A layer potential approach to functional and clinical brain imaging

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Abstract. In this work, we consider the inverse source recovery problem from sEEG, EEG and MEG point-wise data. We regard this as an inverse source recovery problem for \(L^2\) vector-fields normally oriented and supported on the grey/white matter interface, which together with the brain, skull and scalp form a non-homogeneous layered conductor. We assume that the quasi-static approximation of Maxwell’s equation holds for the electro-magnetic fields considered. The electric data is measured point-wise inside and outside the conductor while the magnetic data is measured only point-wise outside the conductor. These ill-posed problems are solved via Tikhonov regularization on triangulations of the interfaces and a piecewise linear model for the current on the triangles. Both in the continuous and discrete formulation the electric potential is expressed as a linear combination of double layer potentials while the magnetic flux density in the continuous case is a vector-surface integral whose discrete formulation features single layer potentials. A main feature of our approach is that these contributions can be computed exactly. Due to the consideration of the regularity conditions of the electric potential in the inverse source recovery problem, the Cauchy transmission problem for the electric potential is inadvertently solved as well. In the problem, we propagate only the electric potential while the normal derivatives at the interfaces of discontinuity of the electric conductivities are computed directly from the resulting solution. This reduces the computational complexity of the problem. There is a direct connection between the magnetic flux density and the electrical potential in conductors such as the one we explore, hence a coupling of the sEEG, EEG and MEG data for solving the respective inverse source recovery problems simultaneously is direct. We treat these problems in a unified approach that uses only single and/or double layer potentials. We provide numerical examples using realistic meshes of the head with synthetic data.

1. Introduction

In this work, we present forward models for the propagation of electric potential in the head and the magnetic flux density outside the head as a result of brain activity. Similar work has been done before notably in [1] and [2] where the so-called boundary elements symmetric method is used. The boundary elements symmetric method uses four boundary integral operators for
the forward models and Galerkin methods which is computationally complex. Instead, we use only two of these four operators and apply the method of fundamental solutions to solve the forward model, see for example [3], [4] for details on the theory and applications of the method of fundamental solutions. We also model brain activity, which may also be referred to as the source or primary current, as $L^2$ vector-fields supported on the grey/white matter interface and we use the formulations introduced by Geselowitz for the electric potential and magnetic flux density, see for example [5]. This result in expressions of the electric potential and magnetic flux density that have strong relations with single and double layer potentials. When considering simple approximations of the head geometry such as approximation by spheres, the forward models have explicit analytic expressions. The situation changes however when considering realistic head geometries as those that can be obtained via segmentation of MRI images, see for example [6]. Analytic expressions are difficult to come by, hence discrete versions of the problem have to be considered. After appropriate discretisations of the brain structures and the vector-fields involved, we arrive at explicit expressions for the electric potential and magnetic flux density that only use single and double layer potentials and do not result in singularities; hence these computations can be done at arbitrary points in space. This offers improved numerical accuracy and the versatility of applying the forward model of the electric potential to both EEG and intra-cranial recordings as in sEEG, and for magnetic flux density for MEG. To further study how EEG, MEG and sEEG work we invite the reader to look into [7], [8] and [9].

When considering the source localisation problem, we solve a Tikhonov regularised problem where we find the source that minimises a functional that involves the forward model. Hence the more accurate the forward model is, the more accurate the source localisation is. In addition to source localisation, the forward models that we employ can be co-opted to solve the so called cortical mapping problem, see [10], given either electrical or magnetic data associated with brain activity. This feature allows for source localisation and cortical mapping in the same instance which in principle should improve the source localisation. The formulations we adopted from [5] provide a natural coupling of electric potential and magnetic flux density and we use this coupling to solve the source localisation problem with simultaneous EEG, sEEG and/or MEG data.

2. Preliminaries

2.1. Maxwell’s equations

According to [11] Maxwell equations are given in differential form as follows

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{B} = \mathbf{J} + \epsilon \frac{\partial \mathbf{E}}{\partial t},$$
$$\nabla \cdot \mathbf{E} = \rho,$$
$$\nabla \cdot \mu \mathbf{B} = 0,$$

where $\mathbf{E} \in \mathbb{R}^3$ is the electric field, $\mathbf{B} \in \mathbb{R}^3$ is the magnetic field both of which are produced by the charge and current densities, $\rho \in \mathbb{R}$ and $\mathbf{J} \in \mathbb{R}^3$, respectively, $\mu \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$ are the magnetic permeability and electric permittivity, respectively.

If the partial derivatives with respect to time in the above expressions are negligible then we
can utilise a quasi-static approximation of the Maxwell equations given as follows

\[ \nabla \times \mathbf{E} = 0, \quad (1) \]
\[ \nabla \times \mathbf{B} = \mathbf{J}, \quad (2) \]
\[ \nabla \cdot \epsilon \mathbf{E} = \rho, \quad (3) \]
\[ \nabla \cdot \mu \mathbf{B} = 0. \quad (4) \]

For EEG and MEG the quasi-static approximation can be made for the electromagnetic dynamics. In the quasi-static regime \( \nabla \times \mathbf{E} = 0 \), thus it follows that \( \mathbf{E} = -\nabla \phi \) for some scalar potential \( \phi \). Thus we can derive a Poisson equation relating the Laplacian of a scalar electrical potential to the charge density by substitution in (3)

\[ \Delta \phi = -\frac{\rho}{\epsilon}, \]

this argument was given in [11, Sec. 1.7]. By the Biot-Savart law \( \mathbf{J} = \mathbf{J}^i - \sigma \nabla \phi \) where \( \mathbf{J}^i \) is the primary current and \( \sigma \) is the electric conductivity which needs not be constant. Using this, a more useful form of the Poisson equation which can be derived from the quasi-static approximation was introduced in [5] and we shall use it in this paper. This form reads

\[ \nabla \cdot (\sigma \nabla \phi) = \nabla \cdot \mathbf{J}^i. \quad (5) \]

From this we have that the potential is harmonic everywhere except at most on the support of \( \mathbf{J}^i \).

2.2. Double and single layer potentials

In this section we assume that \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain with connected boundary. Hence, \( \mathbb{R}^n \setminus \partial \Omega \) has two connected components, namely \( \Omega^- := \text{int} \partial \Omega \) and \( \Omega^+ := \text{ext} \partial \Omega \).

When \( E \subset \mathbb{R}^n \) is Lebesgue-measurable and \( 1 \leq p \leq \infty \), we let \( L^p(E) \) be the space of (equivalence classes of a.e. coinciding) \( \mathbb{R} \)-valued measurable functions on \( E \) whose absolute value to the \( p \)-th power is integrable, with norm \( \|g\|_{L^p(E)} = \left( \int_E |g(y)|^p dy \right)^{1/p} \) (ess. sup \( E |g| \) if \( p = \infty \)).

Given a functional space \( X \), we write \( (X)^m \) for the corresponding space of vector-fields with \( m \) components, each of which lies in \( X \). For example, \( (L^p(E))^m \) is the space of \( \mathbb{R}^m \)-valued vector fields \( \mathbf{M} \) on \( E \) whose components belong to \( L^p(E) \), with norm

\[ \|\mathbf{M}\|_{(L^p(E))^m} = \left( \int_E |\mathbf{M}|^p dy \right)^{1/p} \quad (\text{ess. sup}_E |\mathbf{M}| \text{ if } p = \infty). \quad (6) \]

When \( \Omega \subset \mathbb{R}^n \) is open, \( W^{1,p}(\Omega) \) indicates the Sobolev space of functions lying in \( L^p(\Omega) \) together with their first distributional derivatives. It is a Banach space with norm

\[ \|g\|_{W^{1,p}(\Omega)} = \left( \|g\|_{L^p(\Omega)}^p + \|\nabla g\|_{(L^p(\Omega))^n}^p \right)^{1/p}, \]

where \( \nabla g = (\partial_1 g, \cdots, \partial_n g)^t \) denotes the gradient of \( g \) and \( \partial_j g \) means the derivative with respect to the \( j \)-th variable.

Moreover, when \( 1 < p < \infty \), each \( g \in W^{1,p}(\Omega) \) has a trace on \( \partial \Omega \), say \( \psi \), that lies in the fractional Sobolev space \( W^{1-1/p,p}(\partial \Omega) \), where for \( 0 < s < 1 \) we let

\[ \|\psi\|_{W^{s,p}(\partial \Omega)} := \|\psi\|_{L^p(\partial \Omega)} + \left( \int_{\partial \Omega} \int_{\partial \Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{n-1+sp}} d\sigma(x) d\sigma(y) \right)^{1/p}. \quad (7) \]
The double and single layer potentials of a function $\psi$ defined on $\partial \Omega$, whose smoothness will be made precise later on, are defined by

$$K \psi(x) = \frac{1}{\omega_n} \int_{\partial \Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) \, d\mathcal{H}(y), \quad x \in \mathbb{R}^n \setminus \partial \Omega,$$

and

$$S \psi(x) = \frac{1}{(n-2)\omega_n} \int_{\partial \Omega} \psi(y) \frac{1}{|x-y|^{n-2}} \, d\mathcal{H}(y), \quad x \in \mathbb{R}^n \setminus \partial \Omega,$$

respectively, where $\omega_n$ indicates the area of the unit sphere in $\mathbb{R}^n$ and where $\mathcal{H}$ is the 2-dimensional Hausdorff measure on the surface $\partial \Omega$ and $\nu$ is the outward pointing unit normal to $\partial \Omega$. We call $\psi$ the density of the double or single layer potential, moreover $K \psi$ and $S \psi$ are harmonic in $\mathbb{R}^n \setminus \partial \Omega$. For $x \in \partial \Omega$, the double layer potential is defined as the singular integral

$$K \psi(x) = p.v. \frac{1}{\omega_n} \int_{\partial \Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) \, d\mathcal{H}(y)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int_{|x-y|>\varepsilon} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(y) \, d\mathcal{H}(y).$$

As soon as $\psi \in L^p(\partial \Omega)$ for some $p \in (1, \infty)$, it is well-known that $K \psi$ exists a.e. in $\partial \Omega$ and belongs to $L^p(\partial \Omega)$. Furthermore, for almost every $y \in \partial \Omega$, $K \psi(x)$ converges to $(-\frac{1}{2} I + K) \psi(y)$ (resp. to $(\frac{1}{2} I + K) \psi(y)$), as $x \to y$ non-tangentially in $\Omega^-$ (resp. $\Omega^+$), see [12, Thm 1.10] for a precise statement and references. It follows that the non-tangential limits on $\partial \Omega$ of the double layer potential from inside and outside, differ by the density $\psi$ of the potential. The normal derivatives of the double layer potential are continuous across $\partial \Omega$. For appropriate range of exponents, the double layer potential on $\Omega^+$ is a famous tool to solve the Dirichlet problem for the Laplace equation, which is to find $w : \Omega^- \to \mathbb{R}$ such that

$$\Delta w = 0 \text{ in } \Omega^-,$$

$$w = g \text{ on } \partial \Omega. \tag{11}$$

In fact, $(-\frac{1}{2} I + K) : W^{1/q,p}(\partial \Omega) \to W^{1/q,p}(\partial \Omega)$ is invertible for $p \in [\frac{3}{2}, 3]$ and $1/p + 1/q = 1$, see [13, Thm 8.1]. Hence, the solution to (11) when $g \in W^{1/q,p}(\partial \Omega)$ with $p \in [\frac{3}{2}, 3]$ is given by $w = K(-\frac{1}{2} I + K)^{-1} g$ and belongs to $W^{1,p}(\Omega^-)$. Here, the boundary condition in (11) is satisfied both as a Sobolev trace and as a non-tangential limit a.e. Likewise, the double layer potential on $\Omega^+$ is a tool to solve the exterior Dirichlet problem, which is to find $w : \Omega^+ \to \mathbb{R}$ such that

$$\Delta w = 0 \text{ in } \mathbb{R}^n \setminus \partial \Omega^- \cup \{\infty\},$$

$$w = g \text{ on } \partial \Omega. \tag{12}$$

Here, as $n \geq 3$, harmonicity at infinity means that $\lim_{|x| \to \infty} w(x) = 0$ [14, Thm 4.8]. In fact, $(\frac{1}{2} I + K) : W^{1/q,p}(\partial \Omega)/\{1\} \to W^{1/q,p}(\partial \Omega)/\{1\}$ is invertible for $p \in [\frac{3}{2}, 3]$, where the quotient by $\{1\}$ means “modulo constants”, see [13, Thm 8.1]. Hence, $w = K(\frac{1}{2} I + K)^{-1} g$ will solve the exterior Dirichlet problem up to a constant when $g \in W^{1/q,p}(\partial \Omega)$ and $p \in [\frac{3}{2}, 3]$, with $w|_{\Omega^+ \cap B(0,R)} \in W^{1,p}(\Omega^+ \cap B(0,R))$ for all $R > 0$ and $\nabla w \in [L^p(\Omega^+)]^n$.

Unlike the double layer potential, the single layer potential is continuous across $\partial \Omega$, though its normal derivative is not. Note that the gradient of the single layer potential is given by

$$\nabla S \psi(x) = -\frac{1}{\omega_n} \int_{\partial \Omega} \frac{(x-y)}{|x-y|^n} \psi(y) \, d\mathcal{H}(y), \quad x \in \mathbb{R}^n \setminus \partial \Omega. \tag{13}$$
For $1 < p < \infty$ and $\psi \in L^p(\partial\Omega)$, it holds for a.e. $y \in \partial\Omega$ that $\nu(y) \nabla S\psi(x)$ converges to $(\pm \frac{1}{2}Id - \mathcal{K}^*)\psi(y)$ (resp. $(-\frac{1}{2}Id + \mathcal{K}^*)\psi(y)$) as $x \to y$ non-tangentially in $\Omega^-$ (resp. $\Omega^+$), where $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$ as operators on $L^p(\partial\Omega)$:

$$\mathcal{K}^*\psi(x) = p.v. - \frac{1}{\omega_n} \int_{\partial\Omega} \psi(y) \frac{(x-y)}{|x-y|^n} \cdot \nu(x) \, d\mathcal{H}(y) \quad (14)$$

see [12, Thm 1.11] for a statement and further references.

For an appropriate range of exponents, the single layer potential on $\Omega^-$ allows one to solve the Neumann problem for the Laplace equation

$$\Delta w = 0 \text{ in } \Omega^-$$
$$\nabla w \cdot \nu = g \text{ on } \partial\Omega, \quad (15)$$

where the boundary condition in (15) is meant to satisfy the divergence formula. Likewise, the single layer potential on $\Omega^+$ can be used to solve the exterior Neumann problem:

$$\Delta w = 0 \text{ in } \mathbb{R}^n \setminus \bar{\Omega}^- \cup \{\infty\},$$
$$\nabla w \cdot \nu = g \text{ on } \partial\Omega. \quad (16)$$

More precisely, it follows from [13, Thm 8.1] that $(\pm \frac{1}{2}Id + \mathcal{K}^*)$ extends to an invertible map $\tilde{W}^{-1/p,p}(\partial\Omega) \to \tilde{W}^{-1/p,p}(\partial\Omega)$ for $p \in [\frac{3}{2}, 3]$, where we have set

$$\tilde{W}^{-\frac{1}{p},p}(\partial\Omega) := \{ f \in W^{-\frac{1}{p},p}(\partial\Omega) : \int_{\partial\Omega} f(y) \, d\mathcal{H}(y) = 0 \}.$$

Thus, by [13, Thm 9.2], the solution to (15) (resp. (16)) can be written as $w = S(\pm \frac{1}{2}Id + \mathcal{K}^*)^{-1}g$, up to an additive constant, under the (necessary) condition that $\int_{\partial\Omega} g(y) \, d\mathcal{H}(y) = 0$. Moreover, $w$ belongs to $W^{1,p}(\Omega^-)$ (resp. $w|_{\Omega^+ \cap B(0,R)} \in W^{1,p}(\Omega^+ \cap B(0,R))$) for all $R > 0$ and $\nabla w \in [L^p(\Omega^+)]^n$.

3. Electric potential in unbounded homogeneous domain

In much of the literature of inverse problems of source localisation the “elementary electromagnetic object” that is sought after is a current dipole. In this work, we assume that the electro-magnetic fields observed are generated by vector-field supported on a closed surface. We assume that this vector-field is $(L^2)^3$ on the closed surface. This can however be thought of as a tensor product distribution on the surface in question, hence we have in a sense a continuous distribution of dipoles on this surface.

We first look at how electric potential is generated in an unbounded homogeneous medium, that is, we wish to study how a vector-field on a closed surface normally oriented to the surface generates electrical potential in a unbounded homogeneous domain. This will be useful when we look at nested non-homogeneous bounded domains.

To this end, let $\Sigma \subset \mathbb{R}^3$ be a closed Lipschitz surface. Let $\Omega^- := \text{int } \Sigma$ and $\Omega^+ := \text{ext } \Sigma$. We consider $(L^2(\Sigma))^3$ vector-field supported on $\Sigma$ and normally oriented to $\Sigma$. Let the electrical conductivity of $\mathbb{R}^3$ be equal to a constant $\sigma > 0$.

Let $\phi$ be the potential in $\mathbb{R}^3$ generated by a vector-field on $\Sigma$. For a vector-field $M_\Sigma \in (L^2(\Sigma))^3$, $\phi$ is given by

$$\phi(x) = \frac{1}{4\pi \sigma} \int_{\Sigma} M_\Sigma(y) \cdot \frac{(x-y)}{|x-y|^3} \, d\mathcal{H}(y), \quad x \in \mathbb{R}^3 \setminus \Sigma. \quad (17)$$
It then follows that if $M_\Sigma$ is parallel to $\nu$ at every point on $\Sigma$, we have the following double layer potential representation of the potential

$$\phi(x) = \frac{1}{4\pi\sigma} \int_\Sigma M_\Sigma(y) \frac{(x-y)}{|x-y|^3} \cdot \nu(y) \, dH(y) = \frac{1}{\sigma} K M_\Sigma(x), \quad x \in \mathbb{R}^3 \setminus \Sigma, \tag{18}$$

where $M_\Sigma \in L^2(\Sigma)$ is encodes both the orientation and magnitude of the vector-field on $\Sigma$. When $M_\Sigma$ is positive, it indicates the vector-field, $M_\Sigma$, is pointed outwardly from $\Sigma$ and inwardly for negative $M_\Sigma$. Note that $\sigma \phi$ solves (11) for $g = (-\frac{1}{2} I + K) M_\Sigma$ when $g$ is taken to be the non-tangential limit of the harmonic function in $\Omega^-$. In the sequel, unless stated otherwise, we shall use the double layer potential formulation (18) for the potential associated with the surface vector-field supported on $\Sigma$. For $x \in \Sigma$, we have that by approaching $x$ non-tangentially

$$\sigma \phi(x) = \pm \frac{M_\Sigma(x)}{2} + K M_\Sigma(x), \tag{19}$$

where the $-$ and $+$ are from approaching the boundary from $\Omega^-$ and $\Omega^+$, respectively.

It is well known that $(-\frac{1}{2} I + K) : L^2(\Sigma) \rightarrow L^2(\Sigma)$ and $(\frac{1}{2} I + K) : L^2(\Sigma) \setminus \{1\} \rightarrow L^2(\Sigma) \setminus \{1\}$ are invertible operators, see [12]. Thus, given $\phi$ on $\Sigma$, we can in principle get $M_\Sigma$ directly.

4. Forward models in non-homogeneous domains

We now look at domains that do not have uniform electrical conductivities. To that end consider the following bounded domain. Let $\Omega \subset \mathbb{R}^3$ be a nested non-homogeneous bounded Lipschitz domain. To this end let $\Omega_0, \Omega_1, \ldots, \Omega_n$ be nested Lipschitz domains with $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_n$, where we let the boundaries $\partial \Omega_i \cap \partial \Omega_{i+1} = \Sigma_{i+1}$. The outer boundary of $\Omega_n$ shall be called $\Sigma_n$. Inside $\Omega_0$, there is a closed Lipschitz surface that supports an $(L^2)^3$ vector-field that is normally oriented to this surface, we shall call this surface $\Sigma_0$. Since the electric conductivities of the different domains are different, we shall call them $\sigma_0, \sigma_1, \ldots, \sigma_n$ for $\Omega_0, \Omega_1, \ldots, \Omega_n$, respectively. The conductivity outside $\Sigma_{n+1}$ will be set to zero. For any of the interfaces $\Sigma_i$, $i = 1, 2, \ldots, n+1$, let $\sigma^{-}_i$ and $\sigma^{+}_i$ be the conductivities inside and outside, respectively.

4.1. Forward model for electric potential

For the potential at the interfaces, we shall denote by $\phi^-$ and $\phi^+$ the potential at any interface $\Sigma_i$, $i = 1, 2, \ldots, n+1$ obtain by approaching from inside and outside, respectively.

We shall assume the usual regularity of the potential and normal derivatives of the potential at the interfaces, see [5], that is, on each interface $\Sigma_i$, $i = 1, 2, \ldots, n$

$$\phi^- = \phi^+ \quad \text{and} \quad \sigma^- \partial_\nu \phi^- = \sigma^+ \partial_\nu \phi^+, \tag{20}$$

where the $-$ and $+$ indicate approaching $\Sigma_i$ from inside and outside, respectively. With these assumptions, it is well-known that the potential at any point $x$ in the domain is given by the following formulation, see for example [5],

$$\sigma(x) \phi(x) = \sigma_0 \phi_0(x) - \sum_{i=1}^{n+1} \frac{\sigma^-_i - \sigma^+_i}{4\pi} \int_{\Sigma_i} \phi(y) \nu(y) \cdot \nabla y \left( \frac{1}{|x-y|} \right) dH_i(y), \tag{20}$$

where $\phi_0$ is the electric potential generated by the vector-field supported on $\Sigma_0$ (the formulation is the same as in (18)) and $H_i$ is the 2-dimensional Hausdorff measure on the surface $\Sigma_i$. 

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Figure 1. An example cross-section of a non-homogeneous domain and the placement of intra-cranial electrodes for sEEG, in red; the blue arrows represent the normal orientation of the source term on its support.

4.1.1. Spherical surfaces  If we assume that the $\Sigma_i$ are concentric spheres of radii $r_i$ and we assume that on each interface the potential is $L^2(\Sigma_i)$, we can extend the ideas given in the previous section to solve (20) for a given $(L^2(\Sigma_0))^3$ normally oriented vector-field.

First note that we can write the restrictions of the potentials on the interfaces as linear combinations of spherical harmonics, that is, on each interface $\Sigma_i$, $i = 1, 2, \ldots, n + 1$,

$$\phi_i(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{i lm} Y_l^m(\theta, \varphi),$$

(21)

with $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ and $Y_l^m$ are the spherical harmonics and they form a basis for $L^2(\Sigma_i)$, see for example [14, Chap. 5]. Here we consider the normalised spherical harmonics given in their real form, that is,

$$Y_l^m(\theta, \varphi) = \begin{cases} 
(-1)^m \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - |m|)!}{(l + |m|)!} P_l^{|m|}(\cos \theta) \sin(|m| \varphi) & \text{for } m < 0, \\
\sqrt{\frac{2l + 1}{4\pi}} P_l^{|m|}(\cos \theta) & \text{for } m = 0, \\
(-1)^m \sqrt{\frac{2l + 1}{4\pi}} \frac{(l - m)!}{(l + m)!} P_l^m(\cos \theta) \cos(m \varphi) & \text{for } m > 0,
\end{cases}$$

(22)

where the $P_l^m$ are the associated Legendre functions. In (20) we have to compute the double layer potentials of the restrictions of the potentials at the interfaces hence by representing the potentials as linear combinations of spherical harmonics we have to compute the double layer potentials of spherical harmonics. Hence (20) becomes, at $r \neq r_i$,

$$\sigma(r) \phi(r, \theta, \varphi) = \sigma_0 \phi_0(r, \theta, \varphi) - \sum_{i=1}^{n+1} (\sigma^-_i - \sigma^+_i) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{i lm}^{l} K_i[Y_l^m](r, \theta, \varphi),$$

(23)
where $K_{i}[Y_{l}^{m}]$ is the double layer potential of the spherical harmonic $Y_{l}^{m}$ with respect to the interface $\Sigma_{i}$. It is well-known that for $Y_{l}^{m}$, the double layer potential with respect to $\Sigma_{i}$ is given by

$$K_{i}[Y_{l}^{m}](r, \theta, \varphi) = \begin{cases} \frac{l}{2l+1} Y_{l}^{m}(\theta, \varphi) \left( \frac{r}{r_{i}} \right)^{(l+1)} & \text{for } r > r_{i} \\ \frac{-(l+1)}{2l+1} Y_{l}^{m}(\theta, \varphi) \left( \frac{r}{r_{i}} \right)^{l} & \text{for } r < r_{i} \end{cases}$$

(24)

see for example [15]. Hence, when we approach $\Sigma_{i}$ by the interface $\Sigma_{k}$ where $K_{k}$ and from the interior of the normal derivatives of the double layer potential on the interfaces that the condition

$$(-\frac{1}{2}I + \mathcal{K})[Y_{l}^{m}](r_{i}, \theta, \varphi) = K_{k}^{+}[Y_{l}^{m}](r_{i}, \theta, \varphi) = \frac{l}{2l+1} Y_{l}^{m}(\theta, \varphi)$$

(25)

and from the interior

$$(-\frac{1}{2}I + \mathcal{K})[Y_{l}^{m}](r_{i}, \theta, \varphi) = K_{k}^{-}[Y_{l}^{m}](r_{i}, \theta, \varphi) = \frac{-(l+1)}{2l+1} Y_{l}^{m}(\theta, \varphi)$$

(26)

If one goes on to take normal derivatives in (20) hence in (23) for any $\phi$, therefore any collection of coefficients $\{a_{lm}^{+}\}_{i,l,m}$ that satisfies (20) hence (23), respectively, it follows from the continuity of the normal derivatives of the double layer potential on the interfaces that the condition

$$\sigma^{-} \partial_{n} \phi^{-} = \sigma^{+} \partial_{n} \phi^{+},$$

(27)

is trivially satisfied on all interfaces, $\Sigma_{i}, i = 1, 2, \ldots, n$.

Now due to the continuity condition of the potential across the interfaces we have that (20) hence (23) has to have the same limit on each interface $\Sigma_{k}, k = 1, 2, \ldots, n$ irrespective of the direction of approach, that is,

$$\sigma_{k}^{-} \phi^{-}(r_{k}, \theta, \varphi) = \sigma_{0} \phi_{0}(r_{k}, \theta, \varphi) - \sum_{i=1,i\neq k}^{n+1} \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm}^{1} K_{i}[Y_{l}^{m}](r_{k}, \theta, \varphi)$$

$$- (\sigma_{k}^{-} - \sigma_{k}^{+}) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^{k} K_{k}^{-}[Y_{l}^{m}](r_{k}, \theta, \varphi),$$

and

$$\sigma_{k}^{+} \phi^{+}(r_{k}, \theta, \varphi) = \sigma_{0} \phi_{0}(r_{k}, \theta, \varphi) - \sum_{i=1,i\neq k}^{n+1} \sum_{l=0}^{n} \sum_{m=-l}^{l} a_{lm}^{1} K_{i}[Y_{l}^{m}](r_{k}, \theta, \varphi)$$

$$- (\sigma_{k}^{-} - \sigma_{k}^{+}) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^{k} K_{k}^{+}[Y_{l}^{m}](r_{k}, \theta, \varphi),$$

Noting that

$$K_{k}^{-}[Y_{l}^{m}](r_{k}, \theta, \varphi) + \frac{1}{2}[Y_{l}^{m}](\theta, \varphi) = \frac{-1}{2(2l+1)} Y_{l}^{m}(\theta, \varphi) = \mathcal{K}[Y_{l}^{m}](r_{k}, \theta, \varphi)$$

and

$$K_{k}^{+}[Y_{l}^{m}](r_{k}, \theta, \varphi) - \frac{1}{2}[Y_{l}^{m}](\theta, \varphi) = \frac{-1}{2(2l+1)} Y_{l}^{m}(\theta, \varphi) = \mathcal{K}[Y_{l}^{m}](r_{k}, \theta, \varphi),$$
we obtain the following formulation for the potential at the interface $\Sigma_k$,

$$\frac{\sigma_k^+ + \sigma_k^-}{2} \phi^+(r_k, \theta, \varphi) = \sigma_0 \phi_0(r_k, \theta, \varphi)$$

$$- \sum_{i=1, i\neq k}^{n+1} (\sigma_i^- - \sigma_i^+) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^i K_i^{\mu}(r_k, \theta, \varphi)$$

$$- (\sigma_k^- - \sigma_k^+) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^k K_k^{\mu}(r_k, \theta, \varphi),$$

hence (28) can be written as:

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)\sigma_k^+ + l\sigma_k^-}{2l+1} a_{lm}^k Y_l^m(\theta, \varphi)$$

$$= \sigma_0 \phi_0(r_k, \theta, \varphi)$$

$$- \sum_{i=1, i\neq k}^{n+1} (\sigma_i^- - \sigma_i^+) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^i K_i^{\mu}(r_k, \theta, \varphi),$$

Using (24) for $K_i^{\mu}$, (29) can be written for $k = 1, 2, \ldots, n$ in a compact form as:

$$\sigma_0 \phi_0(r_k, \theta, \varphi) = \sum_{i=1, i\neq k}^{n+1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_{ll}^{ik} a_{lm}^i Y_l^m(\theta, \varphi),$$

with

$$q_{ll}^{ik} = \begin{cases} \frac{(\sigma_i^- - \sigma_i^+)}{2l+1} \frac{l}{r_i} & \text{for } r_k > r_i, \\
\frac{(\sigma_i^- - \sigma_i^+)}{2l+1} \frac{l}{r_i} & \text{for } r_k < r_i, \end{cases}$$

and

$$q_{ll}^{kk} = \frac{(l+1)\sigma_k^+ + l\sigma_k^-}{2l+1},$$

for $k = 1, 2, \ldots, n$. Note that due to (28) a collection of coefficients $\{a_{lm}^i\}_{i,l,m}$ that satisfies (29) also satisfies

$$\phi^-(r_k, \theta, \varphi) = \phi^+(r_k, \theta, \varphi).$$

Note further that a collection of coefficients $\{a_{lm}^i\}_{i,l,m}$, that satisfies (29) automatically satisfies (23) for any $(r, \theta, \varphi)$ hence this collection of coefficients $\{a_{lm}^i\}_{i,l,m}$, will satisfy the jump condition (27) for the normal derivative.

Finally, we have that on $\Sigma_{n+1}$, $\sigma_n \partial_r \phi = 0$ from (20) hence (23) we have that

$$\sigma_0 \partial_r \phi_0(r_{n+1}, \theta, \varphi) = \sum_{i=1}^{n+1} (\sigma_i^- - \sigma_i^+) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}^i \partial_r K_i^{\mu}(r_{n+1}, \theta, \varphi),$$

(30)
note here that the partial derivative of the double layer potential with respect to \( r \) is equivalent to
taking the normal derivative of the double layer potential (the quantities \( \partial_r K_l [Y^m_l] \) are available from (24)).

Thus by solving the linear system that results from (29) and (30) we get the solution to the
forward problem for a given vector-field on \( \Sigma_0 \).

**Remark 1.** The normal derivative of the potential, \( \phi_0 \), given in (30), where we assume spherical
domains, is the partial derivative with respect to the \( r \) once we have expressed it as \( \phi_0 \) linear
combination of spherical harmonics.

### 4.1.2. Non spherical surfaces

In this section we shall study how to numerically implement (20) for non spherical domains. Because we have made the assumption that \( \Omega \) is a Lipschitz domain,
we have that the operator \( K : L^2(\Sigma) \to L^2(\Sigma) \) fails generally to be compact, see for example [12]. It then follows that \( (\pm \frac{1}{2} I + K) : L^2(\Sigma) \to L^2(\Sigma) \) does not have the canonical representation
of compact operators on Hilbert spaces, namely

\[
\left( \pm \frac{1}{2} I + K \right) f = \sum_{j \geq 1} \alpha_j (f, u_j) u_j,
\]

(31)

where \( \{\alpha_j\} \) and \( \{u_j\} \) are the eigenvalues and an orthonormal basis of eigenfunctions of \( (\pm \frac{1}{2} I + K) \),
respectively. In the case in which \( \Sigma \) is a sphere we have that the \( u_j \) are spherical harmonics
as can be observed from (25) and (26). Further, for \( C^1 \) domains \( K \) is a compact operator, see
for example [12], hence a canonical representation as was given above is achievable. Of course
if one can compute the eigenvalues and eigenfunction as we did for the spherical case, one can
have exact formulations of the forward cortical mapping problem as was done for the sphere.
It is also possible to numerically estimate that eigenvalues and eigenfunctions in these smooth
cases, see for example [16].

We propose the following numerical implementation of (20). Given a surface \( \Sigma \), we begin by
triangulating the surface. On this triangulated surface we will represent each function \( f \in L^2(\Sigma) \),
by considering its values on the vertices of the triangulation. We assume that on each triangle
the function can be represented by linear shape functions, that is, on each triangle we have three
basis functions such that each of these three functions will have value one on one of the three
vertices and zero on the other two vertices. So given a function \( f \in L^2(\Sigma) \), on each triangle \( T_k \)
of the triangulation we write,

\[
f(y) = \sum_{j=1}^{3} f_{kj} \psi_j(y),
\]

(32)

where \( y \in T_k \), \( f_{kj} \) is the value of \( f \) on that \( j \)-th vertex of the triangle and \( \psi_j \) is the linear shape
function on \( T_k \) that has value one on the \( j \)-th vertex of the triangle. Now given a point \( x \in \mathbb{R}^3 \)
we wish to compute the double layer potential \( Kf(x) \) for \( f \in L^2(\Sigma) \). We use the numerical
formulation proposed in [17], which enable us to numerically compute the double layer potential
even when \( x \in \Sigma \). With this formulation we write

\[
Kf(x) = \mathbf{H}(x)f,
\]

(33)

where \( \mathbf{H}(x) \) is a row vector that depends on \( x \) where each entry is the sum of the “double layer
potential” over triangles that share a common vertex in the triangulation of the appropriate
linear basis functions and \( f \) is a column vector of the values of \( f \) on the vertices of the
triangulation of \( \Sigma \). Thus depending on where \( x \) is located, \( \mathbf{H}(x)f \) is either \( Kf(x) \) or
\((-\frac{1}{2}I + K)f(x)\), here we have a “-” for \(x \in \Sigma\) as we assume that the approach is from the interior. Using this notation, (20) can then be written as

\[
\sigma(x)\phi(x) = \sigma_0 H_0(x) \Phi_0 - \sum_{i=1}^{n+1} (\sigma_i^- - \sigma_i^+) H_i(x) \Phi_i, \tag{34}
\]

where \(H_i(x)\) is as described above and also depends on the surface \(\Sigma_i, i = 0, 1, 2, \ldots, n + 1\) and \(\Phi_i\) are the values of \(\phi\) at the vertices of the triangulation of the surface \(\Sigma_i, i = 0, 1, 2, \ldots, n + 1\). For \(x \in \Sigma_k, k = 1, 2, \ldots, n + 1\), because of the assumption that we approach the boundary from inside we have

\[
\sigma_k^-(x)\phi(x) + (\sigma_k^- - \sigma_k^+) H_k(x) \Phi_k = \sigma_0 H_0(x) \Phi_0 - \sum_{i=1,i \neq k}^{n+1} (\sigma_i^- - \sigma_i^+) H_i(x) \Phi_i, \tag{35}
\]

which corresponds equation to (29).

We can use the same idea as above to compute the gradient of the double layer potential at any point \(x\) for any function \(f \in L^2(\Sigma)\). To be able to solve the forward cortical mapping problem we only need the normal derivatives of the double layer potential at the outer most surface. Here we again use the method suggested in [17] because of its explicit limiting formulations. We will have the following formulation

\[
\partial_b K f(x) = \mathbf{N}(x) \mathbf{f}, \tag{36}
\]

where \(\mathbf{N}(x)\) is a row vector depending on \(x\) where each entry is the sum of the normal gradients of the “double layer potential” over triangles that share a common vertex in the triangulation of the appropriate linear basis functions and \(\mathbf{f}\) is as above. It then follows that the normal derivative of (20) for \(x \in \Sigma_{n+1}\) becomes

\[
\sigma_0 \mathbf{N}_0(x) \Phi_0 = \sum_{i=1}^{n+1} (\sigma_i^- - \sigma_i^+) \mathbf{N}_i(x) \Phi_i, \tag{37}
\]

Now given the surfaces \(\Sigma_i, i = 1, 2, \ldots, n + 1\), \(\Phi_0\) and a set of points \(X\) on these surfaces we can build the linear system

\[
\left\{ \left(\sigma_k^- I + (\sigma_k^- - \sigma_k^+) H_k(X_k)\right) \Phi_k + \sum_{i=1,i \neq k}^{n+1} (\sigma_i^- - \sigma_i^+) H_i(X_k) \Phi_i = \sigma_0 H_0(X_k) \Phi_0 \right\}_{k=1,2,\ldots,n+1} \tag{38}
\]

where \(X_k\) are those points on the surface \(\Sigma_k, k = 1, 2, \ldots, n + 1\). Solving the above linear system for \(\Phi_k, k = 1, 2, \ldots, n + 1\), solves the forward cortical mapping problem.

4.2. Forward model for magnetic flux density

In this section we look at the MEG problem where we assume that the the measured magnetic flux density is generated by a surface vector-field. To that end let \(\Omega \subset \mathbb{R}^3\) be a homogeneous domain as in Section 3 with magnetic permeability, \(\mu\). We denote the magnetic flux density by \(\mathbf{B}\). Since \(\mathbf{B}\) is divergence free it is the curl of the vector magnetic potential which we denote by \(\mathbf{A}\), that is,

\[
\mathbf{B} = -\nabla \times \mathbf{A}.
\]
Given a vector-field \( \mathbf{M}_\Sigma \in (L^2(\Sigma))^3 \) if follows from discussion in Section 2.2 that its vector magnetic potential is given by

\[
\mathbf{A}(x) = \frac{\mu}{4\pi} \int_\Sigma \mathbf{M}_\Sigma(y) \left( \frac{1}{|x - y|} \right) d\mathcal{H}(y) = S\mathbf{M}_\Sigma, \tag{39}
\]

for \( x \in \mathbb{R}^3 \) with the integral being taken in the principal value sense for \( x \in \partial \Omega \), see for example [11, Eq. (5.32)]. We note that the \( \mathbf{A} \) has components that are equal to the single layer potentials of the corresponding components of \( \mathbf{M}_\Sigma \). From [12, Lem. 1.8] we have that \( \mathbf{A} \in (W^{1,2}(\partial \Omega))^3 \) hence \( \mathbf{A} \in (W^{\frac{1}{2},2}(\Omega))^3 \) and \( \mathbf{A} \in (W^{2,2}(\mathbb{R}^3 \setminus \Omega))^3 \).

Using the relationship above between the vector magnetic potential and the magnetic flux density we have from the regularity of \( \mathbf{A} \) that \( \mathbf{B} \in (W^{\frac{1}{2},2}(\Omega))^3 \) and \( \mathbf{B} \in (W^{\frac{1}{2},2}(\mathbb{R}^3 \setminus \Omega))^3 \). We have that the magnetic flux density associated with a vector-field \( \mathbf{M}_\Sigma \in (L^2(\Sigma))^3 \) is given by

\[
\mathbf{B}(x) = \frac{\mu}{4\pi} \int_\Sigma \mathbf{M}_\Sigma(y) \times \nabla_y \left( \frac{1}{|x - y|} \right) d\mathcal{H}(y) = \nabla \times S\mathbf{M}_\Sigma, \tag{40}
\]

for \( x \in \mathbb{R}^3 \setminus \Sigma \). If we assume that \( \mathbf{M}_\Sigma \) is normally oriented to \( \Sigma \) we have that

\[
\mathbf{B}(x) = \frac{\mu}{4\pi} \int_\Sigma \nu(y) \times \nabla_y \left( \frac{1}{|x - y|} \right) \mathbf{M}_\Sigma(y) d\mathcal{H}(y) = \mathbf{S}\mathbf{M}_\Sigma, \tag{41}
\]

where \( M_\Sigma \in L^2(\Sigma) \) is encodes both the orientation and magnitude of \( \mathbf{M}_\Sigma \) on \( \Sigma \). Due the assumption that \( \mathbf{M}_\Sigma \) is normally oriented to \( \Sigma \) follows from [18, Lem. 4.3] that

\[
\mathbf{B}(x) = \nabla \times \mathbf{S}(\mathbf{M}_\Sigma \nu)(x) = \mathbf{S}(\nu \times \nabla \mathbf{M}_\Sigma), \tag{42}
\]

where \( \nabla \mathbf{M}_\Sigma \) is the gradient of \( \mathbf{M}_\Sigma \) on \( \Sigma \). It follows from [12, Thm. 3.3] that all \( M_\Sigma \in L^2(\Sigma) \) such that the tangential component of \( \nabla \mathbf{M}_\Sigma \) vanishes result in a magnetic flux density that is zero.

If we assume that \( \Omega \) has the same structure and electrical conductivities as was given in Section 4 we have that for any \( x \in \mathbb{R}^3 \) and for a vector-field \( \mathbf{M}_{\Sigma_0} \in (L^2(\Sigma_0))^3 \) that is normally oriented to \( \Sigma_0 \) the magnetic flux density associated with this \( \mathbf{M}_{\Sigma_0} \) is given by

\[
\mathbf{B}(x) = \frac{\mu}{4\pi} \int_{\Sigma_0} \nu(y) \times \nabla_y \left( \frac{1}{|x - y|} \right) M_{\Sigma_0}(y) d\mathcal{H}_0(y) - \frac{\mu}{4\pi} \sum_{i=1}^{n+1} \sigma_i^+ - \sigma_i^- \int_{\Sigma_i} \nu(y) \times \nabla_y \left( \frac{1}{|x - y|} \right) \phi(y) d\mathcal{H}_i(y), \tag{43}
\]

see for example [5, Eq. (17)], where the \( \phi \)'s on the surfaces are surface potentials obtained from solving the forward cortical mapping problem for EEG, see Section 4.1.

Note that MEG measures \( \mathbf{B} \cdot \mathbf{v} \), for some known vector \( \mathbf{v} \). Typically, \( \mathbf{v} \) is taken to be a radial vector, hence if all of the \( \Sigma_i \) above are spherical, the resulting radial component is zero, see [5, Eq. (20)]. Hence the spherical layered model is uninteresting for this case. We therefore look at layered head models in which the surfaces \( \Sigma_i \) are all non-spherical. Formulations for the single layer potentials on spherical surfaces such as those for the double layer potential found in Section 4.1 can be found in [15].

In order to do that, we need to be able to express \( \mathbf{S}\mathbf{M}_{\Sigma_0} \) as a linear combination of the \( \mathbf{S}\mathbf{M}_{\Sigma_0} \)'s where the \( \mathbf{M}_{\Sigma_0} \in L^2(\Sigma_0) \) are elements of the basis of \( L^2(\Sigma_0) \). Note that it only suffices that we look at how to numerically compute the expression given in (41) because to compute (43) we
repeatedly apply the same idea. To that end we begin with a triangulation of the surfaces and discretisation of \( M_\Sigma \in L^2(\Sigma) \) as was done in Section 4.1.2. Note that (41) is approximately
\[
B(x) = \sum_{T_k} \frac{\mu}{4\pi} \int_{T_k} \nu_{T_k}(y) \times \nabla y \left( \frac{1}{|x-y|} \right) M_\Sigma(y) \, d\mathcal{H}(y),
\]
and since \( \nu_{T_k} \) is taken to be constant on each triangle we have that
\[
B(x) = \mu \sum_{T_k} \nu_{T_k} \times \frac{1}{4\pi} \int_{T_k} \nabla y \left( \frac{1}{|x-y|} \right) M_\Sigma(y) \, d\mathcal{H}(y)
= S(x) M_\Sigma,
\]
where \( S(x) \) is a row vector that depends on \( x \) for which each entry is the sum of the “cross of the normals and of a vector of single layer potential” over the triangles that share a common vertex in the triangulation of the appropriate linear basis functions and \( M_\Sigma \) is a column vector of the values of \( M_\Sigma \) on the vertices of the triangulation of \( \Sigma \), where we would use the formula given in [17] to compute the integral above on each triangle. Hence (43) is written in discrete form as
\[
B(x) = S_0(x) \Phi_0 - \sum_{i=1}^{n+1} (\sigma_i^0 - \sigma_i^+) S_i(x) \Phi_i,
\]
where the \( \Phi_i, i = 1, 2, \ldots, n+1 \), are obtained from the forward model for electrical potential.

5. Inverse problems
5.1 EEG, sEEG and cortical mapping
In the inverse source localisation problems that use electric potential, there are two regimes, we can use electrical data measured on the scalp which is what is done for EEG or we can use intra-cranial electric potential recording as is done for sEEG. In both these instances, we wish to solve the problem that given the electric potential \( \phi \) intra-cranial electric potential recording as is done for sEEG. In both these instances, we wish to solve the following problem:
\[
\Phi^* = \min_{\Phi \in \mathbb{R}^n} \left( ||F(X)\Phi - \sigma(X)\phi(X)||_2^2 + \lambda ||\Phi||_2^2 \right),
\]
where \( F \) and \( \Phi \) are the appropriate concatenations of the \( H_i \)'s and \( \Phi_i \)'s found in (34), respectively. \( \phi(X) \) is a vector of the potentials at the points \( X = \{x_j\} \) and \( m \) is the number of vertices in the triangulation of the structures of the head. One observation that can be made here
is that when we solve the inverse source localisation problems of EEG and/or sEEG we in fact have to compute the electric potential on the surfaces $\Sigma_i, i = 1, 2, \ldots, n+1$, this is exactly what cortical mapping aims to achieve. It therefore is natural to add the equations that govern the continuity conditions of electric potential and the jump relations of the normal derivatives of the electric potential to the problem above, that is making an appropriate concatenation of $F(X)\Phi$ and the system (38). For the sake of simplicity we are going to call again this concatenated system $F(X)\Phi$ with $X$ now including the points that were considered for the system (38). Appropriate changes will also need to be made to $\sigma(X)\phi(X)$ namely concatenating it with a zero vector of appropriate size in order to be able to solve the minimisation problem (47).

It is well known that (47) has unique solution for any given $\lambda > 0$, see for example [19], and that the solution depends on $\lambda$. The regularisation term $\lambda \|\Phi\|^2$ can be replaced by $\lambda \|R\Phi\|^2$, where $R$ is a regularisation matrix that is chosen such that $\Phi$ has some desired properties that are based on some assumptions made about the regularity of the potentials on the surfaces and the source. The choice of the appropriate $\lambda$ is an issue that also has to be considered. In the examples we provide, the choice of $\lambda$ is based on the $L$-curve, see for example [20].

5.2. MEG and cortical mapping
In the simplest terms the inverse source localisation problem of MEG can be stated as follows, given $B$ on $\mathbb{R}^3\setminus\mathcal{O}$ and the potential $\phi$ in $\mathcal{O}$ find $M_{\Sigma_0} \in L^2(\Sigma_0)$ such that (43) holds. It follows from (42) and [12, Thm. 3.3(ii)] that those $M_{\Sigma_0} \in L^2(\Sigma_0)$ that are constant are silent hence the recovery of $M_{\Sigma_0}$ in MEG is unique up to additive constants. In practice $B$ is known only point-wise and further only a component of it is considered. All these considerations greatly increases the non-uniqueness of the problem and hence we have to solve a regularised problem, namely the following problem

**Problem 2.** Given point-wise measurements of components of the magnetic flux density, $B(X) \cdot v(X)$, find $\Phi^*$ such that

$$\Phi^* = \min_{\Phi \in \mathbb{R}^m} \left( \|S(X)\Phi \cdot v(X) - B(X) \cdot v(X)\|^2_2 + \lambda \|\Phi\|^2_2 \right),$$

where $S$ and $\Phi$ are the appropriate concatenations of the $S_i$’s and $\Phi_i$’s found in (46), respectively, $B(X) \cdot v(X)$ is a vector of the $v$ components of the magnetic flux density at the points $X = \{x_j\}$, and $m$ is the number of vertices in the triangulation of the structures of the head. Similar observation and consideration about cortical mapping can be made here and appropriate changes akin to those made to the source localisation problem for EEG and/or sEEG can also be made to the MEG source localisation problem.

Comments similar to those made on the EEG and/or sEEG source localisation problem about solvability and properties of the solutions can be made here as well.

5.3. EEG, sEEG, MEG and cortical mapping
There is an obvious connections among the EEG, sEEG and MEG source localisation problem and cortical mapping as can be seen in (20) and (43) hence these problems can be solved in a unified way by making appropriate changes to either (47) and (48) if simultaneous recordings of EEG, sEEG and MEG are available.

6. Numerical Results
We now present some numerical results of the inverse source localisation problem using combined EEG, sEEG and MEG data. These numerical results were obtained using code written in MATLAB. The data we use was generated using OpenMEEG, see [1] and [21], which is based on the symmetric method. OpenMEEG uses current dipoles as the elementary electromagnetic
object, we attempt to recover the locations of the current dipoles associated with these data. These dipoles are assumed to be located on the grey/white matter interface and normally oriented to this surface and pointed outwardly. In addition to the inverse source localisation problem we will solve the cortical mapping problem. The true dipole locations are indicated by the red dots in the bottom pictures of Figures 3, 4 and 5.

Figure 2. The top two figures show the sEEG electrodes positions as red dots in the brain from the left and the right. The bottom figure indicates the EEG electrodes positions as green dots and MEG sensor positions as black dots.

Figure 3 is an example in which the recovery on all surfaces except the cortex are good. The recovery of the potential on the cortex leaves something to be desired in terms of the accuracy of the extrema of the potential but not in the general location of the extrema. We propose that these inaccuracies are a result of the complicated nature of the surface which folds on itself. The respectable recovery of the source location is due to the proximity of the source to an sEEG electrode, there is an sEEG electrode that goes in through the patch on which the dipoles are located, as can be inferred from Figure 2, which results in the small support of the recovered vector-field.

Figure 4 is an example in which an sEEG electrode is close to the dipole patch as can be inferred from Figure 2. Again in this example the recovery of the potential on the scalp and skull are respectable and on the cortex the recovery in terms of the magnitude is not accurate. Even with the sEEG electrode not going through the dipole patch but merely being close, we see that the recovered vector-field has a tight support and the support is correctly placed on the supporting surface.
Figure 3. To locate a source $M_\Sigma = M_\Sigma \nu$ we look for the locations at which $M_\Sigma$ attains its extremal values. The positivity of the extremal value is interpreted as the vector field pointing outward for the surface and inward for negative values. In the bottom most figure, this region is represented in yellow and is very small compared to the whole surface.
Figure 4. To locate a source $M_{\Sigma} = M_{\Sigma} \nu$ we look for the locations at which $M_{\Sigma}$ attains its extremal values. The positivity of the extremal value is interpreted as the vector field pointing outward for the surface and inward for negative values. In the bottom most figure, this region is represented in yellow and is comparatively larger than the one in Figure 3.
Figure 5. To locate a source $M_{\Sigma} = M_{\Sigma} \nu$ we look for the locations at which $M_{\Sigma}$ attains its extremal values. The positivity of the extremal value is interpreted as the vector field pointing outward for the surface and inward for negative values. In the bottom most figure, this region is represented in yellow and is comparatively larger than the ones in the previous two examples.
Figure 5 is an example of a source that is far from any sEEG electrode. In our experiments it seems that the sEEG data does not significantly improve the source localisation problem when the sEEG electrodes are far. We can see that the recovery of the potential on the scalp and the skull are as good as in the previous examples. This is due to the fact that it is the EEG data that does the heavy lifting for the recovery of the scalp potential which further results in the good recovery of the potential of skull. Again the performance of the method on the recovery of the potential on the cortex is not good, in this example even the location of the extrema are not correct. One of the extrema, we suspect, is influenced by one of the sEEG electrodes. In this example we see that it is the sEEG data that improves the spatial resolution of the method, the support of the vector-field we recovered is far too large albeit being vaguely in the correct location.

7. Conclusions
In this paper we have presented a method whose performance is comparable to symmetric method as exemplified in the performance of the recovery using data that was generated using OpenMEEG. This method is less complex computationally and involves fewer operators than the symmetric method. Even in simple geometries such as spheres the method presented here is direct compared to methods that have previously been employed such as rational approximations which involve complicated manipulations of the domain, see [22]. In addition to this, the use of the method of fundamental solutions in the building of the linear systems to be solved opens up to the possibility of using frames/bases of the involved spaces when these are accessible for the surfaces involved which would improve the numerical accuracy of the method or even offer better performance in terms of time.

The experiments we performed also shows that for improved source location recovery it suffices to have sEEG electrodes close to the source. Based on clinical considerations, these electrodes could be placed around areas that EEG and MEG would have isolated in prior test as the example in Figure 5 would suggest.

Further work needs to be done to in order to develop metrics that can be used to compare the performance of existing methods that utilise dipolar source models and the method presented in this work that utilises vector-fields supported on known surfaces.

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