Electrodynamics of Impressed Bound and Free Charge Voltage Sources

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Consideration is given to the electrodynamics of bound charge and free charge alternating current voltage sources. The bound charge problem is equivalent to the idealized bar electret, which is polarized uniformly parallel to its axis with a permanent impressed polarization of $\vec{P}_b(t)$ and in general could be time dependent. For the free charge voltage source, a polarization vector may also be defined based on the separation of free charge, which is equivalent to the force per unit charge converted from the external energy which drives the voltage source. In this work we generalize the electrodynamic model of a bound charge electret voltage source and a free charge voltage source to a quasi-static time varying solution and apply the two-potential formulation. It is shown that an impressed effective magnetic current defines one of the boundary conditions of the voltage sources described by an electric vector potential. This comes about due to a generalization of Faraday’s law, which allows as to define both free and bound magnetic current source terms.

INTRODUCTION

In this work we first consider an ideal dipolar electret, with overall charge neutrality but with a permanent macroscopic electric dipole moment $\vec{D}$. The most common way to make an electret is to heat a polar dielectric material under the influence of a large electric field (thermo electret) [2]. Once cooled and removed from the electric field a net polarization will be maintained. The electret thus becomes a bound charge voltage source, and can supply a current and be discharged in a similar way to a battery [3].

Known electrets exhibit static fields, however they can be configured as AC sources through motion. For example, electret speakers are constructed by placing a mechanical conducting diaphragm near the electret surface, such that motion creates an AC signal. Here we generalise the concept of an electret to a time varying impressed permanent polarization (similar to the impressed electric field defined in Harrington [4] to describe a free charge voltage source), which describes a putative AC voltage source based on oscillating bound charge. The impressed polarization is found to be equivalent to the force per unit charge required to drive the voltage source, which leads to a generalization of Faraday’s Law. Following this we show by equating the force per unit charge supplied by a free charge voltage source, allows the definition of free charge polarization in a similar way to bound charge polarization. Furthermore, by applying a two-potential formulation, the source polarization (or external force per unit charge) may be recognised simply as the non-conservative or non-irrotational component of the electric field sourced by an effective magnetic current in a similar way as detailed in Harrington [4] and Balanis [5]. This realisation leads to the definition of the effective bound and free magnetic current source terms.

TIME VARYING BOUND CHARGE VOLTAGE SOURCE

Impressing a Source Term into Maxwell’s Equations to Describe an Electret

The starting point of this work is to first consider Maxwell’s equations for a dielectric media, represented by the following equations,

$$\nabla \cdot \vec{D} = \rho_f, \quad (1)$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \frac{\partial \vec{D}}{\partial t}, \quad (2)$$

$$\nabla \cdot \vec{B} = 0, \quad (3)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (4)$$

Assuming a lossless dielectric we consider the usual constitutive relationship,

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}. \quad (5)$$

To analyse an idealized dielectric bar electret with a uniformly permanent polarization parallel to its axis an impressed bound polarization of $\vec{P}_b(t)$ when $\vec{E} = 0$, (see fig[1] and [2]) needs to be added. Thus, if we assume a linear dielectric with a permanent impressed polarization, we can write the polarization vector as,

$$\vec{P} = \chi \epsilon_0 \vec{E} + \vec{P}_b. \quad (6)$$

Given $\epsilon_r = 1 + \chi \epsilon$, the $\vec{D}$-field becomes

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E} + \vec{P}_b. \quad (7)$$

In general the curl of both $\vec{D}$ and $\vec{P}_b$ are non-zero for an electret [3], so if we take the curl of equation [7] we obtain,

$$\nabla \times \vec{D} = \epsilon_0 \epsilon_r \nabla \times \vec{E} + \nabla \times \vec{P}_b. \quad (8)$$
As shown in Fig. 1, the permanent polarization is a voltage source, and assuming that the polarization oscillates in time the oscillating bound surface current will source an oscillating AC voltage. For such a physical effect to occur, there would need to be an external force per unit charge, \( \vec{f}_b \), oscillating the charges (for the case of an electret speaker it is the acoustic energy of the oscillating diaphragm). Thus, the total force per unit charge in the system is \( \vec{f}_T = \vec{E} + \vec{f}_b \), where \( \vec{f}_b \) maybe be identified to be related to the source polarization by,

\[
\vec{f}_b = \vec{P}_b^i / (\epsilon_0 \epsilon_r). \tag{13}
\]

As discussed in [6], this is generally true for any voltage source, i.e. a similar force per unit charge vector can be defined whether the separated charges are bound or free (discussed in more detail later in the paper). A voltage source, \( \vec{f}_b \), is ordinarily confined to one portion of the loop (i.e. a battery which converts chemical energy to electromagnetic energy).

One can note upon inspection of equations [9]–[12] that there appear to be no source charges in the equation. This is because the net charge of the system is zero, and there are no volume charges (or losses) in the system. However, in actual fact, the source term is evident in equation (12), which may be more revealing when rewritten as,

\[
\nabla \times \vec{D} = -\epsilon_0 \epsilon_r (\partial \vec{B} / \partial t - \nabla \times \vec{f}_b). \tag{14}
\]

Here, the curl of the oscillating external force per unit charge (\( \vec{f}_b \)) acting on the system, which polarizes the electret, creates a voltage or EMF, which can be represented as an impressed magnetic current source (in a similar way to [4]). This is given by,

\[
\vec{J}_{mb} = -\nabla \times \vec{f}_b = -\nabla \times \vec{P}_b^i / (\epsilon_0 \epsilon_r), \tag{15}
\]

where the \( \partial \vec{B} / \partial t \) term in eqn. (14) can be identified as the magnetic displacement current. Thus, the bound surface charge density separation is represented by the magnetic current term, \( \vec{J}_{mb} \), which also sets the boundary condition for the parallel components of the fields.

The relationship between the surface charges, \( \sigma_b \), created by the external force per unit charge \( \vec{f}_b \), and the magnetic current, \( \vec{J}_{mb} \), can be found by calculating the source EMF, \( \mathcal{E} \), of the permanently polarized material. In the quasi static limit we set \( \partial \vec{B} / \partial t = 0 \) and therefore the EMF may be calculated from,

\[
\mathcal{E} = -I_{mb_{enc}} = \oint_{S} \vec{J}_{mb} \cdot d\vec{a} = \frac{1}{\epsilon_0 \epsilon_r} \oint_{P} \vec{P}_b^i \cdot d\vec{l}, \tag{16}
\]

where \( I_{mb_{enc}} = \oint_{S} \vec{J}_{mb} \cdot d\vec{a} \) is the enclosed magnetic current. As indicated in Fig. 2 the surface charge on the normal faces with respect to the source polarization, \( \vec{P}_b^i \), is given by

\[
\sigma_b = \vec{P}_b^i \cdot \hat{n}, \tag{17}
\]

where \( \hat{n} \) is the normal to the surface.

Directly applying this to the cylindrical bar electret, as shown in fig. [1] and [2] we apply the cylindrical coordinate...
system, and thus on the top surface of the bar electret \( \hat{n} = \hat{z} \) and the polarization \( \hat{P}_b^i \) is in the \( z \) direction, we can then write, \( \sigma_b = P_b^i \). Then, the EMF generated by the electret in the quasi static limit is given by
\[
\mathcal{E} = f_b d = \frac{\sigma_b d}{\varepsilon_0 \varepsilon_r},
\]
(18)
similar to a voltage across a capacitor. However, this is an impressed voltage source creating an EMF and hence \( f_b \) can be interpreted as an EMF per unit length. If we assume \( f_b \) is in the \( z \)-direction, then
\[
f_b \hat{z} = \sigma_b / (\varepsilon_0 \varepsilon_r) \hat{z} \quad \text{and} \quad \hat{P}_b^i \hat{z} = \sigma_b \hat{z}.
\]
(19)
The magnetic surface current per unit length will be apparent at the radial boundary of the bar electret, and will determine the parallel boundary condition, and can be calculated to be
\[
\hat{I}_{mb}^i = -\sigma_b / (\varepsilon_0 \varepsilon_r) \hat{\phi}.
\]
(20)
It should be noted here that the effective magnetic current defined above has nothing to do with magnetic monopoles, and is solely due to the voltage created and the charge distribution caused by the external force per unit charge supplied to the system, given by the fact that \( \nabla \times \hat{P}_b^i = \sigma_b \hat{\phi} \) at the radial boundary of the electret. Thus, the effective magnetic current given by Eqn. (20) is a vector, which manifests due to the force per unit charge conversion of an external energy source within the parallel boundary of the bar electret. The fact that it is present on the boundary impacts the boundary conditions as discussed in the next subsection.

**Boundary Conditions**

The boundary conditions of the fields on the normal and parallel surfaces of the electret can be calculated from the integral equations,
\[
\oint_S \vec{B} \cdot d\vec{a} = 0,
\]
(21)
\[
\int_P \vec{B} \cdot d\hat{l} = \mu_0 \frac{d}{dt} \int_S \vec{B} \cdot d\vec{a},
\]
(22)
\[
\oint_S \vec{B} \cdot d\vec{a} = 0,
\]
(23)
\[
\int_P \vec{D} \cdot d\hat{l} = -\varepsilon_0 \varepsilon_r \frac{d}{dt} \int_S \vec{B} \cdot d\vec{a} - \varepsilon_0 \varepsilon_r I_{mb_{\text{ext}}},
\]
(24)
From these integral equations it is straightforward to derive the modified boundary conditions as follows (subscript “in” refers to inside the bar electret and subscript “out” refers to outside the electret),
\[
\vec{B}_{\text{in}} = \vec{B}_{\text{out}}^/,
\]
(25)
\[
\vec{B}_{\text{in}} = \vec{B}_{\text{out}}^\parallel,
\]
(26)
\[
\vec{B}_{\text{in}} = \vec{B}_{\text{out}}^\perp,
\]
(27)
\[
\vec{D}_{\text{in}} - \vec{D}_{\text{out}}^/ = -\varepsilon_0 \varepsilon_r \iota_{mb}^i \times \hat{n} = \vec{P}_{\text{in}}^i,
\]
(28)
Thus we have derived all the equations required to calculate the fields in a bar electret, assuming the force per unit charge, \( f_b \), driving the charge separation is DC or oscillating in time, as shown in fig 2. With regards to the permanent polarization vector (or impressed force per unit charge) \( P_{\text{in}}^i \), the boundary conditions are:
\[
\vec{P}_{\text{in}}^i = (\vec{P}_{\text{in}}^i \cdot \hat{n}) \hat{n} = \sigma_b \hat{n} \quad \text{and} \quad \vec{P}_{\text{out}}^i = 0
\]
(29)
\[
\vec{P}_{\text{in}}^\parallel = -\varepsilon_0 \varepsilon_r \iota_{mb}^i \times \hat{n} \quad \text{and} \quad \vec{P}_{\text{out}}^\parallel = 0
\]
(30)
In general, an analytical calculation of the fields may only be achieved using an ellipsoid, so the fields in a bar electret must be solved numerically. However, one can look at the limits of various aspect ratios to get an approximation of the fields. First for a thin polarized sheet (approximated by an infinite sheet) there is no field outside (similar to an infinite capacitor). In this approximation, \( \vec{E}_{\text{in}} = -\vec{P}_{\text{in}}^i / (\varepsilon_r \varepsilon_0) \), and \( \vec{D} = 0 \). However, at the edges of the electret, fringing will change the solution. In the opposite limit of a long bar electret the electric field, \( \vec{E}_{\text{in}} \), varies more substantially, and is at a maximum in the middle (where \( \vec{D}_{\text{in}} \) is at a minimum), reducing substantially at the parallel boundary. In general \( \vec{D} \) is continuous (as there are no free charges) and points in the same direction as \( \vec{P}_{\text{in}}^i \) inside the bar electret. However, \( \vec{E}_{\text{in}} \) points in the opposite direction to both \( \vec{P}_{\text{in}}^i \) and \( \vec{D}_{\text{in}} \) inside the electret (as shown in fig 1), but in the same direction outside (\( \vec{E}_{\text{out}} \) is parallel to \( \vec{D}_{\text{out}} = \varepsilon_0 \vec{E}_{\text{out}} \)). The fields in a typical DC bar electret are presented in a problem in Griffiths [6], and are reproduced here in fig 1 and fig 2 from the solution manual.

**TWO POTENTIAL FORMULATION**

The general two potential formulation of impressed current and voltage sources is discussed in detail in Harrington [4] and is included here for completeness. The sum of all possible force per unit charge vectors associated with the electret discussed in the prior subsections is given by, \( f_T = f_b + \vec{E} \), which includes the external energy force per unit charge term and the standard electric
field. From equation (7), we can recognise the total field as discussed in Harrington as
\[ \vec{E}_T = \vec{f}_T = \vec{D}/(\varepsilon_0\varepsilon_r) = \vec{E} + \vec{E}^i \]
where \( \vec{E}^i \) is defined as the impressed electric field and is a non-irrotational term as the curl is non-zero while the divergence is zero.

The non-irrotational electric behaviour also contributes an extra term in the potential, \( \vec{C} \) given by
\[ \vec{E}^i = \vec{E}_C = -\frac{1}{\varepsilon_0\varepsilon_r} \nabla \times \vec{C}, \tag{32} \]
where \( \vec{E}_C \) is the non-conservative component of the electric field driven by the electric vector potential, while the irrotational behaviour contributes the usual term given by
\[ \vec{E}_A = -\frac{\partial \vec{A}}{\partial t} - \nabla V, \tag{33} \]

Where \( \vec{A} \) is the usual magnetic vector potential, \( V \) the electric scalar potential and \( \vec{E}_A \) is the irrotational component of the electric field driven by the magnetic vector potential.

In the quasi-static limit we can ignore \( \frac{\partial \vec{A}}{\partial t} \) and the main source terms are due to the charge distributions defined by the electric charge and the effective magnetic current, with the electric vector potential given by \( 4 \),
\[ \vec{C}(\vec{r}, t) = \frac{\varepsilon_0\varepsilon_r}{4\pi} \int_{\Omega} \frac{\vec{J}_m(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'. \tag{34} \]
and the electric scalar potential given by,
\[ V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0\varepsilon_r} \int_{\Omega} \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3\vec{r}'. \tag{35} \]

Here \( \vec{C} \) and \( V \) at point \( \vec{r} \) and time \( t \) is calculated from magnetic current and charge distribution at distant position \( \vec{r}' \) at an earlier time \( t' = t - |\vec{r} - \vec{r}'|/c \) (known as the retarded time). The location \( \vec{r}' \) is a source point within volume \( \Omega \) that contains the magnetic current distribution. The integration variable, \( d^3\vec{r}' \), is a volume element around position \( \vec{r}' \).

Substituting in the charge and magnetic current distributions of the voltage source we obtain,
\[ \vec{C}(\vec{r}, t) = \frac{\varepsilon_0\varepsilon_r}{4\pi} \int_{S_1} \frac{\vec{J}_m^i(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d^2\vec{r}'. \tag{36} \]
and the electric scalar potential given by,
\[ V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0\varepsilon_r} \int_{S_1, S_2} \frac{\sigma_b(\vec{r}', t')}{|\vec{r} - \vec{r}'|} d^2\vec{r}'. \tag{37} \]

Here the location \( \vec{r}' \) is a source point within the surface that contains the magnetic surface current or surface charge distribution. Here \( S_1 \) is the radial surface of the cylinder, and \( S_2 \) and \( S_3 \) are the top and bottom flat surfaces respectively.

In general by superposition the total electric and magnetic fields may be calculated using the principle of superposition from the potentials by 4, 5:
\[ \vec{E}_T = \vec{E}_A + \vec{E}_C = \frac{\partial \vec{A}}{\partial t} - \nabla V - \frac{1}{\varepsilon_0\varepsilon_r} \nabla \times \vec{C} \tag{38} \]
\[ \vec{B}_T = \vec{B}_A + \vec{B}_C = \nabla \times \vec{A} - \mu_0 \frac{\partial \vec{C}}{\partial t}. \tag{39} \]

**FREE CHARGE AC VOLTAGE SOURCE**

Often the electric field in an electret is compared to a parallel plate capacitor. For the DC case it is essentially the same as a charged capacitor, with the single exception that the charges on the capacitor plate are free. However, if we consider a hypothetical permanent time dependent polarization sourced by an external energy then the generated magnetic field inside the bar electret will be quite different to the capacitor. This is because the \( \vec{D} \) field of the electret (and hence the time dependence of the \( \vec{B} \) field) is different to a capacitor, due to the contributing permanent polarization and the associated magnetic current source term. Using the quasi-static approximation, the \( \vec{B} \)-field can be calculated from eqn. (22). Towards the centre of the electret the \( \vec{D} \)-field is zero (and hence \( \vec{B} \)-field is too), it only becomes significant towards the side boundaries where fringing becomes dominant. The fact is a capacitor is not an actual voltage source, so any analogy will not be complete. However a battery or another voltage source of free charge is analogous to the electret, except that the voltage would be sourced by free charge rather than bound charge. Another difference is that there is a force due to the \( \vec{E} \) field on capacitor plates, where as the force on the surface charge on the voltage source is reduced as the impressed field, \( \vec{E}^i \), supplies a force in the opposite direction to \( \vec{E} \). External to the voltage source and the parallel plate capacitor, the electric field of both are the same form and that of a dipole field.

To analyse a free charge voltage source in a similar way to the electret model in the previous section, we can start with the equations of a voltage source in advanced electrodynamics text books such as Griffiths 6, where the total force per unit charge, \( \vec{f}_T \), in a free charge voltage source is given by,
\[ \vec{f}_T = \vec{f}_f + \vec{E}, \tag{40} \]
where \( \vec{f}_f \) is the force per unit charge supplied by the voltage source from an external energy source. Harrington
presents essentially the same equation as Griffiths, but using different terminology, such as, $\vec{E}_T = \vec{f}_T$ as the total electric field and $\vec{E}_f = \vec{f}_f$ as an impressed electric field, so that

$$\vec{E}_T = \vec{E}_f + \vec{E}$$  \hspace{1cm} (41)

Following this we may define an impressed free charge polarization vector in a similar way as the bound charge polarization (however we assume free-space and no dielectric medium),

$$\vec{P}_f^i = \epsilon_0 \vec{E}_f = \epsilon_0 \vec{f}_f,$$  \hspace{1cm} (42)

where now the $\vec{D}$ has a non-zero value due to a permanent free charge polarization when $E = 0$, such that

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}_f^i,$$  \hspace{1cm} (43)

which is essentially the same equation as equation (41). Defining $\vec{J}_{mf} = \vec{\nabla} \times \vec{P}_f^i / (\epsilon_r \epsilon_0) = \vec{\nabla} \times \vec{E}_f$ and taking the curl of equation (41) we obtain a similar equation to equation (12),

$$\vec{\nabla} \times \vec{E}_T = -\left( \frac{\partial \vec{B}}{\partial t} + \vec{J}_{mf} \right).$$  \hspace{1cm} (44)

Thus, we have defined the effective magnetic current source terms for a free charge voltage source in an analogous way to the bound current source described in the prior sections, as well as calculated the modification to Faraday’s law for an impressed free charge voltage source.

In a similar way we can calculate the boundary conditions, and apply the two-potential formulation, to complete the electrodynamical analysis of a free charge voltage source. This would be simply achieved by assuming a cylindrical volume (as in figures 1 and 2) with free charge at either end instead of bound charge.

**CONCLUSION**

We have explored the electrodynamics of impressed bound and free charge voltage sources. This was represented by a force per unit charge source term, which converts external energy into electromagnetic energy and may be considered as a non-conservative electric field vector with an electric vector potential. The source term is necessarily impressed into Maxwell’s equations as an equivalent magnetic current, due to the energy conversion, the resulting charge distribution and EMF produced by the voltage source, resulting in a modification of Faraday’s law.

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