The point of departure of a particle sliding on a curved surface

Amir Aghamohammadi

Department of Physics, Alzahra University, Tehran 19384, Iran
E-mail: mohamadi@alzahra.ac.ir

Received 2 May 2012, in final form 28 May 2012
Published 28 June 2012
Online at stacks.iop.org/EJP/33/1111

Abstract
A particle is thrown tangentially on a surface. It is shown that for some surfaces and for special initial velocities the thrown particle immediately leaves the surface, and for special conditions it never leaves the surface. The conditions for leaving the surface are investigated. The problem is studied for a surface with the cross-section \( y = f(x) \). The surfaces with the equations \( f(x) = -\alpha x^k (\alpha, k > 0) \) are considered in more detail. Finally, the effect of friction is also considered.

1. Introduction
One of the standard problems in elementary mechanics is finding the point where a sliding particle on a frictionless sphere loses contact with the sphere [1–4]. A particle at the top of a sphere with the radius \( R \) is thrown tangentially with the velocity \( v_0 \). The particle leaves the sphere immediately, provided that \( v_0 \geq v_c := \sqrt{Rg} \). For velocities less than \( v_c \) it will move on the sphere and leave it later. A more realistic variation on this problem is to consider what happens when kinetic friction is present. This problem has been solved exactly [5]. The problem of sliding a particle on a sphere with friction is studied through a perturbation expansion in the coefficient of sliding friction and an exact integration of the equation of motion [6].

The motion of a particle on an arbitrary surface, provided that the particle’s path is always in a vertical plane, is addressed in this paper. It is shown that for some surfaces there is no initial velocity at which the particle leaves the surface immediately, and there are some other surfaces for which the thrown particle with any arbitrary nonzero initial velocity immediately leaves the surface. There is a surface for which—depending on initial velocity—the particle leaves the surface immediately, or never leaves the surface. Finally, there are surfaces for which the particle never leaves the surface for any arbitrary initial velocity. The problem is studied for a surface with the cross-section \( y = f(x) \). The case \( f(x) = -\alpha x^k (\alpha, k > 0) \) is considered in more detail. The effect of friction is considered at the end of the paper. The problem addressed here may be of interest in the framework of an undergraduate course in mechanics.
2. A particle sliding on a parabolic surface

As a simple example, let us first consider the motion of a particle on a frictionless parabolic surface with the cross-section \( y = -\alpha x^2 \), where the \( y \) axis is vertical and the \( x \) axis horizontal (see figure 1). The particle’s initial velocity is taken to be \( v_0 > 0 \), and it is thrown in the \( x \) direction from the origin. Let us consider a circle whose radius is \( R \) passing through the origin, and is tangent to the parabola. The equation of the circle is

\[
x^2 + (y + R)^2 = R^2.
\]

At the vicinity of the origin the equation of the circle is

\[
y = -R + \sqrt{R^2 - x^2} \approx -\frac{x^2}{2R}.
\]

If the second derivative of the two curves at the origin is the same, then the radius of the curvature of the parabola is the same as the radius of circle \( R \). So the radius of the curvature of the parabola at the origin is \( R = \frac{1}{2\alpha} \). The Newtonian equation of motion for the particle at the origin is

\[
mg - N = \frac{mv_0^2}{R} = 2\alpha mv_0^2, \quad \Rightarrow \quad N = m(g - 2\alpha v_0^2),
\]

where \( N \) is the normal force exerted by the surface. \( N \) is positive provided that \( v_0 \) is less than the critical velocity \( v_c := \sqrt{\frac{g}{2\alpha}} \). What happens if \( v_0 = v_c \)? Let us assume that there is no surface; the particle’s path is then a parabola with the equation \( y = -\frac{g}{2v_c^2} x^2 = -\alpha x^2 \). So, if \( v_0 = v_c \), the particle moves tangent to the parabola and \( N \) is always equal to zero. If the initial velocity \( v_0 > v_c \), the particle immediately leaves the surface, and if \( v_0 < v_c \), the particle never leaves the surface.

Figure 1. A particle sliding on a parabolic plane.
3. A particle sliding on a surface whose cross-section is $y = f(x)$

Let us consider the motion of a particle on a frictionless surface whose cross-section is $y = f(x)$, where $f(0) = 0$. The particle is thrown from the origin with the velocity $v_0$ tangential to the surface. Let us assume that the particle leaves the surface at the point $r_d$. At the point of the departure the force exerted by the surface should be equal to zero. Then, at the point $r_d$ the components of the acceleration are

$$\dot{x}|_{r_d} = 0, \quad \dot{y}|_{r_d} = -g.$$  \hfill (4)

Using the equation of the surface cross-section $y = f(x)$, as long as the particle moves on the surface, one may obtain

$$\dot{y} = \dot{x} f'(x)$$

$$\ddot{y} = \dot{x} f'(x) + \dot{x}^2 f''(x),$$ \hfill (5)

where prime means differentiation with respect to $x$. Using equations (4) and (5) at the point of departure, one arrives at

$$\dot{x}|_{x_d} = \sqrt{\frac{-g}{f''(x_d)}},$$ \hfill (6)

$$\dot{y}|_{x_d} = f'(x_d) \sqrt{\frac{-g}{f''(x_d)}}.$$ \hfill (7)

As may be expected at the point where the particle lose contact, $f''(x_d)$ should be negative. So the surfaces for which at all points $f''(x) > 0$, there is no point of departure. If the particle’s velocity at the point $x_d$ is $v_d$, then it will leave the surface

$$v_d = \sqrt{\frac{g(1 + f^2(x_d))}{-f''(x_d)}}.$$ \hfill (8)

If the particle is thrown from the origin with the velocity $v_0$ tangential to the surface, and provided that the surface is frictionless, conservation of energy gives

$$v^2 = v_0^2 - 2gf(x).$$ \hfill (9)

Using equations (6)–(9), knowing the function $f(x)$ one may obtain the point of departure $r_d$ from

$$f''(x_d)(v_0^2 - 2gf(x_d)) + g(1 + f^2(x_d)) = 0.$$ \hfill (10)

The smallest solution of this equation for $x_d$, if one exists, gives the point of departure. The function that solves equation (10) for all $x_d$ is a parabola, subject to the projectile motion. It should be expected, since equation (10) is obtained setting $N = 0$.

As an example, let us consider the standard problem of finding the point where a sliding particle on a frictionless sphere loses contact with the sphere. Then

$$x^2 + (y + R)^2 = R^2, \quad \Rightarrow \quad y = -R + \sqrt{R^2 - x^2},$$ \hfill (11)

where $R$ is the radius of the sphere and the origin is taken to be at the top of the sphere. Using equation (10), one arrives at

$$y_d = \frac{R}{3} + \frac{v_0}{3g}.$$ \hfill (12)
3.1. Example: \( f(x) = -\alpha x^k \), \((\alpha, k > 0)\)

Let us take \( f(x) = -\alpha x^k \), \((\alpha, k > 0)\). We want to obtain the point of departure for different values of \( k \). See figure 2. Equation (10) recasts to

\[
\alpha^2 g k (k-2) x_d^{2k-2} + \alpha v_0^2 k(k-1) x_d^{k-2} = g. \tag{13}
\]

\( x_d \) is the point where the curve \( A(x) := \alpha^2 g k (k-2) x_d^{2k-2} + \alpha v_0^2 k(k-1) x_d^{k-2} \) crosses the line \( B(x) := g \).

3.1.1. \( k > 2 \). For the case \( k > 2 \), \( f''(0) = 0 \), and the radius of curvature at \( x = 0 \) is infinite. Then, at the origin, for any arbitrary tangential initial velocity, \( N = mg + m\ddot{y} > 0 \). So it is not possible to throw the particle with any bounded initial velocity such that it immediately leaves the surface. If \( v_0 = 0 \), the particle remains at rest at the origin. However, \( x = 0 \) is an unstable point.

For \( k > 2 \), \( A(0) = 0 \), and \( A(x) \) is an increasing function of \( x \). So the curve \( A(x) \) should cross the line \( B(x) = g \). For any arbitrary small value of initial velocity \( v_0 \neq 0 \), the particle will leave the surface later. In figure 3 using equation (8) the minimum velocity needed to leave the surface, \( v_d \), is plotted in terms of \( x_d \). As is seen at the origin \( v_d \) approaches to infinity, which supports the above arguments. It is interesting to note that for \( k > 2 \), \( v_d \) has a minimum in terms of \( x_d \). It can easily be shown that at the point on the surface whose \( x \) component is \( \left(\frac{k-2}{\alpha^2 k}\right)^{1/(2k-2)} \), \( v_d \) is minimum.

There is another simple way to analyze the problem. The intersection of two plots for \( v_d \) and \( v = \sqrt{v_0^2 - 2gf(x)} \) gives the point of the departure. In figure 4 three different initial velocities are considered. Particles with smaller initial velocities leave the surface later.

3.1.2. \( k = 2 \). The case \( k = 2 \) corresponds to a parabolic surface that we considered earlier. If \( v_0 = 0 \), the particle remains unstable at the origin. For this case \( A(x) = 2\alpha v_0^2 \) and \( B(x) = g \). If \( v_0 > v_c := \sqrt{\frac{g}{2\alpha}} \) upon throwing, the particle immediately leaves the parabolic surface. If
The point of departure of a particle sliding on a curved surface

"vd" \vdash \frac{1}{2} < k < 2 \quad k = 2 \quad k > 2

Figure 3. Minimum velocity needed to leave the surface, $v_d$, versus $x_d$.

$\nu_0 = \nu_c$ the particle moves tangent to the parabolic surface and $N$ is always equal to zero. Finally, if $\nu_0 < \nu_c$, the particle moves on the parabolic surface and never leaves the surface.

3.1.3. $1 < k < 2$. For the case $1 < k < 2$, $\lim_{x \to 0} f''(x) \to -\infty$, and the radius of curvature at $x = 0$, is equal to zero. So, if the particle moves on the surface, near the origin $\ddot{y} \to -\infty$, which means that $N$ should be $-\infty$, which is impossible. So for any arbitrary nonzero value of initial velocity, the particle immediately leaves the surface. This should be expected as this curve is beneath the parabola which is subject to the projectile motion. If $\nu_0 = 0$, the particle remains unstable at the origin.
3.1.4. $k = 1$. For this case, $A(x) = -\alpha^2 g$ and $B(x) = g$ never cross each other. In fact, this case corresponds to an inclined plane. The particle which is thrown tangentially on the plane never leaves it.

3.1.5. $k < 1$. $A(x) < 0$ and $B(x) = g$ never cross each other. In this case, $f''(x) > 0$ for all $x$ and according to equation (6) there is no point of departure. If $v_0 = 0$, the particle starts moving but never leaves the surface.

4. A particle on a surface with friction

In this section we extend the problem to include frictional effects. The kinetic frictional force is assumed to be $f = \mu N$ opposing the motion in the tangent plane of the surface, where $N$ is the normal force and the coefficient of kinetic friction is $\mu$. The coefficient of static friction is also assumed to be $\mu$.

We put the particle at the point $x_0$ on the surface. It is easy to show that at any point $x_0$ on the surface where $\mu > |f''(x_0)|$, the particle remains there at rest.

4.1. Example; $f(x) = -\alpha x^k$, $(\alpha, k > 0)$

Let us find the point at which the particle will remain at rest if we place it there. For the case $k > 1$, there is a point with $x$ component $a_1 := \left(\frac{\mu}{\alpha^k}\right)^{1/(k-1)}$ on the surface. If the particle is put at any point $x \leq a_1$, it will remain at rest on that point. For the case $k < 1$, there is a point with $x$ component $a_2 := \left(\frac{\alpha^k}{\mu}\right)^{1/(1-k)}$ on the surface. If the particle is put at any point $x \geq a_2$, it will remain at rest on the surface.

4.1.1. $k > 2$. As we saw, it is not possible to throw the particle from the origin with any bounded initial velocity such that it immediately leaves the surface. If $v_0 = 0$, the particle remains at rest at the origin. There is a critical velocity, $v_1c$, If $v_0 \leq v_1c$, the particle moves on the surface until it stops. At most, it reaches the point whose $x$ component is $a_1 := \left(\frac{\mu}{\alpha^k}\right)^{1/(k-1)}$ and will be at rest there. When there is no friction the particle’s velocity is an increasing function and eventually it leaves the surface (see figure 4). When there is friction then the mechanical energy is not conserved, and one should take into account the work done by friction. Then the velocity of the particle $v$ is less than $\sqrt{v_0^2 + 2\alpha x^k}$. So, depending on $\mu$ and $v_0$ the two plots for $v_d(x)$ and $v(x)$ may have no intersection.

4.1.2. $k = 2$. As shown, if $v_0 > v_c := \sqrt{\frac{\alpha}{\mu}}$ upon throwing the particle, it immediately leaves the parabolic surface. There is another critical velocity $v_{2c}$ that should also be less than $\sqrt{\frac{\alpha}{\mu}}$. If $v_0 \leq v_{2c}$, then the particle moves on the surface until it reaches a point $a_1 := \frac{\mu}{\alpha^k}$, and will be at rest there. If $v_{2c} < v_0 < v_c$, then the particle passes $\frac{\mu}{\alpha^k}$, but it never leaves the surface.

4.1.3. $1 < k < 2$. For any arbitrary nonzero initial velocity the particle immediately leaves the surface. If the particle is put at any point $x \leq a_1 := \left(\frac{\mu}{\alpha^k}\right)^{1/(k-1)}$ on the surface, it will remain at rest on that point.
4.1.4. $k = 1$. This case corresponds to an inclined plane. The particle may remain at rest provided that $\mu > \alpha$. If the particle is thrown tangentially with any initial velocity, it will move on the plane for a finite time and will eventually stop. If $\mu < \alpha$ the particle will slide on the plane forever.

4.1.5. $k < 1$. If the particle is thrown tangentially with any initial velocity, it will move on the plane for a finite time and will eventually stop.

Acknowledgments

I would like to thank M Khorrami, and A H Fatollahi for useful comments. This work was supported by the research council of the Alzahra University.

References

[1] Halliday D, Resnick R and Walker J 2010 Fundamentals of Physics Extended 9th edn (New York: Wiley)
[2] Kleppner D and Kolenko R J 1973 An Introduction to Mechanics (Cambridge: Cambridge University Press) p 196
[3] Irodov I E 2002 Fundamental Laws of Mechanics (Moscow: Mir) p 55
Irodov I E 1988 Problems in General Physics (Moscow: Mir) p 28
[4] Morin D 2008 Introduction to Classical Mechanics, With Problems and Solutions (Cambridge: Cambridge University Press) p 77
[5] Mungan C E 2003 Phys. Teach. 41 326–8
[6] Prior T and Mele E J 2007 Am. J. Phys. 75 423–6