LINKS AND HURWITZ CURVES

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Abstract. In the note, we give a proof, based on the Generalized Thom Conjecture, of Bennequin’s Theorem on upper bound for the Euler number of a link which is considered as a closed braid. A lower bound for the Euler number of a link is also given.

1. Introduction

Let $l$ be a link in the three-dimensional sphere $S^3$ consisting of $k$ components. Recall that an oriented surface $S \subset S^3$ is called a Seifert surface of the link $l$ if the boundary $\partial S$ of $S$ coincides with $l$ and $S$ has not a closed component (without boundary). Let $\chi(S)$ be the Euler characteristic of $S$. By definition, the Euler number $e(l)$ of $l$ is

$$e(l) = \max_S \chi(S), \quad (1)$$

where the maximum is taken over all Seifert surfaces of $l$. Note that if $l$ is a knot of genus $g$, then

$$e(l) = 1 - 2g. \quad (2)$$

By Alexander’s theorem (see [Al]), there is a number $m \in \mathbb{N}$ such that a given link $l$ is equivalent to a closed braid $\overline{b}$ (notation: $l \cong \overline{b}$), where $b$ is a braid in the braid group $Br_m$ on $m$ strings.

Below, we fix a set $\{a_1, \ldots, a_{m-1}\}$ of so called standard generators of $Br_m$, i.e., generators being subject to the relations

$$a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1}, \quad 1 \leq i \leq m - 2,$$
$$a_i a_k = a_k a_i, \quad |i - k| \geq 2$$

and extend this set of generators to a set of generators $\{a_{i,j}\}_{1 \leq i < j \leq m}$, where $a_{i,i+1} = a_i$ and

$$a_{i,j} = (a_{j-1} a_{j-2} \ldots a_{i+1}) a_i (a_{j-1} a_{j-2} \ldots a_{i+1})^{-1}$$

The work was partially supported by RFBR (No. 02-01-00786).
for $j - i \geq 2$. An element $b \in \text{Br}_m$ can be presented as a word in the alphabet $\{a_{i,j}, a_{i,j}^{-1}\}_{1 \leq i < j \leq m}$:

$$b = w(a_{1,2}, \ldots, a_{m-1,m}) = \prod_{k=1}^{n_w} a_{i_k,j_k}^{\varepsilon_k},$$

(3)

where $\varepsilon_k = \pm 1$. The minimum

$$| b | = \min_{w(a_{i,j}) = b} n_w,$$

where the minimum is taken over all presentations of $b$ in the form (3) is called the length of $b$.

As is known, if braids $b_1$ and $b_2$ are conjugated in $\text{Br}_m$, then the closed braids $\overline{b}_1$ and $\overline{b}_2$ are equivalent links. The number

$$|| \overline{b} || = \min_{g \in \text{Br}_m} | g^{-1} b g |$$

is called the norm of a closed braid $\overline{b}$.

Let $B_{l,m} = \{ b \in \text{Br}_m \mid l \simeq \overline{b} \}$ be the set of closed braids on $m$ strings equivalent to $l$. If $B_{l,m} \neq \emptyset$, then the number

$$|| l ||_m = \min_{b \in B_{l,m}} || \overline{b} ||$$

is called the $m$-norm of a link $l$.

Denote by $\tilde{\text{Br}}_m^+$ the semigroup generated in the braid group $\text{Br}_m$ by the set $\{a_{i,j}\}_{1 \leq i < j \leq m}$. An element $b \in \text{Br}_m$ is called positive (respectively, negative) if $b \in \tilde{\text{Br}}_m^+$ (respectively, if $b^{-1} \in \tilde{\text{Br}}_m^+$).

Consider the homomorphism $\deg : \text{Br}_m \to \text{Br}_m/[[\text{Br}_m, \text{Br}_m]] \simeq \mathbb{Z}$ sending all $a_{i,j}$ to $1 \in \mathbb{Z}$. The image $\deg b$ of an element $b \in \text{Br}_m$ is called the degree of $b$.

The aim of this note is to give a proof, based on the Generalized Thom Conjecture, of Bennequin’s Theorem \(^1\) (Ben, Ben2) on upper bound for the Euler number $e(l)$ in terms of invariants of a closed braid $\overline{b} \simeq l$ and also to give some lower bound for it.

**Theorem 1.1.** Let a link $l$ be presented as a closed braid $\overline{b}$ for some $b \in \text{Br}_m$. Then

$$m - || \overline{b} || \leq e(l).$$

(4)

**Theorem 1.2.** (Ben, Ben2) Let a link $l$ be presented as a closed braid $\overline{b}$ for some $b \in \text{Br}_m$. Then

$$e(l) \leq m - | \deg b |.$$

(5)
The idea of the proof of Theorem 1.2 is the following. First of all, it is easy to see that the general case can be reduced to the case deg $b \geq 0$. Then for a given link $l \simeq \overline{b}$, where $b \in \mathrm{Br}_m$, deg $b \geq 0$, applying results obtained in [Kh-Ku] about so called Hurwitz curves in the complex Hirzebruch surface $F_N$, we construct smooth real surface $S$ and algebraic curve $C$ lying in $F_N$ for some $N \geq 1$ and having the genera $g(S) = 1 + (Nm(m-1)-m-e(l)-\deg b)/2$ and $g(C) = 1 + (Nm(m-1)-2m)/2$, and such that $[S] = [C]$, where $[C], [S] \in H_2(F_N, \mathbb{Z})$ are the homology classes represented by real two-dimensional surfaces $C$ and $S$. Now, the proof of Theorem 1.2 follows from the Generalized Thom Conjecture proved in [M-S-T] and asserting that $g(C) \leq g(S)$.

Since $\deg b = ||\overline{b}||$ for $b \in \widetilde{\mathrm{Br}}_m$, we have the following corollary.

**Corollary 1.3.** Let a link $l$ be presented as a closed braid $\overline{b}$ for some positive or negative element $b \in \mathrm{Br}_m$. Then

$$||l||_m = ||\overline{b}|| = |\deg b|; \quad (6)$$

$$e(l) = m - ||l||_m. \quad (7)$$

Obviously, $e(l) = k$ for a trivial link $l$ consisting of $k$ connected components. Therefore we have the following corollary.

**Corollary 1.4.** Let a link $l$ consisting of $k$ connected components be presented as a closed braid $\overline{b}$ for some element $b \in \mathrm{Br}_m$. If

$$k > m - |\deg b|$$

then $l$ is a non-trivial link.

**Acknowledgement.** The author thanks I.A. Dynnikov for references and his helpful remarks during the preparation of this paper.

2. **Proof of theorem 1.1**

To prove Theorem 1.1 let us identify the sphere $S^3$ with the boundary $\partial D = (\partial D_1) \times D_2 \cup D_1 \times \partial D_2$ of a bi-disc

$$D = D_1 \times D_2 = \{(z, w) \in \mathbb{C}^2 \mid |z| \leq 1, \ |w| \leq 2\}. $$

Choose $m$ points $w_k = e^{2\pi i k \frac{1}{m}} \in D_2 = \{ |w| \leq 2 \}$, $k = 1, \ldots, m$, and identify the braid group $\mathrm{Br}_m$ with the braid group $\mathrm{Br}[D_2, \{w_1, \ldots, w_m\}]$. In this case the generators $a_{i,j}$ are identified with half-twists along the segments $w = tw_i + (1-t)w_j$, $t \in [0,1]$ (see Fig. 1), and $\overline{b}$ with a closed braid lying in $(\partial D_1) \times D_2.$
Let a link \( l \simeq b \), where
\[
b = \prod_{k=1}^{n_b} a^{\varepsilon_k}_{i_k, j_k} \in \mathrm{Br}_m, \quad \varepsilon_k = \pm 1.
\]

In this case one can construct a Seifert surface \( S \) of the link \( l \) similar to the construction in the standard case when the link \( l \) is represented as a projection of \( l \) to a plane whose image is an immersed curve with simple intersections (Wirtinger presentation). Namely, take \( m \) discs
\[
S_j = \{(z, w) \in \mathbb{S}^3 \mid ||z|| \leq 1, \ w = 2e^{\frac{2\pi \sqrt{-1}}{m} k} \} \subset D_1 \times \partial D_2,
\]
j = 1, \ldots, m, glue each \( S_j \) along a circle
\[
C_j = \{(z, w) \in \mathbb{S}^3 \mid ||z|| = 1, \ w = 2e^{\frac{2\pi \sqrt{-1}}{m} k} \} \subset \partial D_1 \times \partial D_2
\]
with an annulus
\[
A_j = \{(z, w) \in \mathbb{S}^3 \mid ||z|| = 1, \ w = 2te^{\frac{2\pi \sqrt{-1}}{m} k} + (1-t)e^{\frac{2\pi \sqrt{-1}}{m} k}, \ t \in [0, 1] \}
\]
and put \( \overline{S}_j = S_j \cup C_j \). Obviously, each \( \overline{S}_j \) is a disc. Next, in each
\[
(\partial D_1)_k \times D_2 = \{(z, w) \in (\partial D_1) \times D_2 \mid z = e^{\frac{2\pi \sqrt{-1}}{m} k}, \ k - \frac{1}{3} \leq t \leq k + \frac{1}{3} \}
\]
let us attach a band \( B_k \simeq [0, 1] \times [0, 1] \) to \( \overline{S}_{i_k} \) and \( \overline{S}_{j_k} \) in dependence on the sign of \( \varepsilon_k \) as it is depicted in Fig. 2.

As a result, we obtain a surface \( S \) in the sphere \( S^3 \) with the boundary \( \overline{b} \). Obviously, the Euler characteristic \( \chi(S) = m - n_b \). Therefore, Theorem 1.1 is proven.

Fig. 1
3. PROOF OF THEOREM 1.2

To prove Theorem 1.2, let us, in the beginning, briefly recall definitions of topological Hurwitz curves and their braid monodromy factorizations given in [Kh-Ku]. For a group $\text{Br}_m$ one can define the factorization semigroup $S_{\text{Br}_m}$. For this, consider an alphabet

$$X = \{ x_g | \ g \in \text{Br}_m \}$$

and two sets of relations:

- $R_{g_1,g_2;r}$ stands for $x_{g_1} \cdot x_{g_2} = x_{g_2} \cdot x_{g_1}^{-1}g_1g_2$ if $g_2 \neq 1$ and $x_{g_1} \cdot x_1 = x_{g_1}$;
- $R_{g_1,g_2;l}$ stands for $x_{g_1} \cdot x_{g_2} = x_{g_1g_2g_1^{-1}} \cdot x_{g_1}$ if $g_1 \neq 1$ and $x_1 \cdot x_{g_2} = x_{g_2}$.

Now, put

$$\mathcal{R} = \{ R_{g_1,g_2;r}, R_{g_1,g_2;l} | (x_{g_1}, x_{g_2}) \in X \times X, g_1 \neq g_2 \}$$

and introduce the semigroup

$$S_{\text{Br}_m} = \langle x \in X : R \in \mathcal{R} \rangle$$

by means of this relation set $\mathcal{R}$. Introduce also a product homomorphism $\alpha : S_{\text{Br}_m} \to \text{Br}_m$ given by $\alpha(x_g) = g$ for each $x_g \in X$.

Denote by $F_N$ a relatively minimal ruled rational complex surface (a Hirzebruch surface), $N \geq 1$, $\text{pr}: F_N \to \mathbb{C}P^1$ the ruling, $R$ a fiber of $\text{pr}$ and $E_N$ the exceptional section, $E_N^2 = -N$. The variety $F_N \setminus (E_N \cup R)$ is naturally isomorphic to the complex affine plane $\mathbb{C}^2$ with complex coordinates $(z, w)$ such that $\text{pr}(z, w) = z$.

By definition, the image $\hat{H} = f(S) \subset F_N$ of a continuous map $f : S \to F_N \setminus E_N$ of an oriented closed real surface $S$ is called a topological Hurwitz curve (in $F_N$) of degree $m$ if there is a finite subset $Z \subset \hat{H}$ such that:
(i) $f$ is a smooth embedding of the surface $S \setminus f^{-1}(Z)$ and for any
$p \notin Z$, $\bar{H}$ and the fiber $R_{pr}(p)$ of $pr$ meet at $p$ transversely with
positive intersection number;
(ii) the restriction of $pr$ to $\bar{H}$ is a finite map of degree $m$. (We call
a map finite if the preimage of each point is finite.)

Choose a fibre $R = R_\infty$ being in general position with a topological
Hurwitz curve $H$. Put $\mathbb{C}^2 = F_N \setminus (E_N \cup R_\infty)$ and fix complex co-
ordinates $(z, w)$ in $\mathbb{C}^2$ such that $pr(z, w) = z$. At any point $p \in Z$
there is a well-defined $(W$-prepared) germ $(D, H = \bar{H} \cap D, pr)$ of this
curve in a bi-disc $D = D_1 \times D_2$, $D_1 = D_1(\epsilon_1) = \{ | z - z(p) | \leq \epsilon_1 \}$,
$D_2 = D_2(\epsilon_2) = \{ | w - w(p) | \leq \epsilon_2 \}$, $0 < \epsilon_1 << \epsilon_2$, centered at $p$ and
such that the restriction of $pr$ to $\bar{H}$ is a proper map of a finite degree
$k \leq m$. If $\epsilon_1, \epsilon_2$ are sufficiently small, then: $R_{pr(p)} \cap H = p$; the above
degree does not depend on $\epsilon_1, \epsilon_2$; and the link $\partial D \cap H$ defines a unique,
up to conjugation, braid $b \in Br_k \subset Br_m$, where $k$ is the above degree.
So that, we may speak on a $tH$-singularity $(D, H, pr)$ of degree $k$ and
type $b$.

When we are given a link $l \subset \partial D_1 \times D_2$ realizing a braid $b \in B_k$, we
associate with it a standard conical model of a topological singularity
of type $b$. It is given by $H = C(l),$

$$C(l) = \{ (rz, rw) | 0 \leq r \leq 1, (z, w) \in l \}.$$ 

As is known (see, for example, [Kh-Ku]), if $(D, C, pr)$ is a germ of a
$W$-prepared $tH$-singularity then the germ $(D, C, pr)$ is homeomorphic
to the cone singularity of type $b = pr^{-1}(\partial D_1) \cap C$.

Since $\bar{H} \cap E_N = \emptyset$, one can define a braid monodromy factorization
$b(\bar{H}) \in S_{Br_m}$ of $\bar{H}$. For doing this, we fix a fiber $R_\infty$
meeting transversely $\bar{H}$ and consider $\bar{H} \cap \mathbb{C}^2$, where $\mathbb{C}^2 = F_N \setminus (E_N \cup R_\infty)$. Choose
$r_1 >> 1$ such that $pr(Z) \subset D_1(\epsilon_1)$ = $\{ | z | \leq r_1 \} \subset \mathbb{C} = \mathbb{CP}^1 \setminus pr(R_\infty)$.
Denote by $z_1, \ldots, z_n$ the elements of the set $pr(Z)$ and assume that for
each $i$ the intersection $pr^{-1}(z_i) \cap Z$ consists of a single point. Pick $\rho,
0 < \rho << 1$, such that the discs $D_{1,i}(\rho) = \{ z \in \mathbb{C} | | z - z_i | < \rho \}$,
i = 1, $\ldots$, $n$, would be disjoint. Select arbitrary points $u_i \in \partial D_{1,i}(\rho)$
and a point $u_0 \in \partial D_1(r)$. Let $D_2(r_2) = \{ w \in \mathbb{C} | | w | \leq r_2 \}$ be a disc
of radius $r_2 >> 1$ such that $\bar{H} \cap \partial D_{1,i}(\rho) \subset D_1(\epsilon_1) \times D_2(r_2)$.
Put $D_{2,u_0} = \{ (u_0, w) \in \mathbb{C}^2 | | w | \leq r_2 \} \subset pr^{-1}(u_0)$, $K(u_0) = \{ w_1, \ldots, w_n \} = D_{2,u_0} \cap \bar{H}$, and $Br_m = Br[D_{2,u_0}, K(u_0)]$. Choose dis-
joint simple paths $l_i \subset D_{1,i}(\rho) \setminus \bigcup_1^n D_{1,i}(\rho)$, $i = 1, \ldots, n$, starting at $u_0$
and ending at $u_i$ and renumber the points in a way that the product
$\gamma_1 \ldots \gamma_n$ of the loops $\gamma_i = l_i \circ \partial D_{1,i}(\rho) \circ l_i^{-1}$ would be equal to $\partial D_1(\epsilon_1)$
in $\pi_1(D_1(\epsilon_1) \setminus \{ z_1, \ldots, z_n \}, u_0)$. Each $\gamma_i$ defines an element $b_i \in Br_m$
represented by the paths $\text{pr}^{-1}(\gamma_i) \cap \overline{H}$ starting and ending at the points lying in $K(u_0)$. The factorization $b(\overline{H}) = x_{b_1} \cdots x_{b_n} \in S_{Br_m}$ is called a braid monodromy factorization of $\overline{H}$.

Denote by $\Delta^2_m = (a_{1,2}a_{2,3} \cdots a_{m-1,m})^m$ a generator of the center of the group $Br_m$. It is easy to prove the following lemma (see, for example, [Kh-Ku]):

**Lemma 3.1.** For a topological Hurwitz curve $\overline{H} \subset F_N$ it holds

$$\alpha(b(\overline{H})) = \Delta^2_N.$$ 

The converse statement can be also proved in a straightforward way.

**Theorem 3.2.** ([Kh-Ku]) For any $s = x_{b_1} \cdots x_{b_n} \in S_{Br_m}$ such that $\alpha(s) = \Delta^2_N$ there is a topological Hurwitz curve $\overline{H} \subset F_N$ with a braid monodromy factorization $b(\overline{H})$ equal to $s$.

Now we are able to prove inequality (5). First of all, it easy to see that if $l \simeq \overline{b}$ for some $b \in Br_m$, then the link $b^{-1}$ is equivalent to the mirror-image $\overline{l}^{-1}$ of the inverted link $l^{-1}$. Therefore, to prove inequality (5), we can assume that $\deg b \geq 0$, since $e(l) = e(l^{-1}) = e(\overline{l}^{-1})$.

It follows from Theorem 5 in [G] (see, for example, Lemma 1.3 in [Kh-Ku]) that for any $b \in B_m$ there is a positive element $r \in Br_m^+$ and a positive integer $N \geq 1$ such that $rb = \Delta^2_m$. We have $\deg \Delta^2_m = m(m-1)$. Therefore $\deg r = Nm(m-1) - \deg b > 0$. Let

$$r = \prod_{k=1}^{\deg r} a_{i_k,j_k} \quad (8)$$

be a presentation of $r$ as a word in the alphabet $\{a_{i,j}\}^{1 \leq i < j \leq m}$. Factorization (8) defines an element

$$s = \left( \prod_{k=1}^{\deg r} x_{a_{i_k,j_k}} \right) \cdot x_b$$

in the factorization semigroup $S_{Br_m}$. The element $s$ is a braid monodromy factorization of a topological Hurwitz curve $\overline{H} \subset F_N$ whose set $\text{pr}(Z)$ of the critical values consists of points $z_k = \deg r - k + 2$ for $k = 1, \ldots, \deg r$ and $z_{\deg r+1} = 0$, and whose braid monodromy over $z_k, k = 1, \ldots, \deg r$, is equal to the $k$-th factor $a_{i_k,j_k}$ entering in (8), and whose braid monodromy over the point $z_0$ is equal to $b$. Moreover, without loss of generality, we can assume that $\overline{H} \cap \text{pr}^{-1}(\partial D_1) = \overline{b} \subset (\partial D_1) \times D_2$, where $D_1 = \{|z| \leq 1\}$ and $D_2 = \{|w| \leq r\}$ for some
and by definition of topological Hurwitz curves, since 
\( r \gg 1 \). Since all \( a_{i,j} \) are conjugated to \( a_{1,2} \) and the element \( a_{1,2} \) is the monodromy of the critical value of the function given by \( w^2 = z \), then we can assume that the Hurwitz curve \( S_2 = \overline{H} \cap \text{pr}^{-1}(\mathbb{CP}^1 \setminus D_1) \) is a smooth real surface in \( F_N \).

Consider the restriction of \( \text{pr} \) to \( S_2 \):

\[
\text{pr}|_{S_2} : S_2 \to D_{\geq 1} = \mathbb{CP}^1 \setminus D_1.
\]

The Euler characteristic of \( S_2 \) is equal to

\[
\chi(S_2) = m - Nm(m - 1) + \deg b,
\]

since \( D_{\geq 1} \) is a disc, \( \text{pr}|_{S_2} \) has \( \deg r = Nm(m - 1) - \deg b \) simplest critical values, and \( \deg \text{pr}|_{S_2} = m \).

Let \( S_1 \subseteq \partial(D_1 \times D_2) \) be a Seifert surface of the link \( \overline{b} \simeq l \). We can assume that

\[
\chi(S_1) = e(l).
\]

Consider a surface \( S \) in \( F_N \) which is obtained from \( S_1 \) and \( S_2 \) by gluing along \( \overline{b} \). Obviously, \( S \) is a closed real surface. Without loss of generality (after small deformation of \( S \) near \( \overline{b} \)), we can assume that \( S \) is a smooth surface. Since \( \chi(\overline{b}) = 0 \), the Euler characteristic

\[
\chi(S) = \chi(S_1) + \chi(S_2) = e(l) + m - Nm(m - 1) + \deg b. \quad (9)
\]

Consider the class \([S]\) of \( S \) in the homology group \( H_2(F_N, \mathbb{Z}) \). As is known, the group \( H_2(F_N, \mathbb{Z}) \) is generated by the class \([R]\) of a fibre \( R \) of \( \text{pr} \) and the class \([E_N]\) of the exceptional section \( E_N \) which have the following intersection numbers: \([R] \cdot [R] = 0, [R] \cdot [E_N] = 1\), and \([E_N] \cdot [E_N] = -N\). By construction of \( S \), we have \([S] \cdot [R] = \deg \overline{H} = m\) (to see this, one can consider the intersection of \( \overline{H} \) and a fibre \( R_\ell \) lying over a point \( z \in D_{\geq 1} \)) and \([S] \cdot [E_N] = 0\), since \( S_1 \subseteq D \subseteq \mathbb{C}^2 \subseteq F_N \setminus E_N \) and by definition of topological Hurwitz curves, \( \overline{H} \cap E_N = \emptyset \). Therefore

\[
[S] = m[E_N] + Nm[R].
\]

Let \( C \subseteq F_N \) be a non-singular algebraic curve whose class \([C]\) = \( mE_N + Nm[R] \). It is well-known that its genus

\[
g(C) = (Nm(m - 1) - 2m)/2 + 1. \quad (10)
\]

Since \( C \subseteq F_N \) is an algebraic non-singular curve, it follows from the Generalized Thom Conjecture proved in [M-S-T] that \( \chi(S) \leq \chi(C) = 2 - 2g(C) \) for any smooth surface \( S \subseteq F_N \) whose class \([S]\) = \([C]\) in \( H_2(F_N, \mathbb{Z}) \). Therefore, applying (9) and (10), we have

\[
\chi(S) = e(l) + m - Nm(m - 1) + \deg b \leq 2m - Nm(m - 1).
\]

Thus,

\[
e(l) \leq m - \deg b.
\]
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