EQUIVARIANT K-THEORY OF THE SPACE OF PARTIAL FLAGS.

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ABSTRACT. We use Drinfeld style generators and relations to define an algebra \( U_n \) which is a “\( q = 0 \)” version of the affine quantum group of \( \mathfrak{gl}_n \). We then use the convolution product on the equivariant \( K \)-theory of spaces of pairs of partial flags in a \( d \)-dimensional vector space \( V \) to define affine zero-Schur algebras \( S^{aff}_{0}(n, d) \) and to prove that for every \( d \) there exists a surjective homomorphism from \( U_n \) to \( S^{aff}_{0}(n, d) \).

INTRODUCTION

Our paper started as an attempt to study certain strange degenerations of quantum groups and of quantum Schur algebras. Different versions of the famous Schur-Weyl correspondence were the main inspiration for our study. Let us recall various levels of quantum Schur-Weyl correspondence from a geometric perspective.

Finite sets: quantum finite Schur-Weyl correspondence. Let us begin with the original combinatorial setting for the finite Hecke and Schur algebras. Consider a finite field \( k \) with \( q = p^r \) elements. Denote the set of full flags in the vector space \( V = k^n \) by \( B = B(k) \). The group \( GL(V) \) is denoted by \( G \). Notice that \( G \) acts on the set \( B \times B \) diagonally. Consider the \( C \)-vector space of \( G \)-invariant functions on \( B \times B \) denoted by \( \mathcal{H}^\text{fin}_{q} \). It is known and will be recalled below in various contexts that \( \mathcal{H}^\text{fin}_{q} \) is equipped with an associative algebra structure given by convolution. The obtained algebra is called the finite Hecke algebra for the parameter \( q \).

Fix another integer number \( d \). Denote the set of partial \( n \)-step flags in the vector space \( W = k^d \) by \( F^d_n = F^d_n(k) \). Notice that \( G \) acts on the set \( F^d_n \times F^d_n \) diagonally. Consider the \( C \)-vector space of \( G \)-invariant functions on \( F^d_n \times F^d_n \) denoted by \( \mathcal{S}^\text{fin}_{q} = \mathcal{S}^\text{fin}_{q}(n, d) \). Again, the standard construction of convolution makes \( \mathcal{S}^\text{fin}_{q}(n, d) \) into an associative algebra called the quantum finite Schur algebra.

The third geometric datum of the picture is given by the set \( B \times F^d_n \), with the diagonal action of \( G \). The vector space of \( G \)-invariant functions forms a \( \mathcal{H}_{q}^\text{fin} \)-\( \mathcal{S}^\text{fin}_{q}(n, d) \)-bimodule. It is the key ingredient to prove the double centralizer property for the Hecke and Schur algebras known as the quantum finite Schur-Weyl correspondence.

In their groundbreaking work, Beilinson, Lusztig and MacPherson constructed a surjective map from the quantum group \( U_q(\mathfrak{gl}_n) \) to the algebra \( \mathcal{S}^\text{fin}_{q}(n, d) \) (see [11]). The map is explicitly given in terms of quantum Chevalley generators, in particular, the elements \( E_i \) (resp. \( F_i \)) are given by delta-functions of certain closed orbits in \( F^d_n \times F^d_n \) that can be viewed as elementary correspondences. Moreover, the quantum group can be seen as a limit of those for \( d \) going to infinity. Thus a first step to relate quantum groups to the geometry of partial flags started a variety of generalizations.

Finite sets: setting \( q = 0 \). A variation of the picture above was the first motivation for our study. In a series of papers [16, 6, 7, 8] Su, Jensen, and Yang considered the same finite sets \( B(k) \) and \( F^d_n(k) \) and the same \( C \)-vector spaces of \( G \)-invariant functions but changed the definition of convolution: namely they defined formally what it means to take only the “principal part” for the products in the \( q \)-expansion.
This way they obtained the algebras $\mathbb{H}_q^\text{fin}$ and $\mathbb{S}_q^\text{fin}(n, d)$ respectively. Notice that these algebras do not depend on $q$. It turns out that $\mathbb{H}_q^\text{fin}$ is obtained from $\mathbb{H}_0^\text{fin}$ by literally substituting $q = 0$ in the formulas for the product (in a natural basis). The situation on the Schur side is more interesting.

Krob and Thibon (12) defined a version of the algebra $\mathbb{U}_q(\mathfrak{gl}_n)$ for $q = 0$. Notice that the standard quantum Chevalley generators do not allow to substitute 0 literally. The authors performed a certain renormalization to define the finite 0-quantum group $\mathbb{U}_0(\mathfrak{gl}_n)$. The relations on quantum Chevalley generators degenerate, in particular, quantum Serre relations stay cubic but become binomial, known under the name of plactic relations. Jensen, Su, and Yang obtained the same cubic binomial relations in their study of the 0-Schur algebra $\mathbb{S}_0^\text{fin}(n, d)$.

Introducing geometry: quantum affine Schur-Weyl correspondence. Next we replace the finite field $k$ by $\mathbb{C}$. We consider the vector spaces $\mathbb{C}^n$ and $\mathbb{C}^d$. The sets $B$ and $\mathcal{F}_n^d$ become algebraic varieties over complex numbers.

Kazhdan, Lusztig, Ginzburg, and Vasserot considered the Steinberg varieties on the Weyl and Schur sides $S^t$ and $S^t_d$ respectively (see [3, 9, 10, 17]). Those are $G = GL(n)$-invariant Lagrangian subvarieties in $T^*(B)^2$ and $T^*(\mathcal{F})^2$ respectively. Notice that $G_m$ acts on them by dilations of cotangent directions. The complexified $G \times G_m$-equivariant K-group of $S^t$ (resp, of $S^t_d$) is denoted by $\mathbb{H}_q^\text{aff}$ (resp. by $\mathbb{S}_q^\text{aff}(n, d)$). One modifies the convolution product construction for the K-group case. The obtained algebras are called the affine Hecke algebra and the quantum affine Schur algebra respectively.

Kazhdan, Lusztig, and Ginzburg proved that the former is indeed isomorphic to the Hecke algebra for the affine Weyl group, for the parameter $q$ coming from the $G_m$ action. Ginzburg and Vasserot proved that the quantum loop algebra $\mathbb{U}_q(\mathfrak{gl}_n[t, t^{-1}])$, in the Drinfeld realization, maps surjectively onto $\mathbb{S}_q^\text{aff}(n, d)$. Finally an immediate geometric modification of the bimodule construction provides a double centralizer property for the algebras $\mathbb{S}_q^\text{aff}(n, d)$ and $\mathbb{H}_q^\text{aff}$ known as affine quantum Schur-Weyl correspondence.

Introducing geometry: Weyl side for $q = 0$. A modification of the affine Weyl side is due to Kostant and Kumar in [11]. Ignore the cotangent directions and consider the complexified $G$-equivariant K-group of the variety $B \times B$, equipped with the convolution product. The authors proved that the obtained algebra $\mathbb{H}_q^\text{aff}$ is isomorphic to the algebra obtained by substitution $q = 0$ into the definition of the affine Hecke algebra. The algebra is called the affine zero-Hecke algebra. It is the affinization of the algebra considered by Jensen, Su, and Yang.

The main objective: Schur side for $q = 0$ and the affine quantum group. Notice that, similarly to the case of $\mathbb{U}_q(\mathfrak{gl}_n)$, the relations do not allow a direct specialization of the parameter $q$ to 0. One goal of the present paper is to define an algebra $\mathbb{U}_q(\mathfrak{gl}_n[t, t^{-1}])$ by Drinfeld style generators and relations. Then we interpret those generators and relations geometrically via equivariant K-groups of the partial flag varieties.

Following the case studied by Kostant and Kumar, we consider the $G$-equivariant $K$-group of the complex variety $\mathcal{F}_n^d \times \mathcal{F}_d^d$. We introduce the usual convolution product on the vector space. The obtained algebra should be called the affine zero-Schur algebra $\mathbb{S}_0^\text{aff}(n, d)$.

Our main construction provides a surjective map of the algebra $\mathbb{U}_q(\mathfrak{gl}_n[t, t^{-1}])$ onto $\mathbb{S}_0^\text{aff}(n, d)$. The map is given explicitly by the images of the Drinfeld generators. Namely, we modify the Vasserot construction from the previous paragraph to our case and consider explicit K-theory classes supported on the standard elementary correspondences.

One difference both from the Jensen-Su-Yang story and from the Vasserot algebra is that the relations we impose on Drinfeld generators are quadratic. However, the positive part of the Krob-Thibon 0-quantum group can be mapped into our algebra, as the cubic plactic relations on the generators of the constant part follow from our quadratic relations.
Structure of the paper. The rest of the paper is structured as follows. Section 1 is devoted to reviewing the definitions and basic properties as well as formulating the main result. In Subsection 1.1 we define the algebra $U_n := U_0(g_n[t, t^{-1}])$ using generators and relations. In Subsection 1.3 we recall the definitions and basic properties of the equivariant $K$-theory. In Subsection 1.4 we recall the definitions of the space of partial flags $F_n^d$, the correspondence endomorphisms on $K^G(F_n^d)$, and use the convolution product on $K^G(F_n^d \times F_n^d)$ to define the affine $0$-Schur algebra $S_{0 \text{aff}}(n, d)$. Finally, we formulate our main result:

**Theorem A.** (Theorem 1.10). For every positive integer $d$ there exist a surjective morphism

$$\phi_d : U_n \to S_{0 \text{aff}}(n, d).$$

Section 2 is devoted to the combinatorics of the orbit structure of the double partial flag variety $F_n^d \times F_n^d$ under the diagonal action of $G := GL(V)$. In Subsection 2.1 we follow the work of Beilinson, Lusztig, MacPherson ([1]) and Vasserot ([17]) to establish that the orbits in $F_n^d \times F_n^d$ are enumerated by $n \times n$ matrices with non-negative integer entries, with the total sum of entries equal to $d$. In Subsection 2.2 we study the Bruhat order on these matrices given by the adjacency order on orbits. We establish explicit combinatorial conditions for the cover relations and for the order itself (see Theorem 2.7), an elementary combinatorial proof is provided in the Appendix. We further use these results to identify matrices corresponding to closed orbits in $F_n^d \times F_n^d$ and to give a geometric characterization of closed orbits (Corollary 2.8), and also to identify the matrices corresponding to open orbits (Corollary 2.9).

In Subsection 2.3 we study the convolution product on the level of supports of the classes in $K^G(F_n^d \times F_n^d)$. This leads to an associative product on the set of $n \times n$ matrices with non-negative integer entries. Computing the product of two arbitrary matrices is rather complicated and even mysterious, but the multiplication by an almost diagonal matrix (see Definition 2.18) can be described explicitly (see Theorem 2.20). The principal result of this subsection is the following:

**Theorem B** (Corollaries 2.21 and 2.24). Any matrix can factored in to a product of diagonal and almost diagonal matrices (with respect to the product introduced in Definition 2.18).

For most of this subsection we follow [11] and [17], but some of the approaches are new. In addition, we provide multiple examples.

In Section 3 we analyze the structure of the convolution algebra $S_{0 \text{aff}}(n, d)$. In Subsections 3.1 and 3.2 we review the equivariant $K$-theory of the partial flag variety $F_n^d$ and the orbits of the double partial flag variety $F_n^d \times F_n^d$. Those are isomorphic to partially symmetric Laurent polynomials in $d$ variables. In Subsection 3.3 we establish explicit formulas for the push-forward and pull-back morphisms along natural projection between orbits in double partial flag varieties (see Lemma 3.3). The formulas are in terms of the partially symmetric Laurent polynomials, introduced in Subsection 3.2. We further derive explicit corollaries from these formulas for certain specific classes, used in the next Section (see Corollaries 3.9 and 3.10 and Lemma 3.11).

In Subsection 3.4 we prove

**Theorem C** (Theorem 3.17). The convolution algebra $S_{0 \text{aff}}(n, d)$ is generated by the classes supported on the orbits corresponding to diagonal and almost diagonal matrices.

In Subsection 3.5 we introduce special classes $E_{\mu, k}(p)$, $F_{\mu, k}(p)$, and $H_{\mu, n}(r)$ in $S_{0 \text{aff}}(n, d)$, supported on orbits corresponding to diagonal and almost diagonal matrices, and prove a series of key relations on these classes. We further show that

**Theorem D** (Theorem 3.40). Classes $E_{\mu, k}(p)$, $F_{\mu, k}(p)$, and $H_{\mu, n}(r)$ generate the convolution algebra $S_{0 \text{aff}}(n, d)$. 

In Section 4, is devoted the map \( \phi_d : U_n \to S_{\text{aff}}(n, d) \) and the proof of the main theorem. In Subsection 4.1, we introduce the map \( \phi_d \) and use the relations from Subsection 3.5 to prove that it is well defined. In Subsection 4.2, we use Theorem 3.40 to prove surjectivity of the map \( \phi_d \).

1. Definitions and Preliminaries

1.1. The algebra \( U_n := U_0(\mathfrak{g}l_n[t, t^{-1}]) \).

**Definition 1.1.** The algebra \( U_n \) is defined via generators and relations as follows. The set of generators consists of \( E_i(p) \), \( F_i(q) \), and \( H_n(r) \), for \( 0 < i < n \) and \( p, q, r \in \mathbb{Z} \). The generators satisfy the following list of relations.

1. Relations on \( E \) generators:

   \[
   E_i(p)E_i(q) = -E_i(q-1)E_i(p+1),
   \]
   for all \( 0 < i < n \) and \( p, q \in \mathbb{Z} \).

   \[
   E_{i+1}(p)E_i(q) = E_i(q)E_{i+1}(p) - E_i(q+1)E_{i+1}(p-1),
   \]
   for all \( 0 < i < n - 1 \) and \( p, q \in \mathbb{Z} \).

   \[
   E_i(p)E_j(q) = E_j(q)E_i(p),
   \]
   whenever \( |i - j| > 1 \), \( 0 < i, j < n \) and \( p, q \in \mathbb{Z} \).

2. Relations on \( F \) generators:

   \[
   F_i(p)F_i(q) = -F_i(q+1)F_i(p-1),
   \]
   for all \( 0 < i < n \) and \( p, q \in \mathbb{Z} \).

   \[
   F_i(p)F_{i+1}(q) = F_{i+1}(q)F_i(p) - F_{i+1}(q-1)F_i(p+1),
   \]
   for all \( 0 < i < n - 1 \) and \( p, q \in \mathbb{Z} \).

   \[
   F_i(p)F_j(q) = F_j(q)F_i(p),
   \]
   whenever \( |i - j| > 1 \), \( 0 < i, j < n \) and \( p, q \in \mathbb{Z} \).

3. Relations between \( F \) and \( E \) generators:

   \[
   E_i(p)F_j(q) = F_j(q)E_i(p),
   \]
   For all \( i \neq j \), \( 0 < i, j < n \) and \( p, q \in \mathbb{Z} \).

   \[
   E_i(p)F_i(q) - F_i(q)E_i(p) = E_i(p')F_i(q') - F_i(q')E_i(p'),
   \]
   whenever \( p + q = p' + q' \). For convenience, we introduce the notation:

   \[
   H_i(p+q) := E_i(p)F_i(q) - F_i(q)E_i(p).
   \]

4. The \( H \) generators commute:
For all $0 < i, j \leq n$ and $p, q \in \mathbb{Z}$.

5. Relations between $H$ and $E$ generators:

\begin{equation}
H_i(p)E_i(q) = E_i(q)H_i(p),
\end{equation}

for all $0 < i < n$ and $p, q \in \mathbb{Z}$.

\begin{equation}
H_i(p)H_{i-1}(q) = E_{i-1}(q)H_i(p) - E_{i-1}(q + 1)H_i(p - 1),
\end{equation}

for all $0 < i < n - 1$ and $p, q \in \mathbb{Z}$.

\begin{equation}
H_i(p)E_i(q) = -E_i(q - 1)H_i(p + 1),
\end{equation}

for all $0 < i < n$ and $p, q \in \mathbb{Z}$.

6. Relations between $H$ and $F$ generators:

\begin{equation}
F_{i-1}(p)H_i(q) = H_i(q)F_{i-1}(p) - H_i(q - 1)F_{i-1}(p + 1),
\end{equation}

for all $p, q \in \mathbb{Z}$.

\begin{equation}
H_i(p)F_i(q) = -F_i(q + 1)H_i(p - 1),
\end{equation}

for all $0 < i < n$ and $p, q \in \mathbb{Z}$.

\begin{equation}
H_i(p)H_{i}(q) = E_{i}(q)H_{i}(p),
\end{equation}

for all $0 < i < n - 1$ and $p, q \in \mathbb{Z}$.

Remark 1.2. Note that relation (1.2) and the definition of $H_i(p)$ for $0 < i < n$ (formula (3.3)) imply the analogs of relation (5.1):

\begin{equation}
H_i(p)E_{i-1}(q) = E_{i-1}(q)H_i(p) - E_{i-1}(q + 1)H_i(p - 1),
\end{equation}

and

\begin{equation}
E_i(q)H_{i-1}(q) = H_{i-1}(q)E_i(q) - H_{i-1}(q + 1)E_i(q - 1),
\end{equation}

for all $0 < i < n$ and $p, q \in \mathbb{Z}$.

Similarly, one also get the analogs of relation (6.1):

\begin{equation}
F_{i-1}(p)H_i(q) = H_i(q)F_{i-1}(p) - H_i(q - 1)F_{i-1}(p + 1),
\end{equation}

and

\begin{equation}
H_{i-1}(q)F_i(p) = F_i(p)H_{i-1}(q) - F_i(p - 1)H_{i-1}(q + 1),
\end{equation}

for all $0 < i < n$ and $p, q \in \mathbb{Z}$.

Finally, it immediately follows from the definition of $H_i(p)$ for $0 < i < n$ that $H_i(p)$ commutes with both $E_j(q)$ and $F_j(q)$ for $|i - j| > 1$:

\begin{equation}
H_i(p)E_j(q) = E_j(q)H_i(p),
\end{equation}

for all $|i - j| > 1$ and $p, q \in \mathbb{Z}$. 
Hence, using relation 1.2 one gets 

\[ H_i(p)F_j(q) = F_j(q)H_i(p), \]

for all \(|i - j| > 1\) and \(p, q \in \mathbb{Z}\).

**Remark 1.3.** One can think about the extra generators \(H_n(p)\) in the following manner. Let \(\mathcal{U}'_n \subset \mathcal{U}_n\) be the subalgebra generated by \(E_i(p)\) and \(F_j(q)\), \(0 < i, j < n, p, q \in \mathbb{Z}\). One has a natural embedding \(\mathcal{U}'_n \subset \mathcal{U}_{n+1}'\), where \(\mathcal{U}_{n+1}'\) has the extra generators \(E_n(p)\) and \(F_n(q), p, q \in \mathbb{Z}\). Then one also gets

\[ \mathcal{U}'_n \subset \mathcal{U}_n \subset \mathcal{U}_{n+1}', \]

where \(\mathcal{U}_n\) is generated by \(E_i(p), F_j(q), 0 < i, j < n\) and \(p, q \in \mathbb{Z}\), and \(H_n(r), r \in \mathbb{Z}\). This approach allows to shorten the list of relations as relations 5.1 5.3 6.1 and 6.3 become unnecessary.

1.2. **Constant part and Krob-Thibon 0-quantum group.** The positive part \(\mathfrak{U}_0(\mathfrak{gl}_n)^+\) of the Krob-Thibon’s 0-quantum group is generated by elements \(e_i, 0 < i < n\), subject to the plactic relations:

\[ e_i e_j = e_j e_i \text{ for } |i - j| > 1, \quad e_i^2 e_{i+1} = e_i e_{i+1} e_i, \quad e_{i+1} e_i e_{i+1} = e_i e_{i+1}^2. \]

On the other hand, we have

**Lemma 1.4.** In the algebra \(\mathcal{U}_n\) relations \([\mathbb{1}] [\mathbb{2}][\mathbb{3}]\) and \([\mathbb{4}]\) imply

\[ E_i(0)^2 E_{i+1}(0) = E_i(0) E_{i+1}(0) E_i(0), \]

and

\[ E_{i+1}(0) E_i(0) E_{i+1}(0) = E_i(0) E_{i+1}(0)^2. \]

**Proof.** Indeed, using relation \([\mathbb{1}]\) one gets:

\[ E_i(p) E_i(p + 1) = -E_i(p) E_i(p + 1), \]

therefore,

\[ E_i(p) E_i(p + 1) = 0. \]

Hence, using relation \([\mathbb{2}]\) one gets

\[ E_i(0) E_{i+1}(0) E_i(0) = E_i(0)^2 E_{i+1}(0) - E_i(0) E_i(1) E_{i+1}(-1) = E_i(0)^2 E_{i+1}(0), \]

and

\[ E_{i+1}(0) E_i(0) E_{i+1}(0) = E_i(0) E_{i+1}(0)^2 - E_i(1) E_{i+1}(-1) E_{i+1}(0) = E_i(0) E_{i+1}(0)^2. \]

**Definition 1.5.** Let \(\mathcal{U}_n^+ \subset \mathcal{U}_n\) denote the subalgebra of \(\mathcal{U}_n\) generated by elements \(E_i(p), 0 < i < n, p \in \mathbb{Z}\).

**Corollary 1.6.** The map \(i^+ : \mathfrak{U}_0(\mathfrak{gl}_n)^+ \rightarrow \mathcal{U}_n^+\) defined by \(i^+(e_i) := E_i(0)\) is well-defined.

We expect this map to be injective, but do not attempt to prove this here.

**Note** that relations \([\mathbb{1}][\mathbb{2}][\mathbb{3}]\) imply that \(\mathcal{U}_n^+\) has a basis consisting of monomials

\[ E_{i_1}(p_{1,1}) \cdots E_{i_l}(p_{1,k_1}) E_{i_2}(p_{2,1}) \cdots E_{i_l}(p_{2,k_2}) \cdots \cdots \cdots E_{i_1}(p_{l,1}) \cdots E_{i_l}(p_{l,k_l}), \]

where \(i_1 < i_2 < \ldots < i_l\) and for each \(1 \leq j \leq l\) one has \(p_{j,1} \geq p_{j,2} \geq \ldots \geq p_{j,k_j}\). In particular, one can say that \(\mathcal{U}_n^+\) has the size of the polynomial ring in variables \(E_i(p)\).

At the same time, relations \([\mathbb{1}]\) suggest that \(\mathfrak{U}_0(\mathfrak{gl}_n)^+\) is bigger than the polynomial ring in variables \(e_i\). In fact, one can construct a basis in \(\mathfrak{U}_0(\mathfrak{gl}_n)^+\) as follows. For \(i < j\) denote
\[ e_{ij} := e_i e_{i+1} \ldots e_{j-1}. \]

Then there is a basis in \( \Omega_0(\mathfrak{gl}_n)^+ \) consisting of the monomials in \( e_{ij} \) of the form

\[ e_{i_1,j_1} \ldots e_{i_k,j_k}, \]

where \( i_1 \geq i_2 \geq \ldots \geq i_k \), and \( i_k = i_{k+1} \) implies \( j_k \leq j_{k+1} \). In particular, one can say that \( \Omega_0(\mathfrak{gl}_n)^+ \) has the size of the polynomial ring in variables \( e_{ij} \).

1.3. **Equivariant \( K \)-theory.** For a complex linear algebraic group \( G \) and a \( G \)-variety \( X \), let \( \text{Coh}^G(X) \) denote the category of \( G \)-equivariant coherent sheaves on \( X \) and let \( K^G(X) \) denote the complexified Grothendieck group of \( \text{Coh}^G(X) \). Given \( A \in \text{Coh}^G(X) \) let \( [A] \) denote its class in \( K^G(X) \). \( K^G(X) \) then has a natural \( R(G) \)-module structure, where \( R(G) = K^G(pt) \) is the complexified representation ring of \( G \). Here we summarize some basic properties of \( K^G(X) \) that are going to be important for our constructions later in the paper. We refer to [4] for a systematic treatment of the subject. This summary partially follows a similar summary in [17].

**Induction.** For an algebraic subgroup \( H \subset G \) and any \( H \)-variety \( X \) one gets

\[ K^H(X) \simeq K^G(G \times_H X). \]

In particular, for \( X = pt \) one gets

\[ K^G(G/H) \simeq K^H(pt) \simeq R(H). \]

**Reduction.** Any algebraic group \( G \) can be written as a semidirect product \( R \ltimes U \) where \( R \) is reductive and \( U \) is the unipotent radical of \( G \). Then for any \( G \)-variety \( X \) one gets

\[ K^G(X) \simeq K^R(X). \]

**Pushforward.** For any proper \( G \)-equivariant map \( f : X \to Y \) between two \( G \)-varieties \( X \) and \( Y \) there is a derived direct image morphism

\[ Rf_* : K^G(X) \to K^G(Y). \]

Going forward we will use a simplified notation \( f_* := Rf_* \). Note that \( f_* \) is a \( \mathbb{C} \)-linear map.

**Pullback.** For any flat \( G \)-equivariant map \( f : X \to Y \) between two \( G \)-varieties \( X \) and \( Y \) there is an inverse image morphism

\[ f^* : K^G(Y) \to K^G(X). \]

In particular, one gets the pullback map for open embeddings and for smooth fibrations.

**Restriction to closed submanifolds.** Let \( i : N \to M \) be a smooth closed embedding of a \( G \)-submanifold \( N \) into a \( G \)-manifold \( M \). Then there is a restriction morphism

\[ i^* : K^G(M) \to K^G(N). \]

**Tensor product.** If \( M \) is a smooth \( G \)-manifold then \( K^G(M) \) has the structure of a commutative associative \( R(G) \)-algebra with the multiplication given by

\[ f \otimes g := \Delta^*(f \boxtimes g), \]

where \( \boxtimes : K^G(M) \otimes K^G(M) \to K^G(M \times M) \) is the exterior tensor product and \( \Delta : M \to M \times M \) is the diagonal embedding. Note that this multiplication is only defined for a smooth \( G \)-manifold \( M \). Otherwise the restriction \( \Delta^* \) to the diagonal might not be well defined.

**Multiplicativity.** The flat pullback and the smooth closed restriction maps defined above are multiplicative with respect to the tensor product.
Tensor product with supports. One can refine the above definition as follows. Let $M$ be a smooth $G$-manifold and let $X, Y \subset M$ be two closed $G$-invariant subvarieties in $M$. Then one gets the tensor product map

$$\otimes : K^G(X) \otimes K^G(Y) \to K^G(X \cap Y).$$

Note that this map depends on the smooth ambient manifold $M$.

Projection formula. Let $M$ and $N$ be two smooth $G$-manifolds and let $f : N \to M$ be a proper flat map. Then for any $\alpha \in K^G(M)$ and $\beta \in K^G(N)$ one has

$$f_*(\beta \otimes \alpha) = f_*(\beta \otimes f^*(\alpha)).$$

Proper base change. Consider a Cartesian square

$$W = X \otimes_Z Y \xrightarrow{\tilde{g}} Y$$

$$\downarrow \tilde{f} \quad \downarrow f$$

$$X \xrightarrow{g} Z$$

where $f$ is proper and $g$ is flat. It then follows that $\tilde{f}$ is also proper, and $\tilde{g}$ is also flat. One gets

$$g^* \circ f_* = \tilde{f}_* \circ \tilde{g}^*.$$

Conormal bundle. Let $M$ be a smooth $G$-manifold and $N \subset M$ be a smooth closed $G$-submanifold. Let $i : N \to M$ be the embedding. Consider the conormal vector bundle $T^*_N M$. Let

$$\Lambda(T^*_N M) := \sum_{k=0}^{\infty} (-1)^k [\Lambda^k T^*_N M] \in K^G(N)$$

be the $K^G$ class of the exterior algebra of the conormal bundle $T^*_N M$. Then for any $\alpha \in K^G(N)$ one gets

$$i^*(i_* \alpha) = \alpha \otimes \Lambda(T^*_N M).$$

Localization and Lefschetz formula. Let $T \simeq (C^*)^n$ be a complex torus and $t \in T$. In this case the complexified representation ring $R(T)$ can be identified with the ring of regular functions on $T$ by considering the traces of the corresponding operators. Consider the multiplicative set $S_t \subset R(T)$ consisting of elements of $R(T)$ such that the trace does not vanish at $t$. Let $R(T)_t$ denote the localization of $R(T)$ with respect to $S_t$.

Let $M$ be a smooth $T$-manifold, $M_t \subset M$ be the submanifold of $t$-fixed points, and $i : M_t \to M$ be the (smooth, closed) embedding. Consider the localized equivariant $K$ groups

$$K^T(M)_t := R(T)_t \otimes_{R(T)} K^T(M),$$

and

$$K^T(M_t)_t := R(T)_t \otimes_{R(T)} K^T(M_t).$$

Then $\Lambda(T^*_M, M)$ is invertible in $K^T(M)_t$ and $i_*$ induces an isomorphism between $K^T(M)_t$ and $K^T(M_t)_t$.

Let $p : M \to pt$ and $q := p \circ i : M_t \to pt$ be projections to a point, and $\alpha \in K^G(M)$. Then one also gets the Lefschetz formula:

$$p_*(\alpha) = q_*(i^*(\alpha) \otimes \Lambda(T^*_M, M)^{-1}).$$
1.4. The space of partial flags and the convolution algebra. Let $V \cong \mathbb{C}^d$, and $G := \text{Gl}(V)$. We say that $\mu$ is a composition of $d$ of length $n$ if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_{\geq 0}^n$ and $\sum \mu_i = d$. We write $\mu \vdash d$ and $l(\mu) = n$ in that case. Note that the last part $\mu_n$, as well as any other part, is allowed to be zero. We will also use the partial sums $d_k := \sum_{j=1}^{k} \mu_j$.

**Definition 1.7.** The space of $\mu$-partial flags $\mathcal{F}_\mu$ is defined by the following

$$\mathcal{F}_\mu := \{U_1 \subset \ldots \subset U_n \subset V : \dim U_k = d_k\}.$$  

We will also use the notation $\mathcal{F}^d_n := \bigsqcup \mathcal{F}_\mu$.

Consider the space of pairs of partial flags $\mathcal{F}_n^d \times \mathcal{F}_n^d$. The group $G = \text{Gl}(V)$ acts on it diagonally. Let $\alpha \in K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d)$ be an equivariant K-theory class. Consider the diagram

**Definition 1.8.** The correspondence morphism $\phi_\alpha : K^G(\mathcal{F}_n^d) \to K^G(\mathcal{F}_n^d)$ is defined as follows:

$$\phi_\alpha(f) := \pi_2^*(\pi_1^* f \times \alpha).$$

Let $\beta \in K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d)$ be another equivariant $K$-theory class. Consider the composition $\phi_\beta \circ \phi_\alpha : K^G(\mathcal{F}_n^d) \to K^G(\mathcal{F}_n^d)$. We get the following diagram:

Here $\pi_{12} : \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_n^d \times \mathcal{F}_n^d$ and $\pi_{23} : \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_n^d \times \mathcal{F}_n^d$ are the projections onto the first two factors and the second two factors correspondingly. Note that the triple flag variety is the fibered product of the two copies of the double flag variety over the projections $\pi_2$ and $\pi_1$. Therefore, using the base change and the projection formula we get:

$$\phi_\beta \circ \phi_\alpha(f) = \pi_{2*} (\beta \times [\pi_{12}^* (\alpha \times \pi_{12}^* f)]) = \pi_{2*} (\beta \times [\pi_{23}^* (\alpha \times \pi_{12}^* f)]) = p_{3*} (\pi_{12}^* (\alpha) \times \pi_{23}^* (\beta) \times p_{1*} f),$$

where $p_1 = \pi_1 \circ \pi_{12} : \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_n^d$ and $p_3 = \pi_2 \circ \pi_{23} : \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_n^d$ are the projections onto the first and the third factors. Consider the diagram
\[ F_n^d \times F_n^d \times F_n^d \]

where \( \pi_{13} \) is the projection on the first and the third factors. The triangles in the diagram are commutative. Therefore, using the projection formula we get

\[
\phi_\beta \circ \phi_\alpha (f) = p_{34} (\pi_{12}^* \alpha \times \pi_{23}^* \beta \times p_1 f) = \pi_{23} \pi_{12} (\pi_{12}^* \alpha \times \pi_{23}^* \beta \times \pi_{12} \pi_{23} \pi_{13} \pi_4 \alpha \times \pi_{23}^* \beta \times p_1 f) = \pi_{23} \pi_{12} \pi_{13} \pi_{23} (\pi_{12}^* \alpha \times \pi_{23}^* \beta) \alpha \times \pi_{23}^* \beta) \]

This motivates the definition of the convolution product on \( K^G(F_n^d \times F_n^d) \):

**Definition 1.9.** The **convolution product** of the classes \( \alpha, \beta \in K^G(F_n^d \times F_n^d) \) is defined by

\[
\alpha \ast \beta := \pi_{13} (\pi_{12}^* \alpha \times \pi_{23}^* \beta) \]

We conclude that \( S_0 \text{aff} (n, d) := K^G(F_n^d \times F_n^d) \) is an algebra under the convolution multiplication, and it acts on \( K^G(F_n^d) \) via the correspondence morphisms. The main result of this paper is the following Theorem:

**Theorem 1.10.** For every positive integer \( d \) there exists a surjective morphism

\[
\phi_d : \mathcal{U}_n \to S_0 \text{aff} (n, d) .
\]

**Remark 1.11.** Note that for any two orbits \( O_1, O_2 \subset F_n^d \times F_n^d \) such that

\[
\pi_{12}^{-1} (O_1) \cap \pi_{23}^{-1} (O_2) \neq \emptyset ,
\]

the intersection above is transversal. Indeed, since the intersection is non-empty, there exist \( \mu_1, \mu_2, \) and \( \mu_3 \) such that \( O_1 \subset F_{\mu_1} \times F_{\mu_2} \) and \( O_2 \subset F_{\mu_2} \times F_{\mu_3} \). Since \( O_1, F_{\mu_1}, \) and \( F_{\mu_2} \) are all homogeneous spaces, the projection of \( O_1 \) onto \( F_{\mu_1} \) along \( F_{\mu_2} \) is surjective. It follows then that at any point of intersection of the preimages

\[
x \in \pi_{12}^{-1} (O_1) \cap \pi_{23}^{-1} (O_2) = (O_1 \times F_{\mu_3}) \cap (F_{\mu_1} \times O_2)
\]

the sum of tangent spaces to \( O_1 \times F_{\mu_3} \) and \( F_{\mu_1} \times O_2 \) is the whole tangent space to \( F_{\mu_1} \times F_{\mu_2} \times F_{\mu_3} \), so the intersection is transversal.

This remark will be important for computations. In particular, given \( K^G(F_n^d \times F_n^d) \) classes \( \alpha \) and \( \beta \) supported on closed orbits \( O_1 \) and \( O_2 \) respectively, one can compute the convolution product using the restrictions of the projections

\[
\tilde{\pi}_{ij} := \pi_{ij} | (O_1 \times F_n^d) \cap (F_n^d \times O_2) : \]

\[
\alpha \ast \beta = \pi_{13} (\pi_{12}^* \alpha \times \pi_{23}^* \beta) = \tilde{\pi}_{13} (\tilde{\pi}_{12}^* \alpha \times \tilde{\pi}_{23}^* \beta) .
\]

In order to describe the morphisms \( \phi_d \) we need first to explicitly describe the orbit structure of the space of pairs of flags \( F_n^d \times F_n^d \) and the convolution product on \( S_0 \text{aff} (n, d) \).
2. ORBITS STRUCTURE OF THE SPACE OF PAIRS AND TRIPLES OF FLAGS.

2.1. Orbits and Matrices. The orbit structure of the space of pairs of flags was studied in \cite{[17]} and many other sources. In this section we largely follow these sources, although some of the results did not previously appear in literature, up to the authors’ knowledge. For the sake of completeness of the exposition we provide most of the proofs.

Note that the orbits of the $G$ action on the space of partial flags $\mathcal{F}_n^d$ coincide with the connected components $\mathcal{F}_\mu$, i.e. all orbits are open and closed, and enumerated by compositions $\mu$ of $d$ of length $n$. The orbit structure of the diagonal $G$ action on the space of pairs of partial flags $\mathcal{F}_n^d \times \mathcal{F}_n^d$ is more subtle. The key observation here is the following lemma from elementary linear algebra.

**Lemma 2.1.** Let $U := \{0\} = U_0 \subset U_1 \subset \ldots \subset U_n = V \in \mathcal{F}_n^d$ and $W := \{0\} = W_0 \subset W_1 \subset \ldots \subset W_m = V \in \mathcal{F}_n^d$ be two partial flags. Then there exists a basis $\{v_1, \ldots, v_d\}$ such that every element of both flags is a span of basic vectors.

**Proof:** One constructs such a basis recursively, starting from a basis of $U_1 \cap W_1$ and then gradually extending it to other intersections $U_i \cap W_j$.

Suppose that $A$ is a basis of $U_{i-1} \cap W_{j-1}$, $B$ is such that $A \cup B$ is a basis of $U_i \cap W_{j-1}$, and $C$ is such that $A \cup C$ is a basis of $U_{i-1} \cap W_j$, then $A \cup B \cup C$ is linearly independent. Indeed, otherwise one would have

$$A + B + C = 0,$$

where $A$, $B$, and $C$ are linear combinations of elements of $A$, $B$, and $C$ respectively, and at least one of the vectors $A + B$ and $A + C$ is not zero. Without loss of generality, assume that $A + B \neq 0$. But then

$$0 \neq -C = A + B \in (U_i \cap W_{j-1}) \cap (U_{i-1} \cap W_j) = U_{i-1} \cap W_{j-1},$$

which implies that $B = 0$, because $A$ is a basis of $U_{i-1} \cap W_{j-1}$ and $A \cup B$ is linearly independent, which contradicts the linear independence of $A \cup C$. One can then proceed to extend $A \cup B \cup C$ to a basis of $U_i \cap W_j$.

Using the above procedure, one can keep extending the bases of intersections of the elements of the flags $U$ and $W$ until a desired basis of the total space is constructed. \hfill $\square$

**Definition 2.2 (\cite{[17]})**. Given a pair of flags $(U, W) \in \mathcal{F}_n^d \times \mathcal{F}_n^d$ define the corresponding matrix $M := \{m_{ij}\}$ in such a way that it satisfies

$$\dim U_a \cap W_b = \sum_{(i,j) \leq (a,b)} m_{ij},$$

where by $(i, j) \leq (a, b)$ we will always mean that $i \leq a$ and $j \leq b$.

**Corollary 2.3 (\cite{[17]})**. Two pairs of flags belong to the same orbit of the diagonal action of $\text{Gl}(V)$ on $\mathcal{F}_n^d \times \mathcal{F}_n^d$ if and only if the corresponding matrices are the same.

Let $\mathcal{M}(n, d)$ be the set of $n \times n$ matrices with non-negative integer entries and the total sum of entries equal to $d$. For a matrix $M \in \mathcal{M}(n, d)$ let $\mathcal{O}_M \subset \mathcal{F}_n^d \times \mathcal{F}_n^d$ denote the corresponding orbit.

**Example 2.4.** Let $M$ be a diagonal matrix and $(U, W) \in \mathcal{O}_M$. Then for every $i, j$ we get

$$U_i \cap W_j = U_{\min(i, j)} = W_{\min(i, j)},$$

i.e. $U = W$. 

Given a matrix \( M = \{m_{ij}\} \in \mathcal{M}(n, d) \) and a pair of flags \((U, W) \in \mathcal{O}_M\), one gets \( U \in \mathcal{F}_M \) and \( W \in \mathcal{F}_M \), where

\[
R^M := (R_1^M, \ldots, R_n^M), \quad R_k^M := \sum_{i=1}^n m_{ki}, \quad 1 \leq k \leq n,
\]
is the row-sum composition, and

\[
C^M := (C_1^M, \ldots, C_n^M), \quad C_k^M := \sum_{i=1}^n m_{ik}, \quad 1 \leq k \leq n,
\]
is the column-sum composition. Hence one gets \( \mathcal{M}(n, d) = \bigsqcup_{\mu, \nu} \mathcal{M}_{\mu, \nu} \), where

\[
\mathcal{M}_{\mu, \nu} := \{ M \in \mathcal{M}(n, d) \mid R^M = \mu, \quad C^M = \nu \},
\]
and \( \mathcal{O}_M \) and \( \mathcal{O}_N \) belong to the same connected component of \( \mathcal{F}_n^d \times \mathcal{F}_n^d \) if and only if \( R^M = R^N \) and \( C^M = C^N \).

### 2.2. Bruhat order.

The adjacency of the orbits defines a partial order on \( \mathcal{M}(n, d) \). This order is usually called the **Bruhat order**, and it was studied in various sources, see [1, 13, 14, 15, 17] among others. Some of the material in this Section follow these sources, although an explicit description of the cover relations and the characterization of the minimal and maximal elements in terms of the matrices, as well as an explicit characterization of the open orbits appear to be new.

**Definition 2.5.** We say that \( N \prec M \) if \( \mathcal{O}_N \subset \partial \mathcal{O}_M \). Let also \( N \preceq M \) denote the cover relation, i.e. \( N \prec N \) if and only if \( N \prec M \), and there is no matrix \( K \) such that \( N \prec K \prec M \).

Let \( M \in \mathcal{M}(n, d) \) be a matrix and \((U, W) \in \mathcal{O}_M\) be a pair of flags in the corresponding orbit. Let also \( \mathcal{B} = \bigsqcup_{1 \leq i,j \leq n} \mathcal{B}_{ij} \) be a basis of \( V \) such that for any \( 1 \leq k, l \leq n \) one has

\[
U_k \cap W_l = \text{span}(\bigcup_{i=1}^k \bigcup_{j=1}^l \mathcal{B}_{ij}).
\]

Then the dimension of the stabilizer of \((U, W)\) is given by

\[
\dim \text{Stab}(U, W) = \#\{(u, v) \mid u \in \mathcal{B}_{ij}, \quad v \in \mathcal{B}_{op}, \quad i \leq o, \quad j \leq p\} = \sum_{1 \leq i,j \leq n} sm_{ij} m_{ij},
\]
where for all \( 1 \leq k, l \leq n \)

\[
sm_{kl} := \sum_{i=1}^k \sum_{j=1}^l m_{ij}.
\]

Therefore, one gets

\[
\dim(\mathcal{O}_M) = d^2 - \dim \text{Stab}(\mathcal{O}_M) = \#\{(u, v) \mid u \in \mathcal{B}_{ij}, \quad v \in \mathcal{B}_{op}, \quad i > o \text{ or } j > p\} = \sum_{1 \leq i,j \leq n} (d - sm_{ij}) m_{ij}.
\]

**Definition 2.6.** We will use the notation \( E_{ij} \) for the matrix with all entries equal to zero, except for the entry on \( i \)th row and \( j \)th column, which is equal to 1.

**Theorem 2.7.** Let \( N = \{n_{ij}\} \) and \( M = \{m_{ij}\} \) be two non-negative integer matrices. Then \( N \preceq M \) if and only if the following two conditions are satisfied:

1. One has \( R^M = R^N \) and \( C^M = C^N \).
(2) For every \( k, l \) one has
\[ \sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}. \]
Moreover, one can also characterize the cover relations: one has \( N \leq M \) if and only if there exist four integers \( 1 \leq a_1 < a_2 \leq n \) and \( 1 \leq b_1 < b_2 \leq n \) such that the following two conditions are satisfied:

(i) \( M - N = E_{a_1 b_2} + E_{a_2 b_1} - E_{a_1 b_1} - E_{a_2 b_2} \).

(ii) \( m_{ij} = n_{ij} = 0 \) for \( a_1 \leq i \leq a_2 \) and \( b_1 \leq j \leq b_2 \) except for
\[ (i, j) \in \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}. \]

See Figure 1 for an illustration of a cover relation.

**Proof.** Suppose that \( N \leq M \). Then \( R^M = R^N \) and \( C^M = C^N \), because \( O_M \) and \( O_N \) belong to the same connected component. Also, for any \( 1 \leq k, l \leq n \) and any \( (U, W) \in O_M \) one has
\[ \dim(U_k \cap W_l) = \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}. \]
As a family of pairs of flags approaches the boundary of the orbit \( O_M \), the dimension of the intersection can only increase. Therefore, one gets
\[ \sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}. \]
The fact that conditions (1) and (2) imply that \( N \leq M \) follows from work of Bongartz on representations of quivers (see [2, 3]). See also [13] and [14] for a more elementary approach.

It is easy to check that condition (i) implies conditions (1) and (2). Characterizations of cover relations in terms of quiver representations, equivalent to conditions (i) and (ii), can be found in [14].

We include a complete proof of the Theorem using only combinatorics and linear algebra in the Appendix. \( \square \)

**Corollary 2.8.** Suppose that the orbit \( O_M \subset F_n^d \times F_n^d \) is closed. Then there exist sequences \( 1 \leq a_1 \leq \ldots \leq a_k \leq n \) and \( 1 \leq b_1 \leq \ldots \leq b_k \leq n \) such that for all \( 1 \leq i < n \) \( (a_i, b_i) \neq (a_{i+1}, b_{i+1}) \), and \( m_{ij} \neq 0 \) if and only if \( (i, j) = (a_i, b_i) \) for some \( 1 \leq l \leq k \).

Moreover, the orbit \( O_M \) is then the image of the injective map \( \phi_M : F_\mu \to F_n^d \times F_n^d \), where \( \mu = (m_{a_1 b_1}, \ldots, m_{a_k b_k}) \) and the map \( \phi \) is given by \( \phi(V) = (U, W) \) where for all \( i \)
\[ U_i := V_{\max\{j | a_j \leq i\}} \]
and
\[ W_i := V_{\max\{j | b_j \leq i\}}. \]

**Corollary 2.9.** Suppose that the orbit \( O_M \subset F_n^d \times F_n^d \) is open. Then there exist sequences \( 1 \leq a_1 \leq \ldots \leq a_k \leq n \) and \( n \geq b_1 \geq \ldots \geq b_k \geq 1 \) such that for all \( 1 \leq i < n \) \( (a_i, b_i) \neq (a_{i+1}, b_{i+1}) \), and \( m_{ij} \neq 0 \) if and only if \( (i, j) = (a_i, b_i) \) for some \( 1 \leq l \leq k \).
Remark 2.10. Note that for \( \mathcal{M}_{1^d,1^d} \) the partial order considered above coincides with the Bruhat order on the permutation matrices. Moreover, in the general case the order can be obtained from the Bruhat order in the following manner. Let \( \mu = (\mu_1, \ldots, \mu_n) \) and \( \nu = (\nu_1, \ldots, \nu_m) \) be two compositions of \( d \). Consider the map

\[ p : \mathcal{M}_{1^d,1^d} \to \mathcal{M}_{\mu,\nu} \]

given by

\[ p(M)_{kl} = \sum_{\mu_{k-1} \leq i \leq \mu_k, \nu_{l-1} \leq j \leq \nu_l} m_{ij}. \]

The partial order on \( \mathcal{M}_{\mu,\nu} \) is uniquely defined by requiring that the map \( p \) is weakly monotone.

Note that \( p \) is the combinatorial shadow of the forgetful map

\[ \pi : \mathcal{F}_1 \times \mathcal{F}_1 \to \mathcal{F}_\mu \times \mathcal{F}_\nu, \]

i.e. \( \pi(O_M) = O_{p(M)} \) for all \( M \in \mathcal{M}_{1^d,1^d} \).

2.3. The convolution product for supports. In order to study the convolution algebra

\[ S_{0}^{\text{aff}}(n, d) = K^{G}(\mathcal{F}_n \times \mathcal{F}_n) \]

we need to first study the behavior of supports of the K-theory classes under the convolution product. Let \( O_M, O_N \subset \mathcal{F}_n \times \mathcal{F}_n \) be two orbits and \( M, N \) be the corresponding matrices. Let also \( \alpha, \beta \in K^{G}(\mathcal{F}_n) \) be two classes such that \( \text{supp}(\alpha) = \overline{O_M} \) and \( \text{supp}(\beta) = \overline{O_N} \). Then the convolution product \( \alpha \ast \beta \) is supported on the closure of \( \pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N)) \).

Note that \( \pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N)) \) is non-empty if and only if the second flag in \( O_M \) is of the same type as the first flag in \( O_N \), which is equivalent to \( C^M = R^N \).

Lemma 2.11 ([1] [6]). Suppose that \( C^M = R^N \). Then the subvariety \( \pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N)) \subset \mathcal{F}_d \times \mathcal{F}_d \) is irreducible.

The Lemma 2.11 motivates the following definition:

Definition 2.12 ([1] [17] [6]). If \( C^M = R^N \), then we define \( M \circ N \) to be the matrix such that \( O_{M \circ N} \subset \pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N)) \) is the unique orbit that is open in \( \pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N)) \).

If \( C^M \neq R^N \) then one sets \( M \circ N := 0 \).

The product \( M \circ N \) was studied in [1] and [17]. In [6] Jensen and Su studied this product using the quiver representation techniques. Here we establish basic properties of this operation. We mostly follow [1] and [17], although we provide more details and some of the formulations are new.

In general, computing the product \( M \circ N \) is rather complicated. However, if one of the matrices is of a specially simple form, then one can describe \( M \circ N \) explicitly.

Definition 2.13. An orbit \( O \subset \mathcal{F}_d \times \mathcal{F}_d \times \mathcal{F}_d \) of the diagonal \( G(V) \) action is called combinatorial if for every triple of flags \( (U, V, W) \in O \) there exists a basis of \( V \) such that all elements of flags \( U, V, \) and \( W \) are coordinate subspaces.

Remark 2.14. Not all orbits in the triple flag variety are combinatorial: for example, consider three distinct lines in a two dimensional space. Moreover, in most of the cases the triple flag variety contains infinitely many orbits of the diagonal \( G_n \) action. See [13] for a classification of multiple flag varieties with finitely many orbits.

Lemma 2.15. Let \( M = \{m_{ij}\} \in \mathcal{M}(n, d) \) be a matrix such that \( O_M \subset \mathcal{F}_n \times \mathcal{F}_n \) is a closed orbit. Then for any \( (l, m) \in \{(1, 2), (1, 3), (2, 3)\} \) the preimage \( \pi_{lm}^{-1}(O_M) \subset \mathcal{F}_n \times \mathcal{F}_n \times \mathcal{F}_n \) is an union of combinatorial orbits.
such that $R_k$ correspond to the non-zero entries of $M$, and the map

$$\phi : F_\mu \rightarrow O_M,$$

where $\mu = (m_{a_1 b_1}, \ldots, m_{a_k b_k})$, given by $\phi(V) = (U, W)$ where

$$U_i := V_{\max\{j | a_j \leq i\}}$$

and

$$W_i := V_{\max\{j | b_j \leq i\}}$$

is an isomorphism.

Let $(U, W) \in O_M$. Let $V = \phi^{-1}(U, W) \in F_\mu$. Consider any triple of flags $(U, W, T) \in \pi_{12}^{-1}(O_M)$. Let $B$ be a basis of $V$, such that all elements of flags $V$ and $T$ are spans of basic vectors. But then the elements of the flags $U$ and $W$ are also spans of basic vectors, as they coincide with elements of the flag $V$. The preimages under $\pi_{13}$ and $\pi_{23}$ are treated in the same way.

$\square$

Similar to the case of the double flag variety, combinatorial orbits in the triple flag variety can be enumerated by three dimensional arrays of non-negative integers. However, the proof of Lemma 2.15 suggests that the orbits in $\pi_{13}^{-1}(O_M)$ for a closed orbit $O_M$ can be enumerated in an easier way. Let $\phi : F_\mu \rightarrow O_M \subset F_n^d \times F_n^d$ be the equivariant isomorphism from Corollary 2.8.

Let us extend it to the equivariant isomorphism

$$\Phi := (\phi, id) : F_\mu \times F_n^d \rightarrow O_M \times F_n^d \simeq \pi_{13}^{-1}(O_M) \subset F_n^d \times F_n^d \times F_n^d.$$

The orbits in $F_\mu \times F_n^d$ are enumerated by $k \times n$ non-negative integer matrices $N = \{n_{ij}\}$, such that $R^N = \mu$. Furthermore, the following result follows immediately from Corollary 2.8.

**Lemma 2.16.** Let $L = \{l_{kj}\} \in M(n, d)$ be the matrix corresponding to the orbit $\pi_{13}(\Phi(O_N)) \subset F_n^d \times F_n^d$. Then one gets

$$l_{kl} = \sum_{\{i | a_i = k\}} n_{il}.$$

In other words, the $k$th row of the matrix $L$ is obtained from the matrix $N$ by adding up rows corresponding to the non-zero entries of the $k$th row of the matrix $M$.

Similarly, if $K = \{k_{ij}\} \in M(n, d)$ is the matrix corresponding to the orbit $\pi_{23}(\Phi(O_N)) \subset F_n^d \times F_n^d$. Then one gets

$$k_{kl} = \sum_{\{i | b_i = k\}} n_{il},$$

i.e. the $k$th row of the matrix $K$ is obtained from the matrix $N$ by adding up rows corresponding to the non-zero entries of the $k$th column of the matrix $M$.

**Example 2.17.** Consider the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix} \in M(3, 9).$$

Clearly, $M$ satisfies the conditions of Corollary 2.8 with $\mu = (m_{11}, m_{12}, m_{22}, m_{32}, m_{33}) = (2, 1, 2, 1, 3)$. We have the isomorphism $\Phi : F_\mu \times F_3^9 \rightarrow \pi_{12}^{-1}(O_M)$ with orbits in $F_\mu \times F_3^9$. 

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enumerated by $5 \times 3$ matrices with row-sums given by $\mu$. For example, the matrix
\[
N = \begin{pmatrix}
0 & 1 & 1 \\
0 & 1 & 0 \\
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 2
\end{pmatrix}
\]
corresponds to one of the orbits in $\mathcal{F}_\mu \times \mathcal{F}_n$. Furthermore, the orbit $\pi_{13}(\Phi(\mathcal{O}_N)) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d$ corresponds to the matrix
\[
L = \begin{pmatrix}
0 & 2 & 1 \\
1 & 1 & 0 \\
1 & 1 & 2
\end{pmatrix},
\]
obtained from $N$ by summing up the first two rows, and also the last two rows, since $M$ has two non-zero entries in the first row, and two non-zero entries in the last row. Similarly, the orbit $\pi_{23}(\Phi(\mathcal{O}_N)) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d$ corresponds to the matrix
\[
K = \begin{pmatrix}
0 & 1 & 1 \\
2 & 2 & 0 \\
0 & 1 & 2
\end{pmatrix},
\]
obtained from $N$ by summing up rows two, three, and four, since $M$ has three non-zero entries in the second column.

**Definition 2.18.** Let $\mu = (\mu_1, \ldots, \mu_n)$ be a composition of $d-1$, i.e. $\mu \in \mathbb{Z}^n_{\geq 0}$ and $\sum \mu_i = d-1$. For $1 \leq k < n$ we will use the following notations
\[
E(\mu, k) := \text{Diag}(\mu) + E_{k,k+1},
\]
and
\[
F(\mu, k) := \text{Diag}(\mu) + E_{k+1,k},
\]
where $\text{Diag}(\mu)$ is the diagonal $n \times n$ matrix with numbers $\mu_1, \ldots, \mu_n$ along the diagonal. We are going to call such matrices *almost diagonal*.

**Definition 2.19.** We will use the notation $e_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ for the $k$th basic vector.

**Theorem 2.20 (\cite{17}).** Let $E(\mu, k)$ be an almost diagonal matrix, and let $M = \{m_{ij}\} \in \mathcal{M}_{d,n}$. Then $E(\mu, k) \circ M = 0$ unless $R^M = C_{E(\mu, k)} = \mu + e_{k+1}$, in which case one has
\[
E(\mu, k) \circ M = M - E_{k+1,m} + E_{k,m},
\]
where $m = \max\{j|m_{k+1,j} > 0\}$.

Similarly, one gets $F(\mu, k) \circ M = 0$ unless $R^M = C_{F(\mu, k)} = \mu + e_k$, in which case one has
\[
F(\mu, k) \circ M = M - E_{k,m} + E_{k+1,m},
\]
where $m = \min\{j|m_{k,j} > 0\}$.

**Proof.** The theorem follows immediately from Lemma 2.16 and Theorem 2.7. Indeed, let $\nu = (\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_n)$. Then the orbits in the preimage $\pi_{12}^{-1}(\mathcal{O}_{E(\mu, k)}) \simeq \mathcal{F}_\nu \times \mathcal{F}_n^d$ are enumerated by $(n+1) \times n$ non-negative integer matrices $N$ such that $R^N = \nu$. Intersecting with the preimage $\pi_{23}^{-1}(\mathcal{O}_M)$ we get the extra condition that if one replaces $(k+1)$th and $(k+2)$th rows of $N$ by their sum, one obtains $M$.

The orbits in $\pi_{13}(\pi_{12}^{-1}(\mathcal{O}_{E(\mu, k)}) \cap \pi_{23}^{-1}(\mathcal{O}_M))$ are then enumerated by matrices $L$ obtained from $N$ by replacing $k$th and $(k+1)$th rows by their sum. In other words, these orbits are enumerated by
\[
\mathcal{L} := \{M - E_{k+1,j} + E_{k,j}|m_{k+1,j} > 0\}.
\]
By Theorem \[2.7\] the set \( \mathcal{L} \) is totally ordered with the maximal element given by \( M - E_{k+1,m} + E_{k,m} \), where \( m = \max\{j \mid m_{k+1,j} > 0\} \). Therefore,
\[
E(\mu, k) \circ M = M - E_{k+1,m} + E_{k,m}.
\]
The second part of the Theorem is verified in a similar manner.

Corollary 2.21 (\([1, 17]\)). Every non-diagonal matrix \( M = \{m_{ij}\} \in \mathcal{M}(n, d) \) can be factored into a \( \circ \)-product of several almost diagonal matrices.

Proof. Consider the diagonal norm function \( DN : \mathcal{M}(n, d) \to \mathbb{Z}_{\geq 0} \) given by
\[
DN(M) := \sum_{1 \leq i, j \leq n} |i - j|m_{ij}.
\]
The proof proceeds by induction with respect to \( DN(M) \). One has \( DN(M) = 0 \) if and only if \( M \) is diagonal, and \( DN(M) = 1 \) if and only if \( M \) is almost diagonal, in which cases the Corollary is trivial. Suppose that \( DN(M) > 1 \) and the Corollary is proven for all \( N \in \mathcal{M}(n, d) \) such that \( DN(N) < DN(M) \). Then there exist \( i \neq j \) such that \( m_{ij} > 0 \).

Suppose that \( j > i \). Then there exist \( b = \max\{j \mid \exists i < j, m_{ij} > 0\} \) and \( a < b \) such that \( m_{ab} > 0 \). Applying Theorem 2.20 one gets
\[
M = E(R^M - e_a, a) \circ (M + E_{a+1,b} - E_{a,b})
\]
and \( DN(M + E_{a-1,b} - E_{a,b}) = DN(M) - 1 \).

Similarly, if \( j < i \), then there exists \( b = \min\{j \mid \exists i > j, m_{ij} > 0\} \) and \( a > b \) such that \( m_{ab} > 0 \). Applying Theorem 2.20 one gets
\[
M = F(R^M - e_a, a - 1) \circ (M + E_{a-1,b} - E_{a,b})
\]
and \( DN(M + E_{a,b} - E_{a-1,b}) = DN(M) - 1 \). □

Let \( M = \{m_{ij}\}, N = \{n_{ij}\} \in \mathcal{M}(n, d) \) and \( M = E(\mu, k) \circ N \). We know then that
\[
N = M + E_{k+1,b} - E_{k,b},
\]
where \( b \) is such that \( m_{kb} > 0 \) and \( m_{k+1,j} = 0 \) for all \( j > b \). However, given a matrix \( M \), the set
\[
\{L \in \mathcal{M}(n, d) \mid M = E(\mu, k) \circ L\}
\]
might consist of more than one matrix.

Example 2.22. One has
\[
\begin{pmatrix}
1 & 1 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
3 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \circ \begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
3 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \circ \begin{pmatrix}
1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The following Lemma follows immediately from Theorems 2.20 and 2.7.

Lemma 2.23. Let \( M = \{m_{ij}\} \in \mathcal{M}(n, d) \) be a matrix and \( E(\mu, k) \) be such that the set
\[
\mathcal{S} := \{L \in \mathcal{M}(n, d) \mid M = E(\mu, k) \circ L\}
\]
is not empty. Then \( \mathcal{S} \) is totally ordered, and the minimum of \( \mathcal{S} \) is given by
\[
\min \mathcal{S} = M + E_{k+1,b} - E_{k,b},
\]
where \( b \) is the maximal integer satisfying \( m_{kb} > 0 \) and \( m_{k+1,j} = 0 \) for all \( j > b \).

Similarly, if \( F(\mu, k) \) is such that the set
\[
\mathcal{T} := \{ L \in \mathcal{M}(n, d) | M = F(\mu, k) \circ L \}
\]
is not empty, then \( \mathcal{T} \) is totally ordered, and the minimum of \( \mathcal{T} \) is given by
\[
\min \mathcal{T} = M - E_{k+1,b} + E_{k,b},
\]
where \( b \) is the minimal integer satisfying \( m_{k+1,b} > 0 \) and \( m_{k,j} = 0 \) for all \( j < b \).

**Corollary 2.24.** For any matrix \( M \in \mathcal{M}(n, d) \) such that \( DN(M) > 0 \) (i.e. \( M \) is not diagonal) there exists a factorization \( M = D \circ N \) such that \( D \) is almost diagonal, \( DN(N) < DN(M) \), and for any \( L \prec N \) one has \( D \circ L \prec M \).

The following Lemma will be important in the next section:

**Lemma 2.25.** Let \( O_1, O_2, O_3 \subset F_n^{d_1} \times F_n^{d_2} \) be three orbits. Suppose that at least one of them is \( O_1 \). Then the triple intersection \( \pi_{12}^{-1}(O_1) \cap \pi_{23}^{-1}(O_2) \cap \pi_{13}^{-1}(O_3) \subset F_n^{d_1} \times F_n^{d_2} \times F_n^{d_3} \) consists of a single orbit.

**Remark 2.26.** Note that the condition that one of the orbits is closed is essential: see Examples 2.29 and 2.30.

**Proof.** Let \( O_1 \) be the closed orbit, then by Corollary 2.28 we have \( O_1 = O_D \) where \( D = \{ d_{ij} \} \in \mathcal{M}(n, d) \) is such that there exist \( a_1 \leq \ldots \leq a_k \) and \( b_1 \leq \ldots \leq b_k \) and \( \mu := (\mu_1, \ldots, \mu_k) \) such that \( d_{ai,bj} = \mu_l > 0 \) for all \( l \) and all other entries of \( D \) are zero. Here for every \( 1 \leq l < k \) one has \( (a_l, b_l) \neq (a_{l+1}, b_{l+1}) \), i.e. either \( a_l < a_{l+1} \) or \( b_l < b_{l+1} \). Let also \( O_2 = O_M \) and \( O_3 = O_N \) where \( M, N \in \mathcal{M}(n, d) \).

Using Lemma 2.16 we conclude that all orbits in the triple intersection are combinatorial. Moreover, for every such orbit \( O \) there exists a \( k \times n \) matrix \( L \) such that \( O \) is isomorphic to \( O_L \subset F_k^{d_1} \times F_n^{d_2} \) under the embedding
\[
\phi_D \times \text{id} : F_k^{d_1} \times F_n^{d_2} \to F_n^{d_1} \times F_n^{d_2} \times F_n^{d_3},
\]
where \( \phi_D : F_k^{d_1} \to F_n^{d_1} \times F_n^{d_2} \) is given by
\[
\phi_D(V) := (U, W),
\]
where
\[
U_l := V_{\max \{ m | a_m \leq l \}},
\]
and
\[
W_l := V_{\max \{ m | b_m \leq l \}}.
\]

In particular, matrix \( L \) uniquely determines the orbit in the triple intersection, and matrices \( M \) and \( N \) can be obtained from \( L \) as follows. For every \( l \) one gets
\[
R^M_l = \sum_{a_i = l} R^L_i,
\]
and
\[
R^N_l = \sum_{b_i = l} R^L_i.
\]

Let us show that these conditions determine matrix \( L \) uniquely. Suppose that there are two matrices \( L \) and \( L' \) satisfying these conditions. Let \( i \) be the smallest number, such that \( i \)th rows of \( L \) and \( L' \) are not the same \( R^L_i \neq R^{L'}_i \). We have \( (a_i, b_i) \neq (a_{i+1}, b_{i+1}) \) by construction. Without loss of generality, let us assume that \( a_i < a_{i+1} \). Let \( m = \min \{ j | a_j = a_i \} \). Then one gets
\[
R^M_{a_i} = R^L_m + R^L_{m+1} + \ldots + R^L_i,
\]

\[
= R^{L'}_m + R^{L'}_{m+1} + \ldots + R^{L'}_i.
\]
array of non-negative integers $N$ is tricky because the image of the intersection coincides, while the second is different from them. Note that orbit of lines (consisting of triples of distinct lines, and $O$ then the orbit $O$ one gets

Example 2.28. Consider the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \in \mathcal{M}(3, 7).$$

One gets

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \circ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} \circ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

In other word, one gets

$$M = E((2, 1, 3), 1) \circ E((2, 1, 3), 2) \circ E((1, 1, 4), 1) \circ F((1, 2, 3), 2) \circ F((1, 2, 3), 1).$$

Note that on each step the factorization satisfies the condition of Corollary 2.24.

Example 2.28. Consider the matrix

$$M = \{m_{ij}\} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}(2, 2).$$

Then the orbit $O_M \subset \mathcal{F}_2^1 \times \mathcal{F}_2^2$ consists of pairs of distinct lines in a two dimensional space $V$. The intersection of the preimages $\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_M) \subset \mathcal{F}_2^1 \times \mathcal{F}_2^2 \times \mathcal{F}_2^3$ consists of triples of lines $(l_1, l_2, l_3)$, such that $l_1 \neq l_2$ and $l_2 \neq l_3$. There are two orbits in this intersection: $O_1$ consisting of triples of distinct lines, and $O_2$ consisting of triples of lines where the first and the third coincide, while the second is different from them. Note that orbit $O_1$ is not combinatorial. The image of the intersection $\pi_{13}(\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_M)) \subset \mathcal{F}_2^2 \times \mathcal{F}_2^3$ also consists of two orbits: $O_M$ and $O_L$, where

$$L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Clearly, $L \preccurlyeq M$, therefore $M \circ M = M$. Obtaining this result in a direct combinatorial way is tricky because $O_1$ is not a combinatorial orbit. In particular, there does not exist a $2 \times 2 \times 2$ array of non-negative integers $N = \{n_{ijk}\}$, such that for all $1 \leq i, j \leq 2$

$$n_{1ij} + n_{2ij} = n_{ij1} + n_{ij2} = m_{ij}.$$ 

On the other hand, using Theorem 2.20 and the factorization

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

But $R_L^j = R_L^j$ for $j < i$. Therefore, $R_L^i = R_L^i$. Contradiction. \qed
one computes
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left[ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right] \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} 
= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \circ \left[ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Example 2.29. Consider the orbit $O_M$ where
\[
M := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},
\]
i.e. the space of pairs of distinct lines in a three dimensional space $V$. Then the triple intersection $\pi_{12}^{-1}(O_M) \cap \pi_{23}^{-1}(O_M) \cap \pi_{13}^{-1}(O_M)$ consist of triples of distinct lines in $V$, which consists of two orbits: triples of lines that form a direct sum, and triples of distinct lines that lie in the same plane.

Example 2.30. In the previous example one of the orbits in the triple intersection was not combinatorial (triples of lines that lies in the same plane). However, it is not hard to come up with an example of a triple intersection containing more than one combinatorial orbit. This is equivalent to finding two distinct three dimensional arrays of non-negative integers $M = \{m_{ijk}\}$ and $N = \{n_{ijk}\}$ such that for every $a$ and $b$ one has
\[
\sum_i m_{iab} = \sum_i n_{iab},
\]
\[
\sum_j m_{ajb} = \sum_j n_{ajb},
\]
and
\[
\sum_k m_{abh} = \sum_k n_{abh}.
\]
For example, one can take $2 \times 2 \times 2$ arrays given by $m_{ijk} = 1$ for all $i, j, k$, and
\[
n_{ijk} = \begin{cases} 2, & i + j + k \text{ is even}, \\ 0, & i + j + k \text{ is odd}. \end{cases}
\]

3. Equivariant $K$-theory and the convolution algebras $S_0^{\text{aff}}(n, d)$.

3.1. Equivariant $K$-theory of the partial flag variety. Let $\mu$ be a composition of $d$ of length $n$. Consider the partial flag variety $\mathcal{F}_\mu \simeq G/P_\mu$, where the parabolic subgroup $P_\mu \subset G$ is the stabilizer of a partial flag $p \in \mathcal{F}_\mu$. Note that $P_\mu \simeq L_\mu \ltimes U_\mu$, where $L_\mu \simeq GL_{\mu_1} \times \ldots \times GL_{\mu_n} \subset P_\mu$ is the Levi subgroup and $U_\mu$ is the unipotent radical of $P_\mu$. One gets
\[
K^G(\mathcal{F}_\mu) \simeq K^{P_\mu}(pt) \simeq K^{L_\mu}(pt) \simeq R(L_\mu) \simeq C[\mathbf{x}_1^{\pm 1}, \ldots, \mathbf{x}_d^{\pm 1}]^{S_\mu} \simeq \bigotimes_i \Lambda_i^{\pm},
\]
where $S_\mu = S_{\mu_1} \times \ldots \times S_{\mu_n} \subset S_d$ is the corresponding Young subgroup, and $\Lambda_i^{\pm}$ is the ring of symmetric Laurent polynomials in $t$ variables.

This isomorphism can be understood in the following way. Let $\pi : E \to \mathcal{F}_\mu$ be a $G$-equivariant vector bundle. Since $\mathcal{F}_\mu$ is a homogeneous space, the whole bundle is determined by the fiber over just one point, and the action of the stabilizer on that fiber. Let us fix a basis $\mathcal{B} = \{e_1, \ldots, e_d\} \subset V$, let $T \subset G$ be the maximal torus acting by scaling the coordinates in this basis, and let $\{x_1, \ldots, x_d\}$ be the corresponding characters of $T$, forming a basis in the character lattice of $T$. The $T$-fixed points in $\mathcal{F}_\mu$ are the flags consisting of the coordinate subspaces.

Let us introduce the following notations:
\[ \mathcal{B} = \mathcal{B}_1 \sqcup \ldots \sqcup \mathcal{B}_n, \]

where

\[ B_k := \{ e_{d_k+1}, \ldots, e_{d_k} \}. \]

(Here, as before \( d_i = \sum_{a \leq i} \mu_a \) are the partial sums.)

We also set

\[ V_k := \text{span}(B_k), \]

so that \( V = \bigoplus V_k. \)

Let \( p_\mu \in F_\mu \) be the \( T \)-fixed point given by

\[ p_\mu := \{ V_1 \subset V_1 \oplus V_2 \subset \ldots \subset V_1 \oplus \ldots \oplus V_n = V \}. \]

The stabilizer of the fixed point \( p_\mu \) is the parabolic subgroup of \( G \) consisting of block upper triangular matrices with blocks of sizes \( \mu_1, \ldots, \mu_n \), so the \( T \)-character of the fiber \( E_x \subset E \) belongs to \( \mathbb{C}[x_{1}^{\pm 1}, \ldots, x_{d}^{\pm 1}]^{S_\mu} \).

Note that one could have chosen a different torus fixed point. The \( T \)-characters of the fibers of equivariant vector bundles are then related by the following Lemma:

**Lemma 3.1.** Let \( p : E \rightarrow F_\mu \) be a \( G \)-equivariant vector bundle. Let \( x, x' \in F_\mu^T \) be two \( T \)-fixed points. It follows that there exists a permutation \( \omega \in S_d \) such that \( \omega(x) = x' \) (here \( S_d \) acts on \( V \) by permuting the basic vectors \( \{ e_1, \ldots, e_d \} \)). Let \( f \) and \( f' \) be the \( T \)-characters of the fibers \( E_x \) and \( E_{x'} \) correspondingly. Then one has

\[ \omega(f) = f', \]

where \( \omega \) acts by permuting the variables in the Laurent polynomial \( f \).

**Proof.** The operator \( \omega \in S_d \subset G \) acts on the vector bundle \( E \rightarrow F_\mu \) and sends the fiber \( E_x \) to the fiber \( E_{x'} \). It follows that the \( T \)-actions on \( E_x \) and \( E_{x'} \) are related by the automorphism \( t \mapsto \omega^{-1}t\omega \), which implies the formula. \( \square \)

### 3.2. Equivariant K-theory of the double flag variety.

Let \( M = \{ m_{ij} \} \) be an \( n \times k \) matrix with non-negative integer entries, and let \( \mathcal{O}_M \subset F_n^d \times F_k^d \) be the corresponding orbit. Let \( \mathcal{B} = \{ e_1, \ldots, e_d \} = \bigsqcup B_{ij} \) be a decomposition of the basis \( \mathcal{B} \) of \( V \), such that \( zB_{ij} = m_{ij} \), and let \( V_j := \text{span}(B_{ij}) \). Let also \( \{ 1, \ldots, d \} = \bigsqcup I_{ij} \) be the corresponding decomposition of the index set, so that for each \( i \) and \( j \) one has \( B_{ij} = \{ e_a | a \in I_{ij} \} \). Let \( p \in \mathcal{O}_M \) be the pair of flags \( p := (U, W) \), where

\[ U_i := \text{span} \bigoplus_{1 \leq a \leq i, 1 \leq b \leq k} V_{ab}, \quad \text{and} \quad W_i := \text{span} \bigoplus_{1 \leq a \leq n, 1 \leq b \leq i} V_{ab}. \]

The stabilizer \( St_p \subset G \) is then the intersection of two parabolic subgroups \( St_p = St_U \cap St_W \). Also, every element of the stabilizer \( St_p \) can be uniquely factored as \( gh \) where

\[ g \in GL(p) := \bigoplus_{1 \leq a \leq n, 1 \leq b \leq k} GL(V_{ab}), \quad \text{and} \quad h \in N(p) := I_d + \bigoplus_{1 \leq a \leq n, 1 \leq b \leq k} \text{Hom}(V_{ab}, \bigoplus_{c \leq a, d \leq b \atop (c,d) \neq (a,b)} V_{cd}). \]

Therefore, we get a semidirect product decomposition \( St_p = GL(p) \rtimes N(p) \), where \( GL(p) \) is semisimple and \( N(p) \) is the unipotent radical.

Let \( S_p \simeq \bigoplus_{I_{ij}} S_{m_{ij}} \subset S_d \) be the subgroup of permutations preserving the subdivision \( \{ 1, \ldots, d \} = \bigsqcup I_{ij} \). Similar to before, one gets
\[ K^G(\mathcal{O}_M) \simeq K^G(p)(pt) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^S_p \simeq \bigotimes_{i,j} \Lambda_{m_{ij}}^\pm. \]

The isomorphism can be exhibited in the following way. Given an equivariant vector bundle \( W \) over \( \mathcal{O}_M \) the partially symmetric polynomial corresponding to the class \([W] \in K^G(\mathcal{O}_M)\) equals to the \( T \)-character of the fiber \( W_p \) over the point \( p \in \mathcal{O}_M \).

Note that different choices of the subdivision \( B = \bigsqcup B_{ij} \) satisfying the condition \( \# B_{ij} = m_{ij} \) lead to different fixed points in \( \mathcal{O}_M \) and different (but conjugated) subgroups in \( S_p \). The isomorphisms from equation 3.2 for different fixed points are related in a way similar to Lemma 3.1.

**Lemma 3.2.** Let \( p : E \to \mathcal{O}_M \) be a \( G \)-equivariant vector bundle over an orbit \( \mathcal{O}_M \subset F^d_n \times F^d_n \).

Let \( x, x' \in \mathcal{O}_M^r \) be the two \( T \)-fixed points. It follows that there exists a permutation \( \omega \in S_d \) such that \( \omega(x) = x' \) (here \( S_d \) acts on \( V \) by permuting the basic vectors \( \{e_1, \ldots, e_d\} \)). Let \( f \) and \( f' \) be the \( T \)-characters of the fibers \( E_x \) and \( E_{x'} \) correspondingly. Then one has

\[ \omega(f) = f', \]

where \( \omega \) acts by permuting the variable in the Laurent polynomial \( f \).

This Lemma is proved in the same way as Lemma 3.1.

3.3. **Push-forward and pull-back for orbits.** Let \( \mu = (\mu_1, \ldots, \mu_{n+1}) \) and \( \nu = (\nu_1, \ldots, \nu_k) \) be two composition of \( d \), and let \( \mu' = (\mu_1, \ldots, \mu_{l+1}, \ldots, \mu_{n+1}) \). Let \( M \) be an \((n + 1) \times k\) matrix with non negative integer entries such that \( R^M = \mu \) and \( C^M = \nu \), and let the \( n \times k \) matrix \( M' \) be obtained from \( M \) by summing up the \( l \)th and \((l + 1)\)th rows, so that \( R^{M'} = \mu' \) and \( C^{M'} = \nu \). Let \( \mathcal{O}_M \subset F_\mu \times F_\nu \) and \( \mathcal{O}_{M'} \subset F_{\mu'} \times F_{\nu} \) be the corresponding orbits. Then one has a natural equivariant projection \( \pi : \mathcal{O}_M \to \mathcal{O}_{M'} \) given by forgetting the \( k \)th step of the first flag. This gives rise to the pull-back and push-forward operators:

\[ \pi^* : K^G(\mathcal{O}_{M'}) \to K^G(\mathcal{O}_M), \]

and

\[ \pi_* : K^G(\mathcal{O}_M) \to K^G(\mathcal{O}_{M'}). \]

Let \( p \in \mathcal{O}_M \) and \( p' \in \mathcal{O}_{M'} \) be torus fixed points such that \( \pi(p) = p' \), and let \( S_p \subset S_d \) and \( S_{p'} \subset S_d \) be the corresponding subgroups. Note that \( S_p \subset S_{p'} \). One can write explicit formulas for the operators \( \pi^* \) and \( \pi_* \) in terms of the isomorphisms \( K^G(\mathcal{O}_M) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^S_p \) and \( K^G(\mathcal{O}_{M'}) \simeq \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_{p'}} \):

**Lemma 3.3.** Let \( f(x) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_p} \subset \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_{p'}} \), and \( g(x) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_p} \). One has the following formulas:

\[ \pi^* f(x) = f(x), \]

and

\[ \pi_* g(x) = Sym_{S_p}^{S_{p'}} \prod_{1 \leq a < k} \prod_{i \in I_{a,l}} \prod_{j \in I_{a+1,l}} (1 - \frac{x_i}{x_j}), \]

where \( Sym_{S_p}^{S_{p'}}(\cdot) \) is the summation of \( \sigma(\cdot) \) over a choice of representatives \( \sigma \) of the left cosets of \( S_p \) in \( S_{p'} \).

**Remark.** Note that the second formula above does not depend on the choice of representatives, because both \( g(x) \) and \( \prod_{1 \leq a < k} \prod_{i \in I_{a,l}} \prod_{j \in I_{a+1,l}} (1 - \frac{x_i}{x_j}) \) belong to \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_p} \).
Proof. Since \( \pi(p) = p' \), for any vector bundle \( E \to \mathcal{O}_{M'} \), the fiber of \( \pi^*E \) at \( p \) is by definition isomorphic to the fiber of \( E \) at \( p' \), so the first part of the Lemma follows immediately.

We will use the Lefschetz formula to compute the pushforward \( \pi_* \). There are finitely many torus fixed points in both \( \mathcal{O}_M \) and \( \mathcal{O}_{M'} \). Those are precisely the flags consisting of coordinate subspaces. It follows that for a generic point \( t \in T \subset G \) the \( t \)-fixed points coincide with the full torus fixed points. We need to compute the \( T \)-character of the fiber of \( \pi_*f \) at \( p' \). Let \( \tilde{\pi} := \pi|_{\pi^{-1}(p')} : \pi^{-1}(p') \to p' \). We will use the Lefschetz formula to compute \( \tilde{\pi}_*(f|_{\pi^{-1}(p')}) \), which is exactly what we need.

\( S_d \) naturally acts both on the set of torus fixed points in \( \mathcal{O}_M \) and the set of torus fixed points in \( \mathcal{O}_{M'} \). Moreover, both actions are transitive, they commute with the projection \( \pi \), and the stabilizers of \( p \in \mathcal{O}_M \) and \( p' \in \mathcal{O}_{M'} \) are \( S_p \) and \( S_{p'} \) correspondingly. Therefore, \( S_{p'} \) acts transitively on the torus fixed points in the fiber \( \pi^{-1}(p') \), and the stabilizer of \( p \in \pi^{-1}(p') \) is \( S_p \subset S_{p'} \). We conclude that for any collection of representatives \( \{\omega_1, \ldots, \omega_N\} \) of the left cosets of \( S_p \) in \( S_{p'} \) the torus fixed points in the fiber \( \pi^{-1}(p') \) are exactly \( \{\omega_1(p), \ldots, \omega_N(p)\} \), and every point is listed exactly once.

The last ingredient we need is the class of the exterior algebra of the conormal bundle to the fixed point \( p \) in \( K^T(\pi^{-1}(p')) \). Since the fixed points are isolated, this is simply the \( T \)-character of the exterior algebra of the cotangent space to the fiber \( \pi^{-1}(p') \) at \( p \). Note that \( \pi^{-1}(p') \) is isomorphic to the product of Grassmanians:

\[
\pi^{-1}(p') \simeq \prod_{a=1}^k Gr(m_{i,a}, V_{i,a} \oplus V_{i+1,a}),
\]

where \( Gr(m_{i,a}, V_{i,a} \oplus V_{i+1,a}) \) is the Grassmanian of \( m_{i,a} \)-dimensional subspaces in \( V_{i,a} \oplus V_{i+1,a} \). The fixed point \( p \) corresponds to \( (V_{1,1}, \ldots, V_{k,k}) \). For each \( a \) the eigenvalues of the cotangent space of the Grassmanian \( Gr(m_{i,a}, V_{i,a} \oplus V_{i+1,a}) \) at the point \( V_{i,a} \) are \( \frac{x_j}{x_i} \) for \( i \in I_{a,1} \) and \( j \in I_{a,1} \), each with multiplicity 1. Therefore, for the exterior algebra one gets

\[
\prod_{1 \leq a \leq k} \prod_{i \in I_{a,1}, \ j \in I_{a,1}} (1 - \frac{x_i}{x_j})
\]

Using Lemma 3.1 and the Lefschetz formula we get the desired result. \qed

Remark 3.4. Note that if \( \nu = (d) \) one gets \( \mathcal{O}_{M'} = \mathcal{F}_{\nu'} \) and \( \mathcal{O}_M = \mathcal{F}_{\nu} \). Therefore, Lemma 3.3 is also applicable to the natural projections of the partial flag varieties. For \( \pi : \mathcal{F}_{\nu'} \to \mathcal{F}_{\nu} \) one gets

\[
\pi^*f(x) = f(x),
\]

and

\[
\pi_*g(x) = Sym_{\nu'} \prod_{i \in I, \ j \in I+1} \frac{g(x)}{(1 - \frac{x_i}{x_j})},
\]

where \( \{1, \ldots, d\} = \bigcup_{a=1}^{\nu+1} I_a \) is the corresponding subdivision.

Note that since \( \pi : \mathcal{O}_M \to \mathcal{O}_{M'} \) is a smooth fibration with connected fibers, one has \( \pi_*1 = 1 \). One can obtain this from the formula in Lemma 3.3 although it requires some manipulation with determinants. One also gets the projection formula:

\[
\pi_*(\pi^*(f)g) = f \pi_*(g).
\]

This can also be obtained from the formula in Lemma 3.3 directly, since if \( f \in Z[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^S_{\nu'} \) then one has
Lemma 3.8. \[ Sym^S_{S_p} f g = f Sym^S_{S_p} g. \]

In particular, one obtains the following Lemma:

Lemma 3.5. Let \( \pi : \mathcal{O}_M \to \mathcal{O}_{M'} \) be a projection map as above. Then the push-forward operator \( \pi_* : K^G(\mathcal{O}_M) \to K^G(\mathcal{O}_{M'}) \) is surjective.

Proof. Indeed, for any \( f \in K^G(\mathcal{O}_{M'}) \) one has
\[ f = f \cdot 1 = f \cdot \pi_*(1) = \pi_*(\pi^*(f) \cdot 1). \]

We will need to compute the push-forwards in some special cases. Let \( \mu = (\mu_1, \ldots, \mu_n) \) be a composition of \( d \). Let \( \mu' = (\mu_1, \ldots, \mu_{k-1}, 1, \ldots, 1, \mu_{k+1}, \ldots, \mu_n) \) be its refinement, where the part \( \mu_k \) is split into \( \mu_k \) ones. Let \( L_1, \ldots, L_{\mu_k} \) be the tautological line bundles over \( \mathcal{F}_{\mu'} \) corresponding to the one-dimensional steps of the flag, left to right. Let \( \pi : \mathcal{F}_{\mu'} \to \mathcal{F}_\mu \) be the natural projection, and let \( \lambda = (\lambda_1, \ldots, \lambda_{\mu_k}) \) be a partition of length \( \leq \mu_k \) (i.e. \( \lambda_1 \geq \ldots \geq \lambda_{\mu_k} \geq 0 \)). According to Borel-Weil-Bott Theorem, one gets
\[ \pi_*(L_1) = [S_\lambda(T_k)], \]
where \( T_k \) is the tautological vector bundle over \( \mathcal{F}_\mu \) corresponding to the \( k \)th step of the flag, and \( S_\lambda \) is the Schur functor. In terms of the partially symmetric functions one gets
\[ \pi_*(y_1^{\lambda_1} y_2^{\lambda_2} \ldots y_{\mu_k}^{\lambda_{\mu_k}}) = S_\lambda(y_1, \ldots, y_{\mu_k}), \]
where \( y_1, \ldots, y_{\mu_k} \) are the torus characters corresponding to the \( k \)th step of the flag, and \( S_\lambda \) is the Schur function. One can also obtain this formula directly from Lemma 3.3.

Formula (4) also holds in larger generality. Let \( \lambda = (\lambda_1, \ldots, \lambda_{\mu_k}) \in \mathbb{Z}^{\mu_k} \) be an arbitrary integer vector. Let \( S_\lambda \) be the \textit{generalized Schur function} defined as follows:

Definition 3.6. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m \) be an integer vector. Let \( \det_\alpha(y_1, \ldots, y_m) \) denote the determinant of the matrix \( M_\alpha = \{m_{ij}\} = \{y_i^{\alpha_j}\} \). Let also \( \rho^m = (m - 1, m - 2, \ldots, 0) \). In particular, \( \det_{\rho^m}(y_1, \ldots, y_m) \) is the Vandermonde determinant.

Definition 3.7. The Schur function \( s_\lambda(y_1, \ldots, y_m) \) is defined as follows:
\[ s_\lambda = \frac{\det_\lambda + \rho^m(y_1, \ldots, y_m)}{\det_\rho^m(y_1, \ldots, y_m)}. \]

The following relation follows immediately from the above definition:
\[ s_\lambda(y_1, \ldots, y_m) = y_1 \ldots y_m s_{\lambda - (1, \ldots, 1)}, \]
and
\[ s_{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_{i+1}, \ldots, \lambda_m} = -s_{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_{i+1} + 1, \ldots, \lambda_m}. \]

In particular, one gets that for every \( \lambda \in (\mathbb{Z})^n \) the Schur function \( s_\lambda \) is either zero, or equals to \( \pm s_{\lambda_1, \ldots, \lambda_m}(y_1 \ldots y_m)^N \) for a partition \( \nu \) (i.e. \( \nu_1 \geq \ldots \geq \nu_n \geq 0 \)) and a positive integer \( N \). One can use these formulas to define the \textit{generalized Schur functors} \( S_\lambda \) for a general \( \lambda \in \mathbb{Z}^n \). One immediately gets

Lemma 3.8. Formulas (3) and (4) hold for general a general \( \lambda \in \mathbb{Z}^n \).
Formula (5) also implies that if for some \( i \) and \( k \) one has \( \lambda_{i+k} = \lambda_i + k \) then \( s_\lambda = 0 \). In particular, for \( 0 < k < m \) one gets

\[
s_{(-k)}(y_1, \ldots, y_m) = 0.
\]

In some cases it is also convenient to reduce generalized Schur functions to the form

\[
\pm s_\nu \left( \frac{1}{y_1}, \ldots, \frac{1}{y_m} \right)(y_1 \ldots y_m)^N,
\]

where \( \nu \) is a partition. Note that

\[
det_{\rho^n}(y_1, \ldots, y_m) = det_{(0,1,\ldots,m-1)}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})(y_1 \ldots y_m)^{m-1}
\]

\[
= (-1)^{m(m-1)} det_{\rho^n}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})(y_1 \ldots y_m)^{m-1},
\]

and for \( k \geq m \)

\[
det_{(-k)+\rho^n}(y_1, \ldots, y_m) = det_{(k-1,0,\ldots,m-2)}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})(y_1 \ldots y_m)^{m-2}
\]

\[
= (-1)^{(m-1)(m-2)/2} det_{(k-m)+\rho^n}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})(y_1 \ldots y_m)^{m-2}.
\]

Therefore, in this case one gets

\[
s_{(-k)}(y_1, \ldots, y_m) = \frac{(-1)^{m-1}s_{(k-m)}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})}{y_1 \ldots y_m}
\]

\[
= \frac{(-1)^{m-1}h_{k-m}(\frac{1}{y_1}, \ldots, \frac{1}{y_m})}{y_1 \ldots y_m},
\]

where \( h_{k-m} \) is the complete homogeneous symmetric polynomial.

One can use these formulas to define generalized Schur functors \( S_\lambda \) for a general \( \lambda \in \mathbb{Z}^n \). In particular, for \( k \in \mathbb{Z}_{\geq 0} \) one gets

\[
S_{(-k)}(V) = \left\{ \begin{array}{ll} (-1)^{m-1}[\Lambda_m(V^*)][\text{Sym}_{k-m}(V^*)], & 0 < k < m, \\
0, & k = m. \end{array} \right.
\]

**Corollary 3.9.** Let \( \lambda := (\lambda_1, \ldots, \lambda_n) \) and \( \lambda^- := (\lambda_1, \ldots, \lambda_{i-1}, 1, \lambda_i - 1, \lambda_{i+1}, \ldots, \lambda_n) \) be compositions, and let \( \pi^- : F_{\lambda^-} \to F_\lambda \) be the projection. Let \( L^- \) be the tautological line bundle over \( F_{\lambda^-} \) corresponding to the one dimensional step in the flag, and \( T_i \) be the tautological vector bundle over \( F_\lambda \) corresponding to the \( i \)th step. Then

\[
\pi^-([L^-]^p) = \left\{ \begin{array}{ll} (-1)^{\lambda_i-1}[\Lambda_{\lambda_i}(T_i)][\text{Sym}_{p-\lambda_i}(T_i)], & p \geq \lambda_i, \\
0, & 0 < p < \lambda_i, \\
[\text{Sym}_{p}(T_i^*)], & p \leq 0. \end{array} \right.
\]

In terms of the partially symmetric functions one gets

\[
\pi^- (y_i^p) = \left\{ \begin{array}{ll} (-1)^{\lambda_i-1}y_1 \ldots y_{\lambda_i} h_{p-\lambda_i}(y_1, \ldots, y_{\lambda_i}), & p \geq \lambda_i, \\
h_{-p}(\frac{1}{y_1}, \ldots, \frac{1}{y_{\lambda_i}}), & 0 < p < \lambda_i, \\
0, & p \leq 0. \end{array} \right.
\]

**Proof.** Consider the composition \( \lambda' = (\lambda_1, \ldots, \lambda_{i-1}, 1, \ldots, 1, \lambda_{i+1}, \ldots, \lambda_n) \) and the corresponding partial flag variety \( F_{\lambda'} \) with the projection map \( pr : F_{\lambda'} \to F_{\lambda^-} \). Note that \( pr^*(L^-) = L_1 \)
and \( pr_\ast(L^1) = L^- \) where \( L_1 \) is the line bundle on \( \mathcal{F}_\lambda \) corresponding to the first one-dimensional step. Therefore, using projection formula one gets
\[
\pi^-([L^-]^p) = (\pi^\ast \circ pr_\ast) \circ ([L]^p) = [S_{(p, \ldots, p)}(T_i)]
\]
(there are \( \lambda_i - 1 \) zeros).

Now the corollary follows immediately from the computations above.

\[ \square \]

**Corollary 3.10.** Let also \( \lambda^+ := (\lambda_1, \ldots, \lambda_{n-1}, \lambda_1 - 1, 1, \lambda_{i+1}, \ldots, \lambda_n) \), let \( \pi^+ : \mathcal{F}_{\lambda^+} \to \mathcal{F}_\lambda \) be the projection, and let \( L^+ \) be the tautological line bundle over \( \mathcal{F}_{\lambda^+} \) corresponding to the one dimensional step in the flag. Then
\[
\pi^+([L^+]^p) = \left\{ \begin{array}{ll}
[S_{\text{sym}_p}(T_i)], & p \geq 0, \\
0, & -\lambda_i < p < 0,
\end{array} \right.
\]
In terms of the partially symmetric functions one gets
\[
\pi^+(y^p_{\lambda_i}) = \left\{ \begin{array}{ll}
h_{p}(y_1, \ldots, y_{\lambda_i}), & p \geq 0, \\
0, & -\lambda_i < p < 0,
\end{array} \right.
\]

**Proof.** Similar to the above we get
\[
\pi^+([L^+]^p) = [S_{(p)}(T_i)].
\]
Now the corollary follows immediately from the computations above.

\[ \square \]

Finally, the classes derived from the tautological bundles not affected by the projection behave in the natural way:

**Lemma 3.11.** As before, let \( \mu = (\mu_1, \ldots, \mu_{n+1}) \), \( \mu^\prime = (\mu_1, \ldots, \mu_1', \mu_1'', \ldots, \mu_{n+1}) \), where \( \mu_1' + \mu_1'' = \mu_1 \), and let \( \pi : \mathcal{F}_\mu \to \mathcal{F}_{\mu^\prime} \) be the natural projection. Let \( T_k \) and \( T_k' \) be the tautological bundles corresponding to the step \( \mu_k \), \( k \neq l \), over \( \mathcal{F}_\mu \) and \( \mathcal{F}_{\mu^\prime} \) respectively. Then for any \( \lambda \in \mathbb{Z}^n_{\mu} \) and any \( f \in K^G(\mathcal{F}_\mu) \) one has
\[
\pi_\ast(S_\lambda(T_k) \cdot f) = S_\lambda(T_k') \cdot \pi_\ast(f).
\]

**Proof.** One gets \( \pi^* (S_\lambda(T_k')) = S_\lambda(T_k) \) immediately from definitions. Then, applying the projection formula, one gets
\[
\pi_\ast(S_\lambda(T_k) \cdot f) = \pi_\ast(\pi^* (S_\lambda(T_k')) \cdot f) = S_\lambda(T_k') \cdot \pi_\ast(f).
\]

\[ \square \]

**Remark 3.12.** Lemma 3.11 also easily follows from the formulas of Lemma 3.3.

3.4. Convolution algebra.

**Lemma 3.13.** Let \( N \subset M \subset \mathcal{F}^d_n \times \mathcal{F}^d_n \) be two closed invariant subvarieties in the double partial flag variety. Then one gets the short exact sequence
\[
0 \longrightarrow K^G(N) \xrightarrow{i_*} K^G(M) \xrightarrow{j_*} K^G(M \setminus N) \longrightarrow 0,
\]
where \( i : N \hookrightarrow M \) is the closed embedding, and \( j : M \setminus N \hookrightarrow M \) is the open embedding.
Proof. Set $U = M \setminus N$. Then one gets the long exact sequence in higher $K$-theory:

$$
\ldots \longrightarrow K^G_1(U) \longrightarrow K^G_0(N) \longrightarrow K^G_0(M) \longrightarrow K^G_0(U) \longrightarrow 0,
$$

Therefore, it suffices to prove that the map $i_* : K^G(N) \to K^G(M)$ is injective. Suppose that $i_*$ is not injective and $\beta \neq 0$ is such that $i_* \beta = 0$.

Let $O_{B_1} \subset N$ be an open orbit in $N$ such that $\beta|_{O_{B_1}} = 0$, and let $S_1 := N \setminus O_{B_1}$ be the complement. Then it follows from the long exact sequence for $O_{B_1} \subset N$ that there exists $\beta_1 \in K^G(S_1)$ such that $i_{1*} \beta_1 = \beta$, where $i_1 : S_1 \hookrightarrow N$ is the closed embedding. Similarly, let $O_{B_2} \subset S_1$ be an open orbit in $S_1$ such that $\beta_1|_{O_{B_2}} = 0$, and let $S_2 := S_1 \setminus U_1$ be the complement. Then it follows from the long exact sequence for $O_{B_2} \subset S_1$ that there exists $\beta_2 \in K^G(S_2)$ such that $i_{2*} \beta_2 = \beta_1$, where $i_2 : S_2 \hookrightarrow S_1$ is the closed embedding. Since there are only finitely many orbits in $\mathcal{F}^d_n \times \mathcal{F}^d_n$, after repeating the above step several times one gets a closed invariant subvariety $i_S : S \hookrightarrow N$, and a class $\alpha \in K^G(S)$ such that $i_{S*} \alpha = \beta$ and for any orbit $O_{C} \subset S$ open in $S$ one has $\alpha|_{O_{C}} \neq 0$.

Let $L \subset S$ be the union of all orbits open in $S$, $L = O_1 \sqcup \ldots \sqcup O_k$, so that $\overline{L} = S$. Let also $W := \mathcal{F}^d_n \times \mathcal{F}^d_n \setminus \partial L$. Then one gets the following Cartesian square:

$$
\begin{array}{ccc}
L & \xrightarrow{i_L} & W \\
\downarrow{i_L} & & \downarrow{i} \\
S & \xrightarrow{i} & \mathcal{F}^d_n \times \mathcal{F}^d_n
\end{array}
$$

Note that $i : S \hookrightarrow \mathcal{F}^d_n \times \mathcal{F}^d_n$ factors as

$$
S \xrightarrow{i_S} N \xleftarrow{i} M \xleftarrow{i} \mathcal{F}^d_n \times \mathcal{F}^d_n.
$$

Hence one gets

$$
\hat{i}_* \alpha = \hat{i}_* i_* i_{S*} \alpha = \hat{i}_* i_* \beta = 0.
$$

Using base change, one gets

$$
0 = \hat{j}_* \hat{i}_* \alpha = i_{L*} j_* \alpha = i_{L*} \alpha|_L.
$$

Since $L = O_1 \sqcup \ldots \sqcup O_k$, one gets

$$
K^G(L) = \bigoplus_m K^G(O_m), \quad \alpha|_L = (\alpha_1, \ldots, \alpha_k),
$$

and for every $1 \leq m \leq k$, $0 \neq \alpha_m \in K^G(O_m)$. Furthermore, for every $1 \leq m \leq k$, $0 \neq \alpha_m \in K^G(O_m)$. Therefore, for every $1 \leq m \leq k$, $O_m \hookrightarrow W$ is a smooth closed embedding, therefore

$$
i_{L*} \alpha|_L = (\alpha_1 \otimes \Lambda(T^*_O W), \ldots, \alpha_k \otimes \Lambda(T^*_O W)).
$$

Since the torus fixed points are isolated it follows that $\Lambda(T^*_O W) \neq 0$ for every orbit $O$. Also, it follows from equation [3.2] that there are no zero divisors in $K^G(O)$. Therefore,

$$
\alpha_m \otimes \Lambda(T^*_O W) \neq 0
$$

for every $1 \leq m \leq k$. In particular, $i_{L*} \alpha|_L \neq 0$. Contradiction.

Corollary 3.14. For every closed invariant subvariety $M \subset \mathcal{F}^d_n \times \mathcal{F}^d_n$ one gets an injective morphism $K^G(M) \hookrightarrow K^G(\mathcal{F}^d_n \times \mathcal{F}^d_n)$, and the image is a subalgebra under tensor product.
We usually abuse notations by simply writing \( K^G(M) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \).

**Corollary 3.15.** Furthermore, for every orbit \( O_A \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) one gets
\[
K^G(\partial O_A) \subset K^G(\overline{O}_A) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d),
\]
where \( \overline{O}_A \) is the closure of the orbit \( O_A \) and \( \partial O_A = \overline{O}_A \backslash O_A \) is its boundary, and
\[
K^G(O_A) = K^G(\overline{O}_A)/K^G(\partial O_A).
\]

It then follows that classes in \( K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \) have well defines supports:

**Corollary 3.16.** For every class \( \alpha \in K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \) there exists a unique minimal (under inclusion) closed invariant subvariety \( \text{supp}(\alpha) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) such that \( \alpha \in K^G(\text{supp}(\alpha)) \) and for every orbit \( O \subset \text{supp}(\alpha) \) open in \( \text{supp}(\alpha) \) one has \( \alpha|_O \neq 0 \).

**Theorem 3.17.** The convolution algebra \( S^\text{aff}_0(n, d) \) is generated by the classes supported on the orbits \( O_A \) where \( A \) is either a diagonal matrix, or an almost diagonal matrix.

**Proof.** Similar to Corollary 2.21, the proof proceeds by induction in the diagonal norm
\[
DN(M) := \sum_{1 \leq i, j \leq n} |i - j|m_{ij}.
\]

More precisely, we will use induction in \( \max\{DN(N)|O_N \subset \text{supp}(f)\} \) for \( f \in K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \). If \( \max\{DN(N)|O_N \subset \text{supp}(f)\} \leq 1 \) then \( f \) is supported on a union of closed orbits corresponding to diagonal and almost diagonal matrices, therefore the base case of the induction is immediate.

Note that whenever \( N < M \) one has \( DN(N) \leq DN(M) \) (this can be checked by a direct computation). It follows that one can choose an orbit \( O_M \subset \text{supp}(f) \) such that \( DN(M) = \max\{DN(N)|O_N \subset \text{supp}(f)\} \) and \( O_M \) is open in \( \text{supp}(f) \). We will show that there exists a class \( f' \in K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \) such that \( \text{supp}(f') \subset \text{supp}(f) \), \( \text{supp}(f - f') \subset \text{supp}(f) \backslash O_M \), and \( f' \) is generated by classes supported on orbits with lower diagonal norm. Then one can finish the proof by applying this argument several times and using the inductive assumption.

Applying Corollary 2.24 to \( M \) we obtain a factorization \( M = D \circ N \) such that \( D \) is almost diagonal, \( DN(N) < DN(M) \), and for any \( N' < N \) one has \( D \circ N' < M \). Since \( O_D \) is a closed orbit, Lemma 2.25 implies that the triple intersection
\[
O := \pi_{12}^{-1}(O_D) \cap \pi_{13}^{-1}(O_M) \cap \pi_{23}^{-1}(O_N) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d
\]
is a single orbit. According to Lemma 2.16 the orbit \( O \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \) is isomorphic to an orbit \( O_L \subset \mathcal{F}_{n+1}^d \times \mathcal{F}_n^d \), where \( L \) is an \((n + 1) \times n\) matrix. Let us first focus on the convolution map on the orbits:
\[
\hat{x} : K^G(O_D) \otimes K^G(O_N) \xrightarrow{\hat{\pi}_{12} \otimes \hat{\pi}_{23}} K^G(O) \xrightarrow{\hat{\pi}_{13}} K^G(O_M),
\]
where \( \hat{\pi}_{ij} = \pi_{ij}|_O \). We will prove that this multiplication map is surjective. The push-forward map \( \hat{\pi}_{13} \circ \hat{x} : K^G(O_L) \simeq K^G(O) \rightarrow K^G(O_M) \) is surjective by Lemma 3.5 therefore it will suffice to prove that \( \hat{\pi}_{12} \otimes \hat{\pi}_{23} \) is surjective as well.

Suppose that \( D \) is upper-triangular, i.e. \( D = E(\mu, k) \) where \( 1 \leq k < n \) and \( \mu = R^M - e_k \) (the lower-triangular case is done in a similar way). Here, according to Theorem 2.20 Lemma 2.23 and Corollary 2.24 the choice of \( k \) satisfies the following condition: there exists \( l \) such that
\[
m_{k,l+1} = \ldots = m_{k,n} = m_{k+1,l+1} = \ldots = m_{k+1,n} = 0,
\]
and $m_{kl} > 0$. Then the matrix $L$ is obtained from $M$ by reducing $m_{kl}$ by one and inserting the row $(0, \ldots, 0, 1, 0, \ldots, 0)$ between $k$th and $(k + 1)$th rows (here 1 is in the $l$th position):

$$L = \begin{pmatrix}
  m_{11} & \cdots & m_{1,l-1} & m_{1l} & m_{1,l+1} & \cdots & m_{1n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  m_{k1} & \cdots & m_{k,l-1} & m_{kl} - 1 & 0 & \cdots & 0 \\
  0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
  m_{k+1,1} & \cdots & m_{k+1,l-1} & m_{k+1,l} & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  m_{n1} & \cdots & m_{n,l-1} & m_{nl} & m_{n,l+1} & \cdots & m_{nn}
\end{pmatrix},$$

and the matrix $N$ is obtained from $L$ by adding this inserted row to the row below it:

$$N = \begin{pmatrix}
  m_{11} & \cdots & m_{1,l-1} & m_{1l} & m_{1,l+1} & \cdots & m_{1n} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  m_{k1} & \cdots & m_{k,l-1} & m_{kl} - 1 & 0 & \cdots & 0 \\
  m_{k+1,1} & \cdots & m_{k+1,l-1} & m_{k+1,l} + 1 & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  m_{n1} & \cdots & m_{n,l-1} & m_{nl} & m_{n,l+1} & \cdots & m_{nn}
\end{pmatrix}.$$

Pick torus fixed points $p_M \in O_M$, $p_D \in O_D$, and $p_L \in O \simeq O_L$ so that $\tilde{\pi}_{12}(p_L) = p_D$ and $\tilde{\pi}_{23}(p_L) = p_D$. According to equation (3.2) this produces isomorphisms

$$K^G(O_N) \simeq Z[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_N} \simeq \bigotimes_{(i,j) \notin \{(k,l), (k+1,l)\}} \Lambda_{m_{ij}}^{\pm} \otimes \Lambda_{m_{kl-1}}^{\pm} \otimes \Lambda_{m_{k+1,l+1}}^{\pm},$$

$$K^G(O_D) \simeq Z[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_D} \simeq \bigotimes_i \Lambda_{\mu_i}^{\pm} \otimes \Lambda_{1}^{\pm},$$

$$K^G(O_L) \simeq Z[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_L} \simeq \bigotimes_{(i,j) \neq (k,l)} \Lambda_{m_{ij}}^{\pm} \otimes \Lambda_{m_{kl-1}}^{\pm} \otimes \Lambda_{1}^{\pm}.$$

The morphisms $\tilde{\pi}_{12}^*: K^G(O_D) \to K^G(O_L)$ and $\tilde{\pi}_{23}^*: K^G(O_N) \to K^G(O_L)$ are the natural embeddings, where the factors $\Lambda_{m_{ij}}^{\pm}, (i,j) \notin \{(k,l), (k+1,l)\}$, and $\Lambda_{m_{kl-1}}^{\pm}$ of $K^G(O_N)$ are mapped isomorphically to the corresponding factors of $K^G(O_L)$, the factor $\Lambda_{m_{k+1,l+1}}^{\pm}$ of $K^G(O_N)$ is mapped injectively to $\Lambda_{m_{k+1,l}}^{\pm} \otimes \Lambda_{1}^{\pm}$, the factors $\Lambda_{\mu_i}^{\pm}, i \neq k$ of $K^G(O_D)$ are mapped injectively to $\bigotimes_j \Lambda_{m_{ij}}^{\pm}$ (recall that for $i \neq k$ one has $\mu_i = \sum_j m_{ij}$), the factor $\Lambda_{\mu_k}^{\pm}$ is mapped injectively to $\bigotimes_{j \neq k} \Lambda_{m_{mj}}^{\pm} \otimes \Lambda_{m_{k+1,l-1}}^{\pm}$ (recall that $\mu_k = \sum_j m_{kj} - 1$), and the factor $\Lambda_{1}^{\pm}$ of $K^G(O_D)$ is mapped isomorphically to the corresponding factor of $K^G(O_L)$.

It remains to show that the factor $\Lambda_{m_{k+1,l+1}}^{\pm}$ is also generated by the images of $K^G(O_N)$ and $K^G(O_D)$, which follows from the elementary fact that the multiplication map

$$\Lambda_{m_{k+1,l+1}}^{\pm} \otimes \Lambda_{1}^{\pm} \to \Lambda_{m_{k+1,l}}^{\pm} \otimes \Lambda_{1}^{\pm}$$

is surjective (here the factor $\Lambda_{m_{k+1,l+1}}^{\pm}$ is mapped injectively to $\Lambda_{m_{k+1,l}}^{\pm} \otimes \Lambda_{1}^{\pm}$, and $\Lambda_{1}^{\pm}$ is mapped isomorphically to $\Lambda_{1}^{\pm}$).
We have \( f \in K^G(\text{supp}(f)) \subseteq K^G(F^d_n \times F^d_n) \) and \( \emptyset_M \subseteq \text{supp}(f) \), where \( \emptyset_M \) is open in \( \text{supp}(X) \). According to the above, the restriction \( \bar{f} := f|_{\emptyset_M} \) can be written as

\[
\bar{f} = \sum_{i=1}^{T} g_i \ast \tilde{h}_i,
\]

where \( T \) is an integer, \( g_1, \ldots, g_T \in K^G(\emptyset_D) \), and \( \tilde{h}_1, \ldots, \tilde{h}_T \in K^G(\emptyset_N) \).

The classes \( \tilde{h}_1, \ldots, \tilde{h}_T \) can be extended to classes \( h_1, \ldots, h_T \in K^G(\overline{\emptyset_N}) \subseteq K^G(F^d_n \times F^d_n) \). Since \( \emptyset_D \) is closed, one also gets \( g_1, \ldots, g_T \in K^G(\emptyset_D) \subseteq K^G(F^d_n \times F^d_n) \). Consider

\[
f' := \sum_{i=1}^{T} g_i \ast h_i = \pi_{13} \left( \sum_{i=1}^{T} \pi_{12}(g_i) \otimes \pi_{23}(h_i) \right).
\]

For every \( i \), we have

\[
\pi_{12}^* g_i \in K^G(\emptyset_D \times F^d_n) \subseteq K^G(F^d_n \times F^d_n \times F^d_n)
\]

and

\[
\pi_{23}^* h_i \in K^G(\overline{\emptyset_N} \times F^d_n) \subseteq K^G(F^d_n \times F^d_n \times F^d_n).
\]

According to Lemma 2.11, the intersection \((F^d_n \times \emptyset_D) \cap (\overline{\emptyset_N} \times F^d_n)\) is irreducible and \( \emptyset \) is an open orbit in it, therefore \((F^d_n \times \emptyset_D) \cap (\overline{\emptyset_N} \times F^d_n) = \emptyset \) and

\[
\pi_{12}^* g_i \otimes \pi_{23}^* h_i \in K^G(\overline{\emptyset}) \subseteq K^G(F^d_n \times F^d_n \times F^d_n).
\]

Furthermore,

\[
f' = \pi_{13} \left( \sum_{i=1}^{T} \pi_{12}(g_i) \otimes \pi_{23}(h_i) \right) \in K^G(\pi_{13}(\overline{\emptyset})) = K^G(\overline{\emptyset_M}) \subseteq K^G(F^d_n \times F^d_n \times F^d_n),
\]

i.e. \( \text{supp}(f') = \overline{\emptyset_M} \). Let \( U := (F^d_n \times F^d_n \times F^d_n) \setminus (F^d_n \times \partial \emptyset_N) \). Then \( U \subseteq F^d_n \times F^d_n \times F^d_n \) is open, and according to Remark 1.11 \((\emptyset_D \times F^d_n) \cap U \) and \((F^d_n \times \overline{\emptyset_N}) \cap U = F^d_n \times \emptyset_N \) intersect transversely in \( U \). Furthermore,

\[
(\emptyset_D \times F^d_n) \cap (F^d_n \times \overline{\emptyset_N}) \cap U = \emptyset.
\]

Since the restriction commutes with the pull-back, and because of the transversality condition, one gets

\[
\left( \sum_{i=1}^{T} \pi_{12}(g_i) \otimes \pi_{23}(h_i) \right)|_\emptyset = \left( \sum_{i=1}^{T} (\pi_{12}(g_i))|_{\emptyset_D \times F^d_n} \otimes (\pi_{23}(h_i))|_{(F^d_n \times \overline{\emptyset_N}) \cap U} \right)|_\emptyset = \sum_{i=1}^{T} \pi_{12}(g_i) \otimes \pi_{23}(h_i)|_\emptyset = \sum_{i=1}^{T} \tilde{\pi}_{12}(g_i) \otimes \tilde{\pi}_{23}(\tilde{h}_i).
\]

Lemma 2.16, Corollary 2.24, and the choice of the factorization \( M = D \circ \pi \) implies that

\[
\pi_{12}^{-1}(\emptyset_D) \cap \pi_{13}^{-1}(\emptyset_M) \cap \pi_{23}^{-1}(\overline{\emptyset_N}) = \emptyset.
\]
Therefore, one gets
\[ f'|_{\mathcal{O}_M} = \left( \sum_{i=1}^{T} g_i \ast h_i \right)|_{\mathcal{O}_M} = \left[ \hat{\pi}_{13} \left( \sum_{i=1}^{T} \pi_{12}^*(g_i) \otimes \pi_{23}^*(h_i) \right) \right]|_{\mathcal{O}_M} \]
\[ = \hat{\pi}_{13} \left[ \left( \sum_{i=1}^{T} \pi_{12}^*(g_i) \otimes \pi_{23}^*(h_i) \right) \right]|_0 \]
\[ = \hat{\pi}_{13} \left( \sum_{i=1}^{T} \pi_{12}^*(g_i) \otimes \pi_{23}^*(h_i) \right) = \tilde{f} = f|_{\mathcal{O}_M}. \]

Therefore, \( \text{supp}(f - f') \subset \text{supp}(f) \setminus \mathcal{O}_M \), which concludes our proof.

3.5. **Local generators.** Let \( \mu = (\mu_1, \ldots, \mu_n) \) be a composition of \( d - 1 \) of length \( n \) and \( 1 \leq k < n \) be an integer. Consider the matrix \( \mathbf{E}(\mu, k) = \text{Diag}(\mu) + E_{k,k+1} \) and the corresponding (closed) orbit \( \mathcal{O}_{\mathbf{E}(\mu, k)} \subset F^d_n \times F^d_n \). We have an isomorphism \( \mathcal{O}_{\mathbf{E}(\mu, k)} \simeq F_\mu \), where \( \mu = (\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_n) \). Let \( T \rightarrow \mathcal{O}_{\mathbf{E}(\mu, k)} \) be the tautological line bundle corresponding to the codimension one step in the flag.

**Definition 3.18.** For any integer \( p \) define the element \( \mathcal{E}_{\mu,k}(p) \in S_0^{\text{aff}}(n, d) \) by
\[ \mathcal{E}_{\mu,k}(p) := [T]^p \in K^G(\mathcal{O}_{\mathbf{E}(\mu, k)}) \subset S_0^{\text{aff}}(n, d). \]

Similarly, for the matrix \( \mathbf{F}(\mu, k) = \text{Diag}(\mu) + E_{k+1,k} \) the corresponding (closed) orbit \( \mathcal{O}_{\mathbf{F}(\mu, k)} \subset F^d_n \times F^d_n \) is also isomorphic to \( F_\mu \). Let \( T \rightarrow \mathcal{O}_{\mathbf{F}(\mu, k)} \) be the tautological line bundle corresponding to the codimension one step in the flag.

**Definition 3.19.** For any integer \( p \) define the element \( \mathcal{F}_{\mu,k}(p) \in S_0^{\text{aff}}(n, d) \) by
\[ \mathcal{F}_{\mu,k}(p) := [S]^p \in K^G(\mathcal{O}_{\mathbf{F}(\mu, k)}) \subset S_0^{\text{aff}}(n, d). \]

We immediately get the following Lemma:

**Lemma 3.20.** One has \( \mathcal{E}_{\mu,k}(p) \ast \mathcal{E}_{\nu,l}(q) = 0 \) unless
\[ C^{\mathbf{E}(\mu, k)} = R^{E_{\nu,l}} \iff \mu + e_{k+1} = \nu + e_l. \]
Similarly
\[ \mu + e_k \neq \nu + e_{l+1} \Rightarrow \mathcal{F}_{\mu,k}(p) \ast \mathcal{F}_{\nu,l}(q) = 0, \]
\[ \mu + e_k \neq \nu + e_l \Rightarrow \mathcal{F}_{\mu,k}(p) \ast \mathcal{E}_{\nu,l}(q) = 0, \]
and
\[ \mu + e_{k+1} \neq \nu + e_{l+1} \Rightarrow \mathcal{E}_{\mu,k}(p) \ast \mathcal{F}_{\nu,l}(q) = 0. \]

**Lemma 3.21.** Let \( 1 \leq k, l < n \) be such that \(|k - l| > 1\), and \( \mu \) be a composition of \( d - 1 \) of length \( n \). Then
\[ \mathcal{E}_{\mu+e_i,k}(p) \ast \mathcal{E}_{\mu+e_{i+1},l}(q) = \mathcal{E}_{\mu+e_{i+1},k}(p) \ast \mathcal{E}_{\mu+e_i,l}(q), \]
and
\[ \mathcal{F}_{\mu+e_i,k}(p) \ast \mathcal{F}_{\mu+e_{i+1},l}(q) = \mathcal{F}_{\mu+e_{i+1},k}(p) \ast \mathcal{F}_{\mu+e_i,l}(q). \]

Also, for any \( 1 \leq k, l < n \) such that \( k \neq l \) one has
\[ \mathcal{E}_{\mu+e_{i+1},k}(p) \ast \mathcal{F}_{\mu+e_{i+1},l}(q) = \mathcal{F}_{\mu+e_k,l}(q) \ast \mathcal{E}_{\mu+e_i,k}(p). \]
**Proof.** All three formulas are proved very similarly. Let us focus on the first one. On the level of supports one immediately gets

\[ \mathbf{E}(\mu + e_i, k) \circ \mathbf{E}(\mu + e_{k+1}, l) = \mathbf{E}(\mu + e_k, l) \circ \mathbf{E}(\mu + e_{l+1}, k) = \text{Diag}(\mu) + E_{k,l+1}. \]

Note that \( \mathcal{O}_{\text{Diag}(\mu) + E_{k,k+1} + E_{l,l+1}} \) is a closed orbit, isomorphic to \( \mathcal{F}_{(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_l, 1, \mu_{l+1}, \ldots, \mu_n)} \). Furthermore, the intersections \( \pi_{12}^{-1}(\mathcal{O}_{\mathbf{E}(\mu + e_{k+1}, l)}) \cap \pi_{23}^{-1}(\mathcal{O}_{\mathbf{E}(\mu + e_k, l)}) \subset \mathcal{F}^d \times \mathcal{F}^d \) and \( \pi_{12}^{-1}(\mathcal{O}_{\mathbf{E}(\mu + e_{l+1}, k)}) \cap \pi_{23}^{-1}(\mathcal{O}_{\mathbf{E}(\mu + e_k, l)}) \) are both also isomorphic to \( \mathcal{F}_{(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_l, 1, \mu_{l+1}, \ldots, \mu_n)} \), and the restrictions of \( \pi_{13} \) to these orbits are isomorphisms to \( \mathcal{O}_{\text{Diag}(\mu) + E_{k,k+1} + E_{l,l+1}} \). Note that according to Remark [1,11] both these intersections are transversal. Finally, both the left hand side and the right hand side of the equation equal to

\[ z_k^p z_l^q \in K^G(\mathcal{O}_{\text{Diag}(\mu) + E_{k,k+1} + E_{l,l+1}}) \subset K^G(\mathcal{F}^d \times \mathcal{F}^d), \]

where \( z_k \) and \( z_l \) are the classes of the tautological line bundles corresponding to parts of size 1 in the composition (\( \mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_l, 1, \mu_{l+1}, \ldots, \mu_n \)) between \( \mu_k \) and \( \mu_{k+1} \), and between \( \mu_l \) and \( \mu_{l+1} \) correspondingly. \( \square \)

In the remaining cases we get the following relations.

**Lemma 3.22.** Let \( 1 \leq k < n \) be an integer and \( \mu \) be a composition of \( d \) of length \( n \). Then one has

\[ \mathcal{E}_{\mu - e_{k+1}, k}(p) \ast \mathcal{E}_{\mu - e_k, k}(q) = -\mathcal{E}_{\mu - e_{k+1}, k}(q - 1) \ast \mathcal{E}_{\mu - e_k, k}(p + 1). \]

**Proof.** Consider the intersection of preimages

\[ X := \pi_{12}^{-1}(\mathcal{O}_{\mathbf{E}(\mu - e_{k+1}, k)}) \cap \pi_{23}^{-1}(\mathcal{O}_{\mathbf{E}(\mu - e_k, l)}) \subset \mathcal{F}^d \times \mathcal{F}^d \times \mathcal{F}^d. \]

Note that according to Remark [1,11] the intersection is transversal. Furthermore, it consists of triples of flags (\( U, V, W \)) satisfying the following conditions.

- Since \( V \in \mathcal{F}_{\mathbf{E}(\mu - e_{k+1}, k)} = \mathcal{F}_k \), we get \( V_i / V_{i-1} = \mu_i \) for all \( 1 \leq i < n \).
- Since \( (U, V) \in \mathcal{O}_{\mathbf{E}(\mu - e_k, k)} \), we get \( U_i = V_i \) for \( i \neq k \) and \( V_k \subset U_k \), dim \( U_k / V_k = 1 \).
- Since \( (V, W) \in \mathcal{O}_{\mathbf{E}(\mu - e_k, l)} \), we get \( V_i = W_i \) for \( i \neq k \) and \( W_k \subset V_k \), dim \( V_k / W_k = 1 \).

One concludes that \( X \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_k - 1, 1, \mu_{k+1} - 1, \ldots, \mu_n)} \). Moreover, one gets

\[ \pi_{12}^* \mathcal{E}_{\mu - e_{k+1}, k}(p) = l_2^p, \]

and

\[ \pi_{23}^* \mathcal{E}_{\mu - e_k, k}(q) = l_2^q, \]

where \( l_1 \) and \( l_2 \) are the classes of the tautological line bundles on \( X \) corresponding to the two one dimensional steps, left to right correspondingly. On concludes that

\[ \mathcal{E}_{\mu - e_{k+1}, k}(p) \ast \mathcal{E}_{\mu - e_k, k}(q) = \pi_{13}^* (l_2^p l_2^q), \]

and

\[ \mathcal{E}_{\mu - e_{k+1}, k}(q - 1) \ast \mathcal{E}_{\mu - e_k, k}(p + 1) = \pi_{13}^* (l_2^{p+1} l_2^{q-1}). \]

The image \( \pi_{13}(X) \subset \mathcal{F}^d \times \mathcal{F}^d \) is obtained from \( X \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_k - 1, 1, \mu_{k+1} - 1, \ldots, \mu_n)} \) by forgetting the middle flag \( V \). Therefore, one gets \( \pi_{13}(X) \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_k - 2, \mu_{k+1} - 1, \ldots, \mu_n)} \), and the projection \( \pi_{13} | X \) is a natural projection between partial flag varieties. The Lemma now follows immediately from Formulas (4) and (5). \( \square \)

**Lemma 3.23.** Let \( 1 \leq k < n - 1 \) be an integer and \( \mu \) be a composition of \( d - 1 \) of length \( n \). Then one has

\[ \mathcal{E}_{\mu + e_k, k+1}(p) \ast \mathcal{E}_{\mu + e_{k+2}, k}(q) = \mathcal{E}_{\mu + e_{k+1}, k+1}(q) \ast \mathcal{E}_{\mu + e_{k+1}, k+1}(p) - \mathcal{E}_{\mu + e_{k+1}, k+1}(q + 1) \ast \mathcal{E}_{\mu + e_{k+1}, k+1}(p - 1). \]
Proof. Consider the intersection of preimages

\[ X := \pi_{12}^{-1}(\mathcal{O}E(\mu + e_{k+1}, k)) \cap \pi_{23}^{-1}(\mathcal{O}E(\mu + e_{k+1}, k+1)) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d. \]

Note that according to Remark 1.11, the intersection is transversal. Furthermore, it consists of triples of flags \((U, V, W)\) satisfying the following conditions:

- Since \(U \in \mathcal{F}_n^d(\mu + e_{k+1}, k) = \mathcal{F}_{\mu + e_{k+1} + e_k}\), we get
  \[ \dim U_i/U_{i-1} = \mu_i \quad \text{for} \quad 1 \leq i < k \quad \text{and} \quad k + 1 < i < n, \]
  \[ \dim U_k/U_{k-1} = \mu_k + 1, \quad \text{and} \quad \dim U_{k+1}/U_k = \mu_{k+1} + 1. \]

- Since \((U, V) \in \mathcal{O}E(\mu + e_{k+1}, k)\) we get \(U_i = V_i\) for \(i \neq k\) and \(V_k \subset U_k\), \(\dim U_k/V_k = 1\).
- Since \((V, W) \in \mathcal{O}E(\mu + e_{k+1}, k+1)\) we get \(V_i = W_i\) for \(i \neq k + 1\) and \(W_{k+1} \subset V_{k+1}\), \(\dim V_{k+1}/W_{k+1} = 1\).

We use the following configuration space notation for this intersection:

\[ X = \{U_0 \subset \ldots \subset U_{k-1} \subset V_k \subset U_k \subset U_{k+1} \subset \ldots \subset U_n\}. \]

Moreover, one gets

\[ \pi_{12}^*(\mathcal{E}_{\mu + e_{k+1}, k}(q)) = l^q, \]

and

\[ \pi_{23}^*(\mathcal{E}_{\mu + e_{k+1}, k+1}(p)) = m^p, \]

where \(l = [U_k/V_k] \in K^G(X)\) and \(m = [V_{k+1}/W_{k+1}] \in K^G(X)\). Note that \(X\) is a smooth manifold. In fact, it is isomorphic to the \(\mathbb{P}_{k+1}\) bundle over a partial flag variety \(X\). The manifold \(X\) consists of two orbits: the closed orbit \(D\) where \(U_k \subset W_{k+1}\), and the open orbit where \(U_k \cap W_{k+1} = V_k\). Note that the closed orbit \(D \subset X\) has codimension one. We conclude that

\[ \pi_{12}^*(\mathcal{E}_{\mu + e_{k+1}, k}(q))\pi_{23}^*(\mathcal{E}_{\mu + e_{k+1}, k+1}(p)) - \pi_{12}^*(\mathcal{E}_{\mu + e_{k+1}, k}(q + 1))\pi_{23}^*(\mathcal{E}_{\mu + e_{k+1}, k+1}(p - 1)) \]

\[ = l^q m^p - l^{q+1} m^{p-1} = l^q m^p (1 - lm^{-1}) \]

The key observation is that the class \(l^q m^p (1 - lm^{-1})\) is supported on the closed orbit \(D \subset X\). Indeed, consider the short exact sequence

\[ 0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0. \]

where \(\mathcal{L}\) is the line bundle of the divisor \(D \subset X\). Then, in the \(K\)-theory we get

\[ [\mathcal{O}_D] = 1 - [\mathcal{L}]^{-1}, \]

Consider the natural map \(\phi : U_k/V_k \rightarrow U_{k+1}/W_{k+1}\) defined as the composition of the embedding \(U_k/V_k \hookrightarrow U_{k+1}/V_k\) and the projection \(U_{k+1}/V_k \rightarrow U_{k+1}/W_{k+1}\). Note that \(D \subset X\) is the divisor of zeros of \(\phi\). We conclude that

\[ [\mathcal{L}] = ml^{-1}, \]

and

\[ [\mathcal{O}_D] = 1 - lm^{-1}. \]
Let $i : D \to X$ be the inclusion map. Denote $\bar{l} := i^*l$ and $\bar{m} := i^*m$. Note that one also has $i^*[L] = \bar{l}/\bar{m}$ and $i_*\pi^*\beta = [O_D]\beta$ for any $\beta \in K^G(X)$. One gets

\[
\begin{align*}
\mathcal{E}_{\mu+k,1,}^*\mathcal{E}_{\mu,0,1,}^*(\bar{q}) + \mathcal{E}_{\mu,0,1,}^*(\bar{q}+1) - \mathcal{E}_{\mu,0,1,}^*(\bar{q}-1) &= \pi_{13*}\left(l^q m^p\right)(\bar{l}^q m^p) \\
&= \pi_{13*}\left(l^q m^p[O_D]\right) \\
&= \pi_{13*}\left(l^q m^p\right) = (\pi_{13}\bar{l})(\bar{m}).
\end{align*}
\]

Let us now consider the convolution product $\mathcal{E}_{\mu+k,1}^*(\bar{q}) \ast \mathcal{E}_{\mu+k+1}^*(\bar{q})$. The intersection of the preimages

$Y := \pi_{12}^{-1}(O_{E(\mu+k+1)}) \cap \pi_{23}^{-1}(O_{E(\mu+k)}) \subset F^d_n \times F^d_n \times F^d_n.$

Note that according to Remark 1.11, the intersection is transversal. Furthermore, it consists of triples of flags $(U, V, W)$ satisfying the following conditions:

- Since $U \in F_{E(\mu+k+1)} = F_{\mu+k+1+e_k}$, we get
  \[\dim U_i/U_{i-1} = \mu_i \text{ for } 1 \leq i < k \text{ and } k+1 < i < n,\]
  \[\dim U_k/U_{k-1} = \mu_k + 1, \text{ and } \dim U_{k+1}/U_k = \mu_{k+1} + 1.\]

- Since $(U, V) \in O_{E(\mu+k+1)}$ we get $U_i = V_i$ for $i \neq k+1$ and $V_{k+1} \subset U_{k+1}$, $\dim U_{k+1}/U_k = 1$.

In other words, one gets

$Y = \{U_0 \subset \ldots \subset U_{k-1} \subset W_k \subset V_k = U_k \subset V_{k+1} \subset U_{k+1} \subset \ldots \subset U_n\},$

and

$\pi_{12}^*\left(E_{\mu+k,1}(\bar{q})\right) \pi_{23}^*\left(E_{\mu+k+1,1}(\bar{q})\right) = \hat{m}^q \hat{m}^p,$

where $\hat{m} = [U_{k+1}/V_{k+1}] \in K^G(Y)$ and $\hat{l} = [V_k/W_k] \in K^G(Y)$. Note that both $Y$ and the closed orbit $D \subset X$ are partial flag varieties. In fact, one gets

$Y \simeq D \simeq F_{(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, 1, \mu_{k+2}, \ldots, \mu_n)},$

and the classes $\bar{l}$ and $\bar{m}$ correspond to $\hat{l}$ and $\hat{m}$ under the natural isomorphism. Moreover, the isomorphism commutes with the restrictions of the projection $\pi_{13} : F^d_n \times F^d_n \times F^d_n \to F^d_n \times F^d_n$ to $Y$ and to $D$ respectively. One concludes that

$(\pi_{13}|D)_*\left(\hat{l}^q \hat{m}^p\right) = (\pi_{13}|Y)_*\left(\bar{l}^q \bar{m}^p\right),$

which completes the proof.

**Lemma 3.24.** Let $1 \leq k \leq n$ and let $\mu$ be a composition of $d$ of length $n$ such that $\mu_{k+1} = 0$. One has

$0 = \mathcal{E}_{\mu,k}(\bar{q}) + \mathcal{E}_{\mu,k-1}(\bar{q}) - \mathcal{E}_{\mu,k}(\bar{q}+1) \ast \mathcal{E}_{\mu,k-1}(\bar{q}-1),$

for any $p, q \in \mathbb{Z}$.

**Remark 3.25.** This Lemma can considered as a special case of Lemma 3.23, formally speaking, in this case the left hand side of Lemma 3.23 is undefined, as it would require flags with steps of negative codimension, but the right hand side is well defined and is equal to zero.

**Proof.** Consider the preimage

$X := \pi_{12}^{-1}(O_{E(\mu+k)}) \cap \pi_{23}^{-1}(O_{E(\mu+k+1)}) \subset F^d_n \times F^d_n \times F^d_n.$

Note that according to Remark 1.11, the intersection is transversal. Furthermore, it consists of triples of flags $(U, V, W)$ satisfying conditions similar to those in the proof of Lemma 3.23.
Lemma 3.27. Let one has

\[ \dim U_{k+1}/U_k = \mu_{k+1} = 0, \]

which forces \( U_k = U_{k+1}. \) One also gets \( \dim W_{k+1}/W_k = 0, \) or \( W_{k+1} = W_k = V_k. \)

\[ X = \{ U_0 \subset \ldots \subset U_{k-1} \subset V_k = W_k = W_{k+1} \subset U_k = U_{k+1} \subset U_{k+2} \subset \ldots \subset U_n \}. \]

In particular,

\[ \pi_1^*(E_{\mu,k}(q)) = l^q, \quad \text{and} \quad \pi_2^*(E_{\mu,k+1}(p)) = l^p, \]

where \( l = [U_k/V_k] = [V_{k+1}/W_{k+1}] \in K^G(X). \) Therefore,

\[ E_{\mu,k}(q) \ast E_{\mu,k+1}(p) - E_{\mu,k}(q+1) \ast E_{\mu,k+1}(p-1) = \pi_{13*}(l^{q+p} - l^{(q+1)+(p-1)}) = 0. \]

\[ \square \]

Similarly, for the \( F \) classes, one gets

Lemma 3.26. Let \( 1 \leq k < n \) be an integer and \( \mu \) be a composition of \( d-1 \) of length \( n. \) Then one has

\[ \mathcal{F}_{\mu+\epsilon_{k+1},k}(p) \ast \mathcal{F}_{\mu+\epsilon_{k},k}(q) = -\mathcal{F}_{\mu+\epsilon_{k+1},k}(q+1) \ast \mathcal{F}_{\mu+\epsilon_{k},k}(p-1). \]

Proof. We skip the proof as it is very similar to the proof of Lemma 3.22.

Lemma 3.27. Let \( 1 \leq k < n-1 \) be an integer and \( \mu \) be a composition of \( d-1 \) of length \( n. \) Then one has

\[ \mathcal{F}_{\mu+\epsilon_{k+2},k}(p) \ast \mathcal{F}_{\mu+\epsilon_{k+1},k+1}(q) = \mathcal{F}_{\mu+\epsilon_{k+1},k+1}(q) \ast \mathcal{F}_{\mu+\epsilon_{k+1},k}(p) - \mathcal{F}_{\mu+\epsilon_{k+1},k+1}(q-1) \ast \mathcal{F}_{\mu+\epsilon_{k+1},k}(p+1). \]

Proof. We skip the proof as it is very similar to the proof of Lemma 3.23.

Lemma 3.28. Let \( 1 \leq k < n \) and let \( \mu \) be a composition of \( d \) of length \( n \) such that \( \mu_{k+1} = 0. \) One has

\[ 0 = \mathcal{F}_{\mu,k+1}(q) \ast \mathcal{F}_{\mu,k}(p) - \mathcal{F}_{\mu,k+1}(q-1) \ast \mathcal{F}_{\mu,k}(p+1), \]

for any \( p, q \in \mathbb{Z}. \)

Proof. We skip the proof as it is very similar to the proof of Lemma 3.24.

Finally, between the \( E \) and \( F \) classes one gets

Lemma 3.29. Let \( 1 \leq k < n \) be an integer and \( \mu \) be a composition of \( d \) of length \( n \) such that

\( \mu_k > 0 \) and \( \mu_{k+1} > 0. \) Then the class

\[ \mathcal{H}_{\mu,k}(p+q) := \mathcal{E}_{\mu-\epsilon_k,k}(p) \ast \mathcal{F}_{\mu-\epsilon_k,k}(q) - \mathcal{F}_{\mu-\epsilon_k,k}(q) \ast \mathcal{E}_{\mu-\epsilon_k,k}(p) \]

is supported on \( \emptyset_{\text{Diag}(\mu)} \subset \mathcal{F}_n \times \mathcal{F}_n \) and only depends on the sum \( p + q. \) Furthermore,

\[ \mathcal{H}_{\mu,k}(p+q) = \begin{cases} 
(1)_{\mu_k}^{-1}[\Lambda_{\mu_k}(T_k)] [\text{Sym}_{(p+q)-\mu_k}(T_{k+1} \oplus T_k^*)], & p + q \leq -\mu_k, \\
0, & -\mu_k < p + q < \mu_{k+1}, \\
(1)_{\mu_k}^{-1}[\Lambda_{\mu_{k+1}}(T_k)], & p + q \geq \mu_{k+1}, 
\end{cases} \]

where \( T_k \) and \( T_{k+1} \) are the tautological bundles over \( \emptyset_{\text{Diag}(\mu)} \simeq \mathcal{F}_n \) corresponding to the \( k \)th and \( (k+1) \)th steps respectively.

Proof. Consider first the product \( \mathcal{E}_{\mu-\epsilon_k,k}(p) \ast \mathcal{F}_{\mu-\epsilon_k,k}(q). \) The intersection of preimages

\[ X := \pi_{12}^{-1}(\emptyset_{\mathcal{E}_{\mu-\epsilon_k,k}}) \cap \pi_{23}^{-1}(\emptyset_{\mathcal{F}_{\mu-\epsilon_k,k}}) \subset \mathcal{F}_n \times \mathcal{F}_n \times \mathcal{F}_n \]

is transversal according to Remark 1.11 and consists of triples of flags \( (U, V, W) \) satisfying the following conditions:

- Since \( U \in \mathcal{F}_r(\mu-\epsilon_k,k) = \mathcal{F}_\mu, \) we get \( \dim U_i/U_{i-1} = \mu_i \) for all \( 1 < i \leq n. \)
- Since \( (U, V) \in \emptyset_{\mathcal{E}_{\mu-\epsilon_k,k}} \) we get \( U_i = V_i \) for \( i \neq k \) and \( V_k \subset U_k, \) \( \dim U_k/V_k = 1. \)
- Since \( (V, W) \in \emptyset_{\mathcal{F}_{\mu-\epsilon_k,k}} \) we get \( V_i = W_i \) for \( i \neq k \) and \( V_k \subset W_k, \) \( \dim W_k/V_k = 1. \)
In other words, we get:

\[ X = \{ U_0 \subset \ldots \subset U_{k-1} \subset V_k \subset U_{k+1} \subset \ldots \subset U_n \} . \]

We also immediately get

\[ \pi_{12}^*(\mathcal{E}_{\mu-e_k}(p))\pi_{23}^*(\mathcal{E}_{\mu-e_k}(q)) = l_1^*l_2^* \in K^G(X) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d) , \]

where \( l_1 \) and \( l_2 \) are the classes of the line bundles \( U_k/V_k \) and \( W_k/V_k \) respectively. Note that \( X \) is a smooth manifold. In fact, it is isomorphic to the \( \mathbb{P}_{\mu_{k+1}} \) bundle over the partial flag variety \( \mathcal{F}_{(\mu_1, \ldots, \mu_{k-1}, 1, \mu_k-1, \ldots, \mu_n)} \). The manifold \( X \) consists of two orbits: the open orbit, where \( U_k \cap W_k = V_k \) and the closed orbit, where \( U_k = W_k \). Note that the closed orbit has codimension \( \mu_{k+1} \) in \( X \).

The image \( T := \pi_{13}(X) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) of \( X \) is obtained by forgetting the middle flag \( V \). One gets

\[ T := \{ U_0 \subset \ldots \subset U_{k-1} \subset U_k \subset U_{k+1} \subset \ldots \subset U_n \} . \]

Here as before \( U_i = W_i \) for \( i \neq k \), \( \text{dim} \ U_i/U_{i-1} = \mu_i \), \( \text{dim} \ W_k = \text{dim} \ U_k \), and either \( W_k = U_k \) or \( W_k \cap U_k \) has codimension 1 in both \( W_k \) and \( U_k \). Note that \( T \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) also consists of two orbits: the closed orbit where \( U_k = W_k \) and the complement to it. However, \( T \) is not necessarily smooth.

Similarly, for the product \( \mathcal{F}_{\mu-e_{k+1}, k}(q) \times \mathcal{E}_{\mu-e_{k+1}, k}(p) \) we get that the intersection of preimages

\[ Y := \pi_{12}^{-1}(\mathcal{O}_{\mathcal{F}_{(\mu-e_{k+1}, k)}}) \cap \pi_{23}^{-1}(\mathcal{O}_{\mathcal{E}_{(\mu-e_{k+1}, k)}}) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \]

is also transversal according to Remark I.11 and consists of triples of flags \( (U, V', W) \) satisfying the following conditions:

- Since \( U \in \mathcal{F}_{(\mu-e_{k+1}, k)} = \mathcal{F}_\mu \), we get \( \text{dim} \ U_i/U_{i-1} = \mu_i \) for all \( 1 < i < \# \).
- Since \( (U, V') \in \mathcal{O}_{\mathcal{F}_{(\mu-e_{k+1}, k)}} \) we get \( U_i = V'_i \) for \( i \neq k \) and \( U_k \subset V'_k \), \( \text{dim} \ V'_k/U_k = 1 \).
- Since \( (V', W) \in \mathcal{O}_{\mathcal{E}_{(\mu-e_{k+1}, k)}} \) we get \( V'_i = W_i \) for \( i \neq k \) and \( W_k \subset V'_k \), \( \text{dim} \ V'_k/W_k = 1 \).

In other words, we get:

\[ Y = \{ U_0 \subset \ldots \subset U_{k-1} \subset U_k \subset V'_k \subset U_{k+1} \subset \ldots \subset U_n \} . \]

We also immediately get

\[ \pi_{12}^*(\mathcal{F}_{\mu-e_{k+1}, k}(q))\pi_{23}^*(\mathcal{E}_{\mu-e_{k+1}, k}(p)) = m_1 \pi_2^* \in K^G(Y) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d) , \]

where \( m_1 \) and \( m_2 \) are the classes of the line bundles \( V'_k/U_k \) and \( V'_k/W_k \) respectively. Note that \( Y \) is also smooth and isomorphic to a \( \mathbb{P}_{\mu_k} \) bundle over the partial flag variety \( \mathcal{F}_{(\mu_1, \ldots, \mu_{k-1}, 1, \mu_k-1, \ldots, \mu_n)} \).

The manifold \( Y \) also consists of two orbits: the open orbit, where \( U_k + W_k = V'_k \) and the closed orbit, where \( U_k = W_k \). The closed orbit has codimension \( \mu_k \) in \( Y \).

Consider the configuration space.
where as before \( \dim U_i/U_{i-1} = \mu_i \) for \( 1 < i \leq n \), and
\[
\dim U_k/V_k = \dim W_k/V_k = \dim V'_k/U_k = \dim V'_k/W_k = 1.
\]

Similar to \( X \) and \( Y \), \( Z \) is a smooth manifold, isomorphic to a \( \mathbb{P}_1 \) bundle over the partial flag manifold \( \mathcal{F}(\mu_1, ..., \mu_k - 1, 1, \mu_{k+1}, ..., \mu_n) \). It also consists of two orbits: \( U_k \neq W_k \) and \( U_k = W_k \). However, in this case the closed orbit is a hypersurface.

We are abusing notations again by recycling notations for \( U, W, V \), and \( V' \). This is justified by the identifications from the following natural projections

\[
Z = \{ \cdots \subset U_{k-1} \subset V_k \subset U_k \subset V'_k \subset U_{k+1} \subset \cdots \}
\]

\[
X = \{ \cdots \subset U_{k-1} \subset V_k \subset U_k \subset V'_k \subset U_{k+1} \subset \cdots \}
\]

\[
Y = \{ \cdots \subset U_{k-1} \subset V_k \subset U_k \subset V'_k \subset U_{k+1} \subset \cdots \}
\]

where \( b_1 \) forgets \( V'_k \) and \( b_2 \) forgets \( V_k \). The restrictions of \( b_1 \) and \( b_2 \) to the open orbit \( \{ U_k \neq W_k \} \) are both isomorphisms to the open orbits in \( X \) and \( Y \) respectively. Indeed, if \( U_k \neq W_k \), then \( V_k = U_k \cap W_k \) and \( V'_k = U_k + W_k \). In fact, these maps are blow ups of \( X \) and \( Y \) respectively along the closed orbits. It follows that both \( b_1 \) and \( b_2 \) are proper birational maps. In particular, \((b_2)_*(b_1)^* = id \) in K-theory for \( j = 1, 2 \).

The compositions \( \pi_{13} \circ b_1 : Z \to T \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) and \( \pi_{13} \circ b_2 : Z \to T \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) coincide. Indeed, both maps forget both \( V'_k \) and \( V_k \). Denote \( p := \pi_{13} \circ b_1 = \pi_{13} \circ b_2 : Z \to \mathcal{F}_n^d \times \mathcal{F}_n^d \). Also, let

\[
\hat{b}_1 := b_1^*(l_1) = [U_k/V_k] \in K^G(Z),
\]
\[
\hat{b}_2 := b_1^*(l_2) = [W_k/V_k] \in K^G(Z),
\]
\[
\hat{m}_1 := b_2^*(m_1) = [V'_k/U_k] \in K^G(Z),
\]
\[
\hat{m}_2 := b_2^*(m_2) = [V'_k/W_k] \in K^G(Z).
\]

Combining the above, we obtain:

\[
\mathcal{E}_{\mu-e_{k+1},k}(p) * \mathcal{E}_{\mu-e_{k+1},k}(q) - \mathcal{E}_{\mu-e_{k+1},k}(q) * \mathcal{E}_{\mu-e_{k+1},k}(p) = \pi_{13*}(\hat{p}_1 \hat{m}_2 - \hat{m}_1 \hat{p}_2)
\]
\[
= \pi_{13*} \left[ b_{1*}(\hat{p}_1 \hat{m}_2) - b_{2*}(\hat{m}_1 \hat{m}_2) \right] = p_* \left( \hat{p}_1 \hat{m}_2 - \hat{m}_1 \hat{m}_2 \right).
\]

The key observation is that the class \( \hat{p}_1 \hat{m}_2 - \hat{m}_1 \hat{m}_2 \in K^G(Z) \) is supported on the closed orbit:

\[
D := \{ \cdots \subset U_{k-1} \subset V'_k \subset U_k = W_k \subset V_k \subset U_{k+1} \subset \cdots \} \subset Z.
\]

Indeed, consider the short exact sequence

\[
0 \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_D \longrightarrow 0.
\]
where $\mathcal{L}$ is the line bundle of the divisor $D \subset Z$. Then, in the $K$-theory we get

$$|\mathcal{O}_D| = 1 - [\mathcal{L}]^{-1},$$

Consider the natural map $\phi_1 : U_k/V_k \to V_k/W_k$ defined as the composition of the embedding $U_k/V_k \hookrightarrow W_k/V_k$ and the projection $W_k/V_k \to V_k'/W_k$. The natural map $\phi_2 : W_k/V_k \to V_k'/U_k$ is constructed in a similar manner. Note that $D \subset Z$ is the divisor of zeros for both maps. We conclude that

$$[\mathcal{L}] = \hat{m}_2 \hat{i}_1^{-1} = \hat{m}_1 \hat{i}_2^{-1},$$

and

$$\hat{p}_1 \hat{q}_2 - \hat{m}_1 \hat{n}_2 = \hat{p}_1 \hat{q}_2 (1 - \left(\frac{\hat{m}_1}{l_2}\right)^q \left(\frac{\hat{m}_2}{l_1}\right)^p) = \hat{p}_1 \hat{q}_2 (1 - [\mathcal{L}]^{p+q})$$

Finally, $1 - \mathcal{L}^{p+q}$ is always divisible by $|\mathcal{O}_D| = 1 - [\mathcal{L}]^{-1}$ in $K^G(Z)$. Let $i : D \to Z$ be the inclusion map. Denote

$$l := i^* l_1 = i^* l_2,$$

$$m := i^* m_1 = i^* m_2,$$

$$\pi := p|_D = p \circ i.$$

Note that one also has $i^*[\mathcal{L}] = m/l$ and $i_* i^* \beta = |\mathcal{O}_D| \beta$ for any $\beta \in K^G(Z)$. One gets

$$p_* \left(\hat{p}_1 \hat{q}_2 (1 - [\mathcal{L}]^{p+q})\right) = p_* \left(\hat{p}_1 \hat{q}_2 \frac{|\mathcal{O}_D|}{1 - [\mathcal{L}]^{-1}} \frac{1 - [\mathcal{L}]^{-(p+q)}}{1 - [\mathcal{L}]^{-1}}\right) = p_* \left(i_* i^* \left[\hat{p}_1 \hat{q}_2 \frac{1 - [\mathcal{L}]^{-(p+q)}}{1 - [\mathcal{L}]^{-1}}\right]\right) = (p_* i_*) \left(\frac{p^{(p+q)}(1 - (m/l)^{p+q})}{1 - (m/l)^{-1}}\right) = \pi_* \left(-m \frac{p^{p+q} - m^{p+q}}{l-m}\right).$$

Note that the image $\pi(D) \subset T \subset \mathcal{F}_n^d \times \mathcal{F}_n^d$ is the closed orbit

$$\pi(D) = \{\ldots \subset U_{k-1} \subset U_k \subset U_{k+1} \subset \ldots\} = 0_{\text{Diag}(\mu)} \simeq \mathcal{F}_\mu,$$

and

$$D = \{\ldots \subset U_{k-1} \subset V_k \subset U_k \subset V_{k+1} \subset U_{k+1} \subset \ldots\} \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_k-1, 1, 1, \mu_{k+1}-1, \ldots, \mu_n)},$$

with the map $\pi$ forgetting both $V_k$ and $V_k'$. One can factor $\pi = \pi^- \circ \pi^+$ where $\pi^-$ forgets $V_k$ and $\pi^+$ forgets $V_k'$, and apply Corollaries 3.9 and 3.10 and Lemma 3.11 to compute $\pi_* (l^r)$ and $\pi_*(m^r)$ for all $r \in \mathbb{Z}$. In order to complete the computations, we need to consider the cases $p + q = 0$, $p + q > 0$, and $p + q < 0$ separately. Recall that $T_k$ and $T_{k+1}$ be the tautological bundles over $0_{\text{Diag}(\mu)} \simeq \mathcal{F}_\mu$ corresponding to the $k$th and $(k+1)$th steps.

(0) Let $p + q = 0$. In this case one gets $-m \frac{p^{p+q} - m^{p+q}}{l-m} = 0$, therefore

$$\mathcal{E}_{\mu-e_k,k}(p) \ast \mathcal{F}_{\mu-e_k,k}(q) = \mathcal{F}_{\mu-e_{k+1},k}(q) \ast \mathcal{E}_{\mu-e_{k+1},k}(p) = 0.$$
(1) Let $p + q > 0$, then one gets
\[
\pi_* \left( -m \frac{lp^q - mp^{q+1}}{l - m} \right) = -\pi_* \left( \frac{lp^{q-1}m + lp^{q-2}m^2 + \ldots + lm^{p+q-1} + m^{p+q}}{l - m} \right)
\]
\[
= (-1)^{\mu_k+1}[\Lambda_{\mu_k+1}(T_{k+1})] \sum_{j=0}^{p+q - \mu_{k+1}} [\text{Sym}_j(T_k)][\text{Sym}_{p+q-\mu_{k+1} - j}(T_{k+1})]
\]
\[
= (-1)^{\mu_k+1}[\Lambda_{\mu_k+1}(T_{k+1})][\text{Sym}_{p+q-\mu_{k+1}}(T_{k+1} \oplus T_k)].
\]

(2) Let $p + q < 0$ and set $s = -(p + q) > 0$. One gets
\[
\pi_* \left( -m \frac{lp^q - mp^{q+1}}{l - m} \right) = -\pi_* \left( \frac{m^s - l^s}{l^s m^{s-1}(l - m)} \right)
\]
\[
= \pi_* \left( \frac{l^{s-1} + l^{s-2}m + \ldots + m^{s-1}}{l^s m^{s-1}} \right)
\]
\[
= \pi_* \left( \frac{l^{1-s}m^{1-s} + l^{-2}m^{2-s} + \ldots + l^{-s}m^{-1} + l^{-s}}{l^{1-s}m^{1-s}} \right)
\]
\[
= (-1)^{\mu_k-1}[\Lambda_{\mu_k}(T^*_k)] \sum_{j=0}^{s-\mu_k} [\text{Sym}_j(T_{k+1})][\text{Sym}_{s-\mu_k-j}(T^*_k)]
\]
\[
= (-1)^{\mu_k-1}[\Lambda_{\mu_k}(T^*_k)][\text{Sym}_{s-\mu_k}(T^*_{k+1} \oplus T^*_k)]
\]
\[
= (-1)^{\mu_k-1}[\Lambda_{\mu_k}(T^*_k)][\text{Sym}_{-(p+q) - \mu_k}(T^*_{k+1} \oplus T^*_k)]
\]

Note that for $0 < p + q < \mu_{k+1}$ one has $[\text{Sym}_{p+q-\mu_{k+1}}(T_{k+1} \oplus T_k)] = 0$, and for $-\mu_k < p + q < 0$ one has $[\text{Sym}_{-(p+q) - \mu_k}(T^*_{k+1} \oplus T^*_k)] = 0$, which concludes the proof.

The next two Lemmas can be considered as special cases of Lemma 3.29.

**Lemma 3.30.** Let $1 \leq k < n$ be an integer and $\mu$ be a composition of $d$ of length $n$ such that $\mu_k = 0$ and $\mu_{k+1} > 0$. Then the class
\[
H_{\mu,k}(p + q) := -\mathcal{F}_{\mu-e_{k+1},k}(q) \ast \mathcal{E}_{\mu-e_{k+1},k}(p)
\]
is supported on $\mathcal{D}_{\text{diag}(\mu)} \subset \mathcal{F}_n^d \times \mathcal{F}_n^d$ and only depends on the sum $p + q$. Furthermore,
\[
H_{\mu,k}(p + q) = \begin{cases} 
-\text{Sym}_{-(p+q)}(T^*_{k+1}), & p + q \leq 0, \\
(1)^{\mu_{k+1} + 1}[\Lambda_{\mu_{k+1}}(T_{k+1})][\text{Sym}_{p+q-\mu_{k+1}}(T_{k+1})], & 0 < p + q < \mu_{k+1},
\end{cases}
\]
where $T_{k+1}$ is the tautological bundles over $\mathcal{D}_{\text{diag}(\mu)}$ $\simeq \mathcal{F}_n$ corresponding to the $(k + 1)$th step of the flag.

**Remark 3.31.** Note that in the case when $\mu_k = 0$ the first term on the right of the relation in Lemma 3.29 is not well defined. Lemma 3.30 is obtained by replacing it with zero.

**Proof.** Consider the intersection of preimages
\[
Y := \pi_{12}^{-1}(\mathcal{O}_{\mathcal{F}(\mu-e_{k+1},k)}) \cap \pi_{23}^{-1}(\mathcal{O}_{\mathcal{E}(\mu-e_{k+1},k)}) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d.
\]
Note that according to Remark 1.11 the intersection is transversal. Similar to Lemma 3.29 we get triples of flags $(U, V', W)$ satisfying the following conditions:

- Since $U \in \mathcal{F}_{\mathcal{R}(\mu-e_{k+1},k)}$, we get $\dim U_i/U_{i-1} = \mu_i$ for all $1 < i \leq n$. In particular, we get $U_k = U_{k-1}$ as $\mu_k = 0$.
- Since $(U, V') \in \mathcal{O}_{\mathcal{F}(\mu-e_{k+1},k)}$ we get $U_i = V'_i$ for $i \neq k$ and $U_k \subset V'_k$, $\dim V'_k/U_k = 1$. 

• Since \((V', W) \in \mathcal{O}_{E(\mu, e_{k+1}, k)}\) we get \(V'_i = W_i\) for \(i \neq k\) and \(W_k \subset V'_k\), \(\dim V'_k/W_k = 1\).

Moreover, we get \(W_k = W_{k-1} = U_{k-1} = U_k\).

In other words, we get:

\[
Y = \{U_0 \subset \ldots \subset U_k = W_k \subset V'_k \subset U_{k+1} \subset \ldots \subset U_n\} \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}/1, \mu_{k+2}, \ldots, \mu_n)}.
\]

Moreover,

\[
\pi_{12}^* (\mathcal{F}_\mu - e_{k+1}) (q) = l^q,
\]

and

\[
\pi_{23}^* (\mathcal{E}_\mu - e_{k+1}) (p) = l^p,
\]

where \(l = [V'_k/U_k] = [V'_k/W_k] \in K^G(Y)\). The projection \(\pi_{13}|_Y\) forgets the flag \(V'\). One gets

\[
\pi_{13}(Y) = \{U_0 \subset \ldots \subset U_k = W_k \subset U_{k+1} \subset \ldots \subset U_n\} = \mathcal{O}_{\text{Diag}(\mu)} \simeq \mathcal{F}_\mu,
\]

and \(\pi_{13}\) is the natural projection of partial flags forgetting \(V'_k\). Applying Corollary 3.9 one gets

\[
-\mathcal{F}_\mu = \mathcal{E}_\mu - e_{k+1} \mathcal{E}_\mu - e_{k+1} = \pi_{13}(p + q).
\]

Lemma 3.32. Let \(1 \leq k < n\) be an integer and \(\mu\) be a composition of \(d\) of length \(n\) such that \(\mu_k > 0\) and \(\mu_{k+1} = 0\). Then the class

\[
\mathcal{H}_{\mu, k}(p + q) := \mathcal{E}_\mu - e_{k, k}(p) \ast \mathcal{F}_\mu - e_{k, k}(q)
\]

is supported on \(\mathcal{O}_{\text{Diag}(\mu)} \subset \mathcal{F}_n^d \times \mathcal{F}_n^d\) and only depends on the sum \(p + q\). Furthermore,

\[
\mathcal{H}_{\mu, k}(p + q) = \left\{\begin{array}{ll}
(1)^{\mu_k - 1} [\Lambda_{\mu_k}(T_k)][\text{Sym}_{(p+q)} - \mu_k(T_k)], & p + q \leq -\mu_k, \\
0, & -\mu_k < p + q < 0,
\end{array}\right.
\]

where \(T_k\) is the tautological bundles over \(\mathcal{O}_{\text{Diag}(\mu)} \simeq \mathcal{F}_\mu\) corresponding to the \(k\)th step of the flag.

Proof. Consider the intersection of preimages

\[
X := \pi_{12}^{-1}(\mathcal{O}_{E(\mu, e_{k+1}, k)}) \cap \pi_{23}^{-1}(\mathcal{O}_{F(\mu, e_{k+1}, k)}) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d.
\]

Note that according to Remark 1.11 the intersection is transversal. Similar to Lemma 3.29, we get triples of flags \((U, V, W)\) satisfying the following conditions:

• Since \(U \subset \mathcal{F}_{W(\mu, e_{k+1}, k)} = \mathcal{F}_\mu\), we get \(\dim U_i/U_{i-1} = \mu_i\) for all \(1 < i \leq n\). In particular, we get \(U_{k+1} = U_k\) as \(\mu_k = 0\).

• Since \((U, V) \subset \mathcal{O}_{E(\mu, e_{k+1}, k)}\) we get \(U_i = V_i\) for \(i \neq k\) and \(V_k \subset U_k\), \(\dim U_k/V_k = 1\).

• Since \((V, W) \subset \mathcal{O}_{F(\mu, e_{k+1}, k)}\) we get \(V_i = W_i\) for \(i \neq k\) and \(V_k \subset W_k\), \(\dim W_k/V_k = 1\).

Moreover, we get \(W_k = W_{k+1} = U_{k+1} = U_k\).

In other words, we get:

\[
X = \{U_0 \subset \ldots \subset U_{k-1} \subset V_k \subset U_k \subset W_k \subset U_{k+1} \subset \ldots \subset U_n\} \simeq \mathcal{F}_{(\mu_1, \ldots, \mu_{k-1}, \mu_k - 1, 1, \mu_{k+1}, \ldots, \mu_n)}.
\]

Moreover,

\[
\pi_{12}^* (\mathcal{E}_\mu - e_{k, k}) (p) = m^p,
\]

and

\[
\pi_{23}^* (\mathcal{F}_\mu - e_{k, k}) (q) = m^q,
\]
where \( m = [U_k/V_k] = [W_k/V_k] \in K^G(X) \). The projection \( \pi_{13}|_X \) forgets the flag \( V \). One gets
\[
\pi_{13}(X) = \{ U_0 \subset \ldots \subset U_k = W_k \subset U_{k+1} \subset \ldots \subset U_n \} = \emptyset_{\text{Diag}(\mu)} \cong \mathcal{F}_\mu,
\]
and \( \pi_{13} \) is the natural projection of partial flags forgetting \( V_k \). Applying Corollary 3.10 one gets
\[
\mathcal{E}_{\mu - e_{k,k}}(p) \star \mathcal{F}_{\mu - e_{k,k}}(q) = \left\{ \begin{array}{ll}
\text{Sym}_{p+q}(T_k), & p + q \geq 0,
0, & -\mu_k < p < 0,
(-1)^{\mu_k - 1}[\Lambda_{\mu_k}(T_k)][\text{Sym}_{-(p+q)-\mu_k}(T_k)], & p + q \leq -\mu_k.
\end{array} \right.
\]

\[\square\]

**Definition 3.33.** Let \( 1 \leq k < n \) be an integer and \( \mu \) be a composition of \( d \) of length \( n \) such that \( \mu_k = \mu_{k+1} = 0 \). Then we set
\[
\mathcal{H}_{\mu,k} := 0 \in K^G(\emptyset_{\text{Diag}(\mu)}) \subset \mathcal{S}_{\text{aff}}^0(n, d).
\]

**Definition 3.34.** Let \( \mu \) be a composition of \( d \) of length \( n \), and let \( r \in \mathbb{Z} \) be an integer. Suppose that \( \mu_n > 0 \), then define
\[
K^G(\emptyset_{\text{Diag}(\mu)}) \ni \mathcal{H}_{\mu,n}(r) := \left\{ \begin{array}{ll}
(-1)^{\mu_n - 1}[\Lambda_{\mu_n}(T_n^*)][\text{Sym}_{-(r-n)-\mu_n}(T_n^*)], & r \leq -\mu_n,
0, & -\mu_n < r < 0,
[\text{Sym}_r(T_n^*)], & r \geq 0.
\end{array} \right.
\]

If \( \mu_n = 0 \) then set
\[
K^G(\emptyset_{\text{Diag}(\mu)}) \ni \mathcal{H}_{\mu,n}(r) = 0.
\]

Consider the embedding \( i : \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_{n+1}^d \times \mathcal{F}_{n+1}^d \) extending every flag \( U \) in the trivial way \( U_{n+1} = U_n \). It is not hard to see that \( i \) is closed and respects the convolution product, therefore one gets that \( i_* : \mathcal{S}_{\text{aff}}^0(n, d) \to \mathcal{S}_{\text{aff}}^0(n+1, d) \) is an embedding of algebras. Then, according to Lemma 3.32, we get
\[
i_*(\mathcal{H}_{\mu,n}(p + q)) = \mathcal{H}_{\bar{\mu},n}(p + q) = \mathcal{E}_{\bar{\mu} - e_{n,n}}(p) \star \mathcal{F}_{\bar{\mu} - e_{n,n}}(q),
\]
where \( \bar{\mu} := (\mu_1, \ldots, \mu_{n-1}, 0) \). This allows one to deal with \( \mathcal{H}_{\mu,n}(r) \) in the same way as with \( \mathcal{H}_{\mu,k}(r) \) for \( 1 \leq k < n \).

Consider the diagonal \( D \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \) consisting of pairs of identical flags: \( D := \{(U, U)\} \). Clearly, \( D \) is closed and invariant, therefore, we got \( K^G(D) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \).

**Lemma 3.35.** The subspace \( K^G(D) \subset K^G(\mathcal{F}_n^d \times \mathcal{F}_n^d) \) is closed under the convolution product and, moreover, the convolution product on \( K^G(D) \) coincides with the usual tensor product.

**Proof.** Note that the intersection of preimages \( X := \pi_{12}^{-1}(D) \cap \pi_{23}^{-1}(D) \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \) is transversal according to Remark 1.11 and consists of triples of identical flags. Therefore, one gets
\[
\pi_{12}^{-1}(D) \cap \pi_{23}^{-1}(D) \simeq D \simeq \mathcal{F}_n^d,
\]
and the restrictions of the projections \( \pi_{12}|_X, \pi_{23}|_X, \text{ and } \pi_{13}|_X \) are all natural isomorphisms. In particular, for any \( \alpha, \beta \in K^G(D) \) one gets
\[
\alpha \star \beta = \pi_{13*}(\pi_{12*}(\alpha) \times \pi_{23*}(\beta)) = \alpha \times \beta.
\]
\[\square\]

One immediately gets the following Corollary:

**Corollary 3.36.** One gets \( \mathcal{H}_{\mu,k}(p) \star \mathcal{H}_{\nu,l}(q) = 0 \) unless \( \mu = \nu \), and
\[
\mathcal{H}_{\mu,k}(p) \star \mathcal{H}_{\nu,l}(q) = \mathcal{H}_{\mu,l}(q) \star \mathcal{H}_{\mu,k}(p).
\]
Proof. Indeed, all the classes $H_{\mu,k}(p)$ are supported on the diagonal $D \subset F^d_n \times F^d_n$. Therefore, the convolution product coincides with the tensor product, which is commutative in $K^G(F^d_n \times F^d_n)$. Also, for the first statement, note that the supports of the classes $H_{\mu,k}(p)$ and $H_{\nu,l}(p)$ are disjoint for $\mu \neq \nu$. \qed

Lemma 3.37. Let $\mu$ be a composition of $d - 1$ of length $n$. One gets the following relations:

\[ H_{\mu + \epsilon_{n-1,n}}(p) \ast E_{\mu,n-1}(q) = E_{\mu,n-1}(q) \ast H_{\mu + \epsilon_{n-1,n}}(p) - E_{\mu,n-1}(q + 1) \ast H_{\mu + \epsilon_{n-1,n}}(p) - 1, \]

for all $p, q \in \mathbb{Z}$.

For all $1 \leq k < n - 1$ and $p, q \in \mathbb{Z}$.

Similarly, for classes $F$:

\[ F_{\mu,n-1}(p) \ast H_{\mu + \epsilon_{n-1,n}}(q) = H_{\mu + \epsilon_{n-1,n}}(q) \ast F_{\mu,n-1}(p) - H_{\mu + \epsilon_{n-1,n}}(q - 1) \ast F_{\mu,n-1}(p) + 1, \]

for all $p, q \in \mathbb{Z}$.

For all $1 \leq k < n - 1$ and $p, q \in \mathbb{Z}$.

Proof. All four formulas are proved in a similar manner using the embedding $i_* : \mathbb{S}^d_0(n, d) \rightarrow \mathbb{S}^d_0(n + 1, d)$ described above, definition of elements $H$ and the relations on the elements $E$ and $F$. Let us prove the first formula. Applying the embedding $i_*$ and the definition of $H_{\mu + \epsilon_{n-1,n}}(p)$ to the left hand side, one gets (note that $d_{n+1} = 0$):

\[ H_{\mu + \epsilon_{n-1,n}}(p) \ast E_{\mu,n-1}(q) = E_{\mu + \epsilon_{n-1,n}}(p) \ast F_{\mu + \epsilon_{n-1,n}}(0) \ast E_{\mu,n-1}(q) \]

\[ = (E_{\mu,n-1}(q) \ast E_{\mu,n-1}(p) - E_{\mu,n-1}(q + 1) \ast E_{\mu,n-1}(p - 1)) \ast F_{\mu,n-1}(0) \]

\[ = (E_{\mu,n-1}(q) \ast (E_{\mu,n-1}(p) \ast F_{\mu,n-1}(0)) - E_{\mu,n-1}(q + 1) \ast (E_{\mu,n-1}(p - 1) \ast F_{\mu,n-1}(0)) \]

\[ = E_{\mu,n-1}(q) \ast (E_{\mu,n-1}(p) \ast F_{\mu,n-1}(0) - E_{\mu,n-1}(q + 1) \ast H_{\mu + \epsilon_{n-1,n}}(p) \ast H_{\mu + \epsilon_{n-1,n}}(p) - 1). \]

Since $i_*$ is injective, this completes the proof. \qed

Lemma 3.38. Let $\mu$ be a composition of $d - 1$ of length $n$, and let $1 \leq k < n$. Then one gets:

\[ H_{\mu + \epsilon_{k,n}}(p) \ast E_{\mu,k}(q) = -E_{\mu,k}(q - 1) \ast H_{\mu + \epsilon_{k,n-1,k}}(p + 1). \]

Proof. Consider the intersection of preimages

\[ X := \pi_{12}^{-1}(O_{Diag(\mu + e_k)}) \cap \pi_{23}^{-1}(O_{E_{\mu,k}}) \subset F^d_n \times F^d_n \times F^d_n. \]

Note that according to Remark 1.11 the intersection is transversal. Furthermore, it consists of triples of flags $(U, U, V)$ satisfying conditions:

- Since $U \in F_{\mu + e_k}$, one gets $\dim V_i / V_{i-1} = \mu_i$ for $i \neq k$, and $\dim U_k / U_{k-1} = \mu_k + 1$.
- Since $(U, V) \in O_{E_{\mu,k}}$ one gets $V_i = U_i$ for $i \neq k$ and $V_k \subset U_k$, $\dim U_k / V_k = 1$.

In other words, one gets $X \simeq F(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_n)$. The restriction of the projection $\pi_{13}$ to $X$ is an isomorphism to $O_{E_{\mu,k}} \simeq F(\mu_1, \ldots, \mu_{k+1}, 1, \mu_{k+2}, \ldots, \mu_n)$.

Similarly, consider the intersection of preimages

\[ Y := \pi_{12}^{-1}(O_{E_{\mu,k}}) \cap \pi_{23}^{-1}(O_{Diag(\mu + e_{k+1})}) \subset F^d_n \times F^d_n \times F^d_n. \]

It consists of triples of flags $(U, V, V)$, where the flags $U$ and $V$ satisfy the same conditions as before. Therefore, one gets $Y \simeq F(\mu_1, \ldots, \mu_k, 1, \mu_{k+1}, \ldots, \mu_n)$, and the restriction of the projection $\pi_{13}$ to $Y$ is also an isomorphism to $O_{E_{\mu,k}} \simeq F(\mu_1, \ldots, \mu_{k+1}, 1, \mu_{k+2}, \ldots, \mu_n)$. 

\[ F^d_n \times F^d_n \times F^d_n. \]
Let $T_k$ and $T_{k+1}$ be the tautological vector bundles on $O_E(\mu_k)$ corresponding to the steps $\mu_k$ and $\mu_{k+1}$ respectively, and let $L$ be the line bundle corresponding to the one dimensional step, so that $[L] = E_{\mu,k}(1) \in K^G(O_E(\mu_k))$. Then one gets

1. If $p \leq -(\mu_k + 1) \iff p + 1 \leq -\mu_k$, then
   \[
   \mathcal{H}_{\mu+e_k,k}(p) \ast E_{\mu,k}(q) = (-1)^{\mu_k}[L]^q[\Lambda_{\mu_k+1}(T_k^* + L^*)][\text{Sym}_{n-(\mu_k+1)}(T_{k+1} \oplus T_k^* + L^*)]
   \]
   \[
   = (-1)^{\mu_k-1}[L]^{-q-1}[\Lambda_{\mu_k}(T_k^*)][\text{Sym}_{n-(\mu_k+1)}(T_{k+1} \oplus T_k^* + L^*)]
   \]
   \[
   = -E_{\mu,k}(q-1) \ast \mathcal{H}_{\mu+e_{k+1},k}(p+1).
   \]

2. If $-(\mu_k + 1) < p < \mu_{k+1} \iff -\mu_k < p + 1 < \mu_{k+1} + 1$, then
   \[
   \mathcal{H}_{\mu+e_k,k}(p) \ast E_{\mu,k}(q) = 0 = -E_{\mu,k}(q+1) \ast \mathcal{H}_{\mu+e_{k+1},k}(p-1).
   \]

3. If $\mu_{k+1} \leq p + 1 \leq \mu_{k+1} + 1$, then
   \[
   \mathcal{H}_{\mu+e_k,k}(p) \ast E_{\mu,k}(q) = (-1)^{\mu_k+1}[L]^q[\Lambda_{\mu_k+1}(T_{k+1})][\text{Sym}_{n-(\mu_k+1)}(T_{k+1} \oplus T_k + L)]
   \]
   \[
   = (-1)^{\mu_k+1-1}[L]^{-q-1}[\Lambda_{\mu_k+1+1}(T_{k+1} \oplus T_k + L)][\text{Sym}_{n-(\mu_k+1)+1}(T_{k+1} \oplus T_k + L)]
   \]
   \[
   = -E_{\mu,k}(q-1) \ast \mathcal{H}_{\mu+e_{k+1},k}(p+1).
   \]

\[\square\]

**Lemma 3.39.** Let $\mu$ be a composition of $d - 1$ of length $n$, and let $1 \leq k < n$. Then one gets:

\[
\mathcal{H}_{\mu+e_{k+1},k}(p) \ast F_{\mu,k}(q) = -F_{\mu,k}(q+1) \ast \mathcal{H}_{\mu+e_{k+1},k}(p-1).
\]

**Proof.** We skip the proof as it is very similar to the proof of Lemma 3.38. \[\square\]

**Theorem 3.40.** The algebra $\mathbb{S}^{\text{aff}}_G(n,d) = K^G(F_n^d \times F_n^d)$ is generated by the classes $E_{\mu,k}(p)$, $F_{\mu,k}(p)$, and $\mathcal{H}_{\nu,n}(p)$, where $\mu$ runs through all compositions of $d - 1$ of length $n$, $\nu$ run through all compositions of $d$ of length $n$, $k$ runs through $\{1, \ldots, n - 1\}$, and $p$ runs through $\mathbb{Z}$.

**Proof.** According to Theorem 3.17 it is enough to show that by combining classes $E_{\mu,k}(p)$, $F_{\mu,k}(p)$, and $\mathcal{H}_{\nu,n}(p)$ one can obtain any class supported on an orbit $O_M$, where $M$ is either diagonal or almost diagonal.

Let us start with the case when $M$ is diagonal. Let $\nu = (\nu_1, \ldots, \nu_n)$ be a composition of $d$. According to equation 3.2 we have an isomorphism

\[
K^G(O_{\text{Diag}(\nu)}) \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^{S_{\nu}} \cong \bigotimes_{k=1}^{n} \Lambda^\pm[x_k],
\]

where $x_k := (x_{(\sum_{i=1}^{k-1} \nu_i)+1}, \ldots, x_{(\sum_{i=1}^{k} \nu_i)})$, and $\Lambda^\pm[x_k]$ is the ring of symmetric Laurent polynomials of $x_k$. We will prove by induction that elements $\mathcal{H}_{\nu,l}(r)$, $r \in \mathbb{Z}$, $1 \leq l \leq n$, generate $\Lambda^\pm[x_k]$ for all $1 \leq k \leq n$ starting the induction from $k = n$ and proceeding in the decreasing order. Note that according to Corollary 3.35 the convolution product of classes supported on the diagonal coincides with the usual tensor product (i.e. the ordinary product of the corresponding partially symmetric functions).

Under the above isomorphism one has $\mathcal{H}_{\nu,n}(r) = h_r(x_n)$ for $r \geq 0$, and

\[
\mathcal{H}_{\nu,n}(-\nu_n) = (-1)^{\nu_n-1}e_{\nu_n}(x_n^{-1}) = \frac{(-1)^{\nu_n-1}}{x_1(x_{(\sum_{i=1}^{k-1} \nu_i)+1}) \cdots x_k(x_{(\sum_{i=1}^{k} \nu_i)})}.
\]

It follows that $\Lambda^\pm(x_n)$ is generated by $\mathcal{H}_{\nu,p}(r)$, $r \in \{-\mu_n, 0, 1, \ldots\}$.

Suppose now that we proved that $\Lambda^\pm[x_i]$ are generated by $\mathcal{H}_{\nu,l}(r)$ for $i > k$. Then for $r \geq \nu_{k+1}$ one has

\[
\mathcal{H}_{\nu,k}(r) = (-1)^{\nu_{k+1}}e_{\nu_{k+1}}(x_{k+1})h_{r-\nu_{k+1}}(x_k, x_{k+1}),
\]
and for \( r = -\nu_k \) one has
\[
\mathcal{H}_{\nu,k}(-\nu_k) = (-1)^{\nu_k-1}e_{\nu_k}(x_k^{-1}).
\]
Since \( \Lambda^\pm[x_{k+1}] \) is already proven to be generated by elements \( \mathcal{H}_{\nu,l}(r) \) and
\[
e_{\nu_k+\nu_{k+1}}(x_k^{-1}, x_{k+1}^{-1}) = e_{\nu_k}(x_k^{-1})e_{\nu_{k+1}}(x_{k+1}^{-1}),
\]
we conclude that \( \Lambda^\pm[x_k, x_{k+1}] \) is generated too. Finally, we use the elementary fact that \( \Lambda^\pm[x_k, x_{k+1}] \) and \( \Lambda^\pm[x_{k+1}] \) together generate \( \Lambda^\pm[x_k] \times \Lambda^\pm[x_{k+1}] \).

Let now \( \mu = (\mu_1, \ldots, \mu_n) \) be a composition of \( d-1 \) and \( 1 \leq k < n \) be an integer. Consider the orbit \( \mathcal{O}_{E(\mu,k)} \cong \mathcal{F}(\mu_1, \ldots, \mu_k, \mu_{k+1}, \ldots, \mu_n) \). According to equation \((5.2)\) we have an isomorphism
\[
K^G(\mathcal{O}_{E(\mu,k)}) \cong \mathbb{C}[x_1^{\mp 1}, \ldots, x_d^{\mp 1}, y_1^{\pm 1}] S_{\mu_1, \ldots, \mu_n, 1} \cong \left( \bigotimes_{k=1}^n \Lambda^\pm[x_k] \right) \otimes \Lambda^\pm[y],
\]
where for each \( k \) the group of variables \( x_k \) corresponds to the step of dimension \( \mu_k \), and \( y \) corresponds to the off diagonal entry 1 in \( E(\mu, k) \), i.e. \( y = E_{\mu, k}(1) \). We get that the subspace \( \Lambda^\pm[y] \subset K^G(\mathcal{O}_{E(\mu,k)}) \) is generated by \( E_{\mu, k}(p), p \in \mathbb{Z} \). Let
\[
\nu = \mu + e_k = (\mu_1, \ldots, \mu_k + 1, \ldots, \mu_n),
\]
and consider the convolution product
\[
\ast : K^G(\mathcal{O}_{E(\mu,k)}) \otimes \Lambda^\pm[y] \to K^G(\mathcal{O}_{E(\mu,k)}).
\]
Note that similar to the proof of Lemma \((3.38)\), the restriction of the projection \( \pi_{13} : \mathcal{F}_n^d \times \mathcal{F}_n^d \times \mathcal{F}_n^d \to \mathcal{F}_n^d \times \mathcal{F}_n^d \) to the intersection of the preimages \( \pi_{12}^{-1}(\mathcal{O}_{Diag(n)}) \cap \pi_{23}^{-1}(\mathcal{O}_{E(\mu,k)}) \) is an isomorphism to \( \mathcal{O}_{E(\mu,k)} \subset \mathcal{F}_n^d \times \mathcal{F}_n^d \). Therefore, in terms of the partial symmetric functions the convolution product is simply the multiplication map:
\[
\left( \Lambda^\pm[x_k, y] \otimes \bigotimes_{i \neq k} \Lambda^\pm[x_i] \right) \otimes \Lambda^\pm[y] \to \left( \bigotimes_{k=1}^n \Lambda^\pm[x_k] \right) \otimes \Lambda^\pm[y],
\]
which is clearly surjective.

The case when \( M \) is a lower-triangular almost diagonal matrix is done in a similar manner.

\[
4. \ \text{THE MAP } \phi_d : \mathcal{L}_n \to \mathcal{S}_0^{\text{aff}}(n, d)
\]

4.1. Definition of the map \( \phi_d \). Recall that we use notations \( \mu \vdash d, l(\mu) = n \) for a composition of \( d \) of length \( n \). All compositions in this section are of length \( n \).

**Definition 4.1.** Fix positive integers \( n \) and \( d \), and for \( 1 \leq k < n \) and \( p \in \mathbb{Z} \) set
\[
\mathcal{E}_k(p) := \sum_{\mu = d-1}^{\mu = \nu_k} \mathcal{E}_{\mu,k}(p),
\]
and
\[
\mathcal{F}_k(p) := \sum_{\mu = d-1}^{\mu = \nu_k} \mathcal{F}_{\mu,k}(p).
\]
Also, for \( 1 \leq k \leq n \) set
\[
\mathcal{H}_k(p) := \sum_{\nu = d}^{\nu = \mu_k} \mathcal{H}_{\nu,k}(p).
\]

**Theorem 4.2.** The map sending \( \mathcal{E}_k(p) \to \mathcal{E}_k(p) \), \( \mathcal{F}_k(p) \to \mathcal{F}_k(p) \), and \( \mathcal{H}_n(p) \to \mathcal{H}_n(p) \) for all \( 1 \leq k < n \) and \( p \in \mathbb{Z} \) extends to a well defined homomorphism of algebras \( \phi_d : \mathcal{L}_n \to \mathcal{S}_0^{\text{aff}}(n, d) \).
Proof. All we need to do is to check that the elements $\mathcal{E}_k(p)$, $\mathcal{F}_k(p)$, and $\mathcal{H}_n(p)$ satisfy the relations on the generators $E_k(p)$, $F_k(p)$, and $H_n(p)$ of the algebra $\mathfrak{U}_n$. This is done by a direct application of the relations on the local generators $\mathcal{E}_{\mu,k}(p)$, $\mathcal{F}_{\mu,k}(p)$, and $\mathcal{H}_{\nu,n}(p)$ developed in the previous section.

- Let us start with relation 1.1. We use Lemmas 3.20 and 3.22 to get

$$
\mathcal{E}_k(p) \ast \mathcal{E}_k(q) = \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k}(p) \ast \sum_{\nu \vdash d-1} \mathcal{E}_{\nu,k}(q) = \sum_{\mathcal{E}(\mu,k) = \mathcal{E}(\nu,k)} \mathcal{E}_{\mu,k}(p) \ast \mathcal{E}_{\nu,k}(q) = \sum_{\mu+\nu e_k = \nu+e_k} \mathcal{E}_{\mu,k}(p) \ast \mathcal{E}_{\nu,k}(q) = \sum_{\mu \vdash d, \mu_k > 0, \mu_{k+1} > 0} \mathcal{E}_{\mu-e_{k+1},k}(q-1) \ast \mathcal{E}_{\mu-e_{k+1},k}(p+1) = -\mathcal{E}_k(q-1) \ast \mathcal{E}_k(p+1).
$$

- For Relation 1.2 we start by using Lemmas 3.20 and 3.23

$$
\mathcal{E}_{k+1}(p) \ast \mathcal{E}_k(q) = \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k+1}(p) \ast \sum_{\nu \vdash d-1} \mathcal{E}_{\nu,k}(q) = \sum_{\mathcal{E}(\mu,k+1) = \mathcal{E}(\nu,k)} \mathcal{E}_{\mu,k+1}(p) \ast \mathcal{E}_{\nu,k}(q) = \sum_{\mu+\nu e_k = \nu+e_k} \mathcal{E}_{\mu,k+1}(p) \ast \mathcal{E}_{\nu,k}(q) = \sum_{\mu \vdash d-2} \mathcal{E}_{\mu+e_{k+1},k+1}(q) \ast \mathcal{E}_{\mu+e_{k+1},k+1}(p) = -\mathcal{E}_{\mu+e_{k+1},k+1}(q+1) \ast \mathcal{E}_{\mu+e_{k+1},k+1}(p-1).
$$

Recall that according to Lemma 3.24 one has

$$
\mathcal{E}_{\mu,k}(q) \ast \mathcal{E}_{\mu,k+1}(p) - \mathcal{E}_{\mu,k}(q+1) \ast \mathcal{E}_{\mu,k+1}(p-1) = 0,
$$

for any composition $\mu \vdash d-1$ such that $\mu_{k+1} = 0$. In particular, one gets

$$
\sum_{\mu \vdash d-1, \mu_{k+1} = 0} (\mathcal{E}_{\mu,k}(q) \ast \mathcal{E}_{\mu,k+1}(p) - \mathcal{E}_{\mu,k}(q+1) \ast \mathcal{E}_{\mu,k+1}(p-1)) = 0.
$$

Adding this, one continues:

$$
\sum_{\mu \vdash d-2} (\mathcal{E}_{\mu+e_{k+1},k+1}(q) \ast \mathcal{E}_{\mu+e_{k+1},k+1}(p) - \mathcal{E}_{\mu+e_{k+1},k+1}(q+1) \ast \mathcal{E}_{\mu+e_{k+1},k+1}(p-1))
$$

$$
= \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k}(q) \ast \mathcal{E}_{\mu,k+1}(p) - \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k}(q+1) \ast \mathcal{E}_{\mu,k+1}(p-1)
$$

$$
= \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k}(q) \ast \mathcal{E}_{\mu,k+1}(p) - \sum_{\mu \vdash d-1} \mathcal{E}_{\mu,k}(q+1) \ast \mathcal{E}_{\mu,k+1}(p-1)
$$

$$
= \sum_{\mathcal{E}(\mu,k) = \mathcal{E}(\nu,k+1)} \mathcal{E}_{\mu,k}(q) \ast \mathcal{E}_{\nu,k+1}(p) - \sum_{\mathcal{E}(\mu,k) = \mathcal{E}(\nu,k+1)} \mathcal{E}_{\mu,k}(q+1) \ast \mathcal{E}_{\nu,k+1}(p-1)
$$

$$
= \mathcal{E}_k(q) \ast \mathcal{E}_{k+1}(p) - \mathcal{E}_k(q+1) \ast \mathcal{E}_{k+1}(p-1).
$$
For Relation 3.3 we use Lemmas 3.20 and 3.21. Assuming that \(|l - k| > 1\) one gets:

\[
\mathcal{E}_l(p) \star \mathcal{E}_k(q) = \sum_{\mu \in d - 1} \mathcal{E}_{\mu,l}(p) \star \sum_{\nu \in d - 1} \mathcal{E}_{\nu,k}(q) = \sum_{\mu \in d - 1} \mathcal{E}_{\mu,l}(p) \star \mathcal{E}_{\nu,k}(q)
\]

\[
= \sum_{\mu + e_{l+1} = \nu + e_k} \mathcal{E}_{\mu,l}(p) \star \mathcal{E}_{\nu,k}(q) = \sum_{\mu \in d - 2} \mathcal{E}_{\mu + e_{l+1},l}(p) \star \mathcal{E}_{\mu + e_{l+1},k}(q)
\]

\[
= \sum_{\mu \in d - 2} \mathcal{E}_{\mu + e_{l+1},k}(q) \star \mathcal{E}_{\mu + e_{l+1},l}(p) = \mathcal{E}_k(q) \star \mathcal{E}_l(p).
\]

Relations 2.1, 2.2, and 2.3 are proved in a way similar to Relations 1.1, 1.2, and 1.3 above.

For Relation 3.1 we use Lemmas 3.20 and 3.21:

\[
\mathcal{E}_l(p) \star \mathcal{F}_k(q) = \sum_{\mu \in d - 1} \mathcal{E}_{\mu,l}(p) \star \sum_{\nu \in d - 1} \mathcal{F}_{\nu,k}(q) = \sum_{\mu \in d - 1} \mathcal{E}_{\mu,l}(p) \star \mathcal{F}_{\nu,k}(q)
\]

\[
= \sum_{\mu + e_{l+1} = \nu + e_k} \mathcal{E}_{\mu,l}(p) \star \mathcal{F}_{\nu,k}(q) = \sum_{\mu \in d - 2} \mathcal{E}_{\mu + e_{l+1},l}(p) \star \mathcal{F}_{\mu + e_{l+1},k}(q)
\]

\[
= \sum_{\mu \in d - 2} \mathcal{F}_{\mu + e_{l+1},k}(q) \star \mathcal{E}_{\mu + e_{l+1},l}(p) = \mathcal{F}_k(q) \star \mathcal{E}_l(p).
\]

To prove Relations 3.2 and 4.1 it is enough to show that for all \(1 \leq k < n\) and \(p, q \in \mathbb{Z}\) one has

\[
\mathcal{H}_k(p) = \mathcal{F}_k(q) \star \mathcal{E}_k(p) = \mathcal{H}_k(p + q).
\]

Indeed, \(\mathcal{H}_k(p+q)\) clearly only depends on the sum \(p+q\), and the classes \(\mathcal{H}_k(r)\) commute with each other as sums of classes \(\mathcal{H}_{\mu,k}(r)\), which commute by Lemma 3.36. Using Lemma 3.20 one gets:

\[
\mathcal{E}_k(p) \star \mathcal{F}_k(q) - \mathcal{F}_k(q) \star \mathcal{E}_k(p) = \sum_{\mu \in d - 1} \mathcal{E}_{\mu,k}(p) \star \sum_{\nu \in d - 1} \mathcal{F}_{\nu,k}(q) - \sum_{\mu \in d - 1} \mathcal{F}_{\mu,k}(q) \star \sum_{\nu \in d - 1} \mathcal{E}_{\nu,k}(q)
\]

\[
= \sum_{\mathcal{E}^{(\mu,k)} = \mathcal{F}^{(\nu,k)}} \mathcal{E}_{\mu,k}(p) \star \mathcal{F}_{\nu,k}(q) - \sum_{\mathcal{E}^{(\mu,k)} = \mathcal{F}^{(\nu,k)}} \mathcal{F}_{\mu,k}(q) \star \mathcal{E}_{\nu,k}(q)
\]

\[
= \sum_{\mu + e_{l+1} = \nu + e_k} \mathcal{E}_{\mu,k}(p) \star \mathcal{F}_{\nu,k}(q) - \sum_{\mu + e_{l+1} = \nu + e_k} \mathcal{F}_{\mu,k}(q) \star \mathcal{E}_{\nu,k}(q)
\]

\[
= \sum_{\mu \in d - 1} \mathcal{E}_{\mu,k}(p) \star \mathcal{F}_{\mu,k}(q) - \sum_{\mu \in d - 1} \mathcal{F}_{\mu,k}(q) \star \mathcal{E}_{\mu,k}(q)
\]

\[
= \sum_{\mu \in d, \mu_k > 0, \mu_{k+1} > 0} \left( \mathcal{E}_{\mu - e_k,k}(p) \star \mathcal{F}_{\mu - e_k,k}(q) - \mathcal{F}_{\mu - e_k,k}(q) \star \mathcal{E}_{\mu - e_k,k}(q) \right)
\]

\[
+ \sum_{\mu \in d, \mu_k > 0, \mu_{k+1} = 0} \mathcal{E}_{\mu - e_k,k}(p) \star \mathcal{F}_{\mu - e_k,k}(q)
\]

\[
- \sum_{\mu \in d, \mu_k = 0, \mu_{k+1} > 0} \mathcal{F}_{\mu - e_k,k}(q) \star \mathcal{E}_{\mu - e_k,k}(q)
\]
Note that by Definition 3.3 one has $\mathcal{H}_{\mu,k}(p + q) = 0$ whenever $\mu_k = \mu_{k+1} = 0$. Using that and applying Lemmas 3.20, 3.29, and 3.30 we continue:

$$
\begin{align*}
&= \sum_{\mu_k > 0, \mu_{k+1} > 0} \mathcal{H}_{\mu,k}(p + q) + \sum_{\mu_k > 0, \mu_{k+1} = 0} \mathcal{H}_{\mu,k}(p + q) \\
&\quad + \sum_{\mu_k = 0, \mu_{k+1} > 0} \mathcal{H}_{\mu,k}(p + q) + \sum_{\mu_k = \mu_{k+1} = 0} \mathcal{H}_{\mu,k}(p + q) \\
&= \sum_{\mu \geq d} \mathcal{H}_{\mu,k}(p + q) = \mathcal{H}_k(p + q).
\end{align*}
$$

- For the Relation 5.1 we use Lemmas 3.20 and 3.37. One gets:

$$
\mathcal{H}_n(p) \ast \mathcal{E}_{n-1}(q) = \sum_{\nu \geq d} \mathcal{H}_{\nu,n}(p) \ast \sum_{\mu \geq d-1} \mathcal{E}_{\mu,n-1}(q) = \sum_{\nu \geq \mathbb{R}(\mu,n-1)} \mathcal{H}_{\nu,n}(p) \ast \mathcal{E}_{\mu,n-1}(q)
$$

$$
= \sum_{\mu \geq d-1} \mathcal{H}_{\mu+e_{n-1,n}}(p) \ast \mathcal{E}_{\mu,n-1}(q)
$$

$$
= \sum_{\mu \geq d-1} \left( \mathcal{E}_{\mu,n-1}(q) \ast \mathcal{H}_{\mu+e_{n-1,n}}(p) - \mathcal{E}_{\mu,n-1}(q + 1) \ast \mathcal{H}_{\mu+e_{n-1,n}}(p - 1) \right)
$$

$$
= \sum_{\mu \geq d-1} \left( \mathcal{E}_{\mu,n-1}(q) \ast \mathcal{H}_{\nu,n}(p) - \mathcal{E}_{\mu,n-1}(q + 1) \ast \mathcal{H}_{\nu,n}(p - 1) \right)
$$

$$
= \mathcal{E}_{n-1}(q) \ast \mathcal{H}_n(p) - \mathcal{E}_{n-1}(q + 1) \ast \mathcal{H}_n(p - 1)
$$

- For Relation 5.2, we use Lemmas 3.20 and 3.38. One gets:

$$
\mathcal{H}_k(p) \ast \mathcal{E}_k(q) = \sum_{\nu \geq d} \mathcal{H}_{\nu,k}(p) \ast \sum_{\mu \geq d-1} \mathcal{E}_{\mu,k}(q) = \sum_{\nu \geq \mathbb{R}(\mu,k)} \mathcal{H}_{\nu,k}(p) \ast \mathcal{E}_{\mu,k}(q)
$$

$$
= \sum_{\mu \geq d-1} \mathcal{H}_{\mu+e_{k,k}}(p) \ast \mathcal{E}_{\mu,k}(q) = \sum_{\mu \geq d-1} -\mathcal{E}_{\mu,k}(q - 1) \ast \mathcal{H}_{\mu+e_{k+1,k}}(p + 1)
$$

$$
= \sum_{\mathbb{C}(\mu,k) = \nu} -\mathcal{E}_{\mu,k}(q - 1) \ast \mathcal{H}_{\nu,k}(p + 1) = \sum_{\mu \geq d-1} -\mathcal{E}_{\mu,k}(q - 1) \ast \sum_{\nu \geq d} \mathcal{H}_{\nu,k}(p + 1)
$$

$$
= -\mathcal{E}_k(q - 1) \ast \mathcal{H}_k(p + 1).
$$

- For Relation 5.3, we use Lemmas 3.20 and 3.37. For any $1 \leq k < n - 1$ one gets:

$$
\mathcal{H}_n(p) \ast \mathcal{E}_k(q) = \sum_{\nu \geq d} \mathcal{H}_{\nu,n}(p) \ast \sum_{\mu \geq d} \mathcal{E}_{\mu,k}(q) = \sum_{\nu \geq \mathbb{R}(\mu,k)} \mathcal{H}_{\nu,n}(p) \ast \mathcal{E}_{\mu,k}(q)
$$

$$
= \sum_{\mu \geq d-1} \mathcal{H}_{\mu+e_{k,n}}(p) \ast \mathcal{E}_{\mu,k}(q) = \sum_{\mu \geq d-1} \mathcal{E}_{\mu,k}(q) \ast \mathcal{H}_{\mu+e_{k+1,n}}(p)
$$

$$
= \sum_{\mathbb{C}(\mu,k) = \nu} \mathcal{E}_{\mu,k}(q) \ast \mathcal{H}_{\nu,n}(p) = \sum_{\mu \geq d-1} \mathcal{E}_{\mu,k}(q) \ast \sum_{\nu \geq d} \mathcal{H}_{\nu,n}(p) = \mathcal{E}_k(q) \ast \mathcal{H}_k(p).
$$

- We skip the proofs of Relations 6.1, 6.2, and 6.3 as those very similar to Relations 5.1, 5.2, and 5.3 respectively.
4.2. Surjectivity of $\phi_d$.

**Theorem 4.3.** The homomorphism $\phi_d : \mathfrak{U}_n \to \mathbb{S}_{0}^{\text{aff}}(n, d)$ is surjective.

**Proof.** Fix a composition $\mu \subseteq d$, of length $n$. According to the proof of Theorem 3.40, classes $H_{\mu,k}(p)$, $1 \leq k \leq n$, $p \in \mathbb{Z}$, generate $K^{G}(O_{\text{Diag}(\mu)}) \subset \mathbb{S}_{0}^{\text{aff}}(n, d)$. It then follows that there exists an element $a_{\mu} \in \phi_{d}(\mathfrak{U}_n)$ supported on the union of orbits corresponding to the diagonal matrices, and such that $a_{\mu}|_{O_{\text{Diag}(\mu)}} = [\det V]$. (Note that $\det V$ is trivial as an ordinary vector bundle, but carries a non-trivial action of $G$.) In fact, $a_{\mu}$ can be expressed as a polynomial in $H_{k}(p)$, $1 \leq k \leq n$, $p \in \mathbb{Z}$. Recall that according to Lemma 3.15, the convolution product of the elements supported on the diagonal coincides with the usual tensor product.

Consider the element

$$b_{\mu} := H_{1}(-\mu_{1}) \times H_{2}(-\mu_{2}) \times \ldots \times H_{n}(-\mu_{n}) \in \phi_{d}(\mathfrak{U}_n) \subset \mathbb{S}_{0}^{\text{aff}}(n, d).$$

We claim that $b_{\mu}$ is supported on $O_{\text{Diag}(\mu)}$. Indeed, it is clear that $b_{\mu}$ is supported on the orbits corresponding to the diagonal matrices. Let $\nu \not\subseteq d$ be a composition of length $n$, different from $\mu$. Then there exists $1 \leq k \leq n$ such that $\nu_k > \mu_k$. Therefore,

$$H_{k}(-\mu_{k})|_{O_{\text{Diag}(\nu)}} = H_{\nu,k}(-\mu_{k}) = 0,$$

and

$$b_{\mu}|_{O_{\text{Diag}(\nu)}} = 0.$$

Furthermore,

$$b_{\mu}|_{O_{\text{Diag}(\nu)}} = \prod_{k=1}^{n} [\det T_{k}^{*}] = [\det V^{*}] = [\det V]^{-1},$$

where for every $1 \leq k \leq n$, $T_{k}$ is the tautological vector bundle on $O_{\text{Diag}(\mu)} \simeq F_{\mu}$ corresponding to the $k$th step. We conclude that the product $a_{\mu} \times b_{\mu}$ is supported on $O_{\text{Diag}(\mu)}$ and that $(a_{\mu} \times b_{\mu})|_{O_{\text{Diag}(\mu)}} = 1$. Furthermore, for any composition $\mu \subseteq d$ of length $n$ and any $p \in \mathbb{Z}$ on gets

$$(a_{\mu} \times b_{\mu}) \ast H_{n}(p) = H_{\mu,n}(p) \in \phi_{d}(\mathfrak{U}_n),$$

and for any composition $\mu \subseteq d - 1$ of length $n$, any $1 \leq k < n$, and any $p \in \mathbb{Z}$ on gets

$$(a_{\mu+e_k} \times b_{\mu+e_k}) \ast E_{k}(p) = E_{\mu,k}(p) \in \phi_{d}(\mathfrak{U}_n),$$

$$(a_{\mu+e_{k+1}} \times b_{\mu+e_{k+1}}) \ast F_{k}(p) = F_{\mu,k}(p) \in \phi_{d}(\mathfrak{U}_n).$$

We apply Theorem 3.40 to conclude the proof. \qed

5. APPENDIX: PROOF OF THEOREM 2.7

Here we present an elementary proof of Theorem 2.7. Recall the formulation:

**Theorem 2.7.** Let $N = \{n_{ij}\}$ and $M = \{m_{ij}\}$ be two non-negative integer matrices. Then $N \preceq M$ if and only if the following two conditions are satisfied:

1. One has $R^{M} = R^{N}$ and $C^{M} = C^{N}$.
2. For every $k$, $l$ one has $\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}$.

Moreover, one can also characterize the cover relations: one has $N \preceq M$ if and only if there exist four integers $1 \leq a_{1} < a_{2} \leq n$ and $1 \leq b_{1} < b_{2} \leq n$ such that the following two conditions are satisfied:

1. $M - N = E_{a_{1}b_{2}} + E_{a_{2}b_{1}} - E_{a_{1}b_{1}} - E_{a_{2}b_{2}}$, where $E_{ij}$ is the matrix with all entries equal to zero, except for the entry on the $i$th row and the $j$th column, which is equal to 1.
(ii) \( m_{ij} = n_{ij} = 0 \) for \( a_1 \leq i \leq a_2 \) and \( b_1 \leq j \leq b_2 \) except for
\[
(i, j) \in \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}.
\]

**Proof.** Suppose that \( N \leq M \). Then \( R^M = R^N \) and \( C^M = C^N \), because \( O_M \) and \( O_N \) belong to the same connected component. Also, for any \( 1 \leq k, l \leq n \) and any \( (U, W) \in O_M \) one has
\[
\dim(U_k \cap W_l) = \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}.
\]
As a family of pairs of flags approaches the boundary of the orbit \( O_M \), the dimension of the intersection can only increase. Therefore, one gets
\[
\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}.
\]
Suppose now that the matrices \( M \) and \( N \) satisfy condition (i) and \( (U, W) \in O_M \). Let \( B = \bigcup_{1 \leq i,j \leq n} B_{ij} \) be a basis of \( V \) such that for any \( 1 \leq k, l \leq n \) one has
\[
U_k \cap W_l = \text{span}(\bigcup_{i=1}^{k} \bigcup_{j=1}^{l} B_{ij}).
\]
In particular, one gets \( m_{ij} = \sharp B_{ij} \). We know that \( m_{a_1b_2} \) and \( m_{a_2b_1} \) are positive, therefore \( B_{a_1b_2} \) and \( B_{a_2b_1} \) are non-empty. Let \( v \in B_{a_1b_2} \) and \( u \in B_{a_2b_1} \). Consider the family of partial flags
\[
W_k(t) = \text{span}\left(\left[\bigcup_{i=1}^{n} \bigcup_{j=1}^{k} B_{ij} \setminus \{u\}\right] \cup \{u + t(v - u)\}\right),
\]
and \( W_j(t) = W_j \) for all \( j \notin \{b_1, \ldots, b_2 - 1\} \). One gets \( W(0) = W \), and \( (U, W(t)) \in O_M \) for \( 0 \leq t \leq 1 \). Indeed, for \( 1 \leq t < 1 \) the set \( B \setminus \{u\} \cup \{u + t(v - u)\} \) is a basis, and the dimensions of intersections of the elements of the flags \( U \) and \( W(t) \) remain the same. However, at \( t = 1 \) it is not a basis anymore. In fact, one gets \( (U, W(t)) \in O_N \). Indeed, consider another decomposition of the same basis
\[
B = \bigcup_{1 \leq i,j \leq n} \hat{B}_{ij},
\]
where
\[
\hat{B}_{a_1b_1} = B_{a_1b_1} \cup \{v\},
\hat{B}_{a_2b_2} = B_{a_2b_2} \cup \{u\},
\hat{B}_{a_1b_2} = B_{a_1b_2} \setminus \{v\},
\hat{B}_{a_2b_1} = B_{a_2b_1} \setminus \{u\},
\]
and \( \hat{B}_{ij} = B_{ij} \) for \( (ij) \notin \{(a_1b_1), (a_1b_2), (a_2b_1), (a_2b_2)\} \). Then one gets
\[
U_k \cap W_i(1) = \text{span}(\bigcup_{i=1}^{k} \bigcup_{j=1}^{l} \hat{B}_{ij}),
\]
and \( n_{ij} = \sharp \hat{B}_{ij} \) for all \( 1 \leq i, j \leq n \). We conclude that \( N \prec M \).

Suppose now that \( M \) and \( N \) also satisfy (ii). We need to show that if \( N \leq L \leq M \) then either \( L = N \) or \( L = M \). Note that
\[
\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij}
\]
unless \( a_1 \leq k < a_2 \) and \( b_1 \leq l < b_2 \), and
\[
\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} + 1
\]
for \( a_1 \leq k < a_2 \) and \( b_1 \leq l < b_2 \). Since we already proved that \( N \leq L \leq M \) implies
\[
\sum_{i=1}^{k} \sum_{j=1}^{l} n_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} l_{ij} \geq \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij},
\]
we conclude that \( n_{ij} = l_{ij} = m_{ij} \) unless \( a_1 \leq i \leq a_2 \) and \( b_1 \leq j \leq b_2 \). Furthermore, since the row-sums and column-sums are the same for all three matrices and \( M \) and \( N \) satisfy condition (ii), it follows that \( L \) also satisfy (ii), i.e. \( l_{ij} = 0 \) for \( a_1 \leq i \leq a_2 \) and \( b_1 \leq j \leq b_2 \) except for \( (i, j) \in \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\} \).

Finally, one has
\[
n_{a_1b_1} \geq l_{a_1b_1} \geq m_{a_1b_1} = n_{a_1b_1} - 1,
\]
as well as
\[
\begin{align*}
n_{a_1b_1} + n_{a_2b_1} & = l_{a_1b_1} + l_{a_2b_1} = m_{a_1b_1} + m_{a_2b_1}, \\
n_{a_1b_2} + n_{a_2b_2} & = l_{a_1b_2} + l_{a_2b_2} = m_{a_1b_2} + m_{a_2b_2},
\end{align*}
\]
and
\[
n_{a_2b_1} + n_{a_2b_2} = l_{a_2b_1} + l_{a_2b_2} = m_{a_2b_1} + m_{a_2b_2}.
\]
All together this implies that either \( L = N \) or \( L = M \).

To complete the proof of the Theorem it remains to show that if \( N \) and \( M \) satisfy conditions (1) and (2), then there exists a sequence of matrices \( N = L_1, \ldots, L_k = M \) such that for every \( 1 \leq i < k \) the pair \( L_i, L_{i+1} \) satisfies conditions (i) and (ii) (possibly for different values of \( a_1, a_2, b_1, \) and \( b_2 \)).

In fact, it suffices to find just one matrix \( L \) such that \( N \) and \( L \) satisfy conditions (i) and (ii), and \( L \) and \( M \) satisfy conditions (1) and (2). Indeed, one can then proceed by induction as follows. Define the norm of a matrix \( K = \{k_{ij}\} \in \mathcal{M}(n, d) \) as
\[
N(K) := \sum_{1 \leq i, j \leq n} s k_{ij},
\]
where, as before, for all \( 1 \leq p, q \leq n \)
\[
s k_{pq} := \sum_{i=1}^{p} \sum_{j=1}^{q} k_{ij}.
\]
If \( N \) and \( M \) satisfy conditions (1) and (2), then \( N(N) \geq N(M) \) and \( N(N) = N(M) \) if and only if \( N = M \). Also, \( N \) and \( L \) satisfy conditions (i) and (ii) then \( N(N) > N(L) \). We can now proceed by induction with respect to \( N(N) - N(M) \).

**Remark 5.1.** One could also use induction with respect to the difference of dimensions of the orbits \( O_N \) and \( O_M \), however, showing directly that if \( N \) and \( M \) satisfy conditions (1) and (2), and the dimensions are equal, then \( N = M \) is not as easy.

Let \( N \) and \( M \) satisfy conditions (1) and (2). We need to find a matrix \( L \in \mathcal{M}(n, d) \) such that \( N \) and \( L \) satisfy conditions (i) and (ii) and \( L \) and \( M \) satisfy conditions (1) and (2). In order to do so it suffices to find \( 1 \leq a_1 < a_2 \leq n \) and \( 1 \leq b_1 < b_2 \leq n \) such that \( sn_{ij} > sm_{ij} \) for all \( a_1 \leq i < a_2 \) and \( b_1 \leq j < b_2 \), \( n_{a_1b_1} > 0, n_{a_2b_2} > 0, \) and \( n_{ij} = 0 \) for all \( a_1 \leq i \leq a_2 \) and \( b_1 \leq j \leq b_2 \) except for \( (i, j) \in \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\} \). Indeed, then one can take
\[
L := N - E_{a_1b_1} - E_{a_2b_2} + E_{a_1b_2} + E_{a_2b_1}.
\]
It follows from (2) that there exists \( (a, b) \) such that \( n_{ab} > m_{ab} \) and \( n_{ij} = m_{ij} \) for all \( i \leq a, j \leq b, \) and \( (i, j) \neq (a, b) \). It follows that \( n_{ab} > 0 \). Note that if \( n_{a_1b_2} > 0 \) then \( a_1 = a, b_1 = b, a_2 = a + 1, \) and \( b_2 = b + 1 \) satisfy the required conditions.
Let \((r, s) \geq (a, b)\) be such that \(s n_{ij} > s m_{ij}\) for all \(a \leq i < r\) and \(b \leq j < s\), but there exist \(a \leq p < r\) and \(b \leq q < s\) such that \(s n_{pq} = s m_{pq}\) and \(s n_{rq} = s m_{rq}\), i.e. \((r, s)\) is maximal. Note that the choices of \((a, b)\) and \((r, s)\) are not unique. Set

\[
A := \sum_{i=1}^{p} \sum_{j=1}^{q} n_{ij} - m_{ij},
\]

\[
B := \sum_{i=p+1}^{r} \sum_{j=1}^{q} n_{ij} - m_{ij},
\]

\[
C := \sum_{i=1}^{p} \sum_{j=q+1}^{s} n_{ij} - m_{ij},
\]

\[
D := \sum_{i=p+1}^{r} \sum_{j=q+1}^{s} n_{ij} - m_{ij}.
\]

We get \(A = s n_{pq} - s m_{pq} > 0\), \(A + B = s n_{rq} - s m_{rq} = 0\), \(A + C = s n_{ps} - s m_{ps} = 0\), and \(A + B + C + D = s n_{rs} - s m_{rs} \geq 0\), therefore

\[
D = A + B + C + D - (A + B) - (A + C) + A > 0.
\]

In particular, there exist \(p < e \leq r\) and \(q < f \leq s\) such that \(n_{ef} > m_{ef} \geq 0\).

Consider the set

\[
\Lambda := \{(i_1, i_2, j_1, j_2) | a \leq i_1 < i_2 \leq e, b \leq j_1 < j_2 \leq f, s n_{i_1,j_1} > 0, \text{and } s n_{i_2,j_2} > 0\},
\]

and the function \(\phi: \Lambda \to \mathbb{Z}_{>0}\), given by

\[
\phi(i_1, i_2, j_1, j_2) = i_2 - i_1 + j_2 - j_1.
\]

We know that \((a, b, e, f) \in \Lambda\), so \(\Lambda \neq \emptyset\). Then any \((a_1, a_2, b_1, b_2) \in \Lambda\) such that \(\phi\) reaches the minimum at \((a_1, a_2, b_1, b_2)\) satisfies the required conditions. \(\square\)

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