Abstract. In this paper, we consider the initial boundary value problem for the three-dimensional viscous primitive equations of large-scale moist atmosphere which are used to describe the turbulent behavior of long-term weather prediction and climate changes. First, we obtain the existence and uniqueness of global strong solutions of the problem. Second, by studying the long-time behavior of strong solutions, we construct a (weak) universal attractor $\mathcal{A}$ which captures all the trajectories.

Key words: Primitive equations, Navier-Stokes equations, global well-posedness, long-time dynamics.

Mathematics Subject Classification(2000): 35Q30, 35Q35, 86A10.

1 Introduction

In order to understand the mechanism of long-term weather prediction and climate changes, one can study the mathematical equations and models governing the motion of the atmosphere as the atmosphere is a specific compressible fluid (see, e.g., [19, 29]). V. Bjerkness, one of the pioneers of meteorology, said that the weather forecasting can be considered as an initial boundary value problem in mathematical physics. In 1922, Richardson initially introduced the so-called primitive atmospheric equations which consisted of the hydrodynamic, thermodynamic equations with Coriolis force, cf. [30]. At that time, the primitive atmospheric equations were too complicated to be studied theoretically or to be solved numerically. To overcome this difficulty, some simple numerical models were introduced, such as the barotropic model.

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formulated by Neumann etc. in [5] and the quasi-geostrophic model introduced by Charney and Philips in [8]. The 2-D and 3-D quasi-geostrophic models have been the subject of analytical mathematical study, cf., e.g., [2, 3, 6, 7, 13, 14, 28, 37, 38, 39] and references therein.

Due to the considerable improvement in computer capacity and the development of atmospheric science, some mathematicians began to consider the primitive equations of atmosphere in the past two decades (see, e.g., [25, 26, 27, 34] and references therein). In [25], by introducing viscosity terms and using some technical treatment, Lions, Temam and Wang obtained a new formulation of the primitive equations of large-scale dry atmosphere which was amenable to mathematical study. In a p-coordinate system, the new formulation of the primitive equations is a little similar to Navier-Stokes equations of incompressible fluid. By the methods used to solve Navier-Stokes system in [23], they obtained the existence of global weak solutions of the initial boundary value problem for the new formulation of the primitive equations. Moreover, under the assumptions that there exists a unique global strong solution for the problem with vertical viscosity and that $H^1$-norm of the strong solution is uniformly bounded about $t$, they established some physically relevant estimates for the Hausdorff and fractal dimensions of the attractor of the primitive equations with vertical viscosity. Without those assumptions, by the Trajectory Attractors Theory due to Vishik and Chepyzhov (cf. [12, 35]), we obtained the existence of trajectory attractors for the large-scale moist atmospheric primitive equations in [16]. By taking advantage of the geostrophic balance and other geophysical consideration, several intermediate models have been the subject of studying the long-time dynamics and global attractors in order to describe the mechanism of long-term weather prediction and climate dynamics (see, e.g., [9, 10, 21, 36, 37] and references therein).

In recent years, there were some mathematicians who considered the existence of strong solutions for the three-dimensional viscous primitive equations of large-scale atmosphere and ocean (see, e.g., [4, 9, 10, 11, 18, 20, 33, 34] and references therein). In [18], Guillén-González etc. obtained the global existence of strong solutions to the primitive equations of large-scale ocean by assuming that the initial data are small enough, and also proved the local existence of strong solutions to the equation for all initial data. In [34], Temam and Ziane considered the local existence of strong solutions for the primitive equations of the atmosphere, the ocean and the coupled atmosphere-ocean. The papers [9, 10, 11] are devoted to considering the non-dimensional Boussinesq equations or modified models (see, e.g., [29, 33]). In [9], Cao and Titi considered global well-posedness and finite-dimensional global attractor to a 3-D planetary geostrophic model. The paper [11] is devoted to studying the
global well-posedness for the three-dimensional viscous primitive equations of large-scale ocean. In [11], Cao and Titi developed a beautiful approach, by which they obtained the fact that $L^6$-norm of the fluctuation $\tilde{v}$ of horizontal velocity is bounded uniformly about the time $t$. The estimate about $L^6$-norm of the fluctuation $\tilde{v}$ is a key proof in [11]. On the basis of the results of [11], we obtain the existence of (weak) universal attractors for the 3-D viscous primitive equations of the large-scale ocean in [17].

In the present paper we are interested in considering the existence, uniqueness and long-time behavior of global strong solutions to the initial boundary value problem of the new formulation of large-scale moist atmospheric primitive equations (the problem is denoted by (IBVP) and will be given in the section 2). Our main results are Theorem 3.1, Theorem 3.2, Proposition 3.3 and Theorem 3.4. First, we obtain the global well-posedness of the problem (IBVP). Second, by studying the long-time behavior of the strong solution, we prove $H^1$-norm of the strong solution is uniformly bounded about $t$, and also the corresponding semigroup $\{S(t)\}_{t \geq 0}$ possesses a bounded absorbing set $B_\rho$ in $V$ (the definition of the space $V$ will be given in subsection 4.1), by which we construct a (weak) universal attractor $\mathcal{A}$. Since the global well-posedness of the 3-D incompressible Navier-Stokes system is still open, by Theorem 3.1 and Theorem 3.2, we prove rigorously in mathematics that the new formulation of large-scale moist atmospheric primitive equations is simpler than the incompressible Navier-Stokes system, which is consistent with the physical point of view.

Inspired by the methods used in [11], we prove global well-posedness of (IBVP). However, there are two differences between the results of our paper and that in [11]. On one hand, the new formulation of the large-scale moist atmospheric equations is more complicated than the model studied in [11]. If we let $a = 0$, the model considered in this paper is similar to that in [11]. On the other hand, we have studied the long-time dynamics and the existence of the (weak) universal attractors for the large-scale moist atmospheric primitive equations. Here, the results about the universal attractors in this paper is more stronger than that in [16].

In order to study the long-time behavior of strong solutions, we must make three key estimates. First, we must make estimates about $L^3$-norm of the temperature $T$ and the fluctuation $\tilde{v}$ of horizontal velocity $v$ before we study the long-time behavior of strong solutions by the Uniform Gronwall Lemma, without which we only obtain the global well-posedness of (IBVP). Second, we ought to make estimates about $L^4$-norm of $\tilde{v}$, $T$ and the mixing ratio of water vapor in the air $q$. If we only made estimates about $L^6$-norm of $\tilde{v}$, $q$, $T$ as that in [11], we could not study long-time behavior of strong solutions and could not obtain a stronger result than the uniqueness of strong
solutions to (IBVP). Third, since the moist atmospheric equations are more complicated than the oceanic primitive equations, we have to make estimates about $\partial_\xi T$, $\partial_\xi q$ before we prove $H^1$-norm of $v$, $T$, $q$ is bounded.

The paper is organized as follows: In section 2, we pose the primitive equations of large-scale moist atmosphere. Main results of this paper are formulated in section 3. In section 4, we give our working spaces and some preliminaries. We prove main results of our paper in sections 5, 6, 7.

2 The three-dimensional viscous primitive equations of large-scale moist atmosphere

The three-dimensional viscous primitive equations of large-scale moist atmosphere in the pressure coordinate system (for details, we refer the reader to [16, 22, 25, 26] and references therein) is written as

\[
\frac{\partial v}{\partial t} + \nabla_v v + \omega \frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v + \text{grad}\Phi - \frac{1}{Re_1} \Delta v - \frac{1}{Re_2} \frac{\partial^2 v}{\partial \xi^2} = 0,
\]

\[
\text{div} v + \frac{\partial \omega}{\partial \xi} = 0,
\]

\[
\frac{\partial \Phi}{\partial \xi} + \frac{bP}{p} (1 + aq) T = 0,
\]

\[
\frac{\partial T}{\partial t} + \nabla_v T + \omega \frac{\partial T}{\partial \xi} - \frac{bP}{p} (1 + aq) \omega - \frac{1}{Rt_1} \Delta T - \frac{1}{Rt_2} \frac{\partial^2 T}{\partial \xi^2} = Q_1,
\]

\[
\frac{\partial q}{\partial t} + \nabla_v q + \omega \frac{\partial q}{\partial \xi} - \frac{1}{Rq_1} \Delta q - \frac{1}{Rq_2} \frac{\partial^2 q}{\partial \xi^2} = Q_2,
\]

where the unknown functions are $v$, $\omega$, $\Phi$, $q$, $T$, $v = (v_\theta, v_\varphi)$ the horizontal velocity, $\omega$ vertical velocity in $p-$ coordinate system, $\Phi$ the geopotential, $q$ the mixing ratio of water vapor in the air, $T$ temperature, $f = 2 \cos \theta$ Coriolis parameter, $R_0$ the Rossby number, $k$ vertical unit vector, $Re_1$, $Re_2$, $Rt_1$, $Rt_2$, $Rq_1$, $Rq_2$ Reynolds numbers, $P$ an approximate value of pressure at the surface of the earth, $p_0$ pressure of the upper atmosphere and $p_0 > 0$, the variable $\xi$ satisfying $p = (P - p_0)\xi + p_0$ ($0 < p_0 \leq p \leq P$), $Q_1$, $Q_2$ given functions on $S^2 \times (0,1)$ (here we don’t consider the condensation of water vapor), $a$ a positive constant ($a \approx 0.618$), $b$ a positive constant. The definitions of $\nabla_v v$, $\Delta v$, $\Delta T$, $\Delta q$, $\nabla_v q$, $\nabla_v T$, div $v$, grad $\Phi$ will be given in the subsection 4.1. The equations (2.1) – (2.5) are called the 3-D viscous primitive equations of the large-scale moist atmosphere.
The space domain of the equations (2.1) – (2.5) is
\[ \Omega = S^2 \times (0, 1), \]
where \( S^2 \) is two-dimensional unit sphere. The boundary value conditions are given by
\[ \xi = 1(p = P) : \frac{\partial v}{\partial \xi} = 0, \quad \omega = 0, \quad \frac{\partial T}{\partial \xi} = \alpha_s(T_s - T), \quad \frac{\partial q}{\partial \xi} = \beta_s(q_s - q), \quad (2.6) \]
\[ \xi = 0(p = p_0) : \frac{\partial v}{\partial \xi} = 0, \quad \omega = 0, \quad \frac{\partial T}{\partial \xi} = 0, \quad \frac{\partial q}{\partial \xi} = 0, \quad (2.7) \]
where \( \alpha_s, \beta_s \) are positive constants, \( T_s \) the given temperature on the surface of the earth, \( q_s \) the given mixing ratio of water vapor on the surface of the earth. For simplicity and without loss generality we assume that \( T_s = 0 \) and \( q_s = 0 \). If \( T_s \neq 0 \) and \( q_s \neq 0 \), one can homogenize the boundary value conditions for \( T, q \) (cf., e.g., [16]).

Integrating (2.2) and using the boundary conditions (2.6), (2.7), we have
\[ \omega(t; \theta, \varphi, \xi) = W(v)(t; \theta, \varphi, \xi) = \int_0^1 \text{div}(t; \theta, \varphi, \xi') d\xi', \quad (2.8) \]
\[ \int_0^1 \text{div} v \, d\xi = 0. \quad (2.9) \]

Suppose that \( \Phi_s \) is a certain unknown function at the isobaric surface \( \xi = 1 \). Integrating (2.3), we obtain
\[ \Phi(t; \theta, \varphi, \xi) = \Phi_s(t; \theta, \varphi) + \int_0^1 \frac{bP}{p}(1 + aq)T \, d\xi'. \quad (2.10) \]

Then the equations (2.1) – (2.5) can be written as
\[ \frac{\partial v}{\partial t} + \nabla_v v + W(v) \frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v + \text{grad} \Phi_s + \int_0^1 \frac{bP}{p} \text{grad}[(1 + aq)T] \, d\xi' \]
\[ - \frac{1}{R_{e1}} \Delta v - \frac{1}{R_{e2} \partial \xi^2} = 0, \quad (2.11) \]
\[ \frac{\partial T}{\partial t} + \nabla_v T + W(v) \frac{\partial T}{\partial \xi} - \frac{bP}{p}(1 + aq)W(v) - \frac{1}{R_{t1}} \Delta T - \frac{1}{R_{t2} \partial \xi^2} = Q_1, \quad (2.12) \]
\[ \frac{\partial q}{\partial t} + \nabla_v q + W(v) \frac{\partial q}{\partial \xi} - \frac{1}{R_{q1}} \Delta q - \frac{1}{R_{q2} \partial \xi^2} = Q_2, \quad (2.13) \]
\[ \int_0^1 \text{div} v \, d\xi = 0, \quad (2.14) \]

where the definitions of \( \text{grad}[(1+aq)T], \text{grad}\Phi_s \) will be given in the subsection 4.1. The boundary value conditions of the equations (2.11) – (2.14) are given by

\[
\begin{align*}
\xi &= 1: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = -\alpha_s T, \quad \frac{\partial q}{\partial \xi} = -\beta_s q, \\
\xi &= 0: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = 0, \quad \frac{\partial q}{\partial \xi} = 0; \quad (2.15) \\
\end{align*} \]

and the initial value conditions can be given as

\[
U|_{t=0} = (v|_{t=0}, T|_{t=0}, q|_{t=0}) = U_0 = (v_0, T_0, q_0). \quad (2.17)
\]

We call (2.11) – (2.17) as the initial boundary value problem of the new formulation of the 3-D viscous primitive equations of large-scale moist atmosphere, which is denoted by (IBVP).

Now we define the fluctuation \( \tilde{v} \) of horizontal velocity and find the equations satisfied by \( \tilde{v} \) and \( \bar{v} \). By integrating the momentum equation (2.11) with respect to \( \xi \) from 0 to 1 and using the boundary value conditions (2.15) and (2.16), we get

\[
\begin{align*}
\frac{\partial \bar{v}}{\partial t} + \int_0^1 (\nabla v + W(v)) \frac{\partial v}{\partial \xi} d\xi + \frac{f}{R_0} k \times \bar{v} + \text{grad}\Phi_s + \int_0^1 \int_\xi \frac{bP}{p} \text{grad}[(1+aq)T] d\xi' d\xi &= -\frac{1}{Re_1} \Delta \bar{v} = 0 \quad \text{in} \ S^2, \quad (2.18) \\
\end{align*} \]

where \( \bar{v} = \int_0^1 v d\xi \).

Denote the fluctuation of the horizontal velocity by

\[ \tilde{v} = v - \bar{v}. \]

We notice that

\[ \tilde{v} = \int_0^1 \tilde{v} d\xi = 0, \quad \nabla \cdot \tilde{v} = 0. \quad (2.19) \]

By integration by parts and (2.19), we have

\[
\begin{align*}
\int_0^1 W(v) \frac{\partial v}{\partial \xi} d\xi &= \int_0^1 v \text{div} v d\xi = \int_0^1 \tilde{v} \text{div} \tilde{v} d\xi, \\
\int_0^1 \nabla v v d\xi &= \int_0^1 \nabla \tilde{v} \tilde{v} d\xi + \nabla \tilde{v}. \tilde{v}. \quad (2.20)
\end{align*} \]
From (2.18), (2.20) and (2.21), we obtain
\[
\frac{\partial \tilde{v}}{\partial t} + \nabla \tilde{v} + \bar{v} \text{div} \tilde{v} + \nabla \bar{v} + \frac{f}{R_0} k \times \tilde{v} + \text{grad} \Phi + \int_0^1 \int_1^1 \frac{b P}{p} \text{grad}[(1 + aq) T] d\xi' d\xi \\
- \frac{1}{Re_1} \Delta \tilde{v} = 0 \quad \text{in} \ S^2.
\]  
(2.22)

Subtracting (2.22) from (2.11), we know that the fluctuation \( \tilde{v} \) satisfies the following equation and boundary value conditions
\[
\frac{\partial \tilde{v}}{\partial t} + \nabla \tilde{v} + \left( \int_\xi^1 \text{div} \tilde{v} d\xi' \right) \frac{\partial \tilde{v}}{\partial \xi} + \nabla \bar{v} + \nabla \tilde{v} - \left( \text{div} \bar{v} + \nabla \bar{v} \right) + \frac{f}{R_0} k \times \tilde{v} \\
+ \int_\xi^1 \frac{b P}{p} \text{grad}[(1 + aq) T] d\xi' - \int_0^1 \int_\xi^1 \frac{b P}{p} \text{grad}[(1 + aq) T] d\xi' d\xi \\
- \frac{1}{Re_1} \Delta \tilde{v} - \frac{1}{Re_2} \frac{\partial^2 \tilde{v}}{\partial \xi^2} = 0 \quad \text{in} \ \Omega,
\]  
(2.23)
\[\xi = 1 : \frac{\partial \tilde{v}}{\partial \xi} = 0, \quad (2.24)\]
\[\xi = 0 : \frac{\partial \tilde{v}}{\partial \xi} = 0. \quad (2.25)\]

3 Statements of main results

Now we formulate our main results in the present paper.

**Theorem 3.1** Let \( Q_1, Q_2 \in H^1(\Omega), \ U_0 = (v_0, T_0, q_0) \in V \). Then for any \( T > 0 \) given, there exists a strong solution \( U \) of the system (2.11)-(2.17) on the interval \([0, T]\), where the definition of strong solutions to the system (2.11)-(2.17) will be given in subsection 5.1.

**Theorem 3.2** Let \( Q_1, Q_2 \in H^1(\Omega), \ U_0 = (v_0, T_0, q_0) \in V \). Then for any \( T > 0 \) given, the strong solution \( U \) of the system (2.11)-(2.17) on the interval \([0, T]\) is unique. Moreover, the strong solution \( U \) is dependent continuously on the initial data.

**Proposition 3.3** If \( Q_1, Q_2 \in H^1(\Omega), \ U_0 = (v_0, T_0, q_0) \in V \), Then the global strong solution \( U \) of the system (2.11)-(2.17) satisfies \( U \in L^\infty(0, \infty; V) \). Moreover, the corresponding semigroup \( \{S(t)\}_{t \geq 0} \) possesses a bounded absorbing set \( B_0 \) in \( V \), i.e., for every bounded set \( B \subset V \), there exists \( t_0(B) > 0 \) big enough such that
\[S(t)B \subset B_{t_0}, \text{ for any } t \geq t_0,\]
where $B_\rho = \{ U; \| U \| \leq \rho \}$ and $\rho$ is a positive constant dependent on $\| Q_1 \|_1, \| Q_2 \|_1$.

**Theorem 3.4** The system (2.11)-(2.16) possesses a (weak) universal attractor $A = \cap_{s \geq 0} \cup_{t \geq s} T(t)B_\rho$ that captures all the trajectories, where the closures are taken with respect to $V$-weak topology. The (weak) universal attractor $A$ has the following properties:

(i) (weak compact) $A$ is bounded and weakly closed in $V$;

(ii) (invariant) for every $t \geq 0$, $S(t)A = A$;

(ii) (attracting) for every bounded set $B$ in $V$, the sets $S(t)B$ converge to $A$ with respect to $V$-weak topology as $t \to +\infty$, i.e.,

$$\lim_{t \to +\infty} d^w_V(S(t)B, A) = 0,$$

where the distance $d^w_V$ is induced by the $V$-weak topology.

**Remark 3.5** The (weak) universal attractor $A$ has the following additional properties:

(i) By Rellich-Kondrachov Compact Embedding Theorem (cf., e.g., [1]), we know that for any $1 \leq p < 6$ the sets $S(t)B$ converge to $A$ with respect to the $L^p(\Omega) \times L^p(\Omega) \times L^p(\Omega)\times L^p(\Omega)$-norm;

(ii) The (weak) universal attractor $A$ is unique and is connected with respect to $V$-weak topology.

**Remark 3.6** In the forthcoming paper, we shall prove the (weak) universal attractor $A$ have finite fractal and Hausdorff dimensions in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)\times L^2(\Omega)$.

**Remark 3.7** In comparison to the 3-D incompressible Navier-Stokes equations, the 3-D viscous primitive equations of large-scale moist atmosphere have not the time derivative term of the vertical velocity $\omega = W(v)$. Therefore, we can not prove that the bounded absorbing set $B_\rho$ in $V$ is bounded in $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$ as in the case of 3-D incompressible Navier-Stokes equations (for the Navier-Stokes equations, if there exists a bounded absorbing set $B_\rho$ in $H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$, then one can prove $B_\rho$ is bounded in $H^2(\Omega) \times H^2(\Omega) \times H^2(\Omega)$), i.e., we can not prove that the universal attractor $A$ is compact in $V$. In this sense, the primitive equations of the large-scale atmosphere are more complicated than 3-D incompressible Navier-Stokes equations.
4 Preliminaries

4.1 Some function spaces

Let \( e_\theta, e_\varphi, e_\xi \) be the unit vectors in \( \theta, \varphi \) and \( \xi \) directions of the space domain \( \Omega \) respectively,

\[
e_\theta = \frac{\partial}{\partial \theta}, \quad e_\varphi = \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi}, \quad e_\xi = \frac{\partial}{\partial \xi}.
\]

The inner product and norm on \( T(\theta,\varphi,\xi)\Omega \) (the tangent space of \( \Omega \) at the point \( (\theta,\varphi,\xi) \)) are given by

\[
(X,Y) = X \cdot Y = X_1Y_1 + X_2Y_2 + X_3Y_3, \quad |X| = (X,X)^{1/2}
\]

for

\[
X = X_1e_\theta + X_2e_\varphi + X_3e_\xi, \quad Y = Y_1e_\theta + Y_2e_\varphi + Y_3e_\xi \in T(\theta,\varphi,\xi)\Omega.
\]

\( L^p(\Omega) := \{ h; \ h : \Omega \to \mathbb{R}, \int_\Omega |h|^p < +\infty \} \) with the norm \( |h|_p = (\int_\Omega |h|^p)^{1/p}, \quad 1 \leq p < \infty \). \( \int_\Omega |d\Omega \) and \( \int_{S^2} |dS^2| \) are denoted by \( \int_\Omega \) and \( \int_{S^2} \), respectively. \( L^2(T\Omega|TS^2) \) is the first two components of \( L^2 \) vector fields on \( \Omega \) with the norm \( |v|_2 = (\int_\Omega (|v_\theta|^2 + |v_\varphi|^2))^{1/2} \), where \( v = (v_\theta, v_\varphi) : \Omega \to TS^2 \). \( C^\infty(S^2) \) is the function space for all smooth functions from \( S^2 \) to \( \mathbb{R} \). \( C^\infty(\Omega) \) is the function space for all smooth functions from \( \Omega \) to \( \mathbb{R} \). \( C^\infty(T\Omega|TS^2) \) is the first two components of smooth vector fields on \( \Omega \). \( H^m(\Omega) \) is the Sobolev space of functions which are in \( L^2 \), together with all their covariant derivatives with respect to \( e_\theta, e_\varphi, e_\xi \) of order \( \leq m \), with the norm

\[
|h|_m = (\int_\Omega (\sum_{1 \leq k \leq m} \sum_{i_1=1,2,j=1,\cdots,k} |\nabla_{i_1} \cdots \nabla_{i_k} h|^2 + |h|^2))^{1/2},
\]

where \( \nabla_1 = \nabla_{e_\theta}, \nabla_2 = \nabla_{e_\varphi}, \nabla_3 = \nabla_{e_\xi} = \frac{\partial}{\partial \xi} \) (the definitions of \( \nabla_{e_\theta}, \nabla_{e_\varphi} \) will be given later). \( H^m(T\Omega|TS^2) = \{ v; \ v = (v_\theta, v_\varphi) : \Omega \to TS^2, \ |v|^m_\Omega < +\infty \} \), the norm of which is similar to that of \( H^m(\Omega) \), that is, in the above formula of norm, we let \( h = (v_\theta, v_\varphi) = v_\theta e_\theta + v_\varphi e_\varphi \).

The horizontal divergence \( \text{div} \), the horizontal gradient \( \nabla = \text{grad} \), the horizontal covariant derivative \( \nabla_v \) and horizontal Laplace-Beltrami operator \( \Delta \) for scalar and vector functions are defined by

\[
\text{div}v = \text{div}(v_\theta e_\theta + v_\varphi e_\varphi) = \frac{1}{\sin \theta} \left( \frac{\partial v_\theta}{\partial \theta} \sin \theta + \frac{\partial v_\varphi}{\partial \varphi} \right), \quad (4.1)
\]

\[
\nabla T = \text{grad} T = \frac{\partial T}{\partial \theta} e_\theta + \frac{1}{\sin \theta} \frac{\partial T}{\partial \varphi} e_\varphi, \quad (4.2)
\]

\]
\begin{align*}
\text{grad} \Phi_s &= \frac{\partial \Phi_s}{\partial \theta} e_\theta + \frac{1}{\sin \theta} \frac{\partial \Phi_s}{\partial \varphi} e_\varphi, \\
\nabla \mathbf{\tilde{v}} &= (v_\theta \frac{\partial \mathbf{\tilde{v}}_\theta}{\partial \theta} + \frac{v_\varphi}{\sin \theta} \frac{\partial \mathbf{\tilde{v}}_\varphi}{\partial \varphi} - v_\varphi \mathbf{\tilde{v}}_\varphi \cot \theta) e_\theta + (v_\theta \frac{\partial \mathbf{\tilde{v}}_\varphi}{\partial \theta} + \frac{v_\varphi}{\sin \theta} \frac{\partial \mathbf{\tilde{v}}_\varphi}{\partial \varphi} + v_\varphi \mathbf{\tilde{v}}_\theta \cot \theta) e_\varphi, \\
\nabla T &= v_\theta \frac{\partial T}{\partial \theta} + \frac{v_\varphi}{\sin \theta} \frac{\partial T}{\partial \varphi}, \\
\nabla q &= v_\theta \frac{\partial q}{\partial \theta} + \frac{v_\varphi}{\sin \theta} \frac{\partial q}{\partial \varphi}, \\
\Delta T &= \text{div(grad}T) = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 T}{\partial \varphi^2} \right], \\
\Delta q &= \text{div(grad}q) = \frac{1}{\sin \theta} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial q}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 q}{\partial \varphi^2} \right], \\
\Delta v &= (\Delta v_\theta - \frac{2 \cos \theta}{\sin^2 \theta} \frac{\partial v_\varphi}{\partial \varphi} - \frac{v_\theta}{\sin^2 \theta}) e_\theta + (\Delta v_\varphi + \frac{2 \cos \theta}{\sin^2 \theta} \frac{\partial v_\theta}{\partial \varphi} - \frac{v_\varphi}{\sin^2 \theta}) e_\varphi,
\end{align*}

where \( v = v_\theta e_\theta + v_\varphi e_\varphi, \) \( \mathbf{\tilde{v}} = \mathbf{\tilde{v}}_\theta e_\theta + \mathbf{\tilde{v}}_\varphi e_\varphi \in C^\infty(T\Omega|TS^2), \) \( T, q \in C^\infty(\Omega), \) \( \Phi_s \in C^\infty(S^2). \)

Now we can define our working spaces for the problem (IBVP). Let

\begin{align*}
\mathcal{V}_1 &= \{ v; v \in C^\infty(T\Omega|TS^2), \left. \frac{\partial v}{\partial \xi} \right|_{\xi=0} = 0, \left. \frac{\partial v}{\partial \xi} \right|_{\xi=1} = 0, \int_0^1 \text{div} v \, d\xi = 0 \}, \\
\mathcal{V}_2 &= \{ T; T \in C^\infty(\Omega), \left. \frac{\partial T}{\partial \xi} \right|_{\xi=0} = 0, \left. \frac{\partial T}{\partial \xi} \right|_{\xi=1} = -\alpha_s T \}, \\
\mathcal{V}_3 &= \{ q; q \in C^\infty(\Omega), \left. \frac{\partial q}{\partial \xi} \right|_{\xi=0} = 0, \left. \frac{\partial q}{\partial \xi} \right|_{\xi=1} = -\beta_s q \}, \\
V &= \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3, \\
H &= H_1 \times H_2 \times H_2.
\end{align*}
The inner products and norms on $V_1$, $V_2$, $V_3$ are given by

$$(v, v_1)_{V_1} = \int_\Omega (\nabla_{e_\theta} v \cdot \nabla_{e_\theta} v_1 + \nabla_{e_\varphi} v \cdot \nabla_{e_\varphi} v_1 + \frac{\partial v}{\partial \xi} \frac{\partial v_1}{\partial \xi} + v \cdot v_1),$$

$$\|v\| = (v, v)_{V_1}^{\frac{1}{2}}, \quad \forall v, v_1 \in V_1,$n

$$(T, T_1)_{V_2} = \int_\Omega (\text{grad} T \cdot \text{grad} T_1 + \frac{\partial T}{\partial \xi} \frac{\partial T_1}{\partial \xi} + TT_1),$$

$$\|T\| = (T, T)_{V_2}^{\frac{1}{2}}, \quad \forall T, T_1 \in V_2,$n

$$(q, q_1)_{V_3} = \int_\Omega (\text{grad} q \cdot \text{grad} q_1 + \frac{\partial q}{\partial \xi} \frac{\partial q_1}{\partial \xi} + qq_1),$$

$$\|q\| = (q, q)_{V_3}^{\frac{1}{2}}, \quad \forall q, q_1 \in V_2,$n

$$(U, U_1)_H = (v, v_1) + (T, T_1) + (q, q_1),$$

$$(U, U_1)_V = (v, v_1)_{V_1} + (T, T_1)_{V_2} + (q, q_1)_{V_3},$$

$$\|U\| = (U, U)^{\frac{1}{2}}, \quad |U|_2 = (U, U)^{\frac{1}{2}}_H, \quad \forall U = (v, T, q), \quad U_1 = (v_1, T_1, q_1) \in V,$n

where $(\cdot, \cdot)$ denotes the $L^2$ inner products in $H_1$, $H_2$.

### 4.2 Some Lemmas

**Lemma 4.1** Let $u = (u_\theta, u_\varphi)$, $u_1 = ((u_1)_\theta, (u_1)_\varphi) \in C^\infty(T\Omega|TS^2)$, and $p \in C^\infty(S^2)$. Then

$$(1) \quad \int_{S^2} p \text{div} u = -\int_{S^2} \nabla p \cdot u, \quad (4.10)$$

in particular,

$$(2) \quad \int_{S^2} \text{div} u = 0; \quad (4.11)$$

$$(3) \quad \int_{\Omega} (-\Delta u) \cdot u_1 = \int_{\Omega} (\nabla_{e_\theta} u \cdot \nabla_{e_\theta} u_1 + \nabla_{e_\varphi} u \cdot \nabla_{e_\varphi} u_1 + u \cdot u_1). \quad (4.12)$$

**Proof.** We can prove the first part of Lemma 4.1 by using (4.1), (4.2) and Stokes Theorem(cf., e.g., [32, 36]). From (4.4) and (4.9), by direct computation, we can obtain the second part.
Lemma 4.2 (Interpolation Inequality) Let $\Omega_1$ be a bounded domain in $\mathbb{R}^n$, whose boundary $\partial \Omega_1$ satisfies $\partial \Omega_1 \in C^m$. Then for every $u \in W^{m,r}(\Omega_1) \cap L^q(\Omega_1)$, $0 \leq l \leq m$, one has

1) when $m - l - \frac{n}{r}$ is not a non-negative integer,

$$\|D^l u\|_{L^p(\Omega_1)} \leq c \|u\|^\alpha_{W^{m,r}(\Omega_1)} \|u\|^{1-\alpha}_{L^q(\Omega_1)},$$

where $l, p, \alpha, m, r, q$ satisfy

$$\frac{l}{m} \leq \alpha \leq 1, 1 \leq r, q \leq \infty, \frac{1}{p} - \frac{l}{n} = \alpha \left(\frac{1}{r} - \frac{m}{n}\right) + (1 - \alpha) \frac{1}{q};$$

2) when $m - l - \frac{n}{r}$ is a non-negative integer,

$$\|D^l u\|_{L^p(\Omega_1)} \leq c \|u\|^\alpha_{W^{m,r}(\Omega_1)} \|u\|^{1-\alpha}_{L^q(\Omega_1)},$$

where $l, p, \alpha, m, r, q$ satisfy

$$\frac{l}{m} \leq \alpha < 1, 1 < r < \infty, 1 < q < \infty.$$

In particular,

i) for $u \in H^1(S^2)$ (for the definitions of $H^1(S^2)$, $L^p(S^2)$, cf. [24]),

$$\|u\|_{L^4(S^2)} \leq c \|u\|^\frac{1}{4}_{L^4(S^2)} \|u\|^\frac{1}{2}_{H^1(S^2)};$$

(4.13)

$$\|u\|_{L^6(S^2)} \leq c \|u\|^\frac{1}{6}_{L^6(S^2)} \|u\|^\frac{1}{2}_{H^1(S^2)};$$

(4.14)

$$\|u\|_{L^8(S^2)} \leq c \|u\|^\frac{1}{8}_{L^8(S^2)} \|u\|^\frac{1}{2}_{H^1(S^2)};$$

(4.15)

ii) for $u \in H^1(\Omega)$,

$$\|u\|_{L^4(\Omega)} \leq c \|u\|^\frac{1}{4}_{L^2(\Omega)} \|u\|^\frac{1}{2}_{H^1(\Omega)}.$$  (4.16)

Proof. The proof of 1), 2) is similar to that of the case $\Omega_1 = \mathbb{R}^n$ but one must use the Extension Theorem (for the detail of proof, we refer the reader to see, e.g., [1, 15, 31]). For the proof of (4.13), (4.14), (4.15), one can refer to [24, Chapter 1].

Lemma 4.3 For any $h \in C^\infty(S^2), v \in C^\infty(T\Omega|TS^2)$, we have

$$\int_{S^2} \nabla_v h + \int_{S^2} h \text{div} v = \int_{S^2} \text{div}(hv) = 0.$$

Proof. From (4.1), (4.5), (4.11), by direct computation, we can prove Lemma 4.3.

Lemma 4.4 Let $v, v_1 \in V_1, T \in V_2, q \in V_3$. Then we have
1) \( \int_\Omega (\nabla v_1 + (\int_\xi \text{div}\xi') \frac{\partial v_1}{\partial \xi}) v_1 = 0, \)

2) \( \int_\Omega (\nabla T + (\int_\xi \text{div}\xi') \frac{\partial T}{\partial \xi}) T = 0, \)

3) \( \int_\Omega (\nabla q + (\int_\xi \text{div}\xi') \frac{\partial q}{\partial \xi}) q = 0, \)

4) \( \int_\Omega (\int_\xi \frac{bP}{p} \text{grad}[(1 + aq)T] d\xi' \cdot v - \frac{bP}{p}(1 + aq)W(v) \cdot T) = 0. \)

For the detail of proof for Lemma 4.4, we refer the reader to [16, Lemma 3.2].

**Lemma 4.5 (Minkowski Inequality)** Let \((X, \mu), (Y, \nu)\) be two measure spaces and \(f(x, y)\) be a measurable function about \(\mu \times \nu\) on \(X \times Y\). If for a.e. \(y \in Y\), \(f(\cdot, y) \in L^p(X, \mu), 1 \leq p \leq \infty\), and \(\int_Y \|f(\cdot, y)\|_{L^p(X, \mu)} d\nu(y) < \infty\), then

\[ \|\int_Y f(\cdot, y)d\nu(y)\|_{L^p(X, \mu)} \leq \int_Y \|f(\cdot, y)\|_{L^p(X, \mu)} d\nu(y). \]

**Lemma 4.6 (The Uniform Gronwall Lemma)** Let \(\phi, \psi, \varphi\) be three positive locally integrable functions on \([t_0, +\infty)\) such that \(\varphi'\) is locally integrable on \([t_0, +\infty)\), and which satisfy

\[ \frac{d\phi}{dt} \leq \phi \varphi + \psi \quad \text{for } t \geq t_0, \]

\[ \int_t^{t+r} \phi(s)ds \leq a_1, \quad \int_t^{t+r} \psi(s)ds \leq a_2, \quad \int_t^{t+r} \varphi(s)ds \leq a_3 \quad \text{for } t \geq t_0, \]

where \(r, a_1, a_2, a_3\) are positive constants. Then

\[ \varphi(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) \exp(a_1), \quad \forall t \geq t_0. \]

For the detail of proof for Lemma 4.6, we refer the reader to [32, p91].

**5 Global existence of strong solutions**

**5.1 Local existence of strong solutions**

In this subsection we recall the local, in time, existence of strong solutions of the 3-D viscous primitive equations of the large-scale moist atmosphere.
**Definition 5.1** Let $U_0 = (v_0, T_0, q_0) \in V$, and let $T$ be a fixed positive time. $U = (v, T, q)$ is called a strong solution of the system (2.11)-(2.17) on the time interval $[0, T]$ if it satisfies (2.11)-(2.13) in weak sense such that

$$v \in C([0, T]; V_1) \cap L^2(0, T; (H^2(\Omega))^2),$$

$$T \in C([0, T]; V_2) \cap L^2(0, T; H^2(\Omega)),$$

$$q \in C([0, T]; V_3) \cap L^2(0, T; H^2(\Omega)),$$

$$\frac{\partial v}{\partial t} \in L^1(0, T; (L^2(\Omega))^2),$$

$$\frac{\partial T}{\partial t} \in L^1(0, T; L^2(\Omega)),$$

$$\frac{\partial q}{\partial t} \in L^1(0, T; L^2(\Omega)).$$

**Remark 5.2** Since the 3-D viscous primitive equations of large-scale moist atmosphere have not the time derivative term of the vertical velocity $\omega = W(v)$, we cannot prove $\frac{\partial v}{\partial t} \in L^2(0, T; (L^2(\Omega))^2)$, $\frac{\partial T}{\partial t} \in L^2(0, T; L^2(\Omega)).$

**Proposition 5.3** Let $Q_1, Q_2 \in H^1(\Omega), U_0 = (v_0, T_0, q_0) \in V$. Then there exists $T_\ast > 0$, $T_\ast = T_\ast(||U_0||)$, and there exists a strong solution $U$ of the system (2.11)-(2.17) on the interval $[0, T_\ast]$.

**Proof.** The proof of Proposition 5.3 is similar to that for the primitive equations of large-scale ocean given in the papers [18, 34]. So we omit the detail of proof here.

In order to prove the global existence of strong solutions to the system (2.11)-(2.17), we should make a priori estimates about $H^1$-norm of the local solution $U(t)$ obtained in Proposition 5.3, that is, we should show that if $T_\ast < \infty$ then the $H^1$-norm of the strong solution $U(t)$ is bounded over the interval $[0, T_\ast]$.

### 5.2 A priori estimates about local strong solutions

**$L^2$ estimates about $v, T, q$.** Choosing $v$ as a test function in equation (2.11), we obtain

$$
\frac{1}{2} \frac{dv}{dt} + \frac{1}{Re_1} \int_\Omega (|\nabla_{e_x} v|^2 + |\nabla_{e_y} v|^2 + |v|^2) + \frac{1}{Re_2} \int_\Omega |\frac{\partial v}{\partial \xi}|^2
= - \int_\Omega (\nabla_x v + W(v) \frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v + \text{grad}\Phi_s) \cdot v
- \int_\Omega \int_\xi \frac{bP}{p} \text{grad}((1 + aq)T) d\xi' \cdot v.
$$

(5.1)
By Lemma 4.4, we get
\[ \int_{\Omega} (\nabla v + W(v) \frac{\partial v}{\partial \xi}) \cdot v = 0. \] (5.2)

\[ (\frac{f}{R_0} k \times v) \cdot v = 0 \] implies
\[ \int_{\Omega} (\frac{f}{R_0} k \times v) \cdot v = 0. \] (5.3)

By using integration by parts and (2.14), we get
\[ \int_{\Omega} \text{grad}\Phi_s \cdot v = -\int_{\Omega} \Phi_s \text{div}v = -\int_{S^2} \Phi_s (\int_0^1 \text{div}v d\xi) = 0. \] (5.4)

So, from (5.1)-(5.4), we obtain
\[ \frac{1}{2} d |v|^2_2 + \frac{1}{Rc_1} \int_{\Omega} |(\nabla_{es} v)^2 + |\nabla_{es} v|^2 + |v|^2| + \frac{1}{Rc_2} \int_{\Omega} |\frac{\partial v}{\partial \xi}|^2 \]
\[ = -\int_{\Omega} \int_1^1 \frac{bP}{p} \text{grad}((1 + aq)T) d\xi \cdot v. \] (5.5)

Taking the inner product of equation (2.12) with $T$ in $L^2(\Omega)$, we obtain
\[ \frac{1}{2} d |T|^2_2 + \frac{1}{Rt_1} \int_{\Omega} |\nabla T|^2 + \frac{1}{Rt_2} \int_{\Omega} |\frac{\partial T}{\partial \xi}|^2 + \frac{\alpha_s}{Rt_2} |T|_{\xi=1}^2 \]
\[ = -\int_{\Omega} (\nabla_s T + W(v) \frac{\partial T}{\partial \xi}) T + \int_{\Omega} \frac{bP}{p} (1 + aq)TW(v) + \int_{\Omega} Q_1T. \] (5.6)

By Lemma 4.4, we have
\[ \int_{\Omega} (\nabla_s T + W(v) \frac{\partial T}{\partial \xi}) T = 0. \] (5.7)

Combining (5.6) with (5.7), we get
\[ \frac{1}{2} d |T|^2_2 + \frac{1}{Rt_1} \int_{\Omega} |\nabla T|^2 + \frac{1}{Rt_2} \int_{\Omega} |\frac{\partial T}{\partial \xi}|^2 + \frac{\alpha_s}{Rt_2} |T|_{\xi=1}^2 \]
\[ = \int_{\Omega} \frac{bP}{p} (1 + aq)TW(v) + \int_{\Omega} Q_1T. \] (5.8)

Similarly to (5.8), we have
\[ \frac{1}{2} d |q|^2_2 + \frac{1}{Rq_1} \int_{\Omega} |\nabla q|^2 + \frac{1}{Rq_2} \int_{\Omega} |\frac{\partial q}{\partial \xi}|^2 + \frac{\beta_s}{Rq_2} |q|_{\xi=1}^2 \]
\[ = \int_{\Omega} qQ_2. \] (5.9)
By using Lemma 4.4, we have
\[-\int_{\Omega} \left[ \int_{\xi}^{1} \frac{bP}{p} \text{grad}((1 + aq)T) d\xi' \right] \cdot v + \int_{\Omega} \frac{bP}{p} (1 + cq)TW(v) = 0. \tag{5.10}\]

From (5.5) and (5.8)-(5.10), we obtain
\[
\begin{align*}
\frac{1}{2} \frac{d(|v|^2 + |T|^2 + |q|^2)}{dt} &+ \frac{1}{Re_1} \int_{\Omega} (|\nabla e_\theta v|^2 + |\nabla e_\phi v|^2 + |v|^2) + \frac{1}{Re_2} \int_{\Omega} |\frac{\partial v}{\partial \xi}|^2 \\
&+ \frac{1}{R_t_1} \int_{\Omega} |\nabla T|^2 + \frac{1}{R_t_2} \int_{\Omega} |\frac{\partial T}{\partial \xi}|^2 + \frac{\alpha_s}{R_t_2} |T|_{\xi=1}^2 + \frac{1}{R_q_1} \int_{\Omega} |\nabla q|^2 \\
&+ \frac{1}{R_q_2} \int_{\Omega} |\frac{\partial q}{\partial \xi}|^2 + \frac{\beta_s}{R_q_2} |q|_{\xi=1}^2 \\
&= \int_{\Omega} Q_1 T + \int_{\Omega} q Q_2. \tag{5.11}\end{align*}
\]

By \(T(\theta, \varphi, \xi) = -\int_{\xi}^{1} \frac{\partial T}{\partial \xi} d\xi' + T|_{\xi=1}\), using Hölder inequality and Cauchy-Schwarz inequality, we have
\[
|T|_{2}^2 \leq 2 |\frac{\partial T}{\partial \xi}|_{2}^2 + 2 |T|_{\xi=1}^2. \tag{5.12}\]

Similarly to (5.12), we get
\[
|q|_{2}^2 \leq 2 |\frac{\partial q}{\partial \xi}|_{2}^2 + 2 |q|_{\xi=1}^2. \tag{5.13}\]

By Young inequality, we have
\[
\begin{align*}
|\int_{\Omega} Q_1 T| &\leq \varepsilon |T|_{2}^2 + c |Q_1|_{2}^2, \quad |\int_{\Omega} q Q_2| \leq \varepsilon |q|_{2}^2 + c |Q_2|_{2}^2. \tag{5.14}\end{align*}
\]

In this article, \(c\) will denote positive constant and can be determined in concrete conditions. \(\varepsilon\) is a small enough positive constant. Therefore, we obtain
\[
\begin{align*}
\frac{d(|v|^2 + |T|^2 + |q|^2)}{dt} &+ \frac{1}{Re_1} \int_{\Omega} (|\nabla e_\theta v|^2 + |\nabla e_\phi v|^2 + |v|^2) + \frac{1}{Re_2} \int_{\Omega} |\frac{\partial v}{\partial \xi}|^2 \\
&+ \frac{1}{R_t_1} \int_{\Omega} |\nabla T|^2 + \frac{1}{R_t_2} \int_{\Omega} |\frac{\partial T}{\partial \xi}|^2 + \frac{\alpha_s}{R_t_2} |T|_{\xi=1}^2 + \frac{1}{R_q_1} \int_{\Omega} |\nabla q|^2 \\
&+ \frac{1}{R_q_2} \int_{\Omega} |\frac{\partial q}{\partial \xi}|^2 + \frac{\beta_s}{R_q_2} |q|_{\xi=1}^2 \\
&\leq c \int_{\Omega} (Q_1^2 + Q_2^2). \tag{5.15}\end{align*}
\]
By (5.12), (5.13), (5.15) and thanks to the Gronwall inequality, we have
\[ |v|^2 + |T|^2 + |q|^2 \leq e^{-c_0t}(|v_0|^2 + |T_0|^2 + |q_0|^2) + c(|Q_1|^2 + |Q_2|^2) \leq E_0, \quad (5.16) \]
where \( c_0 = \min\{\frac{1}{Rq_1}, \frac{1}{Rq_2}, \frac{\alpha}{Rq_1}, \frac{\beta}{Rq_2}\} > 0, \ t \geq 0 \) and \( E_0 \) is a positive constant. By Minkowski inequality and Hölder inequality, for any \( t \geq 0 \) we have
\[ \|\bar{v}(t)\|_{L^2(S^2)}^2 \leq \|v(t)\|_2^2 \]
\[ \leq e^{-c_0t}(|v_0|^2 + |T_0|^2 + |q_0|^2) + c(|Q_1|^2 + |Q_2|^2) \leq E_0. \quad (5.17) \]

From (5.12), (5.13), (5.15) and (5.16), we get
\[ c_1 \int_t^{t+r} \left[ \int_\Omega (|\nabla v|^2 + |\nabla v|^2 + |\frac{\partial v}{\partial \xi}|^2) + \int_\Omega (|\nabla T|^2 + |\frac{\partial T}{\partial \xi}|^2 + |T|^2) \right. \]
\[ + \left. \int_\Omega (|\nabla q|^2 + |\frac{\partial q}{\partial \xi}|^2 + |q|^2) + |T|_{\xi=1}|^2 + |q|_{\xi=1}|^2 + |U(t)|^2 \right] \]
\[ \leq 2e^{-c_0t}(|v_0|^2 + |T_0|^2 + |q_0|^2) + c(|Q_1|^2 + |Q_2|^2)(2 + r) \leq E_1, \quad (5.18) \]
where \( c_1 = \min\{\frac{1}{Rq_1}, \frac{1}{Rq_2}, \frac{\alpha}{Rq_1}, \frac{\beta}{Rq_2}\} \}, \ t \geq 0, \ 1 \geq r > 0 \)
given, \( E_1 \) is a positive constant and \( \int_t^{t+r} \cdot ds \) is denoted by \( \int_t^{t+r} \cdot \). Since
\[ \int_{S^2} (|\nabla v|^2 + |\nabla v|^2) \leq \int_\Omega (|\nabla v|^2 + |\nabla v|^2), \]
from (5.18), we have
\[ c_1 \int_t^{t+r} \int_{S^2} (|\nabla v|^2 + |\nabla v|^2) + \|\bar{v}\|^2_{L^2(S^2)} \leq E_1, \ \forall t \geq 0. \quad (5.19) \]

**L^4 estimates about q** By taking the inner product of equation (2.13) with \(|q|^2 q\) in \( L^2(\Omega) \), we get
\[ \frac{1}{4} \frac{d|q|^4}{dt} + \frac{3}{Rq_1} \int_\Omega |\nabla q|^2 q^2 + \frac{3}{Rq_2} \int_\Omega |\frac{\partial q}{\partial \xi}|^2 q^2 + \frac{\beta_s}{Rq_2} \int_\Omega |q|_{\xi=1}|^4 \]
\[ = \int_\Omega Q_2|q|^2 q - \int_\Omega (\nabla v q + (\int_\xi^{1} \text{div} v d\xi') \frac{\partial q}{\partial \xi})|q|^2 q. \quad (5.20) \]

By Lemma 4.3, we have
\[ \int_\Omega (\nabla v q + (\int_\xi^{1} \text{div} v d\xi') \frac{\partial q}{\partial \xi})|q|^2 q \]
\[
\frac{1}{4} \int \nabla q^4 + \int_{S^2} \left( \int_0^1 (\int_\xi \text{div}d\xi')d\left(\frac{1}{4} q^4\right)\right)
= \frac{1}{4} \int_\Omega (\nabla v q^4 + q^4 \text{div} v)
= 0.
\]  

(5.21)

Combining (5.20) with (5.21), we obtain

\[
\frac{1}{4} \frac{d|q|^4}{dt} + \frac{3}{Rq_1} \int_\Omega |\nabla q|^2 q^2 + \frac{3}{Rq_2} \int_\Omega \frac{\partial q}{\partial \xi} q^2 + \frac{\beta_s}{Rq_2} \int_{S^2} |q|_{\xi=1|^4} = \int_\Omega Q_2 |q|^2 q.
\]

(5.22)

Since \(q^4(\theta, \varphi, \xi) = -\int_\xi^1 \frac{\partial q^4}{\partial \xi} d\xi' + q^4|_{\xi=1}\), we get by using Hölder inequality and Cauchy-Schwarz inequality,

\[
|q|^4_4 \leq 4 \int_{S^2} \left( \int_0^1 (\int_\xi |q|^3 \frac{\partial q}{\partial \xi} d\xi') d\xi \right) + |q|_{\xi=1|^4}
\leq c \int_{S^2} \left( \int_0^1 |q|^2 \frac{\partial q}{\partial \xi}^2 d\xi \right)^{\frac{1}{2}} \left( \int_0^1 q^4 d\xi \right)^{\frac{1}{2}} + |q|_{\xi=1|^4}
\leq c \left( \int_\Omega |q|^2 \frac{\partial q}{\partial \xi}^2 \right) + \frac{1}{2} \int q^4 + |q|_{\xi=1|^4}.
\]

(5.23)

Since

\[
|\int \Omega Q_2 |q|^2 q| \leq c|Q_2|^4_4 + \varepsilon |q|^4_4,
\]

from (5.22)-(5.23), by choosing \(\varepsilon\) small enough, we obtain

\[
\frac{d|q|^4}{dt} + c_2 |q|^4_4 \leq c|Q_2|^4_4,
\]

(5.24)

where \(c_2\) is a positive constant. By Gronwall inequality, we have

\[
|q(t)|^4_4 \leq e^{-c_2 t} |q_0|^4_4 + c |Q_2|^4_4 \leq E_2,
\]

(5.25)

where \(t \geq 0\), \(E_2\) is a positive constant. From (5.24) and (5.25), we get

\[
c_1 \int_t^{t+r} |q|_{\xi=1|^4} \leq 2E_2, \quad \text{for any } t \geq 0.
\]

(5.26)

**L^3 estimates about T** We take the inner product of equation (2.12) with \(|T| T|\) in \(L^2(\Omega)\) and obtain

\[
\frac{1}{3} \frac{d|T|^3}{dt} + \frac{2}{Rt_1} \int_\Omega |\nabla T|^2 |T| + \frac{2}{Rt_2} \int_\Omega \frac{\partial T}{\partial \xi}^2 |T| + \frac{\alpha_s}{Rt_2} \int_{S^2} |T|_{\xi=1|^3}
\]

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\[
\begin{align*}
= \int_{\Omega} Q_1 |T| \mathcal{T} - \int_{\Omega} (\nabla v \mathcal{T} + \left( \int_{\xi} \mathrm{div} \xi \frac{\partial T}{\partial \xi} \right) |T| \mathcal{T} \\
+ \int_{\Omega} \frac{bP}{p} \left( \int_{\xi} \mathrm{div} \xi \right) |T| \mathcal{T} + \int_{\Omega} \frac{abP}{p} \left( \int_{\xi} \mathrm{div} \xi \right) q |T| \mathcal{T}.
\end{align*}
\]

Using Hölder inequality and Young inequality, we have
\[
| \int_{\Omega} Q_1 |T| \mathcal{T} | \leq c |Q_1|_3^3 + \varepsilon |T|_3^3.
\]

By Hölder inequality, Minkowski inequality and (4.14), we get
\[
\begin{align*}
& | \int_{\Omega} \frac{bP}{p} \left( \int_{\xi} \mathrm{div} \xi \right) |T| \mathcal{T} | \\
\leq & \ c \int_0^1 \left[ \int_{S^2} \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \left( \int_{S^2} |T|^4 \right)^{\frac{1}{2}} \right] d\xi \\
\leq & \ c \left\| \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \right\|_{L^{\infty}_\xi} \int_0^1 \left\| T \right\|_{L^2(S^2)} \left\| T \right\|_{H^1(S^2)} d\xi \\
\leq & \ c \left( |\nabla v|^2 + |\nabla v|^2 \right)^{\frac{1}{2}} |T|_2 |T| \\
\leq & \ c \int_0^1 \left( |\nabla v|^2 + |\nabla v|^2 \right) + c |T|_3^2 |T|_2^2.
\end{align*}
\]

By Hölder inequality, Young inequality and
\[
\| u \|_{L^\frac{10}{3}(S^2)} \leq c \| u \|_{L^2(S^2)}^\frac{5}{3} \| u \|_{H^1(S^2)}^\frac{2}{3}, \text{ for any } u \in H^1(S^2),
\]
we have
\[
\begin{align*}
& | \int_{\Omega} \frac{abP}{p} q \left( \int_{\xi} \mathrm{div} \xi \right) |T| \mathcal{T} | \\
\leq & \ c \int_0^1 \left[ \int_{S^2} q \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \left( \int_{S^2} |T|^4 \right)^{\frac{1}{2}} \right] d\xi \\
\leq & \ c \int_0^1 \left[ \left( \int_{S^2} q^4 \right)^{\frac{1}{2}} \left( \left( \left( |T|^3 \right)^{\frac{2}{3}} \right)^{\frac{3}{2}} \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \right] d\xi \\
\leq & \ c \int_0^1 \left[ \left( \int_{S^2} q^4 \right)^{\frac{1}{2}} \left( \left( \left( |T|^3 \right)^{\frac{2}{3}} \right)^{\frac{3}{2}} \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \right] d\xi \\
\leq & \ c \int_0^1 \left[ \left( \int_{S^2} q^4 \right)^{\frac{1}{2}} \left( \left( \left( \left( |T|^3 \right)^{\frac{2}{3}} \right)^{\frac{3}{2}} \right)^{\frac{3}{2}} \left( \int_{S^2} \left( \int_{\xi} \mathrm{div} \xi \right)^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \right] d\xi.
\end{align*}
\]
Choosing $\varepsilon$ small enough and using an inequality similar to (5.23), we derive from (5.27)-(5.30)

$$
\frac{dT_3^3}{dt} + \frac{2}{R_1} \int |\nabla T|^2 |T| + \frac{2}{R_2} \int \frac{\partial T}{\partial \xi}|^2 |T| + \frac{\alpha_s}{R_2} \int_{S^2} |T|_{\xi=1}^3
\leq c |q_4^3|^3 \left[\int (|\nabla v|^2 + |\nabla v_{\xi}|^2)|T_3^3| + c |Q_1^3|^3 \right] + c (1 + |q_4^3|^3) \left[\int (|\nabla v|^2 + |\nabla v_{\xi}|^2) + c |T_2^2| |T|^2. \right. \quad (5.31)
$$

By Lemma 4.6, (5.18), (5.25) and $|T_3^3| \leq c |T_2^2| |T|^2$, we obtain

$$
|T(t+r)|^3 \leq c (|Q_1^3|^3 + (1 + E_2^3)(1 + E_1) + 2 E_1 + \frac{(E_0 E_1)^3}{r}) \exp(c E_2^3 (1 + E_1)) = E_3, \quad (5.32)
$$

where $E_3$ is a positive constant, $t \geq 0$. By Gronwall inequality, we prove

$$
|T(t)|^3 \leq c (|Q_1^3|^3 + (1 + E_2^3)(1 + E_1) + |T_0^3|^3) \exp(c E_2^3 (1 + E_1)), \text{ for any } 0 \leq t < r. \quad (5.33)
$$

$L^3$ estimates about $T$  We take the inner product of equation (2.12) with $|T|^2 T$ in $L^2(\Omega)$ and obtain

$$
\frac{1}{4} \frac{d |T|^4}{dt} + \frac{3}{R_1} \int |\nabla T|^2 |T|^2 + \frac{3}{R_2} \int \frac{\partial T}{\partial \xi}|^2 |T|^2 + \frac{\alpha_s}{R_2} \int_{S^2} |T|_{\xi=1}^4
= \int Q_1 |T|^2 T - \int (\nabla v T + (\int |T|_{\xi=1}^4) \frac{\partial T}{\partial \xi}) |T|^2 T
+ \int \frac{bP}{p} (\int |\nabla v T + (\int |T|_{\xi=1}^4) \frac{\partial T}{\partial \xi}) q |T|^2 T. \quad (5.34)
$$
Using Hölder inequality and Young inequality, we have
\[
| \int_{\Omega} Q_1 |T|^2 T | \leq c |Q_1|_4^4 + \varepsilon |T|_4^4. \tag{5.35}
\]
By Hölder inequality, Minkowski inequality and (4.14), we get
\[
| \int_{\Omega} \frac{bP}{p} (\int_{\xi}^1 \text{div} v d\xi') |T|^2 T | \\
\leq c \int_0^1 \left[ (\int_{S^2} (\int_{\xi}^1 \text{div} v d\xi')^2 \right]^{\frac{1}{2}} \left( \int_{S^2} |T|^6 \right)^{\frac{1}{2}} d\xi' \\
\leq c \left\| (\int_{S^2} (\int_{\xi}^1 \text{div} v d\xi')^2 \right\|_{L^\infty(S^2)} \left\| |T|^6 \right\|_{H^1(S^2)} d\xi \\
\leq c \left( \int_{\Omega} (|\nabla e_x v|^2 + |\nabla e_x v|^2) \right)^{\frac{1}{2}} |T|_4^{\frac{3}{4}} |T|_4^2. \tag{5.36}
\]
By Hölder inequality, Young inequality and
\[
\| u \|_{L^6(S^2)} \leq c \| u \|_{L^2(S^2)}^\frac{3}{2} \| u \|_{H^1(S^2)}^\frac{9}{2}, \text{ for any } u \in H^1(S^2),
\]
we have
\[
| \int_{\Omega} \frac{abP}{p} q (\int_{\xi}^1 \text{div} v d\xi') |T|^2 T | \\
\leq c \int_0^1 \left[ (\int_{S^2} q^4 \right]^{\frac{1}{2}} \left( \int_{S^2} (|T|^2)^6 \right)^{\frac{1}{2}} \left( \int_{S^2} (\int_{\xi}^1 \text{div} v d\xi')^2 \right)^{\frac{1}{2}} d\xi' \\
\leq c \int_0^1 \left[ (\int_{S^2} q^4 \right]^{\frac{1}{2}} \left( \int_{S^2} (|T|_4^2)^2 \right)^{\frac{1}{2}} \left( \int_{S^2} (\int_{\xi}^1 \text{div} v d\xi')^2 \right)^{\frac{1}{2}} d\xi' \\
\leq c \int_0^1 \left[ (\int_{S^2} q^4 \right]^{\frac{1}{2}} \left( \int_{S^2} |T|_4^2 \right)^{\frac{1}{2}} \left( \int_{S^2} (\int_{\xi}^1 \text{div} v d\xi')^2 \right)^{\frac{1}{2}} d\xi' \\
\leq c |q|_4 |T|_4 \left( \int_{\Omega} |T|^2 |\nabla T|^2 + |T|_4^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} (\int_{\xi}^1 \text{div} v d\xi')^2 \right)^{\frac{1}{2}} d\xi' \\
\leq c |q|_4 |T|_4 \left( \int_{\Omega} (|\nabla e_x v|^2 + |\nabla e_x v|^2) \right)^{\frac{1}{2}} \left( \int_{\Omega} (|\nabla e_x v|^2 + |\nabla e_x v|^2) \right)^{\frac{1}{2}} d\xi' \\
\leq c |q|_4 |T|_4 \int_{\Omega} (|\nabla e_x v|^2 + |\nabla e_x v|^2) + \varepsilon \left( \int_{\Omega} |T|^2 |\nabla T|^2 + |T|_4^2 \right) \\
\leq c (|q|_4^3 + |T|_4^3) \int_{\Omega} (|\nabla e_x v|^2 + |\nabla e_x v|^2) + \varepsilon \left( \int_{\Omega} |T|^2 |\nabla T|^2 + |T|_4^2 \right). \tag{5.37}
\]
Choosing $\varepsilon$ small enough and using an inequality similar to (5.23), we derive from (5.34)-(5.37),
\[
\frac{dT_t^4}{dt} + \frac{3}{Rt_1} \int_\Omega |\nabla T|^2 T^2 + \frac{3}{Rt_2} \int_\Omega \frac{\partial T}{\partial \xi} |2T^2 + \frac{\alpha_s}{Rt_2} \int_{S_2} |T|^4_{\xi=1}\]
\[
\leq c\|T\|^2 + \int_\Omega (|\nabla_{ep} v|^2 + |\nabla_{ep} v|^2) |T|^4_4 + c|Q_4|^4_4
\]
\[
+ c(1 + |q|^4_4) \int_\Omega (|\nabla_{ep} v|^2 + |\nabla_{ep} v|^2).
\]
(5.38)

By Lemma 4.6, (5.18), (5.25), (5.32) and $|T|^4_4 \leq c|T|^2_3 \|T\|^2$, we obtain
\[
|T|t+2r)^4_4 \leq c(|Q_4|^4_4 + E_1E_2 + E_1 + \frac{|E_1E_3|^2}{r}) \exp(cE_1) = E_4,
\]
where $E_4$ is a positive constant and $t \geq 0$. By Gronwall inequality, from (5.38) we prove
\[
|T(t)|t+3r)^4_4 \leq c(|Q_4|^4_4 + E_1E_2 + E_1 + |T|t_4)^4_4 \exp(cE_1) = C_1,
\]
where $C_1 = C_1 ||U_0||, \|Q_4\|_1, \|Q_2\|_1 > 0$ and $0 \leq t < 2r$. From (5.38) and (5.39), we get
\[
c_1 \int_{t+2r}^{t+3r} |T|^4_{\xi=1} \leq E_4^2 + E_4, \text{ for any } t \geq 0.
\]

(5.40)

$L^3$ estimates about $\tilde{v}$  

We take the inner product of equation (2.23) with $|\tilde{v}|\tilde{v}$ in $L^2(\Omega) \times L^2(\Omega)$, and obtain
\[
\frac{1}{3} \frac{d|\tilde{v}|^3}{dt} + \frac{1}{Re_1} \int_\Omega (|\nabla_{ep} \tilde{v}|^2 + |\nabla_{ep} \tilde{v}|^2)|\tilde{v}| + \frac{4}{9} |\nabla_{ep} \tilde{v}|^2 + \frac{4}{9} |\nabla_{ep} \tilde{v}|^2 + |\tilde{v}|^3]
\]
\[
+ \frac{1}{Re_2} \int_\Omega \left( |\tilde{v}|^2 |\tilde{v}| + \frac{4}{9} |\partial_\xi |\tilde{v}|^2 \right)
\]
\[
= - \int_\Omega (\nabla \tilde{v} + (\int_\xi^{1} \text{div}\tilde{v}d\xi') \frac{\partial \tilde{v}}{\partial \xi}) \cdot \tilde{v} \tilde{v} - \int_\Omega (\nabla \tilde{v}) \cdot |\tilde{v}|\tilde{v} - \int_\Omega |\tilde{v}|\tilde{v} \cdot \nabla \tilde{v}
\]
\[
- \int_\Omega \left( \frac{bP}{p} \text{grad}((1 + aq)T)d\xi' - \int_\Omega (\frac{bP}{p} \text{grad}((1 + aq)T)d\xi')d\xi \right) \cdot |\tilde{v}|\tilde{v}
\]
\[
+ \int_\Omega \left( \text{div}\tilde{v} + \nabla \tilde{v} \right) \cdot |\tilde{v}|\tilde{v} - \int_\Omega \left( \frac{\xi}{R_0} k \times \tilde{v} \right) \cdot |\tilde{v}|\tilde{v},
\]
(5.41)

where $\tilde{v} \xi = \partial_\xi \tilde{v}$. By Lemma 4.3 and integration by parts, we have
\[
\int_\Omega (\nabla \tilde{v} + (\int_\xi^{1} \text{div}\tilde{v}d\xi') \frac{\partial \tilde{v}}{\partial \xi}) \cdot |\tilde{v}|\tilde{v} = \frac{1}{3} \int_\Omega |\nabla \tilde{v}|^2 + \int_{S^2} (\int_\xi^{1} \text{div}\tilde{v}d\xi')d|\tilde{v}|^3]
\]

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\[
\begin{align*}
\int_{\Omega} (\nabla_{\tilde{v}}\tilde{v} \cdot |\tilde{v}|^3 + |\tilde{v}|^3 \text{div}\tilde{v}) &= \frac{1}{3} \int_{\Omega} (\nabla_{\tilde{v}}|\tilde{v}|^3 + |\tilde{v}|^3 \text{div}\tilde{v}) \\
&= \frac{1}{3} \int_{\Omega} \text{div}(|\tilde{v}|^3 \tilde{v}) \\
&= 0. 
\end{align*}
\] (5.42)

By Lemma 4.3, we get
\[
\int_{\Omega} \text{div}(|\tilde{v}|^3 \tilde{v}) = \int_{\Omega} \nabla_{\tilde{v}}|\tilde{v}|^3 + |\tilde{v}|^3 \text{div}\tilde{v} = 0.
\]

By (2.19), we derive from above
\[
\int_{\Omega} (\nabla_{\tilde{v}}\tilde{v}) \cdot |\tilde{v}|\tilde{v} = \frac{1}{3} \int_{\Omega} \nabla_{\tilde{v}}|\tilde{v}|^3 = 0. 
\] (5.43)

Using Lemma 4.3, we have
\[
\int_{\Omega} \text{div}((|\tilde{v}|\tilde{v} \cdot \tilde{v})\tilde{v}) = \int_{\Omega} \nabla_{\tilde{v}}(|\tilde{v}|\tilde{v} \cdot \tilde{v}) + \int_{\Omega} |\tilde{v}|\tilde{v} \cdot \tilde{v} \text{div}\tilde{v}
\]
\[
= 0. 
\]

So
\[
\int_{\Omega} |\tilde{v}|\tilde{v} \cdot \nabla_{\tilde{v}}\tilde{v} = \int_{\Omega} \tilde{v} \cdot \nabla_{\tilde{v}}(|\tilde{v}|\tilde{v}) + \int_{\Omega} |\tilde{v}|\tilde{v} \cdot \tilde{v} \text{div}\tilde{v}. 
\] (5.44)

Using integration by parts, we obtain
\[
\int_{\Omega} (\int_{0}^{1} (\tilde{v} \text{div}\tilde{v} + \nabla_{\tilde{v}}\tilde{v})d\xi) \cdot |\tilde{v}|\tilde{v} = \int_{\Omega} (\int_{0}^{1} \tilde{v}_{e\varphi} \tilde{v} d\xi) \cdot \nabla_{e\varphi}(|\tilde{v}|\tilde{v}) \\
+ \int_{\Omega} (\int_{0}^{1} \tilde{v}_{e\varphi} \tilde{v} d\xi) \cdot \nabla_{e\varphi}(|\tilde{v}|\tilde{v}).
\] (5.45)

\((\frac{f}{R_0} k \times \tilde{v}) \cdot |\tilde{v}|\tilde{v} = 0\) implies
\[
\int_{\Omega} (\frac{f}{R_0} k \times \tilde{v}) \cdot |\tilde{v}|\tilde{v} = 0. 
\] (5.46)

By Lemma 4.1, we get
\[
- \int_{\Omega} \int_{\xi} \frac{bP}{p} \text{grad}((1 + aq)T)d\xi' - \int_{\Omega} \int_{\xi} \frac{bP}{p} \text{grad}((1 + aq)T)d\xi \cdot |\tilde{v}|\tilde{v}
\]
\[
= \int_{\Omega} \int_{\xi} \frac{bP}{p} (1 + aq)Td\xi' - \int_{\Omega} \int_{\xi} \frac{bP}{p} (1 + aq)Td\xi \cdot \text{div}(|\tilde{v}|\tilde{v}).
\] (5.47)
From (5.41) to (5.47), we get
\[
\frac{1}{3} \frac{d|\tilde{v}(t)|}{dt} + \frac{1}{Re_1} \int_\Omega [(|\nabla_{\text{e}_1} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)|\tilde{v}| + \frac{4}{9} |\nabla_{\text{e}_1} \tilde{v}|^2 + \frac{4}{9} |\nabla_{\text{e}_2} \tilde{v}|^2 + |\tilde{v}|^3]
\]
\[
+ \frac{1}{Re_2} \int_\Omega \left( (|\tilde{v}|^2|\tilde{v}| + \frac{4}{9} |\tilde{v}|^2) \right) = \int_\Omega (\tilde{v} \cdot \nabla (|\tilde{v}| \tilde{v} + |\tilde{v}| \tilde{v} \cdot \tilde{v} \nabla \tilde{v}))
\]
\[
+ \int \left( \int_0^1 \tilde{v}_\theta \tilde{v} d\xi \right) \cdot \nabla_{\text{e}_0} (|\tilde{v}| \tilde{v}) + (\int_0^1 \tilde{v}_\theta \tilde{v} d\xi) \cdot \nabla_{\text{e}_0} (|\tilde{v}| \tilde{v})
\]
\[
+ \int \left[ \int_0^1 \frac{bP}{p} (1 + aq) T d\xi - \int_0^1 \int_\Omega \frac{bP}{p} (1 + aq) T d\xi d\xi \right] \text{div} (|\tilde{v}| \tilde{v}). \tag{5.48}
\]
By Hölder inequality, we derive from (5.48)
\[
\frac{1}{3} \frac{d|\tilde{v}(t)|}{dt} + \frac{1}{Re_1} \int_\Omega [(|\nabla_{\text{e}_1} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)|\tilde{v}| + \frac{4}{9} |\nabla_{\text{e}_1} \tilde{v}|^2 + \frac{4}{9} |\nabla_{\text{e}_2} \tilde{v}|^2 + |\tilde{v}|^3]
\]
\[
+ \frac{1}{Re_2} \int_\Omega \left( (|\tilde{v}|^2|\tilde{v}| + \frac{4}{9} |\tilde{v}|^2) \right) \leq c \int_0^1 |\tilde{v}| \int_0^1 \tilde{v}(1)|\nabla_{\text{e}_0} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)^{\frac{1}{2}} d\xi
\]
\[
+ \int_0^1 |\tilde{v}|^2 d\xi \left( \int_0^1 |\tilde{v}| (|\nabla_{\text{e}_0} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right)
\]
\[
+ \int_0^1 |\tilde{v}| \left( \int_0^1 |\tilde{v}| (|\nabla_{\text{e}_0} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right)
\]
\[
\leq c \int_0^1 |\tilde{v}| \left( \int_0^1 |\tilde{v}| (|\nabla_{\text{e}_0} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right)
\]
\[
+ c \int_0^1 |\tilde{v}|^2 d\xi \left( \int_0^1 |\tilde{v}| (|\nabla_{\text{e}_0} \tilde{v}|^2 + |\nabla_{\text{e}_2} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right)
\]
\[
+ c \int_0^1 \frac{bP}{p} (1 + aq) T d\xi - \int_0^1 \int_\Omega \frac{bP}{p} (1 + aq) T d\xi d\xi \right] \text{div} (|\tilde{v}| \tilde{v}).
\]

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By Minkowski inequality, Hölder inequality and (4.13), we have
\[
\left( \int_{S^{2}} (\int_{0}^{1} |\tilde{v}|^{3} d\xi)^{\frac{2}{3}} \right)^{\frac{3}{2}} \leq \int_{S^{2}} (\int_{0}^{1} (|\tilde{v}|^{3})^{\frac{2}{3}} d\xi)^{\frac{3}{2}} \\
\leq \int_{S^{2}} (\int_{0}^{1} (\|\nabla |\tilde{v}|^{\frac{2}{3}}\|^{2}_{L^{2}(S^{2})} + \|\nabla |\tilde{v}|^{\frac{2}{3}}\|^{2}_{L^{2}(S^{2})})^{\frac{3}{2}} d\xi \\
\leq c \|\tilde{v}\|_{H^{1}(S^{2})}^{\frac{3}{2}} \left( \int_{S^{2}} (\int_{0}^{1} (\|\nabla |\tilde{v}|^{\frac{2}{3}}\|^{2}_{L^{2}(S^{2})} + \|\nabla |\tilde{v}|^{\frac{2}{3}}\|^{2}_{L^{2}(S^{2})})^{\frac{3}{2}} d\xi \right)^{\frac{5}{2}}.
\]  
(5.50)

By Minkowski inequality, Hölder inequality and
\[
\|u\|_{L^{4}(S^{2})} \leq c \|u\|_{L^{3}(S^{2})}^{\frac{3}{2}} \|u\|_{H^{1}(S^{2})}^{\frac{3}{2}}, \text{ for any } u \in H^{1}(S^{2}),
\]
we get
\[
\int_{S^{2}} (\int_{0}^{1} |\tilde{v}|^{2} d\xi)^{\frac{2}{3}} \leq (\int_{S^{2}} (\int_{0}^{1} |\tilde{v}|^{5} d\xi)^{\frac{3}{5}} d\xi)^{\frac{5}{3}} \\
\leq (\int_{0}^{1} \|\tilde{v}\|_{L^{3}(S^{2})}^{\frac{6}{5}} \|\tilde{v}\|_{H^{1}(S^{2})}^{\frac{3}{5}} d\xi)^{\frac{5}{3}} \\
\leq c \|\tilde{v}\|^{2}_{L^{3}(S^{2})} \|\tilde{v}\|_{H^{1}(S^{2})}^{\frac{3}{2}}.
\]  
(5.51)

By (4.13), we have
\[
\|\tilde{v}\|_{L^{2}(S^{2})} \leq \|\tilde{v}\|_{L^{2}(S^{2})}^{\frac{2}{3}} \|\tilde{v}\|_{H^{1}(S^{2})}^{\frac{1}{3}},
\]  
(5.52)

By Minkowski inequality, (4.13), (4.15) and Hölder inequality, we get
\[
\|\|T\|\|_{L^{4}(S^{2})} = (\int_{S^{2}} (\int_{0}^{1} |T| d\xi)^{4})^{\frac{1}{4}} \leq |T|_{4},
\]  
(5.53)

\[
\|\|qT\|\|_{L^{4}(S^{2})} = (\int_{S^{2}} (\int_{0}^{1} |qT| d\xi)^{4})^{\frac{1}{4}} \\
\leq (\int_{S^{2}} (\int_{0}^{1} |q|^{2} d\xi)^{2} (\int_{0}^{1} |T|^{2} d\xi)^{2})^{\frac{1}{4}} \\
\leq (\int_{S^{2}} (\int_{0}^{1} |q|^{2} d\xi)^{2})^{\frac{1}{2}} (\int_{S^{2}} (\int_{0}^{1} |T|^{2} d\xi)^{2})^{\frac{1}{2}} \\
\leq \left( \int_{S^{2}} (\int_{0}^{1} |q|^{5} d\xi)^{\frac{3}{4}} (\int_{S^{2}} (\int_{0}^{1} |T|^{8} d\xi)^{\frac{1}{4}} d\xi \right)^{\frac{5}{4}} \\
\leq c (\int_{S^{2}} (\int_{0}^{1} \|q\|_{L^{4}(S^{2})} d\xi)^{\frac{3}{4}} (\int_{S^{2}} (\int_{0}^{1} |T|^{8} d\xi)^{\frac{1}{4}} d\xi)^{\frac{5}{4}} \\
\leq c |q|_{L^{2}}^{\frac{3}{4}} \|T\|_{H^{1}(S^{2})}^{\frac{1}{4}} \|T\|_{L^{4}(S^{2})}^{\frac{1}{2}}.
\]  
(5.54)
By Young inequality, we obtain from (5.49)-(5.54)
\[
\frac{d|\tilde{v}|^3}{dt} + \frac{1}{R_1} \int_{\Omega} \left[ \left| \nabla_{e\sigma} \tilde{v} \right|^2 + \left| \nabla_{e\nu} \tilde{v} \right|^2 \right] |\tilde{v}| + \frac{4}{9} \left| \nabla_{e\sigma} |\tilde{v}|^\frac{3}{2} \right|^2 + \frac{4}{9} \left| \nabla_{e\nu} |\tilde{v}|^\frac{3}{2} \right|^2 + |\tilde{v}|^3 \]
+ \frac{1}{R_2} \int_{\Omega} \left( |\tilde{v}|^2 + \frac{4}{9} |\partial_\xi |\tilde{v}|^\frac{3}{2} \right)^2
\leq c(\|\tilde{v}\|_{L^2(S^2)}^2 \|\tilde{v}\|_{H^1(S^2)}^2 + \|\tilde{v}\|^2) |\tilde{v}|^3 + c(\|T\|_4^2 + \|\tilde{v}\|^2 + \|\tilde{v}\|^3 + c|T|_4^4 + c|T|_2^2) \|T\|^2
+ c(1 + \|q\|_{L^2(S^2)}^2 \|\tilde{v}\|_{H^1(S^2)}^2 + \|\tilde{v}\|^2) |\tilde{v}|^3 + c|T|_4^4 + c|T|_2^2 \|T\|^2
\tag{5.55}
\]
By Lemma 4.6, (5.17), (5.19), (5.25), (5.39) and $|\tilde{v}|^3 \leq |\tilde{v}|^\frac{3}{2} |\tilde{v}|^\frac{3}{2}$, we obtain
\[
|\tilde{v}(t+3r)|^3 \leq c(E_4 + E_4^2 E_1 + 2E_0 (1 + E_2^2 E_1) + \left( \frac{4E_0 E_1}{r} \right)^2) \exp c(E_0 E_1 + 2E_1) \leq E_5,
\tag{5.56}
\]
where $E_5$ is a positive constant and $t \geq 0$.

$L^4$ estimates about $\tilde{v}$ We take the inner product of equation (2.23) with $|\tilde{v}|^2 \tilde{v}$ in $L^2(\Omega) \times L^2(\Omega)$, and obtain
\[
\frac{1}{4} \frac{d|\tilde{v}|^4}{dt} + \frac{1}{R_1} \int_{\Omega} \left[ \left| \nabla_{e\sigma} \tilde{v} \right|^2 + \left| \nabla_{e\nu} \tilde{v} \right|^2 \right] |\tilde{v}|^2 + \frac{1}{2} \left| \nabla_{e\sigma} |\tilde{v}|^2 \right|^2 + \frac{1}{2} \left| \nabla_{e\nu} |\tilde{v}|^2 \right|^2 + |\tilde{v}|^4
+ \frac{1}{R_2} \int_{\Omega} \left( |\tilde{v}|^2 |\tilde{v}|^2 + \frac{1}{2} |\partial_\xi |\tilde{v}|^2 \right)^2
= - \int_{\Omega} (\nabla \tilde{v} + (\int_1^1 \text{div} \tilde{v}) \text{d} \xi') \cdot |\tilde{v}|^2 \tilde{v} - \int_{\Omega} (\nabla \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} - \int_{\Omega} |\tilde{v}|^3 |\tilde{v}|^2 \cdot \nabla \tilde{v}
- \int_{\Omega} \left( \int_0^{1} \frac{bp}{p} \text{grad}((1 + aq)T) d \xi' - \int_0^{1} \left( \int_0^{1} \frac{bp}{p} \text{grad}((1 + aq)T) d \xi' \right) d \xi' \right) \cdot |\tilde{v}|^2 \tilde{v}
+ \int_{\Omega} (\tilde{v} \text{div} \tilde{v} + \nabla \tilde{v}) \cdot |\tilde{v}|^2 \tilde{v} - \int_{\Omega} \left( \frac{1}{R_0} |k \times \tilde{v}| \right) \cdot |\tilde{v}|^2 \tilde{v}.
\tag{5.57}
\]
Similarly to (5.48), we derive from (5.57)
\[
\frac{1}{4} \frac{d|\tilde{v}|^4}{dt} + \frac{1}{R_1} \int_{\Omega} \left[ \left| \nabla_{e\sigma} \tilde{v} \right|^2 + \left| \nabla_{e\nu} \tilde{v} \right|^2 \right] |\tilde{v}|^2 + \frac{1}{2} \left| \nabla_{e\sigma} |\tilde{v}|^2 \right|^2 + \frac{1}{2} \left| \nabla_{e\nu} |\tilde{v}|^2 \right|^2 + |\tilde{v}|^4
+ \frac{1}{R_2} \int_{\Omega} \left( |\tilde{v}|^2 |\tilde{v}|^2 + \frac{1}{2} |\partial_\xi |\tilde{v}|^2 \right)^2 = \int_{\Omega} (\tilde{v} \cdot \nabla \tilde{v}) |\tilde{v}|^2 \tilde{v} + |\tilde{v}|^2 \tilde{v} \cdot \text{div} \tilde{v}
+ \int_{\Omega} \left( \int_0^{1} \tilde{v} \nabla \text{d} \xi' \right) \cdot \nabla_{e\sigma} |\tilde{v}|^2 \tilde{v} + \left( \int_0^{1} \tilde{v} \nabla \text{d} \xi' \right) \cdot \nabla_{e\nu} |\tilde{v}|^2 \tilde{v}
+ \int_{\Omega} \left( \int_0^{1} \frac{bp}{p} (1 + aq)T d \xi' - \int_0^{1} \frac{bp}{p} (1 + aq)T d \xi' d \xi' \right) \text{div} |\tilde{v}|^2 \tilde{v}.
\tag{5.58}
\]
By Hölder inequality, we derive from (5.58)

\[
\frac{1}{4} \frac{d|\tilde{v}|^4}{dt} + \frac{1}{Re_1} \int_\Omega \left[ \left( |\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2 \right) |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_\phi} |\tilde{v}|^2|^2 + \frac{1}{2} |\nabla_{e_\varphi} |\tilde{v}|^2|^2 + |\tilde{v}|^4 \right]
\]

\[+ \frac{1}{Re_2} \int_\Omega (|\tilde{v}|^2 |\tilde{v}|^2 + |\partial_\xi |\tilde{v}|^2|^2) \]

\[\leq c \int_{S^2} \int_0^1 |\tilde{v}|^3 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2)^{\frac{1}{2}} d\xi
\]

\[+ c \int_{S^2} \left( \int_0^1 |\tilde{v}|^2 d\xi \right) \left( \int_0^1 |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2)^{\frac{1}{2}} d\xi \right)
\]

\[+ c \int_{S^2} \frac{\overline{T}}{q} \left( \int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}} \left( \int_0^1 |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) d\xi \right)^{\frac{1}{2}}
\]

\[\leq c \|\tilde{v}\|_{L^4(S^2)} \left( \int_\Omega |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) \right)^{\frac{1}{2}} \left( \int_0^1 |\tilde{v}|^4 d\xi \right)^{\frac{1}{2}}
\]

\[+ c \left( \int_\Omega |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) \right)^{\frac{1}{2}} \left( \int_{S^2} \int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}}
\]

\[+ c \|\overline{T}\|_{L^1(S^2)} \left( \int_\Omega |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) \right)^{\frac{1}{2}} \left( \int_{S^2} \int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}}
\]

\[+ c \|q\overline{T}\|_{L^1(S^2)} \left( \int_\Omega |\tilde{v}|^2 (|\nabla_{e_\phi} \tilde{v}|^2 + |\nabla_{e_\varphi} \tilde{v}|^2) \right)^{\frac{1}{2}} \left( \int_{S^2} \int_0^1 |\tilde{v}|^2 d\xi \right)^{\frac{1}{2}} \].

(5.59)

By Minkowski inequality, Hölder inequality and (4.13), we have

\[\left( \int_{S^2} \left( \int_0^1 |\tilde{v}|^4 d\xi \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \leq \int_0^1 \left( \int_{S^2} (|\tilde{v}|^2)^4 d\xi \right)^{\frac{1}{2}} d\xi
\]

\[\leq \int_0^1 \|\tilde{v}\|^2_{L^2(S^2)} (\|\nabla |\tilde{v}|^2\|_{L^2(S^2)} + \|\tilde{v}\|^2_{L^2(S^2)})^{\frac{1}{2}} d\xi
\]

\[\leq c |\tilde{v}|^2_{L^4(S^2)} \left( \int_0^1 (\|\nabla |\tilde{v}|^2\|^2_{L^2(S^2)} + \|\tilde{v}\|^2_{L^2(S^2)}) d\xi \right)^{\frac{1}{2}} \]
By Minkowski inequality, Hölder inequality and (4.14), we get
\[
\int_{S^2} \left( \int_0^1 |\tilde{v}|^2 d\xi \right)^3 \leq \left( \int_0^1 \left( \int_{S^2} |\tilde{v}|^6 d\xi \right)^{\frac{2}{3}} d\xi \right)^3
\]
\[
\leq \left( \int_0^1 (\|\tilde{v}\|_{L^6(S^2)}^2 \|\tilde{v}\|^2_{H^1(S^2)})^{\frac{2}{3}} d\xi \right)^3
\]
\[
\leq c \|\tilde{v}\|^4_{L^4(S^2)}.
\] (5.61)

By (4.13), we have
\[
\|\tilde{v}\|_{L^4(S^2)} \leq \frac{1}{3} \|\tilde{v}\|_{L^2(S^2)} \|\tilde{v}\|_{H^1(S^2)}.
\] (5.62)

By Young inequality, (5.53) and (5.54), we obtain from (5.59)-(4.62)
\[
\frac{d|\tilde{v}|^4_4}{dt} + \frac{1}{Re_1} \int_{\Omega} \left( \left( |\nabla_{e_{\phi}} \tilde{v}|^2 + |\nabla_{e_{\psi}} \tilde{v}|^2 \right) |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_{\phi}} |\tilde{v}|^2|^2 + \frac{1}{2} |\nabla_{e_{\psi}} |\tilde{v}|^2|^2 + |\tilde{v}|^4 \right)
\]
\[
+ \frac{1}{Re_2} \int_{\Omega} \left( ||\tilde{v}_t||^2_2 ||\tilde{v}||^2_{H^1(S^2)} \right)^{\frac{1}{2}} + c \frac{1}{2} ||\tilde{v}||^2_{H^1(S^2)} + c ||\tilde{T}||^2_{H^1(S^2)} + c ||q||^2_{H^1(S^2)} ||\tilde{T}||^2_{H^1(S^2)}
\]
\[
\leq c \frac{1}{2} ||\tilde{v}||^2_{L^2(S^2)} + ||\tilde{T}||^2_{H^1(S^2)} + ||\tilde{v}||^2 + ||\tilde{T}||^2_{H^1(S^2)} ||\tilde{v}||^2 + ||\tilde{v}||^2_{H^1(S^2)} ||\tilde{v}||^2_{H^1(S^2)}
\]
\[
+ c ||\tilde{T}||^2_{H^1(S^2)} ||\tilde{T}||^2_{H^1(S^2)}.
\] (5.63)

By Lemma 4.6, (5.17), (5.18), (5.19), (5.25), (5.39), (5.56) and \( |\tilde{v}|^4 \leq |\tilde{v}|^2 ||\tilde{v}||^2 \), we obtain
\[
|\tilde{v}(t + 4r)|^4_{4} \leq c \left( E_{1}^{\frac{1}{4}} + E_{2}^{\frac{1}{4}} + \frac{E_{3}^{2} E_{4}}{r} \right) \exp \left( cE_{0}E_{1} + E_{2} + \frac{E_{3}^{2}}{r} + E_{4} \right) \leq E_{6},
\] (5.64)

where \( E_{6} \) is a positive constant and \( t \geq 0 \). From (5.63) and (5.64), we have
\[
\int_{t+4r}^{t+5r} \frac{1}{Re_1} \int_{\Omega} \left( \left( |\nabla_{e_{\phi}} \tilde{v}|^2 + |\nabla_{e_{\psi}} \tilde{v}|^2 \right) |\tilde{v}|^2 + \frac{1}{2} |\nabla_{e_{\phi}} |\tilde{v}|^2|^2 + \frac{1}{2} |\nabla_{e_{\psi}} |\tilde{v}|^2|^2 + |\tilde{v}|^4 \right)
\]
\[
+ \frac{1}{Re_2} \int_{\Omega} \left( ||\tilde{v}_t||^2_2 ||\tilde{v}||^2_{H^1(S^2)} \right)^{\frac{1}{2}} \leq E_{6} + E_{6} = E_{7}.
\] (5.65)

By Gronwall inequality, from (5.63) we obtain
\[
|\tilde{v}(t)|^4_{4} \leq C_{2},
\] (5.66)

where \( C_{2} = C_{2}(\|U_{0}\|, \|Q_{1}\|, \|Q_{2}\|_{1}) > 0 \) and \( 0 \leq t < 4r \).
**$H^1$ estimates about $\bar{v}$**  
Taking the inner product of the equation (2.22) with $-\Delta \bar{v}$ in $L^2(S^2) \times L^2(S^2)$, we get

\[
\frac{1}{2} \frac{d}{dt} \left\| \nabla \bar{v} \right\|_{H^1(S^2)}^2 + \frac{1}{2Re_1} \left\| \Delta \bar{v} \right\|_{L^2(S^2)}^2 = \int_{S^2} [\nabla \bar{v} + \int_0^1 (\bar{v} \text{div} \bar{v} + \nabla \bar{v})d\xi] \cdot \Delta \bar{v} \\
+ \int_{S^2} (\text{grad}\Phi_\epsilon + \frac{f}{R_0} k \times \bar{v}) \cdot \Delta \bar{v} + \int_{S^2} \int_0^1 \int_{E} \frac{bP}{p} \text{grad}((1 + aq)T)d\xi' d\xi \cdot \Delta \bar{v}.
\]

By Hölder inequality, (4.13) and Young inequality, we have

\[
| \int_{S^2} (\nabla \bar{v} \cdot \Delta \bar{v}) |
\leq c \int_{S^2} |\bar{v}|(|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} |\Delta \bar{v}|
\leq c \|\bar{v}\|_{L^4(S^2)} \left( \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} \right)^{\frac{3}{4}} \|\Delta \bar{v}\|_{L^2(S^2)}
\leq c \|\bar{v}\|_{L^2(S^2)} \|\bar{v}\|_{L^1(S^2)} \left( \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} \right)^{\frac{3}{4}}
\times \left( \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} + \|\Delta \bar{v}\|_{L^2(S^2)} \right)^{\frac{3}{4}} \|\Delta \bar{v}\|_{L^2(S^2)}
\leq c \|\bar{v}\|_{L^2(S^2)}^2 \|\bar{v}\|_{L^1(S^2)} \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} + \|\Delta \bar{v}\|_{L^2(S^2)}^2.
\]

By Hölder inequality and Minkowski inequality, we obtain

\[
| \int_{S^2} (\bar{v} \text{div} \bar{v} + \nabla \bar{v})d\xi \cdot \Delta \bar{v} |
\leq \int_{S^2} \int_0^1 |\bar{v}|(|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} d\xi |\Delta \bar{v}|
\leq \left( \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} d\xi \right)^{\frac{3}{4}} \|\Delta \bar{v}\|_{L^2(S^2)}
\leq \int_{S^2} (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2)^{\frac{1}{2}} d\xi |\Delta \bar{v}|
\leq c \int_\Omega |\bar{v}|^2 (|\nabla \bar{v}|^2 + |\nabla \bar{v}|^2) + \|\Delta \bar{v}\|_{L^2(S^2)}^2.
\]

(5.67)
Using Lemma 4.1 and (2.19), we have

$$\int_{S^2} \text{grad} \Phi \cdot \Delta \bar{v} = 0,$$

$$\int_{S^2} \left[ \int_0^1 \int_{\xi}^1 \frac{bP}{p} \text{grad}((1 + aq)T) d\xi d\bar{v} \cdot \Delta \bar{v} \right] = 0.$$

(5.70)

\((\frac{L}{R_0} k \times \bar{v}) \cdot \Delta \bar{v} = 0\) implies

$$\int_{S^2} \left( \frac{L}{R_0} k \times \bar{v} \right) \cdot \Delta \bar{v} = 0.$$

(5.71)

From (5.67)-(5.71), choosing \(\varepsilon\) small enough, we obtain

$$\frac{d}{dt} \|\bar{v}\|^2_{H^1(S^2)} + \frac{1}{Re_1} \|\Delta \bar{v}\|^2_{L^2(S^2)} \leq c(\|\bar{v}\|^2_{L^2(S^2)} + \|\bar{v}\|^2_{H^1(S^2)} + \|\bar{v}\|^2_{H^1(S^2)} \|\bar{v}\|^2_{H^1(S^2)})$$

$$+ c \int_{\Omega} |\bar{d}|^2 (|\nabla_{v_\varepsilon} \bar{d}|^2 + |\nabla_{v_\varepsilon} \bar{d}|^2).$$

(5.72)

By Lemma 4.6, (5.17), (5.18), (5.19) and (5.65), we get

$$\|\bar{v}(t + 5r)\|^2_{H^1(S^2)} \leq c \left( \frac{E_1}{r} + E_7 \right) \exp \left( c(E_0E_1 + E_1) \right) \leq E_8,$$

(5.73)

where \(E_8\) is a positive constant. By Gronwall inequality, from (5.72) we obtain

$$\|\bar{v}(t)\|^2_{H^1(S^2)} \leq C_3,$$

(5.74)

where \(C_3 = C_3(\|U_0\|, \|Q_1\|, \|Q_2\|) > 0\) and \(0 \leq t < 5r\).

**L^2 estimates about \(v_\xi\)**

Taking the derivative, with respect to \(\xi\), of equation (2.11), we get the following equation

$$\frac{\partial v_\xi}{\partial t} - \frac{1}{Re_1} \Delta v_\xi - \frac{1}{Re_2} \frac{\partial^2 v_\xi}{\partial \xi^2} + \nabla_v v_\xi + \left( \int_{\xi}^1 \text{div} v d\xi' \right) \frac{\partial v_\xi}{\partial \xi}$$

$$+ \nabla_v v - (\text{div} v) \frac{\partial v}{\partial \xi} + \frac{f}{R_0} k \times v_\xi - \frac{bP}{p} \text{grad}[(1 + aq)T] = 0.$$

(5.75)

Taking the inner product of equation (5.75) with \(v_\xi\) in \(L^2(\Omega) \times L^2(\Omega)\), we obtain

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By integration by parts and Lemma 4.3, we get

\[
\frac{1}{2} \frac{d|v_\xi|^2}{dt} + \frac{1}{Re_1} \int_\Omega (|\nabla e_v v_\xi|^2 + |\nabla e_v v_\xi|^2 + |v_\xi|^2) + \frac{1}{Re_2} \int_\Omega |\nabla v_\xi|^2 \\
= - \int_\Omega (\nabla_v v_\xi \cdot (\int_\xi^1 div v \frac{\partial v_\xi}{\partial \xi}) v_\xi) - \int_\Omega (\nabla_v v - (div v) \frac{\partial v}{\partial \xi}) v_\xi \\
- \int_\Omega (\frac{f}{R_0} k \times v_\xi) \cdot v_\xi + \int_\Omega \frac{bP}{p} grad[(1 + aq)T] \cdot v_\xi. \\
\] (5.76)

By integration by parts and Lemma 4.3, we get

\[
\int_\Omega (\nabla_v v_\xi + (\int_\xi^1 div v \frac{\partial v_\xi}{\partial \xi}) \cdot v_\xi = 0. \\
\] (5.77)

By integration by parts, Hölder inequality, (4.16) and Young inequality, we have

\[
- \int_\Omega (\nabla_v v - (div v) \frac{\partial v}{\partial \xi}) \cdot v_\xi \leq c \int_\Omega |v||v_\xi|(|\nabla e_v v_\xi|^2 + |\nabla e_v v_\xi|^2)^{\frac{1}{2}} \\
\leq c |v|_4 |v_\xi|_4 (\int_\Omega (|\nabla e_v v_\xi|^2 + |\nabla e_v v_\xi|^2)^{\frac{1}{2}} \\
\leq c |v|_4 |v_\xi|_2^2 |v_\xi|_2^2 (\int_\Omega (|\nabla e_v v_\xi|^2 + |\nabla e_v v_\xi|^2))^\frac{1}{2} \\
\leq \varepsilon |v_\xi|^2 + c |v|_4^8 |v_\xi|^2. \\
\] (5.78)

\[\frac{f}{R_0} k \times v_\xi \cdot v_\xi = 0 \text{ implies} \]

\[
\int_\Omega (\frac{f}{R_0} k \times v_\xi) \cdot v_\xi = 0. \\
\] (5.79)

By Lemma 4.1, Hölder inequality and Young inequality, we obtain

\[
\int_\Omega \frac{bP}{p} grad[(1 + aq)T] \cdot v_\xi = - \int_\Omega \frac{bP}{p} (1 + aq)T div v_\xi \\
\leq c \int_\Omega |T||div v_\xi| + c \int_\Omega |q||T||div v_\xi| \\
\leq c |T|^2 + c |q|^2 |T|^4 + \varepsilon |v_\xi|^2. \\
\] (5.80)

Choosing \(\varepsilon\) small enough, we derive from (5.76)-(5.80),

\[
\frac{d|v_\xi|^2}{dt} + \frac{1}{Re_1} \int_\Omega (|\nabla e_v v_\xi|^2 + |\nabla e_v v_\xi|^2 + |v_\xi|^2) + \frac{1}{Re_2} \int_\Omega |\nabla v_\xi|^2 \\
\leq c |v|_4^6 |v_\xi|^2 + c |T|^2 + c |q|^2 |T|^4 \\
\leq c |v + \tilde{v}|_4^6 |v_\xi|^2 + c |T|^2 + c |q|^2 |T|^4 \\
\leq c (|v|_4^6 + |\tilde{v}|_4^6) |v_\xi|^2 + c |T|^2 + c |q|^2 + c |T|^4. \\
\] (5.81)
By Lemma 4.6, (5.18), (5.25), (5.39), (5.64) and (5.73), we get
\[ |v_\xi(t + 6r)|^2 \leq c(E_0 + \frac{E_1}{r} + E_2 + E_4) \exp c(E_0^2 + E_4^4) \leq E_9, \]  
(5.82)
where \( E_9 \) is a positive constant and \( t \geq 0 \). From (5.81) and (5.82), we have
\[ c_1 \int_{t+6r}^{t+7r} \|v_\xi\|^2 \leq E_9^2 + E_9 = E_{10}. \]  
(5.83)
By Gronwall inequality, from (5.81) we obtain
\[ |v_\xi(t)|^2 \leq C_4, \]  
(5.84)
where \( C_4 = C_4(\|U_0\|, \|Q_1\|_1, \|Q_2\|_1) > 0 \) and \( 0 \leq t < 6r \).

**L^2 estimates about** \( T_\xi, q_\xi \) Taking the derivative, with respect to \( \xi \), of the equations (2.12), (2.13), we get the following equations
\[ \frac{\partial T_\xi}{\partial t} - \frac{1}{Rt_1} \Delta T_\xi - \frac{1}{Rt_2} \frac{\partial^2 T_\xi}{\partial \xi^2} + \nabla v \cdot T_\xi + W(v) \frac{\partial T_\xi}{\partial \xi} + \nabla \cdot v - (\text{div}\,v) \frac{\partial T}{\partial \xi} \]  
\[ + \frac{bP}{p} (1 + aq) \text{div}
\[ + \frac{bP}{p} (1 + aq) \text{div} - \frac{abP}{p} q_\xi W(v) + \frac{bP(P - p_0)}{p_2} (1 + aq) W(v) = Q_{1\xi}, \]  
(5.85)
\[ \frac{\partial q_\xi}{\partial t} - \frac{1}{Rq_1} \Delta q_\xi - \frac{1}{Rq_2} \frac{\partial^2 q_\xi}{\partial \xi^2} + \nabla v \cdot q_\xi + W(v) \frac{\partial q_\xi}{\partial \xi} + \nabla \cdot v - (\text{div}\,v) \frac{\partial q}{\partial \xi} = Q_{2\xi}. \]  
(5.86)

We take the inner product of equation (5.85) with \( T_\xi \) in \( L^2(\Omega) \) and obtain
\[ \frac{1}{2} \frac{d}{dt} |T_\xi|^2 + \frac{1}{Rt_1} \int_\Omega |\nabla T_\xi|^2 + \frac{1}{Rt_2} \int_\Omega |\frac{\partial T_\xi}{\partial \xi}|^2 - \frac{1}{Rt_2} \int_{S^2} (T_\xi|_{\xi=1} - 1) \cdot T_\xi|_{\xi=1} \]  
\[ = - \int_\Omega (\nabla v \cdot T_\xi + W(v) \frac{\partial T_\xi}{\partial \xi}) T_\xi - \int_\Omega (\nabla \cdot v - (\text{div}\,v) \frac{\partial T}{\partial \xi}) T_\xi + \int_\Omega Q_{1\xi} T_\xi \]  
\[ + \int_\Omega \left[ - \frac{bP}{p} (1 + aq) (\text{div}) - \frac{bP(P - p_0)}{p} (1 + aq) W(v) + \frac{abP}{p} q_\xi W(v) \right] T_\xi. \]  
(5.87)

Similarly to (5.77), we have
\[ - \int_\Omega (\nabla v \cdot T_\xi + W(v) \frac{\partial T_\xi}{\partial \xi}) T_\xi = 0. \]  
(5.88)

By integration by parts, Hölder inequality, (4.16), Poincaré inequality and Young inequality, we obtain
\[ |\int_\Omega (\nabla v \cdot T - \text{div}(v) \frac{\partial T}{\partial \xi}) T_\xi| \]
Taking the trace on inequality, Young inequality and Lemma 4.2, we obtain
begin{align*}
&\leq c \int_{\Omega} \left( |\nabla q_{\xi}|^2 + |\nabla v_{\xi} v_{\xi}|^2 \right) \frac{1}{2} |T| |T_\xi|_4 + |v_{\xi}| |T| |\nabla T_\xi| + |v| |\nabla T_\xi| |T_\xi|_4 \\
&\leq c \int_{\Omega} \left( |\nabla q_{\xi}|^2 + |\nabla v_{\xi} v_{\xi}|^2 \right) \frac{1}{2} |T|_4 |T_\xi|_4 + c |v_{\xi}|_4 |T|_4 |\nabla T_\xi|_2 + c |v_{\xi}|_4 |\nabla T_\xi|_2 |T_\xi|_4 \\
&\leq c \int_{\Omega} \left( |\nabla q_{\xi}|^2 + |\nabla v_{\xi} v_{\xi}|^2 \right) \frac{1}{2} |T|_4^2 |T_\xi|_4 + c |v_{\xi}|_4^2 |T|_4^2 + c |v_{\xi}|_4^2 |\nabla T_\xi|_2^2 \\
&\leq c |T|_4^2 |v_{\xi}|_4 \left( |v_{\xi}|_2^2 + \int_{\Omega} \left( |\nabla q_{\xi}|^2 + |\nabla v_{\xi} v_{\xi}|^2 \right) \right)^{\frac{1}{2}} \\
&\leq \varepsilon \left( |T_\xi|^2 + |\nabla T_\xi|^2 \right) + c |v_{\xi}|_2^2 + \int_{\Omega} \left( |\nabla q_{\xi}|^2 + |\nabla v_{\xi} v_{\xi}|^2 \right) \\
&\quad + c |T|_4^2 |v_{\xi}|_4^2 + c \left( |T|^4_4 + |v|^2_4 \right) |T_\xi|^2_2. \quad (5.89)
\end{align*}
By integrating by parts, Hölder inequality, Minkowsky inequality, Poincaré inequality, Young inequality and Lemma 4.2, we obtain
begin{align*}
&\leq \frac{1}{p} \varepsilon (1 + a q) (\mathcal{D} v \cdot v) - \frac{b p (P - p_0)}{p} (1 + a q) W(v) + \frac{a b p}{p} q_\xi W(v) |T_\xi| \\
&\leq \frac{1}{p} \varepsilon (1 + a q) (\mathcal{D} v \cdot v) + \frac{b p}{p} (1 + a q) \mathcal{D} v \cdot \nabla T_\xi | + \int_{\Omega} \frac{b p (P - p_0)}{p} (1 + a q) W(v) |T_\xi| \\
&\quad + \int_{\Omega} \frac{a b p}{p} \mathcal{D} q_\xi \left( \int_{\xi} v \right) |T_\xi| + \frac{a b p}{p} q_\xi \left( \int_{\xi} v \right) |\nabla T_\xi| \\
&\leq c \int_{\Omega} \left( |\nabla q|^2 \right)^\frac{1}{2} |v|_4 |T_\xi|_4 + c |q|_4 |v|_4 |\nabla T_\xi|_2 + c |v|_2 |\nabla T_\xi|_2 \\
&\quad + c \left( |T_\xi|^2 + |q||v||v|_4 T_\xi|_4 + c |q||v|_4 |\nabla T_\xi|_2 + c |v||v|^2 \right) + c \left( |v|^2_4 + |q|^2_4 \right) |T_\xi|^2_2 \\
&\quad + c |v|^2_4 |q_\xi|^2_4 + c |q_\xi|^2_4 + c |T_\xi|^2_2 \\
&\leq c \left( |\nabla q_\xi|^2 + |\nabla T_\xi|^2 \right) + c |\nabla q|^2 + c |v|^2 ||v| + c \left( |v|^2_4 + |q|^2_4 \right) |T_\xi|^2_2 \left( |T_\xi|^2 \right)^\frac{1}{2} \\
&\quad + c |v|^2_4 |q_\xi|^2_4 + c |q_\xi|^2_4 + c |T_\xi|^2_2 \\
&\leq c \left( |\nabla q_\xi|^2 + |\nabla T_\xi|^2 \right) + c \left( |\nabla q|^2 + |\nabla T_\xi|^2 \right) + c \left( |v|^2 + |q|^2 \right) |T_\xi|^2_2 \left( |T_\xi|^2 \right)^\frac{1}{2} \\
&\quad + c |v|^2_4 |q_\xi|^2_4 + c |q_\xi|^2_4 + c |T_\xi|^2_2. \quad (5.90)
\end{align*}
By integration by parts, Hölder inequality and Young inequality, we have
begin{align*}
&\left| \int_{\Omega} Q_{1 \xi} T_\xi \right| \leq c |Q_{1 \xi}|^2_2 + c |T_\xi|^2_2. \quad (5.91)
\end{align*}
Taking the trace on $\xi = 1$ of equation (2.12), we have
begin{align*}
\frac{1}{R t_2} T_{\xi} |_{\xi = 1} = \left( \frac{\partial T}{\partial t} |_{\xi = 1} + (\nabla v) T \right) |_{\xi = 1} = \frac{1}{R t_1} \Delta T |_{\xi = 1} - Q_{1 |_{\xi = 1}}. \quad (5.92)
\end{align*}
By Lemma 4.3, we have

\[-\frac{1}{Rt_2} \int_{S^2} (T_{\xi} |\xi=1 T_{\xi} |\xi=1)\]

From (2.15), (5.92), we get

\[-\frac{1}{Rt_2} \int_{S^2} T|\xi=1(\frac{\partial T}{\partial t})|\xi=1 + (\nabla_v T)|\xi=1 - \frac{1}{Rt_1} \Delta T|\xi=1 - Q_1|\xi=1\]

\[= \alpha_s \frac{1}{2} \int_{S^2} T|\xi=1((\nabla_v T)|\xi=1 - Q_1)|\xi=1 + \alpha_s \int_{S^2} T|\xi=1(\nabla_v T)|\xi=1 - Q_1|\xi=1.\] (5.93)

By Lemma 4.3, we have

\[-\alpha_s \int_{S^2} T|\xi=1((\nabla_v T)|\xi=1 - Q_1)|\xi=1\]

\[= -\frac{\alpha_s}{2} \int_{S^2} (\nabla_v T^2)|\xi=1 + \alpha_s \int_{S^2} T|\xi=1Q_1|\xi=1\]

\[= \frac{\alpha_s}{2} \int_{S^2} T^2|\xi=1\div v|\xi=1 + \alpha_s \int_{S^2} T|\xi=1Q_1|\xi=1\]

\[= \frac{\alpha_s}{2} \int_{S^2} T^2|\xi=1(\int_\xi^1 \div v d\xi + \div v) + \alpha_s \int_{S^2} T|\xi=1Q_1|\xi=1\]

\[\leq c|T|\xi=1^4 + c\|v\|^2 + c\|v\|^2 + c|T|\xi=1^2 + c|Q_1|\xi=1^2.\] (5.94)

We derive from (5.87)-(5.94)

\[\frac{1}{2} \frac{d(|T|\xi=1^2 + \alpha_s|T|\xi=1^2)}{dt} + \frac{1}{Rt_1} \int_\Omega |\nabla T_{\xi}|^2 + \frac{1}{Rt_2} \int_\Omega |T_{\xi\xi}|^2 + \frac{\alpha_s}{Rt_1} |\nabla T|\xi=1|^2\]

\[\leq 2\varepsilon(|T_{\xi\xi}|^2 + |\nabla T_{\xi}|^2) + \varepsilon(|\nabla q_{\xi}|^2 + |q_{\xi}|^2) + c(1 + |T|^4 + |q|^4)|T_{\xi}|^2\]

\[+ c|v|^4 |q_{\xi}|^2 + c|v|^6 + c|v|^2 + c \div q_{\xi}^2 + c |T|^4 |v_{\xi}|^2 + c q_{\xi}^2 |v_{\xi}|^2\]

\[+ c|T|\xi=1^4 + c|T|\xi=1^2 + c|Q_1|\xi=1^2 + c|Q_2|\xi=1^2 + c|Q_1|\xi=1^2.\] (5.95)

Similarly to (5.95), we have

\[\frac{1}{2} \frac{d(|q_{\xi}|^2 + \beta_s|q_{\xi}|\xi=1^2)}{dt} + \frac{1}{Rq_1} \int_\Omega |\nabla q_{\xi}|^2 + \frac{1}{Rq_2} \int_\Omega |q_{\xi}|^2 + \frac{\beta_s}{Rq_1} |\nabla q|\xi=1|^2\]

\[\leq \varepsilon(|\nabla q_{\xi}|^2 + |q_{\xi}|^2) + c(|v|^4 + |T|^4)|q_{\xi}|^2\]

\[+ c|v_{\xi}|^2 + c|v|^2 + c q_{\xi}^2 |v_{\xi}|^2\]

\[+ c|q|\xi=1^4 + c|q|\xi=1^2 + c|Q_2|\xi=1^2 + c|Q_2|\xi=1^2 + c|Q_2|\xi=1^2.\] (5.96)

From (5.95) and (5.96), choosing \(\varepsilon\) small enough, we obtain

\[\frac{d(|T|\xi=1^2 + |q_{\xi}|^2 + \beta_s|q_{\xi}|\xi=1^2 + \alpha_s|T|\xi=1^2)}{dt} + \frac{1}{Rt_1} \int_\Omega |\nabla T_{\xi}|^2 + \frac{1}{Rt_2} \int_\Omega |T_{\xi\xi}|^2\]
By Hölder inequality, (4.16) and Young inequality, we have

\[ C_1 \leq \frac{\alpha_1 T}{c} + \frac{1}{R_q} \int_\Omega |\nabla q|_{\xi=1}^2 + \frac{1}{R_q} \int_\Omega |q_{\xi}|^2 + \frac{\beta_1}{R_q} \int_\Omega |\nabla q|_{\xi=1}^2 \]

\[ \leq c(1 + |T_1^\alpha| + |v|_4 + |q|^8)(|T_1|_{1,2}^2 + |q_1|_{1,2}^2) + c\|v\|^2 + c\|q\|^2 \]

\[ + c(|T_1|_{1,2}^2 + |q|^8) + c|q|^2 + c|T_1|_{1,2}^2 + c|q|_{1,2} + c|q_1|_{1,2}^2 + c|q_2|_{1,2}^2 + c|q_3|_{1,2}^2 \]

\[ \leq \left( c(1 + |T_1^\alpha| + |v|_4 + |q|^8)(|T_1|_{1,2}^2 + |q_1|_{1,2}^2) + c\|v\|^2 + c\|q\|^2 \right) \]

\[ + c(|T_1|_{1,2}^2 + |q|^8) + c|q|^2 + c|T_1|_{1,2}^2 + c|q|_{1,2} + c|q_1|_{1,2}^2 + c|q_2|_{1,2}^2 + c|q_3|_{1,2}^2. \]  

(5.97)

By Lemma 4.6, (5.18), (5.25), (5.26), (5.39), (5.40), (5.64), (5.73), (5.82), (5.83), we get

\[ |T_1(t + 7r)|_{1,2}^2 + |q(t + 7r)|_{1,2}^2 \leq E_{11}, \]  

(5.98)

where

\[ E_{11} = c\left( \frac{E_1}{r} + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 \right) + (E_2 + E_3)E_9 \]

\[ + E_{10} + |Q_1|_{1,2}^2 + |Q_2|_{1,2}^2 + |Q_3|_{1,2}^2 + |Q_4|_{1,2}^2 + |Q_5|_{1,2}^2 + |Q_6|_{1,2}^2 \]

\[ \cdot \exp \left( 1 + E_2^2 + E_3^2 + E_4^2 + E_5^2 + E_6^2 + E_8^2 \right). \]

From (5.97) and (5.98), we have

\[ c_1 \int_{t+7r}^{t+8r} (||T_1||^2 + ||q||^2) \leq E_{11}^2 + 2E_{11} + E_1 = E_{12}. \]  

(5.99)

By Gronwall inequality, from (5.97) we obtain

\[ |T_1(t)|_{1,2}^2 + |q(t)|_{1,2}^2 \leq C_5, \]  

(5.100)

where \( C_5 = C_5(\|U_0\|, \|Q_1(11)\|, \|Q_2(11)\|) > 0 \) and \( 0 \leq t < 7r \).

**H^1 estimates about v, T, q** Taking the inner product of equation (2.11) with \(-\Delta v\) in \(L^2(\Omega) \times L^2(\Omega)\), we get

\[ \frac{1}{2} \int_\Omega (|\nabla v|^2 + |\nabla v|^2 + |v|^2) \frac{dt}{dt} + \frac{1}{Re_1} \int_\Omega |\Delta v|_{1,2}^2 \]

\[ + \frac{1}{Re_2} \int_\Omega (|\nabla v|^2 + |\nabla v|^2 + |v|^2) \]

\[ = \int_\Omega (\nabla v \cdot W(v)) v \cdot \Delta v + \int_\Omega \left( \int_\xi \frac{bP}{p} \text{grad}((1 + aq)T) d\xi' \right) \cdot \Delta v \]

\[ + \int_\Omega (\frac{f}{p} k \times v + \text{grad} \Phi) \cdot \Delta v. \]  

(5.101)

By Hölder inequality, (4.16) and Young inequality, we have

\[ \left| \int_\Omega \nabla v \cdot \Delta v \right| \]
By Hölder inequality, Minkowski inequality, (4.13) and Young inequality, we obtain

\[
\begin{align*}
\leq & \int_{\Omega} |v||\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2 |\Delta v| \\
\leq & c|v|^2 \left( \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} + \epsilon |\Delta v| \right)^2 \\
\leq & c|v|^2 \left( \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} \right)^2 \left( \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} + \epsilon |\Delta v| \right)^2 \leq c(\epsilon |\Delta v| + |v|^2) \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} |\Delta v| \\
\leq & c(\epsilon |\Delta v| + |v|^2) \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} \Delta v + \epsilon |\Delta v|^2 \\
\leq & c(\epsilon |\Delta v| + |v|^2) \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} \Delta v + \epsilon |\Delta v|^2 \\
\leq & c(\epsilon |\Delta v| + |v|^2) \int_{\Omega} (|\nabla_{e_\nu} v|^2 + |\nabla_{e_\nu} v|^2)^{\frac{1}{2}} \Delta v + \epsilon |\Delta v|^2 \\
\leq & 2\epsilon |\Delta v|^2 + c(\epsilon |\Delta v|^2 + |v|^2 + \epsilon |\Delta v|^2) \\
\leq & (\epsilon |\Delta v| + |v|^2 + \epsilon |\Delta v|^2) \\
\leq & c(\epsilon |\Delta v| + |v|^2 + \epsilon |\Delta v|^2)
\end{align*}
\]

By Hölder inequality, Young inequality, Minkowski inequality, and (4.13), we have

\[
\int_{\Omega} \int_{\xi} \frac{bP}{P} \text{grad}[(1 + aq)T] d\xi' \cdot \Delta v
\]

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By using Lemma 4.1 and (2.19), we get

\[
\begin{align*}
\leq c \int_{\Omega} \left( \int_{0}^{1} |\nabla T| d\xi |\Delta v| + \int_{0}^{1} |T||\nabla q| d\xi |\Delta v| + \int_{0}^{1} |T||\nabla q| d\xi |\Delta v| \right)
\leq c |\nabla T|_{2} |\Delta v|_{2} + c \int_{\Omega} \left[ (\int_{0}^{1} |q|^{2} d\xi)^{\frac{1}{2}} \left( \int_{0}^{1} |\nabla T|^{2} d\xi \right)^{\frac{1}{2}} |\Delta v| \right]
+ c \int_{\Omega} \left[ (\int_{0}^{1} |T|^{2} d\xi) \frac{1}{2} \left( \int_{0}^{1} |\nabla q|^{2} d\xi \right)^{\frac{1}{2}} |\Delta v| \right]
\leq c |\nabla T|_{2} |\Delta v|_{2} + [(\int_{\Omega} (\int_{0}^{1} |q|^{2} d\xi)^{\frac{1}{2}} (\int_{0}^{1} |\nabla T|^{2} d\xi)^{\frac{1}{2}})^{\frac{1}{2}} + c |\nabla T|_{2} + \varepsilon |\Delta v|_{2}
\leq c (\int_{\Omega} (\int_{0}^{1} |q|^{2} d\xi)^{\frac{1}{2}} (\int_{0}^{1} |\nabla q|^{2} d\xi)^{\frac{1}{2}} |\Delta v|_{2} + (\int_{\Omega} (\int_{0}^{1} |T|^{2} d\xi) \frac{1}{2} (\int_{0}^{1} |\nabla q|^{2} d\xi) \frac{1}{2} + c |\nabla T|_{2} + \varepsilon |\Delta v|_{2}
\leq c (\int_{\Omega} (\int_{0}^{1} |q|^{4} \frac{1}{2} d\xi) (\int_{\Omega} (\int_{0}^{1} |\nabla q|^{4} \frac{1}{2} d\xi)) + c |\nabla T|_{2} + \varepsilon |\Delta v|_{2}
\leq c |q|_{4}^{2} \int_{0}^{1} [||\nabla T||_{L^{2}(S^{2})} (||\nabla T||_{L^{2}(S^{2})}^{\frac{1}{2}} + ||\Delta q||_{L^{2}(S^{2})}^{\frac{1}{2}})] d\xi
+ c |T|_{4}^{2} \int_{0}^{1} [||\nabla q||_{L^{2}(S^{2})} (||\nabla q||_{L^{2}(S^{2})}^{\frac{1}{2}} + ||\Delta q||_{L^{2}(S^{2})}^{\frac{1}{2}})] d\xi + c |\nabla T|_{2} + \varepsilon |\Delta v|_{2}
\leq c |q|_{4}^{2} |\nabla T|_{2} |\Delta T|_{2}^{\frac{1}{2}} + c |T|_{4}^{2} |\nabla q|_{2} (|\nabla q|_{2} + |\Delta q|_{2})^{\frac{1}{2}} + c |\nabla T|_{2} + \varepsilon |\Delta v|_{2}
\leq c |q|_{4}^{2} |\nabla T|_{2}^{\frac{3}{2}} + c |T|_{4}^{2} |\nabla q|_{2}^{\frac{3}{2}} + c |q|_{4}^{2} |\nabla T|_{2}^{\frac{3}{2}} + c |T|_{4}^{2} |\nabla q|_{2}^{\frac{3}{2}} + c |\nabla T|_{2}^{\frac{3}{2}} + \varepsilon |\Delta v|_{2}^{\frac{3}{2}} + \varepsilon |\Delta q|_{2}^{\frac{3}{2}}.
\end{align*}
\]
We derive from (5.101)-(5.106)

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (|\nabla_v v|^2 + |\nabla_{e_v} v|^2 + |v|^2) + \frac{1}{Re_1} |\Delta v|_2^2
\]

\[
+ \frac{1}{Re_2} \int_\Omega \left( |\nabla_{e_v} v_\xi|^2 + |\nabla_{e_v} v_\xi|^2 + |v_\xi|^2 \right)
\leq c(|v|^4 + |v|^2 + 2|v_\xi|^2 + |v_\xi|^4 + (|v_\xi|^2 + 1)||v_\xi||^2) \int_\Omega \left( |\nabla_{e_v} v|^2 + |\nabla_{e_v} v|^2 \right)
\]

\[
+ c(1 + |q|^2 + |q|) |\nabla T|_2^2 + c(|T_\xi|^2 + |T|^2) |\nabla q|_2^2 + 2\varepsilon \int_\Omega \left( |\nabla_{e_v} v_\xi|^2 + |\nabla_{e_v} v_\xi|^2 \right)
\]

\[
+ 5\varepsilon |\Delta v_\xi|^2 + \varepsilon |\Delta T|^2 + \varepsilon |\Delta q_\xi|^2 + c|\nabla T|^2.
\]  

(5.107)

We take the inner product of equation (2.12) with \(-\Delta T\) in \(L^2(\Omega)\) and obtain

\[
\frac{1}{2} \frac{d}{dt} |\nabla T|_2^2 + \frac{1}{Rt_1} |\Delta T|_2^2 + \frac{1}{Rt_2} (|\nabla T|_2^2 + \alpha_s |\nabla T|_{\xi=1}^2)
\]

\[
= \int_\Omega (\nabla_v T + W(v) \frac{\partial T}{\partial \xi}) \Delta T - \int_\Omega \frac{bP}{p} (1 + aq) W(v) \Delta T - \int_\Omega Q_1 \Delta T.
\]  

(5.108)

Similarly to (5.102), we get

\[
|\int_\Omega \Delta T \nabla_v T| \leq c(|v|^2 + |v_\xi|^2) |\nabla T|_2^2 + 2\varepsilon (|\Delta T|^2_2 + |\nabla T|_{\xi}^2).
\]  

(5.109)

We derive as (5.105)

\[
|\int_\Omega W(v) T_\xi \Delta T| \leq \varepsilon |\Delta T|^2_2 + \varepsilon |\Delta v|^2_2 + c|2T_\xi|^2 + |T|^2_2
\]

\[
+ (|T_\xi|^2 + 1) \int_\Omega |\nabla T_\xi|^2 \int_\Omega (|\nabla_{e_v} v|^2 + |\nabla_{e_v} v|^2).
\]  

(5.110)

By Hölder inequality, Minkowsky inequality, (4.13) and Young inequality, we obtain

\[
|\int_\Omega \frac{bP}{p} (1 + aq) W(v) \Delta T|
\]

\[
\leq c \int_\Omega (\int_0^1 |\text{div}v|d\xi) |\Delta T| + c \int_\Omega |q| (\int_0^1 |\text{div}v|d\xi) |\Delta T|
\]

\[
\leq c(\int_\Omega |\text{div}v|^2)^{\frac{1}{2}} |\Delta T|_2 + c|q|_4 (\int_\Omega (\int_0^1 |\text{div}v|^2d\xi)^{\frac{1}{2}} |\Delta T|_2
\]

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By Hölder inequality and Young inequality, we have

\[ |\int_{\Omega} Q_1 \triangle T| \leq c|Q_1|_2^2 + \varepsilon|\triangle T|_2^2. \]  (5.112)

From (5.108)-(5.112), we obtain

\[
\frac{1}{2} \frac{d}{dt}|\nabla T|_2^2 + \frac{1}{R_1} |\triangle T|_2^2 + \frac{1}{R_2} |\nabla T|_{\xi=1}^2 \leq 5\varepsilon|\triangle T|_2^2 + 2\varepsilon|\nabla T|_{\xi}^2 + c(1 + |q|_4^2 + |q|_4^4 + 2|T|_2^2)
+ |T|_2^2 + (|T|_2^2 + 1) \int_{\Omega} |\nabla T|_{\xi}^2 \int_{\Omega} (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2)
+ c(|v|_2^2 + |v|_4^2) |\nabla T|_{\xi}^2 + c|Q_1|_2^2. \]  (5.113)

By taking the inner product of equation (2.13) with \(-\triangle q\) in \(L^2(\Omega)\), we have

\[
\frac{1}{2} \frac{d}{dt}|\nabla q|_2^2 + \frac{1}{R q_1} |\triangle q|_2^2 + \frac{1}{R q_2} (|\nabla q|_{\xi=1}^2 + \beta_s |\nabla q|_{\xi=1}^2) = \int_{\Omega} (\nabla q \cdot W(v) \frac{\partial q}{\partial \xi}) \triangle q - \int_{\Omega} Q_2 \triangle q. \]  (5.114)

Similarly to (5.113), we derive from (5.114)

\[
\frac{1}{2} \frac{d}{dt}|\nabla q|_2^2 + \frac{1}{R q_1} |\triangle q|_2^2 + \frac{1}{R q_2} (|\nabla q|_{\xi=1}^2 + \beta_s |\nabla q|_{\xi=1}^2) \leq 4\varepsilon|\triangle q|_2^2 + 2\varepsilon|\nabla q|_{\xi}^2 + \varepsilon|\nabla q|_{\xi}^2 + c(|v|_4^2 + |v|_4^4) |\nabla q|_{\xi}^2 + c|Q_2|_2^2.
+ (|q|_2^2 + 1) \int_{\Omega} |\nabla q|_{\xi}^2 \int_{\Omega} (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2) + c|Q_2|_2^2. \]  (5.115)

From (5.107), (5.113) and (5.115), choosing \(\varepsilon\) small enough, we obtain

\[
\frac{1}{dt} \int_{\Omega} (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2 + |v|^2 + |\nabla T|_2^2 + |\triangle T|_2^2)
+ \frac{1}{R e_1} |\triangle v|_2^2 + \frac{1}{R t_1} |\triangle T|_2^2 + \frac{1}{R q_1} |\triangle q|_2^2 + \frac{1}{R e_2} \int_{\Omega} (|\nabla e_\theta v|_{\xi}^2 + |\nabla e_\theta v|_{\xi}^2 + |v|_{\xi}^2)
\]  39
By Gronwall inequality, from (5.116) we obtain

\[ C \]

contradiction to prove Theorem 3.1. Indeed, let

**Proof of Theorem 3.1.**

5.3 The global existence of strong solutions

By Lemma 4.6, (5.18), (5.25), (5.39), (5.64), (5.73), (5.82), (5.83), (5.98) and (5.99), we get

\[
|\nabla_{e_x} v(t+8r)|^2 + |\nabla_{e_z} v(t+8r)|^2 + |\nabla T(t+8r)|^2 + |\nabla q(t+8r)|^2 \leq E_{13}, \quad (5.117)
\]

where

\[
E_{13} = c\left( \frac{E_1}{r} + |Q_1|^2 + |Q_2|^2 \right) \exp \left[ c + 1 + E_2 + E_4 + E_6^2 + E_8^2 + E_9^2 + (E_9 + 1) E_{10} + E_{11}^2 + (E_{11} + 1) E_{12} \right].
\]

By Gronwall inequality, from (5.116) we obtain

\[
|\nabla_{e_x} v(t)|^2 + |\nabla_{e_z} v(t)|^2 + |\nabla T(t)|^2 + |\nabla q(t)|^2 \leq C_6, \quad (5.118)
\]

where \( C_6 = C_6(\|U_0\|, \|Q_1\|, \|Q_2\|) > 0 \) and \( 0 \leq t < 8r \).

5.3 The global existence of strong solutions

**Proof of Theorem 3.1.** By Proposition 5.3, we can use the method of contradiction to prove Theorem 3.1. Indeed, let \( U \) be a strong solution to the system (2.11)-(2.17) on the maximal interval \([0, T_*]\). If \( T_* < +\infty \), then

\[
\limsup_{t \to T_*} \|U\| = +\infty,
\]

which is impossible from (5.18), (5.82), (5.84), (5.98), (5.100), (5.117), (5.118). The proof is complete.

6 The uniqueness of strong solutions

**Proof of Theorem 3.2** Let \((v_1, T_1, q_1)\) and \((v_2, T_2, q_2)\) be two strong solutions of (2.11)-(2.17) on the time interval \([0, T]\) with corresponding geopotentials \(\Phi_{e_x}, \Phi_{e_z}\), and initial data \(((v_0)_1, (T_0)_1, (q_0)_1), ((v_0)_2, (T_0)_2, (q_0)_2)\), respectively.

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Define \( v = v_1 - v_2, T = T_1 - T_2, q = q_1 - q_2, \Phi_s = \Phi_{s_1} - \Phi_{s_2}. \) Then \( v, T, q, \Phi_s \) satisfy the following system

\[
\begin{align*}
\frac{\partial v}{\partial t} - \frac{1}{Re_1} \nabla^2 v - \frac{1}{Re_2} \frac{\partial^2 v}{\partial \xi^2} + \nabla_{e_1} v + \nabla_{e_2} v_2 + W(v_1) \frac{\partial v}{\partial \xi} + W(v) \frac{\partial v_2}{\partial \xi} + \frac{f}{R_0} k \times v + & \quad \text{grad} \Phi_s \quad + \int_{\xi}^1 \frac{bP}{p} \text{grad} T d\xi' + \int_{\xi}^1 \frac{abP}{p} \text{grad} q_1 T d\xi' + \int_{\xi}^1 \frac{abP}{p} \text{grad} q_2 T d\xi' = 0, \\
\frac{\partial T}{\partial t} - \frac{1}{Rt_1} \Delta T - \frac{1}{Rt_2} \frac{\partial^2 T}{\partial \xi^2} + \nabla_{e_1} T + \nabla_{e_2} T_2 + W(v_1) \frac{\partial T}{\partial \xi} + W(v) \frac{\partial T_2}{\partial \xi} - \frac{bP}{p} W(v) \quad - \frac{abP}{p} q_1 W(v) - \frac{abP}{p} q_2 W(v) = 0, \\
\frac{\partial q}{\partial t} - \frac{1}{Rq_1} \Delta q - \frac{1}{Rq_2} \frac{\partial^2 q}{\partial \xi^2} + \nabla_{e_1} q + \nabla_{e_2} q_2 + W(v_1) \frac{\partial q}{\partial \xi} + W(v) \frac{\partial q_2}{\partial \xi} = 0, \\
v|_{t=0} = (v_0)_1 - (v_0)_2, \\
T|_{t=0} = (T_0)_1 - (T_0)_2, \\
q|_{t=0} = (q_0)_1 - (q_0)_2, \\
\xi = 1: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = -\alpha_s T, \quad \frac{\partial q}{\partial \xi} = -\beta_s q, \\
\xi = 0: \quad \frac{\partial v}{\partial \xi} = 0, \quad \frac{\partial T}{\partial \xi} = 0, \quad \frac{\partial q}{\partial \xi} = 0.
\end{align*}
\]

We take the inner product of equation (6.1) with \( v \) in \( L^2(\Omega) \times L^2(\Omega) \) and obtain

\[
\begin{align*}
\frac{1}{2} \frac{d|v|^2}{dt} + \frac{1}{Re_1} \int_{\Omega} (|\nabla e_1 v|^2 + |\nabla e_2 v|^2 + |v|^2) + \frac{1}{Re_2} \int_{\Omega} |v_\xi|^2 &= - \int_{\Omega} (\nabla_{e_1} v + W(v_1) \frac{\partial v}{\partial \xi}) \cdot v - \int_{\Omega} v \cdot \nabla_{e_2} v_2 - \int_{\Omega} W(v) \frac{\partial v_2}{\partial \xi} \cdot v \\
&\quad - \int_{\Omega} (\frac{f}{R_0} k \times v + \text{grad} \Phi_s) \cdot v - \int_{\Omega} (\int_{\xi}^1 \frac{bP}{p} \text{grad} T d\xi') \cdot v \\
&\quad - \int_{\Omega} (\int_{\xi}^1 \frac{abP}{p} \text{grad} q_1 T d\xi') \cdot v - \int_{\Omega} (\int_{\xi}^1 \frac{abP}{p} \text{grad} q_2 T d\xi') \cdot v \\
&\quad - \int_{\Omega} (\int_{\xi}^1 \frac{abP}{p} \text{grad} q_1 T d\xi') \cdot v - \int_{\Omega} (\int_{\xi}^1 \frac{abP}{p} \text{grad} q_2 T d\xi') \cdot v.
\end{align*}
\]

By Lemma 4.4, we have

\[
\int_{\Omega} (\nabla_{e_1} v + W(v_1) \frac{\partial v}{\partial \xi}) \cdot v = 0.
\]
Using Lemma 4.3, Hölder inequality, Young inequality and (4.16), we get
\[
|\int_\Omega v \cdot \nabla v| = |\int_\Omega (v_2 \cdot \nabla v + v_2 \cdot \text{div} v)|
\leq c \int_\Omega |v||v_2|(|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2)^{\frac{1}{2}}
\leq \varepsilon \int_\Omega (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2) + c|v|^2|v_2|^2
\leq \varepsilon \int_\Omega (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2) + c|v|^2|v_2|^2|v||^2
\leq 2\varepsilon ||v||^2 + c|v|^2|v_2|^2. \tag{6.9}
\]

By Hölder inequality, Young inequality, Minkowski inequality and (4.13), we obtain
\[
|\int_\Omega W(v) \frac{\partial v_2}{\partial \xi} \cdot v|
\leq \int_{S^2} \left[ \int_0^1 \left( |\nabla e_\theta v|^2 + |\nabla e_\theta v|^2 \right)^{\frac{1}{2}} d\xi \int_0^1 |v_2| |v| d\xi \right]
\leq (\int_\Omega (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2))^{\frac{1}{2}}(\int_{S^2} (\int_0^1 |v|^2 d\xi)^2)^{\frac{1}{2}}(\int_{S^2} (\int_0^1 |v|^2 d\xi)^2)^{\frac{1}{2}}
\leq \varepsilon \int_\Omega (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2) + c \int_{S^2} (\int_0^1 |v|^4)^{\frac{1}{2}} d\xi \int_0^1 (\int |v|^4)^{\frac{1}{2}} d\xi
\leq \varepsilon \int_\Omega (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2) + c \int_0^1 \|v_2\| L^2(S^2)(\int_{S^2} (|\nabla e_\theta v_2|^2 + |\nabla e_\theta v_2|^2))^{\frac{1}{2}} d\xi
\cdot \int_{S^2} (\|v\| L^2(S^2)(|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2 + |v|^2))^{\frac{1}{2}} d\xi
\leq 2\varepsilon ||v||^2 + c[(|v_2|^2 + 1) \int_\Omega (|\nabla e_\theta v_2|^2 + |\nabla e_\theta v_2|^2) + |v_2|^2|v||^2]. \tag{6.10}
\]

By Lemma 4.1, Hölder inequality, Young inequality, Minkowski inequality and (4.13), we have
\[
|\int_\Omega \left( \int_0^1 \frac{abP}{p} \text{grad}(qT_2) d\xi \right) \cdot v|
= |\int_\Omega \left( \int_0^1 \frac{abP}{p} qT_2 d\xi \right) \cdot \text{div} v|
\leq c \int_{S^2} \left[ \int_0^1 |q||T_2| d\xi \int_0^1 (|\nabla e_\theta v|^2 + |\nabla e_\theta v|^2)^{\frac{1}{2}} d\xi \right]
\]
Similarly to (6.13), we get
\[ T \leq c \int_{S^2} (\int_0^1 |q|^2 d\xi)^2 \frac{1}{2} (\int_{S^2} (\int_0^1 |T_2|^2 d\xi)^2 \frac{1}{2} + \varepsilon \int_{\Omega} (|\nabla_{e_5} v|^2 + |\nabla_{e_6} v|^2) \]
\[ \leq c \int_{S^2} (\int_0^1 |q|^2 d\xi)^2 \frac{1}{2} (\int_{S^2} (\int_0^1 |T_2|^2 d\xi)^2 \frac{1}{2} + \varepsilon \int_{\Omega} (|\nabla_{e_5} v|^2 + |\nabla_{e_6} v|^2) \]
\[ \leq c |q|_2 (|\nabla q|_2 + |q|_2) |T_2|^2 \frac{1}{4} + \varepsilon \int_{\Omega} (|\nabla_{e_5} v|^2 + |\nabla_{e_6} v|^2) \]

By Lemma 4.1, we get
\[ \int_{\Omega} (\frac{f}{R_0} k \times v + \text{grad} \Phi_s) \cdot v = 0. \]  

We derive from (6.7)-(6.12)
\[ \frac{1}{2} \frac{d|T|^2}{dt} + \frac{1}{Re_1} \int_{\Omega} (|\nabla_{e_5} v|^2 + |\nabla_{e_6} v|^2 + |v|^2) + \frac{1}{Re_2} \int_{\Omega} |v_\xi|^2 \]
\[ \leq 5\varepsilon |v|^2 + \varepsilon |\nabla q|^2 + c(|T_2|^2 + |T_2|^2) |q|^2 \]
\[ + \int_{\Omega} (\int_\xi^1 \frac{bP}{p} \text{grad} T d\xi) \cdot v - \int_{\Omega} (\int_\xi^1 \frac{abP}{p} \text{grad} (q_1 T) d\xi) \cdot v. \]  

By taking the inner product of equation (6.2) with $T$ in $L^2(\Omega)$ and equation (6.3) with $q$ in $L^2(\Omega)$, we obtain
\[ \frac{1}{2} \frac{d|T|^2}{dt} + \frac{1}{Re_1} \int_{\Omega} (|\nabla T|^2 + \frac{1}{Rt_1} \int_{\Omega} (\nabla v_T + W(v) \frac{\partial T}{\partial \xi} + \int_{\Omega} W(v) \frac{\partial T}{\partial \xi} T \]
\[ = - \int_{\Omega} (\nabla v_T + W(v) \frac{\partial T}{\partial \xi} T + \int_{\Omega} \frac{bP}{p} W(v) T + \int_{\Omega} \frac{abP}{p} q_1 W(v) T, \]  

(6.14)

\[ \frac{1}{2} \frac{d|q|^2}{dt} + \frac{1}{Rq} \int_{\Omega} (|\nabla q|^2 + \frac{1}{Rq_2} \int_{\Omega} (\nabla q_2^2 + \frac{\beta_s}{Rt_2} |q|^2 - \int_{\Omega} q \frac{\partial q}{\partial \xi} q - \int_{\Omega} W(v) \frac{\partial q}{\partial \xi} q. \]  

(6.15)

Similarly to (6.13), we get
\[ \frac{1}{2} \frac{d|T|^2}{dt} + \frac{1}{Rt_1} \int_{\Omega} (|\nabla T|^2 + \frac{1}{Rt_2} \int_{\Omega} |T_\xi|^2 + \frac{\alpha_s}{Rt_2} |T|^2 \]  

(43)
Similarly to (5.104), we have

\begin{equation}
\begin{aligned}
&\leq 3\varepsilon\|v\|^2 + 3\varepsilon\|T\|^2 + c|T_2|^4(|T_2^2 + |v|^2) + c((|T_2|^2 + 1)|\nabla T_2|^2 + |T_2|^2|T_2^2||T_2^2| \\
&+ \int_\Omega \frac{bP}{P}W(v)T + \int_\Omega \frac{abP}{P}q_1W(v)T + \int_\Omega \frac{abP}{P}qTW(v_2),
\end{aligned}
\end{equation}

(6.16)

\begin{equation}
\begin{aligned}
\frac{1}{2}d|q|^2 + \frac{1}{Rq_1} \int_\Omega |\nabla q|^2 + \frac{1}{Rq_2} \int_\Omega |q|^2 + \frac{\beta_s}{Rq_2}|q|_{\xi=1}^2 \\
&\leq 3\varepsilon\|v\|^2_{H^1(\Omega)} + 3\varepsilon\|q\|^2_{H^1(\Omega)} + c((|q_2|^2 + 1)|\nabla q_2|^2 + |q_2|^2|q_2^2|q_2^2 \\
&+ c|q_2|^4(|q|^2 + |v|^2).
\end{aligned}
\end{equation}

(6.17)

By integration by parts, we have

\begin{equation}
-\int_\Omega (\int_\xi^1 \frac{bP}{p}\text{grad}Td\xi') \cdot v + \int_\Omega \frac{bP}{P}W(v)T = 0,
\end{equation}

(6.18)

\begin{equation}
-\int_\Omega (\int_\xi^1 \frac{abP}{p}\text{grad}(q_1T)d\xi') \cdot v + \int_\Omega \frac{abP}{P}q_1W(v)T = 0.
\end{equation}

(6.19)

Similarly to (5.104), we have

\begin{equation}
|\int_\Omega \frac{abP}{p}qW(v_2)T| = |\int_\Omega \int_\xi^1 \frac{abP}{p}\text{grad}(qT)d\xi' \cdot v_2|
\end{equation}

\begin{equation}
\leq c|v_2|^4|q_4|\|\nabla q\|^2 + c|v_2|^4|T_4|\|\nabla q\|^2 \\
\leq c|v_2|^4(|q|^2 + |T|^2) + |v|^2 + \varepsilon|\nabla q|^2 \\
\leq c|v_2|^4(|q|^2 + |T|^2) + 2\varepsilon\|q\|^2 + 2\varepsilon\|T\|^2.
\end{equation}

(6.20)

From (6.13), (6.16)-(6.20), by using (5.12) and (5.13), and choosing \(\varepsilon\) small enough, we obtain

\begin{equation}
\begin{aligned}
&\frac{d}{dt}(|v|_2^2 + |T|_2^2 + |q|_2^2) + \frac{1}{R_1} \int_\Omega (|\nabla e_v|^2 + |\nabla e_v|^2 + |v|^2) + \frac{1}{R_2} \int_\Omega |v|^2 \\
&+ \frac{1}{R_1} \int_\Omega |\nabla T|^2 + \frac{1}{R_2} \int_\Omega |T_\xi|^2 + \frac{\alpha_s}{R_2}|T|_{\xi=1}^2 \\
&+ \frac{1}{R_1} \int_\Omega |\nabla q|^2 + \frac{1}{R_2} \int_\Omega |q|^2 + \frac{\beta_s}{R_2}|q|_{\xi=1}^2 \\
&\leq c|v_2|^4 + |T_2|^4 + |q_2|^4 + |q_2|^4 + |q_2|^2 + (|q_2|^2 + 1) \int_\Omega (|\nabla e_v|^2 + |\nabla e_v|^2)|v|^2 \\
&+ c|v_2|^4 + |T_2|^4 + |q_2|^4 + (|q_2|^2 + 1)|\nabla T_2|^2 + |q_2|^2 \\
&+ c|v_2|^4 + |T_2|^4 + |q_2|^4 + |q_2|^2 + (|q_2|^2 + 1)|\nabla q_2|^2)|q_2|^2.
\end{aligned}
\end{equation}

(6.21)
Proposition 6.1 (The uniqueness of strong/weak solutions) Let $U_1$ be a weak solution to the system (2.11) – (2.17). If there exists a weak solution $U_2$ of the system (2.11)-(2.17) on the interval $[0, T]$ with the same initial conditions, such that

$$U_2 \in L^8(0, T; (L^4(\Omega))^4), \quad U_{2\varepsilon} \in L^\infty(0, T; (L^2(\Omega))^4) \cap L^2(0, T; (H^1(\Omega))^4),$$

Then the solutions $U_1, U_2$ coincide on $[0, T]$.

7 The existence of universal attractors

Proof of Proposition 3.3 From (5.18), (5.82), (5.84), (5.98), (5.100), (5.117), (5.118), we know $U \in L^\infty(0, \infty; V)$. By Theorem 3.1 and Theorem 3.2, we can define the semigroup $\{S(t)\}_{t \geq 0}$ corresponding to the system (2.11)-(2.16) where $S(t) : V \to V, S(t)U_0 = U(t)$. By (5.16), (5.18), (5.25), (5.32), (5.39), (5.56), (5.64), (5.65), (5.73), (5.82), (5.83), (5.93), (5.98), (5.99), (5.117), we prove the corresponding semigroup $\{S(t)\}_{t \geq 0}$ possesses a bounded absorbing set $B_\rho$ in $V$, i.e., for any $U_0 \in V$, there exists $t_0$ big enough such that

$$S(t)U_0 \in B_\rho, \text{ for any } t \geq t_0,$$

where $B_\rho = \{U; \quad U \in V, \quad \|U\| \leq \rho\}$ and $\rho$ is a positive constant dependent on $\|Q_1\|_1, \|Q_2\|_1$.

In order to prove Theorem 3.4, we need the following property about the semigroup $\{S(t)\}_{t \geq 0}$.

Proposition 7.1 For every $t \geq 0$, the mapping $S(t)$ is weakly continuous from $V$ to $V$.

Proof of Proposition 7.1 Let $\{U_n\}$ be a sequence in $V$ such that $U_n \rightharpoonup U$ weakly in $V$. Then $\{U_n\}$ is bounded in $V$. By the priori estimates in section 5, we know that, for every $t \geq 0$, $\{S(t)U_n\}$ is bounded in $V$. So we extract a subsequence $\{S(t)U_{n_k}\}$ such that $S(t)U_{n_k} \to u$ weakly in $V$. Since the embedding $V \hookrightarrow L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ is compact, $U_{n_k} \to U$ strongly in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. By (6.21), we obtain that $S(t)U_{n_k} \to S(t)U$ strongly in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Then $u = S(t)U$. Therefore, the sequence $\{S(t)U_n\}$ satisfies: $S(t)U_{n_k} \to S(t)U$ weakly in $V$. Proposition 7.1 is proved.

Proof of Theorem 3.4 With Proposition 3.3 and Proposition 7.1, we know that the proof of Theorem 3.4 is similar to that of Theorem I.1.1 in
We only need replace ”→ strongly in $H$” by ”→ weakly in $V$” in the course of proof of Theorem I.1.1 in [32]. So the detail of proof for Theorem 3.4 is omitted here.

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