Some aspects of self-consistent higher-order interactions

Hideo Sakamoto
Faculty of Engineering, Gifu University, Gifu 501-1193, Japan
E-mail: sakamoto@gifu-u.ac.jp

Abstract. After a brief review of the formalism of the self-consistent higher-order interactions, applications of a \((Q_3^2 Q_3^2)\) type of three-body interaction to the quadrupole moment of the \(3^-\) state in \(^{208}\text{Pb}\) and the energy splitting of the septuplet of states \((h_9/23^-)I\) with \(I = 3/2, 5/2, \ldots, 15/2\) in \(^{209}\text{Bi}\) are discussed. It is shown that if the contribution of the three-body interaction is included, the theoretical value of the \(Q_{el}(3^-)\) moment becomes rather small compared to the experiment, but the observed small energy splitting of the septuplet can essentially be understood within the particle-vibration coupling model. Roles of non-linear field couplings provided by the self-consistent higher-order interactions are also discussed.

1. Introduction

In a nuclear system, couplings between various modes of motion are not always weak but often become quite strong and can even affect fundamental modes themselves. Therefore, the construction of fundamental modes and the study of the couplings between them have been the important subjects for the theory of nuclear structure [1].

Through the study of giant resonances and low-lying collective states in deformed nuclei, it has been recognized that the concept of nuclear self-consistency (i.e. nuclear saturation and the self-consistency between an average potential and a density distribution) is essential to understand nuclear collective motions [1, 2]. By use of this remarkable concept as an important guiding principle, a general theory of self-consistent effective interactions in nuclei has been developed. In particular, higher-order effective interactions (e.g. a three-body interaction, etc.) are derived for general multipole modes under the condition that the system must observe the nuclear self-consistency in higher-order accuracy even when and especially when more than one mode is simultaneously excited in the system [3, 4, 5]. For the case of quadrupole mode, essentially the same type of many-body interactions were derived by Mrashalek [6].

The self-consistent higher-order interactions have been expected to modify coupling terms significantly compared with the cases when one uses only the two-body multipole-multipole interactions [3, 4, 5]. Among the self-consistent higher-order interactions, however, the higher-order quadrupole interactions have been investigated and applied most extensively [7, 8, 9, 10, 11, 12], while for other higher-order multipole interactions there have been relatively few discussions and realistic applications [5].

In this paper, after a brief review of the formalism of the self-consistent higher-order interactions, applications of a \((Q_3 Q_3)^{(2)}\) type of three-body interaction to the quadrupole moment of the \(3^-\) state in \(^{208}\text{Pb}\) and the energy splitting of the septuplet of states \((h_9/23^-)I\) with
\[ I = 3/2, 5/2, \cdots, 15/2 \text{ in } ^{209}\text{Bi} \text{ are presented, and roles of non-linear field couplings originated from the self-consistent higher-order interactions are discussed.} \]

2. Formalism of the self-consistent higher-order interactions

2.1. Improvement of the multipole interaction model

In the 1970s, difficulties of ordinary multipole interactions were pointed out by several authors \[ [2, 13, 14] \]. For example, as shown by Kishimoto et al. \[ [2] \], splittings of GR in deformed nuclei can not be explained by the conventional multipole-multipole (abbreviated as QQ) interaction model. Then an improved interaction model, the so-called doubly-stretched multipole-multipole (abbreviated as Q"Q") interaction model, was proposed. Since then, the doubly-stretched multipole interaction model has been successfully applied to describe deformed nuclei \[ [2, 3, 5, 14, 15, 16, 17] \].

The Q"Q" interaction model has several advantages over the conventional QQ interaction model \[ [3, 5] \]: (i) It guarantees exact elimination of spurious modes. (ii) It satisfies the self-consistency condition and the saturation condition more rigorously in deformed nuclei. (iii) It has the natural form of the effective interaction responsible for a fluctuation mode about a deformed equilibrium. (iv) The Q"Q" with the same strength for each K component can be approximately converted into the form of the QQ with K-dependent strengths. Such different strengths for each K agree very well with those used empirically in the QQ model for the analyses of low-lying collective quadrupole and octupole states in deformed nuclei.

Furthermore, for deformed nuclei, it has been shown that the double-stretched 2\(^\lambda\)-pole interaction in general is nothing but an improved 2\(^\lambda\)-pole interaction which effectively includes the two-body interaction part of the self-consistent three-body interactions to satisfy the nuclear self-consistency in higher-order accuracy \[ [4, 5] \].

2.2. Higher-order quadrupole interactions

Concerning the origin of the conventional quadrupole-quadrupole interaction in nuclei, there is a famous explanation by Mottelson in terms of the core polarization and effective charge phenomena \[ [18] \]. Along the line of it, higher-order quadrupole interactions have been derived to guarantee the nuclear self-consistency in higher-order accuracy for quadrupole deformed nuclei \[ [3, 4, 5, 6] \]. In the harmonic oscillator model, since the analytic expression of the total energy functional and that of the nuclear self-consistency condition (Mottelson’s relation) are available, by employing the Landau theory of Fermi systems, the effective n-body interaction can be derived by carrying out the nth order derivative of the total energy functional with respect to occupation probabilities. The resultant quadrupole interactions up to the fourth-order are expressed as

\[ V^{(2)}_{\lambda=2} = -\frac{\chi_2}{2} (\hat{Q}_{\lambda=2} \cdot \hat{Q}_{\lambda=2}) \quad \text{with} \quad \chi_2 = \frac{4\pi}{2\lambda + 1} \frac{m\omega_0^2}{A (r^{2\lambda-2})}, \]

\[ V^{(3)}_{\lambda=2} = \frac{1}{3!} \frac{\chi_2}{A (r^2)} \left\{ 3(\hat{Q}_2 \cdot \hat{Q}_2) \hat{R}_0 - \sqrt{\frac{56\pi}{5}} \left( [\hat{Q}_2 \hat{Q}_2]^{(2)} \cdot \hat{Q}_2 \right) \right\}, \]

\[ V^{(4)}_{\lambda=2} = -\frac{1}{4!} \frac{\chi_2}{A (r^2)^2} \left\{ 12(\hat{Q}_2 \cdot \hat{Q}_2) \hat{R}_0^2 - 8 \sqrt{\frac{56\pi}{5}} \left( [\hat{Q}_2 \hat{Q}_2]^{(2)} \cdot \hat{Q}_2 \right) \hat{R}_0 + \frac{48\pi}{5} (\hat{Q}_2 \cdot \hat{Q}_2)^2 \right\}. \]

Applications of the higher-order quadrupole interactions to realistic nuclei have been worked out by several authors \[ [7, 8, 9, 10, 11, 12] \].

2.3. Higher-order interactions for general multipole modes

By use of the nuclear self-consistency as an important guiding principle, a theory of higher-order effective interactions for general multipole modes has been developed, and for shape oscillation...
modes in a harmonic oscillator potential model, an analytical expression of the self-consistent
three-body interactions for general multipole modes has been presented [5]. For example, the
three-body interaction for the dipole mode and that for the octupole mode are expressed as
\[ V^{(3)}_{\lambda=1} = \frac{1}{3!} \chi^1 \left\{ 3(\hat{Q}_1 \cdot \hat{Q}_1)R_0 + 3\sqrt{\frac{24\pi}{5}} \left( [\hat{Q}_1 \hat{Q}_1]^{(2)} \cdot \hat{Q}_2 \right) \right\} , \]  
\[ V^{(3)}_{\lambda=3} = \frac{1}{3!} \chi^3 \left\{ 3(\hat{Q}_3 \cdot \hat{Q}_3)R_0 + \frac{18}{5} \sqrt{\frac{84\pi}{5}} \left( [\hat{Q}_3 \hat{Q}_3]^{(2)} \cdot \hat{Q}_2 \right) \right\} . \] 
For the \( V^{(3)}_{\lambda=1} \) interaction, it has been shown that the \( ([\hat{Q}_1 \hat{Q}_1]^{(2)} \cdot \hat{Q}_2) \) term plays a role to recover
the translational invariance of a quadrupole vibrating system, while the \( (\hat{Q}_1 \cdot \hat{Q}_1)R_0 \) term is
necessary for the particle-vibration coupling model to resolve the problem of divergence in the
self-energy of a single-particle due to the coupling to the translational mode [5]. Some aspects
of the \( ([\hat{Q}_3 \hat{Q}_3]^{(2)} \cdot \hat{Q}_2) \) term in the \( V^{(3)}_{\lambda=3} \) interaction will be discussed in the next section.

3. Quadrupole-octupole coupling in \(^{208}\text{Pb}\) region
In the nucleus \(^{209}\text{Bi}\), there exists a septuplet of states \((h_9/2^3)I\) with \( I = 3/2, 5/2, \ldots, 15/2 \) at
about 2.6 MeV which is almost equal to the excitation energy of the \( 3^- \) state of \(^{208}\text{Pb}\). The
observed small energy splitting of the septuplet components implies that the particle-octupole
coupling here is rather weak.

![Diagrams](image)

\textbf{Figure 1.} The second-order diagrams of the two-body interaction contributing to the energy
of a particle-phonon multiplet [1].

From the late 1960’s to the 1970’s, detailed calculations were performed by Bohr and
Mottelson [1, 18, 19] and Hamamoto [20, 21] from the perturbational point of view. They
employed the particle-vibration coupling Hamiltonian \( H_{\text{coupl}} = -R_0 \frac{\partial}{\partial r} (Y_3 \cdot \alpha_3) \), where the amplitude of the octupole vibration was determined from the experimental \( B(E3) \) value.
Then it was shown that the structure of the above small energy splitting can essentially be
reproduced by estimating the second-order particle-octupole coupling illustrated by the diagrams
in Fig.1, with a correction for the \( I = 3/2 \) level which reflects the coupling to the \((d_{3/2}^+)^0^+\) configuration. They also evaluated the quadrupole moment of the \( 3^- \) state in \(^{208}\text{Pb}\) by counting
the diagrams illustrated in Fig.2. In the evaluation the polarization effect was included. As
the result, Hamamoto got the calculated quadrupole moment of \(-0.13\text{b} (-0.14\text{b})\) for the case of \( e_{eff}^p(E2) = 1.5e \) \( (2e) \) and \( e_{eff}^n(E2) = e \) [20], and in the text book by Bohr and Mottelson the theoretical estimation was reported as \( Q_{el}(3^-) \sim -0.1b \) [1]. Then the effect of the quadrupole
moment of the \( 3^- \) state on the energy splitting of the septuplet in \(^{209}\text{Bi}\) was evaluated by
calculating the diagonal matrix element of the interaction through the quadrupole moments of the phonon and the particle shown in Fig. 3. The resultant correction to the energy shift was estimated as

$$\delta E((h_{9/2}3^-)I) = 1.4Q_{el}(3^-)Q_{el}(h_{9/2})(-1)^{I-1/2}W(3\frac{9}{2}, \frac{9}{2}, I2) \quad \text{keV} \cdot \text{fm}^{-4}$$

(3)

for the case of $\epsilon_{pol}(E2)=0.5e$, and it was indicated that if the value of $Q_{el}(3^-) \sim -1.0b$ is used referring to the reported experimental values at that time [22, 23], the energy splitting of the septuplet becomes too big.

In 1984, Vermerr et al. published an updated value for the quadrupole moment of the $3^-$ state in $^{208}$Pb as $Q_{el}^{exp}(3^-) = -0.34 \pm 0.15 \text{ b}$ [24, 25], and the value was used to reestimate the energy splitting of the septuplet in $^{209}$Bi using coupling constants and energy shifts calculated by Bohr and Mottelson [1] and Hamamoto [21]. As the result, they concluded that the experimental value of $-0.34 \pm 0.15 \text{ b}$ is somewhat larger than theoretical predictions and a value of $Q_{el}(3^-) = 0$ would give a better fit, but can in principle be accommodated within standard nuclear theories.

In the estimation of Fig. 3 by Bohr and Mottelson, by Hamamoto, and probably also by Vermerr et al. accordingly, the value of $Q_{el}(h_{9/2}) \sim -0.4b$ was used considering the experimental data known at that time [26]. Nowadays, however, there are various experimental data accumulated for the $Q_{el}(h_{9/2})$ ranging from $-0.37(4)$ to $-0.71(1)\text{ b}$ [27, 28], which should be confirmed [29]. Therefore, in case the experimental $Q_{el}(h_{9/2})$ is not so small, the above conclusion of Vermerr et al. needs to be reconsidered.

Furthermore, from our present point of view, also the particle-vibration coupling has to be reexamined by including non-linear field couplings coming from the self-consistent higher-order interactions. For the quadrupole-octupole coupling in $^{208}$Pb region under consideration, the three-body ($[Q_3Q_3]^{(2)} \cdot Q_2$) interaction appearing in Eq. (2) is expected to play important roles together with the two-body ($Q_2 \cdot Q_2$) and ($Q_3 \cdot Q_3$) interactions [30]. The first-order diagrams of the three-body interaction contributing to the quadrupole moment of the $3^-$ state are illustrated in Fig. 4.

I have calculated the contribution of the first-order diagrams by use of the modified oscillator model [31]. In the numerical calculation, all the particle-hole excitations within $3 \leq N_p^{\text{osc}} \leq 7$ and $4 \leq N_n^{\text{osc}} \leq 8$ have been included for the quadrupole mode, and for the amplitude of the collective octupole phonon the value of $\alpha_3 = 0.045$, which corresponds to $B(E3) = 39B_{sp}(E3)$ [1], has been adopted. As the result, the correction from the three-body interaction to the $Q_{el}(3^-)$...
moment has been estimated as $\delta Q_{el}(3^-) \sim 0.09b$. The estimation is somewhat preliminary because of the uncertainties involved in the calculation, but if the correctional value is added, for example, to the theoretical value estimated by Bohr-Mottelson [1] and Hamamoto [20, 21], the total $Q_{el}(3^-)$ moment becomes rather small. In this case, the theoretical moment might be too small compared to the experiment, but the observed small energy splitting of the septuplet can essentially be understood within the particle-vibration coupling model.

Thus, for the quadrupole moment of the octupole phonon under consideration, there seems to be a partial cancellation between the contribution of the second-order diagrams of the linear field-coupling, i.e. $H_{coupl}$, coming from the two-body interaction (Fig. 2) and that of the first-order diagrams of the quadratic field-coupling coming from the three-body interaction (Fig.4). Such a cancellation can also be seen in the energy shift of the septuplet. The leading-order contribution of the three-body interaction to the energy shift of the septuplet is illustrated in Fig.5, and is calculated as

$$
\delta E'(h_{9/2}3^-)I = -\chi_3' \langle 3^- || [\hat{Q}_3 \hat{Q}_3]^{(2)} || 3^- \rangle \langle h_{9/2} || \hat{Q}_2 || h_{9/2} \rangle (-1)^I - 1/2 W(9_2 3_2 9_2; I2) \tag{4}
$$

with

$$
\chi_3' = \frac{3}{5} \sqrt{\frac{84\pi}{5}} \frac{\chi_3}{A \langle r^2 \rangle_0}.
$$

$\delta E$ and $\delta E'$ have the same Racah coefficient, and are roughly the same order of magnitude with opposite signs with each other. Thus the contribution of Fig.3 to the energy splitting of the septuplet in $^{209}$Bi is reduced by including the correction coming from the three-body interaction illustrated in Fig.5. As the result, the beautiful explanation for the structure of the energy splitting in terms essentially of the second-order diagrams in Fig.1 can be revived [32].

4. Summary and conclusions

Some properties of the self-consistent higher-order interactions, which have been derived to satisfy the nuclear self-consistency in higher-order accuracy, are sketched. The higher-order interactions have been expected to modify coupling terms significantly compared with the cases when one uses only the two-body multipole-multipole interactions.
To study roles especially of the \([Q_3 Q_3]^{(2)} \cdot Q_2\) type of the self-consistent three-body interaction, the quadrupole moment of the 3\(^{-}\) state in \(^{208}\text{Pb}\) and the energy splitting of the septuplet of states \((h_9/2^+)\) with \(I = 3/2, 5/2, \cdots, 15/2\) in \(^{209}\text{Bi}\) are discussed from the perturbational point of view. It is shown that if the contribution of the three-body interaction is included, the theoretical value of the \(Q_{el}(3^{-})\) moment becomes rather small compared to the experiment, but the observed small energy splitting of the septuplet can essentially be understood within the particle-vibration coupling model.

The above cancellation mechanism seems to be one of the examples of the important role for the higher-order interactions. In general, it can be said that the self-consistent higher-order interactions provide mechanisms to restore the order in the system when more than one mode is simultaneously excited in a system.

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