Systematics of IIB spinorial geometry

U Gran\textsuperscript{1}, J Gutowski\textsuperscript{2}, G Papadopoulos\textsuperscript{1} and D Roest\textsuperscript{1}

\textsuperscript{1} Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK
\textsuperscript{2} DAMTP, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, UK

Received 14 November 2005, in final form 22 December 2005
Published 15 February 2006
Online at stacks.iop.org/CQG/23/1617

Abstract
We reduce the classification of all supersymmetric backgrounds of IIB supergravity to the evaluation of the Killing spinor equations and their integrability conditions, which contain the field equations, on five types of spinors. This extends the work of Gran et al (2005 Class. Quantum Grav. 22 118 (Preprint hep-th/0503046)) to IIB supergravity. We give the expressions of the Killing spinor equations on all five types of spinors. In this way, the Killing spinor equations become a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. This system can be solved to express the fluxes in terms of the geometry and determine the conditions on the geometry of any supersymmetric background. Similarly, the integrability conditions of the Killing spinor equations are turned into a linear system. This can be used to determine the field equations that are implied by the Killing spinor equations for any supersymmetric background. We show that these linear systems simplify for generic backgrounds with maximal and half-maximal number of $H$-invariant Killing spinors, $H \subset \text{Spin}(9, 1)$. In the maximal case, the Killing spinor equations factorize, whereas in the half-maximal case they do not. As an example, we solve the Killing spinor equations of backgrounds with two $SU(4) \ltimes \mathbb{R}^8$-invariant Killing spinors. We also solve the linear systems associated with the integrability conditions of maximally supersymmetric $\text{Spin}(7) \ltimes \mathbb{R}^8$—and $SU(4) \ltimes \mathbb{R}^8$—backgrounds and determine the field equations that are not implied by the Killing spinor equations.

PACS numbers: 11.25.—w, 11.25.Yb

1. Introduction

Supersymmetric solutions of IIB supergravity have found widespread applications in string theory and gauge theories. Some of these solutions have been discovered in the context of branes; see e.g. [1–3] and in the context of AdS/CFT correspondence [4], see e.g. [5–9]. Most of these results rely on Ans"atze appropriately adapted to the requirements of the physical problems. Progress has also been made towards a systematic understanding of
the supersymmetric solutions of IIB supergravity. The maximally supersymmetric solutions of IIB supergravity have been classified in [10] and they have been found to be locally isometric to Minkowski space, $AdS_5 \times S^5$ [5] and a maximally supersymmetric plane wave [9]. In addition, these backgrounds are related by Penrose limits [11]. More recently, the Killing spinor equations of IIB have been solved for one Killing spinor [12, 13], and for all supersymmetric backgrounds with two $Spin(7) \ltimes \mathbb{R}^8$-invariant spinors, and four $SU(4) \ltimes \mathbb{R}^8$- and $G_2$-invariant spinors [13].

In the spinorial geometry approach to supersymmetric backgrounds [14], the Killing spinor equations of $M$-theory and their integrability conditions for any number of supersymmetries turn into linear systems [15]. The linear system of the Killing spinor equations can be solved to express the fluxes of the theory in terms of the geometry and to find the conditions on the geometry imposed by supersymmetry for any number of Killing spinors. The linear system associated with the integrability conditions determines the field equations that are implied by the Killing spinor equations. The main purpose of this paper is to adapt the above results to the Killing spinor equations of IIB supergravity and their integrability conditions. The construction relies on the linearity of the Killing spinor equations and the observation that the IIB Killing spinor equations of any spinor can be determined from those of five types of spinors. These five types of spinors are

$$1, \quad e_{ij}, \quad e_{1234}, \quad e_{i5}, \quad e_{ijk5}, \quad i, j, k = 1, \ldots, 4,$$

which we denote collectively by $\sigma_I$, where we have expressed the spinors in terms of forms. For IIB supergravity, this has been explained in [12], see also appendix A. We evaluate the Killing spinor equations of IIB supergravity on all five types of spinors and express the result in terms of an oscillator basis in the space of spinors. In this way, we can construct a linear system associated with the Killing spinor equations of backgrounds with any number of Killing spinors. This linear system can be used to determine the fluxes in terms of the geometry and the conditions on the geometry imposed by supersymmetry. In IIB supergravity, it is first convenient to solve for the complex fluxes, i.e. the three-form field strength $G$ and the one-form field strength $P$ associated with the two scalars. Then the remaining equations determine some of the components of the five-form flux $F$ and constrain the geometry of spacetime.

The Killing spinor equations of supergravity theories imply some of the field equations. In IIB supergravity, this is related to the computation of the field equations from the commutator of supersymmetry transformations [5]; see also [16]. We identify the integrability conditions $I$ and $I_A$ that contain the field equations and the Bianchi identities\(^3\). Then, we show that the integrability conditions of a Killing spinor can be expressed in terms of those on five types of spinors $\sigma_I$. We evaluate $I \sigma_I$ and $I_A \sigma_I$ in terms of an oscillator basis in the space of spinors and thus derive a linear system. This linear system can be used to determine which field equations and Bianchi identities are implied by the Killing spinor equations for backgrounds with any number of Killing spinors.

The main purpose of this paper is to be used as a manual to solve the Killing spinor equations of IIB supergravity for backgrounds with any number of Killing spinors, and to determine the field equations that are implied from the Killing spinor equations for such backgrounds. Because of this, apart from giving the general formulae of the Killing spinor equations acting on all the spinors $\sigma_I$, we also list the explicit results in the appendices. From these results, one can construct the linear system associated with the Killing spinor equations of any supersymmetric background. The same applies for the linear system associated with the field equations.

\(^3\) The $\Gamma$-matrix algebra has been carried out using the computer program GAMMA [17].
There are several ways to characterize IIB supersymmetric backgrounds. One way is to count the number of Killing spinors $N$ and their stability subgroup $H$ in $\text{Spin}(9, 1) \times U(1)$. The role of the stability subgroup of the Killing spinors in the classification of supersymmetric backgrounds has been stressed in [18]. Backgrounds for which $H$ contains a Berger holonomy group, i.e. $H$ contains $SU(n)$, $G_2$, $\text{Sp}(2)$ and $\text{Spin}(7)$, are of particular interest. The Killing spinors of most of the known solutions have stability subgroups in $\text{Spin}(9, 1) \times U(1)$ which are of Berger type. It has been demonstrated in [13] that for any subgroup $H$ in $\text{Spin}(9, 1)$, there is a basis in the space of $H$-invariant spinors $\Delta^H$ which can be written as $(\eta_j, i\eta_j)$, $j = 1, \ldots, \frac{1}{2} \dim \Delta^H$, where $\eta_j$ are Majorana–Weyl spinors. Moreover it was shown that the Killing spinor equations factorize for some backgrounds which admit $N = \dim \Delta^H$ Killing spinors. Here, we shall show that this is the case for all backgrounds with $N = \dim \Delta^H$ $H$-invariant Killing spinors, i.e. the maximally supersymmetric $H$-backgrounds or maximal $H$-backgrounds.

In addition, we shall examine the backgrounds that admit $N = \frac{1}{2} \dim \Delta^H$ $H$-invariant Killing spinors, i.e. they admit half of the maximal possible number of $H$-invariant Killing spinors. We refer to these backgrounds either as half-maximally supersymmetric $H$-backgrounds or as half-maximal $H$-backgrounds. We show that the Killing spinors of such backgrounds can be written as

$$\epsilon = z\eta$$

where $z$ is a $N \times N$ complex matrix and $\eta$ is a basis of $N$ Majorana–Weyl $H$-invariant spinors. There are two classes of half-maximally supersymmetric $H$-backgrounds. One class consists of those backgrounds for which the Killing spinors are linearly independent over the complex numbers, and so over the real numbers. Such backgrounds are associated with a complex $N \times N$ invertible matrix $z$, $\det z \neq 0$. We refer to these models as generic half-maximal $H$-backgrounds. We shall show that although the Killing spinor equations do not factorize in this case, they simplify. In particular, the gravitino Killing spinor equations can be rewritten so that the only contributions that include terms with more than two gamma matrices are those of the $F$ flux. The dependence on the functions of the Killing spinors is also restricted in the terms that contain up to two gamma matrices. In addition, the solution to the Killing spinor equations gives rise to a parallel transport equation

$$z^{-1} dz + C = 0,$$

where $C$ can be interpreted as the restriction of the supercovariant connection on the bundle of Killing spinors $K$. This is similar to the parallel transport equation\(^4\) that arises in the maximally supersymmetric $H$-backgrounds [13] but in this case $C$ may depend on $z$ and so the resulting first-order system is nonlinear. The other class consists of those backgrounds for which the Killing spinors are linearly independent over the real numbers but linearly dependent over the complex numbers, so $\det z = 0$. We refer to these models as degenerate half-maximal $H$-backgrounds. Clearly this subclass can be further characterized by the rank of $z$. If the rank of $z$ is $r$, then the space of Killing spinors of such backgrounds is of co-dimension $2(N - r)$ in the space of Killing spinors. In particular if the rank of $z$ is $N - 1$, then one of the Killing spinors will be linearly dependent over the complex numbers on the remaining $N - 1$ Killing spinors but linearly independent over the reals.

As an example of our construction, we consider backgrounds with two $SU(4) \times \mathbb{R}^8$-invariant spinors. The dimension of $SU(4) \times \mathbb{R}^8$-invariant spinors in the (complex) chiral representation of $\text{Spin}(9, 1)$ is four. So backgrounds with two $SU(4) \times \mathbb{R}^8$-invariant Killing spinors are $\mathbb{R}$-linear.

\(^4\) The number of Killing spinors is counted over the real numbers because the Killing spinor equations of IIB supergravity are $\mathbb{R}$-linear.

\(^5\) For maximally supersymmetric backgrounds, $H = 1$ and $C$ is the supercovariant connection.
spinors are half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds. We solve the Killing spinor equations for both generic and degenerate backgrounds. For generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds, we find that there are two cases to consider. In both cases, we compute the nonlinear parallel transport equation $z^{-1} dz + C = 0$ but we do not give the most general solution. Instead, we analyse two examples. In one of the examples $z$ is diagonal with complex entries, and in the other $z$ is a real matrix. In both examples, we determine the geometry of the supergravity backgrounds. In particular, we find that the spacetime admits a null Killing vector field and compute all spacetime form bilinears. In the degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds, the second Killing spinor is proportional to the first Killing spinor, $\epsilon_2 = w \epsilon_1$, where $w$ is a complex function with non-vanishing imaginary part, and $\epsilon_1 = (f - g_2 + ig_1)1 + (f + g_2 + ig_1)e_{1234}$ as in [12], $f, g_2 \neq 0$. The geometry of these backgrounds is similar to those with one $SU(4) \ltimes \mathbb{R}^8$-invariant spinor investigated in [12].

We also solve the linear systems associated with the integrability conditions of the Killing spinor equations of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-backgrounds of [13]. We find that in both cases, if the Bianchi identities are imposed, the only field equations that are not implied by the Killing spinor equations are the $E_{--}$ components of the Einstein equations. We explicitly give these equations in terms of the connection and fluxes of the backgrounds.

This paper is organized as follows. In section 2, we review the construction of the bosonic sector of IIB supergravity together with the Killing spinor equations and their integrability conditions. In section 3, we show that the Killing spinor equations and the integrability conditions of any spinor can be determined from those on five types of spinors. We also introduce the maximal and half-maximal supersymmetric $H$-backgrounds and investigate their Killing spinor equations and integrability conditions. In section 4, we construct the linear systems of the Killing spinor equations and the integrability conditions for any supersymmetric background. In section 5, we present two examples of generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds. In section 6, we solve the Killing spinor equations of degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds. In section 7, we solve the linear systems of the integrability conditions of maximally supersymmetric $Spin(7) \ltimes \mathbb{R}^8$- and $SU(4) \ltimes \mathbb{R}^8$-backgrounds, and in section 8 we give our conclusions. In appendix A, we summarize the construction of spinors in terms of forms and compute various spinor bilinears. In appendix B, we explicitly give the Killing spinor equations on all five types of spinors. In appendix C, we explicitly give the integrability conditions of the Killing spinor equations on all five types of spinors. In appendix D, we present the linear system of the Killing spinor equations of generic half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds and give its solution, and in appendix E, we present the linear system of degenerate half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds.

2. Killing spinor equations and integrability conditions

2.1. Killing spinor equations

The bosonic fields of IIB supergravity [5, 19, 20] are the spacetime metric $g$, two real scalars, the axion $\sigma$ and the dilaton $\phi$, which are combined into a complex 1-form field strength $P$, two 3-form field strengths $G^1$ and $G^2$ which are combined to a (complex) 3-form field strength $G$, and a self-dual 5-form field strength $F$. To describe these, we introduce a $SU(1, 1)$ matrix $U = (V^\alpha, V_\alpha), \alpha = 1, 2$ such that

\[ U^{-1} \partial U + i = 0\]
\[ V^\alpha_+ V^\beta_+ - V^\beta_- V^\alpha_- = \epsilon^{\alpha\beta}, \]  
(2.1)

where \( \epsilon^{12} = 1 = \epsilon_{12}, \) \( (V^1_+)^* = V^2_+ \) and \( (V^2_-)^* = V^1_+ \). The signs denote \( U(1) \subset SU(1, 1) \) charge. Then

\[
P_M = -\epsilon_{\alpha\beta} V^\alpha_+ \partial_M V^\beta_+  
\]

\[
Q_M = -i \epsilon_{\alpha\beta} V^\alpha_+ \partial_M V^\beta_+  
\]

(2.2)

The 3-form field strengths \( G^{\alpha}_{MNR} = 3 \partial_M A^\alpha_{NR} \), with \( (A^1_{MN})^* = A^2_{MN} \) combine into the complex field strength

\[
G_{MNR} = -\epsilon_{\alpha\beta} V^\alpha_+ G^{\beta}_{MNR}.  
\]

(2.3)

The 5-form self-dual field strength is

\[
F_{M_1 M_2 M_3 M_4 M_5} = \frac{5}{24} \epsilon_{M_1 \ldots M_5} F^{M_1 \ldots M_5} + \frac{5 i}{8} \epsilon_{M_1 \ldots M_5} A^M_{M_1 \ldots M_5}.  
\]

(2.4)

where \( F_{M_1 \ldots M_5} = -\frac{1}{2} \epsilon_{M_1 \ldots M_5} F^{N_1 \ldots N_5} F_{N_1 \ldots N_5} \) and \( \epsilon_{01\ldots 9} = 1 \). To identify the scalars define the variables \( \rho = V^2_- / V^1_+ \) and

\[
\rho = \frac{1 + i \tau}{1 - i \tau},  
\]

(2.5)

In turn \( \tau = \sigma + i e^{-\phi} \), where \( \sigma \) is the axion (RR scalar) and \( \phi \) is the dilaton.

The Killing spinor equations of IIB supergravity are the parallel transport equations of the supercovariant derivative \( \mathcal{D} \)

\[
\mathcal{D}_M \epsilon = \tilde{\nabla}_M \epsilon + \frac{i}{48} \Gamma^{N_1 \ldots N_5} \epsilon F_{N_1 \ldots N_5 M} = -\frac{1}{96} \left( \Gamma^M_{N_1 N_2 N_3} G_{N_1 N_2 N_3} - 9 \Gamma^{N_1 N_2} G_{M N_1 N_2} \right) (C \epsilon)^* = 0  
\]

(2.6)

and the algebraic condition

\[
\mathcal{A} \epsilon = P_M \Gamma^{M} (C \epsilon)^* + \frac{i}{24} G_{N_1 N_2 N_3} \Gamma^{N_1 N_2 N_3} \epsilon = 0,  
\]

(2.7)

where

\[
\tilde{\nabla}_M = D_M + \frac{i}{4} \Omega_{M,AB} \Gamma^{AB}; \quad D_M = \partial_M - \frac{i}{2} Q_M
\]

is the spin connection, \( \nabla_M = \partial_M + \frac{i}{4} \Omega_{M,AB} \Gamma^{AB} \), twisted with \( U(1) \) connection \( Q_M, Q'_M = Q_M \), \( \epsilon \) is a (complex) Weyl spinor, \( \Gamma^0, \ldots, \epsilon = -\epsilon \), and \( C \) is a charge conjugation matrix. For our spinor conventions\footnote{We use a mostly plus convention for the metric. To relate this to the conventions of [5], one takes \( \Gamma^A \rightarrow i \Gamma^A \) and every time a index is lowered there is also an additional minus sign.} see appendix A. The Killing spinor equations are the vanishing conditions of the supersymmetry transformations of the gravitino, and the supersymmetric partners of the dilaton and axion restricted to the bosonic sector of IIB supergravity, respectively. For a superspace formulation of IIB supergravity see [20]. The recent modification of IIB supergravity with 10-form potentials [21] does not change our analysis below because the Killing spinor equations remain the same.

2.2. Integrability conditions

To determine the field equations which are implied by the Killing spinor equations, one has to investigate the integrability conditions of the Killing spinor equations. This calculation is essentially the same as that of [5] where the field equations of the IIB supergravity were found.
from the commutator of the supersymmetry transformations. However, we cast the results in such a way that is more suitable to our purpose. The integrability conditions are

$$[D_A, D_B] \epsilon = R_{AB} \epsilon = 0,$$  \hspace{1cm} (2.8)

and

$$[D_A, A] \epsilon = 0,$$  \hspace{1cm} (2.9)

where $R$ has been given in [22] and so the expression will not be repeated here. It turns out that some field equations and Bianchi identities of IIB supergravity are contained in the $I_A = \frac{1}{2} \Gamma_A^{BC} R_{BC}$ and $I = \Gamma^{AB}[D_A, A]$ components of the integrability conditions. In particular, we have

$$I_A \epsilon = \left[ \frac{1}{2} \Gamma^B E_{AB} - i \epsilon^{B_1 B_2 B_3} LF_{A B_1 B_2 B_3} \right] \epsilon - \left[ \Gamma^B \Lambda G_{AB} - \Gamma_A^B \Lambda^{B_1 B_2 B_3} B G_{B_1 B_2 B_3} \right] (C \epsilon)^*$$  \hspace{1cm} (2.10)

and similarly, $\Gamma^A[D_A, A] \epsilon$ can be written as

$$I \epsilon = \left[ \frac{1}{2} \Gamma^{AB} \Lambda \Lambda_{AB} + \Gamma^{A_1 \ldots A_4} B \Lambda_{A_1 \ldots A_4} \right] \epsilon + [P + \Gamma^{AB} B P_{AB}] (C \epsilon)^*,$$  \hspace{1cm} (2.11)

where

$$E_{AB} := R_{AB} - \frac{1}{2} \epsilon R_{AB} - \frac{1}{6} F_{A C_1 \ldots C_4} F_B^{C_1 \ldots C_4} - \frac{1}{4} G_{A C_1 C_2} G^{* B C_1 C_2}_B + \frac{1}{24} \epsilon R_{AB} G^{B C_1 C_2} G^{* C_1 C_2}_B - 2 P A P_B^* + g_{AB} P C P^*_C,$$

$$L G_{AB} := \frac{1}{4} \left( \nabla^C G_{A B C} - P^C G_{A B C}^* + \frac{2i}{3} F_{A B C} G^{* C}_C \right),$$

$$L P := \nabla^A P_A + \frac{1}{24} G_{A_1 A_2 A_3} G^{* A_1 A_2 A_3},$$

$$L F_{A_1 \ldots A_6} := \frac{1}{3!} \left( \nabla^B F_{A_1 \ldots A_6 B} + \frac{i}{288} \epsilon A_1 \ldots A_6 B_1 \ldots B_6 G_{B_1 B_2 B_3} G^{* B_1 B_2 B_3}_B \right),$$

$$B \Lambda_{A_1 \ldots A_6} := \frac{1}{5!} \left( \partial_{A_1} F_{A_2 \ldots A_6} - \frac{5i}{12} G_{[A_1 A_2 A_3} G^{* A_4 A_5 A_6]} \right),$$

$$B \Lambda_{A_1 \ldots A_4} := \frac{1}{4!} \left( D_{A_1} G_{A_2 A_3 A_4} + P_{[A_1} G^{* A_2 A_3 A_4]} \right),$$

$$B P_{AB} := D_A P_B.$$  \hspace{1cm} (2.12)

One can show that $L F$ and $B F$ are not independent but are related by the self-duality condition on $F$. The field strengths $P$ and $G$ have different $U(1) \subset SU(1, 1)$ charges. In particular, one has

$$D_A P_B = \partial_A P_B - 2i Q_A P_B$$

$$D_A G_{BCD} = \partial_A G_{BCD} - i Q_A G_{BCD},$$  \hspace{1cm} (2.13)

To derive the above expressions some very painful Dirac algebra is required but we have been assisted by GAMMA [17] to perform most of the computation. The algebraic Killing spinor equation (2.7) has also been used to convert expressions containing $G$ and $P$ fluxes. The above choice of components of integrability conditions that contain the field equations and the Bianchi identities is not unique. For example, the component $\Gamma^B R_{AB}$ may also be used giving identical results. However, it turns out that the computation is more involved.

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8. The integrability conditions of the Killing spinor equations (2.8) and (2.9) may impose further conditions on the Bianchi identities and field equations than those implied by the vanishing of these two components.

9. In terms of $B F$, we have $I_A \epsilon = \left[ \frac{1}{2} \Gamma^B E_{AB} + i \epsilon^{B_1 B_2 B_3} B F_{A B_1 B_2 B_3} \right] \epsilon - \left[ \Gamma^B \Lambda G_{AB} - \Gamma_A^B \Lambda^{B_1 B_2 B_3} B G_{B_1 B_2 B_3} \right] (C \epsilon)^*$. 

3. The five types of spinors

3.1. General case

The spinors that appear in type IIB supergravity are complex Weyl spinors of positive chirality. A direct consequence of this is that the most general Killing spinor of IIB supergravity can be written as

\[ \epsilon = p1 + q \epsilon_{1234} + u^i \epsilon_{i5} + \frac{1}{2} v^{ij} \epsilon_{ij} + \frac{1}{5} w^{ijk} \epsilon_{ijk5}, \]

where \( p, q, u, v \) and \( w \) are complex functions on the spacetime, and \( i, j, k = 1, 2, 3, 4 \). The supercovariant derivative acting on \( \epsilon \) gives

\[ \mathcal{D}_A \epsilon = \partial_A p1 + \partial_A q \epsilon_{1234} + \partial_A u^i \epsilon_{i5} + \frac{1}{2} \partial_A v^{ij} \epsilon_{ij} + \frac{1}{5} \partial_A w^{ijk} \epsilon_{ijk5} + p_0 \mathcal{D}_A 1 \]

\[ + q_0 \mathcal{D}_A \epsilon_{1234} + u_0^i \mathcal{D}_A \epsilon_{i5} + \frac{1}{2} v_0^{ij} \mathcal{D}_A \epsilon_{ij} + \frac{1}{5} w_0^{ijk} \mathcal{D}_A \epsilon_{ijk5} + p_1 \mathcal{D}_A (i1) \]

\[ + q_1 \mathcal{D}_A (i \epsilon_{1234}) + u_1^i \mathcal{D}_A (i \epsilon_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{D}_A (i \epsilon_{ij}) + \frac{1}{5} w_1^{ijk} \mathcal{D}_A (i \epsilon_{ijk5}) \]

and the algebraic Killing spinor equation becomes

\[ \mathcal{A} \epsilon = p_0 \mathcal{A} 1 + q_0 \mathcal{A} \epsilon_{1234} + u_0^i \mathcal{A} \epsilon_{i5} + \frac{1}{2} v_0^{ij} \mathcal{A} \epsilon_{ij} + \frac{1}{5} w_0^{ijk} \mathcal{A} \epsilon_{ijk5} + p_1 \mathcal{A} (i1) \]

\[ + q_1 \mathcal{A} (i \epsilon_{1234}) + u_1^i \mathcal{A} (i \epsilon_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{A} (i \epsilon_{ij}) + \frac{1}{5} w_1^{ijk} \mathcal{A} (i \epsilon_{ijk5}) \]

where \( p = p_0 + i p_1 \) and similarly for the rest of the components. Therefore to compute the Killing spinor equations of the most general spinor in IIB supergravity, it suffices to compute the supercovariant derivative and \( A \) on the ten types of spinors \( 1, \epsilon_{1234}, \epsilon_{i5}, \epsilon_{ij} \) and \( \epsilon_{ijk5} \), and \( i1, i \epsilon_{1234}, i \epsilon_{i5}, i \epsilon_{ij} \) and \( i \epsilon_{ijk5} \). However it is straightforward to see that \( \mathcal{D}_A (i1) \) and \( \mathcal{A} (i1) \) can be easily read from the expressions of \( \mathcal{D}_A 1 \) and \( \mathcal{A} 1 \), respectively, and similarly for the rest of the pairs. The only effect is a sign which appears in those terms of the Killing spinor equation which contain the charge conjugation matrix. Of course the Killing spinor equations acting on 1 should be in addition multiplied by the complex unit \( i \) to recover those acting on \( i1 \), and similarly for the rest of the pairs. Therefore to construct the linear system associated with any number of Killing spinors, it suffices to compute

\[ \mathcal{D}_A 1, \quad \mathcal{D}_A \epsilon_{1234}, \quad \mathcal{D}_A \epsilon_{i5}, \quad \mathcal{D}_A \epsilon_{ij}, \quad \mathcal{D}_A \epsilon_{ijk5}, \]

\[ \mathcal{A} 1, \quad \mathcal{A} \epsilon_{1234}, \quad \mathcal{A} \epsilon_{i5}, \quad \mathcal{A} \epsilon_{ij}, \quad \mathcal{A} \epsilon_{ijk5}, \]

i.e. the Killing spinor equations on five types of spinors.

It remains to show that the integrability conditions \( \mathcal{I} \) on a Killing spinor \( \epsilon \) can also be determined in terms of those on the above five types of spinors. Since these integrability conditions are algebraic, one finds that

\[ \mathcal{I} \epsilon = p_0 \mathcal{I} 1 + q_0 \mathcal{I} \epsilon_{1234} + u_0^i \mathcal{I} \epsilon_{i5} + \frac{1}{2} v_0^{ij} \mathcal{I} \epsilon_{ij} + \frac{1}{5} w_0^{ijk} \mathcal{I} \epsilon_{ijk5} + p_1 \mathcal{I} (i1) \]

\[ + q_1 \mathcal{I} (i \epsilon_{1234}) + u_1^i \mathcal{I} (i \epsilon_{i5}) + \frac{1}{2} v_1^{ij} \mathcal{I} (i \epsilon_{ij}) + \frac{1}{5} w_1^{ijk} \mathcal{I} (i \epsilon_{ijk5}) \]

and similarly for the \( I \) integrability condition. Because the expressions for \( \mathcal{I} (i1) \) can be easily recovered from that of \( \mathcal{I} 1 \), and similarly for the rest, one has to compute

\[ \mathcal{I} 1, \quad \mathcal{I} \epsilon_{1234}, \quad \mathcal{I} \epsilon_{i5}, \quad \mathcal{I} \epsilon_{ij}, \quad \mathcal{I} \epsilon_{ijk5}, \]

\[ \mathcal{I} 1, \quad \mathcal{I} \epsilon_{1234}, \quad \mathcal{I} \epsilon_{i5}, \quad \mathcal{I} \epsilon_{ij}, \quad \mathcal{I} \epsilon_{ijk5}, \]

i.e. the integrability conditions on five types of spinors. In what follows, we shall give the general formulae of the Killing spinor equations and their integrability conditions acting on all five types of spinors. In the appendices, we shall list the various components of these equations in the basis (A.3).
the terms in Killing spinor equations that contain the P shown in [13] that the space of H spinors are invariant under some proper subgroup H. In many cases of interest, the Killing spinor equations of backgrounds with H-invariant Majorana–Weyl spinors. The most general H-invariant Killing spinors in this case are

\[ \epsilon_r = \sum_{j=1}^{k} f_{rj} \eta_j, \quad r = 1, \ldots, N \]  

(3.7)

where \( (f_{rj}) \) is a \( N \times k \) matrix of real functions and \( N \) is the number of Killing spinors of the background. It has also been shown in [13] that the Killing spinor equations of backgrounds with H-invariant Killing spinors whose number of Killing spinors is equal to the real dimension of \( \Delta H \), i.e. of maximally supersymmetric H-backgrounds, dramatically simplify. In particular the terms in Killing spinor equations that contain the \( P \) and \( G \) fluxes factorize from those that contain the \( F \) fluxes and geometry. This was shown in some special cases, here we shall present the proof of the general case.

In the maximally-supersymmetric H-backgrounds, \( f = (f_{rj}) \) is invertible. Because of this, the Killing spinor equations can be written as

\[ \sum_{j=1}^{N} (f^{-1} D_M f) \eta_j + D_M \eta_j = 0, \quad \mathcal{A} \eta_j = 0. \]  

(3.8)

First consider the algebraic Killing spinor equation for \( i = 1 \) and \( i = m + 1 \). In this case

\[ \mathcal{A} \eta_1 = P_A \Gamma^A \eta_1 + \frac{1}{24} \Gamma^{ABC} G_{ABC} \eta_1 = 0, \quad \mathcal{A} \eta_{m+1} = -i P_A \Gamma^A \eta_1 + \frac{i}{24} \Gamma^{ABC} G_{ABC} \eta_1 = 0. \]  

(3.9)

Therefore, we conclude that \( P_A \Gamma^A \eta_1 = 0 \) and \( \Gamma^{ABC} G_{ABC} \eta_1 = 0 \). Applying this for all pairs, we get

\[ P_A \Gamma^A \eta_p = 0, \quad p = 1, \ldots, m, \quad \Gamma^{ABC} G_{ABC} \eta_p = 0, \quad p = 1, \ldots, m. \]  

(3.10)

Similarly, for \( i = p \) and \( i = m + p \) in the first equation in (3.8), we find

\[ \sum_{j=1}^{N} (f^{-1} D_M f) \eta_j + \nabla_M \eta_j + \nabla_M \eta_j + \frac{i}{48} \Gamma^{Ni \cdots N_k} \eta_p F_{Ni \cdots N_k} \]  

\[ = \frac{1}{96} (\Gamma^M_{Ni \cdots N_k} \eta_p G_{Ni \cdots N_k} - 9 \Gamma^N_{Ni \cdots N_k} G_{MN \cdots N_k}) \eta_p = 0, \]

\[ - \frac{i}{96} \sum_{j=1}^{N} (f^{-1} D_M f)_{m+p} \eta_j + \nabla_M \eta_p + \frac{i}{48} \Gamma^{Ni \cdots N_k} \eta_p F_{Ni \cdots N_k} \]  

\[ + \frac{1}{96} (\Gamma^M_{Ni \cdots N_k} \eta_p G_{Ni \cdots N_k} - 9 \Gamma^N_{Ni \cdots N_k} G_{MN \cdots N_k}) \eta_p = 0. \]  

(3.11)

Adding and subtracting these equations, we get

\[ \frac{1}{2} \left[ \sum_{j=1}^{N} (f^{-1} D_M f)_{pj} \eta_j - i \sum_{j=1}^{N} (f^{-1} D_M f)_{m+p} \eta_j \right] + \nabla_M \eta_p + \frac{i}{48} \Gamma^{Ni \cdots N_k} \eta_p F_{Ni \cdots N_k} = 0 \]  

(3.12)

\[ \sum_{j=1}^{N} (f^{-1} D_M f)_{pj} \eta_j + i \sum_{j=1}^{N} (f^{-1} D_M f)_{m+p} \eta_j + \frac{1}{4} G_{MBC} \Gamma^{BC} \eta_p = 0 \]
where we have also used the second equation in (3.10). It is easy to see from (3.10) and (3.12) that, as we have mentioned, the Killing spinor equations factorize.

It has been observed in [13] that the solution to the Killing spinor equation in this case gives rise to a parallel transport equation

\[ f^{-1} \, df + C = 0. \]  

The connection \( C \) can be thought of as the restriction of the supercovariant connection on the bundle of Killing spinors

\[ S \rightarrow S / \mathcal{K} \rightarrow 0, \]  

where \( S \) is the spin bundle of the theory. A necessary condition for the existence of a solution to the parallel transport equation is the vanishing of the curvature

\[ F(C) = dC - C \wedge C = 0. \]  

It is worth pointing out that for maximally supersymmetric backgrounds, \( H = 1 \), \( K = S \) and \( C \) is the supercovariant connection. The curvature \( F(C) \) is the supercovariant curvature \( R = [D, D] \). The vanishing of \( R \) was precisely the condition analysed in [10] to classify the maximally supersymmetric solutions of ten- and eleven-dimensional supergravities.

The integrability conditions of the Killing spinor equations of maximally supersymmetric \( H \)- backgrounds factorize as well. In particular since \( \det f \neq 0 \), it is easy to see that \( \mathcal{I}_A \epsilon_i = 0 \) and \( \mathcal{I}_j \epsilon_i = 0 \) imply that \( \mathcal{I}_A \eta_i = 0 \) and \( \mathcal{I}_i \eta_j = 0 \), \( i = 1, \ldots, 2m \). In turn these two equations give

\[
\begin{align*}
\frac{1}{2} \Gamma^{B} E_{AB} - i \Gamma^{B_i B_j B_k} L F_B B_i B_j B_k \eta_i &= 0, \\
\Gamma^{B} L G_{AB} - \Gamma^{A_i B_j B_k} B G_{B_i B_j B_k} \eta_j &= 0, \\
\frac{1}{2} \Gamma^{A B} L G_{AB} + \Gamma^{A_i \ldots A_k} B G_{A_i \ldots A_k} \eta_i &= 0, \\
L P + \Gamma^{A B} B P_{AB} \eta_j &= 0, 
\end{align*}
\]  

It is clear that if one assumes that the Bianchi identities are satisfied, then the above conditions impose strong conditions on the field equations. We analyse these in detail for the maximally supersymmetric \( SU(4) \times \mathbb{R}^8 \) - and \( \text{Spin}(7) \times \mathbb{R}^8 \)-backgrounds.

### 3.2.2. Half-maximally supersymmetric \( H \)-backgrounds

Apart from the maximally supersymmetric backgrounds above, there is also another class of backgrounds with \( H \)-invariant spinors for which the Killing spinor equations simplify. These are the backgrounds for which the number of Killing spinors is \( N = \frac{1}{2} \dim \Delta^H = m \). We refer to such backgrounds as half-maximal \( H \)-backgrounds. It turns out that if the Killing spinors are generic\( ^{10} \), then the Killing spinor equations of half-maximal \( H \) backgrounds simplify but they do not necessarily factorize as in the maximal case.

The Killing spinors for half-maximal \( H \)-backgrounds can be written as

\[ \epsilon_i = \sum_{j=1}^{m} \tilde{z}_{ij} \eta_j, \quad i, j = 1, \ldots, m, \]  

where \( z = (\tilde{z}_{ij}) \) is an \( m \times m \) matrix of complex functions on the spacetime. This can be easily seen by expressing \( m \) \( H \)-invariant spinors in the basis \( (\eta_j, \eta_i) \) of \( \Delta^H \).

Next suppose that \( \epsilon_i \) are generic, i.e. that \( \det z \neq 0 \). In such case, the algebraic Killing spinor equation can be written as

\[ P_A \Gamma^A \xi \eta + \frac{1}{2m} G_{ABC} \Gamma^{ABC} \xi \eta = 0, \]

\( ^{10} \) We shall make this precise later.
where we have used matrix notation for $z$ and $\eta$. This can be then rewritten as
\begin{equation}
PA \Gamma^A z^{-1} z^* \eta + \frac{1}{24} G_{ABC} \Gamma^{ABC} \eta = 0.
\end{equation}
(3.18)

Acting on this equation with $\Gamma_A$, we find that
\begin{equation}
(p^B \Gamma_{AB} + PA)z^{-1} z^* \eta + \frac{1}{24} G_{BCD} \Gamma^{BCD} \eta + \frac{1}{2} G_{ABC} \Gamma^{BC} \eta = 0.
\end{equation}
(3.19)

Solving for the fourth-order term in the gamma matrices and substituting the result into (2.6), we get
\begin{equation}
D_A z \eta + z \nabla_A \eta + \frac{i}{48} \Gamma^{B_1 \cdots B_n} F_{B_1 \cdots B_n} + \frac{1}{4} z^* z^{-1} z^* [p^B \Gamma_{AB} + PA] \eta
+ \frac{1}{8} G_{ABC} \Gamma^{BC} z^{-1} z^* \eta = 0
\end{equation}
(3.20)
or equivalently,
\begin{equation}
z^{-1} D_A z \eta + \nabla_A \eta + \frac{i}{48} \Gamma^{B_1 \cdots B_n} F_{B_1 \cdots B_n} + \frac{1}{4} z^* z^{-1} z^* [p^B \Gamma_{AB} + PA] \eta
+ \frac{1}{8} G_{ABC} \Gamma^{BC} z^{-1} z^* \eta = 0.
\end{equation}
(3.21)

There is no factorization of the Killing spinor equations in this case, in contrast to the maximal supersymmetric $H$-backgrounds. Nevertheless, the Killing spinor equations simplify because the contribution of the $G$ and the $P$ fluxes in (3.21) is contained in the up to gamma square terms and the $F$ flux term is independent of the spacetime functions $z$. Therefore the effect of the $G$ and $P$ fluxes is to modify the spin connection $\Omega_1$ and the $U(1)$ connection $Q$ with terms that depend on the $P$ and $G$ fluxes and the functions that determine the Killing spinors.

The solution to the Killing spinor equations of generic half-maximally supersymmetric $H$-backgrounds gives rise to a parallel transport equation
\begin{equation}
z^{-1} dz + C = 0.
\end{equation}
(3.22)

The connection $C$ can again be thought of as the restriction of the supercovariant connection on the bundle of Killing spinors $\mathcal{K}$. However unlike the case of maximally supersymmetric $H$-backgrounds, $C$ depends on $z$, $C = C(z)$. To see this observe that some fluxes in (3.21) depend on the functions $z$. We have also confirmed this in an example. Because of this, although it is always possible to solve the linear system associated with the Killing spinor equations, the resulting parallel transport equation may be rather involved.

Next let us consider the special or degenerate cases $\det z = 0$. These cases arise whenever the Killing spinors $\epsilon_i$ are linearly dependent over the complex numbers but linearly independent over the real numbers. These cases are characterized by the rank of $z$. If the rank of $z$ is $m-1$, then it can be arranged such that the first $m-1$ Killing spinors are linearly independent over the complex numbers but the last one is linearly dependent. In such a case, we can write
\begin{equation}
\epsilon_m = w_1 \epsilon_1 + \cdots + w_{m-1} \epsilon_{m-1},
\end{equation}
(3.23)
where at least one of $w_1, \ldots, w_{m-1}$ has a non-vanishing imaginary part. If all the imaginary parts vanish, then $\epsilon_m$ is linearly dependent on $\epsilon_1, \ldots, \epsilon_{m-1}$ over the reals and the background has $m-1$ supersymmetries. One can modify the above argument in the cases for which $z$ has rank $r < m$ for $r = 1, \ldots, m-1$. It appears that the solution of the Killing spinor equations in the degenerate cases requires information on the solutions of the Killing spinor equations for $N < m$ $H$-invariant Killing spinors. We shall see that this is the case in the special case of ($N = 2$) half-maximal $SU(4) \times \mathbb{R}^8$-backgrounds.
The integrability conditions $I_A e_i = I e_i = 0$ also simplify for half-maximally $H$-supersymmetric backgrounds. We shall focus on the case where $\det z \neq 0$. In this case, these integrability conditions can be rewritten as

\[
\begin{align*}
\left[ \frac{1}{2} \Gamma^B L_{AB} - 2 \pi^{B_1 \cdots B_5} L F_{AB_1 \cdots B_5} \right] \eta - \left[ \Gamma^B L G_{AB} - \Gamma_{A} B_{1} \cdots B_{5} B G_{B_{1} \cdots B_{5}} \right] z^{-1} \bar{z}^* \eta = 0, \\
\left[ \frac{1}{2} \Gamma^A B_{1} \cdots B_{5} L G_{A_{1} \cdots A_{5}} + \Gamma^{A_{1} \cdots A_{5}} B_{1} \cdots B_{5} \right] \eta + \left[ L P + \Gamma^A B_{1} \cdots B_{5} P A_{1} \cdots A_{5} \right] z^{-1} \bar{z}^* \eta = 0.
\end{align*}
\]

(3.24)

It is clear that unlike the maximal $H$-backgrounds, the integrability conditions do not factorize in this case.

Combining the maximal and half-maximal cases we have mentioned above, and considering those $H \subset \text{Spin}(9,1)$ that contain a Berger type of group, one can investigate several cases of supersymmetric backgrounds. These include backgrounds with four, eight and sixteen supersymmetries. These cases are summarized in the conclusions in table 1.

4. Linear systems

4.1. The linear system of Killing spinor equations

We have shown that the Killing spinor equations of an arbitrary spinor can be expressed in terms of those on five types of spinors given by

\[
\sigma_i = e_{i_1 \cdots i_5} = \frac{1}{\sqrt{2}} \gamma^{i_1 \cdots i_5} 1,
\]

(4.1)

where the index $i = (1, \ldots, 5)$ contains four holomorphic and one null indices\(^{11}\). Note that $\Gamma^a$ acts as an annihilation or a creation operator on the above spinor, depending on whether the label $i_1 \cdots i_5$ does or does not contain $a$. For this reason, it is convenient to reshuffle ($\alpha, 5$) and ($\bar{\alpha}, \bar{5}$) into indices $p, q, r$ defined by

\[
p = (\tilde{i}_1, \ldots, \tilde{i}_1, i_{l+1}, \ldots, i_5), \quad \bar{p} = (i_1, \ldots, i_1, \tilde{i}_{l+1}, \ldots, \tilde{i}_5),
\]

(4.2)

where the indices $(i_1, \ldots, i_5)$ are some permutation of $(1, \ldots, 5)$, i.e. $e_{i_1 \cdots i_5} = \pm 1$. Thus, $\Gamma^p$ act as annihilation operators on this spinor while $\Gamma^q$ are the creation operators.

With this notation at hand, the expression for the algebraic Killing spinor equation (2.7) on an arbitrary spinor\(^{12}\) can be written as

\[
\begin{align*}
A e_{i_1 \cdots i_5} &= \left[ \frac{1}{4} G_{\bar{q} r} \right] \Gamma^{q} e_{i_1 \cdots i_5} + \left[ \frac{1}{24} G_{\bar{q}_1 \cdots \bar{q}_5} - \frac{s}{12} \epsilon_{\bar{q}_1 \cdots \bar{q}_5} P_{\bar{q}} \right] \Gamma^{\bar{q}_1 \cdots \bar{q}_5} e_{i_1 \cdots i_5} \\
&\quad + \left[ \frac{s}{96} \epsilon_{\bar{q}_1 \cdots \bar{q}_5} P_{\bar{q}} \right] \Gamma^{\bar{q}_1 \cdots \bar{q}_5} e_{i_1 \cdots i_5},
\end{align*}
\]

(4.3)

where the Levi-Civita symbols are defined by $\epsilon_{1334} = +1$ and $\epsilon_{\bar{q}_1 \cdots \bar{q}_5} = 0$, i.e. these are only nonzero when all of its four indices are (anti-)holomorphic. The sign $s$ depends on whether there is a null index in $e_{i_1 \cdots i_5}$:

\[
s = \begin{cases} 
+1, & \text{for } e_{a_{1} \cdots a_{5}}, \\
-1, & \text{for } e_{a_{1} \cdots a_{4}, 5}.
\end{cases}
\]

(4.4)

Observe that

\[
\left\{ e_{i_1 \cdots i_5}, \Gamma^q e_{i_1 \cdots i_5}, \ldots, \Gamma^{\bar{q}_{1} \cdots \bar{q}_{5}} e_{i_1 \cdots i_5} \right\}
\]

is a basis in the space of Dirac spinors and so $e_{i_1 \cdots i_5}$ can be thought of as another Clifford vacuum. Therefore the terms in the square brackets in (4.3) are linearly independent.

\(^{11}\) In this section we will not use the notation ($-\cdots$) for the null indices but rather $(5, \bar{5})$. Thus $\Gamma^5 = \Gamma^+$ and $\Gamma^\bar{5} = \Gamma^-$.  

\(^{12}\) We have used, however, that $I$ is even for all IIB spinors.
We can apply the same procedure to the parallel transport Killing spinor equation (2.6). In particular, we find for $M = p$ that

$$D_pe_{i_1...i_l} = \left[ D_p + \frac{1}{2} \Omega_{p,r} - \frac{i}{4} F_{p r_1 r_2 r_3} \right] e_{i_1...i_l}$$

$$+ \left[ \frac{1}{4} \Omega_{p,q} l + \frac{i}{4} F_{p q r} - \frac{s}{48} g_{p q r} \epsilon_{r_1...r_3} G_{r_1...r_3} - \frac{s}{16} \epsilon_{r_1 r_2 r_3} G_{p r_1 r_3} \right] \Gamma^{q_1 q_2} e_{i_1...i_l}$$

$$+ \left[ \frac{1}{48} g_{p q r} \epsilon_{q_1...q_4} G_{p q_1 q_4} - \frac{s}{192} g_{p q r} \epsilon_{p q_1...q_4} G_{q_1...q_4 r} \right] \Gamma^{q_1...q_4} e_{i_1...i_l}. \quad (4.6)$$

Similarly, the resulting expression for (2.6) with $M = \tilde{p}$ is

$$D_pe_{i_1...i_l} = \left[ D_p + \frac{1}{2} \Omega_{p,r} r - \frac{i}{4} F_{p r_1 r_2 r_3} \right] e_{i_1...i_l}$$

$$+ \left[ \frac{1}{4} \Omega_{p,q} q + \frac{i}{4} F_{p q r} + \frac{s}{32} \epsilon_{p q_1 q_2} G_{q_1 q_2 r} - \frac{s}{32} \epsilon_{q_1 q_2 r} G_{p q_1 q_2} \right] \Gamma^{q_1 q_2} e_{i_1...i_l}$$

$$+ \left[ \frac{1}{48} g_{p q r} \epsilon_{q_1...q_4} G_{p q_1 q_4} + \frac{s}{256} g_{q_1...q_4 r} \epsilon_{p q_1...q_4} G_{p q_1...q_4} \right] \Gamma^{q_1...q_4} e_{i_1...i_l}. \quad (4.7)$$

To derive the linear system associated with the Killing spinor equations of a background with any number of supersymmetries, (4.3), (4.6) and (4.7) must be converted from the oscillator basis (4.5) to the ‘canonical basis’

$$\{1, \Gamma^1, 1, \ldots, \Gamma^{l-1,1}\}. \quad (4.8)$$

To achieve this, we expand the products of $\Gamma^I$ matrices, which are creation operators on $e_{i_1...i_l}$, into a sum of products of $\Gamma^I$ and $\Gamma^J$ matrices, which are annihilation and creation operators, respectively, on 1. Then we act on $e_{i_1...i_l}$ with the annihilation operators. In particular, we have

$$\mathcal{A} e_{i_1...i_l} = \sum_I \left[ \mathcal{A} e_{i_1...i_l} \right] I_I \Gamma^{i_1...i_l} e_{i_1...i_l}$$

$$= \sum_I \sum_{m,n} \frac{1}{m! n!} \left[ \mathcal{A} e_{i_1...i_l} \right]_{j_1...j_m k_1...k_n} \Gamma^{j_1...j_m} \Gamma^{k_1...k_n} e_{i_1...i_l}$$

$$= \sum_I \sum_{m,n} \frac{1}{m! n!} \frac{(-1)^{|m/2| n!}}{2^{l/2 - m} (I - m)!} \epsilon^{j_1...j_m k_1...k_n} \left[ \mathcal{A} e_{i_1...i_l} \right]_{j_1...j_m k_1...k_n} \Gamma^{j_1...j_m} \Gamma^{k_1...k_n} 1, \quad (4.9)$$

with the obvious restrictions $m \leq I$ and $n \leq 5 - I$ and the convention that $\epsilon_{i_1...i_l} = 1$. A similar formula holds for all components of $D_M$. Using these expressions one can easily compute the components of $\mathcal{A} e_{i_1...i_l}$ and $D_M e_{i_1...i_l}$ in the canonical basis (4.8). For convenience we give the explicit expressions for $\mathcal{A} \sigma_I$ and $D \sigma_I$ in the appendices.

4.2. The linear system of integrability conditions

The integrability conditions of the Killing spinor equations (2.11) and (2.10) of a IIB background with any number of supersymmetries can also be expressed in terms of those on five types of spinors $\sigma_I$. To expand $\mathcal{I} \sigma_I$ and $D \sigma_I$ in the canonical basis (4.8), we follow the same procedure as that for the Killing spinor equations in the previous section. In particular,
Similarly for the integrability condition $\mathcal{I}$ in the oscillator basis (4.5) to find
\[
\mathcal{I} e_{i_1 \ldots i_r} = L G_{r_t} + 12 B G_{r_1 r_2} + \left[ \frac{1}{2} L G_{Q_1 Q_2} + 12 B G_{Q_1 Q_2} \right] e_{i_1 \ldots i_r} + \left[ \frac{s}{2} \epsilon_{Q_1 Q_2} + 12 B G_{Q_1 Q_2} \right] e_{i_1 \ldots i_r} - \frac{s}{6} \epsilon_{Q_1 Q_2} B P_{r_t} + \left[ B G_{Q_1 Q_2} + \frac{s}{96} \epsilon_{Q_1 Q_2} L p + \frac{1}{48} \epsilon_{Q_1 Q_2} B P \right] e_{i_1 \ldots i_r}.
\]

Similarly for the integrability condition $\mathcal{I}_A$, we get
\[
\mathcal{I}_A e_{i_1 \ldots i_r} = \left[ \frac{1}{2} E_{p_t} - 6 i L F_{p_t} + 4 s q_{p_t} \epsilon_{r} B G_{r_1 \ldots r_3} - 8 s q_{p_t} \epsilon_{r} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r} + \left[ -i L F_{p_t} + \frac{s}{12} \epsilon_{Q_1 Q_2} L G_{p_t} - 4 s q_{p_t} \epsilon_{Q_1 Q_2} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r} - 2 s \epsilon_{Q_1 Q_2} B G_{p_t} \Gamma_{i_1 \ldots i_r} + \left[ -\frac{s}{96} \epsilon_{Q_1 Q_2} L G_{p_t} + \frac{s}{8} \epsilon_{Q_1 Q_2} B G_{p_t} + \frac{1}{4} \epsilon_{Q_1 Q_2} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r}.
\]

and
\[
\mathcal{I}_B e_{i_1 \ldots i_r} = \left[ \frac{1}{2} E_{p_t} - 6 i L F_{p_t} + 16 s \epsilon_{r} B G_{r_1 \ldots r_3} + 8 s \epsilon_{r} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r} + \left[ -i L F_{p_t} + \frac{s}{12} \epsilon_{Q_1 Q_2} L G_{p_t} - 3 s \epsilon_{Q_1 Q_2} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r} - 9 s \epsilon_{Q_1 Q_2} B G_{p_t} \Gamma_{i_1 \ldots i_r} + \left[ -\frac{s}{96} \epsilon_{Q_1 Q_2} L G_{p_t} + \frac{s}{4} \epsilon_{Q_1 Q_2} B G_{p_t} \right] \Gamma_{i_1 \ldots i_r}.
\]

It remains to convert the above expressions from the oscillator basis (4.5) to the canonical basis (4.8). This can be done as in (4.9) and we shall not repeat the formula here. The explicit expressions of the integrability conditions in the canonical basis can be found in the appendices.

5. Generic $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant spinors

5.1. Preliminaries

Applying the results of section 3.2 to this case, one finds that the two Killing spinors of $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant spinors can be written as
\[
\epsilon = \alpha_i \eta_i
\]

where $\eta_1 = \eta_2 = 1 + \epsilon_{1234}$ and $\eta_2 = i(1 - \epsilon_{1234})$ and $z$ is a complex $2 \times 2$ matrix. There are two classes of such backgrounds. For generic half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds the matrix $z$ is non-degenerate, det $z \neq 0$, whereas the degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds have det $z = 0$ but the two Killing spinors are linearly independent over the real numbers. To solve the Killing spinor equations in the generic case, we adapt the formalism developed in section 3.2. In this section, we shall investigate the generic class of half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds. The degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds will be examined in section 6.
5.2. The solution of the linear system

The linear system and its solution are described in appendix D. It turns out that to solve the linear system for generic half-maximal $SU(4) \ltimes \mathbb{R}^4$-backgrounds one has to consider two cases depending on whether or not $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$, where $A = z^{-1} z^\dagger$. In both cases after solving for the fluxes, the resulting equations can be separated into two classes. One class is the algebraic equations which do not contain spacetime derivatives of the functions $z$, e.g. (D.54) and (D.57). The other case are first-order equations for the functions $z$ which however are nonlinear in $z$, e.g. (D.53), (D.52), (D.61) and (D.62). These first-order equations can be viewed as the parallel transport equations of the restriction of the supercovariant connection on the bundle of the Killing spinors $\mathcal{K}$. However, since the system is nonlinear, the analysis of the general case is rather involved. So instead of solving the system in general, we shall investigate two examples. In the first example $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0$ while in the second $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0$.

5.2.1. Special cases. The matrix $z$ in the example with $(A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0$ is chosen as

$$z = \begin{pmatrix} z_{11} & 0 \\ 0 & z_{22} \end{pmatrix}, \quad z_{11} \neq z_{22} \neq 0,$$

where $z_{11}$ and $z_{22}$ are complex. Without loss of generality, we shall also work in the gauge for which det $z = 1$ and so $z_{11} z_{22} = 1$. This gauge can always be locally attained by an appropriate $U(1)$ gauge transformation and an appropriate Spin(9, 1) transformation along $\Gamma_{05}$. We therefore can set

$$z_{11} = \rho_1 e^{i\varphi}, \quad z_{22} = \rho_2 e^{-i\varphi}, \quad \rho_1 \rho_2 = 1$$

for $\rho_1 = \rho, \rho_2 = \rho^{-1}, \varphi \in \mathbb{R}, \rho > 0$. Consequently, $A = diag(e^{-2\varphi}, e^{2\varphi})$.

We shall not go through a detailed analysis of the solution. This has been done in (D.3). Instead, we shall summarize the conditions on the geometry and fluxes. In particular we find that the conditions on the geometry are

$$\Omega_{\alpha,\beta} = 0, \quad \Omega_{\tau,\rho} = 0, \quad \Omega_{\tau,\alpha} = 0, \quad \Omega_{\tau} = Q = 0, \quad \Omega_{\tau,\tau} = 0$$

$$\partial_\alpha \psi = 0, \quad \partial_\beta \rho = 0, \quad \Omega_{\alpha,\alpha} = 0, \quad \Omega_{\alpha,\alpha} = 0, \quad \Omega_{\alpha,\beta} = 0,$$

$$\Omega_{\phi,\phi} = \frac{3}{\cos(4\psi) + 2} \Omega_{\tau,\tau}, \quad \Omega_{\beta,\beta} = 0$$

$$\Omega_{\alpha,\alpha} + \Omega_{\alpha,\alpha} = 0, \quad \partial_\phi \psi = -\frac{\sin(4\psi)}{\cos(4\psi) + 2} \Omega_{\phi,\phi},$$

$$\Omega_{\alpha,\beta} + \frac{\cos(4\psi)}{\cos(4\psi) + 2} \Omega_{\phi,\phi} = 0, \quad \Omega_{\alpha,\beta} \phi = \frac{2}{\cos(4\psi) + 2} \Omega_{\tau,\phi},$$

the conditions on the $G$ and $P$ fluxes are

$$G_{\tau\alpha} = 0, \quad P_{\tau} = 0, \quad G_{\tau\beta} = G_{\alpha\beta} = 0, \quad G_{\tau\alpha} = -\frac{2}{\cos(2\psi)} \Omega_{\tau,\tau}$$

$$G_{\alpha\beta} = -2 \cos(2\psi) \left( \Omega_{\alpha,\beta} + i F_{\alpha,\beta} \right) + i \sin(2\psi) \epsilon_{\alpha \beta \gamma} \left( \Omega_{\gamma,\beta} - i F_{\gamma,\beta} \right)$$

$$G_{\alpha\beta} = -2 \cos(2\psi) \left( \Omega_{\alpha,\beta} - i F_{\alpha,\beta} \right) + i \sin(2\psi) \epsilon_{\alpha \beta \gamma} \left( \Omega_{\gamma,\beta} + i F_{\gamma,\beta} \right)$$

$$P_{\tau} = \left( P_{\tau} \right)^* = \frac{2}{\cos(4\psi) + 2} \Omega_{\tau,\tau}, \quad \Omega_{\tau,\tau} = (G_{\tau\alpha})^* = \frac{8}{\cos(2\psi)} \cos(4\psi) + 2 \Omega_{\tau,\tau}.$$
\( G_{\alpha\beta} = G_{a\beta} = 0 \), \( \epsilon_{a\beta} \bar{\gamma}_{\beta\mu} G_{\beta\beta\mu} = - (\epsilon_{a\beta} \bar{\gamma}_{\beta\mu} G_{\beta\beta\mu})^* = 24 i \frac{\sin(2\varphi)}{\cos(4\varphi) + 2} \Omega_{-\varphi, \bar{\varphi}} \)

\( G_{a\beta} = 0, \quad G_{a\beta} = -(G_{a\beta})^* = i \frac{1}{\sin(2\varphi)} \Omega_{a, \delta, \bar{\delta}} \epsilon_{\beta\nu} \bar{\gamma}_{\nu, \bar{\delta}} \)

(5.5)

and the conditions on the \( F \) fluxes, in addition to the self-duality, are

\[
F_{-a\beta}, a\beta = 2Q , \quad F_{-a\beta}, a\beta = -12\partial_{-a} - 6\tan(2\varphi)\Omega_{-a, a},
\]

\[
\epsilon_{a\beta} \bar{\gamma}_{\beta\mu} F_{-a\beta}, a\beta = -3\frac{\sin(4\varphi)}{\cos(4\varphi) + 2} \Omega_{-a, \varphi}, \quad F_{-a\beta}, a\beta = 0, \quad F_{a\beta, a\beta} = 0
\]

(5.6)

\[
F_{-a\beta} = -\frac{1}{4} \cotan(2\varphi) \Omega_{a, \gamma}, \bar{\gamma} \epsilon_{\gamma, \beta}, \bar{\beta} + i \text{Im}(F_{a, a\beta, a\beta} \epsilon_{a, a\beta, a\beta}) = -6\tan(2\varphi) \Omega_{-a, a}.
\]

The conditions above have an explicit dependence on the angle \( \varphi \). This is due to the nonlinearity of the Killing spinor equation on the functions \( z \).

The other special case that we shall consider is to take \( z \) to be a real matrix. In this case, \( A = 1_{2 \times 2} \) is the identity matrix and the nonlinear system becomes linear. This case closely resembles the maximally supersymmetric \( SU(4) \times \mathbb{R}^8 \) case that we have investigated in [13]. As in the previous case, we shall simply summarize the solution of the linear system. The conditions on the \( P \) and \( G \) fluxes are

\[
P_{a} = 0, \quad G_{a} = 0, \quad G_{a\beta} = 0, \quad G_{a\beta} = 0, \quad G_{-a} = 0, \quad (G_{-a})^* = 0, \quad G_{a} = -2(\Omega_{a, a} - iF_{-a\nu}, \bar{\nu}), \quad G_{-a} = -2(\Omega_{-a\beta} + iF_{-a\beta}, \bar{\beta}),
\]

\[
G_{a\beta} = \Omega_{a, \beta} - \Omega_{\beta, a}, \quad P_{a} = -\frac{1}{3} G_{a\beta} = \frac{2}{3} \Omega_{\beta, a} - iF_{-a\beta}, \quad G_{a\beta} = G_{a\beta}, \quad G_{a\beta} = 0.
\]

(5.7)

\[
P_{a} = \frac{1}{3} G_{a\beta} = \frac{2}{3} \Omega_{\beta, a} + iF_{-a\beta}, \quad G_{a\beta} = -\frac{1}{3} G_{a\beta} + \frac{8}{3} (\Omega_{\beta, a} + iF_{-a\beta}),
\]

\[
G_{a\beta}, \bar{\beta}_{\varphi} = -2(\Omega_{a, \bar{\beta}_{\varphi} + 2iF_{-a\beta} \bar{\beta}_{\varphi} - \delta_{a, \bar{\beta}_{\varphi}} \left( -\frac{2}{3} G_{\beta_{\varphi}, \bar{\beta}_{\varphi}} - \frac{4}{3} \Omega_{\gamma_{\varphi}, \bar{\beta}_{\varphi} - \frac{8}{3} F_{\beta_{\varphi}, \bar{\beta}_{\varphi}} \right),
\]

\[
G_{a\beta}, \bar{\beta}_{\varphi} = -2(\Omega_{a, \bar{\beta}_{\varphi} + 2iF_{-a\beta} \bar{\beta}_{\varphi} - \delta_{a, \bar{\beta}_{\varphi}} \left( \frac{2}{3} G_{\beta_{\varphi}, \bar{\beta}_{\varphi}} - \frac{4}{3} \Omega_{\gamma_{\varphi}, \bar{\beta}_{\varphi}} + \frac{8}{3} F_{\beta_{\varphi}, \bar{\beta}_{\varphi}} \right).
\]

(5.8)
and the conditions on the $F$ fluxes, in addition to the self-duality, are

\[
F_{a}^{\mu_{1}\mu_{2}}=0, \quad F_{-a}^{\alpha \beta}=2Q_{-}, \quad F^{-\hat{a}_{1}\hat{a}_{2}}=0, \quad F^{-a\hat{y}}=0, \quad F_{+}^{\hat{y}}=0. \quad (5.9)
\]

Observe that the conditions have been expressed in representations of $SU(4) \times \mathbb{R}^{8}$ as may have been expected.

### 5.3. The geometry

To investigate the geometry of generic half-maximally supersymmetric $SU(4) \times \mathbb{R}^{8}$ backgrounds, one has to solve the parallel transport equation

\[
z^{-1} dz + C = 0. \quad (5.10)
\]

However as we have seen that the connection depends on $z$, $C = C(z)$, the first-order system is nonlinear. Because of this, we shall focus on the geometry of two examples we have described in the previous section.

First, let us consider the case for which $z$ is diagonal. The conditions on the geometry (5.4) imply that the Killing spinors can be written as

\[
\epsilon_{1} = \epsilon_{\mu} \eta_{1}, \quad \epsilon_{2} = \epsilon_{\nu} \eta_{2}, \quad (5.11)
\]

where $\varphi$ depends on the spacetime coordinates. In addition $\varphi$ satisfies a parallel transport equation

\[
d\varphi + C = 0 \quad (5.12)
\]

where

\[
C_{+} = 0, \quad C_{-} = \frac{1}{2} \tan(2\varphi) \Omega_{-\alpha} + \frac{1}{12} F^{-\hat{a}_{1}\hat{a}_{2}} e^{\hat{a}_{1}\hat{a}_{2}} e^{\hat{a}_{3}}, \quad C_{0} = \frac{\sin(4\varphi)}{\cos(4\varphi) + 2} \Omega_{-\alpha}. \quad (5.13)
\]

Observe that $C$ depends on the angle $\varphi$ as expected. The dependence of $\epsilon_{1}, \epsilon_{2}$ on the angle $\varphi$ cannot be eliminated with a Spin$(9, 1) \times U(1)$ gauge transformation because we have already used such transformations to simplify the Killing spinors\(^{13}\). The angle $\varphi$ is determined by the field equations.

To investigate further the geometry of these backgrounds, one can compute the spacetime form bilinears associated with the Killing spinors (5.11). This can be easily done using the results in appendix A. For this we introduce a light-cone frame and write the spacetime metric as

\[
dx^{2} = 2e^{-\epsilon^{+}} + \delta_{IJ} e^{\epsilon^{I}} e^{\epsilon^{J}}, \quad I, J = 1, \ldots, 8. \quad (5.14)
\]

Then after an appropriate normalization, one finds that the ring of Killing spinor bilinears is generated by

\[
\kappa = e^{-\epsilon}, \quad \xi = e^{-\epsilon} \wedge \omega, \quad \tau = e^{-\epsilon} \wedge \chi, \quad \tau^{*} = e^{-\epsilon} \wedge \chi^{*}, \quad \lambda = e^{-\epsilon} \wedge \omega \wedge \omega \quad (5.15)
\]

as can be seen from the bilinears $\kappa(\epsilon_{1}, \tilde{\epsilon}_{1}, \tilde{\epsilon}_{2}), \xi(\epsilon_{1}, \epsilon_{2}), \tau(\epsilon_{1}, \epsilon_{2}), \tau(\epsilon_{1}, \tilde{\epsilon}_{1}) \text{ and } \tau(\epsilon_{2}, \tilde{\epsilon}_{2})$. Observe that the remaining bilinears in appendix A depend on the angle $\varphi$. Clearly the ring of bilinears is two step nilpotent\(^{14}\). The 1-form $\kappa$ is associated with a null Killing vector field $X = e_{+}$.

---

\(^{13}\) We do not expect additional Spin$(9, 1) \times U(1)$ gauge transformations that preserve the space spanned by $\epsilon_{1}, \epsilon_{2}$ to exist, apart from the subgroup Spin$(1, 1) \times U(1)$ that we have already used.

\(^{14}\) Compare this with the ring of invariant forms on 2n-dimensional manifolds with an $SU(n)$-structure which is not nilpotent.
However unlike the case of maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds, $X$ is not rotation free. It is straightforward to express the various components of the Levi-Civita connection $\Omega$ that appear in the geometry relations (5.4) in terms of the covariant derivatives of the generators (5.15). Then one could rewrite (5.4) in terms of the covariant derivatives. However, we shall not do this here. Observe that the various geometry conditions in (5.4) depend on the angle $\varphi$ and that the covariant derivatives of (5.15) do not depend on $\varphi$. As a result, the geometry conditions (5.4) are not simply linear relations between the various components of the covariant derivatives of (5.15) as in the case of Hermitian manifolds in [23].

Next, let us take $z$ to be a real invertible matrix. The parallel transport equation for $z$ is given in (5.7). It is easy to see that the connection $C$ can be written as

$$C = \hat{\Omega}^0 t_0 + \hat{\Omega}^1 t_1$$  (5.16)

where $t_0 = 1_{2 \times 2}$, $t_1$ is skew-symmetric with $(t_1)_{12} = 1$, and $\hat{\Omega}^0$ and $\hat{\Omega}^1$ are easily computed from (5.7). Since $t_0$ and $t_1$ commute, $C$ is an Abelian connection. In addition $C$ does not depend on $z$ because $A = 1_{2 \times 2}$ in this example. A necessary condition for the existence of a solution to the parallel transport problem is that the curvature of $C$, $F = dC$, must vanish. In addition, it turns out that $C$ can be trivialized with the $e^{i\Gamma a + b_\infty}$ gauge transformation for suitable choices of the parameters $a, b$. Therefore, we have shown that up to a Spin(9, 1) gauge transformation, we can set $z = 1_{2 \times 2}$ and so the two Killing spinors are

$$\epsilon_1 = \eta_1, \quad \epsilon_2 = \eta_2.$$  (5.17)

Setting $z = 1_{2 \times 2}$ in (5.7), the resulting equations are interpreted either as restrictions on the geometry or as conditions that relate components of the fluxes to the geometry.

To further investigate the geometry, we write the spacetime metric in a light-cone frame as in (5.14). Then it can be easily seen from the results of appendix A that the ring of $SU(4) \ltimes \mathbb{R}^8$ forms is generated by the forms in (5.15). However unlike the previous case, all spinor bilinears are constant in the frame $e^-, e^+, e^\epsilon$. The ring of spinor bilinears is again two step nilpotent. It is easy to see from the conditions on the geometry that $\kappa$ is a null Killing vector field. Unlike the maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$-backgrounds, $\kappa$ is not rotation free. In particular observe that $(d\epsilon)_a\beta$ is proportional to $G_\epsilon a\beta$. One can proceed further to re-express the conditions on the geometry in terms of the Levi-Civita covariant derivatives of the spacetime forms $\kappa, \xi, \tau, \tau^\epsilon$ and $\lambda$ as suggested in [23]. However, we shall not do this here.

### 6. Degenerate $N = 2$ backgrounds with $SU(4) \ltimes \mathbb{R}^8$-invariant spinors

#### 6.1. Preliminaries

The Killing spinors of degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds can be written as

$$\epsilon_1 = (f' - g_2 + ig_1)1 + (f' + g_2 + ig_1)e_{1234}$$
$$\epsilon_2 = w\epsilon_1,$$

where $f, g_1, g_2$ are real functions, $f, g_2 \neq 0$. To see this, it can always be arranged such that $z_{11} \neq 0$. If this is the case, then solving the condition $\det z = 0$ for $z_{22}$ and substituting it into the second spinor, one finds that $\epsilon_2 = w\epsilon_1$, where $w = z_{21}/z_{11}$. If $w$ is real, then $\eta_1$ and

\begin{footnote}{15}It is not always possible to find a gauge such that the Killing spinors of a supersymmetric background are all constant. This has to be shown in each case. An explanation of this has been given in [13]. As an example, it can be shown that the only maximally supersymmetric background of IIB supergravity with constant Killing spinors is locally isometric to Minkowski spacetime but it is known that there are two more maximally supersymmetric backgrounds, the $AdS_5 \times S^5$ and the plane wave.
\end{footnote}
η_2 are linearly dependent and so this case is excluded. Thus, we take w to be complex with Im w \neq 0.

The algebraic Killing spinor equation for \epsilon_2 can be written as

\[ w^{-1}w^* P_A \Gamma^A C \epsilon^*_1 + \frac{i}{16} G_{ABC} \Gamma^{ABC} \epsilon^*_1 = 0. \] (6.2)

Subtracting this from the algebraic Killing spinor equation of \epsilon_1 and using Im w \neq 0, one finds that

\[ P_A \Gamma^A C \epsilon^*_1 = 0, \quad G_{ABC} \Gamma^{ABC} \epsilon^*_1 = 0. \] (6.3)

Similarly using \mathcal{D} \epsilon_1 = 0, the Killing spinor equations of \epsilon_2 associated with the supercovariant derivative \mathcal{D} become

\[ \partial_A w \epsilon_1 - \frac{i}{16} (w^* - w) \left[ \Gamma_A^{B_1 B_2 B_3} G_{B_1 B_2 B_3} - 3 \Gamma^{B_1 B_2} G_{A B_1 B_2} \right] \epsilon^*_1 = 0. \] (6.4)

In turn the Killing spinor equation associated with the supercovariant derivative of \epsilon_1 can be rewritten as

\[ \bar{\nabla}_A \epsilon_1 - (w^* - w)^{-1} \partial_A w \epsilon_1 + \frac{i}{48} \Gamma^{N_1 \ldots N_4} \epsilon_1 F_{N_1 \ldots N_4} = 0. \] (6.5)

Therefore the independent equations that have to be solved in this case are (6.3), (6.4) and (6.5). It is clear that in this special case, the Killing spinor equations factorize in a way similar to that we have encountered for maximally supersymmetric \mathcal{H}-backgrounds. The linear system associated with the above Killing spinor equations is given in appendix E.

6.2. The solution of the linear system

The linear system associated with the Killing spinor equations has been presented in appendix E. We shall not explain in detail the solution of this system. It turns out that it is simpler than that of \( N = 1 \) backgrounds with a SU(4) \times \mathbb{R}^8\)-invariant Killing spinor [12]. First, we shall summarize the conditions that are implied from the equations involving the G and P fluxes and then we shall give the conditions that are implied by the rest of the equations. In particular, we have

\[ \partial_A w = \partial_{-} w = \partial_{a} w = \partial_{\bar{a}} w = 0, \]
\[ P_{a} = G_{-b}^{\delta} = G_{\bar{a}\bar{\delta}} = G_{-a} = G_{a;\partial\bar{\alpha}1} = G_{a\delta} = 0, \]
\[ G_{a\bar{\alpha}} = G_{\bar{a}\bar{\alpha}} = G_{-\bar{\alpha}} = G_{a1\bar{\alpha}1} = G_{\bar{a}\bar{\delta}} = 0, \]
\[ (f + g_2 - ig_1) P_{\bar{a}} = 0, \quad (f - g_2 - ig_1) P_{a} = 0, \]
\[ (f - g_2 - ig_1) G_{\bar{a}\bar{\delta}} = 0, \quad (f + g_2 - ig_1) G_{a\bar{\delta}} = 0. \] (6.6)

The first equation implies that the complex function w is constant. Therefore \epsilon_2 is linearly dependent on \epsilon_1 over the complex numbers as expected. The last three equations require some explanation. If the Killing spinor \epsilon_1 is not pure, then \( P_{a} = P_{\bar{a}} = 0 \) and similarly \( G_{\bar{a}\bar{\delta}} = G_{a\bar{\delta}} = 0 \). However if the Killing spinor \epsilon_1 is pure, then either \( P_{a} = 0 \) or \( P_{\bar{a}} = 0 \) and similarly either the (1,2) or the (2,1) component of G vanishes depending on whether \epsilon_1 is proportional to either 1 or \epsilon_{1234}, respectively. In addition, if \epsilon_1 is a pure spinor, the last equation implies that either \( G_{-a\bar{\delta}} \) or \( G_{-\bar{a}\delta} \) will vanish. The Killing spinor equations do not determine the traceless \( G_{-a\bar{\delta}} \) component of G.
Next, we summarize the conditions on the flux $F$. In addition to the self-duality condition on $F$, we find that

$$iF_{\alpha\beta\gamma\nu} = \frac{f^2 + g_1^2 + g_2^2}{2f g_2} \Omega_{\alpha\beta\gamma\nu},$$

$$iF_{\alpha\beta\gamma\nu} = -\frac{f - (g_2 - ig_1)^2}{4fg_2} \Omega_{\alpha\beta\gamma\nu},$$

$$iF_{\alpha\beta\gamma\nu} = -\frac{1}{2f g_2} \left[ -\left(f^2 + g_1^2 + g_2^2\right) \Omega_{\alpha\beta\gamma\nu} + f^2 \Omega_{\alpha\beta\gamma\nu} + \Omega_{\gamma\nu}\Omega_{\alpha\beta}\Omega_{\alpha\beta}\Omega_{\alpha\beta}\right].$$

$$iF_{\beta\gamma\nu} = \frac{1}{f g_2} \left[ \Omega_{\alpha\beta\gamma\nu} + \Omega_{\alpha\beta\gamma\nu} \right] (\partial f - (g_2 - ig_1)^2).$$

(6.7)

$F_{\alpha\beta\gamma\nu} = 0,$

and if $\epsilon_1$ is not pure

$$iF_{\alpha\beta\gamma\nu} = 0.$$

(6.8)

Alternatively, if $\epsilon_1$ is pure the last equation becomes

$$iF_{\alpha\beta\gamma\nu} = -\Omega_{\alpha\beta\gamma\nu},$$

(6.9)

where the sign depends on whether the pure spinor is proportional to 1 or $\epsilon_{1234}$ respectively.

Taking the complex conjugate of the above relation, we find

$$\Omega_{\alpha\beta\gamma\nu} + \Omega_{\alpha\beta\gamma\nu} = 0.$$  

(6.10)

Observe that in the pure spinor case some of the components of $F$ in (6.7) vanish.

Finally, the conditions on the geometry are

$$\Omega_{\alpha\beta\gamma\nu} = 0, \quad iQ_{\alpha\beta\gamma\nu} = 0,$$

$$2\partial_\alpha f + \Omega_{\alpha\beta\gamma\nu} f = 0,$$

$$2\partial_\alpha g_1 + \Omega_{\alpha\beta\gamma\nu} g_1 = 0,$$

$$2\partial_\alpha g_2 - ig_1 \Omega_{\alpha\beta\gamma\nu} g_2 = 0,$$

$$2\partial_\alpha g_1 = Q_{\alpha\beta\gamma\nu} g_1 + g_1 \Omega_{\alpha\beta\gamma\nu} g_1 = 0,$$

$$\partial (f^2 + g_1^2 + g_2^2) + \Omega_{\alpha\beta\gamma\nu} (f^2 + g_1^2 + g_2^2) = 0,$$

$$\Omega_{\alpha\beta\gamma\nu} = 0, \quad \Omega_{\alpha\beta\gamma\nu} = 0,$$

$$4f^2 g_1^2 \Omega_{\alpha\beta\gamma\nu} + (f^2 + g_1^2 + g_2^2)^2 \Omega_{\alpha\beta\gamma\nu} = 0,$$

$$\frac{1}{2}(f^2 + g_1^2 + g_2^2) \Omega_{\alpha\beta\gamma\nu} + (f^2 + g_1^2 + g_2^2) \Omega_{\alpha\beta\gamma\nu} = 0.$$  

(6.11)

and

$$Q_{\alpha\beta\gamma\nu} = \frac{i}{g_2} \Omega_{\alpha\beta\gamma\nu} - \frac{i}{g_2} (g_2 - ig_1) (f^2 + g_1^2 - ig_1 g_2) \Omega_{\alpha\beta\gamma\nu},$$

$$\partial_\alpha f = -\frac{i g_1 f}{g_2^2} \Omega_{\alpha\beta\gamma\nu} - \frac{1}{2} f \Omega_{\alpha\beta\gamma\nu} + \frac{i(g_2 - ig_1)}{2fg_2^2} (g_1 (f^2 + g_1^2 + g_2^2) - ig_1 (g_1^2 + g_2^2)) \Omega_{\alpha\beta\gamma\nu},$$

$$\partial_\alpha g_1 = \frac{(f^2 + g_2^2)}{g_2^2} \Omega_{\alpha\beta\gamma\nu} - \frac{1}{2} g_1 \Omega_{\alpha\beta\gamma\nu} - \frac{1}{2} (g_2 (f^2 - g_2^2) - ig_1 (f^2 + g_1^2 - 2g_2^2)) \Omega_{\alpha\beta\gamma\nu},$$

$$\partial_\alpha g_2 = \frac{i}{2} g_1 \Omega_{\alpha\beta\gamma\nu} + \frac{1}{2} g_2 \Omega_{\alpha\beta\gamma\nu} + \frac{1}{2} (f^2 + g_1^2 - ig_1 g_2) \Omega_{\alpha\beta\gamma\nu}.$$  

(6.12)
and if \( \epsilon_1 \) is not pure
\[
\Omega_{a, \dot{\gamma} \dot{\beta}} = 0.
\] (6.13)

Observe that some of the conditions on the fluxes \( F \) can be interpreted as conditions on
the geometry because they restrict the \( \partial_- \) derivative of functions \( f, g_1, g_2 \) which determine the
spinor. In turn, the integrability conditions restrict both the fluxes and the geometry.

6.3. The geometry

The linearly independent forms on the spacetime associated with the Killing spinor bilinears
are those that we have computed in [12] for the case of \( N = 1 \) \( SU(4) \times \mathbb{R}^8 \)-backgrounds.
Because of this, we shall not present them here. It turns out that the vector field \( X \) associated
with the form
\[
\kappa = \left(f^2 + g_1^2 + g_2^2\right)e^-
\] (6.14)
is a null Killing vector field. Moreover one can choose the gauge \( f^2 + g_1^2 + g_2^2 = 1 \). This is
achieved by an Spin\((9, 1)\) transformation \( e^{\alpha_1 \alpha_0} \) for an appropriate choice of the parameter \( a \).

The metric then can be put into the form (5.14). Observe that the first-order system for the
functions \( f, g_1, g_2 \) is again nonlinear.

7. Examples of integrability conditions

In this section, we will solve the linear systems associated with the integrability conditions
for maximal16 Spin\((7) \times \mathbb{R}^8 \) and \( SU(4) \times \mathbb{R}^8 \)-backgrounds. This will determine which field
equations are implied by the Killing spinor equations. The remaining field equations, which
still need to be imposed on the supersymmetric background, will be given explicitly.

There are linear systems associated with the Killing spinor equations and with integrability
conditions of any supersymmetric background. However, as was explained in section 3.2, these
systems factorize for maximal \( H \)-backgrounds. In particular, the linear system of integrability
conditions splits up into three separate parts involving only two types of field equations: \( E \)
and \( LF, LP \) and \( BP \) and \( BG \) and \( LG \). This considerably simplifies the analysis of the linear
systems. In what follows, we shall apply the formalism to the linear systems of the maximal
Spin\((7) \times \mathbb{R}^8 \) and \( SU(4) \times \mathbb{R}^8 \)-backgrounds.

7.1. Maximal Spin\((7) \times \mathbb{R}^8 \)-backgrounds

The field equations that are not implied by the Killing spinor equations of maximally
supersymmetric Spin\((7) \times \mathbb{R}^8 \)-backgrounds are (where the tilde indicates traceless components)
\[
E_{-\cdot}, LP, BP_{a,\alpha_1}, BP_{a,\beta}, BP_{\dot{\beta},\dot{\alpha}_2}, BP_{\dot{\alpha}_3}, BP_{\dot{\alpha}_4}, BP_{\dot{\beta},\dot{\alpha}_2}, BG_{a,\alpha_1}, BG_{a,\alpha_2}, BG_{a,\alpha_2}, BG_{\alpha_1,\alpha_2,\beta},\]
\[
BG_{a,\alpha_1,\alpha_2,\beta}, BG_{a,\alpha_1,\alpha_2,\beta}, BG_{a,\alpha_1,\alpha_2,\beta}, BG_{a,\alpha_1,\alpha_2,\beta}, BG_{a,\alpha_1,\alpha_2,\beta},\]
\[
LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta},\]
\[
LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}, LF_{a,\alpha_1,\alpha_2,\beta}.
\] (7.1)

16The Killing spinor equations for these cases were solved in [13]. The same holds for the case with maximal \( G_2 \)
supersymmetry, but since this yielded a purely gravitational solution we will not discuss it.
subject to the relations

\[
LP + 2BP_{-\rho} = BP_{a_{\rho}a_{\rho}} - \frac{1}{2} \varepsilon_{a_{\rho}a_{\rho}} \hat{\beta} \hat{\gamma} B_{\hat{\rho} \hat{\gamma}} B_{\hat{\rho} \hat{\gamma}} = 0,
\]
\[
LG_{a_{\rho}a_{\rho}} + 2BG_{a_{\rho}a_{\rho}} = LG_{a_{\rho}} + 2BG_{a_{\rho}} = 0,
\]
\[
L_{\hat{\rho} \hat{\gamma}} + 2BG_{\hat{\rho} \hat{\gamma}} = LG_{\hat{\rho} \hat{\gamma}} + 2BG_{\hat{\rho} \hat{\gamma}} = 0,
\]
\[
8BG_{\hat{\rho} \hat{\gamma}} = 8 \varepsilon_{\hat{\rho} \hat{\gamma}} \gamma_{\hat{\rho} \hat{\gamma}} = \varepsilon_{\hat{\rho} \hat{\gamma}} \gamma_{\hat{\rho} \hat{\gamma}} = 0.
\]

Similarly, the components of \( L_{\hat{\rho} \hat{\gamma}} \) take values in \( \text{Spin}_{8} \), where

\[
FP_{\hat{\rho} \hat{\gamma}} = \varepsilon_{\hat{\rho} \hat{\gamma}} \gamma_{\hat{\rho} \hat{\gamma}} = \varepsilon_{\hat{\rho} \hat{\gamma}} \gamma_{\hat{\rho} \hat{\gamma}} = 0.
\]

Using these relations, one can show that all the field equations and Bianchi identities are satisfied provided one imposes the vanishing of the equations

\[
E_{\hat{\rho} \hat{\gamma}}, \quad LP_{\hat{\rho} \hat{\gamma}} = 0, \quad LP_{\hat{\rho} \hat{\gamma}} = 0, \quad L_{\hat{\rho} \hat{\gamma}}, \quad LF_{\hat{\rho} \hat{\gamma}}, \quad BF_{\hat{\rho} \hat{\gamma}} = 0.
\]

Since the Bianchi identity and field equation for \( F \) are interchangeable because of the self-duality of \( F \), the only field equation that needs to be imposed is the \( E_{\hat{\rho} \hat{\gamma}} \) component of the field equations.

We now turn to the corresponding supersymmetric background. As has been shown in [13], the metric of Spin(7) \( \times \mathbb{R}^{8} \)-backgrounds can be written as

\[
ds^{2} = 2 du + \alpha du + \beta_{I} dy^{I} + \gamma_{IJ} dy^{I} dy^{J},
\]

with \( \alpha, \beta \) and \( \gamma_{IJ} \) functions of \( v \) and \( y^{I} \) only, and \( I = (1, \ldots, 8) \). This is a pp-wave metric with rotation, see also [24]. A natural frame is given by

\[
e^{-} = dv, \quad e^{+} = du + \alpha dv + \beta_{I} dy^{I}, \quad e^{a} = e^{I} dy^{I}, \quad e^{0} = e^{I} dy^{I}.
\]

The components of the spin connection are

\[
\Omega_{P, Q} = e^{I}(P \alpha_{Q} - \partial_{P} \beta_{Q}), \quad \Omega_{-P} = \partial_{P} \alpha - \partial_{P} \beta_{P}.
\]

and \( \Omega_{P, R, Q} \), where \( P = (\alpha, \beta) \). In addition, the components of \( \Omega_{P, Q} \) and \( \Omega_{-P, Q} \) take values in Spin(7), i.e.

\[
\Omega_{P, a_{\rho}a_{\rho}} = \varepsilon_{a_{\rho}a_{\rho}} \hat{\gamma} \hat{\rho} \hat{\gamma} \Omega_{P, \hat{\rho} \hat{\gamma}}, \quad \Omega_{P, a^{a}} = 0.
\]

The Killing spinor equations restrict the fluxes as follows. The non-vanishing components of the fluxes are \( P_{-}, G_{P_{-}} \) and \( F_{P_{-}} \) which in addition satisfy the following conditions.

\[
G_{P_{-}} \text{ takes values in Spin(7), i.e. in the decomposition } \Lambda^{7}(\mathbb{R}^{8}) \oplus \Lambda_{7}^{8}(\mathbb{R}^{8}) \text{ into Spin(7) representations, only the } \Lambda_{7}^{8}(\mathbb{R}^{8}) \text{ is allowed by the Killing spinor equations. Similarly, the components of } F_{P_{-}} \text{ lie in } \Lambda_{7}^{8}(\mathbb{R}^{8}) \text{ and } \Lambda_{7}^{8}(\mathbb{R}^{8}) \text{ in the decomposition } \Lambda^{7}(\mathbb{R}^{8}) = \Lambda_{7}^{1}(\mathbb{R}^{8}) \oplus \Lambda_{7}^{2}(\mathbb{R}^{8}) \oplus \Lambda_{7}^{3}(\mathbb{R}^{8}) \oplus \Lambda_{7}^{4}(\mathbb{R}^{8}) \text{. The singlet is given by}
\]

\[
F_{P_{-}} = 24 Q_{-},
\]

where \( \psi \) is the Spin(7)-invariant four-form, whose definition can be found in [13].
Among the field equations that still need to be imposed on the solution to the $N = 2$ Spin(7) $\ltimes \mathbb{R}^8$ Killing spinor equations is the Einstein equation $\mathcal{E}_{\cdots\cdots}$, which is given by
\begin{align*}
- (\partial^\alpha + \Omega_{Q}^{\gamma} \partial^\gamma) (\partial \alpha - \partial \beta \beta) + \partial \rho \rho_Q (\partial^\rho \beta^Q - \frac{1}{2} \gamma^{IJ} \partial \gamma_{IJ} - \frac{1}{3} \partial \gamma_{IJ} \partial \gamma_{IJ}) \\
\quad - \frac{1}{2} F_{-} F_{-} + F_{-} - P_{4} P_{4} - \frac{1}{2} G_{4} P_{4} P_{4} - 2 P_{-} P_{-} = 0,
\end{align*}
(7.9)
where $\gamma_{IJ}$ is the inverse of the metric $\gamma_{IJ}$ defined in (7.4). For the special case of $\alpha, \beta$ and $\gamma_{IJ}$ independent of $v$ this equation becomes
\begin{align*}
- \Box_8 \alpha + \partial \rho \rho_Q (\partial^\rho \beta^Q - \frac{1}{2} \gamma^{\alpha} \partial \gamma_{\alpha} - \frac{1}{3} \partial \gamma_{\alpha} \partial \gamma_{\alpha}) \\
\quad - \frac{1}{2} F_{-} F_{-} + F_{-} - P_{4} P_{4} - \frac{1}{2} G_{4} P_{4} P_{4} - 2 P_{-} P_{-} = 0,
\end{align*}
(7.10)
where $\Box_8$ is the Laplacian on the eight-dimensional space and $\partial \rho \rho_Q$ only takes values in Spin(7).

In addition one needs to impose several components of the Bianchi identities on the fluxes $P_{-}, G_{PQ}$ and $F_{PQ}$ for $\cdots$. For example, the remaining $B \beta$ components imply that $P_{-}$ is a function of $v$ only. We will not analyse the rest of these restrictions in detail.

Observe that the contribution of the rotation in the Einstein equations has a different sign from that of the contribution of the fluxes. Because of this and assuming that the transverse space of the pp-wave is a compact Spin(7) manifold, the Einstein equation can be solved provided that the total rotation cancels the contributions from the fluxes. This means that the integral of the expression on the left-hand side of (7.10) must vanish. For a detailed similar argument see [12]. The above solution resembles flux-tube type of configurations [25] but without the backreaction of the branes. There are also similarities with the solutions of [26, 27].

### 7.2. Maximal SU(4) $\ltimes \mathbb{R}^8$ backgrounds

The field equations that are _not_ implied by the Killing spinor equations of maximally supersymmetric SU(4) $\ltimes \mathbb{R}^8$-backgrounds are
\begin{align*}
E_{\cdots\cdots}, \quad LP, \quad B \beta P_{\alpha \beta}, \quad B P_{\alpha -}, \quad B \beta P_{\alpha -}, \quad B P_{-\alpha}, \quad B \beta P_{-\alpha}, \quad L G_{\alpha \beta}, \quad L G_{\alpha -}, \quad L G_{-\alpha}, \\
B \gamma G_{\alpha \beta \gamma \beta}, \quad B G_{\alpha \beta \gamma \gamma}, \quad B G_{\alpha \beta \gamma \gamma}, \quad B G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \\
\gamma G_{\gamma IJ \gamma IJ}, \quad \gamma G_{\gamma IJ \gamma IJ}, \quad \gamma G_{\gamma IJ \gamma IJ}.
\end{align*}
subject to the relations
\begin{align*}
LP + 2B \beta P_{-\alpha} &= L G_{\alpha \beta} + 24 B \gamma G_{\alpha \beta \gamma \beta} = 0, \\
LG_{\alpha -} - 24 B G_{\alpha \gamma \gamma} &= L G_{\alpha -} + 24 B G_{\alpha \gamma \gamma} = 0.
\end{align*}
(7.12)

Using these relations, one can show that all field equations are satisfied provided that the field equations
\begin{align*}
E_{\cdots\cdots}, \quad B \beta P_{\alpha \beta}, \quad B P_{\alpha -}, \quad B \beta P_{\alpha -}, \quad B P_{-\alpha}, \quad B \beta P_{-\alpha}, \quad B \gamma G_{\alpha \beta \gamma \beta}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \quad B \gamma G_{\alpha \beta \gamma \gamma}, \\
B G_{\alpha \beta \gamma \beta}, \quad B G_{\alpha \beta \gamma \gamma}, \quad B G_{\alpha \beta \gamma \gamma}, \quad L F_{\alpha \beta \gamma \beta}, \quad L F_{\alpha \beta \gamma \gamma}, \quad L F_{\alpha \beta \gamma \gamma},
\end{align*}
are satisfied. It is easy to see that apart from the Bianchi identities, the only field equation that one has to impose is the $E_{\cdots\cdots}$ component of the Einstein equation.

The investigation of the field equations of the maximal SU(4) $\ltimes \mathbb{R}^8$-backgrounds is related to that of maximal Spin(7) $\ltimes \mathbb{R}^8$-backgrounds. In particular, there is a gauge for the Killing spinors such that $\Omega_{\alpha -} = 0$ and $\Omega_{\alpha \beta} = 0$ [13]. In addition the metric can be written in Penrose coordinates as in (7.4) and so one can introduce the frame (7.5). One can compute the spin connection which has components $\Omega_{\alpha -}, \Omega_{\alpha \gamma}, \Omega_{\alpha \gamma}$ and $\Omega_{\alpha \gamma \gamma}$. The first three are given in (7.6), and in addition the latter two take values in SU(4), i.e.,
\begin{align*}
\Omega_{P, \alpha \alpha} = 0, \quad \Omega_{P, \alpha} = 0.
\end{align*}
(7.14)
and similarly for $\Omega_{-\rho \Omega}$. The non-vanishing components of the fluxes are

\[ P_{-}, \ G_{\alpha\beta}, \ G^{\alpha}_{\beta}(v), \ F_{\alpha_{1}\alpha_{2}}(v), \ F^{\alpha}_{\beta}(v), \ F^{\alpha}_{\beta}(v) = 2Q_{-}. \quad (7.15) \]

Using these, one can easily compute the Einstein equation $E_{--}$. It is easy to see that it takes the same form as $(7.9)$. In addition the remaining Bianchi identities will impose closure of the remaining fluxes $P_{-}, G_{PQ}$ and $F_{P_{1}...P_{4}}$.

In the case that the components of the metric are independent of the $v$ coordinate, the transverse space of the pp-wave is a Calabi–Yau manifold. To find a solution, one can use the Donaldson theorem for $U(1)$ connections and require cancellation of the rotation and flux charges. Using a similar argument to that in [12], one can show that there is a smooth solution for pp-waves with transverse space a compact Calabi–Yau manifold.

8. Concluding remarks

We have shown that the Killing spinor equations of any IIB supergravity background can be written as a linear system for the fluxes, geometry and spacetime derivatives of the functions that determine the Killing spinors. This has been achieved by using the spinorial geometry techniques of [14]. We have also shown that another linear system, constructed in a similar way, can be used to determine the field equations and Bianchi identities of IIB supergravity that are determined by the Killing spinors for any supersymmetric background. These two linear systems can be used to systematically investigate all supersymmetric backgrounds of IIB supergravity.

For general supersymmetric backgrounds these two linear systems are rather complicated. However, we have shown that these linear systems simplify for backgrounds that admit $H$-invariant spinors, $H \subset \text{Spin}(9,1)$. We have mostly focused on two cases, those for which the background admits a maximal number of $H$-invariant Killing spinors, maximally supersymmetric $H$-backgrounds, and those that admit half the number of maximal $H$-invariant spinors, half-maximally supersymmetric $H$-backgrounds. In the former case, the Killing spinor equations factorize and the resulting linear systems are easy to solve. We have demonstrated that the system associated with the Killing spinor equations gives rise to a flatness condition for the connection which is identified as the restriction of the supercovariant connection on the bundle of Killing spinors $K$.

There are several cases of half-maximal $H$-backgrounds which should be considered. The generic case consists of those backgrounds for which the Killing spinors are linearly independent over the complex numbers. There are also several degenerate cases for which the Killing spinors are linearly dependent over the complex numbers but linearly independent over the real numbers. The degenerate cases are of co-dimension two or more relative to the generic case. We have demonstrated that the Killing spinor equations of half-maximal $H$-backgrounds do not factorize. The linear system of the Killing spinor equations gives rise to a flatness condition for the restriction of the supercovariant connection on the bundle of Killing spinors $K$. However, the restricted connection depends nonlinearly on the functions that determine the Killing spinors.

To give an overview of the current status of the problem in IIB supergravity, we summarize some of the results in table 1. In this table, we indicate the cases that have been investigated as well as the maximal and half-maximal $H$-backgrounds that remain to be tackled.

In table 1, the list of cases that remain to be tackled contains the $N = 4$ and $N = 8 SU(3)$-backgrounds. It is expected that the former includes many important backgrounds which are dual to $N = 1$ ($N = 4$) four-dimensional gauge theories. The list also includes all supersymmetric backgrounds that preserve 1/2 of the supersymmetry ($N = 16$). There are
Table 1. ‘√’ denotes the cases for which the Killing spinor equations have already been solved. ‘⊙’ denotes the remaining cases that correspond to backgrounds with $H$-invariant spinors and can be tackled with the techniques described in this paper. ‘−’ denotes the cases that do not occur, e.g. there are no backgrounds with $N > 2$ and Spin(7) $\ltimes \mathbb{R}^8$-invariant Killing spinors. The remaining entries may occur but it is expected that the associated linear systems are more involved.

| $H$                | $N = 1$ | $N = 2$ | $N = 3$ | $N = 4$ | $N = 6$ | $N = 8$ | $N = 16$ | $N = 32$ |
|--------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| Spin(7) $\ltimes \mathbb{R}^8$ | √       | √       | −       | −       | −       | −       | −       | −       |
| SU(4) $\ltimes \mathbb{R}^8$ | √       | √       | √       | −       | −       | −       | −       | −       |
| $G_2$              | √       | ⊙       | √       | −       | −       | −       | −       | −       |
| Sp(2) $\ltimes \mathbb{R}^8$ | −       | ⊙       | √       | −       | −       | −       | −       | −       |
| $(SU(2) \times SU(2))$ $\ltimes \mathbb{R}^8$ | −       | ⊙       | √       | −       | −       | −       | −       | −       |
| SU(3)              | −       | ⊙       | √       | −       | −       | −       | −       | −       |
| $\mathbb{R}^8$     | −       | ⊙       | √       | −       | −       | −       | −       | −       |
| SU(2)              | −       | ⊙       | √       | −       | −       | −       | −       | −       |
| 1                  | −       | ⊙       | √       | −       | −       | −       | −       | −       |

three classes of 1/2 supersymmetric backgrounds: the maximal $\mathbb{R}^8$-backgrounds, the maximal $SU(2)$-backgrounds and the half-maximal 1-backgrounds. It would be of interest to investigate all these cases.

Acknowledgments

The work of UG is funded by The Swedish Research Council and in addition the research of both UG and DR is funded by the PPARC grant PPA/G/O/2002/00475.

Appendix A. Spinors

The description of IIB supergravity spinors that we used in this paper can be found in [12]. For the general theory see [28–30]. Here for completeness, we shall briefly summarize the main formulae without explanation and then apply them to compute some new spacetime form bilinears.

The space of Dirac spinors of Spin$(9,1)$ is $\Delta_c = \Lambda^*(U \otimes \mathbb{C})$, where $U = \mathbb{R}(e_1,\ldots,e_5)$. The complex chiral representations are $\Delta^+_c = \Lambda^{even}(U \otimes \mathbb{C})$ and $\Delta^-_c = \Lambda^{odd}(U \otimes \mathbb{C})$. The gamma matrices act on $\Delta_c$ as

$$
\Gamma_0 \eta = -e_5 \wedge \eta + e_5 \cdot \eta, \quad \Gamma_5 \eta = e_5 \wedge \eta + e_5 \cdot \eta,
$$

$$
\Gamma_i \eta = e_i \wedge \eta + e_i \cdot \eta, \quad i = 1,\ldots,4
$$

$$
\Gamma_{5+i} \eta = ie_i \wedge \eta - ie_i \cdot \eta,
$$

where $\Gamma_0$ is the gamma matrix along the time direction. The Dirac inner product is $D(\eta,\theta) = \langle \Gamma_0 \eta, \theta \rangle$, where $\langle , \rangle$ is the Hermitian inner product on $U \otimes \mathbb{C}$ and then extended to $\Delta_c$. The Majorana Spin$(9,1)$-invariant inner product that we use is $B(\eta,\theta) = \langle B(\eta^*), \theta \rangle$, where $B = \Gamma_{0789}$. The Majorana–Weyl representations $\Delta^\pm$ of Spin$(9,1)$ are constructed by imposing the reality condition $\eta = -\Gamma_0 B(\eta^*)$ or equivalently

$$
\eta^* = \Gamma_{6789} \eta,
$$

on $\Delta^\pm_c$. The map $C = \Gamma_{6789}$ is the charge conjugation matrix.

The oscillator basis in $\Delta_c$ that we use in this paper is

$$
\Gamma_0 = \frac{1}{\sqrt{2}} (\Gamma_0 + i \Gamma_{0+5}), \quad \Gamma_\pm = \frac{1}{\sqrt{2}} (\Gamma_5 \pm \Gamma_0), \quad \Gamma_\alpha = \frac{1}{\sqrt{2}} (\Gamma_\alpha - i \Gamma_{\alpha+5}).
$$
In addition, we define $\Gamma^B = g^{BA} \Gamma_A$. The 1 spinor is a Clifford vacuum, $\Gamma^\alpha = 1 = \Gamma^{-1} = 0$ and the representation $\Delta_c$ can be constructed by acting on 1 with the creation operators $\Gamma^\alpha, \Gamma^\alpha$. The spacetime form bilinears associated with the pair of spinors $(\eta, \epsilon)$ are

$$\alpha(\eta, \epsilon) = \frac{1}{k!} B(\eta, \Gamma_{\Delta_c \cdots \Delta_c} \eta) \epsilon^{A_1} \wedge \cdots \wedge \epsilon^{A_k}, \quad k = 0, \ldots, 9. \quad (A.4)$$

The spacetime form bilinears of the spinors 1 and $e_{1234}$ have been computed in [12]. Using these, we can easily find the spacetime form bilinears of the half-maximally supersymmetric $SU(4) \ltimes \mathbb{R}^8$ backgrounds. The Killing spinors are $\epsilon = z \eta$, where $\eta_1 = 1 + e_{1234}$ and $\eta_2 = i(1 - e_{1234})$. Since the Killing spinors $\epsilon_1$ and $\epsilon_2$ are generically complex, $\tilde{\eta}_1 = C(\eta_1)^*$ and $\tilde{\eta}_2 = C(\eta_2)^*$ are also defined on the spacetime although they are not necessarily Killing. Because of this, the forms that are defined on the spacetime are associated with the bilinears $(\epsilon_1, \epsilon_2), (\tilde{\eta}_1, \tilde{\eta}_2)$ and $(\epsilon_1, \tilde{\eta}_2), i, j = 1, 2$. It suffices to compute the forms associated with the first two bilinears because the forms of the last bilinear can be easily computed from those of the first. To see this observe that since $\epsilon = z \eta$ and $\eta_1$ are Majorana–Weyl spinors, then $\tilde{\epsilon} = z^* \eta$. Thus the effect of the charge conjugation operation is to replace the matrix $z$ with its complex conjugate $z^*$. In particular, we find the 1-forms

$$\xi(\epsilon_1, \epsilon_2) = -2 \det z (e^0 - e^5) \wedge \omega,$$

$$\xi(\tilde{\eta}_1, \tilde{\eta}_2) = -2 (z_{11} z_{12} - z_{12} z_{11}) (e^0 - e^5) \wedge \omega,$$

and the 3-forms

$$\tau(\epsilon_1, \epsilon_2) = (e^0 - e^5) \wedge [(z_{11} + iz_{12})^2 \chi + (z_{11} - iz_{12})^2 \chi^* - (z_{11} + iz_{12}) \omega \wedge \omega],$$

$$\tau(\tilde{\eta}_1, \tilde{\eta}_2) = (e^0 - e^5) \wedge [(z_{11} + iz_{12})^2 \chi + (z_{11} - iz_{12})^2 \chi^* - (z_{11} + iz_{12}) \omega \wedge \omega],$$

$$\tau(\epsilon_1, \tilde{\eta}_2) = (e^0 - e^5) \wedge [(z_{11} + iz_{12}) (z_{11} + iz_{12}) \chi + (z_{11} - iz_{12}) (z_{11} - iz_{12}) \chi^* - (z_{11} z_{12} + z_{12} z_{11}) \omega \wedge \omega],$$

$$\tau(\tilde{\eta}_1, \epsilon_2) = (e^0 - e^5) \wedge [(z_{11} + iz_{12}) (z_{11} + iz_{12}) \chi + (z_{11} - iz_{12}) (z_{11} - iz_{12}) \chi^* - (z_{11} z_{12} + z_{12} z_{11}) \omega \wedge \omega]. \quad (A.7)$$
where
\[
\omega = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^5 \quad \text{(A.8)}
\]
\[
\chi = (e^1 + ie^5) \wedge (e^2 + ie^7) \wedge (e^3 + ie^8) \wedge (e^4 + ie^6).
\]
Note that \(\chi\) and \(\omega\) are the familiar \(SU(4)\) invariant forms. It is worth mentioning that all the 1-forms point to different directions spanned by \(e_1\) and \(e_5\). The same applies for the 3-forms. However, the 5-forms point to different directions spanned by \((e^0 - e^3) \wedge \chi\), \((e^0 - e^3) \wedge \chi^+\) and \((e^0 - e^3) \wedge \omega \wedge \omega\).

**Appendix B. Killing spinor equations**

**B.1. Killing spinor equations on 1**

The first spinor basis element we consider is 1. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.3), we find

\[
\begin{align*}
1 & : 0, \\
\Gamma^\beta : & \frac{1}{4} G_{\beta \gamma} \gamma + \frac{1}{4} G_{\beta - \gamma}, \\
\Gamma^\alpha : & \frac{1}{4} G_{\gamma \alpha} \gamma, \\
\Gamma^{(2)} : & 0, \\
\Gamma_{\beta \gamma} : & \frac{1}{2} G_{\gamma \alpha} \alpha - \frac{1}{12} \epsilon_{\beta \gamma} \alpha P_{\gamma}, \\
\Gamma_{\beta \gamma}^{(2)} : & \frac{1}{5} G_{\beta \gamma \alpha}, \\
\Gamma^{(4)} : & 0, \\
\Gamma_{\beta \gamma}^{(4)} : & \frac{1}{96} \epsilon_{\beta \gamma \alpha} P_{\alpha}.
\end{align*}
\]

(B.1)

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the \(a\)-frame derivative of the supercovariant connection are

\[
\begin{align*}
1 : D_a & + \frac{1}{2} \Omega_{a, \gamma} \gamma + \frac{1}{2} \Omega_{a, \gamma} \gamma + \frac{i}{4} F_{\alpha \gamma} \gamma \gamma + \frac{1}{2} F_{\alpha \gamma} \gamma, \\
\Gamma^{(1)} : & 0, \\
\Gamma_{\beta \gamma} : & \frac{1}{4} \Omega_{\beta \gamma} \gamma + \frac{i}{4} F_{\alpha \gamma} \gamma \gamma + \frac{i}{4} F_{\alpha \gamma} \gamma - \frac{1}{32} \epsilon_{\beta \gamma} \gamma \gamma G_{\alpha \gamma}, \\
\Gamma_{\beta \gamma}^{(3)} : & \frac{1}{2} \Omega_{\beta \gamma} + \frac{i}{2} F_{\alpha \gamma} \gamma, \\
\Gamma^{(3)} : & 0, \\
\Gamma_{\beta \gamma}^{(3)} : & \frac{i}{4} F_{\alpha \gamma} \gamma - \frac{1}{768} \epsilon_{\beta \gamma} \gamma (G_{\alpha \gamma} \gamma - G_{\alpha - \gamma}), \\
\Gamma_{\beta \gamma}^{(5)} : & \frac{i}{12} F_{\alpha \gamma} \gamma + \frac{1}{96} \epsilon_{\beta \gamma} \gamma G_{\alpha \gamma}, \\
\Gamma^{(5)} : & 0.
\end{align*}
\]

(B.2)

Along the \(\bar{a}\)-frame derivative of the supercovariant connection we find

\[
\begin{align*}
1 : D_a & + \frac{1}{2} \Omega_{\alpha, \gamma} \gamma + \frac{1}{2} \Omega_{\alpha, \gamma} \gamma + \frac{i}{4} F_{\alpha \gamma} \gamma \gamma + \frac{1}{2} F_{\alpha \gamma} \gamma \gamma - \frac{1}{24} F_{\alpha \gamma} \gamma \gamma G_{\alpha \gamma}, \\
\Gamma^{(1)} : & 0.
\end{align*}
\]
The supercovariant derivative with \( M = \) gives

\[
\Gamma^{\hat{\alpha} \hat{\beta} \hat{\gamma}} : \frac{1}{4} \Omega_{\hat{\alpha} \hat{\beta} \hat{\gamma}} + \frac{i}{4} F_{\hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\delta}} + \frac{i}{4} F_{\hat{\alpha} \hat{\gamma} \hat{\delta} \hat{\beta}} - \frac{1}{32} \epsilon_{\hat{\alpha} \hat{\beta} \hat{\gamma} \hat{\delta}} G_{\hat{\gamma} \hat{\delta}} G_{\hat{\alpha} \hat{\beta}},
\]

\( \Gamma^{3} : 0, \)

\[
\Gamma^{\hat{\beta}_{1} \hat{\beta}_{4}} = - \frac{1}{384} \epsilon_{\hat{\beta}_{1} \hat{\beta}_{4}} (G_{\hat{\alpha} \hat{\gamma}} - G_{\hat{\alpha} \hat{\delta}}).
\]

\[
\Gamma^{\hat{\beta}_{1} \hat{\beta}_{3} \hat{\gamma}} : \frac{i}{12} F_{\hat{\alpha} \hat{\beta}_{1} \hat{\beta}_{3} \hat{\gamma}} - \frac{1}{192} \epsilon_{\hat{\beta}_{1} \hat{\beta}_{3} \hat{\gamma}} (4G_{\hat{\alpha} \hat{\gamma}} + 8G_{\hat{\alpha} \hat{\delta}} G_{\hat{\gamma} \hat{\delta}}),
\]

\( \Gamma^{5} : 0. \) (B.3)

Finally, for \( M = + \) we find

\[
1 : \Omega_{+, \gamma} + \frac{1}{2} \Omega_{+, \gamma} + \frac{1}{2} \Omega_{+, \gamma} + \frac{1}{4} F_{+, \gamma} G_{+, \gamma} G_{+, \gamma},
\]

\( \Gamma^{1} : 0, \)

\[
\Gamma^{\hat{\beta}_{1} \hat{\beta}_{2}} : \frac{i}{4} \Omega_{+, \gamma} + \frac{i}{4} F_{+, \gamma} + \frac{i}{4} F_{+, \gamma} G_{+, \gamma} G_{+, \gamma},
\]

\( \Gamma^{2} : 0, \)

\[
\Gamma^{\hat{\beta}_{1} \hat{\beta}_{2} \hat{\gamma}} : \frac{i}{48} F_{+, \gamma} - \frac{1}{384} \epsilon_{\hat{\beta}_{1} \hat{\beta}_{2} \hat{\gamma}} G_{+, \gamma} G_{+, \gamma},
\]

\( \Gamma^{5} : 0. \) (B.4)

Next we consider the basis elements \( \epsilon_{ij} \) with \( i, j \leq 4 \), i.e., without holomorphic indices. We split up \( \alpha \) into \( a = (i, j) \) and \( p \), which contains the remaining two holomorphic indices. Furthermore, two different two-dimensional Levi-Civita tensors will appear, which are defined by \( \epsilon_{ij} = +1 \) and \( \epsilon_{p_{1}p_{2}} = \epsilon_{ij} \epsilon_{p_{1}p_{2}} \). Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.3), we find

\[
\text{B.2. Killing spinor equations on } \epsilon_{ij}
\]
Along the $\bar{a}$-frame derivative of the supercovariant connection we find

$$1: \frac{1}{2} \bar{\epsilon}^{c}(\Omega_{a,c} r + i F_{ac} r + i F_{ac} r') + \frac{1}{8} \bar{\epsilon}^{c} G_{a r},$$

$\Gamma^{(1)}: 0.$
\[ \Gamma_{hj}^{b} = \frac{1}{4} \epsilon_{b} \left( D_{b} - \frac{1}{2} \Omega_{a,c}^{e} + \frac{1}{2} \Omega_{c}^{r} + \frac{1}{2} \Omega_{a}^{r} - \frac{i}{2} F_{d}^{c} r^{e} - \frac{i}{2} F_{d}^{c} r^{r} + \frac{i}{4} F_{d}^{r} r^{e} + \frac{i}{2} F_{d}^{r} r^{r} \right), \]
\[ \Gamma_{hj}^{qb} = -\frac{1}{2} \epsilon_{b} \left( \Omega_{a,c}^{e} - i F_{d}^{a} q^{d} + i F_{d}^{c} q^{r} + i F_{d}^{a} q^{--} + \frac{1}{8} \epsilon_{q} G_{ab} r \right), \]
\[ \Gamma_{hj}^{q+} = -\frac{1}{2} \epsilon_{b} \left( \Omega_{a,c}^{e} - i F_{d}^{a} q^{d} + i F_{d}^{c} q^{r} + i F_{d}^{a} q^{--} \right), \]
\[ \Gamma_{hj}^{q} = \frac{1}{16} \epsilon_{b} \left( \Omega_{a,q}^{e} - i F_{d}^{a} q^{d} + i F_{d}^{e} q^{r} + i F_{d}^{a} q^{--} \right), \]
\[ \Gamma_{hj}^{q+} = \frac{1}{8} \epsilon_{b} \left( \Omega_{a,q}^{e} - i F_{d}^{a} q^{d} + i F_{d}^{e} q^{r} + i F_{d}^{a} q^{--} \right), \]
\[ \Gamma_{hj}^{q} = -\frac{1}{4} \epsilon_{b} \left( F_{d}^{a} q^{d} - \frac{1}{32} \epsilon_{q} G_{ab} r \right), \]
\[ \Gamma^{(3)} = 0, \]
\[ \Gamma^{(5)} = 0. \]

The components along the \( p \)-frame derivative of the supercovariant connection are

\[ 1 = -\frac{1}{2} \epsilon_{c} \left( \Omega_{p,c}^{e} + i F_{p,c} q^{e} + i F_{p,c} q^{--} \right), \]
\[ \Gamma^{(1)} = 0, \]
\[ \Gamma_{hj}^{b} = \frac{1}{4} \epsilon_{b} \left( D_{p} - \frac{1}{2} \Omega_{p,c}^{e} + \frac{1}{2} \Omega_{c}^{r} + \frac{1}{2} \Omega_{c}^{r} + \frac{i}{2} F_{p}^{c} q^{r} - \frac{i}{2} F_{p}^{c} q^{r} + \frac{i}{4} F_{p}^{r} q^{e} + \frac{i}{2} F_{p}^{r} q^{r} \right), \]
\[ \Gamma_{hj}^{q} = -\frac{1}{2} \epsilon_{b} \left( \Omega_{p,q}^{e} + i F_{p,q} q^{e} - i F_{p,q} q^{r} - i F_{p,q} q^{--} - \frac{1}{8} \epsilon_{q} G_{pr} r \right), \]
\[ \Gamma_{hj}^{q+} = -\frac{1}{2} \epsilon_{b} \left( \Omega_{p,q}^{e} + i F_{p,q} q^{e} - i F_{p,q} q^{r} - i F_{p,q} q^{--} \right), \]
\[ \Gamma_{hj}^{q} = \frac{1}{16} \epsilon_{b} \left( \Omega_{p,q}^{e} - i F_{p,q} q^{e} + i F_{p,q} q^{--} \right), \]
\[ \Gamma_{hj}^{q+} = \frac{1}{8} \epsilon_{b} \left( \Omega_{p,q}^{e} - i F_{p,q} q^{e} + i F_{p,q} q^{--} \right), \]
\[ \Gamma_{hj}^{q} = -\frac{1}{4} \epsilon_{b} \left( F_{p,q} q^{e} + \frac{1}{32} \epsilon_{q} G_{bp} r \right), \]
\[ \Gamma^{(3)} = 0, \]
\[ \Gamma^{(5)} = 0. \]
Similarly, the components along the \( \bar{p} \)-frame derivative are
\[
1 = -\frac{1}{2} \epsilon^{c_1 c_2} \left( \Omega_{\rho\varepsilon \zeta} + i F_{\rho \varepsilon \zeta} \right) + \frac{1}{4} \epsilon^{c_1 c_2} (g_{\rho \varepsilon \zeta} + g_{\varepsilon \rho \zeta} + G_{\rho \varepsilon \zeta} + G_{\varepsilon \rho \zeta} - 2G_{\varepsilon \rho \varepsilon})
\]
\[
\Gamma^{(1)} : 0,
\]
\[
\Gamma^{b_b h_b} : \frac{1}{4} \epsilon^{c_1 c_2} \left( D_{\rho} - \frac{1}{2} \Omega_{\rho \varepsilon \zeta} + \frac{1}{2} \Omega_{\varepsilon \rho \zeta} + \frac{1}{4} F_{\rho \varepsilon \zeta} + \frac{1}{4} F_{\varepsilon \rho \zeta} - \frac{1}{2} F_{\rho \varepsilon \zeta} - \frac{1}{2} F_{\varepsilon \rho \zeta} + \frac{1}{4} F_{\varepsilon \rho \varepsilon} + \frac{1}{4} F_{\rho \varepsilon \rho} \right),
\]
\[
\Gamma^{b_b} : -\frac{1}{2} \epsilon^{c_1 c_2} \left( \Omega_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \zeta} \right) + \frac{1}{4} \epsilon^{c_1 c_2} (4G_{\rho \varepsilon \zeta} + g_{\rho \varepsilon \zeta} + G_{\rho \varepsilon \varepsilon} + G_{\varepsilon \rho \zeta} - 2G_{\rho \varepsilon \rho})
\]
\[
\Gamma^{(3)} : 0,
\]
\[
\Gamma^{b_b h_b b_b} : \frac{1}{16} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \zeta} - i F_{\varepsilon \rho \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{b_b b_b} : \frac{1}{8} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \zeta} - i F_{\varepsilon \rho \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{b_b h_b} : \frac{1}{16} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \zeta} - i F_{\varepsilon \rho \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{(5)} : 0.
\]

The supercovariant derivative with \( M = - \) gives
\[
1 = -\frac{1}{2} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \zeta} - i F_{\varepsilon \rho \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{4} \epsilon^{c_1 c_2} G_{\rho \varepsilon \zeta},
\]
\[
\Gamma^{(1)} : 0,
\]
\[
\Gamma^{b_b h_b} : \frac{1}{4} \epsilon^{c_1 c_2} \left( D_{\rho} - \frac{1}{2} \Omega_{\rho \varepsilon \zeta} + \frac{1}{2} \Omega_{\varepsilon \rho \zeta} + \frac{1}{4} F_{\rho \varepsilon \zeta} - \frac{1}{2} F_{\rho \varepsilon \zeta} + \frac{1}{4} F_{\varepsilon \rho \zeta} - \frac{1}{4} F_{\rho \varepsilon \rho} \right),
\]
\[
\Gamma^{b_b} : -\frac{1}{2} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} - i F_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{4} \epsilon^{c_1 c_2} G_{\rho \varepsilon \zeta},
\]
\[
\Gamma^{(3)} : 0,
\]
\[
\Gamma^{b_b h_b b_b} : \frac{1}{16} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} - i F_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{b_b b_b} : \frac{1}{8} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} - i F_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{b_b h_b} : \frac{1}{4} \epsilon^{c_1 c_2} (\Omega_{\rho \varepsilon \zeta} - i F_{\rho \varepsilon \zeta} + i F_{\rho \varepsilon \rho}) + \frac{1}{32} \epsilon^{c_1 c_2} G_{b_b h_b},
\]
\[
\Gamma^{(5)} : 0.
\]

\[
\text{(B.10)}
\]

\[
\text{(B.11)}
\]
Finally, for $M = \gamma$ we find

\[ 1 = -\frac{1}{2} \varepsilon^{c_1 c_2} (\Omega^+_{c_1 c_2} + i F_{c_1 c_2} \gamma') + \frac{1}{8} \varepsilon^{c_1 c_2} G_{c_1 c_2}, \]

\[ \Gamma^{(1)} : 0, \]

\[ \Gamma_{b b} : \frac{1}{4} \varepsilon_{b b} \left( D_3 - \frac{1}{2} \Omega^+_{b b} \gamma + \frac{1}{2} \Omega^+_{b b} \gamma' + \frac{1}{2} \Omega^+_{b b} \gamma'' + \frac{i}{4} F_{b b} c_1 c_2 c_3 = -\frac{i}{2} F_{b b} c_1 c_2 + \frac{i}{4} F_{b b} c_1 c_2 \right). \]

\[ \Gamma_{b q} : -\frac{1}{2} \varepsilon^b \gamma' (\Omega^+_{b q} - i F_{b q} + i F_{b q} \gamma') + \frac{1}{8} \varepsilon^b \gamma' G_{b q}, \]

\[ \Gamma_{b q} : 0, \]

\[ \Gamma^{(3)} : 0, \]

\[ \Gamma_{b b} : \frac{1}{16} \varepsilon_{b b} \left( \Omega^+_{b b} \gamma - i F_{b b} \gamma' \right) - \frac{1}{64} \varepsilon_{b b} G_{b b}, \]

\[ \Gamma_{b b} : 0, \]

\[ \Gamma_{b q} : 0, \]

\[ \Gamma^{(5)} : 0. \]

\[ (B.12) \]

\[ (B.13) \]

**B.3. Killing spinor equations on $\epsilon_{k l}$**

We now consider the basis elements $\epsilon_{k l} = \frac{1}{2} \gamma^k \gamma^l$, with $k \leq 4$. For this purpose we split up $\alpha$ into $\rho$ and $k$, where $\rho$ are the remaining three holomorphic indices: $\rho = (1, \ldots, \hat{k}, \ldots, 4)$. Thus, $k$ is a single element of $\{1, 2, 3, 4\}$ and not an index that should be summed over. Furthermore we will use the three-dimensional Levi-Civita symbol defined by $\varepsilon_{\rho_1 \ldots \rho_5} = \varepsilon_{k \rho_1 \ldots \rho_5}$. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.3), we find

\[ 1 : 0, \]

\[ \Gamma^k : \frac{1}{4} (G_{- k k} - G_{- k k} \gamma'), \]

\[ \Gamma^r : \frac{1}{2} G_{k r -}, \]

\[ \Gamma^+ : \frac{1}{4} (G_{k r} \gamma - G_{k r}), \]

\[ \Gamma^{(2)} : 0, \]

\[ \Gamma^{\bar{\kappa} \bar{t} \bar{r} t} : -\frac{1}{8} G_{t, t \bar{r} \bar{r}}, \]

\[ \Gamma^{\bar{\kappa} r} : \frac{1}{8} (G_{\bar{t} k k} - G_{\bar{t} k k} \gamma + G_{t - t}), \]

\[ \Gamma^{t \bar{t} \bar{t} r} : -\frac{1}{12} \varepsilon_{t \bar{t} \bar{t} r} P_{-}, \]

\[ \Gamma^{t \bar{t} \bar{t} r} : \frac{1}{32} \varepsilon_{t \bar{t} \bar{t} r} P_{-} + \frac{i}{8} G_{t \bar{t} \bar{t} r}, \]

\[ \Gamma^{(4)} : 0, \]

\[ \Gamma^{t \bar{t} \bar{t} r} : \frac{1}{24} \varepsilon_{t \bar{t} \bar{t} r} P_{k} - \frac{1}{48} G_{t \bar{t} \bar{r} t}. \]
Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.16).

The components along the $k$-frame derivative of the supercovariant connection are

$1 : - \Omega_{k,k-}$,

$\Gamma^{(1)} : 0$,

$\Gamma^{k+} : \frac{1}{2} \Gamma^{k+} \Omega_{k,k-} + \frac{i}{2} F_{k^+}^{-i}^{\gamma k} - \frac{1}{16} \epsilon^{\gamma i k} G_{\eta \eta \gamma}$,

$\Gamma^{k-} : \frac{1}{2} \left( D_k - \frac{1}{2} \Omega_{k,kk} + \frac{1}{2} \Omega_{k,\gamma} - \frac{1}{2} \Omega_{k,-} + \frac{i}{2} F_{k^+}^{-i \gamma i k} - \frac{1}{2} F_{k^+}^{-i \gamma i k} \right) + \frac{1}{48} \epsilon^{\gamma i k} G_{\eta \eta \gamma}$,

$\Gamma^{\tau \gamma} : \epsilon^{\tau \gamma \sigma} G_{k \eta \gamma}$,

$\Gamma^{+} : \frac{1}{2} \Omega_{k,k-} - \frac{1}{8} \epsilon^{\gamma i k} G_{k \eta \gamma}$.

Along the $\bar{k}$-frame derivative of the supercovariant connection we find

$1 : - \Omega_{k,k-} + i F_{k^+}^{-i}^{\gamma k}$,

$\Gamma^{(1)} : 0$,

$\Gamma^{k+} : \frac{1}{2} \Gamma^{k+} \Omega_{k,k-} + \frac{i}{2} F_{k^+}^{-i}^{\gamma k}$,

$\Gamma^{k-} : \frac{1}{2} \left( D_k - \frac{1}{2} \Omega_{k,kk} + \frac{1}{2} \Omega_{k,\gamma} - \frac{1}{2} \Omega_{k,-} + \frac{i}{2} F_{k^+}^{-i \gamma i k} - \frac{1}{2} F_{k^+}^{-i \gamma i k} \right)$,

$\Gamma^{\tau \gamma} : \epsilon^{\tau \gamma \sigma} G_{k \eta \gamma}$,

$\Gamma^{+} : \frac{1}{2} \Omega_{k,k-} + \frac{i}{2} F_{k^+}^{-i}^{\gamma k}$.

(B.14)

(B.15)
The components along the $\rho$-frame derivative of the supercovariant connection are

\[
1 : -\Omega_{\rho,k\ldots} + i F_{k\rho\ldots}^\sigma, \\
\Gamma^{(1)} : 0, \\
\Gamma^{k\tau} : \frac{1}{2} \Omega_{\rho,\tau\ldots} - \frac{i}{2} F_{\rho\tau\ldots k\ldots} + \frac{i}{2} F_{\rho\tau\ldots}^\sigma, \\
\Gamma^{k\tau} \left( \frac{1}{2} \left( D_\rho - \frac{1}{2} \Omega_{\rho,k\ldots k\ldots} + \frac{1}{2} \Omega_{\rho,\ldots}^\sigma - \frac{1}{2} \Omega_{\rho,\ldots} - \frac{i}{2} F_{\rho k\ldots}^\sigma + \frac{i}{2} F_{\rho k\ldots} + \frac{i}{4} F_{\rho \sigma_1 \sigma_2 \ldots}^\sigma - \frac{i}{2} F_{\rho \sigma}^\sigma \right) \right), \\
\Gamma^{\tau \rho \sigma} : \frac{1}{2} F_{\rho \tau \ldots} - \frac{1}{16} \epsilon_{\tau \rho \sigma} G_{\rho \sigma \ldots}, \\
\Gamma^{(3)} : 0, \\
\Gamma^{k\tau} : \frac{i}{12} F_{\rho\tau\ldots k\ldots} + \frac{i}{96} \epsilon_{\tau \rho \sigma} G_{\rho \sigma \ldots}, \\
\Gamma^{k\tau} \left( \frac{1}{8} \left( \Omega_{\rho,\tau\ldots} - i F_{k\rho\tau\ldots k\ldots} + i F_{\rho \tau \ldots}^\sigma - i F_{\rho \tau \ldots} - \frac{1}{32} \epsilon_{\tau \rho \sigma} G_{\rho \sigma \ldots} \right) \right), \\
\Gamma^{\tau \rho \sigma} : \frac{1}{12} F_{k \rho \tau \ldots} + \frac{1}{192} \epsilon_{\tau \rho \sigma} (G_{\rho \sigma k\ldots} - G_{\rho \sigma}^\sigma - G_{\rho \sigma \ldots}), \\
\Gamma^{(5)} : 0.
\]

Similarly, the components along the $\bar{\rho}$-frame derivative are

\[
1 : -\Omega_{\bar{\rho},k\ldots} + i F_{k\bar{\rho}\ldots}^\sigma - \frac{1}{8} \epsilon_{\bar{\rho} \sigma \tau} G_{\sigma \tau \ldots}, \\
\Gamma^{(1)} : 0, \\
\Gamma^{\bar{k}\bar{\tau}} : \frac{1}{2} \Omega_{\bar{\rho},\bar{\tau}\ldots} - \frac{i}{2} F_{\rho\tau\ldots k\ldots} + \frac{i}{2} F_{\rho\tau\ldots}^\sigma - \frac{1}{8} \epsilon_{\bar{\rho} \sigma \tau} G_{\rho \sigma \ldots}, \\
\Gamma^{\bar{k}\bar{\tau}} \left( \frac{1}{2} \left( D_\rho - \frac{1}{2} \Omega_{\rho,k\ldots k\ldots} + \frac{1}{2} \Omega_{\rho,\ldots}^\sigma - \frac{1}{2} \Omega_{\rho,\ldots} - \frac{i}{2} F_{\rho k\ldots}^\sigma + \frac{i}{2} F_{\rho k\ldots} + \frac{i}{4} F_{\rho \sigma_1 \sigma_2 \ldots}^\sigma - \frac{i}{2} F_{\rho \sigma}^\sigma \right) \right), \\
\Gamma^{\bar{\tau} \bar{\rho} \sigma} : \frac{i}{2} F_{\bar{\rho} \bar{\tau} \ldots} + \frac{1}{32} \epsilon_{\bar{\tau} \bar{\rho} \sigma} (g_{\bar{\rho} \sigma \bar{\tau} \ldots} - g_{\rho \sigma \bar{\tau} \ldots} - g_{\rho \sigma}^\sigma - 4 G_{\rho \sigma \ldots}), \\
\Gamma^{(3)} : 0, \\
\Gamma^{\bar{k}\bar{\tau}} : \frac{1}{12} F_{k \bar{\rho} \bar{\tau} \ldots} + \frac{1}{48} \epsilon_{\bar{\tau} \bar{\rho} \sigma} (2 G_{\tau \sigma \rho \sigma} + g_{\tau \rho \sigma} (G_{\sigma \rho \sigma k\ldots} + 3 G_{\sigma \rho \sigma}^\sigma - G_{\sigma \rho \sigma \ldots})), \\
\Gamma^{(5)} : 0.
\]
The supercovariant derivative with \( M = - \) gives

\[
1 : - \Omega_{-k-},
\]

\( \Gamma^{(1)} : 0, \)

\( \Gamma^{\xi} : \frac{i}{2} \Omega_{+, t-}, \)

\( \Gamma^{[k} : \frac{i}{2} \left( D_+ - \frac{1}{2} \Omega_{+, -k} + \frac{1}{2} \Omega_{+, -\sigma} - \frac{1}{2} \Omega_{+, -} - \frac{i}{2} \sigma k + \frac{i}{4} F_{-\sigma \partial 1 \partial 2} \right), \)

\( \Gamma^{t_1 t_2} : 0, \)

\( \Gamma^{t_1 t_2} : - \frac{i}{2} \Omega_{+, k t} - \frac{i}{2} F_{k t+\sigma} - \frac{1}{16} \epsilon_{\sigma 1 \sigma 2} G_{\sigma 1 \sigma 2}, \)

\( \Gamma^{(3)} : 0, \)

\( \Gamma^{t_1 t_2} : 0, \)

\( \Gamma^{\xi t_1 t_2} : \frac{1}{8} \Omega_{+, t_2} + i F_{-t_1 t_2} - i F_{-t_1 t_2} + \frac{1}{32} \epsilon_{t_1 t_2} \sigma G_{k\sigma -}, \)

\( \Gamma^{t_1 t_2} : - \frac{i}{12} F_{k t_1 t_2 -} + \frac{1}{192} \epsilon_{t_1 t_2} \sigma (G_{k\sigma -} - G_{-\sigma \gamma}), \)

\( \Gamma^{(5)} : 0. \)

Finally, for \( M = + \) we find

\[
1 : - \Omega_{+, k-}, i F_{k+\sigma} + \frac{1}{24} \epsilon_{\sigma 1 - \sigma 2} G_{\sigma 1 \sigma 2},
\]

\( \Gamma^{(1)} : 0, \)

\( \Gamma^{\xi} : \frac{i}{2} \Omega_{+, t-} = \frac{i}{2} F_{-t+\sigma} + \frac{i}{2} F_{t+\sigma} + \frac{1}{16} \epsilon_{t+\sigma \partial 1 \partial 2} G_{\partial 1 \partial 2}, \)

\( \Gamma^{[k} : \frac{i}{2} \left( D_+ - \frac{1}{2} \Omega_{+, -k} + \frac{1}{2} \Omega_{+, -\sigma} - \frac{1}{2} \Omega_{+, -} - \frac{i}{2} \sigma k + \frac{i}{4} F_{-\sigma \partial 1 \partial 2} \right), \)

\( \Gamma^{t_1 t_2} : - \frac{i}{2} F_{k t_1 t_2 -} - \frac{1}{32} \epsilon_{t_1 t_2} \sigma (G_{\sigma \partial 1 k} - G_{\sigma 1 \partial 2} + 3 G_{\sigma 1 -}), \)

\( \Gamma^{t_1 t_2} : - \frac{i}{2} \Omega_{+, k t} - \frac{i}{2} F_{k t+\sigma} - \frac{1}{8} \epsilon_{\sigma 1 \sigma 2} G_{\sigma 1 \sigma 2}, \)

\( \Gamma^{(3)} : 0, \)

\( \Gamma^{\xi t_1 t_2} : \frac{i}{12} F_{t_1 t_2 -} + \frac{1}{192} \epsilon_{t_1 t_2} \sigma (G_{k\sigma -} - 3 G_{k-\gamma}), \)

\( \Gamma^{t_1 t_2} : \frac{1}{8} (\Omega_{+, t_2} + i F_{k\partial 1 t_2} + i F_{k\partial 2 t_2} + \frac{1}{16} \epsilon_{t_1 t_2} \sigma G_{k\gamma}, \)

\( \Gamma^{t_1 t_2} : - \frac{i}{12} F_{k t_1 t_2 -} + \frac{i}{96} \epsilon_{t_1 t_2} \sigma (G_{k\gamma} - G_{-\gamma}), \)

\( \Gamma^{(5)} : 0. \)

\[B.18\]

**B.4. Killing spinor equations on \( e_{1234} \)**

The next spinor basis element we consider is \( e_{1234} \). Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.3), we find

\[
1 : 0,
\]

\( \Gamma^{[b} : P_{b} + \frac{i}{12} \epsilon_{b} \gamma_{1-\gamma} G_{\gamma_{1-\gamma}}, \)

\( \Gamma^{(3)} : 0, \)

\( \Gamma^{t_1 t_2} : \frac{1}{8} \Omega_{+, t_2} - \frac{i}{2} F_{k\partial 1 t_2} + \frac{i}{8} \epsilon_{t_1 t_2} \sigma G_{k\gamma}, \)

\( \Gamma^{t_1 t_2} : - \frac{i}{12} F_{k t_1 t_2 -} + \frac{i}{96} \epsilon_{t_1 t_2} \sigma (G_{k\gamma} - G_{-\gamma}), \)

\( \Gamma^{(5)} : 0. \)
\[ \Gamma^+ \cdot P_s, \]
\[ \Gamma^{(2)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \cdot \hat{\beta}_3} : \frac{1}{48} \epsilon_{\hat{\beta}_1 \cdot \hat{\beta}_3}^\gamma(G_{\gamma} \delta - G_{\gamma \rightarrow}), \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_3 +} : -\frac{1}{16} \epsilon_{\hat{\beta}_1 \hat{\beta}_3}^{\gamma\gamma_2}G_{\gamma_1\gamma_2^+}, \]
\[ \Gamma^{(4)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \cdot \hat{\beta}_3 +} : -\frac{1}{32i} \epsilon_{\hat{\beta}_1 \cdot \hat{\beta}_3}G_{\gamma} \gamma^\gamma. \]

(B.20)

Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6).

The components along the \( a \)-frame derivative of the supercovariant connection are

\[ 1 : \frac{1}{4} (G_{\alpha} \gamma + G_{\alpha \rightarrow}), \]
\[ \Gamma^{(1)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_2} : \frac{1}{8} \epsilon_{\hat{\beta}_1 \hat{\beta}_2}^{\gamma\gamma_2}(\Omega_{\alpha} \gamma_{\gamma_2} - iF_{\alpha\gamma_1\gamma_2^\delta} + iF_{\alpha\gamma_1\gamma_2 \rightarrow}) + \frac{1}{8} G_{a\hat{\beta}_1 \hat{\beta}_2} - \frac{1}{16 i} G_{a[\hat{\beta}_1 (G_{\beta_2} \gamma)^+ + (G_{\beta_2} \rightarrow)}, \]
\[ \Gamma^{\hat{\beta}_1 +} : \frac{1}{6} \epsilon_{\hat{\beta}_1 \gamma} ^{\gamma_2 \gamma_2} F_{\alpha\gamma_1 \gamma_2^+} + \frac{1}{4} G_{a\hat{\beta}^+} - \frac{1}{16 i} G_{a\hat{\beta}} G_{\gamma} \gamma^\gamma, \]
\[ \Gamma^{(3)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_4} : \frac{1}{96} \epsilon_{\hat{\beta}_1 \hat{\beta}_4} (D_{a} - \frac{1}{2} \Omega_{\alpha} \gamma_{\gamma_2} + \frac{1}{2} \Omega_{\alpha \rightarrow} + \frac{i}{4} F_{\alpha\gamma_1 \gamma_2^\gamma} - \frac{i}{4} F_{\alpha\gamma_1 \gamma_2 \rightarrow}) - \frac{1}{96 i} G_{a[\hat{\beta}_1 (G_{\beta_2} \rightarrow. \]
\[ \Gamma^{\hat{\beta}_1 +} : \frac{1}{24} \epsilon_{\hat{\beta}_1 \gamma} ^{\gamma_2 \gamma_2} F_{\alpha\gamma_1 \gamma_2^+} - \frac{1}{32 i} G_{a[\hat{\beta}_1 (G_{\beta_2} \rightarrow. \]
\[ \Gamma^{(5)} : 0. \]

(B.21)

Along the \( \alpha \)-frame derivative of the supercovariant connection we find

\[ 1 : \frac{i}{12} \epsilon_{\alpha \hat{\beta}_1 \hat{\beta}_4} F_{\alpha \hat{\beta}_1 \hat{\beta}_4} + \frac{1}{8} (G_{\alpha} \gamma + G_{\alpha \rightarrow}), \]
\[ \Gamma^{(1)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_2} : \frac{1}{8} \epsilon_{\hat{\beta}_1 \hat{\beta}_2}^{\gamma\gamma_2}(\Omega_{\alpha} \gamma_{\gamma_2} - iF_{\alpha\gamma_1\gamma_2^\delta} + iF_{\alpha\gamma_1\gamma_2 \rightarrow}) + \frac{1}{16 i} G_{a\hat{\beta}_1 \hat{\beta}_2}, \]
\[ \Gamma^{\hat{\beta}_1 +} : \frac{1}{6} \epsilon_{\hat{\beta}_1 \gamma} ^{\gamma_2 \gamma_2} F_{\alpha\gamma_1 \gamma_2^+} + \frac{1}{8} G_{a\hat{\beta}^+}, \]
\[ \Gamma^{(3)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_4} : \frac{1}{96} \epsilon_{\hat{\beta}_1 \hat{\beta}_4} (D_{a} - \frac{1}{2} \Omega_{\alpha} \gamma_{\gamma_2} + \frac{1}{2} \Omega_{\alpha \rightarrow} + \frac{i}{4} F_{\alpha\gamma_1 \gamma_2^\gamma} - \frac{i}{4} F_{\alpha\gamma_1 \gamma_2 \rightarrow}) \]
\[ \Gamma^{\hat{\beta}_1 +} : \frac{1}{24} \epsilon_{\hat{\beta}_1 \gamma} ^{\gamma_2 \gamma_2} F_{\alpha\gamma_1 \gamma_2^+} - \frac{1}{32 i} G_{a[\hat{\beta}_1 (G_{\beta_2} \rightarrow. \]
\[ \Gamma^{(5)} : 0. \]

(B.22)

The supercovariant derivative with \( M = - \) gives

\[ 1 : \frac{i}{12} \epsilon_{\hat{\beta}_1 \hat{\beta}_4} F_{- \hat{\beta}_1 \hat{\beta}_4} + \frac{1}{4} G_{- \gamma} \gamma, \]
\[ \Gamma^{(1)} : 0, \]
\[ \Gamma^{\hat{\beta}_1 \hat{\beta}_2} : -\frac{1}{8} \epsilon_{\hat{\beta}_1 \hat{\beta}_2}^{\gamma\gamma_2}(\Omega_{-\gamma\gamma_2} - iF_{-\gamma\gamma_2^\delta}) + \frac{1}{8} G_{-\hat{\beta}_1 \hat{\beta}_2}^\gamma. \]}
B.5. Killing spinor equations on $e_{i_1\cdots i_5}$

We now consider the basis elements $e_{i_1\cdots i_5} = \frac{1}{4} \Gamma^{i_1\cdots i_5} \Gamma^+ 1$ with $i_1, i_2, i_3 \leq 4$. For this purpose we split up $\sigma$ into $\rho$ and $k$, where $\rho = (i_1, \ldots, i_3)$ and $k$ is the missing fourth holomorphic coordinate (and is not an index that should be summed over). Again we will use the three-dimensional Levi-Civita symbol defined by $\epsilon_{\rho_1\cdots\rho_5} = \epsilon_{\rho_1\cdots\rho_3}$. Substituting this spinor into the (algebraic) Killing spinor equation (2.7) and expanding the resulting expression in the basis (A.3), we find

$$1 : 0,$$
$$\Gamma^k : -P_-, $$
$$\Gamma^t : \frac{1}{2} \epsilon^t_{\sigma \sigma} G_{\sigma \sigma}, $$
$$\Gamma^+: P_+ = \frac{1}{4} \epsilon^{\gamma_1, \gamma_2} G_{\gamma_1\gamma_2}, $$
$$\Gamma^{(2)} : 0,$$
$$\Gamma^{t_1 t_2} : -\frac{1}{8} \epsilon_{t_1 t_2}^{\bar{\sigma}} G_{\bar{\sigma} \
\bar{\sigma}}, $$
$$\Gamma^{t_1 \bar{t}^\sigma} : -\frac{1}{2} P_+ - \frac{1}{8} \epsilon^t_{\bar{\sigma} \bar{\sigma}} G_{k \sigma}, $$
$$\Gamma^{t_1 \bar{t}^\sigma} = -\frac{1}{8} \epsilon_{t_1 \bar{t}^\sigma}^{\bar{\sigma}} G_{\bar{\sigma} k \sigma},$$
$$\Gamma^{t_1 t_2} : -\frac{1}{16} \epsilon_{t_1 t_2}^{\bar{\sigma} \bar{\sigma}} (G_{\sigma \sigma} - G_{\sigma \sigma}),$$
$$\Gamma^{t_1 \bar{t}^\sigma} : -\frac{1}{48} \epsilon_{t_1 \bar{t}^\sigma}^{\bar{\sigma} \bar{\sigma}} (G_{\sigma \sigma} - G_{\sigma \sigma}),$$
$$\Gamma^{(4)} : 0,$$
$$\Gamma^{t_1 \bar{t}^\sigma} : -\frac{1}{96} \epsilon_{t_1 \bar{t}^\sigma}^{\bar{\sigma} \bar{\sigma}} (G_{\bar{\sigma} \sigma} + G_{\bar{\sigma} \sigma}).$$
Next we turn to the Killing spinor equation associated with the supercovariant derivative (2.6). The components along the \( k \)-frame derivative of the supercovariant connection are

\[
1: \frac{i}{3} \epsilon^{\sigma_{1}\ldots\sigma_{3}} F_{k\sigma_{1}\ldots\sigma_{3}} - \\
\Gamma^{(1)}: 0, \\
\Gamma^{k\ell}: \frac{i}{2} \epsilon_{\sigma_{0} \sigma_{3}} F_{k\kappa_{0}\sigma_{3}} + \frac{1}{8} G_{k\ell}, \\
\Gamma^{k\ell}: -\frac{i}{6} \epsilon^{\sigma_{0} \ldots \sigma_{3}} F_{k\kappa_{0}\sigma_{3}} + \frac{1}{16} (G_{k\sigma} \sigma - G_{k \sigma}), \\
\Gamma^{\tau_{1} \tau_{2}}: \frac{-i}{4} \epsilon_{\tau_{1} \tau_{2}} \sigma_{0} (\Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}}), \\
\Gamma^{+}: -\frac{i}{4} \epsilon^{\sigma_{0}} (\Omega_{k, \sigma_{2}} - i F_{k \sigma_{2}}), \\
\Gamma^{(3)}: 0, \\
\Gamma^{k_{1} k_{2}}: \frac{-i}{24} \epsilon_{k_{1} \ldots k_{2}} \Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}} - G_{k_{1} \sigma_{1}} - G_{k_{2} \sigma_{1}}, \\
\Gamma^{k_{1} k_{2}}: \frac{-i}{8} \epsilon_{k_{1} k_{2}} \sigma_{0} (\Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}}) + \frac{1}{32} G_{k_{1} k_{2}}, \\
\Gamma^{k_{1} k_{2}}: \frac{1}{24} \epsilon_{k_{1} \ldots k_{2}} (D_{k} + \frac{1}{2} \Omega_{k, \kappa} - \frac{1}{2} \Omega_{k, \sigma} - \frac{1}{2} \Omega_{k, \sigma} - \frac{1}{2} F_{k \sigma}, + \frac{i}{4} F_{k \sigma}, + \frac{i}{2} F_{k \sigma}, + \frac{1}{96} G_{l_{1} \ldots l_{2}}), \\
\Gamma^{(5)}: 0.
\]

(B.26)

Along the \( \bar{k} \)-frame derivative of the supercovariant connection we find

\[
1: \frac{i}{3} \epsilon^{\sigma_{1}\ldots\sigma_{3}} F_{\bar{k}\sigma_{1}\ldots\sigma_{3}} + \frac{1}{8} (3 G_{\bar{k} \kappa} + G_{\bar{k} \sigma}), \\
\Gamma^{(1)}: 0, \\
\Gamma^{k\ell}: \frac{i}{4} G_{k\ell}, \\
\Gamma^{k\ell}: \frac{i}{8} (G_{k \sigma} \sigma - G_{k \sigma}), \\
\Gamma^{\tau_{1} \tau_{2}}: \frac{-i}{4} \epsilon_{\tau_{1} \tau_{2}} \sigma_{0} (\Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}}) + \frac{1}{16} G_{\tau_{1} \tau_{2}}, \\
\Gamma^{+}: -\frac{i}{4} \epsilon^{\sigma_{0}} (\Omega_{k, \sigma_{2}} - i F_{k \sigma_{2}}) + \frac{1}{16} (3 G_{\bar{k} \kappa} + G_{\bar{k} \sigma} - G_{\bar{k} \sigma}), \\
\Gamma^{(3)}: 0, \\
\Gamma^{k_{1} k_{2}}: \frac{-i}{24} \epsilon_{k_{1} \ldots k_{2}} \Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}} - G_{k_{1} \sigma_{1}} - G_{k_{2} \sigma_{1}}, \\
\Gamma^{k_{1} k_{2}}: \frac{-i}{8} \epsilon_{k_{1} k_{2}} \sigma_{0} (\Omega_{k, \sigma_{1}} - i F_{k \sigma_{1}}) + \frac{1}{16} G_{k_{1} k_{2}}, \\
\Gamma^{k_{1} k_{2}}: \frac{1}{24} \epsilon_{k_{1} \ldots k_{2}} (D_{k} + \frac{1}{2} \Omega_{k, \kappa} - \frac{1}{2} \Omega_{k, \sigma} - \frac{1}{2} \Omega_{k, \sigma} - \frac{1}{2} F_{k \sigma}, + \frac{i}{4} F_{k \sigma}, + \frac{i}{2} F_{k \sigma}, + \frac{1}{96} G_{l_{1} \ldots l_{2}}), \\
\Gamma^{(5)}: 0.
\]

(B.27)
The components along the $\rho$-frame derivative of the supercovariant connection are

\[
1: \frac{1}{2} G_{k\rho},
\]
\[\Gamma^{(1)}: 0,\]
\[\Gamma^{\xi_t}: -\frac{i}{2} \epsilon^{\sigma_1 \sigma_2} F_{k\rho_1 \sigma_2} - \frac{1}{4} G_{\rho \tau} + \frac{1}{16} g_{\rho \tau} (G_{-kk} - G_{-\sigma}),\]
\[\Gamma^{\xi_\tau}: -\frac{1}{8} (G_{kk} - G_{k\sigma} + G_{\rho \tau}),\]
\[\Gamma^{t_1 t_2}: -\frac{1}{4} \epsilon_{t_1 t_2} \sigma (\Omega_{\rho \sigma \tau} + iF_{\rho_1 \sigma_2 k} - iF_{\rho_1 \sigma_2 -}) + \frac{1}{8} g_{\rho \tau} (G_{t_2 j k} - G_{t_2 j \sigma} + G_{t_2 \sigma}),\]
\[\Gamma^{t_1 t_2 +}: -\frac{1}{4} \epsilon_{t_1 t_2} \sigma (\Omega_{\rho \sigma \tau} + iF_{\rho_1 \sigma_2 k} - iF_{\rho_1 \sigma_2 -}) + \frac{1}{4} G_{k\rho \tau} + \frac{1}{16} g_{\rho \tau} (G_{k\sigma} + G_{k-}),\]
\[\Gamma^{(3)}: 0.\]

Similarly, the components along the \(\bar{\rho}\)-frame derivative are

\[
1: \frac{1}{3} \epsilon^{\rho_1 \rho_2} F_{\rho_1 \rho_2 \rho_3} + \frac{1}{4} G_{k\bar{\rho}},
\]
\[\Gamma^{(1)}: 0,\]
\[\Gamma^{\xi_t}: -\frac{i}{2} \epsilon^{\sigma_1 \sigma_2} F_{k\rho_1 \sigma_2} + \frac{1}{8} G_{\rho \tau},\]
\[\Gamma^{\xi_\tau}: \frac{1}{6} \epsilon^{\rho_1 \rho_2 \rho_3} F_{k\rho_1 \rho_2 \rho_3} - \frac{1}{16} (G_{kk} - G_{\rho \sigma} + G_{\rho \tau}),\]
\[\Gamma^{t_1 t_2}: -\frac{1}{4} \epsilon_{t_1 t_2} \sigma (\Omega_{\rho \sigma \tau} + iF_{\rho_1 \sigma_2 k} - iF_{\rho_1 \sigma_2 -}) + \frac{1}{8} G_{k\rho \tau},\]
\[\Gamma^{(3)}: 0,\]
\[\Gamma^{t_1 t_2 +}: -\frac{1}{24} \epsilon_{t_1 t_2} \sigma (\Omega_{\rho \sigma \tau} + iF_{\rho_1 \sigma_2 k} - iF_{\rho_1 \sigma_2 -}),\]
\[\Gamma^{(5)}: 0.\]
\[ \Gamma_{t_1 \cdots t_5} = \frac{1}{24} \epsilon_{t_1 \cdots t_5} \left( D_\rho + \frac{1}{2} \Omega_{\rho, kk} - \frac{1}{2} \Omega_{\rho, \sigma} - \frac{1}{2} \Omega_{\rho, \tau} + \frac{i}{2} F_{\rho kk} \sigma - \frac{i}{2} F_{\rho kk} \tau + \frac{i}{4} F_{\rho, \sigma_1 \sigma_2} + \frac{i}{2} F_{\rho, \sigma} \right). \]

\( \Gamma^{(5)} = 0. \)  

(B.29)

The supercovariant derivative with \( M = - \) gives

1: 0,  
\( \Gamma^{(1)} = 0, \)

\( \Gamma^{t \tau} = 0, \)

\[ \Gamma^{t_1 \cdots t_5} = \frac{i}{6} \epsilon_{t_1 \cdots t_5} F_{k \sigma_1 \cdots \sigma_3} - \frac{1}{16} (G_{kk} - G_{\sigma \sigma}), \]

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{4} \epsilon_{t \tau t_1 \cdots t_5} \Omega_{\sigma_1 \cdots \sigma_3}, \]

\[ \Gamma^{t_1 \cdots t_5} = -\frac{1}{4} \epsilon_{t \tau t_1 \cdots t_5} (\Omega_{\sigma_1 \cdots \sigma_3} + i F_{\sigma_1 \sigma_2 \cdots \sigma_3} - i F_{\sigma_1 \sigma_2 \cdots \sigma_3}) - \frac{1}{8} \Gamma_{t \tau}, \]

\( \Gamma^{(3)} : 0, \)

\( \Gamma^{t_1 \cdots t_5} = -\frac{1}{24} \epsilon_{t_1 \cdots t_5} \Omega_{k k}, \)

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{8} \epsilon_{t \tau t_1 \cdots t_5} (\Omega_{k, \sigma_1} + i F_{k \sigma_1} - i F_{k \sigma_1} - i F_{k \sigma_1}), \]

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{24} \epsilon_{t \tau t_1 \cdots t_5} \left( D_\rho + \frac{1}{2} \Omega_{\rho, kk} - \frac{1}{2} \Omega_{\rho, \sigma} - \frac{1}{2} \Omega_{\rho, \tau} + \frac{i}{2} F_{\rho kk} \sigma - \frac{i}{2} F_{\rho kk} \tau + \frac{i}{4} F_{\rho, \sigma_1 \sigma_2} + \frac{i}{2} F_{\rho, \sigma} \right), \]

\( \Gamma^{(5)} : 0. \)  

Finally, for \( M = + \) we find

1: \[ \frac{i}{3} \epsilon_{t_1 \cdots t_5} F_{\sigma_1 \cdots \sigma_3} = \frac{1}{8} (G_{k \sigma} + 3G_{k -}), \]

\( \Gamma^{(1)} : 0, \)

\[ \Gamma^{t \tau} = \frac{i}{2} \epsilon_{t \tau t_1 \cdots t_5} F_{k \sigma_1 \cdots \sigma_3} = \frac{1}{16} (G_{kk} = G_{\sigma \sigma} + 3G_{k -}), \]

\[ \Gamma^{t} = \frac{1}{6} \epsilon_{t_1 \cdots t_5} F_{k \sigma_1 \cdots \sigma_3} - \frac{1}{8} (G_{kk} = G_{\sigma \sigma}), \]

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{4} \epsilon_{t \tau t_1 \cdots t_5} (\Omega_{+ \sigma_1} + i F_{+ \sigma_1 \cdots \sigma_3} - i F_{+ \sigma_1 \cdots \sigma_3}) = \frac{1}{16} \Gamma_{t \tau t_1 \cdots t_5}, \]

\[ \Gamma^{t_1 \cdots t_5} = -\frac{1}{4} \epsilon_{t \tau t_1 \cdots t_5} (\Omega_{+ \sigma_1} + i F_{+ \sigma_1 \cdots \sigma_3} - i F_{+ \sigma_1 \cdots \sigma_3}) = \frac{1}{4} \Gamma_{t \tau t_1 \cdots t_5}, \]

\( \Gamma^{(3)} : 0, \)

\( \Gamma^{t \tau t_1 \cdots t_5} = -\frac{1}{24} \epsilon_{t \tau t_1 \cdots t_5} (G_{\omega \omega} - i F_{\omega -}), \]

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{8} \epsilon_{t \tau t_1 \cdots t_5} (G_{\omega \omega} - i F_{\omega -}) + \frac{1}{96} G_{t \tau t_1 \cdots t_5}, \]

\[ \Gamma^{t \tau t_1 \cdots t_5} = \frac{1}{16} G_{t \tau t_1 \cdots t_5}, \]

\[ \Gamma^{t_1 \cdots t_5} = \frac{1}{24} \epsilon_{t \tau t_1 \cdots t_5} \left( D_\rho + \frac{1}{2} \Omega_{\rho, kk} - \frac{1}{2} \Omega_{\rho, \sigma} - \frac{1}{2} \Omega_{\rho, \tau} + \frac{i}{2} F_{\rho kk} \sigma - \frac{i}{2} F_{\rho kk} \tau + \frac{i}{4} F_{\rho, \sigma_1 \sigma_2} + \frac{i}{2} F_{\rho, \sigma} \right), \]

\( \Gamma^{(5)} : 0. \)
Appendix C. Integrability conditions

C.1. Integrability conditions on $I$

The expressions for the integrability condition $I$ on the basis element $1$ read

$$1 : LG_{\beta_1} = LG_{\beta_2} + 12BG_{\gamma_1 \gamma_2 \gamma_3} + 24BG_{\gamma_3},$$

$$g\Gamma_{\beta_1 \beta_2} := \frac{1}{2}LG_{\beta_1 \beta_2} + 12BG_{\beta_1 \beta_2 \gamma_3} + 12BG_{\beta_1 \beta_2 \gamma_3} - \frac{1}{2} \epsilon \beta_1 \beta_2 \gamma_3 BP_{\gamma_3},$$

$$\Gamma_{\beta} := LG_{\beta_1} + 24BG_{\gamma_3},$$

$$\Gamma_{\beta_1 \beta_2} := -3iLF_{\beta_1 \beta_2} + 6\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3},$$

$$\Gamma_{\beta_1 \beta_2 \gamma_3} := -\frac{1}{2} \epsilon \beta_1 \beta_2 \gamma_3 BP_{\gamma_3}.$$

The integrability conditions $I_{\alpha}$ are

$$\Gamma_{\beta} := \frac{1}{2} L_{\alpha 1} - 6iLF_{\alpha \beta} + 8\epsilon \beta \gamma_3 BG_{\alpha \gamma_3},$$

$$\Gamma_{\alpha 1} := \frac{1}{2} L_{\alpha 1} + 6iLF_{\alpha \beta} - 24BG_{\gamma_3},$$

$$\Gamma_{\alpha \gamma_3} := -3iLF_{\alpha \beta} \beta_2 + 6\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3}.$$

The integrability conditions $I_{\alpha}$ read

$$\Gamma_{\beta} := \frac{1}{2} L_{\alpha 1} - 6i(LF_{\alpha \beta} + LF_{\alpha \beta} - 8\epsilon \beta \gamma_3 BG_{\alpha \gamma_3} + 24\epsilon \beta \gamma_3 (BG_{\alpha \gamma_3} - BG_{\gamma_3}),$$

$$\Gamma_{\alpha 1} := \frac{1}{2} L_{\alpha 1} + 6iLF_{\alpha \beta} - 16\epsilon \beta \gamma_3 - 2BG_{\alpha \beta}.$$

$$\Gamma_{\alpha \gamma_3} := -3iLF_{\alpha \beta} \beta_2 + 12\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3} + 6\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3}.$$

Similarly, $I_{\alpha}$ with $A = -$ is given by

$$\Gamma_{\beta} := \frac{1}{2} L_{\alpha 1} - 6iLF_{\beta} - 8\epsilon \beta \gamma_3 BG_{\gamma_3},$$

$$\Gamma_{\alpha 1} := \frac{1}{2} L_{\alpha 1} + 6iLF_{\alpha \beta} - 16\epsilon \beta \gamma_3 - 2BG_{\alpha \beta},$$

$$\Gamma_{\alpha \beta} := -3iLF_{\alpha \beta} \beta_2 - 12\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3}.$$

Finally, the expressions for $I_{\tau}$ read

$$\Gamma_{\beta} := \frac{1}{2} L_{\alpha 1} - 6iLF_{\beta} + 8\epsilon \beta \gamma_3 BG_{\gamma_3},$$

$$\Gamma_{\alpha 1} := \frac{1}{2} L_{\alpha 1} + 6iLF_{\alpha \beta} - 16\epsilon \beta \gamma_3 - 2BG_{\alpha \beta},$$

$$\Gamma_{\alpha \beta} := -3iLF_{\alpha \beta} \beta_2 + 6\epsilon \beta_1 \beta_2 \gamma_3 BG_{\gamma_3}.$$
C.2. Integrability conditions on $e_{ij}$

The expressions for the integrability condition $I$ on the basis element $e_{ij}$ read

\begin{align*}
I_1 &= -\epsilon^{h b_2}_{b_1} \left( L G_{b_2 b_1} + 24 B G_{b_2 b_1 \epsilon'} + 24 B G_{b_2 b_1 \epsilon''} \right) + 2 \epsilon_{i j} B P_{i j, r}, \\
I_2 &= -\epsilon^{h b_2}_{b_1} \left( -L G_{c_1} c_2 + L G_{c_1} + L G_{c_2} + 12 B G_{c_1 c_2} c_3 - 24 B G_{c_1 c_2} \epsilon' + 24 B G_{c_1 c_2} \epsilon'' \right. \\
&\quad \left. + 12 B G_{c_1 \epsilon', c_2 \epsilon'} + 24 B G_{c_1 \epsilon''} \rightarrow \right), \\
I_{b q} &= -\epsilon^{b c}_{i j} \left( L G_{c_1 \epsilon q} - 24 B G_{c_1 c_2 \epsilon q} \epsilon' + 24 B G_{c_1 c_2 \epsilon q} \epsilon'' \right) + 2 \epsilon_{i j} B P_{i j, r}, \\
I_{b q} &= -\epsilon^{b c}_{i j} \left( L G_{c_1 \epsilon q} + 24 B G_{c_1 c_2 \epsilon q} \epsilon' + 24 B G_{c_1 c_2 \epsilon q} \epsilon'' \right), \\
I_{b q} &= -12 \epsilon^{b c}_{i j} B G_{b_2 b_1 \epsilon q, \epsilon q} - \frac{1}{4} \epsilon^{b c}_{i j q, \epsilon q} \left( L P + 2 B P_{i j, \epsilon} - 2 B P_{i j, \epsilon'} + 2 B P_{i j, \epsilon''} \right), \\
I_{b q} &= -24 \epsilon^{b c}_{i j} B G_{b_2 b_1 \epsilon q, \epsilon q} + 2 \epsilon_{i j} B P_{i j, r}, \\
I_{b q} &= \frac{1}{4} \epsilon^{b c}_{i j q, \epsilon q} \left( L G_{q 2} - 24 B G_{q 2 c} \epsilon' + 24 B G_{q 2 c} \epsilon'' \right) - \frac{1}{4} \epsilon^{b c}_{i j q, \epsilon q} B P_{i j, r}, \\
I_{b q} &= -12 \epsilon^{b c}_{i j} B G_{q 2 c, \epsilon q}, - \frac{1}{2} \epsilon^{b c}_{i j q, \epsilon q} B P_{i j, r}.
\end{align*}

The integrability conditions $I_{a b}$ are given by

\begin{align*}
I^{b} &= -\frac{1}{2} \epsilon^{b c}_{i j} \left( E_{a 2 r} + 12 L F_{a 2 r, \epsilon'} - 12 L F_{a 2 r, \epsilon''} \right) - 24 \epsilon_{i j} B G_{a b r, r} \\
&\quad + 24 g_{a b} \epsilon_{i j} \left( B G_{r, r, \epsilon'} + B G_{r, r, \epsilon''} \right), \\
I^{q} &= -\epsilon^{b c}_{i j} \left( L G_{a r, \epsilon' r, \epsilon''} - 24 B G_{a r, \epsilon' r, \epsilon''} + 24 B G_{a r, \epsilon' r, \epsilon''} \right), \\
I^{q} &= -24 \epsilon^{b c}_{i j} B G_{a r, \epsilon' r, \epsilon''}, \\
I_{b q} &= \frac{1}{4} \epsilon^{b c}_{i j b_2 q} \left( E_{a q 2} + 12 L F_{a q 2, \epsilon'} - 12 L F_{a q 2, \epsilon''} \right) \\
&\quad + 24 \epsilon_{i j} g_{a b} b_2 \left( - B G_{b_2 j, r, r} + B G_{b_2 j, r, r} \right), \\
I_{b q} &= \frac{1}{2} \epsilon^{b c}_{i j b_2 q} \left( E_{a 2 q} + 12 L F_{a 2 q, \epsilon'} - 12 L F_{a 2 q, \epsilon''} \right) + 24 \epsilon_{i j} g_{a b} b_2 \epsilon_{i j, r} \left( B G_{c_1 c_2 \epsilon q} - 24 B G_{c_1 c_2 \epsilon q} \epsilon' + 24 B G_{c_1 c_2 \epsilon q} \epsilon'' \right) \left( B G_{c_1 c_2 \epsilon q} - 24 B G_{c_1 c_2 \epsilon q} \epsilon' + 24 B G_{c_1 c_2 \epsilon q} \epsilon'' \right), \\
I_{b q} &= \frac{1}{2} \epsilon^{b c}_{i j q, \epsilon q} \left( L G_{a 2 r} + 24 B G_{a 2 r, \epsilon'} - 24 B G_{a 2 r, \epsilon''} \right), \\
I_{b q} &= \frac{1}{2} \epsilon^{b c}_{i j q, \epsilon q} \left( L G_{a 2 r} + 24 B G_{a 2 r, \epsilon'} - 24 B G_{a 2 r, \epsilon''} \right) - 24 \epsilon_{i j} B G_{a b r, r}.
\end{align*}

Similarly, $I_{a b}$ reads

\begin{align*}
I^{b} &= -\frac{1}{2} \epsilon^{b c}_{i j} \left( E_{a 2 r} + 12 L F_{a 2 r, \epsilon'} - 12 L F_{a 2 r, \epsilon''} \right) + 24 \epsilon_{i j} B G_{b a r, r} \\
I^{q} &= 6 \epsilon^{b c}_{i j} \left( L F_{b 2 q} + \epsilon_{i j} \left( L G_{a 2 c} - 24 B G_{a 2 c} \epsilon' + 24 B G_{a 2 c} \epsilon'' \right) - 24 \epsilon_{i j} B G_{a b r, r} \right), \\
I^{q} &= 6 \epsilon^{b c}_{i j} \left( L F_{a 2 b} + 12 L F_{a 2 b, \epsilon'} - 12 L F_{a 2 b, \epsilon''} \right) + 24 \epsilon_{i j} B G_{a b r, r}, \\
I^{q} &= \frac{1}{8} \epsilon^{b c}_{i j b_2 q} \left( E_{a q 2} + 12 L F_{a q 2, \epsilon'} - 12 L F_{a q 2, \epsilon''} \right), \\
I^{q} &= \frac{1}{8} \epsilon^{b c}_{i j q, \epsilon q} \left( L G_{a 2 q} - 24 B G_{a 2 q, \epsilon'} - 24 B G_{a 2 q, \epsilon''} \right), \\
I^{q} &= \frac{1}{8} \epsilon^{b c}_{i j q, \epsilon q} \left( L G_{a 2 q} - 24 B G_{a 2 q, \epsilon'} - 24 B G_{a 2 q, \epsilon''} \right), \\
I^{q} &= \frac{1}{8} \epsilon^{b c}_{i j b_2 q} \left( E_{a q 2} + 12 L F_{a q 2, \epsilon'} - 12 L F_{a q 2, \epsilon''} \right).
\end{align*}
The integrability condition \( \mathcal{I}_M \) with \( M = p \) is given by

\[
\begin{align*}
\Gamma^b & : -\frac{1}{2} \epsilon^b_c (E_{cipy} - 12iLF_{cicy} + 12iLF_{cipcy} + 12iLF_{cipcy}), \\
\Gamma^q & : 6i\epsilon_{b+p} LF_{b+p+}, \\
\Gamma^* & : 6i\epsilon_{b+p} LF_{b+p+}, \\
\Gamma^b_{b+q2} & : \frac{1}{8} \epsilon_{b+q2} (E_{pq} + 12iLF_{pqcy} - 12iLF_{pqcy}) + 12i_q B G_{b+p+}, \\
\Gamma^b_{b+q2} & : \frac{1}{8} \epsilon_{b+q2} (E_{pq} + 12iLF_{pqcy} - 12iLF_{pqcy}), \\
\Gamma^q_{b+q2} & : -3i_{b+p} LF_{p+q2} - \frac{1}{8} \epsilon_{b+q2} (L G_{bppc} - 24B G_{bpc} + 24B G_{bpc}), \\
\Gamma^b_{b+q2} & : \frac{1}{8} \epsilon_{b+q2} (L G_{pp+} - 24B G_{pp+}), \\
\Gamma^q_{b+q2} & : \frac{1}{2} \epsilon_{b+q2} (L G_{pp+} - 24B G_{pp+}), \\
\Gamma^b_{b+q2} & : 3\epsilon_{b+q2} B G_{b+p+}.
\end{align*}
\]

Furthermore, \( \mathcal{I}_M \) with \( M = \bar{p} \) is given by the expressions

\[
\begin{align*}
\Gamma^b & : -\frac{1}{2} \epsilon^b_c (E_{cipy} - 12iLF_{cicy} + 12iLF_{cipcy} + 12iLF_{cipcy}), \\
\Gamma^q & : -\epsilon_q L G_{pr+} + 6i\epsilon^r_{c+} L F_{c+pq} - 12\epsilon_q (B G_{prc} - B G_{prc}) + 12\epsilon_q B G_{pr}, \\
\Gamma^r & : -12\epsilon_q (B G_{prc} - B G_{prc}), \\
\Gamma^* & : 6i\epsilon_{b+p} LF_{b+p+}, \\
\Gamma^q_{b+p+} & : -\frac{1}{8} \epsilon_{b+p+} (L G_{pp+} - 24B G_{pp+}), \\
\Gamma^q_{b+p+} & : -\frac{1}{8} \epsilon_{b+p+} (L G_{pp+} - 24B G_{pp+}), \\
\Gamma^q_{b+p+} & : 3\epsilon_{b+p+} B G_{b+p+}.
\end{align*}
\]

The integrability conditions \( \mathcal{I}_r \) read

\[
\begin{align*}
\Gamma^b & : -\frac{1}{2} \epsilon^b_c (E_{cipy} - 12iLF_{cicy} + 12iLF_{cipcy} + 12iLF_{cipcy}) + 24\epsilon^r_{c+} B G_{b+}, \\
\Gamma^q & : -6i\epsilon_{b+p} LF_{b+p+} + \epsilon_q L G_{pr}, + 24\epsilon_q (B G_{prc} - B G_{prc}), \\
\Gamma^* & : 6i\epsilon_{b+p} LF_{b+p+} + 24\epsilon^r_{c+} B G_{b+p+}, \\
\Gamma^q_{b+p+} & : \frac{1}{8} \epsilon_{b+p+} (E_{pq} - 12iLF_{pqcy} + 12iLF_{pqcy}) + 12i_q B G_{b+p+}, \\
\Gamma^q_{b+p+} & : \frac{1}{8} \epsilon_{b+p+} (E_{pq} - 12iLF_{pqcy} + 12iLF_{pqcy}), \\
\Gamma^q_{b+p+} & : -3i_{b+p} LF_{p+q2} - \frac{1}{8} \epsilon_{b+q2} (L G_{pp+} + 24B G_{pp+} - 24B G_{pp+}), \\
\Gamma^q_{b+p+} & : 6i\epsilon_{b+p} LF_{b+p+} + 24\epsilon_q (B G_{b+p+} - B G_{b+p+}), \\
\Gamma^q_{b+p+} & : \frac{1}{4} \epsilon_{b+p+} (L G_{pp+} - 24B G_{pp+} - 24B G_{pp+}), \\
\Gamma^q_{b+p+} & : \frac{4}{2} \epsilon_{b+p+} LF_{b+p+} + 3\epsilon_{b+p+} B G_{b+p+}.
\end{align*}
\]
Finally, for $I_*$ we find

\[
\Gamma^b : - \frac{1}{2} \epsilon_\parallel c (E_{c,t} - 12i L F_{c,t} + 12i L F_{c,t}^\prime) - 24 \epsilon_\parallel r_1 B G_{b,t} r_1, \\
\Gamma^q : -6i \epsilon_\parallel b \bar{b} L F_{b,t} \bar{b} \bar{q} \bar{t} + \epsilon_q \left( L G_{t,1} - 24 B G_{t,1} + 24 B G_{t,1}^r \right), \\
\Gamma^r : 0, \\
\Gamma^b \bar{b} \bar{q} : \frac{1}{2} \epsilon_\parallel b \bar{b} E_{b,t} + \epsilon_q \left( L G_{b,t,1} - 24 B G_{b,t} + 24 B G_{b,t}^r \right), \\
\Gamma^r \bar{b} : 0, \\
\Gamma^q \bar{b} : - \frac{3}{2} \epsilon_q \left( L F_{q,b} \bar{t} + \frac{1}{2} \epsilon_q \bar{b} \left( L G_{b,1} - 24 B G_{b,1} + 24 B G_{b,1}^r \right) \right), \\
\Gamma^q \bar{r} : 0, \\
\Gamma^b \bar{b} \bar{q} \bar{t} : 0.
\]

(C.12)

C.3. Integrability conditions on $e_{45}$

The expressions for the integrability condition $I$ on the basis element $e_{45}$ read

\[
1 : -2 L G_{k,-} - 48 L G_{k,-} \sigma^\parallel, \\
\Gamma^k : L G_{t,-} - 24 B G_{k,t,-} + 24 B G_{t,-} \sigma^\parallel, \\
\Gamma^k : - \frac{1}{2} \left( L G_{k,-} - L G_{k,-} + L G_{k,-} \right) - 12 B G_{k,-} \sigma^\parallel + 12 B G_{k,-} + 6 B G_{k,-} \sigma_1 \sigma_2 = -12 B G_{k,-}, \\
\Gamma^r_1 \bar{t}_2 : -24 B G_{k,-} \sigma^\parallel + 24 B G_{k,-} \sigma^\parallel - \epsilon_{r_1 \bar{t}_2 \bar{r}_1} B P_{\sigma^\parallel}, \\
\Gamma^r_1 \bar{t}_2 : -24 B G_{k,-} \sigma^\parallel + 24 B G_{k,-} \sigma^\parallel - \epsilon_{r_1 \bar{t}_2 \bar{r}_1} B P_{\sigma^\parallel}, \\
\Gamma^r_1 \bar{t}_2 : \frac{1}{2} L G_{k,-} - 2 B P_{k,-} + 2 B P_{k,-} - 2 B P_{k,-} + 2 \epsilon_{r_1 \bar{t}_2 \bar{r}_1} B P_{k,-}.
\]

The integrability conditions $I_8$ are given by

\[
\Gamma^k : - \frac{1}{2} L E_{k,-} + 8 i L F_{k,-} \bar{c} + 16 \epsilon_{k,-} \bar{c} B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r : -24 \epsilon_{k,-} \sigma^\parallel B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r : -24 \epsilon_{k,-} \sigma^\parallel B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r : - \frac{1}{2} L E_{k,-} + 3 i L F_{k,-} \bar{c} - 3 i L F_{k,-} \bar{c} + 12 \epsilon_{k,-} \sigma^\parallel B G_{k,-} \sigma_1 \sigma_2 + 24 B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r_1 \bar{t}_2 : - \frac{1}{12} \epsilon_{r_1 \bar{t}_2 \bar{r}_1} \left( L G_{k,-} - 24 B G_{k,-} \sigma^\parallel \right), \\
\Gamma^r_1 \bar{t}_2 : - \frac{1}{12} \epsilon_{r_1 \bar{t}_2 \bar{r}_1} \left( L G_{k,-} - 24 B G_{k,-} \sigma^\parallel \right), \\
\Gamma^r_1 \bar{t}_2 : - \frac{1}{24} \epsilon_{r_1 \bar{t}_2 \bar{r}_1} \left( L G_{k,-} - 12 B G_{k,-} \sigma_1 \sigma_2 = -24 B G_{k,-} \sigma_1 \sigma_2 \right).
\]

Similarly, $I_8$ reads

\[
\Gamma^k : - \frac{1}{2} L E_{k,-} + 6 i L F_{k,-} \bar{c}, \\
\Gamma^r : 12 i L F_{k,-} + 24 \epsilon_{k,-} \sigma_1 \sigma_2 B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r : - \frac{1}{2} L E_{k,-} + 6 i L F_{k,-} \bar{c} - 8 \epsilon_{k,-} \sigma_1 \sigma_2 B G_{k,-} \sigma_1 \sigma_2, \\
\Gamma^r_1 \bar{t}_2 : +3 i L F_{k,-} \bar{c}.
\]
The integrability condition \( \mathcal{I}_M \) with \( M = \rho \) is given by

\[
\Gamma^k := -\frac{1}{2} E_{k^*} - 6 i LF_{k^*}^{\rho \sigma} - 6 i LF_{k^*}^{\rho \sigma},
\]
\[
\Gamma^\tau := 12 i LF_{k^*}^{\rho \sigma} - 24 \epsilon^{\sigma \rho_{j_1} j_2} BG_{\rho_{j_1} j_2} - 6 i LF_{k^*}^{\rho \sigma},
\]
\[
\Gamma^{t_1 t_2} := 3 i LF_{\rho t_1 t_2}^{\rho \sigma} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{\rho \sigma} - 6 i LF_{k^*}^{\rho \sigma},
\]
\[
\Gamma^{t_1 t_2} := -\frac{1}{2} E_{\rho t_1 t_2} - 3 i LF_{\rho t_1 t_2}^{\rho \sigma} - 3 i LF_{\rho t_1 t_2}^{\rho \sigma} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{\rho \sigma},
\]
\[
\Gamma^{t_1 t_2} := 12 i LF_{k^*}^{\rho \sigma} - 24 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*} - 24 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*},
\]
\[
\Gamma^{t_1 t_2} := -\frac{1}{2} E_{\rho t_1 t_2} - 3 i LF_{k^*}^{\rho \sigma} - 3 i LF_{\rho t_1 t_2}^{\rho \sigma} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*}.
\] (C.16)

Furthermore, \( \mathcal{I}_M \) with \( M = \bar{\rho} \) is given by the expressions

\[
\Gamma^k := -\frac{1}{2} E_{k^*} - 6 i LF_{k^*}^{\bar{\rho} \sigma} - 6 i LF_{k^*}^{\bar{\rho} \sigma} - 48 \epsilon_{\bar{\rho} \sigma_{j_1} j_2} BG_{j_1 j_2},
\]
\[
\Gamma^\tau := 12 i LF_{k^*}^{\bar{\rho} \sigma} - 48 \epsilon_{\rho t}^{\sigma \rho_{j_1} j_2} (BG_{k^*}^{j_1 j_2} - BG_{j_1 j_2}) - 24 \epsilon_{\rho t}^{\sigma \rho_{j_1} j_2} BG_{j_1 j_2},
\]
\[
\Gamma^{t_1 t_2} := \frac{1}{12} \epsilon_{t_1 t_2}^{\rho \sigma} (LG_{\rho \rho} + 24 BG_{\rho \sigma} - 24 BG_{k^*}),
\]
\[
\Gamma_{t_1 t_2} := 3 i LF_{k^*}^{\rho \sigma} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*} - 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{k^*}.
\] (C.17)

The integrability conditions \( \mathcal{I}_- \) read

\[
\Gamma^k := -\frac{1}{2} E_{-},
\]
\[
\Gamma^\tau := 0,
\]
\[
\Gamma^{t_1 t_2} := 0,
\]
\[
\Gamma^{t_1 t_2} := -\frac{1}{2} E_{-} + 3 i LF_{k^*}^{\rho \sigma} - 3 i LF_{-}^{\rho \sigma} + 12 \epsilon_{t_1 t_2}^{\rho \sigma} BG_{\rho \sigma},
\]
\[
\Gamma_{t_1 t_2} := 0,
\]
\[
\Gamma_{t_1 t_2} := 3 i LF_{k^*}^{\rho \sigma} + \frac{1}{2} \epsilon_{t_1 t_2}^{\rho \sigma} (LG_{\sigma} - 24 BG_{k^*} + 24 \sigma_{j_1 j_2}),
\]
\[
\Gamma_{t_1 t_2} := -\frac{1}{2} i LF_{k^*}^{\rho \sigma} + \frac{1}{2} \epsilon_{t_1 t_2}^{\rho \sigma} (LG_{k^*} + 24 BG_{k^*}^{\rho \sigma}).
\] (C.18)
Finally, for $I_a$ we find
\[
\Gamma^k : -\frac{1}{4} E_{\gamma} + 6i LF_{k\gamma} = 6i LF_{\gamma} + 16\epsilon_{\delta\gamma\sigma_1} BG_{\delta\gamma_1\sigma_1},
\]
\[
\Gamma^f : - 12i LF_{k\gamma} + 24\epsilon_{\gamma\sigma_1\sigma_2} (BG_{\gamma\sigma_1\sigma_2} - BG_{\sigma_1\sigma_2\gamma})
\]
\[
\Gamma^* : \frac{1}{2} \epsilon_{\gamma\sigma_1\sigma_2} + 8\epsilon_{\gamma\sigma_1\sigma_2} BG_{\gamma_1\sigma_1\sigma_1},
\]
\[
\Gamma^{t_1 t_2} : -3i LF_{t_1 t_2} + 12\epsilon_{t_1 t_2}BG_{\gamma_1\sigma_1\sigma_1},
\]
\[
\Gamma^{t_1 t_2} : - \frac{1}{4} E_{\gamma} + 3i LF_{t_1 t_2} - 3i LF_{t_1 t_2} = 12\epsilon_{t_1 t_2}BG_{\gamma_1\sigma_1\sigma_1},
\]
\[
\Gamma^{k \gamma} : \frac{1}{12} \epsilon_{k \gamma} (LG_{\gamma} + 24BG_{\gamma\gamma\gamma} - 12BG_{\gamma_1\gamma_2\gamma} - 24BG_{\gamma_1\gamma_2\gamma}),
\]
\[
\Gamma^{k \gamma} : - \frac{1}{4} LF_{k\gamma} + \frac{1}{2} \epsilon_{k \gamma} (LG_{\gamma} - 24BG_{k\gamma\gamma} - 12BG_{\gamma_1\gamma_2\gamma}).
\]

\[\tag{C.19}\]

**C.4. Integrability conditions on $e_{1234}$**

The expressions for the integrability condition $I$ on the basis element $e_{1234}$ read
\[
1 : 4\epsilon_{\gamma\gamma\gamma} BG_{\gamma\gamma\gamma} + LP + 2BP_{\gamma\gamma} + 2BP_{\gamma\gamma},
\]
\[
\Gamma^{\beta \gamma} : \frac{1}{4} \epsilon_{\beta \gamma} \gamma_{\gamma\gamma} (LG_{\gamma\gamma\gamma} - 24BG_{\gamma\gamma\gamma} + 24BG_{\gamma_1\gamma_2\gamma} + BP_{\beta \gamma},
\]
\[
\Gamma^{*} : 8\epsilon_{\beta \gamma} \gamma_{\gamma\gamma} BG_{\gamma_1\gamma}, + 2BP_{\beta \gamma},
\]
\[
\Gamma^{\beta \gamma} : \frac{1}{12} \epsilon_{\beta \gamma} \gamma_{\gamma\gamma} (LG_{\gamma} + 24BG_{\gamma\gamma\gamma} - 12BG_{\gamma_1\gamma_2\gamma} - 24BG_{\gamma_1\gamma_2\gamma}).
\]
\[\tag{C.20}\]

The integrability conditions $I_a$ are
\[
\Gamma^{\gamma} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{a\gamma\gamma} = LG_{a\gamma} - 24(BG_{a\gamma\gamma} + BG_{a\gamma\gamma}),
\]
\[
\Gamma^{*} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{a\gamma\gamma} = 12\epsilon_{\gamma} \gamma_{\gamma\gamma} (LG_{\gamma} - 24BG_{\gamma\gamma\gamma} + 24BG_{\gamma_1\gamma_2\gamma}).
\]
\[\tag{C.21}\]

The integrability conditions $I_a$ read
\[
\Gamma^{\gamma} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{a\gamma\gamma} = LG_{a\gamma} - 24(BG_{a\gamma\gamma} + BG_{a\gamma\gamma}),
\]
\[
\Gamma^{*} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{a\gamma\gamma} = 12\epsilon_{\gamma} \gamma_{\gamma\gamma} (LG_{\gamma} - 24BG_{\gamma\gamma\gamma} + 24BG_{\gamma_1\gamma_2\gamma}).
\]
\[\tag{C.22}\]

Similarly, $I_A$ with $A = -$ is given by
\[
\Gamma^{\gamma} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{-\gamma\gamma} = LG_{-\gamma} + 24BG_{-\gamma\gamma},
\]
\[
\Gamma^{*} : -2\epsilon_{\gamma} \gamma_{\gamma\gamma} LF_{-\gamma\gamma} = 12\epsilon_{\gamma} \gamma_{\gamma\gamma} (LG_{\gamma} - 24BG_{\gamma\gamma\gamma} + 24BG_{\gamma_1\gamma_2\gamma}).
\]
\[ \Gamma^{\beta_1 \beta_2} : \frac{3}{2} \epsilon_{\beta_1 \beta_2} \gamma_{\gamma} LF_{\gamma_1 \gamma_2-} + 12BG_{\gamma_1 \gamma_2-}, \]
\[ \Gamma^{\beta_1 \beta_2} : \frac{1}{12} \epsilon_{\beta_1 \beta_2} \left( \frac{1}{2} E_{-+} - 6iLF_{-+\gamma} \right) + BG_{\beta_1 \beta_2}. \quad (C.23) \]

Finally, the expressions for \( I_+ \) read
\[ \Gamma^\beta : 2 \epsilon_{\gamma_1 \gamma_2} LF_{\gamma_1 \gamma_2+} + LG_{\beta+} = 24BG_{\beta+}, \]
\[ \Gamma^\gamma : 0, \]
\[ \Gamma^{\beta_1 \beta_2} : - \frac{1}{12} \epsilon_{\beta_1 \beta_2} \left( \frac{1}{2} E_{++} - 6iLF_{++\gamma} \right) - 4BG_{\beta_1 \beta_2}, \quad (C.24) \]
\[ \Gamma^{\beta_1 \beta_2} : 0, \]
\[ \Gamma^{\beta_1 \beta_2} : \frac{1}{12} \epsilon_{\beta_1 \beta_2} E_{++}. \]

\section*{C.5. Integrability conditions on \( e_{i_1 \cdots i_5} \)}

The expressions for the integrability condition \( I \) on the basis element \( e_{i_1 \cdots i_5} \) read
\[ 1 : 16 \epsilon_{i_1 \cdots i_5} BG_{\sigma_1 \cdots \sigma_5} = 4BP_{k-}, \]
\[ \Gamma^{k} : 24 \epsilon_{i_1 i_2} \sigma_3 \sigma_4 BG_{\sigma_1 \cdots \sigma_5} + 2BP_{k+}, \]
\[ \Gamma^{k} : -8 \epsilon_{i_1 i_2} \sigma_3 \sigma_4 BG_{\sigma_1 \cdots \sigma_5} + \frac{1}{2} LP - BP_{k-} + BP_{\sigma_3 \sigma_4} - BP_{k+}, \]
\[ \Gamma^{i_1 i_2} : - \frac{1}{12} \epsilon_{i_1 i_2} \sigma_4 \left( LG_{\sigma_3 \sigma_4} + 24BG_{k \sigma_5 \sigma_4} - 24BG_{\sigma_3 \sigma_4 \sigma_5} \right), \]
\[ \Gamma^{i_1 i_2} : \frac{1}{2} \epsilon_{i_1 i_2} \sigma_4 \left( LG_{\sigma_3 \sigma_4} + 24BG_{k \sigma_5 \sigma_4} - 24BG_{\sigma_3 \sigma_4 \sigma_5} - 24BG_{\sigma_3 \sigma_4 \sigma_5} \right) - 2BP_{k+}. \quad (C.25) \]
\[ \Gamma^{k+} - \Gamma_{i_1 i_2}^{k} = \frac{1}{2} \epsilon_{i_1 i_2} \sigma_4 \left( LG_{\sigma_3 \sigma_4} + 24BG_{k \sigma_5 \sigma_4} - 24BG_{\sigma_3 \sigma_4 \sigma_5} + 24BG_{\sigma_3 \sigma_4 \sigma_5} \right). \]

The integrability conditions \( I_k \) are given by
\[ \Gamma^k : LG_{k+} = 24BG_{k- \sigma} \]
\[ \Gamma^k : -6 \epsilon_{i_1 \cdots i_5} \sigma_3 \sigma_4 \sigma_5 LF_{k \sigma_1 \sigma_2 \sigma_3}, \]
\[ \Gamma^{i_1 i_2} : \frac{1}{2} \epsilon_{i_1 i_2} \sigma_4 \left( LG_{\sigma_3 \sigma_4} + 24BG_{k \sigma_5 \sigma_4} - 24BG_{\sigma_3 \sigma_4 \sigma_5} + 24BG_{\sigma_3 \sigma_4 \sigma_5} \right) - 2BP_{k+}. \quad (C.26) \]

Similarly, \( I_k \) reads
\[ \Gamma^k : LG_{k+} = 24BG_{k- \sigma \sigma}, \]
\[ \Gamma^k : -6 \epsilon_{i_1 \cdots i_5} \sigma_3 \sigma_4 \sigma_5 LF_{k \sigma_1 \sigma_2 \sigma_3} = 4BG_{k+ \sigma}, \]
\[ \Gamma^{i_1 i_2} : 2 \epsilon_{i_1 \cdots i_5} \sigma_3 \sigma_4 \sigma_5 LF_{k \sigma_1 \sigma_2 \sigma_3} + 48BG_{k+ \sigma}, \]
\[ \Gamma^{i_1 i_2} : 2 \epsilon_{i_1 \cdots i_5} \sigma_3 \sigma_4 \sigma_5 LF_{k \sigma_1 \sigma_2 \sigma_3} + 48BG_{k+ \sigma}, \]
\[ \Gamma^{i_1 i_2} : 12BG_{k+ \sigma}. \]
\[ \Gamma^{\ell+} : \frac{1}{2} LG_{k\ell} + 12\left( BG_{k\ell} \sigma \right)^{\tau} - BG_{k\ell}^{-} \tau, \]
\[ \Gamma^{t-} : -\frac{1}{24} \epsilon_{t \tau} \left( E_{k\ell} + 12iLF_{k\ell} \sigma \right) + 8BG_{k\ell}^{-} t, \]
\[ \Gamma_{\ell+}^{t-} : \frac{1}{8} \epsilon_{\ell+} \left( E_{k\ell} + 12iLF_{k\ell} \sigma \right) + 12\left( BG_{t+} \tau \right)^{\tau} - BG_{t+} \tau, \]
\[ \Gamma_{\ell+}^{t+} : \frac{1}{32} \epsilon_{\ell+} E_{k\ell} + 2BG_{k\ell}^{-} t, \]
\[ \Gamma_{t-}^{t+} : \frac{1}{48} \epsilon_{t-} \left( E_{k\ell} + 12iLF_{k\ell} \sigma \right) + 12\left( BG_{t-} \tau \right)^{\tau} - BG_{t-} \tau, \]

(C.27)

The integrability condition \( I_M \) with \( M = \rho \) is given by

\[ \Gamma_{\ell+}^{\bullet} : LG_{\rho-} - 24BG_{k\rho} - 24BG_{\rho-} \sigma, \]
\[ \Gamma_{t-}^{\bullet} : -6i \epsilon_{t-} \left( LF_{\rho\sigma} \tau - 48BG_{k\rho} \sigma \right), \]
\[ \Gamma_{\ell+}^{t-} : -3i \epsilon_{\ell+} \left( LF_{\rho\sigma} \tau + \frac{1}{2} LG_{\rho\tau} - 12\left( BG_{k\rho} \tau \right)^{\tau} - BG_{k\rho}^{-} \tau \right) + 6g_{\rho\tau} \left( 2BG_{k\rho} \sigma - 2BG_{k\rho}^{-} \sigma \right) - 12BG_{k\rho}^{-} \tau, \]
\[ \Gamma_{t-}^{t+} : -\frac{1}{24} \epsilon_{t-} \left( E_{k\rho} - 12iLF_{k\rho} \sigma \right) + 24g_{\rho\tau} \left( BG_{k\rho} \tau \right)^{\tau} - BG_{k\rho}^{-} \tau - 12BG_{k\rho}^{-} \tau, \]
\[ \Gamma_{\ell+}^{t-} : \frac{1}{32} \epsilon_{\ell+} \left( E_{k\rho} - 12iLF_{k\rho} \sigma \right) + 12iLF_{k\rho} \sigma \right) + 12\left( BG_{t-} \tau \right)^{\tau} - BG_{t-} \tau, \]

(C.28)

Furthermore, \( I_M \) with \( M = \bar{\rho} \) is given by the expressions

\[ \Gamma_{\ell+}^{\bullet} : LG_{\bar{\rho}-} - 24BG_{k\bar{\rho}} \tau - 24BG_{\bar{\rho}-} \sigma, \]
\[ \Gamma_{t-}^{\bullet} : -6i \epsilon_{t-} \left( LF_{\bar{\rho}\sigma} \tau - 48BG_{k\bar{\rho}} \sigma \right), \]
\[ \Gamma_{\ell+}^{t-} : -3i \epsilon_{\ell+} \left( LF_{\bar{\rho}\sigma} \tau + \frac{1}{2} LG_{\bar{\rho}\tau} - 12\left( BG_{k\bar{\rho}} \tau \right)^{\tau} - BG_{k\bar{\rho}}^{-} \tau \right) + 6g_{\bar{\rho}\tau} \left( 2BG_{k\bar{\rho}} \sigma - 2BG_{k\bar{\rho}}^{-} \sigma \right) - 12BG_{k\bar{\rho}}^{-} \tau, \]
\[ \Gamma_{t-}^{t+} : -\frac{1}{24} \epsilon_{t-} \left( E_{k\bar{\rho}} - 12iLF_{k\bar{\rho}} \sigma \right) + 24g_{\bar{\rho}\tau} \left( BG_{k\bar{\rho}} \tau \right)^{\tau} - BG_{k\bar{\rho}}^{-} \tau - 12BG_{k\bar{\rho}}^{-} \tau, \]
\[ \Gamma_{\ell+}^{t-} : \frac{1}{32} \epsilon_{\ell+} \left( E_{k\bar{\rho}} - 12iLF_{k\bar{\rho}} \sigma \right) + 12iLF_{k\bar{\rho}} \sigma \right) + 12\left( BG_{t-} \tau \right)^{\tau} - BG_{t-} \tau, \]

(C.29)

The integrability conditions \( I_\ell \) read

\[ \Gamma_{\ell+}^{\bullet} : 0, \]
\[ \Gamma_{t-}^{\bullet} : 0, \]
\[ \Gamma_{\ell+}^{t-} : -2i \epsilon_{\ell+} \left( LF_{\sigma\tau} \tau - \frac{1}{2} LG_{\sigma\tau} - 24BG_{\sigma\tau}^{-} \tau \right), \]
\[ \Gamma_{t-}^{t+} : 0, \]
\[ \Gamma_{\ell+}^{t+} : -3i \epsilon_{\ell+} \left( LF_{\sigma\tau} \tau - \frac{1}{2} LG_{\sigma\tau} - 12\left( BG_{k\sigma} \tau \right)^{\tau} - BG_{k\sigma}^{-} \tau \right), \]
\[ \Gamma_{t-}^{t+} : -\frac{1}{24} \epsilon_{t-} \left( E_{k\sigma} - 12iLF_{k\sigma} \tau \right)^{\tau} - 12\left( BG_{k\sigma} \tau \right)^{\tau} - 12BG_{k\sigma}^{-} \tau, \]
\[ \Gamma_{\ell+}^{t-} : \frac{1}{32} \epsilon_{\ell+} \left( E_{k\sigma} - 12iLF_{k\sigma} \tau \right)^{\tau} + 2BG_{k\sigma}^{-} \tau, \]

(C.30)
Finally, for $I_s$ we find

\begin{align*}
\Gamma^\ell : & - LG_{++} + 24B G_{k\tilde{k} \sigma}^\sigma - 12 B G_{\sigma_1 \sigma_2}^\sigma, \\
\Gamma^r : & 6i e_\tau^\sigma \sigma_2 L F_{\sigma_1 \sigma_2}^\tau + 48 B G_{k\tilde{k} \sigma}^\sigma, \\
\Gamma^\tau : & -2i e_\tau^\sigma \sigma_2 L F_{\sigma_1 \sigma_2}^\tau + L G_{++} + 24 B G_{k\tilde{k} \sigma}^\sigma, \\
\Gamma^{k_3 t_2} : & -3i e_\tau^\sigma \sigma_2 L F_{k\tilde{k} \sigma}^\tau + 12( B G_{k\tilde{k} t_1 t_2} - B G_{t_1 t_2}^\sigma), \\
\Gamma^{k_3 t_2} : & -3i e_\tau^\sigma \sigma_2 L F_{k\tilde{k} \sigma}^\tau - \frac{1}{2} L G_{++} + 12( B G_{k\tilde{k} t_1 t_2} - B G_{t_1 t_2}^\sigma), \\
\Gamma^{t_1 \ldots t_3} : & - \frac{1}{3} \epsilon_{t_1 \ldots t_3}^\sigma \left( E_{--} + 12i L F_{k\tilde{k} \sigma}^\sigma - 12i L F_{\sigma_1 \sigma_2}^\sigma \right) + 8 B G_{k t_1 \ldots t_3}, \\
\Gamma^{t_1 \ldots t_3} : & \frac{1}{3} \epsilon_{t_1 t_2}^\sigma \left( E_{++} + 12i L F_{k\tilde{k} \sigma}^\sigma - 12i L F_{\sigma_1 \sigma_2}^\sigma \right) + 12 B G_{k t_1 \ldots t_3}, \\
\Gamma^{t_1 \ldots t_3} : & \frac{1}{3} \epsilon_{t_1 t_2}^\sigma \left( E_{++} - 12i L F_{k \sigma_1}^\sigma \right) - 2 B G_{t_1 \ldots t_3}, \tag{C.31}
\end{align*}

\section*{Appendix D. Generic half-maximal SU(4) \times \mathbb{R}^4-backgrounds}

\subsection*{D.1. The linear system}

We decompose the vector $SO(9, 1)$ representation under $SU(4)$. This is equivalent to decomposing the frame indices as $A = (+, -, \alpha, \beta)$. Consequently, the fluxes and geometry decompose into $SU(4)$ representations, i.e. $P_A$ decomposes as $P_+, P_-, P_{\alpha}$ and $P_{\beta}$ and similarly for the other fluxes and geometry\textsuperscript{17}.

Next, we construct the linear system associated with the algebraic and supercovariant connection Killing spinor equations. In particular, the algebraic Killing spinor equations give

\begin{align*}
(A_{11} + i A_{12}) P_\alpha + & \frac{1}{4} G_{+\alpha} + \frac{1}{4} G_{\beta}^\beta + \frac{1}{12} \epsilon_\alpha^\beta \beta_1 \beta_2 \beta_3 G_{\beta_1 \beta_2 \beta_3} = 0, \\
(A_{21} + i A_{22}) P_\alpha + & \frac{1}{4} G_{+\alpha} + \frac{1}{4} G_{\beta}^\beta - \frac{1}{12} \epsilon_\alpha^\beta \beta_1 \beta_2 \beta_3 G_{\beta_1 \beta_2 \beta_3} = 0, \\
(A_{11} - i A_{12}) P_\alpha + & \frac{1}{4} G_{-\alpha} - \frac{1}{4} G_{\beta}^\beta + \frac{1}{12} \epsilon_\alpha^\beta \beta_1 \beta_2 \beta_3 G_{\beta_1 \beta_2 \beta_3} = 0, \\
(A_{21} - i A_{22}) P_\alpha - & \frac{1}{4} G_{-\alpha} + \frac{1}{4} G_{\beta}^\beta + \frac{1}{12} \epsilon_\alpha^\beta \beta_1 \beta_2 \beta_3 G_{\beta_1 \beta_2 \beta_3} = 0, \\
(A_{11} + i A_{12}) P_\alpha + & \frac{1}{4} G_{\alpha}^\alpha = 0, \\
(A_{21} + i A_{22}) P_\alpha + & \frac{1}{4} G_{\alpha}^\alpha = 0, \\
(A_{11} - i A_{12}) P_\alpha - & \frac{1}{4} G_{\alpha}^\alpha = 0, \\
(A_{21} - i A_{22}) P_\alpha + & \frac{1}{4} G_{\alpha}^\alpha = 0,
\end{align*}

and

\begin{align*}
G_{+\alpha \beta} - \frac{1}{2} \epsilon_{\alpha \beta \gamma} G_{+\gamma} = 0, \\
i G_{+\alpha \beta} + \frac{1}{2} \epsilon_{\alpha \beta \gamma} G_{+\gamma} = 0,
\end{align*}

where we have set $A = z^{-1} z^\ast$.

\textsuperscript{17}If the fluxes are complex, like $P$ and $G$, then their various components do not satisfy the ‘naive’ complex conjugate relations, i.e. $(P_\alpha)^\ast \neq P_\beta$ and similarly for $G$. 

The Killing spinor equation associated with the supercovariant derivative (2.6) also decomposes in SU(4) representations. In particular the conditions associated with $D_0$ are

$$(z^{-1}D_0 z)_{11} + i(z^{-1}D_0 z)_{12} + \frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta} + i F_{a-\beta} = 0,$$

$$+ \frac{1}{4} (A_{11} + i A_{12}) G_{a \beta} + \frac{1}{4} (A_{11} + i A_{12}) G_{a-a} = 0,$$  \hfill (D.11)

$$(z^{-1}D_0 z)_{21} + i(z^{-1}D_0 z)_{22} + i \left[ \frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta} + i F_{a-\beta} \right] + \frac{1}{4} (A_{21} + i A_{22}) \left[ G_{a \beta} + G_{a-a} \right] = 0,$$  \hfill (D.12)

$$\Omega_{a, \beta_1 \beta_2} + i F_{a \beta_1 \beta_3} + i F_{a-\beta_1 \beta_3} + \frac{1}{2} (A_{11} + i A_{12}) G_{a \beta_1 \beta_3} + (B_{11} + i B_{12}) \delta_{a \beta_1 \beta_3}$$

$$- \frac{1}{2} \left[ \Omega_{a, \beta_1 \beta_2} + \frac{1}{2} (A_{11} - i A_{12}) G_{a \gamma_1 \gamma_2} \right] \epsilon^{\gamma_1 \gamma_2 \beta_1 \beta_2} = 0,$$  \hfill (D.13)

$$i \left[ \Omega_{a, \beta_1 \beta_2} + i F_{a \beta_1 \beta_3} + i F_{a-\beta_1 \beta_3} + \frac{1}{2} (A_{21} + i A_{22}) G_{a \beta_1 \beta_3} + (B_{21} + i B_{22}) \delta_{a \beta_1 \beta_3} \right]$$

$$- \frac{1}{2} \left[ -i \Omega_{a, \gamma_1 \gamma_2} + \frac{1}{2} (A_{21} - i A_{22}) G_{a \gamma_1 \gamma_2} \right] \epsilon^{\gamma_1 \gamma_2 \beta_1 \beta_2} = 0,$$  \hfill (D.14)

$$(z^{-1}D_0 z)_{11} - i(z^{-1}D_0 z)_{12} + \frac{1}{2} (B_{11} - i B_{12}) P_{a} + \frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta}$$

$$- \frac{1}{4} (A_{11} - i A_{12}) \left[ G_{a \beta} - G_{a-\beta} \right] + \frac{i}{12} F_{a \beta_1 \beta_2 \beta_3} \epsilon^{\beta_1 \beta_2 \beta_3 \beta_4} = 0,$$  \hfill (D.15)

$$(z^{-1}D_0 z)_{21} - i(z^{-1}D_0 z)_{22} + \frac{1}{2} (B_{21} - i B_{22}) P_{a} - i \left[ -\frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta} \right]$$

$$- \frac{1}{4} (A_{21} - i A_{22}) \left[ G_{a \beta} - G_{a-\beta} \right] - \frac{i}{12} F_{a \beta_1 \beta_2 \beta_3} \epsilon^{\beta_1 \beta_2 \beta_3 \beta_4} = 0,$$  \hfill (D.16)

$$\frac{1}{2} \Omega_{a, +\beta} - \frac{1}{4} (B_{11} + i B_{12}) \delta_{a \beta} P_{a} + \frac{1}{2} (A_{11} + i A_{12}) G_{a +\beta} + \frac{i}{2} F_{a +\beta \gamma'} = 0,$$  \hfill (D.17)

$$\left[ \frac{i}{2} \Omega_{a, +\beta} - \frac{1}{4} (B_{21} + i B_{22}) \delta_{a \beta} P_{a} + \frac{1}{2} (A_{21} + i A_{22}) G_{a +\beta} - \frac{i}{2} F_{a +\beta \gamma'} \right] = 0,$$  \hfill (D.18)

$$\frac{i}{12} F_{a +\beta_1 \beta_2 \beta_3} + \frac{1}{12} \left[ \frac{i}{2} \Omega_{a, +\gamma'} + \frac{1}{4} (A_{11} - i A_{12}) G_{a \gamma'} \right] \epsilon^{\gamma' \beta_1 \beta_2 \beta_3} = 0,$$  \hfill (D.19)

$$- \frac{i}{12} F_{a +\beta_1 \beta_2 \beta_3} + \frac{1}{12} \left[ -\frac{i}{2} \Omega_{a, +\gamma'} + \frac{1}{4} (A_{21} - i A_{22}) G_{a \gamma'} \right] \epsilon^{\gamma' \beta_1 \beta_2 \beta_3} = 0,$$  \hfill (D.20)

where $B = A^2$.

The conditions associated with $D_0$ are

$$(z^{-1}D_0 z)_{11} + i(z^{-1}D_0 z)_{12} + \frac{1}{2} (B_{11} + i B_{12}) P_{a} + \frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta}$$

$$+ \frac{i}{12} F_{a \gamma \gamma_1 \gamma_2 \gamma_3} \epsilon^{\gamma \gamma_1 \gamma_2 \gamma_3 \gamma_4} + \frac{1}{4} (A_{11} + i A_{12}) \left[ G_{a \beta} + G_{a-a} \right] = 0,$$  \hfill (D.21)

$$(z^{-1}D_0 z)_{21} + i(z^{-1}D_0 z)_{22} + \frac{1}{2} (B_{21} + i B_{22}) P_{a} + i \left[ \frac{1}{2} \Omega_{a, \beta} + \frac{1}{2} \Omega_{a, -\beta} \right]$$

$$- \frac{i}{12} F_{a \gamma \gamma_1 \gamma_2 \gamma_3} \epsilon^{\gamma \gamma_1 \gamma_2 \gamma_3 \gamma_4} + \frac{1}{4} (A_{21} + i A_{22}) \left[ G_{a \beta} + G_{a-a} \right] = 0.$$  \hfill (D.22)
\[
\Omega_{\alpha,\beta,\gamma} + \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha\beta,\gamma} = -\frac{1}{2} \left[ \Omega_{\alpha,\beta,\gamma} - i F_{\alpha\gamma\delta} + i F_{\alpha-\gamma,\delta} \right] = 0,
\]
\[
\frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\beta,\gamma} + (B_{11} - i B_{12}) \delta_{\alpha,\gamma} P_{\beta} = 0,
\]
\[
\frac{1}{2} \left[ \Omega_{\alpha,\beta,\gamma} - i F_{\alpha\gamma\delta} + i F_{\alpha-\gamma,\delta} \right] = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{11} - i (\gamma - 1) D_{\alpha z} \zeta_{12} = -\frac{1}{2} \Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} \Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} \Omega_{\alpha,\gamma,\gamma} = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{12} = \frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[
\frac{1}{2} \Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha\beta,\gamma} = 0,
\]
\[
\frac{1}{2} \Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{21} = \frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{22} = \frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\gamma,\gamma} = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{22} = \frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\gamma,\gamma} = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{22} = \frac{1}{2} (A_{11} - i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[
The conditions associated with \( D_\gamma \) are
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{11} + i (\gamma - 1) D_{\alpha z} \zeta_{12} = \frac{1}{2} (\Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} \Omega_{\alpha,\gamma,\gamma} + \frac{1}{2} F_{\gamma\delta}) = 0,
\]
\[
\frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0,
\]
\[
(\gamma - 1) D_{\alpha z} \zeta_{21} + i (\gamma - 1) D_{\alpha z} \zeta_{22} = \frac{1}{2} (\Omega_{\alpha,\beta,\gamma} + \frac{1}{2} \Omega_{\alpha,\beta,\gamma} + \frac{1}{2} F_{\gamma\delta}) = 0,
\]
\[
\frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[
\Omega_{\alpha,\beta,\gamma} + i F_{\alpha,\gamma,\gamma} = \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0,
\]
\[
\frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[
\frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = \frac{1}{2} (A_{11} + i A_{12}) G_{\alpha,\gamma,\gamma} = 0.
\]
\[(z^{-1} D_{-} z)_{11} - i(z^{-1} D_{-} z)_{12} = -\frac{1}{2} \Omega^{-\gamma \gamma} + \frac{1}{2} \Omega_{- \rightarrow} + \frac{i}{4} F_{\gamma \gamma}^\delta \delta - \frac{1}{4} (A_{11} - iA_{12}) G_{- \gamma} \gamma \]
\[+ \frac{i}{12} F_{- \beta \beta \beta \beta} \epsilon^{\bar{\beta} \beta \beta \beta \beta} = 0, \quad (D.35)\]
\[(z^{-1} D_{-} z)_{21} - i(z^{-1} D_{-} z)_{22} = i \left[ -\frac{1}{2} \Omega^{-\gamma \gamma} + \frac{1}{2} \Omega_{- \rightarrow} + \frac{i}{4} F_{- \gamma \gamma}^\delta \delta \right] \]
\[- \frac{1}{4} (A_{21} - iA_{22}) G_{- \gamma} \gamma - \frac{1}{12} F_{- \beta \beta \beta \beta} \epsilon^{\bar{\beta} \beta \beta \beta \beta} = 0, \quad (D.36)\]
\[\frac{1}{2} \Omega_{- \rightarrow} + \frac{i}{2} F_{\gamma \gamma}^\delta \delta + \frac{1}{4} (A_{11} + iA_{12}) G_{- \gamma} \gamma + \frac{1}{4} (B_{11} + iB_{12}) P_\beta - \frac{i}{6} F_{\gamma \gamma}^\gamma \gamma \epsilon^{\gamma \gamma \gamma} \gamma \beta = 0, \quad (D.37)\]
\[i \left[ \frac{1}{2} \Omega_{- \rightarrow} + \frac{i}{2} F_{\gamma \gamma}^\delta \delta \right] + \frac{1}{4} (A_{21} + iA_{22}) G_{- \gamma} \gamma + \frac{1}{4} (B_{21} + iB_{22}) P_\beta - \frac{i}{6} F_{\gamma \gamma}^\gamma \gamma \epsilon^{\gamma \gamma \gamma} \gamma \beta = 0, \quad (D.38)\]
\[i F_{\gamma \gamma}^{\beta \beta \beta \beta} + \left[ \frac{1}{2} \Omega_{- \rightarrow} - \frac{i}{2} F_{\gamma \gamma}^\delta \delta \right] \epsilon^{\gamma \gamma \beta \beta \beta} \]
\[+ \frac{1}{4} [(A_{11} - iA_{12}) G_{- \gamma} \gamma + (B_{11} - iB_{12}) P_\beta] \epsilon^{\gamma \gamma \beta \beta \beta} = 0, \quad (D.39)\]
\[-F_{\gamma \gamma}^{\beta \beta \beta \beta} - i \left[ \frac{1}{2} \Omega_{- \rightarrow} - \frac{i}{2} F_{\gamma \gamma}^\delta \delta \right] \epsilon^{\gamma \gamma \beta \beta \beta} \]
\[+ \frac{1}{4} [(A_{21} - iA_{22}) G_{- \gamma} \gamma + (B_{21} - iB_{22}) P_\beta] \epsilon^{\gamma \gamma \beta \beta \beta} = 0. \quad (D.40)\]

The conditions associated with \( D_+ \) are
\[(z^{-1} D_{+} z)_{11} + i(z^{-1} D_{+} z)_{12} = \frac{1}{2} (B_{11} + iB_{12}) P_\gamma + \frac{1}{2} \Omega_{+ \gamma} \gamma + \frac{1}{2} \Omega_{+ \rightarrow} + \frac{1}{4} (A_{11} + iA_{12}) G_{+ \gamma} \gamma \]
\[- \frac{1}{4} (A_{21} + iA_{22}) G_{+ \gamma} \gamma = 0, \quad (D.41)\]
\[(z^{-1} D_{+} z)_{21} + i(z^{-1} D_{+} z)_{22} = \frac{1}{2} (B_{21} + iB_{22}) P_\gamma + i \left[ \frac{1}{2} \Omega_{+ \gamma} \gamma + \frac{1}{2} \Omega_{+ \rightarrow} \right] \]
\[+ \frac{1}{4} (A_{21} + iA_{22}) G_{+ \gamma} \gamma = 0, \quad (D.42)\]
\[\Omega_{+ \beta \beta} + \frac{1}{2} (A_{11} + iA_{12}) G_{+ \beta \beta} - \frac{1}{2} \Omega_{+ \gamma} \gamma \epsilon^{\gamma \gamma \beta \beta} = \frac{1}{4} (A_{11} - iA_{12}) G_{+ \gamma} \gamma \epsilon^{\gamma \gamma \beta \beta} = 0, \quad (D.43)\]
\[i \Omega_{+ \beta \beta} + \frac{1}{2} (A_{21} + iA_{22}) G_{+ \beta \beta} + \frac{1}{4} \Omega_{+ \gamma} \gamma \epsilon^{\gamma \gamma \beta \beta} = \frac{1}{4} (A_{21} - iA_{22}) G_{+ \gamma} \gamma \epsilon^{\gamma \gamma \beta \beta} = 0. \quad (D.44)\]
\[(z^{-1} D_{+} z)_{11} - i(z^{-1} D_{+} z)_{12} = \frac{1}{2} (B_{11} - iB_{12}) P_\gamma - \frac{1}{2} \Omega_{\rightarrow \gamma} \gamma \]
\[+ \frac{1}{2} \Omega_{\rightarrow \rightarrow} - \frac{1}{4} (A_{11} - iA_{12}) G_{\gamma \gamma} = 0, \quad (D.45)\]
\[(z^{-1} D_{+} z)_{21} - i(z^{-1} D_{+} z)_{22} = \frac{1}{2} (B_{21} - iB_{22}) P_\gamma - i \left[ \frac{1}{2} \Omega_{\rightarrow \gamma} \gamma + \frac{1}{2} \Omega_{\rightarrow \rightarrow} \right] \]
\[- \frac{1}{4} (A_{21} - iA_{22}) G_{\gamma \gamma} = 0, \quad (D.46)\]
and
\[ \Omega_{\tau, \tau a} = \Omega_{\tau, \tau \bar{a}} = 0. \]  
(D.47)

As we have already mentioned, all the equations that arise from the Killing spinor equations are linear in the fluxes, geometry and the first derivatives of the functions \( z \) that determine the Killing spinors. The system may appear involved but it can be solved. It also simplifies in some special cases, such as for example whenever \( z \) is a real matrix. In this case \( A = B = 1 \) and so the terms in the linear system above that contain the fluxes and geometry do not depend on the functions \( z \).

D.2. The solution to the linear system

We shall first solve the last six equations of the linear system associated with the algebraic Killing spinor equation. Equations (D.5)–(D.8) imply that
\[ G_{\tau a} = P_\tau = 0, \]  
(D.48)
and (D.9), (D.10) imply that
\[ G_{\tau a \bar{b}} = G_{\tau \bar{a} \bar{b}} = 0. \]  
(D.49)

Next consider equations (D.19), (D.20), (D.27), (D.28). These imply
\[ \Omega_{a, \bar{b} +} = 0, \quad F_{\tau a \bar{b} \bar{c} \bar{d} \bar{e}} = 0 \]  
(D.50)
and (D.43), (D.44) and (D.47) imply that
\[ \Omega_{a, \bar{a} \bar{b}} = 0, \quad \Omega_{a, \tau \bar{a}} = 0. \]  
(D.51)

The remaining components of the \( D_\tau \) equations (D.41), (D.42), (D.45), (D.46) imply that
\[ (z^{-1} D_\tau z)_{11} = (z^{-1} D_\tau z)_{22} = \frac{1}{2} \Omega_{\tau, \tau}, \]  
(D.52)
\[ (z^{-1} D_\tau z)_{12} = -(z^{-1} D_\tau z)_{21} = \frac{i}{2} \Omega_{\tau, a}. \]

Equations (D.31), (D.32), (D.35), (D.36) constrain \( z^{-1} D_\tau z \) via
\[ (z^{-1} D_\tau z)_{11} = -\frac{1}{2} \Omega_{\tau, \tau} - \frac{i}{4} F_{a \bar{a} \bar{b}} \]  
\[ - \frac{i}{24} (F_{-12 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-13 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-14 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}}) = -\frac{1}{4} A_{12} G_{-a}, \]
\[ (z^{-1} D_\tau z)_{22} = -\frac{1}{2} \Omega_{\tau, \tau} + \frac{i}{4} F_{a \bar{a} \bar{b}} \]  
\[ + \frac{i}{24} (F_{-12 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-13 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-14 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}}) = +\frac{1}{4} A_{12} G_{-a}, \]
\[ (z^{-1} D_\tau z)_{12} = \frac{i}{2} \Omega_{\tau, a} - \frac{i}{24} ((F_{-12 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-13 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} - F_{14 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}}) = \frac{i}{4} A_{12} G_{-a}, \]
\[ (z^{-1} D_\tau z)_{21} = -\frac{i}{2} \Omega_{\tau, a} - \frac{i}{24} ((F_{-12 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} + F_{-13 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}} - F_{14 a \bar{a} \bar{b} \bar{c} \bar{d} \bar{e}}) = -\frac{i}{4} A_{12} G_{-a}. \]

And from (D.33) and (D.34), we find
\[ G_{-a \bar{b} \bar{c} \bar{d} \bar{e}} = -\frac{(A_{11} + A_{22} + i(A_{21} - A_{12}))}{A_{11} A_{22} - A_{12} A_{21}} (\Omega_{-a \bar{b} \bar{c} \bar{d} \bar{e}} + i F_{-a \bar{b} \bar{c} \bar{d} \bar{e}} \gamma) \]
\[ + \frac{(A_{22} - A_{11} + i(A_{12} + A_{21}))}{2(A_{11} A_{22} - A_{12} A_{21})} (\Omega_{-a \bar{b} \bar{c} \bar{d} \bar{e}} - i F_{-a \bar{b} \bar{c} \bar{d} \bar{e}} \gamma) \epsilon_{a \bar{b} \bar{c} \bar{d} \bar{e}} \beta_\alpha \beta_\beta \]  
(D.53)
\[ G_{-\alpha_1\alpha_2} = -\frac{(A_{11} + A_{22} - i(A_{22} - A_{12}))}{A_{11}A_{22} - A_{12}A_{21}} (\Omega_{-\alpha_1\alpha_2} - iF_{-\alpha_1\alpha_2} \gamma^\nu) \]
\[ = -\frac{(A_{11} - A_{22} + i(A_{12} + A_{21}))}{2(A_{11}A_{22} - A_{12}A_{21})} (\Omega_{-\beta_1\beta_2} + iF_{-\beta_1\beta_2} \gamma^\nu) \epsilon_{\alpha_1\alpha_2} \hat{\beta}_1 \hat{\beta}_2 \quad \text{(D.54)} \]

In order to proceed we shall consider two separate cases, according as \((A_{11} - A_{22})^2 + (A_{12} + A_{21})^2\) vanishes or not. Observe that \((A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0\) is equivalent to \(A_{11} = A_{22}\) and \(A_{12} = A_{21}\).

First we shall assume that \((A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 \neq 0\). Then (D.17), (D.18), (D.29), (D.30) imply that
\[ G_{\alpha\beta} = \Omega_{\alpha\beta} = 0\]
\[ \text{(D.55)} \]

and
\[ F_{\alpha_1\alpha_2;\beta_1\beta_2} = 0.\]
\[ \text{(D.56)} \]

Next consider the equations which have one free holomorphic index, excluding for the moment those equations involving \(\zeta^{-1}D\zeta\); these are (D.3), (D.4), the trace of the duals of (D.13), (D.14) and the duals of (D.39), (D.40). These fix \(P_\alpha, G_{-\alpha\alpha}, G_{\alpha\beta}, \epsilon_{\alpha_1\beta_2\beta_1}, G_{\bar{\alpha}_1\beta_2\beta_1}, \epsilon_{\alpha_1\beta_2\beta_1}F_{-\bar{\alpha}_1\beta_2\beta_1}, F_{-\alpha\beta}\) in terms of the components of the spin connection \(\Omega_{\beta_1\beta_2}, \epsilon_{\alpha_1\beta_2\beta_1}\Omega_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}, \Omega_{-\alpha\alpha}\) and \(A_{ij}\).

The corresponding equations with one free antiholomorphic constraint are (D.1), (D.4), the traces of (D.13), (D.14) and (D.37), (D.38). These fix \(P_\bar{\alpha}, G_{-\bar{\alpha}\bar{\alpha}}, G_{\bar{\alpha}\bar{\beta}}, \epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}, G_{\bar{\bar{\alpha}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_1}, \epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}F_{-\bar{\bar{\alpha}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_1}, F_{-\bar{\alpha}\bar{\beta}}\) in terms of the components of the spin connection \(\Omega_{\bar{\beta}_1\bar{\beta}_2}, \epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}\Omega_{\bar{\bar{\alpha}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_1}, \Omega_{-\bar{\alpha}\bar{\alpha}}\).

By comparing the complex expressions for \(\epsilon_{\alpha_1\beta_2\beta_1}F_{-\bar{\alpha}_1\beta_2\beta_1}, F_{-\alpha\beta}\) from the former equations with the complex conjugates of the expressions \(\epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}F_{-\bar{\bar{\alpha}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_1}, F_{-\bar{\alpha}\bar{\beta}}\) from the latter, we find the following geometric constraints
\[ \Omega_{\beta_1\beta_2} = -\frac{6A_{11}A_{22} - 4A_{12}A_{21} - A_{12}^2 - A_{21}^2}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}, \]
\[ \epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}\Omega_{\bar{\beta}_1\bar{\beta}_2} = -2i(A_{12} - A_{21})(A_{11} - A_{22} + i(A_{12} + A_{21})) \frac{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}. \]
\[ \text{(D.57)} \]

Using these constraints, we obtain the following simplifications:
\[ P_\alpha = \frac{4}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}, \]
\[ G_{-\alpha\alpha} = -8 \frac{(A_{11} + A_{22})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}, \]
\[ G_{\bar{\alpha}\bar{\beta}} = -8i \frac{(A_{12} - A_{21})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}, \]
\[ \epsilon_{\bar{\alpha}_1\bar{\beta}_2\bar{\beta}_1}G_{\bar{\bar{\alpha}}_1\bar{\bar{\beta}}_2\bar{\bar{\beta}}_1} = -24 \frac{(A_{11} - A_{22} + i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{-\alpha_1\alpha_2}. \]
\[ \text{(D.58)} \]
and

\[ P_\alpha = \frac{4}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{\ldots, \alpha}, \]

\[ G_{-\alpha\beta} = -\frac{8}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \Omega_{\ldots, \alpha\beta}, \]

\[ G_{\alpha\beta} = -\frac{8i}{(A_{12} - A_{21})} \left( \frac{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})}{\Omega_{\ldots, \alpha\beta}} \right), \]

\[ \epsilon_{\alpha \beta \bar{\alpha} \bar{\beta}} G_{\bar{\alpha} \bar{\beta} \bar{\alpha} \bar{\beta}} = -24 \frac{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})}{\Omega_{\ldots, \alpha\beta}}. \]

and

\[ \epsilon_{\alpha \beta \bar{\alpha} \bar{\beta}} F_{-\alpha\beta\bar{\alpha}\bar{\beta}} = -3\frac{(A_{11} + A_{22})(A_{11} - A_{22} + i(A_{12} + A_{21}))}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \frac{1}{\Omega_{\ldots, \alpha\beta}}. \]

\[ F_{-\alpha\beta} = \frac{(A_{11} + A_{22})(A_{11} - A_{22})}{(A_{11}^2 + A_{22}^2 + 4A_{11}A_{22} - 2A_{12}A_{21})} \frac{1}{\Omega_{\ldots, \alpha\beta}}. \]

Substituting these constraints back into (D.21), (D.22), (D.25) and (D.26) we find the following constraints on \( z^{-1} D_{\alpha} z \):

\[ (z^{-1} D_{\alpha} z)_{11} = \frac{1}{2} \Omega_{\alpha, \alpha} - \frac{1}{2} \left( \frac{(A_{11}^2 - 3A_{11}^2 + 4A_{11}A_{22} - 2A_{11}A_{22} + 2A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} D_{\alpha} z)_{22} = \frac{1}{2} \Omega_{\alpha, \alpha} + \frac{1}{2} \left( \frac{(3A_{11}^2 - A_{11}^2 - 4A_{11}A_{22} + 4A_{11}A_{22} + 2A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} \partial_{\alpha} z)_{12} = \frac{i}{2} \Omega_{\alpha, \beta} + \frac{i}{2} \left( \frac{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} \partial_{\alpha} z)_{21} = -\frac{1}{2} \Omega_{\alpha, \beta} - \frac{1}{2} \left( \frac{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}. \]

And from (D.11), (D.12), (D.15) and (D.16) we find the following constraints on \( z^{-1} D_{\alpha} z \):

\[ (z^{-1} D_{\alpha} z)_{11} = \frac{1}{2} \Omega_{\alpha, \alpha} - \frac{1}{2} \left( \frac{(3A_{11}^2 - A_{11}^2 - 4A_{11}A_{22} + 4A_{11}A_{22} + 2A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} D_{\alpha} z)_{22} = \frac{1}{2} \Omega_{\alpha, \alpha} + \frac{1}{2} \left( \frac{(A_{11}^2 - 3A_{11}^2 + 4A_{11}A_{22} - 2A_{11}A_{22} + 2A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} \partial_{\alpha} z)_{12} = \frac{i}{2} \Omega_{\alpha, \beta} - \frac{i}{2} \left( \frac{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}, \]

\[ (z^{-1} \partial_{\alpha} z)_{21} = -\frac{i}{2} \Omega_{\alpha, \beta} + \frac{i}{2} \left( \frac{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \right) \Omega_{\ldots, \alpha}. \]
Note that by means of an appropriate \( U(1) \) transformation, one can work in a gauge for which \( \det z = \det z^* \), so that \( \det A = 1 \). Then (D.61), (D.62) and (D.52) imply that
\[
Q_+ = Q_0 = 0 \tag{D.63}
\]
though in general (D.53) does not constrain \( Q_- \) to vanish.

Finally, we consider equations (D.13), (D.14), (D.23), (D.24). The constraints obtained from taking traces of these equations have already been obtained; the remaining constraints consist of the fixing of two components of the \( G \)-flux via
\[
G_{\alpha\beta\tilde{\alpha}\tilde{\beta}} = \frac{2}{A_{11} - A_{22} + i(A_{12} + A_{21})} \Omega_{\alpha\gamma\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2}\]
\[
= \frac{8i(A_{21} - A_{12})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{\alpha\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2},
\]
\[
G_{\alpha\gamma_2} = \frac{2}{A_{11} - A_{22} - i(A_{12} + A_{21})} \Omega_{\alpha\beta\bar{\gamma}_2} = \frac{8i(A_{12} - A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{\alpha\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2}, \tag{D.64}
\]
and together with the fixing of a component of the \( F \)-flux
\[
F_{+\alpha\beta\tilde{\alpha}\tilde{\beta}} = (A_{11} + A_{22}) \left( \frac{i}{4(A_{11} - A_{22} + i(A_{12} + A_{21}))} \Omega_{\alpha\beta\bar{\gamma}_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2} - \frac{(A_{12} - A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{\alpha\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2} \right), \tag{D.65}
\]
and a geometric constraint
\[
\Omega_{\alpha\beta\bar{\gamma}_2} + \frac{i(A_{12} - A_{21})}{2(A_{11} - A_{22} + i(A_{12} + A_{21}))} \Omega_{\alpha\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2} - \frac{4(A_{11}A_{22} - A_{12}A_{21})}{(A_{11} + A_{22})^2 + 2(A_{11}A_{22} - A_{12}A_{21})} \Omega_{\alpha\gamma_2} \varepsilon_{\tilde{\beta}\tilde{\alpha}\gamma_2} = 0. \tag{D.66}
\]

Next we consider the special case when \( (A_{11} - A_{22})^2 + (A_{12} + A_{21})^2 = 0 \). Then (D.17), (D.18), (D.29), (D.30) imply that
\[
G_{+\alpha\beta} = \frac{1}{A_{11}} (\Omega_{+\alpha\bar{\beta}} - \Omega_{+\bar{\beta}\alpha}), \tag{D.67}
\]
\[
F_{+\alpha\beta\gamma} = -\frac{i}{2A_{11}}((A_{11} - iA_{12})\Omega_{+\alpha\bar{\beta}} + (A_{11} + iA_{12})\Omega_{+\bar{\beta}\alpha}). \tag{D.68}
\]

On comparing (D.68) with its complex conjugate, the geometric constraint
\[
\Omega_{+\alpha\bar{\beta}} + \Omega_{+\bar{\beta}\alpha} = 0 \tag{D.69}
\]
is obtained. Substituting this back into (D.67) and (D.68) we obtain some simplification:
\[
G_{+\alpha\beta} = \frac{2}{A_{11}} \Omega_{+\alpha\bar{\beta}}, \tag{D.70}
\]
\[
F_{+\alpha\beta\gamma} = -\frac{A_{12}}{A_{11}} \Omega_{+\alpha\bar{\beta}}.
\]

Next consider the equations which have one free holomorphic index, excluding for the moment those equations involving \( z^{-1}D_z z \); these are (D.3), (D.4), the trace of the duals of (D.13), (D.14) and the duals of (D.39), (D.40). Unlike the generic case, these do not fix \( P_{\alpha}, G_{+\alpha\gamma}, G_{\alpha\beta\gamma}, \varepsilon_{\alpha\tilde{\beta}\tilde{\gamma}}G_{\tilde{\beta}\tilde{\alpha}\tilde{\gamma}}, \varepsilon_{\alpha\tilde{\beta}\tilde{\gamma}}F_{+\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}, F_{+\bar{\alpha}\bar{\beta}} \) uniquely in terms of the
The corresponding equations with one free antiholomorphic constraint are (D.1), (D.4), the traces of (D.13), (D.14) and (D.37), (D.38). Taking $G_{\alpha\beta}^a$, $F_{-a\beta}^a$ and $\Omega_{\alpha^a}^{\beta} a$ to be arbitrary we find the additional constraints

$$P_{\bar{a}} = \frac{1}{3(A_{11} + iA_{12})} G_{\alpha\beta}^a \beta - \frac{2}{3(A_{11} + iA_{12})^2} (\Omega_{\alpha^a}^{\beta} a + iF_{-a\beta}^a),$$

$$G_{-a\bar{a}} = \frac{1}{3} G_{\alpha\beta}^a \beta + \frac{8}{3(A_{11} + iA_{12})} (\Omega_{\alpha^a}^{\beta} a + iF_{-a\beta}^a),$$

$$G_{\alpha^a\alpha^a} = 0,$$

$$F_{-a\alpha^a\alpha^a} = 0,$$

$$\Omega_{\alpha^a\alpha^a} = 0,$$

$$\Omega_{-a\bar{a}} = -\Omega_{\alpha^a}^{\beta} a.$$ (D.71)

Substituting these constraints back into (D.21), (D.22), (D.25) and (D.26) we find the following constraints on $z^{-1} D_a z$:

$$(z^{-1} D_a z)_{11} = (z^{-1} D_a z)_{22} = -\frac{i}{6} A_{12} G_{\alpha\beta}^a \beta - \frac{1}{2} \Omega_{\alpha^a}^{\beta} a,$$

$$+ \frac{1}{6(A_{11} + iA_{12})} (-3A_{11} + iA_{12}) \Omega_{\alpha^a}^{\beta} a + \frac{i}{3(A_{11} + iA_{12})} (3A_{11} + iA_{12}) F_{-a\beta}^a,$$

$$(z^{-1} d_a z)_{12} = -(z^{-1} d_a z)_{21} = \frac{i}{2} \Omega_{\alpha^a}^{\beta} a + \frac{i}{6} A_{11} G_{\alpha\beta}^a \beta,$$

$$+ \frac{i}{6(A_{11} + iA_{12})} (-A_{11} + iA_{12}) \Omega_{\alpha^a}^{\beta} a - \frac{2A_{11}}{3(A_{11} + iA_{12})} F_{-a\beta}^a. (D.72)$$

And from (D.11), (D.12), (D.15) and (D.16) we find the following constraints on $z^{-1} D_a z$:

$$(z^{-1} D_a z)_{11} = -(z^{-1} D_a z)_{22} = -\frac{1}{2} \Omega_{\alpha^a}^{\beta} a - \frac{i}{6} A_{12} G_{\alpha\beta}^a \beta,$$

$$- \frac{1}{6(A_{11} - iA_{12})^2} (3A_{11} + iA_{12}) \Omega_{\alpha^a}^{\beta} a + \frac{i}{3(A_{11} - iA_{12})} (-3A_{11} + iA_{12}) F_{-a\beta}^a,$$

$$(z^{-1} d_a z)_{12} = -(z^{-1} d_a z)_{21} = \frac{i}{2} \Omega_{\alpha^a}^{\beta} a + \frac{i}{6} A_{11} G_{\alpha\beta}^a \beta,$$

$$+ \frac{i}{6(A_{11} - iA_{12})} (A_{11} + iA_{12}) \Omega_{\alpha^a}^{\beta} a - \frac{2A_{11}}{3(A_{11} - iA_{12})} F_{-a\beta}^a. (D.74)$$

Finally, we consider equations (D.13), (D.14), (D.23), (D.24). These fix two components of the $G$-flux via

$$G_{\alpha^a \beta^2} = -\frac{2}{A_{11} + iA_{12}} (\Omega_{\alpha^a \beta^2} \beta^2 + 2i F_{-a\beta^2} \beta^2)$$

$$- \frac{1}{A_{11} + iA_{12}} \delta_a (\bar{\beta}^2 \left( \frac{2}{3} (A_{11} + iA_{12}) G_{\beta^2 \beta^2} \gamma^2 - \frac{4}{3} \Omega_{\gamma^2 \gamma^2 \beta^2} - \frac{8i}{3} F_{\beta^2 \gamma^2 \gamma^2} \right).$$
$G_{\gamma_1 \gamma_2} = -\frac{2}{A_{11} - i A_{12}} (\Omega_{\gamma_1 \gamma_2} + 2i F_{-\gamma_1 \gamma_2})$

$$= -\frac{1}{A_{11} - i A_{12}} \delta_{\gamma_1 \gamma_2} \left( \frac{2}{3} (A_{11} - i A_{12}) G_{\gamma_2} \rho = -\frac{4}{3} \Omega_{\beta_1 \beta_2 \gamma_2} + \frac{8i}{3} F_{\gamma_2} \rho \right)$$

(D.75)

together with the geometric constraint

$$\Omega_{\alpha,\gamma_1 \gamma_2} = 0.$$  \hspace{1cm} (D.76)

As the first-order equations in spacetime derivatives of $z$ are nonlinear, we shall not investigate the general case here. Instead we shall focus on two examples that illustrate some of the properties of this nonlinear system.

D.3. Special cases

We shall first consider the case that $z$ is diagonal with complex entries. We have seen that it can be arranged such that the Killing spinors can be written as $\epsilon_1 = \rho e^{i \phi} \eta_1$ and $\epsilon_2 = \rho^{-1} e^{-i \phi} \eta_2$. Using this, equations (D.61) and (D.62) imply that

$$\Omega_{\alpha,\beta} = \cos 4\phi \Omega_{\alpha,\beta}, \quad Q_{\alpha} = 0, \quad \Omega_{\alpha,-} + \Omega_{\alpha,+} = 0,$$

(D.77)

and

$$\partial_{\alpha} \rho = 0, \quad \partial_{\alpha} \phi = \frac{4\sin 4\phi}{2 + \cos 4\phi} \Omega_{\alpha,-} = 0.$$  \hspace{1cm} (D.78)

From (D.53) we obtain

$$Q_- = \frac{1}{2} F_{-\alpha} \phi \beta, \quad \Omega_{\alpha, a} = \frac{1}{2} \cos 2\phi G_{\alpha} a,$$

(D.79)

and

$$\partial_{\alpha} \phi = -\frac{1}{\Omega_{\alpha, a} \alpha} \partial_{\alpha} = 0.$$  \hspace{1cm} (D.80)

From (D.52) we obtain

$$\Omega_{\alpha, a} = 0, \quad Q_+ = 0, \quad \Omega_{\alpha,-} = 0,$$

(D.81)

and

$$\partial_{\alpha} \phi = 0, \quad \partial_{\alpha} \rho = 0.$$  \hspace{1cm} (D.82)

For the other example we take $z$ to be real and so $A$ is the identity matrix. Then from (D.73) we find

$$Q_+ = 2 F_{-\alpha} \phi \beta, \quad F_{-\alpha} \phi \beta = \frac{i}{8} (G_{\alpha} \beta + (G_{\alpha} \beta)^*),$$

(D.83)

and

$$\partial_{\alpha} z = \frac{1}{2} \Omega_{\alpha,-} + \frac{1}{2} \Omega_{\beta, \alpha} \beta.$$  \hspace{1cm} (D.84)
From (D.53) we obtain
\[
Q_\alpha = \frac{1}{2} F_{\alpha}{}^{\mu} \tilde{\rho}^\mu, \quad F_{-\alpha,\alpha,\alpha} = 0, \quad (G_{-\alpha})^* + G_{-\alpha} = 0, \quad (D.85)
\]
and
\[
(z^{-1} \partial_x z)_{11} = (z^{-1} \partial_x z)_{22} = -\frac{1}{2} \Omega_{x,--},
\]
\[
(z^{-1} \partial_x z)_{12} = -(z^{-1} \partial_x z)_{21} = \frac{i}{2} \Omega_{x,a}^a + \frac{i}{4} G_{-\alpha}.
\quad (D.86)
\]
From (D.52) we obtain
\[
Q_+ = 0
\quad (D.87)
\]
and
\[
(z^{-1} \partial_x z)_{11} = (z^{-1} \partial_x z)_{22} = -\frac{1}{2} \Omega_{x,--},
\]
\[
(z^{-1} \partial_x z)_{12} = -(z^{-1} \partial_x z)_{21} = \frac{i}{2} \Omega_{x,a}^a.
\quad (D.88)
\]

**Appendix E. The linear system of degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$-backgrounds**

It is straightforward to construct the linear system for the degenerate half-maximal $SU(4) \ltimes \mathbb{R}^8$ case using the linear system of [12] for one $SU(4) \ltimes \mathbb{R}^8$-invariant spinor and those of section 6. In particular, we have that the algebraic Killing spinor equations give

\[
(f + g_2 - ig_1) P_a = 0,
\]
\[
\frac{1}{4} (f - g_2 + ig_1) G_{-\alpha} + \frac{1}{2} (f - g_2 + ig_1) G_{\alpha\beta} + \frac{1}{4} (f + g_2 + ig_1) \epsilon_a^{\beta_1 \beta_2} G_{\beta_1 \beta_2} = 0,
\quad (E.1)
\]
\[
(f - g_2 - ig_1) P_a = 0,
\]
\[
\frac{1}{4} (f + g_2 + ig_1) G_{-\alpha} - \frac{1}{2} (f + g_2 + ig_1) G_{\alpha\beta} + \frac{1}{4} (f - g_2 + ig_1) \epsilon_a^{\beta_1 \beta_2} G_{\beta_1 \beta_2} = 0,
\quad (E.2)
\]
\[
(f + g_2 - ig_1) P_+ = 0,
\]
\[
\frac{1}{4} (f - g_2 + ig_1) G_{a\alpha} = 0,
\quad (E.3)
\]
\[
(f - g_2 + ig_1) P_+ = 0,
\]
\[
-\frac{1}{4} (f + g_2 + ig_1) G_{\alpha a} = 0,
\quad (E.4)
\]
and
\[
(f - g_2 + ig_1) G_{a\beta} - \frac{1}{4} (f + g_2 + ig_1) \epsilon_a^{\gamma \delta} G_{+\gamma \delta} = 0.
\quad (E.5)
\]

The conditions arising from the $D_\alpha$ component of the supercovariant derivative are

\[
\left[ D_\alpha + (w - w^*)^{-1} \partial_\alpha w + \frac{1}{2} \Omega_{\alpha, \beta} \tilde{\rho}^\beta + \frac{1}{2} \Omega_{\alpha, --} + \frac{i}{4} F_{\alpha \beta} \tilde{\rho}^\gamma + \frac{i}{2} F_{\alpha \beta} \right] (f - g_2 + ig_1) = 0,
\quad (E.6)
\]
\[
-(w - w^*)^{-1} \partial_\alpha w (f - g_2 + ig_1) + (f + g_2 - ig_1) \left[ \frac{1}{4} G_{\alpha \beta} \tilde{\rho}^\gamma + \frac{1}{4} G_{-\alpha a} \right] = 0,
\quad (E.7)
\]
\[
(f - g_2 + ig_1) \left[ \Omega_{\alpha, \beta_1 \beta_2} + i F_{\beta_1 \beta_2 \gamma} \tilde{\rho}^\gamma + i F_{-\alpha \beta_1 \beta_2} \right]
\]
\[
- (f + g_2 + ig_1) \left[ \frac{1}{2} \Omega_{\alpha, \gamma_1 \gamma_2} - \frac{i}{2} F_{\alpha \gamma_1 \gamma_2} \tilde{\rho}^\delta + \frac{i}{2} F_{-\alpha \gamma_1 \gamma_2} \right] \epsilon^{\gamma_1 \gamma_2 \beta_1 \beta_2} = 0,
\quad (E.8)
\]
\[ (f + g_2 - i g_1) \left[ \frac{1}{2} G_{\alpha \beta \bar{\gamma}} - \frac{1}{4} g_{a \beta \bar{\gamma}} G_{\alpha \beta \bar{\gamma}} - \frac{1}{4} g_{a \beta \bar{\gamma}} G_{\alpha \beta \bar{\gamma}} \right] \]
\[ - \frac{1}{8} (f - g_2 - i g_1) G_{\alpha \gamma \gamma} \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ \left(D_\alpha + (w - w^*)^{-1} \partial_\alpha w - \frac{1}{2} \Omega_{\alpha \beta} \psi^\beta + \frac{1}{2} \Omega_{\alpha \bar{\gamma}} + \frac{i}{4} F_{\alpha \beta \gamma} \right) (f + g_2 + i g_1) \]
\[ + \frac{i}{12} F_{\alpha \beta \bar{\gamma} \bar{\delta}} (f - g_2 + i g_1) = 0, \]  
\[ (E.9) \]
\[ -(w - w^*)^{-1} \partial_\alpha w (f + g_2 + i g_1) + \left[ -\frac{1}{8} G_{\alpha \gamma \gamma} + \frac{1}{8} G_{\alpha \bar{\gamma} \bar{\gamma}} \right] (f - g_2 - i g_1) \]
\[ - \frac{1}{24} g_{\alpha \beta \bar{\gamma} \bar{\delta}} G_{\alpha \beta \bar{\gamma} \bar{\delta}} (f + g_2 - i g_1) = 0, \]
\[ \left[ \frac{1}{2} \Omega_{\alpha \beta} + \frac{i}{2} F_{\alpha \beta \gamma} \right] (f - g_2 + i g_1) - \frac{i}{6} F_{\alpha \beta \bar{\gamma} \bar{\delta}} G_{\alpha \beta \bar{\gamma} \bar{\delta}} (f + g_2 + i g_1) = 0, \]
\[ \left[ \frac{1}{16} g_{\alpha \beta} G_{\alpha \beta} - \frac{1}{4} G_{\alpha \beta} \right] (f + g_2 - i g_1) = 0, \]
\[ \frac{i}{12} F_{\alpha \beta \bar{\gamma} \bar{\delta}} (f - g_2 + i g_1) + \frac{1}{12} (f + g_2 + i g_1) \left[ \frac{1}{2} \Omega_{\alpha \gamma \gamma} - \frac{i}{2} F_{\alpha \gamma \gamma} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_3 = 0, \]
\[ \frac{1}{32} g_{\alpha \beta \bar{\gamma} \bar{\delta}} G_{\alpha \beta \bar{\gamma} \bar{\delta}} (f + g_2 - i g_1) - \frac{1}{96} (f - g_2 - i g_1) G_{\alpha \gamma \gamma} \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_3 = 0. \]
\[ \text{(E.10)} \]

The conditions arising from the \( D_\alpha \) component of the supercovariant derivative are
\[ \left[D_\alpha + (w - w^*)^{-1} \partial_\alpha w + \frac{1}{2} \Omega_{\alpha \beta} \psi^\beta + \frac{1}{2} \Omega_{\alpha \bar{\gamma}} + \frac{i}{4} F_{\alpha \beta \gamma} \right] (f - g_2 + i g_1) \]
\[ + \frac{i}{12} (f + g_2 + i g_1) F_{\alpha \gamma \gamma \gamma} \epsilon^{\gamma \gamma} = 0, \]
\[ -(w - w^*)^{-1} \partial_\alpha w (f - g_2 + i g_1) + \frac{1}{8} \left[ G_{\alpha \beta \gamma} + G_{\alpha \bar{\gamma}} \right] (f + g_2 - i g_1) \]
\[ - \frac{1}{24} (f - g_2 - i g_1) G_{\alpha \gamma \gamma} \epsilon^{\gamma \gamma} G_{\gamma \gamma \gamma} = 0, \]
\[ \left[ \Omega_{\alpha \beta} \bar{\beta}_2 + i F_{\alpha \beta \bar{\gamma} \bar{\delta}} + i F_{\alpha \gamma \gamma \gamma} \right] (f - g_2 + i g_1) \]
\[ - (f + g_2 + i g_1) \left[ \frac{1}{2} \Omega_{\alpha \gamma \gamma} - \frac{i}{2} F_{\alpha \gamma \gamma} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ \frac{1}{4} G_{\alpha \beta \bar{\gamma} \bar{\delta}} (f + g_2 - i g_1) - (f - g_2 - i g_1) \left[ \frac{1}{8} G_{\alpha \gamma \gamma} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ \left[D_\alpha + (w - w^*)^{-1} \partial_\alpha w - \frac{1}{2} \Omega_{\alpha \gamma} \psi^\gamma + \frac{1}{2} \Omega_{\alpha \bar{\gamma}} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ \text{(E.11)} \]
\[ \left[D_\alpha + (w - w^*)^{-1} \partial_\alpha w - \frac{1}{2} \Omega_{\alpha \gamma} \psi^\gamma + \frac{1}{2} \Omega_{\alpha \bar{\gamma}} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ \left[D_\alpha + (w - w^*)^{-1} \partial_\alpha w - \frac{1}{2} \Omega_{\alpha \gamma} \psi^\gamma + \frac{1}{2} \Omega_{\alpha \bar{\gamma}} \right] \epsilon^{\gamma \gamma} \bar{\beta} \bar{\beta}_2 = 0, \]
\[ -(w - w^*)^{-1} \partial_\alpha w (f + g_2 + i g_1) + \left[ -\frac{1}{4} G_{\alpha \gamma \gamma} + \frac{1}{4} G_{\alpha \bar{\gamma}} \right] (f - g_2 - i g_1) = 0, \]
\[ \text{(E.12)} \]

The conditions arising from the \( D_\alpha \) component of the supercovariant derivative are
\[
\left[ \frac{1}{2} \Omega_{\alpha, \beta} + \frac{i}{2} F_{\alpha + \beta} \gamma \right] (f - g_2 + ig_1) - \frac{i}{6} (f + g_2 + ig_1) F_{\alpha + \gamma, \beta} \gamma = 0, \tag{E.22}
\]
\[
- \frac{1}{8} G_{\alpha, \beta} (f - g_2 - ig_1) - \frac{1}{16} (f - g_2 - ig_1) G_{\gamma, \beta} \gamma = 0, \tag{E.23}
\]
\[
i F_{\alpha + \beta, \gamma, \delta}(f - g_2 + ig_1) + (f + g_2 + ig_1) \left[ \frac{1}{2} \Omega_{\alpha, \gamma} - \frac{i}{2} F_{\alpha + \gamma} \delta \right] \gamma_\beta \delta = 0, \tag{E.24}
\]
\[
(f - g_2 - ig_1) \left[ - \frac{1}{16} g_{\beta \delta} G_{\alpha + \gamma} \gamma + \frac{1}{4} G_{\alpha + \gamma} \gamma \right] \epsilon_\beta_\delta \beta_\beta = 0. \tag{E.25}
\]

The conditions arising from the \( D_+ \) component of the supercovariant derivative are
\[
\left[ D_+ + (w - w^*)^{-1} \partial_- w + \frac{1}{2} \Omega_{-,-} \gamma + \frac{1}{2} \Omega_{-,-} + \frac{i}{4} F_{-,-} \gamma \delta \right] (f - g_2 + ig_1) 
+ \frac{i}{12} (f + g_2 + ig_1) F_{-,-, \gamma, \beta} \gamma \epsilon_{\gamma \beta} = 0, \tag{E.26}
\]
\[
-(w - w^*)^{-1} \partial_- w (f - g_2 + ig_1) + \frac{1}{4} G_{\gamma, \beta} (f + g_2 - ig_1) = 0, \tag{E.27}
\]
\[
\left[ \Omega_{-,-, \beta} + i F_{-,-, \gamma, \delta} \right] (f - g_2 + ig_1) 
- (f + g_2 + ig_1) \left[ \frac{1}{2} \Omega_{-,-, \gamma} - \frac{i}{2} F_{-,-, \gamma} \delta \right] \gamma_\beta \delta = 0, \tag{E.28}
\]
\[
\frac{1}{2} G_{-,-, \beta} (f + g_2 - ig_1) - \frac{1}{4} (f - g_2 - ig_1) G_{-,-, \gamma} \gamma \beta_\delta = 0, \tag{E.29}
\]
\[
\left[ D_+ + (w - w^*)^{-1} \partial_- w - \frac{1}{2} \Omega_{-,-} \gamma - \frac{1}{2} \Omega_{-,-} + \frac{i}{4} F_{-,-} \gamma \delta \right] (f + g_2 + ig_1) 
\+ \frac{i}{12} (f - g_2 + ig_1) F_{-,-, \beta} \beta \delta \gamma \epsilon_{\beta \gamma} = 0, \tag{E.30}
\]
\[
-(w - w^*)^{-1} \partial_- w (f + g_2 + ig_1) - \frac{1}{4} (f - g_2 - ig_1) G_{-,-} \gamma = 0, \tag{E.31}
\]
\[
\left[ \frac{1}{2} \Omega_{-,-} + \frac{i}{2} F_{-,-} \gamma \delta \right] (f - g_2 + ig_1) - \frac{i}{6} (f + g_2 + ig_1) F_{-,-, \gamma, \beta} \gamma = 0, \tag{E.32}
\]
\[
\left[ - \frac{1}{16} G_{\beta, \gamma} \gamma + \frac{3}{16} G_{-,-} \right] (f + g_2 - ig_1) + \frac{1}{48} (f - g_2 - ig_1) G_{\gamma, \beta} \gamma \epsilon_{\gamma \beta} = 0, \tag{E.33}
\]
\[
i F_{-,-, \beta} \beta_\beta (f - g_2 + ig_1) + \left[ \frac{1}{2} \Omega_{-,-, \gamma} - \frac{i}{2} F_{-,-, \gamma} \delta \right] \gamma_\beta \delta (f + g_2 + ig_1) = 0, \tag{E.34}
\]
\[
- \frac{1}{8} G_{-,-, \beta} (f + g_2 - ig_1) + \left[ \frac{1}{16} G_{\beta, \gamma} \delta + \frac{3}{16} G_{-,-, \gamma} \right] (f - g_2 - ig_1) \epsilon_\gamma \beta_\delta = 0. \tag{E.35}
\]

The conditions arising from the \( D_+ \) component of the supercovariant derivative are
\[
\left[ D_+ + (w - w^*)^{-1} \partial_+ w + \frac{1}{2} \Omega_{+,-} \gamma + \frac{1}{2} \Omega_{+,-} + \frac{i}{4} F_{+,-} \gamma \delta \right] (f - g_2 + ig_1) = 0, \tag{E.36}
\]
\[
-(w - w^*)^{-1} \partial_+ w (f - g_2 + ig_1) + \frac{1}{8} G_{+,-} \gamma (f + g_2 - ig_1) = 0. \tag{E.37}
\]
\[
\begin{align*}
\left[ \Omega_{\gamma \delta}^{\alpha \beta} + i F_{\gamma \delta}^{\alpha \beta} \right] (f - g_2 + ig_1) \\
- \left( \frac{1}{2} \Omega_{\gamma \delta}^{\alpha \beta} - \frac{i}{2} F_{\gamma \delta}^{\alpha \beta} \right) (f + g_2 + ig_1) e^{\gamma_5/2} = 0, \\
\end{align*}
\]

(E.38)

\[
\frac{1}{4} (f + g_2 - ig_1) G_{\gamma \delta}^{\alpha \beta} = \frac{1}{8} (f - g_2 - ig_1) G_{\gamma \delta}^{\alpha \beta} e^{\gamma_5/2} = 0, \\
\]

(E.39)

\[
\begin{align*}
D_\alpha + (w - w^*)^{-1} \partial w = \frac{1}{2} \Omega_{\alpha, \gamma}^{\gamma} + \frac{i}{2} \Omega_{\alpha, -\gamma} + \frac{1}{4} F_{\gamma, \gamma}^{\gamma} = 0, \\
\end{align*}
\]

(E.40)

\[
- (w - w^*)^{-1} \partial w (f + g_2 + ig_1) = - \frac{1}{8} G_{\gamma \delta}^{\gamma} (f - g_2 - ig_1) = 0, \\
\]

(E.41)

and

\[
\Omega_{+, \alpha} = \Omega_{+, \alpha^0} = 0. \\
\]

(E.42)

The solution of the above linear system and the conditions on the geometry and the fluxes are summarized in section 6.

References

[1] Dabholkar A, Gibbons G W, Harvey J A and Ruiz Ruiz F 1990 Superstrings and solitons Nucl. Phys. B 340 33
[2] Duff M J and Lu J X 1991 The selfdual type IIB superthreebrane Phys. Lett. B 273 409
[3] Duff M J and Lu J X 1991 Elementary five-brane solutions of $D = 10$ supergravity Nucl. Phys. B 354 141
[4] Horowitz G T and Strominger A 1991 Black strings and P-branes Nucl. Phys. B 360 197
[5] Aharony O, Gubser S S, Maldacena J M, Ooguri H and Oz Y 2000 Large $N$ field theories, string theory and gravity Phys. Rep. 323 183 (Preprint hep-th/9905111)
[6] Schwarz J H 1983 Covariant field equations of chiral $N = 2$ $D = 10$ supergravity Nucl. Phys. B 226 269
[7] Klebanov I R and Strassler M J 2000 Supergravity and a confining gauge theory: duality cascades and chiSB-resolution of naked singularities J. High Energy Phys. JHEP08(2000)052 (Preprint hep-th/0007191)
[8] Maldacena J M and Nunez C 2001 Towards the large $N$ limit of pure $N = 1$ super Yang Mills Phys. Rev. Lett. 86 588 (Preprint hep-th/0008001)
[9] Blau M, Figueroa-O’Farrill J, Hull C and Papadopoulos G 2002 A new maximally supersymmetric background of IIB superstring theory J. High Energy Phys. JHEP01(2002047) (Preprint hep-th/0110242)
[10] Figueroa-O’Farrill J and Papadopoulos G 2003 Maximally supersymmetric solutions of ten-dimensional and eleven-dimensional supergravities J. High Energy Phys. JHEP03(2002)048 (Preprint hep-th/0211089)
[11] Figueroa-O’Farrill J and Papadopoulos G 2004 Pluecker-type relations for orthogonal planes J. Geom. Phys. 49 294–331 (Preprint math-ag/0211170)
[12] Blau M, Figueroa-O’Farrill J, Hull C and Papadopoulos G 2002 Penrose limits and maximal supersymmetry Class. Quantum Grav. 19 L87 (Preprint hep-th/0201081)
[13] Gran U, Gutowski J and Papadopoulos G 2005 The spinorial geometry of supersymmetric IIB backgrounds Class. Quantum Grav. 22 2453 (Preprint hep-th/0501177)
[14] Gran U, Gutowski J and Papadopoulos G 2006 The $G(2)$ spinorial geometry of supersymmetric IIB backgrounds Class. Quantum Grav. 23 143–206 (Preprint hep-th/0505074)
[15] Gran U, Gutowski J and Papadopoulos G 2005 The spinorial geometry of supersymmetric backgrounds Class. Quantum Grav. 22 1033 (Preprint hep-th/0410155)
[16] Gran U, Gutowski J and Papadopoulos G 2005 Systematics of $N$-theory spinorial geometry Class. Quantum Grav. 22 2701 (Preprint hep-th/0503046)
[17] Bellorin J and Ortin T 2005 A note on simple applications of the Killing spinor identities Phys. Lett. B 616 118 (Preprint hep-th/0501246)
[18] Gran U 2001 GAMMA: a mathematica package for performing Gamma-matrix algebra and Fierz transformations in arbitrary dimensions Preprint hep-th/0105086
[19] Schwarz J H and West P C 1983 Symmetries and transformations of chiral $N = 2$ $D = 10$ supergravity Phys. Lett. B 126 301
[20] Howe P S and West P C 1984 The complete \( N = 2, D = 10 \) supergravity \textit{Nucl. Phys.} B \textbf{238} 181

[21] Bergshoeff E A, de Roo M, Kerstan S F and Riccioni F 2005 IIB supergravity revisited \textit{J. High Energy Phys.} JHEP06(2005)088 (Preprint \texttt{hep-th/0506013})

[22] Papadopoulos G and Tsimpis D 2003 The holonomy of IIB supercovariant connection \textit{Class. Quantum Grav.} \textbf{20} L253 (Preprint \texttt{hep-th/0307127})

[23] Gray A and Hervella L M 1980 The sixteen classes of almost Hermitian manifolds and their linear invariants \textit{Ann. Mat. Pura Appl.} \textbf{282} 1

[24] Hull C M 1984 Exact \( pp \) wave solutions of 11-dimensional supergravity \textit{Phys. Lett.} B \textbf{139} 39

[25] Emparan R, Mateos D and Townsend P K 2001 Supergravity supertubes \textit{J. High Energy Phys.} JHEP07(2001)011 (Preprint \texttt{hep-th/0106012})

[26] Cvetic M, Gibbons G W, Lu H and Pope C N 2003 Ricci-flat metrics, harmonic forms and brane resolutions \textit{Commun. Math. Phys.} \textbf{232} 457 (Preprint \texttt{hep-th/0012011})

[27] Chen C M and Vazquez-Poritz J F 2005 Resolving the M2-brane \textit{Class. Quantum Grav.} \textbf{22} 4231–46 (Preprint \texttt{hep-th/0403109})

[28] Wang M Y 1989 Parallel spinors and parallel forms \textit{Ann. Global Anal. Geom.} \textbf{7} 59

[29] Lawson H B and Michelsohn M-L 1989 \textit{Spin Geometry} (Princeton, NJ: Princeton University Press)

[30] Harvey F R 1990 \textit{Spinors and Calibrations} (London: Academic)