On the Parameterized Complexity of 
k-Edge Colouring *

Esther Galby¹, Paloma T. Lima², Daniël Paulusma³, and Bernard Ries¹

¹ Department of Informatics, University of Fribourg, Switzerland, esther.galby,bernard.ries}@unifr.ch
² Department of Informatics, University of Bergen, Norway, paloma.lima@uib.no
³ Department of Computer Science, Durham University, UK, daniel.paulusma@durham.ac.uk

Abstract. For every fixed integer $k \geq 1$, we prove that $k$-Edge Colouring is fixed-parameter-tractable when parameterized by the number of vertices of maximum degree.

1 Introduction

For an integer $k \geq 1$, a $k$-edge colouring of a graph $G = (V, E)$ is a mapping $c : E \to \{1, \ldots, k\}$ such that $c(e) \neq c(f)$ for any two edges $e$ and $f$ of $G$ that have a common end-vertex. The Edge Colouring problem is to decide if a given graph $G$ has a $k$-edge colouring for some given integer $k$. If $k \geq 1$ is fixed, that is, not part of the input, then we denote the problem as:

\[
\begin{align*}
\text{\textbf{k-Edge Colouring}} \\
\text{Instance:} \text{ a graph } G. \\
\text{Question:} \text{ does } G \text{ have a } k \text{-edge colouring?}
\end{align*}
\]

The chromatic index of a graph $G = (V, E)$ is the smallest integer $k$ such that $G$ has a $k$-edge colouring. The degree $d_G(u)$ of a vertex $u \in V$ is the size of its neighbourhood $N(u) = \{v \mid uv \in E\}$. We let $\Delta = \Delta_G$ denote the maximum degree of $G$. Vizing showed the following classical result.

Theorem 1 ([27]). The chromatic index of a graph $G$ is either $\Delta$ or $\Delta + 1$.

Due to Theorem 1 we can make the following observation.

Observation 1 Let $G$ be a graph. If $\Delta \leq k - 1$, then $G$ is a yes-instance of $k$-Edge Colouring. If $\Delta \geq k + 1$, then $G$ is a no-instance of $k$-Edge Colouring.

By Observation 1 we may assume without loss of generality that an input graph of $k$-Edge Colouring has maximum degree $k$. The following well-known hardness result was proven by Holyer for $k = 3$ and by Leven and Galil for $k \geq 4$.

Theorem 2 ([1718]). For every $k \geq 3$, $k$-Edge Colouring is \textsc{NP}-complete even for graphs in which every vertex has degree $k$.

* The second author was supported by the Research Council of Norway via the project CLASSIS and the work was done when the second author was visiting the University of Fribourg funded by a scholarship of the University of Fribourg. The third author was supported by the Leverhulme Trust (RPG-2016-258).
In this note we consider the $k$-Edge Colouring problem from the viewpoint of Parameterized Complexity, where problem inputs are specified by a main part $I$ of size $n$ and a parameter $p$, which is assumed to be small compared to $n$. The main question is to determine if a problem is fixed-parameter tractable (FPT), that is, if it can be solved in time $f(p)n^{O(1)}$, where $f$ is a computable (and possibly exponential) function of $p$. The choice of parameter depends on the context. By the above discussion, the most natural choice of parameter for $k$-Edge Colouring is the number of vertices of maximum degree. We prove the following result (note that $k$-Edge Colouring is polynomial-time solvable for $k \leq 2$).

**Theorem 3.** For every $k \geq 1$, $k$-Edge Colouring can be solved in $O(p^2 k^{2^k} + n + m)$ time on graphs with $n$ vertices, $p$ of which have maximum degree, and $m$ edges. Moreover, it is possible to find a $k$-edge colouring of $G$ in $O(p^2 k^{2^k} + 4 k^2 (n-p) + n + m)$ time (if it exists).

As we assume that $k$ is a fixed constant, Theorem 3 implies that for every $k \geq 3$, the $k$-Edge Colouring problem is fixed-parameter tractable when parameterized by the number of vertices of maximum degree. We prove Theorem 3 in Section 2 by using the alternative proof of Theorem 1 given by Ehrenfeucht, Faber and Kierstead [13]. We first discuss some related work.

**Related Work**

Apart from a number of (classical) complexity results (e.g. [1,2,11,12,15,21,22,23,26]), most results on edge colouring are related to Theorem 1 and are of a more structural nature. That is, they involve the derivation of sufficient or necessary conditions for a graph to be $\Delta$-edge colourable (see, for example [4,8,16,19]), in which case the graph is said to be Class 1 (and Class 2 otherwise). This is also the focus of papers on edge colouring related to the number of maximum-degree vertices (see, for example, [5,10,25,28]). In particular, Fournier [14] proved that every graph $G$ in which the vertices of degree $\Delta$ induce a forest is Class 1. As a consequence, a graph with at most two vertices of maximum degree is Class 1. In [6] and [7], Chetwynd and Hilton gave necessary and sufficient polynomial-time verifiable conditions for a graph with three, respectively, four vertices of maximum degree to be Class 1. The same authors proved an analogous result for $n$-vertex graphs with $r$ vertices of maximum degree in [9], assuming $\Delta \geq \left\lceil \frac{n}{2} \right\rceil + \frac{3}{2} r - 3$.

For more on edge colouring we refer to the recent survey of Cao, Chen, Jing, Stiebitz and Toft [3].

**2 The Proof of Theorem 3**

In this section we prove Theorem 3. Let $G = (V,E)$ be a graph and let $S \subseteq V$. The graph $G[S] = (S, \{ uv \in E(G) \mid u, v \in S \})$ is the subgraph of $G$ induced by $S$. The set $N(S) = \bigcup_{u \in S} N(u)$ is the set of vertices adjacent to at least one vertex in $S$. Let $X$ be the set of maximum-degree vertices of $G$. In this context, the graph $G[X]$ is said to be the core of $G$, whereas the graph $G[X \cup N(X)]$ is said to be the semi-core of $G$.

Machado and de Figueiredo [20] proved the following result.

**Theorem 4 ([20]).** Let $G$ be a graph and $k \geq 0$ be an integer. Then $G$ has a $k$-edge colouring if and only if its semi-core has a $k$-edge colouring.
The proof of Theorem 4 is based on an application of the recolouring procedure of Vizing [27] for proving Theorem 1 and which was also used by Misra and Gries [24] for giving a constructive proof of Theorem 1. As explained by Zate sko [29], the proof of Theorem 4 immediately yields a polynomial-time algorithm for finding a \( k \)-edge colouring of a graph \( G \) given a \( k \)-edge colouring of its semi-core (if it exists). Below we give a new, short algorithmic proof of Theorem 4 with the same time complexity, via a modification of the alternative proof of Theorem 1 by Ehrenfeucht et al. [13], which might be of independent interest. We state this result as a lemma, as we use it to prove our main result.

**Lemma 1.** Let \( G \) be a graph with a core of size \( p \). Given a \( k \)-edge colouring of its semi-core, it is possible to construct in time \( O(k^2(n-p)) \) a \( k \)-edge colouring of \( G \).

**Proof.** Recall that \( X \) denotes the set of vertices of maximum degree \( \Delta \) of \( G \). Let \( c' \) be a \( k \)-edge colouring of the semi-core \( G[X \cup N(X)] \) of \( G \) and denote by \( q \) the size of the semi-core of \( G \). Note that \( \Delta \leq k \). If \( V \setminus (X \cup N(X)) = \emptyset \), then \( c' \) is a \( k \)-edge colouring of \( G \). Assume that \( V \setminus (X \cup N(X)) \neq \emptyset \).

We write \( V \setminus (X \cup N(X)) = \{u_1, \ldots, u_{n-q}\} \). We let \( V_0 = X \cup N(X) \) and \( V_i = X \cup N(X) \cup \{u_1, \ldots, u_i\} \) for \( i = 1, \ldots, n-q \). Note that \( G = G[V_{n-q}] \).

We define \( c_0 = c' \). We now show how to extend \( c_0 \) vertex by vertex until we obtain a \( k \)-edge colouring of the whole graph. That is, for \( i = 1, \ldots, n-q \), we show how to construct a \( k \)-edge colouring \( c_i \) of \( G[V_i] \) given a \( k \)-edge colouring \( c_{i-1} \) of \( G[V_{i-1}] \)

Thus suppose that \( i \geq 1 \) and suppose that we already have a \( k \)-edge colouring \( c_{i-1} \) of \( G[V_{i-1}] \) (note that at the start of the procedure, when \( i = 1 \), this is indeed the case, as we have \( c_0 \)). We say that a vertex \( u \in V_{i-1} \cup \{u_i\} \) misses colour \( \ell \) if none of the edges incident to \( u \) is coloured \( \ell \) by \( c_{i-1} \). We denote the set of colours that \( u \) misses by \( F(u) \). Note that \( F(u_i) = \{1, \ldots, k\} \). We write \( F(uv) = F(u) \cap F(v) \). We now iteratively colour the edges incident to \( u_i \) in \( G[V_i] \) in such a way that at any time, the resulting mapping is a partial \( k \)-edge colouring of \( G[V_i] \) that satisfies the two following properties:

1. for each uncoloured edge \( u_iv \) in \( G[V_i] \), we have \( F(u_iv) \neq \emptyset \);
2. there is at most one uncoloured edge \( u_iv \) in \( G[V_i] \) with \( |F(u_iv)| = 1 \).

Observe that at the start of this process, when no edge \( u_iv \) in \( G[V_i] \) has been coloured, \( c_{i-1} \) satisfies (1) and (2). This can be seen as follows. Vertex \( u_i \) is in \( G[V_i] \) only adjacent to vertices of \( N(X) \cup \{u_1, \ldots, u_{i-1}\} \) (where we define \( \{u_1, \ldots, u_{i-1}\} = \emptyset \) if \( i = 1 \)). Every \( v \in N(X) \cup \{u_1, \ldots, u_{i-1}\} \) that is adjacent to \( u_i \) does not belong to \( X \) and thus has degree at most \( k-1 \). Hence, \( v \) is adjacent to at most \( k-2 \) vertices in \( V_i \setminus \{u_i\} \) and thus has \( |F(v)| \geq 2 \). As \( |F(u_i)| = \{1, \ldots, k\} \), this means that each edge \( u_iv \) in \( G[V_i] \) has \( |F(u_iv)| \geq 2 \), implying that (1) and (2) are satisfied.

We will now describe how we can maintain properties (1) and (2) while colouring the edges incident to \( u_i \) in \( G[V_i] \) one by one. Throughout this process we maintain a set \( W = \{v \in V_{i-1} | u_iv \text{ is an uncoloured edge in } G[V_i]\} \), so at the start \( W \) consists of all neighbours of \( u_i \) in \( G[V_i] \). We distinguish two cases.

**Case 1.** There exists a colour \( \ell \in \bigcup_{v \in W} F(u_iv) \) such that \( \ell \) appears in at most one set \( F(u_iv) \) for some \( v \in W \) with \( |F(u_iv)| \leq 2 \). Then we assign colour \( \ell \) to the edge \( u_iv \) for which \( |F(u_iv)| \) is the smallest over all sets \( F(u_iv) \) that contain \( \ell \). If all these sets have size at least 3, then afterwards (1) and (2) are satisfied. Otherwise, there is a unique smallest set \( F(u_iv) \) of size at most 2, which also implies that afterwards (1) and (2) are satisfied.

**Case 2.** For each colour \( x \in \bigcup_{v \in W} F(u_iv) \), there exists at least two distinct vertices \( w, w' \in W \) such that \( x \in F(u_iw) \cap F(u_iw') \) and both \( F(u_iw) \) and \( F(u_iw') \) have size
Hence, we deduce that \( |\bigcup_{v \in W} F(u,v)| \leq |W| \). As \( u_i \) has degree at most \( k - 1 \) in \( G \), it follows that \( |G[V_i]| \) at most \( k - 1 - |W| \) edges incident to \( u_i \) have already been coloured in this stage. It follows that \( |F(u_i)| \geq k - (k - 1 - |W|) = |W| + 1 \). Hence, there exists a colour \( b \in F(u_i) \setminus \bigcup_{v \in W} F(u_i,v) \). Consequently, each vertex in \( W \) must have an incident edge coloured \( b \). Now choose a vertex \( w \in W \) for which \( |F(u_i,w)| \) is minimum and consider a colour \( a \in F(u_i,w) \). We swap colours \( a \) and \( b \) along the path \( P \) in \( G[V_i] \) that starts in \( w \) and whose edges are alternatingly coloured \( a \) and \( b \). This yields another partial \( k \)-edge colouring of \( G[V_i] \). However, now \( w \) has no incident edge coloured \( b \) anymore, which means that we can colour the edge \( u_i,w \) with colour \( b \). We observe that for all \( v \in W \setminus \{w\} \), the set \( F(u_i,v) \) remains unchanged except the end-vertex \( w^* \) of \( P \) if \( w^* \) is adjacent to \( u_i \); in that case \( F(u_i,w^*) \) is replaced by \( F(u_i,w^*) \setminus \{a\} \). However, by minimality of \( |F(u_i,w)| \), we have that \( |F(u_i,w)| \leq |F(u_i,w^*) \setminus \{a\}| \). Moreover, it is possible to find a \( k \)-edge colouring \( c_i \) of \( G[V_i] \). Hence, after doing this for \( i = n - q \), we have indeed extended \( c \) to a \( k \)-edge colouring \( c \in c_{n - q} \) of \( G = G[V_{n - q}] \).

Finally note that since \( q \geq p \), the algorithm has at most \( n - p \) extension steps and as each takes \( O(k^2) \) time, the total running time is \( O(k^2(n - p)) \). This completes the proof of Lemma 4. \(\Box\)

We are now ready to prove Theorem 3 which we restate below.

**Theorem 3.** For every \( k \geq 1 \), \( k \)-Edge Colouring can be solved in \( O(p^2 k^{\frac{k^2}{2p} + 4} + n + m) \) time on graphs with \( n \) vertices, \( p \) of which have maximum degree, and \( m \) edges. Moreover, it is possible to find a \( k \)-edge colouring of \( G \) in \( O(p^2 k^{\frac{k^2}{2p} + 4} + k^2(n - p) + n + m) \) time (if it exists).

**Proof.** Let \( G = (V, E) \) be an instance of \( k \)-Edge Colouring. We first compute the maximum degree \( \Delta \) of \( G \) and the corresponding set \( X \) of vertices of degree \( \Delta \) in \( O(n + m) \) time (so \( p = |X| \)). Let \( H = G[X \cup N(X)] \) denote the semi-core of \( G \). By Theorem 2 it suffices to check if \( H \) has a \( k \)-edge colouring. Note that we may assume \( \Delta \leq k \) by Observation 1 hence \( |N(X)| \leq \Delta p \leq kp \). This means that

\[
2|E(H)| = \sum_{u \in X} d_H(u) + \sum_{u \in N(X)} d_H(u) \leq \sum_{u \in X} d_G(u) + \sum_{u \in N(X)} d_G(u) \leq kp + kp(k - 1).
\]

Hence, \( H \) has at most \( \frac{k^2p}{2} \) edges. Checking if a mapping \( f : E(H) \rightarrow \{1, \ldots, k\} \) is a \( k \)-edge colouring takes \( O(k^4p^2) \) time. The number of such mappings is at most \( k^{\frac{k^2p}{2}} \).

Hence, brute force checking if \( H \) has a \( k \)-edge colouring takes time \( O(p^2 k^{\frac{k^2}{2p} + 4}) \). Thus the first statement of the theorem follows. We obtain the second statement by applying Lemma 1. \(\Box\)

### 3 Conclusions

We proved that \( k \)-Edge Colouring is fixed-parameter-tractable when parameterized by the number of maximum-degree vertices. We note that our proof does not work to
show this for Edge Colouring, as the set $X \cup N(X)$ may have size $\Omega(k)$. As such, we pose the following open problem: what is the computational complexity of Edge Colouring when parameterized by the number of maximum-degree vertices?

Acknowledgements. We thank Leandro Zatesko for bringing paper [20] to our attention.

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