Positive scalar curvature – constructions and obstructions

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Abstract

This is a survey of the current state of the question “Which closed connected manifolds of dimension $n \geq 5$ admit Riemannian metrics whose scalar curvature function is everywhere positive?” The introduction gives a brief overview of these results, while the body of the paper discusses the methods used in the proofs of these results. We mention the two flavors of topological obstructions to the existence of positive scalar curvature metrics: one is a consequence of the Weizenböck formula for the Dirac operator, the other is obtained by considering stable minimal hypersurfaces. We talk about geometric constructions of positive scalar curvature metrics (the surgery/bordism theorem), which shows that the answer to the question above depends only the bordism class of the manifold in a suitable bordism group. Via the Pontryagin-Thom construction this can be translated into stable homotopy theory, and solved completely in some cases, in particular for simply connected manifolds, or manifolds with very special fundamental groups. The last section discusses some open questions.

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1 Introduction

Among the various flavors of curvature of a Riemannian manifold $M$, the scalar curvature is the most basic: it is a smooth function $s: M \to \mathbb{R}$ whose value at a point $x \in M$ controls the volume of small balls with center $x$ in the sense that $s(x)$ shows up in the first interesting coefficient of the Taylor series expansion

$$\frac{\text{vol } B_r(x, M)}{\text{vol } B_r(0, \mathbb{R}^n)} = 1 - \frac{s(x)}{6(n+2)} r^2 + \ldots$$

Here $\text{vol } B_r(x, M)$ is the volume of the geodesic ball of radius $r$ around $x$ in $M$, and $\text{vol } B_r(0, \mathbb{R}^n)$ is the volume of the ball of the same radius in $\mathbb{R}^n$ for $n = \dim M$.

1.1. Question. Let $M$ be a closed connected manifold. Does $M$ carry a positive scalar curvature metric, i.e., a Riemannian metric such that $s(x) > 0$ for all $x \in M$?

This paper does not contain new results concerning this question, but rather its goal is to give a brief overview of known results addressing this question in the introduction and to outline the methods used in their proof in the body of the paper. In the last section we mention some open problems. We have attempted to give precise references for readers curious about details.

We will restrict our discussion to manifolds of dimension $n \geq 5$. The reason is that the flavor of the answers and the tools used very much depend on $n$, like in the classification of manifolds up to diffeomorphism. The crucial tool for the diffeomorphism classification of manifolds of dimension $n \geq 5$ is the $s$-cobordism theorem, according to which two such manifolds which are $s$-cobordant are in fact diffeomorphic. The analogous tool for studying Question 1.1 is Theorem 1.4 according to which a cobordism between manifolds $M$ and $N$ of dimension $\geq 5$, equipped with suitable additional structures, implies that if $N$ carries a positive scalar curvature metric, then so does $M$. Also, there are diffeomorphism invariants, like the Seiberg-Witten invariant, which are only defined for manifolds of dimension $n = 4$. The Seiberg-Witten invariant is also relevant for understanding positive scalar curvature metrics on 4-manifolds, since it vanishes for any closed 4-manifold that carries a positive scalar curvature metric.

The reader might wonder about the implicit bias in question 1.1 by asking about positive scalar curvature metrics. The reason is that if $s: M \to \mathbb{R}$ is a smooth function on any closed manifold of dimension $n \geq 3$ with $s(x) < 0$ at some point $x \in M$, then there exists a Riemannian metric on $M$ with scalar curvature function $s$. [KW1975]
Thm. 6.4(b)]. Moreover, if a closed manifold $M$ of dimension $n \geq 3$ carries a positive scalar curvature metric, then any smooth function $s: M \to \mathbb{R}$ is realized as the scalar curvature function of some metric [KW1975, Thm. 6.4(a)]. So the situation in dimension $n \geq 3$ is radically different than in dimension 2, where having everywhere positive and everywhere negative functions as the scalar curvature on the same 2-manifold is ruled out by the Gauss-Bonnet Theorem.

Lichnerowicz observed that the Dirac operator on a closed Riemannian spin manifold $M$ with positive scalar curvature is invertible, and hence its index vanishes. By the Atiyah-Singer Index Theorem, the index of the Dirac operator is equal to a topological invariant $\alpha_p(M)$ (known as Hirzebruch’s $A$-genus), and hence $\alpha(M)$ vanishes for if $M$ carries a positive scalar curvature metric [Li1963]. Including a later refinement by Hitchin [Hi1974] leads to the following result.

**Theorem 1.2. (Lichnerowicz, Hitchin)** Let $M$ be a closed Riemannian spin manifold of dimension $n$ with positive scalar curvature metric, then its Atiyah invariant $\alpha(M) \in \text{KO}_n$ vanishes.

Here $\text{KO}_n(\cdot)$ is the generalized homology theory known as real $K$-homology, and $\text{KO}_n = \text{KO}_n(\text{point})$ is the $n$-th real $K$-homology group of a point. The invariant $\alpha(M)$ has a topological as well as an index theory definition. Topologically, the spin structure on $M$ determines a fundamental class $[M]^{\text{Spin}} \in \text{KO}_n(M)$ and $\alpha(M) = p_*[M]^{\text{Spin}}$, where $p_*: \text{KO}_n(M) \to \text{KO}_n(\text{point})$ is the homomorphism induced by the projection map $p: M \to \text{point}$. From an index theory point of view, the invariant $\alpha(M)$ can be viewed as a “refined index” (with values in $\text{KO}_n$ rather than $\mathbb{Z}$) of the Clifford linear Dirac operator on $M$, as explained in section 2.2, see equation (2.10).

For $n \equiv 0 \mod 4$, the invariant $\alpha(M) \in \text{KO}_n \cong \mathbb{Z}$ is equal to $cA(M)$ where $c = 1$ for $n \equiv 0 \mod 8$ and $c = \frac{1}{2}$ for $n \equiv 4 \mod 8$. It is interesting to note that for $n = 1, 2 \mod 8$ and $n \geq 9$ there is an $n$-manifold $\Sigma$ with $0 \neq \alpha(\Sigma) \in \text{KO}_n \cong \mathbb{Z}/2$ which is homeomorphic, but not diffeomorphic to the sphere $S^n$. In particular, $S^n$ carries a positive scalar curvature metric while the homeomorphic manifold $\Sigma$ does not! This shows that Question 1.1 is subtle: the answer in general depends on the smooth structure of $M$.

There is a generalization of the Lichnerowicz-Hitchin Theorem 1.2 due to Rosenberg based on an index invariant $\alpha(M, f) \in \text{KO}_n(C^*\pi)$ associated to a closed spin $n$-manifold $M$ equipped with a map $f: M \to B\pi$ to the classifying space $B\pi$ of a discrete group $\pi$ [Ro1986III, Proof of Thm. 3.4]. Here $\text{KO}_n(C^*\pi)$ is the real $K$-theory of the reduced group $C^*$-algebra $C^*\pi$, which is a completion of the real group ring $\mathbb{R}\pi$. If $\pi$ is the trivial group, this index invariant agrees with $\alpha(M)$.

**Theorem 1.3. (Rosenberg)** If $M$ carries a positive scalar curvature metric, then $\alpha(M, f)$ vanishes.

After discussing obstructions to positive scalar curvature metrics, we now turn to constructions of manifolds which carry such metrics. A major breakthrough was
obtained independently by Gromov-Lawson and Schoen-Yau who proved that if a closed manifold \( M \) is obtained by a surgery of codimension \( \geq 3 \) from a manifold \( N \) which carries a positive scalar curvature metric, than also \( M \) carries a positive scalar curvature metric [GL1980, Thm. A], [SY1979a, Cor. 6]. Gromov and Lawson made the crucial observation that this implies that the answer to Question 1.1 for a simply connected manifold \( M \) of dimension \( n \geq 5 \) depends only on its bordism class in a suitable bordism group [GL1980, Thm. B, Thm. C]. This was later extended by Rosenberg to certain manifolds with non-trivial fundamental group [Ro1986II, Thm. 2.2, Thm. 2.13].

Let \( \Omega^n_{SO} \) resp. \( \Omega^n_{\text{Spin}} \) be the bordism group of \( n \)-dimensional closed manifolds equipped with an orientation resp. spin structure. More generally, for a topological space \( X \), and \( G = SO \) or \( G = \text{Spin} \), let \( \Omega^n_G(X) \) be the bordism classes of pairs \((M, f)\) where \( M \) is an oriented resp. spin \( n \)-manifold and \( f : M \to X \) is a map. Let \( \Omega^n_G^+(X) \) be the subgroup of \( \Omega^n_G(X) \) consisting of those bordism classes that can be represented by pairs \((M, f)\) such that \( M \) carries a positive scalar curvature metric.

**Theorem 1.4. (Gromov and Lawson, Rosenberg).** Let \( M \) be a closed connected \( n \)-manifold, \( n \geq 5 \), with fundamental group \( \pi \), and let \( u : M \to B\pi \) be the classifying map of the universal covering \( \tilde{M} \to M \).

(i) If \( M \) is spin, then it carries a positive scalar curvature metric if and only if \([M, u] \in \Omega^n_{\text{Spin}}^+(B\pi)\).

(ii) If \( M \) is oriented and \( \tilde{M} \) is non-spin, then \( M \) admits a positive scalar curvature metric if and only if \([M, u] \in \Omega^n_{SO}^+(B\pi)\).

More generally, for any closed manifold \( M \) with fundamental group \( \pi \), the pair \((M, u)\) represents an element in a twisted version of the bordism group \( \Omega^n_{\text{Spin}}(B\pi) \) (if \( \tilde{M} \) is spin) resp. \( \Omega^n_{SO}(B\pi) \) (if \( \tilde{M} \) is non-spin). There are generalizations of the theorem above [RS1994, Thm. 3.3 and Examples 3.6 & 3.7].

The oriented bordism ring \( \Omega^*_{SO} \) was determined by Wall [Wa1960], who also provided a list of explicit oriented manifolds whose bordism classes are multiplicative generators for \( \Omega^*_{SO} \). Gromov and Lawson showed that all of these manifolds admit positive scalar curvature metrics, which by the theorem above implies [GL1980, Cor. C]:

**Corollary 1.5. (Gromov and Lawson).** Every closed simply connected non-spin manifold of dimension \( n \geq 5 \) admits a positive scalar curvature metric.

For a closed spin \( n \)-manifold \( M \) the vanishing of the index obstruction \( \alpha(M) \in KO_n \) is a necessary condition for the existence of a positive scalar curvature metric on \( M \) by the Lichnerowicz-Hitchin theorem [L2]. Gromov and Lawson conjectured that it is also a sufficient condition for simply connected manifolds of dimension \( \geq 5 \) [GL1980, p. 424]. This was later proved by the author [St1992, Thm. A]:

**Theorem 1.6. (Stolz).** Let \( M \) be a simply connected closed spin manifold of dimension \( n \geq 5 \). Then \( M \) admits a positive scalar curvature metric if and only if \( \alpha(M) = 0 \).
The spin bordism groups were calculated by Anderson, Brown and Peterson \cite{ABP1967}. However, unlike for the oriented bordism ring, there is no list of explicit spin manifolds which multiplicatively generate $\Omega^\text{Spin}_\ast$ (or the ideal consisting of the bordism classes of spin manifolds with vanishing $\alpha$-invariant). In fact, to the author’s knowledge, the multiplicative structure of $\Omega^\text{Spin}_\ast$ has not been completely determined.

The theorem above is a corollary of a statement about spin bordism, according to which every closed spin manifold $M$ with $\alpha(M) = 0$ is spin bordant to the total space of a fiber bundle, whose fiber is the quaternionic projective plane $\mathbb{HP}^2$, and whose structure group is the isometry group of the standard metric on $\mathbb{HP}^2$ (see Theorem 3.10). This implies Theorem 1.6, since total spaces of such fiber bundles carry positive scalar curvature metrics by Observation 3.1(2).

The bordism statement is proved using the Pontryagin-Thom construction to express bordism groups as homotopy groups of suitable Thom spectra, and using the Adams spectral sequence to calculate with these homotopy groups.

To deal with manifolds with non-trivial fundamental group, we recall that by Theorem 1.4 the classification of manifolds which carry positive scalar curvature metrics boils down to the computation of the subgroups $\Omega^G_\ast(B\pi) \subset \Omega^G_n(B\pi)$, for $G = \text{SO, Spin}$ (and twisted versions thereof). This is difficult, since $\Omega^G_\ast$, the coefficient ring of the generalized homology theory $\Omega^G_\ast( )$ is large (rationally $\Omega^G_\ast$ is a polynomial ring generated by elements $x_4, x_8, \ldots$, where the subscript indicates the degree of the generators). So it is desirable to replace the homology theories $\Omega^G_\ast( )$, $G = \text{Spin, SO}$ by theories with a smaller coefficient rings, which the following result accomplishes.

Given a closed $n$-manifold $M$, an orientation on $M$ determines a homology fundamental class $[M]_{\text{SO}} \in H_n(M)$. Similarly, a spin structure on $M$ determines a ko-theory fundamental class $[M]_{\text{Spin}} \in \text{ko}_n(M)$, where $\text{ko}_n( )$ is a generalized homology theory known as connective real $K$-homology. For a topological space $X$, let $H_n^+(X) \subset H_n(X)$ be the subgroup consisting of homology classes which can written in the form $f_*[M]_{\text{SO}}$, where $M$ is a closed oriented $n$-manifold which carries a positive scalar curvature metric and $f: M \to X$ is a map. The subgroup $\text{ko}_n^+(X) \subset \text{ko}_n(X)$ is defined analogously, requiring a spin structure instead of an orientation on $M$ and replacing $f_*[M]_{\text{SO}}$ by $f_*[M]_{\text{Spin}}$.

**Theorem 1.7.** Let $M$ be a closed connected $n$-manifold, $n \geq 5$, with fundamental group $\pi$, and let $u: M \to B\pi$ be the classifying map of the universal covering $\widetilde{M} \to M$.

(i) If $M$ is spin then it carries a positive scalar curvature metric if and only if $u_*[M]_{\text{Spin}} \in \text{ko}_n^+(B\pi)$.

(ii) If $M$ is oriented, but $\widetilde{M}$ does not admit a spin structure, then $M$ carries a positive scalar curvature metric if and only if $u_*[M]_{\text{SO}} \in H_n^+(B\pi)$.

This was proved by Stolz \cite{St1994} “localized at 2”, using stable homotopy theory, and by Rainer Jung (unpublished) and Führing \cite{Fu2013} with “2 inverted”, using a geometric argument. An outline of the proof is given in section 3.5.
1 INTRODUCTION

The above theorem in particular implies that for a closed connected spin manifold \( M \) of dimension \( n \geq 5 \) with fundamental group \( \pi \) the vanishing of \( u_\ast[M]^{\text{Spin}} \in \text{ko}_n(B\pi) \) is a **sufficient condition** for the existence of a positive scalar curvature metric on \( M \). By Theorem 1.3 the vanishing of the index invariant \( \alpha(M,u) \in \text{KO}_n(C^*\pi) \) is a **necessary condition** for such a metric. The index invariant \( \alpha(M,u) \) is the image of \( u_\ast[M]^{\text{Spin}} \) under the homomorphism

\[
\begin{align*}
\text{ko}_n(B\pi) \xrightarrow{p} \text{KO}_n(B\pi) \xrightarrow{A} \text{KO}_n(C^*\pi),
\end{align*}
\]

where \( p \) is the natural transformation relating connective real \( K \)-homology \( \text{ko}_n(\cdot) \) and periodic real \( K \)-homology \( \text{KO}_n(\cdot) \), and \( A \) is the map known as (real) **assembly map** (it is called the Kasparov map in [Ro1986III, p. 326]).

In particular, if the composition (1.8) is injective, then the vanishing of \( \alpha(M,u) \) is a necessary and sufficient for the existence of a positive scalar curvature metric on closed connected spin \( n \)-manifolds \( M, n \geq 5 \). The latter statement is commonly known as the **Gromov-Lawson-Rosenberg Conjecture** (Rosenberg conjectured this for finite \( \pi \) [Ro1991, Conjecture 0.1]). Gromov and Lawson conjectured that the vanishing of \( p(u_\ast[M]^{\text{Spin}}) \in \text{KO}_n(B\pi) \) is a necessary and sufficient condition for positive scalar curvature metrics [GL1983, p. 89] for “reasonable” \( \pi \).

The GLR Conjecture has been proved for a number of groups. Concerning finite groups \( \pi \), it has been proved for odd order cyclic groups [Ro1986II], for \( \mathbb{Z}/2 \) [RS1994]; more generally, for groups with periodic cohomology by [KwSch1990] (of odd order) and [BGS1997] (in general). It is sufficient to consider \( p \)-groups, since a closed \( n \)-manifold \( M, n \geq 5 \), with finite fundamental group \( \pi \) carries a positive scalar curvature metric if and only if for every prime \( p \) the covering of \( M \) corresponding to the \( p \)-Sylow subgroup of \( \pi \) carries such a metric [KwSch1990, Prop. 1.5]. Botvinnik and Rosenberg proved the GLR Conjecture for spin \( n \)-manifolds \( M \) with fundamental group \( (\mathbb{Z}/p)^r \) for \( p \) odd, \( n < r \) [BR2005, Thm. 2.3], with some corrections of the proof provided by Hanke [Ha2016, section 7].

Concerning infinite groups, the GLR Conjecture has been proved for free and free abelian groups, as well as for fundamental groups of orientable surfaces [RS1994] (for these groups the assembly map is injective, and \( B\pi \) splits stably as a wedge of sphere which implies that \( p \) is injective as well). More generally, it has been verified for groups \( \pi \) for which \( A \) is injective, the classifying space \( B\pi \) is finite dimensional, and manifolds of dimension \( n \geq \dim B\pi - 4 \) [JS2000, Thm. 2.1]. All infinite groups mentioned so far are torsion-free. The GLR-conjecture has also been proved some infinite groups with torsion, for example cocompact Fuchsian groups [DP2003] and crystallographic groups of the form \( \mathbb{Z}^n \times \mathbb{Z}/p \), where \( p \) is a prime and the \( \mathbb{Z}/p \)-action on \( \mathbb{Z}^n \) only fixes the origin [DL2013].

Unfortunately, neither the GLR Conjecture nor the GL Conjecture turn out to be true for a general group \( \pi \): There is a 5-dimensional counterexample to the GLR conjecture with fundamental group \( \pi = \mathbb{Z}^4 \times \mathbb{Z}/3 \) [Sch1998], and there is a 5-dimensional
counterexample to the GL conjecture with a torsion free fundamental group \( \pi \) for which the assembly map is an isomorphism \( \text{DSS2003} \). These examples exhibit 5-dimensional spin manifolds for which all index obstructions vanish, but which do not carry positive scalar curvature metrics, since an obstruction coming from stable minimal hypersurfaces is non-zero (see Cor. 2.13 for the general statement, and \( \S 3.6 \) for a discussion of this counter example).

The stable minimal hypersurface method does not produce any restrictions for the existence of positive scalar curvature metrics on manifolds with \emph{finite} fundamental group, and no counter examples to the GLR Conjecture for finite fundamental groups are known. For general \( \pi \), there is currently no conjectural characterization of the subgroup \( k_{\pi}^n(B\pi) \).

The only known obstructions for the existence of a positive scalar curvature metric on a closed manifolds \( M \) of dimension \( \geq 5 \) come from index theory or the stable minimal hypersurface method. Index theory is not known to produce obstructions if the universal cover \( \hat{M} \) is non-spin, and the stable minimal hypersurface method does not produce obstructions if the fundamental group is finite. So the following is the optimist’s conjecture.

\textbf{Conjecture 1.9.} Let \( M \) be a connected closed manifold of dimension \( n \geq 5 \) with finite fundamental group \( \pi \) whose universal cover does not admit a spin structure. Then \( M \) carries a positive scalar curvature metric.

It suffices to consider \( p \)-groups, since \( M \) carries a positive scalar curvature metric if and only if the coverings of \( M \) whose fundamental groups are the \( p \)-Sylow subgroups of \( \pi \) carry positive scalar curvature metrics by \( \text{KwSch1990} \) Prop. 1.5. Conjecture 1.9 is known for many cases of abelian \( p \)-groups, for example for elementary abelian \( 2 \)-groups \( \pi = (\mathbb{Z}/2)^r \) by combining work of Joachim \( \text{Jo2004} \) and Botvinnik-Rosenberg \( \text{BR2005} \). For elementary abelian \( p \)-groups \( \pi = (\mathbb{Z}/p)^r \) with \( p \) odd, it was proved for \( n > r \) \( \text{BR2005} \) Thm. 2.3 and Thm. 2.4] (with later corrections of the proof provided by \( \text{Ha2016} \)). In fact, they prove a stronger statement, replacing the dimension restriction \( n > r \) by the assumption that the element \( u_\ast[M]^{SO} \in H_n(B\pi) \) is \( p \)-atoral, i.e., for any collection of elements \( \alpha_1, \ldots, \alpha_n \in H^1(B\pi; \mathbb{Z}/p^\ell) \) for any \( \ell > 0 \) the Kronecker product \( \langle \alpha_1 \cup \cdots \cup \alpha_n, u_\ast[M]^{SO} \rangle \in \mathbb{Z}/p^\ell \) vanishes. More recently, Conjecture 1.9 has been proved by Hanke for abelian \( p \)-groups, \( p \) odd, assuming that \( u_\ast[M]^{SO} \in H_n(B\pi) \) is \( p \)-atoral \( \text{Ha2019} \).

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2 Obstructions to positive scalar curvature metrics

This section gives a quick overview of the obstructions to positive scalar curvature metrics derived from considering the Dirac operator, which was pioneered by Lichnerowicz in the early 1960’s [Li1963], further developed by Hitchin [Hi1974] and Rosenberg [Ro1986III]. The last part of this section describes additional obstructions coming from the stable minimal hypersurface method developed by Schoen and Yau [SY1979a], [SY1979b].

2.1 The Dirac operator

The Dirac operator, constructed in this generality by Atiyah and Singer in the context of their proof of the Index Theorem, is a first order elliptic differential operator $D$ acting on the space $\Gamma(S)$ of smooth sections of a vector bundle $S \to M$ called the spinor bundle of a Riemannian manifold $M$. The construction of the spinor bundle requires that $M$ comes equipped with a spin structure (see below). The Dirac operator is closely related to the scalar curvature function $s \in C^\infty(M)$ via the Lichnerowicz formula [LM1989, Ch II, Thm. 8.8]

$$D^2 = \nabla^* \nabla + \frac{1}{4} s. \quad (2.1)$$

Here $\nabla: \Gamma(S) \to \Gamma(TM \otimes S)$ is the connection on the spinor bundle $S$, induced by the Levi-Civita connection on the tangent bundle $TM$, and $\nabla^*: \Gamma(TM \otimes S) \to \Gamma(S)$ is the adjoint of $\nabla$ with respect to the inner product $\langle \cdot, \cdot \rangle$ on the space of sections $\Gamma(S)$ determined by the Riemannian metric on $M$ and the induced inner product on the fibers of $S$. Using the fact that $D$ is self-adjoint, this implies the following inequality for $\psi \in \Gamma(S)$:

$$\|D\psi\|^2 = \langle D\psi, D\psi \rangle = \langle D^2 \psi, \psi \rangle = \langle \nabla^* \nabla \psi + s\psi, \psi \rangle = \|\nabla \psi\|^2 + \langle s\psi, \psi \rangle \geq \langle s\psi, \psi \rangle.$$  

It follows that if the scalar curvature function $s$ is strictly positive on the closed manifold $M$, then the kernel of $D$ is trivial.

As explained below, the spinor bundle $S$ is $\mathbb{Z}/2$-graded, i.e., $S = S^+ \oplus S^-$, and with respect to the induced decomposition of the space of sections $\Gamma(S) = \Gamma(S^+) \oplus \Gamma(S^-)$, the Dirac operator $D$ has the form

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}: \Gamma(S^+) \oplus \Gamma(S^-) \longrightarrow \Gamma(S^+) \oplus \Gamma(S^-). \quad (2.2)$$

Hence the kernel of $D$ decomposes in the form $\ker D = \ker D^+ \oplus \ker D^-$. The fact that $D$ is an elliptic operator implies that $\ker D$ is finite dimensional. Moreover, $D$ is
self-adjoint, which implies that $D^-$ is the adjoint of $D^+$, and hence the index of $D^+$, defined by

$$\text{index}(D^+) := \dim \ker D^+ - \dim \coker D^+$$

is equal to $\text{sdim } \ker D := \dim \ker D^+-\dim \ker D^-$, the superdimension of the $\mathbb{Z}/2$-graded vector space $\ker D = \ker D^+ \oplus \ker D^-$. 

Unlike the dimensions of $\ker D^+$ and $\ker D^-$ which in general depend on the geometry, i.e., the Riemannian metric used in the construction of the Dirac operator, the index of $D^+$ depends only on the topology, i.e., on the closed manifold $M$. In fact, according to the Atiyah-Singer Index Theorem, the index of $D^+$ can be identified with a topological invariant $\hat{A}(M)$, known as Hirzebruch’s $\hat{A}$-genus [LM1989, Ch. III, Thm. 13.10]. This is a characteristic number, that is, a number obtained by evaluating a particular polynomial in the Pontryagin classes of the tangent bundle $TM$ on the fundamental homology class of $M$.

Summarizing, the Lichnerowicz formula (2.1) together with the Atiyah-Singer index theorem for the Dirac operator imply the following result [Li1963].

**Theorem 2.3. (Lichnerowicz).** Let $M$ be a closed spin manifold of dimension $n = 4k$ which admits a positive scalar curvature metric. Then the $\hat{A}$-genus $\hat{A}(M)$ vanishes.

**Example 2.4.** The degree $d$ hypersurface $X^2(d) = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \mid z_0^d + z_1^d + z_2^d + z_3^d = 0\}$ in the complex projective space $\mathbb{CP}^3$ is a simply connected closed manifold of real dimension 4. Its $\hat{A}$-hat genus is $\hat{A}(X^2(d)) = \frac{d(d-2)(d+2)}{24}$ [LM1989, Ch. IV, formula 4.4], and hence $X^2(d)$ does not admit a positive scalar curvature metric if $d \geq 4$ and $d$ is even ($d$ even guarantees that $X^2(d)$ has a spin structure).

### 2.2 The Clifford linear Dirac operator

There is an important refinement of Lichnerowicz’ theorem due to Hitchin [Hi1974, §4]. Following [LM1989, Ch. II, section 7] this can be obtained using a variant of the Dirac operator called the **Clifford linear Dirac operator**. The Clifford algebra $\mathcal{C}l_n$ is the quotient of the tensor algebra generated by $\mathbb{R}^n$ modulo the ideal generated by the elements of the form $v \otimes v + ||v||^2$ for $v \in \mathbb{R}^n$. The natural $\mathbb{Z}$-grading on the tensor algebra induces a $\mathbb{Z}/2$-grading $\mathcal{C}l_n = \mathcal{C}l_n^+ \oplus \mathcal{C}l_n^-$ on the Clifford algebra. An extremely useful property of the Clifford algebra is that $\text{Spin}(n)$, a non-trivial double covering group of $SO(n)$, can be constructed as a subgroup of the group of units in $\mathcal{C}l_n$ (namely $\text{Spin}(n)$ is the intersection of the subgroup generated by unit vectors in $\mathbb{R}^n$ and $\mathcal{C}l_n^+$) [LM1989, Ch I, Thm. 2.9].

To describe the Clifford linear Dirac operator, we first need to be more explicit about the construction of the spinor bundle. Let $M$ be an oriented Riemannian $n$-manifold and let $SO(M) \to M$ be the oriented orthonormal frame bundle of $M$; its fiber $SO(M)_x$ over a point $x \in M$ consists of all orientation preserving isometries $f : \mathbb{R}^n \to T_xM$.

**Definition 2.5.** A **spin structure** on $M$ (see e.g. [LM1989, Ch. II, Def. 1.3]) is a principal $\text{Spin}(n)$-bundle $\text{Spin}(M) \to M$ equipped with a double covering map $q : \text{Spin}(M) \to$
SO(M) which is Spin(n)-equivariant (with Spin(n) acting on SO(M) via the projection map \( p: \text{Spin}(n) \rightarrow \text{SO}(n) \)), and which makes the diagram

\[
\begin{array}{ccc}
\text{Spin}(M) & \xrightarrow{q} & \text{SO}(M) \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\]

commutative.

**Construction 2.6. (Spinor bundle.)** Let \( M \) be a Riemannian spin \( n \)-manifold. Then the *spinor bundle* on \( M \) is a \( \mathbb{Z}/2 \)-graded vector bundle associated to the principal Spin\((n)\)-bundle Spin\((M) \rightarrow M \) provided by the spin structure on \( M \). More precisely, there is a spinor bundle \( S_{\Delta} \) associated to any \( \mathbb{Z}/2 \)-graded left module \( \Delta \) over the Clifford algebra \( C\ell_n \). Since Spin\((n) \subset C\ell_n^\ast \) is a subgroup of the group of invertible elements of \( C\ell_n \), the module \( \Delta \) is in particular a representation of Spin\((n) \), and

\[ S_{\Delta} := \text{Spin}(M) \times_{\text{Spin}(n)} \Delta \rightarrow M \]

is the associated vector bundle. The \( \mathbb{Z}/2 \)-grading of \( \Delta = \Delta^+ \oplus \Delta^- \) is preserved by the Spin\((n)\)-action, and hence it induces a \( \mathbb{Z}/2 \)-grading \( S_{\Delta^+} \oplus S_{\Delta^-} \) on the spinor bundle \( S_{\Delta} \).

There is a conventional choice of an irreducible \( \mathbb{Z}/2 \)-graded module over \( C\ell_n \) (there are one or two such modules, depending on \( n \mod 4 \)); the corresponding spinor bundle is the bundle we denoted by \( S \) in the previous section.

For *any* choice of the \( C\ell_n \)-module \( \Delta \) there is a Dirac operator \( D^M_\Delta: \Gamma(S_{\Delta}) \rightarrow \Gamma(S_{\Delta}) \) with the same formal properties of the Dirac operator \( D \) discussed in the previous section. In particular, \( D^M_\Delta \) is self-adjoint, and it is invertible if the Riemannian metric used in its construction has positive scalar curvature.

At first glance choosing the left \( C\ell_n \)-module \( \Delta \) to be \( C\ell_n \) itself might seem silly, since \( \Delta \) decomposes as a sum of irreducible modules, and so choosing an irreducible module seems less redundant. However, the spinor bundle \( S_{C\ell_n} = \text{Spin}(M) \times_{\text{Spin}(n)} C\ell_n \) has more structure: \( C\ell_n \) is a \( C\ell_n \)-bimodule: its left module structure is used for the construction of \( S_{C\ell_n} \), while its right module structure gives each fiber of \( S_{C\ell_n} \) the structure of a \( \mathbb{Z}/2 \)-graded right module. Hence the space \( \Gamma(S_{C\ell_n}) \) of smooth sections is a right \( C\ell_n \)-module. For that reason, the vector bundle \( S_{C\ell_n} \) is called the \( C\ell_n \)-linear spinor bundle. The Dirac operator

\[ D^M_{C\ell_n}: \Gamma(S_{C\ell_n}) \rightarrow \Gamma(S_{C\ell_n}) \tag{2.7} \]

commutes with the right \( C\ell_n \)-action, and hence \( D_{C\ell_n} \) is known as the \( C\ell_n \)-linear Dirac operator.

To describe the index invariant \( \alpha(M) \in KO_n \), we first recall that the real \( K \)-theory group \( KO_n = KO_n(\text{point}) = KO^{-n}(\text{point}) \) can be described in terms of modules over
Clifford algebras. More precisely, Atiyah, Bott and Shapiro constructed an isomorphism [ABS1963] Thm. 11.5, [LM1989] Ch. I, Thm. 9.27

\[ \mathfrak{M}(\mathbb{C}^{\ell_n})/i^*\mathfrak{M}(\mathbb{C}^{\ell_{n+1}}) \cong K_{0,n}. \] (2.8)

Here \( \mathfrak{M}(A) \) denotes for a \( \mathbb{Z}/2 \)-graded algebra \( A \) the group completion of the semi-group of isomorphism classes of \( \mathbb{Z}/2 \)-graded finitely generated \( A \)-modules (with the semi-group structure provided by the direct sum of modules). The homomorphism

\[ i^* : \mathfrak{M}(\mathbb{C}^{\ell_{n+1}}) \to \mathfrak{M}(\mathbb{C}^{\ell_n}) \] (2.9)

is induced by the inclusion map \( i : \mathbb{C}^{\ell_n} \hookrightarrow \mathbb{C}^{\ell_{n+1}} \). We remark that we do not need to specify whether \( \mathfrak{M}(\mathbb{C}^{\ell_n}) \) consists of left-module or right-modules, since there is an anti-involution \( \beta : \mathbb{C}^{\ell_n} \to \mathbb{C}^{\ell_n} \) (i.e., \( \beta(a^*) = \beta(a')\beta(a) \), without signs), determined by \( \beta(v) = v \) for \( v \in \mathbb{R}^n \subset \mathbb{C}^{\ell_n} \). This anti-involution allows us to turn a right module into a left module and vice versa.

The \( \mathbb{C}^{\ell_n} \)-linearity of the operator (2.7) implies that its kernel \( \ker D_{\mathbb{C}^{\ell_n}} \) is a \( \mathbb{Z}/2 \)-graded \( \mathbb{C}^{\ell_n} \)-module. If the manifold \( M \) is closed, the fact that \( D_{\mathbb{C}^{\ell_n}} \) is an elliptic differential operator implies that its kernel is finite dimensional, and hence \( \ker D_{\mathbb{C}^{\ell_n}} \in \mathfrak{M}(\mathbb{C}^{\ell_n}) \).

The index invariant \( \alpha(M) \) is defined by

\[ \alpha(M) := [\ker D_{\mathbb{C}^{\ell_n}}] \in \mathfrak{M}(\mathbb{C}^{\ell_n})/i^*\mathfrak{M}(\mathbb{C}^{\ell_{n+1}}) \cong K_{0,n}. \] (2.10)

If the Riemannian metric \( g \) used in the construction of the Dirac operator \( D_{\mathbb{C}^{\ell_n}} \) has positive scalar curvature, then by the Lichnerowicz formula (2.1) \( \ker D_{\mathbb{C}^{\ell_n}} \) is trivial, and hence \( \alpha(M) = 0 \). This proves the Lichnerowicz-Hitchin Theorem 1.2.

We note that the construction of the Clifford linear Dirac operator \( D_{\mathbb{C}^{\ell_n}} \) involves the Riemannian metric \( g \) on \( M \) and hence \( \ker D_{\mathbb{C}^{\ell_n}} \) depends on \( g \). For a path of metrics \( g_t \), there is a corresponding path of Clifford linear Dirac operators \( D_t \). The dimension of \( \ker D_t \) might jump at a value \( t_0 \) due to an eigenvalue \( \lambda_t \) of \( D_t \) approaching \( 0 \) for \( t \to t_0 \).

In the rest of this section we explain why \( [\ker D_{\mathbb{C}^{\ell_n}}] \in \mathfrak{M}(\mathbb{C}^{\ell_n})/i^*\mathfrak{M}(\mathbb{C}^{\ell_{n+1}}) \) does not depend on the metric, i.e., why it is an invariant of the smooth manifold \( M \). Let \( E_\lambda \) be the finite dimensional eigenspace of \( D = D_{\mathbb{C}^{\ell_n}} \) with eigenvalue \( \lambda \in \mathbb{R} \setminus \{0\} \) (all eigenvalues are real since \( D \) is self-adjoint). Let \( \tau \) be the grading involution of the \( \mathbb{Z}/2 \)-graded vector space \( \Gamma(S_{\mathbb{C}^{\ell_n}}) \); it anti-commutes with \( D \), i.e., \( \tau D = -D \tau \). This is equivalent to the fact that in the matrix decomposition (2.2) of \( D \) with respect to the grading decomposition \( \Gamma(S_{\mathbb{C}^{\ell_n}}) = \Gamma(S^+_{\mathbb{C}^{\ell_n}}) \oplus \Gamma(S^-_{\mathbb{C}^{\ell_n}}) \) the diagonal entries vanish. In particular, for \( \psi \in E_\lambda \)

\[ D(\tau(\psi)) = -\tau(D(\psi)) = -\tau(\lambda \psi) = -\lambda \tau(\psi), \]

and hence \( \tau(\psi) \in E_{-\lambda} \). The Clifford linearity of \( D \) implies that \( E_\lambda \oplus E_{-\lambda} \) is a \( \mathbb{Z}/2 \)-graded module over \( \mathbb{C}^{\ell_n} \). We claim that it is in the image of the homomorphism (2.9), i.e., that
the $\Cl_n$-module structure on $E_\lambda \oplus E_{-\lambda}$ can be extended to a $\Cl_{n+1}$-module structure. It suffices to construct an endomorphism $e_{n+1}$ of $E_\lambda \oplus E_{-\lambda}$ satisfying $e_{n+1}^2 = -1$ and $e_{n+1} v = -v e_n$ for $v \in \mathbb{R}^n \subset \Cl_n$. It is easy to check that $e_{n+1} := \frac{1}{|\mu|} D \tau$ has these properties.

This show that $\ker D^M_{\Cl_n}$ represents the same class in $\mathfrak{M}(\Cl_n)/i^* \mathfrak{M}(\Cl_{n+1})$ as

$$\bigoplus_{|\lambda| < \mu} E_\lambda = \ker D^M_{\Cl_n} \oplus \bigoplus_{0 < \lambda < \mu} E_{-\lambda} \oplus E_\lambda \quad (2.11)$$

for any $\mu > 0$. If $\mu$ is not an eigenvalue of $D^M_{\Cl_n}$ constructed using a metric $g$, then $\mu$ is also not an eigenvalue for those operators obtained from metrics in an open neighborhood of $g$ in the space of metrics. This implies that the isomorphism class of the $\Cl_n$-module (2.11) is constant in that neighborhood and shows that its class in $\mathfrak{M}(\Cl_n)/i^* \mathfrak{M}(\Cl_{n+1})$ is in fact independent of the metric.

### 2.3 Obstructions associated to coverings

In this section, let $M$ be a closed connected spin manifold of dimension $n$. Let $\pi$ be a discrete group, let $\tilde{M} \to M$ be a principal $\pi$-bundle and let $f : M \to B \pi$ be its classifying map. We are particularly interested in the case where $\pi$ is the fundamental group of $M$ and $\tilde{M} \to M$ is the universal covering, but it is useful to work in this generality. Let $D^M_{\Cl_n}$ be the Clifford linear Dirac operator on $\tilde{M}$. This operator commutes with the action of $\Cl_n$ and the action of $\pi$ by deck transformations on $\tilde{M}$, and hence with the action of $\Cl_n \otimes \mathbb{R} \pi$, where $\mathbb{R} \pi$ is the real group ring of $\pi$. In particular, $\ker D^M_{\Cl_n}$ is a module over $\Cl_n \otimes \mathbb{R} \pi$.

Assuming that $\pi$ is finite, the manifold $\tilde{M}$ is compact, and hence $\ker D^M_{\Cl_n}$ is a finite dimensional vector space. The argument in the previous section implies that the class

$$\alpha(M, f) := \left[ \ker D^M_{\Cl_n} \right] \in \mathfrak{M}(\Cl_n \otimes \mathbb{R} \pi)/i^* \mathfrak{M}(\Cl_{n+1} \otimes \mathbb{R} \pi)$$

depends only on the manifold $M$, not the metric on $M$, and only on the homotopy class of the map $f : M \to B \pi$. The $K$-theory group $KO_n(\mathbb{R} \pi)$ is equal to $\mathfrak{M}(\Cl_n \otimes \mathbb{R} \pi)/i^* \mathfrak{M}(\Cl_{n+1} \otimes \mathbb{R} \pi)$ by definition.

For infinite $\pi$, the construction involves more analytic details. A central role is played by the flat vector bundle

$$\tilde{M} \times_\pi C^* \pi \longrightarrow M,$$

where $C^* \pi \supset \mathbb{R} \pi$ is the reduced group $C^*$-algebra of $\pi$, a suitable completion of $\mathbb{R} \pi$. The dimension of this vector bundle is the cardinality of $\pi$; in particular, it is infinite dimensional if $\pi$ is infinite. Fortunately, it can also be thought of as a line bundle, if we don’t regard it as a real vector bundle, but rather as a bundle of right $C^* \pi$-modules. The Clifford linear spinor bundle on $M$ can be tensored with this flat “line bundle” to obtain a vector bundle whose sections are modules over the $C^*$-algebra
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$\mathcal{C}_n \otimes C^* \pi$. Moreover, there is a twisted Dirac operator acting on this section space which is $\mathcal{C}_n \otimes C^* \pi$-linear. Using Kasparov’s description of $\text{KO}_n(A)$ for real $C^*$-algebras $A$, and the index theory of Fredholm operators acting on Hilbert $A$-modules developed by Mislin-Fomenko, Rosenberg extracts the index of this twisted Dirac operator as an element $\alpha(M,f) \in \text{KO}_n(C^* \pi)$ [Ro1986III, section 3].

2.4 Obstructions coming from stable minimal hypersurfaces

The following result was proved by Schoen-Yau using stable minimal hypersurfaces [SY1979a, Proof of Thm. 1], [SY1979b, Thm. 5.1].

**Theorem 2.12. (Schoen and Yau).** Let $M$ be a closed Riemannian manifold of dimension $n \geq 3$ equipped with a positive scalar curvature metric. If $N \subset M$ is a smooth hypersurface with trivial normal bundle, and if $N$ is a local minimum of the volume functional, then $N$ carries a positive scalar curvature metric.

Results from geometric measure theory guarantee that if $M$ is orientable of dimension $n \leq 7$, and $x \in H_{n-1}(M)$ is a non-vanishing integral homology class, then there is a smooth orientable hypersurface $N \subset M$ which represents $x$ and is a local minimum of the volume functional (see [Sch1998, Thm. 1.3] for relevant references on geometric measure theory). Without the dimension restriction $n \geq 7$, any homology class can be represented by an area-minimizing current $N$ whose singularities have codimension $\geq 7$ in $N$. Hence the condition $n \geq 7$ guarantees that $N$ has no singularities.

For a topological space $X$, let $H^+_n(X)$ be the subgroup of $H_n(X)$ consisting of homology classes of the form $f_*[M]$, where $M$ is a closed oriented $n$-manifold that carries a positive scalar curvature metric, $[M] \in H_n(M)$ is the fundamental homology class of $M$, and $f : M \to X$ is a map. The author observed the following consequence of the results discussed above (see [Sch1998, Cor. 1.5]).

**Corollary 2.13.** For a topological space $X$, let $\alpha \cap : H_n(X) \to H_{n-1}(X)$ be the map given by the cap product with a cohomology class $\alpha \in H^1(X)$ (all (co)homology groups here have integer coefficients). Then for $3 \leq n \leq 7$ this homomorphism maps $H^+_n(X)$ to $H^+_{n-1}(X)$.

**Proof.** If $x \in H_n(X)$ is represented by $f : M \to X$, then $\alpha \cap x \in H_{n-1}(X)$ is represented by $f_*[N] : H \to M$, where $N \subset M$ is a hypersurface in $M$ representing the homology class $f^* \alpha \cap [M] \in H_{n-1}(M)$. If $x$ belongs to $H^+_n(X) \subset H_n(X)$ we can assume that $M$ carries a positive scalar curvature metric and thanks to the dimension restriction $n \leq 7$, we can assume that the hypersurface $N \subset M$ is a local minimizer of the volume function. Hence $N$ carries a positive scalar curvature metric by Theorem 2.12, and so $f|_N : N \to X$ represents a homology class in $H^+_{n-1}(X)$. □
3 Constructions of positive scalar curvature metrics

3.1 Basic examples of positive scalar curvature metrics

There are many manifolds which carry positive scalar curvature metrics. For example, the standard Riemannian metric on the sphere $S^n$ (known as “round metric”) has positive scalar curvature for $n \geq 2$. The complex projective space $\mathbb{CP}^n$ is the quotient of the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with respect to the $S^1$-action given by scalar multiplication. Since $S^1$ acts by isometries, the round metric on $S^{2n+1}$ induces a metric on $\mathbb{CP}^n$ known as Fubini-Study metric (characterized by the property that the projection map $S^{2n+1} \to \mathbb{CP}^n$ is a Riemannian submersion). The Fubini-Study metric on $\mathbb{CP}^n$ has positive scalar curvature. The same construction yields a positive scalar curvature metric on the real projective space $\mathbb{RP}^n$, $n \geq 2$ (the quotient of an isometric $\mathbb{Z}/2\mathbb{Z}$-action on $S^n$) and the quaternionic projective space $\mathbb{HP}^n$ (the quotient of an isometric SU(2)-action on $S^{4n+1} \subset \mathbb{H}^{n+1}$).

The following observation is elementary, but very useful to find many examples of manifolds that carry positive scalar curvature metrics.

3.1. Observation. Let $g$ be a positive scalar curvature metric on a closed manifold $M$. Then the following manifolds admit metrics of positive scalar curvature:

1. The product $M \times N$ for any closed manifold $N$.
2. The total space of any fiber bundle $E \to N$ with fiber $M$ whose structure group is the isometry group of $(M, g)$; i.e., there are local trivializations such that all transition functions are isometries of $(M, g)$.

To prove the first claim, pick a metric $h$ on $N$ (not necessarily with positive scalar curvature) and consider the product metric $g \times h$ on $M \times N$. The scalar curvature at a point $(x, y) \in M \times N$ with respect to the metric $g \times h$ is given by the formula

$$s((x, y); g \times h) = s(x; g) + s(y; h).$$

So without further assumptions on $h$ there is no reason for the scalar curvature of the product metric $g \times h$ to be positive. The trick is to “shrink $M$”, i.e., to replace $g$ by $tg$ for $t > 0$, and letting $t$ approach 0. Then

$$s((x, y); tg \times h) = s(x; tg) + s(y; h) = \frac{1}{t}s(x; g) + s(y; h),$$

and hence this is positive for $t$ sufficiently small thanks to $s(x; g) > 0$. Compactness of $M$ and $N$ guarantee that there is one $t$ that will do the job for all $(x, y) \in M \times N$.

In the case of a fiber bundle $E \to N$ with fiber $M$, the construction of a metric on $E$ requires the choice of a metric $h$ on $N$, and a connection on the fiber bundle. The condition on the structure group guarantees that the connection can be chosen such that the fibers are totally geodesic. Using the O’Neill formulas, which express...
the curvature of $E$ in terms of the connection and the curvatures of $M$ and $N$, then shows that the “shrinking trick” still works for such a fiber bundle: shrinking the fibers $M$ sufficiently makes the scalar curvature function on the total space $E$ everywhere positive.

### 3.2 Surgery results

Surgery is a basic technique to modify the topology of a manifold $M$ by removing a piece of $M$ and replacing it by another piece. The simplest example is the surgery that replaces the disjoint union of two $n$-manifolds $M_1$, $M_2$ by their connected sum $M_1 \# M_2$. Independently Gromov-Lawson [GL1980, Thm. A] and Schoen-Yau [SY1979a, Cor. 6] proved the following result.

**Theorem 3.2 (Surgery Theorem).** Let $M$ be a manifold that carries a positive scalar curvature metric. Then any manifold obtained by a surgery of codimension $\geq 3$ also carries a positive scalar curvature metric.

This result implies in particular that if two manifolds $M_1$, $M_2$ of dimension $\geq 3$ carry positive scalar curvature metrics, then so does their connected sum $M_1 \# M_2$. We first outline the proof for this special case. Then we recall what a $k$-surgery is, and finally we outline the proof of the Surgery Theorem in the general case.

Let $M_1$, $M_2$ be closed connected manifolds of dimension $n$. The connected sum $M_1 \# M_2$ is the closed connected $n$-manifold obtained in the following way: Deleting an open $n$-ball $\hat{D}^n$ from $M_i$ gives a manifold $M_i \setminus \hat{D}^n$ with boundary $S^{n-1}$. Identifying a point on the boundary of $M_1 \setminus \hat{D}^n$ with the corresponding point on the boundary of $M_2 \setminus \hat{D}^n$ yields the connected sum

$$M_1 \# M_2 := (M_1 \setminus \hat{D}^n) \cup_{S^{n-1}} (M_2 \setminus \hat{D}^n).$$

(3.3)

Let $g_1$, $g_2$ be positive scalar curvature metrics on $M_1$ resp. $M_2$. The key for obtaining a positive scalar curvature metric on the connected sum $M_1 \# M_2$ is to modify the metric $g_i$ to obtain a positive scalar curvature metric $g'_i$ on the manifold $M_i \setminus \hat{D}^n$ which

(i) restricts to the standard round metric on the boundary $\partial(M_i \setminus \hat{D}^n) = S^{n-1}$, and

(ii) is a product metric near the boundary.

These conditions guarantee that the metrics $g'_1$ and $g'_2$ fit together to give a positive scalar curvature metric on the connected sum $M_1 \# M_2$. We remark that the dimension restriction $n \geq 3$ ensures that the standard metric on $S^{n-1}$ has positive scalar curvature.

Let $g$ be positive scalar curvature metric on a manifold $M$ of dimension $n \geq 3$ and fix a point $p \in M$. The modified positive scalar curvature metric $g'$ with properties (i) and (ii) on the complement of a disk $D^n \subset M$ containing $p$ is constructed as follows. Think of $M \setminus \hat{D}^n$ as a hypersurface $N$ in the product $M \times [0, 1]$ as shown in the following
The hypersurface \( N \subset M \times [0, 1] \)

More precisely, the hypersurface \( N \) is determined by choosing a smooth curve \( \gamma \) in the \((t, r)\)-plane as shown below:

The curve contains the points on the positive \( r \)-axis for \( r > r_0 \), and the horizontal line segment \( \{(t, \epsilon) \mid t_0 \leq t \leq 1\} \). Such a curve determines a hypersurface \( N \) of the product \( M \times [0, 1] \), defined by

\[
N = \{(x, t) \in M \times [0, 1] \mid (\text{dist}(x, p), t) \in \gamma\},
\]

where \( \text{dist}(x, p) \) is the distance between the points \( x \) and \( p \). Choosing \( r_0 \) to be the injectivity radius of \( g \) guarantees that \( N \) is a smooth hypersurface of \( M \times [0, 1] \). The product metric on \( M \times [0, 1] \) induces a Riemannian metric on \( N \), such that the boundary
3 CONSTRUCTIONS OF POSITIVE SCALAR CURVATURE METRICS

The construction of a positive scalar curvature metric on a boundary of a manifold involves understanding the contributions from the boundary and the interior. Near the boundary, the metric is a product metric, which simplifies the analysis of scalar curvature.

The key point is that a careful choice of the curve $\gamma$ guarantees that the positive contributions to the scalar curvature of $N$ coming from the directions tangent to the $(n-1)$-spheres, given by the intersection of $N$ with $M \times \{t\}$ for fixed $t \in (0, 1)$, dominates the negative contribution coming from the orthogonal direction. After an additional deformation of the metric in a small neighborhood of $\partial N$, this is a positive scalar curvature metric on $N$ which is the standard round metric on $\partial N = S^{n-1}$, as well as a product metric near the boundary.

Next we recall the what a $k$-surgery is, and outline the proof of the Surgery Theorem.

Let $M$ be a manifold of dimension $n$ and assume that there is an embedding of $S^k \times D^{n-k}$ in $M$. Then the complement $M \setminus (S^k \times \hat{D}^{n-k})$ is a manifold with boundary $S^k \times S^{n-k-1}$, which can be glued with $D^{k+1} \times S^{n-k-1}$ along their common boundary to obtain a new $n$-manifold

$$\hat{M} := M \setminus (S^k \times \hat{D}^{n-k}) \cup_{S^k \times S^{n-k-1}} (D^{k+1} \times S^{n-k-1}). \quad (3.4)$$

We say that $\hat{M}$ is obtained from $M$ by a $k$-surgery (or by a surgery of codimension $n-k$). For example, the connected sum $M_1 \# M_2$ of two $n$-manifolds is obtained by a 0-surgery from the disjoint union $M_1 \sqcup M_2$.

The proof of the Surgery Theorem follows the same strategy as in the connected sum case: the positive scalar curvature metric on $M$ is obtained by constructing positive scalar curvature metrics on the two pieces, $D^{k+1} \times S^{n-k-1}$ and $M \setminus (S^k \times \hat{D}^{n-k})$, which agree on the common boundary $S^k \times S^{n-k-1}$, and which are product metrics near the boundary. On $S^k \times \hat{D}^{n-k}$, this is the product metric of the standard round metric on $S^k$, and the hemisphere metric on $D^{n-k} \subset S^{n-k}$ (slightly deformed in order to make it a product metric near the boundary). On $M \setminus (S^k \times \hat{D}^{n-k})$, it is a modification $g'$ of the given positive scalar curvature metric $g$ on $M$. It is obtained by generalizing the technique used above. The curve $\gamma$ as above determines a hypersurface $N \subset M \times [0, 1]$ given by

$$N = \left\{ (x, t) \in M \times [0, 1] \mid (\text{dist}(x, S^k), t) \in \gamma \right\},$$

where $\text{dist}(x, S^k)$ is distance of $x$ from the embedded $k$-sphere $S^k \subset S^k \times D^{n-k} \subset M$ that we do surgery on. The intersection of $N$ with $M \times \{t\}$ for fixed $t \in (0, 1]$ consists of the points in $M$ that have distance $\gamma(t)$ from $S^k$; this is a fiber bundle over $S^k$ whose fibers are spheres of dimension $n - k - 1$. So the codimension restriction $n - k \geq 3$ guarantees that the curvature in the direction tangent to these spheres has a positive contribution to the scalar curvature of the hypersurface. As in the special case, a careful choice of $\gamma$ guarantees that the scalar curvature of $N$ is positive, and an additional modification of the metric near the boundary ensures that its restriction to $\partial N = S^k \times S^{n-k-1}$ is the standard product metric.
3.3 Bordism results

The goal of this section is to outline how the Surgery Theorem 3.2 is used to prove the Bordism Theorem 1.4, which we restate here for the convenience of the reader.

**Theorem 3.5. (Bordism Theorem).** Let \( M \) be a closed connected \( n \)-manifold, \( n \geq 5 \), with fundamental group \( \pi \), and let \( u: M \to B\pi \) be the classifying map of the universal covering \( \tilde{M} \to M \).

(i) If \( M \) is spin, then it carries a positive scalar curvature metric if and only if \( [M, u] \in \Omega^{\text{Spin}}_n(B\pi) \).

(ii) If \( M \) is oriented and \( \tilde{M} \) is non-spin, then \( M \) admits a positive scalar curvature metric if and only if \( [M, u] \in \Omega^{\text{SO}}_n(B\pi) \).

One of the implications of the theorem is tautological: if \( M \) is a closed \( n \)-manifold with a \( G \)-structure (i.e., an orientation if \( G = \text{SO} \) or a spin structure if \( G = \text{Spin} \)) and \( M \) carries a positive scalar curvature metric, then the bordism class \( [M, u] \in \Omega^G_n(B\pi) \), which by definition consists of bordisms classes representable by pairs \((N, f: N \to B\pi)\), where \( N \) carries a positive scalar curvature metric.

The non-trivial statement is that \([M, u] \in \Omega^G_n(B\pi)\) implies that \( M \) carries a positive scalar curvature metric. In other words, if \((M, u)\) is \( G \)-bordant to a pair \((N, f)\) where \( N \) carries a positive scalar curvature metric, then \( M \) itself carries a positive scalar curvature metric. Unwinding the assumption that \((M, u)\) is \( G \)-bordant to \((N, f)\), it means that there is a pair \((W, F)\), where

- \( W \) is a \( G \)-bordism between \( M \) and \( N \), i.e., a manifold of dimension \( n + 1 \) equipped with a \( G \)-structure whose boundary \( \partial W \) is the disjoint union of the \( G \)-manifolds \( M \) and \( -N \), where \( -N \) is the manifold \( N \), equipped with the opposite orientation/spin structure.
- \( F: W \to B\pi \) is a map which makes the following diagram commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{\iota^M} & W \\
\downarrow u & & \downarrow F \\
B\pi & \xleftarrow{\iota^N} & N \\
\end{array}
\]  

(3.6)

The idea for constructing a positive scalar curvature metric on \( M \) is to show that \( M \) can be constructed by a sequence of surgeries starting from \( N \), and to use the Surgery Theorem 3.2 to propagate the given positive scalar curvature metric on \( N \) to one on \( M \). Morse theory shows that \( M \) is obtained by a sequence of surgeries from \( N \): pick a Morse function \( g: W \to [0, 1] \) on \( W \) with \( g|_M = 0 \) and \( g|_N = 1 \) and consider the topology of the level sets \( W_t = g^{-1}(t) \) for \( t \in [0, 1] \):

- If there is no critical value of \( g \) in some interval \([t, t']\), then \( W_t \) is diffeomorphic to \( W_{t'} \).
- If there is one critical point \( x \) of index \( i \) with \( g(x) \in (t, t') \), then \( W_t \) is obtained from \( W_{t'} \) by a surgery of codimension \( i \).
In particular, if all critical points have different values, which can be arranged for, then $M = W_0$ can be obtained from $N = W_1$ by a sequence of surgeries. Moreover, all these surgeries have codimension $\geq 3$ as required by the Surgery Theory 3.2 provided the Morse function $g$ has no critical points of index $\leq 2$.

A Morse function $g$ allows to calculate the relative homology groups $H_*(W, M)$ (with integer coefficients) via the Morse chain complex associated to $g$. That chain complex has one copy of $\mathbb{Z}$ in degree $i$ for any critical point of index $i$, and hence the vanishing of the groups $H_k(W, M)$ for $0 \leq k \leq 2$ is a necessary condition for the existence of a Morse function $g$ without critical points of index $0 \leq i \leq 2$. A sufficient condition is that the inclusion map $i^M: M \hookrightarrow W$ is a 2-equivalence, i.e., the induced map on homotopy $i_*^M: \pi_k(M) \to \pi_k(W)$ is an isomorphism for $k = 0, 1$ and is surjective for $k = 2$.

There is no reason that for the given bordism $W$ the inclusion map $i^M: M \hookrightarrow W$ is a 2-equivalence, but we claim that by surgeries in the interior of $M$ (i.e., without affecting the boundary $\partial W = M \sqcup N$), we can modify the bordism $(W, F: W \to B\pi)$ such that

(i) $F: W \to B\pi$ is a 3-equivalence if $G = \text{Spin}$, and
(ii) $F \times c^{TW}: W \to B\pi \times BSO$ is a 3-equivalence if $G = \text{SO}$.

Here $c^{TW}$ is a classifying map of stable tangent bundle of the oriented manifold $W$.

Assuming the claim above, let us argue that the inclusion map $i^M: M \hookrightarrow W$ to the (modified) bordism $W$ is a 2-equivalence. The classifying map $u: M \to B\pi$ of the universal covering of $\tilde{M}$ is always a 2-equivalence. Hence the fact that $F$ is a 3-equivalence in the case $G = \text{Spin}$ implies that $i^M$ is a 2-equivalence by the commutative diagram (3.6). In the case $G = \text{SO}$, we instead look the commutative (up to homotopy) diagram

$$
\begin{array}{ccc}
M & \xrightarrow{i^M} & W \\
\downarrow{u \times c^{TM}} & & \downarrow{F \times c^{TW}} \\
B\pi \times BSO & & B\pi \times BSO
\end{array}
$$

The condition that the universal covering $\tilde{M}$ of $M$ does not admit a spin structure implies that the classifying map $c^{TM}: M \to BSO$ of the stable tangent bundle of $M$ induces a surjection $c_*^{TM}: \pi_2(M) \to \pi_2(BO) = \mathbb{Z}/2$. Hence the map $u \times c^{TM}$ is a 2-equivalence, and since $F \times c^{TW}$ is a 3-equivalence, it follows that $i^M$ is a 2-equivalence.

To arrange for conditions (i) resp. (ii) by surgery on $W$, we can do 0-surgeries to make $W$ connected which ensures that the map $F_*: \pi_k(W) \to \pi_k(B\pi)$ is an isomorphism for $k = 0$. For $k = 1$, the map $F_*$ is surjective (since $u_*: \pi_1(M) \to \pi_1(B\pi)$ is an isomorphism). Modifying $W$ by surgeries of embedded circles which generate the kernel of $F_*$ ensures that $F_*$ is an isomorphism for $k = 1$. Similarly, if $W$ is spin, the elements of $\pi_2(W)$ can be represented by embedded 2-spheres with trivial normal bundle, allowing to do surgery on these spheres to achieve $\pi_2(W) = 0$. For $G = \text{SO}$, only the elements in the kernel of $c_*^{TW}: \pi_2(W) \to \pi_2(BO)$ can be represented by embedded 2-spheres with trivial normal bundle; doing surgery on those makes the kernel
of $c_T^W$ trivial, and hence $F \times c^W$ is a 3-equivalence.

### 3.4 Simply connected manifolds with positive scalar curvature metrics

The goal of this section is to outline the proofs of Corollary 1.5 and Theorem 1.6 which characterize those simply connected closed manifolds of dimension $n \geq 5$ which carry positive scalar curvature metrics (these are restated as Theorem 3.7 resp. Theorem 3.8 below). The Bordism Theorem 3.5 shows this amounts to determining the subgroup $\Omega^{G,+}_n \subset \Omega^G_n$, $G = \text{SO, Spin}$ of the oriented (resp. spin) bordism group represented by manifolds with positive scalar curvature metrics.

Based on prior work of Thom, Milnor and Dold, the calculation of the oriented bordism groups $\Omega^\text{SO}_n$ was completed by Wall [Wa1960]. In fact, he determined the structure of the $\mathbb{Z}$-graded bordism ring

$$\Omega^\text{SO}_* := \bigoplus_{n=0}^{\infty} \Omega^\text{SO}_n,$$

whose multiplication is given by the cartesian product of manifolds, and provided a list of explicit closed oriented manifolds whose bordism classes multiplicatively generate this ring. Gromov and Lawson noticed that all of these manifolds admit positive scalar curvature metrics, since each one of them can be identified with the total space of a fiber bundle $E \rightarrow N$ whose fibers are complex projective spaces, and whose structure group is the isometry group of the standard positive scalar curvature metric on the complex projective space (see Observation 3.1). By the Bordism Theorem this implies the following result.

**Theorem 3.7.** (Gromov-Lawson, [GL1980, Cor. C]). Every closed simply-connected $n$-manifold, $n \geq 5$, which is not spin, carries a metric of positive scalar curvature.

By contrast, the Lichnerowicz-Hitchin Theorem 1.2 shows that the vanishing of the index invariant $\alpha(M) \in \text{KO}_n$ is a necessary condition for a spin $n$-manifold $M$ to carry a positive scalar curvature metric. In [GL1980] Gromov and Lawson conjectured the following result, which was proved by the author [St1992, Thm. A].

**Theorem 3.8.** Let $M$ be a simply-connected spin manifold of dimension $n \geq 5$. Then $M$ carries a positive scalar curvature metric if and only if $\alpha(M) = 0$.

By the Lichnerowicz-Hitchin Theorem 1.2 there is an inclusion

$$\Omega^{\text{Spin},+}_n \hookrightarrow \ker(\alpha: \Omega^{\text{Spin}}_n \rightarrow \text{KO}_n). \hspace{1cm} (3.9)$$

Here $\Omega^{\text{Spin},+}_n \subset \Omega^{\text{Spin}}_n$ is the subgroup of bordism classes represented by manifolds carrying positive scalar curvature metrics, and $\alpha$ is the well-defined homomorphism given by sending $[M] \in \Omega^{\text{Spin}}_n$ to the index invariant $\alpha(M) \in \text{KO}_n$. By the Bordism Theorem...
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3.5(2), the theorem above is equivalent to showing that the inclusion \( (3.9) \) is an equality. Gromov and Lawson showed that the inclusion map is an isomorphism rationally \([GL1980\text{ Cor. B}]\), and Miyazaki proved it after tensoring with \( \mathbb{Z}[\frac{1}{2}] \) \([Miy1985]\). Rosenberg showed equality for \( n \leq 23 \) by exhibiting explicit generators for the kernel of \( \alpha \) in that range \([Ro1986\text{III Thm. 1.1}]\).

The spin bordism groups \( \Omega_n^{\text{Spin}} \) have been completely determined by Anderson, Brown and Peterson \([ABP1967]\), but unlike for the oriented bordism groups, still today no explicit manifolds are known that generate them. The proof of Theorem 3.8 is based on Observation 3.1(2) that the total space of a fiber bundle \( E \to N \) carries a positive scalar curvature metric, provided the fiber comes equipped with a positive scalar curvature metric, and the structure group is the isometry group of the fiber. A good candidate for the fiber is the quaternionic projective plane \( \mathbb{HP}^2 \), since its spin bordism class \( \mathbb{HP}^2 \in \Omega_8^{\text{Spin}} \) is a generator of the kernel of \( \alpha \) in degree 8, which is the first degree in which the kernel of \( \alpha \) is non-trivial. The standard Riemannian metric on \( \mathbb{HP}^2 \) has positive scalar curvature; its isometry group is the projective-symplectic group \( \text{PSp}(3) \). Hence Theorem 3.8 is a consequence of the following completely topological result.

**Theorem 3.10.** Every element in the kernel of \( \alpha: \Omega_n^{\text{Spin}} \to KO_n \) is represented by a total space of a fiber bundle with fiber \( \mathbb{HP}^2 \) and structure group \( H = \text{PSp}(3) \).

Localized at the prime 2, this was proved by the author \([St1992\text{ Thm. B}]\), and by Kreck and the author in \([KrSt1993\text{ Prop. 4.2}]\) after inverting 2. We explain below what this means, and outline the proof of this result in the rest of this section.

Every fiber bundle \( E \to N \) with fiber \( \mathbb{HP}^2 \) and structure group \( H = \text{PSp}(3) \) is the pull-back of the universal \( \mathbb{HP}^2 \)-bundle \( EH \times_H \mathbb{HP}^2 \to BH \) via some map \( f: N \to BH \). It turns out that a spin structure on \( N \) induces a spin structure on the total space \( E = f^*(EH \times_H \mathbb{HP}^2) \), and hence we can define a homomorphism

\[
\Psi: \Omega_{n-8}^{\text{Spin}}(BH) \to \Omega_n^{\text{Spin}}
\]

by mapping the bordism class of \( f: N \to BH \) to the bordism class of the total space of the pullback bundle \( f^*(EH \times_H \mathbb{HP}^2) \).

We note that the composition

\[
\Omega_{n-8}^{\text{Spin}}(BH) \xrightarrow{\Psi} \Omega_n^{\text{Spin}} \xrightarrow{\alpha} KO_n
\]

is trivial, since the image of \( \Psi \) is represented by total spaces of \( \mathbb{HP}^2 \)-bundles; these carry positive scalar curvature metrics and hence their \( \alpha \)-invariant is trivial by the Lichnerowicz-Hitchin Theorem 1.2. Theorem 3.10 is equivalent to the statement ker \( \alpha = \text{im } \Psi \), i.e., to the exactness of the sequence (3.11) at the middle term.

To prove this, it suffices to show exactness localized at the prime 2, i.e., after tensoring the sequence with \( \mathbb{Z}(2) \), and with 2 inverted, i.e., after tensoring with \( \mathbb{Z}[\frac{1}{2}] \). Here

\[
\mathbb{Z}(2) = \{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ prime to } 2 \} \quad \text{and} \quad \mathbb{Z}[\frac{1}{2}] = \{ \frac{a}{b} \in \mathbb{Q} \mid b \text{ is a power of } 2 \}.
\]

(3.12)
The $\alpha$-invariant is multiplicative, i.e., $\alpha(M \times N) = \alpha(M)\alpha(N)$ for closed spin manifolds $M, N$. Here the product $\alpha(M)\alpha(N) \in KO_\ast$ is given by the tensor product of the Clifford modules representing these elements (see (2.8)). Put another way, the $\alpha$-invariant gives a homomorphism of graded rings

$$\Omega^\text{Spin}_n := \bigoplus_{n \geq 0} \Omega^\text{Spin}_n \xrightarrow{\alpha} KO_n := \bigoplus_{n \geq 0} KO_n.$$  

Explicitly, $KO_n = \mathbb{Z}[\eta, \omega, \mu]/(2\eta, \eta^3, \omega^2 - 4\mu)$, where $\eta, \omega, \mu$ are elements of degree 1, 4, and 8 respectively. In fact,

$$\eta = \alpha(S^1) \quad \omega = \alpha(K) \quad \mu = \alpha(B),$$

where $S^1$ is the circle with the non-bounding spin structure, $K$ is the Kummer surface, a degree 4 hypersurface in $\mathbb{C}P^3$ (see Example 2.4), and $B$ is any closed spin 8-manifold with $\tilde{A}(K) = 1$.

**Proof of Theorem 3.10 with 2 inverted.** The spin bordism ring with 2 inverted, i.e., the ring $\Omega^\text{Spin}_n \otimes \mathbb{Z}[\frac{1}{2}]$ is the polynomial ring $\mathbb{Z}[\frac{1}{2}][x_4, x_8, \ldots]$ with generators $x_{4k}$ of degree $4k$. The generator $x_4$ can be chosen to be the bordism class $[K]$ of the Kummer surface. By a characteristic class calculation, it was shown in [KrSt1993, Prop. 4.2] that for $i \geq 2$ there are manifolds $M^{4i}$ of dimension $4i$ which are $\mathbb{H}P^2$-bundles over a closed spin manifolds with structure group $Sp(3)$ such that the bordism classes

$$[K^4], [M^8], [M^{12}], \ldots$$

are generators of the polynomial ring $\Omega^\text{Spin}_n \otimes \mathbb{Z}[\frac{1}{2}]$.

The classes $[M^{4i}]$ are in the kernel of $\alpha$, since these manifolds carry positive scalar curvature metrics, while $\alpha(K) = \omega$ is the generator of the ring $ko_\ast \otimes \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][\omega]$. This shows that the kernel of $\alpha$ with 2 inverted is the ideal generated by the elements $[M^{4i}]$, for $i = 2, 3, \ldots$. As $\mathbb{H}P^2$-bundles, these generators are in the image of $\Psi$, consisting of all bordism classes represented by total spaces of $\mathbb{H}P^2$-bundles. Since the image of $\Psi$ is an ideal, it contains all of $\ker \alpha$, thus proving Theorem 3.10 with 2 inverted.

**Proof of Theorem 3.10 localized at 2.** The proof at the prime 2 is more involved, due to the intricate structure of spin bordism at the prime 2. Like the computation of the bordism groups $\Omega^\text{SO}_n$ and $\Omega^\text{Spin}_n$, the proof of this statement is based on the Pontryagin-Thom isomorphism which expresses bordism groups as homotopy groups of associated Thom spectra. More precisely, the groups in the sequence (3.11) can be expressed as homotopy groups of spectra by means of a commutative diagram

$$\begin{array}{cccccc}
\Omega^\text{Spin}_n(BH) & \xrightarrow{\Psi} & \Omega^\text{Spin}_n & \xrightarrow{\alpha} & KO_n \\
\cong \downarrow & & \cong \downarrow & & \cong \\
\pi_n(M\text{Spin} \wedge \Sigma^8 BH) & \xrightarrow{T_\ast} & \pi_n(M\text{Spin}) & \xrightarrow{T^\text{Spin}_\ast} & \pi_n(ko)
\end{array}$$

(3.14)
Here

- $MSpin$ is the Thom spectrum whose $n$-th space $MSpin_n$ is the Thom space of the vector bundle over $BSpin(n)$ given by the pullback of the universal $n$-dimensional vector bundle $\gamma^n \to BO(n)$. The middle isomorphism is given by the Pontryagin-Thom construction.

- The left isomorphism is also given by the Pontryagin-Thom isomorphism

$$\Omega_{n-8}^{Spin}(BH) \cong \pi_{n-8}(MSpin \wedge BH_+)$$

composed with the suspension isomorphism

$$\pi_{n-8}(MSpin \wedge BH_+) \cong \pi_n(MSpin \wedge \Sigma^8 BH_+).$$

- The map $T: MSpin \wedge \Sigma^8 BH_+ \to MSpin$ is a transfer map associated to the $\mathbb{H}P^2$-bundle $EH \times_H \mathbb{H}P^2 \to BH$ [St1992 section 2].

- $ko$ is the connective version of the real $K$-theory spectrum, i.e., $\pi_n(ko) \cong KO_n$ for $n \geq 0$ and $\pi_n(ko) = 0$ for $n < 0$. The spectrum map $U^{Spin}: MSpin \to ko$ is the homotopy theoretic incarnation of Atiyah’s real $K$-theory orientation class of spin vector bundles.

The composition $U^{Spin} \circ T$ induces the trivial map on homotopy groups by the diagram above, and the vanishing of $\alpha \circ \Psi$ by the Lichnerowicz-Hitchin Theorem [1.2]. In fact, the composition itself is zero homotopic as an argument based on the Family Index Theorem shows [St1992 Prop. 1.1]. This implies that up to homotopy $T$ factors through a map $\widehat{T}$ from $MSpin \wedge \Sigma^8 BH_+$ to the homotopy fiber $\overline{MSpin}$ of $U^{Spin}$. This yields the commutative diagram whose bottom row is exact.

\[
\begin{array}{ccc}
\pi_n(MSpin \wedge \Sigma^8 BH_+) & \xrightarrow{T_*} & \pi_n(MSpin) \\
\downarrow{\widehat{T}_*} & & \downarrow{U_*^{Spin}} \\
\pi_n(\overline{MSpin}) & \xrightarrow{T_*} & \pi_n(ko) \\
\end{array}
\] (3.15)

It follows that exactness at the middle term of the top row is equivalent to surjectivity of $\widehat{T}_*$.

To prove surjectivity of the map $\widehat{T}_*$ localized at 2, use is made of the mod 2 Adams spectral sequence which for a spectrum $X$ converges to $\pi_*X \otimes \mathbb{Z}/(2)$, the homotopy groups of $X$ localized at 2. Its $E_2$-page is given by Ext-groups built from the homology $H_*(X; \mathbb{Z}/2)$ viewed as comodule over the dual Steenrod algebra $A_*$. The map $\widehat{T}$ induces a map of Adams spectral sequences, which on the $E_2$-page is determined by the map induced by $\widehat{T}$ on $\mathbb{Z}/2$-homology. A calculation shows that this homology map is a split surjection of $A_*$-comodules [St1992 Prop. 1.3], and hence the map induced by $\widehat{T}$ on the $E_\infty$-page is surjective. This does not imply that the map induced by $\widehat{T}$ on the $E_\infty$-page is surjective, since there could be non-trivial differentials in the domain.
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spectral sequence. Fortunately, this is not the case [St1992 Prop. 1.5], and so \( \hat{T} \) does indeed induce a surjection on the \( E_\infty \)-page. This implies that the map induced by \( \hat{T} \) is surjective on 2-local homotopy groups, which proves Theorem 3.10 localized at 2. □

3.5 Reduction to homology and \( \text{ko} \)-homology

This section is an outline of the proof of Theorem 1.7 according to which a closed connected \( n \)-manifold \( M, n \geq 5 \), with fundamental group \( \pi \) carries a positive scalar curvature metric if and only if \( u_*[M]_{\text{Spin}}^+ \in \text{ko}_n^+(B\pi) \) (if \( M \) is spin) resp. \( u_*[M]_{\text{SO}}^+ \in H_n^+(B\pi) \) (if \( M \) is oriented and the universal covering \( \tilde{M} \) is non-spin). The proof is based on the Bordism Theorem 3.5, whose statement is completely analogous, but the condition for the existence of a positive scalar curvature metric is \([M, u] \in \Omega_{n}^{\text{Spin},+}(B\pi)\) resp. \([M, u] \in \Omega_{n}^{\text{SO},+}(B\pi)\). These conditions are related by natural transformations of homology theories

\[
U_{\ast}^{\text{Spin}}: \Omega_{n}^{\text{Spin}}(X) \to \text{ko}_{n}(X) \quad U_{\ast}^{\text{SO}}: \Omega_{n}^{\text{SO}}(X) \to H_{n}(X)
\]

(3.16)

given by sending a bordism class \([f: M \to X] \in \Omega_{n}^{G}(X)\) to \( f_*[M]^G \) for \( G = \text{Spin}, \text{SO} \). By definition, \( \text{ko}_{n}^{+}(X) = U_{\ast}^{\text{Spin}}(\Omega_{n}^{\text{Spin},+}(X)) \) and \( H_{n}^{+}(X) = U_{\ast}^{\text{SO}}(\Omega_{n}^{\text{SO},+}(X)) \), and hence to prove Theorem 1.7 it suffices to show

\[
\ker U_{\ast}^{G} \subset \Omega_{n}^{G,+}(X) \quad \text{for } G = \text{Spin, SO}.
\]

(3.17)

The first step towards proving this is to show that the map \( U_{\ast}^{G} \) is part of an exact sequence, allowing us to think of the kernel \( \ker U_{\ast}^{G} \) as the image of a map. This is achieved in an abstract way, by looking at the spectra representing these homology theories, as well as the spectrum maps inducing the natural transformations (3.16), and taking the homotopy fibers of these maps.

We recall that a spectrum \( E \) determines a (generalized) homology theory \( E_{\ast}(X) \) by defining the \( n \)-th \( E \)-homology group \( E_{n}(X) \) of a topological space \( X \) by \( E_{n}(X) := \pi_{n}(E \wedge X_{+}) \). Conversely, every homology theory comes from a spectrum. For example, the Pontryagin-Thom construction yields isomorphisms

\[
\Omega_{n}^{G}(X) \cong \pi_{n}(MG \wedge X_{+}) = MG_{n}(X) \quad \text{for } G = \text{Spin, SO}.
\]

This shows that homology theories \( \Omega_{n}^{\text{Spin}}(X) \) (resp. \( \Omega_{n}^{\text{SO}}(X) \)) are the homology theories associated to the Thom spectra MS\( n \) resp. MSO. Integral homology \( H_{n}(X) \) is associated to the integral Eilenberg Mac Lane spectrum \( HZ \) (which up to homotopy equivalence is determined by \( \pi_{0}(HZ) \cong \mathbb{Z} \) and \( \pi_{n}(HZ) = 0 \) for \( n \neq 0 \)). The connective real \( K \)-homology \( \text{ko}_{n}(X) \) is by definition the homology theory associated to the real connective \( K \)-theory spectrum \( \text{ko} \).

A map between spectra \( U: E \to F \) determines a natural transformation \( E_{n}(X) \xrightarrow{U_{\ast}} F_{n}(X) \) given by

\[
E_{n}(X) = \pi_{n}(E \wedge X_{+}) \xrightarrow{(U \wedge \text{id}_{X})_{\ast}} \pi_{n}(F \wedge X_{+}) = F_{n}(X).
\]
The natural transformations (3.16) are given by maps of spectra $U^{\text{Spin}} : M\text{Spin} \to K \text{O}$ resp. $U^{\text{SO}} : M\text{SO} \to H\mathbb{Z}$. By taking the homotopy fibers of $U^{\text{Spin}}$ resp. $U^{\text{SO}}$, we obtain homotopy fibrations

$$
\begin{align*}
M\text{Spin} &\xrightarrow{i^{\text{Spin}}} M\text{Spin} \xrightarrow{U^{\text{Spin}}} K \text{O} \\
M\text{SO} &\xrightarrow{i^{\text{SO}}} M\text{SO} \xrightarrow{U^{\text{SO}}} H\mathbb{Z}
\end{align*}
$$

and the associated long exact sequences of homology groups (coming from the long exact sequences of homotopy groups of the fibrations obtained by smashing the fibrations above with $X_+$). In particular, the kernel of $U^G_*$ is the image of $i_*^G$ for $G = \text{Spin}, \text{SO}$, and hence (3.16) is equivalent to

$$\text{image} \left( i^G_* : \widehat{M}G_n(X) \to M\text{G}_n(X) \right) \subseteq \Omega^{G,+}_n(X) \quad \text{for } G = \text{Spin}, \text{SO}. \quad (3.18)$$

It suffices to prove these containment relations localized at 2 and with 2 inverted (see (3.12)). The method of proof for those two cases is quite different:

- Localized at 2, the proof is based on a stable homotopy theoretic understanding of the spectra $M\text{Spin}, M\text{SO}$, and relating the image of $i^G_*$ to $\mathbb{H}^2$-bundles.
- With 2 inverted, it is based on a geometric interpretation of the groups $\widehat{M}G_*(X)$ as bordism groups of manifolds with additional structure.

**Proof of (3.18) for $G = \text{Spin}$ localized at 2.** We recall from the proof of Theorem 3.10 that the transfer map $T$ factors in the form

$$
\begin{align*}
\widehat{T}_n : (M\text{Spin} \wedge \Sigma^8 BH_+)_n(X) &\to M\text{Spin}_n(X) \\
M\text{Spin} \wedge \Sigma^8 BH_+ &\xrightarrow{T} M\text{Spin},
\end{align*}
$$

and that the key for the proof of Theorem 3.10 is the fact that the induced map $\widehat{T}_n$ on homotopy groups is surjective (localized at 2). The key fact needed for the proof here is the stronger statement that $\widehat{T}$ is a split surjection of spectra localized at 2 [St1994, Prop. 8.3], the main technical result of that paper. It implies that the induced map

$$
\widehat{T}_n : (M\text{Spin} \wedge \Sigma^8 BH_+)_n(X) \to M\text{Spin}_n(X)
$$

is surjective (localized at 2) for any space $X$. Hence the image of $i^{\text{Spin}}_*$ is contained in the image of $T_* : (M\text{Spin} \wedge \Sigma^8 BH_+)_n(X) \to M\text{Spin}_n(X) = \Omega^{\text{Spin}}_n(X)$. Geometrically, the image of $T_*$ is represented by total spaces of $\mathbb{H}^2$-bundles. These admit positive scalar curvature metrics, and hence the image of $i^{\text{Spin}}_*$ is contained in $\Omega^{\text{Spin},+}_n(X)$. \qed

**Proof of (3.18) for $G = \text{SO}$ localized at 2.** This case is much simpler. For the convenience of the reader we repeat the argument of [RS2001 proof of Thm. 4.11]). Localized
at 2 the spectra $\text{MSO}$ and $\widehat{\text{MSO}}$ are Eilenberg Mac Lane spectra, which implies that localized at 2,

$$
\text{MSO}_n(X) \cong \bigoplus_{j \geq 0} H_{n-j}(X; \Omega^\text{SO}_j) \quad \text{and} \quad \widehat{\text{MSO}}_n(X) \cong \bigoplus_{j \geq 0} H_{n-j}(X; \Omega^\text{SO}_j).
$$

The summand $H_{n-j}(X; \Omega^\text{SO}_j) \subset \Omega^\text{SO}_n(X)$ is given by bordism classes of the form

$$
[f: M^{n-j} \times P^j \to X],
$$

where $M^{n-j}$, $P^j$ are oriented closed manifolds of the indicated dimension, and $f$ factors through $M$. Since every bordism class in $\Omega^\text{SO}_j$ for $j > 0$ can be represented by a manifold that carries a positive scalar curvature metric, (this is the key fact for the proof of Gromov-Lawson Theorem 3.7), this shows that the image of $i^\text{SO}_*$ is contained in $\Omega^\text{SO,+}_n(X)$.

**Proof of (3.18) with 2 inverted.** Führing has given a geometric interpretation of the homology groups $\text{MSpin}_n(X)$ and $\text{MSO}_n(X)$ with 2 inverted in terms of bordism classes of $n$-dimensional closed $\mathcal{P}$-manifolds equipped with a map to $X$ [Fu2013]. Here $\mathcal{P} = \{P_1, P_2, \ldots\}$ is a family of smooth closed manifolds, and a $\mathcal{P}$-manifold $M$ of dimension $n$ is a smooth $n$-manifold equipped with an additional structure [Fu2013, Def. 2.1]. Very roughly, it consists of a decomposition $M = A_1 \cup \cdots \cup A_k$ of $M$ into $n$-dimensional submanifolds $A_i$ of the form $P_i \times B_i$. Over two-fold intersections $A_i \cap A_j$, these product decompositions are compatible in the sense that $A_i \cap A_j$ has the form $P_i \times P_j \times B_{ij}$; refining the product decompositions of $A_i$ and $A_j$ over the intersection $A_i \cap A_j$, and so forth for higher intersections.

We remark that given a list of manifolds $\mathcal{P}$, there is the Baas-Sullivan theory which considers bordism groups of manifolds with singularities which are built inductively from manifolds on the list $\mathcal{P}$. Removing a small open neighborhood of the singularity from such a manifold with Baas-Sullivan type singularity, leads to a smooth manifold whose boundary has the structure described above.

For any collection $\mathcal{P}$, the bordism classes of $\mathcal{P}$-manifolds with maps to a topological space $X$ give a homology theory $\mathcal{P}_*(X)$ [Fu2013, Prop. 2.11]. Moreover, the homology theories $\text{MSpin}_a(X)$ and $\text{MSO}_a(X)$ are given by specific collections $\mathcal{P}^\text{Spin}_a$, $\mathcal{P}^\text{SO}_a$ of spin (resp. oriented) manifolds [Fu2013, Prop. 2.12 and Prop. 2.11]:

$$
\text{MSpin}_a(X) \cong \mathcal{P}^\text{Spin}_a(X) \quad \text{and} \quad \text{MSO}_a(X) \cong \mathcal{P}^\text{SO}_a(X).
$$

The map $i^G_*: \text{MG}_*(X) \cong \mathcal{P}^G_*(X) \to \text{MG}_*(X) = \Omega^G_*(X)$ for $G = \text{SO,Spin}$ has the simple geometric interpretation of forgetting the additional structure on the smooth spin/oriented $\mathcal{P}^G$-manifolds representing the elements of $\mathcal{P}^G_*(X)$. Moreover, a smooth closed manifolds with a $\mathcal{P}^G$-structure always carries a positive scalar curvature metric, provided all manifolds on the list $\mathcal{P}^G$ do [Fu2013, Thm. 3.1]. This shows that the image of $i^G_*$ is contained in $\Omega^G_+*(X)$ as claimed, provided the collection of manifolds $\mathcal{P}^G$ can be chosen to consist of manifolds which carry positive scalar curvature metrics.
We recall from (3.13) that
\[ \Omega^\text{Spin}_* \otimes \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}][[K], [M^8], [M^{12}], \ldots], \]
where \( K \) is the Kummer surface and \( M^{4i} \) is a spin 4i-manifold which is the total space of an \( \mathbb{HP}^2 \)-bundle and hence carries a positive scalar curvature metric. Führing shows that one can choose \( p^\text{Spin} = \{ M^8, M^{12}, \ldots \} \) [Fu2013, Prop. 2.12]. He furthermore shows that \( p^\text{SO} \) can be chosen to be any collection of oriented manifolds whose bordism classes are generators of the polynomial ring \( \Omega^\text{SO}_* \otimes \mathbb{Z}[\frac{1}{2}] \) [Fu2013, Prop. 2.11]. For example, we could choose \( p^\text{SO} = \{ K, M^8, M^{12}, \ldots \} \), since the natural map \( \Omega^\text{Spin}_* \rightarrow \Omega^\text{SO}_* \) is an isomorphism after inverting 2, but this is not a good idea, since the Kummer surface \( K \) does not carry a positive scalar curvature metric (due to \( p^A K \equiv 0 \)). However, we can replace \( K \) by the complex projective plane \( \mathbb{CP}^2 \), and \( p^\text{SO} = \{ \mathbb{CP}^2, M^8, M^{12}, \ldots \} \) is a family of oriented manifolds which carry positive scalar curvature metrics, and whose bordism classes are polynomial generators of \( \Omega^\text{SO}_* \otimes \mathbb{Z}[\frac{1}{2}] \).

3.6 Positive scalar curvature metrics on manifolds with non-trivial fundamental groups

In this section we illustrate how Theorem 1.7 is used to prove the GLR Conjecture for some fundamental groups \( \pi \) by outlining the proof for finite groups with periodic cohomology [BGS1997]. Finally we discuss Schick’s counter example to the GLR conjecture [Sch1998].

The GLR Conjecture for groups \( \pi \) with periodic cohomology [BGS1997].

To prove the GLR Conjecture for a finite group \( \pi \), it suffices by [KwSch1990, Prop. 1.5] to prove it for the \( p \)-Sylow subgroups of \( \pi \). If \( \pi \) has periodic cohomology, these are either cyclic or quaternion (or rather, generalized quaternion groups, whose order is some power of 2). For odd order cyclic groups the GLR Conjecture was proved in [Ro1986III] (for prime order) and for the general case in [KwSch1990, Thm. 1.8]. Here we outline the argument in [BGS1997] that proves the GLR Conjecture for cyclic groups \( C_\ell \) and quaternionic groups \( Q_\ell \) of order \( \ell = 2^k \).

To prove the GLR Conjecture for a group \( \pi \), it suffices by Theorem 1.7 to show that the inclusion
\[ \text{ker}(\text{ko}_n(B\pi) \rightarrow \text{ko}^+_n(B\pi)) = \text{ker}(A \circ p) \]
is an equality. Since the GLR Conjecture is true for the trivial group, it suffices to show \( \text{ker}(\text{ko}_n(B\pi)) \subset \text{ko}^+_n(B\pi) \) (given by passing to the kernel of the map \( \text{ko}_n(B\pi) \rightarrow \text{ko}_n \)).

The strategy used in [BGS1997] to prove this for \( \pi = C_\ell \) and \( \pi = Q_\ell \), \( \ell = 2^k \), is to consider lens spaces and lens space bundles over \( S^2 \) (in the case \( \pi = C_\ell \)) and lens spaces and quotients of free actions of \( Q_\ell \) on spheres of dimension \( \equiv 3 \) mod 4 (for \( \pi = Q_\ell \)). Let \( M_*(B\pi) \subset \text{\Omega}^\text{Spin}_*(B\pi) \) be the \( \text{\Omega}^\text{Spin}_* \)-submodule generated by these manifolds and their natural maps to \( B\pi \), and consider its image under the natural transformation.
3 CONSTRUCTIONS OF POSITIVE SCALAR CURVATURE METRICS

$U^\text{Spin}_*: \tilde{\Omega}^\text{Spin}_*(\mathcal{B}\pi) \to \tilde{\text{ko}}_s(\mathcal{B}\pi)$. Since the lens spaces and the quaternionic space forms carry positive scalar curvature metrics, there is the following chain of inclusions:

$$U^\text{Spin}_*(\mathcal{M}_n(\mathcal{B}\pi)) \subseteq \tilde{\text{ko}}_n^+(\mathcal{B}\pi) \subseteq \tilde{\ker}_n(A \circ p).$$

Proving the equality $U^\text{Spin}_*(\mathcal{M}_n(\mathcal{B}\pi)) = \tilde{\ker}_n(A \circ p)$ (this is Theorem 2.3 in [BGS1997]) then implies the desired equality $\tilde{\text{ko}}_n^+(\mathcal{B}\pi) = \tilde{\ker}_n(A \circ p)$. Due to the multiplicative structure (given by multiplication by $\eta \in \text{ko}_1$), it suffices to prove $U^\text{Spin}_*(\mathcal{M}_n(\mathcal{B}\pi)) = \tilde{\ker}_n(A \circ p)$ for $n \equiv 3 \mod 4$ for $\pi = Q_\ell$, and for $n$ odd for $\pi = C_\ell$. For these $n$, it is shown that the order of the finite group $U^\text{Spin}_*(\mathcal{M}_n(\mathcal{B}\pi))$ is greater or equal to the order of $\tilde{\ker}_n(A \circ p)$. An upper bound for the order of $\ker_n(A \circ p)$ is obtained by using the Atiyah-Hirzebruch spectral sequence converging to $\text{ko}_*^s(B\pi)$ to obtain an upper bound on $\tilde{\ker}_n(B\pi)$, in conjunction with information about the assembly map $A: \text{KO}_s(B\pi) \to \text{KO}_s(\mathbb{R}\pi)$ for finite 2-groups $\pi$ (which is obtained by dualizing the Atiyah-Segal results concerning $\text{KO}^s(B\pi)$). A lower bound for $U^\text{Spin}_*(\mathcal{M}_n(\mathcal{B}\pi))$ is obtained by calculating the eta-invariants of twisted Dirac operators on the lens spaces resp. quaternionic space forms generating $\mathcal{M}_n(B\pi)$. The eta-invariant associated to an irreducible representation of $\pi$ provide homomorphisms from $\tilde{\text{KO}}_n(B\pi)$ to $\mathbb{R}/\mathbb{Z}$ resp. $\mathbb{R}/2\mathbb{Z}$ (depending on whether the representation of complex, real or quaternionic type). Using all irreducible representations, one obtains a homomorphism $\tilde{\eta}$ from $\text{ko}_n^*(B\pi)$ to a sum of copies of $\mathbb{R}/\mathbb{Z}$ and $\mathbb{R}/2\mathbb{Z}$. An interesting byproduct of these calculations is that the homomorphism

$$\tilde{\text{ko}}_n(B\pi) \xrightarrow{p} \tilde{\text{KO}}_n(B\pi) \xrightarrow{\tilde{\eta}} \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/2\mathbb{Z}$$

is injective if $\pi = C_\ell$ and $n$ is odd, or if $\pi = Q_\ell$ and $n \equiv 3 \mod 4$ [BGS1997].

**Schick’s counter example to the GLR conjecture** [Sch1998]. Let $\rho: \mathbb{Z}^5 \to \mathbb{Z}^4 \times \mathbb{Z}/3 =: \pi$ be the product of the identity map on the first four components, and the projection map $\mathbb{Z} \to \mathbb{Z}/3$ on the last. Then the map $B\rho: T^5 = B\mathbb{Z}^5 \xrightarrow{B\rho} B\pi$ represents an element of the spin bordism $\Omega^\text{spin}_5(B\pi)$ (equip each $S^1$ factor of $T^5$ with the bounding spin structure). This is an element of order 3, and hence its image in $\text{KO}_5(C^*\pi)$ is trivial, since the torsion of $\text{KO}_s(C^*\pi)$ is only 2-torsion [Sch1998, Prop. 2.1] (this is the case for the product of a free abelian group with any finite group). A 1-surgery on $T^5$ on an embedded 1-sphere representing the kernel of $B\rho_*: \pi_1(T^5) \to \pi_1(\mathcal{B}\pi)$ produces a spin manifold $M$ with fundamental group $\pi$ such that $[M,u] = [T^5,B\rho] \in \Omega^\text{spin}_5(B\pi)$. Then the index obstruction $\alpha(M,u) \in \text{KO}_5(C^*\pi)$ is the image of $[T^5,B\rho]$ in $\text{KO}_5(C^*\pi)$ and hence zero.

To show that $M$ does not carry a positive scalar curvature metric, we use Corollary 2.13 (which exploits the obstructions coming from stable minimal hypersurfaces). Let $\alpha_i \in H^1(\mathcal{B}\pi)$ be pullback of the generator of $H^1(S^1)$ via the projection of $B\pi \to S^1$ to the $i$-th factor, $i = 1, \ldots, 4$. Assume that $M$ does carry a positive scalar curvature metric, and hence $u_*[M]^{SO} = B\rho_*(T^5)$ belongs to the positive subgroup $H^+_5(B\pi) \subseteq \text{ko}_5^+(B\pi)$.
Applying Cor. 2.13 three times, then $\alpha_1 \cap (\alpha_2 \cap (\alpha_3 \cap B\rho [T^5]))$ belongs to $H_2^+(B\rho)$. This is the desired contradiction, since this iterated cap product is non-trivial, but by the Gauss-Bonnet Theorem the only closed 2-manifold that carries a positive scalar curvature metric is the 2-sphere, and any homology class in $B\pi$ represented by a map $f : S^2 \to B\pi$ is trivial.

4 Some open questions

1. The focus of this survey is the question which closed manifolds $M$ carry positive scalar curvature metrics. If $M$ is a manifold with boundary $\partial M$, and $h$ is a positive scalar curvature metric on $\partial M$, one can ask the corresponding “relative” question: does $h$ extend to a positive scalar curvature metric $g$ on $M$? We require that $g$ is a product metric near the boundary, a tacit assumption we will make for metrics on manifolds with boundary. If $M$ is a spin $n$-manifold, the Clifford linear Dirac operator $D_{C\ell}$ on $M$ with respect to the Atiyah-Singer-Patodi boundary conditions has a Clifford index $\alpha(M, h) \in KO_n$. The operator $D_{C\ell}$ is constructed using any metric $g$ on $M$ extending $h$. The Clifford index $\alpha(M, h)$ is independent of the choice of $g$, and depends only on the concordance class of the positive scalar curvature metric $h$ on $\partial M$ (two positive scalar curvature metrics $h, h'$ on a closed manifold $N$ are concordant if there is a positive scalar curvature metric on $N \times [0, 1]$ which restricts to $h$ resp. $h'$ on the boundary). Moreover, if the scalar curvature of $g$ is positive, then, as in the case of closed manifolds, the Dirac operator $D_{C\ell}$ is invertible, and hence $\alpha(M, h) = 0$. This shows that the index invariant $\alpha(M, h)$ is an obstruction to extending $h$ to a positive scalar curvature metric $g$ on $M$. According to Theorem 4.2 this is the only obstruction if $M$ is a simply connected compact spin manifold of dimension $n \geq 5$ without boundary. So it is very natural to ask:

**Question 4.1.** Let $M$ be a simply connected compact spin manifold of dimension $n \geq 5$ and $h$ a positive scalar curvature metric on $\partial M$. Does $h$ extend to a positive scalar curvature metric on $M$ if and only if the index obstruction $\alpha(M, h) \in KO_n$ vanishes?

As discussed for closed manifolds, the index invariant $\alpha(M, h)$ has a refinement which lives in $KO_n(C^{*}\pi)$ where $\pi$ is the fundamental group of $M$. The answer to the analogous question is “no” in general, since already for closed manifolds, there are the additional obstructions coming from the stable minimal hypersurface method (see §2.4).

The above question is intimately related to the classification of positive scalar curvature metrics up to concordance. Let $R_n$ be the bordism group of pairs $(M, h)$ consisting of a spin $n$-manifold $M$ and a positive scalar curvature metric $h$ on $\partial M$.

**Theorem 4.2.** Let $M$ be a smooth compact simply connected spin manifold of dimension $n \geq 5$.

(i) **Hajduk [Haj1991]:** A positive scalar curvature metric $h$ on $\partial M$ extends to a positive scalar curvature metric on $M$ if and only if $[M, h] \in R_n$ vanishes.
(ii) \cite[Thm. 3.9]{St1995}, \cite[Thm. 1.1]{St1996}: If $h$ extends to a positive scalar curvature metric on $M$, then the group $R_{n+1}$ acts freely and transitively on the set of concordance classes of such metrics.

This is just the simplest instance of a much more general result. Without assuming simply connectivity for $M$, the bordism group $R_n$ has to be replaced by the bordism group $R_n(\pi)$, $\pi = \pi_1(M)$, where all manifolds come equipped with maps to the classifying space $B\pi$ (\cite[Thms. 3.8 & 3.9]{St1995}); without the spin condition, it needs to be replaced by $R_n(\gamma(M))$ where $\gamma(M)$ is the fermionic fundamental group of $M$ (which depends only on the fundamental group and the first two Stiefel-Whitney classes of $M$ up to isomorphism).

Associating to a pair $(M, h)$ the index obstruction $\alpha(M, h) \in \text{KO}_n$ defines a homomorphism $\alpha: R_n \rightarrow \text{KO}_n$. It is easy to show that $\alpha$ is surjective, and a positive answer to the question above would imply that $\alpha$ is an isomorphism; in particular, $\text{KO}_{n+1}$ would act freely and transitively on the set of concordance classes of positive scalar curvature metrics on any compact manifold $M$ of dimension $n \geq 5$ (and extending a given positive scalar curvature metric $h$ on $\partial M$ if $\partial M \neq \emptyset$), provided there exist such metrics.

Alas, the question above appears out of reach of current methods. For example, consider a simply connected 5-manifold $M$. If the boundary is empty, the answer to the question whether $M$ carries a positive scalar curvature metric depends only on the bordism class of $M$ in $\Omega^\text{Spin}_n$ (if $M$ is spin) or $\Omega^\text{SO}_n$ (if $M$ does not admit a spin structure). Since both bordism groups are trivial, in either case $M$ carries a positive scalar curvature metric.

If $\partial M$ is non-empty, and $h$ is a positive scalar curvature metric on $\partial M$, there is no known obstruction to extending $h$ to a positive scalar curvature metric on $M$, but it seems that some more direct geometric method that uses the metric $h$ in an essential way, is necessary.

2. Let $M$ be a closed connected spin manifold $M$ of dimension $n \geq 5$ with fundamental group $\pi$. According to the Bordism Theorem \cite[3.3]{L4} the answer to the question whether $M$ carries a positive scalar curvature metric depends only on the class it represents in the spin bordism group $\Omega^\text{Spin}_n(B\pi)$. By \cite[3.7]{L7} in fact it depends only on the image of this class under the map

$$U^\text{Spin}_*: \Omega^\text{Spin}_n(B\pi) \rightarrow \text{ko}_n(B\pi).$$

With the weaker hypothesis that the universal covering $\hat{M}$ is spin, the manifold $M$ represents an element in the twisted spin bordism group $\Omega^\text{Spin,}\tau_n(B\pi)$, where the twist $\tau$ depends on the first two Stiefel-Whitney classes of $M$ (the condition that $\hat{M}$ is spin guarantees that $w_2(M) \in H^2(M; \mathbb{Z}/2)$ comes from a unique class in $H^2(B\pi; \mathbb{Z}/2)$.

There is a twisted version of connective real $K$-theory, and a homomorphism

$$U^\text{Spin,}\tau_*: \Omega^\text{Spin,}\tau_n(B\pi) \rightarrow \text{ko}_n^\tau(B\pi),$$
constructed in [HJ2020] by a mixture of homotopy theoretic and operator theoretic methods. As in the non-twisted case, the hope is that it suffices to look at the associated element in $\text{ko}^*_\tau(B\pi)$ to decide whether $M$ admits a positive scalar curvature metric. This boils down to a positive answer to the following question.

**Question 4.3.** Is the kernel of $U^\text{Spin,}\tau_*$ representable by manifolds that carry positive scalar curvature metrics?

The paper [HJ2020] lays the foundation for possible future homotopy theoretic arguments addressing this question.

3. As mentioned in section 2.4, the only known obstructions to positive scalar curvature metrics on a closed manifold with non-spin universal covering comes from the stable minimal hypersurface method. This leads to the restrictions on the subgroup $H^*_n(X) \subseteq H_n(X)$ of homology classes representable by closed oriented manifolds with positive scalar curvature metrics, expressed by Cor. 2.13.

**Question 4.4.** Does the statement of Corollary 2.13 hold without the dimension restriction $n \leq 7$?

The papers [Lo2006] and [SY2017] deal with ways to extend the stable minimal hypersurface method to higher dimension. The technical challenge is to develop techniques to deal with the singularities of stable minimal hypersurfaces.

4. Let $M$ be a closed connected manifold of dimension $n \geq 5$ with finite fundamental group $\pi$ whose universal covering $\tilde{M}$ is non-spin. According to Conjecture 1.9, such a manifold carries a positive scalar curvature metric, and by the induction result [KwSch1990, Prop. 1.5], it suffices to show this for finite $p$-groups. As mentioned in the paragraph following Conjecture 1.9, a lot is known if $\pi$ is an abelian $p$-group. The essential open case is the the following for odd $p$.

**Question 4.5.** Let $\pi$ be the elementary abelian group $(\mathbb{Z}/p)^n$, let $\rho: \mathbb{Z}^n \to (\mathbb{Z}/p)^n$ be the projection map, and let $B\rho: T^n = B\mathbb{Z}^n \to B\pi$ be the induced map of classifying spaces. Does $B\rho_*[T^n]^{SO} \in H_n(B\pi)$ belong to $H^*_n(B\pi)$ for $n \geq 5$? (here $T^n = S^1 \times \cdots \times S^1$ is the $n$-dimensional torus, and $[T^n]^{SO} \in H_n(T^n)$ is its fundamental class).

Joachim has shown that the answer for $p = 2$ is affirmative [Jo2004], but his approach does not work for $p$ odd.

The case non-abelian finite $p$-groups $\pi$ does not seem to have been studied much. The fact that the classifying space $B\pi$ can be “built” from the classifying spaces of its elementary abelian $p$-groups [Dw1997 Thm. 1.4. and 1.12], suggests the following vague question.

**Question 4.6.** Can $H^*_+(B\pi)$ for a $p$-group $\pi$ be determined in terms of $H^*_+$ for its $p$-Sylow subgroups?
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