CONTINUITY OF VOLUMES – ON A GENERALIZATION OF A CONJECTURE OF J.W.MILNOR

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Abstract. In his paper [6] J. Milnor conjectured that the volume of \( n \)-dimensional hyperbolic and spherical simplices, as a function of the dihedral angles, extends continuously to the closure of the space of allowable angles. A prove of this conjecture was recently given by F. Luo [5]. In this paper we give a simple proof of this conjecture, prove much sharper regularity results, and then extend the method to apply to a large class of convex polytopes. The simplex argument works without change in dimensions greater than 3 (and for spherical simplices in all dimensions), so the bulk of this paper is concerned with the three-dimensional argument. The estimates relating the diameter of a polyhedron to the length of the systole of the polar polyhedron are of independent interest.

1. Introduction

Consider the set of simplices in \( \mathbb{H}^n \) or \( \mathbb{S}^n \). It is well-known that this set is parametrized by the (ordered) collection of dihedral angles, and we may call the set of assignments of dihedral angles of geometric simplices in \( \mathbb{H}^n \) as a subset \( \Omega_{\mathbb{H}^n} \subset \mathbb{R}^{n(n+1)/2} \) and similarly, the set of dihedral angle assignments of of geometric simplices in \( \mathbb{S}^n \) as a subset \( \Omega_{\mathbb{S}^n} \subset \mathbb{R}^{n(n+1)/2} \). These sets are open, since they are defined by collections of strict inequalities (which are polynomial in the cosines of the dihedral angles). One may then view the volume \( V \) of a simplex as a function \( V \) on \( \Omega_{\mathbb{X}^n} \). A natural question, then, is that of boundary regularity of \( V \) on \( \overline{\Omega} \). This question was, in fact, asked (in a weaker form of: does \( V \) admit a continuous extension to \( \overline{\Omega} \)) by John Milnor\(^1\) in [6]. An answer to Milnor’s question was given to F. Luo in [5].

\(^1\)Milnor does not attribute the conjecture to himself, so it might be older. W. Thurston (personal communication) opines that the conjecture is, in fact, due to Milnor.
In this paper we begin by answering Milnor’s question in the sharp form (our method gives sharp regularity estimates, though we don’t dwell too much on this). In dimension greater than 3 the argument goes through verbatim to give the same result for general convex polytopes, as long as they are of a combinatorial class where the dihedral angles determine the volume (this includes, eg, all simple polytopes, that is, those with simplicial vertex links). In dimension 3, things are more subtle yet, but we get the result for various classes of convex polyhedra (those with non-obtuse dihedral angles, by using Andreev’s theorem [1] and for “hyperideal” polyhedra, by using the results of Bao-Bonahon [2]). It is quite likely analogous results should hold for hyperbolic cone manifolds. The omnibus form of our results on volumes is:

**Theorem 1.** Let $\Omega$ be one of the following parameter spaces:

1. The space of dihedral angles of polytopes of a fixed combinatorial type in $S^n$, such that the volume is determined by the dihedral angles.
2. The space of dihedral angles of polytopes of a fixed combinatorial type in $H^n$, $n > 3$, such that the volume is determined by the dihedral angles.
3. The space of dihedral angles of tetrahedra (possibly with some hyperideal vertices) in $H^3$.
4. The space of dihedral angles of a fixed combinatorial type of hyperideal polyhedra in $H^3$.
5. The space of non-obtuse dihedral angles of a fixed combinatorial type of polyhedra in $H^3$.

The the volume function on $\Omega$ extends to a $C^{0,1}$ function on $\overline{\Omega}$.

**Corollary 1.** To the above, we can add: The space of dihedral angles of a fixed combinatorial type of ideal polyhedra in $H^3$.

**Proof.** These lie in the boundary of the set of hyperideal polyhedra, and are determined by their dihedral angles (see, eg, [2]). In fact, the same argument works for polyhedra with some ideal and some hyperideal vertices, by the work of [2].

The plan of the paper is as follows.

In Section 2 we give the argument for simplices. This has most of the ingredients of the general result.

In Section 3 we recall the basic characterization results for convex polyhedra in $H^3$ in terms of their dihedral angles (or, more generally, polar metric).

In Section 4 we give quantitative estimates of how close a polyhedron is to degeneracy (boundary of the allowable dihedral angle
space) in terms of its diameter. The results (which are of independent interest) can be loosely summarized as follows:

**Theorem 2.** Let $P$ be a polyhedron with $N$ vertices in $\mathbb{H}^3$ of diameter $\rho \gg 1$. Let $M^*$ be the polar metric of $P$ (as in [10, 8]). The $M^*$ lies within $c_1(N) \exp(-c_2(N)\rho)$ of the boundary of the space of admissible polar metrics, where $c_1, c_2$ are strictly positive functions of $N$.

The constants in the statement of the Theorem above are completely explicit, and can be sharpened by taking into consideration finer invariants of the combinatorics of $P$ than the number of vertices.

2. A simple proof for simplices (among other things)

In dimension 2, the result follows immediately from Gauss’ formula, which states that area is a linear function of the angles, so we will only discuss dimensions 3 or above.

The simple proof relies on the Schlafli differential equality (see [6], which states that in a space of constant curvature $K$, and dimension $n$ the volume of a smooth family of polyhedra $P$ satisfies the differential equation:

\[(1)\quad K dV(P) = \frac{1}{n-1} \sum F V_{n-2}(F) d\theta_F,\]

where the sum is over all codimension-2 faces, $V_{n-2}$ is the $n-2$ dimensional volume of $F$, and $\theta_F$ is the dihedral angle at $F$.

Another way of writing the Schlafli formula is:

\[(2)\quad K \frac{\partial V(P)}{\partial \theta_F} = V_{n-2}(F).\]

This is the form we will use.

The first observation is that $V_{n-2}(F)$ is bounded by a constant (dimensional for $S^n$, , depending on the number of vertices of $F$, but nothing else, and if $n \geq 4$, in $\mathbb{H}^n$).

This immediately shows the continuity of volume for all $S^n$, and for $\mathbb{H}^n$, whenever $n \geq 4$.

We are left with dimension 3. All we really need is noting that the partial derivatives of $V$ with respect to the dihedral angles develop at worst logarithmic singularities as we approach the frontier of $\Omega_{413}$ – this result suffices by the following form of the Sobolev Embedding Theorem (this is [3, Theorem 7.26]):

**Theorem 3.** Let $\Omega$ be a $C^{0,1}$ domain in $\mathbb{R}^n$. Then,
• (i) If $kp < n$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^{p'}(\Omega)$, where $p' = np/(n - kp)$, and compactly imbedded in $L^{q}(\Omega)$ for any $q < p$.
• (ii) If $0 \leq m < k - n/p < m + 1$, the space $W^{k,p}$ is continuously embedded in $C^{m,\alpha}(\Omega)$, $\alpha = k - n/p - m$, and compactly embedded in $C^{m,\beta}(\Omega)$ for any $\beta < \alpha$.

Here, the Sobolev space $W^{k,p}$ is the space of functions whose first $k$ (distributional) derivatives are in $L^p$.

In our case, we know that the domain $\Omega$ is bounded, convex “curvilinear polyhedral” (hence $C^{0,1}$) domain, volume is a bounded function, and we assume that the gradient grows logarithmically as we approach the boundary. This implies that $V$ is in $W^{1,p}$ for all $p > 0$, so we get the following corollary:

**Corollary 2.** Volume is in $C^{0,\alpha}(\Omega)$ for any $\alpha < 1$.

The logarithmic growth of diameter of the simplex as a function of the distance to $\partial \Omega$ can be shown in a completely elementary way using Eq. 2 and elementary reasoning about Gram matrices, as follows:

Let $G$ be “angle Gram matrix” of a simplex $\Delta$, that is, $G_{ij} = -\cos \theta_{ij}$, where $\theta_{ij}$ is the angle between the $i$-th and the $j$-th face. Let $S$ be the matrix whose columns are the normals to the faces of $\Delta$ (all the computations take place in Minkowski space, and we use the hyperboloid model of $\mathbb{H}^n$. It is immediate that $G = S^tS$.

Let now $W$ be the matrix whose columns are the (possibly scaled) vertices of $\Delta$. $W$ satisfies the equation $S^tW = I$, and to get the vertices to lie on the hyperboloid $\langle x, x \rangle = -1$ we must rescale in such a way that the squared norms of the columns of $W$ become $-1$. Call the scaled matrix $W_s$. Since the usual “length” Gram matrix $G^*$ of $\Delta S$ can be written as $W_s^tW_s$, and $G^*_{ij} = -\cosh(d(v_i, v_j))$, a simple computation using Cramer’s rule gives:

$$\cosh d(v_i, v_j) = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}},$$

where $c_{ij}$ is the $ij$-th cofactor of $G$. (see 7 for many related results).

It follows that the distances between the vertices (which are the lengths of the edges, which are the faces of codimension 2.) behave as $|\log c_{ij}|$. Since the cofactors are polynomial in the cosines of the angles, we are done.
It should be noted that this argument works \textit{mutatis mutandis} for \textit{hyperideal} simplices, or simplices with some finite and some hyper-infinite vertices.

3. Convex polytopes

For arbitrary convex polytopes in dimension \( n > 3 \), (and convex \textit{spherical} polytopes in all dimensions) the proof given in Section 2 goes through without change, with the one proviso that it is not currently known whether the volume of a polytope is determined up to congruence by its dihedral angles. Such a uniqueness result is conjectured (indeed, it is conjectured that a polytope is determined up to congruence by the dihedral angles), and is easy to prove for \textit{simple} polytopes – those with simplicial links of vertices – this follows in arbitrary dimension from the corresponding result in 3 dimensions. For the 3-dimensional result see, eg, [8, 10]. In the rest of this paper we will be concerned with the most interesting case of convex polyhedra in \( \mathbb{H}^3 \).

First, we recall the following result of [10, 8]:

\textbf{Theorem 4 ([10, 8])}. A metric space \( (\mathcal{M}, g) \) homeomorphic to \( S^2 \) can arise as the Gaussian image \( G(P) \) of a compact convex polyhedron \( P \) in \( \mathbb{H}^3 \) if and only if the following conditions hold:

\begin{itemize}
\item (a) The metric \( g \) has constant curvature 1 away from a finite collection of cone points.
\item (b) The cone angle at each \( c_i \) is greater than \( 2\pi \).
\item (c) The lengths of closed geodesics of \( (\mathcal{M}, g) \) are all strictly greater than \( 2\pi \).
\end{itemize}

Assume now, for simplicity, that the polyhedron \( P \) is simple (as pointed out above, this also allows us to assume that the geometry is determined by the dihedral angles). In that case, the space of admissible metrics \( \Omega_P \) (as per Theorem 4) is parametrized by the exterior dihedral angles (the cell decomposition dual to that of \( P \) gives a triangulation of the Gaussian image, and the (exterior) dihedral angles are the lengths of edges of the triangulation.) Theorems 9 \textendash 10 immediately imply the following:

\textbf{Theorem 5}. There exists a constant \( L_0 \), such that the maximal length \( \ell_P \) of an edge of \( P \) is bounded as follows:

\[ \ell_P \leq \max(L_0, -2N \log(d(P, \partial \Omega_P)/12N)), \]

where \( N \) is the number of vertices of \( P \).
Proof. Assume the contrary. Then, there exists a sequence of polyhedra \( P_1, \ldots, P_n, \ldots \) with diameter \( \rho(P_i) \geq \ell P_i \) going to infinity, which are farther than \( 12N \exp(-\rho/2N) \). By choosing a subsequence, we may assume that there is a fixed cycle of faces \( F_1, \ldots, F_k \) of \( P \), such that the sum of dihedral angles along the edges \( e_i = F_i \cap F_{i+1} \) is smaller than \( 2\pi + 12N \exp(-\rho/2N) \), (by Theorem 9) and which are a \( 4N \exp(-2\rho) \) quasigeodesic (by Theorem 10). Since the limit point of the \( P_i \) is not in \( \Omega_P \) (by Theorem 4), the result follow. □

The following corollary is immediate (by Schl"afli, see Section 2):

**Corollary 3.** The volume is in \( W^{1,p}(\Omega_P) \) for all \( p > 0 \).

We now have almost enough to show that volume extends to \( \overline{\Omega_P} \), except for the slight matter of not having the required (by Theorem 3) regularity result for \( \partial \Omega_P \). Such a result seems quite non-trivial, since the length of the shortest closed geodesic is a rather badly behaved quantity. However, there are two very important special cases where \( \Omega_P \) is actually a convex polytope, so this problem is finessed. The first is when we restrict the dihedral angles to be non-obtuse. This case is covered by

**Theorem 6** (Andreev’s theorem ([11])). Let \( C \) be an abstract polyhedron with more than 4 faces, and suppose that non-obtuse angles \( \alpha_i \) are given corresponding to each edge \( e_i \) of \( C \). There is a compact convex hyperbolic polyhedron whose faces realize \( C \) with dihedral angle \( \alpha_i \) at each edge \( e_i \) if and only if the following conditions hold:

1. For each edge \( e_i \), \( 0 < \alpha_i \leq \pi/2 \).
2. For any three distinct edges \( e_1, e_2, e_3 \) meeting at a vertex, \( \alpha_1 + \alpha_2 + \alpha_3 > \pi \).
3. Whenever \( \Gamma \) is a prismatic 3-circuit intersecting edges \( e_1, e_2, e_3, e_4 \), then \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 < 2\pi \).
4. Whenever \( \Gamma \) is a prismatic 4-circuit intersecting edges \( e_1, e_2, e_3, e_4, e_5 \), then \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 3\pi \).
5. Whenever there is a four-sided face bounded by edges \( e_1, e_2, e_3, e_4 \) (enumerated successively), with edges \( e_{12}, e_{23}, e_{34}, e_{41} \) entering the four vertices, then:
   
   \[
   \begin{align*}
   \alpha_1 + \alpha_3 & + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi, \\
   \alpha_2 + \alpha_4 & + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi.
   \end{align*}
   \]

The other special case is that of hyperideal polyhedra:

**Theorem 7** (Bao-Bonahon, [2]). Let \( \sigma \) be a cell decomposition of \( S^2 \), and let \( w : \sigma_1 \to (0, \pi) \) be a map on the set of edges of \( \sigma \). Then there exists a
hyperideal polyhedron with combinatorics given by $\sigma$ and exterior dihedral angles given by $w$ if and only if:

1. The sum of the values of $w$ on each circuit in $\sigma_1$ is not smaller than $2\pi$, and is strictly greater if the circuit is non-elementary.
2. The sum of values of $w$ on each simple path in $\sigma_1$ is strictly larger than $\pi$.

The hyperideal polyhedron is then unique.

4. Degeneration estimates

The results of this section are a quantitative version of the results of the compactness results of [10, 8]. First, some key lemmas. The general setup will be as follows: $L$ is a geodesic in $\mathbb{H}^3$, $t$ is a real number (generally large) and $P, P^-, P^+$ are three planes, all orthogonal to $L$, and such that $d(P, P^-) = d(P, P^+) = t$, and $d(P^-, P^+) = 2t$. We denote $x_0 = L \cap P$.

In the sequel, we use the hyperboloid model of $\mathbb{H}^3$, where $\mathbb{H}^3$ is represented by the set $\langle x, x \rangle = -1; x_0 > 0$, in the $\mathbb{R}^4$ equipped with the scalar product $\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^{4} x_iy_i$. The reader is referred to [11] (as well as [10]) for the details (which will be used below).

Returning back to our setup, we can assume, without loss of generality, that

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that

$$P^\perp = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and hence, that $P^+ = \phi(t)P$, while $P^- = \phi(-t)P$, where

$$\phi(r) = \begin{pmatrix} \cosh(r) & \sinh(r) & 0 & 0 \\ \sinh(r) & \cosh(r) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\phi(r)$ is symmetric, it follows that

$$P^{+\perp} = \phi(t)P^\perp = \begin{pmatrix} \cosh(t) \\ \sinh(t) \\ 0 \\ 0 \end{pmatrix},$$
while

\[ P^{-\perp} = \phi(t)P^{\perp} = \begin{pmatrix} \cosh(t) \\ -\sinh(t) \\ 0 \\ 0 \end{pmatrix} \,.

**Lemma 1.** Let \( Q \) be a plane in \( \mathbb{H}^3 \) which intersects both \( P^− \) and \( P^+ \). Then, there exists \( t_0 \), such that \( Q \) intersects \( P \), and the cosine of the angle \( \alpha \) of intersection satisfies \( |\cos(\alpha)| < 3e^{-t}, \) as long as \( t > t_0 \). The number \( t_0 \) can be picked independently of \( Q \).

**Proof.** Let the unit normal \( Q^\perp \) to \( Q \) be \( Q^\perp = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \). Since two planes intersect if and only if the scalar product of their unit normals is less than 1 in absolute value, we have, from the hypotheses of the lemma and the description of the unit normals to \( P^- \) and \( P^+ \) above that:

\[ (5) \quad |a \cosh(t) + b \sinh(t)| < 1 \]
\[ (6) \quad |a \cosh(t) - b \sinh(t)| < 1. \]

Squaring the two inequalities, and adding them together we obtain:

\[ a^2 \cosh^2(t) + b^2 \sinh^2(t) < 1. \]

Since, under the hypotheses of the lemma, \( \min(\cosh(t), \sinh(t)) > e^t/3 \), it follows that

\[ a^2 + b^2 < 3/e^t, \]

and so \( \max(a, b) < 3e^{-t} \). Now, the cosine of the angle between \( Q \) and \( P \) equals \( \langle Q^\perp, P^\perp \rangle = b \), so the result follows. \( \square \)

**Remark 8.** The constant 3 is far from sharp (especially for larger \( t \)).

**Lemma 2.** There exists a \( t_0 \), such that if \( M \) is a line in \( \mathbb{H}^3 \) which intersects both \( P^- \) and \( P^+ \), then \( M \) intersects \( P \), and \( \cosh(d(P \cap M', x_0)) < 4e^{-2t} + 1 \), as long as \( t > t_0 \).

**Proof.** Assume that \( M \cap P^+ = \phi(t)p_1 \), and \( M \cap P^- = \phi(-t)p_1 \), where \( p_{1,2} \in P \). (This is always possible, since \( P^+ = \phi(t)P, P^- = \phi(-t) \).) The intersection of \( M \) with \( P \) is then given by

\[ M \cap P = \frac{x(M \cap P^+) + y(M \cap P^-)}{\|x(M \cap P^+) + y(M \cap P^-)\|} \]

where \( x \) and \( y \) are chosen so that the linear combination is actually in \( P \), or, in other words, the second coordinate of the linear combination vanishes. We abuse notation above by writing \( \|Z\| = \sqrt{-\langle Z, Z \rangle} \).
Let us now compute. Set (for $i = 1, 2$)

$$p_i = \begin{pmatrix} a_i \\ 0 \\ c_i \\ d_i \end{pmatrix}.$$ 

It follows that

$$M \cap P^+ = \begin{pmatrix} a_1 \cosh(t) \\ a_1 \sinh(t) \\ c_1 \\ d_1 \end{pmatrix},$$

while

$$M \cap P^- = \begin{pmatrix} a_2 \cosh(t) \\ -a_2 \sinh(t) \\ c_2 \\ d_2 \end{pmatrix}.$$ 

It follows that we can choose $x = 1/(2a_1)$, $y = 1/(2a_2)$, so that

$$m = xM \cap P^+ + yM \cap P^- = \begin{pmatrix} \cosh(t) \\ 0 \\ \frac{1}{2}(c_1/a_1 + c_2/a_2) \\ \frac{1}{2}(d_1/a_1 + d_2/a_2) \end{pmatrix}.$$ 

It follows that

$$(7) \quad - \cosh(d(M \cap P, x_0)) = \left( \frac{m}{|m|}, x_0 \right) =$$

$$= - \frac{\cosh(t)}{\sqrt{\cosh^2(t) - 1/4 ((c_1/a_1 + c_2/a_2)^2 + (d_1/a_1 + d_2/a_2)^2)}}.$$ 

Since $c_i^2 + d_i^2 + 1 = a_i^2$, for $i = 1, 2$ it follows that $|c_i/a_i| < 1$, and similarly $|d_i/a_i| < 1$, so that

$$\cosh^2(t) \geq \cosh^2(t) - 1/4 \left( (c_1/a_1 + c_2/a_2)^2 + (d_1/a_1 + d_2/a_2)^2 \right) > \cosh^2(t) - 2.$$ 

It follows that

$$\cosh(d(M \cap P, x_0)) \leq \frac{1}{\sqrt{1 - 2/ \cosh^2(t)}},$$

and the assertion of the lemma follows by elementary calculus. \hfill \Box

**Lemma 3.** Let $T$ be a spherical triangle with sides $A, B, C$ and (opposite) angles $\alpha, \beta, \gamma$. Suppose that $|\cos(\beta)| < \varepsilon \ll 1$, $|\cos(\gamma)| < \varepsilon \ll 1$. Then $|\alpha - A| < 2\varepsilon$. 
Proof. The spherical Law of Cosines states that:

\[ \cos(A) = \frac{\cos(\alpha) + \cos(\beta) \cos(\gamma)}{\sin(\beta) \sin(\gamma)}. \]

It follows that

\[ \cos(A) - 2\varepsilon^2 \leq \cos(A)(1 - \varepsilon^2) - \varepsilon^2 \leq \cos(\alpha) \leq \cos(A) + \varepsilon^2. \]

The assertion of the lemma follows immediately. \(\square\)

Corollary 4. Let \(F_1\) and \(F_2\) be two planes intersecting at a dihedral angle \(\alpha\), with both \(F_1\) and \(F_2\) intersecting a third plane \(P\), at angles whose cosines are smaller than \(\varepsilon\). Let \(A\) be the angle between \(F_1 \cap P\) and \(F_2 \cap P\). Then

\[ |\alpha - A| < 2\varepsilon. \]

Proof. Apply Lemma 3 to the link of the point \(F_1 \cap F_2 \cap P\). \(\square\)

Lemma 4. Let \(V\) be a convex polygon in the hyperbolic plane \(\mathbb{H}^2\), such that all the vertices of \(V\) lie within a distance \(r\) of a certain point \(O\). Then, the sum of the exterior angles of \(V\) is smaller than \(2\pi \cosh(r)\).

Proof. The area of a disk of radius \(r\) in \(\mathbb{H}^2\) equals \(4\pi \sinh^2(r/2) = 2\pi(\cosh(r) - 1)\) (see [12]). Since \(V\) is contained in such a disk, its area is at most \(2\pi(\cosh(r) - 1)\), and since the area of \(V\) equals the difference between the sum of the exterior angles and \(2\pi\), the statement of the lemma follows. \(\square\)

Now we are ready to show the following:

Theorem 9. Let \(X\) be a convex polyhedron with \(N\) vertices in \(\mathbb{H}^3\) of diameter \(\rho \gg 1\). Then, there exists a cyclic sequence of faces \(F_1, \ldots, F_k = F_1\), with \(F_i\) sharing an edge \(e_i\), with \(F_{i+1}\) (indices taken \(\mod k\)) so that the sum of exterior dihedral angles at \(e_1, \ldots, e_k\) is smaller than \(2\pi + 12N \exp(-\rho/2N)\).

Proof. Take a diameter \(D\) of \(X\) of length \(\rho\), place points \(p_1, \ldots, p_N\) equally spaced on \(D\). By the pigeonhole principle, one of the segments \(p_i p_{i+1}\) contains no vertices of \(X\). Let \(x_0\) be the midpoint of the segment \(p_i p_{i+1}\). Construct planes orthogonal to \(D\) at \(x_0\) \((P)\) and \(p_i\) \((P^-)\), and at \(p_{i+1}\) \((P^+)\). Let \(t = \rho/(2N)\). The portion of \(X\) contained between \(P^-\) and \(P^+\) is a polyhedral cylinder, consisting of faces \(F_1, \ldots, F_k\). By Lemma 2 the intersection of \(X\) with \(P\) is a polygon \(\mathcal{P}\), whose sum of exterior angles is at most \(2\pi(4 \exp(-2t) + 1)\), and so by Corollary 4 combined with Lemma 1 the sum of the dihedral angles corresponding to pairs \(F_i F_{i+1}\) is at most \(2\pi(4 \exp(-2t) + 1) + 6k \exp(-t)\). Since \(k\) is no greater than the number of faces of \(X\), which, in turn, is at most \(2N - 4\). \(\square\)
Theorem 10. With notation as in Theorem 9, the faces $F_1, \ldots, F_k$ form a curve in the Gaussian image of $X$ with geodesic curvature not exceeding $3k \exp(-\rho/N)$.

Remark 11. The reader is referred to [8, 10] for a more thorough discussion of geodesics on spherical cone manifold, but suffice it to say that the contribution of the face $F_i$ to the geodesic curvature is 0 if the two edges are (hyper)parallel, and equal to the angle of intersection if they intersect.

Proof. Let $e_1$ and $e_2$ be the two edges of $F$. If $e_1$ and $e_2$ do not intersect, there is nothing to prove (by the remark above. If they do intersect at a point $C$, note that $C$ is at a distance at least $\rho/2N$ from $x_0$, while the intersections $A$ and $B$ of $e_1$ and $e_2$ with $P$ are at most arccosh$(4 \exp(-\rho/2N) + 1) \approx \sqrt{8} \exp(-\rho/2N)$ away from $x_0$, and so at most (for large $\rho$) $6 \exp(-\rho/2N)$ away from each other. We will only use the (much cruder) estimate $\cosh(AB) \leq 2$. Now, apply the hyperbolic law of cosines to the triangle $ABC$, to get:

\[
\cos(\gamma) = \frac{-\cosh(AB) + \cosh(AC) \cosh(BC)}{\sinh(AC) \sinh(BC)} \geq 1 - \frac{\cosh(AB)}{\sinh(AC) \sinh(BC)} \geq 1 - \frac{2}{\sinh(AC) \sinh(BC)} \geq 1 - 8 \exp(-\rho/2N).
\]

The estimate now follows.

Remark 12. The argument above is easily modified to show that the curve dual to $F_1, \ldots, F_k$ has small geodesic curvature viewed as a curve in $S^3$, and not just in $X^*$. 

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