Polarization-Induced Interaction as a Perturbation within Orbital Angular Momentum (OAM) Modes of an Optical Fiber

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Abstract: We investigate the role of polarization effects in orbital angular momentum modes of an optical fiber. Specifically, we revisit the removal of the degeneracy of the HE and the EH vector modes that exists within the weakly guiding approximation (WGA); within WGA, these two vector modes coalesce into a single spatial (scalar) orbital angular momentum (OAM) mode, defined by a common topological charge and associated with left or right circular polarization. While the polarization term of the vector wave equation acting as a Hermitian perturbation operator accounts for the splitting of the HE and the EH vector modes, its extension to explain the accompanying field correction, called the polarization-induced field, encounters hurdles; this field is characterized by a topological charge differing by two in magnitude. The problems are critically examined with a possible remedy and its implications.

1. Introduction

In the weakly guiding approximation (WGA), the propagation of orbital angular momentum (OAM) modes within an optical fiber is often described by a scalar wave equation, whose spatial solutions are independent of polarization [1–5]. However, when an externally prepared OAM mode is injected into a fiber, it couples to the vector OAM modes, $HE_{l+1,m}$ or $EH_{l-1,m}$ ($l > 1$) according as the polarization of the input mode is left circular (photon spin $S = h/(2\pi)$) or right circular ($S = -h/(2\pi)$). These vector OAM modes are the solutions of the vector wave equation, expressed in the exponential basis $e^{i\ell\theta}$ as opposed to the $\cos(l\theta)$, $\sin(l\theta)$ basis [1]; $h$ is the Planck’s constant; $l$ and $m$ are the topological charge and the radial index, respectively, of the input OAM mode. The vector modes have a slightly different propagation constant from each other as well as the scalar propagation constant. The vector wave equation includes a polarization-dependent term, or simply a polarization term, which accounts for these slight differences in propagation constants. The polarization term encompasses two effects, one attributable to the spin-orbit (SO) interaction and a second, independent of spin, to a contact interaction in close analogy with atomic physics, where it appears as part of relativistic corrections [3, 6, 7]. While both the SO and the contact interactions contribute to changes in propagation constant, it is the spin-orbit interaction that causes the breaking of the degeneracy between the OAM modes of the same topological charge but opposite spins (see, e.g. [3,4,8]); this degeneracy breaking gives rise to the nondegenerate $HE_{l+1,m}$ and $EH_{l-1,m}$ modes.

In this paper, we critically examine the nature of the polarization effects as a perturbation within the OAM modes of a fiber. In particular, we attempt to understand the presence of the polarization-induced OAM component of the vector modal field as a consequence of the polarization term treated as a perturbation. It is well known that the transverse component of the vector field within a fiber is composed of a modal field, which is the solution of the scalar wave equation (vector wave equation without the polarization term), plus a correction term to account for the absence of the polarization term. The latter field is polarization-induced and is of order $\Delta = (n_1^2 - n_2^2)/(2n_1^2)$, where $n_1$ and $n_2$ are the refractive indices of the core and the cladding, respectively [1]. More recently, it was shown that the transverse component of the vector OAM modal field is in general decomposable into two OAM components, one of which is characterized by a topological charge $l$ and the other by a topological charge $l \pm 2$ [10,11]. The second component is polarization induced and of order $\Delta$ (in the weakly guiding limit, where $n_1 \to n_2$, the parameter $\Delta \to 0$, and the polarization effects discussed above disappear). The question then arises if the presence of this polarization-induced component can be explained within the framework of perturbation theory in the same way as the propagation constant splitting between the vector modes and their scalar counterparts. A recent work [12] motivates this connection, but the treatment is incomplete.

In Section 2, we review the well-known polarization-induced splitting between the $HE_{l+1,m}$ and the $EH_{l-1,m}$ modes as a perturbative effect originating in the polarization term of the vector wave equation. Subsequently,
in Section 3, we extend this concept to explain the existence of the polarization-induced component, which is of topological charge, $l \pm 2$. We find that this extension runs into problems of a physical nature, and suggest a remedy with its ramifications. Section 4 is the summary.

2. Review of polarization effects as a perturbation

In what follows, we drop the radial index $m$ for convenience and clarity, and denote the fields of the $HE_{l\pm1}$ and $EH_{l-1}$ vector modes by $\tilde{\Phi}^{(l,\pm1)}$ and $\tilde{\epsilon}^{(l,-1)}$, respectively. The corresponding transverse fields, $\tilde{\epsilon}_i^{(l,\pm1)}$, satisfy the vector wave equation \[ (\nabla_i^2 + k^2 n^2(r))\tilde{\epsilon}_i^{(l,\pm1)} + \nabla^2_i(\tilde{\epsilon}_i^{(l,\pm1)}) \nabla_i(n^2)) = \beta_i^{(l,\pm1)} \tilde{\epsilon}_i^{(l,\pm1)}, \] where the third term on the left-hand-side is the polarization term; the transverse gradient $\nabla_i$ in cylindrical polar coordinates $(r, \theta, z)$ is $\nabla_i = \partial/\partial r \hat{r} + (1/r) \partial/\partial \theta \hat{\theta}$ and the transverse Laplacian $\nabla_i^2 = \partial^2/\partial r^2 + (1/r) \partial/\partial r + (1/r^2) \partial^2/\partial \theta^2$. The circular fiber is translationally invariant and further assumed to be axisymmetric. The square of the refractive index is given by $n^2(r) = n^2_0(1 - 2\Delta f(r))$, where $f(r)$ is the index profile, $\Delta = (n_2^2 - n_0^2)/(2n_1^2)$ is the profile height parameter, with $n_1$ the refractive index of the core and $n_2$ the refractive index of the cladding as in a step-index fiber or a ring-core fiber \[13, 14\]. As an example, $f(r)$ is a step-function for a step-index fiber, being equal to zero for $r \leq a$ and equal to 1 for $r > a$, where parameter $a$ is the fiber core radius; for the graded-index fibers, the index profile $f(r)$ is parabolic, being $r^2$ in the region $r \leq a$ and $r > a$. Parameter $\beta_{l,\pm1}$ is the propagation constant of the field propagating along the $z$ axis coincident with the fiber axis, and $k = 2\pi/\lambda$.

Because $\Delta << 1$, which is usually the case in commercial fibers, the expression $ln(1 - 2\Delta f(r)) \approx -2\Delta f(r)$, which then yields $\nabla_i(n^2) = -2\Delta \partial f(r)/\partial r$. Thus, the polarization term in Eq. 1 (in lowest order of $\Delta$) is proportional to $\Delta$. In the limit $n_1 \rightarrow n_2$, $\Delta \rightarrow 0$, implying the polarization effects disappear. Eq. 1 then reduces to the scalar wave equation:

\[ (\nabla^2 + k^2 n^2(r))\tilde{\epsilon}_i^{(l,\pm1)} = \beta_i^{(l,\pm1)} \tilde{\epsilon}_i^{(l,\pm1)}, \]

where

\[ \tilde{\epsilon}_i^{(l,\pm1)} = O_l(r, \theta) \tilde{\epsilon}_\pm. \]

The polarization-independent spatial amplitude $O_l(r, \theta)$ is given by

\[ O_l(r, \theta) = (1/\sqrt{N_l}) F_l(r)e^{i\theta}. \]

$\tilde{\epsilon}_\pm = 1/\sqrt{2}(\hat{\epsilon}_+ \pm i\hat{\epsilon}_-)$ are the left/right circularization corresponding respectively to (photon) spin values of $\pm 1$ in units of $\hbar/(2\pi)$. Eq. 3 constitutes an orbital angular momentum (OAM) modal field with total angular momentum $J(= L + S)$ equal to $(l \pm 1)\hbar/(2\pi)$ per photon according as the polarization is left circular (spin $S = +1$) or right circular (spin $S = -1$); $\tilde{\beta}_l$ is the corresponding scalar propagation constant, and $N_l$ is a normalization constant (the reduction of $\tilde{\epsilon}_i^{(l,\pm1)}$ to the expression, Eq. 3 is explicitly shown for the step-index fiber in \[10\]).

Because the polarization term is of order $\Delta$ in Eq. 1, we expect the polarization correction, $\delta \beta_{l,\pm1} = \beta_{l,\pm1} - \tilde{\beta}_l^2$ and the field correction, $\delta \tilde{\epsilon}_i^{(l,\pm1)} = \tilde{\epsilon}_i^{(l,\pm1)} - \tilde{\epsilon}_i^{(l,\pm1)}$ to be each of order $\Delta$ in the first order of perturbation. Therefore, we replace $\tilde{\epsilon}_i^{(l,\pm1)}$ in Eq. 1 with $\tilde{\epsilon}_i^{(l,\pm1)}$ (the unperturbed solution) and substitute Eq. 2 in Eq. 1. We obtain

\[ (\nabla^2 + k^2 n^2(r))\tilde{\epsilon}_i^{(l,\pm1)} = \nabla^2_i(\tilde{\epsilon}_i^{(l,\pm1)}) \nabla_i(n^2)) = \beta_i^{(l,\pm1)} \tilde{\epsilon}_i^{(l,\pm1)} - \beta_i^{(l,\pm1)} \tilde{\epsilon}_i^{(l,\pm1)}. \]

Taking the scalar product of the above with $\tilde{\epsilon}_i^{(l,\pm1)}$, the polarization correction is

\[ \delta \beta_{l,\pm1} = \beta_{l,\pm1} - \tilde{\beta}_l^2 = \int_0^\infty e^{-\rho^2} \tilde{\epsilon}_i^{(l,\pm1)} \nabla_i n^2drd\theta, \]

where $\tilde{\epsilon}_i^{(l,\pm1)}$ is the complex conjugate of $\tilde{\epsilon}_i^{(l,\pm1)}$. We now substitute Eqs. 3, 4, and the expression $\tilde{\epsilon}_\pm = 1/\sqrt{2}(\hat{\epsilon}_+ \pm i\hat{\epsilon}_-)$, $\tilde{\epsilon}_\pm = 1/\sqrt{2}(\hat{\epsilon}_+ \pm i\hat{\epsilon}_-)$ in polar coordinates into Eq. 5. Noting further that $n^2 = n^2(r)$ and fields vanish at the end points of the integral, $r = 0$ and $r = \infty$, we find after performing integration by parts that

\[ \delta \beta_{l,\pm1} = \Delta \int_0^\infty e^{-\rho^2} \tilde{\epsilon}_i^{(l,\pm1)} \nabla_i n^2drd\theta = \frac{2\pi\Delta}{N_l} \int_0^\infty e^{-\rho^2} \tilde{\epsilon}_i^{(l,\pm1)} \nabla_i n^2drd\theta, \]

from which $\delta \beta_{l,\pm1} \approx \beta_{l,\pm1}^2/(2\kappa n_l)$ can be computed. $N_l = 2\pi \int_0^\infty F_l^2(r)ldr$ is the normalization constant (see also \[1\] in the context of Linearly-Polarized (LP) modes). $\delta \beta_{l,\pm1}^2$ is of order $\Delta$, as expected. The first term in
Eq. 6 is identified with the contribution from the contact interaction (this designation arises from atomic physics, where the reduction of the Dirac Hamiltonian for the electron in a central field in the nonrelativistic case yields, in addition to the spin-orbit interaction term, an unknown nonclassical term, later identified with the contact interaction between an electron and the atomic nucleus [6, 7]). This contribution is independent of the spin alignment with the OAM, but can be very large, as demonstrated in the case of a step-index fiber [9]. The second term in Eq. 6 pertains to the spin-orbit interaction, which has a contribution equal in magnitude but opposite in sign for the spin-orbit alignment \((l, 1)\) and nonalignment \((-l, -1)\) cases. The same result is reproduced for the degenerate states, characterized by \(l\) and \(S\) values of opposite signs, i.e., \(\delta \beta_2^l, \mp 1 = \delta \beta_2^l, \pm 1\).

2.1. Polarization term as a perturbation operator

The result of Eq. 5 can be recast as a matrix element

\[
\delta \beta_2^{l, \pm 1}_l = \langle \hat{e}_l^{(l, \pm 1)} | V | \hat{e}_{l}^{(l, \pm 1)} \rangle > ,
\]

where the operator \(V\) represents the perturbative effects of the polarization term in Eq. 1. Subsequently, entire Eq. 1 in conjunction with Eqs. 2 and 7 may be expressed as

\[
\langle \hat{e}_2^{(l, \pm 1)} | (H_0 + V) | \hat{e}_{2}^{(l, \pm 1)} \rangle > = \hat{\beta}_2^l + \delta \beta_2^{l, \pm 1}_l ,
\]

where \(H_0 = \nabla_x^2 + k^2 n^2(r)\); the same result holds for the degenerate OAM-spin combinations: \((-l, -1)\) and \((-l, +1)\). The operator \(H_0 + V\) is Hermitian (as expected) in the space defined by the four OAM states with the OAM-spin combinations: \((l, 1), (l, -1), (-l, 1), (-l, -1)\). The off-diagonal elements of the 4 x 4 matrix in this space are each zero; for example, \(\delta \beta_2^{l, -1}_l = \langle \hat{e}_2^{-1, \pm 1}_l | V | \hat{e}_{2}^{l, \pm 1} \rangle > \) given by

\[
\int_0^{2\pi} \int_0^\infty \hat{e}_2^{-1, \pm 1}_l \hat{e}_2^{l, \pm 1} (r, \theta) \hat{e}_2^{l, \pm 1} (r, \theta)\rangle drd\theta ,
\]

following Eq. 5, evaluates to zero, and so do others. The 4 x 4 \(H_0 + V\) matrix is therefore diagonal; each diagonal element corresponds to a different value of the total angular momentum, \(J = L + S\).

3. Extension of Perturbation Theory to Include the Polarization-Induced Fields

We noted above that the polarization effects also induce a field correction, \(\delta \hat{e}_2^{(l, \pm 1)}\). This field correction is a state of topological charge, \(l \pm 2\), associated with polarization, \(\hat{e}_2\). This is easily seen from a generic form of the transverse vector modal field of Eq. 1: \(\hat{e}_2^{(l, \pm 1)} = (e_2^{(l, \pm 1)}(r) \hat{r} + e_2^{(l, \pm 1)}(r) \hat{\theta}) e^{i(l+1)\theta}\), which, upon substitution of \(\hat{r} = \hat{x} \cos \theta + \hat{y} \sin \theta\) and \(\hat{\theta} = -\hat{x} \sin \theta + \hat{y} \cos \theta\), becomes

\[
\hat{e}_2^{(l, \pm 1)} = (1/\sqrt{2}) \left[ \hat{e}_2^{(l, \pm 1)} e^{i\theta} (e_2^{(l, \pm 1)}(r) + e_2^{(l, \pm 1)}(r)) + \hat{e}_2^{(l, \pm 2)} e^{i2\theta} (e_2^{(l, \pm 1)}(r) \mp e_2^{(l, \pm 1)}(r)) \right].
\]

The field has two components. The second component (of opposite polarization, \(\hat{e}_2\)) is polarization induced and of order \(\Delta\), as discussed in Section 2. Therefore, in the limit \(n_1 \rightarrow n_2\), this component goes to zero while the first component reduces to the traditional scalar mode (see Eq. 3). A detailed analysis of Eq. 9 for a multilayered fiber [10] reveals the novel result that the second term corresponds to a modified scalar mode of topological charge \(l \pm 2\), but for now we will work within the framework of perturbation theory based on the traditional OAM modes of Eqs. 3 and 4 (solutions of the scalar wave equation, Eq. 2). Function \(F_l(r)\) in Eq. 4 is the familiar Bessel function for a step-index fiber and a combination of appropriate Bessel functions in the case of the ring-core fiber [7, 13, 14]. To extend the above perturbative model into the space incorporating the states \((l \pm 2, \hat{e}_2)\), we must consider matrix elements such as \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \) and \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \) for the \(J = L + S = l + 1\) case; and similarly, \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \) and \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \) for the \(J = L + S = l - 1\) case; integrations over \(r\) and \(\theta\) are performed as in Eq. 5 to evaluate them.

In the above expanded space and in particular its subspace spanned by the two OAM states: \((l, 1)\) and \((l + 2, -1)\), each corresponding to total angular momentum, \(J = l + 1\), we now consider the four matrix elements: \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \), \(\langle \hat{e}_2^{(l+2, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l+2, \pm 1)} \rangle > \), \(\langle \hat{e}_2^{(l+2, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l, \pm 1)} \rangle > \), and \(\langle \hat{e}_2^{(l, \pm 1)} | H_0 + V | \hat{e}_{2}^{(l+2, \pm 1)} \rangle > \). The matrix elements respectively equal \(\hat{B}_2^l + \delta \beta_2^l, \hat{B}_2^{l+2} + \delta \beta_2^{l+2}, \delta \beta_2^{l+2, l}, \delta \beta_2^{l+2, l+2}\) (in this abbreviated notation, where we suppress spin \(S\), index \(l\) is equivalent to \((l, 1)\), and index \(l + 2\) is equivalent to \((l + 2, -1)\)). The operator \(H = H_0 + V\) is then represented by the 2 x 2 matrix

\[
H = \begin{bmatrix}
\hat{B}_2^l + \delta \beta_2^l & \delta \beta_2^{l+2, l} \\
\delta \beta_2^{l+2, l+2} & \hat{B}_2^{l+2} + \delta \beta_2^{l+2, l+2}
\end{bmatrix}.
\]
The above matrix is non-Hermitian because

\[
\delta \beta_{i+2,l}^{(l+2,-1)} = \left< \tilde{e}_i^{(l+2,-1)} \right| \beta_{i+2}^{(l+2,-1)} \rangle = \int_0^\infty \int_0^{2\pi} \tilde{e}_i^{(l+2,-1)} \cdot \tilde{\phi}_{i+2}^{(l+2,-1)} \cdot \frac{dF_{i+2}}{dr} \left( \frac{l+2}{r} F_{i+2} \right) rdrd\theta
\]

\[
= 2\pi \Delta \frac{\sqrt{N_{i+2} N_{i+3}}}{\sqrt{N_{i+2} N_{i+3}}} \int_0^\infty \frac{df}{dr} F_{i+2} \left( \frac{l+2}{r} F_{i+2} \right) rdr;
\]

is not equal to

\[
\delta \beta_{i+2,l}^{(l+2,-1)} = \left< \tilde{e}_i^{(l+2,-1)} \right| \beta_{i+2}^{(l+2,-1)} \rangle = \int_0^\infty \int_0^{2\pi} \tilde{e}_i^{(l+2,-1)} \cdot \tilde{\phi}_{i+2}^{(l+2,-1)} \cdot \frac{dF_{i+2}}{dr} \left( \frac{l+2}{r} F_{i+2} \right) rdrd\theta
\]

\[
= 2\pi \Delta \int_0^\infty \frac{df}{dr} F_{i+2} \left( \frac{l+2}{r} F_{i+2} \right) rdr;
\]

the integrations here follow the same steps as in Eq. 5. Same conclusion is reached for \( J = l - 1 \) in a subspace spanned by the two OAM states; \((l, -1)\) and \((l - 2, 1)\). This feature of the \( H \) matrix is highly undesirable as it leads to the violation of the orthogonality of the eigenstates. In the next section, we explicitly demonstrate the nonorthogonality of the eigenstates of \( H \) and point out the related unphysical implications.

### 3.1. Consequences of the non-Hermitian matrix \( H \)

We examine the orthogonality of the eigenstates of matrix \( H \). Its diagonalization yields the eigenvalues: \( \lambda_1 \approx \beta_1^2 + \kappa^2 / \beta_D^2 \), \( \lambda_2 \approx \beta_2^2 - \kappa^2 / \beta_D^2 \), where \( \kappa^2 = \delta \beta_{i+2,l}^{(l+2,-1)} / \beta_{i+2,l}^{(l+2,-1)} \), \( \beta_1^2 = \beta_1^2 + \delta \beta_{i+2,l}^{2} + \delta \beta_{i+2,l}^{2} / \beta_D^2 \), \( \beta_2^2 = \beta_2^2 - \delta \beta_{i+2,l}^{2} - \delta \beta_{i+2,l}^{2} / \beta_D^2 \). The eigenstate corresponding to eigenvalue \( \lambda_1 \) is a 2 x 1 column vector: \( \psi_1 = [1 \ \delta \beta_{i+2,l}^2 / \beta_D^2]^T \), representing the field, \( \tilde{e}_i^{(l+1,1)} \). In terms of the field notation,

\[
\tilde{e}_i^{(l+1,1)} = O_i e_+ + \left( \frac{\delta \beta_{i+2,l}^2}{\beta_1^2 - \beta_2^2} \right) O_{i+2} e_-.
\]

Similarly, the eigenstate corresponding to the eigenvalue \( \lambda_2 \) is found to be \( \psi_2 = [-\delta \beta_{i+2,l}^2 / \beta_D^2]^T \), which translates to

\[
\tilde{e}_i^{(l+2,-1)} = O_{i+2} e_- - \left( \frac{\delta \beta_{i+2,l}^2}{\beta_1^2 - \beta_2^2} \right) O_i e_+.
\]

The scalar product of the fields in Eqs. 13 and 14

\[
\left< \tilde{e}_i^{(l+2,-1)} \right| \tilde{e}_i^{(l+1,1)} > = \frac{\delta \beta_{i+2,l}^2 - \delta \beta_{i+2,l}^2}{\beta_1^2 - \beta_2^2} \neq 0
\]

because \( \delta \beta_{i+2,l}^2 \neq \delta \beta_{i+2,l}^2 \) as seen in Section 3.

The above result violates the condition of orthogonality of the modes in a fiber. Physically, this implies that an \((l, 1)\) mode and an \((l + 2, -1)\) mode injected into an ideal, straight fiber (with no imperfections) will not travel as independent states, but couple into each other. This has the undesired and misleading implication of crosstalk between two independent modes, \( HE_{l+1} \) and \( EH_{l+1} \) in an optical fiber.

Similarly, in the \( J = l - 1 \) case, composed of states, \( O_i e_- \) and \( O_{i+2} e_+ \), we obtain the results:

\[
\tilde{e}_i^{(l-1,1)} = O_i e_- + \left( \frac{\delta \beta_{i+2,l}^2}{\beta_1^2 - \beta_2^2} \right) O_{i-2} e_+.
\]

\[
\tilde{e}_i^{(l-2,1)} = O_{i-2} e_+ - \left( \frac{\delta \beta_{i+2,l}^2}{\beta_1^2 - \beta_2^2} \right) O_i e_-.
\]

The scalar product here also not equal to zero because \( \delta \beta_{i+2,l}^2 \neq \delta \beta_{i+2,l}^2 \). This leads to the spurious result that the modes \( EH_{l+1} \) and \( HE_{l+1} \) mix within a fiber. Prior works [12] are incomplete as they overlook the issue of non-Hermiticity and the associated problem of non-orthogonality and its erroneous implications, as demonstrated here.
3.2. Remedy for the non-Hermitian matrix $H$

A way to get around the above difficulty (that is, to preserve orthogonality) might be to rewrite the non-Hermitian matrix $H$ defined in the space spanned by the states: $(l, 1), (l + 1, -1)$ of the same $J$ value, $l + 1$, as

$$H = M + R$$

where

$$M = (H + H^\dagger)/2$$

is a Hermitian matrix and

$$R = (H - H^\dagger)/2$$

is an anti-Hermitian matrix (with diagonal elements equal to zero). We subsequently consider the eigenstates of $M$, treating the matrix $R$ as an error matrix.

Matrix $M$ has each of its two off-diagonal elements equal to $\delta\beta^2_{l,l+2} = (\delta\beta^2_{l,l+2} + \delta\beta^2_{l+2,l})/2$. The eigenstates of $M$ have the same expression as those in Eqs. 13 and 14, except that $\delta\beta^2_{l,l+2}$ and $\delta\beta^2_{l+2,l}$ are each replaced with their mean value, $\delta\beta^2_{l,l+2}$. Consequently, the right-hand-side of Eq.15 now equals zero, as expected for the eigenstates of a Hermitian matrix. For the $J = l - 1$ case, where the space is spanned by the states, $(l, -1), (l - 2, 1)$, similar results obtain.

Hitherto, we had suppressed the radial index $m$. We assume now that index $m$ equals 1 for the $(l, \pm 1)$ state, while the state $(l \pm 2, \mp 1)$ is permitted to have any value $(1, 2, \ldots p)$ for the radial index denoted $m'$ (the derivation above tacitly assumed one specific value of $m'$). Allowing for more than one value of $m'$ leads to the expansion of the $2 \times 2$ matrix $H$ into a $(p + 1) \times (p + 1)$ matrix, where $p$ denotes the maximum permitted value of $m'$; parameter $p$ is fiber dependent. It can then be shown that the eigenstates specified in Eqs. 13 and 14 and Eqs. 16 and 17 involve a summation over $m'$ on the second term of the right hand side from $m' = 1$ to $m' = p$. This summation over $m'$ is consistent with a perturbation series expansion over a complete set of states, labeled by $m'$ for fixed topological charge. Note further, in Eqs. 11 and 12, $F_l$ will correspond to $m = 1$ and $F_{l+2} \rightarrow f_{l+2}^{(m')}$ will be $m'$ dependent.

3.2.1. Error matrix $R$

The matrix $R$ in Eq. 18 may be deemed an error matrix (Eq. 20) as it represents the deviation from full Hermiticity now captured in the Hermitian matrix $M$. More specifically, error $E$ may be defined as the ratio of the magnitude of the off-diagonal element of the $R$ matrix ($= (\delta\beta^2_{l,l+2} - \delta\beta^2_{l+2,l})/2$) to the magnitude of the diagonal element of the $M$ matrix ($= (\delta\beta^2_{l,l+2} + \delta\beta^2_{l+2,l})/2$). For the step-index fiber, where $n^2(r) = n_1^2[1 - 2\Delta f(r)]$ and $f(r)$ is a step-function at $r=a$, where $a$ is the radius of the fiber core (see Section 2), we can easily evaluate the matrix elements, noting that $\partial f(r)/\partial r = \delta(r-a)$, and $F_l(r)$ is the familiar Bessel function of the first kind, $J_l(pr)$; parameter $p_1 = \sqrt{k^2 n_1^2 - \beta^2}$, where $\beta$ is determined from the characteristic equation [1]. For fiber parameters, $a = 25\mu m$, $n_1 = 1.461, n_2 = 1.444$, and wavelength $\lambda = 1.55\mu m$, we find the error $E$ to increase from 10 to 25% as $l$ is increased from 1 to 4 for $m = 1$ and the single case of $m' = 1$. We further find that numerically $\delta\beta^2_{l+2,l}$ in Eq. 10 is of the same sign as $\delta\beta^2_{l,l+2}$, but consistently larger in magnitude, irrespective of the value of $l$ and $m' (\leq p)$, implying matrix $H$ is always non-Hermitian. The same holds true in the $J = l + 1$ case.

3.3. Discussion

The space spanned by the four states: $\pm l, \pm 1$ (in the $L, S$ notation) has been considered by a number of authors in the past in the context of polarization-induced corrections to the propagation constants of the scalar OAM modes (see, e.g., [3, 4, 8], and also [1] in the context of LP modes). The reason the perturbation formulation fails here is that venturing outside this space triggers a non-Hermitian matrix because the off-diagonal elements like $<\psi_{l+2}^{(1)}|\psi_{l+2}^{(1)}>$ and $<\psi_{l+2}^{(1)}|\psi_{l+2}^{(2)}>$ in the case of $J = l + 1$, are not equal (see Eqs. 11 and 12 above). Casting the polarization term of Eq. 1 as a polarization interaction operator, $V$, appears to be applicable only to the space spanned by the above four OAM modes (with a fixed magnitude of topological charge), where the propagation constants of the individual modes (of the same topological charge) are changed due to different polarizations ($\tilde{E}_x$), in close analogy to atomic physics where the energy levels are changed on account of $\tilde{L}, \tilde{S}$ coupling. However, its extension to generate the polarization-induced field component (see Eqs. 9, 18, and 19) fails as it leads to misleading conclusions and numerical errors that can be significant. In other words, polarization-induced interaction, strictly speaking, is a perturbation internal to an OAM mode within a fiber. While it successfully accounts for
the modal propagation splitting due to the spin orbit interaction, it cannot be treated as an external perturbation acting on the (primary) OAM field of the vector mode to generate the accompanying (secondary) polarization-induced OAM component. In fact, starting from Eq. 9, it is explicitly shown in [10] that the second component in Eq. 9 in fact corresponds to a modified OAM field, where the intensity pattern of the traditional OAM mode, e.g., the radius of the intensity donut ring, expands or shrinks, depending upon whether the polarization is aligned or antialigned with the OAM.

4. Summary

We have reviewed the inherent polarization-induced interaction as a perturbation to explain the differences in the propagation constants between the vector OAM modes $HE_{l+1,m}$ and $EH_{l-1,m}$ in an optical fiber. The transverse fields are eigenstates of the total angular momentum operator, $J = l \pm 1$. Similar results hold for the degenerate vector modes, $HE_{l-1,m}$ and $EH_{l+1,m}$, which correspond to $J = -l \mp 1$. Mathematically, the propagation constant differences between the vector modes can be accounted for by treating the polarization term within the vector wave equation as a perturbation defined in the space of the four OAM states: $(\pm l, \pm 1)$. However, its extension as a perturbation into the expanded space incorporating the polarization-induced OAM states $(l \pm 2, \mp 1)$ generates a non-Hermitian matrix with serious, misleading and physically incorrect implications. We conclude therefore that polarization interaction acts internally within a given vector mode to generate the change in the propagation constant, but cannot strictly be treated as a perturbation operator to account for the polarization-induced field (the secondary component of the vector OAM modal field). Instead, a straightforward, direct analysis of the analytic expression for the polarization induced field reveals it to be a modified version of the traditional scalar OAM field [10].

References

1. Allan W. Snyder and John D. Love, *Optical Waveguide Theory* Chapman and Hall (1983).
2. A.V. Dooghin, N.D. Kindikova, V.S. Lieberman, and B. Ya. Zel’dovich, "Optical Magnus Effect", Phys. Rev. 45, 8204-8208 (1992).
3. A.V. Volyar, V. Z. Zhulaitis, and V. G. Shvedov, "Optical Eddies in Small-Mode Fibers: II. The Spin-Orbit Interaction", Optics and Spectroscopy 86, 593-598 (1999).
4. C.N. Alexeyev, M.S. Soksin, and A.V. Volyar, "Spin-orbit interaction in a generic vortex field transmitted through an elliptic fiber", Semiconductor Phys. Quantum Electron. Optoelectron.3, 501-513 (2000).
5. R. Bhandari, "Orbital Angular Momentum (OAM) Mode Mixing in a Bent Step Index Fiber", IEEE Photonics Journal, vol. 11, no. 3, art. no. 7203421, June 2019 (DOI:10.1109/JPHOT.2019.2920097).
6. A. S. Davydov, *Quantum Mechanics*, Pergamon Press (1965)
7. L.I. Schiff, *Quantum Mechanics*, McGraw Hill (1955)
8. D.L.P. Vitullo, C. C. Leary, P. Gregg, R. A. Smith, D. V. Reddy, S. Ramachandran, and M. G. Raymer, "Observation of Interaction of Spin and Intrinsic Orbital Angular Momentum of Light", Phys. Rev. Lett. 118, 083601 (2017).
9. B. Choudhary, "Spin Orbit and Contact interactions in Orbital Angular Momentum Modes in a Fiber", Proc. of OSA-Frontiers in Optics/Laser Science, September 2019, Paper JW4A.122.
10. Ramasesh Bhandari, "Nature of the orbital angular momentum (OAM) fields in a multilayered fiber," OSA Continuum 4, 1859-1874 (2021)
11. S. Ramachandran, P. Gregg, P. Kristensen and S.E. Golowich, "On the Scalability of Ring Fiber Designs for OAM Multiplexing", Optics Exp, 23, 3721-3730, February, 2015.
12. P. Gregg, P. Kristensen, A. Rubano, S. Golowich, L. Marunucci, and S. Ramachandran, "Enhanced Spin Orbit Interaction of Light in Highly Confining Optical Fibers for Mode Division Multiplexing", Optical Fiber Technology, Oct 2019; https://doi.org/10.1016/j.olfoto.2019.12401-4.
13. B.C. Sarkar, P.K. Choudhary, and T. Yoshino, "On the analysis of a weakly guiding doubly clad dielectric optical fiber with an annular core", Microw. Opt. Techn. vol. 31, no. 6, pp. 435-439, Dec. 2001.
14. C. Brunet, B. Ung, P. Bélanger, Y. Messaddeq, S. LaRochelle and L. A. Rusch, "Vector Mode Analysis of Ring-Core Fibers: Design Tools for Spatial Division Multiplexing," in Journal of Lightwave Technology, vol. 32, no. 23, pp. 4648-4659, 1 Dec.1, 2014, doi: 10.1109/JLT.2014.2361432.