ON THE SCHWARTZ SPACE ISOMORPHISM THEOREM FOR THE
RIEMANNIAN SYMMETRIC SPACES

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Abstract. We deduce a proof of the isomorphism theorem for certain closed subspace $S_p^\Gamma(X)$ of the $L^p$-Schwartz class functions ($0 < p \leq 2$) on a Riemannian symmetric space $X$ where $\Gamma$ is a finite subset of $\hat{K}_M$. The Fourier transform considered is the Helgason Fourier transform. Our proof relies only on the Paley-Wiener theorem for the corresponding class of functions and hence it does not use the complicated higher asymptotics of the elementary spherical functions.

Introduction

Let $X$ be a Riemannian symmetric space realized as $G/K$, where $G$ is a connected, non-compact semisimple Lie group with finite center. Let us fix $K$ is a maximal compact subgroup of $G$. The $L^p$-Schwartz space isomorphism theorem for bi-$K$-invariant functions on the group $G$ under the spherical transform was first proved by Harish-Chandra [HC58a, HC58b, HC66] (for $p = 2$), Trombi and Varadarajan [TV71] (for $0 < p < 2$). Recently Anker [Ank91] gave a remarkable short and elegant proof of the above theorem for $0 < p \leq 2$. Anker’s work does not involve the asymptotic expansion of the elementary spherical functions which has a crucial role in the earlier works. The aim of this paper is to extend Anker’s technique for the $L^p$-Schwartz class functions (for $0 < p \leq 2$) on $X$ and to establish an isomorphism theorem under the Helgason fourier transform (HFT).

The main result of this paper is developed in two theorems Theorem 3.8 and Theorem 5.3. In Theorem 3.8 the basic $L^p$-Schwartz space $S^\delta_0(X)$ is the space of operator valued left-$\delta$-type ($\delta \in \hat{K}_M$), smooth functions on $X$. HFT when restricted to $S^\delta_0(X)$ is identified with the ‘$\delta$-spherical transform’. In Theorem 3.8 we establish an isomorphism between the Schwartz spaces $S^\delta_0(X)$ and $S_\delta(\alpha^*_c)$ under the $\delta$-spherical transform. The image $S_\delta(\alpha^*_c)$ is a space of matrix valued functions with certain decay defined on a closed complex tube $\alpha^*_c = a^* + iC^{c,p}$. Explicit definition of the space $S_\delta(\alpha^*_c)$ and the domain $\alpha^*_c$ will be given in the next section. Restriction of Theorem 3.8 to the rank-one case is a part of the result of Eguchi and Kowata [EK76]. Theorem 3.8 also relaxes the rank restriction of the similar result obtained in [JS07].

Theorem 5.3 further extends the isomorphism obtained in Theorem 3.8 to the space $S^\delta_L(X)$ of scalar valued $K$-finite Schwartz class functions on $X$ for which the left $K$-types lie in a fixed finite subset $\Gamma \subset \hat{K}_M$. The transform considered in Theorem 5.3 is the HFT.

We shall closely follow the notations of [Hel, Hel94]. Some basic definitions and results used in this paper are given in the following section.

Key words and phrases. Schwartz spaces, $\delta$-spherical transform, Helgason Fourier transform.

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1. Notation and Preliminaries

Let $G$ be a connected, noncompact semisimple Lie group with finite center and $K$ be a maximal compact subgroup of $G$. Let $\theta$ be the Cartan involution corresponding to $K$. Let $X$ be a Riemannian symmetric space realized as $G/K$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}_C$ be its complexification. The Hermitian norm of both $\mathfrak{g}$ and $\mathfrak{g}_C$ will be denoted by the notation $\| \cdot \|$. We denote $\mathfrak{k}$ for the Lie algebra of the maximal compact subgroup $K$ of $G$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}$ be the Cartan decomposition of the Lie algebra. We fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{s}$. Let $A$ be analytic subgroup of $G$ with the Lie algebra $\mathfrak{a}$. Let $\mathfrak{a}^+$ and $\mathfrak{a}_C^+$ respectively be the real dual of $\mathfrak{a}$ and its complexification. The Killing form induces a scalar product on $\mathfrak{a}$ and hence on $\mathfrak{a}^+$. We shall denote $\langle \cdot, \cdot \rangle_1$ for the $\mathbb{C}$-bilinear extension of that scalar product to $\mathfrak{a}_C^+$.

A semisimple Lie group $G$ is said to be of ‘real rank-$n$’ if $\dim \mathfrak{a} = n$ and the corresponding symmetric spaces $X$ realized as $X = G/K$ are called ‘rank-$n$’ symmetric spaces.

Let $\Sigma$ be the root system associated with the pair $(\mathfrak{g}, \mathfrak{a})$. For each $\alpha \in \Sigma$ we write $\mathfrak{g}_\alpha$ for the corresponding root space. Let $M'$ and $M$ respectively be the normalizer and the centralizer of $A$ in $K$. The quotient $W = M'/M$ be the Weyl group associated with the root system $\Sigma$.

Let us choose and fix a system of positive roots which we denote by $\Sigma^+$. Let $\mathfrak{a}^+$ be the corresponding positive Weyl chamber and $\mathfrak{a}^+$ be its closer. We denote $\mathfrak{a}^{*+}$ and $\mathfrak{a}^{*+}$ for the similar chambers in $\mathfrak{a}^*$. We put $A^+ = \exp \mathfrak{a}^+$ and $A^F = \exp \mathfrak{a}^+$. The element $\rho \in \mathfrak{a}^*$ is denoted by

$$\rho(H) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(H), \quad \text{where } m_\alpha = \dim \mathfrak{g}_\alpha \text{ and } H \in \mathfrak{a}. \tag{1.1}$$

Let $\Sigma_0 \subset \Sigma$ be the set of all indivisible roots and $\Sigma_0^+ = \Sigma_0 \cap \Sigma^+$. $\mathfrak{n} = \oplus_{\alpha \in \Sigma_0^+} \mathfrak{g}_\alpha$ is a nilpotent subalgebra of $\mathfrak{g}$. Let $N$ be the nilpotent subgroup of $G$ with the Lie algebra $\mathfrak{n}$. The Iwasawa decomposition of the group $G$ is given as $G = KAN$. The map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K \times A \times N$ onto $G$. Let $\mathcal{K} : G \to \mathfrak{a}$ and $A : G \to \mathfrak{a}$ are the $\mathfrak{a}$ projections of $g \in G$ in Iwasawa $KAN$ and $N\mathcal{K}$ decompositions respectively. These two projections are related by $A(g) = -\mathcal{K}(g^{-1})$ for each $g \in G$. Any element $g \in G$ can therefore be written as $g = k \exp \mathcal{K}(g)n = n_1 \exp A(g)k_1$. In the Iwasawa $KAN$ decomposition the Haar measure of the group $G$ is given by

$$\int_G f(g) dg = \text{const.} \int_K dk \int_{\mathfrak{a}^{*+}} d\mathcal{K}(g) \int_N dn \ f(k \exp \mathcal{K}(g)n), \tag{1.2}$$

where const. is a normalizing constant. The ‘Cartan decomposition’ of the group is $G = K\mathcal{K}K = K \exp \mathfrak{a}^+K$. Let $g^+$ denote the $\mathfrak{a}^+$ component of $g \in G$, and we denote $|g| = \|g^+\|$. We have a basic estimate: for some constant $c > 0$

$$\|\mathcal{K}(g)\| \leq c|g|, \quad \text{for each } g \in G. \tag{1.3}$$

The Haar measure for the Cartan decomposition is given by

$$\int_G f(g) dg = \text{const.} \int_K dk \int_{\mathfrak{a}^{*+}} d\Delta(H)dh \int_K dk' f(k \exp Hk'), \quad (H \in \mathfrak{a}^+) \tag{1.4}$$

where $\Delta(H) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha} H$ and const. is a positive normalizing constant. We shall be using the following estimate for the density $\Delta(H)$:

$$0 \leq \Delta(H) \leq ce^{2\rho(H)}, \quad \text{for } H \in \mathfrak{a}^+. \tag{1.5}$$

A function $f$ on $G$ is said to be ‘bi-$K$-invariant’ if $f(k_1gk_2) = f(g)$ for all $k_1, k_2 \in K$ and $g \in G$. We refer a function a function as ‘right-$K$-invariant’ if invariant under the right $K$
action on $G$ that is $f(gk) = f(g)$ for all $k \in K, g \in G$. Although in this paper we shall consider a function on the symmetric space $X = G/K$ as a right-$K$-invariant function on the group $G$. For any function space $\mathcal{F}(G)$ on $G$ or $\mathcal{F}(G/\bar{K})$ on $X$, we shall denote $\mathcal{F}(G//K)$ for the corresponding subspace of bi-$K$-invariant functions.

We denote $C^\infty(G)$ for the set of all smooth functions on $G$. We fix a basis $\{X_j\}$ for the Lie algebra $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{g})$ be the ‘universal enveloping algebra’ over $\mathfrak{g}$. Let $D_1 \cdots D_m, E_1 \cdots E_n \in \mathcal{U}(\mathfrak{g})$, then the action of $\mathcal{U}(\mathfrak{g})$ on a function $f \in C^\infty(G)$ is defined as follows:

$$f(D_1 \cdots D_m, x, E_1 \cdots E_n) =$$

$$\frac{d}{dt_1} |_{t_1=0} \cdots \frac{d}{dt_m} |_{t_m=0} \frac{d}{ds} |_{s_n=0} f(\exp(t_1 D_1) \cdots \exp(t_m D_m) x \exp(s_1 E_1) \cdots \exp(s_n E_n)).$$

Let $b_{ij} = \mathfrak{B}(X_i, X_j)$ and $(b^{ij})$ be the inverse of the matrix $(b_{ij})$. We now define a distinguished element, called the ‘Casimir element’, of $\mathcal{U}(\mathfrak{g})$ by $\Omega = \sum_{i,j} b^{ij} X_i X_j$. The differential operator $\Omega$ lies in the center of $\mathcal{U}(\mathfrak{g})$. The action of the ‘Laplace-Beltrami operator’ $\mathbf{L}$ on $X$ is defined by the action of $\Omega$:

$$(1.6) \quad \mathbf{L}f(xK) = f(x, \Omega), \quad x \in G.$$ 

Let us briefly describe the method of construction of a family of rank-one symmetric spaces, the rank-one reductions, which are totally geodesic submanifolds of the general rank symmetric space $G/K$ (see [Hel01, Ch. IX, §2], [Hel, Ch. IV, §6], [Kna03]. Let $\beta$ be an indivisible root of the system $\Sigma$ and $\mathfrak{g}_\beta$, $\mathfrak{g}_2\beta$, $\mathfrak{g}_{-\beta}$ and $\mathfrak{g}_{-2\beta}$. The subalgebra $\mathfrak{g}_\beta$ is stable under the Cartan involution and it is simple. Let $G_{\beta}(\mathfrak{g})$ be the analytic subgroup of $G$ corresponding to the Lie subalgebra $\mathfrak{g}_\beta$. The Iwasawa decomposition of $G(\beta)$ is $G(\beta) = K(\beta)A(\beta) N(\beta)$ where $K(\beta) = K \cap G(\beta)$, $A(\beta) = A \cap G(\beta)$ and $N(\beta) = N \cap G(\beta)$. Also the centralizer of $A(\beta)$ in $K(\beta)$ is $M(\beta) = M \cap G(\beta)$. The abelian subgroup $A(\beta)$ is one-dimensional and its Lie algebra $\mathfrak{a}_\beta$ is generated by the element $H_0$ in $\mathfrak{a}$ determined by $\lambda(H_0) = \langle \lambda, \beta \rangle$. Hence $G(\beta)$ is of real-rank-one and consequently $G(\beta)/K(\beta)$ is a rank-one symmetric space.

The restricted roots of $G(\beta)$ are $\{\beta, 2\beta, -\beta, -2\beta\}$ or $\{\beta, -\beta\}$ according as $2\beta \in \Sigma$ or not. Let us consider $\beta$ to be a positive root of $G(\beta)$ thus the Lie algebra $\mathfrak{n}_\beta$ of $N(\beta)$ is the sum of the root spaces $\mathfrak{g}_\beta$ and $\mathfrak{g}_{2\beta}$. We write $\rho_\beta$ for the $\rho$-function of $\mathfrak{g}_\beta$. It can be shown that [Hel, B.3, page: 483] $\rho(H_0) \geq \rho(\beta)(H_0)$ for all $\beta \in \Sigma_0^+$. The equality holds only when $\beta$ is simple and in that case $\rho_\beta$ is exactly the restriction of $\rho$ to $\mathfrak{g}_\beta$. For each $\beta \in \Sigma_0$ the restriction $\lambda_\beta$ of $\lambda \in \mathfrak{a}_C^*$ to $\mathfrak{a}_\beta$ is given by the expression $\lambda_\beta = \langle \lambda, \beta \rangle \beta$.

Let $\pi_\lambda (\lambda \in \mathfrak{a}_C^*)$ be the spherical principal series representations of $G$ realized on the Hilbert space $L^2(K/M)$ and given by the following:

$$(1.7) \quad \{\pi_\lambda(g) \zeta\} (kM) = e^{i(\lambda - \rho)\zeta(g^{-1}k)\zeta(K(g^{-1}k)M)},$$

where $\lambda \in \mathfrak{a}_C^*$, $\zeta \in L^2(K/M)$ and $K(g^{-1}k)$ denotes the $K$ part of $g^{-1}k$ in the Iwasawa $KAN$ decomposition. The spherical functions $\varphi_\lambda(\cdot)$, given by the following formula

$$(1.8) \quad \varphi_\lambda(g) = \int_K e^{i(\lambda - \rho)\zeta(g^{-1}k)} dk, \quad \text{where } g \in G, \lambda \in \mathfrak{a}_C^*,$$

are the matrix coefficient of the principal series representations. For each $g \in G$ and $\omega \in W \varphi_{\omega, \lambda}(g) = \varphi_\lambda(g)$. We shall use the following basic estimates for the elementary spherical functions for our purpose.
(i) For each \( H \in \mathfrak{a}^\| \) and \( \lambda \in \mathfrak{a}^* \), we have
\[
0 < |\varphi_{-\lambda}(\exp H)| \leq e^{\lambda(H)} \varphi_0(\exp H),
\]
here, \( \varphi_0(\cdot) \) is the spherical function corresponding to \( \lambda = 0 \). For a proof of the above estimate see [GV88], Proposition 4.6.1.

(ii) For all \( g \in G \), \( 0 < \varphi_0(g) \leq 1 \) [GV88, Proposition 4.6.3]. Also for \( \mathfrak{a}^\| \) we have the following estimate
\[
e^{\rho(H)} \leq \varphi_0(\exp H) \leq \beta(1 + \|H\|^a) e^{\rho(H)},
\]
where, \( \beta, c_a > 0 \) are group dependent constants. This is an work of Harish-Chandra, the above optimal estimate was obtained by Anker [Ank87].

Let \( \mathcal{D}(G) \) be the subspace of \( \mathcal{C}^\infty(G) \) generated by the compactly supported scalar valued smooth functions on \( G \). For each \( f \in \mathcal{D}(G//K) \) the ‘spherical Fourier transform’ \( \mathcal{S}f \) is defined by
\[
\mathcal{S}f(\lambda) = \int_G f(x) \varphi_{-\lambda}(x) dx, \quad \lambda \in \mathfrak{a}^*_C.
\]
The inversion of the spherical transform is given by
\[
f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*_C} \varphi_\lambda(x) \mathcal{S}f(\lambda) |c(\lambda)|^{-2} d\lambda,
\]
where \( |W| \) is the cardinality of the group \( W \) and \( c(\lambda) \) is the ‘Harish-Chandra c-function’. For our purpose we shall use the following estimate [Ank92]: there exists constants \( a, b > 0 \) such that
\[
|c(\lambda)|^{-2} \leq a(\|\lambda\| + 1)^b, \quad \text{for all} \ \lambda \in \mathfrak{a}^*.
\]

For \( f \in \mathcal{D}(X) \), the Helgason Fourier transform (HFT) \( \mathcal{F}f \) is a function on \( \mathfrak{a}^*_C \times K/M \) and it is defined by ( [Hel94], Ch. III, § 1)
\[
\mathcal{F}f(\lambda, kM) = \int_G f(x) e^{i(\lambda-\rho)(H(x, kM))} dx
\]
where the function \( H : G \times K/M \mapsto \mathfrak{a} \) is given by \( H(x, kM) = \mathcal{H}(x^{-1}k) \). For the sake of simplicity we fix the notational convention \( \mathcal{F}f(\lambda, kM) = \mathcal{F}f(\lambda, k) \). We should note that, for a bi-\( K \)-invariant function (that is a left-\( K \)-invariant function) \( g \) on \( X \), the HFT reduces to the spherical transform: \( \mathcal{F}g(\lambda, k) = \mathcal{F}g(\lambda, e) = \mathcal{S}g(\lambda) \).

The inversion formula for HFT [Hel94, Ch.-III, Theorem 1.3] for \( f \in \mathcal{D}(X) \) is as follows:
\[
f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \int_{K} \mathcal{F}f(\lambda, k) e^{-i(\lambda+\rho)(\mathcal{H}(x^{-1}k))} |c(\lambda)|^{-2} d\lambda dk.
\]
Let \( \delta \) be a unitary irreducible representation of \( K \) realized on a finite dimensional vector space \( V_\delta \) with an inner product \( \langle \cdot, \cdot \rangle \). Let us denote \( \dim V_\delta = d_\delta \). We denote by \( \widehat{K} \) the set of equivalence classes of unitary irreducible representations of \( K \) and by customary abuse of notation regard each element of \( \widehat{K} \) as a representation from its equivalence class. For each \( \delta \in \widehat{K} \), let \( \chi_\delta \) stand for the character of the representation \( \delta \) and \( V_\delta^M = \{ v \in V_\delta \mid \delta(m)v = v \text{ for all } m \in M \} \) is the subspace of \( V_\delta \) fixed under \( \delta|_M \). Let \( \widehat{K}_M \) stands for the subset of \( \widehat{K} \) consisting of \( \delta \) for which \( V_\delta^M \neq \{0\} \) and we will mostly be interested in representations \( \delta \in \widehat{K}_M \). We set an orthogonal basis \( \{v_j\}_{1 \leq j \leq d_\delta} \) of \( V_\delta \) and we assume that \( \{v_1, \cdots v_{\ell_\delta}\} \) generates \( V_\delta^M \) where \( \dim V_\delta^M = \ell_\delta \).
We also define a norm for each unitary irreducible representation of $K$. Let $\Theta$ be the restriction of the Cartan-Killing form $\mathcal{B}$ to $\mathfrak{k} \times \mathfrak{k}$. Let $X_1, ..., X_r$ be a basis for $\mathfrak{k}$ over $\mathbb{R}$ orthonormal with respect to $\Theta$. Let $\omega_k = -(X_1^2 + ... + X_r^2)$ be the Casimir element of $K$. Clearly, $\omega_k$ is a differential operator which commutes with both left and right translations of $K$. Thus $\delta(\omega_k)$ commutes with $\delta(k)$ for all $k \in K$. Hence by Schur’s lemma $|\delta(\omega_k)|^2 = c(\delta)$ where $c(\delta) \in \mathbb{C}$. As $\delta(X_i)$ \((1 \leq i \leq r)\) are skew-adjoint operators, $c(\delta)$ is real and $c(\delta) \geq 0$. We define $|\delta|^2 = c(\delta)$, for $\delta \in \hat{K}_M$. As, $\delta \in \hat{K}_M$, $\delta(k)$ is a unitary matrix of order $d_{\delta} \times d_{\delta}$. So $|\delta(k)|_2 = \sqrt{d_{\delta}}$ where $\|\cdot\|_2$ denotes the Hilbert Schmidt norm. Also, from Weyl’s dimension formula we can choose an $r \in \mathbb{Z}^+$ and a positive constant $c$ independent of $\delta$ such that $|\delta(k)|_2 \leq c(1 + |\delta|)^r$ for all $k \in K$. Thus, $d_{\delta} \leq c'(1 + |\delta|)^{2r}$ with $c' > 0$ independent of $\delta$.

For any $f \in \mathcal{C}^\infty(X)$ we put:

\[
(1.16) \quad f^\delta(x) = d_{\delta} \int_K f(kx)\delta(k^{-1})dk.
\]

Clearly, $f^\delta$ is a $\mathcal{C}^\infty$ map from $X$ to $Hom(V_\delta, V_\delta)$ satisfying

\[
(1.17) \quad f^\delta(kx) = \delta(k)f^\delta(x), \text{ for all } x \in X, k \in K.
\]

Any function satisfying the property (1.17) will be referred to as (a $d_\delta \times d_\delta$ matrix valued) left $\delta$-type function. For any function space $\mathcal{E}(X) \subseteq \mathcal{C}^\infty(X)$, we write $\mathcal{E}_\delta(X) = \{f^\delta \mid f \in \mathcal{E}(X)\}$.

We shall denote by $\tilde{\delta}$ the contragradient representation of the representation $\delta \in \hat{K}_M$, and a function $f$ will be called a scalar valued left $\delta$-type function if $f \equiv d_{\delta}^{\chi_\delta} \ast f$, where the operation $\ast$ is the convolution over $K$. For any class of scalar valued functions $\mathcal{S}(X)$ we shall denote

\[
\mathcal{S}(\delta, X) = \{g \in \mathcal{S}(X) \mid g \equiv d_{\delta}^{\chi_\delta} \ast g\}.
\]

The following theorem, due to Helgason, identifies the two classes $\mathcal{D}_\delta(X)$ and $\mathcal{D}(\tilde{\delta}, X)$ corresponding to each $\delta \in \hat{K}_M$.

**Proposition 1.1.** [Helgason [Hel94, Ch.III, Proposition 5.10]]

The map $Q : f \mapsto g$, $g(x) = tr(f(x))$ \((x \in X)\) is a homeomorphism from $\mathcal{D}_\delta(X)$ onto $\mathcal{D}(\tilde{\delta}, X)$ and its inverse is given by $g \mapsto f = g^\delta$.

**Remark 1.2.** For each $\delta \in \hat{K}_M$, the space $\mathcal{D}(X, Hom(V_\delta, V_\delta))$ of $\mathcal{C}^\infty$ functions on $X$ taking values in $Hom(V_\delta, V_\delta)$, carries the inductive limit topology of the Fréchet spaces

\[
\mathcal{D}^R(X, Hom(V_\delta, V_\delta)) = \{F \in \mathcal{D}(X, Hom(V_\delta, V_\delta)) \mid \text{supp} F \subseteq \overline{B^R(0)}\},
\]

for $R = 0, 1, 2, \cdots$. As $\mathcal{D}(\tilde{\delta}, X) \subset \mathcal{D}(X)$, so the natural topology of $\mathcal{D}(\tilde{\delta}, X)$ is the inherited subspace topology.

A consequence of the Peter-Weyl theorem can be stated [Hel, Ch.IV, Corollary 3.4] in the form that any $f \in \mathcal{C}^\infty(X)$ has the decomposition

\[
(1.18) \quad f(x) = \sum_{\delta \in \hat{K}_M} tr(f^\delta(x)).
\]

A function $f \in \mathcal{C}^\infty(X)$ is said to be ‘left-$K$ finite’ if there exists a finite subset $\Gamma(f) \subset \hat{K}_M$ (depending on the function $f$) such that $tr(f^\delta(\gamma)) \equiv 0$ for all $\gamma \in \hat{K}_M \setminus \Gamma(f)$. For any class $\mathcal{S}(X) \subseteq \mathcal{C}^\infty(X)$ of function we shall denote $\mathcal{S}(X)_K$ for its left $K$ finite subclass. Let $\Gamma$ be a fixed subset (finite or infinite) of $\hat{K}_M$. Then we shall use the notation $\mathcal{S}_{\Gamma}(X)$ for the subclass...
of $\mathcal{H}(X)$

\begin{equation}
\mathcal{H}_I(X) = \{ g \in \mathcal{H}(X) \mid g^\delta(\cdot) \equiv 0, \text{ for all } \delta \in \hat{K}_M \setminus \Gamma \}.
\end{equation}

For each $f \in \mathcal{D}(X)$ and $\delta \in \hat{K}_M$, we define the $\delta$ projection of its HFT $\mathcal{F}f$ as follows:

\begin{equation}
(\mathcal{F}f)^\delta(\lambda, k) = d_\delta \int_K \mathcal{F}f(\lambda, k_1 k)\delta(k^{-1}_1)dk_1, \text{ where } \lambda \in \mathfrak{a}_C^* \text{ and } k \in K.
\end{equation}

The HFT $\mathcal{F}(f^\delta)$ of $f^\delta$ is also defined by the formula (1.14), in this case the integration is taken over each matrix entry.

**Lemma 1.3.** For each $f \in \mathcal{D}(X)$ and $\delta \in \hat{K}_M$ the following are true

(i) $(\mathcal{F}f)^\delta(\lambda, k) = \delta(k)(\mathcal{F}f)^\delta(\lambda, e)$,

(ii) $\mathcal{F}(f^\delta)(\lambda, k) = (\mathcal{F}f^\delta)(\lambda, k)$, for all $\lambda \in \mathfrak{a}_C^*$ and $k \in K$.

**Proof.** Part (i) of the Lemma follows trivially from (1.20). Part (ii) can be deduced from the following

\begin{equation}
\mathcal{F}(f^\delta)(\lambda, kM) = \int_X f^\delta(x)e^{i(\lambda-1)H(x^{-1}k)}dx,
\end{equation}

\begin{equation}
= d_\delta \int_X \left\{ \int_K f(k_1 x)\delta(k^{-1}_1)dk_1 \right\} e^{i(\lambda-1)H(x^{-1}k)}dx
\end{equation}

Now the desired result follows from (1.21) by a simple application of the Fubini’s theorem. $\blacksquare$

\section{The $\delta$-spherical transform}

Let us now define the ‘$\delta$-spherical transform’ on $\mathcal{D}_\delta(X)$. Most of the basic analysis was done by Helgason [Hel94] on $\mathcal{D}(\delta, X)$, we shall follow those results closely and prove them on $\mathcal{D}_\delta(X)$ using the homeomorphism $\mathcal{Q}$, defined in Proposition 1.1.

**Definition 2.1.** For $f \in \mathcal{D}_\delta(X)$ the $\delta$-spherical transform $\tilde{f}$ is an operator valued function on $\mathfrak{a}_C^*$ and is given by

\begin{equation}
\tilde{f}(\lambda) = d_\delta \int_G \text{tr}f(x)\Phi_{\lambda,\delta}^*(x)dx
\end{equation}

where for each $\delta \in \hat{K}_M$ and $\lambda \in \mathfrak{a}_C^*$, the function

\begin{equation}
\Phi_{\lambda,\delta}(x) = \int_K e^{-i(\lambda+1)H(x^{-1}k)}\delta(k)dk, \quad x \in G,
\end{equation}

is called the ‘generalized spherical function’ of class $\delta$. For each $x \in G$, $\Phi_{\lambda,\delta}(x)$ is an operator in $\text{Hom}(V_\delta, V_\delta)$. Taking point-wise adjoint leads to the expression

\begin{equation}
\Phi_{\lambda,\delta}^*(x) := \Phi_{\lambda,\delta}^*(x) = \int_K e^{i(\lambda-1)H(x^{-1}k)}\delta(k^{-1})dk, \quad x \in G.
\end{equation}

**Remark 2.2.** From the Iwasawa decomposition, if $x \in G$ and $\tau \in K$, $\mathcal{H}(\tau x) = \mathcal{H}(x)$. Hence, the expressions (2.2) and (2.3) show that both $\Phi_{\lambda,\delta}$ and $\Phi_{\lambda,\delta}^*$ can be considered as functions on the space $X = G/K$.

In the following proposition we list out some basic properties of the generalized spherical functions that we will be using.

**Proposition 2.3.**

(i) For each $x \in X$, the function $\lambda \mapsto \Phi_{\lambda,\delta}(x)$ is holomorphic on $\mathfrak{a}_C^*$. 
(ii) Let $\delta \in \widehat{K}_M$ and $\lambda \in a_C^*$. Then for each $x \in X$ and $k \in K$ we have

$$
\Phi_{\lambda, \delta}(kx) = \delta(k) \Phi_{\lambda, \delta}(x) \quad \text{and} \quad \Phi_{\lambda, \delta}^*(kx) = \Phi_{\lambda, \delta}^*(x) \delta(k^{-1}).
$$

Let $v \in V_\delta$ and $m \in M$ then

$$
\delta(m) \left( \Phi_{\lambda, \delta}^*(x) v \right) = \Phi_{\lambda, \delta}^*(x) v.
$$

(iii) [Helgason [Hel94, Ch.III, Theorem 5.15]] For each $\delta \in \widehat{K}_M$, $\omega \in W$ and for all $\lambda \in a_C^*$, the restrictions $\Phi_{\lambda, \delta}|_A$ and $\Phi_{\lambda, \delta}^*|_A$ satisfy the relations

$$
\Phi_{\lambda, \delta}|_A Q^\delta(\lambda) = \Phi_{\omega \lambda, \delta}|_A Q^\delta(\omega \lambda),
$$

$$
Q^\delta(\lambda)^{-1} \Phi_{\lambda, \delta}|_A = Q^\delta(\omega \lambda)^{-1} \Phi_{\omega \lambda, \delta}|_A
$$

where $Q^\delta(\lambda)$ is a $(\ell_\delta \times \ell_\delta)$ matrix whose entries are certain constant coefficient polynomials in $\lambda \in a_C^*$ (see [Hel94, Ch. III, §2] for details). Furthermore, both sides of

$$
(2.7)
$$

are holomorphic for all $\lambda \in \mathbb{C}$, implying that $\Phi_{\lambda, \delta}|_A$ is divisible by $Q^\delta(\lambda)$ in the ring of entire functions.

(iv) For each fixed $\lambda$ and $\delta$, the function $\Phi_{\lambda, \delta}(x)$ and its adjoint are both joint eigenfunctions of all $G$-invariant differential operators of $X$. Particularly, for the Laplace-Beltrami operator $\mathbf{L}$, the eigenvalues are as follows:

$$
(2.8)
$$

$$
(L \Phi_{\lambda, \delta})(x) = - \left( \langle \lambda, \lambda \rangle_1 + \| \rho \|^2 \right) \Phi_{\lambda, \delta}(x), \quad x \in X.
$$

(v) For each $\delta \in \widehat{K}_M$, the generalized spherical function corresponding to $\delta$ is related with the elementary spherical function by the following differential equation [Helgason, Ch.III, §5, Corollary 5.17]

$$
(2.9)
$$

$$
\Phi_{\lambda, \delta}(gK)|_{V^{\delta}_M} = \left( D^\delta \varphi_\lambda \right)(g) Q^\delta(\lambda)^{-1}
$$

where $D^\delta$ is a differential operator matrix of order $(\ell_\delta \times \ell_\delta)$. Individual matrix entries of $D^\delta$ are certain constant coefficient differential operators on $G$.

(vi) For any $g_1, g_2 \in \mathcal{U}(G_C)$ there exist constants $c = c(g_1, g_2)$, $b = b(g_1, g_2)$ and $c_0 > 0$ so that

$$
(2.10)
$$

$$
\| \Phi_{\lambda, \delta}(g_1, x, g_2) \|_2 \leq c(1 + |\delta|)^b (1 + \| \lambda \|)^b \varphi_\lambda(x) e^{c_0 \| \lambda \| (1 + |x|)}
$$

for all $x \in X$ and $\lambda \in a_C^*$.

**Proof.** Property (2.4) follows trivially from the definition of the generalized spherical function. (2.5) also follows from (2.2) as below:

$$
\delta(m) \left( \Phi_{\lambda, \delta}^*(x) v \right) = \left\{ \int_K e^{(\lambda - 1)\delta(x^{-1}k)} \delta(mk^{-1}) dk \right\} v
$$

$$
= \left\{ \int_K e^{(\lambda - 1)\delta(x^{-1}k')} \delta(k'^{-1}) dk' \right\} v.
$$

The last line follows by a simple change of variable $mk^{-1}$ to $k'^{-1}$. In the last expression above, let $x^{-1}k' = \mathcal{K}(x^{-1}k')(\exp \mathcal{H}(x^{-1}k'))n'$ for some $n' \in N$. As $M$ normalizes $N$ and centralizes $A$ we have

$$
x^{-1}k'm = \mathcal{K}(x^{-1}k')m(\exp \mathcal{H}(x^{-1}k'))n(x^{-1}k').
$$
This shows that $\mathcal{H}(x^{-1}k^\ell) = \mathcal{H}(x^{-1}k^\ell m)$. Thus
\[
\delta(m)\left(\Phi_{\lambda,\delta}(x)v\right) = \left\{ \int_K e^{(i\lambda-1)\mathcal{H}(x^{-1}k^\ell)}\delta(k^{-1})dk^\ell \right\} v = \Phi_{\lambda,\delta}(x)v.
\]

A proof of property (ii) may be found in [Hel94, Ch.III, §1 (6)] and [Hel, Ch.II, Corollary 5.20]. The estimate (2.10) is a work of Arthur [Art79].

**Remark 2.4.** The property (2.5) clearly shows that for each $x \in X$ the operator $\Phi_{\lambda,\delta}(x)$ maps $V_\delta$ to $V^M_\delta$. Hence $\Phi_{\lambda,\delta}(x)$ is a $d_\delta \times d_\delta$ matrix whose only the first $\ell_\delta$ rows can nonzero. Consequently, for each $x \in X$, $\Phi_{\lambda,\delta}(x)$ is a $d_\delta \times d_\delta$ matrix of which only the first $\ell_\delta$ columns can be nonzero. In other words, the operator $\Phi_{\lambda,\delta}(x)$ vanishes identically on the orthogonal complement of the subspace $V^M_\delta$.

**Remark 2.5.** In the case of the rank-one symmetric spaces the Kostant matrix $Q_\delta$ reduces to a constant coefficient polynomial (Kostant polynomial) [Hel94, Theorem 11.2, §11, Ch.III]. The degree of the Kostant polynomials depends on the choice of $\delta \in \hat{K}_M$. Furthermore all the zeros of the Kostant polynomials lie on the open lower half of the imaginary axis.

The general rank analogue of the above result states that $\det Q_\delta(\lambda) \neq 0$ for all $\lambda \in a^* + i\mathbb{R}$. This is an easy consequence of Lemma 2.11 and Proposition 4.1 of [Hel94, Ch.III].

**Lemma 2.6.** If $f \in D_\delta(X)$, where $\delta \in \hat{K}_M$, then $\mathcal{F}f(\lambda, e) = \tilde{f}(\lambda)$ for all $\lambda \in a^*_\mathbb{C}$.

**Proof.** For any $f \in D_\delta(X)$, using the topological isomorphism $Q$ as described in Proposition 1.1, we get $tr f(\cdot) \in D(\delta, X)$ and also $f(x) = d_\delta \int_K tr f(kx)\delta(k^{-1})dk$. Now from the definition of HFT (1.14) we get:

\[
\mathcal{F}f(\lambda, e) = \int_G f(x)e^{(i\lambda-\rho)\mathcal{H}(x^{-1}e)}dx,
\]

\[
= d_\delta \int_G \int_K tr f(kx)\delta(k^{-1})dk e^{(i\lambda-\rho)\mathcal{H}(x^{-1}e)}dx.
\]

(2.11)

A simple application of the Fubini theorem and a substitution $kx = y$ in the integrand of (2.11) gives the following.

\[
\mathcal{F}f(\lambda, e) = d_\delta \int_G tr f(y) \left\{ \int_K e^{(i\lambda-\rho)\mathcal{H}(y^{-1}k)}\delta(k^{-1})dk \right\} dy,
\]

\[
= d_\delta \int_G tr f(y) \Phi_{\lambda,\delta}(y)dy = \tilde{f}(\lambda).
\]

**Lemma 2.7.** Let $f \in D_\delta(X)$, then the inversion formula for the $\delta$-spherical transform (Definition 2.1) is given by:

\[
f(x) = \frac{1}{|W|} \int_{a^*_\mathbb{C}} \Phi_{\lambda,\delta}(x)\tilde{f}(\lambda)|c(\lambda)|^{-2}d\lambda.
\]

Furthermore we get: $\int_G \|f(x)\|^2_{\mathbb{C}}dx = \frac{1}{|W|} \int_{a^*_\mathbb{C}} \|\tilde{f}(\lambda)\|^2_{\mathbb{C}}|c(\lambda)|^{-2}d\lambda$.

**Proof.** The formula (2.12) is derived from the inversion formula (1.15) of the HFT.

\[
f(x) = \frac{1}{|W|} \int_{a^*_\mathbb{C}} \int_K \mathcal{F}f(\lambda, k)e^{-(i\lambda+\rho)(\mathcal{H}(x^{-1}k))}|c(\lambda)|^{-2}dk d\lambda,
\]

\[
= \frac{1}{|W|} \int_{a^*_\mathbb{C}} \int_K \delta(k)\mathcal{F}f(\lambda, e)e^{-(i\lambda+\rho)(\mathcal{H}(x^{-1}k))}|c(\lambda)|^{-2}dk d\lambda,
\]
\[
\begin{align*}
\frac{1}{|W|} \int_{a_*} \left\{ \int_{\mathcal{K}} e^{-(i(\lambda + \rho)(3t(x^{-1}k)))} \delta(k)dk \right\} \tilde{f}(\lambda)|c(\lambda)|^{-2}d\lambda,
= \frac{1}{|W|} \int_{a_*} \Phi_{\lambda,\delta}(x) \tilde{f}(\lambda)|c(\lambda)|^{-2}d\lambda.
\end{align*}
\]

The second and the third line of the above deduction are consequences of Lemma 1.3 and Lemma 2.6 respectively.

The second relation of this Lemma also follows from the Plancherel formula of the HFT ([Hel94], Ch.-III, §1, Theorem 1.5). For \( f \in \mathcal{D}_\delta(X) \), the HFT is defined matrix entry wise, hence in this case the Plancherel formula is given by:

\[
\int_G \|f(x)\|^2 dx = \frac{1}{|W|} \int_{a_*} \int_{\mathcal{K}} \|\mathcal{F}f(\lambda, k)\|^2 |\mathcal{F}c(\lambda)|^{-2} dk d\lambda,
\]

which follows easily from the classical Plancherel formula given in [Hel94]. The required Plancherel formula for the \( \delta \)-spherical transform follows from (2.13) by using relation (i) of Lemma 1.3 and Schur’s Orthogonality relation ([Sug90], Theorem 3.2).

We shall next deduce the deduce a topological Paley-Wiener (P-W) theorem for the \( \delta \)-spherical transform (2.1).

A holomorphic function \( \psi : \mathfrak{a}_C^\delta \rightarrow Hom(V_\delta, V_\delta^M) \) is said to be of ‘exponential type-\( R \)’ \((R \in \mathbb{R}^+)\) if

\[
\sup_{\lambda \in \mathfrak{a}_C^\delta} e^{-R\|3\lambda\|(1 + \|\lambda\|)^N}\|\psi(\lambda)\|_2 < +\infty \quad \text{for each } N \in \mathbb{Z}^+.
\]

Let \( \mathfrak{H}_\delta^C(a^\delta_C) \) be the class of all \( Hom(V_\delta, V_\delta^M) \) valued exponential type-\( R \) functions on \( \mathfrak{a}_C^\delta \) and further let \( \mathfrak{H}_\delta(a^\delta_C) = \bigcup_{R>0} \mathfrak{H}_\delta^R(a^\delta_C) \).

**Theorem 2.8.** For each fixed \( \delta \in \hat{K}_M \), the \( \delta \)-spherical transform given by (2.1) is a homeomorphism between the spaces \( \mathfrak{D}_\delta(X) \) and \( \mathfrak{D}^\delta(a^\delta_C) \), where

\[
\mathfrak{D}^\delta(a^\delta_C) = \left\{ \xi \in \mathfrak{H}_\delta(a^\delta_C) \mid \lambda \mapsto Q^\delta(\lambda)^{-1}\xi(\lambda) \text{ is an } W \text{-invariant entire function} \right\}.
\]

Here \( Q^\delta(\lambda) \) is the matrix of constant coefficient polynomials appeared in the expressions (2.6) and (2.7).

**Proof.** Our proof solely relies on the proof of the topological Paley-Wiener theorem given by Helgason ([Hel94], Ch.-III, Theorem 5.11), where he characterized the image of the space \( \mathfrak{D}(\delta, X) \) under the transform \( f \mapsto \hat{f} \), where

\[
\hat{f}(\lambda) = d_\delta \int_G f(x) \Phi_{\lambda,\delta}(x)dx, \quad (\lambda \in \mathfrak{a}_C^\delta).
\]

Helgason proved that the above transform is a topological isomorphism between the spaces \( \mathfrak{D}(\delta, X) \) and \( \mathfrak{D}^\delta(a^\delta_C) \). From the Proposition 1.1 and the definition (2.1) of the \( \delta \)-spherical transform we get: for each \( f \in \mathfrak{D}_\delta(X) \), \((Q\hat{f})(\lambda) = \hat{f}(\lambda), \quad (\forall \lambda \in \mathfrak{a}_C^\delta) \). The proposition now follows from the fact that both the maps \( Q \) and \( f \mapsto \hat{f} \) are homeomorphisms.

Let us now consider the function space \( \mathfrak{P}^\delta_0(a^\delta_C) = \{ h \in \mathfrak{H}(a^\delta_C) \mid h \text{ is } W \text{-invariant} \} \) with the relative topology. A function \( h \in \mathfrak{P}^\delta_0(a^\delta_C) \) can be written as \( h \equiv (h_{ij})_{\ell_4 \times d_\delta} \) where each of the scalar valued component function \( h_{ij} \) is entire, \( W \)-invariant and of exponential type. Let \( \mathfrak{D}(G//K) \) and \( \mathfrak{D}(G//K, Hom(V_\delta, V_\delta^M)) \) are respectively be the spaces of scalar valued and \( Hom(V_\delta, V_\delta^M) \) valued bi-K-invariant, compactly supported, \( C^\infty \) functions on \( G \). The
operators as mentioned in (Lemma 2.10.\]
\[\text{Lemma 2.9. [Helgason [Hel94, Ch.-III, §5, Lemma 5.12]]} \]
\[\text{The mapping } \psi(\lambda) \mapsto Q^\delta(\lambda)\psi(\lambda) \ (\lambda \in a^*_C) \text{ is a homeomorphism from } P^\delta_0(a^*_C) \text{ onto } P^\delta(\lambda). \]

\[\text{Lemma 2.10. Any function } f \in D_\delta(X) \text{ can be written as } f(x) = \Phi(\lambda) = Q^\delta(\lambda)^{-1}\tilde{f}(\lambda) \text{ in } P^\delta_0(a^*_C). \]

\[\text{Proof. Let } f \in D_\delta(X), \text{ then by Theorem 2.8, its } \delta\text{-spherical transform } \tilde{f} \in P^\delta(a^*_C). \text{ Using the homeomorphism given in Lemma 2.9, we get an unique function } \lambda \mapsto \Phi(\lambda) = Q^\delta(\lambda)^{-1}\tilde{f}(\lambda) \text{ in } P^\delta_0(a^*_C). \]

\[\text{By the PW-theorem for the bi-}\delta\text{-invariant functions we get a function } \phi \in D(G//K, Hom(V_{\delta}, V_{\delta}^M)) \text{ such that:} \]
\[\phi(x) = \frac{1}{|W|} \int_{a^*_C} \varphi_\lambda(x)\Phi(\lambda)|c(\lambda)|^{-2}d\lambda. \]

\[\text{Now by applying the differential operator } D^\delta \text{ on the both sides of (2.16) and by using the absolute convergence of the integral in (2.16) we get:} \]
\[
\left(D^\delta \phi \right)(x) = \frac{1}{|W|} \int_{a^*_C} \left(D^\delta \varphi_\lambda(x)\right)\Phi(\lambda)|c(\lambda)|^{-2}d\lambda,
\]
\[= \frac{1}{|W|} \int_{a^*_C} \varphi_\lambda(x)Q^\delta(\lambda)\Phi(\lambda)|c(\lambda)|^{-2}d\lambda,
\]
\[= \frac{1}{|W|} \int_{a^*_C} \Phi_{\lambda,\delta}(x)\tilde{f}(\lambda)|c(\lambda)|^{-2}d\lambda = f(x). \]

\[\text{The second line of the above calculation is a consequence of (2.9) and the last line is merely the inversion formula (2.12).} \]

We conclude this section with a ‘product formula’, due to Helgason, for the polynomial det \(Q^\delta(\lambda)\) with \(\lambda \in a^*_C\). First we shall characterize the set \(\tilde{K}_{M}^\beta\ (\beta \in \Sigma^+_0)\) of equivalence classes of representations of \(K^\beta\) as certain restrictions of the representations \(\delta \in \tilde{K}_M\). Let \(\delta \in \tilde{K}_M\) and \(V^\delta_\delta\) and \(V^M_\delta\) be as explained earlier. Let \(V^\delta_\delta\) denote the \(K^\beta\)-M-invariant subspace of \(V_\delta\) generated by \(V^\delta_\delta\). Then \(V^\delta_\delta\) decomposes into \(K^\beta\)-M-irreducible subspaces as \(V = \bigoplus_{i=1}^{\ell^\delta} V_i\). Let \(\delta(i, \beta)\) be the representation of \(K^\beta\) on \(V_i\) given by \(\delta\). Then each \(\delta(i, \beta)\) is irreducible except when \(\dim K^\beta = 1\) and \(\text{mkm}^{-1} = k^{-1}\) for all \(k \in K^\beta\) and some \(m \in M\) [Hel94, Ch. III, Lemma 3.9 - 3.11]. In this case \(\delta(i, \beta)\) brakes up into two irreducible one dimensional representations as \(\delta(i, \beta) = \delta(i, \beta)_0 \oplus \delta(i, \beta)_0\) where \(\delta(i, \beta)_0\) is the contragradient representation of \(\delta(i, \beta)_0\) and in this particular case we choose \(\delta(i, \beta)_0\) as \(\delta(i, \beta)\).

\[\text{Lemma 2.11. Helgason [Hel94, Ch. III, Proposition 4.3] For each } \beta \in \Sigma^+_0, \text{ as } \delta \text{ runs through } \tilde{K}_M, \text{ the representations } \delta(i, \beta) \ (1 \leq i \leq \ell) \text{ runs through all of } \tilde{K}_{\beta}^\beta. \]

\[\text{Let } G^\delta_{\beta}(\lambda) \text{ be the Kostant polynomial for the rank-one symmetric space } G^\beta/K^\beta \text{ corresponding to the representation } \delta(i, \beta) \in \tilde{K}_{\beta}^\beta. \text{ Then the determinant of the Kostant matrix} \]
\[ Q^\delta(\lambda) \ (\lambda \in \mathfrak{a}_C^*) \text{ can be represented by a product formula } [\text{Hel94, Ch. III, §3, (50) and §4, Theorem 4.2}] \]

\[
\det Q^\delta(\lambda) = \mathcal{C}_\delta \prod_{\beta \in \Sigma_0^+, 1 \leq i \leq \ell_\delta} Q_{\beta}^{\delta(i,\beta)}(\lambda_\delta), \text{ for all } \lambda \in \mathfrak{a}_C^*
\]

where \( \mathcal{C}_\delta \) is a nonzero constant depending on \( \delta \in \check{K}_M \).

### 3. \( L^p \)-Schwartz spaces

**Definition 3.1.** [Classical \( L^p \) Schwartz space]

A \( C^\infty \) function \( f \) on \( X \) is said to be in the \( L^p \)-Schwartz space \( (0 < p \leq 2) S^p(X) \) if for each nonnegative integer \( n \) and \( D, E \in \mathcal{U}(\mathfrak{g}_C) \) the function \( f \) satisfies the following decay condition:

\[
\mu_{D, E, n}(f) = \sup_{x \in G} |f(D, x, E)| \varphi_0^{-\frac{2}{p}}(x) (1 + |x|)^n < +\infty.
\]

The topology induced by the countable family of seminorms \( \mu_{D, E, n}(\cdot) \) makes \( S^p(X) \) a Fréchet space. It can be shown that \( \mathcal{D}(X) \) is a dense subspace of \( S^p(X) \) for \( 0 < p \leq 2 \). Let \( f \in S^p(X) \), we take its \( \delta \)-projection \( f^\delta \) as defined in (1.16). Then \( f^\delta \) is a left \( \delta \)-type \( \text{Hom}(V_\delta, V_\delta) \) valued function with a decay

\[
\mu_{D, E, n}(f^\delta) = \sup_{x \in G} \|f^\delta(D, x, E)\|_2 \varphi_0^{-\frac{2}{p}}(x) (1 + |x|)^n < +\infty,
\]

where \( \| \cdot \|_2 \) denotes the Hilbert Schmidt norm. Let us denote \( S^p_\delta(X) \) for the class of left \( \delta \)-type \( \text{Hom}(V_\delta, V_\delta) \) valued functions with the decay (3.2). For each \( h \in S^p_\delta(X) \), the scalar valued function \( trh(\cdot) \) satisfies \( trh \equiv d_\delta(\chi_\delta \ast trh) \) and the decay (3.1). Let \( S^p(\check{\delta}, X) \subset S^p(X) \) be the class of all scalar valued left-\( \delta \) type Schwartz class functions. Both the spaces \( S^p_\delta(X) \) and \( S^p(\check{\delta}, X) \) becomes Fréchet spaces with the topologies induced by the family of seminorms given in (3.2) and (3.1) respectively. Moreover, \( \mathcal{D}(X) \) and \( \mathcal{D}(\delta, X) \) are respectively dense subspaces of \( S^p_\delta(X) \) and \( S^p(\check{\delta}, X) \) in the respective Schwartz space topologies. The topological isomorphism \( \mathcal{Q} \) in Proposition 1.1 can be extended between the Schwartz spaces \( S^p_\delta(X) \) and \( S^p(\check{\delta}, X) \).

Next we shall extend the definition of the \( \delta \)-spherical transform to the Schwartz space \( S^p_\delta(X) \) \((0 < p \leq 2)\). The spherical transform (1.11) defined on \( \mathcal{D}(G//K) \) can be extended to the \( L^p \)-Schwartz spaces \( S^p(G//K) \) of bi-\( K \)-invariant functions on the group \( G \). The image \( S(\mathfrak{a}_C^*) \) (defined below) of \( S^p(G//K) \) under the spherical transform is again a Schwartz class of functions defined on the complex tube \( \mathfrak{a}_C^* = \mathfrak{a}^* + iC^{ep} \) where \( \varepsilon = \left( \frac{2}{p} - 1 \right) \) and

\[
C^{ep} = \{ \lambda \in \mathfrak{a}^* \mid \omega \lambda(H) \leq \varepsilon \rho(H) \text{ for all } H \in \overline{\mathfrak{a}^+} \text{ and } \omega \in W \}.
\]

**Definition 3.2.** The space \( S(\mathfrak{a}_C^*) \) consists of the complex valued functions \( h \) on \( \mathfrak{a}_C^* \) such that:

(i) \( h \) is holomorphic in the interior of the tube \( \mathfrak{a}_C^* \) and continues on the closed tube,

(ii) \( h \) is \( W \)-invariant,

(iii) for any polynomial \( P \) in the algebra \( S(\mathfrak{a}) \) of the symmetric polynomials on \( \mathfrak{a}^* \) and any positive integer \( r \)

\[
\tau_{P, r}(h) = \sup_{\lambda \in \text{Int}_C^*} (1 + \|\lambda\|)^r \left| P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \right| < +\infty.
\]
The countable family \( \{ \tau_{P,r} \mid P \in S(\mathfrak{a}), r \in \mathbb{Z}^+ \cup \{0\} \} \) gives a Fréchet norm on the space \( S(\mathfrak{a}_\mathfrak{c}^\ast) \).

**Remark 3.3.** The topology of \( S(\mathfrak{a}_\mathfrak{c}^\ast) \) can also be given by the following equivalent family of seminorms.

\[
(3.4) \quad \tau_{P,r}^+(h) = \sup_{\lambda \in \text{Int}^\ast \cap (\mathfrak{a}^* + i\mathfrak{a}^*')} \left(1 + \|\lambda\|^n \right) \|P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \|_2 < +\infty.
\]

This is an easy consequence of (ii) of Definition 3.2 and the fact \( \|\omega\lambda\| = \|\lambda\| \) for all \( \lambda \in \mathfrak{a}_\mathfrak{c}^* \) and \( \omega \in W \).

We now state the \( L^p \)-Schwartz space isomorphism theorem for the bi-\( K \)-invariant functions.

**Theorem 3.4.** The spherical transform \( f \mapsto \hat{f} \) defined in (1.11) is topological isomorphism between the spaces \( S^p(G//K) \) (\( 0 < p \leq 2 \)) and \( S(\mathfrak{a}_\mathfrak{c}^*) \).

For a proof of this theorem one can see [Ank91]. We are mainly interested in Anker’s proof [Ank91] of the above theorem, which relies on the PW theorem for the spherical transform. Let \( S_0(\mathfrak{a}_\mathfrak{c}^*) \) be the space of all \( Hom(V_\delta, V_\delta^M) \) valued functions \( h \) on \( \mathfrak{a}_\mathfrak{c}^* \) satisfying (i), (ii) of Definition 3.2 along with the decay: for each polynomial \( P \in S(\mathfrak{a}) \) and integer \( n \geq 0 \)

\[
(3.5) \quad \tau_{P,n}^+(h) = \sup_{\lambda \in \text{Int}^\ast \cap (\mathfrak{a}^* + i\mathfrak{a}^*')} \left(1 + \|\lambda\|^n \right) \|P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \|_2 < +\infty.
\]

The countable family \( \{ \tau_{P,n}^+ \} \) induces a a Fréchet structure on \( S_0(\mathfrak{a}_\mathfrak{c}^*) \). The space \( S_0(\mathfrak{a}_\mathfrak{c}^*) \) can also be viewed as a space of \( Hom(V_\delta, V_\delta^M) \) valued functions with each of its matrix entry function in \( S(\mathfrak{a}_\mathfrak{c}^*) \). For our purpose we shall use the following equivalent (inducing the same topology) family of seminorms on \( S_0(\mathfrak{a}_\mathfrak{c}^*) \):

\[
(3.6) \quad \tau_{P,r}^+(h) = \sup_{\lambda \in \text{Int}^\ast \cap (\mathfrak{a}^* + i\mathfrak{a}^*')} \left\|P \left( \frac{\partial}{\partial \lambda} \right) \{ h(\lambda)(|\rho|^2 + (\lambda, \lambda)_1)^1 \} \right\|_2, \quad P \in S(\mathfrak{a}), r \in \mathbb{Z}^+ \cup \{0\}.
\]

Let \( f = (f_{ij}) \in S^p(G//K, Hom(V_\delta, V_\delta^M)) \). Then by defining \( \hat{f} = (\hat{f}_{ij}) \) we can the following extension of the isomorphism of Theorem 3.4.

**Lemma 3.5.** The spherical transform is a topological isomorphism between the spaces \( S^p(G//K, Hom(V_\delta, V_\delta^M)) \) and \( S_0(\mathfrak{a}_\mathfrak{c}^*) \).

This Lemma can be proved easily by using the conclusion of the Theorem 3.4 for each matrix entry of the functions of \( S^p(G//K, Hom(V_\delta, V_\delta^M)) \).

We now define the ambient space for the image of the Schwartz space \( S^p_\delta(X) \) under the \( \delta \)-spherical transform.

**Definition 3.6.** Let \( S_\delta(\mathfrak{a}_\mathfrak{c}^*) \) be the set of all \( Hom(V_\delta, V_\delta^M) \) valued functions \( h \) on the complex tube \( \mathfrak{a}_\mathfrak{c}^* \) such that

(i) \( h \) is holomorphic in the interior of the tube \( \mathfrak{a}_\mathfrak{c}^* \) and extends as a continuous function to the closed tube.

(ii) \( \lambda \mapsto Q^1(\lambda)^{-1} h(\lambda) \) is \( W \)-invariant holomorphic function the interior of the complex tube \( \mathfrak{a}_\mathfrak{c}^* \).

(iii) for each polynomial \( P \in S(\mathfrak{a}) \) and integer \( n \geq 0 \)

\[
(3.7) \quad \nu_{P,n}(h) = \sup_{\lambda \in \text{Int}^\ast} \left(1 + \|\lambda\|^n \right) \left\|P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \right\|_2 < +\infty.
\]
Clearly, \( S_\delta(\mathfrak{a}_\ast^+) \) becomes a Fréchet space with the topology induced by the countable family \( \{ \nu_{P,n} \} \) of seminorms.

**Remark 3.7.** It is easy to observe that the topology of the space \( S_\delta(\mathfrak{a}_\ast^+) \) can also be determined by the following equivalent family of seminorms: for each polynomial \( P \in S(\mathfrak{a}) \) and each nonnegative integer \( n \)

\[
(3.8) \quad h \mapsto \nu_{P,n}^\#(h) = \sup_{\lambda \in \text{Int} \mathfrak{a}_\ast^+} \left| P \left( \frac{\partial}{\partial \lambda} \right) \left\{ \langle \lambda, \lambda \rangle_1 + \| \rho \|^2 \right\}^n h(\lambda) \right| .
\]

Let us now state the first main theorem of this paper.

**Theorem 3.8.** The \( \delta \)-spherical transform \((2.1)\) is a topological isomorphism between the Schwartz spaces \( S_\delta^p(X) \) and \( S_\delta(\mathfrak{a}_\ast^+) \) with \( 0 < p \leq 2 \) and \( \varepsilon = \left( \frac{2}{p} - 1 \right) \).

The following proposition is a key step to prove Theorem 3.8.

**Proposition 3.9.** For each \( \delta \in \widehat{K}_{M,} \), there exists a \( \delta \)-dependent constant \( c_\delta > 0 \) such that

\[
(3.9) \quad \inf_{\lambda \in (\mathfrak{a}^+ + i\mathfrak{a}^+) \mathcal{C}} | \det Q^\delta(\lambda) | \geq c_\delta.
\]

**Proof.** To prove this we use the product formula \((2.17)\) of \( \det Q^\delta(\lambda) \). Each of the factors \( Q^\delta_{\beta}^{(i, \beta)} \) is the Kostant polynomials of he rank-one restrictions \( G_\beta/K_\beta \) and hence it is a polynomial in one complex variable \( \lambda_\beta \in \mathfrak{a}^+_{(\beta)} \mathcal{C} \cong \mathbb{C} \).

It is easy to check that \( \lambda \in \mathfrak{a}^+ + i\mathfrak{a}^+ \mathcal{C} \) if and only if \( \Re \langle i\lambda, \alpha \rangle \leq 0 \) for all \( \alpha \in \Sigma^+ \). Using the definition of the restriction \( \lambda_\beta = \frac{\lambda^{(l)}_\beta}{(\beta, \beta)} \beta \) we get

\[
(3.10) \quad \langle i\lambda_\beta, \beta \rangle_\beta = \left\langle i \left( \frac{\langle \lambda, \beta \rangle_\beta}{(\beta, \beta)} \beta \right) \right\rangle_\beta = \langle i\lambda, \beta \rangle
\]

where \( \langle \cdot, \cdot \rangle_\beta \) denotes the inner product on \( \mathfrak{a}^+_{(\beta)} \) as well as its \( \mathbb{C} \)-bilinear extension to the complexification \( \mathfrak{a}^+_{(\beta)} \mathcal{C} \). Now \((3.10)\) clearly suggests that if \( \lambda \in \mathfrak{a}^+ + i\mathfrak{a}^+ \mathcal{C} \) then for each \( \beta \in \Sigma^+ \) the restriction \( \lambda_\beta \in \mathfrak{a}^+_{(\beta)} + i\mathfrak{a}^+_{(\beta)} \mathcal{C} \). As the polynomial \( Q^\delta_{\beta}^{(i, \beta)}(\lambda_\beta) \neq 0 \) for all \( \lambda_\beta \in \mathfrak{a}^+_{(\beta)} + i\mathfrak{a}^+_{(\beta)} \mathcal{C} \) and \( \mathfrak{a}^+_{(\beta)} + i\mathfrak{a}^+_{(\beta)} \mathcal{C} \) being a closed subset of \( \mathbb{C} \) so we get a positive constant \( c_\delta(i, \beta) \) such that

\[
\inf_{\lambda_\beta \in \mathfrak{a}^+_{(\beta)} + i\mathfrak{a}^+_{(\beta)} \mathcal{C}} | Q^\delta_{\beta}^{(i, \beta)}(\lambda_\beta) | \geq c_\delta(i, \beta).
\]

Since corresponding to each \( \delta \in \widehat{K}_{M} \) the product formula \((2.17)\) of \( \det Q^\delta(\lambda) \) has only finitely many factors hence we get the desired conclusion of the proposition.

The next lemma extends the homeomorphism given in Lemma 2.9 between the P-W spaces to the corresponding Schwartz classes.

**Lemma 3.10.** The map

\[
(3.11) \quad g(\lambda) \mapsto Q^\delta(\lambda)g(\lambda), \quad \text{for all } \lambda \in \mathfrak{a}_\ast^+,
\]

is a homeomorphism from the space \( S_0(\mathfrak{a}_\ast^+) \) onto \( S_\delta(\mathfrak{a}_\ast^+) \).

**Proof.** Let us first take \( g \in S_0(\mathfrak{a}_\ast^+) \). We denote \( h(\cdot) = Q^\delta(\cdot)g(\cdot) \). We shall show that \( h \in S_\delta(\mathfrak{a}_\ast^+) \). As \( Q^\delta(\lambda) \) is an \( \ell_\delta \times \ell_\delta \) matrix of polynomials in \( \lambda \in \mathfrak{a}_\ast^+ \) so \( \lambda \mapsto Q^\delta(\lambda) \) is a holomorphic function on \( \mathfrak{a}_\ast^+ \). Hence the function \( h \) satisfies condition (i) and (ii) of Definition 3.6 which
easily follows from the similar properties of \( g \in \mathcal{S}_0(a^*_\varepsilon) \) and the construction (3.11) of the function \( h \).

To establish the decay condition (3.7) for \( h \) let us take a polynomial \( P \in S(a) \) and \( m \in \mathbb{Z}^+ \cup \{0\} \). Then

\[
\sup_{\lambda \in \text{Int} a^*_{\varepsilon}} \left\| P \left( \frac{\partial}{\partial \lambda} \right) h(\lambda) \right\|_2 (1 + \|\lambda\|)^m \\
\leq \sup_{\lambda \in \text{Int} a^*_{\varepsilon}} \sum_{\kappa} c_\kappa \left\| \left\{ \frac{\partial}{\partial \lambda} P'_\kappa Q^\delta(\lambda) \right\} \left\{ \frac{\partial}{\partial \lambda} P_\kappa g(\lambda) \right\} \right\|_2 (1 + \|\lambda\|)^m \\
\leq \sup_{\lambda \in \text{Int} a^*_{\varepsilon}} \sum_{\kappa} c_\kappa \left\| \left\{ \frac{\partial}{\partial \lambda} P'_\kappa Q^\delta(\lambda) \right\} \left\{ \frac{\partial}{\partial \lambda} P_\kappa g(\lambda) \right\} \right\|_2 (1 + \|\lambda\|)^m \\
\leq \sup_{\lambda \in \text{Int} a^*_{\varepsilon}} \sum_{\kappa} c_\kappa \left\| \left\{ \frac{\partial}{\partial \lambda} P_\kappa g(\lambda) \right\} \right\|_2 (1 + \|\lambda\|)^m 
\]

(3.12)

where, \( m^\delta \) are nonnegative integers and \( c^\delta \) are positive constants both depending on \( \delta \in \widehat{K}_M \).

As \( g \in \mathcal{S}_0(a^*_\varepsilon) \), the right hand side of (3.12) is clearly finite. Moreover, (3.12) shows that the map (3.11) is a continuous function from \( \mathcal{S}_0(a^*_\varepsilon) \) into \( \mathcal{S}_\delta(a^*_\varepsilon) \).

Now let \( \psi \in \mathcal{S}_\delta(a^*_\varepsilon) \) and define \( g(\cdot) := Q^\delta(\cdot)^{-1} \psi(\cdot) \). As, \( \psi \in \mathcal{S}_\delta(a^*_\varepsilon) \), by Definition 3.6 the function

\[
\lambda \mapsto g(\lambda) = \frac{1}{\det Q^\delta(\lambda)} Q^\delta(\lambda)\psi(\lambda)
\]

(here, \( Q^\delta(\lambda) \) is the cofactor matrix of \( Q^\delta(\lambda) \)) is \( W \)-invariant and it is holomorphic in the interior of the tube \( a^*_\varepsilon \). To infer \( g \in \mathcal{S}_0(a^*_\varepsilon) \) all we have to show is that the function \( g \) has certain decay. Let \( P \in S(a) \) and \( t \) be any nonnegative integer, then

\[
\left\{ P \left( \frac{\partial}{\partial \lambda} \right) g(\lambda) \right\} = \left\{ P \left( \frac{\partial}{\partial \lambda} \right) \frac{1}{\det Q^\delta(\lambda)} Q^\delta(\lambda)\psi(\lambda) \right\} \\
= \sum_{\kappa} \frac{P'_\kappa \left( \frac{\partial}{\partial \lambda} \right) Q^\delta(\lambda) P_\kappa \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda)}{(\det Q^\delta(\lambda))^{m_\kappa}}.
\]

(3.13)

The last line of (3.13) follows by an easy application of the Leibniz rule. Here \( P_\kappa, P'_\kappa \) are finite degree polynomials, \( m_\kappa \) is a positive integer depending on \( \kappa \) and the sum is over a finite set. From (3.13) we get:

\[
\left\| P \left( \frac{\partial}{\partial \lambda} \right) g(\lambda) \right\|_2 \leq \sum_{\kappa} \frac{\left\| P'_\kappa \left( \frac{\partial}{\partial \lambda} \right) Q^\delta(\lambda) \right\|_2 \left\| P_\kappa \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda) \right\|_2}{(\det Q^\delta(\lambda))^{m_\kappa}} \\
\leq c(\delta) \sum_{\kappa} \frac{\left\| P'_\kappa \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda) \right\|_2}{(\det Q^\delta(\lambda))^{m_\kappa}} (1 + \|\lambda\|)^{n_\kappa}.
\]

(3.14)

The above inequality is obtained by using the fact that: \( \left\| P'_\kappa \left( \frac{\partial}{\partial \lambda} \right) Q^\delta(\lambda) \right\|_2 \leq c(\delta)(1 + \|\lambda\|)^{n_\kappa} \)

where \( c(\delta) > 0 \) is a \( \delta \)-dependent constant and \( n_\kappa \) is a positive integer depending on the degree of \( P_\kappa \) (it may also depend on \( \delta \)). Now from (3.14) we get the following inequality for any nonnegative integer \( t \).

\[
\sup_{\text{Int}(a^*_\varepsilon \cap (a^* + i\varepsilon + t))} \left\| P \left( \frac{\partial}{\partial \lambda} \right) g(\lambda) \right\|_2 (1 + \|\lambda\|)^t \leq \sum_{\kappa} \sup_{\text{Int}(a^*_\varepsilon \cap (a^* + i\varepsilon + t))} \frac{\left\| P'_\kappa \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda) \right\|_2 (1 + \|\lambda\|)^t}{(\det Q^\delta(\lambda))^{m_\kappa}}
\]
\[ \begin{align*}
&\leq \sum_{\kappa} \frac{1}{G_{\delta}} \sup_{\lambda \in \xi} \left\| P'_{\kappa} \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda) \right\|_2 \left( 1 + \| \lambda \| \right)^{t+n_{\kappa}} \\inf_{\lambda \in \text{Int}(a_{\xi}^* (a^*)^{*} + i \sigma ^{*} \tau))} \| \det Q^\delta(\lambda) \|^{m_{\kappa}} \\
&\leq \sum_{\kappa} \frac{1}{G_{\delta}} \sup_{\lambda \in \xi} \left\| P'_{\kappa} \left( \frac{\partial}{\partial \lambda} \right) \psi(\lambda) \right\|_2 \left( 1 + \| \lambda \| \right)^{t+n_{\kappa}}.
\end{align*} \]  

(3.15)

The last line of the above successive inequalities is a consequence of Proposition 3.9. As \( \psi \in S_{\delta}(a_{\xi}^*) \) so each term of the finite summation on the right hand side of (3.15) is finite. Hence we conclude that \( g \) satisfies the decay (3.5) of the space \( S_{\delta}(a_{\xi}^*) \). The inequality (3.15) further concludes that the continuous map (3.11) from \( S_{\delta}(a_{\xi}^*) \) onto \( S_{\delta}(a_{\xi}^*) \) is injective and also its inverse map is continuous. As both the spaces \( S_{\delta}(a_{\xi}^*) \) and \( S_{\delta}(a_{\xi}^*) \) are Fréchet spaces so the map (3.11) is a homeomorphism.

Lemma 3.11. The Paley-Wiener space \( \mathcal{P}^\delta(a_{\xi}^*) \), defined in (2.14), is a dense subspace of the Schwartz space \( S_{\delta}(a_{\xi}^*) \).

Proof. Let us take any \( H \in S_{\delta}(a_{\xi}^*) \), it is enough to show that, there is a sequence \( \{G_n\} \) \((G_n \in \mathcal{P}(a_{\xi}^*))\) converging to \( H \) in the topology of the space \( S_{\delta}(a_{\xi}^*) \). Let \( H(\lambda) = (H_{ij}(\lambda))_{\ell \times d_\delta} \). By the isomorphism obtained in Lemma 3.10, we get one unique \( G \in S_0(a_{\xi}^*) \) such that

\[ H(\lambda) = (H_{ij}(\lambda))_{\ell \times d_\delta} = Q^\delta(\lambda)G(\lambda), \]

\[ = Q^\delta(\lambda)(G_{ij}(\lambda))_{\ell \times d_\delta}, \]

\[ = \left( \sum_{k=1}^{\ell} Q^\delta(\lambda)_{ik}G_{kj}(\lambda) \right)_{\ell \times d_\delta}. \]

(3.16)

As \( G \in S_0(a_{\xi}^*) \), so from the definition of the Schwartz space \( S_0(a_{\xi}^*) \) it follows that the matrix entry functions \( G_{ij} \in S(a_{\xi}^*) \) for each \( 1 \leq i \leq \ell_\delta \) and \( 1 \leq j \leq d_\delta \). We know that the Paley-Wiener space \( \mathcal{P}(a_{\xi}^*) \) under the spherical transform is dense in the Schwartz class \( S(a_{\xi}^*) \) \([\text{GV88}]\). Therefore we can get a sequence \( \{g_{ij,n}\}_n \subset \mathcal{P}(a_{\xi}^*) \) converging to \( G_{ij} \in S(a_{\xi}^*) \). As, each \( Q^\delta_{ik}(\lambda) \) (\( 1 \leq i \leq \ell_\delta, 1 \leq k \leq d_\delta \)) is a polynomial in \( \lambda \), so the sequence \( \left\{ \sum_{k=1}^{\ell_\delta} Q^\delta_{ik}(\lambda)g_{ij,n}(\lambda) \right\}_n \) converges to \( \sum_{k=1}^{\ell_\delta} Q^\delta_{ik}(\lambda)g_{ij,n}(\lambda) \) (each \( \lambda \) in \( S(a_{\xi}^*) \)). Let \( g_n(\lambda) = (g_{ij,n}(\lambda))_{\ell \times d_\delta} \). As each \( g_{ij,n} \in \mathcal{P}(a_{\xi}^*) \), so from the definition it follows that, the matrix valued function \( g_n \in \mathcal{P}^0(a_{\xi}^*) \). Clearly by Lemma 2.9 for each natural number \( n \), \( Q^\delta(\cdot)g_n(\cdot) \in \mathcal{P}^0(a_{\xi}^*) \). Let \( P \) be any polynomial in \( S(a_{\xi}^*) \) and \( t \) be any nonnegative integer then:

\[ \tau_{P,t} \left( Q^\delta(\cdot)g_n(\cdot) - Q^\delta(\cdot)G(\cdot) \right) \]

\[ = \sup_{\lambda \in \text{Int}a_{\xi}^*} \left\| P \left( \frac{\partial}{\partial \lambda} \right) \left\{ Q^\delta(\lambda)g_n(\lambda) - Q^\delta(\lambda)G(\lambda) \right\} \right\|_2 \left( 1 + \| \lambda \| \right)^t, \]

\[ = \sup_{\lambda \in \text{Int}a_{\xi}^*} \sum_{i=1}^{\ell_\delta} \sum_{j=1}^{d_\delta} P \left( \frac{\partial}{\partial \lambda} \right) \sum_{k=1}^{\ell_\delta} \left\{ Q^\delta(\lambda)_{ik}g_{ik,n}(\lambda) - Q^\delta(\lambda)_{ik}G_{ik}(\lambda) \right\} \left( 1 + \| \lambda \| \right)^t. \]

(3.17)

A suitable choice of \( n \) can be made such that the right hand side of (3.17) arbitrarily small. Hence we get the sequence \( \{Q^\delta(\cdot)g_n\}_n \) in \( \mathcal{P}^\delta(a_{\xi}^*) \) converging to \( H \) in the topology of \( S_{\delta}(a_{\xi}^*) \). This completes the proof of the Lemma.

Next we shall try to extend the definition of the \( \delta \)-spherical transform (2.1) to the Schwartz class \( S^p_{\delta}(X) \) where \( 0 < p \leq 2 \).
Lemma 3.12. For each $f \in S^p_\delta(X)$, the function $\lambda \mapsto \tilde{f}(\lambda)$, where $\tilde{f}$ is given by (2.1), is a holomorphic function in the interior of the complex tube $a^*_\delta$.

Proof. For each $f \in S^p_\delta(X)$ the function $x \mapsto trf(x)$ has the following decay: for each $D, E \in U(\mathfrak{g}_C)$ and integer $n \geq 0$

$$\sup_{x \in G} |trf(D, x, E)|(1 + |x|)^n \varphi_0^{-\frac{2}{p}}(x) < +\infty,$$

which follows easily from (3.2) and the fact that $|trf(x)| \leq \|f(x)\|_2$ ($x \in X$). Using (3.18) and the estimate (2.10) of the generalized spherical functions one can show, by following a standard argument (see [GV88, §6.2], [EK76]), that the $\delta$-spherical transform, defined by the integral (2.1), exists for $\lambda \in a^*_\delta$.

Let $\gamma$ be a closed curve in the interior of the tube $a^*_\delta$. Then for $f \in S^p_\delta(X)$ we get

$$\int_{\gamma} \tilde{f}(\lambda) d\lambda = d_\delta \int_{\gamma} \left\{ \int_G trf(x) \Phi_{X,\delta}(x) dx \right\} d\lambda.$$

As the integral within braces exists absolutely for $\lambda \in a^*_\delta$, so we apply Fubini’s theorem to get: $\int_{\gamma} \tilde{f}(\lambda) d\lambda = d_\delta \int_G trf(x) \left\{ \int_{\gamma} \Phi_{X,\delta}(x) d\lambda \right\} dx$. We also recall that the functions $\lambda \mapsto \Phi_{X,\delta}(\cdot)$ are entire. Hence by an application of Morera’s theorem the desired conclusion of the lemma follows. 

\[\square\]

4. Proof of Theorem 3.8

To show that the $\delta$-spherical transform is a topological isomorphism it is enough to show that it is a continuous surjection from $S^p_\delta(X)$ onto $S_\delta(a^*_\delta)$.

Lemma 4.1. The $\delta$-spherical transform $f \mapsto \tilde{f}$ is a continuous map from the Schwartz space $S^p_\delta(X)$ into $S_\delta(a^*_\delta)$.

Proof. It is enough to show that given any seminorm $\nu$ (or equivalently $\nu^{\#}$) on $S_\delta(a^*_\delta)$ one can find a seminorm $\mu$ on $S^p_\delta(X)$ such that

$$\nu(\tilde{f}) \leq c\mu(f) \quad \text{for all} \quad f \in S^p_\delta(X).$$

With $P \in S(\mathfrak{a})$ and $t \in \mathbb{N} \cup \{0\}$ we get the following by using the integral expression (2.1) of the $\delta$-spherical transform.

$$P \left( \frac{\partial}{\partial \lambda} \right) \left\{ (\lambda, \lambda)_1 + \|\rho\|^2 \right\}^{t} \tilde{f}(\lambda) = P \left( \frac{\partial}{\partial \lambda} \right) \int_G trf(x) \left( (\lambda, \lambda)_1 + \|\rho\|^2 \right)^t \Phi_{X,\delta}(x) dx$$

$$= (-1)^n P \left( \frac{\partial}{\partial \lambda} \right) \int_G trf(x) L^t \Phi_{X,\delta}(x) dx.$$

The last equality is a consequence of the property (2.8) of the generalized spherical function. Now a simple application of integration by parts gives:

$$(4.1) = (-1)^n P \left( \frac{\partial}{\partial \lambda} \right) \int_G L^t trf(x) \Phi_{X,\delta}(x) dx$$

$$= (-1)^n P \left( \frac{\partial}{\partial \lambda} \right) \int_G tr L^t f(x) \Phi_{X,\delta}(x) dx$$

$$= (-1)^n P \left( \frac{\partial}{\partial \lambda} \right) \int_G tr L^t f(x) \left\{ \int_K e^{(i\lambda-\rho)(2t(x^{-1}k))} \delta(k^{-1}) dk \right\} dx.$$
The second line in the above chain of equalities uses the fact \( \mathbf{L} \text{tr} f(\cdot) = \text{tr} \mathbf{L} f(\cdot) \), which is clear as the differential operator \( \mathbf{L} \) acts entry-wise to the operator valued function \( f \).

As \( f \in S^p_0(X) \), so it can also be considered as a right-\( K \)-invariant function on the group \( G \). The action of the Laplace Beltrami operator \( \mathbf{L} \) on \( f \) is the same as the action of the Casimir operator on \( f \) considered as a function on \( G \). Therefore, by the property of the Casimir operator, the action of \( \mathbf{L} \) does not change the left-\( K \)-type of the function \( f \), i.e., the function \( L^r f(\cdot) \) is again of left-\( K \)-type. Moreover, for each nonnegative integer \( n \) the function \( L^r f(\cdot) \in S^p_0(X) \). Hence by Lemma 3.12 the integral on the right hand side of (4.2) exists absolutely. We apply Fubini’s theorem to interchange the integrals and then we put \( x^{-1} k = y^{-1} \) to get:

\[
(4.2) = (-1)^n p \left( \frac{\partial}{\partial \lambda} \right) \int_K \int_G \text{tr} \mathbf{L}^r f(ky) e^{(i\lambda - \rho)\delta(y^{-1})} \delta(k^{-1}) dy dk \\
= (-1)^n p \left( \frac{\partial}{\partial \lambda} \right) \int_G \left\{ \int_K \text{tr} \mathbf{L}^r f(ky) \delta(k^{-1}) dk \right\} e^{(i\lambda - \rho)\delta(y^{-1})} dy \\
= \frac{(-1)^r}{d_0} p \left( \frac{\partial}{\partial \lambda} \right) \int_G \mathbf{L}^r f(y) e^{(i\lambda - \rho)\delta(y^{-1})} dy \\
(4.3) = \frac{(-1)^r}{d_0} p \left( \frac{\partial}{\partial \lambda} \right) \int_G \mathbf{L}^r f(y^{-1}) e^{(i\lambda - \rho)\delta(y)} dy
\]

The third line follows by using the Schwartz space extension of the isomorphism \( \Phi \) of Proposition 1.1 and the last line uses the invariance of the Haar measure under the transformation \( g \rightarrow g^{-1} \). Let us now break up the group \( G \) as well as the Haar measure using the Iwasawa decomposition \( KAN \) decomposition to get:

\[
(4.3) = \frac{-1}{d_0} p \left( \frac{\partial}{\partial \lambda} \right) \int_K \int_{\theta^+} \int_N \mathbf{L}^r f(n^{-1}(\exp H)^{-1}k^{-1}) e^{(i\lambda - \rho)\delta(k(\exp H)n)} dk e^{2\rho(H)} dH dn \\
(4.4) = \frac{-1}{d_0} p \left( \frac{\partial}{\partial \lambda} \right) \int_{\alpha^+} \int_N \mathbf{L}^r f(n^{-1}(\exp H)^{-1}) e^{(i\lambda + \rho)H} dH dn.
\]

Let \( \alpha_1, \alpha_2, \ldots, \alpha_r \) be the set of all positive restricted roots. Let \( \varepsilon_i \in \alpha^* \) \((1 \leq i \leq r)\) be such that \( \langle \alpha_i, \varepsilon_j \rangle = \delta_{ij} \). Then clearly \( \{ \varepsilon_i \}_{1 \leq i \leq r} \) forms a basis of \( \alpha^* \) and thus we introduce a global coordinate on \( \alpha^* \) by \( \lambda = \sum_{i=1}^{r} \lambda_i \varepsilon_i \) \((\forall \lambda \in \alpha^*)\).

Let \( P(\lambda) = \sum_{\theta=0}^{\beta} \sum_{\beta_1 + \cdots + \beta_r = \theta} \alpha_{\beta_1 + \cdots + \beta_r} \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_r^{\beta_r} \). Thus,

\[
P \left( \frac{\partial}{\partial \lambda} \right) = \sum_{\theta=0}^{\beta} \sum_{\beta_1 + \cdots + \beta_r = \theta} \alpha_{\beta_1 + \cdots + \beta_r} \left( \frac{\partial}{\partial \lambda_1} \right)^{\beta_1} \left( \frac{\partial}{\partial \lambda_2} \right)^{\beta_2} \cdots \left( \frac{\partial}{\partial \lambda_r} \right)^{\beta_r}.
\]

Therefore it is easy to check that:

\[
(4.5) \quad P \left( \frac{\partial}{\partial \lambda} \right) e^{(i\lambda + \rho)H} = P(H) e^{(i\lambda + \rho)H},
\]

where \( P(H) = \sum_{\theta=0}^{\beta} \sum_{\beta_1 + \cdots + \beta_r = \theta} \alpha_{\beta_1 + \cdots + \beta_r} \varepsilon_1^{\beta_1} \varepsilon_2^{\beta_2} \cdots \varepsilon_r^{\beta_r} (iH) \). From (4.1) it follows that:

\[
\left\| P \left( \frac{\partial}{\partial \lambda} \right) \left\{ (\lambda, \lambda)_1 + \|\rho\|^2 \tilde{f}(\lambda) \right\} \right\|_2 \leq \frac{1}{d_0} \int_{\alpha^+} \int_N \|L^r f(n^{-1}(\exp H)^{-1})\|_2 \|P(H)\| \left| e^{(i\lambda + \rho)H} \right| dH dn.
\]
(4.6) \[ \leq \frac{1}{d_\delta} \int_{a^+} \int_{A} \|L^f(n^{-1}(\exp H)^{-1})\|_2 \|P(H)\| e^{(|3\lambda|+\rho)(H)} dH dn. \]

Some basic estimates gives the following:

\[ \|e^{\beta_j}(iH)\| \leq \|e^{\beta_j}\| \|H\| \]
\[ \leq c\|e^{\beta_j}\|(|\exp H)n| \]
\[ \leq c\|\beta_j\| (1 + (|\exp H)n|). \]

(4.7)

The above estimate is a consequence of (1.3) and it is true for all \( n \in N \). Using (4.7) one can find \( d_p \in \mathbb{Z}^+ \) such that

\[ \|P(H)\| \leq c_1 (1 + (|\exp H)n|)^{d_p}. \]

As \( f \in S^*_0(X) \) so for each \( m \in \mathbb{Z}^+ \) we have:

\[ \|L^f(n^{-1}(\exp H)^{-1})\| \leq \mu_{L^f,m}(f) (1 + (|\exp H)n|)^{-m} \varphi^{\frac{2}{p}}(n^{-1}(\exp H)^{-1}). \]

(4.9) The above inequality also uses the fact that \( |g| = |g^{-1}| \) for all \( g \in G \). The estimates (4.7) and (4.9) reduce the inequation (4.6) to the following:

\[ \left\| \mu_{L^f,m}(f) \int_{G} \varphi^{\frac{2}{p}}(g^{-1})(1 + |g|)^{-m+d_p} e^{(|\lambda|+\rho)(H)} dH dn \right\|_2 \]
\[ \leq c_1 \mu_{L^f,m}(f) \int_{a^+} \int_{N} \varphi^{\frac{2}{p}}(n^{-1}(\exp H)^{-1}) (1 + (|\exp H)n|)^{-m+d_p} e^{(|3\lambda|+\rho)(H)} dH dn \]
\[ = c_1 \mu_{L^f,m}(f) \int_{K} \int_{a^+} \int_{N} \varphi^{\frac{2}{p}}(n^{-1}(\exp H)^{-1}) (1 + (|\exp H)n|)^{-m+d_p} e^{(|3\lambda|+\rho)(\exp H)} dke^{2\rho(H)} dH dn \]

(4.10)

where \( c_\delta = c_1 \frac{1}{d_\delta} \). Now we use the Cartan decomposition i.e \( g = k_1 \exp |g|k_2 \) and appropriate form of the Haar measure (1.4) to get:

\[ (4.10) = c_\delta \mu_{L^f,m}(f) \int_{K} \varphi^{\frac{2}{p}}(\exp |g^{-1}|)(1 + |g|)^{-m+d_p} e^{(|3\lambda|+\rho)(\exp |g|k_2)} \Delta(|g|)d|g| dk_2, \]
\[ = c_\delta \mu_{L^f,m}(f) \int_{K} \varphi^{\frac{2}{p}}(\exp |g|)(1 + |g|)^{-m+d_p} \left\{ \int_{K} e^{(i|\lambda|+\rho)(\exp |g|k_2)} dk_2 \right\} \Delta(|g|)d|g| \]
\[ = c_\delta \mu_{L^f,m}(f) \int_{K} \varphi^{\frac{2}{p}}(\exp |g|)(1 + |g|)^{-m+d_p} \varphi^{-i|\lambda|}(\exp |g^{-1}|) \Delta(|g|)d|g| \]

\[ (4.11) \leq c_\delta \mu_{L^f,m}(f) \int_{a^+} \varphi^{\frac{2}{p}+1}(\exp |g|)(1 + |g|)^{-m+d_p} e^{(|3\lambda|+\rho)(|g|)} \Delta(|g|)d|g| \]

where the last inequality in this chain follows by using the estimate (1.9) of the elementary spherical function. We take \( \lambda \in \mathfrak{a}_+^* \), therefore \( |3\lambda|(|g|) \leq \varepsilon(|g|) \) where \( \varepsilon = \left( \frac{2}{p} - 1 \right) \). Now by using the another fundamental estimate (1.10) we further reduce the inequality (4.11) to the
We shall show that we use Anker’s \([\Phi]\) with the topologies of the respective Schwartz spaces containing them. We have already µ on \(I_{\delta}\), gives a \(\text{Hom}(\mathcal{S}_\delta, \mathcal{C})\) valued, left-\(\delta\)-type \(C^\infty\) function on \(X\). (From now on we shall denote this function by \(\mathcal{I}h(\cdot)\).)

Proof. Let us take any \(D \in \mathcal{U}(\mathfrak{g}_C)\). Then,

\[
\frac{1}{\omega} \int_{\mathfrak{a}^*} \|\Phi_{\lambda,\delta}(D, x)\|_2 \|h(\lambda)\|_2 \|c(\lambda)\|^{-2} d\lambda \leq c_\delta \varphi_0(x) \int_{\mathfrak{a}^*} (1 + \|\lambda\|)^{b_D + b - n} d\lambda.
\]

The above inequality follows from the fact that \(h \in \mathcal{S}_\delta(a^*_C)\) and by using the decay \((3.7)\), the estimate \((1.13)\) and the estimate \((2.10)\) for the generalized spherical functions. One can choose a suitably large \(n\) so that the integral in the right hand side of \((4.14)\) converges. This proves \(\mathcal{I}h\) is a function on \(X\) and \(D\mathcal{I}h\) exists for all \(D \in \mathcal{U}(\mathfrak{g}_C)\). Hence \(\mathcal{I}h \in \mathcal{C}^\infty(X, \text{Hom}(\mathcal{V}_\delta, \mathcal{V}_\delta))\). As, \(\Phi_{\lambda,\delta}(\cdot)\) is of left-\(\delta\)-type \((2.4)\) of Proposition \(2.3\), so is \(\mathcal{I}h\).

**Lemma 4.3.** If \(h \in \mathcal{S}_\delta(a^*_C)\) then the inverse \(\mathcal{I}h \in \mathcal{S}_\delta^p(X)\).

Proof. To prove this Lemma we shall first consider the spaces \(\mathcal{P}^p(a^*_C)\) and \(\mathcal{D}_\delta(X)\) equipped with the topologies of the respective Schwartz spaces containing them. We have already noticed that \(\mathcal{P}^p(a^*_C)\) and \(\mathcal{D}_\delta(X)\) are dense subspaces of \(\mathcal{S}_\delta(a^*_C)\) and \(\mathcal{S}_\delta^p(X)\) respectively.

We shall show that \(\mathcal{J}\) is a continuous map from \(\mathcal{P}^p(a^*_C)\) onto (by Theorem \(2.8\) \(\mathcal{D}_\delta(X)\) with respect to the Schwartz space topologies. That is for \(h \in \mathcal{P}^p(a^*_C)\) and for each seminorm \(\mu\) on \(\mathcal{D}_\delta(X)\), there exists a seminorm \(\nu\) on \(\mathcal{P}^p(a^*_C)\) such that \(\mu(f) \leq c_\delta \nu(h)\), where \(f = \mathcal{I}h \in \mathcal{D}_\delta(X)\) and \(c_\delta\) is a positive constant depending on \(\delta \in K_M\).

As, \(f \in \mathcal{D}_\delta(X)\), by Lemma \(2.10\), we get a function \(\phi \in \mathcal{D}(G//K, \text{Hom}(\mathcal{V}_\delta, \mathcal{V}_\delta))\) such that \(f \equiv D^\delta \phi\). If \(\Phi\) be the image of \(\phi\) under the spherical transform then it follows easily that \(h = Q^\delta \Phi\). Let \(D, E \in \mathcal{U}(\mathfrak{g}_C)\) and \(n\) be any nonnegative integer, then

\[
\mu_{D,E,n}(f) = \sup_{x \in G} \|f(D, x, E)\|_2 (1 + |x|)^n \varphi_0^{-\frac{2}{p}}(x),
\]

\[
= \sup_{x \in G} \|D^\delta \phi(D, x, E)\|_2 (1 + |x|)^n \varphi_0^{-\frac{2}{p}}(x),
\]

\[
= \mu_{D^\delta D,E,n}(\phi).
\]

(Here \(\mu_0\) denote the seminorms on the Fréchet space \(\mathcal{S}^p(G//K, \text{Hom}(\mathcal{V}_\delta, \mathcal{V}_\delta))\).) At this point we use Anker’s [Ank91] proof of the Schwartz space isomorphism theorem for bi-\(K\)-invariant
functions. For each \( D, E \in \mathcal{U}(\mathfrak{g}_C) \) and \( n \in \mathbb{Z}^+ \) one can find a polynomial \( P \in S(\mathfrak{a}) \) and \( m_\delta \in \mathbb{Z}^+ \) (depending on \( d_\delta \)) such that,

\[
\mu_{D,E,n}(H) \leq c_\delta \sup_{\lambda \in \text{Int}_0^+} \left\| P \left( \frac{\partial}{\partial \lambda} \right) \Phi(\lambda) \right\|_2 \left(1 + \|\lambda\|^{m_\delta}\right),
\]

(4.16)

The last line in (4.16) follows by using the isomorphism, proved in Lemma 3.10, between the Schwartz spaces \( S_0(\mathfrak{a}_\delta^*) \) and \( S_\delta(\mathfrak{a}_\delta^*) \). Hence (4.15) and (4.16) together gives \( \mu_{D,E,n}(f) \leq c_\delta \mu_{1,1}^{m_\delta}(h) \). As we have started with an \( h \in \mathcal{P}(\mathfrak{a}_\delta^*) \subset S_\delta(\mathfrak{a}_\delta^*) \), the right hand side of (4.16) is clearly finite. Hence \( \exists h = f \in S_\delta^0(X) \).

Now we apply the density argument to conclude the Lemma. Let us now take \( h \in S_\delta(\mathfrak{a}_\delta^*) \). As, \( \mathcal{P}(\mathfrak{a}_\delta^*) \) is dense in \( S_\delta(\mathfrak{a}_\delta^*) \), there exists a Cauchy sequence \( \{h_\nu\} \subset \mathcal{P}(\mathfrak{a}_\delta^*) \) converging to \( h \). Then, by what we have proved above, we can get a Cauchy sequence \( \{f_\nu\} \subset D_\delta(X) \) such that \( f_\nu \to h \). As \( \delta_\delta^p(X) \) is a Fréchet space the sequence must converge to some \( f \in S_\delta^0(X) \). Clearly, \( f = \exists h \). This completes the proof of the Lemma.

We note that, the Lemma 4.3 also implies the fact that the \( \delta \)-spherical transform is a injection in the corresponding Schwartz space level.

Finally, Lemma 4.1 and Lemma 4.3 together shows that the \( \delta \)-spherical transform is a continuous surjection of \( S_\delta^0(X) \) onto \( S_\delta(\mathfrak{a}_\delta^*) \) for \( 0 < p \leq 2 \). A simple application of the open mapping theorem concludes that the \( \delta \)-spherical transform is a topological isomorphism between the corresponding Schwartz spaces. This proves the Theorem 3.8.

In the next section we shall extend this result to a slightly larger class of functions.

5. Finite \( K \)-type functions

Let choose and fix a finite subset \( \Gamma \subset \hat{K}_M \). We denote \( D_\Gamma(X) \) for the space of all compactly supported \( C^\infty \) functions on \( X \) with the property that: for \( f \in D_\Gamma(X) \) \( f^\delta \equiv 0 \) for \( \delta \notin \Gamma \). We take the subclass \( S^p_\Gamma(X) \) of the Schwartz class \( S^p(X) \) (for \( 0 < p \leq 2 \)) defined by

\[
S^p_\Gamma(X) = \{ f \in S^p(X) \mid f(X) = \sum_{\delta \in \Gamma} \text{tr} f^\delta(x) \text{ for all } x \in X \}.
\]

The seminorms on \( S^p_\Gamma(X) \) are as follows: for each \( D, E \in \mathcal{U}(\mathfrak{g}_C) \) and \( n \in \mathbb{Z}^+ \),

\[
\mu_{D,E,n}(f) = \sup_{\delta \in \Gamma, x \in X} \left\| f^\delta(D, x, E) \right\|_2 \left(1 + |x|^n \varphi_0^{-\frac{\delta}{p}}(x) \right) < +\infty.
\]

Clearly \( D_\Gamma(X) \) is a dense subset of the Schwartz space \( S^p_\Gamma(X) \) with respect to the Fréchet topology induced by the countable family of seminorms \( \{\mu_{D,E,n}\} \). It also follows easily from the definition 3.1 and (3.2) that, if \( f \in S^p_\Gamma(X) \) then for each \( \delta \in \Gamma \) the projection \( f^\delta \in S^p_\Gamma(X) \). For these classes of functions the transform we shall mainly consider is the Helgason Fourier transform.

For each \( kM \in K/M \) we have \( \lambda \mapsto h(\lambda, kM) \) is holomorphic on \( \text{Int} \mathfrak{a}_\delta^* \), and it extends as a continuous function on the closed complex tube \( \mathfrak{a}_\delta^* \). The function \( h \) is a smooth function in the \( k \in K/M \) variable.

**Definition 5.1.** Let \( S(\mathfrak{a}_\delta^* \times K/M) \) denotes the class of functions \( h \) on \( \mathfrak{a}_\delta^* \times K/M \) satisfying the following properties:

(i) For each \( kM \in K/M \), the function \( \lambda \mapsto h(\lambda, kM) \) is holomorphic on \( \text{Int} \mathfrak{a}_\delta^* \), and it extends as a continuous function on the closed complex tube \( \mathfrak{a}_\delta^* \). The function \( h \) is a smooth function in the \( k \in K/M \) variable.
(ii) For all $\lambda \in \mathfrak{a}^*_x$, $\omega \in W$ and $x \in G$
\begin{equation}
\hat{h}(\lambda, x) = \hat{h}(\omega \lambda, x),
\end{equation}
where $\hat{h}(\lambda, x) = \int_K h(\lambda, k) e^{-i(\lambda + \rho)H(x^{-1}k)} dk$.

(iii) For each $P \in S(\mathfrak{a})$ and for integers $n, m > 0$ the function $h$ satisfies the following decay condition
\begin{equation}
\sup_{(\lambda, k) \in \text{Int } \mathfrak{a}^*_x \times K/M} \left| P \left( \frac{d}{d\lambda} \right) h(\lambda, k, \omega^m_x) \right| (1 + |\lambda|)^n < +\infty.
\end{equation}

(iv) For each $\delta \in \hat{K}_M \setminus \Gamma$ the left-$\delta$-projection $h^\delta$ defined by
\begin{equation}
\hat{h}^\delta(\lambda, k) = d_\delta \int_K h(\lambda, k_1 k) \delta(k_1^{-1}) dk_1,
\end{equation}
is identically a zero function on $\mathfrak{a}^* \times K/M$.

The space $S_\Gamma(\mathfrak{a}^*_x \times K/M)$ is a Fréchet space with the topology induced by the seminorms $(5.4)$. By the theory of smooth functions on compact groups [Sug71], the topology of the space $S_\Gamma(\mathfrak{a}^*_x \times K/M)$ can be given by the following equivalent family of seminorms, for each $P \in S(\mathfrak{a})$ and $m \in \mathbb{Z}^+$ we have
\begin{equation}
\sup_{\lambda \in \text{Int } \mathfrak{a}^*_x, \delta \in F} \left\| P \left( \frac{d}{d\lambda} \right) h^\delta(\lambda, eM) \right\|_2 (1 + |\lambda|)^m < +\infty, \text{ for } h \in S_\Gamma(\mathfrak{a}^*_x \times K/M).
\end{equation}
We denote by $S(\mathfrak{a}^*_x \times K/M)$ the Fréchet space satisfying all the conditions of the Definition 5.1 except condition (iv). The space $S_\Gamma(\mathfrak{a}^*_x \times K/M)$ is a closed subspace of $S(\mathfrak{a}^*_x \times K/M)$. We know that the HFT can extended to the Schwartz class $S^p(X)$ [EK76], furthermore the HFT is a continuous map from $S^p(X)$ into $S(\mathfrak{a}^*_x \times K/M)$. Hence the HFT is a continuous map from $S^p(F, X)$ into $S_\Gamma(\mathfrak{a}^*_x \times K/M)$.

**Lemma 5.2.** Let $h \in S_\Gamma(\mathfrak{a}^*_x \times K/M)$, then for each $\delta \in F$, the left-$\delta$-projection $h^\delta \in S_\delta(\mathfrak{a}^*_x)$.

**Proof.** The function $\lambda \mapsto h^\delta(\lambda, eM)$ trivially satisfies condition (i) of Definition 3.6. The required decay (3.3) is also an easy consequence of (5.4). It can be shown that the $\delta$-projection $h^\delta$ also satisfies the condition
\begin{equation*}
(h^\delta)(\lambda, x) = (h^\delta)(\omega \lambda, x) \text{ for all } \omega \in W.
\end{equation*}
It is easy to check that $h^\delta(\lambda, kM) = \delta(k) h^\delta(\lambda, eM)$, hence for each $(\lambda, a) \in \mathfrak{a}^*_x \times A$ we write $(h^\delta)(\lambda, a)$ as follows
\begin{equation*}
(h^\delta)(\lambda, a) = \Phi_{\lambda, \delta}(a) h^\delta(\lambda, eM).
\end{equation*}
By the property (2.6) of the generalized spherical functions it follows that the function $\lambda \mapsto Q^\delta(\lambda)^{-1} h^\delta(\lambda, eM)$ is $W$-invariant. Hence we conclude that $h^\delta(\cdot, eM) \in S_\delta(\mathfrak{a}^*_x)$.

By using Theorem 3.8, for each $h \in S_\Gamma(\mathfrak{a}^*_x \times K/M)$ we get an unique finite sequence $\{f^\delta\}_{\delta \in F}$ of $C^\infty$ functions on $X$ such that each member $f^\delta \in S_\delta(X)$. We consider the following scalar valued function
\begin{equation}
f(x) = \sum_{\delta \in F} tr f^\delta(x), \; x \in X.
\end{equation}
For each \( \delta \in \Gamma \), \( (\mathcal{F}^{-1}h)^{\delta}(x) = \mathcal{F}(h^{\delta})(x) = f^{\delta}(x) \). Hence, we get \( \mathcal{F}^{-1}h(x) = f(x) \) for all \( x \in X \). The function \( f \in S^p_{\Gamma}(X) \). Furthermore for each \( D, E \in \mathfrak{u}(\mathfrak{g}_C) \) and \( n \in \mathbb{Z}^+ \) we have

\[
\sup_{x \in G} |f(D, x, E)|(1 + |x|)^n \varphi_0^{-\frac{2}{n}}(x)
\leq c \sup_{x \in G, \delta \in \Gamma} \|f^{\delta}(D, x, E)\|_{2}(1 + |x|)^n \varphi_0^{-\frac{2}{n}}(x)
\leq c_1 \sup_{\lambda \in \text{Int}_{\mathfrak{a}^*_c, \delta} \in \Gamma} \left\| P \left( \frac{d}{d\lambda} \right) h^{\delta}(\lambda, eM) \right\|_{2} (1 + |\lambda|)^m
\leq c_2 \sup_{\lambda \in \text{Int}_{\mathfrak{a}^*_c, k} \in K} \left| P_1 \left( \frac{d}{d\lambda} \right) h(\lambda, k, \omega^*_k) \right| (1 + |\lambda|)^{m_1},
\]

for some \( P_1 \in \mathcal{S}(\mathfrak{a}^*) \) and \( r, m_1 \in \mathbb{Z}^+ \). Thus, the HFT is a bijective map from \( S^p_{\Gamma}(X) \) to \( \mathcal{S}_{\Gamma}(\mathfrak{a}^*_c \times K/M) \). Once again, by the open mapping theorem we conclude the following.

**Theorem 5.3.** Let \( \Gamma \) be a finite subset of \( \hat{K}_M \), then the HFT is a topological isomorphism of the space \( S^p_{\Gamma}(X) \) onto the Fréchet space \( \mathcal{S}_{\Gamma}(\mathfrak{a}^*_c \times K/M) \).

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