AUTOMORPHISMS OF $\overline{T}$

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ABSTRACT. Let $\overline{G}$ be the wonderful compactification of a simple affine algebraic group $G$ defined over $\mathbb{C}$ such that its center is trivial and $G \neq \text{PSL}(2, \mathbb{C})$. Take a maximal torus $T \subset G$, and denote by $\overline{T}$ its closure in $\overline{G}$. We prove that $T$ coincides with the connected component, containing the identity element, of the group of automorphisms of the variety $\overline{T}$.

Résumé. Automorphismes de $\overline{T}$

Soit $\overline{G}$ la compactification magnifique d’un groupe algébrique affine $G$ défini sur $\mathbb{C}$, dont le centre est trivial et tel que $G \neq \text{PSL}(2, \mathbb{C})$. Soit $T \subset G$ un tore maximal, et soit $\overline{T}$ son adhérence dans $\overline{G}$. Nous montrons que $T$ est égal à la composante connexe contenant l’élément neutre du groupe d’automorphismes de la variété $\overline{T}$.

1. INTRODUCTION

Let $G$ be a simple affine algebraic group defined over the complex numbers such that the center of $G$ is trivial. De Concini and Procesi constructed a very interesting compactification of $G$ which is known as the wonderful compactification [DP, p. 14, 3.1, THEOREM]. The wonderful compactification of $G$ will be denoted by $\overline{G}$. Fix a maximal torus $T$ of $G$. Let $\overline{T}$ denote the closure of $T$ in $\overline{G}$. The connected component, containing the identity element, of the group of all automorphisms of the variety $\overline{T}$ will be denoted by $\text{Aut}^0(\overline{T})$. For more details about the variety $\overline{T}$ we refer to [BJ, § 1]. Our aim here is to compute $\text{Aut}^0(\overline{T})$.

Using the action of $G$ on $\overline{G}$, we have $T \subset \text{Aut}^0(\overline{T})$; this inclusion does not depend on whether the right or the left action is chosen. We prove that $T = \text{Aut}^0(\overline{T})$, provided $G \neq \text{PSL}(2, \mathbb{C})$; see Theorem 3.1.

Note that $\text{Aut}(\overline{T})$ is not connected since $\overline{T}$ is stable under the conjugation of the normalizer $N_G(T)$ of $T$ in $G$.

If $G = \text{PSL}(2, \mathbb{C})$, then $\overline{T} = \mathbb{P}^1$, and hence $\text{Aut}^0(\overline{T}) = \text{PSL}(2, \mathbb{C})$.

2. LIE ALGEBRA AND ALGEBRAIC GROUPS

In this section we recall some basic facts and notation on Lie algebra and algebraic groups (see [Hu1], [Hu2] for details). Throughout $G$ denotes an affine algebraic group over $\mathbb{C}$ which is simple and of adjoint type. We also assume that the rank of $G$ is at-least two, equivalently $G \neq \text{PSL}(2, \mathbb{C})$.
For a maximal torus $T$ of $G$, the group of all characters of $T$ will be denoted by $X(T)$. The Weyl group of $G$ with respect to $T$ is defined to be $W := N_G(T)/T$, where $N_G(T)$ is the normalizer of $T$ in $G$. By $R \subset X(T)$ we denote the root system of $G$ with respect to $T$. For a Borel subgroup $B$ of $G$ containing $T$, let $R^+(B)$ denote the set of positive roots determined by $T$ and $B$. Let $S = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots in $R^+(B)$. Let $B^-$ denote the opposite Borel subgroup of $G$ determined by $B$ and $T$. For $\alpha \in R^+(B)$, let $s_\alpha \in W$ be the reflection corresponding to $\alpha$. The Lie algebras of $G$, $T$ and $B$ will be denoted by $\mathfrak{g}$, $\mathfrak{t}$ and $\mathfrak{b}$ respectively. The dual of the real form $\mathfrak{t}_\mathbb{R}$ of $\mathfrak{t}$ is $X(T) \otimes \mathbb{R} = \text{Hom}_\mathbb{R}(\mathfrak{t}_\mathbb{R}, \mathbb{R})$.

The positive definite $W$–invariant form on $\text{Hom}_\mathbb{R}(\mathfrak{t}_\mathbb{R}, \mathbb{R})$ induced by the Killing form on $\mathfrak{g}$ is denoted by $(\ , \ )$. We use the notation

$$\langle \nu, \alpha \rangle := \frac{2(\nu, \alpha)}{(\alpha, \alpha)}.$$ 

In this setting one has the Chevalley basis

$$\{x_\alpha, h_\beta \mid \alpha \in R, \beta \in S\}$$

of $\mathfrak{g}$ determined by $T$. For a root $\alpha$, we denote by $U_\alpha$ (respectively, $\mathfrak{g}_\alpha$) the one–dimensional $T$ stable root subgroup of $G$ (respectively, the subspace of $\mathfrak{g}$) on which $T$ acts through the character $\alpha$.

Now, let $\sigma$ be the involution of $G \times G$ defined by $\sigma(x, y) = (y, x)$. Note that the diagonal subgroup $\Delta(G)$ of $G \times G$ is the subgroup of fixed points, while $T \times T$ is a $\sigma$-stable maximal torus of $G \times G$ and $B \times B^-$ is a Borel subgroup having the property that $\sigma(\alpha) \in -R^+(B \times B^-)$ for every $\alpha \in R^+(B \times B^-)$.

Let $\overline{G}$ denote the wonderful compactification of the group $G$, where $G$ is identified with the symmetric space $(G \times G)/\Delta(G)$ (see [DP, p. 14, 3.1. THEOREM]). Let $\overline{T}$ be the closure of $T$ in $\overline{G}$.

3. The connected component of the automorphism group

Recall that if $X$ is a smooth projective variety over $\mathbb{C}$, the connected component of the group of all automorphisms of $X$ containing the identity automorphism is an algebraic group (see [MO, p. 17, Theorem 3.7] and [Gr, p. 268]), which deal also the case when $X$ may be singular or it may be defined over any field). Further, the Lie algebra of this automorphism group is isomorphic to the space of all vector fields on $X$, that is the space $H^0(X, \Theta_X)$ of all global sections of the tangent bundle $\Theta_X$ of $X$ (see [MO, p. 13, Lemma 3.4]).

Let $\text{Aut}(\overline{T})$ denote the group of all algebraic automorphisms of the variety $\overline{T}$. Let

$$\text{Aut}^0(\overline{T}) \subset \text{Aut}(\overline{T})$$
be the connected component containing the identity element. We note that \(\text{Aut}^0(\overline{T})\) is an algebraic group with Lie algebra \(H^0(\overline{T}, \Theta_{\overline{T}})\), where \(\Theta_{\overline{T}}\) is the tangent bundle of the variety \(\overline{T}\); the Lie algebra structure on \(H^0(\overline{T}, \Theta_{\overline{T}})\) is given by the Lie bracket of vector fields.

The subvariety \(\overline{T} \subset \overline{G}\) is stable under the action of \(T \times T\). Further, the subgroup \(T \times 1 \subset T \times T\) acts faithfully on \(\overline{T}\), and \(T \subset \overline{T}\) is a stable Zariski open dense subset for this action of \(T\). Hence, we get an injective homomorphism

\[
\rho : T \longrightarrow \text{Aut}^0(\overline{T}).
\]

**Theorem 3.1.** The above homomorphism \(\rho\) is an isomorphism.

*Proof.* We know that \(T\) is a maximal torus of \(\text{Aut}^0(\overline{T})\) \([\text{De}]\) p. 521, COROLLAIRE 1]. Choose a Borel subgroup \(B' \subset \text{Aut}^0(\overline{T})\) containing the maximal torus \(T\) of \(\text{Aut}^0(\overline{T})\). The action of \(B'\) on \(\overline{T}\) fixes a point in because \(\overline{T}\) is a projective variety (see \([\text{Hu2}]\) p. 134, 21.2, Theorem]). Let \(x \in \overline{T}\) be a point fixed by \(B'\). Clearly, \(n\overline{T}n^{-1} = \overline{T}\) for \(n \in N_G(T)\), and the diagonal subgroup of \(T \times T\) acts trivially on \(\overline{T}\). Hence \(W = N_G(T)/T\) is a subgroup of \(\text{Aut}(\overline{T})\). The diagonal subgroup of \(T \times T\) acts trivially on \(\overline{T}\). So we see that \(T \times T\) fixes the point \(x\). Therefore, by \([\text{BJ}]\) p. 477, (1.2.7) and \([\text{BJ}]\) p. 478, (1.3.8) we have that \(x = w(z)\) for some \(w \in W\), where \(z\) is the unique \(B \times B^{-}\) fixed point in \(\overline{G}\). Using conjugation by \(w^{-1}\), we may assume that \(B'\) fixes \(z\). Let

\[Q \subset \text{Aut}^0(\overline{T})\]

be the stabilizer subgroup for the point \(z\). As \(B' \subset Q\), it follows that \(Q\) is in fact a parabolic subgroup of \(\text{Aut}^0(\overline{T})\).

We first show that \(\text{Aut}^0(\overline{T})\) is reductive. Let \(R_u\) be the unipotent radical of \(\text{Aut}^0(\overline{T})\). Therefore, \(R_u\) is also the unipotent radical of \(\text{Aut}(\overline{T})\). Hence \(wR_uw^{-1} = R_u\) for all \(w \in W\). Consequently, \(R_u \subset B'\) fixes \(w(z)\) for every \(w \in W\).

For \(\chi \in X(B) = X(T)\), let \(L_{\chi}\) be the line bundle on \(\overline{G}\) associated to \(\chi\) (see \([\text{DP}]\) p. 26, 8.1, PROPOSITION]). Take any \(w \in W\). The action of \(R_u\) fixes \(w(z)\), so the fiber \((L_{\chi})_{w(z)}\) of \(L_{\chi}\) over \(w(z)\) is an one dimensional representation of \(R_u\). This \(R_u\)-module \((L_{\chi})_{w(z)}\) is trivial because the group \(R_u\) is unipotent.

Let \(\mathbb{C}[T]\) be the coordinate ring of the affine algebraic group \(T\). We note that \(\mathbb{C}[T]\) is a unique factorization domain, and therefore any line bundle on \(T\) is trivial. As \(T \subset \overline{T}\) is a \(T\) stable open dense subset for the left translation action, we see that the \(T\) module \(H^0(\overline{T}, L_{\chi})\) is a submodule of \(\mathbb{C}[T]\). If \(\chi\) is a dominant character of \(T\), and \(w \in W\), then the weight space \(\mathbb{C}[T]\) of weight \(-w(\chi)\) is one dimensional and spanned by \(t^{-w(\chi)}\). Moreover, we have \(t^{-w(\chi)} \in H^0(\overline{T}, L_{\chi})\), because it is the unique section of weight \(-w(\chi)\) not vanishing at \(w(z)\). Thus, from the above it follows that \(t^{-w(\chi)}\) is fixed by \(R_u\) for every dominant character \(\chi\) of \(T\) and every \(w \in W\).
The set \( \{ t^x \mid \chi \in X(T) \} \) is a basis for the complex vector space \( \mathbb{C}[T] \). Therefore, the action of \( R_u \) on \( H^0(\overline{T}, \mathcal{L}_\chi) \) is trivial for every regular dominant character \( \chi \) of \( T \). We have \( \overline{T} \subset \mathbb{P}(H^0(\overline{T}, \mathcal{L}_\chi)) \), and hence it follows that the action of \( R_u \) on \( \overline{T} \) is trivial, implying that \( R_u \) is trivial. Thus, the group \( \text{Aut}^0(\overline{T}) \) is reductive.

Next we will show that \( Q = B' \). Fix a dominant character \( \chi \) of \( T \subset B \). As \( \text{Aut}^0(\overline{T}) \) is reductive \( \text{Aut}^0(\overline{T})/Z \) is semisimple, where \( Z(\subset Q) \) is the center of \( \text{Aut}^0(\overline{T}) \). Note that \( Q/Z \) is a parabolic subgroup of \( \text{Aut}^0(\overline{T})/Z \) and it fixes \( z \), by the arguments in the proofs of [KKV] p. 81–82, 3.2, Proposition and 3.3, Corollary] there is a positive integer \( a \) such that \( Q/Z \) acts linearly on the fiber of the line bundle \( \mathcal{L}_a^\chi \) over \( z \) through some character \( \chi' \) of \( Q/Z \). Pulling back \( \chi' \) to \( Q \) we see that \( Q \) acts on the fiber the line bundle \( \mathcal{L}_a^\chi \) over \( z \) by a character. The group \( X(T) \) is finitely generated and Abelian, and hence the image of the restriction map

\[
X(Q) \longrightarrow X(B') = X(T)
\]

is of finite index. This implies that the rank of \( X(Q) \) is equal to the the rank of \( X(B') \). Thus, we have \( Q = B' \).

We will now show that \( \text{Aut}^0(\overline{T}) \) is not semisimple. If \( \text{Aut}^0(\overline{T}) \) is semisimple, then

\[
\dim \overline{T} = \dim T \leq \dim B'_u = \dim(\text{Aut}^0(\overline{T})/B'), \tag{1}
\]

where \( B'_u \) is the unipotent radical of \( B' \). Note that by the above observation, \( B' \) is the stabilizer of \( z \) in \( \text{Aut}^0(\overline{T}) \). Since \( B' \) is a Borel subgroup of \( \text{Aut}^0(\overline{T}) \), \( \text{Aut}^0(\overline{T})/B' \) is a closed subvariety of \( \overline{T} \). Thus from (1) we get that \( \text{Aut}^0(\overline{T})/B' = \overline{T} \). This implies that \( \overline{T} = (\mathbb{P}^1)^n \), and \( \text{Lie}(\text{Aut}^0(\overline{T})) = \mathfrak{sl}(2, \mathbb{C})^n \), where \( n = \dim T \). The \( T \)-fixed points of \( (\mathbb{P}^1)^n \) are indexed by the elements of the Weyl group of \( \text{PSL}(2, \mathbb{C})^n \). Therefore, \( \overline{T} \) has \( 2^n \) fixed points for the action of \( T \). On the other hand, by [BJ] p. 477, (1.2.7) and p. 478, (1.3.8)], all \( w(z) \in \overline{T}, w \in W \), are fixed by \( T \), and \( w'(z) = w(z) \) only if \( w' = w \). Consequently, the order of \( W \) is at most \( 2^n \). As \( n = \dim T \), this is possible only if \( W = S_2^n \). Hence it follows that \( G = \text{PSL}(2, \mathbb{C})^n \). But this contradicts the assumption that \( G \) is simple of rank \( n \geq 2 \). So \( \text{Aut}^0(\overline{T}) \) is not semisimple.

The group \( \text{Aut}^0(\overline{T}) \) is reductive but not semisimple, and this implies that the connected component \( Z^0 \), containing the identity element, of the center of \( \text{Aut}^0(\overline{T}) \) is a positive dimensional sub-torus of \( T \). Further, since

\[
w\text{Aut}^0(\overline{T})w^{-1} = \text{Aut}^0(\overline{T}),
\]

it follows that \( wZ^0w^{-1} = Z^0 \) for every \( w \in W \). Thus, the restriction map

\[
r : X(T) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow X(Z^0) \otimes_{\mathbb{Z}} \mathbb{R} \tag{2}
\]

is a nonzero homomorphism of \( W \) modules. Note that \( X(T) \otimes \mathbb{R} \) is an irreducible \( W \) module (this is because \( G \) is simple). So we conclude that the homomorphism \( r \) in (2) is an isomorphism. Consequently, we have \( T = Z^0 \) and \( T = \text{Aut}^0(\overline{T}) \).
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