On Bifurcation Points of the Stationary Vlasov-Maxwell System with Bifurcation Direction

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Let us consider a multicomponent plasma containing electrons and positively charged ions of various kinds and described by the multiparticle distribution function $f_i = f_i(r, v)$, $i = 1, \ldots, N$. Plasma occupies a domain $D \subset \mathbb{R}^3$ with smooth boundary. The particles interact via a self-consistent force fields; collisions of particles are neglected.

The plasma is described by the following version of a nonrelativistic stationary Vlasov-Maxwell(VM) system [1]

\[ v \cdot \partial_r f_i + \frac{q_i}{m_i}(E + \frac{1}{c}v \times B) \cdot \partial_v f_i = 0 \]

\[ r \in D \subset \mathbb{R}^3, \ v \in \mathbb{R}^3, \ i = 1, \ldots, N, \]

\[ \text{curl} E = 0, \]

\[ \text{div} B = 0, \]

\[ \text{div} E = 4\pi \sum_{k=1}^{N} q_k \int_{\mathbb{R}^3} f_k(r, v) dv \overset{\triangle}{=} \rho, \]

\[ \text{curl} B = \frac{4\pi}{c} \sum_{k=1}^{N} q_k \int_{\mathbb{R}^3} v f_k(r, v) dv \overset{\triangle}{=} j. \]
Here $\rho(r), j(r)$ are the densities of the charge and the current, respectively, and $E(r), B(r)$ are the electric and the magnetic field.

We shall search the solution $E, B, f$ of VM system (I) for $r \in D \subset \mathbb{R}^3$ with the boundary conditions on the potentials and the densities

$$U \mid_{\partial D} = u_{01}, \quad (A, d) \mid_{\partial D} = u_{02},$$

$$\rho \mid_{\partial D} = 0, \quad j \mid_{\partial D} = 0,$$

where $E = -\partial_r U$, $B = \text{curl} A$, and $U, A$ are the scalar and the vector potentials.

The solution $E^0, B^0$, for which $\rho^0 = 0, j^0 = 0$ in a domain $D$, will be referred to as the trivial one.

Our aim is to obtain the existence theorem of nontrivial solutions for the boundary-value problem (I), (2), (3).

In this paper we restrict ourselves to distributions of the form

$$f_i(\lambda, r, v) = a(\lambda) \hat{f}_i(-\alpha_i(\lambda)v^2 + \varphi_i(\lambda, r), v \cdot d_i(\lambda) + \psi_i(\lambda, r)) \overset{\Delta}{=} a(\lambda) \hat{f}_i(\lambda, R, G),$$

where the functions $\varphi_i, \psi_i$ generating the corresponding electromagnetic field $(E, B)$ are to be determined. Let us note that similar distribution functions were introduced in [2] and have been used in [3]. We are interested in the dependence of desired functions $\varphi_i, \psi_i$ generating the nontrivial solutions on the parameter $\lambda$ in distribution (4). The case, when the parameters $\alpha_i, d_i$ are independent of $\lambda$ was considered in [4-6]. Here we consider the general case.

Let us give a one preliminary result on reduction of the VM system (I) with conditions (2) to the quasilinear system of elliptic equations for distribution (4) [7]. We assume that the following condition is satisfied:

**A.** $\hat{f}_i(R, G)$ in (4) are fixed, differentiable functions; $\alpha_i, d_i$ are free parameters, $|d_i| \neq 0$; $\varphi_i = c_{1i} + l_i \varphi(r), \psi_i = c_{2i} + k_i \psi(r), c_{1i}, c_{2i}$ are constants; the parameters $l_i, k_i$ are related by equations

$$l_i = \frac{m_1}{\alpha_i q_i} \frac{\alpha_i q_i}{m_i}, \quad k_i \frac{q_i}{m_1} d_i = \frac{q_i}{m_i} d_i;$$

the integrals

$$\int_{R^3} \hat{f}_i \, dv, \quad \int_{R^3} \hat{f}_i v \, dv$$

converge for all $\varphi_i, \psi_i$.

**Theorem 1.** Suppose that the distribution function $f_i$ has the form (4) and condition A is satisfied. Let the vector-function $(\varphi, \psi)$ be a solution of system of equations

$$\Delta \varphi = a(\lambda) \mu \sum_{k=1}^{N} q_k \int_{R^3} f_k(\lambda) dv, \quad \mu = \frac{8\pi \alpha q}{m},$$

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where $E = -\partial_r U$, $B = \text{curl} A$, and $U, A$ are the scalar and the vector potentials.
\( \nabla \psi = a(\lambda)\nu \sum_{k=1}^{N} q_k \int_{R^3} (v, d)f_k(\lambda)dv, \quad \nu = \frac{4\pi q}{mc^2}, \)  

\( \varphi \mid_{\partial D} = -\frac{2\alpha q}{m} u_{01}, \quad \psi \mid_{\partial D} = \frac{q}{mc} u_{02} \) \hfill (6)

in the subspace \( (\partial_r \varphi_i, d_i) = 0, \quad (\partial_r \psi_i, d_i) = 0, \quad i = 1, \ldots, N. \) \hfill (7)

Then the VM system has the solution

\[
E = \frac{m}{2\alpha q} \partial_r \varphi, \quad B = d \left( \beta + \int_{R^2} (d \times J(\text{tr}) \, r) \, dt \right) - \left[ d \times \partial_r \psi \right] \frac{mc}{qd^2}, \]

where

\[
J = \frac{4\pi}{c} \sum_{k=1}^{N} q_k \int_{R^3} v f_k \, dv, \quad \beta - \text{const.} \]

To this solution there correspond potentials

\[
U = -\frac{m}{2\alpha q} \varphi, \quad A = \frac{mc}{qd^2} \psi d + A_1(r), \quad (A_1, d) = 0 \]

with conditions (2).

The proof see in [6].

We introduce the notations

\[
j_i = \int_{R^3} v f_i \, dv, \quad \rho_i = \int_{R^3} f_i \, dv, \quad i = 1, \ldots, N. \]

and impose the following condition:

**B.** There exist vectors \( \beta_i \in R^3 \) such that \( j_i = \beta_i \rho_i, \quad i = 1, \ldots, N. \)

Let condition B holds, then system (5) becomes

\[
\Delta \varphi = a(\lambda)\mu \sum_{i=1}^{N} q_i A_i, \quad \quad (10)\]

\[
\Delta \psi = a(\lambda)\nu \sum_{i=1}^{N} q_i (\beta_i, d) A_i, \]

where

\[
A_i(\lambda, I_i \varphi, k_i \psi) \triangleq \int_{R^3} \hat{f}_i \, dv. \]

From now on, for simplicity, we consider the auxiliary vector \( d \) directed along the axis \( Z \). Then \( \varphi = \varphi(x, y), \quad \psi = \psi(x, y), \quad x, y \in D \in R^2 \) in system (10). Moreover, let \( N \geq 3 \) and \( I_i \neq \text{const.} \).
Let $D$ be a bounded domain in $\mathbb{R}^2$ with boundary $\partial D$ of class $C^{2,\alpha}$, $\alpha \in (0, 1]$.

We introduce a continuous vector-function of parameter $\lambda$

$$\varepsilon(\lambda) = (l_1 \varphi^0, k_1 \psi^0, \alpha_1, d_1, \ldots, l_N \varphi^0, k_N \psi^0, \alpha_N, d_N) \in \mathbb{R}^{4N}$$

(12)

and a contraction of the set (the value of this vector) induced by the vector (12) and the boundary conditions (3) for the local densities of the charge and the current

$$\Omega = \left\{ \varepsilon \mid \sum_{k=1}^{N} q_k A_k(l_k \varphi^0, k_k \psi^0, \alpha_k, d_k) = 0 \right\}$$

(13)

with

$$\varphi^0 = -\frac{2\alpha q}{m} u_{01}, \quad \psi^0 = \frac{q}{mc} u_{02}, \quad N \geq 3.$$  

(14)

We introduce the condition.

C. Let $\varepsilon(\lambda) \in \Omega$ for $\forall \lambda$ from the some open set of real axis for $x \in (-r, r)$.

Then the system (10) with the boundary conditions

$$\varphi |_{\partial D} = \varphi^0, \quad \psi |_{\partial D} = \psi^0$$

(15)

has the trivial solution $\varphi = \varphi^0, \psi = \psi^0$ for $\forall \lambda \in (-r, r)$ by (13), (14). By Theorem 1, the boundary-value problem (1), (2), (3) has the trivial solution $E_0^0 = \frac{m}{2\alpha q} \partial_v \varphi^0 = 0$, $r \in D \subset \mathbb{R}^2$, $B_0^0 = \beta d_1$,

$$f_i^0 = a(\lambda) f_i(-\alpha_i(\lambda) v^2 + c_{1i} + l_i \varphi^0(\lambda), (v, d_1(\lambda) + c_{2i} + k_i \psi^0(\lambda))$$

for all $\lambda \in (-r, r)$. Moreover, $\rho^0 \equiv 0$ and $j^0 = 0$ in a domain $D$.

**Definition 1.** A point $\varepsilon_0 = \varepsilon(\lambda_0) \in \Omega$ is called a bifurcation point of solution of the VM system (I), (2), (3) with a bifurcation direction $\varepsilon = \varepsilon(\lambda)$, where $\lambda : (-r, r) \rightarrow \Omega$ be the continuous vector-function of $\lambda$, if every neighborhood of the vector $(\varepsilon^0, E^0, B^0, f^0)$ corresponding to the trivial solution with $\rho^0 = 0, f^0 = 0$ in a domain $D$ contains a vector $(\varepsilon^*, E, B, f)$ with $\varepsilon^* = \varepsilon^*(\lambda)$, $\lambda \in (-r, r)$ satisfying system (I) with conditions (2), (3) such that

$$\| E - E^0 \| + \| B - B^0 \| + \| f - f^0 \| > 0.$$ 

Moreover, the densities $\rho$ and $j$ interior to domain $D$ need not vanish.

Using the Taylor expansion and singling out the linear terms, we rewrite system (10) in operator form as follows:

$$[L_0 - a(\lambda) L_1(\varepsilon(\lambda))] u - R(\varepsilon, u) = 0,$$

(16)
where $\varepsilon = \varepsilon(\lambda)$ be a bifurcation direction of (12). Here

$$L_0 = \begin{bmatrix} \triangle & 0 \\ 0 & \triangle \end{bmatrix}, \quad u = (\varphi - \varphi^0, \psi - \psi^0)/$$

$$L_1 = \sum_{s=1}^{N} q_s \begin{bmatrix} \mu l_s \frac{\partial A}{\partial x} & \mu k_s \frac{\partial A}{\partial y} \\ \nu l_s (\beta_s, d) \frac{\partial A}{\partial x} & \nu k_s (\beta_s, d) \frac{\partial A}{\partial y} \end{bmatrix}_{x=l_s \varphi^0, y=k_s \psi^0} \triangleq \begin{bmatrix} \mu T_1 & \mu T_2 \\ \nu T_3 & \nu T_4 \end{bmatrix}. \quad (17)$$

Operator $R(\varepsilon(\lambda), u)$ is analytic in neighborhood of the point $u = 0$:

$$R(\varepsilon, u) = \sum_{i \geq 1} \sum_{s=1}^{n} g_{is}(u) b_s,$$

where

$$g_{is}(u) \triangleq \frac{q_s}{i!} (l_s u_1 \frac{\partial}{\partial x} + k_s u_2 \frac{\partial}{\partial y})^i A_s(l_s \varphi^0, k_s \psi^0)$$

are $i$th-order homogeneous forms in $u$ and

$$\frac{\partial^{i_1+i_2}}{\partial x^{i_1} \partial y^{i_2}} A_s(\varepsilon, x, y) |_{x=l_s \varphi^0, y=k_s \psi^0} = 0 \text{ for } 2 \leq i_1 + i_2 \leq l - 1, \ s = 1, \ldots, N,$$

$$b_s \triangleq (\mu, \nu(\beta_s, d))/.$$

The existence problems for a bifurcation point, $\varepsilon \in \Omega$, of the boundary-value problem (10), (15) can be restated as the existence problem for a bifurcation point for the operator equation (16).

Let us introduce the Banach spaces $C^{2,\alpha}(\bar{D})$ and $C^0,\alpha(\bar{D})$ with the norms $\| \cdot \|_{2,\alpha}$ and $\| \cdot \|_{0,\alpha}$, respectively, and let $W^{2,2}(\bar{D})$ be the ordinary Sobolev $L^2$-space in $D$.

Let us introduce the Banach space $E$ of vectors $u \triangleq (u_1, u_2)/$, where $u_i \in L_2(D)$; $L_2$ is the real Hilbert space with inner product $(\cdot, \cdot)$ and the corresponding norm $\| \cdot \|_{L_2(D)}$. We define the domain $D(L_0)$ as the set of vectors $u \triangleq (u_1, u_2)/$ with $u_i \in W^{2,2}(D)$. Here $W^{2,2}(D)$ consists of $W^{2,2}$ functions with zero trace on $\partial D$.

Then $L_0 : D \subset E \longrightarrow E$ is a linear self-adjoint operator. By virtue of the embedding

$$W^{2,2}(D) \subset C^0,\alpha(\bar{D}), \quad 0 < \alpha < 1, \quad (18)$$

the operator $R : W^{2,2} \subset E \longrightarrow E$ is analytic in a neighborhood of the origin. By the embedding (18), any solution of Eq. (16) in $D(L_0)$ is a Hölder function. Moreover, since the coefficients of system (16) are constant, the vector $R(\varepsilon, u)$ is analytic, and
\( \partial D \in C^{2,\alpha} \); it follows from well-known results of regularity theory for weak solutions of elliptic equations [8] that the generalized solutions of Eq. (16) in \( W^{2,2}(D) \) actually belong to \( C^{2,\alpha}(\overline{D}) \).

The operator \( L_1 \in L(E \rightarrow E) \) is bounded and linear. Under the above assumptions on \( L_0 \) and \( L_1 \), all singular points of the operator

\[
L(\varepsilon) \triangleq L_0 - a(\lambda)L_1(\varepsilon(\lambda))
\]

are Fredholm points in sense [9].

Let us introduce the conditions

I. \( T_1 < 0 \);
II. \( T_1T_4 - T_2T_3 > 0 \).

If \( \frac{\partial f_k}{\partial x} |_{x=y=0} > 0 \), then inequality II is satisfied. Let us introduce the matrix \( \| \Theta_{ij} \|_{i,j=1,\ldots,N} \), where \( \Theta_{ij} = q_iq_j(l_jk_i - k_jl_i)(\beta_j - \beta_i, d) \). If the derivatives \( \frac{\partial A_i}{\partial x}, \frac{\partial A_i}{\partial y} \) are positive and equal at the point \( x = y = 0, \Theta_{ij} > 0, i \neq j \), then conditions I, II are satisfied. Evidently, the elements of \( \Theta_{ij} \) are nonnegative, because of identities

\[
sign\frac{q_i}{l_i} = sign q, \quad \frac{(d_i,d)}{\alpha_i} = \frac{d^2k_i}{\alpha l_i}, \quad q < 0, \quad q_i > 0, \quad i = 2, \ldots, N.
\]

Let us denote by \( \Xi \) the matrix generating the operator \( L_1 \).

**Lemma 1.** Let conditions I, II be satisfied. Then the matrix \( \Xi \) in (17) has two simple eigenvalues

\[
\chi^+ (\varepsilon) = \mu T_1 + o(1), \quad \chi^- (\varepsilon) = \eta \frac{T_1T_4 - T_2T_3}{T_1} \frac{1}{c^2} + o(\varepsilon),
\]

for \( \frac{1}{c^2} \to 0 \).

The eigenvectors corresponding to negative eigenvalue \( \chi^- \) of the matrices \( \Xi \) and \( \Xi'/ \), respectively, are

\[
\begin{bmatrix}
  c_1 \\
  c_2
\end{bmatrix} = \begin{bmatrix}
  -\frac{T_4}{T_1} \\
  1
\end{bmatrix} + O\left(\frac{1}{c^2}\right),
\quad \begin{bmatrix}
  c_1^* \\
  c_2^*
\end{bmatrix} = \begin{bmatrix}
  0 \\
  1
\end{bmatrix} + O\left(\frac{1}{c^2}\right).
\]

Proof see in [6, Lemma 4].

Let \( \mu_0 \) is a n-multiple eigenvalue of Dirichlet problem

\[
- \Delta e = \mu e, \quad e |_{\partial D} = 0,
\]

for \( \frac{1}{c^2} \to 0 \).

Let us introduce the condition:

\[
\Delta e = \mu e, \quad e |_{\partial D} = 0.
\]
D. Suppose that condition C is satisfied and there exists $\lambda_0 \in (-r, r)$ such that $a(\lambda_0 \chi_- (\varepsilon (\lambda_0))) + \mu_0 = 0$; moreover, $a(\lambda)\chi_- (\varepsilon (\lambda))$ is the monotone increasing (decreasing) function in neighborhood of $\lambda_0$.

Rewrite the system (16) in the form

$$Bu - B_1(\lambda)u = R(\varepsilon (\lambda), u),$$

(21)

where we introduce the following notations

$$B = L_0 + \mu_0, \quad B_1(\lambda) = L_1(\varepsilon (\lambda)) + \mu_0.$$  

The operator $B$ is Fredholm and it has $n$ eigenfunctions of the form

$$e_i = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e_i, \quad i = 1, \ldots, n.$$  

Vector $(c_1, c_2)' = c$ is defined by (19) up to the constant multiplier and $L_1 c = \chi_- c$. Therefore,

$$B_1 e_i = a(\lambda)L_1(\varepsilon (\lambda))c e_i - a(\lambda_0)L_1(\varepsilon_0)c e_i = (a(\lambda)\chi_- (\varepsilon (\lambda)) - a(\lambda_0\chi_- (\varepsilon_0)))c e_i$$

and by taking into account condition D we obtain the following identities

$$<B_1 e_i, e_j> = (a(\lambda)\chi_- (\varepsilon (\lambda)) + \mu_0) |c|^2 \delta_{ij}, \quad i, j = 1, \ldots, n.$$  

(22)

**Theorem 2.** Let conditions A, B, C, D, I, II, as well one of the following conditions be satisfied:

1) $\mu_0$ is odd-multiple eigenvalue of problem (20);

2) $f_i = a(\lambda)\dot{f}_i(-\alpha_i(\nu^2 + \varphi_i) + v \cdot d_i(\lambda) + \psi_i)$. Then $\varepsilon_0 = \varepsilon (\lambda_0)$ is a bifurcation point of the boundary problem (1), (2), (3) with bifurcation direction $\varepsilon = \varepsilon (\lambda)$.

Proof. In order that to prove Theorem 2, we need to show that $\lambda_0$ is a bifurcation point of Eq. (21). The corresponding finite-dimensional branching system was given in the paper [6, Eq.(5.1)] for the case, when $l_i, d_i$ are independent of $\lambda$ and $a(\lambda) = \lambda \in R^+$. It is shown in the same place that $\lambda_0$ is a bifurcation point if and only if $\lambda_0$ is a bifurcation point of branching system.

In general case, when the values $\alpha_i(\lambda), d_i = d_i(\lambda)$ are dependent of $\lambda$, we can reduce the bifurcating system to the form

$$(a(\lambda)\chi_- (\varepsilon (\lambda)) + \mu_0)(c^2 + c_2^2)\xi_i + r_i(\xi_1, \ldots, \xi_n, \lambda) = 0, \quad i = 1, \ldots, n.$$  

(23)

with $|r(\xi, \lambda)| = o(|\xi|)$, using identities (22) and results from [6]. Here the vector-function $r(\xi, \lambda)$ in equation (23) is potential of $\xi$, if the condition 2) of Theorem 2 be satisfied.

Because the continuous function $a(\lambda)\chi_- (\xi (\lambda)) + \mu_0$ is equal to zero at the point $\lambda_0$, and be monotonic in it neighborhood, then in any neighborhood of point $\lambda_0$, $\xi_0 = 0$ exists a couple $\lambda^*, \xi^*$ with $\xi^* \neq 0$ satisfying the system (23) [6, see the proof of Lemma 8 and terminology in 10]. Hence, $\lambda_0$ is a bifurcation point of the system (23). Theorem 2 is proved.
References

[1] A.A. Vlasov. *Many-particle theory and its application to plasma*. Gordon and Breach, engl. Auflage, 1961.

[2] G.A. Rudykh, N.A. Sidorov, A.V. Sinitsyn. *Stationary solutions of a system of Vlasov-Maxwell equations*. Soviet Phys. Dokl., 33: 673-674, 1986.

[3] P. Braasch. *Semilineare elliptische Differentialgleichungen und das Vlasov-Maxwell-System*. Dissertation an der Fakultät für Mathematik der Ludwig-Maximilians-Universität München. München: Utz, Wiss., 1997.

[4] N.A. Sidorov, A.V. Sinitsyn. *On nontrivial solutions and bifurcation points of the Vlasov-Maxwell system*. Dokl. Akad. Nauk of Russia, V.349, N 1, 26-28, 1996. /in Russian/

[5] N.A. Sidorov, A.V. Sinitsyn. *On bifurcation of solutions of the Vlasov-Maxwell system*. Sibirskii Matemat. Zhurnal, V.37, N 6, 1367-1379, 1996. /in Russian/

[6] N.A. Sidorov, A.V. Sinitsyn. *Analysis of Bifurcation Points and Nontrivial Branches of solutions to the Stationary Vlasov-Maxwell System*. Mathematical Notes, V.62, N.2, 223-243, 1997. /translated from Matematicheskie Zametki, V.62, N 2, 268-292, 1997/.

[7] Yu.A. Markov, G.A. Rudykh, N.A. Sidorov, A.V. Sinitsyn and D.A. Tolstonogov. *Steady-state solutions of the Vlasov-Maxwell system and their stability*. Acta Appl. Math., 28: 253-293, 1992.

[8] O.A. Ladyzhenskaya, N.N. Uralzeva. *Linear and nonlinear equations of elliptic type*. Moscow, Nauka, 1964. /in Russian/

[9] M.M. Vayenberg, V.A. Trenogin. *The theory of branching of solutions of nonlinear equations*. Moscow, Nauka, 1969. /in Russian/

[10] N.A. Sidorov. *The general regularization questions in the problems of bifurcation theory*. Irkutsk Gos. University, Irkutsk. 1982. /in Russian/