Topologizing interpretable groups in $p$-adically closed fields

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Abstract

We consider interpretable topological spaces and topological groups in a $p$-adically closed field $K$. We identify a special class of "admissible topologies" with topological tameness properties like generic continuity, similar to the topology on definable subsets of $K^n$. We show every interpretable set has at least one admissible topology, and every interpretable group has a unique admissible group topology. We then consider definable compactness (in the sense of Fornasiero) on interpretable groups. We show that an interpretable group is definably compact if and only if it has finitely satisfiable generics ($fsg$), generalizing an earlier result on definable groups. As a consequence, we see that $fsg$ is a definable property in definable families of interpretable groups, and that any $fsg$ interpretable group defined over $\mathbb{Q}_p$ is definably isomorphic to a definable group.

1 Introduction

The theory of $p$-adically closed fields, denoted $p$CF, has a cell decomposition theorem [SvdD88 §4, Theorem 1.1'] analogous to the cell decomposition in o-minimal theories. This in turn yields a dimension theory [SvdD88 §3] and topological tameness results. Here are three representative results from the topological tameness of $p$CF:

- If $f : X \to Y$ is a definable function, then $f$ is continuous on a dense open subset of $X$ [SvdD88 §4, Theorem 1.1'].

- If $X$ is definable and non-empty, then the frontier $\partial X$ has dimension strictly less than the dimension of $X$ [CKDL17 Theorem 3.5].

- If $X$ is definable of dimension $n$, then there is an open definable subset $X' \subseteq X$ such that $X'$ is an $n$-dimensional definable manifold and $\dim(X \setminus X') < n$ (see Remark 4.21).

These results in turn allow one to give any definable group $G$ the structure of a definable manifold in a canonical way [Pil89].

Unlike a typical o-minimal theory, $p$CF does not have elimination of imaginaries. Consequently, interpretable sets and groups no longer come with obvious topologies. This is a
shame, as there are some important interpretable sets in \( p\text{CF} \), like the value group and the set of balls. Moreover, interpretable groups arise naturally in the study of definable groups when forming quotient groups.

One can define a general class of *interpretable topological spaces*, that is, definable topological spaces in \( p\text{CF}^{eq} \). However, topological tameness results no longer hold on this general class. For example, if \( X \) and \( Y \) are the home sort \( K \) with the standard topology and discrete topology, respectively, then the identity map \( X \to Y \) is not generically continuous. We need to exclude things like the discrete topology on the home sort.

In this paper, we define a special class of *admissible* interpretable topological spaces (Definition 4.6). The class of admissible topologies excludes cases like the discrete topology on the home sort. The class of admissible topologies has many nice properties. First, there are “enough” admissible topologies:

1. Every interpretable set has at least one admissible topology (Theorem 4.29). Every interpretable group has a unique admissible group topology (Theorem 5.10).
2. A definable set \( D \subseteq K^n \) is admissible, as a topological subspace of \( K^n \).
3. A definable manifold is admissible (Example 4.7).
4. An interpretable subspace of an admissible topological space is admissible (Proposition 4.13).
5. A product or disjoint union of two admissible topological spaces is admissible (Proposition 4.12).
6. Admissibility is preserved under interpretable homeomorphism.

Second, admissible topologies are “tame” or nice:

7. Admissible topological spaces are Hausdorff.
8. If \( X \) is admissible and \( p \in X \), then some neighborhood of \( p \) is interpretable homeomorphic to a definable subset of \( K^n \).
9. If \( X \) is admissible and \( D \subseteq X \) is interpretable, then the frontier \( \partial D \) has lower dimension than \( D \) (Proposition 4.34). See Section 2 for a review of dimension theory on interpretable sets.
10. If \( f : X \to Y \) is interpretable, then there is an interpretable closed subset \( D \subseteq X \) of lower dimension than \( X \), such that \( f \) is continuous on \( X \setminus D \) (Proposition 4.39).
11. If \( X \) is admissible, then there is an interpretable closed subset \( D \subseteq X \) of lower dimension than \( X \), such that \( X \setminus D \) is everywhere locally homeomorphic to \( K^n \), for \( n = \dim(X) \) (Corollary 4.37).
**Remark 1.1.** One interesting corollary of the final point is that if \( S \) is interpretable of dimension \( n \), then there is an interpretable injection from an \( n \)-dimensional ball into \( S \). (This can also be proved directly.)

Admissible interpretable topologies were introduced in [Joh18, Section 4] in the esoteric context of \( \sigma \)-minimal theories without elimination of imaginaries. Luckily, the arguments of [Joh18] carry over to the \( p \)-adic context with almost no changes. Nevertheless, we take the opportunity to clean up some of the proofs and results. We also slightly strengthen the definition of “admissible,” in order to get cleaner theorems.

We apply the theory of admissibility to interpretable groups in \( p \)CF. Classically, Pillay constructed a definable manifold structure on any definable group in \( p \)CF [Pil89]. Admissibility allows us to run the same arguments on interpretable groups, yielding the following:

**Theorem 1.2** (= Theorem 5.10). If \( G \) is an interpretable group, then there is a unique admissible group topology on \( G \).

When \( G \) is definable, the admissible group topology is Pillay’s definable manifold structure on \( G \). Theorem 1.2 feels counterintuitive in the case of the value group \((\Gamma, +)\), which doesn’t admit any obvious topology. In fact, we get the discrete topology:

**Proposition 1.3** (= Remark 5.14). If \( G \) is interpretable, the admissible group topology on \( G \) is discrete if and only if \( \text{dim}(G) = 0 \). In particular, the admissible group topology on \( \Gamma \) is the discrete topology.

Thus, nothing very interesting happens on 0-dimensional groups. Nevertheless, it is convenient to have a canonical topology which works uniformly across both definable groups and 0-dimensional interpretable groups. The admissible group topology is well-behaved in several ways:

**Theorem 1.4.** Let \( G \) be an interpretable group with its admissible group topology, and let \( H \) be an interpretable subgroup.

1. \( H \) is always closed (Proposition 5.16(1)). \( H \) is clopen if and only if \( \text{dim}(H) = \text{dim}(G) \) (Proposition 5.18).
2. The admissible group topology on \( H \) is the subspace topology (Proposition 5.16(2)).
3. If \( H \) is normal, then the admissible group topology on \( G/H \) is the quotient topology (Proposition 5.19(1)).

**Theorem 1.5.** Let \( f : G \to H \) be an interpretable homomorphism.

1. \( f \) is continuous with respect to the admissible group topologies on \( G \) and \( H \) (Proposition 5.17).
2. If \( f \) is injective, then \( f \) is a closed embedding (Corollary 5.17).
3. If \( f \) is surjective, then \( f \) is an open map (Corollary 5.22).
1.1 Application to \textit{fsg} groups

In future work with Yao [JY22a], Theorem 1.2 will be used to generalize some of the results of [JY22b] to interpretable groups. In the present paper, we apply Theorem 1.2 to the study of interpretable groups with \textit{fsg}.

Recall that an interpretable group has \textit{finitely satisfiable generics (fsg)} if there is a small model $M_0$ and a global type $p \in S_G(M)$ such that every left translate $g \cdot p$ is finitely satisfiable in $M_0$. This notion is due to Hrushovski, Peterzil, and Pillay [HPP08], who show that generic sets behave well in \textit{fsg} groups. An interpretable subset $X \subseteq G$ is said to be \textit{left generic} or \textit{right generic} if $G$ can be covered by finitely many left translates or right translates of $X$, respectively.

\textbf{Fact 1.6 ([HPP08 Proposition 4.2]).} Suppose $G$ has \textit{fsg}, witnessed by $p$ and $M_0$.

1. A definable set $X \subseteq G$ is left generic iff it is right generic.

2. Non-generic sets form an ideal: if $X \cup Y$ is generic, then $X$ is generic or $Y$ is generic.

3. A definable set $X$ is generic if and only if every left translate of $X$ intersects $G(M_0)$.

The significance of \textit{fsg} is that it corresponds to “definable compactness” in several settings. For example, if $G$ is a group definable in a nice o-minimal structure, then $G$ has \textit{fsg} if and only if $G$ is definably compact [HP11, Remark 5.3]. In $p$CF, a definable group $G$ has \textit{fsg} if and only if it is definably compact [OP08, Joh21]. With admissible group topologies in hand, the same arguments generalize to interpretable groups:

\textbf{Theorem 1.7 (= Theorem 7.1).} An interpretable group $G$ has \textit{fsg} if and only if it is definably compact with respect to the admissible group topology.

Here, definable compactness is in the sense of Fornasiero [For15]; see Definition 3.2. Using this, we obtain some consequences which have nothing to do with topology:

\textbf{Theorem 1.8.} \begin{enumerate}
\item The \textit{fsg} property is definable in families: if $\{G_a\}_{a \in X}$ is an interpretable family of interpretable groups, and $X_{f_{\text{sg}}}$ is the set of $a \in X$ such that $G_a$ has \textit{fsg}, then $X_{f_{\text{sg}}}$ is interpretable (Corollary 7.2).
\item If $G$ is interpretable over $\mathbb{Q}_p$ and $G$ has \textit{fsg}, then $G$ is isomorphic to a definable group (Corollary 7.3).
\end{enumerate}

Theorem 1.7 says something more concrete for 0-dimensional interpretable groups, such as groups interpretable in the value group $\Gamma$. To explain, we need a few preliminary remarks. The structure $\mathbb{Q}_p^{eq}$ eliminates $\exists^\infty$ (even though the theory $pCF^{eq}$ does not). Consequently, one can define a class of “pseudofinite” interpretable sets, characterized by the two properties:

\begin{itemize}
\item Over $\mathbb{Q}_p$, pseudofiniteness agrees with finiteness.
\item Pseudofiniteness is definable in families.
\end{itemize}
See Proposition 8.5 for a precise formulation. Pseudofiniteness can also be defined explicitly; see Definition 8.3 and Proposition 8.9. For example, \( S \) is pseudofinite if and only if \( S \) is definably compact with respect to the discrete topology. With pseudofiniteness in hand, Theorem 1.7 yields the following for 0-dimensional interpretable groups:

**Theorem 1.9 (= Proposition 8.8).** Let \( G \) be a 0-dimensional interpretable group. Then \( G \) has fsg if and only if \( G \) is pseudofinite.

### 1.2 Relation to prior work

This paper leans heavily on [Joh18], [Pil89], and [Joh21]. On some level, this paper is merely the following three observations:

1. The theory of “admissible topologies” from Sections 3 and 4 of [Joh18] can be transferred from the o-minimal setting to \( p \text{CF} \).

2. Pillay’s construction of definable manifold structures on definable groups [Pil89] can then be applied to interpretable groups, giving a unique admissible group topology on any interpretable group.

3. The proof in [Joh21] that “fsg = definable compactness” then generalizes to interpretable groups.

In the o-minimal context, a similar idea of constructing a topology on interpretable groups appears in the work of Eleftheriou, Peterzil, and Ramakrishnan [EPR14, Theorem 8.7]. Using their topology, the authors show that interpretable groups are definable [EPR14, Theorem 8.22], which implies in hindsight that the topology is Pillay’s topology on definable groups. (This contrasts with \( p \text{CF} \), where there are non-definable interpretable groups such as the value group and residue field.)

It is known that \( p \text{CF} \) eliminates imaginaries after adding the so-called geometric sorts to the language [HMR18, Theorem 1.1]. The \( n \)th geometric sort is a quotient of \( GL_n(K) \) and can be understood as a space of lattices in \( K^n \). It is probably possible to circumvent much of this paper, especially Section 3 through the explicit description of imaginaries, as explained in Section 8.1 below. However, the advantage of the present approach is that it is much more likely to generalize to other settings, such as \( P \)-minimal theories, where an explicit description of imaginaries is unknown.

### 1.3 Outline

In Section 2 we review the dimension theory for imaginaries in geometric structures following [Gag05]. In Section 3 we review the notion of definable and interpretable topological

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1 This is necessary because the use of \( \beta \)-independence and \( U^\beta \)-rank in [Joh18] no longer works in the non-rosy theory \( p \text{CF} \). Note that the use of \( \beta \)-independence and \( U^\beta \)-rank in [Joh18] was overkill; the dimension theory on imaginaries in geometric structures from [Gag05] would have sufficed. In fact, in o-minimal theories, the dimension on imaginaries agrees with \( U^\beta \)-rank.
spaces and Fornasiero’s definition of definable compactness [For15]. In Section 4 we develop the theory of admissible topologies in pCF, showing that they satisfy topological tameness properties akin to definable sets (§4.3), that there are enough of them (§4.2), and that natural operations on topological spaces preserve admissibility (§4.1). In Section 5 we turn to admissible group topologies, showing that each interpretable group has a unique admissible group topology, and checking the topological properties of homomorphisms. In Section 6 we make a few remarks about definable compactness in admissible interpretable topological spaces. In Section 7 we apply this to the study of fsg interpretable groups, and in Section 8 we analyze what happens for 0-dimensional groups. Finally, in Section 9 we consider future research directions, including possible generalizations and open problems.

1.4 Conventions

Open maps are assumed to be continuous. If X is a subset of a topological space, then ∂X denotes the frontier of X and bd(X) denotes the boundary. If E is an equivalence relation on a set X and X′ is a subset, then E ↾ X′ denotes the restriction of E to X′. If E is an equivalence relation on a topological space X, then X/E denotes the quotient topological space. When X′ ⊆ X, we sometimes abbreviate X′/(E ↾ X′) as X′/E.

Following [Bel12], we define a p-adically closed field to be a field elementarily equivalent to Q_p, and we denote the theory of p-adically closed fields by pCF. The term “p-adically closed field” is often used in a more general sense to refer to any field elementarily equivalent to a finite extension of Q_p. For simplicity, we will not consider this more general context. However, all the results in this paper generalize to p-adically closed fields in the broad sense, replacing Q_p with its finite extensions in certain places.

Symbols like x, y, z, a, b, c, . . . can denote singletons or tuples. Tuples are finite by default. Letters A, B, C, . . . are usually reserved for small sets of parameters, and letters M, N are usually reserved for small models. We denote the monster model by M. We maintain the distinction between definable and interpretable sets, as well as the distinction between reals (in M) and imaginaries (in M^eq). “Definable” means “definable with parameters,” and “0-definable” means “definable without parameters.” If D is definable, then “D” denotes “the” code of D, which is well-defined up to interdefinability, but is usually imaginary. We write the acl-dimension in geometric theories as dim(a/B) for complete types and dim(X) for definable sets.

2 Dimension theory in geometric structures

Let M be a monster model of a complete one-sorted theory T. Recall the following definitions from [HP94, Gag05].

Definition 2.1. T is a pregeometric theory if acl(−) satisfies the Steinitz exchange property. T is a geometric theory if it is pregeometric and ∃^∞ is eliminated.
There is a well-known dimension theory on pregeometric structures. This dimension theory assigns a dimension \( \dim(a/B) \) to each complete type \( \text{tp}(a/B) \) and a dimension \( \dim(X) \) to each definable set \( X \). By work of Gagelman [Gag05], the dimension theory extends to \( T^{eq} \). We review this theory below.

For the rest of this section, we assume \( T \) is pregeometric.

**Definition 2.2.** Let \( A \subseteq M \) be a small set of parameters and \( b = (b_1, \ldots, b_n) \) be a tuple in \( M \). The tuple \( b \) is acl-independent (over \( A \)) if \( b_i / A \notin \text{acl}(A \cup \{b_j : j \neq i\}) \) for \( 1 \leq i \leq n \). The dimension of \( b \) over \( A \), written \( \dim(b/A) \), is the length of a maximal subtuple \( c \) of \( b \) such that \( c \) is acl-independent over \( A \). This is independent of the choice of \( c \), assuming the exchange property.

The following properties of dimension are well-known:

1. Automorphism invariance: if \( \sigma \in \text{Aut}(M) \), then \( \dim(\sigma(a)/\sigma(B)) = \dim(a/B) \).
2. Extension: given \( a \) and \( B \subseteq C \), there is \( a' \equiv_B a \) with \( \dim(a'/C) = \dim(a'/B) = \dim(a/B) \).
3. Additivity: \( \dim(a,b/C) = \dim(a/Cb) + \dim(b/C) \).
4. Base monotonicity: if \( B \subseteq C \), then \( \dim(a/B) \geq \dim(a/C) \).
5. Finite character: Given \( a,B \) there is a finite subset \( B' \subseteq B \) with \( \dim(a/B') = \dim(a/B) \).
6. Anti-reflexivity: \( \dim(a/B) = 0 \iff a \in \text{acl}(B) \).

The exchange property is preserved when imaginaries are named as parameters:

**Fact 2.3** ([Gag05 Lemma 3.1]). Suppose \( A \subseteq M^{eq} \) is small and \( b,c \in M \). Then
\[
b \in \text{acl}(Ac) \setminus \text{acl}(A) \implies c \in \text{acl}(Ab). \tag{1}
\]

Therefore Definition 2.2 can be generalized to define \( \dim(b/A) \) for real \( b \in M^n \) and imaginary \( A \subseteq M^{eq} \). The six properties listed above continue to hold. Finally, we consider the case where \( b \) is imaginary:

**Definition 2.4.** Let \( b \) be a tuple of imaginaries and \( A \) be a set of imaginaries. Let \( c \) be a real tuple such that \( b \in \text{acl}^{eq}(Ac) \). Define
\[
\dim(b/A) := \dim(c/A) - \dim(c/Ab).
\]

**Fact 2.5.**

1. In Definition 2.4, \( \dim(b/A) \) is well-defined, independent of the choice of \( c \).
2. When \( b \) is a real tuple, Definition 2.4 agrees with Definition 2.2.
3. \( \dim(-/-) \) satisfies automorphism invariance, extension, additivity, base monotonicity, and finite character.
4. \( \dim(-/-) \) satisfies half of anti-reflexivity:

\[ b \in \acl^{eq}(A) \implies \dim(b/A) = 0. \]

Fact 2.5 is proved in [Gag05, Lemma 3.3, Proposition 3.4]. Gagelman omits the proof of Base Monotonicity, which is mildly subtle, so we review the proof for completeness:

**Proof (of base monotonicity).** Suppose \( a \in M^{eq} \) and \( B \subseteq C \in M^{eq} \). Take a real tuple \( d \in M^{n} \) such that \( a \in \acl^{eq}(Bd) \). By the extension property, we may move \( d \) by an automorphism and arrange \( \dim(d/Ba) = \dim(d/Ca) \). Then

\[
\dim(a/B) = \dim(d/B) - \dim(d/Ba) \geq \dim(d/C) - \dim(d/Ca) = \dim(a/C)
\]

by base monotonicity for real tuples.

**Definition 2.6.** Let \( A \) be a small subset of \( M^{eq} \) and \( X \) be an \( A \)-interpretable set. Then \( \dim(X) := \max_{c \in X} \dim(c/A) \). When \( X = \emptyset \), we define \( \dim(X) = -\infty \).

**Fact 2.7** ([Gag05, p. 320–321]). In Definition 2.6, \( \dim(X) \) does not depend on \( A \).

Fact 2.7 follows formally from base monotonicity and extension. Then Propositions 2.8 and 2.9 below follow formally from Fact 2.5 via the usual proofs.

**Proposition 2.8.** Let \( X,Y \) be interpretable sets.

1. \( \dim(X) \geq 0 \) iff \( X \neq \emptyset \).
2. If \( X \) is finite, then \( \dim(X) \leq 0 \).
3. If \( X \) is definable, then \( \dim(X) \leq 0 \) if and only if \( X \) is finite.
4. If \( X,Y \) are in the same sort, then \( \dim(X \cup Y) = \max(\dim(X), \dim(Y)) \).
5. \( \dim(X \times Y) = \dim(X) + \dim(Y) \).

**Proposition 2.9.** Let \( f : X \rightarrow Y \) be an interpretable function between two interpretable sets.

1. If every fiber has dimension at most \( k \), then \( \dim(X) \leq k + \dim(Y) \).
2. If every fiber has dimension at least \( k \), then \( \dim(X) \geq k + \dim(Y) \).
3. If every fiber has dimension exactly \( k \), then \( \dim(X) = k + \dim(Y) \).
4. If \( f \) is surjective, then \( \dim(X) \geq \dim(Y) \).
5. If \( f \) is injective, or more generally if \( f \) has finite fibers, then \( \dim(X) \leq \dim(Y) \).
6. If \( f \) is a bijection, then \( \dim(X) = \dim(Y) \).
Recall that the pregeometric theory $T$ is geometric if it eliminates $\exists^\infty$.

**Fact 2.10** ([Gag05, Proposition 3.7]).

1. If $a \in M^n$ and $B \subseteq M^{eq}$, then $\dim(a/B)$ is the minimum of $\dim(X)$ as $X$ ranges over $B$-definable sets containing $a$.

2. Suppose $T$ is geometric. If $a \in M^{eq}$ and $B \subseteq M^{eq}$, then $\dim(a/B)$ is the minimum of $\dim(X)$ as $X$ ranges over $B$-interpretable sets containing $a$.

The assumption that $T$ is geometric is necessary in (2), as shown by the following example. Suppose $M$ is an equivalence relation with infinitely many equivalence classes of size $n$, for each positive integer $n$. By saturation, there are also infinite equivalence classes. The following facts are easy to verify:

1. For $A \subseteq M$, the algebraic closure $\text{acl}(A) \subseteq M$ is the union of $A$ and all finite equivalence classes intersecting $A$.

2. $\text{acl}$ satisfies exchange (on $M$).

3. If $C$ is an equivalence class and $b = \lceil C \rceil \in M^{eq}$ is its code, then
   \[
   \dim(b/\emptyset) = \begin{cases} 
   1 & \text{if } C \text{ is finite} \\
   0 & \text{if } C \text{ is infinite}.
   \end{cases}
   
4. If $b$ is the code of an infinite equivalence class and $X$ is an $\emptyset$-interpretable set containing $b$, then $\dim(X) = 1 > \dim(b/\emptyset) = 0$.

Another special property of geometric theories is that dimension is definable in families:

**Fact 2.11** ([Gag05, Fact 2.4]). Suppose $T$ is geometric. Let $\{X_a\}_{a \in Y}$ be a definable family of definable sets. Then for any $k$, the set $\{a \in Y : \dim(X_a) = k\}$ is definable.

This holds for interpretable families as well:

**Proposition 2.12.** Suppose $T$ is geometric. Let $\{X_a\}_{a \in Y}$ be an interpretable family of interpretable sets. Then for any $k$, the set $\{a \in Y : \dim(X_a) = k\}$ is interpretable.

**Proof sketch.** Let $X$ be an interpretable set, defined as $D/E$ for some definable set $D \subseteq M^n$ and definable equivalence relation $E$ on $D$. Let $[a]_E$ denote the $E$-equivalence class of $a \in D$. Let $D_j = \{a \in D : \dim([a]_E) = j\}$. Let $X_j$ be the quotient $D_j/E$. Each set $D_j$ is definable by Fact 2.11. Using Propositions 2.8 and 2.9, one sees that
   \[
   \dim(X) = \dim \left( \bigcup_{j=0}^{n} X_j \right) = \max_{0 \leq j \leq n} \dim(X_j) = \max_{0 \leq j \leq n} (\dim(D_j) - j) .
   
   This calculates $\dim(X)$ in a definable way. \qed
2.1 Dimensional independence

Continue to assume $T$ is pregeometric, but not necessarily geometric.

**Lemma 2.13.** $\dim(a/B) = \dim(a/\acl^e(B))$.

*Proof.* Clear from the definitions. ∎

**Definition 2.14.** Suppose $a, b \in M$ and $C \subseteq M^e$. Then $a \downarrow_C^{\dim} b$ means that $\dim(a, b/C) = \dim(a/C) + \dim(b/C)$.

**Lemma 2.15.** Suppose $a, b \in M$ and $C \subseteq M^e$.

\begin{enumerate}
\item $a \downarrow_C^{\dim} b \iff b \downarrow_C^{\dim} a$.
\item $a \downarrow_C^{\dim} b$ if and only if $\dim(a/C) = \dim(a/Cb)$.
\item If $a' \in \acl^e(Ca)$ and $b' \in \acl^e(Cb)$, then $a \downarrow_C^{\dim} b \implies a' \downarrow_C^{\dim} b'$.
\end{enumerate}

*Proof.* (1) is trivial. (2) holds because

$$\dim(a/C) + \dim(b/C) - \dim(a, b/C) = \dim(a/C) - \dim(a/Cb) \geq 0$$

by additivity and base monotonicity. For part (3), we may assume $a' = a$ by symmetry. Then $a \downarrow_C^{\dim} b \implies \dim(a/C) = \dim(a/Cb)$. By Lemma 2.13 this means

$$\dim(a/\acl^e(C)) = \dim(a/\acl^e(Cb)).$$

But $\acl^e(C) \subseteq \acl^e(Cb') \subseteq \acl^e(Cb)$, so base monotonicity gives

$$\dim(a/\acl^e(C)) = \dim(a/\acl^e(Cb)) = \dim(a/\acl^e(Cb)).$$

By Lemma 2.13 again, $\dim(a/C) = \dim(a/Cb')$, and so $a \downarrow_C^{\dim} b'$. ∎

By Lemma 2.15(3), $a \downarrow_C^{\dim} b$ depends only on $a$ and $b$ as sets.

**Definition 2.16.** If $A, B, C \subseteq M^e$, then $A \downarrow_C^{\dim} B$ means that $A_0 \downarrow_C^{\dim} B_0$ for all finite subsets $A_0 \subseteq A$ and $B_0 \subseteq B$.

This extends Definition 2.14 by Lemma 2.15(3).

**Proposition 2.17.**

\begin{enumerate}
\item $A \downarrow_C^{\dim} B \iff B \downarrow_C^{\dim} A$.
\item If $A' \subseteq \acl^e(CA)$ and $B' \subseteq \acl^e(CB)$, then $A \downarrow_C^{\dim} B \implies A' \downarrow_C^{\dim} B'$.
\item $a \downarrow_C^{\dim} B$ holds iff $\dim(a/C) = \dim(a/CB)$.
\end{enumerate}
4. If \( B_1 \subseteq B_2 \subseteq B_3 \), then
\[
\dim_{B_1} A \downarrow_{B_3} \iff (\dim_{B_1} A \downarrow_{B_2} B_2 \text{ and } \dim_{B_2} A \downarrow_{B_3} B_3).
\]

**Proof.** The first two points are clear from Lemma 2.15. For part (3), use base monotonicity and finite character to reduce to the case where \( B \) is finite, which is Lemma 2.15(2). In part (4), we may assume \( A \) is a finite tuple \( a \), in which case (4) says
\[
\dim(a/B_1) = \dim(a/B_3) \iff (\dim(a/B_1) = \dim(a/B_2) \text{ and } \dim(a/B_2) = \dim(a/B_3)).
\]
This holds because \( \dim(a/B_1) \geq \dim(a/B_2) \geq \dim(a/B_3) \) by base monotonicity. \( \square \)

**Lemma 2.18.** Let \( a \) be a real tuple, possibly infinite. Let \( B \subseteq C \) be sets of imaginaries. Then there is a consistent partial \( \star \)-type \( \Sigma_{a,B,C}(x) \) whose realizations are the tuples \( a' \equiv_B a \) such that \( a' \downarrow^\dim_B C \).

**Proof.** Let \( I \) be a set indexing the tuple \( a = (a_i : i \in I) \). For each finite \( J \subseteq I \), let \( \pi_J \) be the projection map from \( I \)-tuples to \( J \)-tuples. (For example, \( \pi_J(a) \) is the finite subtuple of \( a \) determined by \( J \).) If \( a' \in M^I \), note that
- \( a' \equiv_B a \) holds iff \( \pi_J(a') \equiv_B \pi_J(a) \) for all finite \( J \subseteq I \).
- \( a' \downarrow^\dim_B C \) holds iff \( \pi_J(a') \downarrow^\dim_C \) for all finite \( J \subseteq I \), because of the definition of \( \downarrow^\dim_B \).

Therefore we may reduce to the case where \( I \) is finite. Let \( n = \dim(a/B) \). If \( a' \equiv_B a \), then \( \dim(a'/B) = n \) and so
\[
ad' \downarrow^\dim_B C \iff \dim(a'/B) \leq \dim(a'/C) \iff n \leq \dim(a'/C) \quad \text{(for } a' \equiv_B a\text{)}.
\]
Therefore
\[
\{a' \in M^I : a' \equiv_B a, \ a' \downarrow^\dim_B C\} = \{a' \in M^I : a' \equiv_B a, \ \dim(a'/C) \geq n\}.
\]

The condition \( a' \equiv_B a \) is defined by the type \( \text{tp}(a/B) \). By Fact 2.10(1), the condition \( \dim(a'/C) \geq n \) is defined by the type
\[
\{\neg \varphi(x,c) : \varphi \in L, c \in C, \ \dim(\varphi(M,c)) < n\}.
\]
Therefore \( \{a' \in M^I : a' \equiv_B a, \ a' \downarrow^\dim_B C\} \) is type-definable. Finally, the set is non-empty by the extension property of \( \dim(-/-) \). \( \square \)

**Proposition 2.19.** Let \( A, B, C \) be small subsets of \( M^{eq} \). Then there is \( \sigma \in \text{Aut}(M/C) \) such that \( \sigma(A) \downarrow^\dim_C B \).

**Proof.** Take an infinite real tuple \( a \) such that \( A \subseteq \text{dcl}^{eq}(a) \). By Lemma 2.18 there is some \( a' \equiv_C a \) such that \( a' \downarrow^\dim_C B \). Equivalently, there is \( \sigma \in \text{Aut}(M/C) \) such that \( \sigma(a) \downarrow^\dim_C B \).

Now \( \sigma(A) \subseteq \text{dcl}^{eq}(\sigma(a)) \), so \( \sigma(A) \downarrow^\dim_C B \) holds by Proposition 2.17(2). \( \square \)
3  Interpretable topological spaces and definable compactness

Let $M$ be any structure.

**Definition 3.1.** A topology on an interpretable set $X$ is an *interpretable topology* if there is an interpretable basis of opens, i.e., an interpretable family $\{S_a\}_{a \in Y}$ such that $\{S_a : a \in Y\}$ is a basis for the topology. An *interpretable topological space* is an interpretable set with an interpretable topology.

For example, if $M \models pCF$ then $M^n$ with the standard topology is an interpretable topological space.

A family of sets $\mathcal{F}$ is *downwards-directed* if for any $X,Y \in \mathcal{F}$, there is $Z \in \mathcal{F}$ with $Z \subseteq X \cap Y$. In topology, compactness can be defined as follows: a topological space $X$ is compact if any downwards-directed family of non-empty closed subsets of $X$ has non-empty intersection.

**Definition 3.2.** An interpretable topological space $X$ is *definably compact* (in the sense of Fornasiero) if every downwards-directed interpretable family of non-empty closed subsets of $X$ has non-empty intersection. An interpretable subset $D \subseteq X$ is *definably compact* if it is definably compact with respect to the subspace topology.

Definable compactness in this sense was studied independently by Fornasiero [For15] and the author [Joh18]. Many of the expected properties hold:

**Fact 3.3.** 1. If $X$ is an interpretable topological space and $X$ is compact, then $X$ is definably compact (clear).

2. If $X,Y$ are definably compact, then the disjoint union $X \sqcup Y$ and the product $X \times Y$ are definably compact [Joh18, Lemma 3.5(2), Proposition 3.7].

3. If $X$ is a Hausdorff interpretable topological space and $D \subseteq X$ is a definably compact interpretable subset, then $D$ is closed [Joh18, Lemma 3.8].

4. Finite sets are definably compact (clear).

5. If $X$ is an interpretable topological space and $D_1,D_2$ are definably compact interpretable subsets, then the union $D_1 \cup D_2$ is definably compact [Joh18, Lemma 3.5(2)].

6. If $f : X \rightarrow Y$ is an interpretable continuous map and $X$ is definably compact, then the image $f(X)$ is definably compact as a subset of $Y$ [Joh18, Lemma 3.4].

7. Definable compactness and non-compactness are preserved in elementary extensions (clear).

In $p$-adically closed fields definable compactness has additional nice properties:
Fact 3.4. 1. If \( M \) is a \( p \)-adically closed field, a definable set \( X \subseteq M^n \) is definably compact iff \( X \) is closed and bounded \([JY22b, \text{Lemmas 2.4, 2.5}]\).

2. Consequently, if \( M = \mathbb{Q}_p \), then a definable set \( X \subseteq M^n \) is definably compact iff it is compact.

4 Admissible topologies in \( p \)-adically closed fields

Work in a monster model \( \mathbb{M} \) of \( p\text{CF} \). We give \( \mathbb{M} \) the standard valuation topology and \( \mathbb{M}^n \) the product topology. If \( X \subseteq \mathbb{M}^n \) is definable, give \( X \) the subspace topology. This allows us to regard any definable set as a definable topological space.

Definition 4.1. Let \( X \) be a Hausdorff interpretable topological space.

1. \( X \) is a **definable manifold** if \( X \) is covered by finitely many open subsets \( U_1, \ldots, U_n \), such that \( U_i \) is interpretably homeomorphic to an open definable subset of \( \mathbb{M}^{k_i} \) for some \( k_i \).

2. \( X \) is **locally Euclidean** if for every point \( p \in X \), there is a neighborhood \( U \ni p \) and an interpretable homeomorphism from \( U \) to an open subset of \( \mathbb{M}^n \) for some \( n \) depending on \( p \).

3. \( X \) is **locally definable** if for every point \( p \in X \), there is a neighborhood \( U \ni p \) and an interpretable homeomorphism from \( U \) to a definable subspace of \( \mathbb{M}^n \) for some \( n \) depending on \( p \).

Remark 4.2. 1. In (1) and (2), we allow the dimension to vary from point to point(!)

2. Definable manifolds and locally Euclidean spaces are similar. The difference is that in a definable manifold the atlas is finite, whereas in a locally Euclidean space the atlas can be infinite.

3. Because we are working in a monster model, the atlas in local Euclideanity must be uniformly definable, i.e., the neighborhood \( U \ni p \) and open embedding \( U \to \mathbb{M}^n \) must have bounded complexity. The same holds for local definability.

4. \( \mathbb{M} \) with the discrete topology is a locally Euclidean definable topological space that is not a definable manifold.

5. If \( X \) is a definable manifold, then \( X \) is in interpretable bijection with a definable set. So definable manifolds are essentially definable objects.

Recall our convention that “open map” means “continuous open map.”

Definition 4.3. Let \( X \) be a Hausdorff interpretable topological space.

1. \( X \) is **definably dominated** if there is an interpretable surjective open map \( Y \to X \), for some definable set \( Y \subseteq \mathbb{M}^n \) with the subspace topology.
2. \( X \) is manifold dominated if there is an interpretable surjective open map \( Y \to X \), for some definable manifold \( Y \).

**Lemma 4.4.** Let \( X \to Y \) be an interpretable surjective open map between two Hausdorff interpretable topological spaces \( X \) and \( Y \).

1. If \( X \) is definably dominated, then \( Y \) is definably dominated.
2. If \( X \) is manifold dominated, then \( Y \) is manifold dominated.

**Proof.** We prove (1); (2) is similar. Take a definable set \( W \subseteq \mathbb{M}^n \) and an interpretable surjective open map \( W \to X \) witnessing the fact that \( X \) is definably dominated. Then the composition \( W \to X \to Y \) witnesses that \( Y \) is definably dominated.

**Lemma 4.5.** If \( X \) is manifold dominated, then \( X \) is definably dominated.

**Proof.** By Lemma 4.4, it suffices to show that definable manifolds are definably dominated. Let \( X \) be a definable manifold. By definition there are definable open sets \( U_i \subseteq \mathbb{M}^{n_i} \) for \( 1 \leq i \leq m \) and open embeddings \( f_i : U_i \to X \) which are jointly surjective. The disjoint union \( U_1 \sqcup \cdots \sqcup U_m \) is homeomorphic to a definable set \( D \subseteq \mathbb{M}^N \) for sufficiently large \( N \). The natural map \( U_1 \sqcup \cdots \sqcup U_m \to X \) is an interpretable surjective open map. Therefore \( X \) is definably dominated.

**Definition 4.6.** Let \( X \) be a Hausdorff interpretable topological space.

1. \( X \) is admissible if \( X \) is definably dominated and locally definable.
2. \( X \) is strongly admissible if \( X \) is manifold dominated and locally Euclidean.

We say that an interpretable topology \( \tau \) on an interpretable set \( X \) is admissible or strongly admissible if the space \( (X, \tau) \) is admissible or strongly admissible, respectively.

In [Joh18, Section 4], “admissibility” was used to refer to the weaker condition of definable domination, without local definability. However, the frontier dimension inequality fails without local definability (Proposition 4.34, Remark 4.35).

**Example 4.7.** If \( X \) is a definable manifold, then \( X \) is strongly admissible. (The identity map \( \text{id}_X \) witnesses that \( X \) is manifold dominated.) Similarly, if \( X \subseteq \mathbb{M}^n \) is a definable set with the subspace topology, then \( X \) is admissible.

**Example 4.8.** The discrete topology on the value group \( \Gamma \) is strongly admissible. The valuation map \( \mathbb{M} \to \Gamma \) witnesses manifold-domination. In contrast, the discrete topology on \( \mathbb{M} \) is not admissible, by Proposition 4.32 below.

**Example 4.9.** Let \( \Gamma_\infty \) be the extended value group \( \Gamma \cup \{+\infty\} \). Consider the topology on \( \Gamma_\infty \) with basis

\[ \{\{\gamma\} : \gamma \in \Gamma\} \cup \{[\gamma, +\infty] : \gamma \in \Gamma\}. \]

The valuation map \( \mathbb{M} \to \Gamma_\infty \) shows that \( \Gamma_\infty \) is manifold dominated. However, \( \Gamma_\infty \) is not locally definable. The point \( +\infty \in \Gamma_\infty \) has the property that every neighborhood is infinite and 0-dimensional. This cannot happen in a definable set, where being 0-dimensional is equivalent to being finite.
Example 4.10. Let $D \subseteq \mathbb{M}^2$ be the definable set $\{(x, y) \in \mathbb{M}^2 : x = 0 \lor y \neq 0\}$ with the subspace topology. Then $D$ is admissible. We will see below that $D$ is not manifold dominated (Example 6.3).

4.1 Closure properties

Remark 4.11. Let $X$ be a finite interpretable set with the discrete topology. Then $X$ is strongly admissible. In fact, $X$ is a definable manifold.

In the category of topological spaces, the class of open maps is closed under composition and base change. Consequently, if $f_i : Y_i \to X_i$ is an open map for $i = 1, 2$, then the product map $Y_1 \times Y_2 \to X_1 \times X_2$ is also an open map.

Proposition 4.12. The following classes of interpretable topological spaces are closed under finite disjoint unions and finite products.

1. The class of locally definable spaces.
2. The class of locally Euclidean spaces.
3. The class of definably dominated spaces.
4. The class of manifold dominated spaces.
5. The class of admissible spaces.
6. The class of strongly admissible spaces.

Proof. We focus on binary disjoint unions and binary products. (The 0-ary cases are handled by Remark 4.11)

Cases (2) and (11) are straightforward. For (3), let $X_1, X_2$ be two definably dominated spaces. Let $f_i : Y_i \to X_i$ be a map witnessing definable domination for $i = 1, 2$. Up to definable homeomorphism, $Y_1 \sqcup Y_2$ is a definable set, and so $Y_1 \sqcup Y_2 \to X_1 \sqcup X_2$ witnesses that $X_1 \sqcup X_2$ is definably dominated. Similarly, $Y_1 \times Y_2 \to X_1 \times X_2$ witnesses that $X_1 \times X_2$ is definably dominated. The proof for (4) is similar. Then (1) and (3) give (5), while (2) and (4) give (6).

Say that a class $\mathcal{C}$ of interpretable topological spaces is “closed under subspaces” if $\mathcal{C}$ contains every interpretable subspace of every $X \in \mathcal{C}$.

Proposition 4.13. The following classes of interpretable topological spaces are closed under subspaces:

1. The class of locally definable spaces.
2. The class of definably dominated spaces.
3. The class of admissible spaces.
Proof. (1) is straightforward, and (1) and (2) give (3), so it suffices to prove (2). Suppose $X$ is definably dominated, witnessed by an interpretable surjective open map $Y \to X$. Let $X'$ be an interpretable subspace of $X$. Let $Y'$ be the pullback $X' \times_X Y$:

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}
$$

Then $Y'$ is an interpretable subspace of the definable set $Y \subseteq M^n$, so $Y'$ is also a definable subset of $M^n$. The map $Y' \to X'$ is open and surjective because it is the pullback of the open and surjective map $X' \to X$. Then $Y' \to X'$ witnesses that $X'$ is definably dominated.

The class of strongly admissible spaces is not closed under subspaces; local Euclideanity is not preserved.

Remark 4.14. If $X$ is strongly admissible and $U$ is an interpretable open subset, then $U$ (with the subspace topology) is strongly admissible. Indeed, $U$ is Hausdorff and locally Euclidean, and if $Y$ is a definable manifold with an interpretable surjective open map $f : Y \to X$, then $f^{-1}(U)$ is a definable manifold and $f^{-1}(U) \to U$ shows that $U$ is manifold dominated.

4.2 Construction of strongly admissible topologies

In this section, we show that every interpretable set admits at least one strongly admissible topology. The proofs closely follow the proofs in the $\omega$-minimal case [Joh18, Section 3], and we omit the proofs when no changes are needed.

Definition 4.15. Let $X$ be a topological space and $E$ be an equivalence relation on $X$. Consider $X/E$ with the quotient topology.

1. $E$ is an OQ equivalence relation if $X \to X/E$ is an open map.

2. If $D \subseteq X$, then the $E$-saturation of $D$, written $D^E$, is the union of the $E$-equivalence classes intersecting $D$.

“OQ” stands for “open quotient.” In [Joh18], OQ equivalence relations were called “open equivalence relations,” and the $E$-saturation was called the “$E$-closure.”

Remark 4.16. Let $X$ be a topological space with basis $\mathcal{B}$, and let $E$ be an equivalence relation on $X$. The following are equivalent:

1. $E$ is an OQ equivalence relation on $X$.

2. For any open set $U \subseteq X$, the $E$-saturation $U^E$ is open.

3. For any basic open set $B \in \mathcal{B}$, the $E$-saturation $B^E$ is open.
4. If \( p, p' \in X \) and \( p E p' \) and \( B \in \mathcal{B} \) is a basic neighborhood of \( p \), then there is \( B' \in \mathcal{B} \) a basic neighborhood of \( p' \), such that \( \forall x \in B' \exists y \in B x Ey. \) (That is, \( B' \subseteq B^E \).)

**Fact 4.17** ([Joh18, Lemma 3.12]). Let \( X \) be an interpretable topological space and \( E \) be an interpretable OQ equivalence relation on \( X \). Then the quotient topology on \( X/E \) is interpretable.

When \( E \) is not an OQ equivalence relation, the quotient topology on \( X/E \) can fail to be interpretable. For example, this happens when \( X = \mathbb{M}^2 \) and \( E \) is the equivalence relation collapsing the \( x \)-axis to a point.

**Fact 4.18** ([Joh18, Lemma 3.13]). Let \( X \) be a topological space and \( E \) be an OQ equivalence relation on \( X \). Let \( X' \) be an open subset of \( X \) and let \( E' \) be the restriction \( E \upharpoonright X \).

1. \( E' \) is an OQ equivalence relation on \( X \).
2. \( X'/E' \rightarrow X/E \) is an open embedding.

**Definition 4.19.** Let \( X \) be a non-empty interpretable set. An interpretable subset \( Y \subseteq X \) is a large subset if \( \dim(Y \setminus X) < \dim(X) \).

This terminology follows [Pil88, Definition 1.11] and [EPR14, Definition 8.3]. In [Joh18], large sets were called “full sets.”

**Remark 4.20.**
1. If \( X \) is a large subset of \( Y \), then \( \dim(X) = \dim(Y) \).
2. If \( X \) is a large subset of \( Y \), and \( Y \) is a large subset of \( Z \), then \( X \) is a large subset of \( Z \).
3. If \( \dim(X) = 0 \), the only large subset is \( X \) itself.

**Remark 4.21.** If \( X \subseteq \mathbb{M}^n \) is definable, then there is a large open subset \( X' \subseteq X \) such that \( X' \) is a definable manifold. If \( X \) is \( M \)-definable for a small model \( M \leq \mathbb{M} \), then we can take \( X' \) to be \( M \)-definable. If \( k = \dim(X) \), then we can take \( X' \) to be a \( k \)-dimensional manifold in the strong sense that \( X' \) is covered by finitely many open sets definably homeomorphic to open subsets of \( \mathbb{M}^k \).

This is obvious from cell decomposition results and dimension inequalities, but we give a proof for completeness.

**Proof.** Say that a non-empty set \( C \subseteq \mathbb{M}^n \) is a \( t \)-cell [CKDL] Definition 2.6] if there is a coordinate projection \( \pi : \mathbb{M}^n \rightarrow \mathbb{M}^k \) such that \( \pi(C) \) is open in \( \mathbb{M}^k \), and \( C \rightarrow \pi(C) \) is a homeomorphism. By [SvdD88, §4, Theorem 1.1’], we can write \( X \) as a finite disjoint union of \( t \)-cells \( \bigcup_{i=1}^m C_i \). If \( X \) is \( M \)-definable, we can take the \( C_i \) to be \( M \)-definable. Without loss of generality, \( C_1, \ldots, C_j \) have dimension \( k \) and \( C_{j+1}, \ldots, C_m \) have dimension < \( k \). Let \( D_i \) be the union of the cells other than \( C_i \). For \( i \leq j \), let

\[
U_i = X \setminus \overline{D_i} = C_i \setminus \overline{D_i} = C_i \setminus \partial D_i
\]
and take $X' = \bigcup_{i=1}^{j} U_i$. Each set $U_i$ is an open subset of $X$, so $X'$ is an open subset. For each $i \leq j$, $U_i$ is an open subset of $C_i$, which is definably homeomorphic to an open subset of $\mathbb{M}^k$, and therefore $U_i$ is also definably homeomorphic to an open subset of $\mathbb{M}^k$. Therefore $X'$ is a $k$-dimensional manifold in the strong sense. To see that $X'$ is a large subset of $X$, note that

$$X \setminus X' = \bigcup_{i=1}^{j} (C_i \cap \partial D_i) \cup \bigcup_{i=j+1}^{m} C_i.$$  

For $i > j$, $\dim(C_i) < k$ by assumption. The frontier dimension inequality [CKDL17, Theorem 3.5] shows that $\dim(C_i \cap \partial D_i) \leq \dim(\partial D_i) < \dim(D_i) \leq k$. Therefore $\dim(X \setminus X') < k = \dim(X')$.

**Definition 4.22.** Let $D$ be an $A$-interpretable set. An element $b \in D$ is **generic in $D$ (over $A$)** if $\dim(b/A) = \dim(D)$.

**Warning.** If $D$ is 0-dimensional, then every element of $D$ is generic.

**Lemma 4.23.** If $X \subseteq \mathbb{M}^n$ is non-empty and $A$-definable, and $p$ is generic in $X$ (over $A$), then $X$ is locally Euclidean on an open neighborhood of $p$.

**Proof.** By Remark 4.21 there is a large open subset $X' \subseteq X$ that is a definable manifold. Let $B$ be a set of parameters defining $X'$. Moving $B$ and $X'$ by an automorphism over $A$, we may assume $p \downarrow_{A}^{\dim} B$. Then $\dim(p/B) = \dim(p/A) = \dim(X) > \dim(X \setminus X')$, and so $p$ is not in the $B$-definable set $X \setminus X'$. Then $p \in X'$, and $X'$ is the desired open neighborhood of $p$. \hfill \Box

The following lemma, which parallels [Joh18, Lemma 3.15], collects some useful tools.

**Lemma 4.24.** Let $X \subseteq \mathbb{M}^n$ be $A$-definable, endowed with the subspace topology.

1. Let $D$ be a definable subset of $X$. Let $\partial D$ be the frontier of $D$, within the topological space $X$. Then $\dim(\partial D) < \dim(D)$.

2. Suppose $P \subseteq X$ is definable or $\forall$-definable over $A$, and suppose $P$ contains every element that is generic in $X$. Then there is an $A$-definable large open subset $X' \subseteq X$ such that $X' \subseteq P$.

3. Suppose $b \in X$ and $U$ is a neighborhood of $b$ in the topological space $X$. Let $S \subseteq \mathbb{M}^{eq}$ be a small set of parameters. Then there is a smaller open neighborhood $b \in U' \subseteq U$ such that $\lceil U' \rceil \downarrow_{A}^{\dim} bS$.

**Proof.** Part (1) is [CKDL17, Theorem 3.5]. Part (2) is proved analogously to [Joh18, Lemma 3.15(2)], making use of Fact 2.10(2). Part (3) requires a new argument. Take $U' = X \cap B$, where $B$ is a sufficiently small ball in $\mathbb{M}^n$ around $b$. A simple calculation shows that the set of balls in $\mathbb{M}^n$ is 0-dimensional. Therefore

$$\dim(\lceil B \rceil/\emptyset) = \dim(\lceil B \rceil/A) = \dim(\lceil B \rceil/AbS) = 0,$$

implying $\lceil B \rceil \downarrow_{A}^{\dim} bS$. But $\lceil U' \rceil \in \text{dcl}^{eq}(A\lceil B \rceil)$, and so $\lceil U' \rceil \downarrow_{A}^{\dim} bS$. \hfill \Box
The next three lemmas show that given a definable set $X$ and definable equivalence relation $E$, we can improve the nature of $X/E$ by repeatedly replacing $X$ with a large open definable subset.

**Lemma 4.25.** Let $X \subseteq \mathbb{M}^n$ be $A$-definable and non-empty and let $E$ be an $A$-definable equivalence relation on $X$. Then there is an $A$-definable large open subset $X' \subseteq X$ such that the restriction $E \upharpoonright X'$ is an OQ equivalence relation on $X'$.

**Proof.** The proof of [Joh18, Proposition 3.16] works verbatim, replacing $\downarrow^b$ with $\downarrow^{\dim}$. □

**Lemma 4.26.** Let $X \subseteq \mathbb{M}^n$ be $A$-definable and non-empty and let $E$ be an $A$-definable OQ equivalence relation on $X$. Then there is an $A$-definable large open subset $X' \subseteq X$ such that $X'/E := X'/ (E \upharpoonright X')$ is Hausdorff.

**Proof.** The proof of [Joh18, Proposition 3.18] works verbatim, replacing $\downarrow^b$ with $\downarrow^{\dim}$. □

**Lemma 4.27.** Let $X \subseteq \mathbb{M}^n$ be $A$-definable and non-empty and let $E$ be an $A$-definable OQ equivalence relation on $X$ such that $X/E$ is Hausdorff. Then there is an $A$-definable large open subset $X' \subseteq X$ such that $X'/E$ is locally Euclidean.

**Proof.** The proof of [Joh18, Proposition 3.20] works verbatim, replacing $\downarrow^b$ with $\downarrow^{\dim}$. Definable compactness continues to work properly thanks to Fact 3.3. The use of the $\text{dcl}(-)$ pregeometry on the home sort does not present any problems because of Fact 2.3 and the fact that $\text{acl}^{eq}(A) \cap \mathbb{M} = \text{acl}^{eq}(A) \cap \mathbb{M}$ for $A \subseteq \mathbb{M}^{eq}$ in $\text{pCF}$. □

**Theorem 4.28.** Let $X \subseteq \mathbb{M}^n$ be a non-empty definable set, and $E$ be a definable equivalence relation on $X$. There is a large open subset $X' \subseteq X$ such that

- $X'$ is a definable manifold
- $E \upharpoonright X'$ is an OQ equivalence relation on $X'$
- $X'/E$ is Hausdorff
- $X'/E$ is locally Euclidean.

If $X$ and $E$ are $M$-definable for some small model $M \preceq \mathbb{M}$, then we can take $X'$ to be $M$-definable.

**Proof.** The proof is similar to [Joh18, Theorem 3.14]. Take a small model $M$ defining $X$ and $E$. First apply Remark 4.21 to obtain an $M$-definable large open subset $X_1 \subseteq X$ such that $X_1$ is a definable manifold. Let $E_1 = E \upharpoonright X_1$. Then apply Lemma 4.25 to get an $M$-definable large open subset $X_2 \subseteq X_1$ such that if $E_2 = E \upharpoonright X_2$, then $E_2$ is an OQ equivalence relation on $X_2$. Then apply Lemma 4.26 to get an $M$-definable large open subset $X_3 \subseteq X_2$ such that if $E_3 = E \upharpoonright X_3$, then $X_3/E_3$ is Hausdorff. (The relation $E_3$ is an OQ equivalence relation on $X_3$ by Fact 4.18[1].) Then apply Lemma 4.27 to get an $M$-definable large open subset $X_4 \subseteq X_3$ such that if $E_4 = E \upharpoonright X_4$, then $X_4/E_4$ is locally Euclidean. By Fact 4.18, $E_4$ is an OQ equivalence relation on $X_4$, and the quotient space $X_4/E_4$ is an open subspace of $X_3/E_3$. In particular, $X_4/E_4$ is Hausdorff. By Remark 4.20[2], $X_4$ is a large open subset of $X$. Thus $X_4$ is a definable manifold. Take $X' = X_4$. □
**Theorem 4.29.** Let \( Y \) be an interpretable set. Then \( Y \) admits at least one strongly admissible topology. If \( Y \) is \( M \)-interpretable for a small model \( M \preceq \mathbb{M} \), then \( Y \) admits an \( M \)-interpretable strongly admissible topology.

**Proof.** The proof is similar to [Joh18, Proposition 4.2]. Write \( Y \) as \( X/E \) for some \( M \)-definable set \( X \subseteq \mathbb{M}^n \) and \( M \)-definable equivalence relation \( E \). Proceed by induction on \( \dim(X) \). If \( \dim(X) = -\infty \), then \( X \) and \( Y \) are empty, and the unique topology on \( Y \) is strongly admissible. Suppose \( \dim(X) \geq 0 \), so \( X \) and \( Y \) are non-empty. By Theorem 4.28, there is an \( M \)-definable large open subset \( X' \subseteq X \) such that \( X' \) is a definable manifold, and if \( E' = E \upharpoonright X' \), then \( E' \) is an OQ equivalence relation on \( X' \), and the quotient \( X'/E' \) is Hausdorff and locally Euclidean. Then the quotient topology on \( X'/E' \) is interpretable by Fact 4.17, and the map \( X' \to X'/E' \) is a surjective interpretable open map, because \( E' \) is an OQ equivalence relation. Therefore \( X' \to X'/E' \) witnesses that \( X'/E' \) (with the quotient topology) is strongly admissible.

Let \( Y' = X'/E' \), and let \( Y'' = Y \setminus Y'' \). Then \( Y \) is a disjoint union of \( Y' \) and \( Y'' \). Let \( X'' \) be the preimage of \( Y'' \) in \( X \), and let \( E'' = E \upharpoonright X'' \). Then \( Y'' = X''/E'' \). Moreover, \( X'' \cap X' = \emptyset \), so \( X'' \subseteq X \setminus X' \) and then \( \dim(X'') < \dim(X) \) as \( X' \) is large. By induction, \( Y'' \) admits a strongly admissible topology. So \( Y' \) and \( Y'' \) both admit strongly admissible topologies. The disjoint union topology on \( Y \) is strongly admissible by Proposition 4.12. \( \square \)

### 4.3 Tameness in admissible topologies

In this section, we verify that the topological tameness theorems for definable sets extend to admissible topological spaces.

**Lemma 4.30.** Let \( X \) be a definably dominated interpretable topological space, interpretable over a small set of parameters \( A \). Let \( S \) be a small set of parameters. Let \( b \in X \) be a point. Let \( U \ni b \) be a neighborhood of \( b \) in \( X \). Then there is a smaller neighborhood \( b \in U' \subseteq U \) such that \( \mathcal{V}U'' \upharpoonright A \dim_{A} bS \).

This is similar to but slightly stronger than [Joh18, Lemma 4.5], and nearly the same proof works:

**Proof.** Let \( f : Y \to X \) be a map witnessing definable domination. In particular, \( Y \) is a definable set, and \( f \) is a surjective open map. Let \( A' \supseteq A \) be a small set over which \( X, Y, \) and \( f \) are interpretable. Moving \( f, Y, A' \) by an automorphism over \( A \), we may assume \( A' \downarrow_{A} \dim_{A} pS \) by Proposition 2.19. Take \( \bar{p} \in Y \) with \( f(\bar{p}) = p \). Then \( f^{-1}(U) \) is a neighborhood of \( \bar{p} \). By Lemma 4.24, there is a smaller neighborhood \( U'' \) of \( \bar{p} \) in \( Y \) such that \( \mathcal{V}U'' \upharpoonright A' \dim_{A'} \bar{p}S \). Take \( U' = f(U'') \); this is a neighborhood of \( p \) because \( f \) is an open map. Then \( \mathcal{V}U'' \upharpoonright A' \dim_{A'} pS \). As \( A' \downarrow_{A} \dim_{A} pS \), left transitivity gives \( \mathcal{V}U'' \upharpoonright A \dim_{A} pS \). \( \square \)

**Definition 4.31.** Let \( X \) be an interpretable topological space and \( p \in X \) be a point. The local dimension of \( X \) at \( p \), written \( \dim_{p}(X) \), is the minimum of \( \dim(U) \) as \( U \) ranges over neighborhoods of \( p \) in \( X \). More generally, if \( D \) is an interpretable subset of \( X \), then \( \dim_{p}(D) \) is the minimum of \( \dim(U \cap D) \) as \( U \) ranges over neighborhoods of \( p \) in \( X \).
**Proposition 4.32.** Let $X$ be a definably dominated interpretable topological space. Then $\dim(X) = \max_{p \in X} \dim_p(X)$.

**Proof.** The proof of [Joh18, Proposition 4.6] goes through verbatim, replacing $\downarrow^b$ with $\downarrow^{\dim}$, using Lemma 4.30. \qed

**Corollary 4.33.** Let $X$ be an interpretable set with $n = \dim(X)$. Then there is a non-empty open definable set $D \subseteq M^n$ and an interpretable injection $D \hookrightarrow X$.

**Proof.** By Theorem 4.29, there is a strongly admissible topology on $X$. By Proposition 4.32 there is a point $p \in X$ with $\dim_p(X) = n$. By local Euclideanity, $p$ has a neighborhood that is interpretably homeomorphic to a ball in $M^m$ for some $m$. Clearly $m = \dim_p(X)$. \qed

**Proposition 4.34.** Let $X$ be an admissible interpretable topological space and $D$ be an interpretable subset. Then $\dim \partial D < \dim D$, $\dim \bd(D) < \dim(X)$, and $\dim(D) = \dim(D)$.

**Proof.** The proof of [Joh18, Proposition 4.7] goes through almost verbatim, using Proposition 4.32. \qed

**Remark 4.35.** Proposition 4.34 fails if we replace “admissible” with the weaker condition “definably dominated.” If $\Gamma_\infty$ is as in Example 4.9 then the subset $\Gamma \subseteq \Gamma_\infty$ has the same dimension (zero) as its frontier $\{+\infty\}$.

**Proposition 4.36.** Let $X$ be an admissible interpretable topological space. Then there is a large open subset $X' \subseteq X$ such that $X'$ (with the subspace topology) is locally Euclidean.

**Proof.** The proof is similar to [Joh18, Proposition 4.7(3) and Lemma 4.9]. Let $A$ be a small set of parameters over which $X$ is interpretable. Let $X_{\text{Eu}}$ be the locally Euclidean locus of $X$, the set of points $p \in X$ such that some neighborhood of $p$ is interpretably homeomorphic to an open definable subset of $M^n$ for some $n$. It is easy to see that $X_{\text{Eu}}$ is $\lor$-definable, a small union of $A$-interpretable subsets of $X$.

Let $Y$ be the union of the $A$-interpretable subsets $D \subseteq X$ with $\dim(D) < \dim(Y)$. Like $X_{\text{Eu}}$, the set $Y$ is $\lor$-definable over $A$.

**Case 1: $X_{\text{Eu}} \cup Y \subseteq X$.** Take $b \in X \setminus (X_{\text{Eu}} \cup Y)$. Then $\dim(b/A) = \dim(X)$ by Proposition 2.10[2] and the fact that $b \notin Y$. By local definability and Lemma 4.30 there is a neighborhood $U$ of $b$ such that $U$ is homeomorphic to a definable subset of $M^n$ and $\Gamma U \downarrow^\dim_A b$. Let $f : U \to M^n$ be an interpretable topological embedding. Let $B \supseteq A$ be a small set of parameters defining $U$ and $f$. Moving $B$, $f$ by an automorphism in $\text{Aut}(M^n)$, we may assume $B \downarrow^\dim_{A^\Gamma U} b$. By left transitivity, $B \downarrow^\dim_A b$. Then

$$\dim(X) = \dim(b/A) = \dim(b/B) = \dim(f(b)/B) \leq \dim(f(U)) = \dim(U) \leq \dim(X).$$

Therefore $f(b)$ is generic in the $B$-definable set $f(U)$. By Lemma 4.23, $f(U)$ is locally Euclidean on an open neighborhood of $f(b)$. Then $U$ is locally Euclidean on an open neighborhood of $b$, contradicting the fact that $b \notin X_{\text{Eu}}$. 21
Case 2: $X = X_{Eu} \cup Y$. Then saturation gives $X = X_{Eu} \cup Y_0$ where $Y_0$ is a finite union of $A$-interpretable subsets of $X$ of lower dimension. In particular, $\dim(Y_0) < \dim(X)$. By Proposition 4.34, the closure $\overline{Y_0}$ has lower dimension than $X$. Take $X' = X \setminus \overline{Y_0}$. 

**Corollary 4.37.** If $X$ is an admissible interpretable topological space of dimension $n$, then there is a large open subset $X' \subseteq X$ such that $X'$ is everywhere locally homeomorphic to $M^n$.

*Proof.* By Proposition 4.36, we may pass to a large open subset and assume $X$ is locally Euclidean. Then partition $X$ as $\bigcup_{i=0}^n X_i$, where $X_i = \{ p \in X : \dim_p(X) = i \}$. Local Euclideanity ensures that $p \mapsto \dim_p(X)$ is locally constant, so each $X_i$ is open. By Proposition 4.32, $\dim(X_i) = i$ for non-empty $X_i$. Therefore $X' := X_n$ is non-empty, and its complement has dimension $< n$. 

A slightly different argument from Proposition 4.36 gives the following:

**Lemma 4.38.** Let $X$ be a non-empty definably dominated interpretable topological space. Then $X$ has a non-empty open subset $X' \subseteq X$ that is strongly admissible.

*Proof.* Take $Y \subseteq M^n$ definable and $f : Y \to X$ an interpretable surjective open map. Let $E$ be the kernel equivalence relation $\{(x,y) \in Y^2 : f(x) = f(y)\}$. Then we can identify $X$ with $Y/E$ on the level of sets. Moreover, $X$ is homeomorphic to $Y/E$ with the quotient topology:

- If $U \subseteq X$ is open, then $f^{-1}(U)$ is open because $f$ is continuous.
- If $f^{-1}(U)$ is open then $U = f(f^{-1}(U))$ is open because $f$ is a surjective open map.

Thus we may identify $X$ with the quotient topology $Y/E$. The fact that $Y \to Y/E$ is open means that $E$ is an OQ equivalence relation.

By Theorem 4.28, there is a large (hence non-empty) open subset $Y' \subseteq Y$ such that $Y'$ is a definable manifold, and $Y'/E$ is locally Euclidean. By Fact 4.18, $Y'/E$ is an open subspace of $Y/E = X$, and $Y' \to Y'/E$ is an open map (i.e., $E \upharpoonright Y'$ is an OQ equivalence relation). Then $X' := Y'/E$ is strongly admissible. 

**Proposition 4.39.** Let $X, Y$ be admissible topological spaces. Let $f : X \to Y$ be interpretable.

1. There is a large open subset $X' \subseteq X$ on which $f$ is continuous.
2. We can write $X$ as a finite disjoint union of locally closed interpretable subsets on which $f$ is continuous.

*Proof.* The proof of [Joh18, Proposition 4.12] works, essentially verbatim, using generic continuity of definable functions in pCF. 

**Corollary 4.40.** Let $X, Y$ be $A$-interpretable admissible topological spaces, and $f : X \to Y$ be $A$-interpretable. If $b \in X$ is generic over $A$, then $f$ is continuous at $p$. 

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5 Admissible groups

5.1 Uniqueness

Lemma 5.1. Let $G$ be an interpretable group, and let $\tau$ be an interpretable topology that is invariant under left translations. If $\tau$ is definably dominated, then $\tau$ is admissible and locally Euclidean.

Proof. Lemma 4.38 gives a non-empty open subset of $(G, \tau)$ that is locally Euclidean. By translation invariance, $(G, \tau)$ is locally Euclidean everywhere. Then $(G, \tau)$ is definably dominated and locally definable, i.e., admissible. □

Lemma 5.2. Let $G$ be an interpretable group. Then $G$ admits at most one admissible topology $\tau$ that is invariant under left-translations.

Proof. Suppose $\tau_1$ and $\tau_2$ are left-invariant admissible topologies on $G$. By Proposition 4.39, the map $\text{id}_G : (G, \tau_1) \to (G, \tau_2)$ is continuous at at least one point $a \in G$. If $\delta \in G$ and $f(x) = \delta \cdot x$, then the composition

$$(G, \tau_1) \xrightarrow{f} (G, \tau_1) \xrightarrow{\text{id}_G} (G, \tau_2) \xrightarrow{f^{-1}} (G, \tau_2)$$

is continuous at $\delta^{-1}a$. That is, $\text{id}_G : (G, \tau_1) \to (G, \tau_2)$ is continuous at $\delta^{-1}a$, for arbitrary $\delta$. Then $\text{id}_G : (G, \tau_1) \to (G, \tau_2)$ is continuous (everywhere). Similarly, $\text{id}_G : (G, \tau_2) \to (G, \tau_1)$ is continuous, implying $\tau_1 = \tau_2$. □

Lemma 5.2 shows that an interpretable group admits at most one admissible group topology. In the following section, we will see that every interpretable group admits at least one admissible group topology. For now, we draw a useful consequence of the above lemmas.

Lemma 5.3. Let $G$ be an interpretable group. Let $\tau$ be a definably dominated interpretable topology on $G$ that is invariant under left translations. Then $\tau$ is an admissible group topology on $G$.

Proof. First, $\tau$ is admissible by Lemma 5.1. We claim $\tau$ is invariant under right translations. Take $g \in G$, and let $\tau'$ be the image of $\tau$ under the right translation $x \mapsto g \cdot x$. Then $\tau'$ is invariant under left translations (because left and right translations commute). By Lemma 5.2, $\tau' = \tau$. Thus $\tau$ is right-invariant. In particular, right translations are continuous.

Let $\tau^{-1}$ be the image of $\tau$ under the inverse map. The fact that $\tau$ is right-invariant implies that $\tau^{-1}$ is left invariant. Then $\tau^{-1} = \tau$, implying that the inverse map is continuous.

Finally we show that the group operation $m : G \times G \to G$ is continuous. By Proposition 4.39 (and Proposition 4.12), the group operation $m$ is continuous at at least one point $(a, b)$. For any $\delta, \epsilon \in G$, we have

$$m(x, y) = \epsilon^{-1} \cdot m(\epsilon \cdot x, y \cdot \delta) \cdot \delta^{-1}.$$ 

The right hand side is continuous at $(x, y) = (\epsilon^{-1} \cdot a, b \cdot \delta^{-1})$, and therefore so is the left hand side. As $\epsilon, \delta$ are arbitrary, $m$ is continuous everywhere. □
5.2 Existence

In this section we show that each interpretable group admits at least one strongly admissible group topology.

Lemma 5.4. Let $G$ be an interpretable group and let $U \subseteq G$ be a large subset.

1. Finitely many left-translates of $U$ cover $G$.

2. For any $x, y \in G$, there is a left-translate $g \cdot U$ containing both $x$ and $y$.

Proof. Take a small set of parameters $A$ defining $G$ and $U$.

1. Take $(g_1, \ldots, g_{n+1})$ generic in $G^{n+1}$ over $A$. Note that $\dim(g_i/A, g_1, \ldots, g_{i-1}) = \dim(g_i/A) = \dim(G)$ for each $i$, and so $g_i \downarrow_A^{\dim} g_1, \ldots, g_{i-1}$.

We claim $G \subseteq \bigcup_{i=1}^{n+1} g_i U$. Fix any $h \in U$. For $0 \leq i \leq n+1$ let $d_i = \dim(h/A, g_1, \ldots, g_i)$. Then

$$0 \leq d_{n+1} \leq d_n \leq \cdots \leq d_1 \leq d_0 = \dim(h/A) \leq \dim(G) = n.$$  

It is impossible that $0 \leq d_{n+1} < d_n < \cdots < d_0 \leq n$, so there is some $1 \leq i \leq n+1$ such that $d_{i-1} = d_i$, i.e., $\dim(h/A, g_1, \ldots, g_{i-1}) = \dim(h/A, g_1, \ldots, g_i)$. Then

$$h \downarrow_{A,g_1,\ldots,g_{i-1}}^{\dim} \text{ and } g_1, \ldots, g_{i-1} \downarrow_A^{\dim} g_i,$$

so $h \downarrow_A^{\dim} g_i$ by left transitivity. Then $\dim(g_i/Ah) = \dim(g_i/A) = \dim(G)$, and $g_i$ is generic in $G$ over $Ah$. Then $\dim(g_i^{-1} \cdot h/Ah) = \dim(g_i/Ah) = \dim(G)$, and so $g_i^{-1} \cdot h$ must be in the $Ah$-interpretable large subset $U$, implying $h \in g_i \cdot U$.

2. Take $g \in G$ generic over $Axy$. Then $\dim(g^{-1} \cdot x/Axy) = \dim(g/Axy) = \dim(G)$, and so $g^{-1} \cdot x$ is in the $Axy$-interpretable large subset $U$. Then $x \in g \cdot U$, and similarly $y \in g \cdot U$. \qed

Lemma 5.5. Let $X$ be a Hausdorff interpretable topological space. Let $U_1, \ldots, U_n$ be finitely many interpretable open subsets covering $X$. If each $U_i$ is strongly admissible, then $X$ is strongly admissible.

Proof. The space $X$ is locally Euclidean because it is covered by locally Euclidean spaces. The disjoint union $U_1 \sqcup \cdots \sqcup U_n$ is manifold dominated by Proposition 4.12. The map $U_1 \sqcup \cdots \sqcup U_n \to U_1 \sqcup \cdots \sqcup U_n = X$ is an interpretable surjective open map, and so $X$ is manifold dominated by Lemma 4.4. Then $X$ is Hausdorff, locally Euclidean, and manifold dominated. \qed

Proposition 5.6. Let $G$ be an interpretable group. Then there is a strongly admissible group topology $\tau$ on $G$. 

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Proof. By Theorem 4.29 there is a strongly admissible topology $\tau_0$ on $G$. Let $C$ be a small set of parameters over which everything is defined. If $a, b \in G$, let $a \leq b$ mean that the map $x \mapsto b \cdot a^{-1} \cdot x$ is $\tau_0$-continuous at $a$. Then $\leq$ is a $C$-interpretable preorder on $G$. Let $\approx$ be the associated $C$-interpretable equivalence relation.

Claim 5.7. If $(a, b) \in G^2$ is generic over $C$, then $a \approx b$.

Proof. By symmetry it suffices to show $a \leq b$. Let $\delta = b \cdot a^{-1}$. Then $\dim(a, \delta/C) = \dim(a, b/C) = 2 \dim(G)$, which implies that $a$ is generic in $G$ over $C\delta$. By Corollary 4.40 the map $x \mapsto \delta \cdot x$ is $\tau_0$-continuous at $a$. □Claim

Fix $a_0 \in G$ generic over $C$. If $b \in G$ is generic over $C a_0$, then $b \approx a_0$. Therefore the equivalence class of $a_0$ is a large subset of $G$. Let $U$ be the $\tau_0$-interior of this equivalence class. Then $U$ is a large subset of $G$ by Proposition 4.34. We have constructed a large $\tau_0$-open subset $U \subseteq G$ such that if $a, b \in U$, then the map $x \mapsto b \cdot a^{-1} \cdot x$ is $\tau_0$-continuous at $x = a$.

Consider $G \times U$, where $G$ has the discrete topology(!), $U$ has the topology $\tau_0 | U$, and $G \times U$ has the product topology.

Claim 5.8. Let $E$ be the equivalence relation on $G \times U$ given by

$$(x, y) E (x', y') \iff x \cdot y = x' \cdot y'.$$

Then $E$ is an OQ equivalence relation on $G \times U$ (Definition 4.15).

Proof. Fix an interpretable basis $B$ for $U$. Then $\{ \{a\} \times B : a \in G, \ B \in B \}$ is a basis for $G \times U$.

We verify that $E$ satisfies criterion (4) of Remark 4.16. Fix $p = (a, b)$ and $p' = (c, d)$ in $G \times U$, with $a \cdot b = c \cdot d$. Fix a basic neighborhood $\{a\} \times B$ of $(a, b)$. We must find a basic neighborhood $\{c\} \times B'$ of $(c, d)$ such that every element of $\{c\} \times B'$ is equivalent to an element of $\{a\} \times B$. Note that $b \cdot d^{-1} = a^{-1} \cdot c$. As $b, d \in U$, the map

$$f(x) = b \cdot d^{-1} \cdot x = a^{-1} \cdot c \cdot x$$

is continuous at $x = d$, and maps $d$ to $b$. As $B$ is a neighborhood of $b$, there is a neighborhood $B'$ of $d$ such that $f(B') \subseteq B$. If $(c, x) \in \{c\} \times B'$ then $(c, x) E (a, f(x))$ and $(a, f(x)) \in \{a\} \times B$. So every element of $\{c\} \times B'$ is equivalent to an element of $\{a\} \times B$, as required. □Claim

Let $\tau$ be the quotient equivalence relation on $(G \times U)/E = G$. By Fact 4.17 $\tau$ is an interpretable equivalence relation on $G$ (but not necessarily Hausdorff). For each $a \in G$, the composition

$$U \hookrightarrow G \times U \twoheadrightarrow G$$

$$x \mapsto (a, x) \mapsto a \cdot x$$

is an injective open map, i.e., an open embedding. So we have proven the following:
Claim 5.9. There is an interpretable topology $\tau$ on $G$ such that for any $a \in G$, the map

$$i_a : (U, \tau_0 \upharpoonright U) \to (G, \tau)$$

$$i_a(x) = a \cdot x$$

is an open embedding.

The family of open embeddings $\{i_a\}_{a \in G}$ is jointly surjective, and so the property in Claim 5.9 uniquely determines $\tau$. The property is invariant under left translation, and therefore $\tau$ is invariant under left translations.

We claim $\tau$ is Hausdorff. Given distinct $x, y \in G$, there is some $a \in G$ such that $\{x, y\} \subseteq a \cdot U$, by Lemma 5.4(2). Then $a \cdot U$ is a Hausdorff open subset of $(G, \tau)$ containing $x$ and $y$, so $x$ and $y$ are separated by neighborhoods.

By Remark 4.14, $(U, \tau_0 \upharpoonright U)$ is strongly admissible. By Lemma 5.4(1), finitely many left translates of $U$ cover $G$. Each of these translates is homeomorphic to $(U, \tau_0 \upharpoonright U)$, so by Lemma 5.5, $(G, \tau)$ is strongly admissible.

In summary, $\tau$ is an interpretable topology, $\tau$ is invariant under left translation, $\tau$ is Hausdorff, and $\tau$ is strongly admissible. By Lemma 5.3, $\tau$ is a group topology on $G$. \qed

Theorem 5.10. Let $G$ be an interpretable group.

1. If $\tau$ is an interpretable group topology on $G$, then the following properties are equivalent:

   (a) $\tau$ is strongly admissible.
   (b) $\tau$ is admissible.
   (c) $\tau$ is manifold dominated.
   (d) $\tau$ is definably dominated.

2. There is a unique interpretable group topology $\tau$ satisfying these equivalent conditions.

Proof. The weakest of the four conditions is definable domination, and the strongest is strong admissibility. Proposition 5.6 gives a strongly admissible group topology $\tau_0$. Suppose $\tau$ is definably dominated. Then $\tau$ is admissible by Lemma 5.1, and so $\tau = \tau_0$ by Lemma 5.2. \qed

5.3 Examples

Definition 5.11. An admissible group is an interpretable group with an admissible group topology.

By Theorem 5.10, every interpretable group becomes an admissible group in a unique way.

Example 5.12. If $G$ is a definable group, then Pillay shows that $G$ admits a unique definable manifold topology [Pil89]. Definable manifolds are admissible, so Pillay’s topology agrees with the unique admissible group topology on $G$. 26
Example 5.13. The discrete topology on the value group $\Gamma$ is admissible (Example 4.8). This is necessarily the unique admissible group topology on $\Gamma$.

Remark 5.14. Let $G$ be an interpretable group, topologized with its unique admissible topology. Let $n = \dim(G)$. By Proposition 4.32, there is a point $a \in G$ at which the local dimension is $n$. By translation symmetry, the local dimension is $n$ at every point. As $G$ is locally Euclidean, we see that $G$ is locally homeomorphic to an open subset of $\mathbb{M}^n$ at any point. As a consequence, the admissible group topology on $G$ is discrete if and only if $\dim(G) = 0$.

5.4 Subgroups, quotients, and homomorphisms

In this section, we analyze the topological properties of interpretable homomorphisms.

Proposition 5.15. Let $f : G \to H$ be an interpretable homomorphism between two admissible groups. Then $f$ is continuous.

Proof. Similar to the proof of Lemma 5.2. 

Proposition 5.16. Let $G$ be an interpretable group and let $H$ be an interpretable subgroup. Let $\tau_G$ and $\tau_H$ be the admissible topologies in $G$ and $H$.

1. $H$ is $\tau_G$-closed.
2. $\tau_H$ is the restriction of $\tau_G$.

Proof. 1. Because $\tau_G$ is invariant under translations, the $\tau_G$-frontier $\partial H$ is a union of cosets of $H$. Therefore, one of two things happens: $\partial H = \emptyset$ (meaning that $H$ is $\tau_G$-closed), or $\dim(\partial H) \geq \dim(H)$ (contradicting Proposition 4.34).

2. The restriction $\tau_G \restriction H$ is admissible by Proposition 4.13. The group operation $H \times H \to H$ is continuous with respect to $\tau_G \restriction H$, and so $(H, \tau_G \restriction H)$ is an admissible group. By the uniqueness of the topology, $\tau_G \restriction H = \tau_H$. 

Corollary 5.17. If $f : G \to H$ is an injective homomorphism of admissible groups, then $f$ is a closed embedding.

Proposition 5.18. Let $G$ be an admissible group and $H$ be an interpretable subgroup. Then $H$ is open in the admissible topology on $G$ if and only if $\dim(H) = \dim(G)$.

Proof. $H$ is open if and only if $H$ has non-empty interior (as a subset of $G$). If $\dim(H) = \dim(G)$, then $H$ has non-empty interior by Proposition 4.34. Conversely, suppose $H$ has non-empty interior. By Remark 5.14, $\dim_x(G) = \dim(G)$ for all $x \in G$. Taking $x$ in the interior of $H$, we see $\dim(H) = \dim(G)$. 

Proposition 5.19. Let $G$ be an admissible group and let $H$ be an interpretable normal subgroup. 

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1. The quotient topology on \( G/H \) agrees with the unique admissible group topology on \( G/H \).

2. The map \( G \to G/H \) is an open map.

**Proof.** Let \( E \) be the equivalence relation \( xEy \iff xH = yH \).

**Claim 5.20.** \( E \) is an OQ equivalence relation on \( G \) (**Definition 4.15**).

**Proof.** We verify criterion (1) of Remark 4.16. Suppose \( g_1, g_2 \in G \) are \( E \)-equivalent, and \( B \) is a neighborhood of \( g_1 \). Take \( h \in H \) such that \( g_1 \cdot h = g_2 \). Then \( B \cdot h \) is a neighborhood of \( g_2 \), and every element of \( B \cdot h \) is \( E \)-equivalent to an element of \( B \). \( \square \) Claim

Consider \( G/H \) with the quotient topology \( \tau \). By the claim, \( G \to G/H \) is an open map. By Fact 4.17, \( \tau \) is interpretable. It remains to show that \( \tau \) is the admissible group topology on \( G/H \).

**Claim 5.21.** \( \tau \) is a group topology.

**Proof.** We claim \( (x, y) \mapsto x \cdot y^{-1} \) is continuous with respect to \( \tau \). Fix \( a, b \in G/H \). Take lifts \( \bar{a}, \bar{b} \in G \). Let \( U \) be a neighborhood of \( a \cdot b^{-1} \) in \( G/H \). Let \( \pi : G \to G/H \) be the quotient map. Then \( \pi^{-1}(U) \) is an open neighborhood of \( \bar{a} \cdot \bar{b}^{-1} \) in \( G \). By continuity of the group operations on \( G \), there are open neighborhoods \( W \ni \bar{a} \) and \( V \ni \bar{b} \) in \( G \) such that \( W \cdot V^{-1} \subseteq \pi^{-1}(U) \), in the sense that

\[
x \in W, \ y \in V \implies x \cdot y^{-1} \in \pi^{-1}(U).
\]

As \( \pi \) is an open map, \( \pi(W) \) and \( \pi(V) \) are open neighborhoods of \( a \) and \( b \). Then \( \pi(W) \cdot \pi(V)^{-1} = \pi(V \cdot V^{-1}) \subseteq \pi(\pi^{-1}(U)) = U \). This shows continuity at \((a, b)\). \( \square \) Claim

By Proposition 5.16(1), \( H \) is closed, which implies \( \{1\} \subseteq G/H \) is closed in \( \tau \) by definition of the quotient topology. As \( \tau \) is a group topology, it follows that \( \tau \) is Hausdorff. Then the open map \( G \to G/H \) shows that \( \tau \) is definably dominated, by Lemma 4.4(1). Finally, \( \tau \) is the admissible group topology on \( G/H \) by Theorem 5.10. \( \square \)

**Corollary 5.22.** Let \( f : G \to H \) be a surjective interpretable homomorphism between two admissible groups. Then \( f \) is an open map.

### 6 Definable compactness in strongly admissible spaces

Recall definable compactness from **Definition 3.2**. Let \( X \) be an interpretable topological space in a \( p \)-adically closed field \( M \) with value group \( \Gamma \).

**Definition 6.1.** [JY22b, Definition 2.6] A \( \Gamma \)-exhaustion of \( X \) is an interpretable family \( \{X_\gamma\}_{\gamma \in \Gamma} \), such that

- Each \( X_\gamma \) is a definably compact, clopen subset of \( X \).
- If \( \gamma \leq \gamma' \), then \( X_\gamma \subseteq X_{\gamma'} \).

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• \( X = \bigcup_{\gamma \in \Gamma} X_\gamma. \)

**Proposition 6.2.** Let \( \{X_\gamma\}_{\gamma \in \Gamma} \) be a \( \Gamma \)-exhaustion of an interpretable topological space \( X \). Then \( X \) is definably compact if and only if \( X = X_\gamma \) for some \( \gamma \in \Gamma \).

**Proof.** Each \( X_\gamma \) is definably compact, so if \( X = X_\gamma \) then \( X \) is definably compact. Conversely, suppose \( X \) is definably compact. Then the family \( \{X \setminus X_\gamma\}_{\gamma \in \Gamma} \) is a downward-directed family of closed sets with empty intersection. By definable compactness, some \( X \setminus X_\gamma \) must vanish. \( \square \)

**Lemma 6.3.**

1. Let \( f : X \to Y \) be an interpretable surjective open map between Hausdorff interpretable topological spaces. Suppose \( \{X_\gamma\}_{\gamma \in \Gamma} \) is a \( \Gamma \)-exhaustion of \( X \). Let \( Y_\gamma = f(X_\gamma) \) for each \( \gamma \). Then \( \{Y_\gamma\}_{\gamma \in \Gamma} \) is a \( \Gamma \)-exhaustion of \( Y \).

2. If \( X \) is manifold dominated, then \( X \) has a \( \Gamma \)-exhaustion.

**Proof.**

1. Each \( Y_\gamma \) is open because \( f \) is an open map. Each \( Y_\gamma \) is definably compact as the image of a definably compact set under a definable map. Each \( Y_\gamma \) is closed because \( Y \) is Hausdorff. We have \( \bigcup \gamma Y_\gamma = \bigcup \gamma f(X_\gamma) = f(X) = Y \), because \( f \) is surjective. It is clear that if \( \gamma \leq \gamma' \), then \( Y_\gamma = f(X_\gamma) \subseteq f(X_{\gamma'}) = Y_{\gamma'}. \)

2. Take a definable manifold \( U \) and an interpretable surjective open map \( f : U \to X \). By [JY22b, Proposition 2.8], \( U \) admits a \( \Gamma \)-exhaustion. By the first point, we can push this forward to a \( \Gamma \)-exhaustion on \( X \). \( \square \)

The idea of Lemma 6.3 came out of discussions with Ningyuan Yao and Zhentao Zhang.

**Remark 6.4.** Say that an interpretable topological space \( X \) is “locally definably compact” if for every \( p \in X \), there is a definably compact subspace \( Y \subseteq X \) such that \( p \) is in the interior of \( Y \). Lemma 6.3 shows that any manifold dominated space \( X \) is locally definably compact. Indeed, take a \( \Gamma \)-exhaustion \( \{X_\gamma\}_{\gamma \in \Gamma} \). If \( p \in X \), then there is \( \gamma \in \Gamma \) such that \( p \in X_\gamma \), and we can take \( Y = X_\gamma \).

**Example 6.5.** Let \( D \subseteq \mathbb{M}^2 \) be the definable set \( \{(x, y) \in \mathbb{M}^2 : x = 0 \lor y \neq 0\} \) from Example 4.10. Using Fact 3.4(1), one can see that if \( N \) is a neighborhood of \((0, 0)\) in \( D \), and \( \overline{N} \) is the closure of \((0, 0)\) in \( D \), then \( \overline{N} \) is not definably compact. In other words, local definable compactness fails at \((0, 0)\) \( \in D \). Therefore \( D \) is not manifold dominated, and not strongly admissible (though it is admissible, trivially).

**Theorem 6.6.** Definable compactness is a definable property, on families of strongly admissible spaces, and families of interpretable groups. More precisely:

1. If \( \{(X_i, \tau_i)\}_{i \in I} \) is an interpretable family of strongly admissible spaces, then the set \( \{i \in I : (X_i, \tau_i) \text{ is definably compact}\} \) is an interpretable subset of \( I \).

2. If \( \{G_i\}_{i \in I} \) is an interpretable family of groups, then the set \( \{i \in I : G_i \text{ is definably compact with respect to the admissible group topology}\} \) is an interpretable subset of \( I \).
This follows formally from Proposition [6.2] and Lemma [6.3] using the method of [Joh21, Proposition 4.1]. However, we can give a cleaner proof, using the following lemma.

**Lemma 6.7.** Let $X$ be strongly admissible and definably compact. Then there is a closed and bounded definable set $D \subseteq \mathbb{M}^n$ and a continuous interpretable surjection $f : D \rightarrow X$.

**Proof.** Take $Y$ a definable manifold and $g : Y \rightarrow X$ an interpretable surjective open map. Take $U_1, \ldots, U_m \subseteq Y$ an open cover and interpretable open embeddings $h_i : U_i \rightarrow \mathbb{M}^n_i$. Let $\{U_{i, \gamma}\}_{\gamma \in \Gamma}$ be a $\Gamma$-exhaustion of $U_i$ for each $i$. Then $\bigcup_{i=1}^m U_{i, \gamma}$ is a $\Gamma$-exhaustion of $Y$, and $\{f(\bigcup_{i=1}^m U_{i, \gamma})\}_{\gamma \in \Gamma}$ is a $\Gamma$-exhaustion of $X$ by Lemma [6.3]. As $X$ is definably compact, there is some $\gamma_0 \in \Gamma$ such that $f(\bigcup_{i=1}^m U_{i, \gamma_0}) = X$ by Proposition [6.2].

Each set $U_{i, \gamma_0}$ is homeomorphic to a definable subset of $\mathbb{M}^n_i$, via the embeddings $g_i$. Then the disjoint union $U_{1, \gamma_0} \sqcup \cdots \sqcup U_{m, \gamma_0}$ is homeomorphic to a definable set $D \subseteq \mathbb{M}^n$ for sufficiently large $n$. The natural map

$$U_{1, \gamma_0} \sqcup \cdots \sqcup U_{m, \gamma_0} \rightarrow \bigcup_{i=1}^m U_{i, \gamma_0} \hookrightarrow Y \rightarrow X$$

is surjective by choice of $\gamma_0$. So there is a continuous surjection $D \rightarrow X$. Finally, $D$ is homeomorphic to the disjoint union $U_{1, \gamma_0} \sqcup \cdots \sqcup U_{m, \gamma_0}$, which is definably compact, and so $D$ is closed and bounded by Fact [5.4][1].

Using this we can prove Theorem [6.6].

**Proof (of Theorem 6.6).** 1. Let $S$ be the set of $i \in I$ such that $(X_i, \tau_i)$ is definably compact. It suffices to show that $S$ and $I \setminus S$ are $\forall$-definable, i.e., small unions of $M_0$-definable sets. By Lemma [6.7] $i \in S$ if and only if there is a definable set $D$ and a surjective continuous interpretable map $D \rightarrow X_i$. It is easy to see that the set of such $i$ is $\forall$-definable. Meanwhile, by our definition of definable compactness, $i \notin S$ if and only if there is a downward-directed interpretable family $\{F_j\}_{j \in J}$ of non-empty closed subsets of $X_i$ such that $\bigcap_{j \in J} F_j = \emptyset$. Again, the set of such $i$ is $\forall$-definable.

2. Let $S$ be the set of $i \in I$ such that $G_i$ is definably compact, with respect to the unique admissible group topology on $G_i$. Then $i \in S$ (resp. $i \notin S$) if and only if there is an interpretable topology $\tau$ on $G_i$, a definable set $D \subseteq \mathbb{M}^n$, and an interpretable function $f : D \rightarrow G_i$ such that

- $\tau$ is a Hausdorff group topology on $G_i$
- $f$ is a surjective open map.
- $(G_i, \tau)$ is definably compact (resp. not definably compact).

Again, these conditions are easily seen to be $\forall$-definable, using part [1] for the third point. Thus $S$ and its complement are both $\forall$-definable.

Using different methods, Pablo Andújar Guerrero and the author have shown that definable compactness is a definable property in any interpretable family of topological spaces [AGJ22, Theorem 8.16]. In other words, Theorem [6.6][1] holds without the assumption of strong admissibility.
6.1 Definable compactness in $\mathbb{Q}_p$

Fix a copy of $\mathbb{Q}_p$ embedded into the monster $\mathbb{M} | p\mathbb{CF}$. If $X$ is a $\mathbb{Q}_p$-interpretable topological space, then $X(\mathbb{Q}_p)$ is naturally a topological space.

**Warning.** $X(\mathbb{Q}_p)$ is usually not a subspace of $X(\mathbb{M})$. This is analogous to how if $\mathbb{M}$ is a highly saturated elementary extension of $\mathbb{R}$, then $\mathbb{R}$ with the order topology is not a subspace of $\mathbb{M}$ with the order topology. In fact, if we start with $(\mathbb{M}, \leq)$ and take the induced subspace topology on $\mathbb{R}$, we get the discrete topology.

**Proposition 6.8.** Let $X$ be a strongly admissible topological space in $\mathbb{Q}_p$. Then $X$ is definably compact if and only if $X$ is compact.

**Proof.** If $X$ is compact, then $X$ is definably compact by Fact 3.3(1). Conversely, suppose $X$ is definably compact. By Lemma 6.7 there is a closed bounded definable set $D \subseteq \mathbb{M}^n$ and an interpretable continuous surjection $f : D \to X$. As $\mathbb{Q}_p \preceq \mathbb{M}$ we can take $D$ and $f$ to be defined over $\mathbb{Q}_p$. Then $f : D(\mathbb{Q}_p) \to X(\mathbb{Q}_p)$ is a continuous surjection and $D(\mathbb{Q}_p)$ is compact, so $X(\mathbb{Q}_p)$ is compact. □

**Proposition 6.9.** Let $X$ be $\mathbb{Q}_p$-interpretable strongly admissible topological space. If $X$ is definably compact, then $X$ is a definable manifold. In particular, there is a $\mathbb{Q}_p$-interpretable set-theoretic bijection between $X$ and a definable set.

**Proof.** For each point $a \in X(\mathbb{Q}_p)$, there is an open neighborhood $U_a$ and an open embedding $f_a : U_a \to \mathbb{M}^{n_a}$ where $n_a$ is the local dimension of $X$ at $a$. Because $\mathbb{Q}_p \preceq \mathbb{M}$, we can take $f_a$ and $U_a$ to be $\mathbb{Q}_p$-interpretable. Then $U_a(\mathbb{Q}_p)$ is an open subset of $X(\mathbb{Q}_p)$ containing $a$. By Proposition 6.8 $X(\mathbb{Q}_p)$ is compact, and so there are finitely many points $a_1, \ldots, a_n$ such that $X(\mathbb{Q}_p) = \bigcup_{i=1}^{n} U_{a_i}(\mathbb{Q}_p)$. Then $X = \bigcup_{i=1}^{n} U_{a_i}$ because $\mathbb{Q}_p \preceq \mathbb{M}$. The sets $U_{a_i}$ and maps $f_{a_i} : U_{a_i} \to \mathbb{M}^{n_{a_i}}$ witness that $X$ is a definable manifold. □

7 Interpretable groups and fsg

Recall the definition of fsg (finitely satisfiable generics) from Section 1.1

**Theorem 7.1.** Let $G$ be an interpretable group in a $p$-adically closed field. Then $G$ has finitely satisfiable generics (fsg) if and only if $G$ is definably compact with respect to the admissible group topology.

**Proof.** The arguments from [Joh21, Sections 3, 5–6] work almost verbatim, given everything we have proved so far. The existence of $\Gamma$-exhaustions, used in [Joh21, Section 3], is handled by Lemma 6.3(2). We don’t need to redo the arguments of [Joh21, Section 5]—if $G$ is a definably compact $\mathbb{Q}_p$-interpretable group, then $G$ is already definable by Proposition 6.9, so we can directly apply [Joh21, Proposition 5.1] □ The arguments of [Joh21, Section 6] go through, changing “definable” to “interpretable” everywhere. □

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2In particular, we don’t need to worry about Haar measurability of interpretable subsets of $G$ because $G$ is definable.
Corollary 7.2. If \( \{ G_a \}_{a \in Y} \) is an interpretable family of interpretable groups, then the set
\[ \{ a \in Y : G_a \text{ has fsg} \} \]
is interpretable.

**Proof.** Theorems 7.1 and 6.6.

Corollary 7.3. Let \( G \) be an interpretable group in \( \mathbb{Q}_p \). If \( G \) has fsg, then \( G \) is interpretably isomorphic to a definable group.

**Proof.** Proposition 6.9.

8 Zero-dimensional groups and sets

In Remark 5.14 we saw that an admissible group \( G \) is discrete iff it is zero-dimensional. This holds more generally for admissible spaces, by the following two propositions:

**Proposition 8.1.** If \( X \) is an admissible topological space and \( X \) is discrete, then \( \dim(X) \leq 0 \).

**Proof.** Immediate by considering local dimension (Proposition 4.32).

Conversely, the discrete topology is admissible on zero-dimensional sets:

**Proposition 8.2.** Let \( X \) be a zero-dimensional interpretable set. The discrete topology is strongly admissible, and is the only admissible topology on \( X \).

**Proof.** We claim any admissible topology \( \tau \) on \( X \) is discrete. Indeed, Proposition 4.36 gives a large open subspace \( X' \subseteq X \) that is locally Euclidean. Then \( \dim(X \setminus X') \leq \dim(X) = 0 \), so \( X \setminus X' = \emptyset \) and \( X' = X \). Zero-dimensional locally Euclidean spaces are discrete.

Meanwhile, Theorem 4.29 shows that there is some strongly admissible topology \( \tau \) on \( X \). By the previous paragraph, \( \tau \) must be the discrete topology.

**Definition 8.3.** An interpretable set \( S \) is pseudofinite if \( \dim(S) = 0 \), and \( S \) with the discrete topology is definably compact.

(In Proposition 8.9 we will see that the requirement \( \dim(S) = 0 \) is redundant.)

**Proposition 8.4.** If an interpretable set \( S \) is finite, then \( S \) is pseudofinite.

**Proof.** Suppose \( S \) is finite. Then \( \dim(S) = 0 \) by Proposition 2.8(2). The discrete topology on \( S \) is compact, hence definably compact by Fact 3.3(1).

**Proposition 8.5.** 1. Let \( S \) be a \( \mathbb{Q}_p \)-interpretable set. Then \( S \) is pseudofinite iff \( S \) is finite.

2. If \( \{ S_a \}_{a \in I} \) is an interpretable family, then \( \{ a \in I : S_a \text{ is pseudofinite} \} \) is interpretable.
Proof. 1. If $S$ is finite, then $S$ is pseudofinite by Proposition 8.4. Conversely, suppose $S$ is pseudofinite, so $\dim(S) = 0$ and the discrete topology on $S$ is definably compact. By Proposition 8.2, the discrete topology is strongly admissible. By Proposition 6.8, the discrete topology is compact, so $S$ is finite.

2. Dimension 0 is definable by Proposition 2.12. Assuming dimension 0, the discrete topology is strongly admissible by Proposition 8.2 and then definable compactness is definable by Theorem 6.6(1).

Remark 8.6. Proposition 8.5 characterize pseudofiniteness uniquely—it is the unique definable property which agrees with finiteness over $\mathbb{Q}_p$.

Corollary 8.7. $\mathbb{Q}^\text{eq}_p$ eliminates $\exists^\infty$: for any $L^\text{eq}$-formula $\phi(x,y)$, there is a formula $\psi(y)$ such that $\phi(\mathbb{Q}_p, b)$ is infinite if and only if $b \in \psi(\mathbb{Q}_p)$. (But $\mathcal{M}^\text{eq}$ does not eliminate $\exists^\infty$, of course.)

This could probably also be seen by the explicit characterization of imaginaries in [HMR18, Theorem 1.1].

For 0-dimensional groups, pseudofiniteness is equivalent to $fsg$:

Proposition 8.8. Let $G$ be a 0-dimensional interpretable group. Then $G$ has $fsg$ if and only if $G$ is pseudofinite.

Proof. By Theorem 7.1, $G$ has $fsg$ if and only if the admissible group topology on $G$ is definably compact. By Remark 5.14 or Proposition 8.2, the admissible group topology on $G$ is discrete.

For example, the value group $\Gamma$ does not have $fsg$, but the circle group $[0, \gamma) \subseteq \Gamma$ (with addition “modulo $\gamma$”) does have $fsg$.

We close by giving some equivalent characterizations of pseudofiniteness.

Proposition 8.9. Let $S$ be an interpretable set. The following are equivalent:

1. $S$ is pseudofinite.
2. $S$ with the discrete topology is definably compact.
3. If $D = \{D_a\}_{a \in I}$ is an interpretable family of subsets of $S$, and $D$ is linearly ordered under inclusion, then $D$ has a minimal element.

Proof. Consider two additional criteria:

4. If $D = \{D_a\}_{a \in I}$ is a downwards-directed interpretable family of non-empty subsets of $S$, then $\bigcap_{a \in I} D_a \neq \emptyset$.
5. If $D = \{D_a\}_{a \in I}$ is a linearly ordered interpretable family of non-empty subsets of $S$, then $\bigcap_{a \in I} D_a \neq \emptyset$.

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We claim that $(1) \Rightarrow (2) \iff (4) \Rightarrow (5)$, $(1) \Rightarrow (3) \Rightarrow (5)$, and $(5) \Rightarrow (1)$. First of all, $(1) \Rightarrow (2)$ by our definition of “pseudofinite,” and $(2) \iff (1)$ by definition of definable compactness. The implications $(1) \Rightarrow (3)$ and $(3) \Rightarrow (5)$ are trivial.

If $(1) \not\Rightarrow (3)$, then there is a pseudofinite interpretable set $S$ and an interpretable chain $D$ of subsets of $S$ with no minimum. Because $\mathbb{Q}_p \preceq M$ and pseudofiniteness is a definable property (Proposition 8.5(2)), we can assume $S$ and $D$ are interpretable over $\mathbb{Q}_p$. Then $S$ is actually finite (Proposition 8.5(1)) and so $D$ must have a minimum, a contradiction.

It remains to show $(5) \Rightarrow (1)$. We prove $\neg (5)$ assuming $\neg (1)$. There are two cases, depending on why $(1)$ fails:

$\dim(S) > 0$: Then Corollary 4.33 shows that there is a ball $B$ in $M^n$ for $n = \dim(S)$, and an interpretable injection $B \hookrightarrow S$. Without loss of generality, $B$ is $O^n$, where $O$ is the valuation ring on $M$. Then $B$ contains a linearly ordered definable family of non-empty subsets with empty intersection, namely the family of sets $B(\gamma)^n \setminus \{0\}$, where $B(\gamma) \subseteq M$ is the ball of radius $\gamma$. Using the embedding $B \hookrightarrow S$, we get a similar family in $S$, contradicting $(5)$.

$\dim(S) = 0$ but the discrete topology is not definably compact: Then $S$ with the discrete topology is strongly admissible by Proposition 8.2. By Lemma 6.5 there is a $\Gamma$-exhaustion $\{S_\gamma\}_{\gamma \in \Gamma}$ of $S$. As $S$ is not definably compact, $S \setminus S_\gamma \neq \emptyset$ for each $\gamma$, by Proposition 6.2. Then the family $\{S \setminus S_\gamma\}_{\gamma \in \Gamma}$ contradicts $(5)$. \qed

8.1 Geometric elimination of imaginaries

The $n$th geometric sort is the quotient $S_n := GL_n(\mathbb{M})/GL_n(\mathcal{O})$. The theory $pCF$ has elimination of imaginaries after adding the geometric sorts to the language [HMR18, Theorem 1.1]. Consequently, we may assume any interpretable set $X$ is a definable subset of some product $M^n \times \prod_{i=1}^k S_{m_i}$. The geometric sorts are 0-dimensional, so by Proposition 8.2 the discrete topology on $S_n$ is an admissible topology. Consequently, if we take the the standard topology on $M$, the discrete topology on each $S_{m_i}$, the product topology on $M^n \times \prod_{i=1}^k S_{m_i}$, and the subspace topology on $X \subseteq M^n \times \prod_{i=1}^k S_{m_i}$, we get an admissible topology on $X$, by Section 4.1.

This suggests the following alternative approach to tame topology on interpretable sets in $pCF$. Work in the language $L_G$ with the geometric sorts. Then all interpretable sets are definable, up to isomorphism. Endow the home sort $M$ with the usual valuation topology, and endow each geometric sort $S_n$ with the discrete topology. Endow any $L_G$-definable set $D \subseteq M^n \times \prod_{i=1}^k S_{m_i}$ with the subspace topology (of the product topology). By analogy with the o-minimal “definable spaces” in [PS99], say that a definable topological space $X$ is a geometric definable space if it is covered by finitely many open sets $U_1, \ldots, U_n$, each of which is definably isomorphic to an ($L_G$-)definable set. Note that geometric definable spaces are admissible.

Geometric definable spaces should be an adequate framework for tame topology on interpretable sets and interpretable groups in $pCF$:
1. Every interpretable set is $L_G$-definable by elimination of imaginaries, and therefore admits the structure of a geometric definable space in a trivial way.

2. The topological tameness results of Section 4.3 hold for geometric definable spaces. We can see this from admissibility, but there are probably direct proofs.

3. The methods of Section 5.2 or [Pil89] presumably show that any interpretable group can be given the structure of a geometric definable space.

Geometric definable spaces might provide a simpler alternative to the admissible spaces of the present paper. On the other hand, such an approach is unlikely to generalize beyond $p$CF.

9 Further directions

There are several directions for further research.

9.1 Extensions to $P$-minimal and visceral theories

Many of the results of this paper may generalize from $p$CF to other $P$-minimal theories—expansions of $p$CF in which every unary definable set is definable in the pure field sort [HM97]. The topological tameness results of $p$CF like the cell decomposition and dimension theory are known to generalize to $P$-minimal theories [HM97, CKDL17].

More generally, some of the results of this paper may generalize to visceral theories with the exchange property. Visceral theories were introduced by Dolich and Goodrick [DG15]. Recall the notion of uniformities and uniform spaces from topology. A theory is visceral if there is a definable uniformity on the home sort $M^1$ such that a unary definable set $D \subseteq M^1$ is infinite if and only if it has non-empty interior. The theory $p$CF is visceral, as are $P$-minimal theories, o-minimal expansions of DOAG, $C$-minimal expansions of ACVF, unstable $dp$-minimal theories of fields, and many theories of valued fields. Dolich and Goodrick prove a number of topological tameness results for visceral theories, analogous to those that hold in o-minimal and $P$-minimal theories.

It seems likely that the results of Sections 4 and 5 generalize to visceral theories with the exchange property. (There are some subtleties around the proof of Lemma 4.27, but these problems are not insurmountable.) On the other hand, local definable compactness fails to hold in the visceral setting, so Sections 6–8 probably do not generalize.

9.2 Zero-dimensional $dfg$ groups

An interpretable group $G$ is said to have $dfg$ (definable $f$-generics) if there is a global definable type $p$ on $G$ with boundedly many left translates. In distal theories like $Q_p$, the two properties

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Some of Dolich and Goodrick’s results are dependent on the technical assumption “definable finite choice” (DFC). In forthcoming work, I will show that the assumption DFC can generally be removed from all of Dolich and Goodrick’s results, as one would expect [Joh22].
and are polar opposites, in some sense. For example, if $G$ is an infinite interpretable group, then $G$ can have most one of the two properties $fsg$ and $dfg$ (essentially by [Sim13 Proposition 2.27]). If $G$ has dp-rank 1 and is definably amenable, then $G$ satisfies exactly one of the two properties (essentially by [Sim14 Theorem 2.8] and [Sim15 Proposition 8.21]). By Proposition 6.9 the $fsg$ interpretable groups over $\mathbb{Q}_p$ are definable. This vaguely suggests the following Conjecture:

**Conjecture 9.1.** If $G$ is a $\mathbb{Q}_p$-interpretable 0-dimensional, definably amenable group, then $G$ has $dfg$.

The intuition is that “zero-dimensional” is the opposite of “definable,” for infinite groups. For example, the value group $\Gamma$ has $dfg$.

### 9.3 The adjoint action

Suppose $G$ is interpretable. By Theorem 5.10 the unique admissible group topology on $G$ is locally Euclidean. In particular, $G$ is a manifold in a weak sense. Because of generic differentiability, we can probably endow $G$ with a $C^1$-manifold structure, and then look at how $G$ acts on the tangent space at $1_G$ by conjugation. That is, we can look at the adjoint action of $G$.

Say that a group is “locally abelian” if there is an open neighborhood $U \ni 1_G$ on which the group operation is commutative. If $G$ is locally abelian, then the adjoint action is trivial. The converse should hold by reducing to the case where $G$ is defined over $\mathbb{Q}_p$ and using properties of $p$-adic Lie groups. If $G$ is locally abelian, witnessed by $U \ni 1_G$, then the center of the centralizer of $U$ is an abelian open subgroup. Thus, locally abelian groups have open abelian interpretable subgroups.

For a general interpretable group $G$, the adjoint action gives a homomorphism $G \rightarrow GL_n(M)$, where $n = \dim(G)$. The kernel should be a locally abelian group, and the image is definable. Thus, every interpretable group should be an extension of a definable group by a locally abelian group.

This suggests the question: which groups are locally abelian? Can we classify them? Zero-dimensional interpretable groups are locally abelian, and so are abelian interpretable groups. Are all locally abelian interpretable groups built out of 0-dimensional groups and abelian groups? If $G$ is locally abelian, is there a normal abelian open subgroup?

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References

[AGJ22] Pablo Andújar Guerrero and Will Johnson. Around definable types in \( p \)-adically closed fields. arXiv:2208.05815v1 [math.LO], 2022.

[Bel12] Luc Belair. Panorama of \( p \)-adic model theory. Ann. Sci. Math. Quebec, 36(1), 2012.

[CKDL17] Pablo Cubides-Kovacsics, Luck Darnière, and Eva Leenknegt. Topological cell decomposition and dimension theory in \( P \)-minimal fields. Journal of Symbolic Logic, 82(1):347–358, 2017.

[DG15] Alfred Dolich and John Goodrick. Tame topology over definable uniform structures. arXiv:1505.06455v4 [math.LO], 2015.

[EPR14] Pantelis Eleftheriou, Ya’acov Peterzil, and Janak Ramakrishnan. Interpretable groups are definable. Journal of Mathematical Logic, 14(1), June 2014.

[For15] Antongiulio Fornasiero. Definable compactness for topological structures. Draft, 2015.

[Gag05] Jerry Gagelman. Stability in geometric theories. Annals of Pure and Applied Logic, 132:313–326, 2005.

[HM97] Deirdre Haskell and Dugald Macpherson. A version of \( o \)-minimality for the \( p \)-adics. J. Symbolic Logic, 62(4):1075–1092, December 1997.

[HMR18] Ehud Hrushovski, Ben Martin, and Silvain Rideau. Definable equivalence relations and zeta functions of groups. Journal of the European Mathematical Society, 20(10):2467–2537, July 2018.

[HP94] Ehud Hrushovski and Anand Pillay. Groups definable in local fields and pseudo-finite fields. Israel J. Math., 85:203–262, 1994.

[HP11] Ehud Hrushovski and Anand Pillay. On NIP and invariant measures. J. Eur. Math. Soc., 13(4):1005–1061, 2011.

[HPP08] Ehud Hrushovski, Ya’acov Peterzil, and Anand Pillay. Groups, measures, and the NIP. J. Amer. Math. Soc., 21(2):563–596, April 2008.

[Joh18] Will Johnson. Interpretable sets in dense \( o \)-minimal structures. J. Symbolic Logic, 83:1477–1500, 2018.

[Joh21] Will Johnson. A note on \( fsg \) groups in \( p \)-adically closed fields. arXiv:2108.06092v1 [math.LO], 2021.

[Joh22] Will Johnson. Visceral theories without assumptions. In preparation, 2022.
[JY22a] Will Johnson and Ningyuan Yao. Abelian groups definable in \( p \)-adically closed fields. arXiv:2206.14364v1 [mathЛО], 2022.

[JY22b] Will Johnson and Ningyuan Yao. On non-compact \( p \)-adic definable groups. \textit{J. Symbolic Logic}, 87(1):188–213, 2022.

[OP08] A. Onshuus and A. Pillay. Definable groups and compact \( p \)-adic Lie groups. \textit{Journal of the London Mathematical Society}, 78(1):233–247, 2008.

[Pil88] A. Pillay. Groups and fields definable in \( \mathbb{O} \)-minimal structures. \textit{J. Pure Appl. Algebra}, 53:233–255, 1988.

[Pil89] A. Pillay. On fields definable in \( \mathbb{Q}_p \). \textit{Arch. Math. Logic}, 29:1–7, 1989.

[PS99] Y. Peterzil and C. Steinhorn. Definable compactness and definable subgroups of \( \mathbb{O} \)-minimal groups. \textit{Journal of the London Mathematical Society}, 59(3):769–786, 1999.

[Sim13] Pierre Simon. Distal and non-distal NIP theories. \textit{Annals of Pure and Applied Logic}, 164(3):294–318, March 2013.

[Sim14] Pierre Simon. Dp-minimality: invariant types and dp-rank. \textit{J. Symbolic Logic}, 79(4):1025–1045, December 2014.

[Sim15] Pierre Simon. \textit{A guide to NIP theories}. Lecture Notes in Logic. Cambridge University Press, July 2015.

[SvdD88] Philip Scowcroft and Lou van den Dries. On the structure of semialgebraic sets over \( p \)-adic fields. \textit{J. Symbolic Logic}, 53(4):1138–1164, December 1988.