Hermitian Adjacency Matrices of Mixed Graphs

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Abstract

The traditional adjacency matrix of a mixed graph is not symmetric in general, hence its
eigenvalues may be not real. To overcome this obstacle, several authors have recently defined and
studied various Hermitian adjacency matrices of digraphs or mixed graphs. In this work we unify
previous work and offer a new perspective on the subject by introducing the concept of monographs.
Moreover, we consider questions of cospectrality.

Keywords: mixed graphs, oriented graphs, graph spectra, eigenvalues
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1 Introduction

Algebraic graph theory strives to relate the structural properties of graphs to the algebraic properties
of objects associated with them. Specifically, in spectral graph theory the eigenvalues and eigenvectors
of matrices associated with graphs are studied. Most traditionally, the object of interest would be
the adjacency matrix of some undirected graph, i.e., the square matrix \([a_{uv}]\) such that
\[ a_{uv} = 1 \] if there is an edge between vertices \(u\) and \(v\), otherwise \(a_{uv} = 0\). By construction, the adjacency matrix
of an undirected graph is symmetric. Hence theorems from linear algebra dealing with non-negative
symmetric matrices can be readily applied to obtain a number of desirable spectral properties. Most
notably, the spectrum of the adjacency matrix is real. Moreover, there exists a basis of pairwise
orthogonal eigenvectors.

However, when dealing with directed or mixed graphs the definition of the adjacency matrix needs
to be changed to accommodate the fact that the adjacency relation of vertices is no longer symmetric.
For a digraph we set \(a_{uv} = 1\) if there is an arc from vertex number \(u\) to \(v\) and
\(a_{vu} = 0\) otherwise. The loss of symmetry proves a serious impediment to relating algebraic and structural properties to one
another, cf. the survey [2].

Quite recently, the idea has been presented to modify the definition of the adjacency matrix of
a directed graph, using complex numbers, in such a way that it still properly reflects the adjacency
relation but at the same time constitutes a Hermitian matrix. Let us give an overview of some efforts
and results in this direction. In [3] the authors use the imaginary number \(i\) to specify \(a_{uv} = i\) and
\(a_{vu} = -i\) whenever there is an arc from \(u\) to \(v\), but not vice versa, further \(a_{uv} = 1\) whenever \(u\) and \(v\) are mutually adjacent.
Using this definition of a Hermitian adjacency matrix, it turns out that many results from algebraic graph theory known for undirected graphs also hold for directed graphs
or at least exist in a slightly modified or weaker version. For example, if the underlying undirected
graph of a given oriented graph is bipartite, then the spectrum of the Hermitian adjacency matrix
is symmetric with respect to zero, but – in contrast to the undirected case – the reverse is not true.

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Independently, the authors of [8] introduced the same notion of a Hermitian adjacency matrix and proved many fundamental results. Moreover, they considered the Gutman energy (which is the sum of the absolute values of all eigenvalues) of the Hermitian adjacency matrix. Refer to [6,7,10] for other related work.

The goal of the present paper is as follows. To begin with, we generalize and unify previous results. We will then introduce the concept of monographs, permitting us to view some of these results from a new perspective. Moreover, we will analyze under which conditions a mixed graph has identical spectra for different values of $\alpha$.

## 2 Preliminaries

All graphs considered hereafter shall not contain any loops or multiple edges. A mixed graph $D$ arises from partially orienting an undirected graph $G$, i.e. by turning some of the undirected edges into single arcs. Thus, between any two adjacent vertices $u, v$ of the vertex set $V(G)$ there exists either an arc from $u$ to $v$ (indicated by $u\rightarrow v$), an arc from $v$ to $u$ (indicated by $u\leftarrow v$), or an undirected edge (also called a digon) between $u$ and $v$ (indicated by $u-v$). Altogether, these arcs and digons form the edge set $E(D)$ of $D$. The graph $G$ is called the underlying graph $\Gamma(D)$ of the mixed graph $D$. Much of the traditional terminology (e.g. being regular, being connected, vertex degree $\deg(\cdot)$, maximum degree $\Delta$) that is used for undirected graphs simply carries over to mixed graphs, in the sense that, $D$ is said to have a property whenever $\Gamma(D)$ has this property. In particular, we say that a mixed graph contains a certain undirected subgraph (e.g. the path $P_k$ or the cycle $C_k$ on $k$ vertices) if $\Gamma(D)$ contains this subgraph. A mixed walk in $D$ is a sequence of vertices $v_1, \ldots, v_k$ of $D$ such that there is an edge between any two subsequent vertices $v_iv_{i+1}$ in $D$. The set of all arcs from some vertex $u$ to other vertices $v$ (resp. from other vertices to $u$) is denoted by $N_D^+(u)$ (resp. $N_D^-(u)$). The set of all digons incident with vertex $u$ is denoted by $N^0_D(u)$.

The (traditional) adjacency matrix $A(G) = [a_{ij}]$ of a given, either undirected or directed, graph $G$ on $n$ vertices is the real matrix of order $n \times n$ such that $a_{ij} = 1$ if there is an edge from $v_i$ to $v_j$ and $a_{ij} = 0$ otherwise. For directed graphs, the resulting matrix $A$ is usually non-symmetric, thus losing many desirable algebraic properties. We therefore define the following alternative:

**Definition 1.** Given a mixed graph $D$ and a unit complex number $\alpha$, i.e. $|\alpha| = 1$, we define the $\alpha$-Hermitian adjacency matrix $H^\alpha(D) = [h_{uv}]$ of $D$ by

$$h_{uv} = \begin{cases} 
1 & \text{if } u\rightarrow v, \\
\alpha & \text{if } u\leftarrow v, \\
\bar{\alpha} & \text{if } u-v, \\
0 & \text{otherwise.}
\end{cases} \quad (1)
$$

When there is no ambiguity regarding the reference graph $D$ we will often omit any symbolic reference to $D$, e.g. write $H^\alpha$ instead of $H^\alpha(D)$.

Clearly, the matrix $H^\alpha$ from Definition [1] is Hermitian, i.e. $(H^\alpha)^* = (H^\alpha)$ where $M^*$ denotes the conjugate transpose of matrix $M$. By $\chi_\alpha(D, x) = \det(xI - H^\alpha(D))$, where $I$ is the identity matrix, we denote the characteristic polynomial of the matrix $H^\alpha(D)$, calling this the $\alpha$-characteristic polynomial of $D$. The multiset $\sigma_\alpha(D)$ of all roots of $\chi_\alpha(D, x)$ is called the $\alpha$-spectrum of $D$, as opposed to the (traditional) spectrum $\sigma(\Gamma(D))$ of the underlying undirected graph $\Gamma(D)$. Consequently, we shall refer to the elements of $\sigma_\alpha(D)$ as the $\alpha$-eigenvalues of $D$. Likewise, we speak of $\alpha$-eigenvectors. Note that $\alpha$-eigenvalues are always real.

A direct consequence of Definition [1] is the following summation rule characterizing $\alpha$-eigenvectors:
Proposition 2. Let $D$ be a mixed graph. Then $x$ is an $\alpha$-eigenvector of $D$ corresponding to $\alpha$-eigenvalue $\lambda$ if and only if, for each $u \in V(D)$,

$$\lambda x(u) = \sum_{u \rightarrow v} x(v) + \left(\alpha \sum_{u \rightarrow v} x(v)\right) + \left(\bar{\alpha} \sum_{u \leftarrow v} x(v)\right).$$

(2)

Throughout this paper we shall assume $|\alpha| = 1$, i.e. $\alpha = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Moreover, we make use of the constants $\omega := e^{\frac{2\pi}{3}i}$ (a sixth root of unity) and $\gamma := e^{\frac{2\pi}{3}i}$ (a third root of unity). In the context of Hermitian adjacency matrices, the former constant has been endorsed in [9], whereas the suitability of the latter constant will become evident later on.

3 Characteristic Polynomial of $H^\alpha(D)$

In this section we will expand the determinant of $H^\alpha$ and study the $\alpha$-characteristic polynomial, in particular with respect to the three instances $H^i$, $H^w$ and $H^\gamma$.

A classic result from linear algebra, concerning determinant expansion, is the following:

Theorem 3. If $A = [a_{i,j}]$ is a square matrix of order $n$ then

$$\det(A) = \sum_{\eta \in S_n} \text{sgn}(\eta)a_{1,\eta(1)}a_{2,\eta(2)}a_{3,\eta(3)}\cdots a_{n,\eta(n)}.\tag{3}$$

Decades ago, the above theorem has been applied to adjacency matrices of graphs. The permutations over which the sum ranges can be put into correspondence with certain subgraphs of the given graph. To this end, we define the following terms and notation:

Definition 4. Let $D$ be a mixed graph.

1. $D$ is called elementary if, for every component $C$ of $D$, $\Gamma(C)$ is either isomorphic to $P_2$ or $C_k$ (for some $k \geq 3$).

2. Let $D$ be elementary. The rank of $D$ is defined as $r(D) = n - c$, where $n = |V(D)|$ and $c$ is the number of its components. The co-rank of $D$ is defined as $s(D) = m - r(D)$, where $m = |E(D)|$.

Note that the co-rank $s(D)$ is equal to the number of $C_k$ components of $D$.

Now we are ready to state the following classic theorem by Harary (cf. [4]):

Theorem 5 (Determinant expansion (Harary, 1962)). Let $D$ be a graph with adjacency matrix $A(G)$. Then,

$$\det(A(G)) = \sum_S (-1)^{r(S)}2^{s(S)},\tag{4}$$

where the sum ranges over all spanning elementary subgraphs $S$ of $G$.

Following the classic proof strategy used in Theorem 5 the result readily generalizes to any $\alpha$-Hermitian adjacency matrix. But first we require the following definition:

Definition 6. Let $D$ be a mixed graph and $H^\alpha(D) = [h_{uv}]$. With respect to this, the value $h_\alpha(W)$ of a mixed walk $W$ with vertices $v_1, v_2, \ldots, v_k$ is defined as

$$h_\alpha(W) = (h_{v_1v_2}h_{v_2v_3}h_{v_3v_4}\cdots h_{v_{k-1}v_k}) \in \{\alpha^r\}_{r \in \mathbb{Z}}.\tag{5}$$
Figure 1: A mixed graph where $\sigma_\gamma$, $\sigma_\omega$, $\sigma_i$ are different from one another

**Theorem 7** (Determinant expansion for $H^\alpha$). Let $D$ be a mixed graph. Then

$$\det(H^\alpha) = \sum_{D'} (-1)^{r(D')} 2^{s(D')} \Re \left( \prod_C h_\alpha(\C) \right),$$

where the sum ranges over all spanning elementary mixed subgraphs $D'$ of $D$, the product ranges over all mixed cycles $C$ in $D'$, and $\bar{C}$ is any closed walk traversing $C$.

**Proof.** Consider the matrix $H^\alpha$ and apply the classic proof strategy for determinant expansion on graphs, cf. the proof of Theorem 5 in [1].

Considering specific values of $\alpha$, the formula in Theorem 7 becomes more specific, too. For $\alpha = i$ we may rediscover a result given in [8]. Moreover, Theorem 7 immediately allows us to compute the $\alpha$-characteristic polynomial:

**Corollary 8.** If $\chi_\alpha(D, \lambda) = \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n$ is the $\alpha$-characteristic polynomial of a mixed graph $D$, then

$$(-1)^k c_k = \sum (-1)^{r(D')} 2^{s(D')} \Re \left( \prod_C h_\alpha(\C) \right),$$

where the sum ranges over all elementary mixed subgraphs $D'$ with $k$ vertices, the product ranges over all mixed cycles $C$ in $D'$, and $\bar{C}$ is any closed walk traversing $C$.

**Proof.** This follows immediately from the fact that $(-1)^k c_k$ equals the sum of all principal minors of $H^\alpha(D)$ with $k$ rows and columns.

### 4 Cospectrality

A recurring theme in algebraic graph theory is the hunt for pairs of non-isomorphic graphs having the same spectrum. Such graphs are called cospectral. In contrast to this, we shall look into the question under which conditions the same graph has identical $\alpha$-spectrum for different values of $\alpha$. It comes as no surprise that such spectra may be completely different:

**Example 9.** For the mixed graph shown in Figure 1 we have:

$$\sigma_\gamma = \{2.57083, -2.3222, 1.50413, -1.19239, -0.560369\}$$
$$\sigma_\omega = \{-2.93033, 2.30034, 1.15439, -0.832963, 0.308565\}$$
$$\sigma_i = \{-2.71687, 2.2803, 1.50739, -1.07082, 0.0\}$$

However, there exist mixed graphs exhibiting the same $\alpha$-spectrum for two different values of $\alpha$, say $\alpha_1, \alpha_2$. We call such a mixed graph $\alpha_1$-$\alpha_2$-cospectral. Let us give an example for a $\gamma$-$\omega$-cospectral mixed graph:
Example 10. The mixed graph $D$ shown in Figure 2 is $\gamma$-$\omega$-cospectral, i.e. $\sigma_\gamma(D) = \sigma_\omega(D)$. This is not difficult to see: With respect to Corollary 8 note that $D$ contains only one cycle. Moreover, $h_\gamma(c) \in \{\gamma, \gamma^2\}$ and $h_\omega(c) \in \{\omega^2, \omega^4\}$. Observing $\gamma = \omega^2$ we have $\chi_\gamma(D, \lambda) = \chi_\omega(D, \lambda)$. In contrast, we remark that $\chi_\alpha(D, \lambda) \neq \chi_i(D, \lambda)$, hence $\sigma_\alpha(D) \neq \sigma_i(D)$.

Note that for $\alpha = 1$ we have $H^\alpha = A(\Gamma(D))$, so the special case of $\alpha$-1-cospectrality is equivalent to asking whether the $\alpha$-spectrum of a mixed graph $D$ coincides with the traditional spectrum of its undirected counterpart $\Gamma(D)$. Thus Corollary 8 immediately gives rise to the following result:

Corollary 11. Let $T$ be a mixed tree. Then $\sigma_\alpha(T) = \sigma(\Gamma(T))$.

Proof. Trees do not contain cycles, hence using $\alpha = 1$ in (7) instead of the given value does not change the result. 

Since trees are $\alpha$-1-cospectral for any $\alpha$ we see that they are $\alpha_1$-$\alpha_2$-cospectral for arbitrary values $\alpha_1, \alpha_2$. Now, consider a mixed graph that contains cycles. Obviously, it does not matter for equation (7) if we use $\alpha = 1$ or some other specific value as long as (with respect to that other value) all factors in the involved products are equal to one. This motivates the following definition:

Definition 12. A mixed graph is an $\alpha$-monograph (of 1st kind) if $h_\alpha(\vec{C}) = 1$ for all its cycles $C$.

Trivially, trees are $\alpha$-monographs. By construction, Corollary 11 directly extends to monographs:

Theorem 13. Let $D$ be an $\alpha$-monograph (of 1st kind). Then, $\sigma_\alpha(D) = \sigma(\Gamma(D))$.

Regarding Corollary 8 and (7), note that $h_\alpha(\vec{C}) = \alpha^x\alpha^y = \alpha^{x-y}$, where $x$ (resp. $y$) is the number of forward (resp. backward) edges encountered while traversing $\vec{C}$. We will tacitly make use of this fact hereafter.

Corollary 14. Let $D$ be a connected mixed graph. If, for every cycle in $D$, the difference between the numbers of encountered forward arcs and the number of backward arcs (w.r.t. any traversal direction) is a multiple of the order of $\alpha$, then $D$ is an $\alpha$-monograph.

Corollary 15. A connected mixed graph $G$ is an $\alpha$-monograph for every value $\alpha$ if and only if every cycle in $D$ contains as many forward arcs as backward arcs.

Proof. The forward implication follows directly from Corollary 14. For the converse note that there exist values $\alpha$ of infinite order (i.e. $\alpha^j = \alpha^k$ only if $j = k$).

The properties of $\alpha$-monographs deserve further investigation. But beforehand, we will elaborate more on cospectrality. We start with the question when the $\alpha$-spectra for the special values $\gamma$ and $\omega$ coincide:
Theorem 16. Let \( D \) be a mixed graph. If, for any cycle in \( D \), the difference between the numbers of encountered forward arcs and the number of backward arcs (w.r.t. any traversal direction) is even, then \( D \) is \( \gamma\)-\( \omega \)-cospectral.

Proof. Under the given assumptions there will be only even powers of \( \alpha \) in (7). But \( \text{Re}(\gamma^{2k}) = \text{Re}(\omega^{2k}) \).

Corollary 17. Let \( D \) be a mixed bipartite graph. Suppose that every cycle in \( D \) contains an even number of digons. Then \( D \) is \( \gamma\)-\( \omega \)-cospectral.

Proof. A bipartite graph contains only even cycles. Consider such a cycle \( C \). Subtracting an even number of digons, we conclude that \( C \) contains an even number of arcs. Traversing \( C \), these arcs consist of an even number of forward arcs and an odd number of backward arcs – or vice versa. Hence Theorem 16 can be applied.

Corollary 18. Every oriented bipartite graph is \( \gamma\)-\( \omega \)-cospectral.

As already mentioned in Example 10, the graph in Figure 2 is \( \gamma\)-\( \omega \)-cospectral. This follows from Corollary 18 since it is an oriented bipartite graph.

Using the ideas from the proof of Corollary 17, one can generalize as follows:

Theorem 19. Let \( D \) be a mixed graph. Suppose that every even cycle in \( D \) contains an even number of digons and every odd cycle in \( D \) contains an odd number of digons. Then \( D \) is \( \gamma\)-\( \omega \)-cospectral.

We have shown how Corollary 8 can be used as a tool investigating cospectrality. In view of \( \alpha_1\)-\( \alpha_2 \)-cospectrality it is sufficient to require that, for every elementary mixed subgraph \( D' \), the real part of the associated product in (7) is the same for both values \( \alpha_1 \) and \( \alpha_2 \). As a slightly coarser condition one could require that \( h_{\alpha_1}(\tilde{C}) = h_{\alpha_2}(\tilde{C}) \) for all cycles \( C \) of \( D' \). In view of this, requiring uniform cospectrality for all values of \( \alpha_1, \alpha_2 \) (hence including value 1) amounts to the condition \( h_{\alpha}(\tilde{C}) = 1 \) stated in Definition 12. In view of this, one can devise modifications which, given some \( \alpha \)-monograph \( D \), can be used to construct arbitrarily many derived \( \alpha \)-monographs containing \( D \) as a subgraph:

Theorem 20. Let \( D \) be an \( \alpha \)-monograph. Let \( U \) be a connected undirected subgraph of \( D \). Fix a set \( M \) of new vertices and subsets \( V_x \subset V(U) \), for \( x \in M \). Connect each vertex \( x \in M \) to \( V_x \) such that either \( N^+_{\tilde{D}}(x) = V_x \) or \( N^-_{\tilde{D}}(x) = V_x \). Then the resulting mixed graph \( \tilde{D} \) is an \( \alpha \)-monograph.

Proof. The newly added vertices and their adjacent edges may introduce new cycles. Let \( C \) be such a cycle in \( \tilde{D} \) and \( \tilde{C} \) any traversal of \( C \). By construction, the predecessor \( v_x \) of \( x \) and its successor \( w_x \) (w.r.t. \( \tilde{C} \)) belong to \( U \). Consider the residual graph \( R \) obtained by removing all edges of \( U \) from \( D \). Every path \( W \) in \( R \) joining two vertices \( u_1, u_2 \) that originally belong to the subgraph \( U \) in \( D \) satisfies \( h_{\alpha}(W) = 1 \). To see this, add any path \( W' \) between \( u_1 \) and \( u_2 \) in \( U \) to obtain a cycle \( C' \) in \( D \). Choosing matching traversals for \( W \) and \( W' \), we get \( h_{\alpha}(C'') = h_{\alpha}(W)h_{\alpha}(W') \). Observe \( h_{\alpha}(W') = 1 \) since \( U \) is an undirected subgraph of \( D \). \( D \) is an \( \alpha \)-monograph, so \( h_{\alpha}(C'') = 1 \) and therefore \( h_{\alpha}(W) = 1 \). \( C \) can be segmented into paths of three possible types as follows: Paths within \( U \), paths within \( R \) and the segment from \( v_x \) via \( x \) to \( w_x \). The latter segment can only be \( v_x \rightarrow x \rightarrow w_x \) or \( v_x \leftarrow x \rightarrow w_x \). Both contribute a factor of 1 to the product \( h_{\alpha}(C') \), as do the two segment types mentioned first. Overall, \( h_{\alpha}(\tilde{C}) = 1 \).

Example 21. Figure 3 illustrates the construction mentioned in Theorem 20. The vertices no. 12 and 13 have been newly added.
5 Monograph Structure

Definition 12 characterizes $\alpha$-monographs by a condition concerning the traversal of cycles. In the following, we will render the implications of this condition more tangible, by way of studying mixed walks. To this end, let $D$ be a connected mixed graph. Fix any $u \in V(D)$ and consider some mixed walk $W$ in $D$, say $u = r_1, \ldots, r_k$. Denote the contained partial walks $r_1, \ldots, r_j$ by $W_j$ (for $j = 1, \ldots, k$). Consequently,

$$h_\alpha(W_1) = 1$$  \hspace{1cm} (8)

and

$$h_\alpha(W_{j+1}) = \begin{cases} h_\alpha(W_j) & \text{if } r_j \sim r_{j+1} \\ \alpha h_\alpha(W_j) & \text{if } r_j \rightarrow r_{j+1} \\ \bar{\alpha} h_\alpha(W_j) & \text{if } r_j \leftarrow r_{j+1} \end{cases}$$  \hspace{1cm} (9)

for $j = 1, \ldots, k - 1$.

Example 22. Figure 4 shows the values $h_\gamma(W_j)$ – each value written near the respective $j$-th vertex along the walk – for three different mixed walks $W$ in a mixed example graph.

Next, we state three useful basic properties of $h_\gamma$ with respect to mixed walks (some of which already implicit in the previous section).

Proposition 23. Let $W$ be a mixed walk containing $r$ forward arcs and $s$ backward arcs. Then $h_\alpha(W) = \alpha^r \bar{\alpha}^s$.

Proposition 24. Let $W'$ be a mixed walk and $W''$ the corresponding reverse walk. Let $W$ be the walk resulting from concatenating $W'$ and $W''$. Then $h_\alpha(W) = 1$.

Proposition 25. Let $W', W''$ be two mixed walks such that the final vertex of $W'$ is the start vertex of $W''$. Let $W$ be the walk resulting from concatenating $W'$ and $W''$. Then $h_\alpha(W) = h_\alpha(W') h_\alpha(W'')$.

As can be seen from Figure 4, $h_\alpha$ can be used to assign (possibly multiple) complex numbers to each of the vertices of a mixed graph. In particular, we are concerned about the possible values that get assigned to the start/end vertices of closed walks:

Definition 26. Let $D$ be a mixed graph. The $\alpha$-store $S^\alpha(u)$ of $u \in V(D)$ is defined as $S^\alpha(u) = \{ h_\alpha(W) : W \text{ is a closed walk in } D \text{ from/to } u \}$. Let $s^\alpha(u) = |S^\alpha(u)|$ denote the associated store size.
1 \in S^\alpha(u) \text{ so that } s^\alpha(u) \geq 1. \text{ As the following lemma shows, the store content is independent of } u, \text{ hence we may speak of ‘the’ } \alpha\text{-store of a mixed graph:}

**Lemma 27.** Let \( u,v \in V(D) \). Then \( S^\alpha(u) = S^\alpha(v) \).

**Proof.** Let \( W'' \) be a closed mixed walk from/to \( v \). Clearly, \( h_\alpha(W'') \in S^\alpha(v) \). Now let \( W' \) be a mixed walk from \( u \) to \( v \) and let \( W''' \) its reverse walk. Concatenating \( W', W'', W''' \) one obtains a closed mixed walk \( W \) from/to \( v \). Using Proposition 24 and Proposition 25 we get \( h_\alpha(W) = h_\alpha(W')h_\alpha(W'')h_\alpha(W''') = h_\alpha(W'') \), so that \( h_\alpha(W'') \in S^\alpha(u) \). Repeating the argument with the roles of \( u \) and \( v \) swapped yields \( S^\alpha(v) = S^\alpha(u) \).

**Theorem 28.** Let \( D \) be a connected mixed graph. Then the following statements are equivalent:

(i) \( s^\alpha(u) = 1 \) for at least one \( u \in V(D) \).

(ii) \( s^\alpha(u) = 1 \) for every \( u \in V(D) \).

(iii) \( h_\alpha(W') = h_\alpha(W'') \) for every pair \( W', W'' \) of mixed walks sharing the same start and end vertices.

(iv) \( D \) is an \( \alpha \)-monograph (of 1st kind).

**Proof.** Observing Definition 12 and Definition 26, this follows from Proposition 24, Proposition 25 and Lemma 27.

**Example 29.** The mixed graph shown in Figure 4 is not a \( \gamma \)-monograph, as opposed to the slightly different graph depicted in Figure 3. Notice how, in view of Theorem 28, every mixed walk with the same start/end vertex creates exactly the same walk values along the way.

In view of Theorem 28 one can characterize \( \alpha \)-monographs by the way their vertices can be partitioned:

**Theorem 30.** A connected mixed graph \( D \) is an \( \alpha \)-monograph (of 1st kind) if and only if \( V(D) \) can be partitioned into sets \( V_{\alpha^0}, V_{\alpha^1}, V_{\alpha^2}, \ldots \) (some of which possibly empty) such that there are no digons between any two sets \( V_{\alpha^{j_1}}, V_{\alpha^{j_2}} \) with \( j_1 \neq j_2 \) and every arc starting in a set \( V_{\alpha^j} \) ends in \( V_{\alpha^{j-1}} \).
Using the recursion (9) as well as equations (10) and (11), we deduce:

\[ y = [y_r]_{r \in V(D)} = [h_\alpha(v \rightarrow r)x_r]_{r \in V(D)}, \]

where \( v \rightarrow r \) is an arbitrary mixed walk from \( v \) to \( r \) in \( D \) (cf. Theorem 28 (iii)). Then \( y \) is an \( \alpha \)-eigenvector of \( D \) for eigenvalue \( \lambda \).

Proof. Clearly, for every vertex \( u \), the vector \( x \) satisfies the summation rule

\[ \lambda x_u = \sum_{r \in N_D(u)} x_r. \]

Using the recursion (9) as well as equations (10) and (11), we deduce:

\[ \lambda y_u = \lambda h_\alpha(v \rightarrow u)x_u \]
\[ = \sum_{r \in N_D(u)} h_\alpha(v \rightarrow u)x_r \]
\[ = \sum_{r \in N_D(u)} h_\alpha(v \rightarrow u)x_r + \sum_{r \in N^+_D(u)} h_\alpha(v \rightarrow u)x_r + \sum_{r \in N^-_D(u)} h_\alpha(v \rightarrow u)x_r \]
\[ = \sum_{r \in N_D(u)} h_\alpha(v \rightarrow r)x_r + \alpha \sum_{r \in N^+_D(u)} h_\alpha(v \rightarrow r)x_r + \alpha \sum_{r \in N^-_D(u)} h_\alpha(v \rightarrow r)x_r \]
\[ = \sum_{r \in N_D(u)} y_r + \alpha \sum_{r \in N^+_D(u)} y_r + \alpha \sum_{r \in N^-_D(u)} y_r. \]

Comparing this to the mixed summation rule (2) from Proposition 2, we see that \( y \) is as claimed. \( \square \)

Regarding Theorem 31, note that the construction (10) retains linear independence, i.e., every basis of eigenvectors of \( \Gamma(D) \) can be converted into a basis of \( \alpha \)-eigenvectors of \( D \) (using the same reference vertex \( v \) throughout).
Example 32. Figure 5b depicts an $\alpha$-monograph $D$. With respect to the indicated vertex order, $(-1, -1, 1, 1, 0, -1, 1)^T$ is an eigenvector of $\Gamma(D)$ for eigenvalue 0. Using equation (8) and the store values given in the figure, one obtains the $\gamma$-eigenvector $(-\gamma, -\gamma^2, \gamma^2, 1, 0, -1, \gamma^3)^T$ for eigenvalue 0 of $D$.

In preparation for the following section we define a variant of the function $h_\alpha$. As before, let $D$ be a connected mixed graph. Fix $u \in V(D)$ and let $W$ be a mixed walk $u = r_1, \ldots, r_k$ in $D$. Slightly changing equations (8) and (9), we define $g_\alpha(W_j) = (-1)^{j+1}h_\alpha(W_j)$, \hspace{1cm} (17)
for $j = 1, \ldots, k$. The properties mentioned in Propositions 23 to 25 also hold for $g_\alpha(W)$, thus justifying an alternative notion of $\alpha$-store. Using this notion and the following definition instead of Definition 12, one can check that Lemma 27 and Theorem 28 remain valid for $g_\alpha(W)$ as well.

Definition 33. A mixed graph is an $\alpha$-monograph (of 2nd kind) if $g_\alpha(\tilde{C}) = 1$ for all its cycles $C$.

As a result, we can derive results analogous to (but slightly different from) Theorem 30 and Theorem 31.

Theorem 34. A connected mixed graph $D$ is an $\alpha$-monograph (of 2nd kind) if and only if $V(D)$ can be partitioned into sets $\ldots, V_{-\alpha^2}, V_{-\alpha^1}, V_{-\alpha^0}, V_0, V_{\alpha^1}, V_{\alpha^2}, \ldots$ (some of which possibly empty) such that digons occur only between pairs of sets $V_{\alpha^j}, V_{-\alpha^j}$ and any arc starting in a set $V_{\alpha^j}$ ends in $V_{-\alpha^j-1}$.

Theorem 35. Let $D$ be an $\alpha$-monograph (of 2nd kind). Then, $\sigma_\alpha(D) = -\sigma(\Gamma(D))$.

Example 36. The mixed graph $D$ shown in Figure 6 is an $i$-monograph of 2nd kind. We have $\sigma_i(D) = \{-3, 1^{(3)}\}$ and $\sigma(\Gamma(D)) = \{3, -1^{(3)}\}$. Clearly, $D$ is not an $i$-monograph of 1st kind.

6 Spectral Radius

The spectral radius $\rho(M)$ of a complex matrix $M$ is defined as the largest modulus among its eigenvalues. If $\| \cdot \|$ is any matrix norm, we have $\rho(M) \leq \|M\|$ (cf. Theorem 5.6.9 in [5]). Using the maximum norm $\| \cdot \|_\infty$, one immediately obtains the classic upper bound $\rho(G) \leq \Delta_G$ on the spectral radius $\rho(G) := \rho(A(G))$ of an undirected graph $G$. Supposing $G$ is connected, equality holds if and only if $G$ is regular. Considering a mixed graph $D$ and its $\alpha$-Hermitian adjacency matrix $H_\alpha(D)$ instead, it follows from Definition 1 that $\|H^\alpha(D)\|_\infty = \|A(\Gamma(D))\|_\infty$, since all nonzero entries of $H^\alpha(D)$ have modulus 1. Hence, $\rho_\alpha(D) \leq \Delta_{\Gamma(D)}$. Interestingly, $\alpha$-monographs come into play if one wants to characterize when equality holds:

Theorem 37. Let $D$ be a connected mixed graph. Then, $\rho_\alpha(D) = \Delta_{\Gamma(D)}$ if and only if $D$ is a regular $\alpha$-monograph (of 1st or 2nd kind).
Proof. Let $x = [x_u]_{u \in V(D)}$ be an $\alpha$-eigenvector of $D$ for $\alpha$-eigenvalue $\lambda$. Choose $v \in V(D)$ such that $|x_v|$ is maximal. We may assume $|x_v| = 1$. Using Proposition 2, we can deduce that

$$|\lambda| = |\lambda x_v| \leq \sum_{u \in N(v)} |x_u| + \sum_{u \in N^+(v)} |\alpha x_u| + \sum_{u \in N^-(v)} |\pi x_u|$$

(18)

$$= \sum_{u \in N(v)} |x_u| + \sum_{u \in N^+(v)} |x_u| + \sum_{u \in N^-(v)} |x_u|$$

(19)

$$\leq \sum_{u \in N(v)} |x_v| + \sum_{u \in N^+(v)} |x_v| + \sum_{u \in N^-(v)} |x_v|$$

(20)

$$= \deg_{\Gamma(D)}(v)$$

(21)

$$\leq \Delta_{\Gamma(D)}.$$  

(22)

Suppose that $\lambda$ is an $\alpha$-eigenvalue of $D$ with largest modulus. The condition $\rho_\alpha(D) = \Delta_{\Gamma(D)}$ holds if and only if equality holds in all three conditions (18), (20) and (22). Equality in (22) is achieved if any only if $D$ (resp. $\Gamma(D)$) is regular of degree $\Delta_{\Gamma(D)}$. Since $|x_v|$ is maximal, equality in (20) occurs exactly if $|x_u| = |x_v| = 1$ for all $u \in N_{\Gamma(D)}(v)$. Repeat this argument for all vertices $u \in N_{\Gamma(D)}(v)$, each time taking the role of $v$. Since $\Gamma(D)$ is connected, we successively prove $|x_u| = 1$ for all $u \in V(D)$.

Equality holds in the complex triangle inequality (18) if and only if $\arg(\lambda x_v) = \arg(x_u)$ for all $u \in N_{\Gamma(D)}(v)$. In the following, we shall skip the trivial case $\lambda = 0 = \rho_\alpha(D)$. Let $\arg(\alpha) = \theta \in \mathbb{R}$. Consider the following cases:

(i) Case $\lambda > 0$:

- If $u \in N(v)$, then $\arg(x_u) = \arg(\lambda x_v)$, so that $x_u = x_v$.
- If $u \in N^+(v)$, then $\arg(\alpha x_u) = \arg(\lambda x_v)$, so that $x_u = \alpha x_v$.
- If $u \in N^-(v)$, then $\arg(\alpha x_u) = \arg(\lambda x_v)$, so that $x_u = \alpha x_v$.

Assigning each vertex $u \in V(D)$ to a set $V_\theta$ with $\theta = x_u/x_v$, we obtain a partition as mentioned in Theorem 30.

(ii) Case $\lambda < 0$:

- If $u \in N(v)$, then $\arg(x_u) = \arg(\lambda x_v)$, so that $x_u = -x_v$.
- If $u \in N^+(v)$, then $\arg(\alpha x_u) = \arg(\lambda x_v)$, so that $x_u = -\alpha x_v$.
- If $u \in N^-(v)$, then $\arg(\alpha x_u) = \arg(\lambda x_v)$, so that $x_u = -\alpha x_v$.

Assigning each vertex $u \in V(D)$ to a set $V_\theta$ with $\theta = x_u/x_v$, we obtain a partition as mentioned in Theorem 34.

Conversely, let $D$ be a connected mixed graph having a vertex partition according to one of the cases (i) or (ii). Construct a vector $x = [x_u]_{u \in V(D)}$ as follows. Set $x_u := q$ for any $u \in V_\theta$. It is straightforward to show that $x$ is an $\alpha$-eigenvector for an eigenvalue of modulus $\rho_\alpha(D)$. □

Corollary 38. Let $D$ be a connected mixed graph. Suppose that $\alpha^k \neq -\alpha^l$ for all $k, l \in \mathbb{Z}$. Then, $\rho_\alpha(D) = \Delta_{\Gamma(D)}$ if and only if $D$ is a regular $\alpha$-monograph of 1st kind.

Proof. The given condition on $\alpha$ guarantees that the partition arising in case (ii) in the proof of Theorem 37 can be converted into a bipartition $V(D) = V' \cup V''$, with $V' = V_{\alpha^0} \cup V_{\alpha^1} \cup \ldots$ and $V'' = V_{-\alpha^0} \cup V_{-\alpha^1} \cup \ldots$, such that the subgraphs induced by $V'$ and $V''$ have no edges. So $D$ would be bipartite in this case. Forming pairwise unions $V_{\alpha^k} \cup V_{-\alpha^k}$, it becomes apparent that $D$ must be an $\alpha$-monograph of 1st kind. □
To conclude, let us briefly revisit the three special choices $i$, $\omega$ and $\gamma$ for $\alpha$. Choosing $\alpha = \gamma$, the conditions of Corollary 38 are met, but not for $\alpha \in \{i, \omega\}$. In view of this, $\gamma$ appears to be an interesting choice for $\alpha$. Moreover, since $\gamma$ is a third root of unity, among all candidates satisfying the conditions of Corollary 38 choosing $\alpha = \gamma$ will yield a minimal number of sets in the monograph vertex partition.

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