ON A DIFFERENTIABLE LINEARIZATION THEOREM OF PHILIP HARTMAN

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Dedicated to the memory of Dmitri Anosov and Philip Hartman

Abstract. Given a linear automorphism $L$ of a Banach space $E$, let $\rho(L)$ denote its spectral radius and let $c(L) = \rho(L)\rho(L^{-1})$ denote its spectral condition number. Given the direct sum decomposition $L = A \oplus B$ let $c_h = \max(c(A), c(B))$, $\rho_h = \max(\rho(A^{-1}), \rho(B))$. For $0 < \alpha < 1$, the map $L$ is called $\alpha$-hyperbolic if $L$ can be written as $L = A \oplus B$ so that $c_h \rho_h^\alpha$ is less than one. A fixed point of a $C^1$ diffeomorphism $T$ is called $\alpha$-hyperbolic if its derivative $DT(p)$ is $\alpha$-hyperbolic. A well-known theorem of Philip Hartman states that if $E$ is finite dimensional with an $\alpha$-hyperbolic fixed point and, in addition, the derivative $DT$ is uniformly Lipschitz near $p$, then the map $T$ is locally $C^{1,\beta}$ linearizable near $p$ for some $\beta > 0$. We obtain the same result under the weaker assumption that $DT$ is uniformly $\alpha$-Hölder near $p$. We also extend the result to Banach spaces with $C^{1,\alpha}$ bump functions and obtain continuous dependence of the linearization on parameters. The results apply to give simpler proofs under weaker regularity assumptions of classical theorems of L. P. Shilnikov and others on the existence of horseshoe type dynamics and bifurcations near homoclinic curves.

1. Introduction

Linearization theorems are of fundamental importance in Dynamical Systems. On the one hand they provide simple descriptions of the behavior near critical points and periodic motions, and on the other, they can often be applied with non-local techniques to give substantial information about global structures, e.g. as in [12, 18, 34, 35, 26, 30].

The so-called Grobman-Hartman Theorem states that a $C^r$ (with $r$ a positive integer) diffeomorphism or flow can be (locally) $C^0$ linearized near a hyperbolic fixed point. While this theorem gives topological information about orbits which remain near the fixed point, it is inadequate for the study of orbits which recur near the fixed point after traveling a relatively large distance away from it. The analysis of such orbits often requires estimates of derivatives, and hence, makes good use of linearizations with various amounts of smoothness.

Standard smooth (local) linearization theorems near a hyperbolic fixed point in a finite dimensional manifold have the following form. First, by a suitable choice of coordinates, we may assume that the fixed point is the origin in Euclidean space.

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This theorem, first proved in Euclidean space independently by Hartman and Grobman, was extended to Banach spaces, independently by Palis [29] and Pugh [7].
Next, given a desired smoothness $k$ for the linearization, there is an integer $N(k)$ such that if the $N(k)$-jet of the system at 0 is linear (the non-linear terms up to order $N(k)$ vanish), then there is a linearization of order $k$ near the fixed point. This vanishing of the nonlinear terms is most often guaranteed via a preliminary change of coordinates under so-called *diophantine inequalities* or *non-resonance conditions*. These are polynomial inequalities involving the eigenvalues of the derivative of the map or flow at the fixed point. See Hartman [28], Sternberg [37, 38], Bronstein and Kopanskii [5], and the references contained therein for more details. Several papers of Chaperon [20], [21] describe a beautiful development of Invariant Manifold Theory and its applications to linearization theorems.

In this paper we deal with local linearizations under rather weak smoothness assumptions and, in the bi-circular case defined below, without *diophantine inequalities*. Since the results and arguments hold in Banach spaces with sufficiently smooth bump functions, let us set up the notation in that case.

Let $E$ and $F$ be real Banach spaces and let $L(E, F)$ denote the Banach space of bounded linear maps from $E$ to $F$ with the usual supremum norm, for $L \in L(E, F)$,

$| L | = \sup_{| x | = 1} | Lx |.$

Let $\text{Aut}(E)$ denote the linear automorphisms of $E$; i.e., the invertible elements of $L(E, E)$.

Let $U$ be a non-empty open, connected subset of $E$ and let $\alpha > 0$ be a positive real number.

The map $f : U \to F$ is called $\alpha$–Hölder continuous at a point $x \in U$ (or simply $\alpha$–Hölder at $x$) if

$$\text{Hol}(f, x, U) \overset{\text{def}}{=} \sup_{y \in U, y \neq x} \frac{| f(y) - f(x) |}{| y - x |^\alpha} < \infty. \quad (1)$$

The map $f$ is called $\alpha$–Hölder in $U$ if

$$\text{Hol}(f, U) \overset{\text{def}}{=} \sup_{x, y \in U, y \neq x} \frac{| f(y) - f(x) |}{| y - x |^\alpha} < \infty. \quad (2)$$

The latter condition is sometimes called *uniformly* $\alpha$–Hölder in $U$.

If $\alpha = 1$ in the above inequalities, we say that $f$ is, respectively, Lipschitz at $x$ or Lipschitz in $U$.

We refer to the constant $\text{Hol}(f, U)$ in (1) (or (2)) as the Hölder constant of $f$ at $x$ (in $U$). When $\alpha = 1$, we use the term Lipschitz constant of $f$, and we write this as $\text{Lip}(f, U)$.

The map $f$ is differentiable at a point $x \in U$ if there is a bounded linear map $Df(x) \in L(E, F)$ such that

$$\lim_{| h | \to 0} \frac{| f(x + h) - f(x) - Df(x)h |}{| h |} = 0.$$

Sometimes this concept is referred by saying that $f$ is Fréchet differentiable at $x$. For notational convenience, we sometimes write $Df(x)$ as $Df_x$.

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2 In this paper, unless otherwise stated, all open sets will be assumed to be connected.
If $f$ is differentiable at each $x \in U$, and, in addition, the map $x \to Df(x)$ is continuous from $U$ into $L(E,F)$, then we say that $f$ is \textit{continuously differentiable} or $C^1$ in $U$.

If the $C^1$ map $f : U \to F$ is such that $y \to Df(y)$ is $\alpha$–Hölder at $x$, (resp., in $U$), then we say that $f$ is $C^{1,\alpha}$ at $x$ (resp., in $U$). Unless otherwise stated, we will typically use this for $0 < \alpha < 1$. When $\alpha = 1$, we will say that $Df$ is \textit{Lipschitz} at $x$ (resp., in $U$).

For a $C^{1,\alpha}$ map $f : U \to E$, it is convenient to define its $D$–Hölder constant $\text{Hol}(Df,U)$ by

$$\text{Hol}(Df,U) = \sup_{x \neq y \in U} \frac{|Df(x) - Df(y)|}{|x - y|^\alpha}.$$ 

Recall that two norms $| \cdot |$ and $\| \cdot \|$ on a linear space $E$ are \textit{equivalent} if there is a positive constant $C > 0$ such that

$$C^{-1} \leq \frac{|x|}{\|x\|} \leq C$$

for every $x \neq 0$ in $E$.

Note that differentiability of functions $f : E \to F$ is independent of the choice of equivalent norms on $E$ or $F$.

Throughout this paper, we denote the ball of radius $\delta$ about $0$ in $E$ by $B_\delta = B_\delta(0)$.

Let $0 < \alpha < 1$. We will call a Banach space $(E,| \cdot |)$ a $C^{1,\alpha}$ \textit{Banach space} if there is a $C^{1,\alpha}$ bump function on $E$. This is a $C^{1,\alpha}$ function $\phi : E \to \mathbb{R}$ such that $\phi(E) = [0,1]$, and there are positive numbers $c_1 < c_2$ such that $\phi(x) = 1$ for $x \in B_{c_1}$ and $\phi(x) = 0$ for $x \notin B_{c_2}$. Replacing $\phi(x)$ by $x \to \phi(c_2x)$, we may assume that $c_2 = 1$. This is a somewhat restrictive condition on Banach spaces. For instance, it holds for the $L_p$ spaces with $p > 1$ but fails for $L_1$. If there is an equivalent norm $\| \cdot \|$ on $E$ which is $C^{1,\alpha}$ on open sets which do not contain $0$, then it is easy to see that such bump functions exist. In particular, this is true in Hilbert Spaces since their norms are $C^\infty$ away from $0$. The study of the existence or non-existence of various smooth norms or bump functions in Banach spaces is a central part of the geometric theory of Banach spaces. We refer the reader to [11] and [19] for more information.

Given a linear automorphism $L : E \to E$, the \textit{spectral radius} of $L$, denoted $\rho(L)$, is defined to be

$$\rho(L) = \lim_{n \to \infty} |L^n|^{\frac{1}{n}}$$

To see that this limit exists, observe that the sequence $a_n = \log(|L^n|)$ is sub-additive (i.e. $a_{n+m} \leq a_n + a_m$) and the numbers $\frac{a_n}{n}$ are bounded below (in fact, by the number $-\log(1 - 1/|L^{-1}|))$, so Lemma 1.18 in [3] implies that the quantity

$$h_0 \overset{\text{def}}{=} \lim_{n \to \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}$$

exists. Then, $\rho(L) = \exp(h_0)$.

If $E$ is $n$–dimensional Euclidean space and $L$ is a linear automorphism, then $\rho(L)$ is, of course, the maximum of the absolute values of the eigenvalues of $L$. 

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A linear automorphism $L$ is called contracting if $\rho(L) < 1$. It is expanding if $\rho(L^{-1}) < 1$.

A linear automorphism $L$ is called hyperbolic if it satisfies exactly one of the three conditions

1. $L$ is contracting,
2. $L$ is expanding, or
3. $L$ can be written as a direct sum $L = A \oplus B$ in which $A$ is expanding and $B$ is contracting.

In the last case, we say that $L$ is hyperbolic of saddle type, or, simply of saddle type, and we call $A$ the expanding part of $L$ and $B$ the contracting part of $L$. They are unique.

The spectral condition number of the linear automorphism $L$ is the quantity $c(L) = \rho(L)\rho(L^{-1})$. It is an easy consequence of the spectral radius formula that $c(L) \geq 1$. Indeed, for each positive integer $m$, we have $I = L^{-m}L^m$, so $1 \leq |L^{-m}||L^m|$. Now, take $m$th roots and the limit as $m$ approaches infinity.

A linear automorphism $L$ is called circular if $c(L) = 1$. In the finite dimensional case, this means that all of its eigenvalues lie in a single circle in the complex plane. If $L$ is circular and contracting, then we call it $c$–contracting. Analogously, we say that $L$ is $c$–expanding if $L^{-1}$ is $c$–contracting. We say that $L$ is bi-circular if it is hyperbolic of saddle type, and its expanding and contracting parts are both circular.

Trivially, hyperbolic linear automorphisms of saddle type on the plane are bi-circular as are hyperbolic linear automorphisms of saddle type in $\mathbb{R}^3$ which have a pair of non-real complex conjugate eigenvalues.

An obvious, but important remark, is that every hyperbolic linear automorphism of saddle type on a finite dimensional space is bi-circular on the direct sum of its leading eigenspaces. These are the subspaces on which the eigenvalues are closest to 1 in absolute value (see Section 7 for details on the analogous conditions for vector fields and applications).

Given the real number $\alpha$ with $0 < \alpha < 1$, we say that $L$ is $\alpha$–contracting if $c(L)\rho(L)^\alpha < 1$. Since $c(L)$ is always at least 1, this implies that $\rho(L) < 1$. It is well-known (see Lemma 3.4 below) that this implies the existence of an equivalent norm on $E$ whose induced norm on $L$ is less than 1 (whence the name contracting). In the finite dimensional case, this implies that the absolute values of the eigenvalues lie in a complex annulus whose bounding circles have radii in the open-closed interval $(\rho(L)^{1+\alpha}, \rho(L)]$.

We say that $L$ is $\alpha$-expanding if $L^{-1}$ is $\alpha$-contracting.

Let $L$ be a hyperbolic linear automorphism of saddle type, and let $L = A \oplus B$ with $A$ expanding and $B$ contracting.

Let

$$c_h = c_h(L) = c_h(A, B) = \max(c(A), c(B)).$$

We call this the hyperbolic condition number of $L$.

Similarly, we define the hyperbolic spectral radius of $L$ to be

$$\rho_h = \rho_h(L) = \rho_h(A, B) = \max(\rho(A^{-1}), \rho(B)).$$
We say that $L$ is $\alpha$-hyperbolic if it can be written as a direct sum $L = A \oplus B$ so that

$$c_h \rho_h^\alpha = c_h(L) \rho_h(L)^\alpha < 1.$$  \hfill (6)

**Remark 1.1.** Since definition of $\rho_h$ in (5) requires that the inverse operator appearing is the expanding part in a hyperbolic decomposition of $L$, it follows that $\rho_h(L^{-1}) = \rho_h(L)$.

**Remark 1.2.** Since the condition (6) implies that

we see that the contracting and expanding parts of an $\alpha$-hyperbolic automorphism must, in fact, be $\alpha$-contracting and $\alpha$-expanding, respectively.

Note that if $L$ is bi-circular, then it is $\alpha$-hyperbolic for every $\alpha \in (0, 1)$.

Let $Hyp_\alpha(E)$ be the set of $\alpha$-hyperbolic linear automorphisms of $E$. Using the formula (3) on the contracting and expanding parts of an element of $Hyp_\alpha(E)$ and the techniques in Section 4 of Hirsch and Pugh [16], it can be shown that, for each $\alpha \in (0, 1)$, $Hyp_\alpha(E)$ is an open subset of $L(E, E)$.

Let $r \geq 1, k \geq 1$ be real numbers (not-necessarily integers). Let $[r]$ denote the greatest integer less than or equal to $r$. When $r$ is not an integer, we say that a map $T$ is $C^r$ if it is $C^{[r]}$ and its $[r]$-th derivative is $r-[r]$-Hölder. Setting $r-[r] = \beta$, we will also write this as $C^{[r], \beta}$.

Given the Banach space $E$ and a point $p \in E$, let $T$ be a $C^r$ diffeomorphism from an open neighborhood $U$ of $p$ onto its image such that $T(p) = p$. We say that $p$ is an $\alpha$-contracting fixed point of (or for) $T$ if the derivative $DT(p)$ is $\alpha$-contracting. In a similar way we define $\alpha$-expanding, $\alpha$-hyperbolic, and bi-circular fixed points.

A $C^k$ linearization of $T$ at $p$ (or near $p$) is a $C^k$ diffeomorphism $R$ from a neighborhood of $p$ onto a neighborhood of $0$ such that $RTR^{-1} = L$ on some neighborhood of $0$. An equivalent condition is that $LR = RT$ on some neighborhood of $p$. When such a diffeomorphism $R$ exists, we say that $T$ is $C^k$ linearizable at (or near) $p$. Sometimes the term locally $C^k$ linearizable at (or near) $p$ is used and the map $R$ is called a local linearization at (or near) $p$. At times it will be convenient to use the statement $p$ is $C^k$ linearizable to mean that $p$ is a fixed point of a $C^r$ diffeomorphism $T$ (for some $r \geq k$) which has a $C^k$ linearization at $p$. Since the neighborhoods in the domains of definition of the various maps considered often change, the concept of germ of a map at a point is often used (as in [20], [18]).

As far as we know, S. Sternberg was the first to obtain linearization results for finitely smooth systems without explicit diophantine inequality assumptions. In the paper [37], published in 1957, he proved that, for $k \geq 2$, $C^k$ maps of an interval with contracting or expanding fixed points are locally $C^{k-1}$ linearizable near the fixed points.

In [27], Philip Hartman extended the $C^2$ case of Sternberg’s one dimensional results in the following significant ways.

**Theorem 1.3.** (Hartman). Let $E$ denote the Euclidean space $\mathbb{R}^n$, and let $U$ be an open neighborhood of $0$ in $E$. Let $T$ be a $C^{1,1}$ diffeomorphism from $U$ to its image such that $T(0) = 0$ and $DT(0) = L$ where either
(a) $L$ is contracting, or
(b) $L$ is $\alpha$–hyperbolic for some $\alpha \in (0, 1)$.

Then, $T$ is $C^1$ linearizable at 0.

As is well-known, the more general case in which $T(p) = p$ and $p$ is not necessarily 0 can be reduced to the case of the theorem by replacing $T$ by $S^{-1}TS$ where $S(x) = x + p$ is the translation by $p$.

**Remark 1.4.** In the contracting case, Hartman proved that the linearization $R$ was $C^{1,\beta}$ for some $\beta > 0$. He stated that this was true in the $\alpha$–hyperbolic case, but did not present the proof.

It is natural to ask if one can reduce the regularity assumption on $T$ and still get a $C^1$ linearization.

On page 101 in [37] Sternberg shows that, for any $\alpha \in (0, 1)$, the $C^1$ map $T$ on the real line defined by

$$T(x) = \begin{cases} \frac{ax}{\log(|x|)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has no $C^1$ linearization at 0.

Hence, the most natural assumption is that we consider the case the $DT$ is uniformly Hölder instead of uniformly Lipschitz. That is, we assume that $T$ is $C^{1,\alpha}$ (on some neighborhood of 0) for some positive $\alpha$ with $0 < \alpha < 1$.

It is known (even in arbitrary Banach spaces) that a $C^{1,\alpha}$ diffeomorphism $T$ with an $\alpha$–contracting fixed point at 0 has a $C^{1,\alpha}$ linearization near 0. This is a consequence of Corollary 1.3.3 in Chaperon [20]. Since this result is fundamental for our work here, we will include an elementary proof (see Theorem 3.1) below. In the finite dimensional case, this result was stated (under the stronger assumption that $T$ was $C^{1,1}$) with different notation in the last sentence of the first paragraph in section 8 on page 235 in [24]. There is also a statement on page 222 of [27] that the regularity assumption on $T$ could be weakened to some $C^{1,\gamma}$ with $\gamma > \alpha_0$ where $\alpha_0$ depends on the eigenvalues of $L$. We suspect that this $\alpha_0$ equals our $\alpha$, but, since it is not given explicitly, we cannot be sure of this. In the infinite dimensional case, the weaker result that there is a $C^1$ linearization whose derivative is Hölder at 0 was proved by Mora and Solà-Morales in Theorem 3.1 in [22].

Since any circular linear contraction is $\alpha$–contracting for any $\alpha > 0$, it follows that any $C^{1,\alpha}$ map $T$ with a $c$-contracting fixed point $p$ has a $C^{1,\alpha}$ linearization at $p$.

The next case to consider is contractions which are not circular. Here, in [39], Zhang and Zhang study such $C^{1,\alpha}$ contractions in the plane. They obtain $C^{1,\beta}$ linearizations for various $\beta$ depending on $\alpha$. They also show that any linear non-circular contraction can be perturbed to give a $C^{1,\alpha}$ diffeomorphism fixing the origin which has no $C^{1,\beta}$ linearizations for any $\beta > 0$. It is not known in these examples if there is a $C^1$ linearization.

In [15], Rodrigues and Solà-Morales give examples of $C^{1,1}$ diffeomorphisms in infinite dimensional Banach spaces with contracting fixed points which are not $C^1$ linearizable.

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3Since $(RTR^{-1})^{-1} = RT^{-1}R^{-1}$, the theorem also applies when $L$ is expanding.
An alternative approach to linearizations of $C^2$ flows via the Lie Derivative is in Chicone-Swanson [6].

The main result in the present paper can be paraphrased as follows.

In a $C^{1,\alpha}$ Banach space with $0 < \alpha < 1$, every $\alpha-$hyperbolic fixed point of a $C^{1,\alpha}$ diffeomorphism is $C^{1,\beta}$ linearizable for some $\beta > 0$.

After translating the fixed point to the origin as above, we have the more precise statement.

**Theorem 1.5.** Let $0 < \alpha < 1$ and let $E$ be a $C^{1,\alpha}$ real Banach space. Let $U$ be a neighborhood of $0$ in $E$ and let $T : U \to E$ be a $C^{1,\alpha}$ map such that $T(0) = 0$ and $L = DT(0)$ is an $\alpha-$hyperbolic automorphism of $E$.

Then, there are a subneighborhood $V \subset U$ of $0$, a real number $\beta \in (0, \alpha)$, and a $C^{1,\beta}$ diffeomorphism $R$ from $V$ onto its image such that $R(0) = 0$ and

$$L(R(x)) = R(T(x)) \text{ for all } x \in V \cap T^{-1}V$$

**Remarks.**

(1) In the contracting $C^{1,1}$ case (i.e., part (a) of Theorem 1.3), Hartman proved that the linearization could be made $C^{1,\beta}$ for some $\beta > 0$. In the $\alpha-$hyperbolic case, he stated that his proof could be modified to give a linearization for some $\beta > 0$, but he did not present the proof. Thus, our proof, even in the finite dimensional case, may be the first available proof that such a $C^{1,\beta}$ linearization exists.

(2) Given real numbers $a > b > 1 > c > 0$, and $\epsilon \neq 0$, Hartman shows in [27] that, if $b = ac$, then the quadratic polynomial map

$$T_{a,b,c}(x, y, z) = (ax, b(y + \epsilon xz), cz)$$

has no $C^1$ linearization at $(0, 0, 0)$.

It is interesting to note that Hartman’s example is sharp in the following sense. If $b \neq ac$, and $R(x, y, z) = (x, y + \frac{b}{b-ac}xz, z)$, then $LR = RT$ so the map $R$ gives even a quadratic polynomial linearization of $T$ at $(0, 0, 0)$.

(3) In [13], Rodrigues and Solà-Morales consider $C^{1,1}$ diffeomorphisms having a hyperbolic saddle fixed point at $0$ in a $C^{1,1}$ Banach space. They prove that, under a spectral condition which is equivalent to our notion of $\alpha$-hyperbolicity, the diffeomorphisms are $C^1$ linearizable provided $\alpha$ is sufficiently close to $1$. See Theorem 2 in [13].

(4) Given real numbers $r \geq 1$ and $r \geq k \geq 0$, let us say that a fixed point $p$ of a $C^r$ diffeomorphism $T$ is robustly $(r, k)$-linearizable if, for any $S$ sufficiently $C^r$ close to $T$, there is a pair $(p_S, R_S)$ with $p = p_T$, depending continuously on $S$, such that $S(p_S) = p_S$ and $R_S$ is a $C^k$ linearization of $S$ at $p_S$.

More precisely, let $U$ be an open set in the Banach space $E$. Consider the space $D^r(U, E)$ of $C^r$ diffeomorphisms $T$ from $U$ to $T(U)$ with the uniform $C^r$ topology. For $k = 0$, let $D^k(U, E)$ denote the space of homeomorphisms $T$ from $U$ onto $T(U)$ with the uniform $C^0$ topology. The fixed point $p$ of the diffeomorphism $T \in D^r(U, E)$ is said to be robustly $(r, k)$-linearizable if there are neighborhoods $W$ of $T$ in $D^r(U, E)$ and $V$ of $p$ in $E$ such that, for any $S \in W$, there is a pair $(p_S, R_S)$ such that

(a) $p = p_T$,
(b) $S(p_S) = p_S$, 

(c) $p_S \in V$,  
(d) $R_S \in D^k(V, E)$,  
(e) $R_S(p_S) = 0$,  
(f) $DS_{p_S}(x) = (R_S \circ S \circ R_S^{-1})(x)$ for $x \in R_S(U \cap S^{-1}V)$, and  
(g) the map $S \to (p_S, R_S)$ from $W$ into $E \times D^k(V, E)$ is continuous.

The typical linearization results that we are aware of (e.g. those given by the Grobman-Hartman Theorem, Sternberg’s Theorems, Hartman’s Theorem [1,3] above, etc) yield fixed points which are robustly $(r, k)$—linearizable for some $(r, k)$.

As a consequence of Theorem 6.2 below we also obtain that, for $r = 1+\alpha$, the $\alpha$—hyperbolic and bi-circular fixed points considered in this paper, are, in fact, robustly $(r, 1+\beta)$ linearizable for some $\beta > 0$.

(5) This paper owes much to Hartman’s paper [27]. The scheme of the proof of Theorem 1.5 is similar to that indicated in pages 235-238 in [27] for $C^{1,1}$ maps $T$. In actuality, Hartman only gave the proof that the linearization was $C^0$. He stated that similar methods could be used to prove that it was $C^1$, and, as we have already mentioned, he also stated that the proof could be modified to give a linearization whose derivative was uniformly Hölder continuous.

(6) In the fifty-five years since the paper [27] appeared, two significant developments have occurred that make our arguments possible.

(a) We now know that no smoothness is lost for stable and unstable manifolds at a hyperbolic fixed point. In particular, if $T$ is $C^{1,\alpha}$ with a hyperbolic fixed point at $p$, then the stable and unstable manifolds at $p$ are locally the graphs of $C^{1,\alpha}$ maps.

(b) A $C^{1,\alpha}$ map $T$ with fixed point $p$ such that $DT(p)$ is $\alpha$—contracting has a $C^{1,\alpha}$ linearization. See Theorem 3.1 below.

(7) There are several additional papers in the literature which are relevant to the work presented here. In particular, the papers by Samovol [29], [30], [31], Belitskii [1], [2], [3], and Stowe [9] all contain interesting results on smooth linearizations with finite class of differentiability. Many of these are discussed and generalized in the book of Bronstein and Kopanskii [6].

(8) The contents of this paper are as follows. The proof of Theorem 1.5 will be carried out in Sections (2)-(5). Section 6 considers the dependence of the linearization $R$ in Theorem 1.5 on external parameters, and Section 7 presents some motivation and applications of the results.

Acknowledgements.

(1) Being relatively unaware of the recent literature on the subject, we gave a lecture at Penn State in October, 2014, in which we discussed the generalization to the $C^{1,\alpha}$ case of part (b) in Hartman’s Theorem [1,3] above for bi-circular fixed points. The result was still for finite dimensional systems, including the finite dimensional version of Theorem 3.1 below, and only considered the existence of $C^1$ linearizations. We were not aware that the general result in Theorem 3.1 was known. We wish to thank Misha Guysinsky for subsequently providing many references to the recent literature, including, in particular, [13] and [39]. His remarks and references provided substantial impetus for us to study the recent literature, and eventually extend our results to the infinite dimensional case as presented here.
(2) We have already indicated the importance of Hartman’s paper [27] in connection with this work. This was a special time for him in that, on May 16, 2015, he reached his 100th birthday. We recently learned, sadly, that he passed away on August 28, 2015. We dedicate this paper to both Anosov and Hartman—two icons in Dynamical Systems whose mathematical achievements will have a long lasting legacy.

2. The Associated Algebraic Problem

The standard way to approach the functional equation (7) is to convert it into a related algebraic equation on a suitable Banach space $E$ as follows.

Let $I$ denote the identity map on $E$. Writing $T = L + f$, we try to find $R$ of the form $R = I + \phi$ with $\phi \in E$ so that the equation

$$LR = RT$$

becomes

$$L(I + \phi) = (I + \phi)T = (I + \phi)(L + f)$$

which leads to

$$L + L\phi = L + f + \phi(L + f)$$
$$L\phi = f + \phi(L + f)$$
$$\phi = L^{-1}f + L^{-1}\phi(L + f),$$

or

$$\phi = L^{-1}f$$

Defining the operator $H$ on functions $\phi$ by

$$H(\phi) = L^{-1}\phi(L + f) = L^{-1}\phi \circ T,$$

we see that $H$ is linear, and equation (9) becomes

$$(I - H)(\phi) = L^{-1}f.$$  

The problem, then, is reduced to finding a Banach space $E$ so that the operator $H$ is a well-defined bounded linear map such that $I - H$ has a bounded linear inverse, in which case we obtain

$$\phi = (I - H)^{-1}L^{-1}f.$$  

Theorem 1.5 will be proved by using the map $H$ defined by (10) and the ensuing equations (11) and (12) in several different function spaces and then combining the results.

A rough outline (assuming the hypotheses of Theorem 1.5) is as follows.

**Step 1:** We show that if $L$ is contracting; i.e., $\rho(L) < 1$, then $T = L + f$ can be $C^{1,\alpha}$ linearized near 0. That is, there is a local $C^{1,\alpha}$ diffeomorphism $R$ near 0 satisfying $RTR^{-1} = L$ near 0. It follows, of course, that $RT^{-1}R^{-1} = L^{-1}$ near 0.
Step 2: Assuming $L$ is hyperbolic of saddle type, let $E = E^u \oplus E^s$ be the associated unstable and stable subspaces of $E$. We choose $C^{1,\alpha}$ coordinates near 0 so that the local stable and unstable manifolds are flattened; i.e., are contained in $E^s$ and $E^u$, respectively. Applying the result in Step 1, we can locally $C^{1,\alpha}$ conjugate $T$ to a map $S_1$ which locally preserves $E^s$ and $E^u$ and is linear when restricted to those subspaces. Then, using a bump function, we extend the map $S_1$ (actually its restriction to a small neighborhood of 0) to a $C^{1,\alpha}$ diffeomorphism $S$ from the whole Banach space $E$ onto itself such that $S(0) = 0$, $DS(0) = L$, $\text{Lip}(S-L)$ is small, and $S = L$ on the union of $E^u$, $E^s$, and the complement of a small neighborhood $V$ of 0.

Step 3: For $V$ and $\text{Lip}(S-L)$ small enough, we define an appropriate Banach space $E$ of $C^1$ functions $\phi$ defined on all of $E$ and solve the equation (12) to obtain a global $C^1$ conjugacy $I + \phi$ with $\phi \in E$ from $S$ to $L$ defined on all of $E$.

Step 4: Choosing $V$ small enough, and making use of (12) in two separate Banach spaces of functions, we show that, for appropriately chosen small $\beta > 0$, the map $\phi$ is $C^{1,\beta}$ on bounded subsets of the orbit of $V$.

This gives that $S$ is locally $C^{1,\beta}$ linearizable at 0, and, hence, so is $T$.

The operator $H$ occurs often enough in this kind of problem that it deserves a name. We call it the associated linearization operator for $T$ or the $T-$linearization operator.

3. The $\alpha-$Contracting Case

Let $E$ be a Banach space, and let $U$ be an open neighborhood of 0 in $E$. Let $0 < \alpha < 1$ and consider the set $C^{1,\alpha}_U = C^{1,\alpha}_{0,U}$ of $C^1$ functions $g : U \to E$ such that

\begin{equation}
(13) \quad g(0) = 0, \quad Dg(0) = 0,
\end{equation}

and

\begin{equation}
(14) \quad \text{Hol}(Dg, U) = \sup_{x,y \in U, x \neq y} \frac{|Dg(x) - Dg(y)|}{|x - y|^\alpha} < \infty.
\end{equation}

Using the techniques for statement (8.6.3) in [10] one can verify that the D-Hölder constant of $g$,

\[ |g|_U = |g|_{U,\alpha} = \text{Hol}(Dg, U), \]

defines a norm on $C^{1,\alpha}_U$ making it into a Banach space.

Observe that the usual way of defining a norm on a space of $C^r$ functions into a Banach space involves taking the supremum of the $C^0$ size of the function as well as that of its derivatives. Otherwise one only gets a semi-norm in that the vanishing of this semi-norm does not imply the vanishing of the functions. However, our functions already vanish at 0 (and $U$ is connected), so this issue does not occur.

Note that if $V \subset U$ and $\phi \in C^{1,\alpha}_U$, then the restriction map from $\phi \to \phi|V$ induces a continuous map from $(C^{1,\alpha}_U, |\cdot|_{U,\alpha})$ into $(C^{1,\alpha}_V, |\cdot|_{V,\alpha})$. We will use the same notation for the maps $\phi$ and $\phi|V$ letting the context determine which space we are using at a given moment.

When $U$ is the open ball $B_\delta(0)$ we will denote $C^{1,\alpha}_U$ by $C^{1,\alpha}_\delta$.

Let $I$ denote the identity map on $E$. 
The goal of this section is to prove the following theorem

**Theorem 3.1.** Let $U$ be an open neighborhood of 0, let $\alpha$ be a positive real number with $0 < \alpha < 1$ and let $T$ be a map of the form $T(x) = Lx + f(x)$ where $L : E \to E$ is a linear $\alpha$-contracting automorphism and $f \in C^{1,\alpha}_U$.

Then, for sufficiently small $\delta$, there is a unique map $\phi \in C^{1,\alpha}_{0,\delta}$ such that, setting $R = I + \phi$, we have

\[ L(R(x)) = R(T(x)) \text{ for all } x \in B_\delta(0) \cap T^{-1}B_\delta(0). \]

**Remark 3.2.** For $\delta$ small, the map $R$ will be Lipschitz close to the identity. Therefore, the Inverse Function Theorem implies that $R$ is a $C^{1,\alpha}$ diffeomorphism onto its image with inverse which is also $C^{1,\alpha}$. Also, $L = RTR^{-1}$ implies that $L^{-1} = RT^{-1}R^{-1}$, so Theorem 3.1 also holds when $L$ is a linear $\alpha$-expansion.

**Remark 3.3.** Consider a $C^{1,\alpha}$ diffeomorphism $T : \mathbb{R}^N \to \mathbb{R}^N$ on the Euclidean space $\mathbb{R}^N$ with a hyperbolic fixed point at 0 and derivative $L = DT(0)$. Then, we have the direct sum decomposition $\mathbb{R}^N = E_1 \oplus E_2 \oplus \ldots \oplus E_s$ into $L$-invariant subspaces such that $L | E_i$ is circular for each $1 \leq i \leq s$.

From invariant manifold theory, [17], [8], for some $0 < \alpha < 1$, there are $T$ locally invariant $C^{1,\alpha}$ submanifolds $W_i$ tangent at 0 to $E_i$ for each $i$. These submanifolds are generally not unique, of course, and, even if $T$ is linear, they are not generally $C^{1,1}$. Nevertheless, using Theorem 3.1 Remark 3.2 and the techniques in Section 4.4 below, one can find an $\alpha \in (0, 1)$ and a local $C^{1,\alpha}$ coordinate chart $(U, \zeta)$ near 0 such that $\zeta(0) = 0$, $\zeta(W_i) \subset E_i$ for each $i$, and $\zeta T \zeta^{-1}$ restricted to each $E_i$ is linear near 0.

As we mentioned in the introduction, Theorem 3.1 is not new. It is a consequence of a more general result proved by Chaperon (see Corollary 1.3.3 in [20]). Since the result is needed in an essential way in the present paper and it is relatively simple to prove, we give an alternate proof here for completeness.

We begin with a standard result which shows that, given a Banach space $(E, | \cdot |)$, and a bounded linear operator $L : E \to E$, we can approximate the spectral radius $\rho(L)$ by the supremum norm $| L |_1$ induced by a norm $| \cdot |_1$ on $E$ which is equivalent to $| \cdot |$.

**Lemma 3.4.** Let $(E, | \cdot |)$ be a Banach space and let $L : E \to E$ be a bounded linear map with spectral radius $\rho(L)$. Then, given $\epsilon > 0$ there is a norm $| \cdot |_1$ on $E$ which is equivalent to $| \cdot |$ such that the induced norm $| L |_1$ defined by

\[ | L |_1 = \sup_{| v |_1 = 1} | Lv |_1 \]

satisfies

\[ | L |_1 \leq \rho(L) + \epsilon \]

**Proof.** Setting $\rho_1 = \rho(L) + \epsilon$ and using formula (3), we can choose a positive integer $q$ such that, for $n \geq q$, we have
\[ |L^n| < \rho_1^n. \]
This gives, for any \( v \) and \( n \geq q \),
\[ |L^n v| \leq |L^n| |v| \leq \rho_1^n |v|, \]
which implies that
\[ \rho^{-n}_1 |L^n v| \leq |v|. \]
As usual, we define \( L^0 = I \).
Then, setting
\[ K = \max_{0 \leq n < q} \rho^{-n}_1 |L^n|, \]
we get, for \( n < q \),
\[ \rho^{-n}_1 |L^n v| \leq \rho^{-n}_1 |L^n| |v| \leq K |v|. \]
For \( n = 0 \), and \( |v| \neq 0 \), this gives \( K \geq 1 \).
Putting these inequalities together, gives
\[ \sup_{n \geq 0} \rho^{-n}_1 |L^n v| \leq K |v| \]
for any vector \( v \in E \).
Defining a new norm by
\[ |v|_1 = \sup_{n \geq 0} \rho^{-n}_1 |L^n v|, \]
we then have
\[ |v| \leq |v|_1 \leq K |v| \]
so \( |\cdot|_1 \) is equivalent to \(|\cdot|\).
Moreover, for every \( v \), we have
\[ |Lv|_1 = \sup_{n \geq 0} \rho^{-n}_1 |L^n Lv| \]
\[ = \sup_{n \geq 0} \rho^{-n}_1 |L^{n+1} v| \]
\[ = \rho_1 \sup_{n \geq 0} \rho^{-n-1}_1 |L^{n+1} v| \]
\[ = \rho_1 \sup_{n \geq 1} \rho^{-n}_1 |L^n v| \]
\[ \leq \rho_1 |v|_1. \]
Taking any \( v \) with \(|v|_1 = 1\), this gives \(|Lv|_1 \leq \rho_1 \), whence
\[ |L|_1 = \sup_{|v|_1 = 1} |Lv|_1 \leq \rho_1 \]
as required to complete the proof of Lemma 3.4. \(\square\)
Let us say that a bounded linear map \( H \) of a Banach space \( E \) is \emph{contracting} if its spectral radius is less than 1. Theorem 3.1 will be proved using the method described in section 2. In this case, the corresponding operator \( H \) will, in fact, be contracting. Using Lemma 3.4 we find an equivalent norm in which the induced norm on \( H \) satisfies \( | H | < 1 \). Then, as is well known, \( I - H \) is invertible with inverse given by
\[
(I - H)^{-1} = \sum_{n=0}^{\infty} H^n.
\]

For \( \delta > 0 \) sufficiently small, we take the space \( E \) alluded to in section 2 to be \( C^{1,\alpha}_\delta \), and we use the norm \( | \phi |_\delta = | \phi |_{\delta,\alpha} \) for elements \( \phi \in E \).

Let \( U, L, f, T \) be as in the hypotheses of Theorem 3.1. All numbers \( \delta \) below will be chosen so that \( B_\delta(0) \subset U \), and, hence, \( f \) is defined and \( C^{1,\alpha} \) in \( B_\delta(0) \).

Recalling the procedure in section 2 we consider the functional equation
\[
L(I + \phi) = (I + \phi)(L + f)
\]
and the resulting algebraic equation
\[
(I - H)(\phi) = L^{-1}f.
\]
where \( H(\phi) = L^{-1}\phi(L + f) \).

We will show that, for \( \delta \) small enough, the operator \( H \) has the properties that
\[
H \text{ is defined as a function on } C^{1,\alpha}_\delta,
\]
(17)
\[
H \text{ maps } C^{1,\alpha}_\delta \text{ into itself}
\]
and, for some positive integer \( m \),
(18)
\[
| H^m | < 1.
\]

In then follows that \( \rho(H) < 1 \), and, as we have already noted, the required solution is obtained as \( \phi = (I - H)^{-1}L^{-1}f \).

Before going to the proof of (16)-(18), we need some easy estimates obtained from the Mean Value Theorem and our other assumptions.

As we have already noted, the \( \alpha \)-contracting condition on \( L \) implies that \( \rho(L) < 1 \), so we may use Lemma 3.4 to renorm \( E \) so that
(19)
\[
| L | < 1,
\]
which implies that
(20)
\[
| L^n | < | L |^n < 1
\]
for every positive integer \( n \).

From the fact that \( L \) is \( \alpha \)-contracting, formula 3 gives us a positive integer \( m \) such that
\[
\left( |L^{-m}| |L^m|^{1+\alpha} \right)^{\frac{1}{1+\alpha}} < 1,
\]
which, in turn, implies
\[
|L^{-m}| |L^m|^{1+\alpha} < 1.
\]

Given a linear automorphism \(L\), we recall that its \textit{minimum norm}, denoted by \(m(L)\) (or \(m_L\)) is defined by
\begin{equation}
(21) \quad m(L) = m_L = \inf_{|x|=1} |Lx|.
\end{equation}

It is easily checked that \(m(L) = |L^{-1}|^{-1}\).

Let \(\epsilon_1 > 0\) be such that
\begin{equation}
(22) \quad |L| + \epsilon_1 < 1,
\end{equation}

\begin{equation}
(23) \quad m(L) - \epsilon_1 > 0,
\end{equation}

\begin{equation}
(24) \quad |L^m| + \epsilon_1 < 1,
\end{equation}

and
\begin{equation}
(25) \quad m(L^{-m}) - \epsilon_1 > 0,
\end{equation}

and
\begin{equation}
(26) \quad |L^{-m}|(|L^m| + \epsilon_1)^{1+\alpha} < 1.
\end{equation}

Using \(Df_0 = 0\), choose \(\delta > 0\) small enough so that, for \(x \in B_\delta(0)\),
\begin{equation}
(27) \quad |Df_x| < \epsilon_1.
\end{equation}

In particular,
\begin{equation}
(28) \quad \text{Lip}(f, B_\delta(0)) \leq \epsilon_1.
\end{equation}

Now, for \(|x| < \delta\),
\begin{equation}
(29) \quad |T(x)| = |Lx + f(x)| \leq (|L| + \epsilon_1)|x| \leq |x|,
\end{equation}

and, for \(x \neq y \in B_\delta\), we have
\[
|T(x) - T(y)| = |(L + f)x - (L + f)y| \\
\geq |L(x - y)| - \epsilon_1 |x - y| \\
> (m_L - \epsilon_1) |x - y| > 0.
\]

In particular, since \(T(0) = 0\), it follows that \(|T(x)| > 0\) for \(|x| > 0\).

Further,
\[ | T(x) - T(y) | = | Lx + f(x) - Ly - f(y) | \]
\[ \leq (| L | + Lip(f)) | x - y | \]
\[ \leq (| L | + \epsilon_1) | x - y | , \]
so,
\begin{equation}
Lip(T, B_\delta(0)) \leq | L | + \epsilon_1 < 1.
\end{equation}

This implies that both \( T \) and \( T^m \) map \( B_\delta \) into itself.

An easy induction shows that \( DT^m(0) = L^m \). So, since the composition of \( C^{1,\alpha} \) maps is \( C^{1,\alpha} \), if we set \( f_m = T^m - L^m \), then we have that \( f_m \in C^{1,\alpha}_\delta \).

Now, shrinking \( \delta \), if necessary, we may assume that
\begin{equation}
Lip(f_m, B_\delta) < \epsilon_1,
\end{equation}
and
\begin{equation}
Lip(T^m, B_\delta) < | L^m | + \epsilon_1.
\end{equation}

For \( n = 1 \) or \( n = m \), let
\[ K_{n,1} = | L^{-n} |(| L^n | + \epsilon_1)^\alpha \delta | f_n |_\delta , \]
and
\[ K_{n,2} = | L^{-n} |(| L^n | + \epsilon_1)^{1+\alpha} , \]
and
\[ \tau_n = K_{n,1} + K_{n,2} . \]

From (26), we have \( K_{m,2} < 1 \), so we can choose \( \delta > 0 \) small enough so that \( \tau_m < 1 \).

Let us now, proceed to the proof of (16)-(18).

Let \( \phi \in C^{1,\alpha}_\delta \).

From (30), it follows that \( T(B_\delta(0)) \subset B_\delta(0) \), so \( H(\phi) \) is defined on \( B_\delta(0) \), and this is (16).

Again using that the composition of \( C^{1,\alpha} \) maps is again \( C^{1,\alpha} \), it is clear that \( H(\phi) \) is \( C^{1,\alpha} \). It is also evident that it vanishes at 0 together with its derivative.

For \( x, y \in B_\delta \) with \( | x - y | > 0 \) and \( | x | > 0 \), and \( n = 1 \) or \( n = m \), we have
\begin{align*}
| D(H^n(\phi))(x) - D(H^n(\phi))(y) | &= | L^{-n} D\phi_{T^n(x)} DT^n_x - L^{-n} D\phi_{T^n(y)} DT^n_y | \\
&\leq | L^{-n} | D\phi_{T^n(x)} DT^n_x - D\phi_{T^n(y)} DT^n_y | \\
&\quad + | L^{-n} | D\phi_{T^n(x)} DT^n_y - D\phi_{T^n(y)} DT^n_y | \\
&= (A) + (B)
\end{align*}
where
near the origin. Next, using his earlier result for are flattened (i.e., are contained in the stable and unstable subspaces, respectively) Theorem 1.3. He first changes coordinates so that the stable and unstable manifolds of Theorem 3.1. (33)

Let us begin with an outline of Hartman’s method in the proof of part (b) of $n$

For $n = 1$, this gives (17), and for $n = m$, this gives (18), completing the proof of Theorem 3.1.

4. Preliminary Constructions

Let us begin with an outline of Hartman’s method in the proof of part (b) of Theorem 1.3. He first changes coordinates so that the stable and unstable manifolds are flattened (i.e., are contained in the stable and unstable subspaces, respectively) near the origin. Next, using his earlier result for $C^{1,1}$ contracting $T$, he chooses
\(C^1, \beta\) coordinates (where \(\beta\) is related to the contracting and expanding eigenvalues) so that, near the origin, \(T\) preserves and becomes linear when restricted to the stable and unstable subspaces. Finally, he globalizes the mapping \(T\), using a bump function to replace \(T\) by a map \(T_1\) which equals \(T\) on a ball \(B_{\epsilon_0}(0)\) about zero, equals \(L\) off a slightly larger ball \(B_{\epsilon_1}(0)\), and is globally Lipschitz close to \(L\). This globalization preserves the earlier properties that the stable and unstable manifolds are flattened and \(T_1\) is linear when restricted to the entire stable and unstable subspaces. This allows him to get sufficiently good estimates on the partial derivatives of the non-linear part of \(T_1\) to use the method of successive approximations to obtain the desired local linearization. In fact, he only proves that the linearization is continuous. He states that similar methods will give that it is \(C^1\) and that the proof can be modified to show that it has Hölder continuous derivatives.

We extend and modify his techniques to obtain our result. In addition, we write our solution in a way that it makes use of certain linear contracting maps in suitable Banach spaces. This provides the additional benefit that we can get continuous dependence of the linearization on parameters as in Theorem 6.2 below.

4.1. Estimates of the Lipschitz and D-Hölder constants of inverses and compositions. The proof of Theorem 1.5 requires estimates of the non-linear parts of the diffeomorphisms \(T, T^m, T^{-1}\) and \(T^{-m}\) for a certain positive integer \(m\) in a sufficiently small ball \(B_\delta(0)\) about 0 in the Banach space \(E\). The results in this section can be used to give some information about these estimates in terms of the non-linear part of the original map \(T\). Strictly speaking, they are not needed if one only wants the linearization \(R\) to exist on some small neighborhood of 0 and one does not need to estimate the size of that neighborhood.

Let us begin with a simple lemma relating the Lipschitz and D-Hölder constants of a composition \(S \circ T\) of \(C^{1,\alpha}\) maps in terms of those of the maps \(S\) and \(T\).

**Lemma 4.1.** Let \(U\) and \(V\) be open subsets of the Banach space \(E\), and consider maps \(S \in C^{1,\alpha}(U, E)\) and \(T \in C^{1,\alpha}(V, E)\).

Then,

\[
\text{Lip}(S \circ T, U \cap T(V)) \leq \text{Lip}(S, U) \text{Lip}(T, V),
\]

and

\[
\text{Hol}(D(S \circ T), U \cap T(V)) \leq \text{Lip}(S, U) \text{Hol}(DT, V) + \text{Lip}(T, V)^{1+\alpha} \text{Hol}(DS, U).
\]

**Proof.** Letting \(x, y \in V\), and leaving out the obvious domains of the maps involved, statement (34) is immediate from

\[|S(T(x)) - S(T(y))| \leq \text{Lip}(S)|T(x) - T(y)| \leq \text{Lip}(S)\text{Lip}(T)|x - y|.
\]

For statement (35), we have
\[ |D(S \circ T)(x) - D(S \circ T)(y)| = |DS_{T x} DT x - DS_{T y} DT y| \]
\[ \leq |DS_{T x} DT x - DS_{T y} DT x| + |DS_{T y} DT x - DS_{T y} DT y| \]
\[ \leq |DT x| Hol(DS)|Tx - Ty|^{\alpha} + |DS_{T y}| Hol(DT)|x - y|^{\alpha} \]
\[ \leq Lip(T)^{1+\alpha} Hol(DS)|x - y|^{\alpha} + Lip(S) Hol(DT)|x - y|^{\alpha}. \]

Now, divide both sides by \(|x - y|^{\alpha}\) and take the supremum over \(x, y\) to complete the proof of Lemma 4.1.

\[ \Box \]

**Remark 4.2.** Observe that, if \(T\) has the form \(T = L + f\) where \(L\) is bounded, \(f(0) = 0\), and \(Df(0) = 0\), then, since the derivative of a bounded linear map \(L\) is just \(L\) at every point, it cancels in the calculation of \(DT\). Hence,

\[ Hol(DT) = Hol(Df). \]

That is, the \(D\)-Hölder constant is determined by the nonlinear part of \(T\). In particular, the addition of another bounded linear map to \(T\) does not change \(Hol(DT)\).

**Corollary 4.3.** Let \(S, T, U, V\) be as in the hypotheses of Lemma 4.1, and assume that \(S = L_1 + f_1\) and \(T = L_2 + f_2\) where \(L_1, L_2\) are bounded linear maps on \(E\) and \(f_1, f_2\) vanish at 0 together with their derivatives. Let \(f_3 = S \circ T - L_1 L_2\).

Then,

\[ \text{Lip}(f_3) \leq |L_1| \text{Lip}(f_2) + \text{Lip}(f_1) \text{Lip}(T), \]

and

\[ \text{Hol}(Df_3) \leq \text{Lip}(S) \text{Hol}(DT) + \text{Lip}(T)^{1+\alpha} \text{Hol}(Df_1). \]

**Proof.** We have

\[ ST = (L_1 + f_1)(L_2 + f_2) \]
\[ = L_1 L_2 + f_1(T) \]
\[ = L_1 L_2 + L_1 f_2 + f_1(T), \]

giving \(f_3 = L_1 f_2 + f_1(T)\), and (37) immediately follows using the fact that the function \(\text{Lip}(\cdot)\) is subadditive and submultiplicative.

Statement (38) follows immediately from (35) since

\[ \text{Hol}(Df_3) = \text{Hol}(D(S \circ T)). \]

\[ \Box \]

We will need the following estimate which we first saw in Hirsch and Pugh [16].

**Lemma 4.4.** For any linear automorphisms \(h_1, h_2\), we have

\[ |h_1^{-1} - h_2^{-1}| \leq |h_2^{-1}| |h_1 - h_2| |h_1^{-1}|. \]
Proof. We have
\[
|h_1^{-1} - h_2^{-1}| = |h_2^{-1} h_2 h_1^{-1} - h_2^{-1} h_1 h_1^{-1}|
= |h_2^{-1} (h_2 - h_1) h_1^{-1}|
\leq |h_2^{-1}| |h_2 - h_1| |h_1^{-1}|
= |h_2^{-1}| |h_1 - h_2| |h_1^{-1}|
\]
\[\square\]

Let \(diam(U)\) denote the diameter of a subset \(U\) of a Banach space \(E\).

In Lemma 4.5, we again use the notation
\[m(L) = m_L = \inf_{|x|=1} |Lx| = |L^{-1}|^{-1}\]
for a linear automorphism \(L\).

**Lemma 4.5.** Let \((E, |\cdot|)\) be a real Banach space, and let \(L : E \to E\) be a linear automorphism. Let \(0 < \alpha < 1\), and let \(U\) and \(V\) be neighborhoods of 0 in \(E\) such that
\[(40) \max(\text{diam}(U), \text{diam}(V)) < 1.\]

Let \(T : U \to V\) be a \(C^{1+\alpha}\) diffeomorphism from \(U\) onto \(V\) such that \(T(0) = 0\) and \(DT(0) = L\), and let \(f = T - L\) and \(g = T^{-1} - L^{-1}\) be the nonlinear parts of \(T\) and \(T^{-1}\), respectively.

Assume that
\[(41) \quad \text{Lip}(f, U) = \sup_{z \in U} |Df(z)| < m_L.\]

Then, we have
\[(42) \quad \text{Lip}(T^{-1}, V) \leq (m_L - \text{Lip}(f, U))^{-1};\]
\[(43) \quad \text{Lip}(g, V) \leq |L^{-1}| (m_L - \text{Lip}(f, U))^{-1} \text{Lip}(f, U),\]
\[(44) \quad \text{Hol}(DT^{-1}, V) \leq (m_L - \text{Lip}(f, U))^{-(2+\alpha)} \text{Hol}(Df, U)\]
and
\[(45) \quad \text{Hol}(Dg, V) \leq (m_L - \text{Lip}(f, U))^{-(2+\alpha)} \text{Hol}(Df, U).\]

**Proof.** For notational convenience, we will use the following notation.
\[l_f = \text{Lip}(f, U), \quad h_f = \text{Hol}(Df, U).\]
From (41), we have that, for any \(z \in U\) and non-zero vector \(v\),
\[|DT_z v| = |Lv + Df(z)v| \geq m_L |v| - l_f |v| = (m_L - l_f) |v|.
\]
Now, taking any non-zero \(u\) and setting \(v = DT_{T(z)}^{-1} u\) and \(w = T(z)\), we get
\[
|u| = |DT_wDT_w^{-1}u| \geq (m_L - l_f)|DT_w^{-1}u|,
\]
or
\[
|DT_w^{-1}u| \leq (m_L - l_f)^{-1}|u|.
\]
Thus, for every \(w \in V\), we have
\[
|DT_w^{-1}| \leq (m_L - l_f)^{-1},
\]
and (42) follows.

Again, writing
\[
w = T(z) = Lz + f(z),
\]
and solving for \(z = z(w)\), we get
\[
z(w) = T^{-1}(w) = L^{-1}w - L^{-1}f(z(w)).
\]
Thus,
\[
g(w) = -L^{-1}f(z(w)),
\]
\[
Dg(w) = -L^{-1}Df_z(w)DT_w^{-1},
\]
and
\[
|Dg(w)| \leq |L^{-1}| |Df_z(w)||DT_w^{-1}|
\]
\[
= |L^{-1}| |Df_z(w)|(m_L - l_f)^{-1}
\]
\[
\leq |L^{-1}|l_f(m_L - l_f)^{-1}
\]
This implies (43).

Proceeding to the proof of (44), from (46), we have
\[
DT_w^{-1} = L^{-1} - L^{-1}Df_z(w)DT_w^{-1}
\]
\[
(I + L^{-1}Df_z(w))DT_w^{-1} = L^{-1},
\]
\[
DT_w^{-1} = (I + L^{-1}Df_z(w))^{-1}L^{-1}.
\]
From (41), we have, for each \(w \in V\),
\[
| (I + L^{-1}Df_z(w))^{-1} | \leq (1 - |L^{-1}|l_f)^{-1}
\]
\[
= (|L^{-1}|m_L - |L^{-1}|l_f)^{-1}
\]
\[
= |L^{-1}|^{-1}(m_l - l_f)^{-1}
\]
\[
= m_L(m_l - l_f)^{-1}
\]
This, together with (39) gives
\[ | DT^{-1}_{w_1} - DT^{-1}_{w_2} | \leq m_L^2 (m_L - l_f)^{-2} | L^{-1} |^2 | Df_{z(w_2)} - Df_{z(w_1)} | \]
\[ = (m_L - l_f)^{-2} | Df_{z(w_2)} - Df_{z(w_1)} | \]
\[ \leq (m_L - l_f)^{-2} h_f | z(w_2) - z(w_1) |^\alpha \]
\[ \leq (m_L - l_f)^{-2} h_f | (T^{-1})^\alpha | w_2 - w_1 |^\alpha \]
\[ \leq (m_L - l_f)^{-2} h_f | w_2 - w_1 |^\alpha \]

which is (44).

Also, (45) holds since \( Hol(DT^{-1}, W) = Hol(Dg, W) \). This completes the proof of Lemma 4.5.

The next lemma gives analogous estimates of the non-linear parts of \( T^m \) and \( T^{-m} \) for integers \( m > 1 \).

**Lemma 4.6.** Let \( T, U, V, f, g \) be as in Lemma 4.5, let \( m \) be a positive integer, and assume that \( \text{Lip}(T) \geq 1 \).

Consider the maps \( T^m, T^{-m} \) and \( f_m, g_m \), given by

\[ f_m = T^m - L^m, \quad g_m = T^{-m} - L^{-m}, \]

with corresponding domains \( U_m, V_m \) given by

\[ U_m = \bigcap_{j=0}^{m} T^{-j} U \quad \text{and} \quad V_m = \bigcap_{j=0}^{m} T^j V, \]

respectively.

Then, \( T^m \) and \( T^{-m} \) are defined and \( C^{1, \alpha} \) on \( U_m \) and \( V_m \), respectively, and

\[ \text{Lip}(T^m) \leq \text{Lip}(T)^m, \] (47)
\[ \text{Lip}(f_m) \leq m \text{Lip}(f) \text{Lip}(T)^{m-1}, \] (48)
\[ \text{Hol}(Df_m) \leq m \text{Hol}(Df) \text{Lip}(T)^{(1+\alpha)(m-1)}, \] (49)
\[ \text{Lip}(g_m) \leq m \text{Lip}(g) \text{Lip}(T^{-1})^{m-1}, \] (50)

and

\[ \text{Hol}(Dg_m) \leq m \text{Hol}(Dg) \text{Lip}(T^{-1})^{(1+\alpha)(m-1)}. \] (51)

**Proof.** Since the composition of \( C^{1, \alpha} \) maps is again \( C^{1, \alpha} \), it is clear that \( T^m \) and \( T^{-m} \) are defined and \( C^{1, \alpha} \) on \( U_m \) and \( V_m \), respectively.

Also, the Lipschitz composition formula (34) gives (47) immediately by induction.

Using that \( DT^m(0) = L^n \), we get

\[ DT^{-1}_{w_1} - DT^{-1}_{w_2} | \leq m_L^2 (m_L - l_f)^{-2} | L^{-1} |^2 | Df_{z(w_2)} - Df_{z(w_1)} | \]
\[ = (m_L - l_f)^{-2} | Df_{z(w_2)} - Df_{z(w_1)} | \]
\[ \leq (m_L - l_f)^{-2} h_f | z(w_2) - z(w_1) |^\alpha \]
\[ \leq (m_L - l_f)^{-2} h_f | (T^{-1})^\alpha | w_2 - w_1 |^\alpha \]
\[ \leq (m_L - l_f)^{-2} h_f | w_2 - w_1 |^\alpha \]
\[ T^{n+1} = (L + f)(T^n) \]
\[ = LT^n + f(T^n) \]
\[ = L(L^n + f_n) + f(T^n) \]
\[ = L^{n+1} + Lf_n + f(T^n). \]

This implies that, for each \( n \geq 1, \)

\[ f_{n+1} = Lf_n + f(T^n). \] (52)

We will show that, for different choices of positive numbers \( A_n, B, C, \) the numbers \( Lip(f_m), Lip(g_m), Hol(Df_m) \) and \( Hol(Dg_m) \) all satisfy the recursion relation

\[ A_{n+1} \leq BA_n + B^n C, \quad A_1 = C \] (53)

for each \( 1 \leq n \leq m - 1. \)

This recursion relation has the solution

\[ A_{n+1} \leq (n+1)B^n C \] (54)

as can be seen from

\[
\begin{align*}
A_{n+1} & \leq BA_n + B^n C \\
& \leq B(BA_{n-1} + B^{n-1}C) + B^n C \\
& = B^2 A_{n-1} + B^n C + B^n C \\
& = B^2 A_{n-1} + 2B^n C \\
& \quad \vdots \\
& \leq B^n A_1 + nB^n C \\
& \quad \vdots \\
& = B^n A_1 + nB^n C \\
& = (n+1)B^n C.
\end{align*}
\] (55)

Now, consider \( Lip(f_m). \)

Since \( |L| \leq Lip(T), \) we have

\[ Lip(f_{n+1}) \leq |L|Lip(f_n) + Lip(f)Lip(T^n) \leq Lip(T)Lip(f_n) + Lip(f)Lip(T^n) \]

and we can take \( A_n = Lip(f_n), B = Lip(T) \) and \( C = Lip(f). \)

Next, observe that \( Hol(Df_m) = Hol(DT^m) \) since

\[ |DT^m x - DT^m y| = |L^m + Df_m(x) - (L^m + Df_m(y))| = |Df^m x - Df^m y|. \]

The assumption that \( Lip(T) \geq 1 \) implies that
\[ \text{Lip}(T) \leq \text{Lip}(T)^{1+\alpha}. \]

Using this and (38), we have

\[
\text{Hol}(DT^{n+1}) = \text{Hol}(D(T \circ T^n)) \\
\leq \text{Lip}(T)\text{Hol}(DT^n) + \text{Lip}(T^n)^{1+\alpha}\text{Hol}(DT) \\
\leq \text{Lip}(T)^{1+\alpha}\text{Hol}(DT^n) + \text{Lip}(T^n)^{n(1+\alpha)}\text{Hol}(DT)
\]

and we can take \( A_n = \text{Hol}(DT^n), B = \text{Lip}(T)^{1+\alpha} \) and \( C = \text{Hol}(DT) \).

Similar recursions hold for \( \text{Lip}(g_m) \) and \( \text{Hol}(Dg_m) \).

Thus, formula (54) applied, in turn, to each of \( \text{Lip}(f_m), \text{Hol}(Df_m), \text{Lip}(g_m) \) and \( \text{Hol}(Dg_m) \) gives (48)-(51) to complete the proof of Lemma 4.6.

\[ \square \]

4.2. Global Extension Via Bump Functions. Let \((E, |\cdot|)\) be a \( C^{1,\alpha} \) Banach space. Thus, there is a \( C^{1,\alpha} \) real valued function \( \lambda \) defined on \( E \) and a real number \( c \in (0,1) \) such that

\[
\lambda(E) = [0,1],
\]

\[
\lambda(x) = 1 \text{ for } |x| \leq c,
\]

\[
\lambda(x) = 0 \text{ for } |x| \geq 1,
\]

\[
\text{Lip}(\lambda,E) = \sup_{x \in E} |D\lambda(x)| < \infty,
\]

and

\[
\text{Hol}(D\lambda,E) = \sup_{x \neq y \in E} \frac{|D\lambda(x) - D\lambda(y)|}{|x - y|^\alpha} < \infty.
\]

We call \( \lambda \) a \textit{unit bump function} on \( E \).

Given such a bump function \( \lambda \) and a positive real number \( \delta \), we define the associated \( \delta \)-\textit{scaled} version of \( \lambda \) to be

\[
\lambda_\delta(x) = \lambda\left(\frac{x}{\delta}\right).
\]

Obviously, the function \( \lambda_\delta \) vanishes off \( B_\delta \) and is 1 on \( B_{\delta \delta} \).

The Lipschitz and Hölder constants of \( \lambda_\delta \) are easily computed as follows. We have

\[
D\lambda_\delta(x) = \frac{1}{\delta}D\lambda\left(\frac{x}{\delta}\right),
\]

and
\[ |D\lambda_\delta(x) - D\lambda_\delta(y)| = |\frac{1}{\delta} (D\lambda(\frac{x}{\delta}) - D\lambda(\frac{y}{\delta}))| \]
\[ \leq \frac{1}{\delta} |\text{Hol}(D\lambda, E)| \frac{x}{\delta} - \frac{y}{\delta}|^\alpha \]
\[ = \frac{1}{\delta^{1+\alpha}} |\text{Hol}(D\lambda, E)| |x - y|^\alpha \]

for any \( x \neq y \in E \).

Thus,

\[ (57) \quad \text{Lip}(\lambda_\delta) = \frac{\text{Lip}(\lambda)}{\delta} \quad \text{and} \quad \text{Hol}(D\lambda_\delta) = \frac{\text{Hol}(D\lambda)}{\delta^{1+\alpha}} \]

Recall that, for a \( C^{1,\alpha} \) function \( f : U \to E \), we have defined

\[ \text{Lip}(f, U) = \sup_{x \in U} |Df(x)| \]

and

\[ \text{Hol}(Df, U) = \sup_{x \neq y \in U} \frac{|Df(x) - Df(y)|}{|x - y|^\alpha}. \]

For functions \( g \) defined on all of \( E \), we leave out the domain \( E \) and simply write

\[ \text{Lip}(g) = \text{Lip}(g, E) \quad \text{and} \quad \text{Hol}(Dg) = \text{Hol}(Dg, E). \]

**Lemma 4.7.** Let \( U \) be an open neighborhood of 0 in the \( C^{1,\alpha} \) Banach space \( E \) and let \( f : U \to E \) be a \( C^{1,\alpha} \) map such that \( f(0) = 0 \) and both \( \text{Lip}(f, U) \) and \( \text{Hol}(Df, U) \) are finite.

Let \( \lambda : E \to \mathbb{R} \) be the unit bump function defined above, and define the functions \( C_1(\lambda) \) and \( C_2(\lambda) \) by

\[ C_1(\lambda) = 1 + \text{Lip}(\lambda) \]

and

\[ C_2(\lambda) = 1 + 2\text{Lip}(\lambda) + 3\text{Hol}(D\lambda). \]

For any \( \delta > 0 \) be such that \( B_\delta(0) \subset U \), consider the function \( g \) defined by the product

\[ g(x) = \lambda_\delta(x)f(x) \]

where \( \lambda_\delta \) is the \( \delta \)–scaled version of \( \lambda \) defined in (56).

Then, for \( c \) as in the definition of \( \lambda(\cdot) \), the function \( g \) is defined and \( C^{1,\alpha} \) on all of \( E \) and satisfies the following.

\[ (58) \quad g(x) = f(x) \text{ for } |x| \leq c \delta, \]

\[ (59) \quad g(x) = 0 \text{ for } |x| \geq \delta, \]

\[ (60) \quad \text{Lip}(g) \leq C_1(\lambda)\text{Lip}(f, U) \]
and

\[ \text{Hol}(Dg) \leq C_2(\lambda)\text{Hol}(Df, U) \]

**Proof.** Statements (58) and (59) are obvious.
For statement (60), using \( |x| \leq \delta \) and \( f(0) = 0 \), we have

\[
| D(\lambda \delta(x)f(x)) | = | D\lambda \delta(x)f(x) + \lambda \delta(x)Df(x) | \\
\leq \frac{\text{Lip}(\lambda)}{\delta} | f(x) | + | \lambda \delta(x) || Df(x) | \\
\leq \frac{\text{Lip}(\lambda)}{\delta} \text{Lip}(f, U) | x | + \text{Lip}(f, U) \\
\leq \text{Lip}(\lambda)\text{Lip}(f, U) + \text{Lip}(f, U) \\
= \text{Lip}(f, U)(\text{Lip}(\lambda) + 1)
\]

as needed.
We now proceed to verify (61).
For ease of notation, let us denote \( \lambda \delta(x) \) as \( \gamma(x) \).
It suffices to prove that, for \( x \neq y \in U \),

\[ | D(\gamma f)(x) - D(\gamma f)(y) | \leq C_2(\lambda)\text{Hol}(Df, U) | x - y | ^\alpha. \]

We have

\[
| D(\gamma f)(x) - D(\gamma f)y | = | D\gamma(x)f(x) + \gamma(x)Df(x) - (D\gamma(y)f(y) + \gamma(y)Df(y)) | \\
\leq | D\gamma(x)f(x) - D\gamma(y)f(y) | + | \gamma(x)Df(x) - \gamma(y)Df(y) | \\
\]

Let us estimate the two expressions

\[ R_1 = | D\gamma(x)f(x) - D\gamma(y)f(y) | \]

and

\[ R_2 = | \gamma(x)Df(x) - \gamma(y)Df(y) | \]

separately.
Let \( H_0(Df) \) represent the Hölder constant of \( Df \) at 0.
We have
\[ R_1 = |D\gamma(x)f(x) - D\gamma(y)f(y)| \]
\[ \leq |D\gamma(x)f(x) - D\gamma(y)f(x)| + |D\gamma(y)f(x) - D\gamma(y)f(y)| \]
\[ \leq \text{Hol}(D\gamma)|x - y|^\alpha H_0(Df)|x|^{1+\alpha} \]
\[ + H_0(D\gamma)|y|^\alpha \sup_{0 \leq \tau \leq 1} |Df((1 - \tau)x + \tau y)| |x - y| \]
\[ \leq \frac{\text{Hol}(D\lambda)}{\delta^{1+\alpha}} H_0(Df)|x - y|^\alpha \delta^{1+\alpha} \]
\[ + \frac{\text{Hol}(D\lambda)}{\delta^{1+\alpha}} \delta^\alpha H_0(Df) \max(|x|, |y|)^\alpha |x - y| \]
\[ \leq \text{Hol}(D\lambda)H_0(Df)|x - y|^\alpha + \frac{\text{Hol}(D\lambda)}{\delta} H_0(Df)\delta^\alpha |x - y|^1 + \alpha \]
\[ \leq \text{Hol}(D\lambda)H_0(Df)|x - y|^\alpha + \text{Hol}(D\lambda)H_0(Df)^2 |x - y|^\alpha \]
\[ \leq 3\text{Hol}(D\lambda)H_0(Df)|x - y|^\alpha \]
\[ \leq 3\text{Hol}(D\lambda)\text{Hol}(Df,U)|x - y|^\alpha \]

and

\[ R_2 = |\gamma(x)Df(x) - \gamma(y)Df(y)| \]
\[ \leq |\gamma(x)Df(x) - \gamma(y)Df(x)| + |\gamma(y)Df(x) - \gamma(y)Df(y)| \]
\[ \leq \text{Lip}(\gamma)|x - y||Df(x)| + |Df(x) - Df(y)| \]
\[ \leq \frac{\text{Lip}(\lambda)}{\delta} |x - y|^\alpha |x - y|^{1-\alpha} H_0(Df,U)|x|^\alpha + \text{Hol}(Df,U)|x - y|^\alpha \]
\[ \leq \frac{\text{Lip}(\lambda)}{\delta} |x - y|^\alpha 2^{1-\alpha} \max(|x|, |y|)^{1-\alpha} H_0(Df,U)\delta^\alpha + \text{Hol}(Df,U)|x - y|^\alpha \]
\[ \leq \frac{\text{Lip}(\lambda)}{\delta} |x - y|^\alpha 2\delta^{1-\alpha} H_0(Df,U)\delta^\alpha + \text{Hol}(Df,U)|x - y|^\alpha \]
\[ \leq (2\text{Lip}(\lambda) + 1)\text{Hol}(Df,U)|x - y|^\alpha \]

Hence,

\[ |D(\gamma f)(x) - D(\gamma f)(y)| = R_1 + R_2 \]
\[ \leq (3\text{Hol}(D\lambda) + 2\text{Lip}(\lambda) + 1)\text{Hol}(Df,U)|x - y|^\alpha \]
\[ \leq C_2(\lambda)\text{Hol}(Df,U)|x - y|^\alpha \]

This completes the proof of Lemma 4.7.

4.3. Convention when using direct sum decompositions. When the Banach space \( E \) is written as a direct sum decomposition \( E^a \oplus E^s \), it is often convenient to identify \( T : E \to E \) with the map \( \tilde{T} \) from \( E^a \times E^s \) to \( E^a \times E^s \) defined by taking the unique representations of \( z = x + y, T(z) = x_1 + y_1 \), with \( x, x_1 \in E^a, y, y_1 \in E^s \), and defining.
\[ \tilde{T}(x, y) = (x_1, y_1). \]

The map \( \tilde{T} \) is the conjugate \( RTR^{-1} \) where \( R(z) = (x, y) \). Thus, \( \tilde{T} \) is simply the map obtained using \( R \) as a linear change of coordinates.

Some statements have a more elegant formulation using \( T \), but their proofs are best given in terms of the map \( \tilde{T} \).

We call the map \( \tilde{T} \) the product representation of \( T \). Letting \( \pi^u: E^u \oplus E^s \to E^u \) and \( \pi^s: E^u \oplus E^s \to E^s \) denote the natural projections

\[ \pi^u(z) = x, \pi^s(z) = y, \]

and writing

\[ (\pi^u \circ T)(x, y) = f_1(x, y), \]

and

\[ (\pi^s \circ T)(x, y) = f_2(x, y), \]

we will identify \( T \) with the map \( \tilde{T} \) and simply say we may write \( T \) as

\[ T(x, y) = (f_1(x, y), f_2(x, y)). \]

Similarly, the origin will be written as 0 or \((0,0)\), and balls centered at 0 in \( E \) will be written as \( B_\delta = B_\delta(0) \) or \( B_\delta(0,0) \).

4.4 Flattening and Linearizing on Invariant Manifolds. Let \( E \) be a real Banach space. For a positive real number \( \delta \) we use the notation \( B_\delta = B_\delta(0) \) for the open ball of radius \( \delta \) centered at 0; i.e., the set of points \( x \in E \) such that \( |x| < \delta \).

We will say that a property holds near 0 if it holds in \( B_\delta(0) \) for some small \( \delta > 0 \).

In this section, all neighborhoods of 0 in \( E \) will be assumed to be open and convex. If \( U \) is such a neighborhood and \( f: U \to E \), then the Mean Value Theorem can be applied to show that

\[ \text{Lip}(f, U) = \sup_{x \in U} |Df_x|, \]

and we will often use this fact.

Given a neighborhood \( U \) of 0 in \( E \) and an injective map \( T: U \to E \), define

\[ W^s_U = W^s_U(T) = \bigcap_{n \geq 0} T^{-n}(U) \]

and

\[ W^u_U = W^u_U(T) = \bigcap_{n \geq 0} T^n(U). \]

These sets are called the \( U \)-stable and \( U \)-unstable sets of \( T \), respectively.

From the definitions, it is immediate that

\[ W^s_U(T) = W^s_U(T^{-1}), \quad W^u_U(T) = W^u_U(T^{-1}) \]

\[ T(W^s_U(T)) \subset W^s_U(T), \quad T^{-1}(W^u_U(T)) \subset W^u_U(T). \]
When $U$ is a ball $B_\delta(0)$, we write

$$W^u_U(T) = W^u_\delta(0, T), \quad W^s_U(T) = W^s_\delta(0, T).$$

Let $r \geq 1$ be a real number. A local $C^r$ diffeomorphism at 0 is a pair $(T, U)$ such that $U$ is an open neighborhood of 0 in $E$ and $T$ is a $C^r$ diffeomorphism from $U$ onto its image such that $T(0) = 0$. Note that $T(U)$ is an open set by the Inverse Function Theorem. As usual, we often ignore the domain of the local diffeomorphism, and simply say that $T$ is a local diffeomorphism at 0. When considering compositions and inverses of local $C^r$ diffeomorphisms, we simply shrink domains as needed to make the definitions correct. When $U = E$, we sometimes say that $T$ is a global diffeomorphism to emphasize that we are considering a bijection from $E$ onto $E$.

Two local $C^r$ diffeomorphisms $T$ and $S$ are $C^r$ conjugate if there is a local $C^r$ diffeomorphism $R$ such that $RTR^{-1} = S$ near 0. If $S$ happens to be linear, then, setting $L = DT_0, M = DR_0$, and $P = M^{-1}R$, we have

$$PTP^{-1} = M^{-1}SM, \quad DP(0) = I,$$

and $D(PTP^{-1}) = L$.

Hence,

$T$ is $C^r$ conjugate to some linear map at 0 if and only if it is $C^r$ conjugate to its derivative at 0, and it may be assumed that the conjugacy $R$ has the properties that $R(0) = 0$ and $DR_0 = I$.

To emphasize this concept, we will say that $T$ is strongly $C^r$ conjugate to $S$ if it is $C^r$ conjugate to $S$ and $DS(0) = DT(0)$.

We call the local $C^r$ diffeomorphism $T$ hyperbolic if 0 is a hyperbolic fixed point of $T$.

Let $(T, U)$ be a $C^r$ local hyperbolic diffeomorphism, and let $L = DT(0)$ be its derivative at 0. We assume that $L$ is of saddle type with associated $L$--invariant splitting $E = E^u \oplus E^s$ so that $L|E^u$ is expanding and $L|E^s$ is contracting.

We will say that $T$ is flat in $U$ if

$$T(E^u \cap U) \subset E^u \quad \text{and} \quad T(E^s \cap U) \subset E^s.$$  \hspace{1cm} (65)

We will say that $T$ is locally flat at (or near) 0 if there is some open neighborhood $U$ of 0 such that $T$ is flat in $U$.

It is well-known (see [8], [16]) that, for $\delta > 0$ small, the sets $W^s_\delta(0, T)$ and $W^u_\delta(0, T)$ are $C^r$ embedded submanifolds of $E$ which are tangent at 0 to $E^s$ and $E^u$, respectively.

The following lemma is well-known.

**Lemma 4.8.** Every hyperbolic local $C^r$ diffeomorphism $(T, U)$ is strongly $C^r$ conjugate to a locally flat one.

More precisely, one can find a ball $B_\delta(0) \subset U \cap T(U)$ and a $C^r$ diffeomorphism $R$ from $B_\delta(0)$ onto its image such that $R(0) = 0, DR(0) = I,$ and the map $T_1 = RTR^{-1}$ is flat on $R(T^{-1}B_\delta(0))$.

We call an $R$ as in Lemma 4.8 a flattening map (or diffeomorphism) for $T$.

**Proof.**
We begin by identifying $E$ with $E^u \times E^s$ and take the product representation

\begin{equation}
T(x, y) = (Ax + X(x, y), By + Y(x, y))
\end{equation}

where $A \in \text{Aut}(E^u)$ is expanding, $B \in \text{Aut}(E^s)$ is contracting, and $X : E^u \times E^s \to E^u$ and $Y : E^u \times E^s \to E^s$ are $C^r$ maps which vanish at $(0, 0)$ and have their partial derivatives also vanishing at $(0, 0)$.

Letting $\pi^u : E^u \times E^s \to E^u$ and $\pi^s : E^u \times E^s \to E^s$ be the natural projections, and writing $E^u_\delta = \pi^u(B_\delta(0, 0))$ and $E^s_\delta = \pi^s(B_\delta(0, 0))$, it is proved in [8] and [16] that, for small $\delta$, there are $C^r$ functions $g_u : E^u_\delta \to E^u$ and $g_s : E^s_\delta \to E^s$ such that

\begin{equation}
g_u(0) = (0), \ Dg_u(0) = 0,
\end{equation}

\begin{equation}
g_s(0) = (0), \ Dg_s(0) = 0,
\end{equation}

and $W^u_\delta(0, T)$ and $W^s_\delta(0, T)$ are the graphs of $g_u$ and $g_s$, respectively.

Setting $R(x, y) = (x - g_u(y), y - g_u(x))$, it is readily verified that, for $\delta > 0$ small enough, the map $R$ is $C^r$ diffeomorphism from $B_\delta$ onto its image, and, hence, is a local flattening map for $T$.

**Remark 4.9.** In section 3 we consider continuous dependence of our linearizations on parameters. It will be necessary to have the maps $g_u, g_s$ depend continuously on the parameters as well. The most elegant proof of this result is given in [8] where the Irwin method for proving the existence of invariant manifolds is generalized and simplified.

Our discussion so far shows that, in moving toward the proof of Theorem 1.5, we may assume the $T$ is locally flat after a $C^{1, \alpha}$ coordinate change.

It will be convenient to have a stronger condition which requires another definition.

**Definition 4.10.** Let $U$ be an open neighborhood of 0 in the Banach space $E$, and let $T : U \to V$ be a $C^{1, \alpha}$ diffeomorphism from $U$ onto its image with a hyperbolic fixed point at 0. Let $L = DT(0)$, and let $E = E^u \oplus E^s$ be the hyperbolic splitting associated to $L$.

We say that $T$ is **hyperbolically linear** (or **h-linear**) in $U$ if

\begin{equation}
T(x) = Lx \text{ for } x \in \left[ E^u \cup E^s \right] \cap U.
\end{equation}

Observe that if $T$ is h-linear on $U$ and $U$ is small enough, then it is locally flat in $U$, and $T^{-1}$ is h-linear on $T(U)$. Also, for positive integers $m$, $T^m$ is h-linear on \( \bigcup_{j=0}^m T^{-j} U \), and $T^{-m}$ is h-linear on $\bigcap_{j=0}^m T^j U$.

In general, a $C^r$ local diffeomorphism with a hyperbolic fixed point at 0 will not be conjugate to even a $C^1$ h-linear one.

However, the following lemma shows that a $C^{1, \alpha}$ local diffeomorphism with an $\alpha$—hyperbolic fixed point at 0 is, in fact, strongly $C^{1, \alpha}$ conjugate to an h-linear one.

As in section 3, given an open neighborhood $U$ of 0, consider the space $C^{1, \alpha}_U$ of $C^{1, \alpha}$ maps $f : U \to E$ such that $f(0) = 0$ and $Df(0) = 0$ with finite D-Hölder norm $|f|_U = \text{Hol}(Df, U) < \infty$. 


Lemma 4.11. For $0 < \alpha < 1$, let $E$ be a real Banach space, and let $L \in \text{Aut}(E)$ be an $\alpha-$hyperbolic linear automorphism with hyperbolic splitting $E = E^u \oplus E^s$. Let $U$ be an open neighborhood of 0 in $E$, let $f \in C^{1,\alpha}_U$, and let $T = L + f$.

Then, for any $\epsilon > 0$, there are neighborhoods $V$ and $W$ of 0 in $E$ and a $C^{1,\alpha}$ diffeomorphism $R$ from $V$ onto $W$ such that

\begin{align}
V \cup W & \subset B_\epsilon(0) \subset U \cap T(U) \cap T^{-1}(U), \\
R(0) & = 0, \ DR(0) = I,
\end{align}

and the following conditions are satisfied.

Letting $T_1 = RTR^{-1}$ and $f_1 = T_1 - L$, and defining

$$V_1 = R \left[ T(V) \cap V \cap T^{-1}(V) \right],$$

we have

\begin{align}
T_1 & \text{ is defined and } h\text{-linear on } V_1 \\
\text{Lip}(f_1, V_1) & < \epsilon,
\end{align}

and

\begin{align}
\text{Hol}(Df_1, V_1) & < \infty.
\end{align}

Further, the map $T_1$ is a $C^{1,\alpha}$ diffeomorphism from $V_1$ onto its image $T(V_1) \overset{\text{def}}{=} W_1$, and, setting $g_1 = T_1^{-1} - L^{-1}$, we have

\begin{align}
T_1^{-1} & \text{ is defined and } h\text{-linear on } W_1 \\
\text{Lip}(g_1, W_1) & < \epsilon,
\end{align}

and

\begin{align}
\text{Hol}(Dg_1, W_1) & < \infty.
\end{align}

Proof. Taking $\epsilon > 0$ sufficiently small, we may assume that $T$ is locally flat in the ball $B_\epsilon = B_\epsilon(0)$.

We use the product representation for $T$ and write

$$T(x, y) = (Ax + X(x, y), By + Y(x, y)),$$

with $A$ $\alpha-$expanding, $B$ $\alpha-$contracting, and $X, Y$ $C^{1,\alpha}$ maps vanishing at $(0, 0)$ together with their partial derivatives. In these coordinates, of course,

$$L(x, y) = (Ax, By).$$

It suffices to find a small neighborhoods $V, W$ of $(0, 0)$ contained in $B_\epsilon(0, 0)$ and a local $C^{1,\alpha}$ map $R$ from $V$ onto $W$ such that
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(78) \[ R(0,0) = (0,0), \ DR(0,0) = I, \]
and, setting \( T_1 = RT_1R^{-1} \), we have

(79) \[ T_1(x,0) = (Ax,0) \text{ and } T_1(0,y) = (0,By) \]
for \((x,y) \in W\).

Indeed, it is clear that, for \( V \) and \( W \) small enough, the conditions (73), (74), (76), and (77) will all be satisfied.

Let us proceed to find the small neighborhoods \( V, W \) and the appropriate map \( R \).

Let \( E^u_\epsilon = \pi^u(B_\epsilon) \) and \( E^s_\epsilon = \pi^s(B_\epsilon) \), and consider the maps \( T^u : E^u_\epsilon \to E^u \) and \( T^s : E^s_\epsilon \to E^s \) defined by

\[ T^u(x) = Ax + X(x,0), \quad T^s(y) = By + Y(0,y). \]

Clearly, \( T^u \) has 0 as an \( \alpha \)–expanding fixed point, and \( T^s \) has 0 as an \( \alpha \)–contracting fixed point.

By Theorem 3.1, and the remark following it, there are small neighborhoods \( V^u \) of 0 in \( E^u \), \( V^s \) of 0 in \( E^s \), and \( C^1,\alpha \) diffeomorphisms \( R_1 : V^u \to R_1(V^u) \) and \( R_2 : V^s \to R_2(V^s) \) such that

\[ R_1(0) = 0, \quad DR_1(0) = I, \quad R_2(0) = 0, \quad DR_2(0) = I, \]

and

\[ R_1^{-1}TR_1^{-1} = A \text{ on } R_1(V^u), \]

and

\[ R_2^{-1}TR_2^{-1} = B \text{ on } R_2(V^s). \]

Now, let \( V = V^u \times V^s, \) \( W = R_1(V^u) \times R_2(V^s) \), and let \( R : V \to R(V) \) be the product map

\[ R(x,y) = (R_1(x), R_2(y)). \]

The map \( R \) satisfies (78), and the map \( T_1 = RT_1R^{-1} \) satisfies

\[ T_1(x,0) = R^{-1}TR(x,0) = (Ax,0) \]
and

\[ T_1(0,y) = R^{-1}TR(0,y) = (0,By). \]

This completes the proof of Lemma 4.11. \( \square \)

We will need the analog of Lemma 4.11 for powers of \( T \).

**Lemma 4.12.** Let \( \epsilon > 0, n \) be a positive integer, and let \( R, T_1, f_1, g_1, V_1, W_1 \) be as in the statement of Lemma 4.11.

Then, there are neighborhoods \( V_n \subset V_1 \) and \( W_n \subset W_1 \) such that, \( T_1^n \) is an \( h \)-linear diffeomorphism from \( V_n \) onto \( W_n \), and, defining \( f_n = T_1^n - L^n \) and \( g_n = T_1^{-n} - L^{-n} \), we have

\[ \text{ } \]
(80) \[ \operatorname{Lip}(f_n, V_n) < \epsilon, \]
(81) \[ \operatorname{Hol}(Df_n, V_n) < \infty, \]
(82) \[ \operatorname{Lip}(g_n, W_n) < \epsilon, \]
and
(83) \[ \operatorname{Hol}(Dg_n, W_n) < \infty. \]

Proof. As usual, writing \( B_\delta = B_\delta(0) \) for any positive \( \delta \), first pick \( \delta \in (0, \epsilon) \) so that
\[ B_\delta \subset \bigcap_{k=-n}^n T_k^k(V_1 \cap W_1). \]

Next, choose \( V_n = B_\delta n(0) \) where \( \delta_n \) is small enough so that \( T_j(V_n) \subset B_\delta \) for \( |j| \leq n \).

It is then clear that, setting \( W_n = T_1^n(V_n) \), we have \( V_n \subset V_1, W_n \subset W_1 \), that \( T_n^1 \) is an \( h \)-linear diffeomorphism taking \( V_n \) onto \( W_n \).

The only thing remaining is to get the Lipschitz estimates (80) and (82).

But, an easy induction shows that \( DT_n^1(0) = L_n^1 \), which implies that \( Df_n(0) = 0 = Dg_n(0) \).

Then, it is clear that we can shrink \( \delta_n \) enough so that (80) and (82) hold. In fact, using (43), (48), and (50), one can get explicit estimates of \( \delta_n \) in terms of \( \operatorname{Lip}(f_1) \).

\[ \square \]

4.5. Extension of local \( h \)-linear diffeomorphisms to global diffeomorphisms.

Let \( T_1 \) be the diffeomorphism obtained in Lemma 4.11.

We now wish to use Lemma 4.7 to find an \( h \)-linear \( C^{1, \alpha} \) diffeomorphism \( S \) defined on all of \( E \) which agrees with \( T_1 \) on a small neighborhood of 0 and is linear off a slightly larger ball \( B_\delta \) where \( \delta \) is small relative to \( \operatorname{Hol}(DS, E) \).

Let \( C^{1, \alpha}_0 \) denote the space of \( C^{1, \alpha} \) maps from \( E \) into \( E \) such that \( f(0) = 0 \) and \( Df(0) = 0 \) with the norm
\[ \text{(84)} \operatorname{Hol}(Df) = \sup_{x \neq y} \frac{|Df_x - Df_y|}{|x - y|^{\alpha}} < \infty. \]

Recall that \( \text{Aut}(E) \) is the set of linear automorphisms of \( E \).

Let \( D^{1, \alpha}_0 \) denote the set of maps \( T = L + f \) with \( L \in \text{Aut}(E) \) and \( f \in C^{1, \alpha}_0 \) such that
\[ \text{(85)} \operatorname{Lip}(f, E) \leq \frac{1}{2|L|}. \]

From the Inverse Function Theorem, one sees that the maps \( T \in D^{1, \alpha}_0 \) are bijective \( C^{1, \alpha} \) maps with \( C^{1, \alpha} \) inverses such that \( T(0) = 0 \) and \( DT(0) = L \).
By a $C^{1,\alpha}$ diffeomorphism of $E$, we will mean an element of $D^{1,\alpha}_0$. More general diffeomorphisms can, of course, be defined, but we don’t need them in this paper.

**Definition 4.13.** Given a Banach space $E$, let $T = L + f$ where $L$ is a bounded linear map, and let $f \in C^{1,\alpha}_0(E)$.

We define the **non-linear support of** $T$ to be the set of points $x \in E$ such that $T(x) \neq L(x)$. We denote this by $n\text{supp}(T)$.

The non-linear size of $T$, denoted $N_s(T)$, is defined by

\[ N_s(T) = \inf \{ \xi \in \mathbb{R} : n\text{supp}(T) \subset B(0, \xi) \}. \] (86)

It is clear that $N_s(T) = 0$ if and only if $T = L$ and that, in general, $N_s(T)$ can range from 0 to $\infty$.

We will be interested in $h$-linear $C^{1,\alpha}$ maps $T$ for which $N_s(T)$ is small compared to $\text{Hol}(DT, E)$.

Now, let $C_1(\lambda) < C_2(\lambda)$ be the constants given in Lemma 4.7, and consider the map $T_1 = L + f_1$ defined on the neighborhood $W_1$ obtained in Lemma 4.11.

For positive $\delta > 0$, let

\[ f_2 = \lambda_\delta f_1, \quad T_2 = L + f_2 \] (87)

where $\lambda_\delta$ is the $\delta$-scaled version of the unit bump function $\lambda$ defined in (56).

Let

\[ 0 < \epsilon < \frac{m(L)}{2} = \frac{1}{2|L^{-1}|}. \] (88)

Since $Df_1(0) = 0$, Lemma 4.7 and the Mean Value Theorem say that, for any $\epsilon > 0$, we may choose $\delta > 0$ small enough so that $B_\delta(0) \subset W_1$ and

\[ \text{Lip}(f_2, B_\delta(0)) \leq \delta^\alpha \text{Hol}(Df_2, B_\delta(0)) \leq \delta^\alpha C_2(\lambda) \text{Hol}(Df_1, B_\delta(0)) < \epsilon. \] (89)

From (88) and (89) it is easily seen that $T_2$ is a $C^{1,\alpha}$ global diffeomorphism on $E$.

Let $B_\delta = E \setminus B_\delta$ denote the set-theoretic complement of $B_\delta$ in $E$.

Since $f_1$ vanishes in $B_\delta(0,0) \cap [(E^u \times \{0\}) \cup \{0\} \times E^s]$, and $\lambda_\delta$ vanishes in $B_\delta$, we have that

$f_2$ vanishes on $(E^u \times \{0\}) \cup \{0\} \times E^s \cup B_\delta$.

Thus, $T_2$ is a $C^{1,\alpha}$ $h$-linear global diffeomorphism from $E$ onto $E$ with non-linear support in $B_\delta$.

Next, let $n$ be an integer greater than 1, and consider the powers $T_2^n$ and $T_2^{-n}$.

Let
\[ B^+_n = \bigcap_{k=0}^{n} T^{-k}_2 B^c, \]

and

\[ B^-_n = \bigcap_{k=0}^{n} T^k_2 B^c, \]

In view of (52), we see that for each \( 1 \leq j \leq n - 1 \), and \( x \in B^+_n \), if \( f_j(x) = 0 \), then \( f_{j+1}(x) = 0 \) as well.

In particular,

\[ f_n(x) = 0 \text{ for } x \in B^+_n. \]

A similar argument shows that

\[ g_n(x) = 0 \text{ for } x \in B^-_n, \]

which yields

\[ n\text{supp}(T^n_2) \subset (B^+_n)^c = \bigcup_{k=0}^{n} T^{-k}_2 B^c, \]

and

\[ n\text{supp}(T^{-n}_2) \subset (B^-_n)^c = \bigcup_{k=0}^{n} T^k_2 B^c. \]

Now, defining \( \delta_n \) by

\[ \delta_n = \delta \max(Lip(T^n_2), Lip(T^{-n}_2)), \]

we see that (92) and (93) give

\[ n\text{supp}(T^n_2) \cup n\text{supp}(T^{-n}_2) \subset B_{\delta_n}. \]

Also, obviously, for a fixed positive integer \( n \), we can choose the bump function scaling constant \( \delta \) small enough so that the non-linear sizes of \( T^n_2 \) and \( T^{-n}_2 \) are arbitrarily small.

With the above definitions, let us summarize the results obtained above in the following proposition. For notational convenience, we set \( S = T_2 \).

**Proposition 4.14.** Let \( 0 < \alpha < 1 \), let \( E \) be a \( C^{1,\alpha} \) Banach space, and let \( L \) be an \( \alpha \)-hyperbolic linear automorphism of saddle type. Let \( U \) be an open neighborhood of 0, let \( f : U \to E \) be a function in \( C^{1,\alpha}_U \), and let \( T = L + f \).

Then, for any \( \epsilon > 0 \) and any non-zero integer \( n \), there are a \( \delta_n = \delta_n(L, f, \epsilon) < \epsilon \) and a global \( h \)-linear \( C^{1,\alpha} \) diffeomorphism \( S \in D^{1,\alpha}_0 \) which is strongly \( C^{1,\alpha} \) conjugate to \( T \) on \( B_{\delta_n} \), and setting \( f_n = S^n - L^n \), we have

\[ Lip(f_n, E) < \epsilon \]
and

\[(97) \quad \text{nsupp}(S^n) \subset B_{\delta_n}.\]

We will need the following simple consequence of Proposition 4.14.

**Lemma 4.15.** Let \(S, n, \delta_n, f_n\) be as in Proposition 4.14, and, using the product representation \(E = E^u \times E^s\), write \(f_n(x, y) = (X_n(x, y), Y_n(x, y))\).

Then, for every \((x, y) \in E^u \times E^s\) we have

\[(98) \quad \max(|X_{n,x}(x, y)|, |Y_{n,x}(x, y)|) \leq \text{Hol}(Df_n) \min(\delta_n, |x|^\alpha)\]

and

\[(99) \quad \max(|X_{n,y}(x, y)|, |Y_{n,y}(x, y)|) \leq \text{Hol}(Df_n) \min(\delta_n, |x|^\alpha).\]

**Proof.** We will only give the arguments for \(X_n\) since those needed for \(Y_n\) are similar.

Because we are using the maximum norm on \(E^u \times E^s\), we have

\[\text{Hol}(Df_n) = \max(\text{Hol}(DX_n), \text{Hol}(DY_n)).\]

Since \(X_n(x, 0) = 0\) for each \((x, 0)\) we also have that \(X_{n,x}(x, 0) = 0\) for each \((x, 0)\). Similarly, \(X_n(0, y) = 0\) implies that \(X_{n,y}(0, y) = 0\) for each \((y, 0)\).

Hence,

\[(100) \quad |X_{n,x}(x, y)| = |X_{n,x}(x, y) - X_{n,x}(0, 0)| \leq \text{Hol}(DX_n) |(x, y)|^\alpha,\]

\[(101) \quad |X_{n,y}(x, y)| = |X_{n,y}(x, y) - X_{n,y}(0, 0)| \leq \text{Hol}(DX_n) |(x, y)|^\alpha,\]

\[(102) \quad |X_{n,x}(x, y)| = |X_{n,x}(x, y) - X_{n,x}(x, 0)| \leq \text{Hol}(DX_n) |y|^\alpha,\]

and

\[(103) \quad |X_{n,y}(x, y)| = |X_{n,y}(x, y) - X_{n,y}(0, y)| \leq \text{Hol}(DX_n) |x|^\alpha.\]

The non-linear support condition \((97)\) for \(X_n\) implies that \(X_n(x, y)\) and its partial derivatives vanish at points \((x, y)\) such that \(|(x, y)| > \delta_n\). So, \((100)\) and \((101)\) imply

\[(104) \quad \max(|X_{n,x}(x, y)|, |X_{n,y}(x, y)|) \leq \text{Hol}(DX_n) \delta_n^\alpha\]

This, together with \((102)\), \((103)\) and the fact that \(\text{Hol}(DX_n) \leq \text{Hol}(Df_n)\), gives formulas \((98)\) and \((99)\). \(\square\)
4.6. **Norm Conditions induced by $\alpha$-hyperbolicity.** Assuming $L$ is $\alpha-$hyperbolic, and using the product representation $E = E^u \times E^s$, the map $L$ has the form

$$L(x, y) = (Ax, By).$$

with

$$\max(\rho(A^{-1}), \rho(B)) < 1.\quad (105)$$

By Lemma 4.4, we may assume the maps $A$ and $B$ satisfy

$$|A^{-1}| < 1 \text{ and } |B| < 1.\quad (106)$$

Next, using (6) and formula (3), we choose a positive integer $m$ such that

$$(|A^{-m}||A^m||B^m|^\alpha)^{\frac{1}{m}} < 1,$$

and

$$(|B^{-m}||B^m||A^{-m}|^\alpha)^{\frac{1}{m}} < 1.$$

Taking the $m$-th power, these imply that

$$(107)$$

$$|A^{-m}||A^m||B^m|^\alpha < 1$$

and

$$(108)$$

$$|B^{-m}||B^m||A^{-m}|^\alpha < 1.$$

Following this, we choose $\epsilon > 0$ small enough so that

$$\min(m(A^m) - \epsilon, m(B^{-m}) - \epsilon) > 1,\quad (109)$$

$$\max(|A^{-1}| + \epsilon, |B| + \epsilon) < 1,\quad (110)$$

$$|A^{-m}||A^m| + \epsilon) (|B^m| + \epsilon)^\alpha < 1,\quad (111)$$

and

$$|B^m||B^{-m}| + \epsilon)(|A^{-m}| + \epsilon)^\alpha < 1.\quad (112)$$

The estimates (109) and (111), of course imply that

$$m(A) - \epsilon > 1,\quad (113)$$

and

$$|B^m| + \epsilon < 1.\quad (114)$$

Finally, we choose $0 < \eta < 1$ close enough to 1 so that
Let $E, T, L$ be as in the hypotheses of Theorem 1.5 and let $E = E^u \oplus E^s$ be the hyperbolic splitting associated to $L$.

Applying Proposition 4.14 for $n = 1$ and arbitrary $\epsilon > 0$, there are a $\delta = \delta_1 \in (0, \epsilon)$ and an $h$-linear map $S \in D^{1, \alpha}_1$ of the form $S = L + f_1$ with $f_1 \in C^{1, \alpha}_0$ such that $S$ is strongly $C^{1, \alpha}$ conjugate to $T$ on $B_\delta$ with $\text{nsupp}(S) \subset B_\delta$.

In subsection 5.1 below, we will show that for $\epsilon$ satisfying conditions (109)-(112) and $\delta > 0$ sufficiently small, the map $S$ is globally strongly $C^{1, \alpha}$ conjugate to $L$. That is, there is a $C^{1, \beta}$ diffeomorphism $R$ mapping $E$ onto itself such that $R(0) = 0$, $DR_0 = I$, and $RSR^{-1} = L$. Then, in subsection 5.2 we will show that, adding conditions (115) and (116) and shrinking $\delta$ further, the map $R$ is $C^{1, \beta}$ on bounded subsets of the orbit of $B_\delta(0)$.

Once these things are done, the restriction to a small neighborhood of 0 of the composition of $R$ with the local $C^{1, \alpha}$ conjugacy from $S$ to $T$ provides a local $C^{1, \beta}$ conjugacy from $T$ to $L$ and completes the proof of Theorem 1.5.

5.1. **Global $C^1$ linearization of the map $S$.** To $C^1$ linearize $S = L + f_1$ on $E$, we will employ the method described in section 2. The map $R$ will have the form $R = I + \phi$, where

\[ \phi = (I - H)^{-1}L^{-1}f_1 \]

where $H$ is an automorphism on a suitable Banach space of functions $\mathcal{E} = \mathcal{E}^s \oplus \mathcal{E}^u$ which we now define.

We identify $E$ with the space $E^u \times E^s$ as above.

Given a Banach space $Z$ and a $C^1$ function $\psi : E^u \times E^s \to Z$, consider the following two real-valued functions.

\[ \gamma_1(\psi, x, y, \alpha) = \frac{|\psi_x(x, y)|}{|y|^{\alpha}} \]
\[ \gamma_2(\psi, x, y, \alpha) = \frac{|\psi_y(x, y)|}{|x|^{\alpha}} \]

Each of the preceding functions is defined to have value zero if its denominator vanishes. Otherwise, it is given by the indicated expression.

Consider the corresponding suprema:

\[ |\gamma_i(\psi, \alpha)| = \sup_{x \neq 0, y \neq 0} \gamma_i(\psi, x, y, \alpha) \text{ for } i = 1, 2, \]

and let $C^1_0(E, Z)$ be the set of $C^1$ functions from $\psi : E^u \times E^s$ to $Z$ such that
\( \psi(x, 0) = \psi(0, y) = 0 \) \( \forall (x, y) \),

(121) \( \psi_x(0, 0) = \psi_y(0, 0) = 0 \),

and

(122) \[ | \gamma_i(\psi) | = | \gamma_i(\psi, \alpha) | < \infty \text{ for } i = 1, 2. \]

For such \( \psi \), set

(123) \[ || \psi || = || \psi ||_\alpha = \max_{i=1,2} | \gamma_i(\psi, \alpha) |. \]

One can check that the set \( C^0_1(E, Z) \) is a linear space with the usual point-wise operations of addition and scalar multiplication, and that the function \( || \cdot || \) provides a norm on it so that the pair \( (C^0_1(E, Z), || \cdot ||) \) becomes a Banach space.

Setting \( Z \) to be \( E^u \) and \( E^s \), in turn, gives the two Banach spaces \( E^s = C^0_1(E, E^u) \) and \( E^u = C^0_1(E, E^s) \). For \( \phi = (\phi_s, \phi_u) \in \mathcal{E} = E^s \times E^u \), we define the norm

\[ || \phi || = || (\phi_s, \phi_u) || = \max(|| \phi_s ||, || \phi_u ||). \]

For \( \phi = (\phi_s, \phi_u) \in E^s \times E^u \), the map \( H(\phi)(x, y) = L^{-1}\phi(S(x, y)) \) is expressed as

\[ H(\phi)(x, y) = (H_s(\phi_s)(x, y), H_u(\phi_u)(x, y)) \]

where

(124) \[ H_s(\phi_s)(x, y) = A^{-1}\phi_s(Ax + X_1(x, y), By + Y_1(x, y)) \]

and

(125) \[ H_u(\phi_u)(x, y) = B^{-1}\phi_u(Ax + X_1(x, y), By + Y_1(x, y)). \]

Since \( S \) is \( \lambda \)-linear on \( E \) and the composition of \( C^{1,\alpha} \) maps is again \( C^{1,\alpha} \), we see that, at least as an operator on the function space \( E^s \times E^u \) without consideration of norms, \( H \) is represented as the direct sum of operators \( H = H_s \oplus H_u \).

**Proposition 5.1.** Let \( E, T, L, f \) be as in the statement of Theorem 1.5, and assume that \( L(x, y) = (Ax, By) \) and \( m > 0, \epsilon \) are such that (109)-(114) are valid. There is a \( \delta \in (0, \epsilon) \) such that if \( S \) as in Proposition 4.14, then, the associated maps \( H_s \) and \( H_u \) satisfy

(126) \[ H_s \in \text{Aut}(E^s) \text{ and } | H_s^m | < 1 \]

and

(127) \[ H_u \in \text{Aut}(E^u) \text{ and } | H_u^{-m} | < 1. \]

Hence, \( \rho(H_s) < 1 \) and \( \rho(H_u^{-1}) < 1 \), so \( H \) is hyperbolic.
Assuming Proposition [5.1] we use Lemma [3.4] to find norms \( | \cdot |_u \) and \( | \cdot |_s \) on \( E^u \) and \( E^s \), respectively so that

\[
\left| H_s \right| < 1
\]

and

\[
\left| H_u^{-1} \right| < 1.
\]

Letting \( I_s : E^s \to E^s \), \( I_u : E^u \to E^u \) denote the identity maps on \( E^s \), \( E^u \), respectively, we then have

\[
(I_s - H_s)^{-1} = \sum_{n \geq 0} H_s^n
\]

and

\[
(I_u - H_u)^{-1} = H_u^{-1} H_u (I_u - H_u)^{-1}
\]

\[
= H_u^{-1} (H_u^{-1})^{-1} (I_u - H_u)^{-1}
\]

\[
= H_u^{-1} ((I_u - H_u) H_u^{-1})^{-1}
\]

\[
= H_u^{-1} (H_u^{-1} - I_u)^{-1}
\]

\[
= -H_u^{-1} (I_u - H_u^{-1})
\]

\[
= -H_u^{-1} \sum_{n \geq 0} H_u^{-n}
\]

\[
= -\sum_{n \geq 1} H_u^{-n}
\]

giving

\[
(I - H)^{-1} = (I_s - H_s)^{-1} \oplus (I_u - H_u)^{-1} = \left( \sum_{n \geq 0} H_s^n, -\sum_{n \geq 1} H_u^{-n} \right).
\]

Then, formula (12) in Section 2 shows that a (global) \( C^1 \) linearization of \( S \) is given by \( R = I + \psi \) where

\[
\psi = (I - H)^{-1} L^{-1} f = (\psi_s, \psi_u)
\]

with

\[
\psi_s = (I_s - H_s)^{-1} A^{-1} X_1 = \sum_{n \geq 0} H_s^n (A^{-1} X_1)
\]

and

\[
\psi_u = (I_u - H_u)^{-1} B^{-1} Y_1 = -\sum_{n \geq 1} H_u^{-n} (B^{-1} Y_1).
\]

Let us proceed to the proof of Proposition [5.1].

We first show that the statement (126) is implied by assumptions (110), (111), and (113).
Following this and using the expression

\[(134) \quad H_n^{-m}(\phi_u) = B^m \phi_u(A^{-m}x + X_{-m}(x, y), B^{-m}y + Y_{-m}(x, y)),\]

it is easy to see that the same method works for (127) by replacing \(S, A, B\) by \(S^{-1}, B^{-1}, A^{-1}\), respectively, using the assumptions (110), (112), and (113).

Thus, it suffices to prove (126).

In the sequel, all real numbers \(\delta_n\) will be assumed to be in \((0, \epsilon)\) where \(\epsilon\) satisfies (106)-(112). Also, once a condition is specified by a choice of \(\delta_n\), it will continue to hold for smaller \(\delta_n\), so this will be assumed without further mention.

Moving to (126), for a positive integer \(n\), consider the following real numbers

\[
\begin{align*}
K_{n,1} &= |A^{-n}|(|A^n| + \epsilon)(|B^n| + \epsilon)^\alpha, \\
K_{n,2} &= |A^{-n}|(|A^n| + \epsilon)^\alpha \delta_n \text{Hol}(Df_n), \\
K_{n,3} &= |A^{-n}|(|B^n| + \epsilon)^\alpha \delta_n \text{Hol}(Df_n), \\
K_{n,4} &= |A^{-n}|(|A^n| + \epsilon)(|B^n| + \epsilon),
\end{align*}
\]

and

\[
\tau_n = \max(K_{n,1} + K_{n,2}, K_{n,3} + K_{n,4}).
\]

By (110), we see that \(|B^n| + \epsilon < 1\), so \(K_{n,4} < K_{n,1}\).

Letting \(n = m\) with \(m\) as in (111) and (112), we have

\[
K_{m,4} < K_{m,1} < 1.
\]

So, we may choose \(\delta_m\) small enough so that \(\tau_m < 1\).

Further, for \(n = 1\) or \(n = m\), and \(\delta_n\) small enough, we can guarantee that

\[
\text{(135)} \quad \max(\text{Lip}(X_1), \text{Lip}(Y_1)) < \epsilon,
\]

\[
\text{(136)} \quad \max(\text{Lip}(X_m), \text{Lip}(Y_m)) < \epsilon,
\]

and

\[
\text{(137)} \quad \max(N_s(X_m), N_s(Y_m)) < \delta_m \text{Hol}(Df_m).
\]

In addition, since we are using the \(\delta\)-scaled version of the bump function \(\lambda\) for \(S\), we have that the functions \(X_1, Y_1, X_m, Y_m\) all vanish on

\[
\left( E^n \times \{0\} \right) \bigcup \{0\} \times E^n \bigcup B_\delta^c.
\]

Now, the main step in the proof of (126) is
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Proposition 5.2. Assume that conditions (109)-(112) on $\epsilon$ are satisfied. Then, for any $\psi \in \mathcal{E}^s$, and $n = 1$ or $n = m$, we have

\begin{align*}
\gamma_1(H_s^n(\psi)) &\leq (K_{n,1} + K_{n,2}) \| \psi \| \\
\gamma_2(H_s^n(\psi)) &\leq (K_{n,3} + K_{n,4}) \| \psi \|.
\end{align*}

Since $\| \psi \| = \max(\gamma_1(\psi), \gamma_2(\psi))$, conditions (139) and (140) imply that, for $n = 1$ or $n = m$, we have

$$\| H_s^n(\psi) \| < \tau_n \| \psi \|.$$

Thus, for $n = 1$ and $n = m$, the maps $H_s^n$ are bounded linear maps and $| H_s^m | < 1$. They are also bijective with inverses

$$H_s^{-n}(\psi) = A^n \psi(S^{-n}).$$

By the open mapping theorem, the maps $H_s^n$ are automorphisms (i.e., have bounded inverses) and this proves (129).

Let us proceed to the proof of Proposition 5.2, i.e., to the proofs of (139) and (140).

Define the functions $u_n(x, y), v_n(x, y)$ by

\begin{align*}
u_n &= u_n(x, y) = A^n x + X_n(x, y), \\
v_n &= v_n(x, y) = B^n y + Y_n(x, y).
\end{align*}

Lemma 5.3. For every $(x, y) \in E^n \times E^s$, we have

\begin{align*}
| u_n(x, y) | &\leq (| A^n | + \epsilon) \| x \|, \\
| v_n(x, y) | &\leq (| B^n | + \epsilon) \| y \|, \\
| u_{n,x} | &\leq | A^n | + \epsilon \\
| u_{n,y} | &\leq Hol(Df_n) \min(\delta_n, | x |^\alpha), \\
| v_{n,x} | &\leq Hol(Df_n) \min(\delta_n, | y |^\alpha), \\
| v_{n,y} | &\leq B^n | + \epsilon.
\end{align*}
Proof. The inequalities (145) and (148) follow from (136), and inequalities (146) and (147) are a restatement of (98) and (99).

From (136), the Mean Value Theorem, and the vanishing of \( X_n, Y_n \) on

\[ E^n \times \{0\} \cup \{0\} \times E^n, \]

we get

\[
|u_n(x, y)| = |A^n x + X_n(x, y)| \\
\leq |A^n||x| + |X_n(x, y) - X_n(0, y)| \\
\leq |A^n||x| + \left( \sup_{0<s<1} |X_{n,x}(sx, y)| \right)|x| \\
\leq |A^n||x| + \epsilon |x|,
\]

and

\[
|v_n(x, y)| = |B^n y + Y_n(x, y)| \\
\leq |B^n||y| + |Y_n(x, y) - Y_n(x, 0)| \\
\leq |B^n||y| + \left( \sup_{0<s<1} |Y_{n,y}(x, sy)| \right)|y| \\
\leq |B^n||y| + \epsilon |y|.
\]

Proof of (139).

Since \(|v_{n,x}| = |Y_{n,x}(x, y)| = 0\) if \(|x, y| > \delta_n\), we have that \(|v_{n,x}| \neq 0\) implies that \(|x| < \delta_n^\alpha\).

With \(K_{n,1}\) and \(K_{n,2}\) defined above, we have

\[
|A^{-n}||v_{n/\alpha}|u_{n,x}| \leq |A^{-n}|(|B^n| + \epsilon)^\alpha|y|^\alpha|u_{n,x}| \\
\leq |A^{-n}|(|B^n| + \epsilon)^\alpha|y|^\alpha(|A^n| + \epsilon) \\
\leq K_{n,1}|y|^\alpha
\]

and

\[
|A^{-n}||u_{n/\alpha}|v_{n,x}| \leq |A^{-n}|(|A^n| + \epsilon)^\alpha|x|^\alpha|v_{n,x}| \\
\leq |A^{-n}|(|A^n| + \epsilon)^\alpha \delta_n^\alpha Hol(Df_n)|y|^\alpha \\
\leq K_{n,2}|y|^\alpha.
\]

For notational convenience, let us write \(u = u_n, v = v_n\).

Then, we have
\[ | \partial_x H^n_s(\psi)(x, y) | = | A^{-n} \partial_x \psi(u, v) | \]
\[ \leq | A^{-n} || \psi_u(u, v) u_x + \psi_v(u, v) v_x | \]
\[ \leq | A^{-n} | \left[ \left| \frac{\psi_u(u, v)}{v} \right| | v|^{\alpha} u_x + \left| \frac{\psi_v(u, v)}{u} \right| | u|^{\alpha} v_x \right] \]
\[ \leq | A^{-n} | | \gamma_1(\psi) | v|^{\alpha} u_x | + | A^{-n} | | \gamma_2(\psi) | u|^{\alpha} v_x | \]
\[ \leq K_{n,1} \gamma_1(\psi) y|^{\alpha} + K_{n,2} \gamma_2(\psi) y|^{\alpha} \]
\[ \leq (K_{n,1} + K_{n,2}) || \psi || | y |^{\alpha} \]

Dividing by $| y |^{\alpha}$ and taking the supremum over $x, y$ gives

\[ \gamma_1(H^n_s(\psi)) < (K_{n,1} + K_{n,2}) || \psi ||, \]

and proves (139).

**Proof of (140):**

As above, $| X_{n,y} | = 0$ for $| (x, y) | > \delta_n$.

Thus,

\[ | A^{-n} || v|^{\alpha} u_y | \leq | A^{-n} | (| B^n | + \epsilon)| y|^{\alpha} X_{n,y} | \]
\[ \leq | A^{-n} | (| B^n | + \epsilon) \delta_n^{\alpha} X_{n,y} | \]
\[ \leq | A^{-n} | (| B^n | + \epsilon) \delta_n^{\alpha} H o l(Df_n) x |^{\alpha} \]
\[ = K_{n,3} x |^{\alpha} \]

and, using $(| A^n | + \epsilon) \leq | A^n | + \epsilon$, we have

\[ | A^{-n} | | u|^{\alpha} v_y | \leq | A^{-n} | (| A^n | + \epsilon)| x|^{\alpha} v_y | \]
\[ \leq | A^{-n} | (| A^n | + \epsilon)| x|^{\alpha} (| B^n | + \epsilon) \]
\[ = K_{n,4} x |^{\alpha} \]

Hence,

\[ | \partial_y H^n_s(\psi)(x, y) | = | A^{-n} \partial_y \psi(u, v) | \]
\[ \leq | A^{-n} || \psi_u(u, v) u_y + \psi_v(u, v) v_y | \]
\[ \leq | A^{-n} | \left[ \left| \frac{\psi_u(u, v)}{v} \right| | v|^{\alpha} u_y + \left| \frac{\psi_v(u, v)}{u} \right| | u|^{\alpha} v_y \right] \]
\[ \leq K_{n,3} \gamma_1(\psi) y|^{\alpha} + \gamma_2(\psi) K_{n,4} x |^{\alpha} \]
\[ \leq (K_{n,3} + K_{n,4}) || \psi || | x |^{\alpha} \]

Dividing by $| x |^{\alpha}$ and taking the supremum over $x, y$ gives

\[ \gamma_2(H^n_s(\psi)) \leq (K_{n,3} + K_{n,4}) || \psi ||, \]

and proves (140).

This completes the proof of Proposition 5.1 and gives the desired global $C^1$ linearization $R$ of the diffeomorphism $S$. 
5.2. D-Hölder continuity of $R$ near 0. For a positive real number $\delta > 0$ small, let

\[ O_+(\delta) = \bigcup_{n \geq 0} S^n(B_\delta) \]

and

\[ O_- (\delta) = \bigcup_{n \leq 0} S^n(B_\delta) \]

be the positive and negative $S$–orbits of $B_\delta(0)$, respectively, and let

\[ O(\delta) = \bigcup_{n \in \mathbb{Z}} S^n(B_\delta) \]

be the full $S$–orbit of $B_\delta(0)$.

The main result in this subsection is the following proposition.

**Proposition 5.4.** Let $0 < \eta < 1$ be as in (115) and (116), and let $\beta = \alpha(1 - \eta)$. Then, for $\delta$ sufficiently small,

\[ \psi_s \text{ is } C^{1,\beta} \text{ on } O_+(\delta), \]

and

\[ \psi_u \text{ is } C^{1,\beta} \text{ on } O_-(\delta). \]

Hence, the map $\psi = (\psi_s, \psi_u)$ is $C^{1,\beta}$ on $B_\delta(0) = O_+(\delta) \cap O_-(\delta)$.

Assuming Proposition 5.4, the proof of Theorem 1.5 is completed by simply restricting the map $R = (I_s + \psi_s, I_u + \psi_u)$ to the ball $B_\delta(0)$.

Recalling that finite compositions of D-Hölder maps are again D-Hölder, Proposition 5.4 implies that the map $R = (I_s + \psi_s, I_u + \psi_u)$ (which satisfies $R = L^{-n} R S^n$) is $C^{1,\beta}$ on $S^{-n}(B_\delta(0))$ for any finite integer $n$.

But, any bounded subset of $O(\delta)$ is contained in the set

\[ \bigcup_{|n| \leq N} S^n(B_\delta) \]

for some positive integer $N$, and we obtain

**Proposition 5.5.** For $\delta > 0$ small enough, the map $R$ is $C^{1,\beta}$ on bounded subsets of $O(\delta)$.

To prove Proposition 5.5, we first show that (152) holds.

Then, as in the proof of (127), we obtain (153) with the same method by replacing $S, A, B$ with $S^{-1}, B^{-1}, A^{-1}$, respectively.

Let us use $\psi | Z$ to denote the restriction of the function $\psi$ to a set $Z$.

Our proof of (152) implied that the series

\[ \psi_s = \sum_{n \geq 0} H^n_s(A^{-1}X_1) \]
converges uniformly in the $C^1$ topology, but, of course, it gives no information about D-Hölder convergence. However, observe that the restriction $A^{-1}X_1 \mid O_+(\delta)$ is $C^{1,\alpha}$ on $O_+(\delta)$ since it vanishes off $B_\delta(0)$. Moreover,

$$\psi_s \mid O_+(\delta) = \sum_{n \geq 0} H^n_s(A^{-1}X_1 \mid O_+(\delta)).$$

Of course, we also have that $A^{-1}X_1 \mid O_+(\delta)$ is $C^{1,\beta}$ on $O_+(\delta)$ since $0 < \beta < \alpha$.

We proceed to introduce a Banach space $\mathcal{E}^+ = \mathcal{E}^+ (\delta)$ of $C^{1,\beta}$ functions on $O_+(\delta)$ with the property that if $\delta > 0$ is sufficiently small, then function $H_s(\psi) = A^{-1}\psi \circ S$ is a well-defined bounded linear self-map of $\mathcal{E}^+$ satisfying (166) below.

It will follow from this that the series

$$\sum_{n \geq 0} H^n_s(A^{-1}X_1 \mid O_+(\delta))$$

converges uniformly in $\mathcal{E}^+$.

This will show that $\psi_s \mid O_+(\delta)$ is $C^{1,\beta}$, thus proving (152).

Let us proceed to define the space $\mathcal{E}^+$.

For $(x, y), (h, k) \in E^u \times E^s$, let

(154) \hspace{1cm} M^*_{x,h} = \max(|x|, |h|),

(155) \hspace{1cm} N_{y,k} = \max(|y|, |k|),

(156) \hspace{1cm} M_{x,h} = \min(1, M^*_{x,y}),

and, let

(157) \hspace{1cm} U(x, h, k) = M^*_x |(h, k)|^{\alpha(1-\eta)}

(158) \hspace{1cm} V(y, h, k) = N^*_y |(h, k)|^{\alpha(1-\eta)}.

Given a $C^1$ function $\psi : O_+(\delta) \to E^u$, and points $(x, y) \in O_+(\delta), (h, k) \in E^u \times E^s$, with $(x + h, y + k) \in O_+(\delta)$ and $|(h, k)| > 0$, consider the real-valued functions

(159) \hspace{1cm} \gamma_3(\psi, x, y, h, k) = \frac{|\psi(x + h, y + k) - \psi(x, y)|}{V(y, h, k)}

and

(160) \hspace{1cm} \gamma_4(\psi, x, y, h, k) = \frac{|\psi(x + h, y + k) - \psi(x, y)|}{U(x, h, k)}

and their suprema
\begin{align}
|\gamma_3(\psi)| &= \sup_{\|\|_O+(\delta)} \gamma_3(\psi, x, y, h, k), \\
\text{and} \\
|\gamma_4(\psi)| &= \sup_{\|\|_O+(\delta)} \gamma_4(\psi, x, y, h, k).
\end{align}

As in the case of the functions \(\gamma_1, \gamma_2\), we define the functions in \(\gamma_3\) and \(\gamma_4\) to have the value 0 if the denominators vanish.

We define \(E^+_u\) to be the set of \(C^1\) functions \(\psi: O_+(\delta) \to E^u\) such that

\begin{align}
\psi(x, y) &= 0 \text{ for } (x, y) \in O_+(\delta) \cap \left( E^u \times \{0\} \right) \cup \left( E^s \cup B_\delta^c \right), \\
\text{and} \\
\sup(\gamma_3(\psi), \gamma_4(\psi)) &< \infty.
\end{align}

Defining the norm of \(\psi \in E^+_u\) to be

\begin{align}
||\psi|| &= \max_{i=3,4} ||\gamma_i(\psi)||,
\end{align}

we have that the pair \((E^+_u, ||\cdot||)\) becomes a Banach space.

For \(\psi \in E^+_u\), we use the same formula as in (124):

\[ H_s(\psi) = A^{-1}\psi \circ S. \]

The proof of (152) is obtained from

**Lemma 5.6.** Assume that conditions (109)-(114) are satisfied.

Then, for \(\delta > 0\) small enough, the map \(H_s\) is a well defined bounded linear map from \(E^+_u\) into itself such that

\begin{align}
&|H_s^m| < 1.
\end{align}

Note that, since \(S = L\) off \(B_\delta(0)\) and \(S\) is Lipschitz close to \(L\), we have

\[ S(O_+(\delta) \setminus B_\delta(0)) \subset O_+(\delta) \setminus B_\delta(0) \]

for \(\delta\) small enough.

This, and the linearity of \(S\) on \(E^u \cup \{0\} \cup E^s\), imply that, if \(\psi\) satisfies (163), then \(\psi \circ S\) also satisfies (163). Hence, the \(C^1\) function \(H_s(\psi)\) is well-defined as a map in the function space \(E^+_u\).

We need to show that \(|H_s| < \infty\) and that (166) holds.

These will follow from inequality (207) below applied in the cases \(n = 1\) and \(n = m\), respectively.

Proceeding toward this goal, let \(n \neq 0\) be a non-zero integer, let \(\delta_n, f_n\) be as in Proposition (1.13) and write \(f_n = (X_n, Y_n)\) as in Lemma 4.15.

We will need the following functions.
Define

\begin{align}
\Delta X_n &= \Delta X_n(x, y, h, k) = X_n(x + h, y + k) - X_n(x, y), \\
\Delta Y_n &= \Delta Y_n(x, y, h, k) = Y_n(x + h, y + k) - Y_n(x, y), \\
\Delta X_{n,x} &= \Delta X_{n,x}(x, y, h, k) = X_{n,x}(x + h, y + k) - X_{n,x}(x, y), \\
\Delta X_{n,y} &= \Delta X_{n,y}(x, y, h, k) = X_{n,y}(x + h, y + k) - X_{n,y}(x, y), \\
\Delta Y_{n,x} &= \Delta Y_{n,x}(x, y, h, k) = Y_{n,x}(x + h, y + k) - Y_{n,x}(x, y), \\
\Delta Y_{n,y} &= \Delta Y_{n,y}(x, y, h, k) = Y_{n,y}(x + h, y + k) - Y_{n,y}(x, y),
\end{align}

**Lemma 5.7.** Let $0 < \alpha < 1$, $0 < \eta < 1$, $n \in \{1, m\}$, and let $\epsilon$ satisfy conditions (102), (110). Let $X_n, Y_n$ be as in Proposition (4.15) and let $\delta = \delta_n \in (0, \min(\epsilon, 1/3))$ be small enough that

\begin{equation}
\text{Hol}(Df_n)\delta^\alpha < \frac{\epsilon}{3}.
\end{equation}

Then, for $(x, y) \in O_+(\delta)$, $(x + h, y + k) \in O_+(\delta)$, and $| (x, y) | < \delta$, we have

\begin{align}
\max(| \Delta X_n |, | \Delta Y_n |) &\leq \epsilon \min(| (h, k) |, M_{x,h}, N_{y,k}), \\
\max(| \Delta X_{n,x} |, | \Delta Y_{n,x} |) &\leq 3\text{Hol}(Df_n)\min(| (h, k) |^\alpha, V(y, h, k)) \\
\max(| \Delta X_{n,y} |, | \Delta Y_{n,y} |) &\leq 3\text{Hol}(Df_n)\min(| (h, k) |^\alpha, U(x, h, k))
\end{align}

**Proof.** We only give the arguments for $\Delta X_n$, $\Delta X_{n,x}$ and $\Delta X_{n,y}$ leaving the similar arguments for $\Delta Y_n$, $\Delta Y_{n,x}$, $\Delta Y_{n,y}$ to the reader.

For notational convenience, let $\epsilon_1 = \frac{\epsilon}{3}$.

Since $X_n(x, 0) = 0$ for each $(x, 0)$ we also have that $X_{n,x}(x, 0) = 0$ for each $(x, 0)$. Similarly, $X_n(0, y) = 0$ implies that $X_{n,y}(0, y) = 0$ for each $(0, y)$.

Letting $n = 1$ or $n = m$, (173) and the vanishing of $X_n$ off $B_\delta(0)$ imply that

\begin{equation}
\text{Lip}(X_n) \leq \text{Hol}(Df_n)\delta^\alpha < \epsilon_1.
\end{equation}

From this, the Mean Value Theorem gives

\begin{align}
| \Delta X_n | &= | X_n(x + h, y + k) - X_n(x, y) | \\
&\leq | X_n(x + h, y + k) - X_n(x, y + k) | + | X_n(x, y + k) - X_n(x, y) | \\
&\leq \sup_{0 \leq t \leq 1} | X_n, x(x + th, y + k) || h | + \sup_{0 \leq t \leq 1} | X_n, y(x, y + tk) || k | \\
&\leq 2\epsilon_1 | (h, k) |.
\end{align}
In addition, using the Mean Value Theorem again and the fact that $X_n(0, y) = X_n(x, 0) = 0$ for all $x, y$, we have

$$| \Delta X_n | = | X_n(x + h, y + k) - X_n(x, y) |$$

$$= | X_n(x + h, y + k) - X_n(x + h, y) | + | X_n(x + h, y) - X_n(x, y) |$$

$$\leq \left( \sup_{0 \leq t \leq 1} | X_n(x + th, y + k) | h \right) + | X_n(x + k) - X_n(0, y + k) | + | X_n(x, y) - X_n(0, y) |$$

$$\leq \epsilon_1 | h | + \epsilon_1 | \sup_{0 \leq t \leq 1} X_n(x + th, y + k) | x | + \epsilon_1 | x |$$

$$\leq 3\epsilon_1 \max(| x |, | h |)$$

$$\leq 3\epsilon_1 N_{y,k}$$

$$\epsilon N_{y,k}.$$

Similarly,

$$| \Delta X_n | = | X_n(x + h, y + k) - X_n(x, y) |$$

$$= | X_n(x + h, y + k) - X_n(x, y + k) | + | X_n(x, y + k) - X_n(x, y) |$$

$$\leq \left( \sup_{0 \leq t \leq 1} | X_n(x + th, y + k) | h \right) + | X_n(x + k) - X_n(0, y + k) | + | X_n(x, y) - X_n(0, y) |$$

$$\leq \epsilon_1 | h | + \epsilon_1 | \sup_{0 \leq t \leq 1} X_n(x + th, y + k) | x | + \epsilon_1 | x |$$

$$\leq 3\epsilon_1 \max(| x |, | h |)$$

$$\leq \epsilon M_{x,h}^*.$$

Since $M_{x,h} = \min(1, M_{x,h}^*)$, to get

$$| \Delta X_n | \leq \epsilon M_{x,h},$$

it suffices to show that if $M_{x,h}^* > 1$, then

$$| \Delta X_n | \leq \epsilon.$$

But, if $M_{x,h}^* > 1$, then, since $| (x, y) | < \delta < \frac{1}{4}$, we have

$$| (x + h, y + k) | \geq | x + h | > | h | - \delta > 1 - \delta > \delta,$$

giving $X_n(x + h, y + k) = 0$, and

$$| \Delta X_n | = | X_n(x + h, y + k) - X_n(x, y) | = | X_n(x, y) | \leq \epsilon_1 | (x, y) | \leq \epsilon_1 \delta \leq \epsilon.$$

Moving to (175), we proceed as follows.

First, since $Hol(DX_n) \leq Hol(Df_n)$, we have

$$| \Delta X_n | \leq Hol(Df_n) | (h, k) |^\alpha.$$

Next, using (18) at the points $(x + h, y)$ and $(x, y)$, we have

$$| X_n, x(x + h, y) - X_n, x(x, y) | \leq 2Hol(Df_n) | y |^\alpha$$

for each $x, y, h$.  

Hence,

\[
| \Delta X_{n,x} | = | X_{n,x}(x + h, y + k) - X_{n,x}(x,y) | \\
\leq | X_{n,x}(x + h, y + k) - X_{n,x}(x+h,y) | \\
+ | X_{n,x}(x+h,y) - X_{n,x}(x,y) | \\
\leq Hol(Df_n) | k | |^\alpha + 2Hol(Df_n) | y |^\alpha \\
\leq 3Hol(Df_n) N_{y,k}^{\alpha} \tag{182}
\]

Putting this together with (180) gives

\[
| \Delta X_{n,x} | = | \Delta X_{n,x} |^\eta + 1 - \eta \\
\leq \left[ 3Hol(Df_n) N_{y,k}^{\alpha} \right]^\eta \left[ Hol(Df_n) \right] | (h,k) |^{\alpha(1-\eta)} \\
\leq Hol(Df_n) \left( 3N_{y,k}^{\alpha} \right)^\eta (| (h,k) |^{\alpha(1-\eta)}) \\
= 3^\eta Hol(Df_n) N_{y,k}^{\alpha} (h,k) |^{\alpha(1-\eta)} \\
\leq 3Hol(Df_n) V(y,h,k), \tag{183}
\]

and this proves (175).

The proof of (176) is similar to that for (175), except we have to use \( M_{x,h} = \min(1, M^*_{x,h}) \) instead of \( N_{y,k} \).

Working with \( X_{n,y} \) we have

\[
| \Delta X_{n,y} | \leq Hol(Df_n) | (h,k) |^\alpha, \tag{184}
\]

\[
| X_{n,y}(x,y + k) - X_{n,y}(x,y) | \leq 2Hol(Df_n) | x |^\alpha,
\]

and

\[
| \Delta X_{n,y} | = | X_{n,y}(x + h, y + k) - X_{n,y}(x,y) | \\
\leq | X_{n,y}(x + h, y + k) - X_{n,y}(x,y + k) | \\
+ | X_{n,y}(x,y + k) - X_{n,y}(x,y) | \\
\leq Hol(Df_n) | h |^\alpha + 2Hol(Df_n) | x |^\alpha \\
\leq 3Hol(Df_n) (M^*_{x,h})^\alpha.
\]

As in the proof of (179), if \( M^*_{x,h} > 1 \), then \( | X_{n,y}(x + h, y + k) | = 0, \ M_{x,h} = 1 \), and

\[
| \Delta X_{n,y} | = | X_{n,y}(x,y) | \leq Hol(Df_n) \delta^\alpha \\
\leq 3Hol(Df_n) \\
\leq 3Hol(Df_n) M_{x,h}^\alpha.
\]

Now, replacing \( N_{y,k} \) by \( M_{x,h} \) in the technique used to get (183), we obtain the statement involving \( | \Delta X_{n,y} | \) in (176), and this completes the proof of Lemma 5.7.

Now, set
\begin{align*}
\hat{K}_{n,1} &= |A^{-n}|(|A^n| + \epsilon)(|B^n| + \epsilon)^{\alpha n}(|A^n| + \epsilon)^{\alpha(1 - \eta)}, \\
\hat{K}_{n,2} &= |A^{-n}|(|A^n| + \epsilon)\delta^{\alpha(1 - \eta)}Hol(Df_n), \\
\hat{K}_{n,3} &= |B^{-n}|(|B^n| + \epsilon)^{\alpha n}(|A^n| + \epsilon)^{\alpha(1 - \eta)}Hol(Df_n)\delta^{\alpha(1 - \eta)}, \\
\hat{K}_{n,4} &= |A^{-n}|(|A^n| + \epsilon)(|B^n| + \epsilon), \\
\text{and} \\
\hat{\tau}_n &= \max(\hat{K}_{n,1} + \hat{K}_{n,2}, \hat{K}_{n,3} + \hat{K}_{n,4}).
\end{align*}

Observe that we can choose \(\eta\) close enough to 1 so that 
\[\hat{K}_{m,4} < \hat{K}_{m,1} < 1.\]

Then, we can choose \(\delta\) small enough, depending on \(1 - \eta\), such that

\begin{align*}
\hat{K}_{m,1} + \hat{K}_{m,2} &< 1, \\
\text{and} \\
\hat{K}_{m,3} + \hat{K}_{m,4} &< 1.
\end{align*}

This gives

\begin{equation}
\hat{\tau}_m < 1.
\end{equation}

Next, we define some new functions and make some more estimates.
Write

\[u = u_n = A^n x + X_n(x, y), \quad v = v_n = B^n y + Y_n(x, y),\]

\[u_{1,n} = A^n(x + h) + X_n(x + h, y + k), \quad \text{and} \quad v_{1,n} = B^n(y + k) + Y_n(x + h, y + k)\]

so that

\[T^n(x, y) = (u_n, v_n) \quad \text{and} \quad T^n(x + h, y + k) = (u_{1,n}, v_{1,n}).\]

Then, setting

\[\Delta u = u_{1,n} - u_n = A^n h + \Delta X_n\]

and

\[\Delta v = v_{1,n} - v_n = B^n k + \Delta Y_n,\]

we have, for \(\psi \in \mathcal{E}^+\),

\begin{equation}
H^n_\psi(x + h, y + k) - H^n_\psi(x, y) = A^{-n} \psi(u + \Delta u, v + \Delta v) - A^{-n} \psi(u, v).
\end{equation}
The proof of Lemma 5.6 involves estimating the \( x \) and \( y \) partial derivatives of the left side of (193) in terms of those of \( \psi \) and will be concluded in the inequalities (205) and (206) below.

First, we have the estimates

\[
| \Delta u | = | A^n h + \Delta X_n |
\leq (| A^n | + \epsilon) \min(M_{x,h}, N_{y,k}, | (h, k) |)
\leq (| A^n | + \epsilon) \min(M_{x,h}^*, | (h, k) |),
\]

and

\[
| \Delta v | = | B^n k + \Delta Y_n |
\leq (| B^n | + \epsilon) \min(N_{y,k}, | (h, k) |).
\]

Putting these together with

\[
| u | = | A^n x + X_n(x, y) | \leq (| A^n | + \epsilon_1) | x | \leq (| A^n | + \epsilon_1) M_{x,h}^*,
\]

and

\[
| v | = | B^n y + Y_n(x, y) | \leq (| B^n | + \epsilon_1) | y | \leq (| B^n | + \epsilon_1) N_{y,k},
\]

we have

\[
M_u^{\Delta u} \leq (| A^n | + \epsilon) M_{x,h}^*,
\]

(194)

(195)

and

(196)

\[ | (\Delta u, \Delta v) | \leq (| A^n | + \epsilon) | (h, k) |. \]

From (194), we have

\[
\min(1, M_u^{\Delta u}) \leq \min(1, (| A^n | + \epsilon) M_{x,h}^*). \]

Observe that, if \( a, b, c \) are non-negative real numbers and \( c \geq 1 \), then

\[
\min(a, b, c) \leq c \cdot \min(a, b). \]

Applying this with \( a = 1, b = M_{x,h}^* \) and \( c = | A^n | + \epsilon \), we get

\[
\min(1, M_u^{\Delta u}) \leq (| A^n | + \epsilon) \min(1, M_{x,h}^*),
\]

which, by definition, is

\[
M_u^{\Delta u} \leq (| A^n | + \epsilon) M_{x,h}.
\]

(199)

Consider the compositions of \( U(x, h, k) \) and \( V(y, h, k) \) with \( u, v, \Delta u, \Delta v \):
in turn, would imply that $52\ SHELDON\ E.\ NEWHOUSE$

\[
U_1 = U_1(x, h, k) = U(u, \Delta u, \Delta v) = M_{n, \Delta u}^{\alpha \eta} (\Delta u, \Delta v)^{\alpha(1-\eta)}.
\]

\[
V_1 = V_1(y, h, k) = V(v, \Delta u, \Delta v) = N_{n, \Delta u}^{\alpha \eta} (\Delta u, \Delta v)^{\alpha(1-\eta)}.
\]

Using $(\| A^n \| + \epsilon) > 1$, we have

\[
U_1 \leq (\| A^n \| + \epsilon)^{\alpha \eta} (\| A^n \| + \epsilon)^{\alpha(1-\eta)} U = (\| A^n \| + \epsilon)^{\alpha} U \leq (\| A^n \| + \epsilon) U
\]

and

\[
V_1 \leq (\| B^n \| + \epsilon)^{\alpha \eta} (\| A^n \| + \epsilon)^{\alpha(1-\eta)} V.
\]

Let us observe that, with $0 < \delta \leq \frac{1}{2}$, we have

\[
\max(M_{x,k}, N_{y,k}) \leq 1.
\]

Indeed, the definition of $M_{x,k}$ makes it no larger than 1. If $N_{y,k} > 1$, then, since both $(x, y)$ and $(x + h, y + k)$ are in $O_x(\delta)$, we would get that $|y| < \delta \leq \frac{1}{2}$, which, in turn, would imply that $N_{y,k} = |k| > 1$.

This would give

\[
\delta > |y + k| \geq |k| - |y| > 1 - \delta,
\]

contradicting the condition that $\delta \leq \frac{1}{2}$.

Now, we have $v_x = Y_{n,x}$ and $u_y = X_{n,y}$.

Since $\max(|X_{n,y}|, |Y_{n,x}|) = 0$ when $|(x, y)| > \delta$, we have that $U|v_x| > 0$ or $V|u_y| > 0 \Rightarrow |(x, y)| \leq \delta$.

This, together with (204), gives

\[
| A^{-n} |U_1|v_x| = | A^{-n} |U_1| Y_{n,x} |
\leq | A^{-n} |(| A^n \| + \epsilon)U| Y_{n,x} |
\leq | A^{-n} |(| A^n \| + \epsilon)M_{n,k}^{\alpha \eta} (h, k)^{\alpha(1-\eta)} Hol(Df_n) y |^\alpha
\leq | A^{-n} |(| A^n \| + \epsilon)Hol(Df_n) y |^\alpha (h, k)^{\alpha(1-\eta)}
\leq | A^{-n} |(| A^n \| + \epsilon)Hol(Df_n) y |^{\alpha \eta} y^{\alpha(1-\eta)} (h, k)^{\alpha(1-\eta)}
\leq \tilde{K}_{n,2} |y|^{\alpha \eta} (h, k)^{\alpha(1-\eta)}
\leq \tilde{K}_{n,2} \alpha \eta |(h, k)|^{\alpha(1-\eta)}
\leq \tilde{K}_{n,2} V.
\]

In addition,
\[
V|u_y| = V|X_{n,y}|
\leq N^{(1-\eta)}(h,k) |\alpha x| x^\alpha
\leq \text{Hol}(Df_n)|x| (h,k) |\alpha(1-\eta)\text{Hol}(Df_n)|x||x| (h,k) |\alpha(1-\eta)
= \text{Hol}(Df_n)\delta |\alpha(1-\eta)\text{Hol}(Df_n)|x|\alpha(1-\eta)
\]

which implies

\[
|A^{-n}V|u_y| \leq |A^{-n}||B^n| + \epsilon \alpha \eta (|A^n| + \epsilon)^{(1-\eta)}V|u_y| \leq \tilde{K}_{n,3}U.
\]

We now are in a position to prove that

\[
(205) \quad \gamma_3(H^n_s(\psi)) \leq \tilde{\tau}_n \||\psi||
\]

and

\[
(206) \quad \gamma_4(H^n_s(\psi)) \leq \tilde{\tau}_n ||\psi||.
\]

These inequalities will imply that

\[
(207) \quad ||H^n_s(\psi)|| = \max (\gamma_3(H^n_s(\psi)), \gamma_4(H^n_s(\psi))) \leq \tilde{\tau}^n ||\psi||.
\]

As we mentioned above, applying this, respectively, for \(n = 1\) and \(n = m\), will prove Lemma 5.6 and, hence, complete the proof of Theorem 1.5.

Let us proceed to the proofs of (205) and (206).

For \((x, y) \in O_+(\delta), (h, k) \in E^n \times E^s\) such that \((x + h, y + k) \in O_+(\delta)\) and \(|(h, k)| > 0\), let

\[
\Delta H^n_s(\psi)_x = \partial_x(H^n_s(\psi))(x + h, y + k) - \partial_x(H^n_s(\psi))(x, y)
= \partial_x(H^n_s(\psi))(x + h, y + k) - H^n_s(\psi)(x, y)
\]

and

\[
\Delta H^n_s(\psi)_y = \partial_y(H^n_s(\psi))(x + h, y + k) - \partial_y(H^n_s(\psi))(x, y)
= \partial_y(H^n_s(\psi))(x + h, y + k) - H^n_s(\psi)(x, y).
\]

Then, from (193), we have
Theorem 6.1. Let \( \Phi : \Lambda \times X \to X \) be a uniform contraction map on \( \Lambda \times X \). For each \( \lambda \in \Lambda \), let \( p_\lambda \) be the unique fixed point of the \( \lambda \)-section map \( \Phi_\lambda : X \to X \). Then, the map \( \lambda \to p_\lambda \) is continuous.
For $0 < \alpha < 1$, and a $C^1$ function $f : U \to E$, define

$$|f|_{1,\alpha} = \sup_{x \neq y, x, y \in U} \max \left( |f(x)|, |Df(x)|, \frac{|Df(x) - Df(y)|}{|x - y|^\alpha} \right),$$

and let $C^{1,\alpha}(U, E)$ be the set of $C^1$ functions from $U$ into $E$ such that $|f|_{1,\alpha} < \infty$.

Clearly, the quantity $|f|_{1,\alpha}$ defines a norm in $C^{1,\alpha}(U, E)$ making it into a real Banach space.

We now state the parametrized version of Theorem 1.5

**Theorem 6.2.** Let $0 < \alpha < 1$, let $E$ be a $C^{1,\alpha}$ Banach space, $p \in E$, and let $U$ be an open neighborhood of $p$ in $E$. Let $\Lambda$ be a topological space, let $\lambda_0$ be a point in $\Lambda$, and let $F : \Lambda \times C^{1,\alpha}(U, E)$ be a continuous map such that $p = p_{\lambda_0}$ is an $\alpha$–hyperbolic fixed point of $F_{\lambda_0}$.

Then, there are neighborhoods $\mathcal{L}$ of $\lambda_0$ in $\Lambda$ and $V \subset U$ of $p$ in $E$ and a real number $\beta \in (0, 1)$ such that for each $\lambda \in \mathcal{L}$, the map $F_\lambda$ has a unique $\alpha$–hyperbolic fixed point $p_\lambda$ in $V$ and there is a $C^{1,\beta}$ diffeomorphism $R_\lambda$ from $V$ onto a neighborhood of $0$ in $E$ such that

$$DF_\lambda(p)(R_\lambda(x)) = R_\lambda(F_\lambda(x)) \text{ for } x \in V \cap F_\lambda^{-1}V.$$

Moreover, the map $\lambda \to (p_\lambda, R_\lambda)$ is a continuous map from $\mathcal{L}$ into the product space $V \times C^{1,\beta}(V, E)$.

**Sketch of Proof.**

Replacing $F_{\lambda_0}$ by $F_{\lambda_0}(x + p) - p$, we may assume that $p = 0$, and $U$ is an open connected neighborhood of $0$.

Letting $L = DF_{\lambda_0}(0)$, we write

$$F_{\lambda_0}(x) = Lx + f_{\lambda_0}(x)$$

with $f_{\lambda_0}(0) = 0$ and $Df_{\lambda_0}(0) = 0$.

The definitions and statements below require that $\lambda$ be close to $\lambda_0$ in $\Lambda$, and we assume this without further mention.

Let $f_\lambda$ be the $C^{1,\alpha}$ map defined by

$$f_\lambda = F_\lambda - L.$$

Since the operator $L$ is hyperbolic, the operator $I - L$ has a bounded inverse.

The equation for a fixed point $x$ of $F_\lambda$ has the following forms

$$x = Lx + f_\lambda(x),$$

$$x = (I - L)^{-1}f_\lambda(x),$$

or

$$x = (I - L)^{-1}f_\lambda(x).$$

Let $\bar{B}_\delta$ denote the closed ball of radius $\delta$ about $0$ in $E$. 
Let $G_\lambda = (I - L)^{-1} f_\lambda$.
Given $\epsilon_1 \in (0, 1)$, choose $\delta > 0$ such that, $\bar{B}_\delta \subset U$, and

$$|DG_\lambda| = |(I - L)^{-1} Df_\lambda(x)| < \epsilon_1,$$

for $x \in \bar{B}_\delta$.
Next, let $\delta_1 = (1 - \epsilon_1)\delta$ and let $\mathcal{L}$ be a neighborhood of $\lambda_0$ in $\Lambda$ so that, for $\lambda \in \mathcal{L}$ and $x \in \bar{B}_\delta$, we have

$$(210) \quad |G_\lambda(0)| = |(I - L)^{-1} f_\lambda(0)| < \delta_1,$$

and

$$(211) \quad |DG_\lambda(x)| = |(I - L)^{-1} Df_\lambda(x)| < \epsilon_1. $$

From (211), we see that, for $\lambda \in \mathcal{L}$,

$$(212) \quad \text{Lip}(G_\lambda, \bar{B}_\delta) \leq \epsilon_1. $$

Now, for each $\lambda \in \mathcal{L}$ and each $x \in \bar{B}_\delta$, we have

$$|G_\lambda(x)| = |G_\lambda(x) - G_\lambda(0) + G_\lambda(0)|$$
$$\leq \epsilon_1 |x| + \delta_1$$
$$\leq \epsilon_1 \delta + (1 - \epsilon_1)\delta$$
$$\leq \delta,$$

This and (212) show that the map $(\lambda, x) \to G_\lambda(x)$ is a uniform contraction map on $\mathcal{L} \times \bar{B}_\delta$.

Hence, there is a unique fixed point $p_\lambda$ of $G_\lambda$ in $\bar{B}_\delta(0)$ which depends continuously on $\lambda \in \mathcal{L}$.

Since the set $Hyp_\alpha(E, E)$ of $\alpha-$hyperbolic automorphisms of $E$ is an open subset of the set $L(E, E)$ of bounded linear maps on $E$, we may assume that the point $p_\lambda$ is an $\alpha-$hyperbolic fixed point of $F_\lambda$.

Conjugating $F_\lambda$ with the translations $x \to x + p_\lambda$ we may assume that $F_\lambda(0) = 0$ for each $\lambda$ near $\lambda_0$.

Letting $L_\lambda$ be the derivative of $F_\lambda$ at $0$, it can be shown that the subspace $E_\lambda^u$ is the graph of a bounded linear map $P_\lambda^u : E_{\lambda_0}^u \to E_{\lambda_0}^s$ of small norm. The map $\lambda \to P_\lambda^u$ is continuous with the obvious topologies. A similar statement holds for $E_\lambda^s$.

Next, choose coordinates so that the hyperbolic splitting

$$E_{\lambda_0}^u \times E_{\lambda_0}^s$$

coincides with the coordinate subspaces

$$(213) \quad E^u \times \{0\} \times \{0\} \times E^s. $$

For $\lambda$ near $0$, there is a $\lambda-$dependent family of linear coordinate changes $\{A_\lambda\}$ near the identity, such that $A_\lambda L_\lambda A_\lambda^{-1}$ preserves the splitting (213) for each $\lambda$. Replacing $L_\lambda$ with $A_\lambda L_\lambda A_\lambda^{-1}$, we may assume that $L_\lambda$ preserves this splitting.
Now, all of the constructions in the proof of Theorem 1.5 vary continuously with \( \lambda \). Here we use the Irwin method as in the paper of de la Llave and Wayne [8] to get the local stable and unstable manifolds via the Implicit Function Theorem. Thus, these constructions vary continuously with \( \lambda \).

This leads to continuously varying linearization operators \( \lambda \to H_\lambda \) which are contractions on appropriate function spaces as in the proofs of (126), (127), (152), and (153).

This gives a family \( R_\lambda \) of \( C^{1,\beta} \) linearizations at \( p(\lambda) \) of \( T_\lambda \) depending continuously on \( \lambda \) as required for Theorem 6.2.

7. Vector Fields and Motivation

Our main motivation for the results presented here is concerned with the study of vector fields.

For simplicity, we consider the finite dimensional case.

In this section \( r \) will denote a real number larger than 1.

Let \( M \) be a \( C^{r+1} \) finite dimensional real manifold, and let \( X \) be a \( C^r \) vector field defined in an open subset \( U \subset M \). Let \( p \) be a critical point of \( X \); i.e., \( X(p) = 0 \). Assume that the local flow \( \phi(t, x) \) associated to \( X \) is defined on the product \((-2, 2) \times U\), and let \( T(x) = \phi(1, x) \) be the time one map of \( X \). We choose an open subset \( U_1 \) of \( U \) so that \( T \) and \( T^{-1} \) are defined and \( C^r \) on \( U_1 \).

For \( \alpha \in (0, 1) \), and each of the properties \( \alpha \)–contracting, \( \alpha \)–expanding, \( \alpha \)–hyperbolic, bi-circular, etc., we define \( p \) to have the corresponding property if it has that property as a fixed point of \( T \).

For instance, if \( M \) is the Euclidean space \( R^N \), with \( N \) a positive integer, then, using that \( \exp(DX_p) = DT_p \), it can be seen that \( p \) is bi-circular if the set consisting of the real parts of the eigenvalues of \( DX_p \) is a two element set \( \{a, b\} \) with \( a < 0 < b \).

A well-known technique of Sternberg [38], page 817, implies that a \( C^r \) vector field has a local \( C^k \) (here \( r, k \geq 1 \)) linearization near a critical point \( p \) if and only if its time-one map \( T \) has such a linearization at \( p \).

As a simple application of Theorem 1.5, let us consider the Shilnikov three-dimensional saddle focus theorem as in [39].

One has a vector field \( X \) in \( R^3 \) with a saddle type critical point at 0 such that \( DX_0 \) has a pair of complex conjugate eigenvalues \( a \pm ib \) with \( a < 0, b \neq 0 \) and a real positive eigenvalue \( c > 0 \) such that \( a + c > 0 \). It is also assumed that there is a homoclinic orbit (an orbit which is forward and backward asymptotic to 0).

Shilnikov then showed that there are infinitely many saddle periodic orbits near the homoclinic orbit. In fact, the geometry shows that horseshoe type dynamics (including symbolic systems) appear.

There is a geometric description of this in section 6.5 in [12]. The treatment assumes that the flow is linear in a neighborhood of the critical point. It is not difficult to see that a \( C^1 \) linearization would suffice, so our Theorem 1.5 in the bi-circular case can be applied to show that Shilnikov’s Theorem holds even if the vector field is only \( C^{1,\alpha} \).

More recently, higher dimensional systems with homoclinic orbits involving saddle foci have been studied by Ovsyannikov and Shilnikov in [23] and [24]. For related material, the reader is encouraged to look at the recent books [34] and [35] which contain a beautiful and fairly complete treatment of much of the work carried out by the so-called Shilnikov School in city of Nizhny-Novogorod (formerly
Gorky). See [32] for a detailed description of the work of Shilnikov and his many students and collaborators. Other treatments which include closely related work are in the books of Ilyashenko and Li [18] and Guckenheimer and Holmes [12].

Many of the proofs in works dealing with homoclinic orbits in high dimension (e.g., greater than four) are complicated. We expect that simplifications can be obtained using $C^{1,\beta}$ (or even $C^1$) linearizations on Lyapunov center manifolds as follows.

Consider a $C^r$ vector field $X$ on the Euclidean space $\mathbb{R}^N$ with a hyperbolic critical point of saddle type at 0, with $r > 1$. Thus, the real parts of the eigenvalues of $DX_0$ are non-zero and intersect both the positive and negative sets of reals.

Let $L = DX_0$, and let

$$a_m > a_{m-1} > \ldots > a_1 > 0 > b_1 > b_2 > \ldots > b_n$$

denote the distinct real parts of the eigenvalues of $L$.

We express $\mathbb{R}^N$ in the direct sum decomposition

$$\mathbb{R}^N = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

such that $L(E_i) = E_i$ for $i = 1, 2, 3, 4$, and, writing $L_i$ for the restriction $L|E_i$, we have

1. the eigenvalues of $L_1$ have real parts equal to $a_1$,
2. the eigenvalues of $L_2$ have real parts equal to $b_1$,
3. the eigenvalues of $L_3$ have real parts greater than $a_1$, and
4. the eigenvalues of $L_4$ have real parts less than $b_1$.

Following terminology in [34] we call the eigenvalues with real parts $a_1$ or $b_1$ the leading eigenvalues and the corresponding subspaces $E_1, E_2$ the leading eigenspaces.

The numbers $a_1$ and $b_1$ are the Lyapunov exponents of $L_1$ and $L_2$, respectively. That is, for each $v \in E_1 \setminus \{0\}$ and $w \in E_2 \setminus \{0\}$, we have

$$\chi(v) = \lim_{t \rightarrow \pm \infty} \frac{1}{t} \log |e^{tL}v| = a_1$$

and

$$\chi(w) = \lim_{t \rightarrow \pm \infty} \frac{1}{t} \log |e^{tL}w| = b_1.$$ 

Accordingly, we will call the direct sum $E_1 \oplus E_2$ the Lyapunov Center Subspace, and we denote it by $E^{cc}$.

The theory of invariant manifolds, e.g., as in [8], [17], shows that there are submanifolds $W^{cc}$, invariant by the local flow of $X$, tangent at 0 to the subspace $E^{cc}$. We will call these Lyapunov Center Manifolds. They are not unique, but each will be $C^{1,\alpha}$ near 0 (i.e., the transition maps on local coordinate charts are $C^{1,\alpha}$) provided that

$$(1 + \alpha)a_1 < a_2 \text{ and } (1 + \alpha)b_1 > b_2.$$ 

(214)

The derivative $DX_0$, restricted to $E_1 \times E_2$ is bi-circular, so by Theorem [15] and the Sternberg technique mentioned above, the flow of $X$ restricted to each such $W^{cc}$ is $C^{1,\beta}$ linearizable at 0. Moreover, by Theorem [5,2] the linearizations can be chosen to depend continuously on external parameters.
Note that even for linear differential equations, most of the manifolds $W^{cc}$ may not be smoother than $C^{1,\alpha}$ for some $0 < \alpha < 1$. For instance, consider the three-dimensional linear system $\dot{x} = x, \dot{y} = (1+\alpha)y, \dot{z} = -z$ for $0 < \alpha < 1$. Each surface

$$y = C| x |^{1+\alpha}$$

for some non-zero constant $C$ can be taken as one of the manifolds $W^{cc}$.

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