A LIOUVILLE THEOREM FOR THE SUBCRITICAL LANE-EMDEN SYSTEM

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Abstract. The Lane-Emden conjecture says that the subcritical Lane-Emden system admits no positive solution. In this paper, we present a necessary and sufficient condition to the Lane-Emden conjecture. This condition is an energy-type a priori estimate. The necessity of the condition we found can be easily checked. However, a major difficulty lies in the sufficiency. The proof is quite involving, but the benefit is that it reduces the longstanding problem to obtaining the a priori estimate of energy type.

1. Introduction. This paper is devoted to the following simple looking system, the Lane-Emden system,

\begin{equation}
\begin{aligned}
-\Delta u &= v^p, \\
-\Delta v &= u^q,
\end{aligned}
\end{equation}

where \( u, v \geq 0, \ 0 < p, q < +\infty \).

The hyperbola

\[ \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}, \]

is called a critical curve \([12, 13]\), because it is known that on or above it, i.e.

\[ \frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}, \]

which is called critical and supercritical respectively, the system (1) admits (radial) non-trivial solutions, cf. Serrin and Zou [20] for the critical case, Liu, Guo and Zhang [11] and Li [9] for both critical and supercritical cases. However, for subcritical cases, i.e. \((p, q)\) satisfying,

\[ \frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \]

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people [1, 5, 18] guess that the following statement holds and call it the Lane-Emden conjecture:

**Conjecture.** $u = v \equiv 0$ is the unique nonnegative solution for subcritical system (1).

Mitidieri [12, 13] has proved the nonexistence for radial solution in subcritical case, which settles the radial case of the conjecture. For general case, $n \leq 4$ is solved, and for $n > 4$, only some subregion of (2) is known. While many researchers are making contribution, the full Lane-Emden conjecture is still open, and we shall briefly list some important results.

Denote the scaling exponents of system (1) by
\[
\alpha = \frac{2(p + 1)}{pq - 1}, \quad \beta = \frac{2(q + 1)}{pq - 1}, \quad \text{for } pq > 1.
\]
Then subcritical condition (2) is equivalent to
\[
\alpha + \beta > n - 2, \quad \text{for } pq > 1.
\]

Serrin and Zou [19] showed that if
\[
pq \leq 1, \text{ or } pq > 1 \text{ and } \max\{\alpha, \beta\} \geq n - 2,
\]
(1) admits no positive entire supersolution. This implies the conjecture for $n = 1, 2$.

For $n = 3$, the conjecture is solved by two papers. First, Serrin and Zou [19] confirmed the conjecture assuming $n = 3$ and solution has at most polynomial growth at infinity. Then Poláčik, Quittner and Souplet [16] removed the growth condition. An important feature of system (1) is that, given a pair of solution $(u, v)$, then
\[
(u_\lambda(x), v_\lambda(x)) = (\lambda^{\alpha} u(\lambda x), \lambda^{\beta} v(\lambda x)),
\]
is also a pair of solution. Based on the scaling feature, also called “blow-up” argument, Poláčik et al. [16] proved that no bounded positive solution implies no positive solution. This result has two important consequences. First, it removes the growth condition assumed by Serrin and Zou in the case $n = 3$. Second, to prove the Lane-Emden conjecture one only needs to prove nonexistence of bounded positive solution. Thus, we always assume that the solution $(u, v)$ are bounded.

For $n = 4$, the conjecture is recently solved by Souplet [21]. In [19], Serrin and Zou used the integral estimates of the solution (which can be seen as energy estimates) to derive the nonexistence results. Souplet further developed the approach of integral estimates and solved the conjecture for $n = 4$ along the case $n = 3$. In higher dimensions, this approach provides a new subregion where the conjecture holds, but the problem of full range in high dimensional space still seems stubborn. Souplet has proved that if
\[
\max\{\alpha, \beta\} > n - 3,
\]
then (1) with $(p, q)$ satisfying (2) has no positive solution. Notice that (6) covers (2) only when $n \leq 4$, and when $n \geq 5$ (6) covers a subregion of (2). For further reference of this approach, see also [17, 14, 22, 15].

For $n > 4$, in some other subregion, people has confirmed the conjecture. By the method of moving plane, Busca and Manásevich [2] showed that the conjecture is true for
\[
\min\{\alpha, \beta\} \geq \frac{n - 2}{2}, \quad \text{with } (\alpha, \beta) \neq (\frac{n - 2}{2}, \frac{n - 2}{2}).
\]

(7)
Note that (7) covers the case that both \((p, q)\) are subcritical, i.e. \(\max\{p, q\} \leq \frac{n+2}{n-2}\), with \((p, q) \neq \left(\frac{n+2}{n-2}, \frac{n+2}{n-2}\right)\), which is treated earlier, cf. de Figueiredo and Felmer [6] and Reichel and Zou [18]. Also, Mitidieri [13] has proved that the system admits no radial positive solution. Chen and Li [4] have proved that any solution with finite energy must be radial, therefore combined with Mitidieri [13], no finite-energy positive solution exists.

In this paper, we point out that for \(n \geq 3\) and in the full range of (2), the Lane-Emden conjecture holds true under an assumption of a certain type of energy estimate. This estimate in fact turns out to be a necessary and sufficient condition to the conjecture. The difficulty lies in the sufficiency, where the analysis is involved with various estimates, e.g. energy type \(a\ priori\) estimate of solutions, a varied \(W^{2,p}\)-estimate, comparing principle, etc.

**Theorem 1.1.** Let \(n \geq 3\) and \((u, v)\) be a non-negative bounded solution to subcritical (1), i.e. \(p, q \in (0, +\infty)\) satisfying (2). Without loss of generality, we assume \(p \geq q\). If there exists an \(s > 0\) satisfying \(n - s\beta < 1\) such that

\[
\int_{B_R} v^s \leq CR^{n-s\beta},
\]

(8)

then \(u, v \equiv 0\).

**Remark 1.**
(a) Recently, Villavert [23] applies Theorem 1.1 to \(\Delta u + u^p = 0\), \(0 < p < (n+2)/(n-2)\) and provides a new proof to the classical Liouville result by Gidas and Spruck [7] (see also a simplified proof in [3]).
(b) This energy type estimate (8) is a necessary condition to the Lane-Emden conjecture. One just needs to notice that when \(u, v \equiv 0\), (8) is obviously satisfied. The difficulty of the proof of Theorem 1.1 is to show that (8) is sufficient.
(c) One may wonder if we can substitute (8) with a similar estimate on \(u\). In fact if \(p \geq q\), and we assume similarly for some \(r > 0\), such that \(n - r\alpha < 1\),

\[
\int_{B_R} u^r \leq CR^{n-r\alpha}.
\]

(9)

Then (9) implies (8) by a comparison principle between \(u\) and \(v\), i.e., Lemma 2.5. So (8) is a weaker condition yet it is sufficient for the conjecture.
(d) By taking \(s = p\) Theorem 1.1 recovers the result by Souplet [21].
(e) A technical issue is that the standard \(W^{2,p}\)-estimate used in [21] is not enough to establish Theorem 1.1 (see the footnote of Proposition 2). To overcome this difficulty, a mixed type \(W^{2,p}\)-estimate is introduced in Lemma 2.2.

**Remark 2.**
(a) Consider a more general system

\[
\begin{cases}
-\Delta u = c_1(x)u^p, & \text{in } \mathbb{R}^n, \\
-\Delta v = c_2(x)v^q, & \text{in } \mathbb{R}^n,
\end{cases}
\]

(10)

where \(0 < a \leq c_1(x), c_2(x) \leq b < \infty\) for some constants \(a, b\).

It is worthy to point out an interesting phenomenon that for \(c_1(x), c_2(x)\) such that \(x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) < 0\), there exist non-zero solutions of (10) in some subcritical cases (see Lei and Li [8] for detail).
(b) On the other hand, if \(x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) \geq 0\), one can check that the argument in this paper is also valid for this case. For simplicity, we only prove Theorem
1.1 for (1). The key is to notice that the following Rellich-Pohožaev type identity holds. For some constants \(d_1, d_2 > 0\) such that \(d_1 + d_2 = n - 2\),
\[
\int_{B_R} \left( \frac{nc_1}{p + 1} - d_1 c_1 + \frac{x \cdot \nabla c_1(x)}{p + 1} \right)v^{p+1} + \left( \frac{nc_2}{q + 1} - d_2 c_2 + \frac{x \cdot \nabla c_2(x)}{q + 1} \right)u^{q+1}dx
\]
\[
= R^n \int_{S^{n-1}} \frac{c_1(R)v^{p+1}(R)}{p + 1} + \frac{c_2(R)u^{q+1}(R)}{q + 1}d\sigma
\]
\[
+ R^{n-1} \int_{S^{n-1}} d_1 v' u + d_2 v' v d\sigma + R^n \int_{S^{n-1}} (v' u' - R^{-2} \nabla \theta \cdot \nabla v)d\sigma.
\]

By the constrains on \(c_1(x), c_2(x)\), we can have the left terms (LT) in (11) as
\[
LT \geq \delta_0 \int_{B_R} v^{p+1} + u^{q+1}dx, \quad \text{for some} \quad \delta_0 > 0.
\]  
(12)

Then the rest argument carries similarly with the case \(c_1(x), c_2(x) \equiv 1\).

The complete solution of the Lane-Emden conjecture may be a longstanding work. Hence, it will be interesting to consider the Lane-Emden conjecture under some conditions weaker than (8).

**Open problem 1.** Can we prove the Lane-Emden conjecture under the following pointwise asymptotic:
\[
|v(x)| \leq C|x|^{-\gamma}, \quad \text{for some} \quad 0 < \gamma < \beta.
\]

**Open problem 2.** Can we prove the Lane-Emden conjecture under the following integral asymptotic:
\[
\int_{B_R} v^s \leq CR^\delta, \quad \text{for some} \quad s > 0, \quad 0 < \delta < 1.
\]

Clearly, if problem 2 is solved, problem 1 directly follows by choosing sufficiently large \(s\).

The paper is organized as follows. In Section 2, we provide a few fundamental estimates. Some simplified proofs are given for the completeness and convenience of readers. One of the difficulty in the proof of Theorem 1.1 is to control the embedding index, and a varied form of \(W^{2, p}\)-estimate (see Lemma 16) is needed to solve this problem. In Section 3, we give the proof of Theorem 1.1.

2. **Preliminaries.** Throughout this paper, the standard Sobolev embedding on \(\mathbb{S}^{n-1}\) is frequently used. Here we make some conventions about the notations. Let \(D\) denote the gradient with respect to standard metric on manifold. Let \(n \geq 2, j \geq 1\) be integers and \(1 \leq z_1 < \lambda \leq +\infty, z_2 \neq (n - 1)/j\). For \(u = u(\theta) \in W^{j; z_1}(\mathbb{S}^{n-1})\), we have
\[
\|u\|_{L^{2^*}(\mathbb{S}^{n-1})} \leq C \left( \|D^j u\|_{L^{z_1}(\mathbb{S}^{n-1})} + \|u\|_{L^1(\mathbb{S}^{n-1})} \right),
\]  
(13)
where
\[
\left\{ \begin{array}{ll}
\frac{1}{z_1} = \frac{1}{z_2} - \frac{j}{n-1}, & \text{if } z_1 < (n - 1)/j, \\
z_2 = \infty, & \text{if } z_1 > (n - 1)/j,
\end{array} \right.
\]
and \(C = C(j, z_1, n) > 0\). Although \(C\) may be different from line to line, we always denote the universal constant by \(C\). For simplicity, in what follows, for a function \(f(r, \theta)\), we define
\[
\|f\|_{p}(r) = \|f(\cdot, \cdot)\|_{L^p(\mathbb{S}^{n-1})},
\]  
(14)
if no risk of confusion arises.
At last, we set
\[ F(R) = \int_{B_R} u^{q+1} dx. \]

2.1. Basic inequalities. Let us start with a basic yet important fact. Considering \( L^1 \)-norm on \( B_{2R} \), we can write
\[ \|f\|_{L^1(B_{2R})} = \int_0^{2R} \|f(r)\|_{L^1(S^{n-1})} r^{n-1} dr, \]
then by a standard measurement argument (cf. [19], [21]) one can prove that:

**Lemma 2.1.** Let \( f_i \in L^p_{\text{loc}}(\mathbb{R}^n) \), and \( i = 1, \ldots, N \), then for any \( R > 0 \), there exists \( \tilde{R} \in [R, 2R] \) such that
\[ \|f_i\|_{L^p(S^{n-1})}(\tilde{R}) \leq (N + 1)R^{-\frac{n}{p}} \|f_i\|_{L^p(B_{2R})}, \]
for each \( i = 1, \ldots, N \).

The following lemma is a varied \( W^{2,p} \)-estimate which seems not to appear in any literature, so we give a simple proof.

**Lemma 2.2.** Let \( 1 < \gamma < +\infty \) and \( R > 0 \). For \( u \in W^{2,\gamma}(B_{2R}) \), we have
\[ \|D^2 u\|_{L^\gamma(B_R)} \leq C \left( \|\Delta u\|_{L^\gamma(B_{2R})} + R^{\frac{n}{\gamma} - (n+2)} \|u\|_{L^1(B_{2R})} \right) \]
where \( C = C(\gamma, n) > 0 \).

**Proof.** First we deal with functions in \( C^2(B_2) \cap C^0(\overline{B_2}) \). By standard elliptic \( W^{2,p} \)-estimate, we have
\[ \|D^2 u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_2)} + \|u\|_{L^\gamma(B_2)}). \quad (15) \]

By Lemma 2.1, \( \exists \tilde{R} \in \left[ \frac{R}{2}, 2 \right] \) such that on \( B_{\tilde{R}}, u \) can be written as \( u = w_1 + w_2 \), where respectively \( w_1 \) and \( w_2 \) are solutions to
\[ \begin{cases} \Delta w_1 = \Delta u, & \text{in } B_{\tilde{R}}, \\ w_1 = 0, & \text{on } \partial B_{\tilde{R}}, \end{cases} \]
and
\[ \begin{cases} \Delta w_2 = 0, & \text{in } B_{\tilde{R}}, \\ w_2 = u, & \text{on } \partial B_{\tilde{R}}, \end{cases} \]
and additionally,
\[ \int_{\partial B_{\tilde{R}}} u d\sigma \leq C\|u\|_{L^1(B_2)}. \quad (16) \]

By standard \( W^{2,p} \)-estimate with homogeneous boundary condition, we have
\[ \|w_1\|_{L^\gamma(B_{\frac{3}{2}})} \leq \|w_1\|_{W^{2,\gamma}(B_{\frac{3}{2}})} \leq C\|\Delta w_1\|_{L^\gamma(B_{\tilde{R}})}. \]
Since \( w_2 \) can be solved explicitly by Poisson formula on \( B_{\tilde{R}} \), we see that by (16) for any \( x \in B_{\frac{3}{2}} \subseteq B_{\tilde{R}}, w_2(x) \) can be bounded pointwise by
\[ |w_2(x)| \leq C \int_{\partial B_{\tilde{R}}} |u| \leq C\|u\|_{L^1(B_2)}. \]
So,
\[ \|w_2\|_{L^\gamma(B_{\frac{3}{2}})} \leq C\|u\|_{L^1(B_2)}. \]
Hence, 
\[ \|u\|_{L^\gamma(B_{3/2})} \leq \|w_1\|_{L^\gamma(B_{3/2})} + \|w_2\|_{L^\gamma(B_{3/2})} \leq C(\|\Delta u\|_{L^\gamma(B_R)} + \|u\|_{L^1(B_2)}) . \]

Therefore, (15) becomes 
\[ \|D^2 u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_2)} + \|u\|_{L^1(B_2)}) . \]

Then the lemma follows from a dilation and approximation argument. \(\square\)

**Lemma 2.3** (Interpolation inequality on \(B_R\)). Let \(1 \leq \gamma < +\infty\) and \(R > 0\). For \(u \in W^{2,\gamma}(B_R)\), we have 
\[ \|D_x u\|_{L^1(B_R)} \leq C \left( R^{n(1-\frac{1}{\gamma})+1} \|D_x^2 u\|_{L^\gamma(B_R)} + R^{-1} \|u\|_{L^1(B_R)} \right) \]
where \(C = C(\gamma, n) > 0\).

2.2. Pohožaev identity, comparison principle and energy estimates. For system (1) we have a Rellich-Pohožaev identity, which is the starting point of the proof of Theorem 1.1.

**Lemma 2.4.** Let \(d_1, d_2 \geq 0\) and \(d_1 + d_2 = n - 2\), then 
\[ \int_{B_R} \left( \frac{n}{p+1} - d_1 \right) v^{p+1} + \left( \frac{n}{q+1} - d_2 \right) u^{q+1} \, dx \]
\[ = R^n \int_{S^{n-1}} \frac{v^{p+1}(R)}{p+1} + \frac{u^{q+1}(R)}{q+1} \, d\sigma + R^{n-1} \int_{S^{n-1}} d_1 v' u + d_2 u' v \, d\sigma \]
\[ + R^n \int_{S^{n-1}} (v'u' - R^{-2} \nabla \theta u \cdot \nabla \theta v) \, d\sigma . \]

The following comparison principle is somewhat well known. It is originally proved by [21]. Here we provided an another proof.

**Lemma 2.5** (Comparison Principle). Let \(p \geq q > 0, pq > 1\) and \((u, v)\) be a positive bounded solution of (1). Then we have the following comparison principle, 
\[ v^{p+1}(x) \leq \frac{p+1}{q+1} u^{q+1}(x), \quad x \in \mathbb{R}^n . \]

*Proof.* Let \(l = \left( \frac{p+1}{q+1} \right)^{\frac{1}{p+1}}\), \(\sigma = \frac{q+1}{p+1}\). So \(l^{p+1} \sigma = 1\), and \(\sigma \leq 1\). Denote 
\[ \omega = v - lu^\sigma . \]

We will show that \(\omega \leq 0\).

\[ \Delta \omega = \Delta v - l \nabla \cdot (\sigma u^{\sigma-1} \nabla u) \]
\[ = \Delta v - l \sigma (\sigma - 1) |\nabla u|^2 - l \sigma u^{\sigma-1} \Delta u \]
\[ \geq -u^\sigma + l \sigma u^{\sigma-1} v^p \]
\[ = u^{\sigma-1} \left( \frac{v^p}{l} \right) - u^{\sigma+1-\sigma} \]
\[ = u^{\sigma-1} \left( \frac{v^p}{l} \right) - u^{\sigma+1} . \]
So, $\Delta \omega > 0$ if $w > 0$. Now, suppose $w > 0$ for some $x \in \mathbb{R}^n$, and there are two cases:

**Case 1.** $\exists x_0 \in \mathbb{R}^n$, such that $\omega(x_0) = \max_{\mathbb{R}^n} \omega(x) > 0$, and $\Delta \omega(x_0) \leq 0$. However, when $w > 0$, $\Delta \omega > 0$, a contradiction.

**Case 2.** There exists a sequence $\{x_m\}$ with $|x_m| \to +\infty$, such that $\lim_{m \to +\infty} \omega(x_m) = \max_{\mathbb{R}^n} \omega(x) > c_0 > 0$ for some constant $c_0$.

Let $\omega_R(x) = \phi(\frac{x}{R})\omega(x)$, where $\phi(x) \in C_0^\infty(B_1)$ is a cutoff function and $\phi(x) \equiv 1$ in $B_{\frac{1}{2}}$. Since $\omega_R(x) = 0$ on $\partial B_R$, there exists an $x_R \in B_R$ such that $\omega_R(x_R) = \max_{B_R} \omega_R(x)$ and $\lim_{R \to +\infty} \omega(x_R) = \max_{\mathbb{R}^n} \omega(x) > 0$. Also,

$$0 = \nabla \omega_R(x_R) = \phi(\frac{x_R}{R}) \nabla \omega(x_R) + \frac{1}{R^2} \nabla \phi(\frac{x_R}{R}) \omega(x_R).$$

As $\phi(x_R) \geq 1 > 0$ for some constant $c_1$ (in fact, $\phi(x_R) \to 1$) and $\omega(x_R)$ is bounded since $u, v$ are bounded in $\mathbb{R}^n$, we see that $\nabla \omega(x_R) \to 0$ as $R \to +\infty$. So,

$$0 \geq \Delta \omega_R(x_R) = \frac{1}{R^2} \nabla \phi(\frac{x_R}{R}) \Delta \omega(x_R) + \frac{2}{R^2} \nabla \phi(\frac{x_R}{R}) \cdot \nabla \omega(x_R) + \phi(\frac{x_R}{R}) \Delta \omega(x_R)$$

$$\Rightarrow 0 \geq \Delta \omega(x_R) + O(\frac{1}{R^2})$$

Since $\omega(x_R) > c_0/2$ for sufficiently large $R$, $\Delta \omega(x_R) > c_2 > 0$ for some constant $c_2$, a contradiction.

**Remark 3.** For general Lane-Emden type system (10), we can choose

$$w = v - Clu^p, \quad \text{where} \quad C^{p+1} = \sup_{x \in \mathbb{R}^n} \frac{c_2(x)}{c_1(x)}.$$

By the same arguments, we can also get the desired comparison principle. It is also worth to mention another maximum principle for fractional super-harmonic non-negative functions on a punctured ball in [10].

Next we prove a group of energy estimates which are crucial to the entire argument in this paper. As Theorem 1.1 points out, a stronger energy estimate is the key to the Lane-Emden conjecture. Unfortunately, efforts have been made so far only provide the following inequalities, which are first obtained by Serrin and Zou [19] (1996). Here we give a simpler proof for the convenience of readers.

**Lemma 2.6.** Let $p \geq q > 0$ with $pq > 1$. For any positive solution $(u, v)$ of (1)

$$\int_{B_R} u \leq CR^{n-\alpha}, \quad \text{and} \quad \int_{B_R} v \leq CR^{n-\beta}, \quad (17)$$

$$\int_{B_R} u^q \leq CR^{n-q\alpha}, \quad \text{and} \quad \int_{B_R} v^p \leq CR^{n-p\beta}. \quad (18)$$

**Proof.** Let $\phi \in C_0^\infty(B_R(0))$ be the first eigenfunction of $-\Delta$ in $B_R$ and $\lambda$ be the eigenvalue. By definition and rescaling, it is easy to see that $\phi|_{\partial B_R} = 0$ and $\lambda \sim \frac{1}{R^q}$. By normalizing, one gets $\phi \geq c_0 > 0$ on $B_{R/2}$ for some constant $c_0$ independent of $R$, $\phi(0) = ||\phi||_\infty = 1$. So, multiplying (1) by $\phi$ then integrating by parts on $B_R$ we have,

$$\int_{B_R} \phi u^q = - \int_{B_R} \phi \Delta v = \int_{\partial B_R} v \frac{\partial \phi}{\partial n} d\sigma + \lambda \int_{B_R} \phi v.$$.
By Hopf’s Lemma we know that $\frac{\partial \phi}{\partial n} < 0$ on $\partial B_R$, so
\[ \int_{B_R} \phi u^q \leq \lambda \int_{B_R} \phi v. \]  
(19)

Similarly, we have
\[ \int_{B_R} \phi v^p \leq \lambda \int_{B_R} \phi u. \]  
(20)

Applying Lemma 2.5 to (19), we have
\[ \frac{1}{R^2} \int_{B_R} \phi v \geq C \int_{B_R} \phi v^{\frac{q(p+1)}{q+p+1}}. \]

Notice that $\frac{q(p+1)}{q+p+1} > 1$ as $pq > 1$, so by Hölder inequality
\[ \int_{B_R} \phi v^{\frac{q(p+1)}{q+p+1}} \geq \left( \int_{B_R} \phi \right)^{\frac{q(p+1)}{q+p+1}} \left( \int_{B_R} \phi \right)^{-\frac{(p+1)}{q+p+1}} \geq C \left( \int_{B_R} \phi \right)^{\frac{q(p+1)}{q+p+1}} R^{-n \frac{q-1}{q+p+1}}. \]

So,
\[ \frac{1}{R^2} \int_{B_R} \phi v \geq C \left( \int_{B_R} \phi \right)^{\frac{q(p+1)}{q+p+1}} R^{-n \frac{q-1}{q+p+1}} \Rightarrow \int_{B_R} \phi v \leq CR^{n-\beta}. \]

Therefore, by (19)
\[ \int_{B_R} \phi u^q \leq CR^{n-\beta-2} = CR^{n-q\alpha}. \]

Now, **Case 1.** If $q > 1$, then by Hölder inequality
\[ \int_{B_R} \phi u \leq \left( \int_{B_R} \phi u^q \right)^{\frac{1}{q}} \left( \int_{B_R} \phi \right)^{\frac{1}{q'}} \leq CR^\frac{n}{q} R^{\frac{q-1}{q'}} = CR^\frac{n}{q}, \quad \frac{1}{q} + \frac{1}{q'} = 1. \]

Mean while, by (20)
\[ \int_{B_R} \phi v^p \leq CR^{n-\alpha-2} = CR^{n-p\beta}. \]

This finishes the proof for Case 1.

**Case 2.** Assume that $q \leq 1$. This case is trickier, and first we show that

**Lemma 2.7.** If $\Delta u \leq 0$, then for $\gamma \in (0, 1)$, $\eta \in C_0^\infty(\mathbb{R}^n)$,
\[ \int_{\mathbb{R}^n} \frac{4}{\gamma^2} |D(\eta^2 u^\gamma)|^2 \eta^2 = \int \eta^2 |Du|^2 u^{\gamma-2} \leq C \int |D\eta|^2 u^\gamma. \]  
(21)

**Proof.** Multiply $\eta^2 u^{\gamma-1}$ to $\Delta u \leq 0$ then integrate over the whole space. \qed

We rewrite (21) as
\[ \int_{B_R} |Du|^2 u^{\gamma-2} \leq \frac{C_\gamma}{R^2} \int_{B_{2R}} u^\gamma, \]  
(22)

where $C_\gamma \to +\infty$ as $\gamma \to 1$.

In the following we use the notation,
\[ \| \cdot \|_{L^p(\Omega)} = \| \cdot \|_{L^p(\Omega)}. \]
From Poincaré’s Inequality, we have
\[
|f|_{\|,\|,\Omega} \leq C(n, a, \Omega) \left( |Df|_{\|,\Omega} + |R|^{\frac{n-a}{n-a}} |f_{\Omega R}| \right),
\tag{23}
\]
where
\[
f_{\Omega R} = \int_{\Omega R} f = \frac{1}{|\Omega R|} \int_{\Omega R} f, \quad \Omega R = \{Rx|x \in \Omega\}.
\]

Next we prove a variation of embedding inequality.

**Lemma 2.8.** For any \( l \geq 1 \),
\[
|f|^l \|,\|_{\Omega R} \leq C(n, a, \Omega) \left( |D(f^l)|_{\|,\Omega R} + |R|^{\frac{n-a}{n-a}} |f_{\Omega R}|^l \right).
\tag{24}
\]

**Proof.** By (23),
\[
|f|^l \|,\|_{\Omega R} \leq C(n, a, \Omega) \left( |D(f^l)|_{\|,\Omega R} + |R|^{\frac{n-a}{n-a}} \int_{\Omega R} f^l \right)
\leq C(n, a, \Omega) \left( |D(f^l)|_{\|,\Omega R} + |R|^{\frac{n-a}{n-a}} \left( \int_{\Omega R} f^{l\frac{n-a}{n-a}} \right)^{(1-\theta)\frac{n-a}{n-a}} \right)
\leq C(n, a, \Omega) |D(f^l)|_{\|,\Omega R} + \frac{1}{2} |f|^\frac{n-a}{n-a} |\Omega R| + C(n, a, \Omega) |R|^{\frac{n-a}{n-a}} |f_{\Omega R}|^l,
\]
where \( \theta = \frac{1 - \frac{n-a}{l - \frac{n-a}{n-a}} \in \text{the third inequality. In the last inequality, we have used the Young’s inequality. So we get (24).} \]

Let \( l \geq 1, \theta \leq 2q < 2, \gamma = l\theta < 1, f = u^\frac{q}{2}, a = 2 \). Then
\[
|f|^l \|,\|_{2, BR} \leq C \left( |D(u^\theta)|_{2, BR} + R^{\frac{n+2}{2}} |f_{BR}|^l \right)
\leq C \left( |D(u^\theta)|_{2, BR} + R^{\frac{n+2}{2}} |(u^\theta)_{BR}|^l \right)
\leq C \left( \int_{BR} u^\theta \right)^\frac{1}{\theta} + R^{\frac{n+2}{2}} \left( \int_{BR} u^\frac{q}{2} \right)^l.
\tag{25}
\]
The last term on the right can be estimate by Hölder and the fact that \( \int_{BR} u^q \leq CR^{-q\alpha} \) since \( \frac{q}{2} \leq q \). This yields that
\[
\int_{BR} u^\frac{n-a}{2} \leq C \left( R^{-\frac{2n}{2}} \left( \int_{2BR} u^\theta \right)^\frac{n}{n-2} + R^{n-\frac{n}{n-2} l^{\theta \alpha}} \right).
\tag{26}
\]
This means if \( \int_{BR} u^\theta \leq CR^{-l^{\theta \alpha}} \), we have \( \int_{BR} u^{\frac{n-a}{2}} \leq CR^{-\frac{n}{n-2} l^{\theta \alpha}} \) provided \( l\theta < 1 \). By \( \int_{BR} u^q \leq CR^{-q\alpha} \), we get
\[
\int_{BR} u^s \leq C(s)R^{-s\alpha}, \quad \forall s \in (0, \frac{n}{n-2}), \tag{27}
\]
where \( C(s) \to +\infty \) as \( s \to \frac{n}{n-2} \).

By taking \( s = 1 \), the above inequality immediately leads to
\[
\int_{BR} u \leq CR^{n-\alpha}.
\]
Since $pq > 1$ and we assume that $p \geq q$, $p$ must be greater than 1, then by (20) we get
\[
\int_{B_R} v^p \leq C R^{n-p}.
\]
This finishes the proof of Lemma 2.6.

2.3. Estimates on $S^{n-1}$. Now that we have energy inequalities (18), in our assumption (8) we can always assume $s \geq p$. In the following proof, if no confusion occurs, we always denote
\[
l = \frac{s}{p}, \quad k = \frac{p+1}{p}, \quad m = \frac{q+1}{q}.
\]
The following estimates for quantities on sphere $S^{n-1}$ are necessary to the proof.

**Proposition 1.** For $R \geq 1$, there exists $\tilde{R} \in [R, 2R]$ such that for $l = \frac{s}{p} \geq 1$, $k = \frac{p+1}{p}$ and $m = \frac{q+1}{q}$, we have
\[
\|u\|_1(\tilde{R}) \leq CR^{-\alpha}, \quad \|v\|_1(\tilde{R}) \leq CR^{-\beta},
\]
\[
\|D^2_x u\|_1(\tilde{R}) \leq CR^{-\frac{lp\beta}{p+2}}, \quad \|D^2_x v\|_{1+\epsilon}(\tilde{R}) \leq CR^{-\frac{q\alpha}{q+2}},
\]
\[
\|D_x u\|_1(\tilde{R}) \leq CR^{-\frac{lp\beta}{p+2}}, \quad \|D_x v\|_1(\tilde{R}) \leq CR^{-\frac{q\alpha}{q+2}},
\]
\[
\|D^2_x u\|_k(\tilde{R}) \leq C(R^{-n}F(2R))^{\frac{k}{l}}, \quad \|D^2_x v\|_m(\tilde{R}) \leq C(R^{-n}F(2R))^{\frac{k}{l}}.
\]

In view of Lemma 2.1, to prove Proposition 1, we shall give the corresponding estimates on $B_{2R}$ first. We use the varied $W^{2,p}$-estimate (i.e. Lemma 2.2) to achieve this.

**Proposition 2.** For $R \geq 1$, we have
\[
\left\{ \begin{array}{ll}
\|u\|_{L^1(B_{2R})} \leq CR^{n-\beta}, \\
\|v\|_{L^1(B_{2R})} \leq CR^{n-\alpha},
\end{array} \right. \tag{28}
\]
\[
\left\{ \begin{array}{ll}
\|D^2_x u\|_{L^{1+\epsilon}(B_{2R})} \leq CR^{n-lp\beta}, \quad \text{with } l = \frac{s}{p} \geq 1, \\
\|D^2_x v\|_{L^{1+\epsilon}(B_{2R})} \leq CR^{n-q\alpha},
\end{array} \right. \tag{29}
\]
\[
\left\{ \begin{array}{ll}
\|D_x u\|_{L^1(B_{2R})} \leq CR^{n+1-\frac{lp\beta}{p+2}}, \\
\|D_x v\|_{L^1(B_{2R})} \leq CR^{n+1-\frac{q\alpha}{q+2}},
\end{array} \right. \tag{30}
\]
and let $k = \frac{p+1}{p}$, $m = \frac{q+1}{q}$,
\[
\left\{ \begin{array}{ll}
\|D^2_x u\|_{L^1(B_{2R})} \leq CF(2R), \\
\|D^2_x v\|_{L^m(B_{2R})} \leq CF(2R). \tag{31}
\end{array} \right.
\]

**Proof.** Some frequently used facts include, $q\alpha = \beta + 2$, $p\beta = \alpha + 2$ and hence $n - kp\beta < 0$ (due to (4)) and therefore $l < k$ (since $n - lp\beta \geq 0$).

Estimates (28) directly follow from (17) in Lemma 2.6.

For the first estimate of (29), after applying Lemma 2.2, the mixed type $W^{2,p}$-estimate\(^1\), we get
\[
\|D^2_x u\|_{L^{1+\epsilon}(B_{2R})} \leq C \left( \|\Delta u\|_{L^{1+\epsilon}(B_{2R})} + R^{n-(l+\epsilon)(n+2)}\|u\|_{L^1(B_{2R})} \right).
\]

\(^1\) Notice that with the standard $W^{2,p}$-estimate, we end up with a term of $\|u\|_{l+\epsilon}$ which cannot be estimated by any energy bound.
Then we use the assumed estimate (8) and Lemma 2.6 to get
\[
\|D^2_x u\|_{L^{1+\varepsilon}(B_R)}^{1+\varepsilon} \leq C \left( \int_{B_{2R}} u^{p(l+\varepsilon)} dx + R^{n-(l+\varepsilon)(n+2)} R^{l(n-\alpha)} \right) \\
\leq C \left( R^{n-pl\beta} + R^{n-(l+\varepsilon)(2+\alpha)} \right) \\
\leq CR^{n-pl\beta},
\]
where the last inequality is due to \(\alpha + 2 = p\beta\). For the second estimate of (29),
\[
\|D^2_v\|_{L^{1+\varepsilon}(B_{2R})}^{1+\varepsilon} \leq C \left( \|\Delta v\|_{L^{1+\varepsilon}(B_{2R})} + R^{n-(1+\varepsilon)(n+2)} \|v\|_{L^1(B_{2R})}^{1+\varepsilon} \right) \\
\leq C \left( \int_{B_{2R}} u^{q(1+\varepsilon)} dx + R^{n-(1+\varepsilon)(n+2)} R^{l(n-\beta)} \right) \\
\leq C \left( R^{n-q\alpha} + R^{n-(1+\varepsilon)(\beta+2)} \right) \\
\leq CR^{n-q\alpha}.
\]
For the first estimate of (30), by Lemma 2.3,
\[
\|D_x u\|_{L^1(B_R)} \leq C \left( R^{n\left(1-\frac{1}{1+p}\right)+1} \|D^2_x u\|_{L^{1+\varepsilon}(B_R)} + R^{-1} \|u\|_{L^1(B_R)} \right) \\
\leq C \left( R^{n\left(1-\frac{1}{1+p}\right)+1} R^{n-p\beta} + R^{-1} R^{n-\alpha} \right) \\
\leq CR^{n+1-\frac{2+2}{1+p}}.
\]
The second estimate in (30) can be obtained by a similar process. Last, the fact that \(n-(p+1)\beta < 0\) gives
\[
\|D^2 u\|_{L^k(B_R)} \leq C \left( \int_{B_{2R}} |\Delta u|^k dx + R^{n-k(n+2)} \left( \int_{B_{2R}} |u|^k dx \right)^{1/k} \right) \\
\leq C \left( \int_{B_{2R}} u^{p+1} dx + R^{n-k(n+2)} R^{k(n-\alpha)} \right) \\
\leq C \left( F(2R) + R^{n-(p+1)\beta} \right) \\
\leq CF(2R),
\]
and hence the first estimate in (31) follows, and similarly we get the second estimate. □

Proof of Proposition 1. By Lemma 2.1, \(\exists R \in [R, 2R]\), Proposition 1 follows from Proposition 2 immediately. □

Lemma 2.9. There exists \(M > 0\) such that \(\exists \{R_j\} \to +\infty\),
\[
F(4R_j) \leq MF(R_j).
\]

Proof. Suppose not, then for any \(M > 0\) and any \(\{R_j\} \to +\infty\), we have
\[
F(4R_j) > MF(R_j).
\]
Take \(M > 5^n\) and \(R_{j+1} = 4R_j\) with \(R_0 > 1\). Therefore,
\[
F(R_j) > M^n F(R_0),
\]
which leads to a contradiction with \(F(R_j) \leq CR_j^n \leq C(4^j R_0)^n\). □
Proof of Theorem 1.1. Recall that, without loss of generality, we assume \( p \geq q \). By Lemma 2.5, \( ||v||_{L^{q+1}(B_R)} \leq ||u||_{L^{q+1}(B_R)} \). By the Rellich-Pohožaev type identity in Lemma 2.4, we can denote
\[
F(R) := \int_{B_R} u^{q+1} \leq CG_1(R) + CG_2(R),
\]
where
\[
G_1(R) = R^a \int_{\mathbb{S}^{n-1}} u^{q+1}(R)d\sigma,
\]
\[
G_2(R) = R^a \int_{\mathbb{S}^{n-1}} (|D_xu(R)| + R^{-1}u(R))(|D_xv(R)| + R^{-1}v(R))d\sigma.
\]

Heuristically, we are aiming for estimate as
\[
G_i(R) \leq CR^{-a_i}F^{1-\delta_i}(4R), \quad i = 1, 2.
\]

Then by Lemma 2.9 there exists a sequence \( \{R_j\} \to +\infty \) such that
\[
G_i(R_j) \leq CR^{-a_i}_j F^{1-\delta_i}(R_j), \quad i = 1, 2.
\]

Suppose there are infinitely many \( R_j \)'s such that \( G_1(R_j) \geq G_2(R_j) \), then take that subsequence of \( \{R_j\} \) and still denote as \( \{R_j\} \). We do the same if there are infinitely many \( R_j \)'s such that \( G_1(R_j) \leq G_2(R_j) \). So, there are only two cases we shall deal with: there exists a sequence \( \{R_j\} \to +\infty \) such that

Case 1. \( G_1(R_j) \geq G_2(R_j) \). Then we prove \( a_1 > 0, \delta_1 > 0 \). So, \( F^{\delta_1}(R_j) \leq CR^{-a_1}_j \to 0 \).

Case 2. \( G_1(R_j) \leq G_2(R_j) \). Then we prove \( a_2 > 0, \delta_2 > 0 \). So, \( F^{\delta_2}(R_j) \leq CR^{-a_2}_j \to 0 \).

Then we conclude that \( F(R) \equiv 0 \).

For both cases, Step 1, we show \( a_i = (\alpha + \beta + 2 - n)\delta_i + o(1) \). Recall that (2) is equivalent to (4). Then Step 2, we only need to show \( \delta_i > 0 \).

3.1. Case 1: Estimate for \( G_1(R) \). According to previous discussion in the introduction, we assume that
\[
p \geq q > 0, \quad pq > 1, \quad \beta \leq \alpha < n - 2, \quad n \geq 3,
\]
hence in particular
\[
p > \frac{n}{n-2}.
\]

Remark 4. For systems (10) with double bounded coefficients, (36) is a necessary condition for existence of positive solution, see [8].

In addition to our assumption that \( n - s\beta < 1 \), since we have energy inequalities (18), we can assume \( s \geq p \). Also, if \( n - s\beta < 0 \), (8) implies \( v \equiv 0 \) and hence \( u \equiv 0 \).

So, we assume \( n - s\beta \geq 0 \). Let \( l = \frac{n}{p\beta} \), then
\[
l \geq 1, \quad \text{and} \quad \frac{n-1}{p\beta} < l \leq \frac{n}{p\beta}.
\]

It is worthy to point out that, what the proof of Lane-Emden conjecture really needs is a “breakthrough” on the energy estimate (18). \( s \) in (8) needs not be very large but enough to satisfy \( n - s\beta < 1 \). In other words, \( s \) can be very close to \( \frac{n-1}{\beta} \), and it is sufficient to prove Theorem 1.1.
The strategies of attacking $G_1$ and $G_2$ are the same. Basically, first by Hölder inequality we split the quantities on sphere $S^{n-1}$ into two parts. One has a lower (than original) index after embedding, and the other has a higher one. Then we estimate the latter part by $F(R)$, and thus we get a feedback estimate as (35).

Let

$$k = \frac{p + 1}{p}.$$

Since $p\beta = \alpha + 2, n - (p + 1)\beta = n - 2 - (\alpha + \beta) < 0$ by (4). Thus, $n - kp\beta < 0$ as $n - lp\beta \geq 0$, it follows that $l < k$.

**Subcase 1.1.** $l \geq \frac{2}{n-1} + \frac{1}{q+1}$.

Note that in this subcase, since $l \geq 1$, we must have $n \geq 4$ (i.e., $n \neq 3$). By (36) we see that $k = 1 + \frac{1}{p} < 1 + \frac{n-2}{n} = \frac{2}{n}(n-1) \leq \frac{n-1}{2}$. Take

$$\frac{1}{\mu} = \frac{1}{k} - \frac{2}{n-1}.$$

Then $W^{2,l+\varepsilon}(S^{n-1}) \hookrightarrow L^\lambda(S^{n-1})$.

Direct verification shows that $\frac{1}{\mu} = \frac{1}{l} - \frac{2}{n-1} \leq \frac{1}{q+1}$ which is due to (2), so we have

$$\frac{1}{\mu} \leq \frac{1}{q+1} \leq \frac{1}{\lambda}.$$

Then by Hölder inequality and Sobolev embedding (13), we have (with notation (14))

$$\|u\|_{q+1}(R) \leq \|u\|_\alpha^\theta \|u\|_1^{1-\theta}(R)$$

$$\leq C(R^2\|D^2_xu\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta}(R^2\|D^2_xu\|_k(R) + \|u\|_1(R))^{1-\theta},$$

where $\theta \in [0, 1]$ and

$$\frac{1}{q+1} = \frac{\theta}{\lambda} + \frac{1-\theta}{\mu}.$$ (40)

Since $l$ can be 1 (then $W^{2,p}$-estimate fails for $\|D^2_xu\|_{1}(R)$), we add an $\varepsilon$ to $l$ for later use of $W^{2,p}$-estimate. $\varepsilon$ can be any real positive number and later will be chosen sufficiently small.

To get desired estimate, we have requirements in form of inequalities involving parameters, such as $\alpha, \beta, \varepsilon$ and etc. To verify those requirements very often we just verify strict inequalities with $\varepsilon = 0$ because such inequalities continuously depend on $\varepsilon$.

So, by (33) and (39)

$$G_1(R) \leq CR^n \left( (R^2\|D^2_xu\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta}(R^2\|D^2_xu\|_k(R) + \|u\|_1(R))^{1-\theta} \right)^{q+1}.$$ (41)
Then by Proposition 1, there exists \( \tilde{R} \in [R, 2R] \) such that

\[
G_1(\tilde{R}) \leq CR^n \left( (R^2 R^{-\frac{lp\beta}{l+\varepsilon}} + R^{-2-\alpha})^\theta (R^2 (R^n F(4R))^{\frac{k}{\theta}} + R^{-\alpha})^{1-\theta} \right)^{q+1} \\
\leq CR^n \left( R^{2-\frac{lp\beta}{l+\varepsilon}} - \frac{n(1-\theta)}{\theta} F^{1-\theta}(4R) \right)^{q+1} \\
\leq R^{-\alpha}, F^{1-\delta_1}(4R),
\]

where the last inequality is due to \( R^{-\frac{k}{\theta}} > R^{-\alpha-2} \) and

\[
a_1 = a_1^\varepsilon = (q+1)(\frac{lp\beta}{l+\varepsilon} + \frac{np(1-\theta)}{p+1} - 2 - \frac{n}{1+q}), \quad (42)
\]

\[
1 - \delta_1 = \frac{(1 - \theta)p(q+1)}{p+1}. \quad (43)
\]

Since for sufficiently small \( \varepsilon \), \( a_1^\varepsilon > 0 \) and \( \delta_1 > 0 \) are just a perturbation of

\[
a_1^0 > 0, \quad \text{and} \quad \delta_1 > 0, \quad (44)
\]

we only need to prove \((44)\) is true.

Since \( lp = s, \ p\beta = \alpha + 2 \) and \( q\alpha = \beta + 2 \),

\[
a_1^0 = p\beta\theta(q+1) + (1-\delta_1)n - 2(q+1) - n \\
\quad = (q+1)(p\beta\theta - 2) - \delta_1 n \\
\quad = (q+1)(p\beta(\theta-1) + p\beta - 2) - \delta_1 n \\
\quad = (q+1)(-\alpha(1-\delta_1) + \alpha) - \delta_1 n \\
\quad = \delta_1 ((q+1)\alpha - n) \\
\quad = (\alpha + \beta + 2 - n)\delta_1.
\]

Now we are left to prove \( \delta_1 > 0 \). By \((42)\) and \((40)\) we have

\[
(1 - \theta)p(q+1) < p + 1 \\
\Leftrightarrow \frac{1}{l} - \frac{2}{n-1} - \frac{1}{l+q} (q+1) < k \\
\Leftrightarrow \frac{1}{l} - \frac{2}{n-1} (q+1) - 1 < \frac{k}{l} - 1 \\
\Leftrightarrow \frac{1}{l} (q+1 - 1 - \frac{1}{p}) < \frac{2}{n-1} (q+1) \\
\Leftrightarrow \frac{pq - 1}{s} < \frac{2(q+1)}{n-1} \\
\Leftrightarrow n - 1 < s\beta,
\]

and the last inequality is included in our assumption. So, we have proved subcase 1.1.

**Subcase 1.2.** \( \frac{1}{l} < \frac{2}{n-1} + \frac{1}{q+1} \).

As discussed in the beginning of subcase 1.1, \( k < \frac{n-1}{l} \) if \( n > 3 \). Since \( l < k \), \( \frac{1}{l} > \frac{2}{n-1} \) for \( n > 3 \). When \( n = 3 \), since \( l \geq 1 \) by \((37)\), \( \frac{1}{l} \leq 1 = \frac{2}{n-1} \).

Therefore, for \( n > 3 \), take

\[
\frac{1}{\lambda} = \frac{1}{l} - \frac{2}{n-1} < \frac{1}{q+1}.
\]
and for \( n = 3 \), take
\[
\lambda = \infty,
\]
so we have
\[
W^{2,l+\varepsilon}(S^{n-1}) \hookrightarrow L^\lambda(S^{n-1}), \quad n \geq 3.
\]

So,
\[
\|u\|_{q+1}(R) \leq C\|u\|_\lambda(R) \leq C(R^2\|D_x^2u\|_{l+\varepsilon}(R) + \|u\|_1(R)).
\]

Therefore, by Proposition 1 there exists \( \tilde{R} \in [R, 2R] \) such that
\[
G_1(\tilde{R}) \leq CR^n(R^2\|D_x^2u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{q+1} 
\leq CR^n(R^2R^{-\frac{p\alpha}{p+\beta}} + R^{-\alpha})^{q+1} 
\leq CR^{n+(2-\frac{p\alpha}{p+\beta})(q+1)}.
\]

So,
\[
F(\tilde{R}) \leq CR^{n+(2-\frac{p\alpha}{p+\beta})(q+1)} 
\leq CR^{n(2-\frac{p\beta}{q+1})(q+1)} + \frac{CR^n}{(q+1)}(q+1) 
\leq CR^{-(\alpha + \beta + 2 - n + \frac{p\alpha}{p+\beta})}. 
\]

Since \( \varepsilon \) can be arbitrarily small,
\[
F(\tilde{R}) \leq CR^{-(\alpha + \beta + 2 - n + o(1))}.
\]

Thus, we have proved Case 1.

3.2. Case 2: Estimate for \( G_2(R) \). Let
\[
m = \frac{q+1}{q}.
\]

Subcase 2.1. \( m < n - 1 \).

With \( z, z' > 0 \) and \( \frac{1}{z} + \frac{1}{z'} = 1 \), (34) becomes,
\[
G_2(R) \leq CR^n(\|D_xu\|_z + R^{-1}\|u\|_{z'}\|D_xv\| + R^{-1}\|v\|_{z'}(R)) 
\leq CR^n(\|D_xu\|_z(R) + R^{-1}\|u\|_1(R))\(\|D_xv\|_{z'}(R) + R^{-1}\|v\|_1(R)) 
\leq CR^n(\|D_xu\|_z(R) + R^{-1}\|u\|_1(R))(\|D_xv\|_{z'}(R) + R^{-1}\|v\|_1(R)),
\]

where the last inequality is due to
\[
\|u\|_z(R) \leq C(R)\|D_xu\|_z(R) + \|u\|_1(R),
\]
and \( \|v\|_{z'}(R) \leq C(R)\|D_xv\|_{z'}(R) + \|v\|_1(R)) \).

Assume there exists \( z \) (we shall check the existence later) such that by Sobolev Embedding (13),
\[
\|D_xu\|_z(R) \leq \|D_xu\|_{z\gamma_1}(R)\|D_xu\|_{1-\gamma_1}(R) 
\leq C(R)\|D_xu\|_{1+\varepsilon}(R) + \|D_xu\|_1(R))^{\gamma_1}(R)\|D_x^2u\|_{k}(R) + \|D_xu\|_1(R))^{1-\gamma_1},
\]
\[
\|D_xv\|_{z'}(R) \leq \|D_xv\|_{z'\gamma_2}(R)\|D_x^2v\|_{\gamma_2}(R) 
\leq C(R)\|D_x^2v\|_{1+\varepsilon}(R) + \|D_xv\|_1(R))^{\gamma_2}(R)\|D_x^2v\|_{m}(R) + \|D_xv\|_1(R))^{1-\gamma_2},
\]

(50)
where $\tau_1, \tau_2 \in [0, 1]$ and

\[
\begin{align*}
\frac{1}{z} &= \frac{\tau_1}{\rho_1} + \frac{1 - \tau_1}{\gamma_1}, \\
\frac{1}{z} &= \frac{\tau_2}{\rho_2} + \frac{1 - \tau_2}{\gamma_2},
\end{align*}
\]

(51) and (52)

and since $l < k \leq m < n - 1$, define

\[
\begin{align*}
\frac{1}{\rho_1} &= \frac{1}{l} - \frac{1}{n - 1}, \\
\frac{1}{\rho_2} &= 1 - \frac{1}{n - 1},
\end{align*}
\]

(53) and (54)

So, we have

\[
W^{1,l+\varepsilon}(S^{n-1}) \hookrightarrow L^{m}(S^{n-1}), \quad W^{1,k}(S^{n-1}) \hookrightarrow L^{\gamma_2}(S^{n-1}),
\]

\[
W^{1,k}(S^{n-1}) \hookrightarrow L^{\gamma_1}(S^{n-1}), \quad W^{1,m}(S^{n-1}) \hookrightarrow L^{\gamma_2}(S^{n-1}).
\]

To verify the existence of such $z$, by (51)-(54), we expect that

\[
\max \left\{ \frac{1}{k} - \frac{1}{n - 1}, \frac{1}{n - 1} \right\} \leq \frac{1}{z} \leq \min \left\{ \frac{1}{l} - \frac{1}{n - 1}, \frac{1}{q + 1} + \frac{1}{n - 1} \right\}.
\]

(55)

Thus, we need to verify, (i) $\frac{1}{k} - \frac{1}{n - 1} \leq \frac{1}{l} - \frac{1}{n - 1}$, (ii) $\frac{1}{n - 1} \leq \frac{1}{l} - \frac{1}{n - 1}$, (iii) $\frac{1}{l} - \frac{1}{n - 1} \leq \frac{1}{q + 1} + \frac{1}{n - 1}$, (iv) $\frac{1}{l} - \frac{1}{n - 1} \leq \frac{1}{q + 1} + \frac{1}{n - 1}$.

Since $l < k$, (i) is true. (ii) holds for $n > 3$ as discussed at the beginning of subcase 1.2 $\frac{1}{l} > \frac{1}{k} > \frac{2}{n-1}$; for $n = 3$, take $s = p$ and then $l = 1$, so (ii) still holds. (iii) is obvious. (iv) is equivalent to $\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{n-1}$, which is guaranteed by (2).

So, we put (49) and (50) in (48) and get

\[
G_2(R) \leq CR^{n+2}\left(\|D_x v\|_1(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R)\right)^{\tau_1}
\]

\[
\times \left(\|D_x^2 u\|_1(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R)\right)^{1-\tau_1}
\]

\[
\times \left(\|D_x^2 v\|_1(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R)\right)^{\tau_2}
\]

\[
\times \left(\|D_x v\|_m(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R)\right)^{1-\tau_2}.
\]

(56)

Then by Proposition 1, there exists $\tilde{R} \in [R, 2R]$ such that

\[
G_2(\tilde{R}) \leq CR^{n+2}R^{\alpha+\frac{\alpha+2}{p+\gamma}}\left(\left(R^{-n}F(4R)\right)^{\frac{1}{p+1}} + R^{\frac{\alpha+2}{p+\gamma}} + R^{-\alpha-2}\right)^{1-\tau_1}
\]

\[
\times R^{-\frac{q\alpha\gamma_1}{p+\gamma}}\left(\left(R^{-n}F(4R)\right)^{\frac{1}{p+1}} + R^{\frac{\alpha+2}{p+\gamma}} + R^{-\alpha-2}\right)^{1-\tau_2}
\]

\[
\leq CR^{-\alpha_2^0}F^{-1-\delta_2}(4R),
\]

where the last inequality is due to $R^{-\frac{1}{p}} > R^{-\alpha-2}$ and $R^{-\frac{1}{q}} > R^{-\beta-2}$. Meanwhile,

\[
\begin{align*}
a_2 &= 2 - n + \frac{p\beta\tau_1}{1+\varepsilon/l} + \frac{q\alpha\tau_2}{1+\varepsilon/l} + n - \tau_1 + n \frac{1 - \tau_2}{k},
\end{align*}
\]

(57)

\[
1 - \delta_2 = \frac{1 - \tau_1}{k} + \frac{1 - \tau_2}{m}.
\]

(58)

Similar to subcase 1.1, we only need to prove

\[
\alpha_2^0 > 0, \quad \delta_2 > 0.
\]
Also similarly, Step 1, we prove \( a_2^0 = (\alpha + \beta + 2 - n) \delta_2 \). Then we have \( a_2 = (\alpha + \beta + 2 - n) \delta_2 + \alpha(1) \). Indeed,

\[
a_2^0 = -n - 2 + p\beta(\tau_1 - 1) + p\beta + qa(\tau_2 - 1) + q\alpha + n(1 - \delta_2)
\]

\[
= -n - 2 - p\beta k \frac{1 - \tau_1}{k} - qam \frac{1 - \tau_2}{m} + \alpha + \beta + 4 + n(1 - \delta_2)
\]

\[
= \alpha + \beta + 2 - n - (\alpha + \beta + 2)(1 - \delta_2) + n(1 - \delta_2)
\]

\[
= (\alpha + \beta + 2 - n) \delta_2,
\]

where the third equality above is due to \( p\beta k = (p + 1)\beta = (q + 1)\alpha = qam \) and \( (p + 1)\beta = \alpha + \beta + 2 \). So, we only need to prove \( \delta_2 > 0 \) or equivalently by (51), (52) and (58),

\[
(m - \frac{k}{l}) \frac{1}{z} + \left( \frac{k}{n - 1} + (m - 1)(k - 1) \right) \frac{1}{l} + \frac{m - 2}{n - 1} - (m - 1) > 0,
\]

(59)

To achieve this, we take the upper bound of \( \frac{1}{z} \) in (55) and see whether (59) holds.

**Case 2.1.1.** If \( \frac{1}{l} - \frac{1}{n - 1} \geq \frac{1}{q + 1} + \frac{1}{n - 1} \), then let \( \frac{1}{z} = \frac{1}{q + 1} + \frac{1}{n - 1} \), and (59) becomes,

\[
\left( \frac{1}{pq} - \frac{p + 1}{p(q + 1)} \right) \frac{1}{l} + \frac{q + 1}{q} \left( \frac{1}{n - 1} + \frac{1}{q + 1} \right) + \frac{1}{q(n - 1)} - \frac{1}{q} > 0
\]

\[
\Rightarrow \frac{1}{pq} - \frac{p + 1}{p(q + 1)} \frac{1}{l} + \frac{2}{q(n - 1)} > 0
\]

\[
\Rightarrow -\frac{2}{\beta} + \frac{2}{n - 1} > 0
\]

\[
\Rightarrow s\beta > n - 1.
\]

**Case 2.1.2.** If \( \frac{1}{l} - \frac{1}{n - 1} < \frac{1}{q + 1} + \frac{1}{n - 1} \), then let \( \frac{1}{z} = \frac{1}{l} - \frac{1}{n - 1} \), and (59) becomes,

\[
(m - \frac{k}{l}) \left( \frac{1}{l} - \frac{1}{n - 1} \right) + \left( \frac{k}{n - 1} + (m - 1)(k - 1) \right) \frac{1}{l} + \frac{m - 2}{n - 1} - (m - 1) > 0
\]

\[
\Rightarrow \frac{k}{l^2} + \left( \frac{k}{n - 1} + \frac{2}{n - 1} \right) \frac{1}{l} + \frac{m - 2}{n - 1} - (m - 1) > 0
\]

\[
\Rightarrow \frac{k}{l^2} + \left( \frac{p + 1}{p + 1} + \frac{2}{n - 1} \right) \frac{1}{l} - \frac{2}{n - 1} + \frac{1}{q} > 0
\]

\[
\Rightarrow \left( \frac{k - 1}{l} - \frac{2}{n - 1} + \frac{1}{q} \right) < 0
\]

\[
\Rightarrow \frac{1}{k} < \frac{1}{l} < \frac{2}{n - 1} + \frac{1}{q}.
\]

Notice that \( \frac{1}{l} < \frac{2}{n - 1} + \frac{1}{q} \) holds under the assumption of case 2.1.2, and \( \frac{1}{k} < \frac{1}{l} \) since \( l < k \). In all, (59) always holds under our assumption \( n - s\beta < 1 \).

**Subcase 2.2.** \( m \geq n - 1 \).

First, we have for any \( \gamma \in [1, \infty) \),

\[
W^{1, m}(\mathbb{S}^{n-1}) \hookrightarrow L^\gamma(\mathbb{S}^{n-1}).
\]
Then we claim \( \frac{k}{l} > \frac{1}{n-1} \). Suppose \( \frac{k}{l} \leq \frac{1}{n-1} \), then \( k > l \geq n - 1 \), hence \( p \leq \frac{1}{n-2} \), which is not possible due to (36). Take \( \frac{1}{z} = \frac{1}{l} - \frac{1}{n-1} \) then

\[
W^{1,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^z(\mathbb{S}^{n-1}).
\]

Therefore, by Sobolev embedding and (48)

\[
G_2(R) \leq CR^n(\|D_xu\|_1(R) + R^{-1}\|u\|_1(R))(\|D_xv\|_1 + R^{-1}\|v\|_1(R))
\]

\[
\leq CR^{n+2}(\|D_xu\|_{l+\varepsilon} + R^{-1}\|D_xu\|_1 + R^{-2}\|u\|_1)
\]

\[
\times (\|D_xv\|_m + R^{-1}\|D_xv\|_1 + R^{-2}\|v\|_1).
\]

Similarly to previous work, there exists a \( \tilde{R} \in [R, 2R] \) such that

\[
G_2(\tilde{R}) \leq CR^{n+2}R^{\frac{p\beta}{1+\varepsilon}} \left( (R^{-n} F(4R))^{\frac{1}{m}} + R^{-\beta \frac{2}{1 + \varepsilon}} + R^{-\beta - 2} \right)
\]

\[
\leq CR^{-\alpha_2} F^{1-\delta_2}(4R),
\]

where

\[
a_2 = a_2^* = -n - 2 + \frac{p\beta}{1 + \varepsilon/l} + \frac{n}{m},
\]

\[
1 - \delta_2 = \frac{1}{m}.
\]

Direct verification shows that

\[
a_2^0 = (\alpha + \beta + 2 - n)\delta_2,
\]

and obviously \( \delta_2 > 0 \) so \( a_2^0 > 0 \).

Thus, we have proved Case 2. \( \square \)

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