Monopoles and Solitons

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Abstract
In this paper, we argue that the elusive magnetic monopole arises due to the strong magnetic effects arising from the non commutative space time structure at small scales. If this structure is ignored and we work with Minkowski spacetime, then the magnetic effect shows up as a monopole. This would also explain why the monopole has eluded detection even after seventy years. We next consider another area in which Solitons can be applied, viz., Bose Einstein condensation.

1 Introduction
Non linear wave equations and solitons have been studied for a long time. Let us consider some less well known application of solitons. We start with monopoles. Ever since Dirac deduced theoretically the existence of the monopole in 1931, it has eluded physicists [1]. At the same time the possibility of realising huge amounts of energy using monopoles has been an exciting prospect. In 1980 when the fiftieth Anniversary of the monopole was being commemorated, Dirac himself expressed his belief that the monopole did not exist [2]. Some scholars have indeed dismissed the monopole [3, 4], while in a model based on quantized vortices in the hydrodynamical formulation, the monopole field can be mathematically identified with the momentum vector [5]. Monopoles had also been identified with solitons [6].

In any case, it has been noted that the existence of free monopoles would lead to an unacceptably high density of the universe [7], which in the light of latest observations of an ever expanding universe [8, 9] would be difficult
to reconcile.
We will now show that monopoles arise due to the non commutative structure of space time being ignored, and this would also provide an explanation for their being undetected.

2 The Monopole

Let us start by reviewing Dirac’s original derivation of the Monopole (Cf.ref.[1]). He started with the wave function

$$\psi = Ae^{i\gamma},$$

(1)

He then considered the case where the phase $\gamma$ in (1) is non integrable. In this case (1) can be rewritten as

$$\psi = \psi_1 e^{iS},$$

(2)

where $\psi_1$ is an ordinary wave function with integrable phase, and further, while the phase $S$ does not have a definite value at each point, its four gradient viz.,

$$\kappa^\mu = \partial^\mu S$$

(3)

is well defined. We use natural units, $\hbar = c = 1$. Dirac then goes on to identify $\kappa$ in (3) (except for the numerical factor $\hbar c/e$) with the electromagnetic field potential, as in the Weyl gauge invariant theory.

Next Dirac considered the case of a nodal singularity, which is closely related to what was later called a quantized vortex (Cf. for example ref.[10]). In this case a circuit integral of a vector as in (3) gives, in addition to the electromagnetic term, a term like $2\pi n$, so that we have for a change in phase for a small closed curve around this nodal singularity,

$$2\pi n + e \int \vec{B} \cdot d\vec{S}$$

(4)

In (4) $\vec{B}$ is the magnetic flux across a surface element $d\vec{S}$ and $n$ is the number of nodes within the circuit. The expression (4) directly leads to the Monopole.

Let us now reconsider the above arguments in terms of recent developments.
The Dirac equation for a spin half particle throws up a complex or non Hermitian position coordinate. Dirac identified the imaginary part with Zitterbewegung effects and argued that this would be eliminated once it is realized that in Quantum Mechanics, space time points are not meaningful and that on the contrary averages over intervals of the order of the Compton scale have to be taken to recover meaningful physics [11]. Over the decades the significance of such cut off space time intervals has been stressed by T.D. Lee and several other scholars [12, 13, 14, 15]. Indeed with a minimum cut off length \( l \), it was shown by Snyder [16] that there would be a non commutative space time structure, and infact at the Compton scale we would have (Cf.ref.[13])

\[
[x, y] = 0(l^2)
\] (5)

and similar relations. The Planck scale ofcourse, is the Compton scale for a Planck mass.

Infact starting from the Dirac equation itself, we can deduce directly the non commutativity (5) as recently shown [17]. This non commutative feature has also been recently stressed, both in Quantum Gravity and in Quantum SuperStrings theory [18, 19].

Let us now return to Dirac’s formulation of the monopole in the light of the above comments. As noted above, the non integrability of the phase \( S \) in (4) gives rise to the electromagnetic field, while the nodal singularity gives rise to a term which is an integral multiple of \( 2\pi \). As is well known [20] we have

\[
\tilde{\nabla} S = \tilde{p}
\] (6)

where \( \tilde{p} \) is the momentum vector. When there is a nodal singularity, as noted above the integral over a closed circuit of \( \tilde{p} \) does not vanish. Infact in this case we have a circulation given by

\[
\Gamma = \oint \tilde{\nabla} S \cdot d\tilde{r} = \hbar \oint dS = 2\pi n
\] (7)

It is because of the nodal singularity that though the \( \tilde{p} \) field is irrotational, there is a vortex - the singularity at the central point associated with the vortex makes the region multiply connected, or alternatively, in this region we cannot shrink a closed smooth curve about the point to that point. Infact if we use the fact as seen above that the Compton wavelength is a minimum
cut off, then we get from (7) using (6), and on taking $n = 1$,

$$\int \vec{\nabla} S \cdot d\vec{r} = \int \vec{p} \cdot d\vec{r} = 2\pi mc \frac{l}{2mc} = \frac{h}{2}$$

(8)

($l = \frac{h}{2mc}$ is the radius of the circuit and $h = 2\pi$ in the above natural units). In other words the nodal singularity or quantized vortex gives us the mysterious Quantum Mechanical spin half (and other higher spins for other values of $n$). In the case of the Quantum Mechanical spin, there are $2 \times n/2 + 1 = n + 1$ multiply connected regions, exactly as in the case of nodal singularities. Indeed in the case of the Dirac wave function, which is a bi-spinor $\left( \begin{array}{c} \Theta \\ \phi \end{array} \right)$,

it is well known that far outside the Compton wavelength, it is the usual spinor $\Theta$, preserving parity under reflections that predominates, whereas at and near the Compton scale it is the spinor $\phi$ which predominates, where under a reflection $\phi$ goes over to $-\phi$. This double connectivity of the Dirac spinor was shown to lead immediately to the same electromagnetic potential we had obtained from the nonintegrability of the phase above, which again was identical to that from Weyl’s gauge invariant theory (Cf.ref.[21] for details).

Let us see all this in a little greater detail [22]. We start with a non integrable infinitessimal parallel displacement of a four vector,

$$\delta a^\sigma = -\Gamma^\sigma_{\mu\nu} a^\mu dx^\nu$$

(9)

The $\Gamma$’s are the Christoffel symbols. This represents the extra effect in displacements, due to curvature. In a flat space, all the $\Gamma$’s on the right side would vanish. Considering partial derivatives with respect to the $\mu$-th coordinate, this would mean that, due to (9),

$$\frac{\partial a^\sigma}{\partial x^\mu} \to \frac{\partial a^\sigma}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} a^\nu,$$

(10)

The second term on the right side of (10) can be written as

$$-\Gamma^\lambda_{\mu\nu} g_{\lambda\sigma} a^\sigma = -\Gamma^\nu_{\mu\nu} a^\sigma,$$

where we have linearised the metric,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$
$\eta_{\mu\nu}$ being the Minkowski metric and $h_{\mu\nu}$ a small correction whose square is neglected. From (10) we conclude that,

$$\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\nu}$$

We can identify

$$A_{\mu} = \Gamma^\nu_{\mu\nu}$$

from the above using minimum electromagnetic coupling exactly as in Dirac’s monopole theory.

If we use (11), we will get the commutator relation,

$$\frac{\partial}{\partial x^\lambda} \frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\lambda} \rightarrow \frac{\partial}{\partial x^\lambda} \Gamma^\nu_{\mu\nu} - \frac{\partial}{\partial x^\mu} \Gamma^\nu_{\lambda\nu}$$

Let us now use (12) in (13): The right side does not vanish due to the electromagnetic field (12) and we have a non-commutativity of the momentum components of quantum theory. Indeed the left side of (13) can be written as

$$[p_\lambda, p_\mu] \approx \frac{0(1)}{l^2}$$

$l$ being the Compton wavelength. In (14) we have utilised the fact that at the extreme scale of the Compton wavelength, the Planck scale being a special case, the momentum is $mc$.

From (12), (13) and (14), we have,

$$Bl^2 \sim \frac{1}{e} = \left( \frac{\hbar c}{e} \right),$$

where $B$ is the magnetic field.

Equation (14) is the well-known equation for the magnetic monopole. Indeed it has been shown by Saito and the author [23, 22] that a non-commutative spacetime at the extreme scale shows up as a powerful magnetic field.

To recapitulate, the Monopole was shown by Dirac to arise because of two separate issues. The first was the non integrability of the phase $S$ given in (8), which gave rise to the electromagnetic potential (5) which was equivalent to the Weyl potential (3) which was adhoc. The other issue was that of nodal singularities or alternatively the
multiply connected nature of space which gave rise to a term like $2\pi n$ as in (1). In effect there would be free monopoles. However all this was considered in the context of the usual commutative Minkowski spacetime. Effectively this means that terms $\sim 0(l^2)$ as in (1) are neglected. However once such terms are included, in other words once the non commutative structure of spacetime to this order is recognised, firstly the previously supposedly adhoc Weyl electromagnetic formulation automatically follows as in (12) and furthermore the first term in the monopole expression (1) immediately gives the Quantum Mechanical spin, and the elusive monopole appears as the magnetic effect at the Compton (or Planck scale). Indeed in recent times the fact that non commutative spacetime gives rise to spin has been recognized [24, 17].

3 Non linear Equations

Let us now come to non-linear wave equation. In diverse areas of physical application a non-linearity leads to what may be called auto catalysis or auto production. For example in the well known trimolecular model (Brusselator) a cubic non-linearity is required in the rate equations[25]. In fact the non-linearity has to be at least of the order three. The rate equations are then of the form

$$\frac{\partial X}{\partial t} = K_1 A - (K_2 B + K_4)X + K_3 X^2 Y + D_1 \nabla^2 r X,$$

and a companion equation, where the symbols have the usual significance. Such a process is also well known in the upper atmosphere, in the formation of the triple oxygen molecule, ozone[26]:

$$O + O_2 + m \rightarrow O_3 + m.$$

It is also known that the rate equations in a number of biochemical reactions involving enzyme catalysis also exhibit in some limiting cases, cubic terms, for example the Glycolytic pathway (Cf.[25]).

In fact, in most problems of cooperative phenomena physics, for example plasmas or lasers, at least cubic non-linearities are required for such cooperative behaviour.
In other areas, such as zoology or biology or sociology too non-linearities cause auto production\cite{27,28} (Cf. for a non technical discussion). A very well-known example is the logistic equation. There is a similar situation in quantum field theory also\cite{29}. It is not surprising that auto catalysis or auto production should be a non-linear feature. In fact for a linear system not only $\psi$ but also $\alpha \psi$ is a solution. However if the system is non-linear, it can be written as

$$M(\psi) = L(\psi) + N(\psi) = 0,$$

where $L$ denotes the linear part and $N$ the non-linear part. In a first approximation, we can take

$$L(\psi_0) = 0,$$

and $N$ can be linearized by suitably substituting $\psi_0$ for $\psi$ to get, say, $L^{(0)}(\psi)$. The system would now be approximately described by the linear equation

$$[L + L^{(0)}](\psi) = 0.$$

This process, if convergent can be continued. At each stage, the coefficients of $\psi$ would depend on the linear approximation of up to that stage. This precisely is the characteristic of auto production. A similar technique has been used recently for a Ricatti equation derived from the Schrodinger equation of non-relativistic quantum mechanics\cite{30}.

### 4 The Non-linear Schrodinger Equation

In the light of the above comments, a non-linear Schrodinger equation was deduced and used to argue that the origin of inertial mass lies in self interacting amplitudes within, typically the Compton wavelength\cite{31,32,33,34}. A cubic non-linearity associated with auto catalysis is responsible for the generation of the inertial mass. In this case the equation is given by

$$\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \partial^2 \psi}{\partial x^2} + \int H(x, x')\psi(x')dx'$$

where

$$H(x, x') = \langle \psi(x') | \psi(x) \rangle,$$

\[16\]
All this can be generalized immediately to the three dimensional case.

\[ E\psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \int H(r,r')\psi(r')d^3r' \] (18)

If next in an equation like (17), the amplitude for a particle at \( r' \) to be at \( r \) vanishes outside a small interval, so that a \( \delta \) function can be introduced in (17), then we have the equation

\[ E\psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + g\psi^3 \] (19)

Let us now consider partial wave decomposition equation (18) in spherical symmetry. This gives

\[ \left[ \frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} \right] u = gu^3, k^2 = \frac{2m}{\hbar^2}E \] (20)

Further specializing to the case \( l = 0 \) and \( k^2 \approx 0 \) and in the spirit of the considerations in Section 1, neglecting the cubic term in (19), we have in the zeroeth approximation for \( u, u \approx r \). Now writing in (20), \( u^3 \) as \( u^2 \cdot u \), that is linearizing equation (20) and using the zeroeth approximation we get

\[ \left[ \frac{d^2}{dr^2} + (k^2 - r^2) \right] u = 0(k^2 \approx 0) \] (21)

Equation (21) is the well known Harmonic Oscillator equation with degenerate energy levels[33]. As is well known, a set of Harmonic oscillators as above represents an assembly of Bosons. Thus we have a collection of closely packed nearly zero energy Bosons similar to the Bose-Einstein condensation[36].

Interestingly the link between Solitons arising from the non-linear equations and Bose-Einstein condensation is being investigated, for example by Khaykovich and co-workers at the ENS Laboratory in Paris and also at the European Laboratory for Non-linear Spectroscopy in Italy. In these experiments a Bose-Einstein condensate of a dilute atomic gas of Lithium atoms is used, the inter atomic interaction providing the non-linearity. This in turn ensures the Solitonic propagation[37].

All this could have varied applications in fields ranging from Particle Physics to Non-linear Optics, including the possibility of high speed fibre optic communication.
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