Development of Extreme Thoughts in Calculus and Its Application in Mathematical Analysis

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Abstract. Extreme thought is the basic idea of calculus. A series of important concepts in mathematical analysis, such as the continuity of functions, derivatives and definite integrals, are all defined with the help of extreme thought. The generation and improvement of extreme thought is under the need of society and it has created a new impetus for the development of mathematics, and has become the basis and starting point of modern mathematical thought and methods. In order to deepen the public's understanding of extreme thought, this article has studied the development and application of extreme thought. First of all, the concept and emergence of extreme thought are briefly introduced, and then the development process of extreme thought is analyzed from three stages: the germination period, the development period, and the perfect period, and finally the application of extreme thought in mathematical analysis is explained from three aspects.

1. Introduction
As a kind of philosophical and mathematical thought, extreme thought has sprouted from ancient thoughts to the present complete limit theory. Its long and tortuous course of evolution is filled with the hard work, wisdom, rigorous, earnest, and diligent struggle of many philosophers and mathematicians [1]. The evolution of extreme thoughts is a side reaction of the entire process of human understanding of the world and the transformation of the world for thousands of years. The application of extreme ideas is ubiquitous. Understanding and grasping the rational application of extreme thought allows us to quickly find solutions to problems in the process of solving practical problems and improve practical results [2].

2. Emergence and Development of Extreme Thoughts

2.1 The Concept Of Extreme Thinking
Limits are infinitely approaching a fixed value. Limits can be divided into series limits and function limits [3].

Definition 1: Let \( \{a_n\} \) be a sequence of numbers, a is a fixed number. For any given positive number \( \varepsilon \), there is always a positive integer \( N \), so that when \( n > N \), there is \( |a_n - a| < \varepsilon \).

Then the sequence \( \{a_n\} \) converge to \( a \), the fixed number \( a \) is called a limit of sequence \( \{a_n\} \), and recorded as \( \lim_{n \to \infty} a_n = a \), or \( a_n \to a \; (n \to \infty) \).
Definition 2: Let function \( f(x) \) be defined in a hollow neighborhood of at point \( x_0 \), \( A \) is a fixed number, if \( \varepsilon > 0 \), there is a positive number \( \delta \). So that when \( 0 < |x - x_0| < \delta \), \( |f(x) - A| < \varepsilon \). Then \( A \) is the limit of the function \( f \) as \( x \) tends to \( x_0 \), recorded as

\[
\lim_{x \to x_0} f(x) = A
\]

2.2 Emergence of Extreme Thoughts

Like all scientific thinking methods, extreme thoughts are also a product of social practice. Extreme thoughts can be traced back to ancient times. Dutch mathematician Steve improved the exhaustion method of the ancient Greeks in the process of examining the center of the triangle. He used extreme thoughts to think about the problem and gave up the proof of the fallacy method. Therefore, he inadvertently "proposed to develop the limit method into a method using concepts" [4].

However, due to the limitations of historical conditions in the early days of calculus, people's understanding of his basic concepts and their relationships could not break through the limitations of mechanics and geometrical intuition. Many concepts did not have a precise mathematical definition. The derivation of the formula is still in a chaotic logic.

2.3 Development of Extreme Thoughts

2.3.1 The budding period of extreme thoughts. Far more than 2000 years ago, in the understanding of infinity, people's extreme thoughts and methods were inevitably nurtured in them. In China, the famous book "Zhuangzi · Tianxia Pian" states: "One hammer, take half of the day, endlessly. This can reflect the early limit of material in China. There is a profound understanding of separability and continuity. Although these understandings belong to philosophy, they have reflected the germination of extreme thoughts [5]. Liu Hui used the extreme thought to deal with the problem many times in the commentary "Nine Chapters of Arithmetic", and he was more skilled in using it, indicating that he had a very deep understanding of the extreme thought at that time. He already had the concept and method of the limit application on the basis of intuition. It is based on the theory of "circumcision" that Liu Hui obtained the emblem rate. By the fifth century AD, the great mathematician and scientist Zu Chongzhi (429-500) of the Southern and Northern Dynasties' "Stitching" in the same way, using the "circumcision" method to calculate the 24576 polygons: \( 3.1415926 < \pi < 3.1415927 \). Zu Chongzhi's achievements are nearly 1,000 years ahead of the world.

In foreign countries, when studying the problem of drawing a circle, Antifen thought of using an inscribed regular polygon with an increasing number of sides to approach the area of the circle. When the number of sides of the polygon continued to double, the gap between the inscribed regular polygon and the circle was gradually "exhausted", and Bryson (ca. 450 BC) proposed from the opposite direction to approach the circle by the area of the circumscribed regular polygon. In the 4th century BC, the ancient Greek mathematician Oddox created a stricter general method for determining area and volume-the "exhaustion method". This method assumes the infinite separability of quantities, and the following propositions basis: "If you subtract a part that is not less than half of it from any amount, and subtract another part that is not less than half of it from the rest, and continue, you will end up with a less than any given same amount of quantity ". Applying the exhaustion method, Odoxus, Archimedes used the exhaustion method to find the area of a series of geometric figures. He approached the perimeter with a sufficient number of "inscribed" and "external" fans. The plane figure formed, which is very different from China's "circumcision" theory, is essentially an extreme idea. Archimedes (287-212 BC) was born in Syracuse (now Sicily, Italy)). He is exceptionally intelligent and accomplished, and is known as the greatest
mathematician and scientist in ancient times. His famous masterpieces include "Measurement of Circles", "On Spheres and Cylinders", "On Split Cone Surfaces and Spheres", "Parabolic Bow Quadrature", "On Spiral", "Girt Calculation", etc. He cleverly combined the method of exhaustion of Okodossus with the atomic view of De Mocrit through rigorous calculations, solved a large number of calculation problems. He broke through the traditional finite operation, adopted the idea of infinite approximation, and divided the amount of required product into many tiny units, and then used another groups of small units that are easy to calculate sums for comparison. His concept of infinitesimals was used by Newton as the basis of calculus until the 17th century. Archimedes' outstanding achievements enriched the content of ancient mathematics, and the depth of his thoughts and the rigor of his discourse were extremely rare at the time, so he was called "The God of Mathematics" and the "Four Masters of Mathematics" together with Gauss, Euler and Newton before the 19th century.

From this, we can see that at the beginning of the development of extreme thoughts in mathematics, the ancients have created a glorious starting point in the limit field.

2.3.2 The development period of extreme thoughts. At the end of the 14th century, Europe began to have the germination of capitalism. By the middle of the 15th century, the dissolution of the feudal system, Europe's productive forces developed rapidly, and the "Renaissance" era began. The development of productive forces also promoted the development of science and technology, at that time, a lot of new questions were raised in astronomy, physics, geography and other aspects around mechanics as the center, and the exploration of these questions promoted the development of related sciences. For example, the birth zone of Copernicus' "Heliocentric Theory" is a revolution in natural science; due to the study of celestial mechanics, a group of scientists have emerged, such as Steven, Galileo, Kepler, etc., they have also done a lot of research in mathematics, which has laid the foundation and brought opportunities for the development and application of extreme ideas and methods. After the 16th century, Europe was in the embryonic period of capitalism, and productivity had greatly developed. A large number of variables problems occurred in productivity and science and technology such as curve tangent problems, maximum value problems, speed problems in mechanics, work of force, etc., elementary mathematical methods are becoming more and more powerless for this, and new thoughts and new mathematical method to break through the traditional scope of the study are needed to provide new tools to describe and study movement [6].

Many mathematicians have made unremitting efforts to solve the above problems, such as Descartes, Fermat, Barrow, Cavalieri, Wallis, etc., and have achieved certain results, especially Newton and Leibniz founded calculus in their work, they all use extreme ideas and methods from different angles. Although their work relies too much on intuitiveness and lacks a strict logical foundation, their efforts and achievements have laid a solid foundation for the further improvement of extreme ideas.

2.3.3 The perfect period of extreme thoughts. The improvement of extreme thoughts is closely related to the strictness of calculus. For a long time, many people have tried to solve the problem of calculus theory, but they have not been able to achieve it. This is because the research objects of mathematics have changed from constant extended to variables, and people are not very clear about the specific laws of variable mathematics; they still lack a clear understanding of the differences and connections between variable mathematics and constant mathematics; and they are not clear about the unified relationship of finite and infinite opposition. In this way, people are used to the traditional way of thinking of constant mathematics and cannot adapt to the new needs of variable mathematics. The dialectical relationship between "zero" and "non-zero" mutual transformation cannot be explained by the old concepts alone.

In the 18th century, Robbins, D'Alembert, and Roulire and others have clearly stated that the limit must be used as the basic concept of calculus, and they all have defined limits. Among them, D'Alembert's definition is: "A quantity is the limit of another quantity, if the second quantity is closer
to the first quantity than any given value", it is close to the correct definition of the limit; however, the definition of these people cannot get rid of geometrical intuition. This can only be the case, because most of the concepts of arithmetic and geometry before the 19th century were based on the concept of geometric quantities.

First of all, the definition of the derivative with the concept of limit is the correct definition of Czech mathematician Bolzano. He defined the derivative of the function \( f \) as the limit \( f'(x) \) of the difference quotient \( \Delta y / \Delta x \). He emphasized that \( f'(x) \) is not a zero quotient of the two [7]. Bolzano’s ideas are valuable, but he still hasn’t made clear about the nature of the limits.

By the 19th century, mathematics was plunged into huge contradictions. On the one hand, mathematics has made great achievements in describing and predicting physical phenomena. On the other hand, because a large number of mathematical structures have no logical basis, mathematics cannot be guaranteed to be correct. Advocated by German mathematicians, the mathematical community carried out a critical inspection campaign on mathematics, and rigorous definitions and rigorous proofs of some theories. French mathematician Cauchy was relatively complete on the basis of his predecessors’ work. He expounded the concept of limit and its theory. He pointed out in the "Analysis Tutorial": "When the value that a variable takes sequentially successively approaches a fixed value, the difference between the value of the variable and the fixed value is as small as possible. This fixed value is called the limit value of all other values. In particular, when the value (absolute value) of a variable decreases infinitely to converge to the limit 0, it is said that the variable becomes infinitely small."

In order to eliminate the intuitive traces in the concept of limits, Wilstrass proposed a static definition of limits, which provided a strict theoretical basis for calculus. The so-called \( n = A \) means: "If for any \( \varepsilon > 0 \), there always exists a natural number \( N \), such that when \( n > N \), the inequality \( | n-A | < \varepsilon \) is constant.

This definition, with the help of inequality, quantitatively and specifically characterizes the connection between the two "infinite processes" through the relationship between \( \varepsilon \) and \( n \). Therefore, this definition is strict and can be used as the basis for scientific demonstration. It is still used in mathematical analysis books. In this definition, only the number and its size relationship are involved. In addition, only the words such as given, existence, and optional have been rid of the word "approach" and no longer ask for help of intuitive for movement.

After this, Weirstrass, Dedkin, and Cantor each independently and thoroughly researched the basis of analysis into the theory of real numbers, and each established a complete real number system in the 1970s. Ulstrass’s theory can be attributed to the principle of the existence of increasing bounded sequence numbers; Dedkin established the famous Dedkin segmentation; Cantor proposed the use of the limits of a rational "basic sequence" to define irrational numbers. They established a rigorous limit theory and real number theory, and completed the logical foundational work of analysis [8]. The inconsistency of mathematical analysis was summarized as the inconsistency of real number theory, thus making calculus that the unprecedented and magnificent mansion was built on a solid and reliable foundation. The important and difficult task of reconstructing the foundation of calculus was successfully completed by the efforts of many outstanding scholars. The perfection of the limit theory has made calculus solid basis.

3. Applications of Extreme Thoughts in Mathematical Analysis

3.1 Application in Concepts
The method of thoughts of the limit runs through the course of mathematical analysis. It can be said that almost all concepts in mathematical analysis are inseparable from the limit. In almost all mathematical analysis books, the function theory and the method of thoughts of the limit are first introduced to give concepts of continuity, convergence and divergence of functions, derivatives, definite integrals, poles, double integrals, and curve integral and surface integral.

(1) As continuous definition of a function \( y = f(x) \) at the point \( x_0 \). \( \Delta x = x - x_0 \) is the increment or
change amount of independent variable \( x \) (at the point \( x_0 \)), set \( y_0 = f(x_0) \), the corresponding function \( y \) (at the point \( x_0 \)) is incremented as \( \Delta y = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0) = y - y_0 \). Function \( y = f(x) \) at the point \( x_0 \) continuous equivalent \( \lim_{\Delta x \to 0} \Delta y = 0 \), which is the limit of near-zero when the independent variable \( x \) gain \( \Delta x \) and the function value incrementing \( \Delta y \).

(2) Definition of the derivative of function \( y = f(x) \) at the point \( x_0 \). \( y \) at the point \( x_0 \) is defined in a neighborhood, if the limit \( \lim_{x \to x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \) exists, then we define the function \( f \) is derivative at the point \( x_0 \). Let \( x = x_0 + \Delta x \), \( \Delta y = f(x_0 + \Delta x) - f(x_0) \), and it can be written as \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0) \), so the derivative is the the limit of the ratio \( \frac{\Delta y}{\Delta x} \) of function increment \( \Delta y \) with argument increment \( \Delta x \).

(3) Definition of definite integral of function \( y = f(x) \) in the interval \( [a, b] \). \( f \) is a function defined in \( [a, b] \), \( J \) is a definite real number, if there is always some positive number \( \varepsilon \) towards the recognized positive number \( \varepsilon \) , make as long as \( \| T \| < \sigma \), there is \( \left| \sum_{i=1}^{n} f(\xi_i) \Delta x_i - J \right| < \varepsilon \) towards any division \( T \) of \( [a, b] \) and the set \( \{ \xi_i \} \) of points arbitrarily selected on it, then we define function \( f \) is definite integral in \( [a, b] \) recorded as \( J = \int_{a}^{b} f(x) \, dx \). It is the limit of integral sum \( \sum_{i=1}^{n} f(\xi_i) \Delta x_i \) when the segmentation fineness approaches to zero.

(4) The divergence of several series \( \sum u_n \) is defined by the limits of \( \{ S_n \} \), \( s = \sum u_n \) and so on.

### 3.2 Application In Derivatives

The idea of derivative was originally introduced by French mathematician Fermat to study the extreme value problem, but directly related to the concept of derivative is the following two problems: Knowing the law of motion to find the speed and finding the tangent of the curve.

1. **Instantaneous speed**: Set a mass point for linear motion, and its motion law is \( s = s(t) \). If \( t_0 \) is a certain moment, and \( t \) is adjacent to \( t_0 \), then \( \frac{s(t) - s(t_0)}{t - t_0} \) is the average speed point in time \( [t_0, t] \).

   If limit of average speed \( \bar{v} \) when \( t \to t_0 \) exists, then the limit \( \bar{v} = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0} \) is the Instantaneous speed of particle at the moment \( t_0 \).

2. **Slope of the tangent**: The tangent line PT of \( \mathcal{P}(x_0, y_0) \) which belongs to the curve \( y = f(x) \) is the limit position of the secant line PQ when the moving point Q is infinitely close to the point P along this curve.

   \[ \bar{k} = \frac{f(x) - f(x_0)}{x - x_0} \]

   Since the secant PQ slope is \( \bar{k} \) when \( x \to x_0 \) exists, then the limit \( \bar{k} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) is the slope of the tangent PT.

Give the definition of the derivative: \( y = f(x) \) at the point \( x_0 \) is defined in a neighborhood, if the
limit \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \) exists, then we define the function \( f \) is derivative at the point \( x_0 \).

Make \( x = x_0 + \Delta x \) \( \Delta y = f(x_0 + \Delta x) - f(x_0) \), then the above formula can be rewritten as

\[
\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0).
\]

3.3 Application in Integral

Integral is an important concept in mathematical analysis. The indefinite integral is the inverse operation of the derivative and the definite integral is the limit of some special sum [10]. The following gives the important application of the limit idea in definite integral.

Background proposed by definite integral: curved edge trapezoid is composed of non-negative continuous curve \( y = f(x) \), straight line \( y = a, y = b \) and the x axis, find the area of this curved edge trapezoid?

(1) Dividing the curved edge trapezoid into n small curved trapezoid.

(2) when \( n \) is big and all \( \Delta x_i (i = 1, 2, \ldots, n) \) are very small, every curved trapezoid can be seen as a small rectangular section \( k \) th trapezoidal area \( \Delta s_k \approx f(s_k)\Delta x_k (k = 1, 2, \ldots, n) \), among them \( x_{k-1} \leq s_k \leq x_k \) at this time \( s = \sum_{k=1}^{n} f(s_k)\Delta x_k \).

(3) When \( n \) increases infinitely, that is, when \( \| \Delta \| = \max \{ \Delta x_1, \ldots, \Delta x_n \} \) infinitely approaches 0, \( \sum f(s_k)\Delta x_k (k = 1, 2, \ldots, n) \) infinitely approaching the curved trapezoid area \( s \), so \( s = \lim_{\| \Delta \| \to 0} \sum f(s_k)\Delta x_k \).

Definite integral contains \( n-1 \) points in closed interval \( [a, b] \), in order \( a = x_0 < x_1 < \cdots < x_{n-1} = x_n = b \). They Divide\( [a, b] \) into n small interval \( \Delta_i = [x_{i-1}, x_i] \), \( i = 1, 2, \ldots, n \), these points or these closed subranges constitute a division of pairs \( [a, b] \) and it is recorded as \( T = \{ x_0, x_1, \cdots, x_n \} \) or \( \{ \Delta_1, \Delta_2, \ldots, \Delta_n \} \). Length of inter-cell \( \Delta_i \) is \( \Delta x_i = x_i - x_{i-1} \) and recorded as \( \max_{1 \leq i \leq n} \{ \Delta x_i \} \). \( f \) is a function defined in \( [a, b] \), \( J \) is a definite real number, if there is always some positive number \( \delta \) towards the recognized positive number \( \varepsilon \), make as long as \( \| \Delta \| < \delta \), there is \( \sum_{i=1}^{n} f(\xi_i)\Delta x_i - J < \varepsilon \) towards any division \( T \) of \( [a, b] \) and the set \( \{ \xi_i \} \) of points arbitrarily selected on it, then we define function \( f \) is definite integral in \( [a, b] \) recorded as \( J = \int_{a}^{b} f(x)dx \). It is the limit of integral sum \( \sum_{i=1}^{n} f(\xi_i)\Delta x_i \) when the segmentation fineness approaches to zero.

4. Conclusion

Extreme thought is the basis of calculus. The limit method provides a powerful tool for humans to understand infinity. It prominently shows the characteristics of calculus different from elementary mathematics from the methodology. It is an important idea of modern mathematics. Extreme thought contains rich dialectic thoughts and is an excellent application of the unified laws of opposition of materialist dialectics in the field of mathematics. It will have a certain theoretical significance for some problems in the history of mathematics and the philosophy of mathematics to clear the development context of extreme thoughts. It reveals the core content of extreme thoughts and its
internal connection with philosophical thoughts. It has an excellent promotion role in cultivating people's thinking methods, thinking qualities, and improving their ability to analyze and solve problems.

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