A New Approach to Variational Inequalities of Parabolic Type

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Abstract. This paper is concerned with the weak solvability of fully nonlinear parabolic variational inequalities with time dependent convex constraints. As possible approaches to such problems, there are for instance the time-discretization method and the fixed point method of Schauder type with appropriate compactness theorems. In this paper, our attention is paid to the latter approach. However, there has not been prepared any appropriate compactness theorem up to date that enables us the direct application of fixed point method to variational inequalities of parabolic type. In order to establish it we have to start on the set up of a new compactness theorem for a wide class of parabolic variational inequalities.

1. Introduction

We consider a variational inequality of quasi-linear parabolic type:

\[
2 \sum_{i=1}^{2} \int_{Q} \frac{\partial u_i}{\partial t} (u_i - \xi_i) dx dt + \sum_{i=1}^{2} \sum_{k=1}^{N} \int_{Q} a_i^{(k)} (x, t, u) \frac{\partial u_i}{\partial x_k} \frac{\partial (u_i - \xi_i)}{\partial x_k} dx dt \leq \sum_{i=1}^{2} \int_{Q} f_i (u_i - \xi_i) dx dt,
\]

(1.1)

\[\forall \xi := [\xi_1, \xi_2] \in L^2(0, T; H^1_0(\Omega) \times H^1_0(\Omega)) \text{ with } \xi(t) \in K(t) \text{ a.e. } t \in [0, T],\]

(1.2)

\[u(x, 0) = u_0(x) \text{ in } \Omega, \quad u = 0 \text{ on } \Sigma,\]

(1.3)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(Q := \Omega \times (0, T)\), \(0 < T < \infty\), \(\Gamma := \partial \Omega\), \(\Sigma := \Gamma \times (0, T)\), \(u := [u_1, u_2]\) and the diffusion coefficients \(a_i^{(k)} (x, t, u)\) are strictly positive, bounded and continuous in \((x, t, u) \in \overline{Q} \times \mathbb{R}^2\) as well as constraint set \(K(t)\) is convex and closed in \(H^1_0(\Omega) \times H^1_0(\Omega)\) satisfying some smoothness assumption in \(t \in [0, T]\). Functions
\( f := [f_1, f_2] \) and \( u_0 \) are prescribed in \( L^2(Q) \times L^2(Q) \) and \( K(0) \), respectively, as the data. Our claim is to construct a solution \( u \) of (1.1)-(1.3) in a weak sense such that

\[
    u \in C([0, T]; L^2(\Omega) \times L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega) \times H^1_0(\Omega)), \ u(t) \in K(t), \ \forall t \in [0, T].
\]

In the case without constraint, namely \( K(t) = H^1_0(\Omega) \times H^1_0(\Omega) \), our problem is the usual initial-boundary value problem for parabolic system of quasi-linear PDEs:

\[
    \frac{\partial u_i}{\partial t} - \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left( a^{(k)}_{ij}(x, t, u) \frac{\partial u_i}{\partial x_k} \right) = f_i(x, t) \quad \text{in} \ Q, \ i = 1, 2.
\]

For the solvability a huge number of results have been established (cf. [14, 24]), for instance, the Leray-Schauder principle together with some compactness theorems, such as [3, 26, 31].

In connection with quasi-linear variational inequalities, the concept of nonlinear monotone mappings was generalized to several classes of nonlinear mappings of monotone type, for instance, semimonotone [25], pseudomonotone [5, 11, 17], and furthermore \( L^{-p} \)-pseudomonotone mappings [6, 7]. Especially the last one was introduced as a class of nonlinear perturbations for linear maximal monotone mappings \( L \), which is available for parabolic variational inequalities; in a typical application of this theory, \( L \) is the time-derivative \( \frac{d}{dt} \). However, it seems still difficult to treat directly our model problem (1.1)-(1.3) in these frameworks of nonlinear mappings of monotone type.

Our model problem (1.1)-(1.3) is formally written in the space \( L^2(0, T; X^*) \) with \( X := L^2(0, T; H^1_0(\Omega) \times H^1_0(\Omega)) \) as

\[
    f \in Lu + A(u, u), \ u(0) = u_0,
\]

by taking as \( L \) the mapping \( L := \frac{d}{dt} + \partial I_{K(t)}(\cdot) : D(L) \subset X \rightarrow X^* \) and as \( A \) the mapping \( A(v, u) : D(A) = X \rightarrow X^* \) given by

\[
    \langle A(v, u), [\xi_1, \xi_2] \rangle_{X^*, X} = \sum_{i=1}^{2} \sum_{k=1}^{N} \int_{Q} a^{(k)}_{ii}(x, t, v) \frac{\partial u_i}{\partial x_k} \frac{\partial \xi_i}{\partial x_k} \ dx dt,
\]

for \( u := [u_1, u_2], \ v = [v_1, v_2], \ \xi = [\xi_1, \xi_2] \in X, \)

where \( \langle \cdot, \cdot \rangle_{X^*, X} \) stands for the duality between \( X^* \) and \( X \). We see that \( L \) is maximal monotone from \( D(L) \subset X \) into \( X^* \), but \( L \) is nonlinear in general. Since 1970, it remains to set up an abstract approach to such a quasi-linear parabolic variational inequality as our model problem. In this paper we establish a new approach to parabolic variational inequalities with time-dependent constraints \( \{K(t)\} \), based on a new compactness theorem (see Theorem 2.1) derived from the total variation estimates for solutions of parabolic variational inequalities; this idea was found in a recent work [16] of the authors.

There is a different approach to nonlinear variational inequalities of parabolic type with time-independent convex constraint in [1] where the time-discretization method was employed and a compactness theorem was established to ensure the strong convergence...
of time-discretized approximation schemes in time. This idea seems available to the case of time-dependent convex constraints.

In this paper the following notations are used. For a general (real) Banach space $X$ we denote by $X^*$ the dual space, by $|\cdot|_X$ and $|\cdot|_{X^*}$ the norms in $X$ and $X^*$, respectively, and by $\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_{X^*}$ the duality between $X^*$ and $X$. Especially, when $X$ is a Hilbert space, we denote by $\langle \cdot, \cdot \rangle$ the inner product in $X$.

Let $\varphi(\cdot)$ be a proper, lower semi-continuous (l.s.c.) and convex function on a Banach space $X$. Then the subdifferential $\partial_+ \varphi$ of $\varphi$ is defined by a multivalued mapping from $X$ into $X^*$ as follows: $z^* \in \partial_+ \varphi(z)$ if and only if $z \in D(\varphi) := \{ z \in X \mid \varphi(z) < \infty \}$, $z^* \in X^*$ and

$$\langle z^*, w - z \rangle_{X^*} \leq \varphi(w) - \varphi(z), \quad \forall w \in X.$$ 

The set $D(\varphi)$ is called the effective domain of $\varphi$. The domain of $\partial_+ \varphi$ is the set $D(\partial_+ \varphi) := \{ z \in D(\varphi) \mid \partial_+ \varphi(z) \neq \emptyset \}$ and the range of $\partial_+ \varphi$ is the set $R(\partial_+ \varphi) := \cup_{z \in D(\varphi)} \partial_+ \varphi(z)$.

Let $K$ be a non-empty closed and convex subset of a Banach space $X$. Then the function $I_K(\cdot)$ given by

$$I_K(z) := \begin{cases} 
0, & \text{if } z \in K, \\
\infty, & \text{if } z \in X - K,
\end{cases}$$

is proper, l.s.c. and convex on $X$ and is called the indicator function of $K$ on $X$, and the subdifferential $\partial I_K$ is defined as a multivalued mapping from $X$ into $X^*$. Clearly $D(I_K) = D(\partial I_K) = K$.

For the real function $r \to |r|^{p-1}$ on $\mathbb{R}_+ := \{ r \in \mathbb{R} \mid r \geq 0 \}$ for a fixed number $p > 1$ and a Banach space $X$, we consider the mapping $F : X \to X^*$ which assigns to each $z \in X$ the set

$$Fz := \{ z^* \in X^* \mid \langle z^*, z \rangle_{X^*} = |z|_X^p, |z^*|_{X^*} = |z|_X^{p-1} \}.$$ 

This mapping $F$ is called the duality mapping associated with the gauge function $r \to |r|^{p-1}$. It is well known that $F$ is the subdifferential of the non-negative, continuous and convex function $\psi(z) := \frac{1}{p}|z|_X^p$ on $X$, namely, $Fz = \partial_+ \psi(z)$ for all $z \in D(F) = X$. In particular, if $X$ is reflexive and $X^*$ is strictly convex, then $F$ is singlevalued and continuous from $X$ into $X^*_w$ ($X^*_w$ stands for the space $X^*$ with the weak topology); this continuity is called the “demicontinuity” of $F$. Let $A$ be a (multivalued) mapping from a Banach space $X$ into $X^*$; the graph $G(A)$ of $A$ is the set

$$G(A) := \{ [z, z^*] \in X \times X^* \mid z^* \in Az \}.$$ 

Then $A$ is called monotone from $D(A) \subset X$ into $X^*$, if

$$\langle z_i^* - z_2^*, z_1 - z_2 \rangle \geq 0, \quad \forall [z_1, z_i^*] \in G(A), \ i = 1, 2;$$

in particular, $A$ is called strictly monotone, if the strict positiveness holds whenever $z_1 \neq z_2$ in the above inequality. Moreover, $A$ is maximal monotone, if $A$ is monotone from $D(A) \subset X$ into $X^*$ and $G(A)$ has no proper monotone extension in $X \times X^*$. When $X$ is reflexive, it is well known that $A$ is maximal monotone if and only if the range of $A + F$ is the whole of $X^*$, where $F$ is the duality mapping from $X$ into $X^*$. We refer to [2, 4, 5, 9, 10, 20, 26, 27] for fundamental properties of subdifferentials and monotone mappings.
2. A compactness theorem

In this section, let \( V \) be a (real) reflexive Banach space which is dense and compactly embedded in a Hilbert space \( H \). Identifying \( H \) with its dual space, we have \( V \subset H \subset V^* \) with compact embeddings. Let \( W \) be another reflexive and separable Banach space which is dense and continuously embedded in \( V \); since \( V^* \subset W^* \), it holds that
\[
V \subset H \subset W^* \quad \text{with compact embeddings.}
\]

In order to avoid some irrelevant arguments, suppose that \( V, V^*, W \) and \( W^* \) are strictly convex. We denote by \( C_W \) an embedding constant from \( W \) into \( V \) and \( H \), namely
\[
|z|_V \leq C_W |z|_W, \quad |z|_H \leq C_W |z|_W, \quad \forall z \in W.
\]

For any function \( w : [0, T] \to W^* \), the total variation of \( w \) is denoted by \( \text{Var}_{W^*}(w) \), which is defined by
\[
\text{Var}_{W^*}(w) := \sup_{\eta \in C^1_0(0, T; W^*)} \int_0^T \langle w, \eta' \rangle_{W^*, W} dt.
\]

We refer to [9] or [13] for the fundamental properties of total variation functions.

We fix numbers \( p \) with \( 1 < p < \infty \), \( p' := \frac{p}{p-1} \), and \( T \) with \( 0 < T < \infty \). Given \( \kappa > 0 \), \( M_0 > 0 \) and \( u_0 \in H \), consider the set \( Z(\kappa, M_0, u_0) \) in \( L^p(0, T; V) \cap L^\infty(0, T; H) \) given by:
\[
Z(\kappa, M_0, u_0) := \left\{ u \left| \begin{array}{l}
|u|_{L^p(0, T; V)} \leq M_0, \ |u|_{L^\infty(0, T; H)} \leq M_0, \\
\exists f \in L^{p'}(0, T; V^*) \text{ such that} \\
\int_0^T \langle f, u \rangle dt \leq M_0, \ |f|_{L^1(0, T; W^*)} \leq M_0, \\
\int_0^T \langle \eta' - f, u - \eta \rangle dt \leq \frac{1}{2} |u_0 - \eta(0)|^2_H, \ \forall \eta \in L^p(0, T; V) \\
\text{with} \ \eta' \in L^{p'}(0, T; V^*), \ \eta(t) \in \kappa B_W(0), \ \forall t \in [0, T]
\end{array} \right. \right\},
\]

where \( B_W(0) \) is the closed unit ball in \( W \) with center at the origin and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V^*, V} \).

The variational inequality
\[
\int_0^T \langle \eta' - f, u - \eta \rangle dt \leq \frac{1}{2} |u_0 - \eta(0)|^2_H, \quad \forall \eta \in L^p(0, T; V), \ \eta' \in L^{p'}(0, T; V^*), \ \eta(t) \in \kappa B_W(0), \ \forall t \in [0, T],
\]

in the definition of \( Z(\kappa, M_0, u_0) \), is derived from the time-derivative with the convex constraint \( \kappa B_W(0) \). In fact, for any \( u, \eta \in L^p(0, T; V) \) with \( u', \eta' \in L^{p'}(0, T; V^*) \) with \( u(0) = u_0 \) we have by integration by parts
\[
\int_0^T \langle \eta' - u', u - \eta \rangle dt = \frac{1}{2} |u(0) - \eta(0)|^2_H - \frac{1}{2} |u(T) - \eta(T)|^2_H \leq \frac{1}{2} |u_0 - \eta(0)|^2_H.
\]
Therefore, if $f = u'$ with $u(0) = u_0$, then (2.1) holds. Given $u \in L^p(0, T; V) \cap L^\infty(0, T; H)$ and $u_0 \in H$, the set of all $f$ satisfying (2.1)-(2.2) includes $u'$, provided $u'$ exists in $L^p(0, T; V^*)$ and $u(0) = u_0$. However, in general, it is an extremely large set; note that in the definition of $Z(\kappa, M_0, u_0)$, any differentiability of $u$ in time is not required.

Our main result is stated as follows.

**Theorem 2.1.** Let $\kappa > 0$, $M_0 > 0$ be any numbers and $u_0$ be any element of $H$. Then the set $Z(\kappa, M_0, u_0)$ is relatively compact in $L^p(0, T; H)$. Moreover, the convex closure of $Z(\kappa, M_0, u_0)$, denoted by $\text{conv}[Z(\kappa, M_0, u_0)]$, in $L^p(0, T; V)$ is compact in $L^p(0, T; H)$.

We begin with the following lemmas that are crucial for the proof of Theorem 2.1.

**Lemma 2.1.** Let $\kappa > 0$, $M_0 > 0$ be any numbers and $u_0$ be any element of $H$. Then there exists a positive constant $C^* := C^*(\kappa, M_0, |u_0|_H)$, depending only on $\kappa$, $M_0$ and $|u_0|_H$, such that

$$\text{Var}_{W^*}(u) \leq C^*, \quad (2.3)$$

for all $u \in Z(\kappa, M_0, u_0)$.

**Proof.** Let $u$ be any element in $Z(\kappa, M_0, u_0)$, and take a function $f \in L^p(0, T; V^*)$ satisfying all the required properties in the definition of $Z(\kappa, M_0, u_0)$. Now let $\eta$ be any function in $C_0^1(0, T; W)$ with $|\eta|_{L^\infty(0, T; W)} > 0$. Since $\tilde{\eta} := \pm \frac{\kappa}{|\eta|_{L^\infty(0, T; W)}}$ is a possible test function for (2.1)-(2.2), we have

$$\int_0^T \langle \tilde{\eta}' - f, u - \tilde{\eta} \rangle dt \leq \frac{1}{2} |u_0|_H^2,$$

which shows

$$\int_0^T \langle \tilde{\eta}', u \rangle dt \leq - \int_0^T \langle f, \tilde{\eta} \rangle dt + \int_0^T \langle f, u \rangle dt + \frac{1}{2} |u_0|_H^2,$$

$$\leq |f|_{L^1(0, T; W^*)} |\tilde{\eta}|_{L^\infty(0, T; W)} + M_0 + \frac{1}{2} |u_0|_H^2.$$

Hence,

$$\left| \int_0^T \langle u, \eta' \rangle dt \right| \leq \left\{ M_0 + \frac{M_0}{\kappa} + \frac{|u_0|_H^2}{2\kappa} \right\} |\eta|_{L^\infty(0, T; W)}, \quad \forall \eta \in C^1(0, T; W),$$

so that (2.3) holds with $C^* := M_0 + \frac{M_0}{\kappa} + \frac{1}{2\kappa} |u_0|_H^2$. □

**Lemma 2.2.** Let $M_1$ be any positive number and let $\{u_n\}$ be any sequence of functions from $[0, T]$ into $W^*$ such that $u_n \in L^p(0, T; V) \cap L^\infty(0, T; H)$

$$|u_n|_{L^p(0, T; V)} \leq M_1, \quad |u_n|_{L^\infty(0, T; H)} \leq M_1, \quad \text{Var}_{W^*}(u_n) \leq M_1, \quad n = 1, 2, \ldots. \quad (2.4)$$

Then there are a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u \in L^p(0, T; V) \cap L^\infty(0, T; H)$ such that $u_{n_k} \to u$ weakly in $H$ for every $t \in [0, T]$ as $k \to \infty$. Hence $u_{n_k}(t) \to u(t)$ in $W^*$ for every $t \in [0, T]$ and $u_{n_k} \to u$ in $L^q(0, T; W^*)$ for every $q \in [1, \infty)$ as $k \to \infty$.  

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Proof. Since $W$ is separable, there is a countable dense subset $W_0$ of $W$. Now, we consider a sequence of real valued functions $A_n(t, \xi) := (u_n(t), \xi)_H = (\langle u_n(t), \xi \rangle_{W^*, W})$ on $[0, T]$ for each $\xi \in W_0$. Then, by (2.4) the total variation of $A_n(t, \xi)$ is bounded by $M_1|\xi|_W$. Hence from the Helly selection theorem (cf. [13; Section 5.2.3]) it follows that there is a subsequence $\{A_{nk}(t, \xi)\}$, depending on $\xi \in W_0$, such that $A_{nk}(t, \xi)$ converges to a function $A_0(t, \xi)$ pointwise on $[0, T]$ and its total variation is not larger than $M_1|\xi|_W$.

Since $W_0$ is countable in $W$, by using extensively the above Helly selection theorem we can extract a subsequence, denoted by the same notation as $\{A_{nk}\}$ again, and a function $A_0(t, \xi)$ on $[0, T] \times W_0$ such that

$$A_{nk}(t, \xi) \to A_0(t, \xi) \text{ as } k \to \infty, \quad \forall t \in [0, T], \forall \xi \in W_0. \quad (2.5)$$

Furthermore, by density, this convergence (2.5) can be extended to all $\xi \in W$. Also, the functional $A_{nk}(t, \xi)$ is linear in $\xi$ and uniformly bounded, i.e.

$$|A_{nk}(t, \xi)| \leq M_1|\xi|_H \leq M_1C_W|\xi|_W, \quad \forall t \in [0, T], \forall \xi \in W.$$

This implies that $A_0(t, \xi)$ is linear and bounded in $\xi \in W$ and $|A_0(t, \xi)| \leq M_1|\xi|_H$ for all $\xi \in W$ and $t \in [0, T]$. As a consequence, by Riesz representation theorem, there is a function $u : [0, T] \to H$ with $|u(t)|_H \leq M_1$ for all $t \in [0, T]$ such that

$$A_0(t, \xi) = (u(t), \xi)_H, \quad \forall \xi \in H, \forall t \in [0, T].$$

Now it is clear by (2.5) that $u_{nk}(t) \to u(t)$ weakly in $H$ for $t \in [0, T]$ as $k \to \infty$. Finally, by the compactness of the injection from $H$ into $W^*$, we see that $u_{nk}(t) \to u(t)$ (strongly) in $W^*$ for any $t \in [0, T]$. Hence $u_{nk} \to u$ in $L^q(0, T; W^*)$ for all $q \in [1, \infty)$ as $k \to \infty$. □

Proof of Theorem 2.1. We first note from Lemma 2.1 that

$$Z(\kappa, M_0, u_0) \subset \mathcal{X} := \{u \mid |u|_{L^p(0, T; V)} \leq M_0, \ |u|_{L^\infty(0, T; H)} \leq M_0, \ \text{Var}_{W^*}(u) \leq C^*\},$$

where $C^*$ is the same constant as in Lemma 2.1. Note that $\mathcal{X}$ is closed and convex in $L^p(0, T; V)$. Therefore, in order to obtain Theorem 2.1 it is enough to prove the compactness of $\mathcal{X}$ in $L^p(0, T; H)$.

Let $\{u_n\}$ be any sequence in the set $\mathcal{X}$. Then, by Lemma 2.2, there is a subsequence $\{u_{nk}\}$ and a function $u \in L^\infty(0, T; H)$ such that $u_{nk}(t) \to u(t)$ weakly in $H$ for every $t \in [0, T]$ as $k \to \infty$. By the injection compactness from $H$ into $W^*$ we have that

$$u_{nk} \to u \text{ in } L^p(0, T; W^*) \text{ as } k \to \infty. \quad (2.6)$$

and that $|u|_{L^p(0, T; V)} \leq M_0$ by $|u_{nk}|_{L^p(0, T; V)} \leq M_0$.

Here we recall the Aubin lemma [3] (or [26; Lemma 5.1]): for each $\delta > 0$ there is a positive constant $C_\delta$ such that

$$|z|^p_H \leq \delta|z|^p_V + C_\delta|z|^p_{W^*}, \quad \forall z \in V.$$ 

By making use of this inequality for $z = u_{nk}(t) - u(t)$ and integrating it in time, we get

$$\int_0^T |u_{nk}(t) - u(t)|^p_H \, dt \leq \delta(2M_0)^p + C_\delta \int_0^T |u_{nk}(t) - u(t)|^p_{W^*} \, dt.$$
On account of (2.6), letting $k \to \infty$ gives that
\[
\limsup_{k \to \infty} \int_0^T |u_{n_k} - u|^p_H dt \leq \delta (2M_0)^p.
\]
Since $\delta > 0$ is arbitrary, we conclude that $u_{n_k} \to u$ in $L^p(0, T; H)$. \hfill \Box

**Remark 2.1.** If $f \in L^p(0, T; V^*)$ and (2.1) holds for all $\eta \in L^p(0, T; V)$ with $\eta' \in L^p(0, T; V^*)$, then $u' = f \in L^p(0, T; V^*)$ and $u(0) = u_0$. Therefore, by Theorem 2.1 the set
\[
\{ u \mid |u|_{L^p(0, T; V)} \leq M_0, \quad |u|_{L^\infty(0, T; H)} \leq M_0, \quad |u'|_{L^1(0, T; W^*)} \leq M_0 \}
\]
is relatively compact in $L^p(0, T; H)$ for each finite positive constant $M_0$. Our theorem includes a typical case of Aubin compactness theorem [3].

**Remark 2.2.** A compactness result of the Aubin type was extended in various directions, for instance [12] and [18], and further to a quite general set up [31].

### 3. Time-derivative under convex constraints

Let $H$ be a Hilbert space and $V$ be a strictly convex reflexive Banach space such that $V$ is dense in $H$ and the injection from $V$ into $H$ is continuous. We identify $H$ with its dual space:
\[
V \subset H \subset V^* \quad \text{with continuous embeddings.}
\]

For simplicity, we assume that $V^*$ is strictly convex. Therefore the duality mapping $F$ from $V$ into $V^*$, associated with gauge function $r \to |r|^{p-1}$, is singlevalued and demicontinuous from $V$ into $V^*$, where $p$ is a fixed number with $1 < p < \infty$.

For the sake of simplicity for notation, we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{V^*, V}$ again.

Let $\{K(t)\}_{t \in [0, T]}$ be a family of non-empty, closed and convex sets in $V$ such that there are functions $\alpha \in W^{1,2}(0, T)$ and $\beta \in W^{1,1}(0, T)$ satisfying the following property: for any $s, t \in [0, T]$ and any $z \in K(s)$ there is $\tilde{z} \in K(t)$ such that
\[
|\tilde{z} - z|_H \leq |\alpha(t) - \alpha(s)|(1 + |z|_V^p), \quad |\tilde{z}|_V^p - |z|_V^p \leq |\beta(t) - \beta(s)|(1 + |z|_V^p). \tag{3.1}
\]

We denote by $\Phi(\alpha, \beta)$ the set of all such families $\{K(t)\}$, and put
\[
\Phi_S := \bigcup_{\alpha \in W^{1,2}(0, T), \ \beta \in W^{1,1}(0, T)} \Phi(\alpha, \beta),
\]
which is called the strong class of time-dependent convex sets.

Given $\{K(t)\} \in \Phi_S$, we consider the following time-dependent convex function on $H$:
\[
\varphi'_{K}(z) := \begin{cases} 
\frac{1}{p}|z|^p_V + I_{K(t)}(z), & \text{if } z \in K(t), \\
\infty, & \text{otherwise},
\end{cases} \tag{3.2}
\]
where $I_{K(0)}(\cdot)$ is the indicator function of $K(0)$ on $H$. For each $t \in [0,T]$, $\varphi^t_K(\cdot)$ is proper, l.s.c. and strictly convex on $H$ and on $V$. By the general theory on nonlinear evolution equations generated by time-dependent subdifferentials, condition (3.1) is sufficient in order that for any $u_0 \in \overline{K(0)}$ (the closure of $K(0)$ in $H$) and $f \in L^2(0,T;H)$ the Cauchy problem

$$u'(t) + \partial \varphi^t_K(u(t)) \ni f(t), \quad u(0) = u_0, \text{ in } H,$$

admits a unique solution $u$ such that $u \in C([0,T];H) \cap L^p(0,T;V)$ with $u(0) = u_0$, $t^\frac{1}{2}u' \in L^2(0,T;H)$ and $t \to t\varphi^t_K(u(t))$ is bounded on $(0,T]$ and absolutely continuous on any compact interval in $(0,T]$, where $\partial \varphi^t_K$ denotes the subdifferential of $\varphi^t_K$ in $H$. In particular, if $u_0 \in K(0)$, then $u' \in L^2(0,T;H)$ and $t \to \varphi^t_K(u(t))$ is absolutely continuous on $[0,T]$.

Next, taking constraints of obstacle type into account, we introduce a weak class of time-dependent convex sets. In the sequel, let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

**Definition 3.1.** Let $c_0$ be a fixed constant and $\sigma_0$ be a fixed function in $C([0,T];V)$ with $\sigma^t_0 \in L^p(0,T;V^*)$. Associated with these $c_0$ and $\sigma_0$, for each small positive number $\varepsilon$ a mapping $\mathcal{F}_\varepsilon : [0,T] \times V \to V$ is defined by

$$\mathcal{F}_\varepsilon(t)z = (1 + \varepsilon c_0)z + \varepsilon \sigma_0(t), \quad \forall t \in [0,T], \forall z \in V. \tag{3.3}$$

Then, with $\mathcal{F}_\varepsilon$ the weak class $\Phi_W := \Phi_W(c_0, \sigma_0)$ of time-dependent convex sets is defined by: $\{K(t)\} \in \Phi_W$ if and only if

(a) $K(t)$ is a closed and convex set in $V$ for all $t \in [0,T]$,

(b) there exists a sequence $\{\{K_n(t)\}\}_{n \in \mathbb{N}} \subset \Phi_S$ such that for any $\varepsilon \in (0,\varepsilon_0)$ ($0 < \varepsilon_0 < 1$) there is a positive integer $N_\varepsilon$ satisfying

$$\mathcal{F}_\varepsilon(t)(K_n(t)) \subset K(t), \quad \mathcal{F}_\varepsilon(t)(K(t)) \subset K_n(t), \quad \forall t \in [0,T], \forall n \geq N_\varepsilon.$$

In this case, it is said that $\{K_n(t)\}$ converges to $\{K(t)\}$ as $n \to \infty$, which is denoted by

$$K_n(t) \Rightarrow K(t) \quad \text{on } [0,T] \quad (n \to \infty).$$

We give three typical examples of $\{K(t)\}$ in the weak class $\Phi_W$.

**Example 3.1.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $1 \leq N < \infty$ and $Q := \Omega \times (0,T)$. Let $H := L^2(\Omega)$, $V := W^{1,p}(\Omega)$, $2 \leq p < \infty$. Moreover, let $\rho := \rho(x,t) \in C(\overline{Q})$ and choose a sequence $\{\rho_n\}$ in $C^2(\overline{Q})$ such that $\rho_n \to \rho$ in $C(\overline{Q})$. Now, constraint sets $K(t)$ and $K_n(t)$ are defined by

$$K(t) := \{z \in V \mid z(x) \geq \rho(x,t) \text{ for a.e. } x \in \Omega\}, \quad \forall t \in [0,T],$$

and

$$K_n(t) := \{z \in V \mid z(x) \geq \rho_n(x,t) \text{ for a.e. } x \in \Omega\}, \quad \forall t \in [0,T].$$
Given \( \varepsilon > 0 \), take a positive integer \( N_\varepsilon \) so that

\[
|\rho_n - \rho| \leq \varepsilon \quad \text{on } \overline{Q}, \forall n \geq N_\varepsilon.
\]

In this case, with the choice of \( c_0 = 0 \) and \( \sigma \equiv 1 \) the mapping \( F_\varepsilon(t) \) is of the form

\[
z \rightarrow z + \varepsilon,
\]

which maps \( V \) into itself. Then we have:

(i) \( \{K_n(t)\} \in \Phi_S \). Indeed, for \( z \in K_n(s) \), the function \( \bar{z}(x) := z(x) - \rho_n(x, s) + \rho_n(x, t) \) belongs to \( K_n(t) \) and (3.1) holds with functions \( \alpha(t) = \beta(t) = c_n t \) for a constant \( c_n > 0 \), depending only on \( |\rho_n|_{C^2(Q)} \). Thus \( \{K_n(t)\} \in \Phi_S \).

(ii) \( \{K(t)\} \in \Phi_W \). In fact, for any \( z_n \geq \rho_n(\cdot, t) \) a.e. on \( \Omega \), we have

\[
F_\varepsilon z_n = z_n + \varepsilon \geq \rho_n(\cdot, t) + \varepsilon \geq \rho(\cdot, t) \text{ a.e. on } \Omega,
\]

which implies \( F_\varepsilon(K_n(t)) \subset K(t) \). Similarly, \( F_\varepsilon(K(t)) \subset K_n(t) \). Hence \( K_n(t) \implies K(t) \) on \([0, T]\), and thus \( \{K(t)\} \in \Phi_W \).

**Example 3.2.** Let \( \rho \) and \( \rho_n \) be the same as in Example 3.1, and consider constraint sets

\[
K(t) := \{z \in V \mid z(x) \leq \rho(x, t) \text{ for a.e. } x \in \Omega\}, \quad \forall t \in [0, T]
\]

and

\[
K_n(t) := \{z \in V \mid z(x) \leq \rho_n(x, t) \text{ for a.e. } x \in \Omega\}, \quad \forall t \in [0, T].
\]

Then, just as in Example 3.1, we have \( \{K_n(t)\} \in \Phi_S \) and \( K_n(t) \implies K(t) \) on \([0, T]\) by using the mapping \( F_\varepsilon(t)z = z - \varepsilon \), so that \( \{K(t)\} \in \Phi_W \).

**Example 3.3.** Let \( \Omega \) and \( Q \) be the same as in Example 3.1 and consider the following vectorial case in connection with our model problem (1.1)-(1.3):

\[
H := L^2(\Omega) \times L^2(\Omega), \quad V := H^1_0(\Omega) \times H^1_0(\Omega);
\]

hence \( V^* = H^{-1}(\Omega) \times H^{-1}(\Omega) \).

Let \( \psi = \psi(x, t) \) be an obstacle function prescribed in \( C(\overline{Q}) \) so that \( \psi \geq c_\psi \) on \( \overline{Q} \) for a positive constant \( c_\psi \), and define \( K(t) \) by

\[
K(t) := \{[\xi_1, \xi_2] \in V \mid |\xi_1| + |\xi_2| \leq \psi(\cdot, t) \text{ a.e. in } \Omega\}, \quad \forall t \in [0, T].
\]

Next, choosing a sequence \( \{\psi_n\} \in C^1(\overline{Q}) \) such that

\[
\psi_n \geq c_\psi \text{ on } \overline{Q}, \quad \psi_n \to \psi \text{ in } C(\overline{Q}) \text{ as } n \to \infty
\]

we define

\[
K_n(t) := \{[\xi_1, \xi_2] \in V \mid |\xi_1| + |\xi_2| \leq \psi_n(\cdot, t) \text{ a.e. in } \Omega\}, \quad \forall t \in [0, T].
\]

and the mapping

\[
F_\varepsilon(t)[\xi_1, \xi_2] := (1 - \varepsilon)[\xi_1, \xi_2], \quad \forall \text{ small } \varepsilon > 0;
\]

note that this mapping is obtained by the choice of \( c_0 = -1 \) and \( \sigma_0 \equiv 0 \). Then we have:
Let Theorem 3.1.

(i) \( \{K_n(t)\} \in \Phi_S \). In fact, given \( z = [\xi_1, \xi_2] \in K_n(s) \), we take \( \tilde{z} = [\tilde{\xi}_1, \tilde{\xi}_2] := (1 - \frac{1}{c_\psi})[\psi_n(s) - \psi_n(t)]_{C(\overline{\Omega})}z \). In this case, if \( |s - t| \) is so small that \( \frac{1}{c_\psi}|\psi_n(s) - \psi_n(t))|_{C(\overline{\Omega})} < 1 \), then

\[
|\tilde{\xi}_1| + |\tilde{\xi}_2| = (1 - \frac{1}{c_\psi}|\psi_n(s) - \psi_n(t))|_{C(\overline{\Omega})}(|\xi_1| + |\xi_2|)
\leq (1 - \frac{1}{c_\psi}|\psi_n(s) - \psi_n(t))|_{C(\overline{\Omega})}\psi_n(\cdot, s)
\leq \psi_n(\cdot, s) - \frac{\psi_n(\cdot, s)}{c_\psi}|\psi_n(s) - \psi_n(t)|_{C(\overline{\Omega})} \leq \psi_n(\cdot, t).
\]

Hence \( \tilde{z} \in K_n(t) \) and \( |\tilde{z}|_V \leq |z|_V \). By using extensively this idea we see that (3.1) holds with \( \alpha(t) = c_n t \) for a certain (big) constant \( c_n > 0 \) and \( \beta(t) = 0 \). For the detailed proof we refer to [22; Lemma 3.1] or [15; Example 4.5].

(ii) \( \{K(t)\} \in \Phi_W \), which is proved by using the mapping \( \mathcal{F}_\varepsilon(t)z := (1 - \varepsilon)z \). In fact, for any \( z_n := [\xi_{1n}, \xi_{2n}] \in K_n(t) \) and any small \( \varepsilon > 0 \), we take an integer \( N_\varepsilon \) so that \( |\psi_n - \psi|_{C(\overline{\Omega})} \leq c_\psi \) for all \( n \geq N_\varepsilon \). In this case, we have

\[
(1 - \varepsilon)(|\xi_{1n}(x)| + |\xi_{2n}(x)|) \leq (1 - \varepsilon)\psi_n(x, t) \leq \psi(x, t), \quad \forall n \geq N_\varepsilon.
\]

This shows \( \mathcal{F}_\varepsilon(K_n(t)) \subset K(t) \) for all \( n \geq N_\varepsilon \). Similarly, \( \mathcal{F}_\varepsilon(K(t)) \subset K_n(t) \) for \( n \geq N_\varepsilon \). Hence \( K_n(t) \implies K(t) \) on \([0, T]\) and \( \{K(t)\} \in \Phi_W \).

As is easily seen from the above examples, the class \( \Phi_W \) is strictly larger than \( \Phi_S \). Next, we introduce the time-derivative under constraint \( \{K(t)\} \in \Phi_W \). Put

\[
\mathcal{K} := \{v \in L^p(0, T; V) \mid v(t) \in K(t) \text{ for a.e. } t \in [0, T]\}
\]

and

\[
\mathcal{K}_0 := \{\eta \in \mathcal{K} \mid \eta' \in L^p(0, T; V^*)\}.
\]

**Definition 3.2.** Let \( \{K(t)\} \in \Phi_W \) and \( u_0 \in \text{conv}(K(t)) \). Then we define an operator \( L_{u_0} \) whose graph \( G(L_{u_0}) \) is given in \( L^p(0, T; V) \times L^p(0, T; V^*) \) as follows: \( [u, f] \in G(L_{u_0}) \) if and only if

\[
u \in \mathcal{K}, \quad f \in L^p(0, T; V^*), \quad \int_0^T (\eta' - f, u - \eta)dt \leq \frac{1}{2}|u_0 - \eta(0)|_{L^2}, \quad \forall \eta \in \mathcal{K}_0. \tag{3.4}
\]

We prove the most important property of \( L_{u_0} \) in the next theorems.

**Theorem 3.1.** Let \( \{K(t)\} \in \Phi_W \) and \( u_0 \in \text{conv}(K(t)) \). Then \( L_{u_0} \) is maximal monotone from \( D(L_{u_0}) \subset L^p(0, T; V) \) into \( L^p(0, T; V^*) \), and the domain \( D(L_{u_0}) \) is included in the set \( \{u \in C([0, T]; H) \cap \mathcal{K} \mid u(0) = u_0\} \).

The characterization and fundamental properties of the mapping \( L_{u_0} \) are given in the following theorem.

**Theorem 3.2.** Let \( \{K(t)\} \in \Phi_W \). Then we have:
(1) Let $u_0 \in \overline{K(0)}$. Then $f \in L_{w_0}u$ if and only if there are $\{K_n(t)\} \subset \Phi_S$, $\{u_n\} \subset L^p(0,T;V)$ with $u_n \in K_n := \{v \in L^p(0,T;V) \mid v(t) \in K_n(t) \text{ for a.e } t \in [0,T]\}$ and $u'_n \in L^p(0,T;V^*)$, $\{f_n\} \subset L^p(0,T;V^*)$ such that

$$K_n(t) \Rightarrow K(t) \text{ on } [0,T],$$

$$u_n \to u \text{ in } C([0,T];H) \text{ and weakly in } L^p(0,T;V),$$

$$f_n \to f \text{ weakly in } L^p(0,T;V^*),$$

$$\int_0^T \langle u'_n - f_n, u_n - v \rangle dt \leq 0, \forall v \in K_n, \forall n,$$

$$\limsup_{n \to \infty} \int_{t_1}^{t_2} \langle f_n, u_n \rangle dt \leq \int_{t_1}^{t_2} \langle f, u \rangle dt, \forall t_1, t_2 \text{ with } 0 \leq t_1 \leq t_2 \leq T.$$  

(2) Let $u_0 \in \overline{K(0)}$ and $f \in L_{w_0}u$. Then, for any $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$,

$$\int_{t_1}^{t_2} \langle \eta' - f, u - \eta \rangle dt + \frac{1}{2} |u(t_2) - \eta(t_2)|_H^2 \leq \frac{1}{2} |u(t_1) - \eta(t_2)|_H^2, \forall \eta \in K_0.$$  

(3) Let $u_0 \in \overline{K(0)}$, and $f_i \in L_{w_0}u_i$ for $i = 1, 2$. Then, for any $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$,

$$\frac{1}{2} |u_1(t_2) - u_2(t_2)|_H^2 \leq \frac{1}{2} |u_1(t_1) - u_2(t_2)|_H^2 + \int_{t_1}^{t_2} \langle f_1 - f_2, u_1 - u_2 \rangle dt.$$  

The proofs of these theorems will be given in the next section.

4. Proofs of Theorems 3.1 and 3.2

In this section we use the same notation and assume the same assumptions as in the previous section.

We introduce another mapping $\tilde{L}_{u_0}$ whose graph $G(\tilde{L}_{u_0})$ is given as follows: $f \in \tilde{L}_{u_0}u$ if and only if $u \in \mathcal{K} \cap C([0,T];H)$ with $u(0) = u_0$, $f \in L^p(0,T;V^*)$ and there exist sequences $\{\{K_n(t)\}\} \subset \Phi_S$, $\{u_n\} \subset K_{n,0} := \{v \in \mathcal{K} \mid v' \in L^p(0,T;V^*)\}$ and $\{f_n\} \subset L^p(0,T;V^*)$ and (3.5)-(3.9) are fulfilled.

As to the mapping $\tilde{L}_{u_0}$ we prove:

Lemma 4.1. $\tilde{L}_{u_0}$ is a restriction of $L_{u_0}$, namely $G(\tilde{L}_{u_0}) \subset G(L_{u_0})$, and

$$\int_0^T \langle f - g, u - w \rangle dt \geq 0, \forall [u, f] \in G(\tilde{L}_{u_0}), \forall [w, g] \in G(L_{u_0}).$$  

Proof. Let $f \in \tilde{L}_{u_0}u$ and $\{\{K_n(t)\}\}$, $\{u_n\}$, $\{f_n\}$ be sequences as in the definition of $f \in L_{u_0}u$; (3.5)-(3.9) are fulfilled as well. Then, for any $\eta \in K_0$, we have

$$\int_0^T \langle u'_n - f_n, u_n - F_\varepsilon \eta \rangle dt \leq 0,$$
since \( F \varepsilon \eta \in \mathcal{K}_{n,0} := \{ v \in \mathcal{K}_n \mid v' \in L^p(0, T; V^*) \} \). Substituting the expression \( F \varepsilon \eta = (1 + \varepsilon \sigma_0)\eta + \varepsilon \sigma_0 \) (cf. (3.3)) in the above inequality and using integration by parts, we get
\[
\int_0^T \langle \eta' - f_n, u_n - F \varepsilon \eta \rangle dt + \varepsilon \int_0^T \langle \varepsilon\sigma_0', u_n - F \varepsilon \eta \rangle dt \leq \frac{1}{2} |u_n(0) - F \varepsilon \eta(0)|_H^2.
\]

Now, since \( F \varepsilon \eta \to \eta \) in \( L^p(0, T; V) \cap C([0, T]; H) \) as \( \varepsilon \downarrow 0 \), we have by letting \( n \to \infty \) and (3.7)
\[
\int_0^T \langle \eta' - f, u - \eta \rangle dt \leq \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0.
\]

This implies \( f \in L_{u_0} w. \) Thus \( G(L_{u_0}) \subset G(L_{u_0}). \)

Let \( g \in L_{u_0} w. \) Then, with the same notation as above, it follows from (3.4) that
\[
\int_0^T \langle (F \varepsilon \eta)' - g, w - F \varepsilon u_n \rangle dt \leq \frac{1}{2} |u_0 - F \varepsilon(0)u_n(0)|_H^2,
\]

since \( F \varepsilon u_n = u_n + \varepsilon \sigma_0 u_n + \varepsilon \sigma_0 \in \mathcal{K}_0. \) The above inequality is of the form
\[
\int_0^T \langle u_n' - g + \varepsilon \sigma_0 u_n' + \varepsilon \sigma_0', w - u_n - \varepsilon \sigma_0 u_n - \varepsilon \sigma_0 \rangle dt
\]
\[
\leq \frac{1}{2} |u_0 - u_n(0) - \varepsilon \sigma_0 u_n(0) - \varepsilon \sigma_0(0)|_H^2.
\]

Hence we derive from this inequality that
\[
\int_0^T \langle u_n' - g, w - u_n \rangle dt + \varepsilon \int_0^T \langle \sigma_0', w \rangle dt
\]
\[
\leq \frac{1}{2} |u_0 - u_n(0)|_H^2 + \varepsilon \sup_{n \geq 1} \left\{ |f_n|_{L^p(0, T; V^*)} + |u_n|_{L^p(0, T; V)} + |u_n|_{C([0, T]; H)} \right\} < \infty.
\]

By assumptions (3.6) and (3.7),
\[
\sup_{n \geq 1} \left\{ |f_n|_{L^p(0, T; V^*)} + |u_n|_{L^p(0, T; V)} + |u_n|_{C([0, T]; H)} \right\} < \infty.
\]

Hence, for a constant \( C_1 > 0 \), depending only on \( f, u \) and \( w \) (but independent of \( \varepsilon \) and \( n \)), we infer from (4.2) that
\[
\int_0^T \langle u_n' - g, w - u_n \rangle dt + \varepsilon \sigma_0 \int_0^T \langle u_n', w \rangle dt \leq \frac{1}{2} |u_0 - u_n(0)|_H^2 + \varepsilon C_1.
\]
Also, from (3.8) it follows that
\[
\int_0^T \langle u'_n - f_n, u_n - F(\varepsilon w) \rangle dt = \int_0^T \langle u'_n - f_n, u_n - w - \varepsilon c_0 w - \varepsilon \sigma_0 \rangle dt \leq 0.
\]
In a similar way to (4.3) it follows that
\[
\int_0^T \langle u'_n - f_n, u_n - w \rangle dt \geq -\varepsilon (C_1 + C_2) - \frac{1}{2} |u_0 - u_n(0)|^2_H,
\]
whence (4.1) is obtained by $\varepsilon \downarrow 0$ and $n \to \infty$.

**Corollary 4.1.** $\tilde{L}_{u_0}$ is monotone from $D(\tilde{L}_{u_0}) \subset L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Also, if $u \in D(\tilde{L}_{u_0})$, then $u \in \mathcal{K} \cap C([0, T]; H)$ and $u(0) = u_0$.

Next we show the maximal monotonicity of $\tilde{L}_{u_0}$.

**Lemma 4.2.** Let $\{K(t)\} \in \Phi_W$ and $u_0 \in \overline{\mathcal{K}(0)}$. Then $\tilde{L}_{u_0}$ is maximal monotone from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$; more precisely for any $f \in L^{p'}(0, T; V^*)$ there exists a unique $u \in L^p(0, T; V)$ such that $u(t) \in K(t)$ for a.e. $t \in [0, T]$ and
\[
f \in \tilde{L}_{u_0} u + Fu.
\]

**Proof.** The operator $\tilde{L}_{u_0} + F$ is strictly monotone from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$, so the function $u$ satisfying (4.5) is unique. Now we are to prove the existence of such a function $u$.

Choose $\{\{K_n(t)\}\} \subset \Phi_S$ such that $K_n(t) \to K(t)$ on $[0, T]$ and $\{f_n\} \subset L^2(0, T; H)$ such that $f_n \to f$ in $L^{p'}(0, T; V^*)$ with $|f_n|_{L^{p'}(0, T; V^*)} \leq |f|_{L^{p'}(0, T; V^*)} + 1$. Also, it is easy to take a sequence $\{u_{0,n}\}$ in $V$ such that $u_{0,n} \in \mathcal{K}_n(0)$ and $u_{0,n} \to u_0$ in $H$. For these data we consider the Cauchy problems
\[
u_n'(t) + U_n(t) = f_n(t), \quad U_n(t) \in \partial \varphi_{K_n}^t(u_n(t)) \text{ in } H, \quad \text{a.e. } t \in [0, T], \quad u_n(0) = u_{0,n},
\]
where $\varphi_{K_n}^t(\cdot)$ is a time-dependent proper, l.s.c. and convex functions on $H$ given by (3.2) with $K(t)$ replaced by $K_n(t)$. As was mentioned in section 3, (4.6) possesses a unique solution $u_n \in W^{1,2}(0, T; H)$ such that $t \to \varphi_{K_n}^t(u_n(t))$ is absolutely continuous on $[0, T]$ (hence, $u_n \in \mathcal{K}_n,0$ and $u_n(t) \in K_n(t)$ for all $t \in [0, T]$).

We note here that
\[
U_n(t) \in \partial, I_{K_n,t}(u_n(t)) + Fu_n(t), \quad \text{a.e. } t \in [0, T],
\]
where $I_{K_n,t}(\cdot)$ is the indicator function of $K_n(t)$ defined by (3.3) and $\partial, I_{K_n,t}(\cdot)$ is the subdifferential of $I_{K_n,t}(\cdot)$.
since $\partial \varphi_{K_n}(z) \subset \partial \varphi_{K_n}(z) = \partial I_{K_n(t)}(z) + Fz$ for all $z \in K_n(t)$ (cf. [10; Theorem 2] or [20; Theorem 5.2]), and by the expression

$$\mathcal{F}_\varepsilon^2(t)z := \mathcal{F}_\varepsilon(t)(\mathcal{F}_\varepsilon(t)z) = z + 2\varepsilon c_0 z + \varepsilon^2 c_0^2 z + \varepsilon^2 c_0 \sigma_0(t) + 2\varepsilon \sigma_0(t) \quad (4.8)$$

that $\mathcal{F}_\varepsilon^2(t)K_m(t) \subset K_n(t)$ for any large $n$, $m$ and small $\varepsilon > 0$. Therefore, from (4.6)-(4.8) it follows for any $m$, $n$ and $t \in [0,T]$ that

$$\int_0^t (u'_n, u_n - \mathcal{F}_\varepsilon^2 u_m)_H d\tau + \int_0^t (F u_n, u_n - \mathcal{F}_\varepsilon^2 u_m) d\tau \leq \int_0^t (f_n, u_n - \mathcal{F}_\varepsilon^2 u_m) d\tau, \quad (4.9)$$

and similarly

$$\int_0^t (u'_m, u_m - \mathcal{F}_\varepsilon^2 u_n) d\tau + \int_0^t (F u_m, u_m - \mathcal{F}_\varepsilon^2 u_n) d\tau \leq \int_0^t (f_m, u_m - \mathcal{F}_\varepsilon^2 u_n) d\tau. \quad (4.10)$$

We observe from the energy estimate for (4.6) that $\{u_m\}$ is bounded in $L^p(0,T;V)$ and $C([0,T];H)$, namely there is a constant $C_3 > 0$ such that

$$|u_m|_{L^p(0,T;V)} + |u_m|_{C([0,T];H)} \leq C_3, \quad \forall m. \quad (4.11)$$

In fact, fixing a number $n$, we get an estimate of the form (4.11) by applying the Gronwall’s inequality to (4.10).

Now substitute (4.8) with $z = u_m$ and $z = u_n$ in (4.9) and (4.10), respectively. Then the sum of resultants gives an inequality of the form

$$\int_0^t (u'_n - u'_m, u_n - u_m)_H d\tau + \int_0^t (F u_n - F u_m, u_n - u_m) d\tau \leq \int_0^t (f_n - f_m, u_n - u_m) d\tau + \varepsilon (D_{1,n,m}(t) + D_{2,n,m}(t)), \quad \text{where}$$

$$D_{1,n,m}(t) := (2 + \varepsilon c_0) \int_0^t \{\langle u'_n, c_0 u_m + \sigma_0 \rangle + \langle u'_m, c_0 u_n + \sigma_0 \rangle\} d\tau$$

and

$$D_{2,n,m}(t) := (2 + \varepsilon c_0) \int_0^t \{\langle F u_n - f_n, c_0 u_m + \sigma_0 \rangle + \langle F u_m - f_m, c_0 u_n + \sigma_0 \rangle\} d\tau.$$

Here, by integration by parts, $D_{1,n,m}$ is arranged in the form:

$$D_{1,n,m}(t) = (2 + \varepsilon c_0) c_0 \{\langle u_n(t), u_m(t) \rangle - \langle u_n(0), u_m(0) \rangle\}$$

$$+ (2 + \varepsilon c_0) \{\langle u_n(t) + u_m(t), \sigma_0(t) \rangle - \langle u_n(0) + u_m(0), \sigma_0(0) \rangle\}$$

$$- (2 + \varepsilon c_0) \int_0^t \langle \sigma'_0, u_n + u_m \rangle d\tau$$

for all $t \in [0,T]$, so that it follows from (4.11) that $D_{1,n,m}(t)$ is dominated by a positive constant $D_1$ independent of $m, n$ and $t \in [0,T]$; for instance, $D_1 := 6c_0C_3^2 +
Therefore, putting \( C_4 := D_1 + D_2 \), we obtain
\[
\frac{1}{2} |u_n(t) - u_m(t)|^2_H + \int_0^t \langle Fu_n - Fu_m, u_n - u_m \rangle d\tau \\
\leq \frac{1}{2} |u_{0,n} - u_{0,m}|^2_H + \int_0^t \langle f_n - f_m, u_n - u_m \rangle d\tau + \varepsilon C_4.
\]
By letting \( n, m \to \infty \) and \( \varepsilon \downarrow 0 \) in the above inequality we see that for some \( u \in C([0, T]; H) \cap L^p(0, T; V) \) such that \( u_n \to u \) in \( C([0, T]; H) \) and weakly in \( L^p(0, T; V) \) and
\[
0 \leq \int_{t_1}^{t_2} \langle Fu_n - Fu_m, u_n - u_m \rangle dt \leq \int_0^T \langle Fu_n - Fu_m, u_n - u_m \rangle dt \to 0 \quad \text{as} \quad n, m \to \infty \quad (4.12)
\]
for all \( 0 \leq t_1 \leq t_2 \leq T \). Taking a subsequence of \( \{n\} \) if necessary, we have
\[
Fu_n \to \ell^* \quad \text{weakly in} \quad L^{p'}(0, T; V^*) \quad \text{for some} \quad \ell^* \in L^{p'}(0, T; V^*),
\]
which imply by (4.12)
\[
\limsup_{n \to \infty} \int_{t_1}^{t_2} \langle Fu_n, u_n \rangle dt \leq \int_{t_1}^{t_2} \langle \ell^*, u \rangle dt, \quad \text{for all} \quad 0 \leq t_1 \leq t_2 \leq T.
\]
From the maximal monotonicity of \( F \) we obtain that \( \ell^* = Fu \) and
\[
\lim_{n \to \infty} \int_{t_1}^{t_2} \langle Fu_n, u_n \rangle dt = \int_{t_1}^{t_2} \langle Fu, u \rangle dt, \quad \text{for all} \quad 0 \leq t_1 \leq t_2 \leq T.
\]
Consequently, \( f_n - Fu_n \to f - Fu \) weakly in \( L^{p'}(0, T; V^*) \) and
\[
\lim_{n \to \infty} \int_{t_1}^{t_2} \langle f_n - Fu_n, u_n \rangle dt = \int_{t_1}^{t_2} \langle f - Fu, u \rangle dt, \quad \text{for all} \quad 0 \leq t_1 \leq t_2 \leq T.
\]
Therefore, by definition, \( f - Fu \in \tilde{L}_{uo} u \).

Thus we have seen that the range of \( \tilde{L}_{uo} + F \) is the whole of \( L^{p'}(0, T; V^*) \). Since \( \tilde{L}_{uo} \) is monotone by Corollary 4.1, we conclude that it is maximal monotone from \( L^p(0, T; V) \) into \( L^{p'}(0, T; V^*) \). From the definition of \( \tilde{L}_{uo} \) we see that \( D(\tilde{L}_{uo}) \subset \{ u \in C([0, T]; H) \cap \mathcal{K}, u(0) = u_0 \} \).

Now it follows from the Lemma 4.2 that \( \tilde{L}_{uo} = L_{uo} \), and Theorem 3.1 is obtained.

**Proof of Theorem 3.2.** (1) is obtained from the fact that \( \tilde{L}_{uo} = L_{uo} \). Next, we prove (2). Corresponding to \( f \in L_{uo} u \), choose \( \{ \{ K_n(t) \} \} \subset \Phi_S, \{ u_n \} \) with \( u_n \in \mathcal{K}_{n,0} \) and \( \{ f_n \} \subset L^{p'}(0, T; V^*) \) so that conditions (3.5)-(3.9) hold. Take as a test function \( v \) in (3.8)
\[
v := \begin{cases} 
\mathcal{F}_\varepsilon \eta & \text{on } [t_1, t_2], \\
u_n & \text{on } [0, t_1) \cup (t_2, T],
\end{cases}
\]
for any \( \eta \in \mathcal{K}_0 \) and small \( \varepsilon > 0 \) to obtain

\[
\int_{t_1}^{t_2} \langle u'_n - f_n, u_n - \mathcal{F}_\varepsilon \eta \rangle dt \leq 0.
\]

Applying integration by parts to this inequality and substitute the expression of \( \mathcal{F}_\varepsilon \eta \) in it, we see that

\[
\int_{t_1}^{t_2} \langle \eta' + c_0 \eta' + \varepsilon \sigma_0 - f_n, u_n - \mathcal{F}_\varepsilon \eta \rangle dt + \frac{1}{2} |u_n(t_2) - \mathcal{F}_\varepsilon(t_2)\eta(t_2)|^2_H \leq \frac{1}{2} |u_n(t_1) - \mathcal{F}_\varepsilon(t_1)\eta(t_1)|^2_H.
\]

Therefore, letting \( n \to \infty \) and \( \varepsilon \to 0 \) yield (3.10).

Next we show (3.11). Choose \( \{K_{1,n}\} \in \Phi_S, \{u_{1,n}\} \) and \( \{f_{1,n}\} \) so that conditions (3.5)-(3.9) hold corresponding to \( f_1 \in L_{u_0}u_1 \) as well as \( \{K_{2,m}\} \in \Phi_S, \{u_{2,m}\} \) and \( \{f_{2,m}\} \) corresponding to \( f_2 \in L_{u_0}u_2 \). Noting that \( \mathcal{F}_\varepsilon^2 u_{1,n} \in \mathcal{K}_{2,m} \) as well as \( \mathcal{F}_\varepsilon^2 u_{2,m} \in \mathcal{K}_{1,n} \) for all large \( n, m \), we observe by taking

\[
v := \begin{cases} 
\mathcal{F}_\varepsilon^2 u_{1,n} & \text{on } [t_1, t_2], \\
u_{2,m} & \text{on } [0, t_1) \cup (t_2, T],
\end{cases}
\]

in (3.8) for \( u_{2,m} \) that

\[
\int_{t_1}^{t_2} \langle u'_{2,m} - f_{2,m}, u_{2,m} - \mathcal{F}_\varepsilon^2 u_{1,n} \rangle dt \leq 0. \tag{4.13}
\]

Similarly, for large \( n, m, \)

\[
\int_{t_1}^{t_2} \langle u'_{1,n} - f_{1,n}, u_{1,n} - \mathcal{F}_\varepsilon^2 u_{2,m} \rangle dt \leq 0. \tag{4.14}
\]

Substitute the expression of \( \mathcal{F}_\varepsilon^2 u_{1,n} \) and \( \mathcal{F}_\varepsilon^2 u_{2,m} \) in (4.13) and (4.14) and add them to get

\[
\int_{t_1}^{t_2} \langle u'_{1,n} - u'_{2,m}, u_{1,n} - u_{2,m} \rangle dt \leq \int_{t_1}^{t_2} \langle f_{1,n} - f_{2,m}, u_{1,n} - u_{2,m} \rangle dt + \varepsilon C_5,
\]

where \( C_5 \) is a positive constant independent \( t_1, t_2, \varepsilon \) and \( n, m \). Hence,

\[
\frac{1}{2} |u_{1,n}(t_2) - u_{2,m}(t_2)|^2_H \leq \frac{1}{2} |u_{1,n}(t_1) - u_{2,m}(t_1)|^2_H + \int_{t_1}^{t_2} \langle f_{1,n} - f_{2,m}, u_{1,n} - u_{2,m} \rangle dt + \varepsilon C_5,
\]

and we obtain (3.11) by passing to the limit \( n, m \to \infty \) and \( \varepsilon \to 0 \).

\[\square\]

**Remark 4.1.** In Hilbert spaces similar operators to \( L_{u_0} \) were considered in the time-independent case \( K = K(t) \) (cf. [8]) and it was generalized to the time-dependent case \( K(t) \) (cf. [21]). In the Banach space set-up (cf. [19]), the similar results were discussed, too.

**Remark 4.2.** Theorem 3.1 gives a generalization of the results of [19, 21] in a class of weak variational inequalities. Moreover it is expected to compose \( L_{u_0} \) for various constraint set
5. Perturbations of semimonotone type

We assume that $H$, $V$ and $W$ be the same as in section 2; $V$ is dense in $H$ with compact injection and $W$ is separable and dense in $V$ with continuous injection, and $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $A(t, v, u)$ be a singlevalued mapping from $[0, T] \times H \times V$ into $V^*$, and assume that:

(a) (Boundedness) There are positive constants $c_1$, $c_2$ such that

$$|A(t, v, u)|_{V^*} \leq c_1 |u|^p_{V} + c_2, \quad \forall (t, v, u) \in [0, T] \times H \times V.$$

(b) (Coerciveness) There are positive constants $c_3$, $c_4$ such that

$$\langle A(t, v, u), u \rangle \geq c_3 |u|^p_{V} - c_4, \quad \forall (t, v, u) \in [0, T] \times H \times V.$$

(c) (Semimonotonicity) For each $v \in H$ and $t \in [0, T]$, the mapping $u \to A(t, v, u)$ is demicontinuous from $D(A(t, v, \cdot)) = V$ into $V^*$ and monotone, namely

$$\langle A(t, v, u_1) - A(t, v, u_2), u_1 - u_2 \rangle \geq 0, \quad \forall u_1, u_2 \in V,$$

Moreover, for each $u \in V$ the mapping $(t, v) \to A(t, v, u)$ is continuous from $[0, T] \times H$ into $V^*$.

We derive some properties of $A(t, v, u)$ from the above conditions.

**Lemma 5.1.** Assume that $t_n \to t$ in $[0, T]$, $v_n \to v$ in $H$, $u_n \to u$ weakly in $V$ and

$$A(t_n, v_n, u_n) \to \alpha^* \text{ weakly in } V^*, \quad \limsup_{n \to \infty} \langle A(t_n, v_n, u_n), u_n \rangle \leq \langle \alpha^*, u \rangle. \quad (5.1)$$

Then $\alpha^* = A(t, v, u)$ and $\lim_{n \to \infty} \langle A(t_n, v_n, u_n), u_n \rangle = \langle \alpha^*, u \rangle$.

**Proof.** By condition (c), for each $t \in [0, T]$ and $v \in H$, the mapping $u \to A(t, v, u)$ is monotone and demicontinuous from $V$ into $V^*$ and hence maximal monotone (cf. [20; Theorem 4.2]). Let $\eta$ be any element in $V$. Then, by assumption (5.1) with (c),

$$\langle \alpha^* - A(t, v, \eta), u - \eta \rangle \geq \limsup_{n \to \infty} \langle A(t_n, v_n, u_n) - A(t_n, v_n, \eta), u_n - \eta \rangle \geq 0.$$

The maximal monotonicity of $\eta \to A(t, v, \eta)$ implies that $\alpha^* = A(t, v, u)$ as well as $\langle A(t_n, v_n, u_n), u_n \rangle \to \langle \alpha^*, u \rangle$. □

The above lemma ensures that $A(t, v, u)$ is continuous from $[0, T] \times H \times V$ into $V^*$, so that for every $v \in L^p(0, T; H)$ and $u \in L^p(0, T; V)$ the function $t \to A(t, v(t), u(t))$ is weakly measurable on $[0, T]$ and hence strongly measurable on $[0, T]$ thanks to the
Lemma 5.2. \( A \) satisfies the following properties:

(i) (Boundedness) \(|A(v,u)|_{L^p(0,T;V^*)} \leq c_1|u|_{L^p(0,T;V)}^T + c_2T \|v\|^p_H \) for all \( v \in L^p(0,T;H) \) and \( u \in L^p(0,T;V) \).

(ii) (Coerciveness) \( \int_0^T \langle A(v,u),u \rangle \geq c_3|u|_{L^p(0,T;V)}^p - c_4T \) for all \( v \in L^p(0,T;H) \) and \( u \in L^p(0,T;V) \).

(iii) (Semimonotonicity) \( A \) is demicontinuous from \( L^p(0,T;H) \times L^p(0,T;V) \) into \( L^p(0,T;V^*) \) and for every \( v \in L^p(0,T;H) \) the mapping \( u \to A(v,u) \) is monotone from \( L^p(0,T;V) \) into \( L^p(0,T;V^*) \), namely,

\[
\int_0^T \langle A(v,u_1) - A(v,u_2), u_1 - u_2 \rangle dt \geq 0, \quad \forall u_1, u_2 \in L^p(0,T;V).
\]

(iv) (Continuity in \( v \)) For every \( v \in L^p(0,T;V) \), the mapping \( v \to A(v,u) \) is continuous from \( L^p(0,T;H) \) into \( L^p(0,T;V^*) \).

(v) Assume that \( v_n \to v \) in \( L^p(0,T;H) \), \( u_n \to u \) weakly in \( L^p(0,T;V) \), \( A(v_n,u_n) \to \alpha^* \) in \( L^p(0,T;V^*) \) and

\[
\limsup_{n \to \infty} \int_0^T \langle A(v_n,u_n),u_n \rangle dt \leq \int_0^T \langle \alpha^*,u \rangle dt.
\]

Then,

\[
\alpha^* = A(v,u), \quad \lim_{n \to \infty} \int_0^T \langle A(v_n,u_n),u_n \rangle dt = \int_0^T \langle A(v,u),u \rangle dt.
\]

The statements (i)-(iv) of Lemma 5.2 are straightforwardly obtained from conditions (a), (b) and (c), and (v) is proved in the same way just as Lemma 5.1.

We are now in a position to state a perturbation result of \( L_{u_0} \).

**Theorem 5.1.** Let \( \mathcal{A} := \mathcal{A}(v,u) \) be given by (5.2). Let \( \{K(t)\} \in \Phi_W \) and \( u_0 \in \overline{K(0)} \) and assume that there is a positive number \( \kappa > 0 \) such that

\[
\kappa B_W(0) \subset K(t), \quad \forall t \in [0,T].
\]

Then, for any \( f \in L^p(0,T;V^*) \) there exists a function \( u \in D(L_{u_0}) \) such that

\[
f \in L_{u_0}u + \mathcal{A}(u,u)
\]
Proof. By Lemma 5.2, for each \( v \in L^p(0, T; V) \) the mapping \( u \rightarrow A(v, u) \) is maximal monotone from \( L^p(0, T; V) \) into \( L^{p'}(0, T; V^*) \), bounded and coercive. Therefore, on account of the well-known result of maximal monotone perturbations (cf. [4, 10, 11, 20]) the range of \( L_{u_0} + \mathcal{A}(v, \cdot) \) is the whole of \( L^{p'}(0, T; V^*) \), namely for any \( f \in L^{p'}(0, T; V^*) \) there is an element \( u \in D(L_{u_0}) \) satisfying

\[
\int_0^T \langle \eta' - (f - A(v, u)), u - \eta \rangle dt \leq \frac{1}{2} |u_0 - \eta(0)|^2_{L^2_H}, \quad \forall \eta \in K_0, \tag{5.6}
\]

where \( K_0 = \{ \eta \in L^p(0, T; V) \mid \eta' \in L^{p'}(0, T; V^*), \, \eta(t) \in K(t) \text{ for a.e. } t \in [0, T] \} \).

Next, we show the uniform estimate for solutions \( u \) of (5.5). To do so, use (3) of Theorem 3.2 for \( 0 \in L_0 \) and \( f - A(v, u) \in L_{u_0}u \). Then we get

\[
\frac{1}{2} |u(t)|^2_H + \int_0^t \langle A(v, u), u \rangle d\tau \leq \frac{1}{2} |u_0|^2_H + \int_0^t \langle f, u \rangle d\tau, \quad \forall t \in [0, T].
\]

By condition (b), this inequality implies that

\[
|u(t)|^2_H + 2c_3 \int_0^t |u|^2_V d\tau - 2c_4 t \leq |u_0|^2_H + 2 \int_0^t |f|_{V^*} |u|_V d\tau, \quad \forall t \in [0, T],
\]

and by Young’s inequality

\[
|u(t)|^2_H + c_3 \int_0^t |u|^2_V d\tau \leq 2c_3 T + |u_0|^2_H + c_5 \int_0^T |f|^2_{V^*} d\tau, \quad \forall t \in [0, T],
\]

where \( c_5 \) is a positive constant, for instance \( c_5 := \frac{2^{2p'}}{p' c_3^{p' - 1}} \). We derive from the above estimate that

\[
\sup_{0 \leq t \leq T} |u(t)|_H + |u|_{L^p(0, T; V)} \leq M_2, \tag{5.7}
\]

\( M_2 \) being a positive constant, for instance \( M_2 := 2 \left\{ 1 + 2c_3 T + |u_0|^2_H + c_5 \int_0^T |f|^2_{V^*} d\tau \right\} \). Moreover, with the same notation as above, we have by condition (a)

\[
|f - A(v, u)|_{L^{p'}(0, T; V^*)} \leq |f|_{L^{p'}(0, T; V^*)} + |A(v, u)|_{L^{p'}(0, T; V^*)},
\]

\[
\leq |f|_{L^{p'}(0, T; V^*)} + c_1 |u|^2_{L^p(0, T; V)} + c_2 T^{\frac{1}{p'}} =: M_3,
\]

namely it follows that for all \( v \in L^p(0, T; V) \) and \( u = S_v \)

\[
\inf_{\ell^* \in L_{u_0}u} |\ell^*|_{L^{p'}(0, T; V^*)} \leq M_3 \tag{5.8}
\]
We take as a constant \( M_0 \) of Theorem 2.1 the sum \( M_2 + (1 + M_2 + c_6)M_3 \), where \( c_6 \) is a positive constant satisfying \(| \cdot |_{L^1(0,T;W^\ast)} \leq c_6| \cdot |_{L^p'(0,T;V^\ast)}\), and consider the set

\[
X_0 := \text{conv}[Z(\kappa, M_0, u_0)].
\]

By Theorem 2.1, it is non-empty, closed, convex and compact in \( L^p(0, T; H) \). We are going to apply the Schauder fixed point theorem to \( S \) in \( X_0 \).

First we check that \( S(X_0) \subset X_0 \). In fact, for each \( v \in X_0 \) the solution \( u := Sv \) of (5.5) satisfies (5.6) and estimate (5.7)-(5.8), so that \( u \in Z(\kappa, M_0, u_0) \subset X_0 \). Thus \( S \) maps \( X_0 \) into itself. Next we show the continuity of \( S \) in \( X_0 \) with respect to the topology of \( L^p(0, T; H) \). Assume that \( v_n \in X_0 \) and \( v_n \to v \) in \( L^p(0, T; H) \) (as \( n \to \infty \)). Clearly, \( v_n \to v \) weakly in \( L^p(0, T; V) \) and \( v \in X_0 \). Putting \( u_n = Sv_n \), we infer from the compactness of \( X_0 \) that \( \{v_n\} \) and \( \{u_n\} \) converge in \( L^p(0, T; H) \) and in \( H \) a.e. on \([0, T]\) to some functions \( v \) and \( u \in X_0 \), respectively, for a certain subsequence \( \{n_k\} \) of \( \{n\} \). In this case, with the notations

\[
f = \ell_{n_k}^* + \mathcal{A}(v_{n_k}, u_{n_k}), \quad \ell_{n_k}^* \in L_{u_0}u_{n_k},
\]

we may assume by the boundedness of \( \mathcal{A} \) that

\[
\mathcal{A}(v_{n_k}, u_{n_k}) \to \alpha^* \text{ weakly in } L^{p'}(0, T; V^\ast)
\]

for some \( \alpha^* \in L^{p'}(0, T; V^\ast) \), and

\[
\lim_{k \to \infty} \int_0^T \langle \mathcal{A}(v_{n_k}, u_{n_k}), u_{n_k} - u \rangle dt \text{ exists.}
\]

In order to prove that \( \ell^* := f - \alpha^* \in L_{u_0}u \), we observe

\[
0 = \lim_{k \to \infty} \int_0^T \langle f, u_{n_k} - u \rangle dt
= \lim_{k \to \infty} \left\{ \int_0^T \langle \ell_{n_k}^*, u_{n_k} - u \rangle dt + \int_0^T \langle \mathcal{A}(v_{n_k}, u_{n_k}), u_{n_k} - u \rangle dt \right\}
= \lim_{k \to \infty} \int_0^T \langle \ell_{n_k}^*, u_{n_k} - u \rangle dt + \lim_{k \to \infty} \int_0^T \langle \mathcal{A}(v_{n_k}, u_{n_k}), u_{n_k} - u \rangle dt.
\]

Now we show \( \lim_{k \to \infty} \int_0^T \langle \ell_{n_k}^*, u_{n_k} - u \rangle dt \leq 0 \) by contradiction. Assuming that

\[
\lim_{k \to \infty} \int_0^T \langle \ell_{n_k}^*, u_{n_k} - u \rangle dt > 0,
\]

we have by (iii) and (vi) of Lemma 5.2

\[
0 > \lim_{k \to \infty} \int_0^T \langle \mathcal{A}(v_{n_k}, u_{n_k}), u_{n_k} - u \rangle dt
= \lim_{k \to \infty} \left\{ \int_0^T \langle \mathcal{A}(v_{n_k}, u_{n_k}) - \mathcal{A}(v_{n_k}, u), u_{n_k} - u \rangle dt + \int_0^T \langle \mathcal{A}(v_{n_k}, u), u_{n_k} - u \rangle dt \right\}
\geq \limsup_{k \to \infty} \int_0^T \langle \mathcal{A}(v_{n_k}, u), u_{n_k} - u \rangle dt = 0,
\]

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which is a contradiction. Consequently, we have
\[ \lim_{k \to \infty} \int_0^T \langle \ell_{nk}^*, u_{nk} - u \rangle dt \leq 0. \]
as well as \( \ell_{nk}^* = f - A(v_{nk}, u_{nk}) \leq L u_{nk}, u_{nk} \to u \) weakly in \( L^p(0, T; V) \) and \( \ell_{nk}^* \to \ell^* := f - \alpha^* \) weakly in \( L^p'(0, T; V^*) \). By the maximal monotonicity of \( L_{u_0} \), we obtain that \( \ell^* = f - \alpha^* \in L u_0 u \), namely \( f \in L u_0 u + \alpha^* \), and \( \lim_{k \to \infty} \int_0^T \langle \ell_{nk}^*, u_{nk} \rangle dt = \int_0^T \langle \ell^*, u \rangle dt \) or equivalently \( \lim_{k \to \infty} \int_0^T \langle A(v_{nk}, u_{nk}), u_{nk} \rangle dt = \int_0^T \langle \alpha^*, u \rangle dt \). Therefore it is concluded by (v) of Lemma 5.2 that \( \alpha^* = A(v, u) \) and \( u = S \psi \). By the uniqueness of solution to (5.5), we see that \( u_n \to u \) in \( L^p(0, T; H) \) without extracting any subsequence from \( \{u_n\} \). Thus \( S \) is continuous in \( X_0 \) in the topology of \( L^p(0, T; H) \), so that \( S \) possesses at least one fixed point \( u := S \psi \), which gives a solution of (5.4). \( \square \)

**(Application 1)**

The model problem mentioned in the introduction is here discussed precisely in our framework.

Let \( \Omega \) be a bounded and smooth domain in \( \mathbb{R}^N \) and \( Q := \Omega \times (0, T), \; 0 < T < \infty \). We use our abstract theorems in the set-up

\[ H := L^2(\Omega) \times L^2(\Omega), \; V := H^1_0(\Omega) \times H^1_0(\Omega), \; W := W^{1,q}_0 \times W^{1,q}_0, \; \max\{N, 2\} < q < \infty. \]

Hence \( V^* = H^{-1}(\Omega) \times H^{-1}(\Omega), \; V \subset H \subset V^* \subset W^* \) and \( W \subset C(\overline{\Omega}) \times C(\overline{\Omega}) \) with compact embeddings.

Let \( \psi = \psi(x, t) \) be an obstacle function prescribed in \( C(\overline{\Omega}) \) so that \( \psi \geq c_\psi \) on \( \overline{\Omega} \) for a positive constant \( c_\psi \), and define a constraint set \( K(t) \) by

\[ K(t) := \{ [\xi_1, \xi_2] \in V \mid |\xi_1| + |\xi_2| \leq \psi(\cdot, t) \text{ a.e. in } \Omega \}, \; \forall t \in [0, T]. \]

Then, by virtue of Theorem 3.1, the maximal monotone mapping \( L_{u_0} : D(L_{u_0}) \subset L^2(0, T; V) \to L^2(0, T; V^*) \) is well defined for any given initial datum \( u_0 := [u_{10}, u_{20}] \in K(0) \).

Now, we define a nonlinear mapping \( A(t, v, u) : [0, T] \times H \times V \to V^* \) by

\[ \langle A(t, v, u), \xi \rangle := \int_{\Omega} \{ a_1(x, t, v) \nabla u_1 \cdot \nabla \xi_1 + a_2(x, t, v) \nabla u_2 \cdot \nabla \xi_2 \} dx, \]

for \( v := [v_1, v_2] \in H, \; u := [u_1, u_2] \in V, \; \xi = [\xi_1, \xi_2] \in V, \; t \in [0, T], \)

where \( a_1(x, t, v) \) and \( a_2(x, t, v) \) are functions satisfying the Carathéodory condition on \( \overline{\Omega} \times [0, T] \times \mathbb{R}^2 \) and

\[ a_* \leq a_i(x, t, v) \leq a^* \text{ for a.e. } (x, t) \in \Omega \times (0, T) \text{ and all } v \in \mathbb{R}^2, \; i = 1, 2, \]

for positive constants \( a_*, a^* \). Under the above assumptions, we easily check the conditions (a), (b) and (c) in section 5 as well as condition (5.3) of Theorem 5.1 by the strict positiveness of \( c_\psi \). Accordingly we can apply Theorem 5.1 to solve our model problem.
for given data \( u_0 := [u_{01}, u_{02}] \in \overline{K(0)} \) and \( f = [f_1, f_2] \in L^2(0, T; V^*) \) in the form \( f \in L_{u_0}u + A(u, u) \). This functional inclusion is equivalent to the following weak variational form:

\[
\int_Q \{ \xi_1(t)(u_1 - \xi_1) + \xi_2(t)(u_2 - \xi_2) \} dxdt + \int_Q \{ a_1(x, t, u) \nabla u_1 \cdot \nabla (u_1 - \xi_1) + a_2(x, t, u) \nabla u_2 \cdot \nabla (u_2 - \xi_2) \} dxdt \\
\leq \int_0^T \langle f, u - \xi \rangle dt + \frac{1}{2}\{ |u_{10} - \xi_1(0)|^2_{L^2(\Omega)} + |u_{20} - \xi_2(0)|^2_{L^2(\Omega)} \},
\]

\( \forall \xi = [\xi_1, \xi_2] \in L^2(0, T; V) \cap W^{1,2}(0, T; H), \ |\eta_1| + |\eta_2| \leq \psi \ a.e. \ on \ Q. \)

**Application 2**

Finally, we consider parabolic variational inequalities with gradient constraints. Let \( \Omega \) be a bounded and smooth domain in \( \mathbb{R}^N \) and put

\[
H := L^2(\Omega), \ V := H_0^1(\Omega), \ W := W_0^{2,q}(\Omega), \ \text{max}\{N, 2\} < q < \infty.
\]

For any given obstacle function \( \psi \in C(\overline{Q}) \) such that \( \psi \geq c_\psi \) on \( \overline{Q} \) for a positive constant \( c_\psi \) we define a time-dependent constraint \( K(t) \) by

\[
K(t) := \{ z \in V \mid |\nabla z(x)| \leq \psi(x, t) \text{ for a.e. } x \in \Omega \}, \ \forall t \in [0, T].
\]

Now, choose a sequence \( \{ \psi_n \} \subset C^1(\overline{Q}) \) so that

\[
\psi_n \geq c_\psi \text{ on } \overline{Q}, \ \psi_n \rightarrow \psi \text{ in } C(\overline{Q}) \text{ as } n \rightarrow \infty,
\]

and consider approximate constraint sets

\[
K_n(t) := \{ z \in V \mid |\nabla z(x)| \leq \psi_n(x, t) \text{ for a.e. } x \in \Omega \}, \ \forall t \in [0, T].
\]

Then we have:

(i) \( \{ K_n(t) \} \in \Psi_S. \) The proof is quite similar to (i) of Example 3.3. We refer to [22; Lemma 3.1] for the detailed proof.

(ii) \( \{ K(t) \} \in \Psi_W. \) It is enough to check that \( K_n(t) \implies K(t) \) on \( [0, T], \) Just as in Example 3.3, this is obtained by using the mapping

\[
F_\varepsilon z = (1 - \varepsilon)z, \ \forall z \in V, \ \forall \text{ small } \varepsilon > 0.
\]

(iii) Condition (5.3) is satisfied by the strict positiveness of \( \psi. \) More precisely, if \( z \in W_0^{2,q}(\Omega), \) then \( |\nabla z| \in W_0^{1,q}(\Omega) \subset C(\overline{\Omega}) \) and hence there is a positive constant \( \kappa \) such that \( \kappa |\nabla z| \leq c_\psi \) on \( Q \) for all \( z \in B_W(0). \) Thus (5.3) holds.
Therefore, associated with initial datum \( u_0 \in \overline{K(0)} \), the maximal monotone mapping \( L_{u_0} : D(L_{u_0}) \subset L^2(0,T;V) \to L^2(0,T;V^*) \) is well defined on account of Theorem 3.1. Next we introduce a semimonotone operator \( A := A(t,v,u) : [0,T] \times H \times V \to V^* \) by:

\[
\langle A(t,v,u), \xi \rangle := \int_{\Omega} \{a(x,t,v)\nabla u \cdot \nabla \xi + b(x,t,v)\xi\} dx
\]

\( \forall v \in H, \forall u \in V, \forall \xi \in V, \forall t \in [0,T], \)

where \( a(x,t,v) \) and \( b(x,t,v) \) are functions satisfying the Carathéodory condition on \( \overline{\Omega} \times [0,T] \times \mathbb{R} \) and

\[
a_* \leq a(x,t,v) \leq a^*, \quad -b_* \leq b(x,t,v) \leq b^* \quad \text{for a.e.} \ (x,t) \in \Omega \times (0,T), \quad \text{and all} \ v \in \mathbb{R},
\]

for positive constants \( a_*, a^*, b_*, b^* \).

We easily check conditions (a), (b) and (c) for this operator \( A \). Accordingly we can apply Theorem 5.1 to solve \( f \in L_{u_0}u + A(u,u) \) for given data \( u_0 \in \overline{K(0)} \) and \( f \in L^2(0,T;V^*) \). This is equivalent to the weak variational form:

\[
u \in L^2(0,T;V) \cap C([0,T];H), \ |\nabla u| \leq \psi \ a.e. \ on \ Q, \ u(0) = u_0; \]

\[
\int_Q \xi_t(u - \xi) dx dt + \int_Q \{a(x,t,u)\nabla u \cdot \nabla (u - \xi) + b(x,t,u)(u - \xi)\} dx dt
\leq \int_0^T \langle f, u - \xi \rangle dt + \frac{1}{2}|u(0) - \xi(0)|^2_{L^2(\Omega)},
\]

\( \forall \xi \in L^2(0,T;V) \cap W^{1,2}(0,T;H), \ |\nabla \eta| \leq \psi \ a.e. \ on \ Q. \)

**Remark 5.1.** The similar technique in Application 2 is available for the variational inequalities arising in models of superconductivity with gradient constraints or hydrodynamics with velocity constraints; see [15, 16, 28, 29, 30].

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