On elliptic boundary value problems of order $2m$ in cylindrical domain of large size

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Abstract

We study in this work the convergence of the solution of general elliptic boundary value problems in cylindrical domain, when some directions of the domain go to $+\infty$.

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1 Introduction

The present article generalizes the results of A. Rougirel and M. Chipot in [2], [3] and [1] for the elliptic problems of order 2. We interest in elliptic problems of order $2m$, with conditions of the Dirichlet type on cylindrical domains of $\mathbb{R}^n$. We study the asymptotic behavior of the solution when the cylindrical domain becomes unbounded in several directions. In the second section, we show under certain conditions on data that the solution of such problems converges towards a solution of an elliptic problem in $\mathbb{R}^{n-p}$, in Sobolev space $H^m$, with a speed faster than any power of $\frac{1}{\ell}$; in the third section, we show the same results in higher order Sobolev spaces.

Let $\omega$ be a bounded Lipschitz domain of $\mathbb{R}^{n-p}$ and $n > p \geq 1$. For a positive number $\ell$, we consider the cylinder of $\mathbb{R}^n$

$$\Omega_\ell = (-\ell, \ell)^p \times \omega.$$ 

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we will set

$$X_1 = (x_1, \ldots, x_p), \quad X_2 = (x_{p+1}, \ldots, x_n).$$

We consider the boundary value problems defined by

$$\begin{cases}
    A_\omega u = f & \text{in } \omega, \\
    \partial^k u / \partial \nu^k = 0 & k = 0, \ldots, m - 1 \text{ on } \partial \omega,
\end{cases}$$ (2)

with

$$A_\omega u = \sum_{\alpha, \beta \in \mathbb{N}_2} (-1)^{\vert \alpha \vert} D^\alpha (a_{\alpha \beta} D^\beta u), \quad A u = \sum_{\alpha, \beta \leq m} (-1)^{\vert \alpha \vert} D^\alpha (a_{\alpha \beta} D^\beta u),$$

where $\alpha, \beta \in \mathbb{N}_2$. 

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where we have denoted by $|\alpha|$ the length of the multi-index $\alpha$, $D^\alpha$ the partial derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $\frac{\partial^k u}{\partial v^k}$ the $k$ derivative in the direction $\mathbf{v}$ (the unit outward normal vector on $\partial \Omega_\ell$ or $\partial \omega$), and $N_1$, $N_2$ are given by

$N_1 = \{ \alpha \in \{0,1,\ldots,m\}^p \times \{0\}^{n-p}, \ |\alpha| \leq m \}$, $N_2 = \{ \alpha \in \{0\}^p \times \{0,1,\ldots,m\}^{n-p}, \ |\alpha| \leq m \}$,

$f$ is a function of $L^2(\omega)$ independent of $X_1$

$$f(x) = f(X_2),$$

the coefficients $a_{\alpha\beta}$ satisfy

$$a_{\alpha\beta} \in L^\infty(\mathbb{R}^p \times \omega) \quad \text{for} \quad |\alpha|, |\beta| \leq m,$$

$$a_{\alpha\beta} \in C(\mathbb{R}^p \times \mathbb{S}) \quad \text{for} \quad |\alpha|, |\beta| = m,$$

and

$$a_{\alpha\beta}(x) = a_{\alpha\beta}(X_2) \quad \text{for} \quad \alpha \in N_2$$

i.e. these coefficients are only depending on the last variables $x_{p+1}, \ldots, x_n$. We will impose the usual ellipticity condition, i.e., that for some $\lambda > 0$

$$\sum_{|\alpha|,|\beta|=m} (-1)^m a_{\alpha\beta}(x) \xi^{\alpha+\beta} \geq \lambda |\xi|^{2m}, \text{ a.e. } x \in \mathbb{R}^p \times \omega, \ \forall \xi \in \mathbb{R}^n$$

where $\xi^{\alpha+\beta} = \xi_1^{\alpha_1+\beta_1} \xi_2^{\alpha_2+\beta_2} \ldots \xi_n^{\alpha_n+\beta_n}$, $|\xi|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2$.

The variational problems corresponding to (1) and (2) are the following

$$a(u,v) := \int_{\Omega_\ell} \sum_{|\alpha|,|\beta|\leq m} a_{\alpha\beta} D^\alpha u D^\beta v \, dx = \int_{\Omega_\ell} fv \, dx, \quad \forall v \in H^m_0(\Omega_\ell)$$

$$u \in H^m_0(\Omega_\ell), \tag{7}$$

$$a_\omega(u,v) := \int_{\omega} \sum_{\alpha,\beta \in N_2} a_{\alpha\beta} D^\alpha u D^\beta v \, dx = \int_{\omega} fv \, dx, \quad \forall v \in H^m_0(\omega)$$

$$u \in H^m_0(\omega). \tag{8}$$

where $H^m_0(\Omega_\ell)$ (resp. $H^m_0(\omega)$) is the closure of $\mathcal{D}(\Omega_\ell)$ (resp. $\mathcal{D}(\omega)$) in $H^m(\Omega_\ell)$ (resp. $H^m(\omega)$). Then, it is well known, see for instance [6], that under the above assumptions, the bounded bilinear forms $a(\ldots)$ and $a_\omega(\ldots)$ are coercive on $H^m_0(\Omega_\ell)$ and $H^m_0(\omega)$ respectively, i.e. there exist $C, C_\ell > 0$, $c \in \mathbb{R}$ such that

$$a(u,u) + c \|u\|^2_{L^2(\Omega_\ell)} \geq C_\ell \|u\|^2_{H^m(\Omega_\ell)} \quad u \in H^m_0(\Omega_\ell), \tag{9}$$

$$a_\omega(u,u) + c \|u\|^2_{L^2(\omega)} \geq C \|u\|^2_{H^m(\omega)} \quad u \in H^m_0(\omega), \tag{10}$$

Moreover, if we take $c = 0$, there exists a unique solution $u_0$ in $H^m_0(\Omega_\ell)$ to problem (7) and a unique solution $u_\infty$ in $H^m_0(\omega)$ to problem (8). We will also need to assume that the constant $C_\ell$ in (9) is independent of $\ell$, then

$$a(u,u) \geq C \|u\|^2_{H^m(\Omega_\ell)} \quad u \in H^m_0(\Omega_\ell) \tag{11}$$

**Remark 1** We can only suppose

$$a(u,u) \geq \frac{C}{\ell^k} \|u\|^2_m \quad u \in H^m_0(\Omega_\ell)$$

where $0 < \kappa < 1$ and $|u|^2_m = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u D^\alpha v \, dx$, then we have the same results.
2 The convergence in the space $H^m(\Omega_{\ell_o})$

We start by showing the following result:

**Proposition 2** Let $v$ be an element of $H^m_0(\Omega_{\ell})$. Then

$$v(X_1, \ldots) \in H^m_0(\omega) \text{ for almost all } X_1 \text{ in } (-\ell, \ell)^p.$$  \hspace{1cm} (12)

**Proof.** Using the idea of the proposition (3.1) in [1]. If we take $v \in H^m_0(\Omega_{\ell})$. Then, there exists a sequence $\varphi_n$ of element of $D(\Omega_{\ell})$, such that

$$\int_{\Omega_{\ell}} (D^\alpha (v_n - v))^2 \, dx \to 0 \text{ for } |\alpha| \leq m.$$  \hspace{1cm} (11)

Thus, there exists a subsequence $v_{n'}$, such that

$$\int_{\omega} (D^\alpha (v_{n'} - v)(X_1, \ldots))^2 dX_2 \to 0$$

for almost all $X_1$ in $(-\ell, \ell)^p$ and for $\alpha \in N_2$. Then because $v_{n'}(X_1, \ldots) \in D(\omega)$ for all $X_1$ in $(-\ell, \ell)^p$ and $v_{n'}(X_1, \ldots) \to v(X_1, \ldots) \in H^m_0(\omega)$ for almost all $X_1$ in $(-\ell, \ell)^p$, we have (12). \hfill \blacksquare

**Theorem 3** Under the assumptions (3), (4), (5), (6) and (11), for all $\ell_o > 0$ and $r > 0$, there exists a constant $C > 0$ independent of $\ell$ such that

$$\|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_o})} \leq C \ell^r,$$

where $u_\ell$ and $u_\infty$ are the solutions to (7) and (8) respectively.

**Proof.** We have

$$\int_{\Omega_{\ell}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha u_\ell D^\beta v \, dx = \int_{\Omega_{\ell}} f v \, dx \text{ for all } v \in H^m_0(\Omega_{\ell})$$

and also

$$\int_{\omega} \sum_{\alpha,\beta \in N_2} a_{\alpha\beta} D^\alpha u_\infty D^\beta v \, dx = \int_{\omega} f v \, dx \text{ for all } v \in H^m_0(\omega).$$

Applying the previous proposition, we have

$$\int_{\Omega_{\ell}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha u_\ell D^\beta v \, dx = \int_{(-\ell, \ell)^p} \int_{\omega} \sum_{\alpha,\beta \in N_2} a_{\alpha\beta} D^\alpha u_\infty D^\beta v \, dx \text{ for all } v \in H^m_0(\Omega_{\ell}).$$  \hspace{1cm} (14)

Taking into account the independence of $u_\infty$ from $X_1$, we obtain

$$\int_{\Omega_{\ell}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha (u_\ell - u_\infty) D^\beta v \, dx = -\int_{\Omega_{\ell}} \sum_{\alpha \in N_2, \beta \in N_1} a_{\alpha\beta} D^\alpha u_\infty D^\beta v \, dx \text{ for all } v \in H^m_0(\Omega_{\ell}).$$  \hspace{1cm} (15)
Using Gauss formula, and taking into account the fact that \( u_\infty \) is independent of \( \ell \), the functions \( a_{\alpha \beta} \) for \( \beta \in N_1 \) and \( \alpha \in N_2 \) are independent of \( X_1 \), we obtain, for \( \beta \in N_1 \), and \( |\beta| > 0 \) (i.e. there exists a \( \beta_i \neq 0 \))

\[
\int_{\Omega_\ell} a_{\alpha \beta} D^\alpha u_\infty D^\beta v \, dx = \int_{\Omega_\ell} D^\beta (a_{\alpha \beta} v D^\alpha u_\infty) \, dx = \int_{\partial \Omega_\ell} D^{(\beta_1, \ldots, \beta_i + 1, \ldots, \beta_p, 0, \ldots, 0)} (a_{\alpha \beta} v D^\alpha u_\infty) \nu_i \, dx,
\]

then (15) becomes

\[
\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha \beta} D^\alpha (u_\ell - u_\infty) D^\beta v \, dx = 0 \quad \text{for all } v \in H^m_\alpha(\Omega_\ell) \tag{16}
\]

Let \( \varrho \) be a smooth function of \( \mathbb{R}^p \), such that

\[
0 \leq \varrho \leq 1, \quad \varrho = 1 \text{ on } \left( -\frac{1}{2}, \frac{1}{2} \right)^p, \quad \varrho = 0 \text{ on } \mathbb{R}^p \setminus (-1, 1)^p, \tag{17}
\]

\[
|D^\alpha \varrho| \leq C, \quad |\alpha| \leq m.
\]

where \( C \) is some constant. For \( \ell_1 \leq \ell \), we have

\[
(u_\ell - u_\infty) \varrho^2 \left( \frac{X_1}{\ell_1} \right) \in H^m_\alpha(\Omega_\ell).
\]

We take in (15)

\[
v = (u_\ell - u_\infty) \varrho^2 \left( \frac{X_1}{\ell_1} \right).
\]

We obtain

\[
\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha \beta} D^\alpha (u_\ell - u_\infty) D^\beta \left( (u_\ell - u_\infty) \varrho^2 \left( \frac{X_1}{\ell_1} \right) \right) \, dx = 0. \tag{18}
\]

Using

\[
g D^\alpha (u_\ell - u_\infty) = D^\alpha (g(u_\ell - u_\infty)) - \sum_{\alpha' < \alpha} \frac{1}{\alpha'!} \left( \begin{array}{c} \alpha' \\ \alpha \end{array} \right) D^{\alpha'} (u_\ell - u_\infty) D^{\alpha' \varrho} g
\]

where \( \left( \begin{array}{c} \alpha' \\ \alpha \end{array} \right) = \left( \begin{array}{c} \alpha'_1 \\ \alpha_1 \end{array} \right) \left( \begin{array}{c} \alpha'_2 \\ \alpha_2 \end{array} \right) \ldots \left( \begin{array}{c} \alpha'_n \\ \alpha_n \end{array} \right) \) and \( \left( \begin{array}{c} a' \\ a \end{array} \right) = \frac{a'!}{a!} \), the equality (18) becomes

\[
\int_{\Omega_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha \beta} D^\alpha \left( (u_\ell - u_\infty) g \left( \frac{X_1}{\ell_1} \right) \right) D^\beta \left( (u_\ell - u_\infty) g \left( \frac{X_1}{\ell_1} \right) \right) \, dx
\]

\[
= -\int_{\Omega_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} \sum_{\beta' < \beta} \frac{1}{\beta'! \beta!} \left( \begin{array}{c} \beta \\ \beta' \end{array} \right) a_{\alpha \beta} D^\alpha (u_\ell - u_\infty) D^{\beta'} ((u_\ell - u_\infty) g) D^{\beta - \beta'} g \, dx
\]

\[
+ \int_{\Omega_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} \sum_{\alpha' < \alpha} \frac{1}{\alpha'! a!} \left( \begin{array}{c} \alpha' \\ \alpha \end{array} \right) a_{\alpha \beta} D^{\alpha'} (u_\ell - u_\infty) D^{\alpha' \varrho'} g D^\beta ((u_\ell - u_\infty) \varrho) \, dx. \tag{19}
\]

Using the Cauchy-Schwarz inequality, we can estimate all the terms of right hand side of (19) by

\[
\int_{\Omega_{\ell_1}} \frac{C}{\ell_1^q} a_{\alpha \beta} D^\alpha ((u_\ell - u_\infty) g) D^\beta \varrho \, dx \leq \frac{C}{\ell_1^q} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})} \|((u_\ell - u_\infty) g)\|_{H^m(\Omega_{\ell_1})} \quad (20)
\]
where \( i \geq 1, \ell_1 \geq 1 \) and \( |\alpha|, |\beta|, |\gamma| \leq m \). Using the coercivity of the problem (11) and the estimation (20), we obtain

\[
C' \|(u_\ell - u_\infty)\|_{H^m(\Omega_{\ell_1})}^2 \leq \frac{C}{\ell_1} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})}^2.
\]

It follows

\[
\|(u_\ell - u_\infty)\|_{H^m(\Omega_{\ell_1})} \leq \frac{C}{\ell_1} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})},
\]

where \( C \) is a constant independent of \( \ell_1 \) and \( \ell \). Since \( \varrho = 1 \) on \((-\frac{1}{2}, \frac{1}{2})^p\), we obtain

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\frac{1}{2}\ell})} \leq \frac{C}{\ell_1} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})}.
\]

If we set \( \ell_1 = \frac{\ell}{2^k}, \) \( k \in \mathbb{N}, \) we have

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\frac{1}{2^k}\ell})} \leq \frac{C}{\ell_1} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})},
\]

it follows that

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\frac{1}{2^k}\ell})} \leq \frac{C}{\ell_1} \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})} \quad (21)
\]

where \( C \) is only depending on \( k \). Therefore, it is clear that if we can estimate \( \|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_1})}, \) we have (13).

**Lemma 4** Under the assumption (3), the following estimate holds

\[
\|u_\ell\|_{H^m(\Omega_{\ell_1})} \leq C \ell_1^k \|u_\infty\|_{H^m(\omega)}.
\] \( (22) \)

**Proof.** We set \( v = u_\ell \) in (14),

\[
\int_{\Omega_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha u_\ell D^\beta u_\ell dx = \int \int_{(-\ell,\ell)^p} \sum_{\alpha,\beta \in \mathbb{N}^2} a_{\alpha\beta} D^\alpha u_\infty D^\beta u_\ell dx.
\]

using the ellipticity of the problem (11) and the Cauchy-Schwarz inequality, it follows

\[
C' \|u_\ell\|_{H^m(\Omega_{\ell_1})}^2 \leq C \|u_\ell\|_{H^m(\Omega_{\ell_1})} \\left( \int_{(-\ell,\ell)^p} \|u_\infty\|_{H^m(\omega)}^2 dX \right)^{\frac{1}{2}}.
\]

Then,

\[
\|u_\ell\|_{H^m(\Omega_{\ell_1})} \leq C \ell_1^k \|u_\infty\|_{H^m(\omega)}.
\]

The proof is complete. \( \blacksquare \)

Let us come back now to the proof of the theorem. If we use (22), the inequality (21) implies that

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\frac{1}{2^k}\ell})} \leq \frac{C}{\ell_1} \left( \|u_\ell\|_{H^m(\Omega_{\ell_1})} + \|u_\infty\|_{H^m(\Omega_{\ell_1})} \right) \leq \frac{C}{\ell_1} \left( C' \ell_1^k \|u_\infty\|_{H^m(\omega)} + \ell_1^k \|u_\infty\|_{H^m(\omega)} \right).
\]

from where we get

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\frac{1}{2^k}\ell})} \leq \frac{C}{\ell_1} \left( C' \ell_1^k \|u_\infty\|_{H^m(\omega)} \right).
\]

Choosing then \( k \) such that \( k - \frac{r}{2} > r \), and for \( \ell \) sufficient large such that \( \frac{\ell}{2^k} > \ell_o \), we obtain the desired estimate

\[
\|u_\ell - u_\infty\|_{H^m(\Omega_{\ell_o})} \leq \frac{C}{\ell^r},
\]

which completes the proof of the theorem. \( \blacksquare \)
3 Convergence in higher order Sobolev spaces

In this part we suppose the functions $a_{\alpha\beta}$ verify the following regularity conditions

$$a_{\alpha\beta} \in C^m(\mathbb{R}^p \times \omega) \quad \text{for} \quad |\alpha|, |\beta| \leq m. \tag{23}$$

$$|D^\gamma a_{\alpha\beta}| \leq C \quad \text{on} \quad \mathbb{R}^p \times \omega \quad \text{for} \quad |\gamma|, |\alpha|, |\beta| \leq m \tag{24}$$

where $C$ is constant.

**Theorem 5** Under assumptions (3), (6), (11), (23) and (24), then for any $\ell_0 > 0$, any $r > 0$, and any $\tilde{\Omega}_{\ell_0} \subset \Omega_{\ell_0}$ \(^{1)}\), there exists a constant $C > 0$ independent of $\ell$ such that

$$\|\partial_{x_k} (u_\ell - u_\infty)\|_{H^m(\Omega_{\ell_0})} \leq \frac{C}{\ell^r} \quad \text{for} \quad k = 1, \ldots, p \tag{25}$$

$$\|\partial_{x_k} (u_\ell - u_\infty)\|_{H^m(\tilde{\Omega}_{\ell_0})} \leq \frac{C}{\ell^r} \quad \text{for} \quad k = p + 1, \ldots, n \tag{26}$$

where $u_\ell$ and $u_\infty$ are the solutions of (7) and (8) respectively.

The idea of the proof is based on the use of finite differences, which is possible for any type of functions, instead of derivation. Thus, for $h > 0$ we define the differences of order 1 by

$$\delta_{x_k} v = \delta_{x_k}^h v = \frac{v(x + h e_k) - v(x)}{h}$$

where $e_k = (0, \ldots, 1, \ldots, 0)$, and we define the differences of higher order by

$$\delta^\alpha v = \delta_h^\alpha v = \delta_{x_k}^\alpha \delta_{x_k}^{\alpha_2} \ldots \delta_{x_k}^{\alpha_n} v \tag{27}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\delta_{x_k}^\alpha v = \delta_{x_k}^{\alpha_k-1} v$

We start by giving some properties of the finite differences analogous with those of derivation,

**Lemma 6** Let be $\mathcal{O}$ a bounded domain in $\mathbb{R}^n$, $f \in L^2(\mathcal{O})$, $\eta \in \mathcal{D}(\mathcal{O})$. then

$$\int_{\mathcal{O}} f \delta_h^\alpha \eta dx = (-1)^{|\alpha|} \int_{\mathcal{O}} \delta_{-h}^\alpha f \eta dx. \tag{28}$$

**Proof.** Applying [1, Lemma 3.9], for $h$ sufficiently small, the equality (28) is verified for $|\alpha| = 1$. Thus, by induction on each component of $\alpha$, and using (27), we obtain (28). \(\blacksquare\)

**Lemma 7** Let $f$ and $g$ two functions defined on a part of $\mathbb{R}^n$. Then

$$\delta^\alpha (fg) = \sum_{\beta \leq \alpha} \binom{\beta}{\alpha} \delta^\beta f(x + (\alpha - \beta)h)\delta^{\alpha-\beta} g, \tag{29}$$

**Proof.** This follows by induction on each component $\alpha_k$ of $\alpha$. \(\blacksquare\)

**Lemma 8** Let $f$ be a function of class $C^m$ on the open set $\mathcal{O}$ of $\mathbb{R}^n$. Then for any $x$ in $\mathcal{O}$, $h$ sufficiently small, there exists $\xi^x_h$ of $\mathcal{O}$, such that

$$\delta_h^\alpha f(x) = \frac{1}{\alpha!} D^\alpha f(\xi^x_h) \quad \text{for} \quad |\alpha| \leq m. \tag{30}$$

\(1\) the closure of $\tilde{\Omega}_{\ell_0}$ is in $\Omega_{\ell_0}$.
Proof. This follows immediately from the mean-value theorem. ■

We turn now to the proof of the theorem. Taking $w = u_\ell - u_\infty$ in (16), we obtain

$$\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} D^\alpha w D^\beta v dx = 0 \quad \text{for any } v \in H^m_0(\Omega_\ell).$$

For $v$ in $H^m_0(\Omega_\ell)$ with a support disjoint of $\partial (-\ell, \ell)^p \times \mathbb{R}$ if $\gamma \in N_1$ and with a support in $\Omega_\ell$ if $\gamma \notin N_1$, and $h$ sufficiently small, we can replace in the above expression $v$ by $(-1)^{|\gamma|} \delta_h^\gamma v$, with $|\gamma| \leq m$. Using the permutation of the derivation and the finite difference, and the lemma 6, we obtain

$$\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} \delta_h^\gamma (a_{\alpha\beta} D^\alpha w) D^\beta v dx = 0.$$

The lemma 7 with $f = a_{\alpha\beta}$ and $g = D^\alpha w$ gives

$$\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} (x + \gamma h)\delta^\gamma D^\alpha w D^\beta v dx$$

$$= -\int_{\Omega_\ell} \sum_{|\alpha|,|\beta| \leq m} \sum_{0 < \sigma \leq \gamma} \left( \frac{\alpha}{\beta} \right) \delta^\sigma a_{\alpha\beta} (x + (\gamma - \sigma) h)\delta^{\gamma - \sigma} D^\alpha w D^\beta v dx. \quad (31)$$

Let $\ell_o < \ell_1 \leq \ell$, and $\Omega'_{\ell_1}, \Omega'_{\ell_\ell}$ two bounded domain of $\mathbb{R}^n$, such that $\Omega'_{\ell_o} = \Omega_{\ell_o}, \Omega'_{\ell_1} = \Omega_{\ell_1}$ with $\ell_o < \ell_1 < \ell$ if $\gamma \in N_1$, and $\Omega'_{\ell_1} = \Omega'_{\ell_o} \in \Omega'_{\ell_1} \in \Omega_\ell$ if $\gamma \notin N_1$. Let us denote by $\varrho$ a smooth function with compact support in $(-\ell, \ell)^p \times \mathbb{R}$ if $\gamma \in N_1$, and with compact support in $\Omega'_{\ell_1}$ if $\gamma \notin N_1$, such that in both cases we have

$$0 \leq \varrho \leq 1, \quad \varrho = 1 \quad \text{on } \Omega_{\ell_o}.$$

We take $v = \delta^\gamma w \varrho^{4m}$ in (31) for $h$ small enough , and using equalities

$$D^\beta \left\{ (\delta^\gamma w \varrho^{2m}) \varrho^{2m} \right\} = \varrho^{2m} D^\beta (\delta^\gamma w \varrho^{2m}) + \sum_{\beta' < \beta} \left( \frac{\beta'}{\beta} \right) D^{\beta'} (\delta^\gamma w \varrho^{2m}) D^{\beta - \beta'} \varrho^{2m}$$

$$\varrho^{2m} D^\alpha \delta^\gamma w = D^\alpha (\varrho^{2m} \delta^\gamma w) - \sum_{\alpha' < \alpha} \left( \frac{\alpha'}{\alpha} \right) D^{\alpha'} \delta^\gamma w D^{\alpha - \alpha'} \varrho^{2m}, \quad (32)$$

we see that (31) becomes

$$\int_{\Omega'_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} a_{\alpha\beta} (x + \gamma h) D^\alpha (\delta^\gamma w \varrho^{2m}) D^{\beta} (\delta^\gamma w \varrho^{2m}) dx =$$

$$-\int_{\Omega'_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} \sum_{0 < \sigma \leq \gamma} \sum_{\beta' \leq \beta} \left( \frac{\alpha}{\beta} \right) \delta^\sigma a_{\alpha\beta} (x + (\gamma - \sigma) h) D^\alpha \delta^{\gamma - \sigma} \varrho D^{\beta'} (\delta^\gamma w \varrho^{2m}) D^{\beta - \beta'} \varrho^{2m} dx$$

$$-\int_{\Omega'_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} \sum_{\beta' < \beta} \left( \frac{\alpha}{\beta} \right) a_{\alpha\beta} (x + \gamma h) D^\alpha \delta^\gamma w D^{\beta'} (\delta^\gamma w \varrho^{2m}) D^{\beta - \beta'} \varrho^{2m} dx$$

$$+ \int_{\Omega'_{\ell_1}} \sum_{|\alpha|,|\beta| \leq m} \sum_{\alpha' < \alpha} \left( \frac{\alpha'}{\alpha} \right) a_{\alpha\beta} (x + \gamma h) D^{\alpha'} \delta^\gamma w D^{\beta} (\delta^\gamma w \varrho^{2m}) D^{\alpha - \alpha'} \varrho^{2m} dx. \quad (33)$$
We estimate one by one the three terms of the right hand side. The first term is the sum of terms of the form
\[ \int_{\Omega'_1} C^\delta a_{\alpha\beta}(x + (\gamma - \sigma) h) D^{\alpha} \delta^{\gamma} w D^{\beta'} (\delta^{\gamma} w g^{2m}) D^{\beta} \psi g^{2m} dx \]
such that \( C \) is a constant, \( 0 < \sigma \leq \gamma, \beta' \leq \beta \) and \( |\alpha|, |\beta| \leq m \). Using (30) and the fact that the function \( \psi \) and these derivatives are bounded, and the Cauchy-Schwarz inequality, we can estimate these terms
\[
\left| \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{0 < \sigma \leq \gamma, \beta' \leq \beta} \left( \frac{\beta'}{\beta} \right) (\gamma) \delta^{\alpha} a_{\alpha\beta}(x + (\gamma - \sigma) h) D^{\alpha} \delta^{\gamma} w D^{\beta'} (\delta^{\gamma} w g^{2m}) D^{\beta} \psi g^{2m} dx \right|
\leq C \sum_{0 < \sigma \leq \gamma} \|\delta^{\gamma} w g^{2m}\|_{H^m(\Omega'_1)} \left\| D^{\sigma} w \right\|_{H^m(\Omega'_1)}. \tag{34}
\]

The third term is the sum of terms of the form
\[ \int_{\Omega'_1} C a_{\alpha\beta}(x + \gamma h) D^{\alpha'} \delta^{\gamma} w D^{\beta} (\delta^{\gamma} w g^{2m}) D^{\alpha - \alpha'} \psi g^{2m} dx \]
where \( C \) is a constant, \( \alpha' < \alpha \) and \( |\alpha|, |\beta| \leq m \). Using (23), (24) and the fact that the function \( \psi \) and these derivatives are bounded, and the Cauchy-Schwarz inequality, we obtain
\[
\left| \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta} \left( \frac{\beta'}{\beta} \right) a_{\alpha\beta}(x + \gamma h) D^{\alpha} \delta^{\gamma} w D^{\beta'} (\delta^{\gamma} w g^{2m}) D^{\beta} \psi g^{2m} dx \right|
\leq C \|\delta^{\gamma} w\|_{H^{m-1} (\Omega'_1)} \left\| \delta^{\gamma} w g^{2m} \right\|_{H^m(\Omega'_1)}. \tag{35}
\]

For the second term, a direct estimate as for the other terms is not sufficient. We first write
\[ D^{\beta'} (\delta^{\gamma} w g^{2m}) = \sum_{\beta'' \leq \beta'} \left( \frac{\beta''}{\beta'} \right) D^{\beta''} \delta^{\gamma} w D^{\beta'} - \beta'' \psi g^{2m}, \]
and for \( |\tau| \leq m \)
\[ D^{\tau} g^{2m} = g^{m} \psi \]
where \( \psi \) is a sum and product of \( \psi \) and these derivatives. Then
\[
\int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta} \left( \frac{\beta'}{\beta} \right) a_{\alpha\beta}(x + \gamma h) D^{\alpha} \delta^{\gamma} w D^{\beta'} (\delta^{\gamma} w g^{2m}) D^{\beta} \psi g^{2m} dx
= \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' \leq \beta} \sum_{\beta'' \leq \beta'} \left( \frac{\beta''}{\beta'} \right) \left( \frac{\beta'}{\beta} \right) a_{\alpha\beta}(x + \gamma h) D^{\alpha} \delta^{\gamma} w g^{2m} D^{\beta''} \delta^{\gamma} w \psi \psi g^{2m} dx.
\]
Using (32), we obtain
\[
\int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta} \left( \frac{\beta'}{\beta} \right) a_{\alpha,\beta}(x + \gamma h) D^\alpha \delta^\gamma w D^{\beta'} (\delta^\gamma w \varrho^{2m}) D^{3-\beta'} \varrho^{2m} dx
\]
\[
= \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta, \beta'' \leq \beta'} \left( \frac{\beta''}{\beta} \right) \left( \frac{\beta'}{\beta} \right) a_{\alpha,\beta}(x + \gamma h) D^\alpha (\varrho^{2m} \delta^\gamma w) D^{\beta''} \delta^\gamma w \psi_d dx
\]
\[
- \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta, \beta'' \leq \beta'} \sum_{\alpha' < \alpha} \left( \frac{\alpha'}{\alpha} \right) \left( \frac{\beta'}{\beta} \right) a_{\alpha,\beta}(x + \gamma h) D^{\alpha'} \delta^\gamma w D^{\beta''} \delta^\gamma w \psi_d D^{\alpha'-\alpha'} \varrho^{2m} dx. \tag{36}
\]
Therefore, the second term is a sum of two terms. The first term is a sum of terms of the form
\[
\int_{\Omega'_1} C a_{\alpha,\beta}(x + \gamma h) D^\alpha (\varrho^{2m} \delta^\gamma w) D^{\beta} \delta^\gamma w \psi_d dx
\]
where $|\alpha| \leq m$, $|\beta| < m$. Using (23), (24) and the fact that the function $\varrho$ and these derivatives are bounded, and the Cauchy-Schwarz inequality, we obtain
\[
\left| \int_{\Omega'_1} C a_{\alpha,\beta}(x + \gamma h) D^\alpha (\varrho^{2m} \delta^\gamma w) D^{\beta} \delta^\gamma w \psi_d dx \right| \leq C \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)} \|\delta^\gamma w \varrho^{2m}\|_{H^m(\Omega'_1)} . \tag{37}
\]
The second term can be developed as a sum of terms of the form
\[
\int_{\Omega'_1} C a_{\alpha,\beta}(x + \gamma h) D^{\alpha'} \delta^\gamma w D^{\beta'} \delta^\gamma w \psi_d D^{\alpha'-\alpha'} \varrho^{2m} dx
\]
where $|\alpha'| \leq m$, $|\alpha|, |\beta| < m$, and again using (23), (24) and the fact that the function $\varrho$ and these derivatives are bounded, and the Cauchy-Schwarz inequality, we obtain
\[
\left| \int_{\Omega'_1} C a_{\alpha,\beta}(x + \gamma h) D^{\alpha'} \delta^\gamma w D^{\beta'} \delta^\gamma w \psi_d D^{\alpha'-\alpha'} \varrho^{2m} dx \right| \leq C \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)}^2 . \tag{38}
\]
By (37) and (38) we can estimate the second term of the second member of (33)
\[
\left| \int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} \sum_{\beta' < \beta} \left( \frac{\beta'}{\beta} \right) a_{\alpha,\beta}(x + \gamma h) D^\alpha \delta^\gamma w D^{\beta'} (\delta^\gamma w \varrho^{2m}) D^{3-\beta'} \varrho^{2m} dx \right|
\]
\[
\leq C \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)}^2 + C' \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)} \|\delta^\gamma w \varrho^{2m}\|_{H^m(\Omega'_1)}, \tag{39}
\]
and finally by (34), (35) and (39), we find the desired estimate
\[
\int_{\Omega'_1} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha,\beta}(x + \gamma h) \delta^\gamma D^\alpha (\delta^\gamma w \varrho^{2m}) D^{3} (\delta^\gamma w \varrho^{2m}) dx
\]
\[
\leq C_1 \sum_{\sigma < \gamma} \|\delta^\gamma w \varrho^{2m}\|_{H^m(\Omega'_1)} \|\delta^\gamma w\|_{H^m(\Omega'_1)} + C_2 \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)} \|\delta^\gamma w \varrho^{2m}\|_{H^m(\Omega'_1)}
\]
\[
+ C_3 \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)}^2 + C_4 \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)} \|\delta^\gamma w \varrho^{2m}\|_{H^m(\Omega'_1)} .
\]

Using the coercivity of the problem (11) and the Young inequality, it follows that
\[
\|\delta^\gamma w 2^m\|_{H^m(\Omega'_1)}^2 \leq C_\varepsilon \left( \sum_{\sigma < \gamma} \|\delta^\sigma w\|_{H^m(\Omega'_1)}^2 + \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)}^2 \right) + \varepsilon C \|\delta^\gamma w 2^m\|_{H^m(\Omega'_1)}^2.
\]
Taking \(\varepsilon = \frac{1}{2C}\), we obtain
\[
\|\delta^\gamma w 2^m\|_{H^m(\Omega'_1)}^2 \leq C \left( \sum_{\sigma < \gamma} \|\delta^\sigma w\|_{H^m(\Omega'_1)}^2 + \|\delta^\gamma w\|_{H^{m-1}(\Omega'_1)}^2 \right).
\]
Since \(\varrho = 1\) on \(\Omega'_{\ell_0}\), we have
\[
\|\delta^\gamma w\|_{H^m(\Omega'_{\ell_0})}^2 \leq C \left( \sum_{\sigma < \gamma} \|\delta^\sigma w\|_{H^m(\Omega'_{\ell_0})}^2 + \|\delta^\gamma w\|_{H^{m-1}(\Omega'_{\ell_0})}^2 \right). \tag{40}
\]
If \(|\gamma| = 1\) (i.e. \(\gamma = e_k\)) then for \(\sigma = 0\) in (40) we get
\[
\|\delta_{x_k} w\|_{H^m(\Omega'_{\ell_0})}^2 \leq C \left( \|w\|_{H^m(\Omega'_{\ell_0})}^2 + \|\delta_{x_k} w\|_{H^{m-1}(\Omega'_{\ell_0})}^2 \right) \text{ for } k = 1, \ldots, n.
\]
For another bounded domain verifying the same conditions as \(\Omega'_{\ell_1}\) and containing the closure of \(\Omega'_{\ell_1}\) (we still denote it by \(\Omega'_{\ell_1}\)), and using [1, Lemma 3.10], we obtain
\[
\|\delta_{x_k} w\|_{H^m(\Omega'_{\ell_0})}^2 \leq C \|w\|_{H^m(\Omega'_{\ell_1})}^2 \leq C \|w\|_{H^m(\Omega'_{\ell_1})}^2 \text{ for } k = 1, \ldots, n.
\]
For fixed \(\ell_1\), and by theorem 3, it holds
\[
\|\delta_{x_k}^h w\|_{H^m(\Omega'_{\ell_0})}^2 \leq C \frac{1}{\ell_2} \text{ for } k = 1, \ldots, n,
\]
and thus
\[
\|\partial^\alpha \delta_{x_k}^h w\|_{L^2(\Omega'_{\ell_0})}^2 \leq C \frac{1}{\ell_2} \text{ for } |\alpha| = m, \quad k = 1, \ldots, n.
\]
Then the sequence \((\partial^\alpha \delta_{x_k}^h w)_{k}\) is bounded in \(L^2(\Omega'_{\ell_0})\) and we can extract a subsequence \((\partial^\alpha \delta_{x_k}^h w)_{n \in \mathbb{N}}\) \((h_n \to 0)\) which converges weakly in \(L^2(\Omega'_{\ell_0})\) to some function \(w_{\alpha,k}\) of \(L^2(\Omega'_{\ell_0})\). We then obtain
\[
\|w_{\alpha,k}\|_{L^2(\Omega'_{\ell_0})}^2 = \langle w_{\alpha,k}, w_{\alpha,k} \rangle_{L^2(\Omega'_{\ell_0})} = \lim_{n \to 0} \langle w_{\alpha,k}, \partial^\alpha \delta_{x_k}^h w \rangle_{L^2(\Omega'_{\ell_0})}
\]
\[
\leq \lim_{n \to 0} \|\partial^\alpha \delta_{x_k}^h w\|_{L^2(\Omega'_{\ell_0})} \|w_{\alpha,k}\|_{L^2(\Omega'_{\ell_0})} \leq C \frac{1}{\ell_2} \|w_{\alpha,k}\|_{L^2(\Omega'_{\ell_0})}.
\]
It follows
\[
\|w_{\alpha,k}\|_{L^2(\Omega'_{\ell_0})} \leq C \frac{1}{\ell_2}. \tag{41}
\]
In other way
\[
\partial^\alpha \delta_{x_k}^h w \to \partial^\alpha \partial w \text{ in } D'({\Omega'_{\ell_0}})
\]
\[
\partial^\alpha \delta_{x_k}^h w \to w_{\alpha,k} \text{ in } D'({\Omega'_{\ell_0}}),
\]
and by uniqueness of the limit, we deduce that \(\partial^\alpha \partial w = w_{\alpha,k} \in L^2({\Omega'_{\ell_0}})\), and the proof is completed by (41).
Theorem 9 Under the assumptions (3), (6), (11), (23) and (24), then for any \( \ell_0 > 0 \), any \( r > 0 \) and \( \Omega''_\ell \subseteq \Omega_\ell, \) we have \( u_\ell - u_\infty \in H^{2m}(\Omega''_\ell) \), and there exists a constant \( C > 0 \) independent of \( \ell \) such that

\[
\| u_\ell - u_\infty \|_{H^{2m}(\Omega''_\ell)} \leq \frac{C}{\ell^r}.
\] (42)

Proof. It is enough to show

\[
\| D^\alpha (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)} \leq \frac{C}{\ell^r} \quad \text{for } |\alpha| \leq m,
\] (43)

by the theorem 5, the inequality (43) is verified for \( |\alpha| = 1 \). We show the result by induction. Let us suppose that for \( |\sigma| < |\alpha| \leq m \) we have

\[
\| D^\sigma (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)} \leq \frac{C}{\ell^r},
\] (44)

for any open \( \Omega''_\ell \) such that \( \Omega''_\ell \subseteq \Phi \subseteq \Omega''_\ell \subseteq \Omega_\ell, \) Using (40), we obtain for \( h \) small enough

\[
\| \delta^\alpha (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)} \leq C \left( \sum_{\sigma < \alpha} \| D^\sigma (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)}^2 + \| \delta^\alpha (u_\ell - u_\infty) \|_{H^{m-1}(\Phi)}^2 \right),
\]

and using also [1, Lemma 3.10] several times, we get

\[
\| \delta^\alpha (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)}^2 \leq C \left( \sum_{\sigma < \alpha} \| D^\sigma (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)}^2 + \| \delta^\alpha (u_\ell - u_\infty) \|_{H^{m-1}(\Omega''_\ell)}^2 \right)
\]

\[
\leq C \sum_{\sigma < \alpha} \| D^\sigma (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)}^2.
\]

Thanks to (44), we obtain

\[
\| \delta^\alpha (u_\ell - u_\infty) \|_{H^m(\Omega''_\ell)}^2 \leq \frac{C}{\ell^r},
\]

it holds that

\[
\left\| D^\beta \delta^\alpha (u_\ell - u_\infty) \right\|_{L^2(\Omega''_\ell)}^2 \leq \frac{C}{\ell^{2r}} \quad \text{for } |\beta| \leq m.
\]

The sequence \( (D^\beta \delta^\alpha (u_\ell - u_\infty))_{h>0} \) is bounded in \( L^2(\Omega''_\ell) \), and we can find a subsequence \( (D^\beta \delta^\alpha (u_\ell - u_\infty))_{h_n} \) \( (h_n \to 0) \) converging weakly in \( L^2(\Omega''_\ell) \) to a function \( w_{\alpha,\beta,k} \) of \( L^2(\Omega''_\ell) \). Then, we have

\[
\| w_{\alpha,\beta,k} \|_{L^2(\Omega''_\ell)} = \langle w_{\alpha,\beta,k}, w_{\alpha,\beta,k} \rangle_{L^2(\Omega''_\ell)} = \lim \langle w_{\alpha,k}, D^\beta \delta^\alpha (u_\ell - u_\infty) \rangle_{L^2(\Omega''_\ell)}
\]

\[
\leq \lim \inf \left\| D^\beta \delta^\alpha (u_\ell - u_\infty) \right\|_{L^2(\Omega''_\ell)} \| w_{\alpha,\beta,k} \|_{L^2(\Omega''_\ell)} \leq \frac{C}{\ell^{2r}} \| w_{\alpha,\beta,k} \|_{L^2(\Omega''_\ell)}
\]

which implies that

\[
\| w_{\alpha,\beta,k} \|_{L^2(\Omega''_\ell)} = \frac{C}{\ell^{2r}}.
\]

In other way

\[
\delta^\alpha D^\beta (u_\ell - u_\infty) \to \frac{1}{\alpha!} D^\alpha D^\beta (u_\ell - u_\infty) \quad \text{in } D'(\Omega''_\ell),
\]

\[
D^\beta \delta^\alpha (u_\ell - u_\infty) \to w_{\alpha,\beta,k} \quad \text{in } D'(\Omega''_\ell),
\]
and by uniqueness of the limit, we obtain
\[ \| D^\alpha D^\beta (u_\ell - u_\infty) \|_{L^2(\Omega_{\ell o}')} \leq \frac{C}{\ell^{2r}} \text{ for } |\beta| \leq m. \]
which gives (43), the proof of the theorem is complete.

Derivation in the directions $\alpha$ in $N_1$ does not get any trouble to give an estimate on all $\Omega_{\ell o}$, as show it the following result.

**Theorem 10** Under assumptions (3), (6), (11), (23) and (24), for any $\ell_o > 0$ and $r > 0$, there exists a constant $C > 0$ independent of $\ell$ such that
\[ \| D^\alpha (u_\ell - u_\infty) \|_{H^m(\Omega_{\ell o}')} \leq \frac{C}{\ell^r} \text{ for } \alpha \in N_1. \]

**Proof.** Since (40) is verified for $\Omega_{\ell o}' = \Omega_{\ell o}$, $\Omega_{\ell_1}' = \Omega_{\ell}$ such that $\ell_o < \ell' < \ell_1$ for $\alpha \in N_1$, then we can give the same proof as the previous theorem with $\Omega_{\ell_o}' = \Omega_{\ell_o}$. ■

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