Universal critical coupling constants
for the three–dimensional $n$–vector model
from field theory

A. I. Sokolov$^{1,2}$, E. V. Orlov$^1$, V. A. Ul’kov$^2$, and S. S. Kashtanov$^1$

$^1$Department of Physical Electronics, Saint Petersburg Electrotechnical University,
Professor Popov Street 5, St. Petersburg, 197376, Russia
$^2$Department of Physics, Saint Petersburg Electrotechnical University,
Professor Popov Street 5, St. Petersburg, 197376, Russia

Abstract

The field-theoretical renormalization group (RG) approach in three dimensions is used to estimate the universal critical values of renormalized coupling constants $g_6$ and $g_8$ for the $O(n)$-symmetric model. The RG series for $g_6$ and $g_8$ are calculated in the four-loop and three-loop approximations, respectively, and then resummed by means of the Padé-Borel-Leroy technique. Under the optimal value of the shift parameter $b$ providing the fastest convergence of the iteration procedure, numerical estimates for $g_6^*$ are obtained with the accuracy no worse than 0.3%. The RG expansion for $g_8$ demonstrates a stronger divergence, and results in considerably cruder numerical estimates.

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I. INTRODUCTION

The three-dimensional (3D) \( O(n) \)-symmetric model plays a very important role in the theory of phase transitions. It describes critical phenomena in a variety of physical systems including Ising, XY-like and Heisenberg ferromagnets, simple fluids and binary mixtures, superconductors and Bose superfluids, etc. This model is also relevant to certain asymptotic regimes of the critical behavior of the quark-gluon plasma in quantum chromodynamics \( (n = 4) \). In the critical region, the \( n \)-vector model is known to be thermodynamically equivalent to the 3D Euclidean field theory of \( \lambda \varphi^4 \) type, and may be treated by the field-theoretical renormalization group (RG) technique which proved to be very efficient both for studying the qualitative features of phase transitions and calculating the critical exponents.

On the other hand, for decades the influence of ordering fields upon the critical behavior of various systems attracted permanent attention, being of prime interest both for theorists and experimentalists. Recently, the free energy (effective action) and, in particular, higher-order renormalized coupling constants \( g_{2k} \) for the basic models of phase transitions became the target of intensive theoretical studies. These constants are related to the nonlinear susceptibilities \( \chi_{2k} \) and enter the scaling equation of state, thus playing a key role at criticality. Along with critical exponents and critical amplitude ratios, they are universal, i.e. they possess, under \( T \to T_c \), numerical values that are not sensitive to the physical nature of the phase transition, depending only on the system dimensionality and the symmetry of the order parameter.

Calculation of the universal critical values of \( g_6, g_8 \), etc. for the three-dimensional Ising model by a number of analytical and numerical methods showed that the field-theoretical RG approach in fixed dimensions yields the most accurate numerical estimates for these quantities. It is a consequence of a rapid convergence of the iteration schemes originating from renormalized perturbation theory. Indeed, the resummation of four- and five-loop RG expansions by means of the Borel-transformation-based procedures gave the values for \( g_6^* \), which differ from each other by less than 0.5\% [18,19], while the use of a resummed three-loop RG expansion enabled one to achieve an apparent accuracy no worse than 1.6\% [7,17]. Moreover, the field-theoretical RG approach turns out to be powerful enough even in two dimensions: properly resummed four–loop RG expansions lead to fair numerical estimates for the critical exponents [3] and the renormalized coupling constant \( g_6^* \) [24] of a 2D Ising model, and give reasonable results for its random counterpart [25,26]. It is natural, therefore, to use the field theory for a calculation of renormalized higher-order coupling constants for the 3D \( n \)-vector model. In this paper, the 3D RG expansion for the renormalized coupling constants \( g_6 \) and \( g_8 \) will be calculated, and the numerical estimates for their universal critical values will be obtained.

II. RG EXPANSIONS FOR THE Sextic AND OCTIC COUPLING CONSTANTS

Within the field-theoretical language, the 3D \( O(n) \)-symmetric model in the critical region is described by Euclidean scalar field theory with the Hamiltonian

\[
H = \int d^3 x \left[ \frac{1}{2} (\mathbf{m}_0^2 \varphi^2 + (\nabla \varphi^2) + \lambda (\varphi^2)^2) \right],
\]

where a bare mass squared \( m_0^2 \) is proportional to \( T - T_c^{(0)} \), \( T_c^{(0)} \) being the phase transition temperature in the absence of the order parameter fluctuations. Taking fluctuations into account results in renormalizations of the mass \( m_0 \to m \), the field \( \varphi \to \varphi_R \), and the coupling
constant \( \lambda \to mg_4 \). Moreover, thermal fluctuations give rise to many-point correlations \(< \varphi(x_1)\varphi(x_2)\ldots \varphi(x_{2k}) >\) and, correspondingly, to higher-order terms in the expansion of the free energy in powers of the magnetization \( M \):

\[
F(M, m) = F(0, m) + \sum_{k=1}^{\infty} \Gamma_{2k} M^{2k}.
\] (2)

In the critical region, the coefficients \( \Gamma_{2k} \), being one-particle irreducible \( 2k \)-point vertices taken at zero external momenta, demonstrate well-known scaling behavior:

\[
\Gamma_{2k} = g_{2k} m^{3-k(1+\eta)},
\] (3)

where \( \eta \) is a Fisher exponent, and \( g_{2k} \) are some constants. Let us set as usually \( g_2 = 1/2 \). Then \( g_4, g_6, g_8 \ldots \) will acquire universal values. The asymptotic critical values of \( g_{2}(n) \) have been found by resummation of the six-loop expansion for the RG \( \beta \)-function \([3,4,6,7]\), from strong-coupling series \([27]\), by lattice calculations \([21]\), from the \( \epsilon \)-expansion \([28]\), and are known today with an accuracy which may be considered rather high.

Moreover, via \( g_{2k} \), the non-linear susceptibilities \( \chi_{2k} \) can be expressed. For \( \chi_4 \) and \( \chi_6 \) corresponding formulas are as follows:

\[
\chi_4 = \frac{\partial^3 M}{\partial H^3} \bigg|_{H=0} = -24\chi_2^2 m^{-3} g_4, \quad \chi_6 = \frac{\partial^5 M}{\partial H^5} \bigg|_{H=0} = 720\chi_2^3 m^{-6} (8g_4^2 - g_6).
\] (5)

Their inversion gives the relations

\[
g_4 = -\frac{m^3 \chi_4}{24\chi_2^2}, \quad g_6 = \frac{m^6(10\chi_4^2 - \chi_6\chi_2)}{720\chi_2},
\] (6)

which are widely used for extraction of numerical values of renormalized coupling constants from the results of lattice calculations \([14,16,21,22,29,30]\).

The method of calculating the RG series for the \( g_6 \) and \( g_8 \) we use here is straightforward. Since in three dimensions higher-order bare couplings are irrelevant in RG sense, the renormalized perturbative series to be found can be obtained from conventional Feynman graph expansions for the six-point and eight-point vertices in terms of the only bare coupling constant – \( \lambda \). In the course of calculations the tensor structure of these vertices,

\[
\Gamma_{\alpha\beta\gamma\delta\mu\nu} = \frac{1}{15}(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\mu\nu} + 14 \text{ transpositions})\Gamma_6.
\] (7)

\[
\Gamma_{\alpha\beta\gamma\delta\mu\rho\sigma} = \frac{1}{105}(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\mu\rho}\delta_{\nu\sigma} + 104 \text{ transpositions})\Gamma_8.
\] (8)
should be taken into account. In its turn, $\lambda$ may be expressed perturbatively as a function of the renormalized coupling constant $g_4$. Substituting corresponding power series for $\lambda$ into original expansions, we can obtain the RG series for $g_6$ and $g_8$. The one-, two-, three- and four-loop contributions to $g_6$ are formed by one, three, 16, and 94 one-particle irreducible Feynman graphs, respectively. Their calculation gives:

$$g_6 = \frac{9}{\pi} \left( \frac{\lambda Z^2}{m} \right)^3 \left[ \frac{n + 26}{27} - \frac{9 n^2 + 340 n + 2324}{162\pi} \left( \frac{\lambda Z^2}{m} \right) \right]$$

$$+ (0.0056289546468 n^3 + 0.28932672886 n^2 + 4.0404241235 n + 16.204286853) \left( \frac{\lambda Z^2}{m} \right)^2$$

$$-(0.001493126 n^4 + 0.09961447 n^3 + 2.152320 n^2 + 18.330704 n + 52.830284) \left( \frac{\lambda Z^2}{m} \right)^3. \quad (9)$$

The perturbative expansion for $\lambda$ emerges directly from the normalizing condition $\lambda = m Z^2 Z^{-2} g_4$ and the known series for $Z_4$ [6]:

$$Z_4 = 1 + \frac{n + 8}{2\pi} g_4 + \frac{3 n^2 + 38 n + 148}{12\pi^2} g_4^2$$

$$+ (0.0040314418 n^3 + 0.0679416657 n^2 + 0.466356233 n + 1.24038484) g_4^3. \quad (10)$$

Combining these expressions, we obtain

$$g_6 = \frac{9}{\pi} g_4^3 \left[ \frac{n + 26}{27} - \frac{17 n + 226}{81\pi} g_4 + (0.000999164 n^2 + 0.14768927 n + 1.24127452) g_4^2 \right]$$

$$-(0.00000949 n^3 + 0.00783129 n^2 + 0.34565683 n + 2.14825455) g_4^3. \quad (11)$$

In the case of $g_8$, the one-, two-, and three-loop contributions are given by one, five, and 36 Feynman graphs, respectively. Corresponding ”bare” and renormalized perturbative expansions are found to be:

$$g_8 = -\frac{81}{2\pi} \left( \frac{\lambda Z^2}{m} \right)^4 \left[ \frac{n + 80}{81} - \frac{405 n^2 + 35626 n + 342320}{1312\pi} \left( \frac{\lambda Z^2}{m} \right) \right]$$

$$+ (0.0046907955 n^3 + 0.463650683 n^2 + 8.8681653 n + 45.4769028) \left( \frac{\lambda Z^2}{m} \right)^2. \quad (12)$$

$$g_8 = -\frac{81}{2\pi} g_4^4 \left[ \frac{n + 80}{81} - \frac{81 n^2 + 7114 n + 134960}{1312\pi} g_4 \right]$$

$$+(0.00943497 n^2 + 0.60941312 n + 7.15615323) g_4^2. \quad (13)$$

In Sec.III, the series Eqs. [11], [13] will be used for estimation of the universal numbers $g_6^*$ and $g_8^*$. 
III. RESUMMATION AND NUMERICAL ESTIMATES

Being a field-theoretical perturbative expansions the series of equations (11), (13) have factorially growing coefficients, i.e., they are divergent (asymptotic). Hence, direct substitution of the fixed point value $g_4^*$ into them would not lead to satisfactory results. To get reasonable numerical estimates for $g_6^*$ and $g_8^*$, some procedure making these expansions convergent should be applied. As is well known, the Borel-Leroy transformation

$$f(x) = \sum_{i=0}^{\infty} c_i x^i = \int_0^\infty t^b e^{-t} F(x t) dt, \quad F(y) = \sum_{i=0}^{\infty} \frac{c_i}{(i+b)!} y^i,$$  \hspace{1cm} (14)$$
diminishing the coefficients by the factor $(i+b)!$, can play a role of such a procedure. Since the RG series considered turns out to be alternating the analytical continuation of the Borel transform may be then performed by using Padé approximants.

Let us discuss first the estimation of the sextic coupling constant $g_6^*$. With the four-loop expansion (11) in hand, we can construct, in principle, three different Padé approximants: $[2/1], [1/2]$, and $[0/3]$. To obtain proper approximation schemes, however, only diagonal $[L/L]$ and near-diagonal Padé approximants should be employed \cite{33}. That is why, further, when estimating $g_6^*$ we limit ourselves with approximants $[2/1]$ and $[1/2]$. Moreover, the diagonal Padé approximant $[1/1]$ will be also dealt with, although this corresponds, in fact, to the usage of the lower-order, three-loop RG approximation.

The algorithm of estimating $g_6^*$ we use here is as follows. Since the Taylor expansion for the free energy contains as coefficients the ratios $R_{2k} = g_{2k}/g_{k-1}^*$ rather than the renormalized coupling constants themselves, we work with the RG series for $R_6$. It is resummed in three different ways based on the Borel-Leroy transformation and the Padé approximants just mentioned. The Borel-Leroy integral is evaluated as a function of the parameter $b$ under $g_4 = g_4^*$. For the fixed point coordinate $g_4^*$, the values given by the resummed six-loop RG expansion for the $\beta$-function are adopted \cite{6,7}, which are believed to be the most accurate estimates available today. The optimal value of $b$ providing the fastest convergence of the iteration scheme is then determined. It is deduced from the condition that the Padé approximants employed should give, for $b = b_{opt}$, the values of $R_6^*$ which are as close as possible to each other. Finally, the average over three estimates for $R_6^*$ is found and claimed to be a numerical value of this universal ratio.

To obtain an idea about how such a procedure works, let us use Table I, where the results of corresponding calculations for $n = 1, 3, 10$ are presented. It is seen that for $n = 1$ and 3, $b_{opt}$, providing a coincidence of the estimates given by all three working Padé approximants, is equal to 1.24. For $n = 10$, $b_{opt}$, fixed by the approximants $[1/1]$ and $[2/1]$, is equal to 1, whereas the third approximant $[1/2]$ at $b = 1$ is spoiled by a positive axis pole. Nevertheless, the numerical estimate given by this approximant under the nearest ”safe” (integer) value of $b (b = 2)$ turns out to be very close to that predicted by the pole free approximants for $b_{opt}$. Moreover, as is seen from Table I, with increasing $n$ numerical estimates for $g_6^*$ become less dependent on $b$, i.e., their sensitivity to the type of resummation decreases. This is not surprising. The point is that the RG expansion (11) becomes less divergent when $n$ grows up. To make this property obvious, let us replace $g_4$ in Eq.(11) by the effective coupling constant

$$g = \frac{n+8}{2\pi} g_4,$$  \hspace{1cm} (15)$$
that is known to be only weakly dependent on $n$: it varies from 1.415 to 1 when $n$ goes from 1 to infinity \cite{34,35}. Then we obtain
\[ g_6 = \frac{8\pi^2(n+26)}{3(n+8)^3} g^3 \left[ 1 - \frac{2(17n + 226)}{3(n+8)(n+26)} g + \frac{1.065025n^2 + 157.42454n + 1323.09596}{(n+8)^2(n+26)} g^2 \right. \\
- \left. \frac{-0.0638n^3 + 52.4510n^2 + 2314.9897n + 14387.6460}{(n+8)^3(n+26)} g^3 \right]. \quad (16) \]

One can see now that all the terms in the RG expansion for \( g_6 \) (in square brackets), apart from the first one, decrease monotonically when \( n \to \infty \). This implies that the larger \( n \) is the smaller the contribution of the higher-order terms and, correspondingly, the better the approximating properties of this series.

This conclusion is definitely confirmed by Table II. It contains numerical estimates for \( g_6^* \) resulting from the four-loop RG expansion resummed by the Padé-Borel-Leroy technique described above (column 3) and their analogs given by the Padé-Borel resummed three-loop RG series \([7]\) (column 4). As is seen, with increasing \( n \) the difference between the four-loop and three-loop estimates rapidly diminishes. Being small (0.9\% \( \) even for \( n = 1 \), it becomes negligible at \( n = 10 \) and practically disappears for \( n \geq 14 \).

How close to the exact values of \( g_6^* \) may the numbers in column 3 be? To clear up this point, let us compare our four-loop estimate for \( R_6^* \) at \( n = 1 \) with those obtained recently by the Padé-Borel-Leroy technique using the diagonal Padé approximant \([1/1]\). Other Padé approximants, \([0/2]\) and \([0/1]\), are ignored, since they turn out to lead to quite unsatisfactory numerical results.

In order to estimate \( g_6^*(n) \), we resum the RG expansion for \( g_8 \) by the Padé-Borel-Leroy technique using the diagonal Padé approximant \([1/1]\). Other Padé approximants, \([0/2]\) and \([0/1]\), are ignored, since they turn out to lead to quite unsatisfactory numerical results.

Dealing with a single Padé approximant, in some condition we need to fix the optimal value
of the shift parameter $b$. For the three-dimensional Ising model the estimate $g^*_8 = 0.825$ was recently found [19]. This number has been extracted from the five-loop RG expansion, and may be considered the most accurate known up to the present. It is natural therefore to tune, by proper choice of $b$, a numerical value of $g^*_8(1)$ given by the resummed three-loop RG series with the best estimate available. Such a procedure leads to $b_{opt} = 40$, and this number is adopted as optimal in the course of evaluation of $g^*_8$ for arbitrary $n$.

The results of our calculations are collected in Table III, where the estimates for $g^*_8(n)$, obtained by a constrained analysis of the $\epsilon$-expansion [23], by approximate solution of the exact RG equations [11], and given by the $1/n$-expansion technique, are also presented for comparison. As seen, for $n \geq 8$ the numbers originating from two field-theoretical approaches – $g$-expansion in three dimensions and $\epsilon$-expansion – agree quite well. However, for smaller $n$, especially for $n = 2$, differences between them turn out to be rather large. This is not surprising since overly short perturbative expansions for $g_8$ are available both in $3$ and $4 - \epsilon$ dimensions and they demonstrate a strong divergence preventing accurate numerical estimates from being obtained. At the same time, our three-loop RG estimates are believed to be closer to the true critical values of $g_8$ than those given by the $\epsilon$-expansion, because in three dimensions we have longer perturbative series. A fair agreement between our results and the numbers emerging from the exact RG equations (see Table III) may be considered as an argument in favor of this belief.

IV. CONCLUSION

To summarize, we have calculated the RG expansions for renormalized coupling constants $g_6$ and $g_8$ of the 3D $n$-vector model in four-loop and three-loop orders, respectively. Resummation of the RG series by the Padé-Borel-Leroy method has enabled us to obtain numerical estimates for the universal critical values of these quantities for arbitrary $n$. Having analyzed the sensitivity of the RG estimates for $g^*_6$ to the type of resummation procedure and a character of their dependence on the order of the RG approximation, an apparent accuracy of these numbers has been argued to be no worse than 0.3%. Numerical estimates for $g^*_8$ turned out to be much less accurate because of a smaller length and stronger divergence of the RG expansion obtained. They were found to be consistent, in general, with the values of $g^*_8$ deduced from the exact RG equations and, for $n \geq 8$, with those given by a constrained analysis of corresponding $\epsilon$-expansion.

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TABLE I. The values of $g_0^*$ for $n = 1, 3,$ and $10$ obtained by means of the Padé-Borel-Leroy technique for various $b$ within three-loop (approximant [1/1]) and four-loop (approximants [1/2] and [2/1]) RG approximations. The estimates for several values of $b$ in the middle lines are absent because corresponding Padé approximant turns out to be spoiled by a positive axis pole.

| $n = 1$ |  |  |  |  |  |  |  |  
|---------|---|---|---|---|---|---|---|
| $b$     | 0 | 1 | 1.24 | 2 | 3 | 4 | 5 | 7 |
| [1/1]   | 1.576 | 1.604 | 1.6089 | 1.621 | 1.633 | 1.641 | 1.648 | 1.656 |
| [1/2]   | - | - | 1.6084 | 1.600 | 1.595 | 1.592 | 1.590 | 1.587 |
| [2/1]   | 1.639 | 1.613 | 1.6084 | 1.596 | 1.583 | 1.573 | 1.566 | 1.555 |
| $n = 3$ |  |  |  |  |  |  |  |  
| $b$     | 0.937 | 0.949 | 0.95133 | 0.957 | 0.962 | 0.966 | 0.969 | 0.973 |
| [1/2]   | - | - | 0.95133 | 0.948 | 0.946 | 0.944 | 0.944 | 0.942 |
| [2/1]   | 0.964 | 0.953 | 0.95133 | 0.946 | 0.941 | 0.937 | 0.934 | 0.930 |
| $n = 10$|  |  |  |  |  |  |  |  
| $b$     | 0.2338 | 0.23515 | 0.2360 | 0.2366 | 0.2370 | 0.2373 | 0.2377 |
| [1/2]   | - | - | 0.2348 | 0.2346 | 0.2345 | 0.2344 | 0.2342 |
| [2/1]   | 0.2359 | 0.23515 | 0.2346 | 0.2342 | 0.2339 | 0.2337 | 0.2334 |
TABLE II. Our estimates of universal critical values of the renormalized sextic coupling constant for the 3D $n$-vector model (column 3). The fixed point coordinates $g^*$ are taken from Ref.[3] ($1 \leq n \leq 3$) and Ref.[7] ($4 \leq n \leq 40$). The $g^*_6$ estimates extracted earlier from Padé-Borel resummed three-loop RG expansion (column 4), from the exact RG equations (column 5), obtained by the lattice calculations (column 6), and resulting from a constrained analysis of the $\epsilon$-expansions (column 7) are presented for comparison. Column 8 contains the values of $g^*_6$ given by the $1/n$-expansion technique.

| $n$ | $g^*$ | $g^*_6$ [3] | $g^*_6$ [7] | $g^*_6$ [11] | $g^*_6$ [14] | $g^*_6$ [23] | $g^*_6 (1/n)$ |
|-----|-------|-------------|-------------|-------------|-------------|-------------|----------------|
| 1   | 1.415 | 1.608       | 1.622       | 1.52        | 1.92(24)    | 1.609(9)    |                |
| 2   | 1.406 | 1.228       | 1.236       | 1.14        | 1.27(25)    | 1.21(7)     |                |
| 3   | 1.392 | 0.951       | 0.956       | 0.88        | 0.93(20)    | 0.931(46)   |                |
| 4   | 1.3745| 0.747       | 0.751       | 0.68        | 0.62(15)    | 0.725(29)   | 1.6449         |
| 5   | 1.3565| 0.596       | 0.599       |             |             |             | 1.0528         |
| 6   | 1.3385| 0.483       | 0.485       |             |             |             | 0.7311         |
| 7   | 1.321 | 0.396       | 0.398       |             |             |             | 0.5371         |
| 8   | 1.3045| 0.329       | 0.331       |             | 0.319(4)    |             | 0.4112         |
| 9   | 1.289 | 0.277       | 0.278       |             |             |             | 0.3249         |
| 10  | 1.2745| 0.235       | 0.236       |             |             |             | 0.2632         |
| 12  | 1.2487| 0.174       | 0.175       |             |             |             | 0.1828         |
| 14  | 1.2266| 0.134       | 0.134       |             |             |             | 0.1343         |
| 16  | 1.2077| 0.105       | 0.105       |             | 0.1032(4)   |             | 0.1028         |
| 18  | 1.1914| 0.0845      | 0.0847      |             |             |             | 0.0812         |
| 20  | 1.1773| 0.0693      | 0.0694      |             |             |             | 0.0658         |
| 24  | 1.1542| 0.0487      | 0.0488      |             |             |             | 0.0457         |
| 28  | 1.1361| 0.0360      | 0.0361      |             |             |             | 0.0336         |
| 32  | 1.1218| 0.0276      | 0.0276      |             | 0.0275(1)   |             | 0.0257         |
| 36  | 1.1099| 0.0218      | 0.0218      |             |             |             | 0.0203         |
| 40  | 1.1003| 0.0176      | 0.0176      |             |             |             | 0.0164         |
TABLE III. Three-loop RG estimates of universal critical values of the renormalized octic coupling constant $g_8$ (column 2). The $g_8^*$ estimates resulting from a constrained analysis of the $\epsilon$-expansion (column 3), from the exact RG equations (column 4) and given by the $1/n$-expansion technique (column 5) are presented for comparison.

| $n$ | $g_8^*$ | $g_8^* [23]$ | $g_8^* [11]$ | $g_8^* (1/n)$ |
|-----|---------|--------------|--------------|--------------|
| 1   | 0.825   | 0.82(9)      | 0.721        |              |
| 2   | 0.388   | 0.83(31)     | 0.343        |              |
| 3   | 0.168   | 0.36(17)     | 0.145        |              |
| 4   | 0.057   | 0.15(13)     | 0.042        | -2.151       |
| 6   | -0.021  |              |              | -0.834       |
| 8   | -0.034  | -0.03(2)     |              | -0.0388      |
| 16  | -0.014  | -0.015(2)    |              | -0.0456      |
| 32  | -0.0023 | -0.0023(1)   |              | -0.00395     |
| 48  | -0.00062| -0.00061(2)  |              | -0.00087     |
| 64  | -0.00023|              |              | -0.00029     |
| 100 | -0.000046| -0.000044(2) | -0.000049    | -0.000052    |