NON-SINGULAR GLOBAL STRINGS

Ruth Gregory

Centre for Particle Theory,
University of Durham, Durham, DH1 3LE, U.K.

ABSTRACT

We examine the possibility that time dependence might remove the singular nature of global string spacetimes. We first show that this time dependence takes a specific form – a de-Sitter like expansion along the length of the string and give an argument for the existence of such a solution, estimating the rate of expansion. We compare our solution to the singular Cohen-Kaplan spacetime.

PACS numbers: 04.40.-b, 11.10.Lm

Keywords: gravity, topological defects

Email: dma0rag@gauss.dur.ac.uk
1. Introduction.

Topological defects are ubiquitous in physics, cropping up in some form or another in such widely disparate fields as string theory and low temperature physics. Cosmologists in particular have been attracted to defects as a possible source for the density perturbations which seeded galaxy formation[1,2,3]. A topological defect is a discontinuity in the vacuum, and can be classified according to the topology of the vacuum manifold of the particular field theory being used to model the physical set up: disconnected vacuum manifolds give domain walls, non-simply connected manifolds, strings, and manifolds with non-trivial 2- and 3-spheres give monopoles and textures respectively. Strings and monopoles can be further subdivided into those arising from the breakdown of a local and global symmetry, being called local and global defects respectively. However, apart from the (global) domain wall, global defects do not represent finite energy field configurations, even when we take the modest definition of ‘finite’ to be finite per unit defect area. This indicates that local and global defects will have rather different behaviour. Nowhere does this difference show up more dramatically than in the coupling of defects to gravity. While local strings[4-6] and monopoles[7,8] are well-behaved, and asymptote flat or locally flat spacetimes, global strings and monopoles have strong effects at large distances[9,10], and static global string spacetimes are singular[11,12].

In this paper we are interested in the global string. The global string appears to be in a unique position – whereas the wall and global monopole have well-defined non-singular (though not asymptotically flat) spacetimes, the global string appears to be singular. The domain wall can appeal to finite energy per unit area, but the mass of the global monopole is linearly divergent, therefore it seems strange that the monopole has a well-defined asymptotic structure whereas the global string does not. Previous results on the singularity of the global string metric assumed that the spacetime was static, not an unreasonable assumption since the local string has a static spacetime and in order to call the solution a string we might expect that it has certain features, not least of which a well-defined and constant physical width. However, as the domain wall shows, staticity may be too strong an assumption. When the gravitational field of the domain wall was first explored[4], assuming staticity, it was found to be singular. Rapidly, this assumption was found to be
too restrictive, and the true, non-static, metric for the domain wall was found to be[13,14]

\[ ds^2 = e^{-4\pi G\sigma|z|}(dt^2 - dz^2 - e^{4\pi G\sigma t}(dx^2 + dy^2)) \]

Note the de-Sitter like expansion in the plane of the domain wall. The experience with domain walls then indicates that perhaps staticity is too strong an assumption for global strings. By including some expansion along the length of the string it may be possible to reverse the tendency of the global string spacetime to collapse in at large distances and allow a non-singular metric. Global strings would then be acceptable gravitational sources, although possibly having strong gravitational effects.

A curious corollary of the existence of a non-singular global string metric might be its impact on the thermodynamics of black holes with axion hair[15]. Black holes can carry quantum hair, and discrete quantum hair[16] was shown to have an effect on the thermodynamics of a black hole[17]. The mechanism relies on the fact that the theory admits vortices which interact non-trivially with the ‘fractional’ charge of the discrete hair by acquiring quantum mechanical phases. Virtual string worldsheets[18], sitting on the event horizon of the (Euclidean) black hole then contribute to the Euclidean path integral with different phases yielding a measurable shift in the Hawking temperature of the black hole. The problem with detecting the axion hair of Bowick et.al.[15] is that the appropriate virtual vortices are now global, and give singular Euclidean geometries[18] and therefore do not appear in the path integral. If global strings can be shown to have non-singular spacetimes, then it re-opens the possibility that there might be a way of detecting axion hair.

Very little is known about non-static string spacetimes, most research being directed toward an exploration of the gravity of local strings in an expanding universe[19,20]. The most detailed study of non-stationary strings by Shaver and Lake[21] unfortunately assumes that the radial and azimuthal stresses of the string vanish, which totally precludes its applicability to global strings, which asymptotically have such stresses being equal in magnitude to the energy density. More recently, Banerjee et.al.[22] considered the possibility of non-static global strings, however, as we will show, they missed a large class of potential solutions, and did not comment on the singularity structure or otherwise of their
The purpose of this paper is to address the question of whether a non-singular non-static spacetime exists for a global vortex. The layout of the paper is as follows. In the next section we obtain a set of criteria which a physical global string must satisfy and then derive the general metric and field equations for such a global string. In section three, we show that the metrics contain a free parameter, $b_0$, which can be thought of as an effective cosmological constant along the length of the string, for example, $b_0 > 0$ gives de-Sitter like time dependence along the length of the string. We then exclude regions of parameter space by proving that they lead to singular solutions. In section four we give a dynamical systems analysis of the equations for the global string exterior to the core, and, by reference to the Cohen-Kaplan solution argue that there does indeed exist a unique value of $b_0$ for which a non-singular solution exists. In section five we consider properties of this solution and conclude.

2. Cylindrically symmetric spacetimes.

In this section we derive the metric and field equations appropriate to an isolated gravitating U(1) global vortex. We use a “mostly minus” signature. Since we are looking for a self-gravitating global string spacetime, the energy momentum tensor will be taken to be derived purely from the global string Lagrangian:

$$\mathcal{L} = (\nabla_\mu \Phi)^\dagger \nabla^\mu \Phi - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2$$

(2.1)

By writing

$$\Phi = \eta X e^{i\chi}$$

(2.2)

we reformulate the complex scalar field into two real interacting scalar field, one of which ($X$) is massive, the other ($\chi$) being the massless Goldstone boson.

$$\mathcal{L} = \eta^2 (\nabla_\mu X)^2 + \eta^2 X^2 (\nabla_\mu \chi)^2 - \frac{\lambda \eta^4}{4} (X^2 - 1)^2$$

(2.3)

A vortex solution is characterised by the existence of closed loops in space for which

$$\frac{1}{2\pi} \oint \frac{d\chi}{dl} dl = n \in \mathbb{Z}$$

(2.4)
In other words, the phase of $\Phi$ winds around $\Phi = 0$ as a closed loop is traversed. This in turn implies that $\Phi$ itself has a zero within that loop, and this is the core of the vortex. From now on, we shall assume $n = 1$, and look for a solution describing an infinitely long isolated straight string.

Our starting point will be that the string spacetime will be expected to exhibit cylindrical symmetry, namely, it is invariant under rotation about, and translation along a symmetry axis. Physically, this symmetry axis corresponds to the core of the string, and the massless scalar field corresponds to the azimuthal angle around this symmetry axis.

If we additionally require that the string have fixed proper width, we may then choose coordinates such that the metric is

$$ds^2 = e^{2A} dt^2 + 2Fe^{2A} dt dr - dr^2 - e^{2B} dz^2 - C^2 d\theta^2$$

and $X = X(r)$. Examining the equation of motion for $X$ indicates that $F \dot{B} = 0$, $C$ and $Fe^A$ are functions of $r$, and $A$ and $B$ are separable. The form of the energy-momentum tensor and Einstein tensor then indicates that we must take $F = 0$. Since $A$ is separable, we can always redefine $t$ to absorb any time dependence of $A$, thus obtaining

$$ds^2 = e^{2A(r)} dt^2 - dr^2 - e^{2B(r,t)} dz^2 - C^2(r) d\theta^2$$

as the final form for the general non-static metric for the global string. The Einstein tensor for this metric is

$$G_{rr} = (\ddot{B} + \dot{B}^2)e^{-2A} - A'B' - (A' + B')\frac{C'}{C}$$

$$G_{00} = - \left[ B'' + B'^2 + \frac{C''}{C} + \frac{B'C'}{C} \right]$$

$$G_{zz} = - \left[ A'' + A'^2 + \frac{C''}{C} + \frac{A'C'}{C} \right]$$

$$G_{\theta\theta} = (\ddot{B} + \dot{B}^2)e^{-2A} - \left[ A'' + A'^2 + B'' + B'^2 + A'B' \right]$$

$$G_{rt} = \dot{B}A' - \dot{B}' - \dot{BB}'$$

For the global string fields, recall that $\chi = \theta$, and since we are dealing with an isolated global string, we may, without loss of generality, set $\sqrt{\lambda \eta} = 1$ thereby choosing units in
which the string width is order unity. We will then absorb the symmetry breaking scale into the parameter $\epsilon = 8\pi G\eta^2$ which represents the gravitational strength of the string, generally assumed to be small. The equation of motion for $X$ is then:

$$-X'' - \left[\frac{C'}{C} + A' + B'\right] X' + \frac{X}{C^2} + \frac{1}{2}X(X^2 - 1) = 0$$

(2.8)

and the energy-momentum tensor of the vortex

$$8\pi G T^0_0 = 8\pi G T^z_z = \epsilon \left[ X'^2 + X^2 + \frac{1}{4}(X^2 - 1)^2 \right] = \epsilon \hat{\mathbf{T}}^0_0$$

$$8\pi G T^r_r = \epsilon \left[ -X'^2 + X^2 + \frac{1}{4}(X^2 - 1)^2 \right] = \epsilon \hat{\mathbf{T}}^r_r$$

$$8\pi G T^\theta_\theta = \epsilon \left[ X'^2 - X^2 + \frac{1}{4}(X^2 - 1)^2 \right] = \epsilon \hat{\mathbf{T}}^\theta_\theta$$

(2.9)

In flat space ($\epsilon = 0$), the equation of motion reduces to

$$-X'' - \frac{X'}{r} + \frac{X}{r^2} + \frac{1}{2}X(X^2 - 1) = 0$$

(2.10)

which does not have a closed analytic solution, but can be integrated numerically. In particular, note that as $r \to \infty$, the asymptotic form for $X$ is given by $X \sim 1 - 1/r^2$, and hence $T^0_0 \propto 1/r^2$ giving rise to a logarithmically divergent energy per unit length for the vortex.

Now let us consider coupling in gravity ($\epsilon \neq 0$). Referring to the Einstein equations, it can be seen that boost symmetry of the energy momentum tensor ($T^0_0 = T^z_z$) implies that $A' = B'$. Finally, referring to (2.7e), we see that $\dot{B}' = 0$, consistent with the separability of $B$:

$$B(r, t) = A(r) + b(t)$$

(2.11)

Therefore, collecting all this information, we see that the general form of the global string metric is

$$ds^2 = e^{2A}(dt^2 - e^{2b(t)}dz^2) - dr^2 - C^2(r)d\theta^2$$

(2.12)
with field equations

\[ A'' + \frac{C''}{C} + A'^2 + \frac{A'C'}{C} = -\epsilon \hat{T}_0^0 \]  
(2.13a)

\[(\ddot{b} + \dot{b}^2)e^{-2A} - A'^2 - 2 \frac{A'C'}{C} = \epsilon \hat{T}_r^r \]  
(2.13b)

\[(\ddot{b} + \dot{b}^2)e^{-2A} - 2A'' - 3A'^2 = \epsilon \hat{T}_\theta^\theta \]  
(2.13c)

where the \( \hat{T}_a^b \) are defined in (2.9), together with equation (2.8) for \( X \).

3. The equations of motion.

Now we will analyse the equations derived in the previous section for the isolated global string spacetime and find under what conditions the solutions are singular. Clearly (2.13b,c) imply

\[ \ddot{b} + \dot{b}^2 = b_0, \quad \text{a constant} \]  
(3.1)

and thus

\[ b(t) = \begin{cases} 
\ln \cosh \sqrt{b_0}t ; & b_0 > 0 \\
\frac{b_1}{b_0} \ln t & b_0 = 0, \ b_1 \ \text{a constant} \\
\ln \cos \sqrt{|b_0|}t & b_0 < 0 
\end{cases} \]  
(3.2)

Note that the non-static solutions of Banerjee et.al.[22] correspond to \( b_0 = 0, \ b_1 \neq 0 \). Moreover, since the Einstein equations for \( b_0 = 0 \) are identical in structure whether or not \( b_1 \) is zero, we must expect that solutions with \( b_1 \neq 0 \) are singular since static global strings are singular, and indeed, Banerjee et.al. pointed out that their solutions were a coordinate transformation of the static solution. Therefore, let us take \( b_0 \) to be arbitrary, and consider two cases in turn, \( b_0 \leq 0 \), and \( b_0 > 0 \). In either case, the system of equations we are dealing with is:

\[ [C'e^{2A}]' = -\epsilon Ce^{2A} \left[ \frac{2X^2}{C^2} + \frac{1}{4}(X^2 - 1)^2 \right] \]  
(3.3a)

\[ [A'Ce^{2A}]' = C \left[ b_0 - \frac{1}{4}Ce^{2A}(X^2 - 1)^2 \right] \]  
(3.3b)

\[ A'^2 + \frac{2A'C'}{C} = b_0 e^{-2A} - \epsilon \left[-X'^2 + \frac{X^2}{C^2} + \frac{1}{4}(X^2 - 1)^2 \right] \]  
(3.3c)

\[ [Ce^{2A}X']' = \frac{Xe^{2A}}{C} + \frac{1}{2}Ce^{2A}X(X^2 - 1) \]  
(3.3d)
(i) $b_0 \leq 0$.

Note first that (3.3a,b) together imply that

$$[Ce^{2A}]'' = 2b_0C - \epsilon Ce^{2A} \left[ \frac{2X^2}{C^2} + \frac{3}{4}(X^2 - 1)^2 \right] \leq 0 \quad (3.4)$$

Thus $Ce^{2A} \leq r$ for all $r$. Next we observe that if $(Ce^{2A})' = 0$ at any point, then (3.4) implies that $Ce^{2A} \to 0$ with $(Ce^{2A})'$ strictly negative at some finite $r$, $r_0$ say. Thus $2A' + C'/C \to -\infty$ as $r \to r_0$. Since (3.3b) implies that $A'$ is strictly negative away from $r = 0$, we may conclude that either $A'C'/C$ or $A'$ (or both) become infinite at $r_0$. Therefore

$$R_{abcd}^2 \propto \left( \frac{C''}{C} \right)^2 + 2 \left( \frac{A'C'}{C} \right)^2 + 2(A'' + A'^2)^2 + (A'^2 - b_0 e^{-2A})^2 \quad (3.5)$$

would become infinite at $r_0$ indicating a physical singularity. For a non-singular spacetime, we therefore take $(Ce^{2A})' > 0$. This in turn implies that

$$1 + \int_0^r \left\{ 2b_0C - \epsilon Ce^{2A} \left[ \frac{2X^2}{C^2} + \frac{3}{4}(X^2 - 1)^2 \right] \right\} dr > 0 \quad \forall r \quad (3.6)$$

Therefore the integrals

$$\alpha_1 = \epsilon \int_0^\infty \frac{\epsilon e^{2A}X^2}{C} \, dr \quad (3.7a)$$

$$\alpha_2 = \epsilon \int_0^\infty \frac{1}{4}Ce^{2A}(X^2 - 1)^2 \quad (3.7b)$$

$$\alpha_3 = |b_0| \int_0^\infty C \quad (3.7c)$$

are convergent, and satisfy the inequality

$$2\alpha_1 + 3\alpha_2 + 2\alpha_3 < 1 \quad (3.8)$$

In addition, an examination of (3.3d), together with finiteness of (3.7a-c) shows that

$$\alpha_4 = \epsilon \int_0^\infty Ce^{2A}X'^2 \, dr \quad (3.9)$$
We may now integrate (3.3a,b) out to infinity to obtain

\[ A'C e^{2A} \rightarrow -(\alpha_2 + \alpha_3) \]
\[ C' e^{2A} \rightarrow 1 - 2\alpha_1 - \alpha_2 \] (3.10)

as \( r \rightarrow \infty \), and multiplying (3.3c) by \((Ce^{2A})^2\), we also have

\[(\alpha_2 + \alpha_3)^2 - 2(\alpha_2 + \alpha_3)(1 - 2\alpha_1 - \alpha_2) = Ce^{2A} \left[ b_0 C + \epsilon C e^{2A} \left( X'^2 - \frac{X^2}{C^2} - \frac{1}{4}(X^2 - 1)^2 \right) \right] \] (3.11)

But, finiteness of the integrals in (3.7,9) requires that each of the integrands separately is \( o(r^{-1}) \) as \( r \rightarrow \infty \). Hence \((Ce^{2A})(\text{integrand}) \rightarrow 0 \) as \( r \rightarrow \infty \). Thus the r.h.s. of (3.11) vanishes at infinity and we have

\[(\alpha_2 + \alpha_3) [\alpha_3 + 3\alpha_2 + 4\alpha_1 - 2] = 0 \]
\[ \Rightarrow 2 + 3(\alpha_2 + \alpha_3) = 2[2\alpha_3 + 3\alpha_2 + 2\alpha_1] < 2 \] (3.12)

which cannot be satisfied since all of the \( \alpha_i \) are positive.

Therefore the spacetime must be singular for \( b_0 \leq 0 \). Note that this argument makes no use of the explicit form of \( X \), only the general form of the energy-momentum tensor.

(ii) \( b_0 > 0 \), preliminary considerations.

Note that, irrespective of the sign of \( b_0 \), (3.3a) implies that \([C'e^{2A}]' \leq 0 \) and hence that \( C'e^{2A} \leq 1 \). Therefore \( C'e^{2A} \) either remains positive, or it does not.

If \( C'e^{2A} > 0 \) for all \( r \), then \( 2\alpha_1 + \alpha_2 \leq 1 \), where the \( \alpha_i \) were defined in (3.7). Thus, using the asymptotic properties of the integrands appearing in the \( \alpha_i \) as before, we may conclude from (3.3b,c) that

\[ [A'C e^{2A}]' \sim b_0 C \] (3.13a)
\[ A'^2 \sim b_0 e^{-2A} \] (3.13b)

which is solved by

\[ C = C_0 \ ; \ e^A = \sqrt{b_0} r + A_0 \] (3.13c)
where $C_0$ and $A_0$ are constants. But an examination of the explicit form of the integrals (3.7b,c) shows that this solution is inconsistent with the finiteness of $\alpha_1$ and $\alpha_2$. Therefore $C'e^{2A} = 0$ for some finite $r, r_0$, say.

Suppose first that $C' = 0, e^{2A} \neq 0$. Then (3.3a) implies $C'' < 0$, hence $C'$ becomes negative, and must therefore remain negative. Since an isolated global string has $X \simeq 1$ outside the core, if $C \to 0$ the spacetime will be singular, hence $C$ must be bounded away from zero for all $r$, in which case $C', C'' \to 0$ as $r \to \infty$. (3.3a) (and its first integral) then implies that $A', e^{2A} \to \infty$ which contradicts (3.3c) and in any case would give a singular spacetime.

The only remaining possibility is therefore that $e^{2A} \to 0$ at some finite $r, r_0$ say. Non-singularity of the spacetime requires additionally that $C'(r_0) = 0$. $r_0$ would then represent an event horizon for the global string spacetime, analogous to that of the domain wall spacetime[13,14]. From (3.3a-c), we see that the asymptotic solution near this point would be

$$e^A \sim \sqrt{b_0}(r_0 - r)$$
$$C = C_0 + O(r_0 - r)^2$$
$$X \simeq 1 - \frac{1}{C^2}$$

(3.14)

In other words

$$ds^2 \simeq b_0(r_0 - r)^2[dt^2 - \cosh^2 \sqrt{b_0}tdz^2] - dr^2 - C_0^2d\theta^2$$

(3.15)

near $r = r_0$. The curvature invariants for this metric are all finite and $r = r_0$ does indeed appear to be an event horizon, but before investigating whether (3.14) is indeed possible as an asymptotic solution for the global string, we should verify the coordinate nature of the singularity at $r = r_0$.

Note that the metric (3.15) is reminiscent of that of Harari and Polychronakos[23], who considered static cylindrically symmetric solutions with event horizons. Their metric however had $g_{zz} \equiv 1$ and hence was not appropriate to a boost symmetric source.
Nonetheless, we can define similar Kruskal-like coordinates in the vicinity of $r = r_0$

\[ X = (r_0 - r) \cosh \sqrt{b_0} t \cos \sqrt{b_0} z \]
\[ Y = (r_0 - r) \cosh \sqrt{b_0} t \sin \sqrt{b_0} z \]
\[ T = (r_0 - r) \sinh \sqrt{b_0} t \]

in terms of which (3.15) becomes

\[ ds^2 \simeq dT^2 - dX^2 - dY^2 - C_0^2 d\theta^2 \]

The putative event horizon, $r = r_0$, corresponds to $X^2 + Y^2 = T^2$. Although the coordinate transformation (3.16) is not one to one, it can be made so by restricting the range of $z$, to $(-\frac{\pi}{\sqrt{b_0}}, \frac{\pi}{\sqrt{b_0}})$ for example. Therefore, it provides a coordinate system which extends beyond the event horizon $r = r_0$, and verifies the coordinate nature of that singularity. This situation is more complicated than that of Harari and Polychronakos where $g_{zz} \equiv 1$, and the “Kruskal” diagram is shown in figure 1.

To sum up, for the spacetime to be nonsingular, we have shown that $b_0 > 0$, and there must exist a $(b_0, r_0)$ such that the solution to (3.3) asymptotes (3.14). We will address this question in more detail in the next section, for the moment, we simply note that (3.3a,b) imply

\[ 2\epsilon \int_0^{r_0} \frac{X^2 e^{2A}}{C} dr + \frac{\epsilon}{4} \int_0^{r_0} C e^{2A} (X^2 - 1)^2 dr = 1 \]
\[ b_0 \int_0^{r_0} C = \frac{\epsilon}{4} \int_0^{r_0} C e^{2A} (X^2 - 1)^2 dr \]

Therefore, for $\epsilon \ll 1$, we may approximate $X, C$ and $A$ by their flat space values, at least out to $r = O(\epsilon^{-\frac{1}{2}})$, and thus obtain

\[ b_0 < O(\epsilon^2) \]

4. The exterior spacetime.
We would now like to examine whether (3.14) is indeed admissible as an asymptotic solution of (3.3). In order to address this question, we first recall the analysis of Cohen and Kaplan[9]. Cohen and Kaplan were interested in the form of the global string metric inside its singularity radius. To get the Cohen-Kaplan (CK) metric, we set $b_0 = 0$ and take $X = 1$ outside the core. Since we expect $1 - X \sim 1/C^2$, for $C > e^{-\frac{1}{2}}$ we expect this to be an excellent approximation, and for $C > \text{few}$, it should give a good working approximation. Under these assumptions, (3.3a,b) become

\[ [C' e^{2A}]' = -\frac{2\epsilon e^{2A}}{C} \]  \hspace{1cm} (4.1a)
\[ [A' C e^{2A}]' = 0 \]  \hspace{1cm} (4.1b)

FIGURE (1): The “Kruskal” diagram of the global string spacetime in \{X, Y, T\} coordinates. Region II, exterior to the cone, corresponds to the global string spacetime interior to the event horizon. Regions I and I’ correspond to the spacetime exterior to the event horizon reached by future and past pointing null geodesics respectively.
Thus
\[ (e^{2A})' = -2K\epsilon/C \]  \hspace{1cm} (4.2)

where
\[ K = \int \frac{1}{4} Ce^{2A} (X^2 - 1)^2 dr = O(1) \] \hspace{1cm} (4.3)

Rewriting \( Cdu = -dr \), one obtains the CK solution:
\[ e^{2A} = \frac{u}{u_0} \]  \hspace{1cm} (4.4)
\[ C^2 = \gamma \sqrt{\frac{u_0}{u}} \exp \left\{ \frac{u_0^2 - u^2}{u_0} \right\} \]

Where \( u_0 = \frac{1}{2K\epsilon} \) is given by (4.2), and \( \gamma \) is of order unity (for convenience, we will take \( \gamma = 1 \)). \( u = u_0 \) corresponds roughly to the core of the string, and \( u = 0 \) the singularity, which, can be seen to occur approximately at
\[ r = \int_0^{u_0} Cdu \sim \frac{1}{\epsilon} e^{\frac{1}{2\epsilon}} \]  \hspace{1cm} (4.5)
a very large radius!

Note that the simplicity of the Cohen-Kaplan analysis relied on the vanishing of the right hand side of (4.1b); if \( b_0 \neq 0 \), this r.h.s. is equal to \( b_0 C \). Explicitly, with the same assumptions as Cohen and Kaplan, the asymptotic equations we need to solve are:
\[ [C'e^{2A}]' = -\frac{2\epsilon e^{2A}}{C} \]  \hspace{1cm} (4.6a)
\[ [A'Ce^{2A}]' = b_0 C \]  \hspace{1cm} (4.6b)
\[ A'^2 + \frac{2A'C'}{C} = b_0 e^{-2A} - \frac{\epsilon}{C^2} \]  \hspace{1cm} (4.6c)

In what follows we will take \( \epsilon \ll 1 \), and also recall (3.19), \( b_0 < O(\epsilon^2) \). We choose to write this analysis in a slightly different form. Let \( \rho = \int_0^r e^{-A} dr \), and denote \( \frac{d}{d\rho} \) by a dot. Then, letting \( f = \dot{A} + \dot{C}/C \), and \( g = \dot{C}/C \), we have
\[ f^2 = b_0 + g^2 - \frac{\epsilon e^{2A}}{C^2} \]  \hspace{1cm} (4.7a)
\[ \dot{g} = -fg - \frac{2\epsilon e^{2A}}{C^2} = 2(f^2 - g^2 - b_0) - fg \]  \hspace{1cm} (4.7b)
from (4.6a) and (4.6c). Now, differentiating (4.7b) and using (4.7a), we obtain

$$\dot{f} = f^2 - b_0 - 2g^2$$

(4.7c)

Thus we have reduced our (constrained) coupled second order differential equations to a two-dimensional dynamical system, (4.7c) and (4.7b). Whether or not (3.14) is admissible as an asymptotic solution for the global string will now reduce to a question of whether or not the dynamical system will asymptote the solution appropriate to (3.14) in phase space.

First of all, consider $b_0 = 0$. Then there is just one fixed point, $f = g = 0$, which corresponds to $C$ and $A$ being constant, $C$ infinite. Moreover, since

$$\dot{f} + \dot{g} = (3f - 4g)(f + g)$$

$$\dot{f} - \dot{g} = -f(f - g)$$

(4.8)

$f \pm g$ are separatrices in the phase plane, and the Cohen-Kaplan family of solutions lie entirely in the upper quadrant. We can therefore see that all CK solutions with $g > f > 0$ initially asymptote $g \approx -f \to \infty$, i.e. $\dot{C}/C \approx -\dot{A}/2 \to \infty$, which indeed agrees with the CK solution. A full phase diagram is shown in figure 2.

Now consider $b_0 > 0$. We will first consider the general behaviour of the dynamical system before investigating whether a solution with the asymptotic behaviour (3.14) is possible. First we rescale variables by setting $t = \sqrt{b_0} \rho$, and $(f, g) = \sqrt{b_0}(x, y)$ to obtain

$$\frac{dx}{dt} = x^2 - 2y^2 - 1$$

(4.9a)

$$\frac{dy}{dt} = 2x^2 - 2y^2 - 2 - xy$$

(4.9b)

This system has two saddle points at $(\pm 1, 0)$ and two foci at $(\pm \sqrt{2}, \pm 1/\sqrt{2})$. It is also straightforward to see that the hyperbolae $x^2 - y^2 = 1$ form separatrices in the plane. The analysis is somewhat more complicated than the CK case, but a phase plane diagram for the system is shown in figure 3.

Now consider the critical point $(-1, 0)$, this corresponds to $f = -\sqrt{b_0}$, $g = 0$. Inspecting (4.7), we see that this corresponds to $\dot{C} = 0$, $\dot{A} = -\sqrt{b_0}$, $e^{2A} = 0$, i.e. the asymptotic
form of (3.14). Therefore, asking whether a non-singular solution exists for the global string reduces to asking whether a suitable trajectory exists which terminates on the critical point \((-1, 0)\) in the \((x, y)\)-plane. Now, since \((-1, 0)\) is a saddle point there does indeed exist a unique trajectory approaching \((-1, 0)\) – the stable manifold, however, the question is whether this trajectory is “suitable”, i.e. does it match on to the core of the global string? Therefore, we must now examine the initial conditions for the dynamical system, obtained from integrating out the full equations of motion, in order to see whether we can indeed fit these initial conditions onto the required trajectory.

Remembering that \(b_0 < O(\epsilon^2)\), and letting \(\rho_c = O(1)\) be a suitable value of \(\rho\) representing the transition from core to vacuum or the edge of the vortex, then

\[
\dot{A}(\rho_c) = (\dot{f} - \dot{g})|_{\rho_c} \simeq -\frac{K\epsilon}{\rho_c} \tag{4.10}
\]
from (3.3a), where $K$ is as defined in (4.3). Then (4.7a) gives

$$\frac{g_0}{f_0} = \frac{y_0}{x_0} = 1 + 2K^2 \epsilon$$

(4.11)

independent of $b_0$, and

$$y_0 = \frac{1}{2K\rho_c \sqrt{b_0}}$$

(4.12)

Clearly therefore, the trajectory approaching $(-1, 0)$ in the $(x, y)$ plane will correspond to a global string if it intersects the line $y = (1 + 2K^2 \epsilon)x$ for some (large) $x > 0$.

Now let us examine (4.9) for $x > 0$. By observation, $y \in [1, 2]$ at $x = 0$ for the non-singular trajectory, therefore we can roughly bound $y$ by $[x + 1, 2(x + 1)]$ for general $x > 0$. Therefore, as $t \to -\infty$, $x, y \to \infty$, thus (4.9) asymptotes the CK system as $t \to -\infty$. The solution to leading order for $x$ and $y$ can be written as

$$y \approx x \left[1 + \frac{1}{4 \ln x}\right]$$

(4.13)
(a better approximation can be derived, but this will suffice for our purposes). In particular, note that for non-zero $\epsilon$, there exists an $x_\epsilon$ such that $1 + 1/4 \ln x < 1 + 2K^2 \epsilon$ for all $x > x_\epsilon$, and hence the trajectory will indeed intersect $y = (1 + 2K^2 \epsilon)x$ at some value of $x$. This value of $x$ will then determine $b_0$. An illustration of this process is shown in figure 4, where we have inflated the value of $K^2 \epsilon$ for the purposes of clarity. For $8K^2 \epsilon = 1$, we obtain $b_0 \approx 10^{-3}$.

![Figure 4](image-url)

**FIGURE (4):** An illustration of the determination of $b_0$ from the intersection of the non-singular trajectory (shown as a continuous line) with $y = (1 + 2K^2 \epsilon)x$ (shown as a grey line) for the (rather large) value $2K^2 \epsilon = 0.25$.

The CK trajectory with the same initial conditions is shown as a dashed line.

The value of $b_0$ for this solution is $b_0 = 1/(1028K^2 \rho_c^2) \approx 10^{-3}$.

Thus, what we have shown in this section is that by reducing the far-field equations to a two-dimensional dynamical system, we are able to demonstrate the existence of a trajectory interpolating between the initial conditions at the edge of the vortex and the
asymptotic solution (3.14), the event horizon.

5. Discussion.

We now wish to explore the qualitative features of the solution represented by the trajectory of section four. We will assume $\epsilon \ll 1$. Clearly in the initial stages of the trajectory, for $x, y \gg 1$, we expect a Cohen-Kaplan (CK) trajectory to be a good approximation, and only when $x, y \simeq O(1)$ will the solution significantly differ from CK. By dividing (4.9b) by (4.9a) we get the relation

$$\frac{dy}{dx} = 2 - \frac{y(2y - x)}{2y^2 + 1 - x^2} > 2 - \frac{y(2y - x)}{2y^2 - x^2} \quad \text{for} \quad y > x > 0 \tag{5.1}$$

therefore, at all points in the upper half of the positive $(x, y)$ quadrant, the trajectories of the time dependent global string are steeper than those of the static, CK, string (see figure 4). This allows us to put an upper bound on $b_0$, since the CK trajectory starting at the same initial conditions exterior to the core will always lie above the real trajectory. Working in the $(f, g)$-plane, (4.4) gives

$$f = \frac{e^A}{C} \left[ -\frac{1}{4u} + \frac{u}{u_0} \right] \tag{5.2}$$

$$g = \frac{e^A}{C} \left[ \frac{1}{4u} + \frac{u}{u_0} \right]$$

in terms of the variable $u$, and hence $f = 0$ when $u = \sqrt{u_0}/2$. At this point

$$g \simeq \epsilon \frac{\sqrt{u_0}}{e^{-\frac{1}{2}}} \tag{5.3}$$

Since $g = \sqrt{b_0 y}$, and $y > 1$ for this CK trajectory, we see that

$$b_0 < \epsilon \frac{\sqrt{u_0}}{e^{-1/\epsilon}} \tag{5.4}$$

and indeed, since we expect the CK trajectory to be a good approximation to the real trajectory until $(x, y) \simeq 1$ we expect $b_0$ not to be significantly less than this order of magnitude. Therefore, the rate of expansion along the string, determined by $b_0$, is minutely
small. It is also interesting to note the proper radius at which this transition from a CK solution to the asymptotic form (3.14) occurs

\[ r_k = \int_{u_0}^{u_0} Cdu \simeq r_0(1 - O(\epsilon^{1/2})) \]  \hspace{1cm} (5.5)

For \( r_k < r < r_0 \) we expect that (3.14) will be a good approximation to the spacetime.

From a cosmological point of view it is instructive to estimate these critical radii, \( r_k \) and \( r_0 \), for a typical value of \( \epsilon \) appropriate to a GUT string, \( \epsilon = 10^{-6} \). This gives \( r_0 = O(10^{100.000}) \) with \( r_k \) being of the same order. Even allowing for the fact that \( r_0 \) is measured in units of string width, this value is many many times the current Hubble radius, which is about \( 10^{52} \) in these units! Therefore, cosmologically speaking, not only is the effect of the expansion negligible, but the gravitational field of the string is not appreciably different from that of the singular CK metric. In other words, our solution justifies the use of the CK metric as an approximation to the gravity of a global string on intermediate scales.

It is perhaps more interesting to ask what happens if \( \epsilon \simeq 1 \). Such heavy vortices have relevance to the topological inflation of Linde and Vilenkin [24,25]. Central to their argument is the non-existence of static non-singular supermassive defect solutions, otherwise supermassive defects would not inflate. Certainly, our solution is non-static, however, it is not sufficiently non-static! The existence of a non-singular metric of the general form (2.12) would mean that topological inflation was not possible with global strings, since the “inflation” in (2.12) is occurring only along the length of the string.

Unfortunately, we cannot apply our results directly to this interesting scenario, since for \( \epsilon \simeq 1 \) the analysis of the previous section cannot be straightforwardly applied. It will still be true that there will exist a trajectory with the correct asymptotic behaviour of (3.14), but the appropriate initial conditions can no longer be approximated since we can no longer use the flat space solutions to estimate \( \dot{A}(\rho_c) \). Indeed, whether it is appropriate to be performing an asymptotic \( (X \simeq 1) \) analysis in such a strongly coupled régime is also questionable. It is probably necessary to perform a more detailed numerical investigation in order to resolve this issue.
Finally, it is interesting to return to the question of detecting axion hair. This hair will only be detectable if there exist Euclidean vortices on the event horizons of Schwarzschild black holes which still have asymptotically flat geometries. There are two main reasons why we do not expect the solutions presented here to satisfy this criterion. First, although the solution is non-singular, it does have an event horizon, i.e. a strong asymptotic effect. This tends to indicate that Euclidean global vortices will not be asymptotically flat. However, the second objection is more troubling. In deriving the metric (2.12), and the ensuing analysis, implicit use was made of the non-compactness of the worldsheet directions $z$ and $t$. In particular, the existence of a continuum of choices for what is essentially an eigenvalue, $b_0$, was crucial to our argument. For the global vortex on a black hole, the worldsheet $z$ and $t$ are replaced by $\theta$ and $\phi$, which coordinatise a compact manifold. This means that not only would the corresponding metric have angular dependence, but in addition, we might expect a discrete spectrum of eigenvalues in an analogous metric to (2.12), which means that we can no longer continuously vary $b_0$ to hit the right non-singular trajectory. In other words Euclidean global vortices are quite probably singular no matter what one does. Axion hair, for the moment, must remain undetectable.

Acknowledgements.

It is a pleasure to thank Caroline Santos for useful discussions on dynamical systems. This work was supported by a Royal Society University Research Fellowship.

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