EXACT SOLUTIONS AND PARTICLE CREATION FOR NONCONFORMAL SCALAR FIELDS IN HOMOGENEOUS ISOTROPIC COSMOLOGICAL MODELS

Yu. V. Pavlov

The problem is solved of describing scale factors of a homogeneous isotropic spaces-time such that the exact solution for the scalar field with a nonconformal coupling to curvature can be obtained from solutions for the conformally coupled field by redefining the mass and momentum. Explicit expressions for dependence of time from the large-scale factor are presented in the form of Abelian integrals in these cases. The exact solution for a scalar field with Gauss-Bonnet type coupling with curvature is received and it is shown that the corresponding nonconformal additions can dominate at the particles creation by gravitational field.

Key words: quantum theory in curved space-time, particles creation by gravitational field, scalar field, exact solution

1. Introduction

Quantum effects in a curved space-time, in particular, particle creation by the gravitational field, can have important applications in cosmology and astrophysics [1], [2]. Quantum theory in a gravitational field has been studied intensely since the 1970s. The case studied in particular detail is that of conformally coupled fields in a homogeneous isotropic space-time. Less studied is the case of a nonconformal coupling of a scalar field to curvature. At the same time, nonconformal contributions can dominate both in the effect of particle creation and in the magnitude of vacuum averages of the energy-momentum tensor, which determine the back reaction of a quantized field on the space-time metric (see, e.g., [3], [4]). Until recently, no calculation of quantum effects of a scalar field coupled to a curvature of the Gauss-Bonnet (GB) type have been available.

Here, we study the possibility of obtaining exact solutions for scalar fields with both the standard nonconformal and the GB couplings to curvature from solutions in the case of a conformal coupling in a homogeneous isotropic space-time. We review scaling factors that allow deriving exact solutions for nonconformal fields. We give an exact solution for the scalar field coupled to the GB-type curvature and analyze particle creation in such a
model. We calculate particle creation in the quantum field theory in a curved space-time, where quantized matter fields are considered in an external classical gravitational field described by the space-time metric. The possibility of using such a semiclassical approach relates to a broad domain between Planckian and Compton characteristic values of length, curvature, and density [1], [2].

We use the system of units where $\hbar = c = 1$.

2. Scalar field in a homogeneous isotropic space

We consider a complex scalar field $\phi(x)$ of mass $m$ with the Lagrangian

$$L(x) = \sqrt{|g|} \left[ g^{ik} \partial_i \phi^* \partial_k \phi - (m^2 + V_g) \phi^* \phi \right]$$

(1)

and the corresponding equation of motion

$$(\nabla^i \nabla_i + V_g + m^2) \phi(x) = 0,$$

(2)

where $\nabla_i$ are covariant derivatives in the $N$-dimensional space-time with the metric $g_{ik}$, $g = \det(g_{ik})$, and $V_g$ is a function of invariant combinations of the metric tensor $g_{ik}$ and its partial derivatives. The case $V_g = 0$ corresponds to the minimal coupling of the scalar field to curvature. The case $V_g = \xi_c R$, where $R$ is the scalar curvature and $\xi_c = (N - 2) / [4(N - 1)]$, is called the conformal coupling to curvature ($\xi_c = 1/6$ at $N = 4$). Equation (2) is conformally invariant if $m = 0$ and $V_g = \xi_c R$.

Taking an arbitrary $V_g$ leads to the appearance of third- and higher-order derivatives of the metric in the metric energy-momentum tensor of the scalar field and hence in the Einstein equations. The appearance of additional higher-derivative terms in the equations, even if their coefficients are small, leads to a dramatic restructuring of the theory. If we require that the metric energy-momentum tensor not contain derivatives of the metric of order higher than two, then $V_g$ can be chosen as the function

$$V_g = \xi R + \zeta R^2_{GB},$$

(3)

where

$$R^2_{GB} \overset{\text{def}}{=} R_{lmnp} R^{lmnp} - 4 R_{lm} R^{lm} + R^2.$$

In the four-dimensional space-time, $R^2_{GB}$ coincides with the Euler characteristic density, which by the GB theorem is a topological invariant (in $N = 2$, the corresponding density
is proportional to $R$). Therefore, the coupling to a curvature of form [3], introduced in [5], can be naturally called the GB-type coupling. In lower dimensions $N = 2, 3$, new effects due to $\zeta \neq 0$ are absent, because then $R_{GB}^2 = 0$. For $N = 4$ with constant $\varphi(x)$, the contribution of the $R_{GB}^2$ term to the metric energy-momentum tensor is absent because the corresponding variational derivative vanishes [6]. But for a variable $\varphi(x)$, the contribution of such terms must be taken into account if the constant $\zeta$ of dimension $(mass)^{-2}$ is nonzero.

We write the metric of an $N$-dimensional homogeneous isotropic space-time in the form

$$ds^2 = dt^2 - a^2(t)(d\eta^2 - dl^2),$$

(4)

where $dl^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta$ is the metric of the $(N-1)$-dimensional space of constant curvature $K = 0, \pm 1$. The full system of solutions of Eq. (2) in metric (4) can be found in the form

$$\varphi(x) = a^{-(N-2)/2}(\eta)g_\lambda(\eta)\Phi_J(x),$$

(5)

where

$$g''_\lambda(\eta) + \Omega^2(\eta)g_\lambda(\eta) = 0,$$

(6)

$$\Omega^2(\eta) = m^2 a^2 + \lambda^2 - \Delta \xi a^2 R + \zeta a^2 R_{GB}^2,$$

(7)

$$\Delta_{N-1}\Phi_J(x) = -\left(\lambda^2 - \left(\frac{N-2}{2}\right)^2 K\right)\Phi_J(x),$$

(8)

$\Delta \xi = \xi_c - \xi$, the prime denotes the derivative with respect to the conformal time $\eta$, and $J$ is the set of indices (quantum numbers) that label eigenfunctions of the Laplace-Beltrami operator $\Delta_{N-1}$ in the $(N-1)$-dimensional space. In metric (4), the expressions for the scalar curvature and the GB invariant are [5]

$$R = a^{-2}(N-1)\left[2c' + (N-2)(c^2 + K)\right],$$

(9)

$$R_{GB}^2 = a^{-4}(N-1)(N-2)(N-3)(c^2 + K)\left[4c' + (N-4)(c^2 + K)\right],$$

(10)

where $c = a'/a = \dot{a}(t)$.

In accordance with the method of diagonalization of a Hamiltonian [1] (see [7] for the case of an arbitrary $V_g$), the functions $g_\lambda(\eta)$ must satisfy the initial conditions

$$g'_\lambda(\eta_0) = i\Omega(\eta_0)g_\lambda(\eta_0), \quad |g_\lambda(\eta_0)| = \Omega^{-1/2}(\eta_0).$$

(11)
If the quantized scalar field is in the vacuum state at the instant $\eta_0$, then the number density of the pairs of particles created up to the instant $\eta$ can be evaluated (for $K = 0$) as \[ \text{(12)} \]

\[ n(\eta) = \frac{B_N}{2a^{N-1}} \int_0^\infty S_\lambda(\eta) \lambda^{N-2} d\lambda, \]

where $B_N = \left[ 2^{N-3} \pi^{(N-1)/2} \Gamma((N-1)/2) \right]^{-1}$, $\Gamma(z)$ is the gamma function, and

\[ S_\lambda(\eta) = \frac{|g'_\lambda(\eta) - i\Omega g_\lambda(\eta)|^2}{4\Omega}. \] \[ \text{(13)} \]

As shown in [8], $S_\lambda \sim \lambda^{-6}$, and the integral in \[ \text{(12)} \] converges for $N < 7$.

3. On exact solutions for different types of coupling to curvature

With the substitution $g(\eta) = \exp z(\eta)$, Eq. \[ \text{(6)} \] reduces to the Riccati equation of the general type for $v(\eta) \equiv z'(\eta)$:

\[ v'(\eta) + v^2(\eta) + \Omega^2(\eta) = 0, \] \[ \text{(14)} \]

Therefore, the number of scaling factors $a(\eta)$ admitting exact solutions is relatively small. In the cases where an exact solution can nevertheless be found, it is typically expressed in terms of special functions: hypergeometric, Bessel functions, etc. Several exact solutions are given in [1], [2]. Scaling factors admitting exact solutions are briefly reviewed in [9], [10].

In a homogeneous isotropic space-time, Eq. \[ \text{(6)} \] has a simpler form for the field with the conformal coupling $\Omega^2 = m^2a^2 + \lambda^2$ than for a nonconformal coupling. This does not forbid the existence of exact analytic solutions of a more complicated equation for a nonconformal scalar field. But the majority of the known exact solutions for a nonconformal scalar field are obtained from solutions for the conformal field by replacing the mass and momentum parameters.

We consider the following question. At what scaling factors can the exact solution for the scalar field with a nonconformal coupling to curvature be obtained from a solution for the field with conformal coupling by redefining the values of the field mass $m$ and the dimensionless momentum $\lambda$?
For the scalar field with a nonconformal coupling of the form $\xi R$, in accordance with (7), this is to be the case if there exist constants $\beta_1$ and $\beta_2$ such that

$$\Delta \xi R a^2 = \beta_1 + \beta_2 a^2.$$  \hfill (15)

This condition, with (9) taken into account, is an ordinary second-order differential equation for $a(\eta)$, which reduces to a linear first-order equation for $f(a) = a'^2(\eta)$. Its solution can be represented in the form

$$\int \frac{da}{\sqrt{C_0 a^4 - N + C_1 a^2 + C_2 a^4}} = \pm (\eta - \eta_0),$$ \hfill (16)

where $C_0, C_1, C_2$, and $\eta_0$ are arbitrary real constants such that $C_0 a^4 - N + C_1 a^2 + C_2 a^4 > 0$.

In the general case for $N > 4$, integral (16) is hyperelliptic, and calculating $a(\eta)$ amounts to inverting the hyperelliptic integral [11]. For $N = 4$, obtaining $a(\eta)$ from (16) is the problem of inversion an elliptic integral [12]. In particular cases where integral (16) is pseudoelliptic or $N = 4$ and $C_2 = 0$, it can be expressed in elementary functions. We give examples of such scaling factors.

The first example of a scaling factor is given by the relations

$$C_1 = C_2 = 0, \quad a = a_1 \eta^{2/(N-2)} = a_0 t^{2/N}, \quad a^2 R = (N - 1)(N - 2)K,$$ \hfill (17)

where $a_1, a_0 = \text{const}$. For $N = 4$ with $K = 0$, such a scaling factor corresponds to a case extremely important from the application standpoint, the radiation-dominated universe. Solutions are known for $N = 4$, where they can be expressed in terms of Kummer’s hypergeometric function, and for $N = 6$, where they can be expressed in terms of Bessel functions [9], [13].

The second example of a scaling factor is

$$C_0 = C_2 = 0, \quad a = a_1 e^{\alpha t} = \alpha t, \quad a^2 R = (N - 1)(N - 2)(\alpha^2 + K),$$ \hfill (18)

where $\alpha = \text{const}$. For $\alpha = 1, K = -1$, this is Milne’s universe [14] (i.e., a part of Minkowski space in the appropriate coordinates). A solution of Eq. (6) expressed in terms of a Bessel function was presented in [9].

The third example of a scaling factor is

$$C_0 = C_1 = 0, \quad C_2 = H^2, \quad a = \frac{-1}{H \eta} = \frac{e^{H t}}{H}, \quad a^2 R = (N-1) \left[N H^2 a^2 + (N-2)K\right],$$ \hfill (19)
where $H = \text{const}$. For $K = 0$, corresponding coordinates describe a part of the De Sitter universe. The solution of Eq. (6) can be expressed in terms of a Hankel function \([1]\).

For $C_0 = 0$, solutions of Eq. (16) are also given by

$$\begin{align*}
\frac{a_1}{\cosh \gamma \eta} &= a_1 \sin \frac{\gamma t}{a_1}, \quad \frac{-a_1}{\sinh \gamma \eta} = a_1 \sin \frac{\gamma t}{a_1}, \\
\frac{a_1}{\cos \gamma \eta} &= a_1 \cosh \frac{\gamma t}{a_1},
\end{align*}$$

where $\gamma = \text{const}$. In the four-dimensional space-time, solutions of Eq. (16) with $C_2 = 4C_0C_2$ are

$$\begin{align*}
a_1 \tan \gamma \eta &= a_1 \sqrt{\exp \frac{2\gamma t}{a_1} - 1}, \quad a_1 \tanh \gamma \eta = a_1 \sqrt{1 - \exp \frac{-2\gamma t}{a_1}}, \\
-a_1 \coth \gamma \eta &= a_1 \sqrt{\exp \frac{2\gamma t}{a_1} + 1},
\end{align*}$$

and solutions with $C_2 = 0$ are

$$\begin{align*}
a_1 \sin \gamma \eta &= \sqrt{a_1^2 - \gamma^2 t^2}, \quad a_1 \sinh \gamma \eta = \sqrt{\gamma^2 t^2 - a_1^2}, \quad a_1 \cosh \gamma \eta = \sqrt{\gamma^2 t^2 + a_1^2},
\end{align*}$$

Solutions of Eq. (6) with scaling factors (20) and (21) can be expressed in terms of the hypergeometric functions. Equation (6) with scaling factor (22) reduces to a (modified) Mathieu equation.

We note that if a solution of Eq. (6) with a scaling factor $a(\eta)$ is found, then it can be used to obtain a solution for the scaling factor $\tilde{a} = \sqrt{a^2(\eta) + b^2}$, where $b = \text{const}$, by redefining $\lambda$.

With the coordinate time $t$, solutions of Eq. (15) can be represented in the form

$$\int \frac{d\left(a^{N/2}\right)}{\sqrt{C_0 + C_1 a^{N-2} + C_2 a^N}} = \pm \frac{N}{2} (t - t_0),$$

where $t_0 = \text{const}$. For $N = 4$, integral (23) is expressed in elementary functions. Therefore, in the four-dimensional space-time, the dependence of the coordinate time $t$ on scaling factor under which solutions for the scalar field with a $\xi R$ coupling to curvature can be obtained from solutions for the field with conformal coupling by redefining the mass and momentum values must be expressible in a finite form in terms of elementary functions.

We consider the case of a scalar field with the GB coupling to the curvature in (3) and assume that $N \geq 4$ and $\zeta \neq 0$. An exact solution of Eq. (4) can be obtained from the
solution for the conformally coupled field by redefining $m$ and $\lambda$ if there exist constants $\tilde{\beta}_1$ and $\tilde{\beta}_2$ such that

$$-\Delta \xi R a^2 + \zeta R^2_{GB} a^2 = \tilde{\beta}_1 + \tilde{\beta}_2 a^2.$$  (24)

With formulas (9) and (10), this condition reduces to a first-order linear inhomogeneous differential equation for the function

$$F(a) = \left(\dot{a}^2(t) + K - \frac{\Delta \xi a^2}{2\zeta(N - 2)(N - 3)}\right)^2.$$  (25)

Solving differential equation (24) for $a(t)$, we obtain an expression for the dependence of the scaling factor on the coordinate time:

$$\int -\frac{da}{\sqrt{-K + \frac{\Delta \xi a^2}{2\zeta(N - 2)(N - 3)} \pm \sqrt{D_0 a^4 - N + D_1 a^2 + D_2 a^4}}} = \pm(t - t_0),$$  (26)

where $D_0, D_1,$ and $D_2$ are arbitrary real constants such that the integral is real.

In the general case, finding the explicit form of $a(t)$ amounts to inverting Abelian integral (26). We give examples in particular cases:

$$K = \Delta \xi = D_1 = D_2 = 0, \quad a = \gamma t^{4/N}, \quad R^2_{GB} = 0,$$  (27)

where exact solutions for the conformal coupling are known for $N = 4, 8, 12$ (see formulas (17) and (18));

$$K = D_0 = D_1 = 0, \quad a = a_0 \exp Ht, \quad R^2_{GB} = N(N - 1)(N - 2)(N - 3)H^4$$  (28)

(see formulas (19)); and for $N = 4$, functions (20) are solutions of Eq. (26) with $D_1^2 = 4D_0 D_2$.

In what follows, we consider the exact solution and particle creation for the first scaling factor in (20).

4. **Particle creation in the** $a_1 \sin(\gamma t/a_1) = a_1 / \cosh \gamma \eta$ **model**

Space-time with a scaling factor $a_1 \sin(\gamma t/a_1) = a_1 / \cosh \gamma \eta$ evolves between two singularities at $t = 0$ ($\eta = -\infty$) and $t = T \equiv \pi a_1 / \gamma$ ($\eta = +\infty$), if $\gamma^2 + K \neq 0$, $N \geq 3$. In the case $\gamma = 1$ and $K = -1$, the corresponding coordinates cover a part of the De Sitter space of the second kind [15] with a constant negative curvature $R = -(N - 1)N/a_1^2$. 
Particle creation in this model was previously considered for a conformal coupling
to the curvature in [1], and the case of the $\xi R$ coupling was investigated in [3]. We here
consider the creation of scalar particles with a GB-type coupling to the curvature, Eq. (3),
in a four-dimensional space-time. From (9) and (10) with $N = 4$, we obtain
\[ Ra^2 = 6 \left( \gamma^2 + K \right) - \frac{12\gamma^2}{a_1^2} a^2, \quad a^2 R_{GB}^2 = -\frac{24\gamma^2}{a_1^2} \left[ K + \gamma^2 \left( 1 - \frac{a^2}{a_1^2} \right) \right], \quad (29) \]
and therefore Eq. (6) becomes
\[ g''(\eta) + \left[ \left( m^2 a_1^2 + \Delta \xi 12 \gamma^2 + \zeta \frac{24\gamma^4}{a_1^4} \right) \frac{1}{\cosh^2 \gamma \eta} + \lambda^2 - 6(\gamma^2 + K) \left( \Delta \xi + \zeta \frac{4\gamma^2}{a_1^2} \right) \right] g(\eta) = 0. \quad (30) \]
The solution of Eq. (30) with initial conditions (11) with $\eta \to -\infty$ can be obtained from
the exact solution for the conformally coupled case [1] by redefining the values of the mass
($m \to M$) and momentum ($\lambda \to \Lambda$):
\[ M^2 = m^2 + \frac{12\gamma^2}{a_1^2} \left( \Delta \xi + \zeta \frac{2\gamma^2}{a_1^2} \right), \quad \Lambda^2 = \lambda^2 - 6(\gamma^2 + K) \left( \Delta \xi + \zeta \frac{4\gamma^2}{a_1^2} \right). \quad (31) \]
The solution has the form
\[ g(\eta) = e^{i(\Lambda \eta + \alpha_0)} \sqrt{\Lambda} F \left( A, B; C; \frac{1 + \tanh \gamma \eta}{2} \right), \quad (32) \]
where $F(A, B; C; z)$ is the hypergeometric function, $\alpha_0$ is an arbitrary real constant, and
\[ A = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{M^2 a_1^2}{\gamma^2}}, \quad B = \frac{1}{2} - \sqrt{\frac{1}{4} + \frac{M^2 a_1^2}{\gamma^2}}, \quad C = 1 + i \frac{\Lambda}{\gamma}. \quad (33) \]
The condition $\Omega^2 \geq 0$, which is necessary in the Hamiltonian diagonalization method,
is satisfied for Eq. (30) for any $\lambda$ and $\eta$ if
\[ (\gamma^2 + K) \left( \Delta \xi + \zeta \frac{4\gamma^2}{a_1^2} \right) \leq 0, \quad m^2 a_1^2 + 6(\gamma^2 - K) \Delta \xi - 24\zeta K \frac{\gamma^2}{a_1^2} \geq 0. \quad (34) \]
In particular, $\Omega^2 \geq 0$, at $K = 0$ and
\[ -\zeta \frac{4\gamma^2}{a_1^2} \geq \Delta \xi \geq -\frac{m^2 a_1^2}{6\gamma^2}. \quad (35) \]
From asymptotic expression of (13) for exact solution (32), similarly to the conformal
case [1], we obtain the limit spectrum (as $\eta \to +\infty$) of the created particles:
\[ S_\lambda = \cos^2 \frac{\pi}{2} \sqrt{1 + \frac{4M^2 a_1^2}{\gamma^2}} \frac{\sinh \frac{\pi \Lambda}{\gamma}}{\sinh \frac{\pi \Lambda}{\gamma}}. \quad (36) \]
In accordance with formulas (12), (31), and (36), the number of particle pairs created in a Lagrangian volume $a^3(t)$ at $K = 0$ per the evolution cycle $t \to T$ is given by

$$N = \frac{\gamma^3}{2\pi^5} \cos^2\pi \sqrt{\frac{1}{4} + \frac{m^2a_1^2}{\gamma^2} + 12\Delta\xi + 24\zeta^2 \gamma^2 a_1^2} \int_{x_0}^{\infty} \frac{x \sqrt{x^2 - x_0^2}}{\sinh^2 x} \, dx, \quad (37)$$

where

$$x_0 = \pi \sqrt{-6} \left(\Delta\xi + \frac{4\gamma^2 a_1^2}{a_1^2}\right). \quad (38)$$

Because [16]

$$\int_{0}^{\infty} \frac{x^2}{\sinh^2 x} \, dx = \frac{\pi^2}{6}, \quad (39)$$

it follows from formulas (31), (35), and (38) that

$$\Delta\xi = -\zeta \frac{4\gamma^2 a_1^2}{a_1^2} \geq -\frac{m^2 a_1^2}{6\gamma^2} \quad \Rightarrow \quad N = \frac{\gamma^3}{12\pi^5} \cos^2\pi \sqrt{\frac{1}{2} \left[ 1 + 24\Delta\xi + \frac{4m^2 a_1^2}{\gamma^2} \right].} \quad (40)$$

Formulas (36) and (37) reproduce the corresponding results for the scalar field with the $\xi R$ coupling at $\zeta = 0$ [3] and for conformally coupled particles at $\zeta = 0 = \Delta\xi$ [1]. At $\Delta\xi = 0$, formula (40) gives the number of pairs of conformally coupled particles created per the evolution cycle:

$$N_0 = \frac{\gamma^3}{12\pi^5} \cos^2\pi \sqrt{\frac{1}{2} \left[ 1 + \frac{4m^2 a_1^2}{\gamma^2} \right].} \quad (41)$$

If $\zeta \to -\infty$ for a fixed $\Delta\xi$, then $M^2 \to -\infty$, and the number of created particles increases indefinitely, as is shown, for example, for $ma_1/\gamma = 1$ in Fig. 1.

It follows from (36) that particle creation per evolution cycle is absent if

$$M^2 a_1^2 = n(n + 1)\gamma^2, \quad n = 0, 1, 2, \ldots \quad (42)$$

With expression (31) and inequalities (35) with $K = 0$ taken into account, there is no particle creation per evolution cycle if the parameter $\zeta$ takes the values

$$\zeta_n = \frac{a_1^2}{24\gamma^2} \left( n(n + 1) - 12\Delta\xi - \frac{m^2 a_1^2}{\gamma^2} \right), \quad n = 0, 1, \ldots \leq \sqrt{\frac{m^2 a_1^2}{\gamma^2} + 6\Delta\xi + \frac{1}{4} - \frac{1}{2}.} \quad (43)$$

Hence, depending on the values of $\zeta$, particle creation in this model can either increase without bound or be absent altogether. Therefore, the effect of the parameter $\zeta$ of the
scalar field coupling to the GB invariant can dominate, and it must be taken into account in calculating of the effects of scalar fields in a curved space-time.

Acknowledgments. The author thanks Professor A. A. Grib and the participants in the seminar at the A. Friedmann Laboratory for Theoretical physics for the discussion of this work. The research is done in collaboration with Copernicus Center for Interdisciplinary Studies, Kraków, Poland and supported by the grant from The John Templeton Foundation.

References

[1] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* [in Russian], Énergoatomizdat, Moscow (1988); English transl., Friedmann Laboratory Publ., St. Petersburg (1994).

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*, Cambridge Univ. Press, Cambridge (1982).

[3] V. B. Bezerra, V. M. Mostepanenko, and C. Romero, *Mod. Phys. Lett. A* **12**, 145–154 (1997).

[4] M. Bordag, J. Lindig, V. M. Mostepanenko, and Yu. V. Pavlov, *Int. J. Mod. Phys. D* **6**, 449–463 (1997).
[5] Yu. V. Pavlov, *Teor. Matem. Fiz.* **140**, 241–255 (2004) [English transl.: *Theor. Math. Phys.* **140**, 1095–1108 (2004); arXiv: gr-qc/0409009].

[6] C. Lanczos, *Ann. Math.* **39**, 842–850 (1938).

[7] Yu. V. Pavlov, *Int. J. Mod. Phys. A* **17**, 1041–1044 (2002); arXiv: gr-qc/0202041.

[8] Yu. V. Pavlov, *Teor. Matem. Fiz.* **126**, 115–124 (2001) [English transl.: *Theor. Math. Phys.* **126**, 92–100 (2001); arXiv: gr-qc/0012082].

[9] Yu. V. Pavlov, “Creation of particles in cosmology: Exact solutions” [in Russian], in: *Quantum Theory and Cosmology* (V. Yu. Dorofeev and Yu. V. Pavlov, eds.), Friedmann Laboratory Publ., St. Petersburg (2009), pp. 158–171.

[10] Yu. V. Pavlov, *Uchen. Zap. Kazan. Gos. Un-ta, Ser. Fiz.-Matem. Nauki* **153**, No. 3, 65–71 (2011).

[11] A. I. Markushevich, *Introduction to the Classical Theory of Abelian Functions* [in Russian], Nauka, Moscow (1979); English transl. (Transl. Math. Monogr., Vol. 96), Amer. Math. Soc., Providence, R. I. (1992).

[12] H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 3, *Elliptic and Modular Functions: Lame and Mathieu Functions*, McGraw-Hill, New York (1955).

[13] S. G. Mamaev, V. M. Mostepanenko, and A. A. Starobinskii, *ZhETF* **70**, No. 5, 1577–1591 (1976) [English transl.: *Sov. Phys. JETP* **43**, No. 5, 823–830 (1976)].

[14] E. A. Milne, *Nature* **130**, 9–10 (1932).

[15] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Univ. Press, Cambridge (1973).

[16] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Acad. Press, New York (1980).