Isospectral Deformations of Eguchi-Hanson Spaces as Nonunital Spectral Triples

C. Yang∗

Department of Mathematical Sciences
University of Durham, England, DH1 3LE

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Abstract

We study the isospectral deformations of the Eguchi-Hanson spaces along a torus isometric action in the noncompact noncommutative geometry. We concentrate on locality, smoothness and summability conditions of the nonunital spectral triples, and relate them to the geometric conditions to be noncommutative spin manifolds.

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∗E-mail address: chen.yang2@durham.ac.uk
1 Introduction

As a generalization of Connes’ noncommutative differential geometry [1], noncompact noncommutative geometry is the study of nonunital spectral triples [2, 3]. Various authors also consider the aspect of summability as in [4], [5], [6].

In the unital case, Connes provides a set of axioms for unital spectral triples so to define compact noncommutative spin manifolds. See for example [1]. Rennie and Várilly explicitly reconstruct compact commutative spin manifolds from slightly modified axioms [7]. As to the nonunital case, a complete generalization considering these axioms is not known yet. There are various nonunital examples [8], [3], [9], which may serve the purpose of testing the axioms or geometric conditions suggested. In this article, we obtain another nonunital example by isospectral deformation of Eguchi-Hanson (EH-) spaces [10]. They are geodesically complete Riemannian spin manifolds in the commutative geometry.

Isospectral deformation is a simple method to deform a commutative spectral triple. It traces back to the Moyal type of deformation from quantum mechanics. Rieffel’s insight is to consider Lie group actions on function spaces and hence explain the Moyal product between functions by oscillatory integrals over the group actions [11]. Apart from the well-known Moyal planes and noncommutative tori [12], this scheme allows more general deformations. Connes and Landi in [13] deform spheres and more general compact spin manifold with isometry group containing a two-torus. Connes and Dubois-Violette in [14] observe that this works equally well for noncompact spin manifolds. As in the appendix of [2], it is possible to fit such noncompact examples in the nonunital framework there. The deformation of EH-spaces we will consider in the following is obtained by these methods and serves as an example of a nonunital triple.

The Eguchi-Hanson spaces are of interest in both Riemannian geometry and physics. Geometrically, they are the simplest asymptotic locally Euclidean (ALE) spaces, for which a complete classification is provided by Kronheimer through the
method of hyper-Kähler quotients \[15\]. This construction realizes the family of EH-spaces as a resolution of a singular conifold. In physics, where they first appeared, EH-spaces are known as gravitational instantons. Due to their hyper-Kähler structure, the ADHM construction \[16\], obtaining Yang-Mills’ instantons, is generalized on the EH-spaces in an elegant way \[17\],\[18\]. The nonunital spectral triple from isospectral deformation of Eguchi-Hanson spaces may thus link various perspectives.

Our aim in this article is to concentrate on the locality, smoothness \[2\] and summability conditions of these triples and further see how they fit into the modified geometric conditions for nonunital spectral triples.

The organization of the rest of the article is as follows. In section 2, we describe the Eguchi-Hanson spaces in the spin geometry. In section 3, we consider algebras of functions over EH-spaces, the deformation quantization of algebras, and representations of algebras as operators on the Hilbert space of spinors. We also obtain a projective module description of the spinor bundle. In section 4, we define spectral triples of the deformed EH-spaces and study their summability. In section 5, we discuss how the triple fits into the modified geometric conditions. We conclude in Section 6.

2 Spin geometry of Eguchi-Hanson spaces

In this section, we first describe the metric and the Levi-Civita connection of the Eguchi-Hanson space, and then introduce its spinor bundle, the spin connection and the Dirac operator. Finally, we write down the torus action through parallel propagators on the spinor bundle.

2.1 Metrics, connections and torus isometric actions

The Eguchi-Hanson spaces were originally constructed as gravitational instantons \[10\]. Generalized by Gibbons and Hawking, they fall into a new category of solutions of the Einstein’s equation, known as the multicenter solutions \[19\]. In local coordinates, the metric is

\[ds^2 = \Delta^{-1} dr^2 + r^2 \left[ (\sigma_x^2 + \sigma_y^2) + \Delta \sigma_z^2 \right],\]

where \(\Delta := \Delta(r) := 1 - a^4/r^4\) and \(\{\sigma_x, \sigma_y, \sigma_z\}\) are the standard Cartan basis for three sphere,

\[\begin{align*}
\sigma_x &= \frac{1}{2} \left( -\cos \psi \, d\theta - \sin \theta \sin \psi \, d\phi \right), \\
\sigma_y &= \frac{1}{2} \left( \sin \psi \, d\theta - \sin \theta \cos \psi \, d\phi \right), \\
\sigma_z &= \frac{1}{2} \left( -d\psi - \cos \theta \, d\phi \right),
\end{align*}\]

with \(r \geq a,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi,\ 0 \leq \psi < 2\pi.\)

**Remark 2.1.** The convention that the period of \(\psi\) is \(2\pi\) rather than \(4\pi\) as in the original construction is suggested in \[19\] to remove the singularity at \(r = a\), so that the manifold becomes geodesically complete.
The EH-space is diffeomorphic to the tangent bundle of a 2-sphere $T(S^2)$. Modulo a distortion of the metric, the base as a unit two sphere $S^2$ is parametrised by parameters $\phi$ and $\theta$, with $\theta = 0$ as the south pole and $\theta = \pi$ as the north pole. The angle $\phi$ parametrises the circle defined by a constant $\theta$. Over each point, say $(\theta, \phi)$ on the 2-sphere, the tangent plane is parametrized by $(r, \psi)$. $r$ parametrizes the radial direction with $r = a$ at the origin of the plane. Circles of constant $r$ are parametrised by $\psi$. The identification of $\psi = \psi + 2\pi$ is the identification the antipodal points on the circle of constant radius. Together with the metric, this implies that the space at large enough $r$ is asymptotic to $\mathbb{R}^4/\mathbb{Z}^2$, so that it is an ALE space.

The parameter $a$ in the metric (1) is a non-negative real number parametrizing a family of EH-spaces. When $a = 0$, the metric degenerates to the conifold $\mathbb{R}^4/\mathbb{Z}^2$ and the rest of the family is a resolution of the conifold. This appears as the simplest case in Kronheimer’s classification of ALE spaces [15]. We will only concentrate on the smooth case so that $a$ is assumed to be positive.

Choose the local coordinates $\{x_i\}$ with $x_1 = r$, $x_2 = \theta$, $x_3 = \phi$, $x_4 = \psi$. We will write the coordinates $(r, \theta, \phi, \psi)$ and $(x_1, x_2, x_3, x_4)$ interchangeably throughout the article, because the former give a clear geometric picture while the latter are convenient in tensorial expressions. The corresponding basis on the tangent space $T_x(EH)$ of any point $x \in EH$ are $\{\partial_i := \frac{\partial}{\partial x_i}\}$, and the dual basis on the cotangent space $T^*_x(EH)$ are $\{dx^j\}$. The corresponding metric tensor $g_{ij}(x) dx^i \otimes dx^j$ can be written as entries of the matrix $G = (g_{ij})$ as

$$G(x) = \frac{1}{4} \begin{pmatrix} 4 \Delta^{-1} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & \rho & r^2 \Delta \cos \theta \\ 0 & 0 & r^2 \Delta \cos \theta & r^2 \Delta \end{pmatrix} \quad (2)$$

where $\rho := \rho(r, \theta) := (r^4 - a^4 \cos^2 \theta) / r^2$. We always assume Einstein’s summation convention.

In the same coordinate chart, the Christoffel symbols of the Levi-Civita connection of (1), defined by $\nabla_i \partial_j = \Gamma^k_{ij} \partial_k$, are explicitly,

$$\Gamma^1_{11} = -\frac{\Delta^+}{\Delta}, \quad \Gamma^1_{22} = -\frac{r \Delta}{4}, \quad \Gamma^1_{33} = -\frac{\Delta \rho^+}{4 r},$$

$$\Gamma^1_{34} = -\frac{r \Delta^+ \Delta \cos \theta}{4}, \quad \Gamma^1_{44} = -\frac{r \Delta^+ \Delta}{4},$$

$$\Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^2_{33} = -\frac{a^4 \sin 2\theta}{2 r^4}, \quad \Gamma^2_{34} = \frac{\Delta \sin \theta}{2},$$

$$\Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^3_{23} = \frac{\cot \theta \Delta^+}{2}, \quad \Gamma^3_{24} = -\frac{\Delta}{2 \sin \theta},$$

$$\Gamma^4_{13} = \frac{2 a^4 \cos \theta}{r(r^4 - a^4)}, \quad \Gamma^4_{14} = \frac{\Delta^+}{r \Delta}, \quad \Gamma^4_{23} = \frac{\rho^+}{2 r^2 \sin \theta}, \quad \Gamma^4_{24} = \frac{\cot \theta \Delta}{2}. \quad (3)$$
where
\[ \Delta^+ := \Delta(r) := 1 + \frac{a^4}{r^4}, \quad \Delta' := \frac{\partial \Delta}{\partial r}, \quad \rho^+ := \rho^+(r, \theta) := \frac{r^4 + a^4 \cos^2 \theta}{r^2}. \]

The identity \( \Gamma^i_{jk} = \Gamma^i_{kjis} \), implied by the torsion free property of the connection, generates another set of symbols and all the rest of the Christoffel symbols vanish.

The isometry group of the metric (1) is \((U(1) \times SU(2))/\mathbb{Z}^2\). The Killing vector \( \partial \psi \) generates the group \( U(1)/\mathbb{Z}^2 \). Another Killing vector is \( \partial \phi \). Its action on the restriction of the space at \( r = a \) is analogous to one of the three typical generators of the Lie algebra of the Lie group \( SU(2) \) on a standard two-sphere. These are the two Killing vectors which define a torus action \( \sigma \) on the Eguchi-Hanson space, \( \sigma : U(1) \times U(1) \rightarrow \text{Aut}(EH) \),
\[ \text{(4)} \]
by \( \sigma(\exp(i t_3 \partial_\phi), \exp(i t_4 \partial_\psi))(r, \theta, \phi, \psi) = (r, \theta, \phi + t_3, \psi + t_4) \), where \( 0 \leq t_3 < 2\pi \), \( 0 \leq t_4 < 2\pi \) and for any point \((r, \theta, \phi, \psi) \in EH\). The isometric torus action will determine the isospectral deformation later.

2.2 The stereographic projection and orthonormal basis

We choose an orthonormal basis to trivialize the cotangent bundle of the EH-space and obtain the corresponding transition functions. Since the EH-space is topologically the same as \( T(S^2) \), we may obtain another set of coordinates by taking the stereographic projection of the \( S^2 \) part, while keeping the coordinates on the tangent space unchanged.

The EH-space (1) can be covered by two open neighbourhoods \( U_N \) and \( U_S \), where \( U_N \) covers the whole space except at \( \theta = \pi \) and \( U_S \) covers the whole space except at \( \theta = 0 \). We may define the map \( f_N : U_N \rightarrow \mathbb{C} \times \mathbb{R}^2 \) by taking a stereographic projection of the base two sphere to \( \mathbb{C} \). I.e., \( f_N(\phi, \theta, r, \psi) = (z; r, \psi) \). For the coordinate chart \( U_S \), we similarly define the projection map \( f_S : U_S \rightarrow \mathbb{C} \times \mathbb{R}^2 \), by \( f_S(\phi, \theta, r, \psi) = (w; r, \psi) \), where
\[ z := \cot \frac{\theta}{2} e^{-i\phi}, \quad w := \tan \frac{\theta}{2} e^{i\phi}. \]

For any point \( x \in U_N \cap U_S \), the transition function from the coordinate charts \( U_S \) to \( U_N \) is
\[ (w, r, \psi) = (\frac{1}{z}; r, \psi), \]
and the transition function from \( U_N \) to \( U_S \) is \( (z, r, \psi) = (\frac{1}{w}; r, \psi) \).

The restriction of metric (1) on the \( U_N \) chart with coordinates \((z; r, \psi)\) is,
\[ ds^2 = \frac{r^2}{(1+z^2)^2}dzd\bar{z} + \frac{r^2}{4} \left[ d\psi + \frac{z\bar{z} - 1}{z\bar{z} + 1} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right) \right]^2. \]

To obtain a local orthonormal basis of \( T^*(EH)_{U_N} \) we may simply define
\[ l := \frac{r}{\sqrt{2}(1 + z^2)} \, dz, \quad m := \frac{\Delta^{-1/2}}{\sqrt{2}} \, dr + \frac{r \Delta^{1/2}}{4\sqrt{2}} \left[ \left( \frac{dz}{\Delta} - \frac{d\bar{z}}{\Delta} \right) \frac{1 - z\bar{z}}{1 + z\bar{z}} + 2i \, d\psi \right], \]

with their complex conjugates \( \bar{l}, \bar{m} \) so that the metric tensor over \( U_N \) is \( ds^2 = l \otimes \bar{l} + m \otimes \bar{m} + \bar{m} \otimes m \).

A real orthonormal frame \( \{ \vartheta^n \} \) of \( T^*(EH)_{U_N} \) is thus defined by

\[ \vartheta^1 := \frac{1}{\sqrt{2}}(l + \bar{l}), \quad \vartheta^2 := -\frac{i}{\sqrt{2}}(l - \bar{l}), \quad \vartheta^3 := -\frac{i}{\sqrt{2}}(m - \bar{m}), \quad \vartheta^4 := \frac{1}{\sqrt{2}}(m + \bar{m}). \]

such that the metric on \( U_N \) is diagonalized as \( ds^2 = \delta_{\alpha \beta} \vartheta^\alpha \otimes \vartheta^\beta \). The coordinate transformations \( \vartheta^\alpha = h^\alpha_i dx^i \) are determined by the matrix \( H = (h^\alpha_i) \),

\[ H = \frac{1}{2} \begin{pmatrix} 0 & -r \cos \phi & -r \sin \theta \sin \phi & 0 \\ 0 & r \sin \phi & -r \sin \theta \cos \phi & 0 \\ 0 & 0 & r \Delta^{1/2} \cos \theta & r \Delta^{1/2} \\ \Delta^{-1/2} & 0 & 0 & 0 \end{pmatrix}, \quad (5) \]

whose inverse \( H^{-1} = (\bar{h}^\beta_j) \) from \( dx^j = \bar{h}^\beta_j \vartheta^\beta \) is

\[ H^{-1} = 2 \begin{pmatrix} 0 & \frac{r \cos \phi}{r \sin \theta} & \frac{r \sin \phi}{r \sin \theta} & 0 \\ \frac{r \cos \phi}{r \sin \theta} & 0 & \frac{r \sin \phi}{r \sin \theta} & 0 \\ \frac{r \cos \phi}{r \sin \theta} & \frac{r \sin \phi}{r \sin \theta} & 0 & \frac{1}{r \Delta^{1/2}} \\ \frac{r \cos \phi}{r \sin \theta} & \frac{r \sin \phi}{r \sin \theta} & \frac{1}{r \Delta^{1/2}} & 0. \end{pmatrix}, \quad (6) \]

The above construction on the \( U_N \) chart can be carried out the same way on the \( U_S \) coordinates. We denote orthonormal frames over \( U_S \) by adding *'s to \( l, m, \vartheta^\alpha, x_j \) and etc.

Local frames \( \{ \vartheta^\alpha \} \) on \( U_N \) define a local trivialization of the cotangent bundle, \( F_N : T^*(EH)_{U_N} \to U_N \times \mathbb{R}^4 \) by \( F_N(x; a_1 \vartheta^1 + \cdots + a_4 \vartheta^4) := (x; a_1, \ldots, a_4) \), where \( a_\alpha \)'s are real-valued functions over \( U_N \). In a similar way, the choice of local frames \( \{ \vartheta^\alpha \} \) on \( U_S \) defines a local trivialization of the cotangent bundle, \( F_S : T^*(EH)_{U_S} \to U_N \times \mathbb{R}^4 \).

The transition functions \( f^\alpha_{\beta} 's \) such that \( \vartheta^\alpha = f^\beta_{\alpha} \vartheta^\beta \) are elements of the matrix \( F_{SN} := F_N \circ F_{S}^{-1} \) as

\[ F_{SN} = \begin{pmatrix} \frac{-\pi^2 + z^2}{\pi^2 + 2z} & i \frac{-\pi^2 - z^2}{\pi^2 + 2z} & 0 & 0 \\ i \frac{-\pi^2 - z^2}{\pi^2 + 2z} & \frac{-\pi^2 + z^2}{\pi^2 + 2z} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -\cos 2\phi & \sin 2\phi & 0 & 0 \\ -\sin 2\phi & -\cos 2\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (7) \]

The inverse transition function is given by the inverse of the matrix \( F_{SN} \), \( F_{NS} := F_S \circ F_N^{-1} = F_{SN}^{-1} \). The cotangent bundle is thus

\[ T^*(EH) = (U_N \times \mathbb{R}^4) \cup (U_S \times \mathbb{R}^4) / \sim, \quad (8) \]

where \( (x; a_1, \ldots, a_4) \in U_N \times \mathbb{R}^4 \) and \( (x'; a'_1, \ldots, a'_4) \in U_S \times \mathbb{R}^4 \) are defined to be equivalent if and only if \( x = x' \) and \( F_{NS}(a_1, \ldots, a_4) = (a'_1, \ldots, a'_4)' \).
2.3 Spin structures and spinor bundles

Following a standard procedure from [20], we obtain the spinor bundle of the EH-space. In coordinate charts \( \{ U_N, U_S \} \), the frame bundle \( P_{SO(4)} \) of the EH-space is the \( SO(4) \)-principal bundle with transition functions \( F_{NS} \) in (7) and its inverse \( F_{SN} \).

Recall that the covering map of groups,

\[
\rho : Spin(4) \rightarrow SO(4),
\]

is defined by the adjoint representation of \( Spin(4) \) as \( \rho(w)x := w \cdot x \cdot w^\dagger \) for \( x \in \mathbb{R}^4 \), where \( w = v_1 \cdots v_m \in Spin(4) \), \( m \) is even and \( v_i \in \mathbb{R}^4 \) for \( i = 1, \ldots, m \). Geometrically, \( \rho(w) = \rho(v_1) \circ \cdots \circ \rho(v_m) \), where \( \rho(v_i) \) is the reflection of the space \( \mathbb{R}^4 \) with respect to the hyperplane with normal vector \( v_i \).

Locally, the upper left block of the transition matrix (7) is a rotation in the plane spanned by \( \{ \vartheta^1, \vartheta^2 \} \) through an angle \( 2\phi + \pi \). Such rotation can be decomposed to two reflections say \( \rho(v_2) \circ \rho(v_1) \), with

\[
v_1 := \vartheta^1, \quad v_2 := -\sin \phi \vartheta^1 + \cos \phi \vartheta^2.
\]

**Remark 2.2.** Another choice is \( \rho(-v_2) \circ \rho(v_1) \), which gives the same rotation as an element in \( SO(2) \).

\[v_2 \cdot v_1 \in Spin(4)\] is a lifting of \( \rho(v_2) \circ \rho(v_1) \in SO(4) \) under the covering map (9). Thus, in the local coordinate chart \( U_N \), \( \tilde{F}_{SN} := v_2 \cdot v_1 \) in \( Spin(4) \) defines a lifting of the action \( F_{NS} \in SO(4) \) as in (7) under the double covering (9).

To obtain a global lifting of the frame bundle, we consistently define the transition matrix \( \tilde{F}_{NS} \) as a lifting in the group \( Spin(4) \) over \( x' \in U_S \) by \( \tilde{F}_{NS} = -v_2' \cdot v_1' \), where

\[
v_1' := \vartheta'^1, \quad v_2' := \sin \phi \vartheta'^1 + \cos \phi \vartheta'^2.
\]

The following confirms the consistency of the liftings on two coordinate charts.

**Lemma 2.3.** Transition functions \( \{ \tilde{F}_{NS}, \tilde{F}_{SN} \} \) satisfy the cocycle condition, \( \tilde{F}_{NS} \circ \tilde{F}_{SN} = \tilde{F}_{SN} \circ \tilde{F}_{NS} = 1 \).

**Proof.** Applying the transformation from \( \vartheta^\alpha \)'s to \( \vartheta'^\beta \)'s by (7), we have \( \vartheta'^1 \cdot \vartheta'^2 = \vartheta^1 \cdot \vartheta^2 \). Thus,

\[
\begin{align*}
\tilde{F}_{NS} \circ \tilde{F}_{SN} &= -v_2' \cdot v_1' \cdot v_2 \cdot v_1 \\
&= -(\sin \phi \vartheta'^1 + \cos \phi \vartheta'^2) \cdot \vartheta'^1 \cdot (-\sin \phi \vartheta'^1 + \cos \phi \vartheta'^2) \cdot \vartheta^1 \\
&= \sin^2 \phi - \sin \phi \cos \phi (\vartheta^1 \cdot \vartheta^2 + \vartheta^2 \cdot \vartheta^1) - \cos^2 \phi \vartheta^2 \cdot \vartheta^1 \cdot \vartheta^2 \cdot \vartheta^1 = 1,
\end{align*}
\]

by using identities \( \vartheta^\alpha \cdot \vartheta^\alpha = 1 \) and \( \vartheta^\alpha \cdot \vartheta^\beta = \vartheta^\beta \cdot \vartheta^\alpha \) for \( \alpha \neq \beta \), of elements of the orthonormal bases \( \vartheta^\alpha \)'s and those of \( \vartheta'^\beta \)'s. Similarly, \( \tilde{F}_{SN} \circ \tilde{F}_{NS} = 1 \).

\[\square\]
Therefore, the principal Spin(4)-bundle can be defined by

\[ P_{\text{Spin}(4)} := (U_N \times \text{Spin}(4) \cup U_S \times \text{Spin}(4))/\sim. \]  

(10)

where \((x, g) \in U_N \times \text{Spin}(4)\) and \((x', g') \in U_S \times \text{Spin}(4)\) are defined to be equivalent if and only if \(x = x'\) and \(g' = \tilde{F}_{NS} g\).

The double covering of bundles (10) over the EH-space defines a spin structure of it. We will always assume this choice of spin structure.

The spinor bundle can be defined as an associative bundle of typical fiber \(\mathbb{C}^4\) of the principal Spin(4)-bundle (10), by specifying a representation of Spin(4) on \(GL_\mathbb{C}(4)\). We know that locally, for any \(x \in EH\), there exists a unique irreducible representation space \(\Lambda\) of complex dimension 4 of the Clifford algebra \(C\ell(T_x^*(EH))\) through the Clifford action \(c : C\ell(T_x^*(EH)) \rightarrow \text{End}(\Lambda)\). We define the representation of Spin(4) in \(\text{End}(\Lambda) \cong GL_\mathbb{C}(4)\) simply by the restriction of \(c\) from the Clifford algebra, and obtain the spinor bundle \(S\) of typical fiber \(\Lambda\), with transition functions \(\{c(F_{NS}), c(F_{SN})\}\) in the coordinate charts \(\{U_N, U_S\}\).

With respect to the orthonormal basis, say \(\{\vartheta^\alpha\}\) of \(T^*(EH)_{U_N}\), there exists a unitary frame \(\{f_\alpha\}\) of the representation space \(\Lambda \cong \mathbb{C}^4\), such that the Clifford representations \(\gamma^\alpha := c(\vartheta^\alpha(x))\) for \(\alpha = 1, \ldots, 4\) can be represented as constant matrices,

\[
\gamma^1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},
\]

\[
\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.
\]  

(11)

The fact is that there exists frames \(\{f'_\beta\}\) on the coordinate chart \(U_S\) so that the representation of \(c(\vartheta^\beta\gamma^\beta)\) are also constant matrices \(\gamma^\beta\) as above.

Under the chosen frames \(\{f_\alpha\}\) and \(\{f'_\beta\}\), we may represent the transition functions of the spinor bundle as follows. Define maps \(P, Q : U_N \cap U_S \rightarrow GL_\mathbb{C}(4)\) by

\[
P := c(F_{SN}) = -\sin \phi \gamma^1 \gamma^1 + \cos \phi \gamma^2 \gamma^1 = \text{diag}(-\frac{i\bar{z}}{|z|}, \frac{iz}{|z|}, \frac{i\bar{z}}{|z|}, -\frac{i\bar{z}}{|z|}),
\]

\[
Q := c(F_{NS}) = -\sin \phi \gamma^1 \gamma^1 - \cos \phi \gamma^2 \gamma^1 = \text{diag}(\frac{i\bar{w}}{|w|}, \frac{iw}{|w|}, -\frac{iw}{|w|}, \frac{i\bar{w}}{|w|}).
\]  

(12)

(13)

diag\((a, b, c, d)\) stands for diagonal matrix with diagonal elements \(a, b, c, d\).

The spinor bundle \(S\) is thus,

\[
S := (U_N \times \mathbb{C}^4 \cup U_S \times \mathbb{C}^4)/\sim,
\]  

(14)
where \((x; s_1, \cdots, s_4) \in U_N \times \mathbb{C}^4\) and \((x'; s'_1, \cdots, s'_4) \in U_S \times \mathbb{C}^4\) are defined to be equivalent if and only if \(x = x'\) and \((s'_1, \cdots, s'_4)' = Q(s_1, \cdots, s_4)'\). One can easily see that the cocycle condition of the transition functions \(P \circ Q = Q \circ P = 1\) holds.

The chirality operator is defined by
\[
\chi := c(\vartheta^1) c(\vartheta^2) c(\vartheta^3) c(\vartheta^4) = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = \text{diag}(-1, -1, 1, 1),
\]
such that \(\chi^2 = 1\). The representation space \(\Lambda = \Lambda^+ \oplus \Lambda^-\) is decomposed as \(+1\)-eigenspaces of the operator \(\chi\), with \(\dim_{\mathbb{C}} \Lambda^+ = \dim_{\mathbb{C}} \Lambda^- = 2\). This fiberwise splitting extends to the global decomposition of spinor bundle as subbundles over the EH-space, \(S = S^+ \oplus S^-\), with each of the complex subbundles \(S^+\) and \(S^-\) of rank 2. Therefore, any element \(s \in S\) can be decomposed as \(s = (s^+, s^-)^t\). The charge conjugate operator on the spinor bundle \(J : S \to S\) is defined by
\[
J(s^+, s^-) := \left(\begin{array}{c} -\overline{s}^- \\ \overline{s}^+ \end{array}\right).
\]

### 2.4 Spin connections and Dirac operators of spinor bundles

Following the general procedure in [21], we can induce the spin connection \(\nabla^S\) of the spinor bundle \(S\) from the Levi-Civita connection of the EH-space.

We will only work on the \(U_N\) coordinate chart and the construction on \(U_S\) is similar. In the orthonormal frame \(\{\vartheta^\alpha\}\), the corresponding Levi-Civita connection on the dual tangent bundle, \(T^*(EH)_{U_N}\), can be expressed as \(\nabla^{T^*EH} \vartheta^\beta = -\Gamma^\beta_{i\alpha} dx^i \otimes \vartheta^\alpha\). The metric compatibility of the Levi-Civita connection implies that \(\Gamma^\alpha_{i\beta} = -\Gamma^\alpha_{i\beta}\).

We may represent \(\Gamma^\beta_{i\alpha}\)'s in terms of the Christoffel symbols \(\Gamma^k_{ij}\)'s of \(\nabla\) in the \(dx^i\)'s \([23]\) by
\[
\Gamma^\beta_{i\alpha} = h^\beta_{i\alpha} \left(h^\alpha_{k\beta} \Gamma^k_{ij} - \partial_i h^\beta_{j\alpha}\right),
\]
where \(h^\alpha_{i\beta}\)'s and \(\tilde{h}^\alpha_{i\beta}\)'s are the matrix entries of \(H\) in \([3]\) and \(H^{-1}\) in \([1]\), respectively. Modulo the anti-symmetric condition between \(\alpha\) and \(\beta\) indices, all the nonvanishing Christoffel symbols are
\[
\begin{align*}
\bar{\Gamma}^1_{33} &= \frac{1}{2} \Delta^{1/2} \sin \phi, & \bar{\Gamma}^1_{24} &= -\frac{1}{2} \Delta^{1/2} \cos \phi, \\
\bar{\Gamma}^3_{22} &= \frac{1}{2} \Delta^{1/2} \cos \phi, & \bar{\Gamma}^3_{33} &= -\frac{1}{2} \Delta^{1/2} \sin \phi, \\
\bar{\Gamma}^1_{33} &= -\frac{1}{2} \Delta^{1/2} \sin \theta \cos \phi, & \bar{\Gamma}^1_{34} &= -\frac{1}{2} \Delta^{1/2} \sin \theta \sin \phi, \\
\bar{\Gamma}^3_{32} &= -1 - \frac{1}{2} \Delta^+ \cos \theta, & \bar{\Gamma}^3_{32} &= -\frac{1}{2} \Delta^{1/2} \sin \theta \sin \phi, \\
\bar{\Gamma}^4_{32} &= \frac{1}{2} \Delta^{1/2} \sin \theta \cos \phi, & \bar{\Gamma}^4_{33} &= -\frac{1}{2} \Delta^+ \cos \phi, \\
\bar{\Gamma}^1_{42} &= \frac{1}{2} \Delta, & \bar{\Gamma}^4_{43} &= -\frac{1}{2} \Delta^+.
\end{align*}
\]
We define $\gamma_{\alpha} := \gamma^{\alpha}$, then the spin connection $\nabla^{S} : \mathcal{S} \to \mathcal{S} \otimes \Lambda^{1}(EH)$ is

$$\nabla^{S} := d - \frac{1}{4} \Gamma^{\beta}_{\alpha\delta} dx^{i} \otimes \gamma^{\alpha} \gamma_{\beta}. \quad (19)$$

The covariant derivative $\nabla^{S}_{i} := \nabla^{S}(\partial_{i})$, for $i = 1, \ldots, 4$, equals to $\nabla^{S}_{i} = \partial_{i} - \omega_{i}$, where $\omega_{i} = \frac{1}{4} \Gamma^{\beta}_{\alpha\delta} \gamma^{\alpha} \gamma_{\beta}$. The Dirac operator $D : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ can be defined by

$$D(\psi) := -i \gamma^{j} \nabla^{S}_{j} \psi, \quad \forall \psi \in \Gamma(\mathcal{S}), \quad (20)$$

where $\gamma^{j} := c(dx^{j}) = \tilde{h}^{j}_{\beta} \gamma^{\beta}$. We note that the compatibility of the spin connection with respect to the spin structure implies that the commutativity between the Dirac operator and the charge conjugate operator, i.e. $[D, J] = 0$.

### 2.5 Torus actions on the spinor bundle

A torus action on the spinor bundle $\mathcal{S}$ can be induced from the torus isometric action on a general Riemannian manifold [13], [14]. In this subsection, we will represent such torus action through parallel transporting spinors along geodesics.

Recall that the isometric action $\sigma$ is generated by the two Killing vectors $\partial_{3} = \partial_{\phi}$ and $\partial_{4} = \partial_{\psi}$. Let $c_{k} : \mathbb{R} \to EH$ be the geodesics obtained as integral curves of the Killing vector field $\partial_{k}$ for $k = 3, 4$.

The equation of parallel transportation with respect to the spin connection along any curve $c(t)$ is $\nabla^{S}_{c(t)} \psi = 0$, where $c'(t) := dc(t)/dt$, for $\psi \in \Gamma(\mathcal{S})$. Substituting (19), we obtain

$$\frac{d\psi}{dt} - A(c(t)) \psi = 0, \quad A(c(t)) := \frac{1}{4} \Gamma^{\beta}_{\alpha\delta} dx^{i} (c'(t)) \otimes \gamma^{\alpha} \gamma_{\beta}. \quad (21)$$

When the curve is $c_{3}(t)$, the corresponding matrix $A(c_{3}(t))$ is

$$A(c_{3}(t)) = \frac{1}{2} \begin{pmatrix}
  i & 0 & 0 & 0 \\
  0 & -i & 0 & 0 \\
  0 & 0 & -i (1 + \Delta^{+} \cos \theta) & -\Delta^{1/2} \sin \theta e^{i\phi} \\
  0 & 0 & \Delta^{1/2} \sin \theta e^{-i\phi} & i (1 + \Delta^{+} \cos \theta),
\end{pmatrix} \quad (22)$$

where $r, \theta$, and $\phi$ are understood as components of coordinates on the curve $c_{3}(t)$. When the curve is $c_{4}(t)$, the corresponding matrix $A(c_{4}(t))$ is

$$A(c_{4}(t)) = \frac{i}{2} \text{diag}(-a^{4}, a^{4}, -1, 1), \quad (23)$$

where $r$ is understood as one of the components of coordinates on the curve $c_{4}(t)$.

The corresponding parallel propagator is a map $P_{c(t)}(t_{0}, t_{1}) : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ defined by parallel transporting any section $\psi$ along the curve $c(t)$ with $t \in [t_{0}, t_{1}]$. The propagator can be represented by an iterated integration of the equation (21). For geodesics $c_{k}(t)$, $k = 3, 4$, the corresponding matrix is formally solved as

$$P_{c_{k}(t)}(t_{0}, t_{1}) = \mathcal{P} \exp \left( \int_{t_{i}}^{t_{f}} A_{k}(t) dt \right), \quad (24)$$
where $\mathcal{P}$ is the path-ordering operator.

Let $\mathcal{H}$ be the Hilbert space completion with respect to the $L^2$-inner product on the space of $L^2$-integrable sections of the spinor bundle $S$. The parallel propagators can be extended to families of operators $U_k(t_k - t_0) : \mathcal{H} \to \mathcal{H}$ parametrized by the real number $(t_k - t_0)$ by

$$U_k(t_k - t_0)(\psi)(x) := (P_{c_k(t_0)}(t_0, t_k)\psi)(x), \quad \forall \psi \in \mathcal{H},$$

where we assume $x = c_k(t_0) \in EH$ for $k = 3, 4$. Without loss of generality, we may take $t_0 = 0$ so that the family of operators is parametrized by $t_k$.

Since the spin connection is compatible with the metric of the EH-space, the pointwise inner product of the images of any two sections under parallel transportation along the geodesics $c_k(t)$ remains unchanged. This further implies that their $L^2$-integrations remain the same. Therefore, the operators $U_k(t_k)$ are unitary.

Let $W_k$ be the self-adjoint operators on $\mathcal{H}$ which generate $U_k$ by $U_k(t_k) = \exp(it_k W_k)$, where $t_k \in \mathbb{R}$ for $k = 3, 4$.

We may define a representation of the double cover $p : \tilde{T}^2 \to T^2$ of the two torus by $\tilde{V} : \tilde{T}^2 \to \mathcal{L}(\mathcal{H})$ such that

$$\tilde{V}(\tilde{t}_3, \tilde{t}_4)\psi(x) := \exp(i(\tilde{t}_3 W_3(x) + \tilde{t}_4 W_4(x))\psi(x), \quad \forall \psi \in \mathcal{H}. \quad (25)$$

This action covers the isometric action $\sigma$ of $T^2$ from $T^2$ in the sense that for any $\tilde{v} \in \tilde{T}^2$, $p(\tilde{v}) = v$ implies

$$\tilde{V}_{\tilde{v}}(f\psi) = \alpha_v(f)\tilde{V}_{\tilde{v}}(\psi), \quad \forall \psi \in \mathcal{H},$$

for any bounded continuous function $f \in C_b(EH)$ and the action $\alpha$ on $C_b(EH)$ defined by $\alpha_v(f)(x) := f(\sigma_v(x))$. We assume the choice of the lifting in the double torus is always fixed and omit the $\tilde{\cdot}$ for notational simplicity from now on.

## 3 Smooth algebras and projective modules

We consider algebras of functions over the Eguchi-Hanson spaces, and their deformations as differential algebras. To obtain a $C^*$-norm on the deformed algebra, we consider representations of algebras as operators on the Hilbert space of spinors. Some algebras may be realized as smooth algebras. We also find projective modules from the spinor bundle.

### 3.1 Algebras of smooth functions

We first summarize some related facts on topological algebras of complex-valued functions in $\mathcal{H}$. For a noncompact Riemannian manifold $X$, let $C^\infty_c(X)$ be the space of smooth functions on $X$ of compact support, $C^\infty_0(X)$ be the space of smooth functions vanishing at infinity and $C_b^\infty(X)$ be the space of smooth functions whose derivatives are bounded to all degrees.
In some local coordinate charts with corresponding partition of unity, say \( \mathcal{U} = \{ U_a, h_a \}_{a \in A} \), we may define the family of seminorms on \( C^\infty_b(X) \) by

\[
q^\mathcal{U}_m(f) := \sum_{a \in A} \sup_{|\alpha| \leq m} \left( \sup_{x \in U_a} |h_a(x) \partial^\alpha f(x)| \right),
\]

for any \( f \in C^\infty_b(X) \), \( \alpha \) are multi-indices and \( m \) a non-negative integer. These seminorms restrict on \( C^\infty_b(X) \) and \( C^\infty_c(X) \). The natural topology induced by (26) is the topology of uniform convergence of all derivatives. We can show that two such families of seminorms defined by different coordinate charts are equivalent. Thus the topology defined does not depend on the choice of coordinates \( \mathcal{U} \). We also note that the \( q_0 \) seminorm in the family of seminorms is nothing but the sup norm \( \| \cdot \|_\infty \), which is a \( C^\ast \)-norm with the involution defined by normal complex conjugation.

Algebras \( C^\infty_b(X) \) and \( C^\infty_0(X) \) are both Fréchet in the topology of uniform convergence of all derivatives, while the algebra \( C^\infty_c(X) \) is not complete. However, \( C^\infty_c(X) \) is complete in the topology of inductive limit as the inductive limit of the topology obtained by restriction on a family of algebras \( C^\infty_c(K_n) \), where \( \{ K_n \}_{n \in \mathbb{N}} \) is an increasing family of compact subsets in \( X \).

The algebra \( C^\infty_c(X) \) is dense in the Fréchet algebra \( C^\infty_0(X) \).

To consider algebras of smooth functions of the Eguchi-Hanson spaces, we may use the coordinate charts \( \mathcal{U} = \{ U_N, U_S \} \) defined in Section 2.2 with a partition of unity \( \{ h_N, h_S \} \) subordinated to them. The family of seminorms (26) can be written as

\[
d^\mathcal{U}_m(f) = \sup_{|\alpha| \leq m} \sup_{x \in U_N} |h_N(x) \partial^\alpha f(x)| + \sup_{|\alpha'| \leq m} \sup_{x' \in U_S} |h_S(x') \partial'^\alpha f(x')|.
\]

We obtain the corresponding topological algebras by taking \( X = EH \).

### 3.2 Algebras of integrable functions

Apart from algebras of functions which can be represented as operators, there are algebras of functions which may define projective modules as representation spaces. Decay conditions at infinity and integrability conditions of functions become important when considering noncompact spaces. We consider the following algebras of integrable functions.

The \((k,p)\)-th Sobolev norm of a function \( f \), say in \( C^\infty_b(EH) \), is given as

\[
\| f \|_{H^k_p}^p := \sum_{m=0}^{k} \left( \int_{EH} |\nabla^m f|^p dVol \right)^{1/p},
\]

where \( k \) is a non-negative integer and \( p \) is a positive integer. (We will not consider the case where \( p \) is a real number). We define subspaces in \( C^\infty_b(EH) \) which contain functions with finite Sobolev norm,

\[
C^p_k(EH) := \{ f \in C^\infty_b(EH) : \| f \|_{H^k_p} < \infty \}.
\]

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Let $H^p_k(EH)$ the Banach space obtained by the completion of the algebra $C^p_k(EH)$ with respect to the Sobolev norm. In particular, $H^p_0(EH) \supset \cdots \supset H^p_k(EH) \supset H^p_{k+1}(EH) \supset \cdots$.

**Remark 3.1.** Notice that the algebra $C^\infty_c(EH)$ is contained in $H^p_k(EH)$ for any $k \in \mathbb{N}$. Completion of $C^\infty_c(EH)$ with respect to $\| \cdot \|_{H^p_k}$ gives us the Banach space, $H^p_{k,0}(EH)$ such that $H^p_{k,0}(EH) \subset H^p_k(EH)$. The equality does not hold in general. However, in the circumstances of a complete Riemannian manifold with Ricci curvature bounded up to degree $k-2$, and positive injective radius (which is satisfied by the EH-space), $H^p_{k,0}(EH) = H^p_k(EH)$ when $k \geq 2$.

**Lemma 3.2.** For a fixed non-negative integer $p$, the intersection defined as

$$C^p_c(EH) := \cap_k H^p_k(EH)$$

is a Fréchet algebra in the topology defined by the family of norms $\{ \| \cdot \|_{H^p_k} \}_{k \in \mathbb{N}}$.

**Proof.** The topology is easily seen to be locally convex and metrizable. To show that it is complete, let $\{f_\beta\}$ be any Cauchy sequence in $C^\infty_c(EH)$, then there exists a limit $f^p_k$ of $\{f_\beta\}$ under the norm $\| \cdot \|_{H^p_k}$ in $H^p_k(EH)$ for each $k \in \mathbb{N}$. For any two indices $k_1, k_2$ such that $k_1 \leq k_2$, the norm $\| \cdot \|_{H^p_{k_1}}$ is stronger than the norm $\| \cdot \|_{H^p_{k_2}}$. The Cauchy sequence $\{f_\beta\}$ with the limit $f^p_{k_2}$ in the norm $\| \cdot \|_{H^p_{k_2}}$ is also a Cauchy sequence with the limit $f^p_{k_1}$ in the norm $\| \cdot \|_{H^p_{k_1}}$. Uniqueness of the limit implies that $f^p_{k_2} = f^p_{k_1}$. Since $k_1, k_2$ are arbitrary, the limits $f^p_k$ for any $k \in \mathbb{N}$ agree. We denote the limit as $f$ so that the Cauchy sequence converges to $f \in C^\infty_c(EH)$ with respect to any of the norms. Thus the topology is complete and $C^p_c(EH)$ is a Fréchet algebra.

When $p = 2$, the Fréchet algebra $C^\infty_2(EH)$ belongs to the chain of continuous inclusions,

$$C^\infty_c(EH) \hookrightarrow C^\infty_2(EH) \hookrightarrow C^\infty_0(EH),$$

with respect to their aforementioned topologies.

### 3.3 Deformation quantizations of differentiable Fréchet algebras

Rieffel’s deformation quantization of a differentiable Fréchet algebra in [1] (Chapter 1, 2) can be summarized as follows. Let $\mathcal{A}$ be a Fréchet algebra whose topology is defined by a family of seminorms $\{q_m\}$. We assume that there is an isometric action $\alpha$ of the vector space $V := \mathbb{R}^d$ considered as a $d$-dimensional Lie group acting on $\mathcal{A}$. We also assume that the algebra is smooth with respect to the action $\alpha$, i.e. $\mathcal{A} = \mathcal{A}^\infty$ in the notation of the reference.

Under the choice of a basis $\{X_1, \ldots, X_d\}$ of the Lie algebra of $V$, the action $\alpha_{X_i}$ of $X_i$ defines a partial differentiation on $\mathcal{A}$. One can define a new family of seminorms from $q_m$ by taking into account the action of $\alpha$. For any $f \in \mathcal{A}$,

$$\|f\|_{j,k} := \sum_{m \leq j, |\mu| \leq k} q_m(\delta^\mu f),$$

(30)
where $\mu$ are the multi-indices $(\mu_1, \ldots, \mu_d)$ and $\delta^\mu = \alpha^{\mu_1}_{X_1} \ldots \alpha^{\mu_d}_{X_d}$. The deformation quantization of the algebra $\mathcal{A}$ can be carried out in three steps:

**Step 1.** Let $C_b(V \times V, \mathcal{A})$ to be the space of bounded continuous functions from $V \times V$ to $\mathcal{A}$. One can induce the family of seminorms $\{\| \cdot \|_{j,k}^C\}$ on the space $C_b(V \times V, \mathcal{A})$ by

$$\|F\|_{j,k}^C := \sup_{w \in V \times V} \|F(w)\|_{j,k},$$

for $F \in C_b(V \times V, \mathcal{A})$ and $\| \cdot \|_{j,k}$ on $\mathcal{A}$ as in (30).

Let $\tau$ be an action of $V \times V$ on the space $C_b(V \times V, \mathcal{A})$ defined by translation. That is, $\tau_{w_0}(F)(w) = F(w + w_0)$ for any $w_0, w \in V \times V$ and $F \in C_b(V \times V, \mathcal{A})$. The action $\tau$ is an isometry action with respect to the seminorms (31). We define $\mathcal{B}^A(V \times V)$ to be the maximal subalgebra such that $\tau$ is strongly continuous and whose elements are all smooth with respect to the action of $\tau$.

In the same way as one induces from the family of seminorms $\{q_m\}$ and obtains the seminorms $\| \cdot \|_{j,k}$ of $\mathcal{A}$ in (30), one may induce the family of seminorms on $\mathcal{B}^A(V \times V)$ from (31) by taking into account of the action of $\tau$. For any $F \in \mathcal{B}^A(V \times V)$, let

$$\|F\|_{j,k,l}^B := \sum_{(l,m) \leq (j,k)} \sum_{|\nu| \leq l} \|\delta^\nu F\|_{l,m}^C,$$

where $\nu$ are the multi-indices and $\delta^\nu$ denotes the partial differentiation operator associated to $\tau$ of $V \times V$.

**Step 2.** The following is the fundamental result of the deformation quantization of a differentiable algebra. See Proposition 1.6 in [11]. For any invertible map $J$ on $V$, one can define an $\mathcal{A}$-valued oscillatory integral over $V \times V$ of $F \in \mathcal{B}^A(V \times V)$ by

$$\int_{V \times V} F(u, v)e(u \cdot v) \, dudv,$$

where $e(t) := \exp(2\pi i t)$ for $t \in \mathbb{R}$ and $u \cdot v$ is the natural inner product on $V$ considered as its own Lie algebra.

It is shown to be $\mathcal{A}$-valued by getting the bound of the integral in the family of seminorms $\{\| \cdot \|_{j,k}\}$ on $\mathcal{A}$. Specifically, for large enough $l$, there exists a constant $C_l$ such that

$$\left\| \int_{V \times V} F(u, v)e(u, Jv) \, dudv \right\|_{j,k} \leq C_l \|F\|_{j,k,l}^B < \infty,$$

where the seminorm $\| \cdot \|_{j,k,l}^B$ is defined in (32).

**Step 3.** Any two functions $f, g \in \mathcal{A}$ define an element $F^{f,g} \in \mathcal{B}^A(V \times V)$ by

$$F^{f,g}(u, v) := \alpha_{J}(f)\alpha_v(g) \in \mathcal{A}, \quad \forall(u, v) \in V \times V.$$

The deformed product $f \times_J g$ is thus defined by the integral (33) of $F^{f,g}(u, v)$ as,

$$f \times_J g := \int_V \int_V \alpha_{J}(f)\alpha_v(g)e(u \cdot v) \, dudv.$$
The algebra $A$ with its deformed product $\times_J$, together with its undeformed seminorms $\{\| \cdot \|_{j,k}\}$, defines the deformed Fréchet algebra $A_J$. This is called the deformation of the algebra $A$ (in the direction of $J$) as a differentiable Fréchet algebra.

In the following, we obtain deformation quantizations of various algebras of functions on EH-spaces. We may induce a torus action $\alpha$ similarly on algebras $C^\infty_{\theta}$, from the torus isometric action $\sigma$ of $v \in T^2$ on the EH-space $\mathcal{E}$ by $\alpha_v f(x) = f(\sigma_v(x))$ for any $f \in C^\infty_{\theta}(EH)$ and $x \in EH$.

Under the choice of the covering $\{U_N, U_S\}$, the orbit of any point $x \in EH$ lies in the same coordinate chart as $x$. We assume that the partition of unity $h_N$ and $h_S$ only depend on the coordinate $\theta$ so that they are invariant under the torus action $\alpha$.

One can easily show that the torus action $\alpha$ is isometric with respect to the family of seminorms $\{\| \cdot \|_{\theta}\}$. We also note that each of the Fréchet algebras $C^\infty_{\theta}(EH)$ and $C^\infty_0(EH)$ is already smooth with respect to the action $\alpha$. Thus, each of $C^\infty_{\theta}(EH)$ and $C^\infty_0(EH)$, with the isometric action $\alpha$, regarded as a periodic action of $V = \mathbb{R}^2$, appears exactly as the starting point as $(A, \{q_m\})$ of Rieffel’s deformation quantization. We can carry out step 1 to step 3 and obtain the product $\times_J$ on the respective algebras,

$$ f \times_J g := \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \alpha_J u (f) \alpha_J v (g) e(u \cdot v) \, du \, dv, $$

where the inner product $u \cdot v$ is the one on $\mathbb{R}^2$ and $J$ is a skew-symmetric linear operator on $\mathbb{R}^2$. In the following we assume $J := \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$, for some $\theta \in \mathbb{R}\setminus\{0\}$, and denote $\times_J$ as $\times_\theta$.

The algebra $C^\infty_{\theta}(EH)$ with its deformed product $\times_\theta$, together with its undeformed family of seminorms $\{\| \cdot \|_{\theta,k}\}$ defines the deformed Fréchet algebra $C^\infty_{\theta}(EH)_\theta$ as the deformation quantization of $C^\infty_{\theta}(EH)$. Similarly, $C^\infty_0(EH)_\theta$ is the deformation quantization of the algebra $C^\infty_0(EH)$.

For the Fréchet algebra $C^\infty_2(EH)$, the torus action $\alpha$ is isometric with respect to the family of norms $\{\| \cdot \|_{H^2_k}\}$, because it is isometric with respect to the Riemannian metric. We can similarly obtain the Fréchet algebra $C^\infty_2(EH)_\theta$ as deformation quantization of the algebra $C^\infty_2(EH)$.

**Remark 3.3.** For any of the algebras in our example, the family of seminorms $\| \cdot \|_{j,k}$ induced from $q_m$’s as in Step 1 is equivalent to the original family of seminorms. Indeed, the torus action is defined by the normal differentiation with respect to coordinates.

There follows some immediate observations.

**Lemma 3.4.** The algebra $C^\infty_2(EH)_\theta$ is an ideal of the algebra $C^\infty_{\theta}(EH)_\theta$.

*Proof.* Let $f \in C^\infty_2(EH)$ and $g \in C^\infty_2(EH)$. Considered as elements of the algebra $C^\infty_2(EH)$, they define $F^{f,g} \in \mathcal{B}_2^\infty(EH)(\mathbb{R}^2 \times \mathbb{R}^2)$ by (34). We claim that $F^{f,g}$ lies in $\mathcal{B}_2^\infty(EH)(\mathbb{R}^2 \times \mathbb{R}^2)$ so that its oscillatory integral, or product of $f \times_\theta g$ by definition,
will be finite in the family of seminorms on $C_2^\infty(EH)$ and hence $C_2^\infty(EH)$-valued. In fact,
\[
\int_{EH} |F^f g(u,v)(x)|^2 d\text{Vol}(x) = \int_{EH} |f(Ju + x)g(v + x)|^2 d\text{Vol}(x)
\leq \sup_{x \in EH} |g(x)|^2 \int_{EH} |f(Ju + x)|^2 d\text{Vol}(x)
= \sup_{x \in EH} |g(x)|^2 \int_{EH} |f(x)|^2 d\text{Vol}(x) < \infty.
\]
The last equality is by the invariance of the volume form of the integration with respect to the torus isometric action. The finiteness is because $g$ is a bounded function and $f \in C_2^\infty(EH)$.

Higher orders can be shown as follows. For any non-negative integer $k$, we may expand $\nabla^k f(Ju + x)g(v + x)$ by the Leibniz rule to a summation of terms in the form of $\nabla^l f(Ju + x)\nabla^m g(v + x)$ with $l + m = k$. By the assumption that $\nabla^k f$ is $L^2$-integrable for any $k$ and $\nabla^i g$ is bounded for any $l$, each term in the summation is $L^2$-integrable. Thus $\nabla^k (f(Ju + x)g(v + x))$ is $L^2$-integrable for any $k$ and $F^f g(u,v) \in C_2^\infty(EH)$ for any $(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2$. As a result, the product $f \times_\theta g$ is $C_2^\infty(EH)$-valued and $C_2^\infty(EH)_\theta$ is an ideal. \qed

Restriction of the product \eqref{eq:productRestriction} of the algebra $C_b^\infty(EH)_\theta$ to the algebra $C_c^\infty(EH)_\theta$ gives the deformed algebra $C_c^\infty(EH)_\theta$. We see that it is closed as an algebra as follows. For any $f, g \in C_c^\infty(EH)$, the integral \eqref{eq:productRestriction} vanishes outside the compact set $\text{Orb}(\text{supp}(f)) \cap \text{Orb}(\text{supp}(g))$, where $\text{Orb}(U) := \{ \alpha_\varphi(x) : x \in U \subset EH \}$. Therefore, $f \times_\theta g$ is of compact support and $C_c^\infty(EH)_\theta$ is thus closed. We assign the topology of inductive limit on $C_c^\infty(EH)_\theta$ from that of $C_c^\infty(EH)$. Using definitions, we have

**Lemma 3.5.** $C_c^\infty(EH)_\theta$ is an ideal of the algebras $C_0^\infty(EH)_\theta$ and $C_b^\infty(EH)_\theta$.

**Proof.** For $f \in C_0^\infty(EH)_\theta$ and $g \in C_b^\infty(EH)_\theta$, the integral \eqref{eq:productRestriction} vanishes outside the compact set $\text{Orb}(\text{supp}(f))$. Hence $f \times_\theta g$ is $C_c^\infty(EH)$-valued, so that $C_c^\infty(EH)_\theta$ is an ideal of the algebras $C_0^\infty(EH)_\theta$. The proof for the algebra $C_0^\infty(EH)_\theta$ is the same. \qed

The torus action $\alpha$ as a compact action of an abelian group defines a spectral decomposition of a function $f$ in the algebra $C_b^\infty(EH)$ or $C_0^\infty(EH)$, by
\[
f = \sum_s f_s, \quad f_s(x) = \exp(is_3 \phi) \exp(is_4 \psi) h_s(r, \theta),
\]
where $s = (s_3, s_4) \in \mathbb{Z}^2$, $f_s$ satisfies $\alpha_v f_s = s (s \cdot v) f_s$, $\forall v \in \mathbb{T}^2$, and the series converges in the topology of uniform convergence of all derivatives. Under the decomposition, the product of \eqref{eq:productRestriction} takes a simple form (Chapter 2, [11]). Let $f = \sum_r f_r$ and $g = \sum_s g_s$, in their respective decompositions, be both in the algebra $C_b^\infty(EH)$ (or $C_0^\infty(EH)$), then
\[
f \times_\theta g = \sum_{r,s} \sigma(r,s) f_r g_s.
\]
where $\sigma(r, s) := e(\theta(r_1s_3 - r_3s_4))$ and $r = (r_3, r_4), s = (s_3, s_4) \in \mathbb{Z}^2$. The expression (37) can also be restricted to the algebra $C_c^\infty(EH)$.\[\]

**Lemma 3.6.** $C_0^\infty(EH)_{\theta}$ is an ideal of $C_0^\infty(EH)_{\theta}$.

**Proof.** For any $f \in C_0^\infty(EH)_{\theta}$ and $g \in C_0^\infty(EH)_{\theta}$, it suffices to show that $f \times g \in C_0^\infty(EH)_{\theta}$. For $g$ being zero, this is trivial. We thus assume that $g$ is nonzero. The convergence of the series (37) implies that for any $\varepsilon/2 > 0$, there exists an integer $N$ such that

$$|f \times g(x)| < \sum_{|r|, |s| \leq N} \sigma(r, s)f_r(x)g_s(x) + \frac{\varepsilon}{2},$$

for any $x \in EH$, where $|r| := |r_3| + |r_4|$ and $|s| := |s_3| + |s_4|$.

Since $f_r \in C_0^\infty(EH)$, for each $|r| \leq N$, there exists a compact set $K(f_r) \subset EH$ such that

$$|f_r(x)| < \frac{\varepsilon}{2C}, \quad \forall x \in EH \setminus K(f_r),$$

for any fixed constant $C$.

Therefore, for any $\varepsilon > 0$, we may choose $N$ and $K(f_r)$ as above and define the union of finitely many compact sets as $K := \cup_{|r| \leq N} K(f_r)$, so that $x \in EH \setminus K$ implies that

$$|f \times g(x)| < \sum_{|r|, |s| \leq N} \sigma(r, s)f_r(x)g_s(x) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \frac{\varepsilon A_N}{2C} \sup_{x \in EH} |\sigma(r, s)g(x)| + \frac{\varepsilon}{2},$$

where $A_N$ is a finite non-negative integer counting numbers of indices $r$ and $s$ satisfying $|r|, |s| \leq N$. If we fix the constant $C = \sup_{x \in EH} |\sigma(r, s)g(x)|A_N$, then the above inequalities give $|f \times g(x)| < \varepsilon$, whenever $x \in EH \setminus K$. Therefore, $f \times g$ is $C_0^\infty(EH)$-valued, and $C_0^\infty(EH)_{\theta}$ is an ideal of $C_0^\infty(EH)_{\theta}$. \[\]

We will end this subsection by introducing local algebras.

**Definition 3.7.** [3] An algebra $A_c$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset A_c$, there exists $\phi \in A_c$ such that for each $i$, $\phi a_i = a_i \phi = a_i$.

Let $A$ be a Fréchet algebra such that $A_c \subset A$ is a dense ideal with local units, then $A$ is called a local algebra.

**Lemma 3.8.** The algebra $C_c^\infty(EH)_{\theta}$ has local units and the algebra $C_0^\infty(EH)_{\theta}$ is a local algebra.

**Proof.** For any finite set of elements $\{f_\beta\}_{\beta=1}^n \subset C_c^\infty(EH)_{\theta}$, there exists a compact set $K$ large enough to contain the union of supports $\cup_{\beta} supp(f_\beta)$. Let $\phi$ be a function equal to 1 on $K$ and decaying only with respect to the $r$-variable to zero outside $K$. Thus defined $\phi$ satisfies $\phi = \phi(0,0,0)$ in the spectral decomposition so that $\phi \times f_\beta = f_\beta \times \phi = f_\beta$ for all $\beta$. Thus, $(C_c^\infty(EH), \times \theta)$ is an algebra with units.
The fact that $C_c^\infty(EH)$ is dense in $C_0^\infty(EH)$ with respect to the topology of uniform convergence of all derivatives implies that $C_c^\infty(EH)_\theta$ is dense in $C_0^\infty(EH)_\theta$, since the family of seminorms is not deformed. $C_c^\infty(EH)_\theta$ is an ideal in $C_0^\infty(EH)_\theta$ by Lemma 3.5. Thus $C_0^\infty(EH)_\theta$ is a local algebra. □

Lemma 3 of [2] says that there exists a local approximate unit $\{\phi_n\}_{n \geq 1}$ for a local algebra $(A_c \subset A)$. In this example, we choose a family of compact sets $K_0 \subset K_1 \subset \ldots$ in the $EH$-space, increasing in the $r$-direction. For instance,

$$K_n := \{x \in EH : r \leq n\}, \quad \forall n \in \mathbb{N}.$$

Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a family of functions with compact support $K_n \subset \text{supp}(\phi_n) \subset K_{n+1}$ such that $\phi_n$ is constant 1 on $K_n$ and decays only with respect to $r$ to zero on $K_{n+1}$. This gives a local approximate unit. It is not hard to see that each $\phi_i$ actually commutes with functions in the algebra $C_0^\infty(EH)_\theta$. Furthermore, the union of the algebras $\cup_{n \in \mathbb{N}}[C_0^\infty(EH)_\theta]_n$, where

$$[C_0^\infty(EH)_\theta]_n := \{f \in C_0^\infty(EH)_\theta : \phi_n \times_\theta f = f \times_\theta \phi_n = f\},$$

is the algebra $C_c^\infty(EH)_\theta$.

### 3.4 Algebras of operators and deformations of $C^*$-algebras

**Definition 3.9.** [2] A *-algebra $A$ is smooth if it is Fréchet and *-isomorphic to a proper dense subalgebra $i(A)$ of a $C^*$-algebra $A$ which is stable under the holomorphic functional calculus under suitable representation.

Recall that the $g_0$ seminorm in the family [27] is the suprenorm $\| \cdot \|_\infty$, which defines $C^*$-norms on each of the algebra $C_0^\infty(EH)$ and $C_0^\infty(EH)$. The $C^*$-completion of the former is the algebra $C_b(EH)$ of bounded continuous functions. That $C_0^\infty(EH)$ is stable under the holomorphic functional calculus of $C_b(EH)$ implies that $C_0^\infty(EH)$ is a pre-$C^*$-algebra.

The $C^*$-completion of $C_0^\infty(EH)$ is the algebra $C_0(EH)$ of continuous functions vanishing at infinity. As a nonunital Banach algebra, the holomorphic functional calculus is with respect to its unitization and with respect to holomorphic functions vanishing at 0. $C_0^\infty(EH)$ is stable under the holomorphic functional calculus of $C_0(EH)$ and hence a pre-$C^*$-algebra. Similarly, the Fréchet algebra $C_2^\infty(EH)$ is also a pre-$C^*$-algebra of the $C^*$-completion $C_0(EH)$. We see that $C_0^\infty(EH)$, $C_2^\infty(EH)$ and $C_0^\infty(EH)$ are smooth algebras.

The deformation quantizations of $C_0^\infty(EH)$ and $C_0^\infty(EH)$ obtained before are as differentiable Fréchet algebras. To realize the deformed algebras as pre-$C^*$-algebras of some deformed $C^*$-algebra, we may represent them as operators on some Hilbert space. Following the construction of [13],[14], we may obtain their representations on the Hilbert space $\mathcal{H}$ of spinors, by using the torus isometric action.
Let $C^\infty_s(EH)_\theta$ stand for the algebras $C^\infty_c(EH)_\theta$, $C^\infty_0(EH)_\theta$ or $C^\infty_0(EH)_\theta$. The operator representation of $C^\infty_s(EH)_\theta$ on the Hilbert space $\mathcal{H}$ is defined by

$$L_\theta^\theta := \sum_{r \in \mathbb{Z}^2} M_{f_r} V_{r}^\theta,$$

(38)

where $M_{f_r}$ is the normal multiplication by $f_r$ and $V_{r}^\theta$ is a unitary operator obtained as the evaluation the unitary operator \( \pi_{\theta r} \) at $\tilde{t}_3 = 2\pi\theta r_3$ and $\tilde{t}_4 = -2\pi\theta r_3$. That is,

$$V_{r}^\theta := e(\theta(r_4 W_3 - r_3 W_4)), \quad r = (r_3, r_4) \in \mathbb{Z}^2.$$

(39)

**Remark 3.10.** Geometrically, $V_{r}^\theta$ is the action of parallel transporting any section by $-2\pi\theta r_3$ along the $\psi$ direction followed by a parallel transporting by $2\pi\theta r_4$ along the $\phi$ direction.

With the involution on $C^\infty_s(EH)_\theta$ defined by the complex conjugation of functions, we can use the property $(f^*)_r = (f_{-r})^*$ and $V_{r}^\theta h_s = h_s V_{r}^\theta \sigma(r, s)$ for any simple component $h_s$ from $\sum_s h_s$, to show that the representation (38) is a faithful $\ast$-representation of $C^\infty_s(EH)_\theta$.

We may define the $C^*$-norm of $C^\infty_s(EH)_\theta$ by the operator norm $\| \cdot \|_{op}$ of the representation on $\mathcal{H}$. The series of operators (38) converges uniformly in the operator norm. We denote the $C^*$-completion of the algebra $C^\infty_b(EH)_\theta$ by $C^\ast_b(EH)_\theta$. It is a deformation of $C^\infty_b(EH)$ as a $C^\ast$-algebra. One can also show that $C^\infty_b(EH)$ is stable under the holomorphic functional calculus of $C^\ast_b(EH)_\theta$ and hence a $pre-C^\ast$-algebra.

The $C^*$-completion $C^\ast_0(EH)_\theta$ of the algebra $C^\infty_0(EH)_\theta$ defines a deformation of $C^\ast_0(EH)$ as a $C^\ast$-algebra. This can also be realized as a pre-$C^\ast$-algebra. For similar reasons, the Fréchet algebra $C^\infty_2(EH)_\theta$ can be realized as a pre-$C^\ast$-algebra with the $C^\ast$-completion $C^\ast_0(EH)_\theta$.

In the commutative case, one can show that $\| \cdot \|_{op}$ is bounded by the zero-th seminorm $q_0$ in the family of seminorms (26). Hence the $C^*$-norm is weaker than the family of seminorms (26). To see that the same holds in the deformed case, we note that in Rieffel’s construction, the deformed Fréchet algebras can be represented on the space of Schwarz functions associated with a natural inner product (page 23 [1]) and completed to $C^\ast$-algebras. Furthermore, the correspondent $C^*$-norm is shown to be weaker than the family of seminorms defining the Fréchet topology (Proposition 4.10 [1]). We may induce a $\ast$-homomorphism from the $C^*$-algebra representing on $\mathcal{H}$ to the $C^\ast$-algebra representing on the space of Schwarz functions by the identity map of functions. Since any $\ast$-homomorphism is norm-decreasing, we conclude that the $C^*$-norm on $C^\infty_2(EH)_\theta$ represented on $\mathcal{H}$ is also weaker than the family of seminorms (26) defining the topology of uniform convergence of all derivatives. Both of the algebras $C^\infty_0(EH)_\theta$ and $C^\infty_0(EH)_\theta$ are smooth algebras.

### 3.5 Projective modules of spinor bundles

The link between vector bundles over compact space and projective modules is the Serre-Swan theorem [23]. It is generalized for vector bundles of finite type, of which
there exists a finite number of open sets in the open cover of the base manifold such that the bundle is trivialized on each open set. The smooth version of the result is as follows.

**Theorem 3.11.** The category of complex vector bundles of finite type over $X$ for any differentiable manifold $X$ is equivalent to the category of finitely generated projective $C^\infty_b(X)$-modules.

**Remark 3.12.** There exists an alternative version of the generalized Serre-Swan theorem for vector bundles over noncompact manifolds, proved by using certain compactification of the base manifolds. Since the simplest one-point compactification of the Eguchi-Hanson space gives an orbifold due to the $\mathbb{Z}^2$-identification, it is not straightforward to apply the construction there.

In the following, we will use Theorem 3.11 to find the projective module associated to the spinor bundle $S$ of the EH-space as defined in Section 2.3. In the coordinate charts $U_N$ and $U_S$ of the EH-space, we may choose a partition of unity \( \{ h_N, h_S \} \) by

\[
h_N(x) := \cos^2 \frac{\theta}{2}, \quad h_S(x) := \sin^2 \frac{\theta}{2}, \quad x \in \text{EH}.
\]

(40)

Recall that in the unitary basis \( \{ f_\alpha \} \) of $\mathcal{S}_{UN}$ and \( \{ f'_\beta \} \) of $\mathcal{S}_{US}$, the transition functions $P^\alpha_\beta$’s and $Q^\alpha_\beta$’s, such that $f_\beta = P^\alpha_\beta f_\alpha$ and $f'_\beta = Q^\alpha_\beta f'_\alpha$, are matrix entries of $P$ in (12) and $Q$ in (13), respectively.

The idea is to extend the basis \( \{ f_\alpha \} \) on $U_N$ across the “north pole” $N$ and \( \{ f'_\alpha \} \) on $U_S$ across the “south pole” $S$ so that one can take the summation of both extended global sections to obtain a generating set of the space of smooth bounded sections of the spinor bundle $\Gamma^\infty_b(S)$.

To extend \( \{ f_\alpha \} \) across $N$, we may rescale it by the function $h_N$,

\[
F_\alpha := \begin{cases} f_\alpha h_N & \text{on } U_N, \\ 0 & \text{at } N, \end{cases}
\]

(41)

so that $F_\alpha$’s now decay to zero smoothly at $N$. Similarly, we may rescale the basis \( \{ f'_\alpha \} \) by the function $h_S$ by defining

\[
F'_\alpha := \begin{cases} f'_\alpha h_S & \text{on } U_S, \\ 0 & \text{at } S. \end{cases}
\]

(42)

Note that on the intersection $U_N \cap U_S$, the transition function satisfies $P^\alpha_\beta h_N \to 0$ whenever $h_N \to 0$, and similarly $Q^\alpha_\beta h_S \to 0$ whenever $h_S \to 0$.

**Lemma 3.13.** The set of global sections \( \{ F_\alpha, F'_\alpha \} \), where $\alpha = 1, \ldots, 4$, are the generating set of the space of bounded smooth sections of the spinor bundle $\Gamma^\infty_b(S)$.
Proof. The restriction \( \{F_\alpha|_{U_N}\} \) where \( \alpha = 1, \ldots, 4 \) is a basis for \( S_{U_N} \). Indeed, any section \( \psi \in V^\infty_b(S) \) can be written as \( \psi|_{U_N} = \psi^\alpha f_\alpha = a^\alpha f_\alpha h_N = a^\alpha F_\alpha|_{U_N} \), where \( a^\alpha = \psi^\alpha / h_N \). Similarly, the restriction \( \{F'_\alpha|_{U_S}\} \) gives a basis for \( S_{U_S} \), since any section \( \psi \) can be written as \( \psi|_{U_S} = \psi'^\alpha f'_\alpha = b^\alpha f'_\alpha h_S = b^\alpha F'_\alpha|_{U_S} \), where \( b^\alpha = \psi'^\alpha / h_S \).

On the intersection,

\[
F_\alpha|_{U_N \cap U_S} = h_N P_\alpha h_N^{-1}, \quad F'_\alpha|_{U_N \cap U_S} = h_S Q_\alpha h_S^{-1}.
\]

Let \( \{k_N, k_S\} \) be a new partition of unity such that the \( \text{supp}(k_N) \subseteq U_N \) and \( \text{supp}(k_S) \subseteq U_S \). Furthermore, \( k_N \) (\( k_S \), respectively) is required to decay faster than \( h_N \) around \( N \) (\( h_S \) around \( S \), respectively). We may choose for instance

\[
k_N(x) := \cos^2(\frac{\pi}{2} \sin^2 \theta), \quad k_S(x) := \sin^2(\frac{\pi}{2} \sin^2 \theta), \quad x \in EH.
\]

Therefore, \( a^\alpha k_N \to 0 \) on \( U_N \), whenever \( h_N \to 0 \), and \( b^\alpha k_S \to 0 \) on \( U_S \), whenever \( h_S \to 0 \). Thus, we can extend the coefficient functions \( a^\alpha \)'s and \( b^\alpha \)'s by zero,

\[
A^\alpha := \begin{cases}
    a^\alpha k_N & \text{on } U_N \\
    0 & \text{at } N
\end{cases}, \quad B^\alpha := \begin{cases}
    b^\alpha k_S & \text{on } U_S \\
    0 & \text{at } S
\end{cases}.
\]

so that \( \psi = A^\alpha F_\alpha + B^\alpha F'_\alpha \). In fact,

\[
A^\alpha F_\alpha + B^\alpha F'_\alpha = \begin{cases}
    \psi^\alpha k_N f_\alpha + \psi'^\alpha k_S f'_\alpha & \text{on } U_N \cap U_S \\
    \psi^\alpha k_S f'_\alpha & \text{at } N \\
    \psi^\alpha k_N f_\alpha & \text{at } S
\end{cases} = \begin{cases}
    \psi^\alpha f_\alpha & \text{on } U_N \\
    \psi'^\alpha f'_\alpha & \text{on } U_S
\end{cases} \tag{43}
\]

which is the section \( \psi \) in \( V^\infty_b(S) \). Therefore, \( \{F_\alpha, F'_\alpha\} \) with \( \alpha = 1, \ldots, 4 \) is a generating set of \( V^\infty_b(S) \).

By construction, we may obtain a projection in \( M_S(C^\infty_b(EH)) \) corresponding to the spinor bundle \( S \). Under the standard basis of the free \( C^\infty_b(EH) \)-module \( C^\infty_b(EH)^8 \), we define the matrix,

\[
p := \begin{pmatrix}
    k_N & k_N P \\
    k_S Q & k_S 1
\end{pmatrix} \tag{44}
\]

where \( P \) and \( Q \) are \( 4 \times 4 \) complex matrices from \([12]\) and \([13]\) and \( 1 \) is the four by four identity matrix.

**Proposition 3.14.** \( V^\infty_b(S) \) is a finitely generated projective \( C^\infty_b(EH) \)-module,

\[
C^\infty_b(EH)^8 p \cong V^\infty_b(S). \tag{45}
\]

---

[1] These functions are kindly suggested by Derek Harland.
Proof. It is easy to check that $p^2 = p$ and $p = p^*$. To show that (15) is an isomorphism, any section can be represented as an element in $C_b^∞(EH)^8 p$ by construction. Conversely, the matrix $p$ maps any element $(t_1, \ldots, t_4, t_1', \ldots, t_4')$ of $C_b^∞(EH)^8$ to

$$((t_1 + P_1^β t_β) k_N, (t_2 + P_2^β t_β) k_N, (t_3 + P_3^β t_β) k_N, (t_4 + P_4^β t_β) k_N,$$

$$(t_1' + Q_1^β t_β) k_S, (t_2' + Q_2^β t_β) k_S, (t_3' + Q_3^β t_β) k_S, (t_4' + Q_4^β t_β) k_S).$$

Let $A^α = (t^α + P^α t_β) k_N$ and $B^α = (t^α + Q^α t_β) k_S$, for $α = 1, \ldots, 4$, then the image gives a section in $Γ_b^∞(S)$ in the form of (43). Therefore, (45) is an isomorphism. □

Columns of the matrix $p = (p^β_3)$ give a generating set of $Γ_b^∞(S)$. We may define $P^k = (p^k_1, \ldots, p^k_8) t$ for $k = 1, \ldots, 8$, then any element $ψ ∈ C_b^∞(EH)^8 p$ can be written as $ψ = ψ_k \cdot P^k$ for functions $ψ_k ∈ C_b^∞(EH)$.

### 3.6 Smooth modules

In addition to the description of a vector bundle as a finitely generated projective module, the integrability conditions of the sections become vital when the base manifold is noncompact. The notion of smooth module [2] is proposed to integrate the two aspects. We will give the relevant background from the reference.

Let $A_0$ be an ideal in a smooth unital algebra $A_b$. Suppose that $A_0$ is further a local algebra containing a dense subalgebra of local units $A_c$. Assuming the topology on $A_0$ is the one making it local and the topology on $A_b$ is the one making it smooth, if the inclusion $i : A_0 → A_b$ is continuous, then $A_b$ is a local ideal. It is further called essential if $A_0 b = \{0\}$ for some $b ∈ A_b$ implies $b = 0$.

Let $A_0$ be a closed essential local ideal in a smooth unital algebra $A_b$ and $p ∈ M_n(A_b)$ be a projection. By pulling back the projective modules $E_b$ defined by $A_b^* p$ through inclusion maps $i : A_c → A_b$, one can define the $A_0$-finite projective $A_c$-module $E_c$ by $A_c^* p$. Similarly, one can define the $A_0$-finite projective $A_0$-module $E_0$ by $A_0^* p$.

By using the Hermitian form on the projective modules $(ξ, η) := \sum ξ_β^* η_α$, one may obtain the topology on $E_c$ induced from the topology of inductive limit on $A_c$, the Fréchet topology on $E_0$ induced from the Fréchet topology on $A_0$ and the Fréchet topology on $E_b$ induced from the Fréchet topology on $A_b$. Hence one has the following continuous inclusions of projective modules, $E_c ⊆ E_0 ⊆ E_b$.

**Definition 3.15.** A smooth $A_b$-module $E_2$ is a Fréchet space with a continuous action of $A_b$ such that $E_c ⊆ E_2 ⊆ E_0$, as linear spaces, where the inclusions are all continuous.

Returning to our example, we may choose $A_c$ as $C_c^∞(EH)_θ$, $A_0$ as $C_0^∞(EH)_θ$, $A_2$ as $C_2^∞(EH)_θ$ and $A_b^∞$ as $C_b^∞(EH)_θ$.

**Proposition 3.16.** Assuming that $C_c^∞(EH)_θ$ is the algebra of units, the algebras $C_c^∞(EH)_θ$, $C_2^∞(EH)_θ$ and $C_0^∞(EH)_θ$ are all essential local ideals of $C_b^∞(EH)_θ$ under the topology of uniform convergence of all derivatives.
Proof. $C_0^\infty (EH)_\theta$ is an ideal of $C_b^\infty (EH)_\theta$ by Lemma 3.6. Since the topology on $C_0^\infty (EH)_\theta$ and $C_b^\infty (EH)_\theta$ are both the topology of uniform convergence of all derivatives, the inclusion $C_0^\infty (EH)_\theta \hookrightarrow C_b^\infty (EH)_\theta$ is continuous.

To show that the ideal $C_0^\infty (EH)_\theta$ is essential, we suppose that $f \in C_0^\infty (EH)_\theta$ satisfies $g \times_\theta f = 0$ for all $g \in C_0(EH)_\theta$. Taking $g = 1/r$, $g \times_\theta f = g \times f = 0$. This implies that $f = 0$, since $1/\beta$ is nowhere zero. Thus, $C_0^\infty (EH)_\theta$ is an essential ideal.

$C_2^\infty (EH)_\theta$ is an ideal of $C_b^\infty (EH)_\theta$ by Lemma 3.4. Similar to the proof for $C_0^\infty (EH)_\theta$, $C_2^\infty (EH)_\theta$ is further an essential ideal.

$C_c^\infty (EH)_\theta$ is an ideal of $C_b^\infty (EH)_\theta$ by Lemma 3.5. $C_c^\infty (EH)_\theta$ carrying the topology of inductive limit is a local essential ideal, as is implied by Corollary 7 of [2] directly.

With the differential topologies the same as their commutative restriction, there is a chain of continuous inclusions,

\[ C_c^\infty (EH)_\theta \hookrightarrow C_2^\infty (EH)_\theta \hookrightarrow C_0^\infty (EH)_\theta \hookrightarrow C_b^\infty (EH)_\theta. \tag{46} \]

One may define the following projective modules $C_c^\infty (EH)_\theta^\mathbb{S} p$, $C_2^\infty (EH)_\theta^\mathbb{S} p$ and $C_b^\infty (EH)_\theta^\mathbb{S} p$ by the projection $p$ in the form of (46) while considered as an element in $M_p(C_b^\infty (EH)_\theta)$. It is not hard to see that $p = p^* = p^2$ still holds in this case.

The family of seminorms, say $\{Q_m\}$’s, on the projective modules is induced from the family of seminorms on the algebra, say $\{q_m\}$’s, by composing with the Hermitian form $(\cdot, \cdot)$ on the projective modules as $Q_m(\xi) := q_m((\xi, \xi))$ for any $\xi$ in the projective module. The topologies on the projective modules are defined by the induced family of seminorms. In this way, the chain of algebras (46) induces the chain of projective modules,

\[ C_c^\infty (EH)_\theta^\mathbb{S} p \hookrightarrow C_2^\infty (EH)_\theta^\mathbb{S} p \hookrightarrow C_0^\infty (EH)_\theta^\mathbb{S} p \hookrightarrow C_b^\infty (EH)_\theta^\mathbb{S} p. \]

Note that the action of $C_b^\infty (EH)_\theta$ on $C_2^\infty (EH)_\theta^\mathbb{S} p$ is continuous. Indeed, if a sequence of elements $\{\xi_\beta\}$ in $C_2^\infty (EH)_\theta^\mathbb{S} p$ satisfies that $Q_m(\xi_\beta) \to 0$ as $\beta \to \infty$, then for any $f \in C_b^\infty (EH)_\theta$,

\[ Q_m(\xi_\beta f) = q_m((\xi_\beta f, \xi_\beta f)) = q_m(f^*(\xi_\beta, \xi_\beta) f) = q_m(f^*)Q_m(\xi_\beta)q_m(f) \to 0, \]

where $q_m$ stands for $\|\cdot\|_{H_2^m}$ defined in (20). Therefore, we realize $C_2^\infty (EH)_\theta^\mathbb{S} p$ as a smooth module.

4 Nonunital spectral triples and summability

In this section, we define nonunital spectral triples and consider their summability. We also consider the regularity and measurability of the spectral triples of the isospectral deformations of EH-spaces.

Among normed ideals in the algebra of compact operators $\mathcal{K}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, the Dixmier trace ideal $\mathcal{L}^{1,\infty}(\mathcal{H})$ is the domain of a Dixmier trace $Tr_\omega$, where $\omega$
is some functional on the space of bounded sequences. An operator $T \in L^{1,\infty}(\mathcal{H})$ is *measurable* if its Dixmier trace is independent of $\omega$ and one denotes the Dixmier trace by $Tr^+(T)$. See for example [21]. One may define $fT := Tr^+(T)$ as the noncommutative integral of $T$. Apart from the Dixmier trace ideal, the generalized Schatten ideal $L^{p,\infty}(\mathcal{H})$ for $p > 1$ are the domain of operators where the $(p,\infty)$-summability are considered. They are related to $L^{1,\infty}(\mathcal{H})$ in a similar fashion as various Sobolev spaces are linked. If the operator $T \in L^{p,\infty}(\mathcal{H})$, then $T^p \in L^{1,\infty}(\mathcal{H})$.

Rennie (Theorem 12, [5]) provides a measurability criterion of operators from local nonunital spectral triples. Within the locality framework, a generalized Connes trace theorem over commutative geodesically complete Riemannian manifold is also given (Proposition 15, [5]). The Dixmier trace of such measurable operator agrees with the Wodzicki residue of the operator [4].

Gayral and his coworkers [6] carry out a detailed study on summability of the nonunital spectral triples from isospectral deformations. Their results are also of a local manner.

4.1 Nonunital spectral triples and local $(p,\infty)$-summability

**Definition 4.1.** [2] A nonunital spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by

1. A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a local $\ast$-algebra $\mathcal{A}$, containing some algebra $\mathcal{A}_c$ of local units as a dense ideal, on the Hilbert space $\mathcal{H}$. $\mathcal{A}$ admits a suitable unitization $\mathcal{A}_b$.

2. A self-adjoint (unbounded, densely defined) operator $\mathcal{D} : \text{dom}\mathcal{D} \to \mathcal{H}$ such that $[\mathcal{D},a]$ extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}_b$ and $a(\mathcal{D} - \lambda)^{-1}$ is compact for $\lambda \notin \mathbb{R}$ and all $a \in \mathcal{A}$. This is the compact resolvant condition for nonunital triples.

We omit $\pi$ if no ambiguity arises. The spectral triple is even if there exists an operator $\chi = \chi^\ast$ such that $\chi^2 = 1$, $[\chi,a] = 0$ for all $a \in \mathcal{A}$ and $\chi\mathcal{D} + \mathcal{D}\chi = 0$. Otherwise, it is odd.

To obtain the nonunital spectral triple of the isospectral deformation of the EH-space, let $\mathcal{A}$ be the local $\ast$-algebra $C^{\infty}_0(EH)_\theta$ which contains the algebra of local units $C^{\infty}_c(EH)_\theta$ as a dense ideal. The unitization $\mathcal{A}_b$ is chosen as $C^{\infty}_b(EH)_\theta$. The representation $\pi$ is defined by the representation $L^\theta_f : C^{\infty}_b(EH)_\theta \to \mathcal{B}(\mathcal{H})$ from (38). The boundedness of $L^\theta_f$ where $f = \sum_r f_r$ can be seen as follows,

$$\|L^\theta_f\|_{op} = \left\| \sum_r M_{f_r} V^\theta_r \right\|_{op} \leq \sum_r \|M_{f_r} V^\theta_r\|_{op} \leq \sum_r \|M_{f_r}\|_{op} \leq \sum_r \|f_r\|_{\infty} < \infty,$$

where the summations are over $\mathbb{Z}^2$.

Let $\mathcal{D}$ be extension of the Dirac operator of the spinor bundle to the Hilbert space $\mathcal{H}$. Since the Eguchi-Hanson space is geodesically complete, the extended operator is
The identity $\chi_0$ as a pseudodifferential operator and hence bounded.

The identity $\chi$ is chosen to be the chirality operator defined in (15), such that $\chi = \chi^* + \chi^2 = 1$. Since $\chi$ can be realized as a fiberwise constant matrix operating on the spinor bundle, its commutativity with respect to any $L_\chi$ self-adjoint. We will see in the next subsection that the operator (47) is of degree 0 as a pseudodifferential operator and hence bounded.

Proposition 4.2. For any $f \in C^\infty_c(EH)_\theta$, 

$$L_f^\theta(D - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(\mathcal{H}), \quad \forall \lambda \notin \mathbb{R}. \quad (47)$$

Proof. The proof is a straightforward generalization of Proposition 15 of [5] and references therein.

With respect to the local trivializations $\{U_N, U_S\}$ of the spinor bundle $S$ coming from the stereographic projection as before, we may show the summability of the operator (47) by showing the summability of the restrictions of the operator on each trivialization. Indeed, for any $f \in C^\infty_c(EH)_\theta$, the operator $L_f^\theta = \sum_r M_f V_r^\theta$ is defined by summations of normal multiplications by $f_r$ following parallel transporting in the $\phi$ and $\psi$ directions, so that it is well-defined when restricted on either $U_N$ or $U_S$. We may choose the partition of unity $h_N, h_S$ as (40) so that each function $f$ can be decomposed as $f = f_N + f_S$ with $f_N \in C^\infty_c(U_N)$ and $f_S \in C^\infty_c(U_S)$. It suffices to show that 

$$L_f^\theta(D - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(S_{U_N})), \quad \forall f \in C^\infty_c(U_N), \quad (48)$$

and similarly for $U_S$.

For any fixed $f \in C^\infty_c(U_N)_\theta$, we can find a positive constant $R > a$ big enough, and a constant $\Theta > 0$ small enough such that the compact region defined by 

$$W_{R,\Theta} := \{x \in U_N : r \leq R, \theta \geq \Theta\} \subset U_N,$$

contains the compact support of $f$. Notice that with the restricted metric from the EH-space, the region $W_{R,\Theta}$ is a compact manifold with a boundary $\partial W_{R,\Theta}$ defined by $r = R$ and $\theta = \Theta$. We will fix $R$ and $\Theta$ from now on, and write $W$ instead of $W_{R,\Theta}$ and denote the restriction of the spinor bundle $S$ on $W_{R,\Theta}$ by $S_W$. Because the integral curve starting through any point in $W$ along the $\phi$ or $\psi$ direction still lies within $W$, the action of $L_f^\theta$ can be restricted on sections of the subbundle $S_W$.

To prove (48), it suffices to prove that 

$$L_f^\theta(D - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(S_W)).$$

Let $\tilde{W} := W \cup_{\partial W} (-W)$ be the invertible double of the compact manifold $W$ with boundary $\partial W$, and let the corresponding spinor bundle be $\tilde{S} \rightarrow \tilde{W}$ and the corresponding Dirac operator be $D_I$. Applying the Weyl’s lemma [25] on $\tilde{S} \rightarrow \tilde{W}$ as
a vector bundle over a compact manifold without boundary, we obtain \((\mathcal{D}_I - \lambda)^{-1} \in \mathcal{L}^{4,\infty}(L^2(\tilde{\mathcal{S}}))\), for \(\lambda \notin \mathbb{R}\). That is,

\[
\|(\mathcal{D}_I - \lambda)^{-1}\|_{4,\infty}^{\tilde{W},\tilde{W}} < \infty, \quad \forall \lambda \notin \mathbb{R},
\]

where the norm is the \((4,\infty)\)-Schatten norm and we indicate the domain and image of operators as superscript on the norms.

As to the action of \(L^\theta_f\), we may extend the function \(f \in C^\infty_c(W)\) to a function \(\tilde{f} \in C^\infty_c(\tilde{W})\) by zero. Correspondingly, we may extend the operator \(L^\theta_f : L^2(W,\mathcal{S}) \rightarrow L^2(W,\mathcal{S})\) to

\[
L^\theta_{\tilde{f}} : L^2(\tilde{W},\mathcal{S}) \rightarrow L^2(\tilde{W},\mathcal{S}).
\]

Using the resolvant identity \([L^\theta_f, (\mathcal{D}_I - \lambda)^{-1}] = (\mathcal{D}_I - \lambda)^{-1}[\mathcal{D}_I, L^\theta_f](\mathcal{D}_I - \lambda)^{-1}\), we have

\[
(\mathcal{D}_I - \lambda)^{-1} L^\theta_{\tilde{f}} = L^\theta_f (\mathcal{D}_I - \lambda)^{-1} - (\mathcal{D}_I - \lambda)^{-1}(\mathcal{D}_I L^\theta_f - L^\theta_f \mathcal{D}_I)(\mathcal{D}_I - \lambda)^{-1}. \tag{50}
\]

By composing \(L^\theta_f\) with the restriction of sections of \(L^2(\tilde{W},\mathcal{S})\) to \(L^2(W,\mathcal{S})\), we obtain an operator in the same notation, \(L^\theta_f\) mapping from \(L^2(\tilde{W},\mathcal{S})\) to \(L^2(W,\mathcal{S})\). Let \(\iota : W \hookrightarrow \tilde{W}\) be the inclusion map, the composition of \(\iota\) with the identity \([50]\) then gives,

\[
(\mathcal{D} - \lambda)^{-1} L^\theta_{\tilde{f}} \iota = L^\theta_f (\mathcal{D}_I - \lambda)^{-1} \iota + (\mathcal{D} - \lambda)^{-1}(L^\theta_f D_I - D L^\theta_f)(\mathcal{D}_I - \lambda)^{-1} \iota, \tag{51}
\]

as operators maps from \(L^2(W,\mathcal{S})\) to itself.

Applying \([51]\), we obtain

\[
\|L^\theta_f (\mathcal{D} - \lambda)^{-1}\|_{4,\infty}^{\tilde{W},W} = \|(\mathcal{D} - \lambda)^{-1} L^\theta_f\|_{4,\infty}^{W,\tilde{W}} = \|(\mathcal{D} - \lambda)^{-1} L^\theta_f \iota\|_{4,\infty}^{\overline{W},W} = \|L^\theta_f (\mathcal{D}_I - \lambda)^{-1} \iota + (\mathcal{D} - \lambda)^{-1}(L^\theta_f D_I - D L^\theta_f)(\mathcal{D}_I - \lambda)^{-1} \iota\|_{4,\infty}^{\overline{W},W} \leq \|L^\theta_f (\mathcal{D}_I - \lambda)^{-1} \iota\|_{4,\infty}^{\overline{W},W} + \|(\mathcal{D} - \lambda)^{-1}(L^\theta_f D_I - D L^\theta_f)(\mathcal{D}_I - \lambda)^{-1} \iota\|_{4,\infty}^{\overline{W},W}. \tag{52}
\]

We consider the two terms in the last line separately. Since the inclusion \(\iota\) is an isometry, the first term is bounded as

\[
\|L^\theta_f (\mathcal{D}_I - \lambda)^{-1} \iota\|_{4,\infty}^{\overline{W},W} \leq \|L^\theta_f (\mathcal{D}_I - \lambda)^{-1}\|_{4,\infty}^{\overline{W},W} \leq \|L^\theta_f\|_{op}^{\overline{W},W} \|(\mathcal{D}_I - \lambda)^{-1}\|_{4,\infty}^{\overline{W},W} < \infty, \tag{50}
\]

where \(\|L^\theta_f\|_{op}^{\overline{W},W} < \infty\) is because \(L^\theta_f\) is the trivial extension of the bounded operator \(L^\theta_f\) from \(L^2(W,\mathcal{S})\) to itself and the finiteness of \(\|(\mathcal{D}_I - \lambda)^{-1}\|_{4,\infty}^{\overline{W},W}\) is by \([49]\). The
second term is bounded as

\[ \| (D - \lambda)^{-1}(L_f D_I - D L_f^\theta)(D_I - \lambda)^{-1} \parallel^W_{4,\infty} \]

\[ \leq \| (D - \lambda)^{-1}(L_f^\theta D_I - D L_f^\theta)(D_I - \lambda)^{-1} \parallel^W_{4,\infty} \]

\[ \leq \| (D - \lambda)^{-1} \parallel^W_{op} \| (L_f^\theta D_I - D L_f^\theta) \parallel^W_{op} \| (D_I - \lambda)^{-1} \parallel^W_{4,\infty} < \infty. \]  \tag{53} 

Indeed, the finiteness of \( \| (D - \lambda)^{-1} \parallel^W_{op} \) is by the fact that \( (D - \lambda)^{-1} \) is a bounded operator on \( S \rightarrow W \) as the restriction of the bounded operator on \( L^2(S) \). For the finiteness of \( \| (L_f^\theta D_I - D L_f^\theta) \parallel^W_{op} \), we have

\[ \| (L_f^\theta D_I - D L_f^\theta) \parallel^W_{op} = \| [D, L_f^\theta] \parallel^W_{op} \leq \| [D, L_f^\theta] \parallel^{EH \rightarrow EH}_{op} < \infty, \]

since \( f \) extends \( f \) by zero and \( [D, L_f^\theta] \) is bounded. The the finiteness of \( \| (D_I - \lambda)^{-1} \parallel^W_{4,\infty} \) is again by (49).

Summation of the inequalities (52) and (53) implies that

\[ \| L_f^\theta (D - \lambda)^{-1} \parallel^W_{4,\infty} < \infty. \]

The proof for the coordinate patch \( U_S \) is the same. \( \square \)

As pointed out by Rennie, Proposition 4.2 implies the compact resolvent condition.

**Lemma 4.3.** For any \( f \in C_c^\infty(EH)_\theta \), \( L_f^\theta (D - \lambda)^{-1} \in \mathcal{K}(\mathcal{H}) \) with \( \lambda \notin \mathbb{R} \).

**Proof.** Let \( \{f_\beta\} \) is be a sequence of functions in \( C_c^\infty(EH)_\theta \), which converges to the function \( f \in C_c^\infty(EH)_\theta \) in the topology of uniform convergence, then \( L_f^\theta \) converges to \( L_f^\theta \) in the \( C^* \)-operator norm, for the norm-topology is weaker than the topology of uniform convergence. This further implies that the sequence of operators \( \{L_f^\theta (D - \lambda)^{-1}\} \)

converges uniformly to \( L_f^\theta (D - \lambda)^{-1} \) in the operator norm. The \( (4, \infty) \)-summability of each \( L_f^\theta (D - \lambda)^{-1} \) by (53) implies that they are all compact operators. As the uniform limit of a sequence of compact operators, \( L_f^\theta (D - \lambda)^{-1} \) is also compact. \( \square \)

In summary, the data \((C_0^\infty(EH)_\theta, \mathcal{H}, D)\) of the isospectral deformationss of the Eguchi-Hanson spaces are even nonunital spectral triples as in Definition 4.1.

**Definition 4.4.** A (nonunital) spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) is called local, if there exists a local approximate unit \( \{\phi_n\} \subset \mathcal{A}_c \) for \( \mathcal{A} \) satisfying

\[ \Omega_D(\mathcal{A}_c) = \cup_n \Omega_D(\mathcal{A})_n, \]

where \( \Omega_D(\mathcal{A})_n := \{\omega \in \Omega_D(\mathcal{A}) : \phi_n \omega = \omega \phi_n = \omega\} \).

For \( p \geq 1 \), the local spectral triple is called \( (p, \infty) \)-summable if a \( (D - \lambda)^{-1} \in L^{p,\infty}(\mathcal{H}) \), \( \lambda \notin \mathbb{R} \), for any \( n \in \mathcal{A}_c \).
Local \((p, \infty)\)-summability implies that (Proposition 10 [5])
\[
T(1 + D^2)^{-s/2} \in L^{p/s, \infty}(\mathcal{H}), \quad 1 \leq Re(s) \leq p,
\]
for any \(T \in \mathcal{B}(\mathcal{H})\) such that \(T \phi = \phi T = T\) for some \(\phi \in \mathcal{A}_c\). If \(Re(s) > p\), the operator is of trace class.

In considering the (local) summability of the spectral triples, we restrict ourselves on the spectral triple \((C_c^\infty(EH)_{\theta}, \mathcal{H}, \mathcal{D})\).

**Lemma 4.5.** The spectral triple \((C_c^\infty(EH)_{\theta}, \mathcal{H}, \mathcal{D})\) is local \((4, \infty)\)-summable.

*Proof.* First we show that the spectral triple is local. We may choose the local approximate unit \(\{\phi_n\}\) as defined in Section 3.3 so that each of \(\phi_n\) remains commutative. As operators, they act only by normal multiplication \(M_{\phi_n}\) on spinors.

Define \([C_c^\infty(EH)_{\theta}]_n\) to be the subalgebra of \(C_c^\infty(EH)_{\theta}\) consisting of elements \(L_f^0\) such that \(L_f^0 M_{\phi_n} = M_{\phi_n} L_f^0 = L_f^0\), then \(C_c^\infty(EH)_{\theta} = \bigcup_{n \in \mathbb{N}}[C_c^\infty(EH)_{\theta}]_n\). Thus
\[
\Omega_D(C_c^\infty(EH)_{\theta}) = \Omega_D(\bigcup_{n \in \mathbb{N}}[C_c^\infty(EH)_{\theta}]_n) = \bigcup_{n \in \mathbb{N}}\Omega_D([C_c^\infty(EH)_{\theta}]_n).
\]

We claim that this equals to \(\bigcup_{n \in \mathbb{N}}[\Omega_D(C_c^\infty(EH)_{\theta})]_n\), where
\[
[\Omega_D(C_c^\infty(EH)_{\theta})]_n := \{\omega \in \Omega_D(C_c^\infty(EH)_{\theta}) : \omega M_{\phi_n} = M_{\phi_n} \omega = \omega\}.
\]

By the fact that the orbit of the torus action of any point \(x \in K_n\) remains in \(K_n\), \(M_{\phi_n} L_f^0 = L_f^0 M_{\phi_n}\) whenever \(\text{supp}(f) \subset K_n\). That the Dirac operator preserves support implies
\[
M_{\phi_n} [\mathcal{D}, L_f^0] = [\mathcal{D}, L_f^0] M_{\phi_n} = [\mathcal{D}, L_f^0].
\]
This further gives that \(\bigcup_{n \in \mathbb{N}}\Omega_D([C_c^\infty(EH)_{\theta}]_n) \subset \bigcup_{n \in \mathbb{N}}[\Omega_D(C_c^\infty(EH)_{\theta})]_n\). The other direction is obvious. Therefore, \(\Omega_D(C_c^\infty(EH)_{\theta}) = \bigcup_{n \in \mathbb{N}}[\Omega_D(C_c^\infty(EH)_{\theta})]_n\), and the spectral triple is local.

The local \((4, \infty)\)-summability of the spectral triple \((C_c^\infty(EH)_{\theta}, \mathcal{D}, \mathcal{H})\) is implied by Proposition 4.2.

### 4.2 Regularity of spectral triples

For a given spectral triple \((\mathcal{A}, \mathcal{D}, \mathcal{H})\), we can define a derivation \(\delta\) on the space of linear operators on the Hilbert space \(\mathcal{L}(\mathcal{H})\) by
\[
\delta(T) := [[\mathcal{D}], T], \quad T \in \mathcal{L}(\mathcal{H}).
\]

A linear operator \(T\) is in the domain of the derivation \(\text{dom}\delta \subset \mathcal{L}(\mathcal{H})\), if any \(\psi \in \text{dom}([\mathcal{D}])\) implies \(T(\psi) \in \text{dom}([\mathcal{D}])\). For any positive integer \(k\), \(T\) is in the domain of the \(k\)-th derivation \(\text{dom}\delta^k \subset \mathcal{L}(\mathcal{H})\), if \(\delta^{k-1}(T) \in \text{dom}\delta\), where \(\delta^{k-1}(T) = [[\mathcal{D}], [[\mathcal{D}], \ldots, [[\mathcal{D}], T], \ldots]]\), with \(k - 1\) brackets.

The intersection of domains of \(\delta\) with all possible degree \(\text{dom}^{\infty}\delta := \cap_{k \in \mathbb{N}}\text{dom}\delta^k\) is the smooth domain of the derivation \(\delta\). When \(k = 0\), \(\text{dom}\delta^0\) is simply the space of bounded operator \(\mathcal{B}(\mathcal{H})\). Therefore, an operator \(T \in \text{dom}\delta^k\) if \(\delta^k(T)\) is a bounded operator.
Definition 4.6. A spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\) is regular if \(\Omega_D(\mathcal{A}) \subset \text{dom}^\infty \delta\), where \(\Omega_D(\mathcal{A})\) is the algebra of operators generated by \(\mathcal{A}\) and \([\mathcal{D}, \mathcal{A}]\).

Before considering the regularity of the spectral triple, we collect some related properties of operators \(L^\theta_j\) and \(\mathcal{D}\) as pseudodifferential operators. The Dirac operator \(\mathcal{D}\) on the spinor bundle \(\mathcal{S}\) is a first order differential operator with a principal symbol,
\[
\sigma^\mathcal{D}(x, \xi) = c(\xi_j dx^j).
\]
where \(\xi\) as a section in the cotangent bundle \(T^*(EH)\) is of coordinates \((\xi_1, \ldots, \xi_4)\) with respect to the basis \(\{dx^i\}\), defined in the begin of Section 2.1. The operator \(\mathcal{D}^2\) is a second-order differential operator with a principal symbol
\[
\sigma^\mathcal{D}^2(x, \xi) = g(\xi, \xi) 1,
\]
where \(g\) is the induced metric tensor on the cotangent bundle from that on the tangent bundle [2].

Lemma 4.7. The principal symbol of the pseudodifferential operator \(M_f\) is
\[
\sigma^{M_f}(x, \xi) = M_f(x) = \text{diag}_4(f(x)),
\]
where \(\text{diag}_r(g)\) denotes the \(r \times r\) diagonal matrix of \(g\) on the diagonal. The principal symbol of the pseudodifferential operator \(L^\theta_j\) is
\[
\sigma^{L^\theta_j}(x, \xi) = \sum_{r=(r_3,r_4)} M_{f_x}(x) P^{\theta}(x) e(\theta (r_3\xi_4 - r_4\xi_3)),
\]
where the matrix-valued function \(P^{\theta}(x) = P_{c_3} \circ P_{c_4}(x)\) is defined by the composition of parallel propagators along integral curves of \(\partial_{\phi}\) and \(\partial_{\psi}\).

Proof. Applying \(M_f\) where \(f = \sum_r f_r\) on the inverse Fourier transformation of a spinor \(\psi\),
\[
M_f \left( \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} e^{ix \cdot \xi} \hat{\psi}(\xi) d\xi \right) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \text{diag}_4(f(x)) e^{ix \cdot \xi} \hat{\psi}(\xi) d\xi,
\]
we see that \(M_f\) is an order zero classical pseudodifferential operator with principal symbol (56).

From Remark 3.10, the pointwise evaluation of the operator \(L^\theta_j\) is
\[
L^\theta_j \psi(x) = \sum_r M_{f_r}(P_{c_3} \circ P_{c_4})(\psi(x + (0, 0, -2\pi\theta r_4, 2\pi\theta r_3))),
\]
where \(c_4\) is the integral curve of the Killing field \(\partial_\psi\) starting at \((x_1, x_2, x_3 - 2\pi\theta r_4, x_4 + 2\pi\theta r_3)\) and ending \((x_1, x_2, x_3 - 2\pi\theta r_4, x_4)\), and \(P_{c_4}\) is assumed to be the parallel propagator with respect to the spin connection along the \(c_4\). It is evaluated at the point \((x_1, x_2, x_3 - 2\pi\theta r_4, x_4)\) as a four by four matrix. Similarly, \(c_3\) is the integral curve
of the Killing field $\partial_\theta$ starting at $(x_1, x_2, x_3 - 2\pi \theta r_4, x_4)$ and ending at $(x_1, x_2, x_3, x_4)$. $P_{c_3}$ is assumed to be the parallel propagator with respect to the spin connection along the $c_3$ as defined by \cite{24}. In \cite{55}, their composition is evaluated at the point $(x_1, x_2, x_3, x_4)$ as a four by four matrix.

Applying $L_\theta^0$ on the inverse Fourier transformation of $\psi$,

$$L_\theta^0 \psi(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \sum_r M_f, P_{c_3} P_{c_4} \exp(i((x + (0, 0, -2\pi \theta r_4, 2\pi \theta r_3)) \cdot \xi) \hat{\psi}(\xi) d\xi$$

one obtains the symbol of $L_\theta^0$. With respect to the $\xi$ variable, the complete symbol is bounded by a constant and hence is of degree 0 and it can be chosen to be its principal symbol, which takes the form of \cite{57}.

\[\square\]

**Proposition 4.8.** The spectral triple $(C^\infty_0(EH)_\theta, \mathcal{H}, \mathcal{D})$ is regular.

**Proof.** We write $L_\theta^0$ by $f$ for notational simplicity here. As indicated in the proof of Proposition 20 in \cite{2}, $f, [\mathcal{D}, f] \in \text{dom}^{\infty}\delta$ for any $f \in C^\infty_0(EH)_\theta$ if and only if $f, [\mathcal{D}, f] \in \text{dom}_{k,l \geq 0} L^k R^l$, where

$$L(f) := (1 + D^2)^{-1/2}[D^2, f], \quad R(f) := [D^2, f](1 + D^2)^{-1/2},$$

for the reason that $[\mathcal{D}] - (1 + D^2)^{1/2}$ is bounded. The rest of the proof is a direct generalization of the standard method in the unital case, see for instance \cite{21}. Denote $ad(D^2)^m(\cdot) = [D^2, \ldots, [D^2, \cdot] \ldots]$, with $m$ brackets, so that

$$L^k(f) = (1 + D^2)^{-k/2} ad(D^2)^k(f), \quad R^l(f) = ad(D^2)^l(f)(1 + D^2)^{-l/2},$$

where $k, l \in \mathbb{N}$. Their composition is

$$L^k R^l(f) = (1 + D^2)^{-k/2} ad(D^2)^{k+l}(f)(1 + D^2)^{-l/2}.$$

The operator $ad(D^2)(f) = [D^2, f]$ is of order at most 1, since the commutator of the principal symbols \cite{55} and \cite{57} vanishes. Similarly, the operator $ad(D^2)^{(k+l)}(f)$ is of order at most $k + l$. This implies that the operator $L^k R^l(f)$ is of order at most zero and hence a bounded pseudodifferential operator on $\mathcal{H}$. This holds for any $k$ and $l$ in $\mathbb{N}$. Hence $f \in \text{dom}_{k,l \geq 0} L^k R^l$, for any $f \in C^\infty_0(EH)_\theta$.

Since $[\mathcal{D}, M_f]$ is a bounded operator of degree 0 and $V^0(r)$ is of degree 0, seen from \cite{57}, $[\mathcal{D}, L_\theta^0]$ is also a bounded operator of degree 0. The above proof holds if $f$ is replaced by $[\mathcal{D}, L_\theta^0]$. Thus $[\mathcal{D}, L_\theta^0] \in \text{dom}_{k,l \geq 0} L^k R^l$, for any $f \in C^\infty_0(EH)_\theta$. Since $L^k R^l(T) \in \text{dom}_{k,l \geq 0} L^k R^l = \mathcal{B}(\mathcal{H})$ for any $k, l$ where $T \in \mathcal{B}(\mathcal{H})$ is equivalent to $T \in \text{dom}_{k,l \geq 0} L^k R^l$ for any $k, l$, we obtain $\Omega_{\mathcal{D}}(C^\infty_0(EH)_\theta) \subset \text{dom}^{\infty}\delta$. Hence the spectral triple is regular. \[\square\]
4.3 Measurability in the nonunital case

The following is the measurability criterion of operators from a local nonunital spectral triple [5].

**Theorem 4.9.** Let \((A, \mathcal{H}, \mathcal{D})\) be a regular, local \((p, \infty)\)-summable spectral triple with \(p \geq 1\). Suppose that \(T \in B(\mathcal{H})\) such that \(\psi T = T \psi = T\) for some \(\psi \geq 0\) in \(A\). If the limit
\[
\lim_{s \to \frac{p}{2}^+} \left( s - \frac{p}{2} \right) \text{Trace} \left( T(1 + \mathcal{D}^2)^{-s} \right)
\]
exists, then the operator \(T(1 + \mathcal{D}^2)^{-p/2}\) is measurable and its Dixmier trace equals to the limit up to the a factor of \(2/p\),
\[
\text{Tr}^+ \left( T(1 + \mathcal{D}^2)^{-p/2} \right) = \frac{2}{p} \lim_{s \to \frac{p}{2}^+} \left( s - \frac{p}{2} \right) \text{Trace} \left( T(1 + \mathcal{D}^2)^{-s} \right).
\]

Implied by [5], the operators \(L^\theta_f(1 + \mathcal{D}^2)^{-2}\), for \(f \in C^\infty_c(EH)\), from the spectral triple \((C^\infty_c(EH), \mathcal{D}, \mathcal{H})\) satisfies the measurability criterion [59] and hence the Dixmier trace can be uniquely defined. We include these contents briefly for coherence.

**Lemma 4.10.** The limit
\[
\lim_{s \to 2^+} (s - 2) \text{Trace} \left( L^\theta_f(1 + \mathcal{D}^2)^{-s} \right), \quad \forall f \in C^\infty_c(EH),
\]
exists and the operator \(L^\theta_f(1 + \mathcal{D}^2)^{-2}\) is measurable.

**Proof.** Since the spectral triple satisfies the local \((4, \infty)\)-summability condition, \([54]\) implies that \(L^\theta_f(1 + \mathcal{D}^2)^{-s}\) for \(s > 2\) is of trace class and so is \(M_f(1 + \mathcal{D}^2)^{-s}\). Since both of them are of trace class, their traces agree by Corollary 3.10 of [6]. Thus it suffices to show that the limit
\[
\lim_{s \to 2^+} (s - 2) \text{Trace} \left( M_f(1 + \mathcal{D}^2)^{-s} \right), \quad \forall f \in C^\infty_c(EH),
\]
exists. We may compute the trace of the operator by evaluating the corresponding kernels of operators. The kernel of \(M_f\) is given by
\[
K_{M_f}(x, x') = \sum_{r \in \mathbb{Z}^2} M_f r \delta^\theta(x'),
\]
where \(\delta^\theta(x')\) is defined by requiring \(\psi(x) = \int_{EH} \delta^\theta(x') \psi(x') dVol(x')\) for all \(\psi \in L^2(S)\). For \(s > 2\),
\[
\text{Trace} \left( M_f(1 + \mathcal{D}^2)^{-s} \right)
= \int \int K_{M_f}(x, x') K_{(1+\mathcal{D}^2)^{-s}}(x', x) dVol(x') dVol(x)
= \int \int \text{tr} \left( \sum_{r \in \mathbb{Z}^2} \text{diag}(f_r(r)) \delta^\theta_r(x') \right) K_{(1+\mathcal{D}^2)^{-s}}(x', x) dVol(x') dVol(x)
= 4 \int f(x) K_{(1+\mathcal{D}^2)^{-s}}(x, x) dVol(x),
\]

31
where $\text{tr}$ denotes the trace of a matrix. Applying the method of heat kernel expansion on the Laplacian transformation of the kernel as in the proof of Theorem 6.1 [6], we obtain that for $s > 2$,

$$\lim_{s \to 2^+} (s - 2)\text{Trace}(M_f (1 + D^2)^{-s}) = \frac{4}{(2\pi)^2} \lim_{s \to 2^+} \frac{(s - 2)\Gamma(s - 2)}{\Gamma(s)} \int f(x)dVol(x)$$

$$= \frac{4}{(2\pi)^2} \int f(x)dVol(x) < \infty.$$ 

and this equals to $\lim_{s \to 2^+} (s - 2)\text{Trace}(L_f^\theta (1 + D^2)^{-s})$.

Since $f$ is of compact support, we can always find a function $\phi$ of value one on the compact support of $f$ and decaying to zero only with respect to the $r$ variable so that $L_f^\phi = M \phi$ and hence $L_f^\phi M \phi = M \phi L_f^\phi = L_f^\phi$ holds. By Theorem 4.9 the operator $L_f^\phi (1 + D^2)^{-2}$ is measurable.

(60) further implies that the Dixmier trace is

$$Tr^+ \left( L_f^\phi (1 + D^2)^{-2} \right) = \frac{2}{(2\pi)^2} \int_{EH} f(x)dVol(x). \quad (62)$$

In the reduced commutative case the operator $M_f (1 + D^2)^{-2}$ is measurable, and $Tr^+(M_f (1 + D^2)^{-2})$ equals to the right hand side of (62).

The Connes trace theorem for the unital case (Theorem 7.18 [21]) implies that for a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$,

$$Tr^+ \left( a (1 + D^2)^{-p/2} \right) = \frac{1}{p(2\pi)^p} Wres(a (1 + D^2)^{-p/2}), \quad (63)$$

where $\mathcal{D}$ is the Dirac operator of some $p$-dimensional spin manifold, $a (1 + D^2)^{-p/2}$ is considered as a elliptic pseudodifferential operator on the complex spinor bundle $\mathcal{S}$ and $Wres$ is the Wodzicki residue.

Despite a full understanding of (63) in the noncommutative nonunital case, a Wodzicki residue computation of $M_f^\phi (1 + D^2)^{1/2}$ for $f \in C_c^\infty(EH)$ shows

$$Wres \left( M_f (1 + D^2)^{-2} \right) = 8(2\pi)^2 \int_{EH} f(x)dVol(x), \quad f \in C_c^\infty(EH) \quad (64)$$

Comparing with (62), (63) does hold when taking $a = f$ and $p = 4$. This also serves as an example of Proposition 15 [5] where a geodesically complete manifold is considered.

5 Geometric conditions

In this section, we see how the spectral triples of the isospectral deformations of the EH-spaces fit into the proposed geometric conditions to construct noncompact noncommutative spin manifolds [8],[3].

For a nonunital spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ as in Definition 4.1 the geometric conditions are as follows.
(1) **Metric dimension.** There is a unique non-negative integer \( p \), the metric dimension, for which \( a (1 + D^2)^{-1/2} \) belongs to the generalized Schatten ideal \( L^{p, \infty}(\mathcal{H}) \) for \( a \in A \). Moreover, \( Tr^+ (a (1 + D^2)^{-p/2}) \) is defined and not identically zero. This \( p \) is even if and only if the spectral triple is even.

(2) **Regularity.** Bounded operators \( a \) and \([D, a]\), for \( a \in A \), lie in the smooth domain of the derivation \( \delta = [\cdot \cdot \cdot \cdot D, \cdot \cdot \cdot \cdot] \).

(3) **Finiteness.** The algebra \( A \) and its preferred unitization \( A_b \) are \( C^\ast \)-algebras. There exists an ideal \( A_2 \) of \( A_b \), which is also a \( C^\ast \)-algebra with the same \( C^\ast \)-completion as \( A \), such that the subspace of smooth vectors in \( H = \bigcap_{m \in \mathbb{N}} \text{dom}(D^m) \) is a finitely generated projective \( A_2 \)-module.

(4) **Reality.** There is an antiunitary operator \( J \) on \( H \), such that

\[
[a, J b^* J^{-1}] = 0,
\]

for \( a, b \in A_b \). Thus \( b \mapsto J b^* J^{-1} \) is a commuting representation on \( H \) of the opposite algebra \( A_b^0 \). Moreover, for the metric dimension \( p = 4 \),

\[
J^2 = -1, \quad JD = DJ, \quad J\chi = \chi J.
\]

For other dimensions, we refer to the aforementioned references.

(5) **First order.** The bounded operator \([D, a]\) commutes with the opposite algebra representation: \( [[D, a], J b^* J^{-1}] = 0 \) for all \( a, b \in A_b \).

(6) **Orientation.** There is a Hochschild \( p \)-cycle \( c \) on \( A_b \), with values in \( A_b \otimes A_b^0 \). The \( p \)-cycle is a finite sum of terms like \((a \otimes b^0 \otimes a_1 \otimes \cdots \otimes a_p)\), and its natural representation \( \pi_D(c) \) on \( H \) is defined by

\[
\pi_D((a_0 \otimes b_0^0 \otimes a_1 \otimes \cdots \otimes a_p) := a_0 J b_0^* J^{-1} [D, a_1] \cdots [D, a_k].
\]

The *volume form* \( \pi_D(c) \) solves the equation \( \pi_D(c) = \chi \) in the even case and \( \pi_D(c) = 1 \) in the odd case.

### 5.1 Metric dimensions

One might show \( p = 4 \) for the triples \((C^\infty_0(EH)_\theta, D)\) by considering the measurability of the operator \( L^\theta_f (1 + D^2)^{-2} \) for \( f \in C^\infty_0(EH)_\theta \). However, the algebra \( C^\infty_0(EH)_\theta \) is not integrable, which is necessary for the computation of the Wodzicki residue \([4]\) of the operator \( L^\theta_f (1 + D^2)^{-2} \). Thus \( L^\theta_f (1 + D^2)^{-2} \) may not be measurable. Nonetheless, Lemma \([4.10]\) implies that operators \( L^\theta_f (1 + D^2)^{-2} \) for \( f \in C^\infty_c(EH)_\theta \) are measurable. The Dixmier trace is evaluated as

\[
Tr^+ (L^\theta_f (1 + D^2)^{-2}) = \frac{2}{(2\pi)^2} \int f \, dVol,
\]
which is finite and nonzero. We do not know whether this remains true for some general integrable algebras, for instance $C_2^\infty(EH)_\theta$, lying between $C_c^\infty(EH)_\theta$ and $C_0^\infty(EH)_\theta$.

5.2 Finiteness

By the construction of the ideal $C_2^\infty(EH)$ in Section 3.2, we see that the $C_c^\infty(EH)$-module $C_2^\infty(EH)^8 p$, with $p$ as in (44), is the smooth domain of the Dirac operator in $\mathcal{H}$. In the deformed case, we recall that $C_2^\infty(EH)^8 p$ is a $C_b^\infty(EH)_\theta$-module.

By matching generators, we have the isomorphism between the finitely generated projective modules, $C_2^\infty(EH)^8 p \cong C_2^\infty(EH)^8 p$. Therefore,

$$\bigcap_{m \in \mathbb{N}} \text{dom}(D^m) \cong C_2^\infty(EH)^8 p.$$ 

From Section 3.4, the Fréchet algebra $C_2^\infty(EH)_\theta$ is a pre-$C^*$-algebra with the same $C^*$-completion $C_0(EH)_\theta$ as that of the algebra $C_0^\infty(EH)_\theta$. Hence the finiteness condition is satisfied.

As an application of a general construction considering smooth projective modules in [2], we may define a $\mathbb{C}$-valued inner product on the projective module. Since the Hermitian form on the projective module $C_c^\infty(EH)^8 p$ is $C_c^\infty(EH)_\theta$-valued, composing with the Dixmier trace, one may define an inner product on $C_c^\infty(EH)^8 p$ by

$$\tau(\xi, \eta) := \text{Tr}^+ \left( L^{\theta}(\xi|\eta)(1 + D^2)^{-2} \right) = \frac{2}{(2\pi)^2} \int (\xi|\eta) d\text{Vol},$$

where the equality is by (62). Here the image $(\xi|\eta) = \sum \xi_k^* \times_\theta \eta_k \in C_c^\infty(EH)_\theta$ is considered as a function in $C_c^\infty(EH)$.

One can further take the Hilbert space completion $\overline{C_c^\infty(EH)^8 p}$ with respect to the inner product $\tau$. When restricted to the commutative case, the inner product is simply the $L^2$-inner product on the spinor bundle, and the Hilbert space $\overline{C_c^\infty(EH)^8 p}$ is the Hilbert space $\mathcal{H}$, appearing in the spectral triple.

5.3 Regularity

The regularity condition is implied by Proposition 4.8.

5.4 Reality

The proof of the reality condition is based on the lecture notes [26]. With respect to the decomposition of spinor bundle $S = S^+ \oplus S^-$ as in Section 2.3, we have the corresponding Hilbert space completions under the inner product coming from the $L^2$-norms, and their sum is the Hilbert space completion of $S$, $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. Any element $\psi \in \mathcal{H}$ can thus be decomposed as $\psi = (\psi^+, \psi^-)^t$. The operator $J$ defined on
the spinor bundle \(^{(16)}\) can be extended to the Hilbert space as an antiunitary operator \(J : \mathcal{H} \rightarrow \mathcal{H}\) by

\[
J \left( \begin{array}{c} \psi^+ \\ \psi^- \end{array} \right) := \left( \begin{array}{c} -\overline{\psi}^- \\ -\overline{\psi}^+ \end{array} \right),
\]

satisfying \(J^2 = -1\).

We define the representation of the opposite algebra \(A_b^o\) of \(A_b = C^\infty_b(EH)\) on \(\mathcal{H}\), \(R^0_b : A_b^o \rightarrow \mathcal{B}(\mathcal{H})\) by \(R^0_b := J F^\theta_b J^{-1}\). Specifically, for \(h = \sum_s h_s\), the representation is

\[
R^\theta_b h = \sum_s J M h_s^\theta V_{-s}^\theta J^{-1} = \sum_s M h_s V_{-s}^\theta.
\]

The commutativity of operators \(L^\theta_f\) and \(R^\theta_h\) where \(f = \sum_r f_r\) is seen as follows,

\[
[L^\theta_f, R^\theta_h] = \sum_{r,s} f_r V^\theta_r h_s V_{-s}^\theta - h_s V_{-s}^\theta f_r V^\theta_r
= \sum_{r,s} f_r h_s \sigma(r, s) V^\theta_r V_{-s}^\theta - h_s f_r \sigma(-s, r) V_{-s}^\theta V^\theta_r
= \sum_{r,s} [f_r, h_s] \sigma(r, s) V_{r-s}^\theta = 0,
\]

where identities \(\sigma(r, s) = \sigma(-s, r)\) and \(V^\theta_r V_{-s}^\theta = V_{-s}^\theta V^\theta_r = V_{r-s}^\theta\) are applied.

As in the commutative case, \(D J = J D\) and \(J \chi = \chi J\) where \(\chi\) is the chirality operator \((15)\).

### 5.5 First order

The proof of the first order condition is again from \((20)\). For any \(f = \sum_r f_r\) and \(h = \sum_s h_s\) in \(C^\infty_b(EH)\), the first order property \([[[D, f_r], h_s] = 0\) in the commutative case implies that,

\[
[[[D, L^\theta_f], R^\theta_h] = \sum_{r,s} [[[D, f_r], h_s V^\theta_r] = \sum_{r,s} [[[D, f_r], h_s] \sigma(r, s) V_{r-s}^\theta = 0.
\]

### 5.6 Orientation

In Riemannian geometry, the volume form determines the orientation of a manifold. Translated to the spectral triple language, the volume form is replaced by a Hochschild cycle \(c\) which can be represented on \(\mathcal{H}\) such that \(\pi_D(c) = \chi\) in the even case. For a detailed discussion we refer to \((21)\).

We may obtain a Hochschild 4-cycle of the spectral triple from the classical volume form of the Eguchi-Hanson space. We will only give the construction on the coordinate chart \(U_N\), that for the other chart \(U_S\) is similar and the global construction can be obtained by a partition of unity. We will consider the commutative case first and then the deformed case.
Define a new set of coordinates by \( u_1 = x_1, u_2 = x_2, u_3 = e^{ix_3}, u_4 = e^{ix_4}, \) so that the transition of differential forms \( dx^i = v^i_j du^j \) is given by the diagonal matrix \( V = (v^i_j) := diag(1,1, -\frac{i}{u_3}, -\frac{i}{u_4}) \). Composing with the \( \vartheta^\alpha = h^\alpha_i dx^i \) where \( h^\alpha_i \) are components of the matrix \( H \) in (5), the transition of differential forms \( \vartheta^\alpha = k^\alpha_i du^i \) is given by the matrix \( K = (k^\alpha_i) := HV. \) In components,

\[
k^\alpha_1 = h^\alpha_1, \quad k^\alpha_2 = h^\alpha_2, \quad k^\alpha_3 = h^\alpha_3 \frac{-i}{u_3}, \quad k^\alpha_4 = h^\alpha_4 \frac{-i}{u_4}, \quad \alpha = 1, \cdots, 4.
\]  

(66)

Similarly, the transition \( du^i = \tilde{v}^i_j dx^j \) is given by the inverse matrix \( V^{-1} = (\tilde{v}^i_j) \) of \( V \). Composing with \( d\tilde{u}^i = \tilde{h}^\beta_j \vartheta^\beta \) where \( \tilde{h}^\beta_j \) are elements of the inverse matrix \( H^{-1} \) in (6), we obtain \( d\tilde{u}^i = \tilde{k}^\beta_j \vartheta^\beta \) with \( \tilde{k}^\beta_j \) as the elements of the inverse matrix \( K^{-1} = V^{-1}H^{-1} \). In components,

\[
\tilde{k}^\beta_1 = \tilde{h}^\beta_1, \quad \tilde{k}^\beta_2 = \tilde{h}^\beta_2, \quad \tilde{k}^\beta_3 = i u_3 \tilde{h}^\beta_3, \quad \tilde{k}^\beta_4 = i u_4 \tilde{h}^\beta_4, \quad \beta = 1, \cdots, 4.
\]

To avoid ambiguity, if the \( u \)-coordinates and \( x \)-coordinates appear in the same formula, we will distinguish them by adding \( {}' \) to indices of the \( u \)-coordinates. By tensor transformations, we may obtain the Dirac operator satisfying \( D(s) = -i\gamma^j \nabla_j s \) in the coordinates \( \{u^i\}'s \) from (20) in the coordinates \( \{x_i\}'s \) as,

\[
D = -i \tilde{h}^\gamma_\eta \gamma^\eta \left( \partial^{u_1} - \frac{1}{4} \tilde{\Gamma}^\alpha_{1\alpha} \gamma^\alpha \gamma^\beta \right) - i \tilde{h}^\gamma_{\bar{\eta}} \gamma^{\bar{\eta}} \left( \partial^{u_2} - \frac{1}{4} \tilde{\Gamma}^\alpha_{2\alpha} \gamma^\alpha \gamma^\beta \right) + u_3 \tilde{h}^\gamma_\eta \gamma^\eta \left( \partial^{u_3} + \frac{1}{4} \tilde{\Gamma}^\alpha_{3\alpha} \gamma^\alpha \gamma^\beta \right) + u_4 \tilde{h}^\gamma_\eta \gamma^\eta \left( \partial^{u_4} + \frac{1}{4} \tilde{\Gamma}^\alpha_{4\alpha} \gamma^\alpha \gamma^\beta \right),
\]

where \( \tilde{\Gamma}^\alpha_{i\alpha} \)'s are from (18) and \( \gamma^\alpha \) are from (11).

The volume form of the Eguchi-Hanson space can be represented in the orthonormal basis on \( U_N \) as

\[
\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3 \wedge \vartheta^4 = \vartheta^1 \vartheta^2 \vartheta^3 \vartheta^4 = k^1_{i_1} du^{i_1} \wedge k^2_{i_2} du^{i_2} \wedge k^3_{i_3} du^{i_3} \wedge k^4_{i_4} du^{i_4} = \vartheta^{i_1} \vartheta^{i_2} \vartheta^{i_3} \vartheta^{i_4}.
\]

(67)

We may define a Hochschild 4-cycle \( c_0 \) in \( C_4(\mathcal{A}_b, \mathcal{A}_b \otimes \mathcal{A}_b^\circ) \), with \( \mathcal{A}_b = C_\infty^0(EH) \) and \( \mathcal{A}_b^\circ \) as the opposite algebra of \( \mathcal{A}_b \), by

\[
c_0 := \frac{1}{4!} \sum_{\sigma \in S_4} (-1)^{\vert \sigma \vert} (k^\sigma_{i_1(4)} \otimes 1^\circ)(k^\sigma_{i_2(3)} \otimes 1^\circ)(k^\sigma_{i_3(2)} \otimes 1^\circ)(k^\sigma_{i_4(1)} \otimes 1^\circ) \quad \otimes u^{\sigma(1)} \otimes u^{\sigma(2)} \otimes u^{\sigma(3)} \otimes u^{\sigma(4)}.
\]

(68)

where \( \sigma \) is an element in the permutation group \( S_4 \) and \( (-1)^{\vert \sigma \vert} \) indicates the sign of the permutation. On the \( \mathcal{A}_b \)-bimodule \( \mathcal{A}_b \otimes \mathcal{A}_b^\circ \), \( \mathcal{A}_b \) acts as \( a'(a \otimes b^0)a'' := a' a a'' \otimes b^0 \), for \( a \otimes b^0 \in \mathcal{A}_b \otimes \mathcal{A}_b^\circ \) and \( a', a'' \in \mathcal{A}_b \).

**Lemma 5.1.** The Hochschild 4-chain (68) is a Hochschild cycle. That is, \( b(c_0) = 0 \), where \( b \) is the boundary operator of a Hochschild chain.
Proof. Recall that the Hochschild boundary operator $b$ acts on a simple $n$-chain $a = (a_0 \otimes b_0^i) \otimes a_1 \otimes \cdots \otimes a_n$ in $C_n(A_b, A_b \otimes A_b^\circ)$ by

$$b(a) = (a_0 \otimes b_0^i) a_1 \otimes a_2 \otimes \cdots \otimes a_n$$

$$+ \sum_{j=1}^{n-1} (-1)^j (a_0 \otimes b_0^j) \otimes a_1 \otimes \cdots \otimes a_j \otimes a_{j+1} \otimes \cdots \otimes a_n$$

$$+ (-1)^n a_n (a_0 \otimes b_0^n) \otimes a_1 \otimes \cdots \otimes a_{n-1}. \quad (69)$$

Elements of $b(c_0)$ are of three types.

The first type corresponds to the second line in (69),

$$(-1)^{|\sigma|} (-1)^j (k_{i^\sigma(4)}^\sigma \otimes 1^\circ) (k_{i^\sigma(2)}^\sigma \otimes 1^\circ) (k_{i^\sigma(1)}^\sigma \otimes 1^\circ)$$

$$\otimes u^{i^\sigma(1)} \otimes \cdots \otimes u^{i^\sigma(j)} u^{i^\sigma(j+1)} \otimes \cdots \otimes u^{i^\sigma(n)}.$$

In the summation of all $\sigma \in S_4$, each such term can be cancelled by a term from another $\sigma'$ which obtain from the composition of $\sigma$ by a transition between $\sigma(j)$ and $\sigma(j+1)$, as

$$(-1)^{|\sigma'|} (-1)^j (k_{i^{\sigma'}(4)}^{\sigma'} \otimes 1^\circ) (k_{i^{\sigma'}(2)}^{\sigma'} \otimes 1^\circ) (k_{i^{\sigma'}(1)}^{\sigma'} \otimes 1^\circ)$$

$$\otimes u^{i^{\sigma'}(1)} \otimes \cdots \otimes u^{i^{\sigma'}(j+1)} u^{i^{\sigma'}(j)} \otimes \cdots \otimes u^{i^{\sigma'}(n)}.$$

Indeed, since $(-1)^{|\sigma|} = -(-1)^{|\sigma'|}$ and the elements in the first term from the bimodule are commuting, the summation of such pairs is

$$(-1)^{|\sigma|} (-1)^j (k_{i^\sigma(4)}^\sigma \otimes 1^\circ) (k_{i^\sigma(2)}^\sigma \otimes 1^\circ) (k_{i^\sigma(1)}^\sigma \otimes 1^\circ)$$

$$\otimes u^{i^\sigma(1)} \otimes \cdots \otimes (u^{i^\sigma(j)} u^{i^\sigma(j+1)} - u^{i^\sigma(j+1)} u^{i^\sigma(j)}) \otimes \cdots \otimes u^{i^\sigma(n)} = 0.$$

It vanishes since $u_{i^\sigma(j)} u_{i^\sigma(j+1)} = u_{i^{\sigma}(j+1)} u_{i^{\sigma}(j)}$ as elements in $A_b$.

The second type corresponds to the first line in (69). After the $A_b$-bimodule action from the right, it is in the following form,

$$\left( (k_{i^\sigma(4)}^\sigma k_{i^\sigma(3)}^\sigma k_{i^\sigma(2)}^\sigma k_{i^\sigma(1)}^\sigma u^{i^\sigma(1)}) \otimes 1^\circ \otimes u^{i^\sigma(2)} \otimes u^{i^\sigma(3)} \otimes u^{i^\sigma(4)} \right).$$

The third type of component corresponds to the third line in (69). After the $A_b$-bimodule action from the left, it is in the following form,

$$\left( (u^{i^{\sigma'}(4)} k_{i^{\sigma'}(3)}^{\sigma'} k_{i^{\sigma'}(2)}^{\sigma'} k_{i^{\sigma'}(1)}^{\sigma'}) \otimes 1^\circ \otimes u^{i^{\sigma'}(1)} \otimes u^{i^{\sigma'}(2)} \otimes u^{i^{\sigma'}(3)} \right).$$

By commutativity of $A_b$, the summation of all $\sigma$ of the second type and the third type cancel exactly when the permutation $\sigma'$ differs from $\sigma$ by a transition between $(\sigma(1), \sigma(2), \sigma(3), \sigma(4))$ to $(\sigma(4), \sigma(1), \sigma(2), \sigma(3))$. Indeed, such $\sigma$ and $\sigma'$ are of opposite sign. Therefore, all three types cancel in the summation of $\sigma \in S_4$, and $b(c_0) = 0$. This shows that $c_0$ is a Hochschild 4-cycle. \qed

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We define the representation $\pi_D$ of the Hochschild cycle $c_0$ on the Hilbert space by $\pi_D(a_0 \otimes b_0^* \otimes a_1 \otimes \cdots \otimes a_i) := M_{a_0} M_{b_0} [\mathcal{D}, M_{a_1}][\mathcal{D}, M_{a_2}][\mathcal{D}, M_{a_3}][\mathcal{D}, M_{a_4}]$.

**Proposition 5.2.** The operator $\pi_D(c^0) = \chi$.

**Proof.**

\[
4! \pi_D(c_0) = \sum_{\sigma \in S_4} (-1)^{|\sigma|} K_{i_1\sigma(4)} K_{i_2\sigma(3)} K_{i_3\sigma(2)} K_{i_4\sigma(1)} \gamma_{i_1}^{\sigma(1)} \gamma_{i_2}^{\sigma(2)} \gamma_{i_3}^{\sigma(3)} \gamma_{i_4}^{\sigma(4)} = 4! \gamma_1^{\sigma(1)} \gamma_2^{\sigma(2)} \gamma_3^{\sigma(3)} \gamma_4^{\sigma(4)}.
\]

Thus $\pi_D(c_0) = \chi$. \qed

Now we consider the noncommutative case. Let $A_{b, \theta}$ be $C^\infty_b(\mathcal{E}H)_\theta$ and $A^\circ_{b, \theta}$ be the opposite algebra. On the $A_{b, \theta}$-bimodule $A_{b, \theta} \otimes A^\circ_{b, \theta}$, $A_{b, \theta}$ acts as $a' (a \otimes b^0) a'' := (a' \times \theta a \times \theta a'') \otimes b^0$, for $a \otimes b^0 \in A_{b, \theta} \otimes A^\circ_{b, \theta}$ and $a', a'' \in A_{b, \theta}$.

The Hochschild 4-chain in $C_4(A_{b, \theta}, A_{b, \theta} \otimes A^\circ_{b, \theta})$ is defined by

\[
c := \frac{1}{4!} \sum_{\sigma \in S_4} (-1)^{|\sigma|} K_{i_1\sigma(4)} K_{i_2\sigma(3)} K_{i_3\sigma(2)} K_{i_4\sigma(1)} \otimes u^{i_1\sigma(1)} \otimes u^{i_2\sigma(2)} \otimes u^{i_3\sigma(3)} \otimes u^{i_4\sigma(4)},
\]

where $K_{i}^{\sigma}$ is the corresponding element of $k_{i}^{\sigma}$ in the bimodule $A_{b, \theta} \otimes A^\circ_{b, \theta}$. They are chosen as,

\[
K_{1}^{4} := \Delta(u_{1})^{-1/2} \otimes 1^0, \quad K_{2}^{1} := -\left(\frac{u_{1}}{2} \otimes 1^0\right) \chi(u_3), \quad K_{2}^{2} := \left(\frac{u_{1}}{2} \otimes 1^0\right) \varphi(u_3),
\]

\[
K_{3}^{1} := -\left(\frac{u_{1}}{2} \sin u_2 \otimes 1^0\right) \varphi(u_3) \left(\frac{-i}{u_3} \otimes 1^0\right),
\]

\[
K_{3}^{2} := -\left(\frac{u_{1}}{2} \sin u_2 \otimes 1^0\right) \varphi(u_3) \left(\frac{-i}{u_3} \otimes 1^0\right),
\]

\[
K_{3}^{4} := \left(\frac{u_{1}}{2} \Delta(u_1)^{1/2} \cos u_2 \otimes 1^0\right) \left(\frac{-i}{u_3} \otimes 1^0\right),
\]

\[
K_{4}^{3} := \left(\frac{u_{1}}{2} \Delta(u_1)^{1/2} \otimes 1^0\right) \left(\frac{-i}{u_4} \otimes 1^0\right),
\]

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Notice that \( K \) is the product summation of all the terms of first type is zero. \( \sigma \) is the activity factor coming from the second line. Therefore, it reduces to the commutative activity factor coming from the first line of (71) always cancels with the noncommutative. This also holds when 3 and 4 swap. These observations imply that the noncommutativity factor coming from terms of the first type, \( \sigma \) in the proof Lemma 5.1 for terms of the first type, \( \psi \) only contributions from terms like \( i \) where \( \psi \) and \( \sigma \) swap. In this way, all the \( u_3 \) appearing in the matrix \( H \) of \( K = HV \) can be “commutatized”.

**Lemma 5.4.** The Hochschild 4-chain (70) is a Hochschild cycle in \( Z_4(A_{b,\theta}, A_{b,\theta} \otimes A_{b,\theta}) \). I.e., \( b(c) = 0 \), where \( b \) is the boundary operator of a Hochschild chain.

**Proof.** As in the commutative case, elements of \( b(c) \) are of three types. The first type is,

\[
(-1)^{|\sigma|} (-1)^j (K_{\sigma(4)}^{(4)} \times_{\theta} K_{\sigma(3)}^{(3)} \times_{\theta} K_{\sigma(2)}^{(2)} \times_{\theta} K_{\sigma(1)}^{(1)})
\otimes u^{i_{\sigma(1)}} \otimes \cdots \otimes u^{i_{\sigma(j)}} \otimes_{\theta} u^{i_{\sigma(j+1)}} \otimes \cdots \otimes u^{i_{\sigma(4)}}. \tag{71}
\]

Firstly, from Remark 5.3 we may observe that the noncommutative part of any \( K_i^j \) has only contributions from terms of \( \frac{1}{u_3} \times_{\theta} \frac{1}{u_4} \), for \( i = 3, 4 \). Secondly, any term containing the product \( \frac{-i}{u_3} \times_{\theta} \frac{-i}{u_4} \) contains the product \( u_4 \times_{\theta} u_3 \) and their product is,

\[
\frac{-i}{u_3} \times_{\theta} \frac{-i}{u_4} \times_{\theta} u_4 \times_{\theta} u_3 = e^{-i\theta} \frac{-i}{u_3} \frac{-i}{u_4} e^{i\theta} u_4 u_3 = -1.
\]

This also holds when 3 and 4 swap. These observations imply that the noncommutativity factor coming from the first line of (71) always cancels with the noncommutativity factor coming from the second line. Therefore, it reduces to the commutative case. By the same matching of \( \sigma \)’s in the proof Lemma 5.1 for terms of the first type, summation of all the terms of first type is zero.

The second type is,

\[
\left( K_{\sigma(4)}^{(4)} \times_{\theta} K_{\sigma(3)}^{(3)} \times_{\theta} K_{\sigma(2)}^{(2)} \times_{\theta} K_{\sigma(1)}^{(1)} \right) u^{i_{\sigma(1)}} \otimes u^{i_{\sigma(2)}} \otimes u^{i_{\sigma(3)}} \otimes u^{i_{\sigma(4)}}.
\]

Notice that \( K_{\sigma(1)}^{(1)} \) commutes with \( u_{i_{\sigma(1)}} \). The third type is

\[
u_{i_{\sigma(4)}^{(4)}} \left( K_{\sigma(4)}^{(4)} \times_{\theta} K_{\sigma(3)}^{(3)} \times_{\theta} K_{\sigma(2)}^{(2)} \times_{\theta} K_{\sigma(1)}^{(1)} \right) \otimes u^{i_{\sigma(1)}} \otimes u^{i_{\sigma(2)}} \otimes u^{i_{\sigma(3)}}.
\]

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Notice that \(u_{i\sigma'(4)}\) commutes with \(K_{i\sigma'(4)}\). As in the commutative case, we may pair \(\sigma\) and \(\sigma'\) which are related by \(\sigma'(1) = \sigma(4), \sigma'(2) = \sigma(1), \sigma'(3) = \sigma(2), \sigma'(4) = \sigma(3)\) so that they are canceled through the summation of \(\sigma\). Three cases altogether give us \(b(c) = 0\), and hence the proof. \(\square\)

We represent the Hochschild cycle \(c\) on the Hilbert space \(\mathcal{H}\) by

\[
\pi_D(a_0 \otimes b_0^1 \otimes a_1 \otimes \cdots \otimes a_4) := L_{a_0}^\theta R_{b_0}^\theta [D, \pi_D\theta][D, L_{a_2}^\theta][D, L_{a_3}^\theta][D, L_{a_4}^\theta],
\]

for \(a_0 \otimes b_0^1 \otimes a_1 \otimes \cdots \otimes a_4 \in \mathbb{Z}_4(A_{b,\theta}, A_{b,\theta} \otimes A_{b,\theta}^c)\). A straightforward fact follows,

**Lemma 5.5.** \(\pi_D(\chi(u_3)) = M_\cos \phi\) and \(\pi_D(g(u_3)) = M_\sin \phi\).

**Proposition 5.6.** The operator \(\pi_D(c) = \chi\).

**Proof.** By using the commutativity between the Dirac operator and \(V_{\eta}^\theta\), we can write down the formula for the commutators:

\[
[D, L_{u_i}^\theta] = c(du_i^1), \quad [D, L_{u_3}^\theta] = c(du_3)V_{(1,0)}^\theta, \quad [D, L_{u_4}^\theta] = c(du_4)V_{(0,1)}^\theta,
\]

where \(i = 1, 2\). By Lemma 5.5, all the nonvanishing representation of coefficients in the bimodule of the Hochschild cycle \(c\) are

\[
\pi_D(K_{1}^1) = M_{\Delta(u_1)^{-1/2}}, \quad \pi_D(K_{1}^2) = -M_{\frac{u_1}{u_2}} M_\cos \phi, \quad \pi_D(K_{1}^3) = M_{\frac{u_1}{u_2}} M_\sin \phi,
\]

\[
\pi_D(K_{3}^1) = -M_{\frac{u_1}{u_2}} \sin u_2 M_\sin \phi L_{u_3}^\theta, \quad \pi_D(K_{3}^2) = M_{\frac{u_1}{u_2}} \Delta(u_1)^{1/2} \cos u_2 L_{u_3}^\theta,
\]

\[
\pi_D(K_{3}^3) = -M_{\frac{u_1}{u_2}} \sin u_2 M_\cos \phi L_{u_3}^\theta, \quad \pi_D(K_{4}^3) = M_{\frac{u_1}{u_2}} \Delta(u_1)^{1/2} L_{u_4}^\theta.
\]

The representation \(\pi_D(c)\) is thus

\[
\pi_D(c) = \frac{1}{4!} \sum_{\sigma \in S_4} (-1)^{|\sigma|} \pi_D(K_{i\sigma(4)}^1) \pi_D(K_{i\sigma(3)}^2) \pi_D(K_{i\sigma(2)}^3) \pi_D(K_{i\sigma(1)}^4)
\]

\[
c(du_1^{\sigma(1)})V_{i\sigma(1)}^\theta c(du_2^{\sigma(2)})V_{i\sigma(2)}^\theta c(du_3^{\sigma(3)})V_{i\sigma(3)}^\theta c(du_4^{\sigma(4)})V_{i\sigma(4)}^\theta,
\]

where \(V_{i\sigma(k)}^\theta := V_{\deg(u_{i\sigma(k)})}^\theta\). For any fixed component in the summation we may compare the expression of \(\pi_D(K_{i\sigma}^j)\) and \([D, L_{u_k}^\theta]\). The result is that whenever there is a noncommutative factor generated by some \(\pi_D(K_{i\sigma}^j)\) as \(V_{\deg(1/u_1)}^\theta\) there is a corresponding noncommutative factor generated by \([D, L_{u_k}^\theta]\) as \(V_{\deg(u_1)}^\theta\). Furthermore, these paired noncommutative factors cancel consistently. Thus, each component in the summation is simply the same as that in the commutative case. Applying Proposition 5.2, the summation gives \(\chi\) again and this completes the proof of the orientation condition, \(\pi_D(c) = \chi\). \(\square\)
6 Conclusions

We have obtained the nonunital spectral triples of the isospectral deformations of the Eguchi-Hanson spaces along torus isometric actions and studied analytical properties of the triple. We have also tested the proposed geometric conditions of a noncompact noncommutative geometry on this example.

There are possible generalizations in the following directions. Firstly, we may further consider the Poincaré duality of nonunital spectral triples [27]. Secondly, we may take the conical singularity limit of EH-spaces and consider the spectral triple of the conifold. Thirdly, we may realize the spectral triple as a complex noncommutative geometry defined by [28]. Finally, we may deform the EH-spaces, and possibly for more general ALE-spaces, by using the hyper-Kähler quotient structures.

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