**Abstract.** We define a large class of integrable nonlinear PDE's, \( k \)-symmetric AKS systems, whose solutions evolve on finite dimensional subalgebras of loop algebras, and linearize on an associated algebraic curve. We prove that periodicity of the associated algebraic data implies a type of quasiperiodicity for the solution, and show that the problem of isometrically immersing \( n \)-dimensional Euclidean space into a sphere of dimension \( 2n - 1 \) can be addressed via this scheme, producing infinitely many real analytic solutions.

## 1. Introduction

In this paper we apply the Adler-Kostant-Symes (AKS) theory of integrable systems to the problem of isometrically immersing Euclidean \( n \)-space, \( E^n \), into a sphere of dimension \( 2n - 1 \). We show how the theory can be used to produce families of complete solutions which linearize on an associated algebraic variety, and study the implications of periodicity of the associated solutions. It is the author’s hope that this example will be of value to differential geometers wishing to understand how the AKS theory is applied to problems in geometry, as it demonstrates many of the main features of the theory in a particularly simple and uncluttered setting. It should be mentioned here that there are other methods from the theory of loop groups, such as dressing of solutions, which could be applied to our problem, but are not covered here.

The codimension of \( n - 1 \) for the immersion is of interest because it is critical in the sense that there is no local isometric embedding of \( E^n \) into \( S^k \) for \( k < 2n - 1 \) (see [18], p. 195), whilst for \( k = 2n - 1 \) we at least have the Clifford tori, products \( S^1 \times S^1 \times \cdots \times S^1 \), of \( n \) appropriately scaled circles. In fact the theory described in this paper also applies to codimension greater than \( n - 1 \), if we add the condition that the normal bundle is flat.

The study of isometric immersions from \( E^n \) into \( S^{2n-1} \) has a history going back at least as far as 1896, when Bianchi [3] studied flat surfaces in the 3-sphere. These can be classified roughly using the existence of special “asymptotic coordinates” as well as the group structure on \( S^3 \) (see [17], pp. 139-163). More recently J. Weiner [25] and Y. Kitagawa [13] were able to extend this to classify solutions with two periods, that is 2-tori in \( S^3 \). See also [5].

When \( n > 2 \) however, one cannot use the same methods: although one does have asymptotic coordinates [15], there is no group structure on a higher dimensional sphere. The system of PDE associated with the integrability condition, the so-called ‘generalized wave equation’ has been studied to some extent by Tenenblat [23] and others, though not from the AKS viewpoint.

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On the other hand, over the past two decades, methods based on the work of Adler and Van Moerbeke [1], [2], Kostant [14] and Symes [19] have been applied successfully to problems of producing and sometimes classifying certain special submanifolds. Following work by Hitchin [11], Pinkall and Sterling [16] gave a classification of all constant mean curvature 2-tori in $E^3$ using the so-called “finite gap” method. The finite gap method was also used to complete a classification of minimal 2-tori in $S^4$ in [8]. All this, and other work, such as Uhlenbeck’s study [20] of harmonic maps into Lie groups via loop groups, led to a general scheme for obtaining harmonic tori in symmetric spaces, described in [9]. Recently there has been work by Terng and others on a programme to identify the submanifolds associated to a large class of related integrable systems [21], [22].

Closely related to our target problem, Ferus and Pedit, [6], [7], modified the AKS theory to obtain large families of local isometric immersions of a space form $M^n_c$ into another $M^{2n-1}_{\tilde{c}}$ (the subscript is the constant sectional curvature of the manifold), where $c$ and $\tilde{c}$ are both non-zero. These were special cases of so-called curved flats, which were later also studied in [24], where dressing actions were used to produce families of solutions.

Below we show that the case $c = 0$ can be studied in a similar manner. We begin by introducing some known theory of integrable systems in a form which is intended to be easily applied to problems in differential geometry. Specifically, we define a large class of integrable systems, which we shall call $k$-symmetric AKS systems. These systems are defined by a Lax equation

\[(1.1) \quad dX = [X, A],\]

where $X(t)$, for $t$ in $\mathbb{R}^n$, takes its values in a loop algebra of matrix-valued polynomials in some auxiliary parameter $z$. They have the property that they are of finite type, in the sense that if the initial condition $X(0)$ is a polynomial, then $X(t)$ is also a polynomial of the same degree for all $t$.

There is associated to $X(t)$ a line bundle $L(t)$ over a (constant) algebraic curve $\Sigma$. The evolution of $L(t)$ in the Picard group of line bundles (of fixed degree) over $\Sigma$ is linear, which, in a sense, linearizes the Lax equation.

For applications in submanifold theory, we are actually interested in a map $F$ into a Lie group associated to the loop algebra mentioned above. $F$ can be characterized here by the equation

\[A = F^{-1}dF,\]

\[F(0) = I.\]

We will call $F$ a $k$-symmetric map and $X$ a Killing field for $F$.

Now the question of periodicity is easily addressed for the linearized solution $L(t)$, so we would like to know what this means for the function $F(t)$. In this paper we prove, for a large subclass of $k$-symmetric AKS systems, called simple, the following:

**Proposition 1.1.** Suppose there exists an element $P$ in $\mathbb{R}^n$ such that the line bundle $L(t)$ associated to a solution $X(t)$ of (1.1) satisfies

\[L(t + P) = L(t),\]

for all $t$ in $\mathbb{R}^n$. Then there exists a constant matrix $B \in \text{GL}(2n, \mathbb{C})$, such that the following two equations hold for all $t$ in $\mathbb{R}^n$: 
In Section 5 we study the problem of producing isometric immersions from \( E^n \) into \( S^{2n-1} \) and show that it is a 2-symmetric AKS system. Specifically, we prove Theorem 1.2 below.

To state the result, we first need some definitions. If \( f : \mathbb{R}^n \to S^{2n-1} \) is a smooth map, then we will say \( F : \mathbb{R}^n \to SO(2n) \) is an adapted frame for \( f \) if the \((n + 1)\)th column of \( F \) is \( f \) and if the derivative \( df \) has no component in the directions given by the last \( n \) columns. This means that if \( f \) is an immersion then the first \( n \) columns span the tangent space and the last \( n \) the normal space to the immersion, considered as a map into \( \mathbb{R}^{2n} \).

Let \( so(2n, \mathbb{C}) \) denote the Lie algebra of complex-valued skew symmetric \( 2n \times 2n \) matrices. Consider the involution \( \sigma = Ad(Q) \) on \( so(2n, \mathbb{C}) \), where \( Q \) is the matrix \( \text{diag}(I_{n \times n}, -I_{n \times n}) \), and \( I_{n \times n} \) is the \( n \times n \) identity matrix. Let \( \mathcal{V}_0 \subset so(2n, \mathbb{C}) \) be the fixed point set of \( \sigma \) and \( \mathcal{V}_1 \) the \(-1\) eigenspace. Consider the Lie algebra of matrix valued Laurent polynomials

\[
\mathcal{G}_\sigma := \{ \sum_{i=0}^{\beta} X_i z^i \mid X_i \in \mathcal{V}_i (\text{mod2}) \},
\]

and define for each \( d \geq 0 \) the real vector sub-space

\[
(\mathbb{R}\mathcal{G}_\sigma)_d := \{ \sum_{i=-d}^{1} X_i z^i \mid X_i \in so(2n, \mathbb{R}) \cap \mathcal{V}_i (\text{mod2}) \}.
\]

**Theorem 1.2.** For every initial condition \( X(0) \) in \((\mathbb{R}\mathcal{G}_\sigma)_d\) we have:

1. A family \( f^z \), parameterized by \( z \) in the non-zero real numbers \( \mathbb{R}_* \), of complete real-analytic maps from \( \mathbb{R}^n \) into \( S^{2n-1} \), together with a family of adapted frames \( F^z \) for \( f^z \), which comes from a 2-symmetric AKS system. For a generic initial condition, \( f^z \) is an immersion in a neighbourhood of 0, and wherever \( f^z \) is immersive, the induced curvature on its image is zero.
2. Similar families \( f^z_P \) and \( F^z_P \), which have the additional property that both the tangent and normal frames given by \( F^z_P \) are parallel.

**Remark 1.3.** One obtains a similar result, without the completeness, for flat immersions into hyperbolic space, by substituting the group \( SO_{-1}(2n) \) for \( SO(2n) \) (c.f. [6]).

2. **Lax Equations and Loop Groups**

In this section we outline the AKS theory as it applies to \( k \)-symmetric AKS systems. The main ideas here can be traced to the papers [1] and [2] and their predecessors.

**2.1. Algebraically Completely Integrable Systems.** First we sketch the most important facts about Lax equations and their spectral data, more or less following [8] and [12] for our definitions.
2.1.1. Lax Equations. A Lax equation is a differential equation which can be written in the form:

\[ \frac{dX}{dt} = [X, A], \]

where \( A = \sum_{i=1}^{n} A_i dt_i \), and the square brackets represent the commutator \( XA - AX \). Here we assume \( X \) and \( A_i \) are complex valued matrices.

The integrability condition which ensures the existence of a local solution, Frobenius integrability, is \( \frac{d^2 X}{dt^2} = 0 \) and will be addressed below.

Lax equations are sometimes called isospectral because the spectrum of a solution \( X \),

\[ \{ h \in \mathbb{C} | \det(hI - X) = 0 \}, \]

is constant with respect to \( t \). This well-known fact can be verified by observing that the coefficients of the characteristic equation of \( X \) can all be written in terms of traces of powers of \( X \) and then checking that \( d(\text{Tr}X^k) = k\text{Tr}(X^{k-1}[X, A]) = 0 \).

2.1.2. The Spectral Curve. Let us now allow \( X \) and \( A \) to depend meromorphically on a complex parameter \( z \):

\[ \frac{dX(z, t)}{dt} = [X(z, t), A(z, t)], \quad z \in \mathbb{C}, \quad t \in \mathbb{R}^n. \]

The case we will consider is that \( X \) and \( A_i \) are Laurent polynomials in \( z \), that is \( X(t), A_i(t) \in \tilde{\mathfrak{gl}}(m, \mathbb{C}) := \mathfrak{gl}(m, \mathbb{C})[z, z^{-1}] \).

Rather than a discrete set, the spectrum of \( X \) is now an algebraic curve, called the spectral curve \( \Sigma \) of \( X \), defined as the algebraic completion of the set:

\[ \Sigma_0 = \{(z, w) \in \mathbb{C}^2 | \det(wI - X(z)) = 0 \}. \]

\( X \) is called a regular element of \( \tilde{\mathfrak{gl}}(m, \mathbb{C}) \) if:

1. \( \Sigma \) is a smooth algebraic curve.
2. The projection \( z : \Sigma \to \mathbb{C}P^1 \), given by \([z(w) \mapsto z] \), is an \( m \)-fold branched covering with simple branch points only and no branch points over the points 0 and \( \infty \).

Note that the last requirement implies that if \( X = \sum_{a}^b X_a z^i \) then \( X_a \) and \( X_b \) both have \( m \) distinct eigenvalues. See [8] for more details.

We will always assume that \( X(z, 0) \) (and therefore \( X(z, t) \)) is regular, and this is true for a generic element of \( \tilde{\mathfrak{gl}}(m, \mathbb{C}) \).

2.1.3. The Eigenspace Bundle. Unlike the eigenvalues, the eigenvectors of \( X(z, t) \) are not constant with respect to \( t \). They determine a family of line bundles \( L(t) \) over \( \Sigma \), where \( L(t) \) is the sub-bundle of the trivial bundle \( \Sigma \times \mathbb{C}^m \) whose fibre over \((z, w)\) is the eigenspace corresponding to the eigenvalue \( w \).

2.1.4. Algebraic Complete Integrability. Considering that a matrix is determined by its eigenvalues and eigenvectors, it is reasonable to study the solution \( X(z, t) \) in terms of its spectral data \( \{ \Sigma, L(t), z \} \).

The degree \( d \) of the line bundle \( L(t) \), being an integer, is constant by continuity, so \( L(t) \) takes its values in \( \text{Pic}^d(\Sigma) \), the space of equivalence classes of holomorphic line bundles over \( \Sigma \) of degree \( d \). It is well-known (see [12]) that this is a complex torus \( \mathbb{T}^g \), where \( g \) is the genus of \( \Sigma \), and the Lax equation (2.1) is called algebraically completely integrable if \( L(t) \) follows a straight line motion in this torus. A general study of the question of algebraic complete integrability for (1-dimensional) Lax equations can be found in [10].
2.1.5. **Reconstructing** $X(t)$ **from its Spectral Data.** Suppose given spectral data \( \{ \Sigma, L(t), z \} \) for some (unspecified) regular $X(t)$. The line bundle $L(t) \in \text{Pic}^d(\Sigma)$, is only known up to holomorphic equivalence. In other words, the way $L(t)$ sits as a sub-bundle of $\Sigma \times \mathbb{C}^m$ is not a priori given to us. We need to know this in order to recover the solution $X(t)$. The following result from [8] is essentially what allows one to do this:

**Proposition 2.1.** Let $X \in \widetilde{\text{gl}}(m, \mathbb{C})$ be regular and $L \to \Sigma$ its eigenspace bundle with dual bundle $L^*$, and $\Gamma(L^*)$ the space of holomorphic sections of $L^*$. Then

$$\dim \Gamma(L^*) = m.$$  

This allows us to identify $(\Gamma(L^*))^\ast$ with $\mathbb{C}^m$ and thus think of $L$ as a sub-bundle of $\Sigma \times \mathbb{C}^m$. The identification is not unique, so what we are able to reconstruct in general is of the form:

$$Y(z, t) = G^{-1}(t)X(z, t)G(t),$$

where $G(t)$ is an an element of $\text{GL}(m, \mathbb{C})$, and is independent of $z$. It is therefore reasonable to expect that some kind of normalization condition on $X(t)$ may be needed in order to reconstruct it uniquely from the spectral data.

### 3. $k$-symmetric AKS systems

Consider the Lie algebra $\widetilde{\text{gl}}(m, \mathbb{C}) := \text{gl}(m, \mathbb{C})[z, z^{-1}]$ of Laurent polynomials in $z$, which is given the Lie bracket

$$\left[ \sum_i X_i z^i, \sum_j Y_j z^j \right] := \sum_{i,j} [X_i, Y_j] z^{i+j}.$$  

$k$-symmetric AKS systems are maps into certain subalgebras $\tilde{G}_\sigma$ of $\widetilde{\text{gl}}(m, \mathbb{C})$ which arise as follows: suppose $G$ is a Lie subalgebra of $\text{gl}(m, \mathbb{C})$, and $\sigma$ is an order $k$ automorphism of $G$, that is, $\sigma : G \to G$ is linear and satisfies $[\sigma X, Y] = [\sigma X, \sigma Y]$ and $\sigma^k = \text{id}$. Let $\mathcal{V}_i$ be the eigenspace of $\sigma$ corresponding to the eigenvalue $\zeta^i$, where $\zeta$ is a primitive $k$’th root of unity. If $X_i \in \mathcal{V}_i$, then

$$\sigma[X_i, X_j] = [\sigma X_i, \sigma X_j] = (\zeta)^{i+j} [X_i, X_j],$$

which implies the commutation relations

$$[\mathcal{V}_i, \mathcal{V}_j] \subset \mathcal{V}_{i+j \ (\text{mod } k)}.$$  

It follows that the set

$$\tilde{G}_\sigma := \{ X \in G[z, z^{-1}] \mid X_i \in \mathcal{V}_{i \ (\text{mod } k)} \}$$

is closed under the Lie bracket, and therefore a Lie subalgebra of $\widetilde{\text{gl}}(m, \mathbb{C})$.

Now let $\tilde{G}$ be any subalgebra of $\widetilde{\text{gl}}(m, \mathbb{C})$ and suppose we have a decomposition

$$\tilde{G} = \mathcal{P} \oplus \mathcal{N}$$

into two subalgebras which consist of polynomials in $z$ and $z^{-1}$ respectively. An example of such a decomposition, which we will call *simple*, is

$$\mathcal{P} := \{ X \in \tilde{G} \mid X_i = 0, \ i \leq 0 \},$$

$$\mathcal{N} := \{ X \in \tilde{G} \mid X_i = 0, \ i > 0 \}.$$
Other examples are obtained similarly by stipulating that elements of $\mathcal{P}$ and $\mathcal{N}$ have their constant terms in any pair of complementary subalgebras of $\tilde{\mathcal{G}}_0$, the subalgebra of $\tilde{\mathcal{G}}$ which consists of constant matrices. See Section 5.2 and in particular Remark 5.3.3 for a specific instance of this.

We will construct flows on the finite dimensional vector subspaces

$$\tilde{\mathcal{G}}^1_1 := \{ X = \sum_{i=d}^1 X_i z^i \in \tilde{\mathcal{G}} \},$$

where $d$ is any integer less than or equal to 1. Therefore we can assume that all maps which are to be differentiated are between finite-dimensional subspaces of the infinite-dimensional loop algebras.

Let

$$\tilde{\mathcal{G}}^1 := \bigcup_{d \leq 1} \tilde{\mathcal{G}}^1_d.$$

**Proposition 3.1.** Let $\pi_\mathcal{P} : \tilde{\mathcal{G}} \to \mathcal{P}$ be the projection. Let $p_1, \ldots, p_n \in \mathbb{C}[x, y]$ be a collection of polynomials in two variables. Let $V_i : \mathfrak{gl}(m, \mathbb{C}) \to \mathfrak{gl}(m, \mathbb{C})$ be the map defined by

$$V_i(X) := p_i(X, z^{-1}),$$

and suppose that, for all $i$,

$$V_i(\tilde{\mathcal{G}}^1) \subset \tilde{\mathcal{G}}^1,$$

i.e. $p_i(X, z^{-1})$ is in $\tilde{\mathcal{G}}^1$ whenever $X$ is.

Then:

1. The Lax equation

$$dX = \left[ X, \sum_{i=1}^n \pi_\mathcal{P} V_i(X) dt_i \right],$$

on $\tilde{\mathcal{G}}^1$, is of finite type: if $X \in \tilde{\mathcal{G}}^1_d$ then so are the coefficients of the 1-form $dX$.

2. Equation (3.4) is Frobenius integrable, i.e. $d^2X = 0$.

3. The 1-form $A := \sum_{i=1}^n \pi_\mathcal{P} V_i(X) dt_i$ satisfies the Maurer-Cartan equation

$$dA + A \wedge A = 0.$$

**Definition 3.2.** We will call the equation (3.4) for the case $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_\sigma$ a $k$-symmetric AKS system.

**Remark 3.3.** The condition $3.3\text{a}$ is merely to ensure that the equation (3.4) makes sense on $\tilde{\mathcal{G}}^1$. Finding polynomials which map $\tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ depends on the Lie algebra, but ensuring that the highest power of $z$ in $p_i(X, z^{-1})$ is 1 is easily achieved by multiplying by a sufficiently high power of $z^{-1}$.

We note here the useful fact that for the above-mentioned example $\tilde{\mathcal{G}}_\sigma$, when $\sigma$ is an order $k$ inner automorphism $\sigma(X) = QXQ^{-1}$ for some $Q \in G$ (a Lie group with Lie algebra $\mathcal{G}$), if the monomial $p(X) = X^j$ takes $\mathcal{G} \to \mathcal{G}$ then $p$ also takes $\tilde{\mathcal{G}}_\sigma \to \tilde{\mathcal{G}}_\sigma$, and so does $z^k p$ for any integer $l$.

**Remark 3.4.** Given a Lax equation $dX = [X, A]$, our choice of $A$ is not unique. For example we can always add to $A$ any 1-form whose coefficients commute with $X$, without changing the equation. The fact that, for $k$-symmetric AKS systems,
A satisfies the Maurer-Cartan equation says something about our choice: by standard Lie group theory, the satisfaction of this equation is equivalent (on a simply connected domain) to the fact that there exists \( F : \mathbb{R}^n \to \widetilde{GL}(n, \mathbb{C}) \) such that \( A = F^{-1}dF, F(0) = I \) (here \( F^{-1} \) denotes the matrix inverse). This gives us the useful formula

\[
X(t) = F^{-1}(t)X(0)F(t),
\]

which is verified by checking that the right hand side solves the Lax equation \( dX = [X, F^{-1}dF] \), which defines \( X \).

3.1. **Proof of Proposition 3.1** The fact that the flow is well-defined on \( \widetilde{G}_d^1 \) is easily verified: if \( X \in \widetilde{G}_d^1 \) then the condition (3.3) ensures that \( \pi_P V_i(X) \) are in \( \widetilde{G} \), and so, therefore, are the coefficients of \( dX \), since \( \widetilde{G} \) is a subalgebra. Now if \( X = (... + X_0 + X_1 z) \) then (3.3) also implies that \( V_i(X) = ... + C_1 X_1^k z \), for some non-negative integer \( k \). Thus \( \pi_P V_i(X) = C_0 + C_1 X_1^k z \), where \( C_0 \) and \( C_1 \) are constants, and

\[
[X, \pi_P V_i(X)] = [\sum_{-d}^{1} X_i z^i, C_0 + C_1 X_1^k z] = [\sum_{-d}^{1} X_i z^i, C_0] + [\sum_{-d}^{0} X_i z^i, C_1 X_1^k z] + [X_1 z, C_1 X_1^k z].
\]

The last term is zero, so it follows that \( dX \) takes it’s values in \( \widetilde{G}_d^1 \).

To prove parts 2 and 3, we will use some standard theory for constructing flows from ad-invariant functions on Lie groups, which we now recall.

Let \( \mathcal{G} \) be any Lie algebra.

**Definition 3.5.** An ad-equivariant vector field is a map \( V : \mathcal{G} \to \mathcal{G} \) which satisfies

\[
dV \bigg|_X ([X, Y]) = [V(X), Y].
\]

**Remark 3.6.** If \( \mathcal{G} \) is equipped with an invariant inner product, then ad-equivariant vector fields are (locally) gradients, with respect to this inner product, of ad-invariant polynomials on \( \mathcal{G} \). Thus, for example, if \( \mathcal{G} \) is the Lie algebra of a compact semi-simple Lie group of rank \( k \) then we can expect to find at most \( k \) polynomials which are linearly independent pointwise. This follows from the fact that an ad-invariant function is given by its values on a maximal torus, which has dimension \( k \).

For the purpose of proving Proposition 3.1 we note the following:

**Lemma 3.7.** Any polynomial \( p(X, z, z^{-1}) \) in \( X, z \) and \( z^{-1} \) is an ad-equivariant vector field on \( gl(m, \mathbb{C}) \).

**Proof.** This is easily verified by first checking that \( V(X) = X^n \) satisfies the equation (3.5) as follows:

\[
V(X + t[X, Y]) = X^k + t(\sum_{i=0}^{n-1} X^{n-1-i}(XY - YX)X^i) + o(t^2),
\]
so that
\[ dV\big|_X ([X,Y]) = X^nY - YX^n = [X^n,Y]. \]

Hence the observation that both the left and right hand sides of (3.5) are linear over \( \mathbb{C}[z, z^{-1}] \) proves the lemma. \( \square \)

Now the proof of Proposition 3.1 follows from the following standard result (see, for example, [4]):

**Proposition 3.8.** Suppose we have a collection of \( n \) \( ad \)-equivariant vector fields \( V_i \) on a Lie algebra \( G = \mathcal{P} \oplus \mathcal{N} \), where \( \mathcal{P} \) and \( \mathcal{N} \) are both subalgebras, and consider the projections \( A_i := \pi_P V_i \).

Then:

1. The system
   \[ X_{t_i} = [X, A_i(X)], \quad i = 1...n, \]
   on \( G \) is Frobenius integrable.
2. If \( X : \mathbb{R}^n \supset U \to G \) is a solution to (3.6) then the 1-form
   \[ A := \sum_{i=1}^n A_i dt_i \]
   satisfies the Maurer-Cartan equation \( dA + A \wedge A = 0 \).

**Proof.** Using the Jacobi identity, it is straightforward to reduce the first statement to the equation:
\[ [X, [A_i, A_j]] + \nabla_{[X,A_i]} A_j - \nabla_{[X,A_j]} A_i = 0 \]
for all \( i, j \).

Additionally, the Maurer-Cartan equation of statement two is equivalent to
\[ [A_i, A_j] + \frac{\partial}{\partial t_i} A_j - \frac{\partial}{\partial t_j} A_i = 0 \]
for all \( i, j \).

If the first statement holds, then the Maurer-Cartan equations can be rewritten as
\[ [A_i, A_j] + \nabla_{[X,A_i]} A_j - \nabla_{[X,A_j]} A_i = 0 \]
for all \( i, j \).

Thus it suffices to verify the equations (3.7). This follows from Lemma 3.9 below. \( \square \)

We state here a result which is more general than needed, but is of interest as it can be used to define variants of the AKS theory, as in [6].

**Lemma 3.9.** Suppose \( G = \mathcal{P} \oplus \mathcal{N} \) is a decomposition of a Lie algebra into vector subspaces, and that \( V_i \) are \( ad \)-equivariant vector fields on \( G \). Let \( A_i := \pi_P V_i \). Then:

1. \( [A_i, A_j] + \nabla_{[X,A_i]} A_j - \nabla_{[X,A_j]} A_i = [\pi_P V_i, \pi_P V_j] + \pi_P \{ [V_j, \pi_P V_i] - [V_i, \pi_P V_j] \}. \)
2. If \( \mathcal{P} \) is Lie sub-algebra then
   \[ [A_i, A_j] + \nabla_{[X,A_i]} A_j - \nabla_{[X,A_j]} A_i = \pi_P [\pi_N V_i, \pi_N V_j]. \]
Proof. In both of the above equations, the left hand side is equal to
\[ [\pi PV_i, \pi PV_j] + \nabla_{[X,\pi PV_i]}\pi PV_j - \nabla_{[X,\pi PV_j]}\pi PV_i. \]
Using the decomposition of \( G \), this amounts to
\[ [\pi PV_i, \pi PV_j] + \pi P\{\nabla_{[X,\pi PV_i]}V_j - \nabla_{[X,\pi PV_j]}V_i - \nabla_{[X,\pi PV_i]}\pi NV_j - \nabla_{[X,\pi PV_j]}\pi NV_i\} = [\pi PV_i, \pi PV_j] + \pi P\{\nabla_{[X,\pi PV_i]}V_j - \nabla_{[X,\pi PV_j]}V_i\}. \]
Using the ad-equivariance of \( V_i \) and \( V_j \), this equates to the right hand side of the equation in statement 1.

To get statement 2, we use the assumption that \( P \) is a subalgebra to obtain:
\[ \pi P\{[V_j, \pi PV_i] - [V_i, \pi PV_j]\} = [\pi PV_i, \pi PV_j] + \pi P\{[\pi NV_j, \pi PV_i] + \pi P[\pi PV_j, \pi PV_i] + \pi P[\pi PV_i, \pi NV_j] - \pi P[\pi NV_i, \pi NV_j]\} = -[\pi PV_i, \pi PV_j] + \pi P[\pi NV_i, \pi NV_j]. \]
Here we used the easily verified fact that ad-equivariant vector fields commute. Now substitute this into statement 1 to get statement 2. \( \Box \)

3.2. Linearization of a System of Finite Type. Recall that to describe a holomorphic line bundle over \( \mathbb{C}P^1 \) one merely needs to provide the transition function between the local trivializations over \( \mathbb{C} \) and \( \mathbb{C}^* \cup \{\infty\} \) respectively. Similarly, since \( z : \Sigma \to \mathbb{C}P^1 \) is a smooth covering of \( \mathbb{C}P^1 \), it follows that \( \Sigma^+ = \Sigma - z^{-1}(\infty) \) and \( \Sigma^- = \Sigma - z^{-1}(0) \) is an open cover of \( \Sigma \) by two sets which admit only trivial holomorphic line bundles. Thus a holomorphic line bundle over \( \Sigma \) is defined by a transition function between local trivializations over \( \Sigma^+ \) and \( \Sigma^- \), that is, a holomorphic function \( \rho : \Sigma^+ \cap \Sigma^- \to \mathbb{C}^* \). More specifically, if \( s^\pm(z, w, t) \) is a non-vanishing section of \( L(t) \) over \( \Sigma^\pm \), then \( \rho \) must satisfy the equation \( s^+ = \rho s^- \) on \( \Sigma^+ \cap \Sigma^- \).

The following proposition, says that the evolution of \( L(t) \) in the moduli space of line bundles over \( \Sigma \) is linear.

**Proposition 3.10.** Suppose that \( X(z, t) \) is a solution of the system \( \{\mathbf{X}, \mathbf{F}\} \) with the initial condition \( X(z, 0) = X_0(z) \). Let \( \Sigma \) be the spectral curve of \( X_0 \) and \( \rho_0 \) the transition function of the eigenspace bundle \( L_0 \). Let \( \mu_{i} : \Sigma \to \mathbb{C} \) be the eigenvalue function of the operator \( V_i(X_0(z)) \) defined by the equation
\[ V_i((X_0(z)))s_i(z, w) = \mu_i(z, h)s_i(z, w), \]
where \( w \) is the eigenvalue function of \( X_0(z) \). Denote by \( \mu \) the \( n \)-tuple \( [\mu_1, ..., \mu_n] \).

Then the eigenspace bundle \( L(t) \) of \( X(z, t) \) is defined by the transition function
\[ \rho(t) = \rho_0 \exp(-t \cdot \mu). \]

**Proof.** The proof given here is based on an argument given in [8]. Let \( F(z, t) \) be a solution of
\[ F^{-1}dF = \sum_{i=1}^{n} \pi PV_i dt_i, \quad F(z, 0) = I, \quad z \in \mathbb{C}. \]
Since \( \pi PV_i \) are polynomials in \( z \), \( F \) is a holomorphic function of \( z \) on \( \mathbb{C} \). As described earlier, we know that \( X(t) = F^{-1}(t)X_0F(t) \), so we immediately have
\[ X(t) F^{-1}(t) s^+(0) = w F^{-1}(t) s^+(0) \] (given that \( s^+(0) \) is an eigenvector of \( X_0 \)). In other words, if \( s^+(0) \) is non-vanishing section of \( L_0 \) over \( \Sigma^+ \), then
\[ s^+(t) := F^{-1}(t) s^+(0) \]
is a non-vanishing section of \( L(t) \) over \( \Sigma^+ \).

We need a similar formula for \( s^- \) over \( \Sigma^- \). Set
\[ G := \exp(- \sum_{i=1}^{n} t_i V_i(X_0) F(t)). \]
Then
\[
G^{-1} dG = F^{-1} \left[ \sum_{i=1}^{n} -V_i(X_0) F dt_i + dF \right]
\]
\[ = \sum_{i=1}^{n} [-F^{-1} V_i(X_0) F + \pi_N V_i(X(t))] dt_i \]
\[ = \sum_{i=1}^{n} [-V_i(X(t)) + \pi_P V_i(X(t))] dt_i \]
\[ = \sum_{i=1}^{n} -\pi_N V_i(X(t)) dt_i. \]

We used the fact that \( V_i(X) \) is of the form \( p_i(X, z^{-1}) \) and that \( X(t) = F^{-1} X_0 F \) on the third line.

Since \( \pi_N V_i(X) \) is polynomial in \( z^{-1} \), it follows that \( G \) is holomorphic on \( \mathbb{C}^* \cup \{\infty\} \). Using the fact that \( V_i(X_0) \) commutes with \( X_0 \), we also obtain the formula \( X(t) = G^{-1}(t) X_0 G(t) \), and therefore, as with \( s^+ \),
\[ s^-(t) = G^{-1}(t) s^-(0) \]
is a non-vanishing section of \( L(t) \) over \( \Sigma^- \).

Hence, on the intersection \( \Sigma^+ \cup \Sigma^- \), one obtains
\[
s^+(t) = F^{-1} \rho_0 s^- (0) = \rho_0 G^{-1} G F^{-1} s^- (0) = \rho_0 G^{-1} \exp(- \sum_{i=1}^{n} t_i V_i(X_0)) s^- (0) = \rho_0 G^{-1} \exp(- \sum_{i=1}^{n} t_i \mu_i) s^- (0) = \rho_0 \exp(- t \cdot \mu) s^- (t). \]

\[
\Box
\]

Remark 3.11. It is essential to the argument that the decomposition of the loop algebra \( G = \mathcal{P} \oplus \mathcal{N} \) is into subspaces consisting of functions which are holomorphic in \( z \) and \( z^{-1} \) respectively, and that the functions \( V_i(X) \) are polynomials in \( X \) whose coefficients are meromorphic functions of \( z \). Any such commuting flows on loop algebras will linearize as described here.
4. Simple $k$-symmetric systems and their properties.

In this section we are interested in the question of periodicity of solutions. Since the evolution of the associated line bundle is essentially linear for $k$-symmetric AKS systems, its periodicity can be characterized in terms of the initial data (see [8]). Clearly periodicity of $F$ implies periodicity of the line bundle. Below we will look at what periodicity of the line bundle implies about the corresponding solution $F$.

Throughout this section we will be studying the solutions of a restricted class of AKS systems.

**Definition 4.1.** A $k$-symmetric AKS system will be called simple if the decomposition (3.1) is of the form

$$\tilde{G}_\sigma = \mathcal{P}_a \oplus \mathcal{N}_a,$$

where $\mathcal{P}_a$ and $\mathcal{N}_a$ are defined as in (3.2), so that the whole subalgebra of constants, $\mathcal{V}_0$, is contained in $\mathcal{N}_a$.

For simple systems, the projection of $\tilde{G}_\sigma^1$ onto $\mathcal{P}_a$ is just $\pi_{\mathcal{P}_a}(...+X_0+X_1z) = X_1z$. This allows us to prove the results below more easily.

### 4.1. Recovering $F(t)$ from the Spectral Data

Recall that if $F(0) = I$ then we have the formula

$$X(z,t) = F^{-1}(t)X(0)F(t).$$

For a more general initial condition $F(0) = F_0 \in \tilde{G}$, set

$$Y_0 := F_0X(0)F_0^{-1},$$

and the following formula is similarly deduced:

$$X(t) = F^{-1}(t)Y_0F(t).$$

The following proposition says that for simple $k$-symmetric AKS systems, the spectral data determine $X$ and $F$ uniquely from their initial conditions.

**Proposition 4.2.** Suppose $X(z,t) = F^{-1}(z,t)Y_0(z)F(z,t)$ is a solution corresponding to the regular initial conditions $X(z,0) = F_0^{-1}(z)Y_0(z)F_0(z)$ and $F(z,0) = F_0(z)$, obtained from Proposition 3.3 for the loop algebra decomposition $\mathcal{G} = \mathcal{P}_a \oplus \mathcal{N}_a$. Then both $X(z,t)$ and $F(z,t)$ are determined by the spectral data $\{\Sigma, L(t), z\}$, the initial conditions for $X$ and $F$, and the requirement that $F^{-1}dF \in (\tilde{G}_\sigma)_1^1 \otimes \Omega(\mathbb{R}^n)$.

**Proof.** As described in Section 2.1.3, $X(z,t)$ is determined by its spectral data up to conjugation by an element of GL$(2n, \mathbb{C})$, so suppose we have

$$\tilde{X}(z,t) = B^{-1}(t)X(z,t)B(t)$$

$$= B^{-1}(t)F^{-1}(z,t)Y_0(z)F(z,t)B(t),$$

which has the spectral data $\{\Sigma, L(t), z\}$ and the given initial conditions, that is, $\tilde{X}(z,0) = F_0^{-1}(z)Y_0(z)F_0(z)$ and suppose that $B(t)$ is chosen so that $F(z,0)B(0) = F_0(z)$, and such that $(FB)^{-1}d(FB) \in (\tilde{G}_\sigma)_1^1 \otimes \Omega(\mathbb{R}^n)$. Note that these assumptions mean that $B(0) = \text{Id}$. We show that $B(t) = \text{Id} = \text{constant}$, which proves that $\tilde{X} = X$. Since $F$ is determined by $X(t)$ and $F(0)$ through the equation $F^{-1}dF = \sum_{i=1}^n \pi_p \mathcal{V}_i(X)dt_i$, this proves the proposition.
The requirement that $F^{-1}dF \in (\tilde{G}_\sigma)^1 \otimes \Omega(\mathbb{R}^n)$ simply means we can write $F^{-1}dF = A_1z$, where $A_1 \in V_1 \otimes \Omega(\mathbb{R}^n)$, and then

$$(FB)^{-1}d(FB) = B^{-1}F^{-1}[(dF)B + FdB] = B^{-1}A_1Bz + B^{-1}dB.$$ 

$B$ has no $z$ dependence, so for the right hand side to lie in $(\tilde{G}_\sigma)^1 \otimes \Omega(\mathbb{R}^n)$ we must have $B^{-1}dB = 0$, which implies that $dB = 0$ and $B = \text{constant} = \text{Id}$. \hfill \Box

4.2. Periodicity of Solutions. In this section we study the implications of periodicity of the spectral data in the case of simple $k$-symmetric systems.

4.2.1. A Translation Lemma. Every initial condition $X_0$ in $(\tilde{G}_\sigma)^1$ generates a polynomial Killing field $X(t)$ corresponding to a solution $F(t)$ with initial condition $F(0) = I$. It is of interest to know how $F$ is related to the solution corresponding to that generated by the initial condition $\hat{X}(0) = X(Q)$, for some $Q \in \mathbb{R}^n$.

**Lemma 4.3.** Let $F$ be a simple $k$-symmetric map with Killing field $X$, with the initial conditions $X(0) = X_0$ and $F(0) = I$. Let $\hat{X}$ and $\hat{F}$ be the solutions corresponding to the initial conditions $\hat{X}(0) = X(Q)$ and $\hat{F}(0) = I$. Then, for all $t \in \mathbb{R}^n$,

$$F(t + Q) = F(Q)\hat{F}(t).$$

**Proof.** Since $Y(t) := X(t + Q)$ is a solution of the Lax equation $dY = [Y, A(Y)]$, and $Y(0) = X(Q)$, it follows by uniqueness that $Y = \hat{X}$, which is to say that

$$\hat{X}(t) = X(t + Q).$$

We want to show that

$$\hat{F}(t) = F^{-1}(Q)F(t + Q).$$

It is enough to check that the right hand side satisfies the equations defining $\hat{F}$, namely $\hat{F}(0) = I$ and $\hat{F}^{-1}d\hat{F} = A(\hat{X}) := \sum_{i=1}^n \pi P_i V_i(\hat{X}) dt_i$. The first equation clearly holds, and the second can be verified as follows:

$$F^{-1}(t + Q)F(Q)F^{-1}(Q)dF(t + Q) = F^{-1}(t + Q)dF(t + Q) = A(X(t + Q)) = A(\hat{X}).$$

\hfill \Box

A consequence of Lemma 4.3 is that if $F$ is not injective then it has a period:

**Lemma 4.4.** Let $F$ be a simple $k$-symmetric map. If $F(a) = F(b)$ for some $a \neq b$ then $b - a$ is a period for $F$.

**Proof:** We first assume that $a = 0$ and $F(0) = F(b) = I$. If $X$ is a Killing field for $F$, then $X(b) = F^{-1}(b)X(0)F(b) = X(0)$, so it follows immediately from Lemma 4.3 that $F(t + b) = F(t)$. The general case can be deduced from this one by setting $\tilde{F}(t) := F(a)^{-1}F(t + a)$, so that $\tilde{F}(0) = \tilde{F}(b - a) = I.$
4.2.2. Periodicity of the Killing Field.

**Definition 4.5.** A map \( F \) from \( \mathbb{R}^n \) to a Lie group \( G \) is *type I quasiperiodic* with quasiperiod \( P \in \mathbb{R}^n \) if

\[
F(t + P) = F_0 F(t)
\]

for all \( t \in \mathbb{R}^n \), where \( F_0 \) is some constant element of \( G \).

The translation Lemma of above can be used to show that periodicity for \( X \) is equivalent to type I quasiperiodicity for \( F \). In fact we only need \( X(P) = X(0) \) to guarantee both of these properties:

**Proposition 4.6.** Let \( F \) be a simple \( k \)-symmetric map with initial condition \( F(0) = I \) and Killing field \( X \). The following three conditions are equivalent:

1. \( X(P) = X(0) \).
2. \( F \) is type I quasiperiodic with quasiperiod \( P \), and \( X(P) = X(0) \).
3. \( X(t + P) = X(t) \) for all \( t \in \mathbb{R}^n \).

*Proof.* We prove (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (1).

(1) \( \Rightarrow \) (2) :
This follows from Lemma 4.3.

(2) \( \Rightarrow \) (3) :
If \( F(t + P) = F_0 F(t) \) then \( F^{-1} dF \bigg|_{t+P} = F^{-1} dF \bigg|_t \). But recall that \( F^{-1} dF = \sum_{i=1}^n \pi_{p_n} V_i(X) dt_i \), where, if

\[
X = X_{-d} z^{-d} + ... + X_0 + X_1 z,
\]
then \( \pi_{p_n} V_i(X) := X_i^{2i-1} z \). In other words, we have the formula

\[
F^{-1} dF = \sum_{i=1}^n X_i^{2i-1} z dt_i.
\]

Thus if the connection \( F^{-1} dF \) is periodic, then so, certainly, is the function \( X_1 \). If we now consider the differential equations satisfied by \( X \), namely (3.4), we see this reduces to

\[
\frac{\partial X_i}{\partial t_j} = [X_{i-1}, X_j^{2i-1}], \quad i = (-d + 1), ..., 1,
\]

\[
\frac{\partial X_{-d}}{\partial t_j} = 0.
\]

Thus \( X_{-d} \) is constant, and so \( X_1 \) and \( X_{-d} \) both have a period \( P \). Hence by the equation (4.3), any partial derivative of \( X_{-d+1} \) also has a period \( P \), and so, therefore, does \( X_{-d+1} \). Similarly, by induction, \( X_i \) has a period \( P \) for all \( i \).

(3) \( \Rightarrow \) (1) :
Immediate. \( \square \)

4.2.3. Periodicity of the Eigenspace Bundle. We can now proceed to the main goal of this section, the proof of Proposition 1.1.

**Definition 4.7.** Let \( G \) be a subgroup of \( \tilde{gl}(m, \mathbb{C}) \). A map \( F \) from \( \mathbb{R}^n \) to \( G \) is *type II quasiperiodic* with quasiperiod \( P \in \mathbb{R}^n \) if

\[
F(t + P) = F_0 B^{-1} F(t) B,
\]
for all \( t \in \mathbb{R}^n \), where \( F_0 \) is some constant element of \( G \), and \( B \in \text{GL}(m, \mathbb{C}) \).

**Proposition 4.8.** Suppose that \( F \) is a simple \( k \)-symmetric map with Killing field \( X \) and the initial conditions \( X(0) = X_0 \) and \( F(0) = I \). Suppose further that the eigenspace bundle \( L(t) \) of \( X(t) \) satisfies \( L(t + P) = L(t) \) for all \( t \). Then \( F \) is type II quasiperiodic with quasiperiod \( P \), in particular, there exists a matrix \( B \in \text{GL}(2n, \mathbb{C}) \) such that we have for all \( t \):

\[
F(t + P) = F(P)B^{-1}F(t)B.
\]

Before proving this, we will find out what can be said concerning the Killing field \( X \) under these circumstances.

**Lemma 4.9.** If the hypotheses of Proposition 4.8 hold then

\[
X(t + P) = B^{-1}X(t)B,
\]

for some constant matrix \( B \in \text{GL}(2n, \mathbb{C}) \).

**Proof.** As described in Section 2.1.5, \( X(z, t) \) is determined by its spectral data up to conjugation by an element of \( \text{GL}(2n, \mathbb{C}) \). So we know that there exists a matrix \( B(t) \) such that

\[
X(t + P) = B^{-1}(t)X(t)B(t).
\]

We know that \( Y(t) := X(t + P) \) is the unique solution of the differential equation:

\[
dY = [Y, A(Y)], \quad Y(0) = X(P).
\]

We show that \( \hat{Y}(t) := B^{-1}(0)X(t)B(0) \) also satisfies these equations, which will complete the proof. Clearly the initial condition \( \hat{Y}(0) = X(P) \) holds, so let us check the Lax equation:

\[
d\hat{Y} = B^{-1}(0)dXB(0)
= B^{-1}(0) [X, A(X)] B(0)
= [B^{-1}(0) X B(0), B^{-1}(0) A(X) B(0)]
= [\hat{Y}, B^{-1}(0) \sum_i \pi_{P_a} V_i(X) B(0)]
= [\hat{Y}, \sum_i \pi_{P_a} V_i(B^{-1}(0) X B(0))]
= [\hat{Y}, A(\hat{Y})].
\]

The step on the fourth line was allowed because \( B \) does not depend on the parameter \( z \), and therefore conjugation by \( B \) commutes with projection onto \( P_a \). \( \square \)

**Remark 4.10.** The above proof depends on the fact that, for simple \( k \)-symmetric systems, our decomposition \( \tilde{G} = P_a \oplus N_a \) is such that all the constant terms are in just one of \( P_a \) or \( N_a \). Otherwise conjugation by a constant matrix \( B \) will in general not commute with projection onto \( P_a \).

**Proof.** (of Proposition 4.8). The equation

\[
F(t + P) = F(P)B^{-1}F(t)B
\]
can be proved by checking that both the left and right hand sides are solutions, $H$, for the differential equation:

\begin{align}
(4.6) & \quad H^{-1}dH = B^{-1}F^{-1}dF \, B, \\
(4.7) & \quad H(0) = F(P).
\end{align}

The initial condition (4.7) evidently holds, so we verify the differential equation (4.6) here: setting $H(t) = F(t + P)$ we have

$$H^{-1}dH = F^{-1}(t + P)dF(t + P)$$

$$= A(X(t + P))$$

$$= A(B^{-1}X(t)B)$$

$$= B^{-1}A(X(t))B$$

$$= B^{-1}F^{-1}dFB.$$  

As in the proof of Lemma 4.9 we again used the fact that conjugation by $B$ commutes with projection onto $\mathcal{P}_a$ on the fourth line.

On the other hand, setting $H(t) = F(P)B^{-1}F(t)B$, one obtains:

$$H^{-1}dH = B^{-1}F^{-1}(t)BF^{-1}(P)F(P)B^{-1}dF(t)B$$

$$= B^{-1}F^{-1}dFB.$$  

$\square$

5. Finite Type Flat Immersions in a Sphere

5.1. Admissible Connections. Suppose $f : \mathbb{R}^n \supset U \to S^{2n-1} \subset \mathbb{R}^{2n}$ is an immersion. One can associate to $f$ an immersion $F : U \to SO(2n)$, (called an adapted framing) of the form:

$$F = [e_1, \ldots, e_n, \xi_1, \ldots, \xi_n],$$

where $\xi_1 = f$ and the vectors $\{e_i\}$ and $\{\xi_2, \ldots, \xi_n\}$ are (non-unique) orthonormal bases for the tangent and normal spaces respectively to $f(U) \subset S^{2n-1}$.

Denote by $A = F^{-1}dF \in so(2n) \otimes \Omega^1(U)$ the pull-back by $F$ of the left Maurer-Cartan form of $SO(2n)$. More explicitly,

$$A = \left( \begin{array}{cc} \omega & -\beta^T \\ \beta & \eta \end{array} \right),$$

where $\omega \in so(n) \otimes \Omega^1(U)$ is the Levi-Civita connection form on $f(U)$ and $\eta \in so(n) \otimes \Omega^1(U)$ and $\beta$ are the normal connection and the second fundamental form respectively of the immersion $f$ into $\mathbb{R}^{2n}$.

Since $\xi_1 = f$, the requirement that $\{e_i\}$ span the tangent space to $f$ is equivalent to the statement that the first row and first column of $\eta$ consist of zeros. The map $f$ will be immersive provided that the first row of the second fundamental form $\beta$ (which is the dual frame to $\{e_i\}$) consists of $n$ linearly independent 1-forms.

Given that $A = F^{-1}dF$, we can calculate the Maurer-Cartan equation

$$dA + A \wedge A = 0.$$
For 1-forms $A$ which arise in the way described above, this is equivalent to the structure equations and the Gauss, Codazzi and Ricci equations for the immersion $f$.

Conversely, by standard theory of Lie groups, if $A \in \mathfrak{so}(2n) \otimes \Omega^1(U)$ satisfies the Maurer-Cartan equation then, provided $U$ is simply connected, $A$ can be integrated to give a map $F : U \to SO(2n)$.

Now suppose we have a family $A^z$ of such 1-forms, of the form:

\begin{equation}
A^z = \begin{pmatrix}
\omega \\
z \beta \\
\eta
\end{pmatrix},
\end{equation}

such that the first row and column of $\eta$ are zero and such that the Maurer-Cartan equation $dA^z + A^z \wedge A^z = 0$ is satisfied for all $z \in \mathbb{C}$, and such that $A^z$ is real (i.e. the components of the matrix are real valued one-forms) for $z \in \mathbb{R}$. Then for each non-zero real value of $z$ we can locally integrate $A^z$ to obtain an adapted frame $F^z$ for a map $f^z : M \to S^{2n-1}$. Moreover, writing out the Maurer-Cartan equations explicitly,

\begin{equation}
0 = \begin{pmatrix}
d\omega & -z \beta T \\
z d\beta & d\eta
\end{pmatrix} + \begin{pmatrix}
\omega \wedge \omega - z^2 \beta T \wedge \beta \\
z (\beta \wedge \omega + \eta \wedge \beta)
\end{pmatrix},
\end{equation}

and equating coefficients of like powers of $z$ we obtain an additional condition:

\[d\omega + \omega \wedge \omega = 0,\]

which says that the curvature of the induced metric on $f(U) \subset S^{2n-1}$ is zero.

If $f^z$ are immersions (this doesn’t depend on $z$), then the equations obtained from the other components give nothing further, since all of them follow from the Maurer-Cartan equation at a single value of $z$, with the exception of the flatness of the normal bundle, $d\eta + \eta \wedge \eta = 0$; but it is well known that for any (local) isometric immersion of flat space $E^n$ into $S^{2n-1}$ the normal bundle is necessarily flat.

Now given the flatness of the normal bundle, we could choose a parallel frame $\{\xi_1, ..., \xi_n\}$ for the normal bundle, i.e. one in which $\eta = 0$. This leads to the following:

**Definition 5.1.** An admissible connection on $\mathbb{R}^n$ is a family of $\mathfrak{so}(2n)$-valued 1-forms $A^z$ of the form

\begin{equation}
A^z = \begin{pmatrix}
\omega \\
z \beta \\
0
\end{pmatrix},
\end{equation}

(where $\omega$ is $n \times n$) which satisfy the Maurer-Cartan equation for all $z \in \mathbb{C}$.

Note that an admissible connection can be integrated to get a family of maps $F^z : \mathbb{R}^n \to SO(2n)$, $z \in \mathbb{R}$, and any of the columns $(n+1), ..., 2n$ of $F$ is a flat immersion into $S^{2n-1}$, provided that it is injective. Conversely, any flat immersion of $\mathbb{R}^n$ into $S^{2n-1}$ gives rise to an admissible connection. Therefore, we now consider the problem of producing admissible connections.

**Remark 5.2.** We should note here that in other applications of AKS theory to special submanifolds, it is always the case that one must be able to insert a spectral parameter $z$ into the relevant Maurer-Cartan equation in some such way as we have done here. However, it is not necessary that the satisfaction of the equation for all $z$ need add anything new, as it did here by introducing flatness. See, for example, [8].
5.2. A Scheme for Generating Admissible Connections. Let \( so(2n, \mathbb{C}) \) and \( \sigma = Ad(Q) \) be as in the introduction, so that \( so(2n, \mathbb{C}) = V_0 \oplus V_1 \), where \( V_i \) is the eigenspace of \( \sigma \) corresponding to \((-1)^i\). More explicitly,

\[
V_0 = \left\{ \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix} \right\}, \quad V_1 = \left\{ \begin{pmatrix} 0 & \ast \\ \ast & 0 \end{pmatrix} \right\}.
\]

As in Section 3 we can define a Lie algebra of \( so(2n, \mathbb{C})[z, z^{-1}] \) by

\[
\widetilde{G}_\sigma := \left\{ \sum_{i=0}^{\beta} X_i z^i \mid X_i \in V_i \mod 2 \right\}.
\]

Observe that the Lie subalgebra \( V_0 \subset so(2n, \mathbb{C}) \subset \widetilde{G}_\sigma \) has the Lie subalgebra decomposition:

\[
V_0 = V^U_0 \oplus V^L_0,
\]

\[
V^U_0 = \left\{ \begin{pmatrix} \ast & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad V^L_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \ast \end{pmatrix} \right\}.
\]

Now let

\[
P := \{ X \in \widetilde{G}_\sigma \mid X_0 \in V^U_0, X_i = 0 \text{ for } i < 0 \}.
\]

It is clear that \( P \) is a Lie subalgebra of \( \widetilde{G}_\sigma \). The reason we consider this subalgebra is that, comparing with (5.2), we see that admissible connections are precisely those 1-forms whose coefficients are degree 1 elements of \( P \) and are real-valued for real values of \( z \). The reality condition is captured in the prescription:

\[
(5.3) \quad \bar{X}(\bar{z}) = X(z).
\]

We denote by \( \mathbb{R}\widetilde{G}_\sigma \) the subalgebra of \( \widetilde{G}_\sigma \) consisting of elements which satisfy this condition.

We now have the vector space decomposition:

\[
\widetilde{G}_\sigma = P \oplus N,
\]

where \( P \), and \( N = \{ X \in \widetilde{G}_\sigma \mid X_0 \in V^U_0, X_i = 0 \text{ for } i > 0 \} \) are both Lie subalgebras consisting of polynomials in \( z \) and \( z^{-1} \) respectively. This is exactly the type of decomposition to which the results of Section 3 apply. The reality condition \((5.3)\) is preserved by the Lax equation \((3.4)\) provided the polynomials involved have real coefficients, so we can use Proposition 3.1 to produce admissible connections

\[
A := \sum_{i=1}^{n} \pi_P V_i(X) d_i,
\]

provided we can find polynomials \( V_i(X) = p_i(X, z^{-1}) \in \mathbb{R}[X, z^{-1}] \) which map \( \widetilde{G}_\sigma \) to \( \widetilde{G}_\sigma \) and are of top degree 1 in \( z \). In view of Remark 3.3 this can easily be arranged using the fact that if \( X \) is a skew-symmetric matrix then so is \( X^k \) for any odd positive integer, and we have the following result:

**Proposition 5.3.** The functions \( V_i : \widetilde{G}_\sigma \to \widetilde{G}_\sigma \) given by

\[
V_i(X) = z^{2-2i} X^{2i-1}, \quad i = 1 \ldots n,
\]

satisfy the condition \((5.3)\) of Proposition 3.1 for the loop algebra \( \widetilde{G}_\sigma \) defined above. This choice of \( V_i \) gives for each real initial condition \( X_0(z) = X_0(\bar{z}) \in (\mathbb{R}\widetilde{G}_\sigma)^1_d \) a solution \( X(t) \) of \((2.1)\) such that:
(1) \(A(X)\) is an admissible connection.

(2) \(X : \mathbb{R}^n \rightarrow \mathcal{G}_\sigma\) is complete, and so, therefore is a solution \(F : \mathbb{R}^n \rightarrow SO(2n)\) obtained by integrating \(A(X)\) evaluated at \(z = z_0 \in \mathbb{R}_+\).

5.3. Remarks Concerning Proposition 5.3.

5.3.1. Completeness. Flows of the form \(dX = [X, V(X)]\), where \(V(X)\) takes its values in the same loop Lie algebra \(\mathcal{G}_\sigma\) as \(X\), will always be complete if the underlying real Lie algebra is compact semi-simple (such as \(so(n)\)). This is because in this case there is a positive definite ad-invariant inner product \(<\cdot, \cdot>\) on \(\mathcal{G}_\sigma\), where ad-invariant means
\[
\langle [A, B], C \rangle = \langle [C, A], B \rangle
\]
for any \(A, B\) and \(C\). Thus
\[
\nabla \langle X, X \rangle = 2 \langle \nabla X, X \rangle = 2 \langle [X, V(X)], X \rangle = 2 \langle [X, X], V(X) \rangle = 0,
\]
which is to say that \(\langle X, X \rangle\) is constant. Hence the solution \(X\) is bounded, and therefore complete.

5.3.2. The Immersion Property. Every initial condition \(X(z, 0)\) in \((\mathcal{G}_\sigma)_d\) will generate a family of complete maps \(f_z : \mathbb{R}^n \rightarrow S^{2n-1}\), for \(z \in \mathbb{R}\). The map \(f_z\) fails to be an immersion at points \(t\) where the first \(n\) entries of the \((n+1)\)th row of \(F\) fail to be linearly independent 1-forms. If we define \(M(X)\) to be the \(n \times n\) matrix whose \(i\)-th row is the non-zero part of the \((n+1)\)th row of \(X^{2n-1}_1\), namely
\[
M(X) := \begin{pmatrix}
(X^{(n+1)1}_1) & (X^{(n+1)2}_1) & \cdots & (X^{(n+1)n}_1) \\
(X^{(n+1)1}_2) & (X^{(n+1)2}_2) & \cdots & (X^{(n+1)n}_2) \\
\vdots & \vdots & \ddots & \vdots \\
(X^{(n+1)1}_n) & (X^{(n+1)2}_n) & \cdots & (X^{(n+1)n}_n)
\end{pmatrix},
\]
then the condition for \(f_z\) to be an immersion is
\[
\det(M(X(z, t))) \neq 0.
\]
While most initial conditions \(X(z, 0)\) will ensure that this holds in a neighbourhood of \(t = 0\), in general one should expect to come across values of \(t\) where the condition fails.

5.3.3. Parallel Frames and Curved Flats. As an alternative, one can define
\[
P_a := \{X \in \mathcal{G}_\sigma \mid X_i = 0, \ i < 1\},
\]
\[
N_a := \{X \in \mathcal{G}_\sigma \mid X_i = 0, \ i > 0\}.
\]
These are again subalgebras, and so the same theory can be used to produce admissible connections which have the additional property that \(\omega\) and \(\eta\) are both zero, due to the fact that \(\pi P(\ldots + Y_0 + Y_1 z) = Y_1 z\). Such connections are associated to frames which are parallel.

On the other hand, if we were to set
\[
P_a := \{X \in \mathcal{G}_\sigma \mid X_i = 0, \ i < 0\},
\]
\[
N_a := \{X \in \mathcal{G}_\sigma \mid X_i = 0, \ i \geq 0\},
\]
we would have the \textit{curved flats} which were studied in [7]. These give you moving
frames \( F = [e_1, \ldots, e_n, \xi_1, \ldots, \xi_n] \) which satisfy the flatness conditions \( d\omega + \omega \wedge \omega = 0 \)
and \( d\eta + \eta \wedge \eta = 0 \). Given a solution \( F \) of this system, one can parallelize the
tangent and normal bundle by right multiplication by a matrix \( G \). One can show
that \( G \) is constant in \( z \), but not in \( t \), and that \( FG \) is actually the solution of the
scheme for producing parallel frames corresponding to the same initial condition
\( X(0) \).

5.3.4. Periodicity. As stated in the introduction, it is an interesting question to
look for solutions, \( f \), which have \( n \) linearly independent periods, corresponding to
flat \( n \)-tori in \( S^{2n-1} \). For the case of parallel frames, because the frames, \( F \), are
parallel for the immersion \( f \), it follows from the flatness of \( f \) (parallel transport
is independent of path) that \( f(x) = f(y) \) is equivalent to \( F(x) = F(y) \), thus the
periodicity question for the immersion \( f \) is precisely the periodicity question for the
adapted frame \( F \). Since parallel frames come from simple \( k \)-symmetric systems, we
can use the results of Section 4 to say that periodicity for the line bundle implies
type two quasiperiodicity for \( F \).

\textbf{Example 1} The simplest examples are of the form \( X(0) = X_1z \) which give the
same solutions whether we use the projections to \( P \) or to \( P_n \), namely the constant
solutions \( X(t) = X(0) \), because \([X, \pi_P V_t(X)]\) are zero. For the case \( n = 2 \), the
most general such example is of the form

\[
X(z, t) = X(z, 0) = \begin{pmatrix}
0 & 0 & x_1 & x_2 \\
0 & 0 & y_1 & y_2 \\
-x_1 & -y_1 & 0 & 0 \\
-x_2 & -y_2 & 0 & 0
\end{pmatrix} z.
\]

This generates the connection

\[
A := X dt_1 + z^{-2} X^3 dt_2
\]

\[
= \begin{pmatrix}
0 & 0 & x_1 & x_2 \\
0 & 0 & y_1 & y_2 \\
-x_1 & -y_1 & 0 & 0 \\
-x_2 & -y_2 & 0 & 0
\end{pmatrix} z dt_1 + \begin{pmatrix}
0 & -C \\
C^T & 0
\end{pmatrix} z dt_2,
\]

(5.5)

where \( C \) is the matrix

\[
\begin{pmatrix}
x_1(x_1^2 + y_1^2) + x_2(x_1 x_2 + y_1 y_2) & x_1(x_1 x_2 + y_1 y_2) + x_2(x_2^2 + y_2^2) \\
y_1(x_1^2 + y_1^2) + y_2(x_1 x_2 + y_1 y_2) & y_1(x_1 x_2 + y_1 y_2) + y_2(x_2^2 + y_2^2)
\end{pmatrix}.
\]

The immersion condition (5.4) in this case is

\[
\begin{vmatrix}
x_1 & y_1 \\
x_1(x_1^2 + y_1^2) + x_2(x_1 x_2 + y_1 y_2) & y_1(x_1^2 + y_1^2) + y_2(x_1 x_2 + y_1 y_2) = (x_1 x_2 + y_1 y_2)(x_1 y_2 - y_1 x_2) \\
\end{vmatrix} \neq 0.
\]

Any matrix \( X \) of the above form gives rise to a complete flat immersion \( f : \mathbb{R}^2 \to S^3 \),
provided this condition holds.

Since the second fundamental form is constant for such a connection, one can
perform a global change of coordinates so that it is diagonal and constant every-
where, which means that the submanifold is a product of circles, also known as a
Clifford torus. This argument also applies to \( n > 2 \).
Conversely, we can show that any Clifford 2-torus comes from the scheme in the following way: a Clifford 2-torus has the parameterization
\[ f(s_1, s_2) = [a \cos s_1, a \sin s_1, b \cos s_2, b \sin s_2], \]
where \( a^2 + b^2 = 1 \). We can choose the following adapted frame for \( f \):
\[
F := \begin{pmatrix}
-\sin s_1 & 0 & a \cos s_1 & b \cos s_1 \\
\cos s_1 & 0 & a \sin s_1 & b \sin s_1 \\
0 & -\sin s_2 & b \cos s_2 & -a \cos s_2 \\
0 & \cos s_2 & b \sin s_2 & -a \sin s_2
\end{pmatrix},
\]
and calculate
\[
F^{-1} dF = \begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & 0 & 0 \\
-a & 0 & 0 & 0 \\
-b & 0 & 0 & 0
\end{pmatrix} \, ds_1 + \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & b & -a \\
0 & -b & 0 & 0 \\
0 & a & 0 & 0
\end{pmatrix} \, ds_2.
\]
It is straightforward from here to solve the equation
\[
\begin{pmatrix}
s_1 \\
s_2
\end{pmatrix} = B \begin{pmatrix}
t_1 \\
t_2
\end{pmatrix},
\]
for the matrix \( B \) so that after a linear change of coordinates the connection (5.6) has the form of the right hand side of (5.5).

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Department of Mathematics, Faculty of Science, Kobe University, 1-1, Rokkodai, Nada-ku, Kobe 657-8501, Japan
E-mail address: brander@math.kobe-u.ac.jp