On the Contou-Carrere Symbol for Surfaces

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Abstract

This is a preliminary report on the Contou-Carrére symbol for surfaces. It consists of two parts. In the first part, we recall technical results needed to define the symbol. The second part is where we compute all components of the Contou-Carrére symbol for surfaces, using iterated integrals over membranes.

Key words: reciprocity laws, complex algebraic surfaces, iterated integrals
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0 Introduction

In this paper we give a definition of a two dimensional Contou-Carrere symbol. Previously, Pablos Romo has worked on that topic. He gave a definition whose ingredients resemble powers of the one dimensional Contou-Carrere symbol. Then Osipov and Zhu gave another definition, in which they considered one more key ingredient of the Contou-Carrere symbol. Here, we find one more ingredient for the Contou-Carrere symbol (Case 2, page 15 or $Q_1$ and $Q_2$ in Definition 2.2).

We can consider the results here as a continuation of [H5]. In order to construct the Contou-Carrere symbol, we use iterated integrals over membranes, which are a higher dimensional generalization of iterated path integrals. The geometric constructions that we use here are established in the paper [H5]. They allow us to prove reciprocity laws.
for the symbols. However, in this preliminary version, we only state the Contou-Carrère symbol.

In the last Subsection we compute an example of a symbol from more complicated factors. We are going to prove reciprocity laws for symbols coming from more complicated factors such as rational functions over $\mathbb{P}^2$.

Let $f$ and $g$ be two non-zero meromorphic functions on a Riemann surface $\Sigma$. The tame symbol of $f$ and $g$ at a point $x \in \Sigma$ is given by

\[(f, g)_x = (-1)^{\nu(f)\nu(g)} [g^{\nu(f)}/f^{\nu(g)}](x)\]

where $\nu(f)$ is the valuation of $f$ at $x$, in other words, it is the order of zero of $f$ at $x$, or the opposite of the order of pole of $f$ at $x$. Since the valuation of $g^{\nu(f)}/f^{\nu(g)}$ is zero, the quotient function is holomorphic at $x$ and the symbol is well defined. This symbol is anti-symmetric and, although not obvious, bi-multiplicative and satisfies the Steinberg property:

\[(f, 1 - f)_x = 1\]

for any $f$ not equal to 0 or 1. When the surface $\Sigma$ is compact, this symbol satisfies the following product formula:

\[\prod_{x \in \Sigma} (f, g)_x = 1\]

which generalizes the Weil reciprocity law [Mi, W] $f(\text{div}(g)) = g(\text{div}(f))$, for the case when $f$ and $g$ have disjoint divisors.

This symbol is later greatly generalized by Contou-Carrère symbol [Co] to the case where the base ring for the surface is not the complex number $\mathbb{C}$ but a local artinian ring. Let $A$ be an artinian local ring with maximal ideal $m$. Let $f, g$ be two functions in $A((X))^\times$. Then they can be written uniquely as

\[f = a_0 X^{\nu(f)} \prod_{i = -\infty}^{+\infty} (1 - a_i X^i)\]

\[g = b_0 X^{\nu(g)} \prod_{i = -\infty}^{+\infty} (1 - b_i X^i)\]

where $(f, g)$ is the greatest divisor of $f$ and $g$, $\nu_f, \nu_g \in \mathbb{Z}$, $a_i, b_i \in A$ for $i > 0$, $a_0, b_0 \in A^\times$, $a_i, b_i \in m$ for $i < 0$, and $a_i, b_i$ are zero when $i \ll 0$. Then the symbol is given by

\[
< f, g >_{A((X))^\times} := (-1)^{\nu_f\nu_g} \frac{a_0^{\nu_f}}{b_0^{\nu_g}} \prod_{j = 1}^\infty \prod_{k = 1}^\infty (1 - a_j^{k/(j,k)} b_j^{k/(j,k)})^{(j,k)}
\]

(0.1)

and

\[
\frac{a_0^{\nu_f}}{b_0^{\nu_g}} \prod_{j = 1}^\infty \prod_{k = 1}^\infty (1 - a_j^{k/(j,k)} b_j^{k/(j,k)})^{(j,k)}
\]

(0.2)

Since $a_{-i}, b_{-i}$ are zero when $i$ is large, the product is actually finite, hence the definition makes sense.
There is also an analogous reciprocity law satisfied by this Contou-Carrèr e symbol. Let $A$ be a finite local artinian $C$-algebra, finitely generated as a $C$-module, and let $\Sigma'$ be a surface over $A$. Let $f$ and $g$ be two non-zero meromorphic functions on $\Sigma'$. Then locally at any point $x$, $f$ and $g$ can be identified as elements of the ring $A((x))^\times$. Then we can define the Contou-Carrèr e symbol described above. This symbol satisfies the product formula

$$\prod_{x \in \Sigma'} <f, g>_{x} = 1.$$  

If we take $A = \mathbb{C}$, then the Contou-Carrèr e symbol reduces to the tame symbol and the above product formula becomes the Weil reciprocity formula. Hence the Contou-Carrèr e symbol is a natural extension of the tame symbol.

In the paper [?], the second author constructs the Contou-Carrèr e symbol in terms of Chen iterated integrals and prove the corresponding reciprocity using a geometric property of iterated integrals. Consider a local artinian $C$-algebra $A$, finitely generated by nilpotent elements $\{a_1, a_2, \cdots, a_n\}$ as a $C$-module. Let $\omega$ be a meromorphic 1-form on a compact Riemann surface $X$ with coefficients in $A$, in other words, it is of the form $\sum_{i=0}^{n} a_i \omega_i$, where $a_0 = 1$ and $\omega_i$'s are regular meromorphic 1-forms. Let $\gamma$ be a path on the Riemann surface. Define the integral of $f$ by extending the regular integral linearly:

$$\int_{\gamma} \omega = \sum_{i=0}^{n} a_i \int_{\gamma} \omega_i.$$  

In particular, we consider differential forms of the type $\frac{df}{f}$, where $f$ is a meromorphic function on $X$ with coefficients in $A$. Let $f$ and $g$ be two such functions, and $s$ be a zero or pole of $f$ or $g$ or both. Write $f$ and $g$ in Laurent series in powers of a uniformizer $x_s$, as elements in $A((x_s))^\times$. Thus, as mentioned above, we can express $f$ and $g$ locally as infinite products. Let $\sigma$ be a simple loop that starts and ends at some fixed point $P$, go around the divisor $s$ once in the counterclockwise direction but not any other divisors of $f$ or $g$. Then we have the following result:

$$\exp \left( \frac{1}{2\pi i} \int_{\sigma} \frac{df}{f} \circ \frac{dg}{g} \right) = \left( -1 \right)^{\nu_f \nu_g} g(P)^{\nu_f} a_0^{\nu_f} \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} (1 - a_j^{k/(j,k)} b_{j-k}^{(j,k)})^{(j,k)} f(P)^{\nu_g} b_0^{\nu_g} \prod_{j=1}^{\infty} \prod_{k=1}^{\infty} (1 - a_j^{-k/(j,k)} b_{j-k}^{(j,k)})^{(j,k)}$$  

This formula differs from the Contou-Carrèr e symbol symbol only by $g(P)^{\nu_f} / f(P)^{\nu_g}$, but since the point $P$ is fixed, we can rescale the functions $f$ and $g$ so that $f(P) = g(P) = 1$. Then we can get the usual Contou-Carrèr e symbol . This gives us a new interpretation of the Contou-Carrèr e symbol in terms of exponential of an iterated integral, which is very convenient to use. Many properties of the symbol easily follow from the properties of iterated integrals. For example, it is easy to see from this definition that the symbol is bi-multiplicative and anti-symmetric. It is actually also convenient to prove the Steinberg property using this new description of the Contou-Carrèr e symbol .
We can also reproduce the product formula. If we consider the iterated integral over a loop that represents a relation of the fundamental group, by the homotopy invariance of iterated integral, the result is 0. Using this, we can reprove the reciprocity of Contou-Carrère symbol easily.

In the two dimensional setting we do have a similar infinite product formulas. We have

\[ f_1 = a_1 x^{\nu_1(f_1)} y^{\nu_2(f_1)} \prod_{i_1 > -N} \prod_{j_1 > -N_{i_1}} (1 - a_{i_1,j_1} x^{i_1} y^{j_1}), \]

\[ f_2 = a_2 x^{\nu_1(f_2)} y^{\nu_2(f_2)} \prod_{i_2 > -N} \prod_{j_2 > -N_{i_2}} (1 - a_{i_2,j_2} x^{i_2} y^{j_2}), \]

\[ f_3 = a_3 x^{\nu_1(f_3)} y^{\nu_2(f_3)} \prod_{i_3 > -N} \prod_{j_3 > -N_{i_3}} (1 - a_{i_3,j_3} x^{i_3} y^{j_3}). \]

For each triple of simple factors of the above infinite products, we compute the corresponding symbol in Section 2. We have separated the computation into 8 cases. At the end of the computation we define the Contou-Carrère symbol in Definition 2.2.

1 Algebraic, Geometric and Analytic background

1.1 Infinite product formulas

Let \( A \) be a commutative ring with unit. Let \( I \) be a nilpotent ideal. Let \( \Gamma(A, I) \) be the set of power series \( f = \sum_{i=\infty} \sum_{j=\infty} a_{i,j} t^i \in A[[t]] \) such that for some integer \( \omega = \omega(f) = \omega_{A,I}(f) \) we have \( a_w = 0 \) and \( a_i \in I \) for \( i < \omega \). The set \( \Gamma(A, I) \) is closed under power series multiplication and forms a group. Let \( \Gamma_0(A, I) = \{ f \in \Gamma(A, I) \mid \omega(f) = 0 \} \). It is a subgroup of \( \Gamma(A, I) \). Let \( \Gamma_+(A, I) \) be the subgroup consisting of the elements of the form \( 1 + f \) where \( f \in t^{-1} I[t^{-1}] \). Let \( \Gamma_+(A, I) \) be the subgroup of \( \Gamma_0(A, I) \) consisting elements which have no negative powers of \( t \). Then \( \Gamma_+(A, I) = A[[t]]^\times \). Given rings \( A \) and \( B \) with nilpotent ideal \( I \subset A \) and \( J \subset B \), and a ring homomorphism \( \phi : A \to B \) such that \( \phi(I) \subset J \), then we define the corresponding group homomorphism \( \Gamma(\phi) : \Gamma(A, I) \to \Gamma(B, J) \) by sending \( \sum a_i t^i \) to \( \sum \phi(a_i) t^i \), then the construction \( \Gamma \) becomes a functor. Similarly, \( \Gamma_+ \) and \( \Gamma_0 \) are functors.

The following proposition is proved by Pablos Romo [PR].

**Proposition 1.1** For all \( f \in \Gamma(A, I) \), there exist unique coefficients \( \{ a_i \}_{i=-\infty}^{\infty} \) in \( A \) satisfying \( a_0 = 0, a_i = 0 \) for \( i > 0, a_i = 0 \) for \( i < 0 \), such that

\[ f = a_0 t^{\omega(f)} \prod_{i=1}^{\infty} (1 - a_i t^i) \prod_{i=1}^{\infty} (1 - a_{-i} t^{-i}) \]

The coefficients \( \{ a_i \}_{i=-\infty}^{\infty} \) are called the family of Witt parameters of \( f \in \Gamma(A, I) \). The following two lemmas are used to prove the proposition.
Lemma 1.2 For all \(f \in \Gamma(A, I)\), there exists unique \(g \in \Gamma_+(A, I), h \in \Gamma_-(A, I)\), such that \(f = g \cdot h\).

Lemma 1.3 For all \(f \in \Gamma_+(A, I)\), \(f\) can be written as \(f(0) \prod_{i=1}^{\infty} (1 - a_i t^i)\) for some \(\{a_i\}_{i=1}^{\infty}\) uniquely determined by \(f\). If \(f \in 1 + t^n A[[t]]^\times\), then \(a_1 = a_2 = \cdots = a_{n-1} = 0\). If \(f \in 1 + I[t]\), then \(a_i \in I, \forall i\), and \(a_i = 0\), for \(i < 0\).

See Anderson and Pablo Romo (\[\?\].)

There is a similar result for higher dimensional local fields: Let \(A\) again be a ring, \(I \subset A\) be a nilpotent ideal. Consider the two dimensional local field \(A((t_1)), I((t_1))\) be the subset of \(A((t_1))((t_2))\) that consist of all the elements of the form
\[
\sum_{i=-\infty}^{\infty} g_i(t_1)t_2^i = \sum_{i=-\infty}^{\infty} \left( \sum_{j=-\infty}^{\infty} a_{ij} t_1^j t_2^i \right)
\]
such that there exists \(\omega_2 = \omega_2(f) \in \mathbb{Z}, \omega_1 = \omega_1(f) \in \mathbb{Z}\) satisfying: \(g_{\omega_2}(t_1) \in A((t_1))^\times, g_i(t_1) \in I((t_1)), \forall i < \omega_2\) and \(a_{\omega_2 \omega_1} \in A^\times, a_{\omega j} \in I, \forall j < \omega_1\).

Proposition 1.4 (case of dimension 2) For all \(f \in \Gamma(A((t_1)), I((t_1)))\), there exists a unique family of parameters \(\{a_{i,j}\}_{i,j=-\infty}^{\infty}\) such that
\[
f = a_{0,0} t_1^{\omega_1} t_2^{\omega_2} \prod_{i=-\infty}^{\infty} \prod_{j=-\infty}^{\infty} (1 - a_{i,j} t_1^i t_2^j)
\]

To prove it, we’ll need the following lemmas:

Lemma 1.5 For all \(f \in 1 + t_2 A((t_1))[t_2]\), there exists \(a_{i,j} \in A\) uniquely determined by \(f\), such that
\[
f = \prod_{i=1}^{+\infty} \prod_{j=-N_i}^{+\infty} (1 - a_{i,j} t_1^i t_2^j)
\]

Proof.
Suppose
\[
f_1(t_1) = \sum_{j=-N_1}^{+\infty} a_{1,j} t_1^j.
\]
Let
\[
A_1 = \prod_{j=-N_1}^{+\infty} (1 + a_{1,j} t_1^j t_2).
\]
Then $A_1$ is invertible in $A((t_1))((t_2))$. Define $f^{(1)} := A_1^{-1} f$, then $f = A_1 f^{(1)}$ and

$$f^{(1)} = 1 + \sum_{i=2}^{+\infty} f^{(1)}_i (t_1) t_2^i.$$

Suppose at $n$th step we get $f = (\prod_{i=1}^n A_i)(f^{(n)})$, where:

$$A_i = \prod_{j=-N_i}^{+\infty} (1 + a_{i,j} t_1^i t_2^j), \quad f^{(n)} = 1 + \sum_{i=n+1}^{+\infty} f^{(n)}_i (t_1) t_2^i.$$

If

$$f^{(n)}_{n+1} = \sum_{j=-N_{n+1}}^{+\infty} a^{(n)}_{n+1,j} t_1^j$$

then we can similarly set

$$A_{n+1} = \prod_{j=-N_{n+1}}^{+\infty} (1 + a^{(n)}_{n+1,j} t_1^j t_2^j).$$

Still this is invertible in $A((t_1))((t_2))$ and we can set

$$f^{(n+1)} := A_{n+1}^{-1} f^{(n)} = 1 + \sum_{i=n+2}^{+\infty} f^{(n+1)}_i (t_1) t_2^i.$$

So

$$f = \prod_{i=1}^{n+1} A_i f^{(n+1)}.$$

Continue this process, we’ll get that

$$f = \prod_{i=1}^{+\infty} \prod_{j=-N_i}^{+\infty} (1 - a_{i,j} t_1^i t_2^j).$$

Obviously $a_{ij} \in A$ are uniquely determined by $f$ according to this process.

**Lemma 1.6** For all $f \in 1 - t_2^{-1}I((t_1))[t_2^{-1}]$, then there exist $N, N_i \in \mathbb{Z}$, and $a_{ij} \in I$ uniquely determined by $f$, such that

$$f = \prod_{i=-N}^{+1} \prod_{j=-N_i}^{+\infty} (1 - a_{i,j} t_1^i t_2^j).$$

**Proof.** Induction on the integer $k$ such that $I^k = 0$. If $k = 1$, then $f = 1$, so the statement is true. Now suppose the statement is true for all the ideals $J$ satisfying $J^i = 0$ for some $i < k$ and $I^k = 0, I^{k-1} \neq 0$. Suppose

$$f = 1 - \sum_{i=-n}^{+1} f_i(t_1) t_2^i.$$
where \( f_i(t_1) \in I((t_1)) \). Write \( f_{-1}(t_1) \) as

\[
\sum_{j=-N_1}^{+\infty} a_{1,j} t_1^j,
\]

where \( a_{1,j} \in I \). Then we can similarly set

\[
A_{-1} = \prod_{j=-N_1}^{+\infty} (1 - a_{1,j} t_1^j t_2^{-1}).
\]

Then \( A_{-1} \) is invertible and its inverse is

\[
\prod_{j=-N_1}^{+\infty} (1 + \sum_{l=1}^{+\infty} (a_{1,j} t_1^j t_2^{-1})^l).
\]

Since \( I \) is nilpotent, there exists an integer \( k \), such that \( I^k = 0 \). So every factor of the inverse has only finite terms, ie.

\[
(A_{-1})^{-1} = \prod_{j=-N_1}^{+\infty} (1 + \sum_{l=1}^{k-1} (a_{1,j} t_1^j t_2^{-1})^l).
\]

Let \( f^{-1} = (A_{-1})^{-1} f \). Then \( f^{-1}(t_1) \in 1 - t_2^{-2} I((t_1))[t_2^{-1}] \) and the lowest power of \( t_2 \) in \( f^{-1} \) would be \(-n - k + 1\). Write

\[
f^{-1} = 1 - \sum_{i=-2}^{n-k+1} f_i^{(-1)} (t_1) t_2^i,
\]

then for \(-n \leq i \leq -2\), \( f_i^{(-1)} \in I((t_1)) \), and for \( i < -n \), \( f_i^{(-1)} \in I^{-i-n-1}((t_1)) \). As before, if we write \( f_{-2}^{(-1)} \) as

\[
\sum_{j=-N_2}^{+\infty} a_{-2,j}^{(-1)},
\]

and set

\[
A_{-2} = \prod_{j=-N_2}^{+\infty} (1 - a_{-2,j}^{(-1)} t_1^j t_2^{-2}), f^{-2} = (A_{-2})^{-1} f = 1 + \sum_{i=-3}^{n-k+1} f_i^{-2} (t_1) t_2^i,
\]

then for \(-n \leq i \leq -3\), \( f_i^{-2} \in I((t_1)) \), for \(-n - k + 1 < i < -n \), \( f_i^{-2} \in I^{-i-n-1}((t_1)) \). If we go on with this process, after \( n \) steps, we will get

\[
f = \prod_{i=-n}^{-1} A_if^{(-n)},
\]
where \( f^{(-n)} \in 1 - t_2^{-(n+1)}I^2((t_1))[t_2^{-2}] \). So by induction hypothesis, \( f^{(-n)} \) can be written as
\[
\prod_{i=-N, j=-N'}^{-1} \prod_{j=-N, i=-N'}^{+\infty} (1 - a'_{ij} t_1^i t_2^j).
\]
So \( f \) can be written as
\[
\prod_{i=-N, j=-N}^{-1} \prod_{j=-N, i=-N}^{+\infty} (1 - a_{ij} t_1^i t_2^j),
\]
and \( a_{ij} \in I \) are uniquely determined by \( f \).

Combining these two lemmas and lemma 1.2, we can get proposition 1.4 easily.

Using similar techniques, we can prove the following proposition for any higher dimensional cases:

**Proposition 1.7 (Case of dimension \( n \))**

For any function \( f \in \Gamma((A(t_1)) \cdots (t_n)), I((t_1)) \cdots (t_n)) \) (which is defined similarly), there exist coefficients \( \{a_{i_1,i_2,\ldots,i_n} \} \) uniquely determined by \( f \), such that
\[
f = \prod_{i_n=-N}^{+\infty} \cdots \prod_{i_1=-N}^{+\infty} (1 - a_{i_1,i_2,\ldots,i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}).
\]

### 1.2 Two foliations

The goal of this section is to construct two foliations on a complex projective algebraic surface \( X \) in \( \mathbb{P}^k \). Let \( f_1, f_2, f_3 \) and \( f_4 \) be four non-zero rational functions on the surface \( X \). Let
\[
C \cup C_1 \cup \cdots \cup C_n = \bigcup_{i=1}^{4} |\text{div}(f_i)|.
\]

Let
\[
\{P_1, \ldots, P_N \} = C \cap (C_1 \cup \cdots \cup C_n).
\]

We can assume that the curves \( C, C_1, \ldots, C_n \) are smooth and that the intersections are transversal (normal crossings), by allowing blow-ups on the surface \( X \).

The two foliations have to satisfy the following

**Conditions:**

1. There exists a foliation \( F'_v \) such that
   
   (a) \( F'_v = (f - v)_0 \) are the level sets of a rational function
   
   \[
   f : X \to \mathbb{P}^1,
   \]
   
   for small values of \( v \), (that is, for \( |v| < \epsilon \) for a chosen \( \epsilon \));
   
   (b) \( F'_v \) is smooth for all but finitely many values of \( v \);
   
   (c) \( F'_v \) has only nodal singularities;
(d) $\text{ord}_C(f) = 1$;
(e) $R_i \notin C_j$, for $i = 1, \ldots, M$ and $j = 1, \ldots, n$, where

$$\{R_1, \ldots, R_M\} = C \cap (D_1 \cup \cdots \cup D_m)$$

and

$$F'_0 = (f)_0 = C \cup D_1 \cup \cdots \cup D_m.$$  

2. There exists a foliation $G_w$ such that

(a) $G_w = (g - w)_0$ are the level sets of a rational function

$$g : X \to \mathbb{P}^1;$$

(b) $G_w$ is smooth for all but finitely many values of $w$;

(c) $G_w$ has only nodal singularities;

(d) $g|_C$ is non constant.

3. Coherence between the two foliations $F'$ and $G$:

(a) All but finitely many leaves of the foliation $G$ are transversal to the curve $C$.

(b) $G_{g(P_i)}$ intersects the curve $C$ transversally, for $i = 1, \ldots, N$. (For definition of the points $P_i$ see the beginning of this Subsection.)

(c) $G_{g(R_i)}$ intersects the curve $C$ transversally, for $i = 1, \ldots, M$. (For definition of the points $R_i$ see condition 1(e).)

For the existence of such foliations see [H5].

**Lemma 1.8** With the above notation, for small values of $|v|$, we have that $F_v$ has the homotopy type of $C_0$.

A proof can be found in [H5].

1.3 Iterated integrals

For proofs of theorems of this section, see Chen[Ch] or Goncharov[G].

**Definition 1.9** Let $\omega_1, \omega_2, \cdots, \omega_n$ be holomorphic 1-forms on a simply connected open subset $U$ of the complex plane $\mathbb{C}$. Let $\gamma : [0, 1] \to U$ be a path. Then we call the integral

$$\int_\gamma \omega_1 \circ \cdots \circ \omega_n := \int \cdots \int_{0 \leq t_1 \leq \cdots \leq t_n \leq 1} \gamma^* \omega_1(t_1) \wedge \cdots \wedge \gamma^* \omega_n(t_n)$$

the iterated integral of the differential forms $\omega_1, \omega_2, \cdots, \omega_n$ over the path $\gamma$. 

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Theorem 1.10 Let $\omega_1, \cdots, \omega_n$ be holomorphic 1-forms on a simply connected open subset $U$ of the complex plane $\mathbb{C}$. Let $H : [0, 1] \times [0, 1] \to U$ be a homotopy, fixing the end points, of paths $\gamma_s : [0, 1] \to U$ such that $\gamma_s(t) = H(s, t)$. Then
\[ \int_{\gamma_s} \omega_1 \circ \cdots \circ \omega_n \]
is independent of $s$.

Theorem 1.11 [Shuffle relation] Let $\omega_1, \cdots, \omega_n, \omega_{n+1}, \cdots, \omega_{m+n}$ be differential 1-forms, where some of them could repeat. Let also $\gamma$ be a path that does not pass through any of the poles of the given differential forms. Denote by $\text{Sh}(m, n)$ the shuffles, which are permutations $\tau$ of the set $\{1, \ldots, m, m+1, \ldots, m+n\}$ such that $\tau(1) < \tau(2) < \cdots < \tau(m)$ and $\tau(m+1) < \tau(m+2) < \cdots < \tau(m+n)$. Then
\[ \int_{\gamma} \omega_1 \circ \cdots \circ \omega_n \int_{\gamma} \omega_{n+1} \circ \cdots \circ \omega_{m+n} = \sum_{\tau \in \text{Sh}(m, n)} \int_{\gamma} \omega_{\tau(1)} \circ \omega_{\tau(2)} \cdots \circ \omega_{\tau(m+n)}. \]

Lemma 1.12 (Reversing the path)
\[ \int_{\gamma} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n = (-1)^n \int_{\gamma^{-1}} \omega_n \circ \omega_{n-1} \circ \cdots \circ \omega_1 \]

Theorem 1.13 [Composition of paths] Let $\omega_1, \omega_2, \cdots, \omega_n$ be differential forms, where some of them could repeat. Let $\gamma_1$ be a path that ends at $Q$ and $\gamma_2$ be a path that starts at $Q$. Then
\[ \int_{\gamma_1 \gamma_2} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n = \sum_{i=0}^{n} \int_{\gamma_1} \omega_1 \circ \omega_2 \circ \cdots \circ \omega_i \int_{\gamma_2} \omega_i \circ \omega_{i+1} \circ \cdots \circ \omega_n \]

Let $\tau$ be a simple loop around $C$ in $X - D$, based at $R$. Let $\sigma$ be a loop on the curve $C^0 = C_0 - (D_1 \cup \cdots \cup D_m) \cap C_0$. We define a membrane $m_\sigma$ associated to a loop $\sigma$ in $C^0$ by
\[ m_\sigma : [0, 1]^2 \to X, \]
\[ m_\sigma(s, t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))} \]
\[ m_\sigma(0, 0) = R. \]

Note that for fixed values of $s$ and $t$, we have that
\[ F_{f(\tau(t))} \cap G_{g(\sigma(s))} \]
consists of finitely many points, where $F$ and $G$ are foliations satisfying the Conditions in Subsection 2.1 and Lemma 2.2.

Consider the dependence of $\log(f_i(m(s, t)))$ on the variables $s$ and $t$ via the parametrization of the membrane $m$. We have
\[ d \log(f_i(m(s, t))) = \frac{\partial \log(f_i(m(s, t)))}{\partial s} ds + \frac{\partial \log(f_i(m(s, t)))}{\partial t} dt. \]
In order to use a more compact notation, we will use
\[ \log(f_i)_s(s, t) = \frac{\partial \log(f_i(m(s, t)))}{\partial s} \]
and similarly
\[ \log(f_i)_t(s, t) = \frac{\partial \log(f_i(m(s, t)))}{\partial t}. \]

**Definition 1.14** (Interior Iterated integrals on membranes) Let \( f_1, \ldots, f_{k+l} \) be rational functions on \( X \), where the integers \((k, l)\) will be superscripts. Let \( m \) be a membrane as above. We define:

(a) \( I^{(1,1)}(m; f_1, f_2) = \)
\[
\int_0^1 \int_0^1 \log(f_1)_s(s, t) ds \wedge \log(f_1)_t(s, t) dt;
\]

(b) \( I^{(1,2)}(m; f_1, f_2, f_3) = \)
\[
= \int \int \int_{0 \leq s_1 \leq s_2 \leq 1, 0 \leq t \leq 1} \log(f_1)_{s_1}(s_1, t) ds_1 \wedge \log(f_2)_t(s_1, t) dt \wedge
\]
\[
\wedge \log(f_3)_{s_2}(s_2, t) ds_2;
\]

(c) \( I^{(2,1)}(m; f_1, f_2, f_3) = \)
\[
= \int \int \int_{0 \leq s \leq t_1 \leq t_2 \leq 1, 0 \leq t_1 \leq 1} \log(f_1)_s(s, t_1) ds \wedge \log(f_2)_{t_1}(s, t_1) dt_1 \wedge
\]
\[
\wedge \log(f_3)_{t_2}(s, t_2) dt_2;
\]

Define any smooth metric on \( X \). Let \( \tau \) be a simple loop around the curve \( C \) of distance at most \( \epsilon \) from \( C \). We are going to take the limit as \( \epsilon \to 0 \). Informally, the radius of the loop \( \tau \) goes to zero. Then we have the following lemma.

Using Chen [Ch] we obtain the following Lemma.

**Lemma 1.15** Let \( \alpha \) and \( \beta \) be two loops on the surface \( X \) with a common base. Put
\[ \theta_1 = \frac{df_1}{f_1} \]
and
\[ \theta_2 = \frac{df_2}{f_2} \wedge \frac{df_3}{f_3}. \]

Put \([\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}\). Then
\[ \int_{[\alpha, \beta]} \theta_1 \cdot \theta_2 = \int_\alpha \theta_1 \int_\beta \theta_2 - \int_\beta \theta_1 \int_\alpha \theta_2. \]
Proof. It follows directly from Lemma 1.13 by applying it to each ingredient of the commutator.

As a direct consequence, we obtain the following:

**Corollary 1.16**

\[
\int_{m_{[\alpha,\beta]}} \frac{df_1}{f_1} \cdot \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) \in (2\pi i)^3 \mathbb{Z}.
\]

Following Chen, we obtain the 1-form

\[
\int \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right)
\]
on the loop space is closed since

1. \( \frac{df_1}{f_1} \) and \( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \) are closed and
2. \( \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = 0 \).

Since, we have a closed form it follows that the integral is homotopy invariant. Thus, we can take a relation in the fundamental group of a curve embedded in the surface. More precisely, we take

\[
\delta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \sigma_1 \cdots \sigma_n
\]

for a curve \( C_0 \) of genus \( g \) with \( n \) punctures.

For each of the above loops we associate a torus.

Let \( \tau \) be a simple loop around \( C_0 \) in \( X - C_0 - \left( \bigcup_{i=1}^M G_g(U_i^*) \right) \). Let \( \sigma \) be a loop on the curve \( C_0 \). We define a membrane \( m_\sigma \) associated to a loop \( \sigma \) in \( C^0 \) by

\[
m_\sigma : [0, 1]^2 \to X
\]

and

\[
m_\sigma(s, t) \in F_{f(\tau(t))} \cap G_{g(\sigma(s))}.
\]

Note that for fixed values of \( s \) and \( t \), we have that

\[
F_{f(\tau(t))} \cap G_{g(\sigma(s))}
\]
consists of finitely many points, where \( F \) and \( G \) are foliations satisfying the Conditions in Subsection 1.2.

**Claim:** The image of \( m_\sigma \) is a torus.

Indeed, consider a tubular neighborhood around a loop \( \sigma \) on the curve \( C_0 \). One can take the following tubular neighborhood:

\[
\bigcup_{|v|<\epsilon} F_v \cap G_{g(\sigma)}
\]
of \( F_v \cap G_{g(\sigma)} \). Its boundary is \( F_{f(\tau)} \cap G_{g(\sigma)} \), where \( \tau \) is a simple loop around \( C_0 \) on \( X - \bigcup_{i=1}^n C_i - \bigcup_{j=1}^m D_j \) and \( |f(\tau(t))| = \epsilon \).

In the last section we will associate a Contou-Carrere symbol to a simple loop \( \sigma_i \), namely, \( I_{m_\delta}^{1,2} (f_1, f_2, f_3) \). By the above corollary we have that \( I_{m_\delta}^{1,2} (f_1, f_2, f_3) \) is an integer multiple of \((2\pi i)^3\)

Then \( I_{m_\delta}^{1,2} = 0 \), since \( \delta \) is homotopic to the trivial path. Also, by the Lemma 1.13, we have

\[
0 = I_{m_\delta}^{1,2} (f_1, f_2, f_3) = \sum_{i=1}^n I_{m_{\sigma_i}}^{1,2} (f_1, f_2, f_3) + (2\pi i)^3 \mathbb{Z}.
\]
2 Countou-Carrere symbol for surfaces and its reciprocity laws

2.1 Cocycle on the loop space of a surface

2.2 Semi-local symbol

In this subsection we present computation of the Contou-Carrere symbol for all possible factors from the formal infinite product. Let

\[ f_1 = x^{\nu_1(f_1)} y^{\nu_2(f_1)} \prod_{i_1 > -N} \prod_{j_1 > -N_{i_1}} (1 - a_{i_1,j_1} x^{i_1} y^{j_1}), \]

\[ f_2 = x^{\nu_1(f_2)} y^{\nu_2(f_2)} \prod_{i_2 > -N} \prod_{j_2 > -N_{i_2}} (1 - a_{i_2,j_2} x^{i_2} y^{j_2}), \]

\[ f_3 = x^{\nu_1(f_3)} y^{\nu_2(f_3)} \prod_{i_3 > -N} \prod_{j_3 > -N_{i_3}} (1 - a_{i_3,j_3} x^{i_3} y^{j_3}). \]

We consider an integral over a torus of \( \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \) in the following cases: The function \( f_1 \) is either \( x^{i_1} y^{j_1} \) or \( 1 - x^{i_1} y^{j_1} \). The function \( f_2 \) is either \( x^{i_2} y^{j_2} \) or \( 1 - x^{i_2} y^{j_2} \). The function \( f_3 \) is either \( x^{i_3} y^{j_3} \) or \( 1 - x^{i_3} y^{j_3} \). For each of the functions there are two possibilities. For the triple \((f_1, f_2, f_3)\) there are \(2^3\) possibilities, which we list in the following \(8 = 2^3\) cases. We define the two dimensional Contou-Carrere symbol as a cyclic symmetrization of

\[ \exp \left( \int_{T} \int \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right), \]

where \( T \) is a torus of the type \( m_\sigma \) from Section 1. At the next 8 cases, we examine the logarithm of the Contou-Carrere symbol, when each of the functions \( f_1, f_2, f_3 \) consists of a single factor of the above infinite products.

At the end of the paper, we compute a more complicated case, which will be useful when we consider complex analytic products instead of products coming from Witt parameters.

Case 1: Let \( f_1 = 1 - ax^{i_1} y^{j_1}, f_2 = 1 - bx^{i_2} y^{j_2} \) and \( f_3 = 1 - cx^{i_3} y^{j_3} \). Then
\[
\int \log(f_1) \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} = \int_0^1 \int_0^1 \left( \sum_{n_1=1}^{\infty} -ia^{n_1}b^{n_1i}c^{n_1j} \exp(2\pi \sqrt{-1}i_1j_1^d) \, d\theta_1 + \exp(2\pi \sqrt{-1}i_1j_1^d) \, d\theta_2 \right) \times \\
\times \sum_{n_2,n_3=1}^{\infty} (i_2j_3 - i_3j_2) b^{n_2}c^{n_3}a^{n_2i_2+n_3i_3}b^{n_2j_2+n_3j_3} \times \\
\times \exp(2\pi \sqrt{-1}(n_2i_2+n_3i_3)\theta_1) \exp(2\pi \sqrt{-1}(n_2j_2+n_3j_3)\theta_2) \, d\theta_1 \wedge d\theta_2 = \\
= \int_0^1 \int_0^1 -i_1i_2j_3 \times \\
\times \sum_{n_1,n_2,n_3=1}^{\infty} \frac{1}{i_1n_1} a^{n_1}b^{n_2}c^{n_3}a^{n_1i_1+n_2i_2+n_3i_3}b^{n_1j_1+n_2j_2+n_3j_3} \times \\
\times \int_0^1 \int_0^1 \exp(2\pi \sqrt{-1}(n_2i_2+n_3i_3)\theta_1) \exp(2\pi \sqrt{-1}(n_2j_2+n_3j_3)\theta_2) \, d\theta_1 \wedge d\theta_2 = \\
- (i_2j_3 - i_3j_2) \times \\
\times \int_0^1 \int_0^1 \exp(2\pi \sqrt{-1}(n_1i_1+n_2i_2+n_3i_3)\theta_1) \times \\
\times \exp(2\pi \sqrt{-1}(n_1j_1+n_2j_2+n_3j_3)\theta_2) \, d\theta_1 \wedge d\theta_2 = \\
= - (i_2j_3 - i_3j_2) \sum_{n_1,n_2,n_3=1}^{\infty} \frac{a^{n_1}b^{n_2}c^{n_3}}{n_1} \\
\text{subject to: } n \cdot i = 0, n \cdot j = 0.
\]

Put \( m_k = \begin{vmatrix} i_{k+1} & i_k \\ j_{k+1} & j_k \end{vmatrix} \), where the indices vary modulo 3. Let \( d = \text{gcd}(m_1, m_2, m_3) \).

Then

\[
\begin{align*}
n_1 &= k \left| \frac{m_1}{d} \right|, \\
n_2 &= k \left| \frac{m_2}{d} \right|, \\
n_3 &= k \left| \frac{m_3}{d} \right|.
\end{align*}
\]
Case 2: Let $f_1 = ax^{i_1}y^{j_1}$, $f_2 = 1 - bx^{i_2}y^{j_2}$, $f_3 = 1 - cx^{i_3}y^{j_3}$.

$$
\int \int \int_{T_0} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \int_0^1 \int_0^1 \sum_{n_2,n_3=1}^{\infty} (i_2j_3 - i_3j_2) b^{n_2} e^{n_3x^{n_2i_2+n_3i_3}y^{n_2j_2+n_3j_3}} \frac{dx}{x} \wedge \frac{dy}{y}
$$

If $n_2j_2 + n_3j_3 \neq 0$ then the integral vanishes. Let $n_2j_2 + n_3j_3 = 0$. Then

$$n_2 = k|j_3|/\gcd(j_2,j_3)$$

and

$$n_3 = k|j_2|/\gcd(j_2,j_3).$$

Moreover, we have a geometric series under the integral, namely,

$$g(x) = \sum_{k=1}^{\infty} b^{n_2} e^{n_3x^{n_2i_2+n_3i_3}} = \sum_{k=1}^{\infty} \left( b|j_3|/\gcd(j_2,j_3) \frac{c|j_2|}{\gcd(j_2,j_3)} x^{(|j_3|i_2+|j_2|i_3)/\gcd(j_2,j_3)} \right)^k = b|j_3|/\gcd(j_2,j_3) c|j_2|/\gcd(j_2,j_3) x^{\text{sign}(j_3)m_1/\gcd(j_2,j_3)} \times \left( 1 - b|j_3|/\gcd(j_2,j_3) c|j_2|/\gcd(j_2,j_3) x^{\text{sign}(j_3)m_1/\gcd(j_2,j_3)} \right)^{-1}
$$

Then

$$g(x) \frac{dx}{x} = -d \log \left( 1 - b|j_3|/\gcd(j_2,j_3) c|j_2|/\gcd(j_2,j_3) x^{\text{sign}(j_3)m_1/\gcd(j_2,j_3)} \right)$$
Let
\[ h(x) = 1 - b_{j3} |j2|/\gcd(j2,j3) c_{j3} |j2|/\gcd(j2,j3) x \cdot \text{sign}(j3)m_1/\gcd(j2,j3). \]

Then we have that Case 2 is the logarithm of the 1 dimensional Contou-Carrere symbol of \( x^\mathbf{1} \) and \( h(x) \) times \( i_1(i_2j_3 - i_3j_2) \).

Alternatively, if we sum term by term we can use the following Lemma.

**Lemma 2.1** If \( k \in \mathbb{Z} \) and \( k \neq 0 \) then
\[
\int_0^1 \theta e^{2\pi \sqrt{-1k\theta}} d\theta = \frac{1}{2\pi \sqrt{-1k}}.
\]

**Proof.**
\[
\int_0^1 \theta e^{2\pi \sqrt{-1k\theta}} d\theta = \frac{1}{2\pi \sqrt{-1k}} \int_0^1 \theta e^{2\pi \sqrt{-1k\theta}} = \frac{1}{2\pi \sqrt{-1k}} (\theta e^{2\pi \sqrt{-1k\theta}}|_0^1 - \int_0^1 e^{2\pi \sqrt{-1k\theta}} d\theta) = \frac{1}{2\pi \sqrt{-1k}}.
\]

Then
\[
I^{1,2}(f_1, f_2, f_3) = \int_0^1 \int_0^1 2\pi \sqrt{-1i_1} \theta_1 \times \sum_{n_2,n_3=1}^{\infty} (i_2j_3 - i_3j_2) b_{n_2} c_{n_3} \epsilon_{1,n_2} \epsilon_{1,n_3} \times \exp \left( 2\pi \sqrt{-1}(k|j_3|i_2/d + k|j_2|i_3/d)\theta_1 \right) d\theta_1 \wedge d\theta_2 = \sum_{k_1=1}^{\infty} 2\pi \sqrt{-1i_1(i_2j_3 - i_3j_2)} \left( b_{j3}/d \epsilon_{1,d} |j3|/d + k_1|j2|i_3/d \right)^{k_1} + \text{sign}(j_3) \cdot i_1 \cdot d \cdot \sum_{k_1=1}^{\infty} \left( b_{j3}/d \epsilon_{1,d} |j3|/d + k_1|j2|i_3/d \right)^{k_1} = \text{sign}(j_3) \cdot i_1 \cdot d \cdot \log \left( 1 - b_{j3}/d \epsilon_{1,d} |j3|/d \cdot \text{sign}(j_3)m_1/d \right) = - \text{sign}(j_3) \cdot i_1 \cdot d \cdot \log \left( 1 - b_{j3}/d \epsilon_{1,d} |j3|/d \cdot \text{sign}(j_3)m_1/d \right).
\]
Then

\[ I^{2,1}(f_1, f_2, f_3) = \]
\[ \int_0^1 \int_0^1 2\pi \sqrt{-1} j_1 \theta_2 \times \]
\[ \times \sum_{n_2, n_3 = 1}^{\infty} (i_2j_3 - i_3j_2)b_{n_2}c_{n_3} \epsilon_1^{n_2i_2 + n_3i_3} \times \]
\[ \times \exp \left( 2\pi \sqrt{-1}(n_2j_2 + n_3j_3)\theta_2 \right) d\theta_1 \wedge d\theta_2 = \]
\[ = \sum_{k_2 = 1}^{\infty} 2\pi \sqrt{-1} j_2(i_2j_3 - i_3j_2) \frac{b^{i_3/d}c^{i_2/d}}{2\pi \sqrt{-1}k_2(i_3j_2/d + k_2i_2j_3/d)} \]
\[ = \text{sign}(i_3) \cdot i_1 \cdot d \cdot \sum_{k_2 = 1}^{\infty} \frac{b^{i_3/d}c^{i_2/d}}{k_2} \]
\[ = - \text{sign}(i_3) \cdot j_1 \cdot d \cdot \log \left( 1 - b^{i_3/d}c^{i_2/d} \epsilon_2 \text{sign}(i_3)m_1/d \right) \]

Case 3: Let \( f_1 = 1 - ax^{i_1}y^{j_1}, f_2 = bx^{i_2}y^{j_2}, f_3 = 1 - cx^{i_3}y^{j_3}. \)

\[
\int_0^1 \int_0^1 \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
= \int_0^1 \int_0^1 \left( \int_0^{\theta_1} \sum_{n_1 = 1}^{\infty} -i_1a^{n_1}c_{n_1i_1}^{n_2j_1} \exp \left( 2\pi \sqrt{-1}n_1(i_1\theta_1' + j_1\theta_2) \right) (d\theta_1') \times \\
\times (2\pi \sqrt{-1})(i_2d\theta_1 + j_2d\theta_2) \wedge \\
\wedge \sum_{n_3 = 1}^{\infty} c_{n_3}^{n_1i_1+n_3i_3} \epsilon_2^{n_3j_2} \exp \left( 2\pi \sqrt{-1}(n_3j_3)\theta_1 \right) \exp \left( 2\pi \sqrt{-1}(n_3j_3)\theta_2 \right) (i_3d\theta_1 + j_3d\theta_2) = \\
= \sum_{n_1, n_3 = 1}^{\infty} i_1(i_2j_3 - i_3j_2) a^{n_1}c^{n_1i_1+n_3i_3} \epsilon_2^{n_1j_1+n_3j_3} \]
\[ \times \int_0^1 \int_0^1 \exp \left( 2\pi \sqrt{-1}(n_1i_1 + n_3i_3)\theta_1 \right) \exp \left( 2\pi \sqrt{-1}(n_1j_1 + n_3j_3)\theta_2 \right) d\theta_1 \wedge d\theta_2 = \\
\]

The last double integral vanishes if \( n_1i_1 + n_3i_3 \neq 0 \) or if \( n_1j_1 + n_3j_3 \neq 0. \) If both \( n_1i_1 + n_3i_3 = 0 \) and \( n_1j_1 + n_3j_3 = 0 \) then the two vectors \((i_1, i_3)\) and \((j_1, j_3)\) are linearly dependent. Let \( n_1 = k|i_3|/(i_1, i_3) \) and \( n_3 = k|i_1|/(i_1, i_3). \)
Note that $i_2 j_3 - i_3 j_2 = 0$. Then the contribution from Case 3 becomes

$$I_3 = \sum_{k=1}^{\infty} \frac{i_2 j_3}{n_1} a^{n_1} c^{n_3} =$$

$$= \sum_{k_1=1}^{\infty} \frac{i_2 j_3 (j_1, j_3)}{|j_3| k_1} (a^{j_3|/j_1 j_3} c^{j_1|/j_1 j_3})^{k_1} -$$

$$- \sum_{k_2=1}^{\infty} \frac{i_3 j_2 (i_1, i_3)}{|i_2| k_2} (a^{i_3|/i_1 i_3} c^{i_1|/i_1 i_3})^{k_2} =$$

$$= - \text{sign}(j_3) j_2 (j_1, j_3) \log \left( 1 - a^{j_3|/j_1 j_3} c^{j_1|/j_1 j_3} \right) +$$

$$+ \text{sign}(i_3) j_2 (i_1, i_3) \log \left( 1 - a^{i_3|/i_1 i_3} c^{i_1|/i_1 i_3} \right)$$

Case 4: Let $f_1 = 1 - ax^{i_1} y^{j_1}, f_2 = 1 - bx^{i_2} y^{j_2}, f_3 = cx^{i_3} y^{j_3}$.

The contribution from Case 4 is similar to Case 3.

$$I_4 = \sum_{k=1}^{\infty} \frac{i_2 j_3 - i_3 j_2}{n_1} a^{n_1} b^{n_2} =$$

$$= \sum_{k_1=1}^{\infty} \frac{i_2 j_3 (i_1, i_2)}{|i_2| k_1} (a^{i_2|/i_1 i_2} b^{i_1|/i_1 i_2})^{k_1} -$$

$$- \sum_{k_2=1}^{\infty} \frac{i_3 j_2 (j_1, j_2)}{|j_2| k_2} (a^{j_2|/j_1 j_2} b^{j_1|/j_1 j_2})^{k_2} =$$

$$= - \text{sign}(i_2) j_3 (i_1, i_2) \log \left( 1 - a^{i_2|/i_1 i_2} b^{i_1|/i_1 i_2} \right) +$$

$$+ \text{sign}(j_2) i_3 (j_1, j_2) \log \left( 1 - a^{j_2|/j_1 j_2} b^{j_1|/j_1 j_2} \right)$$

Case 5: Let $f_1 = x^{i_1} y^{j_1}, f_2 = x^{i_2} y^{j_2}, f_3 = C(1 - cx^{i_3} y^{j_3})$. 

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\[
\int \int \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
= \int_0^1 \int_0^1 \left( \int_{(0,0)}^{(\theta_1,\theta_2)} (i_1 d\theta_1' + j_1 d\theta_2') \right) \times \\
\times (2\pi\sqrt{-1})(i_2 d\theta_1 + j_2 d\theta_2) \wedge \\
\wedge \sum_{n_3=1}^{\infty} -c^{n_3} \epsilon_1^{n_3 j_3} \epsilon_2^{n_3 j_3} \exp(2\pi\sqrt{-1}(n_3 i_3) \theta_1) \exp(2\pi\sqrt{-1}(n_3 j_3) \theta_2) (i_3 d\theta_1 + j_3 d\theta_2) = \\
= \int_0^1 \int_0^1 m_1 i_1 \theta_1 \times \\
\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp(2\pi\sqrt{-1}(n_3 i_3) \theta_1) \exp(2\pi\sqrt{-1}(n_3 j_3) \theta_2) d\theta_1 \wedge d\theta_2 + \\
+ \int_0^1 \int_0^1 j_1 m_1 \theta_2 \times \\
\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp(2\pi\sqrt{-1}(n_3 i_3) \theta_1) \exp(2\pi\sqrt{-1}(n_3 j_3) \theta_2) d\theta_1 \wedge d\theta_2
\]

The last integral is different from zero only if \(i_3 = 0\). In that case, we have

\[
\int \int \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
= \int_0^1 \int_0^1 i_1 \theta_1 (-i_3 j_2) \times \\
\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp(2\pi\sqrt{-1}(n_3 i_3) \theta_1) \exp(2\pi\sqrt{-1}(n_3 j_3) \theta_2) d\theta_1 \wedge d\theta_2 + \\
+ \int_0^1 \int_0^1 j_1 \theta_2 (i_2 j_3) \times \\
\times \sum_{n_3=1}^{\infty} c^{n_3} \epsilon_1^{n_3 i_3} \epsilon_2^{n_3 j_3} \exp(2\pi\sqrt{-1}(n_3 i_3) \theta_1) \exp(2\pi\sqrt{-1}(n_3 j_3) \theta_2) d\theta_1 \wedge d\theta_2 = \\
= \sum_{n_3=1}^{\infty} \frac{-i_1 i_3 j_2}{n_3 i_3} c^{n_3} \epsilon_1^{n_3 j_3} + \\
+ \sum_{n_3=1}^{\infty} \frac{i_1 i_2 j_3}{n_3 j_3} c^{n_3} \epsilon_2^{n_3 j_3} = \\
= -\frac{i_1 i_3 j_2}{i_3} \sum_{n_3=1}^{\infty} \left( \frac{\epsilon_1^{n_3 i_3}}{n_3} \right) + \frac{i_1 i_2 j_3}{j_3} \sum_{n_3=1}^{\infty} \left( \frac{\epsilon_2^{n_3 j_3}}{n_3} \right) = \\
= i_1 j_2 \log(1 - \epsilon_1^{i_3}) - i_2 j_1 \log(1 - \epsilon_2^{j_3})
\]
Thus,
\[ I_5 = i_1j_2 \log(1 - c(-P_1)^i_3) - i_2j_1 \log(1 - c(-P_2)^j_3) \]

6: Let \( f_1 = x^{i_1}y^{j_1}, f_2 = B(1 - bx^{i_2}y^{j_2}), f_3 = x^{i_3}y^{j_3} \). Similarly to case 5, we obtain
\[ I_6 = i_3j_1 \log(1 - b(-P_1)^{i_2}) - i_1j_3 \log(1 - b(-P_2)^{j_2}) \]

7: Let \( f_1 = A(1 - ax^{i_1}y^{j_1}), f_2 = x^{j_2}y^{j_2}, f_3 = x^{i_3}y^{j_3} \).

\[
\int \int \int_{T_0} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
= \int_0^1 \int_0^1 \left( \int_0^{\theta_1} \sum_{n_1=1}^{\infty} -ia n_i \epsilon_1^{n_i} \epsilon_2^{n_1j_1} \exp(2\pi \sqrt{-1n_1i_1\theta_1'}) \exp(2\pi \sqrt{-1n_1j_1\theta_2}) d\theta_1' \times \\
\times i_2j_3 1d\theta_1 \wedge d\theta_2 = \\
= \int_0^1 \int_0^1 \sum_{n_1=1}^{\infty} \frac{i_1i_2j_3 a^n \epsilon_1^{n_i} \epsilon_2^{n_1j_1}}{n_1i_1} \times \\
\times (\exp(2\pi \sqrt{-1(n_1i_1)\theta_1}) - 1) \exp(2\pi \sqrt{-1n_1j_1\theta_2}) d\theta_1 \wedge d\theta_2
\]

The integral is zero if \( j_1 \neq 0 \). If \( j_2 = 0 \) then we obtain
\[
\int \int \int_{T_0} \frac{df_1}{f_1} \circ \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \\
= - \int_0^1 \sum_{n_1=1}^{\infty} \frac{i_1i_2j_3 a^n \epsilon_1^{n_i}}{n_1i_1} \exp(2\pi \sqrt{-1(n_1i_1)\theta_1}) d\theta_1 - \\
\sum_{n_1=1}^{\infty} \frac{i_1i_2j_3 a^n \epsilon_1^{n_i}}{n_1i_1} = 0 - i_2j_3 \log(1 - a\epsilon_1^{i_1})
\]

Note that the last integral vanishes if \( i_1 \neq 0 \). However, if \( i_1 = 0 \) and \( j_1 = 0 \) then \( f_1 = 1 \) and the integral vanishes again. Thus,
\[ I_7 = -m_1 \log(1 - aR_1^{i_1}) \]

8: Let \( f_1 = x^{i_1}y^{j_1}, f_2 = x^{i_2}y^{j_2}, f_3 = x^{i_3}y^{j_3} \).

For this case it is better to use the notation \( I_3^{1,2}(f_1, f_2, f_3) \) and \( I_3^{1,1}(f_1, f_2, f_3) \) from Subsection 1.3. Using that the logarithm of the Parshin symbol (see [H5]) can be written as
\[
I_3^{1,2}(f_1, f_2, f_3) + I_3^{1,2}(f_3, f_1, f_2) + I_3^{1,2}(f_2, f_3, f_1) - \\
- I_3^{2,1}(f_1, f_2, f_3) - I_3^{2,1}(f_3, f_1, f_2) - I_3^{2,1}(f_1, f_2, f_3)
\]

\[ I_8 = -m_1 \log(1 - aR_1^{i_1}) \]
The iteration $\frac{df_1}{j_1} \circ (\frac{df_2}{j_2} \land \frac{df_3}{j_3})$ gives 4 from the above 6 terms, namely

$$\frac{(2\pi i)^3}{2} (i_1 i_2 j_3 + i_3 i_1 j_3 - i_3 j_1 j_2 - i_2 j_3 j_1)$$

The ones that are not present are monomials $i_2 i_3 j_1$ and $j_2 j_3 i_1$.

**Definition 2.2 (Contou-Carrere symbol for surfaces)** Let

\[ f_1 = a_1 x^{\nu_1 (f_1)} y^{\nu_2 (f_1)} \prod_{i_1 > -N, j_1 > -N_1} (1 - a_{i_1, j_1} x^{i_1} y^{j_1}), \]

\[ f_2 = a_2 x^{\nu_1 (f_2)} y^{\nu_2 (f_2)} \prod_{i_2 > -N, j_2 > -N_2} (1 - a_{i_2, j_2} x^{i_2} y^{j_2}), \]

\[ f_3 = a_3 x^{\nu_1 (f_3)} y^{\nu_2 (f_3)} \prod_{i_3 > -N, j_3 > -N_3} (1 - a_{i_3, j_3} x^{i_3} y^{j_3}). \]

Let

\[ T(f_1, f_2, f_3) = \prod_{i_1, i_2, i_3, j_1, j_2, j_3} \left( 1 - a_{i_1, j_1}^{[m_1]/d} a_{i_2, j_2}^{[m_2]/d} a_{i_3, j_3}^{[m_3]/d} \right)^{\text{sign}(m_1)d}, \]

where $m_k = i_k + j_k + 2 - i_k + 2 j_{k+1}$ for $k$ modulo 3 and $d$ is the greatest common divisor of $m_1, m_2, m_3$.

\[ Q_1(f_1, f_2, f_3, P_1) = \prod_{\nu_1 (f_1), i_2, i_3, j_2, j_3} \left( 1 - a_{i_2, j_2}^{[j_1]/d} a_{i_3, j_3}^{[j_2]/d} P_1^{\text{sign}(j_3)\nu_1 (f_1) (j_2, j_3)} \right)^{-\text{sign}(j_3)\nu_1 (f_1) (j_2, j_3)} \]

\[ Q_2(f_1, f_2, f_3, P_2) = \prod_{\nu_2 (f_1), i_2, i_3, j_2, j_3} \left( 1 - a_{i_2, j_2}^{[i_3]/d} a_{i_3, j_3}^{[i_2]/d} P_2^{\text{sign}(i_3)\nu_2 (f_1) (i_2, i_3)} \right)^{\text{sign}(i_3)\nu_2 (f_1) (i_2, i_3)} \]

\[ Q_3(f_1, f_2, f_3) = \prod_{\nu_3 (f_1), i_1, i_3, j_1, j_3} \left( 1 - a_{i_1, j_1}^{[j_3]/d} a_{i_3, j_3}^{[j_1]/d} \right)^{\text{sign}(j_3)\nu_2 (f_1) (i_1, i_3)} \]

\[ Q_4(f_1, f_2, f_3) = \prod_{\nu_4 (f_1), i_1, i_3, j_1, j_3} \left( 1 - a_{i_1, j_1}^{[i_3]/d} a_{i_3, j_3}^{[i_1]/d} \right)^{-\text{sign}(j_3)\nu_2 (f_1) (i_1, i_3)} \]

\[ R_1(f_1, f_2, f_3, P_1) = \prod_{\nu_1 (f_2), i_2, i_3, j_2, j_3} \left( 1 - a_{i_1, j_1}^{[j_3]/d} a_{i_3, j_3}^{[j_2]/d} P_1^{\text{sign}(j_3)\nu_1 (f_2) (j_2, j_3)} \right)^{-\text{sign}(j_3)\nu_2 (f_1) (j_2, j_3)} \]

\[ R_2(f_1, f_2, f_3, P_2) = \prod_{\nu_2 (f_2), i_2, i_3, j_2, j_3} \left( 1 - a_{i_1, j_1}^{[i_3]/d} a_{i_3, j_3}^{[i_2]/d} P_2^{\text{sign}(i_3)\nu_2 (f_2) (i_2, i_3)} \right)^{-\text{sign}(j_3)\nu_2 (f_1) (i_2, i_3)} \]

\[ S(f_1, f_2, f_3) = (-1)^A, \text{ where } A = i_1 i_2 j_3 + i_2 i_3 j_1 + i_3 i_1 j_2 - i_1 j_2 j_3 - i_2 j_3 j_1 - i_3 j_1 j_2 \]
Then the Contou-Carrere symbol is a formal product
\[
\{f_1, f_2, f_3\}_C^P = T(f_1, f_2, f_3)S(f_1, f_2, f_3) \times \prod_{\text{cyclic}} Q_1(f_1, f_2, f_3)Q_2(f_1, f_2, f_3, P_1)Q_3(f_1, f_2, f_3, P_2)Q_4(f_1, f_2, f_3) \times \prod_{\text{cyclic}} R_1(f_1, f_2, f_3, P_1)R_2(f_1, f_2, f_3, P_2),
\]
where \(\prod_{\text{cyclic}}\) is a product over a cyclic permutation of the order of the functions \(f_1, f_2, f_3\).

### 2.3 Semi-local formulas

Let \(f_1 = 1 - a_1 x^{i_1} y^{j_1}, f_2 = 1 - a_2 x^{i_2} y^{j_2}, f_1 = 1 - a_3 x^{i_3} y^{j_3} - a_4 x^{i_4} y^{j_4},\)

\[
\int_T \frac{df_1}{f_1} \left( \frac{df_2}{f_2} \wedge \frac{df_3}{f_3} \right) = \int_T \sum_{n_1, n_2, n_3=1}^\infty \sum_{n_4=0}^\infty - \frac{2}{n_1 n_2 n_3 n_4} \prod_{k=1}^4 a_k n_k x^{i_k} y^{j_k} dy^k = \int_T \sum_{n_1, n_2, n_3=1}^\infty \sum_{n_4=0}^\infty \frac{2}{n_1 n_2 n_3 n_4} \prod_{k=1}^4 a_k n_k x^{i_k} y^{j_k} dy^k = \int_T \sum_{n_1, n_2, n_3=1}^\infty \sum_{n_4=0}^\infty - \frac{2}{n_1 n_2 n_3 n_4} \prod_{k=1}^4 a_k n_k x^{i_k} y^{j_k} dy^k =
\]

Let \(\begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} \neq 0\). Then by row reduction we obtain the following

\[
\begin{bmatrix} i_1 & i_2 & i_3 & i_4 \\ j_1 & j_2 & j_3 & j_4 \end{bmatrix} \rightarrow \begin{bmatrix} i_1 & i_3 & i_3 & 0 & i_4 & i_3 \\ j_1 & j_3 & j_3 & j_4 \end{bmatrix} \rightarrow \begin{bmatrix} i_2 & i_1 & 0 & i_2 & i_3 & i_2 & i_4 \\ j_2 & j_1 & j_2 & j_3 & j_2 & j_4 \end{bmatrix}
\]

Then the sums have to vanish, \(\sum k = 1^4 i_k n_k = \sum_{k=1}^4 j_k n_k = 0\). Using the above row
reduction, we obtain

\[ n_1 = \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_1/d \]

\[ n_2 = \left( - \begin{vmatrix} i_4 & i_3 \\ j_4 & j_3 \end{vmatrix} k_4 - \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_1 \right) /d \]

\[ n_3 = \left( - \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix} k_4 - \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} k_1 \right) /d \]

\[ n_4 = \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} k_4/d, \]

where \( d \) is the greatest common divisor of

\[ \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix}, \begin{vmatrix} i_4 & i_3 \\ j_4 & j_3 \end{vmatrix}, \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix}, \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix}. \]

Let

\[ M = \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} /d - \begin{vmatrix} i_2 & i_4 \\ j_2 & j_4 \end{vmatrix} /d \]

and

\[ N = \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} /d. \]

Let also \( \xi_M \) and \( \xi_N \) be a primitive \( M \)-th and \( N \)-th root of unity.
\[ \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} = \int_T \sum_{n_1,n_2,n_3=1}^{\infty} \sum_{n_4=0}^{\infty} \frac{2(n_3 + n_4)!}{n_1 n_2 n_3 n_4!} \prod_{k=1}^{4} a_k^{n_k} x^{i_k} y^{j_k} \frac{dx}{x} \wedge \frac{dy}{y} = \]

\[ = -d \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{a_1 a_2 a_3 a_4} (n_3 + n_4)! = \]

\[ = -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{a_1 a_2} (a_3 + a_4)^{n_3 + n_4} = \]

\[ = -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{a_1 a_2} (a_3 + a_4)^{n_3 + n_4} = \]

\[ = -d \sum_{k_1=1}^{\infty} \sum_{k_4=0}^{\infty} \frac{1}{a_1 a_2} (a_3 + a_4)^{n_3 + n_4} = \]

\[ \times \sum_{M,m_1,m_2=1}^{\infty} (\xi_M^{m_1} a_3 + \xi_M^{m_2} a_4) \left( \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} - \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} \right) \frac{k_1}{d} \]

\[ \times \sum_{n=1}^{N} (\xi_N^{n_1} a_3 + \xi_N^{n_2} a_4) = \]

\[ = \sum_{m_1,m_2=1}^{M} \left( 1 - a_2 \right) \left( \xi_M^{m_1} a_3 + \xi_M^{m_2} a_4 \right) \left( \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} - \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} \right) \frac{k_1}{d} \]

\[ \times \sum_{n=1}^{N} \log \left( 1 - a_2 \right) \left( \xi_N^{n_1} a_3 + \xi_N^{n_2} a_4 \right) \left( \begin{vmatrix} i_2 & i_3 \\ j_2 & j_3 \end{vmatrix} - \begin{vmatrix} i_2 & i_1 \\ j_2 & j_1 \end{vmatrix} \right) \frac{k_1}{d} \]

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