Error estimate of the second-order homogenization for divergence-type nonlinear elliptic equation with small periodic coefficients

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Abstract Second-order two-scale expansions, a unified proof for the regularity of the correctors based on the translation invariant and a lemma for extracting \( O(\varepsilon) \) from the remainder term are presented for the second order nonlinear elliptic equation with rapidly oscillating coefficients. If the data are smooth enough, the error of the zero-order (energy) in \( L^\infty \), first-order in the Hölder norm, second-order’s gradient (flux) in the maximum norm(linear periodic case), are locally \( O(\varepsilon) \). It can be used in the parabolic equation.

Keywords: homogenization, translation invariant, De Giorgi-Nash estimate, error estimate

MSC(2000): 35B27, 35J65

1 Introduction

Consider the homogenization of the following elliptic problem: find \( u_\varepsilon \in H^1_0(\Omega) \),

\[
-\frac{\partial}{\partial x_i} \left( a_{ij}(u_\varepsilon, x, \frac{x}{\varepsilon}) \frac{\partial u_\varepsilon}{\partial x_j} \right) = f(u_\varepsilon, x, \frac{x}{\varepsilon}), \quad \text{in } \Omega; \tag{1}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain and the summation convention is used. \( A_\varepsilon = (a_{ij}) \) is symmetric and positive definite; \( a_{ij}(u_\varepsilon, x, y) \) are 1-periodic in \( y \). In the combined conduction-radiation heat transfer\[^1\], \( a_{ij} = k_{ij}(x, \frac{x}{\varepsilon}) + 4u_\varepsilon^3b_{ij} \). Assume all of the data are smooth enough.

It is difficult to solve this problem numerically because of the rapidly oscillating coefficients. One method is the two-scale homogenization. The error estimate in \( L^\infty \) (\( O(\varepsilon) \)) was presented by J. L. Lions et al\[^2\] and Lin\[^3\]. Oleinik et al\[^4\] proved the \( O(\varepsilon^{1/2}) \) estimate in \( H^1 \). Su et al\[^5\] investigated the quasi-periodic problems; Zhang and Cui\[^1\] gave a numerical example for the Rosseland equation. There are also some other famous methods, such as Multiscale Finite Element Method(MFEM\[^6\]) and Heterogeneous Multiscale Method(HMM\[^7\]).

2 Second-order two-scale expansions

The periodic cell \( Y = [0,1]^n; W_{per}^1(Y) \subset \{ \phi \in H^1(Y) : \int_Y \phi = 0 \} \) consists of the functions with the same traces on the opposite faces of \( Y \). As the same as the linear case\[^2\], we look for a formal asymptotic expansion of the form

\[
u_\varepsilon(x) = u_0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + ... \tag{2}
\]

where \( u_1(\cdot, y) \), \( u_2(\cdot, y) \) is \( Y \)-periodic in \( y \). Let \( y = \frac{x}{\varepsilon} \), then \( \frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial}{\partial y_i} \). Assume the coefficient \( A_\varepsilon \) in (1) has the form \( A_\varepsilon = A(u_0, x, \frac{x}{\varepsilon}) + \varepsilon A_1(u_0, u_1, x, \frac{x}{\varepsilon}) + \varepsilon^2 A_2 + ... \) where \( A = \)

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(a_{ij}(u_0, x, \frac{\varepsilon}{\varepsilon})) is symmetric positive definite and A_1, A_2 are uniform bounded; the righthand side has the form \( f(u_0, x, \frac{\varepsilon}{\varepsilon}) = f(u_0, x, \frac{\varepsilon}{\varepsilon}) + \varepsilon f_1 + \ldots \) with \( f_1 \in L^\infty \). Substituting (2) into (1) and equating the power-like terms of \( \varepsilon \), we introduce the following auxiliary functions to make the term of order \( \varepsilon^{-1} \) equal zero

\[
\int_Y a_{ij}(u_0, x, y) \frac{\partial N_m}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = - \int_Y a_{ij}(u_0, x, y) \frac{\partial \varphi}{\partial y_j} \quad \forall \varphi(y) \in W^1_{\text{per}}(Y), 1 \leq m \leq n; \tag{3}
\]

where \( N_m(u_0, x, y) \in W^1_{\text{per}}(Y)(u_0, x \text{ are parameters}) \). Then \( u_1 = N_1 \partial y u_0 \). The problem for the part of order \( \varepsilon^n \) admits a unique solution iff there exists \( u_0 \in H^1_0(\Omega) \) such that

\[
- \frac{\partial}{\partial x_i}[a_{ij}(\varepsilon u_0) \frac{\partial u_0}{\partial x_j}] = \int_Y f(u_0, x, y)dy, \quad a_{ij}(u_0, x) = \int_Y [a_{ij}(u_0, x, y) + a_{il}(u_0, x, y) \frac{\partial N^l_{ij}}{\partial y_i}]dy. \tag{4}
\]

This system is well-posed because of the compensated compactness\( ^8 \). \( \forall \varphi(y) \in W^1_{\text{per}}(Y), \)

\[
\int_Y a_{ij}(u_0, x, y) \frac{\partial N_l}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = - \int_Y (a_{ij} + a_{km} \frac{\partial N_l}{\partial y_m}) \frac{\partial \varphi}{\partial y_j} + \int_Y a_{km} \frac{\partial N_l}{\partial y_m} \frac{\partial \varphi}{\partial y_i} \tag{5}
\]

\[
\int_Y a_{ij}(u_0, x, y) \frac{\partial Q_k}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = - \int_Y (a_{ija} + A_{1,ik} + A_{1,jl}) \frac{\partial \varphi}{\partial y_i} + \int_Y a_{ik} \frac{\partial N_k}{\partial y_i} - \int_Y (a_{ik} + a_{il}) \frac{\partial \varphi}{\partial y_i} \tag{6}
\]

\[
\int_Y a_{ij}(u_0, x, y) \frac{\partial R}{\partial y_i} \frac{\partial \varphi}{\partial y_j} = \int_Y [f(u_0, x, y) - \int_Y f(u_0, x, y)] \varphi(y). \tag{7}
\]

If \( \varphi(y) = 1 \), the righthand sides of the above equations equal zero. So \( M_{kl}, Q_k, R \in W^1_{\text{per}}(Y)(u_0, x \text{ are parameters}) \). Let \( u_2 = M_{kl} \partial^2_{kl} u_0 + Q_k \partial y u_0 + R \) to make the \( O(\varepsilon^n) \) term equal zero.

3 Error estimate in \( L^\infty \) and \( C^\alpha \)

For the regularity of the following periodic problem (the abstract Eq. from (3),(5)-(7)), it can be unified by the translation invariant. If \( Y_z = Y + z, z \in \mathbb{R}^n \), find \( M \in W^1_{\text{per}}(Y_z) \) such that

\[
\int_{Y_z} A(y) \nabla M \cdot \nabla \varphi = \int_{Y_z} \tilde{B}(y) \cdot \nabla \varphi + d(y) \varphi, \quad \forall \varphi \in W^1_{\text{per}}(Y_z), \tag{8}
\]

where \( A, \tilde{B}, d \) is \( Y \)-periodic and \( \int_{Y_z} d = 0 \). Then \( M = N \in W^1_{\text{per}}(Y) \) after a periodic extension if \( N \) is the solution to (8) in the cell \( Y \). So the boundary estimate for \( N \) is equivalent to the interior estimate for \( M \). If \( \tilde{B} \in L^q, d \in L^{n/2}, q > n \), then \( N \in C^\alpha(\text{Th 8.24}[9]) \).

Notice that Lemma 1.6\( ^4 \) is right for the case \( v = 1, u \in W^1_p, p > 1 \). If \( g(x, y) \in \tilde{L}(\Omega \times \mathbb{R}^n), \int_Y g = 0 \), then by the help of \( W^{-1,p} \) and Meyers estimate (Th 4.2\( ^2 \)) there exists

\[
\nabla \psi_\varepsilon(x) \in L^p(\Omega; \mathbb{R}^n), \quad \text{s. t.} \quad g(x, \frac{\psi_\varepsilon}{\varepsilon}) = \text{div}_x \frac{\psi_\varepsilon}{\varepsilon}, \quad \| \psi_\varepsilon \|_{L^p(\Omega; \mathbb{R}^n)} \leq C_\varepsilon. \tag{9}
\]

Let \( Z_\varepsilon \equiv u_\varepsilon - \hat{u} = u_\varepsilon - u_0 - \varepsilon \psi_\varepsilon - \varepsilon^2 u_2 \). \([A(u_\varepsilon) - A(\hat{u})] \nabla \hat{u} = Z_\varepsilon A'(\xi) \nabla \hat{u} \equiv Z_\varepsilon \tilde{B}(x, \frac{\xi}{\varepsilon}) \). Then

\[
- \varepsilon \frac{\partial}{\partial x_i} [a_{ij}(\varepsilon u_\varepsilon) \frac{\partial Z_\varepsilon}{\partial x_j} + b_{ij}(\varepsilon u_\varepsilon) Z_\varepsilon] = \varepsilon \frac{\partial \psi_\varepsilon}{\partial x_i} + \varepsilon \eta_i, \quad \text{in} \ \Omega, \tag{10}
\]
because \( u_\varepsilon, \tilde{u} \) are known (Th 10.7[9]). \( Z_\varepsilon = -\varepsilon u_1 - \varepsilon^2 u_2 \) is \( O(\varepsilon) \) in \( L^\infty(\partial \Omega) \) and \( O(\varepsilon^{1-\alpha}) \) in \( C^\alpha(\partial \Omega) \). If \( \|a_{ij}\|_\infty, \|b_i\|_\infty, \|\theta_j\|_q, \|\eta\|_{q/2} \leq C, q > n \), then \( \|Z_\varepsilon\|_\infty, [Z_\varepsilon]_{C^\beta(\Omega')} \leq C \varepsilon \) by the maximum principle and De Giorgi-Nash estimate (Th 8.16, Th 8.24[9]) for nonlinear parabolic equation with mixed boundary conditions, see [10].

For the gradient estimate in parabolic case, see Li and Li [11].

\[ \int_\Omega A_i \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_\Omega A_i \nabla \tilde{u}_\varepsilon \cdot \nabla u_\varepsilon \leq C \varepsilon. \]

Consequently, for the energy, \( \int_\Omega A_i \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_\Omega A_i \nabla \tilde{u}_\varepsilon \cdot \nabla u_\varepsilon = 0 \).

If \( \|u_1\|_\infty, \|u_2\|_\infty \leq C, \|u_\varepsilon - u_0\|_\infty \leq C \varepsilon \). Consequently, for the energy, \( \int_\Omega A_i \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_\Omega A_i \nabla \tilde{u}_\varepsilon \cdot \nabla u_\varepsilon \leq C \varepsilon. \)

If \( \|u_2\|_{C^\beta} \leq C \), then \( \|u_\varepsilon - u_0 - \varepsilon u_1\|_{C^\beta(\Omega')} \leq C \varepsilon \). For the case of nonlinear parabolic equation with mixed boundary conditions, see [10].

If in (10) \( a_{ij} = a_{ij}(\varepsilon), a_{ij} \in C^\alpha(\bar{\Omega}) \) (or piecewise smooth), \( b_i = 0, \frac{\partial \theta}{\partial \varepsilon x_i}, \eta \in L^{n+\delta}(\Omega), \delta > 0; \Omega' \) is a open set, \( \Omega' \subset \Omega \). Then by the help of Lemma 16[3], \( \sup_{\Omega'} |\nabla Z_\varepsilon| \leq C \varepsilon. \) If \( |\nabla u_\varepsilon| \leq C \varepsilon^{-1}, \) then \( \sup_{\Omega'} |\nabla (u_\varepsilon - u_0 - \varepsilon u_1)| \leq C \varepsilon. \) It’s also true for the tensor case. For the flux, \( \sup_{\Omega'} |A_i \nabla (u_\varepsilon - u_0 - \varepsilon u_1)| \leq C \varepsilon. \)

Let \( w_\varepsilon(x) = [Z_\varepsilon(\varepsilon x) - Z_\varepsilon(0)]/\varepsilon \), then considering the following equation

\[ -\text{div}[A(\varepsilon x, x)\nabla w_\varepsilon(x) + \varepsilon w_\varepsilon(x) \tilde{b}(\varepsilon x, x)] = \varepsilon F(\varepsilon x) + Z_\varepsilon(0) \text{div}[\tilde{b}(\varepsilon x, x)]. \]

where \( F(\xi) = \varepsilon \frac{\partial F}{\partial \varepsilon x}(\xi) + \varepsilon \eta(\xi). \) Differentiate both sides formally, by De Giorgi-Nash estimate \( [\partial w_\varepsilon]_{C^\gamma(\Omega')} \leq C \varepsilon, \) since \( [Z_\varepsilon(0)] \leq C \varepsilon. \) Then locally \( [\nabla (u_\varepsilon - u_0 - \varepsilon u_1)]_{C^\gamma} \leq C \varepsilon^{1-\gamma}. \)

For the gradient estimate in parabolic case, see Li and Li[11]. Their work was based on the piecewise smooth coefficients so it’s very important for both theory and practice.

Acknowledgements This work is supported by National Natural Science Foundation of China (Grant No. 90916027). The authors thank Professor Yan NingNing and the referees for their careful reading and helpful comments.

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