On Lyapunov-type inequalities for $(p, q)$-Laplacian systems

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Abstract

We establish Lyapunov-type inequalities for a system involving one-dimensional $(p_i, q_i)$-Laplacian operators ($i = 1, 2$). Next, the obtained inequalities are used to derive some geometric properties of the generalized spectrum associated to the considered problem.

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1 Introduction

In this paper, we are concerned with the following system involving one-dimensional $(p_i, q_i)$-Laplacian operators ($i = 1, 2$):

\[ \begin{align*}
-\left(|u'(x)|^{p_1-2}u'(x)\right)' - \left(|v'(x)|^{p_2-2}v'(x)\right)' &= f(x)|u(x)|^{\alpha}u(x), \\
-\left(|v'(x)|^{q_1-2}v'(x)\right)' - \left(|v'(x)|^{q_2-2}v'(x)\right)' &= g(x)|u(x)|^{\alpha}|v(x)|^{\beta}v(x)
\end{align*} \]

on the interval $(a, b)$, under Dirichlet boundary conditions

\[ \begin{align*}
(u(a) = u(b) = v(a) = v(b) = 0).
\end{align*} \]

System (S) is investigated under the assumptions

\[ \alpha \geq 2, \quad \beta \geq 2, \quad p_i \geq 2, \quad q_i \geq 2, \quad i = 1, 2, \]

and

\[ \frac{2\alpha}{p_1 + q_1} + \frac{2\beta}{p_2 + q_2} = 1. \] (1)

We suppose also that $f$ and $g$ are two nonnegative real-valued functions such that $(f, g) \in L^1(a, b) \times L^1(a, b)$. We establish a Lyapunov-type inequality for problem (S)-(DBC). Next, we use the obtained inequality to derive some geometric properties of the generalized spectrum associated to the considered problem.
The standard Lyapunov inequality [1] (see also [2]) states that if the boundary value problem

\[
\begin{cases}
u''(t) + q(t)u(t) = 0, & a < t < b, \\
u(a) = u(b) = 0,
\end{cases}
\]

has a nontrivial solution, where \( q : [a, b] \to \mathbb{R} \) is a continuous function, then

\[
\int_a^b |q(t)| \, dt > \frac{4}{b - a}.
\]

(2)

Inequality (2) was successfully applied to oscillation theory, stability criteria for periodic differential equations, estimates for intervals of disconjugacy, and eigenvalue bounds for ordinary differential equations. In [3] (see also [4]), Elbert extended inequality (2) to the one-dimensional \( p \)-Laplacian equation. More precisely, he proved that, if \( u \) is a nontrivial solution of the problem

\[
\begin{cases}
(|u'|^{p-2}u')' + h(t)|u|^{p-2}u = 0, & a < t < b, \\
u(a) = u(b) = 0,
\end{cases}
\]

where \( 1 < p < \infty \) and \( h \in L^1(a, b) \), then

\[
\int_a^b |h(t)| \, dt > \frac{2^p}{(b-a)^{p-1}}.
\]

(3)

Observe that for \( p = 2 \), (3) reduces to (2). Inequality (3) was extended in [5] to the following problem involving the \( \phi \)-Laplacian operator:

\[
\begin{cases}
(\phi(u'))' + w(t)\phi(u) = 0, & a < t < b, \\
u(a) = u(b) = 0,
\end{cases}
\]

where \( \phi : \mathbb{R} \to \mathbb{R} \) is a convex nondecreasing function satisfying a \( \Delta_2 \) condition. In [6], Nápoli and Pinasco considered the quasilinear system of resonant type

\[
\begin{cases}
-(|u'|^{p-2}u')' = f(x)|u(x)|^{\alpha-2}|v(x)|^{\beta} u(x), \\
-(|v'|^{q-2}v')' = g(x)|u(x)|^{\alpha}|v(x)|^{q-2} v(x)
\end{cases}
\]

(4)

on the interval \((a, b)\), with Dirichlet boundary conditions

\[
u(a) = u(b) = v(a) = v(b) = 0.
\]

(5)

Under the assumptions \( p, q > 1, f, g \in L^1(a, b), f, g \geq 0, \alpha, \beta \geq 0, \) and

\[
\frac{\alpha}{p} + \frac{\beta}{q} = 1,
\]
it was proved (see [6], Theorem 1.5) that if (4)-(5) has a nontrivial solution, then
\[
2^\alpha + \beta \leq (b-a)^{\frac{p}{p-1}} \left( \int_a^b f(x) \, dx \right)^{\frac{p}{q}} \left( \int_a^b g(x) \, dx \right)^{\frac{q}{q}},
\]  
(6)

where \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \). Some nice applications to generalized eigenvalues are also presented in [6]. Different generalizations and extensions of inequality (6) were obtained by many authors. In this direction, we refer the reader to [7–16] and the references therein. For other results concerning Lyapunov-type inequalities, we refer the reader to [17–29] and the references therein.

2 Lyapunov-type inequalities

A Lyapunov-type inequality for problem (S)-(DBC) is established in this section, and some particular cases are discussed.

**Theorem 2.1** If (S)-(DBC) admits a nontrivial solution \((u,v) \in C^2[a,b] \times C^2[a,b]\), then
\[
\left[ \min \left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}} \right\} \right]^{\frac{2^{p_1}}{p_1-1}} \left[ \min \left\{ \frac{2^{p_2}}{(b-a)^{p_2-1}}, \frac{2^{q_2}}{(b-a)^{q_2-1}} \right\} \right]^{\frac{2^{q_2}}{q_2-1}} \leq \left( \frac{1}{2} \int_a^b f(x) \, dx \right)^{\frac{2^{p_1}}{p_1-1}} \left( \frac{1}{2} \int_a^b g(x) \, dx \right)^{\frac{2^{q_2}}{q_2-1}}.
\]  
(7)

**Proof** Let \((u,v) \in C^2[a,b] \times C^2[a,b]\) be a nontrivial solution to (S)-(DBC). Let \((x_0,y_0) \in (a,b) \times (a,b)\) be such that
\[
|u(x_0)| = \max \{|u(x)| : a \leq x \leq b\}
\]
and
\[
|v(y_0)| = \max \{|v(x)| : a \leq x \leq b\}.
\]

From the boundary conditions (DBC), we can write that
\[
2u(x_0) = \int_a^{x_0} u'(x) \, dx - \int_{x_0}^b u'(x) \, dx,
\]
which yields
\[
2|u(x_0)| \leq \int_a^b |u'(x)| \, dx.
\]

Using Hölder’s inequality with parameters \(p_1\) and \(p'_1 = \frac{p_1}{p_1-1}\), we get
\[
2|u(x_0)| \leq (b-a)^{\frac{1}{p_1}} \left( \int_a^b |u'(x)|^{p_1} \, dx \right)^{\frac{1}{p_1}}.
\]
that is,
\[ \frac{2^{p_1}}{(b-a)^{p_1-1}} |u(x_0)|^{p_1} \leq \int_a^b |u'(x)|^{p_1} \, dx. \]  
(8)

Similarly, using Hölder’s inequality with parameters \( q \) and \( q' = \frac{q}{q-1} \), we get
\[ \frac{2^{q_1}}{(b-a)^{q_1-1}} |u(x_0)|^{q_1} \leq \int_a^b |u'(x)|^{q_1} \, dx. \]  
(9)

By repeating the same argument for the function \( v \), we obtain
\[ \frac{2^{p_2}}{(b-a)^{p_2-1}} |v(y_0)|^{p_2} \leq \int_a^b |v'(x)|^{p_2} \, dx \]  
(10)

and
\[ \frac{2^{q_2}}{(b-a)^{q_2-1}} |v(y_0)|^{q_2} \leq \int_a^b |v'(x)|^{q_2} \, dx. \]  
(11)

Now, multiplying the first equation of (S) by \( u \) and integrating over \((a, b)\), we obtain
\[ \int_a^b |u'(x)|^{p_1} \, dx + \int_a^b |u'(x)|^{q_1} \, dx = \int_a^b f(x) |u(x)|^\alpha |v(x)|^\beta \, dx. \]  
(12)

Multiplying the second equation of (S) by \( v \) and integrating over \((a, b)\), we obtain
\[ \int_a^b |v'(x)|^{p_2} \, dx + \int_a^b |v'(x)|^{q_2} \, dx = \int_a^b g(x) |u(x)|^\alpha |v(x)|^\beta \, dx. \]  
(13)

Using (8), (9) and (12), we obtain
\[ |u(x_0)|^\alpha |v(y_0)|^\beta \int_a^b f(x) \, dx \geq \frac{2^{p_1}}{(b-a)^{p_1-1}} |u(x_0)|^{p_1} + \frac{2^{q_1}}{(b-a)^{q_1-1}} |u(x_0)|^{q_1}, \]
which yields
\[ |u(x_0)|^\alpha |v(y_0)|^\beta \int_a^b f(x) \, dx \geq \min \left\{ \frac{2^{p_1}}{(b-a)^{p_1-1}}, \frac{2^{q_1}}{(b-a)^{q_1-1}} \right\} (|u(x_0)|^{p_1} + |u(x_0)|^{q_1}). \]

Using the inequality
\[ A + B \geq 2\sqrt{AB} \]
with \( A = |u(x_0)|^{p_1} \) and \( B = |u(x_0)|^{q_1} \), we get
\[ \min \left\{ \frac{2^{p_1+1}}{(b-a)^{p_1+1}}, \frac{2^{q_1+1}}{(b-a)^{q_1+1}} \right\} \leq |u(x_0)|^{\alpha \frac{p_1+1}{p_1+q_1}} |v(y_0)|^{\beta \frac{q_1}{p_1+q_1}} \int_a^b f(x) \, dx. \]  
(14)

Similarly, using (10), (11) and (13), we obtain
\[ \min \left\{ \frac{2^{p_2+1}}{(b-a)^{p_2+1}}, \frac{2^{q_2+1}}{(b-a)^{q_2+1}} \right\} \leq |u(x_0)|^\alpha |v(y_0)|^\beta \frac{p_2+1}{p_2+q_2} \int_a^b g(x) \, dx. \]  
(15)
Raising inequality (14) to a power $e_1 > 0$, inequality (15) to a power $e_2 > 0$, and multiplying the resulting inequalities, we obtain

\[
\left[ \min \left\{ \frac{2^{p+1}}{(b-a)^{p+1}}, \frac{2^{q+1}}{(b-a)^{q+1}} \right\} \right]^{e_1} \left[ \min \left\{ \frac{2^{p_2+1}}{(b-a)^{p_2+1}}, \frac{2^{q_2+1}}{(b-a)^{q_2+1}} \right\} \right]^{e_2} \leq \left| u(x_0) \right|^{\left[ \frac{(b-x_0)\left| u(x_0) \right|^{p+1} + (x_0-b)\left| u(x_0) \right|^{q+1}}{(b-a)^{p+1}} \right]} \left| u(y_0) \right|^{\left[ \frac{(y_0-x_0)\left| u(x_0) \right|^{p+1} + (x_0-y_0)\left| u(x_0) \right|^{q+1}}{(b-a)^{p+1}} \right]} \left( \int_a^b f(x) \, dx \right)^{e_1} \left( \int_a^b g(x) \, dx \right)^{e_2}.
\]

Next, we take $(e_1, e_2)$ any solution of the homogeneous linear system

\[
\begin{align*}
\alpha & = \alpha - \frac{q}{2} \frac{e_1}{a(e_1 + \alpha e_2)}, \\
\beta e_1 + (\beta - \frac{p+q}{2}) e_2 & = 0.
\end{align*}
\]

Using (1), we may take

\[
\begin{align*}
e_1 & = \alpha, \\
e_2 & = \frac{(\alpha + \beta)}{p+q}.
\end{align*}
\]

Therefore, we obtain

\[
2^{\alpha + \frac{p+q}{2}} \left[ \min \left\{ \frac{2^p}{(b-a)^{p+1}}, \frac{2^q}{(b-a)^{q+1}} \right\} \right]^{\frac{\alpha}{2}} \left[ \min \left\{ \frac{2^{p_2}}{(b-a)^{p_2+1}}, \frac{2^{q_2}}{(b-a)^{q_2+1}} \right\} \right]^{\frac{\alpha}{2}} \left( \int_a^b f(x) \, dx \right)^{\frac{\alpha}{2}} \left( \int_a^b g(x) \, dx \right)^{\frac{\alpha}{2}}.
\]

Using again (1), we get

\[
2 \left[ \min \left\{ \frac{2^p}{(b-a)^{p+1}}, \frac{2^q}{(b-a)^{q+1}} \right\} \right]^{\frac{2\alpha}{p+q}} \left[ \min \left\{ \frac{2^{p_2}}{(b-a)^{p_2+1}}, \frac{2^{q_2}}{(b-a)^{q_2+1}} \right\} \right]^{\frac{2\alpha}{p+q}} \left( \int_a^b f(x) \, dx \right)^{\frac{2\alpha}{p+q}} \left( \int_a^b g(x) \, dx \right)^{\frac{2\alpha}{p+q}},
\]

which proves Theorem 2.1. \qed

As a consequence of Theorem 2.1, we deduce the following result for the case of a single equation.

**Corollary 1** Let us assume that there exists a nontrivial solution of

\[
\begin{align*}
-(\left| u'(x) \right|^p + 2 u''(x))' - (\left| u'(x) \right|^{q-2} u''(x))' & = f(x) |u(x)|^{\frac{p+q}{2}-2} u(x), \quad x \in (a, b), \\
u(a) = u(b) = 0,
\end{align*}
\]

where $p > 1$, $q > 1$, $f \geq 0$, and $f \in L^1(a, b)$. Then

\[
\min \left\{ \frac{2^p}{(b-a)^{p-1}}, \frac{2^q}{(b-a)^{q-1}} \right\} \leq \frac{1}{2} \int_a^b f(x) \, dx.
\]
Proof An application of Theorem 2.1 with
\[ p_1 = p_2 = p, \quad q_1 = q_2 = q, \quad \alpha = \frac{p + q}{2}, \quad \beta = 0, \quad v = u, \quad g = f, \]
yields the desired result. \qed

Remark 1 Taking \( f = 2h \) and \( q = p \) in Corollary 1, we obtain Lyapunov-type inequality (3) for the one-dimensional \( p \)-Laplacian equation.

Remark 2 Taking \( p_1 = q_1 = p \) and \( p_2 = q_2 = q \) in Theorem 2.1, we obtain Lyapunov-type inequality (6).

3 Generalized eigenvalues
The concept of generalized eigenvalues was introduced by Protter [30] for a system of linear elliptic operators. The first work dealing with generalized eigenvalues for \( p \)-Laplacian systems is due to Nápoli and Pinasco [6]. Inspired by that work, we present in this section some applications to generalized eigenvalues related to problem (S)-(DBC).

Let us consider the generalized eigenvalue problem

\[
(S)_{\lambda,\mu}: \begin{cases}
-(|u'(x)|^{p_1-2}u'(x))' - (|v'(x)|^{q_1-2}v'(x))' = \lambda \alpha w(x)|u(x)|^{\alpha-2}|u(x)|^\beta u(x), \\
-(|v'(x)|^{p_2-2}v'(x))' - (|v'(x)|^{q_2-2}v'(x))' = \mu \beta w(x)|u(x)|^\beta |v(x)|^{\beta-2}v(x),
\end{cases}
\]
on the interval \((a, b)\), with Dirichlet boundary conditions (DBC). If problem \((S)_{\lambda,\mu}-(DBC)\) admits a nontrivial solution \((u, v) \in C^2[a, b] \times C^2[a, b]\), we say that \((\lambda, \mu)\) is a generalized eigenvalue of \((S)_{\lambda,\mu}-(DBC)\). The set of generalized eigenvalues is called generalized spectrum, and it is denoted by \(\sigma\).

We assume that
\[ \alpha \geq 2, \quad \beta \geq 2, \quad p_i \geq 2, \quad q_i \geq 2, \quad i = 1, 2, \quad w \geq 0, \quad w \in L^1(a, b), \]
and (1) is satisfied.

The following result provides lower bounds of the generalized eigenvalues of \((S)_{\lambda,\mu}-(DBC)\).

Theorem 3.1 Let \((\lambda, \mu)\) be a generalized eigenvalue of \((S)_{\lambda,\mu}-(DBC)\). Then
\[ \mu \geq h(\lambda), \]
where \( h: (0, \infty) \rightarrow (0, \infty) \) is the function defined by
\[ h(t) = \frac{1}{\beta} \left( \frac{C}{\int_a^b w(x) \, dx} \right)^{\frac{p_2 + q_2}{2}} , \quad t > 0, \]
with
\[
\alpha \frac{2p}{p + q} C = 2 \left[ \min \left\{ \frac{2p_1}{(b-a)^{p_1-1}}, \frac{2q_1}{(b-a)^{q_1-1}} \right\} \right]^{\frac{2p}{p + q}} \\
\times \left[ \min \left\{ \frac{2p_2}{(b-a)^{p_2-1}}, \frac{2q_2}{(b-a)^{q_2-1}} \right\} \right]^{\frac{2q}{p + q}}.
\]
Proof Let \((\lambda, \mu)\) be a generalized eigenpair, and let \(u, v\) be the corresponding nontrivial solutions. By replacing in Lyapunov-type inequality (7) the functions
\[ f(x) = \alpha \lambda w(x), \quad g(x) = \beta \mu w(x), \]
and using condition (1), we obtain
\[ 2M \leq \alpha^{\frac{2\mu}{P_1+q}} \beta^{\frac{2\mu}{P_2+q}} \int_a^b w(x) \, dx, \]
where
\[ M = \left[ \min \left\{ \frac{2^q}{(b-a)^{p_1-1}}, \frac{2^q}{(b-a)^{p_1-1}} \right\} \right]^{\frac{2\lambda}{P_1+q}} \left[ \min \left\{ \frac{2^p}{(b-a)^{p_2-1}}, \frac{2^p}{(b-a)^{p_2-1}} \right\} \right]^{\frac{2\mu}{P_2+q}}. \]
Hence, we have
\[ \mu^{\frac{2\beta}{P_2+q}} \geq \frac{C}{\lambda^{\frac{2\mu}{P_1+q}} \beta^{\frac{2\mu}{P_2+q}} \int_a^b w(x) \, dx}, \]
which yields
\[ \mu \geq \frac{1}{\beta} \left( \frac{C}{\lambda^{\frac{2\lambda}{P_1+q}} \int_a^b w(x) \, dx} \right)^{\frac{P_2+q}{2\beta}}, \]
and the proof is finished. \(\square\)

As consequences of the previous obtained result, we deduce the following Protter’s type results for the generalized spectrum.

**Corollary 2** There exists a constant \(c_{a,b} > 0\) that depends on \(a\) and \(b\) such that no point of the generalized spectrum \(\sigma\) is contained in the ball \(B(0, c_{a,b})\), where
\[ B(0, c_{a,b}) = \{ x = (x_1, x_2) \in \mathbb{R}^2 : \|x\|_\infty < c_{a,b} \}, \]
and \(\| \cdot \|_\infty\) is the Chebyshev norm in \(\mathbb{R}^2\).

Proof Let \((\lambda, \mu) \in \sigma\). From (16), we obtain easily that
\[ \lambda^{\frac{2\mu}{P_1+q}} \mu^{\frac{2\mu}{P_2+q}} \geq \frac{C}{\beta^{\frac{2\mu}{P_2+q}} \int_a^b w(x) \, dx}, \]
On the other hand, using condition (1), we have
\[ \lambda^{\frac{2\lambda}{P_1+q}} \mu^{\frac{2\mu}{P_2+q}} \leq \| (\lambda, \mu) \|_{\infty}^{\frac{2\lambda}{P_1+q}} \| \mu^{\frac{2\mu}{P_2+q}} \|_{\infty}^{\frac{2\mu}{P_2+q}} = \| (\lambda, \mu) \|_{\infty}. \]
Therefore, we obtain
\[ \| (\lambda, \mu) \|_{\infty} \geq c_{a,b}, \]
where
\[ c_{a,b} = \frac{C}{\beta^{\frac{2p}{p+q}} \int_a^b w(x) \, dx}. \]

The proof is finished. \qed

**Corollary 3** Let \((\lambda, \mu)\) be fixed. There exists an interval \(I\) of sufficiently small measure such that, if \(I = [a, b] \subset J\), then there are no nontrivial solutions of \((S)_{\lambda,\mu}^-\) (DBC).

**Proof** Suppose that \((S)_{\lambda,\mu}^-\) (DBC) admits a nontrivial solution. Since \(C \to +\infty\) as \(b - a \to 0^+\), where \(C\) is defined in Theorem 3.1, there exists \(\delta > 0\) such that
\[ b - a < \delta \quad \implies \quad \frac{C}{\int_a^b w(x) \, dx} > \lambda^{\frac{2p}{p+q}} \mu^{\frac{2p}{p+q}} \beta^{\frac{2p}{p+q}}. \]

Let \(J = [a, a + \delta]\). Hence, if \(I \subset J\), we have
\[ \frac{C}{\beta^{\frac{2p}{p+q}} \int_a^b w(x) \, dx} > \lambda^{\frac{2p}{p+q}} \mu^{\frac{2p}{p+q}} , \]
which is a contradiction with (17). Therefore, if \(I \subset J\), there are no nontrivial solutions of \((S)_{\lambda,\mu}^-\) (DBC). \qed

**4 Conclusion**

Lyapunov-type inequalities for a system of differential equations involving one-dimensional \((p_i, q_i)\)-Laplacian operators \((i = 1, 2)\) are derived. It was shown that such inequalities are very useful to obtain geometric characterizations of the generalized spectrum associated with the considered problem.

**Competing interests**
The authors declare to have no competing interests.

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