**FROBENIUS INTEGRABILITY AND FINSLER METRIZABILITY FOR 2-DIMENSIONAL SPRAYS**

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**Abstract.** For a 2-dimensional non-flat spray we associate a Berwald frame and a 3-dimensional distribution that we call the Berwald distribution. The Frobenius integrability of the Berwald distribution characterises the Finsler metrizability of the given spray. In the integrable case, the sought after Finsler function is provided by a closed, homogeneous 1-form from the annihilator of the Berwald distribution. We discuss both the degenerate and non-degenerate cases using the fact that the regularity of the Finsler function is encoded into a regularity condition of a 2-form, canonically associated to the given spray. The integrability of the Berwald distribution and the regularity of the 2-form have simple and useful expressions in terms of the Berwald frame.

1. Introduction

The inverse problem of Lagrangian mechanics requires to decide whether or not a given system of second order ordinary differential equations (SODE or semispray) can be derived from a variational principle. In general, the problem is far from being solved. It has been completely solved in dimension 1 by Darboux [8] and in dimension 2 by Douglas [9].

In the case when the given SODE is homogeneous (i.e., a spray), the problem is known as the Finsler metrizability problem [6, 12, 13, 18]. A solution to this problem has been proposed in the analytic case by Muzsnay [14], by studying the formal integrability of the associated Euler-Lagrange PDE.

In this paper we provide a constructive solution to the 2-dimensional case of the Finsler metrizability problem in the non-flat case. We associate to a given non-flat 2-dimensional spray two canonical geometric structures that contain all the information about the Finsler metrizability of the spray and, in the affirmative case, they give the Finsler function. One of these structures is a 3-dimensional regular distribution, called the *Berwald distribution*, whose integrability provides a candidate for the Finsler function that we look for. The second structure is a 2-form, whose rank gives the information about the regularity of the Finsler candidate. A key aspect in our approach is the use of a canonical frame, called the *Berwald frame*, associated to the given spray that makes easier to express the integrability of the Berwald distribution as well as the rank of the 2-form. Therefore, we reformulate and solve the Finsler metrizability problem in terms of some properties of the Berwald frame. In the integrable case, the Berwald distribution coincides with the holonomy distribution [14] and the number of solutions we provide agrees with the metrizability freedom of a spray introduced in [10].

The Berwald frame has been introduced first, locally, for a 2-dimensional Finsler metric in [3]. An intrinsic formulation of the Berwald frame, in the Finslerian setting, has been provided in [19], see also [17, §9.9.1]. Such a frame has been rediscovered recently for a background Riemannian
metric in [7] to give an alternative proof of the projective Finsler metrizability property of an arbitrary 2-dimensional spray. In our case, we define the Berwald frame directly for an arbitrary spray and use its properties to obtain information about the Finsler metrizability of the given spray.

For the geodesic spray of a Finsler function, it is known that one can always construct an integrable distribution, transverse to the Liouville vector field, that is tangent to the indicatrix of the Finsler function [2]. In dimension 2, our Theorems 4.1 and 4.2 provide a characterization of the Finsler metrizability in terms of the integrability of such a distribution. This distribution is the Berwald distribution and in the integrable case it is tangent to the indicatrix of the Finsler function that metrizes the given spray.

2. Sprays and Finsler functions, a geometric setting

For a 2-dimensional smooth, orientable and real manifold \( M \), we denote by \((T M, \pi, M)\) its tangent bundle and by \( T_0 M := T M \setminus \{0\} \) the total space of the tangent bundle with the zero section removed. Local coordinates \((x^i, y^i)\) on \( M \) induce local coordinates \((x^i, y^i)\) on \( T M \) and \( T_0 M \). There are some canonical structures that naturally live on \( T M \). The vertical distribution, \( u \in T M \mapsto V_u T M := \ker(\pi^* u) \subset T_u T M \) is a regular, integrable, 2-dimensional distribution, and locally generated by \((\partial/\partial y^1, \partial/\partial y^2)\). The Liouville (dilation) vector field \( C = y^i \partial/\partial y^i \) is a vertical vector field whose one-parameter group of diffeomorphisms is generated by the positive homotheties of the fibres. The tangent structure (vertical endomorphism) is given by \( J = \partial/\partial y^i \otimes dx^i \).

2.1. A geometric setting for sprays. A spray (i.e., a homogeneous SODE) on \( M \) is a vector field \( S \in \mathfrak{X}(T_0 M) \) that satisfies \( JS = C \) (is a second order vector field) and \([C, S] = S\) (it is a positively \( 2^+ \)-homogeneous). Locally, a spray can be expressed as follows:

\[
S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},
\]

where the functions \( G^i \) are locally defined, and \( 2^+ \)-homogeneous in the fiber coordinates.

To a given spray \( S \), one can associate, using the Frölicher-Nijenhuis formalism, a geometric setting including a nonlinear connection, dynamical covariant derivative and curvature tensors [5, 11, 12, 16].

A spray \( S \) induces a nonlinear connection through an endomorphism \( \Gamma = [J, S] \) on \( T_0 M \) [11]. The connection \( \Gamma \) is an almost product structure, which means that \( \Gamma^2 = \text{Id} \), and induces two projectors

\[
h = \frac{1}{2} (\text{Id} + [J, S]), \quad v = \frac{1}{2} (\text{Id} - [J, S]).
\]

The projector \( v \) corresponds to the vertical distribution \( V T M \), while \( h \) induces a horizontal distribution \( H T M \), which is supplementary to the vertical distribution. Locally, the two projectors can be expressed as follows:

\[
h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\delta}{\delta y^i} \otimes dy^i, \quad \delta \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j \frac{\partial}{\partial y^j}, \quad \delta \frac{\delta}{\delta y^i} = dy^i + N^j dx^j, \quad N^j := \frac{\partial G^i}{\partial y^j}.
\]

The connection induced by a spray can induces an almost complex structure on \( T_0 M \) given by

\[
\mathcal{F} = h \circ [S, h] - J = \frac{\delta}{\delta x^i} \otimes dy^i - \frac{\partial}{\partial y^i} \otimes dx^i,
\]

see, e.g. [12] (3.14)].
The dynamical covariant derivative $\nabla$, induced by a spray $S$, is a tensor derivation on $T_0 M$, whose action on functions and vector fields is given by, \cite{[5] §3.2},

\[ \nabla f = S(f), \quad \nabla X = h[S, hX] + v[S, vX], \quad f \in C^\infty(T_0 M), \quad X \in \mathfrak{X}(T_0 M). \]

The Jacobi endomorphism, of a spray $S$ is the vector valued 1-form $\Phi = v \circ [S, h]$, whose local expression is

\[ \Phi = R_i^j \frac{\partial}{\partial y^i} \otimes dx^j, \quad R_i^j = 2 \frac{\partial G^i}{\partial x^j} - S \left( \frac{\partial G^i}{\partial y^j} \right) - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j}. \]

A spray $S$ is called isotropic if there exist a function $\rho \in C^\infty(T_0 M)$ and a semi-basic 1-form $\alpha = \alpha_i dx^i \in \Lambda^1(T_0 M)$ such that

\[ \Phi = \rho J - \alpha \otimes C \iff R_i^j = \rho \delta_i^j - \alpha_j y^i. \]

The function $\rho = \operatorname{Tr}(\Phi) = R_1^1 + R_2^2$, which is called the Ricci scalar, is related to the semi-basic 1-form $\alpha$ by $\rho = i_\alpha \alpha = \alpha_1 y^1 + \alpha_2 y^2$.

The homogeneity property of a spray $S$ is inherited by all associated geometric structures. The Jacobi endomorphism is $2^+$-homogeneous and therefore the Ricci scalar $\rho$ is $2^+$-homogeneous, while the semi-basic 1-form $\alpha$ is $1^+$-homogeneous.

It is known that 2-dimensional sprays are always isotropic, see \cite[Lemma 8.1.10]{[15]} or \cite[Corollary 8.3.11]{[17]}, which means that their Jacobi endomorphism is given by formula (2.4), with

\[ \alpha_1 = \frac{R_2^2}{y^1}, \quad \alpha_2 = \frac{R_1^1}{y^2}, \quad \alpha_j = \frac{-R_2^1}{y^j}. \]

In this work we will pay attention to non-flat sprays. This assumption means that the Jacobi endomorphism $\Phi$ is nowhere vanishing, which is equivalent to $\alpha \neq 0$ and it implies that $\rho \neq 0$.

We will use the geometric setting described above to address the following metrizability problem for a given spray $S$. Is there a Finsler function whose geodesic spray is $S$? We will provide the answer to this problem, using two geometric structures. The first structure is a distribution, whose integrability will provide the metric candidate. The second structure is a 2-form that encodes information about the regularity of the metric.

For a spray $S$, the non-flatness assumption implies that the distribution

\[ D = \operatorname{Im}(h) \oplus \operatorname{Im}(\Phi) \]

is a regular, 3-dimensional distribution. We call $D$ the Berwald distribution of the spray $S$ and, we will prove that the spray is metrizable if and only if $D$ is integrable.

Another distribution that can be associated to an arbitrary spray $S$ has been introduced by Muzsnay in \cite{[14]}. It is called the holonomy distribution $D_{\mathcal{H}}$, generated by horizontal vector fields and their successive Lie brackets. The holonomy distribution has been recently used in \cite{[10]} to discuss the metrizability freedom of a spray. The two distributions are related by $D \subset D_{\mathcal{H}}$.

For an isotropic spray $S$ with Jacobi endomorphism $\Phi$ given by formula (2.4), we consider the following 2-form:

\[ \Omega = d \left( \frac{\alpha}{\rho} \right) + 2 i \frac{\alpha}{\rho} \wedge \frac{\alpha}{\rho}. \]

In the case of metrizability, the rank of the 2-form $\Omega$ will provide information about the regularity of the Finsler function which we search for. The idea of considering the 2-form $\Omega$ has the origins in the “Scalar Flag Curvature”-test proposed in \cite[Theorem 3.1]{[6]}.
2.2. A geometric setting for Finsler functions. For the metrizability problem of a given spray we pay attention to the non-Riemannian case, by searching for Finslerian solutions.

Definition 2.1. A continuous function $F : T M \to \mathbb{R}$ is called a Finsler function if it satisfies the following conditions:

i) $F$ is smooth and strictly positive on $T_0 M$;

ii) $F$ is positively $1^+$-homogeneous in the fiber coordinates, which means $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda \geq 0$ and $(x, y) \in TM$;

iii) $F^2$ is a regular Lagrangian, which means that the following metric tensor has maximal rank 2 on $T_0 M$:

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$ 

If in the definition we restrict the domain of the function $F$ to some open cone $A \subset T M$, then we speak about a conic pseudo-Finsler function. Also, if we replace the regularity condition iii) above by the weak regularity condition rank$(g_{ij}) = 1$, then we speak about a degenerate Finsler function.

The regularity conditions can be reformulated in terms of the Hilbert 2-form $\omega_{F^2} := -dd_J F^2$ as follows. The function $F$ satisfies iii), and hence it is a Finsler function, if and only if $\omega_{F^2}$ has maximal rank 4, which means that $\omega_{F^2}$ is a symplectic structure. The function $F$ satisfies the weak regularity condition, and hence it is a degenerate Finsler function if and only if the 2-form $\omega_{F^2}$ has rank 2.

A spray $S$ is Finsler metrizable if there exists a (degenerate, conic pseudo-) Finsler function that satisfies the Euler-Lagrange equation

$$i_S dd_J F^2 = -dF^2. \quad (2.8)$$

Equation (2.8) expresses the fact that the base integral curves of the spray (homogeneous SODE) are solutions to the Euler-Lagrange equations for $F^2$. In other words, $S$ is the geodesic spray, when $F$ is a Finsler function. In the degenerate case, $S$ is one of the geodesic sprays of $F$.

Consider a geodesic spray $S$ of a (degenerate) Finsler function $F$ and let $\Gamma$ be the connection induced by $S$. The Hilbert 2-form of $F$ can be expressed in terms of the connection $\Gamma$ as follows:

$$\omega_{F^2} = 2g_{ij} dx^i \wedge \delta y^j. \quad (2.9)$$

Within this geometric setting, we can associate to a (degenerate) Finsler function $F$ a (semi-)Riemannian metric on $T_0 M$, $[11]$, by

$$G(X, Y) = -\omega_{F^2}(X, FY), \quad \omega_{F^2}(X, Y) = -2G(X, JY) + 2G(JX, Y), \quad G = g_{ij} dx^i \otimes dx^i + g_{ij} \delta y^i \otimes \delta y^j. \quad (2.10)$$

If $S$ is the geodesic spray of a Finsler function then we use the metric $G$ above to construct a distribution that is orthogonal to the Liouville vector field. The integrability of this distribution has been shown in [2], the leaves of the foliation determined by this distribution being given by the indicatrix of the Finsler function.

The main results of our work provide a converse of this result. If for a spray $S$, the Berwald distribution is integrable, then it will be tangent to the indicatrix of a Finsler function that metrizes the given spray.

Consider a (degenerate, conic pseudo-) Finsler function $F$, a spray $S$ satisfying equation (2.8), and its Jacobi endomorphism $\Phi$. We say that $S$ has scalar flag curvature $\kappa \in C^\infty(T_0 M)$ if the
Jacobi endomorphism $\Phi$ is of the form
\begin{equation}
\Phi = \kappa F^2 J - \kappa F d_J F \otimes C.
\end{equation}

When $\kappa$ is a constant, we say that $S$ has constant flag curvature. The non-flatness assumption that we work with is equivalent to $\kappa \neq 0$.

The Euler-Lagrange equation (2.8), satisfied by the geodesic spray $S$ of a (degenerate) Finsler function $F$ is equivalent to $d_h F^2 = 0$. This implies
\begin{equation}
dF^2 = d_v F^2 = i_F d_J F^2.
\end{equation}

It is known that if the geodesic spray of a Finsler function is isotropic then the Finsler function has scalar flag curvature, see [15, Lemma 8.2.2]. We will give here an alternative proof of this result, for the more general case that includes the degenerate Finsler functions.

**Proposition 2.2.** Consider an isotropic spray $S$ that is also a geodesic spray of a (degenerate) Finsler function $F$. Then the spray $S$ has scalar flag curvature.

**Proof.** Consider the Euler-Lagrange equation (2.8), which is equivalent to $d h F^2 = 0$. As a consequence we obtain $S(F^2) = 0$. We also have
\begin{equation}
d_{[S,h]} F^2 = \mathcal{L}_S d_h F^2 - d_h \mathcal{L}_S F^2 = 0.
\end{equation}

Using this equation, the fact that $dF^2 = d_v F^2$ and the definition of the Jacobi endomorphism, we obtain
\begin{equation}
0 = d_{[S,h]} F^2 = i_{[S,h]} dF^2 = i_{[S,h]} d_v F^2 = i_{[S,h]} i_v dF^2 = i_{v_0 [S,h]} dF^2 = i_{d} dF^2 = d_{d} F^2.
\end{equation}

We use now the assumption that $S$ is isotropic, which means that the Jacobi endomorphism is given by formula (2.12)
\begin{equation}
\frac{\alpha}{\rho} = \frac{d_{d} F^2}{2F^2} = \frac{d_J F}{F}.
\end{equation}

Using the function $\kappa = \rho / F^2$ and the formulae (2.12), the Jacobi endomorphism can be written as follows:
\begin{equation}
\Phi = \rho \left( J - \frac{\alpha}{\rho} \otimes C \right) = \rho \left( 2F^2 J - F^2 d_J F \otimes C \right) = \kappa \left( 2F^2 J - F d_J F \otimes C \right).
\end{equation}

This shows that formula (2.11) is true, and hence the spray $S$ has scalar flag curvature $\kappa$. □

For an isotropic spray $S$ that is also a geodesic spray of a (degenerate) Finsler function $F$, in view of formula (2.12), we obtain that the 2-form $\Omega$, (2.7), is related to the Hilbert 2-form $\omega_{F^2}$ by
\begin{equation}
\Omega = d \left( \frac{1}{2F^2} d_J F^2 \right) + \frac{1}{2F^2} dF^2 \wedge d_J F^2 = \frac{1}{2F^2} dd_J F^2 = -\frac{1}{2F^2} \omega_{F^2}.
\end{equation}

Since 2-dimensional sprays are always isotropic, it follows using Proposition 2.2 that Finsler metrizable 2-sprays have scalar flag curvature.
3. Berwald frame.

In [3] Berwald constructed a frame on $T_0M$ canonically associated to a 2-dimensional Finsler manifold and used to characterize projectively flat 2-dimensional Finsler manifold and to classify them in some particular cases: Landsberg spaces and Finsler spaces with the main scalar a function of position only. A detailed analysis of the role played by the Berwald frame for the geometry of a 2-dimensional Finsler space is presented in [1, Section 3.5]. Recently, in [7], Crampin rediscovered such a frame, which he associated to a given Riemannian metric, to give a new constructive proof of the fact that any 2-dimensional spray is projectively Finsler metrizable.

In this section we will show that such a frame, which we will call the Berwald frame, can be associated to any 2-dimensional non-flat spray. In the next section we will prove that a spray is metrizable if and only if the Berwald distribution is integrable and the 2-form $\Omega$ satisfies some regularity conditions. Both the integrability and the regularity conditions can be expressed in terms of the Berwald frame.

In the following subsections we study the regularity conditions and the commutation formulae satisfied by the Berwald frame in three cases: for a spray in general, for a Finsler metrizable spray, and for a spray metrizable by a degenerate Finsler function.

3.1. Berwald frame for a spray. For a 2-dimensional spray $S$, we consider the geometric setting described in the previous section. As we already mentioned, $S$ is isotropic and therefore its Jacobi endomorphism is given by formula (2.4). We make the assumption that $S$ is non-flat and therefore the semi-basic 1-form $\alpha$ and the Ricci scalar $\rho$ are nowhere vanishing on $T_0M$. If we allow to work with conic pseudo-Finsler functions, then we will restrict the domain to some open cone $A \subset T_0M$, where $\alpha$ and $\rho$ are not vanishing.

Consider a vector field $H \in \mathfrak{X}(T_0M)$ that satisfies the following conditions:

\begin{equation}
[C, H] = H, \quad \alpha(H) = 0.
\end{equation}

First two conditions (3.1) express that $H$ is a $2^+\text{-homogeneous}$ horizontal vector field. Last condition above is equivalent to $\Phi(H) = \rho JH$ and means that $H$ is (fibrewise) an eigenvector for the Jacobi endomorphism $\Phi$ corresponding to the non-vanishing eigenvalue $\rho$.

Conditions (3.1) do not determine uniquely the vector field $H$, such vector field is determined only up to a $0^\text{-homogeneous}$ function factor. We can fix such a vector field $H$ by requiring that $\{S, H\}$ is compatible with a fixed orientation of $M$. Since $\alpha(S) = \rho \neq 0$ and $\alpha(H) = 0$ we obtain that $H$ and $S$ are two linearly independent vector fields that generate the horizontal distribution. It follows that $V := JH$ and $\mathcal{C} = JS$ are two linearly independent vector fields that generate the vertical distribution. Consequently, $(H, S, V, \mathcal{C})$ is a frame on $T_0M$, which is called the Berwald frame.

Lemma 3.1. Consider $S$ a spray and let $(H, S, V, \mathcal{C})$ be a fixed Berwald frame.

i) The following formulae are satisfied:

\begin{equation}
[C, V] = 0, \quad [S, H] = \nabla H + \rho V, \quad [S, V] = -H + \nabla V.
\end{equation}

ii) The rank of the 2-form $\Omega$ defined by (2.7) is given by

\begin{equation}
\text{rank}(\Omega) = \begin{cases} 
4, & \text{if } \alpha([H, V]) \neq 0; \\
2, & \text{if } \alpha([H, V]) = 0.
\end{cases}
\end{equation}

Proof. i) The tangent structure $J$ is $0^\text{+}-\text{homogeneous}$, which means that $[C, J] = -J$. If we evaluate both sides of this formula on the horizontal $2^\text{+}-\text{homogeneous}$ vector field $H$ we obtain

\[-V = -J(H) = [C, J](H) = [C, JH] - J[C, H] = [C, V] - V.
\]
Thus formula (3.2) is true, which means that \( V \) is a \( 1^+ \)-homogeneous vector field.

Using formulae (2.3) and (2.4), we have

\[
[S, H] = h[S, hH] + v[S, hH] = \nabla H + \Phi(H) = \nabla H + \rho V.
\]

From the properties of the dynamical covariant derivative, \([5, \text{ Theorem 3.4}]\), we have that \( \nabla J = 0 \), and hence \( J(\nabla H) = \nabla J(H) = \nabla V \). From the second formula in (3.2) we obtain \( J[S, H] = \nabla V \), which was to be shown.

ii) We give a matrix representation of the 2-form \( \Omega \) with respect to the Berwald frame \((H, S, V, C)\).

We evaluate (2.7) on pairs of vector fields \( X, Y \in \{H, S, V, C\} \). Since \( \alpha(\rho(S)) = 1 \), \( \alpha(\rho(X)) = 0 \), \( \forall X \in \{H, V, C\} \), it follows that

\[
d\left( \frac{\alpha}{\rho} \right)(X, Y) = -\frac{\alpha}{\rho}(\{X, Y\}), \forall X, Y \in \{H, S, V, C\}.
\]

Using the commutation formulae (3.2), we obtain the following matrix representation of \( \Omega \) with respect to \((H, S, V, C)\)

\[
\Omega = \begin{pmatrix}
0 & \alpha(\nabla H)/\rho & -\alpha([H, V])/\rho & 0 \\
-\alpha(\nabla H)/\rho & 0 & 0 & 1 \\
\alpha([H, V])/\rho & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

From (3.4) we see that the rank of \( \Omega \) is given by formula (3.3).

**Definition 3.2.** A spray \( S \) is called **regular** if the 2-form \( \Omega \) is an almost symplectic form, which means that it satisfies the condition: \( \text{rank}(\Omega) = 4 \). The spray is called **degenerate** if \( \text{rank}(\Omega) = 2 \).

If we replace \( H \) by \( aH \) where \( a \in C^\infty(T_0M) \) is a nowhere vanishing \( 0^+ \)-homogeneous function, then we have that \( \alpha([aH, aV]) = a^2\alpha([H, V]) \) and hence the regularity condition \( \alpha([H, V]) \neq 0 \) does not depend on the choice of the Berwald frame.

### 3.2. Berwald frame for a Finsler function

Consider the geodesic spray \( S \) of a Finsler function \( F \) and consider the Berwald frame \((H, S, V, C)\), determined the conditions (3.1), with the extra condition that \( H \) has the same length as \( S \), which means that

\[
G(H, H) = G(S, S) = F^2.
\]

The three normalised vector field \((H/F, S/F, V/F)\) were used in [4, §4.3] to provide an orthonormal frame for the projective sphere bundle \( SM \) of a 2-dimensional Finsler metric.

Now, for the Berwald frame, uniquely associated to a Finsler function, we will provide all the commutation formulae, extending the results of Lemma 3.1.

**Lemma 3.3.** Consider \( S \) the geodesic spray of a Finsler function \( F \) and let \((H, S, V, C)\) be its Berwald frame.

i) The Berwald frame satisfies the formulae

\[
H(F^2) = 0, \quad V(F^2) = 0, \quad \nabla H = 0, \quad \nabla V = 0,
\]

\[
[S, H] = \rho V, \quad [S, V] = -H, \quad [H, V] = S + IH + S(I) V.
\]
The geodesic spray is regular.

Proof. We first prove the second part ii) of the lemma. Using formulae (2.13), (2.10) and the
scaling condition (3.3) we obtain
\begin{equation}
\Omega(H,V) = \frac{-1}{2F^2} \omega_{F^2}(H,V) = \frac{-1}{2F^2} \omega_{F^2}(H,-F) = \frac{-1}{2F^2} 2G(H,H) = -1.
\end{equation}
From the matrix representation (3.4) it is clear that rank(Ω) = 4, so the geodesic spray S is regular.

i) Equation (2.8) is equivalent to \( d_h F^2 = 0 \), which is further equivalent to \( h \in X(F^2) = 0 \), for all \( X \in \mathfrak{X}(T_0M) \). It follows that the horizontal vector field \( H \) also satisfies \( H(F^2) = 0 \).

The geodesic spray \( S \) has scalar flag curvature, which means that its Jacobi endomorphism is
given by formula (2.11). It follows that the last condition (3.1), for defining the horizontal vector
field \( H \), can be written as follows
\begin{equation}
0 = 2\alpha(H) = 2\kappa F d_j F(H) = \kappa d_j F^2(H) = \kappa JH(F^2) = \kappa V(F^2),
\end{equation}
which means that \( V(F^2) = 0 \), since \( \kappa \neq 0 \).

From the matrix representation (3.4) we obtain \( \Omega(H,C) = 0 \) and \( \Omega(S,C) = 1 \).

Since the dynamical covariant derivative preserves the horizontal distribution, it follows that \( \nabla H = a_1 S + a_2 H \), for \( a_1, a_2 \in C^\infty(T_0M) \), is a horizontal vector field. It follows that \( \nabla V = \nabla JH = J\nabla H = a_1 C + a_2 V \). Using the properties of the dynamical covariant derivative that were
proven in [3], we have that \( \nabla \Omega = 0 \) and \( \nabla C = 0 \), and therefore
\begin{align*}
0 &= (\nabla \Omega)(H,C) = \nabla(\Omega(H,C)) - \Omega(\nabla H,C) - \Omega(H,\nabla C) = -\Omega(a_1 S + a_2 H,C) = -a_1, \\
0 &= (\nabla \Omega)(H,V) = \nabla(\Omega(H,V)) - \Omega(\nabla H,V) - \Omega(H,\nabla V) = -2a_2 \Omega(H,V) = -2a_2
\end{align*}
and hence \( \nabla H = \nabla V = 0 \). Using these calculations and the last two commutation formulae in
(3.6), it follows that the first two commutation formulae in (3.6) are true.

Using the notation \( [H,V] = b_1 S + b_2 H + b_3 C + b_4 V \), for \( b_1, b_2, b_3, b_4 \in C^\infty(T_0M) \), and
the regularity condition (3.7), we have that \( b_1 = \alpha([H,V])/\rho = -\Omega(H,V) = 1 \).

From the first two commutation formulae (3.6) and the Jacobi identity \([S,[H,V]] + [V,[S,H]] + [H,[V,S]] = 0\), we obtain
\begin{align*}
0 &= [S, S + b_2 H + b_3 C + b_4 V] + [V, \rho V] + [H,H] \\
&= -b_3 S + (S(b_2) - b_4) H + S(b_3) C + (S(b_4) + b_2 \rho + V(\rho)) V.
\end{align*}
Precising calculations imply \( b_3 = 0 \), \( b_4 = S(b_2) \) and \( S(b_4) + b_2 \rho + V(\rho) = 0 \). Using the notation \( I = b_2 \) we obtain that the last commutation formula (3.6) is true and the coefficient function \( I \)
satisfies \( S^2(I) + I \rho + V(\rho) = 0 \).

The three commutation formulae (3.6), viewed as derivations on \( 0^+ \)-homogeneous functions,
were obtained first by Berwald [3, (7.6)]. By comparing the last formula in (3.6) and the first
formula in (3.6) we obtain that the function \( I \) is the main scalar of the Finsler function. In the
Riemannian case, the main scalar \( I \) vanishes and the three commutation formulae (3.6) reduce to
the commutation formulae [7, Lemma 1]. For a different derivation of the Berwald frame with the
commutation formulae (3.6) we refer to [14], see also [17, §9.9.1], where the pull-back formalism is
applied.

3.3. Berwald frame for a degenerate Finsler function. Consider a geodesic spray \( S \) of a
degenerate Finsler function \( F \); then \( S \) is a solution of the equation (2.8). For the spray \( S \), again,
we consider a Berwald frame determined by the conditions (3.1). First we extend the results of
Lemma 3.1.
Lemma 3.4. Consider a geodesic spray $S$ of a degenerate Finsler function $F$ and let $(H, S, V, C)$ be a Berwald frame.

i) The following formulae are valid:
\[
\begin{align*}
H(F^2) &= 0, \quad V(F^3) = 0, \quad \alpha(\nabla H) = 0, \quad d_v F(\nabla V) = 0, \\
\alpha([S, H]) &= d_v F([S, H]) = 0, \quad \alpha([S, V]) = d_v F([S, V]) = 0, \\
\alpha([H, V]) &= d_v F([H, V]) = 0.
\end{align*}
\]
(3.9)

ii) The geodesic spray is degenerate.

Proof. i) We obtain as above that $d_h F^2 = 0$ and hence $H(F^2) = 0$. The condition $\alpha(H) = 0$ implies $V(F^2) = 0$.

Since $F$ is a degenerate Finsler function, $\text{rank}(\omega_{F^2}) = \text{rank}(\Omega) = 2$. Using the matrix representation (3.1) of the 2-form $\Omega$ with respect to the Berwald frame, we find that $\Omega(H, S) = 0$ and $\Omega(H, V) = 0$. So, we have
\[
\alpha([H, V]) = -\rho \Omega(H, V) = 0, \quad \alpha(\nabla H) = \rho \Omega(H, S) = 0.
\]
Therefore, $\alpha(\nabla H) = 0$ and hence $d_v F(\nabla H) = 0$. The dynamical covariant derivative preserves the horizontal distribution, which is spanned by $H$ and $S$. The condition $\alpha(\nabla H) = 0$ means that the vector field $\nabla H$ does not have a component along $S$ and hence $\nabla H = a_1 H$, for some function $a_1 \in C^\infty(T_0 M)$. Using the fact that $\nabla J = 0$, it follows that $\nabla V = a_1 V$, which shows that the vertical vector field $\nabla V$ does not have a component along $C$. The last commutation formulae (3.2) can be written now as follows:
\[
[S, H] = a_1 H + \rho V, \quad [S, V] = -H + a_1 V.
\]
(3.10)

Commutation formulae (3.10) show that the vector fields $[S, H]$ and $[S, V]$ have no components along $S$ and $C$ and therefore the corresponding formulae (3.9) are true.

We have seen already that the degeneracy condition on a Finsler function implies $\alpha([H, V]) = 0$, which means that the vector field $[H, V]$ has no component along $S$ and hence we can write it in the form $[H, V] = a_2 H + a_3 C + a_4 V$, for some functions $a_2, a_3, a_4 \in C^\infty(T_0 M)$. Using the Jacobi identity for the three vector fields $S, H, V$ and the above expressions for the Lie brackets $[S, H]$, $[S, V]$ and $[H, V]$ we obtain $a_3 = 0$ and therefore the last formulae (3.9) are true.

ii) Since $F$ is a degenerate Finsler function, $\text{rank}(\omega_{F^2}) = \text{rank}(\Omega) = 2$, which, in view of (2.5) and (2.11), is equivalent to $\text{rank}(\omega_{F^2}) = \text{rank}(\Omega) = 2$, thus, the spray $S$ is degenerate. \(\square\)

4. Integrability of the Berwald distribution and Finsler metrizability.

In this section we will prove that a 2-dimensional spray $S$ is metrizable if and only if the Berwald distribution (2.6) is integrable. The regularity of the corresponding Finsler function depends on the rank of the 2-form (2.7). We treat separately the regular and degenerate cases.

4.1. Finsler metrizability. For a 2-dimensional, non-flat spray $S$, we consider the Berwald distribution $D$ (2.6) and the 2-form $\Omega$ (2.7). The next theorem provides characterisations for the Finsler metrizability of a regular spray, together with an algorithm that can be used to construct effectively a Finsler function that metrizes the spray.

Theorem 4.1. We consider a 2-dimensional, non-flat spray $S$. The following conditions are equivalent:

i) $S$ is Finsler metrizable.

ii) $S$ is regular and the Berwald distribution $D$ is integrable.
iii) There exists a closed 1-form $\omega \in \mathcal{D}^*$ such that

\begin{equation}
\text{rank}(d_J \omega + 2\omega \wedge i_J \omega) = 4.
\end{equation}

Proof. For the first implication $i) \implies ii)$ we assume that $S$ is the geodesic spray of a Finsler function $F$. We consider the Berwald frame associated to the Finsler function $F$ as it has been described in Section 3.2. The conclusion comes from Lemma 3.3. The spray $S$ is regular and, due to the commutation formulae (3.6), we have that the distribution $\mathcal{D} = \text{span}\{H, S, V\}$ is integrable, by the Frobenius Theorem.

For the second implication $ii) \implies iii)$ we assume that $S$ is a regular spray and the Berwald distribution $\mathcal{D} = \text{span}\{H, S, V\}$ is integrable. Since $\text{rank} \mathcal{D} = 3$, it follows that $\text{rank} \mathcal{D}^* = 1$. This freedom allows us to choose a $0^+$-homogeneous 1-form $\omega \in \mathcal{D}^*$, which means that $\mathcal{L}_C \omega = 0$. We fix this 1-form with the normalisation condition $i_C \omega = 1$. We will prove that the 1-form $\omega$ satisfies the two conditions $iii)$.

Since the Berwald distribution $\mathcal{D}$ is integrable it follows that for any two vector fields $X, Y \in \mathcal{D}$ we have $[X, Y] \in \mathcal{D}$ and hence $\omega([X, Y]) = 0$. Therefore $d\omega(X, Y) = 0$, for all $X, Y \in \mathcal{D}$. Using Cartan’s formula, as well as the defining properties of $\omega$, we have $i_C \omega = \alpha$ and therefore $\omega = \frac{\alpha}{\rho}$.

Using these two formulae we can see that the 2-form in formula (4.1) coincides with the 2-form (2.7). The regularity condition of the spray $S$ implies that condition (4.1) is satisfied.

For the last implication $iii) \implies i)$ consider a closed 1-form $\omega \in \mathcal{D}^*$ that satisfies the condition (4.1). Again, the fact that $\omega \in \mathcal{D}^*$ implies $\omega \circ \Phi = 0$, and as we have just seen formulae (4.2) are true. Locally, these two formulae can be written as follows

\begin{equation}
i_J \omega = \frac{\alpha}{\rho} \quad \text{and therefore} \quad \omega = \frac{\alpha}{\rho}.
\end{equation}

From these two formulae we obtain $i_C \omega = 1$ and $\mathcal{L}_C \omega = 0$.

Since the 1-form $\omega$ is closed, using Poincaré’s Lemma we conclude that there exists a locally defined function $f$ on $T_0 \mathcal{M}$ such that $\omega = df$. We define $F = \exp(f)$ and we show that $F$ is a Finsler function, whose geodesic spray is $S$. Observe first that

$$C(f) = i_C df = i_C \omega = i_C i_C \frac{\alpha}{\rho} = i_C \frac{\alpha}{\rho} = i_S \frac{\alpha}{\rho} = 1,$$

which implies that $C(F) = C(\exp(f)) = \exp(f) = F$, and, therefore, $F$ is $1^+$-homogeneous.

In view of the two formulae (4.2) we have that the 2-form $\Omega$, (2.7), is given by $\Omega = d_J \omega + 2\omega \wedge i_J \omega$. Now, using the assumption (4.1) we obtain $\text{rank}(\Omega) = 4$. First formula (4.2) can be written now as follows

$$\frac{\alpha}{\rho} = i_J \omega = d_J f = \frac{d_J F}{F} = \frac{d_J F^2}{2F^2}.$$
Using this formula, we see that the 2-form $\Omega$ and the function $F$ are related by formula (2.13). It follows that rank$(\omega_F^2) = 4$, which means that $F^2$ is regular and hence $F$ is indeed a Finsler function.

It remains to check that $S$ is the geodesic spray of the Finsler function, which we show by proving that $d_h F = 0$. We use again the condition $\omega \in D^*$, which implies $\omega \circ h = 0$. Since $\omega = df = \frac{dF}{F^2}$ it follows that $d_h f = 0$ and hence $d_h F = 0$, which completes the proof. □

Criterion iii) in Theorem 4.1 shows “where to look” for a Finsler function in the case when a spray $S$ is metrizable. It is enough to pick a closed 1-form $\omega$ from the 1-dimensional annihilator of the Berwald distribution $D$. The two conditions $L_C \omega = 0$ and $i_C \omega = 1$ show that the 1-form $\omega$ is unique. Condition $d\omega = 0$ is equivalent to the integrability of the distribution $D$ and provides the Finsler function $F$ that metrizes the spray $S$, through the condition $\omega = \frac{dF}{F}$. In this case the freedom we have for choosing the Finsler function is 1 and it agrees to the result in [10] for non-flat isotropic spray.

Criterion ii) in Theorem 4.1 is more geometric and it shows that in the integrable case, the Berwald distribution $D$ is tangent to a hyper-surface in $TM$ that represents the indicatrix of the Finsler function that metrizes the spray.

4.2. Degenerate Finsler metrizability. In this section, we pay attention to the case when a 2-dimensional, non-flat spray $S$ is metrizable by a degenerate Finsler function. Similarly with the regular case, we will show in the next theorem that all the information regarding the metrizability of a spray are encoded again into the Berwald distribution (2.6) and the 2-form (2.7).

Theorem 4.2. Let $S$ be a 2-dimensional, non-flat spray. The following conditions are equivalent:

i) $S$ is metrizable by a degenerate Finsler function.

ii) The spray $S$ is degenerate and the Berwald distribution $D$ is integrable.

iii) There exists a closed 1-form $\omega \in D^*$ such that

$$\text{rank} (d_\omega + 2\omega \wedge i_\omega) = 2.$$  (4.3)

Proof. We use similar techniques and ideas that were used in the proof of Theorem 4.1. We will emphasise only the aspects related to the degeneracy of the spray and the corresponding degenerate Finsler function.

For the first implication we make use of Lemma 3.4. According to the second item of this lemma, we have that the spray $S$ is degenerate. From the commutation formulae (3.9) we obtain

$$d_v F ([S, H]) = d_v F ([S, V]) = d_v F ([H, V]) = 0,$$

which show that the vector fields $[S, H]$, $[S, V]$ and $[H, V]$ have no components along the Liouville vector field $C$. Using Frobenius Theorem it follows that the Berwald distribution $D = \text{span}\{H, S, V\}$ is integrable.

To prove the second implication $\text{ii) } \Rightarrow \text{iii) }$, we assume that $S$ is a degenerate spray and the Berwald distribution $D = \text{span}\{H, S, V\}$ is integrable. We can either follow the arguments of Theorem 4.1 for the same implication, or we can just take

$$\omega = i_C \alpha = \frac{\alpha_\rho \delta \rho^i}{\rho} \in D^*.$$  (4.4)

With this choice we have that $i_C \omega = 1$, $L_C \omega = 0$, which implies $i_C d\omega = 0$. Using the assumption that the Berwald distribution $D$ is integrable it follows that there exists $\theta \in A^1(T_0 M)$ such that $d\omega = \omega \wedge \theta$. We evaluate both sides of this formula on the Liouville vector field $C$ to obtain

$0 = i_C d\omega = \theta - (i_C \theta) \omega$, whence $\theta = (i_C \theta) \omega$ and therefore $d\omega = 0.$
Using the expression of the 1-form $\omega$ it follows that the 2-form in formula (4.3) coincides with the 2-form (2.7). The assumption that the spray $S$ is degenerate implies that $\text{rank}(\Omega) = 2$ and hence formula (4.3) is valid.

To prove the last implication, we consider a closed 1-form $\omega \in \mathcal{D}^*$ that satisfies formula (4.3). Condition $\omega \in \mathcal{D}^*$ implies that $\omega$ is actually given by formula (1.3). Similarly as in the proof of Theorem 4.1 we have that there exists a locally defined function $f$ on $T_0M$ such that $\omega = df$. The function $F = \exp(f)$ is $1^+$-homogeneous and satisfies $d_t F = 0$. Formula (4.3) implies that $\text{rank}(\Omega) = 2$, and in view of formula (2.13) it follows that $\text{rank}(\omega_{F^2}) = 2$. Therefore $F$ is a degenerate Finsler function that metrizes the given spray $S$.

One can provide an equivalent reformulation of the Finsler metrizability criteria of Theorems 4.1 and 4.2 using the holonomy distribution $\mathcal{D}_H$ introduced in [14], as follows. The spray $S$ is metrizable by a (degenerate) Finsler function if and only if the Berwald distribution $\mathcal{D}$ is integrable, which is equivalent to $\mathcal{D} = \mathcal{D}_H$. In this case, the metrizability freedom of the spray $S$, [10] Proposition 4.9], is $m_* = \text{codim} \mathcal{D}_H = \text{codim} \mathcal{D} = 1$. Therefore, if a non-flat spray $S$ is metrizable, then the corresponding (degenerate) Finsler function is unique, up to a multiplicative constant.

5. Examples

In this section we exemplify the main results of our work. We will use both criteria $\text{ii)}$ and $\text{iii)}$ in Theorems 4.1 and 4.2 to test the metrizability of the given spray and to show how one can find the corresponding (degenerate) Finsler function in the integrable case.

5.1. Metrizable sprays.

5.1.1. The regular case. We will use the next example to show how the algorithms described in the proof of Theorem 4.1 can be applied to test the Finsler metrizability of a spray and, in the affirmative case, to construct the Finsler function.

On $M = \{(x^1, x^2) \in \mathbb{R}^2, x^2 > 0\}$ we consider the SODE

\begin{equation}
\frac{d^2 x^1}{dt^2} - \frac{2}{x^2} \frac{dx^1}{dt} \frac{dx^2}{dt} = 0, \quad \frac{d^2 x^2}{dt^2} + \frac{1}{x^2} \left( \frac{dx^1}{dt} \right)^2 - \left( \frac{dx^2}{dt} \right)^2 = 0.
\end{equation}

The components of the induced nonlinear connection are given by

\begin{equation}
N_1^1 = -\frac{y^2}{x^2}, \quad N_2^1 = \frac{y^1}{x^2}, \quad N_1^2 = \frac{y^1}{x^2}, \quad N_2^2 = -\frac{y^2}{x^2}.
\end{equation}

The local components of the Jacobi endomorphism are

\[ R_1^1 = -\frac{(y^1)^2}{(x^2)^2}, \quad R_2^1 = R_1^2 = \frac{y^1 y^2}{(x^2)^2}, \quad R_2^2 = -\frac{(y^1)^2}{(x^2)^2}. \]

The spray $S$ is isotropic, the components of the semi-basic 1-form $\alpha$ and the Ricci scalar are

\[ \alpha_1 = \frac{R_2^2}{y^1} = -\frac{y^1}{(x^2)^2}, \quad \alpha_2 = \frac{R_1^2}{y^1} = -\frac{y^2}{(x^2)^2}, \quad \rho = R_1^1 + R_2^2 = -\frac{1}{(x^2)^2} \{(y^1)^2 + (y^2)^2\}. \]

We test first the metrizability of the spray using criterion $\text{iii)}$ of Theorem 4.1. All the information about the Finsler metrizability of the spray are encoded into the 1-form

\[ \omega = \frac{ig}{\rho} \alpha = \frac{\alpha_1}{\rho} dy^1 + \frac{\alpha_2}{\rho} dy^2 = \frac{y^1 dy^1 + y^2 dy^2}{(y^1)^2 + (y^2)^2} - \frac{1}{x^2} dx^2. \]
We have that
\[
\Omega = \frac{-1}{(y^1)^2 + (y^2)^2} \left( dx^1 \wedge dy^1 + dx^2 \wedge dy^2 \right),
\]
so \( \text{rank}(\Omega) = 4 \) and hence \( S \) is a regular spray. Moreover, \( d\omega = 0 \) and hence \( \omega = df \), for \( f(x, y) = \frac{1}{2} \ln \left( \left( y^1 \right)^2 + \left( y^2 \right)^2 \right) - \ln(x^2) \). It follows that \( S \) is metrizable by the Finsler function
\[
F(x, y) = \exp(f(x, y)) = \sqrt{\left( y^1 \right)^2 + \left( y^2 \right)^2 \over x^2},
\]
which is the Poincaré metric on the half-plane \( M \).

Now, we will check again the metrizability of the SODE (5.1) using the second criterion of Theorem 4.1. Consider a vector field \( H \in X(T_0M) \) satisfying conditions (3.1). One can choose such a vector field to be \( H = -y^2 \delta/\delta x^1 + y^1 \delta/\delta x^2 \). Therefore, the Berwald frame \( (H, S, V) = JH \) generates the Berwald distribution \( D \). From the following Lie brackets, we can see directly that this distribution is integrable:
\[
\begin{align*}
[H, V] & = S, \\
[S, V] & = -H, \\
[S, H] & = \rho V.
\end{align*}
\]
We want to find now the integral manifold \( IM \) to the Berwald distribution \( D = \text{Im}(h) \oplus \text{Im}(\Phi) \).

We will search for the manifold \( IM \) using the fact that it contains all horizontal curves and the curves tangent to the vertical vector field \( V \).

A vertical curve \( \gamma_v(t) = (x^1, y^1(t)) \) is tangent to the vector field \( V \) if and only if
\[
\begin{align*}
\frac{dy^1}{dt} & = -y^2, \\
\frac{dy^2}{dt} & = y^1.
\end{align*}
\]
With the initial condition \( \gamma_v(0) = (x^1, y^1) \), we obtain that the curve \( \gamma_v(t) = (x^1, x^2, y^1 \cos t - y^2 \sin t, y^1 \sin t - y^2 \cos t) \) belongs to the family of hyper-surfaces
\[
(y^1)^2 + (y^2)^2 = f(x^1, x^2),
\]
for some arbitrary function \( f \) on the base manifold \( M \).

We will restrict the family of hyper-surfaces (5.3) by requiring them to contain also horizontal curves. A curve \( \gamma_h(t) = (x^1(t), y^1(t)) \) is horizontal if and only if \( \nu(\gamma_h(t)) = 0 \) and therefore it satisfies the system of second order ordinary differential equations
\[
(5.5)
\]
For the nonlinear connection (5.2) the system (5.5) becomes
\[
\begin{align*}
\frac{dy^1}{dt} - \left( y^2 \frac{dx^1}{dt} + y^1 \frac{dx^2}{dt} \over x^2 \right) = 0, \\
\frac{dy^2}{dt} + y^1 \frac{dx^1}{dt} - y^2 \frac{dx^2}{dt} = 0.
\end{align*}
\]
We multiply the first equation by \( y^1 \), the second equation by \( y^2 \), we add them to obtain
\[
y^1 \frac{dy^1}{dt} + y^2 \frac{dy^2}{dt} - \frac{1}{x^2} \left( (y^1)^2 + (y^2)^2 \right) \frac{dx^2}{dt} = 0.
\]
Last equation can be written as
\[
(5.6)
\]

We want the horizontal curves to belong to the family of hyper-surfaces (5.4). Therefore, if we substitute the equations (5.4) into (5.6) we obtain
\[
\frac{d}{dt} \left( f(x^1, x^2) \right) - \frac{2}{x^2} f(x^1, x^2) \frac{dx^2}{dt} = 0,
\]
which implies \( f(x^1, x^2) = c(x^2)^2, \) \( c \in \mathbb{R}^* \). Therefore the integral manifold of the Berwald distribution \( D \) is given by
\[
IM = \{(x^i, y^i) \in T_0M, \frac{1}{(x^2)^4} ((y^1)^2 + (y^2)^2) = c \},
\]
which represents the indicatrix of the Poincaré metric (5.3).

5.1.2. The degenerate case. The next example is a degenerate spray. We will test its metrizability and obtain the corresponding degenerate Finsler function using the methods provided by the two criteria of Theorem 4.2.

On \( M = \mathbb{R}^2 \), consider the SODE
\[
(5.7) \quad \frac{d^2 x^1}{dt^2} + \frac{x^2}{1 + (x^2)^2} \frac{dx^1}{dt} \frac{dx^2}{dt} = 0, \quad \frac{d^2 x^2}{dt^2} = 0.
\]
The coefficients of the nonlinear connection are
\[
(5.8) \quad N_1^1 = \frac{x^2 y^2}{2(1 + (x^2)^2)}, \quad N_1^2 = \frac{x^2 y^1}{2(1 + (x^2)^2)}, \quad N_2^1 = N_2^2 = 0.
\]
The local components of the Jacobi endomorphism and the Ricci scalar are
\[
R_1^1 = \frac{(y^2)^2((x^2)^2 - 2)}{4((x^2)^2 + 1)^2}, \quad R_2^2 = 0, \quad \rho = \frac{(y^2)^2((x^2)^2 - 2)}{4((x^2)^2 + 1)^2}.
\]
The semi-basic 1-form \( \alpha/\rho \) has the components
\[
\frac{\alpha_1}{\rho} = \frac{R_2^2}{y^1 \rho} = 0, \quad \frac{\alpha_2}{\rho} = \frac{R_1^1}{y^2 \rho} = \frac{1}{y^2}.
\]
The information about the metrizability and the regularity of the spray are encoded into the 1-form
\[
\omega = \omega \frac{\alpha}{\rho} \frac{\delta y^1}{\rho} + \alpha \frac{\delta y^2}{y^2} = \frac{1}{y^2} dy^2.
\]
It follows that the corresponding 2-form (2.7) is given by
\[
\Omega = -\frac{1}{(y^2)^2} dx^2 \wedge dy^2,
\]
so \( \text{rank}(\Omega) = 2 \) and hence the spray \( S \) is degenerate. We also have that \( d\omega = 0 \), since \( \omega = df \), for \( f(x, y) = \ln |y^2| \). Hence the degenerate Finsler function that metrizes the given spray is \( F(x, y) = \exp(f(x, y)) = |y^2| \).

We want now to test the metrizability of the system (5.7) using the second criterion of Theorem 4.2. For the given system, we construct a Berwald frame using the conditions (3.1). If we choose \( H = -y^2 \frac{\partial}{\partial x^1} \), then the Lie brackets of the vector fields \( S, H \) and \( V = JH \) are
\[
[H, V] = 0, \quad [S, V] = -H + \frac{y^2 x^2}{2(1 + (x^2)^2)} V, \quad [S, H] = \frac{y^2 x^2}{2(1 + (x^2)^2)} H + \rho V.
\]
It follows that the Berwald distribution is integrable and hence the system (5.7) is metrizable by a degenerate Finsler function. If we compute the integral manifold of the Berwald distribution, we obtain that it is given by

$$\mathcal{IM} = \{(x^1, x^2, y^1, y^2) \in T_0M, y^2 = c\},$$

which represents the indicatrix of the degenerate Finsler function $F(x, y) = |y^2|$. 

5.2. Non-metrizable sprays. The following is an example of a spray whose metrizability freedom is 0 and its has been proposed by Elgendi and Muzsnay in [10]. We will also show that the spray is not Finsler metrizable, using different techniques provided by Theorems 4.1 and 4.2.

On $M = \{(x^1, x^2) \in \mathbb{R}^2, x^2 > 0\}$, we consider the spray

$$S = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - 2 \left( \varphi y^1 + \frac{y^1 y^2}{2x^2} \right) \frac{\partial}{\partial y^1} - 2 \left( \varphi y^2 - \frac{(y^1)^2}{4} \right) \frac{\partial}{\partial y^2},$$

where we use the notation $\varphi = (x^2(y^1)^2 + (y^2)^2)^{1/2}$.

The coefficients of the nonlinear connection are

$$N^1_1 = \frac{y^2}{2x^2} + \varphi + \frac{x^2(y^1)^2}{\varphi}, \quad N^2_1 = \frac{y^1}{2x^2} + \frac{y^1 y^2}{\varphi}, \quad N^1_2 = -\frac{y^1}{2} + \frac{x^2 y^1 y^2}{\varphi}, \quad N^2_2 = \varphi + \frac{(y^2)^2}{\varphi},$$

while the local components of the corresponding Jacobi endomorphism and Ricci scalar are

$$R^1_1 = \frac{(y^2)^2 [4(x^2)^2 + 1]}{4(x^2)^2}, \quad R^2_2 = \frac{(y^1)^2 [4(x^2)^2 + 1]}{4(x^2)^2}, \quad \rho = \frac{(y^1)^2 + (y^2)^2 [4(x^2)^2 + 1]}{4(x^2)^2}.$$

The spray $S$ is isotropic and the semi-basic 1-form $\alpha/\rho$ has the components

$$\frac{\alpha_1}{\rho} = \frac{y^1}{(y^1)^2 + (y^2)^2}, \quad \frac{\alpha_2}{\rho} = \frac{y^2}{(y^1)^2 + (y^2)^2}.$$

Having the form for $\alpha/\rho$, we compute the 1-form

$$\omega = i_\varphi \frac{\alpha}{\rho} = \frac{\alpha_1}{\rho} \delta y^1 + \frac{\alpha_2}{\rho} \delta y^2.$$

In the above formula, we replace $\delta y^i = dy^i + N^i_j dx^j$, and by a direct calculation we find that $d\omega \neq 0$. Therefore, the two Theorems 4.1 and 4.2 tell us that our spray $S$ is not Finsler metrizable.

We can reach the same conclusion by using the Berwald frame. For the given spray $S$, we search for a horizontal vector field that satisfies equations (3.1). In other words we search for $H = H^1 \delta/\delta x^1 + H^2 \delta/\delta x^2$ that satisfies $\alpha(H) = 0$ and $[C, H] = H$. In order to check the second criterion of Theorems 4.1 and 4.2 we can take any solution of the above system. Such a solution is given by $H^1 = -y^2$ and $H^2 = y^1$. We consider the vertical vector field $V = JH$. According to last formula in (3.2) we have that $[S, V] = -H + \nabla V$. If we compute the dynamical covariant derivative $\nabla V$ we obtain that it has a non-vanishing component along the Liouville vector field $C$. Therefore, the Berwald distribution $\mathcal{D} = \text{span}\{H, S, V\}$ is not integrable and hence the spray $S$ is not Finsler metrizable. The same conclusion follows using the metrizability criterion introduced by Muzsnay in [14], since from the above Lie bracket we have that $C \in \mathcal{D}_H$.

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References

[1] Antonelli, P.L., Ingarden, R.S., Matsumoto, M.: The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer, 1993.
[2] Bejancu, A., Farran, H.R.: Finsler geometry and natural foliations on the tangent bundle, Rep. Math. Phys., 58 (2006), 131–146.
[3] Berwald, L.: On Finsler and Cartan geometries. III Two-dimensional Finsler spaces with rectilinear extremals, Annals of Math., 42 (1941), 84–112.
[4] Bao, D., Chern, S.-S., Shen, Z.: An introduction to Riemann–Finsler geometry, Springer, 2000.
[5] Bucataru, I., Dahl, M.: Semi-basic 1-forms and Helmholtz conditions for the inverse problem of the calculus of variations, J. Geom. Mech., 1 (2009), 159–180.
[6] Bucataru, I., Muzsnay, Z.: Finsler Metrizable isotropic sprays and Hilbert’s fourth problem, J. Aust. Math. Soc, 97 (2014), 27–47.
[7] Crampin, M.: Finsler functions for two-dimensional sprays, Publ. Math. Debrecen, 85 (2014), 435–452.
[8] Darboux, G.: Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal, Gauthier–Villars, 1894.
[9] Douglas, J.: Solution of the inverse problem of the calculus of variations, Trans. Amer. Math. Soc. 50 (1941), 71–128.
[10] Elgendi, S.G., Muzsnay, Z.: Freedom of h(2)-variationality and metrizability of sprays, arXiv:1609.06351
[11] Grifone, J.: Structure presque tangente et connections I, Ann. Inst. Fourier, 22 (1972), 287–334.
[12] Grifone, J., Muzsnay, Z.: Variational Principles For Second-Order Differential Equations, World Scientific, Singapore, 2000.
[13] Krupka, D., Sattarov, A.E.: The inverse problem of the calculus of variations for Finsler structures, Math. Slovaca, 35 (1985), 217–222.
[14] Muzsnay, Z.: The Euler-Lagrange PDE and Finsler metrizability, Houston Journal of Mathematics, 32 (2006), 79–98.
[15] Shen, Z.: Differential geometry of spray and Finsler spaces, Springer, 2001.
[16] Szilasi, J.: A setting for spray and Finsler geometry, in ”Handbook of Finsler Geometry” (ed. P.L. Antonelli), Kluwer Acad. Publ., Dordrecht, 2 (2003), 1183–1426.
[17] Szilasi, J., Lovas, R., Kertész, D.: Connections, sprays and Finsler structures, World Scientific, 2014.
[18] Szilasi, J., Vattamány, S.: On the Finsler-metrizabilities of spray manifolds, Period. Math. Hungar., 44 (2002), 81–100.
[19] Vattamány, S., Vincze, C.: Two-dimensional Landsberg manifolds with vanishing Douglas tensor, Ann. Univ. Sci. Budapest., 44 (2001), 11–26.

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