Multifractal properties of visible matter distribution: a stochastic model

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Abstract

Galaxies and clusters distributions show two major properties: (i) the positions of galaxies and clusters are characterized by a power law distribution indicating properties with respect to their positions. (ii) The distribution of masses is also characterized by a power law corresponding to self similarity of different nature. These two properties are naturally unified by the concept of multifractality. This concept naturally arises if the distribution of matter, as given by both positions and masses, has self similar properties. We discuss the experimental situation in this respect and we introduce a simple stochastic model based of the aggregation process of particles. The aim of this model is to understand which characteristic properties of the aggregation probability can gives rise to multifractal distribution. In particular we find that a crucial element in this respect is that the aggregation probability should depend on the environment of the aggregation region.

1 Introduction

The basic characteristic of the observable galaxy distribution is the existence of Large Scale Structures (LSS) having fractal distribution at least up to some length deeper of the available samples: the two point number-number correlation function is a power law with exponent \(\gamma \sim 1.6 - 1.8\) up to the sample limit for galaxy and cluster distributions (Coleman and Pietronero 1992):

\[
G(r) = \langle n(r)n(0) \rangle \sim r^{-\gamma}
\]

A second important observational feature is the galaxy mass function: this function determines the probability of having a mass in the range between \(M\) and \(M + dM\) per unit volume, and can be described by the Press-Schechter function that shows a power law behaviour followed by an exponential cut-off for very large masses (Press and Schechter 1974):

\[
n(M)dM \sim M^{\delta-2} \exp\left(-\left(M/M^*\right)^{2\delta}\right)dM
\]

with \(\delta \sim 0.2\).

These two observational evidences can be naturally linked together by the concept of multifractal (MF) that provides a unified picture of the mass and
spatial distributions. For an introduction to the concept of MF see Coleman and Pietronero (1992), and Sylos Labini and Pietronero (1994a).

Masses of different galaxies can differ by as much as a factor $10^6$: and it is important to include these mass values in order to describe the entire matter distribution and not just the galaxy positions. The distribution of visible matter is described by the density function:

$$\rho(\vec{r}) = \sum_{i=1}^{N} m_i \delta(\vec{r} - \vec{r}_i)$$  \hspace{1cm} (3)

where $m_i$ is the mass of the $i$-th galaxy. This distribution corresponds to a measure defined on the set of points which have the correlation properties described by eq.(1). In this situation the whole distribution of matter may show self-similar properties (Pietronero 1987) and this can be described by the more general concept of multifractal. Therefore the MF distribution discussed here is related to the total distribution of visible matter, including the galaxy masses.

A MF distribution describes systems with local properties of self-similarity (Paladin and Vulpiani 1986) and it is characterised by a continuous set of exponents $\alpha$ each defined on a set with fractal dimension $f(\alpha)$. This distribution implies a strong correlation between spatial and mass distribution (Fig.1) so that the number of objects with mass $M$ in the point $\vec{r}$ per unit volume $\nu(M, \vec{r})$, is a function of space and mass and is not separable in a space density multiplied a mass function (Binggeli et al. 1988). It can be shown (Sylos Labini and Pietronero 1994a) that the mass function of a MF distribution, in a well defined volume, has indeed a Press-Schechter behaviour in which the exponent $\delta$ (eq.(2)) can be related to the properties of $f(\alpha)$. Moreover the fractal dimension of the support is $D(0) = f(\alpha_s) = 3 - \gamma$ (eq.(1)). Hence with the knowledge of the whole $f(\alpha)$ spectrum one obtains information on the correlations in space as well as on the mass function.

Coleman and Pietronero (1992) have performed a MF analysis of the CfA 1 redshift survey, assigning to each galaxy a mass proportional to its luminosity:
clearly this is a crude approximation, however a better relation between luminosity and mass should not change the MF nature of this distribution. They found that the whole distribution of matter provides unambiguous evidence for a MF behaviour.

Apart from the determination of the MF spectrum, there are many other observational consequences of the multifractal behaviour for the galaxy distribution (luminosity function, number counts, luminosity segregation, etc.), that will be discussed in detail in Sylos Labini and Pietronero 1994a.

From a theoretical point of view one would like to identify the dynamical processes that lead to such a MF distribution. In order to gain some insight into this complex problem we have developed a simple stochastic model (Sylos Labini and Pietronero 1994b) that includes the basic properties of the aggregation process and allows us to pose a variety of interesting questions concerning the possible dynamical origin of the MF distribution. The dynamics is characterised by some parameters, that have a direct physical meaning in term of cosmological processes. In this way we can relate the input parameters of the dynamics to the properties of the final configuration and produce a sort of phase diagram. The main point that we want to investigate here, is whether the LSS are generated by an amplification of the small amplitude initial fluctuations or if the generation of such LSS is intrinsically generated by the non linear dynamics, that has an asymptotic critical state. We find the the dependence from the local environment of the aggregation probability is the crucial element in order to give rise a fractal (and multifractal) structure.

2 The model

In our model the formation of structures proceeds by merging of smaller objects. When to particles collide in order to form a bound state, they have to dissipate a certain amount of energy. The basic physical mechanism responsible of energy dissipation in a collisionless and pressurless dustlike particles, interacting only via gravitational force, is the dynamical friction: a test particle moving through a cloud of other background particles, undergoes to a systematic deceleration effect due to the gravitational scattering (Chandrasekhar 1943). Clearly this process depends from the local density, but it depends also from the relative mass ad velocity of the test particle with respect to the background ones. Here we consider only the dependence from the local density: we are currently developing a model in which we take in account also of the other parameters of the dynamical friction (Sylos Labini et al. 1994c).

Due to the effect of the energy dissipation, the aggregation process dependent on the environment in which it takes place, and it is more efficient in more denser region.

In our model when two particles collide they have a probability \( P_a \) of irreversible aggregation and probability \( 1 - P_a \) to scatter. In such a manner the gravitational interaction is simulated only through the aggregation probability \( P_a \) that is made dependent from some parameters, \( P_a = P_a(\alpha, \beta, \ldots) \), that define the dynamics of the aggregation process. We find that environment dependence of the dynamical friction breaks the spatial symmetry of the aggregation process: this is one of the fundamental element that can give rise to a fractal (and multifractal) distribution.
Figure 2: The probability of making an irreversible aggregation $P_a$ during a collision is greater in denser regions than in sparse ones.

We have adopted the following procedure to estimate the local density, and introduce the dissipation effect in the simulation: we assign an influence function to each particle that describes the contribution of the particle to the dissipation effect. The aggregation probability is proportional to the total energy dissipated via dynamical friction. The influence function of the generic particle in the point $\vec{y}$ with mass $m(\vec{y}, t)$ at the time $t$ on the point $\vec{x}$, where the collision occurs, is described by:

$$f(t; \vec{x}, \vec{y}) = \exp\left(-\frac{|\vec{y} - \vec{x}|}{m(\vec{y}, t)\beta}\right)$$  \hspace{1cm} (4)

The multiparticle influence function is:

$$F(t; \vec{x}) = \int_V f(t; \vec{x}, \vec{y})d\vec{y}$$  \hspace{1cm} (5)

where a suitable value for the volume of integration is chosen. We define the aggregation probability as:

$$P_a(t, \vec{x}) = \sim F(t, \vec{x})^\alpha$$  \hspace{1cm} (6)

where we have introduced two free parameters $\alpha$ and $\beta$, that characterized the dynamics, and we have studied their role in the aggregation process.

3 The simulations

We have implemented a simulation in two dimensions in which the mass is conserved. At the beginning $N$ particles with equal mass (typically $N = 10^4$ and $m = 1$) are distributed randomly over a grid (256$^2$) in two dimensions. To each particle is assigned a velocity of equal module ($v = 1$) and random direction. The particles move one step a time along linear trajectories so that the position of the $n$-th particle at time $k$ is defined by:

$$\vec{x}^{(n)}_{(k)} = \vec{x}^{(n)}_{(k-1)} + \vec{v}^{(n)}_{(k-1)}\Delta k$$  \hspace{1cm} (7)
Figure 3: (a): The integrated density-density correlation function: the final states has dimension $D = 1.0$ ($\alpha = 2.5$ and $\beta = 0.5$). (b): the mass function in the same case

where $\Delta k$ is the unitary time step and $\vec{v}_{(k-1)}^{(n)}$ is the velocity of the $n$-th particles at the $k-1$-th time step. At each time step the simulation identifies possible the collision between two or more particles: typically there are mostly binary collisions. Once the collision has been identified we compute the total influence function due to the particles around the collision point, and then the aggregation probability $P_a$ of forming an irreversible aggregation according to eq.(5).

If the aggregation occurs, the two particles merge irreversibly in a single particle with mass $m = m_1 + m_2$. This aggregate has a probability $P(v)$ of having unitary velocity ones move along a linear trajectory, and probability $1 - P(v)$ to be stopped. $P(v)$ depends from momentum conservation,

$$P(v) = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \quad (8)$$

Clearly the heavier aggregates stop with a greater probability. If the aggregation does not occur the two particles will be scattered in random directions with the same incoming velocities.

To study the spatial properties of the simulation we compute the integral density-density correlation:

$$G(\vec{r}) = \int_0^R d\vec{r}' < \rho(\vec{r}_0) \rho(\vec{r} + \vec{r}_0) > \sim R^D \quad (9)$$

if $D = d = 2$ the system is homogeneous while if $D < d$ it is fractal.

At the beginning the aggregation is essentially random and the effect of the local density on the aggregation process is small everywhere. Once a certain mass distribution has been developed, the energy dissipation mechanism becomes dominant for further aggregation and there is a rapid increment of the aggregation probability in the are surrounded by some heavier particles. In a certain region of the parameter-space for the dynamics, there is a spatial symmetry breaking of the aggregation process and the non linear dynamics generates spontaneously the self-similar fractal distribution (Fig.3(a)). The breaking
of the spatial symmetry in the aggregation probability will lead asymptotically to regions that will never filled, (voids) because the density inside is very low. On the contrary aggregation processes are enhanced in dense region. The fractal dimension of the final state depends explicitly on the parameters of the dynamics: \( D = D(\alpha, \beta) \). For a larger value of \( \alpha \) the aggregation occurs in only in very dense regions, so that the clustering is stronger and the fractal dimension is lower. The parameter \( \beta \) tunes the influence of the size of the mass of each particle: if \( \beta \) is enhanced the larger aggregate dominates with respect to the smaller ones. This mechanism will trigger a spontaneous amplification of fluctuations and the whole matter distribution will growth in a in a self-similar way leading to a MF distribution (Fig.3(b)) (Sylos Labini et al. 1994c).

4 Conclusion

We consider an aggregation process in which the formation of structures is a process that depends on the local environment. This dependence to the energy dissipation mechanism (dynamical friction) that strongly depends from the local density. This environment dependent aggregation probability breaks spontaneously the spatial symmetry and leads to the formation of complex structures.

We have implemented a simulation in two dimension in which the mass is conserved and the particles move along linear trajectories. This model shows an asymptotic fractal distribution. The non linear dynamics leads spontaneously the self-similar (multifractal) fluctuations of the asymptotic state, so that there is not any crucial dependence from initial conditions. The fractal dimension of the asymptotic state depends only on the parameters of the non linear dynamics. The necessary ingredients for a dynamics in order to generate a fractal (multifractal including masses) distribution are the breaking of the spatial symmetry, and the Self-Organised nature of the dynamical mechanism.

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