Abstract

We find new inequalities between uniform and individual Diophantine exponents for three-dimensional Diophantine approximations. Also we give a result for two linear forms in two variables. The results improves V.Jarník’s theorem (1954).

1. V.Jarník’s theorem. This paper deals with real numbers only. We consider a matrix

$$
\Theta = \begin{pmatrix}
\theta_1^1 & \cdots & \theta_1^m \\
\vdots & \ddots & \vdots \\
\theta_n^1 & \cdots & \theta_n^m
\end{pmatrix},
$$

and a system of linear forms

$$
L_j(x) = \sum_{i=1}^{m} \theta_j^i x_i, \quad x = (x_1, ..., x_m).
$$

Let

$$
\psi_{\Theta}(t) = \min_{x \in \mathbb{Z}^m: 0 < M(x) \leq t} \max_{1 \leq j \leq n} ||L_j(x)||, \quad M(x) = \max_{1 \leq j \leq m} |x_i|
$$

Suppose that $\psi_{\Theta}(t) > 0$ for all $t > 0$ and define the uniform Diophantine exponent $\alpha(\Theta)$ as the supremum of the set

$$
\{ \gamma > 0 : \limsup_{t \to +\infty} t^\gamma \psi_{\Theta}(t) < +\infty \},
$$

The individual Diophantine exponent $\beta(\Theta)$ is defined as the supremum of the set

$$
\{ \gamma > 0 : \liminf_{t \to +\infty} t^\gamma \psi_{\Theta}(t) < +\infty \}.
$$

In [1] V.Jarník proved the following result.

**Theorem 1.** Suppose that $\psi_{\Theta}(t) > 0$ for all $t > 0$.

(i) Consider the case $m = 1$ and suppose that among numbers $\theta_1^1, ..., \theta_n^1$ there exist at least two numbers which are linearly independent together with 1 over $\mathbb{Z}$. Then

$$
\beta(\Theta) \geq \frac{(\alpha(\Theta))^2}{1 - \alpha(\Theta)}.
$$

In [1] V.Jarník proved the following result.

**Theorem 1.** Suppose that $\psi_{\Theta}(t) > 0$ for all $t > 0$.

(i) Consider the case $m = 1$ and suppose that among numbers $\theta_1^1, ..., \theta_n^1$ there exist at least two numbers which are linearly independent together with 1 over $\mathbb{Z}$. Then

$$
\beta(\Theta) \geq \frac{(\alpha(\Theta))^2}{1 - \alpha(\Theta)}.
$$

---

1 Research is supported by RFBR grant № 09-01-00371a and by Scientific School grant H13-691.2008.1
(ii) Consider the case $m = 2$. Then
\[ \beta(\Theta) \geq \alpha(\Theta)(\alpha(\Theta) - 1). \quad (2) \]

(iii) Consider the case $m > 2$. Suppose that $\alpha(\Theta) > (5m^2)^{m-1}$. Then
\[ \beta(\Theta) \geq (\alpha(\Theta))^m - 3\alpha(\Theta). \quad (3) \]

It is a well-known fact that under the conditions of Theorem 1 one has
\[ \frac{m}{n} \leq \alpha(\Theta) \leq \beta(\Theta) \leq +\infty. \]
Moreover for $m = 1$ one has
\[ \frac{1}{n} \leq \alpha(\Theta) \leq 1. \]

In [2] M. Laurent proved that in the cases $m = 1, n = 2$ and $m = 2, n = 1$ the inequalities of V. Jarník’s theorem cannot be improved.

In the present note we give a sketched proof of an improvement of V. Jarník’s theorem in the cases $m = 1, n = 3$, $m = 3, n = 1$ and $m = n = 2$.

2. Results. For $\alpha > 0$ put
\[ g_1(\alpha) = \frac{\alpha(1 - \alpha) + \sqrt{\alpha^2(1 - \alpha)^2 + 4\alpha(2\alpha^2 - 2\alpha + 1)}}{2(2\alpha^2 - 2\alpha + 1)}. \]

The value $g_1(\alpha)$ is the largest root of the equation
\[ (2\alpha^2 - 2\alpha + 1)x^2 + \alpha(\alpha - 1)x - \alpha = 0. \]

Given $\alpha$ consider a system of equations
\[ \gamma = \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha} \delta = \frac{\alpha}{\gamma(1 - \alpha) - \alpha}. \quad (4) \]

Then there exist a solution of this system with $\delta = g_1(\alpha)$. Note that
\[ g_1(1/3) = g_1(1) = 1 \]
and for $1/3 < \alpha < 1$ one has $g_1(\alpha) > 1$. Let $\alpha_0$ be the unique real root of the equation
\[ x^3 - x^2 + 2x - 1 = 0. \]

In the interval $1/3 < \alpha < \alpha_0$ one has
\[ g_1(\alpha) > \max\left(1, \frac{\alpha}{1 - \alpha}\right). \]

**Theorem 2.** Suppose that $m = 1, n = 3$ and the matrix $\Theta = \begin{pmatrix} \theta_1 \\
\theta_2 \\
\theta_3 \end{pmatrix}$ consists of numbers linearly independent over $\mathbb{Z}$ together with 1. Then
\[ \beta(\Theta) \geq \alpha(\Theta)g_1(\alpha(\Theta)). \quad (5) \]
The inequality (5) is better than (11) in the interval $1/3 < \alpha(\Theta) < \alpha_0$.

For $\alpha \geq 3$ define

$$g_2(\alpha) = \sqrt{\frac{\alpha + \frac{1}{\alpha^2}}{4}} + \frac{1}{\alpha} - \frac{1}{2}, \quad h(\alpha) = \alpha - g_2(\alpha) - 1.$$ 

Here we should note that the functions $g_2(\alpha)$ and $h(\alpha)$ monotonically increase to $+\infty$ when $\alpha \to +\infty$ and

$$g_2(3) = h(3) = 1, \quad g_2(\alpha) \leq \alpha - 2.$$

**Theorem 3.** Suppose that $m = 3, n = 1$ and the matrix $\Theta = (\theta^1, \theta^2, \theta^3)$ consists of numbers linearly independent over $\mathbb{Z}$ together with 1. Then

$$\beta(\Theta) \geq \alpha(\Theta)g_2(\alpha(\Theta)).$$

(6)

The inequality (6) is better than (3) for all values of $\alpha(\Theta)$.

For $\alpha \geq 1$ put

$$g_3(\alpha) = \frac{1 - \alpha + \sqrt{(1 - \alpha)^2 + 4\alpha(2\alpha^2 - 2\alpha + 1)}}{2\alpha}.$$ 

So $g_3(\alpha)$ is a solution of the equation

$$\alpha x^2 + (\alpha - 1)x - (2\alpha^2 - 2\alpha + 1) = 0.$$ 

(7)

We see that $g_3(1) = 1$ and for $\alpha > 1$ one has $g_3(\alpha) > 1$. Moreover in the interval

$$1 \leq \alpha < \left(\frac{1 + \sqrt{5}}{2}\right)^2$$

one has

$$g_3(\alpha) > \max(1, \alpha - 1).$$

**Theorem 4.** Consider four real numbers $\theta^i_j$, $i, j = 1, 2$ linearly independent over $\mathbb{Z}$ together with 1. Let $m = n = 2$ and consider the matrix

$$\Theta = \begin{pmatrix} \theta^1_1 & \theta^2_1 \\ \theta^1_2 & \theta^2_2 \end{pmatrix}$$

Then

$$\beta(\Theta) \geq \alpha(\Theta)g_3(\alpha(\Theta)).$$

(8)

Theorem 4 improves Theorem 1 for $\alpha(\Theta) \in \left(1, \left(\frac{1 + \sqrt{5}}{2}\right)^2\right)$.

3. Best approximations.

For integer vector $x = (x_1, ..., x_m) \in \mathbb{Z}^m$ put

$$\zeta(x) = \max_{1 \leq j \leq n} ||L_j(x)||.$$

A point $x = (x_1, ..., x_m)$ is defined to be a best approximation if

$$\zeta(x) = \min_{x'} \zeta(x'),$$

3
where the minimum is taken over all \( x' = (x'_1, \ldots, x'_m) \in \mathbb{Z}^m \) such that

\[
0 < M(x') \leq M(x).
\]

(For each best approximation \( x \), the point \( -x \) is also a best approximation.) Consider the case when all numbers \( \theta_j, \ 1 \leq i \leq m, \ 1 \leq j \leq n \) are linearly independent over \( \mathbb{Z} \) together with 1. Then all best approximations form the sequences

\[
x_1, x_2, \ldots, x_\nu, x_{\nu+1}, \ldots
\]

\[
M(x_1) < M(x_2) < \ldots < M(x_\nu) < M(x_{\nu+1}) < \ldots,
\]

\[
\zeta(x_1) > \zeta(x_2) > \ldots > \zeta(x_\nu) > \zeta(x_{\nu+1}) > \ldots
\]

We use the notation

\[
M_\nu = M(x_\nu), \quad \zeta_\nu = \zeta(x_\nu).
\]

Define \( y_{1, \nu}, \ldots, y_{n, \nu} \in \mathbb{Z}^n \) to be integers such that

\[
||L_j(x_\nu)|| = |L_j(x_\nu) + y_{j, \nu}|.
\]

We need the notation

\[
z_\nu = (x_1, \nu, \ldots, x_m, \nu, y_{1, \nu}, \ldots, y_{n, \nu}) \in \mathbb{Z}^d, \ d = m + n.
\]

If

\[
\psi(t \Theta) \leq \psi(t)
\]

with continuous and decreasing to zero function \( \psi(t) \) one can easily see that

\[
\zeta_\nu \leq \psi(M_{\nu+1}).
\]

Some useful fact about best approximations one can find in [3].

4. Sketched proof of Theorem 2.

Suppose that

\[
\psi(t \Theta) \leq \psi(t)
\]

with some continuous function \( \psi(t) \) decreasing to zero as \( t \to +\infty \). Moreover we suppose that the function \( t \mapsto t \cdot \psi(t) \) increases to infinity as \( t \to +\infty \).

Consider best approximation vectors \( z_\nu = (x_\nu, y_{1, \nu}, y_{2, \nu}, y_{3, \nu}) \). From the condition that numbers \( 1, \theta_1, \theta_2, \theta_3 \) are linearly independent over \( \mathbb{Z} \) we see that there exist infinitely many pairs of indices \( \nu < k, \nu \to +\infty \) such that

- both triples \( z_{\nu-1}, z_\nu, z_{\nu+1}; \ z_{k-1}, z_k, z_{k+1} \)

consist of linearly independent vectors:

- there exists a two-dimensional linear subspace \( \pi \) such that

\[
z_l \in \pi, \ \nu \leq l \leq k; \ z_{\nu-1} \notin \pi, \ z_{k+1} \notin \pi;
\]

- the vectors

\[
z_{\nu-1}, z_\nu, z_k, z_{k+1}
\]

are linearly independent.
So
\[
1 \leq |\det \begin{pmatrix} y_{1,\nu-1} & y_{2,\nu-1} & y_{3,\nu-1} & x_{\nu-1} \\ y_{1,\nu} & y_{2,\nu} & y_{3,\nu} & x_{\nu} \\ y_{1,k} & y_{2,k} & y_{3,k} & x_{k} \\ y_{1,k+1} & y_{2,k+1} & y_{3,k+1} & x_{k+1} \end{pmatrix}| \leq 24 \zeta_{\nu-1} \zeta_{k} M_{k+1} \leq 24 \psi(M_{\nu}) \psi(M_{\nu+1}) \psi(M_{k+1}) M_{k+1}.
\] (9)

Consider three cases.

1\textsuperscript{o}. Given $\gamma > \frac{\alpha(\Theta)}{1-\alpha(\Theta)}$ there exist infinitely many pairs $(\nu, k)$ such that
\[
M_{k+1} \leq M_{\nu+1}^\gamma.
\]
From (9) we deduce that
\[
\frac{1}{24 \psi(M_{\nu})} \leq M_{\nu+1}^\gamma \cdot \psi(M_{\nu+1}) \cdot \psi(M_{\nu+1})
\]
Suppose the function $t \mapsto t^\gamma \cdot \psi(t) \cdot \psi(t^\gamma)$ to be increasing and let $\rho(t)$ be the inverse function. We see that
\[
\zeta_{\nu} \leq \psi(M_{\nu+1}) \leq \psi \left( \rho \left( \frac{1}{24 \psi(M_{\nu})} \right) \right).
\] (10)

2\textsuperscript{o}. Given $\delta \geq 1$. There exist infinitely many pairs $(\nu, k)$ such that
\[
M_{k+1} \geq M_{k}^\delta.
\]
Then we immediately have
\[
\zeta_{k} \leq \psi(M_{k+1}) \leq \psi(M_{k}^\delta).
\] (11)

3\textsuperscript{o}. There exist infinitely many pairs $(\nu, k)$ such that
\[
M_{\nu+1}^\gamma \leq M_{k+1} \leq M_{k}^\delta.
\]

Lemma 1. Let $z_{\nu}, a \leq \nu \leq b$ lie in a two-dimensional linear subspace $\pi \subset \mathbb{R}^4$. Then for all $\nu_1, \nu_2$ from the interval $a \leq \nu_j \leq b - 1$ one has
\[
\zeta_{\nu_1} M_{\nu_1+1} \asymp \Theta \zeta_{\nu_2} M_{\nu_2+1}.
\]

Sketched proof of Lemma 1. Consider two-dimensional lattice $\Lambda = \pi \cap \mathbb{Z}^4$. The parallelepiped
\[
\{z = (x, y_1, y_2, y_3) : |x| < M_{k+1}, \max_{1 \leq j \leq 3} |\theta_j x - y_j| < \zeta_k \}
\]
has no non-zero integer points inside for every $k$. So for $a \leq \nu \leq b - 1$ we have
\[
\zeta_{\nu} M_{\nu+1} \asymp \Theta \det_2 \Lambda.
\]
Lemma 1 follows.

We apply Lemma 1 and obtain the inequality
\[
\zeta_{k-1} \ll \Theta \frac{M_{\nu+1}^\gamma \psi(M_{\nu+1})}{M_{k}} \ll \Theta \frac{\zeta_{k-1}^\delta \psi(M_{k})}{M_{k}} \leq \frac{\zeta_{k-1}^{\delta-1}}{M_{k-1}} \psi(M_{k-1}) \psi(M_{k-1}^\delta)
\] (12)
(here we suppose the function $t \mapsto t^{\delta-1} \psi(t^\delta)$ to be decreasing).
Given $\varepsilon > 0$ we put $\psi(t) = t^{-\alpha + \varepsilon}$ to deduce (5) from (10)\((11)\)(12). Here we should take into account that $g_1(\alpha)$ satisfies the system (4).

5. Sketched proof of Theorem 3. Suppose that

$$
\psi_\Theta(t) \leq \psi(t)
$$

with some continuous function $\psi(t)$ decreasing to zero as $t \to +\infty$.

First of all consider the case when there exists a 3-dimensional linear subspace $\Pi$ such that all vectors $Z_\nu = (x_{1,\nu}, x_{2,\nu}, x_{3,\nu}, y_{\nu})$ belong to $\Pi$ for all $\nu$ large enough. Then obviously we can apply the statement (ii) of Theorem 1 and (2) gives the bound which is better than (6).

So we can suppose that there exist infinitely many pairs of indices $\nu < k, \nu \to +\infty$ such that

- both triples $Z_{\nu - 1}, Z_\nu, Z_{\nu + 1}$; $Z_{k - 1}, Z_k, Z_{k + 1}$

consist of linearly independent vectors;
  - there exists a two-dimensional linear subspace $\pi$ such that $Z_l \in \pi, \nu \leq l \leq k$; $Z_{\nu - 1} \not\in \pi$, $Z_{k + 1} \not\in \pi$;
  - the vectors $Z_{\nu - 1}, Z_\nu, Z_{\nu + 1}, Z_{k + 1}$

are linearly independent.

Now we see that

$$
1 \leq |\det \begin{pmatrix}
  x_{1,\nu - 1} & x_{2,\nu - 1} & x_{3,\nu - 1} & y_{\nu - 1} \\
  x_{1,\nu} & x_{2,\nu} & x_{3,\nu} & y_{\nu} \\
  x_{1,\nu + 1} & x_{2,\nu + 1} & x_{3,\nu + 1} & y_{\nu + 1} \\
  x_{1,k + 1} & x_{2,k + 1} & x_{3,k + 1} & y_{k + 1}
\end{pmatrix}| \leq 24 \zeta_{\nu - 1} M_\nu M_{\nu + 1} M_{k + 1} \leq 24 \psi(M_\nu) M_\nu M_{\nu + 1} M_{k + 1}. \quad \text{(13)}
$$

Consider three cases.

1. There exist infinitely many pairs $(\nu, k)$ such that

$$
M_{k + 1} \leq M_\nu^{h(\Theta)}.
$$

From (13) we deduce that

$$
M_{\nu + 1} \geq \frac{1}{24 \psi(M_\nu) M_\nu^{1 + h(\Theta)}}, \quad \zeta_{\nu} \leq \psi \left( \frac{1}{24 \psi(M_\nu) M_\nu^{1 + h(\Theta)}} \right).
$$

The inequality (11) follows from the last inequality immediately.

2. There exist infinitely many pairs $(\nu, k)$ such that

$$
M_{k + 1} \geq M_k^{g_2(\Theta)}
$$

Then we immediately have

$$
\zeta_k \leq \psi(M_{k + 1}) \leq \psi(M_k^{g_2(\Theta)}),
$$

and (6) follows.

3. There exist infinitely many pairs $(\nu, k)$ such that

$$
M_\nu^{h(\Theta)} \leq M_{k + 1} \leq M_k^{g_2(\Theta)}.
$$
Lemma 2. Let \( z_\nu, a \leq \nu \leq b \) lie in a two-dimensional linear subspace \( \pi \subseteq \mathbb{R}^4 \). Then for all \( \nu_1, \nu_2 \) from the interval \( a \leq \nu_1, \nu_2 \leq b - 1 \) one has
\[
\zeta_{\nu_1} M_{\nu_1+1} \asymp \zeta_{\nu_2} M_{\nu_2+1}.
\]

Sketched proof of Lemma 2. Consider the projection of the subspace \( \pi \) onto the 3-dimensional subspace
\[
\mathcal{L}(\Theta) = \{ z = (x_1, x_2, x_3, y) : \theta^1 x_1 + \theta^2 x_2 + \theta^3 x_3 + y = 0 \}.
\]
In the general situation this projection is a two-dimensional subspace \( \pi^* \). The intersection \( \ell = \pi \cap \pi^* \) form a one-dimensional subspace. The distance from a point \( z \in \pi \) to the subspace \( \mathcal{L} \) is proportional to the distance from \( z \) to \( \ell \). Let \( \delta \) be the coefficient of this proportionality. The vectors \( z_l \) become the vectors of the best approximations (associated with the induced norm on \( \pi \)) from a lattice \( \Lambda = \mathbb{Z}^4 \cap \pi \) to the one-dimensional subspace \( \ell \). By the Minkowski convex body theorem applied to the two-dimensional lattice \( \Lambda \) we deduce that
\[
\gamma_1(\Theta) \delta \det \Lambda \leq \zeta_\nu M_{\nu+1} \leq \gamma_2(\Theta) \delta \det \Lambda, \quad a \leq \nu \leq b - 1
\]
with some positive constants \( \gamma_i(\Theta), i = 1, 2 \) depending on \( \Theta \). Lemma 2 is proved.

6. Sketched proof of Theorem 4.
Define
\[
R(\Theta) = \min_{\Pi} \dim \Pi \geq 2,
\]
where the minimum is taken over all linear subspaces \( \Pi \subseteq \mathbb{R}^4 \) such that there exists \( \nu_0 \) such that for all \( \nu \geq \nu_0 \) one has \( z_\nu = (x_{1,\nu}, x_{2,\nu}, y_{1,\nu}, y_{2,\nu}) \in \Pi \).

We consider several cases.

1°. Suppose that \( R(\Theta) = 2 \). Then (by Theorem 8 from [2], Section 2.1, applied to the case \( m = n = 2 \)) we see that \( \alpha(\Theta) = 1 \) and there is nothing to prove.

2°. The equality \( R(\Theta) = 3 \) is not possible (see Corollary 4 from the Section 2.1 from [3]).

3°. Suppose that \( R(\Theta) = 4 \). Then there exist infinitely many pairs of indices \( \nu < k, \nu \to +\infty \) such that

- both triples \( z_{\nu-1}, z_\nu, z_{\nu+1}; \quad z_{k-1}, z_k, z_{k+1} \)

consist of linearly independent vectors;

- there exists a two-dimensional linear subspace \( \pi \) such that

\[
z_l \in \pi, \; \nu \leq l \leq k; \quad z_{\nu-1} \notin \pi, \quad z_{k+1} \notin \pi;
\]

- the vectors

\[
z_{\nu-1}, z_\nu, z_k, z_{k+1}
\]

are linearly independent.
So

\[ 1 \leq |\det \begin{pmatrix}
    x_{1,\nu-1} & x_{2,\nu-1} & y_{1,\nu-1} & y_{2,\nu-1} \\
    x_{1,\nu} & x_{2,\nu} & y_{1,\nu} & y_{2,\nu} \\
    x_{1,k} & x_{2,k} & y_{1,k} & x_{2,k} \\
    x_{1,k+1} & x_{2,k+1} & y_{1,k+1} & y_{2,k+1}
  \end{pmatrix} | \leq 24\zeta_{\nu-1}\zeta_{\nu}M_kM_{k+1}. \]

Suppose that a function \( \psi(t) \) decrease to zero as \( t \to +\infty \). Suppose also that the function \( t \mapsto t \cdot \psi(t) \) also decrease to zero. Suppose that

\[ \psi_{\Theta}(t) \leq \psi(t) \]

for all positive \( t \). Then \( \zeta_{l} \leq \psi(M_{l+1}), \ l=1, 2, 3, .. \) and

\[ 1 \leq 24M_{k+1}M_k\psi(M_{\nu+1})\psi(M_{\nu}). \tag{14} \]

We must consider two subcases.

**3.1**. Given \( \gamma > 1 \) there exist infinitely many pairs \((\nu, k)\) such that

\[ M_{k+1} \geq M_k^\gamma. \]

Then we immediately have

\[ \zeta_k \leq \psi(M_{k+1}) \leq \psi(M_k^\gamma). \tag{15} \]

**3.2**. There exist infinitely many pairs \((\nu, k)\) such that

\[ M_{k+1} \leq M_k^\gamma. \]

Then from (14) we see that

\[ M_k \geq (\psi(M_{\nu}))^{-\frac{2}{1+\gamma}}. \tag{16} \]

Consider two-dimensional lattice \( \Lambda = \pi \cap Z^2 \) with the fundamental two-dimensional volume \( \det \Lambda \).

We may suppose that \( \dim \mathcal{L} \cap \pi = 1 \) (the case \( \mathcal{L} \cap \pi = 0 \) can be considered in a similar way).

For any point \( z \in \pi \) the distance from \( z \) to the two-dimensional linear subspace

\[ \mathcal{L} = \{ z = (x_1, x_2, y_1, y_2) : \theta_1^1x_1 + \theta_1^2x_2 + y_1 = \theta_2^1x_1 + \theta_2^2x_2 + y_2 = 0 \} \]

is proportional to the distance from \( z \) to the one-dimensional linear subspace \( \mathcal{L} \cap \pi \). Let \( \delta \) be the coefficient of this proportionality (the "angle" between two-dimensional subspaces \( \pi \) and \( \mathcal{L} \)). The parallelepiped

\[ \{ z = (x_1, x_2, y_1, y_2) : |x| < M_{l+1}, \max_{1 \leq j \leq 3} |\theta_j x - y_j| < \zeta_l \} \]

has no non-zero integer points inside for every \( l \). From the Minkowski convex body theorem e wee that

\[ \gamma_1(\Theta)\delta \det \Lambda \leq \zeta_lM_{l+1} \leq \gamma_2(\Theta)\delta \det \Lambda, \quad \nu \leq l \leq k - 1 \tag{17} \]

with some positive constants \( \gamma_i(\Theta), \ i = 1, 2 \) dependind on \( \Theta \). So from (16) and (17) we see that

\[ \zeta_{\nu} \ll_{\Theta} \psi(M_{k})M_k \ll_{\Theta} M_{\nu+1}^{-1}\psi(M_{\nu})^{-\frac{2}{1+\gamma}} \psi \left( (\psi(M_{\nu}))^{-\frac{2}{1+\gamma}} \right). \tag{18} \]

We should consider \( \psi(t) = t^{-\alpha(\Theta)+\varepsilon} \) for small positive \( \varepsilon \). As \( \gamma = g_3(\alpha(\Theta)) \) satisfies (7) Theorem 2 follows.
References

[1] В.Ярник, К теории однородных линейных диофантовых приближений. // Чехословацкий математический журнал, т. 4 (79), 330 - 353 (1954).

[2] M.Laurent, Exponents of Diophantine approximations in dimension two.// Canad.J.Math. 61, 1 (2009),165 - 189; preprint available at arXiv:math/0611352v1 (2006).

[3] N.G.Moshchevitin, Khintchine’s singular systems and their applications.//Russian Math. Surveys. 65:3 43 - 126 (2010);(2010) Preprint available at arXiv: (2009).