Abstract

We present numerical data of the height-height correlation function and of the avalanche size distribution for the three dimensional Toom interface. The height-height correlation function behaves similarly as the interfacial fluctuation width, which diverges logarithmically with space and time for both unbiased and biased cases. The avalanche size defined by the number of changing sites caused by a single noise process, exhibits an exponentially decaying distribution, which is in contrast to power-law distributions appearing in typical self-organized critical phenomena. We also generalize the Toom model into arbitrary dimensions.

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The Toom model [1] is a dynamical model of Ising spins in which the condition of detailed balance is not satisfied and hence whose phases are not described by equilibrium Gibbs ensembles. Recently, Derrida, Lebowitz, Speer and Spohn (DLSS) [2] studied physical properties of interfaces formed in the two-dimensional Toom model. In low-noise limit, this model leads to a \((1 + 1)\) dimensional solid-on-solid-type (SOS) model, which is in turn much simpler for understanding generic nature of dynamics. In the SOS model, the dynamics of spin-flips may be regarded as a deposition-evaporation process of particles. Due to the nature of the Toom dynamics, the deposition-evaporation process occurs in an avalanche fashion with preferred direction. DLSS found, among others, that the continuum stochastic equation describing such SOS model is the well-known Kardar-Parisi-Zhang (KPZ) equation [3]. In reference [4], present authors proposed a natural generalization of the model in three dimension on the body centered cubic lattice and found that the continuum equation is the anisotropic KPZ (AKPZ) equation given by

\[
\partial_t h = \nu_{\parallel} \partial_{\parallel}^2 h + \nu_{\perp} \partial_{\perp}^2 h + \frac{1}{2} \lambda_{\parallel} (\partial_{\parallel} h)^2 + \frac{1}{2} \lambda_{\perp} (\partial_{\perp} h)^2 + \eta
\]  

(1)

with opposite signs of the coefficients \(\lambda_{\parallel}\) and \(\lambda_{\perp}\). In Eq. (1), \(h\) is the height of the fluctuating surface with respect to a reference (co-moving) plane, \(\partial_{\parallel}\) (\(\partial_{\perp}\)) stands for spatial derivative along the direction parallel (perpendicular) to the avalanche direction, and \(\eta\) is the white noise. AKPZ equation with opposite signs of \(\partial_{\parallel}\) and \(\partial_{\perp}\) is known to renormalize toward the weak coupling limit [5] and consequently, square of the width, \(w^2\), of the Toom interface shows logarithmic dependence both in space and time. This is in contrast to the model studied by Barabási, Araujo, and Stanley [6] which again is described by Eq. (1) but belongs to the strong coupling regime of the AKPZ universality.
In this work, we present numerical data of the height-height correlation function and of the avalanche size distribution for the three-dimensional Toom interface. The height-height correlation function shows the same logarithmic dependences both on space and time as in $w^2$. The avalanche size distribution is found to be exponential. This is in contrast to power law behaviors appearing in typical self-organized critical phenomena and indirectly substantiate the validity of the collective variable approximation used by DLSS.

The $d$-dimensional Toom model we introduced in [4] and generalized here consists of Ising spins ($\sigma(x_1, x_2, \cdots, x_d) = \pm 1$) on $d$-dimensional body-centered-cubic lattice. The spin coordinates $x_i$ takes the values in $\mathbb{Z}$ for the spins on one sublattice and $\mathbb{Z} + \frac{1}{2}$ for those on the other sublattice. At each time step, a randomly selected spin is updated according to the local rule that it becomes, at the next time step, equal to the majority of itself and of its $2^{d-1}$ neighbors relatively situated at $(-\frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \cdots, \pm \frac{1}{2})$ with probability $1 - p - q$, to $+1$ with probability $p$, and to $-1$ with probability $q$. The referencing neighbors are shown in Fig. 1 for $d = 3$. Unbiased (biased) dynamics results when $p = q (p \neq q)$. To produce stationary interface at zero noise, the boundary spins on the three surfaces defined by $x_1 = 0$, $x_2 = 0$ and $x_2 = L$, respectively, need to be fixed to the value $+1 (-1)$ if $x_2 > ( <) L/2$ where $L$ is the system size. Periodic boundary conditions are imposed to other surfaces.

In the low-noise limit, a spin flipped due to noise returns immediately to its original state by the majority rule, if the spin is situated away from the interface. Accordingly, we may assume that spin flips occur only at the interface. We are thus led to study an effective SOS-type model as the generalization of the
(1 + 1)-dimensional stairlike model studied by DLSS. In the SOS model picture, the Toom dynamics is mapped to a particle dynamics on a $d_s ≡ d − 1$ dimensional substrate in the form of deposition-evaporation with avalanche. The $(d_s + 1)$ dimensional SOS model is defined on $d_s$ dimensional body-centered-cubic lattice. For $d_s = 2$, it is the checkerboard lattice, square lattice rotated by 45 degrees. To each lattice point, a relative height corresponding to the $x_2$ coordinate of the original Toom interface is assigned. The height should satisfy the SOS condition, that nearest neighbor heights should differ only by ±1. Initially we begin with a flat surface characterized by height 0 on one sublattice and height 1 on the other. At each time step, we select a random site and start evaporation (deposition) process with probability $\bar{p}$ (probability $1 − \bar{p}$). In the evaporation (deposition) process, the height of the selected site is decreased (increased) by 2 if the SOS condition is satisfied with respect to the $2^{d_s−1}$ spins to the left. By left spins, we mean those nearest neighbor spins whose $x_1$ coordinate is less by $\frac{1}{2}$. If the SOS condition to the left is not satisfied there is no change. Next the avalanche process proceeds to the right (+$x_1$ direction) until all sites satisfy the SOS condition.

When $d_s = 1$, this avalanche dynamics reduces to the spin exchange dynamics of DLSS. The unbiased (biased) dynamics corresponds to the case $\bar{p} = \frac{1}{2}$ ($\bar{p} \neq \frac{1}{2}$). If the avalanche process is not allowed, so that depositions and evaporation occur only on local valleys and mountains respectively, then our model would be equivalent to the deposition-evaporation model proposed by Forrest and Tang [7], a generalization of the Plischke-Rácz-Liu model [8] into higher dimensions. More detailed description of the SOS model for $d_s = 2$ is found in the original paper [4]. Although the model is defined for general $d = d_s + 1$, we confine our attention to $d = 3$ from now on.
The continuum equation for the \((2 + 1)\)-dimensional model is described by Eq. (1), because it selects out a preferred direction, and because the cubic non-linear term derived by DLSS was proved to be marginally irrelevant even in \((1+1)\) dimensions \([9]\). For the unbiased case, both of the nonlinear terms in Eq. (1) disappear due to the symmetry of deposition and evaporation. So the equation reduces to the Edwards-Wilkinson (EW) equation \([10]\) implying that the square of the surface width diverges logarithmically with space and time. For the biased case, average height grows with increasing time, so that \(\lambda \parallel\) and \(\lambda \perp\) are nonzero.

It was shown \([4]\) that \(\lambda \parallel > 0\) and \(\lambda \perp < 0\) by applying the tilt argument \([11]\). Consequently, the model belongs to the weak-coupling regime of the AKPZ universality. Therefore, both the square of the surface width and the height-height correlation function are logarithmic for both unbiased and biased cases.

To check correspondences between the low noise Toom model and the SOS model, we performed numerical simulations for both models. Results for both cases run on small sizes are in complete agreement with each other in low-noise limit. Accordingly, we performed simulations intensively for larger systems using the SOS model. The simulations are done in the range of system size \(L = 20 \sim 140\) for both unbiased \((\bar{\rho} = 0.5)\) and biased \((\bar{\rho} \neq 0.5)\) cases. We measured the surface width \(w^2\) and the height-height correlation functions \(C_\parallel(r_\parallel, t)\) and \(C_\perp(r_\perp, t)\), defined by

\[
C_\parallel(r_\parallel, t) = \langle [h(r_\parallel, t) - h(0, 0)]^2 \rangle, \quad \text{and} \quad C_\perp(r_\perp, t) = \langle [h(r_\perp, t) - h(0, 0)]^2 \rangle, \tag{2}
\]

respectively. Here, \(r_\parallel\) \((r_\perp)\) denotes the spatial separation along the direction parallel \((\text{perpendicular})\) to the avalanche direction. From the theoretical considerations, we expect the height-height correlation functions diverges as \(\sim \ln t\)
before saturation, and $\sim \ln r$ after saturation. Typical data of $C_\parallel(r)$ and $C_\perp(r)$ after saturation are shown in Fig. 2 for the case of unbiased dynamics ($\bar{p} = 0.5$) and in Fig. 3 for the case of biased one ($\bar{p} = 0.3$). Each curve is for system sizes $L = 40, 60, 80, 100,$ and $120$, respectively, from bottom to top and is averaged over 300 configurations. It confirms clearly the logarithmic dependence on $r$. Deviation from straight line for large $r$ is the finite size effect.

Next, we examined the avalanche size distribution $n(s)$. The avalanche size $s$ is defined as the number of successive spin flips by a single noise process. $n(s)$ was measured in two different manners. In the first case, it is measured in the critical state (after saturation), while in the second case, it is measured during the whole time steps. In both cases, the distribution function $n(s)$ is found to be exponential, $n(s) \sim \exp(-s/s^*)$ as shown in Figs. 4 and 5. The characteristic size $s^*$ determined from the slopes of the inset figures is found to be independent of $\bar{p}$ and of the way $n(s)$ is measured. It takes the value $s^* = 1.11 \pm 0.01$.

In conclusion, we have generalized the Toom model into $d$ dimensions and defined its associated SOS-type model. Also we have presented numerical data for the height-height correlation functions in perpendicular and parallel directions, respectively, for unbiased and biased cases for $d = 3$. We have found that for the unbiased case, the interface is described by the EW equation, and for the biased case, it is described by the AKPZ equation with the opposite signs of $\lambda_\parallel$ and $\lambda_\perp$. Consequently, the height-height correlation functions diverge logarithmically with space and time. The avalanche size distribution has also been examined, which exhibits an exponential distribution for unbiased and biased cases instead of a power-law distribution.
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References

[1] A.L. Toom, in Multicomponent Random Systems, edited by R.L. Dobrushin and Ya.G. Sinai (Marcel Dekker, New York, 1980).

[2] B. Derrida, J.L. Lebowitz, E.R. Speer and H. Spohn, Phys. Rev. Lett. 67, 165 (1991); J. Phys. A 24, 4805 (1991).

[3] M. Kardar, G. Parisi and Y. Zhang, Phys. Rev. Lett. 56, 889 (1986).

[4] H. Jeong, B. Kahng, and D. Kim, Phys. Rev. Lett. 71, 747 (1993).

[5] D.E. Wolf, Phys. Rev. Lett. 67, 1783 (1991); J. Villain, J. de Physique I 1, 19 (1991).

[6] A.-L. Barabási, M. Araujo, and H.E. Stanley, Phys. Rev. Lett. 68, 3729 (1992).

[7] B.M. Forrest and L.-H. Tang, Phys. Rev. Lett. 64, 1405 (1990).

[8] M. Plischke, Z. Rácz, and D. Liu, Phys. Rev. B 35, 3485 (1987).

[9] P. Devillard and H. Spohn, J. Stat. Phys. 66, 1089 (1992).
[10] S.F. Edwards and D.R. Wilkinson, *Proc. R. Soc. Lond. A* **381** 17 (1982).

[11] J.M. Kim (private communication).
Figure Captions

Fig. 1  The three dimensional Toom rule on bcc lattice used in this work. The black circled spin in the center is updated with the majority rule of itself and four nearest neighbor spins (the black circled) with probability $1-p-q$, and becomes equal to $+1$ ($-1$) with probability $p$ ($q$). In the low-noise limit, $p, q \rightarrow 0$.

Fig. 2  The height-height correlation functions $C_{||}(r)$ (dotted line) and $C_{\perp}(r)$ (solid line) versus $\ln r$ for unbiased case, after saturation. Each curve is for system sizes $L = 40, 60, 80, 100$ and $120$, respectively, from bottom to top, and is averaged over 300 configurations.

Fig. 3  Same as in Fig. 2 for biased case, $\bar{p} = 0.3$.

Fig. 4  The avalanche size distribution $n(s)$ versus $s$ for unbiased case measured during whole time steps (a) and after saturation (b). The data are obtained from different system size, but they are collapsed into each other, implying that $n(s)$ is independent of system size. Insets show $\ln n(s)$ versus $s$.

Fig. 5  Same as in Fig. 4 for biased case, $\bar{p} = 0.3$. 
