Decay estimates for a class of wave equations

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Abstract In this paper we use a unified way studying the decay estimate for a class of dispersive semigroup given by $e^{it\phi(\sqrt{-\Delta})}$, where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is smooth away from the origin. Especially, the decay estimates for the solutions of the Klein-Gordon equation and the beam equation are simplified and slightly improved.

Keywords: Decay estimates, dispersive wave equations

2000 MS Classification: 42B25, 35F20

1 Introduction

In this paper, we study the decay estimate for a class of dispersive equations:

$$i\partial_t u = -\phi(\sqrt{-\Delta}) u + f, \quad u(0) = u_0(x),$$

where $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ is smooth, $f(x,t), u(x,t) : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{C}$, $n \geq 1$, and $\phi(\sqrt{-\Delta}) u = F^{-1}\phi(|\xi|)F u$. Here $F$ denotes Fourier transform.

Many dispersive wave equations reduce to this type, for instance, the Schrödinger equation ($\phi(r) = r^2$), the wave equation ($\phi(r) = r$), the Klein-Gordon equation ($\phi(r) = \sqrt{1+r^2}$) and the beam equation ($\phi(r) = \sqrt{1+r^4}$). In 1977, Strichartz [15] derived the priori estimates of the solution to (1) in space-time norm by the Fourier restriction theorem of Stein and Tomas. Later, his results was improved via a dispersive estimate and duality argument (cf. [7], [4] and references therein). The dispersive estimate

$$\|e^{it\phi(\sqrt{-\Delta})} u_0\|_X \lesssim |t|^{-\theta}\|u_0\|_{X'},$$

plays a crucial role, where $X'$ is the dual space of $X$. Applying (2), together with a standard argument ([7], [17]), we can immediately get the Strichartz estimates. When $\phi$ is a homogenous function of order $m$, namely, $\phi(\lambda r) = \lambda^m \phi(r)$ for $\lambda > 0$, one can easily obtain a dispersive estimate (2) by a theorem of Littman and dyadic decomposition, which is related to the rank of the Hessian ($\partial^2 \phi / \partial \xi_i^2$). This technique also works very well when $\phi$ is not radial ([9]). However, this issue becomes very complicated when $\phi$ is not homogenous, the main reason is that the scaling constants can not be effectively separated from the time, cf. Brenner [3], Lavandosky [8]. In this paper, we overcome this difficulty via frequency localization by separating $\phi$ between high and low frequency. In higher spatial dimensions, since $\phi$ is radial, we can reduce the problem to an oscillatory integral in one dimension by using the Bessel function. Using the dyadic decomposition and some properties of the Bessel function, we can derive a decay estimate, as desired. Some earlier ideas on this technique can be found in [1], [3], [8].
Since $\phi$ is not homogenous, our idea is to treat the high frequency and the low frequency in different scales. We will assume $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth and satisfies

(H1) There exists $m_1 > 0$, such that for any $\alpha \geq 2$ and $\alpha \in \mathbb{N}$,
\begin{equation*}
|\phi'(r)| \sim r^{m_1-1} \quad \text{and} \quad |\phi^{(\alpha)}(r)| \lesssim r^{m_1-\alpha}, \quad r \geq 1.
\end{equation*}

(H2) There exists $m_2 > 0$, such that for any $\alpha \geq 2$ and $\alpha \in \mathbb{N}$,
\begin{equation*}
|\phi'(r)| \sim r^{m_2-1} \quad \text{and} \quad |\phi^{(\alpha)}(r)| \lesssim r^{m_2-\alpha}, \quad 0 < r < 1.
\end{equation*}

(H3) There exists $\alpha_1$, such that
\begin{equation*}
|\phi''(r)| \sim r^{\alpha_1-2} \quad r \geq 1.
\end{equation*}

(H4) there exists $\alpha_2$, such that
\begin{equation*}
|\phi''(r)| \sim r^{\alpha_2-2} \quad 0 < r < 1.
\end{equation*}

**Remark 1.** (H1) and (H3) reflect the homogeneous order of $\phi$ in high frequency. If $\phi$ satisfies (H1) and (H3), then $\alpha_1 \leq m_1$. Similarly, the homogeneous order of $\phi$ in low frequency is described by (H2) and (H4). If $\phi$ satisfies (H2) and (H4), then $\alpha_2 \geq m_2$. The special case $\alpha_2 = m_2$ happens in the most of time.

Let $\Phi(x) : \mathbb{R}^n \to [0, 1]$ be a even, smooth radial function such that $\text{supp}\Phi \subseteq \{x : |x| \leq 2\}$ and $\Phi(x) = 1$, if $|x| \leq 1$. Let $\psi(x) = \Phi(x) - \Phi(2x)$, and $\triangle_k$ be the Littlewood-Paley projector, namely $\triangle_k f = \mathcal{F}^{-1} \psi(2^{-k} \xi) \mathcal{F} f$. Let $P_{\leq 0} f = \mathcal{F}^{-1} \Phi(\xi) \mathcal{F} f$. Now we state our main result:

**Theorem 1.** Assume $\phi : \mathbb{R}^+ \to \mathbb{R}$ is smooth away from origin, We have the following results.

(a) For $k \geq 0$, $\phi$ satisfies (H1), then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} \triangle_k u_0\|_\infty \lesssim |t|^{-\theta} 2^{(n-m_1 \theta)} \|u_0\|_1, \quad 0 \leq \theta \leq \frac{n-1}{2}.
\end{equation*}

In addition, if $\phi$ satisfies (H3), then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} \triangle_k u_0\|_\infty \lesssim |t|^{-\theta} 2^{k(n-m_1 \theta)} \|	riangle_k u_0\|_1, \quad 0 \leq \theta \leq 1.
\end{equation*}

(b) For $k < 0$, $\phi$ satisfies (H2), then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} \triangle_k u_0\|_\infty \lesssim |t|^{-\theta} 2^{k(n-m_2 \theta)} \|	riangle_k u_0\|_1, \quad 0 \leq \theta \leq \frac{n-1}{2}.
\end{equation*}

In addition, if $\phi$ satisfies (H4), then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} \triangle_k u_0\|_\infty \lesssim |t|^{-\theta} 2^{k(n-m_2 \theta)} \|	riangle_k u_0\|_1, \quad 0 \leq \theta \leq 1.
\end{equation*}

(c) If $\phi$ satisfies (H2), then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} P_{\leq 0} u_0\|_\infty \lesssim (1 + |t|)^{-\theta} \|u_0\|_1, \quad \theta = \min \left( \frac{n}{m_2}, \frac{n-1}{2} \right).
\end{equation*}

In addition, if (H4) holds and $\alpha_2 = m_2$, then
\begin{equation*}
\|e^{it\phi(\sqrt{-\triangle})} P_{\leq 0} u_0\|_\infty \lesssim (1 + |t|)^{-\theta} \|u_0\|_1, \quad \theta = \min \left( \frac{n}{m_2}, \frac{n}{2} \right).
\end{equation*}
Remark 2. If \( m_1 = \alpha_1 \) and (H1) and (H3) hold, one can easily verify that for \( k > 0 \), (3) and (4) are equivalent to
\[
\| e^{it\phi(\sqrt{-\Delta})} \triangle_k u_0 \|_\infty \lesssim |t|^{-\theta} 2^{k(n-m_1)} \| u_0 \|_1, \ 0 \leq \theta \leq \frac{n}{2}.
\]
If \( m_2 = \alpha_2 \) and (H2) and (H4) hold, one can easily verify that for \( k \leq 0 \), (5) and (6) are equivalent to
\[
\| e^{it\phi(\sqrt{-\Delta})} \triangle_k u_0 \|_\infty \lesssim |t|^{-\theta} 2^{k(n-m_2)} \| u_0 \|_1, \ 0 \leq \theta \leq \frac{n}{2}.
\]

Throughout this paper, \( C > 1 \) and \( c < 1 \) will denote positive universal constants, which can be different at different places. \( A \lesssim B \) means that \( A \leq CB \), and \( A \sim B \) stands for \( A \lesssim B \) and \( B \lesssim A \). We denote by \( p^* \) the dual number of \( p \in [1, \infty] \), i.e., \( \frac{1}{p} + \frac{1}{p^*} = 1 \). We will use Lebesgue spaces \( L^p := L^p(\mathbb{R}^n) \), \( \| \cdot \|_p := \| \cdot \|_{L^p} \), Sobolev spaces \( H^s_p = (I - \Delta)^{-s/2} L^p \), \( H^s := H^s_2 \).

Let \( 1 \leq p, q \leq \infty \). Besov spaces are defined in the following way:
\[
B^s_{p,q} = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{B^s_{p,q}} := \left( \sum_{k=0}^{\infty} 2^{ksq} \| \triangle_k f \|_{L^p}^q \right)^{1/q} \right\}.
\]
Some properties of these function spaces can be found in [2, 16].

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1. In Section 3, we derive Strichartz estimate in a general setting. Some applications will be given in Sections 4 and 5.

2 Decay Estimate

In this section we will prove Theorem 1. The proof for the case \( n = 1 \) is direct and simple, but reflects the idea for the higher dimension.

Proof of Theorem 1. Since the proof in the case \( n = 1 \) is slightly different from the case of higher spatial dimensions, we divide the proof into the following two steps.

Step 1. We consider the case \( n = 1 \). First, we prove (a). It follows from Young’s inequality that
\[
\| e^{it\phi(\sqrt{-\Delta})} \triangle_k u_0 \|_\infty \lesssim \| \mathcal{F}^{-1} e^{it\phi(|\xi|)} \psi(2^{-k} |\xi|) \|_\infty \| u_0 \|_1
\]
\[
\lesssim \| I_k \|_\infty \| u_0 \|_1,
\]
where we assume that
\[
I_k(x) = \mathcal{F}^{-1} \left( e^{it\phi(|\xi|)} \psi(2^{-k} |\xi|) \right)(2^{-k} x) = 2^k \int e^{ix\xi} e^{it\phi(2^k \xi)} \psi(\xi) d\xi.
\]
We immediately get that
\[
\| I_k \|_\infty \lesssim 2^k,
\]
which is the result of (3), as desired. Now we assume (H3) holds. Let \( \phi_1(\xi) = x\xi + t\phi(2^k \xi) \), then \( |\phi_1''(\xi)| > |t| 2^{k\alpha_1} \) on the support of \( \psi \). Thus by van der Corput’s Lemma (see [12]) we can get
\[
\| I_k \|_\infty \lesssim |t|^{-\frac{1}{2}} 2^{k(1-\frac{n}{4})}.
\]
By an interpolation between (10) and (11), we get for $0 \leq \theta \leq 1$,
\[
\|I_k\|_{\infty} \lesssim |t|^{-\frac{\theta}{2}2^{k(1-\theta m_2)}},
\]
which completes the proof of (a) in the case $n = 1$.

The proof of (b) is similar to (a) and we omit the details. Now we turn to the proof of (c). First, we consider the case $m_2 < 2$. Fix $0 \leq \theta \leq \min(\frac{1}{m_2}, \frac{1}{2}) = \frac{1}{2}$. Since $\theta < \frac{1}{m_2}$, it follows from (b) that
\[
\|e^{it\phi(\sqrt{-\Delta})}P_{\leq 0}u_0\|_{\infty} \lesssim \sum_{k<0}|t|^{-\theta 2^{k(1-\theta m_2)}}\|u_0\|_1 \lesssim |t|^{-\theta}\|u_0\|_1.
\]
Taking $\theta = 0$ or $\theta = \min(\frac{1}{m_2}, \frac{1}{2})$, we get the result.

Next, we consider the case $m_2 \geq 2$. One easily sees that (12) holds also in the case $m_2 \geq 2$ and $\theta = 0$. So, it suffices to consider the case $m_2 \geq 2$ and $\theta = \min(\frac{1}{m_2}, \frac{1}{2}) = \frac{1}{m_2}$. By simple calculation, we get that for any $m \geq 0$,
\[
\frac{d^m}{d\xi^m}\left(\frac{1}{\phi'(2^k\xi)}\right) \lesssim 2^{-k(m_2-1)}, \quad \xi \in \text{supp}\psi.
\]
Thus, if $|x| \leq 1$, then $|\partial_\xi^m(e^{ix\xi}\psi(\xi))| \lesssim 1$, and integrating by part we can get that for any $q \geq 0$,
\[
|I_k(x)| \lesssim |t|^{-q2^{k(1-m_2q)}}.
\]
If $|x| > 1$, let $k_0$ be the smallest integer such that $|x| \leq |t|2^{k_0m_2}$, then $|x| \approx |t|2^{k_0m_2}$. For $|k-k_0| > C \gg 1$, one has that $|\phi'(\xi)| \geq c|t|2^{km_2}$, integrating by part we can get that for any $q \geq 0$ (see (21) below),
\[
|I_k(x)| \lesssim |t|^{-q2^{k(1-m_2q)}}.
\]
For $|k-k_0| \leq C$, noticing that $|x| > 1$ and $m_2 \geq 2$, we have
\[
|I_k(x)| \lesssim |t|^{-\frac{1}{2}2^{k(1-m_2)}} \lesssim |t|^{-\frac{1}{2}} \left(\frac{|x|}{|t|}\right)^{(1-\frac{m}{m_2})\frac{1}{m_2}} \lesssim |t|^{-\frac{1}{m_2}}.
\]
Therefore, taking $q$ sufficiently large, we have
\[
\left|\sum_{k \leq 0} I_k(x)\right| \lesssim \sum_{|k-k_0| \leq C} |I_k(x)| + \sum_{|k-k_0| \geq C} |I_k(x)| \lesssim \sum_{|k-k_0| \leq C} |t|^{-\frac{1}{m_2}} + \sum_{2^k < |t|^{-\frac{1}{m_2}}} 2^k + \sum_{2^k > |t|^{-\frac{1}{m_2}}} |t|^{-q2^{k(1-m_2q)}} \lesssim |t|^{-\frac{1}{m_2}},
\]
which completes the proof of (c).

**Step 2.** We consider the case $n \geq 2$. Our idea is as follows: First, we reduce the problem to an oscillatory integral in one dimension relating the Bessel function by changing to polar coordinates; Next, we divide the discussion into two cases: in one case we use the vanishing property at the origin and the recurring property for the Bessel function, and in another case we use the decay property of the Bessel function. We denote by $J_m(r)$ the Bessel function:
\[
J_m(r) = \frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}} \int_{-1}^{1} e^{irt} (1-t^2)^{m-1/2} dt, \quad m > -1/2.
\]
We first list some properties of $J_m(r)$ in the following lemma. For their proof we refer the readers to [12], [5].
**Lemma 1** (Properties of the Bessel function). We have for $0 < r < \infty$ and $m > -\frac{1}{2}$

(i) $J_m(r) \leq Cr^m$,
(ii) $\frac{d}{dr}(r^{-m}J_m(r)) = -r^{-m}J_{m+1}(r)$,
(iii) $J_m(r) \leq Cr^{-\frac{m}{2}}$.

It is well known that the Fourier transform of a radial function $f$ is still radial and (cf. [13])

$$\hat{f}(\xi) = 2\pi \int_0^\infty f(r)r^{n-1}(r|\xi|)\frac{n-2}{2} J_{n-2}(r|\xi|) dr,$$

(14)

From (i) and (ii) of Lemma 1, we can easily get that for $0 \leq s \leq 2$ and for any $k \geq 0$,

$$\left| \frac{\partial^k}{\partial r^k} \left( \psi(r)r^{n-1}(rs)^{(n-2)/2} J_{n-2}(rs) \right) \right| \leq c_k.$$  

(15)

If $m = -\frac{n-2}{2}$, $J_m(r)$ is connected to the Fourier transform of the spherical surface measure. It is known that (see [6], Ch. 1, Equation (1.5)),

$$r^{-\frac{n-2}{2}} J_{n-2}(r) = c_n \mathcal{F}(e^{ir}h(r)),$$

(16)

where $h$ satisfies

$$|\partial^k_r h(r)| \leq c_k (1 + r)^{-\frac{n-1}{2} - k}.$$  

(17)

From (17), we get that, for $s > 2$ and for any $k \geq 0$,

$$\left| \frac{\partial^k}{\partial r^k} \left( \psi(r)r^{n-1}h(rs) \right) \right| \leq c_k s^{-\frac{n-1}{2}}.$$  

(18)

We now show the proof of (a). It follows from Young’s inequality that

$$\|e^{it\phi(\sqrt{-\Delta}) \Delta_k u_0}\|_\infty = \|\mathcal{F}^{-1}e^{it\phi(|\xi|)}\psi(2^{-k}|\xi|)\mathcal{F} u_0\|_\infty \lesssim \|\mathcal{F}^{-1}e^{it\phi(|\xi|)}\psi(2^{-k}|\xi|)\|_\infty \|u_0\|_1.$$  

In view of (14) we have

$$\mathcal{F}^{-1}(e^{it\phi(|\xi|)}\psi(2^{-k}|\xi|))(x) = 2^{kn} \mathcal{F}^{-1}(e^{it\phi(|2^k\xi|)}\psi(|\xi|))(2^k|x|) = 2^{kn} \int_0^\infty e^{it\phi(2^kr)} \psi(r)r^{n-1}(r2^k s)^{-\frac{n-2}{2}} J_{n-2}(r2^k s) dr := II_k(2^k s),$$

where $s = |x|$. It suffices to show

$$\|II_k(s)\|_\infty \lesssim |t|^{-\theta} 2^{k(n-m+\theta)}.$$  

From (i) of Lemma 1 we obtain the trivial estimate for $\theta = 0$,

$$\|II_k(s)\|_\infty \lesssim 2^{kn}.\quad(19)$$

We will discuss it in following two cases.

**Case 1.** $s \leq 2$. In this case, we will use the vanishing property of the Bessel function at the origin. Denote $D_r = \frac{1}{i t \phi'(2^k r)^2} \frac{d}{dr}$. We see that

$$D_r(e^{it\phi(2^k r)}) = e^{it\phi(2^k r)}, \quad (D_r)^* f = -\frac{1}{i t 2^k} \frac{d}{dr} \left( \frac{1}{\phi'(2^k r)^2} \right).$$


From (H1), we get that for any $m \geq 0$ and $r \sim 1,$

$$\frac{d^m}{dr^m} \left( \frac{1}{\phi^q(2^k r)} \right) \leq c_m 2^{-k(m_1 - 1)}. \quad (20)$$

Let $\tilde{\psi}(r) = \psi(r) r^{n-1}.$ Using integration by part, we have for any $q \in \mathbb{Z}^+$,

$$II_k(s) = 2^{kn} \int_0^\infty \frac{e^{it\phi(2^k r)} \tilde{\psi}(r)(rs)^{-\frac{nq}{2}} J_{\frac{nq}{2}}(rs)}{\phi^q(2^k r)} \, dr$$

$$= 2^{kn} \int_0^\infty D_r(e^{it\phi(2^k r)}) \tilde{\psi}(r)(rs)^{-\frac{nq}{2}} J_{\frac{nq}{2}}(rs) \, dr$$

$$= -\frac{2^{kn}}{it2^k} \int_0^\infty e^{it\phi(2^k r)} \frac{d}{dr} \left( \frac{1}{\phi^q(2^k r)} \tilde{\psi}(r)(rs)^{-\frac{nq}{2}} J_{\frac{nq}{2}}(rs) \right) \, dr$$

$$= \frac{2^{kn}}{(it2^k)^q} \sum_{m=0}^q \sum_{l_i \in \Lambda_m^q} C_{q,m} \cdot \int_0^\infty e^{it\phi(2^k r)} \prod_{j=1}^q \frac{\partial_j^q}{\phi^q(2^k r)} \partial_j^{q-m} \left( \tilde{\psi}(r)(rs)^{-\frac{nq}{2}} J_{\frac{nq}{2}}(rs) \right) \, dr, \quad (21)$$

where $\Lambda_m^q = \{l_1, \ldots, l_q \in \mathbb{Z}^+ : 0 \leq l_1 \leq \ldots \leq l_q \leq q, l_1 + \ldots + l_q = m \}.$ It follows from (15), (20) and (21) that, for any $q \in \mathbb{Z}^+,$

$$|II_k(s)| \lesssim |t|^{-q2^{k(n-m_1q)}}. \quad (22)$$

Interpolating (22) with (19), we get that for any $\theta \geq 0,$ $|I_k(s)| \lesssim |t|^{-\theta 2^{k(n-m_1\theta)},}$ which completes the proof of (a) in this case.

**Case 2.** $s \geq 2.$ In this case, we will use the decay property of Bessel function. It follows from (16) that

$$II_k(s) = c_n 2^{kn} \int_0^\infty e^{it\phi(2^k r)} \tilde{\psi}(r)(e^{irs} h(r) + e^{-irs} \tilde{h}(r)) \, dr$$

$$= c_n 2^{kn} \int_0^\infty e^{i(t\phi(2^k r)+rs)} \tilde{\psi}(r) h(r) \, dr + c_n 2^{kn} \int_0^\infty e^{i(t\phi(2^k r)-rs)} \tilde{\psi}(r) \tilde{h}(r) \, dr$$

$$= B_1 + B_2.$$ 

Without loss of generality, we can assume that $t > 0$ and $\phi'(r) > 0.$ For $B_1,$ let $\phi_1(r) = t\phi(2^k r) + rs.$ Note that $\phi_1'(r) = t2^k \phi'(2^k r) + s \geq c t2^{kn_1},$ and (20) also holds if we replace $\phi$ by $\phi_1.$ Noticing (18), analogous to Case 1 we can get that for any $\theta \geq 0,$

$$|B_1| \lesssim |t|^{-\theta 2^{k(n-m_1\theta)}}.$$ 

For $B_2,$ let $\phi_2(r) = t\phi(2^k r) - rs.$ We note that if $s = t2^k \phi'(2^k r),$ then $\phi_2'(r) = 0.$ We divide the discussion into the following two cases.

**Case 2a.** $s \geq 2^{sup_{r \in [1/2,2]} t2^k \phi'(2^k r)}$ or $s < \frac{1}{2} inf_{r \in [1/2,2]} t2^k \phi'(2^k r).$ In this case, we see that $|\phi_2'(r)| \geq c t2^{kn_1}$ if $r \sim 1,$ and (20) also holds if one replaces $\phi$ by $\phi_2.$ By (18), we can get that for any $\theta \geq 0,$

$$|B_2| \lesssim |t|^{-\theta 2^{k(n-m_1\theta)}}.$$ 

**Case 2b.** $\frac{1}{2} inf_{r \in [1/2,2]} t2^k \phi'(2^k r) \leq s \leq 2^{sup_{r \in [1/2,2]} t2^k \phi'(2^k r)}.$ It follows from (18) that

$$|B_2| \lesssim 2^{kn} s^{-\frac{n+1}{2}} \lesssim t^{-\frac{n+1}{2}2^{k(n-(n-1)m_1)}}.$$ 

(23)
Interpolating (23) with (19), we get that for \(0 \leq \theta \leq \frac{n-1}{2}\),
\[
|B_2| \lesssim t^{-\theta}2^{k(n-m_1\theta)}.
\] (24)

If (H3) holds in addition, then \(|\phi''(r)| \geq t2^{k\alpha_1}\). It follows from van der Corput’s Lemma that
\[
|B_2| \lesssim (t2^{k\alpha_1})^{-1/2} \int_0^\infty \left| \frac{d}{dr}(\psi(r)h(rs)) \right| dr \lesssim t^{-n/2}2^{k(n-\frac{n}{2}(m_1+\frac{(\alpha_1-m_1)}{n}))}.
\] (25)

Therefore, interpolating (25) with (24) and using the fact that for \(0 \leq \theta \leq 1, \frac{n-1+\theta}{2} = (1-\theta)\frac{n-1}{2} + \theta\frac{n}{2}\), we get
\[
|II_k(s)| \lesssim |t|^{-\frac{n-1+\theta}{2}}2^{k(n-m_1\frac{(n-1+\theta)}{2} - \frac{(\alpha_1-m_1)}{2})}, \quad 0 \leq \theta \leq 1,
\]
which completes the proof of (a).

The proof of (b) is similar to that of (a) and we omit the details. Now we turn to proof of (c). Fix \(0 \leq \theta \leq \min\left(\frac{n}{m_2}, \frac{n-1}{2}\right)\), If \(\theta < \frac{n-1}{m_2}\), then \(n-m_2\theta > 0\). From (b), we have
\[
\begin{align*}
\|e^{it\phi(\sqrt{-\Delta})}P_{\leq 0}u_0\|_\infty & \lesssim \sum_{k=-\infty}^{2} \|e^{it\phi(\sqrt{-\Delta})}\Delta_k P_{\leq 0}u_0\|_\infty \\
& \lesssim \sum_{k=-\infty}^{2} |t|^{-\theta}2^{k(n-m_2\theta)}\|P_{\leq 0}u_0\|_1 \\
& \lesssim |t|^{-\theta}\|P_{\leq 0}u_0\|_1.
\end{align*}
\]

Now we assume \(\frac{n-1}{2} > \frac{n}{m_2}\) and \(\theta = \frac{n}{m_2}\) in the following discussion. From the proof of (b), we know that, if \(k_0 < 0\) and \(s \sim t2^{k_0m_2} \geq 2\), then
\[
|II_{k_0}(s)| \lesssim t^{-\frac{n-1}{2}}2^{k_0(n-\frac{(n-1)m_2}{2})} \\
\lesssim t^{-\frac{n-1}{2}}(\frac{8}{t})^{\frac{(n-1)m_2}{2}} \lesssim t^{-\frac{n}{m_2}}.
\]

If \(|k - k_0| > C \gg 1\), then
\[
|II_k(s)| \lesssim t^{-\alpha}2^{k(n-m_2\alpha)}, \quad \forall \alpha \geq 0.
\]

Therefore, choosing \(\alpha\) large, we have
\[
|II_{\leq 0}(s)| \lesssim \sum_{|k-k_0| \leq C} |II_k(s)| + \sum_{|k-k_0| > C} |II_k(s)| \\
\lesssim t^{-\frac{n}{m_2}} + \sum_{2^k < t} 2^{kn} + \sum_{2^k > t} t^{-\alpha}2^{k(n-m_2\alpha)} \\
\lesssim t^{-\frac{n}{m_2}}.
\]

If in addition (H4) holds, the proof is similar. We omit the details. \(\square\)

**Remark 3.** It’s easy to see that in the case \(n = 1\) we did not use the properties that \(\phi\) is even. Our method is also adapted to more general radial \(\phi\). But it seems difficult to apply for non-radial \(\phi\).
3 Strichartz Estimate

In this section, we show the Strichartz estimate by using the decay estimates obtained in Section 2. We will work in a general setting in this section and apply it to some concrete equations in the next section. Our method is using duality argument. We mention that this argument is quite standard. We will omit most of the proof, and refer the reader to [7] for details. Here we use an argument in [17]. Since the decay rate is different between $|t| > 1$ and $|t| \leq 1$, we will need a variant Hardy-Littlewood-Sobolev inequality.

Lemma 2. Assume $\gamma_1, \gamma_2 \in \mathbb{R}$, let

$$k(y) = \begin{cases} 
|y|^{-\gamma_1}, & |y| \leq 1, \\
|y|^{-\gamma_2}, & |y| > 1, 
\end{cases}$$

Assume that one of the following conditions holds,

(a) $0 < \gamma_1 = \gamma_2 < n$, $1 < p < q < \infty$ and $1 - \frac{1}{p} + \frac{1}{q} = \frac{\gamma_1}{n}$,

(b) $\gamma_1 < \gamma_2$, $0 < \gamma_1 < n$, $1 < p < q < \infty$ and $1 - \frac{1}{p} + \frac{1}{q} = \frac{\gamma_1}{n}$,

(c) $\gamma_1 < \gamma_2$, $0 < \gamma_2 < n$, $1 < p < q < \infty$ and $1 - \frac{1}{p} + \frac{1}{q} = \frac{\gamma_2}{n}$,

(d) $\gamma_1 < \gamma_2$, $1 \leq p \leq q \leq \infty$ and $\frac{\gamma_1}{n} < 1 - \frac{1}{p} + \frac{1}{q} < \frac{\gamma_2}{n}$.

We have

$$\|f \ast k\|_q \lesssim \|f\|_p.$$  

Proof. By splitting $\mathbb{R}^n$ into $|y| \geq 1$ and $|y| \leq 1$, we can easily get the results by following Hardy-Littlewood-Sobolev’s and Young’s inequalities. \hfill \Box

Definition 1. Given $\theta_1 \leq \theta_2$, we say $q$ belongs to $E(\theta_1, \theta_2)$ if one of the following holds:

(a) $0 < \theta_1 = \theta_2 < 1$ and $q = \frac{2}{\theta_1}$,

(b) $\theta_1 < \theta_2$, $0 < \theta_1 < 1$ and $q = \frac{2}{\theta_1}$,

(c) $\theta_1 < \theta_2$, $0 < \theta_2 < 1$ and $q = \frac{2}{\theta_2}$,

(d) $\theta_1 < \theta_2$, $2 \leq q \leq \infty$ and $\theta_1 < \frac{2}{q} < \theta_2$.

We now give the Strichartz estimate. Denote

$$U(t) = e^{it(\sqrt{-\Delta})}, \quad \mathcal{A}f = \int_0^t U(t - \tau)f(\tau, \cdot) d\tau.$$  

We assume that, for $2 \leq p \leq \infty$, $\alpha := \alpha(p) \in \mathbb{R}$, and $\theta_1 \leq \theta_2$,

$$\|U(t)f\|_{\dot{B}_{p,2}^\theta} \lesssim k(t)\|f\|_{\dot{B}^\alpha_{p',2}},$$  

where

$$k(t) = \begin{cases} 
|t|^{-\theta_1}, & |t| \leq 1, \\
|t|^{-\theta_2}, & |t| > 1. 
\end{cases}$$

Using Lemma 2 and standard duality argument, we can prove the following proposition. We omit its proof and refer the reader to see [7], [17].

Proposition 1. Assume $U(t)$ satisfies (26), then we have for $q \in E(\theta_1, \theta_2)$, $\eta \in \mathbb{R}$, and $T > 0$,

$$\|U(t)h\|_{L^q(-T,T;\dot{B}_{p,2}^{\theta+\frac{\alpha}{2}})} \lesssim \|h\|_{H^n},$$

$$\|\mathcal{A}f\|_{L^q(-T,T;\dot{B}^{\theta+\alpha}_{p,2})} \lesssim \|f\|_{L^q(-T,T;\dot{B}^\alpha_{p',2})},$$

$$\|\mathcal{A}f\|_{L^\infty(-T,T;H^n+\frac{\alpha}{2})} \lesssim \|f\|_{L^q(-T,T;\dot{B}^\alpha_{p',2})},$$

$$\|\mathcal{A}f\|_{L^8(-T,T;H^n+\frac{\alpha}{2})} \lesssim \|f\|_{L^q(-T,T;\dot{B}^\alpha_{p',2})}.$$  

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Remark 4. The endpoint case $\theta_i = 1$ also holds by following Keel and Tao’s ideas in [7], but we will not pursue this issue in this paper.

4 Application

In this section we will apply Theorem 1 in Section 2 to some concrete equations. Our results below can cover some known results so far, and make some improvements and provide simple proofs for the Klein–Gordon equation and the Beam equation. A simple case is the semi-group $e^{it(-\Delta)^\rho}$, $\rho > 0$, we do not list its estimates and one can get the desired estimates by using the same way as in the following Klein-Gordon equation.

1 (Klein-Gordon equation). First, we consider the Klein-Gordon equation,

\[
\begin{cases}
\partial_t u - \Delta u + u = F, \\
u(0) = u_0(x), \ u_t(0) = u_1(x).
\end{cases}
\]  

(27)

By Duhamel’s principle, we get

\[u = K'(t)u_0 + K(t)u_1 - \int_0^t K(t - \tau)F(\tau)d\tau,\]

where

\[K(t) = \omega^{-1} \sin(t\omega), \quad K'(t) = \cos(t\omega), \quad \omega = \sqrt{I - \Delta}.
\]

This reduces to the semigroup $K_{\pm}(t) := e^{\pm it(1-\Delta)^{1/2}}$, which corresponds to $\phi(r) = (1 + r^2)^{1/2}$.

By simple calculation,

\[\phi'(r) = r/(1 + r^2)^{3/2}, \quad \phi''(r) = 1/(1 + r^2)^{3/2},\]

we see that $\phi$ satisfies (H1), (H2), (H3) and (H4) with $m_1 = 1$, $\alpha_1 = -1$, $m_2 = \alpha_2 = 2$.

Proposition 2. Assume $2 \leq p \leq \infty$, $1 \leq q \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\delta = \frac{1}{2} - \frac{1}{p}$.

(i) Let $0 \leq \theta \leq 1$, and $(n + 1 + \theta)\delta = 1 + s' - s$, we have

\[\|K(t)g\|_{L^p_{\mu,q}} \lesssim \|g\|_{B^{s'}_{\mu',q}}.\]

(ii) Let $0 \leq \theta \leq n - 1$, and $(n + 1 + \theta)\delta = 1 + s' - s$, we have

\[\|K(t)g\|_{L^p_{\mu,q}} \lesssim \|g\|_{B^{s'}_{\mu',q}}.\]

(iii) In particular, one has that for $\theta \in [0, 1]$, $(n + 1 + \theta)\delta \leq 1 + s' - s$,

\[\|K(t)g\|_{L^p_{\mu,q}} \lesssim k(t)\|g\|_{B^{s'}_{\mu',q}}, \quad k(t) = \begin{cases} |t|^\min(1 + s' - s - 2n\delta, 0), & |t| \leq 1, \\
|t|^{-(n+1+\theta)\delta}, & |t| \geq 1. \end{cases}\]

Proof. First, we show the results of (i). It follows from (c) of Theorem 1 and Plancherel’s identity that,

\[\|K_+(t)P_{\leq 0}u_0\|_\infty \lesssim |t|^{-(n+1+\theta)/2}\|P_{\leq 0}u_0\|_1,\]

\[\|K_+(t)P_{\leq 0}u_0\|_2 \lesssim \|P_{\leq 0}u_0\|_2.\]
From (4) of Theorem 1 and Plancherel’s identity, we can get for $k > 0,$
\[
\|K_+(t)\Delta_k u_0\|_\infty \lesssim |t|^{-\frac{n+1}{2}} 2^{\frac{n+1}{2}+\frac{\theta}{2}} \|\Delta_k u_0\|_1,
\]
\[
\|K_+(t)\Delta_k u_0\|_2 \lesssim \|\Delta_k u_0\|_2.
\]

Thus by Riesz-Thorin theorem,
\[
\|K_+(t)P\leq_0 u_0\|_p \lesssim |t|^{-\delta (n+1+\theta)} \|P\leq_0 u_0\|_{p'},
\]
\[
\|K_+(t)\Delta_k u_0\|_p \lesssim |t|^{-\delta (n+1+\theta)} 2^{\delta (n+1+\theta)} \|\Delta_k u_0\|_{p'}.
\]

Therefore, it follows from $(n+1+\theta)\delta = 1 + s' - s$ that
\[
\|K(t)u_0\|_{B^s_{p,q}} \lesssim |t|^{-\delta (n-1+\theta)} \|u_0\|_{B^s_{p',q}},
\]
which completes the proof of (i).

Next, we prove (ii). Let us rewrite (3) as
\[
\|K_+(t)\Delta_k u_0\|_\infty \lesssim |t|^{-\frac{n-1-\theta}{2}} 2^{\frac{k(n+1+\theta)}{2}} \|\Delta_k u_0\|_1, \quad 0 \leq \theta \leq n-1.
\]

(7) implies that
\[
\|K_+(t)P\leq_0 u_0\|_\infty \lesssim |t|^{-\frac{n-1-\theta}{2}} \|P\leq_0 u_0\|_{1}, \quad 0 \leq \theta \leq n-1.
\]

Hence, using the same way as in the proof of (i), we can easily get the results of (ii).

Following the proofs above, one can get the results of (iii). Indeed, it suffices to consider the case $|t| \leq 1, \theta \in [0, 1]$ and $1+s'-s-2n\delta \geq 0.$ If this case occurs, taking $\theta = n-1$ in (28) and (29) one has the result, as desired.

2 (Beam equation). We now consider the Beam equation, in some literature it is called fourth order wave equations,
\[
\begin{cases}
\partial_{tt}u + \Delta^2 u + u = F, \\
u(0) = u_0(x), \quad u_t(0) = u_1(x).
\end{cases}
\]

By Duhamel’s principle, we have
\[
u(t) = B'(t)u_0 + B(t)u_1 - \int_0^t B(t-\tau)F(\tau) d\tau \tag{30}
\]

where
\[
B(t) = \omega^{-1} \sin(t\omega), \quad B'(t) = \cos(t\omega), \quad \omega = \sqrt{I + \Delta^2}.
\]

This reduces to the semigroup $B_\pm(t) := e^{\pm i(t+\Delta^2)^{1/2}},$ which corresponding to $\phi(r) = (1+r^4)^{1/2}.$ By simple calculation,
\[
\phi'(r) = 2r^3/(1 + r^4)^{\frac{3}{2}}, \quad \phi''(r) = (6r^2 + 2r^6)/(1 + r^4)^{\frac{5}{2}},
\]
we know that $\phi$ satisfies (H1), (H2), (H3) and (H4) with $m_1 = \alpha_1 = 2, m_2 = \alpha_2 = 4.$
Proposition 3. Assume $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq q \leq \infty$, $\delta = \frac{1}{2} - \frac{1}{p}$, and $0 \leq 2 + s' - s$, then
\[
\|B(t)g\|_{B^{p,q}_{p',q}} \leq k(t)\|g\|_{B^{p,q}_{p',q}},
\]
where
\[
k(t) = \begin{cases} 
|t|^{\min\left(1 + \frac{s'}{2} - \frac{s}{2} - n\delta, 0\right)}, & |t| \leq 1, \\
|t|^{-\frac{n\delta}{2}}, & |t| \geq 1.
\end{cases}
\]

Proof. First, we prove the case $|t| \geq 1$. It follows from (c) of Theorem 1 by setting $\theta = \frac{n}{2}$ and Riesz-Thorin interpolation theorem that
\[
\|B_+(t)P_{\leq 0}u_0\|_p \lesssim |t|^{-\frac{n}{2}}\|P_{\leq 0}u_0\|_{p'}.
\]
From (a) of Theorem 1 by setting $\theta = \frac{n}{2}$ and interpolation, we get for $k > 0$
\[
\|B_+(t)\Delta_k u_0\|_p \lesssim |t|^{-n\delta}\|\Delta_k u_0\|_{p'},
\]
and then we have
\[
2^{ks}\|(I + \Delta^2)^{-1/2}B_+(t)\Delta_k u_0\|_p \lesssim |t|^{-\frac{n}{2}}2^{ks}\|\Delta_k u_0\|_{p'}.
\]
Therefore,
\[
\|B(t)g\|_{B^{p,q}_{p',q}} \lesssim |t|^{-\frac{n}{2}}\|g\|_{B^{p,q}_{p',q}},
\]
which completes the proof of the proposition in this case.

For $|t| \leq 1$, from (c) of Theorem 1 by setting $\theta = 0$ and interpolation, we get
\[
\|(I + \Delta^2)^{-1/2}B_+(t)P_{\leq 0}u_0\|_p \lesssim \|P_{\leq 0}u_0\|_{p'}.
\]
For $k > 0$, from (a) of Theorem 1 and interpolation, we get for $0 \leq \theta \leq \frac{n}{2}$,
\[
\|B_+(t)\Delta_k u_0\|_p \lesssim |t|^{-\theta}2^{2k(n-2\theta)\delta}\|\Delta_k u_0\|_{p'},
\]
and then that
\[
2^{ks}\|(I + \Delta^2)^{-1/2}B_+(t)\Delta_k u_0\|_p \lesssim |t|^{-\theta}2^{2k(n-2\theta)\delta}\|\Delta_k u_0\|_{p'}.
\]
If $0 \leq 2 + s' - s \leq 2n\delta$, then we can choose $0 \leq \theta \leq \frac{n}{2}$ such that $2 + s' - s - 2n\delta = -4\delta$. If $2 + s' - s > 2n\delta$, then we choose $\theta = 0$. Thus we get
\[
2^{ks}\|(I + \Delta^2)^{-1/2}B_+(t)\Delta_k u_0\|_p \lesssim |t|^{\min\left(1 + \frac{s'}{2} - \frac{s}{2} - n\delta, 0\right)}2^{ks}\|\Delta_k u_0\|_{p'}.
\]
Therefore,
\[
\|B(t)g\|_{B^{p,q}_{p',q}} \lesssim |t|^{\min\left(1 + \frac{s'}{2} - \frac{s}{2} - n\delta, 0\right)}2^{ks}\|g\|_{B^{p,q}_{p',q}},
\]
which completes the proof of the proposition.

Corollary 1. Assume $2 \leq q < \infty$, $u(t)$ be the solution of (30) with $F = 0$. Then
\[
\|u(t)\|_{L^q(\mathbb{R}^n)} \lesssim k(t)(\|u_0\|_{W^{2,q}} + \|u_1\|_{L^{q'}}),
\]
where
\[
k(t) = \begin{cases} 
|t|^{\min\left(1 + \frac{s'}{2} - \frac{s}{2} - n\delta, 0\right)}, & |t| \leq 1, \\
|t|^{-\frac{n\delta}{2}}, & |t| \geq 1. \end{cases}
\]
Furthermore, if $u_0 = 0$ then for $|t| \leq 1$,
\[
\|u(t)\|_q \lesssim |t|^{1 + \frac{n}{2} - \frac{s}{2}}\|u_1\|_{q'}.\]
Remark 5. Corollary \(\square\) is a slightly modified version of Theorem 2.1 in \([8]\). Actually, our result is slightly stronger except for \(q = \infty\) which is due to the failure of Littlewood-Paley theory.

Proof. From Duhamel’s principle, Proposition \([8]\) by setting \(s = s’ = 0\) and embedding theorem, we immediately get \([31]\). Now we assume \(u_0 = 0\) and \(1 + \frac{n}{q} - \frac{n}{2} \geq 0\). If \(n \geq 2\), then from Proposition \([8]\) we have for \(|t| \leq 1\),

\[
\|\omega^{-1} \sin(t\omega) u_1\|_{B^0_{\infty,2}} \lesssim |t|^{1-\frac{n}{2}} \|u_1\|_{B^0_{1,2}},
\]

and interpolating this with the trivial estimate,

\[
\|\omega^{-1} \sin(t\omega) u_1\|_{2} \lesssim |t| \|u_1\|_{2},
\]

we get that

\[
\|u(t)\| \lesssim |t|^{1+\frac{n}{2} - \frac{n}{4}} \|u_1\|_{p'}.\]

For \(n = 1\), it suffices to show

\[
\|\omega^{-1} \sin(t\omega) u_1\|_{B^0_{\infty,2}} \lesssim |t|^{\frac{1}{2}} \|u_1\|_{B^0_{1,2}}.
\]

Using Young’s inequality, we can easily prove it. We omit the details. \(\square\)

3 (Fourth order Schrödinger equation). Finally, we consider the fourth order Schrödinger equation. It is given by

\[
i\partial_t u + \Delta^2 u - \Delta u = F, \quad u(0) = u_0(x). \tag{32}
\]

By Duhamel’s principle,

\[
u = U(t)u_0 - \int_0^t U(t-\tau) F(\tau) d\tau
\]

where

\[
U(t) = e^{it(\Delta^2 - \Delta)},
\]

which corresponding to \(\phi(r) = r^2 + r^4\). By simple calculation, we know \(\phi\) satisfies \((H1), (H2), (H3) and (H4) with \(m_1 = a_1 = 4, m_2 = a_2 = 2.\)

Proposition 4. Assume \(2 \leq p \leq \infty, \frac{1}{p} + \frac{1}{p'} = 1, 1 \leq q \leq \infty, \delta = \frac{1}{2} - \frac{1}{p'}, -2n\delta \leq s' - s\), then

\[
\|U(t)g\|_{B^p_{q,q}} \leq k(t) \|g\|_{B^{p'}_{q',q}}
\]

where

\[
k(t) = \begin{cases} |t|^{\frac{1}{2} \min(s'-s-2n\delta, 0)}, & |t| \leq 1, \\ |t|^{-n\delta}, & |t| \geq 1.\end{cases}
\]

Proof. For \(|t| \leq 1\), it follows from \((c) of theorem \([1]\) by setting \(\theta = 0\) that

\[
\|U(t)P_{\leq 0} u_0\|_p \lesssim \|P_{\leq 0} u_0\|_{p'}.
\]

For \(k > 0\), from \((a) of Theorem \([1]\) and interpolation, we get for \(0 \leq \theta \leq \frac{n}{2}\),

\[
\|U(t) \Delta_k u_0\|_p \lesssim |t|^{-2\theta\delta} 2^{2k(2\theta - 2\delta)} \|\Delta_k u_0\|_{p'},
\]

and for \(\theta \geq \frac{n}{2}\),

\[
\|U(t) \Delta_k u_0\|_p \lesssim |t|^{-n\delta} 2^{k(n-2\delta)} \|\Delta_k u_0\|_{p'}.
\]
Theorem 2. Let $\kappa(n) < \kappa < 4/n$, $\sigma(n) = \frac{\kappa(n+2)}{2(2+\kappa)}$, $(u_0, u_1) \in H^{\sigma(n)}_{(2+\kappa)/(1+\kappa)} \times H^{\sigma(n)-1}_{(2+\kappa)/(1+\kappa)}$ with sufficiently small norm. Then Eq. (33) has a unique solution
\[ u \in C(\mathbb{R}, H^{\kappa/(2+\kappa)}) \cap L^{1+\kappa}(\mathbb{R}, L^{2+\kappa}(\mathbb{R}^n)). \]

Proof. We present a quite simple proof. Using the basic decay of $K(t)$ and $K'(t)$, we have
\[ \|K'(t)u_0\|_{2+\kappa} \lesssim k(t)\|u_0\|_{H^{\sigma(n)}_{(2+\kappa)/(1+\kappa)}}, \quad \|K(t)u_1\|_{2+\kappa} \lesssim k(t)\|u_1\|_{H^{\sigma(n)-1}_{(2+\kappa)/(1+\kappa)}}, \]
and then
\[ 2^{k_s}\|U(t)\triangle_k u_0\|_{p'} \lesssim |t|^{-2\delta}2^{-k(s' - s - 2(n-4\theta)\delta)}2^{k_s'}\|\triangle_k u_0\|_{p'}. \]
If $-2n\delta \leq s' - s \leq 2n\delta$, then we can choose $0 \leq \theta \leq \frac{n}{2}$ such that $s' - s - 2n\delta = -8\theta\delta$. If $s' - s > 2n\delta$, then we choose $\theta = 0$. Thus we get
\[ 2^{k_s}\|U(t)\triangle_k u_0\|_{p'} \lesssim |t|^\frac{1}{2}\min(s' - s - 2n\delta, 0)2^{k_s'}\|\triangle_k u_0\|_{p'}. \]
Therefore, we get
\[ \|U(t)g\|_{B^s_{p',q}} \lesssim |t|^\frac{1}{2}\min(s' - s - 2n\delta, 0)\|g\|_{B^s_{p',q}'}, \]
which completes the proof in the case $|t| \leq 1$.

For the case $|t| \geq 1$, we can follow the same way as in the proof of Proposition 3 to get the result, which completes the proof. \hfill \Box

5 Nonlinear Klein-Gordon and Beam equations

We consider the Cauchy problem for the nonlinear Klein-Gordon equation (NLKG)
\[ \partial_t u - \Delta u + u = |u|^\kappa+1, \quad u(0) = u_0(x), \quad u_t(0) = u_1(x). \] (33)
By Duhamel’s principle, NLKG is equivalent to
\[ u = K'(t)u_0 + K(t)u_1 - \int_0^t K(t - \tau)|u(\tau)|^{1+\kappa}d\tau. \]
If $\kappa \geq 4/n$, the global well posedness and the scattering with small data in $H^s$ were studied in [10, 11, 14, 18, 19, 21]. When Strauss [14] studied the existence of the scattering operators at low energy, an important critical power $\kappa(n)$ of the following NLKG
\[ \partial_t u + u - \Delta u + |u|^{\kappa+1}u = 0 \] (34)
was discovered, where
\[ \kappa(n) = \frac{2 - n + \sqrt{n^2 + 12n + 4}}{2n}. \]
Strauss [14] obtained the existence of the scattering operators at low energy of Eq. (34) in the case $\kappa(n) < \kappa \leq 4/(n - 1)$. Since Eq. (33) has no conservation of energy, the technique in [14] cannot be directly applied for Eq. (33). However, using the basic decay estimates of the Klein-Gordon equation, we have

Theorem 2. Let $\kappa(n) < \kappa < 4/n$, $\sigma(n) = \frac{\kappa(n+2)}{2(2+\kappa)}$, $(u_0, u_1) \in H^{\sigma(n)}_{(2+\kappa)/(1+\kappa)} \times H^{\sigma(n)-1}_{(2+\kappa)/(1+\kappa)}$ with sufficiently small norm. Then Eq. (33) has a unique solution
\[ u \in C(\mathbb{R}, H^{\kappa/(2+\kappa)}) \cap L^{1+\kappa}(\mathbb{R}, L^{2+\kappa}(\mathbb{R}^n)). \]
By Duhamel's principle, NLB is equivalent to

\[ \begin{cases} 
|t|^\min\left(\frac{2}{2+n}, 0\right), & |t| \leq 1, \\
|t|^{-\frac{n}{2(2+n)}}, & |t| \geq 1. 
\end{cases} \]

Noticing that if \( \kappa(n) < \kappa < 4/n \), then we have

\[(1 + \kappa)\kappa\frac{(n - 2)}{2(2 + \kappa)} < 1, \quad (1 + \kappa)\frac{\kappa n}{2(2 + \kappa)} > 1, \quad \sigma(\kappa) < 1.\]

It follows that \( k(\cdot) \in L^{1+\kappa}(\mathbb{R}^n) \) and

\[
\|K'(t)u_0\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u_0\|_{H^{\sigma(\kappa)}(\mathbb{R}^{2+\kappa})}, \quad \|K(t)u_1\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u_1\|_{H^{\sigma(\kappa)-1}(\mathbb{R}^{2+\kappa})}. 
\]

In view of Young's and Hölder's inequalities,

\[
\left\| \int_0^t K(t - \tau)|u(\tau)|^{1+\kappa}d\tau \right\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u\|^{\kappa+1}_{L^{1+\kappa}(\mathbb{R}, H^{\sigma(\kappa)-1}(\mathbb{R}^{2+\kappa}))} \lesssim \|u\|^{\kappa+1}_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})}. 
\]

Taking \( M = 2C(\|u_0\|_{H^{\sigma(\kappa)}(\mathbb{R}^{2+\kappa})} + \|u_1\|_{H^{\sigma(\kappa)-1}(\mathbb{R}^{2+\kappa})}) \) and

\[ X = \{u \in L^{1+\kappa}(\mathbb{R}, L^{2+\kappa}) : \|u\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \leq M\}. \]

Observing the mapping

\[ \mathcal{F} : u \rightarrow K'(t)u_0 + K(t)u_1 - \int_0^t K(t - \tau)|u(\tau)|^{1+\kappa}d\tau, \]

we have

\[ \|\mathcal{F}u\|_X \leq M/2 + CM^{1+\kappa}. \]

If \( CM^{\kappa} \leq 1/2 \), we see that \( \mathcal{F} : X \to X \) is a contraction mapping. Hence, Eq. (33) has a unique solution \( u \in X \). Moreover,

\[ \|u\|_{L^{\infty}(\mathbb{R}, H^{\kappa/(2+\kappa)})} \lesssim \|u_0\|_{H^{\kappa/(2+\kappa)}} + \|u_1\|_{H^{-2/(2+\kappa)}} + \|u\|^{\kappa+1}_{L^1(\mathbb{R}, H^{-2/(2+\kappa)})}. \]

Using the embedding \( H^{\sigma(\kappa)}_{(2+\kappa)/(1+\kappa)} \subset H^{\kappa/(2+\kappa)} \), we immediately get that \( u \in L^{\infty}(\mathbb{R}, H^{\kappa/(2+\kappa)}) \). \( \square \)

Using the method as in the NLKG, we consider the Cauchy problem for the nonlinear Beam equation (NLB)

\[ \partial_t^2 u + \Delta^2 u + u = |u|^{\kappa+1}, \quad u(0) = u_0(x), \quad u_t(0) = u_1(x). \quad (35) \]

By Duhamel's principle, NLB is equivalent to

\[ u = B'(t)u_0 + B(t)u_1 - \int_0^t B(t - \tau)|u(\tau)|^{1+\kappa}d\tau. \]

If \( \kappa \geq 8/n \), the global well posedness and scattering with small data for the NLB were studied in [8, 20]. Following the same ideas as in the NLKG, we find a critical power

\[ \kappa_B(n) = \frac{4 - n + \sqrt{n^2 + 24n + 16}}{2n}. \]
Theorem 3. Let $\kappa_B(n) < \kappa < 8/n$, $\sigma(\kappa) = \frac{nk}{(2+\kappa)} - \frac{2}{1+\kappa}$, $\sigma(\kappa) < s \leq 2$, $s_2 = s - \frac{\kappa n}{2(2+\kappa)}$, $(u_0, u_1) \in H^s_{(2+\kappa)/(1+\kappa)} \times H^{s-2}_{(2+\kappa)/(1+\kappa)}$ with sufficiently small norm. Then Eq. (35) has a unique solution $u \in C(\mathbb{R}, H^{s\kappa}) \cap L^{1+\kappa}(\mathbb{R}, L^{2+\kappa}(\mathbb{R}^n))$.

Proof. The proof is similar to that of Theorem 2. It is easy to verify that $\sigma(\kappa) \in (0, 2)$. Using the basic decay of $B(t)$ and $B'(t)$, we have

$$
\|B'(t)u_0\|_{2+\kappa} \lesssim k(t)\|u_0\|_{H^{s}_{(2+\kappa)/(1+\kappa)}}, \quad \|B(t)u_1\|_{2+\kappa} \lesssim k(t)\|u_1\|_{H^{s-2}_{(2+\kappa)/(1+\kappa)}},
$$

where

$$
k(t) = \left\{ \begin{array}{ll}
|t|^{\min\left(\frac{2}{1+\kappa} - \frac{n\kappa}{2(2+\kappa)}, 0\right)}, & |t| \leq 1, \\
|t|^{-\frac{n\kappa}{4(2+\kappa)}}, & |t| \geq 1.
\end{array} \right.
$$

Noticing that if $\kappa_B(n) < \kappa < 8/n$ and $\sigma(\kappa) < s \leq 2$, then we have

$$(1 + \kappa)\left(\frac{n\kappa}{2(2+\kappa)} - \frac{s}{2}\right) < 1, \quad (1 + \kappa)\frac{kn}{4(2+\kappa)} > 1.
$$

It follows that $k(\cdot) \in L^{1+\kappa}(\mathbb{R}^n)$ and

$$
\|B'(t)u_0\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u_0\|_{H^{s}_{(2+\kappa)/(1+\kappa)}}, \quad \|B(t)u_1\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u_1\|_{H^{s-2}_{(2+\kappa)/(1+\kappa)}}.
$$

In view of Young's and Hölder's inequalities,

$$
\left\| \int_0^t B(t - \tau)|u(\tau)|^{1+\kappa}d\tau \right\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})} \lesssim \|u\|_{L^{1+\kappa}(\mathbb{R}, H^{s-2}_{(2+\kappa)/(1+\kappa)})} \lesssim \|u\|_{L^{1+\kappa}(\mathbb{R}, L^{2+\kappa})}.
$$

Then, following the same way as in the proof of Theorem 2, we can prove the result, as desired. \hfill \Box

Acknowledgment. This work is supported in part by the NSF of China, grants 10471002, 10571004; RFDP of China, grants 20060001010; and the 973 Project Foundation of China, grant 2006CB805902.

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