SOME GROUP THEORETICAL MASS FORMULAE

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By a mass formula we mean a formula which involves a sum of the form

\[ \sum \frac{1}{|\text{Aut}(G_i)|} \]

where \( G_i \) varies over some collection (finite or infinite) of groups, or other mathematical structures. E.g. in the Smith-Minkowski-Siegel mass formula, going back to 1867, one considers the automorphism groups of certain lattices, associated with a given quadratic form (see [CS], Chap. 16. They mention also mass formulas for codes). This paper aims both to survey results of this type in group theory, and to derive some new ones.

1. Covering groups. To formulate our first result, recall that given a finite group \( G \), a group \( H \) is a covering group (or a representation group) of \( G \) if there exists a normal subgroup \( M \triangleleft H \) such that (i) \( M \leq Z(H) \cap H' \), (ii) \( H/M \cong G \), and (iii) \( |H| \) is maximal among the groups satisfying (i) and (ii). Covering groups exist, they need not be unique, but the kernel \( M \) is determined uniquely up to isomorphism, it is isomorphic to the cohomology group \( H^2(G, \mathbb{C}^*) \) (with respect to the trivial action of \( G \) on the complex field \( \mathbb{C} \)), and is termed the Schur multiplier \( M(G) \) of \( G \) [Hu, V.23]. In certain cases the covering groups are unique (up to isomorphism), e.g. if \( G \) is perfect (i.e. \( G = G' \)).

**Theorem 1.** Let \( G \) be a finite group, and let \( H_1, \ldots, H_n \) be its covering groups. Then

\[ \sum_1^n \frac{1}{|\text{Aut}(H_i)|} = \frac{1}{|\text{Aut}(G)|}. \]

2. Isoclinism. Equalities of the form of (1), the sum of reciprocal orders of several automorphism groups equals the reciprocal order of one automorphism group, were discussed by P.Hall in [H3, H2]. Some of them are related to his notion of isoclinism. Two groups \( G \) and \( H \) are said to be isoclinic, if there exist isomorphisms \( \phi : G/Z(G) \to H/Z(H) \) and \( \theta : G' \to H' \), such that if \( x, y \in G \), then \( \theta[x, y] = [\phi(xZ(G)), \phi(yZ(G))] \). Intuitively, two groups are isoclinic if they have the same commutator function. While this notion is defined to all groups, we will apply it mostly for finite \( p \)-groups. Isoclinism is an equivalence relation, and the set of finite \( p \)-groups in an isoclinism class is termed a family. An autoclinism of \( G \) is an isoclinism of \( G \) with itself, i.e. a pair of automorphisms \( \phi : G/Z(G) \to G/Z(G) \)
and \( \theta : G' \to G' \), which satisfy the above compatibility condition. The set of these autoclinisms form a group, which depends only on the isoclinism class of \( G \), and is termed the autoclinism group \( \text{Acl}(G) \), or \( \text{Acl}(\mathcal{F}) \), where \( \mathcal{F} \) is the family of \( G \). Note that the isomorphism \( \phi \) determines \( \theta \), therefore \( \text{Acl}(G) \) is isomorphic to a subgroup of \( \text{Aut}(G/Z(G)) \).

It is easy to see that an isoclinism between \( G \) and \( H \) induces an isomorphism between \( Z(G) \cap G' \) and \( Z(H) \cap H' \). The factor group \( Z(G)/Z(G) \cap G' \) is termed in [BT] the branch factor of \( G \). Hall proved that each family \( \mathcal{F} \) contains groups whose branch factor is the identity \([H2]\). Taking direct product with any finite abelian \( p \)-group \( Q \) shows that there are groups in \( \mathcal{F} \) with branch factor \( Q \), and the set of all these groups is the \( Q \)-branch \( \mathcal{F}_Q \) of the family.

Hall states, without proof, the following equality:

\[
\sum_{G \in \mathcal{F}_Q} \frac{1}{|\text{Aut}(G)|} = \frac{1}{|\text{Aut}(Q) \times \text{Acl}(\mathcal{F})|}.
\]

For the proof he referred to a planned forthcoming paper by him and J.K.Senior, which apparently was never published. A proof of the case \( Q = 1 \) of (2) was published by R.Reimers and J.Tappe in 1975 [RT], and of the general case by Tappe in 1980 [T] (see also [BT], Proposition 3.8). He defined a notion of strong isoclinism. Two groups \( G \) and \( H \) are strongly isoclinic, if there exist isomorphisms \( \phi : G/Z(G) \cap G' \to H/Z(H) \cap H' \) and \( \theta : G' \to H' \), such that if \( x, y \in G \), then \( \theta[x, y] = [\phi(xZ(G)), \phi(yZ(G))] \). It is clear that strongly isoclinic groups are isoclinic and belong to the same branch. If \( \Phi \) is a strong isoclinism equivalence class, and \( G \in \Phi \), the strong isoclinisms from \( G \) to itself form the strong autoclinism group \( \text{Scl}(G) = \text{Scl}(\Phi) \) of \( G \) (or of \( \Phi \)), which is an invariant of \( \Phi \), and is isomorphic to a subgroup of \( \text{Aut}(G/Z(G) \cap G') \). Tappe proves

\[
\sum_{H \in \Phi} \frac{1}{|\text{Aut}(H)|} = \frac{1}{|\text{Scl}(\Phi)|},
\]

and if \( \Phi \) varies over the strong isoclinism classes contained in the branch \( \mathcal{F}_Q \), then

\[
\sum_{\Phi} \frac{1}{|\text{Scl}(G)|} = \frac{1}{|\text{Aut}(Q) \times \text{Acl}(\mathcal{F})|}.
\]

Combining the two identities yields equation (2).

There are cases in which the formulas (1) and (3) coincide. We need two more notions. A group \( G \) is a capable group, if it is isomorphic to the central factor group \( K/Z(K) \) of some group. Each group contains a characteristic subgroup \( Z^*(G) \), the epicentre of \( G \), defined to be the intersection of all normal subgroups \( N \leq G \) for which \( G/N \) is capable. Then \( G/Z^*(G) \) is itself capable. Naturally \( Z^*(G) \leq Z(G) \), and \( G \) is capable iff \( Z^*(G) = 1 \). Also, if \( H \) is a covering group of \( G \), with kernel \( M \), then, writing \( G = H/M \), we have \( Z^*(G) = Z(H)/M \) ([BT], Section IV.3).
Theorem 2. Two covering groups of the same group are strongly isoclinic. Conversely, if $Z^*(G) \cap G' = 1$, then a group that is strongly isoclinic to a covering group of $G$ is itself a covering group for $G$. In that case, if $H$ covers $G$, then $Sel(H) = Aut(G)$.

The second claim of this theorem gives a partial answer to Problem 2.3 of [JW].

Example 1. A group may be strongly isoclinic to a covering group $H$ of $G$, without being itself a covering group of $G$. Thus, the two non-abelian groups of order $p^3$ are strongly isoclinic, but one of them is its own covering group (the quaternion group if $p = 2$, and the metacyclic one for an odd $p$), and naturally the other one does not cover it.

Example 2. There are groups $G$ with a covering group $H$, such that all the groups that are strongly isoclinic to $H$ are also covering groups of $G$, but $Z^*(G) \cap G' \neq 1$. We start with a perfect group $S$ with a trivial centre and a multiplier of order 2 (many of the finite simple groups have these properties). Since $S$ is perfect, it has a unique covering group, say $G$, satisfying $|Z(G)| = 2$ and $G = G'$. By Theorem 2, any group that is strongly isoclinic to $G$ is isomorphic to it. Moreover, $G$ is its own covering group (see the remark at the beginning of the Proofs section), so we take $H = G$, and $Z^*(G) \cap G' = Z^*(G) = Z(H) = Z(G) \neq 1$.

3. Abelian groups. Infinite sums. We next turn to formulae involving infinitely many groups. Even before [H2, H3], in [H1], Hall published the following remarkable identity, which he described as "rather curious". Fix a prime $p$, and let $A_p$ be the class of all finite abelian $p$-groups. Then

$$\sum_{G \in A_p} \frac{1}{|G|} = \sum_{G \in A_p} \frac{1}{|Aut(G)|}. \quad (5)$$

Since the number of abelian groups of order $p^n$ is $\pi(n)$, the number of partitions of $n$, another way of writing (5) is

$$\sum_{G} \frac{1}{|Aut(G)|} = \sum \frac{\pi(n)}{p^n}. \quad (6)$$

Theorem 1 shows that the sums occurring in (5) and (6) are also equal to the sum of reciprocal orders of all the covering groups of abelian $p$-groups.

Alternative proofs to Hall’s were given by I.G.Macdonald [Mc] (1984, applying Hall algebras), H.Cohen-H.W.Lenstra [CL] (1984, with a number-theoretical motivation), T.Yoshida [Y] (1992), the author [M1] (1996), F.Clarke [C] (2006), and J.Lengler [L] (2008). Also, since (6) can be expressed as a partition identity, or a so-called q-identity, it can be derived from other identities of that type (see e.g. [St]), but in papers doing so the group theoretical connection is often not made explicit. Below we will point out that it can be considered as the limiting form of generalized Rogers-Ramanujan identities. Both [CL] and [Mc] prove a more general result, replacing groups by modules over certain rings. This can also be deduced by the approach of [M1]. The various proofs establish, on the way to the main result, other interesting formulae. Thus Hall deduces his result by summing over $n$ the following equality.
Here the first equality is just the well known fact that the number of partitions of a number $k$ with largest part $n$ is equal to the number of partitions of $k$ to $n$ parts. Note that the group of order 1 is considered as having 0 generators, not 1. In [H3] Hall refers to the last identity as a primary one, from which one derives the secondary identity (2). Moreover, Hall proves (7) by summing another equality. He associates to each relevant group $G$ an infinite collection of abelian $p$-groups, say $S_G$, and proves that

$$\sum_{H \in S_G} \frac{1}{|H|} = \frac{1}{|\text{Aut}(G)|}.$$

Different proofs of (7) occur in [CL, 3.8] and [Y, 4.3]. In [L] a formula similar to (8) is derived, using a collection, say $T_G$, which is different from Hall’s, and (6) is then deduced by summation.

Similarly, the proof in [M1] follows by summing the equality

$$\sum_{G \in \mathcal{A}_p} \frac{1}{|G|} = \frac{1}{|\text{Aut}(G)|} = \frac{p^n}{(p-1)^2 \cdots (p^n - 1)^2}.$$

Theorem 3. Let $\mathcal{C}_p$ be the class of capable abelian $p$-groups. Then

$$\sum_{G \in \mathcal{C}_p} \frac{1}{|G|} = \frac{p-1}{p} \sum_{G \in \mathcal{A}_p} \frac{1}{|G|} = \sum_{G \in \mathcal{A}_p} \frac{1}{|\text{Hol}(G)|}.$$

Writing an abelian group $G$ as a direct product of cyclic subgroups, it is capable if the two largest factors are isomorphic. Let us write $\mathcal{A}_{p,k}$ for the class of abelian $p$-groups, in which, in the above decomposition, the first $k$ factors are isomorphic, and $\mathcal{B}_{p,k}$ for the class of those abelian groups that are direct products of cyclic subgroups of orders at least $p^k$. Theorem 3 is the case $k = 1$ of the following
Theorem 4. For \( k \geq 0 \),

\[
\sum_{G \in A_{p,k+1}} \frac{1}{|G|} = \sum_{G \in B_{p,k+1}} \frac{1}{|G|} = \sum_{G \in A_p} \frac{1}{|G|^k |\text{Aut}(G)|} = \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^2}\right) \ldots \left(1 - \frac{1}{p^k}\right) \sum_{G \in A_p} \frac{1}{|G|}
\]

while the case \( k = 0 \) is Hall’s equation (5) (where we have to ignore the middle term).

The first equality in Theorem 4 is immediate, by the duality of partitions, and the rest is proved by a variation on Hall’s proof of (5), and on the way we find formulae analogous to (7) and (8). E.g. the former one is

\[
\sum_{G \in A_{p,k+1}, \exp(G) = p^n} \frac{1}{|G|} = \sum_{|G|=p^n} \frac{1}{|G|^k |\text{Aut}(G)|}.
\]

A similar variation yields, analogously to (8): if \( G \in A_{p,k+1} \), then

\[
\frac{|G|^k}{|\text{Aut}(G)|} = \sum_{H \in S^G} \frac{1}{|H|},
\]

for some family of groups \( S^G \subseteq A_p \). While the proof makes clear how to describe the family \( S^G \) combinatorially, this class does not seem to have a nice group theoretical description, except for some special cases. E.g. if \( G \) is elementary abelian of order \( p^{k+1} \), the class \( S^G \) consists of the abelian groups of exponent \( p^{k+1} \), and the resulting formula is just the well-known expression for the generating function of the number of partitions to \( k+1 \) parts.

4. More isoclinism. Summing equation (2) over all \( Q \), and applying (6), yields

Theorem 5. Let \( F \) be an isoclinism family of finite \( p \)-groups. Then

\[
\sum_{G \in F} \frac{1}{|\text{Aut}(G)|} = \frac{1}{|\text{Act}(F)|} \sum \frac{\pi(n)}{p^n}.
\]

Neither Hall nor Reimers and Tappe mentioned this identity. Below we give a direct proof of Theorem 5, which includes (6) as a special case, and is independent of (2). That proof utilizes the existence of a free-like object in each family.

Theorem 6. Let \( F \) be an isoclinism family of finite \( p \)-groups. Let \( G \in F \) satisfy \( d(G) = d \). Then there exists a group \( F = F_d \), which can be generated by \( d \) elements, is isoclinic to \( G \), satisfies \( F/F' \cong \mathbb{Z}^d \), and can be mapped onto any \( d \)-generator group in \( F \).

Recall that a variety of groups is a class of groups closed under taking subgroups, factor groups, and cartesian products. Alternatively, it can be described as the class of all groups which satisfy a certain set of laws (identical relations). A variety \( U \) contains free objects, i.e. for each cardinal number \( r \), there exists an \( r \)-generator group \( F_r(U) \) such that an \( r \)-generator group \( G \) lies in \( U \) if and only if there is an epimorphism from \( F_r(U) \) onto \( G \). A variety is locally nilpotent, if the finitely generated groups in it are nilpotent, and it is torsion free, if its free groups are torsion free. Since free objects exist in varieties, we can give similar results for them as well. Given any variety \( U \), we let \( U_p \) denote the set of finite \( p \)-groups in \( U \).
Theorem 7. Let $\mathcal{V}$ be the variety of groups of class 2 with central $p$th powers, and let $s(n)$ be the number of subgroups of an elementary abelian group of order $p^n$.

\[
\sum_{G \in \mathcal{V}, d(G) = d} \frac{1}{|Aut(G)|} = \frac{s\left(\binom{d}{2}\right)}{(p^d - 1)^2(p^{d-1} - 1)^2... (p - 1)^2}.
\]

(14)

\[
\sum_{G \in \mathcal{V}} \frac{1}{s\left(\binom{d(G)}{2}\right)} \frac{1}{|Aut(G)|} = \sum \frac{\pi(n)}{p^n}.
\]

(15)

Note that $s(n)$ is, for a fixed $n$, a polynomial in $p$. Its degree is $\left\lfloor \frac{n^2}{4} \right\rfloor$.

Theorem 8. Let $\mathcal{W}$ be a variety with the following two properties: it is defined by laws which are commutator words, and the finitely generated groups in it have finite commutator subgroups. Let $s_{\mathcal{W}}(n)$ be the number of subgroups in the derived subgroup of the free group of rank $n$ in $\mathcal{W}$. Then

\[
\sum_{G \in \mathcal{W}, d(G) = d} \frac{1}{|Aut(G)|} \leq s_{\mathcal{W}}(d) \cdot \frac{p^d}{(p^d - 1)^2(p^{d-1} - 1)^2... (p - 1)^2}.
\]

(16)

\[
\sum_{G \in \mathcal{W}} \frac{1}{s_{\mathcal{W}}(d)} \frac{1}{|Aut(G)|} \leq \sum \frac{\pi(n)}{p^n}.
\]

(17)

There are a few other cases in which the expression $\sum_{G \in \mathcal{U}} \frac{1}{|Aut(G)|}$ can be evaluated explicitly, e.g. for groups of class 2 with a given number $d$ of generators, but the resulting expressions are long and involved, and we write it down only for the simplest case $d = 2$.

Corollary 9. Let $\mathcal{U}$ be the variety of nilpotent groups of class two. Then

\[
\sum_{G \in \mathcal{U}, d(G) = 2} \frac{1}{|Aut(G)|} = \frac{p^2(2p^4 - 1)}{(p - 1)^3(p + 1)^3(p^2 + 1)}.
\]

(18)

But we can say something about the nature of the relevant sums.

Corollary 10. Let $\mathcal{U}$ be a locally nilpotent torsion free variety. Then for each $d$ the sum $\sum_{G \in \mathcal{U}, d(G) = d} \frac{1}{|Aut(G)|}$ is a rational number.

Corollary 11. Let $\mathcal{U}$ be as in the previous corollary, and assume either that $F_d(\mathcal{U})$ is of class two, or that $d = 2$. Then there exists a rational function $R_d(z)$, with integer coefficients, such that for all primes $p$ we have $\sum_{G \in \mathcal{U}, d(G) = d} \frac{1}{|Aut(G)|} = R_d(p)$.

This corollary is illustrated by Theorem 7 and Corollary 9 above.

For our proofs we need two more technical results. First, write $\pi(n) = \pi^{(r)}(n) + \pi_r(n)$, where $\pi^{(r)}(n)$, $\pi_r(n)$ are the number of partitions of $n$ into more than $r$, and into at most $r$ (positive) parts, respectively.
Proposition 12. Fix a natural number $r$. Then

\begin{equation}
\sum_{s \geq 1, \ d - s = r} \frac{x^{sd}}{(1 - x^d)(1 - x)(1 - x^s)(1 - x)} = \sum \pi^{(r)}(n) x^n.
\end{equation}

Next, if $X$ is a finitely generated group, we let $a_n(X)$ denote the number of subgroups of $X$ of index $p^n$, and define the $p$-local $\zeta$-function of $X$ by $\zeta_{X,p}(z) = \sum a_n(X) p^{-n z}$ (we deviate from the standard notation, where $a_n$ is the number of subgroups of index $n$, because we are considering only indices that are powers of $p$). It is known ([LS], Theorem 15.1) that

\begin{equation}
\zeta_{Z_d,p}(z) = (1 - \frac{1}{p^z})(1 - \frac{1}{p^{z-1}})...(1 - \frac{1}{p^{z-d+1}})
\end{equation}

We also consider the function $\zeta_{\triangleright X,p}(z) = \sum a_{\triangleright n}(X) p^{-n z}$, where $a_{\triangleright n}(X)$ is the number of normal subgroups of index $p^n$ in $X$.

Proposition 13. Let $U$ be any locally nilpotent variety, and let $F_d(U)$ be the free group of rank $d$ in it. Then

\begin{equation}
\zeta_{\triangleright F_d(U),p}(d) = 1 + \sum_{k=1}^{d} (p^d - 1)...(p^d - p^{k-1})p^{-dk} \cdot \sum_{G \in H_p, |G|=p^n, d(G)=k} \frac{1}{|\text{Aut}(G)|}.
\end{equation}

5. The Rogers-Ramanujan identities. Recall that these are

\begin{equation}
1 + \sum_{k=1}^{\infty} \frac{x^{k^2}}{(1 - x)(1 - x^2)...(1 - x^k)} = \prod_{j=0}^{\infty} \frac{1}{(1 - x^{5j+1})(1 - x^{5j+2})},
\end{equation}

\begin{equation}
1 + \sum_{k=1}^{\infty} \frac{x^{k^2+k}}{(1 - x)(1 - x^2)...(1 - x^k)} = \prod_{j=0}^{\infty} \frac{1}{(1 - x^{5j+3})(1 - x^{5j+4})}.
\end{equation}

Here $x$ can be any number such that $|x| < 1$ (or we can consider the identities as formal ones). Various substitutions for $x$ in these equalities yield mass formulae, but first we point out an intriguing relationship to Hall’s formula (5). There seem to be two mysteries in the R-R identities. First, why should an infinite series be equal to an infinite product? and second, why the special role of the number 5? This latter mystery is somewhat dispelled by a generalization, which shows that the R-R identities are just the simplest instance of an infinite series of identities, where 5 is replaced by all odd numbers, taken in turn. These are

\begin{equation}
\sum \frac{x^{k_1^2+...+k_i^2+k_i+...+k_r}}{(1 - x)(1 - x^2)...(1 - x^{k_i-k_2})(1 - x)...(1 - x^{k_r-k_i})(1 - x)...(1 - x^{k_1})}
\end{equation}
Here \( r \geq 1, 1 \leq i \leq r + 1, k_1 \geq k_2 \geq \ldots \geq k_r \geq 0 \) [An, 7.8]. The R-R identities are obtained upon taking \( r = 1 \) and \( i = 2 \) or \( i = 1 \). If we take \( i = k + 1 \) and \( x = 1/p \), the general term on the LHS is \( \frac{1}{\text{Aut}(G)} \), where \( G \) is the abelian \( p \)-group determined by the dual partition to \((k_1, \ldots, k_r)\) (see equation (35) below). If we keep \( r \) fixed, the LHS becomes \( \sum_{|G| \leq p^r} \exp(G) = p^r \frac{1}{|\text{Aut}(G)|} \), a fact that was noticed in [L, 7.1]. If we take \( i = 1 \), the general term on the LHS becomes \( \frac{1}{|G| \text{Aut}(G)} \), where \( G \) is as above. Letting again \( r \) tend to infinity, we get on the RHS \( \prod_{j \geq 2} \frac{1}{1 - x^j} \), which is the generating function for the number of partitions with components at least 2. By duality, the number of such partitions equals the number of partitions with two equal largest components, and thus (23) yields (10) (without the middle term). Presumably there exists a still more general partition identity, which implies (11) and similar equalities.

Now for the aforementioned substitutions. If we take in (21) and (22) \( x = 1/q \), where \( q = p^r \) is a power of the prime \( p \), then the left hand sides become the sum of the reciprocal orders of the general linear groups, which are the automorphism groups of vector spaces, or of the affine groups, respectively:

\[
\sum_{k=1}^{\infty} \frac{1}{|GL(k, q)|} = \prod_{j=0}^{\infty} \frac{q^{10j+5}}{(q^{5j+1} - 1)(q^{5j+4} - 1)}.
\]

\[
\sum_{k=1}^{\infty} \frac{1}{|AGL(k, q)|} = \prod_{j=0}^{\infty} \frac{q^{10j+5}}{(q^{5j+2} - 1)(q^{5j+3} - 1)}.
\]

Similarly, taking \( x = -1/q \) yields the orders of the unitary groups (over the field of size \( q^2 \)). To obtain formulas involving the orders of the orthogonal and symplectic groups, we have to insert some correction factors. Thus, taking \( x = 1/q^2 \) converts the first formula to

\[
\sum_{k=1}^{\infty} \frac{q^{2k^2+k}}{|O(2k+1, q)|} = \prod_{j=0}^{\infty} \frac{q^{20j+10}}{(q^{10j+2} - 1)(q^{10j+8} - 1)}.
\]

\[
\sum_{k=1}^{\infty} \frac{q^{2k^2+k}}{|O^*(2k, q)|} = \prod_{j=0}^{\infty} \frac{q^{20j+10}(q^j - \epsilon)}{(q^{10j+2} - 1)(q^{10j+8} - 1)}.
\]

Here \( \epsilon = 1 \) or \(-1\), and we recall that \( Sp(2k, q) \) and \( O(2k+1, q) \) have the same order.

Combining these last identities with Theorem 5, we can derive identities involving larger classes than isoclinism families.
Theorem 14. Let $G$ vary over all finite $p$-groups with a derived subgroup of order $p$. Then

$$\sum_{|G|=p} \frac{|G : Z(G)|}{|Aut(G)|} = p^{10n+5} \prod_{n=0}^{\infty} \left( \frac{p^{10n+5}}{(p^{5n+2}-1)(p^{5n+3}-1)} \right) - 1.$$

Theorem 15. Let $F^d$, $d \geq 2$ be the family of the groups in which both $G/Z(G)$ and $G'$ are elementary abelian, of orders $p^d$ and $p^{(d)}$, respectively, and let $T^1$ be the class of abelian $p$-groups. Then for $d \geq 2$ we have $\text{Acl}(F^d) \cong \text{GL}(d, p)$, and

$$\sum_{G \in \cup T^d} \frac{1}{|Aut(G)|} = \sum_{n=0}^{\infty} \frac{\pi(n)}{p^n} \cdot \left( \prod_{n=0}^{\infty} \frac{p^{10n+5}}{(p^{5n+1}-1)(p^{5n+4}-1)} \right) - \frac{1}{p-1}.$$

We give two more simple illustrations. Let us write $A_d$ for the direct product of $d$ cyclic groups, one of order $p^2$ and the rest of order $p$ ($d \geq 1$), and let $T^d$ be the class of $p$-groups $G$ of class 2, in which $G/Z(G) \cong A_d$ and $G'$ is elementary abelian of order $p^{(d)}$. This is an isoclinism family. Note that $T^1$ is the class of abelian $p$-groups.

Theorem 16. With these notations

$$\sum_{G \in \cup T^d} \frac{1}{|Out(G)|} = \frac{p}{p-1} \sum_{n=0}^{\infty} \frac{\pi(n)}{p^n} \cdot \prod_{n=0}^{\infty} \frac{p^{10n+5}}{(p^{5n+2}-1)(p^{5n+3}-1)}.$$

Next, let $H_d$ denote the direct product of $d$ cyclic groups of order $p^2$, and let $S^d$ be the isoclinism family of $p$-groups satisfying $G/Z(G) \cong H_d$ and $G' \cong H^{(d)}_1$. Again, $S^1 = T^1 = A_p$ is the class of finite abelian $p$-groups.

Theorem 17. With these notations

$$\sum_{G \in \cup S^d} \frac{|G'|}{|Aut(G)|} = \sum_{n=0}^{\infty} \frac{\pi(n)}{p^n} \cdot \prod_{n=0}^{\infty} \frac{p^{10n+5}}{(p^{5n+2}-1)(p^{5n+3}-1)} - \frac{1}{p(p-1)}.$$

$$\sum_{G \in \cup S^d} \frac{|G' : G : Z(G)|^{\frac{1}{2}}}{|Aut(G)|} = \sum_{n=0}^{\infty} \frac{\pi(n)}{p^n} \cdot \prod_{n=0}^{\infty} \frac{p^{10n+5}}{(p^{5n+1}-1)(p^{5n+4}-1)} - \frac{1}{p-1}.$$
A remark on Example 2. In that example, we applied the following: if $G$ is a perfect group with a covering group $H$, then $H$ is its own covering group, i.e. $M(H) = 1$. This follows from Corollary 2.2 of [IM]. A direct proof is as follows: let $K$ be a covering group of $H$, with kernel $N$. Let $M/N$, with $M \triangleleft K$, be the subgroup corresponding to the kernel of $H$ as a cover of $G$. Then $M \leq Z_2(K)$, and if $z \in M$, then the map $x \to [x, z]$ is a homomorphism of $K$ into $N$. Since $K = K'$, this homomorphism is trivial, i.e. $z \in Z(K)$. Thus $M \leq Z(K) \cap K'$, $K/M \cong G$, and $|K| \geq |H|$. Since $H$ is a covering group of $G$, we get $K = H$.

Proof of Theorem 1. Let $d = d(G)$ be the minimal number of generators of $G$, and let $G$ have $k$ generating $d$-tuples. Let $F$ be a free group of rank $d$, then there are $k$ homomorphisms from $F$ onto $G$, and therefore there are $n = \frac{k |M|^d}{|Aut(G)|}$ normal subgroups $R \triangleleft F$ such that $F/R \cong G$ [LS]. Let $R$ be one of these normal subgroups, and write $K = F/[R, F]$. Then $U := R/[R, F] \leq Z(K)$, $M := R \cap F'/[R, F]$ is the torsion subgroup of $U$, and if $S/[R, F]$ is a complement to $M$ in $U$, then $F/S$ is a covering group for $G$, with kernel $M$. All covering groups of $G$ are obtained in this way. The number of complements to $M$ in $U$ is $|M|^{d-d}$, therefore the number of subgroups $S \triangleleft F$ such that $F/S$ is a covering group of $G$ is $\frac{k |M|^d}{|Aut(G)|}$. On the other hand, let $H$ be any covering group of $G$, with kernel $M$. Then $M \leq Z(H) \cap H' \leq \Phi(H)$, therefore the number of $d$-tuples generating $H$ is $k|M|^d$. Thus the number of $S$ such that $F/S \cong H$ is $\frac{k |M|^d}{|Aut(H)|}$. Summing over all the $H_i$’s yields the theorem. QED

Proof of Theorem 2. Let $H$ and $K$ be covering groups of $G$, with kernels $N$ and $M$, respectively. The proof in [JW] that $H$ and $K$ are isoclinic produces an isomorphism between the two factor groups which induces isomorphisms $H/Z(H) \cong K/Z(K)$ and $H' \cong K'$. Since $N \leq Z(H) \cap H'$ and $Z^*(H/N) = Z(H)/N$, this isomorphism maps $(Z(H) \cap H')/N$ onto $(Z(K) \cap K')$, and thus it is a strong isoclinism. Next, assume that $Z^*(G) \cap G' = 1$, let $H$ be a covering group of $G$, with kernel $N$, let $K$ be strongly isoclinic to $H$, and write $M = Z(K) \cap K'$. Then $N = Z(H) \cap H'$, therefore $K/M \cong G$, and $M \cong N$. Thus $|K| = |H|$ and $K$ is also a covering group of $G$.

Proof of Theorem 4. To prove the second equality we recall Hall’s original proof. Write an abelian group $G$ of order $p^n$ as a direct sum of cyclic subgroups of orders $p^\lambda_1, \ldots, p^\lambda_r$, where $\lambda_1 \geq \ldots \geq \lambda_r$, and $n = \sum \lambda_i$. To the partition $(\lambda_1, \ldots, \lambda_r)$ we associate its Young diagram, which consists of $r$ rows, arranged with the longest one on top, and the $i$th one consists of $\lambda_i$ unit squares. The dual partition $\mu_1, \ldots, \mu_s$ is obtained by counting the number of squares in columns, not rows. Then

\[
|Aut(G)| = \frac{f_{\mu_1-\mu_2}(\rho)f_{\mu_2-\mu_3}(\rho)\ldots}{\rho^{n_1^2+n_2^2+\ldots}},
\]

where $\rho = 1/p$ and $f_k(x) = (1 - x)(1 - x^2)\ldots(1 - x^k)$.\footnote{For this formula, Hall refers to [Sp]. However, the elegant and suggestive form of (35) is due to him.}
partition of \( N \), and let \( n_1 \) be the size of the largest square that can fit in the upper left corner of this diagram. Then delete the first \( n_1 \) columns, and repeat the process with the remaining diagram, to obtain \( n_2 \), etc. Then Hall proves equation (8) of the introduction, with \( S_2 \) being the class of all abelian \( p \)-groups corresponding to diagrams whose associated partitions are the partition \((\mu_1, \ldots, \mu_r)\).

Our variation consists of defining the associated partition by choosing not squares but rather rectangles of size \( n_i \times (n_i + k) \), with the horizontal edge being the narrow one. This replaces the sum \( \sum n_i^2 \) in (35) by \( \sum (\mu_i^2 + k\mu_i) = N + kn \), and therefore replaces \( |\text{Aut}(G)| \) by \( |G|^k|\text{Aut}(G)| \), but the rest of the argument does not change. However, we cannot get all groups \( H \) of order \( p^n \) now, only those in whose young diagram we can fit a rectangle of size at least \( 1 \times (k + 1) \), and then we have to fit the same size rectangle in the remainder diagram, etc. These are just the groups whose diagrams have equal first \( k + 1 \) rows.

To prove the third equality, we associate to each abelian \( p \)-group \( G \) its largest factor group \( H \) which lies in \( A_{p,k+1} \). If \( G \) corresponds to the partition \( n_1, n_2, \ldots, n_k \), then \( H \) corresponds to the partition in which we replace each of \( n_1, n_2, \ldots, n_k \) by \( n_{k+1} \), and \( |G| = p^{(n_1-n_{k+1})-(n_k-n_{k+1})}|H| \). The differences \( m_i = n_i - n_{k+1} \) \((i = 1, \ldots, k)\) form another partition. Fixing \( H \), and summing over all the corresponding \( G \)'s, we first fix \( m_k \). It can have values \( 1, p, p^2, \ldots \), and since \( m_i \geq m_k \), we have to multiply \( \frac{m^{m^{m^{m^k}}}}{p^k} \) by \( \sum_{j=0}^{\infty} \frac{1}{p^j} = \frac{1}{1-p} \). Next we fix \( m_{k-1} \), obtaining a factor \( \frac{p^{m^k}}{p^k} \), etc. Thus \( \sum (1 - \frac{1}{p})(1 - \frac{1}{p^2}) \ldots (1 - \frac{1}{p^k}) \cdot \frac{1}{|H|} = \frac{1}{|G|} \), and we sum this over all \( H \).

The variation referred to in the remark following the statement of Theorem 4 consists in choosing rectangles of size \((n_1 + k) \times n_i\).

**Proof of Theorem 6.** Let \( G \) be a \( d \)-generator group in \( F \). Write \( d = r + s \), where \( |Z(G) : Z(G) \cap \Phi(G)| = p^r \) and \( |G : Z(G)\Phi(G)| = p^s \). Choose generators \( a_1, \ldots, a_r, b_1, \ldots, b_s \) for \( G \), where \( a_1, \ldots, a_r \) generate \( G \) (mod \( Z(G) \)) and \( b_1, \ldots, b_s \) generate \( Z(G) \) (mod \( Z(G) \cap \Phi(G) \)). Let \( u_1, \ldots, u_t \) be defining relations for \( G/Z(G) \), in terms of the generators \( a_iZ(G) \). Since \( a_i \) generate \( G \) (mod \( Z(G) \)), all commutators in \( G \) have the form \( c_i(a_1, \ldots, a_r) = [w_1(a_i), w_2(a_i)] \), where \( w_1, w_2 \) are two words, and \( k \) varies over an appropriate range. These commutators generate \( G' \), and we let \( v_1, \ldots, v_q \) be a set of defining relations for \( G' \) in terms of these generators.

\( F \) is generated by elements \( x_1, \ldots, x_r, y_1, \ldots, y_s \), subject to three sets of defining relations. The first set says that all commutators of one of the \( y_i \)'s with other generators are trivial, i.e. the elements \( y_i \) are central. The next set says that the commutators of the elements \( u_i(x_1, \ldots, x_r) \) with the generators are trivial, i.e. \( u_i(x_1, \ldots, x_r) \) are central elements. The third set consists of the equalities \( v_i(c_i(x_1, \ldots, x_r)) = 1 \).

All these relations are satisfied in \( G \), if we substitute \( a_i, b_j \) for \( x_i, y_j \) respectively. Therefore there exists an epimorphism \( \phi : F \rightarrow G \), mapping \( x_i, y_j \) to \( a_i, b_j \) respectively. The second set of defining relations shows that \( F/Z(F) \) satisfies the relations of \( G/Z(G) \), therefore there exists a map from \( G/Z(G) \) onto \( F/Z(F) \). On the other hand \( \phi(Z(F)) \subseteq Z(G) \), therefore there is a map from \( F/Z(F) \) onto \( G/\phi(Z(F)) \), and from the latter onto \( G/Z(G) \). Thus \( G/Z(G) \) and \( F/Z(F) \) are isomorphic. Similarly, \( \phi(F') = G' \), and on the other hand the generators of \( F' \) satisfy the relations of \( G' \), therefore there exists a map in the reverse direction, \( F' \cong G' \), and \( \phi \) induces an isoclinism between \( F \) and \( G \). In the abelianization \( F/F' \) all the relations of \( F \) become trivial, therefore \( F/F' \cong \mathbb{Z}^d \). Finally, if we replace \( G \) by an isoclinic group \( H \) with the same number of generators, then the defining relations of \( F \) are not changed, i.e. \( G \) and \( H \) define the same \( F \), and \( F \) maps onto both. Thus there
is an epimorphism from $F$ on all $d$-generator groups in $\mathcal{F}$, which induces an isoclinism between $F$ and $G$ (or $H$). Since $F'$ maps isomorphically onto $G'$, we have $F' \cap N = 1$, where $N$ is the kernel of the epimorphism.

**Proof of Proposition 12.** This is proved in the same way as Theorem 351 of [HW] (which is the special case $s = d$), by enumerating the diagrams of partitions by the largest rectangle whose sides differ exactly by $r$ that they contain, i.e. by the number that is denoted by $n_1$ in the proof of Theorem 4 ($k$ there is $r$ here). Such a rectangle can be found only when the partitions have more than $r$ parts, hence the occurrence of $\pi(r)$.

**Remark.** The generating function for $\pi_r$ is $\sum \pi_r(n)x^n = \frac{1}{(1-x)...(1-x^r)}$.

**Proof of Theorem 5.** Let $d$ be a number for which $\mathcal{F}$ contains a $d$-generator group $G$, and construct the group $F = F_d$ of Theorem 15. Let $r, s$ be as before, and write also $|\Phi(G) \cap Z(G)| = p^s$, $|Z(F) \cap F'| = p^r$, and $|G : Z(G)| = p^e$.

Since $|G : \Phi(G) \cap Z(G)| = |G : Z(G)\Phi(G) \cap Z(G)| = p^{s+r}$, we have $|G| = p^{n+s}$.

We count epimorphisms from $F$ to $G$. Each such epimorphism, say $\phi$, maps the generators $x_i, y_j$ onto generators $a_i, b_j$ of $G$ satisfying the defining relations of $F$. First assume that $s > 0$. For $b_1, ..., b_s$ the relations just say that these elements are central. Therefore we can choose them as follows. First we choose $s$ generators for $Z(G) / \Phi(G) \cap Z(G)$.

Fixing one set $b_1, ..., b_s$ of elements in $Z(G)$ generating it (mod $Z(G) \cap \Phi(G)$), we can replace each of them by any element of the coset $b_j(Z(G) \cap \Phi(G))$. This yields $(p^s - 1)...(p^s - p^s - 1)p^r$ possibilities. Next choose some elements $a_1, ..., a_r$, satisfying the required relations. If the elements $d_1, ..., d_r$ satisfy the same relations, then mapping $a_i$ to $d_i$ induces an automorphism of $G / Z(G)$, and mapping commutators of the type $c_i$ above to the corresponding commutators in the $d_i$ induces an automorphism of $G'$. Furthermore, these automorphisms of $G / Z(G)$ and of $G'$ are compatible with each other, i.e. they define an autoclinism of $G$. The number of possible $r$-tuples is then, modulo $Z(G)$, the order $|\text{Aut}(\mathcal{F})|$ of the autoclinism group, and the number of possible $r$-tuples in $G$ is $|\text{Aut}(\mathcal{F})||Z(G)|^r = |\text{Aut}(\mathcal{F})|p^{s+r}$. Thus the number of epimorphisms from $F$ to $G$ is $|\text{Aut}(\mathcal{F})|p^{s+r}(p^s - 1)...(p^s - p^s - 1)$.

If $s = 0$, we skip the stage involving the $b_i$’s, and we find $|\text{Aut}(\mathcal{F})|p^{dn}$ epimorphisms.

Now we want to count the number of kernels $N$ of these epimorphisms. Two epimorphisms onto $G$ have the same kernel if, and only if, they are obtained from each other by multiplication by an automorphism of $G$. Therefore for a given $G$, that number is obtained by dividing the number of epimorphisms above by $|\text{Aut}(G)|$. In $F$ we count the number of these kernels, for all $G$’s of order $p^{n+s}$ and $d$ generators, in a different way. As mentioned at the end of the proof of Theorem 15, we have $N \cap F' = 1$. That shows that $N \leq Z(F)$, and that $N \cong NF'/F' \cong Z^d$, because $N$ has a finite index in $F$. Also $N \leq F'F'$, because $F$ and $F/N$ have the same number of generators. Now $|F : Z(F) \cap F'F'| = |G : Z(G) \cap \Phi(G)| = p^{s+r}$. Like $N$, we have $(Z(F) \cap F'F')/Z(F) \cong Z^d$, and $Z(F) \cap F'F' = (Z(F) \cap F') \times K$, where $K \cong Z^d$. Any subgroup of $Z(F)$ is normal in $F$, therefore we need the number of subgroups of $Z(F) \cap F'F'$ which intersect $F'$ trivially and have index $p^{s+r}$ in $F$. If $N$ is such a subgroup, then $N$ has index $p^s$ in $Z(F) \cap F'F'$ and $N(Z(F) \cap F')$ has index $p^{s+r}$ in $Z(F) \cap F'F'$. The number of products $N(Z(F) \cap F')$ is $a_{n-s}(Z(F) \cap F')/N(Z(F) \cap F') = a_{n-s}(Z^d)$. Given $N(Z(F) \cap F')$, the number of possible $N$’s is the number of complements of $Z(F) \cap F'$ in $N \times (Z(F) \cap F')$. 

which is $|\text{Hom}(N, Z(F) \cap F^*)| = |Z(F) \cap F^*|^d$. Comparing the two expressions for the number of kernels, we have for $s > 0$

$$|\text{Aut}(F)|p^{dn + rs}(p^s - 1)(p^s - p^{s-1}) \sum \frac{1}{|\text{Aut}(G)|} = a_{n - e}(Z^d)(p^r)^d.$$ 

The summation is over all groups of order $p^{n + x + s}$ and $d$ generators in $F$. Transferring the factor $p^nd^d$ to the right and summing over all $n$, we have on the right

$$a_{n - e}(Z^d)p^{-(n - e)d} = \zeta_{Z^d, p}(d) = \frac{1}{(1 - \frac{1}{p^s})(1 - \frac{1}{p^r})}. \text{ Thus,}

$$|\text{Aut}(F)|p^{rs + s^2} \sum \frac{1}{|\text{Aut}(G)|} = \frac{1}{(1 - \frac{1}{p^s})(1 - \frac{1}{p^r})...(1 - \frac{1}{p})},$$

Since $rs + s^n = ds$, we obtain

$$|\text{Aut}(F)| \sum \frac{1}{|\text{Aut}(G)|} = \frac{p^{-ds}}{(1 - \frac{1}{p^s})(1 - \frac{1}{p^r})...(1 - \frac{1}{p})},$$

provided $s > 0$. For $s = 0$ we have

$$|\text{Aut}(F)| \sum_{d(G) = r} \frac{1}{|\text{Aut}(G)|} = \frac{1}{(1 - \frac{1}{p^s})(1 - \frac{1}{p})}.$$ 

Writing $x = 1/p$ in (36) and (37), adding, and quoting Proposition 12, we obtain the theorem.

**Proof of Theorem 14.** If $G$ is a $p$-group satisfying $|G'| = p$, then $\Phi(G) \leq Z(G)$, and we may consider $G/\Phi(G)$ as a vector space with an alternate bilinear form defined on it by commutation. The radical of that form is $Z(G)/\Phi(G)$. We let $|Z(G)/\Phi(G)| = p^s$ and $|G : Z(G)| = p^{2r}$ (note that this notation is different from the previous one). Thus $|G : Z(G)|^\Phi = p^r$. The family of these groups with a fixed value of $r$ is an isoclinism family, and its group of automorphisms is the group preserving the alternate form, up to scalar multiplication, i.e. the general symplectic group $\text{GSSp}(2r, p)$, of order $(p - 1)p^{2r^2}(p^{2r} - 1)(p^{2r - 2} - 1)...(p^2 - 1) = (p - 1)p^{2r^2 + r}(1 - \frac{1}{p^r})...(1 - \frac{1}{p^s})$. Let $A$ denote the sum in (5) and (6). Then, by Theorem 5,

$$\sum_{|G'| = p} \frac{p^r}{|\text{Aut}(G)|} = A(\frac{1}{p - 1} \sum_{r = 1}^\infty \frac{(\frac{1}{p})^{2r^2}}{(1 - \frac{1}{p^r})...(1 - \frac{1}{p^s})}).$$ 

Substituting $x = (\frac{1}{p})^2$ in the first Rogers-Ramanujan identity shows that the sum on the right hand side equals

$$-1 + \prod_{n=0}^{\infty} \frac{1}{(1 - (\frac{1}{p})^{10n+2})(1 - (\frac{1}{p})^{10n+8})} = -1 + \prod_{n=0}^{10} \frac{p^{20n+10}}{(p^{10n+2} - 1)(p^{10n+8} - 1)}.$$
This yields the first equality of Theorem 14. The other one is obtained in the same way, applying the second Rogers-Ramanujan identity.

**Proof of Theorem 15.** This is similar to the previous one. Let \( G \in \mathcal{F}^d \), and let \( x_1, ..., x_d \) generate \( G \) (mod \( Z(G) \)). Then the commutators \( \{[x_i, x_j] \mid i < j \} \) generate \( G' \), and from the given orders it is clear that these commutators are a basis for \( G' \), and that any automorphism of \( G/Z(G) \) induces an automorphism of \( G' \) and an autoclinism of \( G \). Thus \( Aut(\mathcal{F}^d) \cong \text{GL}(d, p) \), and

\[
\sum_{d=2}^{\infty} \frac{1}{|Aut(\mathcal{F}^d)|} = \sum \frac{1}{(p^d-1)...(p^d-p^{d-1})} = \sum \frac{(\frac{1}{p})^d}{(1-p^{-d})...(1-p^{-1})} = \\
\prod \frac{p^{10n+5}}{(p^{5n+1}-1)(p^{5n+4} - 1)} - 1 - \frac{1}{p-1},
\]

where the last equality follows from the Rogers-Ramanujan identities. Combining this with Theorem 5 yields Theorem 15.

**Proofs of Theorem 16 and 17.** Again, for \( G \in \mathcal{F}^d \), let \( x_1, ..., x_d \) generate \( G \) (mod \( Z(G) \)), with \( x_1 \) of order \( p^2 \) and the rest of order \( p \). Again the commutators \([x_i, x_j]\) have order \( p \) and they are a basis of \( G' \), and again all automorphisms of \( G/Z(G) \) induce autoclinisms of \( G \). It remains just to note that \( |Aut(A_d)| = p^{\binom{d}{2}+d(p-1)(p^{n-1}-1)...(p-1)} \), and substitute this value in Theorem 5 and the R-R identities as before, to obtain Theorem 16. For Theorem 17 just note that \( |Aut(H_d)| = p^{d^2}|\text{GL}(d, p)| \).

**Proof of Theorem 7.** In this proof let \( F = F_d \) be the free group of rank \( d \) in \( \mathcal{V} \). Since \( F \) is of class 2 and satisfies \([x, y]^p = 1\), it also satisfies \([x, y]^{p^2} = 1\), therefore \( Z(F) = F'/F'^p \), \( F' \) is elementary abelian of order \( p^\binom{d}{2} \), and \( Z(F)/F' \cong \mathbb{Z}^d \). Let \( G \in \mathcal{V} \) with \( d(G) = d \) and \( |G| = p^{n+d} \), so that \( |\Phi(G)| = p^n \). Then the number of maps of \( F \) onto \( G \) is the number of \( d \)-tuples generating \( G \), i.e. \( |\text{Epi}(F, G)| = (p^d-1)(p^d-p)...(p^d-p^{d-1})p^dn \), and the number of normal subgroups \( N \) of \( F \) such that \( F/N \cong G \) is \( |\text{Epi}(F, G)|/|Aut(G)| \). We count the number of normal subgroups \( N \) of \( F \) of index \( p^{n+d} \) such that \( d(G/N) = d \). Such a subgroup is contained in \( H := \Phi_p(F) = Z(F) \), and on the other hand each subgroup of \( H \) is normal in \( F \), so we have to count the number of subgroups of index \( p^n \) in \( H \). Fix a subgroup \( E \leq F' \) such that \( |F'/E| = p^k \), where \( k \leq n \), and count first the number of subgroups of index \( p^n \) in \( H \) intersecting \( F' \) in \( E \). If \( N \) is one of these subgroups, then \( |H : F'/N| = p^{n-k} \) and \( F'/N = F'/E \times N/E \). The number of products \( F'/N \) is the number of subgroups of index \( p^{n-k} \) in \( H/F' \cong \mathbb{Z}^d \), and the number of subgroups \( N \) giving rise to any product \( F'/N \) is the number of complements to \( F'/E \) in \( F'/N/E \), that number being \( p^{dk} \). Letting the subgroup \( E \) and the index \( p^k \) vary, but keeping \( N \) fixed, we obtain

\[
(p^d-1)(p^d-p)...(p^d-p^{d-1}) \sum_{|G|=p^{n+k}, d(G)=d} \frac{1}{|Aut(G)|} = \sum_{k=0}^{\min(n, \binom{d}{2})} a_k(F') a_{n-k}(\mathbb{Z}^d) p^{d(k-n)}
\]

If we sum this over all \( n \), then for each \( k \) we have \( a_k(F') \) multiplying

\[
\text{...}
\]
\[ \sum_{n=k}^{\infty} a_{n-k}(\mathbb{Z}^d)p^{-d(n-k)} = \zeta_{p,\mathbb{Z}^d}(d) = \frac{1}{1-p^{-d}} \cdot \frac{1}{1-p^{-d+1}} \cdots \frac{1}{1-p^{-1}} = \frac{p^{(d+1)}}{(p^d-1)(p^{d-1} - 1) \cdots (p-1)}. \]

This proves (14). It in turn implies that the left-hand-side of (15) equals

\[ \sum (p^d - 1)^2(p^{d-1} - 1)^2 \cdots (p-1)^2 = \sum (1-p^{-d})^2(1-p^{-d+1})^2 \cdots (1-p^{-1})^2 \]

According to the case \( r = 0 \) of formula (19) this equals \( \sum \frac{\pi(n)}{p^n} \).

The proof of Theorem 8 is obtained by replacing in the proof of Theorem 7 \( \mathcal{V} \) by \( \mathcal{W} \). The differences are that not all subgroups of \( H \) are normal in \( F \), also complements do not always exist, and if they do, their number is different. Therefore we have to replace the equalities of Theorem 7 by inequalities.

The proof of Proposition 13 is just a formalization of part of the argument in the proof of Theorem 5, and Corollary 9 is obtained by substituting in Proposition 13 the formula \( \zeta_{F_2(\mathcal{U}),p}(z) = (1-p^{-1})(1-p^{-2}) \cdots (1-p^{-n}) \) (see [GSS], section 8. We are considering the group denoted there by \( F^2 \)). Note that to apply Proposition 17 we need to consider the groups \( G \) with order \( p^n \) and \( d(G) = 1 \), but these are just the cyclic groups and for them \( |\text{Aut}(G)| = p^{n-1}(p-1). \)

**Proof of Corollary 10.** By Theorem 1 of [GSS] there exists a rational function \( f_p(X) \) over \( \mathbb{Z} \) such that \( \zeta_{F_2(\mathcal{U}),p}^e(z) = f_p(p^{-z}) \), and therefore \( \zeta_{F_2(\mathcal{U}),p}(d) \) is a rational number, and our corollary is implied by Proposition 13.

**Proof of Corollary 11.** For that we need to know that the function \( f_p(X) \) of the previous proof can be described as follows: there exists a rational function \( g(Y,Z) \) over \( \mathbb{Z} \), independent of \( p \), such that \( f_p(X) = g(p,p^{-X}) \). This holds in the case of class two by [GSS], Theorem 2, and for \( d = 2 \) it is an unpublished result of M.P.F.deSautoy and Grunewald (see [LS], p.289).

**Concluding remarks**

1. The construction of \( F_d \) in Theorem 6 depends on the choice of a presentation for \( G/Z(G) \) and for \( G' \), however the resulting group depends only on \( F \). Indeed, if \( T \) is any group such that \( d(T) = d \), \( T \) is isoclinic to \( G/Z(G) \), and \( T/T' \cong \mathbb{Z}^d \), then \( T \) has a set of generators satisfying the relations of \( F \), and hence there is an epimorphism \( \phi : F \to T \). In that epimorphism \( T' \) is mapped isomorphically onto \( T' \), hence \( F/F' \) is mapped onto \( T/T' \), and this map must also be an isomorphism, hence \( \phi \) is an isomorphism.

2. An equality of the type (5) does not hold generally for isoclinism families. E.g. let \( \mathcal{F} \) be a family of groups of class 2. It was proved that if \( c(G) = 2 \), then \( |G| \) divides \( |\text{Aut}(G)| \) [F]. Since there are obviously in \( \mathcal{F} \) groups for which \( |G| < |\text{Aut}(G)| \), an equality of type (5) cannot hold for the groups of \( \mathcal{F} \). It is still possible that some more complicated relation holds between both sides of (5).
may be interesting to look at this question for the families that occur in Theorems 14 and 15. For the former, we mention that it was proved by S.R. Blackburn that, like for abelian groups, the number of $p$-groups of order $p^n$ and $|G'| = p$ depends only on $n$, not on $p$ [B].

3. It is shown in Proposition 17 of [M2] that $\sum \frac{\pi(n)}{p^n}$ is irrational.

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