Deconstructing $^1S_0$ nucleon-nucleon scattering

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Abstract. A distorted-wave method is used to analyse nucleon-nucleon scattering in the $^1S_0$ channel. Effects of one-pion exchange are removed from the empirical phase shift to all orders by using a modified effective-range expansion. Two-pion exchange is then subtracted in the distorted-wave Born approximation, with matrix elements taken between scattering waves for the one-pion exchange potential. The residual short-range interaction shows a very rapid energy dependence for kinetic energies above about 100 MeV, suggesting that the breakdown scale of the corresponding effective theory is only 270 MeV. This may signal the need to include the $\Delta$ resonance as an explicit degree of freedom in order to describe scattering at these energies. An alternative strategy of keeping the cutoff finite to reduce large, but finite, contributions from the long-range forces is also discussed.

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1 Introduction

Much time and effort has been spent pursuing the goal of systematically understanding nuclear forces through the techniques of effective field theory (EFT). Unlike the low-energy interactions among pions, the nucleon-nucleon (NN) interaction is nonperturbative, at least in some channels, and this had led to ongoing debates about the correct power counting to use and about the role of the cutoff or regulator in these theories.

An EFT is built out of fields corresponding to the appropriate low-energy degrees of freedom – pions, nucleons and photons for nuclear physics. Its Lagrangian (or Hamiltonian) contains all possible terms consistent with the symmetries of the underlying dynamics. To have any predictive power, therefore, the theory must be systematically expandable in powers of ratios of low-energy scales, denoted generically by $Q$, to those of the underlying physics. Denoting a generic high-energy scale by $\Lambda_0$, the expansion parameter is $Q/\Lambda_0$ and so the convergence of the theory is expected to break down when $Q$ is of the order of the smallest of these scales. Since the $\Lambda_0$ determine the sizes of the coefficients of the operators in the effective Lagrangian, fitting those coefficients to observables gives a measure of the scale of the underlying physics. This means that an EFT should always provide an indication of its own radius of convergence.

For nuclear physics, the relevant low-energy scales include the momenta of the nucleons and the mass of the pion. The scales of the underlying theory, Quantum Chrododynamics (QCD), include the one associated with the hidden chiral symmetry, $4\pi F_\pi$, as well as the masses of the nucleons and the pions. We might therefore hope that $\Lambda_0$ would be of the order of several hundred MeV and so we could construct an EFT valid for momenta up to about $2m_\pi$. However, there could be other important scales in QCD and so we need to let the data determine the range of validity of our EFT.

The starting point for all applications of EFTs to nuclear forces is Weinberg’s observation \cite{1} that nonrelativistic loop integrals are enhanced relative to those in relativistic theories, being of order $Q$ rather than $Q^2$ as in, for example, mesonic chiral perturbation theory (ChPT) \cite{2}. Naive dimensional analysis (“Weinberg power counting”) indicates that the leading terms in the NN force, one-pion exchange (OPE) and a momentum-independent contact term, are of order $Q^0$. This would imply that these terms are still perturbative, each iteration leading to an extra power of $Q$ from the integral. The resulting theory would not be able to describe bound states – nuclei.

In order to generate low-energy bound or virtual states, we need to identify additional low-energy scales that can promote the leading interactions to order $Q^{-1}$. The first such scales to be noted were provided by the $S$-wave scattering lengths, which lead to EFTs where the leading contact terms must be iterated \cite{7,8,9,10}. At very low energies, where pion-exchange forces can be replaced by contact interactions, this leads to a “pionless” EFT, which provides a field-theoretic realisation of the effective-range expansion \cite{11,12}. This formulation allows that, much older, approach to be extended to systems of three or more par-
tics and to processes involving electromagnetic or weak interactions.

To describe physics on momentum scales of the order of \( m_\pi \) or larger, we need to keep pion-exchange forces explicitly. As already mentioned, OPE is of order \( Q^0 \) in naive dimensional analysis, implying that it should be treated perturbatively. Kaplan, Savage and Wise [10] used this to set up an EFT based on a power counting ("KSW counting") in which the momenta, \( m_\pi \) and the inverse scattering lengths were the low-energy scales. However it was soon shown that the resulting expansion fails to converge in the \( ^3S_1 \) wave [13,14,15,16]. This suggests that there is another low-energy scale related to the strength of OPE that would justifiy iterating that potential as well the leading contact interaction. The perturbative treatment of OPE did seem to be valid in the \( ^1S_0 \) channel, but Fleming, Mehen and Stewart [15] found large changes to the strengths of lower-order contact interactions when higher-order terms were included. They concluded that the expansion parameter of the theory might be as large as \( \sim 1/2 \).

By dividing a factor of \( 1/M_N \) out of the Hamiltonian, we can express the strength of the OPE potential in terms of a momentum scale,

\[
\lambda_{\pi NN} = \frac{16\pi F_π^2}{g_\pi^2 M_N} \approx 290 \text{ MeV},
\]

where, to lowest order in the chiral expansion, the Goldberger-Treiman relation has been used to express the \( \pi N \) coupling constant in terms of \( g_\pi \).

Note that \( \lambda_{\pi NN} \) is composed of high-energy scales in chiral perturbation theory, \( 4\pi F_\pi \) and \( M_N \). Treating it as a high-energy scale leads to the perturbative treatment of OPE and KSW counting for the associated short-range interactions. However the numerical value of \( \lambda_{\pi NN} \) is only about twice \( m_\pi \), which may explain some of the problems with KSW counting found in Refs. [13,14,15,16]. If instead we identify \( \lambda_{\pi NN} \) as a low-energy scale, then the OPE potential is promoted to order \( Q^{-1} \), implying that it should be iterated.

In this paper, I examine the consequences of treating nonperturbatively both OPE and the leading, energy-independent contact interaction in the \( ^1S_0 \) channel. Iterating OPE generates to set of distorted waves (DWs). Further scattering between these waves can be described by a residual \( K \) matrix that contains the effects of short-range interactions and other long-range-forces, such as two-pion exchange (TPE). The short-range interaction is strong in this channel and so this \( K \) matrix can be expanded using a DW or "modified" effective-range expansion [11,17,18,19]. This in contrast to the peripheral channels studied in Refs. [20,21] where the scattering is weak and a distorted-wave Born approximation (DWBA) can be used to determine the effective potentials from empirical phase shifts.

This approach is based on an EFT whose low-energy scales are the nucleon momenta, the inverse of the \( ^1S_0 \) scattering length, \( m_\pi \) and \( \lambda_{\pi NN} \). The power counting for it can be found by analysing the scale dependence of the short-range potential with the help of the renormalisation group (RG) [22]. For present purposes, the RG methods developed in Refs. [23,24,25,26] for nonrelativistic scattering show that the counting follows from the behaviour of the wave functions near the origin which, in turn, is controlled by the singularity of the long-range potential as \( r \to 0 \). The spin-singlet channels see only the central part of OPE, which has the usual Yukawa form, behaving like \( 1/r \) near the origin. Its distorted waves have the same power-law behaviours as free waves and hence iteration of this potential does not alter the power counting for the terms of the short-range interaction [21]. This is quite different from the situation in the spin-triplet channels where the \( 1/r^3 \) singularity of the OPE tensor potential leads to a counting in which the leading contact interactions are substantially promoted [27,28,21].

Unlike the peripheral singlet waves, the \( ^1S_0 \) channel has a virtual state at very low energy and so the scattering is strong near threshold. The appropriate power counting corresponds to an expansion around a nontrivial fixed point of the RG and is similar to the effective-range expansion [11] for pure short-range interactions. The one notable difference in the presence of OPE is a leading-order contact term proportional to \( m_\pi^2/\lambda_{\pi NN} \). This is needed to renormalise a logarithmic divergence, as found first by Kaplan et al. [10] in the context of a perturbative treatment of OPE. Although OPE is iterated to all orders here, the terms of the short-range potential in this channel can still be organised according to a modified version of KSW counting that takes into account the extra scale \( \lambda_{\pi NN} \).

The residual short-range potential of this theory can be determined directly from empirical phase shifts, via a DW effective-range expansion which removes the effects of iterated OPE [21]. This expansion has previously been applied to \( ^1S_0 \) NN scattering by Steele and Furnstahl [19], who noted the appearance of scales of the order of \( 2m_\pi \) in the resulting short-distance coefficients and suggested that removal of two-pion exchange (TPE) might improve the radius of convergence. As a coordinate-space approach, it has close connections to ones developed by other groups [28,29,30,31,32], where interaction strengths are also related to logarithmic derivatives of the wave functions at small radii.

Previous applications of this method to the \( ^1S_0 \) channel [19] and of a related DWBA approach to peripheral partial waves [20,21] have found significant energy dependences in the residual interactions after removal of the effects of OPE. These suggest that other long-range forces are also important. Interactions that have been well studied in ChPT are TPE at orders \( Q^2 \) and \( Q^4 \) (see also Ref. [34]), as well as the order-\( Q^2 \) recoil correction to OPE [35] and the leading \( \pi\gamma \)-exchange force [36]. As terms of

\[ \lambda_{\pi NN} \]

In practice, terms in the present approach that contain powers of the scale \( \lambda_{\pi NN} \) cannot be disentangled from lower-order contributions with the same energy dependences. The present analysis of the \( ^1S_0 \) channel could therefore be viewed as an application of the original KSW counting, the iteration of OPE being purely for computational convenience, to avoid the need for evaluating terms up to fourth order in perturbation theory.
order $Q^2$ or higher, all these should be treated as perturbations in the power counting used here. This means that their effects can be removed using the DWBA, evaluating their matrix elements between DWs of OPE and subtracting them from the residual $K$ matrix, as done in Refs. 20 21.

If we were to iterate these higher-order terms by solving the Schrödinger equation with the full potential, then their singularities would alter the forms of the short-distance wave functions. In general, this destroys any consistent power counting, something that has been observed many times, as in Ref. 37, for example. Such problems can be avoided if we work with a finite cutoff, keeping it within the domain of validity of the EFT as discussed in Refs. 38,39. The price to be paid for this is that the resulting effective potential will contain artefacts of the cutoff, that is, pieces that would vanish if one were able to take the regulator scale to infinity. As a result, the form of the short-distance potential is not determined solely by the physical scales of the system—the finite cutoff also plays a crucial role. In the context of standard few- and many-body techniques, which are designed to solve the Schrödinger equation with a given potential, this price may be unavoidable.

On the other hand, if we treat the higher-order terms in the potential perturbatively, as dictated by the pertinent power counting, then these problems do not arise. To any given order, the EFT contains the necessary counterterms to cancel any divergences. Having done that, we can then make all cutoff artefacts small by taking the regulator to be as large as we like. The only proviso is that, before doing so, we need to have determined the counting that applies to our system 39. The first application of this method, to $L \geq 2$ spin-singlet partial waves, did not require any renormalisation 20. However the higher-order terms of the chiral TPE potential are highly singular as $r \to 0$ and so, in all other waves, their matrix elements are divergent at small radii. For example, the $1P_1$ wave requires a single order-$Q^2$ term to cancel an energy-independent divergence 40, and two terms are needed in each spin-triplet wave 21,22, as expected from the power counting in the presence of tensor OPE 27,28. Once we have renormalised these matrix elements of the higher-order potentials, they can be treated in perturbation theory.

Here, I use this method to “deconstruct” scattering in the $1S_0$ channel. First, the effects of iterated OPE are removed from the empirical phase shifts, and then the DWBA matrix elements of the order-$Q^{2,3}$ chiral potentials are subtracted to leave a residual short-range interaction. In this channel, the presence of a low-energy virtual state means that the leading, energy-independent contact interaction must be treated to all orders. Doing so introduces admixtures of irregular wave functions into the solutions of the Schrödinger equation. These increase the degrees of divergence of the DWBA matrix elements of TPE. Nonetheless, KSW power counting (the counting for perturbations around the nontrivial fixed point) provides the necessary counterterms to renormalise these divergences. As a result, the short-range interaction tends to a form that is independent of regulator, provided that regulator is chosen to be high enough in momentum or, in coordinate space, small enough in radius.

In a closely related approach, Shukla et al. 31 extracted a boundary-condition parameter from the logarithmic derivatives of the $1S_0$ wave functions near the origin, and expressed this in the form of a short-range interaction strength. However they did not strictly follow the power counting advocated here, but either iterated the order-$Q^3$ potential to all orders by solving the Schrödinger equation, or treated the whole long-range potential to second order. The singularities of the TPE potential mean that, for the reasons mentioned above, these calculations were restricted to large cutoff radii, $\geq 1$ fm. More recently, Pavón Valderrama 32 has also applied the DW approach to the $1S_0$ and $3S_1$ and $D_1$ channels, within the framework of the power countings that follow from iterating OPE 24. In that work, he fitted a polynomial form for the short-range interaction to the empirical phase shifts, rather than extracting such a potential directly from them, as done in Ref. 31 and here.

Contrary to the hopes of Steele and Furnstahl 19 and to the situation in the peripheral waves 20,21, but consistent with the observations of Pavón Valderrama 32, the current approach leads to short-range interactions that show an even stronger energy dependence after subtraction of the order-$Q^{2,3}$ potentials. This suggests that the breakdown scale of the resulting EFT may be only of the order of $2m_\pi$. This is a serious problem for hopes of applying the EFT discussed here to larger nuclear systems, where typical momentum scales are of that size. One possibility is to look for an EFT in which at least some parts of the TPE potential are promoted to order $Q^{-3}$ and so should be iterated. However that would require identifying further low-energy scales, by analogy with $\lambda_{\pi NN}$.

An alternative way round this might be to work with a finite regulator, chosen to optimise the convergence of the EFT 43. For example, Pavón Valderrama 32 used relatively large values for the cutoff radius, $\geq 0.6$ fm. This was not essential since the necessary counterterms are present to cancel perturbatively all divergences of the TPE matrix elements, but working with cutoffs in this range does lead to results with a greater apparent radius of convergence. Regulator scales corresponding to even larger radii are commonly used in the “Weinberg scheme” 41,42,43, where the full chiral potential is iterated to all orders. However, as I shall discuss, they would play a quite different role in the present context, taming large but not divergent contributions from the long-range potentials. Similar applications of finite regulators as tools to improve the convergence of EFTs have been suggested by various authors 43,44,46, most recently by Beane et al. 47 for calculations of S-wave NN scattering that treat OPE perturbatively.

This paper is organised as follows. First, in Sec. 2, I outline the DW method used to remove the effects of OPE and hence extract an effective short-range interaction directly from the phase shift. Then, in Sec. 3, I use the
DWBA to subtract perturbatively the effects of the order-$Q^2$ chiral potentials. This leaves a residual short-range interaction whose only “contamination” from long-range forces starts at order $Q^4$. Finally, in Sec. 4, I discuss the implications of the strong energy dependence displayed by this potential and comment on the approaches suggested in Refs. [46–31].

2 One-pion exchange

Having identified $\lambda_{\pi NN}$ as a low-energy scale in NN scattering, we need to operate OPE to all orders. In a coordinate-space approach, this is done by solving the Schrödinger equation in differential form. For the $^1S_0$ channel, this is

$$\left[-\frac{1}{M_N} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + V_{\text{OPE}}(r) \right] \psi(p; r) = \frac{p^2}{M_N} \psi(p; r).$$

(2)

Here $p$ is the on-shell relative momentum which is related to the lab kinetic energy $T$, by $T = 2p^2/M_N$. The central piece of the lowest-order OPE potential is

$$V_{\text{OPE}}(r) = -f_{\pi NN}^2 \frac{e^{-m_{\pi}r}}{r},$$

(3)

with $f_{\pi NN}$ denoting the pseudovector $\pi N$ coupling constant which, in the chiral limit, is given by

$$f_{\pi NN}^2 = \frac{g^2_{\pi NN} m_{\pi}^2}{16\pi F^2_{\pi}}.$$  

(4)

If the short-range interactions are strong enough to generate a low-energy bound or virtual state, then we need both regular and irregular solutions to this equation. Near the origin, these have the expansions

$$\psi_R(p; r) = A_R(p) (1 + O(r)),$$

$$\psi_I(p; r) = A_I(p) \left( \frac{1}{r} - M_N f_{\pi NN}^2 \ln(\lambda r) + O(r^2) \right),$$

(5)

where $A_{R, I}$ are overall normalisation constants. It is convenient to choose these so that as $r \to \infty$ the functions have the asymptotic forms,

$$\psi_R(p; r) \to \frac{1}{pr} \sin(p r + \delta_R(p)),$$

$$\psi_I(p; r) \to \frac{1}{pr} \sin(p r + \delta_I(p)),$$

(6)

where $\delta_R(p)$ is the usual phase shift induced by OPE alone. It is also convenient to introduce a second irregular solution which is asymptotically out of phase with the regular one by $\pi/2$,

$$\psi_{I2}(p; r) \to \frac{1}{pr} \cos(p r + \delta_R(p)).$$

(7)

This is given by the linear superposition

$$\psi_{I2}(p; r) = \frac{\cos(\delta_R(p) - \delta_I(p)) \psi_I(p; r) - \sin(\delta_R(p) - \delta_I(p)) \psi_I(p; r)}{\sin(\delta_R(p) - \delta_I(p))}.$$  

(8)

Linear superposition can then be used to construct a wave function with the observed phase shift $\delta(p)$:

$$\psi_>(p; r) = \cos \delta(p) \psi_R(p; r) + \sin \delta(p) \psi_{I2}(p; r),$$

(9)

where $\delta(p) = \delta(p) - \delta_R(p)$ is the additional shift caused by shorter-range forces than OPE. This wave function is the same as one constructed by Ruiz Arriola and Pavón Valderrama using the variable-phase method [28–30].

Having it in a form that can be decomposed into regular and irregular pieces will be essential for the approach used here.

If these short-range forces are not resolved at the energies of interest, they may be represented by any convenient choice of regularised contact interaction. For example, Shukla et al. [31] use a square well with a sloping floor. Here I take a $\delta$-shell form,

$$V(p; r) = \frac{1}{4\pi R_0^2} \tilde{V}_S(p) \delta(r - R_0),$$

(10)

like that previously used in Refs. [20–21]. In a standard EFT treatment, the strength $\tilde{V}_S(p)$ of the short-range potential is taken to be a low-order polynomial in $p^2$, whose coefficients are then fitted to the empirical phase shifts. Here, I make no assumptions about the form of $\tilde{V}_S(p)$ but instead extracted it directly from the phase shifts. This makes it possible to examine where an expansion in powers of $p^2$ might be valid, and where it breaks down.

Outside the shell $r = R_0$, the wave function has the form $\psi_>(p; r)$, containing the observed phase shift. For $r < R_0$, I take it to be the solution of Eq. (2) which is regular at the origin, $\psi_R(p; r)$ (in contrast to, for example, Ref. [31] where only the short-range potential is present for $r < R_0$). For a solution to the full Schrödinger equation, these pieces must match at $r = R_0$, $\psi_>(R_0) = \psi_R(R_0)$. The discontinuity in their radial derivatives is then related to the strength of the $\delta$-shell potential. We can thus use this discontinuity to determine the strength $\tilde{V}_S(p)$ needed to reproduce the observed phase shifts:

$$\tilde{V}_S^{(2)}(p) = \frac{4\pi R_0^2}{M_N} \frac{\psi_>(p; R_0) - \psi_R(p; R_0)}{\psi_>(p; R_0)}. \quad (11)$$

This shows that the observed phase shift and the short-range effective potential are connected through the logarithmic derivative of the asymptotic wave function, as has previously been noted in other coordinate-space approaches [31–32, 28, 29, 30]. The superscript (2) here signals the fact that long-range forces of order $Q^2$ and higher have not been subtracted and so their effects are subsumed within this short-range potential.

Since the $^1S_0$ channel has a low-energy virtual state, this potential lies close to a nontrivial fixed point. As noted in Refs. [24–25], it is therefore more convenient to look at the behaviour of $1/V_S^{(2)}(p)$. This can be written

$$\frac{1}{V_S^{(2)}(p)} = \frac{M_N}{4\pi} \frac{\psi_>(p; R_0) \psi_R(p; R_0)}{W_>(p)}, \quad (12)$$

where $W_>(p) = M_N \psi_>(p; R_0) / \psi_>(p; R_0)$.
where $W_>(p)$ is the Wronskian

$$W_>(p) = r^2 \left( \psi'_R(p; r) \psi_R(p; r) - \psi'_R(p; r) \psi_>(p; r) \right). \quad (13)$$

In terms of the regular and irregular solutions $\psi_{R,I}$, this becomes

$$\frac{1}{V^{(2)}_S(p)} = -\frac{M_N}{4\pi W_I} \left( \cot \delta_R(p) \sin(\delta_R(p) - \delta_I(p)) \psi_R(p; R_0)^2 + \cos(\delta_R(p) - \delta_I(p)) \psi_R(p; R_0)^2 \right. \left. - \psi_R(p; R_0) \psi_I(p; R_0) \right), \quad (14)$$

where

$$W_I(p) = r^2 \left( \psi'_I(p; r) \psi_R(p; r) - \psi'_R(p; r) \psi_I(p; r) \right). \quad (15)$$

This Wronskian $W_I$ is, of course, independent of $r$ but two useful expressions can be found by evaluating it at very large and very small $r$. Inserting the asymptotic forms of the solutions as $r \to \infty$, Eq. (14), gives

$$W_I(p) = \frac{\sin(\delta_R(p) - \delta_I(p))}{p}, \quad (16)$$

while using the expansions around $r = 0$, Eq. (15), gives

$$W_I = -A_R(p)A_I(p). \quad (17)$$

With these two expressions we can rewrite Eq. (12) as

$$\frac{1}{V^{(2)}_S(p)} = -\frac{M_N}{4\pi} \left( p \cot \tilde{\delta}(p) \psi_R(p; R_0)^2 + p \cot(\delta_R(p) - \delta_I(p)) \psi_R(p; R_0)^2 \right. \left. + \frac{\psi_R(p; R_0) \psi_I(p; R_0)}{A_R(p)A_I(p)} \right). \quad (18)$$

At this point it is useful to introduce the $K$ matrix describing the scattering between the DWs of the OPE potential

$$\tilde{K}(p) = -\frac{4\pi}{M_N p} \cot \tilde{\delta}(p), \quad (19)$$

where, in the numerical results below, $\tilde{\delta}(p)$ is the difference between the empirical phase-shift determined by the Nijmegen group [17] and that generated by OPE alone. This allows us to express Eq. (18) in the form

$$\frac{1}{V^{(2)}_S(p)} = \tilde{K}(p) \left( \frac{\psi_R(p; R_0)^2}{\tilde{K}(p)} - \frac{M_N}{4\pi} \left( p \cot(\delta_R(p) - \delta_I(p)) \psi_R(p; R_0)^2 \right. \left. + \frac{\psi_R(p; R_0) \psi_I(p; R_0)}{A_R(p)A_I(p)} \right) \right). \quad (20)$$

Comparing Eq. (20) with the analogous expressions in Ref. [24], we see that the first term is (the inverse of) the short-range $K$-matrix divided by the square of the DW $\psi_>(p; r)$ at $r = R_0$. The remaining two terms arise from the loop integral over DWs. The first of these can depend nonanalytically on low-energy scales through the phase shifts $\delta_{R,I}$. It removes nonanalytic dependences due to OPE from the empirical $K$ matrix to leave a short-range potential that can be expanded in powers of the low-energy scales, for example with a DW version of the effective-range expansion, as described in Ref. [23].

The last term can be expressed in terms of the power series expansions of $\psi_{R,I}$ around the origin. Since the Wronskian cancels the overall normalisation factors that can contain nonanalytic behaviour, this term is analytic in the low-energy scales. The presence of the irregular solution means that it diverges as the radial cutoff $R_0 \to 0$. Using the expansions [4], we find that the inverse of the potential behaves for small $R_0$ as

$$\frac{1}{V^{(2)}_S(p)} = -\frac{M_N}{4\pi} \left( \frac{1}{R_0} + \frac{m^2}{\lambda_{NN}} \ln(\mu R_0) + \mathcal{O}(R_0^0) \right). \quad (21)$$

If we define a corresponding momentum cutoff scale, $\Lambda \propto 1/R_0$, then we can see that the first term in this expansion is just the familiar linear divergence of the NN loop diagram. The second term is the logarithmic divergence proportional to $m^2$, whose counterterm was identified by Kaplan et al. [9,10] as promoted compared to naive dimensional analysis. Here the term is further promoted by the identification of $\lambda_{NN}$ as a low-energy scale although, in practice, this is not crucial since this term cannot be distinguished from the one that renormalises the linear divergence. The scale $\mu$ in this term depends on the choice of renormalisation scheme.

![Fig. 1. Plots of the inverse of the $^{1}S_0$ short-distance interaction $1/V^{(2)}_S(p)$, in fm$^{-2}$, against lab kinetic energy $T$, in MeV. Results are shown for $R_0 = 1.6$ (shortest dashes), 0.8, 0.4, 0.2 and 0.1 fm (solid line). The divergent terms in Eq. (21) have been removed by subtracting zero-energy values from all of them.](image)

Results for $1/V^{(2)}_S(p)$ are shown in Fig. 1. These were obtained using the $^{1}S_0$ phase shift from the Nijmegen partial-wave analysis PWA93 [17]. Unlike the more peripheral waves studied in Refs. [20,21], the different Nijmegen analyses [17,18] are in good agreement for this channel and so I present only one here. The Nijmegen group’s pre-
ferred value for the πN coupling was used, $f_{NN}^2 = 0.075$. Curves are shown for cutoff radii $R_0$ ranging from 1.6 fm down to 0.1 fm and, to make these easier to compare, I have subtracted the values at zero energy from each. This simple-minded renormalisation scheme removes the divergent terms identified in Eq. (21) but it means that the momentum scales in the leading term of the potential cannot be estimated.

As the radius $R_0$ is taken to zero, artefacts of the cutoff (which are proportional to positive powers of $R_0$) vanish, leaving a result that is independent of $R_0$. The plots show that the $R_0 = 0.1$ fm has nearly converged to its limiting form. In this limit, the expansion of $1/V_S^{(2)}(p)$ in powers of energy (or $p^2$) is just the DW effective-range expansion [21], and the only effect of the radial cutoff $R_0$ is to regulate the linear and logarithmic divergences in the energy-independent term. In contrast, for $R_0$ larger than about 1 fm, the energy dependence of $V_S^{(2)}$ is dominated by artefacts of the cutoff and, by 1.6 fm, and its form is completely different from that of the DW effective-range expansion.

Values of $R_0$ well below 1 fm are obviously outside the expected range of validity of this EFT. Nonetheless the power counting used to organise the potential remains valid because only OPE is iterated and so the scaling behaviour of the wave functions is still controlled by OPE, even in the limit $R_0 \to 0$. This would not have been the case if, for example, the TPE potential or momentum-dependent contact terms were iterated.

The strong energy-dependence of $V_S^{(2)}(p)$ for small $R_0$ reflects the differences between the empirical scattering and that produced by OPE alone. The fact that these can be cancelled by artefacts of the cutoff (as can be seen most clearly for $R_0 = 1.6$ fm) is similar to a recent observation in Ref. [46] for the $^3S_1 - ^3D_1$ channel. The scattering amplitude in that channel is known not to converge for momenta of the order of $m_\pi$ [15] when calculated using dimensional regularisation (which affects only the potentially divergent pieces of the loop integrals, like the small-$R_0$ limit of the approach here). Beane et al. found that using an additional regulator to soften the core of the tensor potential could lead to better convergence. I comment further on this proposal in Sec. 4.

In this context, it is worth noting that for intermediate values of $R_0$, around about 0.8 fm, cutoff-dependent effects are still substantial, leading to a much smaller, and more weakly energy-dependent, interaction. Cutoffs in this regime are often used in applications of EFTs to nuclear forces [31,32,41,42]. For example, the results in Ref. [52] were obtained using radial cutoffs in the range 0.6 to 0.9 fm. The present results suggest that the finite cutoff is crucial to the good fits to the data found there. They may also help to explain why the Nijmegen partial-wave analysis [17], which used boundary conditions with polynomial dependences on energy, could only give good fits for radii in the range 1.4 – 1.8 fm.

Finally, the results for small $R_0$ show that, even after removing OPE, strong energy dependence remains in the short-range potential for energies above about 100 MeV. This matches what Steele and Furnstahl found from their application of a DW effective-range expansion to this channel [19]. Since TPE is known to have important effects on the energy dependence in this region, this should also be removed before drawing any final conclusions about the domain of validity of this EFT.

3 Two-pion exchange

Having used DW methods to remove the effects of OPE from the empirical phase shift, I now turn to TPE. The leading contributions to this appear at orders $Q^2$ and $Q^4$, and expressions for them can be found in Refs. [39,44]. For consistency, as pointed out by Friar [35], the leading recoil correction to OPE should also be removed. In addition, the leading contribution to $\pi \gamma$ exchange has been calculated by Friar et al. [39] and so this can also be dealt with. All of these potentials are perturbations within the power counting used here and so the DWBA is sufficient to remove their effects. This leaves a potential $V_S^{(4)}$ that retains the effects of long-range physics only at order $Q^4$ or above.

Starting from the DW $K$ matrix defined in Eq. (19), the effects of the order-$Q^2,3$ long-range forces can be removed by the subtracting their DWBA matrix elements, taken between the DWs that resum the effects of both OPE and the strong short-range potential. This leaves a modified $K$ matrix,

$$\tilde{K}_S(p) = K(p) - \langle \psi(p) | V_S^{(2)} + V_{\pi\gamma}^T | \psi(p) \rangle. \tag{22}$$

where the DW $\psi(p,r)$ is given by $\psi_>(p,r)$ outside $R_0$ and $\psi_R(p,r)$ inside. In order to apply the effective-range method outlined above, we need the inverse of $\tilde{K}_S$ which, also to first order in the long-range perturbations, is given by

$$\frac{1}{K_S(p)} = \frac{1}{K(p)} + \frac{1}{K(p)} \left( \frac{1}{K(p)} \right)^2 \langle \psi(p) | V_S^{(2)} + V_{\pi\gamma}^T | \psi(p) \rangle. \tag{23}$$

This can then be converted into a residual interaction strength as in Eq. (20).

However, before applying the DWBA to the higher-order long-range potentials, I need to address the fact that they are highly singular as $r \to 0$ and so their matrix elements between $^3S_0$ waves diverge. These must first be made finite, and a convenient way to do this is to cut off the radial integrals at the same radius $R_0$ used to regulate the short-range potential. The divergences can then be identified and cancelled by appropriate counterterms, leaving only finite quantities to be treated in perturbation theory.

The (unrenormalised) DWBA matrix element is given by the radial integral

$$\langle \psi(p) | V | \psi(p) \rangle = 4\pi \int_{R_0}^{\infty} r^2 dr \, V(r) \, \psi_>(p,r)^2. \tag{24}$$
In the present case, its strongest divergences arise from the order-$Q^3$ piece of TPE, which behaves like $1/r^6$ for small $r$, in combination with the irregular parts of the wave functions $\psi_\pi (p; r)$. The latter behave like $1/r$ and so the most divergent term in the radial integral is

$$\int_{R_0} r^2 dr \frac{1}{r^6} \left( \frac{1}{r} \right)^2 \propto \frac{1}{R_0^3}. \quad (25)$$

Higher-order terms in the expansion of $\psi_\pi (r)$ will contain additional powers of $r$ multiplied by low-energy scales. If these scales are $m_\pi$ or $\lambda_\pi$, then the corresponding counterterms cannot in practice be distinguished from the leading one. Terms containing powers of the energy ($p^2$) are therefore of most interest. They arise from terms in the expansion of the irregular solution of orders $p^2 r$, $p^4 r^3$, and so on. These lead to divergences proportional to $p^2/R_0^2$ and $p^4/R_0$. In contrast all terms of order $p^6$ and above are finite. Within the framework of KSW power counting, the required counterterms have orders $Q^0$ and $Q^2$. This is consistent with the treatment of long-range forces here, which includes terms up to order $Q^3$.

To subtract off the divergent pieces of the integrals, I first use Eqs. (4) and (26) to write the matrix element in terms of radial integrals of $V \psi^2$, $V \psi \psi_R$ and $V \psi_R^2$. I then fit the low-energy region ($T = 15–35$ MeV) of each of these with a polynomial of fourth order in $p^2$, and subtract the divergent terms from the integrals. As in the previous section, this rather crude renormalisation scheme is adequate for comparing the forms of the remaining, nondivergent pieces of the potentials but it does not allow an estimate of the scales in the leading terms.

After removing the effects of the order-$Q^2$ long-range potentials from the $K$ matrix using the DWBA, the corresponding short-range potential can be determined as before, using

$$\frac{1}{\tilde{V}_S^{(4)} (p)} = \frac{V_S (R_0)^2}{K_S (p)} - \frac{M_\pi}{4\pi} \left( p \cot (\delta_R (p) - \delta_I (p)) \psi_R (R_0)^2 + \frac{\psi_R (R_0) \psi_I (R_0)}{A_R (p) A_I (p)} \right). \quad (26)$$

The superscript (4) indicates that the only long-range effects that remain in this potential are of order $Q^4$ or higher.

Values from other recent determinations are used. As in Fig. 1, the divergent terms in Eq. (21) have been subtracted to bring the curves for different radii onto the same plot.

The subtraction of terms up to order $p^2$ from the largest contributions (the integrals of $V^{(2,3)}_{\text{TPE}}$) means that the curves are almost identical to those in Fig. 1 at very low energies. Effects of TPE start to become significant for lab energies above about 70 MeV and then grow very rapidly with energy. This rapid increase indicates that any low-energy expansion will break down by about $T \sim 150$ MeV, which corresponds to a momentum $p \sim 270$ MeV. The only curve that does not show this feature is the one for the largest residual cutoff, $R_0 = 1.6$ fm. However in this case, the form of the residual potential is controlled by the large artefacts of the cutoff, like the corresponding curve in Fig. 1. Similar comments to those made in Sec. 2 apply to the use of large cutoff radii.

4 Discussion

The DW method that was previously applied to NN scattering in peripheral waves [20,21] has been applied here to the $^1S_0$ channel. The scattering in this channel is strong at low energies and so the appropriate EFT is based on a DW effective-range expansion, rather than a Born expansion. Such an approach was first applied to this channel by Steele and Furnstahl [19], who noted the need to extend it to take account of the effects of TPE.

The empirical $^1S_0$ phase shift is deconstructed by first using the DW effective-range expansion to remove the effects of OPE to all orders. This also takes account of non-perturbative effects of the leading (energy-independent) contact interaction, and it generates wave functions that are mixtures of the regular and irregular solutions of the Schrödinger equation with the OPE potential. These wave functions are then used to evaluate the DWBA matrix elements of the order-$Q^2$ and $Q^3$ chiral potentials. Subtracting these leaves a residual short-range interaction from which all effects of long-range forces have been removed up to order $Q^4$. 

![Fig. 2. Plots of the inverse of the $^1S_0$ short-distance interaction $1/V^{(4)}(p)$, in fm$^{-2}$, against lab kinetic energy $T$, in MeV. The curves for different $R_0$ are labelled as in Fig. 1.](image-url)
Underlying this treatment is the power counting that follows from identifying the scale controlling the strength of OPE, $\lambda_{NN}$, as a low-energy scale, in addition to the momenta, $m_\pi$ and the inverse scattering length $1/a$. In this scheme, OPE and the leading contact interaction are of order $Q^{-1}$ and hence must be iterated to all orders. All other interactions, including TPE, should be treated perturbatively. The $1/r$ singularity of OPE in this channel means that, even though that potential is iterated, the power counting for the higher-order contact interactions is, for practical purposes, the same as that of KSW [9,10].

The DWBA matrix elements of the TPE potential contain divergent pieces proportional to $p^0$, $p^2$ and $p^4$, but these can be renormalised using energy-dependent contact interactions with orders up to $Q^2$ in the KSW-like counting. The need for counterterms to this order is consistent with the fact that long-range forces have been treated to power counting, and we then respect that counting. That is permissible only if we first determine the appropriate scale of our EFT. However, as stressed in Ref. [39], this corresponds to an arbitrarily large momentum scale.

This exemplifies the point that, so long as we are careful, it is possible to take the cutoff beyond the breakdown scale of our EFT. However, as stressed in Ref. [39], this is permissible only if we first determine the appropriate power counting, and we then respect that counting. That is, we must iterate all terms of order $Q^{-1}$ and we must not iterate any higher-order perturbations (“irrelevant” terms of order $Q^d$ with $d \geq 0$). Otherwise, iteration of singular higher-order potentials changes the forms of the wave functions at short distances and, in general, destroys any consistent power counting.

The present application of the DW method to $^{1}S_0$ NN scattering has mixed implications for EFT descriptions of nuclear forces. The necessary counterterms are available to cancel all the divergence terms that arise from the DWBA matrix elements of the order-$Q^2$, as also noted in Ref. [32]. The remaining dependence on the cutoff consists of artefacts, which vanish like positive powers of $R_0$ as $R_0 \to 0$. For $R_0 \sim 0.1$ fm, the renormalised residual interaction has essentially converged to a cutoff-independent form. In contrast, for $R_0$ around 0.8 fm or larger, cutoff-dependent effects are substantial and both the size and the energy-dependence of the short-range interaction are quite different from its $R_0 \to 0$ limit.

Unfortunately the range of applicability of the resulting EFT is limited since the residual interaction starts to show very strong energy dependence in the region $T \gtrsim 100$ MeV. In fact, this energy dependence strengthens when the higher-order chiral potentials are subtracted, contrary to the hope expressed by Steele and Furnstahl [19] and also to what might have been expected from other channels, where removing long-range forces weakens this dependence [20,21]. Any attempt to expand the residual interaction in powers of the kinetic energy will fail to converge for $T \sim 150$ MeV, implying that the breakdown scale is only of order $\Lambda_0 \sim 270$ MeV.

These results indicate that the EFT developed here should not be used in the $^{1}S_0$ channel for energies above this range. They are consistent with the behaviour shown in Fig. 1 of Ref. [32], which, although not presented in the form of a residual interaction, is based on the same method as used here. That shows that, with cutoff radius of 0.1 fm, a short-range potential consisting of a three-term polynomial in the energy cannot provide a good fit to the $^{1}S_0$ phase shift above about 180 MeV. All this implies that the present EFT, which follows the philosophy of Kaplan et al. [4,10] by treating higher-order terms perturbatively, breaks down at a scale of around $2m_\pi$. This is uncomfortably low for hopes of applying it to other nuclear systems.

Such a low breakdown scale indicates that there are other low-energy scales in this system, in addition to $1/a$, $m_\pi$ and $\lambda_{NN}$. An obvious omission from a low-energy EFT with only pions and nucleons is the $\Delta$ resonance, which has an excitation energy of about 290 MeV. In the present approach effects of the $\Delta$ appear only indirectly, via the coefficients $c_{3,4}$ in the order-$Q^2$ $\pi N$ Lagrangian. The values of these are larger than would be natural if the scale of the omitted physics were, say, the mass of the $\rho$ meson.

For example, the coefficient of $r^{-6} \exp(-2m_\pi r)$ in the order-$Q^3$ TPE potential contains both $\lambda_{NN}$ and $c_3 \simeq -5$ GeV$^{-1}$. This coefficient, although built out of high-energy scales in strict chiral counting, is unnaturally large. This can be seen by defining a scale parameter for it by analogy with $\lambda_{NN}$ for OPE:

$$
\lambda'_{NN} = \left( \frac{(16\pi)^2 f_\pi^2}{144\sigma_A^4 |c_3| |M_N|} \right)^{1/4} \approx 115 \text{ MeV}. \quad (27)
$$

The small size of this and other scales in the TPE potential is a likely cause of the low breakdown scale found here. The important, indirect, contribution of the $\Delta$ to these scales suggests that one possible way to construct a theory with an greater range of validity could be to add the $\Delta$ as an explicit degree of freedom, as done in Refs. [50,51,52]. It will be interesting to see whether the application of the DW method to that EFT can lead to effective interactions with less dramatic energy dependences.

One way forward might be to look for an EFT where at least some of the TPE potential could be iterated to all orders. Working with an EFT containing the $\Delta$ will promote parts of this potential to lower orders but it will still leave them as perturbative effects. Further low-energy scales would need to identified if one wanted to promote these terms to order $Q^{-1}$ and hence to iterate them in the Schrödinger equation. These scales would need to be constructed out of $F_\pi$ and $M_N$, by analogy with $\lambda_{NN}$ in the case of OPE.

A quite different approach to taming this rapid energy dependence could be to leave the cutoff finite, even though this is not required after perturbative renormalisation of the divergences in the DWBA matrix elements. Such treatments have been suggested by various authors [13,44,45,32]. It should be stressed the motivation for doing this is quite different from the standard one for using a finite cutoff to control the unrenormalisable divergences that arise when higher-order terms in the potential are it-
erated by solving the full Schrödinger equation [31,32,33,34]. Instead one would be using cutoff-dependent terms to cancel large but finite contributions that are left after the renormalised contributions of the known long-range interactions have been removed from the observed scattering.

In Ref. [35], a similar additional regulator was applied to the $^3S_1-^3D_1$ channels and this was shown to improve the convergence of the original KSW power counting [9,10], which treats OPE perturbatively. Using the same DWBA approach as here, Ref. [32] shows that cutoff radii in the range 0.6–0.9 fm can lead to better fits to the $^1S_0$ phase shift for energies of 200 MeV and above. The results presented here in Figs. 1 and 2 provide some clues as to why such methods may work: for cutoff radii around 0.8 fm and above, both the size and the energy dependence of the residual short-range interaction are considerably attenuated. As a result the expansion of this potential in powers of low-energy scales will have a wider range of convergence.

From a practical point of view, therefore, such cutoffs can successfully eke out the domain of validity of an EFT with a less than ideal separation of scales. However, the principles that could justify this remain unclear since the use of a cutoff to control contributions that may be large but are nonetheless finite is very different from the normal one of regularising potentially divergent terms in order to renormalise them.

Beane et al. [36] have suggested an analogy with the renormalisation scale $\mu$ of perturbative QCD, which is arbitrary but can be chosen to optimise the convergence of the resulting expansion. Alternatively, it might be viewed as similar to the factorisation scale used to define parton distributions in QCD by separating the nonperturbative regime from the one where perturbation theory applies. Such a role for the radial cutoff of the present approach was in fact suggested in Ref. [24]. It remains to be seen whether either of these interpretations can be placed on a more rigorous basis, presumably with the help of the RG.

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### References

1. S. R. Beane, P. F. Bedaque, W. C. Haxton, D. R. Phillips and M. J. Savage, *At the frontier of particle physics: handbook of QCD*, ed. M. Shifman, vol. 1, p. 133 (World Scientific, Singapore, 2001) [arXiv:nucl-th/0008064].

2. P. F. Bedaque and U. van Kolck, Ann. Rev. Nucl. Part. Sci. 52, 339 (2002) [arXiv:nucl-th/0203055].

3. E. Epelbaum, Prog. Part. Nucl. Phys. 57, 654 (2006) [arXiv:nucl-th/0509032].

4. S. Weinberg, Phys. Lett. B251, 288 (1990).

5. S. Weinberg, Nucl. Phys. B363, 3 (1991).

6. G. Ecker, Prog. Part. Nucl. Phys. 35, 1 (1995) [hep-ph/9501357].

7. P. F. Bedaque and U. van Kolck, Phys. Lett. B428, 221 (1998) [arXiv:nucl-th/9710073].

8. U. van Kolck, Nucl. Phys. A645, 273 (1999) [arXiv:nucl-th/9808007].

9. D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998) [arXiv:nucl-th/9804034].

10. D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. B534, 329 (1998) [arXiv:nucl-th/9802075].

11. H. A. Bethe, Phys. Rev. 76, 38 (1949).

12. J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949).

13. T. D Cohen and J. M. Hansen, Phys. Rev. C 59, 13 (1999) [arXiv:nucl-th/9808038].

14. T. D Cohen and J. M. Hansen, Phys. Rev. C 59, 3047 (1999) [arXiv:nucl-th/9901065].

15. S. Fleming, T. Mehen and I. W. Stewart, Nucl. Phys. A677, 313 (2000) [arXiv:nucl-th/9911001].

16. S. R. Beane, P. F. Bedaque, M. J. Savage and U. van Kolck, Nucl. Phys. A700, 377 (2002) [arXiv:nucl-th/0104030].

17. H. van Haeringen and L. P. Kok, Phys. Rev. A 26, 1218 (1982).

18. A. M. Badalyan, L. P. Kok, M. I. Polikarpov and Yu. A. Simonov, Phys. Rep. 82, 31 (1982).

19. J. V. Steele and R. J. Furnstahl, Nucl. Phys. A645, 439 (1999) [arXiv:nucl-th/9808022].

20. M. C. Birse and J. A. McGovern, Phys. Rev. C 70, 054002 (2004) [arXiv:nucl-th/0307050].

21. M. C. Birse, Phys. Rev. C 76, 034002 (2007) [arXiv:0706.0984].

22. K. G. Wilson and J. G. Kogut, Phys. Rep. 12, 75 (1974).

23. M. C. Birse, J. A. McGovern and K. G. Richardson, Phys. Lett. B464, 169 (1999) [arXiv:hep-ph/9807302].

24. T. Barford and M. C. Birse, Phys. Rev. C 67, 064006 (2003) [arXiv:hep-ph/0206146].

25. T. Barford and M. C. Birse, J. Phys. A: Math. Gen. 38, 697 (2005) [arXiv:nucl-th/0406608].

26. M. C. Birse, Phys. Rev. C 74, 014003 (2006) [arXiv:nucl-th/0507077].

27. A. Nogga, R. G. E. Timmermans and U. van Kolck, Phys. Rev. C 72, 054002 (2005) [arXiv:nucl-th/0505005].

28. M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 72, 054002 (2005) [arXiv:nucl-th/0504067].

29. M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 74, 054001 (2006) [arXiv:nucl-th/0506047].

30. M. Pavón Valderrama and E. Ruiz Arriola, Phys. Rev. C 74, 064004 (2006) [arXiv:nucl-th/0507075].

31. D. Shukla, D. R. Phillips and E. Mortenson, J. Phys. G: Nucl. Part. Phys. 35, 115009 (2008) [arXiv:0803.1190].

32. M. Pavón Valderrama, arXiv:0912.0699.

33. N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A625, 758 (1997) [arXiv:nucl-th/9705045].

34. M. C. M. Rentmeester, R. G. E. Timmermans, J. L. Friar and J. J. de Swart, Phys. Rev. Lett. 82, 4992 (1999) [arXiv:nucl-th/9910054].

35. J. L. Friar, Phys. Rev. C 60, 034002 (1999) [arXiv:nucl-th/9910182].

36. J. L. Friar, U. van Kolck, G. L. Payne and S. A. Coon, Phys. Rev. C 68, 024003 (2003) [arXiv:nucl-th/0303058].
37. D. R. Phillips, S. R. Beane and T. D. Cohen, Ann. Phys. \textbf{263} (1998) 255 [arXiv:hep-th/9706070].
38. E. Epelbaum and J. Gegelia, Eur. Phys. J. A \textbf{41}, 341 (2009) [arXiv:0906.3822].
39. M. C. Birse, in Proc. 6th Int. Workshop on Chiral Dynamics, PoS(CD09)078 (2009) [arXiv:0909.4641].
40. K. L. Ipson, K. Helmke and M. C. Birse, in preparation.
41. D. R. Entem and R. Machleidt, Phys. Rev. C \textbf{68}, 041001 (2003) [arXiv:nucl-th/0304018].
42. E. Epelbaum, W. Glöckle and U.-G. Meissner, Nucl. Phys. \textbf{A747}, 362 (2005) [arXiv:nucl-th/0405048].
43. K. A. Scaldeferri, D. R. Phillips, C.-W. Kao and T. D. Cohen, Phys. Rev. C \textbf{56}, 679 (1997) [arXiv:nucl-th/9610049].
44. D. Djukanovic, M. R. Schindler, J. Gegelia and S. Scherer, Phys. Rev. D \textbf{72}, 045002 (2005) [arXiv:hep-ph/0407170].
45. D. Djukanovic, J. Gegelia, S. Scherer and M. R. Schindler, Few Body Syst. \textbf{41}, 141 (2007) [arXiv:nucl-th/0609055].
46. S. R. Beane, D. B. Kaplan and A. Vuorinen, Phys. Rev. C \textbf{80}, 011001 (2009) [arXiv:0812.3938].
47. V. G. J. Stoks, R. A. M. Klomp, M. C. M. Rentmeester and J. J. de Swart, Phys. Rev. C \textbf{48}, 792 (1993); NN-Online, University of Nijmegen, http://nn-online.org/
48. V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen and J. J. de Swart, Phys. Rev. C \textbf{49}, 2950 (1994) [arXiv:nucl-th/9406039].
49. M. C. M. Rentmeester, R. G. E. Timmermans and J. J. de Swart, Phys. Rev. C \textbf{67}, 044001 (2003) [arXiv:nucl-th/0302080].
50. C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. C \textbf{53}, 2086 (1996) [arXiv:hep-ph/9511380].
51. N. Kaiser, S. Gerstendörfer and W. Weise, Nucl. Phys. \textbf{A637}, 395 (1998) [arXiv:nucl-th/9802071].
52. H. Krebs, E. Epelbaum and U.-G. Meissner, Eur. Phys. J. \textbf{A32}, 127 (2007) [arXiv:nucl-th/0703087].