Four-Impurity Operators and String Field Theory Vertex in the BMN Correspondence

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Abstract

In the context of the Penrose/BMN limit of the AdS/CFT correspondence, we consider four-impurity BMN operators in Yang-Mills theory, and demonstrate explicitly their correspondence to four-oscillator states in string theory. Using the dilatation operator on the gauge-theory side of the correspondence, we calculate matrix elements between four-impurity states. Since conformal dimensions of gauge-theory operators correspond to light-cone energies of string states, these matrix elements may be compared with the string-theory light-cone Hamiltonian matrix elements calculated in the plane-wave background using the string field theory vertex. We find that the two calculations agree, extending the cases of two- and three-impurity operators considered in the literature using BMN gauge-theory quantum mechanics. The results are also in agreement with calculations in the literature based on perturbative gauge-theory methods.

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I. INTRODUCTION

The conjectured AdS/CFT correspondence\cite{1, 2, 3} between Superstring theory and Super Yang-Mills theory has withstood many tests during the years since it was put forward. Since it has not been possible to verify AdS/CFT directly, different special cases have been considered and elucidated. One such limit is the well-known ‘BMN limit,’ proposed by Berenstein, Maldacena and Nastase \cite{4}. On the string side of the correspondence, a Penrose limit of AdS space is taken, producing a plane-wave background\cite{5}. On the Yang-Mills side this corresponds to a specific sector of the theory in which both $N$ and $J$, the $R$-charge, are taken to infinity, holding $N/J^2$ fixed. This is known as the BMN double-scaling limit\cite{6, 7, 8, 9, 10, 11, 12}. A nice review has been given in \cite{13}. To leading order in $g_{YM}^2 N/J^2$ the light-cone gauge energy of a string state is given by

$$E_{LC}/\mu = \Delta - J$$  \hspace{1cm} (1)$$

where $\mu$ is the mass parameter of the AdS space, $\Delta$ is the scaling dimension of the gauge-theory operator corresponding to the string state, and $J$ is its $R$-charge. Writing the exact correspondence in terms of operators, the light-cone string Hamiltonian is given by

$$\mathcal{H}_{LC}/\mu = D - J$$  \hspace{1cm} (2)$$

where $D$ is the dilatation operator in the Yang-Mills theory. In the present paper, we consider the bosonic sector of this correspondence.

On the Yang-Mills side, one may write down a perturbative expansion in the effective t Hooft coupling $\lambda' = g_{YM}^2 N/J^2$ and in the genus-counting parameter $g_s = J^2/N$. The gauge theory sports an $SO(6)$ R-symmetry group with fields $\phi_m$ transforming in the vector representation. One picks two of these, say $\phi_5$ and $\phi_6$, and defines $Z = \phi_5 + i\phi_6$ and $\bar{Z} = \phi_5 - i\phi_6$, which have plus and minus unit $R$-charge respectively. The operators in correspondence with oscillator string states in this limit are the BMN operators, which are products of traces of powers of $Z$, sprinkled with ‘impurities’ which consist of the other four $\phi$ fields\cite{4, 14, 15}. In the BMN limit, these powers are taken to be large and these operators form a basis in which one can investigate the dilatation operator $D$ and obtain information about the light-cone string Hamiltonian. The number of impurities $h$ in such an operator is identified with the excitation level of the corresponding string state, and the total number
of fields \((Z \text{ and } \phi)\) corresponds to \(J + \hbar\), which is the \(R\)-charge and engineering dimension of the operator. Anomalous dimensions of these operators correspond to light-cone string energies.\([4, 16, 17, 18]\) Finding anomalous conformal dimensions amounts to diagonalising the dilatation operator, and this is made non-trivial by operator mixing \([3]\) when non-planar and one-loop contributions are considered. Such investigations may also be carried out by considering two-point functions \([7, 19, 20, 21]\).

In the present paper we investigate four-impurity BMN operators, using the dilatation operator to construct the string Hamiltonian for level-four states. This method, called “BMN Quantum Mechanics” has been used in the two-impurity \([23]\) and three-impurity \([25]\) cases. Since there are only four distinct impurity fields, more impurities cannot be considered without taking into account the combinatoric effects of repeated impurities; thus we complete the analysis of possible distinct-impurity states by considering the maximal four-impurity case. Operators with arbitrary numbers of impurities have been considered using Yang-Mills perturbation theory \([21]\) in which matrix elements can be extracted from three-point functions. This method has been used for various combinations of scalar and vector impurities \([21, 24]\) and for the case of two fermion impurities \([22]\). Our results using BMN Quantum Mechanics will be found to agree with these perturbative Yang-Mills computations. Using the perturbative approach it was possible for the authors of \([24]\) and \([21]\) to consider an arbitrary number of impurities; from our experience with the four-impurity calculation, it seems tedious to consider higher-impurity states using the quantum-mechanical method. In contrast, it should be mentioned that since in the perturbative approach the three-point function is used to obtain matrix elements, the quantum-mechanical approach may lend itself more readily to the consideration of multitrace states with more than two traces.

We construct the string Hamiltonian by first diagonalising the dilatation operator at leading order and planar level, and then using this basis to write genus-one dilatation operator elements at one-loop order. The calculations, although analogous to two- and three-impurity cases, are very lengthy and tedious in comparison. The resulting Hamiltonian, expressed in terms of its matrix elements in the four-impurity state basis is expected to be in correspondence with the three-string light-cone interaction vertex calculated in String Field Theory. We calculate this vertex for level-four string states and find precise agreement with the gauge-theory calculation; the string interaction Hamiltonian of three string states with
a total of four excitations is verified to correspond to the matrix dilatation operator on the
gauge-theory side between the corresponding four-impurity BMN operators.

II. FOUR-IMPURITY BMN OPERATORS

Using the notation

\[ O_{p_1,p_2,...,p_h}^{1,2,...,h} \equiv \text{tr}[\phi_1 Z^{p_1} \phi_2 Z^{p_2} ... \phi_h Z^{p_h}] \]  

for a single-trace operator with \( h \) impurities, \( l \)-trace four-impurity BMN operator basis
elements may be written

'4': \[ O_{p_1,p_2,p_3,p_4}^{1234} \prod_{j=1}^{l} O_{J_j} \]  

'31': \[ O_{p_1,p_2,p_3}^{123} O_{p_4}^{4} \prod_{j=1}^{l} O_{J_j} \]  

'22': \[ O_{p_1,p_2}^{12} O_{p_3,p_4}^{34} \prod_{j=1}^{l} O_{J_j} \]  

'211': \[ O_{p_1,p_2}^{12} O_{p_3}^{3} O_{p_4}^{4} \prod_{j=1}^{l} O_{J_j} \]  

'1111': \[ O_{p_1}^{1} O_{p_2}^{2} O_{p_3}^{3} O_{p_4}^{4} \prod_{j=1}^{l} O_{J_j} \]  

where \( O_p \) simply indicates an operator with no impurities, \( \text{tr}[Z^p] \). We shall use the indicated
labels '4', '31', '22', '211' and '1111' as a short-hand way of referring to these operators. It
may be noted that the superscripts in the above, in addition to indicating which impurities
are present, also denote the order of the impurities. We write \( J_0 = p_1 + p_2 + p_3 + p_4 \) so
that \( J = \sum_{j=1}^{l} J_j \) is the total \( R \)-charge of the operator. Taking the continuum BMN limit
\( J \rightarrow \infty \) with \( x_i = p_i/J \) and \( r_i = J_i/J \), string states corresponding to the above will be

\[ As mentioned in the introduction, for more than four impurities, the same \( \phi \) fields would have to be repeated.\]
denoted as

\[
\langle '4' \rangle := \frac{\sqrt{N J+4}}{J} |x_1, x_2, x_3, x_4\rangle^{1234} \prod_{j=1}^{l} |r_j\rangle
\] (9)

\[
\langle '31' \rangle := \frac{\sqrt{N J+4}}{J} |x_1, x_2, x_3\rangle^{123} |x_4\rangle^4 \prod_{j=1}^{l} |r_j\rangle
\] (10)

\[
\langle '22' \rangle := \frac{\sqrt{N J+4}}{J} |x_1, x_2\rangle^{12} |x_3, x_4\rangle^{34} \prod_{j=1}^{l} |r_j\rangle
\] (11)

\[
\langle '211' \rangle := \frac{\sqrt{N J+4}}{J} |x_1, x_2\rangle^{12} |x_3\rangle^3 |x_4\rangle^4 \prod_{j=1}^{l} |r_j\rangle
\] (12)

\[
\langle '1111' \rangle := \frac{\sqrt{N J+4}}{J} |x_1\rangle^1 |x_2\rangle^2 |x_3\rangle^3 |x_4\rangle^4 \prod_{j=1}^{l} |r_j\rangle
\] (13)

The normalisation of these states may be understood by examining their tree-level planar two-point functions\[\text{25}\].

III. STRING HAMILTONIAN

We wish to find the string Hamiltonian, given by

\[
H = \lim_{N \to \infty, N/J^2 \text{fixed}} (D - J),
\] (14)

in the basis defined in the previous section. As explained in \[\text{25}\] this is not Hermitian, so that \(H\) found in this way may not be directly interpreted as the string Hamiltonian. To remedy this, one begins by defining the inner product using the planar free theory,

\[
\langle a|b \rangle \equiv \langle O_a O_b \rangle_{\text{free, planar}};
\] (15)

\(H\) is not Hermitian with respect to this product, but with respect to the product defined by the full non-planar free correlator

\[
\langle a|b \rangle_{g2} \equiv \langle O_a O_b \rangle_{\text{free, full}} \equiv \langle a|S|b \rangle,
\] (16)

where \(S\) is Hermitian with respect to the original planar product. A new basis state may be defined by the non-unitary transformation

\[
|\tilde{a} \rangle \equiv S^{-1/2} |a \rangle.
\] (17)
Now, a ‘new’ Hamiltonian $\tilde{H}$ may be defined by

$$\langle a|\tilde{H}|b\rangle \equiv \langle a|H|b\rangle_{g_2}$$ (18)

so that

$$\tilde{H} = S^{1/2}HS^{-1/2}$$ (19)

is Hermitian in the original basis and should correspond to the light-cone string Hamiltonian $\mathcal{H}_{LC}$, up to a possible unitary transformation. Now, since $\langle a|S|b\rangle = \langle a|b\rangle_{g_2}$, $S$ is just the full non-planar mixing matrix for basis states. Expanding $S$ and $H$ in the genus-counting parameter $g_2$,

$$S = 1 + g_2\Sigma + \mathcal{O}(g_2^2), \quad H = H_0 + g_2H_1 + \mathcal{O}(g_2^2),$$ (20)

and we may write matrix elements of the string Hamiltonian;

$$\langle a|\tilde{H}|b\rangle = \langle a|(1 + \frac{1}{2}g_2\Sigma)H(1 - \frac{1}{2}g_2\Sigma)|b\rangle = \langle a|H_0|b\rangle + g_2\langle a|(\frac{1}{2}[\Sigma, H_0] + H_1)|b\rangle$$ (21)

Here, $H_0$ is just the planar part of $D_2$, the dilatation operator at one-loop order while $H_1$ is the genus-one part of $D_2$. The dilatation operator is given by

$$D = D_0 + \frac{g_{YM}^2}{16\pi^2}D_2 + \mathcal{O}(g_{YM}^4)$$ (22)

where

$$D_0 = \text{tr}(\phi_m\phi_m), \quad D_2 = -: \text{tr}([\phi_m, \phi_n][\phi_m, \phi_n] + \frac{1}{2}[\phi_m, \phi_n][\phi_m, \phi_n]) :. $$ (23)

Here, $m$ and $n$ are $SO(6)$ indices running from 1 to 6, $\phi = \delta/\delta\phi$ and the normal ordering symbol denotes that the enclosed $\phi$-derivatives only act on fields outside it. For operators containing $Z$ and $\phi_i$ fields (but no $\bar{Z}$ fields), the above expression for $D_2$ becomes

$$D_2 = -: \text{tr}([\phi_i, \phi_j][\phi_i, \phi_j] + 2[\phi_i, Z][\phi_i, Z] + \frac{1}{2}[\phi_i, \phi_j][\phi_i, \phi_j]) :$$ (24)

where $i$ and $j$ run only over impurity fields, that is from 1 to 4. In the following we do not consider ‘boundary’ terms in which the impurities are neighbours\(^2\), so that the first term in eqn.(24), which always produces such states, may be neglected. Since the four impurities

\(^2\) Such terms make diagonalisation of $D$ very difficult in a discrete basis, but in the continuum BMN limit they become unimportant.
we consider are distinct, the third term may also be neglected, since it contains a double derivative of a single \( \phi \) field.\(^3\)

It will be helpful to make note of the following contraction identities.

\[
\text{tr}[1] = N, \tag{25}
\]
\[
\text{tr}[A\phi]\text{tr}[B\phi] = \text{tr}[AB] \quad (\text{"fusion"}), \tag{26}
\]
\[
\text{tr}[A\phi B\phi] = \text{tr}[A]\text{tr}[B] \quad (\text{"fission"}), \tag{27}
\]

and the complete contractions

\[
\text{tr}[Z^p \bar{Z}^q] = \delta_{pq}N^{p+1} + \mathcal{O}(N^{p-1}), \tag{28}
\]
\[
\text{tr}[Z^p]\text{tr}[\bar{Z}^q Z^r] = \delta_{p+r,q}p(r+1)N^{p+r} + \mathcal{O}(N^{p+r-2}), \tag{29}
\]
\[
\text{tr}[Z^p \bar{Z}^q Z^r \bar{Z}^s] = \delta_{p+r,q+s}N^{p+r+1}(\min(p, q, r, s) + 1) + \mathcal{O}(N^{p+r-1}). \tag{30}
\]

Operating with \( H_0 \) on the discrete basis (4)-(8) we find

\[
H_0 \mathcal{O}^{abcd}_{p_a p_b p_c p_d} \prod_j \mathcal{O}_{I_j} = -\frac{g_{\text{YM}}^2 N}{8\pi^2} \left(-2\mathcal{O}^{abcd}_{p_a p_b p_c p_d} + \mathcal{O}^{abcd}_{p_a-1, p_b, p_c, p_d+1} + \mathcal{O}^{abcd}_{p_a+1, p_b, p_c, p_d-1}\right) \prod_j \mathcal{O}_{I_j}
\]
\[
+ \text{(three other cyclic permutations of } abcd) \tag{31}
\]
\[
H_0 \mathcal{O}^{abc}_{p_a p_b p_c} \mathcal{O}^d_{p_d} \prod_j \mathcal{O}_{I_j} = -\frac{g_{\text{YM}}^2 N}{8\pi^2} \left(-2\mathcal{O}^{abc}_{p_a p_b p_c} + \mathcal{O}^{abc}_{p_a-1, p_b, p_c+1} + \mathcal{O}^{abc}_{p_a+1, p_b, p_c-1}\right) \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{I_j}
\]
\[
+ \text{(two other cyclic permutations of } abc) \tag{32}
\]
\[
H_0 \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{cd}_{p_c p_d} \prod_j \mathcal{O}_{I_j} = -\frac{g_{\text{YM}}^2 N}{2\pi^2} \left(-2\mathcal{O}^{ab}_{p_a p_b} + \mathcal{O}^{ab}_{p_a-1, p_b+1} + \mathcal{O}^{ab}_{p_a+1, p_b-1}\right) \mathcal{O}^{cd}_{p_c p_d} \prod_j \mathcal{O}_{I_j}
\]
\[
+ (abcd \rightarrow cdab) \tag{33}
\]
\[
H_0 \mathcal{O}^{ab}_{p_a p_b} \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{I_j} = -\frac{g_{\text{YM}}^2 N}{4\pi^2} \left(-2\mathcal{O}^{ab}_{p_a p_b} + \mathcal{O}^{ab}_{p_a-1, p_b+1} + \mathcal{O}^{ab}_{p_a+1, p_b-1}\right) \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{I_j} \tag{34}
\]
\[
H_0 \mathcal{O}^{a}_{p_a} \mathcal{O}^{b}_{p_b} \mathcal{O}^{c}_{p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{I_j} = 0. \tag{35}
\]

\(^3\) These would contribute in the case of more than four impurities.
In the continuum limit these become \((x_{123} \text{ denotes } x_1 + x_2 + x_3, \text{ etc.})\)

\[
\begin{align*}
H_0 |x_1, x_2, x_3, r_0 - x_{123}\rangle_{1234} & = -\frac{\lambda'}{8\pi^2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \right) |x_1, x_2, x_3, r_0 - x_{123}\rangle_{1234} \\
H_0 |x_1, x_2, x_3\rangle_{123} |r_0 - x_{123}\rangle^4 & = -\frac{\lambda'}{8\pi^2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \right) |x_1, x_2, x_3\rangle_{123} |r_0 - x_{123}\rangle^4 \\
H_0 |x_1, x_2\rangle_{12} |x_3\rangle_{34} |r_0 - x_{123}\rangle^3 & = -\frac{\lambda'}{8\pi^2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \right) |x_1, x_2\rangle_{12} |x_3\rangle_{34} |r_0 - x_{123}\rangle^3 \\
H_0 |x_1, x_2\rangle_{12} |x_3\rangle_{34} |r_0 - x_{123}\rangle^4 & = -\frac{\lambda'}{4\pi^2} \left( \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2 \right) |x_1, x_2\rangle_{12} |x_3\rangle_{34} |r_0 - x_{123}\rangle^4
\end{align*}
\]

(36) to (39)

In the above expressions, we have suppressed the factor \(\prod_{j=1}^4 |r_j\rangle\) since it is unaffected by \(H_0\). The above eigenvalue equations are solved by defining the following momentum-basis states [7, 19].

\[
\begin{align*}
|n_1, n_2, n_3; r_0\rangle & \equiv \frac{1}{\sqrt{r_0^3}} \\
\times \sum_{(abc)} \int_{x_{123} < r_0} d^3 x e^{\frac{2\pi i}{r_0}(n_1 x_1 + n_2 x_2 + n_3 x_3)} O^{1,a+1,b+1,c+1}(x_1, x_2, x_3, r_0 - x_{123})
\end{align*}
\]

(40)

\[
\begin{align*}
|n_1, n_2; r_0 - s\rangle_{1234} |s\rangle^4 & \equiv \frac{1}{r_0 - s} \\
\times \sum_{(ab)} \int_{x_{12} < r_0 - s} d^2 x e^{\frac{2\pi i}{r_0}(n_1 x_1 + n_2 x_2)} O^{1,a+1,b+1}(x_1, x_2, r_0 - s - x_{12}) O^4(s)
\end{align*}
\]

(41)

\[
\begin{align*}
|n_1; r_0 - s\rangle_{12} |n_2; s\rangle_{34} & \equiv \frac{1}{\sqrt{(r_0 - s)s}} \\
\times \int_0^{r_0 - s} dx_1 \int_0^s dx_2 e^{2\pi i \left( \frac{n_1}{r_0 - s} x_1 + \frac{n_2}{s} x_2 \right)} O^{12}(x_1, r_0 - s - x_1) O^{34}(x_2, s - x_2)
\end{align*}
\]

(42)

\[
\begin{align*}
|n; r_0 - s - t\rangle_{12} |s\rangle^3 |t\rangle^4 & \equiv \frac{1}{\sqrt{r_0 - s - t}} \\
\times \int_0^{r_0 - s - t} dx e^{\frac{2\pi i}{r_0 - s - t} nx} O^{12}(x, r_0 - s - t - x) O^3(s) O^4(t)
\end{align*}
\]

(43)

while zero-impurity states are normalised as \(|r\rangle \equiv \frac{1}{\sqrt{r}} O(r)\). Here, \(O(x)\) simply denotes the continuum version of the discrete operator basis [4] \(\otimes [8]\). \(\sum_{(abc)}\) denotes a summation in which \(abc\) takes on each of the six permutations of 123, and similarly for \(\sum_{(ab)}\). The
superscripts now indicate only which impurities are contained in each trace. These above momentum states have the energy eigenvalues

\[ E'_{4} = \frac{\lambda'}{2} \frac{n_{12}^{2} + n_{1}^{2} + n_{2}^{2} + n_{3}^{2}}{r_{0}^{2}}, \]  

(44)  

\[ E'_{31} = \frac{\lambda'}{2} \frac{n_{1}^{2} + n_{2}^{2}}{(r_{0} - s)^{2}}, \]  

(45)  

\[ E'_{22} = \frac{\lambda'}{2} \left( \frac{n_{1}^{2}}{(r_{0} - s)^{2}} + \frac{n_{2}^{2}}{s^{2}} \right), \]  

(46)  

\[ E'_{211} = \frac{\lambda'}{2} \frac{n_{1}^{2}}{(r_{0} - s - t)^{2}}, \]  

(47)  

\[ E'_{1111} = 0. \]  

(48)  

Again, these do not depend on whether or not the states \( \prod |r_{j}\rangle \) are present; \( H_{0} \) will not operate on such factors. The momentum states (44)-(48) are orthonormal, so that

\[ \prod_{j=1}^{l} \langle r_{j}|m_{1}, m_{2}, m_{3}; r_{0}|n_{1}, n_{2}, n_{3}; s_{0}\rangle \prod_{j=1}^{l'} |s_{j}\rangle = \delta_{m_{1}, n_{1}} \delta_{m_{2}, n_{2}} \delta_{m_{3}, n_{3}} \delta_{l, l'} \delta(r_{0} - s_{0}) \sum_{\pi \in S_{l}} \prod_{k=1}^{l} \delta(r_{\pi(k)} - s_{k}) \]  

(49)  

and similarly for the remaining states (49)-(48).

We must now calculate explicitly \( H_{1} \) and \( \Sigma \) in the above \( H_{0}\)-eigenstate basis. For clarity, we follow the convention of writing \( H_{1} = H_{+} + H_{-} \), where \( H_{+} \) increases and \( H_{-} \) decreases the number of traces when acting on a basis state. The procedure is several times more tedious and lengthy than in the three-impurity case \[25\], and we relegate the explicit computation of these quantities to Appendices I and II.

The calculation of \( H_{1} \) in Appendix I leads us to the following observations. Upon writing out the matrix elements of \( H_{+} \) and \( H_{-} \), we see that for matrix elements not involving a ‘4’-state, the calculation effectively reduces to the case of fewer impurities. The matrix elements between our ‘31’, ‘211’, and ‘1111’ states therefore need not be considered further, since they correspond to the three-impurity case already studied in \[25\] or to the two-impurity calculation of \[23\]. Moreover, we note that the ‘22-22’ matrix element of \( H_{1} \) simply involves two copies of the ‘2-2’ element from the two-impurity case studied in \[23\].

We point out that in the following, the notations \( D_{a} \) and \( D_{b}^{a} \) are used differently in each case. We continue to omit the product \( \prod_{j} |r_{j}\rangle \) of zero-impurity states which we omitted in the above, with the understanding that this product may appear in each of the following states without affecting the calculation. Referring again to Appendix I, the ‘4-4’ elements
of $H_+$ and $H_-$ are given by

$$
\langle s | \langle m_1, m_2, m_3; r_0 - s | H_+ | n_1, n_2, n_3; r_0 \rangle = -\lambda' \frac{1}{2\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right)
\mbox{and}

\langle m_1, m_2, m_3; r_0 | H_- | n_1, n_2, n_3; r_0 - s \rangle | s \rangle
= \lambda' \frac{1}{2\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s m_1}{r_0} \right) \sin \left( \frac{\pi s m_2}{r_0} \right) \sin \left( \frac{\pi s m_3}{r_0} \right) \sin \left( \frac{\pi s m_{123}}{r_0} \right)
\times \left[ m_{123} \left( \frac{1}{D_2 D_3} + \frac{1}{D_1 D_2} \right) + \frac{1}{r_0 D_{123}} \left( \frac{m_1}{D_2 D_3} + \frac{m_2}{D_1 D_2} + \frac{m_3}{D_1 D_2} \right) \right]
$$

where $D_a = \frac{n_a}{r_0 - s} - \frac{m_a}{r_0}$.

For the ‘4-31’ elements of $H_+$ and $H_-$ we find

$$
_{123}^{12}(m_1, m_2; r_0 - s) \langle s | H_+ | n_1, n_2, n_3; r_0 \rangle
= -\lambda' \frac{1}{2\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right)
\times \left[ \frac{m_1}{r_0 - s} \left( \frac{1}{D_2^2 D_1} + \frac{1}{D_1 D_2 D_2} \right) + \frac{m_2}{r_0 - s} \left( \frac{1}{D_2^2 D_1} + \frac{1}{D_1^2 D_2} \right) \right]
$$

where $D_a = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s}$.

and

$$
\langle m_1, m_2, m_3; r_0 | H_- | n_1, n_2; r_0 - s \rangle^{123} \langle s \rangle^4
= \lambda' \frac{1}{2\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s m_1}{r_0} \right) \sin \left( \frac{\pi s m_2}{r_0} \right) \sin \left( \frac{\pi s m_3}{r_0} \right) \sin \left( \frac{\pi s m_{123}}{r_0} \right)
\times \left[ \left( \frac{1}{D_{12}^3} - \frac{1}{D_{12} D_{12}^2} - \frac{m_{123}}{D_{12} D_{12}^2} \right) \left( \frac{1}{D_1} + \frac{1}{D_2} \right) - \frac{1}{m_3 D_{12}^{123}} \left( \frac{m_1}{D_2^2} + \frac{m_2}{D_1^2} \right) \right]
$$

where $D_a = \frac{n_a}{r_0 - s} + \frac{m_b}{r_0}$.
Finally, the ‘4-22’ elements are

\[
\begin{align*}
12\langle m_1; r_0 - s | ^{34}m_2; s | H_+ | n_1, n_2, n_3; r_0 \rangle &= -\lambda' \frac{1}{2\pi^4 \sqrt{(r_0 - s)s r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \\
&\times \left[ \frac{m_1}{r_0 - s} D_2^2 D_3^2 \left( \frac{1}{D_1^{-1}} + \frac{1}{D_{123}^{-1}} \right) + \frac{m_2}{s} D_1^{-1} D_{123}^{-1} \left( \frac{1}{D_2^2} - \frac{1}{D_3^2} \right) \right] \\
\text{where } D_1^a &= \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \text{ and } D_2^a = \frac{n_a}{r_0} + \frac{m_2}{s}
\end{align*}
\]  

(54)

and

\[
\begin{align*}
\langle m_1, m_2, m_3; r_0 | H_- | n_1; r_0 - s \rangle^{12} | m_2; s \rangle^{34}
&= \frac{\lambda'}{2\pi^4 \sqrt{(r_0 - s)s r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \\
&\times \left[ \frac{1}{m_2} - \frac{1}{m_3} \right] D_1^{-1} D_{123}^{-1} D_2^{-3} + \frac{1}{m_2} D_1^{-1} D_2^{-3} - \frac{1}{m_3} D_1^{-1} D_2^{-3} + \frac{1}{m_3} \left[ \frac{n_1}{r_0} + \frac{m_2}{r_0} \right] \\
\text{where } D_1^b &= \frac{n_1}{r_0 - s} + \frac{m_2}{r_0} \text{ and } D_2^b = \frac{n_2}{s} + \frac{m_2}{r_0}.
\end{align*}
\]  

(55)

Next the matrix elements of \( \Sigma \), from Appendix II, are given by the following. The ‘4-4’ component of \( \Sigma \) is

\[
\langle s | \langle m_1, m_2, m_3; r_0 - s | \Sigma | n_1, n_2, n_3; r_0 \rangle
\]

\[
= \frac{1}{\pi^4 \sqrt{(r_0 - s)^3 s r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \\
\times \left[ \frac{1}{D_1 D_2 D_3 D_{123}} \right] \\
\text{where } D_a &= \frac{n_a}{r_0} - \frac{m_a}{r_0 - s},
\]  

(56)

while the ‘4-31’ element is found to be

\[
\begin{align*}
^{123}\langle m_1, m_2; r_0 - s | ^4 s | \Sigma | n_1, n_2, n_3; r_0 \rangle
&= \frac{1}{\pi^4 \sqrt{(r_0 - s)^3 s r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \\
&\times \left[ \frac{1}{D_{123} D_3 D_1^{-1} D_2^{-2}} \right] \\
\text{where } D^b &= \frac{n_a}{r_0} + \frac{m_b}{r_0 - s},
\end{align*}
\]  

(57)
and the ‘4-22’ component is

\[ \langle 12 | m_1; r_0 - s | ^{34} m_2; s | \Sigma | n_1, n_2, n_3; r_0 \rangle = \frac{1}{\pi^4 \sqrt{(r_0 - s)s r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \]

\[ \times \frac{1}{D_{123}^1 D_{12}^{-1} D_{23}^2 D_2^2} \]

where

\[ D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \quad \text{and} \quad D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s}. \]  

(58)

In the above formulae, other impurity orderings may be accommodated by considering appropriate permutations of the momenta. These permutations are found using the definitions of the momentum states (40)-(43). Let \( | \rangle ^{abcd} \) be any single- or multi-trace state containing the four impurities \( abcd \). We wish to find the matrix element of some operator \( M \) between this state and the ‘4’-state,

\[ ^{abcd} \langle | M | n_1, n_2, n_3; r_0 \rangle, \]

given that we already have this quantity for the case \( abcd = 1234 ; \)

\[ M_{n_1, n_2, n_3} \equiv ^{1234} \langle | M | n_1, n_2, n_3; r_0 \rangle. \]

(60)

The results are, for \( \pi \in S_3 \) and \( abc = \pi(234) \),

\[ ^{1abc} \langle | M | n_1, n_2, n_3; r_0 \rangle = M_{\pi(n_1, n_2, n_3)} \]

(61)

\[ ^{1abc} \langle | M | n_1, n_2, n_3; r_0 \rangle = M_{\pi(\overline{n_{123}}, n_2, n_3)} \]

(62)

\[ ^{1abc} \langle | M | n_1, n_2, n_3; r_0 \rangle = M_{\pi(-n_{123}, n_2, n_3)} \]

(63)

\[ ^{1abc} \langle | M | n_1, n_2, n_3; r_0 \rangle = M_{\pi(-n_{123}, n_1, n_2)} \]

(64)

Now we are in a position to calculate the string Hamiltonian \( \tilde{H} \), assembling \( H_0, H_1 \) and \( \Sigma \) using eqn. (21). The ‘4-4’ element of the genus-one correction to \( \tilde{H} \) is given by

\[ \tilde{H}_{4-4} |_{g_2} = g_2 \langle s | m_1, m_2, m_3; r_0 - s | \left( \frac{1}{2} [\Sigma, H_0] + H_1 \right) | n_1, n_2, n_3; r_0 \rangle. \]  

(65)

Which may be calculated either using \( \Sigma \) and \( H_+ \) from eqn. (56) and eqn. (50), or \( \Sigma \) and \( H_- \) from the conjugate of eqn. (56) and eqn. (51). It may easily be verified that these give the
same result, showing that $\tilde{H}$ is Hermitian as it should be, and serving as a check on the calculations. The result is

$$
\tilde{H}_{4-4} = \frac{\lambda' g_2}{4 \pi^4} \frac{1}{\sqrt{(r_0 - s)^3 s r_0}} \frac{(D_1)^2 + (D_2)^2 + (D_3)^2 + (D_{123})^2}{D_1 D_2 D_3 D_{123}}
$$

$$
\times \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right)
$$

where $D_a = \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}$. (66)

Of course, the genus-zero ‘4-4’ component is simply given by $H_0$ in eqn. (44). The ‘4-31’ element is similarly obtained using $\Sigma$ from eqn. (57) and $H_{\pm}$ from eqn. (52) or eqn. (53) to calculate

$$
\tilde{H}_{4-31} = g_2^{123} \langle m_1, m_2; r_0 - s | s | \left(\frac{1}{2} [\Sigma, H_0] + H_1 \right) | n_1, n_2, n_3; r_0 \rangle
$$

$$
= \frac{\lambda' g_2}{4 \pi^4} \frac{1}{\sqrt{(r_0 - s)^3 s r_0}} \frac{(D_{123})^2 + (D_1^{-1})^2 + (D_2^{-1})^2 + (D_3^{-1})^2}{D_{123} D_1^{-1} D_2^{-1} D_3^{-1}}
$$

$$
\times \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right)
$$

where $D_a^b = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s}$. (68)

and the ‘4-22’ element by using eqn. (58) and eqn. (54) or eqn. (55), giving

$$
\tilde{H}_{4-22} = g_2^{12} \langle m_1; r_0 - s | s | \left(\frac{1}{2} [\Sigma, H_0] + H_1 \right) | n_1, n_2, n_3; r_0 \rangle
$$

$$
= \frac{\lambda' g_2}{4 \pi^4} \frac{1}{\sqrt{(r_0 - s)^3 s^3}} \frac{(D_{12})^2 + (D_2)^2 + (D_3)^2 + (D_1^{-1})^2}{D_{12} D_2^{-1} D_3^{-1} D_1^{-1}}
$$

$$
\times \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right)
$$

where $D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s}$ and $D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s}$. (70)

With the order-$g_2$ string Hamiltonian now in hand, we turn to the computation of the string field theory vertex with which it is expected to correspond.

**IV. COMPARISON WITH STRING-FIELD VERTEX**

The number of impurities of a BMN operator on the SYM side of the correspondence is identified with the number of oscillator excitations of the corresponding state on the string side. Light-cone String Field Theory in the plane-wave background has been developed in
and the Neumann coefficients necessary for computations have been found in [38] with further results in [39, 40].

We shall consider a three-string interaction, with a total of eight oscillator excitations distributed among the three strings. A multi-string state with $2k$ excitations is given by

$$|A⟩ = \prod_{j=1}^{2k} \alpha_i^{I_j \dagger} |0⟩$$

where $r_j$ are the string numbers, $I_j$ label the transverse AdS directions (i.e. the impurity coordinates), and $m_j$ are the oscillator numbers. For our purposes, we set $k = 4$.

In the case of our ‘4-4’ interaction, we consider the three-string state

$$|4,4⟩ = \alpha_1^{(1) \dagger} \alpha_2^{(2) \dagger} \alpha_3^{(3) \dagger} |0⟩ \otimes |0⟩ \otimes |0⟩,$$

where excitations are absent for string number two; it corresponds to a zero-impurity state. In [15] it is shown how to calculate the interaction vertex between the strings in this state.

We find

$$⟨4,4|H_3⟩ = \frac{\alpha_1^{(1) \alpha_2^{(2)} \alpha_3^{(3)}}}{2} ∏_{j=1}^{4} N_j$$

where

$$N_1 = \left( \frac{\omega_1^{(1) n_1}}{\mu \alpha_1^{(1)}} + \frac{\omega_3^{(3) m_1}}{\mu \alpha_3^{(3)}} \right) \tilde{N}_{n_{1,m_{1}}}^{(1,3)} \tilde{N}_{n_{2,m_{2}}}^{(1,3)} \tilde{N}_{n_{3,m_{3}}}^{(1,3)} \tilde{N}_{n_{123,m_{123}}}^{(1,3)}$$

$$N_2 = \left( \frac{\omega_1^{(1) n_2}}{\mu \alpha_1^{(1)}} + \frac{\omega_3^{(3) m_2}}{\mu \alpha_3^{(3)}} \right) \tilde{N}_{n_{1,m_{1}}}^{(1,3)} \tilde{N}_{n_{2,m_{2}}}^{(1,3)} \tilde{N}_{n_{3,m_{3}}}^{(1,3)} \tilde{N}_{n_{123,m_{123}}}^{(1,3)}$$

$$N_3 = \left( \frac{\omega_1^{(1) n_3}}{\mu \alpha_1^{(1)}} + \frac{\omega_3^{(3) m_3}}{\mu \alpha_3^{(3)}} \right) \tilde{N}_{n_{1,m_{1}}}^{(1,3)} \tilde{N}_{n_{2,m_{2}}}^{(1,3)} \tilde{N}_{n_{3,m_{3}}}^{(1,3)} \tilde{N}_{n_{123,m_{123}}}^{(1,3)}$$

$$N_4 = \left( \frac{\omega_1^{(1) n_{123}}}{\mu \alpha_1^{(1)}} + \frac{\omega_3^{(3) m_{123}}}{\mu \alpha_3^{(3)}} \right) \tilde{N}_{n_{1,m_{1}}}^{(1,3)} \tilde{N}_{n_{2,m_{2}}}^{(1,3)} \tilde{N}_{n_{3,m_{3}}}^{(1,3)} \tilde{N}_{n_{123,m_{123}}}^{(1,3)}$$

where the string frequencies are $\omega_{r,m} = \sqrt{m^2 + \mu^2 \alpha_{r}^2}$. The $\alpha_{r}$ are the fractions of outgoing light-cone momentum carried by each string, and in the present case these are $\alpha_1 = 1 - s$, $\alpha_2 = s$ and $\alpha_3 = -1$. The Neumann coefficients for the plane-wave geometry are given
Substituting into our expression (73) for the ‘4-4’ string amplitude, we obtain

\[ \tilde{N}_{0, n}^{(r, s)} = \tilde{N}_{0, -n}^{(r, s)} = \frac{1}{\sqrt{2}} \tilde{N}_{0, n}^{(r, s)} \]
\[ \tilde{N}_{\pm m, \pm n}^{(r, s)} = \frac{1}{2} \left( \tilde{N}_{m, n}^{(r, s)} - \tilde{N}_{-m, -n}^{(r, s)} \right) \]
\[ \tilde{N}_{\pm m, \pm n}^{(r, s)} = \frac{1}{2} \left( \tilde{N}_{m, n}^{(r, s)} + \tilde{N}_{-m, -n}^{(r, s)} \right) \]
\[ \tilde{N}_{0, n}^{(r, s)} = \frac{1}{2\pi} (-1)^{s(n+1)} s(n) \left| \alpha_{(s)} \right| \sqrt{\frac{\alpha_{(r)} \omega_{(r)} n + \alpha_{(r)} \omega_{(s)} n}{\omega_{(r)} \omega_{(s)} n}} \]
\[ \tilde{N}_{\pm m, \pm n}^{(r, s)} = \pm \frac{1}{2\pi} \frac{(-1)^{r(m+1)+s(n+1)} s(r) s_{(s)n}}{\alpha_{(s)} \omega_{(r)} n + \alpha_{(r)} \omega_{(s)} n} \times \sqrt{\frac{|\alpha_{(r)} | \left[ \omega_{(r)} n \pm \alpha_{(r)} n \pm \alpha_{(s)} n \right]}{\omega_{(r)} \omega_{(s)} n}} \] (78)
\[ s_{(1) m} = s_{(2) m} = 1 \quad s_{(3) m} = 2 \sin \left( \frac{\pi m \alpha(1)}{\alpha(3)} \right) \] (79)

Expanding to leading order in \(1/\mu\), the Neumann matrices become (for the cases we need)

\[ \tilde{N}_{m, n}^{(r, 3)} = \frac{(-1)^{r(m+1)+n} \sin(\pi n s)}{2\pi \sqrt{\alpha(r)} (\frac{m}{\alpha(r)} - n)} + O\left(\frac{1}{\mu^2}\right). \] (80)

Substituting into our expression (73) for the ‘4-4’ string amplitude, we obtain

\[ \langle 4, 4 \vert H_3 \rangle = \frac{s}{2 \left[ 1 + \frac{s}{D_1 D_2 D_3 D_{123}} \sin(\pi s n_1) \sin(\pi s n_2) \sin(\pi s n_3) \sin(\pi s n_{123}) \right]} \times \]
\[ \text{where } D_a = \frac{n_a}{r_0} - \frac{m_a}{r_0 - s}. \] (81)

Apart from normalisation, this is in complete agreement with the ‘4-4’ matrix element (80) of the string Hamiltonian \( \tilde{H} \) calculated on the SYM side, A calculation similar to the above leads to

\[ \langle 4, 3, 1 \vert H_3 \rangle = \frac{s(1 - s)}{2} \left\{ \left( \frac{\omega(3) n_1}{\mu \alpha(3)} + \frac{\omega(2) n_1}{\mu \alpha(2)} \right) \tilde{N}_{3, 0}^{(3)}, \tilde{N}_{4, 0}^{(3)}, \tilde{N}_{5, 0}^{(3)} \right\} \]
\[ + \left( \frac{\omega(3) n_2}{\mu \alpha(3)} + \frac{\omega(1) n_2}{\mu \alpha(1)} \right) \tilde{N}_{3, 1}^{(3)}, \tilde{N}_{4, 1}^{(3)}, \tilde{N}_{5, 1}^{(3)} \]
\[ + \left( \frac{\omega(3) n_3}{\mu \alpha(3)} + \frac{\omega(1) n_3}{\mu \alpha(1)} \right) \tilde{N}_{3, 2}^{(3)}, \tilde{N}_{4, 2}^{(3)}, \tilde{N}_{5, 2}^{(3)} \]
\[ + \left( \frac{\omega(3) n_{123}}{\mu \alpha(3)} + \frac{\omega(1) n_{123}}{\mu \alpha(1)} \right) \tilde{N}_{3, 0}^{(3)}, \tilde{N}_{4, 0}^{(3)}, \tilde{N}_{5, 0}^{(3)} \right\}. \] (82)
which upon substitution of the expanded Neumann matrices (80) leads to
\[
\langle 4,3,1|H_3 \rangle = \frac{1}{2} \sqrt{\frac{s}{(1-s)}} \frac{(D_{123}^{-1})^2 + (D_1^{-1})^2 + (D_0^{-2})^2 + (D_0^{-1})^2}{D_{123} D_1 D_0^{-2} D_3} \\
\times \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right)
\]
where
\[
D_a^b = \frac{n_a}{r_0} + \frac{m_b}{r_0 - s}.
\]

Again, this is in perfect agreement with the gauge-theory result (67). Finally, we calculate the ‘4-22’ interaction
\[
\langle 4,2,2|H_3 \rangle = \frac{s(1-s)}{2} \left\{ \left( \frac{\omega(3)n_1}{\mu_0(3)} + \frac{\omega(1)m_1}{\mu_0(1)} \right) \tilde{N}^{(3,2)}_{-n_1,m_1} \tilde{N}^{(3,1)}_{n_2,-m_1} \tilde{N}^{(3,2)}_{n_3,m_2} \tilde{N}^{(3,2)}_{-n_{123},-m_2} + \left( \frac{\omega(3)n_2}{\mu_0(3)} + \frac{\omega(1)-m_1}{\mu_0(1)} \right) \tilde{N}^{(3,2)}_{n_1,m_1} \tilde{N}^{(3,1)}_{-n_2,-m_1} \tilde{N}^{(3,2)}_{n_3,m_2} \tilde{N}^{(3,2)}_{-n_{123},-m_2} + \left( \frac{\omega(3)n_3}{\mu_0(3)} + \frac{\omega(2)-m_2}{\mu_0(2)} \right) \tilde{N}^{(3,2)}_{n_1,m_1} \tilde{N}^{(3,1)}_{n_2,-m_1} \tilde{N}^{(3,2)}_{-n_3,m_2} \tilde{N}^{(3,2)}_{n_{123},-m_2} + \left( \frac{\omega(3)-n_{123}}{\mu_0(3)} + \frac{\omega(2)-m_2}{\mu_0(2)} \right) \tilde{N}^{(3,2)}_{n_1,m_1} \tilde{N}^{(3,1)}_{n_2,-m_1} \tilde{N}^{(3,2)}_{-n_3,m_2} \tilde{N}^{(3,2)}_{n_{123},-m_2} \right\},
\]
and find
\[
\langle 4,2,2|H_3 \rangle = \frac{1}{2} \sqrt{\frac{1-s}{s}} \frac{(D_{123}^{-1})^2 + (D_0^{-2})^2 + (D_0^{-1})^2}{D_{123} D_0^{-2} D_3} \\
\times \sin\left(\frac{\pi s n_1}{r_0}\right) \sin\left(\frac{\pi s n_2}{r_0}\right) \sin\left(\frac{\pi s n_3}{r_0}\right) \sin\left(\frac{\pi s n_{123}}{r_0}\right)
\]
where \( D_a^1 = \frac{n_a}{r_0} + \frac{m_1}{r_0 - s} \) and \( D_a^2 = \frac{n_a}{r_0} + \frac{m_2}{s} \).

agreement with eqn. (89) obtains.

We see that the string Hamiltonian \( \tilde{H} \) calculated on the Yang-Mills side for four-impurity BMN states reproduces the light-cone string field vertex between four-excitation string states.

V. DISCUSSION

We have used the dilatation operator on the gauge-theory side of the correspondence in the basis of four-impurity BMN operators to derive an expression for the corresponding Hamiltonian on the string side. Four-impurity operators, as in the two- and three-impurity cases, may be considered to have distinct impurities, and this fact has been used to simplify the calculations. Nevertheless, to obtain the gauge-theory matrix elements and transform
to the momentum-state string basis requires many tedious pages of calculation. This is
rewarded by the precise agreement between these and the matrix elements obtained directly
from the string field theory vertex in the plane-wave background. Our results are also
in agreement with the perturbative gauge theory analysis in [21, 24]. One could follow
the analysis presented in [25] and calculate decay widths using our matrix elements of $\tilde{H}$,
although it seems that only special cases may be treated analytically.

Our analysis gives further evidence of the correspondence of BMN operators to string osc-
cillator states, although we have not addressed the potential discrepancy recently uncovered
at three-loop order [12, 41].

It would be interesting to understand in detail what the effect of repeated impurities
would be, and to extend our calculations explicitly to consider more than four. In this case,
there will be more possible contractions of the gauge-theory operators, since there will no
longer be a unique contraction of the impurity fields. These calculations could be compared
against the arbitrary impurity-number calculations of [21, 24].

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Appendix I: Matrix Elements of $H_1$

In this appendix we first calculate the action of $H_1$ on the discrete basis states (1)-(8)
and demonstrate the use of these results to find the matrix elements of $H_1$ in the momentum
basis (10)-(13). We find:
\[ H + \mathcal{O}_{abcd}^{p_a p_b p_c p_d} \prod_j \mathcal{O}_{J_j} = -\lambda' \frac{1}{8\pi^2} \left[ \sum_{i=1}^{p_1-1} \left( \mathcal{O}_{abcd}^{p_a-i-1, p_b, p_c, p_d} + \mathcal{O}_{abcd}^{p_a-i-1, p_b+1, p_c, p_d} - 2\mathcal{O}_{abcd}^{p_a-i, p_b, p_c, p_d} \right) \mathcal{O}_i \right. \\
+ \sum_{i=0}^{p_1-1} \left( (\mathcal{O}_{abcd}^{p_a, p_b+1, p_a-i-1} - \mathcal{O}_{abcd}^{p_a, p_b, p_a-i}) \mathcal{O}^{p_d}_{p_d+i} \\
+ (\mathcal{O}_{abc}^{p_a+i, p_d} - \mathcal{O}_{abc}^{p_a+i, p_d+1}) \mathcal{O}^{bc}_{p_b, p_c, p_a-i-1} \\
+ (\mathcal{O}_{abc}^{p_d, p_b, p_c, p_a-i-1} - \mathcal{O}_{abc}^{p_d, p_b, p_c, p_a-i}) \mathcal{O}^{ad}_{p_a, p_b, p_d, p_c} \\
+ (\mathcal{O}_{abc}^{p_a+i, p_c+1, p_d} - \mathcal{O}_{abc}^{p_a+i, p_c+1, p_d+1}) \mathcal{O}^{ad}_{p_a, p_b, p_c, p_d} \right) \prod_{j \neq i} \mathcal{O}_{J_j} \\
+ (\text{three other cyclic permutations of } abcd), \tag{92} \]

\[ H - \mathcal{O}_{abcd}^{p_a p_b p_c p_d} \prod_j \mathcal{O}_{J_j} = -\lambda' \frac{1}{8\pi^2} \left[ \sum_{i=1}^{l} J_i \left( \mathcal{O}_{abcd}^{p_a+i, p_b, p_c, p_d} - \mathcal{O}_{abcd}^{p_a+i, p_b, p_c, p_d+1} \\
+ \mathcal{O}_{abcd}^{p_a+1, p_b, p_c, p_d+i-1} - \mathcal{O}_{abcd}^{p_a, p_b, p_c, p_d+i-1} \right) \prod_{j \neq i} \mathcal{O}_{J_j} \right] \\
+ (\text{three other cyclic permutations of } abcd). \tag{93} \]

\[ H + \mathcal{O}_{abc}^{p_a p_b p_c} \mathcal{O}^{d}_{p_d} \prod_j \mathcal{O}_{J_j} = -\lambda' \frac{1}{8\pi^2} \left[ \sum_{i=1}^{p_3-1} \left( \mathcal{O}_{abc}^{p_a-i-1, p_b, p_c} + \mathcal{O}_{abc}^{p_a-i-1, p_b+1, p_c} - 2\mathcal{O}_{abc}^{p_a-i, p_b, p_c} \right) \mathcal{O}^d_i \right. \\
+ \sum_{i=0}^{p_3-1} \left( (\mathcal{O}_{abc}^{p_a-i, p_b+1, p_c} - \mathcal{O}_{abc}^{p_a-i, p_b}) \mathcal{O}^a_{p_c} + \mathcal{O}^d_{p_d} \\
+ (\mathcal{O}_{abc}^{p_a+i, p_c+1, p_d} - \mathcal{O}_{abc}^{p_a+i, p_c}) \mathcal{O}^b_{p_d} \right) \prod_{j \neq i} \mathcal{O}_{J_j} \\
+ (\text{two other cyclic permutations of } abc), \tag{94} \]
\[H_+ \mathcal{O}_{p_a p_b p_c}^{ab} \mathcal{O}_{p_d}^{cd} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \left[ \sum_{i=1}^{l} J_i \left( \mathcal{O}_{J_i+p_a-1, p_b, p_c+1}^{ab} - \mathcal{O}_{J_i+p_a, p_b, p_c}^{ab} \right) + \mathcal{O}_{p_a+1, p_b, p_c+J_i-1}^{abc} - \mathcal{O}_{p_a, p_b, p_c+J_i}^{abc} \right] \mathcal{O}_{p_d}^{d} \]
\[+ \sum_{i=0}^{p_a-1} \left( \mathcal{O}_{p_a-i-1, p_b, p_c+p_d+i+1}^{dbca} - \mathcal{O}_{p_a-i, p_b, p_c+p_d+i}^{dbca} \right) \mathcal{O}_{J_i} \]
\[+ \sum_{i=0}^{p_d-1} \left( \mathcal{O}_{p_d-i-1, p_a+i, p_b, p_c+1}^{adbc} - \mathcal{O}_{p_d-i, p_a+i, p_b, p_c}^{adbc} \right) \mathcal{O}_{J_i} \]
\[+ \mathcal{O}_{p_a+1, p_b, p_c+i, p_d+i-1}^{abcd} - \mathcal{O}_{p_a, p_b, p_c+i, p_d-i}^{abcd} \right] \mathcal{O}_{J_i} \right] \prod_{j \neq i} \mathcal{O}_{J_j} + \text{two other cyclic permutations of } abc. \quad (95)\]

\[H_- \mathcal{O}_{p_a p_b}^{ab} \mathcal{O}_{p_c p_d}^{cd} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \sum_{i=1}^{p_a-1} 2 \left( \mathcal{O}_{p_a-i-1, p_b+1}^{ab} - \mathcal{O}_{p_a-i, p_b}^{ab} \right) \mathcal{O}_{p_c p_d}^{cd} \mathcal{O}_i \prod_j \mathcal{O}_{J_j} \]
\[+ \left( \begin{array}{c}
a \leftrightarrow c \\
b \leftrightarrow d
\end{array} \right), \text{ then } + (a \leftrightarrow b), \text{ then } + (c \leftrightarrow d). \quad (96)\]

\[H_- \mathcal{O}_{p_a p_b}^{ab} \mathcal{O}_{p_c p_d}^{cd} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \left[ \sum_{i=1}^{l} J_i \left( \mathcal{O}_{J_i+p_a-1, p_b+1}^{ab} - \mathcal{O}_{J_i+p_a, p_b}^{ab} \right) + \mathcal{O}_{p_a+1, p_b+J_i-1}^{ab} - \mathcal{O}_{p_a, p_b+J_i}^{ab} \right] \mathcal{O}_{p_c p_d}^{cd} \]
\[+ \sum_{i=0}^{p_a-1} \left( \mathcal{O}_{p_a+i, p_b+p_d+i-1}^{cdba} - \mathcal{O}_{p_a+i, p_b+p_d+i}^{cdba} \right) \mathcal{O}_{J_i} \]
\[+ \mathcal{O}_{p_a-i-1, p_b+p_c+i, p_d+1}^{abcd} - \mathcal{O}_{p_a-i, p_b+p_c+i, p_d}^{abcd} \right] \mathcal{O}_{J_i} \right] \prod_{j \neq i} \mathcal{O}_{J_j} + \left( \begin{array}{c}
a \leftrightarrow c \\
b \leftrightarrow d
\end{array} \right), \text{ then } + (a \leftrightarrow b), \text{ then } + (c \leftrightarrow d). \quad (97)\]

\[H_+ \mathcal{O}_{p_a p_b}^{ab} \mathcal{O}_{p_c p_d}^{cd} \prod_j \mathcal{O}_{J_j} = \frac{-\lambda'}{8\pi^2} \left[ -\frac{1}{2} \sum_{i=1}^{p_a-1} \left( \mathcal{O}_{p_a-i, p_b}^{ab} - \mathcal{O}_{p_a-i-1, p_b+1}^{ab} - \mathcal{O}_{p_a-1, p_b-i+1}^{ab} + \mathcal{O}_{p_a, p_b-i}^{ab} \right) \mathcal{O}_{p_c} \mathcal{O}_{p_d} \mathcal{O}_i \right] \prod_j \mathcal{O}_{J_j} + (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b), \quad (98)\]
$$ H - \mathcal{O}_{p_a p_b}^{ab} \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} \prod_{j} \mathcal{O}_{J_j} = -\frac{\lambda'}{8\pi^2} \left[ -\frac{1}{2} \sum_{i=1}^{l} J_i \left( \mathcal{O}_{J_{j_1}+p_a, p_b}^{ab} - \mathcal{O}_{J_{j_1}+p_a-1, p_b+1}^{ab} \right) \right. $$

$$ + \mathcal{O}_{p_a, p_b+j_i}^{ab} - \mathcal{O}_{p_a+1, p_b+1}^{ab} \right) \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} $$

$$ - \sum_{i=0}^{p_c-1} \left( \mathcal{O}_{p_a, p_c-i, p_b+i}^{abc} - \mathcal{O}_{p_a+1, p_c-i-1, p_b+i}^{abc} \right) \mathcal{O}_{p_d}^{d} $$

$$ + \mathcal{O}_{p_a+1, p_b+i, p_c-i}^{abc} \right) \mathcal{O}_{p_a}^{a} \mathcal{O}_{p_b}^{b} \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} \prod_{j \neq i} \mathcal{O}_{J_j} $$

$$ + (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b). \quad \text{(99)} $$

$$ H_+ \mathcal{O}_{p_a}^{a} \mathcal{O}_{p_b}^{b} \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} \prod_{j} \mathcal{O}_{J_j} = 0. \quad \text{(100)} $$

$$ H_+ \mathcal{O}_{p_a}^{a} \mathcal{O}_{p_b}^{b} \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} \prod_{j} \mathcal{O}_{J_j} = -\frac{\lambda'}{8\pi^2} \left[ \sum_{i=0}^{p_b-1} \left( \mathcal{O}_{p_b-i, p_a+i}^{ab} - \mathcal{O}_{p_b-i-1, p_a+i+1}^{ab} \right) \mathcal{O}_{p_c}^{c} \mathcal{O}_{p_d}^{d} \prod_{j \neq i} \mathcal{O}_{J_j} $$

$$ + (11 \text{ other permutations of } abcd), \quad \text{(101)} $$

As with the case of $H_0$, we now write the continuum forms. In order to save space, we will not write out all the permutations, instead using the convention that in the following the permutations must first be carried out, and then $x_d$ set to $r_0 - x_{abc}$ and $\partial_d$ set to zero. Since they are unaffected by $H_+$, we suppress factors of $\prod \mathcal{O}(r_{j})$ in the expressions for $H_+$. We find:
\[
H_+ \mathcal{O}^{abcd}(x_a, x_b, x_c, x_d) = \frac{-\lambda'}{8\pi^2}
\]
\[
\int_0^{x_a} dy [(\partial_d - 2\partial_a + \partial_b)\mathcal{O}^{abcd}(x_a - y, x_b, x_c, x_d)\mathcal{O}(y)
+ (\partial_c - \partial_a)\mathcal{O}^{bcd}(x_b, x_c, x_a - y)\mathcal{O}^a(x_d + y)
+ (\partial_d - \partial_a)\mathcal{O}^{ad}(x_a - y, x_d)\mathcal{O}^{bc}(x_b + x_c + y)
+ (\partial_b - \partial_a)\mathcal{O}^{bc}(x_b, x_a - y)\mathcal{O}^{ad}(x_c + y, x_d)
+ (\partial_c - \partial_a)\mathcal{O}^{acd}(x_a - y, x_c, x_d)\mathcal{O}^b(x_b + y)]
+ (3 \text{ other cyclic permutations of } abcd)
\] (102)

\[
H_- \mathcal{O}^{abcd}(x_a, x_b, x_c, x_d) \prod_j \mathcal{O}(r_j) = \frac{-\lambda'}{8\pi^2}
\]
\[
\sum_{i=1}^{l} r_i (\partial_d - \partial_a)(\mathcal{O}^{abcd}(x_a + r_i, x_b, x_c, x_d) - \mathcal{O}^{abcd}(x_a, x_b, x_c, x_d + r_i)) \prod_{j \neq i} \mathcal{O}(r_j)
+ (3 \text{ other cyclic permutations of } abcd)
\] (103)

\[
H_+ \mathcal{O}^{abc}(x_a, x_b, x_c)\mathcal{O}^d(x_d) = \frac{-\lambda'}{8\pi^2}
\]
\[
\int_0^{x_a} dy [(\partial_c - 2\partial_a + \partial_b)\mathcal{O}^{abc}(x_a - y, x_b, x_c)\mathcal{O}^d(x_d)\mathcal{O}(y)
+ (\partial_b - \partial_a)\mathcal{O}^{ab}(x_a - y, x_b)\mathcal{O}^c(x_c + y)\mathcal{O}^d(x_d)
+ (\partial_c - \partial_a)\mathcal{O}^{ca}(x_c, x_a - y)\mathcal{O}^b(x_b + y)\mathcal{O}^d(x_d)]
+ (2 \text{ other cyclic permutations of } abc)
\] (104)

\[
H_- \mathcal{O}^{abc}(x_a, x_b, x_c)\mathcal{O}^d(x_d) \prod_j \mathcal{O}(r_j) = \frac{-\lambda'}{8\pi^2}
\]
\[
\sum_{i=1}^{l} r_i (\partial_c - \partial_a)(\mathcal{O}^{abc}(x_a + r_i, x_b, x_c) - \mathcal{O}^{abc}(x_a, x_b, x_c + r_i))\mathcal{O}^d(x_d) \prod_{j \neq i} \mathcal{O}(r_j)
+ \int_0^{x_a} dy [(\partial_d - \partial_a)\mathcal{O}^{dca}(x_d + y, x_b, x_c, x_a - y) + (\partial_d - \partial_a)\mathcal{O}^{dca}(x_d - y, x_b, x_c, x_a + y)] \prod_j \mathcal{O}(r_j)
+ \int_0^{x_d} dy [(\partial_c - \partial_d)\mathcal{O}^{adc}(x_d - y, x_a + y, x_b, x_c) + (\partial_a - \partial_d)\mathcal{O}^{adc}(x_a, x_b, x_c + y, x_d - y)] \prod_j \mathcal{O}(r_j)
+ (2 \text{ other cyclic permutations of } abc)
\] (105)
\begin{align*}
H_+ \mathcal{O}^{ab}(x_a, x_b) \mathcal{O}^{cd}(x_c, x_d) &= -\frac{\lambda'}{8\pi^2} \\
\int_0^{x_a} \mathrm{d}y 2(\partial_c - \partial_d) \mathcal{O}^{ab}(x_a - y, x_b) \mathcal{O}^{cd}(x_c, x_d) \mathcal{O}(y) \\
+ (a \leftrightarrow c, b \leftrightarrow d), \text{ then } + (a \leftrightarrow b), \text{ then } + (c \leftrightarrow d) \\
\tag{106} \\
H_- \mathcal{O}^{ab}(x_a, x_b) \mathcal{O}^{cd}(x_c, x_d) \prod_j \mathcal{O}(r_j) &= -\frac{\lambda'}{8\pi^2} \\
\sum_{i=1}^l r_i(\partial_a - \partial_b)(\mathcal{O}^{ab}(x_a, x_b + r_i) - \mathcal{O}^{ab}(x_a + r_i, x_b)) \mathcal{O}^{cd}(x_c, x_d) \prod_{j \neq i} \mathcal{O}(r_j) \\
+ \int_0^{x_a} \mathrm{d}y [(\partial_c - \partial_d) \mathcal{O}^{cda}(x_c, x_d, x_b, x_a) \\
+ (\partial_d - \partial_a) \mathcal{O}^{cda}(x_a - y, x_b, x_c + y, x_d)] \prod_j \mathcal{O}(r_j) \\
+ (a \leftrightarrow c, b \leftrightarrow d), \text{ then } + (a \leftrightarrow b), \text{ then } + (c \leftrightarrow d) \\
\tag{107} \\
H_+ \mathcal{O}^{ab}(x_a, x_b) \mathcal{O}^c(x_c) \mathcal{O}^d(x_d) &= -\frac{\lambda'}{8\pi^2} \\
\frac{1}{2} \int_0^{x_a} \mathrm{d}y (\partial_b - \partial_a)(\mathcal{O}^{ab}(x_a - y, x_b) + \mathcal{O}^{ab}(x_a, x_b - y)) \mathcal{O}^c(x_c) \mathcal{O}^d(x_d) \mathcal{O}(y) \\
+ (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b) \\
\tag{108} \\
H_- \mathcal{O}^{ab}(x_a, x_b) \mathcal{O}^c(x_c) \mathcal{O}^d(x_d) \prod_j \mathcal{O}(r_j) &= -\frac{\lambda'}{8\pi^2} \\
\frac{1}{2} \sum_{i=1}^l r_i(\partial_a - \partial_b)(\mathcal{O}^{ab}(x_a, x_b + r_i) - \mathcal{O}^{ab}(x_a + r_i, x_b)) \mathcal{O}^c(x_c) \mathcal{O}^d(x_d) \prod_{j \neq i} \mathcal{O}(r_j) \\
+ \int_0^{x_c} \mathrm{d}y [(\partial_d - \partial_a)(\mathcal{O}^{abc}(x_a, x_c - y, x_b + y) + \mathcal{O}^{abc}(x_b, x_c + y, x_a - y)) \mathcal{O}^d(x_d) \\
+ \frac{1}{2}(\partial_d - \partial_c)(\mathcal{O}^{dc}(x_c - y, x_d + y) + \mathcal{O}^{dc}(x_d + y, x_c + y, x_c - y)) \mathcal{O}^{ab}(x_a, x_b)] \prod_j \mathcal{O}(r_j) \\
+ \int_0^{x_a} \mathrm{d}y (\partial_a - \partial_c)(\mathcal{O}^{cba}(x_a + y, x_b, x_c - y) - \mathcal{O}^{cba}(x_a - y, x_b, x_c + y)) \mathcal{O}^d(x_d) \prod_j \mathcal{O}(r_j) \\
+ (c \leftrightarrow d), \text{ then } + (a \leftrightarrow b) \\
\tag{109} \\
H_+ \mathcal{O}^a(x_a) \mathcal{O}^b(x_b) \mathcal{O}^c(x_c) \mathcal{O}^d(x_d) \prod_j \mathcal{O}(r_j) &= 0 \\
\tag{110} \\
\end{align*}
\[ H_{-} \mathcal{O}^{a}(x_{a})\mathcal{O}^{b}(x_{b})\mathcal{O}^{c}(x_{c})\mathcal{O}^{d}_{x_{d}} = -\frac{\lambda'}{8\pi^2} \]
\[ \int_{0}^{x_{b}} dy(\partial_{y} - \partial_{a})(\mathcal{O}^{ab}(x_{b} - y, x_{a} + y) + \mathcal{O}^{ab}(x_{a} + y, x_{b} - y))\mathcal{O}^{d}(x_{d})\mathcal{O}^{c}(x_{c}) \]
\[ + (11 \text{ other permutations of abcd}) \quad (111) \]

We point out that the elements not involving four-impurity states are very similar to those calculated in the study of three-impurity states in [25], the only difference being that there appears an additional single-impurity state which, like the zero-impurity states, is unaffected by \( H_{1} \). As discussed in the main text, we need only consider elements involving ‘4’ states. We now restrict our attention to the novel ‘4-4’, ‘4-31’ and ‘4-22’ matrix elements, and express these in the momentum-state basis.

The ‘4-4’ component of \( H_{+} \) may be obtained from the four permutations of the first line of eqn.(102). This may be written
\[ H_{+}\mid_{\text{4-4}}\mid_{n_{1}, n_{2}, n_{3}; r_{0}}^{1234} = -\frac{\lambda'}{8\pi^2} \frac{-2\pi i}{\sqrt{(r_{0} - s)^{3} s_{1} s_{2} s_{3}}} \sum_{n_{1}, n_{2}, n_{3}} \int_{x_{123} < r_{0}} d^{3}x \]
\[ \left\{ - (2n_{1} + n_{2} + n_{3}) \int_{0}^{x_{1}} ds \frac{1}{r_{0} - s} e^{2\pi i - m_{123}s} \right. \]
\[ + (n_{1} - n_{2}) \int_{0}^{x_{2}} ds \frac{1}{r_{0} - s} e^{2\pi i - m_{23}s} \]
\[ + (n_{2} - n_{3}) \int_{0}^{x_{3}} ds \frac{1}{r_{0} - s} e^{2\pi i - m_{3}s} \]
\[ + (n_{1} + n_{2} + 2n_{3}) \int_{r_{0} - x_{123}}^{r_{0} - x_{1}} ds \frac{1}{r_{0} - s} \right\} e^{2\pi i} \left( D_{1}x_{1} + D_{2}x_{12} + D_{3}x_{123} \right) \]
\[ \mid_{m_{1}, m_{2}, m_{3}; r_{0} - s}^{1234}\mid_{s}, \quad (112) \]

where \( D_{a} = \frac{n_{a}}{r_{0}} - \frac{m_{a}}{r_{0} - s} \). Here, we have not yet added states where the impurities 2, 3, 4 are permuted along with the three momenta. We can express this component of \( H_{+} \) in terms of the basis state [10] by adding these permutations. In this case, this involves adding five more terms, in which the impurities 2, 3, 4 are permuted, along with the momenta \( n_{1}, n_{2}, n_{3} \) and also the momenta \( m_{1}, m_{2}, m_{3} \). Performing the \( x \)-integrations and adding the above
permutations, our final result for the ‘4-4’ matrix element of $H_+$ is given by

$$\langle s | m_1, m_2, m_3; r_0 - s | H_+ | n_1, n_2, n_3; r_0 \rangle = \frac{\lambda'}{2\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \times \frac{m_1 D_1 + m_2 D_2 + m_3 D_3 + m_{123} D_{123}}{(r_0 - y)D_1 D_2 D_3 D_{123}}. \tag{113}$$

The other components of $H_+$ and $H_-$ are calculated in similar fashion using the continuum forms (102)-(111), and the results are given in the main text.

**Appendix II: Matrix Elements of $\Sigma$**

Here we consider matrix elements of $\Sigma$ in the momentum-state basis; these are found by first calculating simple two-point functions in the original discrete basis and then transforming. For the present calculation, we need to find the matrix elements of $\Sigma$ corresponding to those calculated in Appendix I for $H_+$ and $H_-$. Since $\Sigma$ is Hermitian in the momentum-state basis, we do not need to distinguish between trace-number increasing and decreasing parts (which other authors have labeled $\Sigma_+$ and $\Sigma_-$) since these are simply related by conjugation.

The ‘4-4’ component of $\Sigma$ is found by considering the correlator

$$\langle \bar{O}^{1234}_{q_1,q_2,q_3,q_4} - k \bar{O}_{p_1,p_2,p_3,p_4}^{1234} \rangle \tag{114}$$

which to genus-one order is given by

$$N^{p_{1234}+3}k(p_4 - k - 1)\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{p_3,q_3}\delta_{p_4,q_4}. \tag{115}$$

Taking the continuum limit, we replace $p$ and $q$ with $x$ and $y$; then transforming to the momentum-state basis requires a tedious procedure of integration similar to that used to calculate the elements of $H_+$ and $H_-$. The result is

$$\langle s | m_1, m_2, m_3; r_0 - s | \Sigma | n_1, n_2, n_3; r_0 \rangle = \frac{1}{\pi^4 \sqrt{(r_0 - s)^3 r_0^3}} \sin \left( \frac{\pi s n_1}{r_0} \right) \sin \left( \frac{\pi s n_2}{r_0} \right) \sin \left( \frac{\pi s n_3}{r_0} \right) \sin \left( \frac{\pi s n_{123}}{r_0} \right) \times \frac{1}{D_1 D_2 D_3 D_{123}} \times \frac{n_a}{r_0 - m_a} - \frac{m_a}{r_0 - s}. \tag{116}$$
Using the same method, the ‘4-31’ element may be found from the correlator
\[
\langle \bar{O}_{q_1,q_2,q_3}^{123} O_{p_1,p_2,p_3,p_4}^{1234} \rangle = N^{p_{1234}+3}(\min(p_3, q_3, p_4, q_4) + 1)\delta_{p_1,q_1}\delta_{p_2,q_2}\delta_{p_3+p_4,q_3+q_4},
\]
and the ‘4-22’ element from the correlator
\[
\langle \bar{O}_{q_1,q_2}^{12} O_{p_1,p_2,p_3,p_4}^{34} \rangle = N^{p_{1234}+3}(\min(p_2, q_2, p_4, q_4) + 1)\delta_{p_1,q_1}\delta_{p_3,q_3}\delta_{p_2+p_4,q_2+q_4}.
\]
The results are given in the main text.

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