Fibonacci-like growth of numerical semigroups of a given genus

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Abstract

We give an asymptotic estimate of the number of numerical semigroups of a given genus. In particular, if \( n_g \) is the number of numerical semigroups of genus \( g \), we prove that

\[
\lim_{g \to \infty} n_g \phi^{-g} = S
\]

where \( S \) is a constant, resolving several related conjectures concerning the growth of \( n_g \). In addition, we show that the proportion of numerical semigroups of genus \( g \) satisfying \( f < 3m \) approaches 1 as \( g \to \infty \), where \( m \) is the multiplicity and \( f \) is the Frobenius number.

1 Introduction

A numerical semigroup is defined to be a cofinite subsemigroup of the non-negative integers. A numerical semigroup \( \Lambda \) is said to have genus \( g = g(\Lambda) \) if \( |\mathbb{N} \setminus \Lambda| = g \). We also define the multiplicity \( m = m(\Lambda) = \min(\Lambda \setminus \{0\}) \). Finally, the Frobenius number \( f = f(\Lambda) \) is defined to be \( \max(\mathbb{N} \setminus \Lambda) \).

Let \( n_g \) be the number of numerical semigroups of genus \( g \). It was observed by Bras-Amorós [2] that \( n_g \) exhibits Fibonacci-like growth; in particular, it was conjectured that

\[
\lim_{g \to \infty} \frac{n_{g+1}}{n_g} = \phi, \quad \text{where } \phi = \frac{1+\sqrt{5}}{2} \text{ is the golden ratio.}
\]

Work towards the resolution of this conjecture has been the subject of a number of recent papers. It is known that \( n_g = \Omega(\phi^g) \) (see [3], [8]), but the only upper bounds for \( n_g \) that have been given are no better than \( O((2-\epsilon)^g) \) (see [8], which summarizes the results of [3] and [5]).

Zhao [8] made the observation that most of the numerical semigroups counted in \( n_g \) seem to be of a certain type. Letting \( t_g \) denote the number of numerical semigroups \( \Lambda \) of genus \( g \) satisfying \( f(\Lambda) < 3m(\Lambda) \), Zhao conjectured the following.

Conjecture 1.

\[
\lim_{g \to \infty} \frac{t_g}{n_g} = 1.
\]
He also gave a formula
\[
\lim_{g \to \infty} t_g \varphi^{-g} = S,
\]
where \( S \) is the value of an infinite sum. It was not determined whether this sum converges, leaving open the possibility that \( S = \infty \). However, Zhao conjectured that this is not the case.\[1\]

**Conjecture 2.**
\[
\sup_{g \in \mathbb{N}} t_g \varphi^{-g} < \infty.
\]

Given that Conjecture\[2\] is true, it follows that \( S \) is finite. Then, if Conjecture\[1\] holds, it follows that \( \lim_{g \to \infty} n_g \varphi^{-g} = S \), proving that \( n_g \) has Fibonacci-like growth. In this paper, we prove Conjectures\[1\] and\[2\], thus establishing that Bras-Amorós’ original conjecture is correct.\[2\] It immediately follows that \( \lim_{g \to \infty} \frac{n_{g-1} + n_{g-2}}{n_g} = 1 \), another conjecture of Bras-Amorós\[2\], and it follows that \( n_{g+1} \geq n_g \) for sufficiently large \( g \), verifying a conjecture of Kaplan\[6\] for all but finitely many cases.

Since the set of numerical semigroups of a given genus does not seem to have much structure, it is difficult to understand exact values of \( n_g \). In order to get around this difficulty, our approach is to use combinatorial arguments to get Fibonacci-like relations on \( n_g \) with some “error terms,” which we then bound. This general idea was already suggested by Bras-Amorós and Bulygin\[4\]. Bras-Amorós defines the semigroup tree\[3\], a combinatorial object that allows us to obtain the main Fibonacci term, which happens to correspond to \( t_g \). Various bounding techniques are then used to complete the proof.

The rest of the paper is divided into several sections. Section 2 provides an introduction to the semigroup tree and provides much of the general setup for our approach. Section 3 gives a proof of the main result under the assumption of a technical lemma. Sections 4 and 5 fill in the details for proving the technical lemma, and Section 6 contains conclusions and further questions.

### 2 The semigroup tree

The semigroup tree is defined in terms of the minimal generators of numerical semigroups. It is well known that any numerical semigroup \( \Lambda \) has a *minimal generating set* \( G \), in the sense that any set that generates \( \Lambda \) contains \( G \) (see Theorem 2.7 of [7]). The elements of \( G \) are called *minimal generators*, and it is evident that no minimal generator can be expressed as the sum of other non-zero elements of the semigroup.

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1 The actual conjecture was phrased in a slightly different but equivalent way.

2 We do not give an explicit estimate of the supremum in Conjecture\[2\], although such an estimate would be theoretically possible. Zhao showed in\[8\] that it is at least 3.78, with numerical evidence suggesting that it is not much larger than that.
Define an effective generator of \( \Lambda \) to be a minimal generator larger than \( f(\Lambda) \). This definition was given in [3], and here we further define the efficacy \( h \) of a semigroup to be the number of effective generators it has. As we will later see, the effective generators correspond to deviations from Fibonacci-like growth.

### 2.1 Defining the semigroup tree

We are now in a position to define the semigroup tree, which was first defined in [3]. Note that by removing any minimal generator from a numerical semigroup, we are left with another numerical semigroup with one higher genus. The idea of the semigroup tree is to characterize all numerical semigroups as a sequence of such removals from the semigroup of genus 0, always removing elements in increasing order.

We say that a semigroup \( \Lambda' \) descends from a semigroup \( \Lambda \) if \( \Lambda' = \Lambda - \{\lambda\} \), where \( \lambda \) is an effective generator of \( \Lambda \). Clearly, \( g(\Lambda') = g(\Lambda) + 1 \). Then, we can consider an infinite tree whose vertices are numerical semigroups, whose root is the semigroup of genus 0, and whose edges are between those pairs of semigroups in which one descends from the other. It can be shown that each numerical semigroup appears in this tree exactly once, and furthermore, it appears at depth \( g \) if \( g \) is its genus (see [3] for a more detailed discussion).

### 2.2 Types of descent

Suppose that \( \Lambda' = \Lambda - \{\lambda\} \) is a numerical semigroup descending from \( \Lambda \). We say that this descent is weak if each effective generator of \( \Lambda' \) is also an effective generator of \( \Lambda \). In other words, no “new” effective generator is created. We say the descent is strong otherwise. If the descent is strong, then we say that \( \Lambda' \) is a strongly descended numerical semigroup. It will be convenient to also consider the genus 0 semigroup to be strongly descended. If a numerical semigroup \( \Lambda'' \) is obtained from \( \Lambda \) by a series of weak descents (and no strong descents), then we say that \( \Lambda'' \) is a weak descendant of \( \Lambda \). We will use the convention that \( \Lambda \) is a weak descendent of itself. These notions are adapted from [3].

Now, for a numerical semigroup \( \Lambda \), let \( N_g(\Lambda) \) denote the number of weak descendents of \( \Lambda \) having genus \( g \). Each numerical semigroup is the weak descendent of a unique strongly descended ancestor (namely, its nearest strongly descended ancestor). Thus, if \( S \) is the set of strongly descended semigroups, then

\[
\sum_{\Lambda \in S} N_g(\Lambda).
\]

In order to bound this sum, it will be useful to make note of two lemmas. The first is an observation from [3] giving a condition for a numerical semigroup to be strongly descended.
Lemma 1 (Bras-Amorós). A numerical semigroup $\Lambda$ is strongly descended if and only if $f(\Lambda) + m(\Lambda)$ is a minimal (hence effective) generator of $\Lambda$.

Proof. See [3].

The second lemma is an upper bound on $N_g(\Lambda)$.

Lemma 2. For any numerical semigroup $\Lambda$, we have $N_g(\Lambda) \leq \binom{h(\Lambda)}{g-g(\Lambda)}$ and $N_g(\Lambda) \leq \varphi^{g-g(\Lambda)+h(\Lambda)}$.

Proof. Let $F_n$ denote the $n$th Fibonacci number. Since $\binom{a}{b} \leq F_{a+b} \leq \varphi^{a+b}$, the second inequality is a consequence of the first. To show the first inequality, note that each weak descendent of $\Lambda$ is obtained by removing several effective generators of $\Lambda$. If the genus of the weak descendent is $g$, then $g - g(\Lambda)$ effective generators must be removed. There are at most $\binom{h(\Lambda)}{g-g(\Lambda)}$ ways to choose $g - g(\Lambda)$ effective generators to remove; consequently, $N_g(\Lambda) \leq \binom{h(\Lambda)}{g-g(\Lambda)}$.

These lemmas give a rough idea of the general approach—use the condition given in Lemma 1 to bound the number of strongly descended numerical semigroups $\Lambda$, and then use Lemma 2 to bound $N_g(\Lambda)$. This will be carried out in detail in the sections that follow.

3 The main result

Recall that the main result of this paper is

Theorem 1. Let $n_g$ be the number of numerical semigroups of genus $g$. Then,

$$\lim_{g \to \infty} \frac{n_g}{\varphi^g} = S,$$

where $S$ is a constant.

In this section, we will prove the main result under the assumption that the following inequality holds:

Lemma 3. Let $S(m, f)$ be the set of all strongly descended numerical semigroups having multiplicity $m$ and Frobenius number $f$. Then,

$$\sum_{\Lambda \in S(m,f)} \varphi^{-g(\Lambda)+h(\Lambda)} \leq 5(f - m) \left( \frac{1.618}{\varphi} \right)^{f-m-1}.$$

Establishing this inequality is actually a key step in showing Theorem 1, but the proof of the inequality is technical and is therefore deferred to the end of the paper.

From the previous section, our task is to estimate
\[ n_g = \sum_{\Lambda \in S} N_g(\Lambda). \]

We do this by partitioning \( S \) into three subsets and summing over the three parts separately. Let \( S_1 \) denote the set of strongly descended semigroups \( \Lambda \) such that \( h(\Lambda) + g(\Lambda) < g \). Let \( S_2 \) denote the set of strongly descended semigroups \( \Lambda \) such that \( h(\Lambda) + g(\Lambda) \geq g \) and \( g(\Lambda) - h(\Lambda) < \frac{g}{3} \). Finally, let \( S_3 \) denote the set of strongly descended semigroups \( \Lambda \) such that \( h(\Lambda) + g(\Lambda) \geq g \) and \( g(\Lambda) - h(\Lambda) \geq \frac{g}{3} \).

It is evident that the \( S_i \) partition \( S \). Thus, we can write

\[ n_g = n_{g,1} + n_{g,2} + n_{g,3}, \]

where

\[ n_{g,i} = \sum_{\Lambda \in S_i} N_g(\Lambda). \]

Note that if \( \Lambda \in S_1 \), then by Lemma 2, \( N_g(\Lambda) = 0 \) because \( h(\Lambda) < g - g(\Lambda) \). It follows that \( n_{g,1} = 0 \). In the next two subsections, we estimate \( n_{g,2} \) and \( n_{g,3} \).

### 3.1 Estimating \( n_{g,2} \)

We will show that \( n_{g,2} = O(\varphi^g) \) and \( n_{g,2} \leq t_g \). The relevant properties of semigroups \( \Lambda \) in \( S_2 \) are that \( \Lambda \) is strongly descended and \( 2h(\Lambda) > g(\Lambda) \). The first property is immediate from the definition \( S \), and the second property follows from manipulating the inequalities defining \( S_2 \):\footnote{Note that \( m(\Lambda) \leq f(\Lambda) + 1 \) for all numerical semigroups \( \Lambda \). In the case that \( m(\Lambda) = f(\Lambda) + 1 \), we take \([m(\Lambda), f(\Lambda)]\) to be empty, and our analysis still carries through.}

\[ 3(g(\Lambda) - h(\Lambda)) < g \leq g(\Lambda) + h(\Lambda) \]
\[ g(\Lambda) < 2h(\Lambda). \]

We will define any semigroup satisfying these two properties to be \textit{orderly}, and rather than work with semigroups in \( S_2 \) directly, it will be more convenient to make observations about orderly semigroups in general and apply them to \( S_2 \). These observations stem from the following proposition:

**Proposition 1.** If \( \Lambda \) is orderly, then \( f(\Lambda) < 2m(\Lambda) \).

**Proof.** First, we observe that for any numerical semigroup \( \Lambda \), the effective generators must lie in the interval \([f(\Lambda) + 1, f(\Lambda) + m(\Lambda)]\). Thus, \( h(\Lambda) \leq m(\Lambda) \).

Since \( \Lambda \) is strongly descended, we know that \( f(\Lambda) + m(\Lambda) \) is an effective generator. Consequently, \( \Lambda \) contains at most half of the integers in the interval \([m(\Lambda), f(\Lambda)]\), since no two elements of \( \Lambda \) can sum to \( f(\Lambda) + m(\Lambda) \). This already forces at least \( \frac{f(\Lambda) - m(\Lambda) + 1}{2} \) elements of the interval \([m(\Lambda), f(\Lambda)]\) to be absent from \( \Lambda \), so

\[ m(\Lambda) - 1 + \frac{f(\Lambda) - m(\Lambda) + 1}{2} \leq g(\Lambda) \leq 2h(\Lambda) - 1 \leq 2m(\Lambda) - 1. \]
Rearranging yields $f(\Lambda) \leq 3m(\Lambda) - 1$.

Note that the number of elements of $\Lambda$ in $[m(\Lambda), f(\Lambda)]$ is $f(\Lambda) - g(\Lambda)$. Since $f(\Lambda) \leq 3m(\Lambda) - 1$, the intervals $[m(\Lambda), 2m(\Lambda) - 1]$ and $[f(\Lambda) - m(\Lambda) + 1, f(\Lambda)]$ cover $[m(\Lambda), f(\Lambda)]$, so at least one of those intervals has at least half of the elements of $\Lambda$ in $[m(\Lambda), f(\Lambda)]$. In other words, one of the intervals contains at least $f(\Lambda) - g(\Lambda)$ elements of $\Lambda$.

Therefore, there are at least $\frac{f(\Lambda) - g(\Lambda)}{2}$ residues $r$ modulo $m(\Lambda)$ for which there exists $\lambda \in \Lambda$ with $\lambda \leq f(\Lambda)$ and $\lambda \equiv r \mod m(\Lambda)$. If such a $\lambda$ exists, it is impossible for $\Lambda$ to have an effective generator congruent to $r$ modulo $m(\Lambda)$, since such a generator would be the sum of $\lambda$ and a multiple of $m(\Lambda)$. Consequently, $\Lambda$ has at most $m(\Lambda) - \frac{f(\Lambda) - g(\Lambda)}{2}$ effective generators.

We thus have

$$g(\Lambda) < 2h(\Lambda) \leq 2m(\Lambda) - (f(\Lambda) - g(\Lambda))$$

$$f(\Lambda) < 2m(\Lambda),$$

as desired.

**Corollary 1.** If $\Lambda$ is an orderly semigroup, then $m(\Lambda) \geq f(\Lambda) + h(\Lambda) - g(\Lambda)$.

*Proof.* By Proposition, for any $\lambda \in \Lambda \cap [m(\Lambda), f(\Lambda)]$, we know that $\lambda + m(\Lambda) \in [f(\Lambda) + 1, f(\Lambda) + m]$, and $\lambda + m(\Lambda)$ cannot be an effective generator.

Note that there are $f(\Lambda) - g(\Lambda)$ elements of $\Lambda$ in $[m(\Lambda), f(\Lambda)]$. Hence, $\Lambda$ has at most $m(\Lambda) - (f(\Lambda) - g(\Lambda))$ effective generators. This gives us the inequality

$$h(\Lambda) \leq m(\Lambda) - f(\Lambda) + g(\Lambda),$$

which is the desired inequality upon rearranging terms.

**Corollary 2.** If $\Lambda'$ is a weak descendent of an orderly semigroup, then $f(\Lambda') < 3m(\Lambda')$.

*Proof.* Let $\Lambda$ be the orderly semigroup for which $\Lambda'$ is the weak descendent of $\Lambda$. By Lemma, the largest effective generator of $\Lambda$ is $f(\Lambda) + m(\Lambda)$. Since $\Lambda'$ is obtained from $\Lambda$ by removing effective generators of $\Lambda$, it follows that $f(\Lambda') \leq f(\Lambda) + m(\Lambda)$. Meanwhile, $m(\Lambda') = m(\Lambda)$, so

$$f(\Lambda') \leq f(\Lambda) + m(\Lambda) < 3m(\Lambda) = 3m(\Lambda').$$

**Corollary 3.** $n_{g,2} \leq t_g$.

*Proof.* By definition, $n_{g,2}$ counts the number of weak genus $g$ descendents of elements of $S_2$. Since all elements of $S_2$ are orderly, all weak descendents of elements of $S_2$ are counted under $t_g$ by the previous corollary. Thus, $n_{g,2} \leq t_g$. 

\[6\]
We next define the function $\tau(\Lambda, \Delta) = \{0\} \cup ((\Lambda \setminus \{0\}) + \Delta)$ for a numerical semigroup $\Lambda$ and $\Delta \in \mathbb{Z}$. In essence, $\tau(\Lambda, \Delta)$ is a shift of the non-zero elements of $\Lambda$ by $\Delta$. We record several basic properties of $\tau$ as lemmas.

**Lemma 4.** Let $\Lambda$ be a numerical semigroup, and suppose that $\Lambda' = \tau(\Lambda, \Delta)$ is also a numerical semigroup. Then, $f(\Lambda') = f(\Lambda) + \Delta$, $m(\Lambda') = m(\Lambda) + \Delta$, and $g(\Lambda') = g(\Lambda) + \Delta$.

*Proof.* These are all immediate from the definition of $\tau$. \hfill \Box

**Lemma 5.** If a numerical semigroup $\Lambda$ and an integer $\Delta$ satisfy $f(\Lambda) < 2m(\Lambda) + \Delta$, then $\tau(\Lambda, \Delta)$ is also a numerical semigroup.

*Proof.* Let $\Lambda' = \tau(\Lambda, \Delta)$. Note that $\min (\Lambda \setminus \{0\}) + \Delta = m(\Lambda) + \Delta \geq f(\Lambda) - m(\Lambda) + 1 \geq 0$. Hence, all elements of $\Lambda'$ are non-negative, and it is easy to see that $\max (\mathbb{N} \setminus \Lambda') = \max (\mathbb{N} \setminus \Lambda) + \Delta = f(\Lambda) + \Delta$.

For any non-zero $\lambda_1, \lambda_2 \in \Lambda'$, we have

$$
\lambda_1 + \lambda_2 \geq 2 \min (\Lambda' \setminus \{0\}) \\
= 2m(\Lambda) + 2\Delta \\
> f(\Lambda) + \Delta \\
= \max (\mathbb{N} \setminus \Lambda').
$$

Hence, $\Lambda'$ is closed under addition, so it is a numerical semigroup. \hfill \Box

**Lemma 6.** For a numerical semigroup $\Lambda$, let $L = L(\Lambda) = \{ x \in [0, f(\Lambda) - m(\Lambda)] \mid m(\Lambda) + x \in \Lambda \}$. If $f(\Lambda) < 2m(\Lambda)$, then $\lambda \in [f(\Lambda) + 1, f(\Lambda) + m]$ is an effective generator if and only if $\lambda - 2m(\Lambda) \notin L + L$.

*Proof.* By definition, $\lambda$ is an effective generator if and only if there do not exist two non-zero elements $\lambda_1, \lambda_2 \in \lambda$ such that $\lambda = \lambda_1 + \lambda_2$. Since $\lambda \leq f(\Lambda) + m(\Lambda)$, and $\lambda_1, \lambda_2 \geq m(\Lambda)$, it follows that we need only consider the situation where $\lambda_1, \lambda_2 \leq f(\Lambda)$.

In other words, we are only concerned with the case $\lambda_1, \lambda_2 \in L + m(\Lambda)$, so $\lambda$ is an effective generator if and only if $\lambda \notin (L + m(\Lambda)) + (L + m(\Lambda))$. Subtracting $2m(\Lambda)$ gives the result. \hfill \Box

**Corollary 4.** Let $\Lambda$ be a numerical semigroup, and suppose that $\Lambda' = \tau(\Lambda, \Delta)$ is also a numerical semigroup. Suppose further that $f(\Lambda) < 2m(\Lambda)$ and $f(\Lambda') < 2m(\Lambda')$. Then, $\Lambda$ is strongly descended if and only if $\Lambda'$ is, and $m(\Lambda) - h(\Lambda) = m(\Lambda') - h(\Lambda')$.

*Proof.* Using the notation of Lemma 6 note that $L(\Lambda) = L(\Lambda')$, so let us use $L$ to denote simultaneously $L(\Lambda)$ and $L(\Lambda')$. Let $K$ denote the set of numbers in $[f(\Lambda) + 1, f(\Lambda) + m(\Lambda)]$ that are not effective generators of $\Lambda$, and similarly, let $K'$ denote the set of numbers in $[f(\Lambda') + 1, f(\Lambda') + m(\Lambda')]$ that are not effective generators of $\Lambda'$. 

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Suppose $\lambda$ is an element of $K$. Then, by Lemma 6, $\lambda - 2m(\Lambda) \in L + L$. This implies first of all that $\lambda \geq 2m(\Lambda)$, and so by Lemma 4, $\lambda + 2\Delta \geq 2m(\Lambda') \geq f(\Lambda') + 1$. Also by Lemma 3, we know from $M(\lambda) \geq \lambda + 2\Delta \leq f(\Lambda') + m(\Lambda')$.

Thus, $\lambda + 2\Delta \in [f(\Lambda') + 1, f(\Lambda') + m(\Lambda')]$, and $\lambda + 2\Delta - 2m(\Lambda') = \lambda - 2m(\Lambda) \in L + L$, which implies by Lemma 8 that $\lambda + 2\Delta$ is not an effective generator of $\Lambda'$, and so $\lambda + 2\Delta \in K'$.

Consequently, $\lambda \mapsto \lambda + 2\Delta$ gives an injection of $K$ into $K'$. Since all of the arguments above still hold when the roles of $\Lambda$ and $\Lambda'$ are reversed, we find that this injection is in fact a bijection.

By Lemma 4, $\Lambda$ is strongly descended if and only if $f(\Lambda) + m(\Lambda) \not\in K$, which occurs if and only if $f(\Lambda) + m(\Lambda) + 2\Delta = f(\Lambda') + m(\Lambda') \not\in K'$. This in turn is equivalent to $\Lambda'$ being strongly descended, proving the first claim of the corollary. The second claim follows upon noting that $m(\Lambda) - h(\Lambda) = |K| = |K'| = m(\Lambda') - h(\Lambda')$.

We next prove two results having to do with counting the number of certain semigroups. Let $M(g, h)$ denote the set of strongly descended numerical semigroups of genus $g$ having $h$ effective generators. Then, the following lemma holds.

**Lemma 7.** $|M(g, h)| = |M(2g - 2h + 1, g - h + 1)|$ whenever $g < 2h$.

*Proof.* Let $\Delta = 2h - g - 1$, and note that $\Delta \geq 0$. It suffices to show that $\Lambda \mapsto \tau(\Lambda, \Delta)$ is a bijection from $M(2g - 2h + 1, g - h + 1)$ to $M(g, h)$, with inverse given by $\Lambda \mapsto \tau(\Lambda, -\Delta)$. Note that the semigroups in $M(g, h)$ and $M(2g - 2h + 1, g - h + 1)$ are orderly.

If $\Lambda \in M(2g - 2h + 1, g - h + 1)$, then $f(\Lambda) < 2m(\Lambda) \leq 2m(\Lambda) + \Delta$ by Proposition 4. Thus, by Lemma 5, $\Lambda' = \tau(\Lambda, \Delta)$ is a semigroup. In addition, $f(\Lambda') = f(\Lambda) + \Delta < 2m(\Lambda') + 2\Delta = 2m(\Lambda')$, so Corollary 4 applies. Thus, $\Lambda'$ is strongly descended, and $h(\Lambda') = m(\Lambda') - m(\Lambda) + h(\Lambda) = \Delta + (g - h + 1) = h$. By Lemma 4, we also know that $g(\Lambda') = g(\Lambda) + \Delta = g$. Hence, $\Lambda' \in M(g, h)$.

Next, suppose that $\Lambda \in M(g, h)$. Then, by Corollary 4 and the general fact that $f(\Lambda) \geq g(\Lambda)$, we find that

$$f(\Lambda) \leq 2f(\Lambda) - g(\Lambda)$$

$$\leq 2m(\Lambda) + g(\Lambda) - 2h(\Lambda)$$

$$< 2m(\Lambda) - \Delta.$$ 

Therefore, Corollary 4 applies, and so it can be verified that $\Lambda' = \tau(\Lambda, -\Delta)$ belongs to $M(2g - 2h + 1, g - h + 1)$ by the same arguments as before. We thus conclude that $M(g, h)$ and $M(2g - 2h + 1, g - h + 1)$ are in bijection, proving the lemma.

**Lemma 8.** $\sum_{i=0}^{\infty} |M(2i + 1, i + 1)| \varphi^{-i}$ converges.
Proof. Let Λ be any semigroup in $M(2i+1, i+1)$. Each number in $[f(\Lambda)+1, 2m(\Lambda)-1]$ is an effective generator of Λ (since it cannot be the sum of two non-zero elements of Λ), so $h(\Lambda) \geq 2m(\Lambda) - f(\Lambda) - 1$. Thus, we have

$$f(\Lambda) \geq g(\Lambda) = 2i + 1 = 2(i + 1) - 1$$

$$= 2h(\Lambda) - 1 \geq 4m(\Lambda) - 2f(\Lambda) - 3,$$

which upon rearranging yields $3 + 4(f(\Lambda) - m(\Lambda)) \geq f(\Lambda)$. We therefore find that

$$\sum_{i=0}^{\infty} |M(2i+1, i+1)| \varphi^{-i} = \sum_{i=0}^{\infty} \sum_{\Lambda \in M(2i+1, i+1)} \varphi^{-g(\Lambda)-h(\Lambda)}$$

$$\leq \sum_{f,m} \sum_{\Lambda \in S(m,f)} \varphi^{-g(\Lambda)-h(\Lambda)}$$

$$\leq \varphi \sum_{f,m} (f - m) \left( \frac{1.618}{\varphi} \right)^{f-m-1}$$

$$\leq \varphi \sum_{k=0}^{\infty} (3 + 4k)k \left( \frac{1.618}{\varphi} \right)^{k-1} < \infty.$$

This proves the lemma. \qed

Having established several results relating to orderly semigroups, we are ready to estimate $n_{g,2}$. We have

$$n_{g,2} = \sum_{\Lambda \in S_2} N_g(\Lambda)$$

$$= \sum_{0 \leq i < \frac{g}{3}} \sum_{\Lambda \in S_2 \atop g(\Lambda) - h(\Lambda) = i} N_g(\Lambda)$$

$$\leq \sum_{0 \leq i < \frac{g}{3}} \sum_{\Lambda \in S_2 \atop g(\Lambda) - h(\Lambda) = i} h(\Lambda) \left( g - g(\Lambda) \right)$$

$$\leq \sum_{0 \leq i < \frac{g}{3}} \sum_{i < h \leq g - i} |M(i + h, h)| \left( \begin{array}{c} h \\ g - i - h \end{array} \right)$$

$$= \sum_{0 \leq i < \frac{g}{3}} |M(2i + 1, i + 1)| \left( \begin{array}{c} h \\ g - i - h \end{array} \right)$$
\[
\sum_{0 \leq i < \frac{g}{3}} |M(2i + 1, i + 1)| F_{g-i+1} \leq \varphi^g \sum_{0 \leq i < \frac{g}{3}} |M(2i + 1, i + 1)| \varphi^{-i}.
\]

The sum in the last expression is bounded, since \(\sum_{i=0}^{\infty} |M(2i+1, i+1)| \varphi^{-i}\) converges. Thus, we find that \(n_{g,2} \varphi^{-g}\) is bounded.

### 3.2 Estimating \(n_{g,3}\)

Consider a numerical semigroup \(\Lambda \in S_3\). We first claim that \(h(\Lambda) \geq 2m(\Lambda) - f(\Lambda) - 1\).

If \(f(\Lambda) + 1 \geq 2m(\Lambda)\), the claim holds trivially. Otherwise, it is easy to check that the numbers in the interval \([f(\Lambda) + 1, 2m(\Lambda) - 1]\) are all effective generators. The interval \([f(\Lambda) + 1, 2m(\Lambda) - 1]\) contains \(2m(\Lambda) - f(\Lambda) - 1\) numbers, so the claim is proven.

Since \(\Lambda \in S_3\), we have \(g(\Lambda) - h(\Lambda) \geq \frac{g}{3}\). Combining this with the claim above yields

\[
g(\Lambda) \geq \frac{g}{3} + 2m(\Lambda) - f(\Lambda) - 1.
\]

Noting that \(g(\Lambda) \leq f(\Lambda)\), we can rearrange this to obtain

\[
f(\Lambda) - m(\Lambda) \geq \frac{g}{6} - \frac{1}{2} > \frac{g}{6} - 1.
\]

Also, note that if \(N_g(\Lambda) > 0\), then we must have \(m(\Lambda) \leq g(\Lambda) + 1 \leq g + 1\). Combining these facts with Lemma 3, we find that

\[
n_{g,3} = \sum_{\Lambda \in S_3} N_g(\Lambda)
\leq \sum_{\Lambda \in S_3, m(\Lambda) \leq g+1} \varphi^{g-g(\Lambda)+h(\Lambda)}
\leq \varphi^g \sum_{k > \frac{g}{6} - 1} \sum_{f-m=k} \sum_{m \leq g+1} \varphi^{-g(\Lambda)+h(\Lambda)}
\leq 5 \varphi^g \sum_{k > \frac{g}{6} - 1} \sum_{f-m=k} \sum_{m \leq g+1} k \left( \frac{1.618}{\varphi} \right)^{k-1}
\leq 5 \varphi^g (g+1) \sum_{k > \frac{g}{6}} k \left( \frac{1.618}{\varphi} \right)^k
= o(\varphi^g).
\]

Since we know that \(n_g\) grows at least as fast as \(\varphi^g\), this shows that \(n_{g,3}\) makes a negligible contribution as \(g \to \infty\).
3.3 Estimating $n_g$

Now that we have estimated $n_{g,2}$ and $n_{g,3}$ separately, it is possible to give an estimate of $n_g$. Recall that we showed

\[ n_{g,2} \leq t_g \]
\[ n_{g,2} = O(\varphi^g) \]
\[ n_{g,3} = o(\varphi^g). \]

The last two bounds give $t_g \leq n_g = n_{g,2} + n_{g,3} = O(\varphi^g)$, which proves Conjecture \[\text{2}\]. Furthermore, we find that $t_g - t_g = n_{g,3} + n_{g,2} - t_g \leq n_{g,3} = o(\varphi^g)$, and it is known that $n_g \geq \varphi^g$. This proves \[\lim_{g \to \infty} \frac{t_g}{n_g} = 1\], which is Conjecture \[\text{1}\]. As noted before, this implies the main result, at least having assumed Lemma \[\text{3}\]. In the next two sections, we set out to prove Lemma \[\text{3}\].

4 Some technical preliminaries

Before proving Lemma \[\text{3}\], we need to establish another inequality not directly involving numerical semigroups. Let $S$ be any finite set of positive integers, and let $m$, $f$, and $d$ be positive integers satisfying $d < f$. (For the purposes of this section, these numbers can be considered to bear no relation to numerical semigroups, but we will later apply our results to the case where $m$ is the multiplicity and $f$ is the Frobenius number of a numerical semigroup.)

We say a subset $U \subset S$ is $(m, f, d)$-admissible if no two elements of $U$ sum to $f + m$, and if $x \in U$ and $x + m \in S$, then $x + m \in U$. Let $A_{(m,f,d)}(S)$ denote the set of all $(m, f, d)$-admissible subsets of $S$. Where there is no risk of confusion, we will drop the $(m, f, d)$ and simply say that $U$ is an admissible subset of $S$, and we will write $A(S)$ for $A_{(m,f,d)}(S)$.

For an admissible subset $U \subset S$, let $E(U, S)$ denote the set of all integers $x \in S$ such that $x, x + m \notin U$, but $x - d \in U$. Define $s(U, S) = |E(U, S)| - |E'(U, S)|$. When it is clear from context what $S$ is, we will simply write $E(U)$, $E'(U)$, and $s(U)$.

Define the $(m, f, d)$-weight of a set $S$ to be

\[ \sum_{U \in A_{(m,f,d)}(S)} \varphi^{-s(U,S)}. \]

We will denote it by $w_{(m,f,d)}(S)$, or simply $w(S)$ when it is clear what the values of $m$, $f$, and $d$ are. If $S$ is empty, we define $w(S)$ to be 1. The main result of this section is the following lemma.

---

4The astute reader may notice that $E'(U, S)$ is actually independent of $S$. However, it is convenient to write it in this way for sake of consistency with the notation for $E(U, S)$ and as a reminder that $U$ is admissible as a subset of $S$. 

Lemma 9. Let $m$, $f$, and $d$ be positive integers such that $d < f$, and let $S = \{m + d + 1, m + d + 2, \ldots, f - 1\}$. Then,

$$w_{m,f,d}(S) \leq 1.618^{|S| + d + 2}.$$  

The rest of this section is devoted to proving Lemma 9. To see how Lemma 9 is used to prove Lemma 10, the reader may wish to skip ahead to the next section. For the remainder of the section, let $m$, $f$, and $d$ be fixed positive integers with $d < f$. We first observe that truncating a set from below can only decrease its weight. More precisely, for any set $S$, define $V_k(S) = \{s \in S \mid s > k\}$. Then, the following inequality holds.

Lemma 10. For any set $S$ of positive integers and any $k$, $w(V_k(S)) \leq w(S)$.

Proof. First of all, since $V_k(S) \subset S$, it is clear that any admissible subset $U$ of $V_k(S)$ is also an admissible subset of $S$.

Clearly, $E(U, V_k(S)) \subset E(U, S)$. The reverse is also true; let $x$ be any element of $E(U, S)$. By definition, $x \in S$ and $x - d \in U$. Since $U \subset V_k(S)$, it follows that $x > k$, and so $x \in V_k(S)$. It then follows that $x \in E(U, V_k(S))$. Hence, $E(U, V_k(S)) = E(U, S)$.

Similarly, it is easy to check that $E'(U, V_k(S)) = E'(U, S)$. Thus,

$$w(V_k(S)) = \sum_{U \in \mathcal{A}(V_k(S))} \varphi^{-s(U, V_k(S))} = \sum_{U \in \mathcal{A}(V_k(S))} \varphi^{-s(U, S)}$$

$$\leq \sum_{U \in \mathcal{A}(S)} \varphi^{-s(U, S)} = w(S),$$

as desired. \qed

Another important observation is that the weight is submultiplicative in a certain sense. In particular, we have the following lemma.

Lemma 11. If $S_1$ and $S_2$ are sets such that $f + m \not\in S_1 + S_2$, and furthermore, for any $s_1 \in S_1$ and $s_2 \in S_2$, we have $s_1 \not\equiv s_2 \pmod{m}$, then $w(S_1 \cup S_2) \leq w(S_1)w(S_2)$.

Proof. Let $S = S_1 \cup S_2$, and note that $S_1$ and $S_2$ are disjoint. It is easy to check that $U \subset S$ is an admissible subset of $S$ if and only if $U \cap S_1$ and $U \cap S_2$ are admissible subsets of $S_1$ and $S_2$, respectively. Thus, there is a bijection between admissible subsets $U$ of $S$ and pairs of admissible subsets $(U_1, U_2)$ of $S_1$ and $S_2$; it is given by $U \mapsto (U \cap S_1, U \cap S_2)$.

Next, for each admissible subset $U$ of $S$, we claim that

$$s(U, S) \geq s(U_1, S_1) + s(U_2, S_2),$$

where $U_i = U \cap S_i$. Note that if $x \in E(U_1, S_1)$, then $x \in S_1$, but $x, x + m \not\in U_1$ and $x - d \in U_1 \subset U$. Since $S_1$ and $S_2$ lie in distinct residue classes modulo $m$, we know that $x, x + m \not\in S_2$. It follows that $x \in E(U, S)$. 

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Hence, \( E(U_1, S_1) \subset E(U, S) \), and analogously, \( E(U_2, S_2) \subset E(U, S) \). Upon observing that \( E(U_1, S_1) \) and \( E(U_2, S_2) \) are disjoint because \( S_1 \) and \( S_2 \) are disjoint, this shows that \( |E(U, S)| \geq |E(U_1, S_1)| + |E(U_2, S_2)| \).

In addition, again using the fact that \( S_1 \) and \( S_2 \) lie in distinct residue classes modulo \( m \), it is easy to see that \( E'(U, S) = E'(U_1, S_1) \cup E'(U_2, S_2) \), and \( E'(U_1, S_1) \) is disjoint from \( E'(U_2, S_2) \). Hence, \( |E'(U, S)| = |E'(U_1, S_1)| + |E'(U_2, S_2)| \).

Subtracting this identity from the previous inequality yields \( s(U, S) \geq s(U_1, S_1) + s(U_2, S_2) \). Then, using the bijection between admissible subsets of \( S \) and pairs of admissible subsets of \( S_1 \) and \( S_2 \), we find that

\[
\begin{align*}
    w(S) &= \sum_{U \in A(S)} \varphi^{-s(U, S)} = \sum_{U_1 \in A(S_1)} \varphi^{-s(U_1 \cup U_2, S)} \leq \sum_{U_1 \in A(S_1)} \varphi^{-s(U_1, S_1) - s(U_2, S_2)} \\
    &= \sum_{U_1 \in A(S_1)} \varphi^{-s(U_1, S_1)} \sum_{U_2 \in A(S_2)} \varphi^{-s(U_2, S_2)} = w(S_1)w(S_2),
\end{align*}
\]

as desired. \( \square \)

Lemma 11 allows us to bound the weight of a set by partitioning it and bounding the different parts separately. To this end, let \( r \) be an integer between 0 and \( m-1 \), and define \( S(r) \) to be the set of all integers in the range \([m, f]\) that are congruent to \( r \) or \( f-r \) modulo \( m \). Let \( I(r) \) denote the number of integers in the interval \([m, f]\) congruent to \( r \) modulo \( m \).

In more explicit terms, \( S(r) \) is the set \( \{r + m, r + 2m, \ldots, r + I(r)m\} \cup \{f - r, f - r - m, \ldots, f - r - (I(r) - 1)m\} \). Assuming that \( r \neq f - r \mod m \), any admissible subset \( U \) of \( S(r) \) takes the form \( \{r + (i+1)m, r + (i+2)m, \ldots, r + I(r)m\} \cup \{f - r, f - r - m, \ldots, f - r - (j - 1)m\} \), where \( i \) and \( j \) are between 0 and \( I(r) \). If \( i = I(r) \), then there are no elements in \( U \) congruent to \( r \) modulo \( m \), and similarly, if \( j = 0 \), there are no elements in \( U \) congruent to \( f - r \) modulo \( m \). In addition, we require that no two elements of an admissible subset sum to \( f + m \), so either \( i = I(r), j = 0 \), or

\[
(r + (i+1)m) + (f - r - (j - 1)m) > f + m \quad i \geq j.
\]

We say that \( U \) has signature \((i, j, I(r))\). The signature of \( U \) completely determines the size of \( E'(U, S(r)) \), as the next lemma shows.

**Lemma 12.** Suppose that an admissible subset \( U \) of \( S(r) \) has signature \((i, j, I(r))\). Then,

\[
|E'(U, S(r))| = \begin{cases} 
    j - 1 & \text{if } i = I(r) \text{ and } 1 \leq j \leq I(r), \\
    I(r) - i - 1 & \text{if } 0 \leq i \leq I(r) - 1 \text{ and } j = 0, \\
    I(r) - i + j - 2 & \text{if } I(r) > i \geq j > 0, \\
    0 & \text{if } i = I(r) \text{ and } j = 0.
\end{cases}
\]
Proof. First, recall that by the preceding discussion, the four cases given in the lemma indeed cover all possible signatures of $U$.

If $U$ has signature $(i, j, I(r))$, then let $A = \{r + (i + 1)m, r + (i + 2)m, \ldots, r + (I(r) - 1)m\}$, and let $B = \{f - r - m, f - r - 2m, \ldots, f - r - (j - 1)m\}$. If $i \geq I(r) - 1$, we take $A$ to be empty, and if $j \leq 1$, we take $B$ to be empty. Then, $E'(U, S(r)) = A \cup B$, and $|E'(U, S(r))| = |A| + |B|$.

Note that $|A| = I(r) - i - 1$ unless $i = I(r)$, in which case $|A| = 0$. Similarly, $|B| = j - 1$ unless $j = 0$, in which case $|B| = 0$. The formula for $|E'(U, S(r))|$ stated in the lemma follows from applying the formulas for $|A|$ and $|B|$ to the four cases. □

Let $\ell$ be the integer between 0 and $m - 1$ congruent to $f - r$ modulo $m$. Define $N(r)$ to be the least non-negative integer such that $r + nd \geq \ell - nd$. Finally, define $T(r) = \bigcup_{i=0}^{N(r)-1} S(r + id)$ (take $T(r)$ to be empty if $N(r) = 0$).

### 4.1 Some bounds on $w(S(r))$ and $w(T(r))$

Let us first bound $S(r)$ where $r \not\equiv f - r \mod m$. In the cases $I(r) = 1$ and $I(r) = 2$, we have the following lemma.

**Lemma 13.** Let $r$ be an integer satisfying $0 \leq r \leq m - 1$ and $r \not\equiv f - r \mod m$. Then, the following bounds hold:

1. If $I(r) = 1$, then $|S(r)| = 2$, and
   
   *(a) $w(S(r)) \leq 3$.
   *(b) $w(S(r) \setminus \{r + m\}) \leq 2 < 0.7726 \cdot 1.618^{|S(r)|}$.
   *(c) $w(S(r) \setminus \{r + m, f - r\}) = 1 < 0.3820 \cdot 1.618^{|S(r)|}$.

2. If $I(r) = 2$, then $|S(r)| = 4$, and
   
   *(a) $w(S(r)) \leq 4 + 2\varphi < 1.0559 \cdot 1.618^{|S(r)|}$.
   *(b) $w(S(r) \setminus \{r + m\}) \leq 4 + \varphi < 0.8198 \cdot 1.618^{|S(r)|}$.
   *(c) $w(S(r) \setminus \{r + m, f - m - r\}) \leq 4 < 0.5837 \cdot 1.618^{|S(r)|}$.

**Proof.** We explicitly determine the possible signatures of admissible subsets $U \subset S(r)$. We then bound $w(S(r))$ using the inequality

$$w(S(r)) = \sum_{U \in \mathcal{A}(S(r))} \varphi^{-s(U)} \leq \sum_{U \in \mathcal{A}(S(r))} \varphi^{|E'(U, S(r))|},$$

where we can compute $|E'(U, S(r))|$ from the signature of $U$ using Lemma 12.

If $I(r) = 1$, then note that $S(r) = \{r + m, f - r\}$. The possible signatures of $U$ are $(0, 0, 1), (1, 0, 1)$, and $(1, 1, 1)$. In each case, $|E'(U, S(r))| = 0$. Hence, $w(S(r)) \leq 3$. It
is also routine to check that \( w(S(r) \setminus \{ r + m \}) \leq 2 \), and \( w(S(r) \setminus \{ r + m, f - r \}) = 1 \). This proves part (i).

If \( I(r) = 2 \), then \( S(r) = \{ r + m, r + 2m, f - r, f - r - m \} \). The possible signatures of \( U \) are \((2, 0, 2), (2, 1, 2), (1, 0, 2), (1, 1, 2), (0, 0, 2), \) and \((2, 2, 2)\). In the first four cases, \(|E(U, S(r))| = 0\), while in the other two, \(|E(U, S(r))| = 1\). Thus, \( w(S(r)) \leq 4 + 2\varphi \).

The admissible subsets of \( S(r) \setminus \{ r + m \} \) are the same as those of \( S(r) \) with the exception of the set having signature \((0, 0, 2)\). The admissible subsets of \( S(r) \setminus \{ r + m, f - r - m \} \) are the same as those of \( S(r) \) except the sets having signature \((0, 0, 2)\) and \((2, 2, 2)\). Therefore, \( w(S(r) \setminus \{ r + m \}) \leq 4 + \varphi \) and \( w(S(r) \setminus \{ r + m, f - r - m \}) \leq 4 \). This proves part (ii), completing the proof. \( \square \)

Using much the same approach, we can also give estimates of \( w(S(r)) \) when \( I(r) \geq 3 \).

**Lemma 14.** Let \( r \) be an integer satisfying \( 0 \leq r \leq m - 1 \) and \( r \not\equiv f - r \mod m \). Also, suppose that \( I(r) \geq 3 \). Then \( |S(r)| = 2I(r) \), and

\[
w(S(r)) \leq 0.8755 \cdot 1.618^{|S(r)|}.
\]

**Proof.** Let us partition \( A(S(r)) = A_1 \cup A_2 \cup A_3 \cup A_4 \) according to the four cases of Lemma 12. More explicitly,

1. \( A_1 \) consists of those subsets \( U \) having signature \((I(r), j, I(r))\) where \( 1 \leq j \leq I(r) \).
2. \( A_2 \) consists of those subsets \( U \) having signature \((i, 0, I(r))\) where \( 0 \leq i \leq I(r) - 1 \).
3. \( A_3 \) consists of those subsets \( U \) having signature \((i, j, I(r))\) where \( I(r) > i \geq j > 0 \).
4. \( A_4 \) consists of the single subset \( U \) having signature \((I(r), 0, I(r))\) (namely, the empty set).

By Lemma 12 we have

\[
\sum_{U \in A_1} \varphi^{|E(U, S(r))|} = \sum_{j=1}^{I(r)} \varphi^{j-1} = \sum_{k=0}^{I(r)-1} \varphi^k.
\]

\[
\sum_{U \in A_2} \varphi^{|E(U, S(r))|} = \sum_{i=0}^{I(r)-1} \varphi^{I(r) - i - 1} = \sum_{k=0}^{I(r)-1} \varphi^k.
\]

\[
\sum_{U \in A_3} \varphi^{|E(U, S(r))|} = \sum_{I(r) > i \geq j > 0} \varphi^{I(r) - i + j - 2}
= \sum_{k=0}^{I(r)-2} (I(r) - k - 1)\varphi^{I(r) - k - 2} = \sum_{k=0}^{I(r)-2} (k + 1)\varphi^k.
\]

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\[
\sum_{U \in A_4} \varphi^{\left|E'(U, S(r))\right|} = 1,
\]

It follows that

\[
w(S(r)) = \sum_{U \in A(S(r))} \varphi^{-s(U)} \leq \sum_{U \in A(S(r))} \varphi^{\left|E'(U, S(r))\right|} = 1 + 2\varphi^{I(r) - 1} + \sum_{k=0}^{I(r)-2} (k + 3)\varphi^k.
\]

Let \(W_N\) denote this final expression when \(I(r) = N\). It remains to show that \(W_N \leq 1.618^{2N}\) when \(N \geq 3\). We can verify this in the cases \(N = 3\) and \(N = 4\) by explicit computation:

\[
W_3 = 1 + 2\varphi^2 + (3 + 4\varphi) = 4 + 4\varphi + 2\varphi^2 < 0.8755 \cdot 1.618^6
\]
\[
W_4 = 1 + 2\varphi^3 + (3 + 4\varphi + 5\varphi^2) = 4 + 4\varphi + 5\varphi^2 + 2\varphi^3 < 0.8755 \cdot 1.618^8.
\]

For all \(N \geq 4\), we also have

\[
\left(\varphi + \frac{1}{\varphi}\right) W_N > 1 + 2\varphi^N + 2\varphi^{N-2} + \sum_{k=0}^{N-2} (k + 3)\varphi^{k+1} + \sum_{k=0}^{N-3} (k + 4)\varphi^k
\]

\[
\geq 1 + 2\varphi^N + \sum_{k=0}^{N-1} (k + 2)\varphi^k + \sum_{k=0}^{N-3} (k + 4)\varphi^k
\]

\[
= 1 + 2\varphi^N + \sum_{k=0}^{N-1} (k + 3)\varphi^k + \left(\sum_{k=0}^{N-3} (k + 3)\varphi^k - \varphi^{N-2} - \varphi^{N-1}\right)
\]

\[
\geq 1 + 2\varphi^N + \sum_{k=0}^{N-1} (k + 3)\varphi^k + \left((N - 1)\varphi^{N-4} + N\varphi^{N-3} - \varphi^{N-2} - \varphi^{N-1}\right)
\]

\[
= 1 + 2\varphi^N + \sum_{k=0}^{N-1} (k + 3)\varphi^k + \left((N - 1) + N\varphi - \varphi^2 - \varphi^3\right)\varphi^{N-4}
\]

\[
> 1 + 2\varphi^N + \sum_{k=0}^{N-1} (k + 3)\varphi^k = W_{N+1}.
\]

Since \(\varphi + \frac{1}{\varphi} < 1.618^2\), the lemma follows by induction.

\[\square\]

**Corollary 5.** Let \(r\) be an integer satisfying \(0 \leq r \leq m - 1\) and \(r \neq f - r \mod m\). Also, suppose that \(I(r) \geq 3\). Then,

\[
w(T(r)) \leq 1.618^{|T(r)|}.
\]
Proof. Recall that \( T(r) = \bigcup_{i=0}^{N(r)-1} S(r + id) \). Thus, by Lemma 11 and Lemma 14, we find that
\[
\nu(T(r)) \leq \prod_{i=0}^{N(r)-1} \nu(S(r + id)) \leq \prod_{i=0}^{N(r)-1} 1.618^{|S(r + id)|} = 1.618^{|T(r)|},
\]
as desired. \( \square \)

Finally, we can give a bound in the case where \( r \equiv f - r \mod m \).

Lemma 15. Let \( r \) be an integer satisfying \( 0 \leq r \leq m-1 \) and \( r \equiv f - r \mod m \) (if such an integer exists). Then, \( |S(r)| = I(r) \), and
\[
\nu(S(r)) \leq 1.618^{|S(r)|}.
\]

Proof. Let \( \ell \) be the remainder when \( f \) is divided by \( m \). Then, \( r = \ell \) or \( r = \frac{m + \ell}{2} \). In either case, \( S(r) = \{m + r, 2m + r, \ldots, I(r)m + r\} \), and so \( |S(r)| = I(r) \).

The non-empty admissible subsets \( U \) of \( S(r) \) take the form \( \{im + r, (i + 1)m + r, \ldots, I(r)m + r\} \), where \( i \leq I(r) \). Since no two elements of \( U \) sum to \( f + m \), we must also have
\[
2(im + r) > f + m = (I(r) + 1)m + 2r, \quad i > \frac{I(r) + 1}{2}.
\]

Note that \( E'(U, S(r)) = \{im + r, (i + 1)m + r, \ldots, (I(r) - 1)m + r\} \), so \( s(U, S(r)) \geq -|E'(U, S(r))| = I(r) - i \). Accounting for the fact that \( s(\emptyset, S(r)) = 0 \), it follows that
\[
\nu(S(r)) = \sum_{U \in \mathcal{A}(S(r))} \varphi^{-s(U, S(r))} \leq 1 + \sum_{I(r) - 1 \leq i \leq I(r)} \varphi^{I(r) - i}
\]
\[
= 1 + \sum_{i=0}^{[\frac{I(r)-1}{2}]-1} \varphi^i = 1 + \varphi^{[\frac{I(r)-1}{2}]+1} - \varphi.
\]

It is routine to check that this quantity is at most \( 1.618^{I(r)} \). \( \square \)

4.2 Bounds on \( \nu(T(r)) \) when \( I(r) = 1 \) or \( I(r) = 2 \)

We now describe another method for bounding the weight of \( T(r) \) in terms of \( N(r) \) and \( I(r) \). Let \( U \) be an admissible subset of \( T(r) \), and let \( U_i = U \cap S(r + id) \) for each \( i < N(r) \). Note that \( I(r + id) = I(r) \) for each \( i < N(r) \).

By Lemma 12, it is clear that the number \( e'_i(U) \) of elements in \( E'(U, T(r)) \) that are congruent to \( r + id \) or \( f - r - id \mod m \) (in other words, are in \( S(r + id) \)) depends only on the signature of \( U_i \). A similar statement holds for the number \( e_i(U) \) of elements in \( E(U, T(r)) \) congruent to \( r + (i + 1)d \) or \( f - r - id \mod m \).
Lemma 16. The value of \( e_i(U) \) depends only on the signatures of \( U_i \) and \( U_{i+1} \). In particular, if their signatures are \((a_i, b_i, I(r))\) and \((a_{i+1}, b_{i+1}, I(r))\), respectively, then

\[
e_i(U) = \begin{cases} 
I(r) - a_i + \max(b_{i+1} - b_i - 1, 0) & \text{if } b_i > 0 \text{ and } a_{i+1} = I(r) \\
\max(a_{i+1} - a_i - 1, 0) + b_{i+1} & \text{if } b_i = 0 \text{ and } a_{i+1} < I(r) \\
\max(a_{i+1} - a_i - 1, 0) + \max(b_{i+1} - b_i - 1, 0) & \text{if } b_i > 0 \text{ and } a_{i+1} < I(r) \\
I(r) - a_i + b_{i+1} & \text{if } b_i = 0 \text{ and } a_{i+1} = I(r).
\end{cases}
\]

Proof. Let \( N_1 \) and \( N_2 \) denote respectively the number of elements in \( E(U, T(r)) \) congruent to \( r + (i + 1)d \) and \( f - r - id \) modulo \( m \).

The number \( r + (i + 1)d + km \) is in \( E(U, T(r)) \) if and only if \( a_i < k \) and either \( a_{i+1} > k \) or \( a_{i+1} = I(r) \). If \( a_{i+1} = I(r) \), then \( a_i < k \leq I(r) \), and so \( N_1 = I(r) - a_i \). Otherwise, \( a_i < k < a_{i+1} \), and \( N_1 = \max(a_{i+1} - a_i - 1, 0) \).

Similarly, \( f - r - id - km \) is in \( E(U, T(r)) \) if and only if \( b_{i+1} > k \) and either \( b_i < k \) or \( b_i = 0 \). If \( b_i = 0 \), then \( 0 \leq k < b_{i+1} \), and so \( N_2 = b_{i+1} \). Otherwise, \( b_i < k < b_{i+1} \), and \( N_2 = \max(b_{i+1} - b_i - 1, 0) \).

Writing out the formula for \( N_1 + N_2 \) in the various cases yields the result.

It follows that for every pair of signatures \((u, v)\), there is a number \( G(u, v) \) such that if \( U_i \) has signature \( u \) and \( U_{i+1} \) has signature \( v \), then \( e_i(U) - e'_i(U) = G(u, v) \). We set aside the task of actually computing \( G(u, v) \) for the moment, but we note that \( G(u, v) \) does not depend explicitly on \( i \) or \( U \).

Let the signature of \( U_i \) be \( u_i \) for each \( i \). We find that

\[
s(U, T(r)) = |E(U, T(r))| - |E'(U, T(r))| \geq \sum_{i=0}^{N(r)-1} (e_i(U) - e'_i(U))
\]

\[
= \sum_{i=0}^{N(r)-2} G(u_i, u_{i+1}) + e_{N(r)-1}(U) - e'_{N(r)-1}(U)
\]

\[
\geq \sum_{i=0}^{N(r)-2} G(u_i, u_{i+1}) - e'_{N(r)-1}(U).
\]

It follows, then, that

\[
w(T(r)) = \sum_{U \in A(T(r))} \varphi^{-s(U, T(r))} \leq \sum_{U \in A(T(r))} \varphi^{-\sum_{i=0}^{N(r)-2} G(u_i, u_{i+1}) + e'_{N(r)-1}(U)}
\]

\[
= \sum_{U \in A(T(r))} \varphi^{e'_{N(r)-1}(U)} \prod_{i=0}^{N(r)-2} \varphi^{-G(u_i, u_{i+1})}.
\]

This bound can be expressed in matrix form.
Lemma 17. Let $r$ be an integer, and let the possible signatures of $T(r) \cap S(r + \text{id})$ be \{s$_1$, s$_2$, \ldots, s$_k$\}. Let $v$ denote the $k$-dimensional vector whose $j$th entry is the value of $\varphi^{e_{N(r^{-1}(U)}}$ when $u_{N(r)}^{-1} = s_j$. Also, let $1$ denote the $k$-dimensional vector all of whose entries are 1. Finally, let $M$ be the $k \times k$ matrix whose $ij$ entry is $\varphi^{-G(s_i,s_j)}$. Then,

$$w(T(r)) \leq 1^T M^{N(r)-1} v.$$ 

We may apply this specifically to the cases $I(r) = 1$ and $I(r) = 2$.

Lemma 18. If $I(r) = 1$, then $w(T(r)) < 1.1460 \cdot 1.618^{||T(r)||}$.

Proof. Using the same notation as above, the possible signatures of the $U_i$ when $U \in \mathcal{A}(T(r))$ and $I(r) = 1$ are $s_1 = (1,0,1)$, $s_2 = (1,1,1)$, and $s_3 = (0,0,1)$. By Lemmas 12 and 16, we find that the matrix $[\varphi^{-G(s_i,s_j)}]$ is

$$
\begin{bmatrix}
1 & 1 & 1 \\
\varphi^{-1} & 1 & 1 \\
\varphi^{-2} & 1 & \varphi^{-1}
\end{bmatrix}.
$$

It is routine to check that $v$ is

$$
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}.
$$

Applying Lemma 17, we have

$$w(T(r)) \leq \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
\varphi^{-1} & 1 & 1 \\
\varphi^{-2} & 1 & \varphi^{-1}
\end{bmatrix}^{N(r)-1} \begin{bmatrix} 1 \\
1 \\
1
\end{bmatrix}.$$ 

Let $W_n$ denote the right hand side of the above inequality when $N(r) = n$. It is not hard to show by induction that $W_n \leq 3 \cdot 1.618^{2n-2}$ (see Appendix for the computation details). This gives

$$w(T(r)) \leq W_{N(r)} \leq 3 \cdot 1.618^{2N(r)-2} < 1.1460 \cdot 1.618^{2N(r)} = 1.1460 \cdot 1.618^{||T(r)||}.$$ 

Lemma 19. If $I(r) = 2$, then $w(T(r)) < 1.0559 \cdot 1.618^{||T(r)||}$.

Proof. The possible signatures of the $U_i$ when $U \in \mathcal{A}(T(r))$ and $I(r) = 2$ are $s_1 = (2,0,2)$, $s_2 = (2,1,2)$, $s_3 = (1,0,2)$, $s_4 = (1,1,2)$, $s_5 = (0,0,2)$, and $s_6 = (2,2,2)$. By Lemmas 12 and 16, we find that the matrix $[\varphi^{-G(s_i,s_j)}]$ is
It is routine to check that \( \mathbf{v} \) is
\[
\frac{1}{1} \varphi^{-1} 1 \varphi^{-1} 1 \varphi^{-2}
\frac{1}{1} 1 1 1 1 
\frac{1}{1} 1 1 1 1
\frac{1}{1} \varphi^{-1} \varphi^{-2} 1 \varphi^{-1} 1 \varphi^{-3}
\frac{1}{\varphi} \varphi \varphi \varphi \varphi \varphi
\]

Applying Lemma 17, we have
\[
w(T(r)) \leq \left[1 \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 \end{array}\right]^{N(r)-1} \left[\begin{array}{cccccc} 1 \\ 1 \\ 1 \\ 1 \\ \varphi \end{array}\right].
\]

Let \( W_n \) denote the right hand side of the above inequality when \( N(r) = n \). This is a closed form for \( W_n \), and straightforward computations yield \( W_n \leq (4 + 2\varphi) \cdot 1.618^{4n-4} \) (for the details of the calculation, see Appendix). This gives
\[
w(T(r)) \leq W_{N(r)} \leq (4 + 2\varphi) \cdot 1.618^{4N(r)-4}
\leq 1.0559 \cdot 1.618^{4N(r)} = 1.0559 \cdot 1.618^{T(r)}.
\]

\[\square\]

**4.3 Proof of Lemma 9**

We have now developed the necessary tools to prove Lemma 9.

**Proof of Lemma 9.** Let \( \ell \) be the remainder when \( f \) is divided by \( m \). For convenience of notation, define \( S(r) \) to be the empty set when \( r \) is not an integer. Note that when \( 0 \leq x < \frac{\ell}{2} \), \( I(x) = I(0) \), and when \( \ell + 1 \leq x < \frac{m+\ell}{2} \), \( I(x) = I(0) - 1 \).

Rather than bounding \( w(S) \) directly, it will be more convenient to bound the weight of a similar set. Define
\[ S' = \left( \bigcup_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} S(x) \right) \cup \left( \bigcup_{x=\ell+1}^{\lfloor \frac{m+\ell}{2} \rfloor} S(x) \right), \]

and define \( S'' = V_{m+d}(S') \). Note that

\[ S' \cup S\left( \frac{\ell}{2} \right) \cup S\left( \frac{m+\ell}{2} \right) = \{m, m+1, \ldots, f\} \]

\[ S'' \cup V_{m+d}(S\left( \frac{\ell}{2} \right)) \cup V_{m+d}(S\left( \frac{m+\ell}{2} \right)) = S \cup \{f\}. \]

We claim that \( w(S'') \leq 1.618^{|S'|} \). To show this, we consider four cases according to the value of \( I(0) \).

**Case** \( I(0) > 3 \). We have

\[ S'' = \left( \bigcup_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{\lfloor \frac{m+\ell}{2} \rfloor} V_{m+d}(S(x)) \right) \]

Note that wherever \( x \) appears in the above equation, \( I(x) \geq 3 \). Thus, Lemma \( 14 \) applies, and

\[ w(S'') \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} w(S(x)) \cdot \prod_{x=\ell+1}^{\lfloor \frac{m+\ell}{2} \rfloor} w(S(x)) \]

\[ \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} 1.618^{|S(x)|} \cdot \prod_{x=\ell+1}^{\lfloor \frac{m+\ell}{2} \rfloor} 1.618^{|S(x)|} = 1.618^{|S'|}. \]

**Case** \( I(0) = 3 \). If \( d \leq \ell \), we again write

\[ S'' = \left( \bigcup_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{\lfloor \frac{m+\ell}{2} \rfloor} V_{m+d}(S(x)) \right), \]

noting that \( I(x) = 2 \) when \( \ell + 1 \leq x \leq \lfloor \frac{m+\ell}{2} \rfloor \). Applying Lemmas \( 14 \) and \( 19 \), we have

\[ w(S'') \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} w(S(x)) \cdot \prod_{x=\ell+1}^{\min\left(\lfloor \frac{m+\ell}{2} \rfloor, \ell+d \right)} w(T(x)) \]
\[\ell \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} 0.8557 \cdot 1.618^{|S(x)|} \cdot \prod_{x=\ell+1}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, \ell+1)} 1.0559 \cdot 1.618^{|T(x)|} \]

\[\leq 0.8557^{\ell/2} \cdot 1.0559^d \cdot 1.618^{|S'|} \leq 1.618^{|S'|}.\]

If \( \ell < d \leq \frac{m+\ell}{2} \), then we use the decomposition

\[S'' = \left( \bigcup_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{d} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=d}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, d)} V_{m+d}(S(x)) \right),\]

and applying Lemmas 14 and 19 and part (ii.b) of Lemma 13, we have

\[w(S'') \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} w(S(x)) \cdot \prod_{x=\ell+1}^{d} w(S(x) \setminus \{m + x\}) \cdot \prod_{x=d+1}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, 2d)} w(T(x)) \]

\[\leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} 0.8557 \cdot 1.618^{|S(x)|} \cdot \prod_{x=\ell+1}^{d} 0.8198 \cdot 1.618^{|S(x)|} \cdot \prod_{x=d+1}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, 2d)} 1.0559 \cdot 1.618^{|T(x)|} \]

\[\leq 0.8557^{\ell/2} \cdot 0.8198^{d-\ell} \cdot 1.0559^d \cdot 1.618^{|S'|} \leq 1.618^{|S'|}.\]

Finally, if \( \frac{m+\ell}{2} < d \), then

\[S'' = \left( \bigcup_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=d}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, d)} V_{m+d}(S(x)) \right),\]

and applying Lemma 14 and part (ii.b) of Lemma 13 yields

\[w(S'') \leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} w(S(x)) \cdot \prod_{x=\ell+1}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, d)} w(S(x) \setminus \{m + x\}) \]

\[\leq \prod_{x=0}^{\lfloor \frac{\ell}{2} \rfloor} 0.8557 \cdot 1.618^{|S(x)|} \cdot \prod_{x=\ell+1}^{\min(\lfloor \frac{m+\ell}{2} \rfloor, d)} 0.8198 \cdot 1.618^{|T(x)|} \]

\[\leq 1.618^{|S'|}.\]
Case $I(0) = 2$. If $d \leq \frac{\ell}{2}$, then we decompose

$$S'' = \left( \bigcup_{x=0}^{d-1} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=d+1}^{\min(\left\lfloor \frac{\ell}{2} \right\rfloor, 2d)} V_{m+d}(T(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} V_{m+d}(T(x)) \right).$$

By Lemmas 18 and 19 and part (ii.b) of Lemma 13, we find that

$$w(S'') \leq \prod_{x=0}^{d-1} w(S(x) \setminus \{m+x\}) \cdot \prod_{x=d+1}^{\min(\left\lfloor \frac{\ell}{2} \right\rfloor, 2d)} w(T(x)) \cdot \prod_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} w(T(x))$$

$$\leq \prod_{x=0}^{d-1} 0.8198 \cdot 1.618^{\left|S(x)\right|} \cdot \prod_{x=d+1}^{\min(\left\lfloor \frac{\ell}{2} \right\rfloor, 2d)} 1.0559 \cdot 1.618^{|T(x)|} \cdot \prod_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} 1.1460 \cdot 1.618^{|T(x)|}$$

$$\leq 0.8198^d \cdot 1.0559^d \cdot 1.1460^d \cdot 1.618^{|S'|} \leq 1.618^{|S'|}.$$

If $\frac{\ell}{2} < d \leq \ell$, then

$$S'' = \left( \bigcup_{x=0}^{\ell-d-1} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell-d}^{\left\lfloor \frac{\ell}{2} \right\rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} V_{m+d}(T(x)) \right),$$

and

$$w(S'') \leq \prod_{x=0}^{\ell-d-1} w(S(x) \setminus \{m+x\}) \cdot \prod_{x=\ell-d}^{\left\lfloor \frac{\ell}{2} \right\rfloor} w(S(x) \setminus \{m+x, m+\ell-x\}) \cdot \prod_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} w(T(x))$$

$$\leq \prod_{x=0}^{\ell-d-1} 0.8198 \cdot 1.618^{|S(x)|} \cdot \prod_{x=\ell-d}^{\left\lfloor \frac{\ell}{2} \right\rfloor} 0.5837 \cdot 1.618^{|S(x)|} \cdot \prod_{x=\ell+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, \ell+d)} 1.1460 \cdot 1.618^{|T(x)|}$$

$$\leq 0.8198^{\ell-d} \cdot 0.5837^{\ell-\ell/2} \cdot 1.1460^d \cdot 1.618^{|S'|} \leq 1.618^{|S'|}.$$

If $\ell < d \leq \frac{m+\ell}{2}$, then

$$S'' = \left( \bigcup_{x=0}^{\left\lfloor \frac{\ell}{2} \right\rfloor} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{d} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=d+1}^{\min(\left\lfloor \frac{m+\ell}{2} \right\rfloor, 2d)} V_{m+d}(T(x)) \right),$$

23.
Finally, if \( m \) and \( w \) are empty, in which case \( w(0) = 1 \).

By part (i.b) of Lemma 13 and Lemma 18, we obtain

\[
\begin{align*}
\sum_{x=0}^{d} w(S(x) \setminus \{m+x, m+\ell-x\}) & \sum_{x=\ell+1}^{d} w(S(x) \setminus \{m+x\}) \\
\prod_{x=0}^{\min(\frac{m+\ell}{2}, 2d)} w(T(x))
\end{align*}
\]

\[
\leq \prod_{x=0}^{\frac{\ell}{2}} 0.5837 \cdot 1.618^{S(x)} \prod_{x=\ell+1}^{d} 0.7726 \cdot 1.618^{S(x)} \prod_{x=d+1}^{\min(\frac{m+\ell}{2}, 2d)} 1.1460 \cdot 1.618^{T(x)}
\]

\[
\leq 0.5837^{\ell/2} \cdot 0.7726^{-\ell} \cdot 1.1460^{d} \cdot 1.618^{S'} \leq 1.618^{S'}.
\]

Finally, if \( \frac{m+\ell}{2} < d \), then

\[
S'' = \left( \bigcup_{x=0}^{\frac{\ell}{2}} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=\ell+1}^{\min(\frac{m+\ell}{2}, 2d)} V_{m+d}(S(x)) \right),
\]

and

\[
w(S'') \leq \prod_{x=0}^{\frac{\ell}{2}} w(S(x) \setminus \{m+x, m+\ell-x\}) \prod_{x=\ell+1}^{\min(\frac{m+\ell}{2}, 2d)} w(T(x))
\]

\[
\leq \prod_{x=0}^{\frac{\ell}{2}} 0.5837 \cdot 1.618^{S(x)} \prod_{x=\ell+1}^{\frac{m+\ell}{2}} 0.7726 \cdot 1.618^{S(x)} \prod_{x=d+1}^{\min(\frac{m+\ell}{2}, 2d)} 1.1460 \cdot 1.618^{T(x)}
\]

\[
\leq 0.5837^{\ell/2} \cdot 0.7726^{\ell/2} \cdot 1.1460^{d} \cdot 1.618^{S'} \leq 1.618^{S'}.
\]

**Case** \( I(0) = 1 \). Note that if \( I(0) = 1 \) then \( f = m + \ell \), and so \( d \leq \ell \) unless \( S'' \) is empty, in which case \( w(S'') \leq 1.618^{S'} \) holds trivially. If \( d \leq \frac{\ell}{2} \), then

\[
S'' = \left( \bigcup_{x=0}^{d} V_{m+d}(S(x)) \right) \cup \left( \bigcup_{x=d+1}^{\min(d, \frac{m+\ell}{2})} V_{m+d}(T(x)) \right).
\]

By part (i.b) of Lemma \[13\] and Lemma \[18\] we obtain

\[
w(S'') \leq \prod_{x=0}^{d} w(S(x) \setminus \{m+x\}) \prod_{x=d+1}^{\min(d, \frac{m+\ell}{2})} w(T(x))
\]

\[
\leq \prod_{x=0}^{d} 0.7726 \cdot 1.618^{S(x)} \prod_{x=d+1}^{\min(d, \frac{m+\ell}{2})} 1.1460 \cdot 1.618^{T(x)}.
\]
\[ \leq 0.7726^d \cdot 1.1460^d \cdot 1.618^{|S'|} \leq 1.618^{|S'|}. \]

If instead \( \frac{d}{2} < d \leq \ell \), then

\[ S'' = \bigcup_{x=0}^{\lfloor \frac{d}{2} \rfloor} V_{m+d}(S(x)). \]

By part (i.b) of Lemma 13, we have

\[ w(S'') \leq \prod_{x=0}^{\lfloor \frac{d}{2} \rfloor} w(S(x) \setminus \{m+x\}) \leq \prod_{x=0}^{\lfloor \frac{d}{2} \rfloor} 0.7726 \cdot 1.618^{|S(x)|} \leq 1.618^{|S'|}. \]

This covers all possible values of \( I(0) \), establishing that \( w(S'') \leq 1.618^{|S'|} \). We now turn to the relatively simple task of bounding \( w(S) \) in terms of \( w(S'') \). First, note that for any admissible subset \( U \) of \( S \), \( U \cup \{f\} \) is an admissible subset of \( S \cup \{f\} \). Furthermore, for any element \( x \in E(U \cup \{f\}, S \cup \{f\}) \), we also have \( x \in E(U, S) \). Finally, if \( x \in E'(U, S) \), then clearly \( x \in E'(U \cup \{f\}, S \cup \{f\}) \). It follows that

\[ s(U, S) = |E(U, S)| - |E'(U, S)| \geq |E(U \cup \{f\}, S \cup \{f\})| - |E'(U \cup \{f\}, S \cup \{f\})| = s(U \cup \{f\}, S \cup \{f\}). \]

Thus,

\[ w(S) = \sum_{U \in A(S)} \varphi^{-s(U, S)} \leq \sum_{U \in A(S)} \varphi^{-s(U \cup \{f\}, S \cup \{f\})} \leq \sum_{U \in A(S \cup \{f\})} \varphi^{-s(U, S \cup \{f\})} = w(S \cup \{f\}). \]

Using the decomposition \( S \cup \{f\} = S'' \cup S(\frac{d}{2}) \cup S(\frac{m+d}{2}) \) and Lemma 15, we find that

\[ w(S) \leq w(S \cup \{f\}) \leq w(S'') \varphi \left( S \left( \frac{\ell}{2} \right) \right) \varphi \left( S \left( \frac{m+\ell}{2} \right) \right) \leq 1.618^{|S'|} \cdot 1.618^{|S(\frac{d}{2})|} \cdot 1.618^{|S(\frac{m+d}{2})|} = 1.618^{f-m+1} = 1.618^{|S|+d+2}, \]

as desired. \( \square \)
5 Proof of Lemma 3

We are now in a position to prove Lemma 3. Recall that $S(m, f)$ denotes the set of all strongly descended numerical semigroups having multiplicity $m$ and Frobenius number $f$. Define $S(m, f, d)$ to be the subset of $S(m, f)$ consisting of those semigroups whose second smallest non-zero element is $m + d$. For any $\Lambda \in S(m, f)$, we know that $f + 1 \in \Lambda$. Hence, $d \leq f - m + 1$, and so $S(m, f) = \bigcup_{d=1}^{f-m+1} S(m, f, d)$. We will prove Lemma 3 by decomposing $S(m, f)$ in this way and applying Lemma 9.

Proof of Lemma 3. As in Lemma 9, fix values for $m$, $f$, and $d$, and define $S = \{m + d + 1, m + d + 2, \ldots, f - 1\}$. For any $\Lambda \in S(m, f, d)$, because $\Lambda$ is strongly descended, $f + m$ is an effective generator by Lemma 1. Hence, no two elements of $\Lambda \cap S$ sum to $f + m$, and furthermore, if $x \in \Lambda$, then $x + m \in \Lambda$, so $\Lambda \cap S$ is an $(m, f, d)$-admissible subset of $S$.

We now give an upper bound on the number of effective generators of $\Lambda$ in terms of $E(\Lambda \cap S, S)$ and $E'(\Lambda \cap S, S)$. First, if $x \in S$ satisfies $x + m \not\in S$, then $x + m \in [f, f + m - 1]$. If in addition we know that $x + m \in \Lambda$, then in fact $x + m \in [f + 1, f + m - 1]$, since $f \not\in \Lambda$.

Now, suppose that $x \in E(\Lambda \cap S, S)$. Then, $x - d \in \Lambda$, so upon noting that $m + d \in \Lambda$, we find that $x + m \in \Lambda$, and $x + m = (x - d) + (m + d)$ is not an effective generator. Furthermore, by the definition of $E(\Lambda \cap S, S)$, $x \in S$ while $x + m \not\in S$. Thus, $x + m \in [f + 1, f + m - 1]$.

For an element $x \in \Lambda \cap S$ that is not in $E'(\Lambda \cap S, S)$, we have $x + m \in \Lambda$ but $x + m \not\in \Lambda \cap S$, so $x + m \not\in S$. Thus, we again have $x + m \in [f + 1, f + m - 1]$, and since $x \in \Lambda$ and $m \in \Lambda$, $x + m$ is not an effective generator.

Note that the sets $E(\Lambda \cap S, S)$ and $(\Lambda \cap S) \setminus E'(\Lambda \cap S, S)$ are disjoint. For any $x$ in either set, $x + m$ is a non-effective generator in the interval $[f + 1, f + m - 1]$. It follows that there are at least

$$|E(\Lambda \cap S)| + |(\Lambda \cap S) \setminus E'(\Lambda \cap S)|$$

$$\leq |E(\Lambda \cap S)| + |\Lambda \cap S| - |E'(\Lambda \cap S)|$$

$$= |\Lambda \cap S| - s(\Lambda \cap S)$$

elements of $\Lambda$ in the interval $[f + 1, f + m - 1]$ that are not effective generators. Since the effective generators of $\Lambda$ are all in the interval $[f + 1, f + m]$, it follows that

$$h(\Lambda) \leq m - |\Lambda \cap S| + s(\Lambda \cap S).$$

Note that $g(\Lambda) = f - |\Lambda \cap [1, f]| = f - |\{m, m + d\} \cup (\Lambda \cap S)|$, and $|S| = f - m - d - 1$. Substituting in these identities, we obtain

$$h(\Lambda) \leq m - (f - 2 - g(\Lambda)) + s(\Lambda \cap S)$$

$$g(\Lambda) - h(\Lambda) \geq f - m - 2 - s(\Lambda \cap S)$$
\[ g(\Lambda) - h(\Lambda) \geq |S| + d - 1 - s(\Lambda \cap S). \]

Using Lemma 9, we find that
\[
\sum_{\Lambda \in \mathcal{S}(m,f,d)} \varphi^{-g(\Lambda) - h(\Lambda)} \leq \varphi^{-|S|} \sum_{\Lambda \in \mathcal{S}(m,f,d)} \varphi^{-s(\Lambda \cap S)}
\leq \varphi^{-|S|} \sum_{U \in \mathcal{A}(S)} \varphi^{-s(U)} = \varphi^{-|S|} w(S)
\leq \varphi \cdot 1.618^2 \cdot \left( \frac{1.618}{\varphi} \right)^{|S| + d} \leq 5 \left( \frac{1.618}{\varphi} \right)^{f-m-1}.
\]

This establishes Lemma 3. \(\square\)

6 Conclusions

The main result of this paper resolves many of the questions surrounding the Fibonacci-like behavior of the number of numerical semigroups of a given genus. However, little has been established concerning the relationship between \(n_g\) and \(n_{g+1}\). In particular, it remains open whether \(n_{g+2} \geq n_{g+1} + n_g\), as conjectured in [2] and the conjecture that \(n_{g+1} \geq n_g\) given in [6] remains unverified for a finite but large number of \(g\).

In addition, we have confirmed Zhao’s conjecture in [8] that the proportion of numerical semigroups \(\Lambda\) of a given genus satisfying \(f(\Lambda) < 3m(\Lambda)\) approaches 1 asymptotically. Thus, in some sense, “most” numerical semigroups satisfy \(f < 3m\). It would be interesting to study whether this is true when counting semigroups by measures of complexity other than genus. For example, [1] have considered the problem of counting the number of numerical semigroups of a given Frobenius number; one might also ask whether most of these semigroups satisfy \(f < 3m\).

In general, it could be considered whether there is some unified sense in which one can take the asymptotic limit of semigroups. For any numerical semigroup \(\Lambda\), we have that \(g(\Lambda) \leq f(\Lambda) + 1\), and \(f(\Lambda) \leq 2g(\Lambda)\). Thus, we might expect the sets \(\{\Lambda \mid f(\Lambda) = n\}\) and \(\{\Lambda \mid g(\Lambda) = n\}\) to behave in similar ways as \(n \to \infty\). Both genus and Frobenius number can be thought of as proxies for the “complexity” of a numerical semigroup, and it would be interesting to explore ways to make this precise.

Finally, the proof of Lemma 3 given here (and in particular the proof of Lemma 9) is quite involved. Although the main idea of bounding the weight of an interval by partitioning it as in Lemma 11 was simple, computations had to be carried out for many specific cases in order to obtain sufficiently strong bounds. We hope that by improving upon the techniques used in this paper, significant simplifications of the proof are possible.
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8 Appendix

We fill in here some of the computational details of the proofs of Lemmas 18 and 19. The matrix calculations of this section were done using Sage.\footnote{www.sagenb.org} Recall that in the proof of Lemma 18 we made the definition

$$W_n = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \varphi^{-1} & 1 & 1 \\ \varphi^{-2} & 1 & \varphi^{-1} \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$  

We now justify in detail the claim that $W_n \leq 3 \cdot 1.618^{2n-2}$. Note that by explicit computation,

$$W_1 = 3$$
$$W_2 = 6 + 2\varphi^{-1} + \varphi^{-2} \leq 7.62 < 3 \cdot 1.618^2$$
$$W_3 = 11 + 9\varphi^{-1} + 6\varphi^{-2} + \varphi^{-3} \leq 19.10 < 3 \cdot 1.618^4.$$  

By the Cayley-Hamilton theorem, we obtain the recurrence

$$W_{n+3} = (2.618 \cdots)W_{n+2} - (0.236 \cdots)W_{n+1} - (0.146 \cdots)W_n$$

for $n \geq 1$. Thus, $W_{n+1} \leq 2.619W_n < 1.618^2W_n$ for each $n \geq 3$, and by induction, $W_n \leq 3 \cdot 1.618^{2n-2}$, as desired.

A similar claim was made in the proof of Lemma 19. In that proof, we defined

$$W_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \varphi^{-1} & 1 & 1 & 1 \\ \varphi^{-1} & \varphi^{-2} & 1 & 1 \\ \varphi^{-1} & \varphi^{-1} & \varphi^{-2} & 1 \\ \varphi^{-1} & \varphi^{-1} & \varphi^{-1} & \varphi^{-2} \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. $$
It was claimed that $W_n \leq (4 + 2\varphi) \cdot 1.618^{4n-4}$. As before, we first verify this for small values of $n$.

\[
\begin{align*}
W_1 &= 4 + 2\varphi \\
W_2 &\leq 41.51 < (4 + 2\varphi) \cdot 1.618^4 \\
W_3 &\leq 226.83 < (4 + 2\varphi) \cdot 1.618^8 \\
W_4 &\leq 1225.28 < (4 + 2\varphi) \cdot 1.618^{12} \\
W_5 &\leq 6599.87 < (4 + 2\varphi) \cdot 1.618^{16}.
\end{align*}
\]

We will prove by induction that $W_{n+1} \leq 6.8W_n$ for all $n \geq 1$. This can be seen by direct verification for $n \leq 4$. Proceeding inductively, for $n > 4$, we have by the Cayley-Hamilton theorem,

\[
W_{n+1} = (7.236 \cdots)W_n - (10.708 \cdots)W_{n-1} + (3.965 \cdots)W_{n-2} - (0.278 \cdots)W_{n-3}
\]
\[
\leq 7.237W_n - (10.707 - 3.966)W_{n-1} \leq \left( 7.237 - \frac{10.707 - 3.966}{6.8} \right) W_n \leq 6.8W_n,
\]

where we have used implicitly the basic inequalities $W_k \geq 0$ and $W_{k+1} \geq W_k$ for all $k \geq 1$. Noting that $1.618^4 > 6.8$, it is now immediate that $W_n \leq (4 + 2\varphi) \cdot 1.618^{4n-4}$ for all $n \geq 1$.

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