Integrated $J_x$ photonic lattices, inspired by the quantum harmonic oscillator and due to their equidistant eigenvalue spectrum, have proven extremely useful for various applications, such as perfect imaging and coherent transfer of quantum states. However, to date their large-scale implementation remains challenging. This study applies concepts from supersymmetry (SUSY) to construct two-dimensional (2D) systems with spectra identical to that of one-dimensional (1D) $J_x$ lattices. While exhibiting different dynamics, these 2D equivalent systems retain the key imaging and state transfer properties of their 1D $J_x$ counterpart. The method extends to all systems with separable spectra and facilitates experimental fabrication of large-scale photonic circuits.

1. Introduction

Evolution dynamics in wave-mechanical systems are governed by the full set of its modes and their respective eigenvalues. In particular, the key task of imaging light between two specific planes can be understood in this framework in ways that readily generalize from conventional free-space settings involving mirrors and lenses to waveguide-based integrated optical systems where crosstalk between the individual channels cannot be globally suppressed: imaging occurs whenever the overall phases of the individual modes simultaneously coincide up to integer multiples of $2\pi$. In discrete settings with symmetric band structures, this can be readily enforced by actively swapping the excitations of mode pairs whose propagation constants are arranged symmetrically with respect to the center of the band, by continuously sweeping the wave packet through the Brillouin zone in reciprocal space by means of a transverse potential gradient (Bloch sweeping the wave packet through the Brillouin zone in reciprocal space) or, resonantly, via dynamic modulations. In contrast to these active methods, image recurrence can also occur naturally as a consequence of the structure of the eigenvalue spectrum: regardless of the specific excitation, any structure with a commensurable set of eigenvalues $\{\lambda_i\}$ features periodic propagation dynamics that yield imaging after distances corresponding to common multiples of the inverse eigenvalues $\{\lambda_i^{-1}\}$. This mechanism is particularly effective if the spectrum is equidistantly spaced similar to that of the harmonic oscillator. In the context of finite-size discrete systems, the so-called $J_x$ lattice fulfills this condition and allows for various operations of relevance to classical and quantum photonic information processing, such as coherent transfer of quantum states, discrete fractional Fourier transforms, and the realization of saturable absorbers. However, despite their inherently finite size, implementing large-scale $J_x$ arrays remains experimentally challenging, as the underlying parabolic coupling profile of an $N$-site $J_x$ array necessitates the precise realization of approximately $N/2$ different nearest-neighbor interaction strengths across a substantial dynamic range of values.

To overcome these limitations, we leverage the concept of supersymmetric (SUSY) photonics to design families of compact 2D systems that share the spectral and imaging characteristics of $J_x$ lattices while requiring dramatically fewer distinct couplings. In its original context of quantum field theory, supersymmetry was developed to facilitate the theoretical description of bosons and fermions in a unified mathematical framework. Subsequently, these methods have been adapted to nonrelativistic quantum mechanics to systematically identify Hamiltonians with identical sets of eigenvalues. Similarly, in photonics, SUSY techniques enable the synthesis of globally phase-matched partner structures as well as the parametric design of families of isospectral and scattering-equivalent refractive index landscapes. Likewise, SUSY transformations of non-Hermitian systems containing gain and loss have been proposed for applications in active systems such as laser arrays. While the factorization techniques typically employed in generating supersymmetric partner structures tend to be restricted to 1D, separation-of-variables methods allow them to be adapted to 2D systems, opening up a second spatial degree of freedom. Here, we present a method to construct compact 2D systems that faithfully reproduce the signature spectral and dynamic features of 1D arrays while dramatically reducing the number of individual structural parameters. We illustrate our method through transforming 1D $J_x$ arrays into equivalent 2D systems using SUSY techniques. We experimentally realize these systems in evanescently coupled waveguide arrays and investigate their imaging properties.
2. Transformation Method

In conventional \( J_x \) arrays, the matrix elements of the quantum-mechanical angular momentum operator’s \( x \) component of a particle with spin \( \frac{1}{2} (N - 1) \) are mapped onto the discrete tridiagonal Hamiltonian of a 1D array of \( N \) sites,\(^9\) resulting in a parabolic distribution of nearest-neighbor coupling coefficients \( c_{n,n+1} = \frac{1}{\sqrt{2}} \sqrt{n(n - 1)} \) between sites \( n \) and \( n + 1 \). The resulting eigenvalues form an equidistant ladder of spacing 1 running from \(-\frac{1}{2}(N - 1)\) to \(+\frac{1}{2}(N - 1)\), illustrating the close connection of this finite system to the continuous quantum-mechanical harmonic oscillator.

In order to illustrate our approach for constructing 2D \( J_x \) equivalent systems, we consider the example of such a system with \( N = 12 \) (see Figure 1), which has eigenvalues \(-\frac{11}{2}, \ldots, +\frac{11}{2}\). Applying a discrete SUSY transformation to the Hamiltonian \( H_0 \) of the \( J_x \) array results in its first-order unbroken superpartner \( H_1 \), which retains the spectrum of the original array with a single removed eigenvalue. In our example, the lowest eigenvalue \( J_{\text{min}} = -\frac{11}{2} \) was removed via Cholesky factorization,\(^13\) through \( H_0 = \lambda_{\text{min}} = A^3 A \) and \( H_1 = \lambda_{\text{min}} = A A^3 \), with the supersymmetry operators \( A, A^\dagger \). Note that, in contrast to the QR factorization method employed to selectively remove states from of \( J_x \) lattices in \(28\), the Cholesky factorization relates \( J_x \) lattices with different numbers of sites, and, in our case, the superpartner turns out to be an 11-site \( J_x \) array subject to a global detuning of \(-\frac{1}{2}\), with eigenvalues \(-\frac{11}{2}, \ldots, +\frac{11}{2}\). A single isolated site with on-site potential \( \beta \) separately hosts the mode corresponding to the removed eigenvalue \(-\frac{11}{2}\). Iterating this procedure on the remaining array results in a sequence of higher-order superpartners and a growing number of isolated detuned sites. Along these lines, the 8th superpartner consists of a four-site \( J_x \) array, globally detuned by -4, and eight individually detuned isolated sites.

Next, we reattach these sites by a series of inverse supersymmetry transformations, transforming \( H_1 \) to \( H_0 \), in the orthogonal \( y \) direction, thereby constructing 2D arrays. The resulting Hamiltonian can be described as the Kronecker sum \( H^{(0)} \otimes H^{(0)} \) of independent components in \( x \) and \( y \) directions, with \( H^{(0)} \) in our case corresponding to the Hamiltonian of the four-site \( J_x \) array detuned by -4 in \( x \) direction and \( H^{(0)} = 0 \) representing the single layer in \( y \) direction. We apply inverse SUSY transformations to \( H^{(0)} \) while keeping \( H^{(0)} \) constant. In a first step, this reconnects the four sites of highest eigenvalue resulting in a two-layer structure, with \( H^{(0)} \) now representing a two-site \( J_x \) array with coupling scaled by a factor of 4 and detuned by -2. Repeating this procedure to reconnect the remaining sites yields a three-site \( J_x \) array with its coupling constants scaled by a factor of 4 and a detuning of -4 for \( H^{(0)} \). The final array of 4 x 3 sites is a 2D \( J_x \) array where coupling in \( y \) direction is increased fourfold compared to the \( x \) direction, a factor corresponding to the number of sites in the \( x \) direction, while the opposite detuning in \( H^{(0)} \) and \( H^{(0)} \) cancels out. This system resembles an anisotropic 2D quantum harmonic oscillator. The newly constructed 2D array shares all eigenvalues of the original 1D \( J_x \) array and can be interpreted as its isospectral (broken) superpartner.

Note that this procedure, although detailed here for the specific case of the \( J_x \) array, holds for all systems with separable eigenvalue spectra \( \Lambda = \Lambda^{(0)} \oplus \Lambda^{(0)} \oplus \cdots \) and thus may be also extended to higher dimensions. Moreover, by virtue of its equidistantly spaced eigenvalues, a \( J_x \) array can be transformed into different 2D arrays, depending on the number \( N \) of involved sites. In fact, for a \( J_x \) array of \( N = N^{(0)} N^{(0)} \) sites, an entire class of isospectral 2D arrays exists with Hamiltonians \( H^{(0)} \oplus N^{(0)} N^{(0)} \) (or, equivalently, \( N^{(0)} H^{(0)} \oplus H^{(0)} \)), each representing an \( N^{(0)} \times N^{(0)} \) 2D \( J_x \) array whose couplings in \( y \) direction have been scaled by the number of sites in the \( x \) direction.

From an experimental point of view, the challenge in constructing large-scale \( J_x \) arrays lies in the requirement of precisely matching approximately \( N/2 \) distinct coupling constants spanning a range of \( \epsilon \in \left( \frac{1}{2} \sqrt{N} \right) \cdots \frac{1}{2} N \) in the limit of \( N \to \infty \). The coupling values needed in the 2D systems similarly fall between \( \epsilon^{(0)} \in \left( \frac{1}{2} \sqrt{N^{(0)}} \right) \cdots \frac{1}{2} N^{(0)} \) in \( x \) direction and \( \epsilon^{(0)} \in \left( \frac{1}{2} N^{(0)} \sqrt{N^{(0)}} \right) \cdots \frac{1}{2} N^{(0)} N^{(0)} \) along \( y \), respectively. While the highest coupling constant that occurs in either system is the same, our 2D \( J_x \)-equivalent systems offer two key advantages for implementation. First, instead of \( N/2 \), the nearest-neighbor couplings only take \( (N^{(0)} + N^{(0)})/2 \) distinct values, a reduction of up to \( 2/\sqrt{N} \). Second, instead of more or less homogeneously covering the entire range, the couplings of the 2D system occur in a well-separated bimodal distribution with a set of strong couplings in one of the spatial directions, and a set of weak coupling in the other. This convenient clustering facilitates a more precise calibration of fabrication parameters around the respective mean values, while also allowing for potential inherent anisotropies of the experimental platform to be taken advantage of, both of which may further simplify the implementation.

To illustrate the imaging dynamics in different isospectral systems, we consider the 2D superpartners of a six-waveguide \( J_x \) lattice and its Hamiltonian \( H_0 \). In line with the prime factor decomposition \( 6 = 2 \cdot 3 \), this structure can either be converted into a 3 x 2 array described by \( H^{(0)} \oplus 3 H^{(0)} \), or a 2 x 3 array with \( H^{(0)} \oplus 2 H^{(0)} \), respectively. As the equidistantly spaced eigenvalue spectrum is invariant under our construction, the individual dynamics can be transformed between them via projections onto their respective eigenvectors. As a result, the imaging properties of the \( J_x \) array naturally carry over to its superpartners, which both display perfect state transfer between opposing sites at integer
a wavelength of 512 nm in fused silica (Corning 7980) through guides were inscribed by focusing 274 fs (FWHM) laser pulses, as well as dependent on the order of inscription, since the properties of inscribed waveguides were depth dependent, through the laser focusing, as well as dependent on the order of inscription, since multiples of the propagation distance $z = \pi$, independent of the structure of the array. Figure 2 shows the numerically calculated intermediate steps of the evolution dynamics resulting from an excitation of the top-left waveguide in a conventional six-waveguide $J_1$ (top) as well as its 2D superpartners (center, bottom). In all three systems, perfect state transfer occurs at integer multiples of a normalized propagation distance of $z = \pi$.

3. Experimental Method

This study experimentally probes the imaging dynamics of these systems in arrays of evanescently coupled waveguides fabricated by femtosecond-laser direct writing technique. The individual guides were inscribed by focusing 274 fs (FWHM) laser pulses from an ultrafast fiber laser amplifier (Coherent Monaco) at a repetition rate of 333 kHz and an average power of 61 mW at a wavelength of 512 nm into fused silica (Corning 7980) through a 50x microscope objective (NA = 0.6). The glass sample was translated at a speed of 100 mm min$^{-1}$ by means of a high-precision stage (Aerotech ALS180). Non-bridgeging oxygen hole centers formed during the inscription process allowed for the intensity dynamics of guided light to be recorded through fluorescence microscopy when excited by a helium-neon laser (633 nm). Inscribing the arrays in a slightly rotated frame of reference allows us to simultaneously observe all waveguides even in 2D lattices (see Supporting Information for the structural parameters of the waveguide arrays). This rotated frame of reference does come with the trade-off of a degraded performance compared to horizontally inscribed 1D arrays, because the properties of inscribed waveguides were depth dependent, through the laser focusing, as well as dependent on the order of inscription, since nearby previously-inscribed waveguides alter the refractive-index environment. In post-processing, the decrease of signal due to propagation losses was compensated by normalizing the measured intensity to the total intensity in the array at any given propagation distance.

4. Results

In a first set of experiments, we implement the conventional six-waveguide $J_1$ array ($H_1$) and its two 2D superpartners ($H_1 \oplus 3H_1$ and $H_1 \oplus 2H_1$, respectively). We excite a single waveguide at the input facet and observe the evolution of light throughout the array by monitoring its fluorescence from the side. The observed light evolution for two distinct input waveguides in each of these systems is shown in Figure 3. In the conventional $J_1$ array (Figure 3a), light follows the well-known transfer trajectories that coalesce to the respective single target waveguide on the opposite side of the array at the imaging distance $z = \pi$, where respectively 80% and 83% of the light intensity is contained. As the evolution continues for larger $z$, light begins to flow in the opposite direction during the second imaging period. The corresponding measurements for the 2D $H_1 \oplus 3H_1$ superpartner array are shown in Figure 3b. Here, the input distribution is mirrored in both $x$ and $y$ directions, with the target waveguides containing 73% and 84% of the light. Note that, as the stronger couplings are assigned to the $y$ dimension of this structure, light here undergoes three transfers in $y$ to sync up with the single transfer in $x$ at $z = \pi$. In contrast, the evolution along $x$ in the $2H_1 \oplus H_1$ system (Figure 3c) proceeds twice as fast as in $y$, such that the input distribution is only mirrored in the $y$ direction. At the imaging distance, 77% and 81% of the light is transferred to the target waveguide. Crucially, while exhibiting systematically different dynamics, each of these configurations supports the desired perfect state transfer enabled by their identical spectra. The measured dynamics are in good agreement with a theoretical coupled-mode model, illustrating that fabrication imperfections in both detuning and coupling coefficients are negligible. These effects as well as the presence of a certain amount of diagonal coupling only will begin to significantly impact the state transfer performance at larger propagation distances.

To demonstrate the versatility of our approach in more extended systems, we turn to a ten-waveguide $J_2$ array ($H_{10}$) and its 2D counterpart, a $5 \times 2$ array described by $2H_1 \oplus H_2$. As shown in Figure 4a, the experimentally observed pattern deviates quite visibly from the desired perfect state transfer, with only 53% of the light contained in the target waveguide at the imaging distance $z = \pi$. This is due to the fact that the wave packet has to sequentially traverse a substantial number of imperfectly implemented links. In contrast, the 2D superpartner is substantially more compact in the $xy$-plane, and thus less susceptible to such perturbations, as can be readily seen for the three exemplary excitations in Figure 4b, where the target waveguides contain approximately 64%, 58% and 60% of the light at $z = \pi$. While these absolute numbers and the relative improvement realized by the 2D superpartner are modest, as a result of the systematic variation in waveguide properties due to the rotated frame of reference, it is clear that the 2D superpartner systematically outperforms the original arrangement.
5. Conclusion

In summary, we have presented a SUSY-based method to transform planar lattices into isospectral two-dimensional systems, which we applied to $J_x$ lattices. The viability of this approach was experimentally demonstrated by observing the imaging dynamics of these structures in laser-written waveguide arrays. The 2D partner arrays faithfully reproduce the hallmark features of the $J_x$ array, such as perfect state transfer and mirroring of input field distributions, albeit on a more compact footprint, while also requiring a systematically lower number of individual structural parameters to be matched during inscription. As such, our method readily allows for increased ease of fabrication and provides the tools to maintain coherence during state transfer in large-scale photonic circuits for imaging through discrete systems or for applications in high-dimensional quantum gates. Beyond the context of $J_x$-type lattices, our method readily translates to any discrete system with a separable eigenvalue spectrum, and can be extended to higher dimensions, e.g., by leveraging the polarization degree of freedom in birefringent waveguides, for even higher packaging density.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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Conflicts of Interest
The authors declare no conflict of interest.

Data Availability Statement
The data that support the findings of this study are available from the corresponding author upon reasonable request.

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