Performance of Dual-Hop Relaying for OWC System Over Foggy Channel with Pointing Errors and Atmospheric Turbulence

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Abstract—Optical wireless communication (OWC) over atmospheric turbulence and pointing errors is a well-studied topic. Still, there is limited research on signal fading due to random fog in an outdoor environment for terrestrial wireless communications. In this paper, we analyze the performance of a decode-and-forward (DF) relaying under the combined effect of random fog, pointing errors, and atmospheric turbulence with a negligible line-of-sight (LOS) direct link. We consider a generalized model for the end-to-end channel with independent and not identically distributed (i.n.i.d.) pointing errors, random fog with Gamma distributed attenuation coefficient, double generalized gamma (DGG) atmospheric turbulence, and asymmetrical distance between the source and destination. We develop density and distribution functions of signal-to-noise ratio (SNR) under the combined effect of random fog, pointing errors, and atmospheric turbulence (FPT) channel and distribution function for the combined channel with random fog and pointing errors (FP). Using the derived statistical results, we present analytical expressions of the outage probability, average SNR, ergonomic rate, and average bit error rate (BER) for both FP and FPT channels in terms of OWC system parameters. We also develop simplified and asymptotic performance analysis to provide insight on the system behavior analytically under various practically relevant scenarios. We demonstrate the mutual effects of channel impairments such as atmospheric turbulence, pointing errors (i.e., misalignment between the transmitter and receiver) and fog. The atmospheric turbulence is caused by the scintillation effect of light propagation whereas the pointing errors happen due to the dynamic wind loads, weak earthquakes, and thermal expansion [4], [5]. The impact of foggy conditions on OWC systems depends on the intensity of fog ranging between light, medium, and dense [6], [7]. Although turbulence and fog may not coexist since both are inversely correlated with each other [4], [8], the effect of turbulence can not be ignored in light foggy conditions. The combined effect of atmospheric turbulence, pointing errors, and fog has a detrimental effect on the signal quality and presents a major challenge in the OWC deployment in outdoor environments.

The use of relaying has been extensively studied to improve the performance of OWC systems under the effect of turbulence and/or pointing errors [9]–[24]. In the aforementioned and related research, the statistical effect of foggy channels combined with pointing errors and atmospheric turbulence has not been considered. Recent studies confirm that the signal attenuation in the fog is not deterministic but follows a probabilistic model [25]–[27]. In [24], authors developed Johnson SB based probability distribution function (PDF) as a model for the random fog channel. They studied numerically the system performance in terms of average bit error rate (BER) and channel capacity. Considering the intractability of the Johnson SB for performance analysis, the authors in [27] proposed Gamma distribution for the signal attenuation in foggy weather and evaluated average signal-to-noise ratio (SNR), ergonomic rate, outage probability, and BER. In our recent paper [28], we proposed a multi-aperture OWC system to mitigate the effect of fog. The authors in [29] presented ultrashort high-intensity laser filaments for high-bit-rate transmissions over the fog. These studies show that the OWC performance is significantly limited in dense fog but can provide acceptable performance in the light fog over shorter links. However, combining the effect of pointing errors with fog shows high degradation in performance even in light foggy conditions [30], [31]. Specifically, the authors in [30] have considered three techniques to mitigate this effect: multihop relay systems using decode-and-forward (DF), active laser selection, and parallel radio frequency/free space optical (RF/FSO) link. Laser selection and hybrid transmission techniques require feedback from the receiver to the transmitter, thereby increasing the overhead. On the other hand, multi-hop relaying requires channel state information (CSI) at each relay to decode the signal, which can be hard in practice. In [30], the outage probability (which requires direct application of
We derive a novel PDF of the SNR of the OWC link for a multi-hop FSO system under the combined effect of fog and pointing errors. However, there is no result available in the literature for other performance metrics such as average SNR, ergodic rate, and average BER even for dual-hop OWC system under the random foggy channel with pointing errors. It is quite involved to derive closed-form expressions of the relay-assisted system since distribution functions of the resultant fading channel consists of an incomplete gamma function with a logarithmic argument. Analyzing the system performance using different performance metrics such as average SNR, ergodic rate, and average BER is desirable for efficient deployment of OWC systems under the combined effect of fog and pointing errors. Most importantly, the atmospheric turbulence has been ignored in the existing literature since deriving the PDF and CDF of the combined channel is complicated and requires novel approaches. It should be mentioned that the atmospheric turbulence can be neglected for shorter links but ignoring the turbulence for longer links may underestimate/overestimate the performance of the OWC system. To the best of the authors’ knowledge, there are no analyses available for average SNR, ergodic rate, BER, and outage probability for the relay-assisted OWC system under the statistical effect of random fog, pointing errors, and atmospheric turbulence. In Table I, we summarize the reported research in the literature on OWC systems under random fog.

### Related Research

1. **A. Related Research**

   Traditionally, signal attenuation due to the fog was assumed to be deterministic and quantified using a visibility range, for example, less attenuation in light fog and more in the dense fog. Kruse model in [6] is based on experimental data, whereas Kim [7] used Mie scattering theory to predict the signal attenuation. The authors in [33] updated the earlier models considering modified Gamma distribution for the particle size of fog. The authors in [35] developed a power delay model for the attenuation coefficient based on extensive measurement data. On the other hand, there are quite a few statistical models for the atmospheric turbulence, for example, log normal [37], exponentiated Weibull (EW) [38], Gamma-Gamma [39], Malaga [40], and F-distribution [41].

   Recently, [42] proposed the DGG distribution model for atmospheric turbulence. It is based on the theory of doubly stochastic scintillation, where irradiance fluctuations are expressed as the product of large-scale and small-scale fluctuations each following the generalized Gamma distribution. The DGG model can be used to model accurately different propagation conditions and it is versatile to include several statistical models for atmospheric turbulence as special cases. The authors in [43], [44] analyzed the performance of the OWC over DGG atmospheric turbulence without the consideration of random fog. It should be noted that the model of

| Reference | System Model | Pointing Errors | Turbulence | Performance Metrics Analysis |
|-----------|--------------|-----------------|------------|-----------------------------|
| [26]      | Direct link, Single-aperture | No              | No         | BER (numerical), Channel Capacity (numerical) |
| [27]      | Direct link, Single-aperture | No              | No         | Outage probability, SNR, Channel Capacity (numerical), BER (numerical) |
| [28]      | Direct link, Multi-aperture | No              | No         | Outage probability, SNR, Channel Capacity |
| [30]      | Multi-hop, Single-aperture | Yes             | No         | Outage probability |
| [31]      | Direct link, Multi-aperture | Yes             | No         | Outage probability, SNR, Channel Capacity |
| [Proposed]| Dual-hop, Single-aperture | Yes             | Yes        | Outage probability, SNR, Channel Capacity, BER |

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- We consider a generalized model for the end-to-end channel of the relay-assisted system with independent and not identically distributed (i.i.d.) pointing errors, random fog with Gamma distributed attenuation coefficient, double generalized gamma (DGG) atmospheric turbulence, and asymmetrical distance between the source and destination. We also analyze the considered system under the effect of random fog and pointing errors (termed as the FP channel) by considering a single DF relaying with no direct transmission to the destination. The major contributions of the paper are as follows:

- We consider a generalized model for the end-to-end channel of the relay-assisted system with independent and not identically distributed (i.i.d.) pointing errors, random fog with Gamma distributed attenuation coefficient, double generalized gamma (DGG) atmospheric turbulence, and asymmetrical distance between the source and destination.
- We also analyze the considered system under the effect of random fog and pointing errors (termed as the FP channel) with negligible atmospheric turbulence for shorter OWC links.
- We derive a novel PDF of the SNR of the OWC link for the FPT channel in terms of a single Meijer’s-G function, which resulted into more elegant performance analysis. We also simplify the existing CDF of the FP channel in terms of a single incomplete gamma function to derived closed-form expressions of the OWC system.
- We use the derived statistical results to develop analytical expressions of the outage probability, average SNR, ergodic rate, and average BER for both FP and FPT channels with single-variate Fox-H function for the FPT channel and standard mathematical functions for the FP channel.
- We present asymptotic analysis in the high SNR regime for outage probability and average BER and derive diversity order depicting the impact of system and channel parameters on the performance of the considered system.
- We use numerical and simulation analysis to show that the dual-hop relaying can mitigate fog, pointing errors, and turbulence-induced fading for high-speed OWC links.

We also demonstrate that there is a significant gap in the performance using the existing visibility range based path-gain computation as compared to the probabilistic modeling of random fog.
pointing errors $\Psi$ is used widely, assuming independent identi-
distributed Gaussian for the elevation and the horizontal
displacement.

Relay-assisted communication is a potential technique to
deal with the channel fading in wireless systems $\Phi$, $\Phi'$. Here, a single relay or many intermediate nodes can assist
data transmission between a single source and destination. In
particular, for OWC systems, there is a vast literature on the
relaying using amplify-and-forward (AF) and DF protocols in
$[9]–[13], [43]$, all-optical relaying in $[16]–[24]$, and relaying for
hybrid RF/FSO systems in $[47]–[54]$. In the seminal work
$[9]$, multi-hop and cooperative relaying using AF and DF
protocols have been considered for an aggregated channel
model which takes into account both path-gain and turbulence-
induced log-normal fading. The authors in $[10]$ investigated a
multi-hop relaying to mitigate the effect of fading in FSO
systems over log-normal atmospheric turbulence channels.
The end-to-end performance in terms of outage probability,
the average BER, and the ergodic capacity for a multi-hop
relaying with AF and DF protocols under the combined effect
of Gamma-Gamma turbulence and pointing errors have been
investigated in $[11]$. Although the complex multi-hop relaying
can provide a better performance, a dual-hop relay system
(with no direct link between the source and the destination)
that selects a single relay opportunistically is considered in
$[12]$. The authors in $[13]$ studied the information-theoretic
performance of parallel relaying for FSO communications
over Gamma-Gamma fading channels with a single relay
but with a line-of-sight link between the source and the
destination. An optimal relay placement scheme for serial
and parallel relaying along with a diversity gain analysis has been
considered in $[14]$. The authors in $[15]$ have considered the
inter-relay cooperation on the outage probability and diversity
order performance of the DF cooperative FSO communication
systems.

An all-optical relaying scheme is efficient since signals are
processed in the optical domain without requiring optical-to-
electrical and electrical-to-optical conversions $[16]–[24]$. The
references $[21], [22]$ provided analytical expressions for the
outage probability, average BER, and ergodic capacity over
strong atmospheric turbulence channels with misalignment-
induced pointing errors by considering a single optical AF
relay with fixed and variable gain. Hybrid RF/FSO systems,
where relays act as an interface between RF and optical links,
have been studied $[47]–[54]$. A dual-hop relay system over the
asymmetric links has been considered for both RF and FSO
environments in $[47]$ and derived exact expression for outage
probability. The BER performance and the capacity analysis
of an AF-based dual-hop mixed RF–FSO is presented in $[48]$,nwhere the RF links are Rayleigh distributed and FSO links are
characterized by the Gamma–Gamma distributed turbulence
and pointing errors. Considering a dual-hop transmission with
a single-relay and ignoring the direct transmission, the authors
in $[50], [52]$ have analyzed the OWC performance under the
turbulence and pointing errors. The authors in $[53]$ have
considered an FSO/RF-FSO link adaptation scheme for hybrid
FSO systems and analyzed different performance metrics like
outage probability, average BER, and ergodic rate. Recently,
a hybrid dual-hop relaying with mmWave and FSO scheme is
studied in $[54]$. 

| TABLE II |
| --- | --- |
| **THE LIST OF MAIN NOTATIONS** |
| $\cdot_1$ | Notation for the first link |
| $\cdot_2$ | Notation for the second link |
| $y$ | Received signal |
| $h$ | Random channel state |
| $R$ | Detector responsivity |
| $x$ | Transmit signal intensity |
| $w$ | AWGN |
| $P_t$ | Transmit power |
| $d_1$ | Distance: transmitter to relay |
| $d_2$ | Distance: relay and destination |
| $d = d_1 + d_2$ | |
| $k$ | Shape parameter of foggy channel |
| $\beta$ | Scale parameter of foggy channel |
| $P_{\text{out}}$ | Outage probability |
| $\gamma_0$ | SNR without fading |
| $\gamma$ | SNR with fading |
| $\bar{\gamma}$ | Average SNR |
| $\bar{\eta}$ | Ergodic rate |
| $P_e$ | Average BER |
| $\Gamma(a)$ | $\int_0^\infty t^{a-1}e^{-t}dt$ |
| $\Gamma(a,t)$ | $\int_t^\infty s^{a-1}e^{-s}ds$ |
| $Q(\gamma)$ | $\frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u^2/\gamma} du$ |
| $\text{erf}(\gamma)$ | $\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du$ |
| $F_1(a;b;z) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a)\Gamma(b+k)} \frac{z^k}{k!}$ |
| $\psi^{(0)}(a)$ | $\frac{d}{da} \ln \Gamma(a)$ |
| $G_{m,n}^{p,q}(\cdot)$ | Meijer-G function |
| $H_{m,n}^{p,q}(\cdot)$ | Fox-H function |

B. Notations and Organization

We list the main notations in Table II. This paper is
organized as follows: In Section II, we discuss the relay
assisted OWC system model. In Section III, we analyze the
OWC system performance by deriving closed-form expres-
sions for the outage probability, average SNR, ergodic rate,
and average BER. In Section IV, the simulation results of
the proposed system are presented. Finally, in Section V, we
provide conclusions.

II. SYSTEM MODEL

We consider an OWC system using intensity modulation/
direct detection (IM/DD). It consists of a single-aperture
transceiver system, with a negligible line-of-sight (LOS) direct
link for the FPT channel under the combined effect of fog,
pointing errors, and atmospheric turbulence, as shown in Fig I.
The signal $y_i$ at the $i$-th receiver aperture is given as

$$y_i = h_{fi}h_{pi}h_{ri}Rx + w_i, \quad (1)$$
where $x$ is the transmitted signal, $R_i$ represents the detector responsivity (in amperes per watt), and $w_i$ represents an additive white Gaussian noise (AWGN) with variance $\sigma_w^2$. The terms $h_{f_i}$, $h_{p_i}$, and $h_{t_i}$ are the random states of the foggy channel, pointing errors, and atmospheric turbulence induced fading, respectively, of the $i$-th link.

We use the PDF of fog as given in [27]:

$$f_{h_f}(h_f) = \frac{z_i^{k_i}}{(k_i)} \left( \ln \frac{1}{h_f} \right)^{k_i-1} h_f^{z_i-1}, \quad (2)$$

where $0 < h_f \leq 1$, $z_i = 4.343/\beta_{\text{fog}} d_i$, $k_i > 0$ is the shape parameter and $\beta_{\text{fog}} > 0$ is the scale parameter. It is noted that different pairs of $k_i, \beta_i$ determine the severity of the foggy channel such as $\{k_i = 2.32, \beta_{\text{fog}} = 13.12\}$, $\{k_i = 5.49, \beta_{\text{fog}} = 12.06\}$, $\{k_i = 6.0, \beta_{\text{fog}} = 23.06\}$ for light, moderate and thick foggy conditions, respectively. The PDF of pointing errors fading $h_{p_i}$ is given in [4]:

$$f_{h_{p_i}}(h_{p_i}) = \frac{R_i^2}{A_i^2} h_{p_i}^{\nu_i-1}, \quad 0 \leq h_{p_i} \leq A_i, \quad (3)$$

where $A_i = \text{erf}(\nu_i)^2$ with $\nu_i = \sqrt{\pi/2} a_i/\omega_{zi}$ and $\omega_{zi}$ is the beam width at the receiver and $\sigma_{p_i}^2$ as the variance of pointing errors displacement characterized by the horizontal sway and elevation [4]. Finally, PDF of the random atmospheric turbulence with DGG distribution combined with zero-bore sight pointing errors [43]:

$$f_{h_{p_{\text{t}}}}(h_{p_{\text{t}}}) = \frac{R_i^2}{A_i^2} \frac{\beta_{i}^{\nu_i-1}}{2} \left( \lambda_i^{\nu_i-1} \sigma_i^{\nu_i-1} \right)^{-rac{1}{2}(\nu_i - 1)} \frac{1}{\Gamma(\nu_i)} G^{0, \lambda_i, \sigma_i+1, 1}, \quad (4)$$

where $\mu_i = 1 - \frac{\sigma_{p_i}^2}{\sigma_{\text{w_i}}^2}$, $\Delta(\sigma_i : 1 - \beta_i)$, $\Delta(\lambda_i : 1 - \varphi_i)$, $\nu_i = \frac{\sigma_{p_i}^2}{\sigma_{\text{w_i}}^2}$, and $\Delta(x : y) = \frac{y}{2} x^{y-1}, \cdots, \frac{y}{2} x+y-1$. The sets $(\alpha_i, \beta_i, \Omega_i)$, $(\phi_i, \varphi_i, \Xi_i)$ are DGG fading parameters and $\rho_i, A_i$ are pointing error parameters.

We denote by $d_1$ the distance between the source and relay and $d_2$ the distance between the relay and destination. For the case of relayed transmission using the DF protocol, the expressions for signals received at the relay and destination when $x$ is the transmitted signal:

$$y_r = h_{f_1} h_{p_1} h_{t_1} R_1 x + w_1 \quad (5)$$

$$y_d = h_{f_2} h_{p_2} h_{t_2} R_2 x + w_2 \quad (6)$$

where $h_{f_1}, h_{p_1}, h_{t_1}$ and $h_{f_2}, h_{p_2}, h_{t_2}$ are random fog, pointing errors, and atmospheric turbulence channel states between source-relay and relay-destination, respectively, each having $w_1$ and $w_2$ as AWGNs. Note that $h_1 = h_{f_1} h_{p_1} h_{t_1}$ is the combined channel between the source and the relay, and $h_2 = h_{f_2} h_{p_2} h_{t_2}$ is the combined channel between the relay and destination.

We use a general relaying scenario $d_2 > d_1$ with different values of pointing errors parameters for both links giving an i.i.d fading model for the FPT channel. To simplify the analysis, we assume random fog to be i.i.d. for the the FP channel since the foggy weather may affect both the links independently generated from the same probabilistic model. As a special case, we also consider that the relay is situated at the midway between the source and destination (i.e., $d_1 = d_2$) and that the channel parameters are same for both links (i.e., i.i.d. condition).

III. PERFORMANCE ANALYSIS

In this section, we analyze the performance of a relay-assisted system. First, we present exact expressions for the PDF and CDF of the SNR for the FPT channel under the combined effect fog, pointing errors, and atmospheric turbulence. We also provide a simplified expression for the FP channel under the effect fog and pointing errors with negligible atmospheric turbulence. Next, we use the derived statistical results to analyze the outage probability, average SNR, ergodic rate, and average BER performance of the OWC system. The derived expressions show the system behavior in a relay-assisted environment under the combined effect of channel impairments that can help the network operator for an efficient design of relay-assisted OWC links.

A. Statistical Results

We define $\gamma_{\text{FPT}} = \gamma_0|h_{t_1}|^2$ as the SNR for the FPT channel, where $h_1 = h_{f_1} h_{p_1} h_{t_1}$, $\gamma_0 = 2P_i^2 R_i^2/\sigma_w^4$ for $i = 1, 2$, and $P_i$ is the average optical transmitted power. In the following Theorem, we provide PDF and CDF for the FPT channel under the combined effect of the random fog, pointing errors, and atmospheric turbulence:

**Theorem 1 (PDF and CDF for FPT):** The PDF and CDF of SNR for the FPT channel under the combined effect of random fog and atmospheric turbulence with pointing errors for the single OWC link is given as

$$f_{\gamma_{\text{FPT}}}(\gamma) = \frac{x_i^{k_i} R_i^2 \beta_i^{k_i-1} \lambda_i^{k_i-1} \sigma_i^{k_i-1} \left( 2 \pi \right)^{k_i-1} \lambda_i^{k_i+1} \sigma_i^{k_i+1}}{\Gamma(k_i)} G^{0, \lambda_i, \sigma_i+1, 1} \left[ \frac{\lambda_i^{k_i} \sigma_i^{k_i} \lambda_i^{k_i} \sigma_i^{k_i}}{\beta_i^{k_i} \sigma_i^{k_i}} \left( \frac{A_i}{R_i} \right)^{\mu_i} \nu_{x_0} \right] \quad (7)$$
where $U_i = \{\mu_i, (1 - \frac{z_i}{\phi_i \lambda_i})^{k_i}\}$ and $V_i = \{\nu_i, (\frac{z_i}{\phi_i \lambda_i})^{k_i}\}$, and

$$F_{\gamma_i}^{FP}(\gamma) = \frac{e^{\phi_i \lambda_i} \gamma^{k_i - 1}}{2^k \phi_i^{k_i}} \left( \Gamma(k_i)(1, \frac{\gamma}{\phi_i \lambda_i}) \right)$$

$$H_{\lambda_i + \sigma_i + k_i + 1} \left[ \frac{\lambda_i^{\frac{1}{2}} \sigma_i^{\frac{1}{2}} (\gamma + \gamma_i)^{\frac{1}{2}}}{\beta_i^{\frac{1}{2}} \varphi_i^{\frac{1}{2}}} \right] \left( A \sqrt{\gamma} \right) \frac{\phi_i \lambda_i}{V_i} \right]$$

(8)

where $\tilde{U}_i = (\{\mu_i, 1\}, (1 - \frac{z_i}{\phi_i \lambda_i})^{k_i}, 1, (\frac{z_i}{\phi_i \lambda_i})^{2})$ and $\tilde{V}_i = (\{0, \frac{z_i}{\phi_i \lambda_i}\}, (\nu_i, 1), (\{\frac{z_i}{\phi_i \lambda_i}, 1\})$.

**Proof:** The PDF of the combined FPT channel $h_i = h_i f_i h_{\text{tpi}}$ can be expressed as

$$f_{h_i}(h_i) = \int \frac{1}{|h_{\text{tpi}}|} f_{h_i}(h_i) f_{h_{\text{tpi}}}(h_{\text{tpi}}) dh_{\text{tpi}}$$

(9)

Substituting (2) and (4) in (9), we get

$$f_{h_i}(h_i) = \frac{e^{\phi_i \lambda_i} \gamma^{k_i - 1}}{2^k \phi_i^{k_i}} \left( \Gamma(k_i)(1, \frac{\gamma}{\phi_i \lambda_i}) \right)$$

$$= \int_0^\infty \frac{1}{2 \pi} \prod_{i=1}^{k_i} \left[ \frac{\lambda_i^{\frac{1}{2}} \sigma_i^{\frac{1}{2}} (\gamma + \gamma_i)^{\frac{1}{2}}}{\beta_i^{\frac{1}{2}} \varphi_i^{\frac{1}{2}}} \right]^{\frac{1}{2}}$$

$$= \int_0^\infty h_{\text{tpi}} z_i^{\frac{1}{2}} \varphi_i \lambda_i \left[ \Gamma \left( \frac{1}{\gamma_i} \right) \left( \frac{1}{\gamma_i} + 1 \right) \right]^{-1} dh_{\text{tpi}}$$

(10)

Using the definition of Meijer-G function and interchanging the order of integration, we represent (10) as

$$f_{h_i}(h_i) = \frac{e^{\phi_i \lambda_i} \gamma^{k_i - 1}}{2^k \phi_i^{k_i}} \left( \Gamma(k_i)(1, \frac{\gamma}{\phi_i \lambda_i}) \right)$$

$$= \int_0^\infty h_{\text{tpi}} z_i^{\frac{1}{2}} \varphi_i \lambda_i \left[ \Gamma \left( \frac{1}{\gamma_i} \right) \left( \frac{1}{\gamma_i} + 1 \right) \right]^{-1} dh_{\text{tpi}}$$

(11)

where $\mu_{i,j} = 1 - \frac{\sigma_i}{\beta_i}, \Delta(\sigma_i : 1 - \beta_i), \Delta(\lambda_i : 1 - \varphi_i)$. Substituting $\ln(h_{\text{tpi}}/h_i) = t$, we solve the inner integral of (11) in terms of Gamma function:

$$I_1 = \int_0^\infty k_i z_i^{\frac{1}{2}} \varphi_i \lambda_i \left[ \Gamma \left( \frac{1}{\gamma_i} \right) \left( \frac{1}{\gamma_i} + 1 \right) \right]^{-1} dh_{\text{tpi}}$$

(12)

Using (12) in (11) and applying the definition of Fox-H function with a transformation of random variable $\gamma_i = \gamma_0 h_i^2$, we get the PDF of SNR in (7).

To find the CDF of the SNR for the FPT channel, we use (7) in $F_{\gamma_i}^{FP}(\gamma) = \int_0^\gamma f_{\gamma_i}^{FP}(x) dx$, apply the definition of Meijer’s G function with the inner integral $\int_0^\gamma \gamma^{-\frac{1}{2}} \varphi_i \lambda_i \left[ \Gamma \left( \frac{1}{\gamma_i} \right) \left( \frac{1}{\gamma_i} + 1 \right) \right]^{-1} d\gamma = \frac{\psi_{\gamma_i}^{\lambda_i \frac{1}{2}}}{\varphi_i \lambda_i} \left[ \Gamma \left( \frac{1}{\gamma_i} \right) \right]$ to get (8), which concludes the proof of Theorem 1.

For shorter OWC links, the effect of atmospheric turbulence can be neglected. Thus, defining $\gamma_i^{\text{FP}} = \gamma_0 |h_i| h_{\text{tpi}}|^2$, the PDF of SNR for the FP channel under the combined effect of random fog and pointing errors for the single OWC link is given as [30]:

$$F_{\gamma_i}^{FP}(\gamma) = \frac{C_1}{\sqrt{\gamma_0}} \left( \sqrt{\gamma} \right)^{\frac{1}{2}} - \frac{C_2}{\sqrt{\gamma_0}} \left( \sqrt{\gamma} \right)^{\frac{1}{2}}$$

where $C_1 = \frac{z_i}{\phi_i \lambda_i}$, $C_2 = \frac{z_i}{\phi_i \lambda_i}$, and $m_i = z_i - \beta_i$.

**Proposition 1:** The CDF of the SNR for the FP channel under the effect of random fog with pointing errors is given as

$$F_{\gamma_i}^{FP}(\gamma) = D(1) \left( \frac{A_i}{\gamma_{\gamma_0}} \right)^{-\frac{1}{2}} - D(2)$$

$$= \sum_{n=0}^{k_i - 1} m_i z_i \ln \left( \frac{A_i}{\gamma_{\gamma_0}} \right)$$

(14)

where $D(1) = \frac{z_i}{m_i}$ and $D(2) = \frac{m_i}{A_i}$ are constants.

**Proof:** Substituting $u = \ln \left( \frac{A_i}{\gamma_{\gamma_0}} \right)$ in the second term of (13), the CDF $F_{\gamma_i}^{FP}(\gamma) = \int_0^\gamma f_{\gamma_i}^{FP}(\gamma) d\gamma$ is given as

$$F_{\gamma_i}^{FP}(\gamma) = D(1) \left( \frac{A_i}{\gamma_{\gamma_0}} \right)^{-\frac{1}{2}} - D(2)$$

$$= \sum_{n=0}^{k_i - 1} m_i z_i \ln \left( \frac{A_i}{\gamma_{\gamma_0}} \right)$$

(15)

Using $\Gamma(a, t) \equiv (a - 1) e^{-t} \sum_{m=0}^{a-1} \frac{m!}{m} t^m$ in (15), and applying the definition of incomplete Gamma function, we get (14).

It should be noted that the derived CDF for the FP channel in (14) is different from [30]: it consists of a single incomplete gamma function without the exponential integral, and can useful for performance analysis in a closed-form using integer $k$. It can be seen that distribution functions in (13) and (14) involves incomplete gamma functions with logarithmic argument requiring novel approaches to performance analysis. The use of simple approximation of incomplete Gamma function $\Gamma[k, m \ln u] \approx u^{-m} (m \ln u)^{k-1}$ can simplify the analysis, but the derived approximate expressions grossly overestimate/underestimate the exact performance.

Finally, we discuss the distribution functions with DF relaying. We assume equal transmit power at the source and relay (i.e., $P_1 = P_2$) to get the instantaneous SNRs of signals received at the relay and receiver as $\gamma_1$ and $\gamma_2$, respectively. Assuming $\gamma_1$ and $\gamma_2$ are independent for analytical tractability, the expression of end-to-end SNR for the DF relaying is given as:

$$\gamma = \min(\gamma_1, \gamma_2)$$

(16)

It is true that $k$ and $\beta$ parameters of the random foggy channel will be identical in both the hops. However, channel realizations at two distinct points separated over several hundred meters might not be the same. The i.i.d. assumption on the foggy channel for the FP channel has been considered in [30].

In general, the CDF and PDF of end-to-end SNR for the DF relaying scheme can be given as [21]:

$$\Psi(\gamma) = \Psi_1(\gamma) + \Psi_2(\gamma) - \Psi_1(\gamma) \Psi_2(\gamma)$$

(17)

$$\psi(\gamma) = \psi_1(\gamma) + \psi_2(\gamma) - \psi_1(\gamma) \Psi_2(\gamma) - \psi_2(\gamma) \Psi_1(\gamma)$$

(18)
where $\psi_1(\gamma), \psi_2(\gamma)$ are the PDF of the first link and second link, respectively. Similarly, $\Psi_1(\gamma)$ and $\Psi_2(\gamma)$ are the CDF of the first link and the second link, respectively.

### B. Outage Probability

Outage probability is a performance measure to demonstrate the effect of the fading channel. It is defined as the probability of failing to reach an SNR threshold value $\gamma_{th}$, i.e., $P_{out} = P(\gamma < \gamma_{th}) = \Psi(\gamma_{th})$. Thus, exact expressions for the outage probability with FPT and FP channels can be obtained by substituting (19) and (21) with $i = 1, 2$ in (17), respectively. To derive the asymptotic analysis for the FPT, we apply (20) to get the outage probability at high SNR $\gamma_0 \to \infty$ for the $i$-th link:

$$P_{out, i, \infty} = \sum_{n=1}^{\lambda_n + \varphi_i + k-1} \frac{\gamma_0^{\lambda_n + \varphi_i + k-1}}{(\lambda_n + \varphi_i + k-1)!} \frac{1}{\psi_i} \frac{1}{2(2\pi)^{1/2}} e^{-\frac{(\gamma_0 - \gamma)^2}{2}} \gamma_0^{\lambda_n + \varphi_i + k-1}$$

where $n = a_j = \{\mu_i, \{1 - \frac{1}{\phi_i}\}_1, 1\}, s_n = s_j = \{1, 2\}$, $b_n = b_j = \{0, 1\}$, and $t_n = t_j = \{\phi_i, 1\}$. The diversity of the FPT channel can be obtained using dominant SNR terms of (17). Using the parameters of $a_n$ and $s_n$ in (19), the diversity order of the system is $M_{out, i} = \min\{\frac{t}{2}, \frac{t}{2}, \frac{t}{2}, \frac{t}{2}\}$. Using $i = 1, 2$, the diversity order for the dual-hop system with outage probability $P_{out, i} = P_{out, i}^\gamma + P_{out, i}^\beta + P_{out, i}^\gamma P_{out, i}^\beta$ can be expressed as $M_{out, i} = \min\{\frac{t}{2}, \frac{t}{2}, \frac{t}{2}, \frac{t}{2}\}$. Thus, the diversity order for the DGG channel with pointing errors as derived in (13), (24) is a special case of the diversity order for the FPT channel. This validates our proposed analysis on the outage probability. Similarly, using the series expansion of incomplete Gamma function $\Gamma(a, t) \equiv (a - 1)! \sum_{n=0}^{\infty} \frac{t^n}{n!}$ in (19), we express the outage probability of the $i$-th link for the FP channel:

$$P_{out, i} = D_i(1) \left(\frac{A_{2, 70}^\gamma}{\gamma_{th}}\right)^{\frac{1}{2}} - D_i(2)(k - 1)! \left(\frac{A_{2, 70}^\gamma}{\gamma_{th}}\right)^{\frac{1}{2}} \sum_{n=0}^{k-1} \sum_{j=0}^{n} \frac{m_n z_j^{n-1}}{\gamma_{th}} \right)$$

Using (20) and following the similar steps of (31), the diversity order for the FP channel can be derived as $M_{out, i} = \min\{\lambda_i, \rho_i^2\}$. The diversity order various possibilities of mitigating the impact of pointing errors and fog. The diversity order provides design criteria of appropriately using the beam width and link distance to mitigate the effect of pointing errors and random fog. As such, the beam-width (associated with pointing errors) and link distance (associated with fog) can be chosen to circumvent the fading due to the atmospheric turbulence. For example, under certain conditions $M_{out} = \frac{1}{2} = \frac{2 \lambda^2}{\beta^2}$ and source-to-relay distance $d_1$.

### C. Average SNR and Ergodic Rate

The average SNR and ergodic rate performance are important parameters for the design of communication systems. In general, expressions for the average SNR and ergodic rate for the IM/DD detector type is given as (10):

$$\bar{\gamma} = \int_0^{\infty} \psi(\gamma) d\gamma$$

where $\psi(\gamma)$ is the limit of integration. It can be easily seen that $\bar{\gamma} \to \infty$ for the FPT channel due to the effect of atmospheric turbulence. However, for the FP channel, $\bar{\gamma} = \min(A_{2, 70}^\gamma, A_{2, 70}^\gamma)$. Further, considering the shorter symbol duration (in the range of few nanoseconds), the OWC channel (19) can be considered a slow-fading channel [27]. We also analyze the ergodic rate performance to provide an estimate on the throughput of the system. We assume that relay requires negligible time to relay the data while computing the ergodic rate. It should be noted that there is a vast literature on the ergodic rate performance on the slow fading FSO channels (See [11], [21], [22], and references therein).

In what follows, we derive closed-form expressions of the average SNR and ergodic capacity for the considered relay-assisted system for both FPT and FP fading channels.

**Lemma 1 (Average SNR for FPT):** If $k_i$ and $\beta_i$ are the parameters of the foggy channel, $A_i$ and $\rho_i$ are the parameters of pointing errors, sets $(a_i, \beta, \Omega_i)$, $(\phi_i, \varphi, \Theta_i)$ are the parameters for DGG atmospheric turbulence, and $d_1 = 3.439 / \beta \cos\theta_1$, then an exact expression of the average SNR for the FPT channel is

$$\bar{\gamma}_1^{FPPT} = \bar{\gamma}_1^{FPPT} + \bar{\gamma}_2^{FPPT} - \frac{\bar{\gamma}_2^{FPPT}}{\beta^{FPPT}}$$

where $\bar{\gamma}_1^{FPPT}$ and $\bar{\gamma}_2^{FPPT}$ are given in (24) with $i = 1$ and $i = 2$ whereas $\gamma_1^{FPPT}$ and $\gamma_2^{FPPT}$ are given in (25) and (26), respectively.

**Proof:** Using (7) in (21) with the substitution $\gamma^{\frac{\phi_i}{\phi_1}} = t$, and applying the identity [59], eq. 07.34.21.0009.01] of the single Meijer-G with few simplifications, we get (24). Next, we use $\psi(\gamma) = f_1^{FPPT}(\gamma) - f_2^{FPPT}(\gamma)$ in (21) with a substitution $\gamma^{\frac{\phi_i}{\phi_1}} = t$ to get

$$\bar{\gamma}_1^{FPPT} = \frac{2 \rho_i A_i}{\phi_i} \int_0^{\infty} t^{-2 - \frac{1}{\phi_i}} e^{-\frac{1}{\phi_i} t} \left[ \begin{array}{c} b_1 t \\ V_1 \end{array} \right]$$

$$H_1^{\Omega_i + \sigma_i + k, k_1 + 1} \left[ \begin{array}{c} U_2 t \end{array} \right]$$

where $b_1 = \frac{A_{2, 70}^\gamma}{2(\phi_i \sigma_1)} \frac{1}{\beta^{FPPT}}$, $b_2 = \frac{2(\phi_i \sigma_1 + 1)(\Omega_i - \sigma_i + k_1) + 2(\phi_i \sigma_1 + 1)(\Omega_i - \sigma_i + 1)}{2(\phi_i \sigma_1 + 1)(\Omega_i - \sigma_i + 1)}$, $U_1 = \{\Omega_i, 1 - \frac{1}{\phi_i} \} \{ k_1 \}$, $V_1 =$
\[ \gamma_{\text{FPT}} = -\frac{k_i}{\rho_i^2} \beta_i \lambda_i^2 \eta_i \left( 2\pi i \right)^{\frac{3}{2}} \frac{1}{\phi_i \lambda_i \Gamma(\beta_i) \Gamma(\phi_i)} \prod_{j=1}^{\nu_i+1} \Gamma \left( 1 + \frac{2}{\phi_i \lambda_i} - \mu_{ij} \right) \left( \frac{\lambda_i^2 \sigma_i^2 \Omega_i \Xi_i}{\beta_i^2 \nu_i} \right) \left( A_i \sqrt{\gamma_i} \phi_i \lambda_i \right) \]

where

\[ \nu_i = \left\{ \left\{ \frac{\lambda_i \rho_i}{\phi_i \lambda_i} \right\}, \left\{ \frac{\lambda_i \rho_i}{\phi_i \lambda_i^2} \right\} \right\}, \quad \Omega_i = \left\{ \left\{ \frac{\lambda_i \rho_i^2}{\phi_i \lambda_i} \right\}, \left\{ \frac{\lambda_i \rho_i^2}{\phi_i \lambda_i^2} \right\} \right\} \]

and

\[ \eta_{\text{FPT}} = \eta_{\text{FPT}} + \eta_{\text{FPT}} - \eta_{\text{FPT}} \]

where \( \eta_{\text{FPT}} \) and \( \eta_{\text{FPT}} \) are given in (2) with \( i = 1 \) and \( i = 2 \) whereas \( \eta_{\text{FPT}} \) and \( \eta_{\text{FPT}} \) are given in (3) and (4), respectively.

**Proof:** Using (7) in (21) with \( \ln(1 + \frac{1}{2\pi} \gamma) = \]

\[ G_{2,2}^{1,1} \left[ \frac{e^{\phi_i \lambda_i \gamma}}{2 \pi} \right] = \left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right], \]

we get

\[ \eta_i = \frac{k_i}{\rho_i^2} \beta_i \lambda_i^2 \eta_i \left( 2\pi i \right)^{\frac{3}{2}} \frac{1}{\phi_i \lambda_i \Gamma(\beta_i) \Gamma(\phi_i)} f_{\text{FPT}} \left( \gamma \right) \]

\[ \int_0^\infty 1 - \frac{1}{2\pi} \gamma f_{\text{FPT}} \left( \gamma \right) \]
Thus, using (60), (61), (62), and (64) of Appendix A in (22), where
\[ \hat{b}_i = \frac{\lambda_i^2 \sigma_i^2 \Gamma(\gamma,1)}{\rho_i^2 \lambda_i} (A_i \sqrt{\gamma}) \hat{V}_i, \hat{W} = (\mu_i, 1), \left\{ (1 - \frac{z_i}{\phi_i \lambda_i}) k_i, 1 \right\}, \left\{ (1 - \frac{z_i}{\phi_i \lambda_i}) k_i, 1 \right\}, (0, -\frac{\phi_i \lambda_i^2}{2}, 0, -\frac{\phi_i \lambda_i^2}{2}), \left\{ (1, -\frac{\phi_i \lambda_i^2}{2}) \right\}, \right\}

\[ \bar{b}_{12} \left( \frac{\phi_i \lambda_i^2 + \phi_j \lambda_j^2}{2} \right) \left( \hat{U}_{12} \right) \]

\[ \bar{b}_{21} \left( \frac{\phi_i \lambda_i^2 + \phi_j \lambda_j^2}{2} \right) \left( \hat{U}_{21} \right) \]

where

\[ \hat{U}_{12} = \left\{ (\mu_1, 1), (\mu_2, 1), \left\{ (1 - \frac{z_1}{\phi_1 \lambda_1}) k_1, 1 \right\}, \left\{ (1 - \frac{z_2}{\phi_2 \lambda_2}) k_2, 1 \right\}, (0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 1, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2} \right\}, \right\}

\[ \hat{V}_{12} = \left\{ (\mu_1, 1), (\mu_2, 1), \left\{ (1 - \frac{z_1}{\phi_1 \lambda_1}) k_1, 1 \right\}, \left\{ (1 - \frac{z_2}{\phi_2 \lambda_2}) k_2, 1 \right\}, (0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 1, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2} \right\}, \right\}

\[ \hat{V}_{21} = \left\{ (\mu_1, 1), (\mu_2, 1), \left\{ (1 - \frac{z_1}{\phi_1 \lambda_1}) k_1, 1 \right\}, \left\{ (1 - \frac{z_2}{\phi_2 \lambda_2}) k_2, 1 \right\}, (0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 1, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2} \right\}, \right\}

\[ \hat{V}_{12} = \left\{ (\mu_1, 1), (\mu_2, 1), \left\{ (1 - \frac{z_1}{\phi_1 \lambda_1}) k_1, 1 \right\}, \left\{ (1 - \frac{z_2}{\phi_2 \lambda_2}) k_2, 1 \right\}, (0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 1, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2} \right\}, \right\}

\[ \hat{V}_{21} = \left\{ (\mu_1, 1), (\mu_2, 1), \left\{ (1 - \frac{z_1}{\phi_1 \lambda_1}) k_1, 1 \right\}, \left\{ (1 - \frac{z_2}{\phi_2 \lambda_2}) k_2, 1 \right\}, (0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 0, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2}, 1, -\frac{\phi_1 \lambda_1^2 + \phi_2 \lambda_2^2}{2} \right\}, \right\}

on ergodic rate are given by

\[ \tilde{\gamma}_{\text{FP}} = \mathcal{F}_\gamma(A_1, A_2, \rho_1, \rho_2, z_1, z_2, k) \quad (35) \]

\[ \tilde{\eta}_{\text{FP}} = \mathcal{F}_\eta(A_1, A_2, \rho_1, \rho_2, z_1, z_2, k) \quad (36) \]

Proof: Assuming longer second link, we use the upper limit of the integration as \( A_2^2 \gamma_0 \) since \( d_2 \geq d_1 \rightarrow A_2 \leq A_1 \). We substitute (13) and (14) in (18) to get \( f(\gamma) \) in terms of system parameters. Since each \( f_1(\gamma) \), \( f_2(\gamma) \), \( F_1(\gamma) \), and \( F_2(\gamma) \) consist of two terms resulting into two integrals in (21).

We use \( m_1 = (z_1 - \rho_1^2) \), \( m_2 = (z_2 - \rho_2^2) \) and the series expansion \( \Gamma(a, t) \propto (a-1) \Gamma(a) \sum_{m=0}^{-1} t^m \), substitute \( u = \frac{A_2}{\sqrt{\gamma_0}} \), and substitute \( u = \sqrt{\gamma_0} \) to simplify the integrations into algebraic functions. Apart from simple integrations, we also encounter the following integration terms:

\[ \int_{1}^{\infty} u^{-n} (\ln(u))^p du, \int_{1}^{\infty} u^{-n-3} (\ln(u))^p du, \int_{1}^{\infty} u^{-n} (\ln(u))^p du \]

\[ \int_{1}^{\infty} u^{-n} (\ln(u))^p du \quad (37) \]

We solve the integration of (37) in closed-forms, as represented in Appendix B. Using (60), (61) and (62) of Appendix B in (21) with some algebraic simplifications, we can get the average SNR in a closed-form (not presented due to the space constraint). Similarly, to obtain the ergodic rate, we use the inequality \( \log_2(1+\gamma) \geq \log_2(\gamma) \) in (22) and follow the same procedure used in deriving the average SNR expression. Here, in addition to terms in (37), we also need to integrate the following:

\[ \int_{1}^{\infty} u^{-n} (\ln(u))^p (\ln(u))^q du \quad (38) \]

A closed-form expression of (38) is given in Appendix A. Thus, using (60), (61), (62), and (64) of Appendix A in (22), we get can get a closed-form expression of the ergodic rate (not presented due to the space constraint).

In the following Lemma 1, we present the closed-form expressions by simplifying the derived expression by considering an i.i.d fading model and when the relay is located in the middle of the source and the destination.

Lemma 4 (Average SNR and ergodic rate for i.i.d FP): If \( k \) and \( \beta_{\text{log}} \) are the parameters of the foggy channel, \( A \) and \( \rho \) are the parameters of pointing errors, and a relay is at the midpoint with \( z = 8.686/\beta_{\text{log}} d \), then a closed-form expression of average SNR and a lower bound on the ergodic rate are:

\[ \gamma_{\text{FP}} = \mathcal{F}_\gamma(A, \rho, z, k) \quad (39) \]

\[ \eta_{\text{FP}} = \mathcal{F}_\eta(A, \rho, z, k) \quad (40) \]

where \( \mathcal{F}_\gamma(A, \rho, z, k) \) and \( \mathcal{F}_\eta(A, \rho, z, k) \) are given in (43) and (44) respectively.

Proof: Using (18) in (21) and (22), and noting that \( A_1 = A_2 = A \), and \( \rho_1 = \rho_2 = \rho \) under the symmetric i.i.d fading model, we get:

\[ \gamma = 2 A^2 \left[ \int_{0}^{\gamma} f_\gamma(\gamma) f_\gamma(\gamma) d\gamma \right] \quad (41) \]

\[ \eta = 2 A^2 \left[ \int_{0}^{\gamma} f_\gamma(\gamma) f_\gamma(\gamma) d\gamma \right] \quad (42) \]

Substituting \( u = \frac{A}{\sqrt{\gamma_0}}, m = (z - \rho^2) \) and using series expansion \( \Gamma(a, t) \propto (a-1) \Gamma(a) \sum_{m=0}^{-1} t^m \) in (41) for the average SNR and (42) with the inequality \( \log_2(1+\gamma) \geq \log_2(\gamma) \) for the ergodic rate, we encounter some simple integration terms along with the first and third integration terms of (37). Using (60) and (62) of Appendix B with some algebraic simplifications, we get (39) and (40) of Lemma 4.

Further, we consider light foggy conditions (i.e., \( k = 2 \)) to derive simple analytical expressions on the average SNR and
Lemma 4 with form expression of average SNR and a lower bound on the wavelength [7].

\[ \text{Corollary 1} \ (\text{Average SNR and ergodic rate for i.i.d FP with} \ k = 2): \text{If} \ k = 2 \ \text{and} \ \beta_{\text{fog}} \ \text{are the parameters of the foggy channel,} \ \text{and} \ \rho \ \text{and} \ \rho^2 \ \text{are the parameters of pointing errors, and}} \ \text{relay is at the mid-point with} \ z = 8.686/\beta_{\text{fog}}d, \text{then a closed-form expression of average SNR and a lower bound on the average rate are given as}

\[ \gamma_{\text{FP}} = 2(A\rho)^2\gamma_0 - \frac{(2(1+z)^{2} + \rho^2(1+2z) + 2(2\rho^2 + z^2))}{4(1+\rho^2)(1+z)^2(2\rho^2 + z^2)} \]

\[ \eta_{\text{FP}} \geq \log_2 \left( \frac{e}{2\pi} \right) + 2 \left( (2z(2\ln(1+\rho^2) - 2\rho^2 + z^2) - 0.36(\rho^2 + z)^{-2} - 2\rho^2 - 5z^{-1} - 3(\rho^2 + z)^{-1} + 4 \ln A + 2 \ln \gamma_0) \right) \]

\[ \text{Proof:} \ \text{The proof follows the similar procedure used in Lemma 4 with} \ k = 2. \]

In our earlier work [31], we have shown that the average SNR without relaying is \( \gamma_{\text{direct}} = \frac{(zA\rho)(1 - 2\rho)}{\text{SNR}_{\text{direct}}^{1/2}} \), where \( z_{\text{direct}} = 4.343/\beta_{\text{fog}}d \) \( d \) is the link distance between the source and destination. Thus, the first term in (45) corresponds to twice of the average SNR without relaying. Since the negative term in (45) is insignificant, we expect a higher average SNR with relaying. Similar conclusions can be made for the ergodic rate performance. These have been extensively studied through numerical analysis in Section IV.

Finally, we demonstrate the impact of randomness in the path gain due to the fog by considering that the fog causes a deterministic path gain \( L = e^{-\tau d} \), where \( d \) is the link distance (in km) and \( \tau \) is the atmospheric attenuation factor, which depends on the visibility range and may depend on the wavelength [7].

Corollary 2 (Average SNR and ergodic rate for i.i.d FP channels with deterministic path gain): If the fog causes a deterministic path gain \( L_r = e^{-\tau d/2} \) with relay-assisted transmission \( d_1 = d_2 = d/2 \) for an i.i.d fading model under random pointing errors, expressions of average SNR and ergodic rate are given as

\[ \gamma_{\text{FP}} = \frac{(AL_r)^2\gamma_0}{(2 + 3\rho^2 + \rho^4)}, \text{and} \]

\[ \eta_{\text{FP}} = \log_2 \left( \frac{e}{2\pi} \right) + \frac{1.4427(\rho^2 \ln((AL_r)^2\gamma_0) - 3)}{\rho^2} \]

\[ \text{Proof:} \ \text{Substituting} \ h_{pt} = \sqrt{\frac{\gamma_0}{\tau}} \ \text{in (3), we get an asymptotic PDF of the SNR for an OWC system under the combined effect of atmospheric turbulence and pointing errors with atmospheric path gain:}

\[ f_{\gamma}(\gamma) = \frac{\rho^2}{2\sqrt{\gamma_0(\tau)}} \left( \frac{\gamma}{\gamma_0} \right)^{\rho^2 - 1}, 0 \leq \gamma \leq (AL_r)^2\gamma_0 \]

Using (48) and the CDF \( F_{\gamma}(\gamma) = \int_0^\gamma f_{\gamma}(\gamma')d\gamma' \) in (41) and (42), it is straightforward to prove (47).

Comparing (45) and (44) with the expressions in (47) shows that the randomness in fog significantly complicates the system analysis. Further, the attenuation coefficient using deterministic path gain may overestimate/underestimate the performance obtained with the random fog distribution.

D. Average BER

In this subsection, we derive the average BER for the proposed relay-assisted scheme. Assuming IM/DD, an unified expressions of the average BER is given as [22]:

\[ P_e = \frac{\delta}{2T(p)} \sum_{n=1}^{N} q_{n} \int_0^\infty \gamma^{p-1} e^{-q_{n}\gamma} \Psi_{\gamma}(\gamma) d\gamma \]

where the set \( \{ N, \delta, \rho, \beta \} \) can specify a variety of modulation schemes.

Lemma 5 (Average BER for FPT): If \( x_i \) and \( \beta_i \) are the parameters of the foggy channel, \( A_i \) and \( \beta_i \) are the parameters of pointing errors, \( (\alpha_i, \beta_i, \Omega_i, (\phi_i, \varphi_i, \Xi_i) \) are the parameters for atmospheric turbulence, and \( z_i = 4.343/\beta_i d_i \), then
an exact expression of the average BER for the FPT channel is

\[ \tilde{P}_e = \tilde{P}_{e,1} + \tilde{P}_{e,2} - 2\tilde{P}_{e,1}\tilde{P}_{e,2} \]  

(50)

where \( \tilde{P}_{e,1} \) and \( \tilde{P}_{e,2} \) are given in (51) with \( i = 1 \) and \( i = 2 \):

\[ \tilde{P}_{e,i} = \frac{\delta z_i^2 p_i^2 \rho_i^2 \beta_{\gamma} \lambda_i^2}{4 \Gamma(p) \rho_i \lambda_i \Gamma(p) \Gamma(\rho_i) (2\pi)^{1-\frac{\lambda_i+\delta}{2}}} \sum_{n=1}^{N} H^{2,1}_{\lambda_i+\sigma_1+k_1+1} \left( \frac{\lambda_i^2 \sigma_1^2 \gamma_i^2 \lambda_i^2}{\beta_1 \gamma_i^2 \lambda_i^2} (A_i \sqrt{q_n \gamma_0}) \phi_i \lambda_i \mid U_i \right) \]

(51)

where \( U_i = \{(\mu_i, 1), (1 - \frac{z_i \lambda_i}{\phi_i \lambda_i}, 1), (1, \frac{\phi_i \lambda_i}{2})\} \) and \( V_i = (p, \frac{\phi_i \lambda_i}{2}, \frac{z_i \lambda_i}{\phi_i \lambda_i}, 1, 1) \).

Proof: Using (5) in (49) and applying the definition of Fox-H function [56], we get the average BER in (51). It is well known that the average BER for the DF relaying with Gray coding can also be expressed as [59]:

\[ P_e = \tilde{P}_{e,1} + \tilde{P}_{e,2} - 2\tilde{P}_{e,1}\tilde{P}_{e,2} \]  

(53)

where \( \tilde{P}_{e,1} \) and \( \tilde{P}_{e,2} \) denote the average BER of the first link and the second link, respectively. Thus, using \( i = 1, 2 \) in (51) with (53), we prove Lemma 5.

We use (56) to present an asymptotic expression at high SNR for the average BER

\[ P_e \approx \frac{\delta z_i^2 p_i^2 \rho_i^2 \beta_{\gamma} \lambda_i^2}{4 \Gamma(p) \rho_i \lambda_i \Gamma(p) \Gamma(\rho_i) (2\pi)^{1-\frac{\lambda_i+\delta}{2}}} \sum_{n=1}^{N} \left( \frac{\lambda_i^2 \sigma_1^2 \gamma_i^2 \lambda_i^2}{\beta_1 \gamma_i^2 \lambda_i^2} (A_i \sqrt{q_n \gamma_0}) \phi_i \lambda_i \right) \]

(54)

where \( a_m = a_j = \{(\mu_i, 1 - \frac{z_i \lambda_i}{\phi_i \lambda_i}, 1), (1, \frac{\phi_i \lambda_i}{2})\} \), \( s_m = \{1, 1, \frac{z_i \lambda_i}{\phi_i \lambda_i}, 1\} \), \( s_m = \{0, p, \mu_i, 1 - \frac{z_i \lambda_i}{\phi_i \lambda_i}, 1\} \) and \( t_m = t_j = \{(\phi_i \lambda_i, 0, \lambda_i, 1), (\phi_i \lambda_i, 1, 1, 1)\} \).

Similar to the outage probability, we can use (54) to obtain the diversity order of the dual-hop relay-assisted system as \( M_{BER}^{FP} = \min\{\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\} \).

Next, we analyze the average BER performance for the FP channel. However, general solution to the integration in (49) with [14] is intractable due to the presence of exponential function and incomplete Gamma function with logarithmic argument raised to the power \( k \) in the CDF function for the FP channel. Thus, we derive a closed-form expression for the average BER for a particular integer value of \( k \) (i.e., \( k = 2 \), \( k = 3 \), and so on). Specifically, we consider light foggy condition (with a shape parameter \( k = 2 \)) and general pointing errors with asymptotic atmospheric turbulence to provide exact closed-form expression on the average BER.

Lemma 6 (Average BER for FP): If \( k = 2 \) and \( \beta_{\log} \) are the parameters of the foggy channel, \( A_1, A_2 \) and \( \rho_1, \rho_2 \) are the parameters of pointing errors and atmospheric turbulence, and \( z_1 = 4.343/\beta_{\log} d_1, z_2 = 4.343/\beta_{\log} d_2 \) with \( d_2 \geq d_1 \), then a closed-form expression of average BER for the FP channel is

\[ \tilde{P}_e = \tilde{P}_{e,1} + \tilde{P}_{e,2} - 2\tilde{P}_{e,1}\tilde{P}_{e,2} \]  

(55)

where \( \tilde{P}_{e,1} \) and \( \tilde{P}_{e,2} \) are given in (56) with \( i = 1 \) and \( i = 2 \).

\[ \tilde{P}_{e,i} = \frac{\delta \rho_{\gamma}^2}{2 \Gamma(p) (p)} \sum_{j=0}^{N} \left[ \frac{D^{(1)}(A_j \sqrt{q_n \gamma_0}) - \rho_{\gamma}^2}{q_{j}^p \rho_{\gamma}^2} \right] \left( \Gamma(p + \frac{\rho_{\gamma}^2}{2}) \right) \]

(56)

Proof: First, we substitute (17) and (14) in (49). We use \( m_1 = (z_1 - \rho_{\gamma}^2) \), \( m_2 = (z_2 - \rho_{\gamma}^2) \) and the series expansion \( \Gamma(a, t) = (a - 1)e^{-t} - \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \) and substitute \( u = \sqrt{\gamma_0/\gamma} \), \( v = \sqrt{\gamma_0/\gamma} \) to simplify the integrations into algebraic functions. Apart from simple integration terms, we also encounter the following terms:

\[ \int_1^{\infty} u^{-n-2} e^{-\frac{u}{2}} du, \int_0^{\infty} u^{-n-2} e^{-\frac{u}{2}} ln(u) du \]  

(57)

We provide closed-form expressions of (57) in Appendix B. Using (65), (66) of Appendix B with some algebraic simplifications, we get (56). Finally, using \( i = 1, 2 \) in (56) with (53), we prove Lemma 6.

Similarly, the average BER expression for other values of \( k \) can be obtained. It should be noted that the use of PDF to derive the average BER results into more terms compared to the CDF based approach of (49). The average BER in (56) is useful in analyzing the system performance using known mathematical functions. To simplify the underlying expressions further, we consider deterministic path gain in the following Corollary.

Corollary 3 (Average BER for FP with deterministic path gain): If the fog causes a deterministic path gain \( L_r = e^{-rd/2} \) with relay-assisted transmission \( d_1 = d_2 = d/2 \) for an i.i.d fading model under random pointing errors and then
expression of average BER is given as

\[
\overline{P}_{e,i}^{FP} = \sum_{n=1}^{N} \frac{\delta}{\Gamma(p)(AL_r^2/qn_0)^{\rho^2}} \left[ \Gamma\left(\frac{p^2}{2} + p\right) - \Gamma\left(\frac{p^2}{2} + p, qn(AL_r^2)^{\gamma_0}\right) 
- \Gamma\left(p^2 + p, qn(AL_r^2)^{\gamma_0}\right) \right]^{(2(AL_r^2/qn_0)^{\rho^2})}
\]

\[\text{(58)}\]

\[\text{Proof: Using (17) in (49), and noting that } A_1 = A_2 = A, \text{ and } \rho_1 = \rho_2 = \rho \text{ under the symmetric i.i.d fading model, we get:}
\]

\[
\overline{P}_{e,i}^{FP} = \frac{\delta}{2\Gamma(p)} \sum_{n=1}^{N} q_0^n \int_{0}^{(AL_r)^{\gamma_0}} \gamma^{p-1} e^{-\gamma} \left[ 2F_\gamma(\gamma) - (F_\gamma(\gamma))^2 \right] d\gamma
\]

\[\text{(59)}\]

Using \[\text{[58]}\] and substituting \(F_\gamma(\gamma) = \int_{0}^{\gamma} f(\gamma)d\gamma\) in \[\text{[59]}\] and applying the definition of Gamma function and incomplete Gamma function, we get \[\text{[58]}\].

Similar to the average SNR and ergodic capacity, it can be seen from \[\text{[58]}\] that the consideration of deterministic path gain simplifies the BER expression compared with the random path gain due to the fog.

### IV. Simulation and Numerical Analysis

In this section, we use numerical analysis and Monte Carlo (MC) simulation (averaged over \(10^7\) channel realizations) to demonstrate the performance of the relay-assisted OWC system under the combined effect of fog with pointing errors. We consider three simulation scenarios: fog, pointing errors, and atmospheric turbulence (FPT), fog and pointing errors (FP), deterministic path gain with pointing errors, and atmospheric turbulence (PT). We also compare the performance of the FP channel with the state-of-the-art paper \[\text{[30]}\]. We validate our derived analytical expressions with numerical and simulation results. Although a direct link between the source and destination may not exist, we also compare the performance of direct and relay-assisted transmissions at various link distances and pointing errors parameters. We consider strong (\(\alpha = 1.8621, \beta = 0.5, \Omega = 1.5074, \phi = 1, \varphi = 1.8, \text{ and } \Xi = 0.928\)), moderate (\(\alpha = 2.169, \beta = 0.55, \Omega = 1.5793, \phi = 1, \varphi = 2.35, \text{ and } \Xi = 0.9671\)), and weak (\(\alpha = 2.1, \beta = 4, \Omega = 1.0676, \phi = 2.1, \varphi = 4.5, \text{ and } \Xi = 1.06\)) turbulence conditions to model the DGG atmospheric turbulence \[\text{[42]}\]. We use \(\lambda = 28, \sigma = 13\) for strong turbulence and \(\lambda = 17, \sigma = 9\) \[\text{[43]}\] for moderate turbulence to compute analytical expressions. We use other standard simulation parameters of the OWC system as given in Table \[\text{III}\].

First, we demonstrate the mutual effects of different channel impairments on the OWC system by plotting the average SNR versus transmit power for a link distance of 800 m, as shown in Fig. \[\text{2}\]. Comparing the PT (where deterministic path gain is considered) and FPT plots in Fig. \[\text{2(a)}\], we can find that the fixed visibility range based path-gain computation overestimate the average SNR by 10 dB with respect to the random path gain consideration. Further, the OWC system performs similarly for FP and FPT channels, as shown in Fig. \[\text{2(a)}\], since the effect of fading due to the atmospheric turbulence is negligible in the presence of fog for shorter links. Moreover, it can also be seen that there is a small difference of around 2 dB in the average SNR between the proposed analysis for FP/FPT channels (which is based on integer-valued \(k = 2\)) with simulation results on real-valued \(k = 2.32\) for light fog. In Fig. \[\text{2(b)}\], we analyze the effect of pointing errors on the average SNR performance over light fog (with strong turbulence) under the FPT channel and moderate fog under the FP channel considering the asymmetric placement of the relay \(d_1 = 300\) m and \(d_2 = 500\) m. It can be seen from the figure that the average SNR performance decreases with an increase in normalized beam width and jitter. It should be noted that the effect of normalized beam width \((w_z/a_r)\) on the average SNR performance is more as compared to the standard deviation of the normalized jitter \((\sigma_s/a_r)\).

The ergodic rate performance in Fig. \[\text{3}\] shows a significant benefit of the relay-assisted system under FP and FPT channels. However, similar to the average SNR, the impact of atmospheric turbulence is found to be negligible on the ergodic rate in the presence of fog. The relay-assisted system gives more increment over moderate fog than the light fog as compared to the no-relay system. Moreover, the slopes at high transmit power show greater improvement with the relay-assisted transmission than with direct transmissions. Comparing Fig. \[\text{3(a)}\] and Fig. \[\text{3(b)}\], it can be seen a greater improvement in the ergodic rate performance using the asymmetric placement of the relay with a longer link length (i.e., 800 m) than a shorter one (i.e., 500 m). It can also be seen from Fig. \[\text{3(b)}\] that the normalized beam width has a significant impact on the ergodic rate performance.

In Fig. \[\text{4(a)}\], we demonstrate the outage probability performance of the OWC system for the FP channel with two foggy conditions (i.e., light and moderate) with different pointing errors conditions and symmetric link distances. For the moderate fog, the normalized jitter of the pointing errors

| Transmitted power | \(P_t\) | 0 to 40 dBm |
|-------------------|---------|-------------|
| Responsivity \(R\) | \(\sigma_w^2\) | 10\(^{-14}\) A\(^2\)/Hz |
| AWGN variance | \(\sigma_w^2\) | 10\(^{-14}\) A\(^2\)/Hz |
| Link distance | \(d\) | 400 m and 1200 m |
| Shape parameter of fog | \(k\) | \{2.32, 6.00\} |
| Scale parameter of fog | \(\beta\) | \{13.12, 12.06, 23.00\} |
| Aperture diameter | \(D\) | \(2a_r\) |
| Normalized beam-width | \(w_z/a_r\) | \{15, 20, 25\} |
| Normalized jitter | \(\sigma_s/a_r\) | \{3, 5\} |
| Refractive index | \(C_n^2\) | 8 \times 10\(^{-14}\) |
| Wavelength | \(\lambda\) | 1550 nm |
| Turbulence parameters | \{(\alpha, \beta, \omega)\} | \[\text{[42]}\] |
is $\sigma_s/r = 3$. We compare the two subplots of the figure to show that an increase in the fog density deteriorates the outage probability performance of the OWC system: transmission in moderate fog requires almost 16dBm more transmit power to achieve the same outage probability $10^{-3}$ for a link distance of 400m in the light fog condition. Further, Fig. 3(a) shows that communication range with the light fog is limited to 800m at a transmit power of 40dBm to achieve an acceptable outage probability of $10^{-3}$.

We consider the channel and system parameters judiciously to demonstrate the diversity order of the system. For the light foggy condition having parameter $\beta_{\text{fog}} = 13.12$, the diversity order for the 1200m link is $M_{\text{out}}^{\text{FP}} = \min\left(\frac{3}{2}, \frac{9}{2}\right) = 0.27$ for each pointing error parameter $\rho^2 = 1, 2, 6$. Similarly, the diversity order for the 800m link is $M_{\text{out}}^{\text{FP}} = 0.42$ for each $\rho^2 = 1, 2, 6$. Thus, the diversity order becomes independent of the pointing error parameter $\rho^2$, which can be confirmed through the slope of plots for varying pointing error parameters for both 1200m and 800m link distances. However, the diversity order for the 400m link is $M_{\text{out}}^{\text{FP}} = 0.5$ with $\rho^2 = 1$ (limited by pointing errors) and $M_{\text{out}}^{\text{FP}} = 0.83$ with $\rho^2 = 2, 6$ (limited by fog). The slope of plots for the 400m link confirms the diversity order behavior. Comparing the plots for 1200m, 800m, and 400m, it can be seen from Fig. 3(a) that there is a change in the slope since the diversity order is different for the considered link distances. Similar observations can be inferred for the moderate fog with $\beta_{\text{fog}} = 12.06$ (see the second subplot of Fig. 4(a)). However, for the moderate fog even for the 400m link, the diversity order is dependent on the foggy condition since the minimum value of pointing error parameter $\rho^2 = 2.84$ (computed from $\sigma_s/a_r = 3$ and $w_z/a_r = 10$) is greater than $z = 1.6551$.

In Fig. 4(b), we demonstrate the performance of outage probability for the FPT channel for light foggy conditions with different pointing errors and turbulence conditions for asymmetric link distances. It can be seen that the intensity
of fluctuations has an impact on the outage probability. For a link distance of 1000m, almost 8dBm higher transmit power is required to achieve the same probability for the strong turbulence when compared to weak turbulence at a transmit power of 60dBm. Similar to the FP channel, slope of plots demonstrate the diversity order of the system. Note that strong, moderate, and weak turbulence can introduce a diversity order of 0.47, 0.60, 4.2, respectively. Considering the link distances, the diversity order (dominated by the fog parameter) can be 0.27 (for $d_2 = 600$m), 0.33 (for $d_2 = 500$m), and 0.55 (for $d_2 = 300$m). Further, the diversity order from pointing errors can be 0.5878 using the parameters $\sigma_s/\alpha_r = 3$ and $w_z/\alpha_r = 8$. Thus, using $M_{\text{out}}^{\text{FPT}} = \min\{z_1, z_2/2, \sigma_s \rho^2, \rho z_{\text{out}}/\alpha_r\}$, the diversity order of the FPT channel is 0.27 (for the link 1000m) and 0.33 (for the link 500m). However, for the 500m, the diversity order is 0.47 as determined from the strong turbulence. It can be seen that the slope of plots in Fig. 5(a) demonstrates the diversity order behavior. When comparing the plots for shorter and longer links, the diversity order depends on the fog parameters for longer links and may depend on the pointing errors and turbulence for shorter links.

In Fig. 5 we demonstrate the average BER performance of the OWC system over the FP channel with the popular on-off keying (OOK) modulation using $\delta = 1$, $N = 1$, $p = 1/2$, and $q = 1/2$ as a function of transmit power. In Fig. 5(a), we consider two foggy conditions (i.e., light and moderate) with different pointing errors conditions and symmetric link distances. Similar to the outage probability, an increase in fog density deteriorates the average BER performance. The diversity order for the BER can be illustrated for the moderate fog condition (see the second subplot Fig. 5(a)). Using $\beta_{\text{log}} = 12.06$, we get $z_1 = z_2 = 0.60$ for the link distance $d = 1200$m, $z_1 = z_2 = 0.90$ for the link distance $d = 800$m, and $z_1 = z_2 = 1.80$ for the link distance $d = 400$m. Using $\min\{z_1/2, \rho^2/2\}$, the diversity order 0.30 depends on fog parameters for link distances 1200m and 800m for each pointing error parameter $\rho^2 = 1, 2, 6$. However, for the link distance 400m and $\rho^2 = 1$, the diversity order 0.5 depends on the pointing error parameter $\rho$. We can observe the slope of plots (see the second subplot of Fig. 4(a)) to confirm the behavior of diversity with different system and channel parameters. Similar observations can be inferred from the average BER for the light fog with $\beta_{\text{log}} = 13.12$ (see the first subplot of Fig. 4(a)).

Finally, we consider a more practical situation for performance evaluation in Fig. 5(b). Specifically, we consider three foggy conditions light, moderate, and thick, which are each associated with corresponding atmospheric turbulence of strong, moderate, and weak, respectively. This is a typical scenario since the atmospheric turbulence and foggy conditions are inversely correlated [4]. We demonstrate the performance of average BER for the considered scenario with different foggy, pointing errors and turbulence conditions for asymmetric link distances. It can be seen that the intensity of fluctuations has an impact on the average BER. The figure shows that the impact of random fog is more pronounced on the performance than the intensity of turbulence. It can be seen that the range for OWC link is limited to 500m for thick fog conditions. The performance under light fog with strong turbulence is acceptable even for longer links. For a link distance of 500m, almost 5 dBm higher transmit power is required to achieve the same BER when the moderate fog (with moderate turbulence) is compared with the light fog (with strong turbulence) at a transmit power of 60dBm. As expected, the moderate fog performs in between the light fog and thick fog. Further, the change in the slope of plots for various channel parameters demonstrates the system’s diversity order.

In all the above plots (Fig. 2 to Fig. 5), we have also verified our derived expressions with the simulation and numerical results. We use MATLAB functions to compute the Meijer-G and Fox-H functions involved in analytical expressions. It can be seen that the derived analytical expressions of the outage...
probability, average SNR, ergodic rate, and average BER have an excellent match with the simulation results. However, there is a gap between simulation and analytical results for the ergodic rate at a lower transmit power due to the use of inequality $\log_2(\gamma) \leq \log_2(1 + \gamma)$.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we investigated the performance of a DF relay-based OWC system under the combined effect of random fog, pointing errors, and the DGG atmospheric turbulence. We have shown that the dual-hop relaying is capable of mitigating fog, pointing errors, and turbulence-induced fading for high-speed OWC links. We provided design criteria of mitigating the impact of pointing errors and random fog by adjusting the beam-width and limiting the communication range. We analyzed the distribution functions of SNR and provided a detailed analysis for the outage probability, average SNR, ergodic rate, and average BER considering a generalized i.i.d fading model with asymmetrical distance for relaying between the source and destination. Using the asymptotic analysis, we have shown that the diversity order depends on the fog parameters for longer links and may depend on the pointing errors and turbulence for shorter links. We have shown that there is a significant gap in the performance using the existing visibility range based path-gain computation as compared to the probabilistic modeling of fog. Further, the effect of normalized beam width on the OWC performance is more than the standard deviation of normalized jitter. It was also demonstrated that the relay-assisted system shows better performance than the direct-link transmissions as a benchmark, providing a more significant benefit in the denser fog, for longer link lengths, and when the relay is located symmetrically between the source and destination. Numerical analysis shows that the derived closed-form expressions excellently match with simulation results, and thus can be implemented for real-time tuning of the system parameters to optimize the OWC performance. We envision that the consideration of the general fading model would be helpful to assess the deployment of OWC system for terrestrial wireless communications under various channel impairments.

The proposed work can be augmented with several research directions. Performance analysis for different system configurations such as amplify-and-forward in dual-hop and multi-hop frameworks would be interesting. Further, the hybrid OWC-RF and multi-aperture systems can be investigated with the generalized fading considered in this paper. It would also be interesting to investigate low complexity all-optical relaying schemes for OWC systems under the combined effect of fog, pointing errors, and atmospheric turbulence.

APPENDIX A
(INTEGRALS OF LEMMA 3)

To solve the first and second integrals of (37), we substitute $\ln u = t$ and apply the definition of Gamma function and incomplete gamma function to get the following closed-form expressions:

$$
\int_1^{\infty} u^{-n} \ln^p(u) \, du = \frac{\Gamma[p+1, (n-1)\mu]}{(n-1)!} \quad (60)
$$

$$
\int_a^{\infty} u^{-n-3} \ln^p(u) \, du = \frac{\Gamma[p+1, (2+n)\ln(a)]}{(2+n)!} \quad (61)
$$

Further, to solve the third integral of (37), we substitute $n \ln u = t$ and use the well-known identity $\int_0^{\infty} e^{-at} \Gamma(b, t) \, dt = a^{-b} \Gamma(b)(1 - (a + 1)^{-b})$ [57–pp.657, eq. 6.451.2] to get the following:

$$
\int_1^{\infty} u^{-n-3} \ln(u) \, du = \frac{1 - n^k(2n - 1)^{-k}}{n - 1} \Gamma[k] \quad (62)
$$

Finally, we solve (38) by substituting $\ln u = t$ and $\ln a + t = v$ and apply the identity $\int_1^{\infty} x^{v-1}(x-u)^{\mu-1} e^{-b\mu} dx = b^{-\mu+1} u^{-\mu+1} \Gamma(\mu) e^{-b} W_{-\mu+1}^{-\mu}(bu)$ [57–pp.348, eq.
and incomplete Gamma function. After a few simplifications, we get
\[
\int_{0}^{\infty} \lambda,\mu \Phi(z,\lambda,\mu) \] 

Using the identity of Meijer's G function \([58], \text{eq. (07.34.21.0084.01)}\), we get the following:
\[
\int_{0}^{\infty} u^{-n} (\ln(u))^p (\ln(u))^d du = \frac{\Gamma(1+p)}{\Gamma(1+d)} \Phi(1+2t;2+p+t(n-1)\ln(u))
\]

For the second integral of (57), we again use \(\int_{0}^{\infty} f(t)dt = \int_{0}^{\infty} f(t)dt - \int_{0}^{\infty} f(t)dt\), and apply the integration by parts on both the terms. Using the identity \(u^{-1} - u^{-1} \ln(t)dt = u^{-1} (1+\gamma) \ln(u)\) \([57], \text{pp.573, eq. 4.352.1)}\), and the identity of Meijer's G function \([58], \text{eq. (07.34.21.0084.01)}\), we get the following:
\[
\int_{0}^{\infty} u^{-n-2} e^{-\frac{x}{u}} \ln(u)du = \frac{1}{2} \left( \frac{1}{p} \right) \left( \frac{1}{2} \right) (\ln(\frac{1}{2}))
\]

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