A Novel Variational Principle arising from Electromagnetism

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Abstract. Analyzing one example of LC circuit in [8], show its Lagrange problem only have other type critical points except for minimum type and maximum type under many circumstances. One novel variational principle is established instead of Pontryagin maximum principle or other extremal principles to be suitable for all types of critical points in nonlinear LC circuits. The generalized Euler-Lagrange equation of new form is derived. The canonical Hamiltonian systems of description are also obtained under the Legendre transformation, instead of the generalized type of Hamiltonian systems. This approach is not only very simple in theory but also convenient in applications and applicable for nonlinear LC circuits with arbitrary topology and other additional integral constraints.

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1 Introduction

The Lagrangian and Hamiltonian formulation of nonlinear inductor-capacitor circuits (LC circuits) has been considered by [3–8], [10] and the many references incited within. van der Schaft, Maschke and coworkers [7] and Bloch and Crouch [3] etc. established the Hamiltonian modeling by utilizing the constant Dirac structure of circuits. Many authors considered the variational approaches to nonlinear LC circuits. Based on the dual extremum principle in [9], Kwatny, Massimo and Bahar [6] realized the Lagrangian modeling. Recently, Moreau and Aeyels [8] obtained the generalized Euler-Lagrange equations and the generalized Hamiltonian system description after applying Pontryagin maximum principle. At the same time, Scherpen, Jeltsema and Klaassens [10] established the Lagrangian modeling of nonlinear LC circuits based on the constrained variational principle from holonomic mechanics, see e.g. [13, 2] etc. Then Clemente-
Gallardo and Scherpen [5] considered the relation between Lagrangian and Hamiltonian formalisms of nonlinear LC circuits via Lie algebroid.

The Lagrangian functional in LC circuits has its own rich properties. As pointed out in [8] (see also [6]), the generalized velocities are not simply the derivatives of the generalized coordinates. In other words as in [5], it lacks of kinetic terms for the capacitors and the potential terms for inductors. Utilizing the notion of tree and cotree in [11], [8] developed one method to consider nonlinear LC circuits with any topology (including with excess elements or without excess elements) through the Lagrangian functional

\[ J(u) = \int_{t_0}^{t_1} L(x, u) \, dt, \]

subject to the dynamics (from Kirchhoff’s current law)

\[ \dot{x} = Au, \]

for some matrix \( A \).

As shown in section 2, this Lagrange problem only have other type critical points except for minimum type and maximum type under many circumstances.

In this paper, we will establish a variational principle for this Lagrange problems. This principle can derive the generalized Euler-Lagrange equation of new type to describe critical points of all types. Meanwhile, the Hamiltonian and the canonical Hamiltonian systems formulation will be given uniformly to describe the critical points of all types under the generalized Legendre transformation.

One of the advantages of this generalized Euler-Lagrange equation formulation is that we can very easily derive the canonical Hamiltonian systems under the generalized Legendre transformation. In this way, the energy function can be explicitly constructed, which is very important especially in applications. It should be pointed out the notion of the generalized Legendre transformation is adapted from the ideas of H. J. Sussmann and J. C. Willems [12].

The second advantage of this generalized Euler-Lagrange equation formulation is that it clearly indicate the distinction between the problems without additional constraints and those with constraints, especially the terminal state constraints. Meanwhile, it will be clearly shown in this canonical Hamiltonian system formulations how the constraints influence the constructions of the Hamiltonian functions – the energy functions.

The third advantage is that, in the applications to nonlinear LC circuits, we will not encounter the technical difficulty to consider the abnormal cases which unavoidably arising in applying Pontryagin maximum principle with additional constraints such as terminal state constraints. Meanwhile, we will also not encounter the complex calculations of pseudo-inverses of matrixes and Lagrangian multipliers, which involve in applying the dual extremal principles of [9].

This paper is organized as follows. In section 2, we analyze the LC examples considered in [8]. It will be shown that the Lagrange problem with constraints has other type critical points except for both minimum and maximum type critical points under many
circumstances; and then put forward a new type of variational problem instead of minimizing problems of the Lagrange functional. In section 3, we will establish one variational principle to derive the generalized Euler-Lagrange equations of new type to describe the critical points of all types. Some illustrative examples will be given. In section 4, we will derive the canonical Hamiltonian systems to describe the critical points of all types under the generalized Legendre transformation. Some illustrative examples will also be given. Last, an appendix will be attached enclosed within the proofs of the results in section 2.

2 A New Variation Problem arising from Inductor-Capacitor Circuits

L. Moreau and D. Aeyels [8] comprehensively considered the dynamic equation of one LC circuit as illustrative examples (see Examples 1, 2, 3 and 4 in [8]).

The associated Lagrange functional consists of magnetic energy (of inductors) minus electric energy (of capacitors)

\[
J(i_3, i_5, i_6) = \int_{t_0}^{t_1} \left[ \frac{1}{2} L_3 i_3^2 + \frac{1}{2} L_4 (i_3 - i_5 - i_6)^2 + \frac{1}{2} L_5 i_5^2 + \frac{1}{2} L_6 i_6^2 \right] dt \\
- \int_{t_0}^{t_1} \left[ \frac{1}{2C_1} q_1^2 + \frac{1}{2C_2} q_2^2 \right] dt,
\]

where \( q_1 \) and \( q_2 \) are described by the dynamics (from Kirchhoff’s current law)

\[
q_1 = i_3, \quad q_2 = i_5 + i_6.
\]

In addition, the following other integral constraints were imposed in [8]:

\[
\int_{t_0}^{t_1} i_3 dt = \lambda_3, \quad \int_{t_0}^{t_1} i_5 dt = \lambda_5, \quad \int_{t_0}^{t_1} i_6 dt = \lambda_6.
\]

In (2.1)–(2.3), \( C_1 > 0 \) and \( C_2 > 0 \) are capacitance, \( L_3 > 0, L_4 > 0, L_5 > 0 \) and \( L_6 > 0 \) are inductance, \( i_3, i_5 \) and \( i_6 \) are the currents. For more information in detailed, please see Examples 1, 2, 3 and 4 in [8].

Remark 2.1. In Examples 2, 3 and 4 of [8], it was also imposed the initial and terminal points constraints: \( q_1(t_0), q_2(t_0), q_1(t_1), q_2(t_1) \) are fixed. It follows from (2.2) and (2.3) that, only \( q_1(t_0) \) and \( q_2(t_0) \) fixed can guarantee \( q_1(t_1) \) and \( q_2(t_1) \) also fixed.

L. Moreau and D. Aeyels [8] considered the following minimum problem:

\[
\text{(MP): To minimize (2.1) subject to (2.2) and (2.3).}
\]

Through defining the augmented state variables

\[
x_1 = q_1, \quad x_2 = q_2, \quad x_3(t) = \int_{t_0}^{t} i_3(s) ds, \\
x_4(t) = \int_{t_0}^{t} i_5(s) ds, \quad x_5(t) = \int_{t_0}^{t} i_6(s) ds,
\]

[8] reformulated (MP) as an optimal control problem with terminal state constraints \( x_3(t_1) = \lambda_3, x_4(t_1) = \lambda_5 \) and \( x_5(t_1) = \lambda_6 \), and then applying Pontryagin maximum
principle to obtain both the generalized Hamiltonian and the generalized Euler-Lagrange model of this LC circuit.

Let us define the symmetric matrix

\[
S_1 := \begin{pmatrix}
L_4 + L_3 - K_1 & -L_4 & -L_4 \\
-L_4 & L_4 + L_5 - 2K_1 & L_4 \\
-L_4 & L_4 & L_4 + L_6 - 2K_1
\end{pmatrix},
\]

(2.4)

\[
S_2 := \begin{pmatrix}
L_4 + L_3 - K_2 & -L_4 & -L_4 \\
-L_4 & L_4 + L_5 - 2K_2 & L_4 \\
-L_4 & L_4 & L_4 + L_6 - 2K_2
\end{pmatrix},
\]

(2.5)

where \( K_1 := \max\{K(C_1), \frac{\tilde{K}(C_2)}{2}\} \) and \( K_2 := \min\{K(C_1), \frac{\tilde{K}(C_2)}{2}\} \). Both \( K(C_1) \) and \( \tilde{K}(C_2) \) are the unique solutions to

\[
\sum_{n=1}^{+\infty} \frac{1}{2\pi^2 C_1 K(C_1)n^2 - \frac{1}{2}} = 1, \quad K(C_1) > \frac{3(t_1 - t_0)^2}{4\pi^2 C_1},
\]

(2.6)

and

\[
\sum_{n=1}^{+\infty} \frac{1}{\pi^2 C_2 \tilde{K}(C_2)n^2 - \frac{1}{2}} = 1, \quad \tilde{K}(C_2) > \frac{3(t_1 - t_0)^2}{2\pi^2 C_2},
\]

(2.7)

respectively.

In the Appendix, we prove the following

**Proposition 2.1.** It holds that

(I) If the matrix \( S_1 \) is positively definite, then the Lagrange functional (2.1) subject to (2.2) and (2.3) has a minimum value at the unique critical point \((i^*_3, i^*_5, i^*_6) \in C([t_0, t_1], \mathbb{R}^3)\);

(II) If the matrix \( S_2 \) is negatively definite, then the Lagrange functional (2.1) subject to (2.2) and (2.3) has neither minimum value nor maximum value.

(III) Let \( C_1 = C_2 \). Then \( \tilde{K}(C_2) = 2K(C_1) \). \( K_1 = K_2 \) and \( S_1 = S_2 \). In these cases, the Lagrange functional (2.1) subject to (2.2) and (2.3) has neither minimum value nor maximum value provided that the matrix \( S_2 \) has at least one negative characteristic root.

Define

\[
M := \begin{pmatrix}
L_4 + L_3 & -L_4 & -L_4 \\
-L_4 & L_4 + L_5 & L_4 \\
-L_4 & L_4 & L_4 + L_6
\end{pmatrix},
\]

(2.8)
(2.9) \[ N := \begin{pmatrix} \frac{1}{c_1} & 0 & 0 \\ 0 & \frac{1}{c_2} & \frac{1}{c_2} \\ 0 & \frac{1}{c_2} & \frac{1}{c_2} \end{pmatrix}. \]

The symmetric matrix \( M \) is positively definite due to the positivity of \( L_3, L_4, L_5 \) and \( L_6 \). Hence, we can define the positively definite matrices \( M^{\frac{1}{2}} \) and \( M^{-\frac{1}{2}} \) uniquely such that

(2.10) \[ M^{\frac{1}{2}} M^{\frac{1}{2}} = M, \quad M^{-\frac{1}{2}} M^{-\frac{1}{2}} = M^{-1}. \]

There exists an orthogonal matrix \( P \) such that

(2.11) \[ M^{-\frac{1}{2}} N M^{-\frac{1}{2}} = P^T \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} P, \]

where both \( h_1 > 0 \) and \( h_2 > 0 \) are the characteristic roots of \( M^{-\frac{1}{2}} N M^{-\frac{1}{2}} \).

**Proposition 2.2.** It holds that

(I) The Lagrange functional (2.1) subject to (2.2) and (2.3) has a unique critical point \((i_3^*, i_5^*, i_6^*) \in C([t_0, t_1], \mathbb{R}^3)\) for any \( \lambda_3, \lambda_5, \lambda_6 \in \mathbb{R} \) if and only if

(2.12) \( (t_1 - t_0)\sqrt{h_1} \neq k\pi, \quad (t_1 - t_0)\sqrt{h_2} \neq k\pi, \quad \forall k \in \mathbb{N}^+; \)

(II) If \( (t_1 - t_0)\sqrt{h_1} = k\pi \) for some \( k \in \mathbb{N}^+ \), or \( (t_1 - t_0)\sqrt{h_2} = k\pi \) for some \( k \in \mathbb{N}^+ \), then the Lagrange functional (2.1) subject to (2.2) and (2.3) has a critical point in \( C([t_0, t_1], \mathbb{R}^3) \) for some \( \lambda_3, \lambda_5, \lambda_6 \in \mathbb{R} \) if and only if there exists \( b \in \mathbb{R}^3 \) such that

(2.13) \[ \Phi(t_1 - t_0)b = (\lambda_3, \lambda_5, \lambda_6)^T - \int_{t_0}^{t_1} \Phi(t_1 - t)M^{-1}a \, dt, \]

where

(2.14) \[ \Phi(t) = M^{-\frac{1}{2}} P^T \begin{pmatrix} \frac{1}{\sqrt{h_1}} \sin(\sqrt{h_1}t) & 0 & 0 \\ 0 & \frac{1}{\sqrt{h_2}} \sin(\sqrt{h_2}t) & 0 \\ 0 & 0 & t \end{pmatrix} P M^{\frac{1}{2}}, \]

(2.15) \[ a := -(\frac{q_1(t_0)}{C_1}, \frac{q_2(t_0)}{C_2}, \frac{q_2(t_0)}{C_2})^T. \]

In these cases, the Lagrange functional (2.1) subject to (2.2) and (2.3) has infinitely many critical points for these \( \lambda_3, \lambda_5, \lambda_6 \in \mathbb{R} \).

The notion of critical points associated with the Lagrange functional (2.1) subject to (2.2) and (2.3) will be precisely given by Definition 3.1 and 3.4 in Section 3.

From these propositions, we known the Lagrange functional (2.1) subject to (2.2) and (2.3) has neither minimum nor maximum value while having critical points in many circumstances. The critical points in these cases we can understand as equilibriums. These facts suggest that we should consider the new variational problem in the next section to replace the minimizing problem.
3 A Novel Variational Principle

In this section, we study the following variational problem:

\[ J(u) := \int_{t_0}^{t_1} L(x(t), u(t)) \, dt = \text{stationary!} \] (3.1)

subject to

\[ x' = f(x(t), u(t)), \quad x(t_0) = x_0, \] (3.2)

where \( x_0 \in \mathbb{R}^n \) is fixed.

It is assumed that

(AI) \( L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) and \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) for \( n, m \in \mathbb{N}^+ \), are continuously differentiable;

(AII) Moreover, for any given \( u \in C([t_0, t_1], \mathbb{R}^m) \), the system (3.2) has a unique solution on the whole interval \([t_0, t_1] \), which will be denoted by \( x(\cdot; u) \).

Obviously, the variational equation of (3.2) at \((x, u) \in C([t_0, t_1], \mathbb{R}^n) \times C^1([t_0, t_1], \mathbb{R}^m) \) with \( x = x(\cdot; u) \) is as follows:

\[ \frac{d(\delta x)}{dt} = \frac{\partial f}{\partial x}(x(t), u(t))\delta x + \frac{\partial f}{\partial u}(x(t), u(t))\delta u, \quad \delta x(t_0) = 0. \] (3.3)

Let \( X(t; u) \) with \( t_0 \leq t \leq t_1 \) be one fundamental solution matrix to the homogeneous equation of (3.3):

\[ \frac{dx}{dt} = \frac{\partial f}{\partial x}(x(t), u(t))x. \] (3.4)

Define \( T(t, s; u) := X(t; u)X^{-1}(s; u) \) with \( t_0 \leq s \leq t \leq t_1 \). By the variation-of-constants formula, the solution to (3.3) is

\[ \delta x(t) = \int_{t_0}^{t} T(t, s; u) \frac{\partial f}{\partial u}(x(s), u(s))\delta u(s) \, ds, \quad t_0 \leq t \leq t_1. \] (3.5)

The variation of the functional \( J \) of (3.1) subject to (3.2) at \( u \in C([t_0, t_1]; \mathbb{R}^m) \) in the direction \( h \in C([t_0, t_1]; \mathbb{R}^m) \) is defined as follows

\[ \delta J(u; h) := \lim_{\varepsilon \to 0} \frac{J(u + \varepsilon h) - J(u)}{\varepsilon}, \] (3.6)

in which \( J(u) \) is defined by (3.1)-(3.2), and

\[ J(u + \varepsilon h) = \int_{t_0}^{t_1} L(x(t), u(t) + \varepsilon h(t)) \, dt, \] (3.7)

subject to

\[ x' = f(x(t), u(t) + \varepsilon h(t)), \quad x(t_0) = x_0. \] (3.8)
Somewhere in the subsequent, for \( \Xi = f, L \) or \( g \), we will denote \( \frac{\partial \Xi}{\partial x}(x(t), u(t)) \) and \( \frac{\partial \Xi}{\partial u}(x(t), u(t)) \) simply by \( \frac{\partial \Xi}{\partial x}(t) \) and \( \frac{\partial \Xi}{\partial u}(t) \) respectively, and analogously for other time variables such as \( s, \tau, \) etc.

Applying (3.5) to (3.7)-(3.8), we can deduce by Fubbi Theorem that

\[
J(u + \varepsilon h) - J(u) = \varepsilon \int_{t_0}^{t_1} \frac{\partial L}{\partial u}(t) T(t, s; u) \frac{\partial f}{\partial u}(s) ds \, dt + \varepsilon \int_{t_0}^{t_1} \frac{\partial L}{\partial u}(t) h(t) dt + o(\varepsilon)
\]

for any \( h \in C([t_0, t_1]; \mathbb{R}^m) \), and then we have

\[
(3.10) \quad \delta J(u; h) = \int_{t_0}^{t_1} \left[ \int_{t_0}^{t} \frac{\partial L}{\partial u}(s) T(s, t; u) \frac{\partial f}{\partial u}(s) ds \, \frac{\partial f}{\partial u}(t) + \frac{\partial L}{\partial u}(t) \right] h(t) dt.
\]

### 3.1 The first case with additional constraints

Let us impose some additional constraints

\[
(3.11) \quad \int_{t_0}^{t_1} [Bu(t) + \alpha] dt = 0,
\]

where \( B \in \mathbb{R}^{l \times m} \) is a matrix and \( \alpha \in \mathbb{R}^l \) is a vector for \( l \in \mathbb{N}^+ \).

**Definition 3.1. (I)** The admissible set is defined as

\[
(3.12) \quad \mathcal{U}_{ad} = \{ u \in C([t_0, t_1], \mathbb{R}^m) | \int_{t_0}^{t_1} [Bu(t) + \alpha] dt = 0 \};
\]

**Definition 3.2.** \( u \in \mathcal{U}_{ad} \) is called a critical point for the Lagrange functional (3.1) subject to the equation (3.2) and the constraints (3.11) provided that

\[
(3.14) \quad \delta J(u, h) = 0, \quad \forall h \in \mathcal{V}_{ad}.
\]

In this case, we call the Lagrange functional (3.1) subject to the equation (3.2) and the constraints (3.11) stationary at this \( u \in \mathcal{U}_{ad} \).

Obviously, \( u \in \mathcal{U}_{ad} \) is a critical point for (3.1) subject to (3.2) and (3.11) provided that (3.1) subject to (3.2) and (3.11) attaches the minimum (or maximum) value at this \( u \).

In classical mechanics, the equation (3.2) is the simplest form as \( x' = u \). \( \delta u = \delta(x') \) uniquely determine \( \delta x \). Conversely, \( \delta x \) also uniquely determine \( \delta u \). So we usually refer the notion of critical points to \( x \) instead of \( u \) in that case. For general cases of (3.2), \( \delta x \) not always uniquely determine \( \delta u \), which can be discovered from the dynamic equation (2.2) of the LC circuit example in Section 2.

The generalization of Hamilton’s principle in classical mechanics to the variational problem of (3.1) subject to (3.2) and (3.11) is as follows:
Definition 3.3. \((x, u)\) with \(x = x(\cdot, u)\) is called a generalized motion of the Lagrange functional (3.1) subject to the equation (3.2) and the constraints (3.11) provided that \(u \in U_{ad}\) is a critical point for (3.11) subject to (3.2) and (3.11).

This principle is might as well called the Hamilton’s type principle.

Theorem 3.1. \((x, u)\) with \(x = x(\cdot, u)\) is a generalized motion of the Lagrange functional (3.1) subject to the equation (3.2) and the constraints (3.11), if and only if \((x, u)\) satisfy both the constraints (3.11) and the generalized Euler-Lagrange equations

\[
\begin{align*}
\frac{\partial L}{\partial u}(x(t), u(t)) + \int_{t_0}^{t_1} \frac{\partial L}{\partial x}(x(s), u(s))T(s, t; u)ds \frac{\partial f}{\partial u}(x(t), u(t)) &= \mu^T B, \\
x'(t) &= f(x(t), u(t)), \quad t_0 \leq t \leq t_1,
\end{align*}
\]

for some \(\mu \in \mathbb{R}^l\).

Proof Let us denote the row vectors of the matrix \(B\) by

\[
b_i := (b_{i1}, b_{i2}, \cdots, b_{im}), \quad i = 1, 2, \cdots, l.
\]

Define that \(l\) functions as follows

\[
\tilde{b}_i(t) \equiv b_i^T, \quad t \in [t_0, t_1],
\]

and

\[
L(B) := \text{span}\{\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_l\},
\]

which is a complete subspace of \(L^2(t_0, t_1; \mathbb{R}^m)\). \(L^2(t_0, t_1; \mathbb{R}^m) = L(B) \oplus L(B)^\perp\) where \(L(B)^\perp\) is the orthogonal complement space of \(L(B)\) in \(L^2(t_0, t_1; \mathbb{R}^m)\). It is well known that \(C([t_0, t_1], \mathbb{R}^m)\) is imbedded in \(L^2(t_0, t_1; \mathbb{R}^m)\) continuously and densely. Similarly, \(V_{ad}\) is also imbedded in \(L(B)^\perp\) continuously and densely. Hence, (3.14) yields that (3.15). \(\square\)

Corollary 3.1. \((x, u)\) with \(x = x(\cdot, u)\) is a generalized motion of (3.1) subject to (3.2) if and only if \((x, u)\) satisfies the generalized Euler-Lagrange equations

\[
\begin{align*}
\frac{\partial L}{\partial u}(x(t), u(t)) + \int_{t_0}^{t_1} \frac{\partial L}{\partial x}(x(s), u(s))T(s, t; u)ds \frac{\partial f}{\partial u}(x(t), u(t)) &= 0, \\
x'(t) &= f(x(t), u(t)), \quad t_0 \leq t \leq t_1.
\end{align*}
\]

Example 3.1 In classical mechanics, \(f(x, u) = u\) yields that \(\frac{\partial f}{\partial u}(t) \equiv T(s, t; u) \equiv I_n\). If the initial state \(x(t_0) = x_0 \in \mathbb{R}^n\) is fixed while the terminal state \(x(t_1)\) is free, then the generalized Euler-Lagrange equation (3.19) reduces to

\[
\frac{\partial L}{\partial x'}(x(t), x'(t)) + \int_{t_0}^{t_1} \frac{\partial L}{\partial x}(x(s), x'(s)) ds = 0,
\]

as one necessary and sufficient condition for the motion \(x \in C^1([t_0, t_1], \mathbb{R}^n)\).
If the terminal state $x(t_1) = x_1 \in \mathbb{R}^n$ is also fixed, which can be reformulated as the constraints (3.11) with $B = I_n$ and $\alpha = -\frac{\omega}{\omega T}$. The trajectory $x$ with $x(t_0) = x_0$ and $x(t_1) = x_1$ is one motion, if and only if $x$ satisfy the equation (3.15), which reduces to

$$(3.21) \quad \frac{\partial L}{\partial x'}(x(t), x'(t)) + \int_{t}^{t_1} \frac{\partial L}{\partial x}(x(s), x'(s)) \, ds = \mu,$$

for some $\mu \in \mathbb{R}^n$.

Differentiating with respect to $t$, both (3.20) and (3.21) yields the classical one

$$(3.22) \quad \frac{d}{dt} \left[ \frac{\partial L}{\partial x'}(x(t), x'(t)) \right] - \frac{\partial L}{\partial x}(x(t), x'(t)) = 0. $$

If some components of the terminal state are fixed while the others are free, then the generalized Euler-Lagrange equation (3.15) is better than the Euler-Lagrange equation (3.22) just as the case without terminal state constraints.

**Example 3.2** For the LC example of (2.1)-(2.2), let $x := (q_1, q_2)^T$, $u := (i_3, i_5, i_6)^T$, $L(x, u) = \frac{1}{2} L_3 i^2_3 + \frac{1}{2} L_4 (i_3 - i_5 - i_6)^2 + \frac{1}{2} L_5 i^2_5 + \frac{1}{2} L_6 i^2_6 - \frac{1}{2c_1} q^2_1 - \frac{1}{2c_2} q^2_2$ and

$$f(x, u) = A \begin{pmatrix} i_3 \\ i_5 \\ i_6 \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} i_3 \\ i_5 \\ i_6 \end{pmatrix}.$$ 

Hence $T(t, s; u) \equiv I_2$. The equation (3.19) is

$$\begin{cases} (L_3 + L_4) i_3(t) - L_4 i_5(t) - L_4 i_6(t) - \int_{t}^{t_1} \frac{q_1(s)}{c_1} \, ds = 0, \\
- L_4 i_3(t) + (L_4 + L_5) i_5(t) + L_4 i_6(t) - \int_{t}^{t_1} \frac{q_2(s)}{c_2} \, ds = 0, \\
- L_4 i_3(t) + L_4 i_5(t) + (L_4 + L_5) i_6(t) - \int_{t}^{t_1} \frac{q_2(s)}{c_2} \, ds = 0; \\
q'_1 = i_3, \\
q'_2 = i_5 + i_6. \end{cases}$$

(3.23)

For the LC example of (2.1)-(2.2) with the terminal state $(q_1(t_1), q_2(t_1))$ fixed, can be reformulated as the constraints (3.11) with $B = A$, the equation (3.15) is

$$\begin{cases} (L_3 + L_4) i_3(t) - L_4 i_5(t) - L_4 i_6(t) - \int_{t}^{t_1} \frac{q_1(s)}{c_1} \, ds = \mu_1, \\
- L_4 i_3(t) + (L_4 + L_5) i_5(t) + L_4 i_6(t) - \int_{t}^{t_1} \frac{q_1(s)}{c_1} \, ds - \int_{t}^{t_1} \frac{q_2(s)}{c_2} \, ds = \mu_2, \\
- L_4 i_3(t) + L_4 i_5(t) + (L_4 + L_5) i_6(t) - \int_{t}^{t_1} \frac{q_1(s)}{c_1} \, ds - \int_{t}^{t_1} \frac{q_2(s)}{c_2} \, ds = \mu_2; \\
q'_1 = i_3, \\
q'_2 = i_5 + i_6. \end{cases}$$

(3.24)

for some $\mu_1, \mu_2 \in \mathbb{R}$.

For the LC example of (2.1)-(2.3), the constraint (2.3) can be reformulated as the
all (3.23), (3.24) and (3.25) yields the same equations

\begin{equation}
\begin{cases}
(L_3 + L_4)i_3(t) - L_4i_5(t) - L_4i_6(t) - \int_{t_1}^{t_1} \frac{q_1(s)}{c_1} \, ds = \mu_1, \\
- L_4i_3(t) + (L_4 + L_5)i_5(t) + L_4i_6(t) - \int_{t_1}^{t_1} \frac{q_2(s)}{c_2} \, ds = \mu_2, \\
- L_4i_3(t) + L_4i_5(t) + (L_4 + L_5)i_6(t) - \int_{t_1}^{t_1} \frac{q_3(s)}{c_3} \, ds = \mu_3;
\end{cases}
\end{equation}

(3.25)

for some \( \mu_1, \mu_2, \mu_3 \in \mathbb{R} \).

Since the matrix \( M \) defined by (2.8) is positively definite, by letting \( i_4 = i_3 - i_5 - i_6 \), all (3.23), (3.24) and (3.25) yields the same equations

\begin{equation}
\begin{cases}
L_3i_3'(t) + L_4i_4'(t) + \frac{q_1(t)}{c_1} = 0, \\
- L_4i_4'(t) + L_5i_5'(t) + \frac{q_2(t)}{c_2} = 0, \\
- L_4i_5'(t) + L_6i_6(t) + \frac{q_3(t)}{c_3} = 0; \\
i_4' = i_3, \\
i_5' = i_5 + i_6 \\
i_4 = i_3 - i_5 - i_6,
\end{cases}
\end{equation}

(3.26)

which is just the generalized Euler-Lagrange equation (3.40)-(3.41) in [8].

**Example 3.3** For the electromechanical system in [8] (see Example 7),

\[ J(u) = \int_{t_0}^{t_1} \left\{ \frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 + \frac{1}{2}ml^2\omega^2 - \frac{q^2}{2C(\theta)} + mgl \cos(\theta) \right\} \, dt \]

subject to

\[ \dot{q} = i_1 + i_2, \quad \dot{\theta} = \omega. \]

where \( x := (q, \theta)^T, \ u := (i_1, i_2, \omega)^T. \) Then \( L(x, u) = \frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 + \frac{1}{2}ml^2\omega^2 - \frac{q^2}{2C(\theta)} + mgl \cos(\theta) \) and

\[ f(x, u) = A \begin{pmatrix} i_1 \\ i_2 \\ \omega \end{pmatrix} := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ \omega \end{pmatrix}. \]

If without additional constraints, then the equation (3.19) is

\begin{equation}
\begin{cases}
L_1i_1(t) - \int_{t_1}^{t_1} \frac{q_1(s)}{C(\theta)} \, ds = 0, \\
L_2i_2(t) - \int_{t_1}^{t_1} \frac{q_2(s)}{C(\theta)} \, ds = 0, \\
ml^2\omega(t) - \int_{t_1}^{t_1} [mgl \sin(\theta) - \frac{q^2(s)}{2C^2(\theta)}C''(\theta)] \, ds = 0; \\
q' = i_1 + i_2, \\
\theta' = \omega.
\end{cases}
\end{equation}

(3.27)
If the terminal point is fixed, which can be reformulated as the constraints \((3.11)\) with \(B = A\), then the equation \((3.15)\) is
\[
\begin{align*}
L_1 i_1(t) - \int_t^{t_1} \frac{q(s)}{C(\theta)} \, ds &= \mu_1, \\
L_2 i_2(t) - \int_t^{t_1} \frac{q(s)}{C(\theta)} \, ds &= \mu_1, \\
m l^2 \omega(t) - \int_t^{t_1} \left[ mgl \sin(\theta) - \frac{q^2(s)}{2C^2(\theta)} C''(\theta) \right] \, ds &= \mu_2;
\end{align*}
\]
for some \(\mu_1, \mu_2 \in \mathbb{R}\).

If with the integral constraints
\[
\int_{t_0}^{t_1} i_1 \, dt = \lambda_1, \quad \int_{t_0}^{t_1} i_2 \, dt = \lambda_2, \quad \int_{t_0}^{t_1} \omega \, dt = \lambda_3,
\]
which can be reformulated as the constraints \((3.11)\) with \(B = I_3\), and can guarantee the terminal point fixed (similar to Remark 2.1), then the equation \((3.15)\) is
\[
\begin{align*}
L_1 i_1(t) - \int_t^{t_1} \frac{q(s)}{C(\theta)} \, ds &= \mu_1, \\
L_2 i_2(t) - \int_t^{t_1} \frac{q(s)}{C(\theta)} \, ds &= \mu_2, \\
m l^2 \omega(t) - \int_t^{t_1} \left[ mgl \sin(\theta) - \frac{q^2(s)}{2C^2(\theta)} C''(\theta) \right] \, ds &= \mu_3;
\end{align*}
\]
for some \(\mu_1, \mu_2, \mu_3 \in \mathbb{R}\).

Differentiating the three equations \((3.27)\), \((3.28)\) and \((3.30)\) yields the same generalized Euler-Lagrange equation \((5.7)\) in [8].

### 3.2 The second case with additional constraints

Consider the Lagrange functional \((3.1)\) subject to the dynamic equation \((3.2)\) and some additional constraints
\[
\int_{t_0}^{t_1} g(x(t), u(t)) \, dt = 0,
\]
where \(g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l\) with \(l \in \mathbb{N}^+\) is continuously differentiable.

Let \(C([t_0, t_1], \mathbb{R}^m)\) be the Banach space equipped with the usual maximum norm \(\| \cdot \|_C\). Denoted by \(u_k \to u\) in \(L^2(t_0, t_1; \mathbb{R}^m)\) the weak convergence of \(u_k\) to \(u\) in \(L^2(t_0, t_1; \mathbb{R}^m)\).

**Definition 3.4.** For arbitrary given \(x_0 \in \mathbb{R}^n\), the admissible set at \(x_0\) is defined as
\[
U_{ad}(x_0) := \{ u \in C([t_0, t_1], \mathbb{R}^m) | \text{ the solution } x(\cdot, u) \text{ to } (3.2) \text{ together with } u \text{ satisfy the constraint } (3.31) \}.
\]
Let $B_1$ be the unit ball in $C([t_0, t_1], \mathbb{R}^m)$, which is a convex set closed under the strong topology of $L^2(t_0, t_1; \mathbb{R}^m)$. It follows from Mazur theorem that $B_1$ is also closed under the weak topology in $L^2(t_0, t_1; \mathbb{R}^m)$. Meanwhile, $B_1$ is weakly precompact in $L^2(t_0, t_1; \mathbb{R}^m)$.

Hence, for any sequence $\{u_k\}_{k=1}^{+\infty} \subset U_{ad}(x_0)$, $\{\frac{u_k-u}{||u_k-u||_C}\}_{k=1}^{+\infty} \subset B_1$ admits a subsequence weakly convergent to some $h \in B_1$. In this way, we can define the notion of allowed variation as follows:

**Definition 3.5.** For any given $x_0 \in \mathbb{R}^n$ and $u \in U_{ad}(x_0)$, $h \in C([t_0, t_1], \mathbb{R}^m)$ is called an allowed variation along $u$ at $x_0$ provided that there exists $\{u_k\}_{k=1}^{+\infty} \subset U_{ad}(x_0)$ such that

\[
\begin{align*}
\left\{ & u_k \rightarrow u \quad \text{in} \quad C([t_0, t_1], \mathbb{R}^m); \\
& \frac{u_k-u}{||u_k-u||_C} \rightharpoonup h \quad \text{in} \quad L^2(t_0, t_1; \mathbb{R}^m).
\end{align*}
\]

The set of all allowed variations along $u$ at $x_0$ is denoted by $\mathcal{V}_{ad}(x_0, u)$.

**Proposition 3.1.** For any given $x_0 \in \mathbb{R}^n$ and $u \in U_{ad}(x_0)$,

\[
\int_{t_0}^{t_1} \left[ \int_t^{t_1} \frac{\partial g}{\partial x}(s)T(s, t; u) ds \frac{\partial f}{\partial u}(t) + \frac{\partial g}{\partial u}(t) \right] h(t) dt = 0, \quad \forall h \in \mathcal{V}_{ad}(x_0, u).
\]

**Proof**

Let $\{u_k\}_{k=1}^{+\infty} \subset U_{ad}(x_0)$ satisfy (3.33), and define $\varepsilon_k := |u_k - u|_C$, $h_k := \frac{u_k-u}{||u_k-u||_C}$, then

\[
\begin{align*}
0 &= \int_{t_0}^{t_1} g(x, u_k) dt - \int_{t_0}^{t_1} g(x, u) dt \\
&= \varepsilon_k \int_{t_0}^{t_1} \frac{\partial g}{\partial x}(t) \int_{t_0}^{t} T(s, t; u) \frac{\partial f}{\partial u}(s) h_k(s) ds dt + \varepsilon \int_{t_0}^{t_1} \frac{\partial g}{\partial u}(t) h_k(t) dt + o(\varepsilon_k) \\
&= \varepsilon_k \int_{t_0}^{t_1} \int_{t}^{t_1} \frac{\partial g}{\partial x}(s)T(s, t; u) ds \frac{\partial f}{\partial u}(t) h_k(t) dt + \varepsilon \int_{t_0}^{t_1} \frac{\partial g}{\partial u}(t) h_k(t) dt + o(\varepsilon_k) \\
&= \varepsilon_k \int_{t_0}^{t_1} \int_{t}^{t_1} \frac{\partial g}{\partial x}(s)T(s, t; u) ds \frac{\partial f}{\partial u}(t) h_k(t) dt + \varepsilon \int_{t_0}^{t_1} \frac{\partial g}{\partial u}(t) h_k(t) dt + o(\varepsilon_k) \\
&= \varepsilon_k \int_{t_0}^{t_1} \int_{t}^{t_1} \frac{\partial g}{\partial x}(s)T(s, t; u) ds \frac{\partial f}{\partial u}(t) h_k(t) dt + \varepsilon \int_{t_0}^{t_1} \frac{\partial g}{\partial u}(t) h_k(t) dt + o(\varepsilon_k).
\end{align*}
\]

Hence we have (3.34). \qed

Analogous to (3.35), let $\{u_k\}_{k=1}^{+\infty} \subset U_{ad}(x_0)$ satisfy (3.33), we have

\[
\int_{t_0}^{t_1} L(x, u_k) dt - \int_{t_0}^{t_1} L(x, u) dt = \varepsilon_k \int_{t_0}^{t_1} \int_{t}^{t_1} \frac{\partial L}{\partial x}(s)T(s, t; u) ds \frac{\partial f}{\partial u}(t) + \frac{\partial L}{\partial u}(t) h(t) dt + o(\varepsilon_k),
\]

for any $h \in \mathcal{V}_{ad}(x_0, u)$.

Thus, we define that

**Definition 3.6.** $u \in U_{ad}(x_0)$ is called a critical point for the Lagrange functional (3.1) subject to the equation (3.2) and the constraints (3.31) provided that

\[
\delta J(u; h) = 0, \quad \forall h \in \mathcal{V}_{ad}(x_0, u).
\]

In this case, we call (3.1) subject to (3.2) and (3.31) is stationary at this $u \in U_{ad}(x_0)$. 

\[12\]
Definition 3.7. \((x, u)\) with \(x = x(\cdot, u)\) is called a generalized motion of the Lagrange functional \((3.1)\) subject to the equation \((3.2)\) and the constraints \((3.3)\) provided that \(u \in U_{ad}(x_0)\) is a critical point for \((3.1)\) subject to \((3.2)\) and \((3.3)\).

Theorem 3.2. \((x, u)\) with \(x = x(\cdot, u)\) is a generalized motion of the Lagrange functional \((3.1)\) subject to the equation \((3.2)\) and the constraints \((3.3)\) provided that \((x, u)\) satisfy \((3.3)\) and the generalized Euler-Lagrange equations

\[
\begin{align*}
\frac{\partial L}{\partial u}(t) - \mu^T \frac{\partial g}{\partial x}(s) T(s, t; u) ds \frac{\partial L}{\partial u}(t) &= 0, \\
x'(t) &= f(x(t), u(t)), \quad t_0 \leq t \leq t_1,
\end{align*}
\]

for some \(\mu \in \mathbb{R}^l\).

Proof Combining the generalized Euler-Lagrange equation \((3.38)\) and \((3.34)\) yields \((3.37)\). The proof is completed. \(\square\)

4 The canonical Hamiltonian systems

4.1 The case without additional constraints

(AIIIa) The equation

\[
0 = p^T \frac{\partial f}{\partial u}(x, u) - \frac{\partial L}{\partial u}(x, u),
\]

admits one smooth solution

\[
u = \varphi(x, p), \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Define the pseudo Hamiltonian

\[
\mathcal{H}(x, p, u) := p^T f(x, u) - L(x, u), \quad \forall (x, p, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m,
\]

and define under the assumption (AIIIa) the Hamiltonian

\[
H(x, p) := \mathcal{H}(x, p, \varphi(x, p)), \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n,
\]

which is called the generalized Legendre transformation of \(\mathcal{H}\).

Remark 4.1. The Legendre transformation of \(\mathcal{H}\) is defined as

\[
H(x, p) := \max_{u \in \mathbb{R}^m} \mathcal{H}(x, p, u), \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

The definition of \((4.4)\) is adapted from (A1) in [12] (p.39), which indicates there may exist different Hamiltonian descriptions for the same equilibrium in nonlinear LC circuits.
Suppose that \((x, u)\) is one solution to the generalized Euler-Lagrange equation \((3.19)\). Let

\[
(4.5) \quad p(t)^T := -\int_{t}^{t_1} \frac{\partial L}{\partial x}(x(s), u(s))T(s, t; u) \, ds, \quad t \in [t_0, t_1],
\]
then \((3.19)\) can be recast as

\[
(4.6) \quad \begin{cases}
 p(t)^T \frac{\partial L}{\partial u}(x(t), u(t)) - \frac{\partial L}{\partial x}(x(t), u(t)) = 0, \\
 x'(t) = f(x(t), u(t)), \quad t \in [t_0, t_1].
\end{cases}
\]

It follows from the first equality of \((4.6)\) and the definitions of \(H\) and \(\mathcal{H}\) that

\[
(4.7) \quad \begin{cases}
 \frac{\partial \mathcal{H}}{\partial p}(x(t), p(t), u(t)) \equiv \frac{\partial H}{\partial x}(x(t), p(t)), \\
 \frac{\partial \mathcal{H}}{\partial x}(x(t), p(t), u(t)) \equiv \frac{\partial H}{\partial x}(x(t), p(t)), \quad t \in [t_0, t_1].
\end{cases}
\]

Meanwhile, the second equality of \((4.6)\) yields that

\[
(4.8) \quad x'(t) = f(x(t), u(t)) = \frac{\partial \mathcal{H}}{\partial p}(x(t), p(t), u(t)),
\]
and differentiating \((4.5)\) yields that

\[
(4.9) \quad [p'(t)]^T = \frac{\partial \mathcal{H}}{\partial x}(x(t), p(t), u(t)) + \int_{t}^{t_1} \frac{\partial L}{\partial x}(x(s), u(s))T(s, t; u) \, ds \frac{\partial f}{\partial x}(x(t), u(t)) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), p(t), u(t)).
\]

Hence, we have

**Theorem 4.1.** Let the assumption (AIIIa) holds. Suppose that \((x, u)\) is one solution to the generalized Euler-Lagrange equation \((3.19)\), then \((x, p)\) given by \((4.5)\) is one solution to the canonical Hamiltonian system

\[
(4.10) \quad \begin{cases}
 x'(t) = \nabla_p H(x, p), \\
 p'(t) = -\nabla_x H(x, p), \\
 x(t_0) = x_0, \\
 p(t_1) = 0.
\end{cases}
\]

Conversely, if \((x, p)\) is one solution to \((4.10)\), then \((x, u)\) given by \((4.2)\) is also one solution to the generalized Euler-Lagrange equation \((3.19)\).

### 4.2 The cases with additional special constraints

In \((3.31)\), let us assume that there exists some matrix \(Q \in \mathbb{R}^{l \times n}\) and \(\beta \in \mathbb{R}^l\) such that

\[
(4.11) \quad g(x, u) = Qf(x, u) + \beta, \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m.
\]

**Remark 4.2.** The terminal state constraints with \(x(t_1) = x_1\) fixed, can be reformulated as \((4.11)\) with \(Q = I_n\) and \(\beta = -\frac{x_1 - x_0}{t_1 - t_0}\).

If there exist \(A \in \mathbb{R}^{n \times m}\), \(B \in \mathbb{R}^{l \times m}\) and \(\alpha \in \mathbb{R}^l\) such that \(f(x, u) = Au\) and \(g(x, u) = Bu + \alpha\), then \((4.11)\) is equivalent to

\[
(4.12) \quad \text{rank}(A) = \text{rank} \begin{pmatrix} A \\ B \end{pmatrix}.
\]
Theorem 4.2. Assume that both (4.11) and the assumption (AIIIa) holds. Suppose that 
(x, u) is one solution to the generalized Euler-Lagrange equations (3.38), then (x, p) given 
by (4.13) is one solution to the Hamiltonian system

\[ \begin{cases} 
  x'(t) = \nabla_p H(x, p), & x(t_0) = x_0, \\
  p'(t) = -\nabla_x H(x, p), & p(t_1) = Q^T \mu, 
\end{cases} \tag{4.15} \]

where the Hamiltonian \( H \) is defined in (4.4).

Conversely, if \( (x, p) \) is one solution to (4.15), then \( (x, u) \) given by (4.2) is also one solution to the generalized Euler-Lagrange equation (3.38) for the parameters \( \mu \in \mathbb{R}^l \).

4.3 The general cases

Define the pseudo Hamiltonian

\[ \mathcal{H}(x, p, u; \mu) := p^T f(x, u) - L(x, u) + \mu^T g(x, u), \quad \forall (x, p, u, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^l. \tag{4.16} \]

(AIIIb) The equation

\[ 0 = p^T \frac{\partial f}{\partial u}(x, u) - \frac{\partial L}{\partial u}(x, u) + \mu^T \frac{\partial g}{\partial u}(x, u), \tag{4.17} \]

admits one smooth solution

\[ u = \varphi(x, p; \mu), \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{4.18} \]

for some parameters \( \mu = (\mu_1, \cdots, \mu_l)^T \in \mathbb{R}^l \).

Under the assumption (AIIIb), let us define the Hamiltonian

\[ H(x, p; \mu) := \mathcal{H}(x, p, \varphi(x, p; \mu); \mu), \quad \forall (x, p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.19} \]

Suppose that \( (x, u) \) is one solution to the generalized Euler-Lagrange equation (3.38) for these parameters \( \mu \in \mathbb{R}^l \). Let

\[ p(t)^T := -\int_{t_0}^{t_1} \left[ \frac{\partial L}{\partial x}(x(s), u(s)) - \mu^T \frac{\partial g}{\partial x}(x(s), u(s)) \right] T(s, t; u) \, ds, \tag{4.20} \]

then the generalized Euler-Lagrange equation (3.38) can be recast as

\[ \begin{cases} 
  p(t)^T \frac{\partial f}{\partial u}(x(t), u(t)) - \frac{\partial L}{\partial u}(x(t), u(t)) + \mu^T \frac{\partial g}{\partial u}(x(t), u(t)) = 0, \\
  x'(t) = f(x(t), u(t)), \quad t \in [t_0, t_1]. 
\end{cases} \tag{4.21} \]

Similar to the proof of Theorem 4.1, we have
Theorem 4.3. Let the assumptions (AIIIb) holds. Suppose that \((x, u)\) is one solution to the generalized Euler-Lagrange equation \((3.38)\) for the parameters \(\mu \in \mathbb{R}^l\), then \((x, p)\) given by \((4.21)\) is one solution to the Hamiltonian system

\[
\begin{align*}
(4.22) \quad & \begin{cases} 
    x'(t) = \nabla_p H(x, p; \mu), & x(t_0) = x_0, \\
    p'(t) = -\nabla_x H(x, p; \mu), & p(t_1) = 0.
\end{cases}
\end{align*}
\]

Conversely, if \((x, p)\) is one solution to \((4.22)\) for the parameters \(\mu \in \mathbb{R}^l\), then \((x, u)\) given by \((4.18)\) is also one solution to the generalized Euler-Lagrange equation \((3.38)\).

Remark 4.3. From Theorem 4.1, Theorem 4.2 and Theorem 4.3, we can find out that the canonical Hamiltonian is not enough to describe the energy function and the dynamic equation when the constraints \((3.31)\) become more complex.

Example 4.1 (Continued from Example 3.1) The Hamiltonian is as usual defined by \(H(x, p) = \max_{u \in \mathbb{R}^n} \{p^T u - L(x, u)\}\), and it follows from Theorem 4.1, that the dynamic system without additional constraints is described by the two-point boundary value problem of the canonical Hamiltonian system.

Two different description of the Hamiltonian system. The first is the classical approach. Applying Theorem 4.2 yields

\[
(4.23) \quad \begin{cases} 
    x'(t) = \nabla_p \tilde{H}(x, p; \mu), & x(t_0) = x_0, \\
    p'(t) = -\nabla_x \tilde{H}(x, p; \mu), & p(t_1) = p_1,
\end{cases}
\]

where the terminal costate \(p_1 \in \mathbb{R}^n\) are the parameters such that the terminal state constraints \(x(t_1) = x_1\) satisfied. The second is applying Theorem 4.3.

\[
(4.24) \quad \begin{cases} 
    x'(t) = \nabla_p \check{H}(x, p; \mu), & x(t_0) = x_0, \\
    p'(t) = -\nabla_x \check{H}(x, p; \mu), & p(t_1) = 0,
\end{cases}
\]

where the Hamiltonian \(\check{H}(x, p; \mu) = \max_{u \in \mathbb{R}^n} \{p^T u - L(x, u) + \mu^T u\}\), and \(\mu \in \mathbb{R}^n\) are the parameters such that the terminal state constraints \(x(t_1) = x_1\) satisfied.

In fact, \(H\) is the energy function and \(\check{H}\) is the energy function with constraints.

Example 4.2 (Continued from Example 3.2) The Hamiltonian is

\[
H(x, p) = \max_{(i_3, i_5, i_6) \in \mathbb{R}^3} \{p_1 i_3 + p_2 (i_5 + i_6) - \frac{1}{2} L_3 i_3^2 - \frac{1}{2} L_4 (i_3 - i_5 - i_6)^2 - \frac{1}{2} L_5 i_5^2 - \frac{1}{2} L_6 i_6^2 + \frac{1}{2c_1} q_1^2 + \frac{1}{2c_2} q_2^2\}
\]

\[
= \frac{1}{2} (p_1, p_2, p_2) M^{-1} \begin{pmatrix} 
    p_1 \\
    p_2 \\
    p_2
\end{pmatrix} + \frac{1}{2c_1} q_1^2 + \frac{1}{2c_2} q_2^2,
\]

where \(M\) is the positively definite matrix defined by \((2.8)\). Then applying Theorem 4.1 and Theorem 4.2 to this Hamiltonian, we obtain the canonical Hamiltonian systems description of this model without additional constraints and with the terminal state constraints, respectively.
The Hamiltonian is
\[
H(x, p; \mu) = \max_{(i_3, i_6) \in \mathbb{R}^2} \{(p_1 + \mu_1)i_3 + (p_2 + \mu_2)i_5 + (p_2 + \mu_3)i_6 - \frac{1}{2}L_3i_3^2 - \frac{1}{2}L_4(i_3 - i_5 - i_6)^2 - \frac{1}{2}L_5i_5^2 - \frac{1}{2}L_6i_6^2 + \frac{1}{2c_1}q_1^2 + \frac{1}{2c_2}q_2^2 \}
= \frac{1}{2}(p_1 + \mu_1, p_2 + \mu_2, p_2 + \mu_3)M^{-1} \left( \begin{array}{c} p_1 + \mu_1 \\ p_2 + \mu_2 \\ p_2 + \mu_3 \end{array} \right) + \frac{1}{2c_1}q_1^2 + \frac{1}{2c_2}q_2^2,
\]
where \( M \) is the positively definite matrix defined by (2.8). Then applying Theorem 4.3 to this Hamiltonian, we obtain the canonical Hamiltonian systems description of this model. Intuitively, the original energy function is not enough in general to describe the dynamic system with constraints since the constraints involves three parameters \( \mu_1, \mu_2, \mu_3 \) while the dimension of the costate is only 2.

**Example 4.3** (Continued from Example 3.3) The Hamiltonian is
\[
H(x, p) = \max_{(i_1, i_2, \omega) \in \mathbb{R}^3} \{(p_1 + \mu_1)i_1 + (p_2 + \mu_2)i_2 + (p_2 + \mu_3)\omega - \frac{1}{2}L_1i_1^2 - \frac{1}{2}L_2i_2^2 - \frac{1}{2}ml^2\omega^2 + \frac{q^2}{2c(\theta)} - mg\cos(\theta) \}
= \frac{1}{2}(\frac{1}{L_1} + \frac{1}{L_2})\omega^2 + \frac{1}{2ml^2}p_2^2 + \frac{q^2}{2c(\theta)} - mg\cos(\theta).
\]
Then applying Theorem 4.1 and Theorem 4.2 to this Hamiltonian, we obtain the canonical Hamiltonian systems description of this model without additional constraints and with the terminal state constraints, respectively.

The Hamiltonian is
\[
H(x, p) = \max_{(i_1, i_2, \omega) \in \mathbb{R}^3} \{(p_1 + \mu_1)i_1 + (p_1 + \mu_2)i_2 + (p_2 + \mu_3)\omega - \frac{1}{2}L_1i_1^2 - \frac{1}{2}L_2i_2^2 - \frac{1}{2}ml^2\omega^2 + \frac{q^2}{2c(\theta)} - mg\cos(\theta) \}
= \frac{1}{2L_1}(p_1 + \mu_1)^2 + \frac{1}{2L_2}(p_1 + \mu_2)^2 + \frac{1}{2ml^2}(p_2 + \mu_3)^2 + \frac{q^2}{2c(\theta)} - mg\cos(\theta).
\]
Then applying Theorem 4.3 to this Hamiltonian, we obtain the canonical Hamiltonian systems description of this model. Similarly, the original energy function is also not enough in general to describe the dynamic system with these constraints (3.29).

## 5 Appendix

Let \( L^2(t_0, t_1; \mathbb{R}) \) be the Hilbert space equipped with the inner product
\[
\langle u, v \rangle := \frac{2}{t_1 - t_0} \int_{t_0}^{t_1} u(t)v(t)dt, \quad \forall u, v \in L^2(t_0, t_1; \mathbb{R}),
\]
and define \( e_0, e_n, \tilde{e}_n \in L^2(t_0, t_1; \mathbb{R}) \) for \( n \in \mathbb{N}^+ \) as follows: \( e_0(t) \equiv \frac{1}{\sqrt{2}} \) and
\[
e_n(t) := \cos \frac{2n\pi(t-t_0)}{t_1 - t_0}, \quad \tilde{e}_n(t) := \sin \frac{2n\pi(t-t_0)}{t_1 - t_0}, \quad t \in [t_0, t_1],
\]
then \( \{e_0, e_1, \tilde{e}_1, e_2, \tilde{e}_2, \cdots, e_n, \tilde{e}_n, \cdots \} \) is an orthonormal basis of \( L^2(t_0, t_1; \mathbb{R}) \).
The constraint (2.3) yields that

\begin{equation}
(5.3) \quad i_k = \frac{\sqrt{2}\lambda_k}{t_1 - t_0} e_0 + \sum_{n=1}^{+\infty} (a_{k,n} e_n + b_{k,n} \tilde{e}_n),
\end{equation}

with \(\sum_{n=1}^{+\infty} (a_{k,n}^2 + b_{k,n}^2) < +\infty\) for \(k = 3, 5, 6\).

Through direct calculations, we have

**Lemma 5.1.** The Lagrange functional (2.1) subject to (2.2) and (2.3) can be represented as follows:

\begin{equation}
(5.4) \quad J(i_3, i_5, i_6) = Q + L + N,
\end{equation}

where

\begin{equation}
(5.5) \quad Q = \left\{ \frac{(t_1-t_0)L_k}{4} \sum_{n=1}^{+\infty} (a_{3,n} - a_{5,n} - a_{6,n})^2 
+ \frac{(t_1-t_0)L_3}{4} \sum_{n=1}^{+\infty} a_{3,n}^2 
+ \frac{(t_1-t_0)L_5}{4} \sum_{n=1}^{+\infty} a_{5,n}^2 
+ \frac{(t_1-t_0)L_6}{4} \sum_{n=1}^{+\infty} a_{6,n}^2 
- \frac{(t_1-t_0)^3}{4\pi^2 c^2} \sum_{n=1}^{+\infty} \left( \frac{a_{5,n} + a_{6,n}}{n} \right)^2 \right\}
\end{equation}

\begin{equation}
(5.6) \quad L = \left[ -\frac{(t_1-t_0)^2}{4\pi^2 c^2} \left( 2q_1(t_0) + \lambda_3 \right) \sum_{n=1}^{+\infty} \frac{b_{3,n}}{n} 
- \frac{(t_1-t_0)^2}{4\pi^2 c^2} \left( 2q_2(t_0) + \lambda_5 + \lambda_6 \right) \sum_{n=1}^{+\infty} \frac{b_{5,n} + b_{6,n}}{n} \right],
\end{equation}

\begin{equation}
(5.7) \quad N = \frac{1}{2(t_1-t_0)} \left[ L_3 \lambda_3^2 + L_5 \lambda_5^2 + L_6 \lambda_6^2 + L_4 (\lambda_3 - \lambda_5 - \lambda_6)^2 \right] 
- \frac{(t_1-t_0)}{6\pi^2 c^2} \left[ 3q_1(t_0)^2 + 3q_1(t_0)\lambda_3 + \lambda_3^2 \right] 
- \frac{(t_1-t_0)}{6\pi^2 c^2} \left[ 3q_2(t_0)^2 + 3q_2(t_0)(\lambda_5 + \lambda_6) + (\lambda_5 + \lambda_6)^2 \right].
\end{equation}

**Lemma 5.2.** Let \(\alpha > 0\) and \(\beta > 0\). Then

\begin{equation}
(5.8) \quad \alpha \sum_{n=1}^{+\infty} \frac{x_n^2}{n^2} + \beta \left( \sum_{n=1}^{+\infty} \frac{x_n}{n} \right)^2 \leq K \sum_{n=1}^{+\infty} \frac{x_n^2}{n},
\end{equation}

where \(K\) is the unique solution to the equation

\begin{equation}
(5.9) \quad \beta \sum_{n=1}^{+\infty} \frac{1}{Kn^2 - \alpha} = 1, \quad K > \alpha + \beta.
\end{equation}

The equality in (5.8) holds if and only if \(x_n = \frac{C}{Kn - \frac{\alpha}{n}}\) for arbitrary \(C \in \mathbb{R}\).

**Proof**

\[ K = \sup_{\sum_{n=1}^{+\infty} x_n^2 = 1} \left[ \alpha \sum_{n=1}^{+\infty} \frac{x_n^2}{n^2} + \beta \left( \sum_{n=1}^{+\infty} \frac{x_n}{n} \right)^2 \right] \geq \alpha + \beta. \]
For $l \in \mathbb{N}^+$, it can be shown by Lagrange multiplier rule that the problem:

\[
\begin{aligned}
\text{To maximize} & \quad \alpha \sum_{n=1}^{l} \frac{x_n^2}{n^2} + \beta (\sum_{n=1}^{l} \frac{x_n}{n})^2 \\
\text{subject to} & \quad \sum_{n=1}^{l} x_n^2 = 1
\end{aligned}
\]

has only two solutions

\[x_{l,n} = \pm \left[ \sum_{n=1}^{l} \frac{1}{(k_l n - \frac{\alpha}{n})^2} \right]^{-\frac{1}{2}} \frac{1}{k_l n - \frac{\alpha}{n}}, \quad n = 1, 2, \ldots, l,
\]

where

\[k_l = \max_{\sum_{n=1}^{l} x_n^2 = 1} \left[ \alpha \sum_{n=1}^{l} \frac{x_n^2}{n^2} + \beta (\sum_{n=1}^{l} \frac{x_n}{n})^2 \right],
\]

satisfies

\[\beta \sum_{n=1}^{l} \frac{1}{k_l n^2 - \alpha} = 1.
\]

Obviously, it follows from $\lim_{l \to +\infty} k_l = K$ that, the equation (5.9) and

\[x_n^* := \lim_{l \to +\infty} x_{l,n} = \pm \left[ \sum_{n=1}^{+\infty} \frac{1}{(Kn - \frac{\alpha}{n})^2} \right]^{-\frac{1}{2}} \frac{1}{Kn - \frac{\alpha}{n}}, \quad n \in \mathbb{N}^+,
\]

\[\alpha \sum_{n=1}^{+\infty} \frac{x_n^*^2}{n^2} + \beta (\sum_{n=1}^{+\infty} \frac{x_n^*}{n})^2 = \lim_{l \to +\infty} \left[ \alpha \sum_{n=1}^{l} \frac{x_{l,n}^2}{n^2} + \beta (\sum_{n=1}^{l} \frac{x_{l,n}}{n})^2 \right] = K,
\]

which implies the sufficiency.

The necessity can be shown directly by Ljusternik Theorem (the Lagrange multiplier rule in infinite dimensional space, see pp.290 in [15]).

Similarly, we have

**Lemma 5.3.** Let $\alpha > 0$ and $\beta > 0$. Then

\[\alpha \sum_{n=1}^{+\infty} \frac{(x_n + y_n)^2}{n^2} + \beta (\sum_{n=1}^{+\infty} \frac{x_n + y_n}{n})^2 \leq \tilde{K} \left( \sum_{n=1}^{+\infty} x_n^2 + \sum_{n=1}^{+\infty} y_n^2 \right),
\]

where $\tilde{K}$ is the unique solution to the equation

\[2 \beta \sum_{n=1}^{+\infty} \frac{1}{Kn^2 - 2\alpha} = 1, \quad \tilde{K} > 2(\alpha + \beta).
\]

The equality in (5.10) holds if and only if $x_n = y_n = \frac{C}{Kn - \frac{\alpha}{n}}$ for arbitrary $C \in \mathbb{R}$.
Proof of Proposition 2.1  Let $Q := Q_1 + Q_2$, where

\begin{align}
Q_1 &= \frac{(t_1 - t_0)L_4}{4} \sum_{n=1}^{+\infty} (a_{3,n} - a_{5,n} - a_{6,n})^2 \\
&+ \frac{(t_1 - t_0)L_3}{4} \sum_{n=1}^{+\infty} a_{2,n}^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_1} \sum_{n=1}^{+\infty} \left(\frac{a_{3,n}}{n}\right)^2 \\
&+ \frac{(t_1 - t_0)L_5}{4} \sum_{n=1}^{+\infty} a_{2,n}^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{a_{3,n}}{n}\right)^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{a_{5,n} + a_{6,n}}{n}\right)^2.
\end{align}

\begin{align}
Q_2 &= \frac{(t_1 - t_0)L_4}{4} \sum_{n=1}^{+\infty} (b_{3,n} - b_{5,n} - b_{6,n})^2 \\
&+ \frac{(t_1 - t_0)L_3}{4} \sum_{n=1}^{+\infty} b_{2,n}^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_1} \sum_{n=1}^{+\infty} \left(\frac{b_{3,n}}{n}\right)^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_1} \sum_{n=1}^{+\infty} \left(\frac{b_{5,n} + b_{6,n}}{n}\right)^2 \\
&+ \frac{(t_1 - t_0)L_5}{4} \sum_{n=1}^{+\infty} b_{2,n}^2 + \frac{(t_1 - t_0)^3}{16\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{b_{3,n} + b_{6,n}}{n}\right)^2 - \frac{(t_1 - t_0)^3}{16\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{b_{5,n} + b_{6,n}}{n}\right)^2.
\end{align}

(I) By Lemma 6.2 and 6.3,

\begin{align}
\frac{1}{t_1 - t_0} Q_2 &= L_4 \sum_{n=1}^{+\infty} (b_{3,n} - b_{5,n} - b_{6,n})^2 \\
&+ \left[ L_3 \sum_{n=1}^{+\infty} b_{3,n}^2 - \frac{(t_1 - t_0)^2}{2\pi^2 C_1} \sum_{n=1}^{+\infty} \left(\frac{b_{3,n}}{n}\right)^2 - \frac{(t_1 - t_0)^3}{2\pi^2 C_1} \sum_{n=1}^{+\infty} \left(\frac{b_{5,n} + b_{6,n}}{n}\right)^2 \right] \\
&+ \left[ L_5 \sum_{n=1}^{+\infty} b_{2,n}^2 + \frac{(t_1 - t_0)^3}{2\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{b_{3,n} + b_{6,n}}{n}\right)^2 - \frac{(t_1 - t_0)^3}{2\pi^2 C_2} \sum_{n=1}^{+\infty} \left(\frac{b_{5,n} + b_{6,n}}{n}\right)^2 \right] \\
&\geq L_4 \sum_{n=1}^{+\infty} (b_{3,n} - b_{5,n} - b_{6,n})^2 \\
&+ \left[ L_3 - K_1 \right] \sum_{n=1}^{+\infty} b_{3,n}^2 + \left[ L_5 - 2K_1 \right] \sum_{n=1}^{+\infty} b_{5,n}^2 + \left[ L_6 - 2K_1 \right] \sum_{n=1}^{+\infty} b_{6,n}^2
\end{align}

If $S_1$ is positively definite, then it follows from (5.14) that $Q_2$ is a coercive quadratic functional, which guarantees $Q_1$ is also coercive. Thus $Q$ is a coercive quadratic functional, which yields that the Lagrange functional (2.1) subject to (2.2) and (2.3) has a minimum value at the unique critical point $(i_3^*, i_5^*, i_6^*) \in C([t_0, t_1]; \mathbb{R}^3)$. Following the approach to indefinite linear quadratic optimal control problems in [14], we can prove any optimal control are continuous, which yields that $(i_3^*, i_5^*, i_6^*) \in C([t_0, t_1]; \mathbb{R}^3)$.

(II) Let

\begin{align}
\hat{i}_3 &= \sum_{n=1}^{+\infty} \hat{b}_{3,n} \hat{e}_n := h \sum_{n=1}^{+\infty} \frac{4\pi^2 C_1 n}{4\pi^2 C_1 n^2 - (t_1 - t_0)^2} \hat{e}_n, \\
\hat{i}_5 &= \sum_{n=1}^{+\infty} \hat{b}_{5,n} \hat{e}_n := h \sum_{n=1}^{+\infty} \frac{2\pi^2 C_2 n}{2\pi^2 C_2 n^2 - (t_1 - t_0)^2} \hat{e}_n, \\
\hat{i}_6 &= \sum_{n=1}^{+\infty} \hat{b}_{6,n} \hat{e}_n := h \sum_{n=1}^{+\infty} \frac{2\pi^2 C_2 n}{2\pi^2 C_2 n^2 - (t_1 - t_0)^2} \hat{e}_n, \quad h \in \mathbb{R}.
\end{align}

Lemma 6.2 and 6.3 yields that

\begin{align}
\frac{4}{t_1 - t_0} \left. Q \right|_{(\hat{i}_3, \hat{i}_5, \hat{i}_6)} &= h^2 \left\{ L_4 \sum_{n=1}^{+\infty} (\hat{b}_{3,n} - \hat{b}_{5,n} - \hat{b}_{6,n})^2 \\
&+ \left[ L_3 - K_1 \right] \sum_{n=1}^{+\infty} \hat{b}_{3,n}^2 + \left[ L_5 - K_1 \right] \sum_{n=1}^{+\infty} \hat{b}_{5,n}^2 + \left[ L_6 - K_1 \right] \sum_{n=1}^{+\infty} \hat{b}_{6,n}^2 \right\} \\
&\leq h^2 \left\{ L_4 \sum_{n=1}^{+\infty} (\hat{b}_{3,n} - \hat{b}_{5,n} - \hat{b}_{6,n})^2 \\
&+ \left[ L_3 - 2K_2 \right] \sum_{n=1}^{+\infty} \hat{b}_{3,n}^2 + \left[ L_5 - 2K_2 \right] \sum_{n=1}^{+\infty} \hat{b}_{5,n}^2 + \left[ L_6 - 2K_2 \right] \sum_{n=1}^{+\infty} \hat{b}_{6,n}^2 \right\} \\
&= h^2 \sum_{n=1}^{+\infty} \left( \hat{b}_{3,n}, \hat{b}_{5,n}, \hat{b}_{6,n} \right) S_2 \left( \hat{b}_{3,n}, \hat{b}_{5,n}, \hat{b}_{6,n} \right)^T < 0,
\end{align}

which implies \( \lim_{n \to \infty} J(\hat{i}_3, \hat{i}_5, \hat{i}_6) = -\infty \). So it follows from the density of $C([t_0, t_1]; \mathbb{R}^3)$ in $L^2(t_0, t_1; \mathbb{R}^3)$ that the Lagrange functional (2.1) subject to (2.2) and (2.3) has no minimum value.
(III) If \( C_1 = C_2 \), then \( \widetilde{K}(C_2) = 2K(C_1) \), \( K_1 = K_2 \) and \( S_1 = S_2 \). If \( S_2 \) has at least one negative characteristic root, then there exist at least one unit vector \((x, y, z)\) such that
\[
(x, y, z)S_2(x, y, z)^T < 0.
\]

Let
\[
\begin{aligned}
\widehat{i}_3 &= \sum_{n=1}^{+\infty} b_{3,n} \widehat{c}_n := h x \sum_{n=1}^{+\infty} \frac{4 \pi^2 C_1 n}{4 \pi^2 C_1 n^2 - (t_1 - t_0)^2} \widehat{c}_n, \\
\widehat{i}_5 &= \sum_{n=1}^{+\infty} b_{5,n} \widehat{c}_n := h y \sum_{n=1}^{+\infty} \frac{4 \pi^2 C_1 n}{4 \pi^2 C_1 n^2 - (t_1 - t_0)^2} \widehat{c}_n, \\
\widehat{i}_6 &= \sum_{n=1}^{+\infty} b_{6,n} \widehat{c}_n := h z \sum_{n=1}^{+\infty} \frac{4 \pi^2 C_1 n}{4 \pi^2 C_1 n^2 - (t_1 - t_0)^2} \widehat{c}_n,
\end{aligned}
\]

Analogous to the proof of (II), Lemma 6.2 and 6.3 yields that
\[
4 \frac{Q}{t_1 - t_0} |\widehat{i}_3, \widehat{i}_5, \widehat{i}_6| = h^2(x, y, z) S_2(x, y, z)^T \sum_{n=1}^{+\infty} \frac{16 \pi^4 C_1^2 n^2}{[4 \pi^2 C_1 K(C_1)n^2 - (t_1 - t_0)^2]^2} < 0,
\]
which implies \( \lim_{h \to 0} J(\widehat{i}_3, \widehat{i}_5, \widehat{i}_6) = -\infty \). So the Lagrange functional (2.1) subject to (2.2) and (2.3) has no minimum value.

**Proof of Proposition 2.2** By Fubini Theorem, it follows from (2.1) and (2.2) that
\[
\begin{aligned}
J(i_3 + \varepsilon \delta i_3, i_5 + \varepsilon \delta i_5, i_6 + \varepsilon \delta i_6) - J(i_3, i_5, i_6) &= \\
&= \varepsilon \int_{t_0}^{t_f} \{[L_3 i_3 + L_4(i_3 - i_5 - i_6)] \delta i_3 - \frac{1}{C_1} (q_1(t_0) + \int_{t_0}^{t_f} i_3 \delta i_3 \, dt) \} \int_{t_0}^{t_f} \delta i_3 \, dt \\
&+ \varepsilon \int_{t_0}^{t_f} \{[L_5 i_5 + L_4(i_5 - i_6)] \delta i_5 \, dt \\
&- \varepsilon \int_{t_0}^{t_f} \frac{1}{C_2} [q_2(t_0) + \int_{t_0}^{t_f} (i_5 + i_6) \, dt] \int_{t_0}^{t_f} \delta i_5 \, dt \\
&+ \varepsilon \int_{t_0}^{t_f} \{L_6 i_6 + L_4(i_6 + i_3) - \frac{1}{C_3} \int_{t_0}^{t_f} [g_2(t_0) + \int_{t_0}^{t_f} (i_5 + i_6) \, dt] \} \delta i_6 \, dt \\
&+ o(\varepsilon).
\end{aligned}
\]

The constraints (2.3) yields that
\[
\delta i_3 \in \mathbb{H}_1, \quad \delta i_5 \in \mathbb{H}_1, \quad \delta i_6 \in \mathbb{H}_1,
\]
where
\[
\mathbb{H}_1 := \left\{ \sum_{n=1}^{+\infty} (a_n e_n + b_n \tilde{c}_n) : \sum_{n=1}^{+\infty} (a_n^2 + b_n^2) < +\infty \right\},
\]
and \( \mathbb{H}_0 := \{ae | a \in \mathbb{R}\} \), then \( L^2(t_0, t_1; \mathbb{R}) = \mathbb{H}_0 \oplus \mathbb{H}_1 \), i.e., \( \mathbb{H}_0 \) is the orthogonal complement space of \( \mathbb{H}_1 \) in \( L^2(t_0, t_1; \mathbb{R}) \).

Hence, we have from (5.17), (5.18) and \( L^2(t_0, t_1; \mathbb{R}) = \mathbb{H}_0 \oplus \mathbb{H}_1 \) that there exist some \( l_3, l_5, l_6 \in \mathbb{R} \) such that \((i_3, i_5, i_6)\) satisfy (2.3) and
\[
\begin{aligned}
L_3 i_3 + L_4(i_3 - i_5 - i_6) &= \frac{1}{C_1} \int_{t_0}^{t_f} [q_1(t_0) + \int_{t_0}^{t_f} i_3 \, dt] \, ds = l_3, \\
L_5 i_5 + L_4(i_5 + i_6 - i_3) &= \frac{1}{C_2} \int_{t_0}^{t_f} [q_2(t_0) + \int_{t_0}^{t_f} (i_5 + i_6) \, dt] \, ds = l_5, \\
L_6 i_6 + L_4(i_6 + i_3 - i_5) &= \frac{1}{C_3} \int_{t_0}^{t_f} [g_2(t_0) + \int_{t_0}^{t_f} (i_5 + i_6) \, dt] \, ds = l_6,
\end{aligned}
\]
\[
(5.19)
\]
if and only if \((i_3, i_5, i_6)\) is a critical point for the Lagrange functional \((2.1)\) subject to \((2.2)\) and \((2.3)\). Through setting

\[
x_1(t) := \int_{t_0}^{t} i_3 \, dt, \quad x_2(t) := \int_{t_0}^{t} i_5 \, dt, \quad x_3(t) := \int_{t_0}^{t} i_6 \, dt,
\]

it follows from the equation \((5.19)\) and \((2.3)\) that

\[
\begin{cases}
(L_4 + L_3)x''_1 - L_4x''_2 - L_4x''_3 + \frac{1}{c_1^4}x_1 + \frac{q_1(t_0)}{c_1} = 0,
-L_4x''_1 + (L_4 + L_5)x''_2 + L_4x''_3 + \frac{1}{c_2^4}x_2 + \frac{q_2(t_0)}{c_2} = 0,
-L_4x''_1 + L_4x''_2 + (L_4 + L_6)x''_3 + \frac{1}{c_2^4}x_2 + \frac{q_2(t_0)}{c_2} = 0.
\end{cases}
\]

(5.20)

with the boundary condition

\[
\begin{align*}
x_1(t_0) &= 0, \quad x_2(t_0) = 0, \quad x_3(t_0) = 0, \\
x_1(t_1) &= \lambda_3, \quad x_2(t_1) = \lambda_5, \quad x_3(t_1) = \lambda_6.
\end{align*}
\]

(5.21)

Through defining

\[
y = (x_1, x_2, x_3, x_1', x_2', x_3')^T,
\]

and due to the positive definiteness of \(M\), the boundary problem \((5.20)-(5.21)\) can be reformulated as follows:

\[
y' = \begin{pmatrix} 0 \\ -M^{-1}N \end{pmatrix} y + \begin{pmatrix} 0 \\ M^{-1}a \end{pmatrix},
\]

(5.22)

with the boundary condition

\[
y(t_0) = \begin{pmatrix} 0 \\ c \end{pmatrix}, \quad y(t_1) = \begin{pmatrix} x(t_1) \\ d \end{pmatrix},
\]

(5.23)

where \(I_3\) is the \(3 \times 3\) identity matrix, \(x(t_1) = (\lambda_3, \lambda_5, \lambda_6)^T\) and the matrice \(M, N\) and \(a\) are defined by \((2.8), (2.9)\) and \((2.15)\) while \(c, d \in \mathbb{R}^3\) are to be known.

By the variation-of-constants formula, the problem \((5.22)-(5.23)\) is equivalent to

\[
\begin{pmatrix} \Phi(t_1 - t_0)c \\ \Psi(t_1 - t_0)c \end{pmatrix} + \int_{t_0}^{t_1} \begin{pmatrix} \Phi(t_1 - t)M^{-1}a \\ \Psi(t_1 - t)M^{-1}a \end{pmatrix} dt = \begin{pmatrix} x(t_1) \\ d \end{pmatrix},
\]

(5.24)

with

\[
\begin{align*}
\Psi(t) &:= I_3 + \sum_{k=1}^{+\infty} \frac{t^k}{(2k)!} (-M^{-1}N)^k, \\
\Phi(t) &:= t[I_3 + \sum_{k=1}^{+\infty} \frac{t^k}{(2k+1)!} (-M^{-1}N)^k].
\end{align*}
\]

In the second equation of \((5.24)\), \(d\) is uniquely determined by \(c\). So we only need to consider the solvability of \(c\) through the first equation of \((5.24)\).

By the definition of the matrice \(M^{-\frac{1}{2}}, M^{-\frac{1}{2}}NM^{-\frac{1}{2}}\) and \(P\) in \((2.10)\) and \((2.11)\), it follows from the definition of \(\Phi(t)\) that

\[
\frac{1}{2}PM^{\frac{1}{2}}\Phi(t)M^{-\frac{1}{2}}P^T = I_3 + \sum_{k=1}^{+\infty} \frac{t^k}{(2k)!} (-PM^{-\frac{1}{2}}NM^{-\frac{1}{2}}P^T)^k
\]

(5.25)

\[
= \begin{pmatrix}
\frac{1}{\sqrt{h_1 t}} \sin(\sqrt{h_1 t}) & 0 & 0 \\
0 & \frac{1}{\sqrt{h_2 t}} \sin(\sqrt{h_2 t}) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
Thus we obtain (I) from the invertibility of $\Phi(t_1 - t_0)$. The proof of (II) can be obtained by direct calculations in this case $\Phi(t_1 - t_0)$ is a singular matrix.

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