Gδ COVERS OF COMPACT SPACES

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Abstract. We solve a long standing question due to Arhangel’skii by constructing a compact space which has a Gδ cover with no continuum-sized (Gδ)-dense subcollection. We also prove that in a countably compact weakly Lindelöf normal space of countable tightness, every Gδ cover has a c-sized subcollection with a Gδ-dense union and that in a Lindelöf space with a base of multiplicity continuum, every Gδ cover has a continuum sized subcover. We finally apply our results to obtain a bound on the cardinality of homogeneous spaces which refines De La Vega’s celebrated theorem on the cardinality of homogeneous compacta of countable tightness.

1. Introduction

Alexandroff and Urysohn asked, in 1923, if the continuum is a bound on the cardinality of compact Hausdorff first-countable spaces. The celebrated solution by Arhangel’skii [1] established that the cardinality of any Hausdorff space is bounded by a function of the Lindelöf degree and character, namely we have the inequality |X| ≤ 2χ(X) L(X). This result was improved in many directions and some still outstanding open problems guide ongoing research. Much of the work in this area is concerned with establishing similar bounds on the cardinality of X from more general cardinal invariants obtained by weakening the Lindelöf degree and character in conjunction with perhaps strengthening the separation axioms. For example, the bound of Arhangel’skii-Sapirovskii that for T2 spaces |X| ≤ 2ψ(X)τ(X)L(X) and the Bell-Ginsburg-Woods inequality for normal spaces that |X| ≤ 2χ(X)wL(X) (see [6]) are in this spirit. Recall that a space is said to be weakly Lindelöf if for each open cover U there is a countable V ⊆ U such that ∪ V is dense and wL(X) (the weak Lindelöf number of X) is defined as the minimum cardinal κ such that every open cover has a subcollection of cardinality ≤ κ with dense

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union. See [14] for a more detailed survey on Arhangel’skii’s solution and subsequent research.

Two questions attributed to Arhangel’skii (see [15] and [11] for published references to these questions) go in a completely different direction, asking about cardinal invariants for the $G_\delta$ topology. Recall that given a space $X$ we denote by $X_{\delta}$ the topology with the same underlying set $X$, generated by the $G_\delta$ subsets of $X$.

**Question 1.** Let $X$ be compact $T_2$.

1. Is $L(X_{\delta}) \leq 2^{\aleph_0}$?
2. Is $wL(X_{\delta}) \leq 2^{\aleph_0}$?

A positive solution to either question would also answer the Alexandroff-Urysohn question. Indeed, for a first countable compactum $X$, the space $X_{\delta}$ would be discrete, and in this case both the Lindelöf and weak Lindelöf degree coincide with the cardinality of $X$.

A negative answer to the first question has been known for some time. For example, Mycielski proved that if $\kappa$ is less than the first inaccessible, then $e(\omega^\kappa) = \kappa$ [21]. Recall that $e(X)$ denotes the extent of $X$, that is the supremum of cardinalities of closed discrete subsets of $X$. Hence $L(\omega^\kappa) = \kappa$ if $\kappa$ is less than the first inaccessible. Therefore, if one considers $X = (2^\omega)^\kappa$, then it follows that $L(X_{\delta}) = \kappa$ as well. Gorelic has similar results for a larger class of cardinals $\kappa$, including that $e(\omega^{2^\kappa}) = \kappa$ if $\kappa$ is less than the first measurable cardinal [13]. As a result, we have that the Lindelöf degree of compacta under the $G_\delta$ topology can be arbitrarily large below the first measurable.

**Question 2.** Is the first measurable a bound on the Lindelöf degree of compacta under the $G_\delta$ topology?

However, Question [11] (2) has remained open until now.

There has been a fair amount of work in a positive direction on Arhangel’skii’s questions. For example, Juhász proved in [15] that $wL(X_{\delta}) \leq 2^{\aleph_0}$ for every compact ccc space $X$, using the Erdős-Rado theorem. The first-named author generalized this in [25] to prove that $wL(X_{\delta}) \leq 2^{\aleph_0}$ for all spaces $X$ such that player II has a winning strategy in $G^{\omega_1}(O,O_D)$, that is the two-player game in $\omega_1$ many innings where at inning $\alpha < \omega_1$, player one chooses a maximal family of non-empty pairwise disjoint open sets $U_\alpha$ and player two chooses $U_\alpha \in U_\alpha$ and player two wins if $\bigcup\{U_\alpha : \alpha < \omega_1\}$ is dense in $X$. In [11], Fleischmann and Williams proved that $L(X_{\delta}) \leq 2^{\aleph_0}$ for every compact linearly ordered space $X$ and Pytkeev proved in [23] that $L(X_{\delta}) \leq 2^{\aleph_0}$ for every compact countably tight space $X$. Carlson, Porter and Ridderbos generalized the latter result in [7] by proving
that $L(X_\delta) \leq 2^{F(X)t(X)L(X)}$, where $F(X)$ is the supremum of the cardinalities of the free sequences in $X$. This is actually an improvement only in the non-compact realm, since for compact spaces $F(X) = t(X)$.

In Section 2 we answer Arhangel’skii’s question by constructing a compact subspace of $(2^\omega)^c$ and a $G_\delta$ cover with no $c$ sized subcollection with dense union in the $G_\delta$ topology.

In Section 3 we provide a few more positive result about covering properties of the $G_\delta$ topology. In particular we prove that in a countably compact weakly Lindelöf normal space of countable tightness, every $G_\delta$ cover has a $c$ sized subcollection with a $G_\delta$-dense union and that in a Lindelöf space with a base of multiplicity continuum, every $G_\delta$ cover has a continuum sized subcover.

In Section 4 we apply one of the results from section 3 to extend De la Vega’s theorem on the cardinality of homogeneous compacta to the realm of countably compact spaces.

In our proofs we will often use elementary submodels of the structure $(H(\mu), \epsilon)$. Dow’s survey [9] is enough to read our paper, and we give a brief informal refresher here. Recall that $H(\mu)$ is the set of all sets whose transitive closure has cardinality smaller than $\mu$. When $\mu$ is regular uncountable, $H(\mu)$ is known to satisfy all axioms of set theory, except the power set axiom. We say, informally, that a formula is satisfied by a set $S$ if it is true when all existential quantifiers are restricted to $S$. A set $M \subset H(\mu)$ is said to be an elementary submodel of $H(\mu)$ (and we write $M \prec H(\mu)$) if a formula with parameters in $M$ is satisfied by $H(\mu)$ if and only if it is satisfied by $M$.

The downward Löwenheim-Skolem theorem guarantees that for every $S \subset H(\mu)$, there is an elementary submodel $M \prec H(\mu)$ such that $|M| \leq |S| \cdot \omega$ and $S \subset M$. This theorem is sufficient for many applications, but it is often useful (especially in cardinal bounds for topological spaces) to have the following closure property. We say that $M$ is $\kappa$-closed if for every $S \subset M$ such that $|S| \leq \kappa$ we have $S \in M$. For large enough regular $\mu$ and for every countable set $S \subset H(\mu)$ there is always a $\kappa$-closed elementary submodel $M \prec H(\mu)$ such that $|M| = 2^\kappa$ and $S \subset M$.

The following theorem is also used often: let $M \prec H(\mu)$ such that $\kappa + 1 \subset M$ and $S \in M$ be such that $|S| \leq \kappa$. Then $S \subset M$.

All spaces are assumed to be Hausdorff. Undefined notions can be found in [19] for set theory and [10] for topology. However, our notation regarding cardinal functions follows Juhász’s book [16].
2. COUNTEREXAMPLES TO ARHANGEL’SKIĬ’S QUESTIONS

We first give a direct alternate proof of the known result that the Lindelöf degree of \((2^\omega)^{c^+}\) under the \(G_\delta\) topology is \(c^+\).

**Theorem 1.** The Lindelöf degree of \((2^\omega)^{c^+}\) under the \(G_\delta\) topology is \(c^+\).

**Proof.** Note first that \((2^\omega)^{c^+}\) with the \(G_\delta\) topology is homeomorphic to \((D(c)^{c^+})_\delta\), where \(D(c)\) is a discrete set of size continuum. We need to exhibit a cover with no subcover of size \(c\).

For any countable partial function \(s : c^+ \to D(c)\), the set

\[
[s] = \{ f \in (D(c))^{c^+} : s \subseteq f \}
\]

is open in the \(G_\delta\) topology. Let

\[ U = \{ [s] : \text{dom}(s) \in [c^+]^{\aleph_0} \text{ and } s : \text{dom}(s) \to D(c) \text{ is not 1-1} \} \]

Note that since any function \(f : c^+ \to D(c)\) fails to be 1-1 on some countable subset, \(((D(c))^{c^+})_\delta\) is covered by \(U\). But if \(V \subseteq U\) has cardinality \(\leq c\), then there is an \(\alpha < c^+\) such that \(\text{dom}(s) \subseteq \alpha\) for all \([s] \in V\). Fix \(g \in (D(c))^{c^+}\) such that \(g \upharpoonright \alpha\) is 1-1, then \(g\) is not covered by \(V\). Thus the Lindelöf degree of \((2^\omega)^{c^+}\) under the \(G_\delta\) topology is \(c^+\).

\(\Box\)

The cover \(U\) could also have been chosen slightly differently. E.g., one could have also considered those \([s]\) with countable support where \(s\) is not finite-to-one, or those \([s]\) with finite support and \(s\) not 1-1 and the proof would also work. Indeed, the latter option gives us an open cover of \((D(c))^{c^+}\) with its usual product topology. Therefore, as a corollary to the proof we obtain Mycielski’s result that \(D(\kappa)^{\kappa^+}\) contains a closed discrete subset of size \(\kappa^+\) [21].

**Corollary 2.** (Mycielski) For any cardinal \(\kappa\), the product topology on \(D(\kappa)^{\kappa^+}\) has Lindelöf degree \(\kappa^+\)

We now construct a compact space \(X\) such that \(wL(X_\delta) > 2^\omega\) in ZFC, which solves Arhangel’skiĭ’s question.

**Theorem 3.** Suppose there is a compact space which has a partition into \(\kappa\) many \(G_\delta\) sets. Then there is a compact space \(X\) admitting a \(G_\delta\)-cover of \(X\) with no \(\kappa\)-sized dense subcollection (in particular, \(wL(X_\delta) > \kappa\)).

**Proof.** Let \(\tilde{K}\) be a compact space having a partition into \(\kappa\) many \(G_\delta\) sets, set \(K = \tilde{K} \times 2\) and let \(\{G_\alpha : \alpha < \kappa\}\) be a partition of \(K\) into \(\kappa\) many \(G_\delta\) sets. Moreover, let \(\{V_0, V_1\}\) be a partition of \(K\) into a pair...
of non-empty clopen sets. Without loss we can assume that \( \{ G_\alpha \cap V_1 : \alpha < \kappa \} \) is a pairwise disjoint family of \( G_\delta \) sets of cardinality \( \kappa \). We define \( X \subset (K)^{\kappa^+} \) as an inverse limit of compacta \( X_\alpha \subset (K)^\alpha \).

The \( X_\alpha \)'s are defined recursively preserving the following two conditions for every \( \alpha < \kappa^+ \):

1. \( X_\alpha \) is a closed subset of \( K_\alpha \).
2. \( (V_1)_{\alpha} \subset X_\alpha \).

The base case is \( X_\kappa = (K)^\kappa \). For \( \alpha > \kappa \) limit, let \( X_\alpha \) be the inverse limit of the previously defined \( X_\beta \)'s. Note that the inductive hypotheses are satisfied.

Suppose now \( \alpha = \beta + 1 \) and \( X_\beta \) has already been defined and use \( (V_1)^\beta \subset X_\beta \) to choose \( f_\beta \in X_\beta \) such that \( f_\beta(\gamma) \) and \( f_\beta(\delta) \) don’t belong to the same \( G_\tau \), whenever \( \gamma < \delta < \alpha \).

Now let \( X_{\beta+1} = \{ g : \beta + 1 \to K : g \upharpoonright \beta \in X_\beta \land (g(\beta) \in V_0 \Rightarrow g \upharpoonright \beta = f_\beta) \} \).

Since \( X_{\beta+1} = (X_\beta \times V_1) \cup \{ \{f_\beta\} \times V_0 \} \), the two inductive hypotheses are preserved.

Finally let \( X = X_\kappa^+ \).

Given a partial function \( s : dom(s) \to K \), where \( dom(s) \in [\kappa^+]^\omega \), let \( < s > = \prod\{W_\alpha : \alpha < \kappa^+ \} \), where \( W_\alpha = G_{s(\alpha)} \) if \( \alpha \in dom(s) \) and \( W_\alpha = K \) otherwise. The set \( \mathcal{U} = \{ < s > : s \text{ is not one-to-one } \} \) is a \( G_\delta \) cover of \( X \) such that no \( \kappa \)-sized subcollection has a dense union. Indeed, let \( \mathcal{V} \subset \mathcal{U} \) be a \( \kappa \)-sized subcollection And let \( \alpha < \kappa^+ \) be an ordinal such that \( dom(s) \subset \alpha \) for every \( < s > \in \mathcal{V} \). Thus \( f_\alpha \notin \bigcup \mathcal{V} \).

Now consider the basic open set \( W := \{ g \in X : g(\alpha) \in V_0 \} \). Then \( W \subset \{ g \in X : g \upharpoonright \alpha = f_\alpha \} \), hence \( W \cap (\bigcup \mathcal{V}) = \emptyset \), as we wanted.

\[ \square \]

**Corollary 4.** There is a compact space \( X \) such that \( wL(X_\delta) = c^+ \).

**Proof.** Simply set \( K = 2^\omega \) in the construction of Theorem 3 and note that every point of \( K \) is a \( G_\delta \) set.
bound on the size of partitions of compacta into $G_δ$ subsets. This suggests the following question:

**Question 4.** Is there in ZFC a cardinal $κ$ so that for any compact space $X$ and any partition $P$ of $X$ into $G_δ$ subsets, we have that $|P| < κ$?

Even if Question 4 had a positive answer, this would not exclude the possibility of the existence of compact spaces with arbitrarily large weak Lindelöf number in their $G_δ$ topologies. So we finish with the following more general question:

**Question 5.** Is there any bound on the weak Lindelöf number of the $G_δ$ topology on a compact space?

We are also intrigued about the possibility of restricting Arhangel’skii’s problem to compact spaces with some additional structure. Every compact group has the countable chain condition, so using Juhász’s result from [15] we get that $wL(X_δ) ≤ c(X_δ) ≤ 2^{ℵ_0}$ for every compact group $X$. Recall that space is homogeneous if for every pair of points $x, y ∈ X$ there is a homeomorphism $f : X → X$ such that $f(x) = y$. Every topological group is a homogeneous space.

**Question 6.** Is there a compact homogenous space $X$ such that $wL(X_δ) > 2^{ℵ_0}$?

Actually, we don’t even know whether the example from Corollary 4 can be made homogenous. If it could, it would provide an answer to van Douwen’s long standing question about the existence of a compact homogenous space of cellularity larger than the continuum (see [18]). As a matter of fact, the cellularity of our example is $c^+$. Indeed, the clopen sets $W_α = π_{(U)}^{-1}(U_0) = \{f ∈ X : f(α) ∈ U_0\}$ are pairwise disjoint. To see this, suppose that $β < α$ and recall that $f_α$ was chosen to be a 1-1 function in $(U_1)^α$. And by the construction, if $f ∈ X$ and $f(α) = 0$ then $f \upharpoonright α = f_α$. And so for any $β < α$ and any $f ∈ W_α$ we have that $f(β) = 1$. I.e., $f \not∈ W_β$ and so $W_α ∩ W_β = ∅$.

### 3. Bounds for the $G_κ$ modification

Given a space $X$ we denote by $X_κ$ the topology on $X$ generated by the $G_κ$-subsets of $X$ (that is, the intersections of $κ$-sized families of open subsets of $X$). It is natural to ask what properties are preserved when passing from $X$ to $X_κ$ and whether cardinal invariants of $X_κ$ can be bound in terms of cardinal invariants of $X$. There has been a fair amount of work in the past on this general question, especially for chain conditions and covering properties (see for example [20], [17], [15], [11], [12], [25]). The aim of this section is to present some preservation
results that are related to Arhangel’skii’s problems mentioned in the introduction.

Recall that \( wL_c(X) \) is defined as the minimum cardinal \( \kappa \) such that for every closed set \( F \subset X \) and for every family \( \mathcal{U} \) of open sets of \( X \) covering \( F \) there is a \( \kappa \)-sized subfamily \( \mathcal{V} \) of \( \mathcal{U} \) such that \( F \subset \bigcup \mathcal{V} \).

It’s well known and easy to prove that \( wL_c(X) = wL(X) \) for every normal space.

A set \( G \subset X \) is called a \( G^c_\kappa \) set if there is a family \( \{U_\alpha : \alpha < \kappa\} \) of open subsets of \( X \) such that \( G = \bigcap \{U_\alpha : \alpha < \kappa\} = \bigcap \{\overline{U_\alpha} : \alpha < \kappa\} \).

Given a space \( X \), we denote with \( X^c_\kappa \) the topology generated by the \( G^c_\kappa \) subsets of \( X \). Clearly, if \( X \) is regular then \( X^c_\kappa = X_\kappa \).

**Theorem 5.** Let \( X \) be an initially \( \kappa \)-compact space such that \( t(X)wL_c(X) \leq \kappa \). Then \( wL(X^c_\kappa) \leq 2^\kappa \).

**Proof.** Let \( \mathcal{F} \) be a cover of \( X \) by \( G^c_\kappa \) sets. Let \( M \) be a \( \kappa \)-closed elementary submodel of \( H(\theta) \) such that \( X, \mathcal{F} \in M \), \( \kappa + 1 \subset M \) and \( |M| \leq 2^\kappa \).

**Claim.** \( \mathcal{F} \cap M \) covers \( \overline{X \cap M} \).

**Proof of Claim.** Fix \( x \in \overline{X \cap M} \) and let \( F \) be an element of \( \mathcal{F} \) containing \( x \). Let \( C \) be a subset of \( X \cap M \) of cardinality \( \kappa \) such that \( x \in \overline{C} \). Let \( \{U_\alpha : \alpha < \kappa\} \) be a \( \kappa \)-sized sequence of open sets such that \( F = \bigcap \{U_\alpha : \alpha < \kappa\} = \bigcap \{\overline{U_\alpha} : \alpha < \kappa\} \). Let \( C_\alpha = U_\alpha \cap C \). Since \( U_\alpha \) is a neighbourhood of \( x \) we have \( x \in \overline{C_\alpha} \). Since \( M \) is \( \kappa \)-closed we have \( C_\alpha \in M \). Moreover, \( \bigcap_{\alpha < \kappa} \overline{C_\alpha} \in M \). Let \( B = \bigcap_{\alpha < \kappa} \overline{C_\alpha} \) and note that \( H(\theta) \models (\exists G \in \mathcal{F})(B \subset G) \). Since \( B \in M \), it follows by elementarity that \( M \models (\exists G \in \mathcal{F})(B \subset G) \). Hence there is \( G \in \mathcal{F} \cap M \) such that \( B \subset G \), and since \( x \in B \subset G \) we get what we wanted. \( \triangle \)

Let us now prove that \( \bigcup (\mathcal{F} \cap M) \) is \( G^c_\kappa \)-dense in \( X \).

If this were not the case, there would be a \( G^c_\kappa \)-subset \( G \subset X \) such that \( G \cap \bigcup (\mathcal{F} \cap M) = \emptyset \).

Let \( \{V_\alpha : \alpha < \kappa\} \) be a sequence of open subsets of \( X \) such that \( G = \bigcap \{V_\alpha : \alpha < \kappa\} = \bigcap \{\overline{V_\alpha} : \alpha < \kappa\} \). Using initial \( \kappa \) compactness and the above claim we can find, for every \( x \in \overline{X \cap M} \) an open neighbourhood \( U_x \in M \) of the point \( x \) and a finite subset \( F_x \subset \kappa \) such that \( U_x \cap \bigcap \{V_\alpha : \alpha \in F_x\} = \emptyset \). For every \( F \in [\kappa]^{<\omega} \) let \( U_F = \bigcup \{U_x : F_x = F\} \). Then \( \{U_F : F \in [\kappa]^{<\omega}\} \) is a \( \kappa \)-sized open cover of the initially \( \kappa \)-compact space \( X \cap M \). Hence we can find a finite subset \( \mathcal{H} \subset [\kappa]^{<\omega} \) such that \( \{U_F : F \in \mathcal{H}\} \) covers \( \overline{X \cap M} \). But then \( \mathcal{U} = \{U_x : F_x \in \mathcal{H}\} \) is an open cover of \( X \cap M \). Using the fact that \( wL_c(X) \leq \kappa \) we can find a subcollection \( \mathcal{V} \in [\mathcal{U}]^\kappa \) such that \( X \cap M \subset \overline{X \cap M} \subset \bigcup \mathcal{V} \). Note that
\[ \mathcal{V} \subset \mathcal{U} \subset M \text{ and } M \text{ is } \kappa \text{-closed, so } \mathcal{V} \in M \text{ and hence } M \models X \subset \bigcup \mathcal{V}. \]

By elementarity it follows that \( H(\theta) \models X \subset \bigcup \mathcal{V}, \) but that contradicts the fact that the open set \( \bigcap \{ \bigcap \{ V_\alpha : \alpha \in F \} : F \in \mathcal{H} \} \) misses every element of \( \mathcal{U}. \)

**Corollary 6.** Let \( X \) be an initially \( \kappa \)-compact regular space such that \( t(X)wL_c(X) \leq \kappa. \) Then \( wL(X_\kappa) \leq 2^\kappa. \)

**Corollary 7.** Let \( X \) be a normal initially \( \kappa \)-compact space such that \( wL(X)t(X) \leq \kappa. \) Then \( wL(X_\kappa) \leq 2^\kappa. \)

Note that all assumptions are essential in Theorem 5:

1. Let \( \kappa \) be a cardinal of uncountable cofinality. To find countably compact spaces of countable tightness \( X \) such that \( wL(X_\delta) \) can be arbitrarily large, let \( X = \{ \alpha < \kappa : cf(\alpha) = \aleph_0 \} \), with the topology induced by the order topology on \( \kappa. \)

2. To find regular spaces \( X \) such that \( wL_c(X)t(X) = \aleph_0 \) and yet \( wL(X_\delta) \) can be arbitrarily large, let \( X \) be Uspenskij’s example of a \( \sigma \)-closed discrete dense subset of a \( \sigma \)-product of \( \kappa \) many copies of the unit interval from [27]. Being dense in \( \mathbb{I}_\kappa, \) the space \( X \) has the countable chain condition and hence \( wL_c(X) = \aleph_0. \) The tightness of \( X \) is countable because a \( \sigma \)-product of intervals is even Fréchet-Urysohn and \( \sigma \)-closed discrete implies points \( G_\delta, \) hence \( X_\delta \) is discrete. Since a \( \sigma \)-product of intervals has density and cardinality \( \kappa \) we actually have \( wL(X_\delta) = \kappa. \)

3. An example of a compact Hausdorff (and hence countably compact normal) space such that \( wL_c(X) = \omega \) and \( wL(X_\delta) > \mathfrak{c} \) is provided by Corollary 4.

**Question 7.** Is there a countably compact normal weakly Lindelöf space of countable tightness such that \( L(X_\delta) > 2^{\aleph_0}. \)

Recall that the **multiplicity** of a base \( \mathcal{B} \) is the minimum cardinal \( \kappa \) such that for every point \( x \in X, \) the set \( \{ B \in \mathcal{B} : x \in B \} \) has cardinality at most \( \kappa. \)

**Theorem 8.** Let \( X \) be a space such that \( L(X) = \kappa \) and \( X \) has a base of multiplicity \( 2^\kappa. \) Then \( L(X_\kappa) \leq 2^\kappa. \)

**Proof.** Fix a base \( \mathcal{B} \) for \( X \) having multiplicity \( \kappa \) and let \( \mathcal{U} \) be a cover of \( X \) by \( G_\kappa \) sets.

Let \( \theta \) be a large enough regular cardinal and let \( M \prec H(\theta) \) be a \( \kappa \)-closed elementary submodel of cardinality \( 2^\kappa \) such that \( X, \mathcal{B}, \mathcal{U} \in M \) and \( 2^\kappa + 1 \subset M. \)

**Claim.** \( \mathcal{U} \cap M \) covers \( X \cap M. \)
Proof of Claim. Fix \( x \in X \cap M \) and let \( U \in \mathcal{U} \) be such that \( x \in U \). Let \( \{ U^x_{\alpha} : \alpha < \kappa \} \) be a sequence of open sets such that \( U = \bigcap \{ U^x_{\alpha} : \alpha < \kappa \} \). For every \( \alpha < \kappa \) let \( B^x_{\alpha} \in \mathcal{B} \) be such that \( x \in B^x_{\alpha} \subset U^x_{\alpha} \). Fix \( x(x) \in B^x_{\alpha} \cap M \) and note that \( \mathcal{B} = \{ B^x_{\alpha} : x(x) \in B^x_{\alpha} \} \) is an element of \( M \) and has size \( 2^\kappa \). Hence \( \mathcal{B} \subset M \). It follows that \( B^x_{\alpha} \in M \) for every \( \alpha < \kappa \) and thus \( \bigcap_{\alpha < \kappa} B^x_{\alpha} \subset M \). Since \( \kappa + 1 \subset M \) by \( \kappa \)-closedness, we actually have \( \mathcal{B} \subset M \). For every \( x \in X \cap M \), there is \( \alpha_x < \kappa \) such that \( p \notin U^x_{\alpha_x} \). Finally \( \mathcal{V} := \{ U^x_{\alpha_x} : x \in X \cap M \} \) is an open cover of the subspace \( X \cap M \), which has Lindelöf number \( \kappa \), and hence there is a \( \kappa \)-sized \( \mathcal{C} \subset \mathcal{V} \) such that \( X \cap M \subset X \cap M \subset \bigcup \mathcal{C} \). But, since \( \mathcal{C} \subset M \) by \( \kappa \)-closedness, this implies that \( M \models X \subset \bigcup \mathcal{C} \) and hence \( H(\theta) \models X \subset \bigcup \mathcal{C} \), which is a contradiction because \( p \notin \bigcup \mathcal{C} \).

\[ \square \]

Corollary 9. Let \( X \) be a Lindelöf space with a point-countable base. Then \( L(X_\delta) \leq 2^{\aleph_0} \).}

4. Applications to homogeneous spaces

De la Vega’s theorem \[^8\] states that the cardinality of every compact homogeneous space of countable tightness is at most the continuum.

We will use the results from Section 3 to extend De La Vega’s theorem to the realm of countably compact spaces. It’s not enough to replace compact with countably compact. Indeed, let \( \kappa \) be an arbitrary cardinal and consider \( 2^\kappa \) with the usual topology and let \( X \subset 2^\kappa \) be the subspace of all functions of countable support. It is easy to see that \( X \) is countably compact and \( X \) is well known to have countable tightness (it is even Fréchet-Urysohn). The space \( X \) is homogeneous because it is a topological group with respect to coordinatewise addition mod 2, yet \( |X| = \kappa^{\omega} \).

A space \( X \) is called power-homogeneous if there is a cardinal \( \kappa \) such that \( X^\kappa \) is homogeneous.
Lemma 10. (Ridderbos, [24]) Let $X$ be a Hausdorff power-homogeneous space. Then $|X| \leq (d(X))^{\pi(X)}$.

The following lemma can be proved by modifying slightly the proof of a result of Shapirovskii (see [16], 3.14).

Lemma 11. Let $X$ be an initially $\kappa$-compact space such that $F(X) \leq \kappa$. Then $\pi\chi(X) \leq \kappa$.

We say that a space $X$ has $G_\kappa$ density at most $\kappa$ at the point $x \in X$ if there is a $G_\kappa$ subset $G$ of $X$ containing $x$ such that $d(G) \leq \kappa$.

Lemmas [12] and [13] are essentially due to Arhangel’skii (see [3] and [15], 3.12 for the proofs of closely related statements).

Lemma 12. Let $X$ be an initially $\kappa$-compact regular space such that $F(X) \leq \kappa$. Then $t(X) \leq \kappa$.

Lemma 13. Let $X$ be an initially $\kappa$-compact regular space such that $F(X) \leq \kappa$. Then $X$ has $G_\kappa$-density at most $\kappa$ at some point.

Lemma 14. (Arhangel’skii, van Mill and Ridderbos, [4]) Let $X = \prod\{X_i : i \in I\}$. Suppose $X$ is homogeneous and for every $i \in I$, the $G_\kappa$-density of $X_i$ does not exceed $\kappa$ at some point. If for some $j \in I$, we have $\pi\chi(X_j) \leq \kappa$ then the $G_\kappa$-density of $X_j$ does not exceed $\kappa$ at all points of $X_j$.

The above lemma, along with Lemma [11] and Lemma [13] implies the following statement.

Corollary 15. Let $X$ be a power-homogeneous initially $\kappa$-compact regular space such that $F(X) \leq \kappa$. Then the $G_\kappa$ density of $X$ does not exceed $\kappa$ at all points.

The following lemma uses an idea of Arhangel’skii from [4].

Lemma 16. Let $X$ be an initially $\kappa$-compact regular power-homogeneous space such that $F(X) \cdot wL_\kappa(X) \leq \kappa$. Then $d(X) \leq 2^\kappa$.

Proof. By Corollary [15] we can choose, for every $x \in X$, a $G_\kappa$ set $G_x$ such that $x \in G_x$ and $d(G_x) \leq \kappa$. Note now that the $G_\kappa$-density of $G_x$ does not exceed its $G_\kappa$-weight, which in turn is at most $w(G_x)$. By regularity of $X$, we have $w(G_x) \leq 2^{d(G_x)}$. Hence the density of $G_x$ in the $G_\kappa$ topology does not exceed $2^\kappa$. Now $\{G_x : x \in X\}$ is a $G_\kappa$-cover of $X$ and hence by Theorem [4] we can find $C \in [X]^{2^\kappa}$ such that $\bigcup\{G_x : x \in C\}$ is dense in $X_\kappa$. For every $x \in C$, fix a set $D_x$, dense in $G_x$ (in the $G_\kappa$ topology). Then $D = \bigcup\{D_x : x \in C\}$ is a dense subset of $X_\kappa$ of cardinality $2^\kappa$. Since the topology of $X_\kappa$ is finer than the topology of $X$, we have that $D$ is also dense in $X$ and hence $d(X) \leq 2^\kappa$. \qed
Theorem 17. Let $X$ be an initially $\kappa$-compact power-homogeneous regular space such that $F(X) \cdot wLc(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.

Proof. By Lemma 11 we have $\pi\chi(X) \leq \kappa$ and using Lemma 16 we obtain that $d(X) \leq 2^\kappa$. Hence, using Lemma 10, we obtain $|X| \leq d(X)^{\pi\chi(X)} \leq 2^\kappa$. \hfill \Box

Corollary 18. (De la Vega) Let $X$ be a compact homogeneous space. Then $|X| \leq 2^{t(X)}$.

Corollary 19. (Arhangel’skii, van Mill and Ridderbos) Let $X$ be a compact power-homogeneous space. Then $|X| \leq 2^{t(X)}$.

Note that, while $F(X) = t(X)$ for every compact space $X$, the cardinal invariants $F(X)$ and $t(X)$ are not related for initially $\kappa$-compact spaces, as the following pair of examples shows. The first example exploits an idea from [22].

Example 20. For every cardinal $\kappa > \omega_1$, there is a countably compact space $X$ such that $F(X) \leq \omega_1$ and $t(X) = \kappa$.

Proof. Let $Y = \{x \in 2^\kappa : |x^{-1}(1)| \leq \aleph_0\}$ and $X = Y \cup \{1\}$, with the topology inherited from $2^\kappa$, where $1$ indicates the function which is identically equal to $1$. Then $X$ is countably compact and it is easy to see that $t(X) = \kappa$.

Since $Y$ is Fréchet-Urysohn, we have $t(Y) = \aleph_0$. It is not too hard to see that $L(Y) = \aleph_1$ (see, for example, the proof of Theorem 3.7 from [26]). If $F$ is a free sequence in $X$, then $F \cap Y$ is a free sequence in $Y$ having the same cardinality. Therefore $F(X) \leq F(Y) \leq t(Y) \cdot L(Y) \leq \aleph_1$. \hfill \Box

Example 21. For every cardinal $\kappa$ of uncountable cofinality, there is a countably compact space such that $t(X) = \omega$ and $F(X) = \kappa$.

Proof. Let $X = \{\alpha < \kappa : cf(\alpha) \leq \omega\}$. Then $X$ is first-countable and hence it has countable tightness. It is also easily seen to be countably compact.

Let $F = \{x_\alpha : \alpha < \kappa\}$ be an increasing enumeration of $Succ(\kappa)$. Then, for every $\beta < \kappa$ we have $\{x_\alpha : \alpha < \beta\} \subset [0, x_\beta)$ and $\{x_\alpha : \alpha \geq \beta\} \subset [x_\beta, \kappa)$. Hence $F$ is a free sequence of cardinality $\kappa$. \hfill \Box

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