ON TROPICAL CYCLE CLASS MAPS

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Abstract. In this paper, we show that tropical cycle class maps introduced by Liu for smooth algebraic varieties over trivially valued fields are isomorphisms. More precisely, we prove that tropical cohomology is isomorphic to Zariski sheaf cohomology of the sheaves of tropical analogs of Milnor $K$-groups. Proof is based on a theorem on “cohomology theories”, developed by many mathematicians, and explicit computation of tropical cohomology of $\mathbb{A}^1$-fibers by non-archimedean geometry.

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1. INTRODUCTION

Tropical geometry is a combinatorial shadow of algebraic geometry. There are many applications by using tropical analogs of classical objects. In particular, cohomology theory in tropical geometry is related to cohomology theory in algebraic geometry. Gross-Siebert ([GS10]) introduced a cohomology theory for a tropical affine manifold $B$ with singularities, and proved that it is isomorphic to log Dolbeault cohomology of a logarithmic Calabi-Yau variety $X_0(B, \mathscr{P}, s)$. Itenberg-Katzarkov-Mikhalkin-Zharkov ([IKMZ17]) introduced tropical cohomology $H^*_{\text{Trop}}(W)$ for a tropical variety $W$, and proved that when $W$ is smooth and is the tropicalization $\text{Trop}(\mathcal{X}_{\mathbb{C}(\!(t)\!)}$) of a family of smooth complex projective varieties $X_t$, they are isomorphic to the weight graded pieces of limit mixed Hodge structure on the singular cohomology of $X_t$. 

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For a smooth algebraic variety $X$ over a field equipped with the trivial valuation, tropical cohomology $H_{\text{Trop}}^{p,q}(X, \mathbb{Q})$ is also defined by tropicalizations of $X$. The aim of this paper is to study it in details. The goal of this paper is the following.

**Theorem 1.1.** For any $p \geq 0$, Liu’s tropical cycle class map

$$\text{CH}^p(X) \otimes \mathbb{Z} \mathbb{Q} \to H_{\text{Trop}}^{p,p}(X, \mathbb{Q})$$

([Liu20, Definition 3.8], Subsection [6.3]) is an isomorphism.

To prove Theorem 1.1, we introduce tropical analogs $K_T$ of Milnor $K$-groups, which are called *tropical* $K$-groups. We shall show a tropical analog of Bloch’s formula

$$\text{CH}^p(X) \otimes \mathbb{Z} \mathbb{Q} \cong H^p(X_{\text{Zar}}, \mathcal{K}_T^p)$$

(Corollary 7.9), where $\mathcal{K}_T^p$ is the Zariski sheaf of $p$-th (rational) tropical $K$-groups. Theorem 1.1 follows from the following (see Remark 8.4).

**Theorem 1.2 (Theorem 8.2).** For any $p, q$, we have

$$H_{\text{Trop}}^{p,q}(X, \mathbb{Q}) \cong H^q(X_{\text{Zar}}, \mathcal{K}_T^p).$$

Most part of this paper is devoted to prove Theorem 1.2.

**Corollary 1.3.** We have $H_{\text{Trop}}^{p,q}(X, \mathbb{Q}) = 0$ for $q \geq p + 1$.

*Proof.* This follows from the Gersten resolution (Corollary 7.9). □

We shall give proof of the following in Subsection 8.1.

**Corollary 1.4 (Corollary 8.6).** When $X$ is proper, we have $H_{\text{Trop}}^{p,0}(X, \mathbb{Q}) = 0$ ($p \geq 1$).

When $X$ is proper and having closed immersions to toric varieties, *tropical homology* $H_{\text{Trop}}^{p,p}(X, \mathbb{Q})$ is defined as the dual of tropical cohomology, and there is another tropical cycle class map (see Subsection 6.3)

$$Z_p(X) \to H_{\text{Trop}}^{p,p}(X, \mathbb{Q}),$$

where $Z_p(X)$ is the group of algebraic cycles of dimension $p$. The following follows from Theorem 1.1 and compatibility of tropical cycle class maps and intersection theory ([Liu20, Theorem 1.1]) (see Subsection 6.3 and Remark 8.4).

**Corollary 1.5.** We assume that $X$ is proper and there is a closed immersion of $X$ to a normal toric variety (e.g., projective). Then, for any $p \geq 0$, the kernel of tropicalization map

$$Z_p(X) \otimes \mathbb{Q} \to H_{\text{Trop}}^{p,p}(X, \mathbb{Q})$$

is the $\mathbb{Q}$-vector subspace generated by algebraic cycles numerically equivalent to $0$.

A character of our result is that it holds for general smooth algebraic varieties. Properties of their tropical cohomology were previously not known so much.

**Remark 1.6.** We remark related works.

- Katzarkov-Kontsevich [Kat09] proposed an approach to the Hodge conjecture for several types of abelian varieties by using tropical geometry and mirror symmetry.
- There is a paper by Zharkov [Zha20] on Kontsevich’s idea to find a counter-example to the Hodge conjecture by finding a counter-example to a tropical analog of the Hodge conjecture.
• Babaee-Huh gave a counter-example to a stronger version of the Hodge conjecture by tropical geometry [BH17].

• When $X$ is a curve, a generalization of Theorem 1.2 is already proved over (not necessarily trivial) non-archimedean valuation fields by Jell-Wanner [JW18] for Mumford curves and Jell [Jel19] for smooth projective curves.

• Many properties (such as Lefschetz (1,1) theorem, Poincare duality, hard Lefschetz property, and Hodge-Riemann relations) of tropical cohomology of smooth (i.e., locally matroidal) tropical varieties are known. (Tropicalizations of smooth algebraic varieties are not necessarily smooth.) See the introduction of [AP20] on these topics. In particular, for rationally triangulable ones, Amini-Piquerez [AP20-2] proved a tropical analog of the Hodge conjecture.

• After the first version of the current paper, Amini-Piquerez [AP21, Theorem 1.3] proved a result similar to Theorem 1.1 for unimodular tropical fans.

Remark 1.7. When the base field is $\mathbb{C}$, we can construct the same tropicalizations of algebraic varieties over $\mathbb{C}$ as the trivial valuation using the usual absolute value of $\mathbb{C}$ ([Jon16]). By using this construction, we have a natural map ([M21, Theorem 1.1])

$$\Trop^*: H^{p,q}_{\text{Trop}}(X, \mathbb{R}) \to H^{p,q}(X(\mathbb{C}))$$

inducing a commutative diagram

$$\begin{array}{ccc}
\text{CH}^p(X) \otimes \mathbb{R} & \xrightarrow{\text{deg}} & H^{p,p}_{\text{Trop}}(X, \mathbb{R}) \\
\downarrow & & \downarrow \Trop^* \\
H^{p,p}(X(\mathbb{C})),
\end{array}$$

where the diagonal arrow is the usual cycle class map. The map $\Trop^*$ is given by a map from tropical differential forms to usual currents of $X(\mathbb{C})$, which is compatible with “weighted tropicalizations” of generic semi-algebraic subsets ([M21, Theorem 1.12]).

Our proof of Theorem 1.2 is based on a theorem for general “cohomology theories”, which is proved by Bloch [Blo74] for $K_2$, Quillen [Qui73] for general algebraic $K$-theory, and developed by many mathematicians including Bloch-Ogus [BO74], Gabber [Gab94], Rost [Ros96], and Collit-Thélène-Hoobler-Kahn [CTHK97]. By this theorem, we reduce Theorem 1.2 to (easy computations and) $\mathbb{A}^1$-homotopy invariance of tropical cohomology.

The most technical part of this paper is proof of $\mathbb{A}^1$-homotopy invariance. This consists of careful computations of tropicalizations of $\mathbb{A}^1$ over valued fields of height 1 and tropical $K$-groups of residue fields of valuations in the adic space $\mathbb{A}_{1,\text{ad}}$.

Throughout this paper, we consider tropical cohomology and tropical $K$-groups of $\mathbb{Q}$-coefficients. For $\mathbb{Z}$-coefficients, at least, we need to modify Section 11.

The organization of this paper is as follows. In Section 2, we fix several notations and terminologies. Section 3, 4, and 6 are devoted to basics of valuations, non-archimedean analytic spaces, tropicalizations, tropical cohomology. In Section 5, we study tropicalizations of valuations of general heights. In Section 7, we introduce and study tropical $K$-groups. In Section 8, we prove the main theorem (Theorem 8.2) of this paper. Proof of key Proposition 8.1 is given in Section 9 (étale excision) and in Section 11 ($\mathbb{A}^1$-homotopy invariance). In Section 10, we give explicit descriptions of analytifications and tropicalizations of the affine line $\mathbb{A}^1$. In Section 11, we prove $\mathbb{A}^1$-homotopy invariance of tropical cohomology over trivially valued fields. In Section 12, we show the existence of corestriction maps for extensions of finite base fields, which is used to prove the main theorem over finite fields.
2. NOTATIONS AND TERMINOLOGIES

For a \( \mathbb{Z} \)-module \( G \) and a commutative ring \( R \), we put \( G_R := G \otimes_{\mathbb{Z}} R \).

Integral separated schemes of finite type over fields are called algebraic varieties. Cones mean strongly convex rational polyhedral cones. Toric varieties are assumed to be normal.

Let \( L/K \) be an extension of fields. We denote algebraic closure of \( L \) by \( L^{\text{alg}} \). We denote transcendental degree of \( L/K \) by \( \text{tr.deg}(L/K) \).

3. VALUATIONS AND NON-ARCHIMEDEAN ANALYTIC SPACES

In this section, we give a quick review on (non-archimedean) valuations (Subsection 3.1) and non-archimedean analytic spaces: Berkovich analytic spaces (Subsection 3.2), Zariski-Riemann spaces (Subsection 3.3), and Huber’s adic spaces (Subsection 3.4). (See Section 10 for analytifications of the affine line, which is used in Section 11.) We refer to [HK94] and [Bou72, Chapter 6] for valuations, [Hub93] and [Hub94] for valuations and Huber’s adic spaces, [Ber90], [Ber93], and [Tem15] for Berkovich analytic spaces, and [Tem11] for Zariski-Riemann spaces.

3.1. Valuations. In this subsection, rings are assumed to be commutative with a unit element.

**Definition 3.1.** We define a valuation \( v \) of a ring \( R \) as a map \( v: R \to \Gamma_v \cup \{\infty\} \) satisfying the following properties:

- \( \Gamma_v \) is a totally ordered abelian group,
- \( v(ab) = v(a) + v(b) \) for any \( a, b \in R \), where we extend the group law of \( \Gamma_v \) to \( \Gamma_v \cup \{\infty\} \) by \( \gamma + \infty = \infty + \gamma = \infty \) for any \( \gamma \in \Gamma_v \),
- \( v(0) = \infty \) and \( v(1) = 0 \),
- \( v(a + b) \geq \min\{v(a), v(b)\} \), where we extend the order of \( \Gamma_v \) to \( \Gamma_v \cup \{\infty\} \) by \( \infty \geq \gamma \) for any \( \gamma \in \Gamma_v \).

The set \( \text{supp}(v) := v^{-1}(\infty) \) is a prime ideal of \( R \), which is called the support of \( v \). The subgroup of \( \Gamma_v \) generated by \( v(R) \setminus \{\infty\} \) is called the value group of \( v \). We denote it by \( \Gamma_v \). The valuation \( v \) extends to a valuation on \( \text{Frac}(R/\text{supp}(v)) \), which is also denoted by \( v \). We put

\[
\mathcal{O}_v := \{ a \in \text{Frac}(R/\text{supp}(v)) \mid v(a) \geq 0\},
\]

which is called the valuation ring of \( v \). We put

\[
\kappa(v) := \text{Frac}(\mathcal{O}_v/\{ a \in \mathcal{O}_v \mid v(a) > 0\}),
\]

which is called the residue field of \( v \). If \( R/\text{supp}(v) \subset \mathcal{O}_v \), we call the image of the maximal ideal under the canonical morphism \( \text{Spec} \mathcal{O}_v \to \text{Spec} R \) the center of \( v \).
Definition 3.2. We call two valuations \( v \) and \( w \) of a ring \( R \) are equivalent if there exists an isomorphism \( \varphi : \Gamma_v \cong \Gamma_w \) of totally ordered abelian groups satisfying \( \varphi \circ v = w \), where \( \varphi' : \Gamma_v \cup \{\infty\} \to \Gamma_w \cup \{\infty\} \) is the extension of \( \varphi \) defined by \( \varphi'(\infty) = \infty \).

We call the rank of a totally ordered abelian group \( \Gamma \) as an abelian group the rational rank of \( \Gamma \). We denote it by \( \text{rank} \Gamma \).

Definition 3.3. Let \( \Gamma \) be a totally ordered abelian group. A subgroup \( H \) of \( \Gamma \) is called convex if every element \( \gamma \in \Gamma \) satisfying \( h < \gamma < h' \) for some \( h, h' \in H \) is contained in \( H \).

When \( H \subset \Gamma \) is a convex subgroup, the quotient subgroup \( \Gamma/H \) has a natural order, i.e., \( \gamma \leq \gamma' \) if \( \gamma \leq \gamma + h \) for some \( h \in H \).

The set of convex subgroups of \( \Gamma \) are totally ordered by inclusions.

Definition 3.4. We call the number of proper convex subgroups of a totally ordered abelian group \( \Gamma \) the height of \( \Gamma \). We denote it by \( \text{ht} \Gamma \).

The following well-known theorem is called the Harn embedding theorem.

Theorem 3.5 (Clifford [Cli54], Hausner-Wendel [HW52]). Every totally ordered abelian group \( \Gamma \) of finite height \( n \) has an embedding into the additive group \( \mathbb{R}^n \) with the lexicographic order.

Remark 3.6. Let \( G \subset \mathbb{R}^n \) be a subgroup. Then the convex subgroups of \( G \) are \( G \cap \{(0, \ldots, 0)\} \times \mathbb{R}^{n-r} \) \( (0 \leq r \leq n) \), where \( (0, \ldots, 0) \in \mathbb{R}^r \). In particular, a totally ordered abelian group \( \Gamma \) of finite height \( n \) can not be embedded in \( \mathbb{R}^{n-1} \).

We call the rational rank (resp. height) of the value group of a valuation \( v \) the rational rank (resp. height) of \( v \). Rational ranks and heights of equivalence classes of valuations are defined as those of representatives.

Definition 3.7. A valued field \((L, v)\) is called trivially valued if \( \Gamma_v = \{0\} \).

Let \( R \) be a ring. We call the set of all equivalence classes of valuations of \( R \) the valuation spectrum of \( R \). We denote it by \( \text{Spv}(R) \). We equip \( \text{Spv}(R) \) with the topology which is generated by the sets

\[
\{v \in \text{Spv}(R) \mid v(a) \geq v(b) \neq \infty \} \quad (a, b \in R).
\]

In this paper, generalizations and specializations of a valuation in (subsets of) \( \text{Spv}(R) \) are in the topological sense.

Let \( v : R \to \Gamma_v \cup \{\infty\} \) be a valuation, \( H \subset \Gamma_v \) a convex subgroup. We define a map

\[
v/H : R \to (\Gamma_v/H) \cup \{\infty\}, \quad a \mapsto \begin{cases} v(a) \mod H & \text{if } v(a) \neq \infty \\ \infty & \text{if } v(a) = \infty. \end{cases}
\]

Lemma 3.8 (Huber-Knebusch [HK94, Lemma 1.2.1]). The map \( v/H \) is a valuation of \( R \), and it is a generalization of \( v \) in \( \text{Spv}(R) \), called a vertical (or primary) generalization.

For a field \( K \), all specializations in \( \text{Spv}(K) \) are vertical [HK94, Proposition 1.2.4].
Remark 3.9 (Bourbaki [Bou72, Proposition 2 in Section 4 in Chapter 6]). For a valuation $v$ of a field $K$, there is a natural bijection between specializations $w$ of $v$ in $\text{Spv}(K)$ and valuations $\overline{w}$ on the residue field $\kappa(v)$ of $v$ given as follows. Let $w \in \text{Spv}(K)$ be a specialization of $v$. Then the image of the valuation ring $\mathcal{O}_w(\subset \mathcal{O}_v)$ under the natural map $\mathcal{O}_v \to \kappa(v)$ is the valuation ring $\mathcal{O}_{\overline{w}}$ of a valuation $\overline{w}$ of $\kappa(v)$. The value group $\Gamma_w$ contains $\Gamma_{\overline{w}}$ as a convex subgroup, and we have $w/\Gamma_{\overline{w}} = v$.

3.2. Berkovich analytic spaces. In [Ber90, Chapter 3], Berkovich introduced the Berkovich analytic space $X^\text{Ber}$ associated to a separated scheme $X$ of finite type over a complete valuation field $(L, v_L: L^\times \to \mathbb{R})$ of height $\leq 1$. Berkovich analytic spaces are, as sets, the sets of bounded multiplicative seminorms. There exists a bijection between multiplicative seminorms $|\cdot|$ and valuations $v$ with target $\mathbb{R}$ defined by $|\cdot| \mapsto -\log |\cdot|$. By this bijection, we consider multiplicative seminorms as valuations.

In this paper, an affinoid algebra $A$ over $L$ means an $L$-affinoid algebra $A$ in the sense of [Ber90, Definition 2.1.1]. We denote the Berkovich analytic space associated to $A$ by $\mathcal{M}(A)$ [Ber90, Section 1.2]. There exists a unique minimal closed subset of $\mathcal{M}(A)$ on which every valuation of $A$ has its minimum [Ber90, Corollary 2.4.5]. It is called the Shilov boundary of $\mathcal{M}(A)$. It is a finite set.

By [Ber90, Proposition 2.4.4], we have the following.

Lemma 3.10. For a pure $d$-dimensional affinoid domain $A$, the Shilov boundary $B(A)$ does not intersect with any Zariski closed subset of dimension $\leq (d - 1)$.

3.3. Zariski-Riemann spaces. For a finitely generated extension $L/K$ of fields, we put $\text{ZR}(L/K) \subset \text{Spv}(L)$ the subspace of equivalence classes of valuations of $L$ which are trivial on $K$. We call $\text{ZR}(L/K)$ the Zariski-Riemann space.

There is another expression. For each $v \in \text{ZR}(L/K)$ and proper algebraic variety $X$ over $K$ with function field $L$, by the valuative criterion of properness, there exists a unique canonical morphism $\text{Spec} \mathcal{O}_v \to X$. This induces a map $\text{ZR}(L/K) \to X$ by taking the image of the maximal ideal of $\mathcal{O}_v$ (i.e., center of $v$). We have a map from $\text{ZR}(L/K)$ to the inverse limit $\varprojlim X$ (as topological spaces) of birational morphisms of proper algebraic varieties $X$ over $K$ whose function fields are $L$.

The following Proposition is well-known, see e.g., [Tem11, Corollary 3.4.7].

Proposition 3.11. The map $\text{ZR}(L/K) \to \varprojlim X$ is a homeomorphism.

Remark 3.12 (Abhyankar’s inequality). For $v \in \text{ZR}(L/K)$, we have

$$\text{tr.deg}(\kappa(v)/K) + \text{rank } \Gamma_v \leq \text{tr.deg}(L/K).$$

The equality holds for some $v$.

Definition 3.13. Let $X$ be an algebraic variety over $K$ whose function field is $L$. We put $(\text{Spec } L/K)^\text{Ber}$ the subspace of the analytification $X^\text{Ber}$ consisting of points whose supports are the generic point of $X$. It is independent of the choice of such a $X$.

Lemma 3.14 ([Duc18, Proposition 7.1.3]). Let $\psi: \text{Spec } L' \to \text{Spec } L$ be a finite extension,

$$U := \{v \in (\text{Spec } L'/K)^\text{Ber} \mid f_i(v) \leq 0 \ (1 \leq i \leq r)\}$$

for some $f_i \in L'$. Then $\psi(U) \subset (\text{Spec } L/K)^\text{Ber}$ is a finite union of subsets of $(\text{Spec } L/K)^\text{Ber}$ of the same form (using elements of $L$ instead of $L'$).
3.4. Huber’s adic spaces. For a separated scheme $X$ of finite type over a trivially valued field $K$, we define the adic space $X^{\text{ad}}$ associated to $X$ as follows. (See [Hub93] and [Hub94] for notations and theory of his adic spaces.) For each affine open subvariety $U = \text{Spec } R \subset X$, we put $U^{\text{ad}} := \text{Spa}(R, R \cap K^{\text{alg}})$ the space of equivalence classes of valuations on $R$ trivial on $K$ (where $R$ is equipped with the discrete topology). We define $X^{\text{ad}}$ by glueing $U^{\text{ad}}_\alpha$ for an affine open covering $\{U_\alpha\}_\alpha$ of $X$.

Remark 3.15. Taking supports of valuations induces a surjective map $X^{\text{ad}} \twoheadrightarrow X$ whose fiber of $x \in X$ is homeomorphic to $\mathbb{Z}R(k(x)/K)$.

Remark 3.16. Taking equivalence classes induces a map $X^{\text{Ber}} \rightarrow X^{\text{ad}}$. This induces a bijection $X^{\text{Ber}}/(\text{equivalence relations as valuations}) \cong X^{\text{ad,ht} \leq 1}$ to the subset $X^{\text{ad,ht} \leq 1}$ of $X^{\text{ad}}$ consisting of equivalence classes of valuations of height $\leq 1$.

For an algebraic variety $Y$ over a complete non-archimedean valuation field $L$ of height $1$, we consider the adic space $Y^{\text{ad}}$ associated to $Y$ in the sense of [Hub94]. What we need to consider in this paper is only the affine line, whose structure will be explained precisely in Section 10.

4. Tropicalizations

In this section, we shall recall basic properties of fans and polyhedral complexes (Subsection 4.1), tropicalizations of Berkovich analytifications of algebraic varieties (Subsection 4.2), and tropical compactifications (Subsection 4.3). For details of tropicalizations of the affine line, see Section 10.

Remind that $M$ is a free $\mathbb{Z}$-module of finite rank $n$ and $N := \text{Hom}(M, \mathbb{Z})$. Let $\Sigma$ be a fan in $N_\mathbb{R}$, and $T_\Sigma$ the normal toric variety over a field associated to $\Sigma$. See [CLS11] for toric varieties. In this paper, cones mean strongly convex rational polyhedral cones. There is a natural bijection between cones $\sigma \in \Sigma$ and torus orbits $O(\sigma)$ in $T_\Sigma$. The torus orbit $O(\sigma)$ is isomorphic to the torus $\text{Spec } K[M \cap \sigma^+]$. We put $N_\sigma := \text{Hom}(M \cap \sigma^+, \mathbb{Z})$. We fix an isomorphism $M \cong \mathbb{Z}^n$, and identify $\text{Spec } K[M]$ with $\mathbb{G}_m^n$.

4.1. Fans and polyhedral complexes. In this subsection, we recall the partial compactification $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$ of $\mathbb{R}^n$ and fans and polyhedral complexes in it.

We define a topology on the disjoint union $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$ as follows. We extend the canonical topology on $\mathbb{R}$ to that on $\mathbb{R} \cup \{\infty\}$ so that $(a, \infty]$ for $a \in \mathbb{R}$ are a basis of neighborhoods of $\infty$. We also extend the addition on $\mathbb{R}$ to that on $\mathbb{R} \cup \{\infty\}$ by $a + \infty = \infty$ for $a \in \mathbb{R} \cup \{\infty\}$. We consider the set of semigroup homomorphisms $\text{Hom}(M \cap \sigma^\vee, \mathbb{R} \cup \{\infty\})$ as a topological subspace of $(\mathbb{R} \cup \{\infty\})^{M \cap \sigma^\vee}$. We define a topology on $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$ by the canonical bijection

$$\text{Hom}(M \cap \sigma^\vee, \mathbb{R} \cup \{\infty\}) \cong \bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}.$$\[Then we define a topology on $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$ by gluing the topological spaces $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$ together.

We shall define fans and polyhedral complex in $\bigsqcup_{\tau \in \Sigma} N_{\tau, \mathbb{R}}$.\]
Definition 4.1. A subset of $\mathbb{R}^n$ is called a polyhedron if and only if it is the intersection of sets of the form
\[ \{ x \in \mathbb{R}^n \mid \langle x, a \rangle \leq b \} \ (a \in \mathbb{Z}^n, b \in \mathbb{R}), \]
here $\langle x, a \rangle$ is the usual inner product of $\mathbb{R}^n$.

Definition 4.2. For a cone $\sigma \in \Sigma$ and a polyhedron (resp. a cone) $C \subset N_{\sigma, \mathbb{R}}$, we call its closure $P := \overline{C}$ in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$ a polyhedron (resp. a cone) in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$. In this case, we put $\text{rel.(P)} := \text{rel.(C)}$, and call it the relative interior of $P$. We put $\dim(P) := \dim(C)$.

For a polyhedron $P$ in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$, we put $\sigma_P \in \Sigma$ the unique cone such that $\text{rel.(P)} \subset N_{\sigma_P, \mathbb{R}}$. A subset $Q$ of a polyhedron (resp. a cone) $P$ in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$ is called a face of $P$ if it is the closure of the intersection $P^a \cap N_{\tau, \mathbb{R}}$ in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$ for some $a \in \sigma_P \cap M$ and some cone $\tau \in \Sigma$, where $P^a$ is the closure of
\[ \{ x \in P \cap N_{\sigma_P, \mathbb{R}} \mid x(a) \leq y(a) \text{ for any } y \in P \cap N_{\sigma_P, \mathbb{R}} \} \]
in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$. A finite collection $\Lambda$ of polyhedra (resp. cones) in $\bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}}$ is called a polyhedral complex (resp. a fan) if it satisfies the following two conditions:

- For all $P \in \Lambda$, each face of $P$ is also in $\Lambda$.
- For all $P, Q \in \Lambda$, the intersection $P \cap Q$ is a face of $P$ and $Q$.

We call the union
\[ |\Lambda| := \bigcup_{P \in \Lambda} P \subset \bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}} \]
the support of $\Lambda$. We say that $\Lambda$ is a polyhedral complex (resp. a fan) structure of $|\Lambda|$.

A polyhedral complex $\Lambda'$ is called a refinement of a polyhedral complex $\Lambda$ (or $\Lambda'$ is finer than $\Lambda$) if their supports are the same and for any $P \in \Lambda'$, there exists a polyhedron $Q_P \in \Lambda$ such that
\[ \text{rel.(P)} \subset \text{rel.int}(Q_P). \]
This induces a surjective map $\Lambda' \to \Lambda$.

4.2. Tropicalizations of Berkovich analytic spaces. We recall basics of tropicalizations of Berkovich analytic spaces, see [Gub13], [GRW16], [GRW17], and [Pay09]. Let $(L, v_L : L^\times \to \mathbb{R})$ be a complete valuation field of height $\leq 1$. In this subsection, every scheme is defined over $L$.

The tropicalization map
\[ \text{Trop} : O(\sigma)^\text{Ber} \to N_{\sigma, \mathbb{R}} = \text{Hom}(M \cap \sigma^\perp, \mathbb{R}) \]
is the proper surjective continuous map given by the restriction
\[ \text{Trop}(v_x) := v_x|_{M \cap \sigma^\perp} : M \cap \sigma^\perp \to \mathbb{R} \]
for $v_x \in O(\sigma)^\text{Ber}$. We define the tropicalization map
\[ \text{Trop} : T^\text{Ber}_\Sigma = \bigsqcup_{\sigma \in \Sigma} O(\sigma)^\text{Ber} \to \bigcup_{\sigma \in \Sigma} N_{\sigma, \mathbb{R}} \]
as their direct sum, which is proper, surjective, and continuous.

For a morphism $\varphi : X \to T^\Sigma$ from a separated scheme $X$ of finite type over $L$, the image $\text{Trop}(\varphi(X^\text{Ber}))$ of $X^\text{Ber}$ is called a tropicalization of $X^\text{Ber}$ (or $X$). For simplicity, we often write $\text{Trop}(\varphi(X))$ instead of $\text{Trop}(\varphi(X^\text{Ber}))$. When $X$ is irreducible, $\varphi : X \to T^\Sigma$ is a closed immersion, and $v_L$ is the trivial valuation, the tropicalization
Trop(φ(X^{\text{Ber}})) is a finite union of (dim $X$)-dimensional cones by \cite[Theorem A]{BGS84} and \cite[Theorem 6.2.18]{MSI15}.

For a toric morphism $\psi: T_{\Sigma'} \to T_{\Sigma}$, there exists a morphism $\operatorname{Trop}(T_{\Sigma'}) \to \operatorname{Trop}(T_{\Sigma})$ inducing a commutative diagram

$$
\begin{array}{ccc}
T_{\Sigma'}^{\text{Ber}} & \xrightarrow{\psi} & T_{\Sigma}^{\text{Ber}} \\
\downarrow \scriptstyle{\text{Trop}} & & \downarrow \scriptstyle{\text{Trop}} \\
\text{Trop}(T_{\Sigma'}) & \longrightarrow & \text{Trop}(T_{\Sigma}).
\end{array}
$$

We also denote it by $\psi: \operatorname{Trop}(T_{\Sigma'}) \to \operatorname{Trop}(T_{\Sigma})$. The restriction $\psi|_{N_{\sigma'}}: N_{\sigma'} \to N_{\sigma}$ of it to each orbits is a linear map.

**Remark 4.3.** Tropicalizations do not change under base extensions, i.e., for an extension $L'/L$ of complete valuation fields of height $\leq 1$, we have a commutative diagram

$$
\begin{array}{ccc}
X_{L'}^{\text{Ber}} & \xrightarrow{\varphi_{L'}} & T_{\Sigma,L'}^{\text{Ber}} \\
\downarrow \scriptstyle{\text{Trop}} & & \downarrow \scriptstyle{\text{Trop}} \\
X^{\text{Ber}} & \xrightarrow{\varphi} & T_{\Sigma}^{\text{Ber}} \\
\downarrow \scriptstyle{\text{Trop}} & & \downarrow \scriptstyle{\text{Trop}} \\
\text{Trop}(T_{\Sigma'}) & \longrightarrow & \text{Trop}(T_{\Sigma}),
\end{array}
$$

where $(-)_{L'}$ means the base change to $L'$. In particular, we have

$$\operatorname{Trop}(\varphi_{L'}(X_{L'})) = \operatorname{Trop}(\varphi(X)) \subset \operatorname{Trop}(T_{\Sigma}).$$

Let $\varphi: X \to T_{\Sigma}$ be a closed immersion of an algebraic variety $X$. We put $\text{Sk}_{\varphi}(X) \subset X^{\text{Ber}}$ the union of the Shilov boundaries of fibers $(\operatorname{Trop} \circ \varphi)^{-1}(a)$ ($a \in \operatorname{Trop}(\varphi(X))$), a tropical skeleton of $X$. When there is no confusion, we simply denote it by $\text{Sk}(X)$.

### 4.3. Tropicalizations and compactifications.

In this subsection, we recall relations between tropicalizations and compactifications, see \cite{Tev07} and \cite{Gub13}. Let $X \subset \mathbb{G}^n_m = \text{Spec} K[M]$ be a closed subvariety over a trivially valued field $K$.

**Definition 4.4.** The closure $\overline{X}$ in the toric variety $T_{\Sigma}$ corresponding to a fan $\Sigma$ in $N_{\mathbb{R}}$ is called a tropical compactification if the multiplication map

$$\mathbb{G}^n_m \times \overline{X} \to T_{\Sigma}$$

is faithfully flat and $\overline{X}$ is proper.

For $\sigma \in \Sigma$, the intersection $\overline{X} \cap O(\sigma)$ is of pure dimension $(\operatorname{dim} X - \operatorname{dim} \sigma)$.

**Theorem 4.5** (Tevelev \cite[Theorem 12.3]{Gub13}). There exists a fan $\Sigma$ such that $\overline{X} \subset T_{\Sigma}$ is a tropical compactification.

**Remark 4.6.** Let $\overline{X} \subset T_{\Sigma}$ be a tropical compactification of $X \subset \mathbb{G}^n_m$.

1. The fan $\Sigma$ is a fan structure of $\operatorname{Trop}(X) \subset N_{\mathbb{R}}$ \cite[Proposition 12.5]{Gub13}.
2. For any refinement $\Sigma'$ of $\Sigma$, the closure of $X$ in $T_{\Sigma'}$ is also a tropical compactification \cite[Proposition 12.4]{Gub13}.

**Proposition 4.7** (Tevelev \cite[Proposition 2.3]{Tev07}). For a fan $\Sigma$ in $N_{\mathbb{R}}$, the closure $\overline{X}$ of $X$ in $T_{\Sigma}$ is proper if and only if $\operatorname{Trop}(X)$ is contained in the support of $\Sigma$.

Let $Y$ be an algebraic variety over $K$.

**Definition 4.8.** We put $Y^o$ the subset of $Y^{\text{Ber}}$ consisting of valuations $v$ such that there exists a natural morphism $\text{Spec} O_v \to Y$. 
Lemma 5.1. Let \( r \). Then we have
\[
Y^\circ = \bigcup_{\text{compact cones } P \in \Lambda} (\Trop \circ \varphi)^{-1}(P).
\]

**Proof.** This easily follows from definitions. \( \square \)

Remark 4.10. Since \( \Trop \circ \varphi \) is proper, by Lemma 4.9, the subset \( Y^\circ \) is compact.

5. Tropicalizations of valuations of general heights

In this section, we introduce and study tropicalizations of Zariski-Riemann spaces (Subsection 5.3) and Huber’s adic spaces (Subsection 5.4), i.e., sets of valuations of any heights. For this purpose, we study group homomorphisms from a free \( \mathbb{Z} \)-module \( M \) of finite rank \( n \) to totally ordered abelian groups (Subsection 5.1) and their relation to limits of fan structures of \( \Hom(M, \mathbb{R}) = N_R \) (Subsection 5.2). (Our tropicalizations of adic spaces are different from Foster-Payne’s adic tropicalizations; see [Fos16].)

### 5.1. Homomorphisms to totally ordered abelian groups

In this subsection, we shall give a description of group homomorphisms from \( M \) to totally ordered abelian groups. Remind that every totally ordered abelian group of height \( r \) can be embedded in \( \mathbb{R}^r \) (Theorem 3.5). For \( l_i \in N_R \), we put \( L_i \) the minimal \( \mathbb{Q} \)-linear subspace of \( N_R \) such that \( L_i \) contains \( l_i \). We put
\[
J_r := J_r(N) := \left\{ (l_1, \ldots, l_r) \in (N_R)^r \mid l_i \notin \sum_{j=1}^{i-1} L_{j,R} \ (1 \leq i \leq r) \right\}.
\]

(For \( i = 1 \), we put \( \sum_{j=1}^{0} L_{j,R} := \{0\} \).

**Lemma 5.1.** Let \( \mathcal{L} = (l_1, \ldots, l_r) \in (N_R)^r \). The abelian subgroup \( \mathcal{L}(M) \subset \mathbb{R}^r \) is of height \( r \) with respect to the lexicographic order if and only if \( \mathcal{L} = (l_1, \ldots, l_r) \in J_r \).

**Proof.** The subgroup \( \mathcal{L}(M) \subset \mathbb{R}^r \) is of height \( r \) if and only if
\[
\mathcal{L}(M) \cap \{0\}^{r-1} \times \mathbb{R}^{r-j+1} \neq \mathcal{L}(M) \cap \{0\}^j \times \mathbb{R}^{r-j}
\]
for any \( j \). Since
\[
(l_1, \ldots, l_s)(M) \cong \mathcal{L}(M)/(\mathcal{L}(M) \cap \{0\}^s \times \mathbb{R}^{r-s}),
\]
the latter is equivalent to that the natural surjection
\[
(l_1, \ldots, l_j)(M) \twoheadrightarrow (l_1, \ldots, l_{j-1})(M)
\]
is not injective for any \( j \), i.e., \( \mathcal{L} \in J_r \). \( \square \)

We say that \( \mathcal{L} = (l_1, \ldots, l_r) \in J_r \) and \( \mathcal{L}' = (l'_1, \ldots, l'_r) \in J_r \) are equivalent if there is an isomorphism \( \varphi : \mathcal{L}(M) \cong \mathcal{L}'(M) \) as totally ordered groups such that \( \varphi \circ \mathcal{L} = \mathcal{L}' \).

We put \( \mathcal{L} \sim_I \mathcal{L}' \) if we have
\[
\mathbb{R}_{>0} \cdot l_i = \mathbb{R}_{>0} \cdot l'_i \in N_R / \sum_{j=1}^{i-1} L_{j,R} \ (1 \leq i \leq r),
\]
where we denote the images of \( l_i \) and \( l'_i \) under the projection also by \( l_i \) and \( l'_i \).

**Lemma 5.2.** \( \mathcal{L} \) and \( \mathcal{L}' \) are equivalent if and only if \( \mathcal{L} \sim_I \mathcal{L}' \).
We have an isomorphism \( \text{Hom}(M \cap L^+) \cong N/N \cap L \) for a \( \mathbb{Q} \)-vector subspace \( L \subset N_\mathbb{Q} \).

**Proof.** We prove the assertion by induction on \( r \geq 1 \). When \( r = 1 \), the assertion is trivial. We assume that \( r \geq 2 \), and one of the two assertions (the equivalence or \( \mathcal{L} \sim_{L} \mathcal{L}' \)) holds. Then by the case of \( r = 1 \), the two assertions hold for \( l_1, l_1' \in J_1 \). In particular, we have \( \text{Ker} l_1 = \text{Ker} l_1' \). By the hypothesis of induction, the two assertions also hold for

\[
(l_2, \ldots, l_r), (l_2', \ldots, l_r') \in J_{r-1}(N/N \cap L_1),
\]

where we denote the restriction of \( l_i \) (resp. \( l_i' \)) \( \in N \) to \( \text{Ker} l_i = M \cap L_1^+ \) also by \( l_i \) (resp. \( l_i' \)) \( \in N/N \cap L_1 \). First, we have \( \mathcal{L} \sim_{L} \mathcal{L}' \) since \( l_1 \sim_{I_1} l_1' \) and \( (l_2, \ldots, l_r) \sim_{I_{r-1}} (l_2', \ldots, l_r') \). Second, there are natural isomorphisms \( \varphi_1: l_1(M) \cong l_1'(M) \) and \( \varphi_2: (l_2, \ldots, l_r)(M \cap L_1^+) \cong (l_2', \ldots, l_r')(M \cap L_1^+) \) as ordered groups. Since \( \text{Ker} l_1' = \text{Ker} l_1 = M \cap L_1^+ \), we have

\[
\text{Ker} \mathcal{L} = \text{Ker} l_1 \cap \text{Ker}(l_2, \ldots, l_r) = \text{Ker}(l_2, \ldots, l_r)|_{M \cap L_1^+} = \text{Ker} \mathcal{L}'.
\]

Hence there exists an isomorphism

\[
\varphi: \mathcal{L}(M) \cong \mathcal{L}'(M)
\]
as abelian groups such that \( \varphi \circ \mathcal{L} = \mathcal{L}' \). The fact that this isomorphism preserves orders easily follows from those for \( \varphi_1 \) and \( \varphi_2 \). \( \square \)

We extend the notion of equivalence to group homomorphisms \( M \rightarrow \Gamma \) to general totally ordered abelian groups \( \Gamma \) (from the case of \( \mathbb{R}^r \)) in the natural way.

We put \( J_0 := J_0/\sim_{J_0}:= \{0\} \).

**Corollary 5.3.** There is a natural bijection between \( J_r/\sim_{J_r} \) and the set of equivalence classes of surjective group homomorphisms from \( M \) to totally ordered abelian groups of height \( r \).

**5.2. Limits of fan structures.** In this subsection, we shall give a bijection between the limit of fan structures of \( N_\mathbb{R} \) and the set of equivalence classes of group homomorphisms from \( M \) to totally ordered abelian groups.

Let \( \varprojlim \Xi \) be the inverse limit of all fan structures \( \Xi \) of \( N_\mathbb{R} \) as sets. (Maps are given by refinements, see Subsection [4.1].)

**Lemma 5.4.** There is a bijection

\[
\bigcup_{r \geq 0} J_r/I_r \cong \varprojlim \Xi
\]
given by \( \{0\} \mapsto (\{0\})_\Xi \) and \( \mathcal{L} = (l_1, \ldots, l_r) \mapsto (P_{\mathcal{L}}\Xi)_\Xi \), where for sufficiently fine fan structure \( \Xi \) (for each \( \mathcal{L} \)), the cone \( P_{\mathcal{L}}\Xi \) is the unique cone in \( \Xi \) satisfying

\[
\mathcal{L}(M \cap P_{\mathcal{L}}\Xi^\vee) \geq 0,
\]

\[
\mathcal{L}((M \cap P_{\mathcal{L}}\Xi^\vee) \setminus P_{\mathcal{L}}\Xi^+) > 0.
\]

**Proof.** We prove the assertion (including the existence and uniqueness of \( P_{\mathcal{L}}\Xi \)) by induction on \( \dim N \). When \( \dim N = 0 \), it is trivial. We assume \( \dim N \geq 1 \). Note that
that there exists a bijection

\[ \bigcup_{r \geq 1} \{ L = (l_1, \ldots, l_r) \in J_r | R_{>0} \cdot l_1 = R_{>0} \cdot l \} / I_r \]

\[ \cong \{(P_\Xi)_{\Xi} \in \varprojlim \Xi | l \in \bigcap_{\Xi} P_\Xi \}. \]

The required bijection is given as disjoint union of these bijections. We put \( L \) the minimal subspace of \( N_\Xi \) such that \( L_{\Xi} \) contains \( l \). Let \( \Xi_0 \) be a sufficiently fine fan structure of \( N_{\Xi_0} \) so that the restriction map \( N_{\Xi_0} \to N_{P_{\Xi_0}, \Xi_0} \) induces a bijective map \( N_{\Xi_0}/L_{\Xi} \cong N_{P_{\Xi_0}, \Xi_0, \Xi}, \) where \( P_{\Xi_0, \Xi_0} \in \Xi_0 \) is the cone whose relative interior contains \( l_{\Xi_0} \).

Then for \( r \geq 1 \), we have a bijection

\[ \{ L = (l_1, \ldots, l_r) \in J_r | R_{>0} \cdot l_1 = R_{>0} \cdot l \} / \sim_{I_{r-1}} \]

by \( (l_1, l_2, \ldots, l_r) \mapsto (l_2, \ldots, l_r). \) By the hypothesis of the induction, we have a bijection

\[ J_{r-1}(N_{P_{\Xi_0}})/I_{r-1} \cong \varprojlim \Lambda, \]

where \( \Lambda \) runs through all fan structures of \( N_{P_{\Xi_0}, \Xi_0}. \) By Lemma 5.5, we get the required bijection (5.1). It remains to show the existence and uniqueness of \( \Xi_{\Xi_0} \) and such \( P_{\Xi_0} \Xi_0, \Xi \) exists for sufficiently fine \( \Xi_0. \) (Note that for \( l = l_1 \), we have \( M \cap l_1 = M \cap L_{\Xi}, \) and \( l \in P \) means \( l_1(M \cap P_{\Xi}) \geq 0. \) Uniqueness of \( P_{\Xi_0} \Xi_0, \Xi \) also follows from by the hypothesis of induction on dim \( N \) and Lemma 5.5 (Note that \( l_1(M \cap P_{\Xi_0, \Xi}) \geq 0 \) means \( l_1 \in P_{\Xi_0, \Xi} \)).

Hence by the hypothesis of induction on dim \( N \) and Lemma 5.5, the bijection (5.1) is given by \( P_{\Xi_0} \Xi_0, \Xi \) and such \( P_{\Xi_0} \Xi_0, \Xi \) exists for sufficiently fine \( \Xi_0. \) (Note that for \( l = l_1 \), we have \( M \cap l_1 = M \cap L_{\Xi}, \) and \( l \in P \) means \( l_1(M \cap P_{\Xi}) \geq 0. \) Uniqueness of \( P_{\Xi_0} \Xi_0, \Xi \) also follows from by the hypothesis of induction on dim \( N \) and Lemma 5.5 (Note that \( l_1(M \cap P_{\Xi_0, \Xi}) \geq 0 \) means \( l_1 \in P_{\Xi_0, \Xi} \)).

Let \( l \in N_{\Xi_0} \) be a non-zero element. Let \( L \) and \( \Xi_0 \) be as in proof of Lemma 5.4. (In particular, we have \( N_{P_{\Xi_0}, \Xi_0} \cong N/N \cap L_{\Xi} \).) For each fan structure \( \Xi \) of \( N_{\Xi_0}, \) we put \( P_{\Xi} \Xi_0, \Xi \) the cone whose relative interior contains \( l \). When \( \Xi \) is finer than \( \Xi_0, \) the natural morphism \( N_{P_{\Xi_0}, \Xi_0} \to N_{P_{\Xi}, \Xi_0, \Xi} \) is a bijection. We identify them. For each \( \Xi_0, \)

\[ \{ \text{pr}_{P_{\Xi}}(P) \mid P \in \Xi, \ P_{\Xi} \subset P \} \]

is a fan structure of \( N_{P_{\Xi_0}, \Xi_0, \Xi} \), where \( \text{pr}_{P_{\Xi}} : N_{\Xi_0} \to N_{P_{\Xi}, \Xi_0, \Xi} \) the projection.

**Lemma 5.5.** \( P \mapsto \text{pr}_{P_{\Xi}}(P) \) for \( \Xi \) finer than \( \Xi_0 \) induces a bijection

\[ \{(P_\Xi)_{\Xi} \in \varprojlim \Xi | l \in \bigcap_{\Xi} P_\Xi \} \cong \varprojlim \Sigma, \]

where \( \Sigma \) runs through all fan structures of \( N_{P_{\Xi_0}, \Xi_0, \Xi}. \)
Proof. Since
\[(P_{\Xi})_{\Xi} \in \varprojlim \Xi | P_{\Xi} \subset P_{\Xi}(\Xi) = ((P_{\Xi})_{\Xi} \in \varprojlim \Xi | l \in \bigcap_{\Xi} P_{\Xi}),\]
the assertion follows from the fact that every fan structure \(\Sigma\) of \(N_{R_{x}, R}\) has a refinement coming from some \(\Xi\), e.g., one given by a hyperplane arrangement. \qed

Remark 5.6. \(\cdot\) Let \(L = (l_1, \ldots, l_r) \in J,\) map to \((P_{\Sigma}, \Xi) \in \varprojlim \Xi\). By proof of Lemma 5.4, for a sufficiently fine fan structure \(\Xi\) of \(N_{R}\), we have
\[
\text{Span}_{\mathbb{R}}(P_{\Sigma}) = \sum_{i=1}^{r} L_{i, \mathbb{R}},
\]
where \(L_i\) is the minimal \(\mathbb{Q}\)-linear subspace of \(N_{Q}\) such that \(L_{i, \mathbb{R}}\) contains \(l_i\).

\(\cdot\) Conversely, for a fan structure \(\Xi\) of \(N_{R}\) and a cone \(P \in \Xi\), there exists \(L = (l_1, \ldots, l_{\dim P}) \in J_{\dim P} \cap (N_{Q})^{r}\) such that \(P_{\Sigma} = P\) and
\[
\text{Span}_{\mathbb{R}} P = \sum_{i=1}^{r} L_{i, \mathbb{R}} = \sum_{i=1}^{r} \mathbb{R} \cdot l_i.
\]
Namely, by taking a subdivision of \(\Xi\), we may assume that \(P = \mathbb{R}_{\geq 0}^{\dim P} \times \{0\}^{n-\dim P}\) for an identification \(N \cong \mathbb{Z}^{n}\), then \(L = (e_1, \ldots, e_{\dim P})\) is a required one, where \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\) is the \(i\)-th coordinate.

5.3. Tropicalizations of Zariski-Riemann spaces. In this subsection, we shall introduce tropicalizations of Zariski-Riemann spaces. Let \(K\) be a trivially valued field. Let \(x \in \mathbb{G}_{m}^{n} = \text{Spec} K[M]\), and \(\overline{\{x\}}\) the closure. We put \(k(x)\) the residue field. For a fan structure \(\Lambda\) of \(\text{Trop}(\overline{\{x\}})\), we put \(\overline{\{x\}}^{\Lambda} \subset T_{\Lambda}\) the closure in the toric variety \(T_{\Lambda}\) corresponding to \(\Lambda\). The algebraic variety \(\overline{\{x\}}^{\Lambda}\) is proper and intersects with any orbit \(O(\lambda)\) (\(\lambda \in \Lambda\)).

Definition 5.7. For \(v \in \text{ZR}(\text{Spec} k(x)/K)\), we put \(\text{Trop}_{\Lambda}^{\text{ad}}(v) \in \Lambda\) the cone such that the image of the maximal ideal under the natural morphism
\[
\text{Spec} \mathcal{O}_v \to \overline{\{x\}}^{\Lambda} \subset T_{\Lambda}
\]
is contained in the orbit \(O(\text{Trop}_{\Lambda}^{\text{ad}}(v))\). This induces a surjective map
\[
\text{Trop}_{\Lambda}^{\text{ad}} : \text{ZR}(k(x)/K) \twoheadrightarrow \Lambda.
\]
We have a surjective map
\[
\text{Trop}_{\Lambda}^{\text{ad}} : \text{ZR}(k(x)/K) \rightarrow \varprojlim \Lambda,
\]
called a tropicalization map of the Zariski-Riemann space \(\text{ZR}(k(x)/K)\), where \(\varprojlim \Lambda\) is the inverse limit of all fan structures \(\Lambda\) of \(\text{Trop}(\overline{\{x\}})\) as sets.

Any fan structure of \(\text{Trop}(\overline{\{x\}})\) has a refinement which is a subfan of a fan structure of \(N_{R}\). Hence we have a natural injective map
\[
\varprojlim \Lambda \hookrightarrow \varprojlim \Xi,
\]
where \(\Xi\) runs through all fan structures of \(N_{R}\). We identify \(\varprojlim \Lambda\) and its image.

For a valuation \(v \in \text{ZR}(k(x)/K)\), the composition
\[
M \to k(x)^{x} \xrightarrow{\nu} \Gamma_{v}
\]
Ryota Mikami is a group homomorphism to a totally ordered group $\Gamma_v$. (Here for simplicity, we identify $v \in ZR(L/K)$ and a representative.) By Corollary 5.3, the group homomorphism $M \to \Gamma_v$ can be considered as an element in $J_r/\sim_r$ for some $r$.

**Lemma 5.8.** For $v \in ZR(L/K)$, the element $\text{Trop}^\text{ad}_v(v) \in \lim\downarrow \Lambda(\subset \lim\downarrow \Xi)$ is the image of $M \to \Gamma_v$ under the bijection in Lemma 5.4.

**Proof.** For $\Lambda$ which is a subfan of a fan structure $\Xi$ of $N_R$, by definition of $\text{Trop}^\text{ad}_\Lambda(v)$, the image $\text{Trop}^\text{ad}_\Lambda(v) \in \Lambda(\subset \Xi)$ satisfies the condition of $P_{M \to \Gamma_v, \Xi}$ in Lemma 5.4. $\Box$

### 5.4. Tropicalizations of adic spaces over trivially valued fields.

In this subsection, we study tropicalizations and tropical skeletons of adic spaces associated with algebraic varieties over a trivially valued field $K$.

We define tropicalizations as the direct sum of tropicalizations of Zariski-Riemann spaces. Let $X$ be a closed subvariety of a torus $G^n_m$ over $K$, and $\Lambda$ a fan structure of $\text{Trop}(X) \subset N_R$. For $x \in X$, since $\text{Trop}(\{x\}) \subset \text{Trop} X$, by Proposition 4.7, we define $\text{Trop}^\text{ad}_\Lambda(X) = \bigcup_{x \in X} \text{Trop}(k(x)/K) \to \Lambda$ in the same way as Definition 5.7.

**Definition 5.9.** We define $\text{Trop}^\text{ad}_\Lambda : X^\text{ad} = \bigcup_{x \in X} \text{ZR}(k(x)/K) \to \Lambda$

the disjoint union of $\text{Trop}^\text{ad}_\Lambda$ on $\text{ZR}(k(x)/K)$.

We have a surjective map $\text{Trop}^\text{ad} : X^\text{ad} \to \lim\downarrow \Lambda$

called a tropicalization map of $X^\text{ad}$, where $\lim\downarrow \Lambda$ is the inverse limit of all fan structures $\Lambda$ of $\text{Trop}(X)$ as sets.

Let $L \in J_r$ map to an element $(P_{L,\Lambda})_\Lambda \in \text{Trop}^\text{ad}(X^\text{ad})$. We put

$$T_L := \text{Spec } K[M \cap L^\vee] = \lim\downarrow T_{P_{L,\Lambda}}(= \text{Spec } K[M \cap P_{L,\Lambda}^\vee]),$$

$$O(L) := \text{Spec } K[M \cap L^\perp] = \lim\downarrow O(P_{L,\Lambda})(= \text{Spec } K[M \cap P_{L,\Lambda}^\perp]),$$

where as usual

$$M \cap L^\vee := \{m \in M \mid m(L) \geq 0\},$$

$$M \cap L^\perp := \{m \in M \mid m(L) = 0\}.$$

For each $\Lambda$, we have a natural morphism $\phi_{L,\Lambda} : T_L \to T_{P_{L,\Lambda}}$. We have

$$(\phi_{L,\Lambda})^{-1}((G^n_m) = G^n_m, \quad (\phi_{L,\Lambda})^{-1}(O(P_{L,\Lambda})) = O(L).$$

Let $\overline{X}^\Lambda$ be the closure of $X$ in $T_L$.

**Lemma 5.10.** Let $\Lambda$ be a fan structure of $\text{Trop}(X)$ such that the closure $\overline{X}^\Lambda \subset T_\Lambda$ is a tropical compactification. Then the morphism $\phi_{L,\Lambda} : T_L \to T_{P_{L,\Lambda}}$ induces a bijection between

- the set of generic points of $\overline{X}^\Lambda \cap O(L)$ and
- the set of generic points of $\overline{X}^\Lambda \cap O(P_{L,\Lambda})$. 

Proof. We define $Y \subset \mathbb{G}^n_m \times T_L$ by a cartesian diagram
\[
\begin{array}{c}
\mathbb{G}^n_m \times X^\Lambda \ar[r] \ar[u] & T_\Lambda \\
Y \ar[u] & T_L. \\
\end{array}
\]
Since $(\phi_{L,\Lambda})^{-1}(\mathbb{G}^n_m) = \mathbb{G}^n_m$, the inverse image of $\mathbb{G}^n_m (\subset T_L)$ in $Y$ is $\mathbb{G}^n_m \times X$. Since the first horizontal arrow is flat, the second one is also flat. Hence $\mathbb{G}^n_m \times X$ is dense in $Y$, i.e., $Y = \mathbb{G}^n_m \times X^C$. Since $(\phi_{L,\Lambda})^{-1}(O(P_{L,\Lambda})) = O(L)$, we have a cartesian diagram
\[
\begin{array}{c}
\mathbb{G}^n_m \times (X^\Lambda \cap O(P_{L,\Lambda})) \ar[r] \ar[u] & O(P_{L,\Lambda}) \\
\mathbb{G}^n_m \times (X^C \cap O(L)) \ar[u] & O(L). \\
\end{array}
\]
Then since $\phi_{L,\Lambda}: O(L) \rightarrow O(P_{L,\Lambda})$ is a surjective morphism between tori, the assertion holds.

By Lemma 5.10, we have an embedding of a finite set
\[
\{\text{generic points of } X^C \cap O(L)\} \hookrightarrow \lim_{\Lambda} X^\Lambda,
\]
where $\Lambda$ runs through all fan structures of $\text{Trop}(X)$.

**Definition 5.11.** We define the tropical skeleton $\text{Sk}(X^{\text{ad}}) \subset X^{\text{ad}}$ as the union of the inverse images of generic points of $X^C \cap O(L)$ ($L \in \text{Trop}^{\text{ad}}(X^{\text{ad}})$) under the map of taking centers $X^{\text{ad}} \rightarrow \lim_{\Lambda} X^\Lambda$.

**Lemma 5.12.** For $v \in \text{Sk}(X^{\text{ad}})$, the support $\text{supp}(v)$ is the generic point of $X$, i.e., $v \in \mathbb{Z}(K(X)/K) \subset X^{\text{ad}}$.

**Proof.** Suppose $v \in Z^{\text{ad}}$ for some proper closed subvariety $Z \subset X$. Then the center of $v$ in each tropical compactification $\overline{X}^\Lambda$ is contained in the closure $\overline{Z}^\Lambda$ of $Z$. For a sufficiently fine $\Lambda$ and a cone $P \in \Lambda$, the subscheme $\overline{Z}^\Lambda \cap O(P)$, which is of pure dimension $(\text{dim } Z - \text{dim } P)$, does not contains the generic point of any irreducible components of $\overline{X}^\Lambda \cap O(P)$, which is of pure dimension $(\text{dim } X - \text{dim } P)$. This is a contradiction.

**Lemma 5.13.** We have a natural bijection
\[
\text{Sk}(X^{\text{ad}}) \cap X^{\text{ht} \leq 1} \cong \text{Sk} X^{\text{Ber}}/(\text{the equivalence relations of valuations}),
\]
where $X^{\text{ht} \leq 1} \subset X^{\text{ad}}$ is the set of equivalence classes of valuations of height $\leq 1$.

**Proof.** This follows from e.g., [Gub13 Remark 12.7].

**Lemma 5.14.** For $v \in \text{Sk}(X^{\text{ad}})$, we have
\[
\text{rank}(v) = \text{rank}(v(M)).
\]
In particular, we have $\text{ht}(v) = \text{ht}(v(M))$. 

Proof. The last assertion follows from the first one. If the first one does not hold, by Remark 5.6 we have
\[
\text{rank} \Gamma_v - 1 \geq \text{rank}(\text{v}(M)) = \min_{\Lambda} \text{dim } P_{\Lambda},
\]
where \(\text{Trop}^{\text{ad}}(v) = (P_{\Lambda})_{\Lambda}\). Hence by Abhyanker’s inequality (Remark 3.12),
\[
\text{tr.deg}(\kappa(v)/K) \leq \dim(\overline{X} \cap O(\Lambda)) - 1
\]
for some \(\Lambda\). This is a contradiction. \(\square\)

Let \(Y\) be a closed subvariety \(Y\) of a toric variety \(T_{\Sigma}\) over \(K\), and \(\Xi\) a fan structure of \(\text{Trop}(Y)\). For a cone \(\sigma \in \Sigma\), we put
\[
\Xi \cap \text{Trop}(O(\sigma)) := \{\xi \cap \text{Trop}(O(\sigma)) \mid \xi \in \Xi, \text{rel.int } \xi \subset \text{Trop } O(\sigma)\},
\]
a fan in \(\text{Trop}(O(\sigma))\). We identify \(\Xi \cap \text{Trop}(O(\sigma))\) and
\[
\{\xi \in \Xi \mid \text{rel.int } \xi \subset \text{Trop } O(\sigma)\}
\]
by taking closures in \(\text{Trop}(T_{\Sigma})\).

**Definition 5.15.** We define
\[
\text{Trop}^{\text{ad}}_{\Xi} : \text{Y}^{\text{ad}} = \bigsqcup_{\sigma \in \Sigma} (Y \cap O(\sigma))^{\text{ad}} \twoheadrightarrow \bigsqcup_{\sigma \in \Sigma} \Xi \cap \text{Trop}(O(\sigma)) = \Xi
\]
the disjoint union of \(\text{Trop}^{\text{ad}}_{\Xi} \cap \text{Trop}(O(\sigma))\).

We have a surjective map
\[
\text{Trop}^{\text{ad}} : \text{Y}^{\text{ad}} \twoheadrightarrow \varprojlim \Xi
\]
called a tropicalization map of \(\text{Y}^{\text{ad}}\), where \(\varprojlim \Xi\) is the inverse limit of all fan structures \(\Xi\) of \(\text{Trop}(Y)\) as sets.

**Definition 5.16.** For a closed immersion \(\varphi : Y \to T_{\Sigma}\) to a toric variety \(T_{\Sigma}\) over \(K\), we put
\[
\text{Sk}_{\varphi}(Y^{\text{ad}}) := \bigsqcup_{\sigma \in \Sigma} \text{Sk}(\varphi(Y^{\text{ad}}) \cap O(\sigma)^{\text{ad}}).
\]

6. **Tropical cohomology**

In this section, we study tropical cohomology, (Subsection 6.1 and 6.2), tropical cycle class maps (Subsection 6.3), and show that the tropical de Rham’s theorem ([JSS17, Theorem 1], [Jel22, Theorem 8.12]) is given by integrations (Subsection 6.4).

Let \(K\) be a trivially valued field. Remind that \(M\) is a free \(\mathbb{Z}\)-module of finite rank and \(N := \text{Hom}(M, \mathbb{Z})\).

6.1. **Tropical cohomology of fans.** We recall tropical cohomology introduced by Itenberg-Katzarkov-Mikhalkin-Zharkov [IKMZ17]. See also [JSS17].

Let \(T_{\Sigma}\) be the toric variety over \(K\) associated with a fan \(\Sigma\) in \(N_\mathbb{R}\). Let \(\Lambda\) be a fan in \(\text{Trop}(T_{\Sigma})\). We put \(A := |\Lambda|\). For a subset \(B \subset \text{Trop}(T_{\Sigma})\), we put
\[
\Lambda \cap B := \{P \cap B\}_{P \in \Lambda}.
\]
Remind that for \(P \in \Lambda\), we put \(\sigma_P \in \Sigma\) the cone such that \(\text{rel.int}(P) \subset N_{\sigma_P, \mathbb{R}}\). Let \(p \geq 0\) be a non-negative integer. We put
\[
\text{Span}(P) := \text{Span}_\mathbb{Q}(P \cap N_{\sigma_P, \mathbb{Q}})
\]
the \( \mathbb{Q} \)-linear subspace of \( N_{\sigma_{P,Q}} \) spanned by \( P \cap N_{\sigma_{P,Q}} \),

\[
F_p(P, \Lambda) := \bigoplus_{P' \in \Lambda \cap N_{\sigma_{P,Q}}} \wedge^p \text{Span}(P') \subset \wedge^p N_{\sigma_{P,Q}},
\]

and

\[
F^p(P, \Lambda) := \wedge^p (M \cap \sigma_{P})_\mathbb{Q} / \{ f \in \wedge^p (M \cap \sigma_{P})_\mathbb{Q} \mid \alpha(f) = 0 \ (\alpha \in F_p(P, \Lambda)) \}.
\]

We have

\[
F^p(P, \Lambda) \cong \text{Hom}(F_p(P, \Lambda), \mathbb{Q}).
\]

Since \( F_p(P, \Lambda) \) and \( F^p(P, \Lambda) \) depends only on the support \( A = |\Lambda| \), we sometimes write \( F_p(P, A) \) (resp. \( F^p(P, A) \)) instead of \( F_p(P, \Lambda) \) (resp. \( F_p(P, \Lambda) \)). When there is no confusion, we simply write \( F_p(P) \) (resp. \( F^p(P) \)).

**Remark 6.1.** Let \( P_1, P_2 \in \Lambda \) with \( P_2 \subset P_1 \). Then we have \( \sigma_{P_1} \subset \sigma_{P_2} \).

- When \( \sigma_{P_1} = \sigma_{P_2} \), there exists a natural injection \( i_{P_2 \subset P_1} : F_p(P_1) \hookrightarrow F_p(P_2) \).
- When \( P_2 = P_1 \cap \text{Trop}(O(\sigma_{P_1})) \), the natural projection \( \text{Trop}(O(\sigma_{P_1})) \twoheadrightarrow \text{Trop}(O(\sigma_{P_2})) \) induces a morphism \( i_{P_2 \subset P_1} : F_p(P_1) \rightarrow F_p(P_2) \).
- In general, we put \( i_{P_2 \subset P_1} := i_{P_2 \subset Q} \circ i_{Q \subset P_1} : F_p(P_1) \rightarrow F_p(Q) \hookrightarrow F_p(P_2) \), where \( Q := P_1 \cap \text{Trop}(O(\sigma_{P_2})) \).

**Definition 6.2.** Let \( B \subset A \) be a subset, and \( \Lambda \) a fan structure of \( A \).

1. For every cone \( P \in \Lambda \), we put \( C_q(B \cap P) \) the free \( \mathbb{Q} \)-vector space generated by continuous maps \( \gamma : \Delta^q \rightarrow B \cap P \) from the standard \( q \)-simplex \( \Delta^q \). We put

\[
C_{p,q}(B, \Lambda) := \bigoplus_{P \in \Lambda} F_p(P, \Lambda) \otimes_\mathbb{Q} C_q(B \cap P) / \text{(equivalence relation)},
\]

where the equivalence relation is generated by

\[
\alpha_{P_1} \otimes \gamma - i_{P_2 \subset P_1} (\alpha_{P_1}) \otimes \gamma
\]

for \( P_1, P_2 \in \Lambda \) with \( P_2 \subset P_1 \), \( \alpha_{P_1} \in F_p(P_1, \Lambda) \), and \( \gamma : \Delta^q \rightarrow B \cap P_2 \subset B \cap P_1 \). We call its elements tropical \((p,q)\)-chains on \((B, \Lambda)\).

2. For \( \gamma \in C_q(B \cap P) \), we denote the usual boundary by \( \partial(\gamma) := \sum_{i=0}^q (-1)^i \gamma^i \). For each \( v \otimes \gamma \in F_p(P, \Lambda) \otimes C_q(B \cap P) \), we put

\[
\partial(v \otimes \gamma) := (-1)^p \sum_{i=0}^q (-1)^i v \otimes \gamma^i \in C_{p,q-1}(B, \Lambda).
\]

We obtain complexes \((C_{p,*}(B, \Lambda), \partial)\).

3. We define the tropical homology groups to be

\[
H^\text{Trop}_{p,q}(B, \Lambda) := H_q(C_{p,*}(B, \Lambda), \partial).
\]

We put \((C^{p,*}(B, \Lambda), \partial)\) the dual complex of \((C_{p,*}(B, \Lambda), \partial)\). We call its cohomology groups

\[
H^p_{\text{Trop}}(B, \Lambda) := H^q(C^{p,*}(B, \Lambda), \partial)
\]

the tropical cohomology groups of \((B, \Lambda)\).
We put
\[ Z^{p,q}(B, \Lambda) := \text{Ker}(\delta: C^{p,q}(B, \Lambda) \to C^{p,q+1}(B, \Lambda)) \]
For a subset \( D \subset B \), we put
\[ C^{p,q}_D(B, \Lambda) := \text{Ker}(C^{p,q}(B, \Lambda) \to C^{p,q}(B \setminus D, \Lambda)) \]
We put \( H^{p,q}_{\text{Trop}, D}(B, \Lambda) \) its cohomology group. We have a natural inclusion
\[ C^{p,q}(B, \Lambda) \to \bigoplus_{P \in \Lambda} F^p(P) \otimes_\Q \text{Hom}(C_q(B \cap P), \Q) \]
For \( \alpha \in C^{p,q}(B, \Lambda) \), \( P \in \Lambda \), and \( \gamma \in C_q(B \cap P) \), we put \( \alpha, \gamma \in F^p(P) \) the restriction of \( \alpha \) to
\[ F_p(P) \otimes_\Q \gamma \xrightarrow{\sim} F_p(P) \]
\[ x \otimes \gamma \xrightarrow{\sim} x \]
We often write \( \alpha = (\alpha, \gamma) \).

**Remark 6.3.** For a refinement \( \Lambda' \) of \( \Lambda \) and any \( p, q \), the natural map
\[ H^{p,q}_{\text{Trop}}(B, \Lambda) \to H^{p,q}_{\text{Trop}}(B, \Lambda') \]
is an isomorphism. This follows from [MZ13 Proposition 2.8] (see [JSS17 Section 3] for a proof). We write \( H^{p,q}_{\text{Trop}}(A) := H^{p,q}_{\text{Trop}}(A, \Lambda) \) when \( A = |\Lambda| \).

### 6.2. Tropical cohomology of algebraic varieties over trivially valued fields.

In this subsection, we recall tropical cohomology of an algebraic variety \( X \) over a trivially valued field \( K \). This is based on tropical charts, which are first given by Chambert-Loir-Ducros [CLD12] over valued fields of height 1. Variations were given by Gubler [Gub16] and Jell [Jel16]. We shall use Jell’s ones. See also [Jel22 Section 4 and 8].

We define a sheaf \( \mathcal{C}^{p,q}_X \) on \( X^\text{Ber} \) by, for each open subset \( V \subset X^\text{Ber} \), putting \( \mathcal{C}^{p,q}_X(V) \) the set of equivalence classes of \( \{(U_i, V_i, \varphi_i, \Lambda_i, \alpha_i)\} \) consisting of
- open coverings \( \{U_i\}_i \) of \( X \) and \( \{V_i\}_i \) of \( V \),
- closed immersions \( \varphi_i: U_i \to \mathbb{A}^{n_i} \) such that \( V_i = (\text{Trop} \circ \varphi_i)^{-1}(\Omega_i) \subset U^\text{Ber}_i \) for some open subset \( \Omega_i \subset \text{Trop}(\varphi_i(U_i)) \),
- fan structures \( \Lambda_i \) of \( \text{Trop}(\varphi_i(U_i)) \), and
- \( \alpha_i \in C^{p,q}(\text{Trop}(\varphi_i(V_i)), \Lambda_i) \)
satisfying the following: for any \( i, j \), there exists
- a covering \( \{U_{i,j,k}\}_k \) of \( U_i \cap U_j \),
- closed immersions \( \varphi_{i,j,k}: U_{i,j,k} \to \mathbb{A}^{n_{i,j,k}} \),
- toric morphisms \( \Psi_{(i,j,k),i}: \mathbb{A}^{n_{i,j,k}} \to \mathbb{A}^{n_i} \) and \( \Psi_{(i,j,k),j}: \mathbb{A}^{n_{i,j,k}} \to \mathbb{A}^{n_j} \), and
- fan structures \( \Lambda_{i,j,k} \) of \( \text{Trop}(\varphi_{i,j,k}(U_{i,j,k})) \)
such that
- for each \( P \in \Lambda_{i,j,k} \) and \( l \in \{i, j\} \), there exists \( Q \in \Lambda_l \) containing \( \Psi_{(i,j,k),l}(P) \),
- the diagrams

\[
\begin{array}{ccc}
U_{i,j,k} & \xrightarrow{\varphi_{i,j,k}} & U_{i,j,k} \\
\downarrow{\varphi_i} & & \downarrow{\varphi_j} \\
\mathbb{A}^{n_i} & \xrightarrow{\Psi_{(i,j,k),i}} & \mathbb{A}^{n_i} \\
\downarrow{\Psi_{(i,j,k),i}} & & \downarrow{\Psi_{(i,j,k),j}} \\
\mathbb{A}^{n_{i,j,k}} & \xrightarrow{\Psi_{(i,j,k),j}} & \mathbb{A}^{n_j}
\end{array}
\]
are commutative, and
Remark 6.4. Let \( X \) be a closed immersion to a toric variety. Then there are many closed immersions of \( X \) to toric varieties. In the rest of this subsection, we assume that there exist \( i(j) \) satisfying the following: for each \( j \), there exist \( i(j) \) and a toric morphism \( \psi_j: \mathbb{A}^{n_j} \to \mathbb{A}^{n(j)} \) such that

- \( U_i(j) \) (resp. \( V_i(j) \)) contains \( U_j' \) (resp. \( V_j' \)),
- the diagram

\[
\begin{array}{ccc}
U_i(j) & \xrightarrow{\psi_i(j)} & \mathbb{A}^{n_i(j)} \\
\uparrow & & \uparrow \\
U_j' & \xrightarrow{\psi_j} & \mathbb{A}^{n_j'}
\end{array}
\]

is commutative,

- for \( P \in \Lambda_j' \), there exists \( Q \in \Lambda_i(j) \) containing \( \psi_j(P) \), and
- \( \alpha_j' = \psi_j^n|_{\text{Trop}(\psi_i'(V_j'))} \).

The coboundary map \( \delta \) in Definition 6.2 induces a complex

\[
\mathcal{G}^{p,0}_X \to \mathcal{G}^{p,1}_X \to \mathcal{G}^{p,2}_X \to \ldots
\]

of sheaves on \( X^{\text{Ber}} \). The cohomology groups

\[
H_{\text{Trop}}^{p,q}(X) := \text{Ker}(\mathcal{G}^{p,q}_X(X^{\text{Ber}}) \to \mathcal{G}^{p,q+1}_X(X^{\text{Ber}}))/\text{Im}(\mathcal{G}^{p,q-1}_X(X^{\text{Ber}}) \to \mathcal{G}^{p,q}_X(X^{\text{Ber}}))
\]

are called the tropical cohomology groups of \( X \). By [JSS17, Proposition 3.15], they are the sheaf cohomology groups of a sheaf \( \mathcal{F}_X := \text{Ker}(\mathcal{G}^{p,0}_X \to \mathcal{G}^{p,1}_X) \) on \( X^{\text{Ber}} \). For a closed subscheme \( Z \subset X \), we put

\[
\mathcal{G}^{p,q}_{Z,X} := \text{Ker}(\mathcal{G}^{p,q}_X \to \pi_\ast\pi_\ast^\ast\mathcal{G}^{p,q}_X),
\]

where \( \pi: X \setminus Z \to X \) is the inclusion, and

\[
H_{\text{Trop},Z}^{p,q}(X) := \text{Ker}(\mathcal{G}^{p,q}_{Z,X}(X) \to \mathcal{G}^{p,q+1}_{Z,X}(X))/\text{Im}(\mathcal{G}^{p,q-1}_{Z,X}(X) \to \mathcal{G}^{p,q}_{Z,X}(X)).
\]

We use another expression of tropical cohomology in [Jel22] by embeddings of \( X \) to toric varieties. In the rest of this subsection, we assume that \( X \) has a closed immersion to a toric variety. Then there are many closed immersions of \( X \) to toric varieties [FGP14, Theorem 1.2]. We define a sheaf \( \mathcal{G}^{p,q}_X \) on \( X^{\text{Ber}} \) in a similar way to \( \mathcal{G}^{p,q}_X \) but the differences are using closed immersions \( X \to T_\Sigma \) to a toric variety \( T_\Sigma \), instead of pairs of open subvarieties \( U_i \subset X \) and closed immersions \( U_i \to \mathbb{A}^{n_i} \).

Remark 6.4. Jell proved that for any \( p, q \), there exists a natural isomorphism \( \mathcal{G}^{p,q}_X \cong \mathcal{G}^{p,q}_{T,X} \) [Jel22, Remark 8.2].

In particular, the tropical cohomology \( H_{\text{Trop}}^{p,q}(X) \) is isomorphic to

\[
H_{T,T}^{p,q}(X) := \text{Ker}(\mathcal{G}^{p,q}_{T,X}(X^{\text{Ber}}) \to \mathcal{G}^{p,q+1}_{T,X}(X^{\text{Ber}}))/\text{Im}(\mathcal{G}^{p,q-1}_{T,X}(X^{\text{Ber}}) \to \mathcal{G}^{p,q}_{T,X}(X^{\text{Ber}})),
\]

and \( H_{T,T}^{p,q}(X) \) is the sheaf cohomology of a sheaf \( \mathcal{F}^{p,q}_{T,X} := \text{Ker}(\mathcal{G}^{p,0}_{T,X} \to \mathcal{G}^{p,1}_{T,X}) \).
Remind that $X^\circ \subset X^{\text{Ber}}$ is the subset consisting of valuations $v$ having a natural morphism $\text{Spec} \, O_v \to X$. We put $\mathcal{C}^p_{T,X^\circ} := \mathcal{C}^p_{T,X}|_{X^\circ}$. For a closed subscheme $Z \subset X$, we also define sheaves

$$\mathcal{C}^p_{T,Z^\circ \subset X^\circ} := \ker(\mathcal{C}^p_{T,X^\circ} \to \pi_*^{\circ,*,*} \mathcal{C}^p_{T,X^\circ}),$$

where $\pi^0 : X^\circ \setminus Z^\circ \to X^\circ$ is the inclusion. We define $H^p_{T,Trop}(X^\circ)$ as the $q$-th cohomology group of $\mathcal{C}^p_{T,X^\circ}(X^\circ)$. We define $H^p_{T,Trop,Z^\circ}(X^\circ)$ similarly.

6.3. Tropical cycle class maps, tropical homology, and intersection theory. In this subsection, we recall tropical cycle class maps to tropical cohomology, tropical homology, and relation to intersection product. Let $X$ be a $d$-dimensional smooth proper algebraic variety over $K$. In this subsection, we assume that there exists a closed immersion of $X$ to a toric variety.

There is a tropical analog $H^p_{Trop,\text{Dol}}(X)$ of Dolbeault cohomology called tropical Dolbeault cohomology. (See [Jel16] Section 3.3 and 3.4. He denote it by $H^p_{d\text{-}\text{an}}(X)$. By [Jel22] Theorem 8.12], we have a tropical analog of de Rham’s theorem

$$H^p_{Trop,\text{Dol}}(X) \cong H^p_{Trop}(X) \otimes_{\mathbb{Q}} \mathbb{R}$$

(for tropical varieties, proved by Jell-Shaw-Smacka [JSS17 Theorem 1]). In the next subsection, we shall show that this isomorphism is given by integration.

The tropical cycle class map to $H^p_{Trop}(X) \otimes_{\mathbb{Q}} \mathbb{R}$ is the composition

$$\text{CH}^p(X) \to H^p_{Trop,\text{Dol}}(X) \cong H^p_{Trop}(X) \otimes_{\mathbb{Q}} \mathbb{R},$$

where the first morphism is Liu’s tropical cycle class map ([Liu20] Definition 3.8]).

We shall define tropical homology. Since $X$ is proper, i.e., $X^{\text{Ber}}$ is compact, every element of $\mathcal{C}^p_{T,X}(X)$ has a representative of the form $(X^{\text{Ber}}, \varphi : X \to T_\Lambda, \alpha)$. Hence we have the following.

**Lemma 6.5.** We have

$$H^p_{Trop}(X) \cong \lim_{\varphi} H^p_{Trop}(\text{Trop}(\varphi(X))),$$

where $\varphi : X \to T_\Lambda$ runs through all closed immersion to toric varieties.

**Definition 6.6.** We put

$$H^p_{Trop}(X) := \lim_{\varphi} H^p_{Trop}(\text{Trop}(\varphi(X))).$$

We call it the tropical homology group of $X$.

We have

$$H^p_{Trop}(X) \cong \text{Hom}(H^p_{Trop}(X), \mathbb{Q}).$$

There is also a natural tropical cycle class map to the tropical homology group. Let $Y \subset X$ be a closed subvariety of dimension $p$. By [MS15 Theorem 3.4.14] and [MZ13 Proposition 4.3], for a closed immersion $\varphi : X \to T_\Lambda$, its tropicalization $\text{Trop}(\varphi(Y))$ with the weight defined in [MS15 Definition 3.4.3] determines an element of $H^p_{Trop}(\text{Trop}(\varphi(X)))$, as follows. There is a fan structure $\Lambda$ of $\text{Trop}(\varphi(Y))$ and the weight $m_P \in \mathbb{Z}_{>0}$ for each $P \in \Lambda$ of dimension $p$. For each $P \in \Lambda$, there is an isomorphism

$$Z \cong \mathbb{Z} \otimes \mathbb{Z} \cong \wedge^p \text{Tan}(\text{rel.int}(P)) \otimes \mathbb{Z} H^p_{\text{sing}}(P, \partial P; \mathbb{Z})(\subset F_p(P) \otimes H^p_{\text{sing}}(P, \partial P, \mathbb{Q})),$$
where \( \text{Tan} \) is the \( Z \)-coefficient tangent space, and the second isomorphism is given by a fixed orientation of \( \text{rel.int}(P) \). This isomorphism is independent of the choice of the orientation. We identify both hand sides. The subvariety \( Y \) determines

\[
[T\text{rop}(\phi(Y))] := (-1)^{\frac{p(p-1)}{2}} (m_P)_{P \in A} \in H^{T\text{rop}}_{p,p}(T\text{rop}(\phi(X))).
\]

This induces a morphism \( Z_p(X) \to H^{T\text{rop}}_{p,p}(X) \) from the set \( Z_p(X) \) of algebraic cycles of dimension \( p \) to the tropical homology group.

Liu \cite[Theorem 1.1]{Liu20} proved compatibility of tropical cycle class maps, intersection products of algebraic cycles, and integrations of superforms on tropicalizations. (He considered only valued fields of height 1, but his proof also works over trivially valued fields.) We will prove that the tropical de Rham’s theorem is given by integration (Corollary 6.15). Consequently, we have the following.

**Theorem 6.7** (Liu \cite[Theorem 1.1, Remark 1.3(1)]{Liu20}). We have a commutative diagram

\[
\begin{array}{ccc}
\text{CH}^p(X)_{\mathbb{Q}} & \times & Z_p(X)_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
H^{T\text{rop}}_{p,p}(X) \otimes_{\mathbb{Q}} \mathbb{R} & \times & H^{T\text{rop}}_{p,p}(X) \rightarrow \mathbb{R},
\end{array}
\]

where the first horizontal arrow is the intersection product.

### 6.4. The tropical analog of de Rham’s theorem

In this subsection, we shall show that the tropical analog of de Rham’s theorem

\[
H^{p,q}_{T\text{rop},Dol}(X) \cong H^{p,q}_{T\text{rop}}(X) \otimes_{\mathbb{Q}} \mathbb{R}
\]

(\cite[Theorem 1]{JSS17} and \cite[Theorem 8.12]{Jel22}) is given by integrations of \((p,q)\)-superforms on general semi-algebraic singular chains (Corollary 6.15). (In \cite[Remark 3.11]{Jel22}, Jell gave integrations of \((p,q)\)-superforms on singular chains, but it is not well defined. We will see that integrations on semi-algebraic singular chains are well-defined. This is due to \cite[Theorem 2.6]{HKT15}.) See \cite{Jel16} and \cite{JSS17} for notations and basics of \((p,q)\)-superforms and tropical Dolbeault cohomology.

Let \( X \) be a proper algebraic variety over a trivially valued field \( K \). In this subsection, we assume that \( X \) has a closed immersion to a toric variety. Let \( \Sigma \) be a fan in \( N_{\mathbb{R}} \).

**Definition 6.8.** A subset \( A \subset T\text{rop}(T\Sigma) \) is said to be semi-algebraic if

\[
A \cap \text{Trop}(O(\sigma)) \subset \text{Trop}(O(\sigma)) = N_{\sigma,\mathbb{R}}
\]

is semi-algebraic in the usual sense for any cone \( \sigma \in \Sigma \).

For example, for \( \phi: X \to T\Sigma \), the tropicalization \( T\text{rop}(\phi(X)) \) is semi-algebraic.

**Definition 6.9.** Let \( S \subset \mathbb{R}^m \) be a semi-algebraic set. A map \( \varphi: S \to \text{Trop}(T\Sigma) \) is said to be semi-algebraic if the graph

\[
\Gamma_{\varphi} \subset \mathbb{R}^m \times \text{Trop}(T\Sigma) = \text{Trop}(\mathbb{G}_m^m \times T\Sigma)
\]

is semi-algebraic.

**Example 6.10.** A map

\[
[0, 1]^n \to [1, \infty]^n \subset (\mathbb{R} \cup \{\infty\})^n = \text{Trop}(\mathbb{A}^n)
\]

given by \( 0 \mapsto \infty \) and \( t \mapsto \frac{1}{t} \) for \( t \neq 0 \) is a semi-algebraic homeomorphism.
We define the semi-algebraic tropical cohomology $H^{p,q}_{\Trop,\text{semialg}}(X)$ in the same way to the usual tropical cohomology $H^{p,q}_{\Trop}(X)$ but using semi-algebraic singular chains.

**Lemma 6.11.** We have a natural isomorphism $H^{p,q}_{\Trop}(X) \cong H^{p,q}_{\Trop,\text{semialg}}(X)$

**Proof.** This follows in the same way as [JSS17, Section 3] by the existence of semi-algebraic triangulations [BCR98, Theorem 9.2.1]. \qed

We shall show that integrations of $(p,q)$-superforms on semi-algebraic singular chains are well-defined. We define integrations of $(p,q)$-superforms by compositions of contractions and integrations of $(0,q)$-superforms, see [JSS17, Section 2] for contractions. It suffices to define integrations of $(0,q)$-superforms. Let $V \subset \Trop(T \Sigma)$ be an open subset, $\varphi: \Delta^q \to V$ a continuous semi-algebraic map, and $w \in \omega^{0,q}(V)$ a $(0,q)$-superform. We may assume that $V \subset \Trop(T \sigma)$ for some $\sigma \in \Sigma$, where $T \sigma$ is the affine toric variety corresponding to the cone $\sigma$. By definition of $\omega^{0,q}$, we may also assume that $w = \Phi_\sigma^*(w|_{\Trop(O(\sigma))})$, where the projection $\Phi_\sigma: \Trop(T \sigma) \to \Trop(O(\sigma))$ is given by $N \to N_\sigma$. We consider $w|_{\Trop(O(\sigma))}$ as an usual smooth $q$-form on $\Trop(O(\sigma)) = N_{\sigma, \mathbb{R}}$. By definition, the composition
$$\Phi_\sigma \circ \varphi: \Delta^q \to \Trop(T \sigma) \to \Trop(O(\sigma))$$
is semi-algebraic. Hence [HKT15, Theorem 2.6], the integration $\int_{\Delta^q} (\Phi_\sigma \circ \varphi)^* w|_{\Trop(O(\sigma))}$ is well-defined and takes value in $\mathbb{R}$. We define the integration of the $(0,q)$-form $w$ by $\int_{\Delta^q} \varphi^* w := \int_{\Delta^q} (\Phi_\sigma \circ \varphi)^* w_\sigma$. Consequently, we have the following.

**Proposition 6.12.** The integrations of $(p,q)$-superforms on semi-algebraic tropical $(p,q)$-chains are well-defined.

Note that our integrations coincides with the one used in [Lim20] when chains are tropicalizations of algebraic subvarieties.

**Lemma 6.13.** (Stokes’ theorem) is a slight generalization of [HKT15, Theorem 2.9].

**Lemma 6.13.** Let $\varphi: \Delta^q \to \mathbb{R}^d$ be a continuous semi-algebraic map, and $w$ a smooth $(q-1)$-form on an open neighborhood of $\varphi(\Delta^q)$. Then we have
$$\int_{\Delta^q} \varphi^*(dw) = \int_{\partial \Delta^q} \varphi^* w,$$
where $d$ is the usual derivation and $\partial \Delta^q$ is the boundary.

**Proof.** When $\varphi$ is smooth on $\text{rel.int}(\Delta^q)$, the assertion is just [HKT15, Theorem 2.9]. (As mentioned in proof, [HKT15, Theorem 2.9] works when $\varphi$ is smooth on the interior.) In general, the assertion follows from the existence of a semi-algebraic triangulation of $\Delta^q$ such that the composition of the triangulation and $\varphi$ is smooth on dense open faces ([HKT15, Corollary 1.12.1] and [BCR98, Theorem 9.2.1]). \qed

There is a differential $d': \omega^{p,q} \to \omega^{p,q+1}$ ([JSS17]).

**Corollary 6.14.** For a closed immersion $\varphi: X \to T \Sigma$, the integrations give a morphism of complexes
$$(\omega^{p,*}(\Trop(\varphi(X))), d') \to (C_{\Trop,\text{semialg}}^p(\Trop(\varphi(X)), \mathbb{R}), \delta).$$

Since both hand sides of Corollary 6.14 are fine or flasque resolutions of the same sheaf (which give tropical de Rham’s theorem), by Lemma 6.15 we have the following.

**Corollary 6.15.** Tropical de Rham’s theorem $H^{p,q}_{\Trop,\text{Dol}}(X) \cong H^{p,q}_{\Trop}(X)_{\mathbb{R}}$ is given by integration of $(p,q)$-superforms on semi-algebraic singular simplices.
ON TROPICAL CYCLE CLASS MAPS

7. Tropical analogs of Milnor $K$-groups

In this section, we shall define and study tropical analog of rational Milnor $K$-groups. Remind that $M$ is a free $\mathbb{Z}$-module of finite rank $n$. Let $K$ be a trivially valued field.

7.1. Definition. Let $L/K$ be a finitely generated extension, and $\varphi: \text{Spec } L \to \mathbb{G}_m^n = \text{Spec } K[M]$ be a morphism over $K$, i.e., a group homomorphism $\varphi: M \to L^\times$. We denote the wedge product of $\varphi \otimes \mathbb{Z} \mathbb{Q}$ by $\varphi: \wedge^p M \mathbb{Q} \to \wedge^p (L^\times) \mathbb{Q}$. A valuation $v \in \text{ZR}(L/K)$ is also a group homomorphism $v: L^\times \to \Gamma_v$. We denote the wedge product of $v \otimes \mathbb{Z} \mathbb{Q}: (L^\times) \mathbb{Q} \to \Gamma_v \mathbb{Q}$ by $\wedge^p v: \wedge^p (L^\times) \mathbb{Q} \to \wedge^p \Gamma_v \mathbb{Q}$.

Let $\varphi(\text{Spec } L) \subset \mathbb{G}_m^n$ be the closure.

Lemma 7.1. We have

$$F^p(0, \text{Trop}(\varphi(\text{Spec } L))) = \wedge^p M \mathbb{Q}/J_M$$

where $J_M$ is the $\mathbb{Q}$-vector subspace generated by $f \in \wedge^p M \mathbb{Q}$ such that $\wedge^p v(\varphi(f)) = 0$ for $v \in \text{ZR}(L/K)$.

Moreover, when $L = k(\varphi(\text{Spec } L))$, where $k(\varphi(\text{Spec } L))$ is the residue field of the structure sheaf at $\varphi(\text{Spec } L) \in \mathbb{G}_m^n$, we have

$$F^p(0, \text{Trop}(\varphi(\text{Spec } L))) = \wedge^p M \mathbb{Q}/J'_M,$$

where $J'_M$ is the $\mathbb{Q}$-vector subspace generated by $f \in \wedge^p M \mathbb{Q}$ such that $\wedge^p v(\varphi(f)) = 0$ for $v \in \text{ZR}(L/K)$ with $\Gamma_v \cong \mathbb{Z}^p$, where $\mathbb{Z}^p$ is equipped with the lexicographic order.

Proof. This follows from Remark 5.6, Lemma 5.8, and the fact that for $w \in \text{ZR}(L/K)$ with $\text{ht}(w) = \text{tr.deg}(L/K)$, by [Bou72, Corollary 3 of Theorem 1 in Section 10.3 of Chapter 6], we have $\Gamma_w = \mathbb{Z}^{\text{tr.deg}(L/K)}$. □

Of course, $\wedge^p M \mathbb{Q} \to F^p(0, \text{Trop}(\varphi(\text{Spec } L)))$ factors through

$$\wedge^p M \mathbb{Q} \to \wedge^p \varphi(M) \mathbb{Q} \to F^p(0, \text{Trop}(\varphi(\text{Spec } L))).$$

Remark 7.2. Consider two morphisms $\varphi_1: \text{Spec } L \to \mathbb{G}_m^r$ and $\varphi_2: \text{Spec } L \to \mathbb{G}_m^l$ over $K$ and a toric morphism $\psi: \mathbb{G}_m^l \to \mathbb{G}_m^r$ (with respect to a fixed toric structures) such that the diagram

$$\begin{array}{ccc}
\text{Spec } L & \xrightarrow{\varphi_1} & \mathbb{G}_m^r \\
\downarrow{\varphi_2} & & \downarrow{\psi} \\
\mathbb{G}_m^l & \xrightarrow{} & \mathbb{G}_m^r
\end{array}$$

is commutative. These induce a pull-back map

$$(7.1) \quad F^p(0, \text{Trop}(\varphi_1(\text{Spec } L))) \to F^p(0, \text{Trop}(\varphi_2(\text{Spec } L))).$$

Note that this pull-back map is injective since

$$\text{Trop}(\varphi_2(\text{Spec } L)) \to \text{Trop}(\varphi_1(\text{Spec } L))$$

is surjective.
Definition 7.3. Let $p \geq 0$ be a non-negative integer. We put
\[ K^p_T(L/K) := \lim_{\varphi: \text{Spec } L \to \mathbb{G}^r_m} F^p(0, \text{Trop}(\varphi((\text{Spec } L)^{\text{Ber}}))), \]
where $\varphi: \text{Spec } L \to \mathbb{G}^r_m$ runs all $K$-morphisms to tori of arbitrary dimensions and morphisms are the pull-back maps under toric morphisms. We call it the $p$-th tropical $K$-group. When there is no confusion, we put $K^p_T(L) := K^p_T(L/K)$.

There is a natural surjective map
\[ \wedge^p(L^x)_\mathbb{Q} \to K^p_T(L/K). \]
Moreover, by Lemma 7.1, we have the following.

Corollary 7.4. We have
\[ K^p_T(L/K) \cong \wedge^p(L^x)_\mathbb{Q} / J \cong \wedge^p(L^x)_\mathbb{Q} / J', \]
where $J$ (resp. $J'$) is the $\mathbb{Q}$-vector subspace generated by $f \in \wedge^p(L^x)_\mathbb{Q}$ such that $\wedge^p v(f) = 0$ for $v \in \text{ZR}(L/K)$ (resp. for $v \in \text{ZR}(L/K)$ with $\Gamma_v \cong \mathbb{Z}^p$, where $\mathbb{Z}^p$ is equipped with the lexicographic order).

Example 7.5. · For any $L/K$, we have $K^0_T(L/K) = \mathbb{Q}$ by definition.
· For any $L/K$, by Corollary 7.4, we have $K^1_T(L/K) = (L^x / (L \cap K^\text{alg})^x)_\mathbb{Q}$.
· For any $L/K$ and any $p \geq \text{tr.deg}(L/K) + 1$, we have $K^p_T(L/K) = 0$.

Wedge product induces a multiplication
\[ K^p_T(L/K) \times K^p_T(L/K) \to K^{p+q}_T(L/K). \]

7.2. Tropical $K$-group is a cycle module. We shall show that tropical $K$-groups satisfy good properties, i.e., they define a cycle module in the sense of Rost [Ros96, Definition 2.1].

We recall the definition and several maps of rational Milnor $K$-groups. For a field $E$, its rational Milnor $K$-group is defined by
\[ K^p_{M,Q}(E) := \wedge^p(E^x)_\mathbb{Q} / I_M, \]
where $I_M$ is the $\mathbb{Q}$-linear subspace of $\wedge^p(E^x)_\mathbb{Q}$ generated by
\[ \{a_1 \wedge \cdots \wedge a_p \mid a_i = 1 - a_j \text{ for some } i \neq j\}. \]
In particular, we have $K^0_{M,Q}(E) = \mathbb{Q}$ and $K^1_{M,Q}(E) = (E^x)_\mathbb{Q}$. The image of $a_1 \wedge \cdots \wedge a_p$ in $K^p_{M,Q}(E)$ is denoted by $(a_1, \ldots, a_p)$.

· A morphism $\varphi: F \to E$ of fields induces a map $\varphi^\ast: K^p_{M,Q}(F) \to K^p_{M,Q}(E)$ by

\[ \varphi^\ast((a_1, \ldots, a_p)) = (\varphi(a_1), \ldots, \varphi(a_p)). \]
· For a finite morphism $\varphi: F \to E$, there is the norm homomorphism $\varphi^\ast: K^p_{M,Q}(E) \to K^p_{M,Q}(F)$.

It is a generalization of the multiplication
\[ \times [E : F]: K^0_{M,Q}(E) = \mathbb{Q} \to \mathbb{Q} = K^0_{M,Q}(F) \]
and the usual norm map $E^x \to F^x$. This is defined by Bass and Tate [BT72] with respect to a choice of generators of $E$ over $F$, and independence of such a choice was proved by Kato [Kat80].
For a normalized discrete valuation \( v : F^\times \to \mathbb{Z} \), there is the residue homomorphism (Milnor [Mil70])
\[
\partial_v : K^{p}_{M,Q}(F) \to K^{p-1}_{M,Q}(\kappa(v)).
\]
It is characterized by
\[
\partial_v((\pi, u_1, \ldots, u_{p-1})) = (\overline{u}_1, \ldots, \overline{u}_{p-1})
\]
\[
\partial_v((u_1, \ldots, u_p)) = 0
\]
for a prime \( \pi \) of \( v \) and \( u_i \in F \) with \( v(u_i) = 0 \) (\( 1 \leq i \leq p \)) and residue class \( \overline{u}_i \).
(Remind that \( \kappa(v) \) is the residue field of \( v \).

**Lemma 7.6.** The canonical surjective \( \mathbb{Q} \)-linear map
\[
\wedge^p(L^\times)_Q \to K^p_T(L/K)
\]
factors through
\[
\wedge^p(L^\times)_Q \to K^p_{M,Q}(L) \to K^p_T(L/K).
\]

**Proof.** For any \( a \in L \), there are no 2-rational-rank valuations of \( K(a) \) which are trivial on \( K \). Hence the assertion follows from Corollary 7.4. \( \square \)

**Lemma 7.7.** Let \( E \) and \( F \) be finitely generated fields over \( K \).

- For a morphism \( \phi : F \to E \) of fields over \( K \), the map
\[
\phi_* : K^p_{M,Q}(F) \to K^p_{M,Q}(E)
\]
induces a map
\[
\phi_* : K^p_T(F/K) \to K^p_T(E/K)
\]
of tropical \( K \)-groups.

- For a finite morphism \( \phi : F \to E \) over \( K \), the norm homomorphism
\[
\phi^* : K^p_{M,Q}(E) \to K^p_{M,Q}(F)
\]
induces a map
\[
\phi^* : K^p_T(E/K) \to K^p_T(F/K),
\]
also called the norm homomorphism.

- For a normalized discrete valuation \( v : F^\times \to \mathbb{Z} \) (i.e., \( v(F^\times) = \mathbb{Z} \)) which is trivial on \( K \), the residue homomorphism
\[
\partial_v : K^p_{M,Q}(F) \to K^{p-1}_{M,Q}(\kappa(v))
\]
induces a map
\[
\partial_v : K^p_T(F/K) \to K^{p-1}_T(\kappa(v)/K),
\]
also called the residue homomorphism.

**Proof.** The assertion on \( \phi_* \) follows from Corollary 7.4 and the existence of extensions of valuations for extensions of fields ([Bou72, Proposition 5 in Section 3.4 of Chapter 6]).

The assertion on \( \partial_v \) follows from Corollary 7.4 and Remark 3.9.

We shall prove the assertion on \( \phi^* \). It suffices to show that for \( \alpha \in K^p_{M,Q}(E) \) whose image in \( K^p_T(E/K) \) is 0, we have
\[
\wedge^p v(\phi^* \alpha) = 0
\]
for any \( v \in ZR(F/K) \) with \( \Gamma_v \cong \mathbb{Z}^p \), where \( \mathbb{Z}^p \) is equipped with the lexicographic order. We shall show this assertion by induction on \( p \). When \( p = 0 \), the assertion is
trivial. We assume \( p \geq 1 \). Let \( v_1 \in \text{ZR}(F/K) \) be the vertical generalization of \( v \) of height 1. Then we have \( \Gamma_{v_1} \cong \mathbb{Z} \). For an extension \( w_i \in \text{ZR}(E/K) \) of \( v_1 \) to \( E \), by the assertion on \( \partial_{w_i} \), the image of \( \partial_{w_i} (\alpha) \in K_{p,1}^{p-1}(\kappa(w_i)) \) in \( K_{p-1}^{p-1}(\kappa(w_i)/K) \) is 0. Hence by applying the assumption of induction to \( \varphi_v : \kappa(v_1) \to \kappa(w_i) \) and \( \partial_{w_i} (\alpha) \in K_{p,1}^{p-1}(\kappa(w_i)) \), the image of \( \varphi_v^* (\partial_{w_i} (\alpha)) \in K_{p-1}^{p-1}(\kappa(v_1)/K) \) is 0. Let \( \nu \in \text{ZR}(\kappa(v_1)/K) \) be the valuation of height \( ht(v) - 1 \) corresponding to \( v \) in the sense of Remark 3.9. Then we have

\[
\wedge^{p-1} \nu (\partial_{v_1} \circ \varphi^* \alpha) = \wedge^{p-1} \nu \left( \sum_{w_i} \varphi_v^* \circ \partial_{w_i} (\alpha) \right) = 0,
\]

where \( w_i \in \text{ZR}(E/K) \) runs through all extensions of \( v_1 \in \text{ZR}(F/K) \) to \( E \), and the second equality follows from a basic property of Milnor \( K \)-groups. Hence \( \wedge^p \nu (\varphi^* \alpha) = 0 \).

We also denote the induced maps of tropical \( K \)-groups by \( \varphi_*, \varphi^*, \partial_v \).

**Theorem 7.8.** The functor

\[
(\text{finitely generated fields over } K) \to (\mathbb{Z}_{\geq 0}\text{-graded abelian group})
\]

\[
L \mapsto \bigoplus_p K_T^p(L/K).
\]

with \( \varphi_*, \varphi^*, \partial_v \) and the natural multiplication

\[
K_{p,1}^p(M,Q)(L) \times K_T^p(L/K) \to K_T^{p+1}(L/K)
\]

is a cycle module in the sense of Rost [Ros96, Definition 2.1].

**Proof.** This follows from Lemma 7.6, Lemma 7.7, and the fact that Milnor \( K \)-groups form a cycle module [Ros96, Theorem 1.4 and Remark 2.4].

We give an explicit resolution of the Zariski sheaf of tropical \( K \)-groups on a smooth algebraic variety \( X \) over \( K \). Let \( X^{(i)} \) be the set of points of the scheme \( X \) of codimension \( i \). For any \( i \) and points \( x \in X^{(i)} \), \( y \in X^{(i+1)} \), Rost defined a map

\[
\partial^y_x : K_T^p(k(x)/K) \to K_T^{p-1}(k(y)/K)
\]

(for cycle modules) [Ros96, Section 2] as follows. When \( y \notin \{x\} \), we put \( \partial^y_x = 0 \). When \( y \in \{x\} \), we put

\[
\partial^y_x := \sum_v \varphi_v^* \circ \partial_v : K_T^p(k(x)/K) \to K_T^{p-1}(k(y)/K),
\]

where \( v \in \text{ZR}(k(x)/K) \) runs through all normalized discrete valuations of \( k(x) \) whose center in \( \{x\} \) is \( y \), and \( \varphi_v : k(y) \to k(v) \) is the induced morphism. Let \( \eta \in X \) be the generic point. We put the sheaf of tropical \( K \)-groups \( \mathcal{K}_T^p \) the sheaf on the Zariski site \( X_{\text{Zar}} \) defined by

\[
\mathcal{K}_T^p(U) := \text{Ker}(K_T^p(K(X)/K) \xrightarrow{d} \bigoplus_{x \in U^{(1)}} K_T^{p-1}(k(x)/K))
\]

for open subsets \( U \subset X \), where \( d := (\partial^x_\eta)_{x \in U^{(1)}} \).
Corollary 7.9. For any $p \geq 0$, the sheaf $\mathcal{K}_T^p$ has the Gersten resolution, i.e., an exact sequence

$$0 \to \mathcal{K}_T^p \to \bigoplus_{x \in X^{(0)}} i_x(K_T^p(k(x)/K)) \overset{d}{\to} \bigoplus_{x \in X^{(1)}} i_xK_T^{p-1}(k(x)/K) \overset{d}{\to} \bigoplus_{x \in X^{(2)}} i_xK_T^{p-2}(k(x)/K) \overset{d}{\to} \cdots$$

where $i_x : \text{Spec}(k(x)) \to X$ are the natural morphisms, we identify the groups $K_T^p(k(x)/K)$ and the Zariski sheaf on $\text{Spec}\, k$, and the natural morphisms, we identify the groups $K_T^p(k(x)/K)$ and the Zariski sheaf on $\text{Spec}\, k(x)$, and $d := (\partial_x)_{\{x \in X^{(0)}, y \in X^{(i+1)}\}}$. In particular, we have

$$H_{\text{Trop}}^p(X, \mathcal{K}_T^p) \cong CH^p(X)_Q.$$

Proof. The first assertion follows from Theorem 7.8 and [Ros96, Theorem 6.1]. The second assertion follows from the fact that for a normalized discrete valuation $v \in \text{ZR}(E/K)$ of a field extension $E/K$, the residue homomorphism

$$\partial_v : K^1_T(E/K) \cong (E^\times)_Q/((E \cap K^{alg})^\times)_Q \to K^0_T(\kappa(v)/K) \cong \mathbb{Q}$$

coincides with the map induced by the valuation $v : E^\times \to \mathbb{Z}$. \qed

Remark 7.10. By construction, residue homomorphisms $\partial_v : K_T^p(F/K) \to K_T^{p-1}(\kappa(v)/K)$ lift to wedge products $\partial_v : \wedge^p F^\times_K \to \wedge^{p-1} \kappa(v)^\times_Q$. Moreover, for a field $L$, the direct sum of residue homomorphisms

$$\wedge^p L(T)^\times_Q \to \bigoplus_{x \in \mathbb{A}_L^{1,(1)}} \wedge^{p-1}\kappa(x)^\times_Q$$

is surjective, where $T$ is the coordinate of $\mathbb{A}_L^1$. This easily follows from induction on $\max_{x \in \mathbb{A}_L^{1,(1)}} \{[x : L]\}$ for elements $(a_x)_x$ of the right hand side.

8. Proof of the main theorem

The aim of this section is to prove the main theorem (Theorem 8.2). It follows from

- Proposition 8.1 (proved in Section 9 and 11),
- easy lemmas on tropical cohomology (Lemma 8.11 and 8.12 proved in Subsection 8.2), and
- a theorem on general "cohomology theories" [CTHK97, Corollary 5.1.11], which is proved by Bloch [Blo74] for $K_2$, Quillen [Qui73] for general algebraic $K$-theory, and developed by many mathematicians including Bloch-Ogus [BO74], Gabber [Gab94], Rost [Ros96], and Collit-Théâne-Hoobler-Kahn [CTHK97].

Let $K$ be a trivially valued field. Tropical $K$-groups in this section are over $K$.

8.1. Proof of the main theorem. Proposition 8.1 is proved in Section 9 and 11

Proposition 8.1. (1) (étale excision) Let $\Phi : X' \to X$ be an étale morphism of smooth quasi-projective varieties over $K$. Let $Z \subset X$ be a closed subscheme. We assume $Z' := \Phi^{-1}(Z) \to Z$ is an isomorphism. Then we have

$$\Phi^* : H_{\text{Trop},Z}^{p,q}(X) \cong H_{\text{Trop},Z'}^{p,q}(X').$$
(2) (*-homotopy invariance) Let \( X \) be a smooth quasi-projective variety over \( K \), and \( Z \subset X \) a closed subscheme. We put \( \pi : X \times \mathbb{A}^1 \to X \) the first projection. Then the pullback map

\[
\pi^* : H_{\text{Trop},Z}^{p,q}(X) \to H_{\text{Trop},Z \times \mathbb{A}^1}^{p,q}(X \times \mathbb{A}^1)
\]

is an isomorphism.

We put \( \mathcal{H}^{p,q} \) the Zariski sheaf on \( X \) associated to the presheaf \( U \mapsto H_{\text{Trop}}^{p,q}(U) \).

**Theorem 8.2.** Let \( X \) be a smooth algebraic variety over \( K \). Then there exist natural isomorphisms

\[
H^{q}_{\text{Zar}}(X, \mathcal{H}^{p,0}) \cong H_{\text{Trop}}^{p,q}(X)
\]

In particular, we have

\[
H^{p,p}_{\text{Trop}}(X) \cong CH^{p}(X)_{\mathbb{Q}}.
\]

**Proof.** The last assertion follows from Corollary 7.9.

For each \( r \), by Proposition 8.1 and [CTHK97, Remarks 5.1.3, Corollary 5.1.11, and Proposition 5.3.2 (a)] (and Corollary 12.2 and [CTHK97, Theorem 6.2.5] when \( K \) is finite), there exists a spectral sequence

\[
E_1^{p,q} = \prod_{x \in X^{(p)}} H_{\text{Trop},x}^{r,p+q}(X) \Rightarrow H_{\text{Trop}}^{r,p+q}(X)
\]

whose \( E_2 \)-terms are \( E_2^{p,q} = H_{\text{Zar}}^{r,p}(X, \mathcal{H}^{r,0}) \), and we have

\[
\mathcal{H}^{r,0}(V) \cong \ker(d_1 : H_{\text{Zar},\eta}^{r,0}(X) \to \bigoplus_{x \in V^{(1)}} H_{\text{Trop},x}^{r+1}(X))
\]

for each open subvariety \( V \subset X \), where

1. \( H_{\text{Trop},x}^{r,p+q}(X) := \lim_{x \in U} H_{\text{Trop},x}^{r,p+q}(U) \), where \( U \subset X \) runs through all open neighborhoods of \( x \),
2. \( V^{(1)} \) is the subset of points of codimension 1,
3. \( d_1 \) is the differential map of the spectral sequence, and
4. \( \eta \in X \) is the generic point.

(Note that we do not need to prove étale excision for smooth algebraic varieties. That for smooth quasi-projective varieties are enough. See [CTHK97, Section 4].)

By Lemma 8.11 we have \( E_2^{p,q} = 0 \) for \( q \geq 1 \). By definition, we have \( E_2^{p,q} = 0 \) for \( q \leq -1 \). Hence

\[
H_{\text{Zar}}^{p}(X, \mathcal{H}^{r,0}) = E_2^{p,0} = E_{\infty}^{p,0} = H_{\text{Trop}}^{r,p}(X).
\]

By Lemma 8.12 and Corollary 7.9, we have

\[
\mathcal{H}^{r,0}(V) \cong \ker(H_{\text{Zar},\eta}^{r,0}(X) \xrightarrow{d_1} \bigoplus_{x \in V^{(1)}} H_{\text{Trop},x}^{r+1}(X))
\]

\[
\cong \ker(K_{r}^{p}(K(X)) \xrightarrow{d = (\partial_{x})_{x}^{T}} \bigoplus_{x \in V^{(1)}} K_{r}^{p-1}(k(x))
\]

\[
\cong \mathcal{H}_{r}^{p}(V).
\]

\end{proof}

**Remark 8.3.** By Theorem 8.2, the complex of Zariski sheaves \( \pi_{*} \mathcal{F}_{X}^{p} \) on \( X \) (where \( \pi : X^{\text{Ber}} \to X \) is the map taking supports) is a flasque resolution of \( \pi_{*} \mathcal{F}_{X}^{p} \cong \mathcal{H}_{r}^{p} \).
Remark 8.4. We have a commutative diagram
\[
\begin{array}{ccc}
CH^p(X)_Q & \xrightarrow{\sim} & H^p_{\text{Zar}}(X, \mathcal{X}_{T}^p) \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
H^p_{\text{Zar}}(X, \mathcal{X}_{T}^p) & \rightarrow & H^p_{\text{Trop, Dol}}(X) \otimes \mathbb{R}
\end{array}
\]

where
\[
\begin{align*}
\mathcal{X}_{T}^p & \text{ is the Zariski sheaf of rational Milnor } K\text{-groups, defined in the same way as that of tropical } K\text{-groups } \mathcal{X}_{T}^p, \\
CH^p(X)_Q & \cong H^p_{\text{Zar}}(X, \mathcal{X}_{M,Q}^p) \text{ is also given by Gersten resolution (Gabber, Rost [Ros96 Corollary 6.5]),} \\
H^p_{\text{Zar}}(X, \mathcal{X}_{M,Q}^p) & \rightarrow H^p_{\text{Trop, Dol}}(X) \text{ is given by a morphism of sheaves [Liu20 Definition 3.3], and} \\
H^p_{\text{Zar}}(X, \mathcal{X}_{M,Q}^p) & \rightarrow H^p_{\text{Trop}}(X, \mathcal{X}_{T}^p) \text{ is given by the natural morphism } \mathcal{X}_{M,Q}^p \rightarrow \mathcal{X}_{T}^p.
\end{align*}
\]

Liu’s tropical cycle class map \(CH^p(X)_Q \rightarrow H^p_{\text{Trop, Dol}}(X)\) ([Liu20, Definition 3.8]) is given as the composition of the first two horizontal arrows. Hence it coincides with the last isomorphism in Theorem 8.2 after \(- \otimes \mathbb{R}\) under tropical de Rham’s isomorphism \(H^p_{\text{Trop, Dol}}(X) \cong H^p_{\text{Trop}}(X) \otimes \mathbb{R}\).

By Theorem 6.7 and 8.2 we have the following Corollary.

Corollary 8.5. Let \(X\) be as in Theorem 8.2. When \(X\) is proper and has a closed immersion to a toric variety (e.g. projective), the kernel of the cycle class map \(Z_p(X) \otimes \mathbb{Q} \rightarrow H^p_{\text{Trop}}(X)\) from the \(\mathbb{Q}\)-vector space \(Z_p(X) \otimes \mathbb{Q}\) of algebraic cycles of dimension \(p\) is the \(\mathbb{Q}\)-vector subspace generated by algebraic cycles numerically equivalent to 0.

Corollary 8.6. Let \(X\) be a smooth proper algebraic variety over \(K\). Then for \(p \geq 1\), we have \(H^p_{\text{Trop}}(X, \mathbb{Q}) = 0\).

Proof. By Theorem 8.2 we have
\[
H^p_{\text{Trop}}(X, \mathbb{Q}) = H^0_{\text{Zar}}(X, \mathcal{X}_{T}^p) = \text{Ker}(K^p_T(K(X))) \xrightarrow{\partial^p_K} \bigoplus_{x \in X^{(1)}} K^p_T(k(x)).
\]

Let \(\alpha \in H^p_{\text{Zar}}(X, \mathcal{X}_{T}^p)\). There exists a \(K\)-morphism \(\varphi: \text{Spec } K(X) \rightarrow \mathbb{G}^n_m\) such that
\[
\alpha \in \text{Im}(F^p(0, \text{Trop}(\varphi(\text{Spec } K(X)))) \hookrightarrow K^p_T(K(X))),
\]

where the closure \(\overline{\varphi(\text{Spec } K(X))}\) is taken in \(\mathbb{G}^n_m\). Let \(\Lambda\) be a fan structure of \(\text{Trop}(\varphi(\text{Spec } K(X)))\). We put \(X_\Lambda\) the closure of \(\varphi(\text{Spec } K(X))\) in the toric variety \(T_\Lambda\). By taking suitable \(\varphi\) and \(\Lambda\), we may assume that there exists a natural proper birational morphism \(f: X_\Lambda \rightarrow X\). By [Ros96 Section 12, p.382] and [Ros96 Lemma 12.8], we have
\[
H^p_{\text{Zar}}(X, \mathcal{X}_{T}^p) \subset H^p_{\text{Zar}}(X_\Lambda, \mathcal{X}_{T}^p)(\cong K^p_T(K(X))).
\]

Hence we have
\[
\alpha \in \text{Ker}(K^p_T(K(X))) \xrightarrow{\partial^p_K} \bigoplus_{y \in X^{(1)}_\Lambda} K^p_T(k(y)),
\]

where
where \( y \in X^{(1)}_\Lambda \) runs through the generic points of irreducible components of \( X_\Lambda \cap O(l) \) for all 1-dimensional cones \( l \in \Lambda \). Consequently, we have

\[
(8.1) \quad \alpha \in \text{Ker}(F^p(0, \text{Trop}(\varphi(\text{Spec} K(X)))) \to \bigoplus_{l \in \Lambda, \dim l=1} F^{p-1}(0_{N_{l,r}}, \text{Trop}(X_\Lambda \cap O(l))))
\]

where \( 0_{N_{l,r}} \in N_{l,r} \) is the zero element, and the map in (8.1) is given as the sum of contractions by the primitive elements \( \nu_l \in l \). (The contractions give residue homomorphisms \( \partial_{\nu_l} \) (up to constant) for elements of

\[
F^p(0, \text{Trop}(\varphi(\text{Spec} K(X))))(\subset K^p_{\text{Trop}}(K(X))),
\]

see Subsection 7.2 for \( \partial_{\nu_l} \).) Since \( \text{Trop}(X_\Lambda \cap O(l)) \) is the intersection of \( \text{Trop}(O(l)) \) and the closure of \( \text{Trop}(\varphi(\text{Spec} K(X))) \), the kernel of (8.1) is 0. \( \square \)

8.2. Easy computation. In this subsection, we shall show Lemma 8.11 and 8.12, which are used to prove Theorem 8.2. Let \( X \) be a smooth algebraic variety over \( K \). Let \( M \) be a free \( \mathbb{Z} \)-module of rank \( n \) and \( N := \text{Hom}(M, \mathbb{Z}) \).

Lemma 8.7. Assume that \( X \) has a closed immersion to a toric variety. Then the pull-back maps induces an isomorphism

\[
H^{p,q}_{\text{Trop}, T}(X) \cong \lim_{\varphi} H^{p,q}_{\text{Trop}}(\text{Trop}(\varphi(X^o))) =: H^{p,q}_{\text{Trop}, T}(X^o),
\]

where \( \varphi \) runs through all closed immersions of \( X \) to toric varieties.

Proof. Since \( X^o \) is compact (Remark 4.10), every element of \( \mathcal{C}^{p,q}_{\text{Trop}, X^o}(X^o) \) is given by a single tropicalization. Hence the morphism in the assertion is well-defined. For each \( \varphi \), a fan structure \( \Lambda \) of \( \text{Trop}(\varphi(X)) \), and a subset \( B \subset \text{Trop}(\varphi(X)) \) containing \( \text{Trop}(\varphi(X^o)) \) and having a strong deformation retraction \( \psi : B \times [0, 1] \to B \) of \( B \) onto \( \text{Trop}(\varphi(X^o)) \) preserving the fan structure, i.e.,

\[
\psi((P \cap B, [0, 1])) \subset P \cap B \quad (P \in \Lambda),
\]

we have

\[
(8.2) \quad H^{p,q}_{\text{Trop}}(\text{Trop}(\varphi(X^o)), \Lambda) \cong H^{p,q}_{\text{Trop}}(B, \Lambda).
\]

Surjectivity follows from (8.2) for \( B = \text{Trop}(\varphi(X)) \). To prove injectivity, we need to construct coboundary on whole \( X^\text{Ber} \). By Lemma 8.8, we can glue local coboundary. Hence injectivity follows from (8.2) for suitable (infinitely many) \( B \). \( \square \)

Lemma 8.8. Let \( T_\Sigma \) be a toric variety, \( \Lambda \) a fan in \( \text{Trop}(T_\Sigma) \), and \( A, B \) be subsets of \( |\Lambda| \) such that \( A \subset B \). We fix \( p \in \mathbb{Z}_{\geq 0} \) and \( q \in \mathbb{Z}_{\geq 1} \). We assume that the pull-back map

\[
(8.3) \quad H^{p,q}_{\text{Trop}, T, X^o}(X^o) = \lim_{\varphi} H^{p,q}_{\text{Trop}, T, \text{Trop}(\varphi(Z^o))}(\text{Trop}(\varphi(X^o))).
\]

is injective for \( \epsilon = 0 \), and surjective for \( \epsilon = 1 \). Then, for \( \alpha \in \mathbb{Z}^{p,q}(B, \Lambda) \) such that \( \alpha|_A = 0 \), there exists \( \beta \in \mathbb{C}^{p,q-1}(B, \Lambda) \) such that \( \beta|_A = 0 \) and \( \alpha = \delta \beta \).

Proof. This is elementary and straightforward. \( \square \)

Since \( X^o \) is compact (Remark 4.10), for closed subscheme \( Z \subset X \), we have

\[
H^{p,q}_{\text{Trop}, T, Z^o}(X^o) \cong \lim_{\varphi} H^{p,q}_{\text{Trop}, T, \text{Trop}(\varphi(Z^o))}(\text{Trop}(\varphi(X^o))).
\]

Corollary 8.9. For a closed subscheme \( Z \subset X \), we have

\[
H^{p,q}_{\text{Trop}, T, Z}(X) \cong H^{p,q}_{\text{Trop}, T, Z^o}(X^o) \cong \lim_{\varphi} H^{p,q}_{\text{Trop}, T, \text{Trop}(\varphi(Z^o))}(\text{Trop}(\varphi(X^o))).
\]
Lemma 8.10. We assume that $X$ is affine. Let $\varphi: X \to T_\sigma$ be a closed immersion to the affine toric variety $T_\sigma$ corresponding to a cone $\sigma \subset N_\mathbb{R}$. Let $x \in X$. We assume that $X \cap \varphi^{-1}(O(\sigma)) = \{x\}$. Let $p_x \in \mathcal{O}(X)$ the prime ideal corresponding to $x$. We put $p = \text{the codimension of } \{x\} \subset X$

Then there exist $f_1, \ldots, f_r \in \mathcal{O}(X) \setminus p_x$ such that

\begin{align}
1 & \quad \text{Trop}((\varphi, (f_i)_i)(X)) \cap (\sigma \times \{0\}) \subset \text{Trop}(T_\sigma) \times (\mathbb{R} \cup \{\infty\})' \\
2 & \quad \text{Trop}((\varphi, (f_i)_i)(X)) \cap (\sigma \times \{0\}' \setminus \text{rel.int}(\sigma \times \{0\}')) \subset \text{Trop}(T_\sigma) \times (\mathbb{R} \cup \{\infty\})'
\end{align}

is a finite union of cones of dimension $\leq p$, and

where $(\varphi, (f_i)_i): X \to T_\sigma \times \mathbb{A}^r$ is a closed immersion.

Proof. Let $\Sigma$ be a fan structure of $\sigma$ such that a subfan $\Lambda \subset \Sigma$ is a fan structure of $\text{Trop}(\varphi(X)) \cap \sigma$, and $\Lambda$ is also a subfan of a fan structure $\Lambda'$ of $\text{Trop}(\varphi(X)) \cap N_\mathbb{R}$ inducing a tropical compactification of $X' \cap \varphi^{-1}(G^m_n)$. This induces a birational toric morphism $\pi: T_\Sigma \to T_\sigma$. We put $X'$ the closure of $\varphi(X) \cap G^m_n$ in $T_\Lambda$. We put $B \subset X$ the union of the image of the generic points of the irreducible components of $X' \cap O(\tau)$ for cones $\tau \in \Lambda$ of dimension $\geq p$ under the natural morphism $X' \to X$. Then any $y \in B$ is of codimension $\geq p$. In particular, every $y \in B \setminus \{x\}$ is not a generalization of $x$. Hence there exist $f_1, \ldots, f_r \in \mathcal{O}(X) \setminus p_x$ such that for any $y \in B \setminus \{x\}$, there exists $f_i$ contained in the prime ideal $p_y$ corresponding to $y$. We shall show that these $f_i$ are required ones. Let $\Xi$ be a sufficiently fine fan structure of $\text{Trop}((\varphi, (f_i)_i)(X) \cap G^m_{n+r})$. Let $\xi \in \Xi$ be a cone of dimension $\geq p$ contained in $\sigma \times \{0\}'$, and $\tau \in \Lambda$ its image under $\Xi \to \Lambda'$ given by the first projection. Suppose that $\dim \xi \geq p$ or $\xi$ does not intersect with $\text{rel.int}(\sigma \times \{0\}')$. Then the generic point of any irreducible component of $X' \cap O(\tau)$ does not map to $x$. Hence the product $f_1 \cdots f_r$ is 0 on $X' \cap O(\tau)$. Hence $f_1 \cdots f_r$ is 0 on $X'' \cap O(\xi)$, where $X''$ is the closure of $((\varphi, (f_i)_i)(X) \cap G^m_{n+r})$ in $T_\Xi$. This is a contradiction. \qed

Lemma 8.11. Let $p \geq 0$, $x \in X^{(p)}$, and $q \geq 1$. Then

$$H^{p+q}_{\text{Trop}, x}(X) = 0.$$

Proof. This follows from Corollary 8.9, Lemma 8.10, and the long exact sequences of relative tropical cohomology of pairs. \qed

Lemma 8.12. There are natural isomorphisms

$$H^{0}_{\text{Trop}, \eta}(X) \cong K^r_T(K(X))$$

and

$$H^{1}_{\text{Trop}, \eta}(X) \cong K^r_T(k(x)) \quad (x \in X^{(1)})$$
such that

\[
\begin{array}{ccc}
H^r_{\text{Trop},q}(X) & \overset{d_1}{\longrightarrow} & H^r_{\text{Trop},x}(X) \\
\downarrow & & \downarrow \\
K_T^r(K(X)) & \overset{\partial_0}{\longrightarrow} & K_T^{r-1}(k(x))
\end{array}
\]

is commutative.

**Proof.** The first isomorphism is given by Corollary 8.9. The second one is given as follows. (By construction, the diagram is commutative.) By Corollary 8.9 and Lemma 8.10, the tropical cohomology \( H^r_{\text{Trop},x}(X) \) is isomorphic to

\[
\lim_{\varphi} H^r_{\text{Trop},I_{\varphi}}(T_{\varphi}, \Lambda_{\varphi}),
\]

where \( \varphi: U_{\varphi} \to T_{\varphi} \) run through all closed immersions from open neighborhood \( U_{\varphi} \subset X \) of \( x \) to the toric varieties \( T_{\varphi} \) associated with 1-dimensional cones \( l_{\varphi} \subset N_{\varphi, \mathbb{R}} \) such that \( \varphi^{-1}(O(l_{\varphi})) = U_{\varphi} \cap \{ x \} \). (Here, the closure \( \overline{T_{\varphi}} \) is taken in \( \text{Trop}(T_{\varphi}) \) and \( \Lambda_{\varphi} \) is a fan structure of \( \text{Trop}(\varphi(U_{\varphi})) \).)

A map \( 
\lim_{\varphi} H^r_{\text{Trop},I_{\varphi}}(T_{\varphi}, \Lambda_{\varphi}) \to K_T^{r-1}(k(x))
\]

given by

\[
H^r_{\text{Trop},I_{\varphi}}(T_{\varphi}, \Lambda_{\varphi}) \ni \alpha_{\varphi} \mapsto (-1)^r (\alpha_{\varphi})_{\gamma_\varphi} (d_{\varphi} \wedge \cdot)
\]

is the required isomorphism, where \( d_{\varphi} \in l_{\varphi} \) is the primitive element, and \( \gamma_{\varphi}: [0,1] \to \overline{T_{\varphi}} \) is a fixed homeomorphism such that \( \gamma_{\varphi}(0) = 0 \in N_{\varphi, \mathbb{R}}. \)

\[
\square
\]

9. Étale excision

We prove étale excision (Proposition 8.1 (1)). Let \( \Phi: X' \to X \) be an étale morphism of smooth quasi-projective varieties over a trivially valued field \( K \). Let \( Z \subset X \) be a closed subscheme. We assume that \( \Phi \) induces an isomorphism \( Z' := \Phi^{-1}(Z) \cong Z \). By Corollary 8.9, it suffices to show that the natural morphism

\[
\mathcal{C}^{p,q}_{Z'^{\circ} \subset X'^{\circ}} \to \Phi_* \mathcal{C}^{p,q}_{Z^{\circ} \subset X^{\circ}}
\]

of sheaves on \( X^{\circ} \) is an isomorphism. We consider an open neighborhood \( X^{\circ} \setminus (X' \setminus Z')^{\circ} \) (resp. \( X^{\circ} \setminus (X' \setminus Z')^{\circ} \) of \( Z^{\circ} \) in \( X^{\circ} \) (resp. of \( Z' \) in \( X^{\circ} \)). Remind that by definition, \( X^{\circ} \setminus (X' \setminus Z)^{\circ} \) (resp. \( X^{\circ} \setminus (X' \setminus Z')^{\circ} \)) is the subset of \( X^{\text{Ber}} \) (resp. \( X^{\text{rBer}} \)) consisting of valuations whose centers are in \( Z \) (resp. \( Z' \)). Since \( \Phi: X' \to X \) is étale and \( Z' \cong Z \), by [Fu15] Lemma 2.8.5], the map

\[
X^{\circ} \setminus (X' \setminus Z')^{\circ} \to X^{\circ} \setminus (X' \setminus Z')^{\circ}
\]

is a homeomorphism. Hence it suffices to show that for any \( v' \in X^{\circ} \setminus (X' \setminus Z')^{\circ} \), the morphism

\[
\mathcal{C}^{p,q}_{Z^{\circ} \subset X^{\circ}, \Phi(v')} \to \mathcal{C}^{p,q}_{Z'^{\circ} \subset X'^{\circ}, v'}
\]

of stalks is an isomorphism. When \( v' \notin Z'^{\circ} \), the both hand sides are 0. We assume \( v' \in Z'^{\circ} \). Injectivity is obvious.

We shall show surjectivity. Let \( f_1, \ldots, f_r \in \mathcal{O}_{X^{\circ}, \text{supp}(v')} \) be elements of the stalks of the structure sheaf \( \mathcal{O}_{X^{\circ}} \).
Lemma 9.1. There exist \( g_1, \ldots, g_r \in \mathcal{O}_{X, \text{supp}(\Phi(v'))} \) such that
\[
\frac{f_i}{g_i} \in \mathcal{O}_{X', \text{supp}(w')}
\]
and
\[
w'(\frac{f_i}{g_i} - 1) > 0
\]
for any \( w' \) in an open neighborhood \( W' \) of \( v' \) in \( X'_{\text{Ber}} \).

Proof. This follows from the fact that the above conditions are open conditions, the morphism \( \Phi: X' \to X \) is étale, and \( Z' \cong Z \).

Hence we have
\[
\text{Trop} \circ (f_i)_i|_{W'} = \text{Trop} \circ (g_i)_i \circ \Phi|_{W'},
\]
here \( (f_i)_i \) and \( (g_i)_i \) are considered as morphisms to the affine space \( \mathbb{A}^r \). Consequently, since \( \{f_i\}_i \) are arbitrary, the morphism
\[
\mathcal{C}^{p,q}_{(Z \subset X'), \Phi(v')} \to \mathcal{C}^{p,q}_{(Z' \subset X')}\vphantom{v'}
\]
is surjective.

10. Analytifications and tropicalizations of the affine line

Let \((L, v_L: L^\times \to \mathbb{R})\) be a valuation field of height 1. We denote its completion by \((\hat{L}, v_L: \hat{L}^\times \to \mathbb{R})\). In this section, we study Berkovich’s and Huber’s analytifications of the affine line \( \mathbb{A}^1_{\hat{L}} = \text{Spec} \hat{L}[T] \) (Subsection 10.1) and their tropicalizations (Subsection 10.2). In Subsection 10.3 we study tropicalizations of \( \mathbb{A}^1 \)-fibers.

We fix an extension of \( v_L \) to an algebraic closure \( L_{\text{alg}} \). We also denote it by \( v_L \). We denote the residue field of \((L, v_L)\) by \( \kappa(L) \).

10.1. Analytifications of the affine line. We describe the structure of the analytifications of the affine line. See [Ber90, 1.4.4] and [Sch11, Example 2.20].

As a set, Huber’s analytification \( \mathbb{A}^{1,\text{ad}}_L \) is the set of equivalence classes of continuous valuations of 5 types. There is a canonical inclusion \( \mathbb{A}^{1,\text{Ber}}_L \subset \mathbb{A}^{1,\text{ad}}_L \) as sets. The Berkovich analytic space \( \mathbb{A}^{1,\text{Ber}}_L \) is identified with the set of points of type 1 – 4.

Definition 10.1. \( x \in \mathbb{A}^{1,\text{ad}}_L \) is said to be of type 1 when there exists \( a \in \hat{L}_{\text{alg}} \) such that \( x \) is defined by the composition
\[
\hat{L}[T] \xrightarrow{T=a} \hat{L}_{\text{alg}} \xrightarrow{v_L} v_L(\hat{L}_{\text{alg}}).
\]
In this case, we say that \( x \) corresponds to \( a \).

\( x \in \mathbb{A}^{1,\text{ad}}_L \) is said to be of type 2 (resp. of type 3) when there exists \( a \in \hat{L}_{\text{alg}} \) and \( r \in v_L((\hat{L}_{\text{alg}})^\times) \) (resp. \( r \in \mathbb{R} \setminus v_L((\hat{L}_{\text{alg}})^\times) \)) such that \( x \) is defined by the restriction of
\[
\hat{L}_{\text{alg}}[T] \ni \sum_i a_i(T-a)^i \mapsto \min_i \{v(a_i) + ir\}
\]
In this case, we say that \( x \) corresponds to \( (a, r) \).
Remark 10.2. Let $x \in \mathbb{A}_L^{1, \text{Ber}}$ be a point of type 1, 2, or 3 corresponding to $a \in \hat{L}^{\text{alg}}$ or $(a, r)$. We put $(x, \infty) \subset \mathbb{A}_L^{1, \text{Ber}}$ the set of valuations defined by

$$\hat{L}^{\text{alg}}[T] \ni \sum_i a_i(T - a)^i \mapsto \min_i \{v(a_i) + ir, ei\} \ (r' \in \mathbb{R}_{< r}),$$

where we put $r := \infty$ when $x$ is of type 1. The map

$$(x, \infty) \ni (\text{the valuation corresponding to } (a, r')) \mapsto r' \in \mathbb{R}_{< r}$$

is a homeomorphism. We put $[x, \infty] := \{x\} \cup (x, \infty)$, which is homeomorphic to $\{r\} \cup \mathbb{R}_{< r}$. 

Remark 10.3. For points $x, y \in \mathbb{A}_L^{1, \text{Ber}}$ of type 1, we have

$$[x, \infty) \cap [y, \infty) = [(\text{the valuation corresponding to } (a, \max_b \{b - a\})), \infty),$$

where $a \in \hat{L}^{\text{alg}}$ is a fixed element corresponding to $x$ and $b \in \hat{L}^{\text{alg}}$ runs through all element corresponding to $y$.

For each $x \in \mathbb{A}_L^{1, \text{Ber}}$, we put $\kappa(x)$ the residue field of the valuation $x$. By Remark 3.9 for a specialization $u$ of $w \in \mathbb{A}_L^{1, \text{ad}}$, we have $\kappa(w) \cap \kappa(L)^{\text{alg}} \subset \kappa(u)$. Note that $\kappa(u) = \kappa(u) \cap \kappa(L)^{\text{alg}}$.

Let $w \in \mathbb{A}_L^{1, \text{ad}}$ (resp. $u \in \mathbb{A}_L^{1, \text{ad}}$) be a point of type 2 or 3 (resp. of type 5). We put

$$\text{mpd}(w) := \min_{(a, r)} \hat{L}(a) : \hat{L}$$

(resp. $\text{mpd}(u) := \min_{(a, r, \epsilon)} \hat{L}(a) : \hat{L}$)
Let $L$ be a point of type 2 corresponding to $(a, r)$. We put $u_{w, \infty} \in \mathbb{A}^1_{\text{L, ad}}$ be the specialization of $w$ corresponding to $(a, r, -1)$. We also put $u_{w, 0} \in \mathbb{A}^1_{\text{L, ad}}$ the specialization of $w$ corresponding to $(0, r, 1)$ when $w$ corresponds to $(0, r)$, and put $u_{w, 0} := u_{w, \infty} \in \mathbb{A}^1_{\text{L, ad}}$ otherwise.

**Lemma 10.5.** [APZSS, Theorem 2.1 and Corollary 2.1] Let $w \in \mathbb{A}^1_{\text{L, ad}}$ be a point of type 2 corresponding to $(a, r)$, and $u \in \mathbb{A}^1_{\text{L, ad}}$ a specialization corresponding to $(a, r, \epsilon)$.

1. The residue field $\kappa(w)$ is isomorphic to the one variable rational function field over $\kappa(w) \cap \kappa(L)^{\text{alg}}$. In particular, by Remark 10.6, we have bijections

\[
\{\text{non-trivial specializations of } w \in \mathbb{A}^1_{\text{L, ad}}\} \\
\cong \{\text{equivalence classes of non-trivial valuations of } \kappa(w) \text{ which are trivial on } \kappa(L)\} \\
\cong \{\text{closed points of } \mathbb{P}^1_{\kappa(w) \cap \kappa(L)^{\text{alg}}}\}.
\]

2. Let $\mu := w$ or $u$. The reductions of polynomials $g \in \hat{L}[T]$ of degree $< \text{mpd}(\mu)$ with $\mu(g) = 0$ generate the multiplicative group $(\kappa(\mu) \cap \kappa(L)^{\text{alg}})^{\times}$, and we also have $v(g(a)) = 0$. This gives a canonical inclusion

\[
\kappa(L)^{\text{alg}} \subset \kappa(\hat{L}(a)).
\]

When $(a, r)$ or $(a, r, \epsilon)$ is minimal, this inclusion is an equality.

3. We have

\[
\kappa(w) \cap \kappa(L)^{\text{alg}} = \kappa(u_{w, \infty}) = \kappa(u_{w, 0}).
\]

**Proof.** [1] is [APZSS, Corollary 2.1].

[2] $\mu(g) = v(g(a))$ can be proved in the same way as proof of [APZSS, Theorem 2.1 a)]. The other parts of [2] can be proved in the same way as proof of [APZSS, Theorem 2.1 d)].

[3] follows [2] since when $(a, r)$ is minimal, $(a, r, -1)$ is also minimal.

For a specialization $u$ of $w$, we put $\overline{u}$ the corresponding valuation of $\kappa(w)$ in Lemma 10.5 [1].

**Lemma 10.6.** Let $w \in \mathbb{A}^1_{\text{L, ad}}$ be a point of type 2 corresponding to $(a, r)$, and $g \in \hat{L}[T]$ the minimal polynomial of $a \in \hat{L}^{\text{alg}}$. Let $b \in \hat{L}$ and $s \in \mathbb{Z}_{>1}$ be such that $w(bg^s) = 0$. We put $\overline{bg^s} \in \kappa(w)$ the reduction.

Then we have

\[
u_{w, \infty}(\overline{bg^s}) < 0, \\
\nu_{(a, r, 1)}(\overline{bg^s}) > 0, \\
\overline{u'(bg^s)} = 0,
\]

where $u_{(a, r, 1)}$ is the specialization of $w$ corresponding to $(a, r, 1)$, and $u' \neq u_{w, \infty}, u_{(a, r, 1)}$ is any specialization of $w$. 

Proof. The inequalities follow from the definition. Let $u'$ correspond to $(a', r, 1)$, and $g = \prod_i (T - a_i)$ ($a_i \in \hat{L}^{\mathrm{alg}}$). Then
\[
\begin{align*}
u'(g) &= \sum_i u'(T - a' + a' - a_i) \\ &= \sum_i \min \{(v(a' - a_i), 0), (r, 1)\} \\ &= \sum_i v(a' - a_i),
\end{align*}
\]
where the last equality follows from Remark 10.2. Hence $\overline{u'(bg^r)} = 0$. \hfill \Box

**Lemma 10.7.** Let $w \in \mathbb{A}^{1,\mathrm{ad}}_L$ be a point of type 2, and $u \in \mathbb{A}^{1,\mathrm{ad}}_L$ a specialization of it. Let $a \in \hat{L}^{\mathrm{alg}}$ be such that $w \notin [a, \infty)$, and $g \in \hat{L}[T]$ the minimal polynomial of it. Then we have $w(g) = u(g)$.

**Proof.** We may assume that $L = \hat{L}^{\mathrm{alg}}$. Then $u$ corresponds to $(b, r, \epsilon)$ for some $b \in L$ and $r > v(b - a)$. Then we have $w(T - a) = v(b - a) = u(T - a)$. \hfill \Box

### 10.2. Tropicalizations and skeletons of the affine line

We shall give explicit descriptions of tropicalizations and tropical skeletons of the affine line.

Let $\varphi: \mathbb{A}^1_L \to T_{\Sigma,L}$ be a closed immersion to the toric variety $T_{\Sigma,L}$ over $L$ corresponding to a fan $\Sigma$ such that $\varphi(\mathbb{A}^1_L)$ intersects with the dense open orbit $\mathbb{G}^n_m = \text{Spec } L[S_i^{\pm 1}]_1 \subset T_{\Sigma,L}$. Let $h_i := \varphi(S_i) \in L(T)$.

**Assumption 10.8.** There exists $r \leq n$ such that $f_i := h_i$ ($1 \leq i \leq r$) is a polynomial and $h_i$ ($l \geq r + 1$) is a quotient of products of copies of $f_i$ ($1 \leq i \leq r$).

We put $f_i = c_i \prod_j (T - a_{i,j})$ ($a_{i,j} \in \hat{L}^{\mathrm{alg}}$).

**Remark 10.9.** Let $a \in \hat{L}$, and $(T - a): \mathbb{A}^1_L \to \hat{L}$ a morphism. Then we have $\text{Sk}_{(T-a)} \mathbb{A}^{1,\mathrm{Ber}}_L = [a, \infty)$, and there is a retraction $\mathbb{A}^{1,\mathrm{Ber}}_L \to [a, \infty)$ giving a commutative diagram
\[
\begin{array}{ccc}
\mathbb{A}^{1,\mathrm{Ber}}_L & \xrightarrow{T \mapsto T - a} & \text{Trop}((T - a)) \\
\downarrow & & \downarrow \\
[a, \infty) & \xrightarrow{\cong} & (-\infty, \infty].
\end{array}
\]

**Lemma 10.10.** We assume Assumption 10.8 and $L = \hat{L}^{\mathrm{alg}}$. We assume that each $f_i$ is a polynomial $c_i(T - a_i)$ of degree 1. Then
\[
\text{Sk}_{(T-a)} \mathbb{A}^{1,\mathrm{Ber}}_L = \bigcup_i [a_i, \infty)
\]
and
\[
\text{Trop} \circ \varphi_L: \text{Sk}_{(T-a)} \mathbb{A}^{1,\mathrm{Ber}}_L \cong \text{Trop}(\varphi_L(\mathbb{A}^{1,\mathrm{Ber}}_L)).
\]

**Proof.** For each $x \in \text{Trop}(\varphi_L(\mathbb{A}^{1,\mathrm{Ber}}_L))$, the fiber $((\text{Trop} \circ \varphi_L)^{-1})(x)$ is a finite union of points of type 1 or type 3, or is isomorphic to $\mathcal{M}(\hat{L}(S, (S - b_l)^{-1}))$ for some $b_l \in \hat{L}$ with $v(b_l) = 0$, where $S$ is an indeterminant. (See [BGR85], Section 6.1) for the affinoid
algebra \( \hat{L}(S, (S - b_i)^{-1})_{i, j} \). In the latter case, its Shilov boundary is a singleton, which is
\[
(T \circ \varphi_{L})^{-1}(x) \cap \bigcup_i [a_i, \infty).
\]
(The other cases is trivial.) The second assertion is trivial. \( \square \)

**Remark 10.11.** In the situation of Lemma 10.10, let \( \mathbb{R} \) be a valuation such that \( v(c_i) = w(c_i), v(a_i) = w(a_i), v(a_i - a_j) = w(a_i - a_j) \) for any \( i, j \). Then by Remark 10.4, the indentifications of \( [a_i, \infty) \) in \( \mathbb{A}^1_{L_w} \) and those in \( \mathbb{A}^1_{L_w} \) induce a commutative diagram
\[
\begin{array}{ccc}
\text{Sk}_{\varphi_{L'}(\mathbb{A}^1_{L'})} & \cong & \text{Sk}_{\varphi_{L_w}(\mathbb{A}^1_{L_w})} \\
\text{Trop}(\varphi_{L'}(\mathbb{A}^1_{L'})) & \cong & \text{Trop}(\varphi_{L_w}(\mathbb{A}^1_{L_w})),
\end{array}
\]
where \( L_w \) is the completion of \( L \) with respect to \( w \).

**Lemma 10.12.** We assume Assumption 10.8. We put
\[
(\varphi_{L'}(\mathbb{A}^1_{L'})) : \mathbb{A}^1_{L'} \to T_{\Sigma, L_{\text{alg}}} \times (\mathbb{A}^1_{L_{\text{alg}}})^s.
\]
Then the natural map
\[
\text{Sk}_{\varphi_{L'}((T - a_{i,j}), i,j)}(\mathbb{A}^1_{L_{\text{alg}}}) \to \text{Sk}_{\varphi_{L'}}(\mathbb{A}^1_{L'})
\]
is surjective.

**Proof.** By [GRW17, Lemma 4.4], we have a surjection \( \text{Sk}_{\varphi_{L'}((T - a_{i,j}), i,j)}(\mathbb{A}^1_{L_{\text{alg}}}) \to \text{Sk}_{\varphi_{L'}}(\mathbb{A}^1_{L'}) \), hence we may assume that \( L = L_{\text{alg}} \). Since the natural map
\[
\phi: \text{Trop}(\varphi_{L'}((T - a_{i,j}), i,j)) \to \text{Trop}(\varphi_{L'}(\mathbb{A}^1_{L'}))
\]
is finite-to-one, for \( x \in \text{Trop}(\varphi_{L'}(\mathbb{A}^1_{L'})) \), the affinoid domain \( (\text{Trop}(\varphi_{L'}(\mathbb{A}^1_{L'})))^{-1}(y) \) is a disjoint union of affinoid domains \( (\text{Trop}(\varphi_{L'}))^{-1}(y) \). Hence we have a bijection \( \text{Sk}_{\varphi_{L'}((T - a_{i,j}), i,j)}(\mathbb{A}^1_{L'}) \cong \text{Sk}_{\varphi_{L'}}(\mathbb{A}^1_{L'}) \). \( \square \)

**Corollary 10.13.** We assume Assumption 10.8. We have
\[
\text{Sk}_{\varphi_{L'}}(\mathbb{A}^1_{L'}) = \bigcup_i [a_i, \infty).
\]

**Lemma 10.14.** We assume Assumption 10.8. There exists a morphism \( \psi: \mathbb{A}^1_{L'} \to (\mathbb{A}^1_{L})^l \) over \( L \) such that \( \text{Trop}(\varphi_{L'}(\mathbb{A}^1_{L'})) \) is injective on \( \text{Sk}_{\varphi_{L'}}(\mathbb{A}^1_{L'}) \).

**Proof.** We take monic irreducible polynomial \( g_{i,j} \in L[T] \) which is irreducible over \( \hat{L} \) and has a zero \( b_{i,j} \in L_{\text{alg}} \) sufficiently near to \( a_{i,j} \). We put \( \psi := (g_{i,j})_{i,j} \). By Remark 10.9 and Corollary 10.13 the assertion holds. \( \square \)

### 10.3 Tropicalizations of trivial line bundles
In this subsection, we describe tropicalizations of fibers of the projection \( \pi: X \times \mathbb{A}^1 \to X \) for algebraic variety \( X \) over a trivially valued field \( K \). Remind that we have
\[
X_{\text{Ber}}/(\text{the equivalence relations of valuations}) \cong X_{\text{ad,ht} \leq 1} \subset X_{\text{ad}}.
\]
Let \( v \in X_{\text{Ber}} \) be a valuation of height 1. We put \([v] \in X_{\text{ad}}\) its equivalence class.
Remark 10.15. There is a natural inclusion

\[ \mathbb{A}_{k(\text{supp}(v))}^{1,\text{ad}} \hookrightarrow \pi^{-1}([v]) \]

whose image is the subset consisting of specializations of valuations in \( \pi^{-1}([v]) \) of height 1, (i.e., specializations of images of \( \pi^{-1}(v) \)). We have the following commutative diagram:

\[
\begin{array}{c}
\mathbb{A}_{k(\text{supp}(v))}^{1,\text{Ber}} \\
\downarrow\
\mathbb{A}_{k(\text{supp}(v))}^{1,\text{ad}}
\end{array} \xrightarrow{\cong} \begin{array}{c}
\pi^{-1}(v) \\
\downarrow\
\pi^{-1}([v])
\end{array} \xrightarrow{\cong} \begin{array}{c}
(X \times \mathbb{A}_{1}^{1})^{\text{Ber}} \\
\downarrow\
(X \times \mathbb{A}_{1}^{1})^{\text{ad}}
\end{array},
\]

see [Ber90 Section 3.1] and [Tem15 Definition/Exercise 4.1.7.1]. We identify elements of the above sets under the above morphisms except for \( (X \times \mathbb{A}_{1}^{1})^{\text{Ber}} \rightarrow (X \times \mathbb{A}_{1}^{1})^{\text{ad}} \).

Let \( \varphi': X \times \mathbb{A}_{1}^{1} \rightarrow T_{\Sigma} \) be a closed immersion over \( K \). By abuse of notation, we put \( \varphi': \mathbb{A}_{1}^{1,\text{Ber}} \rightarrow T_{\Sigma, k(\text{supp}(v))} \) the morphism induced by a natural morphism

\[ \iota: \mathbb{A}_{1}^{1,\text{Ber}} \cong \pi^{-1}(\text{supp}(v)) \rightarrow X \times \mathbb{A}_{1}. \]

We have a commutative diagram

\[
\begin{array}{c}
\mathbb{A}_{k(\text{supp}(v))}^{1,\text{Ber}} \\
\downarrow\text{Trop} \circ \varphi'\
\text{Trop}(\varphi'((\pi^{-1}(v))))
\end{array} \xrightarrow{\cong} \begin{array}{c}
\pi^{-1}(v) \\
\downarrow\text{Trop} \circ \varphi'\
\pi^{-1}([v])
\end{array} \xrightarrow{\cong} \begin{array}{c}
(X \times \mathbb{A}_{1}^{1})^{\text{Ber}} \\
\downarrow\
(X \times \mathbb{A}_{1}^{1})^{\text{ad}}
\end{array}.
\]

The projection \( \pi: X \times \mathbb{A}_{1}^{1} \rightarrow X \) induces a surjective map \( \pi^\circ: (X \times \mathbb{A}_{1}^{1})^{\circ} \rightarrow X^{\circ} \). We will study a subset \( (\pi^\circ)^{-1}(v) \) of the fiber \( \pi^{-1}(v) \) for \( v \in X^{\circ} \). The canonical homeomorphism \( \pi^{-1}(v) \cong \mathbb{A}_{k(\text{supp}(v))}^{1,\text{Ber}} \) induces a homeomorphism

\[
(\pi^\circ)^{-1}(v) \cong \{ w \in \mathbb{A}_{k(\text{supp}(v))}^{1,\text{Ber}} \mid w(T) \geq 0 \}.
\]

We put

\[ \text{Sk}_{\varphi'}((\pi^\circ)^{-1}(v)) := \text{Sk}_{\varphi'}(\pi^{-1}(v)) \cap (\pi^\circ)^{-1}(v). \]

A picture of \( (\pi^\circ)^{-1}(v) \subset \mathbb{A}_{k(\text{supp}(v))}^{1,\text{Ber}} \).
11. \(A^1\)-HOMOTOPY INVARIANCE

In this section, we shall prove Proposition 8.1(2), i.e., \(A^1\)-homotopy invariance of tropical cohomology. By descriptions of \(F_p^p\) by tropical \(K\)-groups (Subsection 11.5), we reduce it to computations of tropical \(K\)-groups of residue fields of valuations in \(A^1,\text{ad}\). Subsection 11.7 and 11.8 are the main parts of the proof. In Subsection 11.2 and Subsection 11.6, we fix notations. In Subsection 11.3, we give technical conditions on tropicalizations.

Let \(X\) be a smooth quasi-projective variety over a trivially valued field \(K\). We put \(\pi: X \times A^1 \to X\) the first projection. Tropical \(K\)-groups are always over \(K\).

11.1. The goal of this section. By five lemma, to prove Proposition 8.1(2), we may assume that \(Z = X\). Proposition 8.1(2) follows from

\[
H^{p,q}_{\text{Trop},T}((X \times A^1)^o) \cong H^{p,q}_{\text{Trop},T}(X^o)
\]

by Remark 6.4 and Corollary 8.7. Since tropical cohomology groups are isomorphic to the sheaf cohomology groups of \(F^\ast\) (see Section 6.2), the Leray spectral sequence

\[
E_2^{p,q} = H^q(X^o, R^p\pi_\ast F^\ast T, (X \times A^1)^o)
\]

for \(\pi^0: (X \times A^1)^o \to X^o\) convergence to the tropical cohomology \(H^{r,p+q}_{\text{Trop},T}((X \times A^1)^o)\).

The isomorphism (11.1) follows from the following, which is the goal of this section.

**Proposition 11.1.** Let \(r \in \mathbb{Z}_{\geq 0}\).

1. We have

\[
R^i\pi_\ast F^r T, (X \times A^1)^o = 0
\]

for \(i \geq 1\).

2. The canonical morphism

\[
F^r T, X^o \to \pi_\ast F^r T, (X \times A^1)^o
\]

is an isomorphism.

In the rest of this section, we fix \(v_0 \in X^o\), and study stalks of sheaves at \(v_0\). We assume that \(v\) is of height 1. When \(v\) is of height 0, the proof is similar and much easier. We omit it.

11.2. Notations. In this subsection, we fix notations and give several assumptions. To compute \(F^p_{T,X^o,v_0}\) and \(R^q\pi_\ast F^p_{T,(X \times A^1)^o,v_0}\), we may assume that \(\text{supp}(v_0)\) is the generic point of \(X\) (by Lemma 11.12 and Lemma 11.10). We can shrink \(X\) to open subvarieties. We consider a commutative diagram

\[
\begin{array}{ccc}
X \times A^1 & \xrightarrow{\pi} & X \\
\downarrow \varphi' & & \downarrow \varphi \\
T_{A^1} & \xrightarrow{\Psi} & \mathbb{G}_m \\
\end{array}
\]

- \(\varphi\) and \(\varphi'\) are closed immersions such that \(\varphi'(X \times A^1)\) intersects with the dense open orbit,
- \(\Psi\) is a toric morphism.

We consider a quadruple \((\varphi, \varphi', \Psi, \Lambda')\) consisting of such a commutative diagram and a fan structure \(\Lambda'\) of \(\text{Trop}(\varphi'(X \times A^1))\) such that

\[
\Xi := \Lambda' \cap \text{Trop}(\varphi'((\pi^0)^{-1}(v_0))) := \{P \cap \text{Trop}(\varphi'((\pi^0)^{-1}(v_0)))\}_{P \in \Lambda'}
\]
is a polyhedral structure of \( \text{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))) \).

We put \( M \) the character lattice of the dense torus of \( T_{\Sigma'} \), \( N := \text{Hom}(M, \mathbb{Z}) \).

**Remark 11.2.** By retractions (Lemma [11.12](#) and Lemma [11.10](#)), every element of \( \mathcal{F}_{T,X}^p \) and \( R^q \pi_* \mathcal{F}_{T,(\mathbb{A}^1)^e}^p \) is represented by an element of \( H^q_{\text{Trop}}(\text{Trop}(\varphi(v_0)), \text{Trop}(\varphi(X))) \)

\[
Z^{p,q}(\text{Trop}(\varphi'((X \times \mathbb{A}^1)^{\circ})) \cap \Psi^{-1}(\text{Trop}(\varphi(v_0))), \Lambda')
\]

for some \( (\varphi, \varphi', \Psi, \Lambda') \).

**Definition 11.3.** We say that another quadruple \( (\varphi_1, \varphi'_1, \Psi_1, \Lambda'_1) \) (using an open subvariety \( Y \subset X \) instead of \( X \)) dominates \( (\varphi, \varphi', \Psi, \Lambda) \) if

- there exists a commutative diagram

\[
\begin{array}{ccc}
Y \times \mathbb{A}^1 & \xrightarrow{\pi} & Y \\
\downarrow{\varphi_1} & & \downarrow{\varphi} \\
T_{\Sigma_1} & \xrightarrow{\varphi_1} & G_m^{r_1} \\
\downarrow{\rho} & & \downarrow{\psi} \\
T_{\Sigma} & \xrightarrow{\psi} & G_m^{r'}
\end{array}
\]

where morphisms between toric varieties are toric morphisms, and

- for a cone \( P'_i \in \Lambda'_1 \), there exists a cone \( P' \in \Lambda' \) containing \( \rho(P'_i) \).

**(When we discuss dominations, we fix toric morphisms \( \rho: T_{\Sigma_1} \to T_{\Sigma} \) and \( G_m^{r_1} \to G_m^{r'} \).)**

The polyhedral complex \( \Xi \) is a graph, i.e., consisting of only 0 and 1-dimensional polyhedra, vertices and edges. To express an element of \( \Xi \) of dimension 0 (resp. of dimension 1), we use character \( l \) (resp. \( e \)) (sometimes, with subscripts). For simplicity, for example, we write \( "l \in \Xi" \) instead of \( "l \in \Xi \) of dimension 0". Note that for \( e \in \Xi \), we have rel.int(\( e \)) \( \subset N^e_\mathbb{R} \). For each \( \xi \in \Xi \), we put \( P_\xi \in \Lambda' \) the cone whose relative interior contains \( \text{rel.int}(\xi) \).

**11.3. Technical conditions.** In this subsection, we introduce conditions on \( (\varphi, \varphi', \Psi, \Lambda') \). In Subsection [11.8](#), we need to replace \( (\varphi, \varphi', \Psi, \Lambda') \) with another one dominating it under the process of induction while keeping conditions. We take \( (\varphi, \varphi', \Psi, \Lambda') \).

The composition

\[
\mathbb{A}^1_{K(X)} = \pi^{-1}(\text{Spec } K(X)) \xrightarrow{\varphi'} T_{\Sigma}
\]

is generically defined by finitely many rational functions \( h_i \in K(X)(T) \) \((1 \leq i \leq n)\).

**Condition 11.4.**

1. There exists a finite set \( \{a_i\}_i \subset O(X)^{\circ} \) such that \( \{v_0(a_i)\}_i \) is a \( \mathbb{Q} \)-basis of \( \mathbb{Q} \cdot \Gamma_{v_0} \).

2. \( \varphi: X \to G_m^{r_0} \) is of the form

\[
(\varphi_0, (a_i)): X \to G_m^{r_0} \cap Q^{\Gamma_{v_0}} \times G_m^{\dim_{Q} Q \cdot \Gamma_{v_0}},
\]

for some \( \varphi_0: X \to G_m^{r_0} \cap Q^{\Gamma_{v_0}} \).

3. We have \( \dim(P_1) = \dim_{Q} Q \cdot \Gamma_{v_0} \) \((l \in \Xi)\) and \( \dim(P_e) = \dim_{Q} Q \cdot \Gamma_{v_0} + 1 \) \((e \in \Xi)\).

4. \( \varphi': X \times \mathbb{A}^1 \to T_{\Sigma_0} \) is of the form

\[
(\varphi'_0, \text{pr}_{\mathbb{A}^1}): X \times \mathbb{A}^1 \to T_{\Sigma_0} \times \mathbb{A}^1
\]

for some toric variety \( T_{\Sigma_0} \) and \( \varphi'_0: X \times \mathbb{A}^1 \to T_{\Sigma_0} \), where \( \text{pr}_{\mathbb{A}^1}: X \times \mathbb{A}^1 \to \mathbb{A}^1 \) is the projection.
Note that Condition 11.4 (3) holds for sufficiently fine \( \Lambda' \) by Condition 11.4 (2). We identify \((\text{Spec } K(X)/K)^{\text{Ber}}\) and its image under the natural map 
\[(\text{Spec } K(X)/K)^{\text{Ber}} \hookrightarrow X^{\text{Ber}}.\]

Remind that for \( v \in X^{\text{Ber}} \), we have \( \pi^{-1}(v) \cong A^{1,\text{Ber}}_{k(\text{supp}(v))} \).

**Condition 11.5.**  
(1) Assumption 10.8 holds for the above \( h_i \).  
(2) The restriction of \( \text{Trop} \circ \phi' \) to \( \text{Sk}_{\phi'}((\pi^o)^{-1}(v_0)) \) is injective.  
(3) For a cone \( \sigma \in \Sigma' \), the image of each irreducible component of \( X \times \mathbb{A}^1 \cap \phi'^{-1}(O(\sigma)) \) under \( \pi: X \times \mathbb{A}^1 \to X \) contains the generic point of \( X \).  
(4) For any \( v \in (\text{Trop} \circ \phi)^{-1}(\text{Trop} \circ \phi)(v_0) \cap (\text{Spec } K(X)/K)^{\text{Ber}}, \)
we have 
\[ \text{Trop}(\phi'(\pi^o)^{-1}(v))) = \text{Trop}(\phi'(\pi^o)^{-1}(v_0))). \]
(5) For \( v \) as above, a vertex \( l \in \Xi, \)
\[ w \in \text{Sk}_{\phi'}((\pi^o)^{-1}(v)) \cap (\text{Trop} \circ \phi')^{-1}(l), \text{ and} \]
\[ w_0 \in \text{Sk}_{\phi'}((\pi^o)^{-1}(v_0)) \cap (\text{Trop} \circ \phi')^{-1}(l), \]
both \( w \) and \( w_0 \) are  
- of type 2 and correspond to the same \((a, r) \in (K(X)^{\text{alg}}, \mathbb{Q} \cdot \Gamma_{v_0}) \) when \( l \in N_R \), and  
- of type 1 and correspond to the same \( a \in K(X)^{\text{alg}} \) when \( l \notin N_R \), where \( a \in K(X)^{\text{alg}} \) is a zero or pole of some \( h_i \).

**Remark 11.6.**  
(1) By Corollary 10.13 and compactness of \((\pi^o)^{-1}(v_0)\), the tropical skeleton \( \text{Sk}_{\phi'}((\pi^o)^{-1}(v_0)) \) is compact. Since it is also simply connected, by Condition 11.5 (2), the set \( \text{Trop}(\phi'(\pi^o)^{-1}(v_0))) \) is simply connected, i.e., the graph \( \Xi \) is a tree.  
(2) By Lemma 3.10,
\[(\text{Spec } K(X)/K)^{\text{Ber}} \to \text{Trop}(\phi(X))\]
is surjective, and by Condition 11.5 (3),
\[\pi^{-1}((\text{Spec } K(X)/K)^{\text{Ber}}) \to \text{Trop}(\phi'(X \times \mathbb{A}^1))\]
is also surjective.  
(3) By (3) and Condition 11.5 (4), we have 
\[\text{Trop}(\phi'(\pi^o)^{-1}(v_0))) = \text{Trop}(\phi'(X \times \mathbb{A}^1)) \cap \Psi^{-1}(\text{Trop}(\phi(v_0))). \]

We shall use Remark 11.7 and Proposition 11.8 when we replace quadruples.

**Remark 11.7.** Let \((\phi, \phi', \Psi, \Lambda')\) be a quadruple satisfying Condition 11.4 and Condition 11.5. Then a quadruple 
\[((\phi, \varphi_1), (\phi', \varphi_1), \Psi \times \text{Id}_{\mathbb{G}^m_1}, \Lambda')\]
dominating it also satisfies Condition 11.4 and Condition 11.5 where \( \varphi_1: X \to \mathbb{G}^m_1 \) is a rational map, \( \Lambda' \) is a sufficiently fine fan structure.

**Proposition 11.8.** Let \((\phi, \phi', \Psi, \Lambda')\) be a quadruple.  
(1) There exists a quadruple dominating \((\phi, \phi', \Psi, \Lambda')\) and satisfying Condition 11.4 and Condition 11.5.
(2) We assume that \((\varphi, \varphi', \Psi, \Lambda')\) satisfies Condition \([11.4]\) and Condition \([11.5]\). Let \(f \in K(X)[T]\) be a polynomial irreducible over \(K(X)_{v_0}\). We consider \(f\) as a rational map \(X \times \mathbb{A}^1 \to \mathbb{A}^1\). Then there exists a quadruple of the form

\[ ((\varphi, \varphi_f), (\varphi', \varphi_f, f), \Psi_f, \Lambda'_f) \]

(for some rational map \(\varphi_f : X \to \mathbb{G}_m^r\)) dominating \((\varphi, \varphi', \Psi, \Lambda')\) and satisfying Condition \([11.4]\) and Condition \([11.5]\).

**Proof.** (1) Condition \([11.4]\) is easy. By Lemma \([10.14]\) there exist finitely many polynomials \(g_j \in K(X)[T]\) as a rational map

\[ (g_j) : X \times \mathbb{A}^1 \to \mathbb{A}^1 \]

such that a quadruple of the form

\[ ((\varphi, \varphi_1), (\varphi', \varphi_1, (g_j)_j), (\Psi \times \text{Id}_{\mathbb{G}_m^r}) \circ \text{pr}, \Lambda'_1) \]

(for some rational map \(\varphi_1 : X \to \mathbb{G}_m^r\) satisfies Condition \([11.5]\) (1) and (2) (and Condition \([11.4]\)), where

\[ \text{pr}: T_{\Sigma} \times \mathbb{G}_m^r \times \mathbb{A}^1 \to T_{\Sigma} \times \mathbb{G}_m^r \]

is the projection. Then by Lemma \([3.14]\), Remark \([10.11]\), Lemma \([10.12]\), Corollary \([10.13]\) and Condition \([11.5]\) (2), there exists \(\varphi_2 : X \to \mathbb{G}_m^r\) such that a quadruple of the form

\[ ((\varphi, \varphi_1, \varphi_2), (\varphi', \varphi_1, (g_j)_j, \varphi_2), ((\Psi \times \text{Id}_{\mathbb{G}_m^r}) \circ \text{pr}) \times \text{Id}_{\mathbb{G}_m^r}, \Lambda'_2) \]

dominates \((\varphi, \varphi', \Psi, \Lambda')\) and satisfies Condition \([11.4]\) and Condition \([11.5]\).

(2) By Corollary \([10.13]\) and Remark \([10.9]\) it is easy to see that a quadruple

\[ (\varphi, (\varphi', f), \Psi \circ \text{pr}, \Lambda'_f) \]

satisfies Condition \([11.4]\) and Condition \([11.5]\) (1) (2) (for some \(\Lambda'_f\)), where \(\text{pr}: T_{\Sigma} \times \mathbb{A}^1 \to T_{\Sigma}\) is the projection. Then in the same way as (1), we get the required quadruple. \(\square\)

We put \(v \in X_{\text{ad}}\) the equivalence class of \(v \in X_{\text{Ber}}\). Remind that \(\mathbb{A}_{K(X),v}^{\text{ad}} \subset \pi^{-1}([v])\) is the subset of specializations of valuations of height 1 in \(\pi^{-1}([v])\), i.e., specializations of \(\mathbb{A}_{K(X),v}^{\text{Ber}} \approx \pi^{-1}(v)\). (See Subsection \([10.3]\))

**Lemma 11.9.** Let \((\varphi, \varphi', \Psi, \Lambda')\) satisfy Condition \([11.4]\) and Condition \([11.5]\). Let \(v, l \in \Xi, w, u_0\) be as in Condition \([11.5]\) (4). We assume that \(l \in N_{\mathbb{R}}\). Let \(e \in \Xi\) be an edge with \(l \subset e\),

\[ u \in \{w\} \cap \pi^{-1}([v]) \cap (\text{Trop}_{e} \circ \varphi')^{-1}(P_e), \quad \text{and} \]

\[ u_0 \in \{w_0\} \cap \pi^{-1}([v_0]) \cap (\text{Trop}_{e} \circ \varphi')^{-1}(P_e). \]

Then both \(u\) and \(u_0\) correspond to the same \((a, r, \epsilon) \in (K(X)^{\text{alg}}, \mathbb{Q}, \Gamma_{v_0}, \{1, -1\})\).

**Proof.** By Condition \([11.5]\) (1) and Corollary \([10.13]\), the valuation \(w\) corresponds to \((a, r)\) for a zero \(a\) of some \(f_i\). By Lemma \([10.6]\) and Lemma \([10.7]\), we can take \(a\) such that \(u\) corresponds to \((a, r, \epsilon)\). By Condition \([11.5]\) (3), similarly, the valuation \(u_0\) also corresponds to \((a, r, \epsilon)\). \(\square\)
11.4. Retractions. We shall show the existence of retractions. Let \((\varphi, \varphi', \Psi, \Lambda')\) be a quadruple.

Lemma 11.10. \([\text{JSS17, Proposition 3.11}]\) For a sufficiently small polyhedron \(A \subset \text{Trop}(T_\Sigma)\) which is a neighborhood of \(\text{Trop}(\varphi(v_0))\), we have
\[
H^p_{\text{Trop}}(\text{rel. int.}(A) \cap \text{Trop}(\varphi(X)), \text{Trop}(\varphi(X))) \\
\cong H^p_{\text{Trop}}(\text{Trop}(\varphi(v)), \text{Trop}(\varphi(X))).
\]

Lemma 11.11. For a sufficiently small polyhedron \(A \subset \text{Trop}(T_\Sigma)\) which is a neighborhood of \(\text{Trop}(\varphi(v_0))\) and a cone \(P \in \Lambda'\) intersecting with \(\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(A)\), the intersection
\[
P \cap \text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(\text{Trop}(\varphi(v_0)))
\]
is nonempty.

Proof. By a commutative diagram
\[
\begin{array}{ccc}
(X \times A^1)^\circ & \xrightarrow{\pi^o} & X^o \\
\downarrow \text{Trop} \circ \varphi' & & \downarrow \text{Trop} \circ \varphi \\
\text{Trop}(\varphi'((X \times A^1)^\circ)) & \xrightarrow{\Psi|_{\text{Trop}(\varphi'((X \times A^1)^\circ))}} & \text{Trop}(\varphi(X^o)),
\end{array}
\]
the restriction
\[
\Psi|_{\text{Trop}(\varphi'((X \times A^1)^\circ))}: \text{Trop}(\varphi'((X \times A^1)^\circ)) \to \text{Trop}(\varphi(X^o))
\]
of \(\Psi\) is proper. \((\pi^o\) and \(\text{Trop} \circ \varphi\) are proper, and \(\text{Trop} \circ \varphi'\) is continuous.) Hence
\[
\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(A)
\]
is compact. Hence its intersection with \(P\) is also compact. Since \(A\) is sufficiently small, the assertion holds.

Lemma 11.12. For a polyhedron \(A\) as in Lemma 11.11, the pull-back map
\[
H^p_{\text{Trop}}(\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(\text{rel. int.}(A)), \Lambda') \\
\to H^p_{\text{Trop}}(\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(\text{Trop}(\varphi(v_0))), \Lambda')
\]
is an isomorphism.

Proof. By Lemma 11.11 there exists a strong deformation retraction \(\psi\) of
\[
\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(\text{rel. int.}(A))
\]
onto
\[
\text{Trop}(\varphi'((X \times A^1)^\circ)) \cap \Psi^{-1}(\text{Trop}(\varphi(v_0)))
\]
which preserving the polyhedral structure, and this gives the isomorphism.

By Lemma 11.12 we have the following.

Corollary 11.13. \((R^q\pi_* \mathcal{F}^p_{T,(X \times A^1)^o})_v = 0\)

for \(q \geq 2\).

Hence what we need to compute are \((R^q\pi_* \mathcal{F}^p_{T,(X \times A^1)^o})_v\) for \(q = 0, 1\).
11.5. **Explicit descriptions of** $F^p$. We describe $F^p$ (see Subsection 6.1 for definition) and maps between them. Let $(\varphi, \varphi', \Psi, \Lambda')$ satisfy Condition 11.4 and Condition 11.5 (by Proposition 11.8).

We shall define $\overline{F}^p$. For a cone $P \in \Lambda'$, we put

$$\overline{F}^p(P) := F_p(P)/\text{Im}(\text{Span}(P) \otimes F_{p-1}(P) \to F_p(P)),$$

where

$$\text{Span}(P) \otimes F_{p-1}(P) \to F_p(P)$$

is a natural morphism, and $\overline{F}^p(P)$ the quotient of the $p$-th wedge product of

$$(M \cap \sigma'_p \cap P^\perp)_Q := \{m \in M \cap \sigma'_p | n(m) = 0 \ (n \in \text{Span}(P)) \} \otimes \mathbb{Z} \mathbb{Q}$$

by the $\mathbb{Q}$-vector subspace generated by elements which are trivial on $\overline{F}^p(P)$, where $\sigma'_p \in \Sigma'$ is the cone such that rel.int$(P) \subset \text{Trop}(O(\sigma'_p))$. We have

$$\overline{F}^p(P) \cong \text{Hom}_\mathbb{Q}(\overline{F}^p(P), \mathbb{Q}).$$

Remind that by Condition 11.4 (1) and (2), a finite set $\{a_j\}_j \subset O(X)^\times$ span $\mathbb{Q} \cdot \Gamma_{v_0}$ and

$$\varphi = (\varphi_0, (a_i)_i): X \to G_m^{p - \dim_\mathbb{Q} \mathbb{Q} \cdot \Gamma_{v_0} \times G_m^\dim_\mathbb{Q} \mathbb{Q} \cdot \Gamma_{v_0}}.$$

By abuse of notation, we denote the $i$-th coordinate function on $\text{Trop}(G_m^{\dim_\mathbb{Q} \mathbb{Q} \cdot \Gamma_{v_0}})$ by $a_i$. For $l \in \Xi$, by Condition 11.4 (3), we have an isomorphism

$$\bigoplus_{k=0}^p \wedge^{p-k}(a_j)_{j,Q} \otimes \overline{F}^k(P_l) \cong F^p(P_l)$$

of $\mathbb{Q}$-vector spaces given by $\alpha \otimes \beta \mapsto \alpha \wedge \beta$. For $l, e \in \Xi$ such that $l \in e \cap N_R$, we fix an element $g_{l,e} \in M \cap P_l^\perp \setminus M \cap P_e^\perp$. Then by Condition 11.4 (3), we have an isomorphism

$$\bigoplus_{k=0}^p \wedge^{p-k}(a_j)_{j,Q} \otimes \overline{F}^k(P_e) \oplus \bigoplus_{k'=1}^p \wedge^{p-k'}(a_j)_{j,Q} \otimes \mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k'-1}(P_e) \cong F^p(P_e)$$

of $\mathbb{Q}$-vector spaces given by

$$\alpha \otimes \beta + \alpha' \otimes g_{l,e} \otimes \beta' \mapsto \alpha \wedge \beta + \alpha' \wedge g_{l,e} \wedge \beta'.$$

The natural map $F^p(P_l) \to F^p(P_e)$ is given by the morphism

$$\iota_{l,e}: \overline{F}^k(P_l) \to \overline{F}^k(P_e) \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F}^{k-1}(P_e))$$

given by a decomposition

$$(M \cap P_l^\perp)_Q = \mathbb{Q} \cdot g_{l,e} \oplus (M \cap P_e^\perp)_Q.$$

**Remark 11.14.** For $P \in \Lambda'$, we have

$$\overline{F}^p(P) \cong F^p(0_{\text{Trop}(O(P \cap N_{\sigma'_p,R})))}, \text{Trop}(\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_p) \cap O(P \cap N_{\sigma'_p,R}))),$$

where the closure $\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_p)$ of $\varphi'(X \times \mathbb{A}^1) \cap O(\sigma'_p)$ is taken in the affine toric variety $T_{\mathbb{P}^n \cap N_{\sigma'_p,R}}$ corresponding to the cone $P \cap N_{\sigma'_p,R}$, and $0_{\text{Trop}(O(P \cap N_{\sigma'_p,R})))}$ is the zero in the tropicalization

$$\text{Trop}(O(P \cap N_{\sigma'_p,R})) \cong N_{\sigma'_p,R}/\text{Span} \mathbb{P}^n \otimes \mathbb{R}$$

and
of the closed orbit $O(P \cap N_{\sigma_P^{1,\text{ad}}})$ of $T_{P \cap N_{\sigma_P^{1,\text{ad}}}}$. In particular, for sufficiently fine $\Lambda'$ and $x \in \text{Trop}^{\text{ad}}(\varphi'(X \times A^1)^{\text{ad}})$ whose image in $\Lambda'$ is $P$, by Lemma 5.10, we have

$$F^k(P) = \wedge^p(M \cap \sigma_P^1 \cap P^\perp_{\varphi}) \big{/} \bigcup_{v \in (\text{Trop}^{\text{ad}} \circ \varphi)^{-1}(x)} \text{Ker}(\wedge^p(M \cap \sigma_P^1 \cap P^\perp_{\varphi}) \rightarrow K^k_T(\kappa(v)))$$

and

$$F^k(P_e) = \wedge^k(M \cap P^\perp_{\varphi}) \big{/} \bigcup_{u, v \in \text{Trop}^{\text{ad}}(\varphi')^{-1}(l)} \text{Ker}(\wedge^k(M \cap P^\perp_{\varphi}) \rightarrow K^k_T(\kappa(v))).$$

Lemma 11.15. We assume that $\Lambda'$ is sufficiently fine. For $l \in \Xi$ (resp. $l, e \in \Xi$ such that $l \in e \cap N_{\Xi}$), we have

$$F^k(P_l) \cong \wedge^k(M \cap \sigma_l^1 \cap P_l^\perp) \big{/} \bigcup_{v \in (\text{Trop}^{\text{ad}} \circ \varphi')^{-1}(\text{supp}(\varphi_v))) \cap (\text{Trop}^{\text{ad}} \circ \varphi')^{-1}(\text{Spec} K(X)/K)^{\text{Ber}}} \text{Ker}(\wedge^k(M \cap \sigma_l^1 \cap P_l^\perp) \rightarrow K^k_T(\kappa(v))).$$

(resp. $F^k(P_e) \cong \wedge^k(M \cap P_e^\perp) \big{/} \bigcup_{u, v \in (\text{Trop}^{\text{ad}} \circ \varphi')^{-1}(l)} \text{Ker}(\wedge^k(M \cap P_e^\perp) \rightarrow K^k_T(\kappa(u))).$

where $w$ runs through

$$w \in (\text{Trop} \circ \varphi)^{-1}(l) \cap \bigcup_{v \in (\text{Trop} \circ \varphi)^{-1}(\text{Trop}(\varphi(v))) \cap (\text{Trop}^{\text{ad}} \circ \varphi')^{-1}(\text{Spec} K(X)/K)^{\text{Ber}}} \text{Sk}_{\varphi'}(\pi^o)^{-1}(v).$$

(resp. $w$ runs through the above set and $u$ runs through all specializations of $w$ in $A_{k(\text{supp}(\varphi_v))) \cap (\text{Trop}^{\text{ad}} \circ \varphi')^{-1}(l).$

By Lemma 5.13 and Lemma 5.14, there exists an element of $\text{Sk}_{\varphi'}(\pi^o)^{-1}(l)$ whose image under $\varphi'(X \times A^1)^{\text{ad}} \rightarrow (X \times A^1)^{\text{ad}}$ is $w$. We also denote it by $w$. Then by Lemma 5.12 and Condition 11.5, we have $\kappa(w) \in \text{Spec} K(X)/K^{\text{Ber}}$. By definition of skeletons, the center of $w$ is the generic point of an irreducible component of $\varphi'(X \times A^1)^{\text{ad}} \cap O(\sigma_P^1) \cap O(P \cap N_{\sigma_P^{1,\text{ad}}})).$

Hence by Remark 12.7, we have $w \in \text{Sk}_{\varphi'}(\pi^o)^{-1}(\kappa(w))$.

Next we consider the case of $l, e$. Let $l \in \text{rel.int.} e. \text{ Consider}$

$$(l, l_e) \in (\text{Trop}^{\text{ad}}(\varphi'(X \times A^1)^{\text{ad}}))$$

via the bijection in Lemma 5.4. Then it maps to $P_e \in \Lambda'$. Let

$$u \in \text{Sk}_{\varphi'}(\pi^o)^{-1}((l, l_e))$$

Then by Lemma 5.12

$$u \in ZR(K(X))(T/K) \subset (X \times A^1)^{\text{ad}}.$$

By Remark 3.9, there exists a generalization $w \in ZR(K(X))(T/K)$ which maps to $l$. Since $u \in \text{Sk}_{\varphi'}(\pi^o)^{-1}(\kappa(w))$, we also have $w \in \text{Sk}_{\varphi'}((X \times A^1)^{\text{ad}}).$ Hence by the case of $l$, we have $w \in \text{Sk}_{\varphi'}(\pi^o)^{-1}(\kappa(w)).$ By Lemma 5.14, we have $\text{rank} \Gamma_u = \text{rank} u(M)$ and $\text{rank} \Gamma_w = \text{rank} w(M)$, hence
rank $\Gamma_{\pi(u)} = \text{rank} \Gamma_{\pi(w)}$. Since $\pi(u)$ is a specialization of $\pi(w) \in \text{ZR}(K(X)/K)$, by Remark 3.9, we have $\pi(u) = \pi(w)$. Hence the assertion holds. \(\square\)

For a specialization $u \in \pi^{-1}(\pi(w))$ of $w \in \text{ZR}(K(X)(T)/K)$, we put $\overline{u}$ the normalized discrete valuation of $\kappa(w)$ corresponding to $u$ in the sense of Lemma 10.5 (2). We denote the first (resp. the second) projection by $\text{pr}_1$ (resp. $\text{pr}_2$).

**Corollary 11.16.** We assume that $\mathcal{N}$ is sufficiently fine. Let $l, e, w, u$ be as in Proposition 11.15, $\zeta \in \Xi$ such that $\zeta \in e \setminus N_\mathbb{R}$, and

\[ \mu \in (\text{Trop} \circ \varphi)^{-1}(\zeta) \cap \text{Sk}_\varphi(\pi^0)^{-1}(\pi(w)) \]

such that $\mu$ corresponds to a and $u$ corresponds to $(a, r, \epsilon)$ for some $a \in K(X)_{\text{alg}}$ ($r$, and $\epsilon \in \{1, -1\}$). We have commutative diagrams

\[
\begin{align*}
\overline{F^k(P_l)} \xrightarrow{\overline{\ell_{l,e}}} \overline{F^k(P_e)} \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F^k(P_e)}) & \xrightarrow{\text{pr}_1} \overline{F^k(P_e)} \\
\quad K^k_T(\kappa(w)) \xrightarrow{\overline{w}_1(g_{l,e}) - \partial a} & \overline{K^k_T(\kappa(u))},
\end{align*}
\]

\[
\begin{align*}
\overline{F^k(P_l)} \xrightarrow{\overline{\ell_{l,e}}} \overline{F^k(P_e)} \oplus (\mathbb{Q} \cdot g_{l,e} \otimes \overline{F^k(P_e)}) & \xrightarrow{\text{pr}_2} \mathbb{Q} \cdot g_{l,e} \otimes \overline{F^k(P_e)} \\
\quad K^k_T(\kappa(w)) \xrightarrow{\partial a} & \overline{K^{k-1}_T(\kappa(u))},
\end{align*}
\]

\[
\begin{align*}
\overline{F^k(P_l)} \xrightarrow{\overline{\ell_{l,e}}} \overline{F^k(P_e)} \\
\quad K^k_T(\kappa(w)) \xrightarrow{\partial a} & \overline{K^k_T(\kappa(u))},
\end{align*}
\]

where $\overline{g_{l,e}} \in \kappa(w)$ is the image of $g_{l,e}$ under $M \cap P^+_l \to \kappa(w)$, we put

\[
\frac{1}{\overline{u}(g_{l,e})} \overline{w}_1(g_{l,e})(a) := \frac{1}{\overline{u}(g_{l,e})} \partial a(g_{l,e} \wedge a),
\]

the second vertical arrow in (11.3) is defined by

\[
1 \cdot g_{l,e} \otimes a \mapsto \overline{u}(g_{l,e}) a,
\]

and $\overline{K^k_T(\kappa(u))} \to \overline{K^k_T(\kappa(\mu))}$ is induced from inclusion $\kappa(u) \subset \kappa(\mu)$ (Lemma 10.5 (2)).

**Corollary 11.17.** For $\zeta, e$ as in Corollary 11.16 the map $\overline{F^k(P_\zeta)} \to \overline{F^k(P_e)}$ is injective.

**Lemma 11.18.** There exists an affinoid subdomain $U$ of $X^\text{Ber}$ containing $v_0$ such that for

- $(l, w)$ (resp. $(l, e, w, u)$) as in Lemma 11.15 such that $\pi(w) \in U \cap (\text{Trop} \circ \varphi)^{-1}(\text{Trop}(\varphi(v_0))) \cap (\text{Spec} K(X)/K)^\text{Ber}$ and $l$ is contained in at least 3 edges or only 1 edge, and
- $(l, w_0)$ (resp. $(l, e, w_0, u_0)$) as in Lemma 11.15 such that $\pi(w_0) = v_0$, for
we have

$$\text{Ker}(\wedge^k(M \cap \sigma_P \cap P_l^\perp)_Q \to \wedge^k(\kappa(w_0)^\times)_Q) \subset \text{Ker}(\wedge^k(M \cap \sigma_P \cap P_l^\perp)_Q \to \wedge^k(\kappa(w)^\times)_Q)$$

(resp.

$$\text{Ker}(\wedge^k(M \cap P_e^\perp)_Q \to \wedge^k(\kappa(u_0)^\times)_Q) \subset \text{Ker}(\wedge^k(M \cap P_e^\perp)_Q \to \wedge^k(\kappa(u)^\times)_Q).$$

**Proof.** It suffices to show that for any $$f \in K(X)[T]$$, there exists an affinoid subdomain $$U_f$$ of $$X^{\text{Ber}}$$ containing $$v_0$$ such that for

$$v \in U_f \cap (\text{Trop} \circ \phi)^{-1}(\text{Trop}(\phi(v_0))) \cap (\text{Spec} K(X)/k)^{\text{Ber}}$$

and $$(l, w, w_0)$$ (resp. $$(l, e, w, w_0, u, u_0)$$) as above such that $$f \in m_{w_0}$$ (resp. $$f \in m_{w_0}$$), we have $$f \in m_{w}$$ (resp. $$f \in m_{w}$$). The existence of such a $$U_f$$ follows from Lemma 3.14 and Condition 11.5 (resp. Lemma 3.14 and Lemma 11.9).

Let $$\phi_1: X \to \mathbb{G}_m^m$$ be a rational map such that

$$(\text{Trop} \circ \phi_1)^{-1}(\text{Trop}(\phi_1(v_0)))$$

is contained in $$U$$ in Lemma 11.18 and $$((\phi, \phi_1), (\phi', \phi_1), \Psi \times \text{Id}_{G_m^m}, \Lambda')$$ a quadruple dominating $$(\phi, \phi', \Psi, \Lambda)$$ satisfying Condition 11.4 and Condition 11.5 (by Remark 11.7). We put

$$\Xi_1 := \Lambda'_1 \cap \text{Trop}((\phi', \phi_1)((\pi^0)^{-1}(v_0))).$$

Since

$$\text{Trop}((\phi', \phi_1)((\pi^0)^{-1}(v_0))) \to \text{Trop}(\phi'((\pi^0)^{-1}(v_0)))$$

is bijective, the tree $$\Xi_1$$ can be considered as a subdivision of $$\Xi$$. For a pair $$l, e \in \Xi$$ with $$l \in e$$, we put $$e_l \in \Xi_1$$ the edge determined by $$l \in e_l \subset e$$. (For $$\zeta, e \in \Xi$$, we define $$e_\zeta$$ similarly.) We put $$P_{s,1} \in \Lambda'_1$$ the unique cone whose relative interior containing $$* = l, \zeta, e_l, e_\zeta$$. For $$* = l$$ or $$\zeta$$ and $$e$$ containing it, we have natural maps

$$\overline{F}^k(P_s) \to \overline{F}^k(P_{s,1})$$

and

$$\overline{F}^k(P_e) \to \overline{F}^k(P_{e,1}).$$

Remind that by construction, the residue homomorphism $$\partial_{w_0}: K^k_T(\kappa(w_0)) \to K^{k-1}_T(\kappa(u_0))$$ has a natural lifting

$$\partial_{w_0}: \wedge^k(\kappa(w_0)^\times)_Q \to \wedge^{k-1}(\kappa(u_0)^\times)_Q.$$
commutative diagrams

\[
\begin{align*}
\text{(11.5)} & \quad \text{Im}(\wedge^k(M \cap P^+_1)_{\mathbb{Q}}) \to \wedge^k(M \cap P^+_1)_{\mathbb{Q}} \\
& \quad \downarrow \quad \downarrow \\
& \quad \overline{F}^k(P_{t,1}) \quad \overline{F}^k(P_{e,1}),
\end{align*}
\]

\[
\begin{align*}
\text{(11.6)} & \quad \text{Im}(\wedge^k(M \cap P^+_1)_{\mathbb{Q}}) \to \wedge^k(M \cap P^+_1)_{\mathbb{Q}} \\
& \quad \downarrow \quad \downarrow \\
& \quad \overline{F}^k(P_{t,1}) \quad \overline{F}^k(P_{e,1}),
\end{align*}
\]

where the second vertical arrow in (11.6) is defined by

\[
a \to \frac{1}{u_0(g_{t,e})} \cdot g_{t,e} \otimes a.
\]

**Proof.** This follows from Lemma 11.15, Corollary 11.16 and Lemma 11.18. \qed

### 11.6. Notations for inductions.

In this section, we give several notations used in the inductions in Subsection 11.7 and 11.8. Let \((\varphi, \varphi', \Psi, \Lambda')\) be a quadruple satisfying Condition 11.4 and Condition 11.5.

Let \(B\) be the edge such that \(B(K(\mathcal{X})_{w_\mathbb{Q}}(T)) = (\pi^n)^{-1}(v_0)\). By abuse of notation, we put \(0 := \text{Trop}(\varphi'(0))\), \(B := \text{Trop}(\varphi'(B))\).

Let \(x \in A_{K(K(\mathcal{X})_{w_\mathbb{Q}})}^1\) be a point of type 1 or 2, we put \([0, x] \in A_{K(K(\mathcal{X})_{w_\mathbb{Q}})}^1\), be the minimal connected subset containing 0 and \(x\), i.e.,

\([0, x] := (0, \infty) \cup [x, \infty) \setminus ((\text{the valuation corresponding to } (0, x(T)))\). \(\infty)\).

For \(l \in \Xi\) and \(pt \in \text{Trop}(\varphi'(\pi^n)^{-1}(v_0)))\), we put

- \(w_{pt,0}\) the unique element of

\[\text{Sk}_{\varphi'}(\pi^n)^{-1}(v_0) \cap (\text{Trop} \circ \varphi')^{-1}(pt),\]

- \(e_{l,\infty} \in \Xi\) the edge such that \(l \in e_{l,\infty} \subset \text{Trop}(\varphi'([w_{l,0}, \infty)))\) when \(l \neq B\),

- \(e_{l,0} \in \Xi\) the edge such that \(l \in e_{l,0} \subset \text{Trop}(\varphi'([0, w_{l,0}]))\) when \(l \neq 0\),

- \(\text{mpd}(pt) := \text{mpd}(w_{pt,0})\),

- \(L_0(l)\) the number of edges \(e \in \Xi\) contained in \(\text{Trop}(\varphi'(\{0, w_{l,0}\}))\),

- \(L_\infty(l)\) the number of edges \(e \in \Xi\) contained in \(\text{Trop}(\varphi'([w_{l,0}, \infty)))\), and

- \(L_{\infty,\text{mpd}}(pt)\) the number of \(l' \in \Xi\) such that

  - \(l'\) is contained in \(\text{Trop}(\varphi'([w_{pt,0}, \infty)))\),

  - \(l'\) is contained in at least 3 edges or only 1 edge in \(\Xi\), and
Example 11.20.

A picture of $\text{Sk}_{\varphi}(\pi^\circ)^{-1}(v_0) \cong \text{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))$ (in the circle)

|   | $l_1$ | $l_2$ | $l_3$ |
|---|---|---|---|
| $L_0$ | 4 | 4 | 2 |
| $L_\infty$ | 1 | 3 | 3 |
| mpd | 1 | 2 | 1 |
| $L_{\infty,\text{mpd}}$ | 1 | 2 | 4 |

Here (mpd = 1)-locus is expressed by fine lines, and (mpd = 2)-locus is expressed by fat lines.

11.7. Proof of $(\pi^\circ_* \mathcal{F}^p_{T,(X \times \mathbb{A}^1)^0})_{v_0} \cong \mathcal{F}^p_{T,X^0,v_0}$. In this subsection, we shall prove the natural map

$$\mathcal{F}^p_{T,X^0,v_0} \rightarrow (\pi^\circ_* \mathcal{F}^p_{T,(X \times \mathbb{A}^1)^0})_{v_0}$$

is an isomorphism. It is injective since the composition

$$\pi \circ s_0: X \rightarrow X \times \mathbb{A}^1 \rightarrow X$$

is the identity map, where $s_0: X \rightarrow X \times \mathbb{A}^1$ is the section at 0. We shall show surjectivity. By Proposition 11.8 (1), it suffices to show that for a quadruple $(\varphi, \varphi', \Psi, \Lambda')$ satisfying Condition 11.4 and Condition 11.5 (1) and

$$\alpha = (\alpha_{pt})_{pt \in \text{Trop}(\varphi'((\pi^\circ)^{-1}(v_0)))} \in Z^{p,0}(\text{Trop}(\varphi'((\pi^\circ)^{-1}(v_0))), \Lambda')$$

such that $\alpha_0 = 0$, we have $\alpha = 0$. We fix such a $\alpha$. We may assume that $\Lambda'$ is sufficiently fine.

For each $e \in \Xi$, we put $\text{Span}(e) := Q \cdot (x_2 - x_1)$, where $x_1 \neq x_2 \in \text{rel.int}(e)$. We denote the image of

$$\text{Span}(e) \otimes F_{p-1}(P_e) \rightarrow F_p(P_e)$$

by $\text{Span}(e) \wedge F_{p-1}(P_e)$. Note that in the situation of Lemma 11.9, we have $\text{Trop}^{\text{ad}}(\varphi'(u)) \neq P_{e_{1,\infty}}$ when $\epsilon = 1$ (by Lemma 10.6 and Condition 11.5 (1)).

Remind that for a valued field $(L,v)$, we put $L_v$ the completion.

Lemma 11.21. Let $l \in \Xi$ with $l \in N_\mathbb{R} \setminus B$. We assume that

$$\alpha_l(\text{Span}(e) \wedge F_{p-1}(P_e)) = 0$$

for any $e \in \Xi \setminus \{e_{1,\infty}\}$ containing $l$.

Then we have

$$\alpha_l(\text{Span}(e_{1,\infty}) \wedge F_{p-1}(P_{e_{1,\infty}})) = 0.$$
Proof. Let $w$ be as in Lemma \[11.15\] for $l$. Since tropical $K$-groups form a cycle module (Theorem \[7.8\]), by Lemma \[10.5\] \[1\] and Ros96 Proposition 2.2, we have an inclusion

$$\text{Ker}(K^k_T(\kappa(w))) \xrightarrow{(\partial_\nu)_{u \neq u_{w,\infty}}} \bigoplus_{u \neq u_{w,\infty}} K^{-1}_T(\kappa(u)) \subset \text{Ker}(K^k_T(\kappa(w))) \xrightarrow{(\partial_\nu)_{u_{w,\infty}}} K^{-1}_T(\kappa(u_{w,\infty}))$$

where $u \in K^1_{k(supp(\pi(w)))_{e(w)}}$ runs through all non-trivial specializations of $w$ except for $u_{w,\infty}$ (defined in Subsection \[10.1\]). Then the assertion follows from Lemma \[11.15\] and Corollary \[11.16\] \[11.3\].

\[\square\]

Lemma \[11.22\]. We have

$$\alpha_l(\text{Span}(e) \wedge F_{p-1}(P_l)) = 0$$

for $l, e \in \Xi$ with $l \subset e$.

Proof. We shall prove the assertion by induction on $L_\infty(l) \in \mathbb{Z}_{\geq 0}$ in the descending order. When $L_\infty(l)$ is the maximal, or more generally, when $l$ is contained in only one edge $e_{l,\infty}$, the image of $\text{Span}(e_{l,\infty}) \subset N_R$ to

$$\text{Trop}(O(\sigma'_P)) = N_R/(\text{Span } \sigma'_P)_R,$$

which contains $l$, is 0. Hence the assertion holds. When $l$ is contained in at least 2 edges, the assertion follows from Lemma \[11.21\] and the assumption of induction. \[\square\]

Lemma \[11.23\]. For any $l \in \Xi$ with $l \subset N_R$, the map

$$(\text{pr}_1 \circ \iota_{l,e_a}), (\text{pr}_2 \circ \iota_{l,e_c}) : \mathcal{F}^k(T_1) \to \mathcal{F}^k(P_{l,e_a} \oplus \bigoplus_{e \neq e_{l,\infty}} (Q \cdot g_{l,e} \otimes \mathcal{F}^{k-1}(P_e))$$

is injective, where $e \in \Xi \setminus \{e_{l,\infty}\}$ runs through all edges containing $l$ except for $e_{l,\infty}$.

Proof. Let $w$ be as in Lemma \[11.15\] for $l$. Since tropical $K$-groups form a cycle module (Theorem \[7.8\]), by Lemma \[10.5\] \[1\] and Ros96 Proposition 2.2, a map

$$K^k_T(\kappa(w)) \xrightarrow{(\partial_\nu)_{u_{w,0}}} K^k_T(\kappa(u_{w,0})) \oplus \bigoplus_{u \neq u_{w,\infty}} K^{-1}_T(\kappa(u))$$

is injective, where $u_{w,0}$ is as in Subsection \[10.1\] and $u \in K^1_{k(supp(\pi(w)))_{e(w)}}$ runs through all non-trivial specializations of $w$ but $u_{w,\infty}$. Hence the assertion follows from Lemma \[11.15\] and Corollary \[11.16\] \[11.2\] \[11.3\]. \[\square\]

Proposition \[11.24\]. We have $\alpha = 0$.

Proof. We show $\alpha_l = 0$ ($l \in \Xi$) by induction on $L_0(l) \in \mathbb{Z}_{\geq 0}$ in the increasing order. When $L_0(l) = 0$, i.e., $l = 0$, we have $\alpha_0 = 0$ by assumption. When $L_0(l) \neq 0$ and $l \in N_R$, by Lemma \[11.22\] \[11.23\] the equation $\delta(\alpha) = 0$, and the assumption of induction, we have $\alpha_l = 0$. When $L_0(l) \neq 0$ and $l \notin N_R$, by Corollary \[11.17\] the equation $\delta(\alpha) = 0$, and the assumption of induction, we have $\alpha_l = 0$. \[\square\]

Corollary \[11.25\]. We have

$$\mathcal{F}^p_{T,X,\kappa^o,v_0} \cong (\pi_\ast^o \mathcal{F}^p_{T,(X \times A^1)^o})_{v_0}.$$
11.8. **Proof of** \((R^1\pi_*^o \mathcal{F}^p_{T, (X \times \mathbb{A}^1)^o})_{\eta_0} = 0\). In this subsection, we shall show that

\[(R^1\pi_*^o \mathcal{F}^p_{T, (X \times \mathbb{A}^1)^o})_{\eta_0} = 0.\]

By Proposition 11.8 (1), it suffices to show that for a quadruple \((\varphi, \varphi', \Psi, \Lambda')\) satisfying Condition 11.4 and Condition 11.5 and

\[
\alpha = (\alpha_\gamma)_{\gamma} \in Z^{p, 1}(\text{Trop}(\varphi'(\pi^o)^{-1}(v_0))), \Lambda',
\]

there is another quadruple \((\phi, \phi', \Phi, \Theta')\) dominating \((\varphi, \varphi', \Psi, \Lambda')\) such that the pull back of \(\alpha\) to \(Z^{p, 1}(\text{Trop}(\varphi'(\pi^o)^{-1}(v_0))), \Theta')\) is a coboundary. We show existence of such a \((\phi, \phi', \Phi, \Theta')\) by induction on \((\text{mpd}(L, \text{mpd})) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}\) with the lexicographic order.

We put

\[
\Xi_{(\theta', \Theta)} := \Theta' \cap \text{Trop}(\varphi'(\pi^o)^{-1}(v_0)).
\]

**Lemma 11.26.** Let \(l \in \Xi\) be contained in at least 3 edges. We assume that

\[
\alpha_\gamma(Span(e) \wedge F_{p-1}(P_e)) = \{0\}
\]

for \(l \in \eta \neq e_{1, \infty}\) and \(\gamma : [0, 1] \to e\) such that \(l \not\in \gamma([0, 1])\). Then there exist a quadruple \((\phi, \phi', \Phi, \Theta')\) satisfying Condition 11.4 and Condition 11.5 dominating \((\varphi, \varphi', \Psi, \Lambda')\) and

\[
\beta_l \in C^{p, 0}(\text{Trop}(\phi_l'(w_{l, 0}))), \Theta_l',
\]

such that

- the natural map

\[
\text{Trop}(\rho_{\phi_l'}) : \text{Trop}(\phi_l'(\pi^o)^{-1}(v_0))) \to \text{Trop}(\varphi'(\pi^o)^{-1}(v_0))
\]

is injective on the inverse image of \(pt\) for \(pt \in \text{Trop}(\varphi'(\pi^o)^{-1}(v_0))\) with \(\text{mpd}(pt) = \text{mpd}(l), \text{L}_\infty, \text{mpd}(pt) \leq \text{L}_\infty, \text{mpd}(l), \text{and} w_{l, 0} \notin [w_{l, 0}, \infty]\), and

- we have

\[
(\text{Trop}(\rho_{\phi_l'})*\alpha + \delta_\beta)_\gamma(Span(e) \wedge F_{p-1}(P_e)) = \{0\}
\]

for \(e \in \Xi_{(\phi, \phi')}(\Theta_l')\) \(\setminus \{e_{\text{Trop}(\phi_l'(w_{l, 0})), \infty}\}\) and \(\gamma : [0, 1] \to e\) such that \(\text{Trop}(\phi_l'(w_{l, 0})) \in \gamma([0, 1])\), where \(e_{\text{Trop}(\phi_l'(w_{l, 0})), \infty}\) is defined similarly to \(e_{1, \infty}\).

**Proof.** By Remark 7.10 and Lemma 10.5 (1), a map

\[
\wedge^k (\kappa(w_{l, 0})^\times)_Q \xrightarrow{\langle \partial_{\pi} \rangle_{u_0 \neq u_{l, 0}, \infty}} \bigoplus_{u_0 \neq u_{l, 0}, \infty} \wedge^{k-1}(\kappa(u_0)^\times)_Q
\]

is surjective, where \(u_0 \in A^1_K(X)_{v_0}\) runs through all non-trivial specializations of \(w_{l, 0}\) except for \(u_{l, 0}, \infty\). The assertion follows from Lemma 10.5 (2), Lemma 10.6, Proposition 11.8 (2), Lemma 11.18 and Corollary 11.19 (11.6). More precisely, we can take \(\phi_l'\) of the form \((\varphi', \phi_l', (g_j)_{j})\), where \(g_j \in K(X)[T]\) is a polynomial irreducible over \(K(X)_{v_0}\) such that \(\deg(g_j) \leq \text{mpd}(l) - 1\) or \(\left[w_{l, 0}, \infty\right] \subset \left[c_j, \infty\right),\) where \(c_j \in K(X)^{alg}\) satisfies \(g_j(c_j) = 0\). 

**Proposition 11.27.** There exists a quadruple \((\phi, \phi', \Phi, \Theta')\) satisfying Condition 11.4 and Condition 11.5 dominating \((\varphi, \varphi', \Psi, \Lambda')\) such that the pull-back \(\text{Trop}(\rho_{\phi_l'})*\alpha\) under the natural map

\[
\text{Trop}(\rho_{\phi_l'}) : \text{Trop}(\phi_l'(\pi^o)^{-1}(v_0))) \to \text{Trop}(\varphi'(\pi^o)^{-1}(v_0))
\]
equals modulo coboundary an element
\[ \alpha' \in \mathbb{Z}^{\ast 1}(\text{Trop}(\phi'_i(\pi^{-1}(v_0))), \Theta'_i) \]
satisfying
\[ \alpha'_i(S_{\text{Span}(e)} \cap F_{p-1}(P_e)) = \{0\} \]
for \( e \in \Xi(\phi'_i, \theta'_i) \) and \( \gamma: [0, 1] \to e \).

**Proof.** This follows from Lemma 11.26 by induction on \((\text{mpd}(l), \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})\) in the descending order. (Computation around \( l \) contained in at most 2 edges is trivial.) \( \square \)

In the following, by Proposition 11.27, we may assume that
\[ \alpha_i(S_{\text{Span}(e)} \cap F_{p-1}(P_e)) = \{0\} \]
for \( e \in \Xi \) and \( \gamma: [0, 1] \to e \).

**Lemma 11.28.** Let \( l \in \Xi \setminus \{B\} \) contained in at least 3 edges or only 1 edge. Assume \( \alpha = 0 \) for \( \gamma: [0, 1] \to e_{1, \infty} \) with \( l \notin \gamma([0, 1]) \).
Then there exist a quadruple \((\phi, \phi'_i, \Phi_l, \Theta'_i)\) satisfying Condition 11.4 and Condition 11.5 dominating \((\varphi, \varphi', \Psi, \Lambda')\) and
\[ \beta_l \in C^{\ast 0}(\text{Trop}(\phi'_i(w_{1,0})), \Theta'_i) \]
such that
- the map
  \[ \text{Trop}(\rho_{\phi'_i}) : \text{Trop}(\phi'_i(\pi)^{-1}(v_0))) \to \text{Trop}(\varphi'(\pi)^{-1}(v_0))) \]
is injective on the inverse image of pt for pt \( \in \text{Trop}(\varphi'(\pi)^{-1}(v_0))) \) with \( \text{mpd}(\text{pt}) \geq \text{mpd}(l) \),
- \( \beta_l(S_{\text{Span}(e)} \cap F_{p-1}(e)) = \{0\} \) for any \( e \in \Xi(\phi'_i, \theta'_i) \) containing \( \text{Trop}(\phi'_i(w_{1,0})) \), and
- we have
  \[ (\text{Trop}(\rho_{\phi'_i})^* \alpha + \delta \beta_l) \gamma = 0 \]
for any \( \gamma: [0, 1] \to e_{\text{Trop}(\phi'_i(w_{1,0}))} \).

**Proof.** When \( l \) is contained in at least 3 edges (resp. only 1 edge), this follows from Lemma 10.5 (2) (3), Lemma 11.18 and Corollary 11.19 (11.5) (11.6) (resp. Lemma 10.5 (2), Lemma 11.18, and Corollary 11.19 (11.7)). More precisely, we can take \( \phi'_i \) of the form \((\varphi', \phi_l(g_j)_j)\) where \( g_j \in K(X)[T] \) is a polynomial irreducible over \( K(X)_{v_0} \) of degree \( \leq \text{mpd}(l) - 1 \). \( \square \)

**Proposition 11.29.** There exist a quadruple \((\phi, \phi'_i, \Phi, \Theta)\) dominating \((\varphi, \varphi', \Psi, \Lambda')\) such that the pull back of \( \alpha \) is equivalent to 0.

**Proof.** This follows from Lemma 11.28 by induction on \((\text{mpd}(l), \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0})\) in the increasing order. \( \square \)

**Corollary 11.30.**
\[ (R^1\pi^\circ_{\ast X} \mathcal{F}^p_{T, l}(X \times A^1)^\circ)_{v_0} = 0. \]
12. Corestriction maps

In this section, we show the existence of corestriction maps, which is used to prove the main theorem (Theorem 8.2) over finite fields. Let $K$ be a trivially valued finite field, and $L$ a trivially valued finite field which is an extension of $K$. Let $X$ be a smooth algebraic variety over $K$. We put $X_L$ the base change. Tropical $K$-groups are over finite fields.

**Proposition 12.1.** Let $X$ be a smooth algebraic variety over $K$. Then there is a $\mathbb{Q}$-linear map

$$\text{cor}: \mathcal{C}^{p,q}_{X_L}(X^\text{Ber}_L) \to \mathcal{C}^{p,q}_X(X^\text{Ber})$$

corestriction map, such that

$$\text{cor}(\mathcal{C}^{p,q}_{Z \subset X_L}(X^\text{Ber}_L)) \subset \mathcal{C}^{p,q}_Z(X^\text{Ber})$$

for any closed subscheme $Z \subset X$, we have $\text{cor} \circ \delta = \delta \circ \text{cor}$, and $\text{cor} \circ \text{res} = [L : K]$, where $\text{res}: \mathcal{C}^{p,q}_X(X^\text{Ber}) \to \mathcal{C}^{p,q}_{X_L}(X^\text{Ber}_L)$ is the base change map.

**Corollary 12.2.** There is a map

$$\overline{\text{cor}}: H^{p,q}_{\text{Trop},Z}(X_L) \to H^{p,q}_{\text{Trop},Z}(X)$$

such that $\overline{\text{cor}} \circ \text{res} = [L : K]$.

We shall construct the corestriction map locally and explicitly. Let $\varphi: X_L \to \text{Spec} L[M']$ be a closed immersion to a torus $\text{Spec} L[M']$. We put $\pi: X_L \to X$ the base change. See Subsection 6.1 for $F^p$, Subsection 11.5 for $\overline{F}^p$. We shall use Remark 11.14 freely. We fix $v \in \text{ZR}(L/K)$.

**Proposition 12.3.** There exist an open subvariety $U_v \subset X$ and a closed immersion $\psi_v: U_v \to \text{Spec} K[M_v]$ to a torus $\text{Spec} K[M_v]$ such that

- the projection

$$\Phi_v: \text{Trop}((\varphi, \psi_v,L)(U_v,L)) \to \text{Trop}(\psi_v(U_v))$$

is finite-to-one, and

- for any $p$, sufficiently fine fan structures $\Lambda_v$ of $\text{Trop}(\psi_v(U_v))$ and $\Lambda'_v$ of $\text{Trop}((\varphi, \psi_v,L)(U_v,L))$

there is a unique $\mathbb{Q}$-linear map

$$\overline{\text{cor}}_{P'_v}: \overline{F}^p(P'_v, \text{Trop}((\varphi, \psi_v,L)(U_v,L))) \to \overline{F}^p(P_v, \text{Trop}(\psi_v(U_v))),$$

such that for a valuation $\mu \in U_v^\text{ad}$ with $\text{Trop}_{\Lambda_v}^\text{ad}(\psi_v(\mu)) = P_v$, a diagram

$$\overline{F}^p(P'_v, \text{Trop}((\varphi, \psi_v,L)(U_v,L))) \to \overline{F}^p(P_v, \text{Trop}(\psi_v(U_v)))$$

is commutative, where the vertical arrows are (sums of) canonical ones, and the last horizontal arrow is the sum of norm homomorphisms.
Remark 12.4. Proposition 12.3 also holds when the source of $\varphi$ is a union of irreducible components of $X_L$. 

Lemma 12.5. There exist a commutative diagram

\[
\begin{array}{ccc}
\text{Spec } L[M'] \times T_{\Lambda_v,0,L} & \xrightarrow{(\varphi, \psi_{v,0,L})} & T_{\Lambda_v,0} \\
U_v \times \Lambda_v & \xrightarrow{\psi_{v,0}} & T_{N_{v,0},L} & \xrightarrow{\psi_v} & U_v
\end{array}
\]

where

- $U_v \subset X$ is an open subvariety, $\psi_v: U_v \to \text{Spec } K[M_v]$ is a closed immersion,
- $T_{\Sigma_v}$ is a toric variety containing $\text{Spec } K[M_v]$ as the dense orbit,
- the variety $T_v$ is the toric variety corresponding to the fan $\Sigma_v$,
- the arrow $\varphi$ is a toric birational map,
- the arrows between toric varieties are toric morphisms,
- the closures $\overline{\psi_{v,0}(U_v)}$ in $T_{\Lambda_v,0}$ and $\overline{\psi_{v,0}'(U_v,L)}$ in $T_{N_{v,0},L}$ are tropical compactifications,
- the image of each map and morphism intersects with the dense torus orbit, and
- $\text{Spec } L[M'] \times T_{\Lambda_v,0,L} \to T_{\Lambda_v,0}$ is the composition of the projection and the base change map.

satisfying the following.

1. We put $M_v,0$ the free $\mathbb{Z}$-module such that $\text{Spec } K[M_v,0] \subset T_{\Lambda_v,0}$ is the dense torus. For $w \in \pi^{-1}(v)$, we have
   \[\dim(\text{Trop}^M_{\Lambda_v,0}(\psi_{v,0}'(w))) = \text{rank } w(M' \oplus M_v,0) = \text{rank } \Gamma_w.\]
2. The natural map
   \[\text{Trop}(\psi_{v,0}'(U_v)) \to \text{Trop}(\psi_{v,0}(U_v))\]
   is finite-to-one.
3. The closure $\overline{\psi_v(U_v)} \subset T_{\Sigma_v}$ is the normalization of $\overline{\psi_{v,0}(U_v)}$.
4. We have
   \[\{x_v\} = \psi_v(U_v) \cap O(\sigma_{x_v}),\]
   where $x_v \in \overline{\psi_v(U_v)}$ is the center of $v$ and $\sigma_{x_v} \in \Sigma_v$ is such that $x_v \in O(\sigma_{x_v})$.
5. For any $p$ and $w \in \pi^{-1}(v)$, we have
   \[\pi^*_w(\Lambda^p\psi_{v,0}'(M' \oplus M_v,0) \cap w^{-1}(\{0\})) \subset \Lambda^p \text{Im}(\Gamma(O(\sigma_{x_v}), O) \to \kappa(v)) \subset K_p^T(\kappa(v)),\]
   where $O$ is the structure sheaf, $\pi^*$ is the images in $K_p^T(\kappa(w))$ and $K_p^T(\kappa(v))$, and $\pi^*_w: K_p^T(\kappa(w)) \to K_p^T(\kappa(v))$ is the norm homomorphism for the natural morphism $\pi_w: \kappa(v) \to \kappa(w)$ of the residue fields.

Proof. First, we take $\Lambda_v,0, N_{v,0}$, and a rational map $\psi_{v,0}: X \dashrightarrow T_{\Lambda_v,0}$ such that $T_{\Lambda_v,0}$ is quasi-projective, and [1] and [2] hold. (In particular, the closure $\overline{\psi_{v,0}(U_v)}$ is projective.) Next, we take $\Sigma_v$ such that [3], [4], and [5] hold. For example, we can take $T_{\Sigma_v} := \prod P^{N_{v,0}} \times T_{\Lambda_v,0}$ and $\psi_v: X \dashrightarrow T_{\Sigma_v}$ given by an embedding of the normalization $\overline{\psi_{v,0}(U_v)}$ into a projective space $P^{N_{v,0}}$ and suitable morphisms from $P^{N_{v,0}}$ to projective spaces $P^{N_{v,1}}$, so that [4] and [5] hold. \qed
Lemma 12.6. There is a natural morphism
\[ \psi_v(U_v)_L \to \psi'_{v,0}(U_v)_L. \]

Proof. By Lemma 12.5(2), there exists a covering of \( \psi_{v,0}(U_v) \) by affinoid subdomains whose inverse images under
\[ \psi'_{v,0}(U_v)_L \to \psi_{v,0}(U_v) \]
are also affinoid subdomains. Hence the proper morphism
\[ \psi'_{v,0}(U_v)_L \to \psi_{v,0}(U_v) \]
is a finite morphism. Hence the assertion follows from Lemma 12.5(3). \( \square \)

We put \( \Phi_v \) as in Proposition 12.3. We fix fan structures \( \Lambda_v \) and \( \Lambda'_v \) of Trop(\( \psi_v(U_v) \)) and Trop(\( (\varphi, \psi_v,L)(U_v,L) \)) such that \( P_v := \text{Trop}^{\text{ad}}_{\Lambda'_v}(\psi_v(v)) \in \Lambda_v \) is of dimension rank \( \Gamma_v \), and \( \Phi^{-1}_v(P_v) \) is the finite union of cones \( P'_v \in \Lambda'_v \) with \( \Phi_v(P'_v) = P_v \). These hold for suitable sufficiently fine \( \Lambda_v \) and \( \Lambda'_v \) by Lemma 12.5(1) and (2).

We fix a cone \( P''_v \in \Lambda'_v \) with \( \Phi_v(P''_v) = P_v \). For \( w \in \pi^{-1}(v) \cap (\text{Trop}^{\text{ad}}_{\Lambda'_v} \circ (\varphi, \psi_v,L))^{-1}(P'_v) \), we put \( y_w \in \overline{\psi_v(U_v)}_L \) the center of \( w \). We put
\[ \Psi: \text{Trop}((\varphi, \psi_v,L)(U_v,L)) \to \text{Trop}((\varphi, \psi_v,0,L)(U_v,L)) \]
the natural map.

Lemma 12.7. There is a natural map
\[ M' \oplus M_{v,0} \cap \Psi(P'_v) \to \mathcal{O}^\times_{y_w} \]
and a unique map
\[ \wedge^p(M' \oplus M_{v,0} \cap \Psi(P'_v)) \to F^p(0_{\text{Trop}(O(\sigma_{x_v}))}, \text{Trop}(\overline{\psi_v(U_v)} \cap O(\sigma_{x_v}))) \]
such that the following diagram commutative:
\[ \begin{array}{ccc}
\wedge^p(M' \oplus M_{v,0} \cap \Psi(P'_v)) & \to & F^p(0_{\text{Trop}(O(\sigma_{x_v}))}, \text{Trop}(\overline{\psi_v(U_v)} \cap O(\sigma_{x_v}))) \\
\oplus_{y = y_w (w \in \pi^{-1}(v) \cap (\text{Trop}^{\text{ad}}_{\Lambda'_v} \circ (\varphi, \psi_v,L))^{-1}(P'_v))} & & \\
\oplus_{w \in \pi^{-1}(v) \cap (\text{Trop}^{\text{ad}}_{\Lambda'_v} \circ (\varphi, \psi_v,L))^{-1}(P'_v)} & K^p_T(k(y)) & K^p_T(k(x_v)) \\
\end{array} \]
where \( \mathcal{O}_{y_w}^\times \) is the multiplicative group of the local ring \( \mathcal{O}_{y_w} \) of the structure sheaf at \( y_w \), \( 0_{\text{Trop}(O(\sigma_{x_v}))} \) is the zero element, and horizontal arrows are the norm homomorphisms.

Commutativity of the lower part of the diagram is [Ros96, Definition 1.1 R1c].

Proof. By Lemma 12.5(1), the cone \( \text{Trop}^{\text{ad}}_{\Lambda'_v}(\psi'_{v,0}(w)) \) spans the same space as \( \Psi(P'_v) \subset \text{Trop}^{\text{ad}}_{\Lambda'_v}(\psi'_{v,0}(w)) \). Hence by Lemma 12.6, we have the first map. By Lemma 12.5(5) and injectivity of the right vertical arrows, we have the unique second map making the diagram commutative. \( \square \)
Let $x \in \{x_v\} \cap O(\sigma_{x_v})$. We put

$$\pi : \overline{\psi_v(U_v)} \to \overline{\psi_v(U_v)}$$

the base change map.

**Lemma 12.8.** There are natural maps

$$M' \oplus M_{v,0} \cap \Psi(P'_v) \to O^\times_y$$

for $y \in \pi^{-1}(x) \cap \{y_w\}$ ($w \in \pi^{-1}(v)$) such that we have a commutative diagram

$$\begin{align*}
\wedge^p(M' \oplus M_{v,0} \cap \Psi(P_v)^{\perp}) & \to F^p(0_{\text{Trop}(O(\sigma_{x_v}))}, \text{Trop}(\psi_v(U_v) \cap O(\sigma_{x_v}))) \\
\bigoplus_{w \in \pi^{-1}(v) \cap (\Psi \circ \text{Trop}_{\psi_v,L})^{-1}(P'_v)} K^p_f(k(y)) & \to K^p_f(k(x)).
\end{align*}$$

**Proof.** We claim that the image of $y$ under the morphism in Lemma 12.6 is contained the orbit $O(\text{Trop}_{\psi_v,0}(\psi'_v(w)))$. This is because the image of $x$ and $x_v$ to $\overline{\psi_v,0(U_v)}$ is contained in the same orbit, and by Lemma 12.5 [2], a cone in $\Lambda_{v,0}$ containing $\text{Trop}_{\psi_v,0}(\psi'_v(w))$ with the same image in $\Lambda_{v,0}$ as $\text{Trop}_{\psi_v,0}(\psi'_v(w))$ is only $\text{Trop}_{\psi_v,0}(\psi'_v(w))$. Hence we have the natural maps in the assertion. When $x \in \{x_v\}$ is of codimension 1, commutativity easily follows from multiplying an element of the structure sheaf $\mathcal{O}_{\{x_v\},x}$ which is 0 at $x$, residue homomorphisms (Subsection 7.2), commutativity of residue homomorphisms with norm homomorphisms (Ros96 Proposition 4.6(1)), and Lemma 12.7. General case follows from the case of codimension 1 and a sequence

$$\{x_v\} = \{x_{v,0}\} \subset \{x_{v,1}\} \subset \cdots \subset \{x\}$$

of closed subvarieties with codim$_{\{x_v\}}\{x\} = i$. 

Let $\mu \in U^\text{ad}_v$ such that $\text{Trop}_{\Lambda_0}(\psi_v(\mu)) = P_v$.

**Lemma 12.9.** We have a commutative diagram

$$\begin{align*}
\wedge^p(M' \oplus M_{v,0} \cap \Psi(P_v)^{\perp}) & \to F^p(0_{\text{Trop}(O(\sigma_{x_v}))}, \text{Trop}(\psi_v(U_v) \cap O(\sigma_{x_v}))) \\
\bigoplus_{\nu \in \pi^{-1}(\mu) \cap (\Psi \circ \text{Trop}_{\psi_v,L})^{-1}(P'_v)} K^p_f(\kappa(\nu)) & \to K^p_f(\kappa(\mu)).
\end{align*}$$

**Proof.** By Lemma 12.5 [4], the center $x_\mu$ of $\mu$ in $\overline{\psi_v(U_v)}$ is in $\{x_v\} \cap O(\sigma_{x_v})$. The center $y_v \in \overline{\psi_v(U_v)}$ of $\nu \in \pi^{-1}(\mu)$ is in $\pi^{-1}(x_\mu) \cap \{y_w\}$ for some $w \in \pi^{-1}(v)$. Hence by Lemma 12.8 and Ros96 Definition 1.1 R1c, it suffices to show that

$$\text{Trop}_{\Lambda_0}(\varphi, \psi_v(L)(\nu)) = \text{Trop}_{\Lambda_0}(\varphi, \psi_v,L)(w).$$

By the definition of $\Lambda_v$ and $\Lambda'_v$, this follows from

$$\text{Trop}_{\Lambda_0}(\psi_v(\mu)) = \text{Trop}_{\Lambda'_0}(\psi_v(v))$$

and

$$\text{Trop}_{\Lambda'_0}(\psi'_v(\nu)) = \text{Trop}_{\Lambda'_0}(\psi'_v,0(w)).$$

(The latter equation is proved in proof of Lemma 12.8). 

\[\square\]
Proof of Proposition 12.3. By Lemma 12.5 and 12.9, the image of $P'_v$ to Trop(Spec $K[M_{v,0}]$) is of dimension rank $\Gamma_v$. Hence the natural map from $P'_v$ to Trop(Spec $K[M_{v,0}]$) is injective. Hence we have

$$
(M' \oplus M_v \cap P'_v) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\cong (M' \oplus M_{v,0} \cap \Psi(P'_v)\perp) \otimes_{\mathbb{Z}} \mathbb{Q} + (M_v \cap P'_R) \otimes_{\mathbb{Z}} \mathbb{Q}
$$

Consequently, by Lemma 12.10 and projection formula ([Ros96, Definition 1.1 R2c]), there exists a map

$$
\land^p(M' \oplus M_v \cap P'_v) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathcal{F}^p(P_v, \text{Trop}(\psi_v(U_v)))
$$

such that for $\mu \in U'_v$ with $\text{Trop}^\text{ad}_L(\psi_v(\mu)) = P_v$, we have a commutative diagram

$$
\begin{array}{ccc}
\oplus_{\nu \in \pi^{-1}(\mu) \cap (\text{Trop}^\text{ad}_L(\psi_v(\mu)))^{-1}(P_v)}^p \mathcal{K}_T^p(\kappa(\nu)) & \to & K^p_T(k(\mu))
\end{array}
$$

Since such a commutative diagram exists for all such $\mu$, by Remark 12.14, this map induces the required map $\text{cor}_P$. (Uniqueness is obvious.)

**Remark 12.10.** Let $v, U_v, \psi_v$, and $\mu$ (for some $\Lambda_v$ and $\Lambda'_v$) be as in Proposition 12.3. Let $v \subset U_v$ be an open subset, and $\psi : U_{v,1} \to \text{Spec } K[M_{v,1}]$. Then by projection formula ([Ros96, Definition 1.1 R2c]), the assertion of Proposition 12.3 holds for $\mu$ (instead of $v$), $U_{v,1}$, and $\psi_\mu := (\psi, \psi_1)$.

Let $\varphi' : X_L \to T_{\Sigma',L}$ be a closed immersion.

**Lemma 12.11.** Let $x \in X$. There exist an open neighborhood $U \subset X$ of $x$ and a closed immersion $\psi : U \to T_{\Sigma}$ to a toric variety $T_{\Sigma}$ such that

- the projection $\Phi : \text{Trop}((\varphi', \psi_L)(U_L)) \to \text{Trop}(\psi(U))$ is finite-to-one, and
- for $p$, sufficiently fine fan structures $\Lambda$ of Trop(\psi(U)) and $\Lambda'$ of Trop((\varphi', \psi_L)(U_L)) for $P' \in \Lambda'$, such that $\Phi(P') \in \Lambda$ ($P' \in \Lambda'$), there is a unique $\mathbb{Q}$-linear map
  $$
  \text{cor}_P^\text{ad} : \mathcal{F}^p(P', \text{Trop}((\varphi', \psi_L)(U_L))) \to \mathcal{F}^p(\Phi(P'), \text{Trop}(\psi(\mu)))
  $$
  such that for a valuation $\mu \in U'_v$ with $\text{Trop}^\text{ad}(\psi(\mu)) = \Phi(P')$, the diagram

$$
\begin{array}{ccc}
\oplus_{\nu \in \pi^{-1}(\mu) \cap (\text{Trop}^\text{ad}_L(\psi(\mu)))^{-1}(P')} \mathcal{K}_T^p(\kappa(\nu)) & \to & K^p_T(k(\mu))
\end{array}
$$

is commutative, where the vertical arrows are canonical ones, and the last horizontal arrow is the sum of norm homomorphism.

**Proof.** We claim that for any $q \geq 1$, there exist an open neighborhood $U_q \subset X$ of $x$, a closed immersion $\psi_q : U_q \to T_{\Sigma_q}$ to a toric variety $T_{\Sigma_q}$, and a closed subscheme $Z_q \subset U_q$ of codimension $\geq q$ of the form $\psi_q^{-1}(V)$ for some closed toric subscheme $V$ of $T_{\Sigma_q}$ such that the properties of the assertion hold for $\psi_q : U_q \setminus Z_q \to T_{\Sigma_q}$. We assume
$q = 1$. By compactness of the Zariski-Riemann space $\text{ZR}(K(X)/K)$, Proposition 12.3 holds for restrictions of $\phi'$ to the inverse images of torus orbits containing generic points of $X_L$ (Remark 12.4, and Remark 12.10) there exist an open subvariety $U_{1,0} \subset X$ and a closed immersion $\psi_1: U_{1,0} \to T_1$ to a torus $T_1$ such that the assertion of Proposition 12.3 holds for the above restrictions of $\phi'$, any $v \in \text{ZR}(K(X)/K)$, and $\psi_1$. By Remark 12.10, we may assume that there is an open neighborhood $U_1$ of $x$ and $\psi_1$ extends to $\psi_1: U_1 \to T_{\Sigma_1}$, where $T_{\Sigma_1}$ is a smooth toric variety containing $T_1$ as the dense torus orbit. Hence $U_1$, $\psi_1$, and the inverse image $Z$ of $T_{\Sigma_1} \setminus T_1$ is a required triple. When $q \geq 2$, the existence of such triples follows from induction, argument similar to Remark 12.10, and applying the case of $q = 1$ to suitable closed subschemes of $X$. (Note that morphisms from closed subschemes to smooth toric varieties lift locally because smooth affine toric varieties are products of affine spaces and algebraic tori.) \qed

We put $M$ the lattice such that $\text{Spec} \ K[M] \subset T_{\Sigma}$ is the dense torus orbit. For each cone $P \in \Lambda$, we take $f_1, \ldots, f_{\dim(P)} \in M \cap \sigma_P^F$ such that the natural map

$$\langle f_1, \ldots, f_{\dim(P)} \rangle_\mathbb{Q} \to \text{Hom}_\mathbb{Q}(\text{Span}(P), \mathbb{Q})$$

is an isomorphism, where $\sigma_P \in \Sigma$ is the cone such that $\text{rel.int}(P) \subset \text{Trop}(O(\sigma_P))$. This induces decompositions

$$F^{p,*}_\mathbb{Q} \cong \bigoplus_{j=0}^p \wedge^j \langle f_1 \rangle_\mathbb{Q} \otimes \overline{F}_p^{-j}(*)$$

for $* = P \in \Lambda$ or $P' \in \Lambda'$ such that $\Phi(P') = P$.

Let $V \subset \text{Trop}(\psi(U))$ be an open subset, and $\alpha = (\alpha_{\gamma,i}', \gamma) \in C^{p,q}(\Phi^{-1}(V))$. For each continuous map $\gamma': \Delta^q \to P' \cap \Phi^{-1}(V) \in C_q(P' \cap \Phi^{-1}(V))$, we write the image of $\alpha_{\gamma'} \in F^p(P')$ under the above isomorphism by

$$\sum_{j=0}^p \sum_{l} b_{j,l} \otimes \alpha_{\gamma',j,l}$$

($b_{j,l} \in \wedge^j \langle f_1 \rangle_\mathbb{Q}$ and $\alpha_{\gamma',j,l} \in \overline{F}_p^{-j}(P')$).

**Definition 12.12.** For each continuous map $\gamma: \Delta^q \to P \cap V \in C_q(P \cap V)$ as in Definition 6.2, we put

$$\text{cor}(\alpha)_\gamma := \sum_{P' \in \Lambda'} \sum_{l=0}^p \sum_{j} b_{j,l} \otimes \text{cor}_{P'}(\alpha(\Phi_{P'})^{-1} \gamma_{j,l} \gamma_{j,l}) \in F^p(P).$$

We put

$$\text{cor}(\alpha) := (\text{cor}(\alpha)_\gamma)_{\gamma} \in C^{p,q}(V).$$

**Proof of Proposition 12.1.** Let $v \in X^{\text{Ber}}$, and $U \subset X$ an affine open subvariety such that $v \in U^{\text{Ber}}$. Then by Remark 6.4, the restriction of an element of $\mathcal{C}_L^{q}(X^{\text{Ber}})$ to a small open neighborhood of $\pi^{-1}(v)$ comes from a single closed immersion $\phi: U_L \to T_{\Sigma}$. Hence by Lemma 12.11, we can define corestriction map locally on $X^{\text{Ber}}$. By uniqueness of these local corestriction maps, we can glue them, and get the corestriction map. By construction, this map satisfies the required properties. (The compatibility $\text{cor} \circ \delta = \delta \circ \text{cor}$ follows from [Ros96, Definition 1.1 R3b].) \qed
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