A Critical Behaviour of Anomalous Currents, Electric-Magnetic Universality and CFT$_4$

D. Anselmi$^a$, M. Grisaru$^b$ and A. Johansen$^{a,3}$

$^a$Lyman Laboratory of Physics, Harvard University
Cambridge, MA 02138, USA
$^b$Physics Department, Brandeis University
Waltham, MA 02254, USA

ABSTRACT

We discuss several aspects of superconformal field theories in four dimensions (CFT$_4$), in the context of electric-magnetic duality. We analyse the behaviour of anomalous currents under RG flow to a conformal fixed point in $N = 1, D = 4$ supersymmetric gauge theories. We prove that the anomalous dimension of the Konishi current is related to the slope of the beta function at the critical point. We extend the duality map to the (nonchiral) Konishi current. As a byproduct we compute the slope of the beta function in the strong coupling regime. We note that the OPE of $T_{\mu\nu}$ with itself does not close, but mixes with a special additional operator $\Sigma$ which in general is the Konishi current. We discuss the implications of this fact in generic interacting conformal theories. In particular, a SCFT$_4$ seems to be naturally equipped with a privileged off-critical deformation $\Sigma$ and this allows us to argue that electric-magnetic duality can be extended to a neighborhood of the critical point. We also stress that in SCFT$_4$ there are two central charges, $c$ and $c'$, associated with the stress tensor and $\Sigma$, respectively; $c$ and $c'$ allow us to count both the vector multiplet and the matter multiplet effective degrees of freedom of the theory.
1. Introduction

A recent insight into quantum field theory has been the discovery by Seiberg of the electric-magnetic duality of non-trivial N=1 superconformal theories in four dimensions that can be realized as infrared fixed points of ordinary N=1 supersymmetric theories \[1\]. Following his approach other examples of such dynamics have been found \[2\]. All these examples have survived various self-consistency tests and therefore one may be convinced that this understanding of their renormalization group flow is correct. However the physical information about the resulting superconformal theories seems to be incomplete. In particular we do not know the exact spectrum and physical correlators of the theory at the conformal point \[3\]. We also have only limited information about duality maps between operators of these theories. A non-trivial duality map of primary chiral operators at the critical point has been constructed in many models \[2\], \[4\] but the extension to nonchiral operators is more problematic \[4\].

In a recent paper \[6\] an analysis of the infrared behaviour of the R-current has been presented. In this paper, we discuss some other aspects of critical behaviour of N=1 supersymmetric theories, focusing on the role and properties of anomalous currents. We shall be mostly interested in the Konishi axial current, contained in a superfield that we call the Konishi superfield: \( J \sim \Phi \bar{\Phi} \) where \( \Phi \) denotes the chiral superfields of these theories. Away from criticality the Konishi current is anomalous - the anomaly is proportional to the gauge \( \beta \)-function - and is not considered in the analysis of the infrared dynamics since it does not generate any symmetry. One may ask what is the fate of this anomalous current at the critical point. A possible answer would be that since the anomaly is proportional to the beta function, the current becomes conserved since the beta function vanishes. This would be similar to the fate of the trace anomaly, which is nonvanishing away from criticality and proportional to the beta function, but disappears at the critical point.

In the case of the Konishi current, if the above scenario were realised the conformal theory would have additional global symmetries, generated by the newly conserved current, as compared to the non-critical theory. Such a scenario would also imply that there is a jump in the cohomology ring of the BRST operator in the twisted N=1 supersymmetric theory \[7\]. (Another possibility is that the anomalous current decouples from the physical correlators, implying that the related degrees of freedom become massive near the conformal point.)

In the present paper we demonstrate that although the anomaly for the renormalized Konishi current is proportional to the beta function, this anomalous current survives at the conformal fixed point and remains anomalous. Moreover, it plays an important role in quantum conformal field theory in four dimensions (CFT\(_4\)) since it appears in the operator product expansions of the stress energy tensor, and defines a new central charge.

As we shall show, at large distances, the two-point correlator of two renormalized Konishi superfields behaves like \(< J(x)J(y) > \sim [\beta(g^2)/g^4]^2 |x - y|^{-4} \) where \( g = g(|x - y|) \)

\[4\] For a recent discussion of some aspects of such an extension see ref. \[6\].
is the running gauge coupling constant at the scale $1/|x - y|$ and $\beta$ is the Gell-Mann-Low function. Consequently, in a theory with a non-trivial conformal fixed point this correlator has a power-like behaviour $\sim 1/|x - y|^{4+2\delta}$ for $|x - y| \to \infty$, where $\delta$, the anomalous dimension of the Konishi current, is given by the slope of the beta function at criticality:

$$\delta = \beta'(\alpha_*)$$  \hspace{1cm} (1.1)

Here $\alpha = g^2/4\pi$. To understand this relation we note that the Konishi current is a relevant deformation of the critical theory (due to the Konishi anomaly, adding the Konishi operator to the action is equivalent to changing the gauge coupling constant), so that Eq. (1.1) can be seen as the generalization of the analogous property that holds in two dimensions \[8\].

In particular in SUSY QCD with the $SU(N_c)$ gauge group and $N_f$ flavours of matter superfields in fundamental representation ($3N_c/2 \leq N_f \leq 3N_c$) without superpotential, we find that for the case $\epsilon = 3 - N_f/N_c << 1$ (which corresponds to a perturbative fixed point \[3\]) the anomalous dimension of the Konishi current is $\delta = \epsilon^2/3$ at the conformal point.

The above result can be generalized to models with superpotential. We use such a generalization to construct a duality map for the (non-chiral) Konishi operator of the electric theory. Using Seiberg electric-magnetic duality \[3\] one can argue that the conformal dimension of the electric Konishi current is equal to that of its magnetic counterpart in the dual formulation. Consequently we find a relation between the slopes of the beta functions in these two formulations, in the conformal window. In particular we compute the dimension of the Konishi current at the strongly interacting conformal point $\sigma = -3/2 + N_f/N_c << 1$ in the electric theory without superpotential, which is small in this regime. We find $\delta = \beta'(\alpha_*) = 28\sigma^2/3$.

Eq. (1.1) is analogous to the fact that the operator of the trace anomaly $\Theta = \beta(\alpha) F_{\mu\nu} F^{\mu\nu}$ acquires the anomalous dimension $\beta'(\alpha) > 0$ (for a recent discussion see ref. \[3\]). Despite the fact that the operator $\Theta$ survives in the infrared the energy-momentum tensor is traceless at the conformal fixed point because the traceless part has canonical dimension which is smaller than that of $\Theta$. The story is different for the Konishi supermultiplet $J$ all components of which have the same anomalous dimension at the fixed point. In particular, the Konishi axial current $a_\mu$ has a non-canonical dimension there. This is compatible with the fact that $a_\mu$ does not generate any new symmetry of the theory at the conformal point; nor is there a jump in the cohomology ring of the BRST operator in the twisted $N=1$ supersymmetric theory. Furthermore, the (component) anomaly operators which enter the chiral superfield $D^2 J$ have conformal dimensions which do not satisfy the relation $d = 3R/2$ between $R$ charge and conformal dimension $d$ which follows from the $N = 1$ superconformal algebra \[10\]. The explanation of this phenomenon is that $D^2 J$ is not a primary chiral operator of the superconformal algebra at the fixed point.

The conserved currents contained in the supercurrent superfield $J_{\alpha\bar{\alpha}}$ have of course their canonical dimensions. However, away from the fixed point, the anomalous axial current that enters the supercurrent superfield $J_{\alpha\bar{\alpha}}$ is a linear combination of a conserved
\( R \) current and the Konishi current. Therefore it contains parts with different scaling behaviour. This is consistent with superPoincaré symmetry due to the reducibility of the superfield \( J_{\alpha\dot{\alpha}} \) at criticality. Furthermore, it follows from the fact that the conformal dimension of the Konishi current is larger than its canonical dimension, that the Seiberg non-anomalous \( R \) current coincides with the current which enters \( J_{\alpha\dot{\alpha}} \) in the infrared. This last fact was first established in ref. \([3]\) by analysing operator equations for anomalous currents.

The importance of anomalous currents at criticality manifests itself also in the non-closure of the operator product expansion of two stress tensors or supercurrent superfields: indeed an additional operator \( \Sigma \) appears that is not conserved and generally coincides with the Konishi superfield (apart from a multiplicative factor that can depend on the coupling). One possible implication of this is an extension of the idea of electric-magnetic duality. Indeed, mapping two conformal theories amounts to identifying their superfields \( J_{\alpha\dot{\alpha}} \). This, in turn, implies the identification of their OPE’s and, in particular, of the operator \( \Sigma \). Consequently, one has a natural mapping between two deformations of the theories, at least in neighborhoods of the critical points.

As mentioned above, away from the fixed point the Konishi current is the only renormalizable deformation of conformal SQCD. With this interpretation we find that the subleading behaviour of the electric and magnetic theories near the conformal fixed point shows an unexpected universality. This universality is a consequence of electric-magnetic duality and means that in the infinitely small neighborhood of a critical point there exists a map from electric off-critical theory to the magnetic one. We discuss this further in the conclusions.

Our paper is organized as follows. In section 2 we review some renormalization group aspects in the electric SUSY QCD and compute the \( \bar{z} \) renormalization factor of the Konishi superfield by considering its matrix elements in an external gauge superfield. In section 3 we apply this result to the computation of the two-point correlator of the Konishi superfield at large distances and determine its conformal dimension at the critical point in terms of the slope of the beta function. In section 4 we consider the magnetic SUSY QCD and construct the duality map for the Konishi superfield from the electric theory to the magnetic one. We observe here a universality in the critical behaviour of the Konishi multiplet and use duality to compute the slope of the beta function in the strong coupling regime. In section 5 we analyse the quantum conformal algebra at the OPE level and point out the basic properties of the axiomatic-algebraic approach to CFT\(_4\). We conclude with a discussion of our results and possible implications such as the extension of the notion of non-Abelian duality.

2. Matrix elements of currents and renormalization

In this section we discuss the renormalization of the Konishi current. We first review the abelian case and then generalize the discussion to the nonabelian case.
We start with a few preliminary comments. In order to determine the renormalization factor of an anomalous current it is convenient to consider its matrix element in an external gauge field. The renormalization factor can be determined from that of the corresponding anomaly. The difference between the operator anomaly equations and the matrix element form of the anomaly equations was first recognized as an essential feature of SUSY theories in ref. [11]. A natural definition of the Konishi current, consistent with supersymmetry, has been shown to lead to a one-loop form of the operator anomaly equation. On the other hand, in ref. [12] it has been demonstrated that the matrix elements of the divergence of anomalous axial currents are always multiloop and do not depend on the definition of the operators. In particular in SUSY theories the matrix elements of the divergence of an anomalous axial current are always proportional to the exact beta function of the coupling constant taken at the scale of typical momenta of external fields.

Actually the one-loop character of the anomaly in operator form is conventional; it is due to a particular choice of regularization. For example, in the nonsupersymmetric case the one-loop form of the operator equation corresponds to a “natural” regularization with the help of Pauli-Villars fermions [13]. However it can be shown that for a different gauge invariant regularization of the current this equation can be of multi-loop form [14], [15]. In the simplest case of non-supersymmetric QED such a change of the definition of the operators is tantamount to adding finite counterterms to the current operator, which are proportional to the current itself.

Regularization of a non-minimal type, leading to the multi-loop operator form of the anomaly, may seem unnatural. However, in a supersymmetric theory this is the only possibility for including the anomalous axial $R$ current and the conserved energy-momentum tensor into the same supercurrent supermultiplet. As for the Konishi anomaly the ambiguity of the operator equation (its one-loop or multi-loop character) has been demonstrated in ref. [12].

It is worth emphasizing that despite the operator equation ambiguity the conditions of cancellation of the axial anomaly are of one-loop nature. The point is that multi-loop corrections to the physical correlators with insertions of the anomalous divergence of the axial current are due to $S$-matrix-like rescattering, while the proper anomaly is due to the one-loop triangle diagram.

### 2.1. Abelian SUSY theories

We first review some of the work of ref. [11], [12]. We consider supersymmetric QED described by a real scalar (vector multiplet) superfield $V$ (the superfield strength $W_\alpha$ contains $F_{\mu\nu}$ and the photino field) and two chiral matter superfields $T$ and $U$. In this model there are two anomalous axial currents, which are the components of two superfields, the Konishi current $J = \bar{U}e^{-V}U + \bar{T}e^{V}T$ and the supercurrent

$$J_{\alpha\dot{\alpha}} = \frac{1}{g^2}W_\alpha \bar{W}_{\dot{\alpha}} + \frac{1}{3}D_\alpha (e^{V}T)e^{-V}\bar{D}_{\dot{\alpha}}(e^{-V}\bar{T}) + \ldots$$

(2.1)
\[-\frac{1}{3}Te^{-V}D_\alpha(e^VD_{\bar{\alpha}}(e^{-V}\bar{T}))+\frac{1}{3}\bar{D}_{\bar{\alpha}}(e^{-V}D_\alpha e^V T)e^{-V}\bar{T}+(T\rightarrow U, V\rightarrow -V)\.\]

We have absorbed the coupling constant in the definition of the superfield $V$. The composite superfield $J_{a\bar{a}}$ contains in addition to the $R$-current $R_\mu$, the energy-momentum tensor $T_{\mu\nu}$ and the supersymmetry current $S_{\mu\bar{\alpha}}$.

At the classical level one has $\bar{D}^2J=0$ and $\bar{D}^{\bar{\alpha}}J_{a\bar{a}}=0$. At the quantum level the operators $\bar{D}^2J$ and $\bar{D}^{\bar{\alpha}}J_{a\bar{a}}$ are proportional to the operator $W^2=W^\alpha W_\alpha$ and $D_\alpha W^2$ respectively. For the supermultiplet $J$ this corresponds to the Konishi anomaly [16]; the anomaly in $\bar{D}^{\bar{\alpha}}J_{a\bar{a}}$ contains in addition to the axial $R$-current anomaly the conformal and superconformal anomalies. The quantity $F_{\mu\nu}\bar{F}^{\mu\nu}$ is contained in the imaginary part of the $F$-term of the chiral superfield $W^2$.

In order to determine the anomalous dimension of the operator $J$ it is sufficient to determine that of the operator $W^2$ and relate the two through the anomaly equation. For the latter operator, it is sufficient to evaluate its matrix elements in the presence of an external (background) gauge field $V_{\text{ext}}$. These matrix elements include multi-loop corrections due to the rescattering of photons.

We consider the generating functional $Z$ in the presence of the external gauge superfield

$$Z = \left(\int DV D U DT \exp[iS(V, V_{\text{ext}}; U, T)]\right)_{1PI},$$

$$S = \int d^4x \left(\frac{1}{4g_0^2} \left[ \int d^2\theta W^2 + \text{h.c.} \right] + \int d^4\theta [\bar{U} e^{-V} U + \bar{T} e^V T] \right).$$

We have omitted the gauge-fixing terms because in an abelian theory they are irrelevant for what follows. Here $g_0$ is the bare coupling constant defined at some ultraviolet cutoff $\Lambda$.

The logarithmic derivative of $Z$ with respect to $g_0^2$ gives the exact expression for the vacuum expectation value $\langle W^2 \rangle$ in the external field $V_{\text{ext}}$, integrated over $d^4xd^2\theta$

$$\int d^4xd^2\theta \langle W^2 \rangle + \text{h.c.} = -4i \frac{\partial}{\partial (1/g_0^2)} \log Z.$$  \hspace{1cm} (2.3)

On the other hand, $\log Z$ is the effective action

$$Z = \exp(iS_{\text{eff}}), \quad S_{\text{eff}} = \int d^4xd^2\theta \frac{1}{4g_{\text{eff}}^2(k)} W_{\text{ext}}^2 + \text{h.c.}.$$ \hspace{1cm} (2.4)

Here $k$ is a typical momentum of the external field; $g_{\text{eff}}^2(k)$ is the effective gauge coupling. In eq. (2.4) we have kept only the terms quadratic in $W_{\text{ext}}$. This equation is valid for $|k|^3 \gg |W_{\text{ext}}^2(k)|$, in an expansion in powers of $|W_{\text{ext}}^2(k)|/|k|^3$. If we now assume that the integration over $x$ and $\theta$ in eq. (2.4) may be omitted, then upon comparison of eqs. (2.3) and (2.4) we obtain separate equations for matrix elements of the chiral superfields $W^2$ and $W^2$

$$\langle W^2 \rangle = \frac{\partial}{\partial (1/g_0^2)} \frac{1}{g_{\text{eff}}^2(k)} W_{\text{ext}}^2 = \frac{g_0^4}{g_{\text{eff}}^2(k)} \cdot \frac{\beta(g_{\text{eff}}^2(k))}{\beta(g_0^2)} W_{\text{ext}}^2.$$ \hspace{1cm} (2.5)
\[ \frac{\beta(g_{\text{eff}})}{\beta_1(g_{\text{eff}})} \cdot \frac{\beta_1(g_0^2)}{\beta(g_0^2)} W_{\text{ext}}. \]

Here \( \beta_1 = g^4/2\pi^2 \) is the one-loop Gell-Mann-Low function. This equation can be easily checked by a direct computation \[12\] to two-loop order.

In the last equality in (2.5) the factor \( \beta_1(g_0^2)/\beta(g_0^2) \), a function of the ultraviolet cutoff \( \Lambda \), can be removed by multiplicative renormalization of the composite operator \( W^2 \):

\[ W^2 = \frac{\beta_1(g_0^2)}{\beta(g_0^2)} W_{\text{ren}} \]

Thus,

\[ <W_{\text{ren}}^2> = \frac{\beta(g_{\text{eff}})}{\beta_1(g_{\text{eff}})} W_{\text{ext}}^2 \]

The factor \( \beta(g_{\text{eff}})/\beta_1(g_{\text{eff}}) \) depends on the external momentum and therefore determines the anomalous dimension of the operator \( W_{\text{ren}}^2 \), and hence, by the anomaly equation, as we discuss below, of \( J \).

As described in ref. \[14\], in the dimensional regularization scheme there is a natural definition of the operator \( J_{\alpha \dot{\alpha}} \) which contains a conserved component of spin 2 corresponding to the energy-momentum tensor. By an explicit computation it has been shown \[14\] that the operator anomaly equation for \( \bar{D}^\dot{\alpha} J_{\alpha \dot{\alpha}} \) has a multi-loop character and is proportional to the exact multi-loop beta function \( \beta \). Since the operator \( J_{\alpha \dot{\alpha}} \) contains conserved currents, it is finite and needs no renormalization.

For the definition of the renormalized Konishi superfield operator \( J \) we have more freedom, since it does not contain any conserved current. For example we can accept a definition \[11\] of \( J \) by the requirement that it coincides with the bare current at the ultraviolet cut-off \( \Lambda \). In this case all radiative corrections to the matrix element of this operator in the presence of external fields vanish at \( k^2 = \Lambda^2 \), where \( k^2 \) is a typical external momentum (we assume that the differences of momenta of individual external legs are negligible as compared to \( k^2 \)). Matrix elements of such an operator will be proportional to a factor \( z_J(g^2(k))/z_J(g_0^2) \), with \( z_J(g) \) to be determined below.

To analyse the renormalization of \( J \) one has to consider a mixing of the operators \( D_\alpha \bar{D}^2 J, \bar{D}^\dot{\alpha} J_{\alpha \dot{\alpha}} \) and \( D_\alpha W^2 \). Similarly to the case of non-supersymmetric QED it can be shown \[12\] that the operator \( D_\alpha \bar{D}^2 J \) can be renormalized by a multiplicative factor \( z_J(g_0^2) \) which does not depend explicitly on the parameter \( \Lambda \), \( (D_\alpha \bar{D}^2 J)_{\text{ren}} = z_J(g_0^2) D_\alpha \bar{D}^2 J \). One also finds that the following operators are invariant under the renormalization group

\[ \bar{D}^\dot{\alpha} J_{\alpha \dot{\alpha}} + \frac{\beta(g_0^2)}{6g_0^2} D_\alpha W^2 = c_1, \quad D_\alpha \bar{D}^2 J - \frac{\beta_1(g_0^2)}{g_0^2} D_\alpha W^2 = c_2, \quad \frac{\beta(g_0^2)}{\beta_1(g_0^2)} D_\alpha \bar{D}^2 J = c_3, \quad (2.6) \]

where \( c_{1,2,3} \) are constants, \( \beta_1 \) is the one-loop beta function, and the other operators and \( g \) are taken at a renormalization point \( m \). The constants \( c_{1,2,3} \) can be fixed by considering the
limit \( g \to 0 \), when obviously \( c_1 = c_2 = 0 \) as a consequence of the one-loop approximation. Thus the corresponding operator equations read [11]

\[
\bar{D}^{\dot{\alpha}} J_{\alpha \dot{\alpha}} = -\frac{\beta(g_0^2)}{6g_0^4} D_\alpha W^2, \quad D_\alpha \bar{D}^2 J = \frac{\beta_1(g_0^2)}{g_0^4} D_\alpha W^2. \quad (2.7)
\]

Finally, for renormalized matrix elements of the operators in the external field \( V_{\text{ext}} \) one obtains [12]

\[
< \bar{D}^{\dot{\alpha}} J_{\alpha \dot{\alpha}} > = -\frac{\beta(g^2)}{6g^4} D_\alpha W_{\text{ext}}^2, \quad < D_\alpha \bar{D}^2 J >_{\text{ren}} = \frac{\beta(g^2)}{g^4} W_{\text{ext}}^2, \quad (2.8)
\]

where \( g = g(k) \) is the effective coupling constant.

Comparing (2.7), (2.5) and (2.8) one verifies that indeed \( J_{\alpha \dot{\alpha}} \) is not renormalized, whereas the renormalization factor for the Konishi current is \( z_J(g_0^2) = \beta(g_0^2)/\beta_1(g_0^2) \).

\[
J_{\text{ren}} = \frac{\beta(g_0^2)}{\beta_1(g_0^2)} J. \quad (2.9)
\]

Equivalently, as established in ref. [11], \((1 - \gamma)J\) is renormalization group invariant, \( \gamma \) being the anomalous dimension of the chiral superfields \( T \) and \( U \).

### 2.2. Non-abelian SUSY theories

We now generalize to the non-Abelian SUSY theory. For definiteness we shall consider SUSY QCD with \( SU(N_c) \) gauge group and \( N_f \) flavours of chiral superfields \( Q_i \) and \( \tilde{Q}^i \), \( i = 1, ..., N_f \), in the fundamental representation. One can define the supercurrent superfield \( J_{\alpha \dot{\alpha}} \) and the Konishi superfield \( J = \sum_i (Q_i e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i) \) which are singlets under the flavour global \( SU(N_f) \times SU(N_f) \) group. These currents are anomalous. One can follow the above analysis to determine the matrix elements of these currents. The analysis here is however less trivial because the external field is charged with respect to the gauge group. Therefore the matrix elements depend on the gauge fixing [12] and hence they are not simply of the form given in eqs. (2.8). However in an appropriate kinematical regime [17] [18] which corresponds to a limit of vanishing momentum of the current these matrix elements reduce to the form of eqs. (2.8). In what follows we shall consider this particular kinematics since, as we shall see, it is relevant for computation of the two-point correlators discussed below.

By differentiating the 1PI effective action \( S_{\text{eff}} \) with respect to the bare coupling constant one gets similarly to the case of the abelian theory the following equation

\[
< \text{Tr} W^2 > = \frac{\beta(\alpha)}{\beta_1(\alpha)} \frac{\beta_1(\alpha_0)}{\beta(\alpha_0)} \frac{1}{1 - \alpha_0 N_c / 2\pi} \text{Tr} W_{\text{ext}}^2. \quad (2.10)
\]
We have adopted a definition of the operator $W^2$ at a scale $m$ (below $\Lambda$) which agrees with that of ref. [11] and is responsible for the factor $1 - \alpha_0 N_c/2\pi$ (see below). The total NSVZ beta function [19] reads

$$
\beta(\alpha) = -\frac{\alpha^2}{2\pi} \cdot \frac{3N_c - N_f + N_f \gamma}{1 - \alpha N_c/2\pi},
$$

where $\alpha = \alpha(k) = g(k)^2/4\pi$ is the effective coupling at the scale of external momenta. Here $\gamma$ is the anomalous dimension of the matter superfields.

$$
\gamma(\alpha) = -\frac{d \log Z}{d \log m} = -\frac{N_c^2 - 1}{2N_c} \cdot \frac{\alpha}{\pi} + O(\alpha^2).
$$

The factor $1/(1 - \alpha_0 N_c/2\pi)$ appears because of the one-loop form of the Wilsonian action [11]

$$
S = \int d^4x \left( \frac{1}{2g_0^2} + b_1 \log \frac{\Lambda}{m} \right) \text{Tr}[W^2]_F + \frac{Z}{4} \int d^4x [\bar{Q}_t e^V Q_t + \bar{Q}_t e^{-V} Q_t]_D
$$

where $b_1$ is the first coefficient of the expansion of the beta function in powers of the coupling constant. The derivative of the effective action (2.13) with respect to $\log \Lambda$ gives the additional factor in (2.10). The operator $\text{Tr} W^2$ used in eq. (2.10) differs by the constant factor $1/(1 - \alpha_0 N_c/2\pi)$ from the bare operator $\text{Tr} W^2$ at the renormalization scale $m = \Lambda$, according to the definitions used in ref. [11].

The analysis of the anomaly operator equations is also more subtle [11] and gives (at a renormalization scale $m$)

$$
\bar{D} \alpha J_{\alpha \hat{\alpha}} = -\frac{3N_c - N_f + N_f \gamma(\alpha_0)}{48\pi^2} D_\alpha W^2, \quad D_\alpha \bar{D} J = \frac{N_f}{2\pi^2} D_\alpha W^2.
$$

Thus, the anomaly of the $J_{\alpha \hat{\alpha}}$ current is proportional to the numerator of the NSVZ beta function (2.11). Consequently, from eqs. (2.10) and (2.14), the renormalized matrix elements of the currents read

$$
< \bar{D} \alpha J_{\alpha \hat{\alpha}} > = \frac{\beta(\alpha)}{24\pi^2} D_\alpha W^2_{\text{ext}}, \quad < \bar{D}^2 J_{\text{ren}} > = \frac{N_f}{2\pi^2} \cdot \frac{\beta(\alpha)}{\beta_1(\alpha)} W^2_{\text{ext}},
$$

which is a generalization of eqs. (2.8) to the nonabelian case. This shows that the $z$ factor for the Konishi superfield is given by $z(\alpha_0) = \beta(\alpha_0)/\beta_1(\alpha_0)$. In the above equation

$$
J_{\text{ren}} = (1 - \frac{\alpha_0 N_c}{2\pi}) z(\alpha_0) J.
$$

This equation deserves a short comment. It is well-known that in an asymptotically free theory the renormalization factor $Z_J(\log \Lambda/\mu, \alpha_0) = \exp \left( -\int_{\alpha_0}^{\alpha} \gamma_J(\alpha') d\alpha'/\beta(\alpha') \right)$ of an anomalous current $J$ admits a smooth limit $Z_J(\alpha_0) = \exp \left( \int_{\alpha_0}^{\alpha} \gamma_J(\alpha') d\alpha'/\beta(\alpha') \right)$ when the cut-off is removed [20], since $\gamma_J$ vanishes at the one-loop order. The finite factor that we have determined can be seen as this limit.
3. Correlators of anomalous currents at large distances

In this section we examine the two-point function of the Konishi current at large distances, and determine its anomalous dimension.

The scaling dimension of an operator at the conformal fixed point is the sum of its classical \((d_0)\) and anomalous \((\gamma)\) dimensions at the critical values of the coupling constants. To determine the scaling dimension of a local operator \(O\) it is sufficient to consider a two-point correlator \(< O(x)O(y) >\) at large distances \(|x - y| \to \infty\). The Callan-Symanzik equation

\[
\left( |x - y| \frac{\partial}{\partial |x - y|} + 2d_0 + 2\gamma(\alpha) + \beta(\alpha) \frac{\partial}{\partial \alpha} \right) < O(x)O(y) > = 0,
\]

implies that the correlator has the following form:

\[
< O(x)O(y) > = (z(|x - y|))^{2\phi(\alpha(|x - y|))}
\]

Here \(z\) is a renormalization factor for the operator \(O\), \(\alpha(|x - y|)\) is the running coupling constant at the scale \(1/|x - y|\) and \(\phi(\alpha)\) is an unknown function. If the function \(\phi(\alpha)\) has a non-vanishing smooth limit at the critical value \(\alpha_*\) of the coupling constant then the leading behaviour of such a correlator is

\[
< O(x)O(y) > \sim \frac{1}{|x - y|^{2d_0+2\gamma_*}},
\]

where \(\gamma_*\) is the anomalous dimension at the critical point. Having established that the renormalization factor for the Konishi current is expressible in terms of the beta function, it will follow that its scaling dimension at the critical point is determined by the slope of the beta function.

To fix the ideas, we review the computation of the scaling dimensions of fundamental fields in the electric formulation of SUSY QCD. More specifically we consider the \(SU(N_c)\) supersymmetric gauge theory with \(N_f\) flavours \(Q_i\) and \(\bar{Q}_i\), \(i = 1,\ldots,N_f\), of chiral quark superfields in the fundamental and anti-fundamental representation of the gauge group respectively.

We consider the propagator of the scalar components of quark superfields at large distances. We will verify that their conformal dimension at the conformal fixed point coincides with \(3R_Q/2\), where \(R_Q = 1 - N_c/N_f\) is the \(R\) charge \([1]\). By standard arguments for elementary fields \(\phi(\alpha(|x - y|)) \to |x - y|^{-2}\) and we have, for large \(|x - y|\),

\[
< Q_i(x)\bar{Q}_j(y) > = \delta_i^j z_Q(\alpha(x - y)) \cdot \frac{1}{|x - y|^2}.
\]

(3.1)

Here \([z_Q(\alpha)]^{1/2}\) is the wave renormalization factor for the field \(Q_i\). At large distances the running gauge coupling \(\alpha(x - y)\) flows to the fixed value \(\alpha_*\) which corresponds to the conformal fixed point. In this regime we have \([3]\)

\[
z_Q(\alpha) = z_Q(\alpha_0) \exp \left( - \int_{\alpha_0}^\alpha \frac{\gamma_Q(\alpha)}{\beta(\alpha)} \right) \sim (\alpha_* - \alpha)^{-\gamma_Q(\alpha_*)/\beta'(\alpha_*)}
\]

(3.2)
where $\gamma_Q(\alpha)$ is the anomalous dimension of the quark field

$$\gamma_Q(\alpha) = -\frac{d \log z_Q(\alpha(m))}{d \log m}.$$ 

Near the critical point the beta function is supposed to have a simple zero, i.e. $\beta(\alpha) = \beta'(\alpha_*)(\alpha - \alpha_*)$. This means that the correlators are power-behaved at the critical point. Higher zeroes would instead produce logarithmic behaviour. Solving the equation $d\alpha(m)/d \log m = \beta(\alpha(m))$ one gets

$$\alpha_* - \alpha = |x - y|^{-\beta'(\alpha_*)} \text{ at } |x - y| \to \infty. \tag{3.3}$$

Here $\alpha$ is taken at the scale $1/|x - y|$. Substituting this expression into eq. (3.1) we get

$$<Q_i(x)\bar{Q}_j(y)> = \delta_{ij} \frac{1}{|x - y|^{2 + \gamma_Q(\alpha_*)}}. \tag{3.4}$$

By using now the fact that $\gamma_Q(\alpha_*) = -(3N_c - N_f)/N_f$ we see that the conformal dimension of the quark field at the fixed point is $d_Q = 1 + \frac{1}{2}\gamma_Q(\alpha_*) = 1 - (3N_c - N_f)/2N_f = 3(1 - N_c/N_f)/2$. The latter value coincides with the value $d_Q = 3R_Q/2$ which is required by the superconformal algebra.

Turning to the operator $J$ whose $z$ factor was determined in the previous section, we consider, at the component level, the two-point correlator of its axial vector component $a_\mu \sim [\bar{D}_\alpha D^\alpha]$. The Callan-Symanzik equation leads to the following form for the correlator:

$$<a_\mu(x)a_\nu(y)> = \left(\frac{\beta(\alpha)}{\beta_1(\alpha)}\right)^2 \left(\frac{\phi_1(\alpha)g_{\mu\nu}}{|x - y|^6} + \frac{(x-y)_\mu(x-y)_\nu\phi_2(\alpha)}{|x - y|^8}\right). \tag{3.5}$$

The presence of the factor $\left(\beta(\alpha)/\beta_1(\alpha)\right)^2$ is our main observation. The functions $\phi_{1,2}(\alpha)$ are not determined by the Callan-Symanzik equation.

We consider the large distance limit of this correlator. At large distances the running gauge coupling $\alpha(|x - y|)$ flows to the fixed value $\alpha_*$ which corresponds to the conformal fixed point. The factor $\left(\beta(\alpha)/\beta_1(\alpha)\right)^2 \to 0$ at $|x - y| \to \infty$ because $\alpha$ goes to the critical value $\alpha_*$. In general the functions $\phi_{1,2}(\alpha)$ could have zeros or poles at the critical point but, as we shall discuss below, this is not generically the case. Using $\beta(\alpha) = \beta'(\alpha_*)(\alpha - \alpha_*)$ and substituting the expression from eq. (3.3) into eq. (3.5) we get

$$<a_\mu(x)a_\nu(y)> \sim \frac{1}{|x - y|^{6 + 2\beta'(\alpha_*)}}. \tag{3.6}$$

Thus, the anomalous dimension of the Konishi current is given by the slope of the beta-function.
Our result can be easily checked in perturbation theory. We outline the calculation in the Appendix.

We discuss now the behaviour of the functions \( \phi_{1,2} \). In perturbation theory we have \( \phi_{1,2}(\alpha) = 1 + O(\alpha^2) \). Therefore at least for the models with \( \alpha_* \ll 1 \) the functions \( \phi_{1,2}(\alpha) \) have a smooth non-vanishing limit at \( \alpha \to \alpha_* \). Thus in these theories the scaling behaviour of the correlator is determined only by the factor \( (\beta(\alpha)/\beta_1(\alpha))^2 \to 0 \). (Note however that the value of \( \phi_{1,2}(\alpha) \) together with the above factor determine the values of some of central charges of the superconformal algebra in section 5.)

For the fixed point in the strong coupling regime we do not have any argument for a smooth non-vanishing limit of \( \phi_{1,2}(\alpha) \) at \( \alpha \to \alpha_* \). One could imagine that a factor \( \sim \exp(-M|x-y|) \) appears due to nonperturbative effects, where \( M \) stands for a mass parameter corresponding to the most relevant intermediate state. If this were so it would mean that there are only massive intermediate states \( |n> \) in the correlator \( <a_\mu(x)a_\nu(y)> \). However in the non-Abelian Coulomb phase the quark fields are massless as manifested in the power-like behaviour of the two point correlator \( <Q_i(x)\bar{Q}^j(y)> \). This suggests that the spectrum of intermediate states in the \( <a_\mu(x)a_\nu(y)> \) channel has no mass gap, and hence the power-like behaviour of this correlator may remain in the infrared even in the strongly coupled regime. Note that this power-like behaviour could be also modified by softer factors \( \phi_{1,2} \) which have a zero or a pole at \( \alpha = \alpha_* \), i.e. \( \phi_{1,2} \sim (\alpha_* - \alpha)^a \) where \( a \) is a constant. This would change the conclusions which we have presented. However by continuity one can argue that this is not the case. Indeed we assume that there is no singularity in the correlator with respect to \( N_c \) and \( N_f \) in the conformal window. Then the exponent \( a \) depends continuously on the numbers \( N_c \) and \( N_f \). Therefore such a non-trivial behaviour of the factors \( \phi_{1,2} \) would extend also to the domain of the weak coupling regime in the space of parameters \( N_c \) and \( N_f \). However we saw that the factors \( \phi_{1,2} \) do not have any zeros or poles in this domain. Therefore we may expect at most that a zero or a pole could appear only outside the conformal window \( 0 < \frac{3N_c-N_f}{N_c} < 1 \). Below we shall argue that this is indeed the case for the theory in the confining phase, i.e. at \( 3N_c/2 \geq N_f \).

It is instructive to comparer the above result with the infrared behavior of non-asymptotically free theories with \( N_f > 3N_c \). Assuming that such a theory flows into a free theory, which corresponds to \( \alpha \to 0 \), we see that the factor \( \beta/\beta_1 \to 1 \) in the infrared. The correlator of Konishi current at large distances has an integral power-like behaviour which implies that all currents have a canonical dimension as expected in a free theory.

So far we discussed perturbative aspects of the infrared behavior of SQCD. The non-perturbative effects are due to instantons and, hence, are proportional to \( \exp(-2\pi/\alpha) \). In a non-asymptotically free theory, the coupling constant in the infrared is small, and hence instanton effects are absent. In a theory which flows to the non-trivial conformal fixed

\footnote{Note that the coefficients in the expansion of the functions \( \phi_{1,2}(\alpha) \) are not singular at \( (3N_c-N_f)/N_c \to 0 \) because these terms are finite without any additional renormalization of the correlator.}
point in the infrared these effects are $\sim \exp(-2\pi/\alpha_*)$. Therefore at least in the regime when the critical value $\alpha_*$ of the coupling constant is small, $\alpha_* << 1$, the instanton effects are exponentially suppressed and, hence, are not important.

We believe that these effects are not important for computation of the conformal dimensions even for strongly coupled conformal points. In particular we saw that there are no instanton corrections to the quark conformal dimension. This follows from the fact that the power-like behaviour of the quark propagator at large distances is determined by the perturbative beta function.

In a confining theory, e.g. N=1 SYM, the perturbative beta function has a pole (in the theories with a non-trivial fixed point such a pole is screened by a zero of the beta function). For N=1 SYM the large distance behaviour of the correlators corresponds to approaching the pole in the beta function, $\alpha \to 2\pi/N_c$. Therefore naively the correlator of two Konishi currents at points with coordinates $x$ and $y$ is proportional to

$$\frac{1}{|x-y|^4 \log(m|x-y|)}$$

in the limit $|x-y| \to 1/m$, where $m$ is a scaling parameter. Such a behaviour is physically meaningless. Therefore the instanton effects and, hence, the factors $\phi_{1,2}$ are crucially important to reproduce a correct behaviour of the correlator. To screen a pole at $|x-y| \to 1/m$ the functions $\phi_{1,2}$ must have a zero at $\alpha \to \alpha_* = 2\pi/N_c$ as mentioned above.

4. Konishi current in the magnetic theory

We consider now the magnetic formulation of the theory [1], which has the $SU(N_f - N_c)$ gauge group, $N_f$ flavours of dual quarks $(q^i, \tilde{q}_j)$ in the fundamental and anti-fundamental representations, respectively, and the meson field $M_{ij}$ in the $(N_f, \bar{N}_f)$ representation of the flavour $SU(N_f) \times SU(N_f)$ group. In contrast to the electric formulation this theory has a superpotential $S = M_{ij}q^i\tilde{q}_j$.

We discuss the determination of the magnetic counterpart of the Konishi current of the electric theory. The conservation of the Konishi current in the electric theory is violated by quantum effects. *A priori* there is no reason for the corresponding current in the magnetic formulation of the theory to be conserved even at the classical level. Its conservation may be violated due to the superpotential. As we shall see below the conservation of the magnetic Konishi current is spoiled both by the superpotential and the gauge anomaly.

To analyse the situation we consider an extension of the electric theory which includes an additional chiral superfield $X$ in the adjoint representation of the gauge group. Such an extension was analysed in ref. [4]. In this model the chiral superfield $X$ of the electric theory has a superpotential

$$S_{el} = s \text{Tr}X^{k+1},$$

where $k$ is a positive integer and $s$ is a coupling constant. We shall focus on $k = 2$, which gives the only interesting renormalizable theory that flows to a non-trivial superconformal
theory (for \(N_c/2 < N_f < 2N_c\), see ref. [4]). On the magnetic side the theory includes the meson fields corresponding to

\[(M_h)^i_j = \tilde{Q}_i X^{-1} Q^j; \quad h = 1, \ldots, k.\]  

(4.1)

It also includes an additional chiral superfield \(Y\) in adjoint representation of the dual gauge group \(SU(kN_f - N_c)\). The magnetic superpotential reads

\[S_{mag} = \tilde{s}\text{Tr}Y^{k+1} + \sum_{h=1}^{k} t_h M_h \tilde{q} Y^{k-h} q,\]

where \(\tilde{s}\) and \(t_h\) are coupling constants. The precise duality map (for primary chiral operators) between these two formulations of the theory (and its deformations) has been constructed in ref. [4]. Here we are interested in a particular aspect of such a map. The strategy that we shall employ is to construct nonanomalous Konishi currents for the dual versions of this theory, including the fields \(X\) and \(Y\), and subsequently, by introducing large mass terms for these fields, recover the original theory and corresponding currents in a low-energy limit. It is convenient to study the correspondence of the electric and magnetic theories at the level of the Wilsonian action.

We consider the Kutasov theory [4] at its critical point (so that in particular the coupling constants have their critical values for this theory \(\alpha = \alpha_\sharp\), etc.) and perform a non-anomalous \(U(1)\) phase transformation of the Kutasov electric theory fields \(Q, \tilde{Q}\) and \(X\) with charges \(q_Q = q_{\tilde{Q}}\) and \(q_X\) respectively. The absence of an anomaly requires that \(N_f q_Q + N_c q_X = 0\) (1/2 for each fundamental representation, \(N_c\) for each adjoint; only the matter sector is relevant, since the Konishi current does not contain the gauge sector). In the magnetic case, the corresponding condition gives \(N_f q_q = (N_c - N_f) q_Y\). The lagrangian of the electric theory is invariant under such a transformation provided we transform the coupling constant \(s\) with the charge \(q_s = -(k + 1)q_X\).

We note that the magnetic lagrangian should be also invariant under the transformation corresponding to that of the electric fields provided the coupling constants are also transformed in an appropriate way. By postulating that the transformation properties of the meson fields are prescribed by the identification (4.1) and due to the holomorphy of the Wilsonian action, we can easily get that the charges of the couplings \(t_h\) and \(\tilde{s}\) are given by

\[q_{t_h} = (2N_c/N_f + 1 - h)q_X + (h - k + 2(N_f - N_c)/N_f)q_Y,\]

(4.2)

\[q_{\tilde{s}} = -(k + 1)q_Y.\]

By considering a matching of deformations of these theories [4] it is easy to see that the charges of the couplings \(t_h\) do not change under a deformation \(N_f \to N_f - 1\). This condition fixes uniquely \(q_X = q_Y\), and hence

\[q_{t_h} = -q_s \frac{3 - k}{k + 1}, \quad q_{\tilde{s}} = q_s.\]
As a byproduct of the above considerations, writing \( t_j = t_j(s) = s^{q_{t_j}/q_\pm} = s^{k-3} \) to denote the dependence of the magnetic coupling on the electric one, we see that holomorphy in the electric parameter \( s \) means that the coupling \( t_j \) is singular in the limit \( s \to 0 \) at \( k < 3 \), for example in the theory with \( S_{el} = s \text{Tr}X^3 \). This suggests that new additional massless bound states appear in the magnetic theory in this limit. This means that for duality to work in the case \( s = 0 \) we have to add a new (massless) field.

In the electric theory we define a non-anomalous current which is a linear combination of the Konishi current and a current constructed from the field \( X \) (the conservation of this current is broken by the superpotential \( S_{el} \))

\[
K_{el} = \sum_i \left[ Q_i^+ e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^+ \right] - \frac{N_f}{N_c} \text{Tr}X^+ e^V X e^{-V}.
\] (4.3)

The corresponding current in the magnetic theory is defined by the \( U(1) \) transformation with the charges found above.

In order to determine the duality map for the Konishi operator in the original SQCD we add now a mass term \( m \text{Tr}X^2 \) and, via integration over \( X \) (or, equivalently, by the Appelquist-Carazzone theorem) go back to the minimal theory with fields \( Q, \tilde{Q} \) on the electric side. Also, by the duality map for chiral operators constructed in ref. \[4\] at the critical point of the Kutasov theory, \( Y \) will have a mass term \( m \text{Tr}Y^2 \) and we will then reproduce the original theory in the low-energy limit with the fields \( q, \tilde{q} \) and \( M \) on the magnetic side. Moreover, to reproduce precisely the initial theory, we have to choose a phase in which the gauge group is broken down to \( SU(N_f - N_c) \). Note that the integration over the heavy fields \( X \) (and \( Y \) on the magnetic side) leads to a non-critical minimal SQCD because, in particular, the coupling constants do not have their critical values, e.g. \( \alpha = \alpha_2 \neq \alpha_* \). Alternatively, we can think of the heavy fields (with masses of order \( m \)) as regulators of the minimal SQCD away from its fixed point. The parameter \( m \) is the only scale in the Kutasov theory and plays the rôle of UV cutoff in this resulting non-critical theory. The effective Konishi operators in this effective non-critical SQCD will be formally obtained by dropping out the heavy fields appearing in the non-anomalous Konishi operators of the Kutasov model. The resulting Konishi operators should be thought of as bare operators defined at this scale. Turning things around, the Konishi operators in the deformed Kutasov theory may be thought as particularly regularized operators of the minimal SQCD.

More specifically, after dropping the fields \( X, Y \), on the electric side the Konishi current \( J_{el} \) has its usual form, and is expected to flow in the infrared to the corresponding current in the critical minimal theory. Its magnetic counterpart is obtained by taking into account that \( q_M = 2q_Q \) for \( h = 1 \) in eq. (4.1). (For \( h > 1 \) \( M_h \) disappears in the low-energy limit, since it contains \( X \)). It can be written as

\[
J_{mag} = 2J_M + \frac{N_f - N_c}{N_c} J_q
\] (4.4)

\[6\] Our normalization is different from that of \[4\]; our \( t_j \) corresponds to their \( s/\mu^2 \).
where

\[ J_M = \text{Tr} \tilde{M}M, \quad J_q = \sum_{f=1}^{N_f} \bar{q}_f e^V q^f + \bar{q}_f e^{-V} \bar{q}, \]

The conservation of this current is broken both by the gauge anomaly and the superpotential \( S = \lambda M^i_j \bar{q}_i \bar{q}_j \), where \( \lambda \sim t_k < Y^2 > \) is a coupling constant. As mentioned above, the operator (4.4) is defined in the theory formulated at the scale \( m \). At this scale the coupling constants in the magnetic SQCD are equal to their values at the critical point in the theory with field \( Y \) present, \( \alpha = \alpha^c, \lambda = \lambda^c \). Below the scale \( m \) the low energy theory is the usual non-critical minimal SQCD. Thus the operator (4.4) is defined in the magnetic minimal SQCD away from criticality on a particular RG trajectory in the \( (\alpha, \lambda) \) phase diagram.

In order to describe the duality map for the Konishi operators in the critical SQCD we have to analyse their RG flow towards the infrared. Equivalently, we are interested in the large distance \( (\gg 1/m) \) behavior of the correlators which is governed by the critical SQCD. Note that in the above discussion we have identified the electric and magnetic Konishi operators in the Kutasov critical theory in the presence of the above mass deformation. Therefore their behaviour at any distance will be identical. This also means that the magnetic Konishi current is multiplicatively renormalizable and does not mix with other operators in the effective off-critical magnetic theory as soon as this is true for the Konishi current on the electric side. Therefore the magnetic Konishi operator can be represented as a linear combination of renormalization group invariant operators in the magnetic version of an off-critical minimal SQCD

\[ J_{mag} = AJ_1 + BJ_2. \]  

(4.5)

Here \( J_1, J_2 \) form a basis of RG invariant operators which diagonalize the matrix of anomalous dimensions and \( A, B \) are constants which depend on the critical values of the coupling constants \( \alpha = \alpha^c, \lambda = \lambda^c \) in the Kutasov theory, and on the scale \( m \). Such a representation is convenient for a consideration of the infrared behaviour near the conformal fixed point.

We will now analyse the matrix of anomalous dimensions of component axial currents in the magnetic SQCD. In the off-critical effective low-energy minimal SQCD theory, one of the RG invariant operators is the non-anomalous conserved \( R \) current defined in ref. [1] which however does not mix with the Konishi current since these are components of the supermultiplets with different superspins. Two other currents are those which are not conserved due to the anomaly and the superpotential. It is convenient to consider the current \( J_W = J_q - 2J_M \) whose (super)divergence is given by the anomaly and the current \( J_{sp} = J_M \) with divergence proportional to the superpotential so that \( J_{mag} = \frac{N_f - N_c}{N_c} J_W + 2 \frac{N_f}{N_c} J_{sp} \). These two currents are mixed under RG flow. One RG invariant combination is given by

\[ J_1 = \frac{2N_f - 3N_c + N_f \gamma_q}{48\pi^2} J_W + \beta_\lambda J_{sp}, \]

where

\[ N_f = 3N_c + \frac{N_f \gamma_q}{48\pi^2} J_W + \beta_\lambda J_{sp}. \]

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which remains invariant under RG flow because its $D_\alpha \bar{D}^2$ divergence is proportional to the anomaly of the supercurrent $J_{\alpha \bar{\alpha}}$ \cite{21,22}

$$\bar{D}^\alpha J_{\alpha \bar{\alpha}} = -\frac{2N_f - 3N_c + N_f \gamma_q}{48\pi^2} D_\alpha \bar{D}^2 \text{Tr} W_{\text{mag}}^2 + \beta_\lambda D_\alpha \bar{D}^2 S.$$ (4.6)

Here $\beta_\lambda = \lambda(\gamma_M + 2\gamma_q)/2$ is the beta function of $\lambda$, and $\gamma_q$ and $\gamma_M$ are anomalous dimensions of the fields $q, \bar{q}$ and $M$ respectively. It is easy to see that this combination is different from (4.4).

Another RG invariant combination, $J_2$, is less trivial to obtain. In order to determine it we have to actually compute the matrix of anomalous dimensions. This can be done by the same procedure we used in section 2 for the electric theory by taking the derivatives of the effective action with respect to the bare coupling constants. As a result the $z$ matrix is determined uniquely by the beta functions via suitable differential equations. After tedious calculations (we do not present them here) one can find that the entries of this $z$ matrix are proportional to linear combinations of beta functions in the infrared limit. We can rewrite the expression (4.4) in the form (4.3) with $A = A(\alpha^\sharp, \lambda^\sharp, m)$, $B = B(\alpha^\sharp, \lambda^\sharp, m)$ uniquely fixed by the beta functions. The values of $A$ and $B$ have well defined limits at $m \to \infty$ which are both non-zero. These particular values will not be important for us.

At this point we wish to make a comment. In an off-critical theory even operators defined in an RG invariant way depend on a particular RG trajectory. This follows from the fact that a correlator of such operators depends on the value of $\Lambda_{\text{QCD}}$ which labels RG trajectories. The construction above used a particular RG trajectory (from the initial critical theory with $X$ and $Y$ integrated out) into the critical SQCD. The information about the particular trajectory is encoded in the parameters $\alpha^\sharp, \lambda^\sharp$, and $m$. One can consider a more general flow along an arbitrary RG trajectory. In this case the correlators of $J_{1,2}$ and hence the operators themselves depend on the chosen trajectory because the running coupling constants do. On the other hand, the operator $J_{\text{mag}}$ at the critical point (of the minimal SQCD) should not depend on the choice of trajectory in the $(\alpha, \lambda)$ plane. Eq. (4.3) still holds, with the coefficients $A$ and $B$ compensating the dependence of the operators $J_{1,2}$ on the trajectory. Thus these coefficients, specifying a particular trajectory, are now to be interpreted as coordinates on the space of non-critical renormalizable deformations of the magnetic SQCD.

As a byproduct of the above discussion one can obtain the magnetic dual description of the operator $\text{Tr} \ W^2_{el}$. Indeed our identification of the magnetic counterpart for the Konishi supermultiplet implies that the operator $\text{Tr} \ W^2$ in the electric theory matches on the magnetic side with a linear combination of the operator $\text{Tr} \ W^2$ and the superpotential, which is proportional to the superdivergence of the operator $J_M$. Thus the duality map that we get for the anomaly multiplet seems to be different from that of ref. \cite{22} where it has been suggested $\text{Tr} \ W^2_{el} = -\text{Tr} \ W^2_{\text{mag}}$. By our analysis this relation is modified as follows: after the integration over the heavy fields in the Kutasov model we have (away from criticality of the minimal SQCD theory, at the scale $m$) $J_{el} \sim J_{\text{mag}} = \frac{N_f - N_c}{N_c} J_W + 2 \frac{N_f}{N_c} J_{sp}$, so that $\text{Tr} W^2_{el} \sim -\text{Tr} W^2_{\text{mag}} + \frac{N_f}{N_c} (\text{Tr} W^2_{\text{mag}} + 2S)$. 16
We now compute the anomalous dimension of the magnetic Konishi current. Consider the two points correlator \( < a_\mu(x) a_\nu(y) > \) at large distances, where \( a_\mu \) is the spin 1 component of \( J_{mag} \). This correlator is essentially determined by the matrix of anomalous dimensions, and hence by the \( z \) factor which takes into account the mixing of the operators \( \text{Tr} \ W_{mag}^2 \) and \( S \). As mentioned above the entries of \( z \) matrix renormalization factor are given by linear combinations of the beta function \( s \). Therefore the correlator \( < a_\mu(x) a_\nu(y) > \) at large distances is given by a bilinear combination of the beta functions of the coupling constants taken at the scale \( 1/|x-y| \).

To determine the asymptotic behaviour of this correlator it is sufficient to consider the asymptotic behaviour of the coupling constants near the critical values \( \alpha = \alpha_* \) and \( \lambda = \lambda_* \). We assume that the beta functions have simple zeros at the critical point i.e.

\[
\beta_\alpha = \beta'_{\alpha\alpha}(\alpha - \alpha_*) + \beta'_{\alpha\lambda}(\lambda - \lambda_*), \\
\beta_\lambda = \beta'_{\lambda\alpha}(\alpha - \alpha_*) + \beta'_{\lambda\lambda}(\lambda - \lambda_*),
\]

(4.7)

where the constants \( \beta'_{\alpha\alpha}, \beta'_{\alpha\lambda}, \beta'_{\lambda\alpha} \) and \( \beta'_{\lambda\lambda} \) are the components of the matrix \( \beta'_{ij} \). An explicit computation shows that

\[
< a_\mu(x) a_\nu(y) > \sim \frac{1}{|x-y|^{2\beta'_{min}+6}},
\]

(4.8)

where \( \beta'_{min} \) is the minimal eigenvalue of the matrix \( \beta'_{ij} \). It follows from eq. (4.8) that the anomalous dimension of the magnetic Konishi current is given by \( \beta'_{min} \) which is to be identified with the anomalous dimension \( \delta \) of the Konishi current in the electric theory. Thus we get

\[
\delta = (\beta'(\alpha_*))_{electric} = (\beta'_{min}(\alpha_*, \lambda_*))_{magnetic}.
\]

(4.9)

Eq. (4.9) is the main result of this section. It is interesting to consider the strong coupling regime \((2N_f - 3N_c)/N_c << 1\). In this case the magnetic theory is weakly coupled, we can compute \( \beta'_{min} \) in perturbation theory [6], and thus we can explicitly obtain the slope of the beta function in the strongly coupled electric theory

\[
\delta = (\beta'(\alpha_*))_{electric} = (\beta'_{min}(\alpha_*, \lambda_*))_{magnetic} = \frac{28}{3} \left( \frac{3}{2} - \frac{N_f}{N_c} \right)^2.
\]

(4.10)

It is instructive to compare the above result with the behavior of the theory outside the conformal window. The electric-magnetic duality implies identical behaviour of the correlators of the Konishi currents at large distances in both electric and magnetic descriptions. Since in the magnetic description the theory for \( N_f < 3N_c \) is free in the infrared, duality predicts that the same is true for the electric formulation where the theory confines. In turn this implies that the functions \( \phi_{1,2} \) in the correlator (3.2) in the electric theory are proportional to \( (\alpha - 2\pi/(N_f - N_c)) \) at \( \alpha \to \alpha_* = 2\pi/(N_f - N_c) \) in order to cancel the pole in the \( z \sim \beta(\alpha) \) factor of the Konishi current. Therefore the Konishi current has a canonical dimension in the infrared, outside the conformal window.
5. Quantum conformal algebra

In this section we collect some simple observations about CFT$_4$ that were stimulated by the investigation carried out so far. By using these observations we shall argue for the existence of a duality map between distinguished deformations (in the infinitely small neighbourhood of the critical point) which follow from the structure of the superconformal algebra. We shall see that such a deformation is a close relative of the Konishi current (and perhaps coincides with the latter) and plays an important role in CFT$_4$. This observation may explain the electric-magnetic universality.

There are two strategies for studying conformal field theories. One is the strategy that we have pursued in the previous sections, starting from the off-critical theories and flowing to the fixed point. We have seen how electric-magnetic duality can be used concretely to compute nontrivial physical quantities that otherwise cannot be computed (in the strong coupling regime, for example). Another strategy to study conformal field theories is to define them in an axiomatic-algebraic way and then classify the unitary theories, the representations, etc., more or less as in two dimensions. In this section we make some remarks about CFT$_4$ that could be the basis for the axiomatic approach to conformal field theories in four dimensions (CFT$_4$). These remarks could also have other consequences in the context of electric-magnetic duality. They give a general picture of the role of anomalous currents at the critical point and allow us to argue that universality is a general property of CFT$_4$. We note that unitary representations of the classical (super) conformal four-dimensional algebras have been studied in ref. [22].

We consider the notion of quantum (super)conformal algebra. The quantum conformal algebra is to the classical conformal group $SO(4,2)$ as, in two dimensions, the centrally-extended Virasoro algebra is to the non-extended one. In other words, we think it is useful to study CFT$_4$ at the level of OPE’s, and not simply at the level of the classical conformal group$^3$. We recall that CFT$_4$ is a useful notion$^2$, namely it is possible to extract information about correlators although the classical conformal algebra is not infinite dimensional as it is in two dimensions.

It turns out that:

i) the OPE of the stress-energy tensor $T_{\mu\nu}$ with itself does not close; another operator $\Sigma$ is brought into the algebra;

ii) there are two central charges, $c$ and $c'$, one related to $T_{\mu\nu}$, the other one to $\Sigma$; in the supersymmetric case they count both the numbers of effective vector multiplets and matter multiplets; $c$ is related to the conformal anomaly in an external gravitational field$^2$;

iii) $\Sigma$ is expected to have an anomalous dimension, in general; in the free supersymmetric case, it is precisely the Konishi current; the results of the previous sections suggest that $\Sigma$ may coincide with the Konishi current in an interacting CFT$_4$.

$^7$ OPE’s in dimensions $2 < d < 4$ have been studied recently by A. Petkou in [23]. We thank A. Petkou for bringing these references to our attention.
We begin with the simplest example of CFT, a free massless scalar. We have
\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi, \quad <\varphi(x)\varphi(0)> = \frac{1}{x^2}, \]
\[ T_{\mu\nu} = \frac{2}{3} \partial_{\mu} \varphi \partial_{\nu} \varphi - \frac{1}{6} \delta_{\mu\nu} (\partial_{\rho} \varphi)^2 - \frac{1}{3} \varphi \partial_{\mu} \partial_{\nu} \varphi. \]
We have written the so-called improved stress energy tensor \( T_{\mu\nu} \), which differs from the canonical one \( T_{\mu\nu}^{c} \) by a term like \( \partial_{\rho} \chi_{\rho\mu\nu} \), with \( \chi_{\rho\mu\nu} = -\chi_{\rho\nu\mu} \). The improved tensor is symmetric, conserved and traceless. Terms proportional to the field equations are omitted here, since they do not affect the arguments to be presented. \( T_{\mu\nu} \) is related to the curved space conformally invariant lagrangian \( \mathcal{L} = \sqrt{g} (1/2 g^{\mu\nu} \partial_{\mu} \varphi \partial_{\nu} \varphi - 1/12 \varphi^2 R) \) [25].

At the level of commutators, \( T_{\mu\nu} \) generates the correct conformal group \( SO(4,2) \) [25]. At the level of OPE’s, however, we note that there is a term containing the unity operator. This term is associated with a notion of central charge in four dimensions and is analogous to the central extension of CFT. In addition a new operator \( \Sigma \) appears, without which the OPE’s do not close. In this case we are dealing with \( \Sigma = \varphi^2 \). Indeed, we find
\[ T_{\mu\nu}(x) T_{\rho\sigma}(0) = -c \frac{1}{360} X_{\mu\nu\rho\sigma} \left( \frac{1}{x^4} \right) - \frac{1}{36} \Sigma(0) X_{\mu\nu\rho\sigma} \left( \frac{1}{x^{4-h}} \right) + \text{less singular}, \quad (5.1) \]
with \( c = 1 \) and \( h = 2 \), where
\[ X_{\mu\nu\rho\sigma} = 2 \Box^2 \delta_{\mu\rho} \delta_{\nu\sigma} - 3 \Box^2 (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\nu\rho} \delta_{\mu\sigma}) - 2 \Box (\partial_{\mu} \partial_{\rho} \delta_{\nu\sigma} + \partial_{\nu} \partial_{\rho} \delta_{\mu\sigma} + \partial_{\mu} \partial_{\sigma} \delta_{\nu\rho} + \partial_{\nu} \partial_{\sigma} \delta_{\mu\rho}) - 2 \Box (\partial_{\mu} \partial_{\rho} \partial_{\nu} \partial_{\sigma} + \delta_{\rho\sigma} \partial_{\mu} \partial_{\nu} + \partial_{\rho} \partial_{\sigma} \delta_{\mu\nu}) - 4 \partial_{\mu} \partial_{\nu} \partial_{\rho} \partial_{\sigma}. \]
\( c \) is the first central charge that we find. Less singular terms are easily workable. The expression \( X_{\mu\nu\rho\sigma} \) is uniquely fixed by the symmetries in the indices and by requirements \( \partial_{\mu} X_{\mu\nu\rho\sigma} = 0 \) and \( X_{\mu\nu\rho\sigma} = 0 \).

To check closure, we examine the OPE’s of \( \Sigma \). We obtain
\[ \Sigma(x) \Sigma(0) = \frac{2c'}{x^{2h}} + \frac{2}{x^h} \Sigma(0) + \cdots \]
\[ T_{\mu\nu}(x) \Sigma(0) = -\frac{h}{3} \Sigma(0) \partial_{\mu} \partial_{\nu} \left( \frac{1}{x^2} \right) + \cdots \quad (5.2) \]
In the case at hand \( c' = c \), but this is not true in general. \( c' \) is a second central charge. The case of \( n \) free massless scalars \( \varphi^i, i = 1, \ldots n \), is straightforward: \( c = c' = n, \quad h = 2 \) and \( \Sigma = \varphi^i \varphi^i \). This is sufficient to show that \( c \) and \( h \) are independent parameters of CFT.

Schematically, we can expect, for bosonic theories, something like
\[ T(x)T(0) = \frac{c}{x^8} + \frac{\Sigma}{x^{8-h}} + \frac{\partial \Sigma}{x^{7-h}} + \frac{T}{x^4} + \cdots, \]
\[ T(x)\Sigma(0) = \frac{h\Sigma}{x^4} + \frac{\partial\Sigma}{x^3} + \frac{\partial^2\Sigma}{x^2} + \frac{T}{x^h} + \cdots, \]
\[ \Sigma(x)\Sigma(0) = \frac{c'}{x^{2h}} + \frac{\Sigma(0)}{x^h} + \frac{\partial\Sigma}{x^{h-1}} \]  
(5.3)
or appropriate superextensions. (5.1) and (5.2), or (5.3), or their superextensions, define what we call the quantum conformal algebra.

In the complex case, \( \Sigma \) becomes \( \bar{\phi}\phi \) and in the supersymmetric case, it becomes the Konishi current. In the free theory, the Konishi current does not depend on the vector multiplets. We shall see that in the absence of matter multiplets the supercurrent \( J_{\alpha\dot{\alpha}} = W_\alpha \bar{W}_{\dot{\alpha}} \) is sufficient for closure; no additional operator appears for vector multiplets. Thus \( c' \) is sensitive to matter only, \( c \) is sensitive to both matter and gauge multiplets. We believe that this is an important property of CFT\(_4\), because it allows us to count separately the matter multiplets and the vector multiplets.

The results of the previous sections suggest that the non-closure of the OPE of \( T_{\mu\nu} \) with itself is a general property of CFT\(_4\), and that in general, the OPE’s are like (5.1) and (5.2) with \( c, c' \) and \( h \) generic. Physical states and primary fields have to be defined with respect to the complete algebra, of course. Defining a primary scalar field \( \phi \) of conformal dimension \( \Delta \) by

\[ T_{\mu\nu}(x)\phi(0) = -\frac{\Delta}{3} \phi(0) \partial_\mu \partial_\nu \left( \frac{1}{x^2} \right) + \text{less singular}, \]  
(5.4)

would imply that the primary fields of a free scalar in CFT\(_4\) are \( \varphi^k \), with weight \( k \), to be compared with \( \partial\varphi \) and \( e^{\alpha\varphi} \) in CFT\(_2\). However, due to the nonclosure of \( T_{\mu\nu} \) with itself, this is not the full story. With respect to \( \Sigma \), \( \varphi^{2k+1} \) mixes with \( \varphi^{2k-1} \). Thus it is not clear what the proper notion of primary field is. Perhaps one has to consider primary sets of fields, such as \( \{ \varphi^{2k+1}, \varphi^{2k-1}, \ldots, \varphi \} \).

For the case of free fermions we have

\[ \mathcal{L} = \frac{1}{2} (\bar{\psi} \dot{\psi} - \partial_\mu \bar{\psi} \gamma_\mu \psi), \quad <\psi(x)\bar{\psi}(0)> = \frac{\bar{\psi}}{x^4}, \]

\[ T_{\mu\nu} = \frac{1}{2} (\bar{\psi}_\gamma \partial_\nu \psi + \bar{\psi}_\nu \partial_\mu \psi - \partial_\mu \bar{\psi}_\gamma \psi - \partial_\nu \bar{\psi}_\gamma \psi). \]

\( \bar{\psi} = x^\mu \gamma_\mu \). The analogue of (5.1) exhibits a central term with \( c = 6 \), so that \( c = \frac{3}{2} \) for any real on-shell component. The \( \Sigma \)-term involves the axial current \( J_5^\mu = \bar{\psi}_5 \gamma_\mu \psi \). We find

\[ T_{\mu\nu}(x)T_{\rho\sigma}(0) = -c \frac{1}{360} X_{\mu\nu\rho\sigma} \left( \frac{1}{x^4} \right) + \sim \varepsilon_{\mu\rho\alpha\beta} J_5^\beta(0) \partial_\nu \partial_\sigma \partial_\alpha \left( \frac{1}{x^2} \right) + \cdots \]  
(5.5)

For the case of vectors we have

\[ T_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} + \frac{1}{4} \delta_{\mu\nu} F^2 + s - \text{exact terms}. \]
One finds \( c = 12 \). The \( s \)-exact terms do not affect \( c \) and only introduce additional \( s \)-exact operators in the OPE’s. There is no \( \Sigma \)-term and closure is achieved. As we shall see below, analogous conclusions hold for the supersymmetric vector multiplet, where the single superfield \( J_{a\dot{a}} = W_a W_{\dot{a}} \) is still sufficient for closure and so again \( \Sigma = 0 \) and \( c' = 0 \). It is also meaningful to speak about primary fields, in this case. For example, \( W_a \) is a primary field with weight \( 3/2 \). It is not clear whether the free vector multiplet is the only CFT\(_4\) with \( \Sigma = 0 \) and \( c' = 0 \). Theories with \( c' = 0 \) would appear effectively as pure Yang-Mills conformal field theories. The other possible case is a topological theory, where \( c \) and \( c' \) both vanish.

In a generic nonsupersymmetric conformal field theory containing scalars, fermions and vectors the central charges are three. They allow to count both the scalar, fermion and vector effective degrees of freedom separately, via the operators \( T_{\mu\nu}, \varphi^2 \) and \( J_5^\mu \). Supersymmetry reduces the number of independent central charges to two, so that for an \( N=1 \) free theory with \( m \) matter multiplets and \( v \) vector multiplets, we have \( c = 15v + 5m, \ c' = 5m \), which encode the effective numbers of matter and vector multiplets.

In free \( N=4 \) supersymmetric Yang-Mills theory, \( c = 30n, \ c' = 15n, \ n = v = m/3 \). Moreover, since \( c \) is related to an anomaly\(^8\) (which obviously does not have any multiloop corrections in this theory), \( c = 30n \) also in the interacting case, at least at the perturbative level (there could be instanton corrections). It is not clear whether \( c' \) is also unaffected by turning on the interaction. S-duality tells us that \( c' = 15n \) also in the infinitely strong coupling regime. Note that a deformation \( \lambda \Sigma \) cannot preserve \( N=4 \) supersymmetry, since the only candidate in that case would be \( \Sigma = \mathcal{L} \), which is not the case in the free theory. Instead, general considerations show that \( \Sigma \) is proportional to the Konishi superfield also in the weakly coupled theory. It would be interesting to compute the corresponding \( c' \). Moreover, \( \Sigma \) has an anomalous dimension \( \beta'_\lambda \neq 0 \). Indeed in the weak coupling limit an explicit computation gives (for \( SU(N_c) \) gauge group) \( \beta'_\lambda = 3\alpha N_c/\pi \). In general due to S duality the parameters \( \beta'_\lambda \) and \( c' \) must be real non-singular functions of \( J(q) \) and \( J(q) \), where \( J(q) \) is a generator of modular functions of \( q = \exp(2\pi i\tau), \tau = \theta/2\pi + 4\pi i/g^2; \theta \) is a theta angle in front of \( FF' \) in the Lagrangian (see, for example, \[^{27}\]). A dependence of \( c' \) on the coupling constant of \( N=4 \) theory would imply that the effective numbers of chiral and vector multiplets (in the \( N=1 \) sense) changes with \( g \) while the total number does not. Further analysis of marginal deformations of SQFT\(_4\) can be found in \[^{28}\].

In conclusion, while the quantum conformal algebra can provide us, through the identification of two CFT\(_4\)’s with a map between the two operators \( \Sigma_{el} \) and \( \Sigma_m \), the considerations of the previous sections show how to map the Konishi currents. It is reasonable to

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\(^8\) We stress here that \( N=4 \) supersymmetric Yang-Mills theory is indeed anomalous in an external background gravitational field. The anomaly vanishes for Ricci-flat backgrounds, but not in general. The importance of this fact is quite evident, since it is reflected in the non-vanishing of \( c \).
expect that in general Σ is related to or even coincides with the Konishi operator. At any rate the understanding of this relation should be a source of additional insight. The maps of anomalous dimensioned operators are also maps of deformations of the conformal theories. The appearance of one anomalous operator in the OPE of the stress-energy tensor helps to clarify the role of anomalous currents in CFT\textsubscript{4}.

For completeness we give the corresponding results in superspace, examining the OPE’s of the supercurrent and Konishi current for a free chiral scalar superfield (scalar multiplet), and for a gauge real scalar superfield (vector multiplet). We use the conventions of Superspace \cite{29}. For a chiral scalar superfield with action ∫ d\textsuperscript{4}x d\textsubscript{4}\θ \Bar{Φ} Φ the propagator, supercurrent, and Konishi current are, respectively, with z ≡ (x, θ, \Bar{θ}),

\begin{align*}
\langle Φ(z) \Bar{Φ}(z') \rangle &= -\Bar{D}^2 D^2 \frac{1}{(x - x')^2} \delta^4(θ - θ') = -\frac{1}{s^2} \\
\langle \Bar{Φ}(z) Φ(z') \rangle &= -D^2 \Bar{D}^2 \frac{1}{(x - x')^2} \delta^4(θ - θ') = -\frac{1}{\Bar{s}^2} \\
J_{α\dot{α}} &= \frac{1}{6} [D_α, \Bar{D}_{\dot{α}}] Φ \Bar{Φ} - \frac{1}{2} Φ i \overleftrightarrow{\Bar{∂}}_{α\dot{α}} Φ \\
J &= Φ \Bar{Φ} \ (5.6)
\end{align*}

while for the vector multiplet with field strength \( W_α = i\Bar{D}^2 D_α V \) and action ∫ d\textsuperscript{4}x d\textsubscript{2}\θ W^2 + h.c., quantized in Feynman gauge, the propagator and supercurrent are

\begin{align*}
\langle V(z) V(z') \rangle &= \frac{1}{(x - x')^2} \delta^4(θ - θ') \\
J_{α\dot{α}} &= W_α \Bar{W}_{\dot{α}} \ (5.7)
\end{align*}

Here \((x - x')^2 = \frac{1}{2} (x - x')^{α\dot{α}} (x - x')_{α\dot{α}}\), while \( s^2 = \frac{1}{2} s^{α\dot{α}} s_{α\dot{α}} \), \( \Bar{s}^2 = \frac{1}{2} \Bar{s}^{α\dot{α}} \Bar{s}_{α\dot{α}} \), where the chiral and antichiral superspace intervals are given by

\begin{align*}
s_{α\dot{α}} &= (x - x')_{α\dot{α}} + \frac{i}{2} [θ_α (\Bar{θ} - \Bar{θ}')_{\dot{α}} + \Bar{θ}'_α (θ - θ')_α] \\
\Bar{s}_{α\dot{α}} &= (x - x')_{α\dot{α}} + \frac{i}{2} [\Bar{θ}_\dot{α} (θ - θ')_α + θ'_α (θ - θ')_\dot{α}]
\end{align*}

We note the relations

\begin{align*}
D_β s^{α\dot{α}} &= iδ_β^α (\Bar{θ} - \Bar{θ}')^{\dot{α}} \ , \quad \Bar{D}_{\dot{β}} s^{α\dot{α}} = 0 \\
\Bar{D}_{\dot{β}} s^{α\dot{α}} &= iδ_{\dot{β}}^\dot{α} (θ - θ')^α \ , \quad D_β s^{α\dot{α}} = 0
\end{align*}

Note also the relations

\begin{align*}
D^2 \frac{1}{s^2} &= δ^4(x - x') δ^2(θ - θ') \\
\Bar{D}^2 \frac{1}{\Bar{s}^2} &= δ^4(x - x') δ^2(θ - θ')
\end{align*}
and, up to contact terms,

$$< \bar{D}_\alpha \Phi D_\alpha \Phi > = -i \partial_{\alpha \dot{\alpha}} \frac{1}{s^2}.$$  

The Taylor expansions are

$$\Phi(z') = \Phi(z) + s^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} \Phi(z) + (\theta - \theta')^\alpha D_\alpha \Phi(z) + \cdots$$

$$\bar{\Phi}(z') = \bar{\Phi}(z) + \bar{s}^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} \bar{\Phi}(z) + (\bar{\theta} - \bar{\theta}')^\dot{\alpha} \bar{D}_{\dot{\alpha}} \bar{\Phi}(z) + \cdots.$$  

We determine now the OPE’s setting, for convenience, $z' = 0$. For the Konishi current, performing double and single contractions of the chiral superfields, we obtain

$$J(z)J(0) = \frac{1}{s^2 \bar{s}^2} - J(0) \left( \frac{1}{s^2} + \frac{1}{\bar{s}^2} \right) + \cdots \quad (5.8)$$

For the OPE of the supercurrent with the Konishi current we obtain

$$J_{\alpha \dot{\alpha}}(z)J(0) = -\frac{J(0)}{3} \partial_{\alpha \dot{\alpha}} \left( \frac{1}{s^2} - \frac{1}{\bar{s}^2} \right) + \cdots \quad (5.9)$$

Finally, for the OPE of two supercurrents we have

$$J_{\alpha \dot{\alpha}}(z)J_{\beta \dot{\beta}}(0) = -\frac{1}{3} s_{\alpha \dot{\beta}} \bar{s}_{\beta \dot{\alpha}} + \frac{J(0)}{9} \partial_{\alpha \dot{\alpha}} \partial_{\beta \dot{\beta}} \left( \frac{1}{s^2} + \frac{1}{\bar{s}^2} \right) + \cdots \quad (5.10)$$

We note the appearance of the Konishi current in the OPE of two supercurrents. Less singular terms involve derivatives of the Konishi current and the supercurrent.

For the vector multiplet the OPE of two supercurrents is simply

$$J_{\alpha \dot{\alpha}}(z)J_{\beta \dot{\beta}}(0) = -\frac{s_{\alpha \dot{\beta}} \bar{s}_{\beta \dot{\alpha}}}{s^2 \bar{s}^2} - J_{\alpha \dot{\alpha}}(0) \left( \frac{i \bar{s}_{\beta \dot{\alpha}}}{\bar{s}^2} \right) + J_{\beta \dot{\beta}}(0) \left( \frac{i s_{\alpha \dot{\beta}}}{s^2} \right) + \cdots \quad (5.11)$$

Note that no term similar to the Konishi term in (5.10) appears here. It is straightforward to check that the above expressions vanish when, e.g., $D^a$ acts on them.

6. Conclusions

In this concluding section we recapitulate the situation and make some further comments.

In this paper we have studied properties of the (anomalous) Konishi current in N=1 SQCD, away from the critical point, and constructed the duality map for the Konishi supermultiplet. We have found that contrary to the duality map constructed previously for the primary chiral ring the present map involves a consideration of the space of off-critical deformations. We have related these observations to general properties of CFT4.
Away from the fixed point the Konishi current is the only renormalizable deformation of electric conformal SQCD. Indeed such a deformation can be reexpressed as a change of the gauge coupling constant due to the Konishi anomaly. Thus the duality map of the Konishi operator of the electric theory implies that a particular renormalizable deformation of the electric theory corresponds to a renormalizable deformation of the magnetic one. This means that in the infinitely small neighborhood of a critical point there exists a map from the electric off-critical theory to the magnetic one. It is worth emphasizing that such a map respects the renormalizability (up to less relevant operators), thus demonstrating a universality of the subleading (non-critical) behaviour of the theory near critical points. This universality manifests itself in the matching of the slopes of the beta functions in the electric and magnetic theories as exhibited in eq. (4.9).

At the level of OPE’s, we have found that an operator $\Sigma$, equal to the Konishi current at least in free supersymmetric theories, mixes with the stress-energy tensor. We believe that quite generally, its existence is related to the presence of anomalous currents in CFT$_4$. We have noted the role of $\Sigma$ in a distinction between matter multiplets and vector multiplets. Finally, in the context of electric-magnetic duality, we note that the matching of $\Sigma_{el}$ and $\Sigma_{mag}$ is also a correspondence between off-critical deformations. We believe that these observations may be useful in an axiomatic-algebraic approach to four-dimensional conformal field theories.

The above observed universality suggests that perhaps one can construct an electric-magnetic duality map even far away from criticality. A priori it is clear that such a map must be very complicated since one cannot in general avoid a non-renormalizable extension of the theory at finite distances away from the fixed point. Indeed, the renormalizable asymptotically free electric and magnetic theories cannot be equivalent at an arbitrary scale since, for example, there is no (at least naively) equivalence at the ultraviolet fixed point. On the other hand it is plausible to expect the existence of a more general map which includes non-renormalizable deformations of the theory. But renormalizability of the theory is a very important constraint in four dimensions since in general there is no way to define a non-renormalizable theory. Therefore, we must understand a non-renormalizable deformation of the theory as the effect of heavy particles which may become relevant in some regime of the low-energy theory. An explicit introduction of such heavy fields can transform the deformed theory into a renormalizable form.

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8. Appendix

We outline here the perturbative checks one can perform on the results of section 3. The leading contribution proportional to $\log \Lambda$ to the correlator $<a_\mu(x)a_\nu(y)>$ is due to the diagram in Fig. 1.

\begin{center}
\begin{tikzpicture}
  \node[vertex] (v1) at (0,0) {$V$};
  \node[vertex] (v2) at (3,0) {$V$};
  \node[vertex] (v3) at (1.5,3) {$V$};
  \node[vertex] (v4) at (1.5,-3) {$V$};
  \node[vertex] (x) at (-1.5,0) {$a_\mu(x)$};
  \node[vertex] (y) at (4.5,0) {$a_\nu(y)$};
  \draw[thick] (x) -- (v1) -- (v2) -- (y);
  \draw[thick] (v3) -- (v4);
\end{tikzpicture}
\end{center}

Fig. 1. Three-loop diagram for the correlator $<a_\mu(x)a_\nu(y)>$.

In order to extract this contribution we have to consider two asymptotic domains in the integral corresponding to the diagram in Fig. 2.

\begin{center}
\begin{tikzpicture}
  \node[vertex] (v1) at (0,0) {$V$};
  \node[vertex] (v2) at (3,0) {$V$};
  \node[vertex] (v3) at (1.5,3) {$V$};
  \node[vertex] (v4) at (1.5,-3) {$V$};
  \node[vertex] (x) at (-1.5,0) {$x$};
  \node[vertex] (y) at (4.5,0) {$y$};
  \draw[thick] (x) -- (v1) -- (v2) -- (y);
  \draw[thick] (v3) -- (v4);
\end{tikzpicture}
\end{center}

Fig 2. Diagrams for two asymptotic domains in the integral for the diagram in Fig. 1.

The short distance contribution of each of the marked subdiagrams in Fig. 2 corresponds to the leading contribution to the anomalous dimension of the current $J$. Therefore these subdiagrams can be effectively shrunk to point-like vertices proportional to $z - 1 = b_2 \alpha^2 \log \Lambda |x - y|$ as shown in Fig. 3, where $b_2$ is the two-loop coefficient of the beta function. Taking into account the tree level diagram we get in this approximation $1 + 2b_2 \alpha^2 \log \Lambda |x - y|$ for the correlator $<a_\mu(x)a_\nu(y)>$. By using the renormalization group we restore the multi-loop expression for the correlator. (Note however that this computation is formally justified only at $g^2 \log |x - y| \leq 1$.) Renormalizing it with the factor $(\beta(\alpha_0)/\beta_1(\alpha_0))^2$ we get eq. (3.3) with $\phi_{1,2}(\alpha) = 1$. The triviality of the function $\phi(\alpha)$ can be of course due to our approximation.
Similarly one can compute the correlators of other currents. In the conformal theory limit we are interested in the operators which are invariant under the renormalization group. Therefore we consider the conserved $\tilde{R}_\mu$ current, which is a linear combination of the current $R_\mu$ which enters the $J_{\alpha\dot{\alpha}}$ superfield and the Konishi axial current $a_\mu$

$$\tilde{R}_\mu = R_\mu + \left(1 - \frac{3N_c}{N_f} - \gamma \right) a_\mu.$$  

It is easy to check by an explicit computation of the correlator $<\tilde{R}_\mu(x)\tilde{R}_\nu(y)>$ that in perturbation theory the anomalous contributions $\sim 1/|x-y|^{6+\beta'(\alpha_*)}$ and $\sim 1/|x-y|^{6+2\beta'(\alpha_*)}$ are cancelled, and hence the correlator is transverse

$$<\tilde{R}_\mu(x)\tilde{R}_\nu(y)> \sim (g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu) \frac{1}{|x-y|^4}.$$  

This is as expected since the conserved current $\tilde{R}_\mu$ should have canonical conformal dimension; there is no $z$ factor for the correlator in question.

The scaling dimension of other components of the operators $J$ and $\text{Tr} W^2$ can be easily determined by supersymmetry. Note that the scaling of $\text{Tr} W^2$ agrees with the fact that this operator is not a primary chiral one. This also follows from the anomaly equation for the Konishi supermultiplet.
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