The uncertainty in $\alpha_s(M_Z^2)$ determined from hadronic tau decay measurements

C.J. Maxwell and D.G. Tonge

Centre for Particle Theory, University of Durham
South Road, Durham, DH1 3LE, UK

Abstract

We show that QCD Minkowski observables such as the $e^+e^- R$-ratio and the hadronic tau decay $R_\tau$ are completely determined by the effective charge (EC) beta-function, $\rho(x)$, corresponding to the Euclidean QCD vacuum polarization Adler $D$-function, together with the next-to-leading order (NLO) perturbative coefficient of $D$. An efficient numerical algorithm is given for evaluating $R, R_\tau$ from a weighted contour integration of $D(se^{i\theta})$ around a circle in the complex squared energy $s$-plane, with $\rho(x)$ used to evolve in $s$ around the contour. The EC beta-function can be truncated at next-to-NLO (NNLO) using the known exact perturbative calculation or the uncalculated N$^3$LO and higher terms can be approximated by the portion containing the highest power of $b$, the first QCD beta-function coefficient. The difference between the $R, R_\tau$ constructed using the NNLO and “leading-b” resummed versions of $\rho(x)$ provides an estimate of the uncertainty due to the uncalculated higher order corrections. Simple numerical parametrizations are given to facilitate these fits. For $R_\tau$ we estimate an uncertainty $\delta\alpha_s(m_\tau^2) \simeq 0.01$, corresponding to $\delta\alpha_s(M_Z^2) \simeq 0.002$. This encouragingly small uncertainty is much less than rather pessimistic estimates by other authors based on analogous all-orders resummations, which we demonstrate to be extremely dependent on the chosen renormalization scheme, and hence misleading.

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1 Introduction

There has been extensive recent interest [1–3] in the possibility of using measurements of $R_\tau$, the total hadronic decay width of the $\tau$ lepton normalized to the leptonic decay width, for precise determination of the renormalized strong coupling $\alpha_s(M_\tau^2)$ (or more fundamentally $\Lambda_{\overline{MS}}$). For this purpose $R_\tau$ apparently possesses a number of advantages compared with other QCD observables. It is an inclusive quantity which can be computed in QCD using the operator product expansion (OPE) supplemented by analyticity [4–7]. It has been calculated in QCD perturbation theory to next-to-next-to-leading order (NNLO) $O(\alpha_s^3)$ [8, 9]. Power corrections are expected to be small [5, 6], and since the $\tau$ mass is below the threshold for charmed hadron production only the light quarks $u, d, s$ are active, so QCD with $N_f = 3$ massless quark flavours should be applicable. $R_\tau$ can be rather accurately determined from the measured electronic branching ratio of the $\tau$ or from the $\tau$ lifetime [10]. In evolving up in energy scale from $\alpha_s(m_\tau^2)$ to $\alpha_s(M_\tau^2)$, which is customarily quoted in global comparisons, the fractional error in $\alpha_s(\Lambda_{\overline{MS}})$ is reduced.

Measurement of the hadronic width of the $Z^0$ to directly determine $\alpha_s(M_\tau^2)$ shares the same advantages of being inclusive, calculated to NNLO in perturbation theory and having small power corrections, but suffers significant corrections from heavy quark masses, and much larger systematic experimental errors.

Despite these undoubted advantages possessed by $R_\tau$ as a means of determining $\alpha_s$, the relatively low energy scale involved, $s = m_\tau^2$, might lead one to expect sizeable corrections from uncalculated $O(\alpha_s^4)$ and higher orders in perturbation theory. To assess the effect of these terms with our present limited state of knowledge one can employ a, necessarily approximate, all-orders resummation of the QCD perturbation series. A well-motivated framework for this is provided by the leading-$b$ approximation [11, 12], also sometimes referred to as naive non-abelianization [13, 14]. In this approach the portion of perturbative coefficients containing the highest power of $b = \frac{1}{6}(11N-2N_f)$, the first beta-function coefficient for SU($N$) QCD with $N_f$ active quark flavours, is resummed to all-orders. This can be accomplished given exact large-$N_f$ all-orders results [15–17]. This technique has been applied to the QCD vacuum polarization Adler $D$-function [13, 15], and its Minkowski continuations, the $e^+e^- R$-ratio and $R_\tau$. Deep inelastic scattering (DIS) sum rules [17] and heavy quark decay widths and pole masses have also been discussed [18, 19].

In several recent papers it has been claimed that applying the leading-$b$ resummation to $R_\tau$ indicates rather large perturbative uncertainties [18, 20, 21]. Indeed the estimated uncertainty in $\alpha_s(M_\tau^2)$ is of the same order as that normally quoted in determinations from jet observables at LEP and SLD [22].

In a recent paper [23] we have pointed out that a straightforward resummation of the leading-$b$ terms of the kind employed in references [18, 20, 21], is renormalization scheme (RS) dependent. This occurs because the compensation mechanism between the renormalization group (RG) improved coupling and the perturbative coefficients is destroyed by retaining only the leading-$b$ terms. As a result the ‘naive’ leading-$b$ resummation is not RS-invariant under the full QCD renormalization group (RG). Whilst at large energies the resulting ambiguities are mild, at $s = m_\tau^2$ this RS dependence is serious and in our view invalidates the rather pessimistic conclusions of these references regarding the likely
uncertainty in $\alpha_s(M_Z^2)$ determined from $R_\tau$.

In reference [23] we proposed an improved RS-invariant resummation based on approximating the unknown effective charge (EC) beta-function coefficients by the portion containing the highest power of $b$. Approximated perturbative coefficients in any RS can then be obtained using the exact QCD RG. The leading-$b$ effective charge beta-function can be resummed using exact all-orders large-$N_f$ results.

The difference between the exact NNLO result for $R_\tau$ in the effective charge RS, and the RS-invariant all-orders resummation indicates a rather small uncertainty due to the approximated higher order terms, and the estimated uncertainty in $\alpha_s(M_Z^2)$ is correspondingly small, $\delta\alpha_s(M_Z^2) \simeq 0.001$.

In this paper we wish to explore the perturbative uncertainty in $R_\tau$ in somewhat more detail. Both $R_\tau$ and $e^+e^- R$-ratio can be represented by a contour integral involving $D(se^{i\theta})$, where $D(-s)$ is the Euclidean Adler $D$-function, around a circle, cut along the positive real axis, $\theta = -\pi$ to $\theta = \pi$, in the complex $s$-plane [4]. Here $s = m_\tau^2$ for $R_\tau$. Conventional perturbation theory involves an expansion in $\alpha_s(s)$ obtained by re-expressing $\alpha_s(se^{i\theta})$ as an expansion in $\alpha_s(s)$ which is then truncated. Alternatively one can simply numerically perform the contour integration for the $\alpha_k^s(se^{i\theta})$ terms up to a given order [1]. This procedure includes in addition to conventional fixed-order perturbation theory a resummation of analytical continuation terms. A subset of these terms involve powers of the first beta-function coefficient, $b$, together with $\pi^2$ factors, and are resummed to all-orders in the leading-$b$ approach. In addition, however, there are potentially large contributions involving higher beta-function coefficients [24]. It would seem sensible, therefore to perform the improved RS-invariant resummation for $D(se^{i\theta})$, and numerically evaluate the contour integral. In this way additional analytical continuation terms not captured in the leading-$b$ resummation are included exactly. This can then be compared with the exact NNLO result for $D(se^{i\theta})$ in the EC scheme, with the contour integral again numerically evaluated. Since in both cases the analytical continuation terms are resummed the difference should be indicative of the effect of the approximated higher effective charge RS invariants for $D$ beyond NNLO.

The plan of the paper is as follows. In section 2 we shall introduce the contour integral representations of the $e^+e^-$ $R$-ratio and $R_\tau$ in terms of the Adler $D$-function. Using Taylor’s theorem we can then expand $R$ and $R_\tau$ in terms of $D(s)$ and its energy derivatives, which in turn can be expressed in terms of the effective charge beta-function for $D$ and its derivatives. These results can be easily used to express the perturbative coefficients of the Minkowski quantities $R$ and $R_\tau$ in terms of those of the Euclidean Adler $D$-function and its effective charge RS invariants. We have compared these with existing expressions available in the literature [24]. We also derive relations between the EC invariants for $R$ and $R_\tau$ and these for $D$. In section 3 we briefly review the basis of the RS-invariant resummation proposal [23]. The contour integrals for the $R$ and $R_\tau$ are evaluated by using Taylor’s theorem successively to evaluate $D$ at a series of values of complex $s$ around the unit circle contour of integration. A Simpson’s rule numerical integration is then performed. The translation of $D$ in complex $s$ involves the effective charge beta-function and its derivatives. This function can be truncated or its leading-$b$ terms resummed [23]. In section 4 fits to the experimental data for $\tilde{R}$ and $\tilde{R}_\tau$ are performed to determine $\alpha_s$ from
fixed-order and resummed perturbation theory. In section 5 we conclude by comparing the resulting values and estimates of the perturbative uncertainty with those suggested by other approaches.

2 Contour integral representation of Minkowski observables

The two quantities with which we shall be concerned are defined as follows.

The $e^+e^- R$-ratio is the observable

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$ (1)

In SU($N$) QCD perturbation theory

$$R(s) = N \sum_f Q_f^2 \left(1 + \frac{3}{4}C_F \tilde{R}\right) + \left(\sum_f Q_f\right)^2 \tilde{R},$$ (2)

where $Q_f$ denotes the electric charge of the quarks and the summation is over the flavours accessible at a given energy. $s$ is the physical timelike Minkowski squared momentum transfer. The SU($N$) Casimirs are defined as $C_A = N$, $C_F = (N^2 - 1)/2N$.

$\tilde{R}$ denotes the perturbative corrections to the parton model result and has the formal expansion

$$\tilde{R}(s) = a + r_1a^2 + r_2a^3 + \cdots + r_ka^{k+1} + \cdots,$$ (3)

where $a \equiv \alpha_s(\mu^2)/\pi$ denotes the renormalization group (RG) improved coupling. The $\overline{\text{MS}}$ scheme with $\mu^2 = s$ is often chosen. The $\tilde{R}$ contribution first enters at $O(a^3)$ due to the existence of diagrams of the “light-by-light” type.

The ratio $R_\tau$ is defined analogously using the total $\tau$ hadronic width as

$$R_\tau \equiv \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons})}{\Gamma(\tau \to \nu_\tau e^-\bar{\nu}_e)}.$$ (4)

Its perturbative expansion has the form

$$R_\tau = N(|V_{ud}|^2 + |V_{us}|^2) S_{\text{EW}} \left[1 + \frac{5}{12} \frac{\alpha(m_\tau^2)}{\pi} + \tilde{R}_\tau\right],$$ (5)

where the $V_{ud}$ and $V_{us}$ are CKM mixing matrix elements, with $|V_{ud}|^2 + |V_{us}|^2 \approx 1$. Since the energy scale, $s = m_\tau^2$, of the process lies below the threshold for charmed hadron production only three flavours $u, d, s$, are active. $\alpha(m_\tau^2)$ denotes the electromagnetic coupling $[25]$ and $S_{\text{EW}} \approx 1.0194$ $[26]$ denotes further electroweak corrections. $\tilde{R}_\tau$ has a perturbative expansion of the form of equation (3) with coefficients which we shall denote $r_k^\tau$. There are no “light-by-light” corrections for $R_\tau$ since $(\Sigma Q_f)^2 = 0$ for $u, d, s$ active quark flavours.
These two Minkowski quantities can both be expressed in terms of the transverse part of the correlator $\Pi(s)$ of two vector currents in the Euclidean region,

$$\Pi(s)(q_\mu q_\nu - g_\mu_\nu q^2) = 4\pi^2 i \int d^4x e^{iq\cdot x} \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | 0 \rangle ,$$  \hspace{1cm} (6)

where $s = -q^2 > 0$. In order to avoid an unspecified constant, it is convenient to consider the related Adler $D$-function,

$$D(s) = -s \frac{d}{ds} \Pi(s).$$  \hspace{1cm} (7)

In perturbation theory $D$ can be written in the form of equation (2) involving perturbative corrections $\tilde{D}$ with an expansion as in equation (3) involving coefficients $d_k$, and “light-by-light” corrections $\tilde{\tilde{D}}$.

$\hat{R}$, $\hat{R}_\tau$ and related Minkowski quantities such as spectral moments \[1\] can all be written in terms of a weighted contour integral of $\tilde{D}(se^{i\theta})$ around a unit circle in the complex $s$-plane \[1\].

Denoting such a generic Minkowski observable as $\hat{R}$ we have

$$\hat{R}(s_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta W(\theta) \tilde{D}(s_0 e^{i\theta})$$  \hspace{1cm} (8)

where $W(\theta)$ is a weight function which depends on the observable $\hat{R}$. For $\hat{R}$ we have $W(\theta) = 1$, and for $\hat{R}_\tau$

$$W_\tau(\theta) = (1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta}) ,$$  \hspace{1cm} (9)

and $s_0 = m^2_\tau$.

A novel representation for $\hat{R}$ can be obtained by using Taylor’s theorem to expand $\tilde{D}(se^{i\theta})$ around $s = s_0$. This yields

$$\hat{R}(s_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta W(\theta) \left\{ \tilde{D}(s_0) + \sum_{n=1}^{\infty} i^n \frac{\theta^n}{n!} \frac{d^n}{d \ln s} \tilde{D}(s) \bigg|_{s = s_0} \right\} .$$  \hspace{1cm} (10)

The derivatives in equation (10) can be recast in terms of the effective charge beta-function $\rho(\tilde{D})$ \[27, 28\], and its derivatives. $\rho(\tilde{D})$ is defined by

$$\frac{d \tilde{D}(s)}{d \ln s} = -\frac{b}{2} \rho(\tilde{D})$$

$$\equiv -\frac{b}{2}(\tilde{D}^2 + c\tilde{D}^3 + \rho_2 \tilde{D}^4 + \cdots + \rho_k \tilde{D}^{k+2} + \cdots) .$$  \hspace{1cm} (11)

Here $b = \frac{1}{6}(11N - 2N_f)$ is the first coefficient of the beta-function, and

$$c = \left[ \frac{7}{8} C_A^2 - \frac{11}{8} \frac{C_A C_F}{b} + \frac{5}{4} C_A + \frac{3}{4} C_F \right] ,$$  \hspace{1cm} (12)

is the second universal beta-function coefficient. The higher coefficients $\rho_2, \rho_3, \cdots$, in equation (11) are RS-invariants and may be expressed in terms of the perturbative coefficients.
of $\tilde{D}$, $d_k$, together with the beta-function coefficients, $c_k$, which define the renormalization scheme employed to define the RG improved coupling $a$ \[29\]. Thus

$$\frac{da(\mu^2)}{d \ln \mu^2} = -\frac{b}{2} \left( a^2 + ca^3 + c_2a^4 + \cdots + c_k a^{k+2} + \cdots \right). \quad (13)$$

The effective charge (EC) scheme corresponds to the choice of coupling $\tilde{D} = a$. The first two EC invariants are given by

$$\rho_2 = c_2 + d_2 - cd_1 - d_1^2$$
$$\rho_3 = c_3 + 2d_3 - 4d_1d_2 - 2d_1\rho_2 - cd_1^2 + 2d_1^3. \quad (14)$$

We note that in references \[23, 28\], to which the reader is referred for additional discussion of the EC beta-function, the dependent energy variable was taken to be $Q$, whereas we are employing $s = Q^2$ in this paper, hence there are additional factors of $\frac{1}{2}$ in equations (11) and (13).

Using equation (11) one can then show that the energy derivatives in equation (10) can be rewritten as

$$\left. \frac{d^n}{d\ln s^n} \tilde{D}(s) \right|_{s=s_0} = \left( -\frac{b}{2} \right)^n \left[ \rho(x) \frac{d}{dx} \rho(x) \right]^{n-1} \left. \frac{d}{dx} \rho(x) \right|_{x=\tilde{D}(s_0)}, \quad n > 0. \quad (15)$$

Thus finally we can write equation (10) in the form

$$\tilde{R}(s_0) = \tilde{D}(s_0) + \sum_{n=1}^{\infty} \left( -\frac{ib}{2} \right)^n \frac{w_n}{n!} \left[ \rho(x) \frac{d}{dx} \rho(x) \right]^{n-1} \left. \frac{d}{dx} \rho(x) \right|_{x=\tilde{D}(s_0)}. \quad (16)$$

Here $w_n$ denotes moments of the weight function $W(\theta)$,

$$w_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \theta^n W(\theta). \quad (17)$$

For $\tilde{R}$ setting $W(\theta) = 1$ yields $w_n = \pi^n/(n+1)$ ($n$ even), $w_n = 0$ ($n$ odd). The first two terms in the sum of equation (16) are then

$$\tilde{R}(s_0) = \tilde{D}(s_0) - \frac{\pi^2 b^2}{24} \rho \rho' + \frac{\pi^4 b^4}{1920} \rho (\rho^3 + 4\rho^2 \rho' + \rho^2 \rho'') + \cdots. \quad (18)$$

Primes denote differentiation of $\rho(x)$ with respect to $x$, evaluated at $x = \tilde{D}(s_0)$. Successive terms are RS-invariants resulting from the resummation to all-orders of analytical continuation terms proportional to $\pi^2 b^2$, $\pi^4 b^4$, \ldots, respectively.

For $\tilde{R}_\tau$ the weight function $W_\tau(\theta)$ of equation (9) has moments $w_1^\tau = \frac{19b}{12}$, $w_2^\tau = \frac{\pi^2}{3} - \frac{265}{72}$, \ldots. Then equation (16) yields

$$\tilde{R}_\tau = \tilde{D}(m_\tau^2) + \frac{19b}{24} \rho - \left( \frac{\pi^2}{3} - \frac{265}{72} \right) \frac{b^2}{8} \rho \rho' + \cdots. \quad (19)$$

From equation (16) we see that Minkowski observables are naturally expressed in terms of the Euclidean Adler $D$-function and its EC beta-function. Given an all-orders definition
of ρ(x) one can discuss the radius of convergence of the sum in equation (16) in $\tilde{D}(s_0)$. This is an interesting question which could be directly addressed using the leading-b resummation of ρ on which the RS-invariant resummations of reference [23] are based. However, one can anticipate a rather restricted radius of convergence by making the one-loop approximation $ρ(x) = x^2$. One then finds

$$\tilde{R}_{\text{one-loop}} = \frac{2}{b\pi} \arctan(\frac{b\pi \tilde{D}}{2}) \quad (20)$$

for the result of resumming the analytical continuation terms which only involve the lowest beta-function coefficient. This suggests that the radius of convergence is limited by $\tilde{D} < \frac{2}{b\pi}$ [11, 23]. For $N_f = 3$ this gives a radius of convergence $\frac{4}{\pi} \approx 0.141 \ldots$, which is to be compared with $\tilde{D}(m_\pi^2) \approx 0.1$ so that the expansion will not be useful for evaluating $\tilde{R}_\tau$ using the leading-b resummation of $ρ(x)$. As we shall discuss in the next section, however, we shall be able to make use of the Taylor’s theorem approach to evaluate $\tilde{D}^{(se^{i\theta})}$ at closely spaced intervals around the integration contour using the resummed $ρ(x)$.

To conclude this section we note that the expansion of equation (16) is of use in straightforwardly relating the $\tilde{R}$, $\tilde{R}_\tau$ Minkowski perturbative coefficients $r_k$, $\tilde{r}_k^\tau$ to the Euclidean $d_k$ coefficients of $\tilde{D}$. One simply expands equation (16) as a power series in $\tilde{D}$ then on substituting $\tilde{D} = a + d_1a^2 + d_2a^3 + \cdots$, and isolating the coefficient of $a^{k+1}$ one can directly obtain $r_k$, $\tilde{r}_k^\tau$ in terms of $d_i |\leq k$, $ρ_i |\leq k$ and $c$. The resulting calculated expressions are in agreement with the results available in the literature for $k \leq 5$ [24], on using equations (14) to re-express beta-function coefficients $c_i$ in terms of $ρ_i$ invariants.

In clarifying the connection between the various versions of fixed-order perturbation theory to be compared in section 4 it will be useful to relate the EC RS-invariants $ρ_k^R$ and $ρ_k^{R\tau}$ corresponding to $\tilde{R}$ and $\tilde{R}_\tau$ to the $ρ_k^D$ (hitherto $ρ_k$) connected with $\tilde{D}$ [23]. This can easily be accomplished by first evaluating $r_k$ and $\tilde{r}_k^\tau$ in the EC scheme for $\tilde{D}$, so that $d_1 = d_2 = \cdots = 0$. These $r_k(d_i = 0)$ and $\tilde{r}_k^\tau(d_i = 0)$ are simply the coefficient of $\tilde{D}^{k+1}$ on the right-hand side of equation (16). One can then use the analogue of equation (14) for $\tilde{R}$, $\tilde{R}_\tau$ with $c_k = ρ_k^D$ to obtain the required relations. Expressions for $ρ_k^R$ and $ρ_k^{R\tau}$ ($k \leq 6$) are included in Appendix A.

### 3 Numerical evaluation of the contour integral

In this section we shall reformulate the Taylor’s theorem expansion approach of the last section to obtain a tractable method for numerically evaluating the contour integral, appropriate not only when $\tilde{D}$ is truncated at fixed-order in perturbation theory but crucially also allowing us to perform the RS-invariant all-orders resummation $\tilde{D}^{(L^\ast)}$ of reference [24].

For ease of reference we shall begin by briefly reviewing the leading-b resummations. The reader is referred to reference [23] for full details.

For a wide range of “quark-initiated” [14] QCD observables the perturbative coefficients can be organized as polynomials in the number of quark flavours, $N_f$. That is assuming such an observable $D(s)$ with the perturbation series

$$D(s) = a + d_1a^2 + d_2a^3 + \cdots + d_ka^{k+1} + \cdots \quad (21)$$
we can write
\[ d_k = d_k^{[k]} N_f^k + d_k^{[k-1]} N_f^{k-1} + \cdots + d_k^{[0]} . \] (22)

By substituting \( N_f = \left( \frac{1}{2} N - 3b \right) \) equation (22) can be recast in powers of \( b \).
\[ d_k = d_k^{(k)} b^k + d_k^{(k-1)} b^{k-1} + \cdots + d_k^{(0)} . \] (23)

Since \( d_k^{(k)} = (-1/3)^k d_k^{(k)} \), exact all-orders large-\( N_f \) results can be used to perform a “leading-\( b \)” resummation,
\[ D^{(L)} \equiv \sum_{k=0}^{\infty} d_k^{(L)} a^{k+1} , \] (24)
where \( d_k^{(L)} \equiv d_k^{(k)} b^k (d_0^{(L)} \equiv 1) \).

\( D^{(L)} \) may be defined as the principal value (PV) regulated Borel sum
\[ D^{(L)}(a) = \text{PV} \int_0^{\infty} dz \, e^{-z/a} B[D^{(L)}](z) , \] (25)

where \( B[D^{(L)}](z) \) denotes the Borel transform, which potentially involves an infinite set of poles at \( z = z_l \equiv \frac{a}{l} \) \((l = 1, 2, 3, \ldots)\) corresponding to infra-red renormalons (IR\(_l\)), and at \( z = -z_l \) corresponding to ultra-violet renormalons (UV\(_l\)). In the specific case of the Adler \( D \)-function, \( D^{(L)} \), IR\(_1\) is not present reflecting the absence of a relevant operator of dimension two in the operator product expansion (OPE) for vacuum polarization [11] [16]. IR\(_2\) is a single pole and the remaining singularities are double poles. Expressions for the residues are given in reference [12]. It is then straightforward to evaluate equation (25) in terms of the exponential integral functions,
\[ \text{Ei}(x) = - \int_{-x}^{\infty} dt \frac{e^{-t}}{t} , \] (26)

where for \( x > 0 \) the Cauchy principal value is taken. The UV\(_l\) singularities may then be expressed in terms of Ei\((-Fz_l)\), where \( F \equiv \frac{1}{a} \), and the IR\(_l\) singularities involve Ei\((Fz_l)\). Corresponding expressions for \( D^{(L)}(F)|_{UV} \) and \( D^{(L)}(F)|_{IR} \) are given in equations (48) and (49) respectively of reference [12].

As pointed out in reference [23] the \( D^{(L)} \) resummation of equation (24) is ambiguous due to its RS-dependence. In particular if, as in the case of \( \bar{D} \), the exact NLO and NNLO coefficients are available it would seem sensible to include them and approximate only the unknown \( d_3, d_4, \ldots \), higher coefficients by \( d_3^{(L)}, d_4^{(L)} \). Unfortunately, however, the resummed result explicitly depends on the RS chosen for evaluating the exact \( d_1, d_2 \) coefficients. This is analogous to the ambiguity encountered in matching leading logarithm resummations of jet observables to exact fixed-order perturbative results [32]. In both cases the difficulty may be avoided by performing a resummation of the EC beta-function [23, 28]. The unknown N\(^3\)LO and higher EC beta-function coefficients \( \rho_3, \rho_4, \ldots \), in equation (11) are approximated by retaining only the portion involving the highest power of \( b \), \( \rho_k^{(L)} \equiv \rho_k^{(k)} b^k \). The \( \rho_k^{(k)} \) can be obtained to all-orders from the large-\( N_f \) result for \( d_k^{(k)} \).

If the NNLO invariant \( \rho_2 \) is known exactly it can be included. One then arrives at the resummed EC beta-function
\[ \rho^{(L \ast)}(x) \equiv x^2 (1 + cx + \rho_2 x^2 + \sum_{k=3}^{\infty} \rho_k^{(L)} x^k) . \] (27)
The improved RS-invariant resummation $D^{(L*)}(s)$ can then be obtained as the solution of the integrated beta-function equation

$$
\frac{1}{D^{(L*)}} + c \ln \frac{c D^{(L*)}}{1 + c D^{(L*)}} = \frac{b}{2} \ln \frac{s}{\Lambda_{\text{MS}} s} d^{(L*)} (\mu^2 = s) - \int_0^{D^{(L*)}} dx \left[ -\frac{1}{\rho^{(L*)}(x)} + \frac{1}{x^2(1 + cx)} \right].
$$

(28)

The exact NLO coefficient $d_1$ occurs in the RS-invariant combination $\rho_0$

$$
\rho_0 = \frac{b}{2} \ln \frac{s}{\Lambda_{\text{RS}}} - d_1^{(L*)} (\mu^2 = s).
$$

(29)

The convention assumed for $\Lambda_{\text{MS}}$ in equation (28) differs from the standard definition by the $N_f$-dependent factor $(2c/b)^{-c/b}$ [24]. If $D^{(L*)}(s)$ is expanded in the coupling $a$ appropriate to some RS one then obtains

$$
D^{(L*)} \equiv \sum_{k=0}^{\infty} d^{(L*)}_k a^{k+1} \quad (d^{(L*)}_0 = 1),
$$

(30)

where now $d^{(L*)}_1 = d_1$ and $d^{(L*)}_2 = d_2$ reproduce the known coefficients and the approximated $d^{(L*)}_3$ and higher coefficients may be obtained in any RS from the approximated $\rho^{(L)}_k$ invariants using the exact QCD RG. For instance if we label the RS by $d_1$ and the beta-function coefficients $c_2, c_3, \ldots$, appearing in equation (13) we have [24]

$$
d^{(L*)}_3 (d_1, c_2, c_3) = d_1^3 + \frac{5}{2} c d_1^2 + (3p_2 - 2c_2)d_1 + \frac{1}{2}(p_3 - c_3).
$$

(31)

This differs from the exact $d_3$ only in the unknown $\rho_3$ term (we note in passing that $c^{\text{MS}}_3$ has now been computed [23]), the known $\rho_2$ has been exactly included. In this approach the maximum available exact information is included in an RS-invariant manner.

It finally remains to perform the resummation $\rho^{(L*)}$ using the explicit expressions for $D^{(L)}(F)$. Defining

$$
\rho^{(L)}(x) \equiv x^2(1 + \sum_{k=2}^{\infty} \rho^{(L)}_k x^k),
$$

(32)

and using the chain rule to relate the beta-function in two different RS’s [28] one has

$$
\rho^{(L)}(x) = \left( a^{(L)}(x) \right)^2 \left. \frac{d D^{(L)}(a)}{da} \right|_{a = a^{(L)}(x)},
$$

(33)

where $a^{(L)}(x)$ is the inverse function to $D^{(L)}(a)$, i.e. $D^{(L)}(a^{(L)}(x)) = x$.

$\rho^{(L)}(x)$ can then be straightforwardly defined from $D^{(L)}(F)$. The first step is to numerically solve

$$
D^{(L)}(F(x)) = x,
$$

(34)

to obtain $F(x)$. Recalling that $F \equiv \frac{1}{a}$ one can then determine

$$
\rho^{(L)}(x) = -\left. \frac{d}{dF} D^{(L)}(F) \right|_{F = F(x)},
$$

(35)
by differentiating the explicit $D^{(L)}(F)$ expressions in terms of $Ei$ functions. Finally on comparing equations (32) and (27) one has

$$\rho^{(L^*)}(x) = \rho^{(L)}(x) + cx^3 + \rho^{(NL)} x^4, \quad (36)$$

where $\rho_2^{(NL)} \equiv \rho_2 - \rho_2^{(L)}$. $\rho^{(L^*)}(x)$ can then be inserted in equation (28) and the integration performed numerically. On solving the transcendental equation the RS-invariant $D^{(L^*)}(s)$ can be evaluated.

We now turn to the problem of evaluating the improved resummations $\tilde{R}^{(L^{**})}$ and $\tilde{R}^{(L^{**})}_{\tau}$ where the contour integration of equation (8) is performed with $D^{(L^*)}(s e^{i\theta})$ in the integrand,

$$\tilde{R}^{(L^{**})}(s_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \, W(\theta) \, \tilde{D}^{(L^*)}(s_0 e^{i\theta}). \quad (37)$$

To perform the contour integration numerically one can split the range from $\theta = 0, \pi$ into $K$ steps of size $\Delta \theta \equiv \frac{\pi}{K}$, and perform a sum over the integrand evaluated at $\theta_n \equiv n\Delta\theta$ $n = 0, 1, \cdots, K$. So that

$$\tilde{R}(s_0) \simeq \frac{\Delta \theta}{2\pi} \left[ W(0) \tilde{D}(s_0) + 2\operatorname{Re} \sum_{n=1}^{K} W(\theta_n) \tilde{D}(s_n) \right], \quad (38)$$

where $s_n \equiv s_0 e^{in\Delta \theta}$. In practice we perform a Simpson’s Rule evaluation.

An efficient strategy [34] is to start with the exact $\tilde{D}(s_0)$ and evolve $\tilde{D}(s_n)$ to $\tilde{D}(s_{n+1})$ using Taylor’s theorem. Thus defining $x_n \equiv \tilde{D}(s_n)$ we have

$$x_{n+1} = x_n - \frac{i\Delta \theta}{2} b \rho(x_n) - \frac{(\Delta \theta)^2}{8} b^2 \rho(x_n) \rho'(x_n)$$

$$+ \frac{i(\Delta \theta)^3}{48} b^3 \left\{ \rho(x_n) \rho'(x_n)^2 + \rho(x_n)^2 \rho''(x_n) \right\} + O(\Delta \theta)^4 + \cdots, \quad (39)$$

analogous to equation (16). If equation (39) is truncated by retaining its first $m$ terms one anticipates an error $\sim \frac{1}{K^{m-2}}$ in equation (38).

To evaluate $\tilde{R}_{\tau}$ to four significant figure accuracy retaining the first four terms in equation (39) we required 100 steps.

The method obviously can also be used to evaluate the contour integral when $\tilde{D}$ is represented by fixed-order perturbation theory in the coupling $a(s_0 e^{i\theta})$,

$$\tilde{D}(s_0 e^{i\theta}) = a(s_0 e^{i\theta}) + d_1 a^2(s_0 e^{i\theta}) + d_2 a^3(s_0 e^{i\theta}). \quad (40)$$

One can start with $a(s_0)$ and evolve $a(s_n)$ to $a(s_{n+1})$ using equation (39) with $\rho(x)$ replaced by the truncated beta-function in the corresponding RS,

$$B(x) = x^2 + cx^3 + c_2 x^4. \quad (41)$$

In standard approaches [21, 31] the contour integral is performed by solving the integrated beta-function equation with complex renormalization scale $s_n$ for $a(s_n)$ at each integration step, and takes much longer to evaluate. Reference [31] considers in some detail the RS dependence of the contour integral.
Determining $\hat{R}^{(L\times s)}$ with this approach is now relatively straightforward. For some given $\Lambda_{\text{MS}}$ one evaluates $\hat{D}^{(L\times s)}(s_0)$ as we have reviewed. The truncated equation (39) is then used to obtain $x_1 = \hat{D}^{(L\times s)}(s_1)$. This requires $\rho^{(L\times s)}(x_0)$ and some number of derivatives. $\rho^{(L\times s)}(x_0)$ can be determined given the $\hat{D}^{(L\times s)}(F)$ expressions of reference [12], and using the numerical inversion route of equations (34) and (35). $\rho^{(L\times s)}$, $\rho^{(L\times s)}'$, $\rho^{(L\times s)}''$, $\ldots$ can then be obtained by successive differentiation of equation (34) with respect to $x$. One finds

$$\rho^{(L\times s)}(x) = -\frac{\hat{D}^{(L\times s)}(F(x))}{\hat{D}^{(L\times s)'}(F(x))}$$

(42)

and

$$\rho^{(L\times s)'}(x) = \frac{\hat{D}^{(L\times s)'}(F(x))}{\hat{D}^{(L\times s)''}(F(x))} - \frac{\hat{D}^{(L\times s)''}(F(x))}{\hat{D}^{(L\times s)'}(F(x))}$$

(43)

where primes denote differentiation with respect to $F$. Thus, once $F(x)$ has been determined from equation (34) no further transcendental equations need to be solved and the explicit expressions for $\hat{D}^{(L\times s)}(F)$ can be repeatedly differentiated to obtain $\rho^{(L\times s)}$, $\rho^{(L\times s)}'$, $\rho^{(L\times s)}''$, $\ldots$. Finally $\rho^{(L\times s)}$, $\rho^{(L\times s)}'$, $\rho^{(L\times s)}''$, $\ldots$ can be obtained using equation (36) and its derivatives. For instance

$$\rho^{(L\times s)'}(x) = \rho^{(L\times s)'}(x) + 3cx^2 + 4\rho^{(N\times L)}(x^3).$$

(44)

The only remaining difficulty is that $x_1$ is now complex, and so at subsequent steps it is unclear how to obtain $\rho^{(L\times s)}(x_n)$, since $\hat{D}^{(L\times s)}(F)$ is only defined for real $F$. One needs to replace the $\text{Ei}(x)$ defined in equation (26) by the generalized exponential integral functions $\text{Ei}(n,w)$ for complex $w$, used to evaluate $\hat{R}^{(L\times s)}$ and $\hat{R}^{(L)}$ in reference [12]. These are defined for $\text{Re } w > 0$ by

$$\text{Ei}(n,w) = \int_1^{\infty} dt \frac{e^{-w t}}{t^n}.$$  

(45)

For $\text{Re } w < 0$ they are defined by analytical continuation to arrive at a function analytic everywhere in the cut complex $w$-plane except at $w = 0$, and with a branch cut running along the negative real axis.

To define $\hat{D}^{(L\times s)}(F)$ correctly for complex $F$ one needs to replace $\text{Ei}(-Fz_i)$ in equation (48) of reference [12] for $\hat{D}^{(L\times s)}(F)|_{UV}$ by $-\text{Ei}(1,Fz_i)$. In equation (49) of reference [12] for $\hat{D}^{(L\times s)}(F)|_{\text{IR}}$ one replaces $\text{Ei}(Fz_i)$ by $-\text{Ei}(1,-Fz_i) + i\pi \text{sign}(\text{Im } Fz_i)$. In this way as $F$ becomes real one avoids $\pm i\pi$ contributions from the discontinuity across the branch cut along the negative real axis and re-obtains $\hat{D}^{(L\times s)}(F)$ for real argument.

With $\hat{D}^{(L\times s)}(F)$ re-defined for complex arguments in this way $x_1, x_2, \ldots$, can be successively obtained. At each step one needs to solve the complex-valued transcendental equation

$$\hat{D}^{(L\times s)}(F_n) = x_n,$$

(46)

and $F_n$ is then used to construct $\rho^{(L\times s)}(x_n)$ and its derivatives using equations (35), (42) and (43). The required computing time is dominated by that required for the solution of equation (46), and is comparable to that needed for the conventional approach in fixed-order perturbation theory, where the complex-valued integrated beta-function equation is numerically solved at each step.
We have checked that evaluating the contour integral with $\tilde{D}^{(L)}$ and $\rho^{(L)}$ reproduces values in numerical agreement with the $\tilde{R}^{(L)}$ and $\tilde{R}^{(L)}_{\tau}$ expressions of reference [12].

In the next section we shall compare the “contour-improved” RS-invariant resummations $\tilde{R}^{(L**)\,n}(EC)$ and $\tilde{R}^{(L**)\,n}_{\tau}(EC)$, with “contour-improved” fixed-order results obtained by truncating $\tilde{D}^{(L*)}$ at $n^{th}$ order in the EC scheme, that is in equation (39) $\rho(x)$ is taken to be the truncation of $\rho^{(L*)}(x)$ in equation (27), retaining terms up to $x^n$, and the input $\tilde{D}^{(L*)\,(n)}(s_0)$ is obtained by solving equation (28) with the truncated $\rho^{(L*)}(x)$. We shall denote these by $\tilde{R}^{(L**)\,(n)}(EC)$ and $\tilde{R}^{(L**)\,(n)}_{\tau}(EC)$ for $n \geq 3$, and for $n = 1, 2$ where the exact $\rho_k$ are used by $\tilde{R}^{(n)}(EC)$ and $\tilde{R}^{(n)}_{\tau}(EC)$.

These “contour-improved” evaluations are to be compared with conventional fixed-order perturbative truncations $\tilde{R}^{(L**)\,(n)}(EC)$ and $\tilde{R}^{(L**)\,(n)}_{\tau}(EC)$ obtained by integrating up the $n^{th}$-order truncated EC beta-functions $\rho^{R,R_\tau,(L**)}$, with coefficients $\rho_k^{R,R_\tau,(L**)}$ obtained using equations (51), (52) in Appendix A, with the exact $\rho_2^D$ and using $\rho_k^{D,(L)}$ for $k > 2$. By truncating $\rho^{R,R_\tau}$ one omits an infinite set of exactly-known and numerically important analytical continuation terms which are included in the “contour-improved” resummations.

4 Numerical results

In Figures 1(a)-(c), for $\sqrt{s} = 91, 5, 1.5$ GeV, respectively, we compare the “contour-improved” RS-invariant resummations $\tilde{R}^{(L**)\,n}(EC)$ and $\tilde{R}^{(L**)\,n}_{\tau}(EC)$ with the RS-invariant resummation $\tilde{R}^{(L**)\,(s_0)}_{\tau}$ (dashed line) at $\sqrt{s_0} = 91$ GeV.

Figure 1(a): Comparison of two versions of fixed-order EC perturbation theory “contour-improved” $\tilde{R}^{(L**)\,n}(EC)$ ("+") and $\tilde{R}^{(L**)\,n}(EC)$ ("×") with the RS-invariant resummation $\tilde{R}^{(L**)\,(s_0)}$ (dashed line) at $\sqrt{s_0} = 91$ GeV.
Figure 1(b): As for Figure 1(a) except at $\sqrt{s_0} = 5$ GeV.

Figure 1(c): As for Figure 1(a) except at $\sqrt{s_0} = 1.5$ GeV
improved” resummation $\tilde{R}^{(L^*)}(s_0)$ with the two versions of fixed-order perturbation theory, “contour-improved” $\tilde{R}^{(L^*)[n]}(EC)$ and $\tilde{R}^{(L^*)[n]}(EC)$ (for $n \geq 2$), described in the last section. Values of $\Lambda^{(5)}_{\overline{MS}} = 200$ MeV, $\Lambda^{(4)}_{\overline{MS}} = 279$ MeV and $\Lambda^{(3)}_{\overline{MS}} = 320$ MeV are used. These assume flavour thresholds at $m_b = 4.5$ GeV and $m_c = 1.25$ GeV.

As can be seen at all energies and in low orders the “contour-improved” fixed-order results (denoted “+”) are significantly closer to the resummation $\tilde{R}^{(L^*)}$ (horizontal line) than the conventional fixed-order results (denoted “x”). This is completely understandable since both the RS-invariant resummations and the contour-improved fixed-order results sum to all-orders known analytical continuation terms, as discussed above. The unnecessary truncation of these terms evidently greatly worsens the performance of $n = 2$ NNLO fixed-order perturbation theory, whilst in higher orders both versions of fixed-order perturbation theory approach each other, and both track the RS-invariant resummation. Eventually, of course, both versions will breakdown as the leading UV singularities asserts itself. Since $n = 2$ represents the highest order for which exact calculations exist at present, “contour-improvement” is clearly essential if reliable NNLO determinations of $\alpha_s(M_Z^2)$ are to be made.

In Figures 2(a)-(c) we plot $\tilde{D}^{(L^*)}(s_0)$ (dashed line) and $\tilde{D}^{(L^*)[n]}(EC)$ (denoted “+”) at $\sqrt{s} = 91, 5, 1.5$ GeV, respectively. These represent the input values of $\tilde{D}(s_0)$ fed into the contour integration to produce the plots in Figures 1(a)-(c). We note that the fixed-order results in Figures 2 show a clear oscillation above and below the resummed result. This is a reflection of the alternating sign factorial behaviour contributed by
Figure 2(b): As for Figure 2(a) except at $\sqrt{s_0} = 5$ GeV.

Figure 2(c): As for Figure 2(a) except at $\sqrt{s_0} = 1.5$ GeV.
the leading UV$_1$ renormalon, which in the case of $\tilde{D}$ is a double pole, in the leading-$b$ approximation. A similar oscillatory behaviour is also evident for the conventional fixed-order perturbative approximants for $\tilde{R}$ in Figures 1(a)-(c), but with much smaller amplitude. This is because for $\tilde{R}$ the UV$_1$ singularity is softened to a single-pole, again in the leading-$b$ approximation. As a result one expects $r_n/d_n \simeq \frac{1}{n}$ asymptotically \[11, 16\], and correspondingly $\rho_n^R/\rho_n^D \simeq \frac{1}{n}$. Notice that the “contour-improved” fixed-order results which partially resum higher-order contributions do not exhibit the simple oscillatory behaviour.

In Figure 3 we give the analogous plot to Figures 1 for $\tilde{R}_\tau$, assuming $\Lambda^{(3)}_{\text{MS}} = 320$ MeV as before, $\sqrt{s_0} = 1.777$ GeV. If we compare with Figure 1(c) for $\tilde{R}$ at the comparable energy $\sqrt{s_0} = 1.5$ GeV, we see a deterioration in the behaviour of both versions of fixed-order perturbation theory. The change of weight function from $W(\theta) = 1$ to $W_\tau = (1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta})$ leads to much less convergent analytical continuation terms and the two versions of fixed-order perturbation theory no longer approach each other in higher orders. The contour-improved results are reasonably close to the resummation. Clearly “contour-improvement” is vital for reliable $\alpha_s(m_Z^2)$ determinations.

We now wish to use the difference between the “contour-improved” $\tilde{R}^{(L,**)}$ and $\tilde{R}^{(2)}(EC)$ to estimate the uncertainty with which $\alpha_s(M_Z^2)$ can be determined for the Minkowski observables. Our main interest will be in $\tilde{R}_\tau$ which potentially gives the most accurate determination. To begin with, however, we consider $\tilde{R}$ at $\sqrt{s} = M_Z$ (i.e. the hadronic decay width of the $Z^0$). As in our fits in reference \[23\] we shall take $\tilde{R}(M_Z^2) = 0.040 \pm 0.004$. 

Figure 3: As for Figure 1(a) except for $\tilde{R}_\tau$ at $\sqrt{s_0} = 1.777$ GeV.
The fits to the three-loop NNLO $\overline{\text{MS}}$ $\alpha_s(M_Z^2)$ are then $\alpha_s(M_Z^2) = 0.122 \pm 0.012$ from both $\tilde{R}^{(L*)}$ and $\tilde{R}^{(L*)}\,(EC)$. This is also the same result as obtained in reference [23] using $\tilde{R}^{(L*)}$. So at this high energy scale the “contour-improvement” has little effect.

For $\tilde{R}_\tau$ we take $\tilde{R}_{\text{data}} = 3.64 \pm 0.01$ [21]. Correcting for the small estimated power corrections [20] then yields $\tilde{R}_\tau = 0.205 \pm 0.006$. Fitting to the “contour-improved” RS-invariant resummation $\tilde{R}_\tau^{(L*)}$ then yields $\alpha_s(m_{\tau}^2) = 0.339 \pm 0.006$, and fitting to the “contour-improved” NNLO EC result $\tilde{R}_\tau^{(EC)}(EC)$ gives $\alpha_s(m_{\tau}^2) = 0.350 \pm 0.008$. Evolving the three-loop coupling $\alpha_s^{\overline{\text{MS}}}$ using Bernreuther-Wetzel matching [35] from $N_f = 3$ to $N_f = 5$ with the flavour thresholds noted above yields $\alpha_s(M_Z^2) = 0.1214 \pm 0.0007$ and $\alpha_s(M_Z^2) = 0.1226 \pm 0.0008$, respectively. Using smaller quark masses ($m_c = 1.0$ GeV and $m_b = 4.1$ GeV) at the bottom of the range quoted in [36] to perform the evolution yields $\alpha_s(M_Z^2) = 0.1222 \pm 0.0007$ and $\alpha_s(M_Z^2) = 0.1234 \pm 0.0008$, respectively. Choosing larger masses ($m_c = 1.6$ GeV and $m_b = 4.5$ GeV) at the top of the quoted range gives $\alpha_s(M_Z^2) = 0.1207 \pm 0.0007$ and $\alpha_s(M_Z^2) = 0.1218 \pm 0.0008$, respectively. Thus one can estimate an uncertainty $\delta \alpha_s(M_Z^2) \simeq 0.002$.

In reference [23] we fitted to the same value of $\tilde{R}_\tau$ using the RS-invariant resummation $\tilde{R}_\tau^{(L*)}$ which only includes analytical continuation terms at the leading-$b$ level, and found $\alpha_s(m_{\tau}^2) = 0.328 \pm 0.005$, similarly fitting to NNLO EC fixed-order perturbation theory gave $\alpha_s(m_{\tau}^2) = 0.320 \pm 0.005$. In both cases the inclusion of exactly known analytical continuation terms involving $c, \rho_2, \ldots$, via the “contour-improvement” serves to significantly increase the fitted $\alpha_s(m_{\tau}^2)$, and hence slightly increase $\alpha_s(M_Z^2)$.  

Figure 4: As for Figure 3 except $\tilde{R}_\tau^{(L*)}$ is fitted to $\tilde{R}_{\tau}\,^{\text{exp}} = 0.205$ at $\sqrt{s_0} = 1.777$ GeV.
In Figure 4 we repeat the plot of Figure 3 but with the increased value of $\Lambda_{\overline{\text{MS}}}^{(3)} = 429$ MeV, which results from fitting $\tilde{R}_\tau^{(L**)}$ to the data for $\tilde{R}_\tau$. A marked deterioration in the performance of both versions of fixed-order perturbation theory is evident, although the “contour-improved” fixed-order results are still significantly closer to the resummation. This serves as a warning that, at this low energy scale, relatively small changes in $m_\tau^2/(\Lambda_{\overline{\text{MS}}}^{(3)})^2$ can produce significant changes in the accuracy of perturbation theory.

In Figure 5 we extend the fits reported earlier. For $0.16 < \tilde{R}_\tau < 0.25$ we plot curves for the $\alpha_s(m_\tau^2)$ obtained by fitting $\tilde{R}_\tau^{(2)}(EC)$ (dotted curve), $\tilde{R}_\tau^{(L**)}$ (solid curve), $\tilde{R}_\tau^{(2)}(EC)$ (dashed curve), to this value. $\delta\alpha_s(m_\tau^2)$ can then be estimated for given $\tilde{R}_\tau$ from the difference between the lower two “contour-improved” curves. Clearly $\delta\alpha_s(m_\tau^2)$ increases very rapidly as $\tilde{R}_\tau$ increases. We are very fortunate that apparently $\tilde{R}_\tau \sim 0.2$, for which $\delta\alpha_s(m_\tau^2) \approx 0.01$.

We can compare the RS-invariant resummation $\tilde{R}_\tau^{(L**)}$ with the results obtained using a Le-Diberder Pich (LP) resummation [41], that is evaluating the contour integral with $\tilde{D}(m_\tau^2 e^{i\theta})$ as in equation (40). Fitting $\tilde{R}_\tau = 0.205 \pm 0.006$ to this $\tilde{R}_\tau^{(2)}(\overline{\text{MS}})$ yields $\alpha_s(m_\tau^2) = 0.359 \pm 0.008$ in reasonable accord with the NNLO EC “contour-improved” value $\alpha_s(m_\tau^2) = 0.350 \pm 0.008$.

Fitting to conventional NNLO fixed-order perturbation theory in the $\overline{\text{MS}}$ scheme $\tilde{R}_\tau^{(2)}(\overline{\text{MS}})$, with $\mu = m_\tau$, yields $\alpha_s(m_\tau^2) = 0.342 \pm 0.006$. No special significance should be ascribed to the numerical coincidence that this is close to the RS-invariant resummation fit, $\alpha_s(m_\tau^2) = 0.339 \pm 0.006$. Crucially these $\overline{\text{MS}}$ fits are strongly dependent on the assumed RS. In Figure 6 we show the $\tilde{R}_\tau$ versus $\alpha_s(m_\tau^2)$ plots with three choices of scale $\mu = 2m_\tau, m_\tau, \frac{1}{2}m_\tau$ (labelled 1, 2, 3, respectively). We also plot the curves for fitting to a LP resummation based on the NNLO $\overline{\text{MS}}$ expansion of $\tilde{D}(m_\tau^2 e^{i\theta})$ in $a(4m_\tau^2 e^{i\theta})$, $a(m_\tau^2 e^{i\theta})$ and $a(\frac{4}{3}m_\tau^2 e^{i\theta})$ (labelled 4, 5, 6, respectively). As can be seen the $\tilde{R}_\tau^{(2)}(\overline{\text{MS}})$ curves for different scales are very widely separated. The scale dependence of the LP resummations is seen to be much reduced compared to conventional fixed-order $\overline{\text{MS}}$ perturbation theory, but is still significant.

For convenience we now present simple numerical parametrizations for the contour-improved resummations $\tilde{R}_\tau^{(L**)}$, $\tilde{R}_\tau^{(2)}(EC)$ and LP (i.e. $\tilde{R}_\tau^{(2)}(\overline{\text{MS}})$ based on an expansion in $a(m_\tau^2 e^{i\theta})$), in terms of $\alpha_s(m_\tau^2)$. We stress that $\alpha_s(m_\tau^2)$ denotes the 3-loop NNLO $\overline{\text{MS}}$ coupling with scale $\mu = m_\tau$.

Given $x = \tilde{R}_\tau$ (data) the fitted $\alpha_s(m_\tau^2)$ is parametrized by

$$\alpha_s(m_\tau^2) = \pi x + A_2 x^2 + A_3 x^3 + A_4 x^4.$$  \hspace{1cm} (47)

The numerical coefficients $A_i$ for the different “contour-improved” versions of perturbation theory are tabulated in Table 1. These coefficients give $\alpha_s(m_\tau^2)$ to a numerical accuracy of three significant figures over the range $0.16 \leq \tilde{R}_\tau \leq 0.25$ covered in Figure 5.

We also present reverse fits. Given $x = \alpha_s(m_\tau^2)$ the different approximations for $\tilde{R}_\tau$ are parametrized by

$$\tilde{R}_\tau = \frac{1}{\pi} x + \tilde{A}_2 x^2 + \tilde{A}_3 x^3 + \tilde{A}_4 x^4.$$  \hspace{1cm} (48)

The numerical coefficients $\tilde{A}_i$ are again tabulated in Table 1. $\tilde{R}_\tau$ is accurate to three significant figures over the range $0.29 < \alpha_s(m_\tau^2) < 0.41$. 
Figure 5: $\tilde{R}_\tau$ versus $\alpha_s(m^2_\tau)$ for $\tilde{R}_\tau^{(2)}(EC)$ (labelled 1), $\tilde{R}_\tau^{(L^*)}$ (labelled 2) and $\tilde{R}_\tau^{(2)}(EC)$ (labelled 3).

Figure 6: $\tilde{R}_\tau$ versus $\alpha_s(m^2_\tau)$ for $\tilde{R}_\tau^{(2)}(\overline{MS})$ and $\tilde{R}_\tau^{(2)}(\overline{MS})$, with three choices of scale $\mu = 2m_\tau, m_\tau, \frac{1}{2}m_\tau$; labelled (1, 2, 3) and (4, 5, 6) respectively.
Finally in this section we wish to examine the performance of a straightforward leading-\(b\) resummation for \(\tilde{R}_\tau\). To emphasise the associated RS ambiguity we shall evaluate it for three \(\overline{\text{MS}}\) scales \(\mu = \lambda m_\tau\), where \(\lambda = \frac{1}{2}, 1, 2\), as before. We then evaluate

\[
\tilde{R}_\tau^{(L)}(F(a)) + r_1^{(NL)} a^2 + r_2^{(NL)} a^3.
\]  

(49)

Here \(a\) denotes the \(\overline{\text{MS}}\) coupling \(a(\lambda^2 m_\tau^2)\). \(\tilde{R}_\tau^{(L)}(F)\) is given by the explicit expressions in equations (69) and (70) of reference [12]: \(F(a) \equiv \frac{1}{a} - b \ln \lambda + \frac{a}{b}\), \(r_1^{(NL)} \equiv r_1^\tau - r_1^{\tau(L)}\). The extra terms ensure that at NLO and NNLO the known exact \(r_1^\tau\), \(r_2^\tau\) are included in the resummation.

Fitting equation (49) to \(\tilde{R}_\tau = 0.205\) as before yields \(\alpha_s(\frac{1}{4} m_\tau^2) = 0.475\), \(\alpha_s(m_\tau^2) = 0.306\), and \(\alpha_s(4 m_\tau^2) = 0.233\), for the three choices of RS. Evolving these all to \(m_\tau\) using the three-loop \(\overline{\text{MS}}\) beta-function, which is presumably appropriate since we are including the exact fixed-order results to NNLO, one obtains \(\alpha_s(m_\tau^2) = 0.297, 0.306, 0.322\), respectively. Even if we restrict the beta-function to the one-loop leading-\(b\) level, as advocated in references [14, 15, 16], we obtain \(\alpha_s(m_\tau^2) = 0.323, 0.306, 0.304\). Only if we leave out the NL correction terms in equation (49) and perform a pure leading-\(b\) resummation do we uniquely obtain \(\alpha_s(m_\tau^2) = 0.305\) for all three RS’s.

\section{Discussion and conclusions}

The essential point which motivates our approach is that the basic ingredient out of which the Minkowski observables \(\tilde{R}\) are built is the EC beta-function \(\rho(\tilde{D}(s))\) defined in equation (11). Using equation (7) one can see that this is proportional to \(\frac{d^2}{d \ln s^2} \Pi(s)\), where \(\Pi(s)\) is the fundamental correlator of two vector currents in the Euclidean region defined in equation (6). If one specifies \(\rho(x)\), then given the NLO perturbative coefficient \(d_1^{\text{MS}}(\mu^2 = s_0)\) and assuming some value of \(\Lambda^{\text{MS}}\), \(\tilde{D}(s_0)\) can be obtained unambiguously on solving equation (28). There is no scale dependence since \(\tilde{D}(s_0)\) only involves the RS-invariant combination \(\rho_0(s_0)\) in equation (29). Using equation (16) \(\tilde{R}(s_0)\) is then also uniquely specified given \(\rho(x)\), where in practice the infinite sum is performed by numerically evaluating the contour integral, using \(\rho(x)\) to evolve \(\tilde{D}(se^{i\theta})\) around the circular contour of integration, as described in section 3.
Of course, the function $\rho(x)$ is not known exactly. From NNLO calculations all that is known is the first three terms in its power series expansion,

$$\rho(x) = x^2 + cx^3 + \rho_2 x^4 + \cdots.$$  \hspace{1cm} (50)

The uncertainty in predicting $\tilde{R}(s_0)$ is then to be estimated by making some approximations for the unknown higher order terms indicated by the ellipsis in equation (50). We have chosen to approximate $\rho_k$ by $\rho_k(L)$ for $k \geq 3$. These leading-$b$ contributions exactly reproduce $\rho_k$ in the large-$N_f$ limit, and for $\rho_2$ are a good approximation in the large-$N$ (or $N_f \simeq 0$) limit \cite{23}. Comparing the predictions for $\tilde{R}_\tau$ constructed from the NNLO $\rho(x)$ in equation (50) and the leading-$b$ resummation indicates a moderate uncertainty $\delta \alpha_s(m^2) \simeq 0.01$ for $\tilde{R}_\tau \simeq 0.2$, which evolves up to $\delta \alpha_s(M^2_Z) \simeq 0.002$ and a central value $\alpha_s(M^2_Z) = 0.122$ in line with other $\alpha_s$ measurements, which indicate a global average $\alpha_s(M^2_Z) = 0.118 \pm 0.005$ \cite{22}.

Our reassuringly small uncertainty $\delta \alpha_s(m^2) \simeq 0.01$ is in stark contrast to other more pessimistic claims in the literature. Application of straightforward leading-$b$ resummations compared to exact NNLO fixed-order perturbation theory leads to a claim of $\delta \alpha_s(m^2) \simeq 0.05$ in reference \cite{21}. As we showed in section 4, however, there is a matching problem if one wishes to include the exactly known NLO and NNLO coefficients. As a result the $\delta \alpha_s(m^2)$ estimate depends strongly on the renormalization scale chosen. This difficulty is avoided in our RS-invariant resummation approach, and originally motivated it.

In reference \cite{37} an overall uncertainty of $\delta \alpha_s(m^2) \simeq 0.06$ is claimed. These authors use an LP resummation together with an acceleration technique applied to the perturbation series to lessen the influence of the leading $UV_1$ renormalon. The resulting uncertainty is dominated by the choice of renormalization scale $\mu$. As we have pointed out above the only uncertainty in $\tilde{R}_\tau$ is due to our lack of knowledge of the uncalculated RS-invariants $\rho_3, \rho_4, \cdots$. Thus there is no scale dependence ambiguity. Since it is $\rho(x)$ which is ambiguous one could attempt to improve the convergence of this series. The corresponding Borel transform has a $UV_1$ renormalon and one could try to use acceleration methods. Crucially, however, the resulting uncertainties would have to do with real ambiguities associated with the singularities of the Borel transform of $\rho(x)$ in the Borel plane, and would not involve the unphysical and irrelevant renormalization scale $\mu$. The same criticism applies to reference \cite{38} which uses similar techniques to assess the perturbative ambiguity in $\tilde{R}$.

We therefore conclude that there is no reason to suppose that $\tilde{R}_\tau$ suffers from serious ambiguities due to $N^3LO$ and higher terms which have yet to be exactly calculated. The techniques on which existing claims to this effect have been based are all severely RS-dependent, and their conclusions can be modified at will by making different ad hoc choices of renormalization scale.

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A Relations between RS-invariants for $\tilde{R}$, $\tilde{R}_\tau$ and the Adler $D$-function

Below we present the analytical continuation terms that link the RS-invariants for the two Minkowski observables $\tilde{R}$, $\tilde{R}_\tau$ to those of the Euclidean Adler $D$-function.

For $\tilde{R}$ we can relate the Minkowski invariants to the Euclidean invariants in the following manner.

\begin{align*}
\rho_2^R &= \rho_2^D - \frac{1}{12} b^2 \pi^2 \\
\rho_3^R &= \rho_3^D - \frac{5}{12} c b^2 \pi^2 \\
\rho_4^R &= \rho_4^D - \frac{1}{12} (8 \rho_2^D + 7 c^2) b^2 \pi^2 + \frac{1}{360} b^4 \pi^4 \\
\rho_5^R &= \rho_5^D - \frac{1}{12} (12 \rho_3^D + 20 \rho_2^D c + 3 c^3) b^2 \pi^2 + \frac{17}{360} c b^4 \pi^4 \\
\rho_6^R &= \rho_6^D - \frac{1}{12} (17 \rho_4^D + 28 \rho_3^D c + 13 (\rho_2^D)^2 + 12 \rho_2^D c b^2 \pi^2 \\
&\quad + \frac{1}{720} (99 \rho_2^D + 137 c^2) b^4 \pi^4 - \frac{1}{20160} b^6 \pi^6 \\
\rho_2^{R_\tau} &= \rho_2^D + I_2 b^2 \\
\rho_3^{R_\tau} &= \rho_3^D + 5 c I_2 b^2 + I_3 b^3 \\
\rho_4^{R_\tau} &= \rho_4^D + (8 \rho_2^D + 7 c^2) I_2 b^2 + 7 c I_3 b^3 + I_4 b^4 \\
\rho_5^{R_\tau} &= \rho_5^D + (12 \rho_3^D + 20 \rho_2^D c + 3 c^3) I_2 b^2 + (12 \rho_2^D + 16 c^3) I_3 b^3 \\
&\quad + \frac{1}{9} (83 I_4 + 28 I_2^3) c b^4 + I_5 b^5 \\
\rho_6^{R_\tau} &= \rho_6^D + (17 \rho_4^D + 28 \rho_3^D c n^2 + 13 (\rho_2^D)^2 + 12 \rho_2^D c^2) I_2 b^2 \\
&\quad + \frac{1}{2} (39 \rho_5^D + 99 \rho_2^D c + 30 c^3) I_3 b^3 \\
&\quad + \frac{1}{36} (612 \rho_2^D + 1081 c^2) I_4 b^4 + \frac{1}{18} (234 \rho_2^D + 277 c^2) I_2^2 b^4 \\
&\quad + \frac{1}{8} (93 I_5 + 60 I_3 I_2) c b^5 + I_6 b^6
\end{align*}

where for convenience we have assigned
\[ I_2 = \frac{169}{576} - \frac{1}{12} \pi^2 \]

\[ I_3 = \frac{1819}{3456} \]

\[ I_4 = \frac{246779}{165888} - \frac{169}{864} \pi^2 + \frac{1}{360} \pi^4 \]

\[ I_5 = \frac{269203}{55296} - \frac{1819}{3456} \pi^2 \]

\[ I_6 = \frac{392305009}{21233664} - \frac{973531}{442368} \pi^2 + \frac{1859}{46080} \pi^4 - \frac{1}{20160} \pi^6 \]
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