Exotic magnetic orders for high-spin ultracold fermions

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Abstract – We study Hubbard models for ultracold bosonic or fermionic atoms loaded into an optical lattice. The atoms carry a high spin $F > 1/2$, and interact on site via strong repulsive Van der Waals forces. Making convenient rearrangements of the interaction terms, and exploiting their symmetry properties, we derive low-energy effective models with nearest-neighbor interactions, and their properties. We apply our method to $F = 3/2$, and $5/2$ fermions on two-dimensional square lattice at quarter, and $1/6$ fillings, respectively, and investigate mean-field equations for repulsive couplings. We find for $F = 3/2$ fermions that the plaquette state appearing in the highly symmetric $SU(4)$ case does not require fine tuning, and is stable in an extended region of the phase diagram. This phase competes with an $SU(2)$ flux state, that is always suppressed for repulsive interactions in the absence of external magnetic field. The $SU(2)$ flux state has, however, lower energy than the plaquette phase, and stabilizes in the presence of weak applied magnetic field. For $F = 5/2$ fermions a similar $SU(2)$ plaquette phase is found to be the ground state without external magnetic field.

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Ultracold atoms in optical lattices provide controllable quantum many-body systems that allow to mimic condensed matter [1,2]. They may in particular serve as quantum simulators of various Hubbard models [3], including those that do not have condensed matter analogues. Prominent examples include Hubbard models for bosons or fermions with high spin $F$. Experimental progress in studies of high-$F$ Bose-Einstein condensates [4] and Fermi gases (cf. [5]) triggered a lot of interest in theoretical studies of such models. These studies go back to fundamental questions of large $N$ limit of $SU(N)$ Heisenberg-Hubbard model [6]; they have continued more recently in the context of ultracold atoms [7–9]. These papers discusses the interplay between the Néel, and valence bond solid (VBS), i.e. P"eierls or plaquette ordering for antiferromagnetic systems. Several other exotic phase should be possible of earth alkali atoms (cf. [10–13]), where two orbital $SU(N)$ magnetism, and even chiral spin liquid states were predicted. Several authors predicted also a variety of novel, exotic phases from effective (generalized Heisenberg) spin Hamiltonians, obtained from spinor Hubbard models (cf. [14,15]). A lot of effort was devoted to the investigations of 1D and 1D ladder systems, where quantum effects are even stronger [16]. While for $F = 1/2$ Hubbard models quantum fluctuations suppress the conductor-Mott insulator transition [17], this is not the case for higher $F$, where dimer (P"eierls) or valence bond crystal (Haldane) order, and in ladders even plaquette order are possible.

Fermi systems with $F = 3/2$ were also intensively studied [18]: first, because this is the simplest case beyond $F = 1/2$, second, because they can be realized with for instance with ultracold $^{132}$Cs, $^{9}$Be, $^{135}$Ba, $^{137}$Ba, and $^{201}$Hg (for an excellent review see ref. [19]; Such systems exhibit a generic $SO(5)$ or isomorphically, $Sp(4)$ symmetry). In 1D there exist the quarteretting phase, a four-fermion counterpart of the Cooper pairing phase. In some situations, counter intuitively quantum fluctuations in spin-3/2 magnetic systems are even stronger than those in spin-1/2 systems.

In this letter we study the two-dimensional Hubbard model for $F = 3/2$ fermions with repulsive singlet and quintet interactions, and for $F = 5/2$ fermions in a 2D plane of its 3D parameter space. First, by rearranging the interaction terms and exploiting their symmetry properties, we derive low-energy effective Hamiltonians. In contrast to the standard approaches (see for instance [14,15]), we do not use the spin representation,
but rather keep the description in terms of fermionic operators. This allows us to formulate mean-field theory, somewhat analogous to slave-boson method [20], and show that the plaquette VBS state is stable in an extended region of the phase diagram, in agreement with the predictions of refs. [18,19]. In the presence of weak applied magnetic field, however, the plaquette phase can be suppressed by an exotic SU(2) flux state. Moreover, for \( F = 5/2 \) fermionic atoms similar SU(2) plaquette phase can be the ground state of the system without external magnetic field.

Let us consider a system described by a Hamiltonian with nearest-neighbor hopping \( H_{\text{kin}} = -t \sum_{\langle i,j \rangle} c_{i,\alpha}^\dagger c_{j,\alpha} \) and strong on-site repulsive interaction
\[
H_{\text{int}} = \sum_i V^\alpha_{\gamma,\delta} c_{i,\alpha}^\dagger c_{i,\alpha} c_{i,\delta} c_{i,\gamma},
\]
where \( S \) is the total spin of the two scattering spin-F\(_1\) and spin-F\(_2\) particles. This means that the scattering processes can happen at different spin channels which are determined by the total spin of the scattering particle. \( P_S \) projects to the total spin-S subspace and \( g_S \) is the coupling constant in the corresponding scattering channel. Due to the on-site interaction the only contributing terms are either antisymmetric \((as)\) or symmetric \((s)\) for an exchange of the spin of the colliding particles depending on their fermionic or bosonic nature. In the following we exploit this property of the on-site interaction which is also preserved for the effective strong repulsion model with nearest-neighbor interaction.

Starting from the fundamental relation between the \( P_S \) projection operator and the product of the \( F\) spin operators for two spin-F fermions:
\[
(2F_1F_2)^l = \sum_S [S(S+1)-2F(F+1)]^l P_S,
\]
we can express the degree of \( 2F \) polynomial of the \( F_1F_2 \) product as
\[
|P_S|_{\gamma,\delta}^\alpha\beta = \sum_{S=[F_1-F_2]} g_S [P_S]_{\gamma,\delta}^\alpha\beta,
\]
where \( S_{\gamma,\delta}^\alpha\beta \) are the usual creation (annihilation) operators of fermions with spin \( \alpha \) at site \( i \), and \( t \) is the hopping amplitude between the neighboring sites. Here, and in the following automatic summation over repeated Greek indices is assumed. The interactions depend on the spin of the scattering particles [21]:
\[
V^\alpha_{\gamma,\delta} = \sum_{S=[F_1-F_2]} g_S [P_S]_{\gamma,\delta}^\alpha\beta,
\]
and the symmetric and antisymmetric projectors can be expressed as follows:
\[
P_S^{(as)} = \sum_{l=0}^{N_{as}-1} b_{\gamma,\delta} [P_S]_{\gamma,\delta}^{(as)},
\]
and
\[
P_S^{(s)} = \sum_{l=0}^{N_{s}-1} c_{\gamma,\delta} [P_S]_{\gamma,\delta}^{(s)}.
\]

In contrast, we will use the expansion of the projector operator in eqs. (1) and (2) in order to collect and treat adequately the two-particle interaction terms that describe different spin exchange and spin flip processes. In this case \( F \) denotes the three generators of the SU(2) Lie algebra in the appropriate representation: for high-spin fermions or bosons they are the SU(2) generators represented by the proper (even/odd)-dimensional matrices. In case of pure boson or fermion system all processes take place only in the symmetric or antisymmetric part of the total spin space, respectively. Therefore, the following decomposition can be used:
\[
(F_1F_2)^l = \left( (F_1F_2)^l \right)^{(as)} + \left( (F_1F_2)^l \right)^{(s)},
\]
and
\[
\text{and the symmetric and antisymmetric projectors can be expressed as follows:}
\]
\[
P_S^{(as)} = \sum_{l=0}^{N_{as}-1} b_{\gamma,\delta} [P_S]_{\gamma,\delta}^{(as)},
\]
Here \( N_{as} \) and \( N_s \) denote the number of antisymmetric and symmetric subspaces of the total spin space, respectively. The symmetric and symmetric part of an operator \( A \) can be constructed by the exchange of two spin indices: \( [A^{(as)}]_{\gamma,\delta} = A_{\gamma,\delta}^s - A_{\gamma,\delta}^a \), and \( [A^{(s)}]_{\gamma,\delta} = A_{\gamma,\delta}^s + A_{\gamma,\delta}^a \). It is obvious that the above decomposition leads to the polynomials eq. (5) having significantly smaller degree than eq. (3). \( N_{as} = 1 \) or \( N_s = 1 \), respectively, determines the minimum degree of the polynomial of the product \( F_1F_2 \) which is equivalent to the interaction eq. (2).

Now let us apply the above procedure to a 2-dimensional \( F = 3/2 \) fermion system. In this case the interaction has to be antisymmetric therefore the only contributing terms are the total spin-0 (singlet) and the spin-2 (quintet) scatterings:
\[
V^\alpha_{\gamma,\delta} = \sum_{S=0}^{3} g_S [P_S]_{\gamma,\delta}^{\alpha\beta} = g_0 [P_0]_{\gamma,\delta}^{\alpha\beta} + g_2 [P_2]_{\gamma,\delta}^{\alpha\beta}.
\]

At quarter filling there is only one particle on each site i) on the average in general, and ii) exactly if the on-site repulsion tends to infinity. Therefore for strong repulsion, the hopping can be considered as a perturbation, and the system can be described by an effective Hamiltonian with nearest-neighbor interaction. The effective model based on perturbation theory up to second (leading) order in the hopping \( t \) is the following:
\[
H_{\text{eff}} = \sum_{\langle i,j \rangle} V^\alpha_{\gamma,\delta} c_{i,\alpha}^\dagger c_{j,\beta} c_{j,\beta} c_{i,\gamma},
\]
where \( V^\alpha_{\gamma,\delta} = \sum_S G_S [P_S^{(as,s)}]_{\gamma,\delta}^{\alpha\beta} \) gives the energy shift to the on-site energies due to the weak nearest-neighbor interaction.
hopping. $G_S = -4t^2/g_S$, ($S = 0, 2$) is the new coupling constant in the spin-$S$ scattering channel. Since the effective model preserves the symmetry of the on-site interaction, it remains antisymmetric for an exchange of two spin indices. Now the components of the $F$ vector are the well known $4 \times 4$ spin matrices and eq. (5) has the following form: $E^{(as)} = F_0 + F_2$, and $(F_1 F_2)^{(as)} = -15 F_0 / 4 - 3 P_2 / 4$. The effective Hamiltonian has the form

$$H_{int} = a_n \sum_{(i,j)} E^{(as)}_{i,j} + a_s \sum_{(i,j)} (F_1 F_2)^{(as)}_{i,j},$$

where $a_n = (5G_2 - G_0)/4$, and $a_s = (G_2 - G_0)/3$. The two-particle nearest-neighbor interaction terms are $E^{(as)}_{i,j} = c_{i,\alpha} c_{j,\beta} c_{i,\gamma}^\dagger E^{(as)}_{i,j} c_{i,\gamma}^\dagger$, and $(F_1 F_2)^{(as)}_{i,j} = c_{i,\alpha} c_{j,\beta} c_{i,\gamma}^\dagger c_{i,\gamma} c_{i,\gamma}^\dagger$, and their explicit spin dependence is $[E^{(as)}_{i,j}]_{\alpha,\beta,\gamma} = \delta_{\alpha,\gamma} \delta_{\beta,\delta} - \delta_{\alpha,\delta} \delta_{\beta,\gamma}$, and $[(F_1 F_2)^{(as)}_{i,j}]_{\alpha,\beta,\gamma} = \delta_{\alpha,\gamma} [F_1]_{\beta,\delta} - [F_1]_{\alpha,\delta} [F_2]_{\beta,\gamma}$. After straightforward calculations one arrives to the following form of the effective Hamiltonian:

$$H_{eff} = V^{(0)} + \sum_{(i,j)} \left[ a_n \left( n_i n_j + x_{i,j} \chi_{i,j} - n_i \right) + a_s \left( S_i S_j + J_{i,j} J_{i,j} - 15 / 4 n_i \right) \right],$$

where $n_i = c_{i,\alpha} c_{i,\alpha}^\dagger$, and $S_i = c_{i,\alpha} F_{\alpha,\beta} c_{i,\beta}^\dagger$ are the usual particle number and spin operators on site $i$, and

$$\chi_{i,j} = c_{i,\alpha} c_{j,\alpha} c_{i,\beta}^\dagger c_{j,\beta}^\dagger,$$

$$J_{i,j} = c_{i,\alpha} F_{\alpha,\beta} c_{i,\beta}^\dagger,$$

are introduced for the $U(1)$, and $SU(2)$ nearest-neighbor link operators, respectively. Note, that in general the $SU(2)$ link operators do not satisfy the spin commutation relations, however, they clearly are related to the bond-centered spin. The competition between the spin and particle fluctuations can be controlled by tuning of $a_n$ and $a_s$. The effective Hamiltonian (9) can be applied for less than quarter filled system too, provided the kinetic term is added to the Hamiltonian (9). $V^{(0)}$ contains the on-site energies and shifts the ground-state energy only, so we do not consider its contribution.

In the following we study the possible phases of the quarter filled system with the constraint $\sum_{\alpha} c_{i,\alpha} c_{i,\alpha}^\dagger = 1$ —only single occupied sites are allowed due to the strong on-site repulsion. Due to this local constraint the Hamiltonian is invariant under a rotation of the phase of the fermions at each sites. This means that the Lagrangian of the system $L = \sum_{\alpha} c_{i,\alpha}^\dagger \partial_\tau c_{i,\alpha} + H$ is invariant under the $U(1)$ gauge transformation $c_{i,\sigma} \rightarrow c_{i,\sigma} e^{i\phi_i}$ reflecting the local constraint for the particle.

Considering the Hamiltonian (9) the terms containing $n_i$ do not give contribution up to an irrelevant constant at quarter filling and the remaining 4-fermion terms can be decoupled via a mean-field treatment by introducing the expectation values of the link operators $\langle \chi_{i,j} \rangle$ and $\langle J_{i,j} \rangle$, and the spin operator $\langle S_i \rangle$. Now the mean-field Hamiltonian is

$$H_{MF} = \sum_{(i,j)} H_{i,j},$$

with

$$H_{i,j} = a_n \left( \langle \chi_{i,j} \rangle c_{i,\alpha} c_{j,\alpha} + \langle J_{i,j} \rangle c_{i,\alpha} c_{j,\alpha} - | \langle \chi_{i,j} \rangle |^2 \right) + a_s \left( \langle J_{i,j} \rangle c_{i,\alpha} F_{\alpha,\beta} c_{j,\beta} - | \langle J_{i,j} \rangle |^2 \right) + \langle S_i \rangle^\dagger c_{i,\alpha} F_{\alpha,\beta} c_{j,\beta} + \langle S_i \rangle c_{i,\alpha} F_{\alpha,\beta} c_{j,\beta} - \langle S_i \rangle^\dagger \langle S_j \rangle).$$

Note that the mean-field Lagrangian also has to remain invariant under the gauge transformation mentioned above. Thus, the link variables must transform as $\langle A_{i,j} \rangle \rightarrow \langle A_{i,j} \rangle e^{-i(\phi_j - \phi_i)}$.

The expectation values of the spin and link operators were determined self-consistently. Anticipating the appearance of a plaquette phase similar to the ground state of the system for $G_0 = G_2$, it is reasonable to split the lattice into 4 sublattices (see fig. 1) leading to the shrinking of the Brillouin zone to the quarter of its original size. We assume different values for the order parameters of the different sublattices and of the alternating links as the only space-dependence of them.

We have found the following four gauge—nonequivalent states with the following non-zero averages to be the solutions of the self-consistent equations: a) Néel order: $\langle S_i \rangle$, b) $U(1)$ plaquette order: $\langle \chi_{i,j} \rangle$, and c) $SU(2)$ plaquette and dimer order: $\langle S_i \rangle$, $\langle \chi_{i,j} \rangle$, $\langle J_{i,j} \rangle$. In the figures the non-zero expectation values are denoted by the following way: $\langle S_i \rangle$: black arrow, $\langle \chi_{i,j} \rangle$: black stripe, and $\langle J_{i,j} \rangle$: light blue stripe with arrow. The phase diagram of the system is shown in fig. 2. If the effective interaction of the singlet channel is significantly stronger than that of the quintet channel, the dominant order is purely antiferromagnetic without any bond order. For $a_n < 0$ and $a_s > 0$ the spin and particle order compete with each other. In ref. [18] a magnetically ordered dimer phase was suggested to appear in this regime, however, we did not find any similar state to be the solution of the self-consistent equations: when the antiferromagnetic...
order of the Néel phase is destroyed, plaquette order appears. The phase boundary is around \( G_0 \approx 1.9G_2 \) or equivalently \(-a_n \approx 2.6a_s\). In the \( U(1) \) plaquette phase the coupling constant \( a_n \) always dominates the interaction independently of the sign of \( a_s \). The non-zero \( U(1) \) links form boxes as shown in fig. 2(b). One can define the \( U(1) \) plaquette variable as \( \Pi = \chi_{i,j,k,l} \), where \( i, j, k \) and \( l \) denote the sites of an elementary plaquette of the square lattice, and \( \chi \) is defined for nearest neighbors only. The \( U(1) \) flux \( \Phi \) is defined by the phase of the plaquette. The plaquette variable and therefore the flux are invariant under the \( U(1) \) gauge transformations mentioned above.

We have found two different gauge–non-equivalent states in the plaquette phase labeled by \( \Phi = 0 \) and \( \Phi = \pi \), respectively, and both states have the same energy. Note that our results are in good agreement with earlier results for the special \( SU(4) \) symmetric case \( G_0 = G_2 \), where similar box state was predicted with zero flux [8]. We have found that this box state does not need fine tuning, it is the ground state in an extended, experimentally reachable region of the parameter space.

In the parameter region \(|G_0| < 1.9|G_2|\), and for \( a_s \neq 0 \) we have found two other solutions on the top of the ground state having 10–15% higher energy: the \( SU(2) \) dimer phase and the \( SU(2) \) plaquette phase, where the latter corresponds to two gauge–non-equivalent states with different fluxes. Both the dimer and the plaquette phases have the same energy. In these states, in addition to weak ferromagnetic order (\( \langle S_i \rangle < 3/2 \) and are equal for all of the 4 sublattices), both types of the link operators \( \chi \) and \( J \) have non-zero expectation values as shown in fig. 3(a) and (b). In the \( SU(2) \) plaquette phase the link operators with non-zero expectation values form plaquettes. These states are completely new RVS orders, therefore we make some notes about their basic properties and comment on their naming. Both states violate spin-rotation invariance, and the \( SU(2) \) dimer state—contrary to the \( SU(2) \) plaquette phases—preserves the translational invariance by one lattice site in one spatial dimension. It is clear that \( J \) is not a member of \( SU(2) \) therefore it is reasonable to ask: why do we use the terms \( SU(2) \) plaquette, flux or dimer for the states where the expectation value of \( J \) is non-zero? To answer this question let us consider the mean-field Hamiltonian eq. (11). The non-local part of the one-particle excitations appears in the Hamiltonian as

\[
\left( a_n \langle \chi_{i,j} \rangle \delta_{\alpha,\beta} + a_s \langle J_{i,j} \rangle F_{\alpha,\beta} \right) c_i^{\dagger}_{\alpha,\beta} c_{j,\beta} + H.c. \tag{12}
\]

From this form it can be read that the excitations consist two branches with two different symmetries: \( \langle \chi_{i,j} \rangle \) relates to the \( U(1) \) excitations, while \( \langle J_{i,j} \rangle F \) to the \( SU(2) \) excitations. In order to define the \( SU(2) \) flux let us introduce the new link parameter according to eq. (12):

\[
U_{i,j} = \langle J_{i,j} \rangle F,
\]

with the usual inner product of the vectors in the 3-dimensional space of the generators \( F \). \( U_{i,j} \) is a member of \( SU(2) \) and a \( 4 \times 4 \) matrix for \( F = 3/2 \) fermionic atoms and the same holds for the \( SU(2) \) plaquette variable \( \Pi^{SU(2)} = U_{i,j} \vec{U}_{i,j} U_{i,j}^{\dagger} \). The flux \( \Phi \) passing through the plaquette defined by the form: \( \Pi^{SU(2)} = e^{i\Phi F} \). In order to determine the ground state it is worth to express the mean-field Hamiltonian with the \( J_{i,j} \) operators, while the excitations and the \( SU(2) \) flux can be expressed with \( U_{i,j} \). Note that the \( SU(2) \) plaquette \( \Pi^{SU(2)} \) is also invariant under the \( U(1) \) gauge transformation defined above: \( c_i^{\dagger}_{\alpha,\sigma} \rightarrow c_i^{\dagger}_{\alpha,\sigma} e^{i\phi_\alpha}, \langle \chi_{i,j} \rangle \rightarrow \langle \chi_{i,j} \rangle e^{i(\phi_j - \phi_i)}, \) and \( U_{i,j} \rightarrow U_{i,j} e^{i(\phi_j - \phi_i)} \). Considering the definition of \( U_{i,j} \), the last relation is obviously equivalent to the transformation \( J_{i,j} \rightarrow J_{i,j} e^{i(\phi_j - \phi_i)} \).

The \( SU(2) \) phases can patently claim to great interest, but they are suppressed by the \( U(1) \) plaquette state. Nevertheless, since the \( SU(2) \) flux, as well as the \( SU(2) \) dimer order coexist with ferromagnetic order, it can be expected that weak magnetic field does not destroy the \( SU(2) \) order, but it can stabilize that. To check this let us investigate the energy of these states in the presence of external magnetic field \( h \) taken into account as a Zeeman term in the Hamiltonian

\[
H^h = H^{MF} + h \sum_i S_i. \tag{13}
\]

![Fig. 2: (Color online) The phase diagram of the spin-3/2 fermion system with strong on-site repulsion on 2D square lattice at quarter filling and the corresponding configurations: (a) Néel order (AFM), and (b) \( U(1) \) plaquette phase.](66005-p4)
Note that for strong magnetic field the quadratic Zeeman term can become important. At this point let us suppose that this term can be neglected and we will check the validity of this assumption at the end of the calculations.

The magnetic-field dependence of the energy of the SU(2) flux state compared to the U(1) plaquette state and to the ferromagnetic state is shown in fig. 3 for a typical value of the couplings in units of the nearest-neighbor hopping. The U(1) plaquette phase remains the ground state in the presence of non-zero, but very small magnetic fields. The SU(2) plaquette state, as well as SU(2) dimer order, have the lowest energy for higher value of the applied magnetic field \( h \), and are the ground state of the system in an extended region of the phase diagram. Both the SU(2) dimer and the plaquette states either 0 or \( \pi \) flux have the same energy. In the presence of even stronger magnetic fields ferromagnetic order suppresses any other order in the system. Now, let us check the validity of our assumption of neglecting the quadratic Zeeman coupling. The linear Zeeman energy (in \( \hbar \) units) is given by the Lamour frequency \( \omega_L = g_F \mu_B B \), and the quadratic Zeeman energy is \( \omega_Q = \frac{a}{\omega_L} \). Here \( \omega_{HF} \) is the hyperfine splitting energy, \( g_F \) is the gyromagnetic factor and \( \mu_B \) is the Bohr magneton. If \( \frac{a}{\omega_{HF}} \gg 1 \) or equivalently \( \frac{\omega_Q}{\omega_L} \gg 1 \), the quadratic Zeeman effect can be neglected. We measure the magnetic field in units of the hopping parameter \( t \). In optical lattices \( t = \omega_R \frac{\omega}{\omega_L} e^{-2\omega^2} \) where \( \omega = (V_0/\omega_R)^{1/4} \), \( V_0 \) is the potential depth, and \( \omega_R \) is the recoil energy. \( t \) has a maximum at \( V_0 \approx \omega_R \), where \( t \approx \omega_R \). \( \omega_R \) is typically in the order of 1–100 kHz (however, \( \omega_R/2\pi \approx 400.98 \) kHz for \(^{9}\text{Be})\). We have found that the SU(2) flux state has the lowest energy if the linear Zeeman energy is around \( \omega_L \approx 0.1–1 \) kHz which means 0.1–100 kHz. The hyperfine frequency is in the order of 1–10 GHz, so \( \omega_{HF}/\omega_L \) is in the order of \( 10^3–10^4 \). Therefore, the quadratic Zeeman effect can be neglected compared to the linear Zeeman term for magnetic fields which are sufficient to stabilize the SU(2) phases. There are some atoms which could be promising candidates for realizing experimentally spin-3/2 fermion systems and have much less recoil energy than the above-mentioned \(^{9}\text{Be}\), namely 1–10 kHz which in our case corresponds to very small magnetic field: \( 10^{-5}–10^{-4} \) G. With these atoms the experimental realization of the SU(2) orders demands quite strong magnetic shielding.

A similar analysis can be easily made for \( F = 5/2 \) fermions at 1/6 filling for the special values of the coupling constants \( G_4 \approx (–7G_0 + 10G_2)/3 \). \( G_4 \) is the coupling of the interaction with 9-fold spin multiplicity that appears in the Hamiltonian eq. (6) for spin-5/2 system in addition to the singlet \( (G_0) \) and quintet \( (G_2) \) scatterings. In the plane of the parameter space defined by \( G_4 = (–7G_0 + 10G_2)/3 \), the structure of the Hamiltonian is exactly the same as eq. (11), there is no term containing higher order of the product \( F_1F_2 \), and the couplings take the values: \( a_n = (–23G_0 + 35G_2)/12 \) and \( a_x = (G_0 + G_2)/3 \). Note that now the components of the spin-3/2 F vectors are the 3 generators of the SU(2) Lie algebra in 6 x 6 representation. During the calculations we have used the same 4-sublattice ansatz as in case of the \( F = 3/2 \) fermions, because the more suitable 6-sublattice ansatz does not respect the symmetries of the original model on square lattice. Similarly we have determined the solutions of the self-consistent equations for the \( (J_{i,j}) \) expectation values and the corresponding energies. The ground states of this system are shown in fig. 4. In case of dominant singlet scatterings the ground state is purely antiferromagnetic, at least while \( a_n > 0 \). A weak negative \( a_n \) seems to lead to an instability in the system, but we could not find any stable solution of the self-consistent equations in this narrow region. Further decreasing \( |G_0| \), a quasi-plaquette phase appears. In this phase the expectation value of the \( U(1) \) link operator \( (\chi_{i,j}) \) is non-zero everywhere, but stronger and weaker links alternate forming a weak plaquette structure. The flux passing through the plaquette is zero and there is no spin order in this phase. For even weaker singlet coupling (increasing the value of \( |a_n| \)) weak ferromagnetic order appears in addition to the plaquette order (the SU(2) plaquette phase in fig. 4). Here the plaquettes are formed not only by the alternating zero and non-zero \( U(1) \) link operators \( (\chi_{i,j}) \), but the SU(2) operators \( (J_{i,j}) \), too. The flux passing through the plaquettes remains zero. This means that while for \( F = 3/2 \) fermions the SU(2) plaquette phase can be the ground state of the system only by applying external magnetic field, for \( F = 5/2 \) fermions a similar SU(2) plaquette order (with zero flux) is the ground state of the system for weak singlet couplings. It is difficult to determine precisely the phase border between the quasi-plaquette and the SU(2) plaquette phases. This is due to the fact that the two orders start to compete around \( 2G_0 \approx G_2 \): both are stable solution of the self-consistent equations but approximately with the same energy. The energy difference between the two phases increases slowly.
for decreasing $|G_0|$, and around $3G_0 \approx G_2$ reaches the 4–5%. Note, that in the same parameter regime we have found another stable solution: a plaquette phase with $\pi$ flux and with stronger ferromagnetic order, however $(S_i)$ remains smaller than $5/2$. In this state the $SU(2)$ order parameter dominates the link variables: the value of $(J_{i,j})$ is twice that of the corresponding $(\chi_{i,j})$. The energy of this spin-ordered $SU(2)$ plaquette state with $\pi$ flux is higher by about 5% than the one with zero flux.

Finally let us discuss the validity of our mean-field results. In two dimensions the mean-field solutions are expected to provide qualitatively reliable results. However, fluctuations around the mean-field results can become relevant, especially in the following two cases: when the system is close to the phase boundary of a continuous transition, the correlation length becomes very large and fluctuations cannot be neglected. The present work focuses on the qualitative description of possible exotic states of matter. The states discussed above are the ground state of the system in an extended region of the parameter space and far enough from the phase boundaries expectedly they can be realized experimentally. Fluctuations could also be important if the mean-field solution is degenerate and the states in the degenerate subspace are not separated by energy barrier. In our case, e.g., the $SU(2)$ dimer and plaquette states have the same energy but they are separated by energy barrier: one cannot arrive from one to the other with continuous (link by link) deformation without increasing the energy. These types of mean-field solutions are expectedly not affected by small fluctuations.

To summarize, we have used a decomposition of the total spin space into its symmetric and antisymmetric part with respect to the exchange of two spin indices of the high-spin scattering particles. This decomposition was used for strongly repulsive systems to derive effective low-energy Hamiltonians. This task was achieved remaining within the two-particle representation. The main advantage of the treatment is that it does not require to introduce complicated effective multiparticle/multispin interactions, but relies only on rearrangements of the usual two-particle interactions. The effectiveness of the treatment does not depend on the statistics of the considered particles, and it allows us to identify the different processes in the spin channel within the concept of site and bond spin. Applying this method to $F = 3/2$ fermions, we have determined the ground-state phase diagram of the system on the mean-field level to complete the earlier results known for some regimes of the couplings. We have found that the VBS state is the ground-state in an extended region of the phase diagram, while in the presence of weak magnetic field an exotic $SU(2)$ flux state has the lowest energy. We have also made some similar calculations for $F = 5/2$ fermions in the plane determined by the condition $G_4 = (-7G_0 + 10G_2)/3$ in the 3-dimensional parameter space of the coupling constants. We have found that novel, exotic $SU(2)$ plaquette phase, similar to one predicted for spin-3/2 fermion system in the presence of external magnetic field, can be the ground state of the system with zero flux, without applied magnetic field.

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