An Adaptive Approach to Recoverable Mutual Exclusion

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Mutual exclusion (ME) is one of the most commonly used techniques to handle conflicts in concurrent systems. Traditionally, mutual exclusion algorithms have been designed under the assumption that a process does not fail while acquiring/releasing a lock or while executing its critical section. However, failures do occur in real life, potentially leaving the lock in an inconsistent state. This gives rise to the problem of recoverable mutual exclusion (RME) that involves designing a mutual exclusion algorithm that can tolerate failures, while maintaining safety and liveness properties.

One of the important measures of performance of any ME algorithm, including an RME algorithm, is the number of remote memory references (RMRs) made by a process (for acquiring and releasing a lock as well as recovering the lock structure after a failure). The best known RME algorithm solves the problem for $n$ processes in sub-logarithmic number of RMRs, given by $O(\log n / \log \log n)$, irrespective of the number of failures in the system.

In this work, we present a new algorithm for solving the RME problem whose RMR complexity gradually adapts to the number of failures that have occurred in the system “recently”. In the absence of failures, our algorithm generates only $O(1)$ RMRs. Furthermore, its RMR complexity is given by $O(\min\{\sqrt{F}, \log n / \log \log n\})$ where $F$ is the total number of failures in the “recent” past. In addition to read and write instructions, our algorithm uses compare-and-swap (CAS) and fetch-and-store (FAS) hardware instructions, both of which are commonly available in most modern processors.

1 INTRODUCTION

One of the most commonly used techniques to handle contention in a concurrent system is to use mutual exclusion (ME). The mutual exclusion problem was first defined by Dijkstra more than half a century ago in [6]. Using locks that provide mutual exclusion enables a process to execute its critical section (part of the program that involves accessing shared resources) in isolation without worrying about interference from other processes. This avoids race conditions, thereby ensuring that the system always stays in a consistent state and produces correct outcome under all scenarios.

Generally, algorithms for mutual exclusion are designed with the assumption that failures do not occur, especially while a process is accessing a lock or a shared resource. However, such failures can occur in the real world. A power outage or network failure might create an unrecoverable situation causing processes to be unable to continue. If such failures occur, traditional mutual exclusion algorithms, which are not designed to operate properly under failures, may deadlock or otherwise fail to guarantee important safety and liveness properties. In many cases, such failures may have disastrous consequences. This gives rise to the recoverable mutual exclusion (RME) problem. The RME problem involves designing an algorithm that ensures mutual exclusion under the assumption that process failures may occur at any point during their execution, but the system is able to recover from such failures and proceed without any negative consequences.

Traditionally, concurrent algorithms use checkpointing and logging to tolerate failures by regularly saving relevant portion of application state to a persistent storage such as hard disk. Accessing a disk is orders of magnitude slower than accessing main memory. As a result, checkpointing and logging algorithms are often designed to minimize disk accesses. Since the advent of NVRAM (non-volatile random access memory), there is an increased focus on designing
Table 1. Comparison of known solutions to recoverable mutual exclusion problem with respect to RMR complexity under three different scenarios.

| Algorithm                                                                 | No failures | F failures | Arbitrarily large number of failures |
|---------------------------------------------------------------------------|-------------|------------|--------------------------------------|
| Golab and Ramaraju’s transformation for recoverability [10, Section 4.1] using MCS lock | $O(1)$       | $O(F)$     | unbounded                             |
| Golab and Ramaraju’s transformation for bounding RMR complexity [10, Section 4.2] using MCS lock | $O(1)$       | $O(n)$     | $O(n)$                               |
| Golab and Hendler’s arbitration tree using k-port MCS lock∗† [8]         | $O(\log n/\log \log n)$ | $O(\log n/\log \log n)$ | $O(\log n/\log \log n)$ |
| Jayanti and Joshi’s wait-free recovery [13]                             | $O(\log n)$ | $O(\log n)$ | $O(\log n)$                         |
| Jayanti, Jayanti and Joshi’s arbitration tree using k-port MCS lock [12] | $O(\log n/\log \log n)$ | $O(\log n/\log \log n)$ | $O(\log n/\log \log n)$ |
| Our algorithm [this work]                                               | $O(1)$       | $O(\sqrt{F})$ | $O(\log n/\log \log n)$ |

∗: It has been recently shown in [12] that the algorithm is prone to deadlocks
†: RMR complexity measures only hold for the CC model

efficient recoverable algorithms for a variety of problems. While existing checkpointing and logging algorithms can be used with NVRAMs instead of disks, thereby yielding better performance; but in doing so, we would not be leveraging the true power of NVRAMs [10, 15]. NVRAMs, when used correctly, can provide near-instantaneous recovery from failures. One can directly store implementation specific variables in the NVRAMs. NVRAMs do not incur as much read-write overhead when compared to the read-write overhead of disks, while guaranteeing data persistence despite process or power failures. Thus application data can be easily recovered after failures. However, recovery of application data alone is not enough. Processor state information such as contents of program counter, CPU registers and execution stack cannot be recovered completely and need to be handled separately. Using innovative methods, with NVRAMs in mind, we aim to design efficient and robust fault-tolerant algorithms for solving mutual exclusion and other important concurrent problems.

The RME problem we study in this work was formally defined a few years ago by Golab and Ramaraju in [10]. Several algorithms have been proposed to solve this problem [8, 10, 12, 13]. One of the most important measures of performance of an RME algorithm is the maximum number of remote memory references (RMRs) made by a process per critical section request in order to acquire and release the lock as well as recover the lock after a failure. Whether or not a memory reference incurs an RMR depends on the underlying memory model. The two most common memory models used to analyze the performance of an RME algorithm are cache-coherent (CC) and distributed shared memory (DSM) models. Different existing RME algorithms have different RMR complexities under different scenarios. For example, one of the RME algorithms presented in [10] has RMR complexity of $O(1)$ in the absence of failures and it grows linearly with the number of failures. As such, the RMR complexity of the algorithm may become arbitrarily large if a process fails repeatedly. On the other hand, the RME algorithm in [12] has RMR complexity of $O(\log n/\log \log n)$, where $n$ is the number of processes in the system, irrespective of how many failures have occurred in the system (including the case when the system has not experienced any failures). To our knowledge, all existing RME algorithms have worst-case RMR complexity of at least $\Omega(\log n/\log \log n)$. A more detailed description of the related work is given later in section 6.
Our Contributions: In this work, we present an RME algorithm with the following desirable properties under both CC and DSM models. First, it has constant RMR complexity in the absence of failures. Second, its RMR complexity grows sub-linearly with (specifically, as square-root of) the number of failures that have occurred in the system in the “recent” past. Third, it has sub-logarithmic worst-case RMR complexity of \( O(\log n/\log \log n) \). We are not aware of any existing RME algorithm that satisfies all of the above three properties. Table 1 compares the performance of different RME algorithms under a variety of situations. The main idea is to use a solution to a weaker variant of the RME problem, in which a failure may cause the mutual exclusion property to be violated temporarily albeit in a controlled manner, repeatedly as a filter to limit contention and achieve adaptability.

In addition to mutual exclusion and starvation freedom properties, our RME algorithm also satisfies bounded exit, bounded recovery and critical section reentry properties [12]. Roughly speaking, an RME algorithm satisfies the bounded exit property if a process is able to leave its critical section within a bounded number of its own steps unless it fails. It satisfies the bounded recovery property if a process is able to recover from a failure within a bounded number of its own steps unless it fails again. Finally, it satisfies the critical section reentry property if, when a process fails inside its critical section, then no other process enters its critical section until it (re)enters its critical section.

Finally, unlike the RME algorithm in [12], our RME algorithm satisfies FCFS (first-come-first-served) property in the absence of failures.

Our approach is general enough that it can be used to transform any non-adaptive RME algorithm with worst-case RMR complexity of \( T(n) \) under a given memory model (CC or DSM) into an adaptive RME algorithm whose worst-case RMR complexity is still \( O(T(n)) \) under the same memory model.

Roadmap: The rest of the text is organized as follows. We describe our system model and formally define the RME problem in section 2. We define the weaker variant of the RME problem and its properties in section 3. We present a highly efficient solution to the weaker variant of the RME problem with constant RMR complexity in section 4. In subsection 5.1, we present a framework to transform a given RME algorithm into a new RME algorithm that preserves the worst-case RMR complexity of the original RME algorithm but has lower RMR complexity in the absence of failures. This transformation uses a solution to the weaker variant of the RME problem as a building block. We then apply recursion to our basic framework in subsection 5.2 to derive an RME algorithm that achieves the desired RMR complexity for each of the three scenarios mentioned earlier (as shown in Table 1). A detailed description of the related work is given in section 6. Finally, in section 7, we present our conclusions and outline directions for future research.

2 SYSTEM MODEL AND PROBLEM FORMULATION

We follow the same model as used by Golab and Ramaraju in their work on recoverable mutual exclusion (RME) [10].

2.1 System model

We consider an asynchronous shared-memory system consisting of \( n \) unreliable processes labeled \( p_1, p_2, \ldots, p_n \). Shared memory is used to store variables that can be accessed by any process. Besides shared memory, each process also has its own private memory that is used to store variables that can only be accessed by that process (e.g., program counter, CPU registers, execution stack, etc.). Processes can only communicate by performing read, write and read-modify-write (RMW) instructions on shared variables. Processes are not assumed to be reliable and may fail.

A system execution is modeled as a sequence of process steps. In each step, some process either performs some local computation affecting only its private variables or executes one of the available instructions (read, write or RMW) on
a shared variable or fails. Processes may run at arbitrary speeds and their steps may interleave arbitrarily. In any execution, between two successive steps of a process, other processes can perform an unbounded but finite number of steps.

To avoid race conditions resulting from multiple processes trying to access the same shared resource, processes synchronize accesses to shared resources using a lock that provides mutual exclusion (ME); at most one process can hold the lock at any time.

2.2 Failure model

We assume crash-recover failure model. A process may fail at any time during its execution by crashing. A crashed process recovers eventually and restarts its execution. A crashed process does not perform any steps until it has restarted. A process may fail multiple times, and multiple processes may fail concurrently.

Note that, upon restarting after a failure, the state of the underlying application as well as the lock needs to be restored to a proper state. In this work, we focus only on recovering the internal structure of a lock. Restoring the application state to its proper state (using logs and/or persistent memory) is assumed to be the responsibility of the programmer and is beyond the scope of this work [8, 10].

On crashing, a process loses the contents of its private variables, including but not limited to the contents of its program counter, CPU registers and execution stack. However, the contents of the shared variables remain unaffected and are assumed to persist despite any number of failures. When a crashed process restarts, all its private variables are reset to their initial values.

Processes that have crashed are difficult to distinguish from processes that are running arbitrarily slow. However, we assume that every process is live in the sense that a process that has not crashed eventually executes its next step and a process that has crashed eventually recovers. In this work, we consider a failure to be associated with a single process. If a failure causes multiple processes to crash, we treat each process crash as a separate failure.

2.3 Process execution model

A process execution is modeled using two types of computations, namely non-critical section and critical section. A critical section refers to the part of the application program in which a process needs to access shared resources in isolation. A non-critical section refers to the remainder of the application program.

If multiple processes access and modify shared resource(s) concurrently, it may lead to race conditions which may prevent the application from working properly and may possibly have disastrous consequences. To avoid such race conditions, a lock (or a mutual exclusion algorithm) is used to enable each process to execute its critical section in isolation. At most one process can hold the lock at any time, and a process can execute its critical section only if it is holding the lock. The lock can be granted to another process only after the process holding the lock releases it after completing its critical section. Hereafter, we use the terms "mutual exclusion algorithm", "ME algorithm" and "lock" interchangeably.

The execution of a process with respect to a lock is depicted in Algorithm 1. As shown, a process repeatedly executes the following five segments in order: NCS, Recover, Enter, CS and Exit. The first segment, referred to as NCS, models the steps executed by a process in which it accesses only private variables. The second segment, referred to as Recover, models the steps executed by a process to perform any cleanup required due to past failures and restore the internal structure of the lock to a consistent state. The third segment, referred to as Enter, models the steps executed by a
Algorithm 1: Process execution model

```
while true do
    Non-Critical Section (NCS)
    Recover
    Enter
    Critical Section (CS)
    Exit
end while
```

A process to acquire the lock so that it can execute its critical section in isolation. The fourth segment, referred to as CS, models the steps executed by a process in the critical section in which it accesses both shared and private variables. Finally, the fifth segment, referred to as Exit, models the steps executed by a process to release the lock it acquired earlier in Enter segment.

We assume that, in NCS segment, a process does not access any part of the lock or execute any computation that could potentially cause a race condition. Moreover, in Recover, Enter and Exit segments, a process accesses shared variables pertaining to the lock (and the lock only).

A process may crash at any point during its execution, including while executing NCS, Recover, Enter, CS or Exit segment. We assume that a crashed process upon restarting starts its execution from the beginning of the loop shown in Algorithm 1, specifically from the beginning of NCS segment. Note that any steps executed by a process to recover the application state are not explicitly modeled here. Specifically, both NCS and CS segments may consist of code in the beginning to recover relevant portions of the application state.

In the rest of the text, by the phrase "acquiring a recoverable lock," we mean "executing Recover and Enter segments (in order) of the associated RME algorithm." Likewise, by the phrase "releasing a recoverable lock," we mean "executing Exit segment of the associated RME algorithm."

Definition 2.1 (passage). A passage of a process is defined as the sequence of steps executed by the process from when it begins executing Recover segment to either when it finishes executing the corresponding Exit segment or experiences a failure, whichever occurs first.

Definition 2.2 (failure-free passage). A passage of a process is said to be failure-free if the process has successfully executed Recover, Enter and Exit segments of that passage without experiencing any failures.

Definition 2.3 (super-passage). A super-passage of a process is a maximal non-empty sequence of consecutive passages executed by the process, where only the last passage of the process in the sequence is failure-free.

A request for critical section by a process \( p \) is said to be satisfied if \( p \) has executed a failure-free passage for that request.

2.4 Problem definition

A history is a collection of steps taken by processes. A process \( p \) is said to be active in a history \( H \) if \( H \) contains at least one step by \( p \). We assume that every critical section is finite.

Definition 2.4 (fair history). A history \( H \) is said to be fair if (a) it is finite, or (b) if it is infinite and every active process in \( H \) either executes infinitely many steps or stops taking steps after a failure-free passage.

Designing a recoverable mutual exclusion (RME) algorithm involves designing Recover, Enter and Exit segments such that the following correctness properties are satisfied.
Mutual Exclusion (ME) For any finite history \( H \), at most one process is in its CS at the end of \( H \).

Starvation Freedom (SF) For any infinite fair history \( H \), if a process \( p \) leaves the NCS segment in some step of \( H \), then eventually \( p \) enters its CS segment, or else there are infinitely many failure steps in \( H \).

Note the mutual exclusion is a safety property, and starvation freedom is a liveness property. Our correctness properties are the same as those used in [8, 10]. We have stated them here for the sake of completeness. In addition to the correctness properties, it is also desirable for an RME algorithm to satisfy the following additional properties.

Bounded Exit (BE) For any infinite history \( H \), any execution of the Exit segment by any process \( p \) either completes in a bounded number of \( p \)'s own steps or ends with \( p \) crashing.

Bounded Recovery (BR) For any infinite history \( H \), any execution of Recover segment by process \( p \) either completes in a bounded number of \( p \)'s own steps or ends with \( p \) crashing.

Critical Section Reentry (CSR) For any history \( H \), if a process \( p \) crashes inside its CS segment, then no other process may enter its CS segment before \( p \) re-enters its CS segment.

If a process fails inside its CS, then a shared object or resource (e.g., a shared data structure) may be left in an inconsistent state. The critical section reentry property allows such a process to "fix" the shared resource before any other process can enter its CS (e.g., [8, 10]).

2.5 Performance measures

We measure the performance of RME algorithms in terms of the number of remote memory references (RMRs) made by the algorithm during a single passage. The definition of a remote memory reference depends on the memory model implemented by the underlying hardware architecture. In particular, we consider the two most popular shared memory models:

Cache Coherent (CC) The CC model assumes a centralized main memory. Each process has access to the central shared memory in addition to its local cache memory. The shared variables, when needed, are cached in the local memory. These variables may be invalidated if updated by another process. Reading from an invalidated variable causes a cache miss and requires the variable value to be fetched from the main memory. Under this model, a remote memory reference occurs each time there is a fetch operation from the main memory or a cached copy is invalidated.

Distributed Shared Memory (DSM) The DSM model has no centralized memory. Shared variables reside on individual process nodes. These variables may be accessed by processes either via the interconnect or a local memory read, depending on where the variable resides. Under this model, a remote memory reference occurs when a process needs to perform any operation on a variable that does not reside in its own node's memory.

In the rest of the text, if not explicitly specified, the RMR complexity measure of an algorithm applies to both CC and DSM models.

We analyze the RMR complexity of an RME algorithm under three scenarios: (a) in the absence of failures (failure free RMR complexity), (b) in the presence of \( F \) failures (limited failures RMR complexity), and (c) in the presence of an unbounded number of failures (arbitrary failures RMR complexity). We identify the following desirable performance measures applicable to an RME algorithm:

PM 1. (Constantness) Failure free RMR complexity of the algorithm is \( O(1) \).

PM 2. (Adaptivity) Limited failures RMR complexity of the algorithm is
Table 2. Comparison of known solutions to recoverable mutual exclusion problem with respect to the four performance measures.

| Algorithm                                           | Performance Measure | Classification       |
|-----------------------------------------------------|---------------------|----------------------|
|                                                     | PM 1 | PM 2(a) | PM 2(b) | PM 3(a) | PM 3(b) |
| Golab and Ramaraju’s transformation for recoverability [10, Section 4.1] using MCS lock | ✓    | ✓       | ×       | ×       | ×       | unbounded adaptive |
| Golab and Ramaraju’s transformation for bounding RMR complexity [10, Section 4.2] using MCS lock | ✓    | ×       | ×       | ✓       | ×       | bounded semi-adaptive |
| Golab and Hendler’s arbitration tree using k-port MCS lock [8] | ×    | ×       | ×       | ✓       | ✓       | sublogarithmic-bounded non-adaptive |
| Jayanti and Joshi’s wait-free recovery [13] | ×    | ×       | ×       | ✓       | ×       | bounded non-adaptive |
| Jayanti and Joshi’s arbitration tree using k-port MCS lock [12] | ×    | ×       | ×       | ✓       | ✓       | sublogarithmic-bounded non-adaptive |
| Our algorithm [this work] | ✓    | ✓       | ✓       | ✓       | ✓       | sublogarithmic-bounded sublinear-adaptive |

*: it has been recently shown in [12] that the algorithm is prone to deadlocks

(a) $O(g(F))$, where $g(x)$ is a monotonically non-decreasing function of $x$.
(b) $o(F)$.

PM 3. (Boundedness) Arbitrarily large number of failures RMR complexity of the algorithm is
(a) $O(h(n))$, where $h(x)$ is a monotonically non-decreasing function of $x$.
(b) $O(\log n)$.

Note that PM 2(a) implies PM 1, PM 2(b) implies PM 2(a) and PM 3(b) implies PM 3(a). A comparison of the known RME algorithms with respect to the above performance measures PM 1 to PM 3 is shown in Table 2. Based on the subset of performance measures an RME algorithm satisfies, given a memory model (CC or DSM), we classify algorithms based on

(1) Adaptivity
- non-adaptive if its failure free RMR complexity is $\Theta$(arbitrary failures RMR complexity).
- semi-adaptive if it satisfies PM 1, but not PM 2(a).
- adaptive if it satisfies PM 2(a) (hence also PM 1).
- sublinear-adaptive if it satisfies PM 2(b) (hence also PM 2(a) and PM 1).

(2) Boundedness
- unbounded if it does not satisfy PM 3(a).
- bounded if it satisfies PM 3(a).
- sublogarithmic-bounded if it satisfies PM 3(b).

As shown in Table 2, all existing RME algorithm are either non-adaptive, semi-adaptive or unbounded adaptive. To our knowledge, there is no bounded-adaptive, let alone sublogarithmic-bounded sublinear-adaptive RME algorithm currently for either memory model. Note that our taxonomy may not be able to classify all possible RME algorithms (or recoverable algorithms in general), but it is sufficient for classifying and comparing existing RME algorithms. Additionally, our taxonomy assumes that there is no algorithm that solves the RME problem with $O(1)$ RMR complexity using only existing hardware instructions.
2.6 Synchronization primitives

We assume that, in addition to read and write instructions, the system also supports fetch-and-store (FAS) and compare-and-swap (CAS) read-modify-write (RMW) instructions.

A fetch-and-store instruction takes two arguments: address and new; it replaces the contents of a memory location (address) with a given value (new) and returns the old contents of that location.

A compare-and-swap instruction takes three arguments: address, old and new; it compares the contents of a memory location (address) to a given value (old) and, only if they are the same, modifies the contents of that location to a given new value (new). It returns true if the contents of the location were modified and false otherwise.

Both instructions are commonly available in many modern processors such as Intel 64 [11] and AMD64 [1].

3 WEAK RECOVERABILITY

To design a sublogarithmic-bounded sublinear-adaptive RME algorithm, we use a solution to the weaker variant of the RME problem as a building block in which a failure may cause the ME property to be violated albeit only temporarily and in a controlled manner. We refer to this variant as the weakly recoverable mutual exclusion problem.

To formally define how long a violation of the ME property may last, we define the notion of consequence interval of a failure.

Definition 3.1 (consequence interval). The consequence interval of a failure \( f \) in a history \( H \) is defined as the interval in time that starts from the onset of the failure and extends to the point when either all requests that were generated before this failure occurred in \( H \) have been satisfied or the last step in \( H \) is performed, whichever happens earlier.

Intuitively, we use the notion of consequence interval to capture the maximum duration for which the impact of a failure may be felt in the system.

Definition 3.2 (weakly recoverable mutual exclusion). We say an algorithm is a weakly recoverable mutual exclusion algorithm if it always satisfies the starvation freedom property and, for any finite history \( H \), if two or more processes are in their critical sections simultaneously at some point in \( H \), then that point overlaps with the consequence interval of some failure.

Roughly speaking, a weakly RME algorithm satisfies the ME property as long as no failure has occurred in the "recent" past. Hereafter, to avoid confusion, we sometimes refer to the traditional recoverable mutual exclusion problem (respectively, algorithm) as defined in subsection 2.4 as strongly recoverable mutual exclusion problem (respectively, algorithm).

Note that bounded exit and bounded recovery properties defined earlier in subsection 2.4 are applicable to weakly RME problem as well. However, the critical section reentry property needs to be redefined.

Bounded Critical Section Re-entry (BCSR): For any history \( H \), if a process \( p \) crashes inside its CS segment, then, until \( p \) has re-entered its CS segment at least once, any subsequent execution of Enter segment by \( p \) either completes within a bounded number of \( p \)'s own steps or ends with \( p \) crashing.

We demonstrate that it is possible to design an optimal weakly RME algorithm using existing hardware instructions whose worst-case RMR complexity is only \( O(1) \) under both CC and DSM models. In contrast, the best known strongly RME algorithm has worst-case RMR complexity of \( \Omega((\log n)/(\log \log n)) \) under both CC and DSM models. We exploit this
gap to design a bounded-adaptive RME algorithm. To prove that our algorithm is sublogarithmic-bounded sublinear-adaptive, we exploit some additional properties of our weakly RME algorithm.

Not all failures may cause the ME property to be violated when using a weakly RME algorithm. To that end, we define the notion of sensitive instruction of an algorithm.

**Definition 3.3 (critical step).** An instruction $s$ of a process is said to be sensitive with respect to a weakly RME algorithm if there exists any finite history $H$ that satisfies the following conditions: (a) it contains exactly one failure in which the process crashes immediately after performing said instruction $s$ and (b) it does not satisfy the ME property; it is said to be non-sensitive otherwise.

**Definition 3.4 (unsafe failure).** A failure is said to be unsafe with respect to a weakly RME algorithm if it involves a process crashing while (immediately before or after) performing a sensitive instruction with respect to the algorithm; it is said to be safe otherwise.

Note that, by definition, every instruction of a strongly RME algorithm is a non-sensitive step. As a result, every failure is safe with respect to a strongly RME algorithm.

The next notion limits the “degree” of violation (of the ME property) by a weakly RME algorithm if and when it occurs.

**Definition 3.5 (responsive weakly recoverable mutual exclusion).** We say that a weakly recoverable mutual exclusion algorithm is responsive if, for all $k \geq 1$, it satisfies the following property: for any finite history $H$, if at least $k + 1$ processes are in their critical sections simultaneously at some point in $H$, then that point overlaps with the consequence intervals of at least $\Omega(k)$ (unsafe) failures.

### 3.1 Composite recoverable locks

The properties defined above are with respect to a single weakly recoverable lock. In order to construct a sublogarithmic-bounded sublinear-adaptive (strongly) recoverable lock with desired performance characteristics, we use multiple weakly recoverable locks. We call a lock as composite if it is employs one or more (weakly or strongly recoverable) locks. Composite locks might have several possible structures. For instance, the Enter segment of one lock could be contained in the Enter or CS segment of another lock or the CS segment of one lock may be contained in the NCS segment of another lock.

Note that, when we have multiple locks, the notions defined in the previous (sub)section, namely consequence interval, sensitive instruction and unsafe failure, become relative to the specific lock. For example, a failure will have a different consequence interval with respect to each lock. An instruction may be sensitive with respect to one lock but non-sensitive with respect to another. Thus, in a composite lock, a failure may be unsafe with respect to one or more weakly recoverable locks.

**Definition 3.6 (locality property).** A composite (weakly or strongly) recoverable lock is said to satisfy the locality property if, for any instruction $s$, $s$ is sensitive with respect to at most one of its component weakly recoverable locks.

A composite lock whose all component locks are strongly recoverable trivially satisfies the locality property.
4 AN OPTIMAL WEAKLY RECOVERABLE LOCK

In this section, we present a weakly recoverable lock whose RMR complexity is $O(1)$ per passage for all three failure scenarios under both CC and DSM models. Our lock is based on the well-known MCS queue-based (non-recoverable) lock [14]. The original lock did not satisfy the bounded exit property. Dvir and Taubenfeld proposed an extension to the original algorithm in [7] to make the Exit segment wait-free. We extend the augmented MCS lock, which satisfies bounded-exit property, to make it weakly recoverable.

4.1 Original MCS queue based lock

Any request in the MCS mutual exclusion algorithm is represented using a node. The algorithm maintains a first-come-first-served (FCFS) queue of outstanding requests using a linked-list of their associated nodes. A node contains two fields: (a) `next`, which is a reference to its successor node in the queue (if any), and (b) `locked`, which is a boolean variable used by a process to spin while waiting for its turn to enter its critical section. The queue itself is represented using a shared variable `tail` that contains reference to the last node in the queue if non-empty and `null` otherwise.

To acquire the lock, a process first initializes its queue node by setting its `next` and `locked` fields to `null` and `true`, respectively. It then appends the node to the queue by performing an FAS instruction on `tail` using the reference to its own node as an argument (to the instruction). Note that the instruction returns the contents of `tail` just before it is modified. If the return value is `null`, then it indicates that the lock is free and the process has successfully acquired the lock. If not, then it indicates that the lock is not free and the return value is the reference to the predecessor of the process’ own node in the queue. In that case, it notifies the owner of the predecessor node of its presence. To that end, it stores the reference to its own node in the `next` field of the predecessor node, thereby creating a forward link between the two nodes. It then starts spinning on the `locked` field of its own node waiting for it to be reset to false by the owner of the predecessor node as part of releasing the lock.

To release the lock, a process first tries to reset the `tail` variable to `null` (if `tail` still contains the reference to this own node) using a CAS instruction. If the instruction succeeds, then it implies that the queue does not contain any more outstanding requests and the lock is now free. On the other hand, if the instruction fails, then it implies that the queue contains at least one outstanding request and its own node is guaranteed to have a successor. It then waits until the `next` field of its own node contains a valid reference (a non-null value) indicating that a link has been created between its own node and its successor. Finally, it follows this link and resets the `locked` field in its successor node to false.

4.2 Adding bounded exit property

The original algorithm as described above does not satisfy the bounded-exit property since a process leaving its critical section may have to wait until a link between its own node and its successor has been created.

To achieve the bounded-exit property, the original algorithm is augmented with a mechanism that allows a leaving process to notify the process slated to next acquire the lock, in case the link from the former’s node to the latter’s node has not been created yet, that the lock is now free. To that end, a process on leaving its critical section attempts to store a special value (e.g., reference to its own node) in the `next` field of its own node using a CAS instruction. Likewise, a link is also created using a CAS instruction instead of a simple write instruction as in the original algorithm. Both CAS instructions are designed to succeed only if the `next` field contains `null` value, thereby ensuring that the `next` field can only be modified once.
Algorithm 2: Pseudocode of weakly recoverable MCS lock with wait-free exit for process $p_i$.

```plaintext
1 struct QNode {  
2     /* location used for spinning while waiting to enter CS */  
3     locked boolean variable;  
4     /* reference to the successor node */  
5     next: reference to QNode;  
6     /* reference to the last node in the queue */  
7     tail: reference to QNode;  
8     /* state of the process with respect to the lock; in the DSM model, the i-th entry is local to process $p_i$ */  
9     state[i] array [1...n] of integer variables;  
10     /* reference to the predecessor node */  
11     pred[i] array [1...n] of reference to QNode;  
12     /* initialization */  
13     Function initialization() {  
14         if (state[i] == locked) then  
15             /* may have a successor; signal it to enter CS */  
16             pred[i] ← mine[i];  
17             /* reset reference to own node */  
18             state[i] ← null;  
19             /* advance the state */  
20             /* may have failed earlier while performing FAS instruction; abort the attempt */  
21             if (state[i] = exit段) then  
22                 /* once FAS step has been performed without any interruption, the two references are guaranteed to be different */  
23                 temp ← pred[i];  
24                 pred[i] ← mine[i];  
25                 /* remove my node from the queue if it has no successor */  
26                 if (state[i] = exit段) then  
27                     /* finish executing Exit segment */  
28                     if (state[i] = locked) then  
29                         /* exit */  
30                         (
31                             mine[i] = create a new node;  
32                             /* reference to my own node; in the DSM model, the i-th entry is local to process $p_i$ */  
33                             mine[i] = null;  
34                             /* state of the process with respect to the lock; in the DSM model, the i-th entry is local to process $p_i$ */  
35                             state[i] = Terms;  
36                             /* append my own node to the queue */  
37                             pred[i] ← mine[i];  
38                             /* persist the result of FAS */  
39                             pred[i] ← temp;  
40                             
41                             /* have a predecessor; create the link */  
42                             if (pred[i] = null) then  
43                                 /* append my own node to the queue */  
44                                 state[i] = Terms;  
45                                 /* process is in CS */  
46                                 state[i] = exit段;  
47                                 /* wait for the predecessor to complete */  
48                                 if (state[i] = locked) then  
49                                     /* may have a successor; signal it to enter CS */  
50                                     state[i] = null;  
51                                     pred[i] = next;  
52                                     mine[i] = next;  
53                                     /* link already created; tell the successor to stop spinning */  
54                                     if (mine[i] = locked) then  
55                                         /* remove my node from the queue if it has no successor */  
56                                         /* may have a successor; signal it to enter CS */  
57                                         state[i] = null;  
58                                         pred[i] = next;  
59                                         mine[i] = next;  
60                                         /* advance the state */  
61 }  
```

Thus, if the CAS instruction performed by a process leaving its critical section fails, then that process can conclude that the forward link has already been created and it then follows this link and resets the locked field of its successor node. On the other hand, if the CAS instruction performed by a process trying to create the link fails, then that process can infer that the lock is free and that it now holds the lock.

With this modification, unlike in the original algorithm, after releasing the lock, a process cannot always reuse its own node for the next request.

4.3 Adding weak recoverability

A pseudocode of the weakly recoverable lock is given in algorithm 2. Our pseudocode uses the following shared variables. The first variable, tail, contains the address of the last node in the queue if the queue is non-empty and null otherwise. The next three variables, state, mine and pred, are arrays with one entry for each process. The $i$-th entry of state, denoted by state[i], contains process $p_i$’s current state with respect to the lock (explained later). The $i$-th entry
Fig. 1. Processes $p_1 \ldots p_8$ successfully append their nodes to the tail of the queue using an FAS instruction. Processes $p_4$ and $p_7$ failed to capture the result value of the FAS and are unable to set the next field of the nodes of $p_3$ and $p_6$. Process $p_3$ has captured the address of the node of $p_2$ and is about to set the corresponding next field on the node of $p_2$. Effectively, three sub-queues are created due to failures of $p_4$ and $p_7$.

of mine, denoted by $\text{mine}_i$, contains the address of the queue node associated with process $p_i$’s most recent request. The $i$-th entry of pred, denoted by $\text{pred}_i$, contains the address of the predecessor node, if any, of process $p_i$ after its node has been appended to the queue.

The state of a process with respect to a lock has five possible values, namely Free, Initializing, Trying, InCS and Leaving. At the beginning, the state of a process, say $p_i$, is set to Free. It is changed to Initializing after $p_i$ has reset $\text{mine}_i$ to null. It is changed to Trying after $p_i$ has initialized $\text{mine}_i$ with the address of a new node, then initialized the two fields of $\text{mine}_i$ and finally initialized $\text{pred}_i$ by setting it equal to $\text{mine}_i$. It is changed to InCS after $p_i$ has acquired the lock. It is changed to Leaving when $p_i$ starts executing the Exit segment. Finally, it is changed to Free again after $p_i$ finishes executing the Exit segment.

Our algorithm has only one sensitive instruction, namely the one involving the FAS instruction (line 40). Recall that a process uses this instruction to append its own node to the queue and also obtain the address of its predecessor node. If a failure occurs while performing this instruction, then a situation may occur in which the process was able to append its node to the queue, but was unable to store the address of its predecessor node to shared memory. This is because the step actually consists of two distinct steps—performing the FAS on shared memory location tail, and storing the result of the FAS to another shared memory location, pred[i] (for persistence). Moreover, there is no easy way to recover this address (of the predecessor) based on the current knowledge of the failed process. The queue continues to grow beyond this node, but it would be disconnected from the previous part of the queue, thereby creating one more sub-queue.

If a process detects that it may have failed while performing the (FAS) step, it “relinquishes” its current node, informs its successor (if any) that the lock is now “free” using the wait-free signalling mechanism described earlier and retries acquiring the lock using a new node. This potentially creates multiple queues (or sub-queues) which may allow multiple processes to execute their critical sections concurrently, thereby violating the ME property. For an example, please refer to Figure 1. All other instructions of our algorithm are non-sensitive. We achieve that by using the following ideas.

First, a process does not use the outcome of the CAS instruction used to modify the next field of a node (line 44 and line 55). After performing the CAS instruction on the next field, it reads the contents of the field again and determines its next step based on what it read. Note that, once initialized, the next field can only be modified once. This makes the two steps involving the CAS instruction on the next field as idempotent; the effect of performing the CAS instruction multiple times if interrupted due to failures is same as performing it once.
Second, portions of Recover and Enter segments are enclosed in if-blocks to be executed conditionally. Intuitively, the guard of an if-block represents the pre-condition that needs to hold before its body can be executed. The outermost if-blocks use guards based on the current state of the process, which is advanced only at the end of the block. The inner if-blocks use guards based on other variables. Except for the if-block containing the FAS instruction (which constitutes a sensitive instruction), all other if-blocks are idempotent and can be executed repeatedly if interrupted due to failures without any adverse impact starting from the evaluation of the guard (lines 16-19, lines 20-22, lines 23-26, lines 29-37 and lines 43-48). Note that if the guard of an if-block does not hold, its body is not executed.

Third, as in case of the next field, a process does not use the outcome of the CAS instruction used to modify the tail pointer of the queue in the Exit segment (line 54). After performing the CAS instruction on the tail pointer, irrespective of the outcome of the instruction, it blindly executes the remainder of the steps pertaining to signalling the successor node (lines 55-58). If the node has no successor, then the steps are redundant, but have no adverse impact even if the node has already been removed from the queue by an earlier CAS instruction.

4.4 Correctness proof and complexity analysis
We refer to the algorithm described in the previous section as WR-Lock. We now prove that WR-Lock is a responsive weakly recoverable ME algorithm.

The following proposition captures the working of the WR-Lock algorithm.

**Proposition 4.1.** Given a history $H$, time $t$ and $k \geq 0$, if at least $k$ processes are in their critical sections simultaneously at time $t$, then (a) the system contains at least $k$ non-empty pairwise disjoint sub-queues at time $t$, and (b) at least one node in each sub-queue is owned by a process that is in its critical section at time $t$.

Note that the sub-queues may be implicit, but can be explicitly constructed using the contents of the shared memory. We use the above proposition to argue that WR-Lock is responsive.

**Theorem 4.2.** Given a history $H$, time $t$ and $k \geq 0$, if at least $k + 1$ processes are in their critical sections simultaneously at time $t$, then time $t$ overlaps with the consequence interval of at least $k$ unsafe failures.

**Proof.** The lemma trivially holds if $k = 0$; therefore assume that $k > 0$. Assume that there are at least $k + 1$ processes in their critical sections simultaneously at time $t$. From Proposition 4.1, the system contains at least $k + 1$ non-empty sub-queues. Only one sub-queue has tail pointing to its last node. Let the set of remaining sub-queues be denoted by $Q = \{Q_1, Q_2, \ldots, Q_k\}$. Note that the first node of each sub-queue is owned by a process that is in its critical section at time $t$.

Consider an arbitrary queue $Q$ from the set $Q$. Let $x$ denote its last node. (Note that the last node of a sub-queue can be deduced by examining the contents of all pred pointers in $H$.) By the way the MCS algorithm works, there exists time $t' \leq t$ such that tail was pointing to $x$ at time $t'$ and some process failed while performing FAS instruction on tail at time $t'$; let the failure be denoted by $f$. Let $p$ denote the process that owns a node of $Q$ and is in its critical section at time $t$. Clearly, $p$ generated its request before $f$ and the request is still pending at time $t$. Thus, the consequence interval of $f$ extends at least until time $t$.

Since $Q$ was chosen arbitrarily, it follows that there exists a unique unsafe failure for each of the $k$ sub-queues in $Q$ whose consequence interval extends until time $t$. □

**Theorem 4.3.** WR-Lock satisfies the SF property.
Let $H$ be an arbitrary infinite fair history. If $H$ has an infinite number of failures, then it trivially satisfies the SF property. Thus, assume that $H$ consists of a finite number of failures. Consider the execution of the system after the last failure has occurred and all processes that are crashed at that time have restarted. The system may contain multiple sub-queues at that time. Only one of these sub-queues can continue to grow, namely the one that contains the node to which $tail$ is pointing. All others sub-queues can only shrink. We can view each of this sub-queue as a separate instance of the MCS lock, which is starvation free. Thus, we can conclude that every current and future request is eventually satisfied. Hence, WR-Lock satisfies the SF property.

**Theorem 4.4.** WR-Lock satisfies the BCSR property.

**Proof.** Assume that some process $p_i$ fails while executing its CS segment, then, at the time of failure, $state[i] = InCS$. When $p_i$ restarts, after executing NCS segment, it executes Recover segment followed by Enter segments. As the code inspection shows, since $state[i] = InCS$, $p_i$ only evaluates a small number of if-conditions, all of which evaluate to false, and then proceeds directly to the CS segment. Hence, WR-Lock satisfies the BCSR property.

It follows from Theorem 4.2, Theorem 4.4 and Theorem 4.3 that

**Theorem 4.5.** WR-Lock is a responsive weakly recoverable mutual exclusion algorithm.

**Theorem 4.6.** WR-Lock satisfies the BR and BE properties.

**Proof.** As the code inspection shows, Recover and Exit segments do not involve any loops. Thus, a process can execute these segments within a bounded number of its own steps. Hence, WR-Lock satisfies the BR and BE properties.

**Theorem 4.7.** The RMR complexity of Recover, Enter and Exit segments of WR-Lock is $O(1)$ each.

**Proof.** As the code inspection shows, Recover and Exit segments do not contain any loop and only contain a constant number of steps. The Enter segment, however has one loop at line 46 of algorithm 2, but otherwise contain a constant number of steps. The loop involves waiting on a boolean variable until it become true and the variable can be written to only once. In the DSM model, this variable is mapped to a location in local memory module. Hence, the RMR complexity of the Enter segment is also $O(1)$.

## 5 A STRONGLY RECOVERABLE SUBLOGARITHMIC-BOUNDED SUBLINEAR-ADAPTIVE LOCK

In this section, we describe a framework that uses other types of recoverable locks with certain properties as building blocks to construct a lock that is not only strongly recoverable but also sublogarithmic-bounded sublinear-adaptive under both CC and DSM models.

Our framework is based on the one used by Golab and Ramaraju in [10, Section 4.2] to construct a strongly recoverable lock that is semi-adaptive. Specifically, in their framework, Golab and Ramaraju use two different types of strongly recoverable locks, referred to as base lock and auxiliary lock, along with two other components to build another strongly recoverable lock, referred to as target lock. The target lock constructed is bounded semi-adaptive based on the base lock that is unbounded adaptive and the auxiliary lock that is non-adaptive. They achieve this by extending the base lock so that, upon detecting a failure, processes can abort their requests and reset the (base) lock. In the presence of failures (even a single failure), the RMR complexity of the target lock is dominated by the overhead of aborting the request for the base lock and then resetting the base lock, thereby making the lock semi-adaptive.
We describe our (sublogarithmic-bounded sublinear-adaptive) lock in two steps. We first describe a basic framework to transform a non-adaptive bounded strongly recoverable lock to a semi-adaptive bounded strongly recoverable lock. We then extend this framework to make the lock sublinear-adaptive while ensuring that it stays strongly recoverable and bounded.

5.1 A sublogarithmic-bounded semi-adaptive RME algorithm

5.1.1 Building blocks. We use four different types of locks as building blocks.

- **Filter lock**: A responsive weakly recoverable lock that provides mutual exclusion in the absence of failures. We use the lock proposed in section 4 to implement this lock, which has $O(1)$ RMR complexity for all three failure scenarios under both CC and DSM models.

- **Splitter**: Used to split processes into fast or slow paths. If multiple processes are navigating the splitter concurrently (which would happen only if an unsafe failure has occurred with respect to the filter lock), only one of them is allowed to take the fast path and the rest are diverted to the slow path. In other words, the splitter is biased. Intuitively, it can be viewed as a strongly recoverable try lock. It is implemented using an atomic integer and a CAS instruction as proposed by Golab and Ramaraju in [10, Section 4.2], which has $O(1)$ RMR complexity for all three failure scenarios under both CC and DSM models.

- **Arbitrator lock**: A dual-port strongly recoverable lock. Each port corresponds to a side; at any time, at most one process should be allowed to attempt to acquire the lock from any side. However, any two of the $n$ processes can compete to acquire the lock. We refer to the two sides as `LEFT` and `RIGHT`. It can be implemented using the dual-port RME algorithm proposed by Golab and Ramaraju in [10, Section 3.1] (a transformation of Yang and Anderson’s ME algorithm to add recoverability), which has $O(1)$ RMR complexity for all three failure scenarios under both CC and DSM models.

- **Core lock**: A (non-adaptive) strongly recoverable lock that provides mutual exclusion among processes taking the slow path. We may use an instance of any of the existing RME algorithms.

5.1.2 The execution flow. In order to acquire the target lock, a process proceeds as follows. It first waits to acquire the filter lock. Once granted, it navigates through the splitter trying to enter the fast path. If successful, it then attempts to acquire the arbitrator lock from the `LEFT` side. If one or more failures occur that are unsafe with respect to the filter lock, then multiple processes may acquire the filter lock simultaneously. If this results in contention at the splitter, then all but one processes are diverted to the slow path. If forced to take the slow path, the process attempts to acquire the core lock. Once granted, it then waits to acquire the arbitrator lock from the `RIGHT` side. Finally, once the process has successfully acquired the arbitrator lock, it is deemed to have acquired the target lock as well, and is now in the CS of the target lock.

In the absence of failures, every process takes the fast path, albeit one at a time. However, some processes do take the fast path even if their super-passage overlaps with the consequence interval of an unsafe failure with respect to the filter lock. Note that at most one process can take the fast path at a time and at most one process can hold the core lock at a time. Any process that takes the fast path will always attempt to acquire the arbitrator lock from the `LEFT` side. Any process that takes the slow path and acquires the core lock will always attempt to acquire the arbitrator lock from the `RIGHT` side. Since the core lock is strongly recoverable, at most one process will try to acquire the arbitrator lock from each side at a time.
In order to release the target lock, a process simply releases its component locks in the reverse order in which it acquired them: the arbitrator lock, followed by the core lock (in case the process took the slow path), followed by the splitter and finally the filter lock.

The RMR complexity of the fast path is given by the sum of the RMR complexities of the filter lock, the splitter and the arbitrator lock. On the other hand, the RMR complexity of the slow path is given by the sum of the RMR complexities of the filter lock, the splitter, the core lock and the arbitrator lock.

For ease of exposition, we use the following terminology. Before a process is assigned a particular path, we refer to it as normal process. It is classified as fast process if it takes the fast path and slow process otherwise. A slow process becomes a medium-slow process once it acquires the core lock.

A pictorial representation of the execution flow is depicted in Figure 2. Note that the pictorial representation depicts the two sides of the arbitrator lock as left and bottom, which actually correspond to the LEFT side and the RIGHT side of the arbitrator lock respectively.

The pseudocode is given in algorithm 3. The pseudocode closely follows the above description in text. A splitter is implemented using an integer (shared) variable, namely owner. The fast path is occupied if and only if owner has non-zero value, in which case the value refers to the identifier of the process currently occupying the fast path. To take the fast path, a process attempts to store its own identifier in owner using a CAS instruction provided its current value is zero (line 20). If the attempt fails, the process changes its path type to SLOW (line 23). Note that a process resets its path type from SLOW to its default value of FAST only after it has executed the Exit segment of the core lock at least once without encountering any failure (line 37).

In Golab and Ramaraju’s framework, even if a process takes the fast path, it may still incur $\Omega(n)$ RMR complexity in the presence of even a single failure because of the overhead of aborting requests and then resetting the base lock, which is an expensive operation. In our framework, on the other hand, a process taking the fast path incurs only $O(1)$ RMR complexity even with arbitrary failures because the RMR complexity of acquiring the filter lock, followed by
Algorithm 3: Pseudocode of the framework for designing semi-adaptive lock for process \( p_i \).

```plaintext
1 shared variables
   /\ filter lock
   /\ core lock
2 \( F \): m-process weakly recoverable lock
3 \( C \): m-process strongly recoverable lock
4 \( \mathcal{A} \): m-process dual-port strongly recoverable lock
5 \( i \) : integer variable
6 \( n \) : integer variable
7 \( \text{side}[1..n] \) of boolean variables (\( \text{FAST}, \text{SLOW} \))
8 \( \text{initialization} \)
9 \( \text{owner} \leftarrow 0 \) // fast path is empty
10 \text{foreach} \( j \in [1, 2, \ldots, n] \) do
11     \text{side}[j] \leftarrow \text{FAST} // default path type
12 \text{end foreach}
13 \text{side}(type) = \begin{cases} 
  \text{Left} & \text{if type = FAST} \\
  \text{Owner} & \text{if type = SLOW} 
\end{cases}
14 \text{Function Enter()}
   /\ In order to follow the execution model of a lock described in section 2 (NC, Recover, Enter, CS, Exit in that order), we execute the Recover segment of each of the recoverable locks (\( F \), \( C \) and \( \mathcal{A} \)) just prior to executing their respective Enter segments
15 \text{end}
16 \text{Function Enter()}
17 \text{Recover();} // recover the filter lock
18 \text{FEnter();} // acquire the filter lock
19 \begin{cases} 
  \text{CEnter();} & \text{if \( \text{type}[i] \neq \text{SLOW} \) then} \\
  \text{FEnter();} & \text{if \( \text{owner} = i \) then} 
\end{cases} // attempt to take the fast path
20 \text{end if}
21 \begin{cases} 
  \text{CEnter();} & \text{if \( \text{type}[i] \rightarrow \text{SLOW} \) then} \\
  \text{FEnter();} & \text{if \( \text{owner} = i \) then} 
\end{cases} // commit to take the slow path
22 \text{end if}
23 \text{release the filter lock}
24 \text{end if}
25 \text{release the core lock}
26 \text{end if}
27 \begin{cases} 
  \text{FEnter();} & \text{if \( \text{type}[i] \rightarrow \text{FAST} \) then} \\
  \text{AEnterSide(side[type](i));} & \text{if \( \text{side}[i] \) \( \neq \text{FAST} \) then} 
\end{cases} // acquire arbitrator lock
28 \text{end if}
29 \text{end}
30 \text{Function Enter()}
31 \text{AEnterSide(side[type](i));} // release the arbitrator lock
32 \begin{cases} 
  \text{CCEnter();} & \text{if \( \text{type}[i] \rightarrow \text{SLOW} \) then} \\
  \text{FEnter();} & \text{if \( \text{owner} = i \) then} 
\end{cases} // release the core lock
33 \text{end if}
34 \begin{cases} 
  \text{CCEnter();} & \text{if \( \text{type}[i] \rightarrow \text{SLOW} \) then} \\
  \text{FEnter();} & \text{if \( \text{owner} \) \( \neq \text{i} \) then} 
\end{cases} // release the filter lock
35 \text{end if}
36 \text{end if}
37 \text{begin}
38 \text{owner} \leftarrow 0; // the fast path is now empty
39 \text{end if}
40 \text{end}
```

Navigating the splitter to take the fast path and finally acquiring the arbitrator lock is only \( O(1) \) irrespective of the number of failures.

5.1.3 Correctness proof and complexity analysis. We refer to the algorithm described in the previous section as \textit{SA-Lock}. When convenient, we use \( F \) and \( C \) to refer to the filter and core locks, respectively, of \textit{SA-Lock}.

**Theorem 5.1.** \textit{SA-Lock} satisfies the ME property.

**Proof.** A process enters the CS segment of \textit{SA-Lock} after acquiring the arbitrator lock from one of the sides. The arbitrator lock satisfies the ME property as long as no more than one process attempts to acquire it from either side \textit{LEFT} or \textit{RIGHT} at any time. The splitter ensures that, at any time, at most one process attempts to acquire the arbitrator lock from the \textit{LEFT} side. The core lock ensures that, at any time, at most one process attempts to acquire the arbitrator lock from the \textit{RIGHT} side.

**Theorem 5.2.** \textit{SA-Lock} satisfies the SF property.

**Proof.** We divide our proof into the following cases, based on whether failures occur or not, and if there is a failure, where does the failure occur.

Case 1. In the absence of failures:

When \( k \) processes try to acquire the lock, exactly one process acquires the filter lock. This process follows the fast path owing to the splitter and then acquires the \textit{arbitrator lock} from the \textit{LEFT} side. To release the lock, this process releases the component locks in the reverse order of acquisition. Since each of the component locks satisfy the SF property individually, we can claim that the lock does not starve in the absence of failures.
Case 2. When failures do occur, let $p_i$ be any arbitrary failed process.

Case 2.1. If $p_i$ fails in the Enter or Exit section of the filter lock:
Process $p_i$ will eventually restart. No process will get starved due to the SF property of the filter lock (Theorem 4.3).

Case 2.2. If $p_i$ fails while navigating through the splitter:
The splitter does not block any process. Hence SF can never be violated in this case.

Case 2.3. If $p_i$ fails in the Enter or Exit section of the core lock:
In this case, process $p_i$ must have already acquired, but not released the core lock and taken the slow path while navigating through the splitter. When $p_i$ eventually restarts, it will attempt to acquire the filter lock again. Since $p_i$ had initially already acquired the filter lock, the BCSR property of the filter lock will ensure that $p_i$ gets reentry into the critical section of the filter lock. Since $p_i$ attempted to acquire the core lock, the variable type[$i$] would have been set to SLOW. Thus, process $p_i$ would retake the slow path and attempt to acquire the core lock. Since the core lock satisfies SF, and each process that fails in the core lock, will always reacquire the core lock, SF will not be violated in this case.

Case 2.4. If $p_i$ fails in the Enter or Exit section of the arbitrator lock from the Right side:
In this case, process $p_i$ must have already acquired, but not released the core lock, taken the slow path while navigating through the splitter. When $p_i$ eventually restarts, it will attempt to acquire the filter lock again. Since $p_i$ had initially already acquired the filter lock, the BCSR property of the filter lock will ensure that $p_i$ gets reentry into the critical section of the filter lock. Since $p_i$ attempted to acquire the arbitrator lock from the Right side, the variable type[$i$] would have been set to SLOW. Thus, process $p_i$ would retake the slow path and attempt to acquire the core lock. Due to CSR property of the core lock, $p_i$ will be able to successfully acquire the core lock, and will attempt to reacquire the arbitrator lock from the Right side. Since the arbitrator lock satisfies SF, and each process that fails in the arbitrator lock will always reacquire the arbitrator lock from the same side, SF will not be violated in this case.

Case 2.5. If $p_i$ fails in the Enter or Exit section of the arbitrator lock from the Left side:
In this case, process $p_i$ must have already acquired, but not released the core lock and taken the fast path while navigating through the splitter. At this point, the value of variable owner will be set to $i$. When $p_i$ eventually restarts, it will attempt to acquire the filter lock again. Since $p_i$ had initially already acquired the filter lock, the BCSR property of the filter lock will ensure that $p_i$ gets reentry into the critical section of the filter lock. The owner variable ensures $p_i$ will retake the fast path. Process $p_i$ will then continue to reacquire the arbitrator lock from the Right side. Since the arbitrator lock satisfies SF, and each process that fails in the arbitrator lock will always reacquire the arbitrator lock from the same side, SF will not be violated in this case.

Thus we have proved that SF is not violated in any case. Hence, the lock satisfies SF property. □

**Theorem 5.3.** SA-Lock satisfies the CSR property.

**Proof.** If some process $p_i$ is in the CS segment of SA-Lock, then it currently holds the filter lock and either (a) acquired the arbitrator lock from the LEFT side by taking the fast path or (b) acquired the core lock first and then acquired the arbitrator lock from the RIGHT side by taking the slow path.
If $p_i$ fails in the CS segment of SA/Lock, it determines the path it took by checking the type[$i$] variable and then retraces the same steps it had executed earlier. Since the filter lock satisfies the BCSR property and the core lock as well as the arbitrator lock satisfies the CSR property, $p_i$ is guaranteed to be able to acquire the requisite locks and reenter the CS segment of SA/Lock within a bounded number of its own steps. Hence, SA/Lock satisfies the CSR property. \hfill \Box

Theorem 5.4. **SA/Lock satisfies the BE and BR properties.**

**Proof.** The Recover segment of SA/Lock is empty and hence it trivially satisfies the BR property.

As part of the Exit segment of SA/Lock, a process executes the Exit segment of the arbitrator lock, optionally followed by the Exit segment of the core lock, followed by the Exit segment of the filter lock. Since each of three locks individually satisfy the BE property, it follows that SA/Lock also satisfies the BE property. \hfill \Box

It follows from theorems 5.1, 5.2, 5.3 and 5.4 that

Theorem 5.5. **SA/Lock is a strongly recoverable lock.**

Theorem 5.6 (SA/Lock is bounded semi-adaptive). **The RMR complexity of SA/Lock is $O(1)$ in the absence of failures and $O(T(n))$ with arbitrary failures, where $T(n)$ denotes the worst-case RMR complexity of the core lock for $n$ processes.**

**Proof.** In the absence of failures, only one process can successfully acquire the filter lock (Definition 3.2). This process navigates the splitter in $O(1)$ steps, takes the fast path and acquires the arbitrator lock from the left side (skipping the core lock along the way). The RMR complexity of the arbitrator lock is $O(1)$. Thus, in this case, the RMR complexity of the target lock is given by $O(1)$.

In the presence of failures, all $n$ processes may be able to successfully acquire the filter lock and proceed to the splitter. Only one of these processes is allowed to take the fast path, which then attempts to acquire the arbitrator lock from the left side. The remaining $(n-1)$ processes are diverted to the slow path and have to acquire the core lock and then acquire the arbitrator lock from the right side. Thus, in this case, the RMR complexity of the target lock is given by $O(T(n))$. \hfill \Box

Theorem 5.7 (SA/Lock is sublogarithmic-bounded semi-adaptive). **Assume that we use Jayanti, Jayanti and Joshi’s RME algorithm [12] to implement the core lock. Then, the RMR complexity of SA/Lock is $O(1)$ in the absence of failures and $O(\log n/\log \log n)$ with arbitrary failures.**

In the rest of this section, we prove an important lemma that is crucial to establishing that the lock described in the next section is sublinear-adaptive.

Intuitively, the set of processes that attempt to acquire the core lock is strictly smaller than the set of processes that attempt to acquire the filter lock. Further, the size of the former set depends on the number of unsafe failures that have occurred with respect to the filter lock. To capture this formally, we first define some notations. Given a lock $l$ and time $t$, let $P(l,t)$ denote the set of processes that have begun executing the Enter segment of the lock $l$ before or at time $t$, but have not begun executing the corresponding Exit segment. Also, let $U(l,t)$ denote the set of all failures that are unsafe with respect to the lock $l$ and whose consequence interval extents at least until time $t$. Further, if a process $p$ has a pending request with respect to the target lock at time $t$, then we use $\Pi(p,t)$ to denote the super-passage of $p$ with respect to the target lock at time $t$. 

Lemma 5.8. Consider a time $t_C$ such that $|\mathcal{F}(C, t_C)| > 0$. Then there exists time $t_F$ with $t_F \leq t_C$ such that the following properties hold.

(a) $\forall p \in \mathcal{P}(C, t_C), \Pi(p, t_F) = \Pi(p, t_C)$.
(b) $\mathcal{P}(C, t_C) \subseteq \mathcal{F}(t_F, t_F)$, and
(c) $|\cup \mathcal{P}(F, t_F)| \geq |\mathcal{P}(C, t_C)|$.

Proof. None of the processes in the set $\mathcal{P}(C, t_C)$ was able to take the fast path while navigating the splitter. Let $q$ be the last process in $\mathcal{P}(C, t_C)$ to read the contents of the variable owner and $t$ denote the time when it performed the read step. Clearly, $t \leq t_C$. Furthermore, let $r$ denote the process whose identifier was stored in owner when $q$ read its contents. We set $t_F$ to $t$. We now prove each property one-by-one.

(i) Consider an arbitrary process $s \in \mathcal{P}(C, t_C)$. Assume, by the way of contradiction, that $\Pi(s, t_F) \neq \Pi(s, t_C)$. This means that process $s$ generated a new request after time $t_F$. Since $s \in \mathcal{P}(C, t_C)$, process $s$ read the contents of the variable owner some time after $t_F$ but before $t_C$. This contradicts our choice of $t_F$. In other words, $\Pi(s, t_F) = \Pi(s, t_C)$. Since $s$ was chosen arbitrarily, it follows that for each $p \in \mathcal{P}(C, t_C)$, $\Pi(r, t_F) = \Pi(p, t_C)$, Thus the property (a) holds.

(ii) Due to the arrangement of the locks, each process in the set $\mathcal{P}(C, t_C)$ holds the lock $\mathcal{F}$ at time $t_F$. Moreover, process $r$ also holds the lock $\mathcal{F}$ at time $t_F$. In other words, $\mathcal{P}(C, t_C) \subseteq \mathcal{F}(t_F, t_F)$, $r \in \mathcal{F}(t_F, t_F)$, and $r \notin \mathcal{P}(C, t_C)$. Thus the property (b) holds.

(iii) Let $|\mathcal{P}(C, t_C)| = k$. Thus, using property (b), we can conclude that $|\mathcal{P}(\mathcal{F}, t_F)| \geq k + 1$. It follows from theorem 4.2 that there exist at least $k$ failures that are unsafe relative to the lock $\mathcal{F}$ and whose consequence interval overlaps with time $t_F$. We have $|\cup \mathcal{P}(\mathcal{F}, t_F)| \geq k = |\mathcal{P}(C, t_C)|$. Thus the property (c) holds. This establishes the result. □

5.2 A sublogarithmic-bounded sublinear-adaptive RME algorithm

5.2.1 The main idea. We use the gap between the (known) worst-case RMR complexity of implementing a weakly recoverable lock and that of implementing a strongly recoverable lock to achieve our goal.

The main idea is to recursively transform the core lock using instances of our semi-adaptive lock. We transform the core lock repeatedly up to a height that is equal to the RMR complexity of the non-adaptive strongly recoverable lock under arbitrary failures. The strongly recoverable lock now becomes the base case of the recursion. For ease of exposition, we refer to the core lock in the base case as the base lock.

A pictorial representation of the execution flow of the recursive framework is depicted in Figure 3.

In order to acquire the target lock, a process starts at the first level as a normal process and waits to acquire the filter lock at level 1. It stays on track to become a fast process until an unsafe failure occurs with respect to the filter lock at the first level as a result of which multiple processes may be granted the (filter) lock simultaneously. All of these processes then compete to enter the fast path by navigating through the splitter. The splitter allows only one process to take the fast path at a time, and the rest are diverted to take the slow path. Note that a slow process is created at the first level only if an unsafe failure occurs with respect to the filter lock at the first level. All slow processes at the first level then move to the second level as normal processes. If no further failure occurs, then no slow process is created at the second level and all processes leave this level one-by-one as fast processes w.r.t.t this level. Thus, only $O(1)$ RMR complexity is added to the passages of all the affected process until the impact of the first failure has subsided. However, if one or more slow processes are created at the second level, then we can infer that a new unsafe failure
must have occurred with respect to the filter lock at the second level. All these slow processes at the second level then move to the third level as normal processes, and so on and so forth. At each level, a slow process, upon either
acquiring the base lock or returning from the adjacent higher level (whichever case applies), becomes a medium-slow process. Irrespective of whether a process is classified as fast or medium-slow, it next waits to acquire the level-specific arbitrator lock. Once granted, it either returns to the adjacent lower level or, if at the initial level, is deemed to have successfully acquired the target lock.

Note that in our algorithm, at least \( k \) unsafe failures are required at any level to force \( k \) processes to be “escalated” to the next level. Each level except for the last one would add only \( O(1) \) RMR complexity to the passages of these process, thus making the target lock adaptive under limited failures. There is no further “escalation” of slow processes at the base level and a bounded non-adaptive strongly recoverable lock is used to manage all slow processes at that point, thus bounding its RMR complexity under arbitrary failures as well.

As before, in order to release the target lock, a process releases its components locks in the reverse order in which it acquired them.

Let \( \text{BA-Lock} \) denote the bounded sublinear-adaptive lock that we wish to construct. \( \text{BA-Lock} = \text{SA-Lock}[1] \) \( \text{SA-Lock}[i].\text{core} = \text{SA-Lock}[i + 1] \quad \forall i \in \{1, 2, \ldots, m - 1\} \)

\( \text{SA-Lock}[m].\text{core} = \text{NA-Lock} \)

To prove that our target lock is sublogarithmic-bounded sublinear-adaptive, we utilize two properties of our framework, namely, our weakly recoverable lock is responsive, and our target lock, which is a composite lock, satisfies the locality property.

5.2.2 Correctness proof and complexity analysis. Let \( F_i \) and \( C_i \) denote the instances of the filter and core locks, respectively, at level \( i \) for \( i = 1, 2, \ldots, m \).

**Theorem 5.9.** For each \( i \) with \( 1 \leq i \leq m \), \( \text{SA-Lock}[i] \) is a strongly recoverable lock.

**Proof.** The proof is by backward induction on the level number of \( \text{SA-Lock} \) starting from level \( m \).

\( \square \) Base case (\( \text{SA-Lock}[m] \) is a strongly recoverable lock). Note that \( \text{SA-Lock}[m] = \text{NA-Lock} \). By construction, \( \text{NA-Lock} \) is a bounded non-adaptive strongly recoverable lock. Thus, \( \text{SA-Lock}[m] \) is a strongly recoverable lock.

\( \square \) Induction hypothesis (\( \text{SA-Lock}[i + 1] \) is a strongly recoverable lock for some \( i \) with \( 1 \leq i < m \)).

Inductive step (\( \text{SA-Lock}[i] \) is also a strongly recoverable lock). Note that \( \text{SA-Lock}[i] \) is an instance of our semi-adaptive lock described in subsection 5.1 with \( \text{SA-Lock}[i + 1] \) as its core lock. By induction hypothesis, \( \text{SA-Lock}[i + 1] \) is a strongly recoverable lock. It follows from Theorem 5.5 that \( \text{SA-Lock}[i] \) is also a strongly recoverable lock.

Thus, by induction, we can conclude that \( \text{SA-Lock}[i] \) is a strongly recoverable lock for each \( i = 1, 2, \ldots, m \). \( \square \)
By construction, BA-LOCK = SA-LOCK[1]. Therefore

**Theorem 5.10.** BA-LOCK is a strongly recoverable lock.

Using induction similar to the one used in Theorem 5.9, we can show that

**Theorem 5.11.** BA-LOCK satisfies the CSR, BR and BE properties.

To analyze the RMR complexity of a passage, we first prove certain results.

**Theorem 5.12.** BA-LOCK satisfies the locality property.

**Proof.** BA-LOCK uses three types of locks, namely filter, arbitrator and base; only filter lock is weakly recoverable. There is one instance of the filter lock at each level. By construction, the Enter segments of any two instances of the filter lock do not overlap. The only sensitive instruction of the filter lock is the FAS instruction in its Enter segment. Therefore, BA-LOCK satisfies the locality property. □

By the construction of our recursive framework, we have

**Proposition 5.13.** For each $i$ and time $t$ with $1 \leq i < m$, $P(C_i, t) = P(SA-LOCK[i + 1], t) = P(F_{i+1}, t)$.

Note that the set of processes that attempt to acquire the filter lock at any level becomes progressively smaller as the level number increases. Furthermore, the number of processes that are escalated to the next level depends on the number of unsafe failures experienced by the filter lock at the current level. This is captured by the next lemma.

**Lemma 5.14.** Consider a process $p$, time $t$ and level $x$, where $1 \leq x \leq m$, such that process $p \in P(F_x, t)$. Then there exist $x$ times $t_1, t_2, \ldots, t_{x-1}, t_x$ with $t_1 \leq t_2 \leq \cdots \leq t_{x-1} \leq t_x = t$ such that the following properties hold. For each $i$ with $1 \leq i < x$, we have

(a) $\Pi(p, t_i) = \Pi(p, t)$,
(b) $P(F_i, t_i) \supseteq P(F_{i+1}, t_{i+1})$, and
(c) $|\bigcup P(F_i, t_i)| \geq |P(F_{i+1}, t_{i+1})|$. 

**Proof.** The proof is by backward induction on $i$ starting from $x - 1$. In order to prove our results, we use the following auxiliary properties, which are part of the induction statement. For each $i$ with $1 \leq i < x$, we have,

(d) $|P(F_i, t_i)| > 0$, and
(e) $p \in P(F_i, t_i)$.

We are now ready to prove the result.

□ Base case (properties (a)-(e) hold for $i = x - 1$). By definition, $t_x = t$. By assumption, $p \in P(F_x, t_x)$. By applying Proposition 5.13, we obtain that $p \in P(C_{x-1}, t_x)$ thereby implying that $|P(C_{x-1}, t_x)| > 0$. We can now apply Lemma 5.8 once to infer that there exists time, say $t_{x-1}$ with $t_{x-1} < t_x$, such that the following properties hold.

(i) $\Pi(p, t_{x-1}) = \Pi(p, t_x)$, which, in turn, implies that $\Pi(p, t_{x-1}) = \Pi(p, t)$ because $t_x = t$ (property (a)).
(ii) $P(F_{x-1}, t_{x-1}) \supseteq P(C_{x-1}, t_x)$, which, in turn, implies that $P(F_{x-1}, t_{x-1}) \supseteq P(F_x, t_x)$ because $P(C_{x-1}, t_x) = P(F_x, t_x)$ (property (b)).
(iii) $|\bigcup P(F_{x-1}, t_{x-1})| \geq |P(C_{x-1}, t_x)|$, which, in turn, implies that $|\bigcup P(F_{x-1}, t_{x-1})| \geq |P(F_x, t_x)|$ (property (c)).
(iv) \(|\mathcal{P}(\mathcal{F}_{t_{x-1}}, t_{x-1})| > 0\) because \(\mathcal{P}(\mathcal{F}_{t_{x-1}}, t_{x-1}) \supseteq \mathcal{P}(\mathcal{F}_x, t_x) \supseteq \{p\}\) (property (d)).

(v) \(p \in \mathcal{P}(\mathcal{F}_{t_{x-1}}, t_{x-1})\) because \(\mathcal{P}(\mathcal{F}_{t_{x-1}}, t_{x-1}) \supseteq \mathcal{P}(\mathcal{F}_x, t_x) \supseteq \{p\}\) (property (e)).

\(\square\) Induction hypothesis (assume that the properties (a)-(e) hold for some \(i \leq 1 < i < x\)).

\(\square\) Inductive step (properties (a)-(e) also hold for \(i - 1\)). Note that, by induction hypothesis, \(|\mathcal{P}(\mathcal{F}_i, t_i)| > 0\).

Thus, We can now apply Lemma 5.8 once to infer that there exists time, say \(t_{i-1}\) with \(t_{i-1} < t_i\), such that the following properties hold.

(i) \(\Pi(p, t_{i-1}) = \Pi(p, t_i)\), which, in turn, implies that \(\Pi(p, t_{i-1}) = \Pi(p, t_i)\) (property (a)).

(ii) \(\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1}) \supseteq \mathcal{P}(\mathcal{C}_{i-1}, t_i)\), which, in turn, implies that \(\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1}) \supseteq \mathcal{P}(\mathcal{F}_i, t_i)\) because \(\mathcal{P}(\mathcal{C}_{i-1}, t_i) = \mathcal{P}(\mathcal{F}_i, t_i)\) (property (b)).

(iii) \(|\cup\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1})| \geq |\mathcal{P}(\mathcal{C}_{i-1}, t_i)|\), which, in turn, implies that \(|\cup\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1})| \geq |\mathcal{P}(\mathcal{F}_i, t_i)|\) (property (c)).

(iv) \(|\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1})| > 0\) because \(\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1}) \supseteq \mathcal{P}(\mathcal{F}_i, t_i) \supseteq \{p\}\) (property (d)).

(v) \(p \in \mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1})\) because \(\mathcal{P}(\mathcal{F}_{t_{i-1}}, t_{i-1}) \supseteq \mathcal{P}(\mathcal{F}_i, t_i) \supseteq \{p\}\) (property (e)).

This establishes the lemma.

The next corollary quantifies the number of processes that must be present at each of the lower levels for some process to be escalated to a certain level.

**Corollary 5.15.** Consider a process \(p\), time \(t\) and level \(x\), where \(1 \leq x \leq m\), such that process \(p \in \mathcal{P}(\mathcal{F}_x, t)\). Let times \(t_1, t_2, \ldots, t_{x-1}, t_x\) be as given by Lemma 5.14. Then, for each \(i \leq 1 < x\), \(|\mathcal{P}(\mathcal{F}_i, t_i)| \geq x - i + 1\).

The next corollary quantifies the number of unsafe failures that must occur with respect to the filter lock at each of the lower levels for some process to be escalated to a certain level.

**Corollary 5.16.** Consider a process \(p\), time \(t\) and level \(x\), where \(1 \leq x \leq m\), such that process \(p \in \mathcal{P}(\mathcal{F}_x, t)\). Let times \(t_1, t_2, \ldots, t_{x-1}, t_x\) be as given by Lemma 5.14. Then, for each \(i \leq 1 < x\), \(|\cup\mathcal{P}(\mathcal{F}_i, t_i)| \geq x - i\).

For the rest of this section, unless otherwise stated, assume that super-passage of a process and consequence interval of a failure are defined relative to the target lock.

**Theorem 5.17.** Suppose a process \(p\) advances to level \(x\) at some time \(t\) during its super-passage, where \(1 \leq x \leq m\). Then, there exist at least \(x(x-1)/2\) failures whose consequence interval overlaps with the super-passage of the process \(p\).

**Proof.** Let \(t_1, t_2, \ldots, t_x\) be the times as given by Lemma 5.14. Since BA-Lock satisfies the locality property, the set of failures that are unsafe with respect to one instance of its filter lock is disjoint from the set of failures that are unsafe with respect to another instance of its filter lock. Formally,

\[ \forall i, j: 1 \leq i, j \leq x \text{ and } i \neq j : \cup\mathcal{P}(\mathcal{F}_i, t_i) \cap \mathcal{P}(\mathcal{F}_j, t_j) = \emptyset \] (pairwise disjoint property)

Let \(\Pi\) be the super-passage of \(p\) at time \(t\). From the property (a) of Lemma 5.14, \(\Pi = \Pi(p, t_1) = \Pi(p, t_2) = \ldots = \Pi(p, t_x)\). In other words, \(p\) is executing the same super-passage during the period \([t_1, t_x]\).

Let \(\Phi\) denote the set of all failures whose consequence interval overlaps with the super-passage \(\Pi\). Note that the consequence interval of any failure with respect to the target lock contains the consequence interval of that failure with respect to any instance of its filter lock. This is because all pending requests for that instance of the filter lock are
also pending requests for the target lock. Thus, \( \forall i : 1 \leq i < x : \bigcup_{i=1}^{x-1} \mathcal{U}(F_i, t_i) \subseteq \Phi \). This in turn implies that

\[
\bigcup_{i=1}^{x-1} \mathcal{U}(F_i, t_i) \subseteq \Phi \quad \text{(containment property)}
\]

We have

\[
|\Phi| \geq |\bigcup_{i=1}^{x-1} \mathcal{U}(F_i, t_i)| \quad \text{(using containment property)}
\]

\[
= \sum_{i=1}^{x-1} |\mathcal{U}(F_i, t_i)| \quad \text{(using pairwise disjoint property)}
\]

\[
= \sum_{i=1}^{x-1} (x - i) \quad \text{(using Corollary 5.16)}
\]

\[
= (x - 1) + \cdots + 2 + 1 \quad \text{(expanding the sum)}
\]

\[
= \frac{x(x-1)}{2} \quad \text{(algebra)}
\]

This establishes the result. \( \square \)

**Theorem 5.18 (BA-Lock is bounded sublinear-adaptive).** If a super-passage of a process overlaps with the consequence interval of at most \( k \) failures, then the RMR complexity of any passage in that super-passage is given by \( O\left(\min\{\sqrt{k}, T(n)\}\right) \), where \( T(n) \) denotes the RMR complexity of the base lock NA-Lock for \( n \) processes.

**Theorem 5.19 (BA-Lock is sublogarithmic-bounded sublinear-adaptive).** Assume that we use an instance of Jayanti, Jayanti and Joshi algorithm \([12]\) to implement the base lock NA-Lock. If a super-passage of a process overlaps with the consequence interval of at most \( k \) failures then the RMR complexity of any passage in that super-passage is given by \( O\left(\min\{\sqrt{k}, \log n/\log \log n\}\right) \).

### 6 RELATED WORK

Bohannon et al. \([4, 5]\) were the first ones to investigate the RME problem. However, their system model is different from the one assumed in this work. Specifically, in their system model, at least one process is reliable while other processes may be unreliable. Once an unreliable process fails, it never restarts. The reliable process is responsible for continuously monitoring the health of all other processes, and, upon detecting that an unreliable process has failed during its passage, it performs recovery by “fixing” the lock. The two RME algorithms differ in the way they implement the lock; the one in \([5]\) uses test-and-set instruction whereas the one in \([4]\) uses MCS queue-based algorithm.

Golab and Ramaraju formally defined the RME problem in \([10]\). We use the same system model as in their work. In \([10]\), Golab and Ramaraju also presented four different RME algorithms—a 2-process RME algorithm and three \( n \)-process RME algorithms. The first algorithm is based on Yang and Anderson’s lock, and is used as a building block to design an \( n \)-process RME algorithm. Both RME algorithms use only read, write and comparison-based primitives. The worst-case RMR complexity of the 2-process algorithm is \( O(1) \) whereas that of the \( n \)-process algorithm is \( O(\log n) \). Both RME algorithms have optimal RMR complexity because, as shown in \([2, 3, 18]\), any mutual exclusion algorithm that uses only read, write and comparison-based primitives has worst-case RMR complexity of \( O(\log n) \). The remaining two algorithms are unbounded adaptive (with \( f(x) = x \)) and semi-adaptive (with \( g(x) = x \)), respectively (where \( f \) and \( g \) are as per the definitions of adaptivity and boundedness from section 2).
Later, Golab and Hendler [8] proposed an RME algorithm with sub-logarithmic RMR complexity of $O(\log n / \log \log n)$ under the CC model using MCS queue based lock [14] as a building block. Note that MCS uses FAS instruction, which is not a comparison-based RMW instruction, and thus the result does not violate the previously mentioned lower bound. Their algorithm does not satisfy the bounded exit property. Moreover, it has been shown to be vulnerable to starvation [12].

Ramaraju showed in [16] that it is possible to design an RME algorithm with $O(1)$ RMR complexity provided the hardware provides a special RMW instruction to swap the contents of two arbitrary locations in memory atomically. Unfortunately, at present, no hardware supports such an instruction to our knowledge.

In [13], Jayanti et al. presented an RME algorithm with $O(\log n)$ RME complexity. Their algorithm satisfies bounded (wait-free) exit and FCFS (first-come-first-served) property.

In [12], Jayanti et al. proposed an RME algorithm that has sub-logarithmic RMR complexity of $O(\log n / \log \log n)$. To our knowledge, this is the best known RME algorithm as far as the worst-case RMR complexity is concerned that also satisfies bounded recovery and bounded exit properties.

In [9], Golab and Hendler proposed an RME algorithm under the assumption of system-wide failure (all processes fail and restart) with $O(1)$ RMR complexity.

Woelfel and Chan [17] present a constant amortized RMR complexity algorithm for solving the RME problem. However, the worst case RMR complexity of a passage in their algorithm depends on the number of failures, which may grow unboundedly. Moreover, as acknowledged by the authors, their algorithm does not satisfy the bounded (stronger) starvation freedom property. In other words, it is possible for a (slow) process to be starved indefinitely even though every process only crashes finitely many times during its super passage. Lastly, their algorithm uses an infinite array, which makes it unsuitable even for those languages that implement their own garbage collector (e.g., Java). It is not clear how it can be modified to use only an array of bounded size while maintaining constant RMR amortized complexity.

7 CONCLUSION AND FUTURE WORK

In this work, we have described a general framework to transform any non-adaptive RME algorithm into a sublinear-adaptive one without increasing its worst-case RMR complexity. In addition to the hardware instructions used by the underlying non-adaptive RME algorithm, our framework uses CAS and FAS RMW instructions, both of which are commonly available on most modern processors. When applied to the non-adaptive RME algorithm proposed in [12], it yields a sublogarithmic-bounded sublinear-adaptive RME algorithm whose RMR complexity is $O(\min\{\sqrt{F}, \log \frac{n}{\log \log n}\})$.

In this work, a failed process, upon restarting, attempts to reacquire all the locks at every level from the beginning. As a result, the worst case RMR complexity of the super-passage of such a process is $O(F_0 + \min\{\sqrt{F}, \log \frac{n}{\log \log n}\})$, where $F_0$ denotes the number of times the process fails while executing its super-passage. However, if a process keeps track of its last known level, the worst case RMR complexity of the super passage of such a process can be reduced to $O(F_0 + \min\{\sqrt{F}, \log \frac{n}{\log \log n}\})$.

In case of system wide failures, if we set $F = n$, it would imply that the RMR complexity of our algorithm would immediately jump to its worst case. However, that would not happen. The worst case RMR complexity result ($O(\min\{\sqrt{F}, \log \frac{n}{\log \log n}\})$) is achieved when failures occur in a particular pattern, and such a pattern cannot be achieved by system wide failures. In future works, we plan to analyze the effect of system wide failures, and more generally batch failures, on our RME algorithm.
Like many other RME algorithms especially those based on MCS lock [8, 12], our work has two limitations. First, it does not satisfy the FCFS property in general. Note that, unlike the RME algorithms in [8, 12], our RME algorithm satisfies the FCFS property in the absence of failures. Second, its space usage may grow unboundedly. A failure may prevent an MCS-queue node from being reused, in which case a separate memory reclamation algorithm is required to determine when it is safe to reuse such a node (and the failing process has to allocate a new node). A technique similar to epoch based reclamation can be used that limits the space complexity to $O(n \log \log n)$. This technique can also provide bounded fairness. We plan to flush out the details of such techniques in future works.

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