Nonperturbative operator quantization of strongly nonlinear fields

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Abstract

At present an algebra of strongly interacting fields is unknown. In this paper it is assumed that the operators of strongly nonlinear field can form a non-associative algebra. It is shown that such algebra can be described as an algebra of some pairs. The comparison of presented techniques with the Green’s functions method in the superconductivity theory is made. A possible application to the QCD and High-Tc superconductivity theory is discussed.

1 INTRODUCTION

The quantization rules in quantum field theory are applied for the noninteracting part of Lagrangian only and then the n-point Green’s functions are calculated using of the Feynman diagram technique. Such procedure is not valid for theories with strong nonlinear fields, for example, for the quantum chromodynamics (QCD) and gravity. This means that in QCD we have a hypothesized flux tube stretched between quark and antiquark and such nonlocal object can not be explained by the use of the perturbative diagram techniques. In the 50’s Heisenberg conceived the difficulties of using an expansion in small parameter for quantum field theories with strong interactions (see, for example, Ref’s [1, 2]). In these papers it was repeatedly underscored that a nonlinear theory with a strong coupling requires the introduction of another quantization procedure. Heisenberg’s basic idea proceeds from the fact that the n-point Green’s functions must be found from some infinite set of differential equations derived from the field equations for the field operators. In this case the n-point Green’s functions and the propagator are not connected with one another by a simple manner as it is in the case of diagram techniques. Later this idea has been abandoned because of big mathematical problems connected to an obtained infinite equations set. Nevertheless it can be shown [3] that the Green’s function

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method and Ginzburg–Landau equation in superconductivity theory is actually a realization of the Heisenberg’s quantization procedure.

For the standard Feynman diagram techniques it is very important that the corresponding classical field equations have wave solutions which become quanta after quantization. But for the strongly nonlinear field theory the situation can be drastically changed: such theory can have (on the classical level) such (possible singular) solutions which after quantization become some nonlocal quantum objects (for example, it can be the flux tube or monopole-like configuration [4]). In this case a nonlinearity leads to a non-locality, i.e. not every quantum object can be described as a cloud of quanta.

In this situation (for strongly nonlinear fields) we have a question: what is the algebra of quantized fields? We know the answer for the noninteracting fields only: it is a noncommutative algebra with canonical commutation relations. The assumption presented here is: in the case of strongly nonlinear fields we should change the algebra of quantized fields to derive functions, which can be connected with the n-point Green’s functions. In this paper we suppose that it can be done if the algebra of quantized fields be much more complicated then for noninteracting fields, for example, it can be a non-associative algebra [8].

Ordinarily the non-associative algebra was used in classical field theories [5], [6], in quantum gravity [7] to describe a discrete spacetime and in a non-associative geometry. In contrast to these approaches the idea, presented here, consists of following regulations

1. The non-associative algebra (of strongly nonlinear quantum fields) can give additional degrees of freedom that is relevant to the description of the n-point Green’s functions which can not be calculated on the language of Feynman diagrams.

2. The Heisenberg’s quantization procedure [1, 2] gives us information to define of the algebra of strongly nonlinear quantum fields.

2 ALGEBRA OF QUANTIZED FIELDS AS A NON-ASSOCIATIVE ALGEBRA

Let us assume that an algebra of quantized fields is a non-associative algebra [9]. It means that

\[ a(bc) \neq (ab)c \]  

(1)

here the operators \( a, b, c \) are the operators of quantized fields. Let us introduce an associator \( Ass(a, b, c) \)

\[ (ab)c - a(bc) = Ass(a, b, c). \]  

(2)
In the general case $\text{Ass}(a, b, c)$ can be an operator. For example, we can calculate following commutator

$$
(ab)(cd) - (cd)(ab) = (c\Delta_{ad})b + \Delta_{ac}(db) + a(c\Delta_{ba}) + a(\Delta_{bc}d)
$$

$$
+ \text{Ass}(c, a, d) - \text{Ass}(c, d, a) + a(\text{Ass}(c, b, d) - \text{Ass}(b, c, d))
$$

$$
+ \text{Ass}(cd, a, b) - \text{Ass}(ca, d, b) + \text{Ass}(a, b, cd) - \text{Ass}(a, c, db)
$$

(3)

here $\Delta_{ab} = ab - ba$ is a commutator that can be an operator. More detailed derivation of this equation see in Appendix A.

The non-associative algebra can be an alternative algebra, i.e.

$$
a(ab) = (a^2)b = a^2b \quad \text{and/or} \quad (ba)a = b(a^2) = ba^2.
$$

(4)

With $b = a = \varphi(x)$ and $c = d = \varphi^*(y)$, where $\varphi^*(y)$ is a Hermitian-conjugated operator. Eq. (3) has following form

$$
2\left(\varphi(x)\varphi^*(y) + \varphi^*(y)\varphi(x)\right)\Delta(x, y) - 2\varphi^*(y)\text{Ass}(\varphi(x), \varphi^*(y), \varphi(x))
$$

$$
+ \text{Ass}(\varphi^*(y), \varphi^2(x), \varphi^*(y)).
$$

(5)

Here we suppose that the commutator $\Delta(x, y) = \varphi(x)\varphi^*(y) - \varphi^*(y)\varphi(x)$, associators $\text{Ass}(\varphi(x), \varphi^*(y), \varphi(x)) = (\varphi(x)\varphi^*(y))\varphi(x) - \varphi(x)(\varphi^*(y)\varphi(x))$ and

$$
\text{Ass}(\varphi^*(y), \varphi^2(x), \varphi^*(y)) = (\varphi^*(y)\varphi^2(x))\varphi^*(y) - \varphi^*(y)(\varphi^2(x)\varphi^*(y))
$$

are $c-$numbers. The first terms of the right-hand side of Eq. (3) is similar to the terms, appearing by time ordering of the canonical quantization of noninteracting fields.

A similar construction can be written for fermions

$$
\left(\psi_\alpha(x)\psi_\alpha(x)\right)\left(\psi_\beta^+(y)\psi_\beta^+(y)\right) - \left(\psi_\beta^+(y)\psi_\beta^+(y)\right)\left(\psi_\alpha(x)\psi_\alpha(x)\right)
$$

$$
= -2\Delta^2 \left(\psi_\alpha(x), \psi_\beta^+(y)\right) + 2\psi_\alpha(x)\text{Ass} \left(\psi_\beta^+(y)\psi_\alpha(x), \psi_\beta^+(y), \psi_\alpha(x)\right)
$$

$$
- \text{Ass} \left(\psi_\alpha(x), \psi_\beta^+(y)\psi_\beta^+(y)\psi_\alpha(x)\right)
$$

(6)

here $\Delta \left(\psi_\alpha(x), \psi_\beta^+(y)\right) = \psi_\alpha(x)\psi_\beta^+(y) + \psi_\beta^+(y)\psi_\alpha(x)$ is the anticommutator.

The most interesting things in Eq.s (3), (5) are the last terms $\text{Ass}(\varphi^*(y), \varphi^2(x), \varphi^*(y))$ and $\text{Ass} \left(\psi_\alpha(x), \psi_\beta^+(y)\psi_\beta^+(y)\psi_\alpha(x)\right)$. On the left-hand side of these equations we have commutators. As well as in the superconductivity theory, the operator $\psi_\beta^+(y)\psi_\beta^+(y)$ can be interpreted as a creation of a pair in the point $y$ (it can be Cooper pair in the superconductivity theory or quark-antiquark pair in the QCD while the distance between fermions tends to zero) and consequently the operator $\psi_\alpha(x)\psi_\alpha(x)$ describes an annihilation of the pair in the point $x$. The first term of the right-hand side of Eq. (3), (5) is the ordinary term appearing
in noninteracting fields. If all quantum particles are combined into pairs (for example, it happens in the ground state of a superconductor for $T = 0$ when all electrons are coupled into Cooper pairs) then $\langle \varphi^*(x) \rangle = 0$ and $\langle \psi_\alpha(x) \rangle = 0$ and second terms of right-hand side of Eqs. (3) (5) are equal to zero after quantum averaging. In this case the last terms in these equations have clear meaning: the origin of this term is the nonlinearity of the field that describes a propagation of a pair like to Cooper pair in the superconductivity or a quark - antiquark pair in the QCD. Let us compare these equations with the expression of average of four $\psi-$ operators in the Green’s function method in the superconductivity theory [10]

\[
\left\langle T \left( \hat{\psi}_\alpha(x_1) \hat{\psi}_\beta(x_2) \hat{\psi}_\gamma^+(x_3) \hat{\psi}_\delta^+(x_4) \right) \rightangle 
\approx - \left\langle T \left( \hat{\psi}_\alpha(x_1) \hat{\psi}_\beta^+(x_3) \right) \right\rangle 
\left\langle T \left( \hat{\psi}_\gamma^+(x_2) \hat{\psi}_\delta^+(x_4) \right) \right\rangle 
+ \left\langle T \left( \hat{\psi}_\alpha(x_1) \hat{\psi}_\beta^+(x_4) \right) \right\rangle 
\left\langle T \left( \hat{\psi}_\gamma^+(x_2) \hat{\psi}_\delta(x_3) \right) \right\rangle 
\right)
\]

(7)

where $|N\rangle$ and $|N+2\rangle$ are ground states of system with $N$ and $N+2$ particles (Cooper pairs), respectively. Comparing this expression with Eq. (6), we see immediately that the last term in Eqs. (5) (6) can be interpreted as a propagator for the pairs $(\varphi(x)\varphi(x))$ or $(\psi_\alpha(x)\psi_\alpha(x))$, i.e. this propagator is connected with the associator. In other words in some situation the nonlinear interaction leads to the appearance of pairs and the non-associative algebra of quantum fields can describe the propagation of these pairs. In Ref. [3] it is shown that the above-mentioned Green's function method in the superconductivity theory realizes the Heisenberg's idea about the calculation of the Green's functions for fields with a nonlinear interaction. In this case the non-associative algebra can give us a possibility to describe field operators with the nonlinear interaction.

3 "COOPER PAIRING" IN NON-ASSOCIATIVE ALGEBRA

Let show that algebra of quantized fields mentioned above can have interesting properties similar to the formation of fermion pairs in the superconductivity theory and in the QCD. In the first case it is a Cooper pair containing two electrons are connected with one another by phonons and in the second case it is the quark and antiquark held by a hypothesized flux tube filled with the SU(3) gauge field.

Let us assume that the commutator/anticommutator $\Delta_{ab}$ and associator $Ass(a, b, c)$ are $c-$numbers for any operators $a, b, c$. This means that

\[
\varphi(x)\varphi^*(y) \mp \varphi^*(y)\varphi(x) = \Delta(x, y),
\]

(8)

\[
(a(x)b(y))c(z) - a(x)(b(y)c(z)) = Ass(a(x), b(y), c(z))
\]

(9)
Analogously (here a, b, c can be changed by the operators \( \varphi(x), \varphi(y), \varphi(z) \) or \( \varphi^*(x), \varphi^*(y), \varphi^*(z) \): the operator \( \varphi(x) \) can have some indices but it is inessential for us; the sign (+) in Eq. (8) is connected with a spinor field. Thus, \( \Delta(x, y) \) and \( \text{Ass}(a(x), b(y), c(z)) \) are some functions.

Let a non-associative algebra of the quantized field has following property

\[
a(x)\left(b(x)c(y)\right) = \left(a(x)b(x)\right)c(y) \tag{10}
\]

and

\[
\left(a(y)b(x)\right)c(x) = a(y)\left(b(x)c(x)\right) \tag{11}
\]

here \( a, b \) and \( c \) are some operators depending on the field operators \( \varphi(x) \) (or \( \varphi^*(x) \)) and \( \varphi(y) \) (or \( \varphi^*(y) \)) respectively. In some way this property is similar to the alternative property (4). If \( a(x) = \varphi(x), b(x) = \varphi^*(x) \) and \( c(y) = \varphi(y) \) then

\[
\varphi(x)\left(\varphi^*(x)\varphi(y)\right) = \left(\varphi(x)\varphi^*(x)\right)\varphi(y). \tag{12}
\]

Analogously

\[
\left(\varphi(y)\varphi^*(x)\right)\varphi(x) = \varphi(y)\left(\varphi^*(x)\varphi(x)\right). \tag{13}
\]

If \( y = x \) then

\[
\left(a(x)b(x)\right)c(x) = a(x)\left(b(x)c(x)\right) = a(x)b(x)c(x), \tag{14}
\]

i.e. the property (14) leads to the associativity of quantized fields in any point \( x \).

To determine the associators we should have some dynamic law. In 50’s Heisenberg has offered a method of quantization of strongly nonlinear fields. He proposed that the dynamic equation for the field operators is the field equation for the classical field where the classical field is replaced by the field operator : \( \varphi(x) \rightarrow \hat{\varphi}(x) \). In this paper we suppose this approach can give us information to determine all associators.

Let us consider a simplest case of a nonlinear spinor field with the following Hamiltonian (this is the Hamiltonian of the electrons system describing the properties of a metal in the superconductivity state [10])

\[
\hat{H} = \int \left[-\left(\hat{\psi}^*_\alpha \frac{\nabla^2}{2m} \hat{\psi}_\alpha\right) + \frac{\lambda}{2} \left(\hat{\psi}^*_\alpha \hat{\psi}^*_\beta \hat{\psi}_\alpha \hat{\psi}_\beta\right)\right] dV, \tag{15}
\]

where \( \hat{\psi}_\alpha \) is the operator of spinor field describing electrons; \( m \) is the electron mass; \( \lambda \) is some constant and \( \alpha, \beta \) are the spinor indices. According to Heisenberg the operators \( \hat{\psi} \) and \( \hat{\psi}^+ \) obey the following operator equations

\[
\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right) \hat{\psi}_\alpha(x) - \lambda \left(\hat{\psi}^*_\beta(x)\hat{\psi}_\beta(x)\right) \hat{\psi}_\alpha(x) = 0, \tag{16}
\]

\[
\left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m}\right) \hat{\psi}^*_\alpha(x) + \lambda \hat{\psi}^*_\alpha(x) \left(\hat{\psi}^*_\beta(x)\hat{\psi}_\beta(x)\right) = 0. \tag{17}
\]
As well as in Heisenberg’s method for nonlinear spinor field we have an equation for the 2-point Green’s function \( G_{\alpha\beta}(x, x') \approx -i\langle T(\hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(x')) \rangle \)

\[
\left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) G_{\alpha\beta}(x, x') + i\lambda T \left( \hat{\psi}_\gamma^+(x)\hat{\psi}_\gamma(x)\hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(x') \right) = \delta(x-x'). \tag{18}
\]

Further we have to write an equation for term \( \langle T(\hat{\psi}_\gamma^+(x)\hat{\psi}_\gamma(x)\hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(x')) \rangle \) and so on. After this we will have an infinite equations set for all Green’s functions. In the textbook \([10]\) it was made the following approximation: the operator \( \hat{\psi}_\alpha(x_1)\hat{\psi}_\beta(x_2)\hat{\psi}_\gamma^+(x_3)\hat{\psi}_\delta^+(x_4) \) contains terms corresponding to the annihilation and creation of bound pairs (Cooper pairs) that allows us to write Eq. (18). We should note that expression (18) is approximate one and following.

To Heisenberg’s idea we can cut off the above-mentioned infinite equations set for Green’s functions. The last term of expression (18) plays the key role here. This expression gives us possibility to split the 4-point Green’s function by the nonperturbative way without using Feynman diagram techniques.

After some simplifications (for details, see Ref. \([10]\)) we can obtain two equations

\[
\left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) G(x-x') - i\lambda F(0+)F^+(x-x') = \delta(x-x'), \tag{19}
\]

\[
\left( i \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - 2\mu \right) F^+(x-x') + i\lambda F^+(0+)G(x-x') = 0 \tag{20}
\]

where

\[
e^{-2i\mu t} F_{\alpha\beta}(x-x') = \left\langle N \left| T \left( \hat{\psi}_\alpha(x)\hat{\psi}_\beta(x') \right) \right| N + 2 \right\rangle, \tag{21}
\]

\[
e^{2i\mu t} F^+_{\alpha\beta}(x-x') = \left\langle N + 2 \left| T \left( \hat{\psi}_\alpha^+(x)\hat{\psi}_\beta^+(x') \right) \right| N \right\rangle, \tag{22}
\]

\[
F_{\alpha\beta}(0+) = e^{2i\mu t} \left\langle N \left| \hat{\psi}_\alpha(x)\hat{\psi}_\beta(x) \right| N + 2 \right\rangle, \tag{23}
\]

\[
F^+_{\alpha\beta}(0+) = e^{-2i\mu t} \left\langle N + 2 \left| \hat{\psi}_\alpha^+(x)\hat{\psi}_\beta^+(x) \right| N \right\rangle, \tag{24}
\]

\[
F_{\alpha\beta}(x-x') = a_{\alpha\beta} F(x-x'), \tag{25}
\]

\[
G_{\alpha\beta}(x-x') = b_{\alpha\beta} G(x-x') \tag{26}
\]

here \( a_{\alpha\beta} \) and \( b_{\alpha\beta} \) are some constant matrixes. Now we can compare this method with the non-associative quantization procedure described above. Let us suppose that the operators \( \hat{\psi}(x) \) form the non-associative algebra and Eq. (18) has following form

\[
\left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) G_{\alpha\beta}(x, x') + i\lambda \left\langle 0 \left| \hat{\psi}_\gamma^+(x)\hat{\psi}_\gamma(x)\hat{\psi}_\alpha(x)\hat{\psi}_\beta^+(x') \right| 0 \right\rangle = \delta(x-x'). \tag{27}
\]

here \( |0\rangle \) is the vacuum state and we consider the case \( x^0 > x'^0 \) since it is not very clear what is the time ordering operator for the non-associative algebra. It
can be shown (see, Appendix [3]) that
\[
\langle 0 \left| \hat{\psi}_\gamma^+(x) \left( \hat{\psi}_\gamma(x) \left( \hat{\psi}_\alpha(x) \hat{\psi}_\beta^+(y) \right) \right) \right| 0 \rangle = \left\langle 0 \left| \hat{\psi}_\gamma(x) \left( \hat{\psi}_\alpha(x) \hat{\psi}_\beta^+(y) \right) \right| 0 \rangle + \text{(combination of commutators)}
\]
\[
= \left\langle 0 \left| \hat{\psi}_\gamma(x) \right| 0 \right\rangle \text{Ass} \left( \hat{\psi}_\gamma(x), \hat{\psi}_\beta^+(y), \hat{\psi}_\alpha(x) \right) - \text{Ass} \left( \hat{\psi}_\gamma(x), \hat{\psi}_\alpha(x) \right) + \text{(combination of commutators)}.
\](29)

If all fermions are combined into pairs then we have \( \langle 0 \left| \hat{\psi}_\gamma^+(x) \right| 0 \rangle = 0 \). In Ref. [10] it is shown that we can omit the combination of commutators since it leads to an additive correction to the chemical potential in the equations for the functions \( G, F, F^+ \). Like to Eq. (7) (if \( x_1 = x_2 = x_3 = x \) and \( x_4 = x' \)) we can assume that (with an accuracy of combination of commutators)
\[
\langle 0 \left| \hat{\psi}_\gamma^+(x) \left( \hat{\psi}_\gamma(x) \left( \hat{\psi}_\alpha(x) \hat{\psi}_\beta^+(y) \right) \right) \right| 0 \rangle = \left\langle 0 \left| \hat{\psi}_\gamma(x) \left( \hat{\psi}_\alpha(x) \hat{\psi}_\beta^+(y) \right) \right| 0 \rangle \]
\[
= \left\langle 0 \left| \hat{\psi}_\gamma(x) \right| 0 \right\rangle \text{Ass} \left( \hat{\psi}_\gamma(x), \hat{\psi}_\beta^+(y), \hat{\psi}_\alpha(x) \right) - \text{Ass} \left( \hat{\psi}_\gamma(x), \hat{\psi}_\alpha(x) \right) + \text{(combination of commutators)}
\]
\[
\approx \left\langle N \left| \hat{\psi}_\gamma(x) \hat{\psi}_\alpha(x) \right| N + 2 \right\rangle \left\langle N + 2 \left| \hat{\psi}_\gamma(x) \hat{\psi}_\beta^+(y) \right| N \right\rangle.
\](30)

This equation allows us to derive Eqs. (9), (10) and others like to Green’s function method in the superconductivity theory.

4 **RENORMALIZATION PROCEDURE AND HEISENBERG’S QUANTIZATION METHOD**

After 40 years of the evolution of quantum field theory we know how we should quantize the fields with a small coupling constant: we draw the Feynman dia-
grams and then summarize them. On this way we have a big problem connected
with singularities of loops. These singularities can be eliminated by a renor-
malization procedure. Let us compare this situation with the Heisenberg’s quanti-
zation method. The most important achievement of the Heisenberg’s method
is the possibility to write the operator field equations for the interacting fields.
And after the Heisenberg’s quantization procedure we have infinite equations
system for the n-point Green’s functions of the interacting fields! This is the
most important difference of the Heisenberg’s method from the perturbative
diagram techniques which allows us to work with the propagators and vertices
of the noninteracting fields. For example, if we apply the Heisenberg’s method
to Green’s functions in the QED then these Green’s functions are a result of the
diagram summation and the renormalization procedure. Probably not without
reason Feynman said that renormalization is a method for sweeping the infinities
of a quantum field theory under the rug.
It allows us to say that the non-associative algebra of quantized fields de-
scribes the propagators and n-points Green’s functions (n ≥ 3) of the interacting
fields. The alternative condition (10) (11) can be interpreted as a consequence
of the field interaction: if Ass(a(x), b(x), c(y) = 0) and Ass(a(y), b(x), c(x) = 0)
then these fields are noninteracting.
The largest problem in the Heisenberg’s quantization procedure is the math-
ematical difficulties: how to cut off the infinite equations system for the Green’s
functions? But it is a price which we should pay to avoid the renormalization
procedure. Nevertheless there are examples of such calculations: (a) the Heisen-
berg’s calculations for a nonlinear spinor field [1] [2]; (b) in the Ref. [3] it is
shown that the Green’s function method and the Ginzburg–Landau equation
in the superconductivity theory are the application of the Heisenberg’s method.

5 CONCLUSIONS
Thus, the main goal of this paper is to show that the quantization rules of the
nonlinear fields can be changed and the Heisenberg’s idea about the quantiza-
tion of strongly interacting fields leads to the change of the canonical operator
quantization rules. It is shown that the non-associative algebra can be used
to describe the quantized fields with a strong nonlinearity and the associators
can be connected with the n-point Green’s functions. In other words the non-
associative algebra can be considered as a candidate to a role of an algebra of
strongly interacting fields. The alternative version of the non-associative alg-
ebra has very similar properties with pairing in the superconductivity theory
(Cooper pair) and probably in the QCD (quark– antiquark attached at the ends
of a hypothesized flux tube filled with the color electric field).
In Ref. [3] an assumption is made that Cooper electrons in a High-Tc super-
conductor is connected with each another much stronger than in an ordinary
superconductor in the consequence of a phonon-phonon interaction, just as the
 gluon-gluon interaction in the QCD leads to the confinement. It is possible
that Heisenberg’s quantization method and the non-associative algebra give us
a description of field operators in the QCD and High-$T_c$ superconductivity.

In the context of this approach many interesting questions arise. The most interesting question is the relation between proposed approach and weakly coupled theories (QED, $\lambda \phi^4$ theories and so on). From the physical point of view the difference between these approaches is connected with the possible expansion of associator into a series in terms of coupling constant. If such possibility be realized then the associator can be written as a series of Feynman diagrams. Another question is connected with the ordinary questions from any quantum field theory: is given theory unitary, causal, local and Lorentz invariant? Evidently these properties of quantum field theory depend on the Lagrangian but not on the quantization rules. An investigation of this subject is much more complicated task in the case of non-associative algebra of field operators.
A THE FIRST EXAMPLE OF NON-ASSOCIATIVE CALCULATIONS

Eq. (3) obtaining

\[
(ab)(cd) = a(b(cd)) + Ass(a, b, cd) \\
= a((bc)d) - aAss(b, c, d) + Ass(a, b, cd) \\
= a((cb)d) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) \\
= a(c(bd)) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
= a(c(db)) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) (31)
\]

\[
(ac)(db) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) - Ass(a, c, db) \\
= (ca)(db) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) - Ass(a, c, db) + \Delta_{ac}(db) - Ass(ca, d, b) \\
= (ca)(db) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) - Ass(a, c, db) + \Delta_{ac}(db) - Ass(ca, d, b) + Ass(c, a, d) b \\
+ (c\Delta_{ad}) b \\
= (ca)(db) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) - Ass(a, c, db) + \Delta_{ac}(db) - Ass(ca, d, b) + Ass(c, a, d) b \\
+ (c\Delta_{ad}) b - Ass(c, d, a) b \\
= (ca)(db) - aAss(b, c, d) + Ass(a, b, cd) + a(\Delta_{bc}d) + aAss(c, b, d) \\
+ a(c\Delta_{bd}) - Ass(a, c, db) + \Delta_{ac}(db) - Ass(ca, d, b) + Ass(c, a, d) b \\
+ (c\Delta_{ad}) b - Ass(c, d, a) b + Ass(cd, a, b).
\]
In these calculations we used following expressions

\[(ab)(cd) = a(b(cd)) + \text{Ass}(a,b,cd),\]  \hspace{1cm} (32)

\[a\left(b(cd)\right) = a\left(bc\right)d - a\text{Ass}(b,c,d),\]  \hspace{1cm} (33)

\[a\left(bc\right)d = a\left(cb\right)d + a\left(\Delta b,c,d\right),\]  \hspace{1cm} (34)

and similar expressions with permutations of \(a,b,c,d\) operators.

**B THE SECOND EXAMPLE OF NON-ASSOCIATIVE CALCULATIONS**

Eq.(28) obtaining. Taking into account properties (8), (9), (10) and (11) we have

\[\psi_\gamma^+(x)\left(\psi_\alpha(x)\psi_\beta^+(y)\right) = \left(\psi_\gamma^+(x)\psi_\alpha(x)\right)\left(\psi_\beta^+(y)\right)
\]

\[= -\left(\psi_\gamma(x)\psi_\alpha^+(x)\right)\left(\psi_\beta(x)\psi_\gamma^+(y)\right) + \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x)
\]

\[= -\psi_\gamma(x)\left(\psi_\alpha^+(x)\left(\psi_\gamma(x)\psi_\beta^+(y)\right)\right) + \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x)
\]

\[= -\psi_\gamma(x)\left(\psi_\alpha^+(x)\psi_\beta^+(y)\right) + \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x)
\]

\[= \psi_\gamma(x)\left(\psi_\alpha^+(x)\psi_\beta^+(y)\right) + \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x)
\]

\[+ \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x)\]  \hspace{1cm} (35)

\[= \left(\psi_\gamma(x)\psi_\alpha(x)\right)\left(\psi_\beta^+(y)\right) + \psi_\alpha(x)\psi_\beta^+(y)\Delta_{\gamma\gamma}(x,x)
\]

\[\quad + \psi_\alpha(x)\psi_\gamma^+(y)\Delta_{\gamma\gamma}(x,x).\]
On the other hand

\[
\psi^+_\alpha(x) \left( \psi_\gamma(x) \left( \psi_\alpha(x) \psi^+_\beta(y) \right) \right) = \left( \psi^+_\gamma(x) \psi_\alpha(x) \right) \left( \psi^+_\alpha(x) \psi^+_\beta(y) \right) \\
= - \left( \psi^+_\gamma(x) \psi_\alpha(x) \right) \left( \psi^+_\beta(y) \psi_\alpha(x) \right) + \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
= - \psi^+_\gamma(x) \left( \psi_\alpha(x) \left( \psi^+_\beta(y) \psi_\alpha(x) \right) \right) + \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
= - \psi^+_\gamma(x) \left( \left( \psi_\gamma(x) \psi^+_\beta(y) \right) \psi_\alpha(x) \right) + \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
+ \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
= \psi^+_\gamma(x) \left( \psi^+_\beta(y) \psi_\alpha(x) \right) + \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
+ \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) - \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
= \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) - \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
- \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) - \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
- \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y).
\]

Comparing these two equations we have

\[
\psi^+_\gamma(x) \left( \psi_\gamma(x) \left( \psi_\alpha(x) \psi^+_\beta(y) \right) \right) = \left( \psi_\gamma(x) \psi_\alpha(x) \right) \left( \psi^+_\gamma(x) \psi^+_\beta(y) \right) \\
- \psi_\gamma(x) \psi^+_\beta(y) \Delta_{\gamma\alpha}(x, x) + \psi_\alpha(x) \psi^+_\beta(y) \Delta_{\gamma\gamma}(x, x) \\
= \left( \psi^+_\gamma(x) \psi^+_\beta(y) \right) \left( \psi_\gamma(x) \psi_\alpha(x) \right) \\
- \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\gamma\beta}(x, y) + \psi^+_\gamma(x) \psi_\gamma(x) \Delta_{\alpha\beta}(x, y) \\
+ \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) - \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y) \\
- \psi^+_\gamma(x) \psi_\alpha(x) \Delta_{\alpha\beta}(x, y).
\]

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