The Bullough-Dodd model coupled to matter fields

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Abstract

The Bullough-Dodd model is an important two dimensional integrable field theory which finds applications in physics and geometry. We consider a conformally invariant extension of it, and study its integrability properties using a zero curvature condition based on the twisted Kac-Moody algebra $A_2^{(2)}$. The one and two-soliton solutions as well as the breathers are constructed explicitly. We also consider integrable extensions of the Bullough-Dodd model by the introduction of spinor (matter) fields. The resulting theories are conformally invariant and present local internal symmetries. All the one-soliton solutions, for two examples of those models, are constructed using an hybrid of the dressing and Hirota methods. One model is of particular interest because it presents a confinement mechanism for a given conserved charge inside the solitons.

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1 Introduction

Solitons play a role in many non-perturbative aspects of field theories. They appear as classical solutions of the equations of motion of some special non-linear theories. In general, their masses and interaction strengths are inversely related to the coupling constant for the original fields of the theory. Therefore, they are weakly interacting and easy to excite at the strong coupling limit, and so are natural candidates to describe the normal modes of the theory in that regime. In the particular case of some supersymmetric gauge theories the solitons (dyons) play a crucial role in electromagnetic duality conjectures involving the weak and strong coupling regimes [1].

Two dimensional integrable field theories possessing soliton solutions constitute, besides their intrinsic beauty, a laboratory to test ideas and develop exact methods on non-perturbative aspects of physical theories. In addition, two dimensional models have direct applications in many areas of physics and non-linear sciences, like condensed matter, non-linear optics, etc. Classical soliton theory in two dimensions is quite well developed [2]. Practically all soliton solutions appear in models that admit a representation of their equations of motion in terms of the so-called zero curvature condition or Lax-Zakharov-Shabat equation [3], with the flat connections living on Kac-Moody algebras [4]. That equation leads to exact methods for constructing solutions and infinite number of conserved charges. An important point to emphasize is that the symmetries responsible for the appearance of solitons are symmetries of the zero curvature equation, and not necessarily of the equations of motions or of the Lagrangean. For that reason the conserved charges are not of the Noether type.

In this paper we consider special integrable extensions of the Bullough-Dodd model [5]. That is a relativistic invariant theory in two dimensions involving just one scalar field, which first appeared in the literature in the context of hyperbolic surfaces in $\mathbb{R}^3$ [6]. It is an integrable field theory with solitons and some of their properties was considered by Zhiber and Shabat [7] and Mikhailov [8]. The integrability properties follow from the fact that the Bullough-Dodd equation admits a zero curvature representation with flat connections taking values on the twisted affine Kac-Moody algebra $A_2^{(2)}$, but with vanishing central term. The first extension we consider is on the lines of [9, 10, 11], and which we call the Conformal Bullough-Dodd model. It involves the addition of two extra fields which makes the theory conformally invariant, and imposes the zero curvature condition to live on the algebra $A_2^{(2)}$, but with non-trivial central term. That fact plays a crucial role in the integrability properties of the model. The representation theory of Kac-Moody algebras changes drastically when the central term is not zero. It admits highest weight state representations which are an important tool in the construction of exact solutions. We then construct the one and two soliton solutions of the Bullough-Dodd using an hybrid of the dressing method [12] and the Hirota method [13] as proposed in [14]. The physically interesting solitons exist when the Bullough-Dodd scalar field is taken to be complex. We also discuss the construction of complex breather solutions.

The second extension of the Bullough-Dodd model that we consider involves the
addition of spinor (matter) fields following the ideas of [15]. That is achieved by considering the principal (integer) gradation of the algebra $A_2^{(2)}$ [4]. The flat connection corresponding to the (Conformal) Bullough-Dodd model has components in the eigensubspaces of grades 0 and $\pm 1$. We extend that connection by allowing it to have components with grades varying from $-l$ to $l$ with $l$ being a positive integer. The equations of motion of the model are obtained by imposing the connections to satisfy the zero curvature condition. Therefore, the models are integrable by construction. As shown in [15], the fields appearing in the non-zero grade components of the connections transform as two dimensional Dirac spinors, and they couple to exponentials of the Bullough-Dodd scalar field. The components of the connection with grades $\pm l$ are taken to be constant, and they play a crucial role in determining the physical properties of the resulting model. We consider two models of that type corresponding to the choices $l = 3$ and $l = 6$. The models with $l = 2, 4, 5$ do not seem to possess solitons. Again using an hybrid of the dressing and Hirota methods we construct all the one-soliton solutions for those two models.

The model corresponding to $l = 6$ however is analysed in much more detail because of its interesting physical properties. Besides, the three scalar fields of the Conformal Bullough-Dodd model, it possesses seven Dirac spinor fields. The equations of motion for them are Dirac equations with interaction terms involving the product of exponentials of the scalar fields with bilinear terms in the spinors. For that reason the model does not possesses a Lagrangean which is local in those fields. The theory however, is conformally invariant and possesses a local $U_L(1) \otimes U_R(1)$ chiral (gauge) symmetry group. Although we can not use the Noether theorem, due to the lack of Lagrangean, there exist two conserved currents corresponding to that symmetry group. In addition, the conservation of those currents is equivalent to the existence of two chiral currents, depending on only one light cone variable each. A remarkable property of that model is that all the one-soliton solutions belong to a submodel where those two chiral currents vanish. On that submodel there exists a conserved current, depending only on the spinor fields, which is exactly equal to the topological current involving the Bullough-Dodd scalar field. Such an equivalence of matter and topological currents leads to a confinement of the corresponding matter charge inside the solitons. As explained in [15] that confinement mechanism is special to some Toda models coupled to matter fields, and have been studied in some interesting cases [16, 17, 14]. In particular, in [16] it was shown that it provides a very interesting way of understanding the quantum equivalence between the sine-Gordon and Tirring models. From the algebraic construction of [15], the model for $l = 6$ that we discuss in this paper stands for the Bullough-Dodd model in the same way as the model considered in [16] stands for the sine-Gordon. We then hope that our work may help understanding if there exist a theory which would be equivalent to the the Bullough-Dodd model, in such way that the elementary excitations of its fields would correspond to the Bullough-Dodd solitons.

Our paper is organized as follows. In section 2 we introduce the Conformal Bullough-Dodd model, discuss its integrability properties and construct its one and two soliton solutions as well as the breathers. The dressing tranformation methos is discussed in section 2.1. The construction of the models involving the matter (spinor) fields through
the extension of the zero curvature connections is discussed in section §3 as well as the symmetries of the resulting theories. The model corresponding to \( l = 6 \) is described in section §4. The symmetries and the equivalence of the matter and topological currents is discussed in section §4.1. The method for constructing the soliton solutions for such model is described in section §4.2 and the particular one-soliton solutions are given in sections §4.3 and §4.4. The model corresponding to \( l = 3 \) is described in section §5 as well as its soliton solutions. Some basic facts about the twisted Kac-Moody algebra \( A_2^{(2)} \) is given in appendix A. The coefficients for the Hirota’s tau functions corresponding to the one-soliton of section §4.3.2 is given explicitly in appendix B and some simplifications of it in the appendix C.

2 The Conformal Bullough-Dodd model

The Bullough-Dodd model is a 1 + 1 dimensional relativistic field theory described by the equation

\[
\partial^2 \varphi = -e^\varphi + e^{-2\varphi} \tag{2.1}
\]

where \( \partial^2 \equiv \partial_\mu \partial^\mu = \frac{1}{c^2} \partial_t^2 - \partial_x^2 \). Together with the sine-Gordon model, it is one of the few relativistic invariant integrable theories with just one scalar field. The integrability properties of such theory were first discovered by Bullough and Dodd [5] who showed the existence of non-trivial conservation laws. Zhiber and Shabat [7] found an infinite Lie-Backlund group of transformations for (2.1), and Mikhailov [8] studied it in the context of the Toda models using inverse scattering method. However, the first time eq. (2.1) appeared in the literature was on a paper by Tzitzica [6] in the context of hyperbolic surfaces in \( \mathbb{R}^3 \). See also [18, 19] for more details.

Here we consider an extension of the Bullough-Dodd model (2.1) on the lines of [9, 10, 11], which we call the Conformal Bullough-Dodd model, and which is defined by the equations of motion\(^2\)

\[
\begin{align*}
\partial^2 \varphi &= -e^\eta \left( e^\varphi - e^{-2\varphi} \right) \\
\partial^2 \eta &= 0 \\
\partial^2 \nu &= -\frac{1}{2} e^{-2\varphi + \eta}
\end{align*} \tag{2.2}
\]

and corresponding Lagrangean

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \partial_\mu \varphi \partial^\mu \eta + 3 \partial_\mu \eta \partial^\mu \nu - \left( e^{\varphi + \eta} + \frac{1}{2} e^{-2\varphi + \eta} \right) \tag{2.3}
\]

The introduction of the field \( \eta \) renders the theory conformally invariant. Indeed, introducing the light cone coordinates

\[
x_\pm \equiv \frac{ct \pm x}{2} \quad \partial_\pm = \frac{1}{c} \partial_t \pm \partial_x \quad \partial_+ \partial_- = \partial^2 = \frac{1}{c^2} \partial_t^2 - \partial_x^2 \tag{2.4}
\]

\(^2\)A coupling constant \( \beta \), mass parameter \( m \), and relative coefficients for the exponential terms can be introduced into the equations by the redefinitions \( \varphi \rightarrow \beta \varphi - \ln \mu, \ x^\mu \rightarrow m x^\mu \), in such way that the eq. for \( \varphi \) would read \( \partial^2 \varphi = -\frac{m^2}{\mu^2} e^n \left( e^{\beta \varphi} - \mu^3 e^{-2\beta \varphi} \right) \)
one can check that (2.2) and (2.3) are invariant under the conformal transformations
\(x_\pm \to f_\pm (x_\pm)\) if the field \(\varphi\) is a scalar under the conformal group and if \(e^{-\eta} \to f'_+ f'_- e^{-\eta}\). The conformal weights of \(\nu\) are arbitrary \([9, 10, 11]\).

The field \(\nu\) is just an expectant in the sense that it does not really influence the dynamics of \(\varphi\) and \(\eta\). However, it plays a crucial role in the integrability properties of the model. As we explain below, the construction of exact solutions under the dressing method makes use of highest weight representations of the Kac-Moody algebra, and that requires the existence of a non-trivial central extension, and the consequent introduction the extra field \(\nu\).

The relevant algebra for the integrability properties of the Bullough-Dodd model (2.1) and its conformal extension (2.2) is the twisted affine Kac-Moody algebra \(A_2^{(2)}\) [4]. The commutation relations and a brief description of its properties are given in appendix \([A]\). What makes the theories (2.1) and (2.2) integrable is the fact that they admit a representation of their eqs. of motion in terms of a zero curvature condition or the Lax-Zakharov-Shabat equation

\[
\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0 \tag{2.5}
\]

where \(\partial_\pm\) are derivatives w.r.t. the light cone coordinates defined in (2.4). For the Conformal Bullough-Dodd model (2.2) the potentials are given by

\[
A_+ = -B \Lambda_+ B^{-1} = -\left(\frac{\sqrt{2}}{2} e^{\varphi+\eta} T_3^0 + e^{-2\varphi+\eta} L^{-1/2}_2\right)
\]

\[
A_- = -\partial_- B B^{-1} + \Lambda_- = -\left(\partial_- \varphi T_3^0 + \partial_- \eta Q + \partial_- \nu C\right) + \Lambda_-
\]

(2.6)

where

\[
B = e^\varphi T_3^0 + e^\eta Q + \nu C
\]

(2.7)

and

\[
\Lambda_+ = \frac{\sqrt{2}}{2} T_3^0 + L^{-1/2}_2 \quad \Lambda_- = \frac{\sqrt{2}}{2} T_3^0 + L^{-1/2}_2
\]

(2.8)

The operators \(T_3^0, T_3^0, L^{\pm 1/2}_\pm, Q, C\) are generators of the twisted affine Kac-Moody algebra \(A_2^{(2)}\) (see appendix \([A]\)). \(Q\) is the grading operator defined in (A.3), defining the so-called principal gradation of \(A_2^{(2)}\) (see (A.2) and (A.5)). Notice that \(B\) is an element of the zero grade subgroup, i.e. the one obtained by exponentiating the zero grade subalgebra \(G_0\) (see (A.2) and (A.5)). The elements \(\Lambda_+\) and \(\Lambda_-\) have grades +1 and −1 respectively, and satisfy

\[
[\Lambda_+, \Lambda_-] = \frac{1}{2} C
\]

(2.9)

One can check that by replacing (2.6) into (2.5) all the non-zero grade components vanish automatically, and the zero grade component leads to the equation

\[
\partial_+ \left(\partial_- B B^{-1}\right) = [\Lambda_-, B \Lambda_+ B^{-1}]
\]

(2.10)
The three components of (2.10), in the direction of $T_3^0$, $Q$ and $C$, are the equations for the fields $\varphi$, $\eta$, and $\nu$ respectively, given in (2.2).

The zero curvature representation for the usual Bullough-Dodd model (2.1) is obtained from (2.6) by setting $\eta = 0$ and working with a representation of the algebra $A_2^{(2)}$ where $C = 0$. Then (2.10) will have just one component in the direction of $T_3^0$ which corresponds to (2.1). The algebra $A_2^{(2)}$ with $C = 0$, a so-called loop algebra, admits finite matrix representations depending upon a complex parameter (the so-called spectral parameter).

Notice that, if one allows the field $\varphi$ to be complex, i.e. $\varphi = \varphi_R + i \varphi_I$, the eqs. (2.1) and (2.2) are invariant under the discrete transformations

$$
\varphi_R \to \varphi_R \quad \varphi_I \to \varphi_I + 2\pi n
$$

with $n$ being an integer. Therefore, the theories have a degenerate vacua which allow the existence of non-trivial topological charges defined by

$$
Q_{\text{top.}} = \int_{-\infty}^{\infty} dx j^0 = \frac{1}{2\pi} [\varphi_I (x = \infty) - \varphi_I (x = -\infty)]
$$

where

$$
j^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu} \frac{\partial \varphi_I}{\partial x^\nu}
$$

with $\varepsilon^{\mu\nu}$ being antisymmetric, $\varepsilon^{01} = 1$, $x^0 = ct$, and $x^1 = x$.

### 2.1 The dressing method and the solitons solutions

The zero curvature (2.5) is invariant under gauge transformations of the form: $A_\mu \to g A_\mu g^{-1} - \partial_\mu g g^{-1}$. The dressing transformations [12] are special types of gauge transformations that constitute maps among the solutions of (2.5) or equivalently (2.2). We shall use a vacuum solution as the seed for the method, and construct the corresponding orbit of solutions under the dressing transformation group. The vacuum solution of (2.2) we take is

$$
\varphi^{\text{vac}} = 0 \quad \eta^{\text{vac}} = 0 \quad \nu^{\text{vac}} = -\frac{1}{2} x^+ x^-
$$

The zero curvature potentials evaluated on such solution can be written as

$$
A^{\text{vac}}_\mu = -\partial_\mu \Psi^{\text{vac}} \Psi^{-1}\text{vac}
$$

with

$$
\Psi^{\text{vac}} = e^{x^+ \Lambda_+} e^{-x^- \Lambda_-}
$$

with $\Lambda_\pm$ defined in (2.8). We now take a constant group element $h$ such that there exist the Gauss type decomposition

$$
\Psi^{\text{vac}} h \Psi^{-1}\text{vac} = G_- G_0 G_+
$$
where \( G_−, G_0 \) and \( G_+ \) are groups elements obtained by exponentiating the generators of \( A_2^{(2)} \) with negative, zero and positive grades respectively, of the principal gradation \( (A.2) \) defined by \( Q \) given in \( (A.3) \). We introduce the element

\[
\Psi_h \equiv G_0^{-1} G_−^{-1} \Psi_{\text{vac}} h = G_+ \Psi_{\text{vac}}
\]

(2.18)

and the transformed potentials

\[
A^h_\mu \equiv -\partial_\mu \Psi_h \Psi_h^{-1}
\]

(2.19)

The fact that \( A^h_\mu \) is of the pure gauge form guarantees that it is a solution of the zero curvature condition \( (2.5) \). In addition, the fact that \( \Psi_h \) can be written in two different ways in terms of \( \Psi_{\text{vac}} \), guarantees that \( A^h_\mu \) has the same grading structure as the potentials \( (2.6) \). Indeed, \( (2.18) \) and \( (2.19) \) implies that

\[
A^h_\mu = G_+ A^\text{vac}_\mu G_0^{-1} - \partial_\mu G_+ G_0^{-1}
\]

(2.20)

Eq. \( (2.20) \) implies that \( A^h_\mu \) has components of grades greater or equal than those of \( A^\text{vac}_\mu \). On the other hand, \( (2.21) \) implies that \( A^h_\mu \) has components of grades smaller or equal than those of \( A^\text{vac}_\mu \). Since \( (2.17) \) and \( (2.18) \) guarantee that both relations hold true it follows that \( A^h_\mu \) has the same grade components as \( A^\text{vac}_\mu \), and so as the potentials \( (2.6) \). By construction we have \( A^h_\mu \) given explicitly in terms of the space-time coordinates \( x_\pm \). Therefore by equating it to \( (2.6) \) we get the solutions for the fields of the model, associated to the choice \( h \) of the constant group element, i.e. a point on the orbit of solutions of the vacuum \( (2.14) \). In order to get the explicit solution we proceed as follows. From \( (2.15) \) and \( (2.21) \) we have that

\[
A^h_- = -\partial_- G_0^{-1} G_0 - G_0^{-1} \partial_- G_−^{-1} G_- G_0 + \frac{1}{2} x_+ C + G_0^{-1} G_−^{-1} \Lambda_- G_- G_0
\]

(2.22)

and so its zero grade part is

\[
\left( A^h_- \right)_0 = -\partial_- G_0^{-1} G_0 + \frac{1}{2} x_+ C = -\partial_- \left( e^{-\frac{1}{2} x_+ C} G_0^{-1} \right) \left( G_0 e^{\frac{1}{2} x_+ C} \right)
\]

(2.23)

Comparing \( (2.23) \) with the zero grade part of \( A_- \) in \( (2.6) \), we get that

\[
B = e^{-\frac{1}{2} x_+ C} G_0^{-1}
\]

(2.24)

and so using \( (2.7) \), we have

\[
G_0 = e^{-\varphi T_0^- \left( \nu + \frac{1}{2} x_+ C \right)} C
\]

(2.25)

Notice that, from the commutation relations of the appendix A, the operator \( D \), and so \( Q \), is never produced by the commutator of any pair of generators of the algebra. Therefore, if one starts with a vacuum solution with \( \eta = 0 \), the dressing method will never produce a solution with \( \eta \neq 0 \). Therefore, \( \eta \) does not appear in \( (2.25) \).
In order to obtain the explicit expression for the solutions for the fields we make use of highest weight representations of $A_2^{(2)}$ and introduce the Hirota’s tau functions as the expectation value of (2.17), i.e.

$$
\tau_{\lambda} \equiv \langle \lambda \mid \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} \mid \lambda \rangle = \langle \lambda \mid G_0 \mid \lambda \rangle \tag{2.26}
$$

where in the second equality we have used the fact that the highest weight state $\mid \lambda \rangle$ satisfies (A.6), and so $G_+ \mid \lambda \rangle = \mid \lambda \rangle$, and $\langle \lambda \mid G_- = \langle \lambda \mid$. We shall use two of such representations and introduce two tau functions. First $\tau_0$ associated to the choice $\mid \lambda \rangle \equiv \mid \lambda_0 \rangle$, and $\tau_1$ associated to the choice $\mid \lambda \rangle \equiv \mid \lambda_1 \rangle \otimes \mid \lambda_1 \rangle$, where $\mid \lambda_0 \rangle$ and $\mid \lambda_1 \rangle$ are defined in (A.7). Then, from (2.25), (2.26) and (A.7), we have that

$$
\tau_0 = e^{-2(\nu + \frac{1}{2}x_+x_-)} \quad \tau_1 = e^{-\varphi - 2(\nu + \frac{1}{2}x_+x_-)} \tag{2.27}
$$

or equivalently

$$
\varphi = \ln \frac{\tau_0}{\tau_1} \quad \nu = -\frac{1}{2} \ln \tau_0 - \frac{1}{2} x_+x_- \tag{2.28}
$$

Replacing into the eqs. of motion (2.2) (with $\eta = 0$) we get the Hirota’s equations

$$
\tau_0 \partial_+ \partial_- \tau_0 - \partial_+ \tau_0 \partial_- \tau_0 = \tau_1^2 - \tau_0^2
$$

$$
\tau_1 \partial_+ \partial_- \tau_1 - \partial_+ \tau_1 \partial_- \tau_1 = \tau_0 \tau_1 - \tau_1^2 \tag{2.29}
$$

The solutions on the orbit of the vacuum (2.14) we are interested in, are those obtained by the so-called solitonic specialization procedure [20]. We take the constant group element $h$ to a product of exponentials of eigenvectors of the operators $\Lambda_\pm$ defined in (2.28), i.e.

$$
h = \prod_{j=1}^{N} e^{V_j} \quad [\Lambda_\pm, V_j] = \beta_j^{(\pm)} V_j \tag{2.30}
$$

Then, using (2.16), one gets

$$
\tau_{\lambda} = \langle \lambda \mid \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} \mid \lambda \rangle = \langle \lambda \mid \prod_{j=1}^{N} e^{V_j} \mid \lambda \rangle \tag{2.31}
$$

with $\Gamma_j = \beta_j^{(+)} x_+ - \beta_j^{(-)} x_-$. The integer $N$ defines the $N$-soliton sector of the solution. For $N = 1$ we get a solution traveling with constant velocity. If $\mid \lambda \rangle$ belongs to an integrable representation [4], the operators $V_j$ are nilpotent and the exponentials $e^{V_j}$ truncate at some finite order. That explains the truncation of the Hirota’s tau functions in the Hirota’s method [13].

The eigenvectors of $\Lambda_\pm$ with non-zero eigenvalues are

$$
V^{(\omega)}(z) = \sum_{n=-\infty}^{\infty} z^{-n} V_n^{(\omega)} \tag{2.32}
$$
with
\[ V_{6n}^\omega = T_3^n - \frac{1}{3} \delta_{n,0} C \]
\[ V_{6n+1}^\omega = -\omega^{-1} \sqrt{3} \left( \frac{\sqrt{2}}{3} T_1^n - 2 L_{-2}^{n+1/2} \right) \]
\[ V_{6n+2}^\omega = \omega^{-2} \sqrt{2} L_{-1}^{n+1/2} \]
\[ V_{6n+3}^\omega = \omega^{-3} \sqrt{2} L_0^{n+1/2} \]
\[ V_{6n+4}^\omega = \omega^{-4} \sqrt{2} L_1^{n+1/2} \]
\[ V_{6n+5}^\omega = -\omega^{-5} \sqrt{3} \left( \frac{\sqrt{2}}{3} T_2^{n+1} - 2 L_2^{n+1/2} \right) \] (2.33)

where \( \omega^6 = -1 \). One can check that
\[ \left[ \Lambda_\pm, V_n^\omega \right] = \omega^{\pm1} \sqrt{3} V_{n\pm1}^\omega \] (2.34)

and so
\[ \left[ \Lambda_\pm, V^\omega (z) \right] = \sqrt{3} (\omega z)^{\pm1} V^\omega (z) \] (2.35)

Therefore the eigenvalues depend upon the product of a complex parameter \( z \) and of a sixth root of \(-1\). However, there is no degeneracy because of the operator equality among degenerate eigenvectors, i.e. \( V^\omega (z) = V^\omega (\gamma z) (z \gamma^{-1}) \), with \( \gamma^6 = 1 \).

Notice that the dependence of the eigenvalues on the free parameter \( z \) comes from a one parameter group acting on the eigenvectors. Indeed, from (2.34) one has the eigenvalue equation
\[ \left[ \Lambda_\pm, \sum_{n=-\infty}^{\infty} V_n^\omega \right] = \omega^{\pm1} \sqrt{3} \sum_{n=-\infty}^{\infty} V_n^\omega \] (2.36)

But since \( \Lambda_\pm \) and \( V_n^\omega \) are eigenvectors of the grading operator, i.e. \([Q, \Lambda_\pm] = \pm \Lambda_\pm\), and \([Q, V_n^\omega] = n V_n^\omega\), we have that
\[ V^\omega (z) = e^{-\ln z Q} \left( \sum_{n=-\infty}^{\infty} V_n^\omega \right) e^{\ln z Q} \] (2.37)

and so, conjugating (2.36) with \( e^{-\ln z Q} \), one gets (2.35).

The final evaluation of the solutions requires the calculation of the expectation values of products of the operators \( V_j \) in the highest weight states \( | \Lambda \rangle \) (see (2.31)). In many cases that may prove to be a laborious task. One can then use an hybrid method (see [14] for details) in which one uses the algebraic dressing method to obtain the relation among the tau functions and the fields (where the Hirota’s method is
helpless), and then apply the Hirota’s method on the equations for the tau functions, with the ansatz provided by the relation (2.31), i.e.

\[
\tau_\lambda = 1 + \sum_{j=1}^{N} \delta^\lambda_j e^{\Gamma_j} + \sum_{j \geq i = 1}^{N} \delta^\lambda_{i,j} e^{\Gamma_i + \Gamma_j} + \ldots \tag{2.38}
\]

with \(\delta^\lambda_j = \langle \lambda | V_j | \lambda \rangle\), \(\delta^\lambda_{i,j} \sim \langle \lambda | V_i V_j | \lambda \rangle\), etc. We shall use such hybrid procedure in this paper, to determine the coefficients \(\delta^\lambda_j, \delta^\lambda_{i,j}\), by the Hirota’s method through a computer algorithm. Since the normalization of the eigenvectors \(V_j\) are not fixed by the above procedure, the coefficients \(\delta^\lambda_j\)'s will be determined up to some rescaling constants.

### 2.2 One soliton solution

The solutions in the one-soliton sector is obtained by taking the constant group element, introduced in (2.17), as \(h = e^{V(\omega)(z')}\), with \(V(\omega)(z')\) given in (2.32). Then solving (2.29) with the ansatz (2.38) for \(N = 1\), one gets the solution

\[
\tau_0 = 1 - 4a e^\Gamma + a^2 e^{2\Gamma} \\
\tau_1 = (1 + a e^\Gamma)^2 \tag{2.39}
\]

with \((z \equiv z'\omega)\)

\[
\Gamma = \sqrt{3} \left( z x_+ - \frac{x_-}{z} \right) = \frac{\sqrt{3}}{\sqrt{1 - \frac{v^2}{c^2}}} \left[ \cos \theta (x - vt) + i \sin \theta \left( ct - \frac{v}{c} x \right) \right] \tag{2.40}
\]

where we have parametrized \(z\) as \(z = e^{-\alpha + i\theta}\), and defined

\[
v = c \tanh \alpha \tag{2.41}
\]

We now write \(a = e^{\beta + i\xi}\) and define \(\tilde{\Gamma} \equiv \Gamma + \beta + i\xi \equiv \Gamma_R + i\Gamma_I\) with

\[
\Gamma_R = \frac{\sqrt{3} \cos \theta}{\sqrt{1 - \frac{v^2}{c^2}}} (x - vt) + \beta \\
\Gamma_I = \frac{\sqrt{3} \sin \theta}{\sqrt{1 - \frac{v^2}{c^2}}} \left( ct - \frac{v}{c} x \right) + \xi \tag{2.42}
\]

Then, using (2.28) and (2.39) one gets \(\varphi = \varphi_R + i \varphi_I\) with

\[
\varphi_R = \frac{1}{2} \ln \left[ 1 + 3 \frac{\left( 1 - 2 \cosh \Gamma_R \cos \Gamma_I \right)}{\left( \cosh \Gamma_R + \cosh \Gamma_I \right)^2} \right] \\
\varphi_I = \arctan \left[ \frac{3 \sinh \Gamma_R \sin \Gamma_I}{\cosh \Gamma_R + \cosh \Gamma_I} \left( \cosh \Gamma_R + \cosh \Gamma_I \right)^2 - 3 \left( 1 + \cosh \Gamma_R \cos \Gamma_I \right) \right] \tag{2.43}
\]

We have the following particular types of solutions (we shall set \(v = 0\) since it can be recovered by a Lorentz boost):
1. There are two types of real solutions \((\varphi_I = 0)\), but both singular:

(a) Take \(\cos \theta = \pm 1, \xi = 0, \beta = 0\) and then

\[
\varphi_R = \frac{1}{2} \ln \left[ 1 + 3 \frac{(1 - 2 \cosh \sqrt{3} x)}{(\cosh \sqrt{3} x + 1)^2} \right]
\]

and so \(\varphi_R \to 0\) for \(x \to \pm \infty\), and \(\varphi_R \to -\infty\) for \(x = \pm \text{ArcCosh}(2)/\sqrt{3}\).

(b) Take \(\sin \theta = \pm 1, \xi = 0, \beta = 0\) and so

\[
\varphi_R = \frac{1}{2} \ln \left[ 1 + 3 \frac{(1 - 2 \cos \sqrt{3} t)}{(1 + \cos \sqrt{3} t)^2} \right]
\]

which is singular whenever \(t = (2n + 1)\pi/(c\sqrt{3})\).

2. The regular static one-soliton solution is obtained when \(\cos \theta = \varepsilon = \pm 1, \beta = 0, \xi \neq 0, \pi\), and then

\[
\varphi_R = \frac{1}{2} \ln \left[ 1 + 3 \frac{(1 - 2 \cosh \sqrt{3} x \cos \xi)}{(\cosh \sqrt{3} x + \cos \xi)^2} \right]
\]

\[
\varphi_I = \text{ArcTan} \left[ \frac{3 \sinh \sqrt{3} x \varepsilon \sin \xi}{(\cosh \sqrt{3} x + \cos \xi)^2 - 3 (1 + \cosh \sqrt{3} x \cos \xi)} \right]
\]

One can check that, as \(x\) varies from \(-\infty\) to \(\infty\), \(\varphi_I\) varies continuously from \(-\pi\) to \(\pi\) for \(\varepsilon \sin \xi < 0\), and from \(\pi\) to \(-\pi\) for \(\varepsilon \sin \xi > 0\). Therefore, the topological charge \((2.12)\) is given by

\[
Q_{\text{top}} = -\text{sign} (\varepsilon \sin \xi)
\]

We give in figure 4 the plots of the one-soliton solution \((2.46)\) for two values of the parameters \(\varepsilon\) and \(\xi\).

3. Denote \(\omega \equiv \sin \theta\), and set \(\beta = \xi = 0\) to get \(\Gamma_R = \sqrt{3} (1 - \omega^2) x\) and \(\Gamma_I = \sqrt{3} \omega c t\). That is a solution which oscillates among the possible forms of the one-soliton given in \((2.46)\), and develops a singularity whenever \(t = n\pi/(\sqrt{3} \omega c)\). At those values of time the topological charge \((2.12)\) flips sign.

### 2.3 Two soliton solution

The solutions on the two-soliton sector are obtained by taking \(h = e^{V(\omega_1)}(z_1) e^{V(\omega_2)}(z_2)\), with \(V(\omega_i)(z_i)\) given in \((2.32)\). Then solving \((2.29)\) with the ansatz \((2.38)\) for \(N = 2\),
one gets the solution (\(z_i = \omega_i z'_i\))

\[
\begin{align*}
\tau_0 &= 1 - 4a_1 e^{\Gamma_1} - 4a_2 e^{\Gamma_2} + a_1^2 e^{2\Gamma_1} + a_2^2 e^{2\Gamma_2} \\
&\quad + 8a_1 a_2 \frac{2z_i^4 - z_i^2 z_{12}^2 + 2z_i^4}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{\Gamma_1 + \Gamma_2} \\
&\quad - 4a_1 a_2 \frac{(z_1 - z_2)^2 (z_i^2 - z_1 z_2 + z_2^2)}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{2\Gamma_1 + \Gamma_2} \\
&\quad - 4a_1 a_2 \frac{(z_1 - z_2)^2 (z_i^2 - z_1 z_2 + z_2^2)}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{\Gamma_1 + 2\Gamma_2} \\
&\quad + a_1^2 a_2^2 \frac{(z_1 - z_2)^4 (z_i^2 - z_1 z_2 + z_2^2)^2}{(z_1 + z_2)^4 (z_1^2 + z_1 z_2 + z_2^2)^2} e^{2\Gamma_1 + 2\Gamma_2} \\
\tau_1 &= 1 + 2a_1 e^{\Gamma_1} + 2a_2 e^{\Gamma_2} + a_1^2 e^{2\Gamma_1} + a_2^2 e^{2\Gamma_2} \\
&\quad + 4a_1 a_2 \frac{z_i^4 + 4z_i^2 z_{12}^2 + z_{22}^4}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{\Gamma_1 + \Gamma_2} \\
&\quad + 2a_1^2 a_2^2 \frac{(z_1 - z_2)^2 (z_i^2 - z_1 z_2 + z_2^2)}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{2\Gamma_1 + \Gamma_2} \\
&\quad + 2a_1 a_2^2 \frac{(z_1 - z_2)^2 (z_i^2 - z_1 z_2 + z_2^2)}{(z_1 + z_2)^2 (z_1^2 + z_1 z_2 + z_2^2)} e^{\Gamma_1 + 2\Gamma_2} \\
&\quad + a_1^2 a_2^2 \frac{(z_1 - z_2)^4 (z_i^2 - z_1 z_2 + z_2^2)^2}{(z_1 + z_2)^4 (z_1^2 + z_1 z_2 + z_2^2)^2} e^{2\Gamma_1 + 2\Gamma_2}
\end{align*}
\] (2.48)

where, as before,

\[
\Gamma_i = \sqrt{3} \left( z_i x_+ - \frac{x_-}{z_i} \right) = \frac{\sqrt{3}}{\sqrt{1 - \frac{v_i}{c^2}}} \left[ \cos \theta_i (x - v_i t) + i \sin \theta_i \left( ct - \frac{v_i}{c} x \right) \right]
\] (2.49)
and we have parametrized \( z_i = e^{-\alpha_i + i\theta_i} \), and defined \( v_i = c \tanh \alpha_i \).

Solutions describing the scattering of two one-solitons can be obtained from (2.48) by setting the parameters \( a_i \) and \( z_i \), \( i = 1, 2 \), to values corresponding to the ones chosen for the one-soliton solutions. For instance, to get the scattering of two regular one-solitons given in (2.46) one has to take \( z_i = e^{-\alpha_i + i\theta_i} \) and \( a_i = e^{i\xi_i} \), with \( \cos \theta_i = \pm 1 \), \( \xi \neq 0, \pi \), and \( \alpha_i \) setting their velocities by \( v_i = c \tanh \alpha_i \). However, we do not explore in this paper such solutions further.

### 2.4 Breathers

Let us now consider the two-soliton solution (2.48) for the case where \( \alpha_1 = \alpha_2 = 0 \), \( \theta_1 = -\theta_2 \equiv \theta \), and \( a_2 = -a_1 \equiv -a \), with

\[
a = \sqrt{-\frac{(z_1 + z_2)^2(z_1^2 + z_2^2)}{(z_1 - z_2)^2(z_1^2 - z_2^2)}} - \cot \frac{\theta}{2} \sqrt{\frac{2 \cos 2\theta + 1}{2 \cos 2\theta - 1}}
\]  

(2.50)

Using (2.28) on then gets that

\[
\varphi = \ln \frac{\cosh 2 \Gamma_R - i 8 a \cosh \Gamma_R \sin \Gamma_I + a^2 \cos 2 \Gamma_I - b}{\cosh 2 \Gamma_R + i 4 a \cosh \Gamma_R \sin \Gamma_I + a^2 \cos 2 \Gamma_I - c}
\]  

(2.51)

with

\[
\Gamma_R = \sqrt{3} \cos (\theta) \quad x \quad \Gamma_I = \sqrt{3} \sin (\theta) \quad c t
\]  

(2.52)

and

\[
b = \frac{4 \cos 4\theta - 1}{\sin^2 \theta (2 \cos 2\theta - 1)} \quad c = \frac{\cos 4\theta + 2}{\sin^2 \theta (2 \cos 2\theta - 1)}
\]  

(2.53)

Therefore, if \( a \) is pure imaginary we have that the argument of the logarithm in (2.51) is real, but it is either positive or negative. Therefore, the solution for \( \varphi \) is not regular since it has an imaginary part that jumps from 0 to \( \pi \).

On the other hand if \( a \) is real we have that the solution is complex, and writing \( \varphi = \varphi_R + i \varphi_I \), we have

\[
\varphi_R = \frac{1}{2} \ln \frac{[\cosh 2 \Gamma_R + a^2 \cos 2 \Gamma_I - b]^2 + [8 a \cosh \Gamma_R \sin \Gamma_I]^2}{[\cosh 2 \Gamma_R + a^2 \cos 2 \Gamma_I - c]^2 + [4 a \cosh \Gamma_R \sin \Gamma_I]^2}
\]

\[
\varphi_I = \arctan \frac{\chi}{\gamma}
\]

with

\[
\chi = 4 a \cosh \Gamma_R \sin \Gamma_I \left[ 2 c + b - 3 \left( \cosh 2 \Gamma_R + a^2 \cos 2 \Gamma_I \right) \right]
\]

\[
\gamma = -32 a^2 \cosh^2 \Gamma_R \sin^2 \Gamma_I
\]

\[
+ \left[ \cosh 2 \Gamma_R + a^2 \cos 2 \Gamma_I - b \right] \left[ \cosh 2 \Gamma_R + a^2 \cos 2 \Gamma_I - c \right]
\]  

(2.54)

Notice that \( \varphi_R \to 0 \) and \( \tan \varphi_I \to 0 \), as \( x \to \pm \infty \). It is a localized solution that oscillates in time, and so it is like a breather, however with a structure much more complex than that of the sine-Gordon breather.
3 The coupling to matter fields

We now consider a generalization of the Conformal Bullough-Dood model (2.2), on the lines of [15 16 14], by the introduction of extra fields. That is done by enlarging the zero curvature potentials (2.6) with components having grades greater than 1 and smaller than −1. We then introduce

\[ A_+ = -B \left( E_{+l} + F_+ \right) B^{-1} \quad A_- = -\partial_- B B^{-1} + E_{-l} + F_- \] (3.1)

where \( B \) is the same group element as in (2.7), i.e. the subgroup obtained by exponentiating the subalgebra of grade zero w.r.t. to the grading operator \( Q \) defined in (A.3). \( E_{\pm l} \) are constant elements of the algebra \( A^{(3)}_2 \) with grades \( \pm l \), and \( F_{\pm} \) have components with grades varying from \( \pm 1 \) to \( \pm (l - 1) \), i.e.

\[ F_\pm = \sum_{m=1}^{l-1} F^m_\pm \quad \left[ Q, F^m_\pm \right] = \pm m F^m_\pm \] (3.2)

The extra fields are contained in the \( F^m_\pm \), and their explicit expression is given in the examples we discuss below.

The equations of motion of the theory is obtained by imposing that the potentials \( A_\pm \) in (3.1), satisfy the zero curvature condition (2.5). Replacing (3.1) into (2.5) and splitting it into their grading components one gets the equations of motion

\[ \partial_+ \left( \partial_- B B^{-1} \right) = \left[ E_{-l}, B E_l B^{-1} \right] + \sum_{n=1}^{l-1} \left[ F^n_-, B F^m_+ B^{-1} \right] \] (3.3)

\[ \partial_- F^m_+ = \left[ E_l, B^{-1} F^{l-m}_- B \right] + \sum_{n=1}^{l-m-1} \left[ F^{n+m}_+, B^{-1} F^n_- B \right] \] (3.4)

\[ \partial_+ F^m_- = -\left[ E_{-l}, B F^{l-m}_+ B^{-1} \right] - \sum_{n=1}^{l-m-1} \left[ F^{n+m}_-, B F^n_+ B^{-1} \right] \] (3.5)

The equations (3.3)-(3.5) have a large symmetry group. Indeed, one can check they are invariant under the conformal transformations

\[ x_+ \rightarrow f_+ \left( x_+ \right) \quad x_- \rightarrow f_- \left( x_- \right) \] (3.6)

with \( f_+ \) and \( f_- \) being analytic functions, and with the fields transforming as

\[ \varphi \left( x_+, x_- \right) \rightarrow \tilde{\varphi} \left( \tilde{x}_+, \tilde{x}_- \right) = \varphi \left( x_+, x_- \right) \]

\[ e^{-\nu(x_+, x_-)} \rightarrow e^{-\nu(\tilde{x}_+, \tilde{x}_-)} = \left( f'_+ \right)^{\delta} \left( f'_- \right)^{\delta} e^{-\nu(x_+, x_-)} \] (3.7)

\[ e^{-\eta(x_+, x_-)} \rightarrow e^{-\eta(\tilde{x}_+, \tilde{x}_-)} = \left( f'_+ \right)^{1/l} \left( f'_- \right)^{1/l} e^{-\eta(x_+, x_-)} \]

\[ F^m_+ \left( x_+, x_- \right) \rightarrow \tilde{F}^m_+ \left( \tilde{x}_+, \tilde{x}_- \right) = \left( f'_+ \right)^{-1+m/l} F^m_+ \left( x_+, x_- \right) \]

\[ F^m_- \left( x_+, x_- \right) \rightarrow \tilde{F}^m_- \left( \tilde{x}_+, \tilde{x}_- \right) = \left( f'_- \right)^{-1+m/l} F^m_- \left( x_+, x_- \right) \]
where the conformal weights of $e^{-\nu}$, namely $\delta$ and $\bar{\delta}$, are arbitrary.

In addition we have local symmetries associated to the generators of the zero grade subalgebra $G_0$ (see (A.2)) which commute with $E_{\pm l}$. Indeed, suppose we have group elements satisfying

$$h_R (x_+) E_l h_R^{-1} (x_+) = E_l \quad h_L (x_-) E_{-l} h_L^{-1} (x_-) = E_{-l}$$

with $h_R$ and $h_L$ being exponentiations of generators of $G_0$. Then the transformations

$$B (x_+, x_-) \to h_L (x_-) B (x_+, x_-) h_R (x_+),$$

$$F_+^m (x_+, x_-) \to h_R^{-1} (x_+) F_+^m (x_+, x_-) h_R (x_+),$$

$$F_-^m (x_+, x_-) \to h_L (x_-) F_-^m (x_+, x_-) h_L^{-1} (x_-)$$

leave the equations (3.3)-(3.5) invariant.

We are interested in models that possess soliton solutions. One of the key ingredients for the appearance of solitons, as explained in [21], is the existence of vacuum solutions such that the zero curvature potentials $A_{\pm}$ evaluated on them, lie on an oscillator (Heisenberg) subalgebra the Kac-Moody algebra. That is an abelian subalgebra up to central terms, i.e. it has generators $b_n$ such that their commutation relations is of the form

$$[b_m, b_n] = \beta m \delta_{m+n,0} C$$

Notice that the constant element $E_{-l}$ will always be present in $A_-$, when evaluated on any solution, since its coefficient is unity. In addition, the term $B E_{+l} B^{-1}$ can never vanish. Therefore, when evaluated on the vacuum solution its commutator with $E_{-l}$ has to produce at most a central term. Since the vacuum value of $B$ can always be absorbed in to the definition of $E_{+l}$, i.e. $B_{\text{vac}} E_{+l} B_{\text{vac}}^{-1} \to E_{+l}$, we conclude that we need

$$[E_l, E_{-l}] \sim C$$

If one looks at the eigensubspaces (A.5) and the commutation relations (A.1) for the $A_2^{(2)}$ algebra, one notices that the only possibilities for $E_{\pm l}$ are

$$E_{6n} \sim T_3^n \quad E_{-6n} \sim T_3^{-n}$$

$$E_{6n+1} \sim \frac{\sqrt{2}}{2} T_1^n + L_0^{n+1/2} \quad E_{-6n-1} \sim \frac{\sqrt{2}}{2} T_1^{-n} + L_0^{-n-1/2}$$

$$E_{6n+3} \sim L_0^{n+1/2} \quad E_{-6n-3} \sim L_0^{-n-1/2}$$

The gradation (A.2) has a period six, and the simplest models occur on the first period. The elements $E_{\pm (6n+1)}$, for $n = 0$ correspond to $A_{\pm}$, given in (2.8), leading to the usual Bullough-Dodd model. We will then consider in this paper the models corresponding to $E_{\pm 6}$, and $E_{\pm 3}$. The first one is more interesting from the physical point of view and we discuss it in detail in section 4. The second model is discussed in section 5.
4 The model for $l = 6$

We now take the potentials (3.1) with $l = 6$, and choose

$$E_{\pm 6} \equiv m T_{3}^\pm$$

where $m$ is a parameter which will set the mass scale for the particles and solitons of the theory. We then have

$$A_+ = -B \left( E_{+6} + \sum_{m=1}^{5} F_{m}^+ \right) B^{-1} \quad A_- = -\partial_+ B B^{-1} + E_{-6} + \sum_{m=1}^{5} F_{m}^-$$

The matter fields are contained in $F_{m}^\pm$, and are defined as

$$F_{5}^+ = \frac{1}{2} \sqrt{m} \psi_{R} \gamma_{5} V_{1} - \sqrt{2m} \bar{\psi}_{R} L_{1}^{1/2} \quad F_{5}^+ = \frac{1}{2} \sqrt{m} \psi_{L} T_{3}^{-1} - \sqrt{2m} \psi_{L} L^{-1/2}$$

$$F_{4}^+ = \sqrt{m} \psi_{R} L_{1}^{1/2} \quad F_{4}^- = -\sqrt{m} \psi_{L} L_{1}^{-1/2}$$

$$F_{3}^+ = \psi_{R} L_{0}^{1/2} \quad F_{3}^- = \psi_{L} L_{0}^{-1/2}$$

$$F_{2}^+ = \sqrt{m} \psi_{R} L_{1}^{1/2} \quad F_{2}^- = \sqrt{m} \bar{\psi}_{L} L_{1}^{-1/2}$$

$$F_{1}^+ = \frac{1}{2} \sqrt{m} \psi_{R} T_{3}^{0} + \sqrt{2m} \psi_{L} L_{2}^{1/2} \quad F_{1}^- = -\frac{1}{2} \sqrt{2m} \bar{\psi}_{L} T_{3}^{0} - \sqrt{2m} \bar{\psi}_{L} L_{2}^{-1/2}$$

Replacing into the equations (3.3)-(3.5) we get the equations of motion of the theory

$$\partial^{2} \varphi = \frac{i}{2} m \bar{\psi}_{R} \gamma_{5} V_{1} \psi_{L} - i 2m \bar{\psi}_{R} W(\eta) \gamma_{5} V^{2} \psi^{2}$$

$$- \frac{i}{2} m \bar{\psi}_{R} W(\eta) \gamma_{5} V^{3} \psi^{3}$$

(4.4)

$$\partial^{2} \nu = -2m^{2} e^{6\eta} - \frac{i}{2} e^{3\eta} \psi_{R} \psi_{R}^{0} + i m e^{6\eta} \bar{\psi}_{L} (1 - \frac{1}{2}) \gamma_{5} V_{1} \psi_{L}$$

$$+ i m \bar{\psi}_{R} W(\eta) V^{2} \psi^{2} + \frac{i}{2} m \bar{\psi}_{R} W(\eta) V^{3} \psi^{3}$$

(4.5)

$$\partial^{2} \eta = 0$$

(4.6)

$$i \gamma^{\mu} \partial_{\mu} \psi^{i} = m_{i} W(\eta) V^{i} \psi^{i} + U^{i}$$

(4.7)

$$i \gamma^{\mu} \partial_{\mu} \bar{\psi}^{i} = m_{i} \bar{W}(\eta) \bar{V}^{i} \bar{\psi}^{i} + \bar{U}^{i}$$

(4.8)

$$i \gamma^{\mu} \partial_{\mu} \psi^{0} = U^{0}$$

(4.9)

where in (4.7) and (4.8) we have $i = 1, 2, 3$, with

$$m_{1} = m_{3} = m \quad \text{and} \quad m_{2} = 2m$$

(4.10)

We have introduced a spinor notation for the matter fields as

$$\psi \equiv \begin{pmatrix} \psi_{R} \\ \psi_{L} \end{pmatrix} \quad U^{i} \equiv \begin{pmatrix} U_{R}^{i} \\ U_{L}^{i} \end{pmatrix}$$

(4.11)
with similar notation for $\bar{\psi}$ and $\bar{U}$. We have defined $\bar{\psi}$ and $\bar{U}$ as

$$\bar{\psi}^i \equiv \left(\bar{\psi}^i\right)^T \gamma_0 \quad \bar{U}^i \equiv \left(\bar{U}^i\right)^T \gamma_0 \quad \text{for } i = 1, 2, 3$$

and

$$\bar{\psi}^0 \equiv \left(\psi^0\right)^T \gamma_0 \quad \bar{U}^0 \equiv \left(U^0\right)^T \gamma_0$$

Notice that in general $\bar{\psi}$ is not the complex conjugate of $\psi$. The representation for the Dirac’s $\gamma$-matrices is

$$\gamma_0 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_1 = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 \equiv \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(4.14)}$$

In addition, we have introduced the quantities

$$W(\eta) \equiv \frac{1 + \gamma_5}{2} + e^{\eta} \frac{1 - \gamma_5}{2} \quad \bar{W}(\eta) \equiv e^\eta \frac{1 + \gamma_5}{2} + \frac{1 - \gamma_5}{2}$$

and the potentials involving the scalar fields$^3$

$$V^1 = e^{(\eta + \varphi)\gamma_5} \quad \bar{V}^1 = e^{-(\eta + \varphi)\gamma_5}$$

$$V^2 = e^{(\eta - \varphi)\gamma_5} \quad \bar{V}^2 = e^{-(\eta - \varphi)\gamma_5}$$

$$V^3 = e^{(2\eta - \varphi)\gamma_5} \quad \bar{V}^3 = e^{-(2\eta - \varphi)\gamma_5}$$

We have also introduced potentials quadratic in the matter fields

$$U^0_R = \sqrt{\frac{3}{2}} \left( e^{\eta + \varphi} \bar{\psi}_L^3 \psi_R^1 + e^{2\eta - \varphi} \bar{\psi}_L^1 \psi_R^3 \right)$$

$$U^0_L = \sqrt{\frac{3}{2}} \left( e^{2\eta - \varphi} \bar{\psi}_L^3 \bar{\psi}_R^1 + e^{\eta + \varphi} \bar{\psi}_L^1 \bar{\psi}_R^3 \right)$$

$$U^1_L = -\sqrt{2} m \left( e^{\eta - \varphi} \bar{\psi}_L^3 \bar{\psi}_R^2 + e^{4\eta + \varphi} \bar{\psi}_L^1 \bar{\psi}_R^2 \right) - \sqrt{\frac{3}{2}} \left( e^{2\eta - \varphi} \bar{\psi}_R^0 \bar{\psi}_L^3 - e^{2\eta + \varphi} \bar{\psi}_L^0 \bar{\psi}_R^3 \right)$$

$$U^2_L = \sqrt{\frac{m}{2}} \left( e^{\eta + \varphi} \bar{\psi}_R^3 \bar{\psi}_L^1 - e^{4\eta + \varphi} \bar{\psi}_R^3 \bar{\psi}_L^1 \right)$$

$$U^3_R = -\sqrt{2} m \left( e^{\eta + \varphi} \bar{\psi}_R^0 \bar{\psi}_L^1 + e^{2\eta - \varphi} \bar{\psi}_R^1 \bar{\psi}_L^0 \right)$$

$$U^3_L = \sqrt{\frac{3}{2}} \left( e^{\eta + \varphi} \bar{\psi}_R^0 \bar{\psi}_L^1 + e^{2\eta + \varphi} \bar{\psi}_R^0 \bar{\psi}_L^1 \right)$$

$$\bar{U}^3_R = \sqrt{\frac{3}{2}} \left( e^{3\eta} \bar{\psi}_R^0 \bar{\psi}_L^1 - e^{\eta + \varphi} \bar{\psi}_R^0 \bar{\psi}_L^1 \right)$$

$$\bar{U}^3_L = -\sqrt{2} m \left( e^{\eta - \varphi} \bar{\psi}_R^1 \bar{\psi}_L^1 + e^{2\eta + \varphi} \bar{\psi}_R^1 \bar{\psi}_L^1 \right)$$

$^3$ Notice that, $W(0) = \bar{W}(0) = 1$, $\bar{W}(\eta) \gamma_0 = \gamma_0 W(\eta)$, and $\bar{V}^i \gamma_0 = \gamma_0 V^i$. 

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\[
\tilde{U}^2_R = \sqrt{\frac{m}{2}} \left( e^{\eta_+ \varphi} \psi^1_R \tilde{\psi}^3_L - e^{\eta_- \varphi} \psi^1_L \tilde{\psi}^3_R \right) \\
\tilde{U}^1_R = -\sqrt{2m} \left( e^{|\eta| \varphi} \psi^2_R \tilde{\psi}^3_L + e^{\eta_+ \varphi} \psi^2_L \tilde{\psi}^3_R \right) + \sqrt{\frac{3}{2}} \left( e^{2\eta \psi^0_R \tilde{\psi}^3_L} + e^{2\eta_+ \varphi} \psi^0_L \tilde{\psi}^3_R \right)
\]
and
\[
U^1_R = U^2_R = \tilde{U}^1_L = \tilde{U}^2_L = 0
\]

### 4.1 Symmetries

As we have discussed in (3.7), the model (4.4)-(4.9) is invariant under conformal transformations on the two dimensional space-time. In addition, it is invariant under local transformations of the type (3.9), since \( T^0_0 \in \mathcal{G}_0 \), commutes with \( E_{x6} \) given in (4.1).

Denoting \( h_R (x_+) = \exp (-\xi_R (x_+)) T^0_3 \), and \( h_L (x_-) = \exp (\xi_L (x_-)) T^0_3 \), we have that the fields transform, under such \( U_R (1) \otimes U_L (1) \) group, as
\[
\varphi \rightarrow \varphi - \xi_R (x_+) + \xi_L (x_-) \quad \eta \rightarrow \eta \quad \nu \rightarrow \nu \\
\psi^0 \rightarrow \psi^0 \\
\psi^i \rightarrow e^{\frac{\eta}{2} (1 + \gamma_3 + \xi_L \frac{1}{2} (1 - \gamma_3))} \psi^i \\
\tilde{\psi}^i \rightarrow e^{-\frac{\eta}{2} (1 + \gamma_3 + \xi_L \frac{1}{2} (1 - \gamma_3))} \tilde{\psi}^i
\]
with the charges being \( q_1 = 1, q_2 = -2 \) and \( q_3 = -1 \). Notice that we can take two special global subgroups of the local \( U_R (1) \otimes U_L (1) \) symmetry group. Taking \( \xi_R = \xi_L \equiv \theta \) with \( \theta \) constant, we have
\[
\varphi \rightarrow \varphi \\
\eta \rightarrow \eta \\
\psi^i \rightarrow e^{q^i \theta} \psi^i \\
\tilde{\psi}^i \rightarrow e^{-q^i \theta} \tilde{\psi}^i \\
\psi^0 \rightarrow \psi^0
\]
Taking now \( \xi_R = -\xi_L \equiv \bar{\theta} \) with \( \bar{\theta} \) constant, we have
\[
\varphi \rightarrow \varphi - 2 \bar{\theta} \\
\eta \rightarrow \eta \\
\psi^i \rightarrow e^{q^i \bar{\theta}} \psi^i \\
\tilde{\psi}^i \rightarrow e^{-q^i \bar{\theta}} \tilde{\psi}^i \\
\psi^0 \rightarrow \psi^0
\]

Therefore, if the theory had a Lagrangean we would expect, due to the Noether theorem, one vector and one axial conserved currents. Those two currents do exist but not quite in the form one would expect. The field \( \psi^0 \), that has charge zero, do contribute to the current as we explain below.

Using the eqs. of motion (4.4)-(4.9) one gets that
\[
\partial_\mu \left( \tilde{\psi}^a \gamma^\mu \psi^a \right) = i \left( \tilde{U}^a \psi^a - \tilde{\psi}^a U^a \right) \quad a = 0, 1, 2, 3 \\
\partial_\mu \left( \psi^i \gamma^\mu \gamma_5 \psi^i \right) = 2 i m_i \tilde{\psi}^i W (\eta) V^i \gamma_5 \psi^i + i \left( \tilde{U}^i \gamma_5 \psi^i + \tilde{\psi}^i \gamma_5 U^i \right) \quad i = 1, 2, 3 \\
\partial_\mu \left( \tilde{\psi}^0 \gamma^\mu \gamma_5 \psi^0 \right) = i \left( \tilde{U}^0 \gamma_5 \psi^0 + \tilde{\psi}^0 \gamma_5 U^0 \right)
\]

(4.22)
with \( m_i \) as in \((1.10)\). One can check that the following identities hold true without the use of the equations of motion

\[
(U^1 \psi^1 - \bar{\psi}^1 U^1) - 2 (U^2 \psi^2 - \bar{\psi}^2 U^2) - (U^3 \psi^3 - \bar{\psi}^3 U^3) + \frac{1}{2 m} \left( \bar{U}^0 \gamma_5 \psi^0 + \bar{\psi}^0 \gamma_5 U^0 \right) = 0
\]

\[(4.23)\]

\[
(U^1 \gamma_5 \psi^1 + \bar{\psi}^1 \gamma_5 U^1) - 2 (U^2 \gamma_5 \psi^2 + \bar{\psi}^2 \gamma_5 U^2) - (U^3 \gamma_5 \psi^3 + \bar{\psi}^3 \gamma_5 U^3) + \frac{1}{2 m} \left( \bar{U}^0 \psi^0 - \bar{\psi}^0 U^0 \right) = 0
\]

\[(4.24)\]

Therefore, we have that the currents

\[
J_\mu = \bar{\psi}^1 \gamma_\mu \psi^1 - 2 \bar{\psi}^2 \gamma_\mu \psi^2 - \bar{\psi}^3 \gamma_\mu \psi^3 + \frac{1}{2 m} \bar{\psi}^0 \gamma_\mu \gamma_5 \psi^0
\]

\[(4.25)\]

and

\[
J_5^\mu = -4 \partial_\mu \varphi + \bar{\psi}^1 \gamma_\mu \gamma_5 \psi^1 - 2 \bar{\psi}^2 \gamma_\mu \gamma_5 \psi^2 - \bar{\psi}^3 \gamma_\mu \gamma_5 \psi^3 + \frac{1}{2 m} \bar{\psi}^0 \gamma_\mu \psi^0
\]

\[(4.26)\]

are conserved

\[
\partial_\mu J_\mu = 0 \quad \partial_\mu J_5^\mu = 0
\]

\[(4.27)\]

as a consequence of the eqs. of motion \((4.4)-(4.9)\).

The conservation laws \((4.27)\) imply that the currents

\[
J^{(\pm)}_\mu = \bar{J}^{(\pm)}_\mu = \partial^\pm J^{(\pm)}_\mu
\]

\[(4.28)\]

\[(4.29)\]

are conserved

\[
\partial^\pm J^{(\pm)}_\mu = 0
\]

\[(4.30)\]

Indeed, if one introduces the currents \(J^{(\pm)}_\mu = (J_\mu \pm J_5^\mu) / 4\) one gets that their light cone components (see \((2.4)\) are

\[
J^{(\pm)}_+ = \bar{\psi}^1 \psi^1 R - 2 \bar{\psi}^2 \psi^2 R - \bar{\psi}^3 \psi^3 R + \frac{1}{2 m} \left( \psi^0 R \right)^2 - \partial_+ \varphi
\]

\[(4.31)\]

\[
J^{(\pm)}_- = -\partial_- \varphi
\]

\[
J^{(\pm)}_+ = \partial_+ \varphi
\]

\[
J^{(\pm)}_- = \bar{\psi}^1 \psi^1 L - 2 \bar{\psi}^2 \psi^2 L - \bar{\psi}^3 \psi^3 L - \frac{1}{2 m} \left( \psi^0 L \right)^2 + \partial_- \varphi
\]

The conservation of \(J^{(\pm)}_\mu\), namely \(\partial^\mu J^{(\pm)}_\mu = \partial_+ J^{(\pm)}_- + \partial_- J^{(\pm)}_+ = 0\), implies \((4.30)\).
The soliton solutions we construct in section 4.2 belong to a submodel of the theory (4.4)-(4.9) where the chiral currents (4.28) and (4.29) vanish, i.e.
\[ \mathcal{J} = 0 \quad \tilde{\mathcal{J}} = 0 \]  
(4.32)
Such conditions can be written in the form of an equivalence between currents
\[ \frac{1}{8 \pi} j^{\text{matter}} = j^{\text{top.}} \]  
(4.33)
with
\[ j^{\text{matter}} = -\bar{\psi}^1 \gamma_\mu \psi^1 + 2 \bar{\psi}^2 \gamma_\mu \psi^2 + \bar{\psi}^3 \gamma_\mu \psi^3 + \frac{1}{2 m} \varepsilon_{\mu \nu} \bar{\psi}^0 \gamma^\nu \psi^0 \]
\[ j^{\text{top.}} = \frac{1}{2 \pi} \varepsilon_{\mu \nu} \partial^\nu \varphi \]  
(4.34)
with \( \varepsilon_{\mu \nu} \) being an antisymmetric symbol such that \( \varepsilon_{01} = 1 \). The fact that \( j^{\text{top.}} \) is trivially conserved, i.e. it is a topological current, then the equivalence (4.33) implies that \( j^{\text{matter}} \) is also conserved. So, we have
\[ \partial^\mu j^{\text{matter}} = 0 \quad \partial^\mu j^{\text{top.}} = 0 \]  
(4.35)
The equivalence (4.33) between matter and topological currents has interesting physical consequence, as already pointed out in similar models in [15, 16, 14]. From (4.33) it follows that the charge density \( j_0^{\text{matter}} \) is proportional to the space derivative of the scalar field \( \varphi \). Therefore, for those solutions where \( \varphi \) is constant everywhere except for a small region in space, like in a kink type solution, the charge density will also have to be localized. That means that the charge gets confined inside the soliton, and outside it we can have only zero charge states. The equivalence (4.33) thus provides a confinement mechanism for the charge associated to the current \( j^{\text{matter}}_\mu \).

### 4.2 Soliton solutions

We construct the solutions for the theory (4.4)-(4.9) using the dressing method [12], in a manner similar to that used in section 2.1 for the Bullough-Dodd model. We shall consider two vacuum solutions as the seeds for the dressing method. The first and simplest vacuum is a solution of (4.4)-(4.9) where
\[ \varphi_{\text{vac}1} = 0 \quad \eta_{\text{vac}1} = 0 \quad \nu_{\text{vac}1} = -2 m^2 x_+ x_- \quad \psi^a_{\text{vac}1} = 0 \quad \bar{\psi}^i_{\text{vac}1} = 0 \]  
(4.36)
with \( a = 0, 1, 2, 3 \) and \( i = 1, 2, 3 \). The second vacuum solution we consider is
\[ \varphi_{\text{vac}2} = 0 \quad \eta_{\text{vac}2} = 0 \]  
(4.37)
\[ \nu_{\text{vac}2} = -2 m^2 x_+ x_- - \frac{1}{2} \rho_+ (x_+) \rho_- (x_-) + \sigma_+ (x_+) + \sigma_- (x_-) \]
\[ \psi^i_{\text{vac}2} = 0 \quad \bar{\psi}^i_{\text{vac}2} = 0 \]
\[ \left( \psi^0_R \right)_{\text{vac}2} = \rho'_+ (x_+) \quad \left( \psi^0_L \right)_{\text{vac}2} = \rho'_- (x_-) \]
The vacuum (4.36) can be considered as a particular case of vacuum (4.37), where one takes \( \rho_\pm = \sigma_\pm = 0 \). The spectrum of soliton solutions obtained, through the dressing method, from the vacuum (4.36), using the solitonic specialization [20], is however much richer than that obtained from vacuum (4.37). But the general analysis of the dressing method can be done for vacuum (4.37), and only when the explicit evaluation of the solution is needed, one specializes to one of the vacua.

The zero curvature potentials (4.2) evaluated on the vacuum (4.37) takes the form

\[
A_{\text{vac}}^+ = -\Omega_+ \quad A_{\text{vac}}^- = \Omega_- - \partial_\nu_{\text{vac}} C \tag{4.38}
\]

where we have denoted

\[
\Omega_+ = E_6 + \rho_+(x_+) L_0^{1/2} \quad \Omega_- = E_6 + \rho_-(x_-) L_0^{-1/2} \tag{4.39}
\]

The potentials (4.38) can be written in the pure gauge form as

\[
A_{\mu}^{\text{vac}} = -\partial_\mu \Psi_{\text{vac}} \Psi_{\text{vac}}^{-1} \tag{4.40}
\]

with

\[
\Psi_{\text{vac}} = e^{x_+ E_0 + \rho_+(x_+) L_0^{1/2}} e^{-x_- E_0 - \rho_-(x_-) L_0^{-1/2}} e^{\sigma_+ C} \tag{4.41}
\]

The dressing method works for the vacuum solutions (4.36) and (4.37), and potentials (4.2), in the very same way as in section 2.1. Indeed, we consider a constant group element \( h \) such that \( \Psi_{\text{vac}} \), given in (4.41), admits the Gauss type decomposition

\[
\Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} = G_+ G_0 G_- \tag{4.42}
\]

such that \( G_-, G_0 \) and \( G_+ \) are group elements obtained by exponentiating the generators of \( A_2^{(2)} \) with negative, zero and positive grades respectively, of the principal gradation (A.2) defined by \( Q \) given in (A.3). Then, we define the group element

\[
\Psi_h \equiv G_0^{-1} G_- \Psi_{\text{vac}} h = G_+ \Psi_{\text{vac}} \tag{4.43}
\]

and the transformed potentials

\[
A_{\mu}^h \equiv -\partial_\mu \Psi_h \Psi_h^{-1} \tag{4.44}
\]

Since \( A_{\mu}^h \) is of the pure gauge it is a solution of the zero curvature condition (2.5) for the potentials (4.2). Replacing (4.43) into (4.44) one gets

\[
A_{\mu}^h = G_+ A_{\mu}^{\text{vac}} G_+^{-1} - \partial_\mu G_+ G_+^{-1} \tag{4.45}
\]

\[
= \left(G_0^{-1} G_-\right) A_{\mu}^{\text{vac}} \left(G_0^{-1} G_-\right)^{-1} - \partial_\mu \left(G_0^{-1} G_-\right) \left(G_0^{-1} G_-\right)^{-1} \tag{4.46}
\]

Notice that (4.45) implies that \( A_{\mu}^h \) has components of grades greater or equal than those of \( A_{\mu}^{\text{vac}} \). In addition, (4.46) implies that \( A_{\mu}^h \) has components of grades smaller or equal than those of \( A_{\mu}^{\text{vac}} \). Since (4.42) and (4.43) guarantee that both relations hold true, it follows that \( A_{\mu}^h \) has the same grade components as \( A_{\mu}^{\text{vac}} \), and so as the
potentials (4.2). By construction we have $A^h_\mu$ given explicitly in terms of the space-time coordinates $x_\pm$. Therefore by equating it to (4.2) we get the solutions for the fields of the model, associated to the choice $h$ of the constant group element, i.e. a point on the orbit of solutions of the vacuum (4.36) or (4.37). However, the procedure of equating (4.2) to $A^h_\mu$ can reveal complex and requires some care, as we now explain.

Taking the zero grade part of the $x_-$ component of (4.46), and using (4.38), we have

$$
\left( A^h_- \right)_0 = -\partial_- G_0^{-1} G_0 - \partial_- \nu_{\text{vac}} C
$$

Comparing with the zero part of $A_-$ given in (4.2),

$$
\left( A_- \right)_0 = -\partial_- B B^{-1}
$$

we get

$$
G_0^{-1} = B e^{-\nu_{\text{vac}} C} = e^{\varphi T_0^h (\nu - \nu_{\text{vac}})} C
$$

where we used (2.7), and have set $\eta$ to zero. The reason is that we have started with vacua (4.36)-(4.37) where the $\eta$ field is zero. Since the grading operator $Q$, defined in (A.3), can not be the result of any commutator (since it contains $D$), it follows that the dressing transformation does not excite $\eta$.

Comparing the positive grade part of $A^h_+$ in (4.46), with $A_+$ in (4.2), which has only positive parts, and using (4.38) and (4.49), we get,

$$
\left( G_-^1 \Omega_+ G_- \right)_{>0} = G^m_+ F^m_+
$$

Comparing the negative part of $A^h_-$ in (4.45), with the negative part of $A_-$ in (4.2) we get

$$
\left( G_+ \Omega_- G_+^{-1} \right)_{<0} = G^m_- F^m_-
$$

Now, comparing the components of grades 1 and 2 of $A^h_+$ in (4.45), with those same grade components of $A_+$ in (4.2) we obtain

$$
\left( \partial_+ G_+ G_+^{-1} \right)_m = B F^m_+ B^{-1}
$$

We do not include the grades 3 and higher, because $A^\text{vac}_+$ (see (4.38)) contains $L_0^{1/2}$ which has grade 3 and so the term $G_+ A^\text{vac}_+ G_+^{-1}$ in (4.45) would have to be considered.

Similarly, comparing the components of grades $-1$ and $-2$ of $A^h_-$ in (4.46), with those same grade components of $A_-$ in (4.2), and using (4.49), we obtain

$$
\left( \partial_- G_- G_-^{-1} \right)_{-m} = -B^{-1} F^m_- B
$$

From (4.41) and (4.42) we have that, given the choice of $h$, $G_0$, $G_+$ and $G_-$ are given explicitly as functions of the the space-time coordinates. Therefore, the relations (4.49)-(4.53), provide ways to get the explicit solutions for the fields contained in $B$, $F^m_+$ and
\( F_m \). Notice that if we write

\[
G_+ = \exp \left( \sum_{n=1}^{\infty} t^{(n)} \right) \quad G_-^{-1} = \exp \left( \sum_{n=1}^{\infty} t^{(-n)} \right) \tag{4.54}
\]

with \( t^{(\pm n)} \) being generators of grades \( \pm n \), we have from (4.50) and (4.51) that

\[
F_+^{6-n} = \left[ t^{(-n)}, E_{+6} \right] + \text{terms involving } t^{(-l)} \text{ with } l < n
\]

\[
F_-^{6-n} = \left[ t^{(n)}, E_{-6} \right] + \text{terms involving } t^{(l)} \text{ with } l < n
\]

Therefore starting with \( n = 1 \), we can recursively relate \( F_+^{6-n} \) to \( t^{(-l)} \), and \( F_-^{6-n} \) to \( t^{(l)} \), with \( l \leq n \). However, at grades \( \pm 3 \) we have \( t^{(-3)} \sim L_0^{-1/2} \) and \( t^{(3)} \sim L_0^{1/2} \), which commute with \( E_{\pm 6} \sim T_3^{\pm 1} \), and so the recursion relations break down. Therefore, our strategy is to use (4.50) and (4.51) to relate \( F_+^5, F_+^4 \) and \( F_-^3 \) to the \( t^{(\pm l)} \)’s, and then to use (4.52) and (4.53) to relate \( F_+^1 \) and \( F_-^2 \) to the same \( t^{(\pm l)} \)’s, but now involving derivatives of them.

In order to extract the explicit space-time dependence of the \( t^{(\pm l)} \)’s we consider matrix elements of both sides of relation (4.42) in highest weight state representations of the \( A_2^{(2)} \) algebra. In fact we use the two fundamental representations of \( A_2^{(2)} \) (see appendix \( \Delta \)). The relevant matrix elements which we need to express the \( t^{(\pm l)} \)’s are

\[
\begin{align*}
\tau_0 &= \langle \lambda_0 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | \lambda_0 \rangle \\
\tau_1 &= \langle 2\lambda_1 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | 2\lambda_1 \rangle \\
\tau_{R,1} &= \langle 2\lambda_1 | T_0^0 \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | 2\lambda_1 \rangle \\
\bar{\tau}_{L,1} &= \langle 2\lambda_1 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} T_0^0 | 2\lambda_1 \rangle \\
\tau_{R,2} &= \langle \lambda_0 | L_{-2}^{1/2} \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | \lambda_0 \rangle \\
\bar{\tau}_{L,2} &= \langle \lambda_0 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} L_{-2}^{1/2} | \lambda_0 \rangle \\
\tau_{R,3(0)} &= \langle \lambda_0 | L_{-1}^{1/2} \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | \lambda_0 \rangle \\
\bar{\tau}_{L,3(0)} &= \langle \lambda_0 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} L_{-1}^{1/2} | \lambda_0 \rangle \\
\tau_{R,3(1)} &= \langle 2\lambda_1 | L_{-1}^{1/2} \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} | 2\lambda_1 \rangle \\
\bar{\tau}_{L,3(1)} &= \langle 2\lambda_1 | \Psi_{\text{vac}} h \Psi_{\text{vac}}^{-1} L_{-1}^{1/2} | 2\lambda_1 \rangle
\end{align*}
\]

Then, using relations (4.49)-(4.53) in the way explained above, we obtain the fields in terms of the tau-functions (or equivalently of the \( t^{(\pm l)} \)’s)

\[
\begin{align*}
\varphi &= \log \frac{\tau_0}{\tau_1} \\
\nu &= -\frac{1}{2} \log \tau_0 - 2m^2 x_+ x_- - \frac{1}{2} \rho_+ (x_+) \rho_- (x_-) + \sigma_+ (x_+) + \sigma_- (x_-) \\
\psi_R^1 &= \frac{1}{\sqrt{m}} \frac{\tau_1}{\tau_0} \tau_1 \frac{\partial_+}{\nabla} \left( \frac{\bar{\tau}_{L,1}}{\tau_1} \right) \\
\psi_L^1 &= -\sqrt{m} \frac{\bar{\tau}_{L,1}}{\tau_1} \\
\psi_R^2 &= \frac{1}{\sqrt{2m}} \frac{\tau_0^2}{\tau_1^2} \tau_1 \frac{\partial_+}{\nabla} \left( \frac{\bar{\tau}_{L,2}}{\tau_0} \right)
\end{align*}
\]
\[
\begin{align*}
\psi^2_L &= -\sqrt{2m} \frac{\tilde{\tau}_{L,2}}{\tau_0} \\
\psi^3_R &= \frac{1}{\sqrt{m}} \frac{\tau_0}{\tau_1} \left[ \frac{\tilde{\tau}_{L,1}}{\tau_1} \partial_+ \left( \frac{\tilde{\tau}_{L,2}}{\tau_0} \right) - \partial_+ \left( \frac{\tilde{\tau}_{L,3(0)}}{\tau_0} \right) \right] \\
\psi^3_L &= -\sqrt{m} \left( \frac{\tilde{\tau}_{L,3(0)}}{\tau_0} - \frac{4 \tilde{\tau}_{L,3(1)}}{\tau_1} \right) \\
\tilde{\psi}^1_R &= -\sqrt{m} \frac{\tau_{R,1}}{\tau_1} \\
\tilde{\psi}^1_L &= -\frac{1}{\sqrt{m}} \frac{\tau_1}{\tau_0} \partial_- \left( \frac{\tau_{R,1}}{\tau_1} \right) \\
\tilde{\psi}^2_R &= -\sqrt{2m} \frac{\tau_{R,2}}{\tau_0} \\
\tilde{\psi}^2_L &= -\frac{1}{\sqrt{2m}} \frac{\tau^2_0}{\tau^2_1} \partial_- \left( \frac{\tau_{R,2}}{\tau_0} \right) \\
\tilde{\psi}^3_R &= \sqrt{m} \left( \frac{\tau_{R,3(0)}}{\tau_0} - \frac{4 \tau_{R,3(1)}}{\tau_1} \right) \\
\tilde{\psi}^3_L &= -\frac{1}{\sqrt{m}} \frac{\tau_0}{\tau_1} \left[ \frac{\tau_{R,1}}{\tau_1} \partial_- \left( \frac{\tau_{R,2}}{\tau_0} \right) + \partial_- \left( \frac{\tau_{R,3(0)}}{\tau_0} \right) \right] \\
\psi^0_R &= \rho_+ (x_+) + \sqrt{6} m \frac{\tau_{R,1} \tau_{R,3(1)}}{\tau^2_1} \\
\psi^0_L &= \rho_- (x_-) - \sqrt{6} m \frac{\tilde{\tau}_{L,1} \tilde{\tau}_{L,3(1)}}{\tau^2_1}
\end{align*}
\]

Now comes the point to make a distinction between the two vacua (4.36) and (4.37). As explained in (2.30) we construct the soliton solutions using the so-called solitonic specialization procedure [20]. According to that, we take \( h \) to be a product of exponentials of eigenvectors of the operators \( \Omega_\pm \) appearing in \( A_\text{vac} \), given in (4.38). Now, for the vacuum (4.36), \( \Omega_\pm \) reduces to \( E_\pm 6 \). Therefore, the set of of eigenvectors associated to the vacua (4.36) and (4.37) are different, and so the spectrum of soliton solutions one obtains through the dressing is also different.

The eigenvectors of \( E_\pm 6 \), with non-vanishing eigenvalues, are given by:

\[
\begin{align*}
V_{\pm 1}^{(T)} (\zeta) &= \sum_{n=\pm \infty} \zeta^{-(6n+1)} T^n_\pm \\
V_{\pm 1}^{(L)} (\zeta) &= \sum_{n=\pm \infty} \zeta^{-(6n+3)} L^{n+1/2}_\pm \\
V_{\pm 2}^{(L)} (\zeta) &= \sum_{n=\pm \infty} \zeta^{-(6n+3)} L^{n+1/2}_\pm
\end{align*}
\]

\(^4\)The free parameter \( \zeta \) has the same origin as the one explained in (2.36).
and satisfying
\[
\begin{align*}
\left[ E_{+6}, V^{(T)}_{\pm 1} (\zeta) \right] &= \pm m \zeta^6 V^{(T)}_{\pm 1} (\zeta) \\
\left[ E_{+6}, V^{(L)}_{\pm 1} (\zeta) \right] &= \pm m \zeta^6 V^{(L)}_{\pm 1} (\zeta) \\
\left[ E_{+6}, V^{(L)}_{\pm 2} (\zeta) \right] &= \pm 2 m \zeta^6 V^{(L)}_{\pm 2} (\zeta)
\end{align*}
\]
\[
\begin{align*}
\left[ E_{-6}, V^{(T)}_{\pm 1} (\zeta) \right] &= \pm m \zeta^{-6} V^{(T)}_{\pm 1} (\zeta) \\
\left[ E_{-6}, V^{(L)}_{\pm 1} (\zeta) \right] &= \pm m \zeta^{-6} V^{(L)}_{\pm 1} (\zeta) \\
\left[ E_{-6}, V^{(L)}_{\pm 2} (\zeta) \right] &= \pm 2 m \zeta^{-6} V^{(L)}_{\pm 2} (\zeta)
\end{align*}
\]

Notice that the eigenvalues \( m \zeta^{\pm 6} \) of \( E_{\pm 6} \), are four fold degenerate, with the corresponding degenerate eigenvectors being \( V^{(T)}_{\pm 1} (\zeta) \), \( V^{(T)}_{-1} (\omega \zeta) \), \( V^{(L)}_{\pm 1} (\zeta) \), and \( V^{(L)}_{\pm 1} (\omega \zeta) \), with \( \omega \) a sixth root of \(-1\), i.e. \( \omega^6 = -1 \). Similarly, the eigenvalues \( 2m \zeta^{\pm 6} \) of \( E_{\pm 6} \), are two fold degenerate, and the corresponding degenerate eigenvectors are \( V^{(L)}_{\pm 2} (\zeta) \), and \( V^{(L)}_{\pm 2} (\omega \zeta) \). Notice that the six possible choices for \( \omega \) do increase the degeneracy. Indeed, if \( \omega \) and \( \gamma \omega \), with \( \gamma^6 = 1 \), are two sixth roots of \(-1\), then \( V^{(T)}_{-1} (\gamma \omega \zeta) = \gamma^{\pm 1} V^{(T)}_{-1} (\omega \zeta) \), and so one gets the same eigenvector. The same holds true for the other type of eigenvectors. Clearly, the eigenvalues \( -m \zeta^{\pm 6} \) and \(-2 m \zeta^{\pm 6} \) are also four fold and two fold degenerate respectively, with the set of degenerate eigenvectors being \( \left( V^{(T)}_{\pm 1} (\omega \zeta), V^{(T)}_{-1} (\omega \zeta), V^{(L)}_{\pm 1} (\omega \zeta), V^{(L)}_{\pm 1} (\omega \zeta) \right) \), and \( \left( V^{(L)}_{\pm 2} (\omega \zeta), V^{(L)}_{\pm 2} (\omega \zeta) \right) \), respectively.

Notice that \( L_0^{\pm 1/2} \) commutes with \( L_{\pm 1/2} \). Therefore, \( V^{(L)}_{\pm 2} (\zeta) \) given in (4.57), has the same eigenvalue of \( \Omega_{\pm} \), given in (4.39), as that of \( E_{\pm 6} \), i.e.
\[
\begin{align*}
\left[ \Omega_{+}, V^{(L)}_{\pm 2} (\zeta) \right] &= \pm 2 m \zeta^6 V^{(L)}_{\pm 2} (\zeta) \\
\left[ \Omega_{-}, V^{(L)}_{\pm 2} (\zeta) \right] &= \pm 2 m \zeta^{-6} V^{(L)}_{\pm 2} (\zeta)
\end{align*}
\]

As for the subspace spanned by \( V^{(T)}_{\pm 1} (\zeta) \) and \( V^{(L)}_{\pm 1} (\zeta) \), we get four non-degenerate eigenvectors of \( \Omega_{\pm} \). Introducing
\[
V^{(\varepsilon_2)}_{\varepsilon_1} (\zeta) = V^{(T)}_{\varepsilon_1} (\zeta) + \varepsilon_2 V^{(L)}_{\varepsilon_1} (\zeta)
\]
with \( \varepsilon_1, \varepsilon_2 = \pm 1 \), one gets
\[
\begin{align*}
\left[ \Omega_{+}, V^{(\varepsilon_2)}_{\varepsilon_1} (\zeta) \right] &= \left[ \varepsilon_1 m \zeta^6 - \varepsilon_2 \frac{3}{2} \zeta^3 \rho'_+ (x_+) \right] V^{(\varepsilon_2)}_{\varepsilon_1} (\zeta) \\
\left[ \Omega_{-}, V^{(\varepsilon_2)}_{\varepsilon_1} (\zeta) \right] &= \left[ \varepsilon_1 m \zeta^{-6} - \varepsilon_2 \frac{3}{2} \zeta^{-3} \rho'_- (x_-) \right] V^{(\varepsilon_2)}_{\varepsilon_1} (\zeta)
\end{align*}
\]

The soliton solutions are constructed taking the constant group element of the dressing method, introduced in (4.42), as the exponential, or product of exponentials, of the eigenvectors of \( E_{\pm 6} \), in the case of vacuum (4.36), or of \( \Omega_{\pm} \), in the case of vacuum (4.37). That is the same solitonic specialization procedure [20] we used for the Bullough-Dodd solitons in (2.31). The explicit soliton solutions are then obtained by evaluating the matrix elements, given in (4.35), in the two fundamental representations of the Kac-Moody algebra \( A_2^{(2)} \).
The evaluation of those matrix elements are in general very laborious, and we use here an hybrid of the dressing and Hirota methods as explained in [14]. The dressing method is very powerful and reveals the mathematical structure behind soliton solutions, namely integrable highest weight state representations of Kac-Moody algebras [4]. The Hirota method is a very direct one, and the calculations in it are easy to implement in a computer algorithm using an algebraic manipulation program like Mathematica or Maple. However, it has a serious drawback, since it does not give a concrete way of finding the transformation among the tau functions and the fields. The main property of the Hirota’s tau functions is that they truncate at some finite order when the Hirota’s ansatz is used. The dressing method provides the algebraic explanation of why that truncation occurs. The matrix elements (4.55) correspond to the Hirota’s tau functions and their truncation comes from the nilpotency of the operators, in the integrable representations of the Kac-Moody algebra, corresponding to the eigenvectors of $E_{\pm 6}$ or of $\Omega_{\pm}$. In addition, the dressing method does give the concrete relations among the tau functions and the fields. For the model under consideration, those are the relations (4.56). Therefore our hybrid method consists of using the dressing method up to the point of obtaining the relations (4.56), then we replace those into the equations of motion (4.4)-(4.9) to obtain the Hirota’s equations for the tau functions. We then solve those equations using Hirota’s recurrence method implemented in a computer algorithm using the Mathematica program. In addition, we use the Hirota’s ansatz for the tau functions, that follows from the choice of $h$ as exponentials of the eigenvectors of $E_{\pm 6}$ or $\Omega_{\pm}$, in a way similar to that done in (2.30), (2.31) and (2.38).

We point out however that the spectrum of soliton solutions we obtain through such hybrid method is richer than that obtained by the solitonic specialization procedure. The Hirota’s recurrence method presents some degeneracies at higher order that makes the solutions to depend upon a number of parameters higher than that expect from the analysis of the degeneracy of the eigenvectors of $E_{\pm 6}$ and $\Omega_{\pm}$, done above. That happens for instance in the soliton solution discussed in section 4.3.2. The one soliton solutions for the abelian affine Toda models also present an enlarged spectrum as reported in [22].

We give in the following sections the various types of soliton solutions. It is worth mentioning that all soliton solutions have the property that the chiral currents (4.28) and (4.29) vanish when evaluated on them. Therefore, in all those solutions it is valid the equivalence (4.33) between the matter and topological currents. Consequently, it holds true the confinement mechanism of the charge associated to the current $j_{\mu}^{\text{matter}}$ given in (4.34), and discussed at the end of section 4.1.
4.3 One-soliton solutions from vacuum \((4.36)\)

4.3.1 One-soliton associated to \(V_{\pm 2}\)

We construct here the solution associated to the choice of the constant group element in \((4.43)\) as \(h = e^V\) with \(V\) being a linear combination of the degenerate eigenvectors \(V_{\pm 2}(\zeta)\) and \(V_{\pm 2}(\omega \zeta)\), defined in \((4.57)\), associated to the eigenvalues \(2m \zeta^\pm 6\) of \(E_{\pm 6}\). Therefore, the solution depends upon two parameters. By replacing that \(h\) into \((4.55)\) with \(\Psi_{\text{vac}}\) given by \((4.41)\) with \(\rho_\pm = \sigma_\pm = 0\), we get the following ansatz for the tau functions

\[
\tau_\ast = \sum_{l=0}^N \delta_\ast^{(l)} e^{l \Gamma_2} \tag{4.61}
\]

with

\[
\Gamma_2 = 2m \left( z x_+ - \frac{x}{z} \right) = \varepsilon \gamma_2 (x - vt) \tag{4.62}
\]

where we have denoted \(\zeta^6 \equiv z = \varepsilon e^{-\theta}\), with \(\varepsilon = \pm 1\), and \(\theta\) real, and so

\[
\gamma_2 = 2m \cosh \theta \quad \quad \nu = c \tanh \theta \tag{4.63}
\]

As explained at the end of section 4.2, we solve the equations for the tau functions, using the Hirota’s method implemented in a computer algorithm, and find that the expansion truncates at order \(N = 4\), with the solution given by

\[
\tau_0 = \left( 1 + \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2} \right) \left( 1 - \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2} \right)
\]

\[
\tau_1 = \left( 1 - \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2} \right)^2
\]

\[
\tilde{\tau}_{L,2} = a_2 e^{\Gamma_2} \left( 1 - \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2} \right)
\]

\[
\tau_{R,2} = \bar{a}_2 e^{\Gamma_2} \left( 1 - \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2} \right)
\]

(4.64)

where \(a_2\) and \(\bar{a}_2\) are the two parameters of the solution mentioned above and associated to the degeneracy of the eigenvalues \(2m \zeta^\pm 6\), of \(E_{\pm 6}\).

Using relations \((4.56)\), one can write the solution in terms of the fields as

\[
\varphi = \log \left( \frac{1 + \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2}}{1 - \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2}} \right)
\]

\[
\nu = -\frac{1}{2} \log \left( 1 - \frac{a_2 \bar{a}_2^2}{16} e^{4 \Gamma_2} \right) - 2m^2 x_+ x_-
\]

\[
\psi^2 = \sqrt{2m a_2 e^{\Gamma_2}} \left( \frac{z}{1 - \frac{a_2 \bar{a}_2^2}{16} e^{4 \Gamma_2}} \right) \quad \quad \tilde{\psi}^2 = \sqrt{-\frac{1}{2m^2 e^{2 \Gamma_2}}} \left( \frac{1 - \frac{a_2 \bar{a}_2^2}{16} e^{4 \Gamma_2}}{1 + \frac{a_2 \bar{a}_2}{4} e^{2 \Gamma_2}} \right)
\]

(4.65)
\[ \eta = \psi^1 = \bar{\psi}^1 = \psi^3 = \bar{\psi}^3 = \psi^0 = 0 \]

From the fact that the solutions excites only the fields \( \varphi, \nu, \psi^2, \) and \( \bar{\psi}^2, \) it is associated to the \( sl(2) \) Kac-Moody subalgebra generated by \( T_3^a, L_{2+1/2}^a \) and \( C \) (see appendix [A]). In fact that solution is the same as the one constructed in section 10.1 of ref. [15], and also studied in [16, 17]. The solution for \( \varphi \) is the same as that for the sine-Gordon model.

The chiral currents (4.28) and (4.29) vanish when evaluated on such solution, and so one has the equivalence (4.33) of the matter and topological currents, i.e.

\[
\varepsilon_{\mu\nu} \partial^\nu \varphi = \frac{1}{2} \bar{\psi}^2 \gamma_\mu \psi^2
\]  

(4.66)

Therefore, we have here a confinement mechanism as discussed at the end of section 4.1, and proposed in [15].

### 4.3.2 One-soliton associated to \( V^{(T)}_{\pm 1} \) and \( V^{(L)}_{\pm 1} \)

We now consider the solution obtained by taking the element \( h, \) defined in (4.43), to be an exponentiation of a linear combination of the four degenerated eigenvectors (4.57) of \( E_{\pm 6}, \) namely \( V^{(T)}_{+1} (\zeta), V^{(T)}_{-1} (\omega \zeta), V^{(L)}_{+1} (\zeta), \) and \( V^{(L)}_{-1} (\omega \zeta), \) with \( \omega^6 = -1. \) By replacing that \( h \) into (4.55) with \( \Psi^{vac} \) given by (4.41) with \( \rho_\pm = \sigma_\pm = 0, \) we get the following ansatz for the tau functions

\[
\tau_s = \sum_{l=0}^{N} \delta^{(l)}_s e^{l \Gamma_1}
\]

(4.67)

with

\[
\Gamma_1 = m \left( z x_+ - \frac{x_+}{z} \right) = \varepsilon \gamma_1 \left( x - v t \right)
\]

(4.68)

where we have denoted \( \zeta^6 \equiv z = \varepsilon e^{-\theta}, \) with \( \varepsilon = \pm 1, \) and \( \theta \) real, and so

\[
\gamma_1 = m \cosh \theta \quad v = c \tanh \theta
\]

(4.69)

Since the solution is associated to four degenerated eigenvectors, it should then depend upon four parameters. However, it depends in fact upon six parameters. The reason is that the Hirota’s recursion method works in a such a way in this case that at second order not all coefficients \( \delta_s^{(2)} \) are determined uniquely in terms of the coefficients \( \delta_s^{(1)} \). There remain two of them free. Such solution with six free parameters truncate at order \( N = 16, \) and the coefficients \( \delta^{(l)}_s \) of the tau functions are given explicitly in appendix [B].

Despite the complexity of such solution, the chiral currents (4.28) and (4.29) vanish when evaluated on it. Therefore, the equivalence (4.33) between the matter and topological currents holds true, and so does the confinement mechanism discussed at the end of section 4.1.
Such solution takes some simpler forms for some special values of the parameters. For instance, the tau functions, with coefficients given in appendix B, truncate at order 8 if one takes

\[ b = \varepsilon_1 \sqrt{z} \quad \bar{b} = i \varepsilon_2 \frac{\bar{a}}{\sqrt{z}} \]  

(4.70)

with \( \varepsilon_a = \pm 1, \ a = 1, 2. \)

However, the solution takes an even simpler form if one takes the parameter to satisfy

\[ b = \varepsilon_1 \sqrt{z} \quad \bar{b} = i \varepsilon_2 \frac{\bar{a}}{\sqrt{z}} \quad c = -2 \varepsilon_1 a^2 \sqrt{z} \quad \bar{c} = i 2 \varepsilon_2 \frac{\bar{a}^2}{\sqrt{z}} \]  

(4.71)

with \( \varepsilon_a = \pm 1, \ a = 1, 2. \) In this case the tau functions truncate at order 4 and are given by

\[
\begin{align*}
\tau_0 &= 1 - 8 a \bar{a} e^{2\Gamma_1} + 8 a^2 \bar{a}^2 e^{4\Gamma_1} \\
\tau_1 &= \left( 1 + 2 \sqrt{2} e^{i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} a \bar{a} e^{2\Gamma_1} \right)^2 \\
\tilde{\tau}_{L,1} &= 4 a e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right) \\
\tilde{\tau}_{L,2} &= 4 i \varepsilon_2 \frac{\bar{a}^2}{\sqrt{z}} e^{2\Gamma_1} \\
\tilde{\tau}_{L,3(0)} &= 4 i \varepsilon_2 \frac{\bar{a}}{\sqrt{z}} e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{-i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right) \\
\tilde{\tau}_{L,3(1)} &= 2 i \varepsilon_2 \frac{\bar{a}}{\sqrt{z}} e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right) \\
\tau_{R,1} &= 4 \bar{a} e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right) \\
\tau_{R,2} &= -4 \varepsilon_1 \sqrt{z} a^2 e^{2\Gamma_1} \\
\tau_{R,3(0)} &= 4 \varepsilon_1 \sqrt{z} a e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{-i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right) \\
\tau_{R,3(1)} &= 2 \varepsilon_1 \sqrt{z} a e^{\Gamma_1} \left( 1 + 2 \sqrt{2} e^{i \varepsilon_1 \varepsilon_2 \frac{3\pi}{4}} \bar{a} a e^{2\Gamma_1} \right)
\end{align*}
\]

(4.72)

The solution for \( \varphi \) can be obtained using the relation between \( \varphi \) and the tau functions given in (4.56). Writing \( \varphi = \varphi_R + i \varphi_I \), and choosing \( a \bar{a} = e^{i \xi}/(2 \sqrt{2}) \), we have

\[
\begin{align*}
\varphi_R &= \frac{1}{2} \ln \frac{\cosh (4\Gamma_1) + 4 + \cos (2\xi) - 4\sqrt{2} \cos (\xi) \cosh (2\Gamma_1)}{\cosh (4\Gamma_1) + 4 + \cos \left( 2 \left( \xi + \varepsilon_1 \varepsilon_2 \frac{3\pi}{4} \right) \right) + 4 \cos \left( \xi + \varepsilon_1 \varepsilon_2 \frac{3\pi}{4} \right) \cosh (2\Gamma_1)} \\
\varphi_I &= \text{ArcTan} \left( \frac{A}{B} \right)
\end{align*}
\]

(4.73)

with

\[
A = \varepsilon_1 \varepsilon_2 e^{4\Gamma_1} + 4 \varepsilon_1 \varepsilon_2 + \sin (2\xi) + \varepsilon_1 \varepsilon_2 \cos (2\xi)
\]

\[
B = (\varepsilon_1 \varepsilon_2) \left( e^{4\Gamma_1} + 4 \varepsilon_1 \varepsilon_2 \right) + 4 \varepsilon_1 \varepsilon_2 \left( 2 \left( \xi + \varepsilon_1 \varepsilon_2 \frac{3\pi}{4} \right) \right) + 4 \varepsilon_1 \varepsilon_2 \left( \xi + \varepsilon_1 \varepsilon_2 \frac{3\pi}{4} \right) \cosh (2\Gamma_1)
\]
Figure 2: Plots of the real and imaginary parts of $\varphi$ against $x$, for the one-soliton solution (4.73), with $\varepsilon = \varepsilon_1 \varepsilon_2 = 1$, $\xi = \frac{-3\pi}{8}$. The real part of $\varphi$ corresponds to the curve that goes to zero as $x \to \pm\infty$.

$$B = e^{-4\Gamma_1} + 4 + \varepsilon_1 \varepsilon_2 \sin (2\xi) + \cos (2\xi)$$

$$- \sqrt{2} \left[ (\varepsilon_1 \varepsilon_2 \sin (\xi) + \cos (\xi)) e^{2\Gamma_1} + (\varepsilon_1 \varepsilon_2 \sin (\xi) + 3 \cos (\xi)) e^{-2\Gamma_1} \right]$$

The real part $\varphi_R$ goes to zero as $x \to \pm\infty$. The imaginary part $\varphi_I$ is very sensitive to variations in $\xi$. We give in figures 2 and 3 the plots of the real and imaginary parts for $\xi = \frac{-3\pi}{8}$ and $\xi = \frac{\pi}{3}$, respectively, and in both we have $\varepsilon = \varepsilon_1 \varepsilon_2 = 1$.

Some other simpler forms of the solution $B$ are given in the appendix C.

### 4.4 One-soliton solutions from vacuum (4.37)

In this case we take the element $h$ introduced in (4.43) to be an exponential of one of the four (non-degenerated) eigenvectors $V_{\varepsilon_1}^{(\varepsilon_2)} (\zeta)$, given in (4.59). By replacing that $h$ into (4.55) with $\Psi_{\text{vac}}$ given by (4.41), we get the following ansatz for the tau functions

$$\tau_* = \sum_{l=0}^{N} \delta^{(l)} e^{\Gamma_{\varepsilon_1,\varepsilon_2}}$$

with

$$\Gamma_{\varepsilon_1,\varepsilon_2} = \varepsilon_1 m \left( z \frac{x_+ - x_-}{z} \right) + \varepsilon_2 \sqrt{\frac{3}{2}} \left( \rho'_+ (x_+) \sqrt{z} - \rho'_- (x_-) \sqrt{z} \right)$$

where we have denoted $\zeta^6 \equiv z$.

We solve the equations for the tau functions by the Hirota’s method implement on a computer (see end of section 4.2), and find that the expansion truncates at order
Figure 3: Plots of the real and imaginary parts of $\varphi$ against $x$, for the one-soliton solution (4.73), with $\varepsilon = \varepsilon_1 \varepsilon_2 = 1$, $\xi = \frac{\pi}{3}$. The real part of $\varphi$ corresponds to the curve that goes to zero as $x \to \pm \infty$.

$N = 2$. However, the dressing method in this case does not excite the fields $\varphi, \nu, \eta$, and $\psi^0$ (and also $\eta$ for the reasons explained below (4.49)). Therefore, in all four solutions these fields have their vacuum values, i.e.

\[
\begin{align*}
\varphi &= 0 \\
\eta &= 0 \\
\nu &= -2 m^2 x_+ x_- - \frac{1}{2} \rho_+ (x_+) \rho_- (x_-) + \sigma_- (x_-) + \sigma_+ (x_+) \\
\psi^0 &= \begin{pmatrix} \rho_+ (x_+) \\ \rho_- (x_-) \end{pmatrix}
\end{align*}
\]

The non-vanishing tau functions in the four solutions are given by

1. For $\varepsilon_1 = -1, \varepsilon_2 = -1$

   \[
   \begin{align*}
   \tau_0 &= 1 \\
   \tilde{\tau}_{L,2} &= -\frac{1}{4} \sqrt{z} a_-- e^{2R--} \\
   \tilde{\tau}_{L,3(1)} &= \frac{1}{2} a-- e^{R--} \\
   \tau_1 &= 1 \\
   \tilde{\tau}_{L,3(0)} &= a-- e^{R--} \\
   \tau_{R,1} &= -\sqrt{z} a-- e^{R--}
   \end{align*}
   \]

2. For $\varepsilon_1 = -1, \varepsilon_2 = 1$

   \[
   \begin{align*}
   \tau_0 &= 1 \\
   \tilde{\tau}_{L,2} &= \frac{1}{4} \sqrt{z} a_+ e^{2R--} \\
   \tilde{\tau}_{L,3(1)} &= \frac{1}{2} a-- e^{R--} \\
   \tau_1 &= 1 \\
   \tilde{\tau}_{L,3(0)} &= a-- e^{R--} \\
   \tau_{R,1} &= \sqrt{z} a-- e^{R--}
   \end{align*}
   \]
3. For $\varepsilon_1 = 1$, $\varepsilon_2 = -1$

\[
\begin{align*}
\tau_0 &= 1 \\
\tilde{\tau}_{L,1} &= \frac{2}{\sqrt{a}} a_{+-} e^{\Gamma_+} \\
\tau_{R,3(0)} &= 2 a_{+-} e^{\Gamma_+} \\
\tau_1 &= 1 \\
\tau_{R,2} &= \frac{1}{\sqrt{2}} a_{+-}^2 e^{2\Gamma_+} \\
\tau_{R,3(1)} &= a_{+-} e^{\Gamma_+}
\end{align*}
\]

4. For $\varepsilon_1 = 1$, $\varepsilon_2 = 1$

\[
\begin{align*}
\tau_0 &= 1 \\
\tilde{\tau}_{L,1} &= \frac{2}{\sqrt{a}} a_{++} e^{\Gamma_+} \\
\tau_{R,3(0)} &= 2 a_{++} e^{\Gamma_+} \\
\tau_1 &= 1 \\
\tau_{R,2} &= -\frac{1}{\sqrt{a}} a_{++}^2 e^{2\Gamma_+} \\
\tau_{R,3(1)} &= a_{++} e^{\Gamma_+}
\end{align*}
\]

5. The model for $l = 3$

As discussed at the end of section 3 we have two types of models defined by the potentials (3.1) which are of more interest. In section 4 we have discussed the first case, and here we present the second one. In this case we take the constant elements $E_{\pm l}$ in (3.1), with $l = 3$ and given by

\[
E_{\pm 3} \equiv m L_0^{1/2}
\]

where again $m$ is a parameter setting the mass scale of the particles and solitons of the theory. The potentials (3.1) are

\[
\begin{align*}
A_+ &= -B \left[ E_{+3} + \sum_{n=1}^{2} F^n_+ \right] B^{-1} \\
A_- &= -\partial_- B B - 1 + E_{-3} + \sum_{n=1}^{2} F^n_- 
\end{align*}
\]

with $B$ being the same group element as in (2.7), and

\[
\begin{align*}
F^1_+ &= \chi_R L_{-3}^{-1/2} + \left( \sqrt{\beta_1/2} \right) \psi_R T^0 \\
F^2_+ &= \sqrt{\beta_1} \psi_+ L_{-3}^{-1/2} \\
F^1_- &= \chi_L L_2^{-1/2} + \left( \sqrt{\beta_1/2} \right) \tilde{\psi}_L T^0 \\
F^2_- &= \sqrt{\beta_1} \psi_+ L_1^{-1/2}
\end{align*}
\]

where $\beta_1$ is a free parameters, rescaling the spinors. By imposing the zero curvature condition (2.5) on the potentials (5.78) we get the equations of motion

\[
\begin{align*}
\partial^2 \varphi - \frac{i}{2} e^{\eta-2\varphi} \bar{\chi} \gamma_5 \chi + \frac{i}{2} \beta_1 \bar{\psi} W (\eta) \gamma_5 V \psi &= 0 \\
\partial^2 \nu + \frac{1}{2} m^2 e^{3\eta} + \frac{i}{4} e^{\eta-2\varphi} \bar{\chi} \gamma_5 \chi + \frac{i}{2} \beta_1 e^{2\eta-\varphi} \bar{\psi} \left( 1 - \gamma_5 \right) \frac{1}{2} \psi &= 0 \\
\partial^2 \eta &= 0
\end{align*}
\]

(5.79)
\[ r_{\gamma}^\mu \partial_\mu \psi + \sqrt{6} m W(\eta) V \psi + U = 0 \]
\[ r_{\gamma}^\mu \partial_\mu \tilde{\psi} + \sqrt{6} m \tilde{W}(\eta) \tilde{V} \tilde{\psi} + \tilde{U} = 0 \]
\[ r_{\gamma}^\mu \partial_\mu \chi + \tilde{U} = 0 \]

where
\[
W(\eta) \equiv \frac{(1 + \gamma_5)}{2} + e^{3\eta} \frac{(1 - \gamma_5)}{2} \quad \tilde{W}(\eta) \equiv e^{3\eta} \frac{(1 + \gamma_5)}{2} + \frac{(1 - \gamma_5)}{2} \quad (5.80)
\]
\[ V = e^{(\eta + \varphi)\gamma_5} \quad \tilde{V} = e^{-(\eta + \varphi)\gamma_5} \quad (5.81) \]

and
\[
U = \begin{pmatrix} 0 \\ e^{\eta - 2\varphi} \tilde{\psi}_R \chi_L \end{pmatrix} \quad \tilde{U} = \begin{pmatrix} e^{\eta - 2\varphi} \psi_L \chi_R \\ 0 \end{pmatrix} \quad U = \begin{pmatrix} \beta_1 e^{\eta + \varphi} \psi_R \psi_L \\ \beta_1 e^{\eta + \varphi} \tilde{\psi}_R \tilde{\psi}_L \end{pmatrix} \quad (5.82)
\]

The spinors have the form
\[
\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \quad \tilde{\psi} = \begin{pmatrix} \tilde{\psi}_R \\ \tilde{\psi}_L \end{pmatrix} \quad \chi = \begin{pmatrix} \chi_R \\ \chi_L \end{pmatrix} \quad (5.83)
\]

and we have defined
\[
\bar{\psi} = \bar{\psi}^T \gamma_0 \quad \bar{\chi} = \bar{\chi}^T \gamma_0 \quad \bar{U} = \bar{U}^T \gamma_0 \quad \bar{U} = \bar{U}^T \gamma_0 \quad (5.84)
\]

5.1 \( \tau \)-functions and solutions

We construct the soliton solutions using the dressing method, as described in sections 2.1 and 4.2 starting from the vacuum solutions
\[ \varphi = \eta = \psi = \tilde{\psi} = \chi = 0 \quad \nu = -\frac{1}{2} m^2 x_+ x_- \quad (5.85) \]

Then we select a set of \( \tau \)-functions appropriate to express the solutions for the fields.

The and the field The \( \tau \)-functions are chosen as
\[
\tau_0 = \langle \lambda_0 | \Psi_{vac} h \Psi_{vac}^{-1} | \lambda_0 \rangle \quad \tau_1 = \langle 2 \lambda_1 | \Psi_{vac} h \Psi_{vac}^{-1} | 2 \lambda_1 \rangle \quad (5.86)
\]
\[
\tau_0^R = \langle \lambda_0 | L_{-2}^R \Psi_{vac} h \Psi_{vac}^{-1} | \lambda_0 \rangle \quad \tau_1^L = \langle 2 \lambda_1 | L_{-2}^R \Psi_{vac} h \Psi_{vac}^{-1} | 2 \lambda_1 \rangle \quad (5.86)
\]

The relations among the \( \tau \)-functions and the fields are
\[
\varphi = \log \frac{\tau_0}{\tau_1} \quad \nu = -\frac{1}{2} \log \tau_0 - \frac{1}{2} m^2 x_+ x_- \quad (5.86)
\]
\[
\psi_R = \frac{1}{\sqrt{\beta_1 \tau_0}} \tau_1 \partial_+ \left( \frac{\tau_1 L}{\tau_1} \right) \quad \psi_L = \sqrt{\frac{3 m}{2 \sqrt{\beta_1 \tau_1}}} \tau_1 L \quad (5.86)
\]
\[
\tilde{\psi}_R = -\sqrt{\frac{3 m}{2 \sqrt{\beta_1 \tau_1}}} \tau_1^R \partial_+ \left( \frac{\tau_1 R}{\tau_1} \right) \quad \tilde{\psi}_L = \frac{1}{\sqrt{\beta_1 \tau_0}} \tau_1 \partial_+ \left( \frac{\tau_1 R}{\tau_1} \right) \quad (5.86)
\]
\[
\chi_R = \frac{\tau_0^2}{\tau_1} \partial_+ \left( \frac{\tau_0 L}{\tau_0} \right) \quad \chi_L = \frac{\tau_0^2}{\tau_1} \partial_- \left( \frac{\tau_0 R}{\tau_0} \right) \quad (5.86)
\]
Replacing those relations into the equations of motion we get the Hirota’s equations for the $\tau$-functions. We then solve those using the Hirota’s recurrence method to get the one-soliton solution

$$
\begin{align*}
\tau_0 &= 1 - \frac{1}{4} a b c e^{2 \Gamma} \\
\tau_1 &= 1 \\
\tau_0^R &= -\frac{1}{4} z b^2 e^{2 \Gamma} \\
\tau_0^L &= \frac{1}{4 z} a^2 e^{2 \Gamma} \\
\tau_1^R &= a e^\Gamma \\
\tau_1^L &= b e^\Gamma
\end{align*}
$$

(5.87)

where

$$
\Gamma = \sqrt{\frac{3}{2}} m(x_+ z - x_- z^{-1})
$$

(5.88)

The solution in terms of the fields read

$$
\begin{align*}
\varphi &= \log \left(1 - \frac{1}{4} a b c e^{2 \Gamma}\right) \\
\nu &= -\frac{1}{2} \log \left(1 - \frac{1}{4} a b c e^{2 \Gamma}\right) - \frac{1}{2} m^2 x_+ x_- \\
\chi_R &= \frac{1}{2} \sqrt{\frac{3}{2}} m a^2 e^{2 \Gamma} \\
\chi_L &= \frac{1}{2} \sqrt{\frac{3}{2}} m b^2 e^{2 \Gamma} \\
\psi_R &= -2 \sqrt{6z} \frac{m}{\sqrt{\beta_1}} \frac{b e^\Gamma}{-4 + a b e^{2 \Gamma}} \\
\psi_L &= \sqrt{\frac{3}{2}} \frac{m}{\sqrt{\beta_1}} b e^\Gamma \\
\tilde{\psi}_R &= -\sqrt{\frac{3}{2}} \frac{m}{\sqrt{\beta_1}} a e^\Gamma \\
\tilde{\psi}_L &= 2 \sqrt{6} \frac{1}{z} \frac{m}{\sqrt{\beta_1}} \frac{a e^\Gamma}{-4 + a b e^{2 \Gamma}}
\end{align*}
$$
A The twisted affine Kac-Moody algebra $A_2^{(2)}$

The generators of the algebra $A_2^{(2)}$ are given by $T_3^m$, $T_3^m$, and $L_j^r$, with $m,n \in \mathbb{Z}$, $r,s \in \mathbb{Z} + \frac{1}{2}$ and $j,k = 0, \pm 1, \pm 2$, and the commutation relations are

$$[T_3^m, T_3^n] = 2m \delta_{m+n,0} C$$
$$[T_3^m, T_3^\pm] = \pm T_3^{m+n}$$
$$[T_3^m, L_k^r] = k L_k^{m+r}$$
$$[T_3^m, L_k^s] = \sqrt{6 - k(k+1)} L_k^{m+r}$$

$$[L_k^r, L_{-k}^s] = (-1)^k \left( \frac{k}{2} T_3^{r+s} + r \delta_{r+s,0} C \right)$$
$$[L_0^r, L_{\pm 1}^s] = -\frac{\sqrt{6}}{4} T_3^{r+s}$$
$$[L_1^r, L_{-2}^s] = \frac{1}{2} T_3^{r+s}$$
$$[L_1^r, L_2^s] = \frac{1}{2} T_3^{r+s}$$

Notice that those commutation relations are compatible with the hermiticity conditions $T_3^m \dagger = T_3^{-m}$, $T_3^\pm \dagger = T_3^{-\pm}$, $L_0^r \dagger = L_0^{-r}$, $L_1^r \dagger = -L_1^{-r}$, and $L_2^r \dagger = L_2^{-r}$.

In the text we have made use of the so-called principal gradation of the algebra $A_2^{(2)}$, i.e.

$$\mathcal{G} = \bigoplus_{n=-\infty}^{\infty} \mathcal{G}_n$$

$$[Q, \mathcal{G}_n] = n \mathcal{G}_n$$

$$[\mathcal{G}_m, \mathcal{G}_n] \subset \mathcal{G}_{m+n}$$

and where the grading operator $Q$ is

$$Q \equiv T_3^0 + 6D$$

The operator $D$ measures the upper index of the generators, i.e.

$$[D, T_j^m] = m T_j^m$$

$$[D, L_p^{m+1/2}] = \left( m + \frac{1}{2} \right) L_p^{m+1/2}$$

with $m \in \mathbb{Z}$, $j = 3, \pm$, and $p = 0, \pm 1, \pm 2$. Therefore the eigensubspaces of $Q$ are

$$\mathcal{G}_0 = \{ T_3^0, Q, C \}$$
$$\mathcal{G}_{6n} = \{ T_3^n \} \quad n \neq 0$$
$$\mathcal{G}_{6n+1} = \{ T_3^m, L_{-2}^{n+1/2} \}$$
$$\mathcal{G}_{6n+2} = \{ L_{-1}^{n+1/2} \}$$
$$\mathcal{G}_{6n+3} = \{ L_0^{n+1/2} \}$$
$$\mathcal{G}_{6n+4} = \{ L_1^{n+1/2} \}$$
$$\mathcal{G}_{6n+5} = \{ T_3^{-1}, L_2^{n+1/2} \}$$
A highest weight representation of $\mathcal{G}$ is one possessing a highest weight state $|\lambda\rangle$ satisfying
\[
\mathcal{G}_n |\lambda\rangle = 0 \quad \langle \lambda | \mathcal{G}_{-n} = 0 \quad \text{for } n > 0 \quad (A.6)
\]
The highest weight states of the two fundamental representations of $A_2^{(2)}$ satisfy
\[
T_3^0 | \lambda_0 \rangle = 0 \quad T_3^0 | \lambda_1 \rangle = \frac{1}{2} | \lambda_1 \rangle \\
C | \lambda_0 \rangle = 2 | \lambda_0 \rangle \quad C | \lambda_1 \rangle = | \lambda_1 \rangle \quad (A.7)
\]

**B  Coefficients of $\tau$-functions of soliton of sec. 4.3.2**

The $\tau$-functions for the solution described in section 4.3.2 truncate at at order 16. We express the $\tau$-functions as
\[
\tau_* = \sum_{l=0}^{16} \delta_*(l) e^{i \Gamma_1} \quad (B.1)
\]
where * stands for any of the labels of the $\tau$-functions defined in (4.55). The quantity $\Gamma_1$ is defined in (1.68). The following coefficients $\delta_*(l)$ vanish
\[
\delta_0^{(2n+1)} = \delta_1^{(2n+1)} = \tilde{\delta}_{L,2}^{(2n+1)} = \delta_{R,2}^{(2n+1)} = 0 \quad n = 1, 2, \ldots 7 \quad (B.2)
\]
\[
\tilde{\delta}_{L,1}^{(2n)} = \tilde{\delta}_{L,3(0)}^{(2n)} = \tilde{\delta}_{L,3(1)}^{(2n)} = \delta_{R,1}^{(2n)} = \delta_{R,3(0)}^{(2n)} = \delta_{R,3(1)}^{(2n)} = 0 \quad n = 0, 1, 2, \ldots 8
\]
The remaining coefficients $\delta_*(l)$ are given by
\[
\delta_0^{(0)} = 1 \\
\delta_0^{(2)} = -8\bar{a}\bar{a} \\
\delta_0^{(4)} = \frac{4}{z} \left( (7\bar{a}^2 z - 3\bar{b}^2) \bar{a}^2 + \bar{b}(4\bar{a}b + c)z \bar{a} + z (-b^2b^2 + 3a^2z\bar{b}^2 - ab \bar{c}) \right) \\
\delta_0^{(6)} = -\frac{8}{z} \left( 7\bar{a}z (\bar{a}^2 + \bar{b}^2) a^3 + 2b z \left( 2\bar{b}^3 + 4\bar{a}^2\bar{b} - a\bar{c} \right) a^2 \\
+ \left( -7b^2\bar{a}^3 + 2\bar{b}c\bar{a}^2 - 3\bar{b}^2\bar{b}^2 z \bar{a} + 2\bar{b}z (\bar{b}^2cz - b^2\bar{c}) \right) a \\
- b(2\bar{a}b - \bar{c}) \left( 2\bar{a}^2 - \bar{b}c z \right) \right) \\
\delta_0^{(8)} = \frac{1}{z^2} \left( 2 \left( 35\bar{a}^2 a^4 - 54b^2z^2a^2 + 19b^4 \right) \bar{a}^4 + 24\bar{b} (4\bar{a}b + c)z \left( a^2 z - b^2 \right) \bar{a}^3 \\
+ 4z \left( b^2 - a^2 \right) \left( 7b^2\bar{b}^2 - 27a^2z\bar{b}^2 - 6ab\bar{c} \right) a^2 \\
+ 4\bar{b}z \left( 4\bar{c}b^4 - 24\bar{a}\bar{b}^2z\bar{b}^3 - 6\bar{b}^2cz\bar{b}^2 + 4az \left( 6a^2z\bar{b}^2 + c\bar{c} \right) \right) b \\
+ cz \left( 6a^2z\bar{b}^2 + c\bar{c} \right) \bar{a}
\right)
\]

35
\[
\begin{align*}
\delta_0^{(10)} &= -\frac{8}{z^2} (b^2 - a^2 z) (a^2 + b^2 z) (-7a z (a^2 + b^2 z) a^3 ) + 2b z (-2z b^3 - 4a^2 b + 6a c) a^2 \\
&+ (7b^3 a^3 - 2b cz a^2 + 3b^2 b^2 z a + 2b z (b^2 c - b c z)) a \\
&+ b(2a b - c)(2a b - b c z)
\end{align*}
\]

\[
\begin{align*}
\delta_0^{(12)} &= -\frac{4}{z^3} (b^2 - a^2 z)^2 (a^2 + b^2 z)^2 ((3b^2 - 7a^2 z) a^2 - b(4ab + c) z a \\
&+ z (b^2 b^2 - 3a^2 z b^2 + ab c))
\end{align*}
\]

\[
\begin{align*}
\delta_0^{(14)} &= -\frac{8a a (a^2 z - b^2)^3 (a^2 + b^2 z)^3}{z^3}
\end{align*}
\]

\[
\begin{align*}
\delta_0^{(16)} &= (b^2 - a^2 z)^4 (a^2 + b^2 z)^4 \\
\delta_1^{(0)} &= 1
\end{align*}
\]

\[
\begin{align*}
\delta_1^{(2)} &= 4b b - 4a a
\end{align*}
\]

\[
\begin{align*}
\delta_1^{(4)} &= 4a^2 a^2 + 4b c a + 4b^2 b^2 - 4a b c - 2c c
\end{align*}
\]

\[
\begin{align*}
\delta_1^{(6)} &= \frac{4}{z} (a z (a^2 + b^2 z) a^3 + b z (z b^3 - 3a^2 b + 2a c) a^2 \\
&+ (-b^2 a^3 - 2b c z a^2 + (3b^2 b^2 + 2c c) z a - 2b^2 b c z) a \\
&= b b (a^2 b^2 + b^2 z b^2 - 2a b c z + c c z))
\end{align*}
\]

\[
\begin{align*}
\delta_1^{(8)} &= \frac{1}{z^2} (-2 (5z^2 a^4 - 6b^2 z a^2 + b^4) a^4 \\
&- 4b^2 z (3b^4 - 10a^2 z b^2 - 4a c z b + z (3a^4 z - c^2)) a^2 \\
&- 4b(2a b + c)^2 c z^2 a \\
&+ z^2 (-10 b^4 b^4 - 2a^4 z^2 b^4 + c^2 c^2 + 4a b c c^2 + 4 a^2 b^2 (3z b^4 + c^2)))
\end{align*}
\]

\[
\begin{align*}
\delta_1^{(10)} &= -\frac{4}{z^2} (b^2 - a^2 z) (a^2 + b^2 z) (a z (a^2 + b^2 z) a^3 + b z (z b^3 - 3a^2 b + 2a c) a^2 \\
&- (b^2 a^3 - 2b c z a^2 - 3b^2 b^2 z a + c c z a + 2b^2 b c z) a \\
&+ b b (a^2 b^2 + b^2 z b^2 + 2a b c z - c c z))
\end{align*}
\]
\[\delta_{L,1}^{(12)} = \frac{2 \left(2a^2\tilde{a}^2 + 2b^2\tilde{b}^2 + 2abc + c (\tilde{c} - 2\tilde{a}\tilde{b})\right) (b^2 - a^2 z)^2 \left(\tilde{a}^2 + \tilde{b}^2 z\right)^2}{z^2}\]

\[\delta_{L,1}^{(14)} = -\frac{4(aa + \tilde{b}\tilde{b}) (a^2 z - \tilde{b}^2)^3 \left(\tilde{a}^2 + \tilde{b}^2 z\right)^3}{z^3}\]

\[\delta_{L,1}^{(16)} = \frac{(b^2 - a^2 z)^4 \left(\tilde{a}^2 + \tilde{b}^2 z\right)^4}{z^4}\]

\[\tilde{\delta}_{L,1}^{(0)} = 0\]

\[\tilde{\delta}_{L,1}^{(1)} = 4a\]

\[\tilde{\delta}_{L,1}^{(3)} = -20a\tilde{a}^2 - 4bc + \frac{12a\tilde{b}^2}{z}\]

\[\tilde{\delta}_{L,1}^{(5)} = \frac{4}{z} \left(3z \left(3a^2 + \tilde{b}^2 z\right) a^3 + b(4 \tilde{a}\tilde{b} - 3\tilde{c}) z a^2 - \left(9a^2\tilde{b}^2 + 7\tilde{b}^2 z b^2 - 4\tilde{a}\tilde{b}cz + c\tilde{c} z\right) a + 4a\tilde{b}^3 \tilde{b} + \tilde{b}^3 \tilde{c} - 2\tilde{b}^2 cz\right)\]

\[\tilde{\delta}_{L,1}^{(7)} = -\frac{1}{z^2} \left(4 \left(\tilde{a}^2 z - b^2\right) \left(z \left(5a^4 + 6\tilde{b}^2 z a^2 + b^4\right) \tilde{a}^3 + 2 \tilde{b}(4ab + 3c) z a^2 z - b^2\right) \tilde{a}^2 + 2 \tilde{a}(4\tilde{a}\tilde{b} - 3\tilde{c}) z a^2 - \tilde{b} \left(2\tilde{b}^4 - 8a\tilde{b}^2 z b^3 + 2a^2 \tilde{c}z b^2\right) + 4az \left(2a^2 z\tilde{b}^2 + \tilde{c} c\right) b + c^2 \tilde{c} z\right)\]

\[\tilde{\delta}_{L,1}^{(9)} = -\frac{1}{z^2} \left(4 \left(\tilde{a}^2 z - b^2\right) \left(z \left(5a^4 + 6\tilde{b}^2 z a^2 + b^4\right) \tilde{a}^3 + 6\tilde{b}cz \left(\tilde{a}^2 + \tilde{b}^2 z\right) a^2 + \left(-5b^2\tilde{a}^4 - 4 \tilde{b}cz \tilde{a}^3 + \left(3cz - 10b\tilde{b}^2 z\right) \tilde{a}^2 + 4\tilde{b}z \left(3b z - 2\tilde{c} z\right) \tilde{a}^2 - 4\tilde{b}z \left(2a^2 z\tilde{b}^2 + \tilde{c}^2 z\right) a + \tilde{b} \left(8b^2 \tilde{a}^3 + 2\tilde{b}^2 cz a^2 + \left(8b^2 \tilde{b}^3 z - 4 \tilde{b}cz\right) \tilde{a} + cz \left(\tilde{c}^2 - 2\tilde{b}^4 z\right) b z\right)\right)\right)\]

\[\tilde{\delta}_{L,1}^{(11)} = \frac{4}{z^3} \left(\tilde{b}^2 - a^2 z\right)^2 \left(\tilde{a}^2 + \tilde{b}^2 z\right) \left(-3 \left(b^2 - 3a^2 z\right) \tilde{a}^3 + \tilde{b}(8ab - c) z a^2 + b \left(\tilde{b}^2 z^2 + 9a^2 z\tilde{b}^2 + \tilde{c} z\right) \tilde{a} + \tilde{b}z \left(-2\tilde{b}^2 + 8ab^2 z b + \tilde{b}^2 cz\right)\right)\]

\[\tilde{\delta}_{L,1}^{(13)} = \frac{4}{z^3} \left(\tilde{b}^2 - a^2 z\right)^3 \left(\tilde{a}^2 + \tilde{b}^2 z\right)^2 \left(5\tilde{a}^2 + 4\tilde{b}a - \tilde{b}^2 + 3a^2 z\right)\]

\[\tilde{\delta}_{L,1}^{(15)} = \frac{4\tilde{a} \left(\tilde{b}^2 - a^2 z\right)^4 \left(\tilde{a}^2 + \tilde{b}^2 z\right)^3}{z^4}\]
\begin{align*}
delta L_{(0)} &= 0 \\
\delta L_{(2)} &= 2\bar{c} \\
\delta L_{(4)} &= 8 \left( b \left( \bar{a}^3 \frac{z}{z} - \bar{b}^2 a + \bar{b} \bar{c} \right) + a \left( -z\bar{b}^3 + \bar{a}^2 \bar{b} - a \bar{c} \right) \right) \\
\delta L_{(6)} &= -\frac{2}{z} \left( -c\bar{a}^4 + 16\bar{b}^2 \bar{a}^2 + \left( 2 \bar{b}^2 cz - 6\bar{b}^2 \bar{c} \right) \bar{a}^2 + 4\bar{b} \left( 4\bar{b}^2 \bar{b}^2 - cc^2 \right) \bar{a} \right) \\
&\quad + 2a^2 (8\bar{a} - 3\bar{c}) (a^2 + \bar{b}^2 z) + z \left( -cz \bar{b}^4 - 6\bar{b}^2 \bar{c}^2 + cc^2 \right) \\
&\quad + 4ab \left( 3 \bar{a}^4 + 10\bar{b}^2 z a^2 - 4\bar{b} \bar{c} \bar{a} + z \left( 3z\bar{b}^4 + c^2 \right) \right) \\
\delta L_{(8)} &= \frac{8}{z^3} \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( -4\bar{b}^3 - 2a^2 z b + acz \right) a^3 + \bar{b} z \left( 6za^3 - 8\bar{b}^2 a - bc \right) \bar{a}^2 \\
&\quad + z \left( -\bar{c}za^3 + 8\bar{b}^2 za^2 + 3\bar{b}^2 \bar{c} a + \bar{b}^2 cz a - 6\bar{b}^3 \bar{b}^2 + bcc \right) \bar{a} \\
&\quad + \bar{b} z \left( \bar{c}b^3 - 2a^2 b \bar{b}^2 + \left( \bar{b}^2 c - 3a^2 \bar{c} \right) zb + az \left( 4a^2 \bar{b}^2 z - cc \right) \right) \\
\delta L_{(10)} &= \frac{2}{z^3} \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( (\bar{c} - 16 \bar{a}b) z^2 a^4 + 8bz \left( \bar{a}^2 - \bar{b}^2 z \right) a^3 \\
&\quad + 2z \left( 3\bar{c}a^2 + 16\bar{b}^2 \bar{a} + \bar{b}^2 \bar{c} - 3 \bar{b}^2 cz \right) a^2 + \left( -8\bar{a}^2 \bar{b}^3 + 8\bar{b}^2 \bar{b}^3 + 4c \bar{c} \bar{b} \right) a \\
&\quad - 16\bar{a}b^4 \bar{b} - 6\bar{a}^2 \bar{b}^2 c + b^4 \bar{c} + 6\bar{b}^2 \bar{b}^2 c + c^2 \bar{c} \right) \\
\delta L_{(12)} &= -\frac{8}{z^3} \left( \bar{b}^2 - a^2 z \right) \left( \bar{a}^2 + \bar{b}^2 z \right)^3 \left( \bar{b} z \left( za^3 - 3b^2 a - bc \right) + \bar{a} \left( b^3 - 3a^2 z b - acz \right) \right) \\
\delta L_{(14)} &= \frac{2(4ab + c) \left( b^2 - a^2 z \right)^2 \left( \bar{a}^2 + \bar{b}^2 z \right)^4}{z^3} \\
\delta L_{(16)} &= 0 \\
\delta L_{(1)} &= 4\bar{b} \\
\delta L_{(3)} &= 4 \left( a(\bar{c} - 4\bar{a}b) + b \left( \frac{3}{z} \bar{a}^2 + \bar{b}^2 \right) \right) \\
\delta L_{(5)} &= \frac{4}{z} \left( ca^3 - 15b^2 \bar{a} \bar{a}^2 + 6\bar{b}^2 \bar{c} \bar{a} + 3\bar{b}^2 cz \bar{a} - 8ab \left( \bar{a}^2 + \bar{b}^2 z \right) \bar{a} - 3\bar{b}^2 b^3 z \bar{a} \right) \bar{a}^2 \\
&\quad - \bar{b} \bar{c} \bar{e} \bar{z} + a^2 z \left( \bar{b} \bar{b}^3 + 9a^2 \bar{b} - 4\bar{a} \bar{c} \right) \bar{a}^2 \\
\delta L_{(7)} &= \frac{1}{z^2} \left( \bar{a} \left( \bar{a}^3 - 3a^2 \bar{b} + 4acz \right) \bar{a}^4 + 4\bar{b} \bar{z} \left( 4za^3 - 2b^2 a + bc \right) \bar{a}^3 \right) \\
&\quad + z \left( 6\bar{c} za^3 + 22 \bar{b} \bar{b}^2 za^2 + 14\bar{b}^2 cz a + 6b^3 \bar{b}^2 - 3bc \bar{c} \right) \bar{a}^2 
\end{align*}
\[ \tilde{\omega}^{(9)}_{L;3(0)} = \frac{1}{z^2} \left( 4 \left( \bar{a}^2 + \bar{b}^2 z \right) \left( z^2 \left( 11 z \bar{b}^3 + 19 \bar{a}^2 \bar{b} - 4 \bar{a} \bar{c} \right) a^4 + 8 b z \left( a^3 + 3 \bar{b}^2 z \bar{a} - \bar{b} \bar{c} z \right) a^3 + z \left( 6 \bar{c} a^3 - 34 b^2 b \bar{a}^2 + 2 \left( 7 c b^2 + 6 \bar{b}^2 c z \right) \bar{a} - 3 \bar{b} \left( 2 b^2 \bar{b}^2 + c \bar{c} \right) z \right) a^2 + 4 b \left( -2 b^2 a^3 + (c \bar{c} z - 6 b^2 \bar{b}^2 z) \bar{a} + 2 b z \left( \bar{c} b^2 + \bar{b}^2 c z \right) \right) a + 15 \bar{a}^2 b^4 \bar{b} - 6 \bar{a}^3 b^2 c + \bar{b} z \left( 3 b^2 b^4 + 3 c \bar{c} b^2 + 2 \bar{b}^2 c^2 z \right) + \bar{a} \left( -6 \bar{b}^4 - 12 \bar{b}^2 c z \bar{b}^2 + c^2 c z \right) \right) \]

\[ \tilde{\omega}^{(11)}_{L;3(0)} = \frac{4}{z^2} \left( b^2 - a^2 z \right) \left( a^2 + b^2 z \right)^2 \left( b^3 + 7 a^2 z b + 4 a c z \right) \bar{a}^2 + 4 b z \left( 3 z a^3 - b^2 a + b c \right) \bar{a} + z \left( -c z a^3 + 9 b b^2 z a^2 - b^2 c a + 6 b^2 c z a + 3 b^3 b^2 - b c \bar{c} \right) \right) \]

\[ \tilde{\omega}^{(13)}_{L;3(0)} = \frac{4 (b^2 - a^2 z)^2 \left( 3 b z a^2 - b^2 \bar{b} + \bar{a} c \right) \left( a^2 + b^2 z \right)^3}{z^3} \]

\[ \tilde{\omega}^{(15)}_{L;3(0)} = -\frac{4 b (b^2 - a^2 z)^3 \left( a^2 + b^2 z \right)^4}{z^4} \]

\[ \tilde{\omega}^{(1)}_{L;3(1)} = 2 \bar{b} \]

\[ \tilde{\omega}^{(3)}_{L;3(1)} = \frac{6 b a^2}{z} + 10 b b^2 - 2 a \bar{c} \]

\[ \tilde{\omega}^{(5)}_{L;3(1)} = -\frac{2}{z} \left( -c a^3 - 9 b^2 b a^2 - 3 \bar{b}^2 c z \bar{a} - 9 b^2 \bar{b}^3 z + \bar{b} c \bar{c} z + a^2 z \left( 3 z \bar{b}^3 + 7 a^2 \bar{b} - 2 \bar{a} \bar{c} \right) \right) + 4 a b \left( a^3 - \bar{b}^2 z \bar{a} + \bar{b} \bar{c} z \right) \right) \]

\[ \tilde{\omega}^{(7)}_{L;3(1)} = \frac{1}{z^2} \left( 2 \left( b^3 - 5 a^2 z b - 2 a c z \right) a^4 + 8 b z \left( z a^3 + b^2 a + b c \right) \bar{a}^3 + z \left( 6 \bar{b}^2 b^3 - 6 a c b^2 - 3 c \bar{b} \bar{c} b^2 + 2 a b^2 c z \right) \bar{a}^2 + 4 b z^2 \left( 2 b^2 z a^3 - 2 b c a^2 + 2 b^2 b^2 a - c \bar{c} a + 2 b b^2 c \right) \bar{a} + z^2 \left( 5 b^3 b^4 - 6 a b^2 c \bar{b}^2 + a c c^2 + b \left( -5 a^2 z b^4 - 3 c \bar{b} \bar{c}^2 + 2 a^2 c^2 \right) \right) \right) \]

\[ \tilde{\omega}^{(9)}_{L;3(1)} = -\frac{1}{z^2} \left( 2 \left( a^2 + \bar{b}^2 z \right) \left( z \bar{b}^3 - 3 a^2 \bar{b} + 2 a \bar{c} \right) a^4 - 4 b z \left( 2 a^3 + \bar{b} \bar{c} z \right) \bar{a}^3 \right) \]
\[+ z \left( -6b^2 z \bar{b}^3 + 6ac z \bar{b}^2 - 10a^2 b^2 \bar{b} - 3c \bar{c} z \bar{b} + 2\bar{a} \bar{b}^2 \bar{c} \right) a^2 \]
\[+ 4b \left( 2b^2 a^3 - 2b c z a^2 + c \bar{c} z \bar{a} + b^2 \bar{c} \bar{z} \right) a + \bar{a} \bar{c} \left( c \bar{c} - 6b^2 \bar{b}^3 \right) z \]
\[+ b^2 \bar{b} \left( 5b^2 \bar{b}^2 + 3c \bar{c} \right) z + \bar{a} \bar{b} \left( 5b^4 - 2c^2 z^2 \right) \right] \right) \right) \]
\[\bar{\delta}_{L,3(1)}^{(11)} = -\frac{2}{z^3} \left( b^2 - a^2 z \right) \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( \left( 3b^3 + a^2 z b + 2a c z \right) \bar{a}^2 + 8a \bar{b} z \left( b^2 - a^2 z \right) \bar{a} \right) \]
\[\bar{\delta}_{L,3(1)}^{(13)} = -2 \frac{2 \left( b^2 - a^2 z \right)^2 \left( -3b z a^2 + 4a b a + 5 b^2 b + a c \right) \left( \bar{a}^2 + b^2 z \right)^3}{z^3} \]
\[\bar{\delta}_{L,3(1)}^{(15)} = -\frac{2b \left( b^2 - a^2 z \right)^3 \left( \bar{a}^2 + b^2 z \right)^4}{z^4} \]
\[\delta_{R,1}^{(1)} = 4\bar{a} \]
\[\delta_{R,1}^{(3)} = -4 \left( 5a a^2 - b\bar{c} + 3a b^2 \bar{z} \right) \]
\[\delta_{R,1}^{(5)} = \frac{4}{z} \left( -3 \left( b^2 - 3a^2 z \right) \bar{a}^3 + \bar{b} \left( 4ab + 3c \right) z \bar{a}^2 \right) \]
\[\delta_{R,1}^{(7)} = -\frac{1}{z} \left( 4 \left( z \left( 5a^4 + 6b^2 z a^2 + b^4 \right) a^3 + 2b \left( 4a \bar{b} - 3\bar{c} \right) z \left( \bar{a}^2 + \bar{b}^2 z \right) a^2 \right) \right) \]
\[\delta_{R,1}^{(9)} = -\frac{1}{z^2} \left( 4 \left( \bar{a}^2 + \bar{b}^2 z \right) \left( 5z^2 a^4 - 6b^2 z a^2 + b^4 \right) \bar{a}^3 + 6b c z \left( b^2 - a^2 z \right) \bar{a}^2 \right) \]
\[\delta_{R,1}^{(11)} = -\frac{4}{z^2} \left( a^2 z - b^2 \right) \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( 3z \left( 3\bar{a}^2 + \bar{b}^2 z \right) \bar{a}^3 + b \left( 8a \bar{b} + \bar{c} \right) z a^2 \right) \]
\[\bar{z} \right) \bar{a} \bar{b} \left( 2c^2 - 5a^4 \bar{z} - 3a^2 c \bar{c} \right) \bar{a} \]
\[\delta_{R,1}^{(11)} = -\frac{4}{z^2} \left( a^2 z - b^2 \right) \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( 3z \left( 3\bar{a}^2 + \bar{b}^2 z \right) \bar{a}^3 + b \left( 8a \bar{b} + \bar{c} \right) z a^2 \right) \]
\[\bar{a} \bar{b} \left( 2c^2 - 5a^4 \bar{z} - 3a^2 c \bar{c} \right) \bar{a} \]
\[\delta_{R,1}^{(11)} = -\frac{4}{z^2} \left( a^2 z - b^2 \right) \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( 3z \left( 3\bar{a}^2 + \bar{b}^2 z \right) \bar{a}^3 + b \left( 8a \bar{b} + \bar{c} \right) z a^2 \right) \]
\[\bar{a} \bar{b} \left( 2c^2 - 5a^4 \bar{z} - 3a^2 c \bar{c} \right) \bar{a} \]
\[
\delta^{(13)}_{R,1} = -\frac{4 \left(b^2 - a^2 z\right)^2 \left(\bar{a}^2 + \bar{b}^2 z\right)^3 \left(\bar{b}(4ab + c)z + \bar{a} \left(5a^2 z - 3b^2\right)\right)}{z^3}
\]
\[
\delta^{(15)}_{R,1} = \frac{4a \left(a^2 z - b^2\right)^3 \left(\bar{a}^2 + \bar{b}^2 z\right)^4}{z^3}
\]
\[
\delta^{(0)}_{R,2} = 0
\]
\[
\delta^{(2)}_{R,2} = 2c
\]
\[
\delta^{(4)}_{R,2} = \frac{8 \left(\bar{b}z \left(za^3 + b^2a + bc\right) - \bar{a} \left(b^3 + a^2zb + acz\right)\right)}{z}
\]
\[
\delta^{(6)}_{R,2} = -\frac{2}{z} \left(\left(12\bar{a}b + \bar{c}\right)z^2 a^4 + 16bz \left(\bar{b}^2 z - \bar{a}^2\right) a^3 + 2z \left(-3c\bar{a}^2 - 20b^2 \bar{b}\bar{a} + b^2\bar{c} + 3\bar{b}^2 cz\right) a^2 + 4b \left(4a^2b^2 - 4\bar{b}^2 zb^2 - 4\bar{a}bcz + c\bar{c} z\right) a + 6\bar{a}^2b^2c + b^4\bar{c} - 6b^2\bar{b}^2cz + c^2 \bar{c}z + 4\bar{a} \bar{b} \left(3b^4 - c^2z\right)\right)
\]
\[
\delta^{(8)}_{R,2} = -\frac{8}{z^2} \left(b^2 - a^2 z\right) \left(4b^3 - 6a^2 zb - acz\right) \bar{a}^3 + \bar{b}z \left(2za^3 - 8b^2a - 3bc\right) a^2 + z \left(\bar{c}za^3 - 8b\bar{b}^2 za^2 + b^2\bar{c}a - 3\bar{b}^2 cz a + 2\bar{b}^3b^2 + bc\bar{c}\right) a + 4b \left(4b^2\bar{b}^2 + \bar{c}\bar{c}\right) z\bar{a} + \bar{b}z \left(4 \bar{b}^2 z^2 a^3 + b\bar{c}za^2 - 6b^2\bar{b}^2za + c\bar{c} z a + b^3\bar{c} - b\bar{b}^2cz\right)
\]
\[
\delta^{(10)}_{R,2} = \frac{2}{z^2} \left(b^2 - a^2 z\right)^2 \left(\bar{c}a^4 + 8b\bar{b}^2 \bar{a}^3 - 2 \left(3\bar{c}\bar{b}^2 + \bar{b}^2 cz\right) a^2 + 4 \bar{b} \left(2b^2\bar{b}^2 + \bar{c}\bar{c}\right) z\bar{a} + 16ab \left(a^2 + \bar{b}^2 z\right)^2 + 2a^2(4\bar{a}b - 3 \bar{c}) z \left(a^2 + \bar{b}^2 z\right) + z \left(cz b^4 - 6b^2\bar{c}b^2 - c\bar{c}^2\right)\right)
\]
\[
\delta^{(12)}_{R,2} = \frac{1}{z^3} \left(8 \left(b^2 - a^2 z\right)^2 \left(\bar{a}^2 + \bar{b}^2 z\right) \left(b\bar{a}^3 + 3a\bar{b}z\bar{a}^2 + \left(3b\bar{b}^2 - a \bar{c}\right) za\right)ight.
\]
\[
\vphantom{\delta^{(12)}_{R,2}} + \bar{b}z \left(4b^2 z^2 - b \bar{c}\right)\right)\right)\right)\right)
\]
\[
\delta^{(14)}_{R,2} = \frac{2(4\bar{a}b - \bar{c}) \left(b^2 - a^2 z\right)^4 \left(\bar{a}^2 + \bar{b}^2 z\right)^2}{z^3}
\]
\[
\delta^{(16)}_{R,2} = 0
\]
\[
\delta^{(1)}_{R,3(0)} = 4b
\]
\[
\delta^{(3)}_{R,3(0)} = -4 \left(3bza^2 + 4aba - b^2\bar{b} + \bar{a}c\right)
\]
\[\begin{align*}
\delta_{R,3(0)}^{(5)} &= \frac{4}{z} \left( (-b^3 + 9a^2zb + 4acz) \bar{a}^2 + 8ab z \left( a^2z - b^2 \right) \bar{a} \\
&+ z \left( cz a^3 + 15b^2 z a^2 - 3b^2 \bar{c}a + 6b^2 cz a - 3b^2 \bar{b}^2 - bc \bar{c} \right) \right) \\
\delta_{R,3(0)}^{(7)} &= -\frac{1}{z} \left( 4 \left( z^2 \left( 11z \bar{b}^3 + 3a^2 \bar{b} + 4 \bar{a}\bar{c} \right) \right) a^4 + 4bz \left( 4a^3 + 2\bar{b}^2 z \bar{a} + \bar{b}cz \right) a^3 \\
&+ z \left( 6c\bar{a}^3 + 22b^2 \bar{b}\bar{a}^2 - 14b^2 \bar{c}a - 6b^2 \bar{b}^3 z + 3 \bar{b}c\bar{e}z \right) a^2 \\
&- 4b \left( 4b^2 a^3 - 4bc z a^2 + \left( 2b^2 \bar{b}^2 + c\bar{c} \right) z \bar{a} + \bar{b}z \left( \bar{b}^2 \bar{c} - 2\bar{b}^2 cz \right) \right) a - 6\bar{a}^3 b^2 c \\
&+ \bar{a}\bar{c} \left( 6b^4 - \bar{c}^2 z \right) + a^2 \left( 4bc^2 z - 9b^4 \bar{b} \right) + \bar{b}z \left( 3\bar{b}^2 b^4 - 3cc\bar{b}^2 + 2 \bar{b}^2 c^2 z \right) \\
\delta_{R,3(0)}^{(9)} &= \frac{1}{z^2} \left( 4 \left( a^2z - b^2 \right) \left( -11b^3 + 19a^2zb + 4acz \right) \right) \bar{a}^4 \\
&+ 8bz \left( -za^3 + 3b^2 a + bc \right) \bar{a}^3 \\
&+ z \left( 6cza^3 + 34bb^2 za^2 - 12b^2 \bar{c}a + 14 \bar{b}^2 cz a - 6b^3 \bar{b}^2 - 3bc\bar{c} \right) \bar{a}^2 \\
&- 4 \bar{b}z \left( 2\bar{c}\bar{b}^3 - 6a\bar{b}^2 zb^2 - 2\bar{b}^2 cz b + az \left( 2a^2 zb^2 + c\bar{c} \right) \right) \bar{a} \\
&+ z \left( \left( 2c^2 - 3\bar{b}\bar{z} \right) \bar{b}^3 - 12a\bar{b}^2 \bar{c}\bar{z} b^2 + 3\bar{b}^2 z \left( 5a^2 \bar{b}^2 z - c\bar{c} \right) \right) b \\
&+ az \left( 6c\bar{b}^4 + 6a^2 \bar{c}z \bar{b}^2 + c \bar{c}^2 \right) \right) \\
\delta_{R,3(0)}^{(11)} &= -\frac{4}{z^2} \left( b^2 - a^2 z \right)^2 \left( \bar{a}^2 + \bar{b}^2 z \right) \left( c\bar{a}^3 + 9b^2 \bar{b}a^2 - \left( 6\bar{c} b^2 + \bar{b}^2 cz \right) \right) \bar{a} \\
&- 3b^2 \bar{b}^3 z + bc \bar{c}z \\
&+ a^2 z \left( \bar{z} \bar{b}^3 - 7a^2 \bar{b}^2 + 4\bar{a} \bar{c} \right) + 4ab \left( \bar{a}^3 + \bar{b}^2 z a + \bar{b} \bar{c}z \right) \\
\delta_{R,3(0)}^{(13)} &= \frac{4 \left( a^2z - b^2 \right)^3 \left( \bar{a}^2 + \bar{b}^2 z \right)^2 \left( 3b\bar{a}^2 + a\bar{b}^2 z + ac\bar{z} \right)}{z^3} \\
\delta_{R,3(0)}^{(15)} &= -\frac{4b \left( b^2 - a^2 z \right)^4 \left( \bar{a}^2 + \bar{b}^2 z \right)^3}{z^3} \\
\delta_{R,3(1)}^{(1)} &= 2b \\
\delta_{R,3(1)}^{(3)} &= 2 \left( -3bza^2 + 5b^2 \bar{b} + \bar{a} \bar{c} \right) \\
\delta_{R,3(1)}^{(5)} &= \frac{2}{z} \left( \left( 3b^3 - 7a^2 zb - 2ac\bar{z} \right) \bar{a}^2 + 4bz \left( za^3 + b^2 a + bc \right) \bar{a} \\
&+ z \left( cz a^3 + 3b^2 \bar{c}a + 9b^3 \bar{b}^2 - b \left( 9a^2 \bar{b}^2 + c \bar{c} \right) \right) \right) \\
\delta_{R,3(1)}^{(7)} &= \frac{1}{z} \left( 2 \left( z^2 \left( z \bar{b}^3 + 5\bar{a}^2 \bar{b} - 2\bar{a} \bar{c} \right) \right) a^4 + 8bz \left( a^3 - \bar{b}^2 z \bar{a} + \bar{b}\bar{c}z \right) a^3 \right)
\end{align*}\]
\[
\begin{align*}
- & \quad z \left(6b^2zb^3 + 6aczb^2 + 2a^2b^2b - 3ccz b + 2\bar{a}b^2\bar{c}\right) a^2 \\
- & \quad 4b \left(2b^2a^3 - 2 \bar{b}cz\bar{a} - 2b^2\bar{z} \bar{a} + c\bar{c}z \bar{a} + 2b^2\bar{c}z\right) a + b^2\bar{b} \left(5b^2b^2 - 3cc\right) z \\
+ & \quad \bar{a}c \left(6b^2b^2 - c \bar{c} \right) z + \bar{a}^2b \left(5b^4 + 2c^2z \right) )
\end{align*}
\]

\[
\delta_{R,3(1)}^{(9)} = -\frac{1}{z^2} \left(2 \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \right)
\]

\[
\delta_{R,3(1)}^{(11)} = -\frac{2}{z^2} \left(2 \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \right)
\]

\[
\delta_{R,3(1)}^{(13)} = 2 \frac{\left(2 \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \right)^2}{z^3} \left(3b\bar{a}^2 + 4a\bar{b}z\bar{a} + 5\bar{b}\bar{b}z - a \bar{c}z \right)
\]

\[
\delta_{R,3(1)}^{(15)} = -2\frac{\left(2 \left(b^2 - a^2z \right) \left(b^2 - a^2z \right) \right)^3}{z^3}
\]

\section{C Simpler forms of one-soliton of section 4.3.2}

Here we present simpler forms of the one-soliton solution discussed in section 4.3.2 and for which the coefficients of the tau functions are given in appendix B.

The choice of the parameters \(a, \bar{a}, b, \bar{b}, c, \) and \(\bar{c},\) appearing the coefficients \(\delta^{(i)}\)’s given in appendix B, for which the solution simplifies, and the corresponding non-vanishing tau functions, are given by

1. For \(b = \bar{b} = c = \bar{c} = 0, a = a_1/4 \) and \(\bar{a} = \bar{a}_1/4\) we get

\[
\tau_0 = \left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)^8
\]

\[
\tau_1 = \left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)^6 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)^2
\]

\[
\tilde{\tau}_{L,1} = a_1 \, e^{\Gamma_1} \left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)^6 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)
\]

\[
\tau_{R,1} = \bar{a}_1 \, e^{\Gamma_1} \left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)^6 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} \right)
\]
2. For \( a = \bar{a} = c = \bar{c} = 0, b = b_1/2 \) and \( \bar{b} = \bar{b}_1/2 \) we get

\[
\tau_0 = \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^4 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^4
\]

\[
\tau_1 = \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^2 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^6
\]

\[
\bar{\tau}_{L,3(0)} = 2 \bar{b}_1 \, e^{\Gamma_1} \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^3 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^4
\]

\[
\bar{\tau}_{L,3(1)} = \bar{b}_1 \, e^{\Gamma_1} \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^2 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^6
\]

\[
\tau_{R,3(0)} = 2 \bar{b}_1 \, e^{\Gamma_1} \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^3 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^4
\]

\[
\tau_{R,3(1)} = b_1 \, e^{\Gamma_1} \left(1 - \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^6 \left(1 + \frac{b_1 \bar{b}_1}{4} e^{2 \Gamma_1}\right)^6
\]

3. For \( a = \bar{a} = b = \bar{b} = 0, c = c_1/2 \) and \( \bar{c} = \bar{c}_1/2 \) we get

\[
\tau_0 = \left(1 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right) \left(1 + \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\]

\[
\tau_1 = \left(1 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)^2
\]

\[
\bar{\tau}_{L,2} = \bar{c}_1 \, e^{2 \Gamma_1} \left(1 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\]

\[
\tau_{R,2} = c_1 \, e^{2 \Gamma_1} \left(1 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\]

4. For \( b = \bar{b} = 0, a = a_1/4, \bar{a} = \bar{a}_1/4, c = c_1/2 \) and \( \bar{c} = \bar{c}_1/2 \) we get

\[
\tau_0 = \left[\left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^4 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right] \left[\left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^4 + \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right]
\]

\[
\tau_1 = \left[\left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^3 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right) - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right]^2
\]

\[
\bar{\tau}_{L,1} = a_1 \, e^{\Gamma_1} \left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^3 \times
\]

\[
\left[\left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^3 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right) - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right]
\]

\[
\bar{\tau}_{L,2} = e^{2 \Gamma_1} \left(\bar{c}_1 \left[\left(1 - \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right)^4 - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right]ight)
\]

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\[
\tau_{R,1} = \bar{a}_1 e^{\Gamma_1} \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\]

\begin{align*}
\tau_{R,2} &= e^{2 \Gamma_1} \left(c_1 \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)\right) \\
&- \frac{\bar{c}_1 a^2_1}{256} e^{4 \Gamma_1} \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\end{align*}

\begin{align*}
\tau_{R,3(0)} &= \frac{1}{2} e^{3 \Gamma_1} \left(\bar{a}_1 c_1 + \frac{a^3_1 \bar{c}_1}{16} e^{2 \Gamma_1}\right) \\
&- \frac{a^3_1 \bar{a}_1}{16} e^{2 \Gamma_1} \left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1} - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right)
\end{align*}

\begin{align*}
\tau_{R,3(1)} &= \frac{1}{4} e^{3 \Gamma_1} \times \\
&\times \left[\left(1 + \frac{a_1 \bar{a}_1}{16} e^{2 \Gamma_1}\right) - \frac{c_1 \bar{c}_1}{4} e^{4 \Gamma_1}\right]
\end{align*}

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