Multi-Round Influence Maximization (Extended Version)

Lichao Sun¹, Weiran Huang², Philip S. Yu¹,², Wei Chen³,*
¹University of Illinois at Chicago, ²Tsinghua University, ³Microsoft Research
lsun29@uic.edu, huang.inbox@outlook.com, psyu@uic.edu, weic@microsoft.com

ABSTRACT
In this paper, we study the Multi-Round Influence Maximization (MRIM) problem, where influence propagates in multiple rounds independently from possibly different seed sets, and the goal is to select seeds for each round to maximize the expected number of nodes that are activated in at least one round. MRIM problem models the viral marketing scenarios in which advertisers conduct multiple rounds of viral marketing to promote one product. We consider two different settings: 1) the non-adaptive MRIM, where the advertiser needs to determine the seed sets for all rounds at the very beginning, and 2) the adaptive MRIM, where the advertiser can select seed sets adaptively based on the propagation results in the previous rounds. For the non-adaptive setting, we design two algorithms that exhibit an interesting tradeoff between efficiency and effectiveness: a cross-round greedy algorithm that selects seeds at a global level and achieves $1/2 - \varepsilon$ approximation ratio, and a within-round greedy algorithm that selects seeds round by round and achieves $1 - e^{-(1-1/\varepsilon)} - \varepsilon \approx 0.46 - \varepsilon$ approximation ratio but saves running time by a factor related to the number of rounds. For the adaptive setting, we design an adaptive algorithm that guarantees $1 - e^{-(1-1/\varepsilon)} - \varepsilon$ approximation to the adaptive optimal solution. In all cases, we further design scalable algorithms based on the reverse influence sampling approach and achieve near-linear running time. We conduct experiments on several real-world networks and demonstrate that our algorithms are effective for the MRIM task.

CCS CONCEPTS
• Information systems → Social advertising; Social networks; • Theory of computation → Probabilistic computation; Submodular optimization and polymatroids;

KEYWORDS
Influence maximization, triggering model, greedy algorithm

1 INTRODUCTION
Most companies need to advertise their products or brands on social networks, through paying for influential people (seed nodes) on Twitter, with the hope that they can promote the products to their followers [1]. The objective is to find a set of most influential people with limited budget for the best marketing effect. Influence maximization (IM) is the optimization problem of finding a small set of most influential nodes in a social network that generates the largest influence spread. It models viral marketing scenario in social networks [2–4], and can also be applied to cascade detection [5], rumor control [6, 7], etc. The standard IM problem and a number of its variants has been extensively studied (c.f. [8]). In most formulations, the IM is formulated as a one-shot task: the seed set is selected by the algorithm at the beginning, and one propagation pass from the seed set activates a subset of nodes in the network. The objective is to maximize the expected number of activated nodes in this one propagation pass. However, in many practical viral marketing scenarios, an advertiser’s viral marketing campaign may contain multiple rounds to promote one product. Each round may be initiated from a different set of influential nodes. The advertiser would like to maximize the total number of users adopting the product over all rounds.

We model the above scenario by the multi-round diffusion model and multi-round influence maximization (MRIM) task. We consider the entire process over $T$ rounds. For each round $t$, an independent diffusion is carried out starting from seed set $S_t$, and a random set of nodes, $I(S_t)$, is activated. Then the total influence spread over $T$ rounds given seed sets $S_1, \ldots, S_T$, $\rho(S_1, \ldots, S_T)$, is the expected total number of activated nodes while considering all rounds together, namely $\rho(S_1, \ldots, S_T) = \mathbb{E}[\bigcup_{t=1}^{T} I(S_t)]$. Note that a node activated in a previous round may be activated again and propagate influence to other nodes in a new round, but it will not be counted again in the final influence spread. The MRIM task is to find seed sets $S_1, \ldots, S_T$, each of which has at most $k$ nodes, so that the final influence spread $\rho(S_1, \ldots, S_T)$ is maximized.

The MRIM task possesses some unique features different from the classical IM task. For example, in the classical IM, it makes no sense to select one seed node multiple times, but for MRIM, it may be desirable to select an influential node in multiple rounds to generate more influence. Moreover, it enables adaptive strategies where the advertiser to adaptive select seeds in the next round based on the propagation results of previous rounds.

We study both the non-adaptive and adaptive versions of MRIM under the Multi-Round Triggering (MRT) model. For the non-adaptive MRIM problem, we design two algorithms that exhibit an interesting tradeoff between efficiency and effectiveness. The first cross-round greedy algorithm selects seeds globally cross different rounds, and achieves an approximation ratio of $1/2 - \varepsilon$, for any $\varepsilon > 0$. The second within-round greedy algorithm selects seeds round by round and achieves an approximation ratio of $1 - e^{-(1-1/\varepsilon)} - \varepsilon \approx 0.46 - \varepsilon$. The higher approximation ratio enjoyed by the cross-round greedy algorithm is achieved by investigating seed candidates in all rounds together in each greedy step, and thus incurs a higher running time cost at a factor proportional to the number of rounds. For the adaptive MRIM problem, we rigorously formulate the problem according to the adaptive optimization framework specified in [9]. We show that our formulation satisfies the adaptive submodularity defined in [9]. Based on the adaptive submodularity, we design the AdaGreedy algorithm that achieves $1 - e^{-(1-1/\varepsilon)} - \varepsilon$ approximation to the optimal adaptive policy.

For both the non-adaptive and adaptive cases, we greatly improve the scalibility by incorporating the state-of-the-art reverse
influence sampling (RIS) approach [10–12]. In each case, the RIS method needs to be carefully revised to accommodate to the multi-round or adaptive situation. In all cases, we show that our scalable algorithms could achieve near-linear running time with respect to the network size, greatly improving the corresponding Monte Carlo greedy algorithms.

To demonstrate the effectiveness of our algorithms, we conduct experiments on real-world social networks, with both synthesized and learned influence parameters. Our experimental results demonstrate that our algorithms are more effective than baselines, and our scalable algorithms run in orders of magnitude faster than their Monte Carlo greedy counterparts while keeping the influence spread at the same level. The results also show some interesting findings, such as the non-adaptive cross-round algorithm could achieve competitive influence spread as the adaptive greedy algorithm. This may suggest that in practice one may need to consider whether spending the cost of collecting feedbacks and doing near-term adaptive strategies based on feedbacks, or spending the upfront cost to do more global planning, and it opens new directions for further investigations.

To summarize, our contributions are: (a) proposing the study of both non-adaptive and adaptive MRIM problems; (b) proposing non-adaptive and adaptive greedy algorithms and showing their trade-offs; (c) designing scalable algorithms in both non-adaptive and adaptive settings; and (d) conducting experiments on real-world networks to demonstrate the effectiveness and the scalability of our proposed algorithms. Due to space constraints, all proofs and supplementary materials are shown in the appendix.

1.1 Related Work

Influence maximization is first studied by Domingos and Richardson [2, 3], and then formulated as a discrete optimization problem by Kempe et al. [4], who also formulate the independent cascade model, the linear threshold model, the triggering model, and provide a greedy approximation algorithm based on submodularity. Since then, a significant number of papers studies improving the efficiency and scalability of influence maximization algorithms [10–16]. At this front, the state of the art is the reverse influence sampling (RIS) approach [10–12], and the IMM algorithm of [12] is among the most competitive ones so far. Our scalable algorithms are based on IMM, but require careful redesigns in the reverse sampling method. Other studies look into different variants, such as community, competitive and complementary influence maximization [6, 7, 17–20], adoption maximization [21], robust influence maximization [22, 23], etc.

Another related work is adaptive seeding [24], which uses the first-stage nodes and their accessible neighbors together as the seed set to maximize the influence, and is quite different from ours. In terms of the multi-round diffusion model and influence maximization, Lin et al. [25] focus on the multi-party influence maximization where there must be at least two parties to compete in networks. Lei et al. [26] use the same formulation of the multi-round diffusion model and the influence maximization objective as in our paper. They focus on the online learning aspect of learning edge probabilities, while we study the offline non-adaptive and adaptive maximization problem when the edge probabilities are known.

Without a rigorous study of such offline problems, it is very difficult to assess the performance of online learning algorithms, and as a result they could only propose heuristic learning algorithms without any theoretical guarantee. From this perspective, our study fills this important gap by providing a solid theoretical understanding of the offline multi-round influence maximization.

There are also a number of studies on the influence maximization bandit problem [27–30]. Their formulations also have multi-rounds, but each round has a separate influence maximization instance, and the total reward is a simple count of activated nodes in all rounds, so one node activated in multiple rounds will be counted multiple times. This makes it qualitatively different from our formulation. Moreover, these studies focus on the online learning aspects of such repeated influence maximization tasks.

The adaptive MRIM study follows the adaptive optimization framework defined by Golovin and Krause [9]. They also study adaptive influence maximization as an application, but the adaptation is at per-node level: seeds are selected one by one. Later seed can be selected based on the activation results from the earlier seeds, but the earlier seeds would not help propagation again for later seeds. This makes it different from our multi-round model.

2 MODEL AND PROBLEM DEFINITION

2.1 Multi-Round Diffusion Model

In this paper, we focus on the well-studied triggering model [4] as the basic diffusion model. A social network is modeled as a directed graph $G = (V, E)$, where $V$ is a finite set of vertices or nodes, and $E \subseteq V \times V$ is the set of directed edges connecting pairs of nodes. The diffusion of information or influence proceeds in discrete time steps $\tau = 0, 1, 2, \ldots$. At time $\tau = 0$, the seed set $S_0$ is selected to be active, and also each node $v$ independently chooses a random triggering set $T(v)$ according to some distribution over subsets of its in-neighbors. At each time $\tau \geq 1$, an inactive node $v$ becomes active if at least one node in $T(v)$ is active by time $\tau - 1$. The diffusion process ends when there is no more nodes activated in a time step. We remark that the classical Independent Cascade (IC) model is a special case of the triggering model. In the IC model, every edge $(u, v) \in E$ is associated with a probability $p_{uv} \in [0, 1]$, and $p_{uv}$ is set to zero if $(u, v) \notin E$. Each triggering set $T(v)$ is generated by independently sampling $(u, v)$ with probability $p_{uv}$ and including $u$ in $T(v)$ if the sampling of $(u, v)$ is successful.

The triggering model can be equivalently described as propagations in live-edge graphs. Given a class of triggering sets $\{T(v)\}_{v \in V}$, we can construct the live-edge graph $L = (V, E(L))$, where $E(L) = \{(u, v) \mid v \in V, u \in T(v)\}$, and each edge $(u, v) \in L$ is called a live edge. It is easy to see that the propagation in the triggering model is the same as the deterministic propagation in the corresponding live-edge graph $L$: if $A$ is the set of active nodes at time $\tau - 1$, then all directed out-neighbors of nodes in $A$ will be active by time $\tau$. An important metric in any diffusion model is the influence spread, the expected number of active nodes when the propagation from the given seed set $S_0$ ends, denoted as $\sigma(S_0)$. Let $\Gamma(G, S)$ denote the set of nodes in graph $G$ that can be reached from the node set $S$. Then, by the above equivalent live-edge graph model, we have $\sigma(S_0) = \mathbb{E}[|\Gamma(L, S_0)|]$, where the expectation is taken over the distribution of live-edge graphs.
A set function $f : V \rightarrow \mathbb{R}$ is called submodular if for all $S \subseteq O \subseteq V$ and $u \in V \setminus T$, $f(S \cup \{u\}) - f(S) \geq f(O \cup \{u\}) - f(O)$. Intuitively, submodularity characterizes the diminishing return property often occurring in economics and operation research. Moreover, a set function $f$ is called monotone if for all $S \subseteq O \subseteq V$, $f(S) \leq f(O)$. It is shown in [4] that influence spread $\sigma$ for the triggering model is a monotone submodular function. A non-negative monotone submodular function allows a greedy solution to its maximization problem subject to a cardinality constraint, with an approximation ratio $1 - 1/e$, where $e$ is the base of the natural logarithm [31]. This is the technical foundation for most influence maximization tasks.

We are now ready to define the Multi-Round Triggering (MRT) model. The MRT model includes $T$ independent rounds of influence diffusions. In each round $t \in [T]$, the diffusion starts from a separate seed set $S_t$ with up to $k$ nodes, and it follows the dynamic in the classical triggering model described previously. Since one node can be repeatedly selected as the seed set in different rounds, to clarify the round, we use pair notation $(v, t)$ to denote a node $v$ in the seed set of round $t$. We use $S_t = \{(v, t) \mid v \in S_t\}$ to represent the seed set of round $t$ in the pair notation. Henceforth, we always use the calligraphic $\mathbb{S}$ for the pair notation and the normal $S$ for node sets. By the equivalence between the triggering model and the live-edge graph model, the MRT model can be viewed as $T$ independent propagations in the $T$ live-edge graphs $L_1, L_2, \ldots, L_T$, which are drawn independently from the same distribution based on the triggering model. The total active nodes in $T$ rounds is counted by $\bigcup_{t=1}^{T} \Gamma(L_t, S_t)$, where $\Gamma(L_t, S_t)$ is the set of final active nodes in round $t$. Given a class of seed set (in pair notation) $S := \bigcup_{t=1}^{T} S_t$, the influence spread $\rho$ in the MRT model is defined as

$$\rho(S) = \rho(S_1 \cup S_2 \cup \cdots \cup S_T) = \mathbb{E}\left[\left|\bigcup_{t=1}^{T} \Gamma(L_t, S_t)\right|\right],$$

where the expectation is taken over the distribution of all live-edge graphs $L_1, L_2, \ldots, L_T$.

### 2.2 Multi-Round Influence Maximization

The classical influence maximization problem is to choose a seed set $S_k$ of size at most $k$ to maximize the influence spread $\sigma(S_k)$. For the Multi-Round Influence Maximization (MRIM) problem, the goal is to select at most $k$ seed nodes of each round, such that the influence spread in $T$ rounds is maximized. We first introduce its non-adaptive version formally defined as follows.

**Definition 2.1.** The non-adaptive Multi-Round Influence Maximization (MRIM) under the MRT model is the optimization task where the input includes the directed graph $G = (V, E)$, the triggering set distribution for every node in the MRT model, the number of rounds $T$, and each-round budget $k$, and the goal is to find $T$ seed sets $S_1, S_2, \ldots, S_T$ with each seed set having at most $k$ nodes, such that the total influence spread is maximized, i.e.,

$$S^* = S_1 \cup S_2 \cup \cdots \cup S_T = \arg\max_{S : |S| \leq k, \forall t \in [T]} \rho(S).$$

The non-adaptiveness refers to the definition that one needs to find $T$ seed sets all at once before the propagation starts. In practice, one may be able to observe the propagation results in previous rounds and select the seed set for the next round based on the previous results to increase the influence spread. This leads to the adaptive version. To formulate the adaptive MRIM requires the setup of the adaptive optimization framework, and we defer to Section 4.1 for its formal definition.

Note that as the classical influence maximization is NP-hard and is a special case of MRIM with $T = 1$, both the non-adaptive and adaptive versions of MRIM are NP-hard.

### 3 Non-Adaptive MRIM

Let $\mathcal{V}_t = \{(v, t) \mid v \in V\}$ be the set of all possible nodes in round $t$ (e.g., $S_t \subseteq \mathcal{V}_t$), and $\mathcal{V} := \bigcup_{t=1}^{T} \mathcal{V}_t$. We first show that the influence spread function $\rho$ for the MRT model is monotone and submodular.

**Lemma 3.1.** Influence spread $\rho(S)$ for the MRT model satisfies (a) monotonicity: for any $S^A \subseteq S^B \subseteq \mathcal{V}$, $\rho(S^A) \geq \rho(S^B)$; and (b) submodularity: for any $S^A \subseteq S^B \subseteq \mathcal{V}$ and any pair $(v, t) \in \mathcal{V} \setminus S^B$, $\rho(S^A \cup \{(v, t)\}) - \rho(S^A) \geq \rho(S^B \cup \{(v, t)\}) - \rho(S^B)$.

**Proof.** The proof of monotonicity is straightforward, so we next consider submodularity. According to the definition of the influence spread, we have

$$\rho(S^A \cup \{(v, t)\}) - \rho(S^A) \geq \rho(S^B \cup \{(v, t)\}) - \rho(S^B)$$

$$= \mathbb{E}\left[\left|\bigcup_{t=1}^{T} \Gamma(L_t, S^A_t) \setminus \Gamma(L_t, S^B_t) \cup \Gamma(L_t, S^A_t \cup \{(v, t)\}) \right|\right] - \mathbb{E}\left[\left|\bigcup_{t=1}^{T} \Gamma(L_t, S^A_t) \setminus \Gamma(L_t, S^B_t) \cup \Gamma(L_t, S^A_t \cup \{(v, t)\}) \right|\right].$$

Then it is sufficient to show that

$$\mathbb{E}\left[\left|\bigcup_{t=1}^{T} \Gamma(L_t, S^A_t) \setminus \Gamma(L_t, S^B_t) \cup \Gamma(L_t, S^A_t \cup \{(v, t)\}) \right|\right] - \mathbb{E}\left[\left|\bigcup_{t=1}^{T} \Gamma(L_t, S^A_t) \setminus \Gamma(L_t, S^B_t) \cup \Gamma(L_t, S^A_t \cup \{(v, t)\}) \right|\right].$$

For a node $u \in \bigcup_{t=1}^{T} \Gamma(L_t, S^B_t) \cup \Gamma(L_t, S^B_t \cup \{(v, t)\}) \setminus \bigcup_{t=1}^{T} \Gamma(L_t, S^B_t)$, $u$ is reachable from $S^B_t \cup \{(v, t)\}$ in $L_t$, but not reachable from $S^B_t$ in $L_i$ for any $i \in [T]$. Thus $u$ is also not reachable from $S^A_t$ in $L_t$ for any $i \in [T]$. Therefore, we conclude that $u \in \bigcup_{t=1}^{T} \Gamma(L_t, S^A_t) \cup \Gamma(L_t, S^A_t \cup \{(v, t)\}) \setminus \bigcup_{t=1}^{T} \Gamma(L_t, S^A_t)$. Thus the submodularity holds.
The monotonocity and submodularity above are the theoretical basis of designing and analyzing greedy algorithms for the non-adaptive MRIM. In the following sections, we will consider two different settings separately for node seed selection: within-round and cross-round. For the within-round setting, one needs to determine the seed sets round by round, while for the cross-round setting, one is allowed to select nodes crossing rounds.

### 3.1 Cross-Round Setting

We design a greedy algorithm for the non-adaptive MRIM under cross-round setting, named CR-Greedy (Algorithm 1). The idea of CR-Greedy is that at every greedy step, it searches all \((v, t)\) in the candidate space \(C\) and picks the one having the maximum marginal influence spread without replacement. If the budget for some round \(t\) exhausts, then the remaining nodes of \(\mathcal{V}_t\) are removed from \(C\). Note that as \(C\) contains nodes assigning to different rounds, CR-Greedy selects nodes crossing rounds.

Given a set \(U\) which is partitioned into disjoint sets \(U_1, \ldots, U_n\) and \(I = \{X \subseteq U : |X \cap U_i| \leq k_i, \forall i \in [n]\}\), \((U, I)\) is called a partition matroid. Thus, the node space \(\mathcal{V}\) with the constraint of MRIM, namely \((\mathcal{V}, \{S : |S| \leq k, \forall t \in [T]\})\), is a partition matroid. This indicates that MRIM under cross-round setting is an instance of submodular maximization under partition matroid, and thus the performance of CR-Greedy has the following guarantee [32].

**Theorem 3.2.** Let \(S^*\) be the optimal solution of the non-adaptive MRIM under cross-round setting. For every \(\epsilon > 0\) and \(\ell > 0\), with probability at least \(1 - \frac{1}{n}\), the output \(S^0\) of CR-Greedy satisfies

\[
\rho(S^0) \geq \left(\frac{1}{2} - \epsilon\right) \rho(S^*),
\]

if CR-Greedy uses \(R = \left[3k^2n^2T \log(2kn^{\ell+1})/\epsilon^2\right]\) as input. In this case, the total running time is \(O(k^3T^4n^2m\log(n)/\epsilon^2)\), assuming each simulation finishes in \(O(m)\) time.

### 3.2 Within-Round Setting

We give the second greedy algorithm (Algorithm 2) for the within-round setting, denoted as WR-Greedy. The idea of WR-Greedy is that seed nodes are selected round by round. More specifically, we greedily select seed nodes for the first round, and only after we selected \(k\) seed nodes for the first round, then we greedily select seed nodes for the next round, and so on. The immediate advantage of WR-Greedy over CR-Greedy is that in each greedy step the WR-Greedy only searches candidates \((v, t)\) with \(t\) being the current round number, while CR-Greedy needs to search \((v, t)\) for all rounds. This would give at least a factor of \(T\) saving on the running time of WR-Greedy. However, as we will show below, it pays a price of a slightly lower approximation ratio.

To analyze WR-Greedy, we utilize the result of Lemma 3.1 in a different way. First, when we fix the seed sets in round 1, \(\ldots, t - 1\), and only vary the seed sets in round \(t\), the influence spread certainly still satisfies the monotonocity and submodularity. Therefore, within round \(t\), the seed set \(S^0\) selected by WR-Greedy for round \(t\) is a \((1 - 1/\epsilon - \epsilon)\)-approximate solution when the previous \(t - 1\) seed sets are fixed. Second, we could view seed set \(S_t\) of each round \(t\) as a unit, and when adding it to the previous units, it would also satisfy the monotonocity and submodularity by Lemma 3.1. Namely, \(\rho(S_1 \cup \ldots \cup S_t) \leq \rho(S_1 \cup \ldots \cup S_t)\) for all \(t < t'\), and \(\rho(S_1 \cup \ldots \cup S_t \cup S_{t'}) - \rho(S_1 \cup \ldots \cup S_t) \leq \rho(S_1 \cup \ldots \cup S_{t'} - S_t)\) for all \(t < t' < t''\). This means that WR-Greedy can be viewed as greedily selecting seed set units \(S_t\) round by round with the monotonocity and submodularity, while within each round, it can not find the optimal \(S_t\) but instead by a \((1 - 1/\epsilon - \epsilon)\)-approximate solution. Together, by the result in [31], we can show that WR-Greedy achieves an approximation factor of \(1 - e^{-(1-\ell)/\epsilon} - \epsilon\), as summarized in the following theorem.

**Theorem 3.3.** Let \(S^*\) be the optimal solution of the non-adaptive MRIM under within-round setting. For every \(\epsilon > 0\) and \(\ell > 0\), with probability at least \(1 - \frac{1}{n}\), the output \(S^0\) of WR-Greedy satisfies

\[
\rho(S^0) \geq \left(1 - e^{-(1-\ell)/\epsilon} - \epsilon\right) \rho(S^*),
\]

if WR-Greedy uses \(R = \left[3k^2n^2T \log(2kn^{\ell+1})/\epsilon^2\right]\) as input. In this case, the total running time is \(O(k^3T^4n^2m\log(nT)/\epsilon^2)\), assuming each simulation finishes in \(O(m)\) time.

**Proof (Sketch).** The proof follows the same structure as the proof of Theorem 3.7 in [8], but it needs to accommodate the new double greedy algorithm structure and the double submodularity property. Let \(\epsilon_0 = 6^{1-1/\ell}\epsilon/2\). From the inner-submodularity

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**Algorithm 1: CR-Greedy: Cross-Round Greedy Algorithm**

**Input:** Graph \(G = (V, E)\), integers \(T\), \(k\) and \(R\), triggering set distributions.

**Output:** \(S^0\).

1. \(S^0 \leftarrow \emptyset; C \leftarrow \mathcal{V};\)
2. \(c_1, c_2, \ldots, c_T \leftarrow \emptyset;\) \hspace{1cm} // node counter for each round
3. **for** \(i = 1, 2, \ldots, kT\) **do**
   4. **for** all \((v, t) \in C \setminus S^0\), estimate \(\rho(S^0 \cup \{(v, t)\})\) by simulating the diffusion process \(R\) times;
   5. \((c_i, t_i) \leftarrow \text{argmax}_{(v, t) \in C \setminus S^0} \rho(S^0 \cup \{(v, t)\});\)
   6. \(S^0 \leftarrow S^0 \cup \{(c_i, t_i)\}; c_t \leftarrow c_t + 1;\)
   7. **if** \(c_t \geq k\) **then** \hspace{1cm} // budget for round \(t\) exhausts
      8. \(C \leftarrow C \setminus \mathcal{V}_{t};\)
4. **return** \(S^0\).

**Algorithm 2: WR-Greedy: Within-Round Greedy Algorithm**

**Input:** Graph \(G = (V, E)\), integers \(T\), \(k\) and \(R\), triggering set distributions.

**Output:** \(S^0\).

1. \(S^0 \leftarrow \emptyset;\)
2. **for** \(t = 1, 2, \ldots, T\) **do**
   3. **for** \(i = 1, 2, \ldots, k\) **do**
      4. **for** all \((v, t) \in \mathcal{V}_t \setminus S^0\), estimate \(\rho(S^0 \cup \{(v, t)\})\) by simulating the diffusion process \(R\) times;
      5. \((u, t) \leftarrow \text{argmax}_{(v, t) \in \mathcal{V}_t \setminus S^0} \rho(S^0 \cup \{(v, t)\});\)
      6. \(S^0 \leftarrow S^0 \cup \{(u, t)\};\)
   7. **return** \(S^0\).
We call (\ref{eq:adaptive_greedy}) an item, where \Si is the seed set chosen in round \t. Let \E be the set of all the possible items. For each item \( (\Si, \t) \), after the propagation, the nodes and edges participated in the propagation are observed as the feedback. Formally, the feedback is referred to as a state, which is the subgraph of live-edge graph \L_i that can be reached by \Si. A realization is a function \( \phi : \E \rightarrow O \) mapping every possible item \( (\Si, \t) \) to a state, where \( O \) is the set of all possible states. Realization \( \phi \) represents one possible propagation from a possible seed set in a round. Let \( \phi(\Si, \t) \) denote the state of \( (\Si, \t) \) under realization \( \phi \). We use \( \Phi \) to denote a random realization, and the randomness comes from random live-edge graphs \( L_1, \ldots, L_T \). For the adaptive MRIM, in each round \t, we pick an item \( (\Si, \t) \), see its state \( \Phi(\Si, \t) \), pick the next item \( (\Si_{t+1}, \t+1) \), see its state, and so on. After each pick, previous observations can be represented as a partial realization \( \psi \), a function from some subset of \( \E \) to its states. For notational convenience, we represent \( \psi \) as a relation, so that \( \psi \subseteq \E \times \O \) equals \( \{(\Si, \t) \in \E, \Phi(\Si, \t) = \o\} \). We define dom(\( \psi \)) = \{\( (\Si, \t) \in \E \) \} as the domain of \( \psi \). A partial realization \( \psi \) is consistent with a realization \( \phi \) if they are equal everywhere in the domain of \( \psi \), denoted as \( \phi \sim \psi \). If \( \psi \) and \( \psi' \) are both consistent with some \( \phi \), and \( \text{dom}(\psi') \subseteq \text{dom}(\psi) \), \( \psi \) is a subrealization of \( \psi' \), also denoted as \( \psi \subseteq \psi' \).

A policy \( \pi \) is an adaptive strategy for picking items based on partial realizations in \( \E \). In each round, \( \pi \) will pick the next set of seeds \( \pi(\psi) \) based on partial realization \( \psi \) so far. If partial realization \( \psi \) is not in the domain of \( \pi \), the policy stops picking items. We use \( \text{dom}(\pi) \) to denote the domain of \( \pi \). Technically, we require that \( \text{dom}(\pi) \) be closed under subrealizations: If \( \psi' \in \text{dom}(\pi) \) and \( \psi \) is a subrealization of \( \psi' \) then \( \psi \in \text{dom}(\pi) \). We use the notation \( E(\pi, \psi) \) to refer to the set of items selected by \( \pi \) under realization \( \psi \). The set of items in \( E(\pi, \phi) \) is always in the form \( \{(\Si_1, \ldots, \Si_t)\} \), so sometimes we also refer to it as sequence of seed sets.

We wish to maximize, subject to some constraints, a utility function \( f : 2^\E \times O^E \rightarrow \mathbb{R}_{\geq 0} \) that depends on the picked items and the states of them. In the adaptive MRIM, \( f(\{(\Si_1), \ldots, (\Si_t)\}, \phi) \) is the total number of active nodes by round \t from the respective seed sets, i.e., \( \bigcup_{r=1}^{t} \{L_r^\phi \} \) where \( L_r^\phi \) is the live-edge graph of round \t. Based on the above notations, the expected utility of a policy \( \pi \) is \( f_{\text{avg}}(\pi) = \mathbb{E}_{\Phi}[f(\E(\pi, \phi), \Phi)] \) where the expectation is taken over the randomness of \( \Phi \). Namely, \( f_{\text{avg}}(\pi) \) is the expected number of active nodes under policy \( \pi \). Let \( \Pi_{T,k} \) be the set of all policies that select seed sets in at most \( T \) rounds and each seed set has at most \( k \) nodes. The goal of the adaptive MRIM is to find the best policy \( \pi \) such that: \( \pi^* = \arg\max_{\pi \in \Pi_{T,k}} f_{\text{avg}}(\pi) \).

### 4.2 Adaptive Submodularity and Greedy Policy

Given a partial realization \( \psi \) of \( t-1 \) rounds with \( \text{dom}(\psi) = \{(\Si_1), (\Si_2), \ldots, (\Si_{t-1}, t-1)\} \), and the seed set \( \Si_t \) for round \t, the conditional expected marginal benefit of item \( (\Si_t, \t) \) conditioned on having observed \( \psi \) is defined as

\[
\Delta((\Si_t, \t), \psi) = \mathbb{E}_\Phi[f(\text{dom}(\psi) \cup \{(\Si_t, \t)\}, \Phi) - f(\text{dom}(\psi), \Phi) \mid \psi].
\]

The conditional expected marginal gain of a policy \( \pi \) is defined as

\[
\Delta(\pi, \psi) = \mathbb{E}[f(\text{dom}(\psi) \cup E(\pi, \Phi), \Phi) - f(\text{dom}(\psi), \Phi) \mid \psi].
\]

The adaptive MRIM satisfies the adaptive monotonicity and submodularity shown as below. The proofs require a careful analysis of the partial realization in the MRT model and is given in the appendix.

**Lemma 4.1.** [Adaptive Monotonicity] For all \( t > 0 \), for all partial realization \( \psi \) with \( t-1 \) rounds and \( \text{Pr}[\Phi \sim \psi] > 0 \), and for all item \( (\Si_t, \t) \), we have:

\[
\Delta((\Si_t, \t), \psi) \geq 0.
\]
AdaGreedy algorithm. For any $\ell > 0$ and $\ell > 0$, if we use $R = \lceil 31k^2n\log(2kn+1+\ell^2)/\epsilon^2 \rceil$ simulations for each influence spread estimation, then with probability at least $1 - \frac{1}{n}$,

$$f_{\text{avg}}(\pi^{\text{NS}}) \geq \left(1 - e^{-1 - \frac{2}{k}} - \epsilon\right) f_{\text{avg}}(\pi^*) .$$

**Algorithm 3: AdaGreedy: Adaptive Greedy for Round t**

**Input:** Graph $G = (V, E)$, integers $T$, $k$, and $R$, triggering set distribution, active node set $A_{t-1}$ by round $t - 1$.

**Output:** Seed set $S_t$, updated active nodes $A_t$.

1. $S_t \leftarrow \text{MC-Greedy}(G, A_{t-1}, k, R)$; // Monte Carlo Greedy
2. Observe the propagation of $S_t$, update activated nodes $A_t$;
3. return $(S_t, t), A_t$.

**Lemma 4.2. [Adaptive Submodularity] For all $t > 0$, for any partial realization $\psi$ with $i - 1$ rounds and partial realization $\psi'$ such that $\psi'$ is a subrealization of $\psi$, i.e., $\psi' \subseteq \psi$, and for all item $(S_t, i)$, we have

$$\Delta((S_t, t)|\psi') \geq \Delta((S_t, t)|\psi).$$

Following the framework of [9], adaptive monotonicity and adaptive submodularity enable an adaptive greedy policy with a constant approximation of the optimal adaptive policy. AdaGreedy (Algorithm 3) is the greedy adaptive policy for MRM. Note that adaptive algorithms operate at per round base — it takes feedback from previous rounds and selects the item for the current round, and then obtain new feedback. Thus we present AdaGreedy for a generic round $t$. Besides the problem input such as the graph $G$, the triggering model parameters, seed set budget $k$, and simulation number $R$, AdaGreedy takes the set of already activated nodes $A_{t-1}$ as the feedback from the previous rounds, and aims at finding the seed set $S_t$ of size $k$ to maximize the expected marginal gain $\Delta((S_t, t)|\psi')$, which is the expected number of newly activated nodes in round $t$. However, this problem is NP-hard, so we use a Monte Carlo greedy approximation MC-Greedy algorithm to find an approximate solution. MC-Greedy greedily finds the seed with the maximum estimated marginal influence spread until $k$ seeds being selected, where the marginal influence spread of adding an unselected seed is estimated by simulating the propagation $R$ times. In AdaGreedy, MC-Greedy won’t count the influence of a node if it has been activated in previous rounds. The rationale is that maximizing the expected marginal gain $\Delta((S_t, t)|\psi')$ is equivalent to the weighted influence maximization task in which we treat nodes in $A_{t-1}$ with weight 0 and other nodes with weight 1, and we maximize the expected total weight of the influenced nodes. By [34], we know that the weighted version is also monotone and submodular, so we could use a greedy algorithm to obtain a seed set $S_t$ as a $(1 - 1/e - \epsilon)$ approximation of the best seed set for round $t$. We could use $R$ Monte Carlo simulations to estimate the weighted influence spread, and $R$ is determined by the desired approximation accuracy $\epsilon$. Once the seed set $S_t$ is selected for round $t$, the actual propagation from $S_t$ will be observed, and the active node set $A_t$ will be updated as the feedback for the next round.

The following theorem summarizes the correctness and the time complexity of AdaGreedy.

**Theorem 4.3.** Let $\pi^{\text{NS}}$ represents the policy corresponding to the AdaGreedy algorithm. For any $\epsilon > 0$ and $\ell > 0$, if we use $R = \lceil 31k^2n\log(2kn+\ell^2)/\epsilon^2 \rceil$ simulations for each influence spread estimation, then with probability at least $1 - \frac{1}{n}$,

$$f_{\text{avg}}(\pi^{\text{NS}}) \geq \left(1 - e^{-1 - \frac{2}{k}} - \epsilon\right) f_{\text{avg}}(\pi^*) .$$

**Algorithm 4: CR-NS: Cross-Round Node Selection**

**Input:** Multi round RR vector sets $M, T, k$

**Output:** seed sets $S^o$

1. Build count array: $c[(u, t)] = \sum_{(u, t) \in M} |(u, t)|, \forall (u, t) \in V$;
2. Build RR set link: $RR[(u, t)], \forall (u, t) \in V$;
3. For all $R \in M$, covered[$R$] = false;
4. $S^o \leftarrow \emptyset, C \leftarrow V, c_1,c_2,...,c_T \leftarrow 0$;
5. for $i = 1$ to $Tk$ do
6. $(u, t) \leftarrow \text{argmax}_{(u', t')} c[(u', t')]$;
7. $S^o \leftarrow S^o \cup [(u, t)]; c_t \leftarrow c_t + 1$;
8. if $c_t = k$ then
9. $C = C \setminus \{(u, t) \mid v \in V\}$;
10. for all $R \in RR[(u, t)] \land \text{covered}[R] = \text{false}$ do
11. $\text{covered}[R] = \text{true}$;
12. for all $(u', t') \in R \land (u', t') \neq (u, t)$ to do
13. $c[(u', t')] = (c[(u', t')] - 1)$;
14. return $S^o$.

In this case, the total running time for $T$-round AdaGreedy is $O(k^3(Tn^2\log(nT)/\epsilon^2))$.

## 5 SCALABLE IMPLEMENTATIONS

In this section, we aim to speed up CR-Greedy, WR-Greedy and AdaGreedy by the reverse influence sampling [10–12].

A Reverse-Reachable (RR) set $R_v$ rooted at node $v \in V$ is the set of nodes which are reached by reverse simulating a propagation from $v$ in the triggering model. Equivalently, $R_v$ is the set of nodes in a random live-edge graph which can reach $v$. We use root($R_v$) to denote its root $v$. We define a (random) RR set $R$ is a RR set rooted at a node picked uniformly at random from $V$, then for any seed set $S \subseteq V$, its influence spread

$$\sigma(S) = n \cdot \mathbb{E}[(1 - S \cap R \neq \emptyset)] ,$$

where $n = |V|$, $\mathbb{I}\{$ is the indicator function, and the expectation is taken over the randomness of $R$: randomness of root node and randomness of live-edge graph. The property implies that we can accurately estimate the influence spread of any possible seed set $S$ by sampling enough RR sets. More importantly, by Eq. (2) the optimal seed set can be found by seeking the optimal set of nodes that intersect with (a.k.a. cover) the most number of RR sets, which is a max-cover problem. Therefore, a series of near-linear-time algorithms are developed [10–12] based on the above observation. All RR-set algorithms have the same structure of two phases. In Phase 1, the number of RR sets needed is estimated, and in Phase 2, these RR sets are generated and greedy algorithm is used on these RR sets to find the $k$ nodes that cover the most number of RR sets. All algorithms have the same Phase 2, but Phase 1 is being improved from one to another so that less and less RR sets are needed. Our algorithms are based on IMM proposed in [12].

### 5.1 Non-Adaptive IMMs

For the non-adaptive MRM, we define the multi-round reverse-reachable (RR) set $R_v$, rooted at node $v$ for the MRT model as
Algorithm 5: CR-NAIMM: Non-adaptive IMM Algorithm for Cross-Round

Input: Graph $G = (V, E)$, round number $T$, budget $k$, accuracy parameters $(\epsilon, \ell)$, triggering set distributions

Output: seed set $S$

// Phase 1: Estimating the number of multi-round RR sets needed, $\theta$
1. $\ell' \leftarrow \ell + \log 2/\log n$; $M \leftarrow \emptyset$; $LB \leftarrow 1$; $\ell' \leftarrow \sqrt{2\ell'}$
2. $\alpha \leftarrow \sqrt{\ln n + \ln 2 + \beta \left(1 - (1/2) \cdot \ell \ln \left(\frac{n}{\epsilon}\right) + \alpha^2\right)}$
3. $\lambda' \leftarrow \left[\frac{(2 + 4\ell')}{(1 + \ell') \cdot \ell \cdot \ln n + \ln \log^2 n} \cdot n^{1/2}\right]^{\ell \cdot \ln n}$
4. $\lambda^* \leftarrow 2nT \cdot \left(\frac{(5 - 1/\ell' \cdot \alpha + \beta)^2 - \epsilon^2}{\ell \cdot \ln n}ight)$
5. for $i = 1$ to $\log_2(n - 1)$ do
   6. $x \leftarrow n/2^i$
   7. $\theta_i \leftarrow \lambda^*/LB$
   8. while $|M| < \theta_i$ do
      9. Select a node $u$ from $V$ uniformly at random;
      10. Generate RR-vector $R$ from $u$, and insert it into $M$;
      11. $S_t \leftarrow$ CR-NS$(M, T, k)$;
      12. if $n \cdot F_M(S_t) \geq (1 + \epsilon') \cdot x$ then
         13. $LB \leftarrow n \cdot F_M(S_t)/(1 + \ell')$
         14. break;
     15. $\theta \leftarrow \lambda^*/LB$
   16. while $|M| \leq \theta$ do
      17. Select a node $u$ from $V$ uniformly at random;
      18. Generate $R$ for $u$, and insert it into $M$;

// Phase 2: Generate $\theta$ RR-vector sets and select seed nodes
19. $S \leftarrow$ CR-NS$(M, T, k)$;
20. return $S$.

$\mathcal{R}_v := \bigcup_{t=1}^T \mathcal{R}_{v, t}$ where $\mathcal{R}_{v, t}$ denotes a RR set rooted at $v$ of round $t$ in pair notation. $\mathcal{R}_v$ is generated by independently reverse simulating the propagation $T$ rounds from $v$ and then aggregating them together. Let $\text{root}(\mathcal{R}_v) := v$. A (random) multi-round RR set $\mathcal{R}$ is a multi-round RR set rooted at a node picked uniformly at random from $V$. We use $\mathcal{M}$ to denote the set of $\mathcal{R}_v$. We are now ready to explain the cross-round and within-round non-adaptive IMM.

5.1.1 Cross-Round Non-Adaptive IMM. In cross-round setting, if $\mathcal{R}$ is a random multi-round RR set, then for any seed set $S$, its influence spread satisfies the following lemma.

Lemma 5.1. For any node-round pair seed set $S$, $\rho(S) = n \cdot \mathbb{E}[\mathcal{S} \cap \mathcal{R} \neq \emptyset]$, where the expectation is taken over the randomness of $\mathcal{R}$.

The Lemma 5.1 implies we can sample enough multi-round RR sets to accurately estimate the influence spread of $S$.

CR-NAIMM is very similar to standard IMM and only has a few differences include several points. First, CR-NAIMM generates multi-round RR sets $\mathcal{R}$ from roots in $V$ (lines 9 and 17). Second, we need to adjust $\ell$ to be $\ell + \log(2)/\log n$, and $\epsilon$. This is to guarantee that we have probability at least $1 - 1/(n^{T/2})$ to have $S_t$ as a $(\frac{1}{m} - \epsilon)$ approximation, so that the result for the total $T$ rounds would come out correctly as stated in the following theorem. Third, we use new $F_M(\cdot)$ denotes the fraction of multi-round RR sets in $\mathcal{R}$ that are covered by a node set $S$ in algorithm CR-NAIMM (lines 12 and 13). Forth, CR-NS returns $T_k$ seeds from the total $T$ rounds (line 5). Last, if the budget for some round $t$ exhausts, then the remaining nodes of $\mathcal{V}_t$ are removed from $C$ in CR-NS (line 9).

By an analysis similar to that of the IMM algorithm [12], we can show that our CR-NAIMM achieves $1/2 - \epsilon$ approximation with expected running time $O(T^2(k + \ell)(m + n) \log(n)/\epsilon^2)$.

Theorem 5.2. Let $S^*$ be the optimal solution of the non-adaptive MRIM. For every $\epsilon > 0$ and $\ell > 0$, with probability at least $1 - \frac{1}{m}$, the output $S_0^*$ of the cross-round algorithm CR-NAIMM satisfies $\rho(S_0^*) \geq \frac{1}{2} \rho(S^*)$.

In this case, the total running time for $T$-round CR-NAIMM is $O(T^2(k + \ell)(m + n) \log(n)/\epsilon^2)$.

5.1.2 Within-Round Non-Adaptive IMM. In the within-round setting, the idea is to use the IMM algorithm in each round to select $k$ seeds. However, seeds selected in earlier rounds may already influence some nodes, so when we select roots for a later round $t$ and generate RR sets for round $t$, the roots should not be selected uniformly at random. Instead, we want to utilize the idea derived from the following lemma. Let $S_{t-1}^* := \bigcup_{t=1}^{t-1} S_t$ be the set of seed pairs in first $t-1$ rounds, and let $S_t$ be a set of seed pairs in round $t$. Similarly, let $\mathcal{R}_{t-1} := \bigcup_{t=1}^{t-1} \mathcal{R}_t$, $\mathcal{R}_t$ be the set of RR sets (in pairs) in first $t-1$ rounds, $\mathcal{R}_t$ be the RR set (in pairs) for round $t$. The marginal influence spread of $S_t$ in round $t$ is

Lemma 5.3. For any node-round pair seed set $S$, $\rho(S_{t-1}^* \cup S_t) - \rho(S_{t-1}^*) = n \cdot \mathbb{E}[\mathcal{S} \cap \mathcal{R}_{t-1} \cup \mathcal{S} \cap \mathcal{R}_t] - \rho(S_{t-1}^*)$.

where the expectation is taken over the randomness of $\mathcal{R}$.

The above lemma suggests that when we want to generate an RR set for round $t$, we should also generate RR sets for earlier rounds and check that if any of them is intersecting with the seed set in the same round, and if so, the RR set for round $t$ is invalid and we need to regenerate an RR set again. By following this, the implementation is similar to the CR-NAIMM, and the only difference between them is that the within-round non-adaptive algorithm WR-NAIMM selects $k$ seeds round by round. The resulting algorithm would have the approximation guarantee of $1 - e^{-(1-\frac{1}{m})} - \epsilon$, but it does not have significant running time improvement over CR-NAIMM, since it wastes many RR set generations. We find a better heuristic to use the roots generated for the previous rounds. In particular, for each $t \geq 2$, after we finished selecting $k$ seeds in round $t - 1$, some RR sets are removed since they are covered by seeds selected. The remaining roots are exactly the ones whose RR sets do not intersect with seed sets in the first $t - 1$ rounds. Therefore, their distribution is close to the distribution of the valid roots satisfying $S_{t-1}^* \cap \mathcal{R}_{t-1} = \emptyset$ in Lemma 5.3. Hence, in round $t$, we
sample roots using the remaining roots from round \( t-1 \). This gives a close estimate of the marginal influence spread. Due to the complicated stochastic dependency of RR sets from round to round, the exact theoretical analysis of this improvement is beyond our reach, and thus we propose it as an efficient heuristic.

The resulting algorithm WR-NAIMM runs almost exactly like running a copy of the standard IMM for each round \( t \). The only difference is that in the standard IMM, the root of an RR set is always sampled uniformly at random from all nodes, but in WR-NAIMM the root of an RR set in round \( t \) is sampled uniformly at random from the remaining roots left in round \( t-1 \). Moreover, for each round \( t \), we set the per-round approximation error bound \( \epsilon_0 = e^{(1-1/\ell)}\epsilon/2 \) and replace \( \ell \) with \( \ell + \log(2T)/\log n \). This is consistent with the adaptive IMM setting in Section 5.2, and with the setting used in the proof of Theorem 3.3 for the within-round greedy algorithm. The details of WR-NAIMM are provided in the Appendix A.2.

5.2 Adaptive IMM

We use \( M \) to denote the set of \( R \). In the adaptive setting, as already mentioned in Section 4.2, some nodes \( A_{t-1} \) are already activated, so they won’t contribute to the influence spread in round \( t \). Therefore, we are working on the weighted influence maximization problem, where nodes in \( A_{t-1} \) have weight 0 and nodes in \( V \setminus A_{t-1} \) has weight 1. Let \( \sigma^{-A_{t-1}}(S) \) be the weighted influence spread according to the above weight. Let \( R^{-A_{t-1}} \) be a random RR set where the root \( v \) is selected from \( V \setminus A_{t-1} \) uniformly at random, and then reverse simulate from \( v \) to get the RR set. Then we obtain the result similar to Eq. (2):

**Lemma 5.4.** For any seed set \( S \subseteq V \),

\[
\sigma^{-A_{t-1}}(S) = (n - |A_{t-1}|) \cdot \mathbb{E}[\{S \cap R^{-A_{t-1}} \neq \emptyset\}],
\]

where the expectation is taken over the randomness of \( R^{-A_{t-1}} \).

**Proof.**

\[
\begin{align*}
\mathbb{E}[\{S \cap R^{-A_{t-1}} \neq \emptyset\}] &= \sum_{v \in V \setminus A_{t-1}} \Pr(v = \text{root}(R^{-A_{t-1}})) \\
&= \mathbb{E}[\{S \cap R^{-A_{t-1}} \neq \emptyset\} | v = \text{root}(R^{-A_{t-1}})] \\
&= \frac{1}{n - |A_{t-1}|} \sum_{v \in V \setminus A_{t-1}} \mathbb{E}[\{S \cap R^{-A_{t-1}} \neq \emptyset\}] \\
&= \frac{1}{n - |A_{t-1}|} \sum_{v \in V \setminus A_{t-1}} \mathbb{E}[\{v \in \Gamma(L, S)\}] \\
&= \frac{1}{n - |A_{t-1}|} \mathbb{E} \left[ \sum_{v \in V \setminus A_{t-1}} \mathbb{I}\{v \in \Gamma(L, S)\} \right] \\
&= \frac{1}{n - |A_{t-1}|} \mathbb{E}[\Gamma(L, S) \setminus A_{t-1}] \\
&= \frac{1}{n - |A_{t-1}|} \sigma^{-A_{t-1}}(S),
\end{align*}
\]

where Eq. (4) is based on the equivalence between RR sets and live-edge graphs, and the expectation from this point on is taken over the random live-edge graphs \( L \).

Therefore, with Eq. (3), the same RR-set based algorithm can be used, and we only need to properly change the RR-set generation process and the estimation process. AdaIMM is based on the IMM algorithm in [12]. The main differences from the standard IMM include several points. First, whenever we generate new RR sets in round \( t \), we only start from roots in \( V \setminus A_{t-1} \) as explained by Lemma 5.4. Second, when we estimate the influence spread, we need to adjust it using \( n_a = n - |A_{t-1}| \) again by Lemma 5.4. Third, we need to adjust \( \ell \) to be \( (1/\ell + \log(2T)/\log n) \) and \( \epsilon \) to \( \epsilon_0 = e^{(1-1/\ell)}\epsilon/2 \). This is to guarantee that in each round we have probability at least \( 1 - 1/(n^f T) \) to have \( S_t \) as a \((1 - 1/\ell - \epsilon_0)\) approximation, so that the result for the total \( T \) rounds would come out correctly as stated in the following theorem. The details are shown in the appendix.

**Theorem 5.5.** Let \( \pi^m \) represents the policy corresponding to the AdaIMM algorithm. For any \( \epsilon > 0 \) and \( \ell > 0 \), with probability at least \( 1 - \frac{1}{n^f} \),

\[
f_{\text{avg}}(\pi^m) \geq \left(1 - e^{-(1-\ell/\ell')}}\epsilon_0 \right) f_{\text{avg}}(\pi^*)
\]

In this case, the total running time for \( T \)-round AdaIMM is \( O(T(k + \ell)(n + m) \log(n T)/\epsilon^2) \).

**Proof (Sketch).** Let \( \epsilon_0 = e^{(1-1/\ell)}\epsilon/2 \), \( \ell' = (1 - 1/\ell - \epsilon_0) \) approximation of the best seed set for this round. Then following the similar arguments as in Theorems 3.3 and 4.3, we know that across all \( T \) rounds, with probability at least \( 1 - 1/n^f \),

\[
f_{\text{avg}}(\pi^m) \geq \left(1 - e^{-(1-\ell/\ell')}}\epsilon_0 \right) f_{\text{avg}}(\pi^*) \geq \left(1 - e^{-(1-\ell/\ell')}}\epsilon \right) f_{\text{avg}}(\pi^*).
\]

Thus, the theorem holds.

Theorem 5.5 clearly shows that the AdaIMM algorithm is near linear time, and its theoretical time complexity bound is much better than the one in Theorem 4.3 for the AdaGreedy algorithm.

In practice, we can further improve the AdaIMM algorithm by incremental computation. In particular, RR sets generated in the previous \( t-1 \) rounds can be used for round \( t \), and we only need to remove those rooted at nodes in \( A_{t-1} \). Searching the lower bound \( LB \) can also be made faster by utilizing the \( S_t \) already obtained in the previous round. Our experiments would use such improvements.

6 EXPERIMENTAL EVALUATION

We conduct experiments on two real-world social networks to test the performance of our algorithms. We use the independent cascade model for the tests. The influence probabilities on the edges are learned from the real-world trace data in one dataset and synthetic in the other dataset, as explained below.

### 6.1 Data Description

**Flixster.** The Flixster dataset is a network of American social movie discovery service (www.flixster.com). To transform the
dataset into a weighted graph, each user is represented by a node, and a directed edge from node \( u \) to \( v \) is formed if \( u \) rates one movie shortly after \( v \) does on the same movie. The dataset is analyzed in [35], and the influence probability are learned by the topic-aware model. We use the learning result of [35] in our experiment, which is a graph containing 29 357 nodes and 212 614 directed edges.

NetHEPT. The NetHEPT dataset [36] is extensively used in many influence maximization studies. It is an academic collaboration network from the “High Energy Physics Theory” section of arXiv from 1991 to 2003, where nodes represent the authors and each edge represents one paper co-authored by two nodes. There are 15 233 nodes and 58 891 undirected edges (including duplicated edges) in the NetHEPT dataset. We clean the dataset by removing those duplicated edges and obtain a directed graph \( G = (V, E) \), \( |V| = 15 233, |E| = 62 774 \) (directed edges).

6.2 Result

We test all six algorithms proposed in the experiment, for the MRIM task of \( T \) rounds with \( k \) seeds in each round. We use \( R = 10000 \) for the Monte Carlo simulation of the influence spread of each candidate seed set for all non-adaptive algorithms. The lazy evaluation technique of [5] to optimize the greedy selection is applied to CR-Greedy, WR-Greedy and AdaGreedy. Besides the six proposed algorithms, we also propose two baseline non-adaptive algorithms (SG and SG-R) using the classical single-round algorithms directly for the multi-round influence maximization problem. SG simply selects \( T \) seed nodes using the single-round greedy algorithm, and then allocates the first \( k \) seeds as \( S_1 \) for the first round, second \( k \) seeds as \( S_2 \) for the second round, and so on. SG-R only selects \( k \) seeds greedily, and then reuse the same \( k \) seeds for all \( T \) rounds.

In the tests, we set \( T = 5 \) and \( k = 10 \) as the default, and we also test different combinations of \( T \) and \( k \) while keeping \( T \) to be the same, to see the effect of different degrees of adaptiveness.

6.2.1 Influence Spread Performance. We test the performance on the influence spread for all algorithms introduced in the above section. For non-adaptive algorithms, for each selected seed set sequence, we do 10 000 forward simulations and take the average to obtain its estimated influence spread. For adaptive algorithms, to obtain their expected influence spread over multiple real-world propagation simulations, we have to re-run the algorithm in each round after obtaining the feedback from the previous rounds. Thus, it would be too time consuming to also run 10 000 adaptive simulations for the adaptive algorithms. In stead, for NetHEPT we use 150 simulations and for Flixster we use 100 simulations. To make fair comparisons, we include confidence intervals in the obtained influence spread results, so that the number of simulations used for the estimation is taken into the consideration.

Tables 1 and 2 show the influence spread results for NetHEPT and Flixster datasets. All five round results are shown, one for each column. Each row is for one algorithm, and the number in the first line of the row is the empirical average of the influence spread, and the line below is the 95% confidence interval. Parameter \( R \) records the number of simulations used to obtain the average spread.

### Table 1: The performance of influence spread on NetHEPT.

| Method/Simulations | 1       | 2       | 3       | 4       | 5       |
|--------------------|---------|---------|---------|---------|---------|
| SG                 | [290.7, 291.6] | [292.1, 293.2] | [294.2, 295.5] | [296.3, 297.5] | [298.4, 299.6] |
| CR-Greedy          | [282.9, 284.2] | [285.1, 286.3] | [287.3, 288.5] | [290.5, 291.7] | [292.7, 293.9] |
| WR-Greedy          | [287.9, 289.2] | [291.0, 292.2] | [294.1, 295.3] | [297.4, 298.6] | [300.5, 301.7] |
| AdaGreedy          | [286.1, 287.3] | [289.3, 290.5] | [292.6, 293.8] | [296.0, 297.2] | [299.3, 300.5] |
| CR-R              | [289.7, 290.9] | [292.1, 293.3] | [295.5, 296.7] | [299.0, 300.2] | [302.5, 303.7] |
| WR-R              | [290.9, 292.1] | [293.3, 294.5] | [296.7, 297.9] | [300.2, 301.4] | [303.8, 305.0] |

Several observations can be made from these results. First, all six proposed algorithms in this paper performs significantly better than the baseline algorithms SG and SG-R. Besides the cross-round non-adaptive algorithms, the confidence intervals do not overlap starting from round 2. In terms of the empirical average, the improvement is obvious: at the end of the 5th round, for NetHEPT, MRIM algorithms is at least 8.8% better than SG, and 7.3% better than SG-R; for Flixster, MRIM algorithms is at least 12.8% better than SG, and 4.3% better than SG-R. Even if we use the ratio between the lower confidence bounds of MRIM algorithms vs. the upper confidence bounds of the baseline algorithms, the result is similar. SG-R performs better than SG, which implies that influential nodes are important in these datasets and it is preferred to re-select them. However, the improvement of MRIM algorithms over SG and SG-R are increasing over rounds, showing that adjusting to MRIM is increasingly important. It is reasonable to see that with more rounds, always sticking to the same seed sets or always changing seed sets would not perform well.

Second, the adaptive algorithms perform better than the within-round non-adaptive algorithms in both dataset: the confidence interval does not overlap for all results in round 4 and 5 and most results in round 3. This conforms with our intuition that adaptive algorithms performs better. The improvement are not very significant in Tables 1 and 2, which means the strength of the adaptiveness has not been fully explored yet.

Third, the cross-round setting is more effective, but less efficient than within-round setting of non-adaptive algorithms, due to a
larger search space. In practice, the cross-round setting only shows outstanding performances overall, but doesn’t guarantee the good performance in every round. In fact, it always performs worst in round 1 comparing to the other algorithms.

Fourth, the cross-round non-adaptive algorithm performs as well as the adaptive algorithm on NetHept dataset. Adaptive algorithm requires the real-life spread between each round comparing to non-adaptive algorithms, but always performs the best in each round.

Fifth, we can see that AdaIMM achieves the same level of influence spread as AdaGreedy: the confidence intervals always overlap, which means AdaIMM performs well in practice.

### 6.2.2 Degree of Adaptiveness
We vary the parameters $T$ and $k$ whiling keeping $Tk$ the same. With smaller $k$, it means each round we select a smaller number of seed sets, and we use more adaptive rounds. Therefore, small $k$ and large $T$ mean a high degree of adaptiveness. We test this using AdaIMM, since it is more efficient than AdaGreedy while providing the same level of influence spread.

Table 3 presents the result on this test, in which we vary $(T, k)$ as $(1, 50), (2, 25), (5, 10)$ and $(10, 5)$. The influence spread significantly increases with the increase of the degree of adaptiveness, and the increase is quite significant. The 10-round 5-seed setup is 36.3% better than 1 round 50-seed setup on the empirical average. This shows that higher adaptive degree indeed improves the performance.

In summary, with same total budget, the higher number of total rounds with higher influence spread performance in general.

### 6.2.3 Running Time
Table 4 reports the running time of all the tested algorithms on the two datasets, when running with $T = 5$ and $k = 10$. One clear conclusion is that all IMM algorithms are much more efficient that others, with two to three orders of magnitude faster than all other algorithms. Among greedy algorithms, SG-R runs faster because it selects only 10 seeds once. SG is the slowest, much slower than E-WR-Greedy and AdaGreedy because it needs to select 50 different seeds, and when it selects more seeds, their marginal influence spread does not differ from one another much and thus the lazy-evaluation optimization is not as effective as selecting the first 10 seeds.

After combining the influence spread and running time performance, our conclusion is that (a) algorithms designed for the MRIM task is better, (b) the cross-round setting is more efficient, but less efficient than the within-round setting of non-adaptive algorithms, and (c) AdaIMM is clearly the best for adaptive MRIM task.

### 7 ACKNOWLEDGE
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APPENDIX

The appendix is organized as follows. In Section B, we give all proofs of lemmas and theorems, i.e., Section B.1 for the non-adaptive MRIM, Section B.2 for the adaptive MRIM, Section B.3 for the CR-NAIMM, and Section B.4 for the WR-NAIMM. In Section A, we give our complete IMM algorithms and compare them with the original IMM.

A IMM ALGORITHMS

A.1 Cross-Round IMM Algorithm

CR-NAIMM is very similar to standard IMM and only has a few differences include several points. We have talked about them in the formal paper already.

A.2 Within-Round IMM Algorithm

WR-NAIMM (Algorithm 7) contains two main phases like standard IMM. In Phase 1, the algorithm estimates the number of RR sets needed by estimating the lower bound LB of the optimal influence spread. It is done by iteratively halving a threshold value $x_j$, and for each $x_j$ value, generating $\theta_j$ RR sets (lines 10–13), and finding the seed set $S_j$ from the current RR sets (line 14), and checking if the current $x_j$ is good enough for the lower bound estimate (line 19), where $F_M(S_j)$ is the fraction of RR sets in $M$ that are covered by $S_j$. Procedure NodeSelection($M,k$) is to greedily select $k$ nodes that covers the most number of RR sets in $M$, and it is the same as in [12], so it is omitted. In Phase 2, based on the $\theta$ RR sets generated, $S_j$ can be selected by procedure NodeSelection($M,k$). Finally, propagation from $S_j$ are observed and the active nodes are used as the feedback for the next round.

Algorithm 6: WR-NS: Within-Round Node Selection

**Input:** Multi round RR vector sets $M,t,k$  
**Output:** seed sets $S_j^o$  
1. Build count array: $c[(u,t)] = \sum_{(u,t) \in M} \delta(u,t)$, $\forall (u,t) \in V_1$;  
2. Build RR set link: $RR(u,t), \forall (u,t) \in V_1$;  
3. For all $R \in M$ covered[R] == false  
4. $S_j^o \leftarrow \emptyset; C \leftarrow V_1$;  
5. for $i = 1$ to $k$  
6. $(u,t) \leftarrow \arg \max_{(u',t') \in C \setminus S_j^o} c[(u',t')];$  
7. $S_j^o \leftarrow S_j^o \cup \{(u,t)\};$  
8. for all $R \in [RR(u,t)] \land $ covered[R] == false do  
9. covered[R] == true;  
10. for all $(u',t') \in R \land (u',t') \neq (u,t)$ do  
11. $c[(u',t')] = c[(u',t')] - 1;$  
12. return $S_j^o$.

A.3 Adaptive IMM Algorithm

Therefore, with Eq. (3), the same RR-set based algorithm can be used, and we only need to properly change the RR-set generation process and the estimation process. Algorithm 8 provides the pseudocode for the AdaIMM algorithm, which is based on the IMM algorithm in [12]. For convenience, we make the part different from IMM in blue. Phase 1 of the algorithm estimate the number $\theta$ of RR sets needed by estimating the lower bound $LB$ of the optimal spread value. This estimation is by iteratively halving a threshold value $x_j$, and for each $x_j$ value, generate $\theta_j$ RR sets (lines 8–11), and find the seed set $S_j$ from the current RR sets (line 12), and check if the current $x_j$ is good enough for the lower bound estimate (line 17), where $F_M(S_j)$ is the fraction of RR sets in $M$ that are covered by $S_j$. Procedure NodeSelection($M,k$) is to greedily select $k$ nodes that covers the most number of RR sets in $M$, and it is the same as in [12], so it is omitted. In Phase 2, $\theta$ RR sets are generated and node $S_j$ is selected greedily by procedure NodeSelection($M,k$). Finally, propagation from $S_j$ are observed and the activated nodes are used as the feedback for the next round.

The main differences from the standard IMM include several points. First, whenever we generate new RR sets in round $t$, we...
only start from roots in $V \setminus A_{t-1}$ (lines 10 and 18), as explained by Lemma 5.4. Second, when we estimate the influence spread, we need to adjust it using $n_a = n - |A_{t-1}|$ (lines 6, 7, 13, 14), again by Lemma 5.4. Third, we need to adjust $\ell$ to be $\ell + \log(2T)/\log n$, and $\e$ to $\e_0 = \e^{(1-1/\e)}T/2$. This is to guarantee that in each round we have probability at least $1 - 1/(n^2T)$ to have $S_t$ as a $(1 - 1/\e - \e_0)$ approximation, so that the result for the total $T$ rounds would come out correctly as stated in the following theorem.

\section{PROOFS FOR LEMMAS AND THEOREMS}

\subsection{B.1 Proofs for Section 3 [Non-Adaptive MRIM]}

\textbf{Theorem 3.2.} Let $S^* \in \mathbb{P}$ be the optimal solution of the non-adaptive MRIM under cross-round setting. For every $0 < \e < 1$, with probability at least 1 - $\frac{1}{n}$, the output $S^0$ of CR-Greedy satisfies

$$p(S^0) \leq \frac{1}{2} - \e p(S^*).$$

\textbf{Proof.} (Sketch). Given a set $U$ which is partitioned into disjoint sets $U_1, \ldots, U_n$, and $I = \{X \subseteq U : |X \cap U_i| \leq k_i, \forall i \in [n]\}$, $(U, I)$ is called a partition matroid. Thus, the node space $V$ with the constraint of MRIM, namely $(V', [S^* : |S^*| \leq k, \forall t \in [T]])$, is a partition matroid. This indicates that MRIM under cross-rounding is an instance of submodular maximization under partition matroid, and thus the performance of CR-Greedy has the following guarantee [32]. Meanwhile, Lemma 3.1 shows that $\rho$ is monotone and submodular in the cross-rounding setting. Therefore, the final output $S^0$ satisfies:

$$\rho(S^0) \geq \frac{1}{2} - \e p(S^*).$$

While $R = [31k^2T^2n \log(2kn^2T)/\e^2]$. Finally, the total running time is simply $O(Tknm) = O(k^2T^3n^2m\log(nT)/\e^2)$. \hfill \Box

\subsection{B.2 Proof for Section 4 [Adaptive MRIM]}

This is the proof of adaptive sub-modularity Lemma 4.2.

**Lemma 4.1.** [Adaptive Monotonicity] For all $t > 0$, for all partial realization $\psi$ with $t - 1$ rounds and $Pr[\Phi \sim \psi] > 0$, and for all item $(S_i, t)$, we have:

$$\Delta((S_i, t)) \psi > 0.$$

\textbf{Lemma 4.2.} [Adaptive Submodularity] For all $t > 0$, for all partial realization $\psi$ with $t - 1$ rounds and partial realization $\psi'$ such that $\psi'$ is a subrealization of $\psi$, i.e., $\psi' \subseteq \psi$, and for all item $(S_i, t)$, we have

$$\Delta((S_i, t)) \psi' > 0.$$

\textbf{Proof.} For all $\psi$ and $\psi'$, such that $\psi' \subseteq \psi'$ and $|\text{dom}(\psi')| = t - 1$. Let $i = |\text{dom}(\psi)| + 1$, then $i \leq t$. We want to show that $\Delta((S_i, t')) \psi > 0$. To prove this rigorously, we define a coupled distribution $\mu$ over pairs of realizations $\phi - \psi \sim \psi' - \psi$. Let $L^\phi_i$ represent the live-edge graph in round $t$ that determines the propagation. Note that in the MRT model, each realization $\phi$ corresponds to a live-edge graph sequence $(L^\phi_1, \ldots, L^\phi_T)$. We use $X$ to denote the random live-edge graph sequence $(L_1, \ldots, L_T)$. We define $\mu$ implicitly in terms of a joint distribution on $X \times X'$, where $\phi = \phi(X)$ and $\phi' = \phi'(X')$ are the realizations induced by the two distinct sets of random live-edge graphs, respectively. Hence, $\mu(\phi(X), \phi'(X')) = \mu(X, X')$. We will construct $\mu$ so that the status of live-edge graph which are unobserved by both $\psi$ and $\psi'$ and are the same in $X$ and $X'$, meaning $L_i = L'_i$ for all such live-edge graphs, or else $\mu(X, X') = 0$.

The above constraints leave us with the following degrees of freedom: we may select $X_i$ which is unobserved by $\psi$. We select them independently. Hence for all $(X, X')$ satisfying the above constraints, where $X(e, L_i(\phi))$ shows the status of $e \in E$ in the $j$-th round with $L_j$ under $\phi$

$$\mu(X, X') = \prod_{j \in [T]} \Pr(L^\phi_j),$$

and otherwise $\mu(X, X') = 0$. Note that only $\phi$ will influence the $\mu(X, X')$. 

We first claim that given a partial realization $\psi'$ with $i$ rounds, and a particular realization $\phi$, the conditional probability of $\phi$ given $\psi$, $p(\phi|\psi)$, satisfies

$$p(\phi|\psi) = \sum_{\phi'} \mu(\phi, \phi'),$$

where $\phi'$ is another particular realization, $\psi \subseteq \psi'$ and $\mu(\cdot)$ is the joint distribution. The proof is as follows.

First, we have fixed $\psi$ and $\psi'$, where $\psi \subseteq \psi'$. By definition of a coupled distribution $\mu$, if and only if $\phi \sim \psi$ and $\phi' \sim \psi'$, $\mu(\cdot) \neq 0$. We want to calculate the conditional probability of $\phi$ given $\psi$. For any fixed $\phi$, let a set $(S_i, t) \in \text{dom}(\psi')$. When $t \leq i' - 1$, if we want $\mu(\cdot) \neq 0$, we have $\phi' \sim \psi'$ and $\phi'((S_i, t)) = \psi'((S_i, t))$. When $t \geq i'$, by definition, if we want $\mu(\cdot) \neq 0$, $(S_i, t) = \psi'((S_i, t))$. Since $\phi$ is fixed, $\phi'$ is consistent with $\phi$ when $t \geq i'$. Finally, we can only find one $\phi'$ that satisfies the constraints for non-zero $\mu(\phi, \phi')$, and we have $\sum_{\phi'} \mu(\phi, \phi') = \mu(\phi(X), \phi'(X')) = \prod_{j \geq t} \Pr(L^j_{\psi'}) = p(\phi | \psi)$, where the last equality holds due to the definition of $\phi$ and $\psi$.

We then claim that given a partial realization $\psi'$ with $i$ rounds, and a particular realization $\phi'$, the conditional probability of $\phi'$ given $\psi'$, $p(\phi'|\psi')$, satisfies that

$$p(\phi'|\psi') = \sum_{\phi} \mu(\phi, \phi'),$$

where $\phi$ is the other a particular realization, $\psi \subseteq \psi'$ and $\mu(\cdot)$ is the joint distribution. We prove this as follows.

Again, we have fixed $\psi$ and $\psi'$, where $\psi \subseteq \psi'$. We want to calculate the conditional probability of $\phi'$ given $\psi'$, where $\phi'$ is a fixed realization. As the same, if and only if $\phi \sim \psi$ and $\phi' \sim \psi'$, $\mu(\cdot) \neq 0$. Consider a set $(S_i, t) \in \text{dom}(\psi')$. When $t \leq i' - 1$, if we want $\mu(\cdot) \neq 0$, $(S_i, t) = \psi'((S_i, t))$. When $t \geq i'$, if we want $\mu(\cdot) \neq 0$, $(S_i, t) = \psi'((S_i, t))$. When $i \leq t \leq i' - 1$, we want $\mu(\cdot) \neq 0$ with any $\phi$. Finally, we can only find multiple $\phi$ satisfies the constraints for non-zero $\mu(\phi, \phi')$, and we have $\sum_{\phi} \mu(\phi, \phi') = \sum_{\phi} \mu(\phi(X), \phi'(X')) = \prod_{j \geq t} \Pr(L^j_{\psi'}) = \prod_{j \geq t} \Pr(L^j_{\psi'}) \sum_{s} \prod_{j \geq t} \Pr(L^j_{\psi'}) = p(\phi'|\psi').$

We next claim that for all $(\phi, \phi') \in \text{support}(\mu)$,

$$f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi') 
\leq f(\text{dom}(\phi') \cup (S_i, i'), \phi) - f(\text{dom}(\phi'), \phi).$$

A sufficient condition for the above inequality is

$$\Gamma(L^j_{\psi'} \backslash S_i) \cap \bigcup_{j=1}^{t-1} \Gamma(L^j_{\psi'} \backslash S_i) \subseteq \Gamma(L^j_{\phi'} \backslash S_i) \cap \bigcup_{j=1}^{t-1} \Gamma(L^j_{\phi'} \backslash S_i).$$

Let $C' = \Gamma(L^j_{\phi'} \backslash S_i)$, $B' = \Gamma(L^j_{\phi'} \backslash S_i)$ denote the active before and after select $S_i$ as the beginning nodes at $i'$ round respect to $\psi'$ and $\phi'$. Similarity, define $C$ a $B$ respect to $\psi$ and $\phi$. Let $D = C \setminus B$ and $D' = C' \setminus B'$. It suffices to show that $B \subseteq B'$ and $D' \subseteq D$ to prove the above inequality, which we will now do. By definition, we know that $C' = C$. Based on adaptive monotonicity, we have $B \subseteq B'$. Then, it is very obviously to prove that $D' \subseteq D$.

Now we proceed to use it to show $\Delta((S_i, i')|\psi) \geq \Delta((S_i, i')|\psi')$.

$$\Delta((S_i, i')|\psi') = \sum_{\phi} \mu(\phi|\psi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi')$$

$$= \sum_{\phi} \sum_{\phi'} \mu(\phi, \phi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi'))$$

$$= \sum_{\phi} \sum_{\phi'} \mu(\phi, \phi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi'))$$

$$\leq \sum_{\phi} \mu(\phi, \phi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi'))$$

$$= \sum_{\phi} \sum_{\phi'} \mu(\phi, \phi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi'))$$

$$= \sum_{\phi} \sum_{\phi'} \mu(\phi, \phi') f(\text{dom}(\phi') \cup (S_i, i'), \phi') - f(\text{dom}(\phi'), \phi'))$$

$$= \Delta((S_i, i')|\psi)$$

This finishes the proof of the adaptive submodularity.

\[\Box\]

**Theorem 4.3.** Let $\pi^{\text{AG}}$ represents the policy corresponding to the AdaGreedy algorithm. For any $\epsilon > 0$ and $\ell > 0$, if we use $R = [31k^2 n \log(2kn^{\ell+1}T)/\epsilon^2]$ simulations for each influence spread estimation, then with probability at least $1 - \frac{1}{T^2}$,

$$f_{\text{avg}}(\pi^{\text{AG}}) \geq \left(1 - e^{-(1-\epsilon)^{1/\ell} - \epsilon}\right) f_{\text{avg}}(\pi^\ast).$$

In this case, the total running time for $T$-round AdaGreedy is $O(k^3 T n m \log(nT)/\epsilon^2)$.

**Proof (Sketch).** From the general result of Theorem 5.2 in [9], we know that if in each round we find an $\alpha$ approximation of the best solution of the round, then the greedy adaptive policy is an $1 - e^{-\alpha}$ approximation of the optimal greedy policy $\pi^\ast$. Then, following essentially the same high probability argument as in the proof sketch of Theorem 3.3, we could conclude that when using $R = [31k^2 n \log(2kn^{\ell+1}T)/\epsilon^2]$ Monte Carlo simulations for each influence spread estimation, with probability at least $1 - \frac{1}{n^2}$, in all rounds the selected $S_i$ is an $1 - 1/e - \epsilon$ approximation, with $\epsilon_i = e^{(1-1/e)^{1/\ell} - \epsilon}$. Together with Theorem 5.2 in [9], we have that with probability at least $1 - \frac{1}{n^2}$, $f_{\text{avg}}(\pi^{\text{AG}}) \geq \left(1 - e^{-(1-\epsilon)^{1/\ell} - \epsilon}\right) f_{\text{avg}}(\pi^\ast)$. The running time is $O(TknRm) = O(k^3 T n^2 m \log(nT)/\epsilon^2)$.

\[\Box\]

**B.3 Proofs for Section 5.1.1 [CR-NAIMM]**

**Lemma 5.1.** For any node-round pair seed set $S$,

$$\rho(S) = n \cdot \mathbb{E}[|S \cap R | \neq 0|].$$

where the expectation is taken over the randomness of $R$. 

Proof.

\[
\mathbb{E}[\{S \cap R \neq \emptyset\}] = \sum_{v \in V} \Pr\{v = \text{root}(R)\} \cdot \mathbb{E}[\{S \cap R \neq \emptyset\} \mid v = \text{root}(R)] \\
= \frac{1}{n} \cdot \sum_{v \in V} \mathbb{E}[\{S \cap R_v \neq \emptyset\}] \\
= \frac{1}{n} \sum_{v \in V} \mathbb{E}\left[\sum_{i=1}^{\left\lfloor \frac{|\Gamma(L_i, S)|}{k} \right\rfloor} I[v \in \bigcup_{i=1}^{k} \Gamma(L_i, S_i)]\right] \\
= \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^{k} |\Gamma(L_i, S_i)|\right] \\
= \frac{1}{n} \cdot \rho(S).
\]

where Eq. (6) is based on the equivalence between multi round RR sets and all T rounds live-edge graphs, and the expectation from this point on is taken over the random live-edge graphs $L_1, L_2, \ldots, L_T$.

Lemma B.1. CR-NS returns the multi-round sets $S$ which covers the multi-round RR set with $\epsilon$-approximate optimal solution. The execution time of CR-NAIMM is $O(T^2 kn + \sum_{R \in M} |R|)$.

Proof. CR-NAIMM is the multi-round RR set implementation of the CR-Greedy. Due to the property of partition matroid, we know the greedy solution returns a $\frac{1}{2}$-approximate solution.

About time complexity, in each iteration, it takes $O(\sum_{R \in M} |M|)$ to generate the count array and RR set link for each node in lines 1 and 2 of CR-NS. In next $Tk$ round, every time it takes $O(Tn)$ for every available seed with the maximum influence, and so the total time is $O(T^2 kn)$. The rest part of CR-NS adjusts the $c((u, t))$ and covered[$R$] (from lines 10 to 12). While $v$ is a chosen seed, each multi-round RR set is traversed at most once for updating both $c((u, t))$ and covered[$R$], so it takes $O(\sum_{R \in M} |R|)$ for the rest part. So, the total execution time of CR-NS is $O(T^2 kn + \sum_{R \in M} |R|)$.

Assume $\hat{\rho}(S, \omega)$ is the random estimation of $\rho(S)$, and $\omega$ is a random sample in space $\Omega$. Let $S^*$ be the optimal solution of $\rho(S)$, so $S^* = \arg\max_{S \subseteq \mathcal{V}, |S| \leq k, t \in \mathcal{T}} \sum_{s \in S} \rho(S, \omega)$, and $OPT = \rho(S^*)$. For a random sample $\omega \in \Omega$, let $\hat{S}(\omega)$ be the solution of the greedy algorithm of $\hat{\rho}(\cdot, \omega)$. For a $\epsilon > 0$, we have a bad solution $S$ under $\epsilon$, if $\rho(S) < (1-1/\epsilon-\epsilon)$-OPT. Next, we give the theoretical guarantee about the approximate solution of the greedy algorithm.

Lemma B.2. For any $\epsilon, \epsilon_1 \in (0, \epsilon/(1/2))$, and any $\delta_1, \delta_2 > 0$, if (a) $Pr\{\hat{\rho}(S^*(\omega), \omega) \geq (1 - \epsilon_1) \cdot OPT\} \geq 1 - \delta_1$, (b) $Pr\{\hat{\rho}(S(\omega), \omega) \geq \frac{1}{2}(1 - \epsilon_1) \cdot OPT\} \leq \delta_2 / \left(\frac{k}{\epsilon_1^2}\right)^T$; (c) for any $\omega \in \Omega$, $\hat{\rho}(S, \omega)$ is monotone and submodular, and the greedy solution $S^*(\omega)$ is a $(\frac{1}{2} - \epsilon)$-approximate solution with at least $1 - \delta_1 - \delta_2$ probability.

Proof. Since $\hat{\rho}(S, \omega)$ holds monotone and submodular, by Lemma B.2, we know

\[
\hat{\rho}(S^*(\omega), \omega) \geq \frac{1}{2} \cdot \hat{\rho}(S^*, \omega).
\]

Due to condition (a), we know

\[
\hat{\rho}(S^*(\omega), \omega) \geq \frac{1}{2} - \epsilon_1) \cdot OPT,
\]

with at least $1 - \delta_1$ probability.

Due to condition (b), we know there is a bad $S$ causing $\hat{\rho}(S, \omega) \geq \frac{1}{2}(1 - \epsilon_1) \cdot OPT$ with at most $\delta_2$ probability. Because of at most $\left(\frac{k}{\epsilon_1^2}\right)^T$ bad samples for $k$ seeds in $T$ rounds, the probability of each bad $S$ is at most $\delta_2 / \left(\frac{k}{\epsilon_1^2}\right)^T$. By using union bound, we know $\hat{\rho}(S^*(\omega)) \geq \frac{1}{2}(1 - \epsilon_1) \cdot OPT$ with at least $1 - \delta_1 - \delta_2$ probability.

In multi-round RR-set-based algorithms, the random sample is a multi-round RR set $R$ by random generation. Let $\theta = |M|$ is the number of multi-round RR sets, and then $\hat{\rho}(S, \omega) = \hat{\rho}(S, M) = n \cdot 1 \cdot (S \cap M)/\theta$ which returns the number of $R \in M$ covered by $S$. The output of CR-NS is $\hat{S} = \hat{S}(M)$.

Definition B.3. (Martingale) A sequence of random variables $X_1, X_2, X_3, \ldots$ is a martingale, for all $i \geq 1$, if and only if $\mathbb{E}[|X_i|] < +\infty$ and $\mathbb{E}[X_{i+1} = |X_1, X_2, \ldots, X_i|] = |X_i|$. Let $Z_i = \sum_{j=1}^{i}(X_j - \mu)$, a sequence $Z_1, Z_2, \ldots, Z_T$ is a martingale.

Lemma B.4. Suppose $X_1, X_2, X_3, \ldots, X_T$ are random variable between $[0, 1]$. There is a $\mu \in [0, 1]$ that $\mathbb{E}[X_i | X_1, X_2, \ldots, X_{i-1}] = \mu$ for all $i \in [T]$. Let $Z_i = \sum_{j=1}^{i}(X_j - \mu)$, a sequence $Z_1, Z_2, \ldots, Z_T$ is a martingale.

Proof. This lemma can be proved directly. First, $\mathbb{E}[Z_{i+1} | Z_1, Z_2, \ldots, Z_i] = \mathbb{E}[\sum_{j=1}^{i+1}(X_j - \mu) | Z_1, Z_2, \ldots, Z_i] = \sum_{j=1}^{i}(X_j - \mu) + \mathbb{E}[X_{i+1} - \mu | Z_1, Z_2, \ldots, Z_i] = Z_i$. Second, $\mathbb{E}[|Z_i|] \leq 2i \leq +\infty$. So, a sequence $Z_1, Z_2, \ldots, Z_T$ is a martingale.

Corollary B.5. Assume $X_1, X_2, X_3, \ldots, X_T$ are random variable between $[0, 1]$. There is a $\mu \in [0, 1]$ that $\mathbb{E}[X_i | X_1, X_2, \ldots, X_{i-1}] = \mu$ for all $i \in [T]$. Let $Y = \sum_{i=1}^{T}X_i$, for any $y > 0$,

\[
\Pr\{Y - y \mu \geq y \cdot \mu\} \leq \exp\left(-\frac{y^2}{2 + \frac{y^2}{2}}\right).
\]

For any $0 < y < 1$,

\[
\Pr\{Y - y \mu \leq -y \cdot \mu\} \leq \exp\left(-\frac{y^2 \mu^2}{2}\right).
\]

Lemma B.6. Assume a set $M = \{R_1, R_2, \ldots, R_B\}$ satisfies the martingale condition. For any $\epsilon > 0, \epsilon_1 \in (0, \frac{1}{2})$, and any $\delta_1, \delta_2 > 0$, let

\[
\theta(1) = 2n \cdot \ln(1/\delta_1), \theta(2) = n \cdot T \ln(\frac{\ln(n)}{\epsilon_1^2}) \cdot \frac{OPT \cdot (\epsilon - \epsilon_1)^2}{\epsilon_1^2}.
\]

If \( \theta \geq \theta(1) \), $Pr\{\hat{\rho}(S^*, R) \geq (1 - \epsilon_1) \cdot OPT\} \geq 1 - \delta_1$; if $\theta \geq \theta(2)$, for any bad $S$ under $\epsilon$, $Pr\{\hat{\rho}(S, R) \geq \frac{1}{2}(1 - \epsilon_1) \cdot OPT\} \leq \delta_2 / \left(\frac{n}{\epsilon_1^2}\right)^T$.
Proof. Since $M$ satisfies the martingale condition, by using Corollary B.5, we have
\[
\Pr\{\hat{\theta}(S^*), \mathcal{R} < (1 - \varepsilon_1) \cdot \text{OPT}\}
\leq \Pr\{n \cdot \sum_{i=1}^{\theta} X_i(M(S^*)) \leq (1 - \varepsilon_1) \cdot \rho(S^*)\}
\leq \Pr\left\{ \sum_{i=1}^{\theta} X_i(M(S^*)) - \cdot \rho(S^*)/n \leq -(\varepsilon_1 \cdot \rho(S^*)/n) \right\}
\leq \exp\left( -\frac{\varepsilon_1^2}{2} \cdot \rho(S^*)/n \right)
\leq \exp\left( -\frac{\varepsilon_1^2}{2} \cdot \frac{2n \cdot \ln(1/\delta_1)}{\text{OPT} \cdot n} \cdot \frac{\rho(S^*)}{n} \right) = \delta_1.
\]
Let $\varepsilon_2 = \varepsilon - \frac{1}{2}\varepsilon_1$. Let a S is bad under $\varepsilon$, thus $\rho(S) < (\frac{1}{2} - \varepsilon) \cdot \text{OPT}$. Then we have
\[
\Pr\{\hat{\theta}(S, \mathcal{R}) \geq \frac{1}{2}(1 - \varepsilon_1) \cdot \text{OPT}\}
\geq \Pr\left\{ \sum_{i=1}^{\theta} X_i(M(S)) - \cdot \rho(S)/n \geq \theta/n \cdot \frac{1}{2}(1 - \varepsilon_1) \cdot \text{OPT} - \rho(S) \right\}
\leq \Pr\left\{ \sum_{i=1}^{\theta} X_i(M(S)) - \cdot \rho(S)/n \geq \theta/n \cdot \varepsilon_2 \cdot \text{OPT} \right\}
\leq \exp\left( -\frac{\varepsilon_2^2}{2} \cdot \rho(S)/n \right)
\leq \exp\left( -\frac{\varepsilon_2^2}{2} \cdot \frac{2n \cdot \ln(1/\delta_2)}{\text{OPT} \cdot n} \cdot \frac{\rho(S)}{n} \right) = \delta_2.
\]
\[\square\]

Corollary B.7. Assume $M$ satisfies the martingale condition. We set the parameters $\delta_1 = \delta_2 = 1/(4n^2)$, and $\varepsilon_1 = \varepsilon \cdot \frac{\alpha}{\alpha + \beta}$ in Lemma B.6, where
\[
\alpha = \sqrt{\ln n + \ln 4}, \beta = \sqrt{\frac{1}{2} \left( \ln \left( \frac{n}{k} \right) + \alpha^2 \right)}.
\]
While $\theta \geq \frac{2\alpha \ln(1/\delta_1) + \beta^2}{\text{OPT} \cdot n}$, the CR-NAIMM returns $(\frac{1}{2} - \varepsilon)$-approximate solution with at least $1 - 1/(2n^2)$ probability.

Proof. While $\delta_1 = \delta_2 = 1/(4n^2)$, we set the $\theta_1(1) = \theta_2(1)$ in Lemma B.6, then we can find $\theta_1(1) = \theta_2(1) = \frac{2n \cdot (\alpha + \beta)^2}{\text{OPT} \cdot n^2}$ by calculation. The condition (a) (b) of Lemma B.6 is satisfied under this setting. For condition (c), it is very easily to prove the monotone and submodular of $\hat{\theta}(S, M) = n \cdot \kappa(S, M)/\theta$. Thus, by Lemma B.2, we know that CR-NS returns $\hat{\theta}$ which is a $1 - 1/e - \varepsilon$-approximate solution with at least $1 - 1/(2n^2)$. $\square$

Lemma B.8. Let $M = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_t\}$ is the multi-round RR sets in CR-NAIMM. For all subset $S \subseteq V$, for all $i \in [t]$, $\mathbb{E}[X_i(M(S)) | X_1^t(M(S), X_2^t(M(S), \ldots, X_{i-1}^t(M(S)) = \rho(S)/n$, where $X_i^t(M(S) = I(S \cap \mathcal{R}_i \neq \emptyset)$. So $M$ satisfies all martingale conditions.

Proof. $X_i^t(M(S)$ is not indepent with $X_i^t(M(S), X_i^t(M(S), \ldots, X_i^t(M(S))$, because the generation of $\mathcal{R}_i$ is decided by $\mathcal{R}_1, \ldots, \mathcal{R}_{i-1}$. However, while the decision of $\mathcal{R}_i$ generation is made, $\mathcal{R}_i$ is generated by random pick the root $v \in V$ and do reverse propagation in each round for multi-round RR set, which is totally independent and not relevant with $\mathcal{R}_1, \ldots, \mathcal{R}_{i-1}$. By Lemma B.1, we have $\mathbb{E}[X_i^t(M(S)| X_1^t(M(S), X_2^t(M(S), \ldots, X_{i-1}^t(M(S)) = \rho(S)/n$. By Lemma B.4, $M$ satisfies all martingale conditions. $\square$

Lemma B.9. For each $i = 1, 2, \ldots, \lfloor \log_2 n \rfloor - 1$, (1) If $x_i = n/2^i > \text{OPT}$, $\hat{\theta}(S, M_i) \geq (1 + \varepsilon) \cdot x_i$ with at most $\frac{1}{2} \log_2 n$ probability; (2) If $x_i = n/2^i \leq \text{OPT}, \hat{\theta}(S, \hat{M}_i) \geq (1 + \varepsilon) \cdot \text{OPT}$ with at most $\frac{1}{2} \log_2 n$ probability.

Proof. Let $M = \{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_t\}$. We know $\hat{\theta}(S, M_i) = n \cdot \sum_{j=1}^{\theta_i} X_j^i(S)/\theta_i$. By Lemma B.8, we know $M$ satisfies the martingale conditions, so we can use Corollary B.5 for rest proof. First, we need to prove (1) Suppose $x_i \geq n/2^i > \text{OPT}$. For any $S$, we have
\[
\Pr\{\hat{\theta}(S, M_i) \geq (1 + \varepsilon) \cdot x_i\}
\leq \text{Pr}\left\{ n \cdot \sum_{j=1}^{\theta_i} X_j^i(S)/\theta_i \geq (1 + \varepsilon) \cdot x_i \right\}
\leq \text{Pr}\left\{ \sum_{j=1}^{\theta_i} X_j^i(S) - \cdot \rho(S)/n \geq \theta_i \cdot (1 + \varepsilon) \cdot x_i/n \cdot \theta_i \cdot \rho(S)/n \right\}
\leq \text{Pr}\left\{ \sum_{j=1}^{\theta_i} X_j^i(S) - \cdot \rho(S)/n \geq \theta_i \cdot \varepsilon \cdot x_i/n \right\}
\leq \text{Pr}\left\{ \sum_{j=1}^{\theta_i} X_j^i(S) - \cdot \rho(S)/n \geq \theta_i \cdot (1 + \varepsilon) \cdot x_i/n \right\}
\leq \text{Pr}\left\{ \sum_{j=1}^{\theta_i} X_j^i(S) - \cdot \rho(S)/n \geq (1 + \varepsilon) \cdot x_i/n \right\}
\leq \text{exp}\left( -\frac{(\varepsilon \cdot x_i)^2}{n} \cdot \frac{n \cdot \ln(n/\delta_2)}{\text{OPT} \cdot n} \cdot \frac{\rho(S) \cdot n}{(1 + \varepsilon)^2} \right) = \delta_2.
\]
\[\square\]
Similarly to prove (1), we show (2) that suppose $x_i \leq n/2^{t'} \cdot OPT$, for any multi-round set $S$, we have

$$Pr(\hat{\rho}(S, M_i) \geq (1 + \epsilon') \cdot OPT)$$

$$= Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S_i)/\theta_i \geq (1 + \epsilon') \cdot OPT \right\}$$

$$= Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S_i) - \theta_i \cdot \rho(S)/n \geq \theta_i \cdot (1 + \epsilon') \cdot OPT/n - \theta_i \cdot \rho(S)/n \right\}$$

$$\leq Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S_i) - \theta_i \cdot \rho(S)/n \geq \theta_i \cdot \epsilon' \cdot OPT/n \right\}$$

$$= Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S_i) - \theta_i \cdot \rho(S)/n \geq (\epsilon' \cdot OPT/\rho(S)) \cdot ((\theta_i / n) \rho(S)/n) \right\}$$

$$\leq \exp\left( -\frac{(\epsilon' \cdot OPT/\rho(S))^2}{2 + \frac{2}{\epsilon'} \cdot OPT/\rho(S)} \left( \frac{\theta_i}{n} \rho(S)/n \right) \right)$$

$$= \exp\left( -\frac{(\epsilon' \cdot OPT)^2}{2 \rho(S) + \frac{2}{\epsilon'} \cdot OPT} \left( \frac{\theta_i}{n} \right) \right)$$

$$\leq \exp\left( \frac{\epsilon'^2 \cdot OPT}{2 + \frac{2}{\epsilon'} \cdot OPT} \left( \frac{\theta_i}{n} \right) \right)$$

$$\leq \exp\left( \frac{\epsilon'^2 \cdot x_i}{n \cdot (2 + \frac{2}{\epsilon'} \cdot OPT)} \right)$$

$$= \exp\left( \frac{\epsilon'^2 \cdot x_i}{n \cdot (2 + \frac{2}{\epsilon'} \cdot OPT)} \left( T \ln \left( \frac{n!}{\sqrt{2\pi n}} \right) + \ell \ln n + \ln 2 + \ln \log_2 n \right) \right)$$

$$= \frac{1}{2 \left( \frac{n}{\epsilon'} \right)^{1/2} \cdot \log \log n}$$

**Lemma B.10.** The lower bound $LB$ in CR-NAIMM is smaller than $OPT$ with at least $1 - \frac{1}{2\epsilon n}$ probability.

**Proof.** Let $LB_i = \rho(S_i^*, M_i)$. First, we discuss the situation while $OPT \geq x_{[\log_2 n]}$. Let $i$ to be smallest round when $OPT \geq x_i$, and then for all $i' \leq i - 1$, $OPT < x_{i'}$. By Lemma B.9 (1), $\hat{\rho}(S_i, M_i) \geq (1 + \epsilon') \cdot x_i$ with at least $\frac{1}{2 \epsilon n}$ probability. By using union bound, we know that for $i' \leq i - 1$, $\hat{\rho}(S_i, M_i) < (1 + \epsilon') \cdot x_i$ with at least $\frac{1}{2 \epsilon n}$ probability. It means the for loop would not break out before $i - 1$ round with at least $1 - \frac{1}{2 \epsilon n}$ probability, so as to $LB = LB_{i''}$, $i'' \geq i$ or $LB = 1$. While $i'' \geq i$ and $x_{i''} < OPT$, by Lemma B.9 (2), we know that $\hat{\rho}(S_i, M_i) > (1 + \epsilon') \cdot OPT$, i.e. $LB_{i''} > OPT$, with at most $\frac{1}{2 \epsilon n}$ probability. Together, we know that $LB > OPT$ with at most $\frac{1}{2 \epsilon n}$ probability.

If $OPT < x_{[\log_2 n]} - 1$, similar to above discussion, we also will know that the for loop would not break out in any round with at least $1 - \frac{1}{2 \epsilon n}$ probability. In this case, $LB = 1$, and $LB \leq OPT$. □

**Lemma B.11.** (Correctness) For any $\epsilon > 0$, $\ell > 0$, the output $S^\ell$ of the cross-round algorithm CR-NAIMM is $(\ell \cdot \epsilon')$-approximate solution with at least $1 - \frac{1}{\ell \epsilon n}$ probability.

**Proof.** Lemma B.10 proves that $LB \leq OPT$ with at least $1 - \frac{1}{2 \epsilon n}$ probability. In this case, we have $\theta \geq \frac{2n(1/2 + \epsilon'^2)}{OPT^2}$.

Lemma B.8 proves the multi-round RR sets $M$ satisfies the martingale condition. By using Corollary B.7, while $\theta \geq \frac{2n(1/2 + \epsilon'^2)}{OPT^2}$, the cross-round node selection CR-NS returns a $(1/2 - \epsilon')$-approximate solution. Overall, CR-NAIMM returns $(\ell \cdot \epsilon')$-approximate solution $S^\ell$ with at least $1 - \frac{1}{\ell \epsilon n}$ probability. □

**Lemma B.12.** For each $i = 1, 2, \ldots, [\log_2 n] - 1$, if $OPT \geq (1 + \epsilon') \cdot x_i/(1/2)$, $\hat{\rho}(S_i, M_i) \geq (1/2) \cdot OPT/(1 + \epsilon')$ and $\hat{\rho}(S_i, M_i) \geq (1 + \epsilon') \cdot x_i$ with at least $1 - \frac{1}{2 \epsilon n}$ probability.

**Proof.** Let $M_i = \{R_1, R_2, \ldots, R_{\theta_i}\}$. By Lemma B.8, we know $M_i$ satisfies the martingale conditions. Then we can use the Corollary B.5 for proof. Assume $OPT \geq (1 + \epsilon') \cdot x_i/(1/2)$, in $i$-th loop. The output $S_i$ is the $\epsilon'$-approximate solution of $M_i$. Let $\rho(S^\ell)$ = OPT, and then we have $\hat{\rho}(S_i, M_i) \geq (S^\ell, M_i) \geq (1/2) \cdot OPT/(1 + \epsilon')$. Now we have

$$Pr\{\hat{\rho}(S_i, M_i) \leq (1/2) \cdot OPT/(1 + \epsilon')\}$$

$$\leq Pr\{\hat{\rho}(S^\ell, M_i) \leq (1/2) \cdot OPT/(1 + \epsilon')\}$$

$$= Pr\left\{ n \cdot \sum_{j=1}^{t_i} X_j^M(S^\ell)/\theta_i \leq OPT/(1 + \epsilon') \right\}$$

$$= Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S^\ell) \leq \frac{1}{1 + \epsilon'} \cdot \theta_i \cdot OPT/n \right\}$$

$$= Pr\left\{ \sum_{j=1}^{t_i} X_j^M(S^\ell) \leq \frac{1}{1 + \epsilon'} \cdot \theta_i \cdot OPT/n - \theta_i \cdot \rho(S^\ell)/n \right\}$$
\[
\theta \leq \max \left( \frac{\lambda^*}{LB}, \frac{\lambda^*}{\log_2 n} \right) \\
\leq \max \left\{ \lambda^*, \lambda^* \cdot (1 + \epsilon) \right\}
\]

By Lemma B.12, we know if the loop break out in i-th round, 
\[ LB = \hat{\rho}(S_i, M_i) \geq (1 - 1/2) \cdot OPT/(1 + \epsilon'). \]
So, 
\[
\theta \leq \max \left[ \frac{\lambda^*}{LB}, \frac{\lambda^*}{\log_2 n} \right] \leq \max \left[ \lambda^* \cdot (1 + \epsilon^2), \lambda^* \cdot (1 + \epsilon)^2 \right] \\
\leq 2 \max\{\lambda^*, \lambda^* \cdot (1 + \epsilon^2)\} \\
\leq \frac{2 \epsilon^2 \cdot OPT}{(1 - 1/2) \cdot OPT} \cdot (1 - 1/2) \cdot OPT \\
\leq \frac{2 \epsilon^2 \cdot OPT}{(1 - 1/2) \cdot OPT} \\
\leq \frac{2 \epsilon^2 \cdot OPT}{(1 - 1/2) \cdot OPT}.
\]

For loop does not break out i-th round with at most 
\[ \frac{1}{2 \epsilon^2} n \log_2 n \]
probability. If we consider the worst case, for loop break out at the end (i.e. ([\log_2 n] - 1)-th round), and LB = 1. Then we have \[ \theta \leq \max\{\lambda^*, \lambda^* \}. \]
Let put two possible cases together, we have 
\[
E[\theta] \leq \frac{4 \epsilon^2}{(1 + \epsilon)^2} \cdot O \left( \frac{(Tk + \ell)Tn \log n}{OPT \cdot \epsilon^2} \right) + 1.
\]

Proof. First, we need to discuss two situations (1) OPT \( (1 + \epsilon^2) \cdot x_1/(1 - 1/2). \) Due to \( x_1/(\log_2 n) \leq 4, \) OPT \( (1 + \epsilon^2)/(1 - 1/2). \) In worst case, \( \theta \leq \max\{\lambda^*/x_1/(\log_2 n) \leq 1}. \]

Next we discuss the second situation (2) OPT \( (1 + \epsilon^2) \cdot x_1/(1 - 1/2). \) Let i be the minimum loop number, while OPT \( (1 + \epsilon^2) \cdot x_1/(1 - 1/2), \) and then we have OPT \( (1 + \epsilon^2) \cdot x_1/(1 - 1/2). \) Finally, \( \theta \leq (1 + \epsilon^2) \cdot x_1, \) with at least \( 2 \epsilon^2 \cdot n \log_2 n \) probability. In this case, it means the loop will break out in i-th for loop round. Thus, the number of multi-round RR sets is 
\[
\theta_i = \frac{\lambda^*}{x_i} \leq \frac{\lambda^* \cdot (2 + \epsilon^2)}{(1 - 1/2) \cdot OPT} \\
\]

For convenience, let us define 
\[
\lambda^* = \frac{2n(1/2 \cdot \alpha + \beta)^2}{\epsilon^2}, \\
\lambda^* = \frac{n(2 + \frac{1}{\epsilon^2}) \cdot (\ln \frac{\epsilon}{\ell} + \ell \ln n + \ln 2 + \ln \log_2 n)}{\epsilon^2}.
\]

Lemma B.13. The lower bound LB \( (1 - 1/2) \cdot OPT \) with at least \( 1 - \frac{1}{2 \epsilon^2} n \log_2 n \) probability. Based on this, if \( k < n \) for each round, we get the upper bound 
\[
E[\theta] \leq \frac{4 \epsilon^2}{(1 + \epsilon)^2} \cdot O \left( \frac{(Tk + \ell)Tn \log n}{OPT \cdot \epsilon^2} \right) + 1.
\]

Finally, we use the \( \lambda^* \) and \( \lambda^* \) defined before and \( \epsilon^* = \sqrt{\epsilon}, \) we have 
\[
E[\theta] \leq \frac{4 \epsilon^2}{(1 + \epsilon)^2} \cdot O \left( \frac{(Tk + \ell)Tn \log n}{OPT \cdot \epsilon^2} \right) + 1.
\]
Due to $E_{\mathcal{R}}[\rho(\mathcal{R})] \leq \text{OPT}$, the total execution time is
\[
O(2kn \log n + \mathbb{E}[\theta] \cdot (EPT + 1))
\]
\[
= O \left( \frac{T^2kn \log n + (Tk + l)Tn \log n}{\text{OPT}} \cdot \frac{m}{n} \cdot \mathbb{E}[\rho(\mathcal{R})] + 1 \right)
\]
\[
= O \left( \frac{T^2(k + l)(m + n) \log n}{\epsilon^2} \right).
\]

\[\square\]

**Theorem 5.2.** Let $S^\ast$ be the optimal solution of the non-adaptive MRIM. For every $\epsilon > 0$ and $\ell > 0$, with probability at least $1 - \frac{\epsilon}{n}$, the output $S^\circ$ of the cross-round algorithm CR-NAIMM satisfies
\[
\rho(S^\circ) \geq \left( \frac{1}{2} - \epsilon \right) \rho(S^\ast).
\]
In this case, the total running time for $T$-round CR-NAIMM is $O(T^2(k + \ell)(m + n) \log n/\epsilon^2)$.

**Proof.** By Lemma B.11, we prove the correctness of the cross-round algorithm CR-NAIMM that the output will hold $1/2 - \epsilon$-approximate solution to multi-round influence maximization. By Lemma B.14, we show the time complexity of the algorithm CR-NAIMM is $O(T^2(k + \ell)(m + n) \log n/\epsilon^2)$. Thus, the theorem holds. \[\square\]

**B.4 Proofs for Section 5.1.2 [WRI-NAIMM]**

**Lemma 5.3.** For any node-round pair seed set $S$,
\[
\rho(S^{t-1} \cup S_t) - \rho(S^{t-1})
\]
\[
= n \cdot \mathbb{E}[\mathbb{I}(L_t(S) = 0) \wedge (S_t \cap R_t \neq \emptyset)]
\]
\[
= n \cdot \Pr[S^{t-1} \cap R^{t-1} = 0] \cdot \mathbb{E}[\mathbb{I}(S_t \cap R_t \neq \emptyset) \mid S^{t-1} \cap R^{t-1} = 0],
\]
where the expectation is taken over the randomness of $R$.

**Proof.**
\[
\mathbb{E}[\mathbb{I}(L_t(S) = 0) \wedge (S_t \cap R_t \neq \emptyset)]
\]
\[
= \sum_{v \in V} \Pr(u = \text{root}(R)) \cdot \mathbb{E}[\mathbb{I}(L_t(S) = 0) \mid u = \text{root}(R)]
\]
\[
= \frac{1}{n} \sum_{v \in V} \mathbb{E}[\mathbb{I}(L_t(S) = 0) \mid u = \text{root}(R)]
\]
\[
= \frac{1}{n} \sum_{v \in V} \left[ \mathbb{I}(v \notin \bigcup_{i=1}^{t-1} \Gamma(L_t, S_i)) \cdot \mathbb{I}(v \in \Gamma(L_t, S_t)) \right]
\]
\[
= \frac{1}{n} \sum_{v \in V} \left[ \mathbb{E}[\bigcup_{i=1}^{t-1} \Gamma(L_t, S_t)] - \mathbb{E}[\bigcup_{i=1}^{t-1} \Gamma(L_t, S_t)] \right]
\]
\[
= \frac{1}{n} \left( \rho(S^t) - \rho(S^{t-1}) \right)
\]
\[
= \frac{1}{n} \left( \rho(S^{t-1} \cup S_t) - \rho(S^{t-1}) \right),
\]
where Eq. (9) is based on the equivalence between multi-round RR sets and the RR sets rooted by the remaining seeds in round $t - 1$, and the expectation from this point on is taken over the random live-edge graph $L_t$. \[\square\]