Scalar field mass in generalized gravity

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Abstract

The notions of mass and range of a Brans–Dicke-like scalar field in scalar–tensor and $f(R)$ gravity are subject to an ambiguity that hides a potential trap. We spell out this ambiguity and identify a physically meaningful and practical definition for these quantities. This is relevant when giving a mass to this scalar in order to circumvent experimental limits on the PPN parameters coming from solar system experiments.

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1. Introduction

The standard theory of gravity, general relativity, is now tested rather accurately in the weak-field, slow-motion regime in the solar system [1], and no deviation from Einstein’s theory has ever been convincingly demonstrated. On the other hand, virtually all high-energy theories attempting to quantize gravity or unifying it with the other interactions predict deviations from general relativity. Sometimes, as in the case of the simplest string theories, these deviations have to be suppressed in order to ensure compatibility with the available experiments [2]. At galactic scales, not only general relativity, but even Newtonian gravity is doubted in attempts to explain away dark matter with MOND and TeVeS theories [3, 4]. In a modern perspective, looking at alternatives to Einstein’s theory seems well justified.

The prototypical theory of gravity alternative to general relativity is Brans and Dicke’s 1961 theory [5], which has since been generalized to a wider class of scalar–tensor theories [6]. The original motivation, i.e., the implementation of Mach’s principle in gravity, is now almost forgotten but these theories have nevertheless become popular because they incorporate key ingredients of string theories, such as the unavoidable presence of a dilaton-like gravitational scalar field, and its non-minimal coupling to the curvature.

More recently, the discovery that the expansion of the universe is accelerated, obtained with the study of type Ia supernovae [7], has prompted cosmologists to postulate the existence
of a mysterious form of dark energy with exotic properties to explain this acceleration within general relativity [8]. However, this assumption seems rather ad hoc, and many authors have tried to address the issue by assuming instead that the Einstein–Hilbert Lagrangian density $R$ receives infrared corrections, being changed to a nonlinear function $f(R)$ with the nonlinearities kicking in at low curvatures (i.e., late in the matter-dominated era) [9, 10] (see [11] for the first models compatible with the post-Newtonian constraints in the solar system). In this framework, the cosmic acceleration observed would be the first signal of deviations from Einstein’s theory. This ‘modified’ or ‘$f(R)$’ gravity contains a massive scalar degree of freedom in addition to the familiar massless graviton. $f(R)$ gravity turns out to be equivalent to a Brans–Dicke theory [12].

Usually, the Brans–Dicke-like scalar field of scalar–tensor theories is given a mass in order to make it short-ranged and evade the weak-field constraints coming from solar system experiments. The concepts of mass and range used are subject to an ambiguity that hides a potential trap and should be clarified. The purpose of this paper is to spell out this ambiguity and clarify the use of these quantities, especially in relation with the Parametrized post-Newtonian formalism used to constrain scalar–tensor gravity [1]. We are not aware of such a discussion in the vast literature on scalar–tensor gravity.

Here we use Brans–Dicke theory [5] for the sake of illustration, but the discussion can be extended to more general scalar–tensor theories. Before proceeding, it is useful to recall the basic ingredients of Brans–Dicke gravity that we use in this paper. The gravitational field is described by both a metric tensor $g_{ab}$ and a scalar field $\phi$ appearing in the (Jordan frame) action
\begin{equation}
S_{BD} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) + L^{(m)} \right],
\end{equation}
where $\omega$ is the constant Brans–Dicke parameter, $V(\phi)$ is a scalar field potential, and $L^{(m)}$ is the matter Lagrangian density. We adopt the notations of [13], but we retain Newton’s constant $G$ in our equations without setting it to unity. Note that the Brans–Dicke field $\phi$ is identified (apart, possibly, from a numerical coefficient) with the inverse effective gravitational coupling $G^{-1} = m_{Pl}^2$ (where $m_{Pl}$ is the Planck mass) and has therefore the dimensions of a mass squared. We restrict ourselves to positive values of $\phi$ to ensure that the graviton carries positive kinetic energy, and to the range of values of the Brans–Dicke parameter $2\omega + 3 > 0$ (the value $\omega = -3/2$ corresponds to a barrier that can not be crossed by a continuous change of this parameter, a theory with a non-dynamical scalar field $\phi$, a Cauchy problem that is ill-posed with most forms of matter [14, 23], and an unphysical dependence of the metric on derivatives of the matter fields of order higher than second [15]). The Brans–Dicke field equations are [5]
\begin{align}
G_{ab} &= \frac{8\pi}{\phi} T^{(m)}_{ab} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{1}{2\phi} g_{ab}, \quad (2) \\
\Box \phi &= \frac{1}{2\omega + 3} \left( 8\pi T^{(m)} + \phi \frac{dV}{d\phi} - 2V \right), \quad (3)
\end{align}
where $T^{(m)}_{ab} = \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-g}} (\sqrt{-g} L^{(m)})$ is the matter energy–momentum tensor and $T^{(m)}$ is its trace

1 The original Brans–Dicke theory [5] did not include a potential $V(\phi)$, and many authors prefer to reserve the name ‘Brans–Dicke theory’ to the case in which $V = 0$. Then, the theory described by (1) could be called ‘scalar–tensor theory with constant $\omega$ and linear coupling function’. For economy of terminology, we still refer to (1) as a Brans–Dicke action.

2
The Jordan frame action (1) can be mapped into its Einstein frame representation by means of the conformal transformation
\[ g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2 g_{ab}, \quad \Omega = \sqrt{G\phi}, \] (4)
accompanied by the scalar field redefinition
\[ \phi \rightarrow \tilde{\phi} = \sqrt{2\omega + 3} \ln \left( \frac{\phi}{\phi_0} \right) \] (5)
to make the kinetic energy of the new scalar assume the canonical form, and where \( \phi_0 \) is an irrelevant constant. Note that the dimensions of the new scalar field \( \tilde{\phi} \) are those of a mass.

The Einstein frame form of the action is
\[ S_{BD} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{16\pi G} - \frac{1}{2} \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} - U(\tilde{\phi}) + \frac{\mathcal{L}_m}{(G\phi)^2} \right], \] (6)
where the new scalar field potential is
\[ U(\tilde{\phi}) = V[\phi(\tilde{\phi})] \frac{(G\phi(\tilde{\phi}))^2}{2\omega + 3} \exp \left( -8 \sqrt{2\omega + 3} \tilde{\phi} \right) \] (7)
and a tilde denotes geometrical quantities associated with \( \tilde{g}_{ab} \).

2. What is the mass of a Brans–Dicke scalar field?

Let us begin with the Jordan frame description of the theory. According to the current terminology used for all kinds of scalar fields, in the presence of a non-trivial potential \( V \) the Brans–Dicke scalar \( \phi \) is massive, with the effective mass \( \mu \) given by
\[ \mu^2(\phi) \equiv \frac{d^2V}{d\phi^2}. \] (8)
However, it is already clear that this definition is problematic because \( \mu \) is dimensionless (\( V \) has the dimensions of an energy density, mass\(^4\), and so does \( \phi^2 \)). Apart from the dimensions, in Brans–Dicke theory other definitions of the effective mass are clearly possible. The Einstein frame effective mass \( \tilde{m} \) is given by
\[ \tilde{m}^2(\tilde{\phi}) \equiv \frac{d^2U}{d\tilde{\phi}^2}. \] (9)
When quantizing linearized scalar–tensor gravity in a canonical approach, the Einstein frame variables are necessary: it is the Einstein frame and not the Jordan frame metric perturbation that must be identified with the physical graviton and leads to the correct propagator. The propagator for the Einstein frame scalar field \( \tilde{\phi} \) in the absence of matter yields again \( \tilde{m} \) as the scalar field mass [16].

We want to express \( \tilde{m} \) in terms of \( \phi \) instead of \( \tilde{\phi} \). Equation (9) yields
\[ \tilde{m}^2 = \frac{d^2U}{d\tilde{\phi}^2} \left[ \frac{V(\phi(\tilde{\phi}))}{(G\phi(\tilde{\phi}))^2} \right] \]
\[ = \frac{1}{G^2\phi^2 \frac{d\phi}{d\tilde{\phi}}^2} \left[ \left( \frac{dV}{d\phi} \right) \frac{d^2\phi}{d\tilde{\phi}^2} + \frac{6V}{\phi^2} \frac{d\phi}{d\tilde{\phi}} + \frac{d^2V}{d\phi^2} \right] \left( \frac{d\phi}{d\tilde{\phi}} \right)^2 \] (10)
and, using
\[ \frac{d\phi}{d\tilde{\phi}} = \left( \frac{16\pi G}{2\omega + 3} \right)^{1/2} \phi, \quad \frac{d^2\phi}{d\tilde{\phi}^2} = \frac{16\pi G}{2\omega + 3} \phi, \] (11)
it leads to

\[ m^2(\phi) = \frac{16\pi}{(2\omega + 3) G\phi} \left( \frac{4V}{\phi} - \frac{3}{\phi} \frac{dV}{d\phi} + \phi \frac{d^2V}{d\phi^2} \right). \]  

(12)

Let us now see, in the Jordan frame, a third possible definition of effective mass of \( \phi \). In general, for a Klein–Gordon field \( \phi \) satisfying \( \Box \phi - \frac{dW}{d\phi} = S \), where \( S \) is a source term, the effective mass is defined by

\[ m^2_{\phi}(\phi) = \frac{d^2W}{d\phi^2}. \]  

In the case of a quadratic potential \( W(\phi) = \frac{\alpha}{2} \phi^2 + \beta \phi + \gamma \) (with \( \alpha, \beta, \) and \( \gamma \) constants), this leads to a constant \( m^2_{\phi} \), and the definition certainly makes sense around a minimum of the potential, i.e., for \( \beta = 0 \) and \( \alpha > 0 \). It is the common use of this definition that inspires equation (8). However, the Brans–Dicke field \( \phi \) does not satisfy the usual Klein–Gordon equation, but obeys equation (3) instead, in which the potential \( V \) appears in an unconventional way. Equation (3) can be turned into the usual Klein–Gordon form by introducing an effective potential \( V_{\text{eff}}(\phi) \) satisfying

\[ \Box \phi - \frac{dV_{\text{eff}}}{d\phi} = \frac{8\pi T_m}{2\omega + 3}, \]  

(13)

\[ \frac{dV_{\text{eff}}}{d\phi} = \frac{1}{2\omega + 3} \left( \frac{dV}{d\phi} - 2V \right). \]  

(14)

Integration with respect to \( \phi \) then yields

\[ V_{\text{eff}}(\phi) = \frac{1}{2\omega + 3} \left[ \phi V(\phi) - 3 \int d\phi \frac{V}{\phi} \right]. \]  

(15)

Note that the addition of a constant to \( V_{\text{eff}}(\phi) \) has no effect on the dynamics of \( \phi \), while adding a constant to \( V(\phi) \) does (more on this later). While \( V \) has the dimensions of an energy density, \( V_{\text{eff}} \) has the dimensions of an energy density times a mass squared.

The relation (15) between \( V \) and \( V_{\text{eff}} \) is rather peculiar: if \( V \) is even (odd), \( V_{\text{eff}} \) is odd (even) and vice versa. If \( V \) is a polynomial of degree \( n \) in \( \phi \), then \( V_{\text{eff}} \) is a polynomial of degree \( (n + 1) \). If \( V \) is positive definite or bounded, the same is not true in general for \( V_{\text{eff}} \).

Now, with the example of the Klein–Gordon equation in mind, one can define a third effective mass \( m \) for the field \( \phi \) by

\[ m^2(\phi) \equiv \frac{d^2V_{\text{eff}}}{d\phi^2} = \frac{1}{2\omega + 3} \left( \frac{\phi}{d\phi} \frac{d^2V}{d\phi^2} - \frac{dV}{d\phi} \right). \]  

(16)

A sufficient condition for \( V_{\text{eff}} \) to have a minimum at \( \phi_0 \) is \( V'_0 = 2V_0/\phi_0 \) and \( V''_0 > \frac{V'}{\phi_0} = \frac{V_0}{\phi^2_0} \). As for the definition (9) of \( \tilde{m} \), it is also inspired by the Klein–Gordon example, but it seems that a proper use of the Klein–Gordon analogy supports the definition (16) rather than (9).

We now have three different definitions (8), (16) and (9) of effective mass of \( \phi \), the first two given using Jordan frame quantities and the third given in the Einstein frame. At least formally, we can retain these definitions even away from minima of the respective (effective) potentials, although their interpretation as ‘masses’ would fail from a strict particle physics point of view.

In principle, infinitely many definitions are possible, corresponding to the infinitely many conformal frames that can be defined based on different choices of the conformal factor \( \Omega(\phi) \) in the conformal transformation of the metric \( \tilde{g}_{ab} = \Omega^2 g_{ab} \) and the corresponding potential

\[ \Omega^2 \approx 1 + \frac{\beta}{\phi^2} + \cdots \]  

2 Already the presence of a tadpole term corresponding to \( \beta \neq 0 \) leads to non-equilibrium and then the quantity \( m^2_{\phi} \) cannot be interpreted as a true mass squared.

\[ \tilde{m}^2 \approx \frac{16\pi}{(2\omega + 3) G\phi} \left( \frac{4V}{\phi} - \frac{3}{\phi} \frac{dV}{d\phi} + \phi \frac{d^2V}{d\phi^2} \right). \]  

(12)

3 This definition is used also in [17].
$U(\varphi) = V(\varphi(\varphi))/\Omega^4$. However, to be practical, we will focus on the three definitions given above.

Let us now see how bad the ambiguity in the concept of ‘mass of $\varphi$’ is by considering various potentials. These are not mere examples: as will be clear below, they correspond to the only cases in which serious ambiguities arise.

2.1. Constant potential $V = V_0$

In this case, it is

$$\mu = 0, \quad m = 0, \quad \tilde{m} = 8 \sqrt{\frac{\pi V_0}{(2\omega + 3)G \phi^4}}.$$  

(17)

The scalar $\varphi$ has infinite range according to all three definitions if $V_0 = 0$, but has finite range according to (9) and infinite range according to both (8) and (16) if $V_0 \neq 0$.

2.2. Linear potential $V = \alpha \varphi + V_0$

In this case, it is

$$\mu = 0, \quad m = \sqrt{\frac{-\alpha}{2\omega + 3}}, \quad \tilde{m} = 4 \sqrt{\frac{\pi (\alpha + 4V_0/\phi)}{(2\omega + 3)G \phi^4}}.$$  

(18)

A natural choice would seem to be $\alpha > 0$ which, for the assumed ranges $\varphi > 0$ and $2\omega + 3 > 0$ of the scalar and of the Brans–Dicke parameter, leads to a potential $V$ that is bounded from below. However, this leads to a tachyonic scalar according to the definition (16), which is based on the dynamical equation actually obeyed by $\varphi$. Aside from this problem, $\varphi$ seems to have infinite range according to $\mu$ but finite range according to $\tilde{m}$.

2.3. Quadratic potential $V = \frac{1}{2} \phi^2 + \beta$

In this case, it is

$$\mu = \sqrt{\alpha}, \quad m = 0, \quad \tilde{m} = 8 \sqrt{\frac{\beta}{(2\omega + 3)G \phi^4}}.$$  

(19)

If $\alpha > 0$, then $\phi$ has a finite range according to $\mu$ but an infinite range according to $m$, and finite or infinite range according to the Einstein frame $\tilde{m}$ if $\beta > 0$ or $\beta = 0$, respectively. Negative values of $\beta$ are ruled out by the requirement that the field be not tachyonic in the Einstein frame. $A$ priori, we identify here the possibility of a field being tachyonic in one conformal frame but not in the other, if the definitions given above make sense.

At this point, one may want to find all the instances in which $\tilde{m} = 0$ while $m \neq 0$, which is equivalent to finding all the functions $V(\phi)$ satisfying the ODE

$$\frac{d^2 V}{d\phi^2} - \frac{3}{\phi} \frac{dV}{d\phi} + \frac{4V}{\phi^2} = 0$$  

(20)

with $m \neq 0$. Equation (20) has the general solution

$$V(\phi) = \phi^2 \left[ A + B \ln \left( \frac{\phi}{\phi_0} \right) \right]$$  

(21)

with $A, B$ and $\phi_0 > 0$ being arbitrary constants. For $B = 0$ one recovers the case of a harmonic potential already considered, in which both $m$ and $\tilde{m}$ vanish. Therefore, the only case in which $\tilde{m} = 0$ while $m \neq 0$ is that of $V(\phi)$ given by equation (21) with $B \neq 0$. Moreover, the only
case in which both $m$ and $\tilde{m}$ vanish is that of a purely quadratic potential $B = 0$ (including, of course, the trivial case $A = 0$). The fact that the choice $V = \mu^2 \phi^2 / 2$ makes the potential disappear from the field equation (3) was noticed, but not pursued, in studies of the phase space of spatially homogeneous and isotropic scalar–tensor cosmology [18–20].

2.4. Potential $V = B \phi^2 \ln(\phi/\phi_0)$

In this case, it is

$$\mu = \sqrt{B \left[2 \ln \left(\frac{\phi}{\phi_0}\right) + 3\right]},$$

$$m = \sqrt{\frac{2B\phi}{2\alpha + 3}},$$

$$\tilde{m} = 0.$$  

If $B > 0$, then $\phi$ has finite range in the Jordan frame (according to both $\mu$ and $m$) but infinite range in the Einstein frame ($\mu$ could also become imaginary).

2.5. Consequences

Needless to say, the ranges (or masses) of the Brans–Dicke scalar can be quite different according to which of the three definitions (8), (16) or (9) is used, and care must be taken when using these quantities. This is particularly important when discussing violations of the experimental constraints in the parametrized post-Newtonian (PPN) formalism. For a free Brans–Dicke field, the experimental limit on the Brans–Dicke parameter is $\omega \geq 40,000$ [21]. To circumvent this constraint, a potential is usually given to $\phi$ to make it short ranged, thereby rendering local (terrestrial and solar system) experiments insensitive to $\phi$. This was done, historically, when it was recognized that one of the first string theories, bosonic string theory, has a low-energy limit corresponding to $\omega = -1$ Brans–Dicke gravity [2, 22]. More recently, a large amount of literature has been devoted to $f(R)$ theories of gravity described by the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [f(R) + \mathcal{L}^{(m)}],$$

in both the metric and Palatini approach (see [23, 24] for reviews). It is well known that these theories are equivalent to an $\omega = 0$ (for the metric case) or an $\omega = -3/2$ (for the Palatini case) Brans–Dicke theory [12]. The constraint on $\omega$ is circumvented by giving an effective mass to the Brans–Dicke scalar through the chameleon mechanism [23, 24]. Further, $\omega = 0$ Brans–Dicke theory is singled out by the area metric approach unifying metric and scalar field into a single geometric structure [25]. In these, and in similar, cases one needs to be clear on what is meant by ‘mass’ and ‘range’ of $\phi$. First, let us focus on the Jordan frame description of scalar–tensor gravity. The range of $\phi$ is determined by the equation of motion satisfied by $\phi$ (in the Jordan frame, equation (3)). This can be written in the Klein–Gordon form (13) and, in the weak-field, slow-motion regime appropriate to the low-density solar system environment, it becomes

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) - \frac{dV_{eff}}{d\phi} \simeq 0$$

4 There are no experimental constraints on gravity at distances smaller than 0.2 mm.
with the further assumption of spherical symmetry $\phi = \phi(r)$. The effective mass is determined by expanding $dV_{\text{eff}}/d\phi$ around the present value $\phi_0$ of $\phi$,

$$
\frac{dV_{\text{eff}}}{d\phi} = \frac{dV_{\text{eff}}}{d\phi} \bigg|_0 + \frac{d^2V_{\text{eff}}}{d\phi^2} \bigg|_0 \phi + \cdots = m_0^2 \phi + \cdots
$$

(27)

if $V_{\text{eff}}$ has a minimum at $\phi_0$, and where $m_0 \equiv m(\phi_0)$. Then, equation (27) has the usual Yukawa solution $\phi(r) \propto e^{-m_0 r} / r$ with range $m_0$ determined by the definition (16), and not by the definition (8) of $\mu$ or (9) of $\bar{m}$. This fact singles out the mass $m$ and the corresponding range of $\phi$ to be used in the PPN analysis. This also leads to some worries because, following the current terminology and usage, violations of the PPN limits are suppressed by giving a short range to $\phi$ according to the definition (8), not (16) (e.g., [26]). If this potential is quadratic, $V(\phi) = \frac{1}{2} \phi^2 + \beta$, the range of $\phi$ according to the correct definition (16) in the Jordan frame, is still infinite! And it will still be infinite also in the Einstein frame if $\beta = 0$. We conclude that, in the Jordan frame, one must be careful to choose $m$ instead of $\mu$ as the $\phi$-mass.

Let us turn now to the Einstein frame picture. The discussion above leads to the potential difficulty that, if $\beta \neq 0$ in the potential $V(\phi) = \frac{1}{2} \phi^2 + \beta$, the fact that $\phi$ has finite or infinite range and evades or not the PPN constraints, seems to depend on the conformal representation of the theory which is adopted. The other potentially dangerous situation is that of the potential $V(\phi) \propto \phi^2 \ln(\phi/\phi_0)$, which yields a massive field according to $\mu$ and $m$ but a free field in the Einstein frame $\bar{m} = 0$.

The violation or non-violation of the PPN constraints seems to depend on the conformal representation of the theory adopted. However, it is still true that a free field in the Jordan frame, defined by $V \equiv 0$, remains a free field in the Einstein frame ($U \equiv 0$). Moreover, from a pragmatic point of view, the problem is not so serious because the calculation of the PPN parameters and metric potentials is done in the Jordan frame (see, e.g., [1, 27])\(^5\). For this reason, we will not discuss this issue further. We only remark that these occurrences do not point to a failure of the physical equivalence between the two frames; they simply mean that care must be taken in defining and calculating quantities correctly when going to the Einstein frame representation. At least at the classical level, the two frames are physically equivalent representations of the same theory [28–30]. It is however, true that certain features of gravitational theories may not be formulated in a representation-independent way, a problem that has been discussed elsewhere [31].

When trying to understand the issue of mass, one should keep in mind that this concept relies on a scalar field potential and its expansion around a minimum. Now, in general, a potential with a given functional form $V(\phi)$ in the Jordan frame corresponds to a very different functional form $U(\phi) = V(\phi(\phi))/(G\phi)^2$ in the Einstein frame. For example, a purely quadratic $V = \mu^2 \phi^2 / 2$ corresponds to the potential $U(\phi) = \mu^2 \phi^2 / 2G\phi$. It is well known that switching to the Einstein frame representation is a mathematical technique which allows one to find exact solutions of the field equations that would be harder, or impossible to find in the Jordan frame. This technique is based on the knowledge of exact solutions of general relativity. However, these solutions correspond to rather odd potentials when mapped back into the Jordan frame. The point is that the functional form of the scalar field potential depends heavily on the conformal frame adopted and, accordingly, so do the fact that the potential has a minimum and the notion of scalar field mass. One has to live with this feature and its consequences if one wants to switch the conformal frame. Therefore, care must be

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\(^5\) This fact happens to be consistent with the fact that the physical choice of the mass that we identify as physical is given by equation (16) in the Jordan frame, but it is just a common practice to use the Jordan frame to perform the PPN limit and there is no fundamental reason to prefer this conformal frame.
taken when specifying the ‘mass’ or ‘range’ of a Brans–Dicke-like scalar field in scalar–tensor gravity.

3. An application to metric $f(R)$ gravity

As an application, we consider metric $f(R)$ gravity described by the action (25). This class of theories was important in building models of inflation in the early universe, and is now widely used to model the current acceleration of the universe as an alternative to a mysterious dark energy [9, 10]. In the metric formalism, in which the metric is the only variable and the connection is the metric one, the field equations are

$$f'(R)R_{ab} - \frac{f(R)}{2}g_{ab} = \nabla_a \nabla_b f'(R) - g_{ab} \Box f'(R) + 8\pi G T_{ab}^{\text{(m)}},$$

(28)

where a prime denotes differentiation with respect to $R$. It is required that $f'(R) > 0$ in order to have a positive effective gravitational coupling $G_{\text{eff}} = G/f'(R)$ and $f''(R) > 0$ for local stability [32].

It is well known that, if $f'' \neq 0$, $f(R)$ gravity is equivalent to a scalar–tensor theory [12]. In fact, by introducing an auxiliary field $\chi$, the action

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}[f(\chi) + f'(\chi)(R - \chi) + L^{(m)}]$$

(29)

is dynamically equivalent to (25). This is trivial if $\chi = R$. Vice versa, variation of (29) with respect to $\chi$ yields $f''(\chi)(R - \chi) = 0$, and $\chi = R$ if $f'' \neq 0$. We can now redefine the field $\chi$ as in $\phi = f'(\chi)$ (this is now dimensionless). By setting

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)),$$

(30)

the action becomes

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}[\phi R - V(\phi) + L^{(m)}],$$

(31)

which describes an $\omega = 0$ Brans–Dicke theory. The dynamical field $\phi = f'(R)$ obeys the trace equation

$$3 \Box \phi + 2V(\phi) - \phi \frac{dV}{d\phi} = 8\pi G T^{(m)}.$$

(32)

One is then led to establish the value of the mass of $\phi$ and to study the dynamics and stability of cosmological models based on this $f(R)$ gravity. Clearly, it would be incorrect to do this by looking at the shape, maxima and minima of $V(\phi)$. The dynamics of $\phi$, through equation (32), are not regulated by $dV/d\phi$ but by the combination $[\phi V'(\phi) - 2V(\phi)]/3$. Consider, for example, the model $f(R) = R + aR^2$ of interest for Starobinsky inflation in the early universe [35, 36]. By naively taking the potential (30), one obtains

$$V(\phi) = aR^2 = \frac{1}{4a}(\phi - 1)^2$$

(33)

and $\mu = 1/\sqrt{2a}$. The true mass $m$ of $\phi$ is instead given by

$$m^2 = \frac{d^2V_{\text{eff}}}{d\phi^2},$$

(34)

6 This is true for both metric and Palatini modified gravity: however, Palatini $f(R)$ gravity has been shown to be non-viable on many grounds [14, 15, 33, 34] and we do not discuss it here.
where

\[
\frac{dV_{\text{eff}}}{d\phi} = \frac{1}{3} \left[ \phi \frac{dV}{d\phi} - 2V(\phi) \right].
\] (35)

This leads to the correct mass \( m = 1/\sqrt{6a} \) (in agreement, e.g., with [36, 37]). For a general \( f(R) \) Lagrangian, it yields

\[
m^2 = \frac{d^2V_{\text{eff}}}{d\phi^2} = \frac{f' - Rf''}{3f''}.
\] (36)

This expression has been derived in different contexts and with different methods: in various studies of stability [38–41], perturbations [42, 43], propagator calculations for \( f(R) \) gravity [44], etc [45–48].

4. Conclusions

The notions of effective mass and range of a Brans–Dicke-like scalar field in scalar–tensor and \( f(R) \) gravity are often used, especially in relation with the PPN formalism. However, the ambiguities related to the usual meaning of these concepts, induced by familiarity with the Klein–Gordon equation instead of the wave equation obeyed by this field, are potentially dangerous. We have clarified this issue and pointed out the correct concept of scalar field effective mass.

A second issue is that of the representation dependence of the scalar field potential that is used to define the effective mass. While, in classical physics, it is perfectly legitimate to switch conformal frames, care must be taken to ensure that the definitions and transformations of geometrical and physical quantities are implemented properly. Even without switching frame, attention has been called to certain potentials (fortunately only a few of them) which can lead to errors. Unfortunately, the terminology currently in use can, and does, lead to misunderstandings. While, at the classical level, we have identified the Jordan frame mass \( m \) as the appropriate definition, when one tries to quantize gravity, one needs to resort to the Einstein frame quantities, and the mass \( \tilde{m} \) is then singled out. It is not surprising that one is faced with this change of prescription: it is well known that, while equivalent at the classical level, these two frames become inequivalent at the quantum level ([30, 31] and references therein).

In addition to the definition of mass of the Brans–Dicke scalar, the definition of cosmological constant deserves a comment. A constant term \( 2\Lambda \) in the scalar field potential \( V \) corresponds to a cosmological constant for non-minimally coupled scalars. However, due to the role played by \( V \) in the Brans–Dicke field equation (3), this is no longer true in scalar–tensor gravity (again, we limit ourselves to Brans–Dicke theory for the sake of illustration). Setting \( V = V_0 = \text{const.} \) does not make \( V \) disappear from this equation. Moreover, if one wants to add a term \( -\Lambda g_{ab} \) to the right-hand side of the field equation (2) for the metric tensor, one should contemplate the linear potential \( V = 2\Lambda \phi \), since the relevant term in this equation is \( -\frac{V}{2\phi}g_{ab} \). It is \( V = 2\Lambda \phi \), and not \( V = 2\Lambda \) that produces a \( -\Lambda g_{ab} \) term in (2). The linear potential produces an effective mass squared \( m^2 = \frac{-2\Lambda}{2\phi^2} \) for the field \( \phi \), as discussed in section 2.

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