. $\square$

The proof of Theorem 1.4 follows the approach of the above proof of Hakimi’s Theorem closely. We will work with a graph $G$ which is a vertex minimal counterexample, and we will pick a forest decomposition which minimizes the number of edges in connected components of one forest, say $F$. As we have a counterexample, there is a connected component in $F$, say $R$, with too many edges, and so we pick a vertex $s$ in this connected component. Then, we can assume that all forests except $F$ are spanning trees, and so we can orient all edges in $E(G) \setminus E(F)$ towards $s$. Then viewing the edges of $F$ as bidirectional, we consider subgraph induced by the vertices which lie on directed paths from vertices in $R$ to $s$. With this, we show that this subgraph has some structure, as otherwise we could perform one of two “flipping” operations to improve our decomposition. Finally, we finish with a counting argument to show that the fractional arboricity bound has been violated.

Before launching into the proof, we mention that the above technique was used to prove a pseudoforest analogue of the Strong Nine Dragon Tree Conjecture (a Nine Dragon Tree analogue was shown by Fan, Li, Song, and Yang in [3]).

**Theorem 1.7** ([4]). Let $k$ and $d$ be integers. If $G$ is a graph where $\text{mad}(G) \leq 2k + \frac{2d}{k+d}$, then $G$ decomposes into $k$ pseudoforests, where one of the pseudoforests has every connected component containing at most $d$ edges.

The bound on the maximum average degree is tight, as shown in [3], even if you only want to show that one of the pseudoforests has bounded maximum average degree.

While this approach is strong enough to prove a pseudoforest analogue to the Strong Nine Dragon Tree Conjecture, and give an approximate version of the Strong Nine Dragon Tree Conjecture, it seems that some new idea is needed to prove the Strong Nine Dragon Tree Conjecture, assuming it is true.

### 2. Picking the Minimal Counterexample

Fix positive integers $k$ and $d$. Let $G$ be a vertex minimal counterexample to Theorem 1.4 for the given values of $k$ and $d$. Let $F$ be the set of decompositions of $G$, $(T_1, \ldots, T_k, F)$ such that $T_i$ is a spanning tree for all $i \in \{1, \ldots, k\}$ and $F$ is a forest.
Lemma 2.1 \cite{6}. For any vertex minimal counterexample Theorem \cite{14}, $F$ is not empty.

Technically this is not shown in \cite{6}. They prove this lemma for a minimal counterexample to the Nine Dragon Tree Theorem. However, the same proof gives the result for a minimal counterexample to the Strong Nine Dragon Tree Theorem, and hence we omit the proof.

It will be convenient to work with (non-proper) edge colourings to keep track of the forest decomposition.

**Definition 2.2.** Given a decomposition $(T_1, \ldots, T_k, F) \in F$, we will colour the edges of $T_i$ blue and the edges of $F$ red. We call such a colouring a **red-blue colouring of $G$**.

Given a red-blue colouring of $G$ and a subgraph $K$ of $G$, we let $e_b(K)$ denote the number of blue edges in $K$, and $e_r(K)$ denote the number of red edges in $K$.

We will focus on a specific subgraph throughout the proof which we define now.

**Definition 2.3.** Let $(T_1, \ldots, T_k, F) \in F$. Let $R$ be a red component of $F$ such that $e_r(R) > d$, and let $s$ be any vertex in $R$. For each $i \in \{1, \ldots, k\}$, orient $T_i$ such that $s$ is the only vertex with outdegree zero. Let $G'$ be the digraph obtained from $G$ where each red edge $xy$ is turned into two arcs $(x, y)$ and $(y, x)$, and each blue edge $uv \in E(T_i)$ is oriented in accordance to the orientation of $T_i$. Let $S$ be the set of vertices $v$ in $G'$ such that there is a directed path from a vertex $u \in V(R)$ to $s$ which contains $v$. Let $H_{R,F,s}$ be the graph in $G$ induced on the vertices of $S$.

We will call $R$ the **root component**. The next observation shows the importance of $H_{R,F,s}$.

**Observation 2.4.** Given a decomposition $(T_1, \ldots, T_k, F) \in F$, a root component $R$, and a vertex $s \in V(R)$, the graph $H_{R,F,s}$ satisfies

\[
\frac{e_b(H_{R,F,s})}{v(H_{R,F,s}) - 1} \leq \frac{d}{d + k + 1}.
\]

**Proof.** Suppose not, then

\[
\Gamma_f(G) \geq \frac{e_b(H_{R,F,s})}{v(H_{R,F,s}) - 1} + \frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} = k + \frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} > k + \frac{d}{d + k + 1}.
\]

Here

\[
\frac{e_b(H_{R,F,s})}{v(H_{R,F,s}) - 1} \geq k,
\]

since $H_{R,F,s}$ is an induced graph, so every vertex other than $s$ has $k$ outgoing blue edges (one for each tree $T_i$), and $s$ has zero outgoing blue edges. The strict inequality follows as we assumed that

\[
\frac{e_r(H_{R,F,s})}{v(H_{R,F,s}) - 1} > \frac{d}{d + k + 1}.
\]

However,

\[
\Gamma_f(G) \leq k + \frac{d}{d + k + 1},
\]

a contradiction. \hfill $\square$

As $H_{R,F,s}$ is defined by vertices reachable from directed paths from $s$, we can put a natural ordering on the red components of $H_{R,F,s}$.
Definition 2.5. Given the graph $H_{R,F,s}$, an ordering of the red components $(R_1, \ldots, R_t)$ is a legal order if all red components in $H_{R,F,s}$ are in the ordering, $R_1 = R$, for any component $R_i$ with $i > 1$, there exists an $R_j$ such that $j < i$ and there is a directed blue arc $(x,y)$ such that $x \in V(R_i)$ and $y \in V(R_j)$.

Naturally arising from the definition of legal order, we can define a notion of parent and child components.

Definition 2.6. Let $(R_1, \ldots, R_t)$ be a legal order. For a component $R_i$, a component $R_j$ is a parent of $R_i$ if $j < i$ and there is a blue directed arc $(x,y)$ where $x \in V(R_j)$ and $y \in V(R_i)$. If $R_j$ is a parent of $R_i$ then we say $R_i$ is a child of $R_j$.

Note that in the above definition, a component can have numerous parents. We will want to compare two different legal orders, and to do this we will use a lexicographic ordering. We recall the definition of a lexicographic ordering.

Definition 2.7. Given two sequences of integers $a = (a_1, \ldots, a_t)$ and $b = (b_1, \ldots, b_t)$, we say that $a$ is lexicographically smaller than $b$ if for the smallest index $i \in \{1, \ldots, t\}$ where $a$ and $b$ differ, we have $a_i < b_i$.

Definition 2.8. Given two legal orders $(R_1, \ldots, R_t)$ and $(R'_1, \ldots, R'_t)$, we say $(R_1, \ldots, R_t)$ is smaller than $(R'_1, \ldots, R'_t)$ if $(v(R_1), \ldots, v(R_t))$ is lexicographically smaller than $(v(R'_1), \ldots, v(R'_t))$.

In the above definition we required the legal orders to be of the same length, however clearly we can have two distinct legal orders with different lengths, and we would still like to be able to compare them. We can still compare two legal orders of different length, by simply extending the sequence with fewer entries with zeros until both sequences are the same length. Since the number of vertices in a component is always at least one, this implies that given two legal orders $(R_1, \ldots, R_t)$ and $(R'_1, \ldots, R'_t)$ such that $t < t'$ but $R_i = R'_i$ for all $i \in \{1, \ldots, t\}$, that $(R_1, \ldots, R_t)$ is smaller than $(R'_1, \ldots, R'_t)$.

We define a residue function, which simply measures how far away we are from a decomposition which satisfies Theorem 1.4.

Definition 2.9. Let $(T_1, \ldots, T_k, F) \in \mathcal{F}$. Let $\mathcal{T}$ be the set of components of $F$. The residue function $\rho$ is defined as

$$\rho(F) = \sum_{C \in \mathcal{T}} \max\{e(C) - d - c(d,k), 0\}.$$ 

Finally, we want to focus on red subgraphs which have few edges relative to the number of vertices, and to this end we make the following definition.

Definition 2.10. A subgraph $K$ is small if

$$\frac{e_n(K)}{v(K)} < \frac{d}{d + k + 1}.$$ 

Observe that when $K$ is connected, a small component satisfies $e(K) < \frac{d}{d + k + 1}$. In particular, as $d \leq k + 1$, the only small tree is an isolated vertex. This is the only fact that we exploit when $d \leq k + 1$.

Now we can describe how we will pick our minimal counterexample. First, we pick our counterexample such that the number of vertices is minimized. Second, pick a decomposition $(T_1, \ldots, T_k, F) \in \mathcal{F}$ which minimizes the residue function. Third, pick a red component $R$
with \( e(R) > d + c(d,k) \), a vertex \( s \in V(R) \), and an ordering of the red components of \( H_{R,F,s} \), so that \((R_1,\ldots,R_t)\) is the smallest legal order.

For the rest of the paper, we will assume we are working on a counterexample picked as described above.

3. Red components with few edges have large children

The purpose of this section is to prove that if a component \( R_j \) is a parent of a component \( R_i \), then \( e(R_j) + e(R_i) \geq d + c(d,k) \). To do so we build up a procedure to reconfigure the forest decomposition if this inequality fails.

The first definition is just notation to keep track of which vertices in a red component have blue arcs to a child component, where the head of the arc is in the parent component.

**Definition 3.1.** Let \((R_1,\ldots,R_t)\) be a legal order. Let \( R_i \) be a component such that \( i > 1 \). For each parent \( R_j \) of \( R_i \), let \( S_{ij} \) be the set of vertices in \( R_j \) such that \( x \in S_{ij} \) if there is a blue arc \((x,y)\) where \( x \in V(R_j) \) and \( y \in V(R_i) \). Let \( P \) denote the set parent components of \( R_i \). Then we define:

\[
S^i = \bigcup_{R_j \in P} S_{ij}.
\]

We say that \( S^i \) is the set of vertices which determine the legal order for \( R_i \).

We will need another definition which is just a data structure that will encode how we can modify the forest decomposition.

**Definition 3.2.** Let \( L = (R_1,\ldots,R_t) \) be a legal order. For each \( i \in \{2,\ldots,t\} \), pick an arbitrary vertex \( x_i \in S^i \). For this choice of vertices, we will say the auxiliary digraph for \( L \) denoted \( \text{Aux}(L) \), has vertex set \( V(H_{R,F,s}) \) and the edge set is obtained by including all red edges in \( H_{R,F,s} \), and for each \( i \in \{2,\ldots,t\} \), we include exactly one arc \((x_i,y)\) where \( y \in V(R_i) \). Then we direct all of the edges towards \( s \).

Note that \( \text{Aux}(L) \) is a tree. We will now run an algorithm on paths in \( \text{Aux}(L) \) to find “special paths” in \( G \) which will be used to modify the forest decomposition. The algorithm is just a convenient way to break up the dipath from a vertex \( x \) to \( s \) in \( \text{Aux}(L) \).

**Algorithm 1** Special Paths Algorithm

**Input**

A red component \( K \) and a red component \( C \) where \( C \) is a child of \( K \). An arc \((x,y)\) \( \in E(T_i) \), where \( x \in V(K) \) and \( y \in V(C) \).

**Output**

A set of paths \( P_1,\ldots,P_q \)

**Initialization**

A path \( Q := x_1,x_2,\ldots,x_q \) such that \( x_1 := x \), \( x_q := s \) and \( Q \) is the unique path from \( x \) to \( s \) in \( \text{Aux}(L) \)

Index counter \( w := i \)

Vertices \( u := x \), \( v := y \)

Path \( P_1 := y,x \)

Paths \( P_2,\ldots,P_q \) defined to be empty paths

**Begin**

for \( j = 1 \) to \( q - 1 \) do
\begin{algorithm}
\begin{algorithmic}
\IF $(x_j, x_{j+1}) \in E(F)$ \AND $(u, v)$ is in the fundamental cycle of $T_w \cup \{u, v\}$\THEN
\STATE $P_w := P_w \cup \{x_{j+1}\}$
\STATE Break and return: $P_1, \ldots, P_q$
\ENDIF
\IF $(x_j, x_{j+1}) \in E(F)$ \AND $(u, v)$ is not in the fundamental cycle of $T_w \cup \{u, v\}$\THEN
\STATE $P_w := P_w \cup \{x_{j+1}\}$
\ENDIF
\IF $(x_j, x_{j+1}) \in E(T_w)$\THEN
\STATE $P_w := P_w \cup \{x_{j+1}\}$
\ENDIF
\IF $(x_j, x_{j+1}) \in E(T_{i'}), i' \neq w$, \AND $(x_j, x_{j+1})$ is in the fundamental cycle of $T_w \cup \{u, v\}$\THEN
\STATE $P_w := P_w \cup \{x_{j+1}\}$
\ENDIF
\ENDFOR
\end{algorithmic}
\end{algorithm}

As the algorithm includes empty paths, we make the following definition which simply takes the non-empty output from the algorithm.

\textbf{Definition 3.3.} Let $(x, y) \in E(T_i)$ be a blue directed arc, \AND suppose that $x \in V(K)$ \AND $y \in V(C)$ \such that $K$ is a parent of $C$. \Let $P_1, \ldots, P_q$ be the paths output by the special paths algorithm, \\AND further suppose that $P_1, \ldots, P_w$ are the non-empty paths output by the algorithm. \Then we say that $P_1, \ldots, P_w$ are \textbf{special paths with respect to $(x, y)$}.

We make a series of claims about special paths and the special paths algorithm.

\textbf{Lemma 3.4.} Let $(x, y) \in E(T_i)$ such that $x \in V(K)$, $y \in V(C)$ \AND $K$ is a parent of $C$. \Let $P_1, \ldots, P_w$ be special paths with respect to $(x, y)$. \Then the last edge added to $P_w$ is $E(F)$.

\textit{Proof.} The algorithm ends either when the for loop ends, \OR if we find a $j$ \such that $(x_j, x_{j+1}) \in E(F)$ \AND $(u, v)$ is in the fundamental cycle of $T_w \cup \{u, v\}$. \If the latter occurs, $(x_j, x_{j+1})$ is in $E(F)$, \AND hence the claim holds. \Otherwise, the last edge added is $(x_{q-1}, x_q)$. \Since $x_q = s$, \AND $s$ has out degree zero in $H_{R,F,s}$ \for each of the spanning trees, $(x_{q-1}, x_q)$ is a red edge (recall the non-red edges have their orientation reversed in $\text{Aux}(L)$).

\textbf{Lemma 3.5.} Let $(x, y) \in E(T_i)$ such that $x \in V(K)$, $y \in V(C)$ \AND $K$ is a parent of $C$. \Let $P_1, \ldots, P_w$ be special paths with respect to $(x, y)$. \Suppose $P_1 = x_{a_0-1}, x_{a_0}, \ldots, x_{a_1}$. \Suppose for $m \in \{2, \ldots, w\}$ we have $P_m = x_{a_{m-1}-1}, x_{a_{m-1}}, \ldots, x_{a_m}$. \Further suppose that for $m \in \{0, \ldots, w-1\}$, that $(x_{a_{m-1}}, x_{a_m}) \in E(T_{a_m})$. \Then for $m \in \{0, \ldots, w-1\}$, $(x_{a_{m-1}}, x_{a_m})$ is in the fundamental cycle of $T_{a_m} \cup \{(x_{a_{m+1}-1}, x_{a_{m+1}})\}$.
Proof. Fix \( m \in \{1, \ldots, w\} \). Note that the algorithm only operates on path \( P_m \) during the iterations \( a_{m-1} \) to \( a_m - 1 \). During these iterations, \( u = x_{am-1} - 1 \) and \( v = x_{am-1} \). Then if \( m < w \), by construction the only if statement which change what path we are extending enforces that \((x_{am-1} - 1, x_{am})\) is in the fundamental cycle of \( T \cup \{(x_{am-1} - 1, x_{am})\} \).

Now suppose that \( m = w \), and the algorithm ended by breaking out of the first if statement. Then the claim holds by the condition of the if statement. Finally, we note that the algorithm always ends by breaking out of the first if statement. To see this, note that if we do not break out of the algorithm earlier, all vertices \( v \in \{x_{a_{w-1}}, \ldots, x_{aw}\} \) must contain \( x_{aw-1} \) and \( x_{aw-1} + 1 \) on the directed path from \( v \) to \( s \) in \( T_{aw-1} \). But \( x_{aw} = s \), and all non-red edges are directed towards \( s \) in \( H_{R,F,s} \), so given the previous conditions, \((x_{aw-1}, x_{aw})\) satisfies the fundamental cycle condition of the first if statement. \( \square \)

Now we can see the main purpose of the special paths algorithm.

**Lemma 3.6.** Let \((x, y) \in E(T_i)\) be a blue directed edge such that \( x \in V(K) \), \( y \in V(C) \) where \( K \) is a parent of \( C \). Let \((T_1, \ldots, T_k, F)\) be the decomposition of \( G \). Then there is a decomposition in \( F \) which can be obtained from \((T_1, \ldots, T_k, F)\) by only modifying edges in the special paths with respect to \((x, y)\), with the property that either this decomposition admits a smaller legal order, or this decomposition splits the root \( R \) into multiple components, such that the resulting component containing \( s \) has at most \( d \) edges.

**Proof.** Let \( P_1, \ldots, P_w \) be special paths with respect to \((x, y)\). Suppose \( P_1 = x_{a_0-1}, x_{a_0}, \ldots, x_{a_1} \) such that \( x_{a_0-1} = y \) and \( x_{a_0} = x \). Suppose for \( j \in \{2, \ldots, w\} \) we have \( P_j = x_{aj-1}, x_{aj}, \ldots, x_{aj+1} \) such that \((x_{aj-1}, x_{aj}) \in E(T_{aj-1}')\).

Now consider the following algorithm: starting with \( j = w \), add \((x_{aj-1}, x_{aj})\) to \( T_{aj-1}' \) and add \((x_{aj-1}, x_{aj})\) to \( F \). By Lemma 3.5 and Lemma 3.4 the result is a decomposition in \( F \). Now repeat the same procedure with \( j \) replaced by \( j - 1 \). Continuing this until \( j = 1 \), we end up adding \((x, y)\) to \( F \) and by appealing to Lemma 3.5 at each iteration, the resulting decomposition is in \( F \). If this series of exchanges results in the root component splitting such that the component containing \( s \) only has \( d \) edges, then we are done (here for each decomposition, we always designate the root component to be the component containing \( s \), and we always pick \( s \) to construct \( H_{F,R,s} \)). Otherwise, to see that this decomposition has a smaller legal order, observe that we can take the same legal order up to the component containing \( x_{aw} \). Further since we changed \((x_{aw-1}, x_{aw})\) from an edge in \( F \) to an edge in one of the spanning trees, we can add the component containing \( x_{aw} \) to the legal order, and this component has strictly fewer edges than before. Now complete this partial legal order arbitrarily. Since \( P_1, \ldots, P_w \) were constructed from a path from \( x \) to \( s \) in \( Aux(L) \), this component is either a subgraph of the component containing \( x \), or a component which is closer to the root in the legal order than the component containing \( x \). In either case, the legal order decreased. \( \square \)

If we can apply Lemma 3.6 without increasing the residue function, then we contradict our choice of decomposition. The next corollary simply points out an important situation when we can do this.

**Corollary 3.7.** There are no red components \( K \) with a child \( C \) such that \( e(K) + e(C) < d + c(d, k) \).

**Proof.** Suppose so. Let \((x, y)\) be an edge in \( T_i \) such that \( x \in V(K) \) and \( y \in V(C) \). Then we can apply Lemma 3.9 to \((x, y)\), \( K \) and \( C \). Since \( e(K) + e(C) < d + c(d, k) \), adding the
edge \((x,y)\) to \(F\) results in a red component which has at most \(d + c(d,k)\) edges. Hence the decomposition obtained by Lemma 3.6 does not increase the residue function. But the new decomposition either reduced the residue function value (as the root has more than \(d + c(d,k)\) edges), or admits a smaller legal order. In either case this is a contradiction. \(\square\)

One important consequence of this corollary is that any small red component has no small children.

4. Bounding the number of small children from a parent

The purpose of this section is to prove that given a red component \(K\), that the component has a bounded number of small children.

As notation, let \(K\) be a red component of \(F\) and for any vertices \(u,v \in V(K)\), we let \(P_{u,v}\) denote the unique path in \(K\) from \(u\) to \(v\). For any tree \(T \in \{T_1,\ldots,T_k\}\), and arbitrary vertices \(u,v\) we let \(P^T_{u,v}\) denote the unique path from \(u\) to \(v\) in \(T\) (note this is well defined as \(T\) is a spanning tree).

The following observation is straightforward and is used implicitly throughout this section.

**Observation 4.1.** Let \(K\) be a red component with two distinct children \(C_1\) and \(C_2\). Further suppose that for some tree \(T \in \{T_1,\ldots,T_k\}\), there are arcs \((x,x'),(y,y') \in E(T)\) such that \(x,y \in V(K)\) and \(x' \in V(C_1)\), \(y' \in V(C_2)\). Then \(x\) and \(y\) are distinct vertices.

**Proof.** Each vertex has outdegree at most one in \(T\), and hence \(x \neq y\). \(\square\)

The next lemma describes an easy situation where we can modify the forest decomposition and obtain a contradiction.

**Lemma 4.2.** Let \(K\) be a red component with two distinct small children \(C_1\) and \(C_2\). Further suppose for some tree \(T \in \{T_1,\ldots,T_k\}\), there are arcs \((x,x'),(y,y') \in E(T)\) such that \(x,y \in V(K)\) and \(x' \in V(C_1)\), \(y' \in V(C_2)\). If there exists an edge \(e \in E(P_{x,y})\) such that \((x,x')\) is in the fundamental cycle of \(T \cup \{e\}\), then the component of \(K - e\) containing \(y\), say \(K'\), satisfies \(e(K') \leq e(C_1)\).

**Proof.** Suppose towards a contradiction that \(e(K') > e(C_1)\). Consider the decomposition where we add \(e\) to \(T\) and \((x,x')\) to \(F\). This is a forest decomposition as we assumed \((x,x')\) is in the fundamental cycle of \(T \cup \{e\}\). Let \(K''\) denote the resulting red component containing \(x\).

First suppose that both \(e(K'')\) and \(e(K')\) have more than \(d\) edges. We claim that the residue function decreased. A quick calculation shows this:

\[
e(K) - d - c(d,k) = e(K'') + e(K') - e(C_1) - d - c(d,k)
\geq e(K'') + e(K') - \frac{d}{k+1} - d - c(d,k)
> e(K'') + e(K') - 2d - 2c(d,k).
\]

Now suppose that \(e(K'')\) has at most \(d + c(d,k)\) edges and \(e(K') > d + c(d,k)\). In this case the residue function decreases as \(e(K') < e(K)\).

Now suppose that \(e(K'')\) has more than \(d + c(d,k)\) edges and \(K'\) has at most \(d + c(d,k)\) edges. Then the residue function decreases as \(e(K'') < e(K)\), since \(e(K') > e(C_1)\).

Finally, suppose that both \(e(K'')\) and \(e(K')\) have at most \(d\) edges. We may assume that \(K\) was not the root, as otherwise the residue function decreased. We claim we can find a smaller
legal order. To see this, consider taking the same legal order up till $K$ (observe this is possible as we do not change any edges in components before $K$), and then replacing $K$ with either $K'$ or $K''$ (at least one of these is possible, as $K$ had a parent component). Now, if we can replace $K$ with $K'$, then as $e(K') < e(K)$ we can extend this legal order to a smaller legal order. Thus we must only be able to replace $K$ with $K''$. But as $e(K'') > e(C_1)$, we have $e(K'') < e(K)$, so again we can extend this legal order to a smaller legal order, a contradiction. 

The next lemma considers an ideal situation for a red component where we can always find a better forest decomposition.

**Lemma 4.3.** Let $K$ be a red component with two distinct small children $C_1$ and $C_2$. Further suppose for some tree $T \in \{T_1, \ldots, T_k\}$, there are arcs $(x, x'), (y, y') \in E(T)$, such that $x, y \in V(K)$ and $x' \in V(C_1)$, $y' \in V(C_2)$. Suppose there is an edge $e \in E(P_{x,y})$ such that $(x, x')$ is in the fundamental cycle of $T \cup \{e\}$, and further there is an edge $e' \in E(P_{x,y})$ such that $(y, y')$ is in the fundamental cycle of $T \cup \{e\}$. Then there is a decomposition in $\mathcal{F}$ with either a smaller residue function value, or a smaller legal order.

**Proof.** We prove the following claim first.

**Claim 4.4.** There is an edge $e = uv \in E(P_{x,y})$ with the following properties. The distance from $u$ to $x$ in $P_{x,y}$ is smaller than the distance from $x$ to $v$ in $P_{x,y}$. There is an edge $e' \in E(P_{x,y})$ such that $(x, x')$ is in the fundamental cycle of $T \cup \{e\}$. There is an edge $e'' \in E(P_{x,y})$ such that $(y, y')$ is in the fundamental cycle of $T \cup \{e''\}$.

**Proof.** Let $P_{x,y} = x_1, x_2, \ldots, x_q$ such that $x_1 = x$ and $x_q = y$. Let $i$ be the smallest index such that $P_{x_i, s}$ does not contain $x$ and $x'$. Such an index exists, as there is an edge $e \in E(P_{x,y})$ such that $(x, x')$ is in the fundamental cycle of $T \cup \{e\}$. Then $(x, x')$ is in the fundamental cycle of $T \cup \{(x_i-1, x_i)\}$ (clearly $i = 1$ is not the smallest index, and so $i - 1 \geq 1$).

We break the argument into a few cases.

First suppose that $P_{x_i, s}$ contains $s$. Observe that if there is no edge $e \in E(P_{y,x_i-1})$ such that $(y, y')$ lies in the fundamental cycle of $T \cup \{e\}$, then for all vertices $v \in V(P_{y,x_i-1})$ we have that $P_{v, s}$ contains $y$ and $y'$. But then $P_{y, x_i}$ does not contain $s$, a contradiction.

Now suppose that $P_{x_i, s}$ contains $x$. If $P_{x_i-1, s}$ contains $x$ but not $x_i$, then $(y, y')$ is in the fundamental cycle of $T \cup \{(x_i-1, x_i)\}$. Thus we can assume that $P_{x_i-1, s}$ contains $x_i$. Therefore if for all of the edges $e \in E(P_{x,y})$ we have that $(y, y')$ is not in the fundamental cycle of $T \cup \{e\}$, then for all $v \in V(P_{x,y})$ we have $P_{v, s}$ contains $y$, and since $i$ was chosen to be the smallest index, this implies that there are no edges in $P_{x,y}$ such that $(y, y')$ lies in their fundamental cycle, a contradiction.

Now suppose that $P_{y, s}$ contains $x_i$. Then if all of the edges $e \in E(P_{y, x_i})$, have the property that $(y, y')$ is not in the fundamental cycle of $T \cup \{e\}$, then every vertex $v \in V(P_{y, x_i})$ has the property that $P_{v, s}$ contains $y$, but $x_i \in V(P_{y, x_i})$, a contradiction. The claim follows.

Let $uv \in E(P_{x,y})$ be an edge guaranteed from the claim. Let $e \in E(P_{x,y})$ where $(x, x')$ is in the fundamental cycle of $T \cup \{e\}$, and let $e' \in E(P_{u,y})$ where $(y, y')$ is in the fundamental cycle of $T \cup \{e\}$.

Let $K'$ be the component of $K - e$ which contains $x$, and $K''$ be component of $K - e$ not containing $x$.

**Case 1:** $e(K'') > e(C_1)$.
This case follows immediately from Lemma 4.2.

**Case 2:** \( e(K'') \leq e(C_1) \)

As \( C_1 \) is small, this implies that \( e(K'') < \frac{d}{d+1} \). Now add \( e' \) to \( T \) and \((y, y')\) to \( F \). Let \( K''' \) be the component comprised of \( K'' \), \((y, y')\) and \( C_2 \). Observe that \( e(K''') < d \) as \( C_2 \) is small and \( e(K'') < \frac{d}{d+1} \). Hence if \( K \) had more than \( d + c(d, k) \) edges, the resulting decomposition has a smaller residue function. If \( K \) had at most \( d + c(d, k) \) edges, and \( e(K''') < e(K) \), then we obtain a smaller legal order simply by taking the same legal order up to \( K \), and then picking either \( K''' \) or \( K' \), whichever one is possible, and completing the order arbitrarily. If \( e(K''') \geq e(K) \) and all vertices which determine the legal order for \( K \) do not lie in \( K''' \), then instead of adding \( e' \) to \( T \) and \((y, y')\) to \( F \), add \( e \) to \( T \) and \((x, x')\) to \( F \). Observe that we do not increase the residue function, and also we can find a smaller legal order by taking the same legal order up to \( K \), then replacing \( K \) with \( K' \) and completing the order arbitrarily. \( \square \)

While unfortunately the situation in Lemma 4.3 does not always occur (if it did, then the Strong Nine Dragon Tree Conjecture would follow), we can obtain something weaker.

**Observation 4.5.** Let \( K \) be a red component with two distinct children \( C_1 \) and \( C_2 \). Further suppose for some tree \( T \in \{T_1, \ldots, T_k\} \), there are arcs \((x, x'), (y, y') \in E(T)\), such that \( x, y \in V(K) \) and \( x' \in V(C_1) \), \( y' \in V(C_2) \). Then there exists an edge \( e \in E(P_{x,y}) \) such that one of \((x, x')\) is in the fundamental cycle of \( T \cup \{e\}\), or \((y, y')\) is in the fundamental cycle of \( T \cup \{e\}\).

**Proof.** Suppose that for any edge \( e \in E(P_{x,y}) \), \((x, x')\) does not lie in the fundamental cycle of \( T \cup \{e\} \). Then every vertex \( v \in V(P_{x,y}) \setminus \{x\} \) has \( v \in V(P_{v,s}^T) \). If for every edge \( e \in E(P_{x,y}) \), \((y, y')\) does not lie in the fundamental cycle of \( T \cup \{e\} \), then similarly we have that for every vertex \( v \in V(P_{x,y}) \setminus \{y\} \), \( v \in V(P_{v,s}^T) \). But then \( P_{x,s}^T \) contains \( y \), and \( P_{y,s}^T \) contains \( x \), a contradiction. \( \square \)

Now we show that any component has at most \( k \) small children which are isolated vertices.

**Lemma 4.6.** Let \( K \) be a red component. Then \( K \) has at most \( k \) small children \( C \) such that \( e(C) = 0 \).

**Proof.** Suppose towards a contradiction that \( K \) has at least \( k + 1 \) small children which are isolated vertices. Then there is a tree \( T \in \{T_1, \ldots, T_k\} \) such that there are two vertices \( x, y \in V(K) \) with arcs \((x, x'), (y, y') \in E(T)\) where \( x' \) and \( y' \) are isolated small children of \( K \). By Lemma 4.3 we can assume without loss of generality that for any edge \( e \in E(P_{x,y}) \), \((y, y')\) is not in the fundamental cycle of \( T \cup \{e\}\). Therefore for every vertex \( v \in V(P_{x,y}) \) we have that \( y \) is in contained in \( P_{v,s}^T \).

By Observation 4.5 there is an edge \( e \in E(P_{x,y}) \) such that \((x, x')\) lies in the fundamental cycle of \( T \cup \{e\} \). By Lemma 4.2 we can assume that \( e = wy \) and that \( y \) is a leaf in \( K \).

Now let \( P_1, \ldots, P_n \) be the special paths with respect to \((y, y')\). Observe that the decomposition obtained when applying Lemma 3.6 to the special paths, we do not modify any vertex on \( P_{x,y} \) and also would not modify any vertex which appears in a red component later in the legal order than \( K \). This follows since for every vertex \( v \in V(P_{x,y}) \) we have that \( y \) is in contained in \( P_{v,s}^T \), and since the special path algorithm will will first consider the tree \( T \).

Now add \( e \) to \( T \), and add \((x, x')\) to \( F \). As \( x' \) is an isolated vertex, the residue function does not increase (if it decreases then we are done, so we can assume the residue function stays the same). Now apply Lemma 3.6 to the decomposition and let \( F' \) be the resulting decomposition.
If the resulting decomposition has a smaller legal order, then since the $K \setminus \{y\} \cup \{x'\}$ was not modified by Lemma 3.6, $F'$ has a smaller legal order than the original decomposition, a contradiction. If Lemma 3.6 modifies the root, then the residue function of $F'$ is smaller than the original decomposition, also a contradiction. The lemma follows. □

Corollary 4.7. Suppose $d \leq k + 1$. Then any red component has at most $k$ small children.

Proof. Note that $\frac{d}{k+1} \leq 1$. Hence all small children are isolated vertices. Hence this follows immediately from Lemma 4.6 □

The above lemmas suffice to prove the Strong Nine Dragon Tree Conjecture when $d \leq k+1$. Now we prove a bit more to obtain the approximate version.

Lemma 4.8. Let $K$ be a red component with two distinct children $C_1$ and $C_2$. Further suppose for some tree $T \in \{T_1, \ldots, T_k\}$, there are arcs $(x, x'), (y, y') \in E(T)$, such that $x, y \in V(K)$ and $x' \in V(C_1)$, $y' \in V(C_2)$. Then there are at most $\left\lceil \frac{d}{k+1} + 2 \right\rceil$ distinct small children $C_i$ of $K$ such that there are distinct vertices $v \in V(P_{x,y})$ and arcs $(v, u) \in E(T)$, where $u \in V(C_i)$.

Proof. Suppose not. Let $P_{x,y} = x_1, \ldots, x_q$ where $x_1 = x$ and $x_q = y$. Let $x_{j_1}, \ldots, x_{j_2}$ be the subsequence of $P_{x,y}$ such that each of $x_{j_i}$ has an edge to a small child of $K$, and $x_{j_1}$ and $x_{j_2}$ are the only vertices of $x_1, \ldots, x_q$ contained in $V(P_{x_{j_1}, x_{j_2}})$. Suppose that the edge $(x_{j_1}, x_{j_2}) \in E(T)$ is an arc from $K$ to a small child of $K$. Observe that $j_2 \geq \left\lceil \frac{d}{k+1} + 2 \right\rceil$

If there is an edge $e \in E(P_{x_{j_1}, x_{j_2}})$ such that $(x_{j_1}, x_{j_2})$ is in the fundamental cycle of $T \cup \{e\}$, then the component of $K - e$ not containing $x_{j_1}$ contains at least $\left\lceil \frac{d}{k+1} + 1 \right\rceil$ vertices, and hence at least $\left\lceil \frac{d}{k+1} \right\rceil$ edges. Thus by Lemma 4.2 we obtain a better decomposition, a contradiction.

By Observation 3.5, there is an edge $e \in E(P_{x_{j_1}, x_{j_2}})$ such that $(x_{j_2}, x_{j_2}')$ is in the fundamental cycle of $T \cup \{e\}$. As we may assume the hypothesis of Lemma 4.3 does not hold, we get that there is an edge $e \in E(P_{x_{j_2-1}, x_{j_2}})$ such that $(x_{j_2}, x_{j_2}')$ is in the fundamental cycle of $T \cup \{e\}$. But then by the same reasoning as above, we can apply Lemma 4.2 and obtain a better decomposition, a contradiction. □

Lemma 4.9. Let $K$ be a red component. There is no vertex $t \in V(K)$ such that there are more than $\left\lceil \frac{d}{k+1} + 2 \right\rceil$ vertices $x_1, \ldots, x_n$ with the following properties. The paths $P_{x_1,t}$ and $P_{x_i,t}$ satisfy $V(P_{x_1,t}) \cap V(P_{x_i,t}) = \{t\}$ if $i \neq j$. There is a tree $T \in \{T_1, \ldots, T_k\}$ where there is an arc $(x_i, x_i') \in E(T)$ so that $x_i' \in E(C_i)$ and $C_i$ is a small child of $K$. Further $C_i$ is distinct from $C_j$ if $i \neq j$.

Proof. Suppose towards a contradiction that such a vertex $t$ exists. By Lemma 4.3, we can assume (up to relabeling indices) that the vertices $x_1, \ldots, x_n$ satisfy $x_{i+1} \in V(P_{x_1,t})$ for $i \in \{1, \ldots, n-1\}$ and that for each $x_i$, $i < n$, there is an edge $e_i \in P_{x_i,x_{i+1}}$ such that $(x_i, x_i')$ is in the fundamental cycle of $T \cup \{e\}$. We may assume that Lemma 4.2 does not apply to any of these edges.

Observe that for any $x_i$, we have $e_i \in E(P_{t,x_{i+1}})$. For if not, the component of $K - e_i$ which does not contain $x_i$ has at least $\left\lceil \frac{d}{k+1} \right\rceil$ vertices, and hence at least $\left\lceil \frac{d}{k+1} - 1 \right\rceil$ edges, and so Lemma 4.2 applies. Hence $e_i \neq e_j$ if $i \neq j$.

Now starting with $i = 1$ till $i = n - 1$, add $(x_i, x_i')$ to $F$, and add $e_i$ to $T$. After this, the resulting red component containing $t$ has at least one fewer edge, as we removed at least $\left\lceil \frac{d}{k+1} \right\rceil$ edges, and gained at most $\left\lceil \frac{d}{k+1} - 1 \right\rceil$ edges, and all other resulting components have less than $d$ edges, by Lemma 4.2 and the fact that small children have less than $\frac{d}{k+1}$ edges. It
follows that either we reduced the residue function, or we can find a smaller legal order. In either case, we obtain a contradiction.

We state the following easy observation.

**Observation 4.10.** Let \( s \) and \( t \) be arbitrary positive integers. Every connected graph on \( s \) vertices either has a vertex of degree \( s \), or a path of length \( t \).

With these we can bound the number of small children any red component has.

**Corollary 4.11.** Let \( K \) be a red component. Then \( K \) has at most \( k(\lceil \frac{d}{k+1} + 2 \rceil) \lceil \frac{d}{k+1} + 2 \rceil \) small children.

**Proof.** Suppose not. Then there is a tree \( T \in \{T_1, \ldots, T_k\} \) such that there are at least \( (\lceil \frac{d}{k+1} + 2 \rceil) \lceil \frac{d}{k+1} + 2 \rceil \) distinct vertices in \( K \) where each vertex has an arc in \( T \) to a distinct small child of \( K \). Contract edges of \( K \) such that in the resulting graph, only vertices with arcs in \( T \) to small children are left. Then by applying Observation 4.10 we have a vertex of degree at least \( \lceil \frac{d}{k+1} + 2 \rceil \) or a path of length \( \lceil \frac{d}{k+1} \rceil \). If the path exists, then in \( K \) we can apply Lemma 4.8. If the vertex of degree at least \( \lceil \frac{d}{k+1} + 2 \rceil \) exists, then we can apply Lemma 4.9. In either case we obtain a contradiction. \( \square \)

5. A counting argument and finishing the proof

This section starts with a counting argument which shows that if we all red components which are not small with its small children, then the resulting subgraph has fractional arboricity at least \( \frac{d}{d+k+1} \). Using this fact plus the fact that the root has at least \( d+1 \) edges, we conclude the theorem. We start with a definition to make the argument less cumbersome.

**Definition 5.1.** Let \( K \) be a red component, and \( K_1, \ldots, K_q \) be the small children of \( K \). Then we let \( K_C \) be the subgraph whose vertices is \( V(K) \cup V(C_1) \cdots \cup V(C_q) \) and contains all red edges on that vertex set.

**Lemma 5.2.** Let \( K \) be a red component which is not small, and has small children \( K_1, \ldots, K_q \). Suppose that \( e(K) + e(K_i) \geq d + c \) for all \( i \in \{1, \ldots, q\} \) for a fixed constant \( c \geq \frac{d(q-k)}{k+1} \). Then \( K_C \) is not small. Further, if \( e(K) > d + \frac{(q-k)}{k+1} \), then

\[
\frac{e(K_C)}{v(K_C)} > \frac{d}{k + d + 1}.
\]

**Proof.** From the hypothesis, we have that for all \( i \in \{1, \ldots, q\} \), \( e(K_i) \geq \max\{0, d + c - e(K)\} \). Expanding the definitions we have:

\[
\frac{e(K_C)}{v(K_C)} = \frac{e(K) + \sum_{i=1}^{q} e(K_i)}{v(K) + \sum_{i=1}^{q} v(K_i)} \\
\geq \frac{e(K) + \sum_{i=1}^{q} \max\{0, d + c - e(K)\}}{e(K) + 1 + q + \sum_{i=1}^{q} \max\{0, d + c - e(K)\}}.
\]

We split into two cases depending on the value of \( \max\{0, d + c - e(K)\} \).

**Case 1:** \( \max\{0, d + c - e(K)\} = 0 \).
Then since $e(K) + e(K_i) \geq d$, it follows that $e(K) \geq d$. Thus it follows that,
\[
\frac{e(K) + \sum_{i=1}^{q} \max\{0, d - e(K_i)\}}{e(K) + 1 + q + \sum_{i=1}^{q} \max\{0, d - e(K_i)\}} \geq \frac{e(K)}{e(K) + q + 1} \geq \frac{d + c}{d + c + q + 1}.
\]

Observe that the inequality
\[
\frac{d + c}{d + c + q + 1} \geq \frac{d}{k + d + 1},
\]
holds if and only if
\[(d + k + 1)(d + c) \geq d(d + c + q + 1).
\]
Simplifying we get
\[kc + c \geq dq - kd.
\]
Solving for $c$, we get that
\[c \geq \frac{dq - kd}{k + 1}.
\]
This holds by the hypothesis, and so the claim follows in this case. Finally, observe that if $e(K) > d + c$, the above inequalities are strict.

**Case 2:** $\max\{0, d + c - e(K)\} = d + c - e(K)$.

Calculating we obtain,
\[
\frac{e(K) + \sum_{i=1}^{q} \max\{0, d + c - e(K_i)\}}{e(K) + 1 + q + \sum_{i=1}^{q} \max\{0, d + c - e(K_i)\}} \geq \frac{e(K) + q(d + c - e(K))}{e(K) + q(d + c - e(K)) + q + 1} \geq \frac{d - e(K) + e(K)}{e(K) + d + c - e(K) + q + 1} = \frac{d + c}{d + c + q + 1}.
\]

By the same argument in case one, $\frac{d + c}{d + c + q + 1} \geq \frac{d}{d + k + 1}$ and again if $e(K) > d + c$, the above inequalities are strict.

Now we finish the proof. Let $\mathcal{R}$ denote the set of red components which are not small. It follows that,
\[V(H) = \bigcup_{K \in \mathcal{R}} V(K_C).
\]

This follows since a small component cannot have a small child by Lemma 3.7. Therefore it follows that:
\[E_r(H_{R,F,s}) = \bigcup_{K \in \mathcal{R}} E(K_C).
\]

When $d \leq k + 1$, by Corollary 4.7 every red component has at most $k$ small children. Thus $K_C$ is not troublesome by Lemma 5.2 (and appealing to Corollary 3.7). Hence the fractional arboricity of $H_{R,F,s}$ must be at least $k + \frac{d}{d+k+1}$. Furthermore, as the root has at least $d + 1$ edges, Lemma 5.2 implies that the fractional arboricity of $H_{R,F,s}$ is strictly larger than $k + \frac{d}{d+k+1}$, contradicting our assumption. The case when $d \leq k + 1$ for Theorem 1.4 follows.

For arbitrary $k$ and $d$, we appeal to Corollary 4.11, Lemma 3.7 and Lemma 5.2 to see that
the fractional arboricity of $H_{R,F,s}$ must be at least $k + \frac{d}{d(k+1)}$. Again, as the root component has at least $d + c(d, k) + 1$ edges, strict inequality holds, and hence Theorem 1.4 follows.

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