SYMMETRY OF INTRINSICALLY SINGULAR SOLUTIONS OF DOUBLE PHASE PROBLEMS

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Abstract. We continue the study of positive singular solutions of PDEs arising from double phase functionals started in [6]. In particular, we consider the case $p < q < 2$, and we relax the assumption on the capacity of the singular set using an intrinsic notion of capacity.

1. Introduction

Double phase functionals are integral functionals of the form
\begin{equation}
|u| \mapsto \int_{\Omega} \left( |\nabla u|^p + a(x)|\nabla u|^q \right) \, dx,
\end{equation}
where $\Omega \subset \mathbb{R}^N$, $1 < p < q < N$ and $a(\cdot) \geq 0$. This class of functionals naturally appear in homogenization theory and in the study of strongly anisotropic materials (see, e.g., [39]), and falls into the framework of the so called functionals with non-standard growth introduced by Marcellini [27, 28]. The literature concerning functionals like (1.1) is pretty vast and concerns as a main topic the regularity of minimizers, see e.g. [2, 11, 12, 23] and the references therein.

In this paper we continue the study started in [6] concerning qualitative properties of positive (and nontrivial) solutions $u \in C^1(\Omega \setminus \Gamma)$ of a class of nonlinear PDEs which are closely related to the functional (1.1), namely
\begin{equation}
\begin{cases}
-\text{div} \left(p|\nabla u|^{p-2}\nabla u + qa(x)|\nabla u|^{q-2}\nabla u\right) = f(u) & \text{in } \Omega \setminus \Gamma, \\
u > 0 & \text{in } \Omega \setminus \Gamma, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{equation}
Here $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, with $N \geq 2$ and $1 < p < q < N$, while $\Gamma \subset \Omega$ is a closed set which is small in a proper (Sobolev)-capacitary sense. Under suitable assumptions on $a$ and $f$ (see below), and only for $2 \leq p < q < N$ we proved in [6] that solutions inherit symmetry from $\Omega$. Our aim is twofold:

- consider the case $1 < p < q < 2$;
- relax the assumptions on $\Gamma$ using a more intrinsic notion of capacity.

2020 Mathematics Subject Classification. 35B06, 35J75, 35J62, 35B51.

Key words and phrases. Double phase problems, singular solutions, moving plane method.

The authors are members of INdAM-GNAMPA. F. Esposito is partially supported by PRIN project 2017JPCAPN (Italy): Qualitative and quantitative aspects of nonlinear PDEs.
We now clarify once for all what is the notion of solution to (1.2) we will work with.

**Definition 1.1.** We say that a function $u \in C^1(\overline{\Omega} \setminus \Gamma)$ is a solution to problem (1.2) if it satisfies the following two properties:

1. $u > 0$ in $\Omega \setminus \Gamma$ and $u = 0$ on $\partial \Omega$;
2. For every $\varphi \in C^1_c(\Omega \setminus \Gamma)$ one has

\[
\int_{\Omega} \left( p|\nabla u|^{p-2} + qa(x)|\nabla u|^{q-2} \right) \langle \nabla u, \nabla \varphi \rangle \, dx = \int_{\Omega} f(u)\varphi \, dx,
\]

where $\langle \cdot , \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^N$.

For results concerning the existence of solutions to (1.2) we refer for example to [13, 30]. We are finally ready to state our main result.

**Theorem 1.2.** Let $\frac{2N}{N+2} < p < q < 2$ and let $\Omega \subseteq \mathbb{R}^N$ be a convex open set, symmetric wrt the $x_1$-direction. Moreover, let $\Gamma \subseteq \Omega \cap \{x_1 = 0\}$ be a closed set such that

(1.4) \[ \text{Cap}_{p,q}(\Gamma) = 0. \]

Finally, we require $a$ and $f$ to satisfy the following assumptions:

1. $a \in L^\infty(\Omega) \cap C^1(\Omega)$ is non-negative and independent of $x_1$;
2. $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function with $f(s) > 0$ for $s > 0$.

Then, any solution $u \in C^1(\overline{\Omega} \setminus \Gamma)$ to (1.2) is symmetric wrt the hyperplane $\{x_1 = 0\}$ and increasing in the $x_1$-direction in $\Omega \cap \{x_1 < 0\}$.

**Remark 1.3.** We explicitly stress that, even if Theorem 1.2 is stated under the assumption $p < q < 2$, by combining the techniques exploited in this paper with the approach carried out in [6] one can easily see that the same result holds when

$$\frac{2N}{N+2} < p < 2 \leq q.$$

We refer to Remark 3.1 for some further detail.

The result is of classical flavor and resembles the seminal papers [26] and [4] where the authors proved symmetry of solutions of semilinear elliptic equations by means of the moving plane method introduced by Alexandrov [1] and Serrin [35]. The power of this very elegant method is witnessed by the huge existing literature: see [17, 37] for the case of cooperative elliptic systems, see, e.g., [14, 15, 16, 24, 25] for the case of quasilinear equations and [18] for the case of non-smooth domains.

The present situation is slightly different in we are considering singular solutions. In order to face the natural technical difficulties which arise in this case, we exploit a rather recent modification of the moving plane method introduced by Sciunzi in [33] in the case of singular solutions. This technique has already proved to be flexible enough to be adapted in the case of unbounded sets [21], $p$-Laplacian operator [29], cooperative elliptic systems [9, 20], fractional Laplacian [29], mixed local–nonlocal elliptic operators [5], cooperative quasilinear systems [7], double phase problems with $p \geq 2$ [6]. Concerning qualitative
properties for *singular solutions* of certain PDEs, it is worth highlighting that there are several works dealing with *point-type singularities*, see, e.g., [8, 10, 36]. As a matter of fact, since our assumption (1.4) is clearly satisfied by this kind of singularities, the above **Theorem 1.2** can be seen as a generalization of [10, 36] to the double phase setting.

Let us now spend a few comments on **Theorem 1.2**. As already mentioned, we use a variant of the moving plane method which needs few classical tools, like a weak/strong comparison principle and Hopf-type Lemma; owing to [32] and [6], these tools are all at our disposal. In order to better explain the delicate technical difference with respect to the case considered in [6], let us recall that already in the pure p-Laplacian case (i.e., \(a(x) \equiv 0\)) when \(1 < p < 2\), the operator may present a pretty singular behavior near the set \(\Gamma\) and hence the inverse of the weight \(\rho := |\nabla u|^{p-2}\) may not have the right summability properties which allow to use a weighted Sobolev inequality of Trudinger [38].

In [22], the authors avoided this problem (for the pure p-Laplacian case) by performing an accurate analysis of the behavior of the gradient of the solution near the set \(\Gamma\) based on previous results contained in [31]. This is currently out of reach for us. Nevertheless, in the recent [7] a similar problem appeared in the case of quasilinear systems with a gradient term and we have been able to circumvent it by a nice use of the classical Hölder inequality. This trick can be adapted to the present setting as well. We believe that the simplicity of the argument is kind of surprising, but we have to pay the price of a not optimal lower bound for \(p\) (i.e. \(p > \frac{2N}{N+2}\)), which we cannot push down to 1.

A second novelty with respect to [6] concerns the use of an intrinsic capacity (denoted by \(\text{Cap}_{p,q}(\cdot)\)) recently used in [19]. First of all, this provides an immediate generalization of our previous result because

\[
\text{Cap}_q(E) = 0 \Rightarrow \text{Cap}_{p,q}(E) = 0.
\]

Moreover, we want to stress that this choice has also a technical impact in the proof of the crucial **Lemma 2.2**. Incidentally, we also point out that the aforementioned Lemma reads exactly as its counterpart [6, Lemma 2.4], but the proof has to be performed once again because we are now considering the case \(p < q < 2\).

The plan of the paper is the following:

- In Section 2 we fix the notations used throughout the paper. Moreover, we recall the notion of \((p, q)\)-capacity and some related theorems. Finally, we prove the key **Lemma 2.2**.
- In Section 3 we prove **Theorem 1.2**.

## 2. Notations and auxiliary results

The aim of this section is twofold: on the one hand, we fix once and for all the relevant notation used throughout the paper; on the other hand, we present some auxiliary results which shall be key ingredients for the proof of **Theorem 1.2**.
2.1. A review of capacities. Let 1 ≤ r ≤ N be fixed, and let K ⊆ R^N be a compact set. We remind that the classical Sobolev r-capacity of K is defined as

\[ \text{Cap}_r(K) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^r \, dx : \varphi \in C_c^\infty(\mathbb{R}^N) \text{ and } \varphi \geq 1 \text{ on } K \right\}. \]

Moreover, if D ⊆ R^N is any bounded set containing K, it is possible to define the r-capacity of the condenser (K, D) in the following way

\[ (2.1) \quad \text{Cap}_r^D(K) := \inf \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^r \, dx : \varphi \in C_c^\infty(D) \text{ and } \varphi \geq 1 \text{ on } K \right\}. \]

More recently there has been a certain interest in the study of capacities connected with Orlicz-Sobolev spaces, see e.g., [3]. In particular, when dealing with the double-phase functionals, one is naturally led to consider a so-called \((p,q)\)-capacity, see [19]: if K ⊂ Ω is a compact set, we define the \((p,q)\)-capacity of K as

\[ \text{Cap}_{p,q}(K) := \inf \left\{ \int_{\mathbb{R}^n} (|\nabla \varphi|^p + a(x)|\nabla \varphi|^q) \, dx : \varphi \in C_c^\infty(\Omega) \text{ and } \varphi \geq 1 \text{ on } K \right\}. \]

Similarly to (2.1), it is then possible to define the \((p,q)\)-capacity of the condenser (K, D) (where D ⊆ Ω is a bounded open set containing K); with this definition at hand, we say that K has vanishing \((p,q)\)-capacity, and we write \(\text{Cap}_{p,q}(K) = 0\), if

\[ \text{Cap}_{p,q}^D(K) = 0 \quad \text{for every open set } D \supseteq K. \]

**Remark 2.1.** We list, for a future reference, a couple of remarks highlighting the relation between the \((p,q)\)-capacity and the classical Sobolev capacities \(\text{Cap}_p(\cdot), \text{Cap}_q(\cdot)\). In what follows, we tacitly understand that K ⊆ Ω is a fixed compact set.

1. If K has vanishing q-capacity, then \(\text{Cap}_{p,q}(K) = 0\). To prove this fact it suffices to observe that, if D ⊆ Ω is a bounded open set containing K and if \(\varphi \in C_c^\infty(D)\) satisfies \(\varphi \geq 1\) on K, by the boundedness of a(⋅) we have

\[ \int_{\mathbb{R}^n} (|\nabla \varphi|^p + a(x)|\nabla \varphi|^q) \, dx \leq |\Omega|^{1-\frac{q}{p}} \left( \int_{\mathbb{R}^n} |\nabla \varphi|^q \, dx \right)^{\frac{p}{q}} + \|a\|_{L^\infty(\Omega)} \int_{\mathbb{R}^n} |\nabla \varphi|^q \, dx, \]

where \(|\cdot|\) denotes the classical Lebesgue measure in \(\mathbb{R}^N\).

2. If K has vanishing \((p,q)\)-capacity, then \(\text{Cap}_p(K) = 0\). To prove this fact it suffices to observe that, if D ⊆ Ω is a bounded open set containing K and if \(\varphi \in C_c^\infty(D)\) satisfies \(\varphi \geq 1\) on K, since \(a(\cdot) \geq 0\) in Ω we have

\[ \int_{\mathbb{R}^n} (|\nabla \varphi|^p + a(x)|\nabla \varphi|^q) \, dx \geq \int_{\mathbb{R}^n} |\nabla \varphi|^p \, dx. \]

In particular, if \(\text{Cap}_{p,q}(K) = 0\) we deduce that \(|K| = 0\).
2.2. Notations for the moving plane method. Let $\Gamma \subseteq \Omega \subseteq \mathbb{R}^N$ be as in the statement of Theorem 1.2, and let $u \in C^1(\overline{\Omega} \setminus \Gamma)$ be a solution of (1.2). For any fixed $\lambda \in \mathbb{R}$, we indicate by $R_\lambda$ the reflection through the hyperplane $\Pi_\lambda := \{x_1 = \lambda\}$, that is,

$$R_\lambda(x) = x_\lambda := (2\lambda - x_1, x_2, \ldots, x_N) \quad \text{for all } x \in \mathbb{R}^N;$$

accordingly, we define the function

$$u_\lambda(x) := u(x_\lambda), \quad \text{for all } x \in R_\lambda(\overline{\Omega} \setminus \Gamma).$$

We point out that, since $u$ solves (1.2) and $a$ is independent of $x_1$, one has

1. $u_\lambda \in C^1(R_\lambda(\overline{\Omega} \setminus \Gamma));$
2. $u_\lambda > 0$ in $R_\lambda(\Omega \setminus \Gamma)$ and $u_\lambda \equiv 0$ on $R_\lambda(\partial \Omega \setminus \Gamma);$  
3. for every test function $\varphi \in C^1_c(R_\lambda(\Omega \setminus \Gamma))$ one has

$$\int_{R_\lambda(\Omega)} (p|\nabla u_\lambda|^p - qa(x)|\nabla u_\lambda|^q) \langle \nabla u_\lambda, \nabla \varphi \rangle \, dx = \int_{R_\lambda(\Omega)} f(u_\lambda) \varphi \, dx.$$

To proceed further, we let

$$a = a_\Omega := \inf_{x \in \Omega} x_1$$

and we observe that, since $\Omega$ is (bounded and) symmetric with respect to the $x_1$-direction, we certainly have $-\infty < a < 0$. Hence, for every $\lambda \in (a, 0)$ we can set

$$\Omega_\lambda := \{x \in \Omega : x_1 < \lambda\}.$$

Notice that the convexity of $\Omega$ in the $x_1$-direction ensures that

$$\Omega_\lambda \subseteq R_\lambda(\Omega) \cap \Omega.$$

Finally, for every $\lambda \in (a, 0)$ we define the function

$$w_\lambda(x) := (u - u_\lambda)(x), \quad \text{for } x \in (\overline{\Omega} \setminus \Gamma) \cap R_\lambda(\overline{\Omega} \setminus \Gamma).$$

On account of (2.4), $w_\lambda$ is surely well-posed on $\overline{\Omega_\lambda} \setminus R_\lambda(\Gamma)$.

2.3. Auxiliary results. From now on, we assume that all the hypotheses of Theorem 1.2 are satisfied. Moreover, we tacitly inherit all the notation introduced so far.

To begin with, we remind some identities between vectors in $\mathbb{R}^N$ which are very useful in dealing with quasilinear operators: for every $s \in (1, 2)$ there exist constants $C_1, C_2 > 0$, only depending on $s$, such that, for every $\eta, \eta' \in \mathbb{R}^N$, one has

$$\langle |\eta|^{s-2} \eta - |\eta'|^{s-2} \eta', \eta - \eta' \rangle \geq C_1 (|\eta| + |\eta'|)^{s-2} |\eta - \eta'|^2,$$

$$|\eta|^{s-2} \eta - |\eta'|^{s-2} \eta' | \leq C_2 |\eta - \eta'|^{s-1}.$$

We refer, e.g., to [14] for a proof of (2.6).

Next, we need to define an ad-hoc family of Sobolev functions in $\Omega$ allowing us to ‘cut off’ of the singular set $\Gamma$. To this end, let $\varepsilon > 0$ be small enough such that

$$B_\varepsilon^\lambda := \{x \in \mathbb{R}^N : \text{dist}(x, R_\lambda(\Gamma)) < \varepsilon\} \subseteq \Omega.$$
As for the classical capacity, being \( R_\lambda \) an affine map, we see that

\[ \text{Cap}_{p,q}(R_\lambda(\Gamma)) = 0. \]

then, by definition, there exists \( \varphi_\varepsilon \in C^\infty_c(\mathcal{B}_\epsilon^\lambda) \) such that

\[ \varphi_\varepsilon \geq 1 \text{ on } R_\lambda(\Gamma) \quad \text{and} \quad \int_{\mathcal{B}_\epsilon^\lambda} (|\nabla \varphi_\varepsilon|^p + a(x)|\nabla \varphi_\varepsilon|^q) \, dx < \varepsilon. \]

We then consider the Lipschitz functions

- \( T(s) := \max\{0; \min\{s; 1\}\} \) (for \( s \in \mathbb{R} \)),
- \( g(t) := \max\{0; -2s + 1\} \) (for \( t \geq 0 \))

and we define, for \( x \in \mathbb{R}^N \),

\[ \psi(x) := g(T(\varphi_\varepsilon(x))). \]

In view of (2.7), and taking into account the very definitions of \( T \) and \( g \), it is not difficult to recognize that \( \psi_\varepsilon \) satisfy the following properties:

1. \( \psi_\varepsilon \equiv 1 \) on \( \mathbb{R}^N \setminus \mathcal{B}_\epsilon^\lambda \) and \( \psi_\varepsilon \equiv 0 \) on some neighborhood of \( R_\lambda(\Gamma) \), say \( \mathcal{V}_\varepsilon^\lambda \subseteq \mathcal{B}_\epsilon^\lambda \);
2. \( 0 \leq \psi_\varepsilon \leq 1 \) on \( \mathbb{R}^N \);
3. \( \psi_\varepsilon \) is Lipschitz-continuous in \( \mathbb{R}^N \), so that \( \psi_\varepsilon \in W^{1, \infty}(\mathbb{R}^N) \);
4. there exists a constant \( C > 0 \), independent of \( \varepsilon \), such that

\[ \int_{\mathbb{R}^N} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx \leq C\varepsilon. \]

With the family \( \{\psi_\varepsilon\}_\varepsilon \) at hand, we can prove the following key lemma.

**Lemma 2.2.** Let \( 1 < p < q < 2 \). For any fixed \( \lambda \in (a, 0) \) we have

\[ \int_{\Omega_\lambda} (p(|\nabla u| + |\nabla u_\lambda|)^{q-2} + qa(x)(|\nabla u| + |\nabla u_\lambda|)^{q-2}) \cdot |\nabla w_\lambda^+|^2 \, dx \leq c_0, \]

where \( c_0 > 0 \) is a constant only depending on \( p, q, \lambda \) and \( \|u\|_{L^\infty(\Omega_\lambda)} \).

**Proof.** For every fixed \( \varepsilon > 0 \), we consider the function

\[ \varphi_\varepsilon(x) := \begin{cases} w_\lambda^+(x) \psi_\varepsilon^{p+q}(x) = (u - u_\lambda)^+(x) \psi_\varepsilon^{p+q}(x), & \text{if } x \in \Omega_\lambda, \\ 0, & \text{otherwise.} \end{cases} \]

We claim that the following assertions hold:

- (i) \( \varphi_\varepsilon \in \text{Lip}(\mathbb{R}^N) \);
- (ii) \( \text{supp}(\varphi_\varepsilon) \subseteq \Omega_\lambda \) and \( \varphi_\varepsilon \equiv 0 \) near \( R_\lambda(\Gamma) \).

In fact, since \( u \in C^1(\overline{\Omega_\lambda}) \) and \( u_\lambda \in C^1(\overline{\Omega_\lambda} \setminus R_\lambda(\Omega)) \), we have \( w_\lambda^+ \in \text{Lip}(\overline{\Omega_\lambda} \setminus V) \) for every open set \( V \supseteq R_\lambda(\Gamma) \); as a consequence, reminding that \( \psi_\varepsilon \in \text{Lip}(\mathbb{R}^N) \) and \( \psi_\varepsilon \equiv 0 \) on a neighborhood of \( R_\lambda(\Gamma) \), we get \( \varphi_\varepsilon \in \text{Lip}(\overline{\Omega_\lambda}) \). On the other hand, since \( \varphi_\varepsilon \equiv 0 \) on \( \partial \Omega_\lambda \), we easily conclude that \( \varphi_\varepsilon \in \text{Lip}(\mathbb{R}^N) \), as claimed. As for assertion (ii), it is a direct consequence of the very definition of \( \varphi_\varepsilon \) and of the fact that

\[ \psi_\varepsilon \equiv 0 \text{ on } \mathcal{V}_\varepsilon^\lambda \supseteq R_\lambda(\Gamma). \]
On account of properties (i)-(ii) of $\varphi_\varepsilon$, a standard density argument allows us to use $\varphi_\varepsilon$ as a test function both in (1.3) and (2.3); reminding that $a$ is independent of $x_1$, this gives

$$
p \int_{\Omega_\lambda} \langle |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \varphi_\varepsilon \rangle \, dx
+ q \int_{\Omega_\lambda} a(x) \langle |\nabla u|^{q-2} \nabla u - |\nabla u_\lambda|^{q-2} \nabla u_\lambda, \nabla \varphi_\varepsilon \rangle \, dx
= \int_{\Omega_\lambda} (f(u) - f(u_\lambda)) \varphi_\varepsilon \, dx.
$$

By unraveling the very definition of $\varphi_\varepsilon$, we then obtain

$$
p \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \cdot \langle |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla w_\lambda^+ \rangle \, dx
+ q \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \cdot a(x) \langle |\nabla u|^{q-2} \nabla u - |\nabla u_\lambda|^{q-2} \nabla u_\lambda, \nabla \psi_\varepsilon \rangle \, dx
+ p(p + q) \int_{\Omega_\lambda} w_\lambda^+ \cdot \psi_\varepsilon^{p+q-1} \cdot \langle |\nabla u|^{p-2} \nabla u - |\nabla u_\lambda|^{p-2} \nabla u_\lambda, \nabla \psi_\varepsilon \rangle \, dx
+ q(p + q) \int_{\Omega_\lambda} w_\lambda^+ \cdot \psi_\varepsilon^{p+q-1} \cdot a(x) \langle |\nabla u|^{q-2} \nabla u - |\nabla u_\lambda|^{q-2} \nabla u_\lambda, \nabla \psi_\varepsilon \rangle \, dx
= \int_{\Omega_\lambda} (f(u) - f(u_\lambda)) w_\lambda^+ \psi_\varepsilon^{p+q} \, dx.
$$

We now observe that the integral in the right-hand side of (2.8) is actually performed on the set $\mathcal{O}_\lambda := \{ x \in \Omega_\lambda : u \geq u_\lambda \} \setminus R_\lambda(\Gamma)$; moreover, for every $x \in \mathcal{O}_\lambda$ we have

$$
0 \leq u_\lambda(x) \leq u(x) \leq \|u\|_{L^\infty(\Omega_\lambda)}.
$$

As a consequence, since $f$ is locally Lipschitz-continuous on $\mathbb{R}$, we have

$$
\int_{\Omega_\lambda} (f(u) - f(u_\lambda)) w_\lambda^+ \psi_\varepsilon^{p+q} \, dx
= \int_{\Omega_\lambda} \frac{f(u) - f(u_\lambda)}{u - u_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^{p+q} \, dx
\leq L \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^{p+q} \, dx,
$$

(2.9)
where \( L = L(f, u, \lambda) > 0 \) is the Lipschitz constant of \( f \) on the interval \([0, \|u\|_{L^\infty(\Omega, \lambda)}]} \subseteq \mathbb{R} \). Starting from (2.8) and using both (2.9) and the estimates in (2.6), we then get

\[
C_1 \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \left\{ p(\|
abla u\| + \|\nabla u_\lambda\|)^{p-2} + qa(x) (\|
abla u\| + \|\nabla u_\lambda\|)^{q-2} \right\} \cdot |\nabla w_\lambda^{+}|^2 \, dx \\
\leq p \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} (\|\nabla u\|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla w_\lambda^{+}) \, dx \\
+ q \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} a(x) (|\nabla u|^{q-2}\nabla u - |\nabla u_\lambda|^{q-2}\nabla u_\lambda, \nabla w_\lambda^{+}) \, dx \\
\leq p(p + q) \int_{\Omega_\lambda} w_\lambda^{+} \psi_\varepsilon^{p+q} |\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda| \|
abla \psi_\varepsilon\| \, dx \\
+ q(p + q) \int_{\Omega_\lambda} w_\lambda^{+} \psi_\varepsilon^{p+q} a(x) \|\nabla u|^{q-2}\nabla u - |\nabla u_\lambda|^{q-2}\nabla u_\lambda| \|
abla \psi_\varepsilon\| \, dx \\
+ L_f \int_{\Omega_\lambda} (w_\lambda^{+})^2 \psi_\varepsilon^{p+q} \, dx \\
\leq C_2 p(p + q) \int_{\Omega_\lambda} |\nabla w_\lambda^{+}|^{p-1}\psi_\varepsilon^{p+q-1}|\nabla \psi_\varepsilon| w_\lambda^{+} \, dx \\
+ C_2 q(p + q) \int_{\Omega_\lambda} a(x)|\nabla w_\lambda^{+}|^{q-1}\psi_\varepsilon^{p+q-1}|\nabla \psi_\varepsilon| w_\lambda^{+} \, dx \\
+ L_f \int_{\Omega_\lambda} (w_\lambda^{+})^2 \psi_\varepsilon^{p+q} \, dx \\
\leq C_0 \left( I_p + I_q + \int_{\Omega_\lambda} (w_\lambda^{+})^2 \psi_\varepsilon^{p+q} \, dx \right),
\]

where \( C_0 = C_0(p, q, \lambda, \|u\|_{L^\infty(\Omega)}, f) > 0 \) is a suitable constant and

\[
I_p := \int_{\Omega_\lambda} |\nabla w_\lambda^{+}|^{p-1}\psi_\varepsilon^{p+q-1}|\nabla \psi_\varepsilon| w_\lambda^{+} \, dx, \\
I_q := \int_{\Omega_\lambda} a(x)|\nabla w_\lambda^{+}|^{q-1}\psi_\varepsilon^{p+q-1}|\nabla \psi_\varepsilon| w_\lambda^{+} \, dx.
\]

We now proceed to estimate both \( I_p \) and \( I_q \).

To begin with, we split the set \( \Omega_\lambda \) as \( \Omega_\lambda = \Omega_\lambda^{(1)} \cup \Omega_\lambda^{(2)} \), where

\[
\Omega_\lambda^{(1)} = \{ x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : |\nabla u_\lambda(x)| < 2|\nabla u| \} \quad \text{and} \\
\Omega_\lambda^{(2)} = \{ x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : |\nabla u_\lambda(x)| \geq 2|\nabla u| \};
\]

accordingly, since Remark 2.1-(2) ensures that \(|R_\lambda(\Gamma)| = 0\), we write

\[
I_p = I_{p,1} + I_{p,2}, \quad \text{with} \quad I_{p,i} = \int_{\Omega_\lambda^{(i)}} \{ \cdots \} \, dx \quad (i = 1, 2).
\]

We then proceed by estimating \( I_{p,1}, I_{p,2} \) separately.
Step I: Estimate of $I_{p,1}$. By definition, for every $x \in \Omega_{\lambda}^{(1)}$ we have

$$|\nabla u_{\lambda}(x)| + |\nabla u(x)| < 3|\nabla u(x)|;$$

Using the Hölder inequality with conjugate exponents $(\frac{p}{p-1}, p)$ and (2.11), we get

$$I_{p,1} \leq \left( \int_{\Omega_{\lambda}^{(1)}} |\nabla w_{\lambda}^+|^{\frac{p(p+q-1)}{p-1}} \psi_{\epsilon} \, d\mathbf{x} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_{\lambda}^{(1)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x} \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{\Omega_{\lambda}^{(1)}} (|\nabla u| + |\nabla u_{\lambda}|)^{p} \psi_{\epsilon}^{\frac{p(p+q-1)}{p-1}} \, d\mathbf{x} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_{\lambda}^{(1)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x} \right)^{\frac{1}{p}}$$

$$\leq 3^{p-1} \left( \int_{\Omega_{\lambda}^{(1)}} |\nabla u|^{p} \psi_{\epsilon}^{\frac{p(p+q-1)}{p-1}} \, d\mathbf{x} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_{\lambda}^{(1)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x} \right)^{\frac{1}{p}}$$

$$\leq C \left( \int_{\Omega_{\lambda}} |\nabla u|^{p} \, d\mathbf{x} \right)^{\frac{p-1}{p}} \left( \int_{\Omega_{\lambda}} (|\nabla \psi_{\epsilon}|^p + a(x)|\nabla \psi_{\epsilon}|^q) \, d\mathbf{x} \right)^{\frac{1}{p}},$$

where $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_{\lambda})}) > 0$.

Step II: Estimate of $I_{p,2}$. By definition, for every $x \in \Omega_{\lambda}^{(2)}$ we have

$$\frac{1}{2} |\nabla u_{\lambda}| \leq |\nabla u_{\lambda}| - |\nabla u| \leq |\nabla w_{\lambda}| \leq |\nabla u_{\lambda}| + |\nabla u| \leq \frac{3}{2} |\nabla u_{\lambda}|.$$ 

Using the weighted Young inequality with conjugate exponents $(\frac{p}{p-1}, p)$ and (2.13), for every $\rho > 0$ we get

$$I_{p,2} \leq C \rho \int_{\Omega_{\lambda}^{(2)}} |\nabla w_{\lambda}^+|^{\frac{p(p+q-1)}{p-1}} \psi_{\epsilon} \, d\mathbf{x} + \frac{C}{\rho^{p-1}} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x}$$

$$\leq C \rho \int_{\Omega_{\lambda}^{(2)}} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla u_{\lambda}|^{2} \psi_{\epsilon}^{\frac{p(p+q-1)}{p-1}} \, d\mathbf{x} + \frac{C}{\rho^{p-1}} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x}$$

$$\leq C \rho \int_{\Omega_{\lambda}^{(2)}} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla u_{\lambda}|^{2} \psi_{\epsilon}^{\frac{p(p+q-1)}{p-1}} \, d\mathbf{x} + \frac{C}{\rho^{p-1}} \int_{\Omega_{\lambda}^{(2)}} |\nabla \psi_{\epsilon}|^{p} (w_{\lambda}^+)^p \, d\mathbf{x}$$

$$\leq C \rho \int_{\Omega_{\lambda}} (p (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_{\lambda}|)^{q-2}) |\nabla w_{\lambda}^+|^{2} \psi_{\epsilon}^{p+q} \, d\mathbf{x}$$

$$+ \frac{C}{\rho^{p-1}} \int_{\Omega_{\lambda}} (|\nabla \psi_{\epsilon}|^p + a(x)|\nabla \psi_{\epsilon}|^q) \, d\mathbf{x},$$
where $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}) > 0$. Arguing similarly, just keeping track of the term $a(x)$, we get an analogous estimate for $I_q$, which reads as follows: for every $\rho > 0$,

$$
I_q \leq C \left( \int_{\Omega_\lambda} |\nabla u|^q \, dx \right)^{\frac{q-1}{q}} \left( \int_{\Omega_\lambda} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx \right)^{\frac{1}{p}} 
+ C \rho \int_{\Omega_\lambda} \left( p (|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_\lambda|)^{q-2} \right) |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q} \, dx
+ \frac{C}{\rho^{p-1}} \int_{\Omega_\lambda} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx,
$$

where $C = C(p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}, \|a\|_{L^\infty(\Omega_\lambda)}) > 0$.

Going back to (2.10), and gathering (2.12), (2.14) and (2.15), we finally derive

$$
C_1 \int_{\Omega_\lambda} \psi_\varepsilon^{p+q} \{ p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_\lambda|)^{q-2} \} \cdot |\nabla w_\lambda^+|^2 \, dx
\leq C \left( \int_{\Omega_\lambda} |\nabla u|^p \, dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega_\lambda} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx \right)^{\frac{1}{p}} 
+ C \left( \int_{\Omega_\lambda} |\nabla u|^q \, dx \right)^{\frac{q-1}{q}} \left( \int_{\Omega_\lambda} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx \right)^{\frac{1}{q}} 
+ C \rho \int_{\Omega_\lambda} \{ p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_\lambda|)^{q-2} \} |\nabla w_\lambda^+|^2 \psi_\varepsilon^{p+q} \, dx
+ C \left( \frac{1}{\rho^{p-1}} + \frac{1}{\rho^{q-1}} \right) \int_{\Omega_\lambda} (|\nabla \psi_\varepsilon|^p + a(x)|\nabla \psi_\varepsilon|^q) \, dx + C_0 \int_{\Omega_\lambda} (w_\lambda^+)^2 \psi_\varepsilon^{p+q} \, dx,
$$

for every $\rho > 0$ and for a suitable positive constant $C$ only depending on $p, \lambda, \|u\|_{L^\infty(\Omega_\lambda)}$ and $\|a\|_{L^\infty(\Omega_\lambda)}$. We are finally ready to conclude the proof: in fact, choosing $\rho > 0$ in such a way that

$$
C_1 - C \rho < \frac{1}{2},
$$

and letting $\varepsilon \to 0$ with the aid of Fatou’s lemma (remind the properties (1)-to-(4) of the function $\psi_\varepsilon$ and that $u \in C^1(\overline{\Omega_\lambda})$ if $\lambda < 0$), we obtain

$$
\int_{\Omega_\lambda} (p(|\nabla u| + |\nabla u_\lambda|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_\lambda|)^{q-2} \} |\nabla w_\lambda^+|^2 \, dx \leq C_0 \int_{\Omega_\lambda} (w_\lambda^+)^2 \, dx =: c_0,
$$

and $c_0$ depends only on $p, q, \lambda$ and the $L^\infty$-norm of $u$. This is ends the proof. \qed

Another key tool for the proof of Theorem 1.2 is the upcoming Lemma 2.3 which has been proved in [6, Lemma 2.5] for every $1 < p < q < N$. To better understand this result, we first introduce a notation: for every fixed $\lambda \in (a, 0)$, we define

$$
Z_\lambda := \{ x \in \Omega_\lambda \setminus R_\lambda(\Gamma) : \nabla u(x) = \nabla u_\lambda(x) = 0 \}.
$$

We also notice that, since $u, u_\lambda \in C^1(\Omega_\lambda \setminus R_\lambda(\Gamma))$, the set $Z_\lambda$ is closed (in $\Omega_\lambda$).
Lemma 2.3. Let $\lambda \in (a, 0)$ and let $C_{\lambda} \subseteq \Omega_{\lambda} \setminus (R_{\lambda}(\Gamma) \cup Z_{\lambda})$ be a connected component of (the open set) $\Omega_{\lambda} \setminus (R_{\lambda}(\Gamma) \cup Z_{\lambda})$. If $u \equiv u_{\lambda}$ in $C_{\lambda}$, then $C_{\lambda} = \emptyset$.

Finally, we will use the following key technical result.

Lemma 2.4. Let $1 < p < q < 2$ be fixed, and let $\lambda \in (a, 0)$ Then, it is possible to find a constant $c > 0$ such that the following estimate holds:

$$
\int_{\Omega_{\lambda}} |\nabla w_{\lambda}^+|^p \, dx \leq c \left( \int_{\Omega_{\lambda}} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla w_{\lambda}^+|^2 \, dx \right)^{\frac{p}{2}}.
$$

Here, $u_{\lambda}$ and $w_{\lambda}$ are as in (2.2) and (2.5), respectively.

Taking into account the previous Lemma 2.2, the proof of Lemma 2.4 is totally analogous to that of [7, Lemma 2.4], and therefore we skip it.

3. Proof of Theorem 1.2

Proof of Theorem 1.2. By assumptions, the singular set $\Gamma$ is contained in the hyperplane $\{x_1 = 0\}$, then the moving plane procedure can be started in the standard way, see e.g [22] for the $p$-laplacian case, by using the weak comparison principle in small domains, see [32, Theorem 4.3]. Indeed, for $a < \lambda < a + \tau$ with $\tau > 0$ small enough, the singularity does not play any role. Therefore, recalling that $w_{\lambda}$ has a singularity at $\Gamma$ and at $R_{\lambda}(\Gamma)$, we have that $w_{\lambda} \leq 0$ in $\Omega_{\lambda}$. To proceed further we define

$$
\Lambda_0 = \{a < \lambda < 0 : u \leq u_t \text{ in } \Omega_t \setminus R_t(\Gamma) \text{ for all } t \in (a, \lambda]\}
$$

and $\lambda_0 = \sup \Lambda_0$, since we proved above that $\Lambda_0$ is not empty. To prove our result we have to show that $\lambda_0 = 0$. To do this we assume that $\lambda_0 < 0$ and we reach a contradiction by proving that $u \leq u_{\lambda_0 + \tau}$ in $\Omega_{\lambda_0 + \tau} \setminus R_{\lambda_0 + \tau}(\Gamma)$ for any $0 < \tau < \bar{\tau}$ for some small $\bar{\tau} > 0$. We remark that $|Z_{\lambda_0}| = 0$, see [16, 32]. Let us take $H_{\lambda_0} \subset \Omega_{\lambda_0}$ be an open set such that

$$
Z_{\lambda_0} \cap \Omega_{\lambda_0} \subset H_{\lambda_0} \subset \subset \Omega.
$$

We note that the existence of such a set is guaranteed by the Hopf lemma, see, e.g., [6, Theorem A.2]. Moreover note that, since $|Z_{\lambda_0}| = 0$, we can take $H_{\lambda_0}$ of arbitrarily small measure. By continuity we know that $u \leq u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. We can exploit the classical strong comparison principle (see [34, Theorem 1] and [6, Theorem A.1]) to get that, in any connected component of $\Omega_{\lambda_0} \setminus Z_{\lambda_0}$, we have

$$
u < u_{\lambda_0} \quad \text{or} \quad u \equiv u_{\lambda_0}.
$$

The case $u \equiv u_{\lambda_0}$ in some connected component $C_{\lambda_0}$ of $\Omega_{\lambda_0} \setminus Z_{\lambda_0}$ is not possible, since by symmetry, it would imply the existence of a local symmetry phenomenon and consequently that $\Omega \setminus Z_{\lambda_0}$ would be not connected, in spite of what we proved in Lemma 2.3. Hence we deduce that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus R_{\lambda_0}(\Gamma)$. Therefore, given a compact set $K \subset \Omega_{\lambda_0} \setminus (R_{\lambda_0}(\Gamma) \cup H_{\lambda_0})$, by uniform continuity we can ensure that $u < u_{\lambda_0 + \tau}$ in $K$ for any $0 < \tau < \bar{\tau}$ for some
small $\bar{r} > 0$. Note that to do this we implicitly assume, with no loss of generality, that $R_{\lambda_0}(\Gamma)$ remains bounded away from $\mathcal{K}$.

Arguing in a similar fashion as in Lemma 2.2, we consider

$$
\varphi_\varepsilon := w_{\lambda_0+\tau}^+\psi_\varepsilon^{p+q} \cdot 1_{\Omega_{\lambda_0+\tau}} = \begin{cases} 
    w_{\lambda_0+\tau}^+\psi_\varepsilon^{p+q}, & \text{in } \Omega_{\lambda_0+\tau}, \\
    0, & \text{otherwise}.
\end{cases}
$$

By density arguments as above, we plug $\varphi_\varepsilon$ as test function in (1.3) and (2.3) so that, subtracting, we get

$$
\int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} \langle |\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla w_{\lambda_0+\tau}^+ \rangle \psi_\varepsilon^{p+q} \, dx 
+ \int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} \langle |\nabla u|^{q-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{q-2}\nabla u_{\lambda_0+\tau}, \nabla w_{\lambda_0+\tau}^+ \rangle \psi_\varepsilon^{p+q} \, dx 
+ (p + q) \int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} \langle |\nabla u|^{p-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{p-2}\nabla u_{\lambda_0+\tau}, \nabla \psi_\varepsilon \rangle \psi_\varepsilon^{p+q-1} w_{\lambda_0+\tau}^+ \, dx 
+ (p + q) \int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} \langle |\nabla u|^{q-2}\nabla u - |\nabla u_{\lambda_0+\tau}|^{q-2}\nabla u_{\lambda_0+\tau}, \nabla \psi_\varepsilon \rangle \psi_\varepsilon^{p+q-1} w_{\lambda_0+\tau}^+ \, dx 
= \int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} (f(u) - f(u_\lambda)) w_{\lambda_0+\tau}^+ \psi_\varepsilon^{p+q} \, dx.
$$

(3.1)

Now we split the set $\Omega_{\lambda_0+\tau}\backslash\mathcal{K}$ as the union of two disjoint subsets $\Omega_{\lambda_0+\tau}^{(1)}$ and $\Omega_{\lambda_0+\tau}^{(2)}$ such that $\Omega_{\lambda_0+\tau}\backslash\mathcal{K} = \Omega_{\lambda_0+\tau}^{(1)} \cup \Omega_{\lambda_0+\tau}^{(2)}$. In particular, we set

$$
\Omega_{\lambda_0+\tau}^{(1)} := \{ x \in \Omega_{\lambda_0+\tau}\backslash\mathcal{K} : |\nabla u_{\lambda_0+\tau}(x)| < 2|\nabla u(x)| \} \quad \text{and} \\
\Omega_{\lambda_0+\tau}^{(2)} := \{ x \in \Omega_{\lambda_0+\tau}\backslash\mathcal{K} : |\nabla u_{\lambda_0+\tau}(x)| \geq 2|\nabla u(x)| \}.
$$

From (3.1) and using (2.6), repeating verbatim arguments along the proof of Lemma 2.2, we get

$$
\int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} (p(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{p-2} + qa(x)(|\nabla u| + |\nabla u_{\lambda_0+\tau}|)^{q-2}) |\nabla w_{\lambda_0+\tau}^+|^2 \, dx 
\leq C_0 \int_{\Omega_{\lambda_0+\tau}\backslash\mathcal{K}} (w_{\lambda_0+\tau}^+)^2 \, dx,
$$

(3.2)
where \( C_0 = C_0(p, q, \lambda_0, \tau, \|u\|_{L^\infty(\Omega)}, f) \). Clearly, the left hand side can estimate from below as follows

\[
\int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda_0 + \tau}|)^{p-2} |\nabla w^+_{\lambda_0 + \tau}|^2 \, dx
\]

(3.3)

\[
\leq \int_{\Omega_{\lambda_0 + \tau} \setminus K} \left( p(|\nabla u| + |\nabla u_{\lambda_0 + \tau}|)^{p-2} + qa(x) (|\nabla u| + |\nabla u_{\lambda_0 + \tau}|)^{q-2} \right) |\nabla w^+_{\lambda_0 + \tau}|^2 \, dx.
\]

For the right hand side, we apply Hölder’s inequality with exponents \((\frac{p^*-2}{p^*}, \frac{p^*}{2})\), the classical Sobolev inequality and Lemma 2.4: this gives

\[
C_0 \int_{\Omega_{\lambda_0 + \tau} \setminus K} (w^+_{\lambda_0 + \tau})^2 \, dx \leq C |\Omega_{\lambda_0 + \tau} \setminus K| \frac{p^*-2}{p^*} \left( \int_{\Omega_{\lambda_0 + \tau} \setminus K} |w^+_{\lambda_0 + \tau}|^p \, dx \right)^\frac{2}{p^*}
\]

(3.4)

\[
\leq C |\Omega_{\lambda_0 + \tau} \setminus K| \frac{p^*-2}{p^*} \left( \int_{\Omega_{\lambda_0 + \tau} \setminus K} |\nabla w^+_{\lambda_0 + \tau}|^p \, dx \right)^\frac{2}{p^*}
\]

\[
= C |\Omega_{\lambda_0 + \tau} \setminus K| \frac{p^*-2}{p^*} \int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla w^+_{\lambda_0 + \tau}|^2 \, dx.
\]

Gathering together (3.2), (3.3) and (3.4), we get

\[
\int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda_0 + \tau}|)^{p-2} |\nabla w^+_{\lambda_0 + \tau}|^2 \, dx
\]

(3.5)

\[
\leq C |\Omega_{\lambda_0 + \tau} \setminus K| \frac{p^*-2}{p^*} \int_{\Omega_{\lambda_0 + \tau} \setminus K} (|\nabla u| + |\nabla u_{\lambda}|)^{p-2} |\nabla w^+_{\lambda_0 + \tau}|^2 \, dx.
\]

For \( \bar{\tau} \) small and \( K \) large, we may assume that

\[
C |\Omega_{\lambda_0 + \tau} \setminus K| \frac{p^*-2}{p^*} < 1.
\]

We can then deduce that

\[
\int_{\Omega_{\lambda_0 + \tau} \setminus K} (w^+_\lambda)^2 \, dx = \int_{\Omega_{\lambda_0 + \tau}} (w^+_\lambda)^2 \, dx = 0
\]

proving that \( u \leq u_{\lambda_0 + \tau} \) in \( \Omega_{\lambda_0 + \tau} \setminus R_{\lambda_0 + \tau}(\Gamma) \) for any \( 0 < \tau < \bar{\tau} \) for some small \( \bar{\tau} > 0 \). Such a contradiction shows that \( \lambda_0 = 0 \).

Since the moving plane procedure can be performed in the same way but in the opposite direction, then this proves the desired symmetry result. The fact that the solution is increasing in the \( x_1 \)-direction in \( \{x_1 < 0\} \) is implicit in the moving plane procedure. \( \square \)

**Remark 3.1.** By carefully scrutinizing the proof of Theorem 1.2, one can easily see that the key point in all the argument is estimate (3.5), which only depends on \( p \). For this
reason, we can prove Theorem 1.2 in the general case

\[ \frac{2N}{N+2} < p \leq q \]

by considering the following facts.

(a) When \( \frac{2N}{N+2} < p < 2 \) and \( q \geq 2 \) one can argue exactly as we did in this paper, up to modifying the proof of Lemma 2.2 to cover the case \( q \geq 2 \); this can be done by adapting the argument exploited in the proof of [6, Lemma 2.4].

(b) When \( q \geq p \geq 2 \), one can proceed exactly as in [6], up to slightly modifying the proof of [6, Lemma 2.4] to take into account the weaker assumption (1.4); this can be done by using the same approach exploited in the proof of Lemma 2.2.

References

[1] A.D. Alexandrov. A characteristic property of the spheres. Ann. Mat. Pura Appl. 58, (1962), 303–354.
[2] P. Baroni, M. Colombo and G. Mingione. Regularity for general functionals with double phase. Calc. Var. Partial Differential Equations 57, (2018), Paper No. 62.
[3] D. Baruah, P. Harjulehto, P. Hästö. Capacities in generalized Orlicz spaces. J. Funct. Spaces (2018), 10 pages.
[4] H. Berestycki and L. Nirenberg. On the method of moving planes and the sliding method. Bol. Soc. Brasil. Mat. (N.S.) 22, (1991), 1–37.
[5] S. Biagi, S. Dipierro, E. Valdinoci and E. Vecchi. Semilinear elliptic equations involving mixed local and nonlocal operators. Proc. Roy. Soc. Edinburgh Sect. A. 151(5), (2021), 1611–1641.
[6] S. Biagi, F. Esposito and E. Vecchi. Symmetry and monotonicity of singular solutions of double phase problems. J. Differential Equations 280, (2021), 435–463.
[7] S. Biagi, F. Esposito, L. Montoro and E. Vecchi. Symmetry and monotonicity of singular solutions to \( p \)-Laplacian systems involving a first order term. Preprint (2022).
[8] S. Biagi, E. Valdinoci and E. Vecchi. A symmetry result for elliptic systems in punctured domains. Commun. Pure Appl. Anal. 18, (2019), 2819–2833.
[9] S. Biagi, E. Valdinoci and E. Vecchi. A symmetry result for cooperative elliptic systems with singularities. Publ. Mat. 64, (2020), 621–652.
[10] L. Caffarelli, Y.Y. Li and L. Nirenberg. Some remarks on singular solutions of nonlinear elliptic equations. II: Symmetry and monotonicity via moving planes. Advances in geometric analysis, 97–105, Adv. Lect. Math. (ALM), 21, Int. Press, Somerville, MA, 2012.
[11] M. Colombo and G. Mingione. Bounded minimisers of double phase variational integrals. Arch. Ration. Mech. Anal. 215, (2015), 443–496.
[12] M. Colombo and G. Mingione. Calderón-Zygmund estimates and non-uniformly elliptic operators. J. Funct. Anal. 270, (2016), 1416–1478.
[13] Á. Crespo-Blanco, L. Gasiński, P. Harjulehto and P. Winkert. A new class of double phase variable exponent problems: Existence and uniqueness. J. Differential Equations 323, (2022), 182–228.
[14] L. Damascelli. Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. Ann. Inst. H. Poincaré Anal. Non Linéaire 15(4), (1998), 493–516.
15

[15] L. DAmascelli and F. Pacella. Monotonicity and symmetry of solutions of $p$-Laplace equations, $1 < p < 2$, via the moving plane method. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4)26, (1998), 689–707.

[16] L. DAmascelli and B. Sciunzi. Regularity, monotonicity and symmetry of positive solutions of $m$-Laplace equations. *J. Differential Equations* 206, (2004), 483–515.

[17] E. N. Dancer. Moving plane methods for systems on half spaces. *Math. Ann.* 342, (2008), 245–254.

[18] E. N. Dancer. Some notes on the method of moving planes. *Bull. Austral. Math. Soc.* 46, (1992), 425–434.

[19] C. De Filippis and G. Mingione. Manifold Constrained Non-uniformly Elliptic Problems. *J. Geom. Anal.* 30, (2020), 1661–1723.

[20] F. Esposito. Symmetry and monotonicity properties of singular solutions to some cooperative semilinear elliptic systems involving critical nonlinearity. *Discrete Contin. Dyn. Syst.* 40, (2020), 549–577.

[21] F. Esposito, A. Farina and B. Sciunzi. Qualitative properties of singular solutions to semilinear elliptic problems. *J. Differential Equations* 265, (2018), 1962–1983.

[22] F. Esposito, L. Montoro and B. Sciunzi. Monotonicity and symmetry of singular solutions to quasilinear problems. *J. Math. Pure Appl.* 126, (2019), 214–231.

[23] L. Esposito, F. Leonetti and G. Mingione. Sharp regularity for functionals with $(p, q)$-growth. *J. Differential Equations* 204, (2004), 5–55.

[24] A. Farina, L. Montoro, G. Riey and B. Sciunzi. Monotonicity of solutions to quasilinear problems with a first-order term in half-spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32, (2015), 1–22.

[25] A. Farina, L. Montoro and B. Sciunzi. Monotonicity of solutions of quasilinear degenerate elliptic equations in half-spaces. *Math. Ann.* 357, (2013), 855–893.

[26] B. Gidas, W. M. Ni and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* 68, (1979), 209–243.

[27] P. Marcellini. Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions. *Arch. Ration. Mech. Anal.* 105, (1989), 267–284.

[28] P. Marcellini. Regularity and existence of solutions of elliptic equations with $(p, q)$-growth conditions. *J. Differential Equations* 90, (1991), 1–30.

[29] L. Montoro, F. Punzo and B. Sciunzi. Qualitative properties of singular solutions to nonlocal problems. *Ann. Mat. Pura Appl.* 4, 197, (2018), 941–964.

[30] N. S. Papageorgiou and P. Winkert. Positive solutions for singular anisotropic $(p, q)$-equations. *J. Geom. Anal.* 31(12), (2021), 11849–11877.

[31] P. Poláčik, P. Quittner and P. Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.* 139, (2007), 555–579.

[32] G. Riey. Regularity and weak comparison principles for double phase quasilinear elliptic equations. *Discrete Contin. Dyn. Syst.* 39, (2019), 4863–4873.

[33] B. Sciunzi. On the moving plane method for singular solutions to semilinear elliptic equations. *J. Math. Pures Appl.* 108, (2017), 111–123.

[34] J. Serrin. On the strong maximum principle for quasilinear second order differential inequalities. *J. Funct. Anal.* 5, (1970), 184–193.

[35] J. Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.* 43, (1971), 304–318.

[36] S. Terracini. On positive entire solutions to a class of equations with a singular coefficient and critical exponent. *Adv. Differential Equations* 1, (1996), 241–264.
[37] W.C. Troy. Symmetry properties in systems of semilinear elliptic equations. *J. Differential Equations* 42, (1981), 400–413.

[38] N.S. Trudinger, Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa* (3)27, (1973), 265–308.

[39] V.V. Zhykov. Averaging of functional of the calculus of variations and elasticity theory. *Izv. Akad. Nauk. SSSR Ser. Mat.*, 50, (1986), 675–710.