THE TRIANGULATED CATEGORIES OF FRAMED BISPECTRA AND FRAMED MOTIVES

GRIGORY GARKUSHA AND IVAN PANIN

ABSTRACT. An alternative approach to classical Morel–Voevodsky stable motivic homotopy theory $\text{SH}(k)$ is suggested. The triangulated category of framed bispectra $\text{SH}^{\text{fr}}_{\text{nis}}(k)$ and effective framed bispectra $\text{SH}^{\text{fr\ eff}}_{\text{nis}}(k)$ are introduced in the paper. Both triangulated categories only use Nisnevich local equivalences and have nothing to do with any kind of motivic equivalences. It is shown that $\text{SH}^{\text{fr}}_{\text{nis}}(k)$ and $\text{SH}^{\text{fr\ eff}}_{\text{nis}}(k)$ recover classical Morel–Voevodsky triangulated categories of bispectra $\text{SH}(k)$ and effective bispectra $\text{SH}^{\text{eff}}(k)$ respectively.

We also recover $\text{SH}(k)$ and $\text{SH}^{\text{eff}}(k)$ as the triangulated category of framed motivic spectral functors $\text{SH}^{\text{fr}}_{\text{S}^1}([\mathcal{R}_0(k)]$ and the triangulated category of framed motives $\mathcal{H}^{\text{fr}}_{\text{fr}}(k)$ constructed in the paper.

1. INTRODUCTION

Stable motivic homotopy theory $\text{SH}(k)$ over a field $k$ was introduced by Morel and Voevodsky in their celebrated paper [12]. One of its equivalent constructions is given by first stabilizing the triangulated category of Nisnevich sheaves of $S^1$-spectra $\text{SH}^{\text{nis}}_{S^1}(k)$ in the $\mathbb{G}^m$-direction arriving at the triangulated category of bispectra $\text{SH}^{\text{nis}}(k)$. Then $\text{SH}(k)$ is defined as the triangulated Bousfield localization of $\text{SH}^{\text{nis}}(k)$ with respect to $\{\text{pr}_X : X \times \mathbb{A}^1 \to X \mid X \in \text{Sm}/k\}$. Because of this localization we cannot control anymore stable homotopy types of bispectra by their stable homotopy sheaves $\pi_{\text{nis}}^*(A)$. Instead, we need to compute their “motivic counterparts” $\pi_{\text{fr}}^*(A)$.

However, it is enormously hard in practice to compute stable motivic homotopy types (and in particular, the sheaves $\pi_{\text{fr}}^*(A)$) as well as the Hom-set $\text{SH}(k)(A,B)$ between two bispectra $A,B$. In [16] Voevodsky introduced framed correspondences, the main purpose of which was to suggest a new approach to stable motivic homotopy theory as such. Namely, Voevodsky writes in his notes [16]: “I hope that the constructions described in the notes will lead to a new model of the stable homotopy theory which will be more friendly for computations questions. Of course one expects that it will be non-trivial to show that the new and the old models agree”.

Using Voevodsky’s theory of framed correspondences [16], the authors introduce and develop the machinery of framed motives and big framed motives in [8] that converts the stable motivic homotopy theory of Morel–Voevodsky into the local theory of framed bispectra.

The purpose of this paper is to give another approach to $\text{SH}(k)$. Precisely, we define a triangulated category of framed bispectra $\text{SH}^{\text{fr}}_{\text{nis}}(k)$ and prove that it recovers $\text{SH}(k)$ (see Theorem 2.2). The main
feature of \( \text{SH}^\text{fr}_{\text{nis}}(k) \) is that its construction is genuinely local in the sense that it does not use any kind of motivic equivalences. In other words, we get rid of motivic equivalences completely making \( \text{SH}^\text{fr}_{\text{nis}}(k) \) more amenable to explicit calculations than \( \text{SH}(k) \) of Morel–Voevodsky. In particular, if \( \mathcal{E}, \mathcal{F} \) are two framed bispectra in \( \text{SH}^\text{fr}_{\text{nis}}(k) \), then a morphism \( f : \mathcal{E} \to \mathcal{F} \) is a stable motivic equivalence in the sense of Morel–Voevodsky if and only if the induced morphisms of Nisnevich sheaves of stable homotopy groups \( f_* : \pi^\text{nis}\left(\mathcal{E}(q)\right) \to \pi^\text{nis}\left(\mathcal{F}(q)\right) \) are isomorphisms in each weight \( q \) (see Corollary 2.7). Therefore stable motivic weak equivalences between framed bispectra coincide with naive local equivalences between them. Furthermore, let \( \text{SH}_{S^1}(k) \) be the stable motivic homotopy category of \( S^1 \)-spectra. Then the canonical functor

\[
\Omega^\circ_{\text{nis}^{\text{fr}}_{S^1}G} : \text{SH}(k) \to \text{SH}_{S^1}(k)
\]

has the following explicit and elementary computation in our language. Namely, one takes a framed bispectrum \( \mathcal{E} \) to its zeroth weight \( \mathcal{E}(0) \) (see Corollary 2.8 for details). A similar explicit computation in terms of framed bispectra is given for the adjoint functor

\[
\Sigma^\circ_{\text{nis}^{\text{fr}}_{S^1}G} : \text{SH}_{S^1}(k) \to \text{SH}(k)
\]

(see Corollary 2.8).

We prove in Theorem 2.9 that the explicit functor of big framed motives \( \mathcal{M}^b_{fr} \), in the sense of \cite{8}, determines a localization functor in \( \text{SH}_{\text{nis}}(k) \). Furthermore, \( \mathcal{M}^b_{fr} \) induces an equivalence of triangulated categories \( \text{SH}_{\text{nis}}(k) / \text{Ker} \mathcal{M}^b_{fr} \simeq \text{SH}^\text{fr}_{\text{nis}}(k) \). As a consequence, \( \text{SH}^\text{fr}_{\text{nis}}(k) \) can also be thought of as the triangulated category of \( \mathcal{M}^b_{fr} \)-local objects in \( \text{SH}^\text{fr}_{\text{nis}}(k) \). As another consequence, we prove that \( \text{Ker} \mathcal{M}^b_{fr} \) coincides with the full subcategory of \( \text{SH}_{\text{nis}}(k) \) compactly generated by the shifted cones of the arrows \( \text{pr}_X : \Sigma_{S^1}^\circ \Sigma^\circ_{\text{nis}^{\text{fr}}_{\text{S}^1}G} (X \times \mathbb{A}^1)_+ \to \Sigma^\circ_{\text{nis}^{\text{fr}}_{\text{S}^1}G} X_+\), \( X \in \text{Sm} / k \).

Next, the triangulated category of effective framed bispectra \( \text{SH}^\text{fr,eff}_{\text{nis}}(k) \) is introduced. It is proved that the big framed motive functor induces a triangle equivalence

\[
\mathcal{M}^b_{fr} : \text{SH}^\text{eff}(k) \to \text{SH}^\text{fr,eff}_{\text{nis}}(k).
\]

The reader can find the relevant definitions and statements in Section 3.

The key ingredient in the theory of framed spectra/bispectra and motivic infinite loop spaces developed by the authors in \cite{8} is the framed motive \( M_{fr}(X) \) of a smooth algebraic variety \( X \in \text{Sm} / k \). \( M_{fr}(-) \) is a functor from smooth algebraic varieties to sheaves of framed \( \mathbb{A}^1 \)-local \( S^1 \)-spectra. In this paper we define the triangulated category \( \text{SH}^\text{fr}_{\text{S}^1}[\mathcal{F}r_0(k)] \) of certain functors from smooth \( k \)-schemes to sheaves of framed \( \mathbb{A}^1 \)-local \( S^1 \)-spectra and show that \( \text{SH}^\text{fr}_{\text{S}^1}[\mathcal{F}r_0(k)] \) is naturally equivalent to \( \text{SH}(k) \) (see Theorem 6.3). This result also shows how to construct the framed motive functor \( \mathcal{M}^b_{fr} \) out of any bispectrum \( E \in \text{SH}(k) \). This construction extends the construction of \( M_{fr}(-) \) to all motivic bispectra. Precisely, \( M_{fr}(X) \) is recovered as \( \mathcal{M}_{fr}(X) \) with \( E \) the motivic sphere bispectrum. In other words, Morel–Voevodsky stable motivic homotopy theory is recovered from such framed motives functors. It is important to note that, by definition, the category \( \text{SH}^\text{fr}_{\text{S}^1}[\mathcal{F}r_0(k)] \) has nothing to do with any sort of motivic equivalences. It is purely of local nature. The reader will find the details in Section 6.
It is worth mentioning that computational advantages of framed bispectra and framed spectral functors introduced and studied in this paper are crucial for the machinery of motivic \( \Gamma\)-spaces developed by authors joint with P. A. Østvær in [9]. Motivic \( \Gamma\)-spaces in the sense of [9] extend the celebrated Segal machine of \( \Gamma\)-spaces [14] to the world of motivic homotopy theory.

We finish this paper by introducing the triangulated category of framed motives \( \mathcal{SH}^{fr}(k) \) and show that the triangulated category of effective bispectra \( \mathcal{SH}^{eff}(k) \) is naturally triangle equivalent to \( \mathcal{SH}^{fr}(k) \) (see Theorem 7.2).

Throughout the paper \( k \) is an infinite perfect field. We denote by \( Sm/k \) the category of smooth separated schemes of finite type over \( k \). We denote by \( (\text{Shv}_*(Sm/k), \wedge, pt_+) \) the closed symmetric monoidal category of pointed Nisnevich sheaves on \( Sm/k \). If there is no likelihood of confusion we often write \([X,Y]\) to denote the internal Hom object \( \text{Hom}(X,Y) \). The category of pointed motivic spaces \( M_* \) is, by definition, the category \( \Delta^\text{op}\text{Shv}_*(Sm/k) \) of pointed simplicial sheaves.

2. THE TRIANGULATED CATEGORY OF FRAMED BISPECTRA

It was shown [8] that Morel–Voevodsky stable motivic homotopy category \( SH(k) \) is equivalent to the full subcategory \( SH^{fr}_{nis}(k) \) consisting of certain framed bispectra. In this section we give a genuinely local model for \( SH^{fr}_{nis}(k) \) recovering \( SH(k) \) in the sense that we shall not operate with any type of motivic equivalences in the definition of \( SH^{fr}_{nis}(k) \).

Let \( SP^{S^1,G_m}_{nis}(k) \) be the category of \( (S^1,G_m^1) \)-bispectra in \( M_* \), where \( G_m^1 \) is the mapping cone of the 1-section \( \text{Spec}(k)_+ \rightarrow G_{m+} \) in \( M_* \). More precisely, \( G_m^1 \) is the pushout of the zigzag

\[
\text{Spec}(k)_+ \cap I \leftarrow I \rightarrow \text{Spec}(k)_+ \rightarrow G_{m+},
\]

in which \( I \) is the pointed simplicial set \( \Delta[1] \) with basepoint 1.

The category \( SP^{S^1,G_m}_{nis}(k) \) comes equipped with the stable projective local model structure defined as follows. By [3] \( M_* \) comes equipped with the projective local monoidal model structure in which weak equivalences are Nisnevich local weak equivalences. Stabilizing the model structure in the \( S^1 \)-direction, we get the category \( SP^{S^1}(k) \) of motivic \( S^1 \)-spectra equipped with the stable projective local monoidal model structure, where weak equivalences are the maps of spectra inducing isomorphisms on sheaves of stable homotopy groups. Now stabilising the model structure on \( SP^{S^1}_{nis}(k) \) in the \( G_m^1 \)-direction, we arrive at the stable projective local model structure on \( SP^{S^1,G_m}_{nis}(k) \). Its triangulated homotopy category is denoted by \( SH_{nis}(k) \).

2.1. Definition. We define \( SH^{fr}_{nis}(k) \) as a full subcategory in \( SH_{nis}(k) \) consisting of those bispectra \( \mathcal{E} \) satisfying the following conditions:

1. each motivic space \( \mathcal{E}_{i,j} \) of \( \mathcal{E} \) is a space with framed correspondences, i.e. a pointed simplicial sheaf defined on the category of framed correspondences \( Fr_s(k) \);
2. the structure maps \( \mathcal{E}_{i,j} \rightarrow \text{Hom}(S^1,\mathcal{E}_{i+1,j}), \mathcal{E}_{i,j} \rightarrow \text{Hom}(G_m^1,\mathcal{E}_{i,j+1}) \) preserve framed correspondences;
3. for every \( j \geq 0 \) the framed presheaves of stable homotopy groups \( \pi_s(\mathcal{E}_{s,j}) \) of the \( S^1 \)-spectrum \( \mathcal{E}_{s,j} \) are stable, radditive and \( k^1 \)-invariant;
4. (Cancellation Theorem) for every \( j \geq 0 \) the structure map \( \mathcal{E}_{s,j} \rightarrow \text{Hom}(G_m^1,\mathcal{E}_{s,j+1}) \) of \( S^1 \)-spectra is a stable local equivalence.
We should stress that the definition of $SH_{\text{nis}}^{fr}(k)$ is local in the sense that its morphisms are computed in $SH_{\text{nis}}(k)$ and has nothing to do with $SH(k)$.

2.2. Theorem. The natural functor $F : SH_{\text{nis}}^{fr}(k) \to SH(k)$, which is the identity on objects, is an equivalence of categories. Its quasi-inverse is given by the big framed motive functor $M_{fr}^{b} : SH(k) \to SH_{\text{nis}}^{fr}(k)$ in the sense of [8].

We postpone the proof but first we need some preparations.

2.3. Notation. Given a bispectrum $E \in Sp_{S_{m}^{1}, G_{m}^{1}}(k)$, we shall also write $E = (E(0), E(1), \ldots)$, where $E(j)$ is the motivic $S^{1}$-spectrum with spaces $E(j)_{i} := E_{i,j}$, $i \geq 0$. We also call $E(j)$ the $j$-th weight of $E$.

For the convenience of the reader we recall from [8, Section 12] the definition of the big framed motive. For any bispectrum $E$, let $E^{c}$ be its cofibrant replacement in the projective model structure. Then $E^{c}$ consists of motivic spaces $E_{i,j}^{c}$ which are sequential colimits of simplicial smooth $k$-schemes. We then take $C_{*}Fr(E_{i,j}^{c})$ at every entry and get a bispectrum $C_{*}Fr(E^{c})$, where $C_{*}Fr$ is the canonical functor from motivic spaces to $A_{*}$-schemes. We then take $\eta : id \to M_{fr}^{b}$, where

\[ \eta : \text{id} \to M_{fr}^{b}, \]

because $\tau_{E}$ is a level weak equivalence by construction. Note that $\eta$ can also be regarded as a natural transformation of endofunctors in $SH_{\text{nis}}(k)$.

If there is no likelihood of confusion, we will also regard $G_{m}^{1} \times \mathbb{A}_{1}$ as a simplicial smooth scheme from $\Delta^{op}Fr_{0}(k)$, where $Fr_{0}(k)$ is the category of framed correspondences of level 0 in the sense of Voevodsky [16]. Recall that the objects of $Fr_{0}(k)$ are those of $Sm/k$ and the morphism sets $\text{Hom}_{Shv}(Sm/k)(U_{+}, V_{+})$.

Here are some examples of objects of $SH_{\text{nis}}^{fr}(k)$.

2.4. Example. (1) A typical example of an object of $SH_{\text{nis}}^{fr}(k)$ is the bispectrum

\[ M_{fr}^{G}(X) = (M_{fr}(X), M_{fr}(X \times G_{m}^{1}), M_{fr}(X \times G_{m}^{2}), \ldots), \quad X \in Sm/k, \]

consisting of twisted framed motives of $X$ (see [8] for details). Recall from [8] that $M_{fr}(X) := C_{*}Fr(\Sigma_{\mathbb{A}_{1}}X_{+})$. By [8, 11.1] the canonical morphism of bispectra $\Sigma_{\mathbb{A}_{1}}\Sigma_{G_{m}^{1}}X_{+} \to M_{fr}^{G}(X)$ is an isomorphism in $SH(k)$. 

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(2) More generally, let $A$ be an $S^1$-spectrum such that every entry $A_j$ of $A$ is a colimit of $k$-smooth simplicial schemes. Then $C_*Fr(S_{km}^1 A)$ is in $\mathcal{SH}^{fr}_{nis}(k)$ (we use here Lemma 4.5).

(3) Another example of an object of $\mathcal{SH}^{fr}_{nis}(k)$ is the bispectrum
\[
M^G(X) := (M(X), M(X \times \mathbb{G}^1_m), M(X \times \mathbb{G}^2_m), \ldots), \quad X \in \text{Sm}/k,
\]
where each weight $M(X \times \mathbb{G}^i_m)$ is the Eilenberg–Mac Lane $S^1$-spectrum associated with the Voevodsky’s motivic complex $C_*\mathbb{Z}_fr(X \times \mathbb{G}^i_m)$. Note that $M^G(pt)$ represents motivic cohomology in $SH(k)$.

(4) More generally, let $\mathcal{A}$ be a strict $V$-category of correspondences in the sense of [6] admitting a functor of categories $F : Fr_*X \to \mathcal{A}$ such that $F$ is identity on objects and $F(\sigma_X) = \text{id}_X$ for all $X \in \text{Sm}/k$. Then the bispectrum
\[
M^G_f(X) = (M_{\mathcal{A}}(X), M_{\mathcal{A}}(X \times \mathbb{G}^1_m), M_{\mathcal{A}}(X \times \mathbb{G}^2_m), \ldots),
\]
in which each weight $M_{\mathcal{A}}(X \times \mathbb{G}^i_m)$ is the Eilenberg–Mac Lane $S^1$-spectrum associated with the complex $C_*\mathcal{A}(\_ \times \mathbb{G}^i_m)_{\text{nis}}$, is an object of $\mathcal{SH}^{fr}_{nis}(k)$.

2.5. **Lemma.** Let $\mathcal{X}$ be a $S^1$-spectrum with presheaves of stable homotopy groups $\pi_*(\mathcal{X})$ being framed stable additive and $k^1$-invariant. Let $\mathcal{X}_f$ be a spectrum obtained from $\mathcal{X}$ by taking a local stable fibrant replacement in the stable local projective model structure of $Sp_{S^1}(k)$. Then $\mathcal{X}_f$ is motivically fibrant.

**Proof.** We have $\mathcal{X}_f = \text{hocolim}_{n \to \infty} (\mathcal{X}_{\geq n})_f$, where $(\mathcal{X}_{\geq n})_f$ is the $n$th truncation of $\mathcal{X}$ in $Sp_{S^1}(k)$. $\mathcal{X}$ has homotopy invariant, stable additive presheaves with framed correspondences of stable homotopy groups $\pi_*(\mathcal{X})$. If $(\mathcal{X}_{\geq n})_f$ is a stable local replacement of $\mathcal{X}_{\geq n}$, the proof of [8, 7.4] shows that $(\mathcal{X}_{\geq n})_f$ is motivically fibrant, and hence so is $\mathcal{X}_f$.

2.6. **Lemma.** Let $\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \ldots)$ be a bispectrum in $\mathcal{SH}^{fr}_{nis}(k)$ and let $\mathcal{E}_f = (\mathcal{E}(0)f, \mathcal{E}(1)f, \ldots)$ be a bispectrum obtained from $\mathcal{E}$ by taking a local stable fibrant replacement of each weight $\mathcal{E}(j)$ in the stable local projective model structure of $Sp_{S^1}(k)$. Then $\mathcal{E}_f$ is motivically fibrant.

**Proof.** By Lemma 2.5 each $S^1$-spectrum $\mathcal{E}(j)_f$ is motivically fibrant. Consider a commutative diagram of $S^1$-spectra
\[
\begin{array}{ccc}
\mathcal{E}(j) & \xrightarrow{\mathcal{Hom}(\mathbb{G}^1_m, \mathcal{E}(j+1))} & \mathcal{E}(j+1) \\
\downarrow & & \downarrow \\
\mathcal{E}(j)_f & \xrightarrow{\mathcal{Hom}(\mathbb{G}^1_m, \mathcal{E}(j+1)_f)} & \mathcal{E}(j+1)_f
\end{array}
\]
The upper arrow and the left vertical arrow are stable local equivalences by assumption. The right vertical arrow is a stable local equivalence by the sublemma of [8, Section 12]. We see that the lower arrow is a stable local equivalence between motivically fibrant $S^1$-spectra. We conclude that it is a sectionwise weak equivalence as well, and hence $\mathcal{E}$ is a motivically fibrant bispectrum.

If $E$ is a bispectrum and $p, q$ are integers, recall that $\mathcal{H}^{-1}_{p,q}(E)$ is the sheaf associated to the presheaf
\[
U \mapsto \mathcal{SH}(k)(\mathcal{S}_{\mathbb{G}^1_m} \mathcal{S}_{\mathbb{G}^q_m} U_+ \wedge S^{p-q} \wedge \mathbb{G}^q_m, E).
\]
Given a presheaf of Abelian groups $F$, denote by $F_{-1}$ the presheaf mapping $U \in Sm/k$ to the kernel of the evaluation at 1: $F(U \times \mathbb{G}_m) \to F(U)$. The presheaf $F_{-q}$, $q > 1$, is defined recursively.

The next statement says that the sheaves $\pi^k_{+1}(E)$, where $E$ is a framed bispectrum, are computed in terms of ordinary Nisnevich sheaves of stable homotopy groups for weighted $S^1$-spectra of $E$.

2.7. Corollary. Let $\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \ldots)$ be a bispectrum in $SH^f_{\text{nis}}(k)$ and $p, q \in \mathbb{Z}$. Then $\pi^k_{p,q}(\mathcal{E}) = \pi^{\text{nis}}_{p-q}(\mathcal{E}(|q|))$ if $q \leq 0$ and $\pi^k_{p,q}(\mathcal{E}) = (\pi^{\text{nis}}_{p-q}(\mathcal{E}(0)))_{-q}$ if $q > 0$, where $|q|$ is the modulus of $q$. In particular, if $\mathcal{F} = (\mathcal{F}(0), \mathcal{F}(1), \ldots)$ is another bispectrum in $SH^f_{\text{nis}}(k)$, then an ordinary morphism of motivic bispectra $f : \mathcal{E} \to \mathcal{F}$ is a stable motivic equivalence in the sense of Morel–Voevodsky if and only if the induced morphisms of Nisnevich sheaves $f_* : \pi^{\text{nis}}(\mathcal{E}(q)) \to \pi^{\text{nis}}(\mathcal{F}(q))$ are isomorphisms in each weight $q$.

Proof. Lemma 2.6 implies $\pi^k_{p,q}(\mathcal{E}) = \pi^{\text{nis}}_{p-q}(\mathcal{E}(|q|))$ if $q \leq 0$ and $\pi^k_{p,q}(\mathcal{E}) = (\pi^{\text{nis}}_{p-q}(\Omega^{\text{cn}_m}(\mathcal{E}(0)))_{-q}$ if $q > 0$. The proof of the sublemma in [8, Section 12] shows that

$$\pi^{\text{nis}}_{p-q}(\Omega^{\text{cn}_m}(\mathcal{E}(0)))_{-q} \cong (\pi^{\text{nis}}_{p-q}(\Omega^{\text{cn}_m}(\mathcal{E}(0)))_{-q},$$

as required. \qed

Proof of Theorem 2.2. Given $\mathcal{E}, \mathcal{E}' \in SH^f_{\text{nis}}(k)$, one has

$$\text{Hom}_{SH(k)}(\mathcal{E}, \mathcal{E}') = \text{Hom}_{SP^f_1, \text{cn}_m(k)}(\mathcal{E}^c, \mathcal{E}'^c_{m,f}) / \sim,$$

where $\sim$ refers to the naive homotopy, $\mathcal{E}^c$ is a cofibrant replacement of $\mathcal{E}$ and $\mathcal{E}'^c_{m,f}$ is a motivically fibrant replacement of $\mathcal{E}'$. By Lemma 2.6 $\mathcal{E}'^c_{m,f}$ can be computed as $\mathcal{E}'^c$, i.e. as a level stable local fibrant replacement of $\mathcal{E}'$. Since $\mathcal{E}'^c_{i,j} \to \mathcal{E}'_{i,j}$ is a weak equivalence in $\mathbb{M}_s$ for all $i, j \geq 0$ and

$$\text{Hom}_{SH^f_{\text{nis}}(k)}(\mathcal{E}, \mathcal{E}') = \text{Hom}_{SP^f_1, \text{cn}_m(k)}(\mathcal{E}^c, \mathcal{E}'^c_{m,f}) / \sim,$$

it follows that $F : SH^f_{\text{nis}}(k) \to SH(k)$ is fully faithful.

Given any $E \in SH(k)$ let $\mathcal{M}^b_f(E)$ be its big framed motive in the sense of [8]. Then $\mathcal{M}^b_f(E) \in SH^f_{\text{nis}}(k)$ and the zigzag $\eta_E$ (1) yields an isomorphism $E \cong \mathcal{M}^b_f(E)$ in $SH(k)$. We conclude that $F$ is an equivalence of categories, as was to be shown. \qed

Denote by $SH^{S^1}(k)$ the stable motivic homotopy category of $S^1$-spectra. There is a canonical pair of adjoint functors

$$\Sigma^\infty_{G_m} : SH^{S^1}(k) \rightleftarrows SH(k) : \Omega^\infty_{G_m^1}.$$

The following result is a consequence of Theorem 2.2 and [8, 12.1].

2.8. Corollary. Let $F : SH^f_{\text{nis}}(k) \to SH(k)$ be the equivalence of Theorem 2.2. Then the composite functor $\Omega^\infty_{G_m^1} \circ F : SH^f_{\text{nis}}(k) \to SH^{S^1}(k)$ is equivalent to the functor taking a bispectrum $\mathcal{E} = (\mathcal{E}(0), \mathcal{E}(1), \ldots) \in SH^f_{\text{nis}}(k)$ to its zeroth weight $\mathcal{E}(0)$. In turn, the composite functor $\mathcal{M}^b_f \circ \Sigma^\infty_{G_m^1} : SH^{S^1}(k) \to SH^f_{\text{nis}}(k)$ is equivalent to the functor taking a motivic $S^1$-spectrum $E \in SH^{S^1}(k)$ to the framed bispectrum $C_+ Fr(\Sigma^\infty_{G_m^1} E^c)$.  


Given a triangulated category \( \mathcal{T} \), we define a localization in \( \mathcal{T} \) as a triangulated endofunctor \( L : \mathcal{T} \to \mathcal{T} \) together with a natural transformation \( \eta : id \to L \) such that \( L \eta_X = \eta_{LX} \) for any \( X \) in \( \mathcal{T} \) and \( \eta \) induces an isomorphism \( LX \cong LLX \). We refer to \( L \) as a localization functor in \( \mathcal{T} \). Such a localization functor determines a full subcategory \( \text{Ker} \eta \) as in \( \mathcal{T} \).

2.9. Theorem. The functor of big framed motives \( \mathcal{M}_f \) determines a localization functor in \( SH_{\text{nis}}(k) \).

The full subcategory \( \text{Ker} \mathcal{M}_f \) is compactly generated by the shifted cones of the arrows \( pr_X : \sum_{G_{\text{Gal}}} (X \times G_1)_+ \to \sum_{G_{\text{Gal}}} X_+ \), \( X \in Sm/k \). Furthermore, \( \mathcal{M}_f \) induces an equivalence of triangulated categories \( SH_{\text{nis}}(k)/\text{Ker} \mathcal{M}_f \cong SH_{\text{nis}}^{fr}(k) \).

Proof. We define a natural transformation \( \eta : id \to \mathcal{M}_f \) as in (2). Consider a commutative diagram of bispectra

\[
\begin{array}{ccc}
E & \xrightarrow{\tau E} & E^c \\
\downarrow{\tau E} & & \downarrow{\tau E^c} \\
E^c & \xrightarrow{(\tau E)^c} & (E^c)^c \\
\downarrow{\alpha E} & & \downarrow{\alpha E^c} \\
\mathcal{M}_f(E) & \xrightarrow{\cdot \mathcal{M}_f(\tau E)} & \mathcal{M}_f((E^c)^c) \\
\end{array}
\]

The left vertical and upper horizontal zigzags equal \( \eta_E \). The lower horizontal zigzag equals \( \mathcal{M}_f(\eta_E) \). In turn, the right vertical zigzag equals \( \eta_{\mathcal{M}_f(E)} \). It follows that \( \eta_{\mathcal{M}_f(E)} \circ \eta_E \) equals \( \mathcal{M}_f(\eta_E) \circ \eta_E \) in \( \text{Mor}(SH(k)) \). As we have noticed above, \( \eta_E \) is an isomorphism in \( SH(k) \), and hence \( \eta_{\mathcal{M}_f(E)} = \mathcal{M}_f(\eta_E) \). Since the functor \( F : SH_{\text{nis}}^{fr}(k) \to SH(k) \) of Theorem 2.2 is an equivalence, we see that \( \eta_{\mathcal{M}_f(E)} = \mathcal{M}_f(\eta_E) \) in \( SH_{\text{nis}}^{fr}(k) \), because both zigzags are images of \( F \). Since \( \mathcal{M}_f \) converts stable motivic equivalences to level local equivalences by \( [8, 12.4] \), it follows that \( \eta_{\mathcal{M}_f(E)} \), \( \mathcal{M}_f(\eta_E) \) are zigzags of level local equivalences. We have also verified that \( \eta \) induces an isomorphism \( \mathcal{M}_f(E) \cong \mathcal{M}_f(\cdot \mathcal{M}_f(E)) \). So \( \mathcal{M}_f \) determines a localization functor in \( SH_{\text{nis}}(k) \).

Next, let \( \mathcal{S} \) denote the full subcategory of \( SH_{\text{nis}}(k) \) compactly generated by the shifted cones of the arrows \( pr_X : \sum_{G_{\text{Gal}}} (X \times G_1)_+ \to \sum_{G_{\text{Gal}}} X_+ \), \( X \in Sm/k \). By definition, \( SH(k) \) is the quotient category \( SH_{\text{nis}}(k)/\mathcal{S} \). Therefore, a bispectrum \( E \) is isomorphic to zero in \( SH(k) \) if and only if it is in \( \mathcal{S} \). By \( [8, 12.4] \) \( \eta : id \to \mathcal{M}_f \) is an isomorphism of endofunctors in \( SH(k) \). It follows that \( \text{Ker} \mathcal{M}_f = \mathcal{S} \) Now the fact that \( \mathcal{M}_f(E) \) induces an equivalence of triangulated categories \( SH_{\text{nis}}(k)/\text{Ker} \mathcal{M}_f \cong SH_{\text{nis}}^{fr}(k) \) follows from Theorem 2.2.

2.10. Corollary. The functor of big framed motives \( \mathcal{M}_f : SH_{\text{nis}}(k) \to SH_{\text{nis}}^{fr}(k) \) is left adjoint to the inclusion functor \( 1 : SH_{\text{nis}}^{fr}(k) \to SH_{\text{nis}}(k) \).

We can summarize the results of the section as follows. We start with the local stable homotopy category of sheaves of \( S^1 \)-spectra \( SH_{\text{nis}}^{S^1}(k) \), which is also closed symmetric monoidal triangulated.
Then stabilizing $SH^\text{nis}_{S^1}(k)$ with respect to the endofunctor $\mathbb{G}_m^{\wedge 1} \wedge -$ we arrive at the triangulated category $SH_{\text{nis}}(k)$. We apply the explicit localization functor of big framed motives $\mathcal{M}_f^b$ to $SH_{\text{nis}}(k)$. Next, we compute the quotient category $SH_{\text{nis}}(k)/\text{Ker} \mathcal{M}_f^b$ or, equivalently saying, the full subcategory of $\mathcal{M}_f^b$-local objects as the full subcategory $SH^\text{fr}_{\text{nis}}(k)$ (see Definition 2.1) and prove that $SH^\text{fr}_{\text{nis}}(k)$ is equivalent to classical Morel–Voevodsky stable motivic homotopy theory $SH(k)$.

3. Effective framed bispectra

In [11] Levine computes slices of motivic $S^1$-spectra and motivic bispectra. Basing on these computations, Bachmann–Fasel [2] give a criterion for effective motivic bispectra. In this section we use Bachmann–Fasel’s [2] technique to describe effective framed bispectra (see Definition 3.5 and Theorem 3.6). Throughout this section by a semi-local scheme we shall mean a localisation of a smooth irreducible scheme at finitely many closed points.

3.1. Lemma. Let $\mathcal{F}$ be a framed radditive $\mathbb{A}^1$-invariant quasi-stable presheaf of Abelian groups. Then $\mathcal{F}(U) \cong \mathcal{F}_\text{nis}(U)$ for any semi-local scheme $U$.

Proof. We claim that the restriction map $\mathcal{F}(U) \to \mathcal{F}(k(U))$ is injective. This was shown for local schemes in [7]. The same proof works over semi-local schemes if we use [13, 2.2, 2.3, 4.3]. Now consider a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & (\text{Ker }\alpha)(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{F}_\text{nis}(U) & \longrightarrow & (\text{Coker }\alpha)(U) & \longrightarrow & 0 \\
0 & \longrightarrow & (\text{Ker }\alpha)(k(U)) & \longrightarrow & \mathcal{F}(k(U)) & \longrightarrow & \mathcal{F}_\text{nis}(k(U)) & \longrightarrow & (\text{Coker }\alpha)(k(U)) & \longrightarrow & 0 \\
\end{array}
$$

with $\alpha : \mathcal{F} \to \mathcal{F}_\text{nis}$ the canonical sheafification map. Since $\mathcal{F}_\text{nis}$ is a framed radditive $\mathbb{A}^1$-invariant quasi-stable presheaf of Abelian groups by [7], then so are the presheaves $\text{Ker }\alpha, \text{Coker }\alpha$. It follows that the vertical maps of the diagram are monomorphisms. But $\alpha(k(U))$ is an isomorphism, and hence $(\text{Ker }\alpha)(k(U)) = (\text{Coker }\alpha)(k(U)) = 0$. We see that $(\text{Ker }\alpha)(U) = (\text{Coker }\alpha)(U) = 0$, and so $\alpha(U)$ is an isomorphism.

3.2. Lemma. Let $\mathcal{X} \in \mathcal{S}P_{S^1}(k)$ be a motivic $S^1$-spectrum with presheaves of stable homotopy groups being framed, radditive, quasi-stable and $\mathbb{A}^1$-invariant. Suppose $\alpha : \mathcal{X} \to \mathcal{X}_f$ is a local stable fibrant replacement of $\mathcal{X}$. Then the induced map of $S^1$-spectra $\alpha(U) : \mathcal{X}(U) \cong \mathcal{X}_f(U)$ is a stable equivalence of ordinary spectra for any semi-local scheme $U$.

Proof. For every $i \in \mathbb{Z}$, consider a commutative diagram of Abelian groups

$$
\begin{array}{cccc}
(\pi_i(\mathcal{X}))(U) & \longrightarrow & (\pi_i(\mathcal{X}_f))(U) \\
\downarrow & & \downarrow \\
(\pi_i^\text{nis}(\mathcal{X}))(U) & \overset{\alpha^i}{\longrightarrow} & (\pi_i^\text{nis}(\mathcal{X}_f))(U). \\
\end{array}
$$

By Lemma 3.1 the left vertical map is an isomorphism. The lower arrow is an isomorphism for obvious reasons. Therefore our assertion will be proved if we show that the right vertical arrow is an isomorphism.
Lemma 3.1. It follows that \( H_{nis}^{p}(U, \pi_{q}^{nis}(\mathcal{X})) \implies (\pi_{q-p}(\mathcal{X})) = 0 \) for all \( p > 0 \). To see this, we use the fact that \( V \mapsto H_{nis}^{p}(V, \pi_{q}^{nis}(\mathcal{X})) \) is a framed quasi-stable radditive \( A^{1} \)-invariant presheaf by [7, Section 16] and the restriction map \( H_{nis}^{p}(U, \pi_{q}^{nis}(\mathcal{X})) \to H_{nis}^{p}(k(U), \pi_{q}^{nis}(\mathcal{X})) \) is a monomorphism by the proof of Lemma 3.1. It follows that \( (\pi_{q}(\mathcal{X}))(U) \cong (\pi_{q}^{nis}(\mathcal{X}))(U) \).

If \( \mathcal{X} \) is not connected, then it is sectionwise weakly equivalent to \( \text{hocolim}_{n} \pi_{n}(\mathcal{X}) \). The following lemma computes its sections \( \pi_{n}(\mathcal{X}) \).

Proof. Suppose \( f_{0} \) is finitely generated fields \( K \). Let \( \mathcal{X}_{\geq n} = \Delta_{n}^{k} \), the \( n \)th truncation of \( \mathcal{X} \) in the category of presheaves of \( S^{1} \)-spectra (see [11, §1.6]). Moreover, \( \mathcal{X}_{f} \) is sectionwise weakly equivalent to \( \text{hocolim}_{n} \pi_{n}(\mathcal{X}_{\geq n}) \), where \( (\mathcal{X}_{\geq n})_{f} \) is a fibrant replacement of \( \mathcal{X} \) in the stable local projective model structure of presheaves of \( S^{1} \)-spectra. Then we have \( \pi_{i}(\mathcal{X}_{f}) = \text{colim} \pi_{i}(\mathcal{X}_{\geq n})_{f} \) and \( \pi_{i}^{nis}(\mathcal{X}_{f}) = \text{colim} \pi_{i}^{nis}(\mathcal{X}_{\geq n})_{f} \). As above, \( \pi_{i}(\mathcal{X}_{f})_{\geq n}(U) \cong \pi_{i}^{nis}(\mathcal{X}_{f})_{\geq n}(U) \), and hence

\[
\pi_{i}(\mathcal{X}_{f})(U) = \text{colim} \pi_{i}(\mathcal{X}_{\geq n})(U) \cong \text{colim} \pi_{i}^{nis}(\mathcal{X}_{\geq n})(U) = \pi_{i}^{nis}(\mathcal{X}_{f})(U),
\]
as required. \( \square \)

Given a field \( K \) and \( \ell \geq 0 \), let \( \mathcal{O}(\ell)_{K, v} \) denote the semi-local ring of the set \( v \) of vertices of \( \Delta_{K}^{\ell} = \text{Spec}(K[t_{0}, \ldots, t_{\ell}]/(t_{0} + \cdots + t_{\ell} - 1)) \) and set

\[
\Delta_{K}^{\ell} := \text{Spec} \mathcal{O}(\ell)_{K, v}.
\]
Then \( \ell \mapsto \Delta_{K}^{\ell} \) is a cosimplicial semi-local subscheme of \( \Delta_{K}^{*} \).

Let \( E \) be an \( A^{1} \)-invariant and Nisnevich excisive \( S^{1} \)-spectrum and let \( s_{0}(E) \) be its zeroth slice. The following lemma computes its sections \( s_{0}(E)(Y) = Y \in S^{1} \), where \( s_{0}(E)(Y) \) is, by definition, the value of a motivically fibrant replacement of \( s_{0}(E) \) at \( Y \).

3.3. Lemma. ([10, 2.2.6]) For \( Y \in S^{1} \), \( s_{0}(E)(Y) \) is weakly equivalent to the total spectrum \( E(\Delta_{K}^{*}(Y)) \) of the simplicial spectrum \( \ell \mapsto E(\Delta_{K}^{\ell}(Y)) \).

3.4. Corollary. \( s_{0}(E) = 0 \) in \( S^{1} \) if and only if the total spectrum \( E(\Delta_{K}^{*}) = 0 \) in \( SH \) for all finitely generated fields \( k \).

Proof. Suppose \( s_{0}(E) = 0 \) in \( S^{1} \). Then \( s_{0}(E)(Y) = 0 \) in \( SH \) for all \( Y \in S^{1} \). Let \( K/k \) be a finitely generated field. Since the base field \( k \) is perfect, then \( K = k(Y) \) for some \( Y \in S^{1} \). By Lemma 3.3 the total spectrum \( E(\Delta_{K}^{*}) = 0 \) in \( SH \). The converse is obvious if we use Lemma 3.3. \( \square \)

3.5. Definition. Let \( E = (E(0), E(1), \ldots) \in SH_{nis}^{1}(k) \) be a framed bispectrum in the sense of Definition 2.1. We say that \( E \) is effective if the total spectrum \( E(j)(\Delta_{K}^{*}) \) is stably trivial, i.e. equals zero in \( SH \), for all positive weights \( j > 0 \) and all finitely generated fields \( k \). The full subcategory of \( SH_{nis}^{1}(k) \) of effective bispectra will be denoted by \( SH_{nis}^{1, eff}(k) \).

The following result is a framed analog of Bachmann–Fasel’s theorem [2, 4.4].

3.6. Theorem. A framed bispectrum \( E \in SH_{nis}^{1}(k) \) is effective in the sense of Definition 3.5 if and only if it is effective as an ordinary motivic bispectrum in \( SH(k) \), i.e. \( E \in SH^{eff}(k) \).
We begin with the following observation. A bispectrum \( E \in \text{SH}_{fr}^f(k) \) satisfies \( \Omega^\infty_{\mathrm{Gm}}(E) = 0 \) in \( \text{SH}_{S_1}(k) \) if and only if \( E = 0 \) in \( \text{SH}(k) \). Indeed, this follows from the isomorphism

\[
\text{SH}_{S_1}(k)(\Sigma^\infty_{\mathrm{Gm}} U_+[i], \Omega^\infty_{\mathrm{Gm}}(E)) \cong \text{SH}(k)(\Sigma^\infty_{\mathrm{Gm}} \Sigma^\infty_{S_1} U_+[i], E), \quad i \in \mathbb{Z}, \ U \in \text{Sm}/k,
\]

and the fact that the objects \( \Sigma^\infty_{\mathrm{Gm}} \Sigma^\infty_{S_1} U_+[i] \) (respectively \( \Sigma^\infty_{S_1} U_+[i] \)) are compact generators of the triangulated category \( \text{SH}_{eff}^f(k) \) (respectively \( \text{SH}_{S_1}(k) \)).

Suppose \( \mathcal{E} \in \text{SH}_{nis}^f(k) \) is effective. We claim that all negative slices \( s_{n<0}(\mathcal{E}) \) are zero in \( \text{SH}(k) \).

As usual, let \( \mathcal{E}_f = (\mathcal{E}(0)f, \mathcal{E}(1)f, \ldots) \) be the motivic bispectrum obtained from \( \mathcal{E} \) by taking stable projective local fibrant replacements of \( S^1 \)-spectra levelwise. By Lemma 3.2 \( \mathcal{E}_f \) is motivically fibrant.

We have that \( \mathcal{E} \wedge \mathbb{G}_m^n \cong (\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots) \) for all \( n > 0 \) and \( s_0(\mathcal{E} \wedge \mathbb{G}_m^n) \cong s_{-n}(\mathcal{E}) \wedge \mathbb{G}_m^n \) in \( \text{SH}(k) \). Since \( - \wedge \mathbb{G}_m^\infty \) is an autoequivalence of \( \text{SH}(k) \), it follows that \( s_{-n}(\mathcal{E}) = 0 \) if and only if \( s_0((\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots)) = 0 \) in \( \text{SH}(k) \).

By Lemma 3.2 \( \alpha : \mathcal{E}(n)(\widehat{\Delta}_k^\bullet) \to \mathcal{E}(n)f(\widehat{\Delta}_k^\bullet) \) is a stable equivalence of ordinary \( S^1 \)-spectra for every \( \ell \geq 0 \) and every finitely generated field \( K/k \). It follows that the map of total spectra \( \alpha : \mathcal{E}(n)(\widehat{\Delta}_k^\bullet) \to \mathcal{E}(n)f(\widehat{\Delta}_k^\bullet) \) is a stable equivalence. But \( \mathcal{E}(n)(\widehat{\Delta}_k^\bullet) \) is stably trivial by assumption, and hence so is \( \mathcal{E}(n)f(\widehat{\Delta}_k^\bullet) \). By Corollary 3.4 \( s_0(\mathcal{E}(n)f) = 0 \) in \( \text{SH}_{S_1}(k) \).

It follows from [11, 7.1.1, 9.0.3] (see [2, 4.1] as well) that for all \( E \in \text{SH}(k) \), \( \Omega^\infty_{\mathrm{Gm}}(s_0(E)) = s_0(\Omega^\infty_{\mathrm{Gm}}(E)) \) in \( \text{SH}_{S_1}(k) \). Using this and the fact that \( \mathcal{E}_f \) is motivically fibrant we have

\[
s_0(\mathcal{E}(n)f) = s_0(\Omega^\infty_{\mathrm{Gm}}(\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots)) = \Omega^\infty_{\mathrm{Gm}}(s_0((\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots))).
\]

Applying the above observation to the bispectrum \( E = s_0((\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots) \) and using the equality \( s_0(\mathcal{E}(n)f) = 0 \) in \( \text{SH}_{S_1}(k) \) we conclude that \( 0 = s_0((\mathcal{E}(n)f, \mathcal{E}(n+1)f, \ldots), \ldots) \) is a chain of isomorphisms in \( \text{SH}(k) \).

Since all negative slices of \( \mathcal{E} \) are zero, it follows that \( f_0(\mathcal{E}) \to f_{-1}(\mathcal{E}) \to f_{-2}(\mathcal{E}) \to \cdots \) is a chain of isomorphisms in \( \text{SH}(k) \). But the canonical map \( \text{hocolim}_{n \to +\infty} f_{-n}(\mathcal{E}) \to \mathcal{E} \) is an isomorphism in \( \text{SH}(k) \) by [2, 4.2], and therefore \( \mathcal{E} \cong f_0(\mathcal{E}) \in \text{SH}_{eff}^f(k) \) is effective as an ordinary motivic bispectrum.

Now assume the converse. Then for every \( n > 0 \) the slice \( s_{-n}(\mathcal{E}) = 0 \) in \( \text{SH}(k) \). We use the above arguments to conclude that \( s_0(\mathcal{E}(n)f) = 0 \) in \( \text{SH}_{S_1}(k) \). By Corollary 3.4 \( \mathcal{E}(n)f(\widehat{\Delta}_k^\bullet) = 0 \) in \( \text{SH} \) for all finitely generated fields \( K/k \). Since \( \alpha : \mathcal{E}(n)(\widehat{\Delta}_k^\bullet) \to \mathcal{E}(n)f(\widehat{\Delta}_k^\bullet) \) is a stable equivalence of ordinary \( S^1 \)-spectra for every \( \ell \geq 0 \) by Lemma 3.2, we see that \( \mathcal{E}(n)(\widehat{\Delta}_k^\bullet) = 0 \) in \( \text{SH} \). Thus \( \mathcal{E} \) is effective in the sense of Definition 3.5. \( \square \)

3.7. **Corollary.** A bispectrum \( E \in \text{SH}(k) \) is effective if and only if the framed bispectrum \( \mathcal{M}^b_{fr}(E) \) is effective in the sense of Definition 3.5.

**Proof.** This follows from Theorem 3.6 and the fact that \( E \cong \mathcal{M}^b_{fr}(E) \) (see [8, 12.4]). \( \square \)

3.8. **Corollary.** The big framed motive functor \( \mathcal{M}^b_{fr} : \text{SH}(k) \to \text{SH}_{nis}^f(k) \) induces an equivalence of triangulated categories \( \mathcal{M}^b_{fr} : \text{SH}_{eff}^f(k) \to \text{SH}_{nis}^f(k) \).

**Proof.** This follows from Theorems 2.2, 3.6 and Corollary 3.7. \( \square \)
4. Useful Lemmas

4.1. Notation. For any bispectrum $A$ such that every entry $A_{i,j}$ of $A$ are sequential colimits of $k$-smooth simplicial schemes denote by $C_\ast Fr(A)$ the bispectrum $(C_\ast Fr(A_{i,j}))_{i,j \geq 0}$ with obvious structure maps

$$C_\ast Fr(A_{i,j}) \to \text{Hom}(S^1, C_\ast Fr(A_{i,j} \otimes S^1)) \xrightarrow{u} \text{Hom}(S^1, C_\ast Fr(A_{i+1,j}))$$

and

$$C_\ast Fr(A_{i,j}) \to \text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(A_{i,j} \otimes G^\Lambda_m^\ast)) \xrightarrow{u} \text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(A_{i,j+1})), $$

where $u$ refers to the structure map in $A$ in the $S^1/G^\Lambda_m^\ast$-direction. Clearly, the family of maps $A_{i,j} \to C_\ast Fr(A_{i,j})$ form an arrow $\zeta_A : A \to C_\ast Fr(A)$ of bispectra. This arrow is natural in $A$.

The proof of [8, 12.1] shows that the following fundamental result is true.

4.2. Theorem. Let $A$ be a bispectrum such that every entry $A_{i,j}$ of $A$ is a sequential colimit of $k$-smooth simplicial schemes. Then the arrow $\zeta_A : A \to C_\ast Fr(A)$ is an isomorphism in $S^1$-spectra. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\zeta_A & & \zeta_B \\
C_\ast Fr(A) & \xrightarrow{C_\ast Fr(\phi)} & C_\ast Fr(B),
\end{array}$$

and Theorem 4.2. \hfill \Box

4.3. Corollary. Let $A, B$ be bispectra such that all entries $A_{i,j}, B_{i,j}$ of $A$ and $B$ are sequential colimits of $k$-smooth simplicial schemes. Let $\phi : A \to B$ be a stable motivic equivalence. Then the induced morphism

$$C_\ast Fr(\phi) : C_\ast Fr(A) \to C_\ast Fr(B)$$

is a stable motivic equivalence.

Proof. This follows from the commutativity of the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\phi} & B \\
\zeta_A & & \zeta_B \\
C_\ast Fr(A) & \xrightarrow{C_\ast Fr(\phi)} & C_\ast Fr(B),
\end{array}$$

and Corollary 4.3. \hfill \Box

4.4. Corollary. Under the hypotheses of Corollary 4.3 the natural map of bispectra

$$\text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(A)) \to \text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(B))$$

induced by the map $C_\ast Fr(\phi)$ is a stable motivic equivalence for all $n \geq 0$.

Proof. Recall that a map of bispectra $f : X \to Y$ is a stable motivic equivalence if and only if $\Theta^\ast_{G^\Lambda_m^\ast}(f) : \Theta^\ast_{G^\Lambda_m^\ast}(X^{mf}) \to \Theta^\ast_{G^\Lambda_m^\ast}(Y^{mf})$ is a sectionwise level equivalence, where “mf” refers to the level stable motivic fibrant replacement functor in the category of motivic $S^1$-spectra. Here $\Theta^\ast_{G^\Lambda_m^\ast}$ is the standard stabilization functor in the $G^\Lambda_m^\ast$-direction.

By Corollary 4.3 the map $C_\ast Fr(\phi) : C_\ast Fr(A) \to C_\ast Fr(B)$ is a stable motivic equivalence. Since both bispectra are such that in each weight stable homotopy presheaves are framed radditive stable and $\mathbb{A}^1$-invariant, $C_\ast Fr(A)^{mf} = C_\ast Fr(A)^f$ and $C_\ast Fr(B)^{mf} = C_\ast Fr(B)^f$ by Lemma 2.5, where “$f$” refers here to the level stable fibrant replacement functor in the category of motivic $S^1$-spectra. It follows from the sublemma of [8, Section 12] that also

$$(\text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(A)))^{mf} = \Omega G^\Lambda_m^\ast(C_\ast Fr(A)^f)$$

and

$$(\text{Hom}(G^\Lambda_m^\ast, C_\ast Fr(B)))^{mf} = \Omega G^\Lambda_m^\ast(C_\ast Fr(B)^f).$$
We see that the map of the statement is a stable motivic equivalence if and only if the map
$$\Theta^{\infty}_{G_m}(\Omega_{G_m}(C,Fr(A)^f)) \to \Theta^{\infty}_{G_m}(\Omega_{G_m}(C,Fr(B)^f))$$
is a sectionwise level equivalence. Since \(\Theta^{\infty}_{G_m}\) commutes with \(\Omega_{G_m}\) on level motivically fibrant bispectra, the latter is equivalent to saying that
$$\Omega_{G_m}\Theta^{\infty}_{G_m}(C,Fr(A)^f) \to \Omega_{G_m}\Theta^{\infty}_{G_m}(C,Fr(B)^f)$$is a sectionwise level equivalence. But the latter arrow is such, because \(\Theta^{\infty}_{G_m}(C,Fr(A)^f) \to \Theta^{\infty}_{G_m}(C,Fr(B)^f)\) is a sectionwise level equivalence. \(\square\)

4.5. **Lemma.** Let \(F\) be an \(S^1\)-spectrum such that every entry \(F_j\) of \(F\) is a sequential colimit of \(k\)-smooth simplicial schemes. Then the map of \(S^1\)-spectra
$$\tau_n: \text{Hom}(G_m^\wedge n, C, Fr(F)) \to \text{Hom}(G_m^\wedge n \land G_m^\wedge 1, C, Fr(F \land G_m^\wedge 1)) = \text{Hom}(G_m^\wedge n, C, Fr(F \land G_m^\wedge 1))$$
is a stable local equivalence for all \(n \geq 0\).

**Proof.** Firstly, prove the case \(n = 0\). The \(S^1\)-spectrum \(F\) has a natural filtration \(F = \colim_n L_n(F)\), where \(L_n(F)\) is the spectrum
$$(F_0, F_1, \ldots, F_n, F_n \land S^1, F_n \land S^2, \ldots).$$Then \(C, Fr(F) = C, Fr(\colim_n L_n(F)) = \colim_n C, Fr(L_n(F))\), where \(C, Fr(L_n(F))\) is the spectrum \((C, Fr(F_0), C, Fr(F_1), \ldots, C, Fr(F_n), C, Fr(F_n \land S^1), C, Fr(F_n \land S^2), \ldots)\). For brevity we write it as \((C, Fr(F_0), C, Fr(F_1), \ldots, C, Fr(F_n), M_{fr}(F_n) [-n]), \)where \(M_{fr}(F_n)\) is the framed motive of \(F_n\) and \(M_{fr}(F_n) [-n]\) for \(r \geq n\). Similarly, \(\text{Hom}(G_m^\wedge 1, C, Fr(F \land G_m^\wedge 1)) = \colim_n \text{Hom}(G_m^\wedge 1, C, Fr(L_n(F) \land G_m^\wedge 1))\), where \(\text{Hom}(G_m^\wedge 1, C, Fr(L_n(F) \land G_m^\wedge 1))\) is the \(S^1\)-spectrum
$$(\text{Hom}(G_m^\wedge 1, C, Fr(F_0 \land G_m^\wedge 1)), \ldots, \text{Hom}(G_m^\wedge 1, C, Fr(F_{n-1} \land G_m^\wedge 1)), \text{Hom}(G_m^\wedge 1, M_{fr}(F_n) \land G_m^\wedge 1)(-n))).$$By the Cancellation Theorem for framed motives [1] the arrow
$$M_{fr}(F_n) \to \text{Hom}(G_m^\wedge 1, M_{fr}(F_n) \land G_m^\wedge 1))$$is a stable local equivalence, and hence so are the arrows
$$C, Fr(L_n(F)) \to \text{Hom}(G_m^\wedge 1, C, Fr(L_n(F) \land G_m^\wedge 1))).$$We conclude that the arrow \(C, Fr(F) \to \text{Hom}(G_m^\wedge 1, C, Fr(F \land G_m^\wedge 1))\) is a sequential colimit of stable local equivalences.

Next take \(n > 0\) and consider a commutative diagram
$$\begin{array}{ccc}
\text{Hom}(G_m^\wedge n, C, Fr(F)) & \to & \text{Hom}(G_m^\wedge n, C, Fr(F)^f) \\
\downarrow & & \downarrow \\
\text{Hom}(G_m^\wedge n, \text{Hom}(G_m^\wedge 1, C, Fr(F \land G_m^\wedge 1))) & \to & \text{Hom}(G_m^\wedge n, \text{Hom}(G_m^\wedge 1, C, Fr(F \land G_m^\wedge 1))^f),
\end{array}$$where “\(f\)” refers to local stable fibrant replacement. By the first part of the proof the right arrow is a sectionwise level equivalence. The horizontal arrows are stable local equivalences by the sublemma of [8, Section 12]. Our statement now follows. \(\square\)

It is also useful to have the following important
4.6. Theorem. If $B$ is an $S^1$-spectrum such that every entry $B_i$ of $B$ is a sequential colimit of $k$-smooth simplicial schemes, then the canonical morphism $C_s Fr(B) 	o \Omega_{G_m}^\infty \Gamma^\infty_{G_m} C_s Fr(\Sigma^\infty_{G_m}) B$ is a stable local equivalence.

Proof. It follows from Lemma 4.5 that $C_s Fr(\Sigma^\infty_{G_m}) B \in SH^j_{nis}(k)$. By Theorem 4.2 the morphism of bispectra

$$\Sigma^\infty_{G_m} B \to C_s Fr(\Sigma^\infty_{G_m}) B$$

is a stable motivic equivalence. Our theorem now follows from Corollary 2.8. □

4.7. Lemma. Let $A, B$ be two bispectra such that every entry $A_{i,j}, B_{i,j}$ of $A, B$ are sequential colimits of $k$-smooth simplicial schemes.

1. For every weight $j \geq 0$, the $S^1$-spectrum $C_s Fr(A_{i,j})$ is $\mathbb{A}^1$-local;
2. If $f : A \to B$ is a stable motivic equivalence of bispectra, then the induced map of $S^1$-spectra $f_j : \Theta^\infty_{G_m}(C_s Fr(A)) (j) \to \Theta^\infty_{G_m}(C_s Fr(B)) (j)$ is a stable local equivalence for every weight $j \geq 0$.

Proof. (1). In each weight $j$, the $S^1$-spectrum $A_{i,j}$ has a natural filtration $A_{i,j} = \text{colim}_n L_n(A_{i,j})$, where $L_n(A_{i,j})$ is the spectrum

$$A_{0,j}, A_{1,j}, \ldots, A_{n,j}, \ldots, \text{colim}_n L_n(A_{i,j}), \ldots$$

Then $C_s Fr(A_{i,j}) = C_s Fr(\text{colim}_n L_n(A_{i,j})) = \text{colim}_n C_s Fr(L_n(A_{i,j}))$, where $C_s Fr(L_n(A_{i,j}))$ is the spectrum

$$C_s Fr(A_{0,j}), C_s Fr(A_{1,j}), \ldots, C_s Fr(A_{n,j}), C_s Fr(A_{n,j} \otimes S^1), C_s Fr(A_{n,j} \otimes S^2), \ldots$$

If we take Nisnevich local replacements of all entries of the latter spectrum, we get that

$$C_s Fr(A_{0,j}), C_s Fr(A_{1,j}), \ldots, C_s Fr(A_{n,j}), C_s Fr(A_{n,j} \otimes S^1), C_s Fr(A_{n,j} \otimes S^2), \ldots$$

is a motivically fibrant $S^1$-spectrum starting from level $n + 1$ by [8, 7.5]. It follows that each $C_s Fr(L_n(A_{i,j}))$ is $\mathbb{A}^1$-local, and hence so is $C_s Fr(A_{i,j})$ (we use here the fact that a sequential colimit of motivically fibrant spectra is motivically fibrant in the stable projective model structure).

(2). By Corollary 4.3 the induced map $C_s Fr(f) : C_s Fr(A) \to C_s Fr(B)$ is stable motivic equivalences of bispectra. Let “$f$” refers to local stable fibrant replacement. Since in each weight $j \geq 0$ the $S^1$-spectra $C_s Fr(A(j)) f, C_s Fr(B(j)) f$ are motivically fibrant, it follows that the induced map $f : \Theta^\infty_{G_m}(C_s Fr(A)) f \to \Theta^\infty_{G_m}(C_s Fr(B)) f$ is a sectionwise level equivalence with $C_s Fr(A) f = (C_s Fr(A(0))) f, C_s Fr(A(1))) f, \ldots$, $C_s Fr(B) f = (C_s Fr(B(0))) f, C_s Fr(B(1))) f, \ldots$ respectively.

Consider a commutative diagram

$$\Theta^\infty_{G_m}(C_s Fr(A)) (j) \xrightarrow{f} \Theta^\infty_{G_m}(C_s Fr(B)) (j)$$

$$\downarrow$$

$$\Theta^\infty_{G_m}(C_s Fr(A)) (j) \xrightarrow{f} \Theta^\infty_{G_m}(C_s Fr(B)) (j)$$

It follows from the sublemma of [8, Section 12] that the vertical arrows are stable local equivalences. Since the bottom arrow is a sectionwise level equivalence, we get that the top arrow is a stable local equivalence, as required. □
We finish the section by the following useful

4.8. Lemma. If $A$ is a bispectrum such that every entry $A_{i,j}$ of $A$ is a sequential colimit of $k$-smooth simplicial schemes, then $\Theta^n_{G_{\text{int}}}(C,A) \in \operatorname{SH}^{eff}(k)$.

5. THE CATEGORY OF SPECTRAL FUNCTORS

In this section we introduce the category of spectral functors, which will be used to give other models for $\operatorname{SH}(k)$ and $\operatorname{SH}^{eff}(k)$. Recall that the category of pointed Nisnevich sheaves $\operatorname{Shv}_{\ast}(Sm/k)$ is closed symmetric monoidal. For brevity, we shall often write $[F,G]$ to denote $\operatorname{Hom}(F,G)$.

5.1. Lemma. For any $U,V,X \in Sm/k$ one has

$$[U_+,V_+](X) = \operatorname{Hom}_{\operatorname{Shv}_{\ast}(Sm/k)}((U \times X)_+,V_+),$$

where and $Fr_0(U \times X,V)$ is the pointed set of framed correspondences of level 0 in the sense of Voevodsky [16].

Proof. One has

$$[U_+,V_+](X) = \operatorname{Hom}_{\operatorname{Shv}_{\ast}(Sm/k)}((U \times X)_+,V_+).$$

The fact that $\operatorname{Hom}_{\operatorname{Shv}_{\ast}(Sm/k)}((U \times X)_+,V_+) = Fr_0(U \times X,V)$ is shown in [16] (see [8] as well).

5.2. Remark. It is worth mentioning that $Fr_0(-,V), V \in Sm/k,$ is the Nisnevich sheaf associated to the presheaf $U \in Sm/k \mapsto \operatorname{Hom}_{Sm/k}(U,V) \sqcup pt$. Whenever $U$ is connected, we have that $Fr_0(U,V) = \operatorname{Hom}_{Sm/k}(U,V) \sqcup pt$.

5.3. Definition. (1) We denote by $F_{\ast}(k)$ the category enriched over $\operatorname{Shv}_{\ast}(Sm/k)$ whose objects are those of $Sm/k$ and $\operatorname{Shv}_{\ast}(Sm/k)$-objects given by $Fr_0(U \times -,V), U,V \in Sm/k$. It is indeed a $\operatorname{Shv}_{\ast}(Sm/k)$-category by Lemma 5.1. Note that $F_{\ast}(k)$ is $\operatorname{Shv}_{\ast}(Sm/k)$-symmetric monoidal with $U \otimes V := U \times V$. Clearly, its underlying category is $Fr_0(k)$, i.e. the category of smooth $k$-schemes and framed correspondences of level 0.

(2) Denote by $\operatorname{Fun}(F_{\ast}(k)) := [F_{\ast}(k),\operatorname{Shv}_{\ast}(Sm/k)]$ the $\operatorname{Shv}_{\ast}(Sm/k)$-category of enriched functors from $F_{\ast}(k)$ to $\operatorname{Shv}_{\ast}(Sm/k)$. It is closed symmetric monoidal by Day’s Theorem [4]. Its monoidal unit $[pt,-]$ is represented by $pt \in F_{\ast}(k)$. By [5, 2.4] $\operatorname{Fun}(F_{\ast}(k))$ is tensored and cotensored over $\operatorname{Shv}_{\ast}(Sm/k)$.

(3) Consider the category $\Delta^\text{op} \operatorname{Fun}(F_{\ast}(k))$ of simplicial objects in $\operatorname{Fun}(F_{\ast}(k))$. By [5, 2.4] $\Delta^\text{op} \operatorname{Fun}(F_{\ast}(k))$ is tensored and cotensored over pointed motivic spaces $M_{\ast} := \Delta^\text{op} \operatorname{Shv}_{\ast}(Sm/k)$. Denote by $Sp_{\ast}^{\text{pt}}[F_{\ast}(k)]$ the category of $S^1$-spectra associated with the category $\Delta^\text{op} \operatorname{Fun}(F_{\ast}(k))$ and call it the category of spectral functors. By definition, the underlying $S^1$-spectrum of a spectral functor $\mathcal{F} \in Sp_{\ast}^{\text{pt}}[F_{\ast}(k)]$ is its evaluation $\mathcal{F}(pt)$ at $pt$.

5.4. Example. Let $Fr_n(U,V), n \geq 0$, be the set of framed correspondences of level $n$ pointed at the empty correspondence. Then $Fr_n(-,-)$ has an action of the category $F_{\ast}(k)$. The assignment

$$F_{\ast}(k)(-,X) : V \in F_{\ast}(k) \mapsto Fr_n(-,X \times V) \in \operatorname{Shv}_{\ast}(Sm/k), \quad X \in Sm/k,$$

together with the maps

$$[U_+,V_+] \mapsto \operatorname{Hom}(Fr_n(-,X \times U),Fr_n(-,X \times V)),$$
induced by the action of $F_{r_0}(k)$ on $F_{r_0}(U,V)$-s, gives an object of $\text{Fun}(\mathcal{F}_{r_0}(k))$ denoted by $\mathcal{F}_{r_0}(-,X)$. Stabilizing over $n$, we get an object of $\text{Fun}(\mathcal{F}_{r_0}(k))$

$$\mathcal{F}_r(-,X) := \text{colim}(\mathcal{F}_{r_0}(-,X) \xrightarrow{\sigma} \mathcal{F}_{r_1}(-,X) \xrightarrow{\sigma} \cdots).$$

We can equally define $\mathcal{F}_r(F \wedge -,X) := \text{Hom}(F,\mathcal{F}_r(-,X)) \in \text{Fun}(\mathcal{F}_{r_0}(k))$ for any pointed sheaf $F \in \text{Shv}_*(\text{Sm}/k)$. If we set $C_*\mathcal{F}_r(-,X) := \mathcal{F}_r(\Delta^* \times -,X)$, we get an object of $\Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k))$.

### 5.5. Definition

Let $\mathbb{M}^{fr}_*$ denote the subcategory of $\mathbb{M}_*$ consisting of the motivic spaces with framed correspondences and the morphisms preserving framed correspondences. A simplicial enriched functor $\mathcal{X} \in \Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k))$ is said to be framed if it factors through $\mathbb{M}^{fr}_*$ in the sense that $\mathcal{X}(U) \in \mathbb{M}^{fr}_*$ for all $U \in \text{Sm}/k$ and the canonical morphism

$$[U_+, V_+] \mapsto \text{Hom}_{\mathbb{M}^{fr}_*}(\mathcal{X}(U), \mathcal{X}(V)(Y \times -)) = \text{Hom}_{\mathbb{M}_*}(\mathcal{X}(U), \mathcal{X}(V))(Y)$$

factors through $\text{Hom}_{\mathbb{M}^{fr}_*}(\mathcal{X}(U), \mathcal{X}(V)(Y \times -))$ for all $U, V, Y \in \text{Sm}/k$.

A typical example of a framed simplicial enriched functor is given by $C_*\mathcal{F}_r(-,X)$.

Voevodsky [15, Section 3] defined a realization functor from simplicial sets to Nisnevich sheaves $| - |: s\text{Sets} \rightarrow \text{Shv}_{nis}(\text{Sm}/k)$ such that $[\Delta[n]] = \Delta_n^\ell$, where $\Delta[n]$ is the standard $n$-simplex. Under this notation the cosimplicial scheme $\Delta^\ell_* \cong \Delta_*$ equals $[\Delta[\bullet]]$. For every $\ell \geq 0$ denote by $sd^\ell \Delta_*$ the cosimplicial Nisnevich sheaf $[sd^\ell \Delta[\bullet]]$. Under this notation we then have a canonical isomorphism in $\Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k))$

$$E^\ell(C_*\mathcal{F}_r(-,X)) = \mathcal{F}_r([sd^\ell \Delta^\bullet][+ \wedge -,X]).$$

It follows that $E^\ell(C_*\mathcal{F}_r(\mathcal{X})) \in \Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k))$ as well as

$$E^\omega(C_*\mathcal{F}_r(\mathcal{X})) := \text{colim}_n E^\ell(C_*\mathcal{F}_r(\mathcal{X})) \in \Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k)).$$

Then $E^\omega(C_*\mathcal{F}_r(-,X))$ is sectionwise fibrant in the first argument and the map $C_*\mathcal{F}_r(-,X) \rightarrow E^\omega(C_*\mathcal{F}_r(-,X))$ is a sectionwise weak equivalence in the first argument.

Likewise, the Segal spectrum $\mathcal{M}_f(X) := E^\omega(C_*\mathcal{F}_r(-,X \otimes S))$, where $S$ is the ordinary sphere spectrum, gives a spectral functor. Observe that the underlying $S^1$-spectrum of $\mathcal{M}_f(X)$ is the framed motive $M_{fr}(X)$ of $X$ in the sense of [8]. We shall refer to $\mathcal{M}_f(X)$ as the framed motive spectral functor of $X$.

There is a natural evaluation functor

$$ev_{\mathcal{G}_m} : S_{Sp}([\mathcal{F}_{r_0}(k)]) \rightarrow S_{Sp}(\mathcal{G}_m(k),)$$

where $S_{Sp}(\mathcal{G}_m(k))$ is the category of $(S^1, \mathbb{G}_m^\wedge)$-bispactra in $\mathbb{M}_*$, defined as follows. For every $U, V \in \text{Sm}/k$ and $\mathcal{X} \in \text{Fun}(\mathcal{F}_{r_0}(k))$ we have natural morphisms

$$V_+ \rightarrow [U_+, (U \times V)_+] \rightarrow \text{Hom}_{\text{Shv}_*}(\mathcal{F}(U), \mathcal{F}(U \times V))$$

inducing a morphism $\mathcal{F}(U) \wedge V_+ \rightarrow \mathcal{F}(U \times V)$ in $\text{Shv}_*(\text{Sm}/k)$. Given $\mathcal{X} \in \Delta^{op}\text{Fun}(\mathcal{F}_{r_0}(k))$, we define $\mathcal{X}(\mathbb{G}_m^\wedge) \wedge \mathbb{G}_m^\wedge \rightarrow \mathcal{X}(\mathbb{G}_m^\wedge+1)$ as the geometric realization of morphisms $[k \rightarrow \mathcal{X}(\mathbb{G}_m(k) \wedge (\mathbb{G}_m^\wedge+1)) \rightarrow \mathcal{X}(\mathbb{G}_m(k) \wedge (\mathbb{G}_m^\wedge+1))].$ The latter morphisms are now easily extended to spectral functors yielding the desired evaluation functor $ev_{\mathcal{G}_m}$. 

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6. RECOVERING $\text{SH}(k)$ FROM FRAMED SPECTRAL FUNCTORS

In Section 2 we recovered the stable motivic homotopy category $\text{SH}(k)$ as the category of framed bispectra. In this section we give another method of reconstructing $\text{SH}(k)$ from framed spectral functors.

6.1. Definition. (1) A spectral functor $\mathcal{X} = (\mathcal{X}_0, \mathcal{X}_1, \ldots)$ in $\text{Sp}_{S^1}[\mathcal{F}_r(k)]$ is called a framed spectral functor if the following properties are satisfied:

- each simplicial enriched functor $\mathcal{X}_i \in \Delta^{\text{op}} \text{Fun}(\mathcal{F}_r(k)), i \geq 0$, of the spectrum $\mathcal{X}$ is framed in the sense of Definition 5.5;
- for every $U \in \text{Sm}/k$ the bispectrum $(3)$ $ev_{G_m}(\mathcal{X}(- \times U)) = (\mathcal{X}(U), \mathcal{X}(\mathbb{G}_m^1 \times U), \mathcal{X}(\mathbb{G}_m^2 \times U), \ldots)$
  is a framed bispectrum in the sense of Definition 2.1, i.e. $ev_{G_m}(\mathcal{X}(- \times U)) \in \text{SH}_{\text{fr}}(k)$;
- for every $U \in \text{Sm}/k$ the canonical map $pr : \mathcal{X}(k^1 \times U) \to \mathcal{X}(U)$ is a stable local equivalence of $S^1$-spectra;
- it is Nisnevich excisive in the sense that for every elementary Nisnevich square in $\text{Sm}/k$
  
  \[
  \begin{array}{ccc}
  U' & \longrightarrow & Y' \\
  \downarrow & & \downarrow \\
  U & \longrightarrow & Y \\
  \end{array}
  \]

  the square of the $S^1$-spectra

  \[
  \begin{array}{ccc}
  \mathcal{X}(U') & \longrightarrow & \mathcal{X}(Y') \\
  \downarrow & & \downarrow \\
  \mathcal{X}(U) & \longrightarrow & \mathcal{X}(Y) \\
  \end{array}
  \]

  is locally homotopy cartesian (=homotopy cocartesian in the stable structure). We also require $\mathcal{X}(\emptyset) = *$.

(2) We define a category $\text{SH}_{S^1}[\mathcal{F}_r(k)]$ as follows. Its objects are the framed spectral functors and morphisms are defined as

$$\text{SH}_{S^1}[\mathcal{F}_r(k)](\mathcal{X}, \mathcal{Y}) := \text{SH}_{\text{fr}}(k)(ev_{G_m}(\mathcal{X}), ev_{G_m}(\mathcal{Y})).$$

We call $\text{SH}_{S^1}[\mathcal{F}_r(k)]$ the category of framed spectral functors.

A basic example of a framed spectral functor is the framed motive spectral functor $\mathcal{M}_{fr}(X)$ of $X \in \text{Sm}/k$. If we apply $\text{Hom}(\mathbb{G}_m^\wedge, -)$ to $\mathcal{M}_{fr}(X), n \geq 0$, we get another framed spectral functor, denoted by $\Omega^\wedge_{\mathbb{G}_m^n} \mathcal{M}_{fr}(X)$. Explicitly, $\Omega^\wedge_{\mathbb{G}_m^n} \mathcal{M}_{fr}(X) = C_* \mathcal{F}_r(\mathbb{G}_m^\wedge, X \otimes S)$. It is important to note that, by definition, the category $\text{SH}_{S^1}[\mathcal{F}_r(k)]$ has nothing to do with any sort of motivic equivalences. It is purely of local nature.
6.2. Definition. Let $\mathcal{F}$ be a symmetric bispectrum. Let $\mathcal{F}[n]$ be the usual $n$th shift of $\mathcal{F}$ in the $G_m^{\wedge 1}$-direction. For each $i \geq 0$ let $\mathcal{F}(i)$ and $\mathcal{F}[1](i)$ be the $i$th weights of $\mathcal{F}$ and $\mathcal{F}[1]$ respectively (see Notation 2.3). Recall that there is a map of symmetric bispectra

$$\nu_{\mathcal{F}} : G_m^{\wedge 1} \wedge \mathcal{F} \to \mathcal{F}[1]$$

defined as follows: for each weight $i \geq 0$ it is the composition of maps of $S^1$-spectra

$$G_m^{\wedge 1} \wedge \mathcal{F}(i) \xrightarrow{\nu} \mathcal{F}(i) \wedge G_m^{\wedge 1} \xrightarrow{\nu} \mathcal{F}(i + 1) = \mathcal{F}[1](i),$$

where $\chi_{i,1}$ is the shuffle permutation. Sometimes we write $\nu$ to denote $\nu_{\mathcal{F}}$ dropping $\mathcal{F}$ from notation. Notice that $\nu_{\mathcal{F}}$ cannot be defined for nonsymmetric bispectra.

The main result of this section is as follows.

6.3. Theorem. The category of spectral framed functors $SH_{fr}^\mathcal{F}[\mathcal{F} r_0(k)]$ is compactly generated triangulated with compact generators $\{\Omega_{G_m^{\wedge 1}} \mathcal{M}_f r(X) \mid X \in Sm/k, n \geq 0\}$ and the shift functor $\mathcal{F}[1] = \mathcal{F}(- \otimes S^1)$. Moreover, the composite functor

$$F \circ ev_{G_m} : SH^\mathcal{F}[\mathcal{F} r_0(k)] \to SH(k)$$

is an equivalence of triangulated categories, where $F : SH_{fr}^\mathcal{F}(k) \to SH(k)$ is the triangulated equivalence of Theorem 2.2.

Proof. The fact that $F \circ ev_{G_m}$ is fully faithful follows from Definition 6.1(2) and Theorem 2.2. We need to show that every bispectrum $E \in SH(k)$ is isomorphic to the evaluation bispectrum of a spectral framed functor. Without loss of generality we may assume that $E$ is a symmetric $(S^1, G_m^{\wedge 1})$-bispectrum which is also fibrant in the stable projective motivic model structure of symmetric bispectra. Then it is also motivically fibrant as a (non-symmetric) bispectrum. Let $\alpha : E^c \to E$ be the cofibrant replacement of $E$ in the category of symmetric bispectra. Then $\alpha$ is an isomorphism in $SH(k)$ and every entry of $E^c$ is a sequential colimit of $k$-smooth simplicial schemes.

Let $R^{n}_{G_m^{\wedge 1}} C_s Fr(E^c), n \geq 0$, be the bispectrum $\text{Hom}(G_m^{\wedge n} C_s Fr(E^c(n)[n]))$, where $E^c[n]$ is the usual $n$th shift of $E^c$ in the $G_m^{\wedge 1}$-direction. In more detail, for each weight $i \geq 0$ it equals the $S^1$-spectrum

$$R^{n}_{G_m^{\wedge 1}} C_s Fr(E^c)(i) := \text{Hom}(G_m^{\wedge n}, C_s Fr(E^c(n + i))),$$

where $E^c(n + i)$ is the $(n + i)$th weight of the bispectrum $E^c$. Each bonding map equals the composition

$$\text{Hom}(G_m^{\wedge n}, C_s Fr(E^c(n + i))) \wedge G_m^{\wedge 1} \to \text{Hom}(G_m^{\wedge n}, C_s Fr(E^c(n + i) \wedge G_m^{\wedge 1})) \xrightarrow{n} \text{Hom}(G_m^{\wedge n}, C_s Fr(E^c(n + i + 1))).$$

It is useful to regard the symmetric bispectrum $E^c = (E^c(0), E^c(1), \ldots)$ as a symmetric $G_m^{\wedge 1}$-spectrum in the category of symmetric motivic $S^1$-spectra. Then there is a map of bispectra

$$R^{n}_{G_m^{\wedge 1}} C_s Fr(E^c) \to R^{n+1}_{G_m^{\wedge 1}} C_s Fr(E^c) \quad (4)$$

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defined at each weight \( i \) as the composition
\[
\text{Hom}(G_m^\wedge, C_s Fr(E^c(n+i))) \to \text{Hom}(G_m^\wedge, \text{Hom}(G_m^\wedge, C_s Fr(E^c(n+i) \wedge G_m^\wedge))) \xrightarrow{\mu} \\
\text{Hom}(G_m^\wedge + 1, C_s Fr(E^c(n+i+1))) \xrightarrow{\chi_{i+1}} \text{Hom}(G_m^\wedge + 1, C_s Fr(E^c(n+1+i))),
\]
where \( \chi_{i+1} \) is the shuffle permutation in \( \Sigma_{n+i+1} \) permuting the last element with preceding \( i \) elements. An equivalent definition is given by the following composition of bispectra
\[
\text{Hom}(G_m^\wedge, C_s Fr(E^c[n])) \xrightarrow{\alpha_n} \text{Hom}(G_m^\wedge, \text{Hom}(G_m^\wedge, C_s Fr(E^c[n] \wedge G_m^\wedge))) \xrightarrow{\mu} \\
\text{Hom}(G_m^\wedge + 1, C_s Fr(G_m^\wedge \wedge E^c[n])) \xrightarrow{\beta_n} \text{Hom}(G_m^\wedge + 1, C_s Fr(E^c[n+1])),
\]
where the arrow \( \beta_n \) is induced by the canonical map of symmetric bispectra
\[
\nu_n = \nu_{E^c[n]} : G_m^\wedge \wedge E^c[n] \to E^c[n][1] = E^c[n+1].
\]

Following Definition 6.2 in each weight \( i \geq 0 \) it is the composition of maps of \( S^1 \)-spectra
\[
G_m^\wedge \wedge E^c[n](i) \xrightarrow{\mu} E^c[n](i) \wedge G_m^\wedge \xrightarrow{\nu} E^c[n](i+1) \xrightarrow{\chi_{i+1}} E^c[n](i+1).
\]

Recall that \( \nu_n \) cannot be defined for nonsymmetric bispectra.

Claim 1. The map \((4)\) is a stable motivic equivalence of bispectra.

To prove this claim, we need the following

6.4. Lemma. Suppose \( \mathcal{F} \) is a motivically fibrant symmetric bispectrum. Then the morphism
\[
\nu : G_m^\wedge \wedge \mathcal{F} \to \mathcal{F}[1]
\]

from Definition 6.2 is a stable motivic equivalence of ordinary nonsymmetric bispectra. More generally, the arrow
\[
\nu_{A \wedge \mathcal{F}} : G_m^\wedge \wedge (A \wedge \mathcal{F}) \to (A \wedge \mathcal{F})[1]
\]
is a stable motivic equivalence of ordinary nonsymmetric bispectra for any projectively cofibrant motivic space \( A \in \mathbb{M}_* \).

Proof. Consider a commutative diagram
\[
\begin{array}{ccc}
\mathcal{F} & \longrightarrow & \text{Hom}(G_m^\wedge, G_m^\wedge \wedge \mathcal{F}) \longrightarrow \Omega_{G_m^\wedge}(\mathcal{F}[1]) \\
\sim & \downarrow & \sim \\
\Omega_{G_m^\wedge}(G_m^\wedge \wedge \mathcal{F})n^f & \longrightarrow & \Omega_{G_m^\wedge}(\mathcal{F}[1])n^f.
\end{array}
\]

Here \( n^f \) refers to the fibrant replacement functor in the category of ordinary nonsymmetric bispectra. The left slanted arrow and the right vertical arrow are stable motivic equivalences of ordinary nonsymmetric bispectra. The upper composite arrow is a stable motivic equivalence between symmetric spectra. Since both spectra are motivically fibrant, it follows that this composite arrow is a naive sectionwise and levelwise weak equivalence. We see that the bottom arrow is a stable motivic equivalence of bispectra. Since \( \Omega_{G_m^\wedge} \) reflects stable motivic equivalences of bispectra, \( (G_m^\wedge \wedge \mathcal{F})n^f \to \mathcal{F}[1]n^f \) is a stable motivic equivalence of bispectra. The latter implies \( G_m^\wedge \wedge \mathcal{F} \to \mathcal{F}[1] \) is a stable motivic equivalence of bispectra as well.
Now let $A$ be a projectively cofibrant motivic space. Then factor $v : G_m^\wedge \mathcal{F} \to \mathcal{F}[1]$ as the composition $G_m^\wedge \mathcal{F} \to Z \to \mathcal{F}[1]$ of a projective stable trivial cofibration followed by a fibration of nonsymmetric bispectra. It follows that $G_m^\wedge (A \wedge \mathcal{F}) \to A \wedge Z$ is a stable trivial cofibration. Since $\mathcal{F}$ is fibrant as a nonsymmetric bispectrum, then so is $Z$. By assumption, $v$ is a stable motivic equivalence. Therefore $Z \to \mathcal{F}[1]$ is a level equivalence. Regarding the entries of $Z$ and $\mathcal{F}[1]$ as injective cofibrant motivic spaces, we conclude that $A \wedge Z \to (A \wedge \mathcal{F})[1]$ is a level equivalence. □

Now let us prove Claim 1. The arrow $G_m^\wedge E^c[n] \xrightarrow{\partial_n} E^c[n+1]$ is a stable motivic equivalence by the preceding lemma. Every entry of $G_m^\wedge E^c[n] \wedge E^c[n+1]$ is a sequential colimit of $k$-smooth simplicial schemes. Thus by Corollary 4.3 the arrow $C_Fr(\partial_n : C_Fr(G_m^\wedge E^c[n]) \to C_Fr(E^c[n+1]))$ is a stable motivic equivalence. By Corollary 4.4 the map of bispectra

$$\beta_n : \text{Hom}(G_m^{n+1}, C_Fr(G_m^\wedge E^c[n])) \to \text{Hom}(G_m^{n+1}, C_Fr(E^c[n+1]))$$

is a stable motivic equivalence for all $n \geq 0$. By Lemma 4.5 the map of $S^1$-spectra

$$\alpha_n : \text{Hom}(G_m^n, C_Fr(E^c[n])) \to \text{Hom}(G_m^n, C_Fr(E^c[n] \wedge G_m^\wedge))$$

is a level stable local equivalence for all $n \geq 0$. The maps $\alpha_n$ are stable motivic equivalences, and hence so is the map (4) of bispectra and Claim 1 follows.

We continue the proof of the theorem by setting

$$R^{\infty}_{G_m} C_Fr(E^c) : = \text{colim}(C_Fr(E^c) \to R^1_{G_m} C_Fr(E^c) \to R^2_{G_m} C_Fr(E^c) \to \cdots).$$

Claim 2. The composition

$$E^c \xrightarrow{\xi_{E^c}} C_Fr(E^c) \to R^{\infty}_{G_m} C_Fr(E^c)$$

is a stable motivic equivalence.

The left arrow is a stable motivic equivalence by Theorem 4.2. The right arrow a stable motivic equivalence by Claim 1 (we use the fact that a sequential colimit of stable motivic equivalences is a stable motivic equivalence).

To complete the proof of the theorem, we need to construct a framed spectral functor such that its evaluation bispectrum is equivalent to $R^{\infty}_{G_m} C_Fr(E^c)$.

For any $n \geq 0$ we define a spectral functor $G_m C_Fr [n] \to C_Fr(X_\wedge \wedge E^c(n)))$ by

$$X \in \mathcal{F}_0(k) \mapsto \text{Hom}(G_m^n, C_Fr(X_\wedge \wedge E^c(n))).$$

Recall that $ev_{G_m}(G_m C_Fr [n]) (i) = G_m C_Fr(G_m^\wedge E^c[n] (pt) = \text{Hom}(G_m^n, C_Fr(G_m^\wedge E^c(n))))$. The bonding maps $ev_{G_m}(G_m C_Fr [n]) (i) \wedge G_m^\wedge \to ev_{G_m}(G_m C_Fr [n]) (i+1)$ are defined as the composite maps

$$\text{Hom}(G_m^n, C_Fr(G_m^\wedge E^c(n))) \wedge G_m^\wedge \to \text{Hom}(G_m^n, C_Fr(G_m^\wedge E^c(n)) \wedge G_m^\wedge) \wedge G_m^\wedge \to \text{Hom}(G_m^n, C_Fr(G_m^\wedge E^c(n)) \wedge G_m^\wedge) \wedge G_m^\wedge \wedge G_m^\wedge \to \text{Hom}(G_m^n, C_Fr(G_m^\wedge E^c(n)) \wedge G_m^\wedge).$$

It follows now from Lemma 4.5 that $ev_{G_m}(G_m C_Fr [n]) \in SH_{nib}^{I_k}(k)$. 

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There is a natural morphism of spectral functors $\mathcal{G}_* Fr^E[n] \to \mathcal{G}_* Fr^E[n+1]$ defined at every $X \in \mathcal{F} r^0(k)$ by the composition
\[
\Hom(G_m^n, C_* Fr(X_+ \wedge E^c(n))) \to \Hom(G_m^n \wedge G_m^1, C_* Fr(X_+ \wedge E^c(n) \wedge G_m^1)) \\
\xrightarrow{\nu} \Hom(G_m^n \wedge G_m^1, C_* Fr(X_+ \wedge E^c(n + 1))) .
\]

6.5. Definition. Set $\mathcal{M}^E_{fr} := \colim(\mathcal{G}_* Fr^E[0] \to \mathcal{G}_* Fr^E[1] \to \cdots)$. By construction, one has $\ev_{G_m}(\mathcal{M}^E_{fr}) = \colim_n \ev_{G_m}(\mathcal{G}_* Fr^E[n])$. Each $\mathcal{G}_* Fr^E[n]$ is a framed spectral functor, and hence so is $\mathcal{M}^E_{fr}$.

Stabilization maps $\ev_{G_m}(\mathcal{G}_* Fr^E[n]) \to \ev_{G_m}(\mathcal{G}_* Fr^E[n+1])$ are given by the compositions
\[
\Hom(G_m^n, C_* Fr(G_m^j \wedge E^c(n))) \to \Hom(G_m^n \wedge G_m^1, C_* Fr(G_m^j \wedge E^c(n) \wedge G_m^1)) \\
\xrightarrow{\nu} \Hom(G_m^n \wedge G_m^1, C_* Fr(G_m^j \wedge E^c(n + 1))) .
\]

For any $n \geq 0$, construct a map of bispectra $f_n : \ev_{G_m}(\mathcal{G}_* Fr^E[n]) \to R_{G_m}^n C_* Fr(E^c)$ as the composition at each level $i \geq 0$
\[
f_n,i : \Hom(G_m^n, C_* Fr(G_m^j \wedge E^c(n))) \xrightarrow{\nu} \Hom(G_m^n, C_* Fr(E^c(n) \wedge G_m^i)) \xrightarrow{\nu} \Hom(G_m^n, C_* Fr(E^c(n + i))).
\]

6.6. Lemma. Each map $f_n, n \geq 0$, is a morphism of bispectra commuting with stabilization maps $\ev_{G_m}(\mathcal{G}_* Fr^E[n]) \to \ev_{G_m}(\mathcal{G}_* Fr^E[n+1])$ and $R_{G_m}^n C_* Fr(E^c) \to R_{G_m}^{n+1} C_* Fr(E^c)$. In particular, they induce a map of bispectra
\[f : \ev_{G_m}(\mathcal{M}^E_{fr}) \to R_{G_m}^\infty C_* Fr(E^c).
\]

Proof. The following diagram commutes:
\[
\begin{array}{ccc}
\Hom(G_m^n, C_* Fr(G_m^j \wedge E^c(n))) \wedge G_m^1 & \xrightarrow{\nu} & \Hom(G_m^n \wedge G_m^1, C_* Fr(E^c(n) \wedge G_m^i)) \\
\downarrow & & \downarrow \\
\Hom(G_m^n, C_* Fr(G_m^j+1 \wedge E^c(n))) & \xrightarrow{\nu} & \Hom(G_m^n \wedge G_m^{i+1}, C_* Fr(E^c(n + i)) \\
\end{array}
\]
where the left vertical arrow is the $i$th bonding map of $\ev_{G_m}(\mathcal{G}_* Fr^E[n])$, and the right vertical map is the $i$th bonding map of $R_{G_m}^n C_* Fr(E^c)$. We see that each map $f_n$ is a morphism of bispectra. Consider a commutative diagram
\[
\begin{array}{ccc}
\Hom(G_m^n, C_* Fr(G_m^j \wedge E^c(n))) \wedge G_m^1 & \xrightarrow{\nu} & \Hom(G_m^n \wedge G_m^1, C_* Fr(E^c(n) \wedge G_m^i)) \\
\downarrow & & \downarrow \\
\Hom(G_m^n, C_* Fr(G_m^j+1 \wedge E^c(n))) & \xrightarrow{\nu} & \Hom(G_m^n \wedge G_m^{i+1}, C_* Fr(E^c(n + i)) \\
\end{array}
\]
\[
\begin{array}{ccc}
\Hom(G_m^{n+1}, C_* Fr(G_m^j \wedge E^c(n + 1))) \wedge G_m^1 & \xrightarrow{\nu} & \Hom(G_m^{n+1} \wedge G_m^1, C_* Fr(E^c(n + 1) \wedge G_m^i)) \\
\downarrow & & \downarrow \\
\Hom(G_m^{n+1}, C_* Fr(G_m^j+1 \wedge E^c(n + 1))) & \xrightarrow{\nu} & \Hom(G_m^{n+1} \wedge G_m^{i+1}, C_* Fr(E^c(n + 1 + i))) \\
\end{array}
\]
in which the left vertical map is the stabilization $\ev_{G_m}(\mathcal{G}_* Fr^E[n])(i) \to \ev_{G_m}(\mathcal{G}_* Fr^E[n+1])(i)$ and the right vertical map is the stabilization map $R_{G_m}^n C_* Fr(E^c)(i) \to R_{G_m}^{n+1} C_* Fr(E^c)(i)$. The middle
vertical arrow equals the composite map

$$\text{Hom}(G^{\wedge n}, C_\ast Fr(E^c(n) \wedge G^{\wedge i})) \to \text{Hom}(G^{\wedge n+1}, C_\ast Fr(E^c(n) \wedge G^{\wedge i+1}))$$

$$(\chi_{1,i})_* \to \text{Hom}(G^{\wedge n+1}, C_\ast Fr(E^c(n) \wedge G^{\wedge i+1})) \to \text{Hom}(G^{\wedge n+1}, C_\ast Fr(E^c(n+1) \wedge G^{\wedge i})),$$

where $(\chi_{1,i})_*$ is induced by the shuffle map $\chi_{1,i} : G^{\wedge i+1} \to G^{\wedge i+1}$. For commutativity of the right square we also use the fact that the diagram

$$
\begin{array}{ccc}
E(n) \wedge G^{\wedge i+1} & \xrightarrow{u} & E(n+1) \wedge G^{\wedge i+1} \\
\downarrow & & \downarrow \chi_{1,i} \\
E(n) \wedge G^{\wedge i+1} & \xrightarrow{u} & E(n+1) \wedge G^{\wedge i+1}
\end{array}
$$

is commutative, because the compositions of horizontal maps are $\Sigma_n \times \Sigma_{i+1}$-equivariant maps. Thus the maps $f_{n,i}$ are compatible with stabilization.

\section{6.7 Lemma.} The map $f$ induces a stable local equivalence of $S^1$-spectra for any $i \geq 0$:

$$f_i : ev_{G_m}(M^F_{Fr})(i) \to R^\infty_{G_m^i} C_\ast Fr(E^c)(i).$$

\textbf{Proof.} The map $f_{n,i} : \text{Hom}(G^{\wedge n}, C_\ast Fr(G^{\wedge i} \wedge E^c(n))) \to \text{Hom}(G^{\wedge n}, C_\ast Fr(E^c(n+i)))$ fits into the following commutative diagram

$$
\begin{array}{ccc}
\text{Hom}(G^{\wedge n}, C_\ast Fr(G^{\wedge i} \wedge E^c(n))) & \xrightarrow{f_{n,i}} & \text{Hom}(G^{\wedge n}, C_\ast Fr(E^c(n+i))) \\
\downarrow \Theta^{n}_{G_m^i} (C_\ast Fr(G^{\wedge i} \wedge E^c))(0) & & \downarrow \Theta^{n}_{G_m^i} (C_\ast Fr(E^c[i]))(0) \\
\Theta^{n}_{G_m^i} (C_\ast Fr(G^{\wedge i} \wedge E^c))(0) & \xrightarrow{u} & \Theta^{n}_{G_m^i} (C_\ast Fr(E^c[i]))(0)
\end{array}
$$

where $\chi_{1,n} \in \Sigma_{i+n}$ is the shuffle map permuting the first $i$ elements with the last $n$ elements and $\Theta^{n}_{G_m^i} (C_\ast Fr(E^c))$ is the bispectrum

$$\Theta^{n}_{G_m^i} (C_\ast Fr(E^c))(i) = \text{Hom}(G^{\wedge n}, C_\ast Fr(E^c(i+n)))$$

with bonding maps being the compositions

$$\text{Hom}(G^{\wedge n}, C_\ast Fr(E^c(i+n))) \to \text{Hom}(G^{\wedge n+1}, C_\ast Fr(E^c(i+n) \wedge G^{\wedge 1})) \to \text{Hom}(G^{\wedge n+1}, C_\ast Fr(E^c(i+n+1))) \cong$$

Note that the maps of the diagram are compatible with stabilization maps, and hence we can pass to the colimit over $n$ arriving at the commutative diagram

$$
\begin{array}{ccc}
ev_{G_m}(M^F_{Fr})(i) & \xrightarrow{f_i} & R^\infty_{G_m^i} C_\ast Fr(E^c)(i) \\
\downarrow & & \downarrow \cong \\
\Theta^{n}_{G_m^i} (C_\ast Fr(G^{\wedge i} \wedge E^c))(0) & \xrightarrow{u} & \Theta^{n}_{G_m^i} (C_\ast Fr(E^c[i]))(0)
\end{array}
$$
We see that \( f_i \) is a stable local equivalence of \( S^1 \)-spectra if and only if \( \nu \) is. The latter is the case by Lemmas 6.4 and 4.7. □

By Lemma 6.7 the arrow \( f : ev_{Gm}(\mathcal{M}^E_{fr}) \to R_{Gm}^{\nu} C_* Fr(E^c) \) is a levelwise stable local equivalence. We have constructed the desired framed spectral functor \( \mathcal{M}^E_{fr} \) and a zigzag of stable motivic equivalences

\[
E^c \to R_{Gm}^{\nu} C_* Fr(E^c) \leftarrow ev_{Gm}(\mathcal{M}^E_{fr}).
\]

Thus \( E \) is isomorphic to \( ev_{Gm}(\mathcal{M}^E_{fr}) \) in \( SH(k) \). This isomorphism is plainly functorial in \( E \).

We claim that \( \mathcal{M}^E_{fr} \) is a framed spectral functor. Clearly, all simplicial enriched functors forming the spectral functor \( \mathcal{M}^E_{fr} \) are framed in the sense of Definition 5.5. Given \( U \in Sm/k \) let \( \mathcal{M}^E_{fr}(\cdot \times U) \) denote the spectral functor

\[
X \in \mathcal{F}_r(k) \mapsto \operatorname{colim}_n \operatorname{Hom}(G^\wedge n, C^\wedge Fr(U^\wedge n \wedge E^c(n))).
\]

We already know that \( ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U)) \in SH_{hfb}(k) \). The above proof shows that

\[
U^\wedge E^c \to R_{Gm}^{\nu} C_* Fr(U^\wedge E^c) \leftarrow ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U))
\]

is a zigzag of stable motivic equivalences, functorial in \( U \) and \( E \).

Given an elementary Nisnevich square

\[
\begin{array}{ccc}
U' & \rightarrow & Y' \\
\downarrow & & \downarrow \\
U & \rightarrow & Y
\end{array}
\]

the square

\[
\begin{array}{ccc}
U'_+ \wedge E^c & \rightarrow & Y'_+ \wedge E^c \\
\downarrow & & \downarrow \\
U_+ \wedge E^c & \rightarrow & Y_+ \wedge E^c
\end{array}
\]

is homotopy cartesian in the stable motivic model structure of bispectra, and hence so is the square of motivically fibrant bispectra

\[
\begin{array}{ccc}
ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U'))_f & \rightarrow & ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times Y'))_f \\
\downarrow & & \downarrow \\
ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U))_f & \rightarrow & ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times Y))_f
\end{array}
\]

(we use here Lemma 2.6). It follows that the square of \( S^1 \)-spectra

\[
\begin{array}{ccc}
ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U'))_f(0) & \rightarrow & ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times Y'))_f(0) \\
\downarrow & & \downarrow \\
ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times U))_f(0) & \rightarrow & ev_{Gm}(\mathcal{M}^E_{fr}(\cdot \times Y))_f(0)
\end{array}
\]
is sectionwise homotopy cartesian in the stable model structure of ordinary $S^1$-spectra. Thus $\mathcal{M}_fr^E$ is Nisnevich excisive (obviously, $\mathcal{M}_fr^E(\emptyset) = +$).

For the same reasons $\text{pr} : \mathcal{M}_fr^E(k^1 \times U) \to \mathcal{M}_fr^E(U)$ is a stable local equivalence of $S^1$-spectra, and hence the claim.

So the functor $F \circ \text{ev}_{G_m} : SH^fr_{S^1}(\mathcal{F}r_0(k)) \to SH(k)$ is an equivalence of categories. The compactly generated triangulated category structure on $SH^fr_{S^1}(\mathcal{F}r_0(k))$ is just the preimage of the structure on $SH(k)$. Clearly, $\{\Omega_{G_m} : \mathcal{M}_fr(X) \mid X \in Sm/k, n \geq 0\}$ are compact generators of $SH^fr_{S^1}(\mathcal{F}r_0(k))$. Since every $\mathcal{X} \in SH^fr_{S^1}(\mathcal{F}r_0(k))$ is Nisnevich excisive, it follows that the canonical map of $S^1$-spectra $\mathcal{X}(U \sqcup V) \to \mathcal{X}(U) \times \mathcal{X}(V)$ is a stable equivalence for all $U, V \in Sm/k$. Therefore the canonical map of spectral functors $\mathcal{X} \land S^1 \to \mathcal{X}(\sim \land S^1)$ is pointwise a stable equivalence. It follows that the triangulated shift $\mathcal{X}[1]$ is computed as $\mathcal{X}(\sim \land S^1)$ in $SH^fr_{S^1}(\mathcal{F}r_0(k))$. This completes the proof of the theorem. □

7. THE TRIANGULATED CATEGORY OF FRAMED MOTIVES

7.1. Definition. (1) We say that a framed spectral functor $\mathcal{X} \in SH^fr_{S^1}(\mathcal{F}r_0(k))$ is effective if for all $n > 0$, $U \in Sm/k$ and finitely generated field $K/k$ the ordinary $S^1$-spectrum $\mathcal{X}(G_m^n \times U)(\Delta^n_*)$ is stably trivial. In other words, $\mathcal{X} \in SH^fr_{S^1}(\mathcal{F}r_0(k))$ is effective if and only if its evaluation bispectrum $\text{ev}_{G_m}(\mathcal{X}(\sim \land U)) = (\mathcal{X}(U), \mathcal{X}(G_m^n \times U), \ldots)$ is effective in the sense of Definition 3.5.

(2) The full subcategory of $SH^fr_{S^1}(\mathcal{F}r_0(k))$ consisting of the effective spectral functors is called the category of framed motives and denoted by $\mathcal{S}\mathcal{H}^fr(k)$.

7.2. Theorem. The category of framed motives $\mathcal{S}\mathcal{H}^fr(k)$ is compactly generated triangulated with compact generators $\{\mathcal{M}_fr(X) \mid X \in Sm/k\}$ and the shift functor $\mathcal{X}[1] = \mathcal{X}(\sim \land S^1)$. Moreover, restriction of the composite functor $F \circ \text{ev}_{G_m} : SH^fr_{S^1}(\mathcal{F}r_0(k)) \to SH(k)$ to $\mathcal{S}\mathcal{H}^fr(k)$ lands in $SH^eff(k)$ and is an equivalence of triangulated categories $G : \mathcal{S}\mathcal{H}^fr(k) \righttwoheadrightarrow SH^eff(k)$, where $F : SH^fr_{nis}(k) \to SH(k)$ is the triangulated equivalence of Theorem 2.2.

Proof. Given $\mathcal{X} \in \mathcal{S}\mathcal{H}^fr(k)$, the evaluation bispectrum $\text{ev}_{G_m}(\mathcal{X})$ is effective in the sense of Definition 3.5. By Theorem 3.6 $\text{ev}_{G_m}(\mathcal{X})$ belongs to $SH^eff(k)$. It follows that restriction of the composite functor $F \circ \text{ev}_{G_m} : SH^fr_{S^1}(\mathcal{F}r_0(k)) \to SH(k)$ to $\mathcal{S}\mathcal{H}^fr(k)$ lands in $SH^eff(k)$. We arrive therefore at a functor $G : \mathcal{S}\mathcal{H}^fr(k) \to SH^eff(k)$, which is fully faithful by construction. Let $E \in SH^eff(k)$ be a symmetric fibrant bispectrum. By the proof of Theorem 6.3 there is a framed spectral functor $\mathcal{M}_fr^E$ such that $E$ is isomorphic to $\text{ev}_{G_m}(\mathcal{M}_fr^E)$ in $SH(k)$. Theorem 3.6 implies $\mathcal{M}_fr^E$ is effective. Since every bispectrum is isomorphic in $SH(k)$ to an associated symmetric fibrant bispectrum, we see that $G$ is an equivalence of categories.

Next, the compactly generated triangulated category structure on $\mathcal{S}\mathcal{H}^fr(k)$ is inherited from $SH^fr_{nis}(k)$. It is also the preimage of the same structure on $SH^eff(k)$. The rest now follows from Theorem 6.3.

In order to formulate the next theorem, we need to define several functors first. Let

$$H : SH^eff(k) \to \mathcal{S}\mathcal{H}^fr(k)$$
be the functor that takes a bispectrum $E$ to the spectral framed functor $\mathcal{M}_f$, where $E_f$ is the symmetric fibrant bispectrum associated to $E$ (we can always do this functorially on the level of bispectra). The proof of the preceding theorem shows that $\mathcal{M}_f$ is effective if and only if $E$ is and that $H$ is quasi-inverse to $G : \mathcal{K}H^f(k) \to SH^f(k)$.

Next, define a triangulated functor

$$\mathcal{M}_f : SH^f(k) \to \mathcal{K}H^f(k)$$

as follows. It takes an $S^1$-spectrum $A$ to the spectral functor

$$X \in \mathcal{F}r_0(k) \mapsto C_*Fr(X_+ \wedge A^c),$$

where $A^c$ is the (functorial) cofibrant replacement of $A$ in the stable projective model structure of $S^1$-spectra. Note that $A^c$ is levelwise a sequential colimit of simplicial smooth schemes. It follows from Lemma 4.5 that $\mathcal{M}_f(A)$ is a framed spectral functor. Given a stable motivic equivalence of $S^1$-spectra $f : A \to B$, consider a commutative diagram of bispectra

$$\begin{array}{ccc}
\Sigma^m_{G_m} A^c & \longrightarrow & C_*Fr(\Sigma^m_{G_m} A^c) \\
\downarrow f & & \downarrow f \\
\Sigma^m_{G_m} B^c & \longrightarrow & C_*Fr(\Sigma^m_{G_m} B^c)
\end{array} \overset{\cong}{\longrightarrow} \begin{array}{ccc}
ev_G(\mathcal{M}_f(A)) & & \ev_G(\mathcal{M}_f(B))
\end{array}$$

Here the right horizontal arrows are isomorphisms given by swappings $G^m \wedge - \cong - \wedge G^m$. By Theorem 4.2 the left horizontal maps are isomorphisms in $SH(k)$, and hence so is the right vertical map. So $\mathcal{M}_f : SH^f(k) \to \mathcal{K}H^f(k)$ is indeed a functor. Clearly, it is triangulated.

Finally using Lemma 2.6, we define a triangulated functor

$$U : \mathcal{K}H^f(k) \to SH^f(k)$$

(“$U$” for underlying) taking $\mathcal{K} \in \mathcal{K}H^f(k)$ to the stable local replacement $\mathcal{K}(pt)_f$ of the underlying $S^1$-spectrum $\mathcal{K}(pt)$.

We now document the above constructions as follows.

7.3. **Theorem.** In the diagram of triangulated functors

$$\begin{array}{ccc}
SH^f(k) & \overset{\Sigma^m_{G_m} \wedge -}{\longrightarrow} & SH^{eff}(k) \\
\downarrow \mathcal{M}_f & & \downarrow \mathcal{K}H^f(k) \\
\mathcal{K}H^f(k) & \overset{H}{\longrightarrow} & \mathcal{K}H^{eff}(k)
\end{array} \overset{\cong}{\longrightarrow} \begin{array}{ccc}
\Omega^m_{G_m} & \longrightarrow & \Omega^m_{G_m} \\
\downarrow U & & \downarrow G \\
\mathcal{K}H^f(k) & \overset{G}{\longrightarrow} & \mathcal{K}H^{eff}(k)
\end{array}$$

all three pairs of functors are adjoint pairs, $G \circ \mathcal{M}_f$ is equivalent to $\Sigma^m_{G_m} \wedge - \cong - \wedge G^m$, $U \circ H$ is equivalent to $\Omega^m_{G_m}$.

We finish the paper with a $G^m$-connectedness result for effective motivic framed spectral functors. Recall that for every $n \geq 0$ one defines the category $f_n(SH^f(k))$ as a full triangulated subcategory of $SH^f(k)$ compactly generated by the suspension spectra $\Sigma^m_{G_m} X_+ \wedge G^m$, $X \in Sm/k$. We say that a $S^1$-spectrum $A \in SH^f(k)$ is $n$-connected in the $G^m$-direction, $n \geq 0$, if $A \in f_n(SH^f(k))$. 24
7.4. **Corollary.** For any effective motivic framed spectral functor \( X \in \mathcal{SH}^{Fr}(k) \) and any \( n \geq 0 \), the \( S^1 \)-spectrum \( X(\mathbb{G}_m^{\wedge n}) \) is \( n \)-connected in the \( \mathbb{G}_m^{\wedge 1} \)-direction.

**Proof.** This follows from Theorem 7.3 and [11, 7.5.1]. \( \square \)

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**DEPARTMENT OF MATHEMATICS, SWANSEA UNIVERSITY, FABIAN WAY, SWANSEA SA1 8EN, UNITED KINGDOM**

*E-mail address: g.garkusha@swansea.ac.uk*

**ST. PETERSBURG BRANCH OF V. A. STEKLOV MATHEMATICAL INSTITUTE, FONTANKA 27, 191023 ST. PETERSBURG, RUSSIA**

**ST. PETERSBURG STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS AND MECHANICS, UNIVERSITETSKY PROSPEKT, 28, 198504, PETERHOF, ST. PETERSBURG, RUSSIA**

*E-mail address: paniniv@gmail.com*