Modeling of the interaction of cylindrical bodies with complex surface properties

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Abstract. The axisymmetric contact problem of interaction between cylindrical bodies with complex surface properties (elastic properties or shapes) is described. For this problem, its mathematical model is constructed. It is mixed integral equation. An analytical solution for the problem of finding contact stresses under the bush is obtained. It allow one to make efficient calculations even when the coating nonuniformity and bush internal profile described by rapidly changing functions.

1. Introduction
Pipes are widely used in industry. Frameworks of buildings and structures can be assembled from pipes. Heat, gas, oil, water is transported to consumers through pipes. However, pipes can be exposed to external and internal aggressive environments, as well as meet safety requirements. Then it is necessary the use multilayered pipes. Each layer of such a pipe performs its function. Due to the various manufacturing processes (see, for example, [1]), the layers may have various complex properties and shapes. Moreover, the properties of the layers change over time, especially in the first month after manufacture, if we use aging materials for pipe layers. To fix and connect the pipes, various fasteners are used. Such fasteners can, for example, compress the pipe, burst it from the inside. As a result of this, the pipe is deformed, and contact stresses occur at the attachment areas. Elevated stress levels can cause pipe failure over time. An accurate calculation, taking into account the complex properties and shapes of the contacting elements, will allow one to pre-adjust the parameters to ensure the durability of such a connection. Such studies will allow the engineers, at the design stage, to take necessary measures to change the design or the conditions of its work, which will lead to economic benefits in the future.

This article is devoted to the formulation and analytical solution of the problem of the interaction of a pipe with nonuniform outer layer and an aging viscoelastic inner layer and a rigid bush with complex internal profile. Similar problems for flat bases were considered in [2–4]. Different methods for solving contact problems were developed, for example, in works [5–12]. This work allows us to obtain an effective analytical solution for one more problem in this area.

2. Mathematical statement of contact problem
The pipe consists of two layers: viscoelastic aging inner layer and nonuniform elastic outer layer (see figure 2). The inner radius of inner layer is equal to \( r_{in} \) and its thickness is equal to \( h_{in} \). The inner radius of outer layer is equal to \( r_{out} = r_{in} + h_{in} \) and its thickness is equal to \( h_{out} \). Properties
of inner layer depend on time $t$ due to it made of viscoelastic aging material. Properties of outer layer depend on longitudinal coordinate $z$. We put on the rigid bush on the described pipe at time $\tau_0$. The inner radius of the bush depends on the $z$-coordinate and does not exceed the outer radius of the outer pipe, i.e. $g(z) \leq r_{out} + h_{out}$. (In this case, the tension depends on the coordinate $z$ and is equal to $\delta(z) = r_{out} + h_{out} - g(z)$. Note that the tension must be equal or greater than zero during the entire time the pipe and bush interact.) We restrict ourselves to the case when the following requirements are satisfied: 1) coating thickness is an order of magnitude less than bush length $2a$ or inner radius of pipe $r_{in}$, i.e. $h_{out} \ll \{2a, r_{in}\}$; 2) outer layer softer then inner layer; 3) there is a smooth contact between layers and between outer layer and bush; 4) the bush is situated at a sufficient distance from the ends of the pipe.

Replace the bush by distributed load $q(z,t)$ on the region $[-a,a]$. Then we obtain the boundary value problem of a distributed load acting on a two-layer pipe. It can be shown that expression for displacements of outer layer for the case when nonuniformity depend only on $z$ coordinate almost coincide with expressions for the case of uniform layers (solution for pipe with

**Figure 1.** Cylindrical contact problem.
uniform layers obtained in [13, 14]):
\[ u_r(r_{out} + h_{out}, z, t) = -\frac{1 - \nu_{out}^2(z)}{E_{out}(z)} h_{out} q(z, t) \]
\[ - \frac{2(1 - \nu_{in}^2)}{\pi} \int_0^a k_{cyl}(\frac{z - \zeta}{r_{in}}) q(\zeta, t) d\zeta \]
\[ - \int_{\tau_0}^t \frac{K(t - \tau_{in}, \tau - \tau_{in})}{E_{in}(\tau - \tau_{in})} \int_0^a k_{cyl}(\frac{z - \zeta}{r_{in}}) q(\zeta, \tau) d\zeta d\tau, \quad z \in [-a, a], \quad t \geq \tau_0. \]

In this equation \( \nu_{out}(z) \) and \( E_{out}(z) \) are the Poisson’s ratio and Young’s modulus of outer layer depending on the longitudinal coordinate; \( \nu_{in} \) and \( E_{in}(t) \) are the Poisson’s ratio and Young’s modulus of inner layer depending on time; \( \tau_{in} \leq \tau_0 \) is moment of inner layer manufacturing; \( K(t, \tau) \) is creep kernel (see, for example, [15, 16])

\[ K(t, \tau) = E_{in}(\tau) \frac{\partial}{\partial \tau} \left[ \frac{1}{E_{in}(\tau)} + C_{in}(t, \tau) \right], \]

\( C_{in}(t, \tau) \) is the tensile creep function; \( k_{cyl}(s) \) is kernel of the cylindrical boundary value problem, which can be calculated by the formulas

\[ k_{cyl}(s) = \int_0^\infty \frac{L(u)}{u} \cos(su) \, du, \quad L(u) = \frac{u |u^2 B_2^2(u) - f(1, u) D_1^2(u) - 1|}{S(u)}, \quad (1) \]

\[ S(u) = f(1, u) + f(k_r, u) + k_r u^2 A_1^2(u) - u^2 f(k_r, u) B_1^2(u) \]
\[ - k_r u^2 f(1, u) C_1^2(u) + f(1, u) f(k_r, u) D_1^2(u), \quad f(r, u) = \frac{2(1 - \nu_{in})}{r} + u^2 r, \quad k_r = \frac{r_{out}}{r_{in}}, \]
\[ A_1(u) = I_0(u) K_0(k_r u) - I_0(k_r u) K_0(u), \quad B_1(u) = I_0(u) K_1(k_r u) + I_1(k_r u) K_0(u), \]
\[ C_1(u) = I_0(k_r u) K_1(u) + I_1(u) K_0(k_r u), \quad D_1(u) = I_1(u) K_1(k_r u) - I_1(k_r u) K_1(u), \]

\( I_0(u), I_1(u), K_0(u), \) and \( K_1(u) \) are Bessel functions.

But radial displacement of the outer face of the such a tube under the bush equal to tension \( \delta(z) = r_{out} + h_{out} - g(z) \), i.e. \( u_r(r_{out} + h_{out}, z, t) = -\delta(z) \). So we obtain integral equation for finding contact pressures

\[ \frac{1 - \nu_{out}^2(z)}{E_{out}(z)} h_{out} q(z, t) + \frac{2(1 - \nu_{in}^2)}{\pi} \int_0^a k_{cyl}(\frac{z - \zeta}{r_{in}}) q(\zeta, t) d\zeta \]
\[ - \int_{\tau_0}^t \frac{K(t - \tau_{in}, \tau - \tau_{in})}{E_{in}(\tau - \tau_{in})} \int_0^a k_{cyl}(\frac{z - \zeta}{r_{in}}) q(\zeta, \tau) d\zeta d\tau \]
\[ = r_{out} + h_{out} - g(z), \quad z \in [-a, a], \quad t \geq \tau_0. \quad (2) \]

It is mixed integral equation with known right side. We will find contact stresses \( q(z, t) \) from this equation.

3. Mixed integral equation in dimensionless form

We bring the resulting equation to a dimensionless form. To this end, we make the change of variables in (2) by the formulas \( E_\infty \) is dimensional modulus; it can be calculated, for example,
as \( E_0 = \max_{z \in (-\infty, +\infty)} E_{out}(z) \)

\[
z^* = \frac{z}{a}, \quad \zeta^* = \frac{\zeta}{a}, \quad t^* = \frac{t}{\tau_0}, \quad \tau_{in}^* = \frac{\tau_{in}}{\tau_0}, \quad g^*(z^*) = \frac{g(z) - (r_{out} + h_{out})}{a},
\]

\[
c^*(t^*) = \frac{E_{in}(t - \tau_{in})}{E_0}, \quad m^*(z^*) = \frac{[1 - \nu^2_{out}(z)]h_{out} E_0}{2(1 - \nu^2_{in})aE_{out}(z)}, \quad q^*(z^*, t^*) = \frac{2(1 - \nu^2_{in})q(z, t)}{E_{in}(t - \tau_{in})},
\]

\[
k_{cyl}^*(z^*, \xi^*) = \frac{1}{\pi} k_{cyl} \left( \frac{z - \zeta}{r_{in}} \right), \quad K^*(t^*, \tau^*) = K(t - \tau_{in}, \tau - \tau_{in})\tau_0.
\]

So the mixed integral equation (2) transforms into a dimensionless form

\[
c^*(t^*)m^*(z^*)q^*(z^*, t^*) + \int_{-1}^{1} k_{cyl}^*(z^*, \zeta^*)q^*(\zeta^*, t^*) d\zeta^*
\]

\[
- \int_{1}^{t^*} K^*(t^*, \tau^*) \int_{-1}^{1} k_{cyl}^*(z^*, \zeta^*)q^*(\zeta^*, \tau^*) d\zeta^* d\tau^* = -g^*(z^*), \quad z^* \in [-1, 1], \quad t^* \geq 1.
\]

Note that functions \( m^*(z^*) \) and \( g^*(z^*) \) connect with properties and forms of contacting bodies. They can be described by a rapidly changing and even discontinuous function.

4. Comparison with similar problems and analytical solution
We consider the obtained integral equation (3). It is similar to equation (1.2) for plane problem from the paper [4] if \( \delta(t) \equiv 0 \) and \( \alpha(t) \equiv 0 \). Moreover kernels of plane and cylindrical problems has same properties. It can be shown that for both problems \( L(u) > 0 \) for all \( u > 0 \) and (properties of function \( L(u) \) see, for example, in [8])

\[
\lim_{u \to 0} \frac{L(u)}{u} = \text{const} > 0, \quad \lim_{u \to \infty} L(u) = 1.
\]

A solution for such an equation will be constructed by a similar method, with the only exception that in the present paper the right-hand side is known: 1) we should use special representation of unknown function:

\[
q^*(z^*, t^*) = \frac{Q(z^*, t^*)}{\sqrt{m^*(z^*)}} - \frac{g^*(z^*)}{c^*(t^*)m^*(z^*)},
\]

where \( Q(z^*, t^*) \) is new unknown function; 2) we should use special basis (see, for example, [17]):

\[
p_j^0(z^*) = \frac{p_j(z^*)}{\sqrt{m^*(z^*)}}, \quad p_j(z^*) = \frac{1}{\sqrt{d_{j-1}d_j}} \begin{vmatrix} J_0 & J_1 & \cdots & J_j & \cdots & J_i & \cdots & J_{j-1} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
J_{j-1} & J_j & \cdots & J_{2j-1} & \cdots & J_i & \cdots & J_{2i} \\
1 & z^* & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \quad d_i = \begin{vmatrix} J_0 & \cdots & J_i \\
\vdots & \ddots & \vdots \\
J_i & \cdots & J_{2i} \end{vmatrix},
\]

\[
J_i = \int_{-1}^{1} (\zeta^*)^id\zeta^*, \quad d_{-1} = 1, \quad p_0(z^*) = \frac{1}{\sqrt{J_0}}, \quad i, j = 0, 1, 2, \ldots.
\]

Finally we can obtain following formulas for function \( q^*(z^*, t^*) \) in dimensionless form:

\[
q^*(z^*, t^*) = \frac{1}{m^*(z^*)} \left[ \sum_{k=0}^{\infty} f_k^*(t^*) \sum_{m=0}^{\infty} q_m^*(z^*) \right] - \frac{g^*(z^*)}{c^*(t^*)}, \quad z^* \in [-1, 1], \quad t^* \geq 1.
\]
where

\[ f_k^*(t^*) = \frac{g_k}{c^*(t^*)} + \int_1^{t^*} \frac{g_k R_k(t^*, \tau^*)}{c^*(\tau^*)} d\tau^*, \]

\[ g_k = \sum_{i=0}^{\infty} \sum_{l=0}^{(k)} R_{il} \int_{-1}^{1} \frac{p_i(z^*) g^*(\zeta^*)}{m^*(z^*)} d\zeta^*, \]

\[ R_{il} = \int_{-1}^{1} \int_{-1}^{1} \frac{k_{cyl}(z^*, \zeta^*) p_i(z^*) p_l(\zeta^*)}{m^*(z^*) m^*(\zeta^*)} dz^* d\zeta^*, \]

\[ R_k(t^*, \tau^*) \] is the resolvent of the kernel

\[ K_k(t^*, \tau^*) = \frac{\gamma_k K(t^*, \tau^*)}{c^*(t^*) + \gamma_k}, \]

i.e. if \( f_2(t^*) = f_1(t^*) - \int_1^{t^*} K_k(t^*, \tau^*) f_1(\tau^*) d\tau^* \) then \( f_1(t^*) = f_2(t^*) + \int_1^{t^*} R_k(t^*, \tau^*) f_2(\tau^*) d\tau^* \).

Coefficients \( \gamma_k \) and \( \psi_m^{(k)} \) can be found from the solution of following spectral problem:

\[ \sum_{l=0}^{\infty} R_{il} \psi_l^{(k)} = \gamma_k \psi_m^{(k)}. \]

Making the inverse change of variables in (4) we can obtain a solution in the dimensional form

\[ q(z, t) = \frac{E_{out}(z) a}{1 - \nu_{out}^2} \left[ \sum_{k=0}^{\infty} f_k(t) \sum_{m=0}^{\infty} \psi_m^{(k)} p_m \left( \frac{z}{a} \right) + r_{out} + h_{out} - g(z) \right], \]

\[ f_k(t) = f_k^* \left( \frac{t}{\tau_0} \right) \frac{E_{in}(t - \tau_{in})}{E_0}, \quad z \in [-a, a], \quad t \geq \tau_0. \]

It can be seen that the functions describing the internal shape of the bush and the variable properties of the outer layer of the pipe are highlighted by separate terms and factors. It allow one to make precision calculations even if these parameters described by rapidly changing functions (elastic properties can be described by discontinuous functions). This cannot be achieved by other known methods. A detailed analysis of proposed approach used for plane and axisymmetric problems was carried out in works [2, 18]. Since the mathematical model of the cylindrical problem is similar, and the Fredholm operators kernels has the same properties, the conclusions about the effectiveness of this approach for solving problems for coated cylindrical bodies are similar. In these works, it was shown that even when calculating the interaction of solids and elastic bodies with inhomogeneous coatings or coatings of variable thickness, significant errors arise due to an ineffective approach. This is due to the fact that standard approaches do not take into account the features of the functions that make up the equation. Moreover, the proposed method allows one to construct the analytical solution for viscoelastic bodies. In other known methods, it is necessary to solve an infinite Volterra linear integral equations system, while in proposed approach, we obtain a diagonal matrix of the system, and its solution can be constructed elementary.

5. Main results and conclusions

The problem of finding the distribution of contact pressures in a two layered pipe clamped by a bush is posed. The mathematical model of this problem is a mixed integral equation. It takes into account the nonuniformity of the outer layer of the pipe, and the changed properties over time of the its inner layer, and the complex inner profile of the bush. The article shows that this
mathematical model is similar to the mathematical model for the problem of the action of a hard punch on a two-layer base, considered earlier in previous articles. This allowed us to obtain a solution for posed problem in a similar way.

In final relation of contact stresses under the bush elastic properties of coating and bush internal shape are represented by separate terms and factor. Such form of solution allows one to perform effective computations for actual coating nonuniformities and bush internal shapes using small number of terms in sequence. This cannot be achieved by other known methods.

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7. References
[1] Parshin D A and Manzhirov A V 2017 Quasistatic problems for piecewise-continuously growing solids with integral force conditions on surfaces expanding due to additional material influx J. Phys.: Conf. Ser. 937 012038 DOI: 10.1088/1742-6596/937/1/012038
[2] Kazakov K E and Manzhirov A V 2008 Conformal contact between layered foundations and punches Mech. Solids 43(3) 512-524 DOI: 10.3103/S0025654408030229.
[3] Kazakov K E 2009 Modeling of contract interaction of solids with inhomogeneous coatings J. Phys.: Conf. Ser. 181 012013 DOI: 10.1088/1742-6596/181/1/012013.
[4] Manzhirov A V and Kazakov K E 2018 Modeling the contact interaction between a nonuniform foundation and a rough punch Math. Models Comput. Simul. 10(3) 314-321 DOI: 10.1134/S2070048218030109.
[5] Johnson K L 1985 Contact Mechanics (Cambridge, New York, New Rochelle, Melbourne, Sydney: Cambridge University Press).
[6] Laursen T A 2002 Computational Contact and Impact Mechanics: Fundamentals of Modeling Interfacial Phenomena in Nonlinear Finite Element Analysis (New York: Springer-Verlag).
[7] Alexandrov V M and Chebakov M I 2002 Analytical Methods for Contact Problems of the Elasticity Theory (Moscow: Fizmatlit).
[8] Alexandrov V M and Chebakov M I 2007 Introduction into Mechanics of Contact Interaction (Rostov-on-Don: Izdat. “TsVVR”).
[9] Soldatenkov I A 2010 Wear-Contact Problem with Applications in Engineering Wear Calculations (Moscow: Fizmatlit).
[10] Konyukhov A and Schweizerhof K 2013 Computational Contact Mechanics. Geometrically Exact Theory for Arbitrary Shaped Bodies (Berlin, Heidelberg: Springer-Verlag).
[11] Konyukhov A and Izi R 2015 Introduction to Computational Contact Mechanics: A Geometrical Approach (Wiley).
[12] Barber J R 2018 Contact Mechanics (Springer).
[13] Manzhirov A V and Chernysh V A 1988 On the interaction of a rigid reinforcing sleeve and inhomogeneous aging high-pressure pipes Mech. Solids 23(6) 104-110.
[14] Manzhirov A V and Chernysh V A 1990 Contact problem for a layered inhomogeneous aging cylinder reinforced by a rigid ring J. Appl. Mech. Tech. Phys. 31(6) 894-900 DOI: 10.1007/BF00854204.
[15] Arutyunyan N Kh 1952 Some Questions in the Creep Theory (Moscow–Leningrad: Gostekhizdat).
[16] Arutyunyan N Kh and Kolmanovskii V B 1983 Creep Theory of Inhomogeneous Solids (Moscow: Nauka).
[17] Szegő G 1959 Orthogonal Polynomials (Providence: Amer. Math. Soc.).
[18] Kazakov K and Kurdina S 2019 Contact problems for bodies with complex coatings Math. Meth. Appl. Sci. DOI: 10.1002/mma.6107.