On long time dynamics of 1D Schrödinger map flows

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Abstract

In this paper, we study the long time dynamics of small solutions to Schrödinger map flows from \( \mathbb{R} \) to Riemannian surfaces. The results are threefold. (i) We prove that for general Riemannian surface targets the points with some geometric condition can be completely divided into two categories according to the sectional curvature so that the long time dynamics of small solutions of 1D Schrödinger map flow near them are described by modified scattering and scattering respectively for the two categories. (ii) If the geometric condition fails, we prove that solutions with slow time growth in frequency space and sharp time decay in physical space, which scatter or scatter by a phrase correction, must be trivial. (iii) We also prove the asymptotic completeness in \( L^2 \) spaces for 1D SMF into general Riemannian surface near points without any geometric assumptions. Compared with our previous works [27, 26] on higher dimensional Schrödinger map flows where resolution to finite numbers of radiation terms in energy space was proved for small solutions, the results of this work reveal the essentially different and diverse dynamical behaviors of 1D Schrödinger map flows.

1 Introduction

Let \((N, J, h)\) be a Kähler manifold. A map \(u(t, x) : \mathbb{R} \times \mathbb{R}^d \rightarrow N\) is called Schrödinger map flow (SMF) if \(u\) satisfies

\[
\begin{cases}
    u_t = J\tau(u) \\
    u|_{t=0} = u_0.
\end{cases}
\tag{1.1}
\]

Here, the tension field \(\tau(u)\) is defined by

\[
\tau(u) = \sum_{j=1}^d \nabla_j \partial_j u,
\]

where \(\nabla\) denotes the induced covariant derivative on the pullback bundle \(u^* TN\). The equation (1.1) with 2D sphere target (introduced by Landau-Lifshitz [24]) describes the evolution of spin fields in continuum ferromagnets and plays a fundamental role in magnetization dynamics. The 1D SMF with sphere target is closely related to the vortex filament equation, and is a completely integrable system, see e.g. [1, 44, 45]. In the late 1990s, geometers and mathematical physicists introduced (1.1) with general targets from different views.

We briefly recall the following non-exhaustive list of works on Cauchy problems and near soliton dynamics of SMF. The local Cauchy theory of SMF was developed by Sulem-Sulem-Bardos [40], Ding-Wang [9], McGahagan [28]. The global theory for Cauchy problem was pioneered by Chang-Shatah-Uhlenbeck [7], Nahmod-Stefanov-Uhlenbeck [32], Bejenaru [2], Ionescu-Kenig [19, 20]. The global well-posedness theory in critical Sobolev spaces for target \(S^2\) was completed by Bejenaru-Ionescu-Kenig [3](\(d \geq 4\)) and Bejenaru-Ionescu-Kenig-Tataru [4] (\(d \geq 2\)). See also Smith [39] for the conditional regularity in critical spaces. In the other direction, the stability/instability of ground state harmonic maps and threshold scattering in equivariant class were studied by works of Gustafson, Kang, Tsai, Nakanish [16, 15], Bejenaru, Ionescu, Kenig, Tataru [5, 6]. Self-similar solutions were studied by Ding-Tang-Zeng [8], Germain-Shatah-Zeng [13], Banica-Vega [1]. The type II singularity formulation was achieved by Merle-Raphael-Rodnianski [30] and Perelman [35].

For the global theory, additional efforts are needed for general targets than the constant curvature case. In \(d = 1\), Rodnianski-Rubinstein-Staffilani [36] proved global regularity for (1.1) and general Kähler manifolds \(N\). In our previous works [26, 27], we proved the global well-posedness and long time dynamics of SMF from \(\mathbb{R}^d\) with \(d \geq 2\) into compact Kähler manifolds with small initial data in critical Sobolev spaces. In this work, we aim to study the long time dynamics of small solutions to (1.1) with \(d = 1\).

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Theorem 1.1. Let $d$ vanishing point in Def. 1.1. Let $K(\cdot)$ be a function with $f(0)$ invariant, i.e., the proof of these facts and other examples are presented in Section 5.

Proposition 1.1. Assume that there exists at least one intrinsic vanishing point.

Lemma 1.1. Assume that $\eta$ be another local complex coordinate near $Q$ such that $\eta = f(z), \text{ where } f$ is a holomorphic function with $f(0) = 0, f_z(0) \neq 0$. Then $Q \in N$ is also an intrinsic vanishing point in the sense of coordinate $\eta$.

In other words, the notion intrinsic vanishing point is truly an intrinsic notion which is free of choice of local complex coordinates.

The following result implies that the existence of intrinsic vanishing point.

Definition 1.1. Assume that $N$ is a Riemannian surface with metric $h(z, \bar{z})dzd\bar{z}$, and $Q \in N$ is a given point which corresponds to $z = 0$. We say $Q$ is an intrinsic vanishing point if

$$[\ln h]_{\bar{z}}(0)[\ln h]_{\bar{z}}(0) - [\ln h]_{z\bar{z}}(0) = 0.$$ (1.3)

We will prove whether $Q$ is an intrinsic vanishing point or not is independent of the choice of local complex coordinates near $Q$. To be precise, we have

Lemma 1.1. Assume that $N$ is a Riemannian surface with metric $h(z, \bar{z})dzd\bar{z}$, and $Q \in N$ is an intrinsic vanishing point in the sense of complex coordinate $z$. Let $\eta$ be another local complex coordinate near $Q$ such that $\eta = f(z)$, where $f$ is a holomorphic function with $f(0) = 0, f_z(0) \neq 0$. Then $Q \in N$ is also an intrinsic vanishing point in the sense of coordinate $\eta$.

In other words, the notion intrinsic vanishing point is truly an intrinsic notion which is free of choice of local complex coordinates.

The following result implies that the existence of intrinsic vanishing point.

Proposition 1.1. Assume that $N$ is a closed oriented Riemannian surface with non-zero Euler-Poincare characteristic. Then there exists at least one intrinsic vanishing point.

We also have many other concrete examples of intrinsic vanishing point. In fact, if the metric near $Q$ is locally rotationally invariant, i.e., $h = h(|z|^2)dzd\bar{z}$ with $z(Q) = 0$, then $Q$ is an intrinsic vanishing point. Particularly, all points of $\mathbb{S}^2$ and $\mathbb{H}^2$ are intrinsic vanishing points. The proof of these facts and other examples are presented in Section 5.

Our main theorems are as follows:

Theorem 1.1. Let $d = 1$. Assume that $N$ is a Riemannian surface with metric $h(z, \bar{z})dzd\bar{z}$, and $Q \in N$ is a given intrinsic vanishing point in Def. 1.1. Let $K(\cdot)$ be the sectional curvature at $Q$.

- Assume that $K(Q) \neq 0$. Then there exist a universal constant $n_\ast \in \mathbb{N}$ and a sufficiently small constant $\epsilon_\ast > 0$ depending only on $N$, and a well chosen local complex coordinate $w$ of $N$ near $Q$ with $w(Q) = 0$, such that if $u_0 : \mathbb{R} \rightarrow N$ satisfies

$$\|w_0\|_{H^{11/2} H^{1/2}} \leq \epsilon_\ast,$$ (1.4)
then (1.1) with initial data \( u_0 \) evolves into a unique global solution \( u(t,x) \), and there exists a \( C \) valued function \( U \in L^\infty(\mathbb{R}, (\xi)^2d\xi) \) such that as \( t \to \infty \) there holds

\[
\lim_{t \to \infty} ||e^{ic\xi^2 t} \theta(\xi) \widehat{\psi}(t,\xi) - e^{-ic\xi^2} U(\xi)||_{L^\infty(\mathbb{R}, (\xi)^2d\xi)} = 0. \tag{1.5}
\]

Here, writing the metric of \( N \) near \( Q \) as \( h(\bar{w}, \bar{w})d\bar{w}d\bar{w} \), the constant \( c \) in (1.5) is given by \( c = -\frac{1}{2}K(Q)h_0 \), \( h_0 = h(0,0) \). Moreover, \( w_0(x) \) and \( w(t,x) \) are the local complex coordinates of \( u_0(x) \) and \( u(t,x) \) respectively.

Moreover, for any local complex coordinate \( z \) of \( N \) near \( Q \) with \( z(Q) = 0 \), we have the sharp decay estimate

\[
||z(t,x)||_{W^2} + ||\nabla_x \partial_t u||_{L^\infty} + ||\partial_x u||_{L^\infty} \lesssim t^{-\frac{3}{4}}. \tag{1.6}
\]

Furthermore, there exist some function \( \psi \in \langle x \rangle^{-2}L^\infty_x(\mathbb{R}) \), and some constant \( \zeta > 0 \) such that \( z(t,x) \) has the refined asymptotic expansion for \( t \geq 1 \)

\[
z(t,x) = \frac{1}{2\pi i} \psi(\frac{x}{2t}) \exp \left( \frac{t|x|^2}{4t} + \frac{ic|x|^3}{12t} \right) + \mathcal{R}(x,t), \tag{1.7}
\]

where the remainder term \( \mathcal{R}(x,t) \) satisfies

\[
||\mathcal{R}(x,t)||_{L^\infty} \lesssim t^{-\frac{3}{4}+\epsilon}, ||\mathcal{R}(x,t)||_{L^2} \lesssim t^{-\zeta}. \tag{1.8}
\]

- Assume that \( K(Q) = 0 \). Let \( z \) be any local complex coordinate of \( N \) near \( Q \) with \( z(Q) = 0 \). Then there exist a sufficiently small constant \( \epsilon > 0 \) and a sufficiently large constant \( n > 0 \) depending only on \( N \) such that if \( u_0 : \mathbb{R} \to N \) satisfies (1.4), then (1.1) with initial data \( u_0 \) evolves into a unique global solution \( u(t,x) \), and there exists a \( C \) valued function \( z_{\infty} \in H^1(\mathbb{R}) \) such that as \( t \to \infty \) there holds

\[
\lim_{t \to \infty} ||z(t,x) - e^{\mu t}z_{\infty}||_{H^1} = 0. \tag{1.8}
\]

Here \( z_0(x) \) and \( z(t,x) \) are the local complex coordinates of \( u_0(x) \) and \( u(t,x) \) respectively. Moreover, the above sharp decay estimates (1.6) hold as well.

If \( Q \in N \) is not an intrinsic vanishing point in Def. 1.1, generically neither modified scattering nor scattering could be expected. To be precise, we have

**Theorem 1.2.** Let \( d = 1 \). Assume that \( N \) is a Riemannian surface, and \( Q \in N \) is not an intrinsic vanishing point in the sense of Def. 1.1. Then there exists a sufficiently small constant \( \epsilon > 0 \) depending only on \( N \), such that, if there exists a local complex coordinate \( z \) of \( N \) near \( Q \) with \( z(Q) = 0 \) under which the solution of (1.1) satisfies

\[
||z_0||_{H^1} + \sup_{t \geq 0} t^\beta ||z(t)||_{H^1_x} \leq \epsilon, \tag{1.9}
\]

\[
\sup_{t \geq 0} \sum_{j=0}^1 \sum_{\xi \in 2^j} ||(x + 2it\xi)_{z}|_{L^\infty_z} ||_{L^2_x} \leq \epsilon, \tag{1.10}
\]

for some \( \beta \in (0, \frac{1}{12}) \), and there exist a \( C \) valued function \( U(\xi) \), a real valued function \( \mathcal{D} \) for which there holds

\[
\lim_{t \to \infty} ||(\zeta^2 [e^{it\xi}z_0(\xi) - e^{-it\xi} U(\xi)])|_{L^2_x} + ||(\zeta^2 [e^{it\xi}z(\xi) - e^{-it\xi} U(\xi)])|_{L^2_x} = 0, \tag{1.11}
\]

then \( z(t) \equiv 0 \). Here \( z_0(x) \) and \( z(t,x) \) are the local complex coordinates of \( u_0(x) \) and \( u(t,x) \) respectively.

For the asymptotic completeness problem, we have the following theorem for general Riemannian surface without assumption (1.3).

**Theorem 1.3.** Let \( d = 1 \). Assume that \( N \) is a Riemannian surface with metric \( h(z, \bar{z})dzd\bar{z} \), and \( Q \in N \) is a given point. Let \( z \) be any given local complex coordinate of \( N \) near \( Q \) with \( z(Q) = 0 \). Then there exist a universal constant \( m \in \mathbb{N} \) and a sufficiently small constant \( \epsilon > 0 \) depending only on \( N \), such that if \( \psi : \mathbb{R} \to \mathbb{C} \) satisfies

\[
\sum_{j=0}^m ||(\zeta)_{z}|_{L^\infty} \leq \epsilon, \tag{1.12}
\]
then there exists an initial data \( u_0 \) which evolves into a global solution \( u(t, x) \) of (1.1) so that as \( t \to \infty \) there holds

\[
\lim_{t \to \infty} \| z(t, x) - \frac{1}{(2it)^\frac{3}{2}} \psi(e^{-\frac{|x|^2}{2t}} e^{2 \nu(\frac{1}{4}|x|^2) \ln(2t)}) \|_{H^1} = 0.
\]

(1.13)

Here \( z(t, x) \) denotes the local complex coordinate of \( u(t, x) \), \( \nu = -\frac{i}{4} K(Q)h_0 \), \( h_0 = h(0, 0) \).

We make a few remarks on the above three theorems.

(i) All the assumptions (1.9), (1.10), (1.11) in Theorem 1.2 are fulfilled by solutions in Theorem 1.1.

(ii) The intrinsic vanishing condition (1.3) indeed corresponds to the vanishing of 4 order terms \( (y^2 + y^2 + y^2)(\partial_y w)^2 \) in the equation (1.21). Theorem 1.2 implies that solutions of slow growth in frequency space and sharp decay in physical space, which scatter or scatter by a phrase correction, must be trivial, if the 4 order terms do not vanish. This fact is not obvious since the leading part of equation (1.21) is the cubic term \( y^2w + y^2w^2(\partial_y w)^2 \), and it seems that the 4 order terms would not have a big effect on the long time dynamics of small solutions at the first glance. In this view, Theorem 1.2 indeed reveals that the 4 order terms also play a vital role in determining the whole dynamics.

(iii) However, Theorem 1.3 shows the existence of wave operators is independent of the intrinsic vanishing condition. A further calculation implies that the key condition the solutions constructed in Theorem 1.3 fail to fulfill is the slow growth assumption (iv) As far as we know, it is a new phenomenon revealed here that high order terms also have a dominant effect on the long time dynamics of small solutions.

Remark 1.1. Note that Theorem 1.1 implies the dynamical behavior is determined by the curvature at the given point. For example, if \( N \) has two intrinsic vanishing points \( Q_1, Q_2 \) with \( K(Q_1) = 0 \) and \( K(Q_2) \neq 0 \), then the 1D SMF near \( Q_1 \) scatters while the 1D SMF near \( Q_2 \) scatters with a phrase correction. Here we have a more concrete example, say \( N \) has a metric \( h(z, z) = e^{2i|z|} \), \( A(z, z) = c_1(z + z) + c_2(z^2 + z^2) + c_3(\bar{z}^2 + \bar{z}^2) + c_4|z|^4 \) with \( z \in \mathbb{C} \). Then both \( z = 0 \) and \( z = 1 \) are intrinsic vanishing points if \( c_1 + 2c_2 + 3c_3 + 4c_4 = 1 \). And \( K(0) = 0 \), \( K(1) \neq 0 \) if \( c_4 \neq 0 \).

Remark 1.2. For the classical 2d sphere target, i.e. the 1D Landau-Lifshitz equation, Theorem 1.1 is still new, since the previous method depending on Hasimoto transform did not give long time dynamics of the map itself. When \( (N, h dz^2) \) is the 2d sphere, the metric writes as \( h(|z|^2) = (1 + |z|^2)^{-2} \) under the stereographic projection. If \( (N, h dz^2) \) is the hyperbolic plane, using the Poincare disk model, the metric is given by \( h(|z|^2) = 4(1 - |z|^2)^{-2} \). Hence, every point of standard \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \) is an intrinsic vanishing point, for which Theorem 1.1 applies. Moreover, for the \( \mathbb{S}^2 \) target and \( \mathbb{H}^2 \) target, the local complex coordinate \( w \) in Theorem 1.1 can be directly chosen to be the stereographic projection coordinate and the coordinate of Poincare disk model respectively.

Remark 1.3. For long time dynamics of small solutions to (1.1) in \( d \geq 2 \), our previous works [26, 27] proved that the map splits into finite numbers of radiation terms plus an asymptotic vanishing error in the energy space, and converges to the constant map in uniform distance. However, the dichotomous result of Theorem 1.1 in this work reveals that both scattering and modified scattering occur for 1D SMF, and it depends on the curvature.

Remark 1.4. Modifications of arguments here can give better regularity index than that stated in (1.4). The choice of \( n_0 \) is largely casual. Besides (1.13), we also have

\[
\| z(t, x) - \frac{1}{(2it)^\frac{3}{2}} \psi(e^{-\frac{|x|^2}{2t}} e^{2 \nu(\frac{1}{4}|x|^2) \ln(2t)}) \|_{H^1} \leq t^{-\frac{3}{2}}.
\]

Remark 1.5. If \( h(z, z) = 1 \) in Theorem 1.1, then \( N \) is the complex plane with the standard metric. In this case, the solution of (1.1) is given by \( u = e^{i\alpha}h_0 \).

1.1 Main ideas of Theorem 1.1

Let’s begin with some known facts on 1D NLS. Recall that for 1D NLS of the form \( i\partial_x v + \Delta v = |v|^{p-1}v \), the solution with small initial data scatters for \( p > 3 \), and for \( 1 < p < 3 \) the only solution that scatters in \( L^2(\mathbb{R}) \) is zero, see e.g. [43]. The case \( p = 3 \) is the critical value. Consider the general NLS with cubic nonlinearity:

\[
i\partial_x v + \Delta v = N(\partial_x^j v^\pm, \partial_x^{j_1} v^\pm, \partial_x^{j_2} v^\pm), \quad j_1, j_2, j_3 \in [0, 1].
\]

(1.14)
where $v^+ = v$, $v^- = \bar{v}$. The typical candidate of (1.14) is
\[
i\partial_t v + \Delta v = a_1\bar{v}v^2 + a_2v^3 + a_3\bar{v}^3 + a_4v^2v.
\] (1.15)

In dimension 1, except for the case $a_2 = a_3 = a_4 = 0$, the long time dynamics of (1.15) are not completely clear even in small data regime, see e.g. [29] and reference therein. The known advantage of $\bar{v}v^2$ compared with other three combinations is that $\bar{v}v^2$ enjoys gauge invariance. And we also remark that the derivative nonlinear terms in (1.14) sometimes give better behaviors in the low frequency regime. For more detailed discussions on model (1.14)-(1.15) see [14, 10, 11, 17, 18, 23, 22, 34, 25, 43, 37].

Let’s recall the intrinsic view of Hasimoto transform. Let $\mathcal{N}$ be a Riemannian surface, and let $\{e, Je\}$ be a moving frame for $u^*TN$. The Hasimoto transform indeed can be viewed as transforming the map $u$ into scalar fields $\psi_{i,s} = (\partial_{x^i} u, e) + i(\partial_{\bar{x}^i} u, Je)$ on the trivial vector bundle over $\mathbb{R}^2$ with fiber $\mathbb{C}$ under the gauge fixing assumption that $\nabla_x e = 0$. If the target $\mathcal{N}$ is a 2d sphere, then $\psi_s$ solves the equation (1.16), see e.g. [33]. By [36] and additional observations, one can check $\psi_s$ solves the general cubic NLS system
\[
i\partial_t \psi + \Delta \psi = C_{\alpha,\beta,\gamma}^{\alpha} \psi^\alpha \psi^\beta \psi^\gamma,
\] (1.16)

if $\mathcal{N}$ is a Hermitian symmetric space. Here, we denote $v^+ = v$, $v^- = \bar{v}$, $C_{\alpha,\beta,\gamma}$ are constant matrices. If $\mathcal{N}$ is a general Kähler manifold without symmetry, [36] deduces that $\psi_s$ satisfies
\[
i\partial_t \psi + \Delta \psi = P + Q.
\] (1.17)

where $|P| \leq |v|^3$, $|Q| \leq (\int |v|^3dx)|v|$. From the conservation of energy, $Q$ is indeed a quadratic term in the view of time decay. According to previous discussions on model (1.15), the gauged equations (1.16), (1.17) shall be very tough to handle in the study of long time dynamics.

Another way to fix the gauge of SMF is the caloric gauge introduced by Tao [42, 41]. It is powerful in the study of critical regularity problem and long time dynamics, see for instance [4, 26, 27]. Let $U(x, t)$ be the solution to heat flow equation with initial data $u(t, x)$, and $\{e_a, Je_a\}$ be the moving frame of $U^*TN$. The caloric gauge assumes $\nabla_x e_a = 0$. Denote $\psi_t, \psi_x$ the differential fields associated with the frame $\{e_a, Je_a\}$ and $A_x, A_t$ the corresponding connection coefficients. In dimension 1, when $s = 0$, $\psi_s$ roughly solves
\[
i(\partial_t + A_x)\psi + (\partial_x + A_t)^2 \psi = 0.
\] (1.18)

(Note that the cubic curvature term vanishes since $d = 1$.) Here, the connection coefficient $A_x, A_t$ is approximately $\int_0^\infty O(\psi_x)(D_x \psi)ds'$. (1.18) suffers from the same structure problem as (1.17). On the other hand, the nonlinearity terms of (1.18) seem to be cubic, but in the study of time decay of $||\psi_x||_{L^T}$, one will find there is no enough decay in $s$ to make the integral $\int_0^\infty O(\psi_x)(D_x \psi)ds'$ and $\int_0^\infty \psi_x ds'$ converge in the desired $L^p$ space. See Appendix A for a concrete discussion.

From the above discussions, assuming neither $\nabla_x e = 0$ nor $\nabla_x e = 0$ is a good choice of gauge fixing for our problem.

Another way to write (1.1) is to use the local complex coordinate. In fact, assume that $\mathcal{N}$ is a Riemannian surface with metric $h(z, \bar{z})dzd\bar{z}$. Then (1.1) is written as
\[
\begin{aligned}
i\partial_t z + \Delta z &= \frac{h(z)}{|h(z)|^2}\partial_z \overline{\partial_z} z \\
|z|_{v=0} &= z_0.
\end{aligned}
\] (1.19)

We observe that (1.19), which is SMF under the complex coordinates (or in the frame $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$), has several key good structures. Expanding $h_z/h$ at $z = 0$ gives
\[
\begin{aligned}
i\partial_t z + \Delta z &= c_0 \partial_z \overline{\partial_z} z + c_1 \overline{\partial_z} \partial_z \overline{\partial_z} z + c_2 \partial_z \overline{\partial_z} \partial_z \overline{\partial_z} z + c_3 \partial_z \overline{\partial_z} \partial_z \overline{\partial_z} z + c_4 \bar{z}\partial_\bar{z} \overline{\partial_z} z \\
&\quad+ c_5 \bar{z}\partial_\bar{z} \partial_z \overline{\partial_z} z + O(|z|^3) \partial_z \overline{\partial_z} z
\end{aligned}
\] (1.20)

The quadratic term $c_0 \partial_z \overline{\partial_z} z$, the cubic term $c_2 \bar{z}(\partial_z z)^2$ and the 4 order term $c_3 \bar{z}(\partial_z z)^2$ can be removed by letting
\[
w := z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4.
\]
for well chosen constants $\gamma_1, \gamma_2, \gamma_3$. Indeed $w$ fulfills
\[
i\partial_t w + \Delta w = v_1 \overline{\partial_z} \partial_z \overline{\partial_z} w + v_2 \overline{\partial_z} \partial_z \partial_z \overline{\partial_z} w + v_3 \bar{z}\overline{\partial_z} \partial_z \overline{\partial_z} w
\]
\[
\quad+ v_4 \bar{z}\partial_\bar{z} \partial_z \overline{\partial_z} w + O(|w|^3) \partial_z \overline{\partial_z} w.
\] (1.21)
Note that (1.19) is invariant under the holomorphic transformation. This is analogous to the aforementioned freedom of gauge fixing.

The leading cubic nonlinearity of (1.21) is \( \mathbb{w} \partial_{x} w_{x}^{2} \). In the physical space, we find that this nonlinearity commutes not that badly with the operator \( L := i x - 2 \partial_{x} \), (the generator of Galilean transformation), although (1.19) is not invariant under the Galilean transformation. This observation inspires us to choose (1.21) as the master equation.

However, (1.21) suffers from two key problems. One is the serious derivative loss compared with (1.16) or (1.17). In fact, to obtain global bounds of solution, we will mainly depend on mass and two vector fields, one is \( S := 2 \partial_{t} + x \partial_{x} \), the infinitesimal generator of scaling, and the other is \( L := i x - 2 \partial_{x} \), arising from the Galilean transformation. Directly calculating \( \| w \|_{L_{2}^{x}} \), \( \| S w \|_{L_{t}^{3}} \), \( \| L w \|_{L_{2}^{x}} \) will face serious derivative loss and time growth. The solution consists of two observations. One observation is that \( S u \) is an intrinsic geometric quantity valued in \( u^{*} T N \). So, instead of estimating \( \| S z \|_{L_{t}^{3}} \), we dominate the geometric quantity \( \| \nabla S u \|_{L_{2}^{x}} \), where \( \nabla \) denotes the induced covariant derivative. The other observation is that for a well chosen weight \( H(w, \tilde{w}) \) the new functional

\[
H(w) := \int_{\mathbb{R}} H(w, \tilde{w}) |w|^{2} dx
\]

fulfills a closed energy estimate without any loss of derivatives. In fact, it satisfies an improved energy estimate

\[
\frac{d}{dt} H(w) \leq \| w \|_{L_{w}^{8}}^{3} H(w). \tag{1.22}
\]

Meanwhile, \( H(w) \) is comparable to \( \| w \|_{L_{2}^{x}}^{2} \). Note that the power of \( \| w \|_{L_{w}^{8}} \) in (1.22) is 3 rather than 2. So it provides a time uniform bound of \( \| w \|_{L_{2}^{x}} \) rather than logarithm upper-bound. The same technique also provides a slow time growth upper-bound of \( \| L w \|_{L_{2}^{x}} \) with additionally assuming the intrinsic vanishing condition. Indeed for the well-chosen weight \( H(w, \tilde{w}) \), the functional, which is comparable to \( \| L w \|_{L_{2}^{x}} \), defined by

\[
\mathcal{L}(w) = \int_{\mathbb{R}} |L w|^{2} H(w, \tilde{w}) dx
\]

satisfies a closed energy estimate without loss of derivative provided that the intrinsic vanishing condition holds.

The other key problem (1.21) suffering from is the 4 order terms \( (\nu_{3} w \tilde{w} + \nu_{4} \tilde{w}^{3})(\partial_{x} w)^{2} \). These two terms have a relatively very large time growth in estimating \( \| L w \|_{L_{2}^{x}} \). And we will see in the proof of Theorem 1.2, that generally one can not expect modified scattering or scattering if these two are not vanishing. Fortunately, we find that the geometric condition (1.3), i.e., intrinsic vanishing point condition, can kill these two terms.

We make several additional remarks on the proof to Theorem 1.1. The proof relies on the framework of space-time resonance analysis developed by Germain-Masmoudi-Shatah[10, 11, 12]. The way of proving modified scattering by stationary phrase method is inspired by Ionescu-Pusateri [21] and Kato-Pusateri [22].

### 1.2 Main idea of Theorem 1.2

Suppose that \( Q \) is not an intrinsic vanishing point, i.e.

\[
[\ln h]_{c}(0)[\ln h]_{c2}(0) - [\ln h]_{c2c}(0) \neq 0. \tag{1.23}
\]

The proof of Theorem 1.2 will be divided into two cases.

**Case 1.** Assume that \( c_{5} \neq 0 \) in (1.20). In this case, letting \( f = e^{-iA_{2}} \), its Fourier transform \( \tilde{f} \) solves the equation (10.10).

When bounding \( \| f \|_{L^{\infty}} \), we observe that the 4 order term associated with \( \phi_{++} \), i.e. \( c_{5} \zeta^2 (\partial_{\xi} \zeta)^2 \) part, implicitly contains a model of logarithmic growth in time. In fact, delicate analysis via stationary phrase method shows

\[
\| \tilde{f}(t, \zeta) \|_{L^{\infty}} \geq C |c_{5}| \int_{1}^{T} \frac{1}{t} \zeta^2 |\tilde{f}(t, \zeta)| d\zeta dt - O_{\| L \|_{L^{\infty}}^{3}}(1),
\]

where \( C > 0 \) is a universal constant. This, together with the convergence claimed in Theorem 1.2 and the almost conservation of mass, implies \( z = 0 \).

**Case 2.** Assume that \( c_{5} = 0 \) in (1.20). In this case, (1.23) shows \( c_{0} \neq 0 \). For well chosen constants \( |k_{i}|_{i=1}^{3} \), letting \( w = z + k_{1} \zeta^2 + k_{2} \zeta^3 + k_{3} \zeta^4 \), \( w \) solves the equation (10.38), i.e. the terms \( (\partial_{\xi} \zeta)^2, z(\partial_{\xi} \zeta)^2, \zeta^2(\partial_{\xi} \zeta)^2, \zeta^4(\partial_{\xi} \zeta)^2 \) vanish together. And...
particularly, $\kappa_1 \neq 0$ because of $c_0 \neq 0$. Let $g = e^{-it\Delta}w$. Its Fourier transform $\hat{g}$ solves (10.39). Then delicate analysis via stationary phase method implies

$$\|\partial_x \hat{g}(t, \xi)\|_L^2 \leq t^{4-\nu+3\beta}$$

with $\nu \in (3\beta, \frac{4}{3})$. This further gives

$$|\kappa_1| \|t\partial_x z\|_L^2 \leq t^{4-\nu+3\beta}.$$ 

Meanwhile, the convergence of Theorem 1.2 implies as $t \to \infty$

$$\|t\partial_x z\|_L^2 \sim t^{4} \|\xi^2 U\|_{L_x^2}^2.$$ 

Then we conclude $U = 0$, and further $z \equiv 0$ by the almost conservation of mass.

### 1.3 Main ideas of Theorem 1.3

Theorem 1.3 aims to prove the asymptotic completeness, i.e. to find an appropriate initial data to match the given final state. The whole proof is divided into two parts, the first part is to prove asymptotic completeness for model (1.21), and the second is to derive asymptotic completeness of (1.19) from (1.21). For model (1.21), we first construct an approximate solution $w_{ap}$, and then solve the perturbation problem around $w_{ap}$. To construct the approximate solution $w_{ap}$, it suffices to choose $w_{ap}$ to make

$$i\partial_t w_{ap} + \Delta w_{ap} - \nu_3 w_{ap}(\partial_x w_{ap})^2 = -\nu_1 w_{ap}(\partial_x w_{ap})^2 - \nu_2 w_{ap}(\partial_x w_{ap})^2$$

decay faster than $t^{-\frac{3}{2}}$ in $L^2_t \cap L^\infty_x$. We remark that the idea of constructing approximate solutions dates back to Ozawa [34], and generalized by many authors to other models, e.g. [31, 38]. The key difficulty to solve the perturbation problem around $w_{ap}$ is to solve the perturbation problem around $w_{ap}$ can not be solved by standard methods such as fixed point argument or bootstrap argument combined with linear estimates of Schrödinger equations (e.g. smoothing estimates, maximal function estimates). The ingredients to solve this problem are the following:

1. Construct a series of solutions to SMF which almost converge to the desired solution in some naive sense;
2. Use geometric Sobolev norms obtained in the Cauchy problem to compensate derivative loss in $O(\partial_x^2 w_{ap}^2 \partial_x(w - w_{ap}))$, $j, k, l \in \{0, 1\}$;
3. Construct a new functional with well chosen weight to compensate derivative loss in $O((\partial_x^k w_{ap})^2 \partial_x(w - w_{ap}))$, $k \in \{0, 1\}$.

In fact, the most serious derivative loss term is the type in (iii), for which we find a weight $W(\cdot)$ and define the functional

$$W(t) := \int_{\mathbb{R}} W(w_{ap}(\bar{w}_{ap}))|w - w_{ap}|^2 |dx,$$

to simultaneously satisfy that $W$ is comparable to $\|w - w_{ap}\|_{L^2}^2$ and $\frac{d}{dt} W(t)$ enjoys closed energy estimate. If $W$ is found, troublesome terms of (iii) cancel with each other, and Theorem 1.3 follows by the above strategy.

**Organization.** The paper is organized as follows. In Section 2, we bound $\|S z\|_{H^2}$ and $\|\partial_x z\|_{H^2}$. In Section 3, we introduce the holomorphic transformation and prove some important algebraic facts. In Section 4, we prove the global bound of mass. In Section 5, we prove several geometric facts on intrinsic vanishing points. In Section 6, we bound $\|L_z\|_{L^2}$. In Section 7, we set up the bootstrap assumption and recall some preliminaries on decay estimates of free Schrödinger equations. In Section 8, we derive bounds in Fourier space. In Section 9, we close bootstrap, prove decay estimate and modified scattering/scattering. In Section 10, we prove Theorem 1.2. In Section 11, we verify Theorem 1.3.

**Notations.** The notation $A \lesssim B$ means there exists some $C > 0$ such that $A \leq CB$.

Let $\varphi$ be a function supported in the annulus $\{\xi \in \mathbb{R} : \frac{1}{2} \leq |\xi| \leq \frac{3}{2}\}$ such that

$$\sum_{k \in \mathbb{Z}} \varphi(\frac{k}{2^k}) = 1, \forall \xi \neq 0.$$ 

Define the Fourier multipliers

$$P_k = \varphi(\frac{D}{2^k}), \quad P_{ck} = \sum_{j < k} \varphi(\frac{D}{2^j}), \quad P_{\geq k} = I - P_{c=k}.$$
Let \((N, h, J)\) denote the target Kähler manifold. The connections of \(TN\) and \(u^*TN\) are all denoted by \(\nabla\) for simplicity. Let \(R\) denote the curvature tensor of \(N\). We make the convention that
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,
\]
and
\[
R(X, Y, Z, W) = h(R(X, Y)Z, W).
\]
In the following, we fix three small constants \(\epsilon, \epsilon, \epsilon\) to fulfill
\[
0 < \epsilon, \epsilon \ll 1.
\]

2 Estimates of Sobolev norms and \(Su\)

In this section, we prove the slow growth of intrinsic Sobolev norms to \(u\) and \(Su\). Throughout Section 2, we essentially only assume \(N\) is a complete Kähler manifold with bounded geometry.

2.1 Preliminaries on Cauchy problem

We recall the global regularity theorem of 1D SMF due to Rodnianski-Rubinstein-Staffilani [36].

**Theorem 2.1 ([36]).** Let \(N\) be a complete Kähler manifold with bounded geometry. Given \(l \geq 2\), assume that \(u_0 \in W^{l, 2}(\mathbb{R}; N)\). Then there exists a unique global solution \(u \in C([0, 1]; W^{l, 2}(\mathbb{R}; N))\) to (1.1).

Recall also that the energy defined by
\[
E(u(t)) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u(t)|^2 dx
\]
conserves along the SMF.

Let \(z\) be any given local complex coordinate of \(N\) near \(Q\) with \(z(Q) = 0\). Assume that
\[
\|z_0\|_{L^2_x} + \|u_0\|_{W^{l, 2}(\mathbb{R}; N)} \ll 1. \tag{2.1}
\]
From Theorem 2.1, initial data \(u_0\) satisfying (2.1) evolves to a global solution of SMF. Moreover, the proof of [36] and assumption (2.1) imply
\[
\|u\|_{C([0, 1]; W^{l, 2}(\mathbb{R}; N))} \leq \|u_0\|_{W^{l, 2}(\mathbb{R}; N)} \ll 1. \tag{2.2}
\]
Let \(\omega \in (0, 1)\) be the radius such that \(\{z \in \mathbb{C} : |z| \leq \omega\}\) lies in the local coordinate chart of \(N\) near \(Q\) which corresponds to \(z = 0\). Assume that \(t_* \in [0, 1]\) be largest time such that
\[
\sup_{t \in [0, t_*]} \|z(t)\|_{L^2_x} \leq \omega.
\]
Theorem 2.1 to the extrinsic function \(z(t, x)\) yields
\[
\sup_{t \in [0, t_*]} \|\partial_z z(t)\|_{L^2_x} \leq \|\partial_z u_0\|_{L^2_x},
\]
\[
\sup_{t \in [0, t_*]} \|\partial_x z(t)\|_{H^1_x} \leq \|u_0\|_{W^{l, 2}(\mathbb{R}; N)}. \tag{2.3}
\]
Moreover, by the SMF equation, one has
\[
\|z(t)\|_{L^2_x}^2 \leq \|z_0\|_{L^2_x}^2 + \int_0^t \|\partial_z z\|_{L^2_x}^2 \|z\|_{L^2_x}^2 ds,
\]
which with (2.3) further shows

\[ \sup_{r \in [0,T]} \|z\|_{L^2_r} \lesssim \|z_0\|_{H^1}. \]

Thus Sobolev embedding yields

\[ \|z\|_{L^p_t(0,T;\mathbb{R}^3)} \lesssim \|z_0\|_{H^1} \lesssim \|z_0\|_{H^1} \ll 1. \]

Therefore, \( T = 1 \), i.e. \( u([0, 1] \times \mathbb{R}) \) lies in the local chart near \( z = 0 \).

Define \( \tilde{T} \in [0, \infty) \) to be the largest time such that

\[ \sup_{r \in [0, \tilde{T}]} \langle t \rangle^k \|z\|_{W^{2,p}} \leq \varepsilon. \]

Then by Sobolev embedding and (2.3), \( \tilde{T} \geq 1 \).

### 2.2 Control of Sobolev norms

**Lemma 2.1.** Let \( u \) be a sufficient regular solution to 1D SMF. Then for each \( k \in \mathbb{N} \) one has

\[
\frac{d}{dt} \int_{\mathbb{R}} |\nabla_x^k \partial_t u|^2 \, dx \leq C_k (\|\partial_t u\|_{L^p_x}^2 + \|\partial_t u\|_{L^{p_j}}^{k+1}) \left( \int_{\mathbb{R}} |\nabla_x^k \partial_t u|^2 \, dx + \left( \int_{\mathbb{R}} |\nabla_x^k \partial_t u|^2 \, dx \right)^{\frac{1}{2}} \right) \tag{2.4}
\]

**Proof.** We compute

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |\nabla_x^k \partial_t u|^2 \, dx = \int_{\mathbb{R}} \langle \nabla_x^k \partial_t u, \nabla_x^k \partial_t u \rangle \, dx.
\]

By the commutating inequality

\[
|\nabla_x^k \partial_t u - \nabla_x^k \partial_t u| \leq \sum_{j \geq 2} \sum_{l_0 + (l_1 + \cdots + (l_{j-1} + k) + 1) \geq 0} |\nabla_x^{l_1} \partial_t u| \cdots |\nabla_x^{l_{j-1}} \partial_t u| |\nabla_x^{l_0} \partial_t u|,
\]

and the equation \( \partial_t u = J \nabla_x \partial_t u \), we get

\[
\int_{\mathbb{R}} \langle \nabla_x^k \partial_t u, \nabla_x^k \partial_t u \rangle \, dx \leq \int_{\mathbb{R}} |\nabla_x^k \partial_t u, J \nabla_x^{k+1} \nabla_x \partial_t u \rangle \, dx + \|\nabla_x^k \partial_t u\|_{L^2_x} \sum_{2 \leq j \geq k+1} \sum_{l_0 + (l_1 + \cdots + (l_{j-1} + k) + 1) \geq 0} |\nabla_x^{l_1} \partial_t u| \cdots |\nabla_x^{l_{j-1}} \partial_t u| \|\nabla_x^{l_0} \partial_t u\|_{L^2_x}. \tag{2.5}
\]

By integration by parts and \( \langle JX, X \rangle = 0 \), (2.5) vanishes. Let \( p_0, p_1, \ldots, p_j \in [1, \infty) \) and \( \theta_0, \ldots, \theta_j \in [0, 1] \) be

\[
\frac{1}{p_0} + \cdots + \frac{1}{p_j} = \frac{1}{2}, \quad \theta_0 = \frac{k + \frac{1}{p_0} - l_0 - 2}{k - \frac{1}{2}}, \quad \theta_b = \frac{k + \frac{1}{p_0} - l_b - 1}{k - \frac{1}{2}}, \quad b = 1, \ldots, j.
\]

Then Hölder inequality and Gagliardo-Nirenberg inequality show (2.6) is dominated by

\[
\|\nabla_x^k \partial_t u\|_{L^2_x} \sum_{2 \leq j \geq k+1} \sum_{l_0 + (l_1 + \cdots + (l_{j-1} + k) + 1) \geq 0} \|\nabla_x^{l_1} \partial_t u\|_{L^{p_0}} \cdots \|\nabla_x^{l_{j-1}} \partial_t u\|_{L^{p_j}} \|\nabla_x^{l_0} \partial_t u\|_{L^{p_j}}^{1 - \theta_j}. \tag{2.6}
\]
It is direct to check
\[
\sum_{b=0}^{j} \theta_b = j + \frac{j - 2}{k - \frac{1}{2}} \in [2, k + 3]
\]
\[
\sum_{b=0}^{j} (1 - \theta_b) = 1 - \frac{j - 2}{k - \frac{1}{2}} \in \left[\frac{1}{2k - 1}, 1\right].
\]

Putting all these together gives our lemma. \hfill \Box

For convenience, let
\[
E_k(u) = \int_{\mathbb{R}} |\nabla_x^k \partial_x u|^2 dx.
\]

Using Lemma 2.1 and Gronwall inequality, we have

**Corollary 2.1.** Let \( u \) be a sufficient regular solution to 1D SMF satisfying
\[
\sup_{0 \leq t \leq T} \langle t \rangle^\frac{k}{2} ||\partial_x u(t)||_{L^\infty_x} \leq \epsilon.
\]

Then for each given \( k \in \mathbb{N} \),
\[
\sup_{t \in [0, T]} \langle t \rangle^{-2k} E_k(u(t)) \leq E_k(u(0)).
\]

**2.3 Control of \( ||S u||_{H^k_x} \)**

Recall that the vector field corresponding to scaling is \( \mathcal{S} := 2t\partial_t + x\partial_x \).

Let’s begin with the evolution of \( ||S u||_{H^k_x} \) along the SMF.

**Lemma 2.2.** If \( u \) solves 1D SMF, then
\[
\frac{d}{dt} ||S u||_{H^k_x}^2 \leq ||\partial_x u||_{L^2_x}^2 ||S u||_{H^k_x}^2. \tag{2.7}
\]

**Proof.** In the following, denote
\[
\langle X, Y \rangle = \int_{\mathbb{R}} h(X, Y) dx,
\]
for \( X, Y \in u^* T N \). Compute
\[
\frac{1}{2} \frac{d}{dt} ||S u||_{H^k_x}^2 = \langle 2\partial_t u, S u \rangle + \langle 2\nabla_x \partial_t u, S u \rangle + \langle x \nabla_x \partial_t u, S u \rangle. \tag{2.8}
\]

By the SMF equation and \( \nabla J = 0 \), the second term of the RHS becomes
\[
\langle 2\nabla_x \partial_t u, S u \rangle = 2t \langle J \nabla_x \partial_t u, S u \rangle = \langle J R(2t\partial_t u, \partial_t u, \partial_t u) \partial_t u, S u \rangle + 2t \langle J \nabla_x \partial_t u, S u \rangle
\]
\[
= \langle J R(2t\partial_t u + x\partial_t u, \partial_t u, \partial_t u) \partial_t u, S u \rangle - \langle J \nabla_x (2t\partial_t u), \nabla_x S u \rangle \tag{2.9}
\]

where in the second line we used \( [\nabla_x, \nabla_x] = R(\partial_t u, \partial_t u) \), and in the third line we applied \( R(\partial_t u, \partial_t u) = 0 \), and integration by parts. Again by integration by parts, \( \nabla J = 0 \), and the SMF equation, the third term on the RHS of (2.8) reduces to
\[
\langle x \nabla_x \partial_t u, S u \rangle = -\langle \partial_t u, S u \rangle + \langle x \partial_t u, \nabla_x S u \rangle
\]
\[
= -\langle \partial_t u, S u \rangle - \langle x \nabla_x \partial_t u, \nabla_x S u \rangle
\]
\[
= -\langle \partial_t u, S u \rangle + \langle J \nabla_x (x \partial_t u), \nabla_x S u \rangle - \langle J \nabla_x (x \partial_t u), \nabla_x S u \rangle - \langle J \nabla_x (x \partial_t u), \nabla_x S u \rangle
\]
\[
= -2\langle \partial_t u, S u \rangle - \langle J \nabla_x (x \partial_t u), \nabla_x S u \rangle. \tag{2.10}
\]
Therefore, (2.8), (2.9), (2.10) together give
\[
\frac{1}{2} \frac{d}{dt} \|S u\|_{L^2_x}^2 = \langle J R(2\partial_t u + x\partial_x u, \partial_x u), S u \rangle - \langle J \nabla_x (x\partial_x u + 2t\partial_t u), \nabla_x S u \rangle \\
= \langle J R(S u, \partial_x u)\partial_x u, S u \rangle - \langle \nabla_x S u, \nabla_x S u \rangle. \tag{2.11}
\]
Since \( \langle J X, X \rangle = 0 \) for any \( X \in u^T N \), the second term of the RHS of (2.11) vanishes. So we conclude
\[
\frac{1}{2} \frac{d}{dt} \|S u\|_{L^2_x}^2 = \langle J R(S u, \partial_x u)\partial_x u, S u \rangle,
\]
from which (2.7) follows.

\[ \square \]

**Lemma 2.3.** If \( u \) solves 1D SMF, then
\[
\frac{d}{dt} \|\nabla_x S u\|_{L^2_x}^2 \leq \|\partial_x u\|_{L^2_x} \|\nabla_x \partial_x u\|_{L^2_x} \|\nabla S u\|_{L^2_x} \|S u\|_{L^2_x} + \|\partial_x u\|_{L^2_x} \|\nabla S u\|_{L^2_x} \|S u\|_{L^2_x} + \|\partial_x u\|_{L^2_x} \|\nabla S u\|_{L^2_x} \|S u\|_{L^2_x}.
\]  

**Proof.** The computation is similar to that of Lemma 2.2. For sake of completeness, we give a detailed proof. As before, in the following, denote
\[
\langle X, Y \rangle = \int_{\mathbb{R}} h(X, Y)dx,
\]
for \( X, Y \in u^T N \). Then one has
\[
\frac{1}{2} \frac{d}{dt} \|\nabla_x S u\|_{L^2_x}^2 = \langle \nabla_x \nabla_x S u, \nabla_x S u \rangle \\
= \langle R(\partial_x u, \partial_x u)S u, \nabla_x S u \rangle + \langle \nabla_x \nabla_x S u, \nabla_x S u \rangle. \tag{2.13}
\]
Since
\[
\nabla_x S u = 2\partial_t u + 2\nabla_x \partial_t u + x\nabla_x \partial_x u,
\]
using integration by parts yields
\[
\langle \nabla_x \nabla_x S u, \nabla_x S u \rangle = 2\langle \nabla_x \partial_t u, \nabla_x S u \rangle + 2\langle \nabla_x \nabla_x \partial_t u, \nabla_x S u \rangle - \langle x\nabla_x \partial_t u, \nabla_x^2 S u \rangle. \tag{2.14}
\]
By the SMF equation and \( \nabla J = 0 \), the second term of the RHS of (2.14) becomes
\[
2t\langle \nabla_x \nabla_x \partial_t u, \nabla_x S u \rangle = 2t\langle J \nabla_x \nabla_x \partial_t u, \nabla_x S u \rangle \\
= -2t\langle J \nabla_x \nabla_x \partial_t u, \nabla_x^2 S u \rangle \\
= -\langle J R(2\partial_x u, \partial_x u)\partial_x u, \nabla_x^2 S u \rangle - 2t\langle J \nabla_x \nabla_x \partial_t u, \nabla_x^2 S u \rangle \\
= -\langle J \nabla_x^2 (2\partial_t u), \nabla_x^2 S u \rangle, \tag{2.15}
\]
where we used \( R(X, X) = 0 \) again, and applied \( \nabla_x \partial_t u = \nabla_x \partial_x u \). Meanwhile, by the SMF equation and \( \nabla J = 0 \), the third term of the RHS of (2.14) can be rewritten as
\[
-\langle x\nabla_x \partial_t u, \nabla_x^2 S u \rangle = -\langle x\nabla_x (x\partial_x u), \nabla_x^2 S u \rangle + \langle \partial_t u, \nabla_x^2 S u \rangle \\
= -\langle J \nabla_x (x\partial_x u), \nabla_x^2 S u \rangle + \langle \partial_t u, \nabla_x^2 S u \rangle + \langle J \nabla_x \partial_x u, \nabla_x^2 S u \rangle \\
= -\langle J \nabla_x^2 (x\partial_x u), \nabla_x^2 S u \rangle + 2\langle \partial_t u, \nabla_x^2 S u \rangle. \tag{2.16}
\]
Combining (2.14), (2.15), (2.16) and (2.13) gives
\[
\frac{1}{2} \frac{d}{dt} \|\nabla_x S u\|_{L^2_x}^2 = -\langle J R(S u, \partial_x u)\partial_x u, \nabla_x^2 S u \rangle + \langle R(\partial_t u, \partial_x u)S u, \nabla_x S u \rangle \\
- \langle J \nabla_x^2 (S u), \nabla_x^2 S u \rangle + 2\langle \partial_t u, \nabla_x^2 S u \rangle + 2\langle \nabla_x \partial_t u, \nabla_x S u \rangle.
\]
The second line of the RHS vanishes by integration by parts and $\langle JX, X \rangle = 0$. Hence one has

$$\frac{1}{2} \frac{d}{dt} \|\nabla_S u\|_{L^2}^2 = -\langle JR(S u, \partial_t u)\partial_t u, \nabla_S^2 u \rangle + \langle R(\partial_t u, \partial_t u)S u, \nabla_S u \rangle.$$ 

For the first term on the RHS, integration by parts shows

$$-\langle JR(S u, \partial_t u)\partial_t u, \nabla_S^2 u \rangle = \langle JR(\nabla_S u, \partial_t u)\partial_t u, \nabla_S u \rangle + \langle JR(S u, \nabla_S u)\partial_t u, \nabla_S u \rangle$$

$$+ \langle JR(S u, \partial_t u)\nabla_S \partial_t u, \nabla_S u \rangle - \langle \nabla_S u, \partial_t u, J \nabla_S u; \partial_t u \rangle.$$

Then (2.12) follows. □

A longer but essentially the same proof of Lemma 2.3 gives the corresponding result of $\nabla_S^2 u$ as follows.

**Lemma 2.4.** If $u$ solves 1D SMF, then we have

$$\frac{d}{dt} \|\nabla_S^2 u\|_{L^2}^2 \leq (\|\partial_t u\|_{L^\infty}^2 + \|\partial_t u\|_{L^2}^2)\|\nabla_S u\|_{L^2}^2 + \|\partial_t u\|_{L^2}^2 \|\nabla_S^2 u\|_{L^2}^2$$

and

$$\frac{d}{dt} \|\nabla_S^3 u\|_{L^2}^2 \leq \|\partial_t u\|_{L^2}^2 \|\nabla_S^3 u\|_{L^2}^2$$

Then by Lemma 2.2, Lemma 2.3, Lemma 2.4 and Gronwall inequality, we have

**Corollary 2.2.** If $u$ solves 1D SMF with sufficient regular initial data, and assume that for all $t \in [0, T]$

$$\|\nabla_S \partial_t u\|_{L^\infty} + \|\partial_t u\|_{L^\infty} \leq \epsilon(t) \leq \epsilon^4,$$  

(2.17)

then for $n_\ast$ sufficiently large there exists a sufficiently small constant $\delta$ depending only on $n_\ast$, $\epsilon$ such that

$$\sum_{i=0,1,2,3} \sup_{t \in [0,T]} (1+t)^{-\delta} \|\nabla_S^i u(t)\|_{L^2} \leq \sum_{j=1}^4 \|\langle x \rangle \nabla_S^j u_0\|_{L^2}.$$  

(2.18)

**Proof.** By Gagliardo-Nirenberg inequality and Kato inequality, one has

$$\|\nabla_S^2 \partial_t u\|_{L^\infty} \leq \|\nabla_S \partial_t u\|_{L^\infty}^2 \|\nabla_S^{n-1} \partial_t u\|_{L^2}^{1-\theta}$$

and

$$\|\nabla_S^3 \partial_t u\|_{L^\infty} \leq \|\nabla_S \partial_t u\|_{L^\infty}^2 \|\nabla_S^{n-1} \partial_t u\|_{L^2}^{1-\gamma},$$

where $\theta = (n_\ast - 3.5)/(n_\ast - 2.5)$, and $\gamma = (n_\ast - 4.5)/(n_\ast - 2.5)$. Using (2.17) and Lemma 2.2, we see if $n_\ast$ is large, then $\|\nabla_S^2 \partial_t u\|_{L^\infty}$ and $\|\nabla_S^3 \partial_t u\|_{L^\infty}$ decay as $t^{-\delta}$. Then (2.18) follows by Lemma 2.3, Lemma 2.4 and Gronwall inequality. □

**3 Holomorphic transformation and Some algebraic facts**

In this section, the assumption (1.3) is not needed. The results hold for general Riemannian targets.

The holomorphic transformation

$$w := z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4$$
plays a vital role in the whole proof. We need to carefully calculate the corresponding coefficients to this transform. Expanding (1.19) gives

\[
(\bar{\partial} \gamma + \Delta \gamma) = c_0(\partial \gamma)^2 + c_1 \bar{\gamma}(\partial \gamma)^2 + c_2 \gamma \partial_\gamma^2 + c_3 \gamma^2(\partial_\gamma)^2 + c_4 \gamma^3(\partial_\gamma)^2 + c_5 \gamma^2(\partial_\gamma)^2 + O(|\gamma|^3)(\partial_\gamma)^2.
\]

(3.1)

Then one has

\[
(\bar{\partial} \gamma + \Delta \gamma) = (i\bar{\partial} \gamma + \Delta \gamma) + 2i\bar{\gamma} \partial_\gamma z + 3i\bar{\gamma} \gamma^2 \partial_\gamma z + \Delta \gamma + \gamma_1 \Delta z^2 + \gamma_2 \Delta z^3
\]

\[
= (i\bar{\partial} \gamma + \Delta \gamma) + 2\gamma_1 \bar{\gamma} \partial_\gamma z + 2\gamma_1 \gamma \Delta z + 2\gamma_1 (\partial_\gamma z)^2
\]

\[
+ 3\gamma_2 \bar{\gamma} z + 3\gamma_2 \gamma^2 \Delta z + 6\gamma_2 (\partial_\gamma z)^2
\]

\[
+ 4\gamma_3 \bar{\gamma} z + 4\gamma_3 \gamma^2 \Delta z + 12\gamma_3 \gamma^2 (\partial_\gamma z)^2
\]

Therefore,

\[
(\bar{\partial} \gamma + \Delta \gamma) = (i\bar{\partial} \gamma + \Delta \gamma) = (c_0 + c_1 \bar{\gamma} + c_2 \gamma + c_3 \gamma^2 + c_4 \gamma^3 + c_5 \gamma^4)(\partial_\gamma z)^2
\]

\[
+ 2\gamma_1 \gamma \gamma_0 + 6\gamma_2 \gamma^2 (\partial_\gamma z)^2 + 2\gamma_1 (\partial_\gamma z)^2 + 3\gamma_0 \gamma \gamma^2 (\partial_\gamma z)^2
\]

\[
+ 6\gamma_2 \gamma^2 (\partial_\gamma z)^2 + 12\gamma_3 \gamma^2 (\partial_\gamma z)^2 + O(|\gamma|^3)(\partial_\gamma z)^2).
\]

In order to cancel quadratic terms and the \(z(\partial_\gamma z)^2\) term, we take

\[
\begin{align*}
2\gamma_1 + c_0 &= 0, \\
c_2 + 2\gamma_1 c_0 + 6\gamma_2 &= 0, \\
12\gamma_3 + c_3 + 2\gamma_0 c_2 + 3\gamma_2 c_0 &= 0.
\end{align*}
\]

(3.2)

Now, for the chosen \(\gamma_1, \gamma_2, \gamma_3\), we conclude

\[
(\bar{\partial} \gamma + \Delta \gamma) = c_1 \bar{\gamma}(\partial_\gamma z)^2 + (c_5 - 2\gamma_1 c_1) \gamma(\partial_\gamma z)^2 + c_4 \gamma^2 (\partial_\gamma z)^2 + O(|\gamma|^3)(\partial_\gamma z)^2).
\]

Since \(w = z + \gamma_1 \gamma^2 + \gamma_2 \gamma^3 + \gamma_3 \gamma^4\), at \(w = 0\) one gets

\[
dz{w} = 1, \quad \frac{d^2 z}{dw^2} = -2\gamma_1.
\]

Thus

\[
z = w - \gamma_1 \gamma^2 + O(w^3).
\]

Then we see

\[
c_1 \bar{\gamma}(\partial_\gamma z)^2 = c_1 \bar{\gamma}(\partial_\gamma w)^2 - c_1 \bar{\gamma} \gamma \gamma^2 (\partial_\gamma w)^2 - 4c_1 \gamma \gamma^2 (\partial_\gamma w)^2 + O(|\gamma|^3)(\partial_\gamma w)^2).
\]

So we have

\[
(\bar{\partial} \gamma + \Delta \gamma) = c_1 \bar{\gamma}(\partial_\gamma w)^2 + (c_5 - 2\gamma_1 c_1) \gamma \gamma^2 (\partial_\gamma w)^2 + (c_4 - c_1 \gamma \gamma^2)(\partial_\gamma w)^2
\]

\[
+ O(|\gamma|^3)(\partial_\gamma w)^2).
\]

(3.3)

Lemma 3.1. Let \(\nu_2 = c_5 - 2\gamma_1 c_1, \nu_3 = c_4 - c_1 \gamma \gamma^2\). Then

\[
\nu_2 = 2\nu_3.
\]
Proof. Note that \( c_4, c_5 \) defined by (3.1) are indeed given by
\[
\begin{align*}
  c_5 &= [\ln h(z)]_{\zbar z}(0, 0); \\
  c_4 &= \frac{1}{2} [\ln h(z)]_{\zbar z}(0, 0).
\end{align*}
\]  
Hence, there holds
\[
c_5 = 2c_4.
\]
And note that
\[
c_1 = [\ln h(z)]_{\zbar z}(0, 0) = \frac{1}{4} (\partial_1^2 + \partial_2^2)(\ln h)(0, 0)
\]
is a real constant. Using these two facts we see
\[
c_5 - 2\gamma_1 c_1 = 2c_4 - 2\gamma_1 c_1, \tag{3.4}
\]
i.e. \( \nu_2 = 2\nu_3 \).
□

**Lemma 3.2.** Let \( N \) be a Riemannian surface and \( Q \in N \). Assume that \( z \) is a local complex coordinate of \( N \) near \( Q \) with \( z(Q) = 0 \), and the metric of \( N \) writes as \( h(z, \zbar z) dz d\zbar z \) near \( Q \). Then the constant \( c_1 \) in the RHS of (3.3) satisfies
\[
c_1 = -\frac{1}{2} K(Q) h_0,
\]
where \( K(Q) \) denotes the sectional curvature at \( Q \), \( h_0 = h(0, 0) \) denotes the metric at \( Q \) under the coordinate \( z \).

**Proof.** The Laplace-Beltrami operator with respect to the metric \( h dz d\zbar z \) is given by
\[
\Delta_N = \frac{4}{h} \frac{\partial}{\partial z} \frac{\partial}{\partial \zbar z}.
\]
The sectional curvature at \( z \) is
\[
K(z) = -\Delta_N (\ln h^\frac{1}{2}).
\]
Note that
\[
c_1 = [\ln h(z)]_{\zbar z}(0, 0).
\]
So one has
\[
c_1 = [\ln h(z)]_{\zbar z}(0, 0) = -\frac{1}{2} K(Q) h_0.
\]
□

**Remark 3.1.** Note that \( h_0 \) in Lemma 3.2 is the same under coordinate \( z \) and coordinate \( w = z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4 \), since \( \frac{\partial w}{\partial c}(0) = 1 \).

### 4 Mass conservation

In this section, the assumption (1.3) is not needed. The results hold for general Riemannian targets.

Recall that \( w(t, x) := z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4 \). For simplicity, write (3.3) as
\[
(i\partial_t + \Delta w) = (v_1 w + v_2 |w|^2 + v_3 w^3)(\partial_{\zbar w} w)^2 + O(|w|^4)(\partial_{\zbar w} w)^2
\]
\[
:= [v_1 \zbar w + g(w, \zbar w)] (\partial_{\zbar w} w)^2 \tag{4.1}
\]
Let
\[
H(w, \zbar w) = 1 - v_1 |w|^2 \tag{4.2}
\]
Lemma 4.1. \(H(w, \overline{w})\) defined by (4.2) is real valued and satisfies
\[
H(w, \overline{w})(v_1 \overline{w} + g(w, \overline{w})) + H_w(w, \overline{w}) = O(|w|^2).
\] (4.3)

Proof. This is a direct calculation. \(\square\)

For the real valued function \(H : \mathbb{C} \to \mathbb{R}\) define in (4.2), consider the functional
\[
\mathcal{H}(w) = \int_{\mathbb{R}} H(w, \overline{w})|w|^2 dx.
\] (4.4)

We will prove for this well chosen \(H\), \(\mathcal{H}(w)\) behaves well along the SMF.

Lemma 4.2. Let \(H(w, \overline{w})\) be defined as above. Assume that \(|w|_{L^\infty_{x, t}(\mathbb{R}^2 \times [0,T])} \leq \omega \ll 1\). Then for all \(t \in [0,T]\)
\[
\frac{d}{dt} \mathcal{H}(w) \leq ||w||^3_{\dot{W}^1_{x, t}} \mathcal{H}(w).
\]

Proof. By definition of \(\mathcal{H}(w)\),
\[
\frac{d}{dt} \mathcal{H}(w) = \int_{\mathbb{R}} (H_w \partial_x w + H_{\overline{w}} \partial_x \overline{w})|w|^2 dx + \int_{\mathbb{R}} H(w, \overline{w})[(\partial_t w)\overline{w} + w \partial_t \overline{w}] dx.
\] (4.5)

(4.1) implies
\[
H_w \partial_x w + H_{\overline{w}} \partial_x \overline{w}
= iH_w \Delta w - iH_{\overline{w}} \Delta \overline{w}
= -iH_w(v_1 \overline{w} + g(w, \overline{w}))(\partial_x w)^2 + iH_{\overline{w}}(v_1 w + \overline{g}(w, \overline{w}))(\partial_x \overline{w})^2.
\] (4.6)

and
\[
\overline{w} \partial_x w + w \partial_x \overline{w}
= i\overline{w} \Delta w - iw \Delta \overline{w}
= -i(g(w, \overline{w}) + v_1 \overline{w})(\partial_x w)^2 \overline{w} + i(\overline{g}(w, \overline{w}) + v_1 w)(\partial_x \overline{w})^2 w.
\] (4.7)

The contributions of the two terms of (4.7) are easy to bound, in fact, one has
\[
\int_{\mathbb{R}} |H_w(v_1 \overline{w} + g(w, \overline{w}))(\partial_x w)^2||w|^2 dx \leq ||w||^3_{\dot{W}^1_{x, t}} ||w||^2_{L_2}.
\]

For the first two terms of RHS of (4.6), integration by parts gives
\[
\int_{\mathbb{R}} [iH_w \Delta w - iH_{\overline{w}} \Delta \overline{w}]|w|^2 dx
= \int_{\mathbb{R}} (-iH_w \partial_x w + iH_{\overline{w}} \partial_x \overline{w})[(\partial_x w)\overline{w} + w \partial_x \overline{w}] dx
- \int_{\mathbb{R}} [i(\partial_x H_w)\overline{w} \partial_x w - i(\partial_x H_{\overline{w}})w \partial_x \overline{w}]|w|^2 dx.
\] (4.10)

Denote the term (4.10) by II. Then II is dominated by
\[
||\text{II}|| \leq ||w||^3_{\dot{W}^1_{x, t}} ||w||^2_{L_2}.
\]

Again by integration by parts, the integral associated with the RHS of (4.8) becomes
\[
\int_{\mathbb{R}} H(w, \overline{w})[i\overline{w} \Delta w - iw \Delta \overline{w}] dx
= \int_{\mathbb{R}} (H_w \partial_x w + H_{\overline{w}} \partial_x \overline{w})[-i(\partial_x w)\overline{w} + iw \partial_x \overline{w}] dx.
\]
Thus the RHS of (4.5) equals

\[
\int_{\mathbb{R}} (H_u \partial_t w + H_{\bar{w}} \partial_t \bar{w}) w \overline{w} dx + \int_{\mathbb{R}} H(w, \bar{w})[(\partial_t w)\overline{w} + \overline{w} \partial_t \overline{w}] dx
\]

\[
= \int_{\mathbb{R}} (-iH_u \partial_t w + iH_{\bar{w}} \partial_t \bar{w})[(\partial_t w)\overline{w} + \overline{w} \partial_t \overline{w}] dx
\]

\[
+ \int_{\mathbb{R}} (H_u \partial_t w + H_{\bar{w}} \partial_t \bar{w})[-i(\partial_t w)\overline{w} + i\overline{w} \partial_t \overline{w}] dx
\]

\[
+ \int_{\mathbb{R}} H(w, \bar{w})[i(g(w, \bar{w}) + \nu_1 w)(\partial_t w)^2] dx + I + II,
\]

where I and II denote the last terms of (4.7) and (4.10) respectively. Notice that (4.3) shows

\[
H(w, \bar{w})(g(w, \bar{w}) + \nu_1 \overline{w}) + H_u(w, \overline{w}) = O(|w|^2).
\]

Then, we arrive at

\[
\int_{\mathbb{R}} (H_u \partial_t w + H_{\bar{w}} \partial_t \bar{w}) w \overline{w} dx + \int_{\mathbb{R}} H(w, \bar{w})[(\partial_t w)\overline{w} + \overline{w} \partial_t \overline{w}] dx
\]

\[
= I + II + \int_{\mathbb{R}} O(|w|^3)|\partial_t w|^2 dx.
\]

Therefore, one obtains

\[
\frac{d}{dt} \mathcal{H}(w) \leq ||w||_{L^3_{t}}^{3} ||w||_{L^2_{t}}^{2}.
\]

(4.11)

Since \( ||w||_{L^3_{t}} \leq \omega \), one has

\[
\mathcal{H}(w) \sim ||w||_{L^2_{t}}^{2}.
\]

Then the lemma follows by (4.11).

Corollary 2.1 and Corollary 2.2 are for the intrinsic quantities. We need to transfer them to the extrinsic function \( w(t, x) \). The bridge is the global bound of mass.

**Corollary 4.1.** If \( u \) solves SMF with initial data fulfilling (1.4) and

\[
\sup_{t \in [0, T]} (1 + t)^{\frac{1}{2}}(||\partial_t u||_{L^p_{t}} + ||\nabla_x \partial_t u||_{L^p_{t}}) \leq \epsilon.
\]

(4.12)

Then for \( n_* \), sufficiently large there exists a sufficiently small \( \delta \) depending only on \( n_* \), \( \epsilon \) such that for all \( t \in [0, T] \)

\[
\sup_{t \in [0, T]} (t)^{-\delta}||S w(t)||_{L^{p_{*}'}_{t}} \leq ||w_0||_{L^{p_{*}'}_{t_{*}}}.
\]

(4.13)

\[
\sup_{t \in [0, T]} (t)^{-\delta}||w(t)||_{L^{p_{*}'}_{t}} \leq ||w_0||_{L^{p_{*}'}_{t_{*}}},
\]

(4.14)

**Proof.** Set

\[
0 < \epsilon_* \ll \epsilon \ll \omega \ll 1.
\]

Let \( T'_* \in [0, T] \) be the maximal time such that

\[
||w(t, x)||_{L^{p_{*}'}_{t}(0, T'_*; \mathbb{R})} \leq \omega.
\]

By (1.4) and the local Cauchy theorem, \( T'_* > 0 \). By Lemma 4.2, assumption (4.12), and Gronwall inequality, one has

\[
||w||_{L^p_{t}(0, T'_*; \mathbb{R})} \leq ||w_0||_{L^p_{t}} \leq \epsilon_*.
\]

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which together with (4.12) gives
\[\|w\|_{L^2_t([0,T]\times\mathbb{R}^3)} \leq \|\partial_x w\|_{L^2_t}^\frac{4}{3}, \|w\|_{L^2}^\frac{3}{2} \leq \epsilon \ll \omega.\]

So \(T' = T\), i.e. \(u([0, T] \times \mathbb{R})\) lies in a local chart of \(\mathcal{N}\) around \(w = 0\). And thus all the Christoffel symbols and their k-th order derivatives with respect to \(x\) are uniformly bounded in \((t, x) \in [0, T'] \times \mathbb{R}\). Then writing \(\nabla_x \partial_i u\) and \(\partial_i u\) in the local coordinate \(z\) shows that (4.12) implies (2.17). Then Corollary 2.2 yields bounds of \(\|\nabla_x^2 u(t)\|_{L^2_t}^2\). Again expressing \(\nabla_x^2 u\) in the local coordinate \(w\) gives the desired result (4.13). Similarly (4.14) follows by Corollary 2.1 and the mass bound. \(\square\)

We will also need the upper-bound of \(\|\langle x \rangle w\|_{H^1}\). The proof is a simple energy argument.

**Lemma 4.3.** Assume that
\[\|w\|_{H^1} + \sup_{t \geq 0} (t)^\frac{3}{2}\|w\|_{W^{2,\infty}} \leq \epsilon. \tag{4.15}\]

Then we have
\[\|xw\|_{L^2_t} \leq ct \tag{4.16}\]
\[\|x\partial_x w\|_{L^2_t} \leq ct^\frac{3}{2}. \tag{4.17}\]

**Proof.** By (4.15) and integration by parts,
\[\frac{d}{dt}\|xw\|_{L^2_t}^2 \leq \|\partial_x w\|_{L^2_t}\|xw\|_{L^2_t} + \|\partial_x w\|_{L^2_t}\|xw\|_{L^2_t}.\]

Then (4.16) follows by Gronwall inequality, (4.15) and energy conservation.

And we also have by (4.15) and integration by parts that
\[\frac{d}{dt}\|x\partial_x w\|_{L^2_t}^2 \leq \|\partial_x^2 w\|_{L^2_t}\|x\partial_x w\|_{L^2_t} + \|\partial_x^2 w\|_{L^2_t}\|x\partial_x w\|_{L^2_t}^2.\]

Thus (4.17) follows by Gronwall inequality, (4.15) and the bound \(\|\partial_x^2 w\|_{L^2_t} \leq ct^\epsilon.\) \(\square\)

## 5 Intrinsic Vanishing condition

First, we prove Lemma 1.1, i.e. the assumption (1.3) is invariant under holomorphic coordinate transformation. Hence, (1.3) is an intrinsic geometric assumption rather than analytic extrinsic assumption.

We first prove the following result.

**Lemma 5.1.** Suppose that \(\mathcal{N}\) is a Riemannian surface and \(Q\) is a given point in \(\mathcal{N}\). Let \(\eta\) be a local coordinate of \(\mathcal{N}\) near \(Q\) with \(\eta(Q) = \eta_0\). The metric in coordinate \(\eta \) writes as \(h(\eta, \bar{\eta})d\eta d\bar{\eta}\). Assume that \(\eta = f(z)\) is a local holomorphic transformation which maps \(z_0\) to \(\eta_0\) and \(f_z(z_0) \neq 0\). Let \(\tilde{h}(z, \bar{z})dzd\bar{z}\) be the metric under the \(z\) coordinate. Then near \(z_0\) one has
\[\left([\ln \tilde{h}], [\ln \tilde{h}]/z \right) = \left(\frac{f_z f_z}{f_z}, \left(\frac{[\ln h]}{f_z} + [\ln h]/\eta f_f\right) \right).\]

**Proof.** The metric under the \(z\) coordinate shall be \(h(f(z), \bar{f}(z))f_z f_{\bar{z}} dzd\bar{z}\). Hence,
\[\tilde{h} = h(f(z), \bar{f}(z))f_z f_{\bar{z}}.\]

By computation, we get
\[\left[\ln \tilde{h}\right] = \frac{h h_f f_z}{f_z}, \quad \left[\ln \tilde{h}\right]/z = \frac{h h_f f_z}{f_z} \frac{f_z}{f_z}.\]
and

\[
[\ln \tilde{h}]_{c_{zz}} = h^{-2} \left( h_{\eta \eta} f_z(f_z)^2 h + h_{\eta \eta} f_z f_z \eta h + h_{\eta \eta} f_z f_z h - h_{\eta \eta} f_z f_z - h_{\eta \eta} h_{\eta \eta} f_z(f_z)^2 \right)
\]

Therefore, we see

\[
[\ln \tilde{h}] [\ln \tilde{h}]_{c_{zz}} - [\ln \tilde{h}]_{c_{zz}} = f_z(f_z)^2 \left( [\ln h]_\eta [\ln h]_{\eta \eta} - [\ln h]_{\eta \eta} \right) + f_z(f_z)^2 \left( [\ln h]_{c_{zz}} - \frac{h_{\eta \eta} f_z f_z h - h_{\eta \eta} h_{\eta \eta} f_z(f_z)^2}{h^2} \right)
\]

\[
= f_z(f_z)^2 \left( [\ln h]_\eta [\ln h]_{\eta \eta} - [\ln h]_{\eta \eta} \right).
\]

Proof of Lemma 1.1. Lemma 1.1 follows directly by Lemma 5.1.

Proof of Proposition 1.1. Let \((U_i, \varphi_i); 1 \leq i \leq n\) be a coordinate chart of \(\mathcal{N}\) for which each \(\varphi_i : U_i \rightarrow \mathbb{C}\) is a homeomorphism from the open subset \(U_i\) of \(\mathcal{N}\) to its image in \(\mathbb{C}\), so that \(\cup U_i\) is a covering of \(\mathcal{N}\), and for each \(i, j \in \{1, ..., n\}\) and \(U_i \cap U_j \neq \emptyset\), the map \(\varphi_j \varphi_i^{-1}\) is a holomorphic map from \(\varphi_j(U_i \cap U_j)\) to \(\varphi_i(U_i \cap U_j)\). In each given open set \(U_i\), define the 1-form

\[
[\ln h] [\ln h]_{c_{zz}} - [\ln h]_{c_{zz}} = \frac{[\ln h]_\eta [\ln h]_{\eta \eta} - [\ln h]_{\eta \eta}}{h(f_0, f_0)} dz + \frac{[\ln h]_z [\ln h]_{c_{zz}} - [\ln h]_{c_{zz}}}{h(f_0, f_0)} dz.
\]

Lemma 1.1 implies that these (real valued) 1-form coincide with each other in \(U_i \cap U_j\). So, one gets a globally defined 1-form \(\omega\) on \(\mathcal{N}\). By duality, we have a globally defined real valued vector field \(X\) on \(\mathcal{N}\). Since the Euler-Poincare characteristic of \(\mathcal{N}\) is non-zero, Poincare-Hopf index theorem implies that there exists at least one point \(Q \in \mathcal{N}\) such that \(X(Q) = 0\). And thus dy duality, \(\omega(Q) = 0\). Let

\[
\frac{[\ln h]_\eta [\ln h]_{\eta \eta} - [\ln h]_{\eta \eta}}{h(f_0, f_0)} = f + ig,
\]

where \(f\) and \(g\) denote the real and respectively imaginary parts. Denote also \(z = x + iy\). Then the 1-form \(\omega\) writes as

\[
2(f dx - g dy).
\]

Thus we infer from \(\omega(Q) = 0\) that \(f(Q) = g(Q) = 0\), i.e.

\[
([\ln h]_\eta [\ln h]_{\eta \eta} - [\ln h]_{\eta \eta})(Q) = 0.
\]

\]

Corollary 5.1. Suppose that \(\mathcal{N}\) is a Riemannian surface and \(Q\) is a given point in \(\mathcal{N}\). Let \(z\) be a local coordinate of \(\mathcal{N}\) near \(Q\) which corresponds to \(z = 0\). Assume further that (1.3) holds. Then the constants \(\nu_2, \nu_3\) in (4.1) fulfills

\[
\nu_2 = 0 = \nu_3.
\]

Proof. By Lemma 3.1, it suffices to prove

\[
\nu_2 = c_5 - 2\gamma_1 c_1 = 0.
\]

The first line of equation (3.2) shows

\[
c_5 - 2\gamma_1 c_1 = c_5 - c_0 c_1.
\]

Note that

\[
c_5 - c_0 c_1 = [\ln h]_{c_{zz}}(0) - [\ln h]_{c_{zz}}(0)[\ln h]_z(0).
\]

So (1.3) implies \(c_5 - c_0 c_1 = 0\), and thus \(\nu_2 = 0 = \nu_3\).
Lemma 5.2. Assume that $N$ is a Riemannian surface with metric $h(z, \bar{z})dzd\bar{z}$.

(i) If $N$ is rotationally symmetric near $Q$, i.e. $h(z, \bar{z}) = g(|z|^2)$ for some function $g$ near $Q$ with coordinate $z = 0$, then $Q$ is an intrinsic vanishing point.

(ii) Each point of $S^2$ or $\mathbb{H}^2$ is an intrinsic vanishing point.

Proof. (i) follows by direct calculation. (ii) follows by the symmetry of $S^2$, $\mathbb{H}^2$ and (i). \hfill \Box

6 Estimates of $Lw$.

In this section, we assume $Q$ is an intrinsic vanishing point. Recall the operator $L := ix - 2i\partial_x$. Recall also that $w(t, x) := z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4$, and it solves (4.1). Under the assumption (1.3), Corollary 5.1 shows the constants $\nu_2, \nu_3$ in (4.1) fulfill $\nu_2 = 0 = \nu_3$.

Thus $w$ satisfies
\begin{equation}
\label{6.1}
i\partial_t w + \Delta w = (v_1 \bar{w} + g(w, \bar{w})) (\partial_x w)^2,
\end{equation}
where $g(w, \bar{w})$ is of order 3 or higher in $w$.

And the function $H(w, \bar{w})$ defined by (4.2) now fulfills
\begin{equation}
\label{6.2}
H(w, \bar{w})(v_1 \bar{w} + g(w, \bar{w})) + H_\nu(w, \bar{w}) = O(|w|^3).
\end{equation}

Applying $L$ to both sides of (6.1) yields
\begin{equation}
\label{6.3}
i\partial_t Lw + \Delta Lw = v_1 Lw\partial_x \partial_t w + 2v_1 \bar{w} \partial_x(Lw)\partial_t w + (g_\nu Lw + g_\bar{w} Lw)(\partial_x w)^2 \\
+ 2g(w, \bar{w})\partial_x \partial_t(Lw) - 2((v_1 \bar{w} + g(w, \bar{w})) \partial_t w + O(|x||w|^3|\partial_x w|^2).
\end{equation}

For the real valued function $H : \mathbb{C} \to \mathbb{R}$ define in (4.2), consider the functional
\begin{equation}
\label{6.4}
\mathcal{L}(w) = \int_{\mathbb{R}} H(w, \bar{w}) Lw \bar{dx}.
\end{equation}

We will prove $\mathcal{L}(w)$ behaves well along the SMF.

Lemma 6.1. Assume that $Q$ is an intrinsic vanishing point. Let $w$ and $H(w, \bar{w})$ be defined as above. Assume that $\|w\|_{L^2([0, T] \times \mathbb{R})} \leq \omega \ll 1$. Then for all $t \in [0, T]$ \begin{equation}
\frac{d}{dt} \|w\|_{L^2([0, T] \times \mathbb{R})} \leq L(w) + \|x\|_{L_2} \|w\|_{L_4} \sqrt{L(w)}
\end{equation}\begin{equation}
+ \|w\|_{L_4} \sqrt{\|\partial_x^2 w\|_{L_2} + \|\partial_t^2 w\|_{L_2} + \|\partial_x \nu \|_{L_2} \sqrt{L(w)}).
\end{equation}

Proof. By definition of $\mathcal{L}(w)$,
\begin{equation}
\frac{d}{dt} \mathcal{L}(w) = \int_{\mathbb{R}} (H_\nu \partial_\bar{w} + H_\bar{w} \partial_x \bar{w}) Lw \bar{dx} + \int_{\mathbb{R}} H(w, \bar{w})(\partial_x Lw) \bar{Lw} + Lw \partial_t \bar{Lw} \bar{dx}.
\end{equation}

(6.1) and (6.3) imply
\begin{equation}
\label{6.5}
H_\nu \partial_\bar{w} + H_\bar{w} \partial_x \bar{w} = iH_\nu \Delta w - iH_\bar{w} \Delta \bar{w} - iH_\nu (v_1 \bar{w} + g(w, \bar{w})) (\partial_x w)^2 + iH_\bar{w} (v_1 \bar{w} + g(w, \bar{w})) (\partial_x \bar{w})^2.
\end{equation}

\begin{equation}
\label{6.6}
\bar{Lw} \partial_t \bar{Lw} + (Lw) \partial_t \bar{Lw} = 2(v_1 \bar{w} + g(w, \bar{w})) \bar{w} \partial_x \bar{w} Lw + 2(v_1 \bar{w} + g(w, \bar{w})) \bar{w} \partial_x \bar{w} Lw
\end{equation}\begin{equation}
\label{6.7} + 2i(g(w, \bar{w}) + v_1 \bar{w}) \partial x_L \bar{Lw} (\partial_x Lw) + 2i(g(w, \bar{w}) + v_1 \bar{w}) \partial_x \bar{w} (\partial_x \bar{Lw}) Lw
\end{equation}\begin{equation}
\label{6.8} - 2i(g(w, \bar{w}) + v_1 \bar{w}) \partial x_L \bar{Lw} (\partial_x Lw) + 2i(g(w, \bar{w}) + v_1 \bar{w}) \partial_x \bar{w} (\partial_x \bar{Lw}) Lw
\end{equation}\begin{equation}
\label{6.9} \label{-i(g(w, \bar{w}) + v_1 \bar{w}) \partial x_L \bar{Lw} (\partial_x Lw) + 2i(g(w, \bar{w}) + v_1 \bar{w}) \partial_x \bar{w} (\partial_x \bar{Lw}) Lw}
\end{equation}\begin{equation}
\label{6.10} + O(|x||w|^3|\partial_x w|^2|Lw|).
\end{equation}
The contributions of (6.10), (6.7), line (6.9) and the last two terms of (6.6) are easy to bound, in fact, one has
\[ \int_{\mathbb{R}} |H_{w}(v_{1}\tilde{w} + g(w, \tilde{w}))|\partial_{1}w|^{2}|Lw|^{2}dx + \int_{\mathbb{R}} |H_{w}((g_{w}Lw + g_{w}\overline{Lw})\partial_{1}w)\partial_{1}w|^{2}\overline{Lw}|dx \
+ \int_{\mathbb{R}} 2|H_{w}(\tilde{w})||v_{1}\tilde{w} + g(w, \tilde{w})|w\partial_{1}w\overline{Lw}|dx + \int_{\mathbb{R}} |x||H_{w}(\tilde{w})||w||\partial_{1}w||Lw|dx \leq ||w||^{2}_{w^{1,\infty}}||Lw||^{2}_{L^{2}} + ||Lw||_{L^{2}}||w||^{2}_{w^{1,\infty}} + ||xw||_{L^{2}}||Lw||||w||^{4}_{w^{1,\infty}}.\]

For the first two terms of RHS of (6.6), integration by parts gives
\[ \int_{\mathbb{R}} [i\partial_{1}w - iH_{\partial_{1}\tilde{w}}]|Lw|^{2}dx = \int_{\mathbb{R}} (-iH_{\partial_{1}w} + iH_{\partial_{1}\tilde{w}})[(\partial_{1}Lw)\overline{Lw} + Lw\partial_{1}\overline{Lw}]dx - \int_{\mathbb{R}} [i(\partial_{1}H_{\partial_{1}w})\tilde{w}\partial_{1}w - i(\partial_{1}H_{\partial_{1}\tilde{w}})w\partial_{1}\overline{\tilde{w}}]|Lw|^{2}dx - \int_{\mathbb{R}} |\partial_{1}w|^{2}[iH_{\partial_{1}w} - iH_{\partial_{1}\tilde{w}}]|Lw|^{2}dx.\]

And three further holds
\[ \int_{\mathbb{R}} [i\partial_{1}w - iH_{\partial_{1}\tilde{w}}]|Lw|^{2}dx = \int_{\mathbb{R}} (-iH_{\partial_{1}w} + iH_{\partial_{1}\tilde{w}})[(\partial_{1}Lw)\overline{Lw} + Lw\partial_{1}\overline{Lw}]dx + I_{1}, \]
where \(I_{1}\) is dominated by
\[ ||I_{1}|| \leq ||w||^{3}_{w^{1,\infty}}||Lw||^{2}_{L^{2}}.\]

Again by integration by parts, the integral associated with the RHS of (6.8) becomes
\[ \int_{\mathbb{R}} H_{w}((v_{1}\tilde{w} = i\tilde{w})Lw - iLw\overline{Lw})dx = \int_{\mathbb{R}} (H_{w}\partial_{1}w + H_{\partial_{1}\tilde{w}})[(\partial_{1}Lw)\overline{Lw} + iLw\partial_{1}\overline{Lw}]dx.\]

Thus the RHS of (6.5) equals
\[ \int_{\mathbb{R}} (H_{w}\partial_{1}w + H_{\partial_{1}\tilde{w}})\overline{Lw}\overline{Lw}dx + \int_{\mathbb{R}} H_{w}((\overline{\partial_{1}Lw})\overline{Lw} + Lw\overline{\partial_{1}\overline{Lw}}dx \\
= \int_{\mathbb{R}} (-iH_{\partial_{1}w} + iH_{\partial_{1}\tilde{w}})[(\partial_{1}Lw)\overline{Lw} + Lw\partial_{1}\overline{Lw}]dx + \int_{\mathbb{R}} (H_{w}\partial_{1}w + H_{\partial_{1}\tilde{w}})[-i(\partial_{1}Lw)\overline{Lw} + iLw\partial_{1}\overline{Lw}]dx \\
+ \int_{\mathbb{R}} H_{w}(\tilde{w})[2i\overline{g}(w, \tilde{w}) + v_{1}\partial_{1}\tilde{w}(\partial_{1}Lw)Lw - 2i(g(w, \tilde{w}) + v_{1}\tilde{w})\partial_{1}w\overline{Lw}(\partial_{1}Lw)]dx \]
+ \(I_{1} + I_{2},\)
where \(I_{2}\) denotes the sum of (6.10), line (6.9), (6.7) and the last two terms of (6.6). Notice that (4.3) now becomes
\[ H_{w}(\tilde{w})(2g(w, \tilde{w}) + v_{1}\tilde{w}) + H_{\partial_{1}\tilde{w}} = O(|w|^{3}).\]

Then we arrive at
\[ \int_{\mathbb{R}} (H_{w}\partial_{1}w + H_{\partial_{1}\tilde{w}})\overline{Lw}\overline{Lw}dx + \int_{\mathbb{R}} H_{w}((\overline{\partial_{1}Lw})\overline{Lw} + Lw\overline{\partial_{1}\overline{Lw}}dx \]
= \(I_{1} + I_{2} + \int_{\mathbb{R}} O(|w|^{3}|\partial_{1}w|^{2}|\partial_{1}w|^{2} + |x\partial_{1}w| + |w||Lw|)dx.\)
Therefore, one obtains
\[
\frac{d}{dt} \mathcal{L}(w) \leq ||w||_{L^2}^2 ||Lw||_{L^2}^2 + ||Lw||_{L^2} ||w||_{L^2} ||w||_{L^2}^2 + \beta ||w||_{L^2} ||Lw||_{L^2} ||w||_{L^2}^4.
\]
(6.11)
Since \( ||w||_{L^2([0,T]\times\mathbb{R})} \leq \omega \), one has
\[
\mathcal{L}(w) \sim ||Lw||_{L^2}^2.
\]
(6.12)
Then the lemma follows by (6.11).
\[\square\]

**Corollary 6.1.** Assume that
\[
\sup_{t \in [0,T]} ||w(t)||_{H^3} \leq \epsilon.
\]
(6.13)
Then under the assumption (1.3) we have
\[
||Lw(t)||_{L^2} \leq \epsilon(t)\epsilon.
\]
(6.14)
**Proof.** By (6.13), Lemma 4.3 and Lemma 6.1, we get
\[
\frac{d}{dt} \mathcal{L}(w) \leq \epsilon^2 (t)^{-1} \mathcal{L}(w) + \epsilon^2 (t)^{-1} \epsilon \sqrt{\mathcal{L}(w)}.
\]
Then (6.14) follows by Gronwall inequality and (6.12).
\[\square\]

7 Setting of bootstrap

Assume that \( T \in (0,\infty) \) is the largest time for which
\[
\sup_{t \in [0,T]} ||w(t)||_{H^3} \leq \epsilon.
\]
(7.1)
Recall that \( w := z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4 \). By Duhamel principle and (1.21), we obtain
\[
w(t) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-\tau)\Delta} G(w, \tilde{w}) d\tau
\]
\[
G(w, \tilde{w}) := \nu_1 \tilde{w} (\partial_4 w)^2 + \nu_2 \tilde{w} (\partial_4 w)^2 + \nu_3 \tilde{w} (\partial_4 w)^2 + O(|w|^3) (\partial_4 w)^2.
\]
We will use the following dispersive estimate.

**Lemma 7.1 ([17]).** Let \( g(t, x) \) be any given \( \mathbb{C} \) valued function with finite norm \( ||g||_{L^\gamma_t(L^\infty_x)} + ||g||_{H^{\gamma+\beta}_t(L^\infty_x)} \). For any \( t \in \mathbb{R} \), and any \( \gamma > \frac{1}{2} + 2\beta \), there holds
\[
||e^{it\Delta} g||_{L^\gamma_t(L^\infty_x)} \leq \tau^{-\frac{1}{2}} ||g||_{L^\gamma_t(L^\infty_x)} + \frac{1}{\tau^{\gamma+\beta}} ||g||_{H^{\gamma+\beta}_t(L^\infty_x)}.
\]

8 Global bounds of the solution

Let \( f = e^{-it\Delta} w \), then (4.1) can be written as
\[
f(t) = f(1) + \int_1^t e^{-ir\Delta} (c\tilde{w} + \nu_2 \tilde{w} w + \nu_3 \tilde{w}^2 + O(|w|^3)) (\partial_4 w)^2 d\tau,
\]
(8.1)
where \( c = -\frac{1}{2} K(Q) h_0 \), see Lemma 3.2. Recall \( \hat{f} \) denotes the Fourier transform of \( f \). (8.1) can be written as

\[
\hat{f}(t, \sigma) = \hat{f}(1) - \frac{i}{2\pi} \int_1^\infty c e^{\tau \phi_0}(\sigma - \xi) f(\sigma - \xi) \hat{f}(\sigma - \xi) f(\sigma - \xi) d\xi \, d\eta \, d\sigma + \int_1^\infty R d\tau
\]

\[
- \sum_{t,i,j} \frac{i}{\pi} \int_1^\infty v_{ij} e^{\tau \phi_{ij}} (\sigma - \xi) f(\hat{f}(\sigma - \xi) f(\sigma - \xi) f(\sigma - \xi) d\xi \, d\eta \, d\sigma \, d\tau.
\]

where we denote

\[
\phi_0 = \sigma^2 - \xi^2 - (\sigma - \xi)^2.
\]

\[
\phi_{ij} = \sigma^2 - t_i \xi^2 - t_j (\xi - \eta)^2 - (\sigma - \xi)^2, \quad t_i, t_j \in \{+, -\}
\]

\[
R = F[e^{-i\tau A}(O(|w|^3)(\partial_x w)^2)]
\]

\[
f^i = f, \text{ if } i = +; \quad f^i = \overline{f}, \text{ if } i = -.
\]

### 8.1 Estimates of \( f \) in \( H^{k,l} \)

Denote the RHS of (4.1) by \( G(w, \tilde{w}) \). Recall that \( S := x\partial_x + 2i\partial_t \) and

\[
i\partial_t w + \Delta w = G(w, \tilde{w}).
\]

With \( f(t) = e^{-\mu \Lambda} w(t) \), one has

\[
(x\partial_x f)(t) = e^{\mu \Lambda} (Sw - 2tG(w, \tilde{w})).
\]

Recall that Corollary 4.1 proves

\[
||S w(t)||_{H^k} \leq (1 + t)^\delta ||w||_{H^{k+1}}.
\]

Thus we have

\[
||x\partial_x f(t)||_{H^l} \leq ||Sw||_{H^k} + t||w||_{H^k} ||w||_{H^{k+1}} \leq (1 + t)^\delta ||w||_{H^{k+1}}.
\]

We summarize the estimates of \( w \) and \( f \) as the following lemma.

**Lemma 8.1.** Assume that \( w_0 \) satisfies (1.4), (7.1) and (1.3) hold. Then for \( n, \) large, there exists a small constant \( \delta \) depending only on \( e, n, \) such that

\[
\sup_{t \in [0,T]} (1 + t)^{-\delta} ||w(t)||_{H^{k+1}} \leq ||w_0||_{H^{k+1}}.
\]

(8.5)

\[
\sup_{t \in [0,T]} (1 + t)^{-\delta} ||f(t)||_{H^{k+1}} \leq ||w_0||_{H^{k+1}}.
\]

(8.6)

**Proof.** (8.5) follows directly by Corollary 4.1. By computation, \( xe^{\mu \Lambda} g = -Lg \), so Corollary 6.1 yields

\[
||f(t)||_{H^{k+1}} \leq ||Lw||_{H^l} \leq t^\delta ||w||_{H^{k+1}}.
\]

And it is easy to check

\[
\frac{3}{3} \sum_{j=1}^3 \||x\partial_x^j f(t)||_{L^2} \leq \||x\partial_x f||_{H^l} + ||f||_{H^l}.
\]

Then (8.6) follows by (8.4) and bounds of \( ||w||_{H^k} \).

\[\square\]
8.2 Estimates in the Fourier Space with assumption (1.3)

Let \( n_* \) be sufficiently large and \( \varepsilon \) be sufficiently small such that \( \delta < 0.01 \).

Under the assumption (1.3), (8.3) can be written as

\[
\hat{f}(t, \sigma) = \hat{f}(t) - \frac{i}{2\pi} \int_{1}^{\tau} e^{i\tau \phi_0}(\eta - \xi)(\zeta - \eta)\hat{f}(\eta - \xi)\hat{f}(\zeta - \eta)d\xi d\eta d\tau + \int_{1}^{\tau} \mathcal{R} d\tau \tag{8.7}
\]

where we denote

\[
\phi_0 = \sigma^2 + \xi^2 - (\eta - \xi)^2 - (\sigma - \eta)^2.
\]

**Space-time resonance analysis of \( \phi_0 \).** By computation, we have the time resonance set of \( \phi_0 \) is

\[
\mathcal{R}_t^{\phi_0} = \{(\xi, \xi, \eta) : \phi_0 = 0\}.
\]

The space resonance sets of \( \phi_0 \) are

\[
\mathcal{R}_{s_0}^{\phi_0} = \{(\xi, \xi, \eta) : \partial_\xi \phi_0 = 0\} = \{(\xi, \xi, \eta) : \eta = 0\}
\]

\[
\mathcal{R}_{s_k}^{\phi_0} = \{(\xi, \xi, \eta) : \partial_\eta \phi_0 = 0\} = \{(\xi, \xi, \eta) : 2\eta - \xi = 0\}.
\]

The space-time resonance set is

\[
\mathcal{R}_{s_{0,\eta}}^{\phi_0} = \{(\xi, \xi, \eta) : \zeta = -\xi, \eta = 0\}.
\]

**Estimates of the leading cubic term.** By the stationary phase method, the RHS of (8.7) can be further decomposed. In fact, similar to [22], by change of variables and Plancherel identity, we find

\[
\frac{i}{2\pi} \int_{1}^{\tau} e^{i\tau \phi_0}(\eta - \xi)(\zeta - \eta)\hat{f}(\eta - \xi)\hat{f}(\xi - \eta)d\xi d\eta d\tau
\]

\[
= \frac{i}{2\pi} \int_{1}^{\tau} e^{i\tau \phi_0}(\zeta - \eta)(\sigma - \xi)\hat{f}(\sigma - \xi)\hat{f}(\xi - \eta)d\xi d\eta d\tau
\]

\[
= \frac{i}{2\pi} \int_{1}^{\tau} \int_{\mathbb{R}^2} \hat{F}\color{red}{-1}\{E(\sigma, \eta, \zeta)\} d\eta d\zeta d\tau
\]

\[
= \frac{i}{2\pi} \int_{1}^{\tau} \int_{\mathbb{R}^2} \frac{1}{2\tau} \hat{F}\color{red}{-1}\{E(\sigma, \eta, \zeta)\} d\eta d\zeta d\tau
\]

\[
- \frac{i}{2\pi} \int_{1}^{\tau} \int_{\mathbb{R}^2} \frac{1}{2\tau} [-\hat{F}\color{red}{-1}\{E(\sigma, \eta, \zeta)\} - 1] d\eta d\zeta d\tau,
\]

where we denote

\[
E(\sigma, \eta, \zeta) := (\sigma - \zeta)(\sigma - \xi)\hat{f}(\sigma - \xi)\hat{f}(\zeta - \eta)\hat{f}(\sigma - \xi).
\]

Observe that

\[
\int_{\mathbb{R}^2} \hat{F}\color{red}{-1}\{E(\sigma, \eta, \zeta)\} d\eta d\zeta = 2\pi E(\sigma, 0, 0) = 2\pi |\sigma \hat{f}(\tau, \sigma)|^2 \hat{f}(\tau, \sigma).
\]

Thus defining \( \hat{F}(t, \sigma) \) by

\[
\hat{F}(t, \sigma) = e^{ic \int_{1}^{\tau} |\tau|^\sigma d\tau} \hat{f}(t, \sigma),
\]

we get

\[
\hat{F}(t, \sigma) = \hat{f}(1, \sigma) + e^{ic \int_{1}^{\tau} |\tau|^\sigma d\tau} \int_{1}^{\tau} \mathcal{R} d\tau
\]

\[
- e^{ic \int_{1}^{\tau} |\tau|^\sigma d\tau} \int_{1}^{\tau} \frac{1}{2\tau} [-\hat{F}\color{red}{-1}\{E(\sigma, \eta, \zeta)\} - 1] d\eta d\zeta d\tau,
\]

(8.10)
where $\mathcal{R}$ denotes 5 order terms and higher order terms.

By computation, we have

$$F^{-1}_{\eta, \zeta}[E(\sigma, \eta, \zeta)] = (2\pi)^2 e^{i\sigma \hat{\eta} \zeta} \int_{\mathbb{R}} e^{-ix_0} \hat{f}(x) \hat{\sigma}(x-\hat{\eta}) \hat{\sigma}(x-\hat{\zeta}) dx.$$  

Hence, for $0 < \nu < \frac{1}{2}$, one has

$$\langle \sigma \rangle^2 \sum_{i_t+i_\xi+i_\eta \leq 2} \left| \int_{\mathbb{R}^3} \partial_i^2 f(x)(|x-\hat{\eta}|^{2\nu} + |x|^{2\nu})(|x-\hat{\zeta}|^{2\nu} + |x|^{2\nu}) \partial_x^{i_\xi+1} f(x-\hat{\eta}) \partial_x^{i_\eta+1} f(x-\hat{\zeta}) dx d\hat{\eta} d\hat{\zeta} \right| \leq \|f\|_{H^1}^3.$$  

Thus, by Lemma 8.1, we conclude that the last term in the RHS of (8.10) satisfies

$$\left\| \langle \sigma \rangle^2 \sum_{i_t+i_\xi+i_\eta \leq 3} \left| \int_{\mathbb{R}^3} \partial_i^2 f(x)(|x-\hat{\eta}|^{2\nu} + |x|^{2\nu})(|x-\hat{\zeta}|^{2\nu} + |x|^{2\nu}) \partial_x^{i_\xi+1} f(x-\hat{\eta}) \partial_x^{i_\eta+1} f(x-\hat{\zeta}) dx d\hat{\eta} d\hat{\zeta} \right| \right\|_{L^\infty} \leq \|f\|_{H^1}^3.$$  

Estimates of the high order terms. Since the high order term $\mathcal{R}$ in the RHS of (8.10) is at least 5 order, by Lemma 8.1, we conclude that $\mathcal{R}$ satisfies

$$\int_1^T \|\langle \sigma \rangle^2 \mathcal{R}\|_{L^\infty} d\tau \leq \int_1^T \|w\|_{H^2}^2 \|\tilde{w}\|_{H^1}^2 d\tau \leq \epsilon^3.$$  

Therefore, we deduce that

**Lemma 8.2.** Under the assumptions (7.1) and (1.3), for any $t \in [0, T]$, we have

$$\|\langle \sigma \rangle^2 \tilde{w}(t)\|_{L^\infty} \leq \epsilon_\sigma + \epsilon^3. \quad (8.13)$$

Moreover, there exists $\kappa > 0$ such that for any $0 \leq t_1 \leq t_2 \leq T$ there holds

$$\|\langle \sigma \rangle^2 (F(t_1, \sigma) - F(t_2, \sigma))\|_{L^\infty} \leq \langle t \rangle^{-\kappa}. \quad (8.14)$$

**Proof.** Observe that $F(t, \sigma)$ defined by (8.9) satisfies $|\tilde{w}(t, \sigma)| = |F(t, \sigma)|$. Then, for $t \in [0, 1]$, Sobolev embeddings and discussions in Section 2.1 give (8.13). For $t \in [1, T]$, (8.13) follows by (8.11) and (8.12). (8.14) results from (8.10) and similar estimates as (8.11)-(8.12) with integration interval replaced by $[t_1, t_2]$. \hfill \Box

### 9 Proof of Theorem 1.1: Decay estimates, modified scattering v.s. scattering

Let $Q$ be an intrinsic vanishing point and $w_0$ be an initial data satisfying Theorem 1.1.

Most parts of this section are standard, we present a detailed proof because some of them are useful in later sections. Lemma 7.1 shows

$$\|w(t)\|_{W^{2,\infty}} \leq \langle \sigma \rangle^{-\frac{1}{2}} \|\langle \sigma \rangle^{\frac{5}{2}} \tilde{w}\|_{L^\infty} + \langle \sigma \rangle^{-\frac{3}{2}} \|f\|_{H^{1,1}}.$$  

So Lemma 8.2 gives

$$\|w(t)\|_{W^{2,\infty}} \leq \epsilon_{\sigma} + \epsilon^3.$$  

Therefore, by bootstrap assumption $T = \infty$. And thus one has the decay estimates

$$\|w(t)\|_{W^{2,\infty}} \leq \langle t \rangle^{-\frac{3}{2}}, \quad \forall \ t > 0.$$
And (8.14) implies that there exists \( U \in (\sigma)^{-2}L^\infty_\sigma \) such that

\[
\| (\sigma)^2 [F(t, \sigma) - U(\sigma)] \|_{L^\infty} \leq t^{-\varepsilon},
\]

(9.1)

from which (1.5) follows.

Let

\[
\Psi(t) := \int_1^t (|\sigma f(\tau, \sigma)|^2 - |\sigma \hat{f}(t, \sigma)|^2) \frac{1}{2\tau} d\tau.
\]

By (8.14), for any \( t_1 < t_2 < \infty \),

\[
\| \Psi(t_1) - \Psi(t_2) \|_{(\sigma)^{-1}L^2} \leq t_1^{-\frac{1}{2}}.
\]

So there exists a real valued function \( \Theta \) such that for \( t \) large

\[
\| \Psi(t, \sigma) - \Theta(\sigma) \|_{(\sigma)^{-1}L^2} \leq t^{-\frac{1}{2} + \varepsilon},
\]

which further shows

\[
\int_1^t |\sigma \hat{f}(\tau, \sigma)|^2 \frac{1}{\tau} d\tau = |\sigma \hat{f}(t, \sigma)|^2 \ln t + \Theta(\sigma) + O(\sigma^{-1}L^2(t^{-\frac{1}{2}})).
\]

Thus by the definition of \( F(t, \xi) \) and (9.1), we have for \( t \) large

\[
\int_1^t |\sigma \hat{f}(\tau, \sigma)|^2 \frac{1}{\tau} d\tau = \Theta(\sigma) + |\sigma U(\sigma)|^2 \ln t + O(\sigma^{-1}L^2(t^{-\frac{1}{2}})).
\]

Hence

\[
\hat{f}(t, \sigma) = e^{\frac{i}{\tau} \int_0^1 |\sigma \hat{f}(\tau, \sigma)|^2 d\tau} F(t, \sigma) = U(\sigma) \exp(i|\sigma U(\sigma)|^2 \ln t + i\Theta(\sigma)) + \tilde{R}(t, \sigma),
\]

(9.2)

where the remainder \( \tilde{R}(t, \sigma) \) fulfills

\[
\| \tilde{R}(t, \sigma) \|_{(\sigma)^{-1}L^2} \leq t^{-\frac{1}{2} + \varepsilon}.
\]

Hayashi-Naumkin [17] has proved that

\[
u(t, x) = \frac{e^{it^2}}{(4it)^2} \mathcal{F}[e^{-it\Delta}u(t)] \left( 2t, \frac{x}{2t} \right) + R(t, x),
\]

(9.4)

where \( R(t, x) \) satisfies

\[
R(t, x) = \frac{e^{it^2}}{(4\pi it)^2} \int_{\mathbb{R}} e^{-\frac{|y|}{2t}} (e^{\frac{|y|^2}{2t}} - 1)(e^{-it\Delta}u(t)) dy.
\]

(9.5)

It is easy to verify by Plancherel identity and change of variables that

\[
\| R(x, t) \|_{L^1_x} \leq t^{-\frac{1}{2}} \| x|e^{-it\Delta}u(t) \|_{L^2_x}.
\]

(9.6)

And [17] proved the point-wise estimate

\[
\| R(x, t) \|_{L^\infty_x} \leq t^{-\frac{1}{2}} \| e^{-it\Delta}u(t) \|_{H^{0,1}}, \forall |t| \geq 1.
\]

(9.7)

Therefore, (9.2) implies

\[
w(t, x) = \frac{e^{it^2}}{(2it)^2} \mathcal{F}[e^{-it\Delta}w(t)] \left( 2t, \frac{x}{2t} \right) + R(t, x)
\]

\[
= \frac{e^{it^2}}{(2it)^2} \mathcal{F}[f] \left( 2t, \frac{x}{2t} \right) + R(t, x)
\]

\[
= \frac{e^{it^2}}{(2it)^2} U(x) \left( e^{\frac{1}{2t}(\|f\|_{L^1}^2 + \ln 2t + 2\Theta(x))} + \frac{e^{it^2}}{(2it)^2} \tilde{R}(2t, \frac{x}{2t}) + R(t, x),
\]
where \( \tilde{R}(t, x) \) satisfies (9.3) and \( R(t, x) \) fulfills (9.6)-(9.7). So if \( \delta > 0 \) is taken to be sufficiently small, for \( t \) large there hold

\[
\|w(t, x) - e^{it\delta^2} U(\frac{x}{2t}) e^{i\delta(x)} \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2}} \| \tilde{R}(2t, x) \|_{L^p_t L^q_x} + \| R(t, x) \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2}} \| f \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2} - \frac{\kappa}{2}} \| w \|_{L^p_t L^q_x},
\]

and

\[
\|w(t, x) - e^{it\delta^2} U(\frac{x}{2t}) e^{i\delta(x)} \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2}} \| \tilde{R}(2t, x) \|_{L^p_t L^q_x} + \| R(t, x) \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2}} \| f \|_{L^p_t L^q_x} \leq t^{-\frac{1}{2} + \kappa} \| w \|_{L^p_t L^q_x}.
\]

And thus (1.7) follows by letting \( \zeta = \tfrac{1}{\delta} \kappa.
\]

\[
\psi(x) = U(x) e^{i\delta(x)},
\]

and noting that

\[
\|w - z\|_{L^p_t L^q_x} \leq \|w^2\|_{L^p_t L^q_x} \leq t^{-1}
\]

\[
\|w - z\|_{L^p_t L^q_x} \leq \|w^2\|_{L^p_t L^q_x} \leq t^{-\frac{3}{2}}.
\]

## 10 Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by contradiction argument.

### 10.1 Multi-linear estimates

To do nonlinear estimates, we need the following lemma due to [21], see also Lemma 5.2 of [22].

**Lemma 10.1** ([21]). Suppose that \( m(\xi, \eta, \zeta) \in L^1(\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \) satisfies

\[
\left\| \int_{\mathbb{R}^3} m(\xi, \eta, \zeta) e^{i\xi x} e^{i\eta y} e^{i\zeta z} d\xi d\eta d\zeta \right\|_{L^p_t L^q_x} \leq A,
\]

for some \( 0 < A < \infty \). Define

\[
FT_m(h_1, h_2, h_3, h_4)(\sigma) = \int_{\mathbb{R}^3} m(\xi, \eta, \zeta) h_1(\xi) h_2(\eta - \xi) h_3(\zeta - \eta) h_4(\sigma - \xi) d\xi d\eta d\zeta.
\]

Then for any \( i_1, i_2, i_3, i_4 \) which are rearrangement of \( \{1, 2, 3, 4\} \) and any \( p_1, p_2, p_3, p_4 \in [1, \infty] \) with \( \sum_{j=1}^4 \frac{1}{p_j} = 1 \), there holds

\[
\|FT_m(h_1, h_2, h_3, h_4)\|_{L^p_t L^q_x} \leq A \|h_1\|_{L^{p_1}_{t,x}} \|h_2\|_{L^{p_2}_{t,x}} \|h_3\|_{L^{p_3}_{t,x}} \|h_4\|_{L^{p_4}_{t,x}}.
\]

The following lemma will be useful in nonlinear estimates of later subsections. In Lemma 10.2, \( \phi \) denotes a quadratic form of \((x, y, \zeta, \sigma) \in \mathbb{R}^4\).

**Lemma 10.2.** Assume that \( g_i \) is a homogenous function such that \( g_i(\lambda y) = \lambda^{y} g_i(y) \) for all \( \lambda > 0, y \in \mathbb{R} \) with \( i = 1, 2, 3, 4 \). Let \( \vartheta_i \) with \( i = 1, 2, 3, 4 \) be a smooth function such that \( g_i(y) \vartheta_i(y) \) is smooth in \( \mathbb{R} \) and

\[
\|g_1(\phi) \vartheta_1(\phi)\|_{W^{2,\infty}(\mathbb{R}^n)} + \sum_{i=2}^4 \|g_i(y) \vartheta_i(y)\|_{W^{2,\infty}(\mathbb{R}^n)} \leq 1. \tag{10.1}
\]

Define

\[
m_i = g_1(\phi) g_2(\vartheta_2 \phi) g_3(\vartheta_3 \phi) g_4(\vartheta_4 \phi) \vartheta_1(\varphi^\mu \phi) \vartheta_2(\varphi^\mu \phi) \vartheta_3(\varphi^\mu \phi) \vartheta_4(\varphi^\mu \phi) e^{i \phi \vartheta_i \phi}.
\]
Then for $\rho > \frac{1}{2}, A > \frac{1}{2}$, there hold

\[
\begin{align*}
| \int_{\mathbb{R}^3} m_{ij}(\xi, \eta, \zeta) \tilde{f}_i(\xi) \tilde{f}_j(\eta - \xi) \tilde{\varphi}_3(\zeta - \eta) \tilde{\varphi}_4(\sigma - \zeta) d\xi d\eta d\zeta | \\
& \leq r^{2\mu(t_1 + \frac{1}{3}(t_2 + t_3 + t_4))} \| F_{ij} \|_{L^\infty} \| F_i \|_{L^2} \| F_j \|_{L^2} \| F_3 \|_{L^2} \| F_4 \|_{L^2}
\end{align*}
\]

(10.2)

Then for

\[
\begin{align*}
| \int_{\mathbb{R}^3} m_{ij}(\xi, \eta, \zeta) \tilde{f}_i(\xi) \tilde{f}_j(\eta - \xi) \tilde{\varphi}_3(\zeta - \eta) \tilde{\varphi}_4(\sigma - \zeta) d\xi d\eta d\zeta | \\
& \leq r^{2\mu(t_1 + \frac{1}{3}(t_2 + t_3 + t_4))} \| F_{ij} \|_{L^\infty} \| F_i \|_{L^2} \| F_j \|_{L^2} \| F_4 \|_{L^2}
\end{align*}
\]

(10.3)

and

\[
\begin{align*}
| \int_{\mathbb{R}^3} m_{ij}(\xi, \eta, \zeta) \tilde{f}_i(\xi) \tilde{f}_j(\eta - \xi) \tilde{\varphi}_3(\zeta - \eta) \tilde{\varphi}_4(\sigma - \zeta) d\xi d\eta d\zeta | \\
& \leq r^{2\mu(t_1 + \frac{1}{3}(t_2 + t_3 + t_4))} \| F_{ij} \|_{L^\infty} \| F_i \|_{L^2} \| F_j \|_{L^2} \| F_4 \|_{L^2} \| F_3 \|_{L^2} \| F_4 \|_{L^2}
\end{align*}
\]

(10.4)

\[
\begin{align*}
| \int_{\mathbb{R}^3} m_{ij}(\xi, \eta, \zeta) \tilde{f}_i(\xi) \tilde{f}_j(\eta - \xi) \tilde{\varphi}_3(\zeta - \eta) \tilde{\varphi}_4(\sigma - \zeta) d\xi d\eta d\zeta | \\
& \leq r^{2\mu(t_1 + \frac{1}{3}(t_2 + t_3 + t_4))} \| F_{ij} \|_{L^\infty} \| F_i \|_{L^2} \| F_j \|_{L^2} \| F_4 \|_{L^2} \| F_3 \|_{L^2} \| F_4 \|_{L^2}
\end{align*}
\]

(10.5)

Proof. First, we verify (10.2). Recall the function $\varphi$ in the definition of Littlewood-Paley decomposition at the end of Section 1. Define

\[
\varphi_0(\xi) = \sum_{j=0} \varphi(2^{-j}\xi)
\]

\[
\varphi_k(\xi) = \varphi(2^{-k}\xi), \quad k \geq 1.
\]

Decompose $f_2, f_3, f_4$ into

\[
f_2 = \sum_{j=0} \mathcal{F}^{-1} \varphi_j(t^\rho \xi) \mathcal{F} f_2; \quad f_3 = \sum_{j=0} \mathcal{F}^{-1} \varphi_k(t^\rho \xi) \mathcal{F} f_3; \quad f_4 = \sum_{j=0} \mathcal{F}^{-1} \varphi_k(t^\rho \xi) \mathcal{F} f_4.
\]

Let $\tilde{\chi}$ be a $C^\infty_c$ cutoff function which equals one in the support of $\varphi$. Define

\[
\tilde{\chi}_k(\xi) = \tilde{\chi}(2^{-k}\xi), \quad k \geq 1.
\]

Let $\tilde{\chi}_0$ be a $C^\infty_c$ cutoff function which equals one in the support of $\varphi_0$.

Consider the integral

\[
I_{kkk} := \int_{\mathbb{R}^3} m_{ij}(\xi, \eta, \zeta) \varphi(\frac{\eta - \xi}{r^\mu}) \varphi_k(\frac{\zeta - \eta}{r^\mu}) \varphi(\frac{\sigma - \zeta}{r^\mu}) \tilde{\chi}(t^\rho(\eta - \xi)) \tilde{\chi}(t^\rho(\zeta - \eta)) \tilde{\chi}(t^\rho(\sigma - \zeta)) \tilde{f}_i(\xi) \tilde{f}_j(\eta - \xi) \tilde{\varphi}_3(\zeta - \eta) \tilde{\varphi}_4(\sigma - \zeta) d\xi d\eta d\zeta.
\]

By scaling, we find the symbol

\[
\tilde{m}_{ij}^{kkk} := m_{ij}(\xi, \eta, \zeta) \varphi(\frac{\eta - \xi}{r^\mu}) \varphi_k(\frac{\zeta - \eta}{r^\mu}) \varphi(\frac{\sigma - \zeta}{r^\mu})
\]

satisfies the bound

\[
\| \mathcal{F}_{\xi,\eta,\zeta} \tilde{m}_{ij}^{kkk} \|_{L^1(\mathbb{R}^3)} \leq r^{-\sigma(t_1 + \frac{1}{3}(t_2 + t_3 + t_4))} \| \mathcal{F}_{\xi,\eta,\zeta} M_{ij}^{kkk} \|_{L^1(\mathbb{R}^3)}
\]

where $M_{ij}^{kkk}$ is defined by

\[
M_{ij}^{kkk} := g_1(\phi) \partial_1(\phi) g_2(\partial_2\phi) \partial_2(\partial_3\phi) g_3(\partial_3\phi) \partial_3(\partial_4\phi) g_4(\partial_4\phi) \partial_4(\partial_5\phi) \varphi(\frac{\eta - \xi}{r^\mu}) \varphi_k(\zeta - \eta) \varphi(\frac{\sigma - \zeta}{r^\mu}).
\]
By Hölder inequality and Plancherel identity, we have
\[ \| F_{\xi,\eta,\zeta} M_{ij}^{jk} \|_{L^2(\mathbb{R})} \leq \| M_{ij}^{jk} \|_{L^2(\mathbb{R})} \leq 2^{4} 2^{\frac{1}{2}} 2^{\frac{1}{2}}. \]

Then by Lemma 10.1 and Bernstein inequality,
\[ \sum_{j,k \geq 0} | I_{j,k,l} | \leq 2^{2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} \| f_{1} \|_{L_{x}^{\infty}} \sum_{k,l,j \geq 0} 2^{\frac{1}{2}} \| \hat{f}_{j} (p^{d}D) f_{2} \|_{L_{x}^{\infty}} 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \]
\[ \cdot 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} T^{2\mu} \| f_{1} \|_{L_{x}^{\infty}} \| f_{2} \|_{W_{x}^{\infty}} \| f_{3} \|_{L_{x}^{\infty}} \| f_{4} \|_{L_{x}^{\infty}}. \]

Second, for (10.3), by Lemma 10.1 and Bernstein inequality, we have
\[ \sum_{j,k \geq 0} | I_{j,k,l} | \leq 2^{2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} \| f_{1} \|_{L_{x}^{\infty}} \sum_{k,l,j \geq 0} 2^{\frac{1}{2}} \| \hat{f}_{j} (p^{d}D) f_{2} \|_{L_{x}^{\infty}} 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \]
\[ \cdot 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} T^{2\mu} \| f_{1} \|_{L_{x}^{\infty}} \| f_{2} \|_{W_{x}^{\infty}} \| f_{3} \|_{L_{x}^{\infty}} \| f_{4} \|_{L_{x}^{\infty}}. \]

Third, for (10.4), by Lemma 10.1 and Bernstein inequality, we get
\[ \sum_{j,k \geq 0} | I_{j,k,l} | \leq 2^{2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} \| f_{1} \|_{L_{x}^{\infty}} \sum_{k,l,j \geq 0} 2^{\frac{1}{2}} \| \hat{f}_{j} (p^{d}D) f_{2} \|_{L_{x}^{\infty}} 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \]
\[ \cdot 2^{\frac{1}{2}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} T^{2\mu} \| f_{1} \|_{L_{x}^{\infty}} \| f_{2} \|_{W_{x}^{\infty}} \| f_{3} \|_{L_{x}^{\infty}} \| f_{4} \|_{L_{x}^{\infty}}. \]

Lastly, for (10.5), decompose \( f_{1}, f_{2}, f_{4} \) into
\[ f_{1} = \sum_{k \geq 0} \mathcal{F}^{-1} \phi_{k} (\xi' \mu) \mathcal{F} f_{1}, \quad f_{2} = \sum_{j \geq 0} \mathcal{F}^{-1} \phi_{j} (\xi' \mu) \mathcal{F} f_{2}, \quad f_{4} = \sum_{j \geq 0} \mathcal{F}^{-1} \phi_{j} (\xi' \mu) \mathcal{F} f_{4}. \]

Consider the integral
\[ I_{j,k,l} := \int \mathbb{R}^{3} m_{j} (\xi, \eta, \zeta) \phi_{j} (\xi' \mu) \phi_{j} (\xi' \mu) \hat{f}_{j} (p^{d}(\sigma - \xi)) \tilde{f}_{j} (p^{d}(\sigma - \xi)) \hat{f}_{j} (p^{d}(\sigma - \xi)) \hat{f}_{j} (p^{d}(\sigma - \xi)) d\xi d\eta d\zeta. \]

By scaling, we find the symbol
\[ \tilde{m}_{j}^{i,j,k,l} := m_{j} (\xi, \eta, \zeta) \phi_{j} (\xi' \mu) \phi_{j} (\xi' \mu) \phi_{j} (\xi' \mu) \phi_{j} (\xi' \mu) \]
satisfies the bound
\[ \| \mathcal{F}_{\xi,\eta,\zeta} \tilde{m}_{j}^{i,j,k,l} \|_{L^{1}(\mathbb{R}^{3})} \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} 2^{\frac{1}{2}} 2^{\frac{1}{2}} 2^{\frac{1}{2}}. \]

Then by Lemma 10.1 and Bernstein inequality,
\[ \sum_{j,k \geq 0} | \tilde{I}_{j,k,l} | \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} \sum_{k,l,j \geq 0} 2^{2\mu} \| \hat{f}_{j} (p^{d}D) f_{1} \|_{L_{x}^{\infty}} 2^{2\mu} \| \hat{f}_{k} (p^{d}D) f_{2} \|_{L_{x}^{\infty}} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \]
\[ \cdot 2^{2\mu} \| \hat{f}_{k} (p^{d}D) \partial_{x} f_{3} \|_{L_{x}^{\infty}} \leq t^{-2\mu(t_{1} + \frac{1}{2}(t_{2} + t_{3} + t_{4}))} T^{2\mu} \| f_{1} \|_{W_{x}^{\infty}} \| f_{2} \|_{W_{x}^{\infty}} \| f_{3} \|_{L_{x}^{\infty}} \| f_{4} \|_{L_{x}^{\infty}}. \]

The following simple lemma will also be useful in nonlinear estimates of later subsections.

**Lemma 10.3.** We have
\[ \| \mathcal{F}_{x} [ e^{-i t \Delta} g (\xi) ] \|_{L_{x}^{\infty}} \leq \| g (\xi) \|_{L_{x}^{\infty}} \]
\[ \| \mathcal{F}_{x} [ e^{-i t \Delta} g (\xi) ] \|_{L_{x}^{\infty}} \leq \| g (\xi) \|_{L_{x}^{\infty}}. \]
10.2 The Proof of Theorem 1.2

In this section, we assume $Q$ is not an intrinsic vanishing point, i.e.

$$[\ln h](0)[\ln h](0) - [\ln h](0) \neq 0. \quad (10.6)$$

We prove Theorem 1.2 by contradiction. Assume that $u$ is a global solution to 1D SMF satisfying (1.9)-(1.11) under the local complex coordinate $z$. Recall that $z$ satisfies

$$\begin{cases}
  i\partial_z z + \Delta z = c_0 \partial_z z \partial_z z + c_1 \partial_z z \partial_z z + c_2 \partial_z z \partial_z z + c_3 z^2 \partial_z z \partial_z z + c_4 z \partial_z z \partial_z z \\
  z \big|_{t=0} = z_0.
\end{cases} \quad (10.7)$$

Define $f = e^{-\delta t}z$. Then $f$ fulfills

$$\begin{align*}
  \hat{f}(t, \sigma) &= \hat{f}(t, \sigma) = \frac{i c_1}{2\pi} \int_1^\infty \int_{\mathbb{R}^2} e^{i|\beta|}(\eta - \xi)(\xi - \eta) \hat{f}(\xi) \hat{f}(\eta - \xi) \xi d\eta d\tau + \int_1^\infty R d\tau \\
  &\quad - \frac{i c_0}{2\pi} \int_1^\infty \int_{\mathbb{R}^2} e^{i|\beta|}(\eta - \xi)(\xi - \eta) \hat{f}(\xi) \hat{f}(\eta - \xi) \eta d\xi d\tau \\
  &\quad - \frac{i c_2}{2\pi} \int_1^\infty \int_{\mathbb{R}^2} e^{i|\beta|}(\eta - \xi)(\xi - \eta) \hat{f}(\xi) \hat{f}(\eta - \xi) \xi d\xi d\tau \\
  &\quad - \sum_{n,s \in \{+, -\}} \frac{1}{2\pi} \int_1^\infty \int_{\mathbb{R}^2} c_{n,s} e^{i|\beta|}(\xi - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \xi d\xi d\eta d\tau, \quad (10.8)
\end{align*}$$

where we denote

$$\begin{align*}
  \phi_0 &= \sigma^2 + \xi^2 - (\eta - \xi)^2 - (\sigma - \eta)^2 \\
  \phi_1 &= \sigma^2 - \xi^2 - (\sigma - \xi)^2 \\
  \phi_2 &= \sigma^2 - \xi^2 - (\eta - \xi)^2 - (\sigma - \eta)^2 \\
  \phi_{n,s} &= \sigma^2 + n \xi^2 + t_2 (\xi - \eta)^2 - (\xi - \eta)^2 - (\sigma - \xi)^2, \ t_1, t_2 \in \{+, -\} \\
  \mathcal{R} &= \mathcal{F}[e^{-ir\alpha}O(|z|^3)(\partial_z z)^2] \\
  f^* &= f, \text{ if } t = +; \quad f^* = \hat{f}, \text{ if } t = -.
\end{align*}$$

Note that $z$ is now a given coordinate, we can not expect any vanishing of $c_0, c_2, c_3$. Moreover, we can not perform the holomorphic transformation $w := z + \gamma z^2 + \gamma z^3 + \gamma z^4$, since the condition (1.10) breaks down for the new coordinate $w$ if $\gamma_1 \neq 0$. Therefore, compared with (8.3), (10.10) has additional quadratic term (10.8), cubic term (10.9) and 4 order term $\phi_{-\ldots}$.

**Lemma 10.4.** Under the assumptions (1.10), (1.9), (1.11), we have

$$\begin{align*}
  ||z(t)||_{H^1} &\leq \epsilon_s \\
  ||z(t)||_{H^2} &\leq \epsilon_s(t) \epsilon \\
  ||f(t)||_{H^2,1} &\leq \epsilon_s(t) \epsilon \\
  ||z(t)||_{L^2} &\leq ||z_0||_{L^2} \leq ||z(t)||_{L^2}. \quad (10.11)
\end{align*}$$

**Proof:** The $H^1$ and $H^2$ bounds follow by Section 2. The $||\partial_z^j f(t)||_{H^{j+1}}$ bound with $j = 0, 1, 2$ follows by the assumption $||L(\partial_z z)||_{L^2} \leq \epsilon_s(t) \epsilon$. It reamins to prove the last inequality on $||z(t)||_{L^2}$. Let $w = z + \gamma_1 z^2 + \gamma_2 z^3 + \gamma_3 z^4$. It follows by Section 4 and assumption (1.9) that

$$\left| \frac{d}{dt} ||w(t)||_{L^2} \right| \leq \epsilon^2_s(t) \epsilon^{-1} \epsilon ||w(t)||_{L^2}^2.$$ 

Thus, we get

$$||w(t)||_{L^2} \leq ||w_0||_{L^2} \leq ||w(t)||_{L^2}.$$
Notice that by the assumption (1.9),
\[ \|w(t)\|_{L^2} \leq \|z(t)\|_{L^2} \leq \|w(t)\|_{L^2}, \forall t \geq 0. \]
So, (10.11) follows. \( \square \)

In the following, we call (10.11) almost conservation of mass.

The proof of Theorem 1.2 will be divided into two cases according to whether \( c_5 \) in (10.7) vanishes or not.

**Proposition 10.1.** Suppose that \( c_5 \neq 0 \) in (10.7). Under the assumptions (10.6), (1.9), (1.10), (1.11), and \( U \neq 0 \), we have as \( t \to \infty \)
\[ \|\tilde{E}(t)\|_{L^\infty} \geq C \ln(t), \]
for some \( C > 0 \).

**Proof.** Let \( \mu > 0, \nu \in (0, \frac{1}{3}) \) satisfy
\[-2\nu + 2\mu\nu + 4\beta < 0 \]
\[-1 + 4\mu + \beta < 0 \]
\[ \nu > 3\beta, \mu > \beta > 2\epsilon. \]
Recall that \( \hat{f} \) satisfies (10.10).

**Estimates of the leading cubic term.** This part is the same as Section 8. Let
\[ E(\sigma, \eta, \zeta) := (\sigma - \zeta)(\sigma - \eta)f(\sigma - \eta - \zeta)\hat{f}(\sigma - \xi). \quad (10.12) \]

Defining \( \hat{F}(t, \sigma) \) by
\[ \hat{F}(t, \sigma) = e^{i \int_0^t \hat{f}(\tau, \sigma') \|dz\| d\tau} \hat{f}(t, \sigma), \quad (10.13) \]
we get
\[ \hat{F}(t, \sigma) = \hat{f}(1, \sigma) \]
\[ + e^{i \int_0^t \hat{f}(\tau, \sigma') \|dz\| d\tau} \left[ -\frac{i}{2\pi} \int_1^\infty \hat{R} d\tau - \frac{i c_1}{2\pi} \int_1^\infty \frac{1}{2\pi} \left| e^{-\frac{c_1}{2\pi}} - 1 \right| E(\sigma, \eta, \zeta) \|d\eta d\zeta d\tau \right]. \quad (10.14) \]
Here, we denote
\[ \hat{R} := R_1 + R_2 + R_+ + R_{++} + R_{++} + R, \]
where \( R_1, R_2 \) denote the terms associated with (10.8) and (10.9) respectively, \( R_+, R_{++}, R_{++} \) denote 4 order terms associated with \( \phi_{++}, \phi_{--} \) and \( \phi_{++} \) respectively, and \( R \) denotes higher order terms. (See (10.10).)

Hence, for \( \nu < \frac{1}{4} \), as Section 8.2 one has
\[ \int_1^\infty \tau^{-1-\nu} d\tau \int_{\mathbb{R}^3} |\hat{R}_{\xi, \eta}|^2 |\hat{F}_{\eta, \zeta}^{-1}(E(\sigma, \eta, \zeta))| d\eta d\zeta \leq \int_1^\infty \tau^{-1-\nu} \|f\|_{H^1}^3 d\tau. \]

Thus the last term in the RHS of (10.14) contributes to \( |\hat{F}(t, \sigma)| = |\hat{f}(t, \sigma)| \) by \( Ce^3 \).

**Estimates of 4 order terms.**

**Space-time resonance analysis of \( \phi_{++} \).** The time resonance set of \( \phi_{++} \) is
\[ \mathcal{R}_{++} = \{ (\xi, \eta, \zeta, \sigma) : \phi_{++} = 0 \}. \]
The space resonance sets are
\[ \mathcal{R}_{\xi, \eta}^{++} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\xi} \phi_{++} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : 2\xi - \eta = 0 \} \]
\[ \mathcal{R}_{\xi, \zeta}^{++} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\xi} \phi_{++} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : \xi - \zeta = 0 \} \]
\[ \mathcal{R}_{\zeta, \eta}^{++} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\zeta} \phi_{++} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : 2\zeta - \sigma - \eta = 0 \}. \]
The space-time resonance set is

\[ \mathcal{R}_{ij}^{++} = \{ (\xi, \eta, \zeta, \sigma) : \eta = 2\xi, \zeta = \xi, \sigma = 0 \}. \]

**Estimates of 4 order term associated with \( \phi_{++} \).** Letting

\[ \xi - \frac{\eta}{2} = \xi' ; \eta - 2\xi = 2\eta', \]

we find

\[ \frac{i}{2\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} \chi (\xi')(\eta')(\xi' + \eta') d\xi' d\eta' d\xi d\eta = \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} (-\xi - \eta)(\xi - \eta) \tilde{f}(\xi + \eta) d\xi d\eta d\xi d\eta. \]

\[ \cdot \tilde{f}(\xi + \eta - \xi') \tilde{f}(\xi - \eta - \xi') \tilde{f}(\xi + \xi' + \eta') \tilde{f}(\xi - \xi' - \eta'). \]

(10.15)

Let \( \chi \) be a smooth cutoff function adapted to the interval \([-1, 1]\). And define \( \chi_{\mu} (\cdot) = \chi (\tau^\mu) \). We decompose (10.15) as the following

\[ \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} \chi_{\mu}(\xi')(\eta')(\xi' + \eta') d\xi' d\eta' d\xi d\eta = \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} \chi_{\mu}(\xi')(\eta')(\xi' + \eta') d\xi' d\eta' d\xi d\eta. \]

(10.16)

\[ \cdot \tilde{f}(\xi - \eta + \xi') \tilde{f}(\xi - \xi') \tilde{f}(\xi + \xi' + \eta') \tilde{f}(\xi - \xi' - \eta'). \]

(10.17)

\[ \cdot \tilde{f}(\xi - \eta + \xi') \tilde{f}(\xi - \xi') \tilde{f}(\xi + \xi' + \eta') \tilde{f}(\xi - \xi' - \eta'). \]

(10.18)

\[ \cdot \tilde{f}(\xi - \eta + \xi') \tilde{f}(\xi - \xi') \tilde{f}(\xi + \xi' + \eta') \tilde{f}(\xi - \xi' - \eta'). \]

(10.19)

**Estimates of (10.16).** To dominate (10.16), we further decompose it as follows,

\[ \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} \chi_{\mu}(\xi')(\eta')(\xi' + \eta') d\xi' d\eta' d\xi d\eta = \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}} e^{i(2\xi^2 - 2\eta^2 + 2\omega)} \chi_{\mu}(\xi')(\eta')(\xi' + \eta') d\xi' d\eta' d\xi d\eta. \]

(10.20)

\[ \cdot \tilde{f}(\xi - \eta + \xi') \tilde{f}(\xi - \xi') \tilde{f}(\xi + \xi' + \eta') \tilde{f}(\xi - \xi' - \eta'). \]

(10.21)

Note that \(|\sigma| \geq C\tau^\mu\) in (10.21). And thus using the fact

\[ e^{i(2\xi^2 - 2\eta^2 + 2\omega)} = \frac{1}{2i\tau \sigma} \partial_\xi e^{i(2\xi^2 - 2\eta^2 + 2\omega)}, \quad |2i\tau \sigma| \geq C|\tau|^{1-\mu}, \]

(10.22)

and integration by parts in \( \zeta \), one obtains

\[ \|(10.21)\|_{L^\infty} \leq \int_1^\infty \tau^{-\mu - 1} \|f\|_{H^\mu}^3 \|f\|_{H^\mu}^3 \|f\|_{H^\mu}^3 d\tau \leq 1. \]

(10.23)
For (10.20), by change of variables and Plancherel identity, we have

\[
(10.20) = \frac{i}{\pi} \int_1^{\infty} \int_{\mathbb{R}^3} \mathcal{F}_{\xi,\eta}^{-1} \left[ e^{i t (2\xi^2 - 2\eta^2)} \right] \mathcal{F}_{\xi,\eta}^{-1} f \, d\xi d\eta d\tau = \frac{i}{(2\pi)^{\frac{3}{2}}} \int_1^{\infty} \int_{\mathbb{R}^3} \frac{e^{-\frac{\xi^2}{4\tau}}}{2\pi} Y_{\sigma,\tau}(\xi, \eta, \zeta) d\xi d\eta d\zeta d\tau,
\]

where \( Y \) is defined by

\[
Y_{\sigma,\tau}(\xi, \eta, \zeta) = \mathcal{F}_{\xi,\eta}^{-1} \left[ e^{2r_{\sigma\tau}^c}(\xi')\nu(\eta') \nu(\sigma) f(\xi' + \zeta + \eta') \tilde{f}(\xi + \eta - \zeta) \tilde{f}(\zeta) \partial_\zeta f(-\zeta - 2\eta') \partial_x f(\sigma - \zeta) \right].
\]

Further splitting \( e^{\frac{2i\xi^2}{4\tau}} \) yields

\[
\frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} e^{i t (\xi - \eta)(\sigma - \zeta)} f(\xi) f(\eta - \xi) \tilde{f}(\zeta - \eta) \tilde{f}(\sigma - \zeta) d\xi d\eta d\zeta d\tau = \frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} \left[ e^{\frac{2i\xi^2}{4\tau}} - 1 \right] Y_{\sigma,\tau}(\xi, \eta, \zeta) d\xi d\eta d\zeta d\tau + \frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} Y_{\sigma,\tau}(\xi, \eta, \zeta) d\xi d\eta d\zeta d\tau
\]

\[
+ \frac{i}{(2\pi)^{\frac{3}{2}}} \int_1^{\infty} \int_{\mathbb{R}^3} e^{2r_{\sigma\tau}^c}(\xi') \tilde{f}(\xi') \partial_x f(-\zeta) \partial_x f(\sigma - \zeta) \tilde{f}(\zeta) \partial_\zeta f(\sigma) d\xi d\eta d\zeta d\tau.
\]

Further splitting \( e^{\frac{2i\xi^2}{4\tau}} \) yields

\[
\frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} e^{i t (\xi - \eta)(\sigma - \zeta)} f(\xi) f(\eta - \xi) \tilde{f}(\zeta - \eta) \tilde{f}(\sigma - \zeta) d\xi d\eta d\zeta d\tau = \frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} \left[ e^{\frac{2i\xi^2}{4\tau}} - 1 \right] Y_{\sigma,\tau}(\xi, \eta, \zeta) d\xi d\eta d\zeta d\tau + \frac{i}{2\pi} \int_1^{\infty} \int_{\mathbb{R}^3} Y_{\sigma,\tau}(\xi, \eta, \zeta) d\xi d\eta d\zeta d\tau
\]

\[
+ \frac{i}{(2\pi)^{\frac{3}{2}}} \int_1^{\infty} \int_{\mathbb{R}^3} e^{2r_{\sigma\tau}^c}(\xi') \tilde{f}(\xi') \partial_x f(-\zeta) \partial_x f(\sigma - \zeta) \tilde{f}(\zeta) \partial_\zeta f(\sigma) d\xi d\eta d\zeta d\tau.
\]

We pick up the main leading part of (10.26) by

\[
(10.26) = \frac{i}{(2\pi)^{\frac{3}{2}}} Y_{\sigma,\tau}(\xi) \int_1^{\infty} \int_{\mathbb{R}} e^{2r_{\sigma\tau}^c}(\xi') \tilde{f}(\xi') \partial_x f(-\zeta) \partial_x f(\sigma - \zeta) \tilde{f}(\zeta) \partial_\zeta f(\sigma) d\xi d\eta d\zeta d\tau \geq \frac{i}{(2\pi)^{\frac{3}{2}}} \int_1^{\infty} \int_{\mathbb{R}} \tilde{f}(\xi') \tilde{f}(\xi') d\xi d\eta d\zeta d\tau.
\]

(10.28) is dominated by

\[
\| (10.28) \|_{L^2_t} \leq \int_1^{\infty} \tau^{-1-\eta} \| \partial_\zeta \tilde{f} \|_{L^2_t} \| \tilde{f} \|_{L^2_t} \| \tilde{f} \|_{L^2_t}^2 d\tau \lesssim 1.
\]

Letting \( \sigma = 0 \), we observe that (10.27) has a lower bound:

\[
\| (10.27) \|_{L^2_t} \geq \frac{1}{(2\pi)^{\frac{3}{2}}} \int_1^{\infty} \int_{\mathbb{R}} \tilde{f}(\xi') \tilde{f}(\xi') d\xi d\eta d\zeta d\tau.
\]

By the assumption (1.11), we see as \( t \to \infty \)

\[
\int_{\mathbb{R}} \tilde{f}(\xi') \tilde{f}(\xi') d\xi \sim \int_{\mathbb{R}} \tilde{f}^2(|U(\zeta)|) d\zeta.
\]

Therefore, we conclude for (10.26) that

\[
\| (10.26) \|_{L^2_t} \geq C|\xi|^2 \| U(\zeta) \|_{L^2_t}^2 \ln(t)
\]

for \( t \) large and some \( C > 0 \).

Moreover, by computation we have

\[
Y_{\sigma,\tau}(\xi, \eta) = [\Theta_{\sigma,\tau}(\xi, \eta) * \Upsilon_{\sigma,\tau}](\xi, \eta)
\]

\[
\Theta_{\sigma,\tau}(\xi, \eta) := c_N e^{2i\xi_0\sigma} \int_{\mathbb{R}^n} e^{-i(\xi - \eta)(\xi' + \eta')} f(y - \xi) \partial_x f(y - \frac{1}{2} \eta - \frac{1}{2} \xi) dy \partial_x f(\sigma - \zeta)
\]

\[
\Upsilon_{\sigma,\tau}(\xi, \eta) := \frac{1}{t^\mu} \bar{\xi}(\xi)(\eta) \bar{\eta}(\eta).
\]
where $\tilde{\chi}$ denotes $\mathcal{F}^{-1} \chi$. Hence, for $\nu \in (0, \frac{1}{4})$, using $|e^{i\nu} - 1| \leq |\nu|$, (10.25) is dominated by

$$
\|(10.25)\|_{L^2} \leq \int_1^\infty \left\| \left( \frac{\partial}{\partial \tau} \right)^2 + \left( \frac{\partial}{\partial \tau} \right)^2 \right\| \frac{1}{\tau} \hat{\chi}(\frac{\tau}{\tau}) \hat{\chi}(\frac{\tau}{\tau}) \right\|_{L^1_{\tau}} \tau^{-\nu} \|\langle \xi \rangle^{2\nu} f(\xi)\|_{L^4_{\xi}}^4 d\tau
\leq \int_1^\infty \tau^{-1-2\nu+2\eta} \|f\|_{H^1_{\eta}}^4 d\tau \lesssim 1.
$$

So we obtain for (10.20) that

$$
\|(10.20)\|_{L^2} \geq C\|\xi^{1/2}|U(\xi)\|_{L^2_{\xi}}^{1/2} \ln(t)
$$

for $t$ large and some $C > 0$. And combining bounds of (10.20) and (10.21), we conclude that (10.16) fulfills

$$
\|(10.16)\|_{L^2} \geq C\|\xi^{1/2}|U(\xi)\|_{L^2_{\xi}}^{1/2} \ln(t)
$$

for $t$ large and some $C > 0$.

Estimates of (10.17). To dominate (10.17), we note that in the support of (10.17), $|\eta'| \geq C r^{-\mu}$. Thus integration by parts in $\eta'$ yields

$$(10.17) = \frac{i}{\pi} \int_1^\infty \int_{\mathbb{R}^3} e^{i(2\xi^2-2\eta^2+2\sigma)} \chi_\mu(\xi')(\sigma - \xi) \hat{f}(\sigma - \xi)
\cdot \frac{1}{4\pi i} \hat{f}(\eta')(\hat{\xi}' - \hat{\xi} - \hat{\xi}')(\hat{\xi}' + \hat{\xi} + \hat{\xi}') \hat{f}(\sigma + \eta - \xi') \hat{f}(\sigma + \eta' - \xi') \hat{\xi}' \hat{\xi}' \hat{\xi}' d\xi' d\eta' d\xi d\tau.
$$

By Lemma 10.2, we thus get

$$
\|(10.17)\|_{L^2} \leq \int_1^\infty \left( \tau^{-1+2\mu}\|e^{i\Delta} f\|_{H^2_{\xi}}^2 + \tau^{-1+2\mu}\|e^{i\Delta}(xf)\|_{H^2_{\xi}}^2\|e^{i\Delta} f\|_{H^2_{\xi}}^2 \right) \|e^{i\Delta} f\|_{H^2_{\xi}}^2 d\tau
\leq \int_1^\infty \tau^{-2+3.5\mu+2\nu+2\eta} d\tau \lesssim 1.
$$

So for some $C > 0$

$$
\|(10.17)\|_{L^2} \leq C.
$$

Estimates of (10.18), (10.19). The estimates of these terms are the same as (10.17). Therefore, we conclude for the 4 order term associated with $\phi_{++}$ that

$$
\| \int_1^\infty R_{++} d\tau \|_{L^2} \geq C\|\xi^{1/2}|U(\xi)\|_{L^2_{\xi}}^{1/2} \ln(t)
$$

for $t$ large and some $C > 0$.

**Estimates of 4 order term associated with $\phi_{+-}$.** The time resonance set of $\phi_{+-}$ is

$$
\mathcal{R}_{+-}^t = \{ (\xi, \eta, \xi, \sigma) : \phi_{+-} = 0 \}.
$$

The space resonance sets are

$$
\mathcal{R}_{+,\xi}^t = \{ (\xi, \eta, \xi, \sigma) : \partial_{\xi}\phi_{+-} = 0 \} = \{ (\xi, \eta, \xi, \sigma) : \eta = 0 \}
$$
$$
\mathcal{R}_{+,\eta}^t = \{ (\xi, \eta, \xi, \sigma) : \partial_{\eta}\phi_{+-} = 0 \} = \{ (\xi, \eta, \xi, \sigma) : 2\eta - \xi = 0 \}
$$
$$
\mathcal{R}_{+,\xi}^t = \{ (\xi, \eta, \xi, \sigma) : \partial_{\xi}\phi_{+-} = 0 \} = \{ (\xi, \eta, \xi, \sigma) : 2\zeta - \sigma = 0 \}.
$$

The space-time resonance set is

$$
\mathcal{R}_{+,\xi,\eta}^t = \{ (\xi, \eta, \xi, \sigma) : \eta = \sigma = \xi = \zeta = 0 \}.
$$
The \( \phi_\pm \) part is easier than \( \phi_{\pm} \). In fact, the phrase \( \phi_\pm \) has a non-degenerate Hessian at critical points, while the Hessian of \( \phi_{\pm} \) discussed above at critical points is degenerate. Due to the non-degenerateness, the stationary phrase analysis indeed gives a \( t^{-\frac{3}{2}} \) decay.

Let
\[
\eta = \frac{1}{2} \xi' + \eta'; \quad \xi = \xi' - \xi' - \frac{\sigma}{2}; \quad \zeta = \frac{\sigma}{2} + \xi'.
\]

Then the inhomogeneous term corresponding to \( \phi_{\pm} \) reads as
\[
\int_1^t R_{\pm}d\tau := c_4 \int_1^t \int_{\mathbb{R}^3} e^{i\tau \phi_{\pm}} (\zeta - \eta)(\sigma - \xi) f(\xi) f(\eta - \xi) f(\xi - \eta) f(\sigma - \xi)d\xi d\eta d\zeta d\tau
\]
\[
= c_4 \int_1^t \int_{\mathbb{R}^3} e^{i\tau \phi_{\pm}} -2(2\xi'^2 + 2\xi'^2) f(\xi - \eta - \xi') f(\eta' + \xi' - \frac{\sigma}{2} + \xi')d\xi d\eta d\zeta d\tau.
\]

Thus, by Plancherel identity, we have
\[
\int_1^t \int_{\mathbb{R}^3} e^{i\tau \phi_{\pm}} (\zeta - \eta)(\sigma - \xi) f(\xi) f(\eta - \xi) f(\xi - \eta) f(\sigma - \xi)d\xi d\eta d\zeta d\tau
\]
\[
= \int_1^t \int_{\mathbb{R}^3} e^{i\tau \phi_{\pm}} \frac{\pi^2}{(2i)^2} \int_{\mathbb{R}^3} e^{\frac{\tau}{\tau^*} i\xi'^2 + \frac{\tau}{\tau^*} \tilde{\xi}'^2 + \frac{1}{\tau} \xi'^2 + \frac{1}{\tau} \tilde{\xi}'^2} X_{\sigma, \tau}(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})d\xi d\eta d\zeta d\tau,
\]
where we denote
\[
X_{\sigma, \tau}(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})
\]
\[
:= \mathcal{F}_{\xi', \eta', \xi'}(-1 f(\xi' - \eta' - \frac{\sigma}{2} + \xi')d\xi d\eta d\zeta.
\]

By computation,
\[
X_{\sigma, \tau}(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta})
\]
\[
= \int_{\mathbb{R}} e^{i\tau(2\xi + 2\tilde{\xi} + \tilde{\eta})} f(y) \partial_{\xi'} f(y - \tilde{\eta}) \partial_{\eta'} f(-\frac{1}{2} \tilde{\eta} + y + \tilde{\xi})d\eta.
\]

Hence, we have
\[
\int_1^t \|R_{\pm}\|_{L^2} d\tau \leq \int_1^t \tau^{-\frac{3}{2}} \|f\|^4_{L^1} d\tau \leq C
\]
for some \( C > 0 \).

Estimates of 4 order term associated with \( \phi_{\pm} \). The time resonance set of \( \phi_{\pm} \) is
\[
R^\pm = \{ (\xi, \eta, \zeta, \sigma) : \phi_{\pm} = 0 \}.
\]

The space resonance sets are
\[
R^\pm_{\xi} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\xi} \phi_{\pm} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : 2\xi = \eta \}
\]
\[
R^\pm_{\eta} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\eta} \phi_{\pm} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : 2\eta - \xi - \zeta = 0 \}
\]
\[
R^\pm_{\zeta} = \{ (\xi, \eta, \zeta, \sigma) : \partial_{\zeta} \phi_{\pm} = 0 \} = \{ (\xi, \eta, \zeta, \sigma) : 2\zeta - \sigma - \eta = 0 \}.
\]

The space-time resonance set is
\[
R^\pm_{\xi, \eta} = \{ (\xi, \eta, \zeta, \sigma) : \eta = \sigma = \xi = \zeta = 0 \}.
\]
We observe that the phrase \( \phi_{-} \) has a non-degenerate Hessian at critical points. Due to the non-degenerateness, the stationary phrase analysis indeed gives a \( r^{-2} \) decay. And the same argument of \( \phi_{+} \) gives

\[
\int_1^\infty \|R_{-}\|_L^2 d\tau \leq \int_1^\infty \tau^{-2} \|f\|^4_{H^3} d\tau \leq C
\]

for some \( C > 0 \).

Estimates of cubic term associated with \( \phi_2 \).

Recall that \( R_2 \) (see (10.9)) writes as

\[
\int_1^\infty R_2 d\tau = c_2 \int_1^\infty e^{i\phi_2}(\eta - \xi)(\sigma - \eta)\bar{f}(\xi)\bar{f}(\eta - \xi)\bar{f}(\sigma - \eta)d\xi d\eta d\tau,
\]

and \( \phi_2 = \sigma^2 - \xi^2 - (\eta - \xi)^2 - (\sigma - \eta)^2 \).

The time resonance set of \( \phi_2 \) is

\[
\mathcal{R}_{t,\xi}^{\phi_2} = \{(\xi, \eta, \zeta, \sigma) : \phi_2 = 0\}.
\]

The space resonance sets are

\[
\mathcal{R}_{s,\xi}^{\phi_2} = \{(\xi, \eta, \sigma) : \partial_{\xi} \phi_2 = 0\} = \{(\xi, \eta, \sigma) : 2\xi = \eta\}
\]

\[
\mathcal{R}_{s,\eta}^{\phi_2} = \{(\xi, \eta, \sigma) : \partial_{\eta} \phi_2 = 0\} = \{(\xi, \eta, \sigma) : 2\eta - \xi - \sigma = 0\}.
\]

The space-time resonance set is

\[
\mathcal{R}_{s,t,\xi}^{\phi_2} = \{(\xi, \eta, \sigma) : \eta = \sigma = \xi = 0\}.
\]

We observe that the phrase \( \phi_2 \) has a non-degenerate Hessian at critical points, and at the critical points the \( (\eta - \xi)(\sigma - \eta) \) term emerging from derivatives vanishes. Let

\[
\xi = \xi' + \frac{1}{3} \sigma + \frac{1}{2} \eta',
\]

\[
\eta = \frac{2}{3}(\sigma + \eta').
\]

Then by Plancherel identity, one has

\[
\int_1^\infty R_2 d\tau = c_2 \int_1^\infty \int_{\mathbb{R}^2} e^{i\xi' + i\sigma} e^{-i\xi(\xi' + i\eta' + i\sigma)} (\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi')(\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi') f(\xi' + \frac{1}{3} \sigma + \frac{1}{2} \eta')
\]

\[
\cdot \bar{f}(\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi') \bar{f}(\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi') d\xi' d\eta' d\tau
\]

\[
= \hat{c} \int_1^\infty \int_{\mathbb{R}^2} \frac{1}{6} e^{-\frac{\pi}{2} i(\xi' + i\eta')} W_{\sigma,\tau}(\xi', \eta') d\xi' d\eta' d\tau
\]

\[
= \hat{c} \int_1^\infty \int_{\mathbb{R}^2} \frac{1}{6} e^{i\xi' + i\sigma} W_{\sigma,\tau}(\xi', \eta') d\xi' d\eta' d\tau + \hat{c} \int_1^\infty \int_{\mathbb{R}^2} \frac{1}{6} e^{i\xi' + i\sigma} \left[ e^{-\frac{\pi}{2} i(\xi' + i\eta')} - 1 \right] W_{\sigma,\tau}(\xi', \eta') d\xi' d\eta' d\tau,
\]

\[(10.30)\]

where

\[
W_{\sigma,\tau}(\xi', \eta) = \mathcal{F}_{\phi_2}^{-1} \left[ \left( \frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi' \right) \left( \frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi' \right) f(\xi' + \frac{1}{3} \sigma + \frac{1}{2} \eta') \bar{f}(\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi') \bar{f}(\frac{1}{3} \sigma + \frac{1}{2} \eta' - \xi') \right].
\]

The first term in the RHS of (10.30) equals

\[
\frac{1}{9} \hat{c} \int_1^\infty \frac{1}{6} e^{i\xi' + i\sigma} \sigma^2 [\bar{f}(\frac{1}{3} \sigma)]^3 d\tau,
\]

which by integration by parts in \( \tau \) reduces to

\[
\int_1^\infty e^{i\xi' + i\tau} \left( \frac{1}{6} e^{i\xi' + i\tau} \frac{1}{3} \sigma \frac{1}{3} \sigma \right) d\tau + O(||f||_{H^{\infty}}^3),
\]

\[(10.31)\]
We claim
\[ ||\widehat{\partial_x f}||_{L^2} \leq \tau^{-\frac{1}{2}} ||f||_{H^{1,1}}^2 + \tau^{-\frac{1}{2}} ||z||_{H^{-1}}^2. \] (10.32)

If (10.32) has been proved, then we see (10.31) is dominated by
\[ ||(10.31)||_{L^2} \leq \int_1^\tau (||f||_{H^{1,1}}^2 + ||z||_{H^{-1}}^2) d\tau + O(||\tilde{f}||_{L^2}^3) \leq 1. \]

The second term in the RHS of (10.30) can be estimated as before by using \(|e^{it} - 1| \lesssim |s|^\nu\). In fact, it is dominated by
\[ \int_1^\tau \tau^{-\nu} ||f||_{H^{1,1}}^2 d\tau \leq 1. \]

Therefore, we get
\[ \| \int_1^\tau R_i d\tau \|_{L^2} \leq 1, \]
provided that (10.32) holds.

**Proof of (10.32).** From the equation of \( \partial_x f \), we observe that
\[ \partial_x f = \text{quadratic term} + \text{cubic terms} + 4 \text{ order terms} + \text{higher order terms}. \]

By Lemma 10.3, the cubic, 4 order and higher order terms can provide at least \( \tau^{-\frac{1}{2}} \) decay. Thus it suffices to prove that the quadratic term fulfills
\[ \| \int_{\mathbb{R}} e^{it\phi_1}(\sigma - \xi) \widehat{f}(\sigma - \xi) \widehat{f}(\xi) d\xi \|_{L^2} \leq \tau^{-\frac{1}{2}} ||f||_{H^{1,1}}^2. \] (10.33)

By change of variables and Plancherel identity,
\[ \int_{\mathbb{R}} e^{it\phi_1}(\sigma - \xi) \widehat{f}(\sigma - \xi) \widehat{f}(\xi) d\xi = \int_{\mathbb{R}} e^{it\phi_1} e^{-2it\xi^2} \partial_x \tilde{f}(\sigma - \xi) \partial_x \tilde{f}(\xi^* + \frac{1}{2} \sigma) d\xi^* \]
\[ = \int_{\mathbb{R}} e^{-it\xi^2} a-t^* \star e^{-\overline{a} \xi^2} [\partial_x \tilde{f}(\sigma - \xi) \partial_x \tilde{f}(\xi^* + \frac{1}{2} \sigma)] d\xi. \]

Then (10.33) follows by Hausdorff-Young inequality for convolution. **Estimates of quadratic term associated with (10.8).**

**Space-time resonance analysis of \( \phi_1 \).** The corresponding quadratic term \( R_1 \) writes as
\[ \int_1^\tau R_1 d\tau = c_0 \int_1^\tau \int_{\mathbb{R}} e^{it\phi_1}(\sigma - \xi) \widehat{f}(\sigma - \xi) \widehat{f}(\xi) d\xi d\tau. \]

Observe that
\[ \partial_x e^{it\phi_1} = 2i(\sigma - \xi). \]

Thus by integration by parts in \( \tau \), we have
\[ \int_1^\tau R_1 d\tau = -\frac{c_0}{2t} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it\phi_1} \partial_\tau \tilde{f}(\xi) \tilde{f}(\sigma - \xi) d\xi d\tau + O(\int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{f}(\xi) \tilde{f}(\sigma - \xi) d\xi d\xi). \] (10.34)

The last term in the RHS is easy to estimate, in fact by Lemma 10.4
\[ \| \int_{\mathbb{R}} \tilde{f}(\xi) \tilde{f}(\sigma - \xi) d\xi \|_{L^2} \leq ||f||_{L^2}^2 \leq \epsilon. \]
To dominate the first term in the RHS of (10.34), by symmetry it suffices to consider
\[
-\frac{1}{2i} \int_1^\infty \int_{\mathbb{R}} e^{i\tau \phi} (\partial_t \hat{f})(\xi) \hat{f}(\sigma - \xi) d\xi d\tau.
\] (10.35)

Recall that
\[
\partial_t \hat{f}(\xi) = -\frac{i c_0}{2\pi} \int_{\mathbb{R}} e^{i(\xi^2 - \xi^2)(\eta - \xi)} f(\eta) \hat{f}(\xi - \eta) d\eta \\
-\frac{i c_1}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi^2 + \xi^2)(\eta - \xi)(\eta - \xi)} f(\eta) \hat{f}(\eta - \xi) d\xi d\eta \\
-\frac{i c_2}{2\pi} \int_{\mathbb{R}^2} e^{i(\xi^2 + \xi^2)(\eta - \xi)} f(\eta) \hat{f}(\eta - \xi) d\xi d\eta
\]
where \( R_{\pm,\pm} \) denotes 4 order terms associated with \( \phi_{\pm,\pm} \). So to bound (10.35), it suffices to estimate
\[
A_1 := \int_1^\infty \int_{\mathbb{R}} e^{i(\xi^2 - \xi^2)(\eta - \xi)} f(\eta) \hat{f}(\xi - \eta) d\xi d\eta d\xi d\tau \\
A_2 := \int_1^\infty \int_{\mathbb{R}^2} e^{i(\xi^2 + \xi^2)(\eta - \xi)(\eta - \xi)} f(\eta) \hat{f}(\eta - \xi) d\xi d\eta d\tau \\
A_3 := \int_1^\infty \int_{\mathbb{R}^2} e^{i(\xi^2 + \xi^2)(\eta - \xi)} f(\eta) \hat{f}(\eta - \xi) d\xi d\eta d\tau \\
A_4 := \int_1^\infty \int_{\mathbb{R}^2} (R_{\pm,\pm} + R(\xi)) \hat{f}(\sigma - \xi) d\xi d\tau.
\]

Let \( \varphi_1 = 2\sigma \xi - 2\xi^2 + \xi^2 - (\xi - \xi)^2 \). The space resonance sets of \( \varphi_1 \) are
\[
\mathcal{R}_{\xi,\xi}^{\varphi_1} = \{ (\xi, \xi, \sigma) : \partial_\xi \varphi_1 = 0 \} = \{ (\xi, \xi, \sigma) : \sigma - 2\xi + \xi = 0 \} \\
\mathcal{R}_{\xi,\xi}^{\varphi_1} = \{ (\xi, \xi, \sigma) : \partial_\xi \varphi_1 = 0 \} = \{ (\xi, \xi, \sigma) : \xi = 2\xi \}.
\]
The space-time resonance set is
\[
\mathcal{R}_{\xi,\xi}^{\varphi_1} = \{ (\xi, \xi, \eta, \sigma) : \sigma = 2\xi = \xi = 0 \}.
\]

We observe that the phrase \( \varphi_1 \) has a non-degenerate Hessian at critical points, and at the critical points the \( \xi (\xi - \xi) \) term emerging from derivatives vanishes. Thus the same argument of \( \phi_2 \) gives
\[
\|A_1\|_{L^\infty} \leq \int_1^\infty \tau^{-1-\gamma} \|f\|_{H^3_x}^3 d\tau \leq C
\]
for some \( C > 0 \).

Let \( \varphi_2 = 2\sigma \xi - 2\xi^2 + \xi^2 - (\eta - \eta)^2 - (\xi - \xi)^2 \). The space resonance sets of \( \varphi_2 \) are
\[
\mathcal{R}_{\xi,\xi}^{\varphi_2} = \{ (\xi, \xi, \eta, \sigma) : \partial_\xi \varphi_2 = 0 \} = \{ (\xi, \xi, \eta, \sigma) : \sigma - 2\xi + \eta = 0 \} \\
\mathcal{R}_{\xi,\xi}^{\varphi_2} = \{ (\xi, \xi, \eta, \sigma) : \partial_\xi \varphi_2 = 0 \} = \{ (\xi, \xi, \eta, \sigma) : \eta = 0 \} \\
\mathcal{R}_{\xi,\xi}^{\varphi_2} = \{ (\xi, \xi, \eta, \sigma) : \partial_\eta \varphi_2 = 0 \} = \{ (\xi, \xi, \eta, \sigma) : \xi + \xi = 2\eta \}.
\]
The space-time resonance set is
\[
\mathcal{R}_{\xi,\xi}^{\varphi_2} = \{ (\xi, \xi, \eta, \sigma) : \sigma = \eta = \xi = \xi = 0 \}.
\]
We observe that the phrase \( \varphi_2 \) has a non-degenerate Hessian at critical points. Then the same argument of \( \phi_{+-} \) gives
\[
\|A_2\|_{L^\infty} \leq \int_1^t \tau^{-\frac{3}{2}} \|f\|_{L^4}^2 d\tau \leq C
\]
for some \( C > 0 \).

Let \( \varphi_3 = 2\sigma \xi - 2\xi^2 + \xi^2 - (\eta - \xi)^2 - (\eta - \xi)^2 \). The space resonance sets of \( \varphi_3 \) are
\[
R_{s,\xi}^{\varphi_3} = \{ (\xi, \zeta, \eta, \sigma) : \partial_\xi \varphi_3 = 0 \} = \{ (\xi, \zeta, \eta, \sigma) : \sigma + \eta = \xi \}
\]
\[
R_{s,\eta}^{\varphi_3} = \{ (\xi, \zeta, \eta, \sigma) : \partial_\eta \varphi_3 = 0 \} = \{ (\xi, \zeta, \eta, \sigma) : 2\eta = \xi + \zeta \}
\]
\[
R_{s,\zeta}^{\varphi_3} = \{ (\xi, \zeta, \eta, \sigma) : \partial_\zeta \varphi_3 = 0 \} = \{ (\xi, \zeta, \eta, \sigma) : \eta = 2\zeta \}.
\]
The space-time resonance set is
\[
R_{s,t}^{\varphi_3} = \{ (\xi, \zeta, \eta, \sigma) : \sigma = \eta = \xi = \zeta = 0 \}.
\]

We observe that the phrase \( \varphi_3 \) has a non-degenerate Hessian at critical points. And the same argument of \( \phi_{+-} \) gives
\[
\|A_3\|_{L^\infty} \leq \int_1^t \tau^{-\frac{3}{2}} \|f\|_{L^4}^2 d\tau \leq C
\]
for some \( C > 0 \).

For \( A_4 \), Lemma 10.3 implies
\[
\|A_4\|_{L^\infty} \leq \int_1^t \tau^{-\frac{3}{2}} \|z\|_{L^4}^2 d\tau \leq C
\]
for some \( C > 0 \).

Therefore, we conclude for \( R_1 \) that
\[
\| \int_1^t R_1 d\tau \|_{L^\infty} \leq C
\]
for some \( C > 0 \).

Estimates of higher order terms \( R \).

The estimates of higher order terms \( R \) are the same as Section 8, and in fact
\[
\| \int_1^t R d\tau \|_{L^\infty} \leq C
\]
for some \( C > 0 \).

Recall \( \hat{F}(t, \sigma) \) defined by (10.13) and its equation given by (10.14). Now, we have proved
\[
\|\hat{f}(t, \sigma)\|_{L^\infty} \geq C \|\xi^4 U(\xi)\|_{L^4}^4 \ln(t)
\]
for \( t \) large and some \( C > 0 \). Then the desired result follows since \( |\hat{f}| = \mathbb{1} \). \( \square \)

**Proposition 10.2.** Suppose that \( c_3 = 0 \) in (10.7). Under the assumptions (10.6), (1.10), (1.9), (1.11), we have as \( t \to \infty \)
\[
\|\tau \partial_\tau z\|_{L^4} \leq t^{\frac{1}{2} - \nu + 3\beta}.
\]

for \( 3\beta < \nu < \frac{1}{4} \).

**Proof.** Recall that \( z \) satisfies (10.7). Since
\[
c_5 = [\ln h]_{C(0)},\ c_4 = 2[\ln h]_{C(0)},\ c_0 = [\ln h]_{C(0)},\ c_1 = [\ln h]_{C(0)},
\]
c_5 = 0 and (10.6) together imply that
\[
c_4 = 0,\ c_0 \neq 0,\ c_1 \neq 0.
\]
So $z$ indeed satisfies

$$
\begin{align*}
\begin{cases}
\partial_t z + \Delta z &= c_0 \partial_x z \partial_y z + c_1 \tilde{z} \partial_z \partial_y z + c_2 \tilde{z} \partial_x \partial_z z + c_3 \tilde{z}^2 \partial_x \partial_y z + O(|z|^3) \partial_z \partial_y z \\
\partial_z z &= 0.
\end{cases}
\end{align*}
$$
(10.36)

Let $w = z + \kappa_1 z^2 + \kappa_2 z^3 + \kappa_3 z^4$, we have

$$
\begin{align*}
i \partial_t w + \Delta w &= \left[(c_0 + 2\kappa_1) + (c_2 + 2\kappa_1 c_0 + 6\kappa_2)\tilde{z} + (c_3 + 2\kappa_1 c_2 + 3c_0 \kappa_2 + 12\kappa_3)c_2^2\right](\partial_z z)^2 \\
&\quad + (c_1 \tilde{z} + 2\kappa_1 c_2 \tilde{z})(\partial_z z)^2 + O(|z|^3) \partial_x \partial_y z.
\end{align*}
$$

Choose $[\kappa_i]_1^{3}$ to fulfill

$$
c_0 + 2\kappa_1 = c_2 + 2\kappa_1 c_0 + 6\kappa_2 = c_3 + 2\kappa_1 c_2 + 3c_0 \kappa_2 + 12\kappa_3 = 0.
$$
(10.37)

Then

$$
i \partial_t w + \Delta w = c_1 \tilde{z}(\partial_z z)^2 + 2\kappa_1 c_2 \tilde{z}(\partial_z z)^2 + O(|z|^3) \partial_x \partial_y z.
$$
(10.38)

Moreover, one has

$$
\kappa_1 \neq 0.
$$

Define $g = e^{-itA} w$, $f = e^{-itA} z$. Then $g$ fulfills

$$
\tilde{g}(\tau, \sigma) = \tilde{g}(1, \sigma) - \frac{i c_1}{2\pi} \int_1^\tau e^{it\phi}(\eta - \xi)(\sigma - \eta)\tilde{f}(\xi)\tilde{f}(\eta - \xi)\tilde{f}(\sigma - \eta)d\eta d\tau + \int_1^\tau \mathcal{R}d\tau
$$

$$
- \frac{i}{2\pi} \int_1^\tau 2\kappa_1 c_2 e^{it\phi - i\xi}(\zeta - \eta)(\sigma - \xi)\tilde{f}(\xi)\tilde{f}(\eta - \xi)\tilde{f}(\zeta - \eta)\tilde{f}(\sigma - \xi)d\eta d\xi d\tau,
$$
(10.39)

where we denote

$$
\phi_0 = \sigma^2 + \xi^2 - (\eta - \xi)^2 - (\sigma - \eta)^2,
$$

$$
\phi_{\pm} = \sigma^2 + \xi^2 - (\eta - \xi)^2 - (\zeta - \eta)^2 - (\sigma - \xi)^2,
$$

$$
\mathcal{R} = \mathcal{F}[e^{-it\Delta} O(|z|^3)(\partial_z z)^2].
$$

We aim to bound $\|\partial_{\sigma} g\|_{L^2}$.

**Estimates of the leading cubic term.** By change of variables,

$$
\int_1^\tau e^{it\phi}(\eta - \xi)(\sigma - \eta)\tilde{f}(\xi)\tilde{f}(\eta - \xi)\tilde{f}(\sigma - \eta)d\eta d\xi
$$

$$
= \int_1^\tau e^{it\phi}(\eta - \xi)(\sigma - \eta)\tilde{f}(\xi')\tilde{f}(\eta - \xi)\tilde{f}(\sigma - \xi') + \tilde{f}(\sigma - \eta)d\xi' d\eta.
$$

Then $\partial_{\sigma}$ will not hit the phase function, and by Plancherel identity, one has

$$
\|\partial_{\sigma} \int_1^\tau e^{it\phi}(\eta - \xi)(\sigma - \eta)\tilde{f}(\xi)\tilde{f}(\eta - \xi)\tilde{f}(\sigma - \eta)d\xi d\eta\|_{L^2}
$$

$$
\leq \int_1^\tau \|\partial_{\sigma} \tilde{g}\|_{L^2}\|\tilde{f}\|_{H^1} d\tau
$$

$$
\leq (t)^{\beta}.
$$

**Estimates of 4 order terms $\mathcal{R}_{\zeta, \xi}$.** We have seen in the proof of Proposition 10.2 that the phrase $\phi_{\pm}$ has a non-degenerate Hessian at critical points, and thus the stationary phase analysis indeed gives a $t^{-\frac{3}{2}}$ decay. The inhomogeneous term
corresponding to \( \phi_+ \) reads as

\[
\partial_\tau \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} (\zeta - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta d\tau \\
= \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} 2i\tau (\zeta - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta d\tau \\
+ \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} \partial_\sigma [((\zeta - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi)) d\xi d\eta d\zeta d\tau \\
:= I + II.
\]

By Plancherel identity, \( II \) is bounded by

\[
||II||_L^2 \leq \int_1^\zeta ||\xi||_{W^{1, \infty}}^2 ||\xi||_{H^1} d\tau \leq \int_1^\zeta \tau^{-\frac{1}{2}} d\tau \leq 1.
\]

By the identity \( \zeta = \xi + (\eta - \xi) + (\zeta - \eta) \), \( I \) further expands as

\[
I = \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} 2i\tau (\zeta - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta d\tau \\
+ \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} 2i\tau (\zeta - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta d\tau \\
+ \int_1^\zeta \int_{\mathbb{R}^3} e^{i\tau \xi} 2i\tau (\zeta - \eta)^2 (\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta d\tau \\
:= I_1 + I_2 + I_3.
\]

Let

\[
\eta = \frac{1}{2} \xi'; \xi = \xi' - \zeta' - \frac{\sigma}{2}; \zeta = \frac{\sigma}{2} + \zeta',
\]

Then by Plancherel identity, we obtain

\[
I_1 = \int_1^\zeta e^{i\tau \xi} \pi^{\frac{3}{2}} (2\pi)^\frac{3}{2} \int_{\mathbb{R}^3} \tau^{-\frac{1}{2}} e^{-\frac{\xi^2}{\tau} + \frac{\eta^2}{\tau} + \frac{\zeta^2}{\tau}} \tilde{X}_{\tau, \xi, \eta, \zeta} d\xi d\eta d\zeta d\tau,
\]

where \( \tilde{X}_{\tau, \xi, \eta, \zeta} \) is given by

\[
\tilde{X}_{\tau, \xi, \eta, \zeta} = \frac{1}{\tau} \theta_{\tau, \xi, \eta, \zeta} \left[ (\xi - \eta)(\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) \right] \\
= \int_\tau^\zeta e^{i\tau \xi - \zeta} \frac{3}{2} (\sigma - \xi) \hat{f}(\xi) \hat{f}(\eta - \xi) \hat{f}(\zeta - \eta) \hat{f}(\sigma - \xi) d\xi d\eta d\zeta.
\]

As before, expanding \( e^{-\frac{\xi^2}{\tau} + \frac{\eta^2}{\tau} + \frac{\zeta^2}{\tau}} \) to 1 and the difference, one has

\[
I_1 = b_1 \left[ \int_1^\zeta e^{i\tau \xi} \tau^{-\frac{1}{2}} \sigma f\left( -\frac{\sigma}{2} \right) \hat{f}\left( \frac{\sigma}{2} \right) d\tau \right] \\
+ b_2 \left[ \int_1^\zeta e^{i\tau \xi} \tau^{-\frac{1}{2}} \int_{\mathbb{R}^3} |e^{-\frac{\xi^2}{\tau} + \frac{\eta^2}{\tau} + \frac{\zeta^2}{\tau}} - 1| \tilde{X}_{\tau, \xi, \eta, \zeta} d\xi d\eta d\zeta d\tau \right] \\
:= I_{11} + I_{12},
\]

where \( b_1, b_2 \) are some universal constants. By integration by parts in \( \tau \), \( I_{11} \) is dominated by

\[
||I_{11}||_L^2 \leq \||\int_1^\zeta e^{i\tau \xi} \tau^{-\frac{1}{2}} \sigma \partial_\tau \hat{f}\left( -\frac{\sigma}{2} \right) \hat{f}\left( \frac{\sigma}{2} \right) d\tau||_{L^1} + ||\sigma \hat{f}\left( -\frac{\sigma}{2} \right)||_{L^1} ||\hat{f}||_{L^2}^2 d\tau \\
\leq \int_1^\zeta \tau^{-1} ||f||_{H^1}^2 ||f||_{L^1} ||\hat{f}||_{L^2} d\tau + \epsilon^4 \\
\leq \langle t \rangle^{2\beta},
\]

40
where we applied (10.32) in the second inequality.

Using $|e^{it} - 1| \leq |s|^\beta$, by Plancherel identity and change of variables, we find $I_{12}$ is controlled by

$$
\|I_{12}\|_{L^2_t} \lesssim \int_1^\infty \tau^{\frac{1}{2} - r} \|f\|_{L^2_t}(\tau)^{2r} \partial_x f \|_{L^2_t}^3 d\tau \lesssim \langle \tau \rangle^{\frac{1}{2} - r + 3\beta}.
$$

The estimate of $I_2$ is the same as $I_1$. For $I_3$, similarly we have

$$
\|I_3\|_{L^2_t} \lesssim \int_1^\infty \tau^{\frac{1}{2} - r} \|f\|_{H^3_t} \|f\|_{H^2_t}^3 d\tau + \int_1^\infty \tau^{-1} \|f\|_{H^1_t} \|f\|_{H^2_t} \|f\|_{L^2_t}^3 d\tau + \|f\|_{L^2_t} \|f\|_{L^2_t}^3 d\tau \lesssim \langle \tau \rangle^{\frac{1}{2} - r + 3\beta}.
$$

Hence, we obtain

$$
\|\partial_\sigma \int_1^\infty R_{\tau -} d\tau\|_{L^2_t} \lesssim \langle \tau \rangle^{\frac{1}{2} - r + 3\beta}.
$$

for some $3\beta < r < \frac{1}{4}$.

And for the high order term $R$, we have

$$
\|\partial_\sigma \int_1^\infty R d\tau\|_{L^2_t} \lesssim \int_1^\infty \|L \left( O(|z|^3)(\partial_x z)^2 \right) \|_{L^2_t} d\tau \lesssim \epsilon^4 \int_1^\infty \langle \tau \rangle^{-1 + \epsilon} + \langle \tau \rangle^{-2 + \beta} d\tau
$$

Combining the above three results on cubic, 4 order and higher order terms, we infer from (10.39) that

$$
\|\partial_\sigma \hat{g}(t, \sigma)\|_{L^2_t} \lesssim \langle t \rangle^{\frac{1}{2} - r + 3\beta},
$$

which further gives

$$
\|L(z + k_1 z^2 + k_2 z^3 + k_3 z^4)\|_{L^2_t} \lesssim \langle t \rangle^{\frac{1}{2} - r + 3\beta}. \quad (10.40)
$$

Since (1.10) implies

$$
\|L(z^2)\|_{L^2_t} \leq \|Lz\|_{L^2_t} \|z\|_{H^1_t}^2 + t \|z\|_{L^2_t} \|z\|_{H^1_t} \leq 1
$$

and

$$
\|L(z^3)\|_{L^2_t} \leq \|Lz\|_{L^2_t} \|z\|_{H^2_t} \|z\|_{H^1_t} \leq 1,
$$

we get from (10.40) that

$$
|k_1| \|L(z^2)\|_{L^2_t} \lesssim \langle t \rangle^{\frac{1}{2} - r + 3\beta}.
$$

And since

$$
L(z^2) = zLz - 2t(\partial_x z)z
$$

$$
\|Lz\|_{L^2_t} \lesssim \langle t \rangle^\beta, \|\partial_x z\|_{L^2_t} \lesssim \langle t \rangle^{-\frac{1}{2}},
$$

we finally obtain that by $k_1 \neq 0$ that

$$
\|2t(\partial_x z)z\|_{L^2_t} \lesssim \langle t \rangle^{\frac{1}{2} - r + 3\beta}.
$$

\[\square\]
10.3 End of Proof to Theorem 1.2

Assume (10.6), (1.9), (1.10), (1.11) and $U \neq 0$. We consider two cases.

Case 1. Suppose that $c_5 \neq 0$ in (10.7). Then Proposition 10.1 shows for $t$ large and some $C > 0$

$$\|\tilde{w}(t)\|_{L^\infty} \geq C\|\zeta^2|U(\zeta)|\|^4 \ln(t).$$

This implies $\|\tilde{w}(t)\|_{L^\infty}$ grows at least as fast as $\ln(t)$. But (1.10) shows $\|\tilde{w}(t)\|_{L^\infty} \leq 1$, thus yielding contradiction. In other words, $U$ must be constantly zero, and thus $w \equiv 0$ by almost conservation of mass.

Case 2. Suppose that $c_5 = 0$ in (10.7). Then Proposition 10.2 shows for $t \geq 1$

$$\|\tilde{z}\partial_\tilde{z}\|_{L^2} \leq t^{-\frac{1}{2} + \nu + 3\beta}$$

with $3\beta < \nu < \frac{1}{2}$. By (9.4), we have

$$z(t, x) = \frac{e^{it}}{(4it)^3} F[e^{-it\Delta}z(t)] \left(2t, \frac{x}{2t}\right) + R(t, x)$$

where $R(t, x)$ and $R(t, x)$ satisfy (9.6), (9.7) as well, i.e.,

$$t^\frac{3}{2} \|R(t, x)\|_{L^\infty} + t^\frac{1}{2} \|\tilde{R}(t, x)\|_{L^\infty} \leq \|e^{-it\Delta}z(t)\|_{H^{6\beta}} \leq \|z\|_{L^2} + \|L\|_{L^2} \leq \langle t \rangle^\beta,$$

$$t^\frac{3}{2} \|R(t, x)\|_{L^\infty} + t^\frac{1}{2} \|\tilde{R}(t, x)\|_{L^\infty} \leq \|e^{-it\Delta}z(t)\|_{H^{6\beta}} \leq \sum_{j=0, 1} \|L_\gamma^{j}z\|_{L^2} + \|\tilde{z}\|_{H^1} \leq \langle t \rangle^\beta.$$}

So (1.10) and (1.11) imply

$$\|\tilde{z}\partial_\tilde{z}\|_{L^2} \sim \|\frac{1}{(4it)} U(\frac{x}{2t})\|_{L^2} \sim \|\frac{1}{4it} U(\frac{x}{2t})\|_{L^2} \sim t^{-\frac{1}{2}} \|x^2 U\|_{L^2},$$

which contradicts with (10.41). Thus $U = 0$, and $z \equiv 0$ by almost conservation of mass.

11 Proof of Theorem 1.3

Since Theorem 1.3 does not assume (1.3), the new function $w := z + \gamma_1 z^2 + \gamma_3 z^3 + \gamma_3^2 z^4$ now solves

$$i\tilde{\partial}w + \Delta w = c\tilde{w}(\tilde{\partial}w)^2 + v_2 \tilde{w}(\tilde{\partial}w)^2 + v_3 \tilde{w}(\tilde{\partial}w)^2 + O(|w|^3)(\tilde{\partial}w)^2,$$

where $c = -\frac{1}{2}K(Q)h_0$.

First, we prove an abstract result which ensures the existence of wave operators under the assumption that there exists a good approximate solution.

Lemma 11.1. Let $N$ be a Riemannian surface. Given $Q \in N$, denote $c = -\frac{1}{2}K(Q)h_0$. Assume that for some sufficiently large $m$, sufficiently small $\epsilon$, and some $\nu > 0$, there exists a function $\nu$ satisfying

$$\|i\tilde{\partial}v + \Delta v - (\tilde{\partial}v)^2(c\tilde{v} + v_2 \tilde{v}^2 + v_3 \tilde{v}^3))\|_{L^\infty} \leq t^{-\frac{1}{2} + \nu},$$

$$\|v(t)\|_{H^1} \leq \epsilon,$$

$$\|v(t)\|_{H^\infty} \leq t^{-\frac{1}{2}} \epsilon,$$

for all $t \geq N_0 \geq 1$. Then there exist a constant $\theta \in \left(\frac{1}{2}, 1\right)$ and an initial data $w_0$ evolving to a global solution of (1.1) so that

$$\sup_{t \in [0, 1]} \|w(t) - v(t, x)\|_{L^2} \leq 1,$$

where $w$ is the well chosen local complex coordinate near $Q$ such that $w(Q) = 0$ and (1.11) holds.
**Proof.** **Step 1.** For simplicity, write the $O(|w|^3)(\partial_i w)^2$ in (11.1) as $K(w)$. Given $N \in \mathbb{Z}_+$, consider the equation

$$w_N(t) = v(t, x) + i \int_t^\infty e^{i(t-\tau)\Delta}[\{c\overline{w_N} + v_2\overline{w_N}w_N + v_3\overline{w_N}^2\}](\partial_i w_N)^2 + K(w_N) - (i\partial_i v + \Delta v)]d\tau.$$  

(11.6)

It is easy to check $w_N(t)$ solves the local equation (1.19) with initial data $w_N(0, x) = v(0, x)$. By the assumption (11.3), for $N \geq N_0$

$$||w_N(0, x)||_{H^1} = ||v(0, x)||_{H^1} \leq \epsilon.$$  

Thus by Section 2, $w_N(t)$ is a global solution of 1D SMF in $t \in \mathbb{R}$.

**Step 2.** For simplicity of notations, we drop $N$ and write $w$ instead of $w_N$. Let

$$R(v) = i\partial_i v + \Delta v - (c\overline{v} + v_2\overline{v}v + v_3\overline{v}^2)(\partial_i v)^2.$$  

Rewrite (11.6) as

$$w(t) = v(t, x) + i \int_t^\infty e^{i(t-\tau)\Delta}[K(w) + R(v) + G(\tau)]d\tau$$

$$G = 2c\overline{v}\partial_i (w-v)\partial_i v + c\overline{w} - v\overline{\partial_i v}\partial_i v + c\overline{z} - v\overline{\partial_i (w-v)\partial_i v}$$

$$+ 2v_2\overline{v}\overline{v}\partial_i (w-v)\partial_i v + v_2\overline{w} - v\overline{\partial_i v}\partial_i v + v_2\overline{v}(w-v)\partial_i v^2\partial_i v$$

$$+ 2v_3\overline{v}^2\partial_i (w-v)\partial_i v + 2v_3\overline{w} - v\overline{v}\partial_i v\partial_i v + \tilde{R}_2 + \tilde{R}_3 + \tilde{R}_4,$$  

where $\tilde{R}_2, \tilde{R}_3, \tilde{R}_4$ denote quadratic, cubic and respectively 4 order terms in $w - v$. Write $G$ as

$$G = 2c\overline{v}\partial_i (z-v)\partial_i v + 2v_2\overline{v}\overline{v}\partial_i (w-v)\partial_i v + 2v_3\overline{v}^2\partial_i (w-v)\partial_i v + \mathcal{G}.$$  

Note that the key troublesome derivative loss occurs in

$$2c_0\overline{v}\partial_i (z-v)\partial_i v + 2v_2\overline{v}\overline{v}\partial_i (w-v)\partial_i v + 2v_3\overline{v}^2\partial_i (w-v)\partial_i v,$$  

and we will see the derivative loss in $\mathcal{G}$ is harmless since it is at least quadratic in $w - v$. Let

$$\mathcal{W} = \int_\mathbb{R} W(v, \overline{v})|w - v|^2dx.$$  

Then we find

$$\frac{d}{dt}\mathcal{W} = \int_\mathbb{R} (W_v \partial_i v + W_{\overline{v}} \partial_i \overline{v})|w - v|^2dx + \int_\mathbb{R} W[\overline{w} - \overline{v}\overline{\partial_i (w-v)\partial_i v}]dx,$$  

which further expands as

$$\int_\mathbb{R} (iW_v \Delta v - i\overline{W_v} \Delta \overline{v})|w - v|^2dx + \int_\mathbb{R} W[\overline{w} - \overline{v}\overline{\partial_i (w-v)\partial_i v}]dx$$

$$+ \int_\mathbb{R} W(-iR(v) + i\overline{R(v)})|w - v|^2dx$$

$$+ \int_\mathbb{R} W[-2ic\overline{w} - \overline{v}(K + R(v) + \mathcal{G}) + i(w-v)(K + R(v) + \mathcal{G})]dx$$

$$+ \int_\mathbb{R} W[-2ic\overline{w} - \overline{v}(\overline{v}\partial_i v\partial_i v(w-v)) + 2ic(w-v)v\overline{\partial_i v}\partial_i v]dx$$

$$+ \int_\mathbb{R} -iW(v-w)\overline{v}^2\partial_i (w-v)\partial_i v + 2v_3\overline{v}^2\partial_i (w-v)\partial_i v$$

$$+ \int_\mathbb{R} iW(v-w)(2v_2\overline{v}^2\partial_i (w-v)\partial_i v + 2v_3\overline{v}^2\partial_i \overline{v}\partial_i v)dx.$$  

(11.7)
By integration by parts,
\[
\int_{\mathbb{R}} (W_i \bar{v} \Delta v - i W_i \bar{v} \Delta \bar{v}) |w - v|^2 \, dx + \int_{\mathbb{R}} W |w - v| \Delta (w - v) - i (w - v) \Delta w - v |\, dx \\
= - \int_{\mathbb{R}} (i W_i \bar{\partial}_x v - i W_i \bar{\partial}_x \bar{v}) [(w - v) \partial_x (\bar{w} - v) + (\bar{w} - v) \partial_x (w - v)] \, dx \\
- \int_{\mathbb{R}} [(i W_i \partial_x (w - v) + i W_i \partial_x \bar{v}) (w - v) \partial_x (w - v) - i (w_i \partial_x v + i W_i) (w - v) \partial_x (w - v)] \, dx \\
- \int_{\mathbb{R}} [i (\partial_x W_i) \partial_x v - i (\partial_x W_i) \partial_x \bar{v}] |w - v|^2 \, dx.
\]

Letting
\[
W(z, \bar{z}) = 1 - c|z|^2 - \frac{1}{2} v_2 z^2 \bar{z} - v_3 \bar{z}^2 z,
\]
from Lemma 3.1, we see
\[
W(z, \bar{z}) (-2c \bar{z} - 2v_2 \bar{z}^2 - 2v_3 \bar{z}^2) - 2W_\xi = O(|z|^3),
\]
and find that
\[
[(11.7) + (11.8) + (11.9) + (11.10)]
\leq \int_{\mathbb{R}} |\partial_x (w - v)||w - v||v|^3 |\partial_x v| \, dx + \int_{\mathbb{R}} |(\partial_x W_i) \partial_x v||w - v|^2 \, dx.
\]

Hence we arrive at
\[
\frac{d}{dt} W \leq \int_{\mathbb{R}} [- i W_i R(v) + i W_i \bar{R}(\bar{v})] |w - v|^2 \, dx + \int_{\mathbb{R}} W |(w - v)(K + R(v) + G)| \, dx \\
+ \int_{\mathbb{R}} |\partial_x (w - v)||w - v||v|^3 |\partial_x v| \, dx + \int_{\mathbb{R}} |(\partial_x W_i) \partial_x v||w - v|^2 \, dx.
\]

Since ||v||_{L^2} \leq 1, we see
\[
W \sim 1.
\]

Thus we conclude
\[
\left| \frac{d}{dt} W \right| \leq ||R(v)||_{L^2}^2 W + ||w - v||_{L^2} ||K||_{L^2} + ||R(v)||_{L^2} ||G||_{L^2} + W ||v||_{L^2} ||\partial_x v||_{L^2}^2 \\
+ ||w - v||_{L^2} ||w - v||_{L^2} ||v||_{L^2}^4 + W ||v||_{L^2}^2.
\]

Noting that ||w - v||_{H^1} \leq \epsilon by energy conservation, we further have
\[
\left| \frac{d}{dt} W \right| \leq ||R(v)||_{L^2}^2 W + \sqrt{W} ||K||_{L^2} + ||R(v)||_{L^2} ||G||_{L^2} + W ||v||_{L^2} ||\partial_x v||_{L^2}^2 \\
+ \sqrt{W} ||v||_{L^2}^2 + W ||v||_{L^2}^2.
\]

**Step 3.** Take \( \theta \in (\frac{1}{2}, \frac{1}{2} + \nu) \) and \( m \) large to fulfill
\[
-\theta + \frac{9}{2m}(\theta + 1) + \frac{1}{2} < 0. \tag{11.11}
\]

Assume that \( S \in [N_0, N] \) is the smallest time such that
\[
\sup_{t \in [S, \infty]} [(t)\|w(t) - v(t)||_{L^2} + (t)^{-1}\|w(t) - v(t)||_{H^1(\Gamma_H^\nu)}] \leq \epsilon.
\]
Thus Section 2.1 implies for all $t$

$$\|w(t) - v(t)\|_{L^2_w} \leq \|w(t) - v(t)\|_{L^2_w}^{1/2} \|\partial_x^m w(t) - v(t)\|_{L^2_w}^{1/2} \leq (t)^{-\frac{1}{2} + \frac{1}{2}(\theta + 1)} \epsilon \leq \epsilon \quad (11.12)$$

$$\|\partial_x w(t) - v(t)\|_{L^2_w} \leq \|w(t) - v(t)\|_{L^2_w}^{1/2} \|\partial_x^m w(t) - v(t)\|_{L^2_w}^{1/2} \leq (t)^{-\frac{1}{2} + \frac{1}{2}(\theta + 1)} \epsilon \leq (t)^{-\frac{1}{2}} \epsilon. \quad (11.13)$$

Since for $t \geq N_0$, $v(t)$ also satisfies (11.4), we obtain by (11.13) and (11.12) that for $t \in [S, N]$

$$\|\partial_x w(t)\|_{L^2_w} \leq \epsilon t^{-\frac{1}{2}}$$

$$\|w(t)\|_{L^2_w} \leq \epsilon.$$

Transferring this bound to $u$, one has

$$\sup_{t \in [S, N]} \epsilon t^\frac{1}{2} \|\partial_x u(t)\|_{L^2_w} \leq \epsilon.$$

Thus Section 2.1 implies for all $t \in [S, N], k \in [1, m],$

$$\|u(t)\|_{W^{k,2}(\mathbb{R}^2)} \leq \epsilon, (1 + t)^r,$$

which further gives

$$\sup_{t \in [S, N]} (1 + t)^{-r} \|w(t)\|_{H^1 \cap H^2_t} \leq \epsilon.$$

Hence, we conclude for Step 3.1 that

$$\sup_{t \in [S, N]} (1 + t)^{-1} \|w(t) - v(t)\|_{H^1 \cap H^2_t} \leq \sup_{t \in [S, N]} (1 + t)^{-r} \|w(t)\|_{H^1 \cap H^2_t} \leq \epsilon, \ll \epsilon.$$

### Step 3.2

Let’s bound the inhomogeneous term. For $t \in [S, N]$, (11.12), (11.13) and bootstrap assumption imply

$$\|K(w)\|_{L^2_w} \leq \|w\|_{W^{1,\infty}_w} \|\partial_x w\|_{L^2_w} \leq \epsilon t^{-2}$$

$$\|G\|_{L^2_w} \leq \|w - v\|_{W^{1,\infty}_w} \|w - v\|_{W^1_{L^2}} + \|v\|_{W^1_{L^2}}^2 \|w - v\|_{L^2_w} \leq \frac{1}{2} \|w - v\|_{W^1_{L^2}} + \|w - v\|_{W^1_{L^2}} \|w - v\|_{W^1_{L^2}} + \|w - v\|_{W^1_{L^2}}^3 \|w - v\|_{L^2_w} \leq \epsilon^2 t^{-2} + \epsilon^3 t^{-1} + \epsilon^4 t^{-3} \leq \epsilon^2$$

where we also used the conservation of energy. Then for $t \in [S, N]$, we get

$$\|\partial_x (t, x)\|_{L^2_w} \approx \mathcal{W}(t)$$

$$\leq \int_1^t \left[ \|\mathcal{R}(v)\|_{L^2_w} + \|K(w)\|_{L^2_w} + \|G\|_{L^2_w} \right] w - v L^2_w d\tau$$

$$\leq \int_1^t \left[ \epsilon t^{-\frac{1}{2} - \frac{1}{2} - \frac{1}{2}} + \epsilon^2 t^{-2} + \epsilon^3 t^{-1} + \epsilon^4 t^{-3} \right] d\tau$$

Therefore, taking $\theta \in (\frac{1}{2}, \frac{1}{2} + \frac{1}{2})$ and letting $m$ be sufficiently large to satisfy (11.11), we arrive at

$$\sup_{t \in [S, N]} \langle t \rangle^\theta \|w(t) - v(t, x)\|_{L^2_w} \leq \epsilon + Ce^2.$$
Lemma 11.2.

Step 4. In Step 3, for each given \( N \geq N_0 \), we have constructed \( w_N(t) \) such that

\[
\sup_{t \in [N_0, N]} \langle t \rangle^\rho \| w_N(t) - v(t, x) \|_{L^2_x} \leq C \epsilon^3
\]

From (11.3) follows.

\[
\| w_N(t) \|_{H_x^{s+}} \leq (1 + t)^\epsilon.
\]

By compactness, \( w_N(t) \) converges to some function \( w(t) \in C([0, T]; H^s) \) strongly in \( C([0, T]; H^{s-1}) \) for any given \( T \in \mathbb{R}^+ \). By Sobolev embedding, \( w \) is at least \( C^2 \) and solves the SMF equation point-wisely. Moreover, \( w(t) \) satisfies

\[
\sup_{t \geq N_0} \langle t \rangle^\rho \| w(t) - v(t, x) \|_{L^2_x} \leq C \epsilon^3,
\]

from which (11.5) follows. \( \square \)

The following four lemmas together prove the existence of approximate solution \( w_{ap} \). We begin with the first time correction.

**Lemma 11.2.** Assume that

\[
\sum_{0 \leq j \leq 2} \| (x) \psi^{(j)}(x) \|_{L^2_x} \leq \epsilon. \tag{11.14}
\]

Let

\[ v_1(t, x) := \frac{1}{(2it)^{\frac{3}{2}}} \psi\left( \frac{x}{2t} \right) e^{it\frac{x^2}{2} i\psi(\frac{x}{2t})} \ln(2t). \]

Then we have

\[
\| i\partial_x v_1 + \Delta v_1 - (\partial_x v)^2 c v_1 \|_{L^2_x} \leq t^{-2} (\ln t)^2. \tag{11.15}
\]

**Proof.** The proof is a straightforward calculation. To avoid the formulas covering too much space, we introduce the notation

\[
\Psi(x, t) := \psi\left( \frac{x}{2t} \right) \bar{\psi}\left( \frac{x}{2t} \right) + \bar{\psi}'\left( \frac{x}{2t} \right) \psi\left( \frac{x}{2t} \right).
\]

For \( \partial_\nu v \), one has

\[
\partial_\nu \left[ \frac{1}{(2it)^{\frac{3}{2}}} e^{it\frac{x^2}{2} i\psi(\frac{x}{2t})} \ln(2t) \right]
\]

\[
= \frac{1}{(2it)^{\frac{3}{2}}} e^{it\frac{x^2}{2} i\psi(\frac{x}{2t})} \ln(2t) \left[ -\frac{1}{2} r^{-1} \psi'(\frac{x}{2t}) - \frac{x}{2t} \psi'(\frac{x}{2t}) - \frac{|x|^2}{4t^2} \psi(\frac{x}{2t}) \right]
\]

\[
+ \frac{1}{(2it)^{\frac{3}{2}}} e^{it\frac{x^2}{2} i\psi(\frac{x}{2t})} \ln(2t) \left[ \psi\left( \frac{x}{2t} \right) \bar{\psi}\left( \frac{x}{2t} \right) \left[ -\frac{i}{2} \frac{x^2}{2t^2} \bar{\psi}(\frac{x}{2t}) \right] \ln(2t) - \frac{i}{2} \frac{x^3}{8t^3} \Psi(x, t) \ln(2t) \right]
\]

\[
+ \frac{1}{(2it)^{\frac{3}{2}}} e^{it\frac{x^2}{2} i\psi(\frac{x}{2t})} \ln(2t) \left[ \psi\left( \frac{x}{2t} \right) \bar{\psi}\left( \frac{x}{2t} \right) \right].
\]
And \( \partial_v^2 \) is given by

\[
\partial_v^2 \left[ \frac{1}{(2it)^{\frac{3}{2}}} e^{i\frac{x^2}{2t}} \phi \left( \frac{x}{2t} \right) \right] = \frac{1}{(2it)^{\frac{3}{2}}} e^{i\frac{x^2}{2t}} \phi' \left( \frac{x}{2t} \right) + \frac{1}{(2it)^{\frac{3}{2}}} e^{i\frac{x^2}{2t}} \phi \left( \frac{x}{2t} \right) \left[ i \frac{x}{2t} \phi \left( \frac{x}{2t} \right) + \frac{ic}{2t} \right] \ln(2t) \psi(x,t)
\]

By computation, \( \bar{v} (\partial_v)^2 \) is

\[
c_v (\partial_v)^2 \left[ \frac{1}{(2it)^{\frac{3}{2}}} e^{i\frac{x^2}{2t}} \phi \left( \frac{x}{2t} \right) \right] = \frac{c}{(2it)^{\frac{3}{2}}} e^{i\frac{x^2}{2t}} \phi \left( \frac{x}{2t} \right) \left[ i \frac{x}{2t} \phi \left( \frac{x}{2t} \right) + \frac{ic}{2t} \right] \ln(2t) + O_{L_2} \left( \frac{\ln t}{t} \right)^2.
\]

Therefore,

\[
|\bar{v} (\partial_v) v_1 + \partial_v v_1 - c v \phi | \leq r^{-\frac{5}{2}} \sum_{j=1,2,3} \sum_{j=0,1,2} (1 + |\ln t|^2 + 1 \frac{c}{2t} \psi \phi \phi^j (\frac{x}{2t}) |^2).
\]

By (11.14), \( v_1 \) satisfies (11.15).

The following lemma is for constructing the second time correction of approximate solution.

**Lemma 11.3.** Let

\[
v_2(x,t) = \frac{iv_2}{8tt} \left[ \phi \left( \frac{x}{2t} \right) + \frac{ic}{2t} \right] \ln(2t) \psi(x,t) \left[ i \frac{x}{2t} \phi \left( \frac{x}{2t} \right) \right] \ln(2t).
\]

Then we have

\[
(i\partial_t + \Delta)(v_1 + v_2) - c \bar{v} (\partial_v)^2 - v_2 \bar{v} (\partial_v)^2 = O_{L_2} \left( r^{-\frac{4}{2}} (\frac{\ln t}{t})^2 \right) + O_{L_2} \left( r^{-\frac{5}{2}} (\ln t)^2 \right)
\]

**Proof.** Let

\[
v_2 = \frac{1}{r^2} \bar{u} \left( \frac{x}{2t} \right) e^{i\frac{x^2}{2t}} \phi \left( \frac{x}{2t} \right) \ln(2t).
\]

By Lemma 11.2, it suffices to find \( k, \gamma \) and \( \bar{u} \) such that

\[
(i\partial_t + \Delta)v_2 - v_2 \bar{v} (\partial_v)^2 = O_{L_2} \left( r^{-\frac{4}{2}} (\frac{\ln t}{t})^2 \right) + O_{L_2} \left( r^{-\frac{5}{2}} (\ln t)^2 \right).
\]

For the sake of simplicity, we denote

\[
\Phi := e^{i\frac{x^2}{2t} + i\frac{xy}{4t^2} \phi (\phi \phi^j (\frac{x}{2t}) \ln(2t))}
\]

\[
\Psi := \psi \left( \frac{x}{2t} \right) \psi \left( \frac{x}{2t} \right) + \psi \left( \frac{x}{2t} \psi \left( \frac{x}{2t} \right) \right).
\]

By computation,

\[
i\partial_t v_2 = -i \frac{k}{r^{2+1}} \bar{u} \left( \frac{x}{2t} \right) \phi + \frac{i}{r^2} \bar{u} \left( \frac{x}{2t} \right) \left( \frac{x}{2t} \right) \Psi
\]

\[
+ \frac{i}{r^2} \bar{u} \left( \frac{x}{2t} \right) \Phi \left( 4 + i\frac{xy}{4t^2} \phi \left( \frac{x}{2t} \right) \ln(2t) - \frac{i\gamma x^3}{16t} \ln(2t) \Psi + \frac{i\gamma x^3}{8t^2} \phi \left( \frac{x}{2t} \right) \right).
\]
And we have

\[
\frac{\nu^2}{\nu^3 + 2} \left( \frac{\nu}{2t} \right) \Phi + \frac{1}{\nu^{k+1}} \nu \left( \frac{\nu}{2t} \right) \left( \frac{i}{y_\nu x} + \frac{i c x y_\nu}{4 t^2} \phi \left( \frac{y_\nu}{2t} \right) \right) \ln(2t) + \frac{i c y x^2}{16 t^3} \psi \ln(2t) \right) \Phi 
\]

Note that

\[
i\partial_t v_2 + \nu^2 v_2 = \frac{\gamma x^2}{4 t^2} \nu \left( \frac{x}{2t} \right) \Phi + O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

Recall also that

\[
v_2 v_1 v_1 (\partial_x v_1)^2 = e^{\frac{i}{2}} v_2 v_1 (\partial_x v_1)^2 + O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

Let \( \gamma = 2, k = 2, \) and \( \dot{U} \) be

\[
\bar{U}(\gamma) = \frac{i v_2}{4} \nu \left( \frac{y_\nu}{2t} \right)^2 \psi^2(y_\nu). 
\]

Then we get

\[
(i\partial_t + \Delta)(v_1 + v_2) - c v_1 v_1 (\partial_x v_1)^2 = O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

The following lemma is the third time correction of approximate solution.

**Lemma 11.4.** Let

\[
v_3 = \frac{1}{t} \bar{Q} \left( \frac{x}{2t} \right),
\]

where

\[
\bar{Q}(y) = -\frac{1}{4} \int_0^\gamma i v_3 \nu \left( \frac{y_\nu}{2t} \right)^2 d\gamma. 
\]

Then we have

\[
(i\partial_t + \Delta)(v_1 + v_2 + v_3) - c v_1 v_1 (\partial_x v_1)^2 = O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

**Proof.** By Lemma 11.3, it suffices to prove

\[
(i\partial_t + \Delta) v_3 - v_3 v_1 v_1 (\partial_x v_1)^2 = O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

Recall also that

\[
v_3 v_1 v_1 (\partial_x v_1)^2 = \frac{v_3 x^2}{16 t^4} \nu \left( \frac{x}{2t} \right)^4 + O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

By computation,

\[
i\partial_t v_3 + \Delta v_3 = \frac{i}{t^2} \bar{Q} \left( \frac{x}{2t} \right) - \frac{i x}{2t^2} \bar{Q} \left( \frac{x}{2t} \right) + O_{L_2^\infty} (r^{-(k+1)}(\ln t)^2) \cap O_{L_2^\infty} (r^{-2}(\ln t)^2). 
\]

Observe that \( \bar{Q} \) solves

\[
\nu \bar{Q}'(y) + \bar{Q} = \frac{v_3 i}{4} \nu \left( \frac{y_\nu}{2t} \right)^2. 
\]

Then we get the desired result. \( \square \)
The following lemma is the forth time correction of approximate solution.

**Lemma 11.5.** Let

\[ v_4 = \frac{1}{r^3} P\left(\frac{x}{2t}\right) e^{\frac{i}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \nu (\phi^2) \ln (2t)} \]

where

\[ P(y) = \frac{c}{4} \overline{Q}(y) \psi^2(y), \quad (11.17) \]

and \( \overline{Q} \) is defined by (11.16). Then we have

\[(\partial_t + \Delta)(v_1 + v_2 + v_3 + v_4) - c\overline{v}(\partial_x v_1)^2 - c\overline{v}(\partial_y v_1)^2 - v_2\overline{v_1}(\partial_x v_1)^2 - v_3\overline{v_1}(\partial_y v_1)^2 \]

\[ = O_{L^2}(r^{-\frac{1}{2}}(\ln t)^2) \cap O_{L^2}(r^{-2}(\ln t)^2). \]

**Proof.** Assume generally that

\[ v_4 = \frac{1}{r^3} P\left(\frac{x}{2t}\right) e^{\frac{i}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \nu (\phi^2) \ln (2t)} \]

By Lemma 11.4, it suffices to prove that \( a = 2 \) and \( P \) defined by (11.17) lead to

\[(\partial_t + \Delta)(v_4) = O_{L^2}(r^{-\frac{1}{2}}(\ln t)^2) \cap O_{L^2}(r^{-2}(\ln t)^2). \]

Recall that

\[ c\overline{v}(\partial_x v_1)^2 = \frac{c^2}{8t^4} \psi^2\left(\frac{x}{2t}\right) \overline{Q}\left(\frac{x}{2t}\right) e^{\frac{i}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \nu (\phi^2) \ln (2t)} + O_{L^2}(r^{-\frac{1}{2}}(\ln t)^2) \cap O_{L^2}(r^{-2}(\ln t)^2). \]

By computation,

\[ i\partial_t v_4 + \partial_x^2 v_4 = \frac{x^2}{2^p+2} P\left(\frac{x}{2t}\right) e^{\frac{i}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \nu (\phi^2) \ln (2t)} + O_{L^2}(r^{-\frac{1}{2}}(\ln t)^2) \cap O_{L^2}(r^{-2}(\ln t)^2). \]

Observe that for \( a = 2 \) and \( P \) defined above, there holds

\[ \frac{x^2}{2^p+2} P\left(\frac{x}{2t}\right) e^{\frac{i}{4} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \nu (\phi^2) \ln (2t)} = 0. \]

Then we get the desired result. \( \square \)

**Lemma 11.6.** Assume that \( \psi \) satisfies (11.12) for \( m \) large enough. Let

\[ v := v_1 + v_2 + v_3 + v_4. \]

Then \( v \) fulfills (11.2), (11.3), (11.4).

\[ \|i\partial_t v + \Delta v - (c\overline{v} + v_2\overline{v} + \nu (\partial_x^2 v)^2)\|_{L^2\cap L^\infty} \leq r^{-\frac{1}{2}}(\ln t)^2 \quad (11.18) \]

\[ \|v(t)\|_{L^\infty} \leq \epsilon_\ast \quad (11.19) \]

\[ \|v(t)\|_{W^{1,\infty}} \leq r^{-\frac{1}{2}}\epsilon_\ast \quad (11.20) \]

**Proof.** To prove (11.20) and (11.19), it suffices to check the explicit formulas of \( \{v_i\}_{i=1}^4 \). We remark that \( \overline{Q} \) defined by (11.16) indeed has no singularity at \( y = 0 \) up to \( H^m \) by expanding \( \psi \) at zero.

Now, let’s prove (11.18). By Lemma 11.4, it suffices to prove

\[ \sum_{j=2,3,4} \|\partial_t \partial_x v_j \partial_x v_j\|_{L^2\cap L^\infty} \leq r^{-\frac{1}{2}}(\ln t)^2 \quad (11.21) \]

\[ \sum_{\text{med}c(i,j)\geq 2} \|\partial_t \partial_x v_j \partial_y v_j\|_{L^2\cap L^\infty} \leq r^{-\frac{1}{2}}(\ln t)^2 \quad (11.22) \]

\[ \sum_{\max(c(i,j,k)\geq 2} \|\partial_t \partial_x v_j \partial_y v_j\|_{L^2\cap L^\infty} \leq r^{-\frac{1}{2}}(\ln t)^2 \quad (11.23) \]

\[ \sum_{\max(c(i,j,k)\geq 2} \|\partial_t v_k \partial_y v_j \partial_x v_j\|_{L^2\cap L^\infty} \leq r^{-\frac{1}{2}}(\ln t)^2 \quad (11.24) \]

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(11.23), (11.22) (11.24) are easy to verify by directly applying the decay of \(\{v_i\}_{i=1}^{4}\). The relatively troublesome term in the LHS of (11.21) is
\[
\|\mathcal{M}\partial_x v_1 \partial_x v_3\|_{L^2_t L^\infty_x}.
\]
But notice that \(\partial_x v_3\) decays as \(t^{-2}\) in \(L^\infty\) which is faster than \(v_3\). So
\[
\|\mathcal{M}\partial_x v_1 \partial_x v_3\|_{L^2_t L^\infty_x} \lesssim t^{-\frac{5}{2}}.
\]

Now, we are ready to prove Theorem 1.3. Choose the approximate solution \(w_{ap}\) to be
\[
w_{ap} = v_1 + v_2 + v_3 + v_4.
\]
By Lemma 11.6 and Lemma 11.1, there exists an initial data \(u_0\) evolving to a global solution of 1D SMF for which
\[
\|w(t) - w_{ap}\|_{L^2_t} \lesssim t^{-\theta},
\]
where \(\theta > \frac{1}{2}\). Meanwhile, Step 4 of Lemma 11.1 also shows
\[
\|w(t)\|_{H^s_t} \lesssim t^\epsilon.
\]
Then by (11.3),
\[
\|w(t) - v(t)\|_{H^s_t} \lesssim t^\epsilon.
\]
By interpolation (11.26) with (11.25), one further obtains
\[
\lim_{t \to \infty} \|w(t) - v(t)\|_{H^s_t} = 0.
\]
Therefore, using
\[
\|w - z\|_{H^s_t} \lesssim \|w\|_{H^s_t}^2 \lesssim t^{-\frac{s}{2}},
\]
and
\[
\sum_{i=2,3,4} \|v_i\|_{H^s_t} \lesssim t^{-\frac{s}{2}},
\]
we get (1.13).

**Appendix A**

We concretely explain the inconvenience of applying caloric gauge in our problem. As discussed in Section 1.1, the differential fields \(\psi_x\) solves (1.18). There are two choices to set up the bootstrap of time decay estimates. One is to assume
\[
(a) \left< t \right>^\frac{1}{2} \|\phi_x\|_{W^1_{p,x}} \leq \epsilon,
\]
the other is to assume
\[
(b) \left< t \right>^{\frac{1}{2}} \left[ \|\psi_x\|_{W^1_{p,x}} + \|z\|_{W^1_{p,x}} \right] \leq \epsilon.
\]
Since the equation of \(z\) always suffers from the troublesome 4 order terms, in order to bound \(\|z\|_{W^1_{p,x}}\), we can only try to use \(\psi_x\). Observe that \(z\) and \(\psi_x\) are related by
\[
\|z(t,x)\|_{L^\infty_t} \leq \|u(t,x) - Q\|_{L^\infty_t} \leq \int_0^\infty \|\phi_x\|_{L^\infty_t} ds \leq \int_0^\infty \|D_x \psi_x\|_{L^\infty_t} ds.
\]
By the decay estimates of heat equation, one can only expect
\[
\int_0^\infty \|\partial_x \psi_x\|_{L^\infty_t} ds \leq \|z(t)\|_{L^p_t},
\]
for \(p \in [1, \infty)\). Note that \(p = \infty\) is not possible since it will lead to an integral \(\int_1^\infty \frac{1}{t} ds\).
Thus, if one assumes (b), the possible time decay of the RHS of (11.27) shall be
\[ \|z(t, x)\|_{L^p} \leq t^{\frac{1}{2}-\frac{1}{2}}, \]
where \( p \) could not take infinity. So the bootstrap of \( (t)^{\frac{n}{2}}\|z\|_{W^{\frac{n}{2}, \infty}} \) could not be closed.

Now, assume that we use assumption (a). Then, the weakest decay one needs, though far from being enough, of \( \|A_s \partial_x \psi_s\|_{L^2} \) shall be \( t^{-1} \). But the integral
\[ \left\| \int_0^\infty |\partial_s \psi_s| ds \right\|_{L^2} \]
does not have enough decay in \( s \) to provide \( t^{-1} \) decay under the assumption (a). For example, let \( p_1, p_2, p_3 \in [2, \infty) \) with \( \sum_i \frac{1}{p_i} = \frac{1}{2} \). Then the decay of heat equations suggests that
\[
\begin{align*}
\left\| \int_0^\infty |\partial_s \psi_s| ds \right\|_{L^2} & \leq \left( \int_1^\infty \left( \int_1^s |\partial_s \psi_s| ds \right) \left| \partial_s \psi_s \right| ds \right)^{\frac{1}{2}} \\
& \leq \left( \int_1^\infty \frac{s}{2} |\partial_s \psi_s| \|\psi_s\|_{L^{p_2}} ds \right)^{\frac{1}{2}} t^{\frac{1}{2}(1-\frac{1}{p_1})}
\end{align*}
\]
In order to reach \( t^{-1} \) decay, we need to obtain \( t^{-\frac{1}{2} + \frac{1}{2p_1} + \frac{1}{2p_2}} \) decay from
\[ \|\int_0^\infty |\partial_s \psi_s| ds \|_{L^2}. \]
The decay estimate of heat equations is likely to give
\[ \|\psi_s\|_{L^p} \lesssim s^{-\frac{1}{2} + \frac{1}{2p} - \frac{1}{2}} \|z\|_{L^p}, \quad 1 \leq p \leq \infty. \]
And under the assumption (a) we see
\[ \|z\|_{L^p} \lesssim \|\psi_s\|_{L^p}^{\frac{1}{2}} \|\psi_s\|_{L^p}^{\frac{1}{2}} \lesssim t^{-\frac{1}{2} + \frac{1}{2}}. \]
So we get
\[ \|\psi_s\|_{L^p} \lesssim s^{-\frac{1}{2} + \frac{1}{2p} - \frac{1}{2}} t^{-\frac{1}{2} + \frac{1}{2}}, \quad 1 \leq p \leq \infty. \]
On the other hand directly applying assumption (a) also gives
\[ \|\psi_s\|_{L^p} \lesssim t^{-\frac{1}{2} + \frac{1}{2}}. \]
Hence,
\[
\int_1^\infty s^{-\frac{1}{2}} \|\psi_s\|_{L^{p_1}} \|\psi_s\|_{L^{p_2}} ds \leq t^{-\frac{1}{2} + \frac{1}{2p_1} + \frac{1}{2p_2}} \int_1^\infty s^{-\frac{1}{2} + \frac{1}{2p} - \frac{1}{2}} ds.
\]
In order to reach \( t^{-\frac{1}{2} + \frac{1}{2p_1} + \frac{1}{2p_2}} \) decay and make the \( s \) integral converge, it requires to set
\[
\begin{align*}
-\frac{1}{2} + \frac{1}{2p_1} + \frac{1}{2p_2} & \leq -\frac{3}{4} + \frac{1}{2p_1} + \frac{1}{2p_2}, \\
-\frac{3}{2} + \frac{1}{p_1} - \frac{1}{q_1} + \frac{1}{p_2} - \frac{1}{q_2} & < -1.
\end{align*}
\]
However, these two inequalities are exactly opposite to each other. So it seems impossible to make \( \|A_s \partial_x \psi_s\|_{L^2} \) have \( t^{-1} \) decay.

From the above discussions, it seems that caloric gauge would not bring additional benefit in our problem.
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