PROGRESS IN CLASSICALLY SOLVING
TEN DIMENSIONAL SUPERSYMMETRIC
REDUCED YANG–MILLS THEORIES

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Abstract

It is shown that there exists an on–shell light cone gauge where half of the fermionic components of the super vector potential vanish, so that part of the superspace flatness conditions becomes linear. After reduction to (1+1) space-time dimensions, the general solution of this subset of equations is derived. The remaining non-linear equations are written in a form which is analogous to Yang equations, albeit with superderivatives involving sixteen fermionic coordinates. It is shown that this non-linear part may, nevertheless, be solved by methods similar to powerful technics previously developed for the (purely bosonic) self–dual Yang Mills equations in four dimensions.

TO THE MEMORY OF MIKHAIL SAVELIEV

This work was being completed when my very good friend and long time collaborator suddenly passed away. His family, his numerous friends and the whole scientific community have suffered an unbearable loss. It is my hope that this article, which describes the state of a part of our research program at the time of his death, will remain as a valuable tribute to his memory.

Jean–Loup Gervais
1 Introduction

It has been known already for more than ten years, see e.g. [1, 2, 3], that ten dimensional supersymmetric Yang–Mills theories may be considered as integrable systems classically, in a “weak” sense, since they admit Lax–type representations in superspace for the equations of motion. The starting point in this direction was the observation [4, 1, 2] that the field equations of these theories are equivalent to the constraint that the purely fermionic components of the supercurvature vanish. So far, the existence of this Lax pair has not been so useful however, since the role of spectral parameter is played by a light-light vector. More recently, the interest was revived into (suitably reduced) ten dimensional supersymmetric Yang–Mills theories in the large $N$ limit since they have been actively considered in the search for the M theory (see e.g. refs[5, 6, 7]). This has motivated us to return to the use of the flatness condition in superspace in order to derive non trivial classical solutions. One may hope, in particular that the problem will be simpler after the reduction process, which we will simply perform by looking for classical solutions that do not depend upon a certain set of space coordinates. We shall indeed make progress after reducing to $1+1$ space–time dimensions, which seems to be the most natural choice at the present time.

Let us first recall some standard formulae in order to establish the notations. In ten dimensions the dynamics is specified by the standard action

$$S = \int d^{10}x \ Tr \left\{ \frac{1}{4} Y_{mn} Y^{mn} + \frac{1}{2} \bar{\phi} \left( \Gamma_m \partial_m \phi + [X_m, \phi]_- \right) \right\}, \quad (1.1)$$

$$Y_{mn} = \partial_m X_n - \partial_n X_m + [X_m, X_n]_. \quad (1.2)$$

The notation is as follows. $X_m(\underline{x})$ is the vector potential, $\phi(\underline{x})$ is the Majorana-Weyl spinor. Both are matrices in the adjoint representation of the gauge group $G$. Latin indices $m = 0, \ldots, 9$ describe Minkowski components. Greek indices $\alpha = 1, \ldots, 16$ denote spinor components. We will use the superspace formulation with odd coordinates $\theta^\alpha$. The super vector potentials, which are valued in the gauge group, are noted $A_m(\underline{x}, \theta), A_\alpha(\underline{x}, \theta)$. As discussed in ref.[4], we may remove all the additional fields and uniquely reconstruct the physical fields $X_m, \phi$ from $A_m$ and $A_\alpha$ if we impose the condition $\theta^\alpha A_\alpha = 0$ on the latter.

With this condition, it was shown in refs[1], [2], that the field equations derived from the Lagrangian [1.1] are equivalent to the flatness conditions

$$F_{\alpha\beta} = 0, \quad (1.3)$$
where $F$ is the supercurvature

$$F_{\alpha \beta} = D_\alpha A_\beta + D_\beta A_\alpha + [A_\alpha, A_\beta] + 2 (\sigma^m)_{\alpha \beta} A_m. \quad (1.4)$$

$D_\alpha$ denote the superderivatives

$$D_\alpha = \partial_\alpha - (\sigma^m)_{\alpha \beta} \theta^\beta \partial_m, \quad (1.5)$$

and we use the Dirac matrices

$$\Gamma^m = \begin{pmatrix} 0_{16 \times 16} & (\sigma^m)_{\alpha \beta} \\ ((\sigma^m)_{\alpha \beta})^T & 0_{16 \times 16} \end{pmatrix}, \quad \Gamma^{11} = \begin{pmatrix} 1_{16 \times 16} & 0 \\ 0 & -1_{16 \times 16} \end{pmatrix}. \quad (1.6)$$

The physical fields appearing in equation (1.1) are reconstructed from the superfields $A_m A_\alpha$ as follows. Using the Bianchi identity on the super curvature one shows that one may write

$$F_{\alpha m} = (\sigma^m)_{\alpha \beta} \chi^\beta. \quad (1.14)$$

Then $X_m, \phi^\alpha$ are, respectively, the zeroth order contributions in the expansions of $A_m$ and $\chi^\alpha$ in powers of the odd coordinates $\theta$.

Throughout the paper, it will be convenient to use the following particular realisation:

$$((\sigma^9)_{\alpha \beta}) = \begin{pmatrix} -1_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix}, \quad (1.7)$$

$$((\sigma^0)_{\alpha \beta}) = -((\sigma^0)_{\alpha \beta}) = \begin{pmatrix} 1_{8 \times 8} & 0_{8 \times 8} \\ 0_{8 \times 8} & 1_{8 \times 8} \end{pmatrix}, \quad (1.8)$$

$$((\sigma^i)_{\alpha \beta}) = -((\sigma^i)_{\alpha \beta}) = \begin{pmatrix} 0 \\ (\gamma^i T)^\mu_{\nu},_{\nu} \end{pmatrix}, \quad i = 1, \ldots, 8. \quad (1.9)$$

The convention for greek letters is as follows: Letters from the beginning of the alphabet run from 1 to 16. Letters from the middle of alphabet run from 1 to 8. In this way, we shall separate the two spinor representations of $O(8)$ by rewriting $\alpha_1, \ldots, \alpha_{16}$ as $\mu_1, \ldots, \mu_8, \nu_1, \ldots, \nu_8$.

Using the above explicit realisations on sees that the equations to solve take the form

$$D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu]_+ = 2 \delta_{\mu \nu} (A_0 + A_9) \quad (1.10)$$

$$D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu]_+ = 2 \delta_{\mu \nu} (A_0 - A_9) \quad (1.11)$$

$$D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu]_+ = -2 \sum_{i=1}^{8} A_i \gamma^i_{\mu \nu} \quad (1.12)$$
A lax pair formalism follows by noticing that if $\lambda^m \lambda_m = 0$, there exists $R[\lambda] \in \text{gauge group}$, such that

$$
\lambda^m (\sigma_m)^{\alpha\beta} \left( A_{\beta} - R[\lambda]^{-1} D_{\beta} R[\lambda] \right) = 0
$$

$$
\lambda^m \left( A_m - R[\lambda]^{-1} \partial_m R[\lambda] \right) = 0.
$$

This allows us to express the solution in terms of pure gauges, but this is not so useful since different components are expressed in terms of $R[\lambda]$ involving different $\lambda$’s. The drawback is that $\lambda$, which plays the role of the spectral parameter, is a vector. In particular, let us choose $\lambda^{(\pm)}$, such that $\lambda^{(\pm)0} = \pm \lambda^{(\pm)0} = 1/2$, $\lambda^{(\pm)i} = 0$, $i = 1, \ldots, 8$. This gives

$$
A_{\mu} = R_{+}^{-1} D_{\mu} R_{+}, \quad A_{+} = R_{+}^{-1} \partial_{+} R_{+}
$$

(1.13)

$$
A_{\mp} = R_{-}^{-1} D_{\mp} R_{-}, \quad A_{-} = R_{-}^{-1} \partial_{-} R_{-}
$$

(1.14)

We let from now on

$$
A_{\pm} = A_{0} \pm A_{9}, \quad \partial_{\pm} = \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^9}.
$$

(1.15)

One sees that $A_{\mu}$, $A_{+}$ (resp. $A_{\mp}$, $A_{-}$) are expressed in terms of $R_{+}$ (resp. $R_{-}$). Moreover, since only the $\mu\nu$ and $\mu\overline{\nu}$ are solved by the above there remain the $\mu\overline{\nu}$ equations. A straightforward computation shows that they become

$$
D_{\overline{\nu}} \left( R^{-1} D_{\mu} R \right) = -2 \sum_{i=1}^{8} \tilde{A}_i \gamma^{ijk}_{\mu, \overline{\nu}}
$$

$$
R \equiv R_{+} R_{-}^{-1}, \quad \tilde{A}_i \equiv R_{-} (A_i + \partial_i) R_{-}^{-1}
$$

We may derive the field $\tilde{A}_i$, if the following conditions hold

$$
\sum_{\mu\overline{\nu}} D_{\overline{\nu}} \left( R^{-1} D_{\mu} R \right) \gamma^{ijk}_{\mu, \overline{\nu}} = 0, \quad 1 \leq i < j < k \leq 8.
$$

(1.16)

These are complicated non linear $\sigma$ model type equations in superspace which so far could not be handled. This is basically why these reasonings and the Lax representation just summarised did not allow yet to construct any explicit nontrivial physically meaningful solution. Conditions [1.16] only provide a procedure for obtaining infinite series of nonlocal, and rather complicated conservation laws.
2 A useful on-shell gauge

Under gauge transformations, we have
\[ R_\pm \rightarrow R_\pm \Lambda, \quad A_m \rightarrow \Lambda^{-1} (A_m + \partial_m) \Lambda, \quad A_\alpha \rightarrow \Lambda^{-1} (A_\alpha + D_\alpha) \Lambda \]
Thus \( R \) is gauge invariant. If \( \Lambda = R^{-1} \), we get
\[ A_+ \rightarrow R^{-1} \partial_+ R, \quad A_\mu \rightarrow R^{-1} D_\mu R, \quad A_i \rightarrow \tilde{A}_i \quad (2.1) \]
\[ A_- \rightarrow 0, \quad A_\mp \rightarrow 0. \]
Thus, if the field equations are satisfied there exists a gauge (on shell) such that \( A_- = A_\mp = 0 \). After this gauge choice, the flatness conditions \[1.10 \quad 1.12\] boil down to
\[ D_\mu A_\nu + D_\nu A_\mu + [A_\mu, A_\nu]_+ = 4\delta_{\mu\nu} A_0 \quad (2.2) \]
\[ 0 = 0 \quad (2.3) \]
\[ D_\mp A_\mu = -2 \sum_1^8 \tilde{A}_i \gamma^i_{\mu, \mp} \quad (2.4) \]

The last mixed ones which in general lead to the complicated conditions \[1.16\] have become linear, and we will next derive their general solution. In order to do so, we will use the following explicit realisation of the \( O(8) \) Dirac matrices,
\[ \gamma^1 = \tau_1 \otimes \tau_3 \tau_1 \otimes 1 \quad \gamma^5 = \tau_3 \otimes \tau_3 \tau_1 \otimes 1 \]
\[ \gamma^2 = 1 \otimes \tau_1 \otimes \tau_3 \tau_1 \quad \gamma^6 = 1 \otimes \tau_3 \otimes \tau_3 \tau_1 \]
\[ \gamma^3 = \tau_3 \tau_1 \otimes 1 \otimes \tau_1 \quad \gamma^7 = \tau_3 \tau_1 \otimes 1 \otimes \tau_3 \]
\[ \gamma^4 = \tau_3 \tau_1 \otimes \tau_3 \tau_1 \otimes \tau_3 \tau_1 \quad \gamma^8 = 1 \otimes 1 \otimes 1. \quad (2.5) \]
Substituting into Eq.\[2.3\] we get
\[ -2 \tilde{A}_1 = f_{77} = f_{28} = f_{53} = f_{64} \]
\[ -2 \tilde{A}_2 = f_{74} = f_{32} = f_{58} = f_{76} \]
\[ -2 \tilde{A}_3 = f_{76} = f_{25} = f_{58} = f_{77} \]
\[ -2 \tilde{A}_4 = f_{78} = f_{55} = f_{53} = f_{72} \]
\[ -2 \tilde{A}_5 = f_{73} = f_{21} = f_{73} = f_{70} \]
\[ -2 \tilde{A}_6 = f_{72} = f_{33} = f_{56} = f_{77} \]
\[ -2 \tilde{A}_7 = f_{75} = f_{62} = f_{57} = f_{54} \quad (2.6) \]
\[ -2\tilde{A}_8 = f_{\bar{\mu} \nu}, \quad \mu = 1, \ldots , 8 \quad (2.7) \]
\[ f_{\bar{\mu} \nu} = - f_{\bar{\nu} \mu} \quad (2.8) \]

where we have let \( f_{\bar{\mu} \nu} = D_{\bar{\mu}} A_\nu \). By convention overlined and non overlined indices with the same letter (such as \( \mu \) and \( \bar{\mu} \)) take the same numerical value.

With the particular realisation of \( \sigma \) matrices displayed on Eqs.1.6–1.9, one has the anticommutation relations
\[ [D_\mu, D_\nu]_+ = 2\delta_{\mu\nu} \partial_+ \]
\[ [D_{\bar{\mu}}, D_{\bar{\nu}}]_+ = 2\delta_{\mu\nu} \partial_- \quad (2.9) \]

Thus it follows from equation 2.8 that there exits a superfield \( \Phi \) such that
\[ A_\nu = D_\nu \Phi \quad (2.10) \]

Then equation 2.7 is automatically satisfied since it becomes \( f_{\bar{\mu} \mu} = D^2_{\bar{\mu}} \Phi = \partial_+ \Phi \) which is indeed independent from \( \bar{\mu} \). On each line, the corresponding component of the vector potential \( \tilde{A}_i \) may be computed iff the three right most equalities are satisfied. Thus we have the consistency equations of the superfield \( \Phi \)

\[ D_\tau D_\nu \Phi = D_\tau D_\sigma \Phi = D_\nu D_\sigma \Phi = D_\sigma D_\mu \Phi \]
\[ D_\tau D_\nu \Phi = D_\tau D_\sigma \Phi = D_\nu D_\sigma \Phi = D_\sigma D_\mu \Phi \]
\[ D_\nu D_\sigma \Phi = D_\nu D_\tau \Phi = D_\mu D_\tau \Phi = D_\nu D_\sigma \Phi \]
\[ D_\tau D_\nu \Phi = D_\tau D_\sigma \Phi = D_\nu D_\sigma \Phi = D_\sigma D_\mu \Phi \]
\[ D_\tau D_\nu \Phi = D_\tau D_\sigma \Phi = D_\nu D_\sigma \Phi = D_\sigma D_\mu \Phi \]
\[ D_\tau D_\nu \Phi = D_\tau D_\sigma \Phi = D_\nu D_\sigma \Phi = D_\sigma D_\mu \Phi \] (2.11)

These consistency conditions take the form
\[ D_{\bar{\mu}} D_\nu \Phi = \sum_{\bar{\rho} < \chi} T_{\bar{\mu} \nu \bar{\rho} \chi} D_{\bar{\rho}} D_\chi \Phi. \]

where \( T_{\bar{\mu} \nu \bar{\rho} \chi} \) is a numerical tensor which is antisymmetric with elements equal to \( \pm 1 \), or 0. Thus we have equations similar to the self–duality relations considered, for bosonic variables in ref. [4]. Here the main difference is that we have superderivatives and that our equations are linear partial differential equations.
3 Solution of the reduced self–duality equations

Let us derive the general solution of equations 2.6 in the particular reduced case where \( \Phi \) does not depend upon \( x^i \), for \( i = 1, \ldots, 8 \). Then the superderivatives take the form

\[
D_\mu = \frac{\partial}{\partial \theta^\mu} + \theta^\mu \partial_+, \quad D_{\bar{\nu}} = \frac{\partial}{\partial \theta^{\bar{\nu}}} + \theta^{\bar{\nu}} \partial_-
\]

(3.1)

In this case equations 2.11 only involve the variables \( x_- \), \( \theta_1 \), \ldots, \( \theta_8 \), and we forget the other variables for the time being. In the forthcoming discussion, we will find it useful to use the following lemma

3.1 Super Cauchy relations:

Consider any pair of different indices, which are selected once for all in this subsection, say \( \mu, \nu \). Given an arbitrary superfield \( \Upsilon(x_-, \theta^1, \ldots, \theta^8) \), there exists a superfield \( \Lambda(x_-, \theta^1, \ldots, \theta^8) \) such that

\[
D_\mu \Upsilon = D_\nu \Lambda, \quad D_\nu \Upsilon = -D_\mu \Lambda
\]

(3.2)

Proof: First one verifies that the consistency of these equations is a consequence of the equations \( D^2_\mu = D^2_{\bar{\nu}} \) and \([D_\mu, D_{\bar{\nu}}]_+ = 0\) which follow from the superalgebra. Next, one may explicitly solve order by order in the expansion in powers of \( \theta^{\bar{\nu}} \). For an arbitrarily given superfield with the expansion

\[
F(x_-, \theta^1, \ldots, \theta^8) = \sum_{p=0}^{8} \sum_{\mu_1, \ldots, \mu_p} \frac{\theta^{\mu_1} \ldots \theta^{\mu_p}}{p!} F^{(p)}_{\mu_1 \ldots \mu_p}(x_-),
\]

(3.3)

the superderivatives act as follows, (with \( p = 1, \ldots, 7 \)),

\[
(D_{\bar{\nu}} F^{(p)}_{\mu_1 \ldots \mu_p})_{\mu_1 \ldots \mu_p} = F^{(p+1)}_{\mu_1 \ldots \mu_p} + \sum_{i=1}^{p} (-1)^{i+1} \delta_{\mu_1 \mu_i} \partial_- F^{(p-1)}_{\mu_2 \ldots \mu_i \mu_{i+1} \ldots \mu_p}
\]

(3.4)

where the extremal values of \( p \) are treated by setting \( F^{(p)} = 0 \) if \( p < 0 \) or if \( p > 8 \). This allows us to solve order by order. One finds the relations

\[
\Upsilon^{(p)}_{\mu_1 \ldots \mu_{p-1}} = \Lambda^{(p)}_{\mu_1 \ldots \mu_{p-1}}, \quad \Upsilon^{(p)}_{\mu_1 \ldots \mu_{p-1}} = -\Lambda^{(p)}_{\mu_1 \ldots \mu_{p-1}}, \quad p = 1, \ldots, 8
\]

\[
\partial_- \Upsilon^{(p)}_{\mu_1 \ldots \mu_p} = -\Lambda^{(p+2)}_{\mu_{p+1} \ldots \mu_p}, \quad \Upsilon^{(p+2)}_{\mu_{p+1} \ldots \mu_p} = \partial_+ \Lambda^{(p)}_{\mu_1 \ldots \mu_p}, \quad p = 0, \ldots, 6,
\]

(3.5)

which determine \( \Lambda \) once \( \Upsilon \) is given.
3.2 Selfduality in four variables

In order to derive the general solution of equations 2.11, we remark that we may re-arrange these relations under the form

\[ D_{17} \Phi = D_{28} \Phi, \quad D_{12} \Phi = D_{57} \Phi, \quad D_{15} \Phi = D_{47} \Phi; \]
\[ D_{12} \Phi = D_{56} \Phi, \quad D_{16} \Phi = D_{57} \Phi, \quad D_{14} \Phi = D_{76} \Phi; \]
\[ D_{14} \Phi = D_{68} \Phi, \quad D_{16} \Phi = D_{57} \Phi, \quad D_{12} \Phi = D_{76} \Phi; \]
\[ D_{12} \Phi = D_{56} \Phi, \quad D_{16} \Phi = D_{57} \Phi, \quad D_{14} \Phi = D_{76} \Phi. \] (3.6)

From now on we let \( D_{\mu \nu} = D_{\mu} D_{\nu} \). Each line forms a closed set of selfduality equations in four variables. Thus we first solve this type of equations. Consider, for fixed \( \mu \neq \nu \neq \sigma \neq \rho \), the equations

\[ (D_{\mu \rho} - D_{\sigma \nu}) \Phi = 0, \quad (D_{\nu \rho} - D_{\sigma \mu}) \Phi = 0, \quad (D_{\mu \sigma} - D_{\rho \nu}) \Phi = 0. \] (3.7)

By using the superalgebra [2,3], one finds that for arbitrary superfield \( \Psi_1 \), \( \Phi = (D_{\tau} D_{\rho} + D_{\rho} D_{\sigma}) \Psi_1 \) is a solution of these last three equations. This form is suspiciously non symmetric. However, we use the above super Cauchy relations to introduce superfields \( \Psi_2, \Psi_3 \) such that

\[ D_{\mu} \Psi_1 = D_{\nu} \Psi_2, \quad D_{\sigma} \Psi_1 = -D_{\tau} \Psi_2; \]
\[ D_{\mu} \Psi_1 = D_{\tau} \Psi_3, \quad D_{\sigma} \Psi_1 = -D_{\tau} \Psi_3. \] (3.8)

Then we see that we may write our solution under three equivalent forms

\[ \Phi = (D_{\pi} D_{\rho} + D_{\rho} D_{\sigma}) \Psi_1 = (D_{\sigma} D_{\rho} + D_{\rho} D_{\tau}) \Psi_2 = (D_{\tau} D_{\rho} + D_{\rho} D_{\sigma}) \Psi_3. \] (3.9)

where the expected symmetry becomes manifest.

3.3 Eight variables, the Cartan basis:

Actually, the superalgebra satisfied by the \( D_{\pi} \) operators, as displayed by equations 2.9 coincides with a Dirac algebra in eight dimensions up to the \( \partial_\mu \) differential operator. It thus follows that the super derivatives \( D_{\pi \nu} \) obey
an $\mathfrak{so}(8)$ Lie algebra\footnote{Since the $\partial_-$ factor will not play a significant role in the forthcoming argument, we do not speak about it any longer for brevity.}. It is useful to organise the superderivatives appearing in the self–duality relations in a Cartan basis. For this we temporarily re-label the indices as follows: $(\overline{1}, \ldots, \overline{8}) \rightarrow (\overline{1}, -\overline{1}, \overline{2}, -\overline{2}, \overline{3}, -\overline{3}, \overline{4}, -\overline{4})$. The roots of $\mathfrak{so}(8)$ may be written under the form $\pm \vec{e}_i \pm \vec{e}_j, 1 \leq j < k \leq 4$. Let us denote $E_{\pm \vec{e}_i \pm \vec{e}_j}$ the step operators and by $h_{\pm \vec{e}_i \pm \vec{e}_j}$ the Cartan generators. One finds the correspondence

\[
E_{\vec{e}_i \pm \vec{e}_j} = \pm \frac{1}{2} \left( D_{-\overline{7} \overline{k}} \pm D_{\overline{7} \overline{k}} \right), \quad E_{\vec{e}_i \pm \vec{e}_j} = \frac{1}{2} \left( D_{\overline{k} \overline{k}} \pm D_{\overline{7} \overline{7}} \right), \quad h_{\vec{e}_i \pm \vec{e}_j} = \frac{i}{2} \left( D_{\overline{7} \overline{7}} \pm D_{\overline{7} \overline{7}} \right)
\]

(3.10)

with the convention that numerically $j = \overline{j}, k = \overline{k}$. Out of the seven self dualities in four variables, six may be written as

\[
D_{7 \overline{k}} \Phi = D_{\overline{k} 7} \Phi, \quad D_{\overline{7} k} \Phi = D_{-7 \overline{7}} \Phi, \quad D_{7 \overline{7}} \Phi = D_{\overline{7} \overline{7}} \Phi \quad (3.11)
\]

with $1 \leq j < k \leq 4$. The corresponding differential operators generate an $\mathfrak{su}(4)$ algebra with simple roots $\vec{e}_1 + \vec{e}_2, \vec{e}_2 + \vec{e}_3, \vec{e}_3 + \vec{e}_4$. Out of these six triplets of relations only the three associated with the simple roots are independent. In total, we are thus left with four triplets of independent relations.

### 3.4 Solving the $\mathfrak{su}(4)$ part

Consider a particular simple root $\vec{e}_i + \vec{e}_{i+1}$. The corresponding self–duality relations take the form

\[
E_{\vec{e}_i \pm \vec{e}_{i+1}} \Phi = E_{\vec{e}_i \pm \vec{e}_{i+1}} \Phi = h_{\vec{e}_i \pm \vec{e}_{i+1}} \Phi = 0.
\]

and the general solution derived above becomes

\[
\Phi = E_{\vec{e}_i \pm \vec{e}_{i+1}} \Psi_1 = E_{\vec{e}_i \pm \vec{e}_{i+1}} \Psi_2 = h_{\vec{e}_i \pm \vec{e}_{i+1}} \Psi_3.
\]

Group theoretically, the super Cauchy relations are seen to allow us to use any of the three generators of the $\mathfrak{su}(2)$ subalgebra with root $\vec{e}_i - \vec{e}_{i+1}$. Returning
to the full $su(4)$ part, we use this liberty to pick up the Cartan generator for each simple root. This leads us to the ansatz
\[ \Phi = h_{e_1-e_2} h_{e_2-e_3} h_{e_3-e_4} \tilde{\Phi}. \] (3.12)

Since the $h$'s commute, it should be clear from the above that, for arbitrary $\tilde{\Phi}$, the last expression obeys all six self-duality relations associated with the $su(4)$ mentioned above. This may of course be checked explicitly from the superalgebra. Let us return to the previous label of indices. At this point it is convenient to introduce the equivalent of the nineth Dirac matrix by writing
\[ D_9 = D_1 D_2 \ldots D_8, \] (3.13)
which is such that $D_9^2 = \partial^8$. After some calculations one may show that above may be written as
\[ \Phi = \{ D_{12} - D_{34} + D_{56} - D_{78} \} \chi, \] (3.14)
where we have let
\[ \chi = \left( D_9 + \partial^4 \right) \tilde{\Phi}. \] (3.15)
This exhibits a chirality projector which commutes with all the $D_\mu \nu$.

### 3.5 Solving the last set of self-duality relations

Looking at the last set of relations, one sees that our task is to determine $\chi$ such that
\begin{align*}
(D_{25} - D_{35}) (D_{12} - D_{34} + D_{56} - D_{78}) \chi &= 0 \quad (3.16) \\
(D_{12} - D_{34}) (D_{12} - D_{34} + D_{56} - D_{78}) \chi &= 0 \quad (3.17) \\
(D_{25} - D_{35}) (D_{12} - D_{34} + D_{56} - D_{78}) \chi &= 0 \quad (3.18)
\end{align*}

Let us first solve the first equation separately. We will see that at the end the other two will also be satisfied. We will use again super Cauchy relations of the type (3.2). One easily verifies that one may at the same time apply two super Cauchy transformations over independent variables. It is convenient to introduce a superfield $\chi_1$ such that
\begin{align*}
D_2 \chi &= D_7 \chi_1, \quad D_7 \chi = -D_2 \chi_1, \quad (3.19) \\
D_4 \chi &= D_5 \chi_1, \quad D_5 \chi = -D_4 \chi_1. \quad (3.20)
\end{align*}
we get
\[(D_{28} - D_{33}) (D_{17} + D_{36}) \chi_1 = 0.\]
Thus the general solution is
\[\chi_1 = (D_{28} + D_{33}) F_1 + (D_{36} - D_{17}) G_1,\]
where \(F_1\) and \(G_1\) are arbitrary chiral superfields. Is it possible to go back to \(\chi\)? We apply Cauchy equations to each term, by letting
\[
\begin{align*}
D_{27} F &= D_{27} F_1, & D_{27} G &= D_{27} G_1, & D_{27} G &= -D_{27} G_1; \\
D_{45} F &= D_{45} F_1, & D_{45} F &= -D_{45} F_1, & D_{45} G &= D_{45} G_1, & D_{45} G &= -D_{45} G_1.
\end{align*}
\]
Then one may verify that equations 3.19–3.20 are satisfied with
\[
\chi = (-D_{28} + D_{33}) F + (D_{17} + D_{46}) G.
\]
At this point, there appears a great simplification since one may check that
\[
\{D_{12} - D_{34} + D_{56} - D_{78}\} (-D_{28} + D_{33})
\]
\[
= \{D_{12} - D_{34} + D_{56} - D_{78}\} (D_{17} + D_{46}).
\]
Thus we may forget the \(G\) term. Turning finally to equations 3.17 and 3.18 one may explicitly verify that they are automatically satisfied for arbitrary \(F\) since one has
\[
(D_{27} - D_{34}) \{D_{12} - D_{34} + D_{56} - D_{78}\} (-D_{28} + D_{33}) = 0,
\]
\[
(D_{27} - D_{34}) \{D_{12} - D_{34} + D_{56} - D_{78}\} (-D_{28} + D_{33}) = 0.
\]
Altogether, we have shown that the general solution of the eight self–duality relations 2.11 is given by
\[
\Phi = \{D_{12} - D_{34} + D_{56} - D_{78}\} (D_{28} - D_{33}) (D_{78} + \partial_-) \Psi,
\]
where we have let \(F = (D_7 + \partial_+) \Psi\) in agreement with equation 3.15, and \(\Psi\) is an arbitrary superfield. At this point, we have to admit that we are unable to explain why the key relations 3.24–3.26 hold, apart from verifying them explicitly. They were discovered using Mathematica on the analogous relations for \(O(8)\) Dirac matrices. The form of the solution is not explicitly symmetric between indices, but here also, as for the above case of four variables, symmetry may be verified from super Cauchy transformations.
4 The non linear equations for $\Phi$.

Equations 2.2 remain to be solved. Make use of equation 2.10 and substitute $A_\mu = D_\mu \Phi$, $A_\nu = D_\nu \Phi$. One gets

$$D_\mu D_\nu \Phi + D_\nu D_\mu \Phi + [D_\mu \Phi, D_\nu \Phi]_+ = 4 \delta_{\mu \nu} A_0.$$  

(4.1)

The super field $A_0$ may be computed from these equations if the following consistency conditions hold. We must verify that, for $\mu \neq \nu$,

$$D_\mu D_\nu \Phi + D_\nu D_\mu \Phi + [D_\mu \Phi, D_\nu \Phi]_+ = 0,$$

(4.2)

and that

$$D_\mu D_\nu \Phi + (D_\nu \Phi)^2 \text{ is independent from } \mu.$$  

(4.3)

If these conditions hold, the superfield $A_0$ may be computed from

$$A_0 = \frac{1}{2} \left\{ D_1 D_\nu \Phi + (D_\nu \Phi)^2 \right\}.$$  

(4.4)

At this point it is interesting to recall the four dimensional Yang equations which arose in solving self–dual (purely bosonic) Yang–Mills in four dimensions. For this, we closely follow the review [10]. There are two bosonic complex coordinates $z, y$ and their conjugate $\bar{z}, \bar{y}$. One may start from the equations (Indices mean derivatives)

$$\left( G_z G^{-1} \right)_\bar{z} + \left( G_y G^{-1} \right)_\bar{y} = 0,$$

where $G$ is in the adjoint representation of the gauge group. This is partially solved by letting

$$G_z G^{-1} = f_{\bar{y}}, \quad G_y G^{-1} = -f_{\bar{z}}.$$  

(4.5)

which leads to the consistency condition

$$f_{zz} + f_{\bar{y}y} + [f_{\bar{y}}, f_z]_- = 0.$$  

(4.6)

In order to draw a parallel with our case, let us recall that, according to equations 2.1, 2.10, we have

$$A_\mu = R^{-1} D_\mu R = D_\nu \Phi.$$  

(4.7)

As before, numerically $\mu = \bar{\mu}, \nu = \bar{\nu}$. 

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There is a similarity between equations 4.1 and 4.6, and between equations 4.5 and 4.7, except that the indices are paired differently. As we will next show, this fact, together with the basic properties of superderivatives allows us to discuss the solution of the present equations for any number of supervariables, while the bosonic case may be handled only with four coordinates.

4.1 A solution of the non linear consistency conditions

In this part we adapt to our problem the perturbative method, valid to all orders in the coupling constant $g$, developed earlier for the bosonic case (see ref.[10] and refs. therein). We expand in powers of $g$ after replacing equation 4.1 by

$$D_\mu D_\nu \Phi + D_\nu D_\mu \Phi + g [D_\mu \Phi, D_\nu \Phi]_+ = 4 \delta_{\mu\nu} A_0.$$  (4.8)

Following a path very similar to the bosonic case (see ref.[10]), one derives the following solution to all orders in $g$. Assume there exists a superfield $F(\lambda, x^+, x^-, \theta^1, \ldots, \theta^8, \bar{\theta}^1, \ldots, \bar{\theta}^8)$, (noted $F(\lambda)$ for brevity) with $\lambda$ an arbitrary (bosonic) parameter, such that

$$D_\mu F(\lambda) = \lambda D_\tau F(\lambda).$$  (4.9)

We shall solve this equation later on explicitly. Then the solution may be written as

$$\Phi = - \sum_{n=0}^{\infty} \frac{g^n}{(n+1)!} \int d\lambda F^{[n]}(\lambda),$$  (4.10)

where $F^{[n]}$ is defined by the recursion

$$F^{[n]}(\lambda) = \sum_{p=0}^{n-1} \binom{n-1}{p} \left[ F^{[p]}(\lambda), \int d\lambda' F^{[n-1-p]}(\lambda') \right]_-.$$  (4.11)

with $F^{[0]}(\lambda) = F(\lambda)$.

**Proof:** The best method is to first derive that the above expression satisfies a first order differential equation of the form

$$D_\mu \Phi = D_\tau \Omega + g \Phi D_\tau \Phi,$$  (4.12)

where $\Omega$ is a superfield which is computed order by order in $g$. Then it is easy to verify that conditions 4.2 and 4.3 follow. Checking equation 4.12 is
straightforward but lengthy. The calculation is almost the same as in the bosonic case. Thus we omit it. Then it is easily verified that equations (4.4) hold with

\[ A_0 = \frac{1}{2} (\partial_- \Omega + g \Phi \partial_- \Phi) \]  

(4.13)

There remains to verify equation (4.9). First, applying \(D_\mu\) to both sides of equation (4.9) one derives the consistency condition

\[ \left( \partial_+ + \lambda^2 \partial_- \right) F(\lambda) = 0 \]  

(4.14)

Next, start from the general expansion.

\[ F(\lambda, x_+, x_-, \theta^1, \ldots, \theta^8, \theta^1, \ldots, \theta^8) = \sum_{p=0}^{8} \sum_{q=0}^{8} \sum_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} \theta^{\mu_1} \ldots \theta^{\mu_p} \theta^{\nu_1} \ldots \theta^{\nu_q} \frac{p!q!}{F(p,q)} \]  

(4.15)

Equations (4.9) give

\[ F(p+1, q)_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} + \sum_{i=1}^{p} (-1)^{i+1} \delta_{\mu_i \mu_i} \partial_- F(p-1, q)_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} = \lambda (-1)^p F(p+1)_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} + \lambda (-1)^p q \sum_{i=1}^{q} (-1)^{i+1} \delta_{\nu_i \nu_i} \partial_- F(p-1, q)_{\mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q}. \]  

(4.16)

The general solution is as follows. Take all indices different, unless they are noted with the same letters with and without overline. Then one has

\[ F(p+k+q)_{\rho_1 \ldots \rho_k \mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} = \lambda^{p+k} (-1)^{p(k+1)/2} F(p,q)_{\rho_1 \ldots \rho_k \mu_1 \ldots \mu_p, \nu_1 \ldots \nu_q} \]  

(4.17)

This, together, with the antisymmetry, allows to express all tensors in terms of \(F(0,n)_{\overline{\mu_1} \ldots \overline{\mu_n}}\) which may be arbitrarily chosen as functions of a single variable.

For later purpose, we derive a compact expression of the solution. It is straightforward to verify that, if we define

\[ \Theta(\lambda) = e^{\lambda \sum_{\mu} (\theta_\mu D_{\overline{\mu}})} \]  

(4.18)

we have

\[ \Theta(\lambda) \frac{\partial}{\partial \theta^\mu} \Theta^{-1}(\lambda) = D_\mu - \lambda D_{\overline{\mu}} - \theta^\mu \left( \partial_+ + \lambda^2 \partial_- \right) \]  

(4.19)

that we will satisfy the equation \(D_\mu F(\lambda) = \lambda D_{\overline{\mu}} F(\lambda)\) if we assume that

\[ F(\lambda) = \Theta(\lambda) \tilde{F}(\lambda), \quad \frac{\partial}{\partial \theta^\mu} \tilde{F}(\lambda) = 0, \quad \left( \partial_+ + \lambda^2 \partial_- \right) \tilde{F} = 0. \]  

(4.20)
5 Combining the linear and the non linear equations

So far we have solved them independently. We have derived the general solution of equation 2.4 and a particular class of solutions of equations 2.2. Since the former is linear, it should be satisfied order by order in $g$ by the expansion 4.10. At this moment we are not able to do so beyond the zeroth order, which is already rather involved. In order that $\Phi[0]$ satisfies the self-consistency condition 2.11 it is sufficient that $F(\lambda)$ satisfies them for any $\lambda$. Thus, according to equation 3.27, we should be able to write

$$F(\lambda) = \{ D_{72} - D_{53} + D_{56} - D_{75} \} (D_{53} - D_{25}) (D_{57} + \partial^1) \Psi(\lambda) = \Theta(\lambda) \tilde{F}(\lambda)$$

This is achieved by letting

$$\tilde{F}(\lambda) = \{ \tilde{D}_{72} - \tilde{D}_{53} + \tilde{D}_{56} - \tilde{D}_{75} \} (\tilde{D}_{53} - \tilde{D}_{25}) (\tilde{D}_{57} + \partial^1) \tilde{\Psi}(\lambda)$$

where $\tilde{\Psi}(\lambda) = \Theta^{-1}(\lambda) \Psi(\lambda)$, and we let systematically

$$\tilde{D}_\mu = \Theta^{-1}(\lambda) D_\mu \Theta(\lambda)$$

Thus the question is whether we may choose $\tilde{\Psi}$ such that equations 4.20 hold. This is complicated but may be verified by expanding over $\lambda$. Rewrite schematically the above as

$$\tilde{F}(\lambda) = \mathcal{P}(\lambda) \tilde{\Psi}(\lambda)$$

At order zero, $\mathcal{P}[0](\lambda)$ is independent from $\theta$, so we simply impose that

$$\partial_{\theta^\mu} \Psi[0](\lambda) = \partial_{\theta^\mu} \Psi[0](\lambda) = 0.$$  

Consider the order one. One has

$$\mathcal{P}[1] = \sum_\mu \theta^\mu \mathcal{P}_\mu^{[1]}, \quad \partial_{\theta^\mu} \mathcal{P}_\mu^{[1]} = 0.$$  

Therefore at order one, we have

$$\tilde{F}[1] = \mathcal{P}[0] \tilde{\Psi}[1] + \sum_\mu \theta^\mu \mathcal{P}_\mu^{[1]} \tilde{\Psi}[0]$$
Since $P^{[0]}$ is independent from $\theta$, we must have
\[ \tilde{\Psi}^{[1]} = \sum_{\mu} \theta^\mu \Psi^{[1]}_{\mu} \]
with
\[ 0 = P^{[0]} \tilde{\Psi}^{[1]}_{\mu} + P^{[1]} \tilde{\Psi}^{[0]}_{\mu} \]
The higher orders in $\lambda$ may be treated similarly.

6 Outlook

It seems fair to say that the equation
\[ \sum_{\mu \nu} D_{\nu} \left( R^{-1} D_{\mu} R \right) \gamma^{ijk}_{\mu \nu} = 0, \]
\[ 1 \leq i < j < k \leq 8. \]
has been the main obstacle in deriving classical solutions. We have been able to derive its general solution by going to a special on–shell light cone gauge, assuming no dependence on $x^i$, $i = 1, \ldots, 8$. After this reduction we found a striking analogy between the other (non linear) equations and some of the equations which arose from self–dual Yang Mills in four bosonic variables. In general there appear interesting novel structures (self–duality, Yang type equations) with superderivatives. They seem to enjoy remarkable properties which, contrary to their purely bosonic counterparts, are not restricted to four variables. It will be probably fruitful to push this aspect further.

There remains the consistency problem between the solutions of the dynamical equations \ref{2.4}, \ref{2.22}. We have transformed the former into equations \ref{2.6}, \ref{2.8} and the latter into equations \ref{4.1}. After introducing $\Phi$ by equation \ref{2.10}, we have solved equations \ref{2.7}, \ref{2.8} as well as equations \ref{4.1}. Thus the difficulty which remains is really the consistency between equations \ref{2.6} and the others. At this time we are not able to go beyond the zeroth order in solving this problem. We do not know whether the self–duality conditions may be imposed to the higher orders, but the equations are not too promising. Note that we have derived the general solution of the linear part of the equations, but only a subclass of solutions of the non-linear part. The situation may be improved either by deriving a more general solution of the latter (equations \ref{4.1}), or perhaps by only considering the $N \to \infty$ limit. This problem is left for the future.
Finally, the geometrical significance of the flatness conditions is based on the super light-like lines formulation of integrability along these lines. In other words, the construction of solutions to ten dimensional supersymmetric Yang–Mills equations can be related, via the Radon–Penrose–type transformation, to the construction of holomorphic bundles with some trivialisation on the space of light–like lines. Various algebraic geometry constructions of the solutions to the problem, in particular by the forward images method and the Baker functions were described in [8]. There, briefly speaking, one deals with the following objects. Let $X$ be a complex supermanifold (twistor superspace) of dimension $(17|8)$ parametrising super–light–like rays of dimension $(1|8)$, and is endowed with some family of closed superspaces $Y(u) \subset X$. These superspaces consist of 8 dimensional quadrics with points $u \in U$, $U$ is a superdomain in $\mathbb{C}^{10|16}$ whose even part $\mathbb{C}^{10}$ is a complexification of a fundamental representation of $O(1,9)$. Then locally free bundles on $X(U)$ being trivial under restriction on $Y(u)$ correspond to the solutions of the ten dimensional supersymmetric Yang–Mills equations. It would be interesting to understand the geometrical meaning of the present work along these lines.

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