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**H*-ALGEBRAS AND QUANTIZATION OF PARA-HERMITIAN SPACES**

**GERRIT VAN DIJK, MICHAEL PEVZNER**

**Abstract.** In the present note we describe a family of H*-algebra structures on the set \( L^2(X) \) of square integrable functions on a rank-one para-Hermitian symmetric space \( X \).

**Introduction**

Let \( X \) be a para-Hermitian symmetric space of rank one. It is well-known that \( X \) is isomorphic (up to a covering) to the quotient space \( SL(n, \mathbb{R})/GL(n-1, \mathbb{R}) \), see [4] for more details. We shall thus assume throughout this note that \( X = G/H \), where \( G = SL(n, \mathbb{R}) \) and \( H = GL(n-1, \mathbb{R}) \).

The space \( X \) allows the definition of a covariant symbolic calculus that generalizes the so-called convolution-first calculus on \( \mathbb{R}^2 \), see ([2, 7, 8]) for instance. Such a calculus, or quantization map \( Op_{\sigma} \), from the set of functions on \( X \), called symbols, onto the set of linear operators acting on the representation space of the maximal degenerate series \( \pi_{-\frac{n}{2} + i\sigma} \) of the group \( G \), induces a non-commutative algebra structure on the set of symbols, that we suppose to be square integrable. On the other hand, the taking of the adjoint of an operator in such a calculus defines an involution on symbols. It turns out that these two data give rise to a H*-algebra structure on \( L^2(X) \).

According to the general theory, ([1, 5, 6]), every H*-algebra is the direct orthogonal sum of its closed minimal two-sided ideals which are simple H*-algebras. The main result of this note is the explicit description of such a decomposition for the Hilbert algebra \( L^2(X) \) and its commutative subalgebra of \( SO(n, \mathbb{R}) \)-invariants.

**1. Definitions and basic facts**

**1.1. H*-algebras.**

**Definition 1.1.** A set \( R \) is called a H*-algebra (or Hilbert algebra) if

(1) \( R \) is a Banach algebra with involution;

(2) \( R \) is a Hilbert space;

(3) the norm on the algebra \( R \) coincides with the norm on the Hilbert space \( R \);

(4) For all \( x, y, z \in R \) one has \( (xy, z) = (y, x^*z) \);

(5) For all \( x \in R \) one has \( \|x^*\| = \|x\| \);

(6) \( xx^* \neq 0 \) for \( x \neq 0 \).

An example of a Hilbert algebra is the set of Hilbert-Schmidt operators \( HS(I) \) that one can identify with the set of all matrices \((a_{\alpha\beta})\), where \( \alpha, \beta \) belong to a fixed set of indices \( I \), satisfying the condition \( \sum_I |a_{\alpha\beta}|^2 < \infty \).

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\textbf{Theorem 1.2.} [9], p. 331. Every Hilbert algebra is the direct orthogonal sum of its closed minimal two-sided ideals, which are simple Hilbert algebras.

Every simple Hilbert algebra is isomorphic to some algebra \( HS(I) \) of Hilbert-Schmidt operators.

\textbf{Definition 1.3.} [5], p. 101 An idempotent \( e \in R \) is said to be irreducible if it cannot be expressed as a sum \( e = e_1 + e_2 \) with \( e_1, e_2 \) idempotents which annihilate each other: \( e_1e_2 = e_2e_1 = 0 \).

\textbf{Lemma 1.4.} [5], p. 102. A subset \( I \) of a Hilbert algebra \( R \) is a minimal left (right) ideal if and only if it is of the form \( I = R \cdot e \) (\( I = e \cdot R \)), where \( e \) is an irreducible self-adjoint idempotent. Moreover \( e \cdot R \cdot e \) is isomorphic to the set of complex numbers and \( R \) is spanned by its minimal left ideals.

Observe that any minimal left ideal is closed, since it is of the form \( R \cdot e \).

\textbf{Corollary 1.5.} If \( R \) is a commutative Hilbert algebra, then any minimal left (or right) ideal is one-dimensional.

\textbf{1.2. An algebra structure on} \( L^2(X) \). Let \( G = SL(n, \mathbb{R}) \), \( H = GL(n-1, \mathbb{R}) \), \( K = SO(n) \) and \( M = SO(n-1) \). We consider \( H \) as a subgroup of \( G \), consisting of the matrices of the form \(
\begin{pmatrix}
  (\det h)^{-1} & 0 \\
  0 & h
\end{pmatrix}
\) with \( h \in GL(n-1, \mathbb{R}) \).

Let \( P^- \) be the parabolic subgroup of \( G \) consisting of \( 1 \times (n-1) \) lower block matrices \( P = \begin{pmatrix} a & 0 \\ c & A \end{pmatrix} \), \( a \in \mathbb{R}^n \), \( c \in \mathbb{R}^{n-1} \) and \( A \in GL(n-1, \mathbb{R}) \) such that \( a \cdot \det A = 1 \). Similarly, let \( P^+ \) be the group of upper block matrices \( P = \begin{pmatrix} a & b \\ 0 & A \end{pmatrix} \) \( a \in \mathbb{R}^n \), \( b \in \mathbb{R}^{n-1} \) and \( A \in GL(n-1, \mathbb{R}) \) such that \( a \cdot \det A = 1 \).

The group \( G \) acts on the sphere \( \tilde{S} = \{ s \in \mathbb{R}^n, \|s\|^2 = 1 \} \) and acts transitively on the set \( \tilde{S} = S/\sim \), where \( s \sim s' \) if and only if \( s = \pm s' \), by \( g.s = \frac{g(s)}{\|g(s)\|} \), where \( g(s) \) denotes the linear action of \( G \) on \( \mathbb{R}^n \). Clearly the stabilizer of the equivalence class of the first basis vector \( e_1 \) is the group \( P^+ \), thus \( \tilde{S} \simeq G/P^+ \). If \( ds \) is the usual normalized surface measure on \( \tilde{S} \), then \( d(g.s) = \|g(s)\|^{-n}ds \).

For \( \mu \in \mathbb{C} \), define the character \( \omega_\mu \) of \( P^\pm \) by \( \omega_\mu(P) = |a\mu| \). Consider the induced representations \( \pi_\mu^\pm = \text{Ind}_{P^\pm}^G \omega_\mu \).

Both \( \pi_\mu^+ \) and \( \pi_\mu^- \) can be realized on \( C^\infty(\tilde{S}) \), the space of even smooth functions \( \phi \) on \( S \). This action is given by
\[
\pi_\mu^\pm(\phi)(s) = \phi(g^{-1}.s) \cdot \|g^{-1}(s)\|^\mu.
\]

Let \( \theta \) be the Cartan involution of \( G \) given by \( \theta(g) = g^{-1} \). Then
\[
\pi_\mu^-(\phi)(s) = \phi(\theta(g^{-1}).s) \cdot \|\theta(g^{-1})(s)\|^\mu.
\]

Let \( (\cdot, \cdot) \) denote the usual inner product on \( L^2(S) : (\phi, \psi) = \int_S \phi(s)\overline{\psi}(s)ds \). Then this sesqui-linear form is invariant with respect to the pairs of representations \( (\pi_\mu^+, \pi_{\lambda-\mu-n}^-) \) and \( (\pi_\mu^-, \pi_{\mu-n}^-) \). Therefore the representations \( \pi_\mu^\pm \) are unitary for \( \Re \mu = -\frac{n}{2} \).

The group \( G \) acts also on \( \tilde{S} \times \tilde{S} \) by
\[
g(u, v) = (g.u, \theta(g)v).
\]

This action is not transitive: the orbit \( (\tilde{S} \times \tilde{S})o = G.(\tilde{e}_1, \tilde{e}_1) = \{(u, v) : (u, v) \neq 0\}/\sim \) is dense (here \( (\cdot, \cdot) \) denotes the canonical inner product on \( \mathbb{R}^n \)). Moreover \( (\tilde{S} \times \tilde{S})o \simeq X \).
The map \( f \mapsto f(u,v)\langle u,v \rangle^{-\frac{n}{2}+i\sigma} \), with \( \sigma \in \mathbb{R} \) is a unitary \( G \)-isomorphism between \( L^2(X) \) and \( \pi^+_{\frac{n}{2}+i\sigma} \hat{\otimes}_2 \pi^-_{\frac{n}{2}+i\sigma} \) acting on \( L^2(S \times \tilde{S}) \). The latter space is provided with the usual inner product.

Define the operator \( A_\mu \) on \( C^\infty(\tilde{S}) \) by the formula:

\[
A_\mu \phi(s) = \int_S |\langle s,t \rangle|^{-\mu-n}\phi(t)dt.
\]

This integral converges absolutely for \( \Re \mu < -1 \) and can be analytically extended to the whole complex plane as a meromorphic function of \( \mu \). It is easily checked that \( A_\mu \) is an intertwining operator, that is, \( A_\mu \pi^+_{\mu}(g) = \pi^+_{-\mu-n}(g)A_\mu \).

The operator \( A_{-\mu-n} \circ A_\mu \) intertwines the representation \( \pi^+_{\mu} \) with itself and is therefore a scalar \( c(\mu) \text{Id} \) depending only on \( \mu \). It can be computed using \( K \)-types.

Let \( c(\mu) = \int_S |\langle s,t \rangle|^{-\mu-n}dt \), then \( c(\mu) = c(\mu)e(-\mu-n) \). But on the other hand side \( c(\mu) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(-\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(-\frac{n}{2}\right)\Gamma\left(-\frac{n+1}{2}\right)} \). One also shows that \( A^*_\mu = A_\mu \). So that, for \( \mu = -\frac{n}{2} + i\sigma \) we get (by abuse of notations):

\[
c(\sigma) = \frac{\left(\frac{n}{2}\right)^2}{\pi} \frac{\Gamma\left(-\frac{n+1}{2}\right)\Gamma\left(-\frac{n+1}{2}+i\sigma\right)}{\Gamma\left(-\frac{n+1}{2}+2i\sigma\right)} ,
\]

and moreover \( A_{-\frac{n}{2}+i\sigma} \circ A^*_{-\frac{n}{2}+i\sigma} = c(\sigma) \text{Id} \), so that the operator \( d(\sigma)A_{-\frac{n}{2}+i\sigma} \), where \( d(\sigma) = \frac{\sqrt{\pi}}{\Gamma\left(-\frac{n+1}{2}+i\sigma\right)} \) is a unitary intertwiner between \( \pi^-_{\frac{n}{2}+i\sigma} \) and \( \pi^+_{\frac{n}{2}-i\sigma} \).

We thus get a \( \pi^+_{\frac{n}{2}+i\sigma} \hat{\otimes}_2 \pi^-_{\frac{n}{2}+i\sigma} \) invariant map from \( L^2(X) \) onto \( L^2(S \times \tilde{S}) \) given by

\[
f \mapsto d(\sigma) \int_S f(u,w)|\langle u,w \rangle|^{-\frac{n}{2}+i\sigma}|\langle v,w \rangle|^{-\frac{n}{2}-i\sigma}dw =: (T_\sigma f)(u,v), \forall \sigma \neq 0.
\]

This integral does not converge absolutely, it must be considered as obtained by analytic continuation.

**Definition 1.6.** A symbolic calculus on \( X \) is a linear map \( Op_\sigma : L^2(X) \to \mathcal{L}(L^2(\tilde{S})) \) such that for every \( f \in L^2(X) \) the function \( (T_\sigma f)(u,v) \) is the kernel of the Hilbert-Schmidt operator \( Op_\sigma(f) \) acting on \( L^2(\tilde{S}) \).

**Definition 1.7.** The product \( \#_\sigma \) on \( L^2(X) \) is defined by

\[
Op_\sigma(f \#_\sigma g) = Op_\sigma(f) \circ Op_\sigma(g), \forall f,g \in L^2(X).
\]

We thus have

- The product \( \#_\sigma \) is associative.
- \( \|f \#_\sigma g\|_2 \leq \|f\|_2 \cdot \|g\|_2 \), for all \( f,g \in L^2(X) \).
- \( Op_\sigma(L_x f) = \pi^+_{\frac{n}{2}+i\sigma}(x) Op_\sigma(f) \pi^+_{\frac{n}{2}+i\sigma}(x^{-1}) \), so \( L_x (f \#_\sigma g) = (L_x f) \#_\sigma (L_x g) \), for all \( x \in G \), where \( L_x \) denotes the left translation by \( x \in G \) on \( L^2(X) \).

This non-commutative product can be described explicitly:

\[
(f \#_\sigma g)(u,v) = d(\sigma) \int_S \int_S f(u,x)g(y,v)|\langle u,y,x,v \rangle|^{-\frac{n}{2}+i\sigma}d\mu(x,y),
\]

where \( d\mu(x,y) = |\langle x,y \rangle|^{-n}dxdy \) is a \( G \)-invariant measure on \( S \times \tilde{S} \) for the \( G \)-action \( \Box \), and

\[
[u,y,x,v] = \frac{\langle u,x \rangle \langle y,v \rangle}{\langle u,v \rangle \langle x,y \rangle}.
\]
On the space $L^2(X)$ there exists an (family of) involution $f \to f^*$ given by: $Op_{\sigma}(f^*) = Op_{\sigma}(f)^*$. Notice that the correspondence $f \to Op_{\sigma}(f^*)$ is what one calls in pseudo-differential analysis "anti-standard symbolic calculus". The link between symbols of standard and anti-standard calculus in the setting of the para-Hermitian symmetric space $X$ has been made explicit in [7] Corollary 1.4, see also Section 3.

Obviously we have $(f \sharp_{\sigma} g)^* = g^* \sharp_{\sigma} f^*$ and with the above product and involution, the Hilbert space $L^2(X)$ becomes a Hilbert algebra.

2. The structure of the subalgebra of $K$-invariant functions in $L^2(X)$

Let $A$ be the subspace of all $K$-invariant functions in $L^2(X)$.

**Theorem 2.1.** The subset $A$ is a closed subalgebra of $L^2(X)$ with respect to the product $\sharp_{\sigma}$.

This statement clearly follows from the covariance of the symbolic calculus $Op_{\sigma}$, namely: $L_x(f \sharp_{\sigma} g) = (L_x f) \sharp_{\sigma} (L_x g)$, for all $x \in G, f, g \in L^2(X)$.

**Theorem 2.2.** Let $n > 2$, then the subalgebra $A$ is commutative.

**Proof.** For a function $f \in L^2(X)$ we set $\tilde{f}(u, v) = f(v, u)$. The map $f \to \tilde{f}$ is a linear involution. Indeed,

$$
(f \sharp_{\sigma} g)(u, v) = d(\sigma) \int_S \int_S \tilde{f}(x, u) \tilde{g}(v, y) |[u, y, x, v]|^{-\frac{n}{2} + i\sigma} d\mu(x, y).
$$

Permuting $x$ and $y$ and $u$ and $v$ respectively, we get

$$
(f \sharp_{\sigma} g)(v, u) = d(\sigma) \int_S \int_S \tilde{g}(u, x) \tilde{f}(y, v) |[v, x, u, y]|^{-\frac{n}{2} + i\sigma} d\mu(x, y).
$$

But $|[v, x, y, u]| = |[u, y, x, v]|$, therefore $(f \sharp_{\sigma} g) = \tilde{g} \sharp_{\sigma} \tilde{f}$.

On the other hand, given a couple $(u, v) \in \tilde{S} \times \tilde{S}$ there exists an element $k \in K$ such that $k.(u, v) = (v, u)$. Geometrically $k$ can be seen as a rotation of angle $\pi[2\pi]$ around the axis defined by the bisectrix of vectors $u$ and $v$ in the plane they generate. Of course, such a $k$ exists for an arbitrary couple $(u, v)$ only if $n > 2$.

Hence for every $f \in A$ we have $f = \tilde{f}$ and therefore $f \sharp_{\sigma} g = g \sharp_{\sigma} f$, for $f, g \in A$. $\square$

3. Irreducible self-adjoint idempotents of $A$

We begin with a reduction theorem for the multiplication and involution in $L^2(X)$.

As usual, we shall identify $L^2(X)$ with $L^2(\tilde{S} \times \tilde{S}; |\langle x, y \rangle|^{-n} dx dy)$. If $\phi \in L^2(X)$ we shall write $\phi(u, v) = |\langle u, v \rangle|^{n/2 - i\sigma} \phi_o(u, v)$. Then $\phi_o \in L^2(\tilde{S} \times \tilde{S}; ds dt) = L^2(\tilde{S} \times \tilde{S})$, and therefore the map $\phi \to \phi_o$ is an isomorphism.

**Theorem 3.1.** Under the isomorphism $\phi \to \phi_o$ the product $\#_{\sigma}$ translates into

$$
\phi_o \#_{\sigma} \psi_o(u, v) = d(\sigma) \int_S \int_S \phi_o(u, x) \psi_o(y, v) |\langle x, y \rangle|^{-n/2 + i\sigma} dx dy
$$

and the involution becomes:

$$
\phi_o^*(u, v) = d(\sigma)^2 \int_S \int_S \tilde{\phi}_o(x, y) |\langle x, y \rangle| |\langle u, y \rangle|^{-n/2 + i\sigma} dx dy.
$$
for some constant \(a\).

Let \(\phi\) be an irreducible self-adjoint idempotent in \(A\). We shall give an explicit formula for the \(\phi_o\)-component of \(\phi\).

Consider the decomposition of the space \(L^2(\mathbb{S} \times \mathbb{S}) = \oplus_{\ell \in 2\mathbb{N}} V_\ell\), where \(V_\ell\) is the space of harmonic polynomials on \(\mathbb{R}^n\), homogeneous of even degree \(\ell\).

Then the space \(L^2(\mathbb{S} \times \mathbb{S})\) decomposes into a direct sum of tensor products \(\oplus_{\ell,m \in 2\mathbb{N}} V_\ell \otimes \bar{V}_m\) and consequently \(L^2_K(\mathbb{S} \times \mathbb{S}) = \oplus_{\ell \in 2\mathbb{N}} (V_\ell \otimes \bar{V}_\ell)^K\), where the sub(superscript)-script \(K\) means: “the \(K\)-invariants in”.

Let \(\dim V_\ell = d\) and \(f_1, \ldots, f_d\) be an orthonormal basis of \(V_\ell\). Then the function \(\theta_\ell(u, v) = \sum_{i=1}^d f_i(u)f_i(v)\), that is the reproducing kernel of \(V_\ell\), is, up to a constant, the \(K\)-invariant element of \(V_\ell \otimes \bar{V}_\ell\).

**Theorem 3.2.** Let \(\phi(u, v) = |\langle u, v \rangle|^{\frac{n}{2} - i\sigma}\phi_o(u, v)\) be an irreducible self-adjoint idempotent in \(A\). Then there exist complex numbers \(c(\sigma, \ell)\) such that for any \(\ell \in 2\mathbb{N}\) one has

\[
\phi_o(u, v) = c(\sigma, \ell) \theta_\ell(u, v).
\]

For different \(\ell\) and \(\ell'\) the idempotents annihilate each other. Moreover they span \(A\).

**Proof.** Firstly we shall show that \(\theta_\ell\) satisfies the condition

\[
\theta_\ell \#_o \theta_\ell = a(\sigma, \ell) \theta_\ell
\]

for some constant \(a(\sigma, \ell)\). Indeed,

\[
d(\sigma) \int_\mathbb{S} \int_\mathbb{S} \theta_\ell(u, x) \theta_\ell(y, v) |\langle x, y \rangle|^{-\frac{n}{2} - i\sigma} dx dy
\]

\[= d(\sigma) e_\ell(\sigma) \int_\mathbb{S} \theta_\ell(u, y) \theta_\ell(y, v) dy = d(\sigma) e_\ell(\sigma) \theta_\ell(u, v)\]

by the intertwining relation (apply \(A_{-\frac{n}{2} + i\sigma}\) to \(\theta_\ell(., x)\)):

\[
\int_\mathbb{S} \theta_\ell(u, x) |\langle x, y \rangle|^{-\frac{n}{2} - i\sigma} dx = e_\ell(\sigma) \theta_\ell(u, y)
\]

where \(e_\ell(\sigma) = \int_\mathbb{S} \frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)} |x_1|^{-\frac{n}{2} - i\sigma} dx\).

Observe that \(\frac{\theta_\ell(e_1, x)}{\theta_\ell(e_1, e_1)}\) is a spherical function on \(\mathbb{S}\) with respect to \(M\) of the form

\[a_\ell C_{\ell}^{\frac{n-2}{2}}(|x_1|)\)

where \(C_{\ell}^{\frac{n-2}{2}}(u)\) is a Gegenbauer polynomial and

\[a_\ell^{-1} = C_{\ell}^{\frac{n-2}{2}}(1) = 2^{\ell} \frac{\Gamma(\frac{n-2}{2} + \ell)}{\Gamma(\frac{n-2}{2})} \ell!\]

See for instance [3], Chapter IX, §3. Notice that \(\theta_\ell(e_1, e_1) = \dim V_\ell = \frac{(n + \ell - 1)!}{(n - 1)!\ell!} \neq 0\).

The integral defining \(e_\ell(\sigma)\) does not converge absolutely, but has to be considered as the meromorphic extension of an analytic function. Poles only occur in half-integer points on the real axis. So we have to restrict (and we do) to \(\sigma \neq 0\).

So we have \(\theta_\ell \#_o \theta_\ell = d(\sigma) e_\ell(\sigma) \theta_\ell\) and hence \(\varphi_\ell = [d(\sigma) e_\ell(\sigma)]^{-1} \theta_\ell\) is the \(\phi_o\)-component of an idempotent in \(A\). Furthermore \(\theta_\ell \#_o \theta_\ell = 0\) if \(\ell \neq \ell'\). Clearly \(\varphi_\ell\) is self-adjoint, since

\[|d(\sigma)|^{-2} = |e_\ell(\sigma)|^2,\]

being equal to the constant \(c(\sigma)\) from Section 1.
So the $\varphi_\ell$ are mutually orthogonal idempotents in the algebra $L^2_v((\tilde{S} \times \tilde{S}); dsdt)$ and span this space. The theorem now follows easily. □

**Remark** The constant $e_\ell(\sigma)$ can of course be computed. Apply e.g. [3], Section 7.31, we get, by meromorphic continuation:

$$
e_\ell(\sigma) = a_\ell \int_S C^{n-2}_\ell(|x_1|)|x_1|^{-\frac{n}{2}-i\sigma} \, dx
= 2a_\ell \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-2}{2})} \sqrt{\pi} \int_0^1 u^{-\frac{n}{2}-i\sigma} (1-u^2) \frac{\pi^{n-2}}{2} C^{n-2}_\ell(u) \, du
= 2^{-2\ell} \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}} \frac{\Gamma(n-2+\ell)}{\Gamma(n-2)} \frac{\Gamma(n-2)}{\Gamma(-\frac{n}{2}+\ell)} \frac{\Gamma(-\frac{n}{2}-i\sigma+1)}{\Gamma(-\frac{n}{2}-i\sigma-\ell+1)} \frac{\pi^{n-2}}{2}.$$

4. **The structure of the Hilbert algebra $L^2(X)$**

We now turn to the full algebra $L^2(X)$. We again reduce the computations to $L^2(\tilde{S} \times \tilde{S})$. In a similar way as for $A$ we get:

**Lemma 4.1.** If $\phi_o \in V_\ell \otimes \tilde{V}_m$, $\psi_o \in V_{\ell'} \otimes \tilde{V}_{m'}$ then

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } m \neq \ell' \\ \text{in } V_\ell \otimes \tilde{V}_{m'} & \text{if } m = \ell'. \end{cases}$$

More precisely we have the following result. Let $(f_i, (g_j), (k_l))$ be orthonormal bases of $V_\ell, V_m$ and $V_{m'}$ respectively, and $\phi_o(u, v) = f_i(u)\overline{g_j(v)}$, $\psi_o(u, v) = g_{j'}(u)\overline{k_l}(v)$, then

$$\phi_o \#_\sigma \psi_o = \begin{cases} 0 & \text{if } j \neq j' \\ d(\sigma) e_m(\sigma) f_i(u)\overline{k_l}(v) & \text{if } j = j'. \end{cases}$$

The proof is again straightforward and uses the intertwining relation:

$$\int_S |\langle x, y \rangle|^{-n/2-i\sigma} g_{j'}(y)dy = e_m(\sigma) g_{j'}(x).$$

**Theorem 4.2.** The irreducible self-adjoint idempotents of $L^2(\tilde{S} \times \tilde{S})$ are given by

$$e_\ell^f(u, v) = \{d(\sigma) e_\ell(\sigma)\}^{-1} f(u)\overline{f}(v)$$

with $f \in V_\ell$, $\|f\|_{L^2(\tilde{S})} = 1$ and $\ell$ even. The left ideal generated by $e_\ell^f$ is equal to $L^2(\tilde{S}) \otimes \overline{\mathcal{T}}$.

The proof is by application of Lemma 4.1.

**Remarks**

1. The minimal right ideals are obtained in a similar way.
2. The minimal two-sided ideal generated by $L^2(\tilde{S} \times \tilde{S}) \cdot e_\ell^f$ is the full algebra $L^2(\tilde{S} \times \tilde{S})$.
3. The closure of $\bigoplus_{\ell \in \mathbb{N}} V_\ell \otimes \overline{\mathcal{T}}$ is a $H^*$-subalgebra of $L^2(\tilde{S} \times \tilde{S})$. The minimal left ideals are here $V_\ell \otimes \overline{\mathcal{T}}$ ($f \in V_\ell$, $\|f\|_{L^2(\tilde{S})} = 1$); they are generated by the $e_\ell^f$ as above. The minimal two-sided ideal generated by $V_\ell \otimes \overline{\mathcal{T}}$ is equal to $V_\ell \otimes \overline{\mathcal{T}}$.

5. **The case of a general para-hermitian space**

It is not necessary to assume rank $X = 1$ in order to show that $A$ is commutative. Theorem 3.2 is also valid mutatis mutandis in the general case since $(K, K \cap H)$ is a Gelfand pair, and it clearly implies the commutativity of $A$. To the general construction of the product and the involution we shall return in another paper.
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