Moment categories and operads

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CRM Barcelone Seminar
Higher Homotopical Structures
March 9, 2021
### Summary (active/inert factorisation system)

| moments \(\sim\) | moment category | units \(\sim\) | operad-type | plus \(\sim\) | Segal presheaf |
|------------------|-----------------|----------------|-------------|--------------|----------------|
| \(\mathbb{C}\)   | \(\mathbb{C}\)-operad | \(\mathbb{C}\)-monoid | \(\mathbb{C}_\infty\)-monoid |              |                |
| \(\Gamma\)      | sym. operad     | comm. monoid   |              | \(E_\infty\)-space |                |
| \(\Delta\)      | non-sym. operad | assoc. monoid  |              | \(A_\infty\)-space |                |
| \(\Theta_n\)    | \(n\)-operad   | \(n\)-monoid   | \(E_n\)-space |              |                |
| \(\Omega\)      | tree-hyperoperad| sym. operad    | \(\infty\)-operad |              |                |
| \(\Gamma\updownarrow\) | graph-hyperoperad | properad | \(\infty\)-properad |              |                |

### Related concepts (replacing “inert part” with \(\sim\))

- Operator category (Barwick \(\sim\) pullback structure)
- Operadic category (Batanin-Markl \(\sim\) fibre structure)
- Feynman category (Kaufmann-Ward \(\sim\) sym. monoidal structure)
- Categorical pattern (Chu-Haugseng \(\sim\) \(\infty\)-categorical context)
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Related concepts (replacing “inert part” with \( \rightsquigarrow \))

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|---------|----------------|-------|-------------|------|----------------|
| C       | C-operad       | C-monoid | C∞-monoid  |      |                |
| Γ       | sym. operad    | comm. monoid | E∞-space  |      |                |
| Δ       | non-sym. operad| assoc. monoid | A∞-space  |      |                |
| Θ_n    | n-operad       | n-monoid | E_n-space  |      |                |
| Ω       | tree-hyperoperad| sym. operad | ∞-operad  |      |                |
| Γ↑      | graph-hyperoperad| properad | ∞-properad |      |                |
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| C         | C-operad        | C-monoid| C-∞-monoid  |        |                |
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Related concepts (replacing “inert part” with ▲)

Operator category (Barwick ▲ pullback structure)
Operadic category (Batanin-Markl ▲ fibre structure)
Feynman category (Kaufmann-Ward ▲ sym. monoidal structure)
Categorical pattern (Chu-Haugseng ▲ ∞-categorical context)
A moment category is a category $\mathcal{C}$ with an active/inert factorisation system $(\mathcal{C}_{act}, \mathcal{C}_{in})$ such that

1. each inert map admits a unique active retraction;
2. if the left square below commutes then the right square as well

where $r, r'$ are the active retractions of $i, i'$ provided by (1).
Definition (moment category)

A *moment category* is a category $\mathcal{C}$ with an *active/inert* factorisation system $(\mathcal{C}_{\text{act}}, \mathcal{C}_{\text{in}})$ such that

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\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^{i} & & \downarrow^{i'} \\
A' & \xrightarrow{g} & B'
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^{r} & & \downarrow^{r'} \\
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1. each inert map admits a unique active retraction;
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where $r, r'$ are the active retractions of $i, i'$ provided by (1).
Lemma (inert subobjects vs moments)

For each object $A$ of a moment category $\mathbb{C}$ there is a bijection between \textit{inert subobjects} of $A$ and \textit{moments} of $A$, i.e. endomorphisms $\phi : A \to A$ s.th. $\phi = \phi_{\text{in}}\phi_{\text{act}} \implies \phi_{\text{act}}\phi_{\text{in}} = 1_A$.

Put $m_A = \{ \phi \in \mathbb{C}(A, A) \mid \phi_{\text{act}}\phi_{\text{in}} = 1_A \}$

For $f : A \to B$ define $f_* : m_A \to m_B$ by

$A \xrightarrow{\phi_{\text{act}}} A_{\phi} \xleftarrow{f'} B_{\psi} \xleftarrow{\psi_{\text{act}}} B \xrightarrow{\psi_{\text{in}}} B_{\psi}$

with $f_*(\phi_{\text{in}}\phi_{\text{act}}) = \psi_{\text{in}}\psi_{\text{act}}$. 
Lemma (inert subobjects vs moments)

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A & \xrightarrow{f} & B \\
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A & \xrightarrow{f} & B \\
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& & \\
\psi_{in} & \downarrow & \psi_{act} \\
& & \\
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\text{with} \quad f_*(\phi_{in}\phi_{act}) = \psi_{in}\psi_{act}.
$$
**Proposition (left regular band – skew-commutativity)**

The moment set $m_A$ is a submonoid of $C(A, A)$ consisting of idempotent elements satisfying the relation $\phi \psi \phi = \phi \psi$.

**Example (Segal’s category $\Gamma \rightsquigarrow \Gamma^{\text{op}} = $ finite sets and partial maps)**

- $m^{(n_1, \ldots, n_m)} \to n$ active provided $n_1 \cup \cdots \cup n_m = n$. (partition)
- $m^{(n_1, \ldots, n_m)} \to n$ inert provided all $n_i$ are singleton. (embedding)

**Example (simplex category $\Delta$)**

- $[m] \xrightarrow{f} [n]$ is active provided $f$ is endpoint-preserving, i.e. $f(0) = 0, f(m) = n$.
- $[m] \xrightarrow{f} [n]$ is inert provided $f$ is distance-preserving, i.e. $f(i + 1) = f(i) + 1$ for all $i$. 
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Definition (units, elementary moments, nilobjects)

- A moment $\phi$ is centric if $\phi_{in}$ is the only inert section of $\phi_{act}$.
- A unit is an object $U$ sth. $1_U$ is the only centric moment but $m_U \neq \{1_U\}$, and every active map with target $U$ admits exactly one inert section.
- A moment is elementary if it splits over a unit. The set of elementary moments of $A$ is denoted $e{l}_A \subset m_A$.
- An object without elementary moments is called a nilobject.

Example ($\Gamma$ and $\Delta$)

- $0$ is the nilobject, and $1$ the unit of $\Gamma$. Elementary inert subobjects $1 \rightarrow n$ are elements. Cardinality of $e{l}_n$ is $n$.
- $[0]$ is the nilobject, and $[1]$ the unit of $\Delta$. Elementary inert subobjects $[1] \rightarrow [n]$ are segments. Cardinality of $e{l}_{[n]}$ is $n$. 
Definition (units, elementary moments, nilobjects)

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Example (Γ and Δ)

- $0$ is the nilobject, and $1$ the unit of Γ. Elementary inert subobjects $1 \rightarrowtail n$ are elements. Cardinality of $\text{el}_n$ is $n$.
- $[0]$ is the nilobject, and $[1]$ the unit of Δ. Elementary inert subobjects $[1] \rightarrowtail [n]$ are segments. Cardinality of $\text{el}_{[n]}$ is $n$. 
Definition (units, elementary moments, nilobjects)

- A moment $\phi$ is **centric** if $\phi_{in}$ is the only inert section of $\phi_{act}$.
- A **unit** is an object $U$ s.t. $1_U$ is the only centric moment but $m_U \neq \{1_U\}$, and every active map with target $U$ admits exactly one inert section.
- A moment is **elementary** if it splits over a unit. The set of elementary moments of $A$ is denoted $\text{el}_A \subset m_A$.
- An object without elementary moments is called a **nilobject**.

Example ($\Gamma$ and $\Delta$)

- $0$ is the nilobject, and $1$ the unit of $\Gamma$. Elementary inert subobjects $1 \to n$ are elements. Cardinality of $\text{el}_n$ is $n$.
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### Definition (units, elementary moments, nilobjects)

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### Definition (units, elementary moments, nilobjects)

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- A moment is *elementary* if it splits over a unit. The set of elementary moments of \( A \) is denoted \( \text{el}_A \subset m_A \).
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### Example (\( \Gamma \) and \( \Delta \))

- \( 0 \) is the nilobject, and \( 1 \) the unit of \( \Gamma \). Elementary inert subobjects \( 1 \rightarrow n \) are elements. Cardinality of \( \text{el}_n \) is \( n \).
- \( [0] \) is the nilobject, and \( [1] \) the unit of \( \Delta \). Elementary inert subobjects \( [1] \rightarrow [n] \) are segments. Cardinality of \( \text{el}_{[n]} \) is \( n \).
Definition (units, elementary moments, nilobjects)

- A moment $\phi$ is **centric** if $\phi_{in}$ is the only inert section of $\phi_{act}$.
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- A moment is **elementary** if it splits over a unit. The set of elementary moments of $A$ is denoted $el_A \subset m_A$.
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- $0$ is the nilobject, and $1$ the unit of $\Gamma$. Elementary inert subobjects $1 \rightarrow n$ are elements. Cardinality of $el_n$ is $n$.
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**Definition (units, elementary moments, nilobjects)**

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- $[0]$ is the nilobject, and $[1]$ the unit of $\Delta$. Elementary inert subobjects $[1] \twoheadrightarrow [n]$ are segments. Cardinality of $\text{el}_{[n]}$ is $n$. 
Definition (\(\mathcal{C}\)-operads for unital moment categories \(\mathcal{C}\))

A \(\mathcal{C}\)-operad \(\mathcal{O}\) in a symmetric monoidal category \((\mathbb{E}, \otimes, I_\mathbb{E})\) assigns to each object \(A\) of \(\mathcal{C}\) an object \(\mathcal{O}(A)\) of \(\mathbb{E}\), together with

- a unit \(I_\mathbb{E} \rightarrow \mathcal{O}(U)\) in \(\mathbb{E}\) for each unit \(U\) of \(\mathcal{C}\);
- a unital, associative and equivariant composition
  \(\mathcal{O}(A) \otimes \mathcal{O}(f) \rightarrow \mathcal{O}(B)\) for each active \(f : A \longrightarrow B\), where
  \(\mathcal{O}(f) = \otimes_{\alpha \in \text{el}_A} \mathcal{O}(B_{f_*}(\alpha)).\)

Example (\(\Gamma\) and \(\Delta\))

- \(\Gamma\)-operads—symmetric operads:
  \(\mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \rightarrow \mathcal{O}_{n_1 + \cdots + n_m}\) for each \(m \longrightarrow n\).

- \(\Delta\)-operads—non-symmetric operads:
  \(\mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \rightarrow \mathcal{O}_{n_1 + \cdots + n_m}\) for each \([m] \longrightarrow [n]\).
Definition (C-operads for unital moment categories C)

A C-operad O in a symmetric monoidal category (E, ⊗, I_E) assigns to each object A of C an object O(A) of E, together with

- a unit I_E → O(U) in E for each unit U of C;
- a unital, associative and equivariant composition O(A) ⊗ O(f) → O(B) for each active f : A → B, where O(f) = ⊗_{α ∈ elA} O(B_{f*}(α)).

Example (Γ and Δ)

- Γ-operads—symmetric operads:
  \[ O_m ⊗ O_{n_1} ⊗ \cdots ⊗ O_{n_m} → O_{n_1+\cdots+n_m} \text{ for each } m → n. \]
- Δ-operads—non-symmetric operads:
  \[ O_m ⊗ O_{n_1} ⊗ \cdots ⊗ O_{n_m} → O_{n_1+\cdots+n_m} \text{ for each } [m] → [n]. \]
Definition (\(\mathbb{C}\)-operads for unital moment categories \(\mathbb{C}\))

A \(\mathbb{C}\)-operad \(O\) in a symmetric monoidal category \((\mathbb{E}, \otimes, I_{\mathbb{E}})\) assigns to each object \(A\) of \(\mathbb{C}\) an object \(O(A)\) of \(\mathbb{E}\), together with

- a unit \(I_{\mathbb{E}} \rightarrow O(U)\) in \(\mathbb{E}\) for each unit \(U\) of \(\mathbb{C}\);

- a unital, associative and equivariant composition \(O(A) \otimes O(f) \rightarrow O(B)\) for each active \(f : A \rightarrow B\), where \(O(f) = \bigotimes_{\alpha \in \text{el}_A} O(B_{f*}(\alpha))\).

Example (\(\Gamma\) and \(\Delta\))

- \(\Gamma\)-operads = symmetric operads:
  \[O_m \otimes O_{n_1} \otimes \cdots \otimes O_{n_m} \rightarrow O_{n_1 + \cdots + n_m}\] for each \(m \rightarrow n\).

- \(\Delta\)-operads = non-symmetric operads:
  \[O_m \otimes O_{n_1} \otimes \cdots \otimes O_{n_m} \rightarrow O_{n_1 + \cdots + n_m}\] for each \([m] \rightarrow [n]\).
**Definition (C-operads for unital moment categories C)**

A \( \mathbb{C} \)-operad \( \mathcal{O} \) in a symmetric monoidal category \( (\mathbb{E}, \otimes, I_{\mathbb{E}}) \) assigns to each object \( A \) of \( \mathbb{C} \) an object \( \mathcal{O}(A) \) of \( \mathbb{E} \), together with

- a unit \( I_{\mathbb{E}} \rightarrow \mathcal{O}(U) \) in \( \mathbb{E} \) for each unit \( U \) of \( \mathbb{C} \);
- a unital, associative and equivariant composition \( \mathcal{O}(A) \otimes \mathcal{O}(f) \rightarrow \mathcal{O}(B) \) for each active \( f : A \rightarrow B \), where \( \mathcal{O}(f) = \bigotimes_{\alpha \in \text{id}_A} \mathcal{O}(B_{f^*}(\alpha)) \).

**Example (Γ and Δ)**

- Γ-operads = symmetric operads:
  \( \mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \rightarrow \mathcal{O}_{n_1+\cdots+n_m} \) for each \( m \rightarrow n \).
- Δ-operads = non-symmetric operads:
  \( \mathcal{O}_m \otimes \mathcal{O}_{n_1} \otimes \cdots \otimes \mathcal{O}_{n_m} \rightarrow \mathcal{O}_{n_1+\cdots+n_m} \) for each \( [m] \rightarrow [n] \).
Definition (C-operads for unital moment categories C)

A C-operad O in a symmetric monoidal category (E, ⊗, I_E) assigns to each object A of C an object O(A) of E, together with

- a unit I_E → O(U) in E for each unit U of C;
- a unital, associative and equivariant composition

O(A) ⊗ O(f) → O(B) for each active f : A → B, where

O(f) = \bigotimes_{\alpha \in \text{el}_A} O(B_{f*}(\alpha)).

Example (Γ and ∆)

Γ-operads=symmetric operads:

O_m \otimes O_{n_1} \otimes \cdots \otimes O_{n_m} → O_{n_1+\cdots+n_m} for each m → n.

Δ-operads=non-symmetric operads:

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**Definition (C-operads for unital moment categories C)**

A $\mathbb{C}$-operad $O$ in a symmetric monoidal category $(\mathbb{E}, \otimes, I_\mathbb{E})$ assigns to each object $A$ of $\mathbb{C}$ an object $O(A)$ of $\mathbb{E}$, together with

- a unit $I_\mathbb{E} \to O(U)$ in $\mathbb{E}$ for each unit $U$ of $\mathbb{C}$;
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**Example (Γ and Δ)**

- $\Gamma$-operads = symmetric operads:
  \[ O_m \otimes O_{n_1} \otimes \cdots \otimes O_{n_m} \to O_{n_1+\cdots+n_m} \] for each $m \otimes \to n$.

- $\Delta$-operads = non-symmetric operads:
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Definition (unital moment categories)

For every object $A$, $\text{el}_A$ has finite cardinality and receives an essentially unique active morphism $U_A \rightarrow A$ from a unit.

Proposition (universal role of $\Gamma$)

For every unital moment category $\mathcal{C}$ there is an essentially unique cardinality preserving moment functor $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \Gamma$.

Definition (wreath product of unital moment categories $A, B$)

$\text{Ob}(A \wr B) = \{(A, B_\alpha) \mid A \in \text{Ob}(A), \alpha \in \text{el}_A, B_\alpha \in \text{Ob}(B)\}$

$(f, f_\alpha^\beta) : (A, B_\alpha) \rightarrow (A', B'_\beta)$ where $f_\alpha^\beta$ for each $\beta \leq f_* (\alpha)$.

Proposition

Joyal’s category $\Theta_n$ is an iterated wreath product $\Delta \wr \cdots \wr \Delta$. $\Theta_n$-operads are Batanin’s $(n - 1)$-terminal $n$-operads.
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Proposition
Joyal’s category $\Theta_n$ is an iterated wreath product $\Delta \wr \cdots \wr \Delta$. $\Theta_n$-operads are Batanin’s $(n - 1)$-terminal $n$-operads.
Remark (moment category structure on $\Theta_n$)

- Objects of $\Theta_n$ correspond to $n$-level trees.
- There is a unique unit $U_n$, the linear tree of height $n$.
- $\gamma_{\Theta_n} : \Theta_n \to \Gamma$ takes $n$-level tree to its set of height $n$ vertices.
- Active maps $S \to T$ correspond to Batanin’s $S_\ast$-indexed decompositions of $T_\ast$, where $T_\ast$ is the $n$-graph defined by the inert subobjects of $T$ whose domains are subobjects of $U_n$.

Example (inert substructure of $[2]|[2],[0]$ in $\Delta \wr \Delta = \Theta_2$)
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**Definition (C-monoids for C with single rigid unit U)**

- $\mathcal{E}_X(A) = \text{hom}_E(X \otimes^\text{el}_A, X)$ (endomorphism-$\mathbb{C}$-operad of $X$).
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- $\mathbb{C}$-monoid = algebra over the unit-$\mathbb{C}$-operad.

**Lemma (presheaf presentation for closed symmetric monoidal E)**

$\mathbb{C}$-monoids are presheaves $X: \mathbb{C}^{\text{op}}_{\text{act}} \to E$ such that

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Definition (hypermoment category)

A hypermoment category $\mathcal{C}$ comes equipped with an active/inert factorisation system and $\gamma_\mathcal{C} : \mathcal{C} \to \Gamma$ such that

- $\gamma_\mathcal{C}$ preserves active (resp. inert) morphisms;
- for each $A$ and $1 \xrightarrow{} \gamma_\mathcal{C}(A)$, there is an ess. unique inert lift $U \xrightarrow{} A$ in $\mathcal{C}$ such that $U$ satisfies the second unit-axiom.

Example (dendroidal category $\Omega$ of Moerdijk-Weiss)

- objects (dendrices) are finite rooted trees with leaves.
- every morphism decomposes into a degeneracy followed by active mono followed by inert mono.
- active mono = inner face = dendrix insertion
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### Example (dendroidal category $\Omega$ of Moerdijk-Weiss)

- objects (dendrices) are finite rooted trees with leaves.
- every morphism decomposes into a degeneracy followed by active mono followed by inert mono.
- active mono = inner face = dendrix insertion
- inert mono = outer face = dendrix embedding
- $\gamma_{\Omega} : \Omega \to \Gamma$ takes a dendrix to its vertex set.
- units = corollas $C_n$, one for each $n \in \mathbb{N}$.
Definition (hypermoment category)

A hypermoment category $\mathcal{C}$ comes equipped with an active/inert factorisation system and $\gamma_\mathcal{C} : \mathcal{C} \to \Gamma$ such that

- $\gamma_\mathcal{C}$ preserves active (resp. inert) morphisms;
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- $\gamma_{\Gamma \downarrow \Gamma} : \Gamma \downarrow \Gamma \to \Gamma$ takes a graphix to its vertex set.
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Remark (hypermoment embeddings $\Delta \subset \Omega \subset \Gamma \downarrow \Gamma$)

- $\Omega / \Gamma \downarrow \Gamma$-operads = tree/graph-hyperoperads (Getzler-Kapranov)
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- A $\mathcal{C}$-tree $([m], A_0 \rightarrow \cdots \rightarrow A_m)$ consists of $[m]$ in $\Delta$ and a functor $A_\bullet : [m] \rightarrow \mathcal{C}_{act}$ such that $A_0$ is a unit in $\mathcal{C}$.
- A $\mathcal{C}$-tree morphism $(\phi, f)$ consists of $\phi : [m] \rightarrow [n]$ and a nat. transf. $f : A \rightarrow B\phi$ sth. $f_i : A_i \rightarrow B\phi(i)$ is inert for $i \in [m]$.
- $\mathcal{C}^+$ is the category of $\mathcal{C}$-trees and $\mathcal{C}$-tree morphisms.
- A vertex is given by $([1], U \rightarrow \rightarrow A) \rightarrow \rightarrow ([m], A_\bullet)$.

Theorem (cf. Baez-Dolan)

$\mathcal{C}^+$ is a unital hypermoment category such that $\mathcal{C}$-operads get identified with $\mathcal{C}^+$-monoids.
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$\mathbb{C}^+$ is a unital hypermoment category such that $\mathbb{C}$-operads get identified with $\mathbb{C}^+$-monoids.
Proposition \((\Omega \subseteq \Gamma^+, \text{cf. Chu-Haugseng-Heuts})\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 \\
\downarrow \downarrow \\
\bullet \\
\downarrow \\
1
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2 \\
\downarrow \\
\bullet \\
\downarrow \\
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
3 \\
\downarrow \\
\bullet \\
\downarrow \downarrow \\
\{1,2\},\{3\}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Remark (reduced dendrices)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\Gamma_{\text{mono}}^+ \leftarrow \equiv \Omega_{\text{open, pruned}} \Rightarrow \Omega_{\text{reduced}}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]
Proposition ($\Omega \supset \Gamma^+$, cf. Chu-Haugseug-Heuts)

\[
\begin{array}{c}
\bullet \\
\downarrow^{1} \\
\bullet \\
\downarrow^{2} \\
\bullet \\
\downarrow^{3} \\
\end{array}
\quad
\begin{array}{c}
3 \\
\mapsto \{1,2\}, \{3\} \\
2 \\
\mapsto \{1,2\} \\
1 \\
\end{array}
\]

Remark (reduced dendrices)

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\begin{array}{c}
\Gamma_{mon}^+ \cong \Omega_{open, pruned} \\
\cong \Omega_{reduced} \\
\end{array}
\]
Definition (extensionality)
A hypermoment category $\mathcal{C}$ is \textit{extensional} if pushouts of inert maps along active maps exist, are inert and preserved by $\gamma_{\mathcal{C}}$.

Proposition ($\mathcal{C}$-tree insertion for extensional $\mathcal{C}$)
$\mathcal{C}$-trees can be inserted into vertices of $\mathcal{C}$-trees. There exists a Feynman category $\mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{C}\text{-operads})\simeq (\mathcal{F}_{\mathcal{C}}\text{-algebras})$.

Theorem (monadicity for extensional $\mathcal{C}$)
The forgetful functor from $\mathcal{C}$-operads to $\mathcal{C}$-collections is monadic.

Remark
$\mathcal{F}_{\Gamma}$ is the coloured symmetric operad of finite rooted trees whose algebras are symmetric operads.
**Definition (extensionality)**

A hypermoment category $\mathcal{C}$ is *extensional* if pushouts of inert maps along active maps exist, are inert and preserved by $\gamma_{\mathcal{C}}$.

**Proposition (C-tree insertion for extensional C)**

$\mathcal{C}$-trees can be inserted into vertices of $\mathcal{C}$-trees. There exists a Feynman category $\mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{C}\text{-operads}) \simeq (\mathcal{F}_{\mathcal{C}}\text{-algebras})$.

**Theorem (monadicity for extensional C)**

The forgetful functor from $\mathcal{C}$-operads to $\mathcal{C}$-collections is monadic.

**Remark**

$\mathcal{F}_\Gamma$ is the coloured symmetric operad of finite rooted trees whose algebras are symmetric operads.
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\( \mathcal{F}_\Gamma \) is the coloured symmetric operad of finite rooted trees whose algebras are symmetric operads.
Moment categories and operads

Monadicity

Definition (Segal core for strongly unital $\mathcal{C}$)

The Segal core $\mathcal{C}_{\text{Seg}}$ is the subcategory of $\mathcal{C}_{\text{in}}$ spanned by nil- and unit-objects. $\mathcal{C}$ is strongly unital if $\mathcal{C}_{\text{Seg}}$ is dense in $\mathcal{C}_{\text{in}}$.

| $\mathcal{C}$ | $\Delta$ | $\Theta_n$ | $\Omega$ | $\Gamma\uparrow$ |
|---------------|----------|------------|----------|-----------------|
| $\mathcal{C}_{\text{Seg}}$ | $[0] \Rightarrow [1]$ | cell-incl. of glob. $n$-cell | edge-incl. of corollas | edge-incl. of dir. corollas |
| $\mathcal{C}$-$\text{gph}$ | graph | $n$-graph | multigraph | dir. multigraph |
| $\mathcal{C}$-$\text{cat}$ | category | $n$-category | col. operad | col. properad |

Theorem (coloured monadicity for strongly unital $\mathcal{C}$)

The forgetful functor from $\mathcal{C}$-categories to $\mathcal{C}$-graphs is monadic.

Thanks for your attention!
**Definition (Segal core for strongly unital $\mathcal{C}$)**

The *Segal core* $\mathcal{C}_{\text{Seg}}$ is the subcategory of $\mathcal{C}_{\text{in}}$ spanned by nil- and unit-objects. $\mathcal{C}$ is *strongly unital* if $\mathcal{C}_{\text{Seg}}$ is dense in $\mathcal{C}_{\text{in}}$.

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|---------------------|---------------|---------------------------------|------------------------------|-------------------|
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| $\mathcal{C}$-$\text{cat}$ | category    | $n$-category                   | col. operad                 | col. properad     |

**Theorem (coloured monadicity for strongly unital $\mathcal{C}$)**

The forgetful functor from $\mathcal{C}$-categories to $\mathcal{C}$-graphs is monadic.

*Thanks for your attention!*
Definition (Segal core for strongly unital $\mathcal{C}$)

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|-----------------|-----------|-----------------------------|--------------------------------|---------------------------|
| $\mathcal{C}_{\text{Seg}}$ | $[0] \Rightarrow [1]$ | cell-incl. of glob. $n$-cell | edge-incl. of corollas | edge-incl. of dir. corollas |
| $\mathcal{C}$-gph | graph    | $n$-graph                   | multigraph                    | dir. multigraph          |
| $\mathcal{C}$-cat | category | $n$-category                | col. operad                   | col. properad             |

Theorem (coloured monadicity for strongly unital $\mathcal{C}$)

The forgetful functor from $\mathcal{C}$-categories to $\mathcal{C}$-graphs is monadic.

Thanks for your attention!
Definition (Segal core for strongly unital $\mathcal{C}$)

The *Segal core* $\mathcal{C}_{\text{Seg}}$ is the subcategory of $\mathcal{C}_{\text{in}}$ spanned by nil- and unit-objects. $\mathcal{C}$ is *strongly unital* if $\mathcal{C}_{\text{Seg}}$ is dense in $\mathcal{C}_{\text{in}}$.

| $\mathcal{C}$   | $\Delta$       | $\Theta_n$             | $\Omega$                | $\Gamma \updownarrow$ |
|-----------------|----------------|------------------------|-------------------------|------------------------|
| $\mathcal{C}_{\text{Seg}}$ | [0] $\Rightarrow$ [1] | cell-incl. of glob. $n$-cell | edge-incl. of corollas  | edge-incl. of dir. corollas |
| $\mathcal{C}$-gph | graph         | $n$-graph              | multigraph              | dir. multigraph        |
| $\mathcal{C}$-cat | category      | $n$-category           | col. operad             | col. properad          |

Theorem (coloured monadicity for strongly unital $\mathcal{C}$)

The forgetful functor from $\mathcal{C}$-categories to $\mathcal{C}$-graphs is monadic.

*Thanks for your attention!*