EXACT MAXIMUM LIKELIHOOD ESTIMATOR FOR DRIFT FRACTIONAL BROWNIAN MOTION AT DISCRETE OBSERVATION

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This paper deals with the problems of consistence and strong consistence of the maximum likelihood estimators of the mean and variance of the drift fractional Brownian motions observed at discrete time instants. A central limit theorem for these estimators is also obtained by using the Malliavin calculus.

1. Introduction. Long memory processes have been widely applied to various fields, such as finance, hydrology, network traffic analysis and so on. Fractional Brownian motions are one special class of long memory processes when the Hurst parameter $H > 1/2$. The stochastic calculus for these processes has now been well-established (see [2]). When a long memory model is used to describe some phenomena, it is important to identify the parameters in the model. In this paper, we shall consider the following simple model

$$Y_t = \mu t + \sigma B^H_t, \quad t \geq 0,$$

(1.1)

where $\mu$ and $\sigma$ are constants to be estimated from discrete observations of the process $Y$. Our method works for fractional Brownian motions of all parameters. So in this paper we assume that $(B^H_t, t \geq 0)$ is a fractional Brownian motion of Hurst parameter $H \in (0, 1)$. But we do not discuss the case $H = 1/2$, the standard Brownian motion case since it is known. This means, $(B^H_t, t \geq 0)$ is a mean 0 Gaussian process with the following covariance structure:

$$\mathbb{E} \left( B^H_t B^H_s \right) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right).$$

We assume that the process is observed at discrete time instants $(t_1, t_2, \cdots, t_N)$. To simplify notation we assume $t_k = kh, k = 1, 2, \cdots, N$ for some fixed length

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Thus the observation vector is $\mathbf{Y} = (Y_{t_1}, Y_{t_2}, \cdots, Y_{t_N})'$. We will obtain the maximum likelihood estimators $\hat{\mu}_N$ and $\hat{\sigma}^2_N$ of $\mu$ and $\sigma^2$ respectively and study their asymptotic behaviors. In particular, the almost sure convergence and the central limit type theorem.

The first reason we chose to study (1.1) is because it is simple and we can obtain explicit estimators. The second reason is that it is also widely applied in various fields. The logarithm of a widely used geometric fractional Brownian motion, which is popular in finance, is of the form (1.1). This paper is also complementary to the work [6], where the parameter estimation problem (with continuous time observation) for fractional Ornstein-Uhlenbeck processes is studied.

The parameter estimation problem for long memory processes have been well-studied (see [1], [3], [4], [9], [11]). Although most work requires the process to be stationary, we may still adapt their idea to analyze above model (1.1). But we shall use the method of [6] which seems to be the simplest one to us. This method is based on a result of (8]) and uses the idea of Malliavin calculus.

We introduce notation

$$Y = \mu t + \sigma \mathbf{B}_t^H,$$

where and for the rest of the paper $t = (h, 2h, \cdots, Nh)'$ and $\mathbf{B}_t^H = (B^H_{ih}, \cdots, B^H_{Nh})'$. The joint probability density function of $\mathbf{Y}$ is

$$h(\mathbf{Y}) = (2\pi\sigma^2)^{-\frac{N}{2}} |\Gamma_H|^{-\frac{1}{2}} \exp \left( -\frac{1}{2\sigma^2}(\mathbf{Y} - \mu t)' \Gamma^{-1}_H (\mathbf{Y} - \mu t) \right),$$

where

$$\Gamma_H = \left[ \text{Cov}[B^H_{ih}, B^H_{jh}] \right]_{i,j=1,2,\ldots,N} = \frac{1}{2} h^{2H} (i^{2H} + j^{2H} - |i-j|^{2H}).$$

The maximum likelihood estimators of $\mu$ and $\sigma^2$ from the observation $\mathbf{Y}$ are given by

$$\hat{\mu} = \frac{\mathbf{t}' \Gamma^{-1}_H \mathbf{Y}}{\mathbf{t}' \Gamma^{-1}_H \mathbf{t}},$$

$$\hat{\sigma}^2 = \frac{1}{N} \frac{(\mathbf{Y}' \Gamma^{-1}_H \mathbf{Y})(\mathbf{t}' \Gamma^{-1}_H \mathbf{t}) - (\mathbf{t}' \Gamma^{-1}_H \mathbf{Y})^2}{\mathbf{t}' \Gamma^{-1}_H \mathbf{t}}.$$

In Section 2, we shall show that $\hat{\mu}$ and $\hat{\sigma}^2$ converge to $\mu$ and $\sigma^2$ both in mean square and almost surely. In Section 3, we prove central limit type theorem. In Section 4, we give some simulation to demonstrate our estimators $\hat{\mu}$ and $\hat{\sigma}^2$.

2. Consistence. In this section we will consider the $L^2$ consistency and the strong consistency of both MLE $\mu$ and $\sigma^2$.

Now, let us first consider the $L^2$ consistency of (1.3).
Theorem 2.1  The estimator $\hat{\mu}$ (defined by (1.3)) of $\mu$ is unbiased and it converges in probability to $\mu$ as $N \to \infty$.

PROOF. Substituting $Y$ by $\mu t + \sigma B^H_t$ in (1.3), we have

$$\hat{\mu} = \mu + \sigma \frac{t' \Gamma^{-1}_H B^H_t}{t' \Gamma^{-1}_H t}.$$  \hfill (2.1)

Thus $E[\hat{\mu}] = \mu$ and hence $\hat{\mu}$ is unbiased. On the other hand, we have

$$\text{Var}[\hat{\mu}] = \sigma^2 E\left[ \frac{t' \Gamma^{-1}_H B^H_t (B^H_t)' \Gamma^{-1}_H t}{(t' \Gamma^{-1}_H t)^2} \right] = \sigma^2 \frac{t' \Gamma^{-1}_H \Gamma_H \Gamma^{-1}_H t}{(t' \Gamma^{-1}_H t)^2} = \frac{\sigma^2}{t' \Gamma^{-1}_H t}.$$  

Denote

$M = \begin{pmatrix} \lambda_{ij} \end{pmatrix}_{i,j=1,\ldots,N}$, where $\lambda_{ij} = \frac{1}{2} (i^{2H} + j^{2H} - |i-j|^{2H})$,

and denote by $m^{-1}_{i,j}$ the entry of the inverse matrix $M^{-1}$ of $M$. Then we may write

$$\text{Var}[\hat{\mu}] = h^{-2H} \sigma^2 \frac{t' M^{-1} t}{t' M^{-1} t} = h^{-2H} \lambda_{\max} \frac{\sigma^2}{\sum_{i,j=1}^{N} \lambda_{ij} m^{-1}_{i,j}} = \frac{\sigma^2 h^{-2H-2}}{\sum_{i,j=1}^{N} \lambda_{ij} m^{-1}_{i,j}}.$$  

We shall use the following inequality (with $x = N = (1, 2, \ldots, N)$)

$$x' M^{-1} x \geq \frac{\|x\|_2^2}{\lambda_{\max}},$$  

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $M$. Thus we have

$$\text{Var}[\hat{\mu}] \leq \sigma^2 h^{-2H-2} \frac{\lambda_{\max}}{\|N\|_2^2}.$$  

Since $\|N\|_2^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ we know that $\|N\|_2^2 \approx N^3$. On the other hand we have by the Gerschgorin Circle Theorem (see [7], Theorem 8.1.3)

$$\lambda_{\max} \leq \max_{i=1,\ldots,N} \sum_{j=1}^{N} \lambda_{ij} \leq CN^{2H+1},$$  

where $C$ a positive constant whose value may be different in different occurrences. Consequently, we have

$$\text{Var}[\hat{\mu}] \leq C \sigma^2 h^{-2H-2} N^{-3} N^{2H+1} = C N^{2H-2},$$  

which converges to zero as $N \to \infty$.

Next we study the estimator $\hat{\sigma}^2$ defined by (1.4).
Theorem 2.2 We have
\[
\mathbb{E} (\hat{\sigma}^2) = \frac{N - 1}{N} \sigma^2 \quad \text{and} \quad \text{Var}[\hat{\sigma}^2] \xrightarrow{N \to \infty} 0. \tag{2.2}
\]

PROOF. By replacing \( Y \) with \( \mu t + \sigma B_t^H \) in (1.4), we have
\[
\hat{\sigma}^2 = \frac{\sigma^2}{N} \left[ (B_t^H)' \Gamma_H^{-1} B_t^H - \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right) \right].
\]
Thus
\[
E[\hat{\sigma}^2] = \frac{\sigma^2}{N} E \left[ (B_t^H)' \Gamma_H^{-1} B_t^H - \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right) \right]
\]
\[
= \frac{\sigma^2}{N} \left( N - \frac{t' \Gamma_H^{-1} E[(B_t^H)' \Gamma_H^{-1} B_t^H] \Gamma_H^{-1} t}{t' \Gamma_H^{-1} t} \right) = \frac{N - 1}{N} \sigma^2. \tag{2.3}
\]

To compute the variance of \( \hat{\sigma}^2 \) we also need to compute \( E[(\hat{\sigma}^2)^2] \):
\[
E[(\hat{\sigma}^2)^2] = \frac{\sigma^4}{N^2} E \left[ \left( (B_t^H)' \Gamma_H^{-1} B_t^H - \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right) \right)^2 \right]
\]
\[
= \frac{\sigma^4}{N^2} \left( E[(B_t^H)' \Gamma_H^{-1} B_t^H]^2 - 2 E[(B_t^H)' \Gamma_H^{-1} B_t^H] \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right) \right.
\]
\[
\left. + \frac{\sigma^4}{N^2} E \left[ \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right)^2 \right] \right)
\]
\[
= \frac{\sigma^4}{N^2} \left( E[(B_t^H)' \Gamma_H^{-1} B_t^H]^2 - 2 E[(B_t^H)' \Gamma_H^{-1} B_t^H] \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right) \right)
\]
\[
+ \frac{\sigma^4}{N^2} \left( \frac{(t' \Gamma_H^{-1} B_t^H)^2}{t' \Gamma_H^{-1} t} \right)^2 \right). \tag{2.4}
\]

Denote \( X = \Gamma_H^{-1/2} B_t^H \). Then \( \mathbb{E} (XX') = E[(\Gamma_H^{-1/2} B_t^H)' \Gamma_H^{-1/2} B_t^H] = I \). Therefore, \( X \) is a standard Gaussian vector of dimension \( N \). For any \( \lambda \) small enough and \( \varepsilon \in \mathbb{R} \) let us compute the following .
\[
E[\exp(\lambda (B_t^H)' \Gamma_H^{-1} B_t^H + \varepsilon t' \Gamma_H^{-1} B_t^H)] = E[\exp(\lambda |X|^2 + \varepsilon t' \Gamma_H^{-1/2} X)]
\]
\[
= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|X|^2}{2} + \lambda X^2 + \varepsilon t' \Gamma_H^{-1/2} X} dx \).
\]

A standard technique of completing the squares yields
\[
E[\exp(\lambda (B_t^H)' \Gamma_H^{-1} B_t^H + \varepsilon t' \Gamma_H^{-1} B_t^H)] = (1 - 2\lambda)^{-N/2} \exp \left\{ \frac{\varepsilon^2 t' \Gamma_H^{-1} k}{2(1 - 2\lambda)} \right\} =: f(\lambda, \varepsilon).
\]

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We are only interested in the coefficient of \( \lambda^2 \) and \( \lambda \varepsilon^2 \) in the above expression \( f(\lambda, \varepsilon) \). We have

\[
f(\lambda, \varepsilon) = (1 + N\lambda + N(N + 2)\lambda^2 + \cdots) \left[ 1 + \frac{\varepsilon^2 t'\Gamma_H^{-1}t}{2}(1 + 2\lambda + \cdots) + \cdots \right]
\]

\[
= 1 + N\lambda + N(N + 2)\lambda^2 + \cdots + (N + 2)\lambda \varepsilon^2 t'\Gamma_H^{-1}t + \cdots.
\]

Comparing the coefficients of \( \lambda^2 \) and \( \lambda \varepsilon^2 \) we have

\[
E \left[ (B_H^t)^{-1} B_H^t \right] = N(N + 2), \quad (2.5)
\]

\[
E[\left( (B_H^t)^{-1} B_H^t (t'\Gamma_H^{-1}t)^2 \right)] = (N + 2)(t'\Gamma_H^{-1}t).
\]

Hence, we have

\[
E[\left( (B_H^t)^{-1} B_H^t (t'\Gamma_H^{-1}t)^2 \right)] = (N + 2)(t'\Gamma_H^{-1}t) = N + 2. \quad (2.6)
\]

Using (2.3), (2.4), (2.5) and (2.6), we obtain

\[
\text{Var}[\hat{\sigma}^2] = E[\left( \hat{\sigma}^2 \right)^2] - (E[\hat{\sigma}^2])^2
\]

\[
= \frac{\sigma^4}{N^2} [N(N + 2) - 2(N + 2) + 3 - (N - 1)^2]
\]

\[
= 2(N - 1) \sigma^4, \quad (2.7)
\]

which is convergent to 0. Thus we prove the theorem.

Now we can show the strong consistence of the MLE \( \hat{\mu} \) and \( \hat{\sigma}^2 \) as \( N \to \infty \).

**Theorem 2.3** The estimators \( \hat{\mu} \) and \( \hat{\sigma}^2 \) defined by (1.3) and (1.4), respectively, are strongly consistent, that is,

\[
\hat{\mu} \to \mu \quad \text{a.s. as} \quad N \to \infty \quad (2.8)
\]

\[
\hat{\sigma}^2 \to \sigma^2 \quad \text{a.s. as} \quad N \to \infty \quad (2.9)
\]

**PROOF.** Let’s prove the convergence for \( \hat{\mu} \) first. We will use a Borel-Cantelli lemma. To this end, we will show that

\[
\sum_{N \geq 1} P\left( |\hat{\mu} - \mu| > \frac{1}{N^\epsilon} \right) < \infty \quad (2.10)
\]

for some \( \epsilon > 0 \).
Take $0 < \epsilon < 1 - H$. Then from the Chebyshev’s inequality and the Nelson’s hypercontractivity inequality [5], we have
\[
P(\left| \hat{\mu} - \mu \right| > \frac{1}{N^{\epsilon}}) = N^{2p\epsilon} E(\left| \hat{\mu} - \mu \right|^p) \leq C_p N^{2p\epsilon} (E(\left| \hat{\mu} - \mu \right|^2))^{p/2} \\
\leq C_p' \sigma^p H^{-1} \epsilon^p (2H+2) N^{2p\epsilon + (2H-2)p}.
\]
For sufficiently large $p$, we have $2p\epsilon + (2H - 2)p < -1$. Thus (2.10) is proved, which implies (2.8) by Borel-Cantelli lemma.

In the same way, we can show (2.9).

3. **Asymptotic.** Now we are interested in the central limiting type theorem for the estimators $\hat{\mu}$ and $\hat{\sigma}^2$. First from (2.1), it is easy to see that
\[
\sqrt{\frac{N}{2}} (\hat{\sigma}^2 - \sigma^2) \xrightarrow{L^2} \mathcal{N}(0, \sigma^4) \quad \text{as } N \text{ tends to infinity}
\]
We want to study $\hat{\sigma}^2$

**Theorem 3.1** We have
\[
\sqrt{\frac{N}{2}} (\hat{\sigma}^2 - \sigma^2) \xrightarrow{L^2} \mathcal{N}(0, \sigma^4) \quad \text{as } N \to \infty. \quad (3.1)
\]

**PROOF.** To simplify notation we assume $H > 1/2$. The case $H < 1/2$ is similar. We define
\[
G_N = \sqrt{\frac{N}{2}} (\hat{\sigma}^2 - \sigma^2) = \frac{\sigma^2}{\sqrt{2N}} \left[ (B_H^t)^t \Gamma_H^{-1} B_H^t - \frac{(t' \Gamma_H^{-1} B_H^t)^2}{t' \Gamma_H^{-1} t} \right] - \sqrt{\frac{N}{2}} \sigma^2.
\]
From (2.7), it is obvious that $E[G_N^2]$ converges to $\sigma^4$. Thus from Theorem 4 of [8] to show (3.1), it suffices to show that $\| DG_N \|_{L^2(\Omega)} \to C$.

First, using the definition of Malliavin calculus, we obtain
\[
D_s G_N = \sqrt{\frac{2}{N}} \sigma^2 [D_s (B_H^t)^t \Gamma_H^{-1} B_H^t - \frac{t' \Gamma_H^{-1} B_H^t \cdot t' \Gamma_H^{-1} D_s B_H^t} {t' \Gamma_H^{-1} t}],
\]
where \( D_s(B_t^H)' = (1_{[0,h]}(s), 1_{[0,2h]}(s), \ldots, 1_{[0,Nh]}(s)) \). Therefore, we have

\[
\| D_s G_N \|_2^2 \leq \frac{2\sigma^4}{N} \alpha_H \int_0^t \int_0^t |u - s|^{2H-2} \left[ D_s(B_t^H)' \Gamma_H^{-1} B_t^H - \frac{t' \Gamma_H^{-1} B_t^H \cdot t' \Gamma_H^{-1} D_s B_t^H}{t' \Gamma_H^{-1} t} \right] duds
\]

\[
= \frac{2\sigma^4}{N} 4\alpha_H \int_0^t \int_0^t |u - s|^{2H-2} [D_s(B_t^H)' \Gamma_H^{-1} B_t^H \cdot D_u(B_t^H)' \Gamma_H^{-1} B_t^H - D_s(B_t^H)' \Gamma_H^{-1} B_t^H \cdot t' \Gamma_H^{-1} D_s B_t^H - \frac{t' \Gamma_H^{-1} B_t^H \cdot t' \Gamma_H^{-1} D_s B_t^H}{t' \Gamma_H^{-1} t}] duds
\]

\[
= 2\sigma^4 [A_T^{(1)} - 2A_T^{(2)} + A_T^{(3)}].
\]

Since both \( D_s(B_t^H)' \Gamma_H^{-1} B_t^H \) and \( D_u(B_t^H)' \Gamma_H^{-1} B_t^H \) are Gaussian random variables we can write

\[
E(\| A_T^{(1)} - E A_T^{(1)} \|_2^2)
\]

\[
= \frac{2}{N^2} \alpha_H \int_{[0,T]^4} E[D_s(B_t^H)' \Gamma_H^{-1} B_t^H \cdot D_r(B_t^H)' \Gamma_H^{-1} B_t^H] \cdot E[D_u(B_t^H)' \Gamma_H^{-1} B_t^H \cdot D_r(B_t^H)' \Gamma_H^{-1} B_t^H] |s - u|^{2H-2} |r - v|^{2H-2} dsdrduds
\]

\[
= \frac{2}{N^2} \int_{[0,T]^4} [D_s(B_t^H)' \Gamma_H^{-1} D_r B_t^H \cdot D_u(B_t^H)' \Gamma_H^{-1} D_u B_t^H] \cdot |s - u|^{2H-2} |r - v|^{2H-2} dsdrduds.
\]

Let \( \Gamma^{-1} = (\Gamma^{-1}_{ij})_{i,j=1,...,N} \), \( \Gamma_H = (\Gamma^{-1}_{ij})_{i,j=1,...,N} \) and \( \delta_{ik} \) be the Kronecker symbol. We shall use \( \int_0^t \int_0^t |s - u|^{2H-2} dsdu = \Gamma_{ii'} \) and \( \sum_{j=1}^N \Gamma_{ij}^{-1} \Gamma_{ji'} = \delta_{i'i} \).
Then we have

\[
E(|A_T^{(1)} - EA_T^{(1)}|^2) = \frac{2}{N^2} \int_{[0,T]^4} E[D_r(B_t^H)\Gamma^{-1}_H B_t^H \cdot t'\Gamma^{-1}_H t] + \frac{t'\Gamma^{-1}_H D_v B_t^H}{t'\Gamma^{-1}_H t} + 1_{\{0,\bar{h}\}}(s) \Gamma^{-1}_{ij} 1_{\{0,\bar{h}\}}(r) \cdot 1_{\{0,\bar{h}\}}(u) \Gamma^{-1}_{i'j'} 1_{\{0,\bar{h}\}}(v) \cdot \alpha_H |s - u|^{2H-2} \alpha_H |r - v|^{2H-2} dsdrdudv
\]

\[
= \frac{2}{N^2} \sum_{i,j=1}^N \sum_{i',j'=1}^N \Gamma^{-1}_{ij} \Gamma^{-1}_{i'j'} \cdot \Gamma_{i'j'}
\]

\[
= \frac{2}{N^2} \sum_{i,j=1}^N \delta_{ij} = \frac{2}{N},
\]

which converges to 0 as \( N \to \infty \).

Now we deal with \( A_T^{(2)} \).

\[
E(|A_T^{(2)} - EA_T^{(2)}|^2) = \frac{2\alpha_H^2}{N^2} \int_{[0,T]^4} E[D_r(B_t^H)\Gamma^{-1}_H B_t^H \cdot t'\Gamma^{-1}_H t] + \frac{t'\Gamma^{-1}_H D_v B_t^H}{t'\Gamma^{-1}_H t} + 1_{\{0,\bar{h}\}}(s) \Gamma^{-1}_{ij} 1_{\{0,\bar{h}\}}(r) \cdot 1_{\{0,\bar{h}\}}(u) \Gamma^{-1}_{i'j'} 1_{\{0,\bar{h}\}}(v) \cdot \alpha_H |s - u|^{2H-2} \alpha_H |r - v|^{2H-2} dsdrdudv
\]

\[
= \frac{2\alpha_H^2}{N^2} \int_{[0,T]^4} \frac{D_r(B_t^H)\Gamma^{-1}_H t \cdot t'\Gamma^{-1}_H D_v B_t^H}{t'\Gamma^{-1}_H t} + 1_{\{0,\bar{h}\}}(s) \Gamma^{-1}_{ij} 1_{\{0,\bar{h}\}}(r) \cdot 1_{\{0,\bar{h}\}}(u) \Gamma^{-1}_{i'j'} 1_{\{0,\bar{h}\}}(v) \cdot \alpha_H |s - u|^{2H-2} \alpha_H |r - v|^{2H-2} dsdrdudv
\]

\[
= \frac{2}{N^2} \sum_{i,j=1}^N \sum_{i',j'=1}^N \Gamma^{-1}_{ij} \Gamma^{-1}_{i'j'} \cdot \Gamma_{i'j'}
\]

where the summation is over \( 1 \leq i, j, i', j', k, l, k', l' \leq N \). Sum first over \( 1 \leq
As for $A^{(3)}_T$, we have

$$E(\|A^{(3)}_T - E A^{(3)}_T\|^2) = \frac{2\alpha^2_H}{N^2} \left( \int_{[0,T]^4} \frac{\left( t^t \Gamma^{-1}_H D_s B^H \cdot t^t \Gamma^{-1}_H D_{s'} B^H \right) \cdot \left( t^t \Gamma^{-1}_H D_u B^H \cdot t^t \Gamma^{-1}_H D_{u'} B^H \right)}{(t^t \Gamma^{-1}_H t)^2} ds \right) |s-u|^{2H-2} ds du$$

which converges to 0 as $N \to \infty$.

By triangular inequality, we have that

$$E(\|DG_N \|^2_{L^H} - E \|DG_N \|^2_{L^H})^2$$

$$= E(A^{(1)}_T + A^{(2)}_T + A^{(3)}_T - E(A^{(1)}_T + A^{(1)}_T + A^{(3)}_T))^2$$

$$\leq 9[E(A^{(1)}_T - E(A^{(1)}_T))^2 + E(A^{(2)}_T - E(A^{(2)}_T))^2 + E(A^{(3)}_T - E(A^{(3)}_T))^2] \to 0.$$

This completes the proof of the theorem.
4. Simulation. This section contains numerical simulations of the estimators obtained in this paper. The fractional Brownian motions are simulated by the Paxson’s method [10].

| Table 1 | The means and standard deviations of estimators ($\mu=0.7880$, $\sigma^2=0.8116$) |
|---------|---------------------------------|
| $H=0.25$ | $H=0.45$ | $H=0.55$ | $H=0.75$ |
| $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ |
| Mean | 0.7862 | 0.8152 | 0.7884 | 0.8153 | 0.7911 | 0.8126 | 0.7678 | 0.7910 |
| Std.dev. | 0.0116 | 0.0830 | 0.0112 | 0.0937 | 0.0514 | 0.0692 | 0.0974 | 0.0736 |

| Table 2 | The means and standard deviations of estimators ($\mu=1.5880$, $\sigma^2=1.8116$) |
|---------|---------------------------------|
| $H=0.25$ | $H=0.45$ | $H=0.55$ | $H=0.75$ |
| $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ |
| Mean | 1.5863 | 1.8694 | 1.5882 | 1.8719 | 1.5961 | 1.8647 | 1.5925 | 1.7864 |
| Std.dev. | 0.0148 | 0.1724 | 0.0456 | 0.1786 | 0.0710 | 0.1567 | 0.1879 | 0.1644 |

| Table 3 | The means and standard deviations of estimators ($\mu=3.5880$, $\sigma^2=5.8116$) |
|---------|---------------------------------|
| $H=0.25$ | $H=0.45$ | $H=0.55$ | $H=0.75$ |
| $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ | $\mu$ | $\sigma^2$ |
| Mean | 3.5861 | 5.8133 | 3.5810 | 5.8192 | 3.5837 | 5.8229 | 3.5834 | 5.8346 |
| Std.dev. | 0.0314 | 0.1648 | 0.0792 | 0.1737 | 0.0905 | 0.1031 | 0.0526 | 0.1026 |

From these numerical computations, we see the estimators are excellent both for $H > 1/2$ and $H < 1/2$.

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