THE GEOMETRIC REALIZATION OF A NORMALIZED
SET-THEORETIC YANG-BAXTER HOMOLOGY OF Biquandles

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Abstract. Biracks and biquandles, which are useful for studying the knot theory, are special families of solutions of the set-theoretic Yang-Baxter equation. A homology theory for the set-theoretic Yang-Baxter equation was developed by Carter, Elhamdadi, and Saito in order to construct knot invariants. In this paper, we construct a normalized (co)homology theory of a set-theoretic solution of the Yang-Baxter equation. We obtain some concrete examples of non-trivial $n$-cocycles for Alexander biquandles. For a biquandle $X$, its geometric realization $BX$ is discussed, which has the potential to build invariants of links and knotted surfaces. In particular, we demonstrate that the second homotopy group of $BX$ is finitely generated if the biquandle $X$ is finite.

1. Introduction

The Yang-Baxter equation has played an important role in various fields such as quantum group theory, braided categories, and low-dimensional topology since it was first introduced independently in a study of theoretical physics by Yang [31] and statistical mechanics by Baxter [1]. In particular, since the discovery of the Jones polynomial [14] in 1984, it has been extensively studied in knot theory [4]. As a homological approach, Carter, Elhamdadi, and Saito [2] defined a (co)homology theory for set-theoretic Yang-Baxter operators, from which they provided a method to generate link invariants, and further developments were made by Przytycki [25]. Meanwhile, Joyce [15] and Matveev [20] independently introduced a self-distributive algebraic structure, called a quandle, which satisfies axioms motivated by the Reidemeister moves, and it has been generalized as a biquandle. Quandles and biquandles are solutions of the set-theoretic Yang-Baxter equation, which have been used to define homotopical and homological invariants of knots and links [23, 24, 32, 2, 4, 3]. This paper describes the study of a normalized homology theory of a set-theoretic solution of the Yang-Baxter equation and its geometric realization.

1.1. Preliminary. Let $k$ be a commutative ring with unity and $X$ be a set. We denote by $V$ the free $k$-module generated by $X$. Then, a $k$-linear map $R: V \otimes V \to V \otimes V$ is called a pre-Yang-Baxter operator if it satisfies the equation of the following maps $V \otimes V \otimes V \to V \otimes V \otimes V$:

$$(R \otimes \text{Id}_V) \circ (\text{Id}_V \otimes R) \circ (R \otimes \text{Id}_V) = (\text{Id}_V \otimes R) \circ (R \otimes \text{Id}_V) \circ (\text{Id}_V \otimes R).$$

We call a pre-Yang-Baxter operator $R$ a Yang-Baxter operator if it is invertible.

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$^1$It is known that a certain solution of the Yang-Baxter equation gives rise to the Jones polynomial [14, 29].

$^2$As homology theories of set-theoretic Yang-Baxter operators, they coincide as shown in [27].
The classification of the solutions of the Yang-Baxter equation has been actively studied. Following the study by Drinfel’d [3], the set-theoretic solutions of the Yang-Baxter equation have been the focus of various studies [7, 8, 17, 18, 30].

**Definition 1.1.** For a given set $X$, a function $R : X \times X \rightarrow X \times X$ satisfying the following equation (called a set-theoretic Yang-Baxter equation)

$$(R \times \text{Id}_X) \circ (\text{Id}_X \times R) \circ (R \times \text{Id}_X) = (\text{Id}_X \times R) \circ (R \times \text{Id}_X) \circ (\text{Id}_X \times R)$$

is called a set-theoretic pre-Yang-Baxter operator or a set-theoretic solution of the Yang-Baxter equation. In addition, if $R$ is invertible, then we call $R$ a set-theoretic Yang-Baxter operator.

Special families known as biracks and biquandles are strongly related to the knot theory. Their precise definitions are as follows:

**Definition 1.2.** For a given set $X$, let $R$ be a set-theoretic Yang-Baxter operator denoted by

$$R(A_1, A_2) = (R_1(A_1, A_2), R_2(A_1, A_2)) = (A_3, A_4),$$

where $A_i \in X$ ($i = 1, 2, 3, 4$), and $R_j : X \times X \rightarrow X$ ($j = 1, 2$) are binary operations. We consider the following conditions:

1. For any $A_1, A_3 \in X$, there exists a unique $A_2 \in X$ such that $R_1(A_1, A_2) = A_3$. In this case, $R_1$ is left-invertible.
2. For any $A_2, A_4 \in X$, there exists a unique $A_1 \in X$ such that $R_2(A_1, A_2) = A_4$. In this case, $R_2$ is right-invertible.
3. For any $A_1 \in X$, there is a unique $A_2 \in X$ such that $R(A_1, A_2) = (A_1, A_2)$.

The algebraic structure $(X, R_1, R_2)$ is called a birack if it satisfies the conditions (1) and (2). A birack is a biquandle if the condition (3) is also satisfied.

**Remark 1.3.** The condition (3) in Definition 1.2 implies that for any $A_2 \in X$, there is a unique $A_1 \in X$ such that $R(A_1, A_2) = (A_1, A_2)$. See Remark 3.3 in [2].

**Example 1.4.**

1. Let $C_n$ be the cyclic rack of order $n$, i.e., the cyclic group $\mathbb{Z}_n$ of order $n$ with the operation $i \ast j = i + 1 \pmod{n}$. Then the function $R : X \times X \rightarrow X \times X$ defined by

$$R(i, j) = (R_1(i, j), R_2(i, j)) = (j + 1, i - 1)$$

forms a set-theoretic Yang-Baxter operator. Moreover, $(C_n, R_1, R_2)$ is a biquandle, called a cyclic biquandle.

2. Let $k$ be a commutative ring with unity 1 and with units $s$ and $t$ such that $(1 - s)(1 - t) = 0$. Then the function $R : k \times k \rightarrow k \times k$ given by

$$R(a, b) = (R_1(a, b), R_2(a, b)) = ((1 - s)a + sb, ta + (1 - t)b)$$

is a set-theoretic Yang-Baxter operator, and $(k, R_1, R_2)$ forms a biquandle, called an Alexander biquandle. For example, let $k = \mathbb{Z}_m$ with units $s$ and $t$ such that $m = |(1 - s)(1 - t)|$, then the function $R$ defined as above forms a set-theoretic Yang-Baxter operator and $\mathbb{Z}_{m; s, t} := (\mathbb{Z}_m, R_1, R_2)$ is a biquandle.
2. Normalized homology of a set-theoretic solution of the Yang-Baxter equation

In this section, we study a normalized homology theory for set-theoretic solutions of the Yang-Baxter equation, defined in a similar way as to obtain the quandle homology [1] from the rack homology [11, 12]. We construct concrete examples of non-trivial n-cocycles for the Alexander quandles.

First, we review the homology theory for the set-theoretic Yang-Baxter equation based on [2]. For a set $X$, let $R : X \times X \to X \times X$ be a set-theoretic Yang-Baxter operator on $X$. For each integer $n > 0$, we define the $n$-chain group $C_n^{YB}(X)$ to be the free abelian group generated by the elements of $X^n$ and the $n$-boundary homomorphism $\partial_n^{YB} : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ by $\sum_{i=1}^n (-1)^{i+1} (d_{i,n}^r - d_{i,n}^l)$, where the two face maps $d_{i,n}^l, d_{i,n}^r : C_n^{YB}(X) \to C_{n-1}^{YB}(X)$ are given by

$$d_{i,n}^l = (R_2 \times \text{Id}_X^{(n-2)}) \circ (\text{Id}_X \times R \times \text{Id}_X^{(n-3)}) \circ \cdots \circ (\text{Id}_X^{(i-2)} \times R \times \text{Id}_X^{(n-i)}),$$

$$d_{i,n}^r = (\text{Id}_X^{(n-2)} \times R_1) \circ (\text{Id}_X^{(n-3)} \times R \times \text{Id}_X) \circ \cdots \circ (\text{Id}_X^{(i-1)} \times R \times \text{Id}_X^{(n-i-1)}).$$

Then $C_n^{YB}(X) := (C_n^{YB}(X), \partial_n^{YB})$ forms a chain complex, and the yielded homology $H_n^{YB}(X)$ is called the set-theoretic Yang-Baxter homology of $X$.

Consider the subgroup $C_n^D(X)$ of $C_n^{YB}(X)$ defined by

$$C_n^D(X) = \text{span}\{(x_1, \ldots, x_n) \in C_n^{YB}(X) \mid R(x_i, x_{i+1}) = (x_i, x_{i+1}) \text{ for some } i = 1, \ldots, n-1\},$$

if $n \geq 2$, otherwise we let $C_n^D(X) = 0$.

**Proposition 2.1.** $(C_n^D(X), \partial_n^{YB})$ is a sub-chain complex of $(C_n^{YB}(X), \partial_n^{YB})$.

**Proof.** We need to show that $\partial_n^{YB}(C_n^D(X)) \subseteq C_n^{D}(X)$ for every $n > 0$.

Let $(x_1, \ldots, x_n) \in C_n^D(X)$. Then there exists $j \in \{1, \ldots, n-1\}$ such that $R(x_j, x_{j+1}) = (x_j, x_{j+1})$, and we denote such $x_{j+1}$ by $\overline{x}_j$.

We first show that the image of $d_{i,n}^l$ is contained in $C_{n-1}^D(X)$.

1. Clearly $d_{i,n}^l(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n) \in C_{n-1}^D(X)$ if $i < j$.
2. The terms for $d_{j,n}^l$ and $d_{j+1,n}^l$ cancel each other because $R(x_j, \overline{x}_j) = (x_j, \overline{x}_j)$; thus,

$$d_{j+1,n}^l(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n) = d_{j,n}^l(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n),$$

and there is a sign difference.

3. When $i > j$, denote by $d_{i,n}^l(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n) = (y_1, \ldots, y_j, y_{j+1}, \ldots, y_{i-1}, x_{i+1}, \ldots, x_n)$. We prove that $y_{j+1} = \overline{y}_j$ so that $d_{i,n}^l(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n) \in C_{n-1}^D(X)$. By the definition of the face map $d_{i,n}^l$, we have

$$(R \times \text{Id}_X) \circ (\text{Id}_X \times R)(x_j, \overline{x}_j, z) = (w, y_j, y_{j+1}),$$

where $z$ is the $(j+2)$th coordinate of $(\text{Id}_X^{(j+1)} \times R \times \text{Id}_X^{(n-j-3)}) \circ \cdots \circ (\text{Id}_X^{(i-2)} \times R \times \text{Id}_X^{(n-i-1)})(x_1, \ldots, x_j, \overline{x}_j, \ldots, x_n)$ and $w$ is the $j$th coordinate of $(\text{Id}_X^{(j-1)} \times R \times \text{Id}_X^{(n-j-1)} \circ$
\[ \cdots \circ (\text{Id}_X^{(i-2)}) \times R \times \text{Id}_X^{(n-i)})(x_1, \ldots, x_j, \overline{t_j}, \ldots, x_n). \]

Then
\[
(w, y_j, y_{j+1}) = (R \times \text{Id}_X) \circ (\text{Id}_X \times R)(x_j, \overline{t}_j, z)
\]
\[
= (R \times \text{Id}_X) \circ (\text{Id}_X \times R)(x_j, \overline{t}_j, z) \text{ because } R(x_j, \overline{t}_j) = (x_j, \overline{t}_j)
\]
\[
= (\text{Id}_X \times R) \circ (R \times \text{Id}_X)(x_j, \overline{t}_j, z) \text{ by the Yang-Baxter equation}
\]
\[
= (\text{Id}_X \times R)(w, y_j, y_{j+1}).
\]

Therefore, \(R(y_j, y_{j+1}) = (y_j, y_{j+1}), \) i.e., \(y_{j+1} = \overline{y}_j\) as desired.

In the same way as above, one can show that the image of \(d_{i,n}^1\) is also contained in \(C_{n-1}^D(X)\).

The homology \(H_n^D(X) = H_n(C_n^D(X))\) is called the degenerate set-theoretic Yang-Baxter homology groups of \(X\). Consider the quotient chain complex \(C^\text{NYB}_n(X) := (C_n^\text{NYB}(X), \partial_n^\text{NYB})\), where \(C_n^\text{NYB}(X) = C_n^Y(X)/C_n^D(X)\), and \(\partial_n^\text{NYB}\) is the induced homomorphism. For an abelian group \(A\), define the chain and cochain complexes \(C^\text{NYB}_n(X; A) := (C_n^\text{NYB}(X; A), \partial_n^\text{NYB})\) and \(C^\text{NYB}_n(X; A) := (C_n^\text{NYB}(X; A), \delta_n^\text{NYB})\), where
\[
C_n^\text{NYB}(X; A) = C_n^\text{NYB}(X) \otimes A, \quad \partial_n^\text{NYB} = \partial_n^\text{NYB} \otimes \text{Id}_A,
\]
\[
C^\text{NYB}_n(X; A) = \text{Hom}(C_n^\text{NYB}(X; A), \partial_n^\text{NYB}), \quad \delta_n^\text{NYB} = \text{Hom}(\partial_n^\text{NYB}, \text{Id}_A).
\]

**Definition 2.2.** Let \(R\) be a set-theoretic Yang-Baxter operator on \(X\). For a given abelian group \(A\), the homology group and cohomology group
\[
H_n^\text{NYB}(X; A) = H_n(C^\text{NYB}_n(X; A)) = Z_n^\text{NYB}(X; A)/B_n^\text{NYB}(X; A),
\]
\[
H^n_{\text{NYB}}(X; A) = H^n(C^\text{NYB}_n(X; A)) = Z^n_{\text{NYB}}(X; A)/B^n_{\text{NYB}}(X; A)
\]
are called the \(n\)th normalized set-theoretic Yang-Baxter homology group of \(X\) with coefficient group \(A\) and the \(n\)th normalized set-theoretic Yang-Baxter cohomology group of \(X\) with coefficient group \(A\).

**Example 2.3.** The following are some computational results for (normalized) set-theoretic Yang-Baxter homology groups:

1. Homology groups of some cyclic biquandles and Alexander biquandles are provided in Table [1] and Table [2].
2. For a given rack (respectively, a given quandle) \((X, *)\), one can obtain the birack (respectively, the biquandle) \(\tilde{X}\) by defining \(R_1(A_1, A_2) = A_2\) and \(R_2(A_1, A_2) = A_1 * A_2\) for all \(A_1, A_2 \in X\). In this case, the rack homology (respectively, quandle homology) of \(X\) and the set-theoretic Yang-Baxter homology of \(\tilde{X}\) coincide.
3. For a given shelf \((X, *)\) (Laver tables, for example) one can construct the set-theoretic solution of the Yang-Baxter equation in the same way as in (2) above, and it is neither a birack nor a biquandle, in general. Homology groups of some shelves are given in Table [3]. In Table [3], \(L_2\) denotes the second Laver table and \(T_2\) and \(T_3\) are the shelves provided in [5].

It is natural to ask whether the set-theoretic Yang-Baxter homology groups can be split into the normalized and degenerated parts. General degeneracies and decompositions in the set-theoretic Yang-Baxter homology of a semi-strong skew cubical structure have been discussed in [16]. However, the above is not a semi-strong skew cubical structure in general.
Table 1. Homology of cyclic biquandles

| $n$ | 1       | 2       | 3       | 4       |
|-----|---------|---------|---------|---------|
| $H_n^Y(B(C_3))$ | $\mathbb{Z} \oplus \mathbb{Z}_3$ | $\mathbb{Z}^3$ | $\mathbb{Z}^9 \oplus \mathbb{Z}_3$ | $\mathbb{Z}^{27}$ |
| $H_n^D(C_3)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}^5$ | $\mathbb{Z}^{19}$ |
| $H_n^{NYB}(C_3)$ | $\mathbb{Z} \oplus \mathbb{Z}_3$ | $\mathbb{Z}^2$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_3$ | $\mathbb{Z}^8$ |
| $H_n^Y(B(C_5))$ | $\mathbb{Z} \oplus \mathbb{Z}_5$ | $\mathbb{Z}^5$ | $\mathbb{Z}^{25} \oplus \mathbb{Z}_5$ | $\mathbb{Z}^{125}$ |
| $H_n^D(C_5)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}^9$ | $\mathbb{Z}^{61}$ |
| $H_n^{NYB}(C_5)$ | $\mathbb{Z} \oplus \mathbb{Z}_5$ | $\mathbb{Z}^4$ | $\mathbb{Z}^{16} \oplus \mathbb{Z}_5$ | $\mathbb{Z}^{64}$ |
| $H_n^Y(B(C_8))$ | $\mathbb{Z} \oplus \mathbb{Z}_8$ | $\mathbb{Z}^8$ | $\mathbb{Z}^{64} \oplus \mathbb{Z}_8$ |
| $H_n^D(C_8)$ | 0 | $\mathbb{Z}$ | $\mathbb{Z}^{15}$ |
| $H_n^{NYB}(C_8)$ | $\mathbb{Z} \oplus \mathbb{Z}_8$ | $\mathbb{Z}^7$ | $\mathbb{Z}^{49} \oplus \mathbb{Z}_8$ |

Table 2. Homology of Alexander biquandles

| $n$ | 1       | 2       | 3       | 4       |
|-----|---------|---------|---------|---------|
| $H_n^Y(B(\mathbb{Z}_{8;3,5}))$ | $\mathbb{Z}^2$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_2^2$ | $\mathbb{Z}^8 \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_8^2$ |
| $H_n^D(\mathbb{Z}_{8;3,5})$ | 0 | $\mathbb{Z}^2$ | $\mathbb{Z}^6 \oplus \mathbb{Z}_2^2$ |
| $H_n^{NYB}(\mathbb{Z}_{8;3,5})$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$ | $\mathbb{Z}^2 \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_8^2$ |
| $H_n^Y(B(\mathbb{Z}_{8;5,5}))$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_2$ | $\mathbb{Z}^{24} \oplus \mathbb{Z}_2^3$ | $\mathbb{Z}^{160} \oplus \mathbb{Z}_2^{15} \oplus \mathbb{Z}_4$ |
| $H_n^D(\mathbb{Z}_{8;5,5})$ | 0 | $\mathbb{Z}^4$ | $\mathbb{Z}^{14} \oplus \mathbb{Z}_2^4$ |
| $H_n^{NYB}(\mathbb{Z}_{8;5,5})$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_2$ | $\mathbb{Z}^{20} \oplus \mathbb{Z}_2^3$ | $\mathbb{Z}^{116} \oplus \mathbb{Z}_2^{11} \oplus \mathbb{Z}_4$ |
| $H_n^Y(B(\mathbb{Z}_{16;13,13}))$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_4$ | $\mathbb{Z}^{24} \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_4^3$ |
| $H_n^D(\mathbb{Z}_{16;13,13})$ | 0 | $\mathbb{Z}^4$ |
| $H_n^{NYB}(\mathbb{Z}_{16;13,13})$ | $\mathbb{Z}^4 \oplus \mathbb{Z}_4$ | $\mathbb{Z}^{20} \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_4^3$ |

It was proven in [26] that the set-theoretic Yang-Baxter homology of a cyclic biquandle can be split into the normalized and degenerated parts.

2.1. The $n$-cocycles of the Alexander biquandles. We investigate some non-trivial cocycles of the Alexander biquandles, which could later be used to compute the homology groups and classify knots and links.
Lemma 2.4. For an Alexander biquandle $X$, the face maps $d^i_{t,n}, d^r_{t,n} : C^Y_B(X) \to C^Y_B(X)$ have the formulas:

\[
d^i_{t,n}(x_1, \ldots, x_n) = (tx_1 + (1-t)x_i, \ldots, tx_{i-1} + (1-t)x_i, x_{i+1}, \ldots, x_n),
\]

\[
d^r_{t,n}(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, (1-s)x_i + sx_{i+1}, \ldots, (1-s)x_i + sx_n).
\]

Proof. Since $(1-s)(1-t) = 0$, we have the identities $s(1-t) = 1-t$ and $t(1-s) = 1-s$. By direct computation, one can obtain the above formulas.

Table 3. Homology of shelves

| $n$ | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|
| $H^Y_B(L_2)$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $H^NY_B(L_2)$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| $H^Y_B(T_2)$ | $\mathbb{Z}^2$ | $\mathbb{Z}^4$ | $\mathbb{Z}^8$ | $\mathbb{Z}^{16}$ |
| $H^NY_B(T_2)$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ | $\mathbb{Z}^2$ |
| $H^Y_B(T_3)$ | $\mathbb{Z}^3$ | $\mathbb{Z}^9$ | $\mathbb{Z}^{27}$ | $\mathbb{Z}^{81}$ |
| $H^NY_B(T_3)$ | $\mathbb{Z}^3$ | $\mathbb{Z}^6$ | $\mathbb{Z}^{12}$ | $\mathbb{Z}^{24}$ |

Theorem 2.5. Let $X = \mathbb{Z}_{m;s,t}$ be an Alexander biquandle. For $n \geq 2$, the map $\theta_n \in C^N_YB(X; \mathbb{Z}_m)$ defined by

\[
\theta_n(x_1, \ldots, x_n) = \prod_{i=1}^{n-1} (x_i - x_{i+1})
\]

and extending linearly to all elements of $C^N_YB(X)$ is an n-cocycle.

Proof. (i) Note that $\mathcal{F} = x$ in $\mathbb{Z}_{m;s,t}$ as $s$ and $t$ are units. Thus, $\theta_n(x_1, \ldots, x_n) = 0$ for every $(x_1, \ldots, x_n) \in C^D_n(X)$.

(ii) We next show that $\theta_n \circ \partial^Y_{n+1} = \theta_n \circ \left( \sum_{i=1}^{n+1} (-1)^{i+1} (d_{i,n+1}^i - d_{i,n+1}^r) \right) = 0$ for each $n \geq 2$.

Let $(x_1, \ldots, x_{n+1}) \in C^Y_B(X)$. By Lemma 2.4 we have

\[
\theta_n \circ d^i_{t,n+1}(x_1, \ldots, x_{n+1}) = \theta_n(tx_1 + (1-t)x_i, \ldots, tx_{i-1} + (1-t)x_i, x_{i+1}, \ldots, x_{n+1})
\]

\[
= t^{i-1}(x_1 - x_2) \cdots (x_{i-2} - x_{i-1})(x_{i-1} - x_i)(x_{i+1} - x_{i+2}) \cdots (x_n - x_{n+1})
\]

\[
+ t^{i-2}(x_1 - x_2) \cdots (x_{i-2} - x_{i-1})(x_i - x_{i+1})(x_{i+1} - x_{i+2}) \cdots (x_n - x_{n+1}).
\]

\[\text{For example, } \left[ \theta_2 \right] \text{ is non-trivial when } X = \mathbb{Z}_{8;3,5}, \mathbb{Z}_{9;4,7}, \mathbb{Z}_{15;11,7} \]
Therefore, $\theta_n \circ \left( \sum_{i=1}^{n+1} (-1)^{i+1} d^i_{n+1} \right) (x_1, \ldots, x_{n+1}) = 0$. Similarly, one can obtain

$\theta_n \circ \left( \sum_{i=1}^{n+1} (-1)^i d^i_{n+1} \right) (x_1, \ldots, x_{n+1}) = 0$, as desired.

3. Biquandle spaces and their homotopy groups

A pre-simplicial set $\mathcal{X} = (X_n, d_i)$ is a collection of sets $X_n$, $n \geq 0$ together with face maps $d_i : X_n \to X_{n-1}$, which are defined for $0 \leq i \leq n$ and satisfy the relation

$$d_i d_j = d_{j-1} d_i \text{ for } i < j.$$  

The dependencies with respect to $n$ are typically omitted from its notation. A pre-simplicial set can be turned into a chain complex $(C_n, \partial_n)$, where $C_n = \mathbb{Z} X_n$ is the free abelian group generated by the elements of $X_n$, and $\partial_n$ is the linearization of $\sum_{i=0}^{n} (-1)^i d_i$.

The geometric realization $|\mathcal{X}|$ of a pre-simplicial set $\mathcal{X} = (X_n, d_i)$ is the cell complex constructed by gluing together standard simplices with the instruction provided by $\mathcal{X}$. A detailed construction is as follows:

$$|\mathcal{X}| = \coprod_{n \geq 0} (X_n \times \Delta^n) / \sim_s,$$

where

- each set $X_n$ is endowed with the discrete topology;
- $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}$ is the $n$-simplex with its standard topology;
- the equivalence relation $\sim_s$ is defined by $(\mathbf{x}, d^i(\mathbf{t})) \sim_s (d_i(\mathbf{x}), \mathbf{t})$,
  where $\mathbf{x} \in X_n$, $\mathbf{t} \in \Delta^{n-1}$ and $d^i : \Delta^{n-1} \to \Delta^n$ are the coface maps given by $d^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 1, t_i, \ldots, t_{n-1})$ for $0 \leq i \leq n$ and satisfy the relation $d^i d^{i+1} = d^{i+1} d^i$ if $0 \leq i < j \leq n$.

Note that $|\mathcal{X}|$ is a cell-complex with one $n$-cell for each element of $X_n$, and the homology of $|\mathcal{X}|$ coincides with that of the chain complex $(C_n, \partial_n)$.

The cubical category can be developed in a similar way to the simplicial one. A pre-cubical set $\mathcal{X} = (X_n, d_i)$ is a collection of sets $X_n$, $n \geq 0$ with face maps $d_i : X_n \to X_{n-1}$, which are defined for $1 \leq i \leq n$, $\varepsilon \in \{0,1\}$ and satisfy the relation

$$d_i^\delta d_i^\varepsilon = d_j^{\delta+1} d_i^\varepsilon \text{ if } i < j \text{ for } \delta, \varepsilon \in \{0,1\}.$$  

A pre-cubical set can also be turned into a chain complex $(C_n, \partial_n)$, where $C_n = \mathbb{Z} X_n$ is the free abelian group generated by the elements of $X_n$, and $\partial_n$ is the linearization of $\sum_{i=1}^{n} (-1)^i (d_i^\delta - d_i^\varepsilon)$.

\[Eilenberg and Zilber [9] introduced the notion of a simplicial set under the name of complete semi-simplicial complex. A semi-simplicial complex in [9] is now called a pre-simplicial set [19, 21].
\[The cubical category was considered first by J.-P. Serre in his PhD thesis [22]. See [12] for details.
One can obtain the geometric realization $|\mathcal{X}|$ of a pre-cubical set $\mathcal{X} = (X_n, d^n_i)$ by gluing together standard cubes with the instruction provided by $\mathcal{X}$:

$$|\mathcal{X}| = \coprod_{n \geq 0} (X_n \times [0,1]^n) \bigg/ \sim_c,$$

- each set $X_n$ is endowed with the discrete topology;
- $[0,1]^n$ is the $n$-cube with its standard topology;
- the equivalence relation $\sim_c$ is defined by $(x, d^n_0(t)) \sim_c (d^n_i(x), t)$, where $x \in X_n$, $t \in [0,1]^n$ and $d^n_0 : [0,1]^n \rightarrow [0,1]^n$ are the coface maps defined by $d^n_0(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, t_i, \ldots, t_{n-1})$ for $1 \leq i \leq n$, $t \in \{0,1\}$ and satisfy the relation $d^nijd^n_{i-1} = d^n_jd^n_i$ if $1 \leq i < j \leq n$, $t \in \{0,1\}$.

Again, $|\mathcal{X}|$ is a cell-complex with one $n$-cell for each element of $X_n$, and the homology of $|\mathcal{X}|$ coincides with that of the chain complex $(C_n, \partial_n)$.

3.1. **Biquandle spaces.** For a given set $X$, let $R = (R_1, R_2) : X \times X \rightarrow X \times X$ be a set-theoretic Yang-Baxter operator. We define the face maps $d^n_i, d^n_i : X^n \rightarrow X^{n-1}$ by

$$d^n_i = (\text{Id}_X^{(n-2)} \times R_1) \circ (\text{Id}_X^{(n-3)} \times R \times \text{Id}_X) \circ \cdots \circ (\text{Id}_X \times R \times \text{Id}_X^{(n-1)}),$$

$$d^n_i = (R_2 \times \text{Id}_X^{(n-2)}) \circ (\text{Id}_X \times R \times \text{Id}_X^{(n-3)}) \circ \cdots \circ (\text{Id}_X^{(i-2)} \times R \times \text{Id}_X^{(n-1)}).$$

Then $\mathcal{X} = (X^n, d^n_i, d^n_i)$ forms a pre-cubical set, where $X^0$ is a singleton set $\{\ast\}$. In this case the homology of its geometric realization $|\mathcal{X}|$ is the homology for the set-theoretic Yang-Baxter equation in [2] (see Figure 3.1). When $(X, R_1, R_2)$ is a birack, its geometric realization is called a **birkack space**. If $(X, R_1, R_2)$ is a biquandle, the birack space can be transformed into a more interesting space, called a **biquandle space**\(^6\). Let $(X, R_1, R_2)$ be a biquandle, and let $|\mathcal{X}|_n$ denote the $n$-skeleton of the birack space $|\mathcal{X}|$ of the biquandle $X$. For each $x \in X$, we denote the unique element $y \in X$ such that $R(x, y) = (x, y)$ by $\overline{x}$, that is, $R(x, \overline{x}) = (x, x)$. Let $D^m = \{(x_1, \ldots, x_m) \in X^m \mid x_i = x_{i0} = x_{i0}+1 \text{ for some } 1 \leq i_0 \leq m-1\}$ be the subset of $X^m$. Note that $D^0 = \emptyset = D^1$.

In analogy to the way quandle spaces in [23] were obtained from rack spaces [11], one can construct the $n$-skeleton of the biquandle space $BX_n$ by attaching extra cells inductively that bound degenerate cells labeled by the elements of $\bigcup_{m=2}^{n-1} D^m$ to $|\mathcal{X}|_n$.

\(^6\)The classifying space of a biquandle was first discussed by Fenn [10].
The 4-skeleton $BX_4$ of a biquandle space is especially important for classical and surface-knot-theoretic applications, and is given particular attention in this paper. We provide a detailed description of $BX_4$ as follows. Let $|\mathcal{X}|_4$ be the 4-skeleton of the birack space of a biquandle $X$. For each rectangle labeled by $(a, \bar{a}) \in D^2$, we glue two edges $\{d_1^e(a, \bar{a})\} \times \square^1$ and $\{d_2^e(a, \bar{a})\} \times \square^1$ of the rectangle together for every $e \in \{l, r\}$. Then it becomes a 2-sphere with labeling (see Figure 3.2), and we denote the cone over it by $B_{(a, \bar{a})}^3$. Let $x = (x_1, x_2, x_3) \in D^3$. Then there exists $i_0 \in \{1, 2\}$ so that $x_{i_0} = x_{i_0+1}$. When we glue two faces $\{d_i^e(x)\} \times \square^2$ and $\{d_{i_0+1}^e(x)\} \times \square^2$ of the cube labeled by $x$ for every $e \in \{l, r\}$, it becomes a $S^2 \times [0, 1]$ with labeling, denoted by $S_x$. Then $S_x \cup B_{(x)}^3 \cup B_{(x)}^3$ is a 3-sphere, where $i \neq i_0, i_0 + 1$. See Figure 3.3 for details. The cone over $S_x \cup B_{(x)}^3 \cup B_{(x)}^3$ is denoted by $B_{(x)}^4$. The space $\left(\frac{|\mathcal{X}|_4 \cup \bigcup_{x \in D^2} B_{x}^3 \cup \bigcup_{x \in D^3} B_{x}^4}{\sim}\right)$ is the 4-skeleton $BX_4$ of the biquandle $X$.

Figure 3.2. Degenerate 2-chains

Figure 3.3. Degenerate 3-chains

Remark 3.1. Ishikawa and Tanaka [13] also gave a rigorous definition of a biquandle space, denoted by $B^QX$, using different face maps and different degeneracies. It is not yet known
whether the biquandle space defined above is homotopy equivalent to theirs. However, we
do not think they are equivalent in general because in the face map
\[ x \mapsto (x_1 \# x_i, \ldots, x_{i-1} \# x_i, x_i, x_{i+1} \# x_i, \ldots, x_n \# x_i) \]
defined in [13], we see that each coordinate is acted by only single element \( x_i \).

However, for some special biquandles, one can find a continuous map between these two
spaces which shows that they are homotopy equivalent to each other. For a cyclic biquandle
\( X \), the cellular map \( BX \to B^Q X \) assigning each cell labeled by \( (x_1, x_2, \ldots, x_n) \) to the cell
labeled by \( (x_1, x_2 + 1, \ldots, x_n + (n-1)) \) is an example.

3.2. Homological and homotopical link invariants. The rack homotopy invariant of
framed oriented links, obtained from the classifying spaces of racks, was introduced previously
[11]. It was transformed into the quandle homotopy invariant [23, 24] and the shadow
homotopy invariant [32] of oriented links using the classifying spaces of quandles, which are
constructed by adding extra cells to the classifying spaces of racks. In a similar manner, we
construct a homotopy invariant of oriented links using the classifying spaces of biquandles.

Let \( K \) be an oriented link (respectively, an oriented closed knotted surface). Let \( (X, R_1, R_2) \)
be a biquandle. We call a generic projection of \( K \) into \( \mathbb{R}^2 \) (respectively, \( \mathbb{R}^3 \)) a diagram of \( K \).
For each \( n = 2, 3 \), consider the one-point compactification \( S^n \) of \( \mathbb{R}^n \) with base point \( \infty \). A
biquandle coloring by \( X \) of an oriented diagram of \( K \) is an assignment of the elements of
\( X \) to the semi-arcs (respectively, faces) of the oriented diagram with the convention depicted in
Figure 3.4 (respectively, in Figure 3.5). Note that an \( X \)-colored crossing (respectively, an \( X \)-
colored triple point) of \( K \) represents a chain \( \pm (a, b) \) (respectively, \( \pm (a, b, c) \)) in \( C_n^{NYB}(X; \mathbb{Z}) \)
(see Figures 3.4, 3.5, and compare them to each other). The signs of the chain are deter-
mined by the orientations of the corresponding cells. An \( X \)-colored diagram of \( K \) represents
a cycle in \( Z_n^{NYB}(X; \mathbb{Z}) \) (for \( n = 2, 3 \)) that is the signed sum of the chains represented by all
crossings of the diagram of \( K \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\caption{Coloring convention for oriented links}
\end{figure}

Proposition 3.2. The homology class in \( H_n^{NYB}(X) \) (for \( n = 2, 3 \)) determined by the repre-
sented cycle of a diagram of \( K \) is independent of the choice of the diagram.

Proof. Let \( D_1 \) and \( D_2 \) be two diagrams representing \( K \). Note that one diagram can be
transformed into the other by a finite sequence of Reidemeister moves or Roseman moves.
One can show that each move either does not affect the represented cycle or changes its
homology class by a boundary (cf. [4] Theorem 5.6), i.e., the representative cycles of \( D_1 \) and
\( D_2 \) are homologous.
\endproof
We let $BX$ be (the 4-skeleton of) the biquandle space of a given biquandle $X$. It is well-known that two diagrams represent the same oriented link (respectively, the same closed knotted surface) if and only if they are related by a finite sequence of Reidemeister moves (respectively, Roseman moves). Suppose that the diagram $D$ of an oriented link (respectively, an oriented closed knotted surface) is placed inside $I^2$ (respectively, $I^3$), where $I = [0, 1]$ is the unit interval. By considering $D$ as a decomposition of $I^2$ (respectively, $I^3$) by an immersed curve (respectively, immersed surface), one can obtain its dual decomposition of $I^2$ (respectively, modified dual decomposition of $I^3$) (see, e.g., [23, 24, 32] for the definitions). Then each $X$-colored crossing (respectively, each $X$-colored triple point) of $D$ is enveloped in a rectangle (respectively, a rectangular box) labeled by the chain represented by the crossing. The union of the rectangles (respectively, rectangular boxes) is mapped to $BX$ and the boundary of $I^2$ (respectively, $I^3$) is mapped to the base point $*$ of the biquandle space $BX$ (see Figure 3.6 for example). Then the homotopy class of the map $(I^n, \partial I^n) \to (BX, *)$ is an invariant under Reidemeister moves (respectively, Roseman moves) since every diagrammatic equivalence corresponds to cells in $BX$.

![Figure 3.5. Coloring convention for oriented knotted surfaces](image)

![Figure 3.6. The maps $(I^n, \partial I^n) \to (BX, *)$ for $n = 2, 3$](image)

**Proposition 3.3.** (cf. [23, 32, 24]) The homotopy class of the maps $(I^n, \partial I^n) \to (BX, *)$ in $\pi_n(BX)$ (for $n = 2, 3$) is independent of the representative diagram.

By using Proposition 3.2 and Proposition 3.3, one can construct homological and homotopical link invariants in a similar manner to [4, 23, 24]. For example, we define the homotopical state-sum invariants of oriented links and oriented closed knotted surfaces as follows:
Let $K$ be an oriented link or an oriented closed knotted surface, and let $D$ be its diagram. For a given finite biquandle $X$, we denote by $\text{Col}_X(D)$ the set of biquandle colorings of $D$ by $X$. Then for each $C \in \text{Col}_X(D)$, the homotopy class $\Psi_X(D;C)$ of the map $\psi_X(D;C) : (S^n, \infty) \to (BX, \ast)$ (for $n = 2, 3$) is independent of the representative diagram by Proposition 3.3. Therefore, $\Psi_X(K) = \sum_{C \in \text{Col}_X(D)} \Psi_X(D;C) \in \mathbb{Z}[\pi_n(BX)]$ is a link invariant. Here, $n = 2$ in the case of an oriented link, and $n = 3$ in the case of an oriented closed knotted surface.

3.3. The second homotopy groups of biquandle spaces. Some properties of the homotopy groups of quandle spaces were discussed in [23, 24]. In a way similar to the idea shown in [11, 23, 24], we prove that the second homotopy group of a biquandle space (respectively, a birack space) is finitely generated if the biquandle (respectively, the birack) is finite.

It was shown in [11] that every rack space is a simple space, i.e., $\pi_1$ acts trivially on $\pi_n$ for each $n > 1$. We can generalize it on birack spaces as follows.

Let $X$ be a set, and let $R$ be a set-theoretic Yang-Baxter operator on $X$. The associated group or enveloping group of $(X, R)$, denoted by $\text{AS}(X, R)$ or simply by $\text{AS}(X)$, is the group with $X$ as the set of generators and defining relations $xy = R_1(x, y)R_2(x, y)$ for all $x, y \in X$. Note that if $X$ is a birack, then $\text{AS}(X)$ is isomorphic to $\pi_1(|X|)$ by the definition of the birack space $|X|$.

Lemma 3.4. Let $X$ be a finite birack. Consider the set $\overline{X}$, which consists of the elements in $X$ and their inverse elements in the free group $F_X$ generated by $X$. We denote the symmetric group on $\overline{X}$ by $\text{Sym}(\overline{X})$. Then there exists a homomorphism $\phi : \text{AS}(X) \to \text{Sym}(\overline{X}) \times \text{Sym}(\overline{X})^{\text{op}}$ such that

1. $\text{Ker}(\phi)$ is finitely generated and abelian;
2. $\text{Im}(\phi)$ is finite.

Proof. (1) Based on the construction in [18], we let $\xi$ and $\eta$ be the left and right actions of $\text{AS}(X)$ on itself that extend $R_1$ and $R_2$ respectively. Consider the homomorphism $\phi : \text{AS}(X) \to \text{Sym}(\overline{X}) \times \text{Sym}(\overline{X})^{\text{op}}$ defined by $\phi(a) = (\xi_X(a), (a)\eta_X)$, where $\xi_X, \eta_X : \text{AS}(X) \to \text{Sym}(\overline{X})$ are the homomorphism induced by $\xi$ and the anti-homomorphism induced by $\eta$ respectively (see [18] for further details). It is clear that $\text{Ker}(\phi) = \text{Ker}(\xi_X) \cap \text{Ker}(\eta_X)$. Then Proposition 6 in [18] implies that $\text{Ker}(\phi)$ is finitely generated and abelian.

(2) Since $X$ is finite and $\text{Im}(\phi)$ is a subgroup of $\text{Sym}(\overline{X}) \times \text{Sym}(\overline{X})^{\text{op}}$, $\text{Im}(\phi)$ is finite.

Definition 3.5. [11] A cobordism by moves between two labelled diagrams is a sequence of the following moves:

1. Legal Reidemeister $\Omega_2$ and $\Omega_3$ moves.
2. Introduction and deletion of unknotted and unlinked circle components in the diagram $D \leftrightarrow D \cup \bigcirc$.
3. A bridge move between adjacent arcs with the same label and opposite orientations.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{bridge_move.png}
\caption{A bridge move}
\end{figure}
Proposition 3.6. [11] Homotopy classes of maps $(I^2, \partial I^2) \to (|X|, *)$ are in bijective correspondence with equivalence classes of diagrams in $S^2$ labelled by $X$ under cobordism by moves.

Let us consider the homomorphism $\xi_X : AS(X) \to Sym(\overline{X})$ introduced in the proof of Lemma 3.4. Let $\tilde{M}_X$ be the connected covering space of $M_X := |X|$ having $\text{Ker} \xi_X$ as the fundamental group, i.e., $\pi_1(\tilde{M}_X) \cong \text{Ker} \xi_X < AS(X) \cong \pi_1(|X|)$. Note that $\tilde{M}_X \to M_X$ is a finite covering because $Sym(\overline{X})$ is of finite order.

Lemma 3.7. For any birack $X$, the canonical action of $\pi_1(\tilde{M}_X)$ onto $\pi_2(\tilde{M}_X)$ is trivial.

Proof. Let $[\gamma] \in \pi_1(\tilde{M}_X)$ and $[f] \in \pi_2(\tilde{M}_X)$. Consider the covering $p : \tilde{M}_X \to M_X$ and its induced homomorphisms $(p_i)_* : \pi_i(\tilde{M}_X) \to \pi_i(M_X)$ $(i = 1, 2)$. Then we have

\[
(p_2)_*([\gamma] \cdot [f]) = [p \circ \gamma] \cdot [p \circ f] = (p_1)_*([\gamma]) \cdot (p_2)_*([f]) \quad (\text{See Figure 3.8})
\]

\[
= [p \circ f] \quad \text{because} \quad [p \circ \gamma] \in \text{Ker} \xi_X \quad (\text{See Figure 3.9})
\]

\[
= (p_2)_*([f]).
\]

Figure 3.8. The canonical action of $\pi_1(\tilde{M}_X)$ onto $\pi_2(\tilde{M}_X)$

\[
K_x \quad \text{is the diagram obtained after the circle labelled by} \ x \ \text{passing under the labelled diagram} \ K.
\]

If $x \in \text{Ker} \xi_X$, then the diagram does not change in this process, i.e., $K_x = K$.

Figure 3.9. A cobordism by moves between $K$ and $K_x$

Since $(p_2)_*$ is an isomorphism, $[\gamma] \cdot [f] = [f]$. Therefore, the action is trivial.

The following proposition gives us a long exact sequence which is used in the proof of Theorem 3.9.

Proposition 3.8. [22] Let $M$ be a connected CW-complex with the trivial canonical action of $\pi_1(M)$ on $\pi_2(M)$. Then we have the following long exact sequence:

\[
\cdots \to H_3(\pi_1(M); \mathbb{Z}) \xrightarrow{\tau} \pi_2(M) \xrightarrow{h} H_2(M; \mathbb{Z}) \to H_2(\pi_1(M); \mathbb{Z}) \to 0,
\]
where $H_3(\pi_1(M); \mathbb{Z})$ is the group homology, $\tau$ is the transgression map, and $\mathfrak{h}$ is the Hurewicz homomorphism.

**Theorem 3.9.** For any finite birack $X$, $\pi_2(|X|)$ is finitely generated. If $X$ is a finite biquandle, then $\pi_2(BX)$ is finitely generated.

**Proof.** (i) Since $X$ is a birack, the sequence

$$
H_3(\pi_1(\tilde{M}_X); \mathbb{Z}) \xrightarrow{\tau} \pi_2(\tilde{M}_X) \xrightarrow{\mathfrak{h}} H_2(\tilde{M}_X; \mathbb{Z})
$$

is exact by Lemma 3.7 and Proposition 3.8.

(ii) Since $X$ is finite, its birack space $|X|$ contains only finitely many 2-cells, and so does $\tilde{M}_X$ as $\tilde{M}_X \to M_X = |X|$ is a finite covering. Thus, $H_2(\tilde{M}_X; \mathbb{Z})$ is finitely generated.

(iii) Let us consider the homomorphism $\phi : AS(X) \to \text{Sym}(X) \times \text{Sym}(X)^{op}$ defined in Lemma 3.4 and its restriction $\tilde{\phi} : \ker \xi_X \to \text{Sym}(X) \times \text{Sym}(X)^{op}$.

Consider the canonical short exact sequence

$$0 \to \ker(\tilde{\phi}) \to \ker \xi_X \to \text{Im}(\tilde{\phi}) \to 0.$$ 

Then the Lyndon-Hochschild-Serre spectral sequence of the group extension above takes the form

$$E_p^2 \simeq H_p(\text{Im}(\tilde{\phi}); H_q(\ker(\tilde{\phi}); \mathbb{Z})) \Rightarrow H_{p+q}(\ker \xi_X; \mathbb{Z}).$$

Note that $\ker(\phi)$ is finitely generated and abelian by Lemma 3.4(1). Since $\ker(\tilde{\phi}) < \ker(\phi)$, $\ker(\tilde{\phi})$ is finitely generated and abelian. Then $H_q(\ker(\tilde{\phi}); \mathbb{Z})$ is finitely generated, and so is $H_p(\text{Im}(\tilde{\phi}); H_q(\ker(\tilde{\phi}); \mathbb{Z}))$ because $\text{Im}(\tilde{\phi}) < \text{Im}(\phi)$ is finite by Lemma 3.4(2), i.e., $H_*(\ker \xi_X; \mathbb{Z})$ is finitely generated. Accordingly, $H_*(\pi_1(\tilde{M}_X); \mathbb{Z})$ is finitely generated since $\ker \xi_X$ is isomorphic to $\pi_1(\tilde{M}_X)$.

Therefore, $\pi_2(|X|) = \pi_2(M_X) \cong \pi_2(\tilde{M}_X)$ is finitely generated by (i), (ii), and (iii).

Moreover, for a biquandle $X$, the inclusion map $|X| \hookrightarrow BX$ induces the epimorphism $\pi_2(|X|) \to \pi_2(BX)$. Hence, $\pi_2(BX)$ is also finitely generated if $X$ is finite. \qed

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**References**

[1] R. J. Baxter, *Partition function of the eight-vertex lattice model*, Ann. Physics 70 (1972), 193-228.

[2] J. S. Carter, M. Elhamdadi, and M. Saito, *Homology theory for the set-theoretic Yang-Baxter equation and knot invariants from generalizations of quandles*, Fund. Math. 184 (2004), 31-54.

[3] J. Ceniceros, M. Elhamdadi, M. Green, and S. Nelson, *Augmented biracks and their homology*, Internat. J. Math. 25 (2014), no. 9, 1450087, 19 pp.

[4] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. 355 (2003), no. 10, 3947-3989.
[5] A. S. Crans, S. Mukherjee, and J. H. Przytycki, *On homology of associative shelves*, J. Homotopy Relat. Struct. 12 (2017), no. 3, 741-763.

[6] V. G. Drinfel’d, *On some unsolved problems in quantum group theory*, Quantum groups (Leningrad, 1990), 1-8, Lecture Notes in Math., 1510, Springer, Berlin, 1992.

[7] P. Etingof and S. Gelaki, *A method of construction of finite-dimensional triangular semisimple Hopf algebras*, Math. Res. Lett. 5 (1998), no. 4, 551-561.

[8] P. Etingof, T. Schedler, and A. Soloviev, *Set-theoretical solutions to the quantum Yang-Baxter equation*, Duke Math. J. 100 (1999), no. 2, 169-209.

[9] S. Eilenberg and J. A. Zilber, *Semi-simplicial complexes and singular homology*, Ann. of Math. (2) 51 (1950), 499-513.

[10] R. Fenn, *Tackling the trefoils*, J. Knot Theory Ramifications 21 (2012), no. 13, 1240004, 20 pp.

[11] R. Fenn, C. Rourke, and B. Sanderson, *An introduction to species and the rack space*, Topics in knot theory, 33-55, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, 1993.

[12] R. Fenn, C. Rourke, and B. Sanderson, *Trunks and classifying spaces*, Appl. Categ. Structures 3 (1995), no. 4, 321-356.

[13] K. Ishikawa and K. Tanaka, *Quandle colorings vs. biquandle colorings*, Preprint; e-print: arxiv.org/abs/1912.12917.

[14] V. F. R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. of Math. (2) 126 (1987), no. 2, 335-388.

[15] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), no. 1, 37-65.

[16] V. Lebed and L. Vendramin, *Homology of left non-degenerate set-theoretic solutions to the Yang-Baxter equation*, Adv. Math. 304 (2017), 1219-1261.

[17] J.-H. Lu, M. Yan, and Y. Zhu, *On Hopf algebras with positive bases*, J. Algebra 237 (2001), no. 2, 421-445.

[18] J.-H. Lu, M. Yan, and Y. Zhu, *On the set-theoretical Yang-Baxter equation*, Duke Math. J. 104 (2000), no. 1, 1-18.

[19] J.-L. Loday, *Cyclic Homology* (2nd ed.), Grund. Math. Wissen., Vol. 301 (Springer, 1998).

[20] S. V. Matveev, *Distributive groupoids in knot theory*, (Russian) Mat. Sb. (N.S.) 119(161) (1982), no. 1, 78-88, 160. English translation: Math. USSR-Sb. 47 (1984), no. 1, 73-83.

[21] J. P. May, *Simplicial Objects in Algebraic Topology*, Chicago Lectures in Mathematics (University of Chicago Press, 1967).

[22] J. McCleary, *A user’s guide to spectral sequences* (Second edition), Cambridge Studies in Advanced Mathematics, 58, Cambridge University Press, Cambridge, 2001.

[23] T. Nosaka, *On homotopy groups of quandle spaces and the quandle homotopy invariant of links*, Topology Appl. 158 (2011), no. 8, 996-1011.

[24] T. Nosaka, *Quandle homotopy invariants of knotted surfaces*, Math. Z. 274 (2013), no. 1-2, 341-365.

[25] J. H. Przytycki, *Knots and distributive homology: from arc colorings to Yang-Baxter homology*, New ideas in low dimensional topology, 413-488, Ser. Knots Everything, 56, World Sci. Publ., Hackensack, NJ, 2015.

[26] J. H. Przytycki, P. Vojtěchovský, and S. Y. Yang, *Set-theoretic Yang-Baxter (co)homology theory of involutive non-degenerate solutions*, Preprint; e-print: arxiv.org/abs/1911.03009.

[27] J. H. Przytycki and X. Wang, *Equivalence of two definitions of set-theoretic Yang-Baxter homology and general Yang-Baxter homology*, J. Knot Theory Ramifications 27 (2018), no. 7, 1841013, 15 pp.

[28] J.-P. Serre, *Homologie singuliè re des espaces fibrés*. Applications, PhD Thesis 1951 Université Paris IV-Sorbonne.

[29] V. G. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. 92 (1988), no. 3, 527-553.

[30] A. Weinstein and P. Xu, *Classical solutions of the quantum Yang-Baxter equation*, Comm. Math. Phys. 148 (1992), no. 2, 309-343.

[31] C. N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. 19 (1967), 1312-1315.

[32] S. Y. Yang, *Extended quandle spaces and shadow homotopy invariants of classical links*, J. Knot Theory Ramifications 26 (2017), no. 3, 1741010, 13 pp.
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