Orbital stability of peakon solutions for a generalized higher-order Camassa–Holm equation

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Abstract. In this paper, we investigate the orbital stability issue of a generalized higher-order Camassa–Holm (HOCH) equation, which is a higher-order extension of the quadratic CH equation. Firstly, we show that the HOCH equation admits a global weak peakon solution by paring it with some smooth test function. Secondly, with the help of two conserved quantities and the non-sgn-changing condition, we prove the orbital stability of this peakon solution in the energy space in the sense that its shape remains approximately the same for all times. Our results enrich the research of the orbital stability for the CH-type equations and are useful to better understand the impact of higher-order nonlinearities on the dispersion dynamics.

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1. Introduction

In this paper, we would like to focus on the orbital stability of the peakon solutions to the following generalized high-order Camassa–Holm (HOCH) equation [1]

\[
m_t + (u^2 - u_x^2)^{n-1} u_x m + \partial_x \left[(u^2 - u_x^2)^{n-1} u m\right] = 0, \quad m = u - u_{xx},
\]

where \( n \in \mathbb{N}^+ \). Equation (1.1) was shown to admit the single peakon solution and two conserved quantities [1]

\[
H_1(u) = \int_{\mathbb{R}} (u^2 + u_x^2) \, dx, \quad \hat{H}_2(u) = \int_{-\infty}^{\infty} \frac{1}{2n} (u^2 - u_x^2)^n \, m \, dx.
\]

Expanding the integrand and dropping the factor \( \frac{1}{2n} \) in \( \hat{H}_2(u) \) give its equivalence as

\[
\hat{H}_2(u) = \int_{\mathbb{R}} \left( u^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} \binom{k}{n} u^{2n-2k+1} u_x^{2k} \right) \, dx.
\]

Qu–Fu [28] first established the local well-posedness of the corresponding Cauchy problem of Eq. (1.1) in the setting of Besov spaces \( B_{p,r}^s \) with \( s > \max\{2 + 1/p, 5/2\} \). Chen–Deng–Qiao [4] first proved that Eq. (1.1) with the special case \( n = 2 \) admits global peakon and periodic peakon solutions, and then established their orbital stability in the energy space.

At \( n = 1 \), Eq. (1.1) reduces to the remarkable CH equation [3]

\[
m_t + 2u_x m + um_x = 0, \quad m = u - u_{xx}.
\]
It is widely known that the CH equation was originally put forward as a bi-Hamiltonian system by Fokas–Fuchssteiner [16] and was later re-obtained as an approximation to the Euler equations of hydrodynamics by Camassa–Holm [3]. It can represent the propagation of axially symmetric waves in hyperelastic rods [12,14]. The CH equation is completely integrable such that it admits a Lax pair, bi-Hamiltonian structure, and infinitely many conservation laws [3,15,16]. The Lax pair enables one to solve the Cauchy problem of (1.4) by the inverse scattering transform (IST) [9]. The local or global well/ill-posedness of the Cauchy problem of Eq. (1.4) has been discussed in [2,5,7,8,13,20,23]. An important feature of the CH equation is that it can model the wave-breaking phenomena, namely, the solution itself remains bounded but its slope becomes unbounded in finite time [5–8]. This phenomena cannot be described by the classical integrable systems, such as the KdV equation and the Schrödinger equation. Another significant property of the CH equation is that it admits exact weak peakon solutions of the form
\[ u(t, x) = c e^{-|x-ct|}, c \in \mathbb{R}, \]
which are peaked traveling waves with a discontinuous derivative at the crest. Peakons are a kind of solitons. When interacting with another one, the peakon will keep its shape and speed unchanged, so it is reasonable to expect the stability of the peakons, an elementary question for nonlinear wave equations. Here, for CH-type equations, the appropriate concept of stability is orbital stability. Namely, if a wave is close to a solitary wave initially, then it will remain close to some translate of the solitary wave at all later times, or to put it briefly, the shape of the wave remains approximately the same for all times [11]. The orbital stability issue of single peakon for Eq. (1.4) has been tackled by Constantin–Strauss [11] with the aid of the conserved densities and specific structure of the peakons. Constantin–Molient [10] introduced a variational approach to settle the orbital stability issue.

The nonlinearity in Eq. (1.4) is quadratic. However, there do exist other CH-type equations with cubic or higher-order nonlinearities. For example, the following cubic modified Camassa–Holm equation
\[ m_t + \left( \left( u^2 - u_x^2 \right) m \right)_x = 0, \quad m = u - u_{xx}, \] (1.5)
which is also called the FORQ equation [22,29]. It was first deduced by Fuchssteiner [18] and Olver–Rosenau [24] as a new generalization of integrable system by applying tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation. Later, Qiao [25] discussed its integrability and the structure of solutions. Like the CH equation, it is also completely integrable and admits a Lax pair [24,26]. The Cauchy problem in the Besov setting as well as the blow-up data was established by Fu–Gui–Qu–Liu [17]. Gui–Liu–Olver–Qu [19] addressed the blow-up criterion and another type of wave-breaking data as well as the explicit form of the single peakon and the dynamical systems satisfied by the multi-peakons. The form of the single peakon is
\[ \varphi_c(t, x) = a e^{-|x-ct|}, c = \sqrt{3c/2e^{-|x-ct|}}, c \in \mathbb{R} \] whose orbital stability has been tackled by Qu–Liu–Liu [27].

An instance of the modified CH equation with higher-order nonlinearities is written as
\[ m_t + \left( \left( u^2 - u_x^2 \right)^n m \right)_x = 0, \quad m = u - u_{xx}, \] (1.6)
which was deduced by Anco–Recio [1]. Its local well-posedness of the Cauchy problem in Besov spaces was established by Yang–Li–Zhao [29]. The single peakon solution
\[ \varphi_c(t, x) = a e^{-|x-ct|}, c = \left( 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \right) a^{2n}, \]
was first obtained by Anco–Recio [1], and later re-derived by Guo–Liu–Liu–Qu [21]. By combining the idea of Constantin–Strauss [11] and Qu–Liu–Liu [27], the authors [21] also discussed the orbital stability of the single peakon, where the following inequality
\[ \frac{n(2 - c_1)}{n + 1} M^{2n+2} - \frac{2 - c_1}{2} M^{2n} E(u) + \tilde{F}(u) \leq 0 \] (1.7)
plays a crucial role in the proof, where \( M(t) = \max_{x \in \mathbb{R}} \{ u(x,t) \} \), \( E(u) = H_1(u) \) is given by Eq. (1.2), and

\[
\tilde{F}(u) = \int_{\mathbb{R}} \left( u^{2(n+1)} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_{n+1}^k u^{2(n-k+1)} u_x^{2k} + \frac{(-1)^n}{2n+1} u_x^{2(n+1)} \right) dx.
\]

(1.8)

Another example of the CH-type equation with higher-order nonlinearities is just Eq. (1.1). Some properties in the setting of Besov spaces \( B^s_{p,r} \) with \( s > \max\{2+1/p, 5/2\} \) of Eq. (1.1) are listed in “Appendix A.”

Motivated by the works [4, 11, 21], we will explore the orbital stability of the single peakon of Eq. (1.1) in this paper. For this purpose, we invoke similar method analogous to that employed in [11, 21]. Of course, the two conserved quantities \( E(u) \) and \( F(u) \) will play a vital role in this issue. We will prove the following significant inequality (see Proposition 1.3)

\[
\frac{(2n-1) (2-c_1)}{2n+1} M^{2n+1}(t) - \frac{2-c_1}{2} M^{2n-1}(t) H_1(u) + H_2(u) \leq 0,
\]

(1.9)

which relates \( H_{1,2}(u) \) and the maximal value of approximate solutions \( u \). One can compare (1.9) with (1.7). To obtain (1.9), we first define the same functional \( g \) as that in [11] or in the CH and mCH equations due to the same conserved quantity \( H_1(u) \). Note that the constructed function \( g \) needs to vanish at the peakons. We also need to find the other function \( h \) as previous literatures did. This can be done by first assuming the expression of \( h \) with some coefficients \( c_k, d_k \) to be determined and then comparing the expansion of \( \int g^2 h \) with the formula \( H_2(u) \) and finally obtaining the expression of \( c_k, d_k \) by solving recursive formulas of the two sequence. The function \( h \) is also required to vanish at the peakons. The third ingredient in establishing (1.9) is to prove

\[
h(t,x) \leq \frac{2-c_1}{2} M^{2n-1}(t,x) \quad \text{for } \forall (t,x) \in [0,T) \times \mathbb{R}
\]

(1.10)

with \( M(t) = \max_{x \in \mathbb{R}} \{ u(x,t) \} \). Notice that in the case of the CH equation, the function \( h \) satisfy \( h = u \leq M \), while in the case of mCH equation, the function \( h \) satisfy \( h \leq \frac{4}{3} M^2 \). This results from the higher-order conserved quantity \( F(u) \) in our case. One can also compare (1.10) with that in [21], where \( h \) satisfy \( h \leq \frac{2-c_1}{2} M^{2n} \). This is due to the fact that the conserved quantity \( H \) here is a degree lower than that in [21], where the corresponding \( \tilde{H}(u) = \frac{1}{2(n+1)} \int_{-\infty}^{\infty} (u^2 - u_x^2)^n \mu dx \). To obtain (1.10), we need to carefully analyze the properties of the coefficients \( c_k, d_k \) of the function \( h \) under the non-sgn-changing condition.

Having established (1.9), it is then ready to settle the orbital stability issue. One first expands the conserved quantity \( E(u) \) around the peakon \( \varphi_c \), then uses the error term \( |M - \max_{x \in \mathbb{R}} \varphi_c| \) to control the term \( |u - \varphi_c(\cdot - \xi(t))|_{H_1} \). Finally, the root structure of the polynomial inequality \( Q(y) \) defined By Eq. (3.31) will be analyzed to estimate the error term.

To state our main results, we need to define the weak solutions associated with Eq. (1.1). For this purpose, we recast it as

\[
u_t - u_{txx} + \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k u^{2n-2k-2} u_x^{2k+1} [(2n-2k+1)u + 2(2n-2k-1)u_{xx}] + 2 \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k (n-k-1) u^{2n-2k-3} u_x^{2k+1} u_{xx} + \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k u^{2n-2k-1} u_x^{2k} u_{xxx} = 0.
\]

(1.11)
Then, operating \((1 - \partial_x^2)^{-1}\) to (1.11) gives rise to

\[
u_t + \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} C_n^k u^{2n-2k-1} u_x^{2k+1} + (1 - \partial_x^2)^{-1} \partial_x \left( u^{2n} + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n-2k+1)}{2n(2k-1)} C_n^k u^{2n-2k} u_x^{2k} \right) + (1 - \partial_x^2)^{-1} \left( \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k-1} u_x^{2k+1} \right) = 0. \tag{1.12}
\]

Let \(p(x) = \frac{1}{2} e^{-|x|} (x \in \mathbb{R})\). Then, there holds \(p * f = (1 - \partial_x^2)^{-1} f\) for \(f \in L^2\). We use this combined with (1.12) to define weak solutions of (1.1) as follows.

**Definition 1.1.** For the given initial data \(u_0 \in W^{1,2n}(\mathbb{R}), u(t,x) \in L^\infty_{loc} ([0,T); W^{1,2n}_{loc}(\mathbb{R})\)) is called a weak solution to the HOCH equation (1.1) if it satisfies

\[
\begin{aligned}
\int_0^T \int_\mathbb{R} &\left[ u \phi_t + \frac{1}{2n} u^{2n} \phi_x + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k-1} u_x^{2k+1} \right] \phi \\
&+ p * \left( u^{2n} + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n-2k+1)}{2n(2k-1)} C_n^k u^{2n-2k} u_x^{2k} \right) \cdot \partial_x \phi \\
&- p * \left( \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_n^k u^{2n-2k-1} u_x^{2k+1} \right) \cdot \phi \\
&\text{dxdt} + \int_\mathbb{R} u_0(x) \phi(0,x) \text{dx} = 0
\end{aligned}
\]

for any test function \(\phi(t,x) \in C_c^\infty([0,T) \times \mathbb{R})\). In particular, \(u(t,x)\) is called a global weak solution to Eq. (1.1) if it is a weak solution on \([0,T)\) for every \(T > 0\).

Then comes our first result concerning the explicit formula of the weak peakon solution to (1.1):

**Proposition 1.1.** For any \(c \in \mathbb{R}\), the peaked function of the form

\[
u(t,x) = \varphi_c(t,x) = ae^{-|x-ct|}, \quad c = \left( 1 + \frac{1}{2n} \right) \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} C_n^k a^{2n-1}
\]

is a global weak solution to Eq. (1.1) in the sense of Definition 1.1.

**Remark 1.1.** If \(u(t,x)\) is a solution of Eq. (1.1), then so is \(-u(-t,x)\). Therefore, without loss of generalization, one can assume that \(a > 0\).

**Proposition 1.2.** For every \(u \in H^1(\mathbb{R})\) and \(\xi \in \mathbb{R}\), one has

\[
H_1(u) - H_1 (\varphi_c (-\xi)) = \| u - \varphi_c (-\xi) \|_{L^1}^2 + 4a(u(\xi) - a),
\]

where \(a\) and \(c\) are defined by \(c = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} C_n^k a^{2n-1}\).

**Proposition 1.3.** Assume \(u_0 \in H^s(\mathbb{R}), s > 5/2\), and \(m_0 \geq 0\). Let \(u(t,x)\) be the positive solution of the Cauchy problem of the HOCH equation (1.1) with initial data \(u_0\). Then,

\[
\frac{(2n-1)(2-c_1)}{2n+1} M^{2n+1}(t) - \frac{2-c_1}{2} M^{2n-1}(t) H_1(u) + H_2(u) \leq 0,
\]

where \(M(t) \triangleq \max_{x \in \mathbb{R}} \{ u(t,x) \} \), and \(c_1 = \frac{1}{2} + \sum_{j=1}^{n} (-1)^{j+1} \frac{j-3}{2(2j-1)} C_n^j\).

**Proposition 1.4.** For \(u \in H^s(\mathbb{R}), s > \frac{5}{2}\), if \(\| u - \varphi_c \|_{H^1(\mathbb{R})} < \varepsilon\), with \(0 < \varepsilon < (\gamma - 2\sqrt{2})a, \gamma > 2\sqrt{2}\), then

\[
|H_1(u) - H_1 (\varphi_c)| \leq a\gamma \varepsilon, \quad |H_2(u) - H_2 (\varphi_c)| \leq G(n,c,\|u\|_{H^s}) \varepsilon,
\]

where the constant \(G(n,c,\|u\|_{H^s}) > 0\) depends only on the wave speed \(c, n \in \mathbb{N}^+\) and \(\|u\|_{H^s}\).
Proposition 1.5. For \( 0 < u \in H^s(\mathbb{R}), s > \frac{5}{2} \), let \( M = \max_{x \in \mathbb{R}} \{u(x)\} \). If \( H_{1,2}(u) \) satisfy Eq. (1.16) then
\[ |M - a| \lesssim \sqrt{G(n,c,\|u\|_{H^s})\varepsilon}. \]

We then establish the orbital stability of the peakon solutions (1.13), that is

Theorem 1.1. The peakon solution \( \varphi_c(t,x) \) defined in (1.13) with the traveling wave speed \( c \) isorbitally stable in the following sense. If \( u_0(x) \in H^s(\mathbb{R}) \), for some \( s > 5/2, m_0(x) = (1-\partial^2_x)u_0 \neq 0 \) is nonnegative, and
\[ \|u(0,\cdot) - \varphi_c\|_{H^1(\mathbb{R})} < \varepsilon, \quad \text{for} \quad 0 < \varepsilon < (\gamma - 2\sqrt{2})a, \quad \gamma > 2\sqrt{2}, \quad a > 0 \]

Then, the corresponding solution \( u(t,x) \) of Eq. (1.1) satisfies
\[ \sup_{t \in [0,T]} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} \lesssim \sqrt{a\gamma\varepsilon + 4a\sqrt{G(n,c,\|u_0\|_{H^s})\varepsilon}}, \]
where \( T > 0 \) is the maximal existence time, \( \xi(t) \in \mathbb{R} \) is the maximum point of function \( u(t,\cdot) \) and the constant \( G(n,c,\|u_0\|_{H^s}) > 0 \) depends only on the wave speed \( c, n \in \mathbb{N}^+ \) and the norm \( \|u_0\|_{H^s}. \)

The rest of this paper is organized as follows. In Sect. 2, we verify that (1.13) is a global weak solution to equation (1.1) in the sense of Definition 1.1 and show \( F(u) \) is independent of time. Section 3 will be used to prove the orbital stability result—Theorem 1.1 by showing Propositions 1.2–1.5.

Notation 1. A \( \lesssim B \) represents \( A \leq CB \) for some constant \( C > 0 \).

2. Peakons and conserved quantities

In this section, we give the proof of Proposition 1.1 After that, we will prove the quantity \( F(u) \) is independent of time.

Proof of Proposition 1.1:. We consider the peakon solution of Eq. (1.1) in the form
\[ u(t,x) = \varphi_c(t,x) = ae^{-|x-ct|}, \quad (2.1) \]
where \( a, c \in \mathbb{R} \) are constants to be determined later.

First, in light of integration by parts, one easily verifies
\[ \partial_x \varphi_c(t,x) = -\text{sgn}(x-ct)\varphi_c(t,x), \quad \partial_t \varphi_c(t,x) = c \partial_x \varphi_c(t,x) \in L^\infty \quad (2.2) \]
for all \( t \geq 0 \) in the sense of distribution. Set \( \varphi_{0,c}(x) \equiv \varphi_c(0,x) \). Then, there holds
\[ \lim_{t \to 0^+} \|\varphi_c(t,\cdot) - \varphi_{0,c}(\cdot)\|_{W^{1,\infty}} = 0. \quad (2.3) \]
Invoking (2.2)–(2.3), one derives for any test function \( \phi(t,x) \in C_c^\infty([0,\infty) \times \mathbb{R}) \) that
\[ \int_{0}^{+\infty} \int_{\mathbb{R}} \left[ \varphi_c \partial_t \phi + \frac{1}{2n} \varphi^2_c \phi_x + \left( \sum_{k=1}^{n-1} \frac{(-1)^k+1}{2k+1} C_{n-1}^k \varphi^{2n-2k-1}_c (\partial_x \varphi_c)^{2k+1}_c \right) \phi \right] dxdt + \int_{\mathbb{R}} \varphi_{0,c}(x) \phi(0,x)dx \]
\[ = -\int_{0}^{+\infty} \int_{\mathbb{R}} \left[ \partial_t \varphi_c + \varphi^{2n-2}_{c} \partial_x \varphi_c - \left( \sum_{k=1}^{n-1} \frac{(-1)^k+1}{2k+1} C_{n-1}^k \varphi^{2n-2k-1}_c (\partial_x \varphi_c)^{2k+1}_c \right) \phi \right] dxdt \quad (2.4) \]
\[ = -\int_{0}^{+\infty} \int_{\mathbb{R}} \phi \text{sgn}(x-ct) \varphi_c \left[ c - \left( 1 - \sum_{k=1}^{n-1} \frac{(-1)^k+1}{2k+1} C_{n-1}^k \right) \varphi^{2n-1}_c \right] dxdt. \]
Substituting (1.13) into the above integrands, one derives that

\[ \text{sgn}(x-ct)\varphi_c\left[c - \left(1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\right)\varphi_c^{2n-1}\right] = a^{2n} \left(1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\right)\left(e^{ct-x} - e^{2n(ct-x)}\right) \]

for \( x > ct \) and

\[ \text{sgn}(x-ct)\varphi_c\left[c - \left(1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\right)\varphi_c^{2n-1}\right] = -a^{2n} \left(1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\right)\left(e^{x-ct} - e^{2n(x-ct)}\right) \]

for \( x \leq ct \), where one can easily verify that \( \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n}^k = 1 - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k \).

Substituting \( \varphi_c \) in the remainder terms in Definition 1.1 and employing (2.2) lead to

\[
\int_{0}^{\infty} \int_{\mathbb{R}} \left[p * \left(\varphi_c^{2n} + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n - 2k + 1)}{2n(2k-1)} C_{n}^k\varphi_c^{2n-2k}(\partial_x \varphi_c)^{2k}\right) \cdot \partial_x \phi \right] \, dx \, dt 
- p * \left(\sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\varphi_c^{2n-2k-1}(\partial_x \varphi_c)^{2k+1}\right) \cdot \phi \right] \, dx \, dt 
+ p * \left(2n \varphi_c^{2n-1}\partial_x \varphi_c + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^k\varphi_c^{2n-2k-1}(\partial_x \varphi_c)^{2k+1}\right) \cdot \phi \right] \, dx \, dt 
= - \int_{0}^{\infty} \int_{\mathbb{R}} \phi \cdot p_x * \left[\sum_{k=1}^{n} \frac{(-1)^{k-1}(2n - 2k + 1)}{2n(2k-1)} C_{n}^k\varphi_c^{2n-2k}(\partial_x \varphi_c)^{2k}\right] \, dx \, dt 
+ \left(1 + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2n(2k+1)} C_{n-1}^k\right) \varphi_c^{2n} \, dx \, dt.
\]

Using \( \partial_x p(x) = -\frac{1}{2} \text{sgn}(x)e^{-|x|} \) for \( x \in \mathbb{R} \), one finds the integrand in the above(after dropping the factor \( \phi \)) is

\[
-\frac{a^{2n}}{2} \left(1 + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2n(2k+1)} C_{n-1}^k + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n - 2k + 1)}{2n(2k-1)} C_{n}^k\right) \int_{\mathbb{R}} \text{sgn}(x-y)e^{-|x-y|-2n|y-ct|} \, dy (2.7)
\]

When \( x > ct \), we have

\[
(2.7) = -\frac{a^{2n}}{2} \left(1 + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2n(2k+1)} C_{n-1}^k + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n - 2k + 1)}{2n(2k-1)} C_{n}^k\right) \times \left(\int_{-\infty}^{ct} + \int_{ct}^{+\infty} \right) \text{sgn}(x-y)e^{-|x-y|-2n|y-ct|} \, dy. (2.8)
\]
Direct computation yields
\[ \int_{-\infty}^{x} \text{sgn}(x-y)e^{-|x-y|-2n|y-ct|}dy = \frac{1}{2n+1}e^{ct-x}, \]
\[ \int_{ct}^{x} \text{sgn}(x-y)e^{-|x-y|-2n|y-ct|}dy = \frac{1}{2n-1}[e^{ct-x} - e^{2n(ct-x)}], \]
\[ \int_{x}^{+\infty} \text{sgn}(x-y)e^{-|x-y|-2n|y-ct|}dy = -\frac{1}{2n+1}e^{2n(ct-x)}. \]

Plugging the above into (2.8) yields for \( \pm (x-ct) \geq 0 \)
\[ (2.7) = \frac{a^{2n}}{\pm(1-4n^2)} \left( 2n + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k} + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n-2k+1)}{2k-1} C_{n}^{k} \right) [e^{\pm(ct-x)} - e^{\pm2n(ct-x)}]. \]

Moreover, one can prove the identity
\[ \frac{1}{4n^2-1} \left( 2n + \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k} + \sum_{k=1}^{n} \frac{(-1)^{k-1}(2n-2k+1)}{2k-1} C_{n}^{k} \right) = 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k}, \]
(2.10)
according to the relations [21]
\[ \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} C_{n}^{k} = \frac{(2n)!!}{(2n-1)!!} - 1, \quad \sum_{k=0}^{n} \frac{(-1)^{k}}{2k+1} C_{n}^{k} = 1 - \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k} = \frac{(2n)!!}{(2n+1)!!}. \]

Gathering (2.5)–(2.6) and (2.9)–(2.10), one deduces for any \( \phi(t,x) \in C_{c}^{\infty}([0,\infty) \times \mathbb{R}) \) that
\[ \int_{0}^{+\infty} \int_{\mathbb{R}} \varphi_{c} \phi_{t} + \frac{1}{2n} \varphi_{c} \varphi_{x} + \left( \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k} \varphi_{c}^{2n-2k-1}(\partial_{x} \varphi_{c})^{2k+1} \right) \phi \]
\[ + p \left( \varphi_{c}^{2n} + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}(2n-2k+1)}{2n(2k-1)} C_{n}^{k} \varphi_{c}^{2n-2k-1}(\partial_{x} \varphi_{c})^{2k} \right) \cdot \partial_{x} \phi \]
\[ - p \left( \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{2k+1} C_{n-1}^{k} \varphi_{c}^{2n-2k-1}(\partial_{x} \varphi_{c})^{2k+1} \right) \cdot \phi \]dxdt + \int_{\mathbb{R}} \varphi_{0,c}(x) \phi(0,x)dx = 0.

We thus complete the proof of Proposition 1.1. \( \square \)

We next prove that the quantity \( F(u) \) given by Eq. (1.8) is independent of time.
To do this, we set \( v(x,t) = \int_{-\infty}^{x} u_{t}(z,t)dz \) and employ integration by parts to obtain
\[ \frac{d}{dt} \int_{\mathbb{R}} u^{2n+1}dx = (2n+1) \int_{\mathbb{R}} u^{2n}v_{x}dx = -(2n+1) \int_{\mathbb{R}} (2n)u^{2n-1}u_{x}vdx, \]
(2.11)
and
\[
\frac{d}{dt} \int_{\mathbb{R}} nu^{2n-1}u_x^2 \, dx = n \int_{\mathbb{R}} [(2n-1)u^{2n-2}u_x^2 v_x + 2u^{2n-1}u_x v_{xx}] \, dx
\]
\[
= n \int_{\mathbb{R}} [(2n-1)(2n-2)u^{2n-3}u_x^4 + 4(2n-1)u^{2n-2}u_x u_{xx} + 2u^{2n-1}u_{xxx}] \, vdx.
\] (2.12)

In a similar manner,
\[
\frac{d}{dt} \int_{\mathbb{R}} \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k u^{2n-2k+1}u_x^{2k} \, dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \int_{\mathbb{R}} [(2n-2k+1)u^{2n-2k}u_x^{2k} u_t + 2ku^{2n-2k+1}u_x^{2k-1}u_{tx}] \, dx
\]
\[
= \sum_{k=2}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \int_{\mathbb{R}} [(2n-2k+1)(2n-2k)u^{2n-2k-1}u_x^{2k+1}
\]
\[+ 4k(2n-2k+1)u^{2n-2k}u_x^{2k-1}u_{xx} + 2k(2k-2)u^{2n-2k+1}u_x^{2k-3}u_{xx}^2 + 2ku^{2n-2k+1}u_x^{2k-2}u_{xxx}] \, vdx.
\] (2.13)

Combining (2.11)–(2.13), one finds
\[
\frac{dF(u)}{dt} = \int_{\mathbb{R}} \left[ \sum_{k=2}^{n-1} (-1)^{k+1} C_n^k (2n-2k+1)(2n-2k)u^{2n-2k-1}u_x^{2k+1}
\]
\[+ \sum_{k=1}^{n-1} (-1)^k C_n^{k+1} 4(k+1)(2n-2k-1)u^{2n-2k-2}u_x^{2k+1}u_{xx}
\]
\[+ \sum_{k=0}^{n-2} (-1)^{k+1} C_n^{k+2} 2(k+2)(2k+2)u^{2n-2k-3}u_x^{2k+1}u_{xx}^2
\]
\[+ \sum_{k=1}^{n-1} (-1)^k C_n^{k+1} 2(k+1)u^{2n-2k-1}u_x^{2k}u_{xxx} \right] \, vdx
\]
\[
= \int_{\mathbb{R}} \left[ \sum_{k=0}^{n-1} (-1)^{k+1} C_{n-1}^k (2n-2k+1)2nu^{2n-2k-1}u_x^{2k+1}
\]
\[+ \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k 4n(2n-2k-1)u^{2n-2k-2}u_x^{2k+1}u_{xx}
\]
\[+ \sum_{k=0}^{n-1} (-1)^{k+1} C_{n-1}^k 4n(n-k-1)u^{2n-2k-3}u_x^{2k+1}u_{xx}^2
\]
\[+ \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k 2nu^{2n-2k-1}u_x^{2k}u_{xxx} \right] \, vdx,
\] (2.14)
which along with (1.11) leads to
\[
\frac{dF(u)}{dt} = 2n \int_{\mathbb{R}} (u_t - u_{txx}) v dx = \int_{\mathbb{R}} 2n (vv_x - vv_{xxx}) dx = 0.
\]

3. Orbital stability of peakon solutions

In this section, we prove the orbital stability of peakons for Eq. (1.1). The proof will be divided into several lemmas. One should first notice that under the assumption of Proposition 1.1, Eq. (1.1) admits a unique local positive solution of by the use of Lemmas A.1–A.3. Also, it is easily to see that the function \( \varphi_c(x) = a\varphi(x) = ae^{-|x|} \in H^1(\mathbb{R}) \) takes its maximum at \( x = 0 \), i.e.,
\[
\max_{x \in \mathbb{R}} \{ \varphi_c(x) \} = \varphi_c(0) = a.
\]

Simple calculation yields
\[
E(\varphi_c) = \| \varphi_c \|_{H^1}^2 = a^2 \int_{\mathbb{R}} (\varphi^2 + \varphi_x^2) dx = 2a^2
\]
and
\[
F(\varphi_c) = a^{2n+1} \int_{\mathbb{R}} \left( \varphi^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^{2n-2k+1} \varphi_x^{2k} \right) dx = \frac{2a^{2n+1}}{2n+1} \left( 1 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^{2k} \right),
\]
where \( a \) is determined implicitly by \( c = \sum_{k=0}^{n} \frac{(-1)^{k}}{2k+1} \frac{2n+1}{2n} C_n^{2n-2k-1} \) provided by Theorem 1.1.

Proof of Proposition 1.2. First, one easily verifies \( \varphi - \partial_x^2 \varphi = 2\delta \) with \( \delta \) being the Dirac distribution. Consequently, we find with the help of integration by parts and (3.2) that
\[
\| u - \varphi_c(\cdot - \xi) \|_{H^1}^2 = \int_{\mathbb{R}} \left( u^2 + u_x^2 \right) dx + \int_{\mathbb{R}} \left( \varphi_c^2 + (\partial_x^2 \varphi_c)^2 \right) dx - 2a \int_{\mathbb{R}} u(x) \varphi(x - \xi) dx - 2a \int_{\mathbb{R}} u(x) \varphi(x - \xi) dx
\]
\[
= E(u) + E(\varphi_c(\cdot - \xi)) - 2a \int_{\mathbb{R}} \left( 1 - \partial_x^2 \right) \varphi(x - \xi) u(x) dx
\]
\[
= E(u) + E(\varphi_c(\cdot - \xi)) - 4a \int_{\mathbb{R}} \delta(x - \xi) u(x) dx
\]
\[
= E(u) + E(\varphi_c(\cdot - \xi)) - 4au(\xi)
\]
\[
= E(u) - E(\varphi_c(\cdot - \xi)) - 4a(u(\xi) - a).
\]

This completes the proof of this Proposition 1.2.

Proof of Proposition 1.3. Let the maximus of \( u(t, x) \) be taken at \( x = \xi(t) \), i.e., \( M(t) = u(t, \xi(t)) \). We define the same function \( g \) as in [11]
\[
g(t, x) \triangleq \begin{cases} u(t, x) - u_x(t, x), & x < \xi(t), \\ u(t, x) + u_x(t, x), & x > \xi(t). \end{cases}
\]
Then, it is easy to know that
\[
\int_{\mathbb{R}} g^2(t, x) dx = E(u) - 2M^2(t).
\]
Let a function $h(t, x)$ be

$$h(t, x) \triangleq \begin{cases} u^{2n-1} + \sum_{k=1}^{2n-2} c_k u^{2n-1-k} u_x^k (t, x), & x < \xi(t), \\ u^{2n-1} + \sum_{k=1}^{2n-2} d_k u^{2n-1-k} u_x^k (t, x), & x > \xi(t), \end{cases}$$

where $c_k, d_k (k = 1, 2, \cdots, 2n - 2)$, are constants given by

$$\begin{cases} c_1 = -d_1 = \frac{1}{2} + \sum_{j=1}^{n} (-1)^{j+1} \frac{2j-3}{2j-1} C_n^j, \\ c_{2m} = d_{2m} = \sum_{j=m+1}^{n} (-1)^{j+1} \frac{2j-(2m+1)}{2j-1} C_n^j, & m = 1, 2, \cdots, n - 1, \\ c_{2m-1} = -d_{2m-1} = \sum_{j=m+1}^{n} (-1)^{j+1} \frac{2j-m}{2j-1} C_n^j, & m = 1, 2, \cdots, n - 1. \end{cases} \tag{3.6}$$

With $c_k$ and $d_k (k = 1, 2, \cdots, 2n - 2)$ defined above, one easily verifies that

$$\begin{cases} c_{2n-2} - \frac{(-1)^{n+1}}{2n-1} = 0, \\ c_{2n-3} - 2c_{2n-2} = 0, \\ c_k - 2c_{k-1} + c_{k-2} = 0, & k = 2j + 1, (j = 1, 2, \cdots, n - 2), \\ c_k - 2c_{k-1} + c_{k-2} = \frac{(-1)^{j+1}}{2j-1} C_n^j, & k = 2j, (j = 2, 3, \cdots, n - 1), \\ c_2 - 2c_1 + 1 = C_n^1 \end{cases} \tag{3.7}$$

and

$$\begin{cases} \frac{(-1)^{n+1}}{2n-1} + d_{2n-2} = 0, \\ d_{2n-3} + 2d_{2n-2} = 0, \\ d_k + 2d_{k-1} + d_{k-2} = 0, & k = 2j + 1, (j = 1, 2, \cdots, n - 2), \\ d_k + 2d_{k-1} + d_{k-2} = \frac{(-1)^{j+1}}{2j-1} C_n^j, & k = 2j, (j = 2, 3, \cdots, n - 1), \\ d_2 + 2d_1 + 1 = C_n^1 \end{cases} \tag{3.8}$$

Simple calculation then yields

$$\begin{align*}
(u^2 - 2uu_x + u_x^2) \left( u^{2n-1} + \sum_{k=1}^{2n-2} c_k u^{2n-1-k} u_x^k \right) & = u^{2n+1} + (c_2 - 2c_1 + 1) u^{2n-1} u_x + \sum_{k=3}^{2n-2} (c_k - 2c_{k-1} + c_{k-2}) u^{2n-k+1} u_x^k \\
& + (c_{2n-3} - 2c_{2n-2}) u^2 u_x^{2n-1} + c_{2n-2} u_x^{2n} \end{align*} \tag{3.9}$$

and

$$\begin{align*}
(u^2 + 2uu_x + u_x^2) \left( u^{2n-1} + \sum_{k=1}^{2n-2} d_k u^{2n-1-k} u_x^k \right) & = u^{2n+1} + (d_1 + 2) u^{2n} u_x + (d_2 + 2d_1 + 1) u^{2n-1} u_x + \sum_{k=3}^{2n-2} (d_k + 2d_{k-1} + d_{k-2}) u^{2n-k+1} u_x^k \\
& + (d_{2n-3} + 2d_{2n-2}) u^2 u_x^{2n-1} + d_{2n-2} u_x^{2n} \end{align*} \tag{3.10}$$
Combining (3.7)–(3.10), one finds
\[
\int_{\mathbb{R}} h(t, x)g^2(t, x)\,dx
\]
\[
= \int_{-\infty}^{\xi} (u - u_x)^2 \left( u^{2n-1} + \sum_{k=1}^{2n-2} c_k u^{2n-1-k} u_x^k \right) \,dx + \int_{\xi}^{\infty} (u + u_x)^2 \left( u^{2n-1} + \sum_{k=1}^{2n-2} d_k u^{2n-1-k} u_x^k \right) \,dx
\]
\[
= \int_{-\infty}^{\xi} \left( u^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k u^{2n-2k+1} u_x^{2k} \right) \,dx
\]
\[
+ (c_1 - 2) \int_{-\infty}^{\xi} u^{2n} u_x \,dx + \int_{\xi}^{\infty} \left( u^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k u^{2n-2k+1} u_x^{2k} \right) \,dx + (d_1 + 2) \int_{\xi}^{\infty} u^{2n} u_x \,dx
\]
\[
= F(u) + \frac{2(c_1 - 2)}{2n+1} u^{2n+1}(t, \xi)
\]
\[
= F(u) - \frac{2(2 - c_1)}{2n+1} M^{2n+1}(t). \quad (3.11)
\]

It follows from Lemma A.3 that one knows
\[
u(t, x) \geq 0, \quad \text{and} \quad (u \pm u_x)(t, x) \geq 0 \quad \text{for} \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \quad (3.12)
\]

We will next prove
\[
h(t, x) \leq \frac{2 - c_1}{2} u^{2n-1}(t, x) \quad \text{for} \quad \forall (t, x) \in [0, T) \times \mathbb{R}, \quad (3.13)
\]
or equivalently,
\[
\sum_{k=1}^{2n-2} c_k u^{2n-1-k} u_x^k \leq -\frac{c_1}{2} u^{2n-1}, \quad \sum_{k=1}^{2n-2} d_k u^{2n-1-k} u_x^k \leq -\frac{c_1}{2} u^{2n-1}. \quad (3.14)
\]

Let \( z = u_x / u \). Then, it suffices to show the nonpositivity of the following function \( f(z) \)
\[
f(z) = \sum_{k=1}^{2n-2} c_k z^k + \frac{c_1}{2} \leq 0, \quad \text{for} \quad z \in [-1, 1] \quad (3.15)
\]
recalling (3.12).

Using \( c_{2j+1} - 2c_{2j} + c_{2j-1} = 0 \) \( j = 1, 2, \cdots, n - 2 \) and \( c_{2n-2} = \frac{c_{2n-3}}{2} \) stated in (3.7), we recast \( f(z) \) as
\[
f(z) = \sum_{j=1}^{n-1} \left( c_{2j} z^{2j} + c_{2j-1} z^{2j-1} \right) + \frac{c_1}{2}
\]
\[
= \frac{c_{2n-3}}{2} z^{2n-2} + \sum_{j=1}^{n-2} \left( \frac{c_{2j+1}}{2} z^{2j} + \frac{c_{2j-1}}{2} z^{2j} \right) + \sum_{j=1}^{n-1} c_{2j-1} z^{2j-1} + \frac{c_1}{2}
\]
\[
= \frac{(1 + z)^2}{2} \sum_{k=1}^{n-1} c_{2k-1} z^{2k-2}.
\]
Accordingly, the proof of (3.15) is equivalent to

\[ \phi(z) = \sum_{k=1}^{n-1} c_{2k-1} z^{2k-2} \leq 0, \quad z \in [-1, 1]. \] (3.16)

Since \( \phi(z) \) is even and continuous at \( z = 1 \), one only needs to prove (3.16) for \( z \in [0, 1) \). Using the expression of \( c_{2k-1}(k = 1, 2, \ldots, n-1) \) provided in (3.6) and exchanging the order of summation lead to

\[ \phi(z) = c_1 + \sum_{k=2}^{n-1} \sum_{j=k+1}^{n} (-1)^{j+1} \frac{2j - 2k}{2j - 1} C_n^j z^{2k-2} \]

(3.17)

One calculates \( \tilde{\phi}_1(z) \) as

\[ \tilde{\phi}_1(z) = \sum_{j=3}^{n} (-1)^{j+1} C_n^j \left( \frac{z^2 (1 - z^{2(j-2)})}{1 - z^2} \right) \]

(3.18)

Further computation generates

\[ \tilde{\phi}_{1,1}(z) = -\frac{z^2}{1 - z^2} \left( \sum_{j=0}^{n} (-1)^j C_n^j - \sum_{j=0}^{2} (-1)^j C_n^j \right) = \frac{z^2}{1 - z^2} \cdot \sum_{j=0}^{2} (-1)^j C_n^j = \frac{n^2 - 3n + 2}{2} \cdot \frac{z^2}{1 - z^2} \]

(3.19)

and

\[ \tilde{\phi}_{1,2}(z) = \frac{z^{-2}}{1 - z^2} \left( \sum_{j=0}^{n} (-z^2)^j C_n^j - \sum_{j=0}^{2} (-z^2)^j C_n^j \right) = \frac{z^{-2}}{1 - z^2} \left( (1 - z^2)^n - 1 + nz^2 - z^4 C_n^2 \right) \]

(3.20)

Consequently, combining Eqs. (3.18)–(3.20) produces

\[ \tilde{\phi}_1(z) = \frac{n^2 - 3n + 2}{2} \cdot \frac{z^2}{1 - z^2} + \frac{z^{-2}}{1 - z^2} \rho(z). \] (3.21)
For $\tilde{\phi}_2(z)$, direct computation yields

$$
\tilde{\phi}_2(z) = \frac{d}{dz} \left( \sum_{j=3}^{n} \sum_{k=2}^{j-1} (-1)^j C_n^j \frac{z^{2k-1}}{2j-1} \right)
$$

$$
= \frac{d}{dz} \left( \sum_{j=3}^{n} (-1)^j C_n^j \frac{z^3 (1 - z^2(j-2))}{1 - z^2} \right)
$$

$$
= \frac{d}{dz} \left( \frac{z^3}{1 - z^2} \cdot \sum_{j=3}^{n} (-1)^j C_n^j \frac{z^{2j-2}}{2j-1} - \frac{1}{1 - z^2} \sum_{j=3}^{n} (-1)^j C_n^j \frac{z^{2j-1}}{2j-1} \right)
$$

$$
\triangleq \frac{d}{dz} \left( \tilde{\Phi}_{2,1}(z) + \tilde{\Phi}_{2,2}(z) \right).
$$

Using similar method as (3.20), one derives

$$
\tilde{\Phi}_{2,1}(z) = \frac{z^3}{1 - z^2} \cdot \int_{0}^{1} \sum_{j=3}^{n} (-1)^j C_n^j z^{2j-2} dz
$$

$$
= \frac{z^3}{1 - z^2} \cdot \int_{0}^{1} \sum_{j=3}^{n} C_n^j (-z^2)^j \frac{dz}{z^2} = \frac{z^3}{1 - z^2} \int_{0}^{1} \frac{\rho(z)}{z^2} dz
$$

and

$$
\tilde{\Phi}_{2,2}(z) = \frac{1}{z^2 - 1} \cdot \int_{0}^{z} \sum_{j=3}^{n} (-z^2)^j C_n^j \frac{dz}{z^2} = \frac{1}{z^2 - 1} \cdot \int_{0}^{z} \frac{\rho(z)}{z^2} dz.
$$

Gathering (3.22)–(3.24) leads to

$$
\tilde{\phi}_2(z) = \frac{3z^2 \int_{0}^{1} \frac{\rho(s)}{s^2} ds - \frac{\rho(z)}{z^2}}{1 - z^2} + \frac{2z \left( z^3 \int_{0}^{1} \frac{\rho(s)}{s^2} ds - \int_{0}^{z} \frac{\rho(s)}{s^2} ds \right)}{(1 - z^2)^2}.
$$

Similar as (3.23), we invoke (3.6) to compute $c_1$ to find

$$
c_1 = \frac{1}{2} + \sum_{j=1}^{n} (-1)^{j+1} C_n^j \frac{C_n^j}{2j - 1} - \sum_{j=1}^{n} (-1)^{j+1} C_n^j \frac{C_n^j}{2j - 1}
$$

$$
= 1 + \sum_{j=0}^{n} (-1)^{j+1} C_n^j \frac{C_n^j}{2j - 1} + \sum_{j=1}^{n} (-1)^{j} C_n^j \frac{C_n^j}{2j - 1} = 1 - n + C_n^2 + \int_{0}^{1} \frac{\rho(s)}{s^2} ds.
$$

Employing the definition of $\omega(z)$ in (3.20), one finds

$$
\int_{0}^{1} \frac{\rho(s)}{s^2} ds = -\int_{0}^{n-1} (1 - z^2)^k dz + \int_{0}^{1} \left( n - z^2 C_n^2 \right) dz
$$

$$
= -\sum_{k=1}^{n-1} \int_{0}^{1} (1 - z^2)^k dz + (n - 1) - \frac{C_n^2}{3} \triangleq -B + (n - 1) - \frac{C_n^2}{3}.
$$
It follows from (3.17), (3.21) and (3.25)–(3.27) that one has

\[
\phi(z) = -B + \frac{2(n+1)}{1-z^2} z^2 + \frac{2z^4 (-B + (n-1) - \frac{1}{3} C_n^2)}{(1-z)^2} + \frac{2z \int_0^z \left( \sum_{k=0}^{n-1} (1-s^2)^k - n + s^2 C_n^2 \right) ds}{(1-z)^2}
\]

\[
\begin{align*}
&= -B - B z^2 + 2z \sum_{k=1}^{n-1} (1-s^2)^k ds \\
&\leq -B - B z^2 + \frac{2z B}{(1-z)^2} = -B \frac{(1+z)^2}{(1-z)^2} \leq 0
\end{align*}
\]

implying (3.16) for \( z \in [0, 1] \). Then, using (3.6) and (3.8), after a similar discussion, one finds that (3.14) is equivalent to \( \sum_{k=1}^{n-1} c_{2k-1} z^{2k-2} (z-1)^2 \leq 0 \), which is obviously true by (3.16). Therefore, we complete the proof of (3.13). Gathering (3.5), (3.11) and (3.13) generates

\[
F(u) - \frac{2(2-c_1)}{2n+1} M^{2n+1}(t) = \int h(t, x) g^2(t, x) dx
\]

\[
\leq \frac{2-c_1}{2} M^{2n-1}(t) \int g^2(t, x) dx = \frac{2-c_1}{2} (E(u) - 2M^2(t)) M^{2n-1}(t)
\]

indicating (1.15). This completes the proof of Proposition 1.3.

\[\square\]

**Proof of Proposition 1.4.** Since \( 0 < \varepsilon < (\gamma - 2\sqrt{2})a, \gamma > 2\sqrt{2} \) and \( \|\varphi_c\|_{H^1}^2 = 2a^2 \) given by Eq. (3.2), one easily deduces

\[
|E(u) - E(\varphi_c)| = |(\|u\|_{H^1} - \|\varphi_c\|_{H^1}) (\|u\|_{H^1} + \|\varphi_c\|_{H^1})|
\]

\[
\leq \|u - \varphi_c\|_{H^1} (\|u - \varphi_c\|_{H^1} + 2\|\varphi_c\|_{H^1}) \leq \varepsilon (\varepsilon + 2\sqrt{2}a) < a\gamma \varepsilon.
\]

Moreover, we find

\[
|F(u) - F(\varphi_c)|
\]

\[
= \left| \int u^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k u^{2n-2k+1} x^2 \right| dx - \left| \int \varphi_c^{2n+1} + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \varphi_c^{2n-2k+1} (\partial_x \varphi_c)^{2k} \right| dx
\]

\[
\leq \left| \int u^{2n+1} + nu^{2n-1} x - \varphi_c^{2n+1} + n\varphi_c^{2n-1} (\partial_x \varphi_c)^2 \right| dx
\]

\[
+ \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \left| \int u^{2n-2k+1} x^2 - \varphi_c^{2n-2k+1} (\partial_x \varphi_c)^{2k} \right| dx
\]

\[
\leq \left| \int u^{2n-1} - \varphi_c^{2n-1} \right| \left| u^2 + nu^2 \right| dx + \left| \int \varphi_c^{2n-1} \left( \left( u^2 - \varphi_c^2 \right) + n \left( u^2 - (\partial_x \varphi_c)^2 \right) \right) dx
\]

\[
+ \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \left| \int u^{2n-2k+1} x^2 - (\partial_x \varphi_c)^{2k} \right| dx + \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \left| \int (\partial_x \varphi_c)^{2k} \left( u^{2n-2k+1} - \varphi_c^{2n-2k+1} \right) dx
\]

\[\triangleq A_1 + A_2 + A_3 + A_4.\]
We will estimate $A_1 - A_4$ term by term. We employ Eq. (3.28) to control the term $A_1$ as

$$A_1 \leq n \int \left| u - \varphi_c \right| \cdot \left| u^{2n-2} + u^{2n-3} \varphi_c + \cdots + u \varphi_c^{2n-2} + \varphi_c^{2n-2} \right| \cdot \left( u^2 + u_x^2 \right) \, dx$$

$$\leq n E(u) \| u - \varphi_c \|_{L^\infty} \left( \| u \|_{L^\infty}^{2n-2} + \| u \|_{L^\infty}^{2n-3} \| \varphi_c \|_{L^\infty} + \cdots + \| u \|_{L^\infty} \| \varphi_c \|_{L^\infty}^{2n-3} + \| \varphi_c \|_{L^\infty}^{2n-2} \right)$$

$$\leq \frac{n}{2n-1} \left( E(\varphi_c) + a \gamma e \right) \| u - \varphi_c \|_{H^1} \left( \| u \|_{H^1}^{2n-2} + \| u \|_{H^1}^{2n-3} \| \varphi_c \|_{H^1} + \cdots + \| u \|_{H^1} \| \varphi_c \|_{H^1}^{2n-3} + \| \varphi_c \|_{H^1}^{2n-2} \right)$$

$$\leq \frac{n}{2n-1} \left( E(\varphi_c) + a \gamma e \right) \left( \| u - \varphi_c \|_{H^1} + 2 \| \varphi_c \|_{H^1} \right)^{2n-2} \| u - \varphi_c \|_{H^1}$$

$$\leq \frac{n}{2n-1} \varepsilon \left( 2a^2 + a \gamma e \right) (\varepsilon + 2\sqrt{2a})^{2n-2},$$

where we have also used the following inequality

$$\sup_{x \in \mathbb{R}} |v(x)| \leq \frac{\sqrt{E(v)}}{\sqrt{2}} \leq \frac{\|v\|_{H^1}}{\sqrt{2}}, \quad \text{for} \quad v \in H^1(\mathbb{R})$$

coming from (3.5).

A similar argument applied for $A_2$ leads to

$$A_2 \leq \| \varphi_c \|_{H^1}^{2n-1} \left( \int \left( \left( \frac{2n-2}{2n-1} \right) \left( u - \varphi_c \right)^2 + n \left( u_x - \partial_x \varphi_c \right)^2 + 2 \varphi_c \left( u - \varphi_c \right) + 2n \partial_x \varphi_c \left( u_x - \partial_x \varphi_c \right) \right) \, dx \right)$$

$$\leq \left( \frac{\| \varphi_c \|_{H^1}}{\sqrt{2}} \right)^{2n-1} \left( (n+1) \| u - \varphi_c \|_{H^1}^2 + 2(n+1) \| \varphi_c \|_{H^1} \| u - \varphi_c \|_{H^1} \right)$$

$$\leq (n+1) a^{2n-1} \varepsilon (\varepsilon + 2\sqrt{2a}).$$

The term $A_3$ can be handled by

$$A_3 \leq \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \|u\|_{L^\infty}^{2n-2k+1} \left( \int \left( u_x^k + \partial_x \varphi_c \right)^{k-1} \left( u_x^{k-1} + u_x^{k-2} \partial_x \varphi_c + \cdots + u_x \left( \partial_x \varphi_c \right)^{k-2} \right) \, dx \right)$$

$$\leq \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \|u\|_{L^\infty}^{2n-2k+1} \left( \int u_x^{4k-2} + 2u_x^{4k-3} \partial_x \varphi_c + \cdots + (2k-1) u_x^{2k} \left( \partial_x \varphi_c \right)^{2k-2} + 2ku_x^{2k-1} \left( \partial_x \varphi_c \right)^{2k-1} + (2k-1) u_x^{2k-2} \left( \partial_x \varphi_c \right)^{2k} + \cdots + 2u_x \left( \partial_x \varphi_c \right)^{4k-3} \right)$$

$$\leq \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \|u\|_{L^\infty}^{2n-2k+1} \cdot \sqrt{2k} \left( \int u_x^{4k-2} \, dx + \int \left( \partial_x \varphi_c \right)^{4k-2} \, dx \right)^{\frac{1}{2}} \cdot \| u - \varphi_c \|_{H^1}$$

$$\leq \frac{n}{2n-1} \varepsilon \left( 2a^2 + a \gamma e \right) (\varepsilon + 2\sqrt{2a})^{2n-2},$$

$$\leq G \left( \|u\|_{H^1}, n, c \right) \varepsilon,$$
where we have used the Gagliardo–Nirenberg inequality \( \|u_x\|_{L^{4k-2}}^{4k-2} \leq C \|u\|_{L^2}^{k} \|u_{xx}\|_{L^2}^{3k-2} \) and the fact \( \|\partial_x \varphi_c\|_{L^{4k-2}} = a_k^{4k-2} \).

For the term \( A_4 \), the Hölder inequality produces

\[
A_4 \leq \sum_{k=2}^{n} \frac{C_n^k}{2k-1} \left( \|u\|_{L^2}^{2n-2k} + \|u\|_{L^\infty}^{2n-2k-1} \|\varphi_c\|_{L^\infty} + \cdots + \|u\|_{L^\infty} \|\varphi_c\|_{L^\infty}^{2n-2k-1} + \|\varphi_c\|_{L^\infty}^{2n-2k} \right)
\]

This completes the proof of Proposition 1.4.

**Proof of Proposition 1.5.** We first derive from (1.15) that

\[
(2n-1)M^{2n+1} - \frac{2n+1}{2} M^{2n-1} E(u) + \frac{2n+1}{2} F(u) \leq 0. \tag{3.31}
\]

Let \( Q(y) = (2n-1)y^{2n+1} - \frac{2n+1}{2} y^{2n-1} E(u) + \frac{2n+1}{2} F(u) \). For the case of \( E(u) = E(\varphi_c) = 2a^2 \) and \( F(u) = F(\varphi_c) = \frac{2}{2n+1} a^{2n+1} \left( 1 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \right) \), \( Q(y) \) reduces to

\[
\dot{Q}(y) = \frac{2}{2n+1} a^{2n+1} \left( 1 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \right). \tag{3.32}
\]

where the relation

\[
2 - c_1 = 1 + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} C_n^k \tag{3.33}
\]

is used since \( \sum_{k=1}^{n} (-1)^{k+1} C_n^k = 1 \).

Then, it follows from (3.31) and (3.32) that one has

\[
\dot{Q}(M) \leq \dot{Q}(M) - Q(M) = \frac{2n+1}{2} M^{2n-1} (E(u) - E(\varphi_c)) - \frac{2n+1}{2} (F(u) - F(\varphi_c)),
\]

which along with (3.32) generates

\[
2a^{2n-1}(M - a)^2 \leq \frac{2n+1}{2} M^{2n-1} (E(u) - E(\varphi_c)) - \frac{2n+1}{2} (F(u) - F(\varphi_c)). \tag{3.34}
\]
Thanks to (3.5) and the assumption of this lemma, one derives for $0 < \varepsilon < (\gamma - 2\sqrt{2})a$ ($\gamma > 2\sqrt{2}$) that

$$0 < M^2 \leq \frac{E(u)}{2} \leq \frac{2a^2 + a\gamma \varepsilon}{2} < \frac{(\gamma - \sqrt{2})a^2}{2}. \tag{3.35}$$

Gathering (3.34) and (3.35) gives rise to

$$\sqrt{2}a^{n-1/2}|M - a| \lesssim \sqrt{\frac{(2n + 1)(\gamma - \sqrt{2})^n}{2^n\sqrt{2(\gamma - \sqrt{2})}}} a^{2n\varepsilon} + \frac{2n + 1}{2} G(n, c, \|u\|_{H^s}) \varepsilon. \tag{3.37}$$

Combing this inequality and the relation $c = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} C_k^2 C^2 a^{2n-1}$, one concludes that there exists a constant, still expressed via $G(n, c, \|u\|_{H^s})$, such that

$$|M - a| \lesssim \sqrt{G(n, c, \|u\|_{H^s})}\varepsilon.$$

This completes the proof of Proposition 1.5. \hfill $\Box$

We finally prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $u \in C([0, T); H^s(\mathbb{R})) \ (s > \frac{5}{2})$ satisfy Eq. (1.1) with the initial condition $u_0 \in H^s(\mathbb{R})$. Moreover, one can know

$$E(u(t, \cdot)) = E(u_0), \quad F(u(t, \cdot)) = F(u_0), \quad \forall t \in [0, T). \tag{3.36}$$

Since $\|u(0, \cdot) - \varphi_c\|_{H^s} < \varepsilon$, with $0 < \varepsilon < (\gamma - 2\sqrt{2})a$, and $0 \neq (1 - \partial_x^2)^{T} u_0 \geq 0$, in view of (3.36) and Proposition 1.4, the hypotheses of Proposition 1.5 are satisfied for $u(t, \cdot)$ with a chosen positive constant $G(n, c, \|u_0\|_{H^s})$ depending only on the wave speed $c$, $n \in \mathbb{N}^+$ and $\|u_0\|_{H^s}$. Accordingly

$$|u(t, \eta(t)) - a| \lesssim \sqrt{G(n, c, \|u_0\|_{H^s})} \varepsilon, \quad \forall t \in [0, T). \tag{3.37}$$

where $x = \eta(t) \in \mathbb{R}$ stands for the maximum point of function $u(t, x)$. Consequently, we conclude from (3.37) and Proposition 1.4 that for $t \in [0, T]$,

$$\|u - \varphi_c(\cdot - \eta(t))\|_{H^1} \leq \sqrt{|E(u_0) - E(\varphi_c)| + 4a|u(t, \eta(t)) - a|} \lesssim \sqrt{a\gamma \varepsilon + 4a \sqrt{G(c, \|u_0\|_{H^s})} \varepsilon.}$$

We thus complete the proof of Theorem 1.1. \hfill $\Box$

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**Appendix A:** Some basics properties of Eq. (1.1) with the initial condition $u(0, x) = u_0(x)$

**Lemma A.1.** \cite{28} Let $u_0(x) = u(0, x) \in H^s(\mathbb{R})$ with $s > 5/2$. Then, there exists a time $T > 0$ such that the Cauchy problem of Eq. (1.1) with initial data $u_0$ has a unique strong solution $u(t, x) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ and the map $u_0 \mapsto u$ is continuous from a neighborhood of $u_0$ in $H^s(\mathbb{R})$ into $C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).$
For the characteristic equation
\[
\begin{cases}
\frac{dr(t,x)}{dt} = \left( u \left( u^2 - u_x^2 \right)^{n-1} \right) (t, r(t,x)), x \in \mathbb{R}, \quad t \in [0, T) \\
r(0,x) = x, x \in \mathbb{R},
\end{cases}
\]
there exists the following lemma:

Lemma A.2. [28] Suppose \( u_0 \in H^s(\mathbb{R}) \) with \( s > 5/2 \), and let \( T > 0 \) be the maximal existence time of the strong solution \( u(t, x) \in C \left( [0, T), H^s(\mathbb{R}) \right) \cap C^1 \left( [0, T), H^{s-1}(\mathbb{R}) \right) \) to the Cauchy problem (1.1). Then, (A.3) has a unique solution \( r \in C^1 \left( [0, T) \times \mathbb{R}, \mathbb{R} \right) \) such that \( r(t, \cdot) \) is an increasing diffeomorphism over \( \mathbb{R} \) with
\[
r_x(t, x) = \exp \left( \int_0^t \left( u_x \left( u^2 - u_x^2 \right)^{n-1} + 2(n-1)uu_x \left( u^2 - u_x^2 \right)^{n-2} m \right) (s, r(s,x)) ds \right) > 0
\]
for all \((t, x) \in [0, T) \times \mathbb{R}\).

Furthermore, the momentum density \( m = u - u_{xx} \) satisfies
\[
m(t, r(t,x))r_x(t,x) = m_0(x) \exp \left[ -\int_0^t \left( u_x \left( u^2 - u_x^2 \right)^{n-1} \right) (s, r(s,x)) ds \right],
\]
for all \((t, x) \in [0, T) \times \mathbb{R}\), which implies that the \text{sgn} and zeros of \( m \) are preserved under the flow.

Based on the above lemma, there was the following property:

Lemma A.3. [28] Assume \( u_0(x) \in H^s(\mathbb{R}), s > 5/2 \). If \( m_0(x) = (1 - \partial_x^2)u_0(x) \) does not change \text{sgn}, then \( y(t,x) \) will not change \text{sgn} for all \( t \in [0, T) \). It follows that if \( m_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of equation (1.1) is positive for \((t, x) \in [0, T) \times \mathbb{R}\). Furthermore, if \( m_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of equation (1.1) satisfies
\[
(1 \pm \partial_x)u(t,x) \geq 0, \quad \text{for} \quad \forall (t,x) \in [0, T) \times \mathbb{R}.
\]

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