The CPT-violating effects on neutron’s gravitational bound state

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In this paper, the CPT-violating (CPTV) spin interactions on neutron’s gravitational bound state are studied. The helicity-dependent phase evolution due to $\vec{\sigma} \cdot \Delta$ and $\vec{\sigma} \cdot \vec{p}$ couplings is transparently demonstrated with analytical solutions. The consequent phenomena include not only spin precession and the azimuthal angle $\Theta_\pi$ and $\phi$-dependent probability variation, but also transition-frequency shifts between different gravitational bound states. The $\Theta$ and $\phi$ time-dependences due to the Earth motion may lead to sidereal variation of the probability profile, a clear signal of Lorentz violation. We also utilize the gravitational transition frequencies measured precisely in the qBounce experiment [26] to obtain the rough bound $|b| < 6.9 \times 10^{-21}$GeV. Incorporating known systematic errors or using polarized neutron beams may lead to more robust and tighter constraints.

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I. INTRODUCTION

Symmetry and its breaking pattern are the main theme of physics in the last century. SU(3)$_c$ $\otimes$ SU(2)$_L$ $\otimes$ U(1)$_Y$ symmetry is responsible for the observed electroweak and strong forces dominant in microscopic world, and Local Lorentz and diffeomorphism invariance are responsible for the gravity dominant in macroscopic world. All the fundamental forces are closely embedded into the gauge structure. On the other hand, scalar bosons arising from spontaneously breaking of certain (approximate) global symmetries can also account for forces we seen in nature, such as the nuclear force due to exchange of $\pi$-meson. However, dark matter and dark energy still remain challenge to our current paradigm of understanding, not to mention the struggle in marrying quantum mechanics with gravity. An very interesting scenario proposed in [1] is that photon and graviton may be collective modes originating in other exotic scenarios, such as the˜$\cdot$-meson. Moreover, viewed in a broader perspective, SME also contains operators with essentially the same forms as those originating in other exotic scenarios, such as the˜$\cdot$-type coefficients, and help to fill this gap. As is well-known, spin-gravity effects for subatomic particles are very tiny even in the Lorentz invariant (LI) context [16][30]. Experimental searches will be quite challenging. On the other hand, first of all, the amazing development in precision experiments is quite impressing [17][18]; secondly, the exotic scenarios such as LV and spin-dependent new forces may provide more promising signals testable in the near future; at last, besides energy-momentum, another matter source gravity may couple, and may couple quite differently from the prediction of GR is spin [19], which makes the probe of spin-gravity effect [20] indispensable.

Moreover, viewed in a broader perspective, SME also contains operators with essentially the same forms as those originating in other exotic scenarios, such as torsion and nonmetricity [19][21][22]. As an example, spin-dependent $b^\mu$ term can also entail the structure of minimal torsion $T^\lambda_{\mu
u}$ coupling if we identify $b^\mu_{\text{eff}} = b^\mu - \frac{1}{6} s^\alpha\beta\gamma T_{\alpha\beta\gamma}$ [10]. Effective LV backgrounds also emerge...
from certain slow varying dynamical fields, such that the test of tiny LV effects may partially overlap with the search of fifth force. A subset of operators induced by new spin-dependent forces mediated by axion or axionlike particles (ASP) [18][23] can also appear in the study of spin-dependent LV couplings, such as $\vec{\sigma} \cdot \vec{r}$, $\vec{\sigma} \cdot \vec{p}$ and $\vec{\sigma} \cdot (\vec{r} \times \vec{p})$. Relaxing Lorentz invariance admits more interesting possibilities, such as $\vec{\partial} \cdot \vec{b}$ and $\vec{\partial} \cdot \vec{\sigma}$ ($\vec{v} \equiv \frac{\vec{v} \times \vec{p}}{2m}$ is the mean velocity), which violates boost invariance apparently [24]. As far as we known, these apparently LV scalar mediated potentials have never been completely analyzed experimentally. In a word, test of LV and searching for fifth force can be cross fertilized and be beneficial from each other’s progress. In this paper, we mainly focus on $\vec{\sigma} \cdot \vec{b}$ type operators in slow neutron experiment, with the motivating background in [13][25]. Due to the neutrality and very tiny polarizability under external electromagnetic fields [27], neutron has long been an excellent candidate to the test of interplay between gravity and quantum mechanics [28], we hope it may also shed light on the test of LV spin-gravity couplings.

The organization of this work is the following. By focusing on the spin-dependent terms in [11], we obtain an effective vertical Hamiltonian in Sec. II using the reduction ansatz from [12]. As a slight glimpse on the LV spin-gravity effects, tiny LV correction to the Larmor frequency from the spin evolution equation is obtained. In Sec. III, we review the eigen-solution in the linear gravitational potential, which is the basis for our proceeding analysis. In Sec. IV, with analytical solutions of gravitational bound states, we discuss the spin precession and polarization probability variation due to the $\vec{\sigma} \cdot \vec{b}$ coupling. Analysis on other CPTV operators and the frequency shifts due to the CPTV spin-gravity couplings utilizing perturbation theory are also given. Interestingly, the energy shifts coincide with the approximation of the eigen-energy obtained in an analytical form.

In Sec. V, we summarize our results and speculate on possible experiments which may be potentially viable to the test of the CPTV spin-gravity couplings, such as the weak equivalence principle test with polarized matter.

II. THE HAMILTONIAN WITH LV FERMION-GRAVITY COUPLINGS

A. Hamiltonian in uniform gravitational field

As a first step, we briefly review the procedure of getting the effective Hamiltonian to describe the motion of neutron near the Earth surface. The general action [3][4] describing the LV neutron-gravity coupling is

$$S_{\psi} = \int d^4x e^{\frac{i}{2} e_{\mu} \omega_{\mu} \Gamma^a D_{\mu} \phi_{a} - \bar{\psi} M \psi},$$

(1)

where $\Gamma^a D_{\mu} \phi_{a} = \chi D_{\mu} \phi_{a} - (\chi D_{\mu}) \Gamma_{\mu} \psi$, $D_{\mu} \psi = \partial_{\mu} \psi + \frac{1}{2} \epsilon_{\mu\nu\lambda} \phi_{\nu} \Gamma_{\lambda} \psi$, and $M = m + 2a_{\mu} e_{\mu} \gamma_{5} + b_{\mu} \epsilon_{\mu\nu\lambda} (\chi \delta_{\nu\lambda} - \frac{1}{2} \chi^{\nu\lambda}) \psi / 2H_{\mu\nu} e_{\nu\lambda} \sigma_{\mu\lambda}$ and $\Gamma_{\mu} = \gamma_{\mu} [\epsilon_{\mu\nu\lambda} + \frac{1}{2} \frac{1}{2} \epsilon_{\mu\nu\lambda} \epsilon_{\sigma} - \epsilon_{\sigma\nu\lambda}] e_{\sigma\nu\lambda} - \frac{1}{2} g_{\mu\nu\lambda} e_{\lambda\sigma\nu\lambda} \epsilon_{\sigma\nu\lambda}$ for minimal SME. Assuming torsion free $T_{\mu\nu} = 0$ and choosing the Schwinger gauge $e_{\rho}^0 = V^0\theta^0$ and $e_{\rho} = W\theta^0$, with the test particle assumption one can get the Hamiltonian (13) in [11] for the uniform gravitational field. Though the metric $ds^2 = V^2dt^2 - W^2dx^2$ with $V = 1 + \Phi = 1 + \vec{g} \cdot \vec{x}$, $W = 1$ is essentially flat, it is still a good approximation to describe the gravitational field near the Earth surface. From the relativistic Hamiltonian, the non-relativistic spin-dependent Hamiltonian, Eqn. (38) in [11] can be obtained by Foldy-Wouthuysen transformation [29][10][11][33] order by order. As the spin-independent LV operators have already been extensively studied [10][12], we only list the spin-dependent counterparts in this work.

Following the spirit of the definition (27) in [42] and also for notational simplicity, we redefine

$$\hat{B}_k \equiv b_k + m \epsilon_0 \epsilon^0_{lm} \hat{g}_{lm} \hat{g}^0 \hat{g}_{lm} \hat{g}^0 + m \epsilon^0_{lm} \hat{g}_{lm} \hat{g}^0 \hat{g}_{lm} \hat{g}^0 \equiv - m \left[ \frac{1}{3} \hat{g}^0_{lm} \hat{g}^0_{lm} + \frac{1}{3} \hat{g}^0_{lm} \hat{g}^0_{lm} \right],$$

(2)

where the hat on LV coefficients with $n$ 0-index means multiplication by $1 - n\Phi$ [11], so $\hat{g}^0_{\mu\nu}$ is just the hat-corrected effective LV coefficients $\hat{g}^0_{\mu\nu}$. The $\hat{g}$ means equal up to a normalization constant, which is not relevant here. In the same spirit, we can also define $\hat{d}^\mu_\nu \equiv \hat{d}^\mu_\nu - \frac{1}{2m} \epsilon^\mu_{\nu\rho} \hat{H}^\rho_\sigma$ such that

$$\hat{\chi}^0 \equiv \hat{\chi}^0 + \epsilon_0 \epsilon^0_{lm} \hat{H}_{lm}, \quad \hat{\chi}^i = \hat{\chi}^i + \epsilon^0_{lm} \hat{H}_{0lm}, \quad \hat{\sigma}^0 = \hat{\sigma}^0.$$
With these redefined coefficients, we rewrite the LI and the spin-dependent LV Hamiltonian as

\[
\hat{H}_{\text{LI}} = m(1 + \Phi) + \frac{(1 + \Phi)\vec{p}^2}{2m} - i\frac{\hbar}{2m}\vec{\nabla}\Phi \cdot \vec{p} + \frac{(1 + \Phi)}{2m}\vec{\sigma} \cdot (\vec{\nabla} \times \vec{p})
\]

\[
\hat{H}_{\text{LV}} = -[\vec{B}_k + m\hat{d}^{00}\sigma^k(1 + \Phi)] + \left(\delta^i_k\hat{d}^{00} + \hat{d}^i_k\right)\frac{-\vec{B}_0^i}{m}(1 + \Phi)\vec{p}_i - \frac{i\hbar}{2}\nabla_i \Phi - \frac{1 + 3\Phi}{2m^2}\vec{B}_j\vec{p}_j(\vec{\sigma} \cdot \vec{p})
\]

Note for clarity, we have intentionally separated \([\vec{B}_k + m\hat{d}^{00}\sigma^k(1 + \Phi)]\) in (5) into the traceless part \([\vec{B}_0^i = \frac{\hat{d}^i_k}{m}(1 + \Phi)\vec{p}_i - \frac{1}{2}\nabla^k \Phi]\) and the trace part \([\hat{d}^{ik}]\). To be consistent with the standard notation in the literature [36][43], we denote \(\hat{d}(\hat{\Phi})\) instead of \(\hat{d}[\hat{\Phi}]\) of \(\hat{\Phi}\). We have to stress that \(\hat{d}(\hat{\Phi})\) is the intrinsic curvature effects [31], so interesting phenomena induced by LV spin-orbit couplings are expected. Our starting metric is intrinsically flat. Further investigation with curved metric is needed, however, this is beyond of our simple approximation, and we will focus on the the first two terms in (5) in the following.

B. Reduced Hamiltonian for vertical Motion

Keep only the first two terms in (5) leaves us the Hamiltonian

\[
(\hat{H}_{\text{LV}})_{\parallel} = -\vec{\sigma} \cdot \vec{b}(1 + \Phi) - \hat{b}_{\text{eff}}(1 + \Phi)(\vec{\sigma} \cdot \vec{p})
\]

\[
-\frac{i}{2}(\vec{\sigma} \cdot \vec{\nabla}\Phi) - \frac{\hat{b}_{\text{eff}}}{m}(1 + \Phi)\vec{p} - \frac{i}{2}\nabla_i \Phi.
\]

Note for clarity, we have intentionally separated \([\vec{B}_k + m\hat{d}^{00}\sigma^k(1 + \Phi)]\) in (5) into the traceless part \([\vec{B}_0^i = \frac{\hat{d}^i_k}{m}(1 + \Phi)\vec{p}_i - \frac{1}{2}\nabla^k \Phi]\). To be consistent with the standard notation in the literature [36][43], we denote \(\hat{d}(\hat{\Phi})\) instead of \(\hat{d}[\hat{\Phi}]\) of \(\hat{\Phi}\). We have to stress that \(\hat{d}(\hat{\Phi})\) is the intrinsic curvature effects [31], so indeed denotes LV spin-gravity couplings (even just the approximation of linear potential). The signature also differs in the Minkowski limit by \(\eta_{\mu\nu} = \text{diag}(1, 1, 1, 1)\). For generality, the physical meanings of the above operators may not necessarily be confined to LV, despite the search for LV is definitely our starting motivation. While the last term in (6) necessarily violates LI, the first two terms may be not, and can be regarded as arising from broader scenarios, such as axion or axionlike particles. For example, if we identify \(\hat{b}_{\text{eff}} \Phi \rightarrow -\frac{\hat{d}(\hat{\Phi})}{m}\) and \(\lambda \rightarrow +\infty\), the \(\Phi\)-dependent part of \(\hat{b}_{\text{eff}}\) operator can be identified as the

12, 13 operators in [18][23], i.e.,

\[
f_{12+13} = \frac{1}{8\pi m} \frac{e^{-r/\lambda}}{r} = \frac{1}{8\pi m} \frac{e^{-r/\lambda}}{r} (\vec{\sigma} \cdot \vec{p})
\]

\[
+ \frac{i}{8\pi m} \frac{e^{-r/\lambda}}{r^2} (1 + r/\lambda)(\vec{\sigma} \cdot \vec{r}.
\]

Before averaging over the horizontal degree of freedom (d.o.f.) to get the vertical Hamiltonian, we can use (6) to calculate the LV corrections to neutron's spin precession in a weak magnetic field \(\vec{B}\). From the Heisenberg equation, we get

\[
\frac{d\vec{S}}{dt} = i\left[-\mu \cdot \vec{B} + (\hat{H}_{\text{LV}})_{\parallel}, \vec{S}\right]
\]

\[
= \vec{S} \times \left[\gamma \vec{B} + 2(1 + \Phi) \left(\vec{b} + \frac{\hat{b}_{\text{eff}}}{m} - \frac{\hat{b}_T}{m}\right) \vec{p}\right].
\]
(\hat{H}_{L1})_z = m \left[ \Phi_0 + gz - \frac{g z^2}{R^2} \right] + \frac{(1 + \Phi_0 + gz) \hat{p}_z^2}{2m} + 1 \frac{1}{2m} \left[ \sum_{a=1}^{2} p_a^2 + \frac{1}{\sigma^2} \right] + \frac{1}{2m} [\Phi_0 + gz] \left( \sum_{a=1}^{2} p_a^2 \right) \\
+ \frac{g}{2m} \left\{ \left[ \frac{1}{R} - i \hat{p}_z + 2i \frac{\hat{z}}{\hat{p}_z} - (\hat{\sigma} \times \hat{p})_z \right] - (\hat{\sigma} \times \hat{p})_z \left[ (1 - \frac{2z}{R}) + \frac{1}{R} \right] \right\} ,
(9)

(\hat{H}_{LV})_z = -\hat{\sigma} \cdot \hat{b} (1 + \Phi_0 + gz) - \frac{\bar{b}_{\text{eff}}}{m} (1 + \Phi_0 + gz) \left[ \sigma_z \hat{p}_z + \sum_{a=1}^{2} \sigma_a p_a \right] + \frac{i \bar{b}_{\text{eff}}}{2m} \sigma_z g (1 - \frac{2z}{R}) \\
- \frac{2}{m} \left[ \frac{|b_T|^a}{m} \sigma^k (1 + \Phi_0 + gz) p_a - \frac{|b_T|^k}{m} \sigma^k \left( (1 + \Phi_0 + gz) \hat{p}_z - \frac{i}{2} \nabla \Phi \right) \right] ,
(10)

and R is the Earth mean radius. For technical details, see appendix A. Since we want our discussions to be helpful also for the search of fifth force, we disregarded the off-diagonal terms proportional to $|b_T|^k$ in (10) in the following.

III. REVIEW FOR A QUANTUM PARTICLE IN LINEAR POTENTIAL

For completeness, we first review the solution to the Schrödinger equation [32]

$$i\hbar \frac{\partial}{\partial t} \Phi(z,t) = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + mGz \right] \Phi(z,t) ,
(11)$$

where $\Phi(z,t) = \int dE e^{-iEt/\hbar} \rho(E) \phi_E(z)$, with $\phi_E(z)$ the solution to $\phi''_E(z) + \frac{2mE}{\hbar^2} (1 - \frac{m}{\hbar^2} g z) \phi_E(z) = 0$ and $\rho(E)$ the weighting factor for the Fourier expansion of the wave-packet $\Phi(z,t)$. Defining $\tilde{z} \equiv z/z_c$, $\tilde{k} \equiv \frac{2m}{\hbar} k z_c$, where $z_c \equiv \frac{E}{mG}$ is the classical turning height in z-axis, the stationary equation can be casted into the dimensionless form

$$\phi'' + \tilde{k}^2 (1 - \tilde{z}) \phi(\tilde{z}) = 0 ,
(12)$$

where for simplicity, the subscript in $\phi_E(z)$ is suppressed. The solution to (12) is the famous Airy function

$$\phi(\tilde{z}) = \sqrt{\frac{3\pi}{k^3}} c_3 \text{Ai}[^{2}\tilde{z} - 1]$$

since $\phi(\tilde{z})$ must be finite at $\tilde{z} \to +\infty$. For details, see appendix B or the textbooks on quantum mechanics [32]. If no requirement is imposed on the $\tilde{z} < 0$ region, (13) can be a wave-function describing either scattering or bouncing particle. As a trampoline or a mirror can be idealized as an infinitely high barrier, which prohibits $\phi(\tilde{z})$ from penetrating into the $\tilde{z} < 0$ zone, we can further impose $\phi(\tilde{z})|_{\tilde{z}=0} = 0$ to get a bouncer solution.

If we denote the zeros of $\text{Ai}[-x] = 0$ as $x_{n+1}$, when $n=0,1,2,\ldots$, then $\phi(\tilde{z})|_{\tilde{z}=0} = 0$ gives us the eigen-energies

$$E_n = x_{n+1} \frac{(m_G g \hbar)^{2/3}}{(2m)^{1/3}} , \quad \{x_1, x_2, x_3, \ldots\} = \{2.33811, 4.08795, 5.52056, \ldots\} ,
(14)$$

where we intentionally offset n by 1 to make $E_0$ the ground state eigen-energy. The eigen-function is given by

$$\phi_n(z) = \frac{1}{L_c^{\frac{1}{2}} |\text{Ai}[-x_{n+1}]|} \text{Ai}[^{2}\frac{z}{L_c} - x_{n+1}] ,
(15)$$

where $L_c \equiv (2m \bar{m}_G g/\hbar^2)^{-1/3}$ ($L_c = 5.871 \mu m$ for neutron) is the characteristic length scale for the neutron bouncing problem. We keep distinctive notations between inertial mass $m_I$ and gravitational mass $m_G$ to emphasis that the quantum test of equivalence principle utilizing neutron beams may be quite promising [33]. Note we have restored $\hbar$ in (14) temporarily, and $x_{n+1} = x_{n+1} L_c$ corresponds to the classical turning height for $\phi_n(z)$. In contrast to the classical case, the turning height $x_{n+1}$ is also quantized [13], a natural consequence of the energy quantization condition. However, to detect this quantization is extremely difficult. Largely due to the extreme weakness of gravity, $g = 9.818 m/s^2 = 2.154 \times 10^{-32} \text{GeV}$ in HEP units, the eigen-energies of neutron's gravitational bound states are only at the peV level, for example, $E_0 = 1.408, E_1 = 2.462, E_2 = 3.324, E_3 = 4.087 \text{ peV}, \ldots$, and the corresponding turning heights are $z_0 = 13.72, z_1 = 23.98, z_2 = 32.39, z_3 = 39.81 \mu m$. The effective thermal temperature is around $\sim 10$ nK, and the neutron beam must be ultracold [27], which is technically quite challenging. However, with sophisticated experimental designs, the electric neutrality and relatively long lifetime of the ultracold neutron (UCN) still enable the observation of the extremely delicate quantum nature of the weakly bounded states by the Earth gravitational field [13]. Next, we consider the corrections to the bound state due to the spin-dependent LV operators in (10).

IV. SPIN-DEPENDENT LV CORRECTIONS TO GRAVITATIONAL BOUND STATE

For UCN experiment, the energy scales for various operators in (9) and (10) are separated far away from each other. To give a glance of these diverse energy scales, we...
FIG. 1: Diverse Energy Scales for operators in UCN Experiments. The relevant parameters are choosen from the Neutron Experiment in ILL[26], and the unit is in peV $(10^{-12} eV)$.

utilize the parameters in [26] to give a naive estimate. The energy scales are shown in Fig.1. Clearly, the energy budget is horizontal motion dominate. However, these horizontal d.o.f. have been integrated out and may only contribute to an irrelevant phase factor, which will be ignored in the following discussions.

The leading order LI contribution comes from the first two terms in (9), and has been mostly incorporated in (11). We may suppose the other LI contributions are subdominant, as indicated in Fig.1. However, neither of those terms are CPT odd, nor do we have a clear experimental verification on them, so we simply disregard them in the study of exotic spin-gravity couplings, and phenomenologically regard the first two terms in (10) as the main perturbations.

The leading order perturbation in (10) is $-\vec{\sigma} \cdot \vec{b}$, where $\vec{b}$ behaves like an effective magnetic field and can mimic either neutron magnetic interaction $-\vec{\mu} \cdot \vec{B}$, or spin-rotation coupling $\vec{\sigma} \cdot \Omega$. However, assuming ideal magnetic screen or alternating the direction of external magnetic field intentionally, we can study the effective LV perturbations or certain hypothetical spin-dependent interactions described by $\vec{\sigma} \cdot \vec{b}$. In general, we can even absorb those classical factors into $\vec{b}$, i.e., $(1 + \Phi_0) \left[ (\vec{b})^i + \frac{2m}{\hbar} \delta_{ai} p^a \right] \rightarrow (\vec{b})^i$. In principle, $(\vec{b})^i$ can be time-dependent (LV sidereal effect) and may even be horizontal momentum-dependent. However, we note the sidereal period is much greater than the characteristic time scale for UCN experiment, i.e., $T = 23.56h \gg \tau \simeq 23ms$ [26]. To make life simpler, we can regard $\vec{b}$ as a constant vector, and take the time dependence into account only at the final stage.

A. The $-\vec{\sigma} \cdot \vec{b}$ correction

Here we have already made the identification of $(1 + \Phi_0) \left[ (\vec{b})^i + \frac{2m}{\hbar} \delta_{ai} p^a \right]$ as $(\vec{b})^i$. First we consider the time independent case of $\vec{b} \equiv B_0 \vec{n}$, with $\vec{n} \equiv (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ a constant unit vector, then we take the horizontal momentum $p_n$-dependence into account, and consider the simple time dependent case with $\vec{b} \equiv B_0 (\sin \theta \cos[\omega t + \phi], \sin \theta \sin[\omega t + \phi], \cos \theta)$.

1. The case of time independent $\vec{b}$

Considering the case of $\vec{b} = B_0 \vec{n}$ by ignoring the sidereal time dependence temporarily, we diagonalize the full Hamiltonian $\hat{H}_1 \equiv \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + mgz - B_0 \vec{\sigma} \cdot \vec{n} \right]$ with

$$U = \left( \begin{array}{cc} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\ \sin(\theta/2)e^{i\phi} & -\cos(\theta/2) \end{array} \right).$$

Performing a unitary transformation $\Psi = U \tilde{\Psi}$, the Schrödinger equation $i\frac{\partial}{\partial t} \tilde{\Psi} = \hat{H}_1 \tilde{\Psi}$ can be cast into the form

$$i\frac{\partial}{\partial t} \left( \begin{array}{c} \tilde{\xi} \\ \tilde{\eta} \end{array} \right) = \left( \begin{array}{cc} \hat{H}_0 - B_0 & 0 \\ 0 & \hat{H}_0 + B_0 \end{array} \right) \left( \begin{array}{c} \tilde{\xi} \\ \tilde{\eta} \end{array} \right),$$

where $\tilde{\psi} \equiv \left( \begin{array}{c} \tilde{\xi} \\ \tilde{\eta} \end{array} \right)$ and $\hat{H}_0 \equiv -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + mgz$. This can be viewed as two decoupled equations of the same form of Eqn.(11) with oppositely shifted eigen-energies, so we get

$$\tilde{\xi} = c_1 \psi_n(z)e^{-i(E_n - B_0)t}, \quad \tilde{\eta} = c_2 \psi_n(z)e^{-i(E_n + B_0)t},$$

where $\psi_n(z) = A_1 \left[ x_n - x_{n+1} \right]$ and $E_n$ is given by (14) with $n = 0, 1, 2, 3, \ldots$. Substituting (17) back into $\Psi$, we get the initial wave function

$$\Psi(0, z) = \left( \begin{array}{c} c_1 \cos(\theta/2) \psi_n(z) + c_2 \sin(\theta/2)e^{-i\phi} \psi_n(z) \\ c_1 \sin(\theta/2)e^{i\phi} \psi_n(z) - c_2 \cos(\theta/2) \psi_n(z) \end{array} \right).$$

(18)
Assuming the initial neutron state is spin-up or spin-down leads to

\[
c_1/c_2 = \cot(\theta/2)e^{-i\phi} \frac{\psi_m[z]}{\psi_n[z]}, \quad \text{spin } \uparrow ;
\]
\[
c_1/c_2 = -\tan(\theta/2)e^{-i\phi} \frac{\psi_m[z]}{\psi_n[z]}, \quad \text{spin } \downarrow . \tag{19}
\]

Clearly the constant ratio \(c_1/c_2\) requires \(m = n\), otherwise this ratio would be position-dependent, and cannot

we find the eigen-solutions corresponding to the initial spin-up \(\uparrow\) and spin-down \(\downarrow\) states are

\[
\Psi_\uparrow(t, z) = \begin{pmatrix} \cos(B_0 t) + i \sin(B_0 t) \cos \theta \ e^{-i \phi} \\ i \sin \theta \sin(B_0 t) \end{pmatrix} \phi_n(z) e^{-i E_n t}, \tag{21}
\]
\[
\Psi_\downarrow(t, z) = \begin{pmatrix} i \sin \theta \sin(B_0 t) e^{-i \phi} \\ \cos(B_0 t) - i \sin(B_0 t) \cos \theta \end{pmatrix} \phi_n(z) e^{-i E_n t}, \tag{22}
\]

where \(\phi_n(z) = \frac{\Lambda_1[z - x_{n+1}]}{L_\pi \Lambda'[-x_{n+1}]}\) is given by (15). If the initial state is horizontally polarized, say, \(\Psi(0, z) \propto \sqrt{2} (1, 1)^T\), the eigen-solution is given by

\[
\Psi_\perp(t, z) = \frac{e^{i\pi}}{\sqrt{2}} \begin{pmatrix} \cos(B_0 t) + i \sin(B_0 t) \cos \theta \ + \sin \theta \ e^{-i \phi} \\ [\cos(B_0 t) - i \sin(B_0 t) \cos \theta \ - \sin \theta \ e^{-i \phi}] \end{pmatrix} \phi_n(z) e^{-i E_n t}. \tag{23}
\]

From (23), we can immediately obtain the probability difference in finding the particle in spin-up and -down states as

\[
\frac{P_\uparrow - P_\downarrow}{P_\uparrow + P_\downarrow} = \int_0^{+\infty} dz \left| \langle \uparrow | \Psi_\perp(t, z) \rangle \right|^2 - \left| \langle \downarrow | \Psi_\perp(t, z) \rangle \right|^2 = \sin(2\theta) \cos \phi \sin^2(B_0 t) + \sin \theta \sin \phi \sin(2B_0 t). \tag{24}
\]

Similarly, the horizontal spin preserving probability is given by

\[
P_{\sigma_z} = \int_0^{+\infty} dz \left| \frac{1}{\sqrt{2}} (1, 1)^T \cdot \Psi_\perp(t, z) \right|^2 = \cos^2(B_0 t) + \sin^2(B_0 t) \sin^2 \theta \cos^2 \phi. \tag{25}
\]

2. The simple case with time dependent \(\vec{b}\)

Taking horizontal \(\vec{p}_n\)-dependence into account means the horizontal part of \(\vec{b}\) could be time dependent.

For simplicity, we assume \(\vec{b} = B_0 (\sin \theta \cos \omega t + \phi), \sin \theta \sin(\omega t + \phi), \cos \theta)\). The Hamiltonian is in the form

\[
\hat{H}_2 = \begin{pmatrix} \hat{H}_0 - B_0 \cos \theta & -B_0 \sin \theta e^{-i(\omega t + \phi)} \\ -B_0 \sin \theta e^{i(\omega t + \phi)} & \hat{H}_0 + B_0 \cos \theta \end{pmatrix}, \tag{26}
\]

and the state vector can take the form

\[
\Psi = \begin{pmatrix} c_1(t) \psi_n e^{-i E_n t} + B_0 \cos \theta \ \\
0 \psi_m e^{-i E_m t} + B_0 \cos \theta \end{pmatrix}, \tag{27}
\]

according to the discussion of the last section. Since \(\xi_{n\pm}(t, z) = \psi_n(z) e^{-i E_n t} + B_0 \cos \theta \) satisfies the equation \(i \partial_t \xi_{n\pm}(t, z) = \left[ \hat{H}_0 \mp B_0 \cos \theta \right] \xi_{n\pm}(t, z)\), substituting (27) into the Schrödinger equation \(i \partial_t \Psi = \hat{H}_2 \Psi\), we satisfy the Schrödinger equation. Imposing the normalization condition

\[
i \left[ c_1 e^{i E_0 t} - B_0 \sin \theta \left( c_2 e^{i (B_0 \cos \theta t + \omega t + \phi)} \right) \right] = -B_0 \sin \theta \left( c_1 e^{i (B_0 \cos \theta t + \omega t + \phi)} \right), \tag{28}
\]

where we have used the position independence of \(c_i(t) (i = 1, 2)\) to impose \(m = n\). The solutions to \(c_i(t)\) (i = 1, 2) to impose \(m = n\). The solutions to \(c_i(t)\) are

\[
c_1(t) = \left[ d_1 e^{-i 2 \Omega t} + d_2 e^{i 2 \Omega t} \right] e^{-(B_0 \cos \theta t + \Omega)} \tag{29}
\]
\[
c_2(t) = \frac{-e^{i \phi + i (B_0 \cos \theta) t}}{2B_0 \sin \theta} \left[ d_1 (\omega + 2B_0 \cos \theta + \Omega) e^{i \phi \Omega t} + d_2 (\omega + 2B_0 \cos \theta - \Omega) e^{-i \phi \Omega t} \right], \tag{30}
\]

where \(\Omega = \sqrt{\omega^2 + 4B_0 \cos \theta} + 4B_0^2\), and \(d_1, d_2\) are two constants to be determined. Substituting (29), (30) back into the ansatz (27) and imposing the normalization condition
\[
\int_0^{+\infty} dz \Psi(t,z)^2 = \left( \left| d_1 \right|^2 + \left| d_2 \right|^2 \right) \Omega^2 + \left( \left| d_1 \right|^2 - \left| d_2 \right|^2 \right) \Omega (\omega + 2B_0 \cos \theta) \int_0^{+\infty} dz |\psi_n(z)|^2 = 1.
\] (31)

we find that for the state initially in the spin-up and spin-down states respectively, the time evolved states are

\[
\Psi_\uparrow(t,z) = \left( \cos \left( \frac{\Omega t}{2} \right) + i \frac{\omega + 2B_0 \cos \theta}{\Omega} \sin \left( \frac{\Omega t}{2} \right) \right) e^{-i t} \phi_n(z) e^{-iE_n t},
\]

\[
\Psi_\downarrow(t,z) = \left( \cos \left( \frac{\Omega t}{2} \right) - i \frac{\omega + 2B_0 \cos \theta}{\Omega} \sin \left( \frac{\Omega t}{2} \right) \right) e^{i t} \phi_n(z) e^{-iE_n t}.
\]

(32)

(33)

Inspection of (32), (33) shows that, by setting \( \omega = 0 \) they can be reduced to (21), (22) respectively. For an initially spin-up state, the probability of conserving its helicity is

\[
\text{Prob}_{\uparrow}(t) = \int_0^{+\infty} dz |(1,0) \cdot \Psi_\uparrow(t,z)|^2 = 1 - 4 \left( \frac{B_0 \sin \theta}{\Omega} \right)^2 \sin^2 \left( \frac{\Omega}{2} t \right).
\]

(34)

If instead, the initial state is in a superposition of the spin-up and -down states, say, the eigenstate of \( \sigma_z \), then the time evolved state is

\[
\Psi_{\sigma}(t,z) = \frac{1}{\sqrt{2}} \left( \cos \left( \frac{\Omega t}{2} \right) + i \frac{\omega + 2B_0 \cos \theta}{\Omega} \sin \left( \frac{\Omega t}{2} \right) \right) e^{-i t} \phi_n(z) e^{-iE_n t}.
\]

(35)

Similar to the time-independent case shown before, we can also obtain

\[
P_{\sigma} - \sigma = \cos^2 \left( \frac{\Omega t}{2} \right) \cos \left( \frac{\Omega t}{2} \right) + \left[ \cos^2 \left( \frac{\Omega t}{2} \right) + \frac{\omega + 2B_0 \cos \theta}{\Omega} \sin^2 \left( \frac{\Omega t}{2} \right) \right] \sin^2 \left( \frac{\Omega t}{2} \right)
\]

\[
\sin^2 \left( \frac{\Omega t}{2} \right) = 1 - 4 \left( \frac{B_0 \sin \theta}{\Omega} \right)^2 \sin^2 \left( \frac{\Omega}{2} t \right)
\]

(36)

(37)

The equations (36), (37) can also be reduced to (24), (25) respectively by setting \( \omega = 0 \), proving the consistency of our calculations. A more realistic consideration of the \( p_n \)-dependence in \( \bar{b} \) requires more detailed information about the horizontal momentum distribution, which is beyond the scope of our current work.

3. Translating to Sun-centered Frames

Now it is time to consider the effect caused by the Earth’s rotation (the time scale of orbital motion is much longer). The \( \bar{b} \) with indices of lower case letters (the Lab frame) in the above calculations has to be replaced by

\[
\begin{pmatrix}
\bar{b}_x \\
\bar{b}_y \\
\bar{b}_z
\end{pmatrix} = \begin{pmatrix}
\cos(\chi) \left( \bar{b}_x \cos(\phi_\odot) + \bar{b}_y \sin(\phi_\odot) \right) - \bar{b}_z \sin(\chi) \\
\bar{b}_y \cos(\phi_\odot) - \bar{b}_x \sin(\phi_\odot) \\
\sin(\chi) \left( \bar{b}_x \cos(\phi_\odot) + \bar{b}_y \sin(\phi_\odot) \right) + \bar{b}_z \cos(\chi)
\end{pmatrix}
\]

where \( \bar{b} \) with indices of capital letters represents constant LV coefficients in the Sun-centered frame [35]. We expect the \( \theta, \phi \) in the Lab frame also have to be replaced by

\[
\phi_{\text{Lab}} \equiv \tan^{-1} \left[ \frac{\bar{b}_y}{\bar{b}_x} \right] = \cot^{-1} \left( \cos \chi \cot(\phi_\odot - \omega_\odot T) \right)
\]

\[
- \sin \chi \cot(\phi_\odot - \omega_\odot T)
\]

(38)

\[
\theta_{\text{Lab}} \equiv \sin^{-1} \left[ \sqrt{\bar{b}_x^2 + \bar{b}_y^2} \right] = \sin^{-1} \sqrt{\bar{b}_x^2 + \bar{b}_y^2},
\]

(39)

\[
\bar{b}_z = \{ \cos \chi \sin \theta_\odot \cos(\phi_\odot - \omega_\odot T) - \sin \chi \cos \theta_\odot \}^2,
\]

\[
\bar{b}_y = \sin^2 \theta_\odot \sin^2(\phi_\odot - \omega_\odot T)
\]

which are obviously time dependent. Note \( \theta_\odot \) and \( \phi_\odot \) are the azimuthal angles of the LV \( \bar{b} \) in the Sun-centered Frame, so are supposed to be constant, or vary very slowly to be regarded approximately as constants. The
attributed to the improper cotangent function. Away from singularities, Fig.2(b) may be caused by the multivalueness of the inverse sine function as well as the inverse cotangent function. However, unlike the cotangent function, sine function does not have any singularities, so the steep cliff is absent from the inverse sine function, see Fig.2(a). The time-dependence of \( \theta_{Lab} \) as a function of time \( T \) and \( \theta_{\odot} \) is shown in Fig.2. The discontinuity shown in the steep cliff in Fig.2(b) may be attributed to the improper cot\(^{-1}\) behavior at certain arguments. However, unlike the cotangent function, sine function does not have any singularities, so the steep cliff is absent from the inverse sine function, see Fig.2(a).

The parameter \( \omega_{\odot} \approx 2\pi/(23h56m) \) is the Earth rotation frequency, \( \chi = 45.206\degree \) is the cotidate of the laboratory in ILL in Grenoble [34]. From the standard convention [36], \( T \) is measured in the Sun-centered frame from the time when \( y \) and \( Y \) axes coincide, and is chosen for convenience for each experiment [35].

The time-dependence of \( \theta_{Lab} \), \( \phi_{Lab} \) for \( \phi_{\odot} = 0\degree \) (by a proper rearrangement of coordinates, \( \phi_{\odot} \) can be assigned any value in \( [0, \pi) \)) are shown in Fig.2. The discontinuity shown in the steep cliff in Fig.2(b) may be attributed to the improper cot\(^{-1}\) behavior at certain arguments. However, unlike the cotangent function, sine function does not have any singularities, so the steep cliff is absent from the inverse sine function, see Fig.2(a). The non-smoothness in Fig.2(b) may be caused by the multivalueness of the inverse sine function as well as the inverse cotangent function. Away from singularities, Fig.2(b) can still demonstrate that the variation of \( \phi_{Lab} \) can be as large as dozens of degree. Since the spin preserving probability \( P_{\sigma_x\rightarrow} \) depends on \( \phi_{Lab} \), as clearly shown in Fig.3(a), and can also be seen from (25) and (37), \( P_{\sigma_x\rightarrow} \) evolves periodically with \( \phi_{Lab} \). Similar time dependence also happens for other probability functions such as (24) and (36), a clear manifestation of the sidereal effect.

However, note the period of the probability is given by \( T_{Prob} = 2\pi/\Omega = \pi/B_0 \). Taking the relatively conservative value of the neutron CPTV \( |\tilde{B}| < 10^{-20}\text{GeV} \) in table D12 in [36], we find \( T_{Prob} > 206684s = 2.39\frac{2\pi}{\Omega_{\odot}} > 2\frac{\pi}{\omega_{\odot}} \), so the LV induced probability variation must be strongly suppressed. We also note that if \( B_0 = 0 \), (37) is identically 1, and even very tiny nonzero \( B_0 \) can induce a small deviation of \( P_{\sigma_x\rightarrow} \) from 0, see (36) and Fig.3(b). As time evolves, the polarization variation is manifested as spin precession, see Fig.3(b). Since spin precession frequency can be measured very accurately, the tiny spin precession caused by CPTV spin-gravity couplings may be testable either by comagnetometer [38], or through the comparison of the ratio of Zeeman level frequency difference by intentionally reversing the reference magnetic field [39], or the gravitational resonance techniques [26][36][37] mentioned below.
B. The corrections with \((\vec{\sigma} \cdot \vec{b})g_z\) and \(\sigma_z \hat{p}_z\)

1. The \(-\vec{\sigma} \cdot \vec{b}(1 + \Phi_0 + g_z)\) correction

Identifying \(\vec{b} + \frac{b}{m_L} \hat{p} \cdot \vec{p}\) as the new \(\vec{b}\), the \((\hat{H}_{LV})_z\) in (10) with only the first two terms can be rewritten as

\[
\delta \hat{H}_z = -\vec{\sigma} \cdot \vec{b}(1 + \Phi_0 + g_z) - \frac{b}{m_L} [(1 + \Phi_0 + g_z) \sigma_z \hat{p}_z].
\]

(40)

Now temporarily ignoring the second mass suppressed term in (40), we can solve the Schrödinger equation with

\[
\hat{H} = \vec{b}^{\dagger} + m_g g_z - \vec{\sigma} \cdot \vec{b}(1 + \Phi_0 + g_z).
\]

Still assuming \(\vec{b} \equiv B_0(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)\) and performing a unitary transformation with

\[
U = \begin{pmatrix}
\cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\
\sin(\theta/2)e^{i\phi} & -\cos(\theta/2)
\end{pmatrix},
\]

we get

\[
\psi_u = c_a e^{iB_0(1 + \Phi_0)t} e^{-iE_n t} \sqrt[3]{\frac{E_n g_z}{2m_I}} \theta (x_{n+1}),
\]

(42)

where \(a = u, d, L_{C \pm} \equiv [2m_I(m_g + B_0)g_z]^{1/3}\) and

\[
E_{n \pm} = \left[\frac{(m_g + B_0)g_z}{2m_I}\right]^{1/3} x_{n+1} \simeq \left[\frac{m_g g_z}{2m_I}\right]^{1/3} x_{n+1} \left[1 + \frac{2B_0}{3m_g}\right].
\]

(43)

Note that (42) can be identified as a particle with oppositely shifted gravitational mass \(m_a = (m_g + B_0)\) moving near the Earth surface with oppositely shifted constant energy \(B_0(1 + \Phi_0)\). An approximate solution is given by

\[
\Psi = U \begin{pmatrix}
c_a e^{-iB_0(1 + \Phi_0)t} e^{-iE_n t} \sqrt[3]{\frac{E_n g_z}{2m_I}} \theta (x_{n+1}) \\
c_a e^{iB_0(1 + \Phi_0)t} e^{-iE_n t} \sqrt[3]{\frac{E_n g_z}{2m_I}} \theta (x_{n+1})
\end{pmatrix}
\]

\[
\simeq \frac{1}{2} \begin{pmatrix}
\left[ e^{-i(E_n + B_0)t} + e^{-i(E_n + B_0)t} \right] + \cos \theta \left[ e^{-i(E_n - B_0)t} - e^{-i(E_n + B_0)t} \right] e^{-i\phi} \\
\sin \theta \left[ e^{-i(E_n + B_0)t} - e^{-i(E_n - B_0)t} \right]
\end{pmatrix} \phi_n(z), \]

(44)

where for notational simplicity, we define \(B_0 \equiv (1 + \Phi_0)B_0\) in the second equality. At the last step we assume the state is initially in a spin-up eigenstate, and approximate the spatial wave-function \(\sqrt[3]{\frac{E_n g_z}{2m_I}} \theta (x_{n+1})\) by \(\sqrt[3]{\frac{E_n g_z}{2m_I}} \theta (x_{n+1})\) as \(B_0 \ll m_g\). Since most experiments transfer the phase difference measurement to frequency (or energy shift) measurement, we keep the phase factors unaltered in the approximation. If we probe the spin-down state later, the probability is

\[
\text{Proba}(t) = \int_0^{+\infty} dz |\langle 0, 1 |\Psi|^2 | \simeq \frac{1}{2} \sin^2 \theta \left( 1 - \cos \left[ 2B_0(1 + \Phi_0 + \frac{2E_n}{3m_g}) \right] \right)
\]

\[
= \sin^2 \theta \sin^2 \left[ B_0(1 + \Phi_0 + \frac{2E_n}{3m_g}) \right] \simeq \sin^2 \theta \left[ B_0(1 + \Phi_0 + \frac{2E_n}{3m_g}) \right]^2,
\]

(45)

where \(E_n = \left[\frac{m_g g_z}{2m_I}\right]^{1/3} x_{n+1}\). At the last approximation we have utilized the fact that the exotic LV or spin-dependent couplings \(B_0\) must be very small, which indicates that the spin-flip probability is also extremely small, at the order of \(B_0^2\). The analysis also indicates that a nonzero \((\vec{\sigma} \cdot \vec{b})g_z\)-type spin-gravity coupling can mimic itself in the spin-dependent effective gravitational mass \(m_g \equiv |\vec{b}|\), hence may be revealed in the ultra-precision test of weak equivalence principle.
2. The $-\sigma_z \hat{p}_z$ correction

As another example, we may disregard $\vec{b}$ and consider $\hat{b}_{\text{eff}}$ separately. The eigen-solution to the Schrödinger equation with $\hat{H}_4 = \frac{\hat{p}^2}{2m_I} + m_I g_z - \frac{\hat{b}^2_{\text{eff}}}{\hbar^2} (1 + \Phi_0) \sigma_z \hat{p}_z$ is

$$\Psi(t, z) = \left( \begin{array}{c} c_1 e^{i\theta z} \\ c_2 e^{-i\theta z} \end{array} \right) A_0 \left[ \frac{z}{L_c} - x_{n+1} \right] e^{-i(E_n - \frac{\tilde{b}^2}{\hbar^2})t}, (46)$$

where $\tilde{b}' \equiv \hat{b}_{\text{eff}} (1 + \Phi_0)$. As $\hat{H}_4 = \frac{\hat{p}^2}{2m_I} + m_I g_z - \frac{\hat{b}^2_{\text{eff}}}{\hbar^2}$, which is simply $\hat{H}_0 - \frac{\hat{b}^2_{\text{eff}}}{\hbar^2}$, with a momentum shift $\hat{p}_z \rightarrow \tilde{p}_z - \sigma_z \tilde{b}_{\text{eff}} (1 + \Phi_0)$, the solution is obtained by applying the momentum shifting operator $e^{i\theta x_{\text{eff}} (1 + \Phi_0) z}$ to the Airy function, which is just (46). Imposing the normalization condition $\int_0^{\infty} dx |\Psi(0, z)|^2 = 1$, we get

$$|c_1|^2 + |c_2|^2 = \frac{1}{L_c^{-1}} |A_0| (x_{n+1})^{-2}. \quad (47)$$

Up to a phase difference, the undetermined constants $c_1$ and $c_2$ can be further pinned down by imposing the initial condition on (46). As the wave-function (46) is sufficient to clearly demonstrate the opposite $z$-dependent phase evolutions, which is attributed to the CPT-odd nature of $\frac{\hat{p}}{m_I} \sigma_z \hat{p}_z$ operator, we leave more detailed exploration into future. Furthermore, keeping the much tiny $g_z \sigma_z \hat{p}_z$ term in $\hat{H}_4$ can also lead to analytically solvable solutions. These solutions involve Hermite and confluent Hypergeometric functions, which are more complicated than the Airy function, so we also prefer to leave to the future.

3. Perturbative calculation with both $\vec{a} \cdot \vec{b}$ and $\sigma_z \hat{p}_z$

For a combined analysis of $\hat{H}_{LV} = -\vec{a} \cdot \vec{b} (1 + \Phi_0 + g_z) - \frac{\hat{b}_{\text{eff}}}{\hbar^2} (1 + \Phi_0) \sigma_z \hat{p}_z$, we can apply the degenerate perturbation theory. We calculate the matrix elements with the unperturbed eigen-state $|n, \pm\rangle$ of $\hat{H}_0$, i.e.,

$$\langle z | n, + \rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \phi_n(z), \quad \langle z | n, - \rangle = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \phi_n(z), \quad (48)$$

where $\phi_n(z)$ is given by (15). The matrix elements are given by

$$\left[ \begin{array}{c} \cos \theta \\ \sin \theta e^{i\phi} \end{array} \right] \left[ \begin{array}{c} \cos \theta e^{i\phi} \\ -\cos \theta \end{array} \right]. \quad (49)$$

where we have used the fact that $\int_0^{\infty} dx \phi_n(z) \frac{d\phi_n(z)}{dz} = 0$, $\int_0^{\infty} dx \phi_n(z) \bar{z} \phi_n(z) = \frac{2}{3} L_c x_{n+1}$. As mentioned in previous subsections, $\theta, \phi$ also vary very slowly with side-real period. The matrix (49) can be diagonalized with eigen-solutions given by $|\tilde{n} + \rangle = \left( \begin{array}{c} \cos (\theta/2) e^{-i\phi} \\ \sin \theta/2 \end{array} \right)$ and $|\tilde{n} - \rangle = \left( \begin{array}{c} \sin \theta/2 \\ -\cos \theta/2 e^{i\phi} \end{array} \right)$, and the energy-eigen to leading order of LV is

$$\left( \begin{array}{c} E_{n+} \\ E_{n-} \end{array} \right) = E_0 \pm \delta E_n = \left[ \frac{(m_I g)^2}{2m_I} \right]^{1/3} x_{n+1} \times (1 + \Phi_0 + \frac{2}{3} L_c x_{n+1}) \quad (50)$$

We note that $E_{n+}, E_{n-}$ in (50) are exactly the first order approximation of the exact energy $E_{n \pm}$ obtained in (43), since the extra term $-\hat{b}_{\text{eff}} (1 + \Phi_0) \sigma_z \hat{p}_z$ does not contribute. However, it does contribute to higher order approximation of eigen-energies, as $\int_0^{\infty} dx \phi_n(z) \frac{d\phi_n(z)}{dz} \neq 0$ for $n \neq m$. Note (50) leads to tiny corrections to the transition-frequency between different gravitational bound states,

$$\nu_{mn} = \frac{(E_{m+} - E_{n+})}{\hbar} = \nu_{0mn} \pm \frac{2g}{3h} [\tilde{b}] L_c (x_{m+1} - x_{n+1}), \quad (51)$$

$$\nu_{\sigma_{mn}} = \frac{(E_{m\pm} - E_{n\pm})}{\hbar} = \nu_{0mn} \pm \frac{2}{3} \frac{[\tilde{b}]}{h} \left[ (1 + \Phi_0) + \frac{2}{3} L_c (x_{m+1} + x_{n+1}) \right], \quad (52)$$

where $\nu_{0mn} \equiv \left[ \left( \frac{m_I g}{{2m_I}} \right)^{1/3} x_{m+1} - x_{n+1} \right] / h$ is the transition frequency without exotic contributions, and we have borrowed the notation from atom optics by using $\pi$ and $\sigma \pm$ to mark the spin-conserving and spin-flip transitions. Substituting the experimental results $\nu_{02} = (464.8 \pm 1.3) \text{Hz}$, $\nu_{03} = (649.8 \pm 1.8) \text{Hz}$ and the derived local acceleration $g = (9.866 \pm 0.042) \text{m/s}^2 [26]$ into (51), we get an upper bound $[\tilde{b}] < 6.946 \times 10^{-21} \text{GeV}$, where we have averaged the bounds obtained from the uncertainty of $\nu_{02}, \nu_{03}$ respectively. Estimated from the energy resolution $\Delta E = 2 \times 10^{-15} \text{eV} [26]$ gives $[\tilde{b}] < 1.399 \times 10^{-26} \text{GeV}$. If the energy resolution is able to reach $\Delta E \sim 10^{-21} \text{eV} [41]$ in the future, the bound can be more stringent by at least 5 $\sim$ 6 orders of magnitude, i.e., $[\tilde{b}] < (10^{-25} \sim$...
In this paper, we calculate the tiny CPT-violating (CPTV) effects on neutron’s gravitational bound states in minimal SME [4]. Following the spirit of [42], we re-define the LV coefficients and rewrite the spin-dependent LV Hamiltonian (5) in uniform gravity [11]. The two effective CPTV coefficients \( \tilde{b}_k \equiv b_k + md_k^{(0)} + \frac{e_k}{k} (m\tilde{g}_{mn0} + H_{kn0}) \) and \( \tilde{b}_{\text{eff}} = \tilde{b}_0 + \frac{e_{\text{eff}}}{k} (m\tilde{g}_{w0} - m(d_{00} + \frac{e_{\text{eff}}}{k} d_{0k})) \) are our main concerns. The former reminiscent of the \( \tilde{b}_0 \) coefficient [43] defined in flat space, and also appears naturally in the non-relativistic (NR) Foldy-Wouthuysen transformation of the relativistic LV fermion Hamiltonian [43]. Though \( \tilde{b}_0 \) has already been tightly constrained by magnetometer experiment [38], \( \tilde{b}_k \) receives tiny corrections from linear gravitational potential indicated by the hat in its definition, and represents LV spin-gravity couplings. As far as we know, very rare tests have been done in this region.

From the NR Hamiltonian (6), we obtain the operator equation of motion (8) for spin evolution by focusing on \( \tilde{b}_k \), \( \tilde{b}_{\text{eff}} \) and \( [b_T]_k \) only. A tiny correction to the Larmor frequency, \( \delta \omega_L \equiv -2(1 + \Phi) \left[ \tilde{b} + \left( \tilde{b}_{\text{eff}} \tilde{p} + [b_T] \cdot \tilde{p} \right) / m \right] \) can be obtained. This precession frequency depends in a complicated way on both position and momentum, and may be testable in magnetometer experiment [38]. We guess that the \( \beta \)-dependence of \( \delta \omega_L \) may magnify the tiny spin-precession effect in high energy region. However, in ultrarelativistic situation, helicity \( \Sigma \tilde{p}_f / |\tilde{p}| \) turns out to be the proper observable rather than spin. So exploration of LV spin-gravity effect in relativistic situation may be formally quite different, and be interesting both in theory and experiment.

By applying the method in [12], we get an effective LV Hamiltonian (10) for vertical motion. To make our formulas applicable to the search of fifth force [23] as well as LV, we disregard the off-diagonal \( [b_T]_k \) term. The remain spin-dependent operators share the form similarity with \( V_{12+13} \) in [18][23]. By considering LV coefficients in (10) separately, we obtain several analytical solutions, where the CPT-odd nature is manifested from the helicity-dependent phase evolution of the wave-function components. This is also a common feature in the search of the P-odd exotic interactions in slow neutron experiments [22]. Associated to the helicity-dependent phase evolution is the spin-polarized detection probability variation, see Fig.3(a). For \( \tilde{b} \) and \( \tilde{b}_{\text{eff}} \) originating from LV scenario, the probability profile varies with a sidereal period signaling the Earth motion, a characteristic phenomenon in the search of LV. However, the sidereal period is much larger than the characteristic time scale for UCN experiment [26], which makes the probability variation practically undetectable. A complementary way is to search for the spin precession, another consequence of helicity-dependent phase evolution, as shown in equation (8) and demonstrated clearly in Fig.3(b). With sophisticated design, magnetometer experiment may be capable to probe the \( z \)- and \( \rho \)-dependent spin precession, and hence to tell the negligibly small difference of the precession effect caused by the \( \Phi \)-corrected \( \tilde{b} \)-coupling from that caused by flat space \( \tilde{b}_{\text{eff}} \)-coupling. Finally, using perturbation theory, we calculate the frequency shifts (51) and (52) due to the tiny CPTV couplings. From the precisely measured spin-conserving transition-frequency between different gravitational bound states [26], we obtain a rough bound \(|\tilde{b}| < 6.946 \times 10^{-21} \text{GeV} \) with (51). Systematic error analysis of the underline physics with experimental details may be able to place a more robust and stringent bound. If the spin-flip transition-frequency is attainable for polarized neutron beam in future, it may also improve the bounds significantly.

Aside from the UCN experiments, the CPTV spin-gravity couplings can also lead to violation of universal free fall (UFF), indicated in the main context. However, the bound is much weaker than that extracted from [26]. For example, if we naively interpret the potential of spin-1/2 \( ^{87}\text{Sr} \) as \( V = (m_{87}\tilde{\sigma} \cdot \tilde{b}) (1 + \Phi) \) and assume no exotic corrections to spin-0 \( ^{88}\text{Sr} \), then from the Eötvös parameter \( \eta = (0.2 \pm 1.6) \times 10^{-7} \) [47], we can get \(|\tilde{b} | < 2 \times 10^{-8} m_{87} \approx 1.62 \times 10^{-6} \text{GeV} \), consistent with the bound \(|\tilde{b} | < 2.5 \times 10^{-8} m_{87} \) extracted from the spin-gravity coupling strength \( k = (0.5 \pm 1.1) \times 10^{-7} \). The comparison of UFF for \(^{87}\text{Sr} \) with opposite spin states at the same accuracy can improve the bound by a factor of 2. To enhance the CPTV effects significantly to obtain much tighter constraints, torsion pendulum experiment with large bunch of spin-polarized macroscopic matter [49] may be a promising candidate [48].

In addition to those rotational invariant operators such as \( \tilde{\sigma} \cdot \tilde{b} \), many LV operators, such as \( (\tilde{\sigma} \cdot \tilde{p}) d_{00} \tilde{p} \) and \( \frac{2\pi m}{k} \left[ \tilde{b}_{\text{eff}} \delta_{ij} + [b_T]_k \right] \sigma^i \nabla_k \Phi \) have yet to be probed. The latter LV dipole spin-gravity operator may be testable in tests of the Kobzarev-Okun relation [45], a manifestation of the equivalence principle [39][44]. Taking the LV fermion-gravity operators into account can large enrich the field in the study of spin-gravity couplings. Actually, even in the simple setting of this work, the scattering states in the tunneling region [46] with LV spin-dependent gravitational couplings are still open to explore.
VI. ACKNOWLEDGEMENT

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Appendix A: Various operators in tracing over horizontal motion

The effective vertical operator is obtained as \( \langle \hat{O} \rangle \equiv \int d\vec{r}_\perp \psi^*(\vec{r}_\perp) \hat{O} \psi(\vec{r}_\perp) \), where \( \psi(\vec{r}_\perp) \equiv \frac{1}{\sqrt{2\pi}} e^{i\vec{p}_\perp \cdot \vec{r}_\perp} \) represents the horizontal wave-packet [12]. The various terms after tracing over the horizontal d.o.f. are listed in the following:

\[
\langle \Phi \rangle = -\frac{\sqrt{\pi}}{\sigma} G_N M f_0[\xi],
\]

\[
\langle \Phi^2 \rangle \equiv \frac{\pi}{2m^2} [\xi + \sqrt{\pi^2 (1 + 2\xi^2)} f_0(\xi) - 2\xi]
\]

\[
\langle \Phi^2 \rangle \equiv 2G_N M \left\{ \frac{\xi + \sqrt{\pi^2 (1 + 2\xi^2)}}{2\sigma^2} \right\} f_0(\xi)
\]

\[
\langle \Phi \rangle \equiv \frac{1}{2m} \hat{\Phi} \cdot \vec{p} = \frac{G_N M}{2m\sigma^2} \left\{ \left[ \sqrt{\pi^2 (1 + 2\xi^2)} f_0(\xi) - 2\xi \right] - i \left[ \sigma - \sqrt{\pi^2 (R + z)} f_0(\xi) \right] \right\}
\]

\[
\langle \Phi \rangle \equiv \frac{1}{2m} \hat{\Phi} \cdot \vec{p} = \frac{G_N M}{2m\sigma^2} \left[ \xi - \xi e^{i\xi^2} Ei(-\xi^2) \right]
\]

\[
\langle \Phi \rangle \equiv G_N M\sigma \frac{\xi}{\sigma} \left[ \sqrt{\pi} f_0(\xi)(1 + 2\xi^2) - 2\xi \right]
\]

where \( f_0(\xi) \equiv e^{\xi^2} \text{erfc}[\xi], \xi \equiv (R + z)/\sigma, \text{erfc}[\xi] \) is the complementary error function and \( \text{Ei}[\xi] \) is the exponential integral function.

Appendix B: Airy function and linear potential

1. Airy function and Bessel function

In this section, we review the properties of Airy function and its relation to the Schrödinger equation with linear potential. Airy function is a solution to the differential equation

\[
\frac{d^2y}{dx^2} - xy = 0,
\]

and can be related to the Bessel functions. To see this, starting with the Bessel equation

\[
\frac{d^2Z_\nu}{d\rho^2} + \frac{1}{\rho} \frac{dZ_\nu}{d\rho} + (1 - \nu^2/\rho^2)Z_\nu[\rho] = 0,
\]

and performing a variable transformation \( u(z) = z^\alpha Z_\nu[\rho] \) with \( \rho = \lambda_2^\beta \), (B2) can be cast into the form

\[
z^2 \frac{d^2u}{dz^2} + (1 - 2\alpha)z \frac{du}{dz} + [(\lambda \beta)^2 z^{2\beta} + \alpha^2 - \nu^2 \beta^2]u = 0
\]

Setting \( \alpha = \frac{1}{2}, \beta = 3/2, \nu = 1/3 \), Eqn.(B3) reduces to the same form as (B1),

\[
u''[z] + \frac{3\lambda^2}{2} z^2 u[z] = 0,
\]

and \( Z_\nu[\rho] \) satisfy the \( \frac{1}{3} \)-order Bessel equation (B2), with \( \rho = \lambda_2^{3/2} \) and \( \nu = \frac{1}{3} \not\in Z \). Accordingly, the general solution of (B4) is \( u[z] = z^{-\frac{1}{2}} \left[ c_1 Z_{\alpha_2}^\pm(\lambda z^{3/2}) + c_2 Z_{\beta_2}^\pm(\lambda z^{3/2}) \right] \).

In general, we can choose \( Z_{\pm} \) either as \( J_{\pm} \) or the linear combinations as \( Y_{\pm}(z) \equiv J_{\pm}(z) \) or \( H_{\pm}(i)(z) \equiv J_{\pm}(z) \). For details, see [40].

2. Dimensionless Schrödinger equation with linear potential

By a variable shift \( \tilde{z} = z - 1 \), we can transform the dimensionless Schrödinger equation (12) into

\[
\tilde{\phi}'' - \frac{\tilde{\phi}'}{\tilde{z}} + \tilde{\phi}(\tilde{z}) \nabla^2 \tilde{\phi}(\tilde{z}) = 0.
\]

According to (B4), the solution is \( \tilde{\phi}(\tilde{z}) = \phi(\tilde{z}) = (\tilde{z} - 1)^{\frac{3}{4}} \left[ c_1 Z_{\frac{3}{4}} \pm \frac{2}{3} \frac{k}{\sqrt{3}} (\frac{\tilde{z} - 1}{3})^{1/2} \right] + c_2 Z_{-\frac{3}{4}} \pm \frac{2}{3} \frac{k}{\sqrt{3}} (\frac{\tilde{z} - 1}{3})^{1/2} \].

To fix the two integral constant \( c_1 \) and \( c_2 \), we have to consider the asymptotic behavior of the wave-function at boundary. As a physical wavefunction, \( \phi(\tilde{z}) \) must be finite when \( |\tilde{z}| \to +\infty \). Then \( \tilde{z} - 1 < 0 \), the argument of \( Z_{\pm} \), \( \pm \frac{2}{3} k (\tilde{z} - 1)^{3/2} \), is real. We can choose \( J_{\pm} \) for convenience. The asymptotic from in the limit of
$|1 - \tilde{z}| \to \infty$ is

$$J_{\pm} (\rho) \sim \sqrt{\frac{2}{\pi \rho}} \left[ \cos(\rho - \frac{5\pi}{12}) \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{5}{3} - 2m)}{(2\rho)^{2m}} - \sin(\rho - \frac{5\pi}{12}) \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{3} - 2m + 1)}{(2\rho)^{2m+1}} \right], \quad (B6)$$

where $\left(\frac{x}{3}, 0\right) = 1$ and $\left(\frac{1}{3}, n\right) = \frac{\Gamma\left(\frac{x}{3} + n\right)}{\sqrt{\pi} \Gamma\left(\frac{x}{3} - n\right)}$ with $n = 1, 2, 3, ..., \text{ see } [40]$. The asymptotic form (B6) has a sine-like oscillating behavior as expected, since in the $z < z_c$ region, particle kinematic energy is still positive. So we can write down the explicit form of wave-function in $\tilde{z} < 1 \ (z < z_c)$ as

$$\phi(\tilde{z})|_{\tilde{z} < 1} = e^{i\frac{\pi}{2}}(1 - \tilde{z})^\frac{1}{2} \left[ c_1 J_{\pm} \left(\frac{2}{3}k(1 - \tilde{z})^{3/2}\right) + c_2 J_{\pm} \left(\frac{2}{3}k(1 - \tilde{z})^{-3/2}\right) \right]. \quad (B7)$$

When $\tilde{z} > 1$, the argument in $Z_\pm(\rho)$ is imaginary and we can replace the variable $\rho \to ip$. Inspection of (B2), this variable transformation makes $Z_\pm$ a solution to the modified Bessel equation. When $|\tilde{z}| \to \infty$, the asymptotic behavior of $K_{\nu}(\rho)$ and $I_{\nu}(\rho)$ for $|\nu| \leq 1$ are

$$K_{\nu}(\rho) \sim \sqrt{\frac{\pi}{2p}} e^{-\rho} \left[ 1 + \sum_{n=0}^{\infty} \frac{(\nu, n)}{(2\rho)^n} \right],$$

$$I_{\nu}(\rho) \sim \sqrt{\frac{\pi}{2p}} e^{+\rho} \left[ 1 + \sum_{n=0}^{\infty} \frac{(-\nu, n)}{(2\rho)^n} \right], \quad (B8)$$

By a proper variable transformation, we can always make the argument positive, $\rho \to \frac{2}{3}k(\tilde{z} - 1)^{3/2} > 0$. Scrutinizing (B8) and (B9), we can choose the proper solution as $K_{\frac{1}{2}}$, since $I_{\frac{1}{2}}$ blows up when $\tilde{z} \to \infty$. In the region $\tilde{z} > 1$, we tentatively write down the solution

$$\phi(\tilde{z})|_{\tilde{z} > 1} = c_3(\tilde{z} - 1)^{\frac{3}{2}} K_{\frac{1}{2}} \left[ \frac{2}{3}k(\tilde{z} - 1)^{3/2}\right]. \quad (B10)$$

Now we can use the continuity condition $\phi(\tilde{z})|_{\tilde{z} = 1} = \phi(\tilde{z})|_{\tilde{z} = 1 - 0}$ to match the three undetermined constants. This can be achieved by exploring the behavior of (B7) and (B10) when $\tilde{z}$ is infinitesimally close to $1 \pm 0$. Up to $O((\tilde{z} - 1)^3)$, from (B10) we have

$$\phi(\tilde{z})|_{\tilde{z} < 1} \sim c_3 e^{\mp\frac{2\pi}{3k}} \left[ \begin{array}{c} (3/k)^{1/3} \\
+ \frac{k(\bar{k}/3)^{2/3}}{4} \frac{\bar{k}^{z^4}}{\zeta^{3}} \end{array} \right] \Gamma[1/3],$$

$$+ e^{\pm\frac{2\pi}{3k}}(\bar{k}/3)^{1/3} \left[ \begin{array}{c} \frac{\bar{z}}{2} \\
+ \frac{k^{z^4}}{24} \end{array} \right] \Gamma[-1/3] \right) \right), \quad (B11)$$

where $\bar{z} = \tilde{z} - 1$. Similarly, up to the same order, from (B7) we have

$$\phi(\tilde{z})|_{\tilde{z} > 1} \sim \left\{ \begin{array}{c} -e^{\pm\frac{2\pi}{3k}} c_1 \left( \frac{\bar{k}^{3/4}}{12} \right) \frac{\bar{k}^{z^4}}{12} \\
+ c_2 \Gamma[1/3] \end{array} \right\}, \quad (B12)$$

where we have taken into account of the sign change, $1 - \bar{z} = -\tilde{z}$. Matching (B12) with (B11), we immediately get

$$c_1 = i \frac{\Gamma[-1/3] \Gamma[1/3]}{2} c_3 = -\frac{i\pi}{\sqrt{3}} c_3,$$

$$c_2 = -\frac{i\pi}{\sqrt{3}} c_3. \quad (B13)$$

In combination, the wave-function is

$$\phi(\tilde{z}) = \left\{ \begin{array}{c} c_3(\tilde{z} - 1)^{\frac{3}{2}} K_{\frac{1}{2}} \left[ \frac{2}{3}k(\tilde{z} - 1)^{3/2}\right], \quad \tilde{z} < 1 \\
- \frac{\sqrt{3\pi}}{k^{1/2}} c_3 \text{Ai}[k^{1/2}(\tilde{z} - 1)]. \quad (B14) \end{array} \right.$$
