Incompressible limit of strong solutions to 3-D Navier-Stokes equations with Navier’s slip boundary condition for all time*

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Abstract

This paper studies the incompressible limit of global strong solutions to the three-dimensional compressible Navier-Stokes equations associated with Navier’s slip boundary condition, provided that the time derivatives, up to first order, of solutions are bounded initially. The main idea is to derive a differential inequality with decay, so that the estimates are bounded uniformly both in the Mach number $\epsilon \in (0, \epsilon_0]$ (for some $\epsilon_0 > 0$) and the time $t \in [0, +\infty)$.

1 Introduction

The motions of highly subsonic viscous fluids in a bounded domain $\Omega \subset \mathbb{R}^3$ are described by the following non-dimensionalized Navier-Stokes equations:

\begin{align}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}S + \frac{1}{\epsilon^2} \nabla p &= 0,
\end{align}

where the first equation represents the conservation of mass and the second one denotes the conservation of momentum. The unknowns $\rho$, $u$ and $p$ are the density, the velocity and the

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pressure, respectively. And the matrix \( S \equiv S(u) = 2\mu D(u) + \lambda \text{div} u I_3 \) is the viscous stress tensor for Newtonian fluids, where \( D(u) = (\nabla u + \nabla u^t)/2 \). The constant \( \epsilon \in (0, 1] \) is the Mach number of the highly subsonic fluids. The constants \( \mu, \lambda \) are viscosity coefficients with \( \mu > 0, \mu + 3\lambda/2 \geq 0 \). In this paper, we suppose that the pressure \( p = p(\rho) \) is a \( C^3 \) function satisfying that \( p'(\rho) > 0 \) for \( \rho > 0 \).

In the physical viewpoint, the motions of highly subsonic compressible fluids would behave similarly to the incompressible ones (see [17]). Formally, as the Mach number \( \epsilon \) tends to zero, the solutions to (1.1)-(1.2) will converge to the solution \( (u, \pi) \) of the incompressible Navier-Stokes equations, namely

\[
\begin{align*}
&u_t + u \cdot \nabla u - \mu \Delta u + \nabla \pi = 0, \\
&\text{div}u = 0.
\end{align*}
\]

It is known as the incompressible limit, which is one of the fundamental hydrodynamic limits. However, the rigorous justification of the limit poses challenging problems mathematically since singular phenomena usually occur in this process. To be precise, both the uniform estimates in Mach number and the convergence to the incompressible model are usually difficult to obtain. In the following, we restrict the discussion in the isentropic regime only.

The general framework for studying the incompressible limit for local strong or smooth solutions was established by S. Klainerman and A. Majda in [14, 15]. In these works, they proved the incompressible limit of local smooth solutions to the Navier-Stokes equations (or the Euler equations) with “well-prepared” - some smallness assumption on the divergence of initial velocity - initial data, in \( \mathbb{R}^n \) or \( \mathbb{T}^n \). Indeed, by analyzing the rescaled linear group generated by the penalty operator of order \( \epsilon^{-1} \) (see [23, 28] for instance), the incompressible limit can also be verified for the cases of general data that the velocity of incompressible fluid is just the limit of Leray projection for the velocities in compressible fluids. This method also applies to global weak solution of the isentropic Navier-Stokes equations with general initial data and various boundary conditions [5, 6, 18]. Especially, P.-L. Lions and N. Masmoudi [18] studied the incompressible limit for the weak solutions to the Navier-Stokes equations with a slip boundary condition, that is, on the boundary \( \partial \Omega \) of \( \Omega \subset \mathbb{R}^n \),

\[
\begin{align*}
&u \cdot n = 0, \; \text{curl} u = 0 \quad \text{for} \quad n = 2, \quad \text{or} \quad 1.3 \\
&u \cdot n = 0, \; n \times \text{curl} u = 0 \quad \text{for} \quad n = 3, \quad 1.4
\end{align*}
\]

where \( \text{curl} u = (\partial_2 u, -\partial_1 u)^t \) for \( n = 2 \) and \( \text{curl} u = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1)^t \) for \( n = 3 \). Recently, D. Donatelli, E. Feireisl, A. Novotný, etc. have also obtained a series of important progresses on incompressible limits of weak solutions to compressible Navier-Stokes equations associated with slip boundary conditions (see [8, 9], for instance). For other interesting results on the incompressible limit in a finite time interval, which may be independent of the initial data, for isentropic fluids, the reader may refer to [4, 11, 12, 16, 19, 24, 25] and many others.

Although numerous significant progresses on incompressible limit had been achieved during the last four decades, only a few results were concerned with global strong or classical solutions for the time \( t \in [0, +\infty) \). In this situation, one needs to show the uniform estimates with respect to both \( \epsilon \in (0, \epsilon_0] \) (for some small constant \( \epsilon_0 > 0 \)) and \( t \in [0, +\infty) \).
Thus additional difficulties arise. D. Hoff [10] verified the incompressible limit for the global solutions in \( \mathbb{R}^3 \times [0, +\infty) \) with general initial data, provided that the background solution to the incompressible Navier-Stokes equations is sufficiently smooth. For regular solutions with no-slip boundary conditions, i.e., \( u|_{\partial \Omega} = 0 \), where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain, and slightly compressible initial data, H. Bessaih [1] established the uniform estimates both in the Mach number and \( t \in [0, +\infty) \), and showed the strong convergence to the solution of incompressible Navier-Stokes equations. In [21], the author studied the incompressible limit of regular solutions to the compressible Navier-Stokes equations (1.1)-(1.2) with slightly compressible initial data in a 2-D bounded domain with the boundary condition in (1.3).

The aim of this paper is to extend the result in [21] to three spatial dimensions, that is, to study the incompressible limit of global strong solutions to the 3-D compressible Navier-Stokes equations (1.1)-(1.2) with Navier’s slip boundary condition

\[
\begin{align*}
    u \cdot n &= 0, \quad \tau \cdot S(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]  

where \( n, \tau \) are the unit outer normal and tangential vector to the boundary, respectively. This is a non-trivial generalization since on the boundary of a 3-D bounded domain, the information on the normal component of the vorticity curl\( u \) is unavailable (see Lemma 2.6 for instance), thus the classical regularity theory doesn’t apply.

At the same time, this paper also generalizes the result in [1] in the sense that all the second-order spatial derivatives are uniformly bounded with respect to the Mach number and the time. Moreover, it is worthy to note that the result in this paper can not be covered by the ones in [8, 9] since our estimates are uniformly bounded for all the time in \( [0, +\infty) \), instead of a fixed finite interval. Furthermore, the method in the current paper can also simplify the proof of local existence results in [30].

To simplify the proof, we convert the equations into the anti-symmetric form by setting \( \rho = 1 + \epsilon \sigma \). Then the Navier-Stokes equations (1.1)-(1.2) are equivalent to

\[
\begin{align*}
    \sigma_t + \text{div}(\sigma u) + \frac{1}{\epsilon} \text{div}u &= 0, \\
    \rho(u_t + u \cdot \nabla u) + \frac{1}{\epsilon} p'(1 + \epsilon \sigma) \nabla \sigma &= 2\mu \text{div}(D(u)) + \lambda \nabla \text{div}u.
\end{align*}
\]  

(1.6) \quad (1.7)

For the new unknowns \( (\sigma, u) \), we impose the following initial condition

\[
(\sigma, u)|_{t=0} = (\sigma_0, u_0)(x), \quad x \in \Omega \quad (1.8)
\]

and the slip boundary condition

\[
\begin{align*}
    u \cdot n &= 0, \quad \tau \cdot D(u) \cdot n + \alpha u \cdot \tau = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]  

(1.9)

which is equivalent to (1.5). One may refer to [30], for instance, for the description and the background on this boundary condition.

First, the local existence of the solution \( (\sigma, u) \) to the problem (1.6)-(1.9) is indeed established by W.M. Zajaczkowski [30] in the framework of [20, 29].

\[3\]
Theorem 1.1 (Local existence) Let $\epsilon \in (0, 1]$ be a fixed constant and $\Omega \subset \mathbb{R}^3$ be a simply connected, bounded domain with smooth boundary $\partial \Omega$. Suppose that the initial datum $(\sigma_0, u_0)$ satisfies the following conditions,

$$(\sigma_0, u_0) \in H^2(\Omega), \quad (\sigma_t(0), u_t(0)) \in H^1(\Omega), \quad (\sigma_{tt}(0), u_{tt}(0)) \in L^2(\Omega),$$

with $\int_{\Omega} \sigma_0 dx = 0$ and $1 + \epsilon \sigma_0 \geq m$ for some positive constant $m$. Assume the following compatibility conditions are satisfied:

$$\partial_t^i u(0) \cdot n = \tau \cdot S(\partial_t^i u(0)) \cdot n + \alpha \tau \cdot \partial_t^i u(0) = 0 \quad \text{on } \partial \Omega, \ i = 0, 1, 2.$$

Then there exists a positive constant $T = T(\sigma_0, u_0, m, \epsilon)$ such that the initial-boundary problem (1.6)-(1.9) admits a unique solution $(\sigma, u)$ satisfying that $1 + \epsilon \sigma_0 > 0$ in $\Omega \times (0, T)$, and

$$\sigma \in C([0, T], H^2), \quad u \in C([0, T], H^1_0 \cap H^2) \cap L^2(0, T; H^3),$$

$$\sigma_t \in C([0, T], H^1), \quad u_t \in C([0, T], H^1_0) \cap L^2(0, T; H^2),$$

$$\sigma_{tt} \in C([0, T], L^2), \quad u_{tt} \in C([0, T], L^2) \cap L^2(0, T; H^1).$$

Remark 1.1 To simplify the statement, we use the notation “$u_t(0)$” to signify the quantity $u_t|_{t=0} := -u_0 \cdot \nabla u_0 + (-p'(1 + \epsilon \sigma_0)\nabla \sigma/\epsilon + \mu \Delta u_0 + \nu \nabla \text{div } u_0)/(1 + \epsilon \sigma_0)$ obtained from the equation (1.7). And the notation “$\partial_t^i u(0)$” is given by differentiating (1.7) $i - 1$ times with respect to $t$ and then letting $t = 0$. The same rule applies to the notations $\partial_t^i \sigma(0)$, $\partial_t^i \rho(0)$, etc.

The purpose of this paper is to prove the following uniform estimates with respect to $\epsilon \in (0, \epsilon_0]$ (for some $0 < \epsilon_0 \leq 1$) and $t \in (0, +\infty)$, thus apply Theorem 1.1 to obtain the global existence theorem and the corresponding incompressible limits. In order to state the theorem precisely, we introduce the following notation

Definition 1.1

$$\phi^s(t) := \max_{0 \leq s \leq t} (\| (\sigma^s, u^s) \|_{H^2} + \| (\sigma_t^s, u_t^s) \|_{H^1} + \epsilon \| (u_{tt}^s, \sigma_{tt}^s) \|_{L^2}(s)).$$

Then the main results of this paper is stated as follows.

Theorem 1.2 (Global-in-time existence and incompressible limit). Let all the assumptions in Theorem 1.1 be satisfied and $\epsilon \in (0, \overline{\epsilon}]$ for some sufficiently small constant $\overline{\epsilon} \in (0, 1]$. Moreover, we assume that

$$\phi^s(0) \leq \theta,$$

for some sufficiently small positive constant $\theta$. Then there exists a unique solution $(\sigma^s, u^s)$ to the initial-boundary value problem (1.6)-(1.9) in $\Omega \times \mathbb{R}^+$, such that

$$\sigma^s \in C(\mathbb{R}^+, H^2), \quad u^s \in C(\mathbb{R}^+, L^2 \cap H^2) \cap L^2(\mathbb{R}^+; H^3),$$

$$\sigma_t^s \in C(\mathbb{R}^+, H^1), \quad u_t^s \in C(\mathbb{R}^+, H^1_0) \cap L^2(\mathbb{R}^+; H^2),$$

$$\sigma_{tt}^s \in C(\mathbb{R}^+, L^2), \quad u_{tt}^s \in C(\mathbb{R}^+, L^2) \cap L^2(\mathbb{R}^+; H^1).$$
where \( \mathbb{R}^+ = [0, +\infty) \). Furthermore, the following uniform estimate in \( \epsilon \in (0, \bar{\epsilon}] \) holds:

\[
\phi^\epsilon(t) \leq C \theta, \quad \forall t \in \mathbb{R}^+,
\]

for some positive constant \( C \). Thus \( u^\epsilon \to v \) strongly in \( C(\mathbb{R}_+^{1n}; H^s) \) as \( \epsilon \to 0 \) for any \( 0 \leq s < 2 \). And there exists a function \( P(x, t) \), such that \((v, P)\) is the unique solution of the following initial-boundary value problem of the incompressible Navier-Stokes equations:

\[
\text{div} v = 0, \quad v_t + v \cdot \nabla v + \nabla P = \mu \Delta v, \quad \text{in} \quad \Omega \times (0, +\infty),
\]

\[
v \cdot n = \tau \cdot S(v) \cdot n + \alpha \tau \cdot v = 0 \quad \text{on} \partial \Omega,
\]

\[
v|_{t=0} = v_0(x), \quad x \in \Omega,
\]

where \( \|u_0^\epsilon - v_0\|_{H^2} + \|u_0^\epsilon - v_0\|_{H^1} + \|\epsilon(u_0^\epsilon - v_0)\|_{L^2} \to 0 \) as \( \epsilon \to 0 \).

**Proof.** This theorem was shown by Lemma 3.17 and the same arguments as in [1, 21].

**Remark 1.2** Although the time derivatives up to second order are estimated, however, only the derivatives up to first order are required to be bounded initially.

The rest of this paper is organized as follows. In Section 2, we present some lemmas which will be used in estimating the Sobolev norms in a bounded domain and dealing with the slip boundary condition. In Section 3, we show the uniform-in-\( \epsilon \) estimates by deriving a differential inequality with certain decay property. We first show the \( L^2 \) estimate of the solutions, next the low-order spatial, temporal or mixed derivatives, and then the high-order derivatives. The strategy for estimating derivatives is to treat the vorticity and the divergence of velocity respectively, based on the decomposition \( \Delta = \nabla \text{div} - \text{curl curl} \) and the slip boundary condition. Moreover, to overcome the difficulty in estimating the vorticity, due to the loss of information on the normal component, we take the advantage of the isothermal coordinates to estimate it in local regions near the boundary. By combining carefully all the spatial-temporal estimates, we obtain the uniform estimate with respect to both \( \epsilon \in (0, \bar{\epsilon}] \) (\( 0 < \bar{\epsilon} \leq 1 \)) and \( t \in [0, +\infty) \).

## 2 Preliminaries

Throughout this paper, we will use the following lemmas from time to time.

**Lemma 2.1** (See [2]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and outward normal \( n \). Then there exists a constant \( C > 0 \) independent of \( u \), such that

\[
\|u\|_{H^s(\Omega)} \leq C(\|\text{div} u\|_{H^{s-1}(\Omega)} + \|\text{curl} u\|_{H^{s-1}(\Omega)} + \|u \cdot n\|_{H^{s-\frac{1}{2}}(\partial \Omega)} + \|u\|_{H^{s-1}(\Omega)}),
\]

for any \( u \in H^s(\Omega)^N \).

**Lemma 2.2** (See [2]). Assume \( f \in C([0, T]; W^{k,p}(\Omega, \mathbb{R}^N)) \) with

\[
k > \frac{N}{p} + 1 \quad \text{and} \quad 1 \leq p \leq +\infty.
\]
Then the problem
\[ \frac{du}{dt}(x,t) = f(u(x,t),t), \quad u(x,0) = x \]
has a solution \( u \in C^1([0,T]; D^{k,p}(\Omega)) \), where
\[ D^{k,p}(\Omega) = \{ \eta \in W^{k,p}(\Omega) \mid \eta \text{ is a bijective from } \overline{\Omega} \text{ onto } \overline{\Omega}, \eta^{-1} \in W^{k,p}(\Omega) \}. \]

**Lemma 2.3** (See [2]). Let \( k \geq 2 \) be an integer, and let \( 1 \leq p \leq q \leq +\infty \) be such that \( p < +\infty \) and \( k > \frac{N}{p} + 1 \). Let \( f \in W^{k,p}(\Omega) \), then the mapping \( g \mapsto g \circ f \) is continuous from \( D^{k,p}(\Omega) \) into \( W^{k,p}(\Omega) \).

**Lemma 2.4** (See [26]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial U \) and outward normal \( n \). Then there exists a constant \( C > 0 \) independent of \( u \), such that
\[ \| u \|_{H^s(\Omega)} \leq C(\| \text{div} u \|_{H^{s-1}(\Omega)} + \| \text{curl} u \|_{H^{s-1}(\Omega)} + \| u \times n \|_{H^{s-1}(\partial\Omega)} + \| u \|_{H^{s-1}(\Omega)}) \]
for all \( u \in H^s(\Omega)^N \).

**Lemma 2.5** (See [7]). Let \( \Omega \subset \mathbb{R}^3 \) be a open bounded domain with \( C^2 \) boundary \( \partial\Omega \). Moreover, we assume that \( \Omega \) is simply connected and non-axisymmetric. Then for any \( u \in H^1(\Omega) \) satisfying \( u \cdot n|_{\partial\Omega} = 0 \), one has
\[ \| u \|_{H^1(\Omega)} \leq C(\| \text{Div} u \|_{L^2(\Omega)} + \| u \|_{L^2(\partial\Omega)}) \] (2.1)
and
\[ \| \nabla u \|_{L^2(\Omega)} \leq C(\| \text{Div} u \|_{L^2(\Omega)} + \| \text{curl} u \|_{L^2(\Omega)}), \] (2.2)
where \( C \) is a constant independent of \( u \).

The following lemma is a variant of Theorem 3.10 in [27] in the case of Navier’s slip boundary condition. It plays a key role in proving the vorticity estimates.

**Lemma 2.6**. Let \( \Omega \subset \mathbb{R}^3 \) be a open bounded domain with \( C^2 \) boundary \( \partial\Omega \). If \( u \in H^2(\Omega)^2 \) with (1.9) being satisfied, then
\[ \tau \cdot (n \times w) = 2\tau \cdot (\alpha u - \nabla nu). \] (2.3)

**Proof.** Using the density in \( \{ u \in H^2(\Omega)^2 | u \cdot n = 0 \text{ on } \partial\Omega \} \) of the velocity fields \( u \in C^\infty(\Omega)^2 \) such that \( u \cdot n = 0 \text{ on } \partial\Omega \), and the continuity of the trace operators, it suffices to handle the case where \( u \) is a smooth velocity field on \( \Omega \). Now, after extending \( n(x) \) to a tubular neighbourhood of \( \partial\Omega \), we obtain
\[ \partial_n u = \frac{1}{2} w \times n + D(u)n \]
\[ \partial_r u = \frac{1}{2} w \times \tau + D(u) \tau, \]

which yield
\[ 2(D(u)n) \cdot \tau = \partial_r u \cdot n + \partial_n u \cdot \tau \]
and
\[ (n \times w) \cdot \tau = \partial_r u \cdot n - \partial_n u \cdot \tau. \]

It follows that
\[ 2(D(u)n) \cdot \tau + (n \times w) \cdot \tau = 2 \partial_r u \cdot n = -2u \cdot \partial_r n = -2\tau \cdot (\nabla nu), \]
due to the boundary condition \( u \cdot n = 0 \). By use of (1.9), we easily get (2.3).

### 3 Energy estimates

In this section, we shall derive the uniform estimates with respect to both the time \( t \in [0, +\infty) \) and the Mach number \( \epsilon \in (0, \bar{\epsilon}] \) for some \( \bar{\epsilon} \in (0, 1] \), which is stated as in Lemma 3.1. We will drop the superscript \( \epsilon \) of \( \sigma^\epsilon, u^\epsilon, p^\epsilon \) and so on, for the sake of simplicity. From now on, the positive constants \( C, C_i \) for \( i = 0, 1, \ldots \) below depend only on \( \Omega, \mu, \lambda, \) and \( p \), but not on \( T \) and \( \epsilon \). We will use \( \delta, \eta, \) and \( \eta_i \) for \( i = 1, 2, \ldots \) to denote various small positive constants and \( C_\delta, C_\eta \) to denote various positive constants depending on \( \delta \) and \( \eta \) respectively. For the sake of simplicity, we denote the partial derivatives \( \partial_{x_i} \) by \( \partial_i \), \( \partial_{x_i} \partial_{x_j} \) by \( \partial_{ij} \), and so on.

Suppose that \( (\sigma, u) \) solves the initial-boundary value problem (1.6)-(1.9) in \( \Omega \times (0, T) \), for \( 0 < T < +\infty \). In the energy estimates, we always assume that \( \frac{1}{4} \leq 1 + \epsilon \sigma \leq 4 \) in any \( (x, t) \in \Omega \times (0, T) \) where \( \epsilon \in (0, 1] \).

We will derive a differential inequality in the form that,
\[ \frac{d}{dt} \Phi(t) + \Psi(t) \leq C\Psi(t)(\Phi(t) + \Phi^2(t)), \quad \forall 0 \leq t \leq T, \]
where \( C \geq 1 \) is a constant, and \( \Phi(t) \) is an equivalent norm to \( \phi(t) \). Here \( \Psi(t) \) and \( \Phi(t) \) are both non-negative quantities with \( \Psi(t) \geq C\Phi(t) \) for some constant \( C \in (0, 1] \). The above inequality is equivalent to
\[ \frac{d}{dt} \Phi(t) \leq -\Psi(t)(1 - C(\Phi(t) + \Phi^2(t))), \quad \forall 0 \leq t \leq T. \]

Thus, if \( \Phi(0) \) is small enough, \( \Phi(t) \) will be dominated by \( \Phi(0) \).

#### 3.1 The basic estimate

**Lemma 3.1** There exist positive constants \( C_0 \) and \( C_1 \), such that
\[ \frac{d}{dt} \Phi_0(t) + \Psi_0(t) \leq C_1 ||\sigma||_{H^2}||\sigma||_{H^1}^2 + \epsilon ||\sigma||_{L^2}^2, \quad (3.1) \]
where $\Phi_0(t) = \| \sqrt{\rho u} \|_{L^2}^2$ and $\Psi_0(t) = C_0 \| u \|_{H^1}^2$.

**Proof.** Due to the boundary conditions (1.3) and Lemma 2.5, we have

\[
- \int_\Omega (2\mu \text{div}(D(u)) + \lambda \nabla \text{div} u) \cdot u \, dx = \int_\Omega (2\mu |D(u)|^2 + \lambda (\text{div} u)^2) \, dx + \int_{\partial \Omega} \alpha |u|^2 dS \geq \gamma_0 \| u \|_{H^1}^2.
\]

Multiplying (1.6) by $p'(1 + \epsilon \sigma)\sigma$ and (1.7) by $u$, we get

\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho u} \|_{L^2}^2 + \gamma_0 \| u \|_{H^1}^2 \leq \frac{1}{\epsilon} \int_\Omega (p'(\rho) \text{div} u) \, dx + \int_\Omega |p'(\rho)\sigma \text{div}(\sigma u)| \, dx + \frac{\epsilon}{2} \int_\Omega |p''(\rho)\sigma \sigma'| \, dx := A_1 + A_2 + A_3.
\]

With the aid of the boundary condition $u \cdot n = 0$, we have

\[
A_1 = \left| \int_\Omega p''(\rho)\sigma \nabla \sigma \cdot u \, dx \right| \leq \eta \| \sigma \|_{H^1}^2 + C_\eta \| \sigma \|_{H^1}^2 \| \nabla \sigma \|_{H^1}^2.
\]

On the other hand,

\[
A_2 \leq \eta \| \sigma \|_{H^1}^2 + C_\eta \| \sigma \|_{H^1}^2 \| \nabla \sigma \|_{H^1}^2
\]

and

\[
A_3 \leq \epsilon (\| \sigma \|_{L^2}^2 + \| \sigma \|_{H^1}^4).
\]

As a result, we finish the proof of this lemma. \(\square\)

**3.2 The first-order estimate**

**Lemma 3.2** There exists a positive constant $C_2$ such that

\[
\frac{d}{dt} \left( \int_\Omega (2\mu |D(u)|^2 + \lambda (\text{div} u)^2) + \rho u \cdot u \right) \, dx + \int_{\partial \Omega} \alpha |u|^2 dS + \| \sqrt{p'(\rho)\sigma_t} \|_{L^2}^2 \leq C_2 (\| u_t \|_{L^2}^2 + \| u \|_{H^1}^4 + \| \sigma_t \|_{H^1}^2 (\| \sigma \|_{H^2}^4 + \| u \|_{H^2}^4) + \| u_t \|_{H^1}^2 \| u_t \|_{H^1}).
\]

**Proof.** By differentiating (1.7) with respect to $t$, we have

\[
(pu_t - 2\mu \text{div}(D(u_t))) - \lambda \nabla \text{div} u_t + \frac{1}{\epsilon} p'(\rho) \nabla \sigma_t = -(\epsilon \sigma_t u \cdot \nabla u + \rho u_t \cdot \nabla u + u \cdot \nabla u_t) + p''(\rho) \sigma_t \nabla \sigma_t. \tag{3.2}
\]

Then we integrate the product of (3.2) and $u$ to get

\[
\frac{d}{dt} \left( \int_\Omega (2\mu |D(u)|^2 + \lambda (\text{div} u)^2) + 2\rho u \cdot u \right) \, dx + \int_{\partial \Omega} \alpha |u|^2 dS + \frac{1}{\epsilon} \int_\Omega p'(\rho) \nabla \sigma_t \cdot u \, dx \leq \| \sqrt{\rho u_t} \|_{L^2}^2 + \eta \| u \|_{H^1}^2 + C_\eta (\| \sigma_t \|_{L^2}^2 (\| \sigma \|_{H^2}^4 + \epsilon^2 \| u \|_{H^2}^4) + \| u_t \|_{H^1}^2 \| u_t \|_{H^1}). \tag{3.3}
\]

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On the other hand, we multiply (1.6) by $p'(\rho)\sigma_t$ and integrate to get
\[
\|\sqrt{p'(\rho)}\sigma_t\|_{L^2}^2 - \frac{1}{\epsilon} \int_{\Omega} p'(\rho)\nabla \sigma_t \cdot u dx \leq \eta\|u\|_{H^1}^2 + C_\eta\|\sigma_t\|_{H^1}^2\|\sigma\|_{H^1}^2.
\] (3.4)

Thus we summarize (3.3) and (3.4) to get the lemma. 

From now on we may often use the following relations
\[
2\text{div}(D(u)) = (\Delta u + \nabla \text{div})u, \\
\Delta u = \nabla \text{div} - \text{curl curl} u,
\]
for any vector function $u = (u_1, u_2, u_3)^t$.

**Lemma 3.3** There exists a positive constant $C_3$ such that
\[
\frac{d}{dt}\|\nabla \sigma\|_{L^2}^2 + (2\mu + \lambda)\|\sqrt{p'(\rho)}^{-1}\nabla \text{div} u\|_{L^2}^2
\leq C_3\|u_t\|_{L^2}^2 + C_3\|u\|_{H^2}^2(\|u\|_{H^1}^2 + \|\sigma\|_{H^2}^2) + \eta_1\|\nabla^2 \text{div} u\|_{L^2}^2 + C_4\|u\|_{H^1}^2,
\]
where $\eta_1(< (2\mu + \lambda)/(8p'(4)))$ and $C_4 = C_4(\eta_1)$ are to be determined later.

**Proof.** Applying $\nabla$ to (1.6), we obtain
\[
(\nabla \sigma)_t + \nabla^2 \sigma u + \nabla u \nabla \sigma + \sigma \nabla \text{div} u + \text{div} \nabla \sigma + \frac{1}{\epsilon} \nabla \text{div} u = 0,
\] (3.5)
then multiply the equation by $\nabla \sigma$ and integrate
\[
\frac{1}{2}\frac{d}{dt}\|\nabla \sigma\|_{L^2}^2 + \frac{1}{\epsilon} \int_{\Omega} \nabla \sigma \cdot \nabla \text{div} u dx \\
\leq C \int_{\Omega} (\|\nabla u\|\|\nabla \sigma\|^2 + \|\sigma\|\|\nabla \text{div} u\|\|\nabla \sigma\|) dx \\
\leq \frac{2\mu + \lambda}{8p'(4)}\|\nabla \text{div} u\|_{L^2}^2 + \eta_1\|u\|_{H^1}^2 + C_\eta\|\sigma\|_{H^2}^2.
\] (3.6)

Multiplying (1.7) by $p'(\rho)^{-1}\nabla \text{div} u$ and integrating over $\Omega$ immediately yield
\[
(2\mu + \lambda)\|\sqrt{p'(\rho)}^{-1}\nabla \text{div} u\|_{L^2}^2 - \frac{1}{\epsilon} \int_{\Omega} \nabla \sigma \cdot \nabla \text{div} u dx \\
= \int_{\Omega} p'(\rho)^{-1}(\mu \text{curl} \text{curl} u \cdot \nabla \text{div} u + \rho(u_t + u \cdot \nabla u) \cdot \nabla \text{div} u) dx.
\] (3.7)

Let $\text{curl} u = w$. Then
\[
\int_{\Omega} p'(\rho)^{-1}\text{curl} \text{curl} u \cdot \nabla \text{div} u dx
\leq \int_{\Omega} \epsilon p'(\rho)^{-2}p''(\rho)(\nabla \sigma \times w) \cdot \nabla \text{div} u dx + \int_{\partial \Omega} p'(\rho)^{-1}(n \times w) \cdot \nabla \text{div} u S.
\] (3.8)
Therefore, using (2.3) and the trace theorem, the boundary integral in (3.8) can be dominated by
\[
C \int_{\partial \Omega} |(n \times w) \cdot \tau| \| \nabla \text{div} u \| dS \\
\leq C \int_{\partial \Omega} 2|\alpha u - 2\nabla nu| \| \nabla \text{div} u \| dS \\
\leq \eta \| \nabla \text{div} u \|_{H^1}^2 + C_\eta \| u \|_{H^1}^2.
\]
Thus, we can get the following inequality from (3.7) and (3.8):
\[
(2\mu + \lambda)\| \sqrt{p'(\rho)^{-1}} \nabla \text{div} u \|_{L^2}^2 - \frac{1}{\epsilon} \int_{\Omega} \nabla \sigma \nabla \text{div} u \, dx \\
\leq \eta \| \nabla \text{div} u \|_{H^1}^2 + C_\eta \| u \|_{H^1}^2 + C\| \sigma \|_{H^2}^2(\| u \|_{H^1}^2 + \| \sigma \|_{H^2}^2) \\
+ \frac{2\mu + \lambda}{8p'(4)} \| \nabla \text{div} u \|_{L^2}^2 + C\| u \|_{L^2}^2.
\tag{3.9}
\]
Then we summarize (3.6) and (3.9) to get the lemma.

\[\square\]

**Lemma 3.4** There exist positive constants $C_5$ and $C_6$ such that
\[
\frac{d}{dt}(\| \sqrt{p u_t} \|_{L^2}^2 + \| p(\rho) \sigma_t \|_{L^2}^2) + C_5 \| u_t \|_{H^1}^2 \\
\leq C_6(\| u_t \|_{H^1}^2(\| u_t \|_{L^2}^2 + \| u \|_{H^1}^2 + \| \sigma \|_{H^2}^2 + \| \sigma_t \|_{H^1}^2) \\
+ C_6(\epsilon^2\| \sigma_t \|_{L^2}^2 + \| u \|_{H^1}^2)).
\]

**Proof.** Note that
\[
-\int_{\Omega} (2\mu \text{div}(D(u_t)) + \lambda \nabla \text{div} u_t) \cdot u_t \, dx \\
= \int_{\Omega} (2\mu |D(u_t)|^2 + \lambda(\text{div} u_t)^2) \, dx + \int_{\partial \Omega} \alpha |u_t|^2 \, dS \geq \gamma_1 \| u_t \|_{H^1}^2,
\]
where $\gamma_1$ is a positive constant.

We multiply (3.2) by $u_t$ and integrate to obtain
\[
\frac{1}{2} \frac{d}{dt}(\| \sqrt{\rho u_t} \|_{L^2}^2 + \frac{1}{\epsilon} \int_{\Omega} p'(\rho) \nabla \sigma_t \cdot u_t \, dx + \gamma_1 \| u_t \|_{H^1}^2 \\
\leq \eta \| u_t \|_{H^1}^2 + C_\eta(\| \sigma_t \|_{H^1}^2(\| u_t \|_{L^2}^2 + \| u \|_{H^1}^2)) + \| u_t \|_{H^1}^2 \| u \|_{H^1}^2).
\tag{3.10}
\]
Applying $\partial_t$ to (1.6) gives
\[
\sigma_{tt} + u \cdot \nabla \sigma_t + u_t \cdot \nabla \sigma + \sigma_t \text{div} u + \sigma \text{div} u_t + \frac{1}{\epsilon} \text{div} u_t = 0.
\tag{3.11}
\]
Due to the boundary condition $u \cdot n = 0$, we multiply the above equality by $p'(\rho)\sigma_t$ and integrate to get
\[
\frac{1}{2} \frac{d}{dt}(\| \sqrt{\rho} p'(\rho) \sigma_t \|_{L^2}^2 - \frac{1}{\epsilon} \int_{\Omega} p'(\rho) \nabla \sigma_t \cdot u_t \, dx \\
\leq \eta \| u_t \|_{H^1}^2 + C_\eta(\| \sigma \|_{H^2}^2 \| \sigma_t \|_{L^2}^2 + \| \sigma_t \|_{H^1}^2) + C\epsilon^2 \| \sigma_t \|_{L^2}^2 + C\| u \|_{H^1}^2.
\tag{3.12}
\]
Thus we summarize (3.10) and (3.12) and choose \( \eta \) to be sufficiently small to get the lemma.

Next, we should estimate the vorticity \( \text{curl}\ u \).

**Lemma 3.5** Let \( w := \text{curl}\ u \). Then

\[
\frac{d}{dt} \|\sqrt{\rho} w\|^2_{L^2} + \mu \|\text{curl}\ w\|^2_{L^2} \leq \eta_2 \|\nabla \text{curl}\ w\|^2_{L^2} + C_7 \|u\|^2_{H^2} + \|\sigma\|^2_{H^2} + C_7 \|u\|^2_{H^1},
\]

where \( \eta_2(< \mu) \) and \( C_7 = C_7(\eta_2) \) are to be determined.

**Proof.** We rewrite (1.7) as

\[
u_t + u \cdot \nabla u + \frac{1}{\epsilon} \nabla G(\sigma) = \frac{1}{1 + \epsilon \sigma} (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u),
\]

where \( \nabla G(\sigma) \equiv \frac{1}{\epsilon} \frac{p'(1 + \epsilon \sigma)}{1 + \epsilon \sigma} \nabla \sigma \) for some scalar function \( G \). Applying “curl” to (3.14), we obtain

\[ho(w_t + u \cdot \nabla w) - \mu \Delta w = g,
\]

where

\[
g := \rho h - \epsilon \rho^{-1} \nabla \times (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u),
\]

with

\[
h := (\partial_2 u_j \partial_j u_3 - \partial_3 u_j \partial_j u_2, \partial_3 u_j \partial_j u_1 - \partial_1 u_j \partial_j u_3, \partial_1 u_j \partial_j u_2 - \partial_2 u_j \partial_j u_1)^t.
\]

Here and in the sequel we adopt the Einstein convention about summation over repeated indices. Observing that \( \Delta w = -\text{curl}\text{curl}\ w \), we have

\[
\frac{d}{dt} \|\sqrt{\rho} w\|^2_{L^2} + 2\mu \|\text{curl}\ w\|^2_{L^2} = \int_\Omega 2g \cdot w dx + 2\mu \int_{\partial \Omega} (\text{curl}\ w \times n) \cdot w dS.
\]

From (2.3), we have

\[
|\int_{\partial \Omega} (\text{curl}\ w \times n) \cdot w dS| = |\int_{\partial \Omega} (w \times n) \cdot \text{curl}\ w dS| \\
\leq C \int_{\partial \Omega} |\text{curl}\ w| |(w \times n) \cdot \tau| dS \\
\leq \eta \|\text{curl}\ w\|^2_{H^1} + C\eta \|u\|^2_{H^1}.
\]

Thus we can easily get this lemma. \( \Box \)

Next, we introduce the following two notations:

\[
\Phi_1(t) := \int_\Omega (2\mu |D(u)|^2 + \lambda (\text{div} u)^2 + |\nabla \sigma|^2 + 2C_8 (\rho |u_t|^2 + p'(\rho) \sigma^2_t) + 2\rho u_t \cdot u + \rho |\text{curl}\ u|^2) dx + \int_{\partial \Omega} \alpha |u|^2 dS,
\]
\[ \Psi_1(t) := \frac{1}{2} p'(1/4) \| \sigma_t \|^2_{L^2} + (2\mu + \lambda) \| \sqrt{p'(\rho)} \| \nabla \text{div} u \|_{L^2}^2 + C_8 C_5 \| u_t \|^2_{H^1} + \mu \| \text{curl} \text{curl} u \|_{L^2}^2, \]

where \( C_8 \) is a positive constant such that \( C_8 > C_5^{-1}(C_2 + C_3) + 1 \). We remark that it is important to determine the constants \( C_i \)'s sequentially. First, we choose \( C_0, C_1, C_2, C_3, C_5 \) and \( C_6 \) to be fixed positive constants. Next, once \( \eta_1 \) and \( \eta_2 \) are fixed, the constants \( C_4 \) and \( C_7 \) are determined. Then it follows from Lemmas 3.2, 3.3, 3.4 and 3.5 that

**Lemma 3.6** Let \( \epsilon_1 = \min(1, \frac{1}{2} \sqrt{C_8 C_0 p'(1/4)}) \). Then for any \( \epsilon \in (0, \epsilon_1] \), there exists a positive constant \( C_9 := C_9(C_2, C_3, C_6, C_7, C_8) \) such that

\[
\frac{d}{dt} \Phi_1(t) + \Psi_1(t) \leq C_9(\| \sigma_t \|^2_{H^1} + \| u \|_{H^2}^2)(\| u_t \|^2_{H^1} + \| u \|^4_{H^2} + \| \sigma_t \|^2_{H^1} + \| (\sigma, u) \|^2_{H^2}) + (C_2 + 2C_8 C_6 + C_4 + C_7) \| u \|^2_{H^1} + \eta_1 \| \nabla \text{div} u \|_{L^2}^2 + \eta_2 \| \text{curl} \text{curl} u \|_{L^2}^2, \]

where \( \eta_1, \eta_2, C_4(\eta_1) \) and \( C_7(\eta_2) \) are positive constants to be determined later.

\[ \square \]

### 3.3 The second-order estimates

We need to estimate the spatial and temporal derivatives of second order to close the energy estimates. The strategy is similar as that in the first-order estimates, namely, estimating the vorticity and the divergence of velocity fields respectively. However, boundary estimates are required to complete the estimates for the derivatives of highest order. We evaluate these derivatives one by one as follows.

**Lemma 3.7** For the positive constants \( \eta_3 \) and \( C_{10} := C_{10}(\eta_3) \), which are to be determined later, we have

\[
(2\mu + \lambda) \frac{d}{dt} \| \nabla \text{div} u \|_{L^2}^2 - \frac{d}{dt} \int_{\Omega} \rho u_t \cdot \nabla \text{div} u dx + \| \sqrt{p'(\rho)} \nabla \sigma_t \|_{L^2}^2 \leq \eta_3(\| \nabla \text{div} u \|_{L^2}^2 + \| \nabla \text{div} u \|_{H^1}^2) + C_{10}(\| u_t \|^2_{H^1} + \| \sigma_t \|^2_{H^1} + \| \sigma \|^2_{H^2}).
\]

**Proof.** Note that \( \text{curl} \nabla = 0 \), thus

\[
\int_{\Omega} \text{curl} \text{curl} u_t \cdot \nabla \text{div} u dx = \int_{\partial \Omega} (n \times w_t) \cdot \nabla \text{div} u dS \\
\leq C \int_{\partial \Omega} \| (n \times w_t) \cdot \tau \| \nabla \text{div} u dS \\
\leq \eta \| \nabla \text{div} u \|_{H^1}^2 + C_\eta \| u_t \|^2_{H^1}.
\]

Multiplying both sides of (3.2) by \( \nabla \text{div} u \) and integrating, we obtain

\[
\frac{2\mu + \lambda}{2} \frac{d}{dt} \| \nabla \text{div} u \|_{L^2}^2 - \frac{d}{dt} \int_{\Omega} \rho u_t \cdot \nabla \text{div} u dx - \frac{1}{\epsilon} \int_{\Omega} p'(\rho) \nabla \sigma_t \cdot \nabla \text{div} u dx \leq \eta(\| \nabla \text{div} u_t \|_{L^2}^2 + \| \nabla \text{div} u \|_{H^1}^2) + C_\eta \| u_t \|^2_{H^1} \]

\[
+ C_\eta(\| u \|^2_{H^2}(\epsilon^2 \| \sigma_t \|^2_{H^1} + u_t \| u \|_{H^2}^2) + \| u_t \|^2_{H^1} + \| \sigma_t \|^2_{H^1} \| \nabla \sigma \|^2_{H^1}).
\]

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We multiply (3.3) by $p'(\rho) \nabla \sigma_t$ and integrate to get

$$
\|\sqrt{p'(\rho)} \nabla \sigma_t\|_{L^2}^2 + \frac{1}{\epsilon} \int_\Omega p'(\rho) \nabla \sigma_t \cdot \nabla \text{div} u d\Omega \leq \frac{1}{2} \|\sqrt{p'(\rho)} \nabla \sigma_t\|_{L^2}^2 + C \|u\|_{H^2}^2 \|\sigma\|_{H^2}^2. \tag{3.20}
$$

Combining the above two inequalities, we get this lemma.

\begin{proof}

\end{proof}

**Lemma 3.8** For the positive constants $\eta_4$ and $C_{11} := C_{11}(\eta_4)$, which are to be determined later, we have

\begin{align*}
\frac{d}{dt} \|\nabla^2 \sigma\|_{L^2}^2 + (2\mu + \lambda) \|\sqrt{p'(\rho)}^{-1} \nabla^2 \text{div} u\|_{L^2}^2 \\
& \leq \eta_4 \|u\|_{H^3}^2 + C_{11}(\|\nabla^2 \text{curl} u\|_{L^2}^2 + \|u_t\|_{H^1}^2 + \|(\sigma, u)\|_{H^2}^4 + \|\sigma\|_{H^2}^2(\|u_t\|_{H^1}^2 + \|u\|_{H^2}^4)). \tag{3.21}
\end{align*}

**Proof.** The following calculations are done in the form of Einstein’s convention. Applying $\partial_{ij}$ to (3.16) for $i, j = 1, 2, 3$, where $\partial_{ij}$ denotes $\partial_{x_i x_j}$, then multiplying both sides by $\partial_{ij} \sigma$ and integrating on $\Omega$, we have

$$
\frac{1}{2} \frac{d}{dt} \|\partial_{ij} \sigma\|_{L^2}^2 + \frac{1}{\epsilon} \int_\Omega \partial_{ijk} u_k \partial_{ij} \sigma d\Omega \leq C \|\sigma\|_{H^2}^2 \|u\|_{H^3} \leq \eta_4 \|u\|_{H^3}^2 + C_\eta \|\sigma\|_{H^2}^4.
$$

Next, we differentiate (1.7), multiply the resulting equality by $p'(\rho)^{-1} \partial_{ijk} u_k$ and integrate to obtain

$$
(2\mu + \lambda) \|\sqrt{p'(\rho)}^{-1} \partial_{ijk} u_k\|_{L^2}^2 - \frac{1}{\epsilon} \int_\Omega \partial_{ijk} u_k \partial_{ij} \sigma d\Omega \\
\leq \frac{2\mu + \lambda}{2} \|\sqrt{p'(\rho)}^{-1} \partial_{ijk} u_k\|_{L^2}^2 + C \|\nabla^2 \text{curl} u\|_{L^2}^2 + \|u_t\|_{H^1}^2 \\
+ \|(\sigma, u)\|_{H^2}^4 + \epsilon^2 \|\sigma\|_{H^2}^2(\|u_t\|_{H^1}^2 + \|u\|_{H^2}^4)).
$$

Summarizing the above two inequalities, we show this lemma.

\begin{proof}

\end{proof}

**Lemma 3.9** There exists a positive constant $\tilde{C}_{12}$ such that

\begin{align*}
\frac{d}{dt}(\|\text{div} u_t\|_{L^2}^2 + \|\sqrt{\rho^{-1} p'(\rho)} \nabla \sigma_t\|_{L^2}^2) + (2\mu + \lambda) \|\sqrt{\rho^{-1} \nabla \text{div} u_t}\|_{L^2}^2 \\
\leq \eta_5(\|u\|_{H^3}^2 + \|u_t\|_{H^2}^2) + \tilde{C}_{12}\|\text{curl} u_t\|_{L^2}^2 \\
+ C_{12}(\|\sigma, u_t\|_{H^2}^2(\|(\sigma, u)\|_{H^2}^2 + \|u\|_{H^2}^4 + \|(\sigma, u_t)\|_{H^2}^4)), \tag{3.22}
\end{align*}

where $\eta_5$ and $C_{12} := C_{12}(\eta_5)$ are to be chosen.

**Proof.** Note that with the Young inequality we have

$$
\int_\Omega \rho^{-1} \text{curl} \text{curl} u_t \cdot \nabla \text{div} u_t d\Omega = \eta \|\nabla \text{div} u_t\|_{L^2}^2 + C_{\eta} \|\text{curl} u_t\|_{L^2}^2.
$$
By integrating the product of (3.2) and $\nabla \text{div} u$, we obtain the following inequality
\[
\frac{1}{2} \frac{d}{dt} \| \nabla \text{div} u \|_{L^2}^2 + (2\mu + \lambda) \| \sqrt{\rho} \chi \nabla \text{div} u \|_{L^2}^2 - \frac{1}{\epsilon} \int_{\Omega} \rho^{-1} p'(\rho) \nabla \sigma \cdot \nabla \text{div} u \, dx \\
\leq \frac{2\mu + \lambda}{8} \| \nabla \text{div} u \|_{L^2}^2 + C \| \text{curl} w \|_{L^2}^2 + C \| (\sigma_t, u_t) \|_{H^1}^2 \| (\sigma, u) \|_{H^2}^2 + \| u \|_{H^2}^2 + \| u_t \|_{H^1}^2.
\]
(3.23)

Applying $\partial_t \nabla$ to (1.6) and integrating the product of the resulting identity and $\rho^{-1} p'(\rho) \nabla \sigma$, we get
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho^{-1} p'(\rho)} \nabla \sigma \|_{L^2}^2 + \frac{1}{\epsilon} \int_{\Omega} \rho^{-1} p'(\rho) \nabla \sigma \cdot \nabla \text{div} u \, dx \\
= \frac{1}{2} \int_{\Omega} ((\rho^{-1} p'(\rho))_{\rho} + \text{div}(\rho^{-1} p'(\rho) u)) \| \nabla \sigma \|_{L^2}^2 \, dx \\
- \int_{\Omega} \rho^{-1} p'(\rho) \nabla \sigma \cdot (\nabla u \nabla \sigma + \nabla^2 \sigma u_t + \nabla u_t \sigma + \nabla \sigma \text{div} u) \\
+ \sigma \nabla \text{div} u_t + \sigma \text{div} u + \sigma_t \nabla u \, dx.
\]
By (1.1), the first term on the right-hand side of the above inequality reads
\[
\frac{1}{2} \int_{\Omega} (G_1(\rho) - G_1'(\rho) \rho) \text{div} u \| \nabla \sigma \|_{L^2}^2 \, dx \leq \eta \| \text{div} u \|_{H^2}^2 + C_\eta \| \nabla \sigma \|_{L^2}^2,
\]
where $G_1(\rho) := \rho^{-1} p'(\rho)$. Then it follows that
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho^{-1} p'(\rho)} \nabla \sigma \|_{L^2}^2 + \frac{1}{\epsilon} \int_{\Omega} \rho^{-1} p'(\rho) \nabla \sigma \cdot \nabla \text{div} u \, dx \\
\leq \eta \| \nabla u \|_{H^2}^2 + \| u_t \|_{H^2}^2 + C_\eta \| \nabla \sigma \|_{L^2}^2 \| (\sigma_t) \|_{H^1}^2 + \| \sigma \|_{H^2}^2.
\]
(3.24)

We summarize (3.23) and (3.24) to get this lemma.

Next, we should derive the estimates of the vorticity $w$, which is the key of the energy estimates.

**Lemma 3.10** There exists a positive constant $C_{13}$ such that
\[
\| w \|_{H^2}^2 \leq C_{13} (\| \Delta w \|_{L^2}^2 + \| u \|_{H^2}^2).
\]
(3.25)

**Proof.** Using Lemmas 2.1 and 2.4 we have
\[
\| w \|_{H^2} \leq C (\| \text{curl} w \|_{H^1} + \| w \times n \|_{H^2(\partial \Omega)} + \| w \|_{H^1})
\]
and
\[
\| \text{curl} w \|_{H^1} \leq C (\| \Delta w \|_{L^2} + \| \text{curl} w \times n \|_{H^2(\partial \Omega)} + \| \text{curl} w \|_{L^2}).
\]
(3.27)

From (2.3) and the trace theorem, we obtain
\[
\| w \times n \|_{H^2(\partial \Omega)} \leq C \| u \|_{H^2(\partial \Omega)} \leq C \| u \|_{H^2}.
\]
(3.28)
We construct the local coordinates by the isothermal coordinates \( \lambda(\psi, \varphi) \) to derive an estimate near the boundary (see [13] for instance), where \( \lambda(\psi, \varphi) \) satisfies
\[
\lambda_\psi \cdot \lambda_\psi > 0, \quad \lambda_\varphi \cdot \lambda_\varphi > 0 \quad \text{and} \quad \lambda_\psi \cdot \lambda_\varphi = 0.
\]
We cover the boundary \( \partial \Omega \) by a finite number of bounded open sets \( W^k \subset \mathbb{R}^3, k = 1, 2, ..., L \), such that for any \( x \in W^k \cap \Omega \),
\[
x = \lambda^k(\psi, \varphi) + r n(\lambda^k(\psi, \varphi)) = \Lambda^k(\psi, \varphi, r),
\]
where \( \lambda^k(\psi, \varphi) \) is the isothermal coordinate and \( n \) is the unit outer normal to \( \partial \Omega \). For simplicity, in what follows we will omit the superscript \( k \) in each \( W^k \). Then we construct the orthonormal system corresponding to the local coordinates by
\[
e_1 = \frac{\lambda_\psi}{|\lambda_\psi|}, \quad e_2 = \frac{\lambda_\varphi}{|\lambda_\varphi|}, \quad e_3 = n(\lambda) = e_1 \times e_2.
\]
By a straightforward calculation, we see that \( J \in C^2 \) and
\[
J = \det \text{Jac} \Lambda = (\Lambda_\psi \times \Lambda_\varphi) \cdot e_3
\]
\[
= |\lambda_\psi||\lambda_\varphi| + r(\lambda_\psi n_\varphi \cdot e_2 + |\lambda_\varphi| n_\psi \cdot e_1) + r^2[(n_\psi \cdot e_1)(n_\varphi \cdot e_2) - (n_\psi \cdot e_2)(n_\varphi \cdot e_1)] > 0,
\]
for sufficiently small \( r > 0 \). Obviously, \( \text{Jac}(\Lambda^{-1}) = (\text{Jac} \Lambda)^{-1} \). Moreover, we can easily derive the following relations (see also [29]):
\[
[\nabla(\Lambda^{-1})^1] \circ \Lambda = \frac{1}{J}(\Lambda_\psi \times e_3),
\]
\[
[\nabla(\Lambda^{-1})^2] \circ \Lambda = \frac{1}{J}(e_3 \times \Lambda_\varphi),
\]
\[
[\nabla(\Lambda^{-1})^3] \circ \Lambda = \frac{1}{J}(\Lambda_\varphi \times \Lambda_\psi),
\]
where the notation ‘\( \circ \)’ stands for the composition of operators. Set \( y := (y_1, y_2, y_3) := (\psi, \varphi, r) \), \( a_{ij} = ((\text{Jac} \Lambda)^{-1})_{ij} \). Then \( n = (a_{31}, a_{32}, a_{33}) \), the tangential directions \( \tau_i = (a_{i1}, a_{i2}, a_{i3})(i = 1, 2) \), and
\[
a_{ij}a_{3j} = 0, \quad \text{for} \quad i = 1, 2.
\]
(3.29)
Then we denote by \( D_i \) the partial derivative with respect to \( y_i \) in local coordinates. To be precise, \( D_3 \) is the normal derivative and \( D_i \) for \( i = 1, 2 \) are the tangential derivatives in the original coordinates. Moreover, we have
\[
\partial_{x_j} = a_{kj}D_k.
\]
Next, we denote the vorticity near the boundary as \( \tilde{\omega} := (\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)^t := w(t, \Lambda(y)) \). By
There exists a positive constant $\eta$ such that

$$\int_{\partial \Omega} |\nabla w|^2 dS = \eta \int_{\partial \Omega} |\nabla w_t|^2 dS.$$ 

**Lemma 3.11** There exists a positive constant $C_{14}$ such that

$$\mu \frac{d}{dt} \|\text{curl} w\|^2_{L^2} + \mu^2 \frac{d}{dt} \|\Delta w\|^2_{L^2} + \frac{\mu}{20} \|\sqrt{\sigma} w_t\|^2_{L^2}$$

$$\leq \eta_0 \|u\|^2_{H^2} + C_{14} (\|u\|^4_{H^2} + \|\sigma\|^2_{H^2} \|u\|^2_{H^3}) + C_{15} \|u_t\|^2_{H^1},$$

where $\eta_0$ and $C_{15} := C_{15}(\eta_0)$ are to be chosen.

**Proof.** Note that

$$\int_{\Omega} \Delta w w_t dx = -\frac{1}{2} \int_{\Omega} \frac{d}{dt} |\text{curl} w|^2 dx + \int_{\partial \Omega} (n \times \text{curl} w) \cdot w_t dS$$

and

$$\int_{\partial \Omega} (n \times \text{curl} w) \cdot w_t dS = \int_{\partial \Omega} \text{curl} \cdot (n \times w_t) dS$$

$$\leq \eta \|\text{curl} w\|^2_{L^2(\partial \Omega)} + C_n \|n \times w_t\|^2_{L^2(\partial \Omega)}$$

$$\leq \eta \|\text{curl} w\|^2_{H^1} + C_n \|u_t\|^2_{H^1}.$$
invoking of Lemma 2.6. Multiplying (3.19) by \( w_t - \delta \Delta w \), where \( \delta \) is a positive constant to be chosen, and integrating, we get
\[
\frac{\mu}{2} \frac{d}{dt} \| \text{curl}w \|_{L^2}^2 + \mu \| \Delta w \|_{L^2}^2 + \| \rho w_t \|_{L^2}^2 \\
\leq \frac{1}{4} \| \sqrt{\rho} w_t \|_{L^2}^2 + C(\| g \|_{L^2}^2 + \| u \|_{H^2}^2 \| \nabla w \|_{L^2}^2) + \frac{1}{4} \| \sqrt{\rho} w_t \|_{L^2}^2 + \delta^2 \| \rho \|_{L^\infty} \| \Delta w \|_{L^2}^2 \\
+ \delta^2 \| \Delta w \|_{L^2}^2 + C(\| u \|_{H^2}^2 \| \nabla w \|_{L^2}^2 + \| g \|_{L^2}^2) + \eta \| \text{curl}w \|_{H^1}^2 + C_9 \| u_t \|_{H^1}^2.
\]
In virtue of (3.16), (3.17) and (3.31), we get this lemma by choosing \( \eta \) small enough and \( \delta = \frac{\mu}{16} \).

\[ \square \]

**Lemma 3.12** There exists a positive constant \( C_{16} \) such that
\[
\frac{d}{dt} \| \sqrt{\rho} w_t \|_{L^2}^2 + \mu \| \text{curl}w_t \|_{L^2}^2 \\
\leq \eta_7(\| u_t \|_{H^1}^2 + \| u_t \|_{H^1}^2) + C_{16} \| u_t \|_{H^1}^2 \\
+ C_{17}(\| \sigma_t \|_{H^1}^2(\| u_t \|_{H^1}^2 + \| u \|_{H^2}^2 + \| u \|_{H^2}^2 + \| \sigma \|_{H^2}^2 \| u \|_{H^2}^2) + \| u_t \|_{H^2}^2 \| (u, \sigma) \|_{H^2}^2),
\]
where \( \eta_6 \) and \( C_{17} := C_{17}(\eta_6) \) are to be chosen.

**Proof.** From (3.15), we have
\[
\rho(w_t + u \cdot \nabla w_t) - \mu \Delta w_t = \tilde{g},
\]
where \( \tilde{g} := g_t - \epsilon \sigma_t(w_t + u \cdot \nabla w) - \rho u_t \cdot \nabla w \) with
\[
|g_t| \leq C(\epsilon |\sigma_t| \| \nabla u \|^2 + \| \nabla u_t \| \| \nabla u \| + \epsilon^2 |\sigma_t| \| \nabla \sigma \| \| \nabla^2 u \| \\
+ \epsilon (|\sigma_t| \| \nabla^2 u \| + \| \nabla \sigma \| \| \nabla^2 u_t \|)).
\]
Multiplying (3.34) by \( w_t \) and integrating over \( \Omega \), we have
\[
\frac{d}{dt} \| \sqrt{\rho} w_t \|_{L^2}^2 + 2 \mu \| \text{curl}w_t \|_{L^2}^2 = \int_\Omega 2\tilde{g} \cdot w_t dx + 2 \mu \int_{\partial \Omega} \text{curl}w_t \cdot (w_t \times n) dS.
\]
Similar as (3.18) and (3.27), one has
\[
\int_{\partial \Omega} \text{curl}w_t \cdot (w_t \times n) dS \leq \eta \| \text{curl}w_t \|_{H^1}^2 + C_9 \| u_t \|_{H^1}^2.
\]
and
\[
\| \text{curl}w_t \|_{H^1}^2 \leq C(\| \Delta w_t \|_{L^2}^2 + \| \text{curl}w_t \cdot n \|_{H^1(\partial \Omega)}^2 + \| \text{curl}w_t \|_{L^2}^2).
\]
From (3.34) again we get
\[
\mu \| \Delta w_t \|_{L^2}^2 \leq C(\| u_t \|_{H^1}^2 + \| \tilde{g} \|_{L^2}^2 + \| u \|_{H^2}^2 \| u_t \|_{H^1}^2).
\]
Moreover, similar as (3.30), we can derive that
\[
\| \text{curl}w_t \cdot n \|_{H^1(\partial \Omega)} \leq C \| u_t \|_{H^2}.
\]
Collecting all the above estimates, this lemma is shown.

\[ \square \]

To close the energy estimates, we estimate \( \sigma_{tt} \) and \( u_{tt} \) in the following two lemmas.
Lemma 3.13. There exists a positive constant $C_{18}$ such that
\[
\frac{d}{dt}\|\epsilon \sqrt{\rho'(\rho)} \sigma_{tt}\|_{L^2}^2 + C_{18}\|\epsilon u_{tt}\|_{H^1}^2 \\
\leq \eta_{\delta}\|\text{div}(u_t, u_t)\|_{H^2}^2 + C_{19}[\|\sigma\|_{H^1}^2(1 + \|\sigma\|_{H^2}^2) + \|u_t\|_{H^2}^2(\|\sigma\|_{H^1}^2 + \|\sigma\|_{H^2}^2) + \|u_t\|_{H^2}^2 + \|\epsilon u_{tt}\|_{L^2}^2],
\]
where $\eta_{\delta}$ and $C_{19} := C_{19}(\eta_{\delta})$ are to be determined.

**Proof.** We differentiate (1.6) twice with respect to $t$, multiply the resulting equality by $\epsilon^2 p'(\rho)\sigma_{tt}$ and then integrate over $\Omega$. Finally we get
\[
\frac{1}{2} \frac{d}{dt}\|\epsilon \sqrt{\rho'(\rho)} \sigma_{tt}\|_{L^2}^2 - \epsilon \int \rho'(\rho)\nabla \sigma_{tt} : u_t dx \\
\leq \eta\|\text{div}(u_t, u_t)\|_{H^2}^2 + \delta\|u_t\|_{H^1}^2 + C_{18}[\|\sigma\|_{L^2}^2(\|\sigma\|_{H^1}^2 + \|\sigma\|_{H^2}^2) + \|u_t\|_{H^2}^2].
\]

On the other hand, we get the following equality by differentiating (3.2) in temporal variable:
\[
\rho(u_{tt} + u \cdot \nabla u_t) + \frac{1}{\epsilon} p'(\rho)\nabla \sigma_{tt} - 2\mu\text{div}(D(u_t)) - \lambda \nabla \text{div} u_t = f,
\]
where
\[
-f = 2p''\sigma_t \nabla \sigma_t + (\epsilon p''\sigma_t^2 + p''\sigma_{tt}) \nabla \sigma + \epsilon\sigma_{tt} u_t + 2\epsilon\sigma_{tt} u_t \\
+ (\rho u_{tt} + \epsilon u_{tt} u_t + 2\epsilon\sigma_t u_t) \cdot \nabla u_t + 2(\epsilon\sigma_t u + \rho u_t) \cdot \nabla u_t.
\]
Multiplying (3.37) by $\epsilon^2 u_{tt}$, then integrating on $\Omega$, we get
\[
\frac{1}{2} \frac{d}{dt} \int \rho u_{tt}^2 dx + \epsilon \int \rho'(\rho)\nabla \sigma_{tt} : u_t dx \\
+ \int \Omega (2\mu\|\text{div}(u_{tt})\|^2 + \lambda\|\nabla \text{div} u_{tt}\|^2) dx + \int_{\partial \Omega} \alpha |u_t|^2 dS \\
\leq \delta\|\sigma_{tt}\|_{L^2}^2 + C_{9}[\|\sigma_t\|_{H^1}^2(\|\sigma_t\|_{H^1}^2 + \|\sigma\|_{H^2}^2) + \|u_t\|_{H^2}^2].
\]
Note that all the above calculations can be verified rigorously by regularization arguments. Summarizing (3.36) and (3.38) and selecting $\eta$ small enough, we can prove this lemma.

Lemma 3.14. There exists a positive constant $C_{20}$ such that
\[
\frac{d}{dt} \int \Omega \epsilon^2 \rho u_{tt} : u_t dx + \frac{\epsilon^2}{2} \frac{d}{dt} \int \Omega (2\mu |D(u_t)|^2 + \lambda\|\nabla \text{div} u_{tt}\|^2) dx + \int_{\partial \Omega} \alpha |u_t|^2 dS + C_{20}\|\epsilon u_{tt}\|_{L^2}^2 \\
\leq \eta_{\delta}\|u_t\|_{H^1}^2 + 5\|u_{tt}\|_{H^1}^2 + C_{21}\epsilon(\|\sigma_t\|_{H^1}^2(1 + \|\sigma\|_{H^2}^2) + \|\sigma_t\|_{H^1}^2(1 + \|\sigma\|_{H^2}^2) + \|\sigma_t\|_{H^1}^2(1 + \|\sigma\|_{H^2}^2),
\]
where $\eta_{\delta}$ and $C_{21} := C_{21}(\eta_{\delta})$ are to be determined.

**Proof.** We integrate $\epsilon^2 p'(\rho)\sigma_{tt}$ times (3.11) to get
\[
\|\epsilon \sqrt{\rho'(\rho)} \sigma_{tt}\|_{L^2}^2 + 2\epsilon \int \rho'(\rho)\text{div} u_t \sigma_{tt} dx \\
\leq C\epsilon(\|u_t\|_{H^2}^2 \|\sigma_t\|_{H^1}^2 + \|u_t\|_{H^1}^2 \|\sigma_t\|_{H^2}^2).
Multiplying (3.37) by $\varepsilon^2 u_t$ and integrate over $\Omega$, we obtain
\[
\frac{d}{dt} \int_{\Omega} \varepsilon^2 \rho u_t \cdot u_t dx + \varepsilon \int_{\Omega} p'(\rho) \nabla \sigma_t \cdot u_t dx + \int_{\Omega} \varepsilon^2 p''(\rho) \nabla \sigma \cdot u_t u_t dx \\
+ \frac{\varepsilon^2}{2} \frac{d}{dt} \left( \int_{\Omega} (2\mu |D(u_t)|^2 + \lambda (\text{div} u_t)^2) dx + \int_{\partial \Omega} \alpha |u_t|^2 dS \right)
\leq \int_{\Omega} \rho |u_t|^2 dx + \varepsilon^3 \int_{\Omega} \sigma_t u_t \cdot u_t dx + \int_{\Omega} \varepsilon^2 p''(\rho) \nabla \sigma \cdot u_t u_t dx + \int_{\Omega} \varepsilon^2 f \cdot u_t dx
\leq 5 \varepsilon |u_t|_{H^1}^2 + \eta \|\sigma u_t\|_{H^1}^2 + C \eta \varepsilon (\|u_t\|_{H^2}^2 + \|\sigma\|_{H^2}^2 + \|u_t\|_{H^2}^2 + \|u\|_{H^2}^2) + \|u_t\|_{H^1}^2 (\|\sigma_t\|_{H^1}^2 + \|\sigma\|_{H^2}^2 + \|u\|_{H^2}^2).
\]

Thus summarizing the above inequalities we get this lemma. \hfill \Box

**Lemma 3.15** There exists a positive constant $C_{22}$ such that
\[
\|\sigma\|_{H^2}^2 \leq C_{22} \varepsilon (\|u_t\|_{H^1}^2 + \|u\|_{H^2}^2 + \|u\|_{H^3}^2) (1 + \|\sigma\|_{H^1}^2).
\] \hfill (3.40)

**Proof.** From (1.6) and (1.8), we deduce that
\[
\frac{d}{dt} \int_{\Omega} \sigma dx = - \int_{\partial \Omega} (\sigma + \frac{1}{\varepsilon}) u \cdot n dS = 0.
\]

By the assumption $\int_{\Omega} \sigma_0 dx = 0$, we have $\int_{\Omega} \sigma dx = 0$. Then this lemma follows from (1.7) and the Poincaré inequality
\[
\|\sigma\|_{H^2} \leq C \|\nabla \sigma\|_{H^1}.
\] \hfill \Box

We introduce the following notations:
\[
\Phi(t) := C_{23} \Phi_0(t) + C_{24} \Phi_1(t) + \Phi_2(t),
\]
\[
\Psi(t) := C_{23} \Psi_0(t) + C_{24} \Psi_1(t) + \Psi_2(t),
\]
where
\[
\Phi_2(t) := (2\mu + \lambda) \|\nabla \text{div} u\|_{L^2}^2 - 2 \int_{\Omega} \rho u_t \cdot \nabla \text{div} u dx \\
+ \|\nabla^2 \sigma\|_{L^2}^2 + \|\text{div} u_t\|_{L^2}^2 + \|\sqrt{\rho^{-1}p'(\rho)} \nabla \sigma_t\|_{L^2}^2 \\
+ \frac{C_{12} + 2}{\mu} \|\nabla \text{curl} u_t\|_{L^2}^2 + \frac{20(C_{11} + 2)C_{13}}{\mu} \|\text{curl} \text{curl} u_t\|_{L^2}^2 \\
+ \int_{\Omega} \varepsilon^2 (2\mu |D(u_t)|^2 + \lambda (\text{div} u_t)^2) dx + \int_{\partial \Omega} \alpha |u_t|^2 dS
\]
and
\[ \Psi_2(t) := (2\mu + \lambda) \left( \| \nabla^2 \text{div} u \|_{L^2}^2 + \| \nabla \text{div} u_t \|_{L^2}^2 \right) + \left( \| \nabla \nabla \text{curl} u \|_{L^2}^2 + 2 \| \text{curl} u_t \|_{L^2}^2 \right) + \frac{20(C_{11} + 1)\mu}{\mu^2} \| \text{curl} u \|_{L^2}^2 + 2 \| \text{curl} u_t \|_{H^2}^2 + C_{20}\| \sigma \|_{L^2}^2 + 2 \| e \|_{H^1}^2 + \| \sigma \|_{H^2}^2. \]

Here we choose \( K \) satisfying \( K > 7C_{18}^{-1} \) and the constants \( C_{23} \) and \( C_{24} \) are to be determined. We summarize (3.19), (3.21), (3.22), (3.25), 20\(\mu^{-2}(C_{11} + 1)C_{13} \times (3.32) \), \( \mu^{-1}(\tilde{C}_{12} + 2) \times (3.33) \), \( K \times (3.35) \), (3.39) and (3.40), with the aid of Lemmas 2.1 and 2.4, and then choose \( \eta(i = 3, 4, 5, 6, 7, 8, 9) \) and \( \epsilon \) small enough in the resulting inequality. Then we get the estimates of highest order as follows:

**Lemma 3.16** There exist positive constants \( \epsilon_2 \in (0, \epsilon_1] \) and \( C_{26} \) such that
\[
\frac{d}{dt}\Phi_2(t) + \frac{3}{4}\Psi_2(t) \\
\leq (C_{10} + C_{11} + \frac{20(C_{11} + 1)C_{13}}{\mu^2} \tilde{C}_{12} + \frac{2}{\mu} C_{16} + 1)\| u_t \|_{H^1}^2 + ((C_{11} + 1)\| u \|_{H^2}^2 + C_{26}\Psi(t)(\Phi(t) + \Phi(t)^2)).
\]

Next we redefine the constant \( C_8 \) to close the energy estimates:
\[
C_8 \geq 4C_5^{-1}(C_2 + C_3 + C_5(\frac{32}{2\mu + \lambda} + \frac{4}{K}) + C_{10} + C_{11} + \frac{20(C_{11} + 1)C_{13}}{\mu^2} \tilde{C}_{12} + \frac{2}{\mu} C_{16} + 2).
\]

Let
\[
C_{24} \geq 4\overline{C}(1 + (C_{11} + 2)C_{13})/\min(2\mu + \lambda, \mu),
\]
for some constant \( \overline{C} \) depending on the best constants in Lemmas 2.1 and 2.4, next choose \( \eta_1 \) and \( \eta_2 \) small enough in Lemma 3.6 and then set
\[
C_{25} := C_{24}(C_2 + 2C_8C_6 + C_4 + C_7 + \overline{C}(1 + (C_{11} + 2)C_{13})).
\]

Thus we have
\[
\frac{d}{dt}(C_{24}\Phi_1 + \Phi_2) + \frac{3}{4}(C_{24}\Psi_1 + \Psi_2) \\
\leq C\Psi(t)(\Phi(t) + \Phi(t)^2) + C_{25}\| u \|_{H^1}^2.
\]

Finally, we choose \( C_{23} \) such that
\[
C_{23}C_0 \geq C_{25},
\]
and next \( \epsilon \) small enough in Lemma 3.1. Then there exist positive constants \( \epsilon_3 \in (0, \epsilon_2] \) and \( C \in [1, +\infty) \) such that
\[
\frac{d}{dt}\Phi(t) + \frac{1}{2}\Psi(t) \leq \frac{1}{2}C\Psi(t)(\Phi(t) + \Phi(t)^2), \quad \forall 0 \leq t \leq T, \quad 0 \leq \epsilon \leq \epsilon_3.
\]
That is,
\[
\frac{d}{dt} \Phi(t) \leq -\frac{1}{2} \Psi(t) (1 - C \Psi(t) (\Phi(t) + \Phi(t)^2)), \quad \forall 0 \leq t \leq T, \quad 0 \leq \epsilon \leq \epsilon_3. \quad (3.42)
\]

Then we can obtain the following lemma, which can be shown exactly in the same way as in [29, 21]. Thus the details are omitted.

**Lemma 3.17 (Uniform Estimates)** Let \( \Omega \subset \mathbb{R}^3 \) be a simply connected, bounded domain with smooth boundary \( \partial \Omega \). Let \((u, \sigma)\) be a solution to \((1.6)-(1.9)\) in \( \Omega \times (0, T) \) with \( \frac{1}{4} \leq 1 + \epsilon \sigma \leq 4 \), \( \forall (x, t) \in \Omega \times (0, T) \), \( \epsilon \in (0, \epsilon_3] \). Suppose that
\[
\Phi(0) \leq \frac{\beta}{\Theta}, \quad \beta \in (0, \frac{1}{2}],
\]
for some constant \( \Theta > 0 \). Then we have
\[
\Phi(t) \leq \frac{\beta}{\Theta}, \quad \forall t \in [0, T].
\]

\(\Box\)

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