MINIMAL SURFACE SYSTEM IN EUCLIDEAN FOUR-SPACE

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Abstract. We construct generalized Cauchy-Riemann equations of the first order for a pair of two $\mathbb{R}$-valued functions to deform a minimal graph in $\mathbb{R}^3$ to the one parameter family of the two dimensional minimal graphs in $\mathbb{R}^4$. We construct the two parameter family of minimal graphs in $\mathbb{R}^4$, which include catenoids, helicoids, planes in $\mathbb{R}^3$, and complex logarithmic graphs in $\mathbb{C}^2$. We present higher codimensional generalizations of Scherk’s periodic minimal surfaces.

1. Introduction

Extending Bernstein’s Theorem that the only entire minimal graphs in $\mathbb{R}^3$ are planes, Osserman [24, Theorem 5.1] proved that any entire two dimensional minimal graph in $\mathbb{R}^4$ should be degenerate, in the sense that its generalized Gauss map lies on a hyperplane of the complex projective space $\mathbb{CP}^3$. Landsberg [15] investigated the systems of the first order whose solutions induce minimal varieties. The classical Cauchy–Riemann equations $(f_x, f_y) = (g_y, -g_x)$ satisfies the minimal surface system of the second order

\[
\begin{align*}
0 &= (1 + f_y^2 + g_y^2) f_{xx} - 2(f_x f_y + g_x g_y) f_{xy} + (1 + f_x^2 + g_x^2) f_{yy}, \\
0 &= (1 + f_y^2 + g_y^2) g_{xx} - 2(f_x f_y + g_x g_y) g_{xy} + (1 + f_x^2 + g_x^2) g_{yy}.
\end{align*}
\]

We construct the Osserman system of the first order, whose solution graphs become degenerate minimal surfaces in $\mathbb{R}^4$.

**Theorem 1.1** (Osserman system as a generalization of Cauchy–Riemann equations). Let

\[
\Sigma = \left\{ \begin{bmatrix} x \\
y \\
f(x, y) \\
g(x, y) \end{bmatrix} \in \mathbb{R}^4 \mid (x, y) \in \Omega \right\}
\]

be the graph in $\mathbb{R}^4$ of the pair $(f(x, y), g(x, y))$ of height functions defined on the domain $\Omega$. Let $g_\Sigma = Edu^2 + 2Fdu^2 + Gdy^2$ denote the induced metric on $\Sigma$.

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If the pair \((f(x,y), g(x,y))\) obeys the Osserman system with \(\mu \in \mathbb{R} - \{0\}\):

\[
\frac{f_x}{f_y} = \mu \left[ \frac{E}{F} \frac{F_y}{G} - g_x \right], \quad \text{or equivalently,} \quad \left[ \begin{array}{c} g_x \\ g_y \\
 \end{array} \right] = -\frac{1}{\mu} \left[ \begin{array}{c} E \\ F \\
 \end{array} \right] \left[ \begin{array}{c} f_x \\ -f_y \\
 \end{array} \right],
\]

where \(\omega = \sqrt{EG - F^2}\), then the two dimensional graph \(\Sigma\) is minimal in \(\mathbb{R}^4\).

The Lagrange potentials (Lemma 4.1 and Remark 4.2) on minimal graphs in \(\mathbb{R}^3\) play a critical role in the Jenkins-Serrin construction [11, Section 3] of minimal graphs with infinite boundary values. We use the Lagrange potentials to construct explicit examples of two dimensional minimal graphs in \(\mathbb{R}^4\) and three dimensional minimal graphs in \(\mathbb{R}^6\).

**Theorem 1.2** (Two applications of Lagrange potentials of the height functions on minimal surfaces in \(\mathbb{R}^3\)). Let

\[
\Sigma_0 = \left\{ \left[ \begin{array}{c} x \\ y \\ p(x,y) \\
 \end{array} \right] \in \mathbb{R}^3 \mid (x,y) \in \Omega \right\}
\]

be the minimal graph of the function \(p : \Omega \to \mathbb{R}\) defined on a domain \(\Omega \subset \mathbb{R}^2\). Let \(q : \Omega \to \mathbb{R}\) denote the Lagrange potential of \(p : \Omega \to \mathbb{R}\) such that

\[
\left[ \begin{array}{c} q_x \\ q_y \\
 \end{array} \right] = \left[ \begin{array}{c} \sqrt{1 + p_x^2 + p_y^2} \\ \frac{p_x}{p_x^2 + p_y^2} \\
 \end{array} \right].
\]

(a) For a constant \(\lambda \in \mathbb{R} - \{0\}\), we consider the graph of the pair \((f(x,y), g(x,y))\):

\[
\Sigma_\lambda = \left\{ \left[ \begin{array}{c} x \\ y \\ f(x,y) \\
 \end{array} \right] = \left[ \begin{array}{c} x \\ y \\ (\cosh \lambda) p(x,y) \\
 \end{array} \right] \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}. 
\]

Then, the pair \((f(x,y), g(x,y))\) satisfies the Osserman system (1) in Theorem 1.1 with \(\mu = \coth \lambda\). In particular, the graph \(\Sigma_\lambda\) is a minimal surface in \(\mathbb{R}^4\). Also, we obtain the invariance of the conformally changed induced metric

\[
\frac{1}{\sqrt{\det(g_{\Sigma_\lambda})}} g_{\Sigma_\lambda} = \frac{1}{\sqrt{\det(g_{\Sigma_0})}} g_{\Sigma_0}.
\]

(b) For any constant \(\lambda \in \mathbb{R} - \{0\}\), the three dimensional graph

\[
\left\{ \left[ \begin{array}{c} x \\ y \\ z \\
 p_x + \lambda q_x \\ p_y + \lambda q_y \\
 \end{array} \right] \in \mathbb{R}^6 \mid (x,y) \in \Omega, z \in \mathbb{R} \right\}
\]

is minimal in \(\mathbb{R}^6\). Moreover, it is a special Lagrangian graph in \(\mathbb{C}^3\).
We present examples of minimal graphs of codimension two in \( \mathbb{R}^4 \). In Example 2.3, we construct the two parameter family of minimal graphs in \( \mathbb{R}^4 \), which include catenoids, helicoids, planes in \( \mathbb{R}^3 \), and complex logarithmic graphs in \( \mathbb{C}^2 \). In Example 4.7, we give a family of codimension two minimal graphs in \( \mathbb{R}^4 \), which contains Scherk's doubly periodic minimal graphs in \( \mathbb{R}^3 \). We present higher codimensional generalizations of Scherk’s periodic minimal surfaces.

2. Minimal surface system in \( \mathbb{R}^4 \) and Cauchy–Riemann equations

Our ambient space is the Euclidean space \( \mathbb{R}^4 \) equipped with the flat metric \( dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \).

**Proposition 2.1** (Two dimensional minimal graphs in \( \mathbb{R}^4 \)). Let \( \Sigma \) be the graph

\[
\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}.
\]

The induced metric \( g_\Sigma \) on the surface \( \Sigma \) reads

\[
g_\Sigma = E dx^2 + 2F dx dy + G dy^2,
\]

where the coefficients of the first fundamental form are determined by

\[
E = \Phi_x \cdot \Phi_x = 1 + f_x^2 + g_x^2, \quad F = \Phi_x \cdot \Phi_y = f_x f_y + g_x g_y, \quad G = \Phi_y \cdot \Phi_y = 1 + f_y^2 + g_y^2.
\]

Let \( \omega = \sqrt{EG - F^2} \). We introduce the minimal surface operator \( L_\Sigma \) and Laplace-Beltrami operator \( \triangle_\Sigma \) acting on functions on \( \Omega \):

\[
L_\Sigma = G \frac{\partial^2}{\partial x^2} - 2F \frac{\partial^2}{\partial x \partial y} + E \frac{\partial^2}{\partial y^2},
\]

\[
(2) \quad \triangle_\Sigma = \triangle g_\Sigma = \frac{1}{\omega} \left[ \frac{\partial}{\partial x} \left( \frac{G}{\omega} \frac{\partial}{\partial x} - \frac{F}{\omega} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial y} \left( -\frac{F}{\omega} \frac{\partial}{\partial x} + \frac{E}{\omega} \frac{\partial}{\partial y} \right) \right].
\]

Then, the following three conditions are equivalent.

(a) The height functions \( f(x,y) \) and \( g(x,y) \) are harmonic on the graph \( \Sigma \):

\[
\triangle_\Sigma f = 0 \quad \text{and} \quad \triangle_\Sigma g = 0.
\]

(b) The graph \( \Sigma \) is minimal in \( \mathbb{R}^4 \).

(c) The height functions \( f(x,y) \) and \( g(x,y) \) solve the minimal surface system:

\[
L_\Sigma f = 0 \quad \text{and} \quad L_\Sigma g = 0.
\]

**Proof.** Though the equivalences of (a), (b), (c) are well-known, we sketch the proof for the convenience of the readers. The equivalence of (a) and (b) follows from [24, Equation (3.14) in Section 2], which indicates that the Euler-Lagrange system for the area functional of the graph is

\[
\frac{\partial}{\partial x} \left( \frac{G}{\omega} \left[ \begin{array}{c} f_x \\ g_x \end{array} \right] - \frac{F}{\omega} \left[ \begin{array}{c} f_y \\ g_y \end{array} \right] \right) + \frac{\partial}{\partial y} \left( -\frac{F}{\omega} \left[ \begin{array}{c} f_x \\ g_x \end{array} \right] + \frac{E}{\omega} \left[ \begin{array}{c} f_y \\ g_y \end{array} \right] \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

where
which is equivalent to
\[ \triangle \Sigma \begin{bmatrix} f_x \\ g_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

There are several ways to establish the equivalence of (b) and (c): [24, Section 2, p. 16–17], [17, Section 2], [1, Section 1.2] (for arbitrary codimension), [20, Appendix: The minimal surface system], and [7, Example 1] (for more general ambient spaces). Here, we adopt the argument in the proof of [22, Theorem 2.2]. We use the formula (2) and introduce
\[(P, Q) := \left( \frac{\partial}{\partial x} \left( G \omega \right) - \frac{\partial}{\partial y} \left( F \omega \right), \frac{\partial}{\partial y} \left( E \omega \right) - \frac{\partial}{\partial x} \left( F \omega \right) \right) \]
to obtain the identity for the mean curvature vector \( H(x,y) \):
\[ H = \triangle \Sigma \begin{bmatrix} x \\ y \\ f(x,y) \\ g(x,y) \end{bmatrix} = \frac{1}{\omega} \begin{bmatrix} P \\ Q \\ \frac{1}{\omega} \mathcal{L}_\Sigma f \\ \frac{1}{\omega} \mathcal{L}_\Sigma g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathcal{L}_\Sigma f \\ \mathcal{L}_\Sigma g \end{bmatrix}. \]

First, we assume (b). Since the mean curvature vector \( H(x,y) \) vanishes on the minimal surface, the four quantities \( P, Q, \mathcal{L}_\Sigma f, \mathcal{L}_\Sigma g \) vanish. So, (c) holds. Second, we assume (c). Since \( \mathcal{L}_\Sigma f = 0 \) and \( \mathcal{L}_\Sigma g = 0 \), the mean curvature vector \( H(x,y) \) is equal to the tangent vector \( \frac{P}{\omega} \Phi_x + \frac{Q}{\omega} \Phi_y \). As the mean curvature vector \( H(x,y) \) is normal to the graph \( \Sigma \), \( H(x,y) \) vanishes. So, (b) holds. \( \square \)

Remark 2.2 (Minimal surface operator \( \mathcal{L}_\Sigma \) and Laplace-Beltrami operator \( \triangle \Sigma \)). We assume that the two dimensional minimal graph \( \Sigma \) is minimal in \( \mathbb{R}^4 \). Then,
\[ \triangle \Sigma = \frac{1}{\omega^2} \mathcal{L}_\Sigma. \]
Indeed, the minimality of the graph \( \Sigma \) implies the two interesting identities
\[ \frac{\partial}{\partial y} \left( \frac{F}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{G}{\omega} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{E}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{F}{\omega} \right), \]
which imply
\[ \triangle \Sigma = \frac{1}{\omega^2} \mathcal{L}_\Sigma + \left[ \frac{\partial}{\partial x} \left( \frac{G}{\omega} \right) - \frac{\partial}{\partial y} \left( \frac{F}{\omega} \right) \right] \frac{\partial}{\partial x} + \left[ \frac{\partial}{\partial y} \left( \frac{E}{\omega} \right) - \frac{\partial}{\partial x} \left( \frac{F}{\omega} \right) \right] \frac{\partial}{\partial y} \]
\[ = \frac{1}{\omega^2} \mathcal{L}_\Sigma. \]
A geometric meaning of (3) is given in Rado’s book [26, p. 108]. A variational proof of (3) can be found in Osserman’s book [24, Chapter 3]. An interpretation of (3) (via the conjugate minimal surface) is illustrated in Remark 4.2.

Example 2.3 (Two parameter family of minimal graphs in \( \mathbb{R}^4 \) connecting complex logarithmic graphs in \( \mathbb{C}^2 \), catenoids, and helicoids in \( \mathbb{R}^3 \)). Given a
pair \((\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+\), we define the two dimensional graph \(\Sigma(\alpha, \beta)\) in \(\mathbb{R}^4\):

\[
\Sigma(\alpha, \beta) = \left\{ \begin{array}{c}
\alpha \ln \left( \sqrt{x^2 + y^2 + \sqrt{x^2 + y^2 + \beta^2 - \alpha^2}} \right) \\
\beta \arctan \left( \frac{y}{x} \right)
\end{array} \right\} \in \mathbb{R}^4 \mid (x, y) \in \Omega.
\]

The domain \(\Omega\) depends on the choice of \((\alpha, \beta)\). We distinguish the three cases.

(a) We consider the case when \(\alpha > \beta > 0\). So, \(\sqrt{\alpha^2 - \beta^2} > 0\). Observing that \(\left( \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \right)^2 - \left( \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \right)^2 = 1\), we can take the constant \(\lambda > 0\) with \((\cosh \lambda, \sinh \lambda) = \left( \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}}, \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \right)\). We introduce the new coordinates \((\tilde{x}, \tilde{y}) = \left( \frac{x}{\sqrt{\beta^2 - \alpha^2}}, \frac{y}{\sqrt{\beta^2 - \alpha^2}} \right)\). Recalling the identity \(\text{arcosh} \, r = \ln \left( r + \sqrt{r^2 - 1} \right), \, r \geq 1\), we find that, up to translations, the rescaled graph \(\frac{1}{\sqrt{\beta^2 - \alpha^2}} \Sigma(\alpha, \beta)\) is congruent to the surface

\[
\left\{ \begin{array}{c}
\tilde{x} \\
\tilde{y}
\end{array} \right\} \in \mathbb{R}^4 \mid \tilde{x}^2 + \tilde{y}^2 \geq 1, \, \tilde{x} \neq 0
\]

(b) When \(\alpha = \beta > 0\), we take \(\lambda = \alpha = \beta\), the minimal surface \(\Sigma(\alpha, \beta)\) in \(\mathbb{R}^4\) can be identified as the complex logarithmic graph in \(\mathbb{C}^2\):

\[
\left\{ \begin{array}{c}
\zeta \\
\lambda \log \zeta
\end{array} \right\} \in \mathbb{C}^2 \mid \zeta = x + iy \in \mathbb{C} - \{0\}
\]

The limit case \(\alpha = \beta = 0\) (or \(\lambda = 0\)) recovers a plane in \(\mathbb{R}^3\).

(c) We assume that \(\beta > \alpha > 0\). So, \(\sqrt{\beta^2 - \alpha^2} > 0\). Observing that

\[
\left( \frac{\beta}{\sqrt{\alpha^2 - \beta^2}} \right)^2 - \left( \frac{\alpha}{\sqrt{\alpha^2 - \beta^2}} \right)^2 = 1,
\]

we can take the constant \(\lambda > 0\) with \((\cosh \lambda, \sinh \lambda) = \left( \frac{\beta}{\sqrt{\beta^2 - \alpha^2}}, \frac{\alpha}{\sqrt{\beta^2 - \alpha^2}} \right)\). We introduce the new coordinates \((\tilde{x}, \tilde{y}) = \left( \frac{x}{\sqrt{\beta^2 - \alpha^2}}, \frac{y}{\sqrt{\beta^2 - \alpha^2}} \right)\). Recalling
the identity \( \arcsinh r = \ln \left( r + \sqrt{r^2 + 1} \right) \), \( r \in \mathbb{R} \), we find that, up to translations, the rescaled graph 
\[
\frac{1}{\sqrt{\beta^2 - \alpha^2}} \Sigma_{(\alpha, \beta)}
\]
is congruent to
\[
\left\{ \begin{array}{l}
\tilde{x} \\
\tilde{y}
\end{array} \right\} 
\]
\[
\left( \sinh \lambda \right) \arcsinh \left( \sqrt{\tilde{x}^2 + \tilde{y}^2} \right)
\]
\[
\left( \cosh \lambda \right) \arctan \left( \frac{\tilde{y}}{\tilde{x}} \right)
\]
\[
\in \mathbb{R}^4 \left| \tilde{x} \in \mathbb{R} \setminus \{0\}, \tilde{y} \in \mathbb{R} \right.
\]
The limit case \( \alpha = 0 \) (or \( \lambda = 0 \)) recovers a helicoid in \( \mathbb{R}^3 \).

**Proposition 2.4** (Cauchy–Riemann equations on the minimal graph). Let

\[
\Sigma = \left\{ \begin{array}{l}
x \\
y \\
f(x,y) \\
g(x,y)
\end{array} \right\} 
\]

be the two dimensional minimal graph in \( \mathbb{R}^4 \). If the system

\[
\begin{pmatrix}
A_x \\
A_y
\end{pmatrix} = \begin{pmatrix}
\frac{E}{\omega} & \frac{F}{\omega} \\
\frac{F}{\omega} & \frac{G}{\omega}
\end{pmatrix} \begin{pmatrix}
B_y \\
-B_x
\end{pmatrix}, \quad \text{or equivalently,} \quad \begin{pmatrix}
B_x \\
B_y
\end{pmatrix} = \begin{pmatrix}
\frac{E}{\omega} & \frac{F}{\omega} \\
\frac{F}{\omega} & \frac{G}{\omega}
\end{pmatrix} \begin{pmatrix}
A_y \\
-A_x
\end{pmatrix}
\]

holds on \( \Omega \), then the function \( A(x, y) + iB(x, y) \) is holomorphic on \( \Sigma \).

*Proof.* We observe the two identities (3) in Remark 2.2:

\[
\frac{\partial}{\partial y} \left( \frac{F}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{G}{\omega} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{E}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{F}{\omega} \right).
\]

Hence, we can find the potential functions \( M(x, y) \) and \( N(x, y) \) so that

\[
(M_x, M_y) = \left( \frac{E}{\omega}, \frac{F}{\omega} \right) \quad \text{and} \quad (N_x, N_y) = \left( \frac{F}{\omega}, \frac{G}{\omega} \right),
\]

in a simply connected neighborhood of any point in the domain \( \Omega \). Then,

\[
(x, y) \rightarrow (\xi_1, \xi_2) = (x + M(x, y), y + N(x, y))
\]
is a local diffeomorphism [24, Lemma 4.4]. The induced conformal metric on the minimal graph \( \Sigma \) in \( \mathbb{R}^4 \) is given by

\[
g_{sc} = \frac{\omega}{2 + \frac{E}{\omega} + \frac{F}{\omega}} \left( d\xi_1^2 + d\xi_2^2 \right).
\]

The function \( A(x, y) + iB(x, y) \) is holomorphic with respect to the conformal coordinates \( (\xi_1, \xi_2) \) if and only if the Cauchy–Riemann equations holds:

\[
\begin{bmatrix}
\frac{\partial}{\partial \xi_2} (A \circ \Xi^{-1}) \\
\frac{\partial}{\partial \xi_1} (A \circ \Xi^{-1})
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial \xi_2} (B \circ \Xi^{-1}) \\
\frac{\partial}{\partial \xi_1} (B \circ \Xi^{-1})
\end{bmatrix}.
\]

It could be transformed to the desired system (4) via the chain rule. \( \square \)
Remark 2.5. The Beltrami equations [2] associated to the metric

\[ g_\Sigma = E dx^2 + 2F dx dy + G dy^2 \]

is the system

\[
\begin{bmatrix}
B_x \\
B_y
\end{bmatrix} = - \begin{bmatrix}
\frac{E}{\omega} & \frac{F}{\omega} \\
\frac{F}{\omega} & \frac{G}{\omega}
\end{bmatrix} \begin{bmatrix}
A_y \\
-A_x
\end{bmatrix}, \quad \text{where} \quad \omega = \sqrt{EG - F^2}.
\]

3. Generalized Gauss map and Osserman system of the first order

To define the generalized Gauss map [6,10,22,24] of minimal surfaces in \( \mathbb{R}^4 \), we prepare the complex hyperquadric \( Q_2 \) in the complex projective space \( \mathbb{CP}^3 \):

\[ Q_2 = \{ [z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^3 | z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \} \]

Definition (Generalized Gauss map of minimal surfaces in \( \mathbb{R}^4 \), [24, Section 2]). We consider a conformal harmonic immersion \( X : \Sigma \to \mathbb{R}^4 \), \( \xi \to X(\xi) \). The generalized Gauss map of \( \Sigma \) is the map \( G : \Sigma \to Q_2 \subset \mathbb{CP}^3 \) defined by

\[ G(\xi) = \begin{bmatrix}
\frac{\partial X}{\partial \xi_1} \\
\frac{\partial X}{\partial \xi_2} + i \frac{\partial X}{\partial \xi_2}
\end{bmatrix} \in Q_2. \]

The conformality of the immersion \( X \) guarantees that the generalized Gauss map is a well-defined \( Q_2 \)-valued function. The harmonicity of the immersion \( X \) guarantees that the generalized Gauss map is anti-holomorphic.

Lemma 3.1 (Generalized Gauss map of two dimensional minimal graphs in \( \mathbb{R}^4 \)). We consider the minimal graph \( \Sigma \) in \( \mathbb{R}^4 \)

\[ \Sigma = \left\{ \begin{bmatrix}
x \\
y \\
f(x,y) \\
g(x,y)
\end{bmatrix} \in \mathbb{R}^4 | (x,y) \in \Omega \right\}. \]

The induced metric on \( \Sigma \) is \( E dx^2 + 2F dx dy + G dy^2 \). Let \( \omega = \sqrt{EG - F^2} \). Its generalized Gauss map \( G : \Omega \to Q_2 \subset \mathbb{CP}^3 \) in terms of the coordinates \( x,y \) is

\[ G(x,y) = [z_1 : z_2 : z_3 : z_4] = \begin{bmatrix}
\frac{G}{\omega} & i - \frac{F}{\omega} \\
\frac{G}{\omega} f_x + \left( i - \frac{F}{\omega} \right) f_y & \frac{G}{\omega} g_x + \left( i - \frac{F}{\omega} \right) g_y \\
1 - i \frac{F}{\omega} & i \frac{E}{\omega} \\
1 - i \frac{F}{\omega} f_x + i \frac{E}{\omega} f_y & 1 - i \frac{F}{\omega} g_x + i \frac{E}{\omega} g_y
\end{bmatrix}. \]

Proof. For the details of the deduction of Lemma 3.1, we refer to [18, Proposition 6], which was inspired by the equality in [23, Lemma, p. 290].

Definition (Degenerate minimal surfaces in \( \mathbb{R}^4 \), [24, Section 2]). We say that a minimal surface \( \Sigma \) in \( \mathbb{R}^4 \) is degenerate if the image of its \( Q_2 \)-valued generalized Gauss map lies in a hyperplane of the complex projective space \( \mathbb{CP}^3 \).
Remark 3.2 (Degeneracy of entire two dimensional minimal graphs in arbitrary codimensions). Extending Bernstein’s Theorem that the only entire minimal graphs in \( \mathbb{R}^3 \) are planes, Osserman [24, Chapter 5] showed that the generalized Gauss map of entire two dimensional minimal graphs in \( \mathbb{R}^{n+2} \geq 4 \) are degenerate. For a geometric illustration of generalized Gauss map of degenerate minimal surfaces, see [5, Figure 1]. As in [10, Theorem 4.7], degenerate minimal surfaces in \( \mathbb{R}^4 \) can be described by the Enneper-Weierstrass type representation formula.

**Definition** (Osserman system for minimal graphs in \( \mathbb{R}^4 \)). Let \( \Sigma \) be the graph in \( \mathbb{R}^4 \) of the pair \((f(x, y), g(x, y))\) of height functions defined on the domain \( \Omega \):

\[
\Sigma = \left\{ \Phi(x, y) = \begin{bmatrix} x \\ y \\ f(x, y) \\ g(x, y) \end{bmatrix} \in \mathbb{R}^4 \mid (x, y) \in \Omega \right\}.
\]

The induced metric \( g_\Sigma \) and the area element on the surface \( \Sigma \) are given by

\[
g_\Sigma = Edx^2 + 2Fdxdy + Gdy^2, \quad dA_\Sigma = \omega dx dy, \quad \omega = \sqrt{EG - F^2},
\]

where the coefficients of the first fundamental form are determined by

\[
E = \Phi_x \cdot \Phi_x = 1 + f_x^2 + g_y^2, \quad F = \Phi_x \cdot \Phi_y = f_x f_y + g_x g_y, \quad G = \Phi_y \cdot \Phi_y = 1 + f_y^2 + g_x^2.
\]

Given a constant \( \mu \in \mathbb{R} - \{0\} \), we introduce

\[
\left(\begin{array}{c}
  f_x \\
  f_y \\
\end{array}\right) = \mu \left(\begin{array}{c}
  E \\
  F \\
\end{array}\right) \left(\begin{array}{c}
  g_y \\
  -g_x \\
\end{array}\right),
\]

or equivalently,

\[
\left(\begin{array}{c}
  g_x \\
  g_y \\
\end{array}\right) = -\frac{1}{\mu} \left(\begin{array}{c}
  E \\
  F \\
\end{array}\right) \left(\begin{array}{c}
  f_y \\
  -f_x \\
\end{array}\right),
\]

which will be called the Osserman system with the coefficient \( \mu \in \mathbb{R} - \{0\} \).

Remark 3.3. To prove the equivalence of two systems (5) and (6), one may use

\[
\left(\begin{array}{c}
  E \\
  F \\
\end{array}\right) \left(\begin{array}{c}
  E \\
  F \\
\end{array}\right)^{-1} = \left(\begin{array}{c}
  G \\
  -E \\
\end{array}\right) \left(\begin{array}{c}
  -F \\
  E \\
\end{array}\right)
\]

**Theorem 3.4** (Minimality and degeneracy of Osserman minimal graphs in \( \mathbb{R}^4 \)). If the pair \((f(x, y), g(x, y))\) satisfies the Osserman system (5) with \( \mu \in \mathbb{R} - \{0\} \), then the graph

\[
\Sigma = \left\{ \begin{bmatrix} x \\ y \\ f(x, y) \\ g(x, y) \end{bmatrix} \in \mathbb{R}^4 \mid (x, y) \in \Omega \right\}
\]

is minimal in \( \mathbb{R}^4 \). Moreover, its generalized Gauss map lies on the hyperplane \( z_3 + i\mu z_4 = 0 \) of the complex projective space \( \mathbb{C}P^3 \).
Proof. To show the minimality of the graph $\Sigma$, we employ Proposition 2.1. Indeed, we use the equalities (6) to obtain

$$\Delta_{\omega} f = \frac{1}{\omega} \left[ \frac{\partial}{\partial x} \left( \frac{G}{\omega} f_x - \frac{F}{\omega} f_y \right) + \frac{\partial}{\partial y} \left( -\frac{F}{\omega} f_x + \frac{E}{\omega} f_y \right) \right]$$

$$= \frac{1}{\omega} \left[ \frac{\partial}{\partial x} (\mu g_y) + \frac{\partial}{\partial y} (-\mu g_x) \right] = 0,$$

and use the equalities in (5) to obtain

$$\Delta_{\omega} g = \frac{1}{\omega} \left[ \frac{\partial}{\partial x} \left( \frac{G}{\omega} g_x - \frac{F}{\omega} g_y \right) + \frac{\partial}{\partial y} \left( -\frac{F}{\omega} g_x + \frac{E}{\omega} g_y \right) \right]$$

$$= \frac{1}{\omega} \left[ \frac{\partial}{\partial x} \left( -\frac{1}{\mu} f_y \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} f_x \right) \right] = 0.$$

To prove the degeneracy of the minimal graph $\Sigma$, we exploit Lemma 3.1. It follows from the Osserman systems (5) and (6) that

$$(f_y, g_y) = \left( \mu \left( \frac{F}{\omega} g_y - \frac{G}{\omega} g_x \right), \frac{1}{\mu} \left( \frac{G}{\omega} f_x - \frac{F}{\omega} f_y \right) \right),$$

which can be complexified to

$$\frac{G}{\omega} f_x + \left( i - \frac{F}{\omega} \right) f_y = -i\mu \left( \frac{G}{\omega} g_x + \left( i - \frac{F}{\omega} \right) g_y \right).$$

We conclude that the generalized Gauss map $G : \Omega \to \mathbb{Q}_2 \subset \mathbb{C}P^3$, which can be explicitly given in terms of the coordinates $(x,y)$,

$$G(x,y) = [z_1 : z_2 : z_3 : z_4]$$

$$= \left[ \frac{G}{\omega} : i - \frac{F}{\omega} : \frac{G}{\omega} f_x + \left( i - \frac{F}{\omega} \right) f_y : \frac{G}{\omega} g_x + \left( i - \frac{F}{\omega} \right) g_y \right]$$

lies on the hyperplane $z_3 = -i\mu z_4$. □

4. Applications of Lagrange potentials on minimal graphs in $\mathbb{R}^3$

It would be not easy to construct explicit examples of non-holomorphic minimal graphs in $\mathbb{R}^4$ by directly solving the minimal surface system of the second order. We solve the Osserman system of the first order to construct explicit examples of two dimensional minimal graphs in $\mathbb{R}^4$.

Lemma 4.1 (Existence of Lagrange potentials on minimal graphs in $\mathbb{R}^3$). Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. We consider the two dimensional graph

$$\Sigma = \left\{ \left[ \begin{array}{c} x \\ y \\ p(x,y) \end{array} \right] \in \mathbb{R}^3 \mid (x,y) \in \Omega \right\}$$

of the $C^2$ height function $p : \Omega \to \mathbb{R}$. Then, the following two statements are equivalent:

(a) The graph $\Sigma$ is a minimal surface in $\mathbb{R}^3$. 

(b) There exists a function \( q : \Omega \to \mathbb{R} \) satisfying the Lagrange system

\[
\begin{bmatrix}
q_y \\
-q_x
\end{bmatrix} = \begin{bmatrix}
p_y \\
n_{p_y}
\end{bmatrix} \frac{1}{\sqrt{1+p_x^2+p_y^2}}
\]

and the gradient estimate

\[
q_x^2 + q_y^2 < 1.
\]

**Proof.** The graph \( \Sigma \) is minimal in \( \mathbb{R}^3 \) if and only if the function \( p(x, y) \) satisfies

\[
0 = \frac{\partial}{\partial x} \left( \frac{p_x}{\sqrt{1+p_x^2+p_y^2}} \right) + \frac{\partial}{\partial y} \left( \frac{p_y}{\sqrt{1+p_x^2+p_y^2}} \right), \quad (x, y) \in \Omega,
\]

which indicates that the following one form is closed:

\[
\omega = -\frac{p_y}{\sqrt{1+p_x^2+p_y^2}} \, dx + \frac{p_x}{\sqrt{1+p_x^2+p_y^2}} \, dy.
\]

Since \( \Omega \) is simply connected, by Poincaré Lemma, the one form \( \omega \) is exact. So, we can find a potential function \( q : \Omega \to \mathbb{R} \) such that

\[
-q_y \frac{p_y}{\sqrt{1+p_x^2+p_y^2}} \, dx + \frac{p_x}{\sqrt{1+p_x^2+p_y^2}} \, dy = dq = q_x \, dx + q_y \, dy.
\]

The inequality (8) follows from the equality \( 1 - q_x^2 - q_y^2 = \frac{1}{1+p_x^2+p_y^2} \). □

**Remark 4.2** (Lagrange potentials and conjugate surfaces of minimal graphs in \( \mathbb{R}^3 \)). The exactness of the one form \( \omega \) in (10) on the minimal graph is discovered by Lagrange [14], who deduced the minimal surface equation (9). When we have the Cauchy-Riemann equations

\[
\begin{bmatrix}
(x_k^x) x \\
(x_k^y) y
\end{bmatrix} = \begin{bmatrix}
p_x p_y \\
p_y
\end{bmatrix} \frac{1}{\sqrt{1+p_x^2+p_y^2}} - \frac{1+p_y^2}{\sqrt{1+p_x^2+p_y^2}} \begin{bmatrix}
p_x \\
p_y
\end{bmatrix} \frac{1}{\sqrt{1+p_x^2+p_y^2}} \begin{bmatrix}
p_x^x \\
p_x^y
\end{bmatrix},
\]

on the minimal graph

\[
\Sigma = \left\{ \begin{bmatrix} x_1(x, y) \\ x_2(x, y) \\ x_3(x, y) \end{bmatrix} \in \mathbb{R}^3 \left| (x, y) \in \Omega \right. \right\},
\]

the function \( x_k + ix_k^x \) is holomorphic on \( \Sigma \) for each \( k \in \{1, 2, 3\} \). It is straightforward to check that this observation is a particular case of Proposition 2.4 with the pair \( (f(x, y), g(x, y)) = (p(x, y), 0) \).

(a) The conjugate surface

\[
\Sigma^* = \left\{ \begin{bmatrix} x_1^*(x, y) \\ x_2^*(x, y) \\ x_3^*(x, y) \end{bmatrix} \in \mathbb{R}^3 \left| (x, y) \in \Omega \right. \right\}
\]

is a minimal surface locally isometric to the minimal surface \( \Sigma \).
(b) Taking \( k = 3 \) in the Cauchy-Riemann equations (11) yields the Lagrange system (7), which reduces to

\[
\begin{bmatrix}
q_x \\
q_y
\end{bmatrix} = \begin{bmatrix}
\frac{p_x p_y}{\sqrt{1 + p_x^2 + p_y^2}} & -\frac{1 + p_x^2}{\sqrt{1 + p_x^2 + p_y^2}} \\
\frac{p_x p_y}{\sqrt{1 + p_x^2 + p_y^2}} & \frac{p_x p_y}{\sqrt{1 + p_x^2 + p_y^2}}
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y
\end{bmatrix}.
\]

The function \( p + iq \) is holomorphic on \( \Sigma \) with respect to the classical conformal coordinates constructed in Proposition 2.4.

(c) Combining the Lagrange system (7) and the gradient estimation (8) yields

\[
\frac{\partial}{\partial x} \left( \frac{q_x}{\sqrt{1 - q_x^2 - q_y^2}} \right) + \frac{\partial}{\partial y} \left( \frac{q_y}{\sqrt{1 - q_x^2 - q_y^2}} \right) = 0,
\]

which reduces to

\[
(1 - q_y^2) q_{xx} + 2q_x q_y q_{xy} + (1 - q_x^2) q_{yy} = 0.
\]

As a historical remark, the dual equation (13) is reported in 1855 by Catalan [4, Equation (C), p. 1020], where he discovered minimal surfaces generated by a one parameter family of parabolas. Calabi [3] observed that (8) and the dual equation (12) indicates that the graph \( z = q(x, y) \) is a maximal surface (spacelike surface with zero mean curvature) in Lorentz-Minkowski space \( \mathbb{L}^4 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2) \).

(d) Taking \( k = 1 \) and \( k = 2 \) in the Cauchy-Riemann equations (11) yields two identities

\[
\frac{\partial}{\partial y} \left( \frac{p_x p_y}{\sqrt{1 + p_x^2 + p_y^2}} \right) = \frac{\partial}{\partial x} \left( \frac{1 + p_y^2}{\sqrt{1 + p_x^2 + p_y^2}} \right),
\]

and

\[
\frac{\partial}{\partial y} \left( \frac{1 + p_x^2}{\sqrt{1 + p_x^2 + p_y^2}} \right) = \frac{\partial}{\partial x} \left( \frac{p_x p_y}{\sqrt{1 + p_x^2 + p_y^2}} \right).
\]

Following previous notations, these two equalities can be rewritten as

\[
\frac{\partial}{\partial y} \left( \frac{F}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{G}{\omega} \right) \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{E}{\omega} \right) = \frac{\partial}{\partial x} \left( \frac{F}{\omega} \right).
\]

**Theorem 4.3** (Degenerate minimal graphs in \( \mathbb{R}^4 \) derived from minimal graphs in \( \mathbb{R}^3 \)). Let \( \Sigma_0 \) be the minimal graph

\[
\Sigma_0 = \left\{ \left[ \begin{array}{c} x \\
y \\
p(x, y)
\end{array} \right] \in \mathbb{R}^3 \mid (x, y) \in \Omega \right\}
\]
of the $C^2$ height function $p : \Omega \to \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^2$. Let $q : \Omega \to \mathbb{R}$ be the Lagrange potential of the function $p$, which solves the Lagrange system

\begin{equation}
\begin{pmatrix}
q_y \\
-q_x
\end{pmatrix} = \frac{p_y}{\sqrt{1 + p_x^2 + p_y^2}} - \frac{p_x}{\sqrt{1 + p_x^2 + p_y^2}}.
\end{equation}

For a constant $\lambda \in \mathbb{R} - \{0\}$, we associate the two dimensional graph in $\mathbb{R}^4$:

\[\Sigma_\lambda = \left\{ \begin{pmatrix} x \\
y \\
f(x,y) \\
g(x,y) \end{pmatrix} \in \mathbb{R}^4 \mid (x,y) \in \Omega \right\}.\]

Then, the graph $\Sigma_\lambda$ is minimal in $\mathbb{R}^4$. Also, we obtain the conformal invariance of the conformally changed induced metric

\begin{equation}
\frac{1}{\sqrt{\det(g_{\Sigma_\lambda})}} g_{\Sigma_\lambda} = \frac{1}{\sqrt{\det(g_{\Sigma_0})}} g_{\Sigma_0}.
\end{equation}

**Proof.** We want to show that the pair $(f, g) = ((\cosh \lambda) p, (\sinh \lambda) q)$ satisfies the Osserman system (5) in Theorem 3.4 with the coefficient $\mu = \coth \lambda$:

\begin{equation}
\begin{pmatrix}
f_x \\
g_y
\end{pmatrix} = \coth \lambda \begin{pmatrix} E \\
g \end{pmatrix} \begin{pmatrix} F \\
E \end{pmatrix} \begin{pmatrix} g_y \\
-f_x
\end{pmatrix}, \quad \text{where} \quad \omega = \sqrt{EG - F^2}.
\end{equation}

Taking $W = \sqrt{1 + p_x^2 + p_y^2} \geq 1$ and using the system (14), we deduce

\begin{equation}
(q_x, q_y) = \left( -\frac{p_y}{W}, \frac{p_x}{W} \right) \quad \text{and} \quad q_x^2 + q_y^2 = \frac{W^2 - 1}{W^2}.
\end{equation}

We recall the definition $(f, g) = ((\cosh \lambda) p, (\sinh \lambda) q)$ and deduce

\[f_x g_y - f_y g_x = \cosh \lambda \sinh \lambda \frac{W^2 - 1}{W}.
\]

We use the definition $\omega = \sqrt{EG - F^2}$ to obtain

\[\omega^2 = (1 + f_x^2 + g_x^2) (1 + f_y^2 + g_y^2) - (f_x f_y + g_x g_y)^2
\]
\[= 1 + (f_x^2 + f_y^2) + (g_x^2 + g_y^2) + (f_x g_y - f_y g_x)^2
\]
\[= 1 + \cosh^2 \lambda (W^2 - 1) + \sinh^2 \lambda + \left( \cosh \lambda \sinh \lambda \frac{W^2 - 1}{W} \right)^2
\]
\[= \left( (\cosh^2 \lambda) W - \frac{\sinh^2 \lambda}{W} \right)^2.
\]

However, it follows from $W \geq 1$ that

\[(\cosh^2 \lambda) W - \frac{\sinh^2 \lambda}{W} - 1 = (\cosh^2 \lambda) (W - 1) + \frac{\sinh^2 \lambda}{W} (W - 1) \geq 0,
\]
which implies that
\[(\cosh^2 \lambda) W - \frac{\sinh^2 \lambda}{W} \geq 1 > 0.\]

We conclude that
\[(18) \quad \omega = (\cosh^2 \lambda) W - \frac{\sinh^2 \lambda}{W}.\]

We use (18) and (17) to deduce the first row equality in (16):
\[
\frac{E}{\omega} g_y - \frac{F}{\omega} g_x = \frac{1}{\omega} \left[ (1 + f_x^2 + g_x^2) g_y - (f_x f_y + g_x g_y) g_x \right]
= \frac{1}{\omega} \left[ (1 + f_x^2) g_y - f_x f_y g_x \right]
= \frac{1}{\omega} \left[ (1 + (\cosh^2 \lambda) p_x^2) (\sinh \lambda) g_y - (\cosh^2 \lambda) p_x p_y (\sinh \lambda) q_x \right]
= \frac{\sinh \lambda}{\omega} \cdot \frac{p_x}{W} \cdot [1 + (\cosh^2 \lambda) (p_x^2 + p_y^2)]
= \frac{\sinh \lambda}{\omega} \cdot \frac{p_x}{W} \cdot [-\sinh^2 \lambda + (\cosh^2 \lambda) W^2]
= (\sinh \lambda) \frac{1}{\omega} \left[ (\cosh^2 \lambda) W - \frac{\sinh^2 \lambda}{W} \right]
= (\sinh \lambda) \frac{f_x}{\coth \lambda}.
\]

We omit a similar verification of the second row equality in (16). Finally, one can use the equalities
\[
\begin{pmatrix} E & F & G \\ \omega & \omega & \omega \end{pmatrix} = \begin{pmatrix} 1 + p_x^2 & p_x p_y & 1 + p_y^2 \\ W & W & W \end{pmatrix}
\]
to check the conformal invariance (15). \(\square\)

**Remark 4.4 (Holomorphic null curves lifted from degenerate minimal graphs in \(\mathbb{R}^4\)).** In Theorem 4.3, if the initial minimal graph \(\Sigma_0\) in \(\mathbb{R}^3\) is induced by the holomorphic null curve
\[
\phi = (\phi_1(\zeta), \phi_3(\zeta), \phi_3(\zeta))
\]
in \(\mathbb{C}^3\) with \(\phi_1^2 + \phi_2^2 + \phi_3^2 = 0\) and a local conformal coordinate \(\zeta\) on \(\Sigma_0\), the minimal graph \(\Sigma_\lambda\) in \(\mathbb{R}^4\) is induced by
\[
\phi_\lambda = (\phi_1(\zeta), \phi_2(\zeta), (\cosh \lambda) \phi_3(\zeta), (i \sinh \lambda) \phi_3(\zeta))
\]
with the conformal coordinate \(\zeta\) on \(\Sigma_\lambda\). The identity \(\cosh^2 \lambda - \sinh^2 \lambda = 1\) implies the nullity of the induced holomorphic curve \(\phi_\lambda\) in \(\mathbb{C}^4\):
\[
\phi_1^2 + \phi_2^2 + [(\cosh \lambda) \phi_3]^2 + [(i \sinh \lambda) \phi_3]^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0.
\]

For a survey of various deformations of holomorphic null curves in \(\mathbb{C}^n\) lifted from minimal surfaces in \(\mathbb{R}^n\), we invite readers to refer to [19, Section 2].
We apply Theorem 4.3 to classical minimal graphs in \(\mathbb{R}^3\) to find explicit examples of old and new minimal graphs in \(\mathbb{R}^4\).

**Example 4.5** (Minimal surfaces in \(\mathbb{R}^4\) foliated by hyperbolas or lines). We consider the fundamental piece of the helicoid

\[
\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} : \begin{bmatrix} x \\ y \\ x \tan y \end{bmatrix} \in \mathbb{R}^3 \right\} = \mathbb{R} \times \left( \frac{\pi}{2}, \frac{\pi}{2} \right),
\]

Solving the induced Lagrange system (7) in Lemma 4.1

\[
\begin{bmatrix} q_y \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_y}{\sqrt{\cosh^2 \lambda \sin^2 y + \cos^2 y_y + \cos^2 y}} \\ \frac{p_x}{\sqrt{\cosh^2 \lambda \sin^2 y + \cos^2 y}} \end{bmatrix},
\]

we obtain \(q(x,y) = -\sqrt{\cos^2 y + x^2}\), up to an additive constant. Let \(\lambda \in \mathbb{R}\) be a constant. Theorem 4.3 yields the two dimensional minimal graph \(\Sigma_\lambda^\perp\) in \(\mathbb{R}^4\):

\[
\Sigma_\lambda^\perp = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{bmatrix} x \\ y \\ (\cosh \lambda)x \tan y \\ \sinh \lambda \sqrt{\cos^2 y + x^2} \end{bmatrix} \in \mathbb{R}^4 \right\}.
\]

(a) When \(\lambda = 0\), the graph \(\Sigma_\lambda^\perp\) recovers the helicoid in \(\mathbb{R}^3\) foliated by lines.

(b) Let \(\lambda \neq 0\). Fix \(y_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\). The intersection \(\mathcal{C}_{y_0}\) of the surface \(\Sigma_\lambda^\perp\) and the hyperplane \(x_2 = y = y_0\) is a hyperbola. Indeed, letting the new orthogonal coordinates \((\tilde{x}_1, \tilde{x}_3)\) in the \(x_1 x_3\)-plane defined by

\[
\tilde{x}_1 + i\tilde{x}_3 = \frac{\cosh \lambda \sin y_0 + i \cos y_0}{\sqrt{\cosh^2 \lambda \sin^2 y_0 + \cos^2 y_0}} \left(x_1 + i x_3\right),
\]

we can check that the level curve \(\tilde{C}_{y_0}\) in the \(\tilde{x}_1 \partial_1 \tilde{x}_3 \partial_4\)-space lies on \(x_2 = y_0\), \(\tilde{x}_1 = 0\), and

\[
\left(\frac{x_4}{\sinh \lambda \cos y_0}\right)^2 - \left(\frac{\tilde{x}_3}{\sqrt{\cosh^2 \lambda \sin^2 y_0 + \cos^2 y_0}}\right)^2 = 1.
\]

Under the coordinate transformation \((x,y) = (\sinh \mathcal{U} \cos \mathcal{V}, \mathcal{V}) \rightarrow (\mathcal{U}, \mathcal{V})\), we obtain the conformal harmonic patch for the minimal surface \(\Sigma_\lambda^\perp\) in \(\mathbb{R}^4\):

\[
F_{\mathcal{U}},\mathcal{V} = \begin{bmatrix} \sinh \mathcal{U} \cos \mathcal{V} \\ \mathcal{V} \\ \cosh \lambda \sinh \mathcal{U} \sin \mathcal{V} \\ \sinh \lambda \cosh \mathcal{U} \cos \mathcal{V} \end{bmatrix}.
\]

The graph \(\Sigma_\lambda^\perp\) belongs to the family of minimal surfaces discovered by the author [19, Example 6.1]. It was originally discovered by an application of the so called parabolic rotations of holomorphic null curves in \(\mathbb{C}^3 \subset \mathbb{C}^4\) lifted from helicoids in \(\mathbb{R}^4\).
Example 4.6 (Hoffman-Osserman’s minimal surfaces in $\mathbb{R}^4$). Over the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leq \cosh^2 y\},$$

we consider a half of the catenoid

$$\Sigma_0 = \left\{ \begin{bmatrix} x \\ y \\ p(x, y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ \sqrt{-x^2 + \cosh^2 y} \end{bmatrix} \in \mathbb{R}^3 \mid (x, y) \in \Omega \right\}.$$

The pair $(p(x, y), q(x, y)) = \left( \sqrt{-x^2 + \cosh^2 y}, x \tanh y \right)$ solves the Lagrange system (7) in Lemma 4.1:

$$\begin{bmatrix} q_y \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_y}{\sqrt{1 + p_x^2 + p_y^2}} \\ \frac{p_x}{\sqrt{1 + p_x^2 + p_y^2}} \end{bmatrix} = \begin{bmatrix} -x \cosh y \\ \sinh y \end{bmatrix}.$$ 

Let $\lambda$ be a constant. Theorem 4.3 yields the minimal graph $\Sigma^+_{\lambda}$ in $\mathbb{R}^4$:

$$\Sigma^+_{\lambda} = \left\{ \begin{bmatrix} x \\ y \\ f(x, y) \\ g(x, y) \end{bmatrix} = \begin{bmatrix} x \\ y \\ \cosh \lambda \sqrt{-x^2 + \cosh^2 y} \\ ((\sinh \lambda) x \tanh y \end{bmatrix} \in \mathbb{R}^4 \mid (x, y) \in \Omega \right\}.$$

Under the coordinate transformation $(x, y) = (\cosh \mathcal{U} \cos \mathcal{V}, \mathcal{U}) \rightarrow (\mathcal{U}, \mathcal{V})$, we obtain the conformal harmonic patch for the minimal surface $\Sigma^+_{\lambda}$ in $\mathbb{R}^4$:

$$F^+_{\lambda} (\mathcal{U}, \mathcal{V}) = \begin{bmatrix} \cosh \mathcal{U} \cos \mathcal{V} \\ \mathcal{U} \\ \cosh \lambda \cosh \mathcal{U} \sin \mathcal{V} \\ \sinh \lambda \sinh \mathcal{U} \cos \mathcal{V} \end{bmatrix}.$$ 

This recovers Osserman-Hoffman’s minimal annuli in $\mathbb{R}^4$ ([10, Proposition 6.6 and Remark 1] and [19, Example 6.2 and Theorem 6.3]). The conformal harmonic patches (19) in Example 4.5 and (20) in Example 4.6 represent conjugate minimal surfaces in $\mathbb{R}^4$. For the notion of associate family of locally isometric minimal surfaces in $\mathbb{R}^{n+2 \geq 3}$, we invite interested readers to refer to [16].

Example 4.7 (Doubly periodic minimal graphs in $\mathbb{R}^4$ derived from Scherk’s doubly periodic graph in $\mathbb{R}^3$). Over the open square

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \},$$

we define the fundamental piece of the doubly periodic graph in $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} x \\ y \\ \ln \left( \frac{\cos x}{\cos y} \right) \end{bmatrix} \in \mathbb{R}^3 \mid (x, y) \in \Omega \right\},$$

(21)
which was originally discovered by Scherk [27, p. 196]. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{array}{l}
\begin{bmatrix}
x \\
y \\
(\cosh \lambda) \ln \left( \frac{\cos x}{\cos y} \right) \\
(\sinh \lambda) \arcsin (\sin x \sin y)
\end{bmatrix} \\
\in \mathbb{R}^4 \quad \left( x, y \right) \in \Omega
\end{array} \right\}.$$

Remark 4.8 (Jenkins-Serrin type minimal graphs). Inspired by the existence of Scherk’s first surfaces, Jenkins and Serrin [11] offers a powerful analytic method to extend Scherk’s construction. The fundamental piece of Scherk’s first surface can be obtained as a Jenkins-Serrin graph by solving the Dirichlet problem for the minimal surface equation over a square and taking boundary values plus infinity on two opposite sides and minus infinity on the other two opposite sides.

Example 4.9 (Doubly periodic minimal graphs in $\mathbb{R}^4$ derived from sheared Scherk’s doubly periodic graph in $\mathbb{R}^3$). Scherk [27, p. 187] showed that his surface (21) in Example 4.7 belongs to a one parameter family of minimal graphs. Following [21, p. 70], for an angle constant $2\alpha \in (0, \pi)$ and a dilation constant $\rho > 0$, we define Scherk’s doubly periodic minimal graph $\Sigma^2\rho$ by

$$z = p(x, y) = \frac{1}{\rho} \ln \left[ \cos \left( \frac{\rho}{2} \left( \frac{x}{\cos \alpha} - \frac{y}{\sin \alpha} \right) \right) \right],$$

where its domain is an infinite chess board-like net of rhomboids $\Omega = \bigcup_{i,j \in \mathbb{Z}} \mathcal{R}_{ij}$. Here, we define the rhomboid domain $\mathcal{R}_{ij}$ with the length $\frac{\pi}{\rho}$ as follows

$$\mathcal{R}_{ij} = \left\{ (x, y) \in \mathbb{R}^2 \left| \begin{array}{l}
\frac{x}{\cos \alpha} - \frac{y}{\sin \alpha} - \frac{4i}{\rho} \pi < \frac{\pi}{\rho}, \\
\frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} - \frac{4j}{\rho} \pi < \frac{\pi}{\rho}
\end{array} \right. \right\}.$$

Taking $\alpha = \frac{\pi}{4}$ and $\rho = 2$ in (22), the graph $\Sigma^2_{\frac{\pi}{4}}$ is congruent to the minimal graph (21) in Example 4.7, after the $\frac{\pi}{4}$-rotation. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{array}{l}
\begin{bmatrix}
x \\
y \\
(\cosh \lambda) p(x, y) \\
(\sinh \lambda) q(x, y)
\end{bmatrix} \\
\in \mathbb{R}^4 \quad \left( x, y \right) \in \Omega
\end{array} \right\},$$

where we take the Lagrange potential

$$q(x, y) = \frac{1}{\rho} \arccos \left( \cos^2 \alpha \cos \left( \frac{\rho x}{\cos \alpha} \right) - \sin^2 \alpha \cos \left( \frac{\rho y}{\sin \alpha} \right) \right).$$

Example 4.10 (Minimal graphs in $\mathbb{R}^4$ derived from Scherk’s saddle tower in $\mathbb{R}^3$). Over the domain $\Omega = \{ (x, y) \in \mathbb{R}^2 \mid -1 < \sinh x \sinh y < 1 \}$, we consider
a fundamental piece of the singly periodic multi-valued graph in $\mathbb{R}^3$:

$$\left\{ \begin{array}{c}
\begin{bmatrix} x \\
y \\
\arcsin(\sinh x \sinh y) 
\end{bmatrix} 
\in \mathbb{R}^3 \mid (x, y) \in \Omega
\end{array}\right\},$$

which was originally discovered by Scherk [27, p. 198]. Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{array}{c}
\begin{bmatrix} x \\
y \\
(cosh \lambda) \arcsin(\sinh x \sinh y) \\
(sinh \lambda) \ln \left( \frac{cosh x}{cosh y} \right)
\end{bmatrix} 
\in \mathbb{R}^4 \mid (x, y) \in \Omega
\end{array}\right\}.$$

**Remark 4.11 (Scherk’s saddle tower in $\mathbb{R}^3$ and its influences).** Geometrically, Scherk’s saddle tower is a smooth minimal desingularization of two perpendicular vertical planes. Scherk’s saddle tower plays a fundamental role in the modern theory of desingularizations and gluing construction for surfaces with constant mean curvature and solitons to various curvature flows. Karcher [13] discovered embedded minimal surfaces in $\mathbb{R}^3$ derived from Scherk’s examples, and Pacard [25] showed the existence of $(N-2)$-periodic embedded minimal hypersurfaces in $\mathbb{R}^{N \geq 4}$ with four hyperplanar ends.

**Example 4.12 (Minimal graphs in $\mathbb{R}^4$ derived from Scherk’s generalized tower in $\mathbb{R}^3$).** As in [25, Section 1] and [21, p. 74], we take the fundamental piece of the singly periodic multi-valued graph in $\mathbb{R}^3$:

$$z = p(x, y) = \frac{1}{\rho} \arccos \left( \cos^2 \alpha \cosh \left( \frac{px}{\cos \alpha} \right) - \sin^2 \alpha \cosh \left( \frac{py}{\sin \alpha} \right) \right).$$

Let $\lambda \in \mathbb{R}$ be a constant. Theorem 4.3 yields the minimal graph in $\mathbb{R}^4$:

$$\left\{ \begin{array}{c}
\begin{bmatrix} x \\
y \\
(cosh \lambda) p(x, y) \\
(sinh \lambda) q(x, y)
\end{bmatrix} 
\in \mathbb{R}^4 \mid (x, y) \in \Omega
\end{array}\right\},$$

where we take the Lagrange potential

$$q(x, y) = \frac{1}{\rho} \ln \left( \frac{\cosh \left( \frac{x}{\cos \alpha} - \frac{y}{\sin \alpha} \right)}{\cosh \left( \frac{x}{\cos \alpha} + \frac{y}{\sin \alpha} \right)} \right).$$

**5. Minimal graphs in $\mathbb{R}^3$ and special Lagrangian graphs in $\mathbb{C}^3$**

Fu [8], Jost-Xin [12], Tsui-Wang [28], Yuan [29] established Bernstein type results for entire special Lagrangian graphs in even dimensional Euclidean space. Here, we construct non-entire special Lagrangian graphs in $\mathbb{C}^3$. 

Proposition 5.1 (Special Lagrangian equation in \( \mathbb{C}^3 \), [9, Theorem 2.3 and (4.8)]). Let \( S \) be the gradient graph of the \( \mathbb{R} \)-valued function \( F(x,y,z) \) in \( \mathbb{R}^3 \). Then, the 3-fold \( S \) in \( \mathbb{R}^6 \) admits an orientation making it into a special Lagrangian graph in \( \mathbb{C}^3 \) under the complexification

\[
(x_1 + iy_1, x_2 + iy_2, x_3 + iy_3) = (x_1, x_2, x_3, y_1, y_2, y_3)
\]

when the function \( F(x,y,z) \) satisfies the special Lagrangian equation

\[
\det \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix} = \text{tr} \begin{bmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{bmatrix}.
\]

Remark 5.2. In [9, III.4.B. Degenerate projections and harmonic gradients], Harvey and Lawson investigated the interesting special case when the solution of the equation (23) is affine with respect to the coordinate \( z \). The function \( F(x,y,z) = p(x,y) + z q(x,y) \) satisfies the special Lagrangian equation (23) if and only if the pair \( (p(x,y), q(x,y)) \) solves the system

\[
\begin{align*}
(1 + p_y^2) p_{xx} - 2 p_x p_y p_{xy} + (1 + p_x^2) p_{yy} = 0, \\
(1 + p_y^2) q_{xx} - 2 p_x p_y q_{xy} + (1 + p_x^2) q_{yy} = 0.
\end{align*}
\]

The first equation means that the graph of \( p(x,y) \) is a minimal surface in \( \mathbb{R}^3 \). The second equation means that \( q(x,y) \) is harmonic on the graph of \( p(x,y) \).

Combining the harmonicity of the Lagrange potentials of height functions of the minimal graph in \( \mathbb{R}^3 \) and the Harvey-Lawson reduction [9, Theorem 4.9], we immediately deduce the following result.

Corollary 5.3 (Lagrange potential construction of special Lagrangian graphs in \( \mathbb{R}^6 = \mathbb{C}^3 \)). Let

\[
\Sigma = \left\{ \begin{bmatrix} x \\ y \\ p(x,y) \end{bmatrix} \in \mathbb{R}^3 \mid (x,y) \in \Omega \right\}
\]

be the minimal graph of the height function \( p(x,y) : \Omega \to \mathbb{R} \) on the domain \( \Omega \subset \mathbb{R}^2 \). Let \( q : \Omega \to \mathbb{R} \) be the Lagrange potential of the function \( p \) such that

\[
\begin{bmatrix} q_y \\ -q_x \end{bmatrix} = \begin{bmatrix} \frac{p_y}{\sqrt{1+p_x^2+p_y^2}} \\ \frac{p_x}{\sqrt{1+p_x^2+p_y^2}} \end{bmatrix}.
\]

For any constant \( \lambda \in \mathbb{R} \), we obtain the special Lagrangian graph in \( \mathbb{C}^3 \) :

\[
\Sigma_\lambda = \left\{ \begin{bmatrix} x \\ y \\ z \\ p_x + \lambda z q_x \\ p_y + \lambda z q_y \end{bmatrix} \in \mathbb{R}^6 \mid (x,y) \in \Omega, z \in \mathbb{R} \right\}.
\]
Proof. By the item (b) in Remark 4.2, the function $p + iq$ is holomorphic on the minimal graph $\Sigma$. Since $p$ and $q$ are harmonic on the minimal graph $\Sigma$, by Remark 2.2, we obtain the system of equations

$$
\begin{align*}
(1 + p_y^2) p_{xx} - 2p_x p_y p_{xy} + (1 + p_x^2) p_{yy} &= 0, \\
(1 + p_y^2) (\lambda q)_{xx} - 2p_x p_y (\lambda q)_{xy} + (1 + p_x^2) (\lambda q)_{yy} &= 0.
\end{align*}
$$

(24)

Applying the Harvey-Lawson reduction [9, Theorem 4.9] to the system (24) yields that the gradient graph of the function $F(x, y, z) = p(x, y) + \lambda z q(x, y)$ becomes a special Lagrangian 3-fold in $\mathbb{C}^3$. □

Example 5.4 (Doubly periodic special Lagrangian graph in $\mathbb{C}^3$). Let $\lambda$ be a constant. We apply Corollary 5.3 to the fundamental piece of Scherk’s doubly periodic graph on the domain $\Omega$ in Example 4.7 to have the one parameter family of special Lagrangian graph $\Sigma_{\lambda}$ in $\mathbb{C}^3$:

$$
\Sigma_{\lambda} = \left\{ \begin{array}{l}
x \\
y \\
z \\
\sin x \cos y + \lambda z \frac{\sin x \cos y}{\sqrt{1 - \sin^2 x \sin^2 y}} \\
\frac{\sin y}{\cos y} + \lambda z \frac{\sin y}{\sqrt{1 - \sin^2 x \sin^2 y}} \\
\lambda \arcsin (\sin x \sin y)
\end{array} \right\} 
\in \mathbb{R}^6 
\quad \left| \begin{array}{l}
(x, y) \in \Omega, \\
z \in \mathbb{R}
\end{array} \right. 
$$

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